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Annales mathématiques Blaise Pascal, tome 2, n° 2 (1995), p. 15-23

<http://www.numdam.org/item?id=AMBP_1995__2_2_15_0>
ABSOLUTE VALUES ON ALGEBRAS $H(D)$

by Kamal Boussaf and Alain Escassut

Abstract. Let $K$ be an algebraically closed complete ultrametric field, and let $D$ be an infraconnected set in $K$ such that the set $H(D)$ of the analytic elements on $D$ is a ring. Among the continuous multiplicative semi-norms on $H(D)$, we look for the ones that are absolute values. They are characterized by the location of the $T$-filters on $D$. Besides, we characterize the sets $D$ such that $H(D)$ admits at least one continuous absolute value $|\cdot|$. 

Notations: Let $K$ be an algebraically closed field complete for an ultrametric absolute value.

Given $a \in K$ and $r > 0$, $d(a, r)$ (resp. $d(a, r^-)$, resp. $C(a, r)$) denotes the disk $\{x \in K \mid |x - a| \leq r\}$ (resp. $\{x \in K \mid |x - a| < r\}$, resp. the circle $\{x \in K \mid |x - a| = r\}$).

Given $a \in K$, $r' > 0$ and $r'' > r'$, $\Gamma(a, r', r'')$ denotes the annulus $\{x \in K \mid r' < |x - a| < r''\}$.

Given a set $A$ in $K$ and a point $a \in K$, we denote by $\delta(a, A)$ the distance from $a$ to $A$. Let $E$ be an infinite set in $K$, and let $a \in E$. If $E$ is bounded of diameter $r$, we denote by $\tilde{E}$ the disk $d(a, r)$, and if $E$ is not bounded, we put $\tilde{E} = K$. Then, $\tilde{E} \setminus E$ is known to admit a partition of the form $(d(a_i, r_i^-))_{i \in J}$, with $r_i = \delta(a_i, D)$ for each $i \in J$. The disks $d(a_i, r_i^-)_{i \in J}$, are named the holes of $E$.

$R(E)$ denotes the set of rational functions $h \in K(x)$ with no poles in $E$. This is a $K$-subalgebra of the algebra $K^D$ of all functions from $E$ into $K$. Then $R(E)$ is provided with the topology $\mathcal{U}_E$ of uniform convergence on $E$, and is a topological group for this topology. $H(E)$ denotes the completion of $R(E)$ for this topology and its elements are named the analytic elements on $E$ [1], [2], [3], [9].

By [3], we remember that $H(E)$ is a $K$-subalgebra of the algebra $K^D$ if and only if $E$ satisfies the following conditions:

A) $\tilde{E} \setminus E$ is bounded,

B) $E \setminus E \subset \tilde{E}$. 
Henceforth, $D$ will denote an infraconnected set satisfying Conditions A) and B).

In [7], [4], the continuous multiplicative semi-norms of an algebra $H(D)$ were characterized by means of the circular filters on $D$. So we have to recall the definitions of monotonous and circular filters.

**Definitions and notations:** The set of those multiplicative semi-norms that are continuous with respect to the topology of uniform convergence on $D$ is denoted by $\text{Mult}(H(D), \mathcal{U}_D)$. Given a continuous multiplicative semi-norm $\psi \in \text{Mult}(H(D), \mathcal{U}_D)$ we denote by $\text{Ker}\psi$ the closed prime ideal of the $f \in H(D)$ such that $\psi(f) = 0$.

$\psi$ will be said to be *punctual* if $\text{Ker}\psi$ is a maximal ideal of codimension 1 of $H(D)$. We know that there exists a bijection $M$ from $D$ onto the set of maximal ideals of codimension 1 of $H(D)$, defined as $M(a) = \{ f \in H(D) | f(a) = 0 \}$ (indeed, this was shown in [3], Proposition II.6, when $D$ is closed and bounded, and it is easily extended to all sets $D$ satisfying Conditions A) and B)). As a consequence, there exists a bijection $S$ from $D$ onto the set of punctual continuous multiplicative semi-norms of $H(D)$ defined as $S(a)(f) = |f(a)|$, whenever $f \in H(D)$.

In order to recall the characterization of the continuous multiplicative semi-norms of $H(D)$, we first have to recall the definition of monotonous and circular filters.

Given a filter $\mathcal{F}$ on $D$, we will denote by $\mathcal{I}(\mathcal{F})$ the ideal of the $f \in H(D)$ such that $\lim_{x \in \mathcal{F}} f(x) = 0$.

Let $a \in \tilde{D}$ and $S \in \mathbb{R}^*_+$ be such that $\Gamma(a, r, S) \cap D \neq \emptyset$ whenever $r \in [0, S]$ (resp. $\Gamma(a, S, r) \cap D \neq \emptyset$ whenever $r > S$). We call an increasing (resp. a decreasing) filter of center $a$ and diameter $S$ on $D$ the filter $\mathcal{F}$ on $D$ that admits for base the family of sets $\Gamma(a, r, S) \cap D$ (resp. $\Gamma(a, S, r) \cap D$). For every sequence $(r_n)_{n \in \mathbb{N}}$ such that $r_n < r_{n+1}$ (resp. $r_n > r_{n+1}$) and $\lim_{n \to \infty} r_n = S$, it is seen that the sequence $\Gamma(a, r_n, S) \cap D$ (resp. $\Gamma(a, S, r_n) \cap D$) is a base of $\mathcal{F}$ and such a base is called a canonical base.

We call a decreasing filter with no center of canonical base $(D_n)_{n \in \mathbb{N}}$ and diameter $S > 0$, on $D$ a filter $\mathcal{F}$ on $D$ that admits for base a sequence $(D_n)_{n \in \mathbb{N}}$ of the form $D_n = d(a_n, r_n) \cap D$ with $D_{n+1} \subset D_n$, $r_{n+1} < r_n$, $\lim_{n \to \infty} r_n = S$, and $\bigcap_{n \in \mathbb{N}} d(a_n, r_n) = \emptyset$.

Given an increasing (resp. a decreasing) filter $\mathcal{F}$ on $D$ of center $a$ and diameter $r$, we will denote by $\mathcal{P}(\mathcal{F})$ the set $\{ x \in D | x - a \geq r \}$ (resp. the set $\{ x \in D | |x - a| \leq r \}$ and by $\mathcal{C}(\mathcal{F})$ the set $\{ x \in D | x - a < r \}$ (resp. the set $\{ x \in D | |x - a| > r \}$). Besides $\mathcal{C}(\mathcal{F})$ will be named the body of $\mathcal{F}$ and $\mathcal{P}(\mathcal{F})$ will be named the beach of $\mathcal{F}$.

We call a monotonous filter on $D$ a filter which is either an increasing filter or a decreasing filter (with or without a center).

Given a monotonous filter $\mathcal{F}$ we will denote by $\text{diam}(\mathcal{F})$ its diameter.
The field $K$ is said to be spherically complete if every decreasing filter on $K$ has a center in $K$. (The field $\mathbb{C}_p$ for example is not spherically complete). However, every algebraically closed complete ultrametric field admits a spherically complete algebraically closed extension [10], [11].

Two monotonous filters $\mathcal{F}$ and $\mathcal{G}$ are said to be complementary if $\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G}) = D$. Let $\mathcal{F}$ be an increasing (resp. a decreasing) filter of center $a$ and diameter $S$ on $D$. $\mathcal{F}$ is said to be pierced if for every $r \in ]0, S[,$ (resp. $r < S$), $\Gamma(a, r, S)$ (resp. $\Gamma(a, S, r)$) contains some hole $T_m$ of $D$. A decreasing filter with no center $\mathcal{F}$, and canonical base $(D_n)_{n \in \mathbb{N}}$, on $D$ is said to be pierced if for every $m \in \mathbb{N}$, $\overline{D}_m \setminus \overline{D}_{m+1}$ contains some hole $T_m$ of $D$.

Let $a \in \overline{D}$, let $\rho = \delta(a, D)$ be such that $\rho \leq S \leq \text{diam}(D)$. We call circular filter of center $a$ and diameter $S$ on $D$ the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a, S), r' < S < r''$, i.e. $\mathcal{F}$ is the filter which admits for base the family of sets of the form $D \cap \left(\bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i)\right)$ with $\alpha_i \in d(a, S), r'_i < S < r''_i$ $(1 \leq i \leq q, \, q \leq \mathbb{N})$.

A decreasing filter with no center, of canonical base $(D_n)_{n \in \mathbb{N}}$ is also called circular filter on $D$ with no center, of canonical base $(D_n)_{n \in \mathbb{N}}$.

Finally the filter of the neighbourhoods of a point $a \in D$ will be called circular filter of the neighbourhoods of $a$ on $D$. It will be also named circular filter of center $a$ and diameter 0.

A circular filter on $D$ will be said to be large if it has diameter different from 0.

Given a circular filter $\mathcal{F}$, its diameter will be denoted by $\text{diam}(\mathcal{F})$.

The set of the circular filters on $D$ will be denoted by $\Phi(D)$.

Now let $\mathcal{F}$ be a circular filter on $D$. By [7], [4], we have the following characterization of continuous multiplicative semi-norms of $H(D)$.

**Theorem 0:** Let $\mathcal{F}$ be a circular filter on $D$. For every $f \in H(D)$, $|f(x)|$ admits a limit along $\mathcal{F}$, and this limit, denoted by $\varphi_{\mathcal{F}}(f)$, defines a continuous multiplicative semi-norm $\varphi_{\mathcal{F}}$ on $H(D)$. Further, the mapping $\Theta$ from $\Phi(D)$ into $\text{Mult}(H(D), \mathcal{U}_D)$ defined as $\Theta(\mathcal{F}) = \varphi_{\mathcal{F}}$ is a bijection.

**Notations:** For convenience, when $\mathcal{F}$ is the circular filter of center $a$ and diameter $r$, we also denote by $\varphi_{a, r}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$.

Here, assuming $H(D)$ to be a $K$-algebra, we study what continuous multiplicative semi-norms of $H(D)$ are norms, i.e. are absolute values on $H(D)$. Of course, this requires $H(D)$ to have no divisors of zero. But then, as a transcendental extension of the field $K$, the field of quotients $L$ of $H(D)$ does admit absolute values extending the
one of $K$. Hence so does $H(D)$. The problem, here, is whether such absolute values are continuous with respect to the topology of $H(D)$, i.e. are defined by circular filters on $D$. So, we will give the condition a circular filter has to satisfy in order that its continuous multiplicative semi-norm be an absolute value, and next, we will characterize the sets $D$ such that at least one of the continuous multiplicative semi-norms is an absolute value.

All this study involves $T$-filters, and now we have to introduce them.

**Definition:** Let $F$ be an increasing (resp. a decreasing) filter on $D$, of center $a$ and diameter $s$. An element $f \in I(F)$ is said to be strictly vanishing along $F$ if there exists $t < s$ (resp. $t > s$) such that $\varphi_{a, r}(f) > 0$ for all $r \in [t, s]$, (resp. $]s, t]$).

Let $F$ be a decreasing filter on $D$, with no center, of diameter $r$, of canonical base $(D_n)_{n \in \mathbb{N}}$, with $D_n = d(a_n, r_n) \cap D$. Then an element $f \in I(F)$ is said to be strictly vanishing along $F$ if there exists $t > s$ such that $\varphi_{a_n, r}(f) > 0$ for all $r \in [r_n, t]$, for every $n \in \mathbb{N}$.

Let $\mathcal{F}$ be a filter on $D$, and let $A \subset D$. $\mathcal{F}$ will be said to be secant with $A$ if for every $F \in \mathcal{F}$, $A \cap F$ is not empty.

$T$-filters are certain pierced monotonous filters satisfying particular properties linked to the holes of $D$, and were defined in [1], [2], [5]. Here we will only use the following characterization:

A monotonous filter $\mathcal{F}$ is a $T$-filter if and only if there exists $f \in H(D)$ strictly vanishing along $\mathcal{F}$.

From [2], [5], [6]. we can easily deduce the following theoretical propositions that will be indispensable.

**Proposition P:** Let $b \in D$, $l > 0$ and let $f \in H(D)$ satisfy $f(b) \neq 0$ and $\varphi_{b, l} = 0$. There exists an increasing $T$-filter $\mathcal{F}$ of center $b$ and diameter $t \in ]0, l]$ such that $f$ is strictly vanishing along $\mathcal{F}$ and satisfies $\varphi_{b, s}(f) > 0$ for every $s \in ]0, t]$.

Let $a \in \tilde{D}$ and let $r, s \in \mathbb{R}$ satisfy $\delta(a, D) \leq r \leq \text{diam}(D)$. If $f$ satisfies $f(x) = 0$ whenever $x \in d(a, r) \cap D$, then $f$ is strictly vanishing along a $T$-filter $\mathcal{F}$ such that $d(a, r) \cap D \subset \mathcal{P}(\mathcal{F})$ and $b \in \mathcal{C}(\mathcal{F})$.

Now, we can characterize absolute values among continuous multiplicative semi-norms.

**Theorem 1:** Let $\mathcal{F}$ be a large circular filter on $D$. Then $\varphi_{\mathcal{F}}$ is not an absolute value if and only if it satisfies one of the following conditions:

a) There exists a $T$-filter $\mathcal{G}$ on $D$ such that $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$,

b) $\mathcal{F}$ is a $T$-filter.

**Proof:** First, suppose that there exists a $T$-filter $\mathcal{G}$ satisfying a). By Lemma 1.6 A of [5], there exists $f \in H(D)$, strictly vanishing along $\mathcal{G}$, equal to 0 in all of $\mathcal{P}(\mathcal{G})$. Hence we have $\varphi_{\mathcal{F}}(f) = 0$ and then $\varphi_{\mathcal{F}}$ is not a norm. Now if $\mathcal{F}$ is a $T$-filter, then there exists
$f \in H(D)$ strictly vanishing along $\mathcal{F}$ and therefore we have $\lim_{x \to x_0} f(x) = 0$, hence $\varphi_x$ is not a norm.

Now we suppose that there exists no $T$-filter $\mathcal{G}$ satisfying a) and that $\mathcal{F}$ is not a $T$-filter, and we suppose that $\varphi_x$ is not a norm. Let $f \in H(D) \setminus \{0\}$ satisfying $\varphi_x(f) = 0$. Let $S = \text{diam}(\mathcal{F})$. Let $b \in D$ be such that $f(b) \neq 0$.

We first assume that $\mathcal{F}$ has a center $a$.

On the first hand, we suppose that $b \in d(a, S)$. Since $\varphi_{a,r}(f) \neq 0$, when $r$ approaches 0 there does exist $s \in ]0, S[$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in ]0, s[$. Hence $f$ is strictly vanishing along an increasing $T$-filter $\mathcal{G}$ of center $b$ and diameter $s$, and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$.

On the second hand, we suppose that $|a - b| > S$. Let $t = |a - b|$. If $\varphi_{b,t}(f) = 0$, there exists $s \in ]0, t]$ such that $\varphi_{b,s}(f) = 0$ and $\varphi_{b,r}(f) \neq 0$ whenever $r \in ]0, s[$, hence $f$ is strictly vanishing along an increasing $T$-filter $\mathcal{G}$ of center $b$ and diameter $s$ and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$. Now we may assume $\varphi_{b,t}(f) \neq 0$. But $\varphi_{b,t}(f) = \varphi_{a,t}(f)$ and therefore there exists $s \in [S, t]$ such that $\varphi_{a,s}(f) = 0$ and $\varphi_{a,r}(f) \neq 0$ whenever $r \in ]s, t]$. Hence $f$ is strictly vanishing along a decreasing $T$-filter $\mathcal{G}$ of center $a$ and diameter $s$, and therefore $\mathcal{F}$ is secant with $\mathcal{P}(\mathcal{G})$.

Now, assume that $\mathcal{F}$ is a decreasing filter with no center. Let $(D_n)_{n \in \mathbb{N}}$ be a canonical base of $\mathcal{F}$, and for each $n \in \mathbb{N}$, let $D_n = d(a_n, r_n) \cap D$ and let $u_n = \varphi_{a_n, r_n}(f)$. If there exists $q \in \mathbb{N}$ such that $u_q = 0$, by Proposition $P$, $D$ admits a $T$-filter $\mathcal{G}$ such that $D_q$ is included in $\mathcal{P}(\mathcal{G})$, and therefore, $\mathcal{F}$ is obviously secant with $\mathcal{P}(\mathcal{G})$. Hence we can assume that $u_n > 0$ for every $n \in \mathbb{N}$. Now, suppose that there exist $q \in \mathbb{N}$ and $r \in [r_{q+1}, r_q]$ such that $\varphi_{a_{q+1}, r_q}(f) = 0$. As we just saw, there exists a $T$-filter $\mathcal{G}$ on $D$ such that the circular filter $\mathcal{F}_q$ is secant with $\mathcal{P}(\mathcal{G})$, and therefore so is $\mathcal{F}$.

Thus, without loss of generality, we can assume that $\varphi_{a_{q+1}, r_q}(f) > 0$ for every $r \in [r_{q+1}, r_q]$, for every $q \in \mathbb{N}$. Hence $f$ is just strictly vanishing along the decreasing filter $\mathcal{F}$ and therefore $\mathcal{F}$ is a $T$-filter. This ends the proof of Theorem 1.

**Corollary a:** All the not punctual continuous multiplicative semi-norms of $H(D)$ are absolute values if and only if $D$ has no $T$-filter.

**Definitions and notations:** Let $\text{inc} T(D)$ (resp. $\text{dec} T(D)$) be the set of increasing (resp. decreasing) $T$-filters on $D$. We will denote by $\preceq$ the relation defined on $\text{inc} T(D)$ (resp. $\text{dec} T(D)$) by $\mathcal{F}_1 \preceq \mathcal{F}_2$ if $\mathcal{C}(\mathcal{F}_2) \subset \mathcal{C}(\mathcal{F}_1)$. This relation is obviously seen to be an order relation on $\text{inc} T(D)$ (resp. $\text{dec} T(D)$).

An increasing (resp. decreasing) $T$-filter $\mathcal{F}$ will be said to be maximal if it is maximal in $\text{inc} T(D)$ (resp. in $\text{dec} T(D)$) with respect to this relation.

We will denote by $\prec$ the strict order associated to $\preceq$ by $\mathcal{F}_1 \prec \mathcal{F}_2$ if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_1 \neq \mathcal{F}_2$.

We will call an ascending chain of increasing (resp. decreasing) $T$-filters a sequence of increasing (resp. decreasing) $T$-filters $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \prec \mathcal{F}_{n+1}$ whenever $n \in \mathbb{N}$.
Let \((F_n)_{n \in \mathbb{N}}\) be an ascending chain of increasing \(T\)-filters. For each \(n \in \mathbb{N}\) let 
\[ r_n = \text{diam}(F_n). \]
Since the sequence \((r_n)_{n \in \mathbb{N}}\) is decreasing, we put 
\[ r = \lim_{n \to \infty} r_n, \]
and then \(r\) will be named the \textit{diameter of the chain}.

We put \(A := \bigcap_{n \in \mathbb{N}} C(F_n)\) and for each \(n \in \mathbb{N}, D_n := C(F_n) \setminus A.\) The sequence \((D_n)_{n \in \mathbb{N}}\) is then a base of a filter \(\mathcal{F}\) on \(D\) of diameter \(r\).

If \(r = 0\), since \(D\) belongs to \(A\), by Condition (B), \(A\) is a point \(a\) of \(D\), hence \(\mathcal{F}\) is the filter of the neighbourhoods of \(a\) in \(D\).

If \(r > 0\), \(\mathcal{F}\) is a decreasing filter on \(D\) of diameter \(r\).

In both cases \(\mathcal{F}\) will be called the \textit{returning filter of the ascending chain} \((F_n)_{n \in \mathbb{N}}\).

Now let \((F_n)_{n \in \mathbb{N}}\) be an ascending chain of decreasing \(T\)-filters and let \(a \in \mathcal{P}(F_n)\) for some \(n \in \mathbb{N}\). The sequence \((r_n)_{n \in \mathbb{N}}\) is an increasing sequence of limit \(r \in ]0, +\infty]\), and \(r\) will be named the \textit{diameter of the chain}. Since \(D\) belongs to \(A\), by Condition (A) we notice that \(r < +\infty\), and then we will call the \textit{returning filter of the ascending chain} \((F_n)_{n \in \mathbb{N}}\) the increasing filter \(\mathcal{F}\) of center \(a\) and diameter \(r\) (it is seen that \(\mathcal{F}\) does not depend on the point \(a \in \mathcal{P}(F_n)\), whenever \(n \in \mathbb{N}\)).

**Lemma 1:** Let \(H(D)\) have no divisors of zero. Then \(\text{inc}\, T(D)\) is totally ordered with respect to the order \(\preceq\).

**Proof:** Suppose that \(\mathcal{F}, \mathcal{G}\) are increasing \(T\)-filters on \(D\) that are not comparable. We put \(A = C(\mathcal{F})\) and \(B = C(\mathcal{G})\). Then \(A, B\) are two disks of \(D\) which satisfy neither \(A \subseteq B\), nor \(B \subseteq A\). Hence we have \(A \cap B = \emptyset\). As a consequence, \(\mathcal{P}(\mathcal{F}) \cup \mathcal{P}(\mathcal{G})\) is equal to \(D\) and then by \([6]\) \(H(D)\) has divisors of zero. Hence this contradicts the hypothesis and ends the proof.

**Corollary b:** Let \(H(D)\) have no divisors of zero, and let \(\mathcal{F}\) (resp. \(\mathcal{G}\)) be an increasing \(T\)-filter on \(D\), of diameter \(r\) (resp. \(s\)). If \(r > s\) then \(\mathcal{F} \prec \mathcal{G}\). If \(r = s\), then \(\mathcal{F} = \mathcal{G}\).

We are now able to characterize the sets \(D\) such that \(H(D)\) admits continuous absolute values. Let us recall the following theorem of \([6]\):

\textit{The algebra} \(H(D)\) \textit{has no divisors of zero if and only if} \(D\) \textit{does not admit two complementary} \(T\)-\textit{filters}.

We will use comparison between filters. Here, a filter \(\mathcal{F}\) will be said \textit{thinner than} a filter \(\mathcal{G}\) every element of \(\mathcal{G}\) belongs to \(\mathcal{F}\).

**Theorem 2:** Let \(H(D)\) have no divisors of zero. Then \(\text{Mult}(H(D), \mathcal{U}_D)\) contains no norm if and only if \(D\) admits an ascending chain of \(T\)-filters \((F_n)_{n \in \mathbb{N}}\) whose returning filter is either a \(T\)-filter or a Cauchy filter.

**Proof:** On the first hand, we suppose that \(D\) admits an ascending chain of \(T\)-filters \((F_n)_{n \in \mathbb{N}}\) whose returning filter \(\mathcal{F}\) is either a \(T\)-filter or a Cauchy filter and we will prove that \(\text{Mult}(H(D), \mathcal{U}_D)\) contains no norm. We denote by \(r\) the diameter
of this ascending chain \( (F_n)_{n \in \mathbb{N}} \). Let \( G \) be a circular filter on \( D \), of diameter \( s > 0 \).

By Theorem 1, we only have to show that either \( G \) is a \( T \)-filter, or \( G \) is secant with the beach of a \( T \)-filter.

First we suppose \( F \) is a Cauchy filter. Then the \( F_n \) are increasing \( T \)-filters. Let \( q \in \mathbb{N} \) be such that \( r_q < s \). Then \( G \) is clearly secant with \( P(F_q) \).

Now we suppose that \( F \) is a \( T \)-filter. Then we just have to consider the case when \( G \) is secant with \( C(F) \) and is not equal to \( F \).

First we suppose \( F \) increasing, of center \( b \) and diameter \( r \). Hence we have \( C(F) = d(b, r^-) \cap D \). Then \( G \) has a diameter \( s \in ]0, r[ \), and then it admits elements \( E \) of diameter \( t \in ]s, r[ \), included in \( d(b, r^-) \cap D \). Since \( E \cap C(F) \neq \emptyset \), given a point \( a \in E \cap C(F) \) we have \( E \subseteq d(a, t) \). Let \( q \in \mathbb{N} \) be such that \( r_q > \max(t, |a - b|) \). Then \( E \) is included in \( d(b, r_q) \) and therefore is included in \( P(F_q) \). Hence \( G \) is secant with \( P(F_q) \).

Finally we suppose \( F \) decreasing, of diameter \( r \). Since \( G \) is secant with \( C(F) \), and is not less thin than \( F \), we have \( r < s \), and there exists \( t \in ]r, s[ \) and \( a \in D \) such that \( G \) is secant with \( (K \setminus d(a, t^-)) \cap D \) while \( F \) is secant with \( d(a, t^-) \cap D \). Let \( q \in \mathbb{N} \) be such that \( r_q < t \) and let \( a_q \) be a center of \( F_q \). Then \( a_q \) belongs to \( d(a, t) \) and we have \( \mathcal{P}(F_q) = (K \setminus d(a, r_q^-)) \cap D \). Hence \( G \) is secant with \( \mathcal{P}(F_q) \) and this finishes showing that \( \text{Mult}(H(D), \mathcal{U}_D) \) contains no norm.

On the second hand, reciprocally, we suppose that \( \text{Mult}(H(D), \mathcal{U}_D) \) contains no norm and we will show that \( D \) admits an ascending chain of \( T \)-filters \( (F_n)_{n \in \mathbb{N}} \) whose returning filter is either a \( T \)-filter or a Cauchy filter.

We denote by \( \mathcal{R}' \) the set of the diameters of the \( F \in \text{incT}(D) \), by \( \mathcal{R}'' \) the set of the diameters of the \( F \in \text{decT}(D) \), and we put \( \mathcal{R} = \mathcal{R}' \cup \mathcal{R}'' \). Since \( H(D) \) has no norm, by Theorem 1 \( \mathcal{R} \) is not empty. Since \( D \) belongs to \( \mathcal{A} \), by Condition A) \( \mathcal{R} \) is obviously bounded. We put \( t = \sup(\mathcal{R}) \). Let \( a \in D \). We will show that \( \text{incT}(D) = \emptyset \).

Indeed, suppose \( \text{incT}(D) = \emptyset \). First let \( D \) be bounded, of diameter \( S \). Any decreasing filter on \( D \) has a diameter \( r < S \), and therefore the circular filter \( G \) of center \( a \) and diameter \( S \) is secant with \( D \), but (of course) is not a \( T \)-filter on \( D \), and is not secant with the beach of any decreasing \( T \)-filter on \( D \). As a consequence, by Theorem 1 \( \varphi_G \) is a norm. Thus we see that \( D \) is not bounded. Then, any circular filter \( G \) of center \( a \) and diameter \( r > t \) is not a \( T \)-filter and is not secant with the beach of any \( T \)-filter. Finally this shows that \( \varphi_G \) is a norm again. Thus we see that \( \text{incT}(D) \) is not empty, and neither is \( \mathcal{R}' \).

Now, we put \( s = \inf(\mathcal{R}') \). First we suppose \( s \in \mathcal{R}' \). Let \( T \in \text{incT}(D) \) satisfy \( \text{diam}(T) = s \), and let \( b \in C(T) \). Then for every \( r \in ]0, s[ \), the circular filter of center \( b \) and diameter \( r \) is not a decreasing \( T \)-filter, and therefore is secant with the beach of a decreasing \( T \)-filter. Hence there exists a decreasing \( T \)-filter \( F \) of center \( b \) and diameter \( \ell \geq s \). But since \( H(D) \) has no divisors of zero, \( F \) is not complementary with \( T \), hence we have \( s \leq \ell \), i.e. \( s \leq \ell < r \). So, we clearly deduce the existence of a sequence of decreasing \( T \)-filters \( (F_n)_{n \in \mathbb{N}} \), such that each one admits \( b \) as a center and has a diameter \( r_n \) satisfying \( r_n < r_{n+1} < s \), \( \lim_{n \to \infty} r_n = s \). Therefore the sequence \( (F_n)_{n \in \mathbb{N}} \)
is an ascending chain of decreasing $T$-filters such that $\bigcap_{n\in\mathbb{N}} \mathcal{P}(\mathcal{F}_n) = \mathcal{C}(T)$, hence the returning filter of the ascending chain $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a $T$-filter.

Now, we suppose $s \notin \mathcal{R}$. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence in $incT(D)$ such that $\lim_{n \to \infty} diam(\mathcal{T}_n) = s$, with $diam(\mathcal{T}_{n+1}) < diam(\mathcal{T}_n)$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we put $A_n = \mathcal{C}(\mathcal{T}_n)$, and $r_n = diam(A_n)$. By Lemma 1 the sequence $\mathcal{C}(\mathcal{T}_n)_{n\in\mathbb{N}}$ is strictly decreasing, and so is the sequence $(A_n)_{n\in\mathbb{N}}$. Besides, the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is an ascending chain of increasing $T$-filters. Obviously each $A_n$ contains a hole $T_n$ of $D$.

If $s = 0$, then we have $\delta(b,T_n) \leq r_n$, and therefore $b$ does not belong to $\overline{D}$, but then, by Condition A) $b$ must belong to $\overline{D}$. Thus, the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is an ascending chain of increasing $T$-filters that converges to $b$, and therefore the returning filter of the ascending chain $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is a Cauchy filter.

Finally, it only remains to consider the case when $s > 0$, with $s \notin \mathcal{R}$. Let $\mathcal{F}$ be the returning filter of the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$. If $\mathcal{F}$ were a $T$-filter, or were secant with the beach of an increasing $T$-filter, this increasing $T$-filter would have a diameter inferior or equal to $s$. Hence $\mathcal{F}$ either is a decreasing $T$-filter or is secant with the beach of a decreasing $T$-filter. Of course, if $\mathcal{F}$ is a $T$-filter, it is just the returning filter of the chain $(\mathcal{T}_n)_{n\in\mathbb{N}}$. Finally if $\mathcal{F}$ is secant with the beach of a decreasing $T$-filter $\mathcal{G}$, then we have $diam(\mathcal{G}) \leq s$ because if $diam(\mathcal{G})$ were strictly superior to $s$, then $\mathcal{G}$ would be complementary to $\mathcal{T}_n$ when $n$ is big enough, and therefore $H(D)$ would have divisors of zero. Hence we have $diam(\mathcal{G}) = s$, and therefore $\mathcal{G}$ is just the returning filter of the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$. This finishes proving that $D$ admits an ascending chain of $T$-filters $(\mathcal{F}_n)_{n\in\mathbb{N}}$ whose returning filter is either a $T$-filter or a Cauchy filter, and this ends the proof of Theorem 2.

REFERENCES

[1] ESCASSUT, A. Algèbres d'éléments analytiques au sens de Krasner dans un corps valué non archimédien complet algébriquement clos, C.R.A.S.Paris, A 270, p.758-761 (1970).

[2] ESCASSUT, A. Algèbres d'éléments analytiques au sens de Krasner dans un corps valué non archimédien complet algébriquement clos, Thèse de Doctorat de spécialité, Faculté des Sciences de Bordeaux, 1970.

[3] ESCASSUT, A. Algèbres d'éléments analytiques en analyse non archimédienne, Indag. math.,t.36, p. 339-351 (1974).

[4] ESCASSUT, A. Elements analytiques et filtres percés sur un ensemble infraconnexe , Ann. Mat. Pura Appl. t.110 p. 335-352 (1976).

[5] ESCASSUT, A. T-filtres, ensembles analytiques et transformation de Fourier
$p$-adique, Ann. Inst. Fourier 25, n 2, p. 45-80, (1975).

[6] ESCASSUT, A. Algèbres de Krasner intègres et noetheriennes, Indag. Math. 38, p. 109-130, (1976).

[7] GARANDEL, G. Les semi-normes multiplicatives sur les algèbres d'éléments analytiques au sens de Krasner, Indag. Math., 37, n4, p.327-341, (1975).

[8] GUENNEBAUD, B. Sur une notion de spectre pour les algèbres normées ultramétriques, thèse Université de Poitiers, (1973).

[9] KRASNER, M. Prolongement analytique uniforme et multiforme dans les corps valués complets. Les tendances géométriques en algèbre et théorie des nombres, Clermont-Ferrand, p.94-141 (1964). Centre National de la Recherche Scientifique (1966) , ( Colloques internationaux de C.N.R.S. Paris , 143).

[10] VAN ROOIJ, A.C.M. Non-Archimedean Functional Analysis, Marcel Decker, inc. (1978).

[11] SCHIKHOF, W.H. Ultrametric calculus. An introduction to $p$-adic analysis, Cambridge University Press (1984).

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