D-Branes and Derived Categories

Yuri Malyuta

Institute for Nuclear Research
National Academy of Sciences of Ukraine
03022 Kiev, Ukraine
e-mail: interdep@kinr.kiev.ua

Abstract

The digest of ideology interpreting D-branes on Calabi-Yau manifolds as objects of the derived category is given.

Keywords: D-branes, Derived category, Triangulated structure,
Monodromy.
1 Introduction

Recently there has been substantial progress \cite{1, 2, 3} in understanding D-branes on Calabi-Yau manifolds in context of derived categories \cite{4}.

The purpose of the present paper is to give the digest of this ideology.

2 Sheaves

In this section we shall introduce the definitions of presheaves and sheaves \cite{5}.

A \textit{presheaf} $F$ over a topological space $X$ is

1) An assignment to each nonempty open set $U \subset X$ of a set $F(U)$ (\textit{sections} of a presheaf $F$);

2) A collection of mappings (called restriction homomorphisms)

$$r_{UV} : F(U) \rightarrow F(V)$$

for each pair of open sets $U$ and $V$ such that $V \subset U$, satisfying

$$r_{UU} = 1_U , \quad r_{VW} r_{UV} = r_{UW} \quad \text{for} \quad W \subset V \subset U .$$

A presheaf $F$ is called a \textit{sheaf} if for every collection $U_i$ of open subsets of $X$ with $U = \bigcup_{i \in I} U_i$ the following axioms hold:

a) If $s, t \in F(U)$ and $r_{UU_i}(s) = r_{UU_i}(t)$ for all $i$, then $s = t$;

b) If $s_i \in F(U_i)$ and if for $U_i \cap U_j \neq \emptyset$ we have

$$r_{U_i, U_i \cap U_j}(s_i) = r_{U_j, U_i \cap U_j}(s_j)$$
for all $i$, then there exists an $s \in F(U)$ such that $r_{U,U_i}(s) = s_i$ for all $i$.

If $F$ and $G$ are presheaves over $X$, then a *morphism* of presheaves $f : F \to G$ is a collection of maps $f(U) : F(U) \to G(U)$, satisfying the relation $r_{UV} f(U) = f(V) r_{UV}$.

Morphisms of sheaves are simply morphisms of the underlying presheaves.

Let $(X, \mathcal{O})$ be a complex manifold. A sheaf $B$ over $X$ is called a *coherent sheaf* of $\mathcal{O}$-modules if for each $x \in X$ there is a neighborhood $U$ of $x$ such that there is an exact sequence of sheaves over $U$,

$$0 \to B|_U \to \mathcal{O}^{\oplus p_1}|_U \to \mathcal{O}^{\oplus p_2}|_U \to \ldots \to \mathcal{O}^{\oplus p_k}|_U \to 0.$$ 

3 Complexes

Let $B^\bullet$ denote a *complex* of coherent sheaves $\mathcal{H}$

$$B^\bullet : \ldots \xrightarrow{d^{i-2}} B^{i-1} \xrightarrow{d^{i-1}} B^{i} \xrightarrow{d^i} B^{i+1} \xrightarrow{d^{i+1}} \ldots ,$$

where $d^i d^{i-1} = 0$.

*Cohomology groups* of the complex $B^\bullet$ are defined as

$$H^i(B^\bullet) = \text{Ker } d^i/\text{Im } d^{i-1}.$$ 

A morphism of complexes $f : B^\bullet \to C^\bullet$ induces a morphism of cohomology groups $H(f) : H^\bullet(B^\bullet) \to H^\bullet(C^\bullet)$.

If $H(f)$ is an isomorphism, the morphism $f$ is said to be a *quasi-isomorphism*.

If morphisms $f$ and $g$ are *homotopy equivalent*, then $H(f) = H(g)$. 

2
4 Categories

In this section we shall give some formal definitions [6].

A category $C$ consists of the following data:

1) A class $\text{Ob } C$ of objects $A, B, C, \ldots$;
2) A family of disjoint sets of morphisms $\text{Hom}(A, B)$, one for each ordered pair $A, B$ of objects;
3) A family of maps

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

one for each ordered triplet $A, B, C$ of objects.

These data obey the axioms:

a) If $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$, then composition of morphisms is associative, that is, $h(gf) = (hg)f$;

b) To each object $B$ there exists a morphism $1_B : B \rightarrow B$ such that $1_Bf = f$, $g1_B = g$ for $f : A \rightarrow B$ and $g : B \rightarrow C$.

An additive category is a category in which each set of morphisms $\text{Hom}(A, B)$ has the structure of an abelian group, subject to the following axioms:

A1 Composition of morphisms is distributive, that is,

$$(g_1 + g_2)f = g_1f + g_2f, \quad h(g_1 + g_2) = hg_1 + hg_2$$

for any $g_1, g_2 : B \rightarrow C$, $f : A \rightarrow B$, $h : C \rightarrow D$;

A2 There is a null object $0$ such that $\text{Hom}(A, 0)$ and $\text{Hom}(0, A)$ consist of one morphism for any $A$;
To each pair of objects $A_1$ and $A_2$ there exists an object $B$ and four morphisms

\[
p_1 \quad p_2
\]
\[
A_1 \iff B \iff A_2
\]
\[
i_1 \quad i_2
\]

which satisfy the identities

\[
p_1 i_1 = 1_{A_1}, \quad p_2 i_2 = 1_{A_2}, \quad i_1 p_1 + i_2 p_2 = 1_B, \quad p_2 i_1 = p_1 i_2 = 0.
\]

An abelian category $\mathcal{A}$ is an additive category which satisfies the additional axiom:

A4 To each morphism $f : A \to B$ there exists the sequence

\[
K \xrightarrow{k} A \xrightarrow{i} I \xrightarrow{j} B \xrightarrow{c} K'
\]

with the properties

a) $ji = f$,

b) $K$ is a kernel of $f$, $K'$ is a cokernel of $f$,

c) $I$ is a cokernel of $k$ and a kernel of $c$.

The category of coherent sheaves is the abelian category $\mathcal{A}$.

5 The derived category

The derived category $D(\mathcal{A})$ is constructed in three steps \[4\] : 1st step. The category of complexes of coherent sheaves $\text{Kom}(\mathcal{A})$ is determined as follows

- $\text{Ob } \text{Kom}(\mathcal{A}) = \{\text{complexes } B^\bullet \text{ of coherent sheaves}\}$,
- $\text{Hom}(B^\bullet, C^\bullet) = \{\text{morphisms of complexes } B^\bullet \to C^\bullet\}$;
2nd step. The homotopy category $\mathcal{K}(\mathcal{A})$ is determined as follows

\[ \text{Ob} \mathcal{K}(\mathcal{A}) = \text{Ob Kom}(\mathcal{A}) , \]
\[ \text{Mor} \mathcal{K}(\mathcal{A}) = \text{Mor} \text{Kom}(\mathcal{A}) \text{ modulo homotopy equivalence} ; \]

3rd step. The derived category $\mathcal{D}(\mathcal{A})$ is determined as follows

\[ \text{Ob} \mathcal{D}(\mathcal{A}) = \text{Ob} \mathcal{K}(\mathcal{A}) , \]
\[ \text{The morphisms of} \mathcal{D}(\mathcal{A}) \text{ are obtained from morphisms in} \]
\[ \text{K}(\mathcal{A}) \text{ by inverting all quasi-isomorphisms}. \]

The derived category $\mathcal{D}(\mathcal{A})$ is the additive category.

6 Triangulated structure

The derived category $\mathcal{D}(\mathcal{A})$ admits a triangulated structure \[^4\] with shift functor $[n]$ defined by

\[ (B[n])^i = B^{n+i} \]

and with a class of distinguished triangles

\[ \begin{array}{ccc}
A & \rightarrow & B \\
\uparrow & & \downarrow \\
C & \rightarrow & C = A[1] \oplus B
\end{array} \]

These data satisfy a number of axioms. The octahedral axiom is an essential ingredient in the study of D-brane stability \[^1\]. The octahedral axiom states that there exists the octahedron consisting of a top cap and a bottom cap:
Interpreting D-branes as vertices of the octahedron, it is possible to describe \textit{D-brane decays}: if \( C \) is stable against decay into \( A \) and \( B \), but that \( B \) itself is unstable with respect to a decay into \( E \) and \( F \), than \( C \) will always be unstable with respect to decay into \( F \) and some bound state \( G \) of \( A \) and \( E \).

7 The quintic

Let \( X \) be the quintic hypersurface in \( CP^4 \). The mirror \( Y \) is defined as the orbifold \( X/Z_5^3 \). In virtue of \textit{mirror symmetry} the \textit{Kähler moduli space} of \( X \) is identified with the \textit{complex structure moduli space} of \( Y \). The complex structure moduli space of \( Y \) is described by the Picard-Fuchs equation

\[
\{ \theta_z^4 + 5z(5\theta_z + 4)(5\theta_z + 3)(5\theta_z + 2)(5\theta_z + 1) \} \omega_k(z) = 0 ,
\]

where \( \theta_z = z \frac{d}{dz} \), the complex variable \( z \) spans the complex structure moduli space of \( Y \).
The Landau-Ginzburg point of the moduli space of $X$ is mirror to $z = \infty$, the large radius limit of $X$ is mirror to $z = 0$, the conifold point of $X$ is mirror to $z = 1$. The periods $\omega_k(z)$ are singular at these three points.

8 Monodromy

Acting on the derived category $D(A)$, the monodromy is induced by a Fourier-Mukai transform associated to some generator $K^\bullet \in D(A)$. The formula for the monodromy action on a complex $B^\bullet$ is

$$B^\bullet \mapsto Rp_{1*}(K^\bullet \overset{L}{\otimes} p_2^*(B^\bullet)) .$$

Geometry associated to this monodromy action is

$$\begin{array}{c}
\bigtriangleup \\
\cap
\end{array}
\begin{array}{c}
X \times X \\
p_1 \\
\leftarrow \downarrow \\
x \\
p_2 \\
\rightarrow X
\end{array}
\begin{array}{c}
\bigtriangleup \subset X \times X
\end{array}$$

where $\bigtriangleup \subset X \times X$ is the diagonal embedding of $X$.

In the formula for the monodromy action, we

1) Take a complex of sheaves $B^\bullet$ on $X$, ”pull it back” to the inverse-image complex of sheaves $p_2^*(B^\bullet)$ on $X \times X$; 

2) Take the tensor-product with the generator $K^\bullet$ and construct the left-derived complex of sheaves; 

3) ”Push-forward” to the direct image complex $p_1(\cdot)$ and construct the right-derived complex of sheaves on $X$. 

7
The most obvious monodromy is that about the Landau-Ginsburg point in the Kähler moduli space of the quintic. This monodromy is generated by \[ K_{LG}^* = 0 \rightarrow \mathcal{O} \boxtimes \mathcal{O}(1) \rightarrow \mathcal{O}_\triangle(1) \rightarrow 0. \]

The monodromy calculations for \( \mathcal{O} \in D(A) \) yield the result

\[
M_{LG}(\mathcal{O}) = 0 \rightarrow \mathcal{O}^{\oplus 5} \rightarrow \mathcal{O}(1) \rightarrow 0
\]

\[
(M_{LG})^2(\mathcal{O}) = 0 \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \mathcal{O}(1)^{\oplus 5} \rightarrow \mathcal{O}(2) \rightarrow 0
\]

\[
(M_{LG})^3(\mathcal{O}) = 0 \rightarrow \mathcal{O}^{\oplus 10} \rightarrow \mathcal{O}(1)^{\oplus 10} \rightarrow \mathcal{O}(2)^{\oplus 5} \rightarrow \mathcal{O}(3) \rightarrow 0
\]

\[
(M_{LG})^4(\mathcal{O}) = \mathcal{O}(-1)[4]
\]

\[
(M_{LG})^5(\mathcal{O}) = \mathcal{O}[2]
\]

9 Boundary linear \( \sigma \)-model

*Boundary linear \( \sigma \)-model* \[2\] is determined by the Lagrangian

\[
L = \sum_n \left( i\beta^{(2n)} \partial_0 \beta^{(2n)} + i\rho^{(2n+1)} \partial_0 \rho^{(2n+1)} + \right.
\]

\[
+ \frac{1}{2} \beta^{(2n)} (|\kappa^{(2n+1)}|^2 c_{2n} c_{2n} + |\kappa^{(2n)}|^2 c_{2n-1} \bar{c}_{2n-1}) \beta^{(2n)} + \right.
\]

\[
+ \frac{1}{2} \rho^{(2n+1)} (|\kappa^{(2n+2)}|^2 c_{2n+1} c_{2n+1} + |\kappa^{(2n+1)}|^2 c_{2n} \bar{c}_{2n}) \rho^{(2n+1)} \right),
\]

which involves superfields \( \beta^{(2n)}, \rho^{(2n+1)}, c_k \).

Consider the complex of direct sums of holomorphic line bundles

\[
\ldots \rightarrow \bigoplus_i \mathcal{O}(m_i^{(2n-1)}) \xrightarrow{c_{2n-1}} \bigoplus_i \mathcal{O}(m_i^{(2n)}) \xrightarrow{c_{2n}} \bigoplus_i \mathcal{O}(m_i^{(2n+1)}) \xrightarrow{c_{2n+1}} \ldots
\]
Sections of holomorphic line bundles describe superfields $\beta^{(2n)}$, $\rho^{(2n+1)}$; differentials describe superfields $c_k$.

**Acknowledgement**

This material was presented in the lecture given by the author at the Institute of Mathematics (Kiev, Ukraine). The author thanks audience for questions and comments.
References

[1] P.S. Aspinwall and M.R. Douglas, *D-brane stability and monodromy*, hep-th/0110071.

[2] J. Distler, H. Jockers and H. Park, *D-brane monodromies, derived categories and boundary linear sigma models*, hep-th/0206242.

[3] P.S. Aspinwall, R.L. Karp and R.P. Horja, *Massless D-branes on Calabi-Yau threefolds and monodromy*, hep-th/0209161.

[4] S.I. Gelfand and Yu.I. Manin, *Homological algebra*, Springer-Verlag, Berlin, 1994.

[5] R.O. Wells, *Differential analysis on complex manifolds*, Springer-Verlag, Berlin, 1980.

[6] S. MacLane, *Categorical algebra*, Bull. Amer. Math. Soc. 71 (1965) 40.

[7] P. Candelas, X.C. de la Ossa, P.S. Green and L. Parkes, *A pair on Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B359 (1991) 21.

[8] M.R. Douglas, *D-branes, categories and N=1 supersymmetry*, hep-th/0011017.