Trions in a periodic potential

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The group-theoretical classification of trion states (charged excitons $X^\pm$) is presented. It is based on considerations of products of irreducible projective representations of the 2D translation group. For a given BvK period $N$ degeneracy of obtained states is $N^2$. Trions $X^\pm$ consist of two identical particles (holes or electrons), so the symmetrization of states with respect to particles transposition is considered. There are $N(N+1)/2$ symmetric and $N(N-1)$ antisymmetric states. Completely antisymmetric states can be constructed by introducing antisymmetric and symmetric spin functions, respectively. Two symmetry adapted bases are considered: the first is obtained from a direct conjugation of three representations, whereas in the second approach the states of a electrically neutral pair hole-electron are determined in the first step. The third possibility, a conjugation of representations corresponding to identical particles in the first step, is postponed for the further investigations.

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I. INTRODUCTION

The quantum Hall effect and high temperature superconductivity have raised interest in properties of the two-dimensional electron gas subjected to electric and magnetic fields. The observation of (negatively) charged excitons has recalled a forty-year old concept of excitons “trions” or “charged excitons” introduced by Lampert in 1958. Recently, such excitons, consisting of two holes and an electron or two electrons and hole (denoted $X^\pm$, respectively), have been investigated both experimentally and theoretically.

In this paper classification based on translational symmetry in the presence of a periodic potential and an external magnetic field is presented. To perform this task the so-called magnetic translation operators, introduced by Brown and Zak, are used. These operators commute with the standard Hamiltonian of an electron in the magnetic field $H = \nabla \times A$ and a periodic potential $V(r)$

$$\mathcal{H} = \frac{1}{2m} \left( p + \frac{e}{c} A \right)^2 + V(r).$$

Brown and Zak’s concepts can be generalized to a local gauge of the vector potential $A$, $N$-dimensional lattices, and a spatially inhomogeneous magnetic field. This paper exploits the fact that after imposing the Born–von Karmán (BvK) periodic conditions the magnetic translations form a finite-dimensional projective representation of the 2D translation group. Kronecker products of irreducible projective representations can be applied to description of multi-particle states.

Considering problems, which involve the magnetic field determined by the vector potential $A$, one has to keep in mind that some results may depend on a chosen gauge, though physical properties should be gauge-independent. Two gauges are most frequently used in description of the 2D electron systems: the Landau gauge with $A = [0, xH, 0]$ and the antisymmetric one with $A = (H \times r)/2$. The relations between these gauges was discussed in the earlier article, so this problem is left out in the present considerations. However, it should be underlined that a form representation matrices depends on chosen gauge and, moreover, obtained representations are inequivalent, what means, among others, that their bases are not related by a unitary transformation. Since it is a symmetry adapted basis what results from presented material, then it is important to stress that the Landau gauge is assumed and obtained results can not be immediately applied to other gauges.

II. IRREDUCIBLE PROJECTIVE REPRESENTATIONS OF THE 2D TRANSLATION GROUP

All finite-dimensional irreducible projective representations of the 2D translation for a given BvK period $N$ are labeled by numbers $n, l, k_1, k_2$, where $n$ is a divisor of $N$, $0 < l < n$ is mutually prime with $n$, and $k = (k_1, k_2)$ labels
irreducible representations of 2D translation group with the BvK period \( N/n \), so \( k_1, k_2 = 0, 1, \ldots, (N/n) - 1 \). In the special case \( n = 1 \) matrix elements are given by the following formula \[ D^j_{jk}[n_1, n_2] = \delta_{j,k-n_2} \omega_N^{l_n j}, \] where \( l \) is mutually prime with \( N, \omega_N = \exp(2\pi i/N) \), \( j, k, n_1, n_2 = 0, 1, \ldots, N - 1 \) (so all expressions are calculated modulo \( N \)), and \([n_1, n_2]\) denotes a vector of the 2D translation group (strictly speaking, their coordinates in the lattice basis \( \{a_1, a_2\} \)). For the sake of simplicity the considerations are limited to this case and the presented results correspond to the limit of high magnetic fields, \( i.e. \) there is no Landau level mixing. The other special case \( n = N \) leads to the standard (vector) representations of the translation group, \( i.e. \)

\[ D^k[n_1, n_2] = \omega_N^{k_1 n_1 + k_2 n_2}. \]

It has been shown in the earlier paper [13] that for a given magnetic field \( \mathbf{H} \) the indices \( l \) and \( n \) are related with the charge of a moving particle. In the considered case it is assumed that the representation \( D^l \) corresponds to a hole, whereas \( D^{-l} \) — to an electron. The vector representation \( D^k \) corresponds to a neutral particle or to an electron-hole pair, what is a case considered here. Hence, trions \( X^\pm \) are related with a Kronecker product of three representations:

\[ D^\pm_l \otimes D^\mp_l \otimes D^{\mp l}. \]

III. TRION STATES AND THEIR SYMMETRIZATION

The trions \( X^\pm \) are charged excitons with the charge \( \pm e \) equal to that of a single hole or electron, so, from the group-theoretical point of view, their states have to transform as vectors of the irreducible projective representation \( D^\pm_l \). Since representations \( D^\pm_l \) are \( N \)-dimensional then the following decomposition is true (see also [13])

\[ D^\pm_l \otimes D^\pm_l \otimes D^{\mp l} = N^2 D^\pm_l. \]

This relation expressed in terms of the basis vectors has a form

\[ |w\rangle_{pq}^{\pm} = \sum_{stu} a^p_{stu,w} |stu\rangle_{\pm \mp \mp}, \]

where \( s, t, u, w, p, q = 0, 1, \ldots, N - 1 \), \( |w\rangle_{\pm} \) is a state of the trion \( X^\pm \), \( |stu\rangle_{\pm \mp \mp} \) is a three-particle state with \( s, t \) labeling states of two holes (electrons), \( u \) — a label of a single electron (hole), and the pair \( p, q \) plays a role of a repetition index. The state \( |w\rangle_\pm \) should behave as a basis vector of the representation \( D^\pm_l \), so

\[ D^\pm_l[n_1, n_2]|w\rangle_\pm = \omega^{\pm n_1(w-n_2)}|w-n_2\rangle_\pm. \]

It is satisfied if \( a^p_{stu,w} = \delta_{s,w+p}\delta_{t,w+q}\delta_{u,w+p+q} \), or

\[ |w\rangle_{pq}^{\pm} = |(w+p)(w+q)(w+p+q)|_{\pm \mp \mp}. \]

Namely, for each pair \( p, q \) the product \( \{\} \) acts on a the ket \( |w\rangle_{pq}^{\pm} \) as follows

\[ D^\pm_l \otimes D^\pm_l \otimes D^{\mp l}[n_1, n_2]|w\rangle_{pq}^{\pm} = D^\pm_l[n_1, n_2]|w+p\rangle D^\pm_l[n_1, n_2]|w+q\rangle D^{\mp l}[n_1, n_2]|w+p+q\rangle = \omega^{\pm n_1(w-n_2)}|w+p-n_2\rangle|w+q-n_2\rangle(w+p+q-n_2)|_{\pm \mp \mp} = \omega^{\pm n_1(w-n_2)}|w-n_2\rangle_{\pm}. \]

For \( p = q \) the obtained states are symmetric with respect to the transposition of identical particles (holes or electrons). In the other cases \( (p \neq q) \) it is easy to form symmetric and antisymmetric combinations

\[ |w\rangle_{pq}^{\pm} = 2^{-1/2}(|w\rangle_{pq}^{\pm} \pm |w\rangle_{qp}^{\pm}), \]

where now \( q > p = 0, 1, \ldots, N - 1 \). Completely antisymmetric states can be constructed by introducing antisymmetric and symmetric spin functions, respectively. Non-interacting trions, therefore, will be describe by a similar Hamiltonian as for free holes (electrons) (\( c.f. \) Eq. [1]) with a modified effective mass and, if necessary, an appropriate potential \( V(r) \). However, the degeneracy of energy levels is \( N^2 \) times larger.

Another form of the irreducible basis can be obtained when one consider at first conjugation of two representations and next conjugation of the resultant representation with the third one. The first step can be done in two ways: (i) two identical representations are conjugated, \( i.e. \) one considers a product \( D^l \otimes D^l \) or \( D^{-l} \otimes D^{-l} \) or (ii) states of a pair hole-electron are determined. The first method is more interesting since it can be used in problems where pairs of identical particles come into play (\( e.g. \) superconducting states). However, the considerations are a bit more complicated since the parity of \( N \) has to be taken into account [14]. Hence, in this paper the second possibility will be investigated, whereas the first is left for the further works.
A. States of a hole-electron pair

To begin with a hole-electron pair (in a general case: a particle-antiparticle pair) is taken into account. Since such a pair has the charge zero, then its behavior in an external magnetic field should be similar (up to effective mass etc.) to that of a non-charged particle. It means that in the algebraic picture ordinary (vector) representations of the translation group should appear. This is confirmed by the brief outlook presented above: projective representations used to labeling of electron and hole states differ in the sign of $l$ only. When both particles move in the same magnetic field then corresponding representations are $D^{+l}$ and $D^{-l}$ (of course, the BvK period $N$ is identical in both cases). In this case matrix elements of these representations are given by (2) and those of their product are used to labeling of electron and hole states differ in the sign of $l$. This is confirmed by the brief outlook presented above: projective representations etc.) to this of a non-charged particle. It means that in the algebraic picture ordinary (vector) representations of the translation group should appear. This is confirmed by the brief outlook presented above: projective representations used to labeling of electron and hole states differ in the sign of $l$ only. When both particles move in the same magnetic field then corresponding representations are $D^{+l}$ and $D^{-l}$ (of course, the BvK period $N$ is identical in both cases).

The product of representations $D^{+l} \otimes D^{-l}$ is a reducible representations which decomposes into irreducible one-dimensional vector representations (3)

\begin{equation}
D^{+l} \otimes D^{-l} = \bigoplus_k D^k.
\end{equation}

There is no need to use a repetition index, because each representations appears only ones. We are looking for such $N^2$ linear combinations

\[ |1\rangle_k = \sum_{s,t} a^k_s |s\rangle_+ |t\rangle_- \]

that each behaves as a basis vector of a given irreducible representations $D^k$. It can be shown that $a^k_s = \delta_{s,-xk} \omega_N^{sk_2}$, where $x$ is the inverse of $l$ modulo $N$ (since $l$ is mutually prime with $N$, then $x$ is well-determined), i.e. $xl = 1 \mod N$. Namely one obtains

\begin{equation}
(D^{+l} \otimes D^{-l})|n_1,n_2\rangle = \sum_s \omega_N^{sk_2} D^{+l}[n_1,n_2]|s\rangle_+ D^{-l}[n_1,n_2]|s-xk_1\rangle_-
= \sum_s \omega_N^{sk_2} \omega_N^{in_1(s-n_2)}|s-n_2\rangle_+ \omega_N^{ln_1(n_2+xk_1-s)}|s-xk_1-n_2\rangle_- \\
= \omega_N^{k_1n_1+k_2n_2} \sum_{s'} \omega_N^{s'k_2}|s'\rangle_+ |s' - xk_1\rangle_- = D^k[n_1,n_2]|1\rangle_k.
\end{equation}

B. Trion states

The results presented in the previous section yield

\[ |1\rangle_k = \sum_s \omega_N^{sk_2}|s(s-xk_1)\rangle_+ , \quad xl = 1 \mod N. \]

Therefore,

\[ D^{+l} \otimes D^{-l} \otimes D^{\pm l} = \bigoplus_k D^k \otimes D^{\pm l} \]

and trion states $|u\rangle^k_{\pm}$, $w = 0,1,\ldots, N-1$, can be written as

\[ |u\rangle^k_{\pm} = \sum_{s,t,u} b_{stu}^{w} |stu\rangle_{++\pm}, \]

where $s,t$ label states of a pair and $u$ — those of the second hole (electron) for a trion $X^{\pm}$. The repetition index $k$ follows from the way in which final states are constructed: at first states $|s\rangle_+$ and $|t\rangle_-$ are conjugated to the states $|1\rangle_k$ according with Eq. (11) and next linear combinations of pairs $|1\rangle_k|u\rangle_\pm$ are considered. Therefore one can write

\[ |u\rangle^k_{\pm} = \sum_{u} c_{u}^{w} |1\rangle_k|u\rangle_\pm. \]
Since for each \( k = (k_1, k_2) \), \( k_1, k_2 = 0, 1, \ldots, N - 1 \) one has \[ D^k \otimes D^{\pm l} = D^{\pm l} \],
then each such product yields states \(|w\rangle_\pm \), but the coefficients \( c^w_u \) depend on \( k \) and are given as

\[ c^w_u = \omega_N^{-w_kz} \delta_{u, w \mp x_k1} \]

where, again, \( x_l = 1 \, \text{mod} \, N \), so

\[ |w\rangle_\pm = \omega_N^{-w_kz} |1\rangle_k |w \mp x_k1\rangle_\pm . \tag{12} \]

Taking into account equations (6) and (3) one obtains

\[
(D^k \otimes D^{\pm l})[n_1, n_2]|w\rangle^k_\pm = \omega_N^{-w_kz} D^k[n_1, n_2]|1\rangle_k D^{\pm l}[n_1, n_2]|w \mp x_k1\rangle_\pm
= \omega_N^{-(w-n_kz)2} \omega_N^{\pm l(n_1(w-n_2))}|1\rangle_k |w \mp x_k1 - n_2\rangle_\pm
= \omega_N^{\pm l(n_1(w-n_2))}|w - n_2\rangle_\pm = D^{\pm l}[n_1, n_2]|w\rangle_\pm .
\]

Equations (11) and (12) lead to the final expression \((x_l = 1 \, \text{mod} \, N, \, N^{-1/2} \, \text{is a normalization factor})\)

\[ |w\rangle^k_\pm = N^{-1/2} \omega_N^{-w_kz} \sum_s \omega_N^{skz} s (s - x_k1)(w \mp x_k1)_{++-} , \tag{13} \]

In such a state there is a kind of symmetry between an electron and a hole forming the neutral pair electron-hole, but there is no symmetry between two holes (electrons) in a trion \( X^\pm \). Since there are \( N^2 \) trion states labeled by \( w \) then it is possible to construct states symmetric and antisymmetric with respect to particles transposition. One of possible ways is presented in Sec. III and it is easy to determined a transformation between the obtained bases.

Let us consider states of a trion \( X^+ \). The results of Sec. III read

\[ |w\rangle^pq_+ = |(w + p)(w + q)(w + p + q)\rangle_{++-} , \tag{14} \]

where the first two indices correspond to hole states and third to a state of an electron. In the above presented formula \( 11 \) holes are labeled by the first and the third indices, whereas the middle one corresponds to an electron. Therefore, to calculate scalar products the order of indices has to be changed in one of these formulae. Having this done one obtains

\[
\kappa_+ \langle w | w \rangle^pq_+ = N^{-1/2} \omega_N^{-p_kz} \sum_s \omega_N^{skz} (s - x_k1)(w - x_k1)(w + p)(w + p + q)(w + q)_{++-}
= N^{-1/2} \omega_N^{-p_kz} \delta_{q+x_k1,0} \tag{15} \]

In the simplest case \( N = 2 \) this formula reads (the unique representation is obtained for \( l = x = 1; \, k = (k_1, k_2) \))

\[
|w\rangle^{00}_+ = 2^{-1/2} \left( |w\rangle^{(0,0)}_+ + |w\rangle^{(1,0)}_+ \right) , \\
|w\rangle^{01}_+ = 2^{-1/2} \left( |w\rangle^{(0,0)}_+ - |w\rangle^{(1,0)}_+ \right) , \\
|w\rangle^{10}_+ = 2^{-1/2} \left( |w\rangle^{(0,1)}_+ + |w\rangle^{(1,1)}_+ \right) , \\
|w\rangle^{11}_+ = 2^{-1/2} \left( |w\rangle^{(0,1)}_+ - |w\rangle^{(1,1)}_+ \right) .
\]

The second and the third formulae can be symmetrized what yields the following expressions

\[
|w\rangle^{01+}_+ = 2^{-1} \left( |w\rangle^{(0,0)}_+ - |w\rangle^{(1,0)}_+ + |w\rangle^{(0,1)}_+ + |w\rangle^{(1,1)}_+ \right) , \\
|w\rangle^{01-}_+ = 2^{-1} \left( |w\rangle^{(0,0)}_+ - |w\rangle^{(1,0)}_+ - |w\rangle^{(0,1)}_+ - |w\rangle^{(1,1)}_+ \right) .
\]
IV. FINAL REMARKS

The presented considerations have shown that free trions should behave in similar way as free electrons or holes. However, due to their internal structure the degeneracy is higher and there are many possibilities to construct states |$w\rangle _{\pm}$, two of which have been discussed above. In these simplified considerations there are no interactions between trions or Landau level mixing and, moreover, the spin or angular momentum numbers. Taking into account spins will allow to construct states completely antisymmetric with respect to the permutational symmetry. Such problem has been discussed lately by Dzyubenko et al. for the case of free trions (i.e. without a periodic potential, so there is no discrete translational symmetry). A sum of indices in the RHS of (14), taking into account signs of charges, is $(2w + p + q) - (w + p + q) = w$, what is equal to the index in the LHS of this equation. This is the same result as presented in , where the total angular momentum projection of a trion equals $(n_1 - m_1) + (n_2 - m_2) - (n_3 - m_3)$, where $(n_j - m_j)$ is the total angular momentum projection for holes ($j = 1, 2$) and an electron ($j = 3$), with $n$ and $m$ being the Landau level and the oscillator quantum numbers, respectively. It is interesting that Dzyubenko et al. obtained their results in the antisymmetric gauge $A = (\mathbf{H} \times \mathbf{r})/2$, whereas in the presented considerations the Landau gauge has been used. It confirms that the physical properties are gauge-independent. On the other hand, the actual form of wave functions is not discussed here, but the relations between representations and their product are taken into account only. These relations are independent of the matrix representations and, similarly, the form of resultant basis is independent of the function form: for a given BvK period $N$ and any gauge irreducible projective representations are $N$-dimensional and their action on basis vectors are similar (up to a factor system).

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