Quasinormal frequencies and thermodynamic quantities for the Lifshitz black holes

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Abstract

We find the connection between thermodynamic quantities and quasinormal frequencies in Lifshitz black holes. It is shown that the globally stable Lifshitz black holes have pure imaginary quasinormal frequencies. We also show that by employing the Maggiore’s method, both the horizon area and the entropy can be quantized for these black holes.

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1 Introduction

The Lifshitz black holes \cite{1, 2, 3, 4, 5, 6, 7, 8} have received considerable attentions since these may provide a model of generalizing AdS/CFT correspondence to non-relativistic condensed matter physics as the Lif/CFT correspondence \cite{9, 10, 11}. Although their asymptotic spacetimes are known to be Lifshitz, it is a non-trivial task to find an analytic solution. One of the known solutions is a four-dimensional topological black hole which is asymptotically Lifshitz with the dynamical exponent $z = 2$ \cite{12}. Analytic black hole solutions with planar horizon were found in the Einstein-scalar-massive vector theory \cite{13} and in the Einstein-scalar-Maxwell theory \cite{14}. Another analytic solution has been recently found in the Lovelock gravity \cite{15}. The $z = 3$ Lifshitz black hole \cite{16} was derived from the new massive gravity (NMG) \cite{17}. Numerical solutions and thermodynamic property of Lifshitz black hole were explored in \cite{18, 19}.

A thermodynamic study is important to understand the Lifshitz black hole because heat capacity and free energy determine the global stability of the Lifshitz black hole. A positive (negative) heat capacity imply thermally stable (unstable) black hole, while a positive (negative) free energy means unfavorable (favorable) configuration in given ensemble. Hence, a black hole with positive heat capacity and negative free energy is considered as a globally stable black hole (GSBH). However, the thermodynamic study on Lifshitz black holes was limited because it was difficult to compute their conserved quantities in Lifshitz spacetimes. Recently, there was a progress on computation of mass and related thermodynamic quantities by using the ADT method \cite{20, 21} and the Euclidean action approach \cite{22}. Concerning the mass of 3D Lifshitz black hole, there is an apparent discrepancy between $M = \frac{7r^4}{8G_3}\ell^4$ obtained from the ADT method \cite{20} and $M = \frac{r^4}{4G_3}\ell^4$ from other methods \cite{22, 23, 24}. Phase transitions between Lifshitz black holes and other configurations were investigated by using on-shell and off-shell free energies \cite{25}.

On the other hand, quasinormal modes (QNMs) of a perturbed field contain important information about the black hole. Their quasinormal frequencies (QNFs) are given by $\omega = \omega_R - i\omega_I$ whose real part represents the perturbation oscillation and whose imaginary part denotes the rate at which this oscillation is damped, because of the presence of black hole horizon. In this sense, one requires $\omega_I > 0$ which is consistent with the stability condition of the black hole. Since in asymptotically AdS (Lifshitz) spacetimes, spacelike infinity acts like a reflecting boundary, Dirichlet, Neumann, or mixed boundary condition have to be imposed there. The QNFs could be obtained by solving the Klein-Gordon equation for a minimally coupled scalar by imposing the boundary conditions: ingoing mode near the horizon and Dirichlet condition at infinity.
Importantly, all known QNFs \[26, 27, 28\] have no real part, which implies that they are purely imaginary. If the purely imaginary frequency \((\omega = -i\omega_I, \omega_I > 0)\) represents an interesting feature of the Lifshitz black hole, it is very curious to explore its connection to thermodynamic property. We note that the imaginary part of QNFs involves the temperature of black hole and a GSBH may provide an analytic form of QNFs.

According to the Hod’s conjecture \[29\], the asymptotic QNFs is related to the quantized black hole area. Identifying the vibrational frequency \(\omega(E)\) with the real part \(\omega_R\), it leads to an area quantization of \(\Delta A_n = 4 \ln[3]t_p^2\) which is not universal for all black holes. For a large damped case, Maggiore \[30\] has proposed that the identification of \(\omega(E)\) with the imaginary part \(\omega_I\) might lead to the Bekenstein universal quantization of \(\Delta A_n = 8\pi t_p^2\) \[31\]. Hence, the analytic computation of QNFs is crucial for extracting an important information on Lifshitz black holes obtained from different gravitational theories.

In this work, we investigate Lifshitz black holes by exploring the connection between thermodynamic property and quasinormal frequencies. It is shown that globally stable Lifshitz black holes provide purely imaginary quasinormal frequencies. We also find that by using the Maggiore’s method, the horizon area and entropy can be equally spaced for these black holes.

The organization of our work is as follows. In section 2, we study the 3D Lifshitz black hole by exploring a connection between its thermodynamic quantities and QNFs, where QNFs are already known in \[26\]. We investigate the 2D Lifshitz black hole by obtaining its purely imaginary QNFs in section 3. In section 4, we study two 4D Lifshitz black holes: one obtained from the Einstein-scalar-massive vector theory and the other from the Einstein-scalar-Mawxwll theory. QNFs of the former black hole were found in \[27, 28\] when replacing a radial coordinate \(r\) by \(1/r\), while we obtain newly QNFs of the latter. Finally, we find the area spectrum of two 4D Lifshitz black holes after reviewing the 3D Lifshitz black hole in \[26\].

## 2 3D Lifshitz black hole

The NMG \[17\] composed of the Einstein gravity with a cosmological constant \(\Lambda\) and higher-order curvature terms is given by

\[
S^{(3)}_{NMG} = - \left[ S^{(3)}_{EH} + S^{(3)}_{HC} \right], \tag{1}
\]

\[
S^{(3)}_{EH} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-G} \left( \mathcal{R} - 2\Lambda \right), \tag{2}
\]

\[
S^{(3)}_{HC} = - \frac{1}{16\pi G_3 m^2} \int d^3x \sqrt{-G} \left( \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} - \frac{3}{8} \mathcal{R}^2 \right), \tag{3}
\]
where $G_3$ is a three-dimensional Newton constant and $\tilde{m}^2$ a parameter with mass dimension 2. We mention that to avoid negative mass and entropy, it is necessary to take “−” sign in the front of $[S_{EH}^{(3)} + S_{HC}^{(3)}]$. The field equation is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} - \frac{1}{2\tilde{m}^2}K_{\mu\nu} = 0,$$

where

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_{\mu}\nabla_{\nu}R - \frac{1}{2}\Box g_{\mu\nu} + 4\nabla_{\mu}R_{\rho\sigma}R^{\rho\sigma} - \frac{3}{2}R_{\mu\rho}R_{\nu\sigma}g_{\rho\sigma} + \frac{3}{8}R_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu}. $$

\[(5)\]

We have to choose $\tilde{m}^2 = -\frac{1}{2\ell^2}$ and $\Lambda = -\frac{13}{2\ell^2}$ to obtain the $z = 3$ Lifshitz black hole solution. Here $\ell$ the curvature radius of Lifshitz spacetimes. Explicitly, we find the Lifshitz black hole solution [16] as

$$ds^2_{3D} = g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{r^2}{\ell^2}\right)^z \left(1 - \frac{M\ell^2}{r^2}\right)dt^2 + \frac{dr^2}{r^2 - M} + r^2 d\phi^2,$$

\[(6)\]

where $M$ is an integration constant related to the mass of black hole. From the condition of $g^{\tau\tau} = 0$, the event horizon is determined to be $r = r_+ = \ell\sqrt{M}$. This line element is invariant under the anisotropic scaling of

$$t \to \lambda^2 t, \quad \phi \to \lambda \phi, \quad r \to \frac{r}{\lambda}, \quad M \to \frac{M}{\lambda^2}. $$

\[(7)\]

For $z = 1$, the ADM mass is determined to be $M = \frac{r^2}{\ell^2}$, while for $z = 3$, the ADM mass is not yet fixed completely.

All thermodynamic quantities were driven by using the Euclidean action approach. Its Hawking temperature ($T_H$), mass ($M$), heat capacity ($C = \frac{dM}{dT_H}$), Bekenstein-Hawking entropy ($S_{BH}$), and Helmholdt free energy ($F = M - T_H S_{BH}$) are given by

$$T_H = \frac{r_+^3}{2\pi \ell^4}, \quad M = \frac{r_+^4}{4G_3\ell^4}, \quad C = \frac{4\pi r_+}{3G_3}, \quad S_{BH} = \frac{2\pi r_+}{G_3}, \quad F = -\frac{3r_+^4}{4G_3\ell^4}.$$ \[(8)\]

At this stage, we mention a global structure of Lifshitz black hole. Its Penrose diagram is figured out to be $\bigcirc$ where a light-like curvature singularity is located at $r = 0$ (top and bottom), while Lifshitz asymptote is at $r = \infty$ (two sides). Generally, a black hole is increasing by absorbing radiations in the heat reservoir, while a black hole is decreasing by Hawking radiation as evaporation process. In studying the phase transition, two important quantities are the heat capacity $C$ which shows thermal stability (instability) for $C > 0(C <$
0) and free energy $F$ which indicates the global stability for $F < 0$. For the case of positive heat capacity and negative free energy, we call it the globally stable black hole (GSBH). Here it is observed from (8) that the Lifshitz black hole belongs to the GSBH because of $C > 0$ and $F < 0$.

In order to make a connection to the QNMs, we consider the minimally coupled scalar described by the Klein-Gordon equation
\[
\Box_{3D} - m^2 \varphi = 0 \tag{9}
\]
in the background of Lifshitz black hole (6) for $z = 3$. Decomposing $\varphi$ with $y = r_+ / r$ as
\[
\varphi(t, y, \phi) = R(y)e^{-i\omega t + i\kappa \phi}, \tag{10}
\]
the radial equation takes the form
\[
R'' + \frac{y^2 - 3}{y(1 - y^2)} R' + \frac{\ell^2}{1 - y^2} \left[ \frac{\omega^2 y^4}{M^3 (1 - y^2)} - \frac{m^2}{y^2} - \frac{\kappa^2}{M\ell^2} \right] R = 0. \tag{11}
\]
Here we note that $r \in [r_+, \infty)$ is mapped inversely to $y \in [1, 0)$. The solution to (11) is given by the confluent Heun (HeunC) functions as
\[
R(y) = C_1 y^{2+\alpha}(1 - y^2)^{\frac{\beta}{2}} \text{HeunC}[0, \alpha, \beta, -\frac{\beta^2}{4}, \frac{\alpha^2}{4} + \frac{\kappa^2}{4M}; y^2] + C_2 y^{2-\alpha}(1 - y^2)^{\frac{\beta}{2}} \text{HeunC}[0, -\alpha, \beta, -\frac{\beta^2}{4}, \frac{\alpha^2}{4} + \frac{\kappa^2}{4M}; y^2], \tag{12}
\]
where $C_1$ and $C_2$ are arbitrary constants and
\[
\alpha = 2 \sqrt{1 + \frac{m^2 \ell^2}{4}} > 2, \quad \beta = -i \frac{\omega \ell}{M^{3/2}} = -i \frac{\omega}{2\pi T_H}. \tag{13}
\]
Requiring the Dirichlet condition at infinity ($y = 0$) leads to $C_2 = 0$ because of $2 - \alpha < 0$. In order to impose the ingoing mode at horizon, one uses the connection formula \[32\]:
\[
\text{HeunC}[0, \alpha, \beta, \gamma, \delta; z] = \frac{\Gamma(\alpha + 1)\Gamma(-\beta)}{\Gamma(1 - \beta + K)\Gamma(\alpha - K)} \text{HeunC}[0, \beta, \alpha, -\gamma, \gamma + \delta; 1 - z] + \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(1 + \beta + S)\Gamma(\alpha - S)} (1 - z)^{-\beta} \text{HeunC}[0, -\beta, \alpha, -\gamma, \gamma + \delta; 1 - z], \tag{14}
\]
where $K$ and $S$ are determined by solving two algebraic equations
\[
K^2 + (1 - \alpha - \beta)K - \alpha - \beta - \epsilon + \frac{\gamma}{2} = 0, \quad S^2 + (1 - \alpha + \beta)S - \alpha - \alpha\beta - \epsilon + \frac{\gamma}{2} = 0
\]
with \( \epsilon = [1 - (\alpha + 1)(\beta + 1)]/2 - \delta \). Near the horizon \((y \to 1)\), using (14), the solution (12) can be written by

\[
R_{y \to 1} = C_1 \left[ \xi_1 (1 - y^2)^{\frac{\beta}{2}} + \xi_2 (1 - y^2)^{-\frac{\beta}{2}} \right],
\]

where \( \xi_1 \) and \( \xi_2 \) are given by

\[
\xi_1 = \frac{\Gamma(1 + \alpha)\Gamma(-\beta)}{\Gamma(\alpha - K)\Gamma(1 - \beta + K)}, \quad \xi_2 = \frac{\Gamma(1 + \alpha)\Gamma(\beta)}{\Gamma(\alpha - S)\Gamma(1 + \beta + S)}.
\]

For the real \( \omega = \omega_R \) in (15), the former (latter) correspond to ingoing mode \( \rightarrow |y = 1 \) (outgoing mode \( \leftarrow |y = 1 \)) because the scalar field \( \varphi \) behaves as

\[
\varphi = \xi_1 e^{-i\omega t} (1 - y^2)^{\frac{\beta}{2}} + \xi_2 e^{-i\omega t} (1 - y^2)^{-\frac{\beta}{2}}
\]

\[
\sim \xi_1 e^{-i\omega \left[ t + \frac{\pi}{4} \ln(1 - y) \right]} + \xi_2 e^{-i\omega \left[ t - \frac{\pi}{4} \ln(1 - y) \right]}
\]

near the horizon. Hence, to obtain the ingoing mode at the horizon, one imposes \( \xi_2 = 0 \) which implies that \( \Gamma(\alpha - S) \to \infty \) or \( \Gamma(1 + \beta + S) \to \infty \). This could be done by requiring

\[
\alpha - S = -n, \quad 1 + \beta + S = -n, \quad n = 0, 1, 2, \cdots
\]

which lead to the same expression of quasinormal frequency.

Finally, one recovers the purely imaginary frequency [26]

\[
\frac{\omega_{\pm}}{4\pi T_H} = -i \left[ -1 - 2n - (4 + m^2 \ell^2)^{1/2} \pm \left( 7 + \frac{3m^2 \ell^2}{2} + \frac{\kappa^2}{2M} + (3 + 6n)(4 + m^2 \ell^2)^{1/2} + 6n(n + 1) \right)^{1/2} \right].
\]

Its asymptotic frequency takes the form

\[
\frac{\omega_{\pm}^\infty}{4\pi T_H} = -i (\sqrt{6} - 2) n,
\]

when choosing “\( \omega_+ \)” because selecting “\( \omega_- \)” leads to unstable quasinormal modes. At this stage, we note that the expression (19) has already appeared in [26]. However, it seems that in deriving this expression, they have made two mistakes: one was to choose \( C_1 = 0 \), instead of \( C_2 = 0 \), and the other was to use a wrong connection formula, instead of a correct one (14). Two mistakes happen to provide a correct expression (19). In comparison to the BTZ quasinormal frequency

\[
\omega_{\pm}^{BTZ} = \pm \frac{\kappa}{\ell} - i 4\pi T_H \left[ n + \frac{1}{2} (1 + \sqrt{1 + m^2 \ell^2}) \right],
\]

the imaginary frequency involves the angular quantum number \( \kappa \) in the Lifshitz black hole.

Consequently, we confirm that the QNFs of 3D Lifshitz black hole, which is a GSBH, are purely imaginary. We check the connection between purely imaginary quasinormal frequency and positive heat capacity and negative free energy for the 3D Lifshitz black hole.
3 2D Lifshitz black hole

It is necessary to study a lower dimensional Lifshitz black hole because the exact computation of QNFs is available for the lower dimensional gravitational system. The only known 2D Lifshitz black hole could be obtained when applying the Achucarro-Ortiz (AO) dimensional reduction to the NMG action \([33]\)

\[
ds^2_{(3)} = g_{ij} dx^i dx^j + \ell^2 \Phi^2 d\theta^2
\]

with the dilaton \(\Phi\). Integrating over \(\theta\) on \(S^1\), the action \([1]\) reduces to the 2D effective dilaton action as \([24]\)

\[
S_{NMG} = -\left[ S_{EH} + S_{HC} \right],
\]

where

\[
S_{EH} = \ell \int d^2x \sqrt{-g} \Phi (R_{(2)} - 2\Lambda),
\]

\[
S_{HC} = -\frac{\ell}{2m^2} \int d^2x \sqrt{-g} \Phi \left[ \frac{1}{4} R_{(2)}^2 + \frac{1}{\Phi} R_{(2)} \nabla^2 \Phi + \frac{2}{\Phi^2} \nabla_i \Phi \nabla^i \nabla^j \Phi - \frac{1}{\Phi^2} (\nabla^2 \Phi)^2 \right].
\]

\(S_{HC}\) contains fourth-order derivatives as the dilatonic kinetic term. It turned out that for \(\Phi = r/\ell\) and \(z = 3\), equations of motion for 2D metric tensor \(g^{ij}\) and dilaton \(\Phi\) admit the 2D Lifshitz black hole solution

\[
ds^2_{2D} = g_{ij} dx^i dx^j = -\left(\frac{r^2}{\ell^2} \right)^3 \left(1 - \frac{M\ell^2}{r^2}\right) dt^2 + \frac{dr^2}{r^2 - M}. \tag{24}
\]

All thermodynamic quantities of 2D Lifshitz black hole are the same as \([8]\) of the 3D Lifshitz black hole because the AO-dimensional reduction preserves all thermodynamic properties of 3D Lifshitz black hole. Hence, the 2D Lifshitz black hole is also a GSBH.

Now we introduce a minimally coupled scalar equation

\[
\left[ \Box_{2D} - m^2 \right] \psi = 0 \tag{25}
\]

in the background of 2D Lifshitz black hole \([24]\) to find the QNMs. Decomposing \(\psi\) with \(y = \ell \sqrt{M}/r = r_+/r\) as

\[
\psi(t, \rho) = \rho(y) e^{-i\omega t}, \tag{26}
\]

the radial equation becomes

\[
\rho'' - \frac{2}{y(1-y^2)} \rho' + \frac{\ell^2}{1-y^2} \left[ \frac{\omega^2 y^4}{M^3 (1-y^2)} - m^2 y^2 \right] \rho = 0, \tag{27}
\]
which is similar to the 3D radial equation (11) with \( \kappa = 0 \) (s-mode). Also we note a coordinate mapping: \( r \in [r_+, \infty) \rightarrow y \in (0, 1] \). Solving the equation (27) leads to the solution which is expressed in terms of the HeunC functions as

\[
\rho(y) = \tilde{C}_1 y^{\frac{3}{2} + \gamma}(1 - y^2)^{\frac{\delta}{2}} \text{HeunC}\left[0, \gamma, \beta, -\frac{\beta^2}{4}, \frac{\gamma^2}{4} + \frac{3}{16}, y^2\right] + \tilde{C}_2 y^{\frac{3}{2} - \gamma}(1 - y^2)^{\frac{\delta}{2}} \text{HeunC}\left[0, -\gamma, \beta, -\frac{\beta^2}{4}, \frac{\gamma^2}{4} + \frac{3}{16}, y^2\right],
\]

(28)

where \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are arbitrary constants and

\[
\gamma = \frac{3}{2} \sqrt{1 + \frac{4m^2\ell^2}{9}} > \frac{3}{2}, \quad \beta = -i \frac{\omega \ell}{M^{3/2}} = -i \frac{\omega}{2\pi T_H}.
\]

(29)

We observe that at infinity \( (y \rightarrow 0) \), imposing the Dirichlet condition leads to \( \tilde{C}_2 = 0 \). In order to obtain the ingoing mode at horizon, we use the connection formula (14). Near the horizon \( (y \rightarrow 1) \), the solution (28) takes the form

\[
\rho_{y \rightarrow 1} = \tilde{C}_1 \left[\tilde{\xi}_1 (1 - y^2)^{\frac{\delta}{2}} + \tilde{\xi}_2 (1 - y^2)^{-\frac{\delta}{2}}\right],
\]

(30)

where the coefficients \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) are given by

\[
\tilde{\xi}_1 = \frac{\Gamma(1 + \gamma)\Gamma(-\beta)}{\Gamma(\gamma - K)\Gamma(1 - \beta + K)}, \quad \tilde{\xi}_2 = \frac{\Gamma(1 + \gamma)\Gamma(\beta)}{\Gamma(\gamma - S)\Gamma(1 + \beta + S)}.
\]

(31)

In these expressions, \( \tilde{K} \) and \( \tilde{S} \) are determined by solving two equations

\[
\tilde{K}^2 + (1 - \gamma - \beta)\tilde{K} - \gamma - \beta - \tilde{\epsilon} + \frac{\tilde{\delta}}{2} = 0,
\]

\[
\tilde{S}^2 + (1 - \gamma + \beta)\tilde{S} - \gamma - \gamma\beta - \tilde{\epsilon} + \frac{\tilde{\delta}}{2} = 0
\]

with \( \tilde{\delta} = -\beta^2/4 \) and \( \tilde{\epsilon} = \frac{1}{2} \left(1 - (\gamma + 1)(\gamma + 1)/2 - \gamma^2/4 - 3/16\right) \).

For the real \( \omega = \omega_R \) in (30), the former (latter) correspond to ingoing mode (outgoing mode) because near the horizon, the scalar field \( \psi \) takes the form

\[
\psi = \tilde{\xi}_1 e^{-i\omega t}(1 - y^2)^{\frac{\delta}{2}} + \tilde{\xi}_2 e^{-i\omega t}(1 - y^2)^{-\frac{\delta}{2}} \sim \tilde{\xi}_1 e^{-i\omega t + \frac{i\omega}{\pi T_H} \ln(1-y)} + \tilde{\xi}_2 e^{-i\omega t - \frac{i\omega}{\pi T_H} \ln(1-y)}.
\]

(32)

To obtain the ingoing mode at the horizon, one has to impose \( \tilde{\xi}_2 = 0 \) which means that \( \Gamma(\gamma - \tilde{S}) \rightarrow \infty \) or \( \Gamma(1 + \beta + \tilde{S}) \rightarrow \infty \). This is achieved by requiring

\[
\gamma - \tilde{S} = -n, \quad 1 + \beta + \tilde{S} = -n, \quad n = 0, 1, 2, \cdots
\]

(33)
which lead to the same QNFs.

Consequently, one obtains the purely imaginary frequency

\[
\frac{\omega_{\pm}}{4\pi T_H} = -i \left[ -1 - 2n - \left( \frac{9}{4} + m^2 \ell^2 \right)^{1/2} \pm \left( \frac{19}{4} + \frac{3m^2 \ell^2}{2} + (3 + 6n) \left( \frac{9}{4} + m^2 \ell^2 \right)^{1/2} + 6n(n+1) \right)^{1/2} \right].
\] (34)

We note that its asymptotic frequency takes the same form as that of 3D Lifshitz black hole in (20)

\[
\frac{\omega_{\pm}^\infty}{4\pi T_H} = -i(\sqrt{6} - 2)n
\] (35)

which shows that 2D Lifshitz black hole is very similar to 3D Lifshitz black hole. In comparison, we introduce the QNFs for the AdS$_2$ black hole \[34\]

\[
\frac{\omega_{AO}}{2\pi T_H} = -i \left( n + 1 + \sqrt{1 + 4m^2 \ell^2} \right)
\] (36)

which is purely imaginary.

As a result, it is shown that a scalar propagating in the 2D Lifshitz spacetimes provides the purely imaginary quasinormal frequency. We find the connection between purely imaginary quasinormal frequency and positive heat capacity and negative free energy for 2D Lifshitz black hole.

4 4D Lifshitz black holes

Up to now, we have found the connection between purely imaginary quasinormal frequency and positive heat capacity and negative free energy for 2D and 3D Lifshitz black holes. In order to confirm this suggested connection, it is necessary to investigate 4D Lifshitz black holes. Two known Lifshitz black holes were obtained from the Einstein-scalar-massive vector theory and the Einstein-scalar-Maxwell theory.

4.1 Einstein-scalar-massive vector theory

First we study the 4D1 Lifshitz black hole obtained from the Einstein-scalar-massive vector theory \[13\]

\[
S_{ESMV} = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left[ R - 2\Lambda - \tilde{m}^2 A_\mu A^\mu - 2(e^{-2\phi} - 1) - \frac{1}{2} e^{-2\phi} F_{\mu\nu} F^{\mu\nu} \right],
\] (37)

where

\[
\Lambda = -\frac{z^2 + z + 4}{2}, \quad \tilde{m}^2 = 2z, \quad F = dA
\] (38)

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with $L^2 = 1$. The Lifshitz black hole solution for $z = 2$ is given by

$$ds_{4D1}^2 = -r^2 f(r) dt^2 + r^2 (dx_1^2 + dx_2^2) + \frac{dr^2}{f(r) r^2},$$

$$e^{-2\phi} = 1 + \frac{r_+^2}{r^2}, \quad A = \frac{f(r) r^2 dt}{\sqrt{2}},$$

(39)

with the metric function

$$f(r) = 1 - \frac{r_+^2}{r^2}.$$  

It is important to note that the above solution could be obtained by replacing $r$ by $1/r$ in the original solution appeared in [1].

Applying the Euclidean action approach to this theory, one finds the thermodynamic quantities

$$T_H = \frac{r_+^2}{2\pi}, \quad M = \frac{V_2 r_+^4}{16\pi G_4}, \quad C = \frac{V_2 r_+^2}{4G_4}, \quad S_{BH} = \frac{r_+^2 V_2}{4G_4}, \quad F = -\frac{V_2 r_+^4}{16\pi G_4},$$

(40)

where $V_2$ is the volume of the transverse directions. The 4D1 Lifshitz black hole is a GSBH because of $C > 0$ and $F < 0$. We note that the thermodynamic quantities in (40) are obtained by replacing $r_+ + 1/r_+$ in the original expressions as

$$T_H = \frac{1}{2\pi r_+^2}, \quad M = \frac{V_2}{16\pi G_4 r_+^2}, \quad C = \frac{V_2}{4G_4 r_+^2}, \quad S_{BH} = \frac{V_2}{4G_4 r_+^2}, \quad F = -\frac{V_2}{16\pi G_4 r_+^4}. $$

(41)

Here, the limit $r_+ \to 0$ of all thermodynamic quantities goes to infinity which indicates that the original radial coordinate is not appropriate for describing thermodynamics of the Lifshitz black hole. In addition, the computation of QNFs has been performed by using the original coordinate appeared in [27], showing four types of QNFs: $\omega_i, i = 1, 2, 3, 4$. This is the reason why we use the new radial coordinate as the inverse of original coordinate.

In order to compute QNFs for the $z = 2$ Lifshitz black hole (39), we consider a massive scalar field given by

$$\left[\Box_{4D1} - m^2\right] \Psi_1 = 0.$$  

(42)

Assuming $\Psi_1 = H_1(r)e^{-i\omega t}e^{-i(k_1 x_1 + k_2 x_2)}$ and introducing $y = r_+/r$, equation (42) becomes

$$H''_1 + \frac{y^2 - 3}{y(1 - y^2)} H'_1 + \frac{y^4 \omega^2 - r_+^2 (r_+^2 m^2 + k^2 y^2)(1 - y^2)}{r_+^4 y^2 (1 - y^2)^2} H_1 = 0,$$

(43)

where $k^2 = k_1^2 + k_2^2$ and the prime (’) denotes the differentiation with respect to $y$. It is found that the solution to Eq.(43) is given by the hypergeometric functions

$$H_1(y) = D_1 y^\beta (1 - y^2)^\beta \, _2F_1\left[a, b, c; y^2\right] + D_2 y^{4 - \alpha} (1 - y^2)^\beta \, _2F_1\left[a - c + 1, b - c + 1, 2 - c; y^2\right],$$

(44)
where \( _2F_1 \) is the hypergeometric function, \( D_1 \) and \( D_2 \) are arbitrary constants, and

\[
\alpha = 2 \left[ 1 + \sqrt{1 + \frac{m^2}{4}} \right] > 4, \quad \beta = -\frac{i\omega}{4\pi T_H}.
\] (45)

In the hypergeometric function, the arguments of \( a, b, \) and \( c \) are given by

\[
a = \beta + \frac{\alpha}{2} - \frac{1}{2} - \frac{1}{2r_+^2} \sqrt{r_+^4 - \omega^2 - r_+^2 k^2},
\] (46)

\[
b = \beta + \frac{\alpha}{2} - \frac{1}{2} + \frac{1}{2r_+^2} \sqrt{r_+^4 - \omega^2 - r_+^2 k^2},
\] (47)

\[
c = \alpha - 1.
\] (48)

At infinity \( (y \to 0) \), we see that \( D_2 \) of (44) should be zero because it corresponds to non-normalizable mode \( (4 - \alpha < 0) \). At the horizon \( (y = 1) \), using the connection formula which connects \( y^2 \) to \( 1 - y^2 \) for the hypergeometric function, the solution (44) becomes

\[
H_1(y) = D_1 y^a \left[ (1 - y^2)^\beta \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} _2F_1 \left[ a, b, a + b - c + 1; 1 - y^2 \right] 
+ (1 - y^2)^{-\beta} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} _2F_1 \left[ c - a, c - b, c - a - b + 1; 1 - y^2 \right] \right].
\] (49)

It is worth to note that at the horizon, the first term in (49) corresponds to the ingoing mode, while the second corresponds to the outgoing mode. The ingoing mode near the horizon \( (y \to 1) \) is obtained by imposing the condition

\[
\frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} = 0,
\] (50)

which implies that

\[
a = -n, \quad b = -n, \quad n = 0, 1, 2, \ldots.
\] (51)

Consequently, we find the purely imaginary quasinormal frequency as

\[
\frac{\omega_{\text{QNM}}}{2\pi T_H} = -i \left[ \frac{(2n + 1) \left( (4n^2 + 4n - 4 - m^2) + \frac{k^2}{2\pi T_H} \right)}{2(4n^2 + 4n - 3 - m^2)} 
+ \frac{\left( (4n^2 + 4n - 2 - m^2) - \frac{k^2}{2\pi T_H} \right) \sqrt{4 + m^2}}{2(4n^2 + 4n - 3 - m^2)} \right]
\] (52)
which is the same expression as those of $\omega_2$ in [27] and $\omega$ in [28]. The $n = 0$ case leads to $\omega_1 > 0$, which means that the 4D1 Lifshitz black hole are stable against the external perturbations. Also, we observe that in the large $n$ limit, $\omega_{4D1}$ becomes

$$\frac{\omega_{4D1}^n}{2\pi T_H} = -in,$$

which will be used to derive the area quantization in section 5.

Finally, we confirm that a scalar propagating in the 4D1 Lifshitz black hole has purely imaginary quasinormal frequency. We find the connection between purely imaginary quasinormal frequency and positive heat capacity and negative free energy for the 4D1 Lifshitz black hole obtained from the Einstein-scalar-massive vector theory.

### 4.2 Einstein-scalar-Maxwell theory

We introduce the action of Einstein-scalar-Maxwell theory [14]

$$S_{ESM} = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} [R - 2\Lambda - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{4} e^{\tilde{\lambda}\phi} F_{\mu\nu} F^{\mu\nu}],$$  \hspace{1cm} (54)

where $\Lambda$ is the cosmological constant and two fields are a massless scalar and a Maxwell field. It admits the 4D2 Lifshitz black hole with $z = 2$ as solution to equations of motion [36]

$$ds^2_{4D2} = L^2 \left[ -r^{2z} f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{2} dx_i^2 \right],$$

$$f(r) = 1 - \frac{r^{z+2}}{r^{z+2}}, \quad e^{\tilde{\lambda}\phi} = \frac{1}{r^4}, \quad \tilde{\lambda}^2 = \frac{4}{z-1},$$

$$F_{rt} = qr^{z+1}, \quad \Lambda = -\frac{(z+1)(z+2)}{2L^2},$$

$$q^2 = 2L^2 (z-1)/(z+2),$$ \hspace{1cm} (55)

where the event horizon is located at $r = r_+$. This line element is invariant under the anisotropic scaling of $t \to \lambda^z t, x_i \to \lambda x_i, r \to r/\lambda$, and $r_+ \to r_+/\lambda$. It is important to note from the last relation of (55) that the charge $q$ is not an independent hair because it is determined by the curvature radius $L$ of Lifshitz black hole and its dynamical exponent $z$, which contrasts to the Reissner-Nordström-AdS black hole. A similar case was found in the charged MTZ black hole [37][38]

The temperature and Bekenstein-Hawking entropy are determined by

$$T_H^z = \left[ \frac{z+2}{4\pi} \right] r_+^z, \quad S_{BH} = \frac{L^2 V_2}{4G_4} r_+^2,$$

(56)
where $V_2$ denotes the volume of two-dimensional spatial directions. Mass, heat capacity, and Helmholtz free energy are obtained by using Euclidean action approach as

$$M^z = \frac{2L^2V_2r_+^{z+2}}{16\pi G_4}, \quad C^z = \frac{dM^z}{dT_H} = \frac{2L^2V_2r_+^2}{4\pi G_4}, \quad F^z = -\frac{zL^2V_2r_+^{z+2}}{16\pi G_4},$$

(57)

where $F^z$ is different from the Gibbs free energy defined by $\tilde{F}^z = \frac{L^2V_2r_+^{z+2}}{16\pi G_4}$. We stress that the 4D Lifshitz black hole is also a GSBH because of $C^z > 0$ and $F^z < 0$.

In order to compute QNFs for the $z = 2$ Lifshitz black hole (55), we consider a massive scalar field given by

$$\square_{4D^2} - m^2 \Psi_2 = 0.$$  \hspace{1cm} (58)

Assuming $\Psi_2 = H_2(r)e^{-\omega t}e^{-i(k_1 x_1 + k_2 x_2)}$ and introducing $y = r_+/r$, equation (58) becomes

$$H_2'' - \frac{3 + y^4}{y(1 - y^4)}H_2' + \frac{y^4\omega^2 - r_+^2(k^2y^2 + r_+^2L^2m^2)(1 - y^4)}{r_+^4y^2(1 - y^4)^2}H_2 = 0,$$

(59)

where $k^2 = k_1^2 + k_2^2$ and the prime (') denotes the differentiation with respect to $y$. Importantly, the solution to (59) is given by the general Heun (HeunG) function as follows:

$$H_2(y) = \tilde{D}_1y^{2+\alpha}(1 + y^2)^{-\frac{\alpha}{2}}(1 - y^2)^{\frac{\beta}{2}}\text{HeunG}\left[-1, \ -\gamma, \ 1 + \frac{\alpha}{2}, \ 1 - \frac{\alpha}{2}; \ 1 + \alpha, \ 1 + \beta; \ y^2\right]$$

$$+ \tilde{D}_2y^{-\alpha}(1 + y^2)^{-\frac{\alpha}{2}}(1 - y^2)^{\frac{\beta}{2}}\text{HeunG}\left[-1, \ -\gamma + 2\alpha\beta, \ 1 - \frac{\alpha}{2}, \ 1 + \alpha, \ 1 + \beta; \ y^2\right],$$

(60)

where $\tilde{D}_1$ and $\tilde{D}_2$ are arbitrary constants, and

$$\alpha = 2\sqrt{1 + \frac{m^2L^2}{4}} > 2, \quad \beta = -i\frac{\omega}{2\pi T_H}, \quad \gamma = (1 + \alpha)\beta + \frac{k^2}{4\pi T_H}.$$  \hspace{1cm} (61)

At infinity ($y = 0$), we see that $\tilde{D}_2$ should be zero because it corresponds to non-normalizable mode $(2 - \alpha < 0)$. In order to derive the asymptotic form of the HeunG function near the horizon ($y \to 1$), we use the formula [39, 40]

$$\text{HeunG}\left[b_1, \ b_2, \ a_1, \ a_2, \ a_3, \ a_4; \ z\right]$$

$$= (1 - z)^{1-a_4}\text{HeunG}\left[b_1, \ b_2 - (a_4 - 1)a_3b_1, \ a_2 - a_4 + 1, \ a_1 - a_4 + 1, \ a_3, \ 2 - a_4; \ z\right]$$

$$= E_1\text{HeunG}\left[1 - b_1, \ -b_2 - a_1a_2, \ a_1, \ a_2, \ 1 + a_1 + a_2 - a_3 - a_4, \ a_4; \ 1 - z\right] +$$

$$(1 - z)^{a_2+a_4-a_1-a_2}E_2\text{HeunG}\left[1 - b_1, \ -b_2 - a_1a_2 - (a_3 + a_4 - a_1 - a_2)(a_3 + a_4 - b_1a_3), \ a_3 + a_4 - a_1, \ a_3 + a_4 - a_2, \ 1 - a_1 - a_2 + a_3 + a_4, \ a_4; \ 1 - z\right].$$  \hspace{1cm} (62)
where $E_1$ and $E_2$ are given by

$$E_1 = \text{HeunG}\left[ b_1, b_2, a_1, a_2, a_3, a_4; 1 \right],$$

$$E_2 = \text{HeunG}\left[ b_1, b_2 - b_1a_3(a_3 + a_4 - a_1 - a_2), a_3 + a_4 - a_1, a_3 + a_4 - a_2, a_3, a_4; 1 \right].$$

Using the formula (62), the solution (60) takes the form near the horizon

$$H_2(y) = \tilde{D}_1 y^{2+\alpha} (1 + y^2)^{\frac{\beta}{2}} \left[ (1 - y^2)^{1 - \frac{\beta}{2}} E_1 + (1 - y^2)^{\frac{\beta}{2}} E_2 \right].$$

(65)

We note that the first term in (65) corresponds to the outgoing mode, while the second corresponds to the ingoing mode. To obtain the ingoing mode for the solution (65) near the horizon ($y \to 1$), we impose $E_1 = 0$ which implies

$$\text{HeunG}\left[ -1, \frac{k^2}{4r_H^2}, 1 + \frac{\alpha}{2} - \beta, 1 + \frac{\alpha}{2} - \beta, 1 + \alpha, 1 - \beta; 1 \right] = 0.$$ (66)

At this stage, we have to mention that a little bit of general Heun function and its connection formula is known, in comparison with the HeunC function and hypergeometric function. Hence, it seems to be a formidable task to obtain QNFs from a general condition of (66). Fortunately, we observe that for $k^2 = 0$ (s-mode), (66) reduces to a condition for the hypergeometric function

$$2F_1\left[ \frac{1}{2} + \frac{\alpha}{4} - \frac{\beta}{2}, \frac{1}{2} + \frac{\alpha}{4} - \frac{\beta}{2}, 1 + \frac{\alpha}{2}; 1 \right] = 0.$$ (67)

In deriving this condition, we used the reduction formula [41]

$$\text{HeunG}\left[ -1, 0, a_1, a_2, a_3, \frac{a_1 + a_2 - a_3 + 1}{2}; z \right] = 2F_1\left[ \frac{a_1}{2}, \frac{a_2}{2}, \frac{1 + a_3}{2}; z^2 \right].$$ (68)

Considering a relation for the hypergeometric function

$$2F_1[c_1, c_2, c_3; 1] = \frac{\Gamma(c_3)\Gamma(c_3 - c_1 - c_2)}{\Gamma(c_3 - c_1)\Gamma(c_3 - c_2)}.$$ (69)

the condition of $2F_1[c_1, c_2, c_3; 1] = 0$ implies that

$$c_3 - c_1 = -n, \quad \text{or} \quad c_3 - c_2 = -n, \quad n = 0, 1, 2, \cdots.$$ (70)

Applying the condition (70) to (67) leads to the purely imaginary quasinormal mode as

$$\frac{\omega_{k^2=0}}{4\pi T_H} = -i\left[ n + \frac{1 + \sqrt{1 + \frac{m^2 k^2}{4}}}{2} \right],$$ (71)
which is similar to (36) of the AdS$_2$ black hole. In the large $n$ limit, the QNFs behave as

\[
\frac{\omega_{\infty}^{2} = 0}{4\pi T_H} = -in,
\]

which will be used to derive the area quantization of 4D2 Lifshitz black hole.

Finally, to confirm the QNFs of $s$-mode, we start with equation (59) with $k^2 = 0$,

\[
\tilde{H}_2'' - \frac{3 + y^4}{y(1 - y^4)} \tilde{H}_2' + \frac{(y^4 \omega^2 - r^4 L^2 m^2)(1 - y^4)}{r^4 y^2 (1 - y^4)^2} \tilde{H}_2 = 0.
\]

(73)

The solution to (73) is given by the hypergeometric functions

\[
\tilde{H}_2(y) = d_1 y^{2 - \alpha} (1 - y^4)^{-\frac{\alpha}{4}} \, _2F_1 \left[ -\frac{\beta}{2} - \frac{\alpha}{4} + \frac{1}{2}, -\frac{\beta}{2} - \frac{\alpha}{4} + \frac{1}{2}; 1 - \frac{\alpha}{2}; y^4 \right]
\]

\[
+ d_2 y^{2 + \alpha} (1 - y^4)^{-\frac{\alpha}{4}} \, _2F_1 \left[ -\frac{\beta}{2} + \frac{\alpha}{4} + \frac{1}{2}, -\frac{\beta}{2} + \frac{\alpha}{4} + \frac{1}{2}; 1 + \frac{\alpha}{2}; y^4 \right]
\]

(74)

The condition of (77) provides us purely imaginary QNFs as

\[
\omega = -in^2 \left[ 4n + 2 + \sqrt{4 + L^2 m^2} \right],
\]

(78)

which is exactly the same form as in (71). Consequently, we have shown that $s$-mode QNFs of 4D2 Lifshitz black hole are purely imaginary. Our question is “Can QNFs of Lifshitz black hole remain purely imaginary form even if $k^2 \neq 0$?” At this stage, we could not answer to this question. However, the observation of QNFs of Lifshitz black holes suggests that $k^2$-term includes as a term in $\omega_I$ with $\omega_R = 0.$
5 Area spectrum of Lifshitz black holes

It was suggested that black holes could provide a test bed for any proposed scheme for quantum theory of gravity. In this direction, Hod has combined the perturbation of black holes with the quantum mechanics and statistical physics to derive the quantum of the black hole area spectrum [29]. For a highly excited Schwarzschild black hole, Hod has used the real part \( \omega_R \) of QNFs to obtain the area quanta of \( \Delta A_n = 4 \ln[3]l_p^2 \). However, it is not consistent with \( \Delta A_n = 8\pi l_p^2 \) which was obtained from the fact that the black hole area is adiabatically invariant by Bekenstein [31]. Kunstatter has shown that the area spectrum is equally spaced for higher dimensional Schwarzschild black holes when using the adiabatically invariant integral [42]

\[
I = \int \frac{dE}{\omega(E)} \rightarrow \int \frac{dM}{\omega_R},
\]

where \((E, \omega)\) are (energy, vibrational frequency) and \((M, \omega_R)\) are (black hole mass, real part of QNFs). On later, Maggiore has proposed that a black hole perturbed by external field is considered as a collection of damped harmonic oscillators [30]. He has regarded \( \omega_0 = \sqrt{\omega_R^2 + \omega_I^2} \) as a physically proper frequency and thus, \( \omega_0 = \omega_I \) was used to derive \( \Delta A_n = 8\pi l_p^2 \) for highly excited QNFs of \( \omega_I \gg \omega_R \) by considering the transition from \( n \) to \( n - 1 \).

Since QNFs of Lifshitz black holes are known to be purely imaginary and all thermodynamic quantities of Lifshitz black holes are known, we could use these to derive area spectrum of Lifshitz black holes by using Maggiore’s method solely. In other words, the Hod’s method is inapplicable to extracting information on Lifshitz black holes. Especially, we have their asymptotic frequencies which are given by (20) for 3D Lifshitz black hole, (53) and (72) for 4D Lifshitz black holes. We review how to derive the quantization of horizon area for the 3D Lifshitz black hole [26]. We compute the adiabatic invariant \( I \)

\[
I = \int \frac{dM}{\omega_c} = \frac{r_+}{2G_3(\sqrt{6} - 2)},
\]

where

\[
\omega_c = i\left[(\omega_+^\infty)_n - (\omega_+^\infty)_{n-1}\right] = 4\pi T_H(\sqrt{6} - 2) = 2(\sqrt{6} - 2)\frac{r_+^3}{l_4}.
\]

Using the Bohr-Sommerfeld quantization condition of \( I \approx n\hbar \) and considering the horizon area \( A = 2\pi r_+ \), the quantized area spectrum is given by

\[
A_n = 4\pi G_3(\sqrt{6} - 2)n\hbar,
\]

which is not the universal area spectrum of \( A_n^u = 8\pi n\hbar \). The entropy spectrum takes the
form

\[ S_{BH}^n = \frac{A_n}{G_3} = 4\pi(\sqrt{6} - 2)n\hbar. \]  

(83)

For 4D1 Lifshitz black hole, the adiabatic invariant \( I \) is given by

\[ I_{4D1} = \int \frac{dM}{\omega_c} = \frac{V_2r_+^2}{4\pi G_4}, \]

(84)

where

\[ \omega_c = i\left((\omega_{4D1}^\infty)_n - (\omega_{4D1}^\infty)_{n-1}\right) = 2\pi T_H = r_+^2. \]  

(85)

Considering the horizon area \( A = V_2r_+^2 \), the quantized area spectrum is given by

\[ A_n = 4\pi G_4 n\hbar, \]

(86)

which is the universal area spectrum of \( A_n^u = 8\pi n\hbar \) with \( G_4 = 2 \). In addition, the entropy spectrum can be obtained by

\[ S_{4D1}^n = \frac{A_n}{4G_4} = \pi n\hbar. \]  

(87)

For other 4D2 Lifshitz black hole with \( L^2 = 1 \), the adiabatic invariant \( I \) takes the form

\[ I_{4D2} = \int \frac{dM}{\omega_c} = \frac{V_2r_+^2}{8\pi G_4}, \]

(88)

where

\[ \omega_c = i\left((\omega_{4D2}^\infty)_{k=0}^\infty - (\omega_{4D2}^\infty)_{k=0}^{n-1}\right) = 4\pi T_H = 4r_+^2. \]  

(89)

Considering the horizon area \( A = V_2r_+^2 \), the quantized area spectrum is given by

\[ A_n = 8\pi G_4 n\hbar, \]

(90)

which is not the universal area spectrum of \( A_n^u = 8\pi n\hbar \) with \( G_4 = 2 \). In this case one finds that the entropy spectrum is given by

\[ S_{4D2}^n = \frac{A_n}{4G_4} = 2\pi n\hbar. \]  

(91)

6 Discussions

First of all, the purely imaginary QNFs show that scalar perturbation has no considerable oscillation stage around the Lifshitz black hole. This implies that the equilibrium is stable and thus, it is difficult to deviate the black hole from its equilibrium configuration. If the Lif/CFT correspondence exists really, the thermalization timescale of the boundary conformal
field theory is given by $\tau = \frac{1}{4\pi T}$ which implies that at high temperature, the field theory time scale is very small, indicating that a perturbation in the boundary conformal field theory is not long-lived and it decreases exponentially to zero.

We regard the purely imaginary QNFs as an interesting feature of the Lifshitz black hole, in comparison to the AdS black hole. In this work, we have connected it to thermodynamic properties of Lifshitz black hole. All heat capacities of Lifshitz black holes are positive and all their free energies are negative, which means that all Lifshitz black holes belong to the GSBH. These globally stable Lifshitz black holes provide purely imaginary QNFs when choosing a scalar perturbation, which means that it is hard to take the black hole out off the equilibrium.

Although we have tested a few of Lifshitz black holes, we suggest that most of Lifshitz black holes provide purely imaginary QNMs even if one uses different physical field (being not a minimally coupled scalar) as a perturbation field.

As a byproduct, we have computed area spectrum of Lifshitz black holes by using the Maggiore’s method. The constant $\gamma$ (or $\sigma$) of $A_n = \gamma n\hbar$ ($S_n = \sigma n\hbar$) is given by $\gamma = \sigma = 4(\sqrt{6} - 2)\pi$ for 3D Lifshitz black hole with $G_3 = 1$, $\gamma = 8\pi$ ($\sigma = \pi$) for 4D1 Lifshitz black hole, and $\gamma = 16\pi$ ($\sigma = 2\pi$) for 4D2 Lifshitz black hole. Even though all are not universal area spectrum, the area quantum spectrum could be derived from QNFs of Lifshitz black holes. Consequently, this result shows that both the horizon area and the entropy can be quantized for the globally stable Lifshitz black holes.

We conclude with a comment on the recent issue related to scale covariant metric [43] which leads to a violation of hyperscaling of the dual field theory [44]. When a hyperscaling exponent $\theta$ is zero, a scale covariant metric reduces to a scale invariant metric of Lifshitz metric. It would be interesting to answer to the question on the possible computation of QNFs of general Lifshitz black holes with scale covariant metric. Unfortunately, we could not obtain an analytic solution to the Klein-Gordon equation in these backgrounds (e.g., Eq.(5.2) of the X. Dong et al. work [43]). However, according to the AdS/CFT correspondence, it is known that the poles of the retarded Green’s function in the momentum space correspond to QNFs [45]. If one computes such poles for the general Lifshitz black holes using the Lif/CFT correspondence, it might shed some light on finding QNFs. This is surely beyond the scope of the present paper, but nevertheless it is worthwhile to be explored in future work.

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