GLOBAL HYPOELIPTICITY OF THE KOHN-LAPLACIAN $\Box_b$ ON PSEUDOCONVEX CR MANIFOLDS

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Abstract. Let $X$ be a complex manifold and $M \subset X$ a compact, smooth, pseudoconvex CR manifold of dimension $2n - 1$. (Here $n \geq 3$ or, in case $n = 2$, it is made the extra assumption that $\bar{\partial}_b$ has closed range on functions.) Assume that there exists a strictly CR-plurisubharmonic function in a neighborhood of $M$ in $X$. In this situation, there are here proved

(i) The global existence of $C^\infty$ solutions to the tangential Cauchy-Riemann operator $\bar{\partial}_b$.

(ii) The global hypoellipticity of the Kohn-Laplacian $\Box_b$, under the additional condition of “weak Property (P)”.

The related result to (ii) for the complex Laplacian $\Box$ in a domain/annulus of $X$ is also obtained (as for $\bar{\partial}$ it was already stated by Kohn and Shaw in [K73] and [Sh85]).

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1. Introduction

Let $M$ be a smooth compact manifold of odd dimension $2n - 1$ in a complex manifold $X$, $\dim_C X = N$ ($n \leq N$). Denote by $T^C M$ the distribution of complex tangent vectors; we assume that $M$ is a CR manifold of hypersurface type (or CR manifold for short), that is, $\dim_C T^C M \equiv n - 1$ and therefore $\dim \left( \frac{T M}{T^C M} \right) \equiv 1$. We decompose the complexified bundle $\mathbb{C}TM$ as the direct sum of the bundles of vector fields of type $(1, 0)$ and $(0, 1)$ respectively, that is, $\mathbb{C}TM = T^{1,0} M \oplus T^{0,1} M$. We choose a local basis $L_1, \ldots, L_{n-1}$ of $(1, 0)$ vector fields,
the basis $L_1, \ldots, L_{n-1}$ of conjugated (0, 1) vector fields, and supplement them by a purely imaginary vector field $T \in TM$ transversal to $T^{1,0}M \oplus T^{0,1}M$ in such a way that a full basis of $\mathbb{C}TM$ is obtained. We denote by $\omega_1, \ldots, \omega_{n-1}, \bar{\omega}_1, \ldots, \bar{\omega}_{n-1}, \gamma$ the dual basis of 1-forms. We define the Levi form $\mathcal{L}_M$ of $M$ as the Hermitian form $\mathcal{L}_M(L, L') := d\gamma(L, L')$ for $L, L' \in T^{1,0}M$. Let $(c_{ij})_{i,j=1,\ldots,n-1}$ be the matrix of $\mathcal{L}_M$ in the above basis; Cartan formula yields

$$[L_i, \bar{L}_j] = c_{ij}T \mod T^{1,0}M \oplus T^{0,1}M.$$ 

We assume, throughout the paper, that $M$ is pseudoconvex, that is, the Levi form is non-negative. We denote by $\mathcal{B}^k$ the space of $(0, k)$-forms (or $k$-forms for short) with smooth coefficients; if $u \in \mathcal{B}^k$, then $u$ is locally expressed as a combination

$$u = \sum_{|J|=k} u_{j\bar{j}} \bar{\omega}_j,$$

of basis forms $\bar{\omega}_j = \bar{\omega}_1 \wedge \ldots \wedge \bar{\omega}_{j_k}$ for ordered indices $j_1 < \ldots < j_k$ with $C^\infty$-coefficients $u_{j\bar{j}}$.

We choose a Riemannian metric $\langle \cdot, \cdot \rangle_z$ at each point $z \in M$ associated to the splitting

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T.$$

If $dV$ is the element of volume for this metric, there is induced a $L_2$-inner product and a norm on $\mathcal{B}^k$ defined by

$$(u, v) = \int_M \langle u, v \rangle_z dV, \quad \|u\|^2 = (u, u).$$

In general, $\|\cdot\|_s$ denotes the Sobolev $H^s$-norm. We also define $L^k_2$ to be the Hilbert space obtained by completing $\mathcal{B}^k$ under the $L_2$-norm.

There is a natural restriction to the $\mathcal{B}^k$’s of the de-Rham exterior complex of derivatives $d$ that we denote $\partial_b : \mathcal{B}^k \to \mathcal{B}^{k+1}$. We denote by $\bar{\partial}_b^* : \mathcal{B}^{k+1} \to \mathcal{B}^k$ the transposed complex of derivatives and define the Kohn-Laplacian by $\square_b := \bar{\partial}_b \partial_b^* + \partial_b^* \bar{\partial}_b$. We call Levi form of a smooth function $\phi$ in $M$, the alternate form $\frac{1}{2}(\partial_b \partial_b^* - \bar{\partial}_b \bar{\partial}_b)\phi$ on $T^{1,0}M \oplus T^{0,1}M$; we denote by $(\phi_{ij})$ the matrix, in the above basis, of the Hermitian form induced over $T^{1,0}M \times T^{0,1}M$.

**Definition 1.1.** Let $M$ be a CR manifold. A smooth function $\lambda$ defined in the neighborhood of $M$ is said to be strictly CR-plurisubharmonic if $\frac{1}{2}(\partial_b \partial_b - \bar{\partial}_b \bar{\partial}_b)\phi + ad\gamma > 0$ for suitable constant $a$.

We remark that if either $M$ is strictly pseudoconvex or $X$ is a Stein manifold, then there always exists a strictly CR-plurisubharmonic function. But it does not always exist in an abstract model $M$ in a complex manifold $X$, (see Grauert’s example in [G]).

We begin our work by stating in Theorem 2.1 of Section 2 a version of the basic Kohn-Morrey-Hörmander estimate with weight adapted to $M$. Once the basic estimate is settled,
there is a well established $L^2$ theory for $\bar{\partial}_b$ over a pseudoconvex CR manifold in presence of a strictly CR-plurisubharmonic function. Further, one can lift the estimate from $L^2$ to $H^s$ along the guidelines of Kohn [K73] to get existence of global $C^\infty$ solutions for $\bar{\partial}_b$. The most general results in this respect have been obtained by Kohn and Nicoara.

- In [K86] Theorem 5.3, Kohn showed that if $M$ is a boundary of a smooth complex manifold such that $M$ is compact and pseudoconvex and if there exists a strictly plurisubharmonic function defined in a neighborhood of $M$, then $\bar{\partial}_b$ has global $C^\infty$ solutions.

- In [N06] and [KN06], Nicoara and Kohn proved existence of global $C^\infty$ solutions on a compact, orientable, pseudoconvex CR manifold of dimension $(2n-1)$, embedded in $\mathbb{C}^N$, $(n \leq N)$, (for $n \geq 3$ and, in the case $n = 2$, under the additional hypothesis that $\bar{\partial}_b$ has closed range on functions).

As a byproduct of our preparation, we can restate in a slightly more general form the above conclusions (without the orientability assumption and for an abstract CR manifold).

**Theorem 1.2.** Let $M$ be a CR manifold of dimension $2n-1$ endowed with a strongly CR-plurisubharmonic function (here $n \geq 3$, or in the case $n = 2$ with the additional hypothesis that $\bar{\partial}_b$ has closed range on functions). Then the operator $\bar{\partial}_b$ has closed range in $H^s$ for any $s \geq 0$. Moreover, for $1 \leq k \leq n-1$ the equation

$$\bar{\partial}_b u = f \quad \text{for any } f \in \mathcal{B}^k \text{ satisfying } \begin{cases} \bar{\partial}_b f = 0 & \text{if } 1 \leq k \leq n-2 \\ f \perp \ker(\bar{\partial}_b^*) & \text{if } k = n-1, \end{cases}$$

has a solution $u \in \mathcal{B}^{k-1}$.

The proof is contained in Section 4 below. It consists in using a basic estimate with a weight which has a big Levi form so that these are lifted from $L^2$ to $H^s$ and in applying a standard approximation procedure.

The space of harmonic $k$-forms $\mathcal{H}^k := \ker \square_b$ coincides with $\ker \bar{\partial}_b \cap \ker \bar{\partial}_b^*$ and is finite-dimensional with $1 \leq k \leq n-2$. Moreover both $\bar{\partial}_b$ and $\bar{\partial}_b^*$ have closed range and we have the estimate

$$\|u\| \leq c(\|\bar{\partial}_b u\| + \|\bar{\partial}_b^* u\|) \quad \text{for } u \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}_b^*) \cap \mathcal{H}^k \perp,$$

(1.1)

for any $0 \leq k \leq n-1$. This implies in particular that $\square_b$ maps $\mathcal{H}^k$ into itself and has a complex bounded inverse there.

**Definition 1.3.** The inverse of the induced operator by $\square_b$ over $\mathcal{H}^k \perp$ is denoted by $G^k$ and is called the Green operator.

As for the global hypoellipticity of $\square_b$, or the regularity of its inverse $G^k$, when $M$ is the boundary of a domain in $\mathbb{C}^n$, it was proved by Boas-Straube [BS91] in case $M$ admits a plurisubharmonic defining function. A further result has been obtained by Raich [R10].
who proved global regularity if $M$ is a smooth, orientable, pseudoconvex CR manifold of hypersurface type of real dimension at least five that satisfies a pair of potential-theoretical conditions that he names properties $(CR-P_q)$ and $(CR-P_{n-1-q})$. With independent technique, Straube gets in [S10] the same conclusion without assuming that $M$ is orientable.

The main goal of this paper consists in stating regularity under a weaker hypothesis. For this, notice that along with the $T$-components of commutators $[L_i, \bar{L}_j]$ which represent the Levi form of $M$, a crucial role is played by the $T$-components of $[L_i, T]$ that we denote by $\theta_i$:

$$[L_i, T] = \theta_i T \mod T^{1,0}M \oplus T^{0,1}M; \quad (1.2)$$

we also introduce the notation $|\theta u|^2 = \sum'_{|K|=k-1} \sum_{j \leq n-1} \theta_{ij} u_j K \bar{u}_K$ for $u \in B^k$. With this preliminary, we are ready to introduce the “Property $(\theta - P_q)$” inspired to the classical theory by Catlin [C84] and its further development by the author in [Kh10].

**Definition 1.4.** Let $M$ be a CR manifold and let $z_o$ a point of $M$, $q$ an index $1 \leq q \leq n-1$. We say that $M$ has property $(\theta - P_q)$ at $z_o$ if for any $\epsilon > 0$, there is a vector field $\theta = (\theta_i)$ defined by (1.2) under a choice of $T$ in a neighborhood $U$ of $z_o$ on $M$ and a function $\phi^\epsilon \in C^\infty(U)$ such that for any $u \in B^k$ we have

$$\begin{cases}
|\phi^\epsilon| \leq 1, \\
\epsilon \sum'_{|K|=q-1} \sum_{ij} \theta_{ij} u_i K \bar{u}_K \geq |\theta u|^2 \quad \text{on } U.
\end{cases} \quad (1.3)$$

We say that the Green operator $G_k$ is regular when it preserves $C^\infty$ smoothness; we say that it is exactly regular if it preserves $H^s$-smoothness for any integer $s$. Here is the central result of the paper.

**Theorem 1.5.** Let $M$ be a pseudoconvex, compact, smooth, CR manifold endowed with a strictly CR-plurisubharmonic function and of dimension $2n-1$ (with $n \geq 3$ or, in case $n = 2$ with the additional hypothesis that $\bar{\partial}_b$ has closed range). Assume that $(\theta - P_q)$ property is satisfied for $1 \leq q \leq \frac{n-1}{2}$; then the Green operator $G_k$ is exactly regular for any degree $q \leq k \leq n-1 - q$.

The proof follows in Section 7 below. The idea which underlies the whole proof is the combination of global hypoellipticity of $\bar{\partial}_b$ in Section 4 and the “Boas-Straube formula on a CR manifold” in Section 5 with the “$\theta$-compactness estimate” in Section 6. We have a final result which is about the $\bar{\partial}$-Neumann problem. We consider a bounded pseudoconvex smooth domain $\Omega$ of a complex manifold $X$, or an annulus $\Omega_1 \setminus \Omega_2$ between two bounded pseudoconvex boundaries, the complex Laplacian $\square = \bar{\partial}\partial^* + \bar{\partial}^*\partial$ and its inverse $N_k$ on $k$-forms.

**Theorem 1.6.** (i) Let $\Omega$ be a pseudoconvex domain. Assume that $X$ is endowed with a strictly plurisubharmonic function and that $b\Omega$ has $(\theta - P_q)$ property. Then $N_k$ is exactly globally regular for any $k \geq q$. 

(ii) Let \( \Omega = \Omega_1 \setminus \Omega_2 \) be an annulus where \( \Omega_1 \) and \( \Omega_2 \) are pseudoconvex domains. Assume that \( X \) is endowed with a strictly plurisubharmonic function, and that \( b\Omega_1 \) and \( b\Omega_2 \) satisfy \((\theta - P_q)\) and \((\theta - P_p)\) property, respectively. Then \( N_k \) is exactly globally regular for any \( k \) with \( q \leq k \leq n - 1 - p \).

**Proof.** This is a consequence of the theorem about \( \theta \)-compactness estimates in microlocalized version (Proposition 6.4) in combination with the equivalence between estimates in \( \Omega \) and microlocalized estimates on \( b\Omega \). This latter goes back to Kohn [K02] and was recently developed by the author in [Kh10], Corollary 4.13. Finally, for passing from a \( \theta \)-compactness estimate for the \( \bar{\partial} \)-Neumann problem to regularity one uses the same argument as in Section 7 of the present paper in its original version by Straube [S08]. \( \square \)

2. Weighted basic estimate on a CR manifold

We choose a basis \( L_1, ..., L_{n-1}, \bar{L}_1, ..., \bar{L}_{n-1}, T \) of tangent vector fields to \( M \) and the dual basis \( \omega_1, ..., \omega_{n-1}, \bar{\omega}_1, ..., \bar{\omega}_{n-1}, \gamma \) of 1-forms. We express the operator \( \bar{\partial}_b : B^k \to B^{k+1} \) and its adjoint \( \bar{\partial}_b^* : B^{k+1} \to B^k \) in the local basis as follows. Denote \( B^k_c(U) = B^k \cap C^\infty_c(U) \). Let \( u \in B^k_c(U) \), be written as

\[
    u = \sum' u_{J} \bar{\omega}_J, \tag{2.1}
\]

where \( \sum' \) denotes summation over strictly increasing indices \( J = j_1 < ... < j_k \) and where \( \bar{\omega}_J \) denotes the wedge product \( \bar{\omega}_J = \bar{\omega}_{j_1} \wedge ... \wedge \bar{\omega}_{j_k} \). When the multiindices are not ordered, the coefficients are assumed to be alternant. Thus, if \( J \) decomposes as \( J = jK \), then \( u_{jK} = \epsilon_{jK}^J u_J \) where \( \epsilon_{jK}^J \) is the sign if the permutation \( jK \mapsto J \). Then we have

\[
\bar{\partial}_b u = \sum'_{|K|=k-1} (\sum_{i \leq j} \bar{L}_i u_{jK} - \bar{L}_j u_{iK}) \bar{\omega}_i \wedge \bar{\omega}_j \wedge \bar{\omega}_K \tag{2.2}
\]

and

\[
\bar{\partial}_b^* u = -\sum'_{|K|=k-1} \sum_j (L_j u_{jK} + a_{K} u_K) \bar{\omega}_K \tag{2.3}
\]

where \( a_{jK}^J, a_K \in C^\infty(U) \).

Recall that the coefficient \( c_{ij} \) of the Levi form \( d\gamma \) is identified to the \( T \)-component of \([L_i, \bar{L}_j]\). We further define \( -\bar{a}_{ij}^k \) to be the \( \bar{L}_k \)-component of \([L_i, \bar{L}_j]\). We have

\[
\bar{a}_{ij}^k := -\langle \bar{\omega}_k, [L_i, \bar{L}_j] \rangle = \langle \partial_b \bar{\omega}_k, L_i \wedge \bar{L}_j \rangle \tag{2.4}
\]
It follows
\[ \partial_b \bar{\omega}_k = \sum_{ij} a_{ij}^{\bar{k}} \omega_i \wedge \bar{\omega}_j. \] (2.5)
Taking conjugation yields
\[ \bar{\partial}_b \omega_k = - \sum_{ij} a_{ji}^{k} \omega_i \wedge \bar{\omega}_j. \] (2.6)
Using again Cartan’s formula we conclude that \( a_{ji}^{k} \) coincides with the \( L_k \)-component of \([L_i, \bar{L}_j] \). Thus the full commutator is expressed by
\[ [L_i, \bar{L}_j] = c_{ij} T - \sum_k \bar{a}^k_{ij} \bar{L}_k + \sum_k a^{i}_{jk} L_k. \] (2.7)

For a smooth function \( \phi \) we want to describe the matrix (\( \phi_{ij} \)) of the Hermitian form
\[ \frac{1}{2} \left( \partial_b \bar{\partial}_b - \partial_b \partial_b \right) \phi. \] Now, \( \bar{\partial}_b \phi = \sum_k \bar{L}_k(\phi) \bar{\omega}_k \) and therefore
\[ \partial_b \bar{\partial}_b \phi = \partial_b \left( \sum_k \bar{L}_k(\phi) \bar{\omega}_k \right) \]
by (2.8)
\[ = \sum_{ij} \left( L_i \bar{L}_j(\phi) + \sum_k \bar{a}^k_{ij} \bar{L}_k(\phi) \right) \omega_i \wedge \bar{\omega}_j. \]
Similarly,
\[ \bar{\partial}_b \partial_b \phi = \bar{\partial}_b \left( \sum_k L_k(\phi) \omega_k \right) \]
by (2.9)
\[ = \sum_{ij} \left( -\bar{L}_j L_i(\phi) - \sum_k a^{i}_{jk} L_k(\phi) \right) \omega_i \wedge \bar{\omega}_j. \]
Combining (2.8) with (2.9) we get
\[ \phi_{ij} = \langle \frac{1}{2} \left( \partial_b \bar{\partial}_b - \partial_b \partial_b \right) \phi, L_i \wedge \bar{L}_j \rangle \]
\[ = \frac{1}{2} \left( L_i L_j(\phi) + L_j L_i(\phi) + \sum_k \left( \bar{a}^k_{ij} \bar{L}_k(\phi) + a^{i}_{jk} L_k(\phi) \right) \right) \]
\[ = \bar{L}_j L_i(\phi) + \frac{1}{2} \left( [L_i, \bar{L}_j](\phi) + \sum_k \left( \bar{a}^k_{ij} \bar{L}_k(\phi) + a^{i}_{jk} L_k(\phi) \right) \right) \] (2.10)
by (2.7)
\[ = \bar{L}_j L_i(\phi) + \frac{1}{2} c_{ij} T(\phi) + \sum_k a^{i}_{jk} L_k(\phi). \]

We will work with the weighted \( L^p_2 \)-norm defined by
\[ \| u \|_{\phi}^2 = (u, u)_{\phi} := \| u e^{-\frac{\phi}{2}} \|^2 = \int_M \langle u, u \rangle e^{-\phi} dS. \]
Let \( \bar{\partial}_{b,\phi}^* \) be the \( L_2^\phi \)-adjoint of \( \bar{\partial}_b \). It is easy to see that

\[
\bar{\partial}_{b,\phi}^* u = - \sum'_{|K|=k-1} \sum_j (\delta_j^\phi u_K + a_K u_K) \bar{\omega}_K
\]

(2.11)

where \( \delta_j^\phi \varphi := e^\phi L_j(e^{-\phi} \varphi) \). Notice that

\[
\bar{\partial}_{b,\phi}^* u = \bar{\partial}_b^* u + \sum'_{|K|=k-1} \sum_j L_j(\phi) u_K \bar{\omega}_K
\]

(2.12)

In particular,

\[
[\delta_i^\phi, \bar{L}_j] = \bar{L}_j L_i(\phi) + [L_i, \bar{L}_j]
\]

by (2.10) and (2.7)

\[
= \phi_{ij} - \frac{1}{2} c_{ij} T(\phi) - \sum_k a_k^i L_k(\phi) + c_{ij} T - \sum_k a_k^j \bar{L}_k + \sum_k a_k^j \bar{L}_k
\]

(2.13)

By developing the equalities (2.2) and (2.11), we get the Kohn-Morrey-Hömander inequality for a CR manifold.

**Theorem 2.1.** Let \( z_0 \in M \), \( \phi \) be a real \( C^2 \) function, and let \( q_o \) be an integer with \( 0 \leq q_o \leq n - 1 \). Then, there exists a neighborhood \( U \) of \( z_0 \) and a constant \( C \) (independent of \( \phi \)) such that

\[
\| \bar{\partial}_b u \|^2_\phi + \| \bar{\partial}_{b,\phi}^* u \|^2_\phi + C \| u \|^2_\phi
\]

\[
\geq \sum'_{|K|=k-1} \sum_{ij} (\phi_{ij} u_{iK}, u_{jK})_\phi - \sum'_{|J|=k} \sum_{j=1}^{q_o} (\phi_{jj} u_{J}, u_{J})_\phi
\]

\[
+ \sum'_{|K|=k-1} \sum_{ij} (c_{ij} T u_{iK}, u_{jK})_\phi - \sum'_{|J|=k} \sum_{j=1}^{q_o} (c_{jj} T u_{J}, u_{J})_\phi
\]

\[
+ \frac{1}{2} \left( \sum_{j=1}^{q_o} \| \delta_j^\phi u \|^2_\phi + \sum_{j=q_o+1}^n \| \bar{L}_j u \|^2_\phi \right)
\]

(2.14)

for any \( u \in \mathcal{B}_b^k(U) \).

**Proof.** Let \( Au \) denote the sum in (2.2); we have

\[
\| Au \|^2_\phi = \sum'_{|J|=k} \sum_j \| \bar{L}_j u_j \|^2_\phi - \sum'_{|K|=k-1} \sum_{ij} (\bar{L}_i u_{iK}, \bar{L}_j u_{jK})_\phi.
\]

(2.15)
Let $Bu$ denote the sum in (2.11); we have

$$
\|Bu\|_\phi^2 = \sum'_{|K|=k-1} \sum_{ij} (\delta^\phi_i u_{iK}, \delta^\phi_j u_{jK})_\phi.
$$

(2.16)

Remember that $Au$ and $Bu$ differ from $\bar{\partial}u$ and $\bar{\partial}_\phi u$ by terms of order 0 which do not depend on $\phi$. We then have

$$
\|\bar{\partial}u\|_\phi^2 + \|\bar{\partial}_\phi u\|_\phi^2
= \|Au\|_\phi^2 + \|Bu\|_\phi^2 + R
= \sum'_{|J|=k} \sum_j \|\bar{L}_j u_J\|_\phi^2 + \text{Re}(\sum'_{|K|=k-1} \sum_{ij} (\delta^\phi_i u_{iK}, \delta^\phi_j u_{jK})_\phi - (\bar{L}_j u_{iK}, \bar{L}_i u_{jK})_\phi + R),
$$

(2.17)

where $R$ is an error coming from the scalar product of 0-order terms with terms $\bar{L}_j u_J$, $\bar{L}_j u_{iK}$ or $u$.

We want to apply now integration by parts to the term $(\delta^\phi_i u_{iK}, \delta^\phi_j u_{jK})_\phi$ and $(\bar{L}_j u_{iK}, \bar{L}_i u_{jK})_\phi$. Notice that for each $\varphi, \psi \in C^1_c(U \cap M)$, we have

$$
\begin{align*}
(\varphi, \delta^\phi_j \psi)_\phi &= -(\bar{L}_j \varphi, \psi)_\phi + (a_j \varphi, \psi)_\phi \\
- (\varphi, \bar{L}_i \psi)_\phi &= (\delta^\phi_i \varphi, \psi)_\phi - (b_i \varphi, \psi)_\phi
\end{align*}
$$

and for some $a_j, b_i \in C^1(U \cap M)$ independent of $\phi$.

This immediately implies

$$
\begin{align*}
(\delta^\phi_i u_{iK}, \delta^\phi_j u_{jK})_\phi &= -(\bar{L}_j \delta^\phi_i u_{iK}, u_{jK})_\phi + R \\
- (\bar{L}_j u_{iK}, \bar{L}_i u_{jK})_\phi &= (\delta^\phi_i \bar{L}_j u_{iK}, u_{jK})_\phi + R
\end{align*}
$$

(2.18)

From here on, we denote by $R$ terms involving the product of $u$ by $\delta^\phi_j u$ or $\bar{L}_j u$ for $j \leq n-1$ but not twice a derivative of $u$. Taking the sum of two terms in the right side of (2.18), we get

$$
(\delta^\phi_i u_{iK}, \delta^\phi_j u_{jK})_\phi - (\bar{L}_j u_{iK}, \bar{L}_i u_{jK})_\phi = ([\delta^\phi_i, \bar{L}_j] u_{iK}, u_{jK})_\phi + R.
$$

(2.19)

Notice that (2.18) is also true if we replace both $u_{iK}$ and $u_{jK}$ by $u_J$ for indices $i = j \leq q_0$. Then we obtain

$$
\|\bar{L}_j u_J\|_\phi^2 = \|\delta^\phi_j u_J\|_\phi^2 - ([\delta^\phi_j, \bar{L}_j] u_J, u_J)_\phi + R.
$$

(2.20)
Applying (2.19) and (2.20) to the last line in (2.17), we have
\[
\|\bar{\partial}u\|_{\phi}^2 + \|\bar{\partial}^* u\|_{\phi}^2
= \text{Re} \left( \sum_{|K|=k-1}^{+} \sum_{i,j=1}^{+} (\delta^\phi_j, \bar{L}_j) u_{iK}, u_{jK})_\phi \right)
- \text{Re} \left( \sum_{|J|=k}^{+} \sum_{j=1}^{q_o} (\delta^\phi_j, \bar{L}_j) u_{J}, u_{J})_\phi + R \right)
+ \sum_{|J|=k}^{+} \|\bar{\delta}^\phi_j u_{J}\|_{\phi}^2 + \sum_{j=q_0+1}^{n} \|\bar{L}_j u_{J}\|_{\phi}^2 \right).
\]

We replace in (2.21) the expression (2.13) for \([\delta^\phi_i, \bar{L}_j]\) and remark that the last two sums in the commutator produce an expression \((\sum_{k}(-\delta^\phi_j \bar{L}_k + \delta^\phi_j \bar{L}_k) u_{iK}, u_{jK})_\phi\) which is an error term \(R\). This yields
\[
\|\bar{\partial}u\|_{\phi}^2 + \|\bar{\partial}^* u\|_{\phi}^2 = \sum_{|K|=k-1}^{+} \sum_{i,j=1}^{+} (\phi_{ij} u_{iK}, u_{jK})_\phi - \sum_{|J|=k}^{+} \sum_{j=1}^{q_o} (\phi_{jj} u_{J}, u_{J})_\phi
+ \text{Re} \left( \sum_{|K|=k-1}^{+} \sum_{i,j=1}^{+} (c_{ij} T u_{iK}, u_{jK})_\phi \right)
- \text{Re} \left( \sum_{|J|=k}^{+} \sum_{j=1}^{q_o} (c_{jj} T(\phi) u_{J}, u_{J})_\phi \right)
- \frac{1}{2} \text{Re} \left( \sum_{|K|=k-1}^{+} \sum_{i,j=1}^{+} (c_{ij} T(\phi) u_{iK}, u_{jK})_\phi \right)
+ \sum_{|J|=k}^{+} \|\delta^\phi_j u_{J}\|_{\phi}^2 + \sum_{j=q_0+1}^{n} \|\bar{L}_j u_{J}\|_{\phi}^2 \right) \tag{2.22}
\]

Notice that \(\text{Re} \bar{z} = \text{Re} \bar{z}^*\) and that \((c_{ij})\) is a hermitian metric, so that the fourth line in (2.22) is identically zero. We denote by \(S\) the sum in the last line in (2.22). To conclude our proof, we only need to prove that for a suitable \(C\) independent of \(\phi\) we have
\[
R \leq \frac{1}{2} \sum_{|J|=k}^{+} \left( \sum_{j=1}^{q_o} \|\delta^\phi_j u_{J}\|_{\phi}^2 + \sum_{j=q_0+1}^{n} \|\bar{L}_j u_{J}\|_{\phi}^2 \right) + C \|u\|_{\phi}^2. \tag{2.23}
\]

In fact, if we point our attention at those terms which involve \(\delta^\phi_j u\) for \(j \leq q_o\) or \(\bar{L}_j u\) for \(q_o+1 \leq j \leq n-1\), then (2.23) is clear since \(S\) carries the corresponding square \(\|\delta^\phi_j u\|_{\phi}^2\) and \(\|\bar{L}_j u\|_{\phi}^2\). Otherwise, we note that we may interchange \(\bar{L}_j\) and \(\delta^\phi_j\) by means of integration by parts. This concludes the proof of Theorem 2.1.

Furthermore, from (2.22) and (2.23) we also obtain
for any \( u \) and \( \tilde{x} \) the (10 T. V. KHANH)

such that if we set \( z = (x_1, \ldots, x_n) \) be the dual coordinates to the \((x_1, \ldots, x_{2n-1})\)'s.

Let \( C^+, C^-, C^0 \) be a covering of \( \mathbb{R}^{2n-1} \) such as

\[
C^+ = \{ \xi | \xi_{2n-1} > \frac{1}{4} |\xi'| \} \cap \{ |\xi| \geq 1 \};
\]

\[
C^- = \{ \xi - \xi \in C^+ \};
\]

\[
C^0 = \{ \xi | |\xi_{2n-1}| < \frac{3}{4} |\xi'| \} \cup \{ \xi : |\xi| < 3 \}. \tag{3.1}
\]

Thought out this paper, we fix a smooth function \( \psi \) such that \( \psi = 1 \) in \( \{ \xi | |\xi_{2n-1}| > \frac{1}{3} |\xi'| \} \cap \{ |\xi| \geq 2 \} \) and \( \text{supp}\psi \subset C^+ \); it follows that \( \text{supp}(d\psi) \subset C^0 \). Set \( \psi^+(\xi) := \psi(\xi) \); \( \psi^-(\xi) := \psi(-\xi) \) and \( \psi^0(\xi) := \sqrt{1 - (\psi^+(\xi))^2 - (\psi^-(\xi))^2} \). Define \( \tilde{\psi}^0 \) so that \( \text{supp}\tilde{\psi}^0 \subset C^0 \) and \( \tilde{\psi}^0 = 1 \) on a neighborhood of \( \text{supp}\psi^0 \cup \text{supp}(d\psi^+) \cup \text{supp}(d\psi^-) \).

Associated to a function \( \psi \) there is a pseudodifferential operator \( \Psi \) with symbol \( \sigma(\Psi) = \psi \) whose action on a function \( \varphi \in C^\infty_c(U) \) is defined by

\[
\widehat{\Psi \varphi}(\xi) = \psi(\xi) \hat{\varphi}(\xi),
\]
where \(\hat{\cdot}\) denotes the Fourier transform. The operators \(\Psi^+, \Psi^-, \Psi^0\) and \(\tilde{\Psi}^0\) are defined as above with symbols \(\psi^+, \psi^-, \psi^0\) and \(\tilde{\psi}^0\), respectively.

By definition of \(\psi^+, \psi^-, \psi^0\) and \(\tilde{\psi}^0\), we have that \(\Psi^{\pm*}\Psi^+ + \Psi^-\Psi^- + \Psi^0\Psi^0 = id\) and \(\|\Psi^+, \alpha\|\varphi\|, \|\Psi^-, \alpha\|\varphi\|, \|\Psi^0, \alpha\|\varphi\|\) are estimated by \(\|\tilde{\psi}^0\varphi\|\), where \(\alpha \in C^\infty\) and \(\varphi \in C^\infty_c(U)\).

Finally, for \(A \geq 0\) to be chosen later, we define the symbols \(\psi^0_A(\xi) = \psi^0(\xi/(A + 1))\), \(\tilde{\psi}^0_A(\xi) = \tilde{\psi}^0(\xi/(A + 1))\) and denote by \(\Psi^\pm_A, \tilde{\Psi}^0_A\) the corresponding operators.

**Lemma 3.1.** Let \(M\) be a CR manifold and \(z_0 \in M\). Then, there exists a neighborhood \(U\) of \(z_0\) such that for any \(k = 0, 1, \ldots, n - 1\), we have
\[
\|\tilde{\psi}^0_A u\|^2 + Q_0(\zeta \Psi^0_A u, \zeta \Psi^0_A u) \geq \tilde{\psi}^0_A u\|^2
\]
for any \(u \in \mathcal{B}_c(U)\), where \(\zeta\) is a cut-off function such that \(\zeta|_{\text{supp}u} \equiv 1\).

**Proof.** The proof of lemma is obtained by applying Theorem 2.1 once for \(q_0 = 0\) and a second time for \(q_0 = n - 1\) and by using the estimate for symbols \(|\sigma(T)| \leq \sum_{j \leq n - 1} (|\sigma(L_j)| + |\sigma(L_j)|)\) in the elliptic region \(C^0\).

The following lemma is the analog of Lemma 4.12 and Lemma 4.13 in [N06].

**Lemma 3.2.** Let \(M\) be a pseudoconvex CR manifold and and \(\phi\) a real \(C^2\) function. Then, for any sufficiently small set \(U \subset M\) and any form \(u \in \mathcal{B}_c(U)\), we have
\[
\begin{align*}
(i) \quad & \text{Re}\left( \sum_{|K|=k-1} (c_{ij} T \zeta \Psi^+_A u_{iK}, \zeta \Psi^+_A u_{jK} \phi) \right) \\
& \geq \sum_{|K|=k-1} (Ac_{ij} \zeta \Psi^+_A u_{iK}, \zeta \Psi^+_A u_{jK} \phi) - c \|\zeta \Psi^+_A u\|_\phi^2 - c_\phi \|\tilde{\psi}^0_A u\|^2
\end{align*}
\]
for any \(k = 1, \ldots, n - 1\); and
\[
\begin{align*}
(ii) \quad & \text{Re}\left( \sum_{|K|=k-1} (c_{ij} T \zeta \Psi^-_A u_{iK}, \zeta \Psi^-_A u_{jK} \phi) - \sum_{|J|=k} \sum_{|K|=k-1} (c_{ij} T \zeta \Psi^-_A u_{jK}, \zeta \Psi^-_A u_{jL} \phi) \right) \\
& \geq - \sum_{|K|=k-1} (Ac_{ij} \zeta \Psi^-_A u_{iK}, \zeta \Psi^-_A u_{jK} \phi) + \sum_{|J|=k} \sum_{|K|=k-1} (Ac_{ij} \zeta \Psi^-_A u_{jK}, \zeta \Psi^-_A u_{jL} \phi) \\
& - c \|\zeta \Psi^-_A u\|_\phi^2 - c_\phi \|\tilde{\psi}^0_A u\|^2
\end{align*}
\]
for any \(k = 0, \ldots, n - 2\), where \(\zeta|_{\text{supp}u} = 1\); here \(c\) (resp. \(c_\phi\)) is a positive constant independent (resp. dependent) of \(\phi\).
In combination with the basic estimate, Lemma 3.2 yields

**Corollary 3.3.** Let $z_0 \in M$ and $\phi$ be a real $C^\infty$ function. Then there exist a neighborhood $U$ of $z_0$ and constants $c$ and $c_\phi$ such that for any $u \in \mathcal{B}_c^k(U)$

\[(i) \quad c_\phi \| \bar{\Psi}_0 A u \|^2 + c \| \zeta \Psi_0^+ u \|^2_\phi + \| \bar{\partial}_b \zeta \Psi_0^+ u \|^2_\phi + \| \bar{\partial}_{b,\phi} \zeta \Psi_0^+ u \|^2_\phi \geq \sum_{|K|=k-1} \sum_{ij} ((\phi_{ij} + Ac_{ij}) \zeta \Psi_0^+ u_{iK}, \zeta \Psi_0^+ u_{jK})_\phi \quad (3.4)\]

for any $k = 1, \ldots, n - 1$; and

\[(ii) \quad c_\phi \| \bar{\Psi}_0 A u \|^2 + c \| \zeta \Psi_0^- u \|^2_\phi + \| \bar{\partial}_b \zeta \Psi_0^- u \|^2_\phi + \| \bar{\partial}_{b,\phi} \zeta \Psi_0^- u \|^2_\phi \geq \sum_{|K|=k-1} \sum_{ij} ((\phi_{ij} - A c_{ij}) \zeta \Psi_0^- u_{iK}, \zeta \Psi_0^- u_{jK})_\phi \quad (3.5)\]

for any $k = 0, \ldots, n - 2$.

4. **Global hypoellipticity for $\bar{\partial}_b$**

Throughout this section, we assume $M$ endowed a nonnegative strictly CR-plurisubharmonic function $\lambda$ in a neighborhood of $M$; then there is a constant $a \geq 0$ such that

\[\frac{1}{2} (\partial_b \bar{\partial}_b - \bar{\partial}_b \partial_b) \lambda + ad\gamma > 0. \quad (4.1)\]

From (4.1), we obtain

\[\sum'_{|K|=k-1} \sum_{ij} (\lambda_{ij} + ac_{ij}) u_{iK} \bar{u}_{jK} \geq |u|^2 \quad (4.2)\]

for $k = 1, 2, \cdots, n - 1$; and

\[\sum'_{|J|=k} \sum_{j} (\lambda_{jj} + ac_{jj}) |u_j|^2 - \sum'_{|K|=k-1} \sum_{ij} (\lambda_{ij} + ac_{ij}) u_{iK} \bar{u}_{jK} \geq |u|^2. \quad (4.3)\]

for $k = 0, \cdots, n - 2$.

We use Corollary 3.3 for $A = ta$. We choose $\phi = t\lambda$ combined with (4.2) on positive microlocalization and $\phi = -t\lambda$ combined with (4.3) on negative microlocalization. What we get is
 Proposition 4.1. Let $M$ be a pseudoconvex CR manifold endowed with a strictly CR-plurisubharmonic function $\lambda$ as in (4.1). Then for any point $z_0 \in M$, there exist a neighborhood $U$ of $z_0$ and constants $c$ and $c_t$ such that for any $u \in B^k_1(U)$

\[(i) \quad c_t\|\tilde{\Psi}_t^0 u\|^2 + \|\tilde{\partial}_b \Xi u\|^2_\lambda + \|\tilde{\partial}_b^* \Xi u\|^2_\lambda > t\|\Psi_t^+ u\|^2_\lambda \quad (4.4)\]

for any $k = 1, \ldots, n - 1$; and

\[(ii) \quad c_t\|\tilde{\Psi}_t^0 u\|^2 + \|\tilde{\partial}_b \Xi u\|^2_{-\lambda} + \|\tilde{\partial}_b^* \Xi u\|^2_{-\lambda} > t\|\Psi_t^- u\|^2_{-\lambda} \quad (4.5)\]

for any $k = 0, \ldots, n - 2$.

In the following we define global norms from norms over local patches and give estimates on these norms.

For a choice of a partition of the unity $\{\zeta^0_\nu\}$ subordinate to a covering $\{U_\nu\}$, we set $u_\nu = \zeta^0_\nu u$. The global $L^2$-inner product and the global $H^s$-Sobolev norm for $u, v \in B^k$ are defined by

\[(u, v) = \sum_\nu (\Psi^0 u_\nu, \Psi^0 v_\nu) + (\Psi^+ u_\nu, \Psi^+ v_\nu) + (\Psi^- u_\nu, \Psi^- v_\nu),\]

and

\[\|u\|_s^2 = \sum_\nu \|T^s \Psi^0 u_\nu\|^2 + \|\Lambda^s \Psi^0 u_\nu\|^2 + \|T^s \Psi^- u_\nu\|^2.\]

It is natural to define the global weighted $L^2$-inner product and $H^s_t$-Sobolev norm by

\[\langle u, v \rangle_t = \sum_\nu (\Psi^+_{ta} u_\nu, \Psi^+_{ta} v_\nu)_{\lambda t} + (\Psi^0_{ta} u_\nu, \Psi^0_{ta} v_\nu) + (\Psi^-_{ta} u_\nu, \Psi^-_{ta} v_\nu)_{-\lambda t}, \quad (4.6)\]

and

\[\langle u \rangle_{t,s}^2 = \sum_\nu \|T^s \Psi^+_{ta} u_\nu\|^2_{\lambda t} + \|\Lambda^s \Psi^0_{ta} u_\nu\|^2 + \|T^s \Psi^-_{ta} u_\nu\|^2_{t\lambda} \quad (4.7)\]

It is easy to check that $\langle u, v \rangle_0 \sim (u, v)$ and that there exist constants $c_{s,t}$ and $c'_{s,t}$ such that

\[c'_{s,t}\|u\|_s^2 \leq \langle u \rangle_{t,s}^2 \leq c_{s,t}\|u\|_s^2 \quad \text{for any } u \in B^k.\]
Let $\{\zeta_{\nu}\}$ be a sequence of cutoff functions such that for each $\nu$, \[
\zeta_{\nu} = 1 \text{ on supp} c_{\nu}^{0}, \quad \text{supp} \zeta_{\nu} \subset U_{\nu}.
\]
We have
\[
\langle |\bar{\partial}u| \rangle_{t,0}^{2} = \sum_{\nu} \langle |\bar{\partial}_{\nu}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + \langle |\bar{\partial}_{\nu}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + \langle |\bar{\partial}_{\nu}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + \langle |\bar{\partial}_{\nu}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + E(u)
\]
(4.8)
where $E(u)$ is an error term that can be estimated by
\[
|E(u)| \lesssim \langle |u| \rangle_{t,0}^{2} + C_{t} \sum_{\nu} \langle |\bar{\partial}_{\nu}\zeta_{\nu}^{0}u| \rangle_{t,\lambda}^{2}.
\]
(4.9)
Let $\bar{\partial}_{b}^{*, t}$ be the adjoint of $\bar{\partial}_{b}$ with respect to $\langle |\cdot, \cdot| \rangle_{t,0}$. Similarly as for $\bar{\partial}_{b}$, we have
\[
\langle |\bar{\partial}_{b}^{*, t}u| \rangle_{t,0}^{2} = \sum_{\nu} \langle |\bar{\partial}_{b}^{*, t}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + \langle |\bar{\partial}_{b}^{*, t}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + \langle |\bar{\partial}_{b}^{*, t}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + \langle |\bar{\partial}_{b}^{*, t}\zeta_{\nu}^{0}\bar{u}| \rangle_{t,\lambda}^{2} + E(u).
\]
(4.10)
We define
\[
Q_{b}\langle |u, v| \rangle_{t} = \langle |\bar{\partial}_{b}u, \bar{\partial}_{b}v| \rangle_{t,0} + \langle |\bar{\partial}_{b}^{*, t}u, \bar{\partial}_{b}^{*, t}v| \rangle_{t,0}.
\]
To control the term involving $\bar{\Psi}$, we use Lemma 3.1 which yields

**Lemma 4.2.** For any $\epsilon > 0$, there exists a constant $c_{\epsilon}$ such that
\[
\sum_{\nu} \langle |\bar{\partial}_{\nu}\zeta_{\nu}^{0}u| \rangle_{t,\lambda}^{2} \leq \epsilon (Q_{b}\langle |u, u| \rangle_{t} + \langle |u| \rangle_{t,0}^{2}) + c_{\epsilon} \langle |u| \rangle_{t,\lambda - 1}^{2}
\]
(4.11)
for any $u \in B^{k}$ with $0 \leq k \leq n - 1$.

Combination of Proposition 4.1 with (4.8), (4.9), (4.10) and Lemma 4.2 yields

**Proposition 4.3.** Let $M$ be a pseudoconvex CR manifold of dimension $2n - 1$ with $n \geq 3$ and endowed with a strictly CR-plurisubharmonic function. Then for any $t$ suitably large there is a constant $c_{t}$ such that
\[
c_{t} \langle |u| \rangle_{t,\lambda - 1}^{2} + Q_{b}\langle |u, v| \rangle_{t} \geq t \langle |u| \rangle_{t,0}^{2}
\]
(4.12)
for any $u \in B^{k}$ with $1 \leq k \leq n - 2$.

For $0 \leq k \leq n - 1$, we define the space of harmonic $k$-forms $H_{t}^{k}$ by
\[
H_{t}^{k} = \{ u \in B^{k} \mid \bar{\partial}_{b}u = 0 \text{ and } \bar{\partial}_{b}^{*, t}u = 0 \}
\]
(4.13)
\[
= \{ u \in B^{k} \mid Q_{b}\langle |u, u| \rangle_{t} = 0 \}.
\]
On $\mathcal{H}_t^{k\perp}$, we define
\[ \square_{b,t} = \overline{\partial}_b \overline{\partial}^{*,t}_b + \overline{\partial}^{*,t}_b \overline{\partial}_b. \]
If the inverse of $\square_{b,t}$ exists on $\mathcal{H}_t^{k\perp}$, we define the Green operator
\[ G_{k,t} = \begin{cases} (\square_{b,t})^{-1} & \text{on } \mathcal{H}_t^{k\perp} \\ 0 & \text{on } \mathcal{H}_t^k. \end{cases} \]

We carry on our discussion for the case $n \geq 3$. Since $L_{2,t}$ is compact in $H_t^{-1}$, Proposition 4.3 implies that there exists a constant $c_t$ such that
\[ \langle |u| \rangle_{t}^2 \leq c_t Q_b \langle |u,u| \rangle_{t}^2 \]
for $1 \leq k \leq n - 2$. Thus, we have the following results:

1. For $1 \leq k \leq n - 2$, $\mathcal{H}_t^k$ is finite dimensional and its dimension is independent of $t$.
2. The ranges of both $\overline{\partial}_b$ and $\overline{\partial}^{*,t}_b$ are closed on $k$-forms for $1 \leq k \leq n - 2$. This also implies that $\overline{\partial}_b$ has closed range at the bottom degree $k = 0$ and $\overline{\partial}^{*,t}_b$ at the top degree $k = n - 1$.
3. $\square_{b,t}$ has closed range and $G_{k,t}$ exists and continuous on $\mathcal{H}_t^{k\perp}$ for any $k = 0, \cdots, n - 1$.
4. For $1 \leq k \leq n - 1$, let $f \in \mathcal{B}^k$ satisfy
\[ \begin{cases} \overline{\partial}_b f = 0 & \text{for } k \leq n - 2 \\ f \perp \ker(\overline{\partial}^{*,t}_b) & \text{for } k = n - 1. \end{cases} \]

Set $u_t = \overline{\partial}^{*,t}_b G_{k,t} f$, then $u$ is a solution of equation $\overline{\partial}_b u = f$.
5. For $0 \leq k \leq n - 2$, let $f \in \mathcal{B}^k$ satisfy
\[ \begin{cases} \overline{\partial}^{*,t}_b f = 0 & \text{for } k \geq 0 \\ f \perp \ker(\overline{\partial}_b) & \text{for } k = 0. \end{cases} \]

Set $u_t = \overline{\partial}_b G_{k,t} f$; then $u$ is a solution of equation $\overline{\partial}^{*,t}_b u = f$.

Together with the above enumerated results, we also have

**Proposition 4.4.** Let $M$ be a pseudoconvex CR manifold of dimension $2n - 1$ (with $n \geq 3$ and, in the case $n = 2$, with the extra assumption that $\overline{\partial}_b$ has closed range on functions) and endowed with a strictly CR-plurisubharmonic function. Then for any large $t$ and for a suitable $c_t$ we have for the two degrees $k = 0$ and $k = n - 1$ which are missing in Proposition 4.3
\[ c_t \langle |u| \rangle_{t,0}^2 + Q_b \langle |u, v| \rangle_{t} \geq t \langle |u| \rangle_{t,0}^2 \]
for any $u \in \mathcal{B}^k \cap \mathcal{H}_t^{k\perp}$. 
Proof. 

Case 1, $n > 2$. Let $f$ be smooth function orthogonal to ker($\bar{\partial}_b$); as above, if we set $u_t = \partial_b G_{0,t} f$ then $\bar{\partial}_b^{*t} u_t = f$. Using the estimate (4.12) for the 1-form $u_t$, we get

$$t \langle |u_t| \rangle^2_{t, 0} \lesssim \langle |\partial_b u_t| \rangle^2_{t, 0} + \langle |\bar{\partial}_b^{*t} u_t| \rangle^2_{t, 0} + c_t \langle |u_t| \rangle^2_{t, -1}$$

(4.16)

Hence, we have the estimate for $f$

$$t \langle |f| \rangle^2_{t, 0} \lesssim t \langle |\partial_b f| \rangle^2_{t, 0} + c_t \langle |f| \rangle^2_{t, -1}$$

(4.17)

For $\epsilon$ small, we obtain the $L_2$-norm estimate for any function $f \perp \ker(\partial_b)$

$$t \langle |f| \rangle^2_{t, 0} \lesssim |\partial_b f|_{t, 0}^2 + c_t \langle |f| \rangle^2_{t, -1}$$

(4.18)

Case 2, $n = 2$. There is no estimate provided by Proposition 4.3. We observe that we have the estimate for postive microlocalization at degree 1 and negative microlocalization at degree 0. We prove how to get the estimate for postive microlocalization at degree 0. We use the extra assumption that $\bar{\partial}_b$ has closed range on functions. Thus the operator $\bar{\partial}_b^{*t} \bar{\partial}_b$ has closed range on ker($\bar{\partial}_b$) and we have $G_{0,t} = (\bar{\partial}_b^{*t} \bar{\partial}_b)^{-1}$ on ker($\bar{\partial}_b$).

Let $f$ be a smooth function on ker($\bar{\partial}_b$)$^\perp$; we set $u_t = \partial_b G_{0,t} f$; then $\bar{\partial}_b^{*t} \bar{\partial}_b u_t = f$. The proof goes through in the same way as (4.16) and (4.17) with $\langle |\cdot| \cdot \rangle$ replaced by $\sum_\nu (\Psi^+ \zeta_\nu, \Psi^+ \zeta_\nu)_{t, 0}$ and we obtain

$$t \sum_\nu \| \Psi^{t,\nu}_0 f \|^2_{t, 0} \lesssim \sum_\nu \| \partial_b \Psi^{t,\nu}_0 f \|^2_{t, 0} + c_t \sum_\nu \| \bar{\Psi}^{t,\nu}_0 f \|^2_{t, 0}$$

(4.19)

Combination of (4.19) with Proposition 4.1 yields

$$t \langle |f| \rangle^2_t \lesssim \langle |\partial_b f| \rangle^2_t + c_t \langle |f| \rangle^2_{t, -1}$$

for any $f \perp \ker(\partial_b)$.

The same argument applies to the case of top degree $k = n - 1$ (both for $n \geq 3$ and $n = 2$).

□

One applies the $L_2$ estimate in (4.12) and (4.15) to the $s$-derivatives of $u$ and controls error terms coming from commutators by the aid of a big constant $t$ depending on $s$. In this way one obtains the proof of the theorem below about $H^s$-estimates for the system $(\partial_b, \bar{\partial}_b^{*t} \bar{\partial}_b)$.

Theorem 4.5. Let $M$ be a pseudoconvex CR manifold of dimension $2n - 1$ (where $n \geq 3$ and, in case $n = 2$, with the extra assumption that $\partial_b$ has closed range on functions).
and endowed with a strictly CR-plurisubharmonic function. Then, for any integer $s$ there exists a constant $T_s$ such that for any $t \geq T_s$, we have

$$\langle |u| \rangle_{t,s}^2 \leq c_{t,s} (\langle |\bar{\partial}_b u| \rangle_{t,s}^2 + \langle |\bar{\partial}_b^{*t} u| \rangle_{t,s}^2)$$

(4.21)

for any $u \in B^k \cap \mathcal{H}_t^k$ with $0 \leq k \leq n - 1$ where $c_{t,s}$ is a constant depending on $t$ and $s$.

Also, an immediate consequence of Theorem 4.5 is the proof of Theorem 1.2 for whose completion only minor details must be clarified.

**Proof of Theorem 1.2.** By the method of the elliptic regularization one passes from a priori estimates (4.21) to actual estimates and obtains $H^s$ solvability. By an approximation argument, one obtains a $C^\infty$ solution as a sum of a convergent series $u = \sum_s (u_{s+1} - u_s)$ ($s$ integer). The proof of Theorem 1.2 is complete. \qed

The following corollary to Theorem 4.5 is one key ingredient for the proof of Theorem 1.5 in Section 6 below.

**Corollary 4.6.** Under the hypothesis in Theorem 1.2, we have

$$\|\Psi^+ T^s \bar{\partial}^{*t} G_{k,t} u\| \lesssim \|\Psi^+ T^s u\|^2 + \|\Psi^0 \Lambda^s u\|^2, \quad \|\Psi^+ T^s \bar{\partial} G_{k,t} u\| \lesssim \|\Psi^+ T^s u\|^2 + \|\Psi^0 \Lambda^s u\|^2, \quad \|\Psi^+ T^s \bar{\partial}^{*t} G_{k,t} u\| \lesssim \|\Psi^+ T^s u\|^2 + \|\Psi^0 \Lambda^s u\|^2, \quad \|\Psi^+ T^s \bar{\partial} G_{k,t} u\| \lesssim \|\Psi^+ T^s u\|^2 + \|\Psi^0 \Lambda^s u\|^2,$$

(4.22)

for any $u \in B^k$.

5. The Straube-Boas formula for Green operators

In this section, we shall show the expression of the Green operator by Szego projection and anti-Szego-projection (similar as to Straube-Boas formula in [BS90] for the $\bar{\partial}$-Neumann operator).

From Proposition 4.4, the operator $\bar{\partial}_b$ has closed range on $k$-forms for $0 \leq k \leq n - 2$. Thus $\bar{\partial}_b$ (the unweighted adjoint of $\bar{\partial}_b$) has also closed range on $k$-forms for $1 \leq k \leq n - 1$. This implies

$$\|u\|^2 \lesssim \|\bar{\partial}_b u\|^2 + \|\bar{\partial}_b^* u\|^2$$

(5.1)

for any $u \in B^k \cap \mathcal{H}_t^{k\perp}$ and the Green operator $G_k$ exists and is $L_2$ continuous for any $k = 0, 1, \ldots, n - 1$. Let $S_k$ be the Szego projection, the operator that projects $L_2^k$ to the space of holomorphic $k$-forms $\{u \in \text{Dom}(\bar{\partial}) \cap L_2^k | \bar{\partial} u = 0\}$. By Kohn’s formula we have

$$S_k = \begin{cases} I - \bar{\partial}_b^* G_{k+1} \bar{\partial}_b & \text{for } 0 \leq k \leq n - 2, \\ I & \text{for } k = n - 1. \end{cases}$$
Let $A_t$ be the operator defined by 

$$(A_t u, v) = \langle |u, v| \rangle_t.$$ 

Denote $A_t^{-1}$ be the inverse of $A_t$. Notice that both $A_t$ and $A_t^{-1}$ are self-adjoint in $L_2$. Define

$$M_k = (I - S_k)\bar{\partial}_b^*G_{k+1,t}S_{k+1}A_tG_{k,t}\bar{\partial}_bA_t^{-1}(I - S_k) \quad \text{for} \quad k = 0, 1, ..., n - 2,$$

$$N_k = S_kA_tG_{k,t}\bar{\partial}_bA_t^{-1}(I - S_{k-1})\bar{\partial}_b^*G_{k,t}S_k \quad \text{for} \quad k = 1, ..., n - 1. \quad (5.2)$$

**Theorem 5.1.** There exists $T_0 \geq 0$ such that for any $t \geq T_0$ the following equalities hold

$$G_k = \begin{cases} 
M_k & \text{for } k = 0, \\
M_k + N_k & \text{for } 1 \leq k \leq n - 2, \\
N_k & \text{for } k = n - 1. 
\end{cases} \quad (5.3)$$

**Proof.** We only prove the theorem in the cases $1 \leq k \leq n - 2$, the proof of the cases $k = 0$ and $k = n - 1$ being similar.

For $1 \leq k \leq n - 2$, since

$$G_k = G_k(\bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b)G_k = (G_k\bar{\partial}_b)(\bar{\partial}_b^*G_k) + (\bar{\partial}_b^*G_{k+1})(G_{k+1}\bar{\partial}_b)$$

$$= (\bar{\partial}_b^*G_k)^*(\bar{\partial}_b^*G_k) + (\bar{\partial}_b^*G_{k+1})(\bar{\partial}_b^*G_{k+1})^* \quad (5.4)$$

it suffices to write $\bar{\partial}_b^*G_k$ and $\bar{\partial}_b^*G_{k+1}$ in terms of Szego projections.

Let $g \in B^k$ and set $f = S_kg$; then $g$ is a $\bar{\partial}_b$-closed form. By Theorem 1.2 there exists a constant $t$ (sufficiently large) such that $u = \bar{\partial}_b^*tG_{k,t}f \in B^{k-1}$ is a solution of equation $\bar{\partial}_b u = f$. So we have

$$(I - S_{k-1})\bar{\partial}_b^*G_{k,t}S_kg = \bar{\partial}_b^*G_k\bar{\partial}_b\bar{\partial}_b^*G_{k,t}\bar{\partial}_bu$$

$$= \bar{\partial}_b^*G_k\bar{\partial}_bu = \bar{\partial}_b^*G_kS_kg = \bar{\partial}_b^*G_kg. \quad (5.5)$$

Moreover, we have

$$(u, G_k\bar{\partial}_b v) = (\bar{\partial}_b^*G_ku, v)$$

$$= ((I - S_{k-1})\bar{\partial}_b^*G_{k,t}S_ku, v)$$

$$= (\bar{\partial}_b^*tG_{k,t}S_ku, (I - S_{k-1})v)$$

$$= \langle |\bar{\partial}_b^*tG_{k,t}S_ku, A_t^{-1}(I - S_{k-1})v| \rangle$$

$$= \langle |S_ku, G_{k,t}\bar{\partial}_bA_t^{-1}(I - S_{k-1})v| \rangle$$

$$= (u, S_kA_tG_{k,t}\bar{\partial}_bA_t^{-1}(I - S_{k-1})v)$$

for any $u \in B^k(U); v \in B^{k-1}(U)$. 

---

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Therefore, we obtain the equalities
\[
\begin{align*}
\bar{\partial}_b^* G_k &= (I - S_{k-1}) \bar{\partial}_b^{*t} G_{k,t} S_k, \\
G_k \bar{\partial}_b &= S_k A_t G_{k,t} \bar{\partial}_b A_t^{-1} (I - S_{k-1}).
\end{align*}
\] (5.7)
Combining (5.4) and (5.7) we get the proof of Theorem 5.1.

Similarly, we define the anti-Szego-projection
\[
R_k = \begin{cases} 
I - \bar{\partial}_b G_{k-1} \bar{\partial}_b^t & \text{for } 1 \leq k \leq n - 1, \\
I & \text{for } k = 0.
\end{cases}
\]
We can check that this operator projects $L^2_k$ to the space of anti-holomorphic forms $\{u \in \text{Dom}(\bar{\partial}) \cap L^k_2 | \bar{\partial}^* u = 0\}$. Let
\[
P_k = R_k \bar{\partial}_b^{*t} G_{k+1,t} (I - R_{k+1}) A_t G_{k,t} \bar{\partial}_b A_t^{-1} R_k \quad \text{for } k = 0, 1, ..., n - 2, \\
Q_k = (I - R_k) A_t G_{k,t} \bar{\partial}_b A_t^{-1} R_k \bar{\partial}_b^{*t} G_k (I - R_k) \quad \text{for } k = 1, ..., n - 1.
\] (5.8)

**Theorem 5.2.** For large $t$ the following equalities hold
\[
G_k = \begin{cases} 
P_k & \text{for } k = 0, \\
P_k + Q_k & \text{for } 1 \leq k \leq n - 2, \\
Q_k & \text{for } k = n - 1.
\end{cases}
\] (5.9)

**Proof.** The proof of this Theorem is entirely analogous to the case of the Szego projection.

\[\square\]

6. **$\theta$-COMPACTNESS ESTIMATES**

Let $\theta = (\theta_1, \ldots, \theta_{n-1})$ be a vector field. We define the operator $\bar{\Theta} : \mathcal{B}^k \to \mathcal{B}^{k+1}$ by
\[
\bar{\Theta} u = \sum_{|\mathcal{J}|=k} \sum_{i} \bar{\theta}_j u_{jK} \bar{\omega}_j \wedge \bar{\omega}_K.
\] (6.1)
We denote by $\bar{\Theta}^* : \mathcal{B}^{k+1} \to \mathcal{B}^k$ the $L_2$-adjoint of $\bar{\Theta}$, that is
\[
\bar{\Theta}^* u = \sum_{|\mathcal{K}|=k-1} \sum_{i} \theta_j u_{jK} \bar{\omega}_K.
\] (6.2)
Notice that $|\bar{\Theta}^* u|$ coincides with $|\theta u|$ formerly defined.

We define now “microlocalized $\theta$-compactness estimates”.
Definition 6.1. A $\theta$-compactness estimate is said to hold for $\Box_b$ on positively (resp. negatively) microlocalized $k$-forms at $z_0$ if there exists a neighborhood $U$ of $z_0$ such that for any $\epsilon$ and suitable $c_\epsilon$ we have
\[
(\theta - C)^+ \cdot \| \Theta^* \Psi^+ u \|^2 \leq \epsilon (\| \bar{\partial}_b \zeta \Psi^+ u \|^2 + \| \bar{\partial}_b^* \zeta \Psi^+ u \|^2) + c_\epsilon \| \tilde{\Psi}^0 u \|_{2-}, \quad (6.3)
\]
(resp.
\[
(\theta - C)^- \cdot \| \bar{\Theta} \zeta \Psi^- u \|^2 \leq \epsilon (\| \bar{\partial}_b \zeta \Psi^- u \|^2 + \| \bar{\partial}_b^* \zeta \Psi^- u \|^2) + c_\epsilon \| \tilde{\Psi}^0 u \|_{2-}. \quad (6.4)
\]
for any $u \in \mathcal{B}^k_b(U)$ such that $\zeta |_{\text{supp } u} \equiv 1$.

Positively and negatively microlocalized estimates are related, in complementary degree, by the aid of the star-Hodge theory.

Proposition 6.2. We have that $(\theta - C)^-_- \equiv (\theta - C)^+_{n-1-k}$.

Proof. We define the local conjugate-linear duality map $F^k : \mathcal{B}^k \to \mathcal{B}^{n-1-k}$ as follows. If $u = \sum_{|J|=k} u_J \tilde{\omega}_J$ then
\[
F^k u = \sum \epsilon_{\{J,J'\}}^\{\{1,\ldots,n-1\}\} \tilde{u}_J \tilde{\omega}_{J'},
\]
where $J'$ denotes the strictly increasing $(n-k-1)$-tuple consisting of all integers in $[1, n-1]$ which do not belong to $J$ and $\epsilon_{\{1,n-1\}}$ is the sign of the permutation $\{J, J'\} \rightarrow \{1, \ldots, n-1\}$.

Since $\bar{\zeta} \Psi^+ u = \zeta \Psi^- u$, then
\[
F^k \zeta \Psi^+ u = \sum \epsilon_{\{J,J'\}}^\{\{1,\ldots,n-1\}\} \zeta \Psi^- \tilde{u}_J \tilde{\omega}_{J'}.
\]
Also, $F^{n-1-k} F^k \zeta \Psi^+ u = \zeta \Psi^+ u; \| F^k \zeta \Psi^+ u \| = \| \zeta \Psi^- u \|$
\[
\left\{
\begin{array}{l}
\partial_b F^k \zeta \Psi^+ u = F^{k-1} \tilde{\partial}_b \zeta \Psi^- u + \cdots \\
\bar{\partial}_b F^k \zeta \Psi^+ u = F^{k+1} \tilde{\bar{\partial}}_b \zeta \Psi^- u + \cdots
\end{array}
\right.
\]
where dots refers the term in which $u$ is not differentiated; and finally
\[
\left\{
\begin{array}{l}
\tilde{\Theta} F^k \zeta \Psi^+ u = F^{k-1} \tilde{\Theta}^* \zeta \Psi^- u \\
\tilde{\Theta}^* F^k \zeta \Psi^+ u = F^{k+1} \tilde{\Theta} \zeta \Psi^- u.
\end{array}
\right.
\]
Thus we obtain the proof of Proposition 6.2. \hfill \Box

We also remark that if a positively (resp. negatively) microlocalized $\theta$-compactness estimate holds on $q$-forms, then it also holds in higher (resp. lower) degree of forms.

Proposition 6.3. Let $M$ be CR manifold; then
(i) $(\theta - C)^q_+ \equiv (\theta - C)^q_+$ for any $k \geq q$.
(ii) $(\theta - C)^q_- \equiv (\theta - C)^q_-$ for any $k \leq q$. 

Proof. We first discuss the positive microlocalization case; in particular we only need to show that $(\theta - C)^k_+$ implies $(\theta - C)^k_+$. Let $u = \sum'_{|L|=k+1} u_L \omega_L$ have degree $k + 1$. We rewrite $u$ as a non-ordered sum

$$u = \frac{1}{(k + 1)!} \sum_{|L|=k+1} u_L \omega_L = \frac{(-1)^k}{k + 1} \sum_{l=1}^{n-1} \left( \frac{1}{k!} \sum_{|J|=k} u_{l,j} \omega_j \right) \wedge \omega_l.$$  

For $l = 1, \ldots, n - 1$, we define a set of $k$-forms $v_l$ by $v_l := \sum'_{|J|=k} u_{l,j} \omega_j$. It is easy to see that

$$\sum_{l=1}^{n-1} |v_l|^2 = (k + 1)|u|^2 \quad \text{and} \quad \sum_{l=1}^{n-1} \sum'_{|K|=k-1} c_{ij} T(v_l)_{iK} \overline{(v_l)_{jK}} = k \sum'_{|J|=k} c_{ij} T v_{i,j} \tilde{\omega}_{j,l} \quad \text{for each pair } (i, j).$$

Using (2.24) for the case $\phi = 0$, we have

$$\sum_{l=1}^{n} Q(\zeta \Psi^+ v_l, \zeta \Psi^+ v_l) \leq \frac{3}{2} \sum_{l=1}^{n} \left( \frac{3}{2} \sum_{j=1}^{n} \left| L_j \zeta \Psi^+ v_l \right|^2 + \sum_{ij} \sum'_{|K|=k-1} (c_{ij} T(\zeta \Psi^+ v_l)_{iK}, (\zeta \Psi^+ v_l)_{jK}) + c \| \zeta \Psi^+ v_l \|^2 \right)$$

$$= (k + 1) \left( \frac{3}{2} \sum_{j=1}^{n} \left| L_j \zeta \Psi^+ u \right|^2 + c \| \zeta \Psi^+ u \|^2 \right) + k \sum'_{|J|=k} \sum_{ij} (c_{ij} T(\zeta \Psi^+ u_{i,l}), (\zeta \Psi^+ u_{j,l}))$$

$$\leq \frac{3}{4} (k + 1) \left( \frac{1}{2} \sum_{j=1}^{n} \left| L_j \zeta \Psi^+ u \right|^2 + \sum'_{|J|=k} \sum_{ij} (c_{ij} T(\zeta \Psi^+ u_{i,l}), (\zeta \Psi^+ u_{j,l})) \right)$$

$$+ c \| \zeta \Psi^+ u \|^2 + c' \| \tilde{\Psi}^0 u \|^2$$

$$\leq \frac{3}{4} (k + 1) Q(\zeta \Psi^+ u, \zeta \Psi^+ u) + c \| \zeta \Psi^+ u \|^2 + c' \| \tilde{\Psi}^0 u \|^2$$

where $c$ and $c'$ are constants (different in each line). Here the second inequality follows from Lemma 3.2 and the last one from Theorem 2.11. Moreover, we have

$$\sum_{l=1}^{n-1} |\Theta^* v_l|^2 = k |\Theta^* u|.$$  

This concludes the proof of the positively microlocalized case.

We discuss now the negatively microlocalized case. Let $u = \sum'_{|K|=k-1} u_K \omega_K$ have degree $k - 1$. For $l=1,\ldots,n$, define $v_l = \sum'_{|K|=k-1} u_K \omega_K \wedge \omega_l$; this has degree $k$. Then the proof follows the same lines as in the positive microlocalization case. 

We prove now the $\theta$-compactness estimate (6.3) and (6.4) when Property $(\theta - P_q)$ holds.
Proposition 6.4. Let $M$ be a pseudoconvex CR manifold. Assume that property $(\theta - P_q)$ holds at $z_0$. Then

(i) A $\theta$-compactness estimate on positively microlocalized $k$ forms $(\theta - C)^k_+$ holds for any $k \geq q$.

(ii) A $\theta$-compactness estimate on negatively microlocalized $k$ forms $(\theta - C)_{-k}$ holds for any $k \leq n - 1 - q$.

Proof. (i) We assume that we have family $\phi^\varepsilon$ with properties

\[
\begin{cases} 
|\phi^\varepsilon| \leq 1 \\
\varepsilon \sum'_{|K|=q} \phi^\varepsilon_{ij} u_{iK} \bar{u}_{jK} \geq |\theta u|^2;
\end{cases}
\] (6.6)

and we first show that $(\theta - C)^q_+$ holds.

We will apply the first inequality of Corollary 3.3 for $A = 0$ and $\phi = \chi(\phi^\varepsilon)$ where $\chi$ is a function to be chosen later. First, we notice that

\[
(\chi(\phi^\varepsilon))_{ij} = \dot{\chi} \phi^\varepsilon_{ij} + \ddot{\chi} |\phi^\varepsilon|_2 \kappa_{ij},
\] (6.7)

(\text{where $\kappa_{ij}$ is the Kronecker symbol}) and

\[
|\tilde{\partial}_{b,\phi}^* u|^2 \leq 2 |\tilde{\partial}_{b}^* u|^2 + 2 \dot{\chi}^2 \sum'_{|K|=k-1} |\sum_j \phi_{jK}|^2.
\] (6.8)

Thus we get from (6.9) and taking into account (6.7) and (6.8)

\[
C \varepsilon \|
\begin{array}{c}
\tilde{\Psi}^0 u \\
\varepsilon \sum'_{|K|=q} \phi^\varepsilon_{ij} u_{iK} \bar{u}_{jK} \tilde{\Psi}^+ \bar{u}
\end{array}
\|_{X(\phi^\varepsilon)}^2 + 2 \|
\begin{array}{c}
\tilde{\partial}_{b}^* \tilde{\Psi}^+ u \tilde{\Psi}^+ \bar{u}
\end{array}
\|_{X(\phi^\varepsilon)}^2
\geq \int_M \dot{\chi} e^{-\chi(\phi^\varepsilon)} \phi_{ij} \tilde{\Psi}^+ u_{iK} \tilde{\Psi}^+ \bar{u}_{jK} dV + \int_M (\ddot{\chi} - 2 \dot{\chi}^2) e^{-\chi(\phi^\varepsilon)} \sum'_{|K|=k-1} |\sum_j \phi_{jK} \tilde{\Psi}^+ u_{jK}|^2 dV.
\] (6.9)

We choose $\chi$ such that $\ddot{\chi} \geq 2 \dot{\chi}^2$ for $t < 1$ (for example: $\chi(t) = \frac{1}{2} t^{t-1}$). Combination of (6.6) and (6.9) yields $(\theta - C)^q_+$. Next, by Proposition 6.3 we obtain $(\theta - C)^k_+$ for any $k \geq q$.

(ii) is a consequence of (i) in addition to Proposition 6.2.

\[
\square
\]

7. Global hypoellipticity for $\Box_b$

We write the commutator $[L_j, T]$ as

$$[L_j, T] = \theta_j T + V_j \text{ where } V_j \text{ are error terms in } T^{1,0}M \oplus T^{0,1}M.$$
Associated to the vector-operator \((\theta_jT)\) there are the operators \(\tilde{\Theta}\) and \(\tilde{\Theta}^*\) defined by (6.1) and (6.2). Using the vector-operator \((V_j)\) defined above, we introduce
\[
\mathcal{V} u = \sum' \sum_{|J|=kj\leq n-1} V_j u J \tilde{w}_J,
\]
and its adjoint
\[
\mathcal{V}^* u = \sum' \sum_{|K|=k-1j\leq n-1} V_j u K \tilde{w}_K.
\]
We have, in other words
\[
\begin{cases}
[\tilde{\partial}_b, T] = \tilde{\Theta} T + \tilde{\mathcal{V}}, \\
[\tilde{\partial}_b^*, T] = \tilde{\Theta}^* T + \tilde{\mathcal{V}}^*.
\end{cases}
\]
By induction, we get the equalities on higher power of \(T\)
\[
\begin{cases}
[\tilde{\partial}_b, T^s] = s\tilde{\Theta} T^s + \sum_{m=0}^{s-1} a_m T^m \tilde{\mathcal{V}}, \\
[\tilde{\partial}_b^*, T^s] = s\tilde{\Theta}^* T^s + \sum_{m=0}^{s-1} b_m T^m \tilde{\mathcal{V}}^*.
\end{cases}
\]
If we consider Kohn’s microlocalization, we have
\[
\begin{cases}
[\tilde{\partial}, (\Psi^+ T^s)^*(\Psi^+ T^s)] u = (\Psi^+ T^s)^*(2s\zeta \tilde{\Theta} \Psi^+ T^s u + \text{good}^+ u) \\
[\tilde{\partial}^*, (\Psi^- T^s)^*(\Psi^- T^s)] u = (\Psi^- T^s)^*(2s\zeta \tilde{\Theta}^* \Psi^- T^s u + \text{good}^- u)
\end{cases}
\]
for any \(u \in \mathcal{B}^k(U)\), where \(\zeta \in C^\infty_c(U)\) is a cut-off function such that \(\zeta|\text{supp } u \equiv 1\) and “good\textsuperscript{±}” denotes a term which satisfies
\[
||\text{good}^\pm u||^2 \lesssim ||\Psi^\pm T^{s-1} \mathcal{V} u||^2 + ||\Psi^\pm T^{s-1} \mathcal{V}^* u||^2 + ||\tilde{\Psi}^0 u||_s^2
\]
This is just a consequence of (7.2) by the aid of Jacobi identity. Now, we claim the first term in the right hand side of (7.4) can be estimated as
\[
||\Psi^\pm T^{s-1} \mathcal{V} u||^2 + ||\Psi^\pm T^{s-1} \mathcal{V}^* u||^2 \lesssim ||\Psi^\pm T^{s-1} \tilde{\partial}_b u||^2 + ||\Psi^\pm T^{s-1} \tilde{\partial}_b^* u||^2 + ||\Psi^\pm T^s u|| (||\Psi^\pm T^{s-1} u|| + ||\tilde{\Psi}^0 u||_s),
\]
for any \(u \in \mathcal{B}^k(U)\). In fact, using twice Theorem 2.1 for \(q_0 = 0\) and \(q_0 = n-1\) and taking summation, we get
\[
||\mathcal{V} u||^2 + ||\mathcal{V}^* u||^2 \lesssim ||\tilde{\partial}_b u||^2 + ||\tilde{\partial}_b^* u|| + ||T u|| ||u||
\]
for any \(u \in \mathcal{B}^k(U)\). Replacing \(u\) in (7.5) by \(\Psi^\pm T^{s-1} u\), we get the proof of the claim. We are ready for the central result

**Theorem 7.1.** Let \(q\) be an index \(1 \leq q \leq n-1\). Assume that property \((\theta - P_q)\) holds at \(z_0\) and that \(M\) is endowed with a strictly CR-plurisubharmonic function; then there exists a neighborhood \(U\) of \(z_0\) such that
\[
||\Psi^\pm T^s S_{k-1} u||^2 \lesssim ||\Psi^\pm T^s u||^2 + ||\tilde{\Psi}^0 u||_s^2
\]
for any \( u \in \mathcal{B}_c^k(U) \) with \( k \geq q \), where \( s \geq 0 \) is integer. Moreover, we have
\[
\|\Psi^+T^sG_ku\|^2 \lesssim \|\Psi^+T^su\|^2 + \|\bar{\Psi}^0u\|^2_s
\]  
(7.7)
for any \( u \in \mathcal{B}_c^k(U) \) with \( k \geq q \).

Proof. We want to use induction on \( k = n, \ldots, q \). We start on \( n+1 \) forms; since \( S_{n-1} = I \), then (7.6) trivially holds.

Now, we assume by induction that (7.6) holds on \((0, k)\)-forms. Since \( k < n \), we use Kohn’s formula for Szego projection \( S_{k-1} = I - \bar{\partial}_b^*G_k\bar{\partial}_b \) to get
\[
\|\Psi^+T^sS_{k-1}u\|^2 = (\Psi^+T^sS_{k-1}u, \Psi^+T^su) - (\Psi^+T^sS_{k-1}u, \Psi^+T^s\bar{\partial}_bG_k\bar{\partial}_bu)
\]
\[
= (\Psi^+T^sS_{k-1}u, \Psi^+T^su) - ((\Psi^+T^s)^*\Psi^+T^s\bar{\partial}_bS_{k-1}u, G_k\bar{\partial}_bu)
\]
\[
+ ([\bar{\partial}_b, (\Psi^+T^s)^*\Psi^+T^s]S_{k-1}u, G_k\bar{\partial}_bu)
\]
(7.8)
We describe how to estimate the second two lines. The first term is controlled by the “small/large constant” argument. The second is 0. Hence we only need to estimate the last term. By (7.3), we have
\[
([\bar{\partial}_b, (\Psi^+T^s)^*\Psi^+T^s]S_{k-1}u, G_k\bar{\partial}_bu) = 2s(\Psi^+T^sS_{k-1}u, \bar{\partial}_b^*\zeta\Psi^+T^sG_k\bar{\partial}_bu)
\]
\[
+ (\text{good}^+S_{k-1}u, \Psi^+T^sG_k\bar{\partial}_bu)
\]
\[
\leq \epsilon(\|\Psi^+T^sG_k\bar{\partial}_bu\|^2 + \|\Psi^+T^sS_{k-1}u\|^2)
\]
\[
+ c_s(\|\text{good}^+S_{k-1}u\|^2 + \|\bar{\partial}_b^*\zeta\Psi^+T^sG_k\bar{\partial}_bu\|^2).
\]
(7.9)
To estimate the first term in the last line of (7.9), we combine (7.4) and Proposition 7.4. We obtain:
\[
\|\text{good}^+S_{k-1}u\|^2 \lesssim \epsilon\|\Psi^+T^sS_{k-1}u\|^2 + c_s, \text{admissible},
\]
(7.10)
where “admissible” stands for a term of type \( \|\Psi^+T^su\|^2 + \|\bar{\Psi}^0u\|^2_s + \|\Psi^+T^{s-1}S_{k-1}u\|^2 \) and where \( \epsilon \) and \( c_s \) are possibly new constants.

For the second one in the last line of (7.9), we use the hypothesis of the theorem, apply Proposition 6.3(i) and use the \((\theta - C)^k_u\) estimate in degree \( k \geq q \). This yields
\[
\|\bar{\partial}_b^*\zeta\Psi^+T^sG_k\bar{\partial}_bu\|^2 \leq \epsilon(\|\bar{\partial}_b^*\zeta\Psi^+T^sG_k\bar{\partial}_bu\|^2 + \|\bar{\partial}_b^*\zeta\Psi^+T^sG_k\bar{\partial}_bu\|^2) + c_s\|\Psi^0T^sG_k\bar{\partial}_bu\|^2
\]
\[
\leq \epsilon(\|\Psi^+T^sG_k\bar{\partial}_bu\|^2 + \|\Psi^+T^sS_{k-1}u\|^2) + c_s, \text{admissible}.
\]
(7.11)
Combination of (7.8), (7.9), (7.10) and (7.11) yields
\[
\|\Psi^+T^sS_{k-1}u\|^2 \leq \epsilon\|\Psi^+T^sG_k\bar{\partial}_bu\|^2 + c_s, \text{admissible}.
\]
(7.12)
We apply now Theorem 5.1
\[
G_k\bar{\partial}_bu = S_kA_kG_k\bar{\partial}_bA_k^{-1}(I - S_{k-1})u.
\]
We then get
\[ \| \Psi^+ T^s G_k \bar{\partial} u \|^2 = \| \Psi^+ T^s S_k A_t G_k, t \bar{\partial} A_t^{-1} (I - S_{k-1}) u \|^2 \leq C_s \| \Psi^+ T^s A_t G_k, t \bar{\partial} A_t^{-1} (I - S_{k-1}) u \|^2 + \text{admissible} \]
\[ \leq C_s \| \Psi^+ T^s (I - S_{k-1}) u \|^2 + \text{admissible} \leq C_s \| \Psi^+ T^s S_{k-1} u \|^2 + \text{admissible}. \]

(7.13)

Here the first inequality follows from the inductive hypothesis and the second one from Corollary 4.6. Combination of (7.12) and (7.13) yields the regularity of \( S_{k-1} \) which concludes the proof of inequality (7.6).

The proof of the second inequality, that is (7.7), is immediate consequence of Corollary 4.6, Theorem 5.1 and the first inequality, that is (7.6), in both cases \( n = 2 \) or \( n \geq 3 \).

□

Using the anti-Szego-projection and induction over increasing degree starting from 0 degree, we obtain a perfectly symmetric statement for the negative microlocalization.

**Theorem 7.2.** Let \( q \) be an index \( 1 \leq q \leq n - 1 \). Assume that property \((\theta - P_q)\) holds at \( z_0 \) and \( M \) is endowed with a strictly CR-plurisubharmonic function; then there exists a neighborhood \( U \) of \( z_0 \) such that
\[ \| \Psi^+ T^s R_k u \|^2 \precsim \| \Psi^+ T^s u \|^2 + \| \tilde{\Psi}^0 u \|^2 \] (7.14)
for any \( u \in \mathcal{B}_c^k(U) \) with \( k \leq n - q \), where \( s \geq 0 \) is integer. Moreover, we have
\[ \| \Psi^+ T^s G_k u \|^2 \precsim \| \Psi^+ T^s u \|^2 + \| \tilde{\Psi}^0 u \|^2 \] (7.15)
for any \( u \in \mathcal{B}_c^k(U) \) with \( k \leq n - q - 1 \).

**Corollary 7.3.** Let \( q \) be an index \( 1 \leq q \leq \frac{n-1}{2} \). Assume that Property \((\theta - P_q)\) holds at \( z_0 \) and \( M \) is endowed with a strictly CR-plurisubharmonic function; then there exists a neighborhood \( U \) of \( z_0 \) such that
\[ \| G_k u \|^2 \precsim \| u \|^2 \] (7.16)
for any \( u \in \mathcal{B}_c^k(U) \) with \( q \leq k \leq n - q - 1 \).

**Proof.** This is just a combination of Theorem [7.1] for \( \Psi^+ u \), Theorem [7.2] for \( \Psi^- u \) and Lemma [3.1] for \( \Psi^0 u \).
Proof of Theorem 1.5. Let $u \in \mathcal{B}^k$ and $\{\zeta_\nu\}$ be a partition of the unity subordinate to the covering $\{U_\nu\}$ of $M$ satisfying $\sum_\nu \zeta_\nu = 1$. Using Corollary 7.3 on each $U_\nu$, we get.

$$\|G_k u\|_s = \|G_k \sum_\nu \zeta_\nu u\|_s \lesssim \sum_\nu \|G_k \zeta_\nu u\|_s \lesssim \sum_\nu \|\zeta_\nu u\|_s \lesssim \|u\|_s. \quad (7.17)$$

Using the method of elliptic regularization as in [FK72] the a priori estimate (7.17) becomes an actual estimate in $H^s$ which yields the conclusion of the proof of Theorem 1.5. □

Appendix

The following is the Hodge decomposition theorem for $\bar{\partial}_b$.

Theorem 7.4. Let $M$ be the CR manifold of dimension $2n - 1$ with $n \geq 2$. Assume that $\bar{\partial}_b$ has closed range on $k$-forms with $0 \leq k \leq n - 2$. Then for any $0 \leq k \leq n - 1$, there exists a linear operator $G_k : L^2(M) \to L^2(M)$ such that

1. $G_k$ is bounded and $\mathcal{R}(G_k) \subset \text{Dom}(\Box_b)$.
2. We have

$$S_k = \begin{cases} I - \bar{\partial}_b^* G_{k+1} \bar{\partial}_b & \text{for } 0 \leq k \leq n - 2, \\ I & \text{for } k = n - 1, \end{cases}$$

$$R_k = \begin{cases} I - \bar{\partial}_b G_{k-1} \bar{\partial}_b^* & \text{for } 1 \leq k \leq n - 1, \\ I & \text{for } k = 0. \end{cases}$$

3. For any $L_2$ form $\alpha$ of degree $k$, we have

$$\alpha = \begin{cases} \bar{\partial}_b \bar{\partial}_b^* G_k \alpha \oplus \bar{\partial}_b^* \bar{\partial}_b G_k \alpha & \text{if } 1 \leq k \leq n - 2, \\ \bar{\partial}_b \bar{\partial}_b^* G_k \alpha \oplus S_k \alpha & \text{if } k = 0, \\ \bar{\partial}_b \bar{\partial}_b^* G_k \alpha \oplus R_k \alpha & \text{if } k = n - 1. \end{cases}$$

4. If $1 \leq k \leq n - 2$, we have

$$G_k \Box_b = \Box_b G_k = I \quad \text{on} \quad \text{Dom}(\Box_b),$$

$$\bar{\partial}_b G_k = G_{k+1} \bar{\partial}_b \quad \text{on} \quad \text{Dom}(\bar{\partial}_b),$$

$$\bar{\partial}_b^* G_k = G_{k-1} \bar{\partial}_b^* \quad \text{on} \quad \text{Dom}(\bar{\partial}_b^*),$$

$$S_k + R_k = I.$$  

5. We have

$$G_0 = \bar{\partial}_b^* G_1^2 \bar{\partial}_b \quad \text{and} \quad G_{n-1} = \bar{\partial}_b G_{n-2}^2 \bar{\partial}_b^*.$$
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