An Algebraic-Geometric Approach to Shuffled Linear Regression

Manolis C. Tsakiris*, Liangzu Peng*, Aldo Conca†, Laurent Kneip*, Yuanming Shi*, Hayoung Choi *

* School of Information Science and Technology, ShanghaiTech University, Shanghai, China.
† Department of Mathematics, University of Genova, Genova, Italy.

Abstract—Shuffled linear regression is the problem of performing a linear regression fit to a dataset for which the correspondences between the independent samples and the observations are unknown. Such a problem arises in diverse domains such as computer vision, communications and biology. In its simplest form, it is tantamount to solving a linear system of equations, for which the entries of the right hand side vector have been permuted. This type of data corruption renders the linear regression task considerably harder, even in the absence of other corruptions, such as noise, outliers or missing entries. Existing methods are either applicable only to noiseless data or they are very sensitive to initialization and work only for partially shuffled data. In this paper we address both of these issues via an algebraic geometric approach, which uses symmetric polynomials to extract permutation-invariant constraints that the parameters $x \in \mathbb{R}^n$ of the linear regression model must satisfy. This naturally leads to a polynomial system of $n$ equations in $n$ unknowns, which contains $x$ in its root locus. Using the machinery of algebraic geometry we prove that as long as the independent samples are generic, this polynomial system is always consistent with at most $n!$ complex roots, regardless of any type of corruption inflicted on the observations. The algorithmic implication of this fact is that one can always solve this polynomial system and use its most suitable root as initialization to the Expectation Maximization algorithm. To the best of our knowledge, the resulting method is the first working solution for small values of $n$ able to handle thousands of fully shuffled noisy observations in milliseconds.

Index Terms—Shuffled Linear Regression, Linear Regression Without Correspondences, Unlabeled Sensing, Algebraic Geometry.

1 INTRODUCTION

In the span of more than 200 years since the work of Legendre [1] and Gauss [2], linear regression has grown to be a cornerstone of statistics, with applications in almost every branch of science and engineering that involves computing with data. In its simplest form, classical linear regression considers a data model whose output $y$ is a linear function of known functions of the input data $u \in \mathbb{R}^n$. More precisely, given correspondences $\{u_j, y_j\}_{j=1}^m$, one seeks to find real numbers $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$, such that

$$y_j \approx a_j^\top x, \quad \forall j = 1, \ldots, m,$$

$$a_j := [a_1(u_j), \ldots, a_n(u_j)]^\top,$$

where $a_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are known functions. Between the classic least-squares solution of Gauss and modern approaches designed to deal with highly corrupted data [3], [4], [5], [6], a literature too vast to enumerate has been devoted to progressively more complicated versions of the linear regression problem [7].

1.1 Shuffled Linear Regression

In this paper we are interested in a particular type of data corruption, i.e., lack of correspondences. In such a case, one is given the input samples, or more precisely functions of the input samples as in (2), i.e.,

$$A = [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m \times n},$$

and a shuffled version $y = [y_{j_1}, \ldots, y_{j_m}]^\top \in \mathbb{R}^m$, of the observations $y_1, \ldots, y_m$, where the shuffling indices $j_1, \ldots, j_m$ are unknown. This problem of shuffled linear regression, also known as linear regression without correspondences [8], linear regression with shuffled data [9], unlabeled sensing [10] or permuted linear model [11], can be stated, in the absence of any other data corruptions, as follows:

Problem 1. Suppose we are given a matrix $A \in \mathbb{R}^{m \times n}$ with $n > m$, and a vector $y = (\Pi^*)^\top A \xi^* \in \mathbb{R}^m$, where $\xi^* \in \mathbb{R}^n$ is some vector and $\Pi^* \in \mathfrak{S}_m$ an $m \times m$ permutation matrix. We wish to efficiently compute $\xi^*$ when $\Pi^*$ is unknown and $n \gg m$. In other words, without knowing $\Pi^*$, we want to solve the linear system

$$\Pi^* y = A x.$$

Problem 1 naturally arises in many applications, such as in 1) computer vision, e.g., for simultaneous localization and mapping [12], [13], [14], multi-target tracking [15], and pose/correspondence estimation [16], [17], in 2) biology, e.g., for cell tracking [18], genome-assembly [19], and identical tokens in signaling pathways [20], in 3) communication networks, e.g., for data de-anonymization [21], [22] and low-latency communications in Internet-Of-Things networks [11], in 4) signal processing, e.g., when dealing with signals sampled in the presence of time jitter [23], as well as in 5) archaeology, e.g., for the estimation of the chronological ordering of archaeological deposits [24].

1.2 Prior Art

Over the past years there has been a considerable amount of work on special instances of Problem 1 in diverse contexts from com-
computes an estimate for $\Pi^*$ or $\Pi^*$ (9), (10), (25), §3, (26), §1, (7), (27), (28).

Letting $A$ be drawn at random from any continuous probability distribution, the work of [10] proves that any such $\Xi^*$ can be uniquely recovered with probability 1 as long as $m \geq 2n$. If on the other hand $m < 2n$, then $\Xi^*$ is not unique with probability 1. Further considering additive random noise on $y$, the authors in [8] establish lower bounds on the SNR, below which for any estimator there is a $\Xi^*$ whose estimation error is large. With $A$ drawn from the standard normal distribution (without loss of generality), $\Xi^*$ unknown but fixed and $y$ corrupted by additive random noise, [9] shows that, as long as the SNR exceeds a threshold, $\Pi^*$ coincides with high probability with the Maximum Likelihood Estimator (MLE), which they define as (also considered in [26], [25])

$$\hat{\Pi}_{\text{ML}}, \hat{x}_{\text{ML}} = \arg \min_{\Pi, x} \| P y - A x \|_2;$$

where $\Pi$ in [5] is constrained to be a permutation matrix. If on the other hand the SNR is not large enough, $\hat{\Pi}_{\text{ML}}$ differs from $\Pi^*$ with high probability, in agreement with the results of [8]. This is further complemented by [10], which shows that $\hat{x}_{\text{ML}}$ is locally stable under noise, in the sense that as the SNR tends to infinity $\hat{x}_{\text{ML}}$ tends to $\xi^*$. However, according to [26], for SNR fixed, $\hat{x}_{\text{ML}}$ is asymptotically inconsistent.

On the algorithmic front of solving Problem 1 the case $n = 1$ is well-understood and solved at a complexity of $O(m \log(m))$ by sorting ([9], [26]). For $n > 1$ the obvious approach is by brute force: for each permutation $\Pi$ among the $m!$ permutations of the $m$ entries of $y$ check whether the linear system $P y = A x$ is consistent, and if yes, solve it; this is an algorithm of complexity $O(n^2(m + 1)!)$). In [10] they authors write, ...although we showed that recovery of the unknown $\xi^*$ is possible from unlabeled measurements, we do not study the problem of designing an efficient algorithm to recover $\xi^*$. Our solution is to consider all possible permutations of the unlabeled observations which might be prohibitively complex in large dimensional problems. An algorithm that is more efficient than the brute force one is that of [8], which is able to reduce the complexity as a function of $m$ from $(m + 1)!$ to a factor of at least $m^7$. However, as the authors of [8] write, their algorithm strongly exploits the assumption of noiseless measurements and is also very brittle and very likely fails in the presence of noise; the same is true for the $O(n^4)$ complexity algorithm of [29]. Finally, the authors in [8] write we are not aware of previous algorithms for the average-case problem in general dimension $n$. In the same paper a $(1 + \epsilon)$ approximation algorithm with theoretical guarantees and of complexity $O((m / \epsilon)^n)$ is proposed, which however, is not meant for practical deployment, but instead is intended to shed light on the computational difficulty of the least squares problem [5]. Indeed, as per [9] this is an NP-hard problem for $n > 1$.

On the other hand, the approach that seems to be the predominant one in terms of practical deployment is that of solving [5] via alternating minimization [25]: given an estimate for $\Xi^*$ one computes an estimate for $\Pi^*$ via sorting; given an estimate for $\Pi^*$ one computes an estimate for $\xi^*$ via ordinary least-squares. Nevertheless, this approach is very sensitive to initialization and generally works only for partially shuffled data; see [25] for a soft variation of this alternating scheme.

In conclusion, to the best of our knowledge, there does not yet exist an algorithm for solving Problem [1] that is theoretically justifiable, efficient and robust to even mild levels of noise.

1.3 Our contributions

In this work, we contribute to the study of Problem 1 on both theory [23] and algorithms [3]. On the theoretical level, which is our main focus, we show that for generic noiseless data $y, A$, there is a unique solution $\Pi^*, \xi^*$, as soon as $m > n$. We show that $\xi^*$ is contained in the root locus of a system of $n$ polynomial equations in $n$ unknowns. Using advanced tools from algebraic geometry, we show that this polynomial system is always consistent with at most $n!$ complex solutions, regardless of any noise that may further corrupt the observations $y$.

The algorithmic implication of our theoretical development complements the current state-of-the-art: we solve the polynomial system and use a simple criterion to identify its most appropriate root to be used as an initialization for computing the MLE estimator [3] via alternating minimization. Even though solving the polynomial system entails in principle an exponential complexity in $n$, its complexity in $m$ is linear for any $n$. Furthermore, we use methods from automatic algebra-geometric solver generation to obtain highly efficient solvers for $n = 3, 4$. Overall, for $n \leq 5$, our approach is the first working solution to shuffled linear regression that remains stable under noise and has manageable complexity. As an example, for $n = 4, m = 10, 000$, and 1% additive noise, our method computes in 313 milliseconds a solution that is within 0.6% error from the ground truth.

2 Theoretical Contributions

The main contribution of this paper is to develop the theory to an algebraic geometric approach towards solving Problem 1. The key idea of this approach, described in detail in [22], uses symmetric power-sum polynomials to eliminate the unknown permutation, thus resulting in a polynomial system $P$ of $n$ equations in $n$ unknowns. These polynomials were considered implicitly in the statistical approach of [26], towards constructing a self-moment estimator. The authors of that paper wrote, ...in fact there may not be a solution to the system, which led them to compute their estimator via gradient descent on a highly non-convex objective function, a procedure lacking any theoretical guarantees and very sensitive to initialization, thus requiring a large number of restarts. The geometric significance of the polynomial system $P$ was also recognized by the last two authors of the present paper in the short conference paper [11], but important questions such as 1) “does $P$ have finitely many solutions?” or 2) “does $P$ have any solutions in the presence of noise?”; where left as open problems.

In this paper we answer these two questions in the affirmative in [23] via Theorems 2 and 3 respectively. The main message is that if the data $A \in \mathbb{R}^{m \times n}, y \in \mathbb{R}^m$ are generic (to be made precise in [21]), then using $n$ power-sum polynomials of degrees 1, 2, ..., $n$ as constraints, defines an algebraic variety that consists of at most $n!$ points, among which lies the solution $\xi^*$ of the shuffled linear regression Problem 1. In addition the same conclusion holds true in the case where the observation vector $y$ has undergone any type of corruption: the variety defined by the noisy polynomials is non-empty and consisting of at most $n!$ complex points. This guarantees that the equations are almost
always consistent even if the data are imperfect, which enables algorithmic development and applications.

The proofs of Theorems 2.3 given in 2.5 require a thorough understanding of the notion of dimension of polynomial systems of equations. We describe the necessary notions in 2.4 in an expository style for the benefit of the reader who is familiar with linear algebra but not with algebraic geometry.

2.1 Genericity and well-posedness

Before we are in a position to state our main results, i.e., Theorems 2.3 described in 2.3 we need to clarify what we mean when we refer to $A, y$ as being generic (2.1.1), and also settle the well-posedness of Problem 1 (2.1.2).

2.1.1 The notion of generic $A, y$

We start with an example.

Example 1. Consider the $2 \times 2$ matrix

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

where the entries of $B$ are real numbers. Then $B$ is invertible if and only if $b_{11}b_{22} - b_{12}b_{21} \neq 0$. Now consider the polynomial ring $\mathbb{R}[x_{11}, x_{12}, x_{21}, x_{22}]$ in four variables, with each of them corresponding to an entry of $B$. The equation

$$x_{11}x_{22} - x_{12}x_{21} = 0$$

defines a hypersurface $\mathcal{V}(x_{11}x_{22} - x_{12}x_{21})$ of $\mathbb{R}^4 \cong \mathbb{R}^{2 \times 2}$, and $B \in \mathcal{V}(x_{11}x_{22} - x_{12}x_{21})$ if and only if $B$ is non-invertible. This hypersurface has measure zero, say, under the standard normal distribution of $\mathbb{R}^4$, and hence if one samples $B$ at random from this distribution, $B$ will lie outside of $\mathcal{V}(x_{11}x_{22} - x_{12}x_{21})$ with probability 1. We express this by saying that “if $B$ is a generic $2 \times 2$ matrix, then $B$ is invertible”.

As Example 1 suggests, we usually attach the attribute generic to an object $\mathcal{O}$ (matrix $B$ in Example 1) with respect to some property $\mathcal{P}$ of $\mathcal{O}$ (invertibility of $B$ in Example 1). We say that “if $\mathcal{O}$ is generic then $\mathcal{P}$ is true”, and mean that the set of objects for which $\mathcal{P}$ is not true forms a zero-measure set of the underlying space that parametrizes that object under some continuous probability distribution. Hence sampling $\mathcal{O}$ at random from that probability distribution results in $\mathcal{O}$ having the property $\mathcal{P}$ with probability 1. Finally, if there are finitely many properties $\mathcal{P}_1, \ldots, \mathcal{P}_k$ of interest with regard to the object $\mathcal{O}$, and if $\mathcal{O}$ is generic with respect to each of the $\mathcal{P}_i$, then $\mathcal{O}$ is generic with respect to all of them; this follows from the fact that the union of finitely many zero-measure sets is a zero-measure set. The connection between algebraic geometry and measure theory that the reader should keep in mind for our purpose here, is that algebraic varieties have zero measure and that the union of finitely many algebraic varieties is an algebraic variety.

2.1.2 Uniqueness of $\Pi^*, \xi^*$

Our first result, Theorem 1 guarantees that Problem 1 is well-posed for generic data, in which case it makes sense to talk about a unique permutation $\Pi^*$ and a unique solution $\xi^*$.

Theorem 1. Suppose that $m > n$. Then as long as $A \in \mathbb{R}^{m \times n}$ is a generic matrix and $y$ the permutation of a generic vector in $\mathcal{V}(A)$, $\Pi^*$ and $\xi^*$ in Problem 1 are unique.

It is interesting to compare Theorem 1 with a simplified version of the main result of [10], stated next for convenience.

Theorem A (Theorem 1 in [10]). Suppose that $m \geq 2n$. Then as long as $A \in \mathbb{R}^{m \times n}$ is a generic matrix and $y$ the permutation of any vector in $\mathcal{V}(A)$, $\Pi^*$ in Problem 1 is unique.

Both Theorems 1 and A are concerned with a generic matrix $A$ and the permutation $y$ of some vector $v$ in $\mathcal{V}(A)$. In Theorem 1 the vector $v$ is taken to be a generic vector in $\mathcal{V}(A)$, and as it turns out it is enough that $m > n$ for $v$ to be uniquely defined from the data $A, y$. On the other hand, in Theorem A the vector $v$ is allowed to be any vector in $\mathcal{V}(A)$. This is a considerably more difficult setting, and the remarkable proof of [10] reveals that $v$ is still uniquely defined from the data, as long as now $m \geq 2n$.

Finally, we note that in the setting of Theorem A unique recovery of the permutation $\Pi^*$ is in principle not possible, as per Theorem 10 in [10]. Instead, one has to either allow for the observed vector $y$ to be generic (i.e., the permutation of a generic vector of $\mathcal{V}(A)$) in which case $\Pi^*$ is uniquely recoverable by Theorem 1, or consider unique recovery with high probability, which is indeed possible even when $y$ is corrupted by noise [9].

2.2 Eliminating $\Pi^*$ via symmetric polynomials

To describe the main idea of the algebraic-geometric approach to solving Problem 1, let $\mathcal{V}[z] := \mathbb{R}[z_1, \ldots, z_m]$ be the ring of polynomials with real coefficients over variables $z := [z_1, \ldots, z_m]^\top$. A polynomial $p \in \mathcal{V}[z]$ is called symmetric if it is invariant to any permutation of the variables $z$, i.e.,

$$p(z) := p(z_1, \ldots, z_m) = p(z_{\pi(1)}, \ldots, z_{\pi(m)}) =: p(\Pi z),$$

where $\pi$ is a permutation on $\{1, \ldots, m\} =: [m]$ (i.e., a bijective function from $[m]$ to itself) and $\Pi$ is an $m \times m$ matrix representing this permutation, with $i$th row the canonical vector $e_{\pi(i)}$ of all zeros, except a 1 at position $\pi(i)$.

Now let $A, y$ be as in Problem 1 and let $(\Pi^*, \xi^*)$, $\xi^* = [\xi^*_1, \ldots, \xi^*_m]^\top$ be a solution. Let $p \in \mathcal{V}[z]$ be a symmetric polynomial. Then the key observation is that the equality $\Pi^* y = A \xi^*$ implies the equality $p(\Pi^* y) = p(A \xi^*)$, and since $p$ is symmetric, this in turn implies

$$\Pi^* y = A \xi^* \implies p(y) = p(\Pi^* y) = p(A \xi^*).$$

That is, the symmetric polynomial $p$ serves in eliminating the unknown permutation $\Pi^*$ and providing a constraint equation that depends only on the known data $A, y$.

$$p(x) := p(A x) - p(y) = 0,$$

and which the solution $\xi^*$ that we seek needs to satisfy.

Example 2. Consider the data

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad y = \begin{bmatrix} -18 \\ 19 \\ 6 \end{bmatrix}.$$
It is simple to check that there is only one permutation

\[
\Pi^* = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]  

(12)

that results in a consistent linear system of equations

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-18 \\
19 \\
6 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 2 \\
0 & 3 \\
0 & -9 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
x_1 \\
x_2 \\
\end{bmatrix}
\]  

(13)

with solution \( \xi_1^* = 15 \), \( \xi_2^* = 2 \); every other permutation results in inconsistent equations, since it forces \( \xi_2^* \) to have different values. Now consider the symmetric polynomial

\[ p_1(z_1, z_2, z_3) = z_1 + z_2 + z_3, \]

(14)

which we may use as in (11) to generate the constraint

\[ (x_1 + 2x_2)^2 + 3x_2 - 9x_2 = -18 + 19 + 6 \]

(15)

\[ \Leftrightarrow \tilde{p}_1(x) := p_1(Ax) - p_1(y) = 0. \]

(17)

Indeed, we see that this constraint \( \xi^* = [15, 2]^\top \) satisfies (17).

The polynomial \( \tilde{p} \) in (10) is an element of the polynomial ring \( \mathbb{R}[x] \) in \( n \) variables \( x := [x_1, \ldots, x_n]^\top \), and the set of its roots, denoted as \( \mathcal{V}(\tilde{p}) := \{ \xi \in \mathbb{R}^n : \tilde{p}(\xi) = 0 \} \) and called an algebraic variety, in principle defines a hypersurface of \( \mathbb{R}^n \). Since the solution \( \xi^* \) to (11) is an element of the \( n \)-dimensional space \( \mathbb{R}^n \), and \( \xi^* \in \mathcal{V}(\tilde{p}) \) for any such \( \tilde{p} \), one expects that using \( n \) sufficiently independent such polynomials, will yield a system of \( n \) equations in \( n \) unknowns,

\[ \tilde{p}_1(x) = \cdots = \tilde{p}_n(x) = 0, \]

(18)

that has a finite number of solutions. Geometrically, these solutions are the intersection points of the corresponding \( n \) hypersurfaces \( \mathcal{V}(\tilde{p}_1), \ldots, \mathcal{V}(\tilde{p}_n) \), which contain all solutions to Problem [1], as well as potentially other irrelevant points.

**Example 3.** Continuing with Example 2, suppose we further use the symmetric polynomial

\[ p_2(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 \]

(19)

in (11) to obtain the additional constraint

\[ (x_1 + 2x_2)^2 + (3x_2)^2 + (-9x_2)^2 = (-18)^2 + 19^2 + 6^2, \]

(20)

\[ \Leftrightarrow \tilde{p}_2(x) := p_2(Ax) - p_2(y) = 0. \]

Solving (17) with respect to \( x_1 \) and substituting to (20), gives a quadratic equation in \( x_2 \) with solutions \( \xi_2 = 2 \) and \( \xi_2 \approx -2.67 \). Solving (17) for \( x_1 \) gives (up to second decimal precision)

\[ \mathcal{V}(\tilde{p}_1, \tilde{p}_2) = \left\{ \begin{array}{c}
15 \\
2 \\
-3.67 \\
-2.67 \\
\end{array} \right\}. \]

(21)

We see that \( \mathcal{V}(\tilde{p}_1, \tilde{p}_2) \) contains the solution of the linear system (4) but also an additional irrelevant point.

We note here that one may use \( n + 1 \) polynomials in order to remove the irrelevant points, e.g., as was done in (11). However, such an approach is of theoretical interest only, since a system of \( n + 1 \) (sufficiently independent) equations in \( n \) unknowns is bound to be inconsistent even in the slightest presence of noise. Instead, here we study a system of \( n \) equations in \( n \) unknowns and later show (see 3) how one can filter its roots of interest.

### 2.3 Main Results: zero dimension theorems

We present our main results in [2.3.1] (Theorem 3) and [2.3.2] (Theorem 4) for exact and corrupted data, respectively.

#### 2.3.1 Exact data

As Examples 2.3 suggest, a natural choice for our \( n \) symmetric polynomials is the first \( n \) power sum \( p_k(z) \in \mathbb{R}[z] := \mathbb{R}[z_1, \ldots, z_m], k \in [n] := \{ 1, \ldots, n \} \), defined as

\[ p_k(z) := z_1^k + \cdots + z_m^k. \]

(22)

The above discussion has already established that any solution \( \xi^* \) of (4) must satisfy the polynomial constraints

\[ \tilde{p}_k(x) = 0, \quad k \in [n], \]

(23)

\[ \tilde{p}_k(x) := p_k(Ax) - p_k(y) = \sum_{i=1}^m (a_i^\top x)^k - \sum_{j=1}^m y_j^k, \]

(24)

and \( a_i^\top \) denotes the \( i \)-th row of \( A \). The next major result guarantees that there can only be a finite number of other irrelevant solutions.

**Theorem 2.** If \( A \) is generic and \( y \) is some permutation of some vector in \( \mathbb{R}(A) \), then the algebraic variety \( \mathcal{V}(\tilde{p}_1, \ldots, \tilde{p}_n) \) contains all \( \xi_1, \ldots, \xi_n \in \mathbb{R}^n \) such that there exist permutations \( \Pi_1, \ldots, \Pi_n \) with \( \Pi_i^\top y = A\xi_i, \forall i \in [n] \), while it may contain at most \( n! - 1 \) other points of \( \mathbb{C}^n \). If in addition \( y \) is some permutation of a generic vector in \( \mathbb{R}(A) \), then \( \ell = 1 \).

Theorem 2 guarantees that the system of polynomial equations

\[ \tilde{p}_1(x) = \cdots = \tilde{p}_n(x) = 0, \]

(25)

always has a finite number of solutions in \( \mathbb{C}^n \) (at most \( n! \)), among which lie all possible solutions \( \xi_1, \ldots, \xi_n \in \mathbb{R}^n \) of Problem [1]. The importance of the solutions being finite in \( \mathbb{C}^n \) is computational: even if one is interested only in the real roots, knowing that the system has finitely many complex roots allows one to use much more efficient solvers. On the other hand, there exist many pathological cases where a system of polynomial equations has finitely many real roots but an infinity of complex roots, as the next example demonstrates; Theorem 2 guarantees that such a pathological case can not occur.

**Example 4.** The polynomial equation \( x_1^2 + x_2^2 = 0 \) has only one real root \( [0, 0]^\top \), while over the complex numbers it defines a union of two lines in \( \mathbb{C}^2 \).

#### 2.3.2 Corrupted data

Our next result addresses the issue of corrupted data. As is common in practical applications, we consider the case where we are given data \( A, y \) that are corrupted versions of \( A, y \). Even in the presence of the slightest corruptions the linear system

\[ \Pi y = \tilde{A} x \]

(26)

is expected to be inconsistent for every permutation matrix \( \Pi \). On the other hand, if the level of corruption is sufficiently small, there will exist a permutation \( \Pi = \Pi^* \) such that (26) is approximately consistent, in the sense that its least-squares solution \( \xi^* \) has small residual error. In such a case, we still want to use the concepts described so far to get an approximate solution of (26). Towards that end, define the corrupted power-sum polynomials as

\[ \tilde{p}_k(x) := p_k(Ax) - p_k(y), \quad k \in [n], \]

(27)
and consider the polynomial system \( \tilde{P} \) given by
\[
\tilde{p}_1 = \cdots = \tilde{p}_n = 0. 
\] (28)

These are \( n \) equations of degrees 1, 2, \ldots, \( n \) in \( n \) unknowns. Since \cite{26} has in principle no exact solution \((\mathbf{P^\ast}, \mathbf{P})\), it is entirely unclear whether \( \tilde{P} \) has any solutions at all, even in \( \mathbb{C}^n \).

**Example 5.** Let \( \tilde{P} \) as in (27)-28 with \( m \geq 5 \), \( n = 3 \), given by
\[
\begin{align*}
  x_1 + x_2 + x_3 &= 0 \quad (29) \\
  x_1x_2 + x_2x_3 + x_3x_1 &= 0 \quad (30) \\
  2x_1^2 + 2x_2^2 + 2x_3^2 &= 0. \quad (31)
\end{align*}
\]

Then \( \tilde{P} \) has no solutions in \( \mathbb{C}^3 \) because multiplying the first equation with \( 2x_1x_2 \) and subtracting it from the third equation gives \( (x_1 - x_2)^2x_3 = 0 \), which contradicts the second equation.

In principle, given \( n \) polynomial equations in \( n \) unknowns it is very complicated to determine whether a solution exists or not; a general criterion is given by Hilbert’s Nullstellensatz (Proposition \cite{4}), which can be checked algorithmically via the device of Groebner basis. Even if \( \tilde{P} \) has a solution, it might be the case that it has infinitely many of them as the next example shows.

**Example 6.** Let \( \tilde{P} \) as in (27)-28 with \( m \geq 5 \), \( n = 3 \), given by
\[
\begin{align*}
  x_1 + x_2 &= 0 \quad (32) \\
  x_2x_3 + x_1^2 &= 0 \quad (33) \\
  x_1^2 + x_2^2 + x_3^2 &= 0. \quad (34)
\end{align*}
\]

Notice that the third equation is equal to the product of the first equation with \( x_1^2 + x_2^2 \). Thus the third equation represents the union of two surfaces, one of them being the surface defined by the first equation. Hence, every point in the curve that arises as the intersection of the surfaces defined by the first two equations is a solution to \( \tilde{P} \), i.e., \( \tilde{P} \) has infinitely many solutions in \( \mathbb{C}^3 \).

Theorem \cite{3} below essentially states that the pathological situations of Examples \cite{5,6} can only occur for \( \mathbf{A} \) taking values on a subset of \( \mathbb{R}^{m \times n} \) of measure zero, regardless of what \( \mathbf{y} \) is.

**Theorem 3.** If \( \mathbf{A} \) is generic and \( \tilde{y} \in \mathbb{R}^n \) is any vector, then \( \mathcal{V}(\tilde{p}_1, \ldots, \tilde{p}_n) \) is non-empty containing at most \( n! \) points of \( \mathbb{C}^n \).

Theorem \cite{4} is important for at least two reasons. First, it guarantees that the system of polynomial equations \cite{28} is almost always consistent, i.e., there exists at least one solution. In the absence of this property this is immediate simply because \( \xi^\ast \) is a root to any of the noiseless polynomials \( \tilde{p}_j \). However, for noisy data the consistency of \cite{28} is far from obvious; for example, the authors of \cite{26} write it is generally impossible to solve these equations analytically; in fact there may not be a solution to the system. Theorem \cite{3} guarantees that such an issue is of no concern. Secondly, it guarantees that the system \cite{28} has a finite number of solutions in \( \mathbb{C}^n \); solving \cite{28} with any standard algebraic geometry software will yield a finite number of points, among which lies an approximate solution to the shuffled linear system \cite{26}. The algorithmic implication of Theorem \cite{3} is further pursued in \cite{3}.

### 2.4 The notion of dimension in algebraic geometry

#### 2.4.1 Geometric characterization of dimension

Let \( f_1, \ldots, f_s \) be polynomials in \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \) and consider their common root locus \( \mathcal{V}_{\mathbb{R}^n}(f_1, \ldots, f_s) \), called an algebraic variety, defined as
\[
\mathcal{V}_{\mathbb{R}^n}(f_1, \ldots, f_s) := \{ \xi \in \mathbb{R}^n : f_k(\xi) = 0, \forall k \in [s] \}. \quad (35)
\]

What is the dimension of \( \mathcal{V}_{\mathbb{R}^n}(f_1, \ldots, f_s) \)? If \( s = 1 \), then we have a single equation and one intuitively expects \( \mathcal{V}_{\mathbb{R}^n}(f_1) \) to be a hypersurface of \( \mathbb{R}^n \) having dimension \( n - 1 \); this is in analogy with linear algebra, where a single linear equation defines a linear subspace of dimension one less than the ambient dimension. However, as Example \cite{4} shows, it may be the case that \( \mathcal{V}_{\mathbb{R}^n}(f_1) \) consists of a single point (or even no points at all), in which case \( \dim \mathcal{V}_{\mathbb{R}^n}(f_1) \) should be zero (or \(-1 \) if the variety is empty), in analogy with linear algebra where a linear subspace has zero dimension only if it contains a single point (the origin 0).

To resolve the above issue and have a consistent definition of dimension that generalizes the linear algebraic one, it is necessary that we consider the common root locus of the polynomials in the algebraic closure \( \mathbb{C} \) of \( \mathbb{R}^n \):
\[
\mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) := \{ \xi \in \mathbb{C}^n : f_k(\xi) = 0, \forall k \in [s] \}. \quad (36)
\]

In that case, there is a well developed theory \cite{30,31,32} that leads to consistent characterizations of the dimension of the geometric object \( \mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \subset \mathbb{C}^n \) and that of its algebraic counterpart \( \{ f_1, \ldots, f_s \} \subset \mathbb{C}[x] \). The next definition provides the geometric characterization of \( \dim \mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \).

**Definition 1.** Defining \( \mathcal{Y} \subset \mathbb{C}^n \) to be closed if it is of the form \( \mathcal{Y} = \{ g_1, \ldots, g_r \} \) for some polynomials \( g_1, \ldots, g_r \in \mathbb{C}[x] \), and irreducible if it is not the union of two proper closed subsets, \( \dim \mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \) is defined to be the largest non-negative integer \( d \) such that there exists a chain of the form
\[
\mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \supset \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \mathcal{Y}_2 \supset \cdots \supset \mathcal{Y}_d,
\]
where each \( \mathcal{Y}_i \) is a closed irreducible subset of \( \mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \).

**Definition 1** is a generalization of the notion of dimension in linear algebra: if \( \mathcal{Y} \) is a linear subspace of \( \mathbb{C}^n \), then \( \dim \mathcal{Y} \) is precisely equal to the maximal length of a descending chain of linear subspaces that starts with \( \mathcal{Y} \); one can get such a chain by removing a single basis vector of \( \mathcal{Y} \) at each step.

**Example 7.** With \( e_i \) the vector with zeros everywhere except a 1 at position \( i \), and \( \mathcal{Y} = \text{Span}(e_1, \ldots, e_{n-1}) \subset \mathbb{C}^n \) admits a chain
\[
\mathbb{C}^n = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \mathcal{Y}_2 \supset \cdots \supset \mathcal{Y}_{n-1} \supset \mathcal{Y}_n := \{ 0 \}. \quad (38)
\]

A very important structural fact about algebraic varieties is the following decomposition theorem.

**Proposition 1.** Let \( \mathcal{Y} = \mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \) for some \( f_1 \in \mathbb{C}[x] \). Then \( \mathcal{Y} \) can be written uniquely as \( \mathcal{Y} = \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_s \) for some positive integer \( \ell \), where the \( \mathcal{Y}_i \) are irreducible closed sets of \( \mathbb{C}^n \) (see Definition \cite{4}), and they are minimal, in the sense that if one removes one of the \( \mathcal{Y}_n \), the resulting union is a strictly smaller set than \( \mathcal{Y} \). The \( \mathcal{Y}_i \) are called the irreducible components of \( \mathcal{Y} \).

**Definition 1** together with Proposition \cite{4} ensure that the only algebraic varieties \( \mathcal{V}_{\mathbb{C}^n}(f_1, \ldots, f_s) \) that have dimension zero are the ones that consist of a finite number of points; these are precisely the varieties of interest in this paper.

4. The acute reader may notice that there is a-priori no guarantee that such a maximal integer exists. However, this is true because \( \mathbb{C}^n \) is a Noetherian topological space, a technical notion that is beyond the scope of this paper.

5. For more information on the algebraic geometric structure of linear subspaces the reader is referred to Appendix C in \cite{13}.
Proposition 2. Let $\mathcal{Y} = \mathcal{V}_C(f_1, \ldots, f_s)$. Then $\dim \mathcal{Y} = 0$ if and only if $\mathcal{Y}$ consists of a finite number of points of $C^n$.

2.4.2 Algebraic characterization of dimension

Even though Definition 1 is perfectly acceptable from an intuitive point of view, it is not as convenient to use in practice, since one is usually given polynomials $f_1, \ldots, f_s$ of interest and wants to determine whether $\mathcal{V}_C(f_1, \ldots, f_s)$ has zero dimension, without having to solve the polynomial system. This is the case in this paper, where, e.g., to prove Theorem 2 we need to show that $\mathcal{V}_C(p_1, \ldots, p_m)$ has zero dimension for any suitable $A, y$: clearly, computing the common root locus of $p_1, \ldots, p_m$, as a function of $A, y$, is extremely challenging if not impossible. This is precisely where the algebraic characterization of $\dim \mathcal{V}_C(f_1, \ldots, f_s)$ comes in handy, since it allows its computation solely from the algebraic structure of $f_1, \ldots, f_s$.

To introduce this algebraic notion of dimension we first need the notion of an ideal of $C[x]$. Given polynomials $f_1, \ldots, f_s \in C[x]$, the ideal generated by these polynomials, denoted by $(f_1, \ldots, f_s)$, is the set of all linear combinations of the $f_i$, but in contrast to linear algebra, the coefficients of the linear combination are allowed to be polynomials themselves:

$$(f_1, \ldots, f_s) := \left\{ \sum_{i=1}^s g_i f_i, \forall g_i \in C[x] \right\}$$  \hspace{1cm} (39)

Next we need the notion of a prime ideal. An ideal $P \subseteq C[x]$ is called prime if it satisfies the following property: whenever the product of two polynomials is inside $P$, then at least one of these polynomials must be inside $P$. With that we have:

Proposition 3. $\dim \mathcal{V}_C(f_1, \ldots, f_s)$ is the largest non-negative integer $d$ such that there exists a chain of the form $P_0 \supsetneq P_1 \supsetneq P_2 \supsetneq \cdots \supsetneq P_d \supset (f_1, \ldots, f_s)$, where each $P_i$ is a prime ideal of $C[x]$.

Example 8. Continuing with Example 8, $C^n = \mathcal{V}(0)$ and $\mathcal{Y}_i := \text{Span}(e_1, \ldots, e_{n-i}) = \mathcal{V}(x_{n-i+1}, \ldots, x_n)$  \hspace{1cm} (41)

Since every ideal of the form $(x_{n-i+1}, \ldots, x_n)$ is prime, we have the ascending chain of prime ideals of length $n$:

$$(x_1, \ldots, x_n) \supset (x_2, \ldots, x_n) \supset \cdots \supset (x_n) \supset (0)$$  \hspace{1cm} (42)

We close this section by noting that the main tool behind the proof of Proposition 3 is the famous Hilbert’s Nullstellensatz, stated next, which holds over $C$ but not over $R$. This is why we need to work over $C$ to get consistent geometric and algebraic characterizations of the dimension of an algebraic variety.

Proposition 4. Let $f_1, \ldots, f_s$ be polynomials of $C[x]$. Then

- $\mathcal{V}_C(f_1, \ldots, f_s) = \emptyset$ if and only if $1 \in (f_1, \ldots, f_s)$.
- Suppose that $\mathcal{V}_C(f_1, \ldots, f_s) = \emptyset$ and let $f$ be a polynomial such that $f(\xi) = 0$, $\forall \xi \in \mathcal{V}_C(f_1, \ldots, f_s)$. Then $f \in (f_1, \ldots, f_s)$ for some positive integer $\ell$.

2.4.3 Dimension and homogenization

A monomial of degree $d$ is a polynomial of the form $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = [\alpha_1, \ldots, \alpha_n]^T$ is a vector of non-negative integers such that $\alpha_1 + \cdots + \alpha_n = d$. Every polynomial $f$ can be uniquely written as a linear combination of monomials:

$$f = \sum_{\alpha \in A} c_\alpha x^\alpha \in C[x]$$  \hspace{1cm} (43)

where $A$ is a finite set of multi-exponents and $c_\alpha \in C$ are the corresponding coefficients. Then $f$ is called homogeneous of degree $d$ if all its monomials have the same degree $d$.

Let $f_1, \ldots, f_s$ be a set of polynomials of $C[x]$. For rather subtle reasons beyond the scope of this paper, further characterizing the dimension of $\mathcal{V}_C(f_1, \ldots, f_s)$ beyond Proposition 3 is simpler when all the $f_i$ are homogeneous (this will be discussed in the next section). When this is not the case, there is a simple procedure called homogenization, through which we can convert non-homogeneous polynomials to homogenous.

Example 9. A non-homogeneous polynomial $f$ of degree 6 and its homogenization $f^{(h)}$:

$$f = x_1^3 x_2^2 x_3 + x_2^3 + x_3 \in C[x_1, x_2, x_3]$$  \hspace{1cm} (45)

$$f^{(h)} = x_1^3 x_2^2 x_3 + x_2^3 x_3^3 \in C[x_1, x_2, x_3, t]$$  \hspace{1cm} (46)

Now, let $I$ be the ideal generated by some polynomials $f_1, \ldots, f_s \in C[x]$ and consider the homogenization of this ideal

$$I^{(h)} = \{ f^{(h)} : f \in I \} \subset C[x, t]$$  \hspace{1cm} (47)

We note here the subtle fact that $I^{(h)}$ certainly contains $f_1^{(h)}, \ldots, f_s^{(h)}$, but in principle it is larger than the ideal generated by $f_1^{(h)}, \ldots, f_s^{(h)}$, as the next example illustrates.

Example 10. Let $f_1 = x_1^2 + x_2$, $f_2 = x_2^3 + x_3$ be polynomials of $C[x_1, x_2, x_3]$. Then $f_1^{(h)} = x_1^2 + x_2 t$, $f_2^{(h)} = x_2^3 + x_3 t$. Now, the polynomial $x_2 x_3 - x_1$ is in the ideal $I = (f_1, f_2)$, and it is already homogeneous so that $x_2 x_3 \in I^{(h)}$. However $x_2 x_3$ is not inside the ideal $I^{(h)}$ since $f_1^{(h)} + f_2^{(h)} = (x_1^2 + x_2 t, x_2^3 + x_3 t)$, since the latter only contains elements of degree 2 and higher.

Since the elements of $I^{(h)}$ are polynomials in $n+1$ variables, they define an algebraic variety $\mathcal{Y}^{(h)} = \mathcal{V}_{C^{n+1}}(I^{(h)})$ of $C^{n+1}$. What is the relationship between $\mathcal{Y}$ and $\mathcal{Y}^{(h)}$? It is actually not hard to see that if $[\xi_1, \ldots, \xi_n]^T$ is a point of $\mathcal{Y}$, then $\lambda [\xi_1, \ldots, \xi_n]^T$ is a point of $\mathcal{Y}^{(h)}$, for any $\lambda \in C$. Hence any non-zero point $\xi$ of $\mathcal{Y}$ gives rise to an entire line inside $\mathcal{Y}^{(h)}$, this line passes through the origin and $\xi$, and its intersection with the hyperplane $t = 1$ can be used to recover the original point $\xi$.

Hence $\mathcal{Y}^{(h)}$ is called the affine cone over $\mathcal{Y}$ with vertex $0 \in C^{n+1}$. Moreover, the variety $\mathcal{Y} \subset C^n$ is embedded inside the affine cone through a mapping that takes points to lines. In addition, $\mathcal{Y}^{(h)}$ contains so-called points at infinity, which are obtained by setting $t = 0$. As it turns out, there is a tight topological relationship between $\mathcal{Y}$ and $\mathcal{Y}^{(h)}$ and the important fact for our analysis is the following dimension theorem; see [34] for a detailed discussion for the non-expert reader in the context of subspace clustering.

Proposition 5. Let $\mathcal{Y}$ be an algebraic variety of $C^n$ and let $\mathcal{Y}^{(h)} \subset C^{n+1}$ be its affine cone. Then $\dim \mathcal{Y} = \dim \mathcal{Y}^{(h)} - 1$.

6. Let $\mathcal{I} \subset C[x]$ be an ideal. Then Hilbert’s Basis Theorem guarantees that $\mathcal{I}$ always has a finite set of generators, i.e., there is a positive integer $\ell$ and polynomials $g_1, \ldots, g_{\ell} \in C[x]$ such that $\mathcal{I} = (g_1, \ldots, g_{\ell})$. 
Example 11. Let $Y$ be an affine line of $C^2$ given by the equation $\alpha x_1 + \beta x_2 + \gamma = 0$. Then $Y^{(h)}$ is a plane through the origin in $C^3$ given by the equation $\alpha x_1 + \beta x_2 + \gamma t = 0$.

The next fact, known as Bezout’s Theorem, will be used in bounding the number of points of the zero-dimensional variety of Theorem 2.

Proposition 6. Let $h_1, \ldots, h_n$ be homogeneous polynomials of $C[x, t]$ of degrees $\deg(h_i) = d_i$, $i \in [n]$. If $V_{C^n}(h_1, \ldots, h_n)$ is a finite union of lines through the origin, then the number of these lines is at most $d_1 d_2 \cdots d_n$.

2.4.4 Regular sequences

In [2,2] we argued that if the polynomials $\hat{p}_1, \ldots, \hat{p}_n \in C[x]$ are sufficiently independent, then $\dim V_{C^n}(\hat{p}_1, \ldots, \hat{p}_n) = 0$, i.e., the dimension of the algebraic variety drops precisely by the number $n$ of its defining equations. More generally, the precise notion of what sufficiently independent should mean for polynomials $f_1, \ldots, f_s$, $s \leq n$, so that $\dim V_{C^n}(f_1, \ldots, f_s) = n - s$, is easier to characterize when all the $f_i$ are homogeneous. The right notion is that of a regular sequence.

Definition 2. Let $f_1, \ldots, f_s$ be polynomials of $C[x]$. Then $f_1, \ldots, f_s$ is a regular sequence if $(f_1, \ldots, f_s) \subseteq C[x]$, and for every $i = 2, \ldots, s$ the following property is true: whenever there is a polynomial $g$ such that $f_1 g \in (f_1, \ldots, f_{i-1})$, then we must have $g \in (f_1, \ldots, f_{i-1})$.

The crucial fact for our analysis is the following.

Proposition 7. Let $f_1, \ldots, f_s$, $s \leq n$, be non-constant homogeneous polynomials of $C[x]$. Then $\dim V_{C^n}(f_1, \ldots, f_s) = n - s$, if and only if $f_1, \ldots, f_s$ is a regular sequence.

Given a regular sequence of polynomials $f_1, \ldots, f_s$ in $C[x]$ of length $s < n$, it is of interest to be able to augment this sequence to a regular sequence $f_1, \ldots, f_s, g$ of length $s + 1$. The simplest type of a homogeneous polynomial $g$ that one may consider is a linear form $g = \ell \cdot x$, which represents a hyperplane with normal vector $\ell \in C^n$. As it turns out, almost all such hyperplanes qualify, with the exception of those with normal vector $\ell$ that lies inside an algebraic variety of $C^n$ determined by $f_1, \ldots, f_s$.

Proposition 8. Let $f_1, \ldots, f_s$, $s < n$, be a regular sequence of homogeneous polynomials of $C[x]$. If $\ell \in C^n$ is a generic vector, then $f_1, \ldots, f_s, \ell \cdot x$ is a regular sequence.

2.4.5 Initial ideals

The notion of the initial ideal $\in_{\prec}(I)$ of an ideal $I \subseteq C[x]$ with respect to a monomial order $\prec$ is a central one in computational algebraic geometry [35]. A more advanced object that is needed for our analysis in this paper is the initial ideal $\in_{\prec}(I)$ of $I$ with respect to a weight-order [36], [37], which we introduce next.

Let $w = (w_1, \ldots, w_n)$ be a vector of positive integers. To each variable $x_i$ of $C[x]$ we assign the weight $w_i$, and to each monomial $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ the weighted degree $d_w(x^{\alpha}) := w_1 \alpha_1 + \cdots + w_n \alpha_n$. A polynomial $f$ is called $w$-homogeneous, if all its monomials have the same weighted degree.

Example 12. Let $w = [1, 2, 3]^\top$ and let $f = x_1 x_2 + x_3$. Then $f$ is not homogeneous in the usual sense (see 2.4.3), but it is $w$-homogeneous of degree 3.

Now let $f \in C[x]$ be any polynomial. Then $f$ can be uniquely written as $f = f^{(d_1)} + f^{(d_2)} + \cdots + f^{(d_t)}$, with $d_1 > d_2 > \cdots > d_t > 0$, where each $f^{(d_i)}$ is a $w$-homogeneous polynomial of degree $d_i$. We define the initial form of $f$ with respect to $w$ as $\in_{w}(f) := f^{(d_1)}$. Given an ideal $I = (f_1, \ldots, f_s)$, we define $\in_{w}(I)$ to be the ideal generated by all initial forms $\in_{w}(f)$ for all $f \in I$. That is, $h \in \in_{w}(I)$ if and only if there exist polynomials $g_i \in C[x]$, $i \in [s]$, such that $h = \in_{w}(g_1 f_1 + \cdots + g_s f_s)$.

Example 13. Let $f = x_1 x_2 + x_3 + x_2^2$ and $w$ as in Example 2. Then $f = f^{(3)} + f^{(2)} + f^{(1)}$ with

$$ f^{(3)} = x_1 x_2 + x_3, \quad f^{(2)} = x_1^2 + x_2, \quad f^{(1)} = x_1. \quad (48) $$

Moreover, $\in_{w}(f) = x_1 x_2 + x_3$.

The initial ideal $\in_{w}(I)$ is certainly a significantly simpler object than the ideal $I$ itself, since it retains only the initial information about $I$, so to speak. What is remarkable though, is that many structural properties of $I$ are inherited from those of $\in_{w}(I)$. For this paper, the most important relationship is that the varieties defined by these two ideals have the same dimension:

Proposition 9. Let $I \subseteq C[x]$ be an ideal, $w \in Z_{\geq 0}^n$ a weight, and $\in_{w}(I) \subseteq C[x]$ the initial ideal of $I$ with respect to $w$. Then

$$ \dim V_{C^n}(I) = \dim V_{C^n}(\in_{w}(I)). \quad (49) $$

Hence to compute the dimension of the algebraic variety defined by an ideal $I = (f_1, \ldots, f_s)$, we may instead use the simpler object $\in_{w}(I)$. But how can we efficiently compute a set of generators for $\in_{w}(I)$ given $f_1, \ldots, f_s$? Note here that $(\in_{w}(f_1), \ldots, \in_{w}(f_s)) \subseteq \in_{w}(I)$ but equality does not hold in general, as the next example shows.

Example 14. Let $w = [1, 2, 3]^\top$ and $I = (f_1, f_2) \subset C[x]$ with

$$ f_1 = x_1^3 + x_2 + x_3, \quad f_2 = x_1^2 + x_2 x_3. \quad (50) $$

Then $\in_w(f_1) = \in_w(f_2) = x_1^3$ and so $(\in_w(f_1), \in_w(f_2)) = (x_1^3) = \{x_1^3 g : \forall g \in C[x]\}$. On the other hand $f_2 - f_1 = x_2 x_3 - x_2 - x_1 \notin I$ and so $\in_w(f_2 - f_1) = x_2 x_3 \notin \in_w(I)$. But clearly, $x_2 x_3 \notin (x_1^3)$. Hence $(\in_w(f_1), \in_w(f_2)) \nsubseteq \in_w(I)$.

The next fact, well known to experts in Gröbner basis theory and otherwise much less known, describes a situation where we can directly get the generators of $\in_{w}(I)$ from the generators of $I$.

Proposition 10. Let $w \in Z_{\geq 0}^n$ be a weight, and $f_1, \ldots, f_s$ be polynomials of $C[x]$, such that $\in_{w}(f_1), \ldots, \in_{w}(f_s)$ is a regular sequence. Then $\in_{w}(I) = (\in_{w}(f_1), \ldots, \in_{w}(f_s))$.

2.5 Proofs

2.5.1 Proof of Theorem 7

Let $A$ be a generic $m \times n$ matrix and $\xi$ an $n \times 1$ generic vector. Let $\Pi$ be any $m \times m$ permutation matrix different than the identity. Since $A$ is generic and $m > n$, the rank of $A$ is equal to $n$. Thus, to prove the proposition we need to show that

$$ \text{rank}[A \ \Pi A \xi] = n + 1. \quad (51) $$

Since $A$ is a generic, it can be written as the product $A = LU$ of an $m \times m$ generic lower triangular matrix $L$ and an $m \times n$ matrix $U$, whose top $n \times n$ block is a generic upper triangular matrix and its $(m - n) \times n$ bottom part is the zero matrix. Then

$$ L^{-1}[A \ \Pi A \xi] = [U \ L^{-1} \Pi L \xi]. \quad (52) $$


Because of the structure of $U$ it is enough to show that one of the last $m-n$ entries of the vector $L^{-1} \Pi LU\xi$ is non-zero. But because $\xi$ is generic, it is enough to show that one of the last $m-n$ rows of the matrix $L^{-1} \Pi LU$ is non-zero. Towards that end, we will show that the $(m,1)$ entry of this matrix is non-zero. This entry is zero if and only if

$$L_{m,1}\Pi L_{1,1} = 0,$$

(53)

where $L_{m,1}$ denotes the last row of $L^{-1}$ and $L_{1,1}$ denotes the first column of $L$. This last equation says that $L_{m,1}$ must be orthogonal to $\Pi L_{1,1}$. But by definition, $L_{1,1}$ is orthogonal to all the columns $L_{1,1}, \ldots, L_{m,1}$ of $L$ except the last one. Put together, we have that $L_{m,1}$ must be orthogonal to $L_{1,1}, \ldots, L_{m,1}, \Pi L_{1,1}$, which is possible only if

$$\text{rank}(L_{1,1} \cdots L_{m,1} \Pi L_{1,1}) = m - 1.$$  

(54)

Hence, it is enough to show that

$$\det[L_{1,1} \cdots L_{m,1} \Pi L_{1,1}] \neq 0.$$  

(55)

Towards that end, we view the entries $\ell_{1,i}$ of $L$ as variables of a polynomial ring $\mathbb{C}[\ell_{1,1}, \ldots, \ell_{m,m}]$ in $(m+1)/2$ variables. Then the determinant in (55) is a polynomial of $\mathbb{C}[\ell_{1,1}, \ldots, \ell_{m,m}]$. It is enough to show that this polynomial is non-zero. For in that case, it defines a hypersurface of $\mathbb{C}^{(m+1)/2}$, which will not contain $L$ since $L$ is generic. To show that the determinant is indeed a non-zero polynomial we consider an ordering of the variables

$$\ell_{11} > \ell_{21} > \ell_{22} > \ell_{31} > \ell_{32} > \cdots > \ell_{m-1,m} > \ell_{m,m},$$  

(56)

and then consider the induced lexicographic order on all the monomials of $\mathbb{C}[\ell_{1,1}, \ldots, \ell_{m,m}]$.

First, suppose that $\ell_{1,i}$ occurs in the $k$th entry of $\Pi L_{1,1}$ where $k > 1$. We will show that the largest monomial appearing in the definition of $\det[L_{1,1} \cdots L_{m,1} \Pi L_{1,1}]$ only occurs in one way; this guarantees that the determinant is non-zero. Indeed, the largest monomial occurs in the following unique way: choose the largest element from each row, starting from the rows that contain the largest elements. That is, pick element $\ell_{11}$ from row 1, element $\ell_{11}$ from row $k$, element $\ell_{11}$ from row $i$ for $i = 2, \ldots, k-1$, and elements $\ell_{1,i}$ from row $i + 1$ for $i = k + 1, \ldots, m - 1$.

Next, suppose that $k \geq 1$ is the largest index such that the $i$th entry of $\Pi L_{1,1}$ is equal to $\ell_{11}$ for $i = 1, \ldots, k$. In that case, we apply an elementary column operation by subtracting the first column of $[L_{1,1} \cdots L_{m,1} \Pi L_{1,1}]$ from its last, to obtain a new matrix $M$ of the same rank. Then the first $k$ entries of the last column of $M$ are zero, while its $(i,m)$ entry for $i > k$ is of the form $\ell_{s_{i-1}} - \ell_{1,i}$, with $s_i > k$, $\forall i = k + 1, \ldots, m$, and with $s_{k+1} \neq k + 1$. Let $t > 1$ be such that $s_{k+t} = k + 1$. Then the largest monomial in $\det(M)$ occurs in a unique way as the first term of the expanded product of elements $\ell_{1i}$ from row $i$ for $i = 1, \ldots, k + t - 1$, element $(\ell_{k+1,i} - \ell_{k+1,t})$ from row $k + t$, and elements $\ell_{i,k+t}$ from row $i + 1$ for $i = k + t, \ldots, m - 1$. That is, $\det(M)$ is a non-zero polynomial, which concludes the proof.

#### 2.5.2 Power-sums and regular sequences

Recall the power-sums polynomials

$$p_k = z_1^k + \cdots + z_m^k, \quad k : \text{positive integer},$$

(57)

that were used in 2.3 as the base symmetric polynomials towards eliminating the unknown permutation $\Pi^*$ associated with the shuffled linear regression Problem 1. These are polynomials in $m$ variables $z_1, \ldots, z_m$, i.e., $p_k \in \mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_m]$. The next two lemmas establish that $p_1, \ldots, p_m$ form a regular sequence.

**Lemma 1.** Let $\sigma_1, \ldots, \sigma_m$ be the elementary symmetric polynomials in $m$ variables $z$, defined as

$$\sigma_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} z_{i_1} z_{i_2} \cdots z_{i_k}.$$  

(58)

Then $\sigma_1, \ldots, \sigma_m$ form a regular sequence of $\mathbb{C}[z]$.

**Proof.** We will show that $\mathcal{V}_{\mathbb{C}}(\sigma_1, \ldots, \sigma_m) = \{0\}$, in which case the statement will follow from Proposition 7 together with the fact that each $\sigma_k$ is homogeneous (of degree $k$). We proceed by induction on $m$. If $m = 1$, $\mathbb{C}[z] = \mathbb{C}[z_1]$ and the only elementary symmetric polynomial is $z_1$. Clearly, $z_1$ vanishes only on $0 \in \mathbb{C}$. Suppose now that $m > 1$. Let $\zeta = [\zeta_1, \ldots, \zeta_m]^T \in \mathcal{V}(\sigma_1, \ldots, \sigma_m)$. Then $\sigma_m(\zeta) = \zeta_1 \cdots \zeta_m = 0$, and without loss of generality we can assume that $\zeta_m = 0$. Then $[\zeta_1, \ldots, \zeta_{m-1}]^T$ is in the variety generated by the $m - 1$ elementary symmetric polynomials in $m - 1$ variables and the induction on $m$ gives $\zeta_1 = \cdots = \zeta_{m-1} = 0$. \hfill $\square$

**Lemma 2.** The first $m$ power-sum polynomials $p_1, \ldots, p_m$ defined in (57) form a regular sequence of $\mathbb{C}[z]$.

**Proof.** Newton’s identities

$$k\sigma_k = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i} p_i, \quad \forall k \in [m],$$

(59)

show that $\sigma_1, \ldots, \sigma_m$ can be obtained inductively in terms of $p_1, \ldots, p_m$. On the other hand, the fundamental theorem of symmetric polynomials states that every symmetric polynomial can be written as a polynomial in $\sigma_1, \ldots, \sigma_m$. This implies that we have an equality of ideals $(p_1, \ldots, p_m) = (\sigma_1, \ldots, \sigma_m)$, hence an equality of algebraic varieties $\mathcal{V}_{\mathbb{C}}(p_1, \ldots, p_m) = \mathcal{V}_{\mathbb{C}}(\sigma_1, \ldots, \sigma_m)$. Thus Lemma 1 gives that $\mathcal{V}_{\mathbb{C}}(p_1, \ldots, p_m)$ consists of a single point and Proposition 7 together with the fact that each $p_k$ is homogeneous (of degree $k$) establishes that $p_1, \ldots, p_m$ is a regular sequence. \hfill $\square$

Now, recall that in 2.3 each base polynomial $p_k$ was used to furnish a polynomial equation $p_k(x) = p_k(Ax) - p_k(y) = 0$ that has the unique solution $\xi^*$ to Problem 4 should satisfy. Notice here how we are passing from polynomials $p_k \in \mathbb{C}[z]$ in $m$ variables to polynomials $\hat{p}_k \in \mathbb{C}[z]$ in $n$ variables. Notice further that even though the $p_k$ are symmetric and homogeneous, the $\hat{p}_k$ are in principle neither symmetric nor homogeneous. In fact,

$$\hat{p}_k = \bar{p}_k + p_k(y),$$

(60)

$$\bar{p}_k := p_k(\Pi^*Ax) = \sum_{i=1}^{m} (\alpha_i^*x)^k, \quad k = 1, \ldots, n,$$

(61)

is a homogeneous polynomial of degree $k$ and $p_k(y) \in \mathbb{R}$ is a constant. As it turns out, the homogeneous parts of $\bar{p}_k$ are the bridge for passing properties of the base polynomials $p_k$ to the polynomials of interest $\hat{p}_k$. The next two lemmas are the two required steps towards building that bridge.

7. This also follows from a more general theorem in 13, which states that any $m$ consecutive such polynomials $p_{k_1}, p_{k_1+1}, \ldots, p_{k_1+m-1}$, where $k_1$ is any positive integer, form a regular sequence of $\mathbb{C}[z]$. To make the paper more accessible and self-contained, we have taken the liberty of giving the rather simple case $k_1 = 1$ its own proof.
Lemma 3. Let $\ell_1^T, \ldots, \ell_{m-n}^T \in \mathbb{R}^{1 \times m}$ be a basis for the left nullspace of $A$. Then the algebraic varieties $V_{C^n}(p_1, \ldots, p_n)$ and $V_{C^n}(\ell_1^T z, \ldots, \ell_{m-n}^T z, p_1, \ldots, p_n)$ are isomorphic. In particular, the two varieties have the same dimension.

Proof. Let $X := V_{C^n}(\ell_1^T z, \ldots, \ell_{m-n}^T z, p_1, \ldots, p_n)$ and $Y := V_{C^n}(p_1, \ldots, p_n)$. To show that $X$ and $Y$ are isomorphic as algebraic varieties, it is enough to find a bijective map $f : X \rightarrow Y$ such that both $f$ and $f^{-1}$ are given by polynomials. First, we define $f$ by specifying its image on an arbitrary point $\xi$ of $Y$:

$$\xi \in Y \subset C^n \xrightarrow{f} A\xi \in C^m.$$  

(62)

We need to show that $f(\xi) \in X$. To do that, we need to check that $f(\xi)$ satisfies the defining equations of $X$. Towards that end, note that $\ell_j^T f(\xi) = \ell_j^T A\xi = 0$, for all $j \in [m-n]$, by definition of the $\ell_j$. Moreover, $p_k(f(\xi)) = p_k(A\xi) = \tilde{p}_k(\xi) = 0$, for all $k \in [n]$, where the last equality is true because $\xi \in Y$. Hence $f$ is indeed a map of the form $f : X \rightarrow Y$. Moreover, $f$ is given by linear polynomials, i.e., the coordinates of $f(\xi)$ are linear polynomials of the coordinates of $\xi$.

Next, we define a map $g : X \rightarrow Y$ as follows. Let $\xi \in X$. Then $\ell_j^T \xi = 0$, for all $j \in [m-n]$. Since the $\ell_j$ form a basis for $\mathcal{N}(A^T)$, this means that $\xi \in \mathcal{N}(A^T)$. But from basic linear algebra, $\mathcal{N}(A^T) = \mathcal{R}(A)$. Hence there exists some $\xi \in C^n$ such that $\xi = A\xi$. Moreover, this $\xi$ is unique because $A$ has full rank by the assumption of Problem 1. This allows us to define the map $g$ as $g(\xi) = (A^T A)^{-1} A^T \xi$. Then

$$\tilde{p}_k(g(\xi)) = p_k(A g(\xi)) = p_k(A A^T A)^{-1} A^T \xi = p_k(A\xi) = \psi_k(\xi),$$

(63)

(64)

(65)

(66)

where the last equality is true because $\xi \in X$. Hence $g(\xi) \in Y$. Moreover, $g$ is given by polynomials, since each coordinate of $g(\xi)$ is a linear polynomial in the coordinates of $\xi$. Finally, it is simple to check that $f$ and $g$ are inverses of each other.

Lemma 4. If $A$ is generic, then the polynomials $\tilde{p}_1, \ldots, \tilde{p}_n$ defined in (61) form a regular sequence of $C[x]$.

Proof. The space of all linear subspaces of dimension $n$ of $\mathbb{R}^m$ is an algebraic variety of $\mathbb{R}^m$, $M = (m \times n)$, called Grassmannian and denoted $G(n, m)$. The space $G(m-n, m)$ of all linear subspaces of dimension $m-n$ of $\mathbb{R}^m$ is also an algebraic variety of $\mathbb{R}^m$. These two varieties are isomorphic under a mapping that takes a linear subspace $S$ of dimension $n$ to its orthogonal complement $S^\perp$. Hence, $S$ is generic if and only if $S^\perp$ is generic. Choosing $A$ generic is the same as choosing $n$ generic vectors in $\mathbb{R}^m$, which is the same as choosing a generic subspace $S = \mathcal{R}(A)$ of $\mathbb{R}^m$. This is the same as choosing $S^\perp = \mathcal{N}(A^T)$, which is a set of $m-n$ generic vectors of a generic $(m-n)$-dimensional linear subspace of $\mathbb{R}^m$. Hence, with respect to any property that does not depend on $A$, the vectors $\ell_1, \ldots, \ell_{m-n}$ behave as generic vectors of $\mathbb{R}^m$.

So let $L = [\ell_1 \cdots \ell_{m-n}]$ be a matrix containing in its columns a set of generic vectors of $\mathcal{N}(A^T)$. By Lemma 2, $p_1, \ldots, p_m$ is a regular sequence of $C[z]$. By the definition of regular sequence, the subsequence $p_1, \ldots, p_n$ is also regular. By inductive application of Proposition 8

$$p_1, \ldots, p_n, \ell_1^T z, \ldots, \ell_{m-n}^T z$$

(67)

is also a regular sequence of $C[z]$. By Proposition 7 we have that

$$\dim V_{C^n}(\ell_1^T z, \ldots, \ell_{m-n}^T z, p_1, \ldots, p_n) = 0.$$  

(68)

By Lemma 3 we have that $\dim V_{C^n}(p_1, \ldots, p_n) = 0$. Then by Proposition 7 we have that $\dim V_{C^n}((\tilde{p}_1), \ldots, (\tilde{p}_n))$ is a regular sequence of $C[x]$.

2.5.3 Proof of Theorem 2

Having developed the machinery that led to Lemma 4, the task of proving Theorem 2 is a matter of putting this machinery to work.

Let $I = (p_1, \ldots, p_n)$ be the ideal generated by our polynomials $p_1, \ldots, p_n$. Consider the weight order of $C[z]$ defined by the vector $w = [1, \ldots, 1]^T \in \mathbb{Z}^n$ (see §2.4.3). Since $\tilde{p}_1, \ldots, \tilde{p}_n$ is a regular sequence of homogeneous polynomials (Lemma 4) and $\tilde{p}_k = \tilde{p}_k + \text{constant}$, we can obtain $\text{in}_w(I)$ just from the leading homogeneous terms of the generators of $I$, i.e., $\text{in}_w(I) = (p_1, \ldots, p_n)$ (Proposition 10). Since $\tilde{p}_1, \ldots, \tilde{p}_n$ is a regular sequence of length $n$ in the polynomial ring $C[x]$ of $n$ variables, we have $\dim V_{C^n}(\text{in}_w(I)) = 0$ (Proposition 2). Since $V_{C^n}(I)$ and $V_{C^n}(\text{in}_w(I))$ have the same dimension (Proposition 9), we have $\dim V_{C^n}(I) = 0$. Since zero-dimensional varieties have a finite number of points (Proposition 2), we have that $V_{C^n}(p_1, \ldots, p_n)$ consists of finitely many points.

We now bound from above the number of points of $Y := V_{C^n}((\tilde{p}_1), \ldots, (\tilde{p}_n))$. Consider the affine cone $Y^{(h)} = V_{C^{n+1}}((\tilde{p}_1), \ldots, (\tilde{p}_n))$ of $Y$ (see §2.4.3). Since $\dim V_{C^n} = 0$, we have $\dim Y^{(h)} = 1$ (Proposition 5). Let $Y^{(h)} = \bigcup W_i$ be the irreducible decomposition of $Y^{(h)}$ (Proposition 1). Since $Y^{(h)}$ is generated by homogeneous polynomials, it is the union of lines through the origin. Hence, each irreducible component $W_i$ is the union of lines through the origin. We argue that each $W_i$ is a single line through the origin. For if some $W_i$ is not a single line, let $\xi \in W_i$ be a point different than the origin $0$. Letting $L$ be the line through the origin and $\xi$, we have a chain $W_i \supseteq L \supseteq \{0\}$ of irreducible subsets of $Y^{(h)}$ of length 2. But this contradicts the fact that $\dim Y^{(h)} = 1$ (Definition 1). This shows that $Y^{(h)}$ is the union of $L$ lines through the origin. Then $\ell \leq \deg((\tilde{p}_1), \ldots, \deg((\tilde{p}_n)) = n!$ (Proposition 9). Finally, the points of $Y$ are in $1 - 1$ correspondence with the intersection points of $Y^{(h)}$ with the hyperplane $t = 1$ of $C^{n+1}$ (§2.4.3). But the number of these intersection points can not exceed the number of lines of $Y^{(h)}$.

2.5.4 Proof of Theorem 3

The proof is almost the same as the proof of Theorem 2 with an additional twist: we need to show that the variety $Y := V_{C^n}((\tilde{p}_1), \ldots, (\tilde{p}_n))$, where the $\tilde{p}_k$ are defined in (27), is non-empty. So suppose that $Y = \emptyset$. Then $1 \in \text{in}_w(I) = (\tilde{p}_1, \ldots, \tilde{p}_n)$ (Proposition 4). Hence $1 = \text{in}_w(1) \in \text{in}_w(I)$. But since $A$ is generic, the polynomials $\tilde{p}_k := p_k(A\xi), \ k \in [n]$, form a regular sequence (Lemma 4). Since $\tilde{p}_k := p_k(A\xi) = -p_k(\tilde{y})$, we must have that $\text{in}_w(I) = (\tilde{p}_1, \ldots, \tilde{p}_n)$ (Proposition 10). But all the $\tilde{p}_k$ are non-constant homogeneous polynomials and it is impossible for the equation $1 = g_1 \tilde{p}_1 + \cdots + g_n \tilde{p}_n$ to be true for any $g_1, \ldots, g_n \in C[z]$. This is a contradiction on the hypothesis $1 \in I$, which then implies that $Y$ is non-empty (Proposition 4). The rest of the proof is the same as that of Theorem 2.
3 ALGORITHMIC IMPLICATIONS

3.1 Algebraically-Initialized Expectation-Maximization

Assuming for simplicity (and without much loss of generality as per Theorem 1) that there is a unique solution \( \xi^* \) to the Shuffled Linear Regression (SLR) Problem 1, Theorem 2 guarantees that \( \xi^* \) is one of the finitely many complex roots of the polynomial system (23) of \( n \) equations in \( n \) unknowns. Even if the data \( y, A \) are corrupted, Theorem 3 further guarantees that the system remains consistent with \( L \leq n! \) complex roots; if in addition the corruption level is mild, we expect one of the roots to be a good approximation to the Maximum Likelihood Estimator (MLE) \( \xi_{\text{ML}} \) given in (35). Our goal is to isolate that root and refine it.

To begin with, solving the polynomial system of equations (23) can be done using any among several state-of-the-art polynomial system solvers [39]. Nevertheless, the complexity of solving such a system is known to be exponential in \( n \), even for well-behaved cases [40]. Practically speaking, this currently limits us to the regime \( n \leq 6 \), since for \( n = 6 \) a standard homotopy-based solver such as Bertini [41] takes about 37 minutes on a machine with specifications CPU:Intel(R) Core(TM) i7-8650U CPU @ 1.9GHz, Memory: 16GB. On the other hand, for \( n = 3, 4 \) using an automatic solver generator [42] along the lines of [43] for the specific structure of our system, we are able to obtain very efficient linear algebra based solvers that run in milliseconds.

Having obtained the roots \( \xi_1, \ldots, \xi_L \in \mathbb{C}^n, L \leq n! \), of the polynomial system, we retain only their real parts \( \hat{\xi}_1, \ldots, \hat{\xi}_L \), and identify the root that can serve as a first approximation to the ML estimator \( \hat{\xi}_{\text{ML}} \). We do this by selecting the root that yields the smallest \( \ell_2 \) error among all possible permutations \( \Pi \):

\[
\hat{\xi}_{\text{AI}} := \arg\min_{\Pi} \left\{ \min_{i} \left\| \Pi y - A(\hat{\xi}_i) \right\|_2 \right\}.
\]

(69)

We note that each inner minimization \( \min_{\Pi} \left\| \Pi y - A(\hat{\xi}_i) \right\|_2 \) in (69) can be solved by sorting (see (26) and so the computation of \( \hat{\xi}_{\text{AI}} \) is of complexity \( O(Ln \log(n)) \). Finally, we use the algebraic initialization \( \hat{\xi}_{\text{AI}} \) as an initialization to the Expectation Maximization algorithm of (35), which as noted in (12) consists of solving (5) via alternating minimization. The complete algorithmic listing, which we refer to as Algebraically-Initialized Expectation-Maximization (AI-EM) is given in Algorithm 1.

Algorithm 1 Algebraically-Initialized Expectation-Maximization

1: procedure AI-EM(y \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, T \in \mathbb{N}, \epsilon \in \mathbb{R}_+^+)
2: \quad \hat{p}_k(z) := \sum_{j=1}^{n} z_j^k, \hat{p}_k := \hat{p}_k(Ax) - \hat{p}_k(y), k \in [n];
3: \quad \text{Compute roots } \{\hat{\xi}_i\}_{i=1}^{L} \subseteq \mathbb{C}^n \text{ of } \{\hat{p}_k = 0, k \in [n]\};
4: \quad \text{Extract the real parts } \{\langle \hat{\xi}_i \rangle \}_{i=1}^{L} \subseteq \mathbb{R}^n;
5: \quad \{\xi_0, \Pi_0\} \leftarrow \arg\min_{\xi \in \langle \hat{\xi}_i \rangle_{i=1}} \left\| \Pi y - A\xi \right\|_2;
6: \quad t \leftarrow 0, \Delta \mathcal{J} \leftarrow \infty, \mathcal{J} \leftarrow \left\| \Pi y - A\xi_0 \right\|_2;
7: \quad \text{while } t < T \text{ and } \Delta \mathcal{J} > \epsilon \Delta \mathcal{J} \text{ do}
8: \quad \xi_t \leftarrow \arg\min_{\xi \in \mathbb{R}^n} \left\| \Pi_{t-1} y - A\xi \right\|_2;
9: \quad \Pi_t \leftarrow \arg\min_{\Pi} \left\| \Pi_{t-1} y - A\xi_t \right\|_2;
10: \quad \Delta \mathcal{J} \leftarrow \mathcal{J} - \left\| \Pi_t y - A\xi_t \right\|_2;
11: \quad \mathcal{J} \leftarrow \left\| \Pi_t y - A\xi_t \right\|_2;
12: \quad \text{end while}
13: \quad \text{Return } \xi_t, \Pi_t.
14: end procedure

3.2 Numerical evaluation

Algorithms. We perform a numerical evaluation of our proposed AI-EM Algorithm 1 which relies on solving the polynomial system by homotopy continuation in Bertini [41] for \( n \geq 5 \), and on our C++ elimination-template-base custom solvers for \( n = 3, 4 \) (see [42], [43] for more details). We also compare with two variations of EM algorithms that were proposed in (25) towards computing the Maximum Likelihood Estimator of (5), which are the only existing algorithms that we are aware of with the potential of operating on corrupted data. The first, referred to as LS-EM, computes the MLE via alternating minimization exactly as in Algorithm 1 except that it uses as initialization the vector that best fits the data \( y, A \) in the least-squares sense:

\[
\xi_{0,\text{LS}} := \arg\min_{\xi \in \mathbb{R}^n} \| y - Ax \|_2.
\]

(70)

The second variation, referred to as Soft-EM, uses the same initialization as LS-EM, but it replaces the brute search over all possible permutations by a dynamic empirical average of permutation matrices drawn from a suitable Markov chain. For all algorithms we use a maximal number of iterations \( T = 50 \) and for AI-EM and LS-EM a convergence parameter of \( \epsilon = 0.01 \).

Data. We use the following generative model with additive noise. We randomly sample \( A \in \mathbb{R}^{m \times n} \) and \( \xi^* \in \mathbb{R}^n \) from the corresponding standard normal distributions and perform a random permutation on the entries of \( A\xi^* \) to obtain a vector \( y \in \mathbb{R}^m \). We further corrupt \( y \) by adding to it a vector \( w \in \mathbb{R}^m \) sampled from the zero-mean normal distribution with covariance matrix \( \sigma^2 I_m \). The input data then consist of \( A \) and \( y := y + w \).

Metrics. We assess all methods by measuring the relative estimation error of \( \xi^* \), e.g., if \( \xi_{\text{AI-EM}} \) is the output of AI-EM, we report

\[
100\| \xi^* - \xi_{\text{AI-EM}} \|_2 / \| \xi^* \|_2.
\]

(71)

For AI-EM we further report the estimation error that corresponds to the best root \( \xi_{\text{AI}} \) of the polynomial system, defined as

\[
\xi_{\text{AI}} := \arg\min_{\xi \in \langle \hat{\xi}_i \rangle_{i=1}} \| \xi - (\hat{\xi}_i)_r \|_2,
\]

(72)

as well as that of our estimated best root \( \hat{\xi}_{\text{AI}} \) computed as in (69).

Results. Figures 14-15 depict the estimation error of the compared methods for fully shuffled data, for \( n = 3, 4, 5, \sigma = 0 : 0.01 : 0.1 \) and \( m = 500 \) fixed, averaged over 100 independent trials. Evidently, both LS-EM and Soft-EM fail. This is not surprising, since, when the data are fully shuffled, the least-squares initialization (70) used by both LS-EM and Soft-EM is rather far from the ground truth \( \xi^* \). On the other hand, \( \xi_{\text{AI}} \) remains relatively close to \( \xi^* \) as \( \sigma \) increases and our actual initialization \( \xi_{\text{AI}} \) coincides with \( \xi_{\text{AI}} \) for \( \sigma \leq 0.04 \) and is slightly worse than \( \xi_{\text{AI}} \) otherwise. Regardless, the alternating minimization further refines \( \xi_{\text{AI}} \) leading to even lower errors than \( \xi_{\text{AI}} \).

Fig. 17 depicts the estimation error for different percentages 0% : 10% : 100% of partially shuffled data, and for \( n = 4, m = 500, \sigma = 0.01 \) fixed. In such a case, only a subset of the entries of the vector \( A\xi^* \) is shuffled according to a random permutation. As seen, both LS-EM and Soft-EM perform much better than for fully shuffled data. In fact, LS-EM is comparable to AI-EM for up
to 50% shuffled data, upon which percentage it starts deteriorating and eventually breaks down for 90% shuffled data. On the other hand, Soft-EM starts breaking down as soon as 20% of the data have been shuffled. Fig. 3 shows a more detailed behavior of the methods for 0% : 1% : 10% shuffled data in the same setting. We see that both LS-EM and AI-EM are almost perfectly accurate, while Soft-EM presents already an error of 6% for 4% shuffled data, and 20% for 10% shuffled data, i.e., Soft-EM seems to be accurate only when the percentage of shuffled data is small. On the other hand, its true advantage is that it can handle large regression dimensions (\( n \geq 7 \)), a regime in which AI-EM is currently not applicable due to computational reasons.

Table 1 provides numerical data for the estimation error and running times for different values of \( n = 3, 4, 5, 6 \). We see that for \( n = 3, 4 \) solving the polynomial system \( \mathcal{P} \) via our custom solvers requires only a few milliseconds. On the other hand, the running time increases exponentially with 45 seconds required to solve the \( n = 5 \) system and about 37 minutes for \( n = 6 \). As expected, the running time of the Alternating Minimization (AM) remains practically unaffected by the values of \( n \) that we are considering. Interestingly, the estimation error of the AI-EM algorithm is also stable and of the order of 0.1% regardless of \( n \).

Table 2 attests to the scalability of our algorithm AI-EM in terms of \( m \). Fixing \( n = 4 \) and \( \sigma = 0.01 \), the table reports running times and estimation errors for different values of \( m \), ranging from \( m = 10^3 \) to \( m = 10^5 \). Indeed, solving the polynomial system requires only 10 msec for \( m = 10^3 \) and only 268 msec for \( m = 10^5 \). The increase in the running time of the alternating minimization from 15 msec to 1.3 sec is more noticeable due to the complexity \( O(m \log m) \) of the sorting operation required to compute the optimal permutation at each iteration. For \( m = 1000 \) the estimation error of AI-EM is only 0.3%, but as \( m \) increases,
the device (69) for selecting the best root of \( P \) becomes less accurate; e.g., for \( m = 10^9 \), \( \xi_{AM}^* \) is 2.2% away from \( \xi^* \), as opposed to 4.7% for \( \xi_{AI} \).

| \( m \) | \( \mathcal{P} \) | AM | \( \xi_{AM} \) | \( \xi_{AI} \) | AI-EM |
|---|---|---|---|---|---|
| 1,000 | 10 | 15 | 2.8% | 2.8% | 0.3% |
| 5,000 | 21 | 105 | 2.2% | 2.4% | 0.6% |
| 10,000 | 32 | 281 | 1.8% | 1.8% | 0.6% |
| 25,000 | 66 | 357 | 2.3% | 3.4% | 2.6% |
| 50,000 | 126 | 613 | 2.5% | 3.5% | 3.1% |
| 100,000 | 268 | 1,271 | 2.2% | 4.7% | 4.5% |

### 4 Conclusions

In this paper we primarily focused on the theory of an algebraic geometric solution to the problem of Shuffled Linear Regression (SLR). The main object of interest was a polynomial system \( \mathcal{P} \) of \( n \) equations in \( n \) unknowns constraining the parameters \( \mathbf{x} \in \mathbb{R}^n \) of the linear regression model. The main theoretical contribution was establishing that \( \mathcal{P} \) is almost always consistent even for corrupted data with at most \( n! \) complex roots. Our algorithmic proposal consisted of solving \( \mathcal{P} \) and using its most appropriate root as initialization to the Expectation-Maximization algorithm. This yielded an efficient solution for small values of \( n \), stable under mild levels of noise and outperforming existing state-of-the-art methods. Applications to real-world problems as well as solving \( \mathcal{P} \) efficiently for large \( n \) are the subject of current research.

### References

[1] A. M. Legendre, *Nouvelles méthodes pour la détermination des ordres des côtes.* Firmin Didot, Paris, 1805.

[2] C. F. Gauss, *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solent Ambientum,* 1809.

[3] R. A. Maronna, “Principal components and orthogonal regression based on robust scales,” *Technometrics,* vol. 47, no. 3, pp. 264–273, 2005.

[4] Y. Wang, C. Dicle, M. Szaiaier, and O. Camps, “Self scaled regularized robust regression,” in *IEEE Conference on Computer Vision and Pattern Recognition (CVPR),* 2015.

[5] G. Lerman and T. Maunu, “An overview of robust subspace recovery,” arXiv:1803.01013v1, 2018.

[6] M. C. Tsakiris and R. Vidal, “Dual principal component pursuit,” *Journal of Machine Learning Research,* vol. 19, no. 18, pp. 1–49, 2018.

[7] R. A. Maronna, R. D. Martin, and V. J. Yohai, *Robust statistics: theory and methods.* Wiley Series in Probability and Statistics, 2006.

[8] D. J. Hsu, K. Shi, and X. Sun, “Linear regression without correspondence and using its most appropriate root,” *Mathematical and Computer Modelling,* vol. 53, no. 1-2, pp. 145 – 153, 2011.

[9] P. David, D. DeMenthon, R. Duraiswami, and H. Samet, “Softposit: Simultaneous pose and correspondence determination,” *International Journal of Computer Vision,* vol. 59, no. 3, pp. 259–284, 2004.

[10] M. Marques, M. Stosic, and J. Costa e Silva, “Subspace matching: Unique solution to point matching with geometric constraints,” in *International Conference on Computer Vision (ICCV),* 2009, pp. 1288–1294.

[11] C. Rose, I. S. Mian, and R. Song, “Timing channels with multiple identical quanta,” arXiv:1208.1070v2, 2018.

[12] S. Thrun and J. J. Leonard, *Simultaneous Localization and Mapping.* Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 871–889.

[13] J. Ummikrishnan and M. Vetterli, “Sampling and reconstruction of spatial fields using mobile sensors,” *IEEE Transactions on Signal Processing,* vol. 61, no. 9, pp. 2328–2340, May 2013.