On Non-degenerate Chaos Processes

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Abstract

We consider a process \( \{X_t\}_{0 \leq t \leq 1} \) in a fixed Wiener chaos \( \mathcal{H}_n \). We establish some non-degenerate properties and related results for \( \{X_t\}_{0 \leq t \leq 1} \). As an application, we show that solution to SDE driven by \( \{X_t\}_{0 \leq t \leq 1} \) admits a density. Our approach relies on an interplay between Malliavin calculus and analysis on Wiener space.

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1 Introduction

Over the last two decades, tremendous progresses have been achieved in the study of differential equations driven by Gaussian rough paths (\[3\] \[6\] \[4\] \[2\] \[1\] \[10\] \[5\]). Central to this direction is the non-degenerate property of Gaussian processes. It was first appeared in \[4\]; a non-degenerate property for Gaussian processes in the following form: we say non-degeneracy holds for a Gaussian process \( \{X_t\}_{0 \leq t \leq 1} \) if for any \( g \in C^\infty([0,1]) \), we have

\[
\left\{ \int_0^1 g_s dh_s = 0, \forall h \in \mathcal{H} \right\} \Rightarrow \{g \equiv 0\},
\]

(1)
where $\mathcal{H}$ is the Cameron-Martin space associated with $\{X_t\}_{0 \leq t \leq 1}$. Condition (1) has many interesting consequences. Most notably, (1) implies any Gaussian variable of the form
\[
\int_0^1 g_s dX_s, \quad g \in C^\infty([0,1]), \quad g \neq 0
\]
has positive variance (hence a density). In fact, (1) also implies the existence of the density for Gaussian rough differential equations (see [3]).

It is well known that Gaussian processes belong to the larger family of processes that live in a fixed Wiener chaos (also known as chaos processes). More specifically, they all live in the first Wiener chaos. The study of chaos vectors and processes have gained a lot of success in recent years (for instance, [16] [15] [12] [14] and references therein). It is a bit surprising that the non-degenerate property of chaos processes received little attention during this period. It is a natural question to ask that, to what extend can we generalize characterizations (1) and (2) from Gaussian processes to chaos processes.

The goal of this paper is to prove the abovementioned two characterizations of non-degeneracy for chaos processes. More precisely, we will define
\[
X_t = I_n(f_t), \quad \forall t \in [0,1],
\]
where $I_n$ is the $n$-th multiple Wiener integral, and look for a class of such processes that exhibit similar non-degenerate behaviors (1) and (2). In this endeavor, we first need to clarify what would be the proper generalization of (1) to a general chaos process. Indeed, the role of a non-Gaussian $\{X_t\}_{0 \leq t \leq 1}$ in (1) is not clear. Moreover, for an abstract Wiener space, which will be our setting, elements from $\mathcal{H}$ might not be processes, and the integral in (1) is not well defined. To answer this question, let us observe that we can rewrite (1) as
\[
\left\{ \int_0^1 g_s d\langle DX_s, h \rangle_{\mathcal{H}} = 0, \forall h \in \mathcal{H} \right\} \Rightarrow \{ g \equiv 0 \},
\]
where $DX_s$ is the Malliavin derivative of $X_s$. Thus, a reasonable generalization of (1) should be of the form
\[
\left\{ \int_0^1 g_s dDX_s = 0 \right\} \Rightarrow \{ g = 0 \}.
\]

More precisely, we will prove that
\[
\mathbb{P} \left\{ \int_0^1 g_s dDX_s = 0, g_t \neq 0 \right\} = 0.
\]

To sum up, our goal is to prove (3) and the existence of density for variables of the form given by (2).

We expect many arguments can be directly borrowed from previous studies in Gaussian processes, but there are two fundamental difficulties that one must overcome.

1. Unlike a Gaussian process, whose distribution is completely determined by the first two moments, the distribution of a general chaos process cannot be characterized by its moments up to a finite order. As a result, one must find a reasonable assumption that gives enough control over the process. We will see in the next section that one such option is to look into the finer structures generated by partially integrating the kernels.
2. The Malliavin derivative of a Gaussian process is deterministic. The study of (1) is a pure analysis question and many analysis tools, such as the Hardy–Littlewood inequality, can thus be used. However, a chaos process living in the \( n \)th-homogeneous chaos has Malliavin derivatives in terms of the \((n - 1)\)th-homogeneous chaos, which in general are random. This is much more than just a small nuisance, as one will have to find a condition that can be applied to all sample paths of \( \{DX_t\}_{0 \leq t \leq 1} \) in a uniform way.

The rest of the paper is organized as follows: Section 2 is a detailed discussion of our assumptions with their motivations, followed by the statements of our main results. Section 3 provides some necessary preliminary materials. Section 4 is devoted to the study of non-degenerate property of chaos processes, while Section 5 proves the existence of density for differential equations driven by chaos processes.

2 Statements of assumptions and main theorems

Our first assumption provides regularity for the sample paths of \( \{X_t\}_{0 \leq t \leq 1} \). Thanks to the fact that \( \{X_t\}_{0 \leq t \leq 1} \) belongs to a fixed chaos, we will see that the Malliavin derivative of \( \{X_t\}_{0 \leq t \leq 1} \) has the same regularity.

Assumption 1. Let \( \{f_t\}_{0 \leq t \leq 1} \subset \mathcal{H}^\otimes n \) be the kernels of \( X_t \). We assume \( f_0 = 0 \) and that we can find constants \( C > 0 \) and \( \theta > 1 \), such that for any \( 0 \leq s < t \leq 1 \)

\[
0 < \|f_t - f_s\|_{\mathcal{H}^\otimes n} \leq C|t - s|^\frac{\theta}{2}.
\]

By the moment equivalence of Wiener chaos, for any \( p > 1 \)

\[
\mathbb{E}|X_t - X_s|^p \leq C_{2,p} \left( \mathbb{E}|X_t - X_s|^2 \right)^{\frac{p}{2}} \leq C_{2,p,n}|t - s|^\frac{p\theta}{2}.
\]

It is then a consequence of Kolmogorov continuity theorem that the sample paths of \( \{X_t\}_{0 \leq t \leq 1} \) are \( \rho \)-Hölder continuous, for any \( \rho < \theta/2 \), almost surely. Moreover, we have by Meyer’s inequality (see proposition 3.2)

\[
\mathbb{E}||DX_t - DX_s||_{\mathcal{H}^\otimes n}^p \leq C_p \mathbb{E}|X_t - X_s|^p \leq C'_{2,p,n}|t - s|^\frac{p\theta}{2}.
\]

Another application of Kolmogorov continuity theorem gives \( \{DX_t\}_{0 \leq t \leq 1} \) the same regularity as \( \{X_t\}_{0 \leq t \leq 1} \).

Remark 2.1. As a rather straightforward consequence of assumption 1, integrals like

\[
\int_0^t X_s dX_s, \quad \int_0^t X_s dX_s
\]

are well-defined as Young’s integrals. In the language of the rough paths theory, we are now in the regular case.

In order to state and explain our second and most important assumption as transparent as possible, let us first briefly discuss its motivation. We know that it is standard, when studying integrals like

\[
\int_0^t Y_s dX_s,
\]
to consider its discrete Riemann sum approximation. The latter is nothing but linear combinations of finite-dimensional distributions of \( \{X_t\}_{0 \leq t \leq 1} \). If \( \{X_t\}_{0 \leq t \leq 1} \) is centered Gaussian, its distribution is completely determined by its covariance function. In other words, assumption \( \textbf{I} \) is sufficient for us, should \( \{X_t\}_{0 \leq t \leq 1} \) be Gaussian. But as we explained in the previous section, the situation is drastically different when \( \{X_t\}_{0 \leq t \leq 1} \) lives in a general chaos. We need an assumption that can give us enough control on the finite-dimensional distributions of \( \{X_t\}_{0 \leq t \leq 1} \).

It helps to look at the Gaussian case for hints. It is well-known that if \( \{X_t\}_{0 \leq t \leq 1} \) is Gaussian, then its finite dimensional distributions are non-degenerate Gaussian if and only if any finite collection of \( \{f_t\}_{0 \leq t \leq 1} \) are linearly independent. Since our goal is to make \( X_t = I_n(f_t) \) as non-degenerate as possible, we should try to formulate a similar but stronger version of linearly independence in terms of \( \{f_t\}_{0 \leq t \leq 1} \). It turns out, one such formulation can be carried out as follows.

For each \( f_t \), let us consider

\[
F_t = \text{Span} \left\{ \xi = (f_t, e_{t_1} \otimes e_{t_2} \otimes \cdots \otimes e_{t_{n-1}}) \mathcal{H}^{(a_{n-1})}, \ i_1, i_2, \cdots, i_{n-1} \geq 1 \right\},
\]

where \( \{e_k\}_{k \geq 1} \) is an orthonormal basis of \( \mathcal{H} \). We define \( F_{s,t} \) in the same way with \( f_t \) replaced by \( f_t - f_s \). One can extract plenty of information about \( I_n(f_t) \) from \( F_t \). For instance, \( I_n(f_t) \) and \( I_n(f_s) \) are independent if and only if \( F_t \perp F_s \) (see \([18],[11])\). We will also see later that the Malliavin derivative of \( X_t \) lives in the subspace \( F_t \) (lemma \[4N\]). Intuitively, \( F_t \) as a subspace of \( \mathcal{H} \) contains all the “building blocks” for \( f_t \).

Our previous discussions motivate our next

**Assumption 2.** We assume that there exists a constant \( 0 < \alpha < 1 \) such that for any \( m \in \mathbb{N}^+, k, l \in \mathbb{N} \), any nonzero vector \( (a_1, \cdots, a_m) \), any \( \{t_1 < t_2 < \cdots < t_m\} \subset [0,1] \), any \( \{s_1, \cdots, s_k\} \subset [0,1] \) and any \( \{r_1, r_2, \cdots, r_l\} \subset (l, 1] \), we have

\[
\| (a_1 \xi_{t_1, t_2} + \cdots + a_m \xi_{t_{m-1}, t_m}) - P_{[s_1, s_k] \cup [r_1, r_l]} (a_1 \xi_{t_1, t_2} + \cdots + a_m \xi_{t_{m-1}, t_m}) \|_{\mathcal{H}}^2 > \alpha \| a_1 \xi_{t_1, t_2} + \cdots + a_m \xi_{t_{m-1}, t_m} \|_{\mathcal{H}}^2,
\]

where \( \xi_{t_i, t_{i+1}} \in F_{t_i, t_{i+1}} \) and \( P_{[s_1, s_k] \cup [r_1, r_l]} \) is the projection onto

\[
\text{Span}\{F_{0,s_1}, \cdots, F_{s_k,t_1}, F_{t_{l_1},r_1}, \cdots, F_{r_{l-1},r_l}\}.
\]

**Remark 2.2.** We notice that, given \( f_0 = 0 \), \([4]\) is nothing but a quantitative way of saying that the any finite collection of \( \xi_t \) (hence \( f_t \)) are linearly independent. It is essentially a non-determinism condition, reminiscent of condition 2 of \([5]\). The main difference, is that we take projections of linear combinations of \( \xi_{s,t} \) rather than just a single term. This is for the need of controlling finite-dimensional distributions of \( \{X_t\}_{0 \leq t \leq 1} \) as we discussed before.

**Remark 2.3.** It is worth mentioning that there is another way to see why considering linear combinations of \( \xi_{s,t} \) is necessary in our setting. One often needs to use conditioning in the study of Gaussian processes. Conditional variance has a natural algebraic expression given by the Schur complement (see \([5]\) section 6) for Gaussian random variables. However, It is very difficult, if not impossible, to get such explicit conditioning formulas for general chaos random variables (see \([17]\)).

**Remark 2.4.** Assumption \( \textbf{III} \) also relates to locally non-determinism and locally approximately independent increments property introduced in \([13]\). We instead formulated our assumption in a global manner.
Although we do not need the next two assumptions for our main results, they are closer related to the assumptions used for Gaussian processes. Moreover, they enable us to apply techniques that will give a special version of theorem 2.5 when the integrand is deterministic, which also highlights some unique properties of chaos processes.

Our next assumption is just assumption 2 at the kernel level.

**Assumption 3.** We can find $0 < \beta < 1$ such that for any $m \in \mathbb{N}^+, k, l \in \mathbb{N}$, any nonzero vector $(a_1, \ldots, a_m)$, any $t_1 < t_2 < \cdots < t_m \in [0, 1]$, any $s_1, \ldots, s_k \in [0, t_1)$ and any $r_1, r_2, \cdots, r_l \in (t_m, 1]$, we have

$$\left\| (a_1f_{t_1, t_2} + \cdots + a_mf_{t_{m-1}, t_m}) - P_{[s_1, s_k] \cup [r_1, r_l]} (a_1f_{t_1, t_2} + \cdots + a_mf_{t_{m-1}, t_m}) \right\|_H^2 > \beta \left\| a_1f_{t_1, t_2} + \cdots + a_mf_{t_{m-1}, t_m} \right\|_H^2,$$

where $P_{[s_1, s_k] \cup [r_1, r_l]}$ is the projection onto

$$\text{Span}\{f_{s_1, t_1}, \cdots, f_{s_k, t_1}, f_{t_1, r_1}, \cdots, f_{r_l-1, r_l}\}.$$

Finally, our last assumption

**Assumption 4.** We assume that for any $[u, v] \subset [s, t] \subset [0, 1]$, we have

$$(f_v - f_u, f_t - f_s)_H \geq 0.$$

This assumption will function in the exact same way as condition 3 of [5]; they both ensure the covariance matrices of $X_t$ have non-negative row sums.

### 2.1 Main results

Now we are ready to state our main results.

**Theorem 2.5.** Let $\{X_t\}_{0 \leq t \leq 1} = I_n(f_t)$ be a continuous process in the $n$-th homogeneous Wiener chaos. If $\{X_t\}_{0 \leq t \leq 1}$ satisfies assumptions 4 and 2 then for any process $\{g_t\}_{0 \leq t \leq 1}$ whose sample paths are $\tau$-Hölder continuous almost surely, with $\tau + \rho > 1$, we have

$$\mathbb{P} \left\{ \int_0^1 g_t dD X_t = 0, \ g_t \neq 0 \right\} = 0.$$

As an application, we have

**Theorem 2.6.** Let $\{X_t\}_{0 \leq t \leq 1} = I_n(f_t)$ be a continuous process in the $n$-th homogeneous Wiener chaos satisfies assumptions 7 and 2. Consider the following SDE

$$dY_t = \sum_{i=1}^d V_i(Y_t) dX_t^i + V_0(Y_t) dt, \ Y_0 = y_0 \in \mathbb{R}^d.$$

If $\{V_i\}_{0 \leq i \leq d} \subset C_0^\infty(\mathbb{R}^d)$ and $\{V_i\}_{1 \leq i \leq d}$ form an elliptic system, then, for any $0 < t \leq 1$, $Y_t$ has a density with respect to the Lebesgue measure on $\mathbb{R}^d$. 

5
3 Preliminaries

3.1 Wiener chaos

Let $\mathcal{H}$ be a real separable Hilbert space. We say $X = \{W(h) : h \in \mathcal{H}\}$ is an isonormal Gaussian process over $\mathcal{H}$, if $X$ is a family of centered Gaussian random variables defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}(W(h)W(g)) = \langle h, g \rangle_{\mathcal{H}}.$$ 

We will further assume that $\mathcal{F}$ is generated by $W$.

For every $k \geq 1$, we denote by $H^k$ the $k$-th homogeneous Wiener chaos of $W$ defined as the closed subspace of $L^2(\Omega)$ generated by the family of random variables $\{H_k(W(h)) : h \in \mathcal{H}\}$ where $H_k$ is the $k$-th Hermite polynomial given by

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} \left(e^{-\frac{x^2}{2}}\right).$$

$H_0$ is by convention defined to be $\mathbb{R}$.

For any $k \geq 1$, we denote by $\mathcal{H}^\otimes k$ the $k$-th tensor product of $\mathcal{H}$. If $\phi_1, \phi_2, \cdots, \phi_n \in \mathcal{H}$, we define the symmetrization of $\phi_1 \otimes \cdots \otimes \phi_n$ by

$$\phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(n)},$$

where $\Sigma_n$ is the symmetric group of $\{1, 2, \cdots, n\}$. The symmetrization of $\mathcal{H}^\otimes k$ is denoted by $\mathcal{H}^\otimes k$. We consider $f \in \mathcal{H}^\otimes n$ of the form

$$f = e^{\hat{\otimes} k_1} \hat{\otimes} e^{\hat{\otimes} k_2} \cdots \hat{\otimes} e^{\hat{\otimes} k_n},$$

where $\{e_i\}_{i \geq 1}$ is an orthonormal basis of $\mathcal{H}$ and $k_1 + \cdots + k_n = n$. The multiple Wiener-Itô integral of $f$ is defined as

$$I_n(f) = H_{k_1}(W(e_{j_1})) \cdots H_{k_n}(W(e_{j_n})).$$

If $f, g \in \mathcal{H}^\otimes n$ are of the above form, we have the following isometry

$$\mathbb{E}(I_n(f)I_n(g)) = n!(f, g)_{\mathcal{H}^\otimes n}.$$ 

For general elements in $\mathcal{H}^\otimes n$, the multiple Wiener-Itô integrals are defined by $L^2$ convergence with the previous isometry equality.

Let $\mathcal{G}$ be the $\sigma$-algebra generated by $\{W(h) : h \in \mathcal{H}\}$, then any random variable $F \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ admits an orthonormal decomposition (Wiener chaos decomposition) of the form

$$F = \sum_{k=1}^{\infty} I_k(f_k),$$

where $f_0 = \mathbb{E}(F)$ and $f_k \in \mathcal{H}^\otimes k$ are uniquely determined by $F$.

We end this subsection with a useful lemma.

**Lemma 3.1.** For $b \geq 1$, and let $f \in \mathcal{H}^\otimes n$ be such that $\|f\|_{\mathcal{H}^\otimes n} > 0$. Then $I_n(f)$ has a density with respect to the Lebesgue measure on $\mathbb{R}$. 


3.2 Malliavin calculus

Let $\mathcal{FC}^\infty$ denote the set of cylindrical random variables of the form

$$F = f(W(h_1), \cdots, W(h_n)),$$

where $n \geq 1$, $h_i \in \mathcal{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$; that means $f$ is a smooth function on $\mathbb{R}^n$ bounded with all derivatives. The Malliavin derivative of $F$ is a $\mathcal{H}$-valued random variable defined as

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n))h_i.$$

By iteration, one can define the $k$-th Malliavin derivative of $F$ as a $\mathcal{H}^\otimes k$-valued random variable. For $m, p \geq 1$, we denote $\mathbb{D}^{m,p}$ the closure of $\mathcal{FC}^\infty$ with respect to the norm

$$\|F\|_{m,p} = \mathbb{E}(\|F\|^p) + \sum_{k=1}^{m} \mathbb{E}\left\| D^k F \right\|^p_{\mathcal{H}^\otimes k}.$$

For any $F \in L^2(\Omega)$ with Wiener chaos decomposition $F = \sum_{n=0}^{\infty} I_n(f_t)$, we define the operator $C = -\sqrt{-L}$ by

$$CF = -\sum_{n=0}^{\infty} \sqrt{n} I_n(f_t),$$

provided the series is convergent. Here $L$ is the Ornstein-Uhlenbeck operator. Now we can state the Meyer’s inequality

**Proposition 3.2.** For any $p > 1$ and any integer $k \geq 1$ there exist positive constants $C_{p,k}, C'_{p,k}$ such that for any $F \in \mathcal{FC}^\infty$

$$C_{p,k} \mathbb{E}\|D^k F\|^p_{\mathcal{H}^\otimes k} \leq \mathbb{E}\|C^k F\|^p \leq C'_{p,k} \left\{ \mathbb{E}\|F\|^p + \mathbb{E}\|D^k F\|^p_{\mathcal{H}^\otimes k} \right\}.$$

For a random vector $F = (F_1, \cdots, F_n)$ such that $F_i \in \mathbb{D}^{1,2}$, we denote $C(F)$ the Malliavin matrix of $F$, which is a non-negative definite matrix defined by

$$C(F)_{ij} = \langle DF_i, DF_j \rangle_{\mathcal{H}}.$$

If $\det(C(F)) > 0$ almost surely, then the law of $F$ admits a density with respect to the Lebesgue measure on $\mathbb{R}^n$. This is a basic result that we will use in later sections.

3.3 SDE with Young’s setting

Suppose that $\{X_t\}_{0 \leq t \leq 1}$ is a process whose sample paths are $\beta$-Hölder continuous almost surely for some $\beta \in (1/2, 1)$. We consider the following stochastic differential equation

$$Y_t = y_0 + \sum_{i=1}^{d} \int_{0}^{t} V_i(Y_s) dX_s + \int_{0}^{t} V_0(Y_s) ds, \quad y_0 \in \mathbb{R}^d. \quad (6)$$

It is well known that if $\{V_i\}_{0 \leq i \leq d} \subset C_b^2(\mathbb{R}^d)$, then (6) admits a unique solution. A standard way to prove this fact is to regard $\{Y_t\}_{0 \leq t \leq 1}$ as a fixed point of the map

$$\mathcal{M} : C^\beta([0, 1]) \rightarrow C^\beta([0, 1])$$

$$Y \mapsto y_0 + \sum_{i=1}^{d} \int_{0}^{t} V_i(Y_s) dX_s + \int_{0}^{t} V_0(Y_s) ds,$$
where the integrals are understood as Young’s integrals. By Young’s maximal inequality, one has the estimate
\[ \| Y \|_\beta \leq C \| V \|_{C^2} \| X \|_\beta, \]
where the constant \( C \) only depends on \( \beta \). (There is a subtlety in the actual proof of these results; one usually needs to replace \( \beta \) by \( \beta' \in (1/2, \beta) \). We refer to [7] chapter 8 for more details.)

The differential equation (6) is a smooth map with respect to the initial point. We can define the Jacobian process of \( Y_t \) as
\[ J_{t=0} = \frac{\partial Y_t}{\partial y_0}. \]

By differentiating both sides of (6) we can obtain a non-autonomous linear differential equation governing \( J_{t=0} \):
\[ dJ_{t=0} = \sum_{i=1}^d DV_i(Y_t)J_{t=0}dX_t + DV_0(Y_t)J_{t=0}dt, \quad J_{0=0} = I_{d \times d}. \] (7)

The inverse \( J_{t=0}^{-1} := J_{0-t} \) of the Jacobian process can be found by solving the SDE
\[ J_{0-t} = -\sum_{i=1}^d J_{0-t}DV_i(Y_t)dX_t - J_{0-t}DV_0(Y_t)dt, \quad J_{0=0} = I_{d \times d}. \] (8)

4 Non-degenerate properties of \( X_t \)

4.1 Deterministic integrand

Let us first consider the case where \( \{g_t\}_{0 \leq t \leq 1} \) is deterministic. In the rest of the paper we will often use \( X_{s,t} \) as a short notation for \( X_t - X_s \).

**Proposition 4.1.** Let \( \{X_t\}_{0 \leq t \leq 1} = I_n(f_t) \) be a continuous process in the \( n \)-th homogeneous Wiener chaos. Let \( g_t \) be a \( \tau \)-Hölder continuous function with \( \tau + \rho > 1 \) and Hölder norm \( \|g\|_\tau \neq 0 \). Then under assumptions [4], [5] and [4] one of the following two inequalities is always true:
\[ \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 \geq \frac{\beta}{4} \left( \sup_{t \in [0,1]} |g_t| \right)^2 \mathbb{E}(X_1)^2, \] (9)

or,
\[ \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 \geq \frac{\beta}{4} \left( \sup_{t \in [a,b]} |g_t| \right)^2 \mathbb{E}(X_{a,b})^2, \] (10)

for a certain interval \( [a, b] \subset [0, 1] \) such that
\[ \left( \frac{\sup_{t \in [0,1]} |g_t|}{2\|g\|_\tau} \right)^{\frac{1}{\rho}} \leq |b - a|. \]

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Proof. We have
\[ \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 = \int_0^1 \int_0^1 g_s g_t d\mathbb{E}(X_s X_t) = n! \int_0^1 \int_0^1 g_s g_t d(f_s, f_t)_{\mathcal{H}^\otimes n}. \]
Since \( g \) is continuous, we can always find an interval \([a, b] \in [0, 1]\) such that
\[ \inf_{s \in [a, b]} |f_s| \geq \frac{1}{2} \sup_{s \in [0, 1]} |f_s|. \]
Consider the dyadic partitions \( \{D_k\}_{k \geq 1} \) of \([0, 1]\). For each \( k \geq 1 \), we write
\[ D_k = \{t_0, t_1, \ldots, t_{2^k}\}. \]
The discrete approximation of the above double integral along \( D_k \) is given by
\[ \int_{D^n \times D^n} g_s g_t d\mathbb{E}(X_s X_t) = \mathbb{E} \left( g_{t_0} X_{t_0, t_1} + g_{t_1} X_{t_1, t_2} + \cdots + g_{t_{2^k}} X_{t_{2^k}, t_{2^k+1}} \right)^2. \]
Since \( \{X_t\}_{t \in [0, 1]} \) belongs to the \( n \)-th homogeneous Wiener chaos, we have
\[ \int_{D^n \times D^n} g_s g_t d\mathbb{E}(X_s X_t) = n! \left\| \sum_{(t_i, t_{i+1}) \subseteq [a, b]} g_{t_i} f_{t_i, t_{i+1}} + \sum_{(t_i, t_{i+1}) \not\subseteq [a, b]} g_{t_i} f_{t_i, t_{i+1}} \right\|_{\mathcal{H}^\otimes n}^2. \]
Define
\[ \Gamma_k = \sum_{(t_i, t_{i+1}) \subseteq [a, b]} g_{t_i} f_{t_i, t_{i+1}}, \]
\[ \mathcal{S}_k = \text{Span} \left\{ f_{t_i, t_{i+1}} : (t_i, t_{i+1}) \not\subseteq [a, b] \right\}. \]
We use \( P_{\mathcal{S}_k} \) to denote the projection onto \( \mathcal{S}_k \), then by assumption 3 we have
\[ \int_{D^k \times D^k} g_s g_t d\mathbb{E}(X_s X_t) \]
\[ = n! \left\| \left( \Gamma_k - P_{\mathcal{S}_k}(\Gamma_k) \right) + P_{\mathcal{S}_k}(\Gamma_k) + \sum_{(t_i, t_{i+1}) \in [a, b] \cup (b, 1]} g_{t_i} f_{t_i, t_{i+1}} \right\|_{\mathcal{H}^\otimes n}^2 \]
\[ \geq n! \left\| \left( \Gamma_k - P_{\mathcal{S}_k}(\Gamma_k) \right) \right\|_{\mathcal{H}^\otimes n}^2 \geq \beta n! \|\Gamma_k\|_{\mathcal{H}^\otimes n}^2. \] (11)
On the other hand, we have
\[ \|\Gamma_k\|_{\mathcal{H}^\otimes n}^2 = (gt_{k_1}, gt_{k_2}, \ldots) \cdot (f_{k_1, k_{i+1}}, f_{k_{i+1}, k_{i+2}})_{\mathcal{H}^\otimes n} \cdot (gt_{k_1}, gt_{k_2}, \ldots)^T, \]
where \( \{t_{k_i}\} \subset [a, b] \cap D_k \). Since the last expression is quadratic in \( (gt_{k_1}, gt_{k_2}, \ldots) \) and \( g_s \) does not change sign in \([a, b]\), we may assume without loss of generality that
\[ g_s \geq \frac{1}{2} \sup_{r \in [0, 1]} |g_r|, \forall s \in [a, b]. \]
To bound $\|\Gamma_k\|_{\mathcal{H}^\otimes n}^2$ from below, it is equivalent to solving the optimization problem

$$
\inf_{x \in C^k} x^T \cdot M^k \cdot x,
$$

where

$$
C^k = \{ x \in \mathbb{R}^{d_k} : x_i = \frac{1}{2} \sup_{r \in [0,1]} |g_r|, 1 \leq i \leq d_k \},
$$

$$
M^k_j = \langle f_{t_{k_j}}^1, f_{t_{k_j+1}} \rangle_{\mathcal{H}^\otimes n},
$$

and $d_k$ is the row number of $M^k$. By assumption 4 for any $k \geq 1$ the matrix $M^k$ has non-negative row sums. Thus, by lemma 6.2 of [5], the infimum of (12) is achieved when all the components of $x$ equal to $\frac{1}{2} \sup_{r \in [0,1]} |g_r|$. We therefore deduce

$$
\|\Gamma_k\|_{\mathcal{H}^\otimes n}^2 \geq \frac{1}{4} (\sup_{r \in [0,1]} |g_r|)^2 \|f_{t_{k_1}, t_{k_{d_k}}}\|_{\mathcal{H}^\otimes n}^2 = \frac{1}{4n!} (\sup_{r \in [0,1]} |g_r|)^2 \mathbb{E}(X_{t_{k_1}, t_{k_{d_k}}})^2.
$$

Sending $k$ to infinity, we infer from (11) that

$$
\int_0^1 \int_0^1 g_s g_t d\mathbb{E}(X_s X_t) \geq \frac{\beta}{4} (\sup_{r \in [0,1]} |g_r|)^2 \mathbb{E}(X_{a,b})^2.
$$

Finally, let us give estimate to the interval $[a, b]$. There are two possibilities which are mutually exclusive. In the first case, we have

$$
|g(x)| > \frac{1}{2} \sup_{r \in [0,1]} |g_r|, \forall x \in [0,1].
$$

We can thus choose $[a, b] = [0, 1]$, and deduce (9) from (13). In the second case, there exists $x \in [0, 1]$ such that $|g(x)| = \frac{1}{2} \sup_{r \in [0,1]} |g_r|$. Then, we can choose $b$ such that $|g(b)| = \sup_{r \in [0,1]} |g_r|$ and define

$$
a = \sup \{ x : |f(x)| \leq \frac{1}{2} \sup_{r \in [0,1]} |g_r|, 0 \leq x < b \}.
$$

By Hölder continuity of $g_s$, we have

$$
\frac{1}{2} \sup_{r \in [0,1]} |g_r| \leq \|g\|_r |b - a|^{\tau}.
$$

Inequality (10) now follows from (13). Our proof is now complete.

If one had a Hölder type lower bound on the covariance function (as often the case with Gaussian processes), we may incorporate the information on the interval $[a, b]$ into the main inequality.

**Corollary 4.2.** Under the assumptions of proposition 4.1 if, in addition, we have for some $\eta, C > 0$ and any $0 \leq s < t \leq 1$ that

$$
\mathbb{E}(X_{s,t})^2 \geq C|t - s|^{\eta}.
$$

Then

$$
\sup_{r \in [0,1]} |g_r| \leq \max \left\{ \frac{2}{\sqrt{\beta}} \left( \mathbb{E}(X_1)^2 \right)^{-\frac{1}{2}} \left( \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 \right)^{\frac{1}{2}}, \frac{2^{2\tau - \eta}}{2^{2\tau + \eta}} \left( \frac{2}{(\beta C)^{2\tau + \eta}} \right)^{\frac{1}{2}} \left( \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 \right)^{\frac{\eta}{2\tau + \eta}} \|g\|_r^{\frac{\eta}{2\tau + \eta}} \right\}.
$$
Remark 4.3. The previous inequality is of the same form as lemma A.3 from [9] and corollary 6.10 from [5].

With a little more effort, we have the following

Corollary 4.4. Under the assumptions of corollary 4.2, we have

\[ \sup_{r \in [0,1]} |g_r| \leq \max \left\{ \frac{2}{n \sqrt{\beta}} \left( \mathbb{E}(X_1)^2 \right)^{-\frac{1}{2}} \left( \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 \right)^{\frac{1}{2}}, \frac{2^{2r-\eta}}{n(\beta C)^{2r-\eta}} \left( \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 \right)^{\frac{\tau}{2\tau+\eta}} \right\}. \]  

(14)

Proof. This is the consequence of

\[ \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2 = \int_0^1 \int_0^1 g_t g_s d\mathbb{E}(X_t X_s) = \frac{1}{n} \int_0^1 \int_0^1 g_t g_s d\mathbb{E}(DX_t, DX_s)_{\mathcal{H}} = \frac{1}{n} \mathbb{E} \left( \int_0^1 g_t dX_t \right)^2. \]

\[ \square \]

Remark 4.5. Corollary 4.4 is a quantitative version of

\[ \left\{ \int_0^1 g_s dX_s = 0, \forall \omega \in \Omega \right\} \Rightarrow \{ g \equiv 0 \}. \]

This implication is weaker than (3) when setting \( f_t \) to be deterministic, but the inequality (14) itself is quite interesting in its own.

Now we move on to prove the existence of density for variables defined in (2) under assumptions 1, 3 and 4.

Proposition 4.6. Let \( \{X_t\}_{0 \leq t \leq 1} = I_n(f_t) \) be a continuous process in the \( n \)-th homogeneous Wiener chaos. Let \( g_t \) be any \( \tau \)-Hölder continuous function with \( \tau + \rho > 1 \). If \( g_t \) is not identically zero, then under assumptions 1, 3 and 4, the random variable

\[ Y = \int_0^1 g_t dX_t \]

has a density with respect to the Lebesgue measure on \( \mathbb{R} \).

Proof. Since \( Y \) is a \( \mathbb{R}^1 \) valued, by definition, the Malliavin matrix of \( Y \) is just

\[ C(Y) = \langle DY, DY \rangle_{\mathcal{H}}. \]

Thanks to \( g_t \) being deterministic, \( Y \) also belongs to the \( n \)-th homogeneous Wiener chaos. By theorem 3.1 of [15], it suffices to show that

\[ \mathbb{E}(\det C(Y)) = \mathbb{E}(\langle DY, DY \rangle_{\mathcal{H}}) > 0. \]  

(15)

A standard computation gives

\[ DY = \int_0^1 g_t dX_t, \]
where the integral is understood as a Young integral. Thus,

\[ \mathbb{E}(\langle DY, DY \rangle_H) = \int_0^1 \int_0^1 g_s g_t d\mathbb{E} \langle D_r X_s, D_r X_t \rangle = n \int_0^1 \int_0^1 g_s g_t d\mathbb{E} (X_s X_t). \]  

(16)

By proposition 4.1 we have

\[ \int_0^1 \int_0^1 g_s g_t d\mathbb{E} (X_s X_t) = \mathbb{E} (\langle D_r X_s, D_r X_t \rangle)^2 \geq \frac{\beta}{4} \left( \sup_{r \in [0,1]} |g_r| \right)^2 \{ \mathbb{E}(X_{a,b})^2 \land \mathbb{E}(X_1)^2 \}. \]  

(17)

Since \( g_t \) is not identically zero, we have\( \sup_{r \in [0,1]} |g_r| > 0 \), and \( a < b \).

Combing (16), (17) and the lower bound from assumption 1, we conclude that

\[ \mathbb{E}(\det C(Y)) = \mathbb{E}(\langle DY, DY \rangle_H) > 0. \]

The proof is now complete.

\[ \square \]

**Remark 4.7.** Notice we used in (15) the fact that if \( F \) is in a fixed Wiener chaos then

\[ \mathbb{P}\{DF = 0\} = 0 \iff \mathbb{E}(DF, DF)_H = 0. \]

We can thus take expectation and transform \( DF \) into \( F \) as we did. This equivalence fails radically when \( F \) is not in a finite Wiener chaos (in our setting, when \( g_t \) is not deterministic).

### 4.2 Random integrand

Now we turn to more the general case where \( \{g_t\}_{0 \leq t \leq 1} \) is random. We first prepare a lemma, which helps reveal the relationship between the kernel \( f_t \) and the subspace \( F_t \) we used in assumption 2.

**Lemma 4.8.** For every \( t \in [0,1] \), we have \( DX_t \in F_t \) almost surely.

**Proof.** Let \( \{e_i\}_{i \geq 1} \) be an orthonormal basis of \( H \). Then for each kernel \( f_t \), we may write

\[ f_t = \sum_{i_1, i_2, \ldots, i_n \geq 1} a_{i_1, i_2, \ldots, i_n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}. \]

For any set of \( \{e_{k_1}, \cdots, e_{k_{n-1}}\} \), we have by definition

\[ \langle f_t, e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{n-1}} \rangle_{H^{\otimes (n-1)}} = \sum_{i_1 \geq 1} a_{i_1, k_1, k_2, \ldots, k_{n-1}} e_{i_1} \in F_t. \]

Let \( \{\xi^i_t\}_{i \geq 1} \) be an orthonormal basis of \( F_t \), then we may write

\[ \sum_{i_1 \geq 1} a_{i_1, k_1, k_2, \ldots, k_{n-1}} e_{i_1} = \sum_{i_1 \geq 1} b_{i_1, k_1, k_2, \ldots, k_{n-1}} \xi^i_t. \]
We thus infer that
\[ f_t = \sum_{k_1,k_2,\ldots,k_{n-1} \geq 1} b_{i_1,i_2,\ldots,i_{n-1}} c_{i_1,i_2,\ldots,i_{n-1}} \xi_{i_1} \otimes \cdots \otimes e_{k_{n-1}}. \]

We can repeat this process and get
\[ f_t = \sum_{i_1,i_2,\ldots,i_{n} \geq 1} c_{i_1,i_2,\ldots,i_{n}} \xi_{i_1} \otimes \cdots \otimes \xi_{i_{n}}. \]

As a result, we have \( f_t \in F_t \) and
\[ DI_n(f_t) = nI_{n-1}(f_t) \in F_t. \]

\[ \square \]

**Proposition 4.9.** Let \( \{X_t\}_{0 \leq t \leq 1} = I_n(f_t) \) be a continuous process in the \( n \)-th homogeneous Wiener chaos. Let \( g_t \) be any process whose sample paths are \( \tau \)-Hölder continuous almost surely, with \( \tau + \rho > 1 \). Then under assumptions \( \square \) and \( \square \), we have
\[ \left\| \int_0^1 g_t dDX_r \right\|_{\mathcal{H}} \geq \sqrt{\alpha} \cdot \sup_{t \in [0,1]} \left\| \int_0^t g_t dDX_r \right\|_{\mathcal{H}}. \]

**Proof.** Let \( \{D_k\}_{k \geq 1} \) be an increasing sequence of partitions of \([0,1]\). It suffices to prove that for any \( t \in [0,1], \)
\[ \left\| \int_{D_k \cap [0,1]} g_t dDX_r \right\|_{\mathcal{H}} \geq \sqrt{\alpha} \left\| \int_{D_k \cap [0,t]} g_t dDX_r \right\|_{\mathcal{H}}. \]

We have
\[ \int_{D_k \cap [0,1]} f_t dDX_r = \int_{D_k \cap [0,t]} g_t dDX_r + \int_{D_k \cap (t,1]} g_t dDX_r. \]

Define
\[ G^k_t = \text{Span} \{ (DX_{r_{i+1}} - DX_{r_i}), \forall r_i, r_{i+1} \in D_k \cap (t,1) \}. \]

By previous lemma, we have
\[ \int_{D_k \cap (t,1]} g_t dDX_r \in C^k_t \subset \overline{\text{Span}} \{ F_{r_i,r_{i+1}}, \forall r_i, r_{i+1} \in D_k \cap (t,1) \}. \]

To ease notations, we simply denote the projection onto \( \overline{\text{Span}} \{ F_{r_i,r_{i+1}}, \forall r_i, r_{i+1} \in D_k \cap (t,1) \} \) by \( P \). Now we can deduce from assumption \( \square \) that
\[ \left\| \int_{D_k \cap [0,1]} g_t dDX_r \right\|_{\mathcal{H}}^2 = \left\| \int_{D_k \cap [0,t]} g_t dDX_r + \int_{D_k \cap (t,1]} g_t dDX_r \right\|_{\mathcal{H}}^2 \]
\[ = \left\| \int_{D_k \cap [0,t]} g_t dDX_r - P (\int_{D_k \cap [0,t]} g_t dDX_r) \right\|_{\mathcal{H}}^2 + \left\| \int_{D_k \cap (t,1]} g_t dDX_r + P (\int_{D_k \cap [0,t]} g_t dDX_r) \right\|_{\mathcal{H}}^2 \]
\[ \geq \left\| \int_{D_k \cap [0,t]} g_t dDX_r - P (\int_{D_k \cap [0,t]} g_t dDX_r) \right\|_{\mathcal{H}}^2 \]
\[ \geq \alpha \left\| \int_{D_k \cap [0,t]} g_t dDX_r \right\|_{\mathcal{H}}^2. \]

\[ \square \]
Remark 4.10. In fact, with the same argument, we can show

\[ \left\| \int_0^1 g_r dDX_t \right\|_{\mathcal{H}} \geq \sqrt{\alpha} \cdot \sup_{[s, l] \subset [0, 1]} \left\| \int_s^l g_r dDX_r \right\|_{\mathcal{H}}. \]

Now we are ready for the proof of theorem 2.5.

Proof of theorem 2.5. By the maximal inequality of Young’s integrals, we can find \( C(\rho, \tau) > 0 \) independent of \( g(\omega) \) such that for any \([a, a + \epsilon] \subset [0, 1]\)

\[ \left\| \int_a^{a+\epsilon} g_r dDX_r - g_a \cdot (DX_{a+\epsilon} - DX_a) \right\|_{\mathcal{H}} \leq C \left\| g_r(\omega) \right\|_{\mathcal{H}} \| DX_r(\omega) \|_{\mathcal{H}}. \] (18)

From proposition 4.9, we know that

\[ \left\{ \int_0^1 g_t dDX_t = 0 \right\} \Rightarrow \left\{ \int_0^{a+\epsilon} g_r dDX_r = 0 \right\}. \]

Combining this with (18) gives

\[ \left\{ \int_0^1 g_t dDX_t = 0 \right\} \Rightarrow \left\{ \| g_a \cdot (DX_{a+\epsilon} - DX_a) \|_{\mathcal{H}} \leq C \left\| g_r(\omega) \right\|_{\mathcal{H}} \| DX_r(\omega) \|_{\mathcal{H}}. \] (19)

If \( g_t(\omega) \neq 0 \), by continuity we can find \([l(\omega), u(\omega)] \subset [0, 1]\) such that \( |g_t(\omega)| > c(\omega) > 0 \) on \([l, u]\). Hence, we have for any \([a, a + \epsilon] \subset [l, u]\)

\[ \left\{ \int_0^1 g_t dDX_t = 0, \ g_t \neq 0 \right\} \Rightarrow \left\{ c(\omega) \cdot DX_{a+\epsilon} - DX_a \|_{\mathcal{H}} \leq C \left\| g_t(\omega) \right\|_{\mathcal{H}} \| DX_t(\omega) \|_{\mathcal{H}}. \] (20)

The right-hand side of the above implication says that \( DX_t \) is \((\tau + \rho)-\)Hölder continuous on \([l, u]\). Moreover, since \( \tau + \rho > 1 \), \( DX_t \) must remain constant on \([l, u]\), which means that \( DX_t - DX_s = 0 \) for any \([s, t] \subset [l, u]\). However, since \( DX_s \in F_s, DX_t \in F_t \), assumptions [1] and [2] imply that \( DX_s, DX_t \) are linearly independent unless they are both zero. Hence, we infer that \( DX_s = DX_t = 0 \).

Gathering everything we have proved so far gives

\[ \left\{ \int_0^1 g_t dDX_t = 0, \ g_t \neq 0 \right\} \Rightarrow \{ DX_t = 0, \ \forall t \in [l(\omega), u(\omega)] \}. \] (21)

The event on the right-hand side depends on the sample paths \( \omega \). We need the following uniform estimate.

\[ \{ DX_t = 0, \ \forall t \in [l(\omega), u(\omega)] \} \Rightarrow \{ \exists t \in \mathbb{Q} \cap [0, 1], \ DX_t = 0 \} \]

\[ \Rightarrow \bigcup_{t \in \mathbb{Q} \cap [0, 1]} \{ DX_t = 0 \}. \] (22)

Now we resort to use lemma 3.1. Since \( \| f_t \|_{\mathcal{H} \oplus n} > 0 \), it is always possible to find \( h \in \mathcal{H} \) such that \( \| (f_t, h)_{\mathcal{H}} \|_{\mathcal{H} \oplus (n-1)} > 0 \). As a result,

\[ \{ DX_t = 0 \} \Rightarrow \{ (DX_t, h)_{\mathcal{H}} = 0 \} = \{ nI_{n-1}((f_t, h)_{\mathcal{H}}) = 0 \}. \]
By lemma \[3.1\] \( I_{n-1}(f_t, h) \) has a density. So, we have
\[
\mathbb{P}\{DX_t = 0\} \leq \mathbb{P}\{I_{n-1}(f_t, h) = 0\} = 0,
\]
which immediately gives
\[
\mathbb{P}\left\{\bigcup_{t \in \mathbb{Q} \cap [0,1]} \{DX_t = 0\}\right\} = 0. \tag{23}
\]
Combining (21), (22) and (23) finishes the proof. \( \square \)

**Remark 4.11.** Let \( \nu \in (0, 1) \) be a positive constant. If \( \{DX_t\}_{0 \leq t \leq 1} \) has the so called \( \nu \)-Hölder roughness property (see definition 6.7 of [7]), then we for any \( s \in [0,1] \) and \( \epsilon \in [0,1/2] \), we can find \( L_\nu(DX)(\omega) > 0 \) and \( t \in [0,1] \) such that \( 0 < |t-s| < \epsilon \) and
\[
\|DX_t - DX_s\|_H \geq L_\nu(DX)\epsilon^\nu.
\]
Then (18) gives
\[
g_\alpha \cdot L_\nu(DX)\epsilon^\nu \leq 2 \sup_{t \in [0,1]} \left\| \int_0^t g_r dDX_r \right\| + C\|g_r(\omega)\|_{\mathcal{T}}\|DX_r(\omega)\|_\rho \epsilon^{\tau + \rho}.
\]
We can recast it as
\[
|g_\alpha| \leq \frac{C'}{L_\nu(DX)} \left( \sup_{t \in [0,1]} \left\| \int_0^t g_r dDX_r \right\| \epsilon^{-\nu} + \|g_r(\omega)\|_{\mathcal{T}}\|DX_r(\omega)\|_\rho \epsilon^{\tau + \rho - \nu} \right).
\]
Optimizing the right-hand side with respect to \( \epsilon \) and taking supreme of the left-hand side, one verifies
\[
\sup_{r \in [0,1]} |g_r| \leq \frac{C''}{L_\nu(DX)} \left( \sup_{t \in [0,1]} \left\| \int_0^t g_r dDX_r \right\| \left\| g_r(\omega)\right\|_{\mathcal{T}} \|DX_r(\omega)\|_\rho \right). \tag{24}
\]
Inequality (24) can be regarded as a Norris’ type lemma for \( \{DX_t\}_{0 \leq t \leq 1} \), which (taking proposition 4.9 into account) is another quantitative version of (3). The Hölder roughness of \( \{X_t\}_{0 \leq t \leq 1} \) will be an interesting topic to investigate.

A straightforward consequence of theorem 2.5 is the following zero-one law.

**Corollary 4.12.** Let \( g_t \) be any \( \tau \)-Hölder continuous function with \( \tau + \rho > 1 \). Then under assumptions \[4\] and \[2\] we have
\[
\mathbb{P}\left\{\int_0^1 g_t dDX_t = 0\right\} = 1 \text{ or } 0.
\]

**Proof.** We can regard \( g \) as a constant random process, then by theorem 2.5
\[
\mathbb{P}\left\{\int_0^1 g_t dDX_t = 0\right\} = \mathbb{P}\left\{\int_0^1 g_t dDX_t = 0, \ g_t \neq 0\right\} + \mathbb{P}\left\{\int_0^1 g_t dDX_t = 0, \ g_t = 0\right\}
= \mathbb{P}\left\{\int_0^1 g_t dDX_t = 0, \ g_t = 0\right\}
= \mathbb{P}\{ g_t = 0\}. \tag{25}
\]
We conclude with the fact that \( g_t \) is deterministic. \( \square \)
We can now give the proof for the existence of density for variables defined in (2) under assumptions 1 and 2.

**Proposition 4.13.** Let \( \{X_t\}_{0 \leq t \leq 1} = I_n(f_t) \) be a continuous process in the \( n \)-th homogeneous Wiener chaos. Let \( g_t \) be any \( \tau \)-Hölder continuous function with \( \tau + \rho > 1 \). If \( g_t \) is not identically zero, then under assumptions 1 and 2, the random variable

\[
Y = \int_0^1 g_t dX_t
\]

has a density with respect to the Lebesgue measure on \( \mathbb{R} \).

**Proof.** Note that by (25)

\[
P\{DY = 0\} = P\left\{ \int_0^1 g_t dX_t = 0 \right\} = P\{g_t = 0\} = 0.
\]

\[\square\]

5 Application to SDE

In this section, we use \( \{X_t\}_{0 \leq t \leq 1} \) to denote a \( d \)-dimensional chaos process for a fixed positive integer \( d \). More explicitly, for \( 1 \leq i \leq d \), \( \{X^i_t\}_{0 \leq t \leq 1} \) is an independent copy of \( \{X_t\}_{0 \leq t \leq 1} \) we defined in previous sections. We consider the following SDE

\[
dY_t = \sum_{i=1}^d V_i(Y_t) dX^i_t + V_0(Y_t) dt, \ Y_0 = y_0 \in \mathbb{R}^d.
\]

Since \( \{X_t\}_{0 \leq t \leq 1} \) is \( \rho \)-Hölder continuous for some \( \rho > 1/2 \), the above SDE is understood in Young’s sense. By Duhamel’s principle (see [8] chapter 4), we have

\[
\langle DY_t, h \rangle = \sum_{i=1}^d \int_0^t J_{t-s}V_i(Y_s) d\langle DX^i_s, h \rangle,
\]

where \( J_{t-s} \) is the Jacobian process defined in (7). Note that in multi-dimensional case, we have

\[
DX^i_t = (D^1 X^i_t, D^2 X^i_t, \ldots, D^d X^i_t)^T
\]

where \( D^j X^i_t \) is the Malliavin derivative of \( X^i_t \) with respect to the underlying Brownian motion of \( X^j_t \). Thanks to the fact that different components of \( \{X_t\}_{0 \leq t \leq 1} \) are independent, \( D^j X^i_t = \delta_{ij} \cdot D^i X^i_t \), where \( \delta_{ij} \) is the Kronecker delta function.

**Proof of theorem 2.6** We have by definition

\[
D^j Y_t = \sum_{i=1}^d \int_0^t J_{t-s}V_i(Y_s) dD^j X^i_s = \int_0^t J_{t-s}V_j(Y_s) dD^j X^j_s.
\]

The Malliavin matrix of \( Y_t \) is given by

\[
C^{ij}_t = \langle Y^i_t, Y^j_t \rangle_{\mathcal{H}^d}.
\]
We can write
\[ P\{\det(C_t) = 0 \} = P\left\{ v^T \cdot C_t \cdot v = 0, \text{ for some vector } v \neq 0 \right\}. \]

Since
\[ v^T \cdot C_t \cdot v = \left\| \sum_{i=1}^{d} v_i D^i Y_t \right\|^2_{H^d}. \]

We can infer that
\[ P\left\{ v^T \cdot C_t \cdot v = 0, \text{ for some vector } v \neq 0 \right\} = P\left\{ \int_0^t v^T J_{t^-} V_j(Y_s) dD^j X_s^j = 0, 1 \leq i \leq d, \text{ for some vector } v \neq 0 \right\}. \] (26)

However, by theorem 2.5, we have
\[ \left\{ \int_0^t v^T J_{t^-} V_j(Y_s) dD^j X_s^j = 0, 1 \leq j \leq d \right\} \Rightarrow \left\{ v^T J_{t^-} V_j(Y_s) = 0, 1 \leq j \leq d \right\} \Rightarrow \left\{ v^T J_{t^-} V(Y_s) = 0 \right\}, \]

where \( V \in \mathbb{R}^{d \times d} \) whose columns are given by \( \{V_i\}_{1 \leq i \leq d} \). Since \( V(Y_s) \) is elliptic by our assumption and \( J_{t^-} \) is invertible almost surely with inverse given by (22), we see that
\[ \left\{ v^T J_{t^-} V(Y_s) = 0 \right\} \Rightarrow \{ v = 0 \}. \]

Plugging this back to (26) gives
\[ P\{v^T \cdot C_t \cdot v = 0, \text{ for some vector } v \neq 0 \} = P\{v = 0, \text{ for some vector } v \neq 0 \} = 0. \]

Our proof is complete. \( \square \)

**Remark 5.1.** If \( \{X_t\}_{0 \leq t \leq 1} \) is Hölder rough, we can apply the Norris's lemma (24) and prove the existence of density with a weaker parabolic Hörmander's condition, instead of ellipticity, on the vector fields \( \{V_i\}_{0 \leq i \leq d} \).

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