Bounded Max-Colorings of Graphs

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Abstract

In a bounded max-coloring of a vertex/edge weighted graph, each color class is of cardinality at most \( b \) and of weight equal to the weight of the heaviest vertex/edge in this class. The bounded max-vertex/edge-coloring problems ask for such a coloring minimizing the sum of all color classes’ weights.

In this paper we present complexity results and approximation algorithms for those problems on general graphs, bipartite graphs and trees. We first show that both problems are polynomial for trees, when the number of colors is fixed, and \( H_b \)-approximable for general graphs, when the bound \( b \) is fixed. For the bounded max-vertex-coloring problem, we show a \( \frac{17}{11} \)-approximation algorithm for bipartite graphs, a PTAS for trees as well as for bipartite graphs when \( b \) is fixed. For unit weights, we show that the known \( \frac{4}{3} \) lower bound for bipartite graphs is tight by providing a simple \( \frac{4}{3} \) approximation algorithm. For the bounded max-edge-coloring problem, we prove approximation factors of \( \frac{3-2}{\sqrt{2}}b \), for general graphs, \( \min\{e,\frac{3-2}{\sqrt{5}}\} \), for bipartite graphs, and 2, for trees. Furthermore, we show that this problem is NP-complete even for trees. This is the first complexity result for max-coloring problems on trees.

1 Introduction

The bounded max-vertex-coloring (resp. bounded max-edge-coloring) problem takes as input a graph \( G = (V,E) \), a weight function \( w : V \rightarrow N \) (resp. \( w : E \rightarrow N \)) and an integer \( b \); the question of this problem is to find a proper vertex- (resp. edge-) coloring of \( G \), \( C = \{C_1,C_2,\ldots,C_k\} \), where each color \( C_i \), \( 1 \leq i \leq k \), has weight \( w_i = \max\{w(u) \mid u \in C_i\} \) (resp. \( w_i = \max\{w(e) \mid e \in C_i\} \)), cardinality \( |C_i| \leq b \), and the sum of colors’ weights, \( W = \sum_{i=1}^{k} w_i \), is minimized.

We shall denote the vertex and edge bounded max-coloring problems by \( VC(w,b) \) and \( EC(w,b) \), respectively. These problems, without the presence of the cardinality bound \( b \), have been already addressed in the literature as max-vertex-coloring [20] and max-edge-coloring [18] problems; here we denote them by \( VC(w) \) and \( EC(w) \), respectively. For unit weights we get the bounded vertex-coloring [2] and bounded edge-coloring [11] problems, denoted by \( VC(b) \) and \( EC(b) \), respectively. For both unbounded colors cardinalities and unit weights, we have the classical vertex-coloring (VC) and edge-coloring (EC) problems.

Motivation. Max-coloring problems have been well motivated in the literature. Max-vertex-coloring problems arise in the management of dedicated memories, organized as buffer pools, which is the case for wireless protocol stacks like GPRS or 3G [20][19]. Max-edge-coloring problems arise in switch based communication systems, like SS/TDMA [4][17], where messages are to be transmitted through direct connections established by an underlying network. Moreover, max-coloring problems correspond to scheduling jobs with conflicts into a batch scheduling environment [9][13].

In all applications mentioned above, context-related entities require their service by physical resources for a time interval. However, there exists in practice a natural constraint on the number

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of entities assigned the same resource or different resources at the same time. Indeed, the number of memory requests assigned the same buffer is determined by strict deadlines on their completion times, while the number of messages and jobs assigned, at the same time, to different channels and machines, respectively, is bounded by the number of the available resources. The existence of such a constraint motivates the study of the bounded max-coloring problems.

**Related Work.** It is well known that for general graphs it is NP-hard to approximate the VC problem within any constant factor and the EC problem within a factor less than 4/3; for bipartite graphs both problems become polynomial.

The complexity of the VC(b) problem (known also as Mutual Exclusion Scheduling problem [2]) on special graph classes has been extensively studied (see [14] and the references therein). It is polynomial for trees [16], but NP-complete for bipartite graphs even for three colors [3]. This last result implies also a 4/3 inapproximability bound for the VC(b) problem on bipartite graphs.

The VC(w) problem is not approximable within a factor less than 8/7 even for planar bipartite graphs, unless P=NP [9, 19]. This bound is tight for general bipartite graphs, as an 8/7-approximation algorithm is also known [7, 19]. On the other hand, the complexity of the problem in trees is an open question, while a PTAS for this case has been presented in [19, 12]. Other results for the VC(w) problem on several graph classes have been also presented in [9, 4, 20, 19, 12, 11].

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The EC(b) problem is polynomial for bipartite graphs [4] as well as for general graphs if b is fixed [1]. Moreover, it is implied by the results in [14] that there is a 4/3 approximation algorithm for the EC(b) problem on general graphs.

The EC(w) problem is not approximable within a factor less than 7/6 even for cubic planar bipartite graphs with edge weights \( w(e) \in \{1, 2, 3\} \), unless P=NP [7]. A simple greedy 2-approximation algorithm for general graphs has been proposed in [17]. The complexity of the EC(w) problem on trees remains also open, while a 3/2-approximation algorithm has been recently presented [13].

Known results for the VC(w, b) and EC(w, b) problems have been appeared in the context of batch scheduling for complements of special graph classes (see e.g. [13]). In this context both problems have been shown to be polynomial for general graphs and \( b = 2 \) [5].

In Table 1 we summarize the best known results for bipartite graphs and trees together with our contribution.

| Problem | Bipartite graphs | Trees |
|---------|-----------------|-------|
| VC(b)   | 4/3 [9]         | 4/3 OPT [16] |
| VC(w)   | 8/7 [9, 11, 19] | open* PTAS [19, 12] |
| VC(w, b)| 4/3 [3]         | 17/11 open* PTAS |
| EC(b)   | OPT [4]         | OPT [4] |
| EC(w)   | 7/6 [7]         | 2 [17] open* 3/2 NP-complete |
| EC(w, b)| 7/6 [2] \*\* | \( \min\{3 - 2/\sqrt{2b}, H_b\} \)* NP-complete |

Table 1: Known and ours (in bold) approximability results for bounded and/or max coloring problems. *Even the complexity of the problem is unknown. **We also show a ratio of \( \min\{3 - 2/\sqrt{2b}, H_b\} \) for general graphs. In both cases, the ratio \( H_b \) holds only if \( b \) is fixed.

**Our results and organization of the paper.** In this paper we deal with bounded max-coloring problems on general graphs, bipartite graphs and trees. Our interest in bipartite graphs and trees is two-fold. Despite their simplicity, these classes of graphs are important in their own right both from a theoretical point of view but also from the applications’ perspective [17, 19].

In the next section, we relate our problems with two well known problems, namely the list coloring and the set cover problems. We also introduce some useful notation. In Section 3, we deal with the VC(w, b) problem and we give a simple 2-approximation algorithm for bipartite graphs. As a byproduct, we show that this algorithm becomes a 4/3-approximation algorithm for
the VC($b$) problem, which matches the $4/3$ inapproximability bound. Then, we present a generic scheme that we show to be a 17/11-approximation algorithm for bipartite graphs, while it becomes a PTAS for trees as well as for bipartite graphs when $b$ is a fixed constant. In Section 4, we deal with the EC($w, b$) problem and we present approximation algorithms of ratios $\min\{3 - 2/\sqrt{2b}, H_b\}$, for general graphs, and $\min\{e, 3 - 2/\sqrt{b}, H_b\}$, for bipartite graphs. More interestingly, we prove that the EC($w, b$) problem on trees is NP-complete. Given that the complexity question of VC($w$), EC($w$) and VC($w, b$) problems for trees remains open, this is the first max-coloring problem on trees proven to be NP-hard. Finally, we propose a 2-approximation algorithm for the EC($w, b$) problem on trees.

2 Preliminaries and Notation

We first establish a relation between our problems and bounded list coloring.

**Bounded List Vertex (resp. Edge) Coloring problem**

**Instance:** A graph $G = (V, E)$, a set of colors $C = \{C_1, C_2, \ldots, C_k\}$, a list of colors $\phi(u) \subseteq C$ for each $u \in V$ (resp. $\phi(e) \subseteq C$ for each $e \in E$), and integers $b_i$, $1 \leq i \leq k$.

**Question:** Is there a $k$-coloring of $G$ such that each vertex $u$ (resp. edge $e$) is assigned a color in its list $\phi(u)$ (resp. $\phi(e)$) and every color $C_i$ is used at most $b_i$ times?

Clearly, the bounded list coloring problems, denoted by VC($\phi, b_i$) and EC($\phi, b_i$), are generalizations of the VC($b$) and EC($b$) problems, as well as of the VC and EC problems, respectively. In the next theorem we summarize some of the known results for the VC($\phi, b_i$) and EC($\phi, b_i$) problems that we shall use in the rest of the paper.

**Theorem 1**

(i) The VC($\phi, b_i$) problem is NP-complete even for chains, $|\phi(u)| \leq 2$, for all $u \in V$, and $b_i \leq 5$, $1 \leq i \leq k$ [10].

(ii) Both VC($\phi, b_i$) and EC($\phi, b_i$) problems are polynomial for trees if the number of colors $k$ is fixed [15, 8].

(iii) The VC($\phi, b_i$) problem is polynomial for general graphs if $k = 2$ [15].

Using an exhaustive transformation of an instance of the VC($w, b$) and EC($w, b$) problems to an instance of the VC($\phi, b$) and EC($\phi, b$) problems (where $b = b_i$, $1 \leq i \leq k$), respectively, and Theorem 1(ii) ([15, 8]) we get next proposition.

**Proposition 1** For a fixed number of colors $k$, both the VC($w, b$) and EC($w, b$) problems on trees are polynomial.

**Proof:** We give here the proof for the VC($w, b$) problem; the proof for the EC($w, b$) problem is quite similar. Given a vertex weighted graph $G = (V, E)$ we generate all $\binom{|V|}{k}$ possible combinations for the weights of the $k$ colors. Let $w_1 \geq w_2 \geq \ldots \geq w_k$ be the colors’ weights in such a combination. For each one of these combinations we construct an instance of the VC($\phi, b$) problem on the graph $G$: is there a $k$-coloring of the vertices of $G$ such that each color is used at most $b$ times and each vertex $v \in V$ is assigned a color in $\phi(u) = \{C_i : w(u) \leq w_i, 1 \leq i \leq k\}$? A “yes” answer to this instance of the VC($\phi, b$) problem corresponds to a feasible solution for the VC($w, b$) problem of weight $W = \sum_{i=1}^{k} w_i$. An optimal solution to the VC($w, b$) problem corresponds to the combination where $W$ is minimized.

There are $O(|V|^k)$ combinations of weights to be generated. For a fixed $k$, by Theorem 1(ii) ([15, 8]), the VC($\phi, b$) and EC($\phi, b$) problems are polynomial and the proposition follows. $\blacksquare$

Next proposition is based on a relation between VC($w, b$) and EC($w, b$) problems and the set cover problem.
Proposition 2 For a fixed bound b, there is an $H_b$-approximation algorithm for both VC($w,b$) and EC($w,b$) problems on general graphs.

Proof: In the set cover problem, we are given a universe $U$ of elements, and a collection, $S = \{S_1,S_2,\ldots,S_m\}$, of subsets of $U$, each one of positive cost $c_i, 1 \leq i \leq m$, and we ask for a minimum cost subset of $S$ that covers all elements of $U$. For an instance of the VC($w,b$) (resp. EC($w,b$)) problem on a graph $G = (V,E)$ we consider the set of vertices $V$ (resp. edges $E$) corresponding to the universe set $U$. For each $j, 1 \leq j \leq b$, we generate all $\binom{|V|}{j}$ (resp. $\binom{|E|}{j}$) possible subsets of cardinality $j$ of vertices (resp. edges) of $G$. From all these subsets we get rid those containing adjacent vertices (resp. edges) and we consider the rest corresponding to the set $S$. For each such subset $S_i \subseteq S$ we set $c_i = \max\{w(u)|u \in S_i\}$ (resp. $c_i = \max\{w(e)|e \in S_i\}$).

The cardinality of $S$ is $O(|V|^b)$, since $b$ is $O(|V|)$, and as an $H_b$-approximation algorithm is known for the set cover problem [3], the proposition follows.

Our notation. Given a set $S$ and a positive integer weight $w(s)$ for every element $s \in S$, we denote by $(S) = (s_1,s_2,\ldots,s_{|S|})$ an ordering of $S$ such that $w(s_1) \geq w(s_2) \geq \ldots \geq w(s_{|S|})$. For such an ordering of $S$ and a positive integer $b$, let $k_S = \lceil \frac{|S|}{b} \rceil$. We define the ordered $b$-partition of $S$, denoted by $P_S = \{S_1,S_2,\ldots,S_{k_S}\}$, to be the partition of $S$ into $k_S$ subsets, such that $S_i = \{s_{j_1},s_{j_2},\ldots,s_{\min(j+b-1,|S|)}\}, i = 1,2,\ldots,k, j = (i-1)b+1$. In other words, $S_1$ contains the $b$ heaviest elements of $S$, $S_2$ contains the next $b$ heaviest elements of $S$ and so on; clearly, $S_{k_S}$ contains the $|S| \mod b$ lightest elements of $S$.

By $OPT = w_1^* + w_2^* + \ldots + w_k^*$, we denote the weight of an optimal solution to the VC($w,b$) or EC($w,b$) problem, where $w_i^*, 1 \leq i \leq k^*$, is the weight of the $i$-th color class. By $\Delta$ we denote the maximum degree of a graph.

3 Bounded Max-Vertex-Coloring

In this section we first present a simple 2-approximation algorithm for the VC($w,b$) problem on bipartite graphs. The unweighted variant of this algorithm gives a $\frac{4}{3}$ approximation ratio for the VC($b$) problem on bipartite graphs, which closes the approximability question for this case. Then, we give a generic scheme which becomes a $\frac{17}{11}$-approximation algorithm for bipartite graphs, a PTAS for bipartite graphs and fixed $b$, as well as a PTAS for trees. Recall also that by Proposition 2 there is an $H_b$ approximation ratio for general graphs, if $b$ is fixed.

3.1 A simple split algorithm

Let $G = (U \cup V, E), |U \cup V| = n$, be a vertex weighted bipartite graph. Our first algorithm colors the vertices of each class of $G$ separately, by finding the ordered $b$-partitions of classes $U$ and $V$. For the minimum number of colors $k^*$ it holds that $k^* \geq \lceil \frac{|U|+|V|}{b} \rceil$ and, therefore, $k = \lceil \frac{|U|}{b} \rceil + \lceil \frac{|V|}{b} \rceil \leq \lceil \frac{|U|+|V|}{b} \rceil + 1 \leq k^* + 1$.

Algorithm Split
1. Let $P_U = \{U_1, U_2, \ldots, U_{k_U}\}$ be the ordered $b$-partition of $U$;
2. Let $P_V = \{V_1, V_2, \ldots, V_{k_V}\}$ be the ordered $b$-partition of $V$;
3. Return the coloring $C = P_U \cup P_V$;

Theorem 2 Algorithm Split returns a solution of weight $W \leq 2 \cdot w_1^* + w_2^* + \ldots + w_k^* \leq 2 \cdot OPT$ for the VC($w,b$) problem in bipartite graphs.

Proof: Let $(C) = (C_1,C_2,\ldots,C_k)$ be the colors constructed by Algorithm Split, that is $w_1 \geq w_2 \geq \ldots \geq w_k$. Assume, w.l.o.g., that $U_x, 1 \leq x \leq k_U$, is the $i$-th color in $(C)$. Let also $u$ be the heaviest vertex of $U_x$, that is $w(u) = w_i$.

The ordered $b$-partition of $U$ and $V$ implies that colors $U_1, U_2, \ldots, U_{x-1}$ and colors $V_1, V_2, \ldots, V_y$, $y = i-x$, appear before color $U_x$ in $(C)$. Then, all $(x-1) \cdot b$ vertices of colors $U_1, U_2, \ldots, U_{x-1}$
Lemma 1

7. Concatenate the two solutions found in Lines 5 and 6;

6. Run Algorithm 4. If there is a solution for $G$, find an optimal solution for $G_{j} = \langle 1 \rangle$. Let $U = \{1, 2, \ldots, n\}$, $V = U$, and $E = \emptyset$. For $j = 1, 2, \ldots, n$, do

- $G_{1,j}$ induced by vertices $u_1, u_2, \ldots, u_j$
- $G_{j+1,n}$ induced by vertices $u_{j+1}, u_{j+2}, \ldots, u_n$

4. If there is a solution for $G_{1,j}$ with at most $p - 1$ colors then

5. Find an optimal solution for $G_{1,j}$ with at most $p - 1$ colors;

6. Run Algorithm $\text{Split}$ for $G_{j+1,n}$;

7. Concatenate the two solutions found in Lines 5 and 6;

8. Return the best solution found;

Lemma 1 Algorithm $\text{Scheme}(p)$ achieves a $\left(1 + \frac{1}{p^2} \right)$ approximation ratio for the $\text{VC}(w, b)$ problem.

The complexity of Algorithm $\text{Split}$ is dominated by the sorting needed to obtain the ordered $b$-partitions of $U$ and $V$ in Lines 1 and 2, that is $O(n \cdot \log n)$.

Algorithm $\text{Split}$ applies also to the $\text{VC}(b)$ problem on bipartite graphs. Moreover, the absence of weights in the $\text{VC}(b)$ problem allows a tight analysis with respect to the $\frac{4}{3}$ inapproximability bound.

Theorem 3 There is a $\frac{4}{3}$-approximation algorithm for the $\text{VC}(b)$ problem on bipartite graphs.

Proof: Assume, first, that $|U| + |V| \geq 2b + 1$. Then, $k^* \geq \lceil \frac{2b+1}{b} \rceil = 3$ and, since $k \leq k^* + 1$, we get $\frac{1}{k} \leq \frac{1}{3}$.

Assume, next, that $b < |U| + |V| \leq 2b$. In this case the optimal solution consists of two or three colors and it is polynomial to decide between them. In fact, it is polynomial to decide if such a bipartite graph can be colored with two colors even for the generalized $\text{VC}(\phi, b)$ problem (see also Theorem 1(iii) ([15])).

Assume, finally, that $|U| + |V| \leq b$. Then, an optimal solution consists of either two colors (if $E \neq \emptyset$) or one color (if $E = \emptyset$).

3.2 A generic scheme

To obtain our scheme we split a bipartite graph $G = (U \cup V, E)$, $|U \cup V| = n$, into two subgraphs $G_{1,j}$ and $G_{j+1,n}$ induced by the $j$ heaviest and the $n - j$ lightest vertices of $G$, respectively (by convention, we consider $G_{1,0}$ as an empty subgraph). Our scheme depends on a parameter $p$ such that all the vertices of $G$ of weights $w_1^*, w_2^*, \ldots, w_{p-1}^*$ are in a subgraph $G_{1,j}$. This is always possible for some $j \leq b(p - 1)$, since each color of an optimal solution for $G$ contains at most $b$ vertices. In fact, for every $j$, $1 \leq j \leq b(p - 1)$, we obtain a solution for the whole graph by concatenating an optimal solution of at most $p - 1$ colors for $G_{1,j}$, if there is one, and the solution obtained by Algorithm $\text{Split}$ for $G_{j+1,n}$.

Algorithm $\text{Scheme}(p)$
1. Let $(U \cup V) = \langle u_1, u_2, \ldots, u_n \rangle$;
2. For $j = 0, 1, \ldots, b \cdot (p - 1)$ do
3. Split the graph into two vertex induced subgraphs:
   - $G_{1,j}$ induced by vertices $u_1, u_2, \ldots, u_j$
   - $G_{j+1,n}$ induced by vertices $u_{j+1}, u_{j+2}, \ldots, u_n$
4. If there is a solution for $G_{1,j}$ with at most $p - 1$ colors then
5. Find an optimal solution for $G_{1,j}$ with at most $p - 1$ colors;
6. Run Algorithm $\text{Split}$ for $G_{j+1,n}$;
7. Concatenate the two solutions found in Lines 5 and 6;
8. Return the best solution found;

Lemma 1 Algorithm $\text{Scheme}(p)$ achieves a $(1 + \frac{1}{p^2})$ approximation ratio for the $\text{VC}(w, b)$ problem.

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Proof: Consider the iteration $j$, $j \leq b \cdot (p - 1)$, of the algorithm where the weight of the heaviest vertex in $G_{j+1,n}$ equals to the weight of the $i$-th color of an optimal solution, i.e. $w(u_{j+1}) = w_i$, $1 \leq i \leq p$.

The vertices of $G_{1,j}$ are a subset of those appeared in the $i - 1$ heaviest colors of the optimal solution. Thus, an optimal solution for $G_{1,j}$ is of weight $OPT_{1,j} \leq w_i + w_{i+1} + \ldots + w_n$.

The vertices of $G_{j+1,n}$ are a superset of those appeared in the $k^* - (i - 1)$ lightest colors of the optimal solution. The extra vertices of $G_{j+1,n}$ are of weight at most $w_i$ and appear in an optimal solution into at most $i - 1$ colors. Thus, an optimal solution for $G_{j+1,n}$ is of weight $OPT_{j+1,n} \leq w_i + w_{i+1} + \ldots + w_k + (i - 1) \cdot w_i = i \cdot w_i + w_{i+1} + \ldots + w_k$. By Theorem 2, Algorithm SPLIT returns a solution for $G_{j+1,n}$ of weight $W_{j+1,n} \leq (i + 1) \cdot w_i + w_{i+1} + \ldots + w_k$.

Therefore, the solution found in this iteration $j$ for the whole graph $G$ is of weight $W_i = OPT_{1,j} + W_{j+1,n} \leq w_i^* + w_{i+1}^* + \ldots + w_n + (i + 1) \cdot w_i^* + w_{i+1}^* + \ldots + w_k^*$.

In all the iterations of the algorithm we obtain $p$ such inequalities for $W$. By multiplying the $i$-th, $1 \leq i \leq p$, inequality by $\frac{1}{i(H_p+1)}$ and adding up all of them, we have $\left(\sum_{i=1}^{p} \frac{1}{i(H_p+1)}\right)W \leq OPT$, that is $\frac{W}{\sum OPT} \leq \frac{H_p+1}{H_p} = 1 + \frac{1}{H_p}$.

The complexity of the Algorithm SCHEME($p$) is $O(bp(f(p) + n \log n))$, where $O(f(p))$ is the complexity of checking for the existence of solutions with at most $p - 1$ colors for $G_{1,j}$ and finding an optimal one among them, while $O(n \log n)$ is the complexity of Algorithm SPLIT. Algorithm SCHEME(1) coincides with Algorithm SPLIT. Algorithm SCHEME(2) has simply to check if the $j \leq b$ vertices of $G_{1,j}$ are independent from each other and, therefore, it derives a $\frac{2}{b}$ approximate solution in polynomial time. Algorithm SCHEME(3) has to check and find, a two color solution for $G_{1,j}$, if any. This can be done in polynomial time by Theorem III (iii) (15). Thus, Algorithm SCHEME(3) is a polynomial time $\frac{17}{11}$-approximation algorithm for the VC($w,b$) problem on bipartite graphs.

However, when $p \geq 4$ and $b$ is a part of the instance, finding an optimal solution in $G_{1,j}$ is an NP-hard problem (even for the VC($b$) problem [3]). Hence, we consider that $b$ is a fixed constant.

In this case, we run an exhaustive algorithm for finding, if any, an optimal solution in $G_{1,j}$ of at most $p - 1$ colors. The complexity of such an exhaustive algorithm is $O((p - 1)^b(p - 1))$ and thus, the complexity of Algorithm SCHEME($p$), $p \geq 4$, becomes $O(bp^p + n^2 \log n)$, since $bp$ is $O(n)$. Choosing $\epsilon = \frac{1}{H_p}$, we get $p = O(2^{\frac{1}{2}})$. Consequently, for fixed $b$, we have a PTAS for the VC($w,b$) problem on bipartite graphs, that is an approximation ratio of $1 + \frac{1}{H_p} = 1 + \epsilon$ within $O(b(2^{\frac{1}{2}})^{b^2(2^{\frac{1}{2})} + n^2 \log n})$ time.

Furthermore, in the particular case of trees, checking the existence of solutions with at most $p - 1$ colors for $G_{1,j}$, and finding an optimal one among them, can be done, by Proposition II in polynomial time for fixed $p$. The complexity of our scheme in this case becomes $O(b^{2+2}(n^{2+2} + n^2 \log n))$. Therefore, the following theorem holds.

**Theorem 4** For the VC($w,b$) problem, Algorithm SCHEME($p$) is a
(i) polynomial time $\frac{17}{11}$-approximation algorithm for bipartite graphs (for $p = 3$),
(ii) PTAS for bipartite graphs if $b$ is fixed,
(iii) PTAS for trees.

4 Bounded Max-Edge-Coloring

In this section we deal with the complexity and approximability of the EC($w,b$) problem. We present, first, approximation results for general and bipartite graphs. Then, we prove that the problem is NP-complete for trees and we give a 2-approximation algorithm for this case.

4.1 General and bipartite graphs

We first adapt the greedy 2-approximation algorithm presented in [17] for the EC($w$) problem to the EC($w,b$) problem.
Algorithm GREEDY
1. Let \(\langle E\rangle = \langle e_1, e_2, \ldots, e_{|E|}\rangle\);
2. For \(j = 1, 2, \ldots, |E|\) do
   3. Insert edge \(e_j\) in the first color of cardinality less than \(b\) which does not contain other edges adjacent to \(e_j\);

The analysis of Algorithm GREEDY is based on tight bounds on the number of colors in a solution to the EC\((w, b)\) problem.

Proposition 3 Algorithm GREEDY achieves approximation ratios of \((3 - \frac{2}{\sqrt{2b}})\), on general graphs, and \((3 - \frac{2}{\sqrt{2b}})\), on bipartite graphs, for the EC\((w, b)\) problem.

Proof: We call a solution \((C) = \langle C_1, C_2, \ldots, C_k\rangle\) to the EC\((w, b)\) problem nice if each color \(C_i\), \(1 \leq i \leq k\), is of cardinality \(|C_i| = b\) or \(C_i\) is maximal in the subgraph induced by the edges \(E \setminus \bigcup_{j=1}^{i-1} C_j\). We first bound the number of colors in such a solution.

Claim. For the number of colors \(k\) in any nice solution to the EC\((w, b)\) problem it holds that:

\[
\max\{\Delta, \lceil \frac{|E|}{b} \rceil\} \leq k \leq \left\lceil \frac{|E|}{b} \rceil - \left\lceil \frac{\Delta^2}{b} \right\rceil + (2\Delta - 1), \text{ for general graphs}
\]

\[
\left\lceil \frac{|E|}{b} \rceil - \left\lceil \frac{\Delta^2}{b} \right\rceil + (2\Delta - 1), \text{ for bipartite graphs}
\]

The lower bounds follow trivially. For the upper bounds, let \((C) = \langle C_1, C_2, \ldots, C_k\rangle\) be a nice solution, \(e = (u, v)\) be an edge in the last color \(C_k\), and \(E_u\) and \(E_v\) be the sets of edges adjacent to vertices \(u\) and \(v\), respectively. By the niceness of the solution \(C\) it follows that edge \(e\) does not appear in any color \(C_i\), \(1 \leq i \leq k - 1\), because \(|C_i| = b\) or \(C_i\) contains at least one edge in \(E_u\) or \(E_v\). Let \(W, X, Y \subseteq \{C_1, C_2, \ldots, C_{k-1}\}\) such that \(W = \{C_i : |C_i| = b\}\), \(X = \{C_i : |C_i| < b\}\) and \(C_k\) contains an edge \(e \in E_u\) and \(Y = \{C_i : |C_i| < b\}\) and \(C_k\) contains an edge \(e \in E_v\). Let \(E_1\) be the set of edges in the colors in \(W\) and \(E_2 = E \setminus E_1\) be the set of edges in the colors in \(X \cup Y \cup \{C_k\}\). Then, \(k = \frac{|E_1|}{b} + x + y + 1\), where \(x = |X|\) and \(y = |Y|\).

Assume, w.l.o.g., that \(X_1Y_1X_2Y_2\ldots X_lY_l\) is the order of colors in the nice solution \((C)\), where \(X_i \subseteq X, Y_i \subseteq Y, 1 \leq i \leq l,\) and \(X_1\) is possibly empty. Let \(x_i = |X_i|\) and \(y_i = |Y_i|, 1 \leq i \leq l\).

For general graphs, consider a color \(C \subseteq X_i\) and let \(S_i = \bigcup_{j=i}^l Y_j\) and \(s_i = \sum_{j=i}^l y_j\). The edge \((u, z) \in C \cap E_u\) prevents at most one edge \((v, z) \in S_i \cap E_v\) from being into \(C\). Moreover, each other edge \((p, q) \in C\) prevents at most two edges \((v, p), (v, q) \in S_i \cap E_v\) from being into \(C\). As the colors in \(S_i\) contain exactly \(s_i\) edges from \(E_v\) and all of them are prevented from being into color \(C\), it follows that \(|C| \geq \left\lceil \frac{s_i - 1}{2} \right\rceil + 1\). Therefore, there exist at least \(x_i \cdot \left\lceil \frac{s_i - 1}{2} \right\rceil + x_i\) edges in \(X_i\). In a similar way, by considering a color \(C \subseteq Y_i\) there exist at least \(y_i \cdot \left\lceil \frac{s_i - 1}{2} \right\rceil + y_i\) edges in \(Y_i\), where \(t_i = \sum_{j=i+1}^l x_j\). Summing up these bounds, and taking into account that \(y_i - 1 \geq 0\) (since \(Y_i\) is not empty), it follows that

\[
|E_2| \geq \sum_{i=1}^l \left( x_i \cdot \left\lceil \frac{s_i - 1}{2} \right\rceil + x_i \right) + \sum_{i=1}^l \left( y_i \cdot \left\lceil \frac{t_i - 1}{2} \right\rceil + y_i \right)
\]

\[
= \frac{x(x - 1)}{2} + \frac{y(y - 1)}{2} + x + y + 1
\]

\[
\geq \frac{(x - 1)(y - 1)}{2} + x + y + 1 \\
\geq \frac{(x + 1)(y + 1)}{2} + 1 \geq \frac{(x + 1)(y + 1)}{2}.
\]

Therefore, \(k = \frac{|E_1|}{b} + x + y + 1 \leq \frac{|E_1|}{b} - \left\lceil \frac{2\Delta^2}{b} \right\rceil + x + y + 1\). If \(\Delta \leq 2b\) then this quantity is maximized when \(x = y = \Delta - 1\) and hence \(k \leq \frac{|E_1|}{b} - \left\lceil \frac{2\Delta^2}{b} \right\rceil + (2\Delta - 1)\). If \(\Delta > 2b\) then the above quantity is maximized when \(x = \Delta - 1\) and \(y = 0\) and hence \(k \leq \frac{|E_1|}{b} - \left\lceil \frac{2\Delta^2}{b} \right\rceil + \Delta \leq \frac{|E_1|}{b} - \left\lceil \frac{2\Delta^2}{b} \right\rceil + (2\Delta - 1)\).
For bipartite graphs, the proof is similar. The structure of a bipartite graph allows a tighter bound on the number of edges in the colors in \( X_i \) and \( Y_i \). Consider, again, a color \( C \in X_i \). For the edge \((u, z) \in C \cap E_u\), there is no edge \((v, z) \in S_i \cap E_v\), while each other edge \((p, q) \in C\) prevents at most one edge \((v, p)\) or \((v, q)\) in \( S_i \cap E_v\) from being into \( C\). Thus, \(|C| \geq s_i + 1\) and, hence, there exist at least \(x_i(s_i + 1)\) edges in \( X_i\). Similarly there exist at least \(y_i(t_i + 1)\) edges in \( Y_i\). The rest of the proof is along the same lines, but using these bounds.

We return, now, to the solution, \((C) = (C_1, C_2, \ldots, C_k)\), derived by Algorithm Greedy. Consider the color \( C_i \) and let \( e_j \) be the first edge inserted in \( G_i\), i.e. \( w_i = w(e_j)\). Let \( E_i = \{e_1, e_2, \ldots, e_j\}\), \( G_i \) be the subgraph of \( G \) induced by the edges in \( E_i\), and \( \Delta_i \) be the maximum degree of \( G_i\).

The solution \((C)\) is a nice one, since it is constructed in a First-Fit manner. Moreover, an optimal solution can be also easily transformed into a nice one of the same total weight. For general graphs, by the bounds above, it follows that \((i)\) \( i \leq \left\lceil \frac{|E_i|}{b} \right\rceil - \left\lfloor \frac{x_i^2}{2b} \right\rfloor + (2\Delta_i - 1)\), and \((ii)\) in an optimal solution the edges of \( G_i\) appear in at least \( i^* \geq \max\{\Delta_i, \left\lceil \frac{|E_i|}{b} \right\rceil\} \) colors, each one of weight at least \( w_i\). Therefore, \( \frac{i^*}{i} \leq \frac{\left\lceil \frac{|E_i|}{b} \right\rceil - \left\lfloor \frac{x_i^2}{2b} \right\rfloor + (2\Delta_i - 1)}{\max\{\Delta_i, \left\lceil \frac{|E_i|}{b} \right\rceil\}} \). By distinguish between \( \Delta_i \geq \left\lceil \frac{|E_i|}{b} \right\rceil \) and \( \Delta_i < \left\lceil \frac{|E_i|}{b} \right\rceil \) it follows that in either case \( \frac{i^*}{i} \leq 3 - \frac{x_i^2 + 3b}{2b\Delta_i} \). This bound is maximized when \( \Delta_i = \sqrt{2b}\), that is \( \frac{i^*}{i} \leq 3 - \frac{2}{\sqrt{2b}} \). Thus, \( w_i \leq w_i^+ \leq w_i^* \leq \frac{x_i^2}{3 - \frac{2}{\sqrt{2b}}} \). Summing up these inequalities for all \( i\)'s, \( 1 \leq i \leq k \), we obtain the \((3 - \frac{2}{\sqrt{2b}})\) ratio for general graphs.

A similar analysis yields the \((3 - \frac{2}{\sqrt{b}})\) ratio for bipartite graphs. We present here an example for which the algorithm performs a ratio of exactly \(3 - \frac{2}{\sqrt{b}}\) for bipartite graphs. There is, also, an analogous example for general graphs. Consider the bipartite graph shown in Figure 1(a), where \( C \gg \epsilon \), and \( b = 9\). The weight of the optimal solution shown in Figure 1(b) is \(3C + 3\epsilon\). The weight of the solution obtained by Algorithm Greedy, shown in Figure 1(c), is \(7C - \epsilon\). Thus, the ratio for this instance is \(\frac{7C - \epsilon}{3C + 3\epsilon}\) \(\approx\frac{7}{3} = 3 - \frac{2}{\sqrt{9}}\).

![Figure 1](image_url)

Figure 1: (a) An instance of the EC\((w, b)\) problem on bipartite graphs, \( C \gg \epsilon, b = 9\). (b) An optimal solution. (c) The solution obtained by Algorithm Greedy.

Another approximation result for the EC\((w, b)\) problem is obtained by exploiting a general framework, presented in [11], which allows to convert a \(\rho\)-approximation algorithm for a coloring problem into an \(e \cdot \rho\)-approximation one for the corresponding max-coloring problem, for hereditary classes of graphs. In fact, this framework has been presented for such a conversion from the VC to the VC\((w)\) problem, but it can be easily seen that this applies also for conversions from the EC, VC\((b)\) and EC\((b)\) problems to the EC\((w)\), VC\((w, b)\) and EC\((w, b)\) problems, respectively.
However, this conversion leads to ratios greater than those shown in Table 1 for the EC($w$, $b$) and VC($w$, $b$) problems. For the EC($w$, $b$) problem on general graphs this approach gives a ratio of at least $\frac{4}{3}$, $e > 3$, as the EC, and hence the EC($b$), problem cannot be approximated within a ratio less than $\frac{4}{3}$. On the other hand, the EC($w$, $b$) problem on bipartite graphs can be approximated, this way, with a ratio of $e$, as the EC($b$) problem is polynomial in this case (see Table 1).

Combining the discussion above with Propositions 3 and 2 it follows that

**Theorem 5** The EC($w$, $b$) problem can be approximated with a ratio of $\min\{3 − 2/\sqrt{2b}, H_b\}$, for general graphs, and $\min\{e, 3 − 2/\sqrt{b}, H_b\}$, for bipartite graphs.

Note that, the $H_b$ ratio outperforms the other only for $b \leq 5$, for general graphs, and $b = 3$, for bipartite graphs, and, hence, $b$ can be considered as fixed. These ratios are shown in Table 2, for several values of $b$.

### 4.2 NP-completeness for trees

We prove first that the bounded list edge-coloring, EC($\phi$, $b$), problem is NP-complete even if the graph $G = (V, E)$ is a set of chains, $|\phi(e)| = 2$, for all $e \in E$, and $b = 5$. We denote this problem as EC(chains, $|\phi(e)| = 2$, $b = 5$).

**Proposition 4** The EC(chains, $|\phi(e)| = 2$, $b = 5$) problem is NP-complete.

**Proof:** By Theorem 1(ii) (III), the VC(chains, $|\phi(v)| \leq 2$, $b_i \leq 5$) problem is NP-complete. Given that the line-graph of a chain is also a chain, it follows that the EC(chains, $|\phi(e)| \leq 2$, $b_i \leq 5$) problem is also NP-complete. The later problem can be easily reduced to the EC(chains, $|\phi(e)| \leq 2$, $b = 5$) problem, where $b_i = b = 5$ for all colors: for every color $C_i$, with $b_i < 5$, add $5 − b_i$ independent edges with just $C_i$ in their lists. This last problem reduces to the EC(chains, $|\phi(e)| = 2$, $b = 5$) problem, where $|\phi(e)| = 2$ for all edges. This can be done by transforming an instance of EC(chains, $|\phi(v)| \leq 2$, $b = 5$) as following: (i) add two new colors $C_{k+1}$ and $C_{k+2}$, both with cardinality bound $b = 5$, (ii) add color $C_{k+1}$ to the list of every edge $e$ with $|\phi(e)| = 1$, (iii) add ten independent edges and put in their lists both colors $C_{k+1}$ and $C_{k+2}$.  

**Theorem 6** The EC($w$, $b$) problem on trees is NP-complete.

**Proof:** Our reduction is from EC(chains, $|\phi(e)| = 2$, $b = 5$) problem. We construct an instance of the EC($w$, $b$) problem on a forest $G' = (V', E')$ as follows.

We replace every edge $e = (u, v) \in E$ with a chain of three edges: $e_1 = (u, u')$, $e_2 = (u', v')$ and $e_3 = (v', v)$, where $w(e_1) = w(e_2) = w(e_3) = 1$. Moreover, we create $k − |\phi(e)| = k − 2$ stars of $k − 1$ edges each. We add edges $(u', s_t)$, $1 \leq t \leq k − 2$, between $u'$ and the central vertex $s_t$ of each of these $k − 2$ stars; thus every star has now exactly $k$ edges. Let $\phi(e) = \{C_i, C_j\}$. The $k − 2$ edges $(u', s_t)$ take different weights in $\{1, 2, \ldots, k\} \setminus \{i, j\}$. Let $q$ be the weight taken by an

| $b$ | General graphs | Bipartite graphs |
|-----|----------------|------------------|
| 3   | 1.833 $H_b$    | 1.833 $H_b$      |
| 4   | 2.083 $H_b$    | 2.000 $3 − 2/\sqrt{b}$ |
| 5   | 2.283 $H_b$    | 2.106 $3 − 2/\sqrt{b}$ |
| 6   | 2.423 $3 − 2/\sqrt{2b}$ | 2.184 $3 − 2/\sqrt{b}$ |
| ... | ... $3 − 2/\sqrt{2b}$ | ... $3 − 2/\sqrt{b}$ |
| 50  | 2.800 $3 − 2/\sqrt{2b}$ | 2.717 $3 − 2/\sqrt{b}$ |
| 51  | 2.802 $3 − 2/\sqrt{2b}$ | 2.718 $e$ |
| ... | ... $3 − 2/\sqrt{2b}$ | ... $e$ |

Table 2: Approximation ratios for the EC($w$, $b$) problem.
edge \((u', s_t)\). The remaining \(k - 1\) edges of the star \(t\) take different weights in \(\{1, 2, \ldots, k\} \setminus \{q\}\). In the same way, we add \(k - 2\) stars connected to \(v'\). In Figure 2 is shown the \(u''\)'s part of this edge-gadget for \(e = (u, v)\). For every edge \(e\) of \(G\), we add \(2(k - 2)\) stars and \(2(k - 2)k + 2\) edges.

![Figure 2: The gadget for an edge \(e = (u, v)\) with \(\phi(e) = \{C_i, C_j\}\).](image)

The number of stars in the forest \(G'\) we have constructed is \(2|E|(k - 2) + \sum_{i=1}^{k} (F - f_i)(k - 1) = k(k - 1)F - 2|E|\), since \(\sum_{i=1}^{k} f_i = 2|E|\). By setting \(b' = k(k - 1)F - 2|E| + 5 + F\), we prove that:

*There is a \(k\)-coloring for EC\((\phi, b)\) (chains, \(|\phi(e)| = 2, b = 5\), if and only if, \(G'\) has a bounded max-edge-coloring of total weight \(\sum_{i=1}^{k} i\) such that every color is used at most \(b'\) times*.

Consider, first, a solution \(C\) to the EC\((\phi, b)\) problem. We construct a solution \(C'\) for the EC\((w, b)\) problem as following. Let \(e = (u, v) \in E\) be an edge with \(\phi(e) = \{C_i, C_j\}\), which, w.l.o.g., appears in the color \(C_i\) of \(C\). Put the edges \(e_1\) and \(e_3\) of the edge-gadget for \(e\) in color \(C_i\), while the edge \(e_2\) in color \(C_j\). After doing this for all edges in \(E\), each color \(C'_i\) contains at most \(2 \cdot 5 + 1 \cdot (f_i - 5) = f_i + 5\) edges. Next, put the edges with weight \(i, 1 \leq i \leq k\), from the \(k(k - 1)F - 2|E|\) stars into \(C'_i\). Each color \(C'_i\) in \(C'\) constructed so far contains at most \(k(k - 1)F - 2|E| + f_i + 5 = b' - (F - f_i)\) edges and, by the construction of \(G'\), \(C'\) is a proper coloring. In the \(F - f_i\) color-gadgets for \(C_i\) there are \(F - f_i\) remaining \((x, y)\) edges of weight \(i\), which can still be inserted into color \(C'_i\). Thus, we get a solution for the EC\((w, b)\) problem of \(k\) colors, each one of at most \(b'\) edges, and total weight \(\sum_{i=1}^{k} i\).

Conversely, consider a solution \(C'\) to the EC\((w, b)\) problem. \(C'\) consists of exactly \(k\) colors of weights \(1, 2, \ldots, k\), since each star in \(G'\) has \(k\) edges and each edge has a different weight in the range \(\{1, 2, \ldots, k\}\). Thus, all edges of the same weight, say \(i\), should belong in the same color \(C'_i\) of \(C'\). Therefore, \(C'_i\) contains one edge from each one of the \(k(k - 1)F - 2|E|\) stars as well as the \(F - f_i\) remaining \((x, y)\) edges of the color-gadgets having weight \(i\). Consider, now, the edges of \(G'\) corresponding to the edges \(e_1, e_2\) and \(e_3\) of the edge-gadget for an edge \(e\) with \(\phi(e) = \{C_i, C_j\}\).
By the construction of $G'$ and the choice of edge weights, the edges $e_1$, $e_2$ and $e_3$ should appear into colors $C'_i$ and $C'_j$. Thus, edges $e_1$ and $e_3$ should appear, w.l.o.g., into color $C'_i$, while $e_2$ into color $C'_j$. Therefore, the edge $e \in E$ can be colored by color $C_i \in \phi(e)$. Finally, a color $C_i$ contains at most $5$ edges of type $e_1$ (or $e_3$), corresponding to at most 5 edges of $E$; otherwise $|C'_i| \geq k(k - 1)F - 2|E| + (F - f_i) + (2 \cdot 6 + 1 \cdot (f_i - 6)) > b'$, a contradiction.

To prove our proof for the EC($w, b$) problem on trees, let $p$ be the number of trees in $G'$. We add a set of $p - 1$ edges of weight $\epsilon < 1$ to transform the forest $G'$ into a single tree $T$. This can be easily done since every tree of $G'$ has at least two vertices. By keeping the same bound $b'$, it is easy to see that there is a solution for the EC($w, b$) problem on $G'$ of weight $\sum_{i=1}^{k} \frac{1}{i} p_i$, if and only if, there is a solution for the EC($w, b$) problem on $T$ whose weight is equal to $\sum_{i=1}^{k} i + \frac{1}{2^{k-1}} \epsilon$. 

4.3 A 2-approximation algorithm for trees

In [18] a 2-approximation algorithm for the EC($w$) problem on trees has been presented, which is also exploited to derive a ratio of 3/2 for that problem. This algorithm yields to a solution of $\Delta$ colors, $\mathcal{M} = \{M_1, M_2, \ldots, M_\Delta\}$. Starting from this solution we obtain a solution to the EC($w, b$) problem by finding the ordered $b$-partition of each color in $\mathcal{M}$. For the sake of completeness we give below the whole algorithm.

Algorithm Convert
1. Let $T_r$ be the tree rooted in an arbitrary vertex $r$;
2. For each vertex $v$ in pre-order traversal of $T_r$ do
3. \hspace{0.5cm} Let $\langle E_v \rangle = \langle e_1, e_2, \ldots, e_{d(v)} \rangle$ be the edges adjacent to $v$, and $(v, p)$ be the edge from $v$, $v \neq r$, to its parent;
4. \hspace{0.5cm} Using ordering $\langle E_v \rangle$, insert each edge in $E_v$, but $(v, p)$, into the first matching which does not contain an edge in $E_v$;
5. Let $\mathcal{M} = \{M_1, M_2, \ldots, M_\Delta\}$ be the colors constructed;
6. For $i = 1$ to $\Delta$ do
7. \hspace{0.5cm} Let $\mathcal{P}_i = \{M_1, M_2, \ldots, M_i\}$ be the ordered $b$-partition of $\langle M_i \rangle$;
8. \hspace{0.5cm} Return a solution $\langle C \rangle = \langle C_1, C_2, \ldots, C_k \rangle$, $\mathcal{C} = \bigcup_{i=1}^{\Delta} \mathcal{P}_i$.

Theorem 7 Algorithm Convert is a 2-approximation one for the EC($w, b$) problem on trees.

Proof: Consider the color $C_j$ in the solution $\langle C \rangle$ and let $e$ be the heaviest edge in $C_j$, i.e., $w(e) = w_j$. Let $X \subseteq C_j = \{C_1, C_2, \ldots, C_{j-1}\}$ such that each color $C_p \in X$ has (i) $|C_p| = b$, and (ii) all edges of weight at least $w(e)$. Let also $Y = C_j \setminus X$, $|X| = x$ and $|Y| = y$. Clearly, $x + y = j - 1$. Let $j^*$ be the number of colors in an optimal solution of weight at least $w(e)$, that is $w_j = w_j^*$.

There are at least $x \cdot b + y + 1$ edges of weight at least $w(e)$. These edges in an optimal solution appear in at least $\lfloor \frac{x \cdot b + y + 1}{k} \rfloor \geq x + 1$ colors, that is, $j^* \geq x + 1$.

We show, next, that all colors in $Y \cup \{C_j\}$ come from $y + 1$ different colors in $\mathcal{M}$. Assume that two of these colors, $C_q$ and $C_r$, come from the ordered $b$-partition of the same color $M_i \in \mathcal{M}$. Assume, w.l.o.g., that $w_q \geq w_r$, and let $f$ be the heaviest edge in $C_r$. Note that $C_r$ may coincide with $C_j$, while $C_q$ cannot. As $C_q \in Y$, it follows that $|C_q| = b$ and there is an edge $f' \in C_q$ with $w(f') < w(e) \leq w(f)$, a contradiction to the definition of the ordered $b$-partition of $M_i$. Therefore, $C_j$ comes from a color $M_i \in \mathcal{M}$, $i \geq y + 1$, that is $e \in M_i$. By the construction of the coloring $\mathcal{M}$, there are at least $i - 1$ edges, adjacent to each other, of weight at least $w(e)$ (i.e., $i - 2$ of them adjacent to $e$ and $e$ itself). These $i - 1$ edges appear in different colors in an optimal solution, that is, $j^* \geq y$.

Combining the two lower bounds for $j^*$ and taking into account that $x + y = j - 1$ we get $j^* \geq \lceil \frac{1}{2} \rceil$. Therefore, $w_j = w_j^* \leq w_j^* \leq \lceil \frac{1}{2} \rceil$. For large values of $b$ the EC($w, b$) coincides with the EC($w$) problem.
By a careful analysis, the complexity of both Lines 2 and 6 of the algorithm is $O(n \log n)$, where $n$ is the number of vertices of the tree.

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