Inference for the proportional odds cumulative logit model with monotonicity constraints for ordinal predictors and ordinal response

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Abstract

The proportional odds cumulative logit model (POCLM) is a standard regression model for an ordinal response. Ordinality of predictors can be incorporated by monotonicity constraints for the corresponding parameters. It is shown that estimators defined by optimization, such as maximum likelihood estimators, for an unconstrained model and for parameters in the interior set of the parameter space of a constrained model are asymptotically equivalent. This is used in order to derive asymptotic confidence regions and tests for the constrained model, involving simple modifications for finite samples. The finite sample coverage probability of the confidence regions is investigated by simulation. Tests concern the effect of individual variables, monotonicity, and a specified monotonicity direction. The methodology is applied on real data related to the assessment of school performance.

Keywords: Ordinal data, Monotonic regression, Monotonicity direction, Monotonicity test, Constrained Maximum Likelihood Estimation.

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1 Introduction

Regression problems are treated in which the response variable is ordinal and one or more of the explanatory variables are ordinal as well. When considering regression relations between variables, monotonicity has a special
status regarding ordinal variables. The ordinal scale type implies that the ordering of the possible values is meaningful, whereas there is no meaningful quantitative distance between them. Changing values of the explanatory variable in a meaningful manner means to increase or decrease them. Assuming a general monotonic relationship between explanatory variable and response means that such a meaningful change is connected to the response in the same way (meaning either decreasing or increasing) everywhere on the measurement scale of the explanatory variable. This gives the monotonicity assumption a special status between ordinal variables, although of course not all relationships between ordinal variables are monotonic. Note that monotonicity for ordinal variables can be seen as analogous to linearity for interval scaled measurements, which implies that the same quantitative difference between values of the explanatory variable (defined as meaningful at interval scale level) has the same quantitative effect on the response everywhere on the measurement scale.

In this sense the monotonic relationship between ordinal variables in a regression is a key reference point, as is the linearity assumption between interval scaled variables. A monotonicity assumption can be seen as default, unless contradicted by the data or specific background knowledge.

We treat a maximum likelihood estimator (MLE) in a proportional odds cumulative logit model (POCLM, McCullagh (1980)), in which relationships between ordinal variables are constrained to be monotonic. Such constraints have been proposed by Espinosa and Hennig (2019). We pay special attention to the relation between the constrained and the unconstrained MLE, and to confidence regions (CRs), which based on the unconstrained MLE may be used to reject the monotonicity assumption where contradicted by the data, and for monotonicity direction detection. Espinosa and Hennig (2019) proposed to decide about the appropriateness of the monotonicity assumption and monotonicity directions based on one-dimensional confidence intervals (CIs) for parameters, which can lose a lot of power compared to using multivariate CRs taking into account dependence between parameters.

An alternative approach for regression with ordinal predictors would be to represent them by latent variables (Moustaki (2000)) or optimal scaling (De Leeuw et al. (2009)). Work on isotonic regression (Stout (2015)) is also related to the present approach.

There is a lot of work on constrained inference with some results concerning monotonicity constraints as a special case of linear constraints, see, e.g., Shapiro (1988); Dupacova and Wets (1988); Silvapulle and Sen (2005). In particular, in many settings based (potentially asymptotically) on Gaussian distributions, the distribution of the loglikelihood ratio test statistic involving constrained MLEs in presence of linear constraints for at least one of
the null hypothesis or the alternative is a mixture of $\chi^2$-distributions with degrees of freedom smaller or equal to the $\chi^2$-distribution that applies in the corresponding unconstrained case (Gouriéroux et al. (1982)). Unfortunately, these results do not apply here, because (a) we allow for both isotonic and antitonic relationships, so that the constrained parameter space is not convex, and (b) the distribution of the constrained loglikelihood ratio statistic for one ordinal predictor of interest will depend on the unknown parameter values for other ordinal predictors if present, which we do not exclude. Self and Liang (1987); Kopylev and Sinha (2011) demonstrate that constraints in nuisance parameters may spoil the theory based on mixtures of $\chi^2$-distributions. This also implies that straightforward simulation cannot be used to find critical values. Some modern alternatives as suggested by Ketz (2018); Al Mohamad et al. (2020); Cox and Shi (2022); Chernozhukov et al. (2023) seem attractive but cannot be applied to the situation here without much theoretical effort, the success of which cannot currently be taken for granted. Bootstrap may in principle be available, but can be inconsistent in such a situation (Andrews (2000)). Cohen et al. (2000) treats a further issue with constrained likelihood inference.

Overall, the adaptation of existing approaches for constrained inference seems hard to impossible here, and we take a different direction. Whereas much of the constrained inference literature focuses on tests, the starting point for the inference presented here is CRs, and we explore what can be done based on the standard asymptotics for unconstrained inference. Tests can then be constructed by the equivalence of tests and CRs. It has been noted in the literature (e.g., Shapiro (1988); Kopylev and Sinha (2011)) that in many cases the standard $\chi^2$-asymptotics apply to parameters in the interior of the parameter space, whereas the cited theory of constrained inference focuses on parameters on the boundary, as is required for computing p-values under “worst case” conditions in the null hypothesis. Focusing on CRs means that “most” of the considered parameter values are in the interior, and boundary behaviour is rather exceptional. We do not deny that the behaviour on the boundary of the parameter space is relevant. Sometimes it is of interest to test a boundary null hypothesis, and also, for finite samples, boundary effects are relevant if the true parameter is close to the boundary but not necessarily on it.

Coverage probabilities of CRs ignoring constraints are still valid for boundary parameters. The problem with these CRs is (a) their overlap with the constrained parameter set may be empty or very small in case that the data are to some extent in conflict with the constraints, and (b) that otherwise they can be too big, i.e., associated tests are too conservative and less powerful than they could be if based on distribution theory that take the constraints
properly into account (as illustrated in the Introduction of Silvapulle and Sen (2005)). Without having such a theory available, we investigate in Section 4.3 how well modifying unconstrained inference by just using the constrained estimator works. Given that all inference is based on asymptotic theory, one result there is that coverage probabilities can be slightly lower than nominal for finite samples, particularly small ones. This suggests that the higher power to be potentially achieved by proper boundary theory, which here can also only be asymptotic, may come at the price of even worse coverage and anticonservativity of tests, as the resulting CRs may be smaller.

Section 2 introduces the unconstrained POCLM and reviews its inference and asymptotic theory. Not only are CRs for the unconstrained POCLM a basis for the CRs for the constrained POCLM, they are of interest in their own right when making statements about whether the data are compatible with none, one, or both possible monotonicity directions for the ordinal predictors. Section 3 introduces the monotonicity constraints for ordinal predictors. It is possible to leave the monotonicity direction unspecified, or to constrain the parameters belonging to a variable to be isotonic, or antitonic. In order to use the unconstrained asymptotic theory for the constrained model, a general theorem is shown that constrained estimators defined by optimization are asymptotically equivalent to their unconstrained counterparts on the interior of the constrained parameter space. This result is used in Section 4 in order to define confidence regions and tests for the constrained POCLM. Confidence regions based on unconstrained asymptotics may in finite samples contain parameters that do not fulfill monotonicity constraints, and adaptations for this case are proposed, and their coverage probabilities compared in a simulation study. The methodology is applied to a data set on school performance in Chile in Section 5, and Section 6 concludes the paper.

2 The unconstrained POCLM

Consider a regression with an ordinal response $z_i$, $i = 1, \ldots, n$, $n$ being the number of independent observations, with $k$ categories. There are $t$ ordinal predictors (OPs; $x_{i,s,h}$ refers to a dummy variable for ordinal predictor $s = 1, \ldots, t$ and category $h_s = 1, \ldots, p_s$) and $v$ non-ordinal quantitative predictors $x_{n,u}$, $u = 1, \ldots, v$. According to the POCLM (McCullagh (1980)),

$$\logit[P(z_i \leq j | x_i)] = \alpha_j + \sum_{s=1}^{t} \sum_{h_s=2}^{p_s} \beta_{s,h_s} x_{i,s,h_s} + \sum_{u=1}^{v} \beta_{u} x_{i,u}, \quad j = 1, \ldots, k - 1,$$

(1)
with
\[-\infty = \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < \alpha_k = \infty,\]  
and where the first category for each predictor is the baseline category, i.e., \(\beta_{s,1} = 0\).

The dimensionality of the parameter space is \(p = (k-1) + \sum_{s=1}^{t} (p_s - 1) + v\), defining the \(p\)-dimensional parameter vector as
\[
\gamma^\prime = (\alpha^\prime, \beta^\prime) \\
= (\alpha^\prime, \beta_{(ord)}^\prime, \beta_{(nonord)}^\prime) \\
= (\alpha^\prime, \beta_1^\prime, \ldots, \beta_t^\prime, \beta_1, \ldots, \beta_v),
\]
where \(\beta_{(ord)}\) is the parameter vector associated with the ordinal predictors and \(\beta_{(nonord)}\) is the one associated with the non-ordinal predictors.

With \(z = (z_1, \ldots, z_n)\), \(X_n = (x_1, \ldots, x_n)\), the log-likelihood function is
\[
\ell(\gamma) = \sum_{i=1}^{n} \sum_{j=1}^{k} y_{ij} \log \pi_j(x_i),
\]
where \(y_{ij} = 1(z_i = j)\), and \(1(\bullet)\) denotes the indicator function, and \(\pi_j(x_i) = P(z_i = j|x_i)\), i.e.,
\[
\pi_j(x_i) = \frac{e^{\alpha_j + \sum_{s=1}^{t} \sum_{h_s=2}^{p_s} \beta_{s,h_s} x_{i,s,h_s} + \sum_{u=1}^{v} \beta_{u} x_{i,u}}}{1 + e^{\alpha_j + \sum_{s=1}^{t} \sum_{h_s=2}^{p_s} \beta_{s,h_s} x_{i,s,h_s} + \sum_{u=1}^{v} \beta_{u} x_{i,u}}} - \frac{e^{\alpha_j-1 + \sum_{s=1}^{t} \sum_{h_s=2}^{p_s} \beta_{s,h_s} x_{i,s,h_s} + \sum_{u=1}^{v} \beta_{u} x_{i,u}}}{1 + e^{\alpha_j-1 + \sum_{s=1}^{t} \sum_{h_s=2}^{p_s} \beta_{s,h_s} x_{i,s,h_s} + \sum_{u=1}^{v} \beta_{u} x_{i,u}}}. \tag{4}
\]
The parameter space for model (1) is
\[
U_{UM} = \{ \gamma^\prime = (\alpha^\prime, \beta_1^\prime, \ldots, \beta_t^\prime, \beta_{(nonord)}^\prime) \in \mathbb{R}^p : -\infty < \alpha_1 < \cdots < \alpha_{k-1} < \infty \}, \tag{5}
\]
where \(p = k - 1 + \sum_{s=1}^{t} (p_s - 1) + v\). We refer to this as the “unconstrained model” (thus “\(U_{UM}\)”) despite the constraints on the \(\alpha\)-parameters, because in Section 3 constraints will be introduced for the \(\beta\)-vectors of ordinal predictors.

The unconstrained maximum likelihood estimator (UMLE) is
\[
\hat{\gamma} = \arg \max_{\gamma \in U_{UM}} \ell(\gamma), \tag{6}
\]
then $\hat{\gamma}$ is the vector of UMLEs belonging to the parameter space $U_{UM}$.

The score function and the Fisher information matrix of the first $n$ observations are

$$s_n(\gamma) = \partial \log L(\gamma|z_n, X_n)/\partial \gamma,$$

$$F_n(\gamma) = \text{cov}_\gamma s_n(\gamma).$$

2.1 Asymptotic results for the unconstrained POCLM

Asymptotic theory for the unconstrained POCLM is given in Fahrmeir and Kaufmann (1986), applying results for generalized linear models (GLMs) proved in Fahrmeir and Kaufmann (1985); Fahrmeir (1987). This theory is summarized here. The unconstrained POCLM is treated as a multivariate generalized linear model (GLM), for which the following structure is assumed:

(i) The $\{y_i\}$ are $k$-dimensional independent random variables with densities

$$f(y_i|\theta_i) = c(y_i) \exp(\theta_i' y_i - b(\theta_i)), \quad i = 1, 2, \ldots,$$

with parameter vector $\theta_i \in \Theta^0$, which is the interior of the convex set $\Theta$ of all parameters $\theta$ for which (9) defines a valid density. For $\theta_i \in \Theta^0$: $E_{\theta_i}(y_i) = \partial b(\theta_i)/\partial \theta_i = \mu(\theta_i)$.

(ii) The matrix $X_i$ influences $y_i$ in form of a linear combination $\eta_i = X_i' \gamma$ where $\gamma$ is a $p$-dimensional parameter.

(iii) The linear combination is related to the mean $\mu(\theta)$ of $y_i$ by the injective link function $g : M \rightarrow \mathbb{R}^k$, $\eta_n = g(\mu(\theta_i))$, where $M$ is the image $\mu(\Theta^0)$ of $\Theta^0$.

Model (1) can be connected to (9) by setting $k = k - 1$, $y_i = (1(z_i = j))_{j=1,\ldots,k}$. (1) applies the logit (which is the natural link function for a multinomial model for $y_i$) to $(1(z_i \leq j))_{j=1,\ldots,k}$ instead, and the link function also needs to involve $1(z_i = j) = 1(z_i \leq j) - 1(z_i \leq j - 1)$, which makes it non-natural for the POCLM.

Making reference to the results for non-natural link functions in Fahrmeir and Kaufmann (1985), the following conditions, of which only one has to be fulfilled, for asymptotic results for the POCLM are given in Fahrmeir and Kaufmann (1986):

(B) There is a $c < \infty$ such that $\|x_i\| < c$ for $i \in \mathbb{N}$, and for $n \rightarrow \infty$, the smallest eigenvalue of $\sum_{i=1}^n x_i x_i'$ goes to $\infty$. 

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(G) \( \|x_n\| = o(\log n) \) and for some \( a > 0, c > 0, n_1 \in \mathbb{N}, \) for all \( n > n_1 : \sum_{i=1}^n x_i x'_i > cn^a. \)

Although somewhat restrictive, these conditions are conditions on the observed predictors only, not on unobservable model parameters. They are stronger than what is required for standard linear regression, see Fahrmeir and Kaufmann (1986), but can be argued to be realistic in many cases.

\( \xrightarrow{\Delta} \) denotes convergence in distribution, \( \mathcal{N}_p(0, I_p) \) is the \( p \)-variate Gaussian distribution with mean vector \( 0 \) and unit matrix \( I_p \) as covariance matrix, and \( F_n(\gamma)^{-1/2} \) is the Cholesky square root of \( F_n(\gamma) \). The following results then hold:

**Theorem 2.1.** Assuming (B) or (G), the probability that a unique MLE \( \hat{\gamma} \) exists converges to one. Any sequence of MLEs \( (\hat{\gamma}_n)_{n \in \mathbb{N}} \) is weakly consistent and asymptotically normal,

\[
(F_n(\gamma)^{-1/2})'(\hat{\gamma}_n - \gamma) \xrightarrow{\mathcal{L}} \mathcal{N}_p(0, I_p). \tag{10}
\]

Fahrmeir and Kaufmann (1986) also state an optimality result for the MLE. See Fahrmeir and Kaufmann (1985) for an additional condition that allows for strong consistency.

Now consider a standard loglikelihood ratio test for the linear hypothesis

\( H_0 : C\gamma = \xi \) against \( H_1 : C\gamma \neq \xi, \tag{11} \)

where \( C \) is an \( r \times p \)-matrix of full row rank \( r, \xi \in \mathbb{R}^r, \) and \( H_0 \) has a nonempty intersection with the parameter space. Let

\[
R_n = 2[\ell(\hat{\gamma}) - \ell(\tilde{\gamma})] \tag{12}
\]

be the standard loglikelihood ratio statistic, where \( \hat{\gamma} \) is the MLE within \( H_0, \) fulfilling \( C\hat{\gamma} = \xi, \) and

\[
W_n = (C\hat{\gamma} - \xi)' [CF_n^{-1}(\hat{\gamma})C']^{-1} (C\hat{\gamma} - \xi) \tag{13}
\]

be the Wald statistics.

**Theorem 2.2.** Assuming (B) or (G), if \( H_0 \) in (11) is true, then \( R_n \) and \( W_n \) are asymptotically equivalent, and

\[
R_n, W_n \xrightarrow{\mathcal{L}} \chi^2_r. \]
Under additional conditions, Fahrmeir (1987) derives an asymptotic \( \chi^2_r(\delta) \) distribution with noncentrality parameter \( \delta \) for \( W_n \) under certain sequences of parameters from \( H_1 \).

We will mainly consider asymptotic CRs based on \( R_n \) here. One-dimensional CIs for a parameter \( \beta \) of level \( 1 - \nu \) (as used in Espinosa and Hennig (2019)) are usually constructed as

\[
\hat{\beta} \pm z_{1-\nu/2}(SE_{\hat{\beta}}),
\]

where \( z_{1-\nu/2} \) is the \( 1 - \nu/2 \)-quantile of the Gaussian distribution, and the estimated standard error \( SE_{\hat{\beta}} \) can be obtained from the asymptotic covariance matrix. In Theorem 2.1, \( F_n(\gamma) \) depends on the true parameter and replacing it by \( \hat{\gamma} \) requires additional conditions, see Fahrmeir and Kaufmann (1985). However, the resulting CI is equivalent to the one that can be obtained from the Wald statistic \( W_n \) with \( r = 1 \), and therefore it is valid according to Theorem 2.2. Otherwise, we normally prefer the loglikelihood ratio statistic \( R_n \) for inference, see Harrell (2001) for a discussion.

### 3 Monotonicity constraints for the POCLM

#### 3.1 The constrained parameter space

When ordinal predictors are treated as of nominal scale type, no monotonicity constraints are imposed on any of the \( t \) vectors \( \beta_s \), \( s = 1, \ldots, t \), and the parameter space is \( U_{UM} \).

In order to take their ordinality into account, Espinosa and Hennig (2019) proposed monotonic constraints on the \( p_s - 1 \) parameters associated with the categories of an ordinal predictor \( s \). Monotonicity is allowed to be either isotonic or antitonic. The isotonic constraints are

\[
0 \leq \beta_{s,2} \leq \cdots \leq \beta_{s,p_s}, \quad \forall s \in \mathcal{I},
\]

where \( \mathcal{I} \subseteq \mathcal{S} \), with \( \mathcal{S} = \{1, 2, \ldots, t\} \), and the antitonic constraints are

\[
0 \geq \beta_{s,2} \geq \cdots \geq \beta_{s,p_s}, \quad \forall s \in \mathcal{A},
\]

where \( \mathcal{A} \subseteq \mathcal{S} \).

The model under monotonicity constraints, i.e., requiring \( s \in \mathcal{A} \cup \mathcal{I} \) for all ordinal variables \( s \) in (1) will be referred to as the constrained model. Even if a variable is in fact ordinal, a researcher may have reasons to not enforce constraints, in which case it can be treated as non-ordinal in the model.
\( \tilde{U}_{CM} \) is the parameter set that constrains all ordinal variables to fulfill monotonicity constraints but allows both monotonicity directions:
\[
\tilde{U}_{CM} = \{ \gamma' = (\alpha', \beta_{1}', \ldots, \beta_{t}', \beta_{(nonord)}) \in \mathbb{R}^p : -\infty < \alpha_1 < \cdots < \alpha_{k-1} < \infty, \\
(\beta_{s,2} \geq 0, \beta_{s,h_s} \geq \beta_{s,h_s-1}) \text{ or } (\beta_{s,2} \leq 0, \beta_{s,h_s} \leq \beta_{s,h_s-1}) \}
\]
\[\forall (s, h_s) \in \mathcal{S} \times \{3 \ldots, p_s\}. \]
\[\text{(17)}\]

\( U_{CM} \) is the parameter set that fixes the monotonicity directions for all ordinal variables (normally assuming that these are correct):
\[
U_{CM} = \{ \gamma' = (\alpha', \beta_{1}', \ldots, \beta_{t}', \beta_{(nonord)}) \in \mathbb{R}^p : -\infty < \alpha_1 < \cdots < \alpha_{k-1} < \infty, \\
(\beta_{s,2} \geq 0, \beta_{s,h_s} \geq \beta_{s,h_s-1}) \forall (s, h_s) \in \mathcal{I} \times \{3 \ldots, p_s\}, \\
(\beta_{s,2} \leq 0, \beta_{s,h_s} \leq \beta_{s,h_s-1}) \forall (s, h_s) \in \mathcal{A} \times \{3 \ldots, p_s\}. \}
\]
\[\text{(18)}\]

Note that in practice equality for the \( \beta \)-parameters should be allowed, because in case the unconstrained optimizer of the likelihood violates monotonicity constraints, the constrained optimizer will only exist if equality is allowed. This is different from equality for the \( \alpha \)-parameters, because equality of subsequent \( \alpha \) parameters would imply that there is a category \( j \in \{1, \ldots, k\} \) with \( P(z_i = j) = 0, i = 1, \ldots, n \), in which case this category can be dropped from the model.

The constrained maximum likelihood estimator (CMLE) is
\[
\hat{\gamma}_c = \arg \max_{\gamma \in \tilde{U}_{CM}} \ell(\gamma).
\]
\[\text{(19)}\]

One may also be interested in direction constrained MLEs (DMLEs) with fully prespecified monotonicity directions with \( \mathcal{I} \) and \( \mathcal{A} \) chosen suitably,
\[
\hat{\gamma}_d = \arg \max_{\gamma \in U_{CM}} \ell(\gamma),
\]
\[\text{(20)}\]
or even partially direction constrained MLEs (PMLEs) that optimize \( \gamma \) over a parameter space with some monotonicity directions fixed and others left open. These estimators can be computed using the R-function \text{maxLik} from the package of the same name (Henningsen and Toomet (2011)).

### 3.2 Asymptotics for constrained estimators

The asymptotic theory for the UMLE presented above makes sure that for large enough \( n \) it is arbitrarily close to the true parameter. In case that
monotonicity constraints indeed hold, the UMLE can therefore be expected to be in $U_{CM}$ with arbitrarily large probability as long as the true parameter vector is in the set $U^o_{CM}$, where for a set $S \subseteq \mathbb{R}^p$, $S^o$ denotes the interior set, i.e.,

$$S^o = \{x : \exists \epsilon > 0 : \|x - y\| < \epsilon \Rightarrow y \in S\}.$$ 

$U^o_{CM}$ is $U_{CM}$ with all “$\geq$” and “$\leq$” replaced by “$>$” and “$<$”, respectively. In this case, the UMLE and the CMLE will be equal with probability approaching one, and asymptotically equivalent.

We will state this rather obvious fact as a general theorem for constrained estimators, which we have not been able to find in the literature; Liew (1976) showed a corresponding result for the least squares estimator; Silvapulle and Sen (2005); Shapiro (1988); Kopylev and Sinha (2011) mention the applicability of standard theory in the interior of the parameter space in various situations.

For observations $x_i, i \in \mathbb{N}$ in some space $\mathcal{X}$, consider estimators $(T_n)_{n \in \mathbb{N}}$ of parameters $\theta$ from some parameter set $\Theta \subseteq \mathbb{R}^p$ that are defined by optimization of a function $h_n$:

$$T_n(x_1, \ldots, x_n) = \arg \max_{\theta \in \Theta} h_n(x_1, \ldots, x_n; \theta). \quad (21)$$

In case the argmax does not exist or is not unique, $T_n$ can be defined as taking any value. The issue of interest here is whether the following properties hold for a constrained estimator assuming that they hold for the unconstrained $T_n$:

(C1) The probability that the argmax in the definition of $T_n$ exists and is unique converges to 1 for $n \to \infty$.

(C2) $T_n$ is weakly consistent for $\theta$ (the results below also hold if “weakly” is replaced by “strongly”).

(C3) For given functions $G_n$ and distribution $Q$, assuming $\theta_0 \in \Theta$ as the true parameter,

$$G_n(T_n; \theta_0) \xrightarrow{\mathcal{D}} Q.$$ 

Constrained estimation is defined by assuming

$$\theta \in \tilde{\Theta} \subset \Theta, \quad \tilde{T}_n(x_1, \ldots, x_n) = \arg \max_{\theta \in \tilde{\Theta}} h_n(x_1, \ldots, x_n; \theta).$$

If the argmax in the definition of $T_n$ exists uniquely, and $T_n \in \tilde{\Theta}$, then $T_n = \tilde{T}_n$. 

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Theorem 3.1. With the notation introduced above, assuming that the true \( \theta_0 \in \tilde{\Theta}^o \),

(a) If \( T_n \) fulfills (C1) and (C2), then \( \tilde{T}_n \) fulfills (C1) and (C2), and

\[
\lim_{n \to \infty} P\{T_n = \tilde{T}_n\} = 1.
\]

(b) If \( T_n \) fulfills (C1), (C2), and (C3), then \( \tilde{T}_n \) also fulfills (C3).

Proof. (C2) for \( T_n \) means that for all \( \epsilon > 0 \) :

\[
P\{\|T_n - \theta_0\| < \epsilon\} \to 1.
\]

Consider \( \theta_0 \in \tilde{\Theta}^o \) and \( \epsilon > 0 \) so that

\[
U_\epsilon = \{\theta \in \Theta : \|\theta - \theta_0\| < \epsilon\} \subseteq \tilde{\Theta}.
\]

If the argmax exists and is unique (which happens with \( P \to 1 \) because of (C1)), if \( T_n \in U_\epsilon \), then \( \tilde{T}_n = T_n \), therefore

\[
P\{T_n = \tilde{T}_n\} \to 1 \quad \text{and} \quad P\{\|\tilde{T}_n - \theta_0\| < \epsilon\} \to 1.
\]

This holds for all \( \epsilon > 0 \), as \( U_\epsilon \subseteq \tilde{\Theta} \) for arbitrarily small \( \epsilon \), and if \( \epsilon^* \) is so large that \( U_{\epsilon^*} \not\subseteq \tilde{\Theta} \), then \( \epsilon^* > \epsilon \) and

\[
P\{\|\tilde{T}_n - \theta_0\| < \epsilon^*\} \geq P\{\|\tilde{T}_n - \theta_0\| < \epsilon\} \to 1.
\]

Therefore (C2) holds for \( \tilde{T}_n \), as does (C1), because

\[
P\left(\{\text{argmax is unique and exists}\} \cap \{T_n \in U_\epsilon \subseteq \tilde{\Theta}\}\right) \to 1.
\]

This proves (a).

(C3) for \( T_n \) means that for all continuity points \( x \) of the cdf of \( Q \),

\[
P\{G_n(T_n; \theta_0) \leq x\} \to Q(-\infty, x].
\]

Consider again \( U_\epsilon \subseteq \tilde{\Theta} \). Ignoring the possibility that the argmax might not exist or be unique (the probability of which vanishes due to (C1)), let

\[
U_{1n} = \{T_n \in U_\epsilon, \ G_n(\tilde{T}_n; \theta_0) \leq x\} = \{T_n \in U_\epsilon, \ G_n(T_n; \theta_0) \leq x\}, \quad (22)
\]

\[
U_{2n} = \{T_n \in U_\epsilon^c, \ G_n(\tilde{T}_n; \theta_0) \leq x\}. \quad (23)
\]

Then \( \{G_n(\tilde{T}_n; \theta_0) \leq x\} = U_{1n} \cup U_{2n} \). Because of (C2), \( P\{T_n \in U_\epsilon\} \to 1 \), and therefore

\[
|P(U_{1n}) - P\{G_n(T_n; \theta_0) \leq x\}| \to 0, \ P(U_{2n}) \to 0.
\]

Therefore,

\[
\lim_{n \to \infty} P\{G_n(\tilde{T}_n; \theta_0) \leq x\} = \lim_{n \to \infty} P(U_{1n}) = \lim_{n \to \infty} P(G_n(T_n; \theta_0) \leq x) = Q(-\infty, x],
\]

(24)

proving (C3) for \( \tilde{T}_n \) and therefore (b). \[\square\]
Theorem 3.1 implies that Theorems 2.1 and 2.2 do not only hold for the UMLE but also for the CMLE, and in fact the DMLE and PMLEs, meaning that asymptotic inference for them is the same, unless \( \theta_0 \) is on the border of \( \tilde{U}_{CM} \), in which case the probability for the CMLE to equal the UMLE may not go to zero. Weak consistency also implies that for \( \theta_0 \in \tilde{U}_{CM}^0 \) and \( n \to \infty \) all monotonicity directions will be estimated correctly with probability converging to 1, i.e., UMLE and CMLE will be in \( U_{CM} \).

4 Inference based on asymptotics

4.1 Confidence regions

Due to the asymptotic equivalence of UMLE and CMLE, inference about the parameters of the constrained POCLM can be based on the theory for the unconstrained POCLM, particularly (12) and Theorem 2.2. As opposed to Espinosa and Hennig (2019), the results here allow for inference regarding multiple parameters. Simultaneous inference may be of particular interest regarding all parameters belonging to a variable. Unconstrained inference can be relevant for testing monotonicity. If the monotonicity assumption for ordinal variables is indeed true, one should expect higher efficiency for inference that assumes the monotonicity constraints into account. Regarding constrained inference, CRs are probably most interesting; they may be interpreted as quantifying some kind of “effect size”.

A major issue regarding Theorem 3.1 is that its proof is based on the fact that asymptotically UMLE and CMLE are the same. CRs based on Theorem 2.2 will be, with probability converging to one, subsets of arbitrarily small neighborhoods of the MLE and therefore also of the true parameters, meaning that if the monotonicity constraints are indeed fulfilled with strict inequalities, the whole CR will only contain parameters indicating the correct monotonicity direction.

In practice, for finite \( n \), it is neither guaranteed that UMLE and CMLE are the same, nor that all parameters in such a CR indicate the same monotonicity direction, or at least fulfill any monotonicity constraint. The validity of asymptotic CRs may then be doubted.

For the sake of simplicity, in the following discussion, we consider CRs for a vector of parameters of interest, usually those belonging to a single ordinal variable, although ideas also apply to more general vectors fulfilling linear constraints as in (11). For a vector with \( r \) parameters of interest, \( \beta_r \), the overall parameter vector \( \gamma' = (\alpha', \beta') \) is partitioned as \( (\beta', \phi') \), where \( \phi \) is a vector with the remaining \( (p - r) \) parameters. The unconstrained MLE is
now denoted as \((\hat{\beta}_r', \hat{\phi}')\) accordingly, and the constrained MLE is \((\hat{\beta}_{c,r}', \hat{\phi}_{c}')\).

Unconstrained CRs have the form

\[
\text{UCR} = \left\{ \beta_{0r} : 2[\ell(\hat{\beta}_r, \hat{\phi}) - \ell(\beta_0, \tilde{\phi})] \leq \chi^2_{r:1-\alpha}, \beta_{0r} \in U_{UM} \right\}, \tag{25}
\]

where \(\tilde{\phi}\) is defined by

\[
\ell(\beta_{0r}, \tilde{\phi}) = \max_{\phi \in U_{UM}} \ell(\beta_{0r}, \phi). \tag{26}
\]

Using the UCR, considering a single ordinal variable, the following cases can occur:

1. UMLE and CMLE are the same, and all parameter vectors in the CR indicate the same monotonicity direction.

2. UMLE and CMLE are the same; the CR contains parameter vectors indicating the same monotonicity direction as the MLE, but also parameter vectors violating monotonicity.

3. UMLE and CMLE are the same; the CR contains parameter vectors indicating the same monotonicity direction as the MLE, but also parameter vectors indicating the opposite monotonicity direction, in which case there will normally also be parameters violating monotonicity in the CR.

4. UMLE and CMLE are not the same; the CR contains parameter vectors indicating the same monotonicity direction as the CMLE on top of parameters vectors that are non-monotonic, or that potentially also indicate another monotonicity direction.

5. UMLE and CMLE are not the same, and the CR does not contain any parameter vector fulfilling the monotonicity constraints.

6. In case that there are further variables in the model, there is a further possibility, namely that the UMLE fulfills monotonicity constraints, but due to enforcement of constraints for another variable that is potentially dependent on the variable in question, UMLE and CMLE differ on that variable as well. In that case it is possible that all parameter vectors in the CR indicate the same monotonicity direction, but the other cases listed above can in principle also occur.

Only the first case indicates a behavior in line with the asymptotic theory. In the other cases the validity of the asymptotic CR for the given finite sample
can be doubted. Assuming that monotonicity in fact holds, the other cases indicate that the true parameter is so close to the boundary of the constrained parameter space that for the given sample size one can expect an asymmetric distribution of the CMLE, because even if UMLE and CMLE are actually the same, parameters that do not satisfy the constraints are also compatible with the data. In the third and potentially also the fourth case, the distribution of the CMLE may be bimodal, as the true parameter may be so close to the boundary between monotonicity directions that they may “switch”; for datasets generated by such parameters, the UMLE may fall within both monotonicity direction domains and also may violate monotonicity, in which case the CMLE may be in either possible domain.

One could use techniques such as bootstrap to find constrained finite sample CRs, but here we explore what can be done based on the asymptotic CRs. Section 4.3 will then explore these CRs empirically.

The easiest way to obtain a constrained CR from an unconstrained one is to just remove all parameter vectors that do not satisfy monotonicity constraints:

$$\text{UCCR} = \left\{ \beta_{0r} : 2[\ell(\tilde{\beta}_{r}, \hat{\phi}) - \ell(\beta_{0r}, \tilde{\phi})] \leq \chi_{r;1-\alpha}^2, \beta_{0r} \in \tilde{U}_{CM} \right\}. \quad (27)$$

This may be seen as legitimate, because if the confidence level is (approximately) valid in the unconstrained case, this will also hold for the constrained case, as the constrained parameter set is a subset of the unconstrained one. This may however result in very small CRs that suggest more precision than the given sample size can provide. In case five above, the CR will even be empty. This is not incompatible with the general theory of CRs, but can be seen as undesirable.

An alternative is to center the CR at the CMLE rather than the UMLE (using the unconstrained theory otherwise), which guarantees it to be nonempty. Asymptotically, at least in the interior of the parameter space, UMLE and CMLE are equal, so the UCR, UCCR, and CCR are asymptotically equivalent, where

$$\text{CCR} = \left\{ \beta_{0r} : 2[\ell(\beta_{c,r}, \tilde{\phi}_{c}) - \ell(\beta_{0r}, \tilde{\phi}_{c})] \leq \chi_{r;1-\alpha}^2, \beta_{0r} \in \tilde{U}_{CM} \right\}. \quad (28)$$

On the boundary, the CCR is not even asymptotically guaranteed to have the nominal confidence level. We additionally take into account potential constraints on other parameters by using $\tilde{\phi}_{c}$ instead of $\tilde{\phi}$. $\tilde{\phi}_{c}$ is the vector of constrained maximum likelihood estimators as a function of the value of $\beta_{0r}$, i.e., defined according to (26), but requiring $\tilde{\phi} \in \tilde{U}_{CM}$ rather than $U_{UM}$. This can be expected to improve the power but may incur anticonservativity
(Self and Liang (1987); Kopylev and Sinha (2011)), which in the definition of the ACR below based on the CCR is not an issue. The UCCR with $\hat{\phi}$ is asymptotically trivially conservative but not with $\tilde{\phi}_c$. For the CCR this would not even necessarily hold with $\tilde{\phi}$. Where standard constrained inference results apply (see Chapter 4 of Silvapulle and Sen (2005)) and the constrained loglikelihood ratio follows a mixture of $\chi^2$-distributions, using the CMLE and the plain $\chi^2$-distribution will be conservative. In the given situation, this cannot be taken from granted.

A third option would be to use the aggregated CR:

$$\text{ACR} = \text{UCCR} \cup \text{CCR}.$$  

This guarantees a nonempty CR that is asymptotically conservative.

### 4.2 Tests

Three kinds of null hypotheses may be of special interest for an ordinal variable $s \in \{1, \ldots, t\}$:

1. $H_0 : \beta_s = 0$; no impact of variable $s$.

2. $H_0 : \text{either } 0 \leq \beta_{s,2} \leq \cdots \leq \beta_{s,p_s}, \text{ or } 0 \geq \beta_{s,2} \geq \cdots \geq \beta_{s,p_s}$; variable $s$ fulfills monotonicity constraints.

3. $H_0 : 0 \leq \beta_{s,2} \leq \cdots \leq \beta_{s,p}$; the impact of variable $s$ is isotonic (or, analogously, antitonic); a specific monotonicity direction obtains.

Only the first $H_0$ is a linear hypothesis of the type (11). This can be tested by a standard $\chi^2$ test using $R_n$, but there is a subtlety. $\hat{\gamma}$ in (12), the definition of $R_n$, may be taken as the UMLE, or the CMLE (or even DMLE or PMLE), respectively, in case these are different. The choice between these relies on how confident the user is to impose a monotonicity assumption, even in case that the UMLE indicates against it.

The second $H_0$ implies that monotonicity is not assumed for the alternative. An asymptotically valid test can be defined by rejecting $H_0$ in case that no parameter vector in the UCR fulfills monotonicity constraints. In fact, it only needs to be checked whether the CMLE is in the UCR (equivalent to the UCCR not being empty), because the CMLE maximizes the likelihood among the parameter vectors that fulfill the constraints, so if the CMLE does not fulfill $2[\ell(\hat{\beta}_r, \hat{\phi}) - \ell(\beta_{0r}, \hat{\phi})] \leq \chi^2_{r,1-\alpha}$ with $\beta_{0r} = \hat{\beta}_{c,r}$, no element $\beta_{0r} \in \tilde{U}_{CM}$ can. The correspondence between CRs and tests is used here to define these tests, but $p$ fulfilling

$$2[\ell(\hat{\beta}_r, \hat{\phi}) - \ell(\beta_{0r}, \hat{\phi})] = \chi^2_{r,1-p}$$

(29)
for $\beta_{or} = \hat{\beta}_{c,r}$ does not define a valid p-value, because asymptotically, under $H_0$, UMLE and CMLE are the same, so that a p-value defined in this way will converge to 1 under $H_0$. The opposite $H_0$, namely whether variable $s$ does not fulfill monotonicity constraints, can be tested in an analogous way by checking whether non-monotonic parameters are in the UCR.

The third $H_0$ can be tested by asking whether the corresponding PMLE (or DMLE) constraining the monotonicity direction of interest accordingly is in a suitable CR; if not, the $H_0$ is rejected. Once more, it depends on what assumptions the user is willing to make otherwise, whether the UCCCR (or, here equivalently, UCR), CCR, or ACR should be used. As long as the true parameter is in the interior of the constrained space, asymptotically these are all the same, as is the $\chi^2$-distribution used to define the CR.

4.3 Finite sample coverage probabilities of confidence regions

Coverage probabilities (CPs) of the UCCR, CCR, and ACR, will be simulated and compared under different scenarios in order to see whether these asymptotically motivated CRs perform well on finite samples; see Morris et al. (2019) for some general considerations regarding such experiments.

Consider model (1) with four ordinal predictors with 3, 4, 5, and 6 ordered categories each, one categorical (non-ordinal) predictor with 5 categories and one interval-scaled predictor. For every $i$th observation, each of the four ordinal predictors ($s = 1, \ldots, 4$) is represented in the model by dummy variables denoted as $x_{i,s,h_s}$, with $h_s = 2, \ldots, p_s$ and where $p_1 = 3$, $p_2 = 4$, $p_3 = 5$, and $p_4 = 6$; the nominal predictor is denoted as $x_{i,5,h_5}$ with $h_5 = 2, \ldots, 5$; and the interval-scaled predictor as $x_{i,1}$. The first category of the categorical variables is considered as the baseline. Thus, the simulated model is

$$
\text{logit}[P(z_i \leq j | x_i)] = \alpha_j + \sum_{h_1=2}^{3} \beta_{1,h_1}x_{i,1,h_1} + \sum_{h_2=2}^{4} \beta_{2,h_2}x_{i,2,h_2} + \sum_{h_3=2}^{5} \beta_{3,h_3}x_{i,3,h_3} \\
+ \sum_{h_4=2}^{6} \beta_{4,h_4}x_{i,4,h_4} + \sum_{h_5=2}^{5} \beta_{5,h_5}x_{i,5,h_5} + \beta_{1}x_{i,1},
$$

(30)

where the number of categories of the ordinal response is $k = 4$, i.e., $j = 1, 2, 3$. This model was fitted for 500 data sets that were simulated using the following true parameters: for the intercepts $\alpha_1 = -2$, $\alpha_2 = 2$, and $\alpha_3 = 5.5$; for the non-ordinal categorical predictor $\beta'_{5} = (0.7, 1.4, -0.3, -1.2)$; and for the interval-scaled predictor $\beta_1 = 0.3$. The values of the ordinal predictors were drawn from the population distributions illustrated in Figure 1. The
simulated values for the non-ordinal categorical predictor were drawn from the following population distribution: 0.2, 0.2, 0.3, 0.1, 0.2 for categories 1, 2, 3, 4, and 5, respectively. The interval-scaled covariate \( x_1 \) was randomly generated from \( \mathcal{N}(1, 4) \).

The current simulation design offers 12 different scenarios defined by two factors: (i) distances between parameter values of adjacent ordinal categories and (ii) sample sizes. The true parameter vectors of the ordinal predictors were chosen to represent three different levels (large, medium, and small) of distances between the parameter values of their adjacent ordinal categories as shown in Figure 2. These will be referred to as “monotonicity degrees”. Four different sample sizes were considered: \( n = 50, 100, 500, \) and 1,000.

Table 1 shows the results for the three CRs defined in the previous section. The nominal confidence level was 95%. All parameters were assessed simultaneously \( (r = p) \). The “Total” row shows the final coverage percentages. Coverage percentages are also differentiated according to whether the UMLE and CMLE are the same or not. Note that simulated coverage probabilities below 93.4\% (467/500) are significantly smaller than the confidence level of 95%.

The key results are:

- For \( n = 500, \) all overall coverage probabilities are close to 95\% except for “Medium distances”, which slightly not close to 95\% when UMLE and CMLE are different (11.6\% of the cases) but it is close to 95\% when the UMLE and CMLE are the same (442 cases out of 500, 88.4\%). Except for “Small distances” and \( n = 500, \) the UMLE and CMLE are equal in most cases, as suggested by the asymptotic.
Figure 2: True parameter values for the simulation of coverage probabilities. Different line styles represent different distances between the parameter values associated with adjacent ordinal categories: large, medium, small.

Table 1: Coverage percentages (%) for different sample sizes, definitions of CRs, and according to whether the UMLE and CMLE are the same or not. The row “Same MLE” has the number of cases (out of 500) in brackets for which UMLE and CMLE were the same.

| CR | 50  | 100 | 500 | 1000 |
|----|-----|-----|-----|------|
| n  | UCCR | CCR | ACR | UCCR | CCR | ACR | UCCR | CCR | ACR | UCCR | CCR | ACR |
| 50 | 92.3 | 96.0 | 96.0 | 94.4 | 95.2 | 95.2 | 94.4 | 95.2 | 95.2 | 95.4 | 95.4 | 95.4 |
| 100| 95.0 | 93.2 | 93.2 | 93.4 | 93.4 | 93.4 | 93.4 | 93.4 | 93.4 | 93.4 | 93.4 | 93.4 |
| 500| 97.4 | 97.6 | 97.6 | 95.4 | 95.4 | 95.4 | 95.4 | 95.4 | 95.4 | 95.4 | 95.4 | 95.4 |
| 1000| 97.8 | 98.6 | 98.6 | 95.2 | 95.2 | 95.2 | 95.2 | 95.2 | 95.2 | 95.2 | 95.2 | 95.2 |

Small distances between parameter values of adjacent ordinal categories

| Same MLE | Different MLE | Total |
|----------|---------------|-------|
| 90.0 | 94.9 | 95.0 |
| 83.2 | 88.5 | 88.6 |
| 96.2 | 98.6 | 98.6 |
| 85.7 | 90.5 | 95.2 |
| 94.5 | 94.9 | 95.6 |
| 96.6 | 96.0 | 94.6 |
| 92.3 | 82.8 | 94.3 |
| 96.0 | 82.8 | 95.1 |
| 96.0 | 82.8 | 96.8 |
| 93.8 | 93.2 | 93.2 |
| 96.2 | 93.4 | 93.4 |
| 96.2 | 93.4 | 93.4 |
| 98.4 | 93.4 | 93.4 |

Medium distances between parameter values of adjacent ordinal categories

| Same MLE | Different MLE | Total |
|----------|---------------|-------|
| 94.9 | 97.7 | 95.0 |
| 88.5 | 85.7 | 88.6 |
| 98.6 | 99.4 | 99.4 |
| 90.5 | 94.3 | 95.2 |
| 94.9 | 95.1 | 95.1 |
| 95.6 | 95.1 | 95.1 |
| 96.0 | 94.3 | 95.4 |
| 96.0 | 94.6 | 94.6 |
| 97.4 | 95.4 | 95.4 |

Large distances between parameter values of adjacent ordinal categories

| Same MLE | Different MLE | Total |
|----------|---------------|-------|
| 94.9 | 97.7 | 95.0 |
| 88.5 | 85.7 | 88.6 |
| 98.6 | 99.4 | 99.4 |
| 90.5 | 94.3 | 95.2 |
| 94.9 | 95.1 | 95.1 |
| 95.6 | 95.1 | 95.1 |
| 96.0 | 94.3 | 95.4 |
| 96.0 | 94.6 | 94.6 |
| 97.4 | 95.4 | 95.4 |
• For low \( n \), coverage probabilities of both UCCR and CCR can be substantially below 95%, whereas the ACR is conservative.

• With higher monotonicity degree, coverage probabilities tend to be larger, and UMLE and CMLE tend to be the same more often.

• For \( n = 50 \), the UCCR has higher coverage probabilities than the CCR, whereas for \( n = 100 \) the opposite is true. We believe that the reason is that for \( n = 50 \) the estimators still have such a large variability that imposing the constraints may occasionally make matters worse by enforcing the wrong monotonicity direction, whereas for \( n = 100 \) constraining starts to help.

• In reality researchers don’t know the degree of monotonicity, and for \( n = 50 \) only the ACR ensures good coverage for all degrees. For \( n = 100 \) and above, the CCR achieves satisfactory results, being less conservative than the ACR.

A further simulation addresses a case in which there is an OP with parameters at the border of the constrained parameter space \( \tilde{U}_{CM} \). Model (1) with two ordinal predictors is simulated:

\[
\logit[P(z_i \leq j | x_i)] = \alpha_j + \sum_{h_1 = 2}^{3} \beta_{1,h_1} x_{i,1,h_1} + \sum_{h_2 = 2}^{4} \beta_{2,h_2} x_{i,2,h_2}. \tag{31}
\]

OP1 has \( \beta_{1,1} = \beta_{1,2} = \beta_{1,3} = 0 \), and OP2 has four categories using the different degrees of monotonicity from the previous setting (small, medium and large). The number of categories of the ordinal response and the remaining characteristics of the simulation setting are the same as in the first simulation above. The results are shown in Table 2.

Different from the first simulation, for all \( n \) and monotonicity degrees, both UMLE=CMLE and UMLE\neq CMLE happen with substantial probability not apparently converging to zero. This was to be expected, as in any neighbourhood of the true parameters there are both monotonic and non-monotonic parameters. The argument for the asymptotic validity of the UCCR does not hold here. Nevertheless, except for one particular combination of factors (\( n = 100 \), “Small distances”, and UCCR), all overall coverage probabilities are close to 95%, none is significantly smaller, and only for large distances between the parameter values of OP2 and low \( n \) there is a certain tendency to conservativity.

Certain results on constrained inference (e.g., Gouriéroux et al. (1982)) suggest that with the true parameter on the border of the parameter space,
Table 2: Simulation with two OPs, where OP1 has $\beta_{1,1} = \beta_{1,2} = \beta_{1,3} = 0$. Coverage percentages for different sample sizes, definitions of CRs, and according to whether the UMLE and CMLE are the same or not. The row “Same MLE” has the number of cases (out of 500) in brackets for which UMLE and CMLE were the same.

| n   | UCCR  | CCR  | ACR  | UCCR  | CCR  | ACR  | UCCR  | CCR  | ACR  | UCCR  | CCR  | ACR  |
|-----|-------|------|------|-------|------|------|-------|------|------|-------|------|------|
| 50  | 97.8  | 96.0 | 96.9 | 97.9  | 97.1 | 96.9 | 98.3  | 97.3 | 95.9 | 96.3  | 98.5 | 98.0 |
| 100 | 93.6  | 91.4 | 94.4 | 93.6  | 90.6 | 93.6 | 97.0  | 97.6 | 94.8 | 96.3  | 98.1 | 98.0 |
| 500 | 94.4  | 92.8 | 95.0 | 94.4  | 92.8 | 95.0 | 97.0  | 97.4 | 95.6 | 94.8  | 98.4 | 98.4 |
| 1000| 96.8  | 96.6 | 96.6 | 97.8  | 98.0 | 97.4 | 97.2  | 97.0 | 95.4 | 96.0  | 96.0 | 96.0 |

the distribution of the loglikelihood ratio statistic is a mixture of $\chi^2$-distributions, quantiles of which could be bounded from above by a $0.5\chi^2 + 0.5\chi^2_{r-1}$-mixture, even though known theory does not apply to our setup with more than one potentially constrained OP (see the Introduction). For the setups discussed above, we also simulated coverage probabilities using $1 - \alpha$-quantiles from this mixture instead of the plain $\chi^2_r$; for CCR and ACR (although there is no theoretical basis for doing this with ACR). The mixture quantiles are a bit smaller, and therefore the resulting overall coverage probabilities (results not shown) are slightly lower. The difference is quite small. The mixture helps a bit with the too conservative cases in Table 2, but makes matters worse in those situations where coverage probabilities were already too low in Table 1. In reality it is not known whether the true parameter is on the boundary, and therefore our results overall do not indicate that using mixtures of $\chi^2$-distributions improve on the plain $\chi^2$-distribution.

4.3.1 Finite sample rejection probabilities

Consider model (31) with settings and parameters values as used for the simulation behind Table 2. Rejection probabilities based on 500 replicates were calculated testing the three proposed hypotheses defined in Section 4.2, which in Tables 3, and 4 are referred to as “first”, “second” and “third”
\( H_0 \). We only show results “small” (Table 3) and “large” (Table 4) distances between parameter values of OP2, as all results for “medium” are between these and do not add interesting insight. Each one of these tables show the rejection probabilities for different ordinal predictors, hypotheses, confidence regions and sample sizes.

Comparisons between Table 2 of Section 4.3 and those of the current section are not straightforward, because coverage probabilities in Section 4.3 were computed using confidence regions for the whole parameter vector \((r = p)\), whereas rejection probabilities were computed using only two degrees of freedom when testing OP1 and three when testing OP2. Therefore, the quantiles \( \chi^2_{r,1-a} \) in (27) for UCCR and in (28) for CCR differ between the Sections.

Results are only shown for CCR and UCCR; results for the ACR are largely identical to those for the CCR, because parameters in UCCR \( \setminus \) CCR are not normally in the \( H_0 \).

For interpreting the results in Tables 3 and 4 keep in mind that out of the eight tests (“First”, “Second”, “Third” in two directions, for both OP1 and OP2), OP1 fulfills all null hypotheses, and for OP2 two null hypotheses are violated, namely “First (all zero)” and “Third (isotonic)”. The power for all tests of these two null hypotheses is 100% for OP2. Most of the rejection probabilities for the true null hypotheses are smaller than 5% or at least not significantly larger. For any monotonicity degree, rejection probabilities for “Second” and “Third (antitonic)” of OP2 and “Second” of OP1 are zero or close to it; for “OP1: First” and “OP1: Third” they are somewhat more conservative, less strongly so based on the CCR. Only the rejection probabilities based on the UCCR for “OP1: First” seem to stabilise around the nominal 5%.

This means that these tests can generally be trusted to not reject too easily. Loss of power due to conservativity may be a worry, although this is not confirmed for the null hypotheses that are violated. Regarding OP2, as its monotonicity is in some distance from the border of the parameter space, conservativity is to be expected (particularly for a high degree of monotonicity), and can probably not be avoided. Regarding OP1, exact distribution theory at the border of the parameter space may make the CCR smaller and reject more often in turn.

5 Application to school performance data

We apply the introduced methodology to data regarding the performance assessment of educational institutions in Chile. The observations are schools
| H₀    | n   | Monotonicity Direction | 50      | 100     | 500     | 1000    |
|-------|-----|------------------------|---------|---------|---------|---------|
|       |     |                        | UCCR    | CCR     | UCCR    | CCR     | UCCR    | CCR     |
| OP1   | First | None (all zero)         | 1.8     | 1.4     | 2.2     | 1.6     | 2.6     | 2.0     | 5.4     | 2.8     |
|       | Second| Isotonic or Antitonic   | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     |
|       | Third | Isotonic                | 1.2     | 0.8     | 1.2     | 0.8     | 1.2     | 1.0     | 2.2     | 1.2     |
|       |      | Antitonic               | 0.4     | 0.4     | 1.0     | 0.8     | 1.2     | 0.8     | 2.0     | 1.6     |
| OP2   | First | None (all zero)         | 100     | 100     | 100     | 100     | 100     | 100     | 100     | 100     |
|       | Second| Isotonic or Antitonic   | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     |
|       | Third | Isotonic                | 100     | 100     | 100     | 100     | 100     | 100     | 100     | 100     |
|       |      | Antitonic               | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     |

Table 3: Rejection percentages (%) based on different confidence regions for “small” distances between parameter values of adjacent ordinal categories.

| H₀    | n   | Monotonicity Direction | 50      | 100     | 500     | 1000    |
|-------|-----|------------------------|---------|---------|---------|---------|
|       |     |                        | UCCR    | CCR     | UCCR    | CCR     | UCCR    | CCR     |
| OP1   | First | None (all zero)         | 1.8     | 1.0     | 2.2     | 0.8     | 4.4     | 3.4     | 4.4     | 2.4     |
|       | Second| Isotonic or Antitonic   | 0.0     | 0.0     | 0.2     | 0.0     | 0.0     | 0.0     | 0.0     | 0.0     |
|       | Third | Isotonic                | 0.8     | 0.6     | 1.0     | 0.6     | 2.0     | 1.6     | 1.6     | 1.0     |
|       |      | Antitonic               | 0.6     | 0.4     | 0.8     | 0.2     | 2.2     | 2.0     | 2.4     | 1.6     |
| OP2   | First | None (all zero)         | 100     | 100     | 100     | 100     | 100     | 100     | 100     | 100     |
|       | Second| Isotonic or Antitonic   | 0.0     | 0.0     | 0.0     | 0.0     | 0.2     | 0.0     | 0.2     | 0.0     |
|       | Third | Isotonic                | 100     | 100     | 100     | 100     | 100     | 100     | 100     | 100     |
|       |      | Antitonic               | 0.0     | 0.0     | 0.0     | 0.0     | 0.2     | 0.0     | 0.2     | 0.0     |

Table 4: Rejection percentages based on different confidence regions for “large” distances between parameter values of adjacent ordinal categories.

that provide educational services at primary and/or secondary level. The database was built using freely accessible information¹, which is published by the Education Quality Agency, an autonomous public service that interacts with the presidency of Chile through its Ministry of Education.

In Chile, the educational system considers 12 years of schooling, eight at the primary level and four at the secondary level. The performance of each school (zᵢ) is recorded as an ordinal variable of four ordered categories: “Insufficient”, “Medium-Low”, “Medium”, and “High”. There are two extra categories representing lack of information or low number of registered students (overall 28.3% of the schools), which were excluded from the analysis. The analysed performance assessment results belong to 5,333 schools in both 2019 and 2016, specifically from their fourth year students of primary level, with national coverage. The ordinal predictors are:

¹http://informacionestadistica.agenciaeducacion.cl/#/bases
- **perf2016**: school performance results in 2016, measured in the same way as the one of 2019,

- **funding**: the way the students pay for their studies with three categories: “Public”, “Mixed” and “Private”. This is an ordinal predictor in the sense that “Private” is associated to schools where the students need to pay by themselves and the fees are higher than in schools belonging to the other classes. This also indicates the level of financial resources that each school is able to reach, being the class “Private” the highest and “Public” the lowest, and

- **regisRat**: the ratio of number of registered students in 2019 over the one of 2016.

The parameter estimators from the unconstrained POCLM are given in Table 5. The variable **perf2016** is estimated as antitonically related to \( z_i \); in fact, it has negative coefficients, meaning that a better performance in 2016 is related to a better performance in 2019. This is because (1) models probabilities to be smaller than certain values as dependent on \( x \). The coefficients associated with the categories of **funding** are not estimated to be monotonic according to the results of the UMLE, although monotonicity is only very slightly violated. Taking the meaning of the variables into account, it can be considered highly likely that a monotonic relationship holds; “Mixed” funding is properly between “Public” and “Private”, and reasons for potential non-monotonicity are hard to imagine. In order to improve interpretability, these parameter estimates were therefore constrained to be monotonic, and the CMLE was computed. The UMLE of **funding** is so close to being antitonic that not only is the CMLE for its parameters very close to the UMLE, also parameters of the other variables do not change by constraining **funding** to be monotonic, although theoretically this may happen.

Figure 3 visualizes the parameter estimators with 95% confidence intervals. The difference between the UMLE and CMLE is hardly visible. Figure 4 shows a two-dimensional 95% CR for the two free parameters of **funding**. Here the UMLE and the CMLE are visibly slightly different. CRs are visualized by checking whether two-dimensional projections of \( \beta_{0r} \) (for **funding** with \( r = 2 \) this is just \( \beta_{0r} \)) are in the CR, for a dense grid of values. The UCR comprises of monotonic and non-monotonic parameters. The latter (colored grey) belong to the UCR, but not to the UCCR and CCR. The former (colored orange) belong to UCR, UCCR and CCR.

We highlight two differences between the UCCR and CCR for this application. The first one is that UCCR is centered at the UMLE (even though all the non-monotonic parameters are removed) whereas the CCR is centered at
Table 5: Unconstrained (UMLE) and constrained (CMLE) estimates for school performance data.

|                | UMLE      | CMLE      |
|----------------|-----------|-----------|
| Intercepts     |           |           |
| 1              | -0.62759  | -0.62759  |
| 2              | 1.83259   | 1.83260   |
| 3              | 5.87701   | 5.87701   |
| perf2016       |           |           |
| Medium-Low     | -1.23255  | -1.23255  |
| Medium         | -3.20697  | -3.20697  |
| High           | -5.81422  | -5.81422  |
| funding        |           |           |
| Mixed          | 0.00609   | 0.00000   |
| Private        | -0.73117  | -0.73117  |
| regisRat       |           |           |
|                | -0.34234  | -0.34233  |

the CMLE. The second is that UCCR uses $\hat{\phi}$ when computing $\ell(\beta_{0r}, \hat{\phi})$ as defined in (27) whereas CCR uses $\tilde{\phi}_c$ for $\ell(\beta_{0r}, \tilde{\phi}_c)$ as defined in (28). In the case of the school performance data, $\phi$ and $\hat{\phi}$ are the same for every given $\beta_{0r}$ used in the grid because the (unconstrained) parameters of perf2016 in $\phi$ turned out to show an antitonic pattern and therefore it was not necessary to impose monotonicity (antitonic) constraints on its parameters when assessing whether $\beta_{0r}$ was in CCR. Hence, for the school performance data the only relevant difference between UCCR and CCR was the first one. According to the results of the real data application, the difference between these two CRs is not large; on the grid of parameter values for which we checked whether they are in the CR, only two (red) points belong to the CCR but not the UCCR.

Regarding the tests in Section 4.2, the CR shows that the $H_0: \beta_{2,3} = \beta_{2,2} = 0$ can be rejected, as this pair of parameter values is not in the CRs. There are monotonic parameters in the UCR, therefore the $H_0$ of monotonicity of funding cannot be rejected, neither can non-monotonicity be rejected. However, it can be rejected that funding is isotonic. From a subject matter perspective, antitonicity seems likely for funding, and it is also well compatible with the data, therefore it makes sense to base further inference on the CMLE.

For perf2016 there are three free parameters, so not everything can be shown simultaneously in two dimensions. Figure 5 shows two selected views. On the left side, joint CRs for $\beta_{1,4}$ and $\beta_{1,2}$ are shown. Given that every $\beta_{0r} \in UCR$ is also in $\tilde{U}_{CM}$, UCR and UCCR are the same. These two CRs are the same as CCR as well, because UMLE and CMLE here are the same and $\ell(\beta_{0r}, \tilde{\phi}_c) < \ell(\beta_{0r}, \hat{\phi})$. The plot on the left side of Figure 5 is of interest because the difference between $\beta_{1,4}$ and $\beta_{1,2}$ is the largest possible for this variable. This difference would need to change sign in order to revert
Figure 3: Coefficients estimated by UMLE and CMLE and one-dimensional 95% confidence intervals based on the UMLE for school performance data.

Figure 4: 95% two-dimensional confidence regions (UCR, UCCR, and CCR) for the two parameters of funding.
Figure 5: Left side: 95% two-dimensional confidence regions (UCR, UCCR, and CCR are the same here) with different degrees of freedom for the $\beta_{1,4}$ vs. $\beta_{1,2}$-parameters of perf2016 for school performance data. Right side: 95% two-dimensional confidence regions (UCR, UCCR, and CCR are the same here) with different degrees of freedom for $\beta_{1,4} - \beta_{1,3}$ vs. $\beta_{1,3} - \beta_{1,2}$ of perf2016.

the monotonicity direction of perf2016. No such parameter pair is in the CR, which suggests that the $H_0$ of isotonicity can be rejected, which can be confirmed by checking that $\hat{\beta}_{d,r}$, the three-dimensional MLE assuming isotonicity, is not in the three-dimensional CR. An important consideration here is the number of degrees of freedom for the $\chi^2$-distribution used in the computation of the CR. If the researcher were only interested in $\beta_{1,4}$ and $\beta_{1,2}$, there should be two degrees of freedom. However, taking into account that this plot was picked because $\beta_{1,4}$ and $\beta_{1,2}$ are the most distant parameters for this variable, choosing the degrees of freedom as 3 (number of free parameters for perf2016) takes into account possible variation on all three parameters for the resulting parameter pairs of $\beta_{1,4}$ and $\beta_{1,2}$. Figure 5 also shows the CR based on 9 degrees of freedom, which takes into account variation of all free parameters in the model. Note that CRs are not perfectly elliptical, as the loglikelihood ratio can have irregular variations in line with asymmetries in the data, as opposed to the Wald test statistic. This is actually an advantage of the loglikelihood ratio statistic, as it reflects the information in the data more precisely.

The CR on the right side of Figure 5 concerns the parameter differences $\beta_{1,4} - \beta_{1,3}$ vs. $\beta_{1,3} - \beta_{1,2}$. These differences are smaller in absolute value than $\beta_{1,4} - \beta_{1,2}$, and can therefore more easily indicate potential non-monotonicity. In fact, no non-monotonic pairs of differences (let alone isotonic ones) are in the CR, so that non-monotonicity can be rejected, as can be the $H_0$ that perf2016 does not have any impact. On top of these tests researchers of
course may be interested in “effect sizes”, i.e., how far the parameter values are away from non-monotonicity or isotonicity relative to the variation of the parameter estimators, which the CRs visualize as well.

In general, for variables with more than three categories, and consequently \( r > 2 \) free parameters, the most interesting two-dimensional views may be the one that shows the most extreme parameter pair (left side of Figure 5), which helps to assess how far the data are from the inverse monotonicity direction, and one or more plots of pairs of differences that either in terms of the UMLE run counter to the dominating monotonicity direction, or are the closest to violating it (right side of Figure 5; even though for this variable all plots comfortably confirm antitonicity).

6 Conclusion

We believe that the monotonicity assumption plays a key role in regression problems with ordinal predictors as well as an ordinal response. Obviously, monotonicity cannot be taken for granted in every application, but as is the case for the linear assumption for interval scaled variables, it means that the impact of a predictor on the response works in the same way over the whole range of the scale.

Espinosa and Hennig (2019) already proposed an approach to decide about whether monotonicity is appropriate at all, and what monotonicity direction to choose for which variable. This was based on one-dimensional confidence intervals, not taking the potential dependence between parameters appropriately into account, resulting in weaker power than possible. Here we show that looking at higher dimensional CRs the asymptotic theory of the unconstrained POCLM applies to the model with monotonicity constraints as well, unless parameters are on the boundary of the constrained space. Finite samples can highlight issues with the asymptotic approximation, namely where UMLE and CMLE are different, and where non-monotonic parameter vectors are in the UCR. For testing monotonicity, the unconstrained theory can be used, but assuming monotonicity, CRs need to be adapted. We propose some simple ways of doing that, and some tests of standard hypotheses of interest that are easy to evaluate. Finite sample theory is not easily available, neither is asymptotic theory on the boundary, therefore the CRs can only be validated empirically. Alternatives worthy to explore are bootstrap tests and confidence regions (Hall (1987); Olive (2018)), although multivariate bootstrap CRs are often computationally expensive and rarely applied in the literature. It may also be possible if probably hard to adapt existing approaches to constrained inference (in particular potentially the approach
in Al Mohamad et al. (2020); Cox and Shi (2022)) to the situation studied here.

As opposed to most work on constrained inference, we focus on CRs and define tests indirectly by correspondence to CRs. Disadvantages are the lack of automatically produced p-values, and loss of power compared with potential procedures based on constrained theory for parameters on the boundary, although our simulation study did in most situations not result in too large overconservative CRs; for small samples they could even be anticonservative. But there are also advantages. There is currently much controversy regarding significance tests and p-values and their endemic abuse. A major issue with them is that researchers often focus on significance only and neglect effect sizes (Wasserstein and Lazar (2016); Stahel (2021)), which CRs enforce to take into account. Furthermore, basing the CRs on unconstrained theory may help in situations in which a researcher is not sure about imposing monotonicity; information is always implied to assess this assumption.

7 Data Availability

Datasets for the real data application on school performance are of public access. They are available in the Studies Portal of the Agency of Education Quality, an autonomous public service that interacts with the presidency of Chile through its Ministry of Education. The link of its website is https://informacionestadistica.agenciaeducacion.cl/#/bases. Two datasets were used, one of 2016 and another of 2019. They can be found using the following options: choose “Categorías de desempeño” for “Proceso”, 2016 or 2019 for “Año”, and “Educación Básica” for “Grado”.

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