Scalable Bell inequalities for multiqubit systems

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received 12 May 2015; accepted in final form 3 August 2015
published online 1 September 2015

PACS 03.67.-a - Quantum information
PACS 03.65.Bz - Entanglement and quantum nonlocality (e.g. EPR paradox, Bell’s inequalities, GHZ states, etc.)

Abstract – Based on Clauser-Horne-Shimony-Holt inequality, we show a fruitful method to exploit Bell inequalities for multipartite qubit systems. These Bell inequalities are designed with a simpler architecture tailored to experimental demonstration. With the point of view of the stabilizer formalism, we suggest a method to investigate quantum nonlocality for multipartite systems. Under the optimal setting we derive a set of compact Mermin-type inequalities and then discuss quantum violations for generalized Greenberger-Horne-Zeilinger (GGHZ) states and two kinds of mixed states. Also, as an example, we reveal relationship between quantum nonlocality and four-particle entanglement for four-qubit GGHZ states.

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Introduction. – In 1964, the first Bell inequality was exploited [1]. Based on the local hidden variable model (LHVM), Bell inequalities can always give upper bounds on certain quantities. In quantum mechanics, if a value higher than these bounds is obtained, it is commonly referred to as a quantum violation [2–5]. Also, since Bell inequalities allow one to characterize the possible connection between quantum violation and entanglement [6–9], they are of great importance in investigating quantum entanglement [10–16].

Inspired by Bell’s paper, then, Clauser, Horne, Shimony, and Holt (CHSH) [17] derived a correlation inequality, which involves a bipartite correlation function (average over many runs of experiment) for two alternative dichotomic observables. It has been pointed out that [18,19] any pure entangled state of two alternative dichotomic observables. It has been pointed out that [18,19] any pure entangled state of two alternative dichotomic observables. It has been pointed out that [18,19] any pure entangled state of two alternative dichotomic observables. It has been pointed out that [18,19] any pure entangled state of two alternative dichotomic observables. In this letter, we present a method to construct the scalable Bell inequalities for multiqubit systems. Based on the CHSH inequality, we establish a set of Bell inequalities, it reveals the standard MABK inequality for n = 3, the Mermin inequality for n = 4, and then a set of compact Mermin type inequalities is proposed under the specified setting.

A method: constructing multipartite Bell inequalities. – The well-known Bell-type inequality for bipartite systems, the CHSH inequality [17], is given by

\[ |\langle A_1 A_2^2 + A_1 A_2^2 + A_2 A_3^2 - A_2 A_3^2 \rangle| \leq 2, \quad (1) \]

where \(A_1, A_2, A_3^1\) and \(A_3^2\) denote any physical variables with value ±1. We here consider a partition of the CHSH polynomial, \(A_1 A_2^2 - A_1 A_2^2\) and \(A_3 A_3^2 + A_3 A_3^2\), which can be referred to as two blocks. Obviously, based

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on the absolute LHVM, there exist two possible outcomes: $A_1^i A_2^i - A_1^i A_2^i = 0$ and $A_1^i A_2^i + A_1^i A_2^i = \pm 2$, or $A_1^i A_2^i - A_1^i A_2^i = \pm 2$ and $A_1^i A_2^i + A_1^i A_2^i = 0$. In terms of this particular relevance, then, we suggest an efficient algorithm for constructing the scalable Bell inequalities for multiqubit systems.

Consider an $n$-qubit system, where each of the parties is allowed to choose independently between two dichotomic observables $A_1^i$, $A_2^i$ for the $i$-th observer, $i = 1, 2, \ldots, n$, and each outcome can either take value $+1$ or $-1$. A set of Bell polynomials for multiqubit systems is defined by

$$B = \frac{1}{\sqrt{2^n}} \left[ \prod_{j \text{ is odd}} \left( A_1^i A_2^{i+1} - A_2^i A_1^{i+1} \right) \right] + \left( -1 \right)^k \prod_{j \text{ is odd}} \left( A_1^i A_2^{i+1} + A_2^i A_1^{i+1} \right) \tag{2}$$

for even $n$, and

$$B = \frac{1}{\sqrt{2^{n-2}}} \left[ A_1^i \prod_{j \text{ is even}} \left( A_1^i A_2^{j+1} - A_2^i A_1^{j+1} \right) \right] + \left( -1 \right)^k A_2^i \prod_{j \text{ is even}} \left( A_1^i A_2^{j+1} + A_2^i A_1^{j+1} \right) \tag{3}$$

for odd $n$, with $k$ a factor dependent on $n$ (let $k = \lfloor \frac{n-1}{4} \rfloor$ for the following compact Mermin-type inequalities, for example). Actually, for odd $n$, the products can be viewed as a considerable generalization by multiplying a pair of trivial items $A_1^i$ and $A_2^i$ (or $(A_1^i + A_2^i)/2$ and $(A_1^i - A_2^i)/2$) on any of the observers $i$, letting $i = 1$, for example. Note here that in this architecture, elementary units of the present inequalities are paired blocks of the CHSH polynomial. In the terms of the peculiarity of two blocks $A_1^i A_1^{i+1} - A_2^i A_2^{i+1}$ and $A_1^i A_2^{i+1} + A_2^i A_1^{i+1}$, algebraically, we have $|B| \leq 1$. Then, letting $\langle B \rangle$ denote the mean value of $B$, with the LHVM one can obtain the inequalities $|\langle B \rangle| \leq 1$.

**Quantum violations and examples.** – We now investigate quantum violations of the present Bell inequalities. As quantum counterpart of the Bell polynomial, the Bell operator $B$ should satisfy the inequality

$$\text{tr}(\rho B) = |\langle B \rangle| \leq 1, \tag{4}$$

where the density operator $\rho$ is used to characterize an $n$-partite quantum system. In general, an arbitrary quantum state of $n$ qubits can be represented by $\rho = \frac{1}{2^n} \sum_{x_1, \ldots, x_n = 0}^{2^n} T_{x_1, \ldots, x_n} \sigma_{x_1} \otimes \cdots \otimes \sigma_{x_n}$ with correlation coefficients $T_{x_1, \ldots, x_n} = \text{tr}([\sigma_{x_1} \otimes \cdots \otimes \sigma_{x_n}]),$ where $I$ is the identity operator and $\sigma_i$ ($i = 1, 2, 3$) are Pauli operators. Quantum mechanically, one can express the observables as $A_1^i = a_1^i \cdot \sigma_i, A_2^i = a_2^i \cdot \sigma_i, i = 1, 2, \ldots, n$, where $a_1^i, a_2^i$ are unit vectors and $\sigma_i = (\sigma_x, \sigma_y, \sigma_z)$ denotes a vector of Pauli matrices. A quantum violation of the Bell inequality refers to the result of the left-hand side of expression (4) larger than 1. For simplicity, instead of calculating the formulas of the maximal violations we here study the present Bell inequalities for both pure and mixed multiqubit systems with the point of view of the stabilizer formalism.

An $n$-qubit Greenberger-Horne-Zeilinger (GHZ) state

$$|\text{GHZ}_n \rangle = \frac{1}{\sqrt{2^n}} (|0\rangle^\otimes n + |1\rangle^\otimes n) \tag{5}$$

is an eigenstate of operator

$$A_n = \frac{1}{2} \left( \prod_{j=1}^{n} (\sigma_x + i \sigma_y) \right) + \frac{1}{2} (\sigma_x - i \sigma_y) \left( \prod_{j=1}^{n} (\sigma_x + i \sigma_y) \right) \tag{6}$$

with eigenvalue $2^{n-1}$. As is well known, GHZ states maximally violate Mermin inequalities [23], which have $2^{n-1}$ terms and bound the values of quantum violation $\alpha^{(n-1)/2}$ for odd $n$, and $2^{n/2-1}$ for even $n$. It is worth noting that the present inequalities involve a series of compact Mermin-type inequalities, which have $2^{(n+1)/2}$ terms for odd $n$ and $2^{n/2+1}$ for even $n$. Under the optimal setting, we show that the compact Mermin-type inequalities are maximally violated by $n$-qubit GHZ states with a certain constant visibility $\sqrt{2}$ or 2. This can be seen in the following manner. In detail, for $n = 4k - 2$, $k = 1, 2, \ldots$, taking the experimental setting $a_1^i = (1/\sqrt{2}, 1/\sqrt{2}, 0), a_2^i = (-1/\sqrt{2}, 1/\sqrt{2}, 0), a_1^i = (1, 0, 0), a_2^i = (0, 1, 0), i = 2, 3, \ldots, n$, we have $\text{tr}(\rho(B)|\text{GHZ}_n \rangle \langle \text{GHZ}_n |) \geq \sqrt{2}$ for $n = 4k - 1$ and $n = 4k$, taking $a_1^i = (1, 0, 0), a_2^i = (0, 1, 0), i = 1, 2, \ldots, n$ and for $n = 4k + 1$, taking $a_1^i = a_2^i = (1, 0, 0), a_1^i = (1, 0, 0), a_2^i = (0, 1, 0), i = 2, 3, \ldots, n$, then, we have $\text{tr}(\rho(B)|\text{GHZ}_n \rangle \langle \text{GHZ}_n |) = 2$.

We next give some examples to investigate quantum violations of the present inequalities with the above experimental setting. Firstly, consider the $n$-qubit generalized GHZ (GGHZ) states

$$|G_n(\alpha) \rangle = \cos \alpha |0\rangle^\otimes n + \sin \alpha |1\rangle^\otimes n. \tag{7}$$

A direct computation shows that

$$\text{tr}(\rho(G_n(\alpha) | B \rangle B) = 2 \sin 2\alpha \tag{8}$$

for $n \neq 4k - 2$, $k = 1, 2, \ldots,$ and

$$\text{tr}(\rho(G_n(\alpha) | B \rangle B) = \sqrt{2} \sin 2\alpha \tag{9}$$

for $n = 4k - 2$. That is, when $\sin 2\alpha > 1/2$ (or $\sin 2\alpha > 1/\sqrt{2}$ for $n = 4k - 2$) the present inequalities are violated. Compared with the threshold $\sin 2\alpha = 1/\sqrt{2^{n-1}}$ described in [37], one can find that the decrease of items of correlation functions is sometimes at the cost of detecting precision.

Other examples for quantum violations are typically two kinds of mixed states: the Werner states

$$\rho_W = p|\text{GHZ}_n \rangle \langle \text{GHZ}_n | + (1-p)\rho_{\text{Noise}}, \tag{10}$$
where \( \varrho_{\text{Noise}} = I/2^n \) represents completely uncorrelated noise contribution, and the mixtures of two maximally entangled GHZ states \( |GHZ^+_{n}\rangle = \frac{1}{\sqrt{2}} (|0^n\rangle + |1^n\rangle) \) read
\[
\varrho_{GHZ^\pm} = q|GHZ^+_{n}\rangle\langle GHZ^+_{n}| + (1-q)|GHZ^-_{n}\rangle\langle GHZ^-_{n}|.
\] (11)

With a straightforward calculation, for \( n \neq 4k - 2, k = 1, 2, \ldots \), one can obtain the results \( \text{tr}[\varrho_{W}\mathcal{B}] = 2p \) and \( \text{tr}[\varrho_{GHZ^\pm}\mathcal{B}] = 4q - 2 \). Obviously, the quantum violations occur with \( p > 1/2 \) for the Werner states and \( q > 3/4 \) for the states \( \varrho_{GHZ^\pm} \). For \( n = 4k - 2 \), the results are given by \( \text{tr}[\varrho_{W}\mathcal{B}] = 2\sqrt{p} \) and \( \text{tr}[\varrho_{GHZ^\pm}\mathcal{B}] = \sqrt{2}(2q - 1) \), and quantum violations occur with \( p > 1/\sqrt{2} \) and \( q > (2 + \sqrt{2})/4 \), respectively. Especially, when \( q = 1/2 \) we note that \( \text{tr}[\varrho_{GHZ^\pm}\mathcal{B}] = 0 \). In view of the fact that neither mixture can lead to nonlocal correlations, it is an eminently reasonable result, because when \( q = 1/2 \) the state \( \varrho_{GHZ^\pm} \) is completely equivalent to a mixture of the product states \( |0^n\rangle\langle 0^n| + |1^n\rangle\langle 1^n| \) with equal probability.

At last, let us discuss the relationship between quantum nonlocality and entanglement for GHZ states. Note that how to extend the entanglement measure to multipartite systems is still an open question and, therefore, we here focus on the situation of \( n = 4 \), for example. Now, we recast the four-qubit Bell polynomial as
\[
\mathcal{B} = \frac{1}{4} \sum_{\text{perm}} \sigma_x \sigma_x \sigma_x \sigma_x + \sigma_y \sigma_y \sigma_y \sigma_y - \sum_{\text{perm}} \sigma_x \sigma_y \sigma_y \sigma_y
\] (12)
and thus the expression (8) holds. On the other hand, consider the four-tangle of four-qubit GHZ states
\[
\tau(|G_4(\alpha)\rangle) = \sin^2 2\alpha,
\] (13)
where the four-tangle \( \tau \) quantifies four-partite entanglement [38]. Therefore, one can reveal the entanglement-nonlocality relationship as
\[
\text{tr}[\varrho(|G_4(\alpha)\rangle)\mathcal{B}] = 2\sqrt{\tau}.
\] (14)

By the way, we also note that, for a generalized slice state [39]
\[
|\psi_k\rangle = \cos \alpha |0000\rangle + \sin \alpha |11\rangle (\cos \beta |0\rangle + \sin \beta |1\rangle)
\times (\cos \gamma |0\rangle + \sin \gamma |1\rangle)(\cos \delta |0\rangle + \sin \delta |1\rangle),
\] (15)
the relationship (14) holds. In fig. 1, we show the numerical results of quantum nonlocality vs. entanglement for the four-qubit GGHZ states. The green line is the threshold above which the inequalities are violated. It has been shown that, for \( \tau > 1/4 \), the quantum violation of the present inequality occurs and varies with four-tangle, see blue line. Also, we plot the relationship between entanglement and the maximum expectation value of a generalized Svetlichny operator [9] denoted by the red line, where the maximum expectation value \( M_{\text{max}} = \sqrt{2}\tau \) for \( \tau > 1/8 \) or \( M_{\text{max}} = 1/2 \) for \( \tau \leq 1/8 \), equivalently, and thus the inequality is violated with \( \tau > 1/2 \). Therefore, in a sense, the present inequality is more sensitive to detecting the four-qubit GGHZ states.

\[ \text{Discussion and summary. –} \] Having constructed a set of multipartite Bell inequalities with a simpler architecture, we now discuss the possible experimental test of these inequalities with quantum optical systems [40]. In the decades following the famous experimental tests by Aspect et al. [41,42], based on optical systems there have been many reports [5,43] for testing the original and improved Bell inequalities. However, in experimental tests, there exist some critical loopholes, locality loopholes and detection loopholes, for example. Although simultaneously closing all such loopholes is still an open challenge, by now, these loopholes can be individually closed [44–46]. On the other hand, consider the preparations of multiphoton entanglement by using linear optics [43,47–50]. Especially, more recently, based on linear optics and available weak nonlinearity we describe a method to obtain positive \( n \)-photon GHZ states in a near deterministic way [51]. In a word, these results allow us to suggest the experimental test for the present inequalities with optical systems.

In summary, we have shown a fruitful method for constructing a series of correlation Bell inequalities for multipartite qubit systems. In the architectures of these Bell inequalities, paired blocks of the CHSH inequality are first exploited. With the point of view of the stabilizer formalism, under the specified setting we have reported a set of compact Mermin-type inequalities and discussed quantum violations of them for some multiqubit systems including \( n \)-qubit GHZ states and two kinds of mixed states. Also, we have obtained a useful relationship between quantum nonlocality and four-partite entanglement for four-qubit GGHZ states. Of course, mathematically, the maximal violations of the present inequalities may be calculated by optimizing the mean values of the Bell operators over all measurement directions [34]. In contrast to the previous inequalities [23,26,27], the total numbers of correlation functions of the present inequalities have been greatly reduced, growing exponentially with half of the number of
qubits rather than growing exponentially with the number of qubits. The compact form, together with the assumption of two dichotomic observables per site, makes it much easier to explore scalable schemes for testing the quantum formalism against the LHVM in experiments.

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