Definition of the Riesz Derivative and its Application to Space Fractional Quantum Mechanics

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Abstract

We investigate and compare different representations of the Riesz derivative, which plays an important role in anomalous diffusion and space fractional quantum mechanics. In particular, we show that a certain representation of the Riesz derivative, $R_\alpha^x$, that is generally given as also valid for $\alpha = 1$, behaves no differently than the other definition given in terms of its Fourier transform. In the light of this, we discuss the $\alpha \to 1$ limit of the space fractional quantum mechanics and its consistency.

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I. Introduction

Fractional calculus is an effective tool to study nonlocal and memory effects in physics. Successful applications to anomalous diffusion and evolution problems [1–4] were immediately followed by applications to quantum mechanics [5–12]. In particular, Laskin’s space fractional quantum mechanics is intriguing since it also follows from Feynman’s path integral approach over Lévy paths [5]. One of the first solutions of the space fractional Schrödinger equation was given for the infinite square well problem [6]. Despite its simplicity, this solution is important since it is basically the prototype of a quantum detector with internal degrees of freedom. Recently, this solution has also been the subject of some controversy and was used to point to the potential existence of an inconsistency in the solutions obtained by the piecewise method [10–13, 15, 16]. The proposed inconsistency argument was based on the evaluation of a certain integral, which resulted when the solution for the box problem obtained by the piecewise method was substituted back into the Schrödinger equation [15, 16]. One of the crucial elements of the space fractional quantum mechanics is the Riesz derivative. We have shown that a particular representation of the Riesz derivative that accommodates analytic continuation can be used to evaluate the integral in question, thus resolving the so-called inconsistency problem [10–13]. Misleading conclusions regarding the Laskin’s space fractional quantum mechanics [5] often results when one ignores the basic assumptions and restrictions involved in the use of the Riesz derivative [15–21].

A crucial part of the space fractional quantum mechanics is the Riesz derivative operator, $R_x^\alpha$, which satisfies the fractional diffusion equation:

$$\frac{\partial p_L(x,t;\alpha)}{\partial t} - \sigma_\alpha R_x^\alpha p_L(x,t;\alpha) = 0,$$

(1)

where $p_L(x,t)$ is the $\alpha$–stable Lévy distribution and $\alpha$, $0 < \alpha \leq 2$, is called the Lévy index. The $\alpha$–stable Lévy distribution with $0 < \alpha < 2$ has finite moments of order $\mu < \alpha$ and infinite moments for higher orders [5]. The Gaussian distribution, $\alpha = 2$, is also stable with moments of all orders. In space fractional quantum mechanics, the existence of average position, $\langle x \rangle$, and momentum, $\langle p \rangle$, of a particle demands that the moments of first order exist [5]. In this regard, $\alpha$ has to be restricted to the range $1 < \alpha \leq 2$.

Even though the problems regarding a particular integral in the infinite box solution, which was the basis of the inconsistency arguments, has been resolved for the range $1 < \alpha \leq 2$, potential issues regarding the $\alpha \to 1$ limit of the solutions and its connection with the particular solution of the space fractional Schrödinger equation for $\alpha = 1$ need to be clarified [20]. As the upper bound, $\alpha = 2$, is approached, the space fractional Schrödinger equation approaches smoothly to the ordinary Schrödinger equation for the classical particle. However, we can not say the same thing as $\alpha$ approaches the lower bound. We will discuss in the last section that within the context of the Schrödinger theory, the interpretation of the ordinary derivative operator for the lower bound, $\alpha = 1$, and the corresponding Hamiltonian is dubious [17, 18].
In literature, arguments about the $\alpha = 1$ case are usually carried over another representation of the Riesz derivative, which is usually given as also valid for $\alpha = 1$ [3, 21–26]. To investigate this in detail, in Section II we start with a brief review of the Riesz derivative, which is generally given in terms of its Fourier transform and valid for the range $0 < \alpha \leq 2$, $\alpha \neq 1$. In Section III, we continue with another representation of the Riesz derivative, which is generally given in literature as valid over the entire range, $0 < \alpha \leq 2$, including $\alpha = 1$ [3, 21–26]. We scrutinize its derivation and its Fourier transform, and on the contrary to common opinion, we show that its behavior at and near $\alpha = 1$ is no different than the previous definition. Finally, in Section IV we have conclusions and discuss the implications of our results in terms of Laskin’s space fractional quantum mechanics. We argue that the nonlocality implied by the Riesz derivative is of different nature than the nonlocality of the particular solution of the Schrödinger equation for $\alpha = 1$.

II. Definition of the Riesz derivative

Riesz derivative, $R_\alpha^\alpha f(x)$, is usually defined in terms of its Fourier transform as

$$\mathcal{F} \{ R_\alpha^\alpha f(x) \} = -|\omega|^\alpha F(\omega), \ \alpha > 0.$$ (2)

The fact that $|\omega|^\alpha$ is not an analytic function does not allow one to use complex contour integral theorems. In space fractional quantum mechanics, one usually encounters real singular integrals like

$$I = \int_{-\infty}^{\infty} |\omega|^\pm\alpha f(\omega) d\omega,$$ (3)

where $f(\omega)$ is a complex valued even function with finite number of singular points on the real axis. One can also write $I$ as

$$I = 2 \int_{0}^{\infty} |\omega|^\pm\alpha f(\omega) d\omega.$$ (4)

However, a common source of error is in dropping the absolute value sign and then evaluating the integral:

$$I' = 2 \int_{0}^{\infty} \omega^\pm\alpha f(\omega) d\omega,$$ (5)

via the complex contour integral theorems [27]. The last step naturally alters the analytic structure of the Riesz derivative and as in the box problem leads to misleading results [15, 16, 28]. In such situations, using the original expression of the Riesz derivative [10–13] that accommodates analytic continuation allows correct implementation of the contour integral theorems, thus resolving the controversy.
Using the Fourier transforms [13]:

\[ F\left\{ \int_{-\infty}^{\alpha} f(x) \right\} = (i\omega)^\alpha F(\omega), \quad \alpha > 0 \]  
(6)

\[ F\left\{ \int_{\infty}^{\alpha} f(x) \right\} = (-i\omega)^\alpha F(\omega), \quad \alpha > 0, \]  
(7)

where \( F(\omega) = F\{ f(x) \} \), we define the derivative

\[ D_x^\alpha f(x) = (\int_{-\infty}^{\alpha} + \int_{\infty}^{\alpha}) f(x), \]  
(8)

the Fourier transform of which is

\[ F\left\{ D_x^\alpha f(x) \right\} = ((i\omega)^\alpha + (-i\omega)^\alpha) F(\omega), \quad \alpha > 0. \]  
(9)

For real \( \omega \), the Fourier transform, \( F\{ D_x^\alpha f(x) \} \), can be written as

\[ F\left\{ D_x^\alpha f(x) \right\} = |\omega|^\alpha 2\cos(\alpha\pi/2) F(\omega), \quad \alpha > 0. \]  
(10)

From here, it is seen that \( D_x^\alpha f(x) \) does not have the desired Fourier transform for neither \( \alpha = 1 \) nor \( \alpha = 2 \), that is,

\[ F\left\{ D_x^1 f(x) \right\} \neq i\omega F(\omega), \]  
(11)

\[ F\left\{ D_x^2 f(x) \right\} \neq (i\omega)^2 F(\omega) = -|\omega|^2 F(\omega). \]  
(12)

In this regard, the Riesz derivative is defined with a minus sign as [13, 24–26]

\[ R_x^\alpha f(x) = -\frac{(\int_{-\infty}^{\alpha} + \int_{\infty}^{\alpha}) f(x)}{2\cos(\alpha\pi/2)}, \]  
(13)

where its Fourier transform becomes

\[ F\left\{ R_x^\alpha f(x) \right\} = -\frac{(i\omega)^\alpha + (-i\omega)^\alpha}{2\cos(\alpha\pi/2)} F(\omega). \]  
(14)

This form of the Riesz derivative allows analytic continuation and thus the correct implementation of the complex contour integral theorems becomes possible [10–13, 27]. For real \( \omega \), \( F\{ R_x^\alpha f(x) \} \) [Eq. (14)] can be written as

\[ F\left\{ R_x^\alpha f(x) \right\} = -|\omega|^\alpha F(\omega). \]  
(15)

This definition of the Riesz derivative has the desired Fourier transform for \( \alpha = 2 \), but it still does not reproduce the standard result for \( \alpha = 1 \). Therefore, the above definition is generally written as valid for \( 0 < \alpha \leq 2, \alpha \neq 1 \).

In space fractional quantum mechanics, the \( \alpha = 2 \) case corresponds to the Schrödinger equation for a massive nonrelativistic particle, while the \( \alpha = 1 \) case needs to be scrutinized carefully both on physical and mathematical grounds. In the following section, we investigate another representation of the Riesz derivative that is given in literature as also valid for \( \alpha = 1 \) and thus written as good for the full range \( 0 < \alpha \leq 2 \) [3, 21–26].
III. Another Representation of the Riesz Derivative

We start with the formula [1–3, 27]

\[ -\infty D_\alpha^x f(x) = \frac{d^2}{dx^2} \left[ -\infty I_\alpha^x f(x) \right], \quad 1 < \alpha < 2, \]  

(16)

and write

\[ -\infty D_\alpha^x f(x) = -\infty I_\alpha^x f(x) \]

(17)

\[ = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_{-\infty}^{x} \frac{f(x')}{(x - x')^{\alpha - 1}}dx', \]

(18)

\[ = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_{0}^{\infty} \xi^{-\alpha + 1} f(x - \xi)d\xi, \]

(19)

Using the relations:

\[ \xi^{-\alpha + 1} = (\alpha - 1) \int_{\xi}^{\infty} \frac{d\eta}{\eta^{\alpha}}, \]

(20)

\[ \frac{\partial^2 f(x - \xi)}{\partial x^2} = \frac{\partial^2 f(x - \xi)}{\partial \xi^2}, \]

(21)

we can write

\[ -\infty D_\alpha^x f(x) = \frac{(\alpha - 1)}{\Gamma(2 - \alpha)} \int_{0}^{\infty} \frac{\partial^2 f(x - \xi)}{\partial \xi^2} \left[ \int_{\xi}^{\infty} \frac{f(x')}{\eta^{\alpha}}d\eta \right]d\xi, \quad 1 < \alpha < 2. \]

(22)

Integrating by parts twice yields

\[ -\infty D_\alpha^x f(x) = -\frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} \frac{f(x - \xi) - f(x)}{\xi^{\alpha + 1}}d\xi, \quad 1 < \alpha < 2. \]

(23)

Following similar steps, we obtain

\[ \infty D_\alpha^x f(x) = -\frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} \frac{f(x + \xi) - f(x)}{\xi^{\alpha + 1}}d\xi, \quad 1 < \alpha < 2. \]

(24)

Combining these in Equation [13], we obtain another representation of the Riesz derivative:

\[ R_\alpha^x f(x) = \frac{\Gamma(1 + \alpha) \sin \alpha \pi / 2}{\pi} \int_{0}^{\infty} \frac{f(x + \xi) - 2f(x) + f(x - \xi)}{\xi^{1 + \alpha}}d\xi, \quad 1 < \alpha < 2. \]

(25)

Using similar steps, one can show that an identical relation results for the range 0 < \alpha < 1 [25]. In Literature, this representation is commonly used as another representation of the Riesz derivative that is also regular at \alpha = 1, thus written as good for the entire range 0 < \alpha \leq 2 [3, 21–26]. Since this point has been a source of major misunderstanding and misuse of the Riesz derivative, we will analyze the end points carefully and hope to clear any misconceptions that exists.
A. Riesz Derivative for $0 < \alpha < 1$ and $1 < \alpha < 2$

Before we discuss the behavior of the Riesz derivative at the end points, $\alpha = 1$ and $\alpha = 2$, we concentrate on the derivative, $\tilde{D}_x^\alpha f(x)$:

\[
\tilde{D}_x^\alpha f(x) = \frac{(-\infty D_x^\alpha + \infty D_x^\alpha) f(x)}{2 \cos(\alpha \pi/2)} = -\frac{\Gamma(1 + \alpha) \sin \alpha \pi / 2}{\pi} \int_0^\infty \frac{f(x + \xi) - 2f(x) + f(x - \xi)}{\xi^{1+\alpha}} d\xi, \tag{26}
\]

which has been obtained for the ranges $0 < \alpha < 1$ and $1 < \alpha < 2$, separately. Note that we have taken out the minus sign in front of Equation (13) and Equation (25) which was inserted by hand in the first place. In other words, $-\tilde{D}_x^\alpha f(x)$ is the second representation of the Riesz derivative (25):

\[
R_x^\alpha f(x) = -\tilde{D}_x^\alpha f(x). \tag{28}
\]

Evaluating the Fourier transform, $\mathcal{F}\{\tilde{D}_x^\alpha f(x)\}$, with respect to $x$, and using the relations:

\[
\frac{\alpha}{\Gamma(1 - \alpha)} = -\frac{1}{\Gamma(-\alpha)} = \Gamma(1 + \alpha) \frac{\sin \alpha \pi}{\pi}, \tag{29}
\]

\[
\mathcal{F}\{f(x - \xi)\} = e^{-i\omega \xi} F(\omega), \tag{30}
\]

we can write

\[
\mathcal{F}\{\tilde{D}_x^\alpha f(x)\} = -\frac{\alpha}{\Gamma(1 - \alpha)} \left[ \mathcal{F}\left\{ \int_0^\infty f(x - \xi) \frac{d\xi}{\xi^{1+\alpha}} \right\} + \mathcal{F}\left\{ \int_0^\infty \frac{f(x + \xi)}{\xi^{1+\alpha}} d\xi \right\} - 2 \mathcal{F}\left\{ \int_0^\infty \frac{f(x)}{\xi^{1+\alpha}} d\xi \right\} \right], \tag{31}
\]

\[
= -\frac{\alpha}{\Gamma(1 - \alpha)} \left[ \int_0^\infty \mathcal{F}\left\{ f(x - \xi) \right\} \frac{d\xi}{\xi^{1+\alpha}} + \int_0^\infty \mathcal{F}\left\{ f(x + \xi) \right\} \frac{d\xi}{\xi^{1+\alpha}} - 2 \int_0^\infty \mathcal{F}\left\{ f(x) \right\} \frac{d\xi}{\xi^{1+\alpha}} \right], \tag{32}
\]

\[
= -\frac{\alpha}{\Gamma(1 - \alpha)} \left[ \int_0^\infty \frac{\mathcal{F}\{f(x - \xi)\}}{\xi^{1+\alpha}} d\xi + \int_0^\infty \frac{\mathcal{F}\{f(x + \xi)\}}{\xi^{1+\alpha}} d\xi - 2 \int_0^\infty \frac{\mathcal{F}\{f(x)\}}{\xi^{1+\alpha}} d\xi \right], \tag{33}
\]

\[
= -\frac{\alpha}{\Gamma(1 - \alpha)} \frac{F(\omega)}{1 - \alpha} \left[ I_1 + I_2 - I_3 \right], \tag{34}
\]

where

\[
I_1 = \int_0^\infty \frac{e^{-i\omega \xi}}{\xi^{1+\alpha}} d\xi, \quad I_2 = \int_0^\infty \frac{e^{i\omega \xi}}{\xi^{1+\alpha}} d\xi, \quad I_3 = 2 \int_0^\infty \frac{1}{\xi^{1+\alpha}} d\xi. \tag{35}
\]

We have assumed that the integral over $\xi$ and the integral from the Fourier transform with respect to $x$ can be interchanged. The three integrals [Eq. (36)] are singular and do not exist in the Riemann sense. However, using analytic
continuation and along with an appropriate contour, we can regularize these divergent integrals and show that a meaningful result for their sum exists.

Starting with $I_1$, we use the contour in Fig. 1 and evaluate the contour integral

$$\oint_C e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}},$$

where $\xi$ is now in the complex $\xi-$plane and the contour $C$ has the parts $C_i$, $C_0$, $L_1$ and $L_2$. Since there are no singularities inside $C$, we can write [27]

$$\oint_C e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} = 0,$$

$$\oint_{C_0} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} + \oint_{C_i} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} + \oint_{L_1} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} + \oint_{L_2} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} = 0,$$

where the integral over $C_0$ is to be evaluated in the limit $R \rightarrow \infty$ and the integral $C_i$ is to be evaluated in the limit $\delta \rightarrow 0$. In these limits, the integral over $L_1$:

$$\oint_{L_1} \frac{e^{-i\omega \xi}}{\xi^{1+\alpha}} d\xi,$$

is the needed integral, $I_1$, while the integral over $L_2$ can be evaluated as

$$\oint_{L_2} \frac{e^{-i\omega \xi}}{\xi^{1+\alpha}} d\xi = -(i\omega)^\alpha \Gamma(-\alpha).$$
For the integral over $C_0$ we can put an upper bound as

$$I_{C_0} = \oint_{C_0} e^{-i\omega \xi} \xi^{1+\alpha} d\xi$$

(41)

$$= \int_{-\pi/2}^{\pi/2} \frac{e^{-i\alpha \theta}}{R^\alpha} e^{-i\omega R \cos \theta + \omega R \sin \theta} d\theta$$

(42)

$$\leq \int_{-\pi/2}^{\pi/2} \left| \frac{e^{-i\alpha \theta}}{R^\alpha} e^{-i\omega R \cos \theta + \omega R \sin \theta} \right| d\theta$$

(43)

$$\leq \frac{1}{R^\alpha} \int_{-\pi/2}^{\pi/2} e^{i\omega R \sin \theta} d\theta \leq \frac{1}{R^\alpha} \int_{-\pi/2}^{\pi/2} e^{i\omega R (2\theta/\pi)} d\theta$$

(44)

$$\leq \frac{1}{R^\alpha} \frac{\pi}{2\omega R} \left[ e^{(2\omega R/\pi)\theta} \right]_{-\pi/2}^{\pi/2} \leq \frac{1}{R^\alpha+1} \frac{\pi}{2\omega} \left[ e^{-\omega R} - 1 \right],$$

(45)

where in Equation (44) we have used the inequality [27]

$$2\theta/\pi \geq \sin \theta, \; \theta \in [-\pi/2, 0].$$

Thus, in the limit as $R \to \infty$, the integral $|I_{C_0}|$ goes to 0. On the other hand, the integral over $C_i$:

$$I_{C_i} = \oint_{C_i} e^{-i\omega \xi} \xi^{1+\alpha} d\xi$$

(46)

$$= \int_{-\pi/2}^{\pi/2} \frac{e^{-i\alpha \theta}}{\delta^\alpha} e^{-i\omega \delta \cos \theta + \omega \delta \sin \theta} d\theta$$

(47)

$$= \frac{i}{\delta^\alpha} \int_{-\pi/2}^{\pi/2} e^{-i\alpha \theta} \left[ e^{\omega \delta (\sin \theta - i \cos \theta)} \right] d\theta,$$

(48)

diverges as $\delta \to 0$. Expanding the exponential inside the square brackets and integrating term by term we can write $I_{C_i}$ explicitly in terms of $\delta$ as

$$I_{C_i} = \frac{1}{\delta^\alpha \alpha} \left[ (e^{i\alpha \pi/2} - 1) - i(\omega \delta) \frac{\sin(\alpha - 1)\pi/2}{(\alpha - 1)} + O(\omega^2 \delta^2) \right].$$

(49)

The divergent part of $I_{C_i}$ is now explicitly written in terms of $\delta$ as

$$\lim_{\delta \to 0} I_{C_i} = -\frac{1}{\delta^\alpha \alpha} (1 - e^{i\alpha \pi/2}).$$

(50)

Combining these results in Equation (39), we obtain $I_1$ as

$$I_1 = (i\omega)^\alpha \Gamma(-\alpha) + \frac{1}{\delta^\alpha \alpha} (1 - e^{i\alpha \pi/2}).$$

(51)

Similarly, but with the contour in Fig. 2, and in the limit $\delta \to 0$, we obtain $I_2$ as

$$I_2 = (-i\omega)^\alpha \Gamma(-\alpha) + \frac{1}{\delta^\alpha \alpha} (1 - e^{-i\alpha \pi/2}).$$

(52)
Figure 2: Contour for $I_2$.

Since the results for $I_1$ and $I_2$ are valid for both $1 < \alpha < 2$ and $0 < \alpha < 1$, separately, we can deduce the value of $I_3$ from $I_1$ and $I_2$ by setting $\omega$ to zero. However, we first add the two graphs (Fig. 3) to write

$$I_{L_1+L_2+C_0+C_i} + I_{L'_1+L'_2+C'_0+C'_i} = 0. \quad (53)$$

When $\omega = 0$, the integrals over $L_1$ and $L'_1$ are equal and thus give the needed integral $I_3$ as

$$I_3 = \left[ \oint_{L_1} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} + \oint_{L'_1} e^{i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} \right]_{\omega=0} = 2 \int_0^\infty \frac{1}{\xi^{1+\alpha}} d\xi. \quad (55)$$

Therefore,

$$I_3 = - \left[ \oint_{L_2} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} + \oint_{L'_2} e^{i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} \right] - \left[ \oint_{C'_i} e^{-i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} + \oint_{C_i} e^{i\omega \xi} \frac{d\xi}{\xi^{1+\alpha}} \right]. \quad (56)$$

Using Equations (40, 50–52) we write

$$I_3 = [(i\omega)^\alpha + (-i\omega)^\alpha] \Gamma(-\alpha) + \left[ \frac{1}{\delta^{\alpha \alpha}} (1 - e^{i\alpha \pi/2}) + \frac{1}{\delta^{\alpha \alpha}} (1 - e^{-i\alpha \pi/2}) \right] + 0(\omega^2 \delta^2). \quad (57)$$

Setting $\omega = 0$ and taking the limits $R \to \infty$ and $\delta \to 0$, we finally obtain $I_3$ as

$$I_3 = \frac{1}{\delta^{\alpha \alpha}} (2 - e^{i\alpha \pi/2} - e^{-i\alpha \pi/2}). \quad (58)$$

Combining these in Equation (54) and using the relation
\[
\Gamma(-\alpha)\Gamma(1+\alpha) = -\frac{\pi}{\sin \pi \alpha}
= -\frac{\pi}{2 \sin \pi \alpha/2 \cos \pi \alpha/2},
\]
we finally obtain
\[
\mathcal{F}\left\{\tilde{D}_\alpha^\alpha f(x)\right\} = \frac{[i\omega]^\alpha + (-i\omega)^\alpha}{2\cos \pi \alpha/2} F(\omega), \quad 0 < \alpha < 1, \quad 1 < \alpha < 2. \tag{61}
\]
Note that the divergences as \( \delta \to 0 \) in \( I_1 \) and \( I_2 \) are cancelled by the divergence in \( I_3 \), thus yielding a finite result [Eq. (61)] for the transform \( \mathcal{F}\left\{\tilde{D}_\alpha^\alpha f(x)\right\} \). For real \( \omega \), this can also be written as
\[
\mathcal{F}\left\{\tilde{D}_x^\alpha f(x)\right\} = |\omega|^\alpha F(\omega), \quad 0 < \alpha < 1, \quad 1 < \alpha < 2. \tag{62}
\]

**B. The \( q = 1 \) case and the \( q \to 1^\pm \) limits**

We now investigate the \( \alpha = 1 \) case. From Equation (61) it is seen that the Fourier transform \( \mathcal{F}\left\{\tilde{D}f(x)\right\} \) at \( \alpha = 1 \) is 0/0, thus undefined. Since Equation (61) is valid for the ranges \( 0 < \alpha < 1 \) and \( 1 < \alpha < 2 \), separately, we can investigate the \( \alpha = 1 \) case as 1 is approached from both directions. Since the numerator, \([i\omega]^\alpha + (-i\omega)^\alpha\], and the denominator, \( \cos \pi \alpha/2 \), are analytic functions of \( \alpha \), we can write their Taylor series expansions about \( \alpha = 1 \):
\[
(i\omega)^\alpha = i\omega \left[ 1 + \ln(i\omega)(\alpha - 1) \right] + 0((\alpha - 1)^2), \tag{63}
\]
\[
(-i\omega)^\alpha = -i\omega \left[ 1 + \ln(-i\omega)(\alpha - 1) \right] + 0((\alpha - 1)^2), \tag{64}
\]
\[
\cos \pi \alpha/2 = -\frac{\pi}{2} (\alpha - 1) + 0((\alpha - 1)^2). \tag{65}
\]
For real $\omega$, we write

$$i\omega = \pm i|\omega|,$$

+ for $\omega > 0$ and $-$ for $\omega < 0,$

and regardless of the direction of approach, we obtain

$$\lim_{\alpha \to 1^\pm} [(i\omega)^\alpha + (-i\omega)^\alpha] \simeq -|\omega|\pi(\alpha - 1).$$

Using Equations (65) and (67) in Equation (61) we finally obtain

$$\lim_{\alpha \to 1^\pm} F\{\tilde{D}^\alpha_x f(x)\} \rightarrow -|\omega|\pi(\alpha - 1)\right.$$ (68)

$$= |\omega| F(\omega).$$ (69)

Therefore, the Fourier transform, $F\{\tilde{D}^\alpha_x f(x)\}$, can be written for the entire range, $0 < \alpha < 2$, including $\alpha = 1$, as

$$F\{\tilde{D}^\alpha_x f(x)\} = |\omega|^\alpha F(\omega), \ 0 < \alpha < 2, \ \omega \ \text{real.}$$ (70)

IV. Conclusions and Space Fractional Quantum Mechanics

Even though the Fourier transform of $\tilde{D}^\alpha_x f(x)$ exists in the entire range, $0 < \alpha < 2$, including $\alpha = 1$, it does not reduce to the expected ordinary derivatives neither at $\alpha = 1$ nor at $\alpha = 2$, that is,

$$\tilde{D}_x^1 f(x) \neq \frac{df(x)}{dx}, \ \tilde{D}_x^2 f(x) \neq \frac{d^2f(x)}{dx^2}. $$ (71)

Introducing a minus sign rectifies the situation at $\alpha = 2$:

$$-\tilde{D}_x^2 f(x) = \frac{d^2f(x)}{dx^2}$$ (72)

but the problem at $\alpha = 1$ remains. In conclusion, the second representation of the Riesz derivative [Eqs. (25) and (28)] :

$$\tilde{D}_x^\alpha f(x) = -\tilde{D}_x^\alpha f(x),$$ (73)

has the same Fourier transform as the first definition [Eqs. (13)–(15)] in the entire range $0 < \alpha \leq 2$:

$$F\{R_x^\alpha f(x)\} = F\{-\tilde{D}_x^\alpha f(x)\} = -|\omega|^\alpha F(\omega), \ 0 < \alpha \leq 2. $$ (74)

On the contrary to the prevailing opinion, both representations exist at $\alpha = 1$ with the Fourier transform:

$$F\{R_x^{\alpha=1} f(x)\} = -|\omega| F(\omega).$$ (75)

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But unlike the $\alpha = 2$ case, neither representation reduces to the ordinary derivative at $\alpha = 1$, that is,

$$R_{\alpha}^{x=1} f(x) \neq \frac{df(x)}{dx}.$$  \hfill (76)

In this regard, the Riesz derivative is usually defined as [25]

$$R_{\alpha} f(x) = -\frac{(-\infty D_{\alpha}^x + \infty D_{\alpha}^x)f(x)}{2 \cos(\alpha \pi/2)}, \quad 0 < \alpha \leq 2, \quad \alpha \neq 1,$$  \hfill (77)

$$R_{\alpha}^x f(x) = \frac{df(x)}{dx} \text{ for } \alpha = 1.$$  \hfill (78)

Note that for $\alpha = 1$, one can also use the integral representation [25]

$$R_{\alpha}^{x=1} f(x) = \frac{d}{dx} \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{x - \xi} d\xi \right].$$  \hfill (79)

In this definition [Eqs. (77) and (78)] the Riesz derivative, $R_{\alpha} f(x)$, has a discontinuity at $\alpha = 1$, which can become a source of confusion in applications to space fractional quantum mechanics [20]. In space fractional quantum mechanics, the $\alpha = 2$ case corresponds to the Schrödinger equation for nonrelativistic particles with mass. However, the $\alpha = 1$ case deserves special attention.

First of all, the Riesz derivative, $R_{\alpha} f(x)$, in Equation (77) does not reduce to the ordinary derivative $df(x)/dx$ as $\alpha \to 1$. On the other hand, for the operator $d/dx$, the Schrödinger equation:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = H \Psi(x,t),$$  \hfill (80)

and the corresponding Hamiltonian for a free particle in momentum and configuration spaces are given, respectively, as

$$H = pc$$  \hfill (81)

$$= -i\hbar c \frac{d}{dx}.$$  \hfill (82)

In the early days of relativistic quantum mechanics, in analogy with the classical theory, the following Hamiltonian for the relativistic free particle was considered [29, 30]:

$$H = \sqrt{p^2c^2 + (m_0c^2)^2},$$  \hfill (83)

where $m_0$ is the rest mass. In configuration space, the Hamiltonian operator becomes

$$H = \sqrt{-\hbar^2c^2 \left( \frac{d}{dx} \right)^2 + m_0^2c^4},$$  \hfill (84)

where one immediately faces the problem of interpreting the square root operator. Expanding the square root gives an expression that contains derivatives of all orders, thus giving a nonlocal theory. In the classical limit, the higher
derivatives disappear thereby reducing $H$ to the well known classical Hamiltonian operator:

$$ H = -\frac{\hbar^2}{2m_0} \frac{d^2}{dx^2}. $$

(85)

Such theories are not only very difficult to handle but also the unsymmetric appearance of the time and the space coordinates eventually led to their demise and opened the path to Dirac theory. In this regard, if one wants to investigate fractional relativistic quantum mechanics, one has to start with the Dirac theory [29, 30]. Note that it is not possible to interpret Equation (82) as the Hamiltonian for photons (massless particles or ultra relativistic particles) either, since it implies a nonlocal expression for the energy density [29, 30].

In this regard, in space fractional quantum mechanics, unlike the upper bound, $\alpha = 2$, the physical meaning of the Hamiltonian [Eq. (82)] corresponding to $\alpha = 1$ is at most dubious. Besides, even if one could surmount the above mentioned difficulties and manage to find a solution to the fractional Schrödinger equation for $\alpha = 1$:

$$ i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -i\hbar c \frac{d \Psi(x, t)}{dx}, $$

(86)

it will not be the $\alpha \to 1$ limit of the solution of the Laskin’s space fractional quantum mechanics. This should not be interpreted as an inconsistency, since the nature of the nonlocality will be different [20, 15 – 18, 28].

The space fractional quantum mechanics via the Riesz derivative is a nonlocal theory. However, the Riesz derivative corresponds to a particular sampling of the function, thus it is not the only possible nonlocal theory that one could consider. Among all possible nonlocal theories, the space fractional quantum mechanics via the Riesz derivative has the intriguing feature that it also follows from Feynman’s path integral approach over Lévy paths.

The $\alpha$–stable Lévy distribution, $p_L(x, t; \alpha)$, satisfies the fractional diffusion equation with the Riesz derivative [5]:

$$ \frac{\partial p_L(x, t; \alpha)}{\partial t} - \sigma_\alpha R^\alpha_x p_L(x, t; \alpha) = 0, $$

(87)

where $0 < \alpha \leq 2$ is called the Lévy index. The $\alpha$–stable Lévy distribution with $0 < \alpha < 2$ has finite moments of order $\mu < \alpha$, but infinite moments for higher orders. The Gaussian distribution corresponds to $\alpha = 2$ and is also stable with moments of all orders. For applications to space fractional quantum mechanics, it is essential that the moments of first order exist. In other words, the existence of average position and momentum of the physical particle demands that $\alpha$ be restricted to the range $1 < \alpha \leq 2$ [5]. In this regard, comparison of the $\alpha = 1$ solution with the $\alpha \to 1$ limit of the Laskin’s space fractional Quantum mechanics not only violates the basic premises of the theory but also could be misinterpreted as an inconsistency.
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