Null hypersurface caustics for high dimensional superentropic black holes

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A black hole is superentropic if it violates the reverse isoperimetric inequality. Recently, some studies have indicated that some four dimensional superentropic black holes have the null hypersurface caustics (NHC) outside their event horizons. In this paper, we extend to explore whether or not the NHC exists in the cases of high dimensional superentropic black holes. We consider the singly rotating Kerr-AdS superentropic black holes in arbitrary dimensions and the singly rotating charged superentropic black hole in five-dimensional minimal gauged supergravity, and find that the NHC exists outside the event horizons for these superentropic black holes. Furthermore, the spacetime dimensions and other parameters of black hole, such as the electric charge, have important impact on the NHC inside the horizon. Our results indicate that when the superentropic black hole has the Cauchy horizon, the NHC will also exist inside the Cauchy horizon.

I. INTRODUCTION

Black hole is one of the most remarkable and quite fascinating objects predicted by the Einstein’s general relativity. It has been listened by the LIGO-Virgo observation [1, 2] and seen through the Event Horizon Telescope [3, 4], which spurs an increasing investigation on the properties of all kinds of black hole. Among them, the asymptotically anti de-Sitter (AdS) black hole becomes particular interest since it is very significant in various gauge-gravity dualities. Almost ten years ago, Cvetic et al. conjectured that the AdS black hole satisfies the reverse isoperimetric inequality (RII) [5]

\[ \mathcal{R} = \left( \frac{(d-1)V}{\mathcal{A}_{d-2}} \right)^{1/(d-1)} \left( \frac{\mathcal{A}_{d-2}}{A} \right)^{1/(d-2)} \geq 1, \]  

where \( V \) is the thermodynamic volume of black hole, \( \mathcal{A}_{d-2} = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2] \) is the area of the unit \((d-2)\)-sphere with \( d \) being the number of spacetime dimensions and \( A \) is the area of the outer horizon. Equality is attained for the Schwarzschild-AdS black hole, which indicates that the Schwarzschild-AdS black hole has the maximum entropy. In other words, it implies that for a specified entropy, the Schwarzschild-AdS black hole occupies the least volume.

However, if taking the ultraspinning limit for the rotating AdS black holes, which leads to that black holes have finite area but noncompact event horizons, Hennigar et al. [6] found that the area entropy of ultraspinning Kerr-Newman-AdS\textsubscript{4} black hole [7, 8] exceeds the maximum entropy limit since \( \mathcal{R} < 1 \). Thus, this kind of black holes is dubbed “superentropic”. The superentropic black hole is the first example violating the RII. Soon afterward, a lot of new ultraspinning AdS black hole solutions [9–14] from some known rotating AdS black holes have been generated successfully. The singly rotating Kerr-AdS ultraspinning black holes in arbitrary dimensions as well as singly rotating ultraspinning black hole of five-dimensional minimal gauged supergravity satisfy the relation \( \mathcal{R} < 1 \) [9]. However some new ultraspinning AdS black holes [9, 10, 14] violates the RII only in some ranges of values of the solution parameters. Thus, one naturally has a question: what causes this new family of ultraspinning black holes to be superentropic?

Recently, it has been found that, although there is the null hypersurface caustics (NHC) only inside the Cauchy horizon for the usual Kerr(-Newman)-(A)dS black hole [15], after taking the ultraspinning limit the NHC can exist outside the event horizon of the Kerr(-Newman)-AdS superentropic black hole [16]. The existence of NHC means that the causal structure of spacetime has some pathologies. The work in Ref. [16] gives rise to an indication: the presence of the NHC outside the event horizon may be related with the superentropy. Subsequently, the NHC of ultraspinning Kerr-Sen-AdS\textsubscript{4} black hole was studied in Ref. [17], and it was found that for the ultraspinning Kerr-Sen-AdS\textsubscript{4} black hole, whether it is superentropic or not, the NHC always appears both out and inside of the horizon. Obviously, the existence of NHC outside the horizon is only investigated for several four dimensional superentropic black holes, and whether the high dimensional superentropic black holes have the NHC outside their horizons or not needs to be explored, which motivates us to undertake the present work.

In this paper, we investigate the NHC for two high dimensional superentropic black holes, including the singly rotating Kerr-AdS superentropic black holes in arbitrary dimensions [6] and the singly rotating charged superentropic black hole in five-dimensional minimal gauged supergravity theory [9]. We find that for these superentropic black holes, there is NHC outside the event horizon. But the presence of NHC inside the horizon seems to require the existence of Cauchy horizon. The remaining part of this paper is organized as follows. Sec. II is a brief review of the NHC of the Kerr-AdS\textsubscript{4} superentropic black hole. In Sec. III and Sec. IV, we investigate the NHC of the singly rotating Kerr-AdS superentropic black holes in arbitrary dimensions and the singly rotating charged superentropic black hole in five-dimensional minimal gauged supergravity theory, respectively. Finally, our conclusions are
II. THE NHC OF KERR-ADS\textsubscript{4} SUPERENTROPIC BLACK HOLE

For the four-dimensional Kerr-AdS superentropic black hole, its metric has the form \[6–9\]:

\[
ds^2 = -\frac{\Delta}{\Sigma} (dt - l \sin^2 \theta d\phi)^2 + \Sigma \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\sin^2 \theta}\right) + \frac{\sin^4 \theta}{\Sigma} \left[l dt - (r^2 + l^2) d\phi\right]^2,
\]

(2)

where \(l\) is the cosmological scale, and

\[
\Delta_r = (r^2 + l^2)^2/l^2 - 2mr, \quad \Sigma = r^2 + l^2 \cos^2 \theta.
\]

Here, \(m\) is the mass parameter. The azimuthal coordinate \(\phi\) is noncompact and must be compactified by requiring \(\phi \sim \phi + \mu\) with \(\mu\) being a dimensionless parameter related to a new chemical potential \(K\).

The horizon of black hole satisfies the equation

\[
\Delta_r = 0.
\]

(3)

It is worth to note that the existence of the inner and outer horizons requires the mass to be larger than a critical value \(m_c\)

\[
m_c \equiv 2r_c \left(\frac{r_c^2}{l^2} + 1\right) = \frac{8l}{3\sqrt{3}},
\]

(4)

where

\[
r_c = \frac{l}{\sqrt{3}}.
\]

When \(m = m_c\), the superentropic Kerr-AdS\textsubscript{4} black hole is extremal. While for \(m < m_c\), it has a naked singularity.

Since outside the black hole horizon \(g_{\phi\phi}\) satisfies

\[
g_{\phi\phi} = \frac{2mr^4 \sin^4 \theta}{\Sigma} \geq 0,
\]

(5)

and thus is strictly positive, the spacetime is free of closed timelike curves, which is consistent with the result given in Ref. \[9\].

Now, we turn to investigate the NHC for superentropic Kerr-AdS\textsubscript{4} black hole. Following Refs. \[15–18\], we introduce the outgoing and ingoing Eddington-Finkelstein coordinates defined in terms of the "generalized tortoise coordinate" \(r_+(r, \theta)\):

\[
u = t + r_+(r, \theta).
\]

(6)

The null hypersurfaces are described by

\[
\begin{align*}
u &= \text{const}, \\ \lambda &= \text{const},
\end{align*}
\]

(7)

which are dubbed the outgoing and ingoing null congruences of the hypersurfaces, respectively. It is easy to obtain that the null hypersurfaces defined by \(u = \text{const}\) or \(\nu = \text{const}\) satisfy the equation

\[
g^{\mu\nu} \partial_\mu (t + r_+) \partial_\nu (t + r_+) = g^{tt} + g^{r\theta} (\partial_r r_+) + g^{\theta\theta} (\partial_\theta r_+) = 0.
\]

(8)

Solving this partial differential equation can give the null hypersurfaces for the superentropic Kerr-AdS\textsubscript{4} metric (2).

Using the contravariant components \(g^{\mu\nu}\), \(g^{r\theta}\), and \(g^{\theta\theta}\) of the metric (2), the partial differential equation (8) can be re-expressed as

\[
l^2 - \frac{(r^2 + l^2)^2}{\Delta_r} + \Delta_r (\partial_r r_+)^2 + \sin^2 \theta (\partial_\theta r_+)^2 = 0.
\]

(9)

Introducing the so-called "constant of separation" \(l^2 \lambda\) (hereinafter, \(\lambda\) is referred to as the separation constant), from Eq. (9) one can have

\[
(\partial_r r_+)^2 = \frac{Q^2(r)}{\Delta_r}, \quad (\partial_\theta r_+)^2 = \frac{P^2(\theta)}{\sin^2 \theta}.
\]

(10)

where

\[
Q^2(r) = (r^2 + l^2)^2 - l^2 \lambda \Delta_r, \quad P^2(\theta) = (\lambda - 1) l^2 \sin^2 \theta.
\]

(11)

with \(\lambda > 1\). Here a complex \(\theta\)-dependent form of \(P(\theta)\) is chosen in order to give the elegant expression for the following equation (18). Using the relations given in (10), one can express the total differential \(dr_+ = \partial_r r_+ dr + \partial_\theta r_+ d\theta\) to be

\[
dr_+ = \frac{Q(r)}{\Delta_r} dr + \frac{P(\theta)}{\sin^2 \theta} d\theta.
\]

(12)

If treating \(\lambda\) as a variable, the exact differential Eq. (12) can be generalized to be

\[
dr_+ = \frac{Q(r, \lambda)}{\Delta_r} dr + \frac{P(\theta, \lambda)}{\sin^2 \theta} d\theta + c_1 F(r, \theta, \lambda) d\lambda,
\]

(13)

where \(c_1\) is an arbitrary constant and \(F(r, \theta, \lambda)\) is an arbitrary function. Since Eqs. (12) and (13) are functionally equivalent, the condition

\[
F(r, \theta, \lambda) = 0
\]

(14)

needs to be satisfied, which indicates \(dF(r, \theta, \lambda) = 0\) and thus yields

\[
[\partial_\lambda F(r, \theta, \lambda)] d\lambda + [\partial_r F(r, \theta, \lambda)] dr + [\partial_\theta F(r, \theta, \lambda)] d\theta = 0.
\]

(15)

From the Poincaré lemma in the external differentiation theory, namely \(d(dr_+) = 0\), one can obtain the following integrable conditions

\[
\frac{\partial_\lambda Q(r, \lambda)}{\Delta_r} = c_1 \partial_r F(r, \theta, \lambda),
\]

(16)

\[
\frac{\partial_\lambda P(\theta, \lambda)}{\sin^2 \theta} = c_1 \partial_\theta F(r, \theta, \lambda).
\]
From the definitions of \( Q(r, \lambda) \) and \( P(\theta, \lambda) \), one can obtain straightforwardly
\[
\partial_r Q(r, \lambda) = -\frac{l^2 \Delta_r}{2Q(r, \lambda)}, \quad \partial_\theta P(\theta, \lambda) = -\frac{l^2 \sin^2 \theta}{2P(\theta, \lambda)},
\] (17)
and then rewrite Eq. (15) as
\[
vd\lambda = -\frac{dr}{Q(r, \lambda)} + \frac{d\theta}{P(\theta, \lambda)},
\] (18)
after choosing the constant \( c_1 \) to be \( l^2/2 \) and defining \( \nu = -\partial_r F(r, \theta, \lambda) \). Making use of Eq. (12) and Eq. (18), the metric of the superentropic Kerr-AdS\(_4\) can be re-expressed in terms of the coordinates \((t, r, \theta, \lambda)\)
\[
ds^2 = \frac{\Delta_r \sin^2 \theta}{R^2}(dr^2 - dt^2) + R^2 \sin^2 \theta (d\phi - \Omega dt)^2 + \frac{\nu^2 P^2(\theta, \lambda)Q^2(r, \lambda)}{R^2}d\lambda^2,\]
where
\[
\Delta_r = \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{2mr^2 \sin^2 \theta}{\Sigma},
\]
\[
\Omega = -\frac{\nu \sin^2 \theta}{\Sigma R^2} = \frac{(r^2 + l^2) \sin^2 \theta - \Delta_r}{R^2}.
\] (20)

Obviously, \( \Omega \) is the Zero-Angular-Momentum-Observer (ZAMO) angular velocity of the superentropic Kerr-AdS\(_4\) black hole [6, 9, 19]. Then, using Eqs. (12) and (18), one has
\[
dr = \frac{Q(r, \lambda)\Delta_r}{\Sigma R^2} \left[ \sin^2 \theta dr_s - P^2(\theta, \lambda)vd\lambda \right], \quad d\theta = \frac{P(\theta, \lambda) \sin^2 \theta}{\Sigma R^2} \left[ \Delta_r dr_s + Q^2(r, \lambda)vd\lambda \right].
\] (22)

Since the outgoing and ingoing null congruences are defined to be \( du = dv = 0 \), which means \( dr_s^2 = dt^2 \), the metric of the superentropic Kerr-AdS\(_4\) black hole on the null hypersurface is reduced to be
\[
dl^2 = R^2 \sin^2 \theta (d\phi - \Omega dt)^2 + \frac{\nu^2 P^2(\theta, \lambda)Q^2(r, \lambda)}{R^2}d\lambda^2.
\] (24)

One can see that the null congruences are rotating with the ZAMO angular velocity \( \Omega \). Since that the volume element of a null hypersurface is the square root of the determinant of the induced metric, the points where the induced metric determinant goes to zero correspond to the NHC. Thus, the condition of the NHC is
\[
vP(\theta, \lambda)Q(r, \lambda) \sin \theta = \nu \sqrt{\lambda - 1}Q(r, \lambda) \sin^2 \theta = 0,
\] (25)
which is the determinant of the induced metric of the superentropic Kerr-AdS\(_4\) black hole.

Let us consider the outgoing null hypersurface case (same for ingoing null hypersurface case), namely, \( \lambda = \text{const} \) and a increasing \( r \), as an example to study NHC. We need to analyze each factor in Eq. (25). First of all, \( l \) is the AdS radius and assumed to be always positive. According to Eq. (23), \( \theta \) can be expressed as \( \theta(r, \lambda) \). Combining Eqs. (22) and (23) yields
\[
\left( \frac{\partial \theta}{\partial r} \right) = \frac{P(\theta)}{Q(r)} \geq 0,
\] (26)
which means that \( \theta \) increases, or remains to be a constant, as the radius of the superentropic Kerr-AdS\(_4\) black hole increases. According to Eq. (18), one can easily obtain
\[
\frac{\partial v}{\partial r} = -\frac{\partial F(r, \theta, \lambda)}{\partial r} = \frac{l^2 \Delta_r}{2Q^3(r, \lambda)},
\]
\[
\frac{\partial v}{\partial \theta} = -\frac{\partial F(r, \theta, \lambda)}{\partial \theta} = \frac{l^2 \sin^2 \theta}{2P^3(\theta, \lambda)}.
\] (27)

Hence, we have
\[
\left( \frac{\partial v}{\partial r} \right) = \frac{\partial v}{\partial \theta} \left( \frac{\partial \theta}{\partial r} \right) = \frac{mv^2}{(\lambda - 1)Q^3(r)}.
\] (29)

Since \( \lambda > 1 \), Eq. (29) is always positive, which means that once the radius of the superentropic Kerr-AdS\(_4\) black hole increases, so does \( v \). When \( r \to \infty \), \( v \) is not divergent since \( (\partial v/\partial r)_{\lambda} \to r^{-3} \). Clearly, if \( Q(r) = 0 \), Eq. (25) is always satisfied. Thus, \( Q(r) = 0 \) is a sufficient condition that leads to the existence of NHC. \( Q(r) = 0 \) means that
\[
(r^2 + l^2)^2 - l^2 \lambda \Delta_r = 0,
\] (30)
which is a quartic equation of \( r \) and is hard to give the analytical results.

![FIG. 1](image)

FIG. 1. Contours of the caustics on the \((\bar{r}, \bar{m})\)-plane with different values of \( \lambda \) for the superentropic Kerr-AdS\(_4\) black hole, where \( \bar{m} = m/1 \) and \( \bar{r} = r/l \). The red line is the horizon equation and the black dot represents the extremal black hole case. Two intersection points between the dashed black line
and the solid red line represent the Cauchy horizon and the event horizon of the black hole, respectively. So, it is easy to find that the NHC appears both inside the Cauchy horizon and outside the event horizon of the superentropic Kerr-AdS black hole, which is consistent with the conclusion given in Ref. [16].

### III. SINGLY ROTATING KERR-ADS SUPERENTROPIC BLACK HOLES IN ARBITRARY DIMENSIONS

In this section, we will extend the above discussion to the case of superentropic black hole with higher dimensions by considering the arbitrary dimensional singly rotating Kerr-AdS superentropic black holes. The singly rotating Kerr-AdS superentropic black holes in arbitrary dimensions are strictly superentropic [9] and the metric has the form [6, 19]

\[
\begin{align*}
    ds^2 &= -\frac{\Delta_r}{\Sigma} \left( dt - l \sin^2 \theta d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 \\
    &+ \frac{\Sigma}{\sin^2 \theta} d\theta^2 + \frac{\sin^4 \theta}{\Sigma} \left[ l dt - (r^2 + l^2) d\phi \right]^2 \\
    &+ r^2 \cos^2 \theta d\Omega_d^2,
\end{align*}
\]

where

\[
\Delta_r = (r^2 + l^2)^2 / l^2 - 2m \phi^d - d^2, \quad \Sigma = r^2 + l^2 \cos \theta,
\]

and the horizon equation is

\[
\Delta_r = 0.
\]

As did in the last section, we can derive that the null hypersurfaces equation is

\[
I^2 - \left( \frac{r^2 + l^2}{\Delta_r} \right)^2 + \Delta_r (\partial_0 r_e)^2 + \sin^2 \theta (\partial_0 r_e)^2 = 0.
\]

Introducing a constant \( \lambda \) with \( \lambda > 1 \) for the separation of variables, i.e. \( I^2 \lambda \), the above equation (33) yields

\[
\begin{align*}
    (\partial_0 r_e)^2 &= \left( \frac{r^2 + l^2}{\Delta_r} \right)^2 - I^2 \lambda \Delta_r \\
    (\partial_0 r_e)^2 &= \frac{(\lambda - 1)I^2 \sin^2 \theta}{\sin^4 \theta}.
\end{align*}
\]

After a similar procedure in the previous section, one can find that the caustics condition is

\[
(r^2 + l^2)^2 - I^2 \lambda \Delta_r = 0.
\]

To study the effect of dimensions on the caustics condition (35) and the horizon equation (32), we consider the five and six dimensions, respectively, as examples. For singly rotating Kerr-AdS5 superentropic black hole, one can obtain that the caustics condition and horizon equation become

\[
\begin{align*}
    (r^2 + l^2)^2 - I^2 \lambda \left[ (r^2 + l^2)^2 / l^2 - 2m \right] &= 0, \\
    (r^2 + l^2)^2 / l^2 - 2m &= 0.
\end{align*}
\]

These two identifies are shown in FIG. 2 with the horizon equation being represented by the red solid line. One can see that the black dashed line and the red line only has one intersection point when \( \bar{r} > 0 \), which means that the singly rotating Kerr-AdS5 superentropic black hole only has the event horizon and the Cauchy horizon vanishes. This can also be known from Eq. (37) since it has only one positive real root \( r = \sqrt{-I^2 + l\sqrt{2m}} \). This property is the same as the singly rotating Kerr-AdS6 black hole [20], but is different from the case of the superentropic Kerr-AdS5 black hole discussed in the above section. Furthermore, comparing FIG. 2 with FIG. 1, we find that the NHC inside the horizon vanishes, and the NHC only exists outside the event horizon when the spacetime dimension is increased from four to five.

![FIG. 2. Contours of the caustics on the (\( \bar{r}, \bar{m} \))-plane for the singly rotating Kerr-AdS5 superentropic black hole, where \( \bar{m} = m / l^2 \) and \( \bar{r} = r / l \). The red line is the horizon equation.](image)

For the singly rotating Kerr-AdS6 superentropic black hole, its caustics and horizon equations, respectively, satisfy

\[
\begin{align*}
    (r^2 + l^2)^2 - I^2 \lambda \left[ (r^2 + l^2)^2 / l^2 - 2m / r \right] &= 0, \\
    (r^2 + l^2)^2 / l^2 - 2m / r &= 0,
\end{align*}
\]

which are plotted in FIG. 3. In this figure, the horizon equation is shown by the red solid line. FIG. 3 is very similar to FIG. 2, and the same conclusion that the NHC only exists outside the event horizon of superentropic black hole can be obtained. Thus, increasing the spacetime dimension from five to six has negligible effect on the NHC. Comparing FIG. 1, FIG. 2 and FIG. 3, one can infer that the existence of the NHC outside the event horizon of superentropic black hole, but once the spacetime dimension is larger than four, then the Cauchy horizon of superentropic black hole disappears and the NHC existing inside the horizon will disappear too. Therefore, the dimension of spacetime only has an important effect on the property of NHC inside the horizon.
FIG. 3. Contours of the caustics on the (\(\bar{r}, \bar{m}\))-plane for the singly rotating Kerr-AdS\(_6\) superentropic black hole, where \(\bar{m} = m/l^3\) and \(\bar{r} = r/l\). The red line is the horizon equation.

IV. SINGLY ROTATING CHARGED SUPERENTROPIC BLACK HOLE IN FIVE-DIMENSIONAL MINIMAL GAUGED SUPERGRAVITY

To investigate whether the conclusion obtained in Sec. III is general or not, in this section we turn to analyze the NHC of the singly rotating charged superentropic black hole in five-dimensional minimal gauged supergravity theory. According to the solution of the double rotating charged superentropic black hole in five-dimensional minimal gauged supergravity theory [9], we obtain the singly rotating charged superentropic black hole solution by taking one of the angular velocity, which is not boosted to the speed of light, to be zero. Then the metric and the Abelian gauged potential are

\[
\begin{align*}
  ds^2 &= -\frac{\Delta_r}{\Sigma} \left( dt - l \sin^2 \theta d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 \\
  &\quad + \frac{\Sigma}{\sin^2 \theta} d\theta^2 + \sin^4 \theta \left[ l dt - (r^2 + l^2) d\phi \right]^2 \\
  &\quad + r^2 \cos^2 \theta \left[ d\psi - \frac{ql dt - l \sin^2 \theta d\phi}{r^2 \Sigma} \right]^2, \\
  \hat{\Lambda} &= \frac{\sqrt{3}q}{2\Sigma} (dt - l \sin^2 \theta d\phi), \\
\end{align*}
\]

where

\[
\begin{align*}
  \hat{\Delta}_r &= \left( r^2 + l^2 \right)^2 / l^2 - 2m + q^2 / r^2, \\
  \hat{\Sigma} &= r^2 + l^2 \cos^2 \theta,
\end{align*}
\]

with \(q\) being the electric charge parameter. The horizon equation is

\[
\hat{\Delta}_r = 0.
\]

Obviously, it can reduce consistently that of the singly rotating Kerr-AdS\(_5\) superentropic black hole given in (31) when \(q = 0\).

To discuss the NHC, we first obtain the null hypersurfaces equation

\[
l^2 - \left( \frac{r^2 + l^2}{\Delta_r} \right)^2 + \hat{\Delta}_r (\partial_r r_s) + \sin^2 \theta (\partial_\theta r_s) = 0.
\]

After introducing a constant for the separation of variables as \(l^2 \hat{\Lambda}\) with \(\hat{\Lambda} > 1\), the above equation (43) yields

\[
\begin{align*}
  (\partial_r r_s)^2 &= \left( \frac{r^2 + l^2}{\hat{\Delta}_r^2} \right)^2 - l^2 \hat{\Lambda}\hat{\Delta}_r, \\
  (\partial_\theta r_s)^2 &= \left( \frac{\lambda - 1}{\sin^4 \theta} \right).
\end{align*}
\]

After a similar procedure as what was done in Sec. II, one can obtain that the caustics condition is

\[
\begin{align*}
  (r^2 + l^2)^2 - l^2 \hat{\Lambda}\hat{\Delta}_r &= 0. \\
\end{align*}
\]

FIG. 4. Contours of the caustics on the (\(\bar{r}, \bar{m}\))-plane with a fixed \(\bar{q}\), and on the (\(\bar{r}, \bar{q}\))-plane with a fixed \(\bar{m}\) for the singly rotating charged superentropic black hole of five-dimensional minimal gauged supergravity, where \(\bar{m} = m/l^3\), \(\bar{q} = q/l^2\) and \(\bar{r} = r/l\). The red line is the horizon equation and the black dot represents the extremal black hole case.

FIG. 4 shows the numerical results of the caustic condition (45) and the horizon equation (42) (red dashed/solid lines).
In this figure, the extremal black hole case is represented by black spots. One can see that the black dashed line and the red dashed/solid line have two intersection points, which represent, respectively, the position of the Cauchy and event horizons of the singly rotating charged superentropic black hole of five-dimensional minimal gauged supergravity theory. One can see that the NHC exists both inside its Cauchy horizon and outside its event horizon. This result is different from that of five-dimensional black hole obtained in the Sec. III. Thus, the electric charge of superentropic black hole has also an important impact on the property of the NHC.

V. CONCLUSIONS

The superentropic black hole, which violates the RII, has spurred an increasing deal of interest. Recently, the authors in Refs. [16, 17] found that there is the NHC outside the event horizons for some four-dimensional superentropic black holes. In this paper, we have generalized these researches presented in [16, 17] to the case of high dimensional superentropic black holes. We first consider the singly rotating Kerr-AdS superentropic black holes in arbitrary dimensions, and focus on the cases of four, five and six dimensions as examples. When the spacetime dimension is larger than four, the superentropic black hole has only the event horizon while its Cauchy horizon vanishes. Only in the case of four-dimensional black hole, the NHC also exists inside the Cauchy horizon. This result seems to indicate that the spacetime dimension has an effect on the NHC inside the horizon. After studying the singly rotating charged superentropic black hole in five-dimensional minimal gauged supergravity, we find that the NHC exists outside the event horizon, and the electric charge parameter leads to the presence of NHC inside the Cauchy horizon. Thus, the spacetime dimensions and other parameters of black hole, such as the electric charge, have important impacts on whether the NHC exists inside the black hole horizon or not. Our results indicate that when the superentropic black hole has the Cauchy horizon, the NHC exists also inside this Cauchy horizon. From the results of this paper and previous works [16, 17], one can find that there is the NHC outside the event horizon of the superentropic black hole, but the existence of the NHC outside the event horizon of black hole does not mean necessarily that this black hole is superentropic. Thus, there should be some relations between the presence of the NHC outside the event horizon and the superentropy of black hole. However, the origin of the super-entropy of black hole still needs to be further explored.

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