The General Class of Accelerating, Rotating and Charged Plebanski-Demianski Black Holes as Heat Engine

Ujjal Debnath*

Department of Mathematics,
Indian Institute of Engineering Science and Technology,
Shibpur, Howrah-711 103, India.

(Dated: June 5, 2020)

We first review the general class of accelerating, rotating and charged Plebanski-Demianski (PD) black holes in presence of cosmological constant, which includes the Kerr-Newman rotating black hole and the Taub-NUT spacetime. We assume that the thermodynamical pressure may be described by the negative cosmological constant and so the black hole represents anti-de Sitter (AdS) PD black hole. The thermodynamic quantities like surface area, entropy, volume, temperature, Gibb’s and Helmholtz’s free energies of the AdS PD black hole are obtained due to the thermodynamic system. Next we find the critical point and corresponding critical pressure, critical temperature and critical volume for AdS PD black hole. Due to the study of specific heat capacity, we obtain $C_V = 0$ and $C_P \geq 0$. From this result, we conclude that the AdS PD black hole may be stable. We examine the Joule-Thomson expansion of PD black hole and by evaluating the sign of Joule-Thomson coefficient $\mu$, we determine the heating and cooling nature of PD black hole. Putting $\mu = 0$, we find the inversion temperature. Next we study the heat engine for AdS PD black hole. In Carnot cycle, we obtain the work done and its maximum efficiency. Also we describe the work done and its efficiency for a new engine. Finally, we analyze the efficiency for the Rankine cycle in PD black hole heat engine.

* ujjaldebnath@gmail.com
I. INTRODUCTION

The black hole thermodynamics has become an important topic of intensive research since Hawking’s radiation of black holes [1, 2] and considered as a crucial topic to gaining insight into the quantum nature of gravity. From the early discoveries that black hole area and surface gravity behave as thermodynamic entropy [3, 4] and temperature [1] respectively. Gibbons and Hawking [5] have studied the physics of anti-de Sitter (AdS) black hole due to AdS/CFT correspondence. Hawking and Page [6] have studied the thermodynamic properties of static Schwarzschild-AdS black hole. After few years, Chamblin et al. [7, 8] have investigated the physical properties of charged Reissner-Nordstrom-AdS black hole. If one considers charge and/or rotation of the AdS black hole, the nature of the AdS black hole is similar to the Van der Waals fluid [9, 10]. In the black hole chemistry [11, 12], the negative cosmological constant ($\Lambda < 0$) is considered as a thermodynamic pressure $P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi \ell^2}$ ($\ell$ is the length of AdS black hole) [13, 22], has recently started to attract a growing deal of interest. In the thermodynamic system, the first law of black hole thermodynamics gives $\delta M = T \delta S + V \delta P + ...$ with the black hole thermodynamic volume $V = \left(\frac{\partial M}{\partial T}\right)s_{...}$, where, $M$ is the mass, $S$ is the entropy and $T$ is the temperature of the AdS black hole. The geometry of AdS black hole thermodynamics has been extensively studied by several authors [23, 39].
In the context of black hole chemistry, the concept of holographic heat engine has been proposed by Johnson [40] for AdS black hole, where he has considered the cosmological constant as a thermodynamic variable. Based on the holographic heat engine proposal, Setare et al [41] have studied polytropic black hole as a heat engine. Subsequently, Johnson [42-44] has analyzed the heat engine phenomena for the Gauss-Bonnet black holes and Born-Infeld AdS black holes. Holographic heat engines for different types of black holes have been studied in [45-53]. Zhang et al [54] have studied the $f(R)$ black holes as heat engines. Heat engines of AdS black hole have been analyzed in the frameworks of massive gravity [46, 55-57], gravity’s rainbow [58] and conformal gravity [59]. Heat engine efficiency has been studied for Hayward [60] and Bardeen [61, 62] black holes. Heat engine in three dimensional BTZ black hole has been obtained in [63, 64]. Heat engine for dilatonic Born-Infeld black hole has been analyzed in [65]. For charge rotating dyonic black hole, the thermodynamic efficiency has been studied in [66]. Till now, several authors also have studied the heat engine phenomena of black holes in various occasions [67-80].

Using the standard black hole thermodynamics, it was found that the Hawking temperature of accelerating black holes is more than Unruh temperature of the accelerated frame. Thermodynamics nature of accelerating black holes have been discussed in [81-86]. The thermodynamics properties of accelerating and rotating black holes have been investigated in [87, 88]. Also charged accelerating black hole thermodynamics have been analyzed in [89-91]. Charged rotating and accelerating black hole thermodynamics have been studied in [92, 93]. The entropy bound of horizons for accelerating, rotating and charged Plebanski-Demianski black hole have been discussed in [94]. Zhang et al [95] have studied the accelerating AdS black holes as the holographic heat engines in a benchmarking scheme. Also Zhang et al [96] have studied the thermodynamics of charged accelerating AdS black holes and holographic heat engines. Recently Jafarzade et al [97] have investigated the thermodynamics and heat engine phenomena of charged rotating accelerating AdS black holes. Motivated by the above works, here we’ll study the thermodynamics, $P-V$ criticality, stability, Joule-Thomson expansion and heat engine for more general class of accelerating, rotating and charged Plebanski-Demianski (PD) black hole in AdS system. In section II, we write the general class of accelerating, rotating and charged AdS Plebanski-Demianski black hole metric. Then we obtain the thermodynamic quantities, critical point, specific heat, stability and Joule-Thomson expansion. In section III, we study the phenomena of heat engine for PD black hole and study the Carnot cycle, Rankine cycle, work done and their efficiency. Finally, we provide the result of the whole work in section IV.
II. THERMODYNAMICS OF PLEBANSKI-DEMIAŃSKI BLACK HOLE

A. Plebanski-Demianski Black Hole Metric

Plebanski and Demianski [98] have presented a large class of Einstein-Maxwell electro-vacuum (algebraic type $D$) solutions, which includes Kerr-Newman black hole, Taub-NUT spacetime, (anti-)de Sitter (AdS) metric and their arbitrary combinations. The general class of accelerating, rotating and charged Plebanski-Demianski (PD) black hole metric in AdS system is given by [94, 99-103]

$$ds^2 = \frac{1}{\Omega^2} \left[ -\frac{Q}{\rho^2} \left\{ dt - \left( a \sin^2 \theta + \frac{4}{\omega} \sin^2 \theta \right) d\phi \right\}^2 + \frac{\rho^2}{Q} dr^2 
+ \frac{\mathcal{P}}{\rho^2} \left\{ a dt + (r^2 + (a + l)^2) d\phi \right\}^2 + \frac{\rho^2}{\mathcal{P}} \sin^2 \theta d\theta^2 \right]$$

where $\Omega, \rho^2, Q$ and $\mathcal{P}$ are given by

$$\Omega = 1 - \frac{\alpha}{\omega} (l + a \cos \theta) r$$

$$\rho^2 = r^2 + (l + a \cos \theta)^2$$

$$Q = (\omega^2 k + e^2 + g^2) - 2Mr + \epsilon r^2 - \frac{2\alpha n}{\omega} r^3 - \left( \alpha^2 k + \frac{\Lambda}{3} \right) r^4$$

$$\mathcal{P} = (1 - a_1 \cos \theta - a_2 \cos^2 \theta) \sin^2 \theta$$

$$a_1 = \frac{2\alpha a}{\omega} M - \frac{4\alpha^2 al}{\omega^2} (\omega^2 k + e^2 + g^2) - \frac{4}{3} a\Lambda,$$

$$a_2 = -\frac{\alpha^2 a^2}{\omega^2} (\omega^2 k + e^2 + g^2) - \frac{a^2}{3} \Lambda,$$

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + \frac{4al}{\omega} M - (a^2 + 3l^2) \left[ \frac{\alpha^2}{\omega^2} (\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3} \right],$$

$$n = \frac{\omega^2 kl}{a^2 - l^2} - \frac{\alpha(a^2 - l^2)}{\omega} M + l(a^2 - l^2) \left[ \frac{\alpha^2}{\omega^2} (\omega^2 k + e^2 + g^2) + \frac{\Lambda}{3} \right],$$

$$k = \left[ 1 + \frac{2al}{\omega} M - \frac{3\alpha^2 l^2}{\omega^2} (e^2 + g^2) - l^2 \Lambda \right] \left[ \frac{\omega^2}{a^2 - l^2} + 3\alpha^2 l^2 \right]^{-1}$$
Here, \( a = J/M \), \( l \), \( e \), \( g \), \( \alpha \), \( \omega \) are angular momentum, NUT parameter, electric charge, magnetic charge, acceleration parameter and rotation parameter respectively. Also \( M \) is the mass of the PD black hole and \( \Lambda \) is the cosmological constant. In particular, the PD black hole metric can be reduced to the following black hole metrics: (i) C-metric \( (a = l = 0) \) \[104\], (ii) Kerr-Newman-Taub-NUT black hole \( (\alpha = g = 0) \) \[105, 106\], (iii) Kerr-Taub-NUT black hole \( (\alpha = e = g = 0) \) \[107\], (iv) Kerr-Newman black hole \( (\alpha = l = g = 0) \) \[109\], (v) Kerr black hole \( (\alpha = l = e = g = 0) \) \[110\], (vi) Riessner-Nordstrom black hole \( (\alpha = a = l = g = 0) \) and (vii) Schwarzschild black hole \( (\alpha = a = l = e = g = 0) \).

### B. Thermodynamic Quantities

Here, we'll obtain the thermodynamic quantities for PD black hole in AdS system. For this purpose, the cosmological constant \( (\Lambda) \) can be written in terms of thermodynamic pressure \( (P) \) as \( \Lambda = -8\pi P \).

The horizon radius \( r_h \) of PD black hole can be obtained from the following equation (putting \( Q = 0 \))

\[
(\omega^2 k + e^2 + g^2) - 2Mr_h + e r_h^2 - \frac{2\alpha n}{\omega} r_h^3 - \left( \frac{\alpha^2 k - 8\pi P}{3} \right) r_h^4 = 0
\]

From above equation, pressure \( P \) can be expressed in terms of \( r_h \) as

\[
P = \frac{3}{8\pi\omega^3} \left( a^2 r_h^2 \alpha^2 - (\omega - lr_h\alpha)^2 \right) \left[ 2l^4 Mr_h\alpha^2 - 2l^3(M + r_h)\alpha\omega + 2lr_h \left( e^2 + g^2 + Mr_h \right) \alpha \omega \\
+ \left( e^2 + g^2 - 2Mr_h + r_h^2 \right) \omega^2 - l^2 \left( 3e^2 r_h^2 \alpha^2 + 3g^2 r_h^2 \alpha^2 + \omega^2 \right) + a^2 \left( -2l^2 Mr_h \alpha^2 + 2l(M + r_h)\alpha\omega + \omega^2 \right) \right] \\
\times \left[ -8l^3 r_h^3 \alpha - 3l^4 \omega + 6l^2 r_h^2 \omega + r_h^4 \omega + a^2 \left( 2l^3 r_h^3 \alpha + 3l^2 \omega + r_h^2 \omega \right) \right]^{-1}
\]

Now the surface area of the PD black hole in AdS system is obtained in the form \[111\]

\[
A = \int \int \sqrt{g_{\theta\theta}g_{\phi\phi}} \ d\theta d\phi = \frac{4\pi\omega^2(r_h^2 + (a + l)^2)}{(\omega - l\alpha r_h)^2 - a^2\alpha^2 r_h^2}
\]

So the Bekenstein-Hawking entropy \[3, 112\] on the horizon is given by

\[
S = \frac{A}{4} = \frac{\pi\omega^2(r_h^2 + (a + l)^2)}{(\omega - l\alpha r_h)^2 - a^2\alpha^2 r_h^2}
\]

From this, we can obtain the horizon radius in terms of entropy as

\[
r_h = \frac{\omega\sqrt{f_1(S)} - l\omega S}{f_2(S)}
\]

where

\[
f_1(S) = \alpha^2 l^2 S^2 + \left( S - (a + l)^2 \pi \right) f_2(S)
\]

\[
f_2(S) = \frac{\alpha^2 l^2 S^2}{(S - (a + l)^2 \pi)^2} - \frac{\alpha^2 l^2}{(S - (a + l)^2 \pi)}
\]
The volume of the PD black hole is given by
\[
V = \left( \frac{\partial M}{\partial P} \right)_{S,\omega} = \frac{4\pi \omega \left[ \omega (r_h^2 + 6l^2r_h - 6l^4) - 8\alpha \omega^2 + a^2 (2l \alpha r_h^2 + 6l^2 \omega + \omega r_h) \right]}{3(3a^2l^2 \alpha^2 - 3l^4 \alpha^2 + \omega^2)}
\] (18)

which can be written in terms of entropy as in the following form
\[
V = \frac{4\pi \omega^2 \left[ 6l^2 (a^2 - l^2) f_3^2(S) + \omega f_2(S) f_3(S) \left( \sqrt{f_1(S)} - l \alpha S \right) + \omega^3 \left( \sqrt{f_1(S)} - l \alpha S \right)^3 \right]}{3f_2^3(S) \left[ 3l^2 \alpha^2 (a^2 - l^2) + \omega^2 \right]}
\] (19)

where
\[
f_3(S) = a^2 \left[ f_2(S) + 2l \alpha \left( \sqrt{f_1(S)} - l \alpha S \right) \right] + 2l^2 \left[ 3f_2(S) - 4l \alpha \left( \sqrt{f_1(S)} - l \alpha S \right) \right]
\] (20)

Now the surface gravity on the horizon of black hole is defined by
\[
\kappa = \frac{1}{2 \sqrt{-h}} \frac{\partial}{\partial x^a} \left( \sqrt{-h} h^{ab} \partial_r \right) \bigg|_{r=r_h}
\] (21)

where \( h_{ab} \) is the second order metric constructed from the \( t-r \) components of the metric and \( h = \det(h_{ab}) \).

The temperature on the horizon of the PD black hole is given by
\[
T = \frac{\kappa}{2\pi} = \frac{\left[ \omega - \alpha (l + a) r_h \right]}{2\pi \omega [r_h^2 + (l + a)^2]} \left[ -2M + 2\epsilon r_h - \frac{6\alpha n}{\omega} r_h^2 - 4 \left( \alpha^2 k - \frac{8\pi P}{3} \right) r_h^3 \right]
\] (22)

The temperature can be written in terms of entropy as in the following form
\[
T = \frac{\left[ f_2(S) + \alpha (a + l) \left( \sqrt{f_1(S)} - l \alpha S \right) \right] f_5(S)}{3\pi f_2^3(S) f_4(S)}
\] (23)

where
\[
f_4(S) = (a + l)^2 f_2^2(S) + \omega^2 \left( \sqrt{f_1(S)} - l \alpha S \right)^2
\] (24)

and
\[
f_5(S) = 3M f_2^3(S) + \omega \left( \sqrt{f_1(S)} - l \alpha S \right) \left( 9n \alpha f_2(S) \left( \sqrt{f_1(S)} - l \alpha S \right) \right.
\]
\[+ 2\omega^2 \left( \sqrt{f_1(S)} - l \alpha S \right)^2 (3k \alpha^2 - 8\pi P) - 3f_2^3(S) \left( \right) \]

The Gibb’s free energy is given by
\[
G = M - TS = \frac{\left( \omega^2 + e^2 + g^2 \right) f_2(S)}{2\omega \left( \sqrt{f_1(S)} - l \alpha S \right)} + \frac{e \omega \left( \sqrt{f_1(S)} - l \alpha S \right)}{2f_2(S)} - \frac{\alpha n \omega \left( \sqrt{f_1(S)} - l \alpha S \right)^2}{f_2^3(S)}
\]
\[-\frac{\omega^3}{6f_2^3(S)} \left( 3\alpha^2 k - 8\pi P \right) \left( \sqrt{f_1(S)} - l \alpha S \right)^3 - \frac{S \left[ f_2(S) + \alpha (a + l) \left( \sqrt{f_1(S)} - l \alpha S \right) \right] f_5(S)}{3\pi f_2^3(S) f_4(S)} \]

(26)
FIG. 1: Figure represents the plot of entropy $S$ against PD black hole horizon radius $r_h$.

Also the Helmholtz’s free energy is obtained as \[ F = G - PV = \frac{(\omega^2 + e^2 + g^2) f_2(S)}{2\omega \left(\sqrt{f_1(S)} - \alpha S \right)} + \frac{\epsilon \omega \left(\sqrt{f_1(S)} - \alpha S \right)}{2f_2(S)} - \frac{\alpha \omega \left(\sqrt{f_1(S)} - \alpha S \right)^2}{f_2^2(S)} \]

\[
- \frac{\omega^3}{6f_2^3(S)} (3\alpha^2 k - 8\pi P) \left(\sqrt{f_1(S)} - \alpha S \right)^3 - \frac{
\frac{S f_2(S) + \alpha (a + l) \left(\sqrt{f_1(S)} - \alpha S \right) x_5(S)}{3\pi f_2^2(f_4(S))}
\right)  
\]

\[
4\pi \omega^2 P \left[ 6l^2(a^2 - l^2) f_2^3(S) + \omega f_2(S) \left(\sqrt{f_1(S)} - \alpha S \right) f_3(S) + \omega^3 \left(\sqrt{f_1(S)} - \alpha S \right)^3 \right]  
\]

\[
\frac{3f_2^3(S)}{3l^2 \omega^2 (a^2 - l^2) + \omega^2}  
\]

(27)

We have drawn the entropy $S$, pressure $P$, temperature $T$, volume $V$, Gibb’s free energy $G$ and Helmholtz’s free energy $F$ against PD black hole horizon radius $r_h$ in figures 1-6 respectively for the parameters $a = 1.5$, $l = 1.2$, $\omega = 0.5$, $\alpha = 1.8$, $e = 1$, $g = 1$, $M = 10$. From figure 1 we see that the entropy $S$ first sharply decreases upto $r_h \approx 2$ and then slowly decreases as PD black hole horizon radius $r_h$ increases. We’ll choose the same values of the parameters in all the figures. From figure 2 we see that the pressure $P$ increases as $r_h$ increases. From figure 3 we see that the temperature $T$ decreases with equal slope as $r_h$ increases. From figure 4 we see that the volume $V$ increases but maintains with nearly equal slope as $r_h$ grows. From figure 5 and 6 we see that the Gibb’s free energy $G$ and Helmholtz’s free energy $F$ first sharply increase upto $r_h \approx 2$ and then slowly increase but maintain with nearly equal slope as $r_h$ grows.

C. P-V Criticality

Following [20] the idea of critical behaviour of charged AdS black holes, here we will study the critical behavior of PD black hole. The critical points for PD black hole can be found from the following conditions:

\[
\left( \frac{\partial P}{\partial r_h} \right)_{cr} = 0, \quad \left( \frac{\partial^2 P}{\partial r_h^2} \right)_{cr} = 0 \quad (28)
\]
FIG. 2: Figure represents the plot of pressure $P$ against PD black hole horizon radius $r_h$.

FIG. 3: Figure represents the plot of temperature $T$ against PD black hole horizon radius $r_h$.

FIG. 4: Figure represents the plot of volume $V$ against PD black hole horizon radius $r_h$.

From these conditions, we obtain the critical point $r_{cr}$ as

$$r_{cr} = M + 2^{\frac{2}{3}} X^{-\frac{1}{3}} \left[(a + l)^2 + M^2\right] + 2^{-\frac{2}{3}} X^{\frac{1}{3}}$$  \hspace{1cm} (29)

where

$$X = M(5a^2 + 12al + 4M^2) + \sqrt{M^2(5a^2 + 12al + 4M^2)^2 - 16((a + l)^2 + M^2)^3}$$  \hspace{1cm} (30)
At the critical point, the critical values $S_{cr}$, $P_{cr}$, $T_{cr}$ and $V_{cr}$ are obtained as

$$S_{cr} = \frac{\pi \omega^2 (r_{cr}^2 + (a + l)^2)}{(\omega - l \alpha r_{cr})^2 - a^2 \alpha^2 r_{cr}^2},$$

(31)

$$P_{cr} = \frac{3 [M - r_{cr} + \pi (a + l)^2 T_{cr} + \pi r_{cr}^2 T_{cr}]}{8 \pi r_{cr} (a^2 + 6l^2 + 2r_{cr}^2)},$$

(32)

$$T_{cr} = \frac{4r_{cr}^3 - (a^2 + 6l^2 + 6r_{cr}^2)M}{\pi [a^4 + 2a^3l + 12al(l^2 + r_{cr}^2) + a^2(7l^2 + 5r_{cr}^2) + 2(3l^4 + r_{cr}^4)]},$$

(33)

$$V_{cr} = \frac{4\pi \omega [\omega (r_{cr}^3 + 6l^2 r_{cr} - 6l^4) - 8\alpha l^2 r_{cr}^2 + \alpha^2 (2l \alpha r_{cr}^2 + 6l^2 \omega + \omega r_{cr})]}{3(3a^2l^2 \alpha^2 - 3l^4 \alpha^2 + \omega^2)}$$

(34)

If we choose the values of the parameters $a = 1.5$, $l = 1.2$, $\omega = 0.5$, $\alpha = 1.8$, $e = 1$, $g = 1$, $M = 10$, then we obtain the critical point $r_{cr} = 3.915$. At this critical point, the critical values of entropy, pressure, temperature and volume are $S_{cr} = 0.21$, $P_{cr} = 24.89$, $T_{cr} = 382.56$ and $V_{cr} = 7.95$. 

FIG. 5: Figure represents the plot of Gibb’s free energy $G$ against PD black hole horizon radius $r_h$.

FIG. 6: Figure represents the plot of Helmholtz’s free energy $F$ against PD black hole horizon radius $r_h$. 
FIG. 7: Figure represents the plot of specific heat capacity $C_P$ against PD black hole horizon radius $r_h$.

### D. Stability

The specific heat capacity of the black hole thermodynamical system is the key quantity to determine the stability of the black hole and can be written as

$$ C = T \left( \frac{\partial S}{\partial T} \right) $$

(35)

If $C \geq 0$ then the black hole will be stable and if $C < 0$ then the black hole will be unstable. If volume $V$ is constant (i.e., entropy $S$ is constant), then the specific heat capacity $C_V = 0$. But for constant pressure (i.e., $P$ constant), we can obtain the specific heat capacity as in the form:

$$ C_P = \left( 2\sqrt{f_1(S)f_2(S)} \right) \left( -(a + l)\alpha \left( l\alpha S - \sqrt{f_1(S)} \right) + f_2(S) \right) f_4(S)f_5(S) \\
\times \left[ (a + l)\alpha f_2(S)f_4(S)f_5(S)f_4'(S) - 2(a + l)\alpha f_1(S) (f_2(S)f_5(S)f_4'(S) + f_4(S) (2f_5(S)f_2'(S) \\
- f_2(S)f_4'(S) - 2\sqrt{f_1(S)} \left( -2l(a + l)S\alpha^2f_4(S)f_5(S)f_4'(S) + f_2^2(S) (f_5(S)f_4'(S) - f_4(S)f_5'(S)) \\
+ f_2(S) \left( -l(a + l)S\alpha^2f_5(S)f_4'(S) + f_4(S) \left( f_5(S) \left( l(a + l)\alpha^2 + f_4'(S) \right) + l(a + l)S\alpha^2f_5'(S) \right) \right) \right) \right]^{-1} $$

(36)

where dash represents derivative with respect to $S$. We have drawn $C_P$ against horizon radius $r_h$ in figure 7. We see that the specific heat capacity $C_P$ first sharply increases from some positive value upto $r_h \approx 3$ and then slowly increases as $r_h$ grows. So our considered PD black hole is stable in nature since from graph $C_P > 0$.

### E. Joule-Thomson Expansion

Joule-Thomson expansion \[113, 114\] is irreversible process, which explains that the temperature changes from high pressure region to low pressure region, while the enthalpy remains constant. Since in
We observe that \( \mu \) has been drawn for PD black hole. The Joule-Thomson coefficient \( \mu \) has been studied in [116, 117]. Many authors have studied the Joule-Thomson expansion for several AdS black holes [57, 61, 80, 118–131]. Here, we’ll examine the Joule-Thomson expansion for AdS PD black hole. The Joule-Thomson coefficient \( \mu \) is the slope of the isenthalpic curve, given by

\[
\mu = \left( \frac{\partial T}{\partial P} \right)_M
\]

which can be written as

\[
\mu = \frac{1}{C_P} \left[ T \left( \frac{\partial V}{\partial T} \right)_P - V \right]
\]

By evaluating the sign of \( \mu \), we can determine the cooling or heating nature of the black hole. During the Joule-Thomson expansion process, the pressure always decreases, so the change of pressure is always negative while the change of temperature may be positive or negative. So the temperature determines the sign of \( \mu \). If the change of temperature is positive then \( \mu \) is negative and so heating process occurs. But if the change of temperature is negative then \( \mu \) is positive and therefore cooling process occurs. Now for AdS PD black hole, we obtain

\[
\mu = \frac{4\pi\omega^2}{3(3l^2(a^2 - l^2)\alpha^2 + \omega^2)f_2(S)} \left( -3\left( \omega^3 \left( -l\alpha S + \sqrt{f_1(S)} \right)^3 + 6l^2 \left( a^2 - l^2 \right) f_2^3(S) \right) \right.
\]

\[
+ \omega \left( -l\alpha S + \sqrt{f_1(S)} \right) f_2(S)f_3(S) f_2'(S) + f_2(S) \left( 3\omega^3 \left( -l\alpha S + \sqrt{f_1(S)} \right)^2 \left( -l\alpha + \frac{f_1'(S)}{2\sqrt{f_1(S)}} \right) \right)
\]

\[
+ \omega f_2(S)f_3(S) \left( -l\alpha + \frac{f_1'(S)}{2\sqrt{f_1(S)}} \right) + 18l^2 \left( a^2 - l^2 \right) f_2^3(S)f_2'(S) + \omega \left( -l\alpha S + \sqrt{f_1(S)} \right) f_3(S)f_2'(S)
\]

\[
+ \omega \left( -l\alpha S + \sqrt{f_1(S)} \right) f_2(S)f_3'(S) - \left( \omega^3 \left( -l\alpha S + \sqrt{f_1(S)} \right)^3 + 6l^2 \left( a^2 - l^2 \right) f_2^3(S) \right)
\]

\[
+ \omega \left( -l\alpha S + \sqrt{f_1(S)} \right) f_2(S)f_3(S) - 2 \left( (a + l)\alpha \left( -l\alpha S + \sqrt{f_1(S)} \right) + f_2(S) \right) f_4(S)f_5(S)f_2'(S)
\]

\[
+f_2(S)f_4(S)f_5(S) \left( (a + l)\alpha \left( -l\alpha S + \sqrt{f_1(S)} \right) + f_2(S) \right) \left( -l\alpha + \frac{f_1'(S)}{2\sqrt{f_1(S)}} \right) + f_2(S) \left( (a + l)\alpha \left( -l\alpha S + \sqrt{f_1(S)} \right) + f_2(S) \right) f_4(S)f_5'(S)
\]

\[
\times \left( (a + l)\alpha \left( -l\alpha S + \sqrt{f_1(S)} \right) + f_2(S) \right) f_4(S)f_5(S) \right)
\]

We have drawn the Joule-Thomson coefficient \( \mu \) against PD black hole horizon radius \( r_h \) in figure 8. We observe that \( \mu \) first keeps nearly parallel to \( r_h \) axis up to \( r_h \approx 3 \) but keeps positive sign and then increases as \( r_h \) increases. Since \( \mu > 0 \) throughout the expansion of \( r_h \), so the change of temperature is negative and therefore cooling process occurs for PD black hole.
For \( \mu = 0 \), we can determine the expansion process of inversion curve in a small infinite pressure. At the inversion temperature, put \( \mu = 0 \) in (38) and inversion temperature is given by

\[
T_i = V \left( \frac{\partial T}{\partial V} \right)_P
\]

(40)

So for PD black hole, we obtain the inversion temperature

\[
T_i = \left[ 3\pi S \alpha^3 f_1(S) - \omega^3 f_1^{3/2}(S) - \sqrt{f_1(S)} \left( 3 \int S^2 \alpha^2 \omega^3 + \omega f_2(S) f_3(S) \right) \\
+ l \left( F S \alpha^3 \omega^3 + \left( -6a^2 l + 6l^3 \right) f_2^3(S) + S \alpha \omega f_2(S) f_3(S) \right) \right] ((a + l) \alpha f_2(S) f_4(S) f_5(S) f_5(S) f_5(S) f_5(S))
- 2(l + 1) \alpha f_1(S) (f_2(S) f_5(S) f_4(S) f_5(S) f_5(S) f_5(S))
- 2\sqrt{f_1(S)} \left( -2l(a + l) \alpha S^2 f_5(S) f_4(S) f_2(S) + f_2(S) f_5(S) f_4(S) (f_5(S) f_4(S) f_4(S) - f_2(S) f_5(S)) \right)
+ f_2(S) \left( -l(a + l) \alpha S^2 f_5(S) f_4(S) + f_4(S) (f_5(S) \left( l(a + l) \alpha^2 + f_2(S) \right) + l(a + l) \alpha^2 f_5(S) \right) \right]
\times \left[ 3\pi \alpha f_2^2(S) f_4^2(S) - f_2(S) \left( 3 \int S^2 \alpha^2 \omega^2 + f_2(S) f_3(S) \right) f_4(S) + 6\omega^2 f_1^2(S) f_2^2(S) \right]
+ 6l\alpha^2 f_1^{3/2}(S) (f_2(S) - 3S f_2(S)) + f_1(S) \left( 18l S^2 \alpha^2 \omega^2 f_2^2(S) + f_2(S) \left( -12l S^2 \alpha^2 \omega^2 - 3\omega^2 f_1^2(S) \\
+ 4f_3(S) f_5(S) \right) - 2f_2^2(S) f_5(S) \right) + 2l \alpha \sqrt{f_1(S)} \left( -3l S^3 \alpha^2 \omega^2 f_2^2(S) + S f_2(S) \left( 3l S^2 \alpha^2 \omega^2 + 3\omega^2 f_1^2(S) \right.ight.
- 2f_3(S) f_5(S) \right) + f_2^2(S) \left( f_3(S) + S f_3^2(S) \right) \right]^{-1}
\]

(41)

We have drawn the inversion temperature \( T_i \) against PD black hole horizon radius \( r_h \) in figure 9. We observe that \( T_i \) decreases slowly as \( r_h \) increases but slopes of the curve at all points are almost same throughout the evolution of \( r_h \).

III. HEAT ENGINE

A heat engine is a thermodynamic system that converts thermal energy (or heat) and chemical energy to mechanical energy to do mechanical work. So physically, a heat engine carries heat from hot reservoir in
FIG. 9: Figure represents the plot of inversion temperature $T_i$ against PD black hole horizon radius $r_h$.

which part of the heat converts into the physical works while the remaining part of the heat is transferred to cold reservoir. The working substance in a black hole heat engine is thought to be the black hole fluid or black hole molecules. The heat engine brings a working substance from a higher state temperature to a lower state temperature. Then the working substance generates work while transferring heat to the cold reservoir until it reaches a low temperature state. During this process in the heat engine, some of the thermal energy is converted into work where the working substance has non-zero heat capacity. Therefore, the heat engine operates in a cyclic manner by adding energy (heat) in one part of the cycle and using that energy to do work in another part of the cycle. In the following subsections, we’ll study the Carnot cycle and Rankine cycle of the heat engine for AdS PD black hole.

A. Carnot Cycle

Carnot cycle is theoretical ideal thermodynamic cycle which was proposed by N. L. S. Carnot in 1824. A Carnot heat engine is a classical thermodynamic engine that operates on the Carnot cycle which can be achieved during the conversion of heat into work. There are two heat reservoirs (hot and cold) forming part of the heat engine of the Carnot cycle. Here we assume, $T_H$ and $T_C$ are temperatures of hot and cold reservoirs respectively. These are included upper and lower isothermal processes with two adiabatic processes. The $P$-$V$ diagram has been shown in Ref. [40] for Carnot heat engine which forms a closed path. From the diagram, it was shown that along the upper isotherm process, the heat flows are generated from 1 to 2 and which is given by [40]

$$Q_H = T_H \Delta S_{1 \rightarrow 2} = T_H(S_2 - S_1)$$

and the exhausted heat produced from 3 to 4 along lower isothermal process is given by [40]

$$Q_C = T_C \Delta S_{3 \rightarrow 4} = T_C(S_4 - S_3)$$
Here PD black hole entropies $S_i$'s are related to volumes $V_i$'s satisfying as

\[
V_i = \frac{4\pi \omega^2 \left[ 6l^2(a^2 - l^2)f_2^3(S_i) + \omega f_2(S_i)f_3(S_i) \left( \sqrt{f_1(S_i)} - l\alpha S_i \right) + \omega^3 \left( \sqrt{f_1(S_i)} - l\alpha S_i \right)^3 \right]}{3f_2^3(S_i) \left[ 3l^2\alpha^2(a^2 - l^2) + \omega^2 \right]} , \quad i = 1, 2, 3, 4, \tag{44}
\]

where $f_1(S_i)$, $f_2(S_i)$ and $f_3(S_i)$ can be calculated from the following relations

\[
f_1(S_i) = a^2l^2S_i^2 + \left( S_i - (a + l)^2\pi \right)f_2(S_i) , \quad i = 1, 2, 3, 4, \tag{45}
\]

\[
f_2(S_i) = a^2(a^2 - l^2)S_i + \pi\omega^2 , \quad i = 1, 2, 3, 4 \tag{46}
\]

and

\[
f_3(S_i) = a^2 \left[ f_2(S_i) + 2l\alpha \left( \sqrt{f_1(S_i)} - l\alpha S_i \right) \right] + 2l^2 \left[ 3f_2(S_i) - 4l\alpha \left( \sqrt{f_1(S_i)} - l\alpha S_i \right) \right] , \quad i = 1, 2, 3, 4. \tag{47}
\]

Also the heat engine flow has been shown in Ref. [40]. The total mechanical work done by the heat engine is the difference of the amount of heat energies between upper and lower isotherm processes as

\[
W = Q_H - Q_C \tag{48}
\]

A central quantity, the efficiency of a Carnot heat engine is defined by the ratio of total mechanical work and the amount of heat energy along the upper isotherm process and is given by

\[
\eta_{Car} = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H} \tag{49}
\]

Since for Carnot cycle, $V_4 = V_1$ and $V_3 = V_2$, so we have the maximum efficiency for Carnot cycle and is given by

\[
(\eta_{Car})_{max} = 1 - \frac{T_C(S_3 - S_4)}{T_H(S_2 - S_1)} \bigg|_{V_4 = V_1, V_3 = V_2} \tag{50}
\]

which simplifies to the following form

\[
(\eta_{Car})_{max} = 1 - \frac{T_C}{T_H} \tag{51}
\]

Since $T_H > T_C$, so we have $0 < (\eta_{Car})_{max} < 1$. It should be noted that the $(\eta_{Car})_{max}$ is the maximum efficiency of all possible cycles between higher temperature $T_H$ and lower temperature $T_C$. Since we know that the Stirling cycle consists of two isothermal processes and two isochores processes, so this maximally efficient Carnot engine is also Stirling engine. From the figure [3] if we choose $T_C = 250$ and $T_H = 400$ then the maximum efficiency for Carnot cycle is obtained as $(\eta_{Car})_{max} = 0.375$. 


Now we discuss a new engine of black hole which consists of two isobars and two isochores/adiabats described in Ref [40], where the heat flows show along the top and bottom. The total work done along the isobars is given by

\[ W = \Delta P_{4\rightarrow1} \Delta V_{1\rightarrow2} = (P_1 - P_4)(V_2 - V_1) \]  

(52)

where \( V_1 \) and \( V_2 \) are described in [44]. The upper isobar gives the net inflow of heat which is given by

\[
Q_H = \int_{T_1}^{T_2} C_P(P_1, T) dT
= \int_{S_2}^{S_1} \frac{f_2(S) + \alpha(a + l) \left( \sqrt{f_1(S)} - l\alpha S \right)}{3\pi f_2^2(S)f_4(S)} \left| \frac{P = P_1}{P = P_1} \right| dS
\]  

(53)

The lower isobar gives the exhaust of heat which is given by

\[
Q_C = \int_{T_1}^{T_4} C_P(P_4, T) dT
= \int_{S_4}^{S_3} \frac{f_2(S) + \alpha(a + l) \left( \sqrt{f_1(S)} - l\alpha S \right)}{3\pi f_2^2(S)f_4(S)} \left| \frac{P = P_1}{P = P_4} \right| dS
\]  

(54)

The thermal efficiency for the new heat engine is described by

\[
\eta_{new} = \frac{W}{Q_H} = \frac{(P_1 - P_4)(V_2 - V_1)}{Q_H}
= (P_1 - P_4)(V_2 - V_1) \left[ \int_{S_2}^{S_1} \frac{f_2(S) + \alpha(a + l) \left( \sqrt{f_1(S)} - l\alpha S \right)}{3\pi f_2^2(S)f_4(S)} \left| \frac{P = P_1}{P = P_1} \right| dS \right]^{-1}
\]  

(55)

Since the above integration is very complicated, so we may find the value of the efficiency of the new engine for particular values of the parameters, say \( a = 1.5, l = 1.2, \omega = 0.5, \alpha = 1.8, e = 1, g = 1, M = 10. \)

If we choose \( S_1 = 0.25, S_2 = 0.40, P_1 = 15, P_4 = 25, V_1 = 5, V_2 = 15 \) from the figures [1] [2] and [3] then the value of the thermal efficiency of the new engine is obtained as \( \eta_{new} = 0.907. \)

**B. Rankine Cycle**

A Rankine cycle [71] is an idealized thermodynamic cycle of a heat engine, which converts heat into mechanical work for undergoing phase change. The Rankine cycle for black hole heat engine is shown in Ref [72]. From the Ref [72], we see that the working substance starts from \( A \) to \( B \) with increasing temperature and adiabatic pressure. Next the working substance follows from \( B \) to \( E \) and within these states a phase transition occurs from \( C \) to \( D \) with constant temperature. Then the working substance reduces the temperature from \( E \) to \( F \) and returns to \( A \) by reducing its volume. For constant pressure \( P, \)
we have \( dP = 0 \). From the first law of the black hole thermodynamics \( dH_P = TdS \) for constant pressure, we have the enthalpy function \( H_P(S) = \int TdS \). Now according to the formalism of Wei et al [71, 72], the efficiency for Rankine cycle for black hole heat engine can be expressed as

\[
\eta_{Ran} = 1 - \frac{T_A(S_F - S_A)}{H_{P_2}^B(S_F) - H_{P_2}^B(S_A)} = 1 - \frac{T_1(S_3 - S_1)}{H_{P_2}^B(S_3) - H_{P_2}^B(S_1)}
\]  

(56)

where the subscripts \( A, B, F \) are denoted by 1, 2, 3 respectively. Here \( T_1 \) and \( H_{P_2} \) for PD black hole are given by

\[
T_1 = \frac{f_2(S_1) + \alpha(a + l)\left(\sqrt{f_1(S_1)} - l\alpha S_1\right)}{3\pi f_2^2(S_1)f_4(S_1)}
\]  

(57)

and

\[
H_{P_2}(S_i) = \int_{S_0}^{S_i} \left[\frac{f_2(S) + \alpha(a + l)\left(\sqrt{f_1(S)} - l\alpha S\right)}{3\pi f_2^2(S)f_4(S)}\right] f_5(S) \, dS \bigg|_{P=P_2}
\]  

(58)

So equation (56) can be expressed in the following form:

\[
\eta_{Ran} = 1 - \frac{(S_3 - S_1)}{3\pi f_2^2(S_1)f_4(S_1)} \left[ f_2(S_1) + \alpha(a + l)\left(\sqrt{f_1(S_1)} - l\alpha S_1\right) \right] f_5(S_1) \times \left[ \int_{S_1}^{S_3} \left[\frac{f_2(S) + \alpha(a + l)\left(\sqrt{f_1(S)} - l\alpha S\right)}{3\pi f_2^2(S)f_4(S)}\right] f_5(S) \, dS \bigg|_{P=P_2} \right]^{-1}
\]  

(59)

Since the above integration is very complicated, so we may find the value of the efficiency of the Rankine cycle for particular values of the parameters, say \( a = 1.5, \ l = 1.2, \ \omega = 0.5, \ \alpha = 1.8, \ e = 1, \ g = 1, \ M = 10 \). If we choose \( S_1 = 0.25, \ S_3 = 0.45, \ P_2 = 15 \) from the figures [1] and [2] then the value of the efficiency of the rankine cycle is obtained as \( \eta_{Ran} = 0.968 \).

**IV. DISCUSSIONS AND CONCLUDING REMARKS**

We have assumed the general class of accelerating, rotating and charged Plebanski-Demianski (PD) black holes in presence of cosmological constant, which includes the Kerr-Newman rotating black hole and the Taub-NUT spacetime. We have assumed that the thermodynamical pressure \( P \) may be described as the negative cosmological constant \( \Lambda < 0 \) by the relation \( \Lambda = -8\pi P \) and so the black hole may be formed in anti-de Sitter (AdS) PD black hole. Using the horizon radius \( r_h \), the thermodynamic quantities like surface area \( \kappa \), entropy \( S \), volume \( V \), temperature \( T \), Gibb’s free energy \( G \) and Helmholtz’s free energy \( F \) of the AdS PD black hole have been obtained due to the thermodynamic system. We have drawn the entropy \( S \), pressure \( P \), temperature \( T \), volume \( V \), Gibb’s free energy \( G \) and
Helmholtz’s free energy $F$ against PD black hole horizon radius $r_h$ in figures 1-6 respectively for the parameters $a = 1.5$, $l = 1.2$, $\omega = 0.5$, $\alpha = 1.8$, $e = 1$, $g = 1$, $M = 10$. From figure 1 we have observed that the entropy $S$ first sharply decreases upto $r_h \approx 2$ and then slowly decreases as PD black hole horizon radius $r_h$ increases. From figure 2 we have seen that the pressure $P$ increases as $r_h$ increases. From figure 3 we have seen that the temperature $T$ decreases with equal slope as $r_h$ increases. From figure 4, we have observed that the volume $V$ increases but maintains with nearly equal slope as $r_h$ grows. From figure 5 and 6 we have seen that the Gibb’s free energy $G$ and Helmholtz’s free energy $F$ first sharply increase upto $r_h \approx 2$ and then slowly increase but maintain with nearly equal slope as $r_h$ grows.

Next we found the critical point and corresponding critical entropy, critical pressure, critical temperature and critical volume for AdS PD black hole. In particular, for the chosen the values of the parameters $a = 1.5$, $l = 1.2$, $\omega = 0.5$, $\alpha = 1.8$, $e = 1$, $g = 1$, $M = 10$, we have obtained the critical point $r_{cr} = 3.915$ and the corresponding critical values of entropy, pressure, temperature and volume are $S_{cr} = 0.21$, $P_{cr} = 24.89$, $T_{cr} = 382.56$ and $V_{cr} = 7.95$. We have drawn $C_P$ against horizon radius $r_h$ in figure 7. We have seen that the specific heat capacity $C_P$ first sharply increases from some positive value upto $r_h \approx 3$ and then slowly increases as $r_h$ grows. Due to the study of specific heat capacity, we have obtained $C_V = 0$ and from the graph, we have obtained $C_P > 0$. From this result, we have concluded that the AdS PD black hole may be stable. We have also examined the Joule-Thomson expansion of PD black hole and by evaluating the sign of Joule-Thomson coefficient $\mu$, we have determined the heating and cooling nature of PD black hole. We have drawn the Joule-Thomson coefficient $\mu$ against PD black hole horizon radius $r_h$ in figure 8. We have observed that $\mu$ first keeps nearly parallel to $r_h$ axis upto $r_h \approx 3$ but keeps positive sign and then increases as $r_h$ increases. Since $\mu > 0$ throughout the expansion of $r_h$, so the change of temperature is negative and therefore cooling process occurs for PD black hole. Putting $\mu = 0$, we have obtained the inversion temperature $T_i$. We have drawn the inversion temperature $T_i$ against PD black hole horizon radius $r_h$ in figure 9. We have observed that $T_i$ decreases slowly as $r_h$ increases but slopes of the curve at all points are almost same throughout the evolution of $r_h$.

Next we have studied the heat engine phenomena for AdS PD black hole. We have analyzed the heat flows from upper and lower isotherms process. For the heat engine in Carnot cycle, we have calculated the work done and maximum efficiency. If we choose $T_C = 250$ and $T_H = 400$ then the maximum efficiency for Carnot cycle is obtained as $(\eta_{car})_{max} = 0.375$. Also for a new engine, we have assumed another cycle which consists of two isobars and two isochores. Then we have calculated the net inflow of
heat in upper isobar, work done and its efficiency of the new heat engine for this cycle. In particular, for $S_1 = 0.25$, $S_2 = 0.40$, $P_1 = 15$, $P_4 = 25$, $V_1 = 5$, $V_2 = 15$ the value of the thermal efficiency of the new engine is obtained as $\eta_{\text{New}} = 0.907$. Finally, we have analyzed the efficiency for the Rankine cycle in PD black hole heat engine. In particular, for $S_1 = 0.25$, $S_3 = 0.45$, $P_2 = 15$ the value of the efficiency of the rankine cycle is obtained as $\eta_{\text{Ran}} = 0.968$.

[1] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975); Erratum: [Commun. Math. Phys. 46, 206 (1976)].
[2] S. Hawking, Phys. Rev. D 13, 191 (1976).
[3] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[4] J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974).
[5] G. W. Gibbons and S. W. Hawking, Phys. Rev. D 15, 2738 (1977).
[6] S. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).
[7] A. Chamblin, R. Emparan, C. Johnson and R. Myers, Phys. Rev. D 60, 064018 (1999).
[8] A. Chamblin, R. Emparan, C. Johnson and R. Myers, Phys. Rev. D 60, 104026 (1999).
[9] M. Cvetic and S. Gubser, JHEP 9904, 024 (1999).
[10] C. Niu, Y. Tian and X. -N. Wu, Phys. Rev. D 85, 024017 (2012).
[11] D. Kubiznak and R. B. Mann, Can. J. Phys. 93, 999 (2015).
[12] R. B. Mann, Springer Proc. Phys. 170, 197 (2016).
[13] C. Teitelboim, Phys. Lett. B 158, 293 (1985).
[14] J. Creighton and R. B. Mann, Phys. Rev. D 52, 4569 (1995).
[15] M. M. Caldarelli, G. Cognola and D. Klemm, Class. Quant. Grav. 17, 399 (2000).
[16] D. Kastor, S. Ray and J. Traschen, Class. Quant. Grav. 26, 195011 (2009).
[17] B. P. Dolan, Class. Quant. Grav. 28, 125020 (2011).
[18] B. P. Dolan, Class. Quant. Grav. 28, 235017 (2011).
[19] M. Cvetic, G. W. Gibbons, D. Kubiznak and C. N. Pope, Phys. Rev. D 84, 024037 (2011).
[20] D. Kubiznak and R. B. Mann, JHEP 1207, 033 (2012).
[21] S. Gumasekaran, R. B. Mann and D. Kubiznak, JHEP 1211, 110 (2012).
[22] D. Kubiznak, R. B. Mann and M. Teo, Class. Quant. Grav. 34, 063001 (2017).
[23] A. Sahay, T. Sarkar and G. Sengupta, JHEP 1004, 118 (2010).
[24] A. Sahay, T. Sarkar and G. Sengupta, JHEP 1007, 082 (2010).
[25] R. Banerjee, S. K. Modak and S. Samanta, Phys. Rev. D 84, 064024 (2011).
[26] R. Banerjee, S. Ghosh and D. Roychowdhury, Phys. Lett. B 696, 156 (2011).
[27] C. Niu, Y. Tian and X. -N. Wu, Phys. Rev. D 85, 024017 (2012).
[28] A. Lala and D. Roychowdhury, Phys. Rev. D 86, 084027 (2012).
[29] S. -W. Wei and Y. -X. Liu, Phys. Rev. D 87, 044014 (2013).
[30] J. -L. Zhang, R. -G. Cai, and H. Yu, Phys. Rev. D 91, 044028 (2015).
[31] A. Sheykhi, S. H. Hendi, S. Panahiyan and B. E. Panah, Can. J. Phys. 94, 1045 (2016).
[32] G. -Q. Li and J. -X. Mo, Phys. Rev. D 93, 124021 (2016).
[33] P. Chaturvedi and G. Sengupta, Phys. Lett. B 765, 67 (2017).
[34] P. Chaturvedi and G. Sengupta, Eur. Phys. J. C 77, 110 (2017).
[35] B. -B. Ye, J. -H. Chen and Y. -J. Wang, Chin. Phys. B 26, 090202 (2017).
[36] S. -W. Wei, Q. -T. Man and H. Yu, Commun. Theor. Phys. 69, 173 (2018).
[37] A. Ghosh and C. Bhamidipati, Phys. Rev. D 100, 126020 (2019).
[38] P. Wang, H. Wu and H. Yang, Eur. Phys. J. C 80, 216 (2020).
[39] J. D. Bairagya, K. Pal, K. Pal and T Sarkar, [arXiv:2004.06498 [hep-th]].
[40] C. V. Johnson, Class. Quant. Grav. 31, 205002 (2014).
[41] M. R. Setare and H. Adami, Gen Rel. Grav. 47, 133 (2015).
[42] C. V. Johnson, Class. Quant. Grav. 33, 135001 (2016).
[43] C. V. Johnson, Class. Quant. Grav. 33, 215009 (2016).
[44] C. V. Johnson, Entropy 18, 120 (2016).
[45] R. A. Hennigar, F. McCarthy, A. Ballon and R. B. Mann, Class. Quant. Grav. 34, 175005 (2017).
[46] J. -X. Mo and G. -Q. Li, JHEP 1805, 122 (2018).
[47] L. -Q. Fang and X. -M. Kuang, Sci. China Phys. Mech. Astron. 61, 080421 (2018).
[48] F. Rosso, Int. J. Mod. Phys. D 28, 1950030 (2019).
[49] C. V. Johnson and F. Rosso, Class. Quant. Grav. 36, 015019 (2019).
[50] S. -Q. Hu and X. -M. Kuang, Sci. China-Phys. Mech. Astron. 62, 060411 (2019).
[51] W. Ahmed, H. Z. Chen, E. Gesteau, R. Gregory and A. Scoins, Class. Quant. Grav. 36, 214001 (2019).
[52] H. Ghaffarnejad, E. Yaraie, M. Farsam and K. Bamba, Nucl. Phys. B 952, 114941 (2020).
[53] C. V. Johnson, Class. Quant. Grav. 37, 034001 (2020).
[54] M. Zhang and W. -B. Liu, Int. J. Theor. Phys. 55, 5136 (2016).
[55] S. H. Hendi, B. E. Panah, S. Panahiyan, H. Liu and X. -H. Meng, Phys. Lett. B 781, 40 (2018).
[56] S. Fernando, Mod. Phys. Lett. A 33, 1850177 (2018).
[57] C. H. Nam, [arXiv:1906.05557 [gr-qc]].
[58] B. E. Panah, Phys. Lett. B 787, 45 (2018).
[59] H. Xu, Y. Sun and L. Zhao, Int. J. Mod. Phys. D 26, 1750151 (2017).
[60] S. Guo, Q. Q. Jiang and J. Pu, [arXiv:1908.01712 [gr-qc]].
[61] V. Rajuni, C. L. A. Rizwan, A. N. Kumara, D. Vaid and K. M. Ajith, [arXiv:1904.06914 [gr-qc]].
[62] Y. Runqian, J. Zheng, J. Chen and Y. Wang, Commun. Theor. Phys. 72, 035401 (2020).
[63] J. -X. Mo, F. Liang and G. -Q. Li, JHEP 1703, 010 (2017).
[64] L. Balart and S. Fernando, Phys. Lett. B 795, 638 (2019).
[65] B. Chandrasekhar and P. K. Yerra, Eur. Phys. J. C 77, 534 (2017).
[66] J. Sadeghi, Int. J. Theor. Phys. 56, 3387 (2017).
20

[67] A. Chakraborty and C. V. Johnson, Int. J. Mod. Phys. D 28, 1950012 (2019).
[68] R. A. Hennigar, F. McCarthy, A. Ballon and R. B. Mann, Class. Quant. Grav. 34, 175005 (2017).
[69] H. Liu and X. -H. Meng, Eur. Phys. J. C 77, 556 (2017).
[70] C. V. Johnson, Class. Quant. Grav. 35, 045001 (2018).
[71] S. -W. Wei and Y. -X. Liu, Commun. Theor. Phys. 71, 711 (2019).
[72] S. -W. Wei and Y. -X. Liu, Nucl. Phys. B 946, 114700 (2019).
[73] A. Chakraborty and C. V. Johnson, Int. J. Mod. Phys. D 28, 1950006 (2019).
[74] J. -X. Mo and S. -Q. Lan, Eur. Phys. J. C 78, 666 (2018).
[75] J. F. G. Santos, Eur. Phys. J. Plus 133, 321 (2018).
[76] C. V. Johnson, arXiv:1907.05883 [hep-th].
[77] U. Debnath and B. Pourhassan, arXiv:1910.00466 [gr-qc].
[78] S. Guha and S. Chakraborty, arXiv:1908.06763 [gr-qc].
[79] M. Rizwan and K. Saifullah, Astrophys. Space Sci. 343, 165 (2013).
[80] M. Sharif and W. Javed, Can. J. Phys. 91, 236 (2013).
[81] A. Anabalon, R. Gregory, D. Kubiznak and R. B. Mann, JHEP 1904, 096 (2019).
[82] P. Pradhan, Annals Phys. 372, 449 (2016).
[83] J. Zhang, Y. Li and H. Yu, Eur. Phys. J. C 78, 645 (2018).
[84] J. Zhang, Y. Li and H. Yu, JHEP 02, 144 (2019).
[85] K. Jafarzade and B. E. Panah, arXiv:1906.09478 [hep-th].
[86] J. B. Griffiths and J. Podolsky, Class. Quantum Grav. 22, 3467 (2005).
[87] J. B. Griffiths and J. Podolsky, Class. Quantum Grav. 23, 555 (2006).
[88] J. B. Griffiths and J. Podolsky, Int. J. Mod. Phys. D 15, 335 (2006).
[89] J. Podolsky and J. B. Griffiths, Phys. Rev. D 73, 044018 (2006).
[103] J. Podolsky and H. Kadlecova, Class. Quantum Grav. 26, 105007 (2009).
[104] K. Hong and E. Teo, Class. Quantum Grav. 20, 3269 (2003).
[105] J. G. Miller, J. Math. Phys. 14, 486 (1973).
[106] D. Bini and C. Cherubini, R. T. Jantzen and B. Mashhoon, Class. Quantum Grav. 20, 457 (2003).
[107] D. Bini and C. Cherubini, R. T. Jantzen and B. Mashhoon, Phys. Rev. D 67, 084013 (2003).
[108] T. Iwai and N. Katayama, J. Geom. Phys. 12, 55 (1993).
[109] C. H. Lee, Phys. Lett. B 68, 152 (1977).
[110] J. M. Bardeen, Nature 226, 64 (1970).
[111] P. Pradhan, Mod. Phys. Lett. A 30, 1550170 (2015).
[112] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
[113] D. E. Winterbone, “Advanced Thermodynamics for Engineers”, 1st edn. (Butterworth-Heinemann, Oxford, 1997).
[114] D. C. Johnston, “Advances in Thermodynamics of the Van der Waals Fluid”, (Morgan & Claypool, San Rafael, CA, 2014).
[115] D. Kastor, S. Ray and J. Traschen, Class. Quant. Grav. 26, 195011 (2009).
[116] O. Okcu and E. Aydiner, Eur. Phys. J. C 77, 24 (2017).
[117] O. Okcu and E. Aydiner, Eur. Phys. J. C 78, 123 (2018).
[118] J. -X. Mo, G. -Q. Li, S. -Q. Lan and X. -B. Xu, Phys. Rev. D 98, 124032 (2018).
[119] M. Chabab, H. El Moumni, S. Iraoui, K. Masmar and S. Zhizeh, LHEP 02, 05 (2018).
[120] S. -Q. Lan, Phys. Rev. D 98, 084014 (2018).
[121] C. L. A. Rizwan, A. N. Kumara, D. Vaid, K. M. Ajith, Int. J. Mod. Phys. A 33 35 (2018).
[122] A. Cisterna, S. -Q. Hu and X. -M. Kuang, Phys. Lett. B 797, 134883.
[123] A. Haldar and R. Biswas, EPL 123, 40005 (2018).
[124] C. Li, P. He, P. Li and J. -B. Deng, arXiv:1904.09548 [gr-qc].
[125] J. Pu, S. Guo, Q. -Q. Jiang and X. -T. Zu, Chin. Phys. C 44, 035102 (2020).
[126] S. Guo, J. Pu and Q. -Q. Jiang, arXiv:1905.03604 [gr-qc].
[127] D. M. Yekta, A. Hadikhani and O. Okcu, Phys. Lett. B 795, 521 (2019).
[128] M. Rostami, J. Sadeghi, S. Miraboutalebi, A. A. Masoudi, B. Pourhassan, arXiv:1908.08410 [gr-qc].
[129] S. -Q. Lan, Nucl. Phys. B 948, 114787 (2019).
[130] S. Guo, Y. Han and G. P. Li, arXiv:1912.09590 [hep-th].
[131] J. Sadeghi and R. Toorandaz, Nucl. Phys. B 951, 114902 (2020).