1-FACtorizations of Cayley Graphs

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Abstract. In this note we prove that all connected Cayley graphs of every finite group $Q \times H$ are 1-factorizable, where $Q$ is any non-trivial group of 2-power order and $H$ is any group of odd order.

1. Introduction and Results

Let $G$ be a non-trivial group, $S \subseteq G \setminus \{1\}$ and $S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph $\Gamma(S : G)$ of the group $G$ with respect to the set $S$ has the vertex set $G$ and the edge set $\{\{g, sg\} : g \in G, s \in S \cup S^{-1}\}$.

A $j$-factor of a graph is a spanning subgraph which is regular of valence $j$. In particular, a 1-factor of a graph is a collection of edges such that each vertex is incident with exactly one edge. A 1-factorization of a regular graph is a partition of the edge set of the graph into disjoint 1-factors. A 1-factorization of a regular graph of valence $v$ is equivalent to a coloring of the edges in $v$ colors (coloring each 1-factor a different color). This enables us to use a very helpful result: Any simple, regular graph of valence $v$ can be edge-colored in either $v$ or $v + 1$ colors. This is a specific case of Vizing’s theorem (see [2, pp. 245-248]).

We study the conjecture that says all Cayley graphs $\Gamma(S : G)$ of groups $G$ of even order are 1-factorizable whenever $G = \langle S \rangle$. There are some partial results on this conjecture obtained by Stong [1]. Here we prove

Theorem. Let $H$ be a finite group of odd order and let $Q$ be a finite group of order $2^k$ ($k > 0$). Then the Cayley graph $\Gamma(S : Q \times H)$ is 1-factorizable for all generating sets $S$ of $Q \times H$.

As a corollary we prove that all connected Cayley graphs of every finite nilpotent group of even order are 1-factorizable which has been proved by

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Stong in [1] Corollary 2.4.1] only for Cayley graphs on minimal generating sets.

2. Proof of the Theorem

We need the following lemma whose proof is more or less as Lemma 2.1 of [1] with some modifications.

Lemma 2.1. Let $H$ be a finite group of odd order. Then the Cayley graph $\Gamma(S: \mathbb{Z}_2 \times H)$ is 1-factorizable, for any generating set $S$ of $\mathbb{Z}_2 \times H$ containing exactly one element of even order.

Proof. Let $a$ be the only element of $S$ of even order. Then $a = zh$, where $z \in \mathbb{Z}_2$ and $h \in H$ and $z$ of order 2. If $a^2 = 1$, then $h = 1$ and $S \setminus \{a\} \subseteq H$ and so $axa^{-1} = x$ for all $x \in S \cap H$. Thus, in this case, Theorem 2.3 of [1] completes the proof. Therefore we may assume that $a^2 \neq 1$. Let $\Gamma' = \Gamma(S \setminus \{a\}: \mathbb{Z}_2 \times H)$ and $\Gamma_1$ and $\Gamma_2$ be the induced subgraphs of $\Gamma'$ on the sets $H$ and $zH$, respectively. It can be easily seen that the map $x \mapsto zx$ is an graph isomorphism from $\Gamma_1$ to $\Gamma_2$. By Vizing’s theorem the edges in both $\Gamma_1$ and $\Gamma_2$ can be edges-colored in the same manner in $|S \setminus \{a\}| + 1$ colors (by “the same manner” we mean that the edge $\{h_1, h_2\}$ in $\Gamma_1$ has “the same” color as $\{zh_1, zh_2\}$ in $\Gamma_2$, and vice versa). Then all that remains to be done is to color the edges from $H$ to $zH$, that is the following two ‘disjoint’ 1-factors of $\Gamma(S: \mathbb{Z}_2 \times H)$ (here we use $a^2 \neq 1$):

$$\{\{x, ax\} \mid x \in H\} \text{ and } \{\{x, a^{-1}x\} \mid x \in H\}.$$ (note that the edges of $\Gamma(S: \mathbb{Z}_2 \times H)$ are exactly the edges of $\Gamma_1$, $\Gamma_2$ and those in the above 1-factors). Now since both $x \in H$ and $zx \in zH$ have edges (in $\Gamma_1$ and $\Gamma_2$, respectively) of the same $|S \setminus \{a\}|$ colors to them, there are ‘two’ colors (note that here we again use $a^2 \neq 1$) that can be used to color 1-factors in (*). This completes the proof. \qed

Proof of the Theorem. Let $G = Q \times H$ and $S$ be any generating set of $G$. We argue by induction on $|S|$. If $|S| = 1$, then $G$ is a cyclic group of even order and Corollary 2.3.1 of [1] completes the proof. Now assume that $|S| > 1$ and for any non-trivial group $Q_1$ of 2-power order and subgroup $H_1$ of $H$ the Cayley graph $\Gamma(S_1 : Q_1 \times H_1)$ is 1-factorizable for any generating set $S_1$ of $Q_1 \times H_1$ with $|S_1| < |S|$. Since the set of elements of odd order in $G$ is the subgroup $H$ and $G = \langle S \rangle$, $S$ has at least one element $a$ of even order.

First assume that $S$ has another element distinct from $a$ of even order. Consider the subgroup $G_1$ generated by $S \setminus \{a\}$ of $G$. Then $G_1 = Q_1 \times H_1$ for some subgroups $Q_1 \leq Q$ and $H_1 \leq H$ such that $Q_1 \neq 1$. Therefore the induction hypothesis implies that $\Gamma(S \setminus \{a\} : G_1)$ has a 1-factorization. Since $\Gamma(S \setminus \{a\}, G)$ consists of disjoint copies of $\Gamma(S \setminus \{a\} : G_1)$ which are 1-factorizable, $\Gamma(S \setminus \{a\}, G)$ has a 1-factorization. Now since the only element
of \( S \backslash (S \backslash \{a\}) \) has even order, Lemma 2.2 of [1] shows that \( \Gamma(S : G) \) is 1-factorizable.

Hence we may assume that \( a \) is the only element of \( S \) of even order. Since \( a = a_1 a_2 \) for some \( a_1 \in Q \) and \( a_2 \in H \), we have

\[
G = \langle S \rangle = \langle S \backslash \{a\}, a_1 a_2 \rangle = \langle a_1 \rangle \times \langle S \backslash \{a\}, a_2 \rangle.
\]

It follows that \( Q = \langle a_1 \rangle \). Consider the subgroup \( N = \langle a_2 \rangle \). Then \( N \) is a normal subgroup of \( G \) such that \( N \cap S = \emptyset \). It is easy to see that when \( s, t \in S \) with \( s \neq t \pm 1 \), neither \( st \) nor \( st^{-1} \) belongs to \( N \). Now by Lemma 2.4 of [1], it is enough to show that \( \Gamma(S N : G N) \) is 1-factorizable. Since \( N \cong Z_2 \times H \), it follows from Lemma [2.1] that \( \Gamma(S N : G N) \) is 1-factorizable. This completes the proof. \( \square \)

**Corollary 2.2.** If \( G \) is a finite nilpotent group of even order, then \( \Gamma(S : G) \) is 1-factorizable for all generating sets \( S \) of \( G \).

**Proof.** It follows from the Theorem and the fact that every finite nilpotent group is the direct product of its Sylow subgroups. \( \square \)

**References**

[1] R. A. Stong, *On 1-factorizability of Cayley graphs*, Journal of Combinatorial Theory, Series B, 39, 298-307 (1985).

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