Abstract. We give a concise introduction to the Farrell-Jones Conjecture in algebraic $K$-theory and to some of its applications. We survey the current status of the conjecture, and we illustrate the two main tools that are used to attack it: controlled algebra and trace methods.

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1. Introduction

The classification of manifolds and the study of their automorphisms are central problems in mathematics. For manifolds of sufficiently high dimension, these problems can often be successfully solved using algebraic topological invariants in the algebraic $K$-theory and $L$-theory of group rings.

In an article published in 1993 [FJ93a], Tom Farrell and Lowell Jones formulated a series of Isomorphism Conjectures about the $K$ and $L$-theory of group rings, which became universally known as the Farrell-Jones Conjectures. On the one hand these conjectures represented the culmination of decades of seminal work by Farrell, Jones, and Wu Chung Hsiang, e.g. [FH78], [FH81a], [FH81b], [Hsi84], [FJ86], [FJ89]. On the other hand they have motivated and continue to motivate an impressive body of research.

In this article we focus only on the Farrell-Jones Conjecture for algebraic $K$-theory, and mention briefly some of its variants in Subsection 2.6. We give a concise introduction to this conjecture and to some of its applications, survey its current status, and most importantly we explain the main ideas and tools that are used to attack the conjecture: controlled algebra and trace methods.

Section 2 begins with some fundamental conjectures in algebra and geometric topology, which can be reformulated in terms of $K_0$ and $K_1$ of group rings. These conjectures are all implied by the Farrell-Jones Conjecture, but they are more accessible and elementary; moreover, their importance and appeal do not require algebraic $K$-theory, but may serve as motivation to study it.

In Subsections 2.3 and 2.4 we define assembly maps and use them to formulate the Farrell-Jones Conjecture. Then we discuss how the Farrell-Jones Conjecture implies all other conjectures discussed in this article.

In Section 3 we collect most of what is known today about the Farrell-Jones Conjecture in algebraic $K$-theory. We invite the reader to compare that section to the corresponding Section 2.6 in the survey article [LR05] from 2005, to appreciate the tremendous amount of activity and progress that has taken place since then.

The last two sections focus on proofs. In Section 4 we introduce the basic concepts of controlled algebra and see them at work. In particular, we give an almost complete proof of the Farrell-Jones Conjecture in the simplest nontrivial case, that of the free abelian group on two generators. Many ingenious ideas, mainly going back to Farrell and Hsiang, enter the proof already in this seemingly basic case. This section is meant to be an accessible introduction to controlled algebra. We do not even mention the very important flow techniques, and highly recommend Arthur Bartels’s survey article [Bar16].

In Section 5 we illustrate how trace methods are used to prove rational injectivity results about assembly maps. We give a complete proof of an elementary but illuminating statement about $K_0$ in Subsection 5.1 and then explain how this idea can be generalized using more sophisticated tools like topological Hochschild homology and topological cyclic homology. The complicated technical details underlying the construction of these tools are beyond the scope of this article, and we refer the reader to [DGM13], [Hes05], and [Mad94] for more information. However, we carefully explain the structure of the proof of the algebraic $K$-theory Novikov Conjecture due to Marcel Bökstedt, Hsiang, and Ib Madsen [BHM93]. We follow the point of view used by the authors in joint work with Wolfgang Lück and John Rognes [LLRVT11], leading to a generalization of this theorem for the
Farrell-Jones assembly map. In particular, we highlight the importance of a variant of topological cyclic homology, Bökstedt-Hsiang-Madsen’s functor $C$, which has seemingly disappeared from the literature since [BHM93].

We tried to make our exposition accessible to nonexperts, and no deeper knowledge of algebraic $K$-theory is required. However, we expect our reader to have seen the basic definitions and properties of $K_0$ and $K_1$, and to be willing to accept the existence of a spectrum-valued algebraic $K$-theory functor. Classical and less classical sources for the $K$-theoretic background include [Bas68], [Cor11], [DGM13], [Mil71], [Ros94], and [Wei13].

There are other survey articles about the Farrell-Jones and related conjectures: [Bar16], [LR05], and [Mad94], which we already recommended, and also [Lüc10] and the voluminous book project [Lüc]. Our hope is that this contribution may serve as a more concise and accessible starting point, preparing the reader for these other more advanced surveys and for the original articles.

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2. Conjectures

In this section we discuss many conjectures related to group rings and their algebraic $K$-theory. These conjectures are all implied by the Farrell-Jones Conjecture, which we formulate in Subsection 2.4. All of these conjectures are known in many cases but open in general, as we review in Section 3.

2.1. Idempotents and projective modules. An element $p$ in a ring is an idempotent if $p^2 = p$. The trivial examples are the elements 0 and 1.

Conjecture 1 (trivial idempotents). Let $k$ be a field of characteristic zero and let $G$ be a torsion-free group. Then every idempotent in the group ring $k[G]$ is trivial.

The assumption that $G$ is torsion-free is necessary: if $g \in G$ is an element of finite order $n$, then $\frac{1}{n}\sum_{i=0}^{n-1} g^i$ is a nontrivial idempotent in $\mathbb{Q}[G]$.

A counterexample to the conjecture above would be in particular a zero-divisor in $k[G]$, and hence a counterexample to Problem 6 in Irving Kaplansky’s famous problem list [Kap57], which is reproduced in [Kap70].

It is interesting to notice that the analog of Conjecture 1 for the integral group ring is true for all groups, even for groups with torsion. The proof that we give below uses operator algebras, as suggested in [Kap70, page 451], and therefore it is very different from the rest of this paper, even though the idea of using traces plays a central role in Section 5.

Theorem 2. For any group $G$, every idempotent in the integral group ring $\mathbb{Z}[G]$ is trivial.

Proof. The integral group ring embeds into the reduced complex group $C^*$-algebra $C^*_r G$, and the map $\mathbb{Z}[G] \rightarrow \mathbb{Z}$, $\sum a_g g \mapsto a_e$ extends to a positive faithful trace $tr: C^*_r G \rightarrow \mathbb{C}$. Let $p \in \mathbb{Z}[G]$ be an idempotent, i.e., $p = p^2$. It is known that in the $C^*$-algebra $C^*_r G$ every idempotent is similar to a projection, i.e., there exist $q, u \in C^*_r G$ such that $q = q^2 = q^*$, $u$ is invertible, and $p = u^{-1}q u$; see for example [CMR09] Proposition 1.8, Lemma 1.18]. Therefore $tr(p) = tr(q)$. Applying the
trace to $1 = q + (1 - q) = q^*q + (1 - q)^*(1 - q)$ and using positivity one sees that the trace of $q$ lies in $[0, 1]$. The trace of $p$ is clearly an integer. Therefore $\text{tr}(q) = 0$ or $\text{tr}(q) = 1$. By faithfulness of the trace this implies that $q = 0$ or $q = 1$, and then the same holds for $p = u^{-1}qu$.

The module $Rp$ for an idempotent $p = p^2$ in the ring $R$ is an example of a finitely generated projective left $R$-module. In view of the conjecture and the result above it seems natural to ask whether all finitely generated projective modules over group rings of torsion-free groups are necessarily free. Again, the assumption that $G$ is torsion-free is necessary: if $g \in G$ is an element of finite order $n$, then for the non-trivial idempotent $p = \frac{1}{n} \sum_{i=0}^{n-1} g^i \in \mathbb{Q}[G]$ the module $\mathbb{Q}[G]p$ is projective but not free.

**Examples 3.**

(i) Over fields and over principal ideal domains, hence in particular over the polynomial and Laurent polynomial rings $k[t]$ and $k[t^{\pm 1}]$ with coefficients in a field $k$, all projective modules are free.

(ii) The question whether finitely generated projective modules over the polynomial ring $k[t_1, \ldots, t_n]$ for $n \geq 2$ are necessarily free was raised by Jean-Pierre Serre in [Ser55], and was answered affirmatively only 21 years later independently by Dan Quillen and Andrei Suslin. The wonderful book [Lam06] gives a detailed account of this exciting story.

The polynomial ring $R[t_1, \ldots, t_n]$ is the monoid algebra $R[A]$ of the free abelian monoid $A$ generated by $\{t_1, \ldots, t_n\}$. The statement in (ii) was generalized as follows to monoid algebras.

(iii) If $R$ is a principal ideal domain, then every finitely generated projective module over the monoid algebra $R[A]$ is free provided that $A$ is a semi-normal, abelian, cancellative monoid without nontrivial units [Gub88, Swa92]. Free abelian groups are examples of monoids satisfying these conditions.

(iv) If $R$ is a principal ideal domain and $F$ a finitely generated free group, then every finitely generated projective module over the group ring $R[F]$ is free [Bas64].

At this point one could over-optimistically conjecture that every finitely generated projective $\mathbb{Q}[G]$-module is free if $G$ is a torsion-free group. However:

(v) Martin Dunwoody constructed in [Dun72] a torsion-free group $G$ and a finitely generated projective $\mathbb{Z}[G]$-module $P$ which is not free but has the property that $P \oplus \mathbb{Z}[G] \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G]$. There are also finitely generated projective modules over $\mathbb{Q}[G]$ with analogous properties.

A weakening of the question above is whether all finitely generated projective $R[G]$-modules are induced from finitely generated projective $R$-modules when $G$ is torsion-free. Recall that $K_0(R)$ is defined as the group completion of the monoid of isomorphism classes of finitely generated projective $R$-modules. The surjectivity of the natural map

$$K_0(R) \longrightarrow K_0(R[G])$$

induced by $[M] \mapsto [R[G] \otimes_R M]$ studies the stable version of this question: is every finitely generated projective $R[G]$-module $P$ stably induced? I.e., is there an $n \geq 0$ such that $P \oplus R[G]^n$ is induced from a finitely generated projective $R$-module? Notice that this is true for Dunwoody’s example (v) above. The stable version of
Serre’s Conjecture \[\text{[ii]}\] above is a lot easier to prove and was established much earlier in \[\text{[Ser58, Proposition 10]}\].

This discussion leads to the following conjecture. In order to formulate it, we need to recall some notions from the theory of rings. A ring \( R \) is called left Noetherian if submodules of finitely generated left modules are always finitely generated, and it is said to have finite left global dimension if every left module has a projective resolution of finite length. If both properties hold, then \( R \) is called left regular. In the sequel we only consider left modules and therefore simply say regular instead of left regular. The ring of integers \( \mathbb{Z} \), all PIDs, and all fields are examples of regular rings.

**Conjecture 4.** Let \( R \) be a regular ring, and assume that the orders of all finite subgroups of \( G \) are invertible in \( R \). Then the map

\[
\colim_{H \in \text{obj} \mathbf{SubG}(\text{Fin})} K_0(R[H]) \overset{\cong}{\longrightarrow} K_0(R[G])
\]

is an isomorphism. In particular, if \( G \) is torsion-free, then for any regular ring \( R \) there is an isomorphism

\[
K_0(R) \overset{\cong}{\longrightarrow} K_0(R[G]).
\]

Here the colimit is taken over the finite subgroup category \( \mathbf{SubG}(\text{Fin}) \), whose objects are the finite subgroups \( H \) of \( G \) and whose morphisms are defined as follows. Given finite subgroups \( H \) and \( K \) of \( G \), let \( \text{conhom}_G(H, K) \) be the set of all group homomorphisms \( H \to K \) given by conjugation by an element of \( G \). The group \( \text{inn}(K) \) of inner automorphisms of \( K \) acts on \( \text{conhom}_G(H, K) \) by postcomposition. The set of morphisms in \( \mathbf{SubG}(\text{Fin}) \) from \( H \) to \( K \) is then defined as the quotient \( \text{conhom}_G(H, K)/\text{inn}(K) \). Since inner conjugation induces the identity on \( K_0(R[-]) \), this is indeed a well defined functor on \( \mathbf{SubG}(\text{Fin}) \). In the special case when \( G \) is abelian, the category \( \mathbf{SubG}(\text{Fin}) \) is just the poset of finite subgroups of \( G \) ordered by inclusion.

**Proposition 5.** Conjecture \[\text{[4]}\] implies Conjecture \[\text{[7]}\]

**Proof.** Let \( k \) be a field of characteristic zero and let \( G \) be a torsion-free group. Let \( \epsilon : k[G] \longrightarrow k \) denote the augmentation and write \( \epsilon_*M = k \otimes_{k[G]} M \). If \( p \in k[G] \) is an idempotent, then

\[
k[G] \cong k[G]p \oplus k[G](1 - p) \quad \text{and} \quad k \cong \epsilon_*k[G] \cong \epsilon_*k[G]p \oplus \epsilon_*k[G](1 - p).
\]

Since \( k \) is a field, either \( \epsilon_*k[G]p \) or \( \epsilon_*k[G](1 - p) \) is the zero module. Replacing \( p \) by \( 1 - p \) if necessary, let us assume that \( \epsilon_*k[G]p \) is zero. The assumption \( \mathbb{Z} \cong K_0(k) \cong K_0(k[G]) \) implies that there exist \( n \) and \( m \) such that

\[
k[G]p \oplus k[G]^n \cong k[G]^m.
\]

Applying \( \epsilon_* \) we see that \( n = m \), and from this we conclude that \( k[G]p \) is zero as follows. Recall that a ring \( R \) is called stably finite if \( M \oplus R^n \cong R^n \) always implies that \( M \) is zero; see \[\text{[Lam99, Section 1B]}\]. Kaplansky showed that, if \( k \) is a field of characteristic 0, then any group ring \( k[G] \) is stably finite; compare \[\text{[Mon69]}\].
2.2. h-Cobordisms. Recall that a smooth cobordism over a closed \( n \)-dimensional smooth manifold \( M \) consists of another closed \( n \)-dimensional smooth manifold \( N \) and an \((n+1)\)-dimensional compact smooth manifold \( W \) with boundary \( \partial W \) together with a diffeomorphism \((f,g) : M \amalg N \to \partial W \). This is called an \( h \)-cobordism if both \( \text{incl} \circ f \) and \( \text{incl} \circ g \) are homotopy equivalences, where \( \text{incl} \) denotes the inclusion of \( \partial W \) in \( W \). Two cobordisms \( W \) and \( W' \) over \( M \) are called isomorphic if there exists a diffeomorphism \( F : W \to W' \) such that \( F_{|\partial W} \circ f = f' \). A cobordism over \( M \) is called trivial if it is isomorphic to the cylinder \( M \times [0,1] \) (and this in particular implies that \( M \) and \( N \) are diffeomorphic).

**Conjecture 6** (trivial \( h \)-cobordisms). Let \( M \) be a closed, connected, smooth manifold of dimension at least \( 5 \) and with torsion-free fundamental group. Then every \( h \)-cobordism over \( M \) is trivial.

Surprisingly, this conjecture can be reinterpreted in terms of algebraic \( K \)-theory. In fact, the celebrated \( s \)-Cobordism Theorem of Stephen Smale, Barry Mazur, John Stallings, and Dennis Barden (e.g., see [Mil65], [KL05]), states that there is a bijection
\[
\{ \text{\( h \)-cobordisms over \( M \}) \}/\text{iso} \cong \text{Wh}(\pi_1(M))
\]
between the set of isomorphism classes of smooth cobordisms over \( M \) and the Whitehead group \( \text{Wh}(\pi_1(M)) \) of the fundamental group of \( M \), whose definition we now review.

Recall that, given a ring \( R \), invertible matrices with coefficients in \( R \) represent classes in \( K_1(R) \). Given any group \( G \), the elements \( \pm g \in \mathbb{Z}[G] \) are invertible for any \( g \in G \), and hence represent elements in \( K_1(\mathbb{Z}[G]) \). By definition the Whitehead group \( \text{Wh}(G) \) is the quotient of \( K_1(\mathbb{Z}[G]) \) by the image of the map that sends \( (\pm 1, g) \) to the element represented by \( \pm g \) in \( K_1(\mathbb{Z}[G]) \). This map factors over \( \{ \pm 1 \} \oplus G_{ab} \), where \( G_{ab} \) is the abelianization of \( G \), and the induced map \( \{ \pm 1 \} \oplus G_{ab} \to K_1(\mathbb{Z}[G]) \) is in fact injective; see for example [LR05, Lemma 2]. So there is a short exact sequence
\[
0 \to \{ \pm 1 \} \oplus G_{ab} \to K_1(\mathbb{Z}[G]) \to \text{Wh}(G) \to 0.
\]

For Whitehead groups there is the following well-known folklore conjecture. The cases of the infinite cyclic group [Hig40], of finitely generated free abelian groups [BHS64], and of finitely generated free groups [Sta65] provided early evidence for this conjecture.

**Conjecture 8.** If \( G \) is a torsion-free group, then \( \text{Wh}(G) = 0 \).

By the \( s \)-Cobordism Theorem recalled above, the connection between the last two conjectures is as follows.

**Proposition 9.** Let \( M \) be a closed, connected, smooth manifold of dimension at least \( 5 \) and with torsion-free fundamental group. Then Conjecture 6 for \( M \) is equivalent to Conjecture 8 for \( G = \pi_1(M) \).

For groups with torsion, the situation is much more complicated. For example, if \( C_n \) is a finite cyclic group of order \( n \not\in \{1,2,3,4,6\} \), then \( \text{Wh}(C_n) \neq 0 \), and in fact even \( \text{Wh}(C_n) \otimes \mathbb{Z} \mathbb{Q} \neq 0 \). The analog of Conjecture 8 for arbitrary groups is the following.
Conjecture 10. For any group $G$ the map
\[
\colim_{H \in \text{obj} \text{Sub}_G(F_{\text{fin}})} \text{Wh}(H) \otimes \mathbb{Q} \longrightarrow \text{Wh}(G) \otimes \mathbb{Q}
\]
is injective.

We highlight two differences with the corresponding Conjecture 4 for $K_0$. First, Conjecture 10 is only a rational statement, i.e., after applying $- \otimes \mathbb{Q}$. Second, it is only an injectivity statement. In order to obtain a rational isomorphism conjecture for $\text{Wh}(G)$ one needs to enlarge the source of the map (11). This requires some additional explanations and is postponed to Conjecture 24 below.

2.3. Assembly maps. The Farrell-Jones Conjecture, which we formulate in the next subsection, generalizes Conjectures 4, 8, and 10 from statements about the abelian groups $K_0$ and $\text{Wh}$ to statements about the non-connective algebraic $K$-theory spectra $K(R[G])$ of group rings, for arbitrary coefficient rings and arbitrary groups. In order to formulate the Farrell-Jones Conjecture, we need to first introduce the fundamental concept of assembly maps.

Fixing a ring $R$, algebraic $K$-theory defines a functor $K(R[-])$ from groups to spectra. In fact, it is very easy to promote this to a functor $K(R[-]): \text{Groupoids} \longrightarrow \text{Sp}$ from the category of small groupoids (i.e., small categories whose morphisms are all isomorphisms) to the category of spectra. Moreover, this functor preserves equivalences, in the sense that it sends equivalences of groupoids to $\pi_*$-isomorphisms (i.e., weak equivalences) of spectra. For any such functor we now proceed to construct assembly maps, following the approach of [DL98]. It is not enough to work in the stable homotopy category of spectra, but any point-set level model would work.

Let $T: \text{Groupoids} \longrightarrow \text{Sp}$ be a functor that preserves equivalences. Given a group $G$, consider the functor $G[-]: \text{Sets}^G \longrightarrow \text{Groupoids}$ that sends a $G$-set $S$ to its action groupoid $G[S]$, with $\text{obj} G[S] = S$ and $\text{mor}_{G[S]}(s,s') = \{ g \in G \mid gs = s' \}$. Restricting to the orbit category $\text{Or} G$, i.e., the full subcategory of $\text{Sets}^G$ with objects $G/H$ as $H$ varies among the subgroups of $G$, we obtain the horizontal composition in the following diagram.

Now we take the left Kan extension [Mac71, Section X.3] of $T(G[-])$ along the full and faithful inclusion functor $i: \text{Or} G \hookrightarrow \text{Top}^G$ of $\text{Or} G$ into the category of all $G$-spaces. The left Kan extension evaluated at a $G$-space $X$ can be constructed as the coend [Mac71, Sections IX.6 and X.4]

\[ (\text{Lan}_G T(G[-]))(X) = X_+ \wedge_{\text{Or} G} T(G[-]) \]

of the functor

\[ (\text{Or} G)^{\text{op}} \times \text{Or} G \longrightarrow \text{Sp}, \]

\[ (G/H, G/K) \mapsto \text{map}(G/H, X^G_+ \wedge T(G[f(G/K)]) \cong X^H_+ \wedge T(G[f(G/K))]. \]
There are natural isomorphisms $G/H \wedge_{\text{OrG}} T(Gf-) \cong T(G/(G/H))$, and the fact that $T$ preserves equivalences implies that these spectra are $\pi_*\text{-isomorphic to } T(H)$. Notice that for $pt = G/G$ we even have an isomorphism $pt \wedge_{\text{OrG}} T(Gf-) \cong T(G)$.

To define the assembly map we apply this construction to the following $G$-spaces. Consider a family $\mathcal{F}$ of subgroups of $G$ (i.e., a collection of subgroups closed under passage to subgroups and conjugates) and consider a universal $G$-space $EG(\mathcal{F})$. This is a $G$-CW complex characterized up to $G$-homotopy equivalence by the property that, for any subgroup $H \leq G$, the $H$-fixed point space
\[(EG(\mathcal{F}))^H = \begin{cases} \text{empty} & \text{if } H \not\in \mathcal{F}; \\ \text{contractible} & \text{if } H \in \mathcal{F}. \end{cases}\]

The assembly map is by definition the map
\[\text{asbl}_\mathcal{F}: EG(\mathcal{F})_+ \wedge_{\text{OrG}} T(Gf-) \to T(G)\]
induced by the projection $EG(\mathcal{F}) \to pt$ (where, in the target, we use the isomorphism $pt \wedge_{\text{OrG}} T(Gf-) \cong T(G)$).

**Remark 12.**

(i) In the special case of the trivial family $\mathcal{F} = 1$, a universal space $EG(1)$ is by definition a free and non-equivariantly contractible $G$-CW complex, i.e., the universal cover of a classifying space $BG$. In this case, there is an identification $EG(1)_+ \wedge_{\text{OrG}} T(Gf-) \cong BG_+ \wedge T(1)$ and therefore we obtain the so-called classical assembly map
\[\text{asbl}_1: BG_+ \wedge T(1) \to T(G).\]

(ii) Any $G$-CW complex whose isotropy groups all lie in the family $\mathcal{F}$ has a map to $EG(\mathcal{F})$, and this map is unique up to $G$-homotopy. This applies in particular to $EG(\mathcal{F}')$ when $\mathcal{F}' \subseteq \mathcal{F}$, and we refer to the induced map
\[\text{asbl}_{\mathcal{F}' \subseteq \mathcal{F}}: EG(\mathcal{F}')_+ \wedge_{\text{OrG}} K(R[Gf-]) \to EG(\mathcal{F})_+ \wedge_{\text{OrG}} K(R[Gf-])\]
as the relative assembly map.

(iii) The source of the assembly map is a model for
\[\text{hocolim}_{G/H \in \text{obj OrG}} T(Gf/(G/H)) \]
the homotopy colimit of the restriction of $T(Gf-)$ to the full subcategory of $\text{OrG}$ of objects $G/H$ with $H \in \mathcal{F}$; compare [DL98 Section 5.2].

(iv) Taking the homotopy groups of $X_+ \wedge_{\text{OrG}} T(Gf-)$ defines a $G$-equivariant homology theory for $G$-CW complexes $X$. This is an equivariant generalization of the well-known statement that $\pi_*(X_+ \wedge E)$ gives a non-equivariant homology theory for any spectrum $E$. The Atiyah-Hirzebruch spectral sequence converging to $\pi_{*+t}(X_+ \wedge E)$ with $E^2_{s,t} = H_s(X; \pi_t E)$ also generalizes to a spectral sequence converging to $\pi_{*+t}(X_+ \wedge_{\text{OrG}} T(Gf-))$ with
\[E^2_{s,t} = H^G_s(X; \pi_t T(Gf-)),\]
the Bredon homology of $X$ with coefficients in $\pi_t T(Gf-): \text{OrG} \to \text{Ab}$; compare [DL98 Theorem 4.7]. Using this we see that, if $\text{asbl}_\mathcal{F}$ is a $\pi_*$-isomorphism, then in general all $\pi_t T(H)$ with $H \in \mathcal{F}$ and $-\infty < t \leq n$ contribute to $\pi_n T(G)$. 

We conclude with a historical comment. The classical assembly map asbl_1 from Remark [4][4] for algebraic K-theory was originally introduced in Jean-Louis Loday’s thesis [Lod76, Chapitre IV] using pairings in algebraic K-theory and the multiplication map

$$G \times GL(R) \to GL(R[G]).$$

Friedhelm Waldhausen [Wal78a, Section 15] characterized this map as a universal approximation by a homology theory evaluated on a classifying space. This point of view was nicely explained by Michael Weiss and Bruce Williams in [WW95]. In their original work [FJ93a], Farrell and Jones used the language developed by Frank Quinn [Qui82, Appendix]. Later, Jim Davis and Wolfgang Lück [DL98] gave an equivariant version of the point of view of [WW95], clarifying and unifying the underlying principles. Their approach leads to the concise description of the assembly map given above. The different approaches are compared and shown to agree in [HP04].

2.4. The Farrell-Jones Conjecture. We begin by formulating the Farrell-Jones Conjecture in the special case of torsion-free groups and regular rings.

Farrell-Jones Conjecture 13 (special case). For any torsion-free group G and for any regular ring R the classical assembly map

$$\text{asbl}_1 : BG_+ \wedge K(R) \to K(R[G])$$

is a $\pi_*$-isomorphism.

On $\pi_0$ the classical assembly map produces the map $K_0(R) \to K_0(R[G])$ induced by the inclusion $R \to R[G]$. So we see that the Farrell-Jones Conjecture [13] implies the torsion-free case of Conjecture [4].

On $\pi_1$, in the special case when $R = \mathbb{Z}$, we have

$$\pi_1(BG_+ \wedge K(\mathbb{Z})) \cong H_0(BG; K_1(\mathbb{Z})) \oplus H_1(BG; K_0(\mathbb{Z})) \cong \{\pm 1\} \oplus G_{ab}.$$  

The first isomorphism comes from the Atiyah-Hirzebruch spectral sequence, which is concentrated in the first quadrant because regular rings have vanishing negative $K$-theory. The second isomorphism comes from the computations $K_1(\mathbb{Z}) \cong \{\pm 1\}$ and $K_0(\mathbb{Z}) \cong \mathbb{Z}$. Under the isomorphism (14), it can be shown [Wal78a, Assertion 15.8] that the classical assembly map produces on $\pi_1$ the left-hand map in (7). whose cokernel is by definition the Whitehead group $Wh(G)$. So we see that the Farrell-Jones Conjecture [13] implies Conjecture [8].

From these identifications and computations of $K_0(\mathbb{Z}[G])$ and $Wh(G)$ for finite groups we see that $\pi_0(\text{asbl}_1)$ and $\pi_1(\text{asbl}_1)$ may not be surjective for groups with torsion, even when $R = \mathbb{Z}$. The classical assembly map may also fail to be injective on homotopy groups if we drop the assumption torsion-free. This happens for example for $\pi_2(\text{asbl}_1)$ if $R = \mathbb{F}$ is a finite field of characteristic prime to 2 and $G$ is the non-cyclic group with 4 elements [UW17].

The regularity assumption cannot be dropped either. For example, consider the case when $G = C_\infty$ is the infinite cyclic group. Then of course $BC_\infty = S^1$ and $R[C_\infty] = R[t, t^{-1}]$, and it can be shown that on $\pi_n$ the classical assembly map produces the left-hand map in the short exact sequence

$$0 \to K_n(R) \otimes K_{n-1}(R) \to K_n(R[t, t^{-1}]) \to NK_n(R) \oplus NK_{n+1}(R) \to 0$$

given by the Fundamental Theorem of algebraic $K$-theory; see for example [BHS64, in low dimensions, Swa95, Section 10], and [Wal78a, Theorem 18.1]. Recall that the
groups $NK_n(R)$ are defined as the cokernel of the split injection $K_n(R) \to K_n(R[t])$ induced by the natural map $R \to R[t]$. It is known that $NK_n(R) = 0$ for each $n$ if $R$ is regular [Swa95, Theorem 10.1(1) and 10.3], but $NK_n(R)$ can be nontrivial for arbitrary rings. So we see that the classical assembly map for the infinite cyclic group is a $\pi_*$-isomorphism if the ring $R$ is regular, but otherwise it may fail to be surjective on homotopy groups.

For arbitrary groups and rings, the generalization of Conjecture 13 is the following.

Farrell-Jones Conjecture 15. For any group $G$ and for any ring $R$ the Farrell-Jones assembly map

$$\text{asbl}_{\text{VCyc}}: EG(\text{VCyc})_+ \wedge_{OrG} K(R[G^{-}]) \to K(R[G])$$

is a $\pi_*$-isomorphism.

Here $\text{VCyc}$ denotes the family of virtually cyclic subgroups of $G$. A group is called virtually cyclic if it contains a cyclic subgroup of finite index.

The Farrell-Jones Conjectures 13 and 15 are related as follows.

Proposition 16. If $G$ is a torsion-free group and $R$ is a regular ring, then the Farrell-Jones Conjectures 13 and 15 are equivalent.

Proof. This is an application of the following principle, which is proved in [LR05, Theorem 65].

Transitivity Principle 17. Let $\mathcal{F}$ and $\mathcal{F}'$ be families of subgroups of $G$ with $\mathcal{F} \subseteq \mathcal{F}'$. Assume that for each $H \in \mathcal{F}'$ the assembly map

$$EH(\mathcal{F}|H)_+ \wedge_{OrH} K(R[H^{-}]) \to K(R[H])$$

is a $\pi_*$-isomorphism, where $\mathcal{F}|H = \{ K \leq H \mid K \in \mathcal{F} \}$. Then the relative assembly map explained in Remark 12(ii), i.e., the left vertical map in the following commutative triangle, is a $\pi_*$-isomorphism.

$$EG(\mathcal{F})_+ \wedge_{OrG} K(R[G^{-}]) \xrightarrow{\text{asbl}_\mathcal{F}} K(R[G])$$

Therefore, asbl$_F$ is a $\pi_*$-isomorphism if and only if asbl$_{F'}$ is a $\pi_*$-isomorphism.

We now apply the transitivity principle in the case $\mathcal{F} = 1$ and $\mathcal{F}' = \text{VCyc}$. Any nontrivial torsion-free virtually cyclic group is infinite cyclic. Recall that asbl$_1$ can be identified with the classical assembly map in Conjecture 13. So it is enough to show that the classical assembly map is a $\pi_*$-isomorphism for the infinite cyclic group $C$. The fact that this is true in the case of regular rings is explained above, before the statement of Conjecture 15.

The next result shows, as promised, that the Farrell-Jones Conjecture implies all the other conjectures introduced in the first two subsections; the case of Conjecture 10 is considered right after Conjecture 24 below.

Proposition 18. The Farrell-Jones Conjecture 15 implies Conjectures 4 and 8, and so also Conjectures 2 and 6 by Propositions 5 and 9.
Theorem 19

This gives the isomorphism

then follows by standard properties of colimits.

The case of Conjecture 8 and the torsionfree case of Conjecture 4 is explained in the source of the Farrell-Jones assembly map.

Proof. The case of Conjecture 8 and the torsionfree case of Conjecture 4 is explained above, directly after the statement of Conjecture 13. The general case of Conjecture 4 follows from the following isomorphisms.

\[
\pi_0 \left( EG(\mathbb{VCyc}) + \wedge_{\text{Or}_G} K(R[G[-]]) \right) \\
\cong \pi_0 \left( EG(\mathbb{Fin}) + \wedge_{\text{Or}_G} K(R[G[-]]) \right) \\
\cong \pi_0 \left( EG(\mathbb{Fin}) + \wedge_{\text{Or}_G(\mathbb{Fin})} K(R[G[-]]) \right) \\
\cong H_0 \left( C(EG(\mathbb{Fin})) \otimes_{\mathbb{Z} \text{Or}_G(\mathbb{Fin})} K_0(R[G[-]]) \right) \\
\cong \mathbb{Z} \otimes_{\text{Or}_G(\mathbb{Fin})} K_0(R[G[-]]) \\
\cong \text{colim}_{\text{Sub}_G(\mathbb{Fin})} K_0(R[G[-]]) \\
\cong \text{colim}_{\text{Sub}_G(\mathbb{Fin})} K_0(R[-])
\]

Theorem 14(ii) yields the isomorphism 1. Since \( EG(\mathbb{Fin})^H = \emptyset \) if \( H \) is not finite, the isomorphism 2 follows immediately by inspecting the construction of the coend. The assumptions that \( R \) is regular and that the order of every finite subgroup \( H \) of \( G \) is invertible in \( R \) imply that also \( R[H] \) is regular. For regular rings the negative \( K \)-groups vanish \[Ros94, 3.3.1\], and therefore the equivariant Atiyah-Hirzebruch spectral sequence explained in Remark 12(iv) is concentrated in the first quadrant. This gives the isomorphism 3. The singular or cellular chain complex \( C(EG(\mathbb{Fin})) \), considered as a contravariant functor \( G/H \mapsto C(EG(\mathbb{Fin})^H) \), resolves the constant functor \( \mathbb{Z} \), therefore 4 follows from right exactness of \( - \otimes_{\mathbb{Z} \text{Or}_G(\mathbb{Fin})} M \) for any fixed \( M : \text{Or}_G(\mathbb{Fin}) \rightarrow \text{Ab} \). The coend with the constant functor \( \mathbb{Z} \) is one possible construction of the colimit in abelian groups, hence 5. Since \( K_n(R[G/G[H]]) \cong K_n(R[H]) \) and since inner automorphisms induce the identity on \( K \)-theory, the functor \( K_n(R[G/-]) \) factors over \( \text{Or}_G(\mathbb{Fin}) \rightarrow \text{Sub}_G(\mathbb{Fin}) \), the functor sending \( G/H \rightarrow G/K, gH \mapsto gaH \) to the class of \( H \rightarrow K, h \mapsto a^{-1}ha \). The isomorphism 6 then follows by standard properties of colimits. \( \square \)

The next result deals with the passage from finite to virtually cyclic subgroups in the source of the Farrell-Jones assembly map.

Theorem 19 (finite to virtually cyclic).

(i) The relative assembly map

\[
\text{asbl}_{\mathbb{Fin} \subseteq \text{VCyc}} : EG(\mathbb{Fin}) + \wedge_{\text{Or}_G} K(R[G[-]]) \rightarrow EG(\text{VCyc}) + \wedge_{\text{Or}_G} K(R[G[-])
\]

is always split injective.

(ii) If \( R \) is regular and the order of every finite subgroup of \( G \) is invertible in \( R \), then \( \text{asbl}_{\mathbb{Fin} \subseteq \text{VCyc}} \) is a \( \pi_* \)-isomorphism.

(iii) If \( R \) is regular then \( \text{asbl}_{\mathbb{Fin} \subseteq \text{VCyc}} \) is a \( \pi_*^Q \)-isomorphism, i.e., it induces isomorphisms on \( \pi_n(-) \otimes_{\mathbb{Z}} \mathbb{Q} \) for all \( n \in \mathbb{Z} \).
Proof. Part (i) is the main result of [Bar03a]. Part (ii) is shown in [LR05, Proposition 70]. Part (iii) is proved in [LS16, Theorem 0.2] and generalizes [Gru08, Corollary on page 165]. □

2.5. Rational computations. After tensoring with the rational numbers, the Farrell-Jones Conjecture [15] for regular rings can be reformulated in a more concrete and computational fashion as follows.

Assume that $R$ is a regular ring. Recall from Theorem [19(ii)] that the relative assembly map $\text{asbl}_{\mathcal{F}in \subseteq \mathcal{VCyc}}$ induces isomorphisms

$$\pi_n\left(\mathcal{E}G(\mathcal{F}in)_+ \wedge_{\mathcal{O}_{\mathcal{G}}} K(R[Gf[-]])\right) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_n\left(\mathcal{E}G(\mathcal{VCyc})_+ \wedge_{\mathcal{O}_{\mathcal{G}}} K(R[Gf[-]])\right) \otimes \mathbb{Q}.$$

The theory of equivariant Chern characters developed by Lück in [Lüc02] yields the following isomorphisms:

$$\bigoplus_{(C) \in (\mathcal{FCyc})} \bigoplus_{s+t=n} H_s(BZ; C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C\left(K_t(R[C]) \otimes \mathbb{Q}\right) \xrightarrow{\cong} \pi_n\left(\mathcal{E}G(\mathcal{F}yc)_+ \wedge_{\mathcal{O}_{\mathcal{G}}} K(R[Gf[-]])\right) \otimes \mathbb{Q}.$$

Before we explain the notation, notice the analogy with the well-known isomorphism

$$\bigoplus_{s+t=n} H_s(BG; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C\left(K_t(R[C]) \otimes \mathbb{Q}\right) \xrightarrow{\cong} \pi_n(BG_+ \wedge K(R)) \otimes \mathbb{Q},$$

whose source corresponds to the summand in (21) indexed by $C = 1$.

Given a subgroup $H$ of $G$, we denote by $N_G H$ the normalizer and by $Z_G H$ the centralizer of $H$ in $G$, and we define the Weyl group as the quotient $W_G H = N_G H / (Z_G H : H)$. Notice that the Weyl group $W_G H$ of a finite subgroup $H$ is always finite, since it embeds into the outer automorphism group of $H$. We write $\mathcal{FCyc}$ for the family of finite cyclic subgroups of $G$, and $(\mathcal{FCyc})$ for the set of conjugacy classes of finite cyclic subgroups. Furthermore, $\Theta_C$ is an idempotent endomorphism of $K_t(R[C]) \otimes \mathbb{Q}$, which corresponds to a specific idempotent in the rationalized Burnside ring of $C$, and whose image is a direct summand of $K_t(R[C]) \otimes \mathbb{Q}$ isomorphic to

$$\text{coker}\left(\bigoplus_{D \subseteq C} \text{ind}_D : \bigoplus_{D \subseteq C} K_t(R[D]) \otimes \mathbb{Q} \rightarrow K_t(R[C]) \otimes \mathbb{Q}\right).$$

The Weyl group acts via conjugation on $C$ and hence on $\Theta_C(K_t(R[C]) \otimes \mathbb{Q})$. The Weyl group action on the homology groups in the source of (21) comes from the fact that $EN_G C/Z_G C$ is a model for $BZ_G C$. 

Farrell-Jones Conjecture 23 (rationalized version). For any group $G$ and for any regular ring $R$ the composition of the Farrell-Jones assembly map and the isomorphisms (21) and (20)

$$\bigoplus_{(C) \in (\mathcal{FCyc})} H_2(BZ_G; \mathbb{Q}) \otimes \Theta_C \left( K_1(R[C]) \otimes \mathbb{Q} \right) \to K_n(R[G]) \otimes \mathbb{Q}$$

is an isomorphism for each $n \in \mathbb{Z}$.

Analogously one obtains the following conjecture for Whitehead groups, which is the correct generalization of Conjecture 10 mentioned at the end of Subsection 2.2

Conjecture 24. For any group $G$ there is an isomorphism

$$\bigoplus_{(C) \in (\mathcal{FCyc})} \left( \mathbb{Q} \otimes \Theta_C \left( Wh(C) \otimes \mathbb{Q} \right) \oplus H_2(BZ_G; \mathbb{Q}) \otimes \Theta_C \left( K_{-1}(Z[C]) \otimes \mathbb{Q} \right) \right) \cong Wh(G) \otimes \mathbb{Q}.$$ 

Conjecture 24 implies Conjecture 10 because in fact

$$\text{colim}_{H \in \text{obj SubG}(\mathcal{FCyc})} Wh(H) \otimes \mathbb{Q} \cong \bigoplus_{(C) \in (\mathcal{FCyc})} \mathbb{Q} \otimes \Theta_C \left( Wh(C) \otimes \mathbb{Q} \right)$$

and the map (11) coincides with the restriction to this summand of the map in Conjecture 24.

Remark 25. For finite groups $H$ we have that $Wh(H) \otimes \mathbb{Q} \cong K_1(\mathbb{Z}[H]) \otimes \mathbb{Q}$ by the exact sequence (7). The only difference between the sources of the maps in Conjectures 23 and 24 is the absence from 24 of the summands with $(s, t) = (1, 0)$. For finite groups $H$ the natural map $\mathbb{Q} \cong K_0(\mathbb{Z}) \otimes \mathbb{Q} \to K_0(\mathbb{Z}[H]) \otimes \mathbb{Q}$ is an isomorphism, and hence it follows from (22) that the only non-vanishing summand among these is $H_1(BG; \mathbb{Q}) \cong G_{ab} \otimes \mathbb{Q}$ corresponding to $C = 1$. This is consistent with the exact sequence (7).

Finally, we note that in the special case when $R = \mathbb{Z}$ the dimensions of the $\mathbb{Q}$-vector spaces in (22) for any $t$ and any finite cyclic group $C$ can be explicitly computed as follows.

Theorem 26. Let $C$ be a cyclic group of order $c$. Then

$$\dim_{\mathbb{Q}} \Theta_C \left( K_t(\mathbb{Z}[C]) \otimes \mathbb{Q} \right) = \begin{cases} s(c) - 1 & \text{if } t = -1; \\ \varphi(c)/2 - 1 & \text{if } t = 1 \text{ and } c > 2; \\ 1 & \text{if } t > 1, t \equiv 1 \mod 4, \text{ and } c = 2; \\ \varphi(c)/2 & \text{if } t > 1, t \equiv 1 \mod 2, \text{ and } c > 2; \\ 0 & \text{otherwise.} \end{cases}$$

Here $\varphi(c) = \# \{ x \in C \mid x \text{ generates } C \}$ is Euler’s $\varphi$-function, $c = \prod_{i=1}^a p_i^{e_i}$ is the prime factorization of $c$, and $s(c) = \sum_{i=1}^a \varphi(n/p_i^{e_i})/f_{p_i}$, where $f_{p_i}$ is the smallest number such that $p_i^{f_{p_i}} \equiv 1 \mod n/p_i^{e_i}$.

This result is proved in [Pat14, Theorem on page 9], and more details will appear in [PRV].
2.6. Some related conjectures. We now survey very briefly some other conjectures that are analogous to Conjecture 15. For details and further explanations we recommend [FRR95], [KL05], [Lüc], [LR05], and [MV03].

In [FJ93a], Farrell and Jones formulated Conjecture 15 not only for algebraic $K$-theory, but also for $L$-theory; more precisely, for $L^{(-\infty)}(R[G])$, the quadratic algebraic $L$-theory spectrum of $R[G]$ with decoration $-\infty$, for any ring with involution $R$. The corresponding assembly map is constructed completely analogously, by applying the machinery of Subsection 2.3 to the functor $L^{(-\infty)}(R[-])$. In the special case of torsion-free groups $G$, this conjecture is equivalent to the statement that the classical assembly map $BG_+ \wedge L^{(-\infty)}(R) \to L^{(-\infty)}(R[G])$ is a $\pi_*$-isomorphism, for any ring $R$, not necessarily regular.

If $G$ is a torsion-free group and the Farrell-Jones Conjectures hold for both $K(Z[G])$ and $L^{(-\infty)}(Z[G])$, then the Borel Conjecture is true for manifolds with fundamental group $G$ and dimension at least 5. The Borel Conjecture states that, if $M$ and $N$ are closed connected aspherical manifolds with isomorphic fundamental groups, then $M$ and $N$ are homeomorphic, and every homotopy equivalence between $M$ and $N$ is homotopic to a homeomorphism. In short, the Borel Conjecture says that closed aspherical manifolds are topologically rigid. Recall that a connected CW complex $X$ is aspherical if its universal cover is contractible, or equivalently if $\pi_n(X) = 0$ for all $n > 1$.

We also mention that the Farrell-Jones Conjecture in algebraic $L$-theory implies the Novikov Conjecture about the homotopy invariance of higher signatures.

Furthermore, Farrell and Jones also formulated an analog of Conjecture 15 for the stable pseudo-isotopy functor, or equivalently for Waldhausen’s $A$-theory, also known as algebraic $K$-theory of spaces. We refer to [ELP+16] for a modern approach to this conjecture and in particular for its many applications to automorphisms of manifolds.

Finally, the analog of the Farrell-Jones Conjecture 15 for the complex topological $K$-theory of the reduced complex group $C^*$-algebra of $G$ is equivalent to the famous Baum-Connes Conjecture, formulated by Paul Baum, Alain Connes, and Nigel Higson in [BCH94]. For the Baum-Connes Conjecture, the relative assembly map $asbl_{\eta_{\infty} \subseteq \psi_{\infty}}$ is always a $\pi_*$-isomorphism; compare and contrast with Theorem 19. Also the Baum-Connes Conjecture implies the Novikov Conjecture. For more information on the relation between the Baum-Connes Conjecture and the Farrell-Jones Conjecture in $L$-theory we refer to [LN17] and [Ros95].

3. State of the art

We now overview what we know and don’t know about the Farrell-Jones Conjecture 15 to the best of our knowledge in January 2017. We aim to give immediately accessible statements, which may not always reflect the most general available results. We restrict our attention to algebraic $K$-theory and ignore the related conjectures mentioned in the previous subsection.

3.1. What we know already. The following theorem is the result of the effort of many mathematicians over a long period of time. The methods of controlled algebra and topology that underlie this theorem (and that we illustrate in the next section) were pioneered by Steve Ferry [Fer77] and Frank Quinn [Qui79], and were then applied with enormous success by Farrell-Hsiang [FH78], [FH81b], [FH83] and Farrell-Jones [PJ86], [PJ89], [PJ93b], [PJ93a]. Many ideas in the proofs of
Theorem 27. Let \( \mathcal{G} \) be the smallest class of groups that satisfies the following two conditions.

(1) The class \( \mathcal{G} \) contains:
   (a) hyperbolic groups \([\text{BLR08}]\);
   (b) finite-dimensional CAT(0)-groups \([\text{BL12a}, \text{Weg12}]\);
   (c) virtually solvable groups \([\text{FW14}, \text{Weg15}]\);
   (d) Baumslag-Solitar groups and graphs of abelian groups \([\text{FW14}, \text{GMR15}]\);
   (e) lattices in virtually connected Lie groups \([\text{BFL14}, \text{KLR16}]\);
   (f) arithmetic and S-arithmetic groups \([\text{BLRR14}, \text{Rüp16}]\);
   (g) fundamental groups of connected manifolds of dimension at most 3 \([\text{Ron08}]\);
   (h) Coxeter groups;
   (i) Artin braid groups \([\text{AFR00}]\);
   (j) mapping class groups of oriented surfaces of finite type \([\text{BB16}]\).

(2) The class \( \mathcal{G} \) is closed under:
   (A) subgroups \([\text{BR07a}]\);
   (B) overgroups of finite index \([\text{BLRR14}, \text{Section 6}]\);
   (C) finite products;
   (D) finite coproducts;
   (E) directed colimits \([\text{BCL08}]\);
   (F) graph products \([\text{GR13}]\);
   (G) if \( 1 \rightarrow N \rightarrow G \xrightarrow{p} Q \rightarrow 1 \) is a group extension such that \( Q \in \mathcal{G} \) and \( p^{-1}(C) \in \mathcal{G} \) for each infinite cyclic subgroup \( C \leq Q \), then \( G \in \mathcal{G} \);
   (H) if \( G \) is a countable group that is relatively hyperbolic to subgroups \( P_1, \ldots, P_n \) and each \( P_i \in \mathcal{G} \), then \( G \in \mathcal{G} \).[Bar17]

Then the Farrell-Jones Conjecture \([13]\) holds for any ring \( R \) and for any group \( G \in \mathcal{G} \).

Proof. In order to have the inheritance properties formulated in (2) one needs to work with a slight generalization of the Farrell-Jones Conjecture. First, one needs to allow coefficients in arbitrary additive categories with \( G \)-actions \([\text{BR07a}]\); then, one says that the conjecture with finite wreath products is true for \( G \) if the conjecture holds not only for \( G \), but also for all wreath products \( G \wr F \) of \( G \) with finite groups \( F \) \([\text{Ron07}, \text{BLRR14}, \text{Section 6}]\). The Farrell-Jones Conjecture with coefficients and finite wreath products is true for all groups listed under (1) and has all the inheritance properties listed under (2).

Some of the earlier references given above omit the discussion of the version with finite wreath products; consult \([\text{BLRR14}, \text{Section 6}]\) and \([\text{GR13}, \text{Proposition 1.1}]\) for the corresponding extensions.

We discuss the statements \([\text{h}], [\text{C}], [\text{D}], [\text{G}]\) for which no reference was provided above. Coxeter groups \([\text{h}]\) are known to fall under \([\text{b}]\) by a result of Moussong; compare \([\text{Dav08}, \text{Theorem 12.3.3}]\). For \([\text{C}]\) use \([\text{BR07a}, \text{Corollary 4.3}]\) applied to the projection to the factors, the Transitivity Principle \([17]\) and the fact that the Farrell-Jones Conjecture is known for finite products of virtually cyclic groups. The extension to the version with finite wreath products uses the fact that \( (G_1 \times G_2) \wr F \) is a subgroup of \( (G_1 \wr F) \times (G_2 \wr F) \). Finite coproducts are treated
similarly using property \([G]\) and the natural map from the coproduct to the product; compare [GR13, Proposition 1.1]. Statement \([G]\) itself is simply a combination of [BR07a, Corollary 4.3] and the Transitivity Principle [17]. □

3.2. What we don’t know yet. At the time of writing the Farrell-Jones Conjecture [15] seems to be open for the following classes of groups:

(i) Thompson’s groups;
(ii) outer automorphism groups of free groups;
(iii) linear groups;
(iv) (elementary) amenable groups;
(v) infinite products of groups (satisfying the Farrell-Jones Conjecture).

However, for some of these groups there are partial injectivity results, as we explain in Remark 29 below.

3.3. Injectivity results. The next theorem gives two examples of injectivity results for assembly maps in algebraic \(K\)-theory. Part (i) is proved using the trace methods explained in Section 5 below, where more rational injectivity results are described. Part (ii) is based on a completely different approach using controlled algebra, the descent method due to Gunnar Carlsson and Erik Pedersen [CP95]. For this method to work, the group has to satisfy some mild metric conditions, which are not needed for the weaker statement in part (i). One such condition goes back to [FH81a]. The condition of finite asymptotic dimension appeared in the context of algebraic \(K\)-theory in [Bar03b] and [CG04, CG05], and was later generalized to finite decomposition complexity in [RTY14]. The extension to non-classical assembly maps appeared in [BR07b, BR17] and [Kas15]. The statement in part (ii) below is from [KNR18] and further improves and combines these developments. We also mention [FW91] for yet another approach to injectivity results.

Recall from Theorem 19 that the relative assembly map

\[
\pi_n \left( EG(Fin)_+ \wedge_{Or_G} K(Z[Gf-]) \right) \rightarrow \pi_n \left( EG(\mathcal{VC}yc)_+ \wedge_{Or_G} K(Z[Gf-]) \right)
\]

is always split injective, and it becomes an isomorphism after applying \(- \otimes_{\mathbb{Z}} \mathbb{Q}\) if \(R\) is regular, e.g., if \(R = \mathbb{Z}\). Therefore the results below would follow if we knew the Farrell-Jones Conjecture [15].

**Theorem 28.** Assume that there exists a finite-dimensional \(EG(Fin)\), and that there exists an upper bound on the orders of the finite subgroups of \(G\).

(i) If \(R = \mathbb{Z}\), then there exists an integer \(L > 0\) such that for every \(n \geq L\) the rationalized assembly map

\[
\pi_n \left( EG(Fin)_+ \wedge_{Or_G} K(Z[Gf-]) \right) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q}
\]

is injective.

(ii) Assume furthermore that \(G\) has regular finite decomposition complexity. Then for any ring \(R\) the assembly map

\[
EG(Fin)_+ \wedge_{Or_G} K(R[Gf-]) \rightarrow K(R[G])
\]

is split injective on \(\pi_*\).
Proof. (i) is a consequence of Theorem 69 below, or rather of its more general version in [LRRV17a Main Technical Theorem 1.16]; see Remark 70(iv) and [LRRV17a, Theorem 1.15], where the result is only stated for cocompact $EG(F_{in})$, but the proof given on page 1015 only uses finite-dimensionality and the existence of a bound on the order of the finite cyclic subgroups. (ii) is [KNR18 Theorem 1.3]. □

Remark 29. Theorem 28 applies to groups for which no isomorphism results were known at the time of writing:

(i) The existence of an upper bound on the orders of the finite subgroups of $G$ follows from the existence of a cocompact $EG(F_{in})$. For example, this is the case for outer automorphism groups of free groups, to which Theorem 28(i) then applies.

(ii) Regular finite decomposition complexity is a property shared by all groups that are either (a) of finite asymptotic dimension, (b) elementary amenable, (c) linear, or (d) subgroups of virtually connected Lie groups.

4. Controlled algebra methods

As noted in the previous section, most proofs of the Farrell-Jones Conjecture use the ideas and technology of controlled algebra, which are the focus of this section. The ultimate goal is to explain the Farrell-Hsiang Criterion for assembly maps to be $\pi_*(-)$-isomorphisms. The criterion goes back to [FH78] and has been successfully applied in many cases, e.g. [FH81b, FH83, Qui12], and plays an important role in the proof of Theorem 27(1)(e) [BFL14]. The formulation that we give here in Theorem 57 is due to [BL12b].

Our goal is to keep the exposition as concrete as possible, and to work out the main details of the proof of the following result, establishing the first nontrivial case of the Farrell-Jones Conjecture.

Theorem 30. The Farrell-Jones Conjecture holds for finitely generated free abelian groups, i.e., for any $n \geq 2$ and for any ring $R$, the assembly map

$$EZ^n(Cyc)_{+} \wedge_{Or\mathbb{Z}_n} K(R[\mathbb{Z}^n/\mathbb{Z}]) \longrightarrow K(R[\mathbb{Z}^n])$$

is a $\pi_*(-)$-isomorphism.

Before we get to the proof, we want to show how Theorem 30 leads to a simple formula for the Whitehead groups of $\mathbb{Z}^n$; the article [LR14] contains many similar but way more general explicit computations. The Whitehead groups of $G$ over $R$ are defined as $Wh^R_k(G) = \pi_k(Wh^R(G))$, where $Wh^R(G)$ is the homotopy cofiber of the classical assembly map $asb_1: BG_+ \wedge K(R) \longrightarrow K(R[G])$ appearing in Conjecture. Of course, $Wh^R_1(G) = Wh(G)$.

Corollary 31. For any $n \geq 2$ and $k \in \mathbb{Z}$ there are isomorphisms

$$Wh^R_k(\mathbb{Z}^n) \cong \bigoplus_{C \in MaxCyc} Wh^R_k(C) \cong \bigoplus_{C \in MaxCyc} NK_k(R) \oplus NK_k(R),$$

where $MaxCyc$ denotes the set of maximal cyclic subgroups of $\mathbb{Z}^n$.

Observe that the set of maximal cyclic subgroups of $\mathbb{Z}^n$ can be identified with $\mathbb{P}^{n-1}(\mathbb{Q})$, the set of all 1-dimensional subspaces of the $\mathbb{Q}$-vector space $\mathbb{Q}^n$. 
Proof. There is a $\mathbb{Z}^n$-equivariant homotopy pushout square

$$\begin{array}{ccc}
\bigcap_{C \in \text{Max cyc}} \mathbb{Z}^n \times EC_C & \longrightarrow & E\mathbb{Z}^n \\
\downarrow & & \downarrow \\
\bigcap_{C \in \text{Max cyc}} \mathbb{Z}^n \times \text{pt}_C & \longrightarrow & E\mathbb{Z}^n(\text{cyc}).
\end{array}$$

Applying $(\ ? )_+ \wedge_{\Omega(\mathbb{Z}^n)} K(R[\mathbb{Z}^n f -])$ preserves homotopy pushout squares, and the induced left vertical map can be identified with a wedge sum of copies of the classical assembly map $\text{asbl}_1$ for $C$, using induction isomorphisms. The homotopy cofibration sequence

$$BC_+ \wedge K(R) \cong EC_+ \wedge_{\Omega(\mathbb{Z}^n)} K(R[C f -]) \xrightarrow{\text{asbl}_1} K(R[C]) \longrightarrow \text{Wh}^R(C)$$

is known to split, and $\text{Wh}^R_1(C) \cong NK_n(R) \oplus NK_n(R)$; compare [Swa95, Section 10] and [Wal78a, Theorem 18.1]. Therefore we obtain the following homotopy pushout square.

$$\begin{array}{ccc}
\text{pt} & \longrightarrow & E\mathbb{Z}^n_+ \wedge_{\Omega(\mathbb{Z}^n)} K(R[\mathbb{Z}^n f -]) \\
\downarrow & & \downarrow \\
\bigvee_{C \in \text{Max cyc}} \text{Wh}^R(C) & \longrightarrow & E\mathbb{Z}^n(\text{cyc})_+ \wedge_{\Omega(\mathbb{Z}^n)} K(R[\mathbb{Z}^n f -]).
\end{array}$$

Theorem [30] identifies the bottom right corner with $K(R[Z^n])$, and therefore the homotopy cofiber of the right vertical map agrees with the homotopy cofiber of the classical assembly map for $\mathbb{Z}^n$, completing the proof.

Working with the Farrell-Jones Conjecture with coefficients mentioned in the proof of Theorem [27], we can use induction and reduce the proof of Theorem [30] to the case $n = 2$, by applying the inheritance property formulated in Theorem [27][2][G] to a surjective homomorphism $\mathbb{Z}^n \longrightarrow \mathbb{Z}^2$. Notice that for $\mathbb{Z}^2$ itself Theorem [27][2][G] is useless.

However, even in the case $n = 2$ the full proof of Theorem [30] involves many technicalities that obscure the underlying ideas. For this reason, we concentrate on the following partial result.

**Proposition 32.** The assembly map

$$\pi_1 \left( E\mathbb{Z}^2(\text{cyc})_+ \wedge_{\Omega G} K(R[\mathbb{Z}^2 f -]) \right) \longrightarrow K_1(R[\mathbb{Z}^2])$$

is surjective for any ring $R$.

In the rest of this section we give a complete proof of this proposition modulo Theorem [37], which we use as a black box. The proof is completed right after the statement of Claim [50].
4.1. Geometric modules. The main characters of controlled algebra are defined next.

Definition 34 (geometric modules). Given a ring $R$ and $G$-space $X$, the category $\mathcal{C}(X) = \mathcal{C}(X; R)$ of geometric $R[G]$-modules over $X$ is defined as follows. The objects of $\mathcal{C}(X)$ are cofinite free $G$-sets $S$ together with a $G$-map $\varphi: S \to X$. Notice that, given a cofinite free $G$-set $S$, the $R$-module $R[S]$ is in a natural way a finitely generated free $R[G]$-module. The morphisms in $\mathcal{C}(X)$ from $\varphi: S \to X$ to $\varphi': S' \to X$ are simply the $R[G]$-linear maps $R[S] \to R[S']$.

The category $\mathcal{C}(X)$ is additive and depends functorially on $X$, in the sense that a $G$-map $f: X \to X'$ induces an additive functor $f_*: \mathcal{C}(X) \to \mathcal{C}(X')$ which sends the object $\varphi$ to $f \circ \varphi$. Let $\mathcal{F}(R[G])$ be the category of finitely generated free $R[G]$-modules. The functor $U: \mathcal{C}(X) \to \mathcal{F}(R[G])$ (where $U$ stands for underlying) that sends $\varphi: S \to X$ to $R[S]$ is obviously an equivalence of additive categories, since $\varphi$ does not enter the definition of the morphisms in $\mathcal{C}(X)$. Therefore we obtain a $\pi_*$-isomorphism

\begin{equation}
K(\mathcal{C}(X)) \xrightarrow{\sim} K(R[G]) .
\end{equation}

However, the advantage of $\mathcal{C}(X)$ is that morphisms have a geometric shadow in $X$, and if $X$ is equipped with a metric we can talk about their size.

Definition 36 (support and size). Let $\alpha: R[S] \to R[S']$ be a morphism in $\mathcal{C}(X)$ from $\varphi: S \to X$ to $\varphi': S' \to X$. Let $(\alpha_{s', s})_{(s', s) \in S' \times S}$ be the associated matrix. Define the support of $\alpha$ to be

$$\text{supp} \alpha = \{ (\varphi'(s'), \varphi(s)) \in X \times X \mid \alpha_{s', s} \neq 0 \} \subseteq X \times X .$$

If $X$ is equipped with a $G$-invariant metric $d$, define the size of $\alpha$ to be

$$\text{size} \alpha = \sup \{ d(\varphi'(s'), \varphi(s)) \mid \alpha_{s', s} \neq 0 \} .$$

Figure 1. Support of a morphism with $G = \mathbb{Z}$ acting via shift on a band.

Note that the supremum is really a maximum, since $\alpha$ is $G$-equivariant, $d$ is $G$-invariant, and $S$ is cofinite. As we will see, sometimes it is convenient to work with extended metrics, i.e., metrics for which $d(x, x') = \infty$ is allowed. Being of finite size is then a severe restriction on $\alpha$. In the support picture no arrow is allowed between points at distance $\infty$; compare Figure 2 on page 23.

The main idea now is that assembly maps can be described as forget control maps. Proving that an element is in the image of an assembly map can be achieved by proving that it has a representative of small size. Before making this precise we introduce some more conventions and definitions.
Recall that a point \( a \) in a simplicial complex \( Z \) can be written uniquely in the form
\[
a = \sum_{v \in V} a_v v,
\]
where \( V \) is the set of vertices of the underlying abstract simplicial complex, \( a_v \in [0, 1] \), and
\[
\sum_{v \in V} a_v = 1.
\]
The point \( a \) lies in the interior of the realization \( \Delta_v \) of the unique abstract simplex given by \( \{ v \mid a_v \neq 0 \} \). The \( l_1 \)-metric on \( Z \) is defined as
\[
d_1(a, b) = \sum_{v \in V} |a_v - b_v|.
\]
Observe that the distance between points is always \( \leq 2 \), and that every simplicial automorphism is an isometry with respect to the \( l_1 \)-metric.

**Theorem 37** (small elements are in the image). For any integer \( n > 0 \) there is an \( \varepsilon = \varepsilon(n) > 0 \) such that for every \( G \)-simplicial complex \( Z \) of dimension \( n \) the following is true. Let \( x \in K_1(R[G]) \) and consider the assembly map \( \text{asbl}_Z \) induced by \( Z \to \text{pt} \).

\[
K_1(C(Z)) \ni [\alpha] \quad \Upsilon \cong \quad \pi_1 \left( Z_+ \wedge_{\text{Or} G} K(R[G_f]) \right) \xrightarrow{\text{asbl}_Z} K_1(R[G]) \ni x
\]
Then \( x \in \text{im}(\text{asbl}_Z) \) if there exists an automorphism \( \alpha \) in \( C(Z) \) with \( \Upsilon([\alpha]) = x \) and
\[
\text{size}(\alpha) \leq \varepsilon \quad \text{and} \quad \text{size}(\alpha^{-1}) \leq \varepsilon.
\]

**Corollary 38.** Retain the notation and assumptions of Theorem 37. If all isotropy groups of \( Z \) belong to the family \( F \), then \( x \) is also in the image of the assembly map
\[
(39) \quad \pi_1 \left( EG(F) \wedge_{\text{Or} G} K(R[G_f]) \right) \xrightarrow{\text{asbl}_F} K_1(R[G]).
\]

**Proof.** The universal property of \( EG(F) \) in Remark 12(ii) gives a \( G \)-equivariant map \( Z \to EG(F) \). Hence the assembly map \( \text{asbl}_Z \), which is induced by \( Z \to \text{pt} \), factors over the assembly map \( \text{asbl}_F \), which is induced by \( EG(F) \to \text{pt} \). \( \square \)

The sufficient condition for surjectivity on \( \pi_1 \) from the preceding two results is generalized in Theorem 55 below to a necessary and sufficient condition for assembly maps to be \( \pi_* \)-isomorphisms. In Remark 56 we explain how and where in the literature Theorem 37 is proved.

**4.2. Contracting maps.** In view of Theorem 37 and Corollary 38 a possible strategy to prove surjectivity of \( \text{asbl}_F \) is to look for contracting maps. This leads to the following criterion.

**Criterion 40.** Fix \( G, R, F \), and a word metric \( d^G \) for \( G \). Suppose that there is an \( N > 0 \) such that for any arbitrarily large \( D > 0 \) there exist a simplicial complex \( Z_D \) with a simplicial \( G \)-action and a \( G \)-equivariant map \( f_D : G/1 \to Z_D \) satisfying the following conditions:
(i) \( \dim Z_D \leq N \);
(ii) all isotropy groups of \( Z_D \) lie in \( F \);
(iii) the map \( f_D \) is \( D \)-contracting with respect to the \( l^1 \)-metric in the target and the word metric in the source, i.e., for all \( g, g' \in G \) we have

\[
d^1(f_D(g), f_D(g')) \leq \frac{1}{D}d^G(g, g').
\]

Then the map (39) is surjective.

The projection map to a point always satisfies (i) and (iii) but not (ii). The \( N \)-skeleton of a simplicial model for \( EG(F) \) always satisfies (i) and (ii). But how can we produce contracting maps \( f_D \) that satisfy all three conditions? In Remark 41 below we explain why the assumptions of the criterion are too strong to be useful. Nevertheless, we spell out the proof of the criterion as a warm-up exercise.

**Proof.** Set \( \epsilon = \min \{ \epsilon(n) \mid n \leq N \} \), where \( \epsilon(n) \) comes from Theorem 37. Given any \( x \in K_1(R[G]) \) consider the following diagram.

\[
\begin{array}{ccc}
K_1(C(Z_D)) & \xrightarrow{f_D} & K_1(C(G/1)) \\
\cup \cong & \searrow \cong & \downarrow \cong \\
K_1(R[G]) & \ni x
\end{array}
\]

Choose an automorphism \( \alpha \) in \( C(G/1) \) whose class \( [\alpha] \in K_1(C(G/1)) \) maps to \( x \) under \( \cup \). Determine the sizes of \( \alpha \) and \( \alpha^{-1} \), and then choose \( D \) so large that

\[
\text{size } f_{D\ast}(\alpha) \leq \frac{1}{D} \text{size } \alpha < \epsilon
\]

and analogously for \( \alpha^{-1} \). Then Corollary 38 implies that \( x \) is in the image of the assembly map asbl in (39). \( \square \)

**Remark 41.** The case of Proposition 32 is when \( G = \mathbb{Z}^2 \) and \( F = \text{Cyc} \). Unfortunately, the conditions of Criterion 40 cannot possibly be satisfied in this case. To explain why, we need the following lemma.

**Lemma 42.** Let \( s \) be a simplicial automorphism of a simplicial complex \( Z \) with \( \dim Z \leq N \). If \( \bar{x} = \sum x_v v \in Z \) is such that

\[
d^1(x, sx) < \frac{1}{(N+1)^2},
\]

then a barycenter of a face of the simplex \( \Delta_x \) spanned by \( V_x = \{ v \mid v \neq 0 \} \) is fixed under \( s \).

**Proof.** For a vertex \( v \) with \( x_v \neq 0 \) set \( A_v = \{ v, sv, s^2v, \ldots \} \). If \( A_v \subset V_x \) then \( s \) permutes the finitely many elements in \( A_v \) and in particular fixes the barycenter of the face spanned by \( A_v \).

Suppose that for no vertex \( v \) with \( x_v \neq 0 \) we have \( A_v \subset V_x \). Then for all \( v \in V_x \) there exists a smallest \( n(v) \geq 1 \) such that \( s^{n(v)}v \notin V_x \) and hence \( x_{s^{n(v)}v} = 0 \). Since \( V_x \) contains at most \( N + 1 \) vertices we know that \( n(v) \leq N + 1 \) for all \( v \in V_x \). Write \( \epsilon = \frac{1}{(N+1)^2} \); then from

\[
d^1(x, s^{-1}x) = d^1(s^{-1}x, s^{-2}x) = \cdots = d^1(s^{-N}x, s^{-(N+1)}x) < \epsilon
\]
and $x_{s(n(v))} = 0$ we conclude that

$$x_{s(n(v)) - 1} < \epsilon, \quad x_{s(n(v)) - 2} < 2\epsilon, \quad \ldots, \quad x_{n(v)} < n(v)\epsilon \leq \frac{1}{N+1}.$$ 

However, since $\sum_{v \in V_x} x_v = 1$, the last inequality cannot be true for all vertices in $V_x$. □

Now suppose we can arrange (i) and (iii) from Criterion 40. If $S$ is a (very large) finite subset of $G$, then by (iii) there exists a $G$-equivariant map $f : G/1 \rightarrow Z$ to a $G$-simplicial complex that is contracting enough in order to have $d^1(f(1), f(s)) < \frac{1}{(N+1)^2}$ for all $s \in S$. The lemma implies that for each $s \in S$ a barycenter of a face of the simplex $\Delta_{f(1)}$ determined by $f(1)$ is fixed under $s$. Let $b(N)$ be the number of vertices in the barycentric subdivision of an $N$-simplex. Then there exists a subset $T \subset S$ with cardinality $\#T \geq \frac{b(N)}{l}^2$ and a point in $\Delta_{f(1)}$ fixed by all elements of $T$. The subgroup generated by $T$ must lie in $F$ if we require (ii). Since $S$ can be arbitrarily large it seems difficult to keep $F$ small.

In the case $G = \mathbb{Z}^2$ we can choose

$$S = S_l = \left\{ (x_1, x_2) \mid |x_1| \leq \frac{l}{2} \text{ and } |x_2| \leq \frac{l}{2} \right\}.$$ 

Then if $l > b(N)$ we have $\#T \geq \frac{b^2(N)}{l^2} > l$, and since a subset of $S_l$ with more than $l$ elements cannot be contained in a line, the set $T$ generates a finite index subgroup. Hence we can never arrange $F = C_{yc}$ as desired, proving the claim in Remark 41.

4.3. The Farrell-Hsiang Criterion. The trick to obtain sufficiently contracting maps is to relax the requirement that the maps are $G$-equivariant, and instead only ask for equivariance with respect to (finite index) subgroups. We first illustrate this phenomenon in an example that is too simple to be useful.

**Example 43.** Consider the standard shift action of the infinite cyclic group $G = \mathbb{Z}$ on the real line:

$$\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (z, x) \mapsto z + x.$$ 

This is a simplicial action if we consider $\mathbb{R}$ as 1-dimensional simplicial complex with set of vertices $\mathbb{Z} \subset \mathbb{R}$. The map

$$f_D : \mathbb{Z} \rightarrow \mathbb{R}, \quad z \mapsto \frac{1}{D} z$$ 

is $D$-contracting but not $\mathbb{Z}$-equivariant. It becomes $\mathbb{Z}$-equivariant if we change the action on $\mathbb{R}$ to the action given by

$$\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (z, x) \mapsto \frac{1}{D} z + x.$$ 

However, this action is no longer simplicial. If we restrict the action to the subgroup $D\mathbb{Z} \subset \mathbb{Z}$ or to any subgroup $H$ with $H \leq D\mathbb{Z}$, then the $H$-action on $\text{res}_H^\mathbb{R} \mathbb{R}$ is simplicial, and

$$f_D : \text{res}_H^\mathbb{R} \mathbb{Z} \rightarrow \text{res}_H^\mathbb{R} \mathbb{R}$$ 

is a $D$-contracting $H$-equivariant map.

Assume for a moment that for a subgroup $H \leq G$ of finite index we have an $H$-equivariant map

$$f_D : \text{res}_H^G G/1 \rightarrow E_H$$
to an \( H \)-simplicial complex \( E_H \) that is \( D \)-contracting with respect to a word metric in the source and the \( l^1 \)-metric in the target. Let us see what happens when we induce up to \( G \).

If \((X,d)\) is a metric space with an isometric \( H \)-action, then \( \text{ind}_H^G X = G \times_H X \) has an isometric \( G \)-action with respect to the extended metric

\[
d([g,x],[g',x']) = \begin{cases} 
    d_X(x,g^{-1}g'x) & \text{if } g^{-1}g' \in H; \\
    \infty & \text{if } g^{-1}g' \notin H.
\end{cases}
\]

Applying this to \( f_D \) we obtain a map

\[
\text{ind}_H^G f_D: \text{ind}_H^G \text{res}_H^G G/1 \to \text{ind}_H^G E_H
\]

which is still \( D \)-contracting. However, observe that \( \frac{1}{D}\infty = \infty \), and that a pair of points at distance \( \infty \) in the source is mapped to a pair of points still at distance \( \infty \) in the target. Hence the map can be used to diminish the size of a morphism between geometric modules only if the morphism over \( \text{ind}_H^G \text{res}_H^G G/1 \) is of finite size, i.e., only if it has no components that connect points at distance \( \infty \).

The usual induction homomorphism \( \text{ind}_H^G: K_1(R[H]) \to K_1(R[G]) \) given by the functor \( R[G] \otimes_R R[H] \) can be easily lifted to the categories of geometric modules, i.e., for any metric space \( X \) the functor

\[
\text{ind}_H^G: C(X) \to C(\text{ind}_H^G X), \quad (\varphi: S \to X) \mapsto (\text{ind}_H^G \varphi: \text{ind}_H^G S \to \text{ind}_H^G X)
\]

induces the upper horizontal map in the following commutative diagram.

\[
\begin{array}{ccc}
K_1(C(X)) & \xrightarrow{\text{ind}_H^G} & K_1(C(\text{ind}_H^G X)) \\
\downarrow \cong & & \downarrow \cong \\
K_1(R[H]) & \xrightarrow{\text{ind}_H^G} & K_1(R[G])
\end{array}
\]

If \( X \) is a metric space in the usual sense (where \( \infty \) is not allowed), then morphisms in the image of \( \text{ind}_H^G \) have the desired property: the size of \( \text{ind}_H^G \alpha \) is finite even though \( \text{ind}_H^G X \) is a metric space in the extended sense. Moreover

\[
\text{size} \text{ind}_H^G \alpha = \text{size} \alpha.
\]

Therefore, using the map \( \text{ind}_H^G f_D \) we can hope to show that, maybe not arbitrary elements, but at least elements of the form \( \text{ind}_H^G [\beta] \) belong to the image of \( \text{asbl}_F \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Support of \( \alpha \) and \( \text{ind}_H^G \alpha \) in an index 3 situation.}
\end{figure}

The reason why this is useful is the following theorem of Swan. Recall that a finite group \( E \) is called hyperelementary if it fits into a short exact sequence

\[1 \to C \to E \to P \to 1\]

where \( C \) is cyclic and the order of \( P \) is a prime power.
Theorem 44 (Swan induction). Let $F$ be a finite group, $\pr : G \rightarrow F$ a surjective homomorphism, and
\[
\mathcal{H}_{pr} = \{ \pr^{-1}(E) \mid E \text{ is a hyperelementary subgroup of } F \}.
\]
Then for every $H \in \mathcal{H}_{pr}$ there exist $\mathbb{Z}[H]$-modules $M^+_H$ and $M^-_H$ that are finitely generated free as $\mathbb{Z}$-modules, and such that, for each $n \in \mathbb{Z}$ and each $x \in K_n(R[G])$, we have
\[
(45) \quad x = \sum_{n \in \mathcal{H}_{pr}} \text{ind}^G_H \left( [M^+_H] \cdot \text{res}^G_H x \right) - \text{ind}^G_H \left( [M^-_H] \cdot \text{res}^G_H x \right).
\]

Here, for $y \in K_n(R[H])$ and a $\mathbb{Z}[H]$-module $M$ which is finitely generated free as a $\mathbb{Z}$-module, we write $[M] \cdot y = \ell_M(y)$ for the image of $y$ under the map induced in $K$-theory by the functor $\ell_M$ that sends the $R[H]$-module $P$ to $M \otimes_{\mathbb{Z}} P$ equipped with the diagonal $H$-action.

Proof. The Swan group $Sw(H; \mathbb{Z})$ is by definition the $K_0$-group of $\mathbb{Z}H$-modules that are finitely generated free as $\mathbb{Z}$-modules. The relation is the usual additivity relation for (not necessarily split) short exact sequences. Tensor products over $\mathbb{Z}$ equipped with the diagonal $H$-actions induce the structure of a unital commutative ring on $Sw(H; \mathbb{Z})$, and also define an action of $Sw(H; \mathbb{Z})$ on $K_n(R[H])$. Swan showed in [Swa60] that for a finite group $F$ there exist $\mathbb{Z}[E]$-modules $N^+_E$ and $N^-_E$, where $E$ runs through all hyperelementary subgroups of $F$, such that in $Sw(F; \mathbb{Z})$ we have
\[
(46) \quad 1 = [Z] = \sum_{E} \text{ind}^F_E[N^+_E] - \text{ind}^F_E[N^-_E].
\]

The natural isomorphisms
\[
\text{ind}^G_H \left( M \otimes_{\mathbb{Z}} \text{res}^G_H P \right) \cong \left( \text{ind}^G_H M \right) \otimes_{\mathbb{Z}} P \quad \text{and} \quad \text{ind}^G_{pr^{-1}(E)} \text{res}_{pr} N \cong \text{res}_{pr} \text{ind}^F_E N,
\]
given by $g \otimes m \otimes p \mapsto g \otimes m \otimes gp$ and $g \otimes n \mapsto \pr(g) \otimes n$, respectively, yield the following identity in $K_n(R[G])$ for $H = \pr^{-1}(E)$: \[
(\text{res}_{pr} \text{ind}^F_E[N]) \cdot x = (\text{ind}^G_H \text{res}_{pr} N) \cdot x = \text{ind}^G_H ([\text{res}_{pr} N] \cdot \text{res}^G_H x).
\]
Using this and $\text{res}_{pr} 1 = [\text{res}_{pr} Z] = [Z] = 1$ one derives the statement in the theorem with $M_{pr^{-1}(E)} = \text{res}_{pr} N_E$ from (46).

If we want to use $H$-equivariant contracting maps, as explained above, to show that each of the summands in (45) is in the image of $\text{asbl}_E$, we need to control the size of a geometric representative of $[M^+_H] \cdot \text{res}^G_H x$ in terms of the size of a representative of $x$.

This is indeed easy. Similarly to induction, also the functors restriction $\text{res}^G_H$ and $\ell_M = M \otimes_{\mathbb{Z}} -$ can be lifted to categories of geometric modules. For restriction simply send the object given by $\phi : S \rightarrow Z$ to $\text{res}^G_H \phi : \text{res}^G_H S \rightarrow \text{res}^G_H Z$. For $\ell_M$ observe that if $B$ is a finite $\mathbb{Z}$-basis for the $\mathbb{Z}[H]$-module $M$, then there are isomorphisms of $\mathbb{Z}[H]$-modules
\[
(47) \quad \mathbb{Z}[B] \otimes_{\mathbb{Z}} R[\prod H/1] \cong \mathbb{Z}[B] \otimes_{\mathbb{Z}} R[\prod H/1] \cong R[B \times \prod H/1].
\]
Here the first isomorphism is given by $m \otimes h \mapsto h^{-1}m \otimes h$, where in the source one uses the diagonal $H$-action, and in the target the $H$-action on the right tensor factor. The second isomorphism is the obvious one. One constructs the desired functor by working only with objects of the form $\phi : \prod H/1 \rightarrow Z$ and sending such
a \phi to \phi \circ \pr, where \pr: B \times \coprod H/1 \to \coprod H/1 is the projection onto the second factor. The behaviour on morphisms is determined by the isomorphism (47): one defines \( l_M \alpha \) between the objects on the right in (47) in such a way that on the left it corresponds to \( \id \otimes \alpha \). One then checks easily that

\[
\text{size } \res^G_H \alpha = \text{size } \alpha \quad \text{and} \quad \text{size } l_M \alpha = \text{size } \alpha.
\]

In summary, given a finite index subgroup \( H \leq G \), a \( \mathbb{Z}[H] \)-module \( M \) that is finitely generated free as a \( \mathbb{Z} \)-module, and an \( H \)-equivariant \( D \)-contracting map \( f_D: \res^G_H G/1 \to Z \) to an \( H \)-simplicial complex, we have a commutative diagram (48)

\[
\begin{array}{ccc}
K_1(\mathcal{C}(G/1)) & \longrightarrow & K_1(\mathcal{C}(\res^G_H G/1)) \\
\vert & & \vert \\
\vert & & \vert \\
K_1(\mathcal{R}(\mathcal{G})) & \longrightarrow & K_1(\mathcal{R}(H)) \\
\end{array}
\]

and the estimate (49)

\[
\text{size } \left( \ind_H^G f_D \right) \left( \ind_H^G l_M \res^G_H \alpha \right) \leq \frac{1}{D} \text{size } \left( \ind_H^G l_M \res^G_H \alpha \right) = \frac{1}{D} \text{size } \alpha < \infty.
\]

In order to prove surjectivity of \( \text{asbl}_F \) it remains to find suitable finite quotients \( \pr: G \to F \) and suitable \( H \)-equivariant contracting maps for each \( H \in \mathcal{H}_{pr} \). This leads to the criterion formulated in Theorem 57 below for arbitrary groups \( G \).

Groups that meet this criterion have been named Farrell-Hsiang groups in [BL12b].

### 4.4. \( \mathbb{Z}^2 \) is a Farrell-Hsiang group.

Now we concentrate on the concrete situation where \( G = \mathbb{Z}^2 \), and explain how the criterion is met in this special case.

**Claim 50** (\( \mathbb{Z}^2 \) is a Farrell-Hsiang group with respect to \( \text{Cyc} \)). Fix a word metric \( d^{2 \times 2} \) on \( \mathbb{Z} \times \mathbb{Z} \). Consider \( \mathbb{R} \) as a simplicial complex with vertices \( \mathbb{Z} \subset \mathbb{R} \) and with the corresponding \( \ell^1 \)-metric \( d^1 \). For any arbitrarily large \( D > 0 \) there exists a surjective homomorphism \( \pr_D: \mathbb{Z} \times \mathbb{Z} \to F \) to a finite group \( F \) with the following property. For each \( H \in \mathcal{H}_{pr,D} = \{ \pr_D^{-1}(E) \mid E \text{ is a hyperelementary subgroup of } F \} \)

there exist:

(i) a simplicial \( H \)-action on \( \mathbb{R} \) with only cyclic isotropy,

(ii) a map \( f_H: \res^H_H (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{R} \) that is \( H \)-equivariant and \( D \)-contracting, i.e.,

\[
d^1(f_H(g), f_H(g')) \leq \frac{1}{D} d^{2 \times 2}(g, g')
\]

for all \( g, g' \in \mathbb{Z} \times \mathbb{Z} \).

We first show that this implies Proposition 32.

**Proof of Proposition 32**. The simplicial complex \( \mathbb{R} \) is 1-dimensional. Let \( \epsilon = \epsilon(1) \) be as in Theorem 37. Given \( x \in K_1(\mathcal{R}(\mathcal{G})) \) choose an automorphism \( \alpha \) in \( \mathcal{C}(G/1) \) such that \( [\alpha] \) maps to \( x \) under the forgetful map \( \cup: K_1(\mathcal{C}(G/1)) \to K_1(\mathcal{R}(\mathcal{G})) \). Choose \( D > 0 \) so large that \( \frac{1}{D} \max\{\text{size}(\alpha), \text{size}(\alpha^{-1})\} \leq \epsilon \). Use Claim 50 in order to find a finite quotient \( \pr_D : \mathbb{Z} \times \mathbb{Z} \to F \) and \( H \)-equivariant \( D \)-contracting maps \( f_H: \res^H_H G/1 \to \mathbb{R} \) for every \( H \in \mathcal{H}_{pr,D} \).
For each $H \in \mathcal{H}_{pr_H}$, let $M = M_H^+$ be as in Theorem 44, and send $[\alpha]$ through the upper row in diagram (48). Use estimate (49) to conclude that

$$\text{size} \left( (\text{ind}_H^G f_H)_* (\text{ind}_H^G l_M \text{res}_H^G \alpha) \right) \leq \epsilon.$$  

By Corollary 38 and the commutativity of (48), we see that $\text{ind}_H^G l_M \text{res}_H^G x$ is in the image of the map (33). Because of the decomposition (45) in Theorem 44, also $x$ is in the image.

Proof of Claim 54: We begin with some simplifications. With respect to the standard generating set $\{ (\pm 1, 0), (0, \pm 1) \}$, the word metric is Lipschitz equivalent to the Euclidean metric on $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$. On $\mathbb{R}$ the simplicial $\ell^1$ metric and the Euclidean metric satisfy

$$d^1(x, y) \leq C d^{\text{Eucl}}(x, y)$$

for some fixed constant $C$. Therefore it is enough to establish (51) with respect to the Euclidean metrics on $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$ and on $\mathbb{R}$, instead of the word and $\ell^1$-metrics. Moreover, it is enough to consider only maximal hyperelementary subgroups of $F$, because then for any $H' < H$ we can take $f_{H'} = \text{res}_{H'} f_H$.

Let us start to look for suitable finite quotients $F$ of $\mathbb{Z} \times \mathbb{Z}$. If $F$ itself were hyperelementary, then we would have to find a contracting map $f_{\mathbb{Z} \times \mathbb{Z}}$ to a $(\mathbb{Z} \times \mathbb{Z})$-simplicial complex with cyclic isotropy that is $(\mathbb{Z} \times \mathbb{Z})$-equivariant. But in Remark 41 we saw that this is impossible.

Every finite quotient $F$ of $\mathbb{Z} \times \mathbb{Z}$ is isomorphic to $\mathbb{Z}/a \times \mathbb{Z}/ab$, which is hyperelementary if and only if $a$ is a prime power. Hence a simple choice of $F$ which is not itself hyperelementary is $\mathbb{Z}/pq \times \mathbb{Z}/pq$ for distinct primes $p$ and $q$. In order to achieve the contracting property we will later choose the primes to be very large.

Let $\text{pr}_{pq} : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}/pq \times \mathbb{Z}/pq$ be the projection. A maximal hyperelementary subgroup $E$ of $\mathbb{Z}/pq \times \mathbb{Z}/pq$ has order $pq^2$ or $p^2q$. By symmetry it is enough to consider the case where the order of $E$ is $pq^2$. Let $H = \text{pr}_{pq}^{-1}(E)$. Now we need to construct $f_H$.

For every $v \in \mathbb{Z} \times \mathbb{Z}$ with $v \neq 0$, consider the map

$$f_v : \mathbb{Z} \times \mathbb{Z} \xrightarrow{\ell_v (v, -)} \mathbb{Z} \xrightarrow{-/p} \mathbb{R} , \quad w \longmapsto \frac{1}{p} \langle v, w \rangle$$

where $\langle -, - \rangle$ is the standard inner product on $\mathbb{R}^2$. If we equip $\mathbb{R}$ with the $(\mathbb{Z} \times \mathbb{Z})$-action given by

$$(\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R} \longrightarrow \mathbb{R} , \quad (w, x) \longmapsto x + \frac{1}{p} \langle v, w \rangle$$

then $f_v$ is $(\mathbb{Z} \times \mathbb{Z})$-equivariant. More importantly, we have that:

(A) $f_v$ is $p/\|v\|$-contracting, i.e., $|f_v(w) - f_v(w')| \leq \frac{\|w\|}{p} \|w - w'\|$. This follows immediately from the linearity of $f_v$ and the Cauchy-Schwarz inequality.

(B) The isotropy group at every point of $\mathbb{R}$ is $\ker(\ell_v) = \{ w \in \mathbb{Z} \times \mathbb{Z} \mid \langle v, w \rangle = 0 \}$, and hence cyclic since we assumed that $v \neq 0$.

(C) The action restricts to a simplicial $H$-action if $\ell_v(H) \subseteq p\mathbb{Z}$.
Let us reformulate the last condition. Consider the following commutative diagram.

\[
\begin{array}{ccccccc}
H & = & \text{pr}^{-1}(E) & \longrightarrow & \mathbb{Z} \times \mathbb{Z} & \longrightarrow & Z \\
\downarrow & & \downarrow & & \downarrow \text{pr} & & \downarrow \\
E & \longrightarrow & \mathbb{Z}/pq \times \mathbb{Z}/pq & \longrightarrow & \mathbb{F}_p \times \mathbb{F}_p & \longrightarrow & \mathbb{F}_p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{pr}(E) & = & \mathbb{F}_p, & \pi & \longrightarrow & \mathbb{F}_p \times \mathbb{F}_p & \longrightarrow & \mathbb{F}_p \\
\end{array}
\]

(52)

Here \( \pi \in \mathbb{F}_p \times \mathbb{F}_p \) is a generator of the \( \mathbb{F}_p \)-vector space \( \text{pr}(E) \). Observe that \( \text{pr}(E) \neq \mathbb{F}_p \times \mathbb{F}_p \) because the order of \( E \) is \( pq^2 \). Then the last condition above is equivalent to saying that the composition in diagram (52) from \( H \) to \( \mathbb{F}_p \) is trivial, i.e., that \( \ell_{\pi}(\pi) = 0 \).

Hence, if we can find a vector \( v \in \mathbb{Z} \times \mathbb{Z} \) such that

\[
0 < \|v\| \leq 4\sqrt{p} \quad \text{and} \quad \ell_{\pi}(\pi) = 0,
\]

then from (53) we get that \( f_v \) is a \( (p/4\sqrt{p} = \sqrt{p}/4) \)-contracting \( H \)-equivariant map to \( \mathbb{R} \), where \( \mathbb{R} \) is equipped with a simplicial \( H \)-action by (53) and has cyclic isotropy by (53).

The existence of such a vector \( v \) is established by the following counting argument. Consider the set

\[
S = \{ v = (x_1, x_2) \in \mathbb{Z} \times \mathbb{Z} \mid |x_1| \leq \sqrt{2p} \text{ and } |x_2| \leq \sqrt{2p} \}.
\]

This set has more than \( p \) elements, and therefore the map

\[
S \longrightarrow \mathbb{F}_p, \quad v \mapsto \ell_{\pi}(\pi)
\]

is not injective, where \( \pi \) was defined right after diagram (52). If \( v_0 \) and \( v_1 \) are two distinct vectors in \( S \) with \( \ell_{\pi}(\pi) = \ell_{\pi}(\pi) \), then \( v = v_0 - v_1 \) is a vector which satisfies the equality in (53). For the inequality in (53) we estimate

\[
\|v\| \leq \|v_0\| + \|v_1\| \leq 2\sqrt{2}\sqrt{2p} = 4\sqrt{p}.
\]

So we define \( f_H = f_v \) for such a \( v \) and finish the argument using Euclid’s Theorem: since there are infinitely many primes, for any given \( D > 0 \) we can find distinct primes \( p \) and \( q \) such that both \( \sqrt{p}/4 \geq D \) and \( \sqrt{q}/4 \geq D \), and hence for every

\[
H \in \mathcal{H}_{\text{pr}pq} = \{ \text{pr}^{-1}(E) \mid \text{hyperelementary } E < \mathbb{Z}/pq \times \mathbb{Z}/pq \}
\]

the map \( f_H \) is \( D \)-contracting.

\[\square\]

4.5. The Farrell-Hsiang Criterion (continued). We now indicate how the ideas developed in this section can be used to prove isomorphism results in all dimensions instead of just surjectivity results for \( K_1 \). In [BLR08] the authors introduce, for an arbitrary \( G \)-space \( X \), the additive categories \( T^G(X) \), \( O^G(X) \), and \( D^G(X) \), and establish in [BLR08, Lemma 3.6] a homotopy fibration sequence

\[
\mathbf{K}(T^G(X)) \longrightarrow \mathbf{K}(O^G(X)) \longrightarrow \mathbf{K}(D^G(X)).
\]

(54)

The category \( T^G(X) \) is a variant of the category denoted \( C(X) \) in this section. The functor \( X \mapsto \mathbf{K}(D^G(X)) \) is a \( G \)-equivariant homology theory on \( G \)-CW complexes
and the value at $G/H$ is $\pi_*$-isomorphic to $\Sigma K(\mathcal{R}[H])$ [BFJR04, Section 6]. Therefore the general principles in [WW95] and [DL98] identify the map

$$K(D^G(EG(F))) \to K(D^G(pt))$$

with the (suspended) assembly map $\text{asbl}_F$.

A variant of the category $O^G(X)$ can be defined as follows. Objects are $G$-equivariant maps $\varphi: S \to X \times [1, \infty)$, where now the free $G$-set $S$ is allowed to be cocountable instead of only cofinite. Moreover we require that $\varphi^{-1}(X \times [1, N])$ is cofinite for every $N$.

A morphism $\alpha$ from $\varphi$ to $\varphi'$ is again an $R[G]$-linear map $\alpha: R[S] \to R[S']$, but now there is a severe restriction on the support of a morphism: towards $\infty$ the arrows representing non-vanishing components must become smaller and smaller. Notice though that $X$ is only a topological and not a metric space, and “small” has no immediate meaning. We refer to [BFJR04, Definition 2.7] for the precise definition of this condition, which is known as equivariant continuous control at infinity.

The following result explains the choice of notation: the category $O^G(X)$ is the obstruction category.

**Theorem 55.** The assembly map

$$EG(F)_+ \wedge_{O^G} K(R[G]) \to K(R[G])$$

is a $\pi_*$-isomorphism if and only if $K_*(O^G(EG(F))) = 0$.

**Proof.** The map $EG(F) \to pt$ and the homotopy fibration sequence (54) induce the following commutative diagram with exact rows.

$$\cdots \to K_n(O^G(EG(F))) \to K_n(D^G(EG(F))) \to K_{n-1}(T^G(EG(F))) \to \cdots$$

$$\cdots \to K_n(O^G(pt)) = 0 \to K_n(D^G(pt)) \to K_{n-1}(T^G(pt)) \to \cdots$$

The map $\circlearrowright$ is an isomorphism, because source and target are both isomorphic to $K_{n-1}(R[G])$ via the forgetful map (53). Using the shift map $[1, \infty) \to [1, \infty)$, $x \mapsto x + 1$, it is not difficult to prove that $O^G(pt)$ admits an Eilenberg swindle, and so $K_*(O^G(pt)) = 0$. Therefore also the map $\circlearrowright$ is an isomorphism. Since the map $\circlearrowright$ is identified with the assembly map, the result follows. \qed
Remark 56 (Proof of Theorem 37). Consider the ladder diagram in the previous proof, but replace $EG(\mathcal{F})$ with a simplicial complex $Z$. The maps $\oplus$ and $\otimes$ are still isomorphisms. The maps $\oplus$ and $\otimes$ for $n = 2$ are both models for the assembly map $\text{asbl}_Z$ in Theorem 37. Exactness implies that $[\alpha] \in K_1(T^G(Z))$ is in the image of the assembly map if it maps to $0 \in K_1(O^G(Z))$. The statement of Theorem 37 is now a special case of [BL12a, Theorem 5.3(i)].

With some additional work, the program carried out above to decompose an arbitrary $K_1$-element into summands with sufficiently small representatives can be generalized to show that the $K$-theory of the obstruction category in Theorem 55 vanishes. This leads to the following theorem, which is the main result of [BL12b].

Theorem 57 (Farrell-Hsiang Criterion). Let $\mathcal{F}$ be a family of subgroups of $G$. Fix a word metric on $G$. Assume that there exists an $N > 0$ such that for any arbitrarily large $D > 0$ there exists a surjective homomorphism $\text{pr}_D: G \to F$ to a finite group $F$ with the following property. For each $H \in \mathcal{H}_{\text{pr}_D} = \{ \text{pr}_D^{-1}(E) \mid \text{hyperelementary } E \leq F \}$

there exist:

(i) an $H$-simplicial complex $Z_H$ of dimension at most $N$ and whose isotropy groups are all contained in $F$;

(ii) a map $f_H: \text{res}_H G \to Z_H$ that is $H$-equivariant and $D$-contracting, i.e., $d^1(f_H(g), f_H(g')) \leq \frac{1}{D} d^G(g, g')$ for all $g, g' \in G$.

Then the assembly map

$$EG(\mathcal{F})_+ \wedge_{O^G} K(R[G/\{-\}]) \to K(R[G])$$

is a $\pi_*$-isomorphism.

5. Trace methods

Trace maps are maps from algebraic $K$-theory to other theories like Hochschild homology, topological Hochschild homology, and their variants, which are usually easier to compute than $K$-theory. These trace maps have been used successfully to prove injectivity results about assembly maps in algebraic $K$-theory. In fact, the most sophisticated trace invariant, topological cyclic homology, was invented by Bökstedt, Hsiang, and Madsen specifically to attack the rational injectivity of the classical assembly map for $K(\mathbb{Z}[G])$, as explained in Subsection 5.2 below. In joint work with Lück and Rognes, we applied similar techniques to the Farrell-Jones assembly map, and in particular we obtained the following partial verification of Conjecture [10] see [LRR15] Theorem 1.1].

Theorem 58. Assume that, for every finite cyclic subgroup $C$ of a group $G$, the first and second integral group homology $H_1(BZ\mathbb{Z}C; \mathbb{Z})$ and $H_2(BZ\mathbb{Z}C; \mathbb{Z})$ of the centralizer $Z_G C$ of $C$ in $G$ are finitely generated abelian groups. Then $G$ satisfies Conjecture [10], i.e., the map

$$\text{colim}_{H \in \text{obj Sub}(\mathcal{G}(\mathbb{F}_n))} \text{Wh}(H) \otimes \mathbb{Q}_Z \to \text{Wh}(G) \otimes \mathbb{Q}_Z$$

is injective.
In this section we want to explain the ideas and the structure of the proofs of Bökstedt-Hsiang-Madsen’s Theorem 66 and its generalization, suppressing some of the technical details. We first consider a $K_0$-analog of Theorem 58 and explain in full detail its proof, which is an illuminating example of the trace methods.

5.1. A warm-up example.

**Proposition 59.** Let $k$ be any field of characteristic zero. Then for any group $G$ the map
\[
\colim_{H \in \text{obj} \text{Sub}_G(F_{\text{fin}})} K_0(k[H]) \otimes \mathbb{Q} \to K_0(k[G]) \otimes \mathbb{Q}
\]
is injective.

This is closely related to Conjecture 4 for $R = k$, but observe that, even though $K_0(k[H])$ is a finitely generated free abelian group for each finite group $H$, the colimit in the source of the map in Conjecture 4 may contain torsion [KM91]. Therefore Proposition 59 does not imply the injectivity of the map in Conjecture 4.

The key ingredient in the proof of Proposition 59 is the trace map
\[
\text{tr}: K_0(R) \to R/[R,R],
\]
where $[R,R]$ denotes the subgroup of the additive group of $R$ generated by commutators. The trace map is defined as follows. The projection $R \to R/[R,R]$ extends to a map
\[
\text{tr}: M_n(R) \to R/[R,R], \quad a = (a_{ij}) \to \text{tr}(A) = \sum_{i=1}^n [a_{ii}],
\]
which is easily seen to be the universal additive map out of $M_n(R)$ with the trace property: $\text{tr}(ab) = \text{tr}(ba)$. If $p$ is an idempotent matrix in $M_n(R)$, then $\text{tr}(p)$ only depends on the isomorphism class of the projective $R$-module $R^n p$. Since the trace sends the block sum of matrices to the sum of the traces, it induces a group homomorphism
\[
\text{tr}: K_0(R) \to R/[R,R], \quad [p_{ij}] \to \sum_i [p_{ii}].
\]

Now consider the case of group algebras. We denote by $\text{conj} G$ the set of conjugacy classes of elements of $G$. The map $R[G] \to R[\text{conj} G]$ induced by the projection sends $[R[G], R[G]]$ to zero, and it induces an isomorphism
\[
R[G]/[R[G], R[G]] \cong R[\text{conj} G].
\]
The composition of the trace map $\text{tr}$ from (60) with this isomorphism gives a map
\[
\text{tr}: K_0(R[G]) \to R[\text{conj} G],
\]
which is known as the Hattori-Stallings rank. In the special case of group algebras of finite groups with coefficients in fields of characteristic zero we have the following result.
Lemma 61. Suppose that the group $G$ is finite and that $R = \mathbb{k}$ is a field of characteristic zero. Let $R_k(G)$ be the representation ring of $G$ over $\mathbb{k}$, and consider the map
\[
\chi : R_k(G) \longrightarrow \mathbb{k}[\text{conj } G], \quad \rho \mapsto \left( \chi_\rho : g \mapsto \text{tr}_k(\rho(g)) \right)
\]
that sends each representation to its character. Then there is a commutative diagram
\[
\begin{array}{ccc}
K_0(\mathbb{k}[G]) & \xrightarrow{\text{tr}} & \mathbb{k}[\text{conj } G] \\
\cong & & \\
R_k(G) & \xrightarrow{\chi} & \mathbb{k}[\text{conj } G] \\
\end{array}
\]
whose vertical maps are isomorphisms.

In other words, the Hattori-Stallings rank can be identified up to isomorphism with the character map $\chi$. Notice, though, that unlike $\chi$ the Hattori-Stallings rank is natural in $G$.

Proof of Lemma 61. Since $G$ is finite and $\mathbb{k}$ has characteristic zero, a finitely generated projective $\mathbb{k}[G]$-module $V$ is the same as a finite-dimensional $\mathbb{k}$-vector space equipped with a linear $G$-action $\rho : G \rightarrow GL(V)$. This explains the left vertical isomorphism in the diagram above. It is well known that every irreducible representation is contained as a direct summand in the regular representation $\mathbb{k}[G]$. Therefore we can assume that the idempotent $p = p^2 = \sum_{k \in G} p_k k$ lies in $\mathbb{k}[G]$. Let $\langle \cdot , \cdot \rangle$ be the $\mathbb{k}$-bilinear form on $\mathbb{k}[G]$ that is determined on group elements by $\langle g, h \rangle = \delta_{gh}$. Then
\[
\chi_\rho(g) = \text{tr}_k(\mathbb{k}[G]p \rightarrow \mathbb{k}[G]p, \ x \mapsto gx) = \text{tr}_k(\mathbb{k}[G] \rightarrow \mathbb{k}[G], \ x \mapsto gxp) = \\
= \sum_{h \in G} \langle h, ghp \rangle = \sum_{h \in G} \sum_{k \in G} p_k \langle h, gkh \rangle = \sum_{h \in G} p_{h^{-1}g^{-1}h} = \sum_{x \in [g^{-1}]} \#(Z_G(\langle g^{-1} \rangle))p_x.
\]
For the last equality observe that the stabilizer of $g \in G$ under the action of $G$ on itself via conjugation is the centralizer $Z_G(\langle g \rangle)$. For the Hattori-Stallings rank we have $\text{tr}(p)([g]) = \sum_{x \in [g]} p_x$. □

We are now ready to prove Proposition 59.

Proof of Proposition 59. It suffices to prove the injectivity of the map in Proposition 59 with $- \otimes \mathbb{Q}$ replaced by $- \otimes \mathbb{k}$. We explain the proof in the case $\mathbb{k} = \mathbb{C}$. Consider the following commutative diagram.

The vertical maps are induced by the $\mathbb{C}$-linear extension of the Hattori-Stallings rank. For each finite group $H$ this extension is an isomorphism by Lemma 61.
and Corollary 1 in §12.4], and so the map \( \Theta \) is an isomorphism. The map \( \Theta \) is an isomorphism because the functor \( C[-] \) is left adjoint and hence preserves colimits. Since conjugation with elements in \( G \) represents morphisms in \( \text{Sub}G(\text{Fin}) \), the map \( \Theta \) is easily seen to be injective already before applying \( C[-] \).

The proof for an arbitrary field \( k \) of characteristic zero is completely analogous, but the set \( \text{conj} G \) needs to be replaced by the set \( \text{conj}_k G \) of \( k \)-conjugacy classes, a certain quotient of \( \text{conj} G \).

Notice that for each finite group \( H \) the Hattori-Stallings rank itself (before \( k \)-linear extension) is always injective. But we cannot leverage this fact to prove integral injectivity results because colimits need not preserve injectivity.

### 5.2. Bökstedt-Hsiang-Madsen’s Theorem

The map \( \text{tr} \) in (60) is just the first (or rather the zeroth) and the easiest trace invariant of the algebraic \( K \)-theory of \( R \).

We now briefly overview how it can be generalized, starting with the Dennis trace with values in Hochschild homology.

Consider the simplicial abelian group

\[
\cdots R \otimes R \otimes R \xrightarrow{d} R \otimes R \xrightarrow{d} R
\]

whose face maps are

\[
d_i(r_0 \otimes \cdots \otimes r_n) = \begin{cases} 
  r_0 \otimes \cdots \otimes r_ir_{i+1} \otimes \cdots \otimes r_n & \text{if } i < n; \\
  r_nr_0 \otimes r_1 \otimes \cdots \otimes r_{n-1} & \text{if } i = n.
\end{cases}
\]

The geometric realization of the simplicial abelian group (62) is the zeroth space of an \( \Omega \)-spectrum denoted \( \text{HH}(R) = \text{HH}(R|\mathbb{Z}) \), whose homotopy groups

\[
\text{HH}_s(R) = \pi_s \text{HH}(R)
\]

are the Hochschild homology groups of \( R \).

In particular, we see that \( \text{HH}_0(R) \) is the cokernel of the map \( r \otimes s \mapsto rs - sr \), and hence

\[
\text{HH}_0(R) \cong R/[R, R].
\]

The trace map \( \text{tr}: K_0(R) \to \text{HH}_0(R) \) in (60) lifts to a map of spectra

\[
\text{tr}_d: K^{\geq 0}(R) \to \text{HH}(R)
\]

called the Dennis trace, such that \( \pi_0 \text{tr}_d = \text{tr} \). We use \( K^{\geq 0} \) to denote connective algebraic \( K \)-theory, the \((-1)\)-connected cover of the functor \( K \) we used throughout.

Following ideas of Goodwillie and Waldhausen, Bökstedt [Bök86] introduced a far-reaching generalization of \( \text{HH}(R) \), called topological Hochschild homology and denoted \( \text{THH}(R) \). We omit the technical details of the definitions, and we rather explain the underlying ideas and structures.

The key idea in the definition of topological Hochschild homology is to pass from the ring \( R \) to its Eilenberg-Mac Lane ring spectrum \( \mathbb{H}R \), and to replace the tensor products (over the initial ring \( \mathbb{Z} \)) with smash products (over the initial ring spectrum \( \mathbb{S} \)). In order to make this precise, one needs to work within a symmetric monoidal model category of spectra (e.g., symmetric spectra), or with ad hoc point-set level constructions (as Bökstedt did, long before symmetric spectra and the like were discovered). Once these technical difficulties are overcome, one obtains a simplicial spectrum.
whose geometric realization is $\text{THH}(R) = \text{HH}(\mathbb{H}R \mid S)$. Notice that of course this definition applies not only to Eilenberg-Mac Lane ring spectra $\mathbb{H}R$ but to arbitrary ring spectra $A$.

Bökstedt also lifted the Dennis trace to topological Hochschild homology for any connective ring spectrum $A$:

\[ \text{THH}(A) \xrightarrow{\text{trd}} \mathbb{H}_0^\geq(A) \xrightarrow{\text{trc}} \text{HH}(\pi_0 A). \]

Cyclic permutation of the tensor factors in (62) or smash factors in (63) makes those simplicial objects into cyclic objects, thus inducing a natural $S^1$-action on their geometric realizations; see for example [Jon87, Section 3] and [Dri04]. Bökstedt, Hsiang, and Madsen [BHM93] discovered that topological Hochschild homology has even more structure, which Hochschild homology lacks. Fix a prime $p$. As $n$ varies, the fixed points of the induced $C_p^p$-actions are related by maps

\[ \text{THH}(A)^{C_p^n} \xrightarrow{\text{R}} \text{THH}(A)^{C_p^{n-1}}, \]

called Restriction and Frobenius. The map $F$ is simply the inclusion of fixed points, whereas the definition of the map $R$ is much more delicate and specific to the construction of $\text{THH}$. The homotopy equalizer of (64) is denoted $\text{TC}^{n+1}(A; p)$. One important property of the maps $R$ and $F$ is that they commute, and therefore they induce a map $\text{TC}^{n+1}(A; p) \to \text{TC}^n(A; p)$. The topological cyclic homology of $A$ at the prime $p$ is then defined as the homotopy limit

\[ \text{TC}(A; p) = \text{holim}_n \text{TC}^n(A; p). \]

Bökstedt, Hsiang, and Madsen lifted the Bökstedt trace to topological cyclic homology, thus obtaining the following commutative diagram for any connective ring spectrum $A$:

\[ \text{TC}(A; p) \xrightarrow{\text{trc}} \text{THH}(A) \xrightarrow{\text{trd}} \mathbb{H}_0^\geq(A) \xrightarrow{\text{trb}} \text{HH}(\pi_0 A). \]

The map trc is called the cyclotomic trace map.

They then used this technology to prove the following striking theorem, which is often referred to as the algebraic $K$-theory Novikov Conjecture; see [BHM93, Theorem 9.13] and [Mad94, Theorem 4.5.4].
Theorem 66 (Bökstedt-Hsiang-Madsen). Let $G$ be a group. Assume that the following condition holds.

\[ A_1 \] For every $s \geq 1$, the integral group homology $H_s(BG; \mathbb{Z})$ is a finitely generated abelian group.

Then the classical assembly map
\[
\operatorname{asbl}_1 : BG_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[G])
\]
is $\pi_*^\mathbb{Q}$-injective, i.e., $\pi_n(\operatorname{asbl}_1) \otimes \mathbb{Q}$ is injective for all $n \in \mathbb{Z}$.

We now explain the structure of the proof of Theorem 66 following the approach of [LRRV17a]. As mentioned above, the idea is to use the cyclotomic trace map. However, it is not enough to work with topological cyclic homology, and one needs a variant of it that we proceed to explain. Instead of taking the homotopy equalizer of $R$ and $F$ in (64), we may consider just the homotopy fiber of $R$ and define
\[
C^{n+1}(A; p) = \text{hofib} \left( THH(A)_{hC_p^n} \to THH(A)^{C_p^{n-1}} \right).
\]
The map $F$ induces a map $C^{n+1}(A; p) \to C^n(A; p)$, and we define
\[
C(A; p) = \text{holim}_n C^n(A; p).
\]
A fundamental property, also established in [BHM93], is that $C^{n+1}(A; p)$ can be identified with $\text{THH}(A)_{hC_p^n}$, up to a zigzag of $\pi_*$-isomorphisms. In [LRRV17a Section 8] we provided a natural zigzag of $\pi_*$-isomorphisms between $\text{THH}(A)_{hC_p^n}$ and $C^{n+1}(A; p)$, natural even before passing to the stable homotopy category of spectra. The key tool here is the natural Adams isomorphism for equivariant orthogonal spectra developed in [RV16].

In the special case when $A = S[G]$ is a spherical group ring, then the maps $R$ split, and these splittings can be used to construct a map
\[
\text{TC}(S[G]; p) \to C(S[G]; p).
\]
The crucial advantage of using $C$ instead of $\text{TC}$ is that more general rational injectivity statements can be proved for the assembly maps for $C$; compare Remark 73 below.

In order to prove Theorem 66 one studies the following commutative diagram.

\[
\begin{array}{ccc}
BG_+ \wedge K(\mathbb{Z}) & \xrightarrow{\text{asbl}_1} & K(\mathbb{Z}[G]) \\
\uparrow & & \uparrow \\
BG_+ \wedge K^{\geq 0}(\mathbb{Z}) & \to & K^{\geq 0}(\mathbb{Z}[G]) \\
\uparrow & & \uparrow \\
BG_+ \wedge K^{\geq 0}(S) & \to & K^{\geq 0}(S[G]) \\
\uparrow & & \uparrow \\
BG_+ \wedge \text{TC}(S; p) & \to & \text{TC}(S[G]; p) \\
\downarrow & & \downarrow \\
BG_+ \wedge (\text{THH}(S) \times C(S; p)) & \xrightarrow{(6)} & \text{THH}(S[G]) \times C(S[G]; p)
\end{array}
\]
The horizontal maps are all classical assembly maps, and we want to prove that the one at the top of the diagram is $\pi_\mathbb{Q}^*$-injective. The maps $\circled{1}$ and $\circled{6}$ are induced by the natural maps from connective to non-connective algebraic $K$-theory. Since $\mathbb{Z}$ is regular, $\circled{1}$ is a $\pi_\mathbb{Q}^*$-isomorphism. The maps $\circled{2}$ and $\circled{3}$ come from the linearization (or Hurewicz) map $\mathbb{S} \rightarrow \mathbb{Z}$, and they are both $\pi_\mathbb{Q}^*$-isomorphisms by a result of Waldhausen [Wal78b, Proposition 2.2]. The maps $\circled{5}$ and $\circled{4}$ are given by the cyclotomic trace map, and $\circled{7}$ and $\circled{3}$ by the natural maps in (65) and (67).

So, in order to prove that the top horizontal map in diagram (68) is $\pi_\mathbb{Q}^*$-injective, it is enough to show that:

(a) The assembly map $\circled{2}$ is $\pi_\mathbb{Q}^*$-injective.
(b) The composition $\circled{3} \circ \circled{5}$ is $\pi_\mathbb{Q}^*$-injective.

The assumption $[A_1]$ is then shown to imply [a] and in fact not just for $\mathbb{S}$ but for arbitrary connective ring spectra $\mathbb{A}$. This is the special case $\mathcal{F} = 1$ of Theorems [71 and 72] below. The difficult part in proving (b) is the analysis of the map $\circled{5}$.

We remark that little is known about the rationalized homotopy groups of $K_{\geq 0}(\mathbb{Z}_p)$ without $p$-completion; compare [Wei05, Warning 60].

We conclude our explanation of the proof of Theorem 66.

5.3. Generalizations. The following result generalizes Theorem 66 from the classical to the Farrell-Jones assembly map, and is a special case of [LRRV17a, Main Technical Theorem 1.16].
Theorem 69. Let $G$ be a group and let $\mathcal{F} \subseteq \mathcal{FCyc}$ be a family of finite cyclic subgroups of $G$. Assume that the following two conditions hold.

[A$_F$] For every $C \in \mathcal{F}$ and every $s \geq 1$, the integral group homology $H_s(BZC; \mathbb{Z})$ of the centralizer of $C$ in $G$ is a finitely generated abelian group.

[B$_F$] For every $C \in \mathcal{F}$ and every $t \geq 0$, the natural homomorphism

$$K_t(\mathbb{Z}[\zeta_c]) \otimes \mathbb{Q} \to \prod_{p \text{ prime}} K_t\left(\mathbb{Z}_p \otimes \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\right) \otimes \mathbb{Q}$$

is injective, where $c$ is the order of $C$, $\zeta_c$ is any primitive $c$-th root of unity, and $K_t(R; \mathbb{Z}_p) = \pi_t(R(R)_p^\text{yc})$.

Then the assembly map

$$\text{asbl}_F: EG(\mathcal{F})_+ \wedge_{\mathcal{G} \in \mathcal{G}} K^\geq_0(\mathbb{Z}[G/\mathcal{F}]) \to K^\geq_0(\mathbb{Z}[G])$$

is $\pi^\text{Q}_*$.injective.

Remark 70. Several comments are in order.

(i) When $\mathcal{F} = 1$ is the trivial family, Theorems 66 and 69 coincide. This is because assumption [A$_1$] of Theorem 69 is literally the same as assumption [A$_1$] of Theorem 66 and assumption [B$_1$] follows at once from the corresponding true statement explained at the end of the previous subsection.

(ii) When $\mathcal{F} = \mathcal{FCyc}$, then the rationalized assembly map for connective algebraic $K$-theory studied in Theorem 69 can be rewritten as in Conjecture 23 because the isomorphisms (20) and (21) hold for both connective and non-connective algebraic $K$-theory. The only difference is that the summands indexed by $t = -1$ in the source of the map in Conjecture 23 are now missing. Notice that the negative $K$-groups $K_t(\mathbb{Z}[C])$ are known to vanish for any $t < -1$ if $C$ is finite or even virtually cyclic [FJ95].

(iii) As noted above, assumption [A$_F$] implies and is the obvious generalization of assumption [A$_1$]. For any $\mathcal{F} \subseteq \mathcal{FCyc}$, assumption [A$_F$] is satisfied if there is a universal space $EG(\mathcal{Fin})$ of finite type, i.e., whose skeletons are all cocompact. Hyperbolic groups, finite-dimensional CAT(0)-groups, cocompact lattices in virtually connected Lie groups, arithmetic groups in semisimple connected linear $\mathbb{Q}$-algebraic groups, mapping class groups, and outer automorphism groups of free groups are all examples of groups that even have a finite-dimensional and cocompact $EG(\mathcal{Fin})$. Among these groups, outer automorphism groups of free groups do not appear in Theorem 27 and for them Theorem 69 gives the first result about the Farrell-Jones Conjecture. An interesting example of a group that satisfies [A$_{FCyc}$] without having an $EG(\mathcal{Fin})$ of finite type is given by Thompson’s group $T$ of orientation preserving, piecewise linear, dyadic homeomorphisms of the circle; see [GV17].

(iv) Conjecturally assumption [B$_F$] of Theorem 69 is always satisfied; in fact, it is implied by a weak version of the Leopoldt-Schneider Conjecture for cyclotomic fields, as explained carefully in LRRV17a Sections 2 and 18. When $t = 0$ or $t = 1$, i.e., for $K_0$ and $K_1$, the map in [B$_F$] is injective for arbitrary $c$ by direct computation; compare LRRV17a Proposition 2.4. For any fixed $c$ it is known that injectivity may fail for at most finitely many values of $t$. These two facts allow to deduce Theorems 58 and 24.1 from Theorem 69, or rather from its more general version in LRRV17a Main Technical Theorem 1.16], as explained in loc. cit., Section 17 and page 1015. Notice that, on the other
hand, Theorem 66 cannot be used to deduce information about the Whitehead group $\text{Wh}(G)$, which is the cokernel of the map induced on $\pi_1$ by the classical assembly map $\text{asbl}_1$.

The proof of Theorem 69 follows the same strategy as the proof of Theorem 66 outlined above. We consider the analog of diagram (68) for the generalized assembly map $\text{asbl}_F$; compare [LRRV17a, "main diagram" (3.1)]. The key results about assembly maps are summarized in the following two theorems [LRRV17a, Theorem 1.19, parts (i) and (ii)]. We point out that all the following results hold for arbitrary connective ring spectra $A$.

**Theorem 71.** For any group $G$ and for any family $F$ of subgroups of $G$, the assembly map

$$\text{asbl}_F : \text{EG}(F)_+ \wedge_{\text{Or}_G} \text{THH}(A[G]) \to \text{THH}(A[G])$$

induces split monomorphisms on $\pi_*$, and it is a $\pi_*$-isomorphism if and only if $F$ contains all cyclic subgroups of $G$, i.e., $F \supseteq \text{Cyc}$.

**Theorem 72.** Let $G$ be a group and let $F \subseteq \text{FCyc}$ be a family of finite cyclic subgroups of $G$. Assume that the following condition holds.

$[A_F]$ For every $C \in F$ and every $s \geq 1$, the integral group homology $H_s(B\mathbb{Z}_G C; \mathbb{Z})$ of the centralizer of $C$ in $G$ is a finitely generated abelian group.

Then the assembly map

$$\text{asbl}_F : \text{EG}(F)_+ \wedge_{\text{Or}_G} \text{C}(A[G]; p) \to \text{C}(A[G]; p)$$

is $\pi_0$-injective.

**Remark 73.** In order to establish an analog of Theorem 72 for the assembly map

(74) $$\text{asbl}_F : \text{EG}(F)_+ \wedge_{\text{Or}_G} \text{TC}(A[G]; p) \to \text{TC}(A[G]; p)$$

in topological cyclic homology, we need to assume not only condition $[A_F]$ but also the following two conditions:

$[A'_F]$ the family $F$ contains only finitely many conjugacy classes of subgroups;

$[A''_F]$ for every $g \in G$, $(g) \in F$ if and only if $(g^p) \in F$.

The fact that the assembly map (74) is $\pi_0$-injective under assumptions $[A_F]$, $[A'_F]$, and $[A''_F]$ is a special case of [LRRV17b, Theorem 1.8]. Notice the following facts.

(i) When $F = 1$, assumption $[A'_1]$ is vacuously true, but $[A''_1]$ is not satisfied if $G$ has $p$-torsion. This is the reason why, in the proof of Bökstedt-Hsiang-Madsen’s Theorem 66, we need to work with $C$ and not just $\text{TC}$.

(ii) As pointed out in Remark 71(iii) Thompson’s group $T$ satisfies $[A_{F_{\text{Cyc}}}']$ and obviously also $[A_{F_{\text{Cyc}}}'']$. However, $T$ contains finite cyclic subgroups of any given order, and therefore does not satisfy $[A'_{F_{\text{Cyc}}}']$. It is an interesting open question whether the assembly map (74) is $\pi_0$-injective for $G = T$.

(iii) Without homological finiteness assumptions on $G$, the assembly map (74) is not rationally injective in general. For example, if $G = \mathbb{Q}$ and $F = 1 = F_{\text{Cyc}}$, then (74) is essentially trivial after applying $\pi_*(\_ \otimes_{\mathbb{Q}} \mathbb{Q})$. This is explained in [LRRV17a, Remark 3.7]. Of course, the group $G = \mathbb{Q}$ does not satisfy $[A_1]$.
Finally, we mention the following two additional results about assembly maps for topological cyclic homology, which we proved in \cite{LRRV17b} Theorems 1.1, 1.4(ii), and 1.5.

One should view Theorem 75 as a cyclic induction theorem for the topological cyclic homology of any finite group, with coefficients in any connective ring spectrum. It allows to reduce the computation of TC of any finite group to the case of the finite cyclic subgroups; this is carried out explicitly in \cite{LRRV17b} Proposition 1.2 for the basic case of the symmetric group on three elements.

Theorem 76 studies the analog for TC of the Farrell-Jones Conjecture \cite{LRRV17a}. For a large class of groups (for which Conjecture \cite{LRRV17a} is already known; see Theorem \ref{thm:farrell-jones}), we prove that \(\text{asbl}_{\text{VCyc}}\) is injective, but surprisingly not surjective.

**Theorem 75.** For any finite group \(G\) the assembly map
\[
\text{asbl}_{\text{Cyc}}: EG(\text{Cyc})_+ \wedge_{\text{Or}G} \text{TC}(\mathbb{A}[G]; p) \to \text{TC}(\mathbb{A}[G]; p)
\]
is a \(\pi_*\)-isomorphism.

**Theorem 76.** Assume that \(G\) is either hyperbolic or virtually finitely generated abelian. Then the assembly map
\[
\text{asbl}_{\text{VCyc}}: EG(\mathcal{VCyc})_+ \wedge_{\text{Or}G} \text{TC}(\mathbb{A}[G]; p) \to \text{TC}(\mathbb{A}[G]; p)
\]
is always injective but in general not surjective on homotopy groups. For example, it is not surjective on \(\pi_{-1}\) if \(\mathbb{A} = \mathbb{Z}(p)\) and \(G\) is either finitely generated free abelian or torsion-free hyperbolic, but not cyclic.

**References**

[AAR00] C. S. Aravinda, F. Thomas Farrell, and S. K. Roushon. Algebraic K-theory of pure braid groups. *Asian J. Math.*, 4(2):337–343, 2000. DOI 10.4310/AJM.2000.v4.n2.a4 MR 1797585

[Bar03a] Arthur Bartels. On the domain of the assembly map in algebraic K-theory. *Algebr. Geom. Topol.*, 3:1037–1050, 2003. DOI 10.2140/agt.2003.3.1037 MR 2012963

[Bar03b] Arthur Bartels. Squeezing and higher algebraic K-theory. *K-Theory*, 28(1):19–37, 2003. DOI 10.1023/A:1024166521174 MR 1988817

[Bar16] Arthur Bartels. On proofs of the Farrell-Jones conjecture. In *Topology and geometric group theory*, volume 184 of *Springer Proc. Math. Stat.*, pages 1–31. Springer, Cham, 2016. DOI 10.1007/978-3-319-43674-6_1 MR 3598160

[Bar17] Arthur Bartels. Coarse flow spaces for relatively hyperbolic groups. *Compos. Math.*, 153(4):745–779, 2017. DOI 10.1112/S0010437X16008216 MR 3631229

[Bas64] Hyman Bass. Projective modules over free groups are free. *J. Algebra*, 1:367–373, 1964. DOI 10.1016/0021-8693(64)90016-X MR 0178032

[Bas68] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968. MR 0249491

[BB16] Arthur Bartels and Mladen Bestvina. The Farrell-Jones conjecture for mapping class groups. Preprint, available at [arXiv:1606.02844](http://arxiv.org/abs/1606.02844) 2016.

[BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C*-algebras. In *C*-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994. DOI 10.1090/conm/167/1292018 MR 1292018
[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in $K$- and $L$-theory. *K-Theory*, 15(3):201–252, 1998. DOI 10.1023/A:1007841068777 MR 1659969

[Dri04] Vladimir Drinfeld. On the notion of geometric realization. *Mosc. Math. J.*, 4(3):619–626, 2004. DOI 10.1142/S0219887404000353 MR 2119142

[Dun72] Martin J. Dunwoody. Relation modules. *Bull. London Math. Soc.*, 4:151–155, 1972. DOI 10.1112/blms/4.2.151 MR 0327915

[ELP+16] Nils-Edvin Enkelmann, Wolfgang Lück, Malte Pieper, Mark Ullmann, and Christoph Winges. On the Farrell-Jones Conjecture for Waldhausen’s $A$-theory. Preprint, available at arXiv:1607.06395, 2016.

[Fer77] Steven C. Ferry. The homeomorphism group of a compact Hilbert cube manifold is an ANR. *Ann. of Math. (2)*, 106(1):101–119, 1977. DOI 10.2307/1971161 MR 0461536

[ELP78] F. Thomas Farrell and Wu Chung Hsiang. The topological-Euclidean space form problem. *Invent. Math.*, 45(2):181–192, 1978. DOI 10.1007/BF01390272 MR 0482771

[FH81b] F. Thomas Farrell and Lowell E. Jones. The Whitehead group of poly-(finite or cyclic) groups. *J. London Math. Soc. (2)*, 24(2):308–324, 1981. DOI 10.1112/jlms/s2-24.2.308 MR 631942

[FH83] F. Thomas Farrell and Wu Chung Hsiang. Topological characterization of flat and almost flat Riemannian manifolds $M^n$ $(n \neq 3, 4)$. *Amer. J. Math.*, 105(3):641–672, 1983. DOI 10.2307/2374318 MR 704219

[FJ86] F. Thomas Farrell and Lowell E. Jones. K-theory and dynamics. I. *Ann. of Math. (2)*, 124(3):531–569, 1986. DOI 10.2307/2007092 MR 866708

[FJ93a] F. Thomas Farrell and Lowell E. Jones. Isomorphism Conjectures in algebraic $K$-theory. *J. Amer. Math. Soc.*, 6(2):249–297, 1993. DOI 10.2307/2152801 MR 1179537

[FJ93b] F. Thomas Farrell and Lowell E. Jones. Topological rigidity for compact non-positively curved manifolds. In *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 229–274. Amer. Math. Soc., Providence, RI, 1993. MR 1216629

[FJ95] F. Thomas Farrell and Lowell E. Jones. The lower algebraic $K$-theory of virtually infinite cyclic groups. *K-Theory*, 9(1):13–30, 1995. DOI 10.1007/BF00966547 MR 1340838

[GMR15] Giovanni Gandini, Sebastian Meinert, and Henrik Rüping. The Farrell-Jones Conjecture for fundamental groups of graphs of abelian groups. *Groups Geom. Dyn.*, 9(3):783–792, 2015. DOI 10.4171/ggd/227 MR 3420543

[GR13] Giovanni Gandini and Henrik Rüping. The Farrell-Jones Conjecture for graph products. *Algebr. Geom. Topol.*, 13(6):3651–3660, 2013. DOI 10.2140/agt.2013.13.3651 MR 3248744
[Gru08] Joachim Grunewald. The behavior of Nil-groups under localization and the relative assembly map. *Topology*, 47(3):160–202, 2008. DOI [10.1016/j.top.2007.03.007]. MR 2414359.

[Gub88] Joseph Gubeladze. The Anderson conjecture and a maximal class of monoids over which projective modules are free. *Mat. Sb. (N.S.),* 135(177)(2):169–185, 271, 1988. MR 937805.

[GV17] Ross Geoghegan and Marco Varisco. On Thompson’s group *T* and algebraic *K*-theory. In *Geometric and Cohomological Group Theory*, volume 444 of *London Math. Soc. Lecture Note Ser.*, pages 34–45. Cambridge Univ. Press, Cambridge, 2017. DOI [10.1017/9781316771327.005].

[Hes05] Lars Hesselholt. *K*-theory of truncated polynomial algebras. In *Handbook of *K*-theory. Vol. 1*, pages 71–110. Springer, Berlin, 2005. DOI [10.1007/3-540-27855-9_3]. MR 2181821.

[Hig40] Graham Higman. The units of group-rings. *Proc. London Math. Soc. (2)*, 46:231–248, 1940. DOI [10.1112/plms/s2-46.1.231]. MR 0002137.

[HM97] Lars Hesselholt and Ib Madsen. On the *K*-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997. DOI [10.1016/0040-9383(96)00003-1]. MR 1410465.

[HP04] Ian Hambleton and Erik K. Pedersen. Identifying assembly maps in *K*- and *L*-theory. *Math. Ann.*, 328(1-2):27–57, 2004. DOI [10.1007/s00208-003-0454-5]. MR 2030369.

[Hsi84] Wu Chung Hsiang. Geometric applications of algebraic *K*-theory. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 99–118. PWN, Warsaw, 1984. MR 804679.

[Jon87] John D. S. Jones. Cyclic homology and equivariant homology. *Invent. Math.*, 87(2):403–423, 1987. DOI [10.1007/BF01389424]. MR 870737.

[Kap57] Irving Kaplansky. *Problems in the theory of rings*. Report of a conference on linear algebras, June, 1956, pp. 1-3. National Academy of Sciences-National Research Council, Washington, Publ. 502, 1957. MR 0096696.

[Kap70] Irving Kaplansky. “Problems in the theory of rings” revisited. *Amer. Math. Monthly*, 77:445–454, 1970. DOI [10.2307/2317376]. MR 0258865.

[Kas15] Daniel Kasprowski. On the *K*-theory of groups with finite decomposition complexity. *J. Topol. Anal.*, 2018. DOI [10.1142/s1793525319500286]. MR 3342098.

[KL05] Matthias Kreck and Wolfgang Lück. *The Novikov Conjecture. Geometry and algebra*, volume 33 of *Oberwolfach Seminars*. Birkhäuser, Basel, 2005. DOI [10.1007/b137100]. MR 2117411.

[KLR16] Holger Kammeyer, Wolfgang Lück, and Henrik Rüping. The Farrell–Jones Conjecture for arbitrary lattices in virtually connected Lie groups. *Geom. Topol.*, 20(3):1275–1287, 2016. DOI [10.2140/gt.2016.20.1275]. MR 3523058.

[KM91] Peter H. Kropholler and Boaz Moselle. A family of crystallographic groups with 2-torsion in *K*0 of the rational group algebra. *Proc. Edinburgh Math. Soc. (2)*, 34(2):325–331, 1991. DOI [10.1017/S0013091500007215]. MR 1115651.

[KNR18] Daniel Kasprowski, Andrew Nicas, and David Rosenthal. Regular finite decomposition complexity. *J. Topol. Anal.*, 2018. DOI [10.1142/s1793525319500286]. MR 3342098.

[Lam99] Tsit-Yuen Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer, New York, 1999. DOI [10.1007/978-1-4612-0525-8]. MR 1653294.

[Lam06] Tsit-Yuen Lam. *Serre’s problem on projective modules*. Springer Monographs in Mathematics. Springer, Berlin, 2006. DOI [10.1007/978-3-540-34575-6]. MR 2235330.

[LN17] Markus Land and Thomas Nikolaus. On the relation between *K*- and *L*-theory of *C*-algebras. *Math. Ann.*, 2017. DOI [10.1007/s00208-017-1617-8].

[Lod76] Jean-Louis Loday. *K-théorie algébrique et représentations de groupes*. *Ann. Sci. École Norm. Sup. (4)*, 9(3):309–377, 1976. DOI [10.24033/asens.1312]. MR 0447373.

[LR05] Wolfgang Lück and Holger Reich. The Baum-Connes and the Farrell-Jones Conjectures in *K*- and *L*-theory. In *Handbook of *K*-theory. Vol. 2*, pages 703–842. Springer, Berlin, 2005. DOI [10.1007/978-3-540-27855-9_15]. MR 2181833.
[LR14] Wolfgang Lück and David Rosenthal. On the $K$- and $L$-theory of hyperbolic and virtually finitely generated abelian groups. *Forum Math.*, 26(5):1565–1609, 2014. DOI 10.1515/forum-2011-0146 MR 3334038.

[LRRV17a] Wolfgang Lück, Holger Reich, John Rognes, and Marco Varisco. Algebraic $K$-theory of group rings and the cyclotomic trace map. *Adv. Math.*, 26(5):1565–1609, 2014. DOI 10.1016/j.aim.2016.09.004 MR 3588224.

[LRRV17b] Wolfgang Lück, Holger Reich, John Rognes, and Marco Varisco. Assembly maps for topological cyclic homology of group algebras. *J. Reine Angew. Math.*, 2017. DOI 10.1515/crelle-2017-0023 MR 3936432.

[Lüc] Wolfgang Lück. Isomorphism Conjectures in $K$- and $L$-Theory. In preparation, preliminary version available at him.uni-bonn.de/lueck/.

[Lüc02] Wolfgang Lück. Chern characters for proper equivariant homology theories and applications to $K$- and $L$-theory. *J. Reine Angew. Math.*, 543:193–234, 2002. DOI 10.1515/crll.2002.015 MR 1887884.

[Lüc10] Wolfgang Lück. *K- and L-theory of group rings*. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 1071–1098. Hindustan Book Agency, New Delhi, 2010. DOI 10.1007/978-3-0348-8089-3_0087 MR 2827832.

[Mac71] Saunders MacLane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, New York, 1971. DOI 10.1007/978-1-4612-9839-7 MR 0354798.

[Mad94] Ib Madsen. Algebraic $K$-theory and traces. In *Current developments in mathematics, 1995 (Cambridge, MA)*, pages 191–321. Int. Press, Cambridge, MA, 1994. MR 1474979.

[Mil65] John Milnor. *Lectures on the $h$-cobordism theorem*. Princeton University Press, Princeton, N.J., 1965. MR 0190942.

[Mil71] John Milnor. *Introduction to algebraic $K$-theory*. Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J., 1971. MR 0349811.

[Mon69] M. Susan Montgomery. Left and right inverses in group algebras. *Bull. Amer. Math. Soc.*, 75:539–540, 1969. DOI 10.1090/S0002-9904-1969-12234-2 MR 0238967.

[MV03] Guido Mislin and Alain Valette. *Proper group actions and the Baum-Connes Conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser, Basel, 2003. DOI 10.1007/978-3-0348-9839-7 MR 2027168.

[Pat14] Dimitrios Patronas. *The Artin defect in algebraic $K$-theory*. PhD thesis, Freie Universität Berlin, 2014. Available at diss.fu-berlin.de/diss/receive/FUDISS_thesis_000000097442.

[PRV] Dimitrios Patronas, Holger Reich, and Marco Varisco. In preparation.

[Qui79] Frank Quinn. Ends of maps. I. *Ann. of Math. (2)*, 110(2):275–331, 1979. DOI 10.2307/1971262 MR 549490.

[Qui82] Frank Quinn. Ends of maps. II. *Invent. Math.*, 68(3):353–424, 1982. DOI 10.1007/BF01389410 MR 669423.

[Qui12] Frank Quinn. Algebraic $K$-theory over virtually abelian groups. *J. Pure Appl. Algebra*, 216(1):170–183, 2012. DOI 10.1016/j.jpaa.2011.06.001 MR 2826431.

[Ros94] Jonathan Rosenberg. *Algebraic $K$-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer, New York, 1994. DOI 10.1007/978-1-4612-4314-4 MR 1282290.
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