UNIQUENESS OF NONZERO POSITIVE SOLUTIONS OF
LAPLACIAN ELLIPTIC EQUATIONS ARISING IN
COMBUSTION THEORY

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Abstract. Uniqueness of nonzero positive solutions of a Laplacian elliptic
equation arising in combustion theory is of great interest in combustion theory
since it can be applied to determine where the extinction phenomenon
occurs. We study the uniqueness whenever the orders of the reaction rates are
in (−∞, 1]. Previous results on uniqueness treated the case when the orders
belong to [0, 1). When the orders are negative or 1, it is physically meaning-
ful and the bimolecular reaction rate corresponds to the order 1, but there
is little study on uniqueness. Our results on the uniqueness are completely
new when the orders are negative or 1, and also improve some known results
when the orders belong to (0, 1). Our results provide exact intervals of the
Frank-Kamenetskii parameters on which the extinction phenomenon never oc-
curs. The novelty of our methodology is to combine and utilize the results from
Laplacian elliptic inequalities and equations to derive new results on uniqueness
of nonzero positive solutions for general Laplacian elliptic equations.

1. Introduction. One of important Laplacian elliptic equations arising in com-
bustion theory is of the form

\[
\begin{aligned}
-\Delta u(x) &= \nu(1 + \varepsilon u)^m e^{\frac{u}{1+\varepsilon u}} := \nu f(u) \quad \text{almost every (a.e.) on } \Omega, \\
u(x) &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

where $\Omega$ is a bounded open set in $\mathbb{R}^n$ ($n \geq 1$) with $\mu := \text{meas}(\Omega) > 0$ and with
smooth boundary $\partial\Omega$, $u(x)$ presents the dimensionless temperature at location $x$,
$\varepsilon > 0$ denotes the reciprocal activation energy, $\nu > 0$ is the Frank-Kamenetskii
parameter, the reaction term $f(u)$ is the reaction rate with $m$th-order which obeys
the general Arrhenius law of reaction rate, and $m \in (-\infty, \infty)$ is a parameter.
Physically, the reaction rate with \( m = 0 \) is called the Arrhenius reaction rate while the bimolecular reaction rate corresponds to \( m = 1 \), see [3, 7, 28, 30]. The reaction rate with \( m < 0 \) is physically meaningful and has been widely studied, see [21, 24, 27, 31] for \( m = -1 \) or \( m = -2 \), [27] for \( m \in [-2, 2.67] \) and [11, 21] for \( m \in [-10.31, 2.81] \).

The study on uniqueness of positive solutions of Eq. (1.1) is of great interest in combustion theory since the extinction phenomenon occurs at the critical value \( \nu_E \), where

\[
\nu_E = \sup \{ \nu^* > 0 : (1.1) \text{ has a unique solution for each } \nu \in (0, \nu^*) \},
\]

see [26]. There has been progress in finding an approximation value smaller than or equal to \( \nu_E \) for suitable \( n, \varepsilon \) and \( m \). Taira [26] showed that when \( n \geq 2 \) and \( m \in [0, 1) \), \( \nu_E = \infty \) for \( \varepsilon \in \left(1 + \sqrt{1 - m}\right)^{-2}, \infty \) (see the second result of [26, Theorem 1.1]), and that Eq. (1.1) has a unique solution for \( \nu \in (0, \Lambda(\varepsilon, m)) \) and \( \varepsilon \in \left(0, 1 + \sqrt{1 - m}\right)^{-2} \), where \( \Lambda(\varepsilon, m) \) is a small and computable constant (refer to [26, Theorem 1.5]). The latter case implies that \( \Lambda(\varepsilon, m) \leq \nu_E \) and \( \Lambda(\varepsilon, m) \) is an approximation value to \( \nu_E \). When \( n = 1, 2 \), \( m \in [0, 1) \) and \( \Omega \) is the unit open ball in \( \mathbb{R}^n \), Du [7] proved that \( \nu_E = \infty \) for \( \varepsilon \in \left(1 + \sqrt{1 - m}\right)^{-2}, \infty \) [7, Theorem 3.5].

We refer to [5, 7, 8, 29, 30] for the study on the \( S \)-shaped solution curves of Eq. (1.1) when \( n = 1 \) or \( n = 2 \), \( m \in (-\infty, 1) \) and \( \varepsilon \) is sufficiently small. The \( S \)-shaped solution curves of Eq. (1.1) imply that there exists a number \( \lambda_1(\varepsilon) \in (0, \infty) \) such that Eq. (1.1) has a unique nonzero positive solution for \( \nu \in (0, \lambda_1(\varepsilon)) \). When \( n \geq 2 \) and \( m \in (-\infty, 0) \cup \{1\} \), there is little study on \( \nu_E \) and its approximations. Indeed, the existence of \( \lambda_1(\varepsilon) \) does not provide a good approximation to the critical value \( \nu_E \).

In this paper, we prove that when \( n \geq 3 \) and \( \varepsilon \in (0, \infty) \), \( \nu_E = \infty \) for \( m \in (-\infty, -\varepsilon^{-1}] \), and Eq. (1.1) has a unique nonzero positive solution for \( m \in (-\varepsilon^{-1}, 1] \) and \( \nu \in (0, \nu_1(\varepsilon, m)) \), where the expression of \( \nu_1(\varepsilon, m) \), a computable real number, will be given. This implies that the value \( \nu_1(\varepsilon, m) \) is a computable approximation to the critical value \( \nu_E \) and the extinction phenomenon never occurs for \( \nu \in (0, \nu_1(\varepsilon, m)) \). Our results are new when \( n \geq 3 \), \( \varepsilon \in (0, \infty) \) and \( m \in (-\infty, 0) \cup \{1\} \). Moreover, when \( n \geq 3 \), \( \varepsilon \in (0, \infty) \) and \( m \in (0, 1) \), our results improve the results in [26].

Our methodology is to study uniqueness of nonzero positive solutions for a more general Laplacian elliptic equations of the form

\[
\begin{cases}
-\Delta u(x) = \nu f(u(x)) & \text{a.e. on } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.3)

It is known that Eq. (1.3) has a uniqueness positive solution if the function

\[
g(u) = f(u)/u
\] (1.4)

is either strictly decreasing in [2, 4, 6, 10, 20, 25] or strictly increasing in [9, 22, 23]. We refer to [1] for other monotonicity conditions imposed on \( f \). However, the reaction rate function \( f \) given in (1.1) does not necessarily satisfy these monotonicity conditions. Indeed, with some specific parameters, the function \( g \) with this reaction rate function \( f \) is neither decreasing nor increasing on \( \mathbb{R}_+ = (0, \infty) \). A concrete example is shown in Fig. 1.
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Figure 1. The non-monotonicity of the function $g(u)$. Here, $g(u) = f(u)/u$ in which the parameters of the reaction rate function $f$ in Eq. (1.1) are taken as $\varepsilon = 0.08$ and $m = -2$.

To deal with the case when $g$ is neither decreasing nor increasing on $\mathbb{R}_+$, we prove a new result on uniqueness of positive weak solutions for the following Laplacian elliptic inequality

$$
\begin{cases}
-\Delta u(x) \geq \nu f(u(x)) & \text{a.e. on } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.5)

We refer to [13, 14, 15, 16, 17, 18] for the theories of variational inequalities and applications to inequality (1.5).

The novelty of our method is to combine the above result and a known result on existence of nonzero positive solutions for the Laplacian elliptic equation (1.3) obtained in [14] to derive a new result on uniqueness of nonzero positive solutions for Eq. (1.3). In particular, we show that if $f$ is decreasing (may not be strictly decreasing) on $\mathbb{R}_+$, Eq. (1.3) has a unique nonzero positive solutions for each $\nu \in (0, \infty)$, and further that if $f$ is not decreasing on $\mathbb{R}_+$, there exists a computable constant $\nu_1(\varepsilon, m) > 0$ such that Eq. (1.3) has a unique nonzero positive solutions for each $\nu \in (0, \nu_1(\varepsilon, m))$.

The difficulty of applying the new result on uniqueness of nonzero positive solutions of Eq. (1.3) directly to Eq. (1.1) is to find an upper bound of the derivative of the reaction rate function $f$ given in Eq. (1.1). We find out the upper bound (see Lemma 3.2) which enables us to obtain the results on uniqueness of nonzero positive solutions of Eq. (1.1).

2. Uniqueness of nonzero positive solutions for Laplace elliptic equations.

In this section, we study uniqueness of nonzero positive solutions for Laplacian elliptic equations of the form

$$
\begin{cases}
-\Delta u(x) = \nu f(u(x)) & \text{a.e. on } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(2.1)

where $\Delta u(x) = \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2}(x)$, $\Omega$ is a bounded open set in $\mathbb{R}^n$ ($n \geq 1$) with a measure $\mu := \text{meas}(\Omega) > 0$ and with a smooth boundary $\partial \Omega$, $\nu > 0$ is a parameter, and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.
Uniqueness of positive solutions of Eq. (2.1) was studied under the assumption that the function \( g \) defined in (1.4) is either strictly decreasing [2, 4, 6, 10, 20, 25] or strictly increasing [9, 22, 23]. However, there are functions \( f \) arising in real applications which do not satisfy these conditions. The reaction rate function \( f \) in Eq. (1.1) is an example of such functions, see section 3 for detailed study and Figure 1 in [26], where it is showed that the function \( g \) with the reaction rate function \( f \) in Eq. (1.1) is neither decreasing nor increasing on \( \mathbb{R}_+ \).

In the following we shall prove new results on uniqueness under a different condition imposed on \( f \) [see (2.7)], which is weaker than the well-known Lipschitz condition and is satisfied by the reaction rate function \( f \) in Eq. (1.1).

We denote by \( C(\Omega) \) the Banach space of continuous functions defined in \( \Omega \). Let \( P = \{ z \in C(\Omega) : z(x) \geq 0 \text{ for } x \in \Omega \} \) be the standard positive cone in \( C(\Omega) \).

By a positive solution to Eq. (2.1), we mean a function \( u \) in \( P \) which satisfies Eq. (2.1).

We denote by \( \mu_1 \) the largest characteristic value of the linear Laplacian elliptic equation

\[
\begin{cases}
-\Delta u(x) = \mu u(x) & \text{on } \Omega, \\
u(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(2.2)

By [19, (2.18)], \( \mu_1 \) can be computed by

\[
\mu_1 = \inf \left\{ \int_{\Omega} |\nabla u(x)|^2 \, dx / \int_{\Omega} |u(x)|^2 \, dx : u \in P_1 \setminus \{0\} \right\},
\]  

(2.3)

where \( P_1 = \{ u \in H^1_0 : u(x) \geq 0 \text{ a.e. on } \Omega \} \) is the positive cone in the Sobolev space \( H^1_0 := H^1_0(\Omega) \). Let

\[
f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u} \quad \text{and} \quad f^\infty = \limsup_{u \to \infty} \frac{f(u)}{u}.
\]  

(2.4)

We first state the following result on existence of nonzero positive solutions of Eq. (2.1), which is a special case of [12, Corollary 2.2].

**Lemma 2.1.** Assume that \( f \) satisfies the following condition:

\[
0 \leq f^\infty < f_0 \leq \infty,
\]  

(2.5)

where \( f^\infty \) and \( f_0 \) are the same as those defined in (2.4). Then Eq. (2.1) has a nonzero positive solution in \( P \) for \( n \geq 1 \) and \( \nu \in \left( \frac{\mu_1}{f_0}, \frac{\mu_1}{f^\infty} \right) \).

Let

\[
\nu_0 = \mu_1^{-\frac{2}{n}} \left[ \frac{2(n-1)}{(n-2)\sqrt{n}} \right]^{-2} \quad \text{for } n \geq 3.
\]  

(2.6)

We prove the following result on uniqueness of Laplacian elliptic inequalities.

**Lemma 2.2.** Assume that \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and there exists a number \( b \in (0, \infty) \) such that\n
\[
f(u) - f(v) \leq b(u - v) \quad \text{for } 0 \leq v \leq u.
\]  

(2.7)
By Lemma 2.1, Eq. (2.1) has a nonzero positive solution $u$. By (2.5) and (2.10),

\begin{equation}
\int_{\Omega} [\nabla u(x) \nabla (u(x) - v(x)) - \nu f(u(x)) [u(x) - v(x)] \, dx \leq 0 \quad \text{for } v \in P_1,
\end{equation}

where \( \nabla u(x) = \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right) \).

\textbf{Proof.} Let $\nu \in \left( 0, \frac{\mu_1}{\nu_0} \right)$. We verify that $\nu f$ satisfies all the conditions of [14, Theorem 3.2]. Since $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f$ satisfies (P$_1$) and (P$_2$)$_\sigma$ with $\sigma \in \left( 0, \frac{n+2}{n-2} \right)$ of [14, Theorem 3.2]. It is obvious that $\nu f(0) \in L^\infty(\Omega)$. This, together with condition (2.7), implies that (P$_3$)$_\sigma$, of [14, Theorem 3.2] holds for $s \in \left[ \frac{n}{2}, \infty \right]$. The rest of the proof is to verify $k(s) < 1$, where

\begin{equation}
k(s) = \|\nu b\|_{L^2} [C(\zeta(s))]^2, \quad C(p) = \mu \left( \frac{1}{2} - \frac{n+2}{p} \right) \frac{2(n-1)}{2(n-2)} \frac{1}{\sqrt{n}},
\end{equation}

and

\begin{equation}
\zeta(s) = \begin{cases} \frac{2s}{s-1}, & \text{if } s \in \left[ \frac{n}{2}, \infty \right), \\ 2, & \text{if } s = \infty.
\end{cases}
\end{equation}

By computation, we have for $s \in \left( \frac{n}{2}, \infty \right]$,

\begin{equation}
C(\zeta(s)) = \mu^{-\frac{1}{2}} \left( \frac{2}{p+1} + \frac{2(n-1)}{2(n-2)} \right) = \mu^{-\frac{1}{2}} \frac{1}{\sqrt{n}},
\end{equation}

and

\begin{equation}
k(s) = \|\nu b\|_{L^2} [C(\zeta(s))]^2 = \nu b \mu^{-\frac{1}{2}} [C(\zeta(s))]^2 = \frac{\nu b}{\nu_0} < 1.
\end{equation}

The result follows from [14, Theorem 3.2]. \hfill \Box

Now, we give our main result on uniqueness of nonzero positive solutions in $P$ of Eq. (2.1).

\textbf{Theorem 2.3.} Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies conditions (2.5) and (2.7) for some $b \in (0, \infty)$. Assume further that $n \geq 3$ and the following inequality holds:

\begin{equation}
\frac{\mu_1}{\nu_0} < \frac{\nu_0}{b}.
\end{equation}

Then (2.1) has a unique nonzero positive solution in $P$ for $\nu \in \left( \frac{\mu_1}{\nu_0}, \min \left\{ \frac{\nu_0}{\nu_0}, \frac{\nu_1}{\nu_0} \right\} \right)$.

\textbf{Proof.} By (2.5) and (2.10), \( \left( \frac{\mu_1}{\nu_0}, \min \left\{ \frac{\nu_0}{\nu_0}, \frac{\nu_1}{\nu_0} \right\} \right) \neq \emptyset \). Let $\nu \in \left( \frac{\mu_1}{\nu_0}, \min \left\{ \frac{\nu_0}{\nu_0}, \frac{\nu_1}{\nu_0} \right\} \right)$.

By Lemma 2.1, Eq. (2.1) has a nonzero positive solution $u$ in $P$ since $\nu \in \left( \frac{\mu_1}{\nu_0}, \min \left\{ \frac{\nu_0}{\nu_0}, \frac{\nu_1}{\nu_0} \right\} \right)$. By [12, Lemma 2.2], we see that the solution $u$ belongs to $C^1(\Omega)$ and thus, $u \in P_1$. Moreover, $u$ satisfies the inequality (2.9) since $u$ satisfies Eq. (2.1). Hence, $u$ is a positive weak solution of Eq. (2.8) in $P_1$. By Lemma 2.2, Eq. (2.8) has only one positive weak solution in $P_1$. The uniqueness implies that $u$ is the unique nonzero positive solution of Eq. (2.1). \hfill \Box
Proof. Let \( \nu \in \left( \frac{\mu_+}{b}, \frac{\mu_0}{b} \right) \). Let \( b \in (0, \frac{\mu_0}{\nu}) \) if \( f^\infty = 0 \) and \( b \in \left( 0, \frac{\nu_0 f^\infty}{\mu_1} \right) \) if \( f^\infty > 0 \). Since \( f \) is decreasing on \( \mathbb{R}_+ \),

\[
f(u) - f(v) \leq 0 \leq b(u - v) \quad \text{for } 0 \leq v \leq u,
\]

and condition \((2.7)\) holds. Moreover, if \( f^\infty = 0 \), then

\[
\frac{\mu_1}{f_0} < \nu < \frac{\nu_0}{b} = \min \left\{ \frac{\nu_0}{b}, \frac{\mu_1}{f^\infty} \right\}.
\]

If \( f^\infty > 0 \), then \( \frac{\mu_1}{f^\infty} < \frac{\nu_0}{b} \) and

\[
\frac{\mu_1}{f^\infty} < \nu < \frac{\mu_1}{f^\infty} = \min \left\{ \frac{\nu_0}{b}, \frac{\mu_1}{f^\infty} \right\}.
\]

Hence, condition \((2.10)\) holds and \( \nu \in \left( \frac{\mu_0}{b}, \min \left\{ \frac{\nu_0}{b}, \frac{\mu_1}{f^\infty} \right\} \right) \). By Theorem 2.3, Eq. \((2.1)\) has a unique nonzero positive solution in \( P \).

3. Laplacian elliptic equations in combustion theory. In this section, we apply the results obtained in section 2 to study existence and uniqueness of positive solutions of the Laplacian elliptic equation \((1.1)\) in combustion theory.

Let \( \varepsilon > 0 \) and \( m \in \mathbb{R} \). We define a continuous function \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) by

\[
f(u) := f_{\varepsilon, m}(u) = (1 + \varepsilon u)^m e^{\frac{u}{1 + \varepsilon^2}}.
\]

We first give a result on existence of nonzero positive solutions of Eq. \((1.1)\), where we allow \( n \geq 1 \) and \( m \in (-\infty, 1] \).

**Theorem 3.1.** Assume that \( n \geq 1 \) and \( \varepsilon > 0 \). Then the following assertions hold.

1. If \( m \in (-\infty, 1) \), Eq. \((1.1)\) has a nonzero positive solution in \( P \) for each \( \nu \in (0, \infty) \).

2. If \( m = 1 \), Eq. \((1.1)\) has a nonzero positive solution in \( P \) for each \( \nu \in \left( 0, \mu_1(\varepsilon e^1)^{-1} \right) \), where \( \mu_1 \) is the same as in Eq. \((2.3)\).

**Proof.** Since \( m \in (-\infty, 1] \) and \( \varepsilon > 0 \), we have \( \lim_{u \to 0^+} \frac{f(u)}{u} = \infty \) and

\[
f^\infty = \lim_{u \to \infty} \frac{f(u)}{u} = \begin{cases} 0, & \text{if } m \in (-\infty, 1), \\ \varepsilon e^1, & \text{if } m = 1. \end{cases}
\]

(3.2)

Therefore, the results (1) and (2) follow directly from Lemma 2.1.

**Remark 1.** When \( n = 1, m \in (-\infty, 1) \) and \( \Omega = [-1, 1] \), Wang \([29, 30]\) proved that the solution curve of Eq. \((1.1)\) is \( S \)-shaped for \( \varepsilon \in (0, \varepsilon_0(m)) \), where \( \varepsilon_0(m) \) is a computable small constant (see \([29, \text{Theorem 1}]\) and \([30, \text{Theorems 1.4 and 1.5}]\)). When \( \Omega \) is the unit open ball in \( \mathbb{R}^2 \), the solution curve of Eq. \((1.1)\) is \( S \)-shaped for sufficiently small \( \varepsilon > 0 \) if \( m = 0, n = 2 \) (see Du and Lou \([8, \text{Theorem 1}]\)), and if \( 0 \leq m < 1, n = 1, 2 \) (see Du \([7, \text{Theorem 3.5}]\)). This implies that for sufficiently small \( \varepsilon > 0 \) and each \( \mu \in (0, \infty) \), Eq. \((1.1)\) has at least one nonzero positive solution. Our Theorem 3.1 allows \( n \geq 1, \varepsilon > 0 \) and \( m \in (-\infty, 1] \). Hence, when \( n = 1, 2 \), Theorem 3.1 (1) improves \([29, \text{Theorem 1}], [30, \text{Theorems 1.4 and 1.5}], [8, \text{Theorem 1}] \) and \([7, \text{Theorem 3.5}]\).
Remark 2. By using [12, Corollary 2.2], Theorem 3.1 can be generalized to

\[\begin{align*}
-Lu(x) &= \nu(1 + \epsilon u)^m e^{\frac{\nu}{1 + \epsilon u}} \quad \text{a.e. on } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}\]

(3.3)

where \(L\) is a strongly uniformly elliptic differential operator and the boundary condition involves a first order boundary operator. Note that Theorem 3.1 (1) allows \(n \geq 1\) and \(m < 0\). Hence, Theorem 3.1 (1) and its generalization to (3.3) generalize the first result of [26, Theorem 1.1], where \(n \geq 2\) and \(m \in [0, 1)\). The generalization of Theorem 3.1 (2) to (3.3) is new. Note that the condition \(m \in [0, 1)\) is used in an essential way in [26], so the method in [26] cannot be used to treat Theorem 3.1 (2), where \(m = 1\) corresponds to the bimolecular reaction rate.

Now, we turn our attention to uniqueness of nonzero positive solutions of Eq. (1.1). We need to suppose \(n \geq 3\). Let

\[\nu_E = \sup \{\nu^*: \text{Eq. (1.1) has a unique solution for each } \nu \in (0, \nu^*)\}.\]

(3.4)

It is known that the extinction phenomenon occurs at the critical value \(\nu_E\) if \(\nu_E < \infty\) and never occurs if \(\nu_E = \infty\) (refer to [26]). In the following, we find the regions of \((\epsilon, m)\) on which \(\nu_E = \infty\). For some \((\epsilon, m)\), it seems difficult to find the exact value \(\nu_E\). In the latter case, we provide a subinterval \((0, \nu(\epsilon, m))\) of \((0, \nu_E)\) and so, the computable value \(\nu(\epsilon, m)\) is an approximation value to \(\nu_E\).

We define two functions \(m_1, m_2: (0, \infty) \to \mathbb{R}\) by

\[m_1(\epsilon) = \frac{(\epsilon - 2) - \sqrt{\epsilon^2 + 4\epsilon}}{2\epsilon} \quad \text{and} \quad m_2(\epsilon) = \frac{(\epsilon - 2) + \sqrt{\epsilon^2 + 4\epsilon}}{2\epsilon}.\]

(3.5)

The graph of \(m_2(\epsilon)\) is depicted in Figure 2. It is easy to verify that \(m_1(\epsilon)\) and \(m_2(\epsilon)\) are solutions of the following equation

\[m^2 - \frac{(\epsilon - 2)}{\epsilon} m + \frac{1 - 2\epsilon}{\epsilon^2} = 0,\]

(3.6)

and \(m_1(\epsilon) \leq m \leq m_2(\epsilon)\) if and only if

\[m^2 - \frac{(\epsilon - 2)}{\epsilon} m + \frac{1 - 2\epsilon}{\epsilon^2} \leq 0.\]

(3.7)

Moreover, \(m_1(\epsilon)\) and \(m_2(\epsilon)\) satisfy

\[-\infty < m_1(\epsilon) < -\epsilon^{-1} < m_2(\epsilon) < 1\]

for \(\epsilon \in (0, \infty)\), and

\[m_2(\epsilon) \begin{cases} < 0 & \text{if } \epsilon \in (0, \frac{1}{2}), \\
 = 0 & \text{if } \epsilon = \frac{1}{2}, \\
 > 0 & \text{if } \epsilon \in (\frac{1}{2}, \infty).\end{cases}\]

These properties of \(m_1(\epsilon)\) and \(m_2(\epsilon)\) can be used to show the signs of the functions \(u_1\) and \(u_2\) defined in (3.13).

We partition the set \((0, \infty) \times (-\infty, 1]\) into six parts as follows:

\[\Omega_1 = \left\{ (\epsilon, m): \epsilon \in (0, \infty) \text{ and } m \in (-\infty, -\epsilon^{-1}] \right\};\]

\[\Omega_2 = \left\{ (\epsilon, m): \epsilon \in (0, \infty) \text{ and } m \in (-\epsilon^{-1}, m_2(\epsilon)] \right\};\]
Figure 2. The six regions $\Omega_i$ for $i = 1, \cdots, 6$ in the $\epsilon$-$m$ plane.

$\Omega_3 = \{ (\epsilon, m) : \epsilon \in \left(0, \frac{1}{2}\right) \text{ and } m \in (m_2(\epsilon), 0) \};$

$\Omega_4 = \{ (\epsilon, 0) : \epsilon \in \left(0, \frac{1}{2}\right) \};$

$\Omega_5 = \{ (\epsilon, m) : \epsilon \in \left(0, \frac{1}{2}\right) \text{ and } m \in (0, 1) \text{ or } \epsilon \in \left[\frac{1}{2}, \infty\right) \text{ and } m \in (m_2(\epsilon), 1) \};$

$\Omega_6 = \{ (\epsilon, 1) : \epsilon \in (0, \infty) \}.$

See Fig. 2 for the six regions. Further set

$$\eta(\epsilon, m) = \begin{cases} 0, & \text{if } (\epsilon, m) \in \Omega_1, \\ 1 + \epsilon m, & \text{if } (\epsilon, m) \in \Omega_2, \\ 4\epsilon^2e^{(\epsilon^{-1})}, & \text{if } (\epsilon, m) \in \Omega_4, \\ 4\epsilon^2e^{(\epsilon^{-1})}, & \text{if } (\epsilon, m) \in \Omega_6, \\ \epsilon^2m \left[1 - \frac{1}{m}\right]^{m-2} \left(1 - m\right)^{\frac{1{-m}}{1+\epsilon u}} e^{\frac{1-(1+\epsilon m)}{1+\epsilon u} \frac{1}{1+\epsilon u}}, & \text{if } (\epsilon, m) \in \Omega_3 \cup \Omega_5. \end{cases}$$

(3.8)

To apply Theorem 2.3, we need to find an upper bound for the derivative $f'$ of the reaction rate function $f$ in Eq. (3.1). The following result provides an upper bound or a maximum value of $f'$.

Lemma 3.2. Let $f$ be the same as in Eq. (3.1). Then

$$f'(u) \leq \eta(\epsilon, m) \quad \text{for } (\epsilon, m) \in (0, \infty) \times (-\infty, 1],$$

where $\eta(\epsilon, m)$ is same as in (3.8).

Proof. Let $\epsilon \in (0, \infty)$ and $m \in (-\infty, 1]$. By computation, we have for $u \geq 0$,

$$f'(u) = f(u) \left[ \frac{\epsilon m}{1 + \epsilon u} + \frac{1}{(1 + \epsilon u)^2} \right].$$

(3.9)

By the derivative (3.9), if $m = 0$, then

$$f''(u) = \frac{f(u)}{(1 + \epsilon u)^4} (1 - 2\epsilon - 2\epsilon^2 u) = -\frac{f(u)}{2(1 + \epsilon u)^4} \left[ (\epsilon - \frac{1}{2}) + \epsilon^2 u \right].$$

(3.10)
and if $m \neq 0$, then
\[ f''(u) = \frac{f(u)}{(1 + \varepsilon u)^2} g(u), \tag{3.11} \]
where the function $g : \mathbb{R} \to \mathbb{R}$ is defined by
\[ g(u) = m^{-1} - \varepsilon^4 m(1 - m)(u - u_*)^2 \tag{3.12} \]
and $u_* := u_*(\varepsilon, m) = -(1 + \varepsilon m)\varepsilon^{-2}m^{-1}$. The quadratic equation $g(u) = 0$ has two solutions:
\[ u_1 = u_* - \left(\varepsilon^2 |m|\sqrt{1 - m}\right)^{-1} \quad \text{and} \quad u_2 = u_* + \left(\varepsilon^2 |m|\sqrt{1 - m}\right)^{-1}. \tag{3.13} \]
It is obvious that $u_1 \leq u_* \leq u_2$ and it is easy to verify that when $m < 0$, the following assertions hold.

(p1) $g$ is decreasing on $(-\infty, u_*)$ and increasing on $[u_*, \infty)$.

(p2) $g(u) > 0$ for $u \in (-\infty, u_1)$.

(p3) $g(u) \leq \max\{g(u_1), g(u_2)\} = 0$ for $u \in [u_1, u_2]$.

(p4) $g(u) > 0$ for $u \in (u_2, \infty)$.

(p5) For $\varepsilon \in (0, \infty)$, $u_1 \leq 0$ if $m \leq m_2(\varepsilon)$ and $u_2 > 0$ if $m \in (-\varepsilon^{-1}, 0)$.

The properties of $m_1(\varepsilon)$ and $m_2(\varepsilon)$ we have mentioned above are needed to prove the assertion (p5) and we leave the proof to the reader.

By the derivative (3.9), $f''(0) = 1 + \varepsilon m$ and by computations we have for $\varepsilon \in (0, \infty)$,
\[ f'(u_0) = \begin{cases} 4\varepsilon^2e^{(\varepsilon^{-1}-2)}, & \text{if } m = 0, \\ \varepsilon^{2-m}\left[1 - \frac{\varepsilon^2}{m}\right]^{m-2}(1 - m)^{1-m-2}\varepsilon^m - \frac{1-(1+\varepsilon m)}{\sqrt{1-\varepsilon^2m}} & \text{if } m \in (-\infty, 0) \cup (0, 1), \end{cases} \]
where
\[ u_0 = \begin{cases} 1 - \frac{2\varepsilon}{2\varepsilon^2}, & \text{if } m = 0, \\ u_* - \left(\varepsilon^2 m \sqrt{1 - m}\right)^{-1}, & \text{if } m \in (-\infty, 0) \cup (0, 1). \end{cases} \]

We consider the following six cases.

(1) If $(\varepsilon, m) \in \Omega_1$, then by (3.9), we have for $u \in \mathbb{R}_+$,
\[ f'(u) = \frac{e^{-\frac{\varepsilon u}{1+\varepsilon u}}}{(1 + \varepsilon u)^{1-m}} \left[\varepsilon m + \frac{1}{1 + \varepsilon u}\right] \leq \frac{e^{-\frac{\varepsilon u}{1+\varepsilon u}}}{(1 + \varepsilon u)^{1-m}}(\varepsilon m + 1) \leq 0. \tag{3.14} \]

(2) Let $(\varepsilon, m) \in \Omega_2$. We consider the following three cases.

(2-i) If $\varepsilon \in (0, \frac{1}{2})$ and $m \in (-\infty, m_2(\varepsilon))$ or $\varepsilon \in (\frac{1}{2}, \infty)$ and $m \in (-\varepsilon^{-1}, 0)$, then by (p5), $u_1 \leq 0 < u_2$. By the derivative (3.11), (p3) and (p4), $f''(u) \leq 0$ for $u \in [0, u_2]$ and $f''(u) > 0$ for $u \in [u_2, \infty)$. Hence, $f'$ is decreasing on $[0, u_2]$ and increasing on $[u_2, \infty)$. This implies that
\[ f'(u) \leq \max\{f'(0), f'(\infty)\} = f'(0) = \varepsilon m + 1 \quad \text{for } u \in \mathbb{R}_+. \tag{3.14} \]
where $f'(\infty) = \lim_{u \to \infty} f'(u) = 0$.

(2-ii) If $\varepsilon \in (\frac{1}{2}, \infty)$ and $m \in (0, m_2(\varepsilon))$, then $u_1 \leq u_* \leq u_2 \leq 0$. Since $m > 0$, $g$ is increasing on $(-\infty, u_*]$ and decreasing on $[u_*, \infty)$. It follows that $g(u) \leq g(u_2) = 0$ for $u \in [u_2, \infty)$. By the derivative (3.11), $f''(u) \leq 0$ for $u \in [u_2, \infty)$. Since $u_2 < 0$, $f''(u) \leq 0$ for $u \in \mathbb{R}_+$ and $f'$ is decreasing on $\mathbb{R}_+$. Hence
\[ f'(u) \leq f'(0) = \varepsilon m + 1 \quad \text{for } u \in \mathbb{R}_+. \tag{3.14} \]

(2-iii) If $(\varepsilon, m) \in \left[\frac{1}{2}, \infty\right) \times \{0\}$, then by the derivative (3.10), $f''(u) \leq 0$ for $u \in \mathbb{R}_+$. Hence, $f'$ is decreasing on $\mathbb{R}_+$ and
\[ f'(u) \leq f'(0) = \varepsilon m + 1 = 1 \quad \text{for } u \in \mathbb{R}_+. \tag{3.14} \]
It follows that (2.5) of Theorem 2.3 holds. We consider the following three cases:

(3) If $(\varepsilon, m) \in \Omega_3$, then $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $m \in (m_2(\varepsilon), 0)$. This implies $u_1 > 0$. The derivative (3.11), together with $(p_2)$, $(p_3)$, $(p_4)$, yields $f''(u) \geq 0$ for $u \in [0, u_1] \cup [u_2, \infty)$ while $f''(u) \leq 0$ for $u \in [u_1, u_2]$. Hence, $f'$ is increasing on either $[0, u_1]$ or $[u_2, \infty)$, but decreasing on $[u_1, u_2]$. This, together with $f'(\infty) = 0$, implies

$$f'(u) \leq \max \left\{ f'(u_1), f'(\infty) \right\} = f'(u_1) = f'(u_0) = \eta(\varepsilon, m) \quad \text{for } u \in \mathbb{R}_+.$$ 

(4) If $(\varepsilon, m) \in \Omega_4 = (0, \frac{1}{2}) \times \{0\}$, then by the derivative (3.10), we see that $f''(u_0) = 0$, $f''(u) > 0$ for $u \in [0, u_0]$ and $f''(u) < 0$ for $u \in (u_0, \infty)$, where $u_0 = \frac{1-\varepsilon}{2\varepsilon} > 0$. Hence,

$$f'(u) \leq f'(u_0) = \eta(\varepsilon, m) \quad \text{for } u \in \mathbb{R}_+.$$ 

(5) If $(\varepsilon, m) \in \Omega_5(\varepsilon)$, then $m \in (0, 1)$ and $u_1 < 0 < u_2$. Note that $g$ is decreasing on $[u_1, \infty)$. Hence, $g(u) \geq g(u_2) = 0$ for $u \in [0, u_2]$ and $g(u) \leq g(u_2) = 0$ on $[u_2, \infty)$. It follows from the derivative (3.11) that $f''(u) \geq 0$ for $u \in [0, u_2]$ and $f''(u) \leq 0$ on $[u_2, \infty)$. This implies that $f'$ is increasing on $[0, u_2]$ and decreasing on $[u_2, \infty)$. Hence,

$$f'(u) \leq f'(u_2) = f'(u_0) = \eta(\varepsilon, m) \quad \text{for } u \in \mathbb{R}_+.$$ 

(6) If $(\varepsilon, m) \in \Omega_6 = (0, \infty) \times \{1\}$, then by (3.11) with $m = 1$, we have for $u \in [0, \infty)$,

$$f''(u) = \frac{f(u)}{(1+\varepsilon u)^4} \left[ \varepsilon^4 m(m-1) \left( u + \frac{1+\varepsilon m}{\varepsilon m} \right)^2 + \frac{1}{m} \right] = \frac{f(u)}{(1+\varepsilon u)^4} > 0.$$ 

It follows that $f'$ is increasing on $[0, \infty)$. By the derivative (3.9) with $m = 1$, we have

$$\lim_{u \to \infty} f'(u) = \lim_{u \to \infty} \varepsilon e^{\frac{u}{\varepsilon}} \left[ \varepsilon + \frac{1}{1+\varepsilon u} \right] = \frac{\varepsilon}{\varepsilon} = \eta(\varepsilon, m) \quad \text{for } u \in \mathbb{R}_+$$

and $f'(u) \leq \varepsilon e^{\frac{u}{\varepsilon}} = \eta(\varepsilon, m)$ for $u \in \mathbb{R}_+$. \hfill \Box

**Theorem 3.3.** Assume that $n \geq 3$, $\varepsilon > 0$ and $m \in (-\infty, 1]$. Then Eq. (1.1) has a unique nonzero positive solution in $P$ for each $\nu \in (0, \nu_1(\varepsilon, m))$, where

$$\nu_1(\varepsilon, m) = \begin{cases} \infty, & \text{if } (\varepsilon, m) \in \Omega_1, \\ \frac{\nu_0}{\eta(\varepsilon, m)}, & \text{if } (\varepsilon, m) \in \cup_{i=2}^6 \Omega_i, \\ \min\{\nu_0, \mu_1\}(\varepsilon e^{\frac{1}{\varepsilon}})^{-1}, & \text{if } (\varepsilon, m) \in \Omega_6. \end{cases} \quad (3.15)$$

**Proof.** Note that $f_0 = \infty$ and by Eq. (3.2),

$$f^\infty = \begin{cases} 0, & \text{if } m \in (-\infty, 1), \\ \varepsilon e^{\frac{1}{\varepsilon}}, & \text{if } m = 1. \end{cases} \quad (3.16)$$

It follows that (2.5) of Theorem 2.3 holds. We consider the following three cases:

(i) If $(\varepsilon, m) \in \Omega_1$, then by Lemma 3.2, $f'(u) \leq 0$ for $u \in \mathbb{R}_+$ and $f$ is decreasing on $\mathbb{R}_+$. By Eq. (3.16), $f^\infty = 0$ and $f_0 = \infty$. The result follows from Corollary 1.

(ii) If $(\varepsilon, m) \in \cup_{i=2}^6 \Omega_i$, then by Lemma 3.2,

$$f'(u) \leq \eta(\varepsilon, m) \quad \text{for } (\varepsilon, m) \in (0, \infty) \times (-\infty, 1].$$

Let $0 \leq v < u < \infty$. Then there exists $\zeta \in (v, u)$ such that

$$f(u) - f(v) = f'(\zeta)(u - v) \leq \eta(\varepsilon, m)(u - v).$$

Hence, $f$ satisfies (2.7) of Theorem 2.3 with $b = \eta(\varepsilon, m)$. By Eq. (3.16), we obtain $\frac{\nu_1}{f_0} = 0$ and $\min\left\{ \frac{\nu_0}{f_0}, \frac{\mu_1}{f^\infty} \right\} = \nu_1(\varepsilon, m)$. The result follows from Theorem 2.3. \hfill \Box
Remark 3. Theorem 3.3 is new when \( n \geq 3, \varepsilon \in (0, \infty) \) and \( m \in (-\infty, 0) \cup \{1\} \). When \( n \geq 3, \varepsilon \in (0, \infty) \) and \( m \in [0, 1) \), Theorem 3.3 improves [26, Theorems 1.1 and 1.5]. To see this, we refer to Fig. 3, where \( m_3(\varepsilon) = (2\sqrt\varepsilon - 1)\varepsilon^{-1} \). Note that \((\varepsilon, m) \in A_1\) if and only if \( m_3(\varepsilon) < m < 1 \) for \( \varepsilon \in (0, 1) \) if and only if \( 0 < \varepsilon < \left(1 + \sqrt{1 - m}\right)^{-2} \) for \( m \in (0, 1) \).

When \( n \geq 3 \) and \((\varepsilon, m) \in A_1\), Theorem 3.3 shows that Eq. (1.1) has a unique nonzero positive solution in \( P \) for each \( \nu \in (0, \nu_1(\varepsilon, m)) \) while [26, Theorem 1.5] shows that Eq. (1.1) has a unique nonzero positive solution in \( P \) for each \( \nu \in (0, \lambda(\varepsilon, m)) \), where \( \lambda(\varepsilon, m) > 0 \) is a small computable constant. Note that if \( \mu = \text{meas}(\Omega) \) is small, then \( \nu_1(\varepsilon, m) > \lambda(\varepsilon, m) \) since when \( m \in [0, 1) \), \( \nu_1(\varepsilon, m) \) goes to \( \infty \) as \( \mu \) goes to 0. Hence, if \( \mu = \text{meas}(\Omega) \) is small, Theorem 3.3 gives a larger interval \((0, \nu_1(\varepsilon, m))\) for \( \nu \) than that of [26, Theorem 1.5] and generalizes [26, Theorem 1.5].

When \( n \geq 3 \) and
\[
(\varepsilon, m) \in A_2 \cup \left\{ (\varepsilon, m) : \varepsilon \in (1/2, \infty), m \in (0, m_2(\varepsilon)) \right\},
\]
Theorem 3.3 is contained in the second result of [26, Theorem 1.1] which shows that Eq. (1.1) has a unique nonzero positive solution in \( P \) for each \( \nu \in (0, \infty) \).

4. Conclusion remarks. As mentioned in the Introduction, the bimolecular reaction rate corresponds to \( m = 1 \) and the reaction rate with \( m < 0 \) is of physical significance [11, 21, 24, 27, 31]. When \( n \geq 3, \varepsilon \in (0, \infty) \) and \( m \in (-\infty, 0) \cup \{1\} \), Theorem 3.3 provides completely new criteria for the extinction phenomenon never to occur. Indeed, from Theorem 3.3, we conclude that the extinction phenomenon never occurs when \((\varepsilon, m) \in \Omega_1 \) since \( \nu_E = \infty \) and when \( \nu \in (0, \nu_1(\varepsilon, m)) \) and \((\varepsilon, m) \in \bigcup_{i=2}^5 \Omega_i \). We conjecture that the set
\[
\Omega_1 \cup A_2 \cup \left\{ (\varepsilon, m) : \varepsilon \in (1/2, \infty), m \in (0, m_2(\varepsilon)) \right\}
\]
is not the largest set for Eq. (1.1) to have a unique nonzero positive solution in \( P \) for each \( \nu \in (0, \infty) \). It would be interesting subjects to seek regions larger than (4.1) on which the extinction phenomenon never occurs.
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