Heat kernels on regular graphs and generalized Ihara zeta function formulas

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Abstract
We establish a new formula for the heat kernel on regular trees in terms of classical $I$-Bessel functions. Although the formula is explicit, and a proof is given through direct computation, we also provide a conceptual viewpoint using the horocyclic transform on regular trees. From periodization, we then obtain a heat kernel expression on any regular graph. From spectral theory, one has another expression for the heat kernel as an integral transform of the spectral measure. By equating these two formulas and taking a certain integral transform, we obtain several generalized versions of the determinant formula for the Ihara zeta function associated to finite or infinite regular graphs. Our approach to the Ihara zeta function and determinant formula through heat kernel analysis follows a similar methodology which exists for quotients of rank one symmetric spaces.

1 Introduction
Let $X$ be a $(q+1)$-regular graph for an integer $q > 0$. There is an associated heat kernel $K_X(t, x_0, x)$ corresponding to the Laplacian formed by considering the adjacency matrix on $X$. We show that in a natural way the building blocks of $K_X$ are the functions

$$q^{-r/2}e^{-(q+1)t}I_r(2\sqrt{qt}),$$

where $r \in \mathbb{Z}_{\geq 0}$, time $t \in \mathbb{R}_{\geq 0}$, and $I_r$ is the classical $I$-Bessel function of order $r$. The expression for the heat kernel on $X$ comes from a new formula for the heat kernel on regular trees (Proposition 3.1) which we prove in this article. Our expression is quite different from a previous formula due to F. Chung and S.-T. Yau [CY99], which we describe in subsection 3.1 Equations (5a, 5b). If we write the functions in the above stated building block as

$$q^{-r/2} \cdot e^{-(\sqrt{q-1})^2 t} \cdot e^{-(\sqrt{q})^2 / 4t} I_r(2\sqrt{qt}),$$

then there is moreover a near-perfect analogy with the building blocks of typical heat kernel expressions on Riemannian symmetric spaces, which have the form

$$F(r) \cdot e^{-at} \cdot \frac{1}{\sqrt{4\pi t}} e^{-r^2/4t}$$

for certain constants $a, d$ and function $F$ that allow further interpretations; we refer to the survey article [JL01] and the references therein for further discussion.

More precisely, we prove the following result.

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*The first and second authors acknowledge support provided by grants from the National Science Foundation and the Professional Staff Congress of the City University of New York. The third author received support from SNSF grant 200021_132528/1. Support from Institut Mittag-Leffler (Djursholm, Sweden) is also gratefully acknowledged by all authors. We thank Pierre de la Harpe for a number of corrections.
Theorem 1.1. The heat kernel on a \((q+1)\)-regular graph \(X\) is given by

\[ K_X(t, x_0, x) = e^{-(q+1)t} \sum_{m=0}^{\infty} b_m(x)q^{-m/2}I_m(2\sqrt{qt}), \]

where \(I_m\) is the I-Bessel function of order \(m\), \(b_m(x) = c_m(x) - (q-1)(c_{m-2}(x) + c_{m-4}(x) + \ldots)\) and \(c_m(x)\) is the number of geodesics from a fixed base point \(x_0\) to \(x\) of length \(m \geq 0\).

To be specific, we define a **geodesic** in a graph to be a path without back-tracking. The terminology is consistent with concepts from Riemannian geometry where a geodesic is a path which locally is distance minimizing. Moreover, a **closed geodesic** is a closed path without back-tracking or tails. We defer until section 2 for more details on these definitions and for a precise axiomatic characterization of the heat kernel. In view of a combinatorial observation, Proposition 2.1, we may formulate the following result.

Corollary 1.2. In addition to the stated assumptions, suppose that \(X\) is vertex transitive. Let \(N^0_m\) denote the number of closed geodesics of length \(m\) in \(X\) with base point \(x_0\). Then

\[ K_X(t, x_0, x_0) = K_{q+1}(t, x_0, x_0) + e^{-(q+1)t} \sum_{m=1}^{\infty} N^0_m q^{-m/2}I_m(2\sqrt{qt}) \]

where \(K_{q+1}\) denotes the heat kernel of the \((q+1)\)-regular tree.

For finite, not necessarily vertex transitive, graphs \(X\), similar formulas were previously proved by Ahumada [Ahu87], Terras-Wallace [TW03] and Mnëv [Mnë07]. Note that our formula holds for infinite graphs as well, and is therefore more general than the finite graph case. With our methods, one can also deduce the formula for the finite non-vertex transitive case using Proposition 2.2.

There is a second expression for the heat kernel coming from spectral considerations. Equating the two expressions for the heat kernel, as in known approaches to the Poisson summation formula or the Selberg trace formula, one obtains an identity which is a type of theta inversion formula. From the identity, we will apply a certain integral transform, which amounts to a Laplace transform with a change of variables from which we obtain the logarithmic derivative of the Ihara zeta function. This procedure is motivated by McKean [McK72] in his approach to the Selberg zeta function and was axiomatized in [JL01] to abstract settings. In the end, one obtains determinantal formulas for Ihara zeta-like functions. In a special case, we recover the standard formula stemming from Ihara’s work [Iha66], which in turn is generalized in [Has89] and [FZ99] for finite graphs; see subsection 4.3.

We now describe one sample outcome which comes from the above described sequence of calculations. Let \(X\) be a vertex transitive \((q+1)\)-regular graph. We define the associated **Ihara zeta function** of \(X\) by

\[ \zeta_X(u) = \exp \left\{ \sum_{m=1}^{\infty} \frac{N^0_m}{m} u^m \right\}, \]

where \(N^0_m\) is the number of closed geodesics of length \(m\) starting at a fixed vertex \(x_0\). For finite graphs, the classical Ihara zeta function is just our Ihara zeta function raised to the power equaling the number of vertices.

Theorem 1.3. Let \(X\) be a vertex transitive \((q+1)\)-regular graph with spectral measure \(\mu\) for the Laplacian. Then

\[ \zeta_X(u)^{-1} = (1 - u^2)\frac{(q-1)/2}{\exp \left( \int \log (1 - (q+1 - \lambda)u + qu^2) \, d\mu(\lambda) \right)}. \]

Again, we defer to section 2 for the definitions of the Laplacian and spectral measure. There are some papers in the literature defining Ihara zeta functions for infinite graphs, in particular Clair and Mokhtari-Sharghi [CMS01], Grigorchuk-Zuk [GZ04] and Guido, Isola and Lapidus [GIL08]. Their definitions are at least a priori somewhat different in that they typically look at approximations by finite graphs and also using von Neumann trace for group operator algebras. In many
cases it coincides with our definition and thereby the formula in Theorem 1.3 can be recovered in those references.

A number of interesting examples of Ihara zeta functions for infinite Cayley graphs can be found in [GZ04]. Additionally, we refer to the articles [Sun86], [Sun94] and [Sun08] in which the author provides a fascinating discussion in which the spectral and zeta function analysis on graphs is compared to similar studies in spectral theory on symmetric spaces and zeta functions from number theory.

In summary, we have given a new expression for the heat kernel associated to any regular graph. One can quickly deduce the Ihara determinant formula and a number of interesting extensions, not the least of which is to infinite transitive graphs. As is well-known, one main application of such formulas is to the study of counting closed geodesics. For us, it is also significant that our analysis provides yet another instance of when the heat kernel yields zeta functions together with their main functional relation, just as in the case of Riemann, Selberg, and beyond. In particular, the present paper can be viewed in the context of the last section of [JL01].

2 Preliminaries

2.1 Graphs

We follow the definitions in Serre’s book [Ser03]. A graph \( X \) consists of a set \( V_X \) which are called vertices, a set \( E_X \) which are called edges, and two maps

\[
E_X \rightarrow V_X \times V_X, \quad y \mapsto (o(y), t(y))
\]

and

\[
E_X \rightarrow E_X, \quad y \mapsto \overline{y}
\]

such that for each \( y \in E_X \) we have that \( \overline{y} = y \), \( \overline{y} \neq y \) and \( o(y) = t(\overline{y}) \).

The vertices \( o(y) \) and \( t(y) \) are the extremities of the edge \( y \). Two vertices are adjacent if they are extremities of an edge. The degree of a vertex \( x \) is

\[
\deg x = \text{Card}\{y \in E_X : o(y) = x\}.
\]

A graph is \( d \)-regular if each vertex has degree \( d \).

There is an obvious notion of morphism. Let \( PATH_n \) denote the graph with vertices \( 0, 1, 2, \ldots, n \) and (half of) the edges are given by \([i, i+1], i = 0, \ldots, n-1\). A path (of length \( n \)) is a morphism \( c \) from \( PATH_n \) into the graph. The sequences of edges \( y_i = c([i, i+1]) \) (such that \( t(y_i) = o(y_{i+1}) \)) determines the path. In particular a path is oriented. There is a backtracking if for some \( i \) that \( y_{i+1} = \overline{y}_i \) and there is a tail if \( y_0 = \overline{y}_{n-1} \). A path is closed if \( c(0) = c(n) \). A geodesic is a path without backtracking. A geodesic loop (or circuit in Serre’s terminology) is a closed path that is a geodesic. A closed geodesic is a closed path with no tail and without backtracking. (This is analogy with Riemannian geometry where a closed geodesic, as opposed to a geodesic loop, is required to be smooth also at the start/end point). The path of length zero counts as a closed geodesic and, therefore, is a geodesic loop. Additionally, every closed path with one edge counts as a closed geodesic. Any length two geodesic loop is also a closed geodesic, but the closed path \( y, \overline{y} \) is neither.

A prime geodesic is an equivalence class of closed geodesics \([c]\), where the equivalence class is forgetting the starting point and which is primitive in the sense that it is not a power of another closed geodesic. The latter means by definition that there is no closed geodesic \( d \) and integer \( n > 1 \) such that \([c] = [d^n]\), which says in words that \( c \) is not just a geodesic that traverses another one \( n \) number of times.

An orientation is a subset \( E_X^+ \) of edges such that \( E_X \) is the disjoint union of \( E_X^+ \) and \( \overline{E_X}^+ \). With the data of a graph one can associate a geometric realization: start with the discrete topology, take \( V_X \times [0, 1] \) and make identification based on the maps \( o \) and \( t \).

A tree is a connected nonempty graph without geodesic loops.
We will in particular consider vertex transitive graphs, that means that there is a group of automorphisms which is transitive on the vertices. In particular such a graph is of course regular. A rich source of such graphs is provided by Cayley graphs of groups: Let $G$ be a group and let $S$ be a subset of $G$. We denote by $X(G, S)$ the oriented graph having $G$ as vertices and $EX_+ = G \times S$ with $o(g, s) = g$ and $t(g, s) = gs$ for each edge $(g, s)$.

Let $X$ be a graph on which a group $G$ acts. An inversion is a pair consisting of an element $g$ and an edge $y$ such that $gy = y$. If $G$ acts without inversions (which is the same as saying that there is an orientation of $X$ preserved by $G$) we can define the quotient graph $G \setminus X$ in an obvious way; the respective edge and vertex sets are the corresponding quotients. (To get rid of inversions one may pass to a barycentric division.) As in topology we say that $X$ is a regular covering of $Y$ if there is a group which acts on $X$ freely and without inversion with quotient $Y$.

Let $X$ be a $(q + 1)$-regular graph. Then its universal covering is the $(q + 1)$-regular tree, and the covering group acts freely on the tree without inversion. The covering group is a free group.

### 2.2 Path counting in graphs

We fix a base vertex $x_0$ in a graph $X$ and define the following counting functions which will be used in this paper:

- $a_k(x)$ is the number of paths of length $k$ from $x_0$ to $x$,
- $c_k(x)$ is the number of geodesics of length $k$ from $x_0$ to $x$,
- $c_k^0 = c_k(x_0)$ is the number of geodesic loops of length $k$ starting at $x_0$,
- $c_k$ the number of geodesic loops of length $k$, from some starting point with a distinct direction,
- $N_k^0$ is the number of closed geodesics of length $k$ starting at $x_0$,
- $N_k$ is the number of closed geodesics of length $k$, from some starting point with a distinct direction,
- $\pi_k$ is the number of prime geodesics of length $k$.

The sequences $\{c_k\}$, $\{N_k\}$ and $\{\pi_k\}$ only make sense for finite graphs, since if the graph is infinite, these values are typically infinite. For finite vertex transitive graphs, the sequences $\{N_k^0\}$ and $\{N_k\}$ are related by the number of vertices, i.e. starting points. Specifically, if the graph $X$ has $n$ vertices, then $N_k^0 \cdot n = N_k$ for all $k$. Also, $N_k$ and $\pi_k$ have a precise relationship, see e.g. [Ter11] or [GZ04]. Finally, we recall the conventions that $c_0^0 = N_0^0 = 1$, $c_1^0 = N_1^0$ and $c_2^0 = N_2^0$.

**Proposition 2.1.** Let $X$ be a transitive $(q + 1)$-regular graph. Then for $k \geq 3$, following relation holds true

$$N_k^0 = c_k^0 - (q - 1)(c_{k-2}^0 + c_{k-4}^0 + \ldots)$$

the last term being $c_0^0$ or $c_2^0$ depending on the parity of $k$.

**Proof.** This is similar to an argument in [Ser97]. A geodesic loop of length $k \geq 3$ which is not a closed geodesic has the form $y_1 \cdots y_1$ where $z$ is a geodesic loop of length $k - 2$. There are two possibilities, either $z$ is a closed geodesic or not. If we fix $z$, then the number of possibilities for $y_1$ is $q - 1$ in the first case and $q$ in the second case. Since $X$ is vertex transitive, we may freely change the starting point of any loop, namely $z$. With this in mind, we obtain the recursive relation that

$$c_k^0 - N_k^0 = (q - 1)N_{k-2}^0 + q(c_{k-2}^0 - N_{k-2}^0) = (c_{k-2}^0 - N_{k-2}^0) + (q - 1)c_{k-2}^0,$$

which we can write as

$$N_k^0 - N_{k-2}^0 = c_k^0 - qc_{k-2}^0.$$

Using that $c_1^0 = N_1^0$ and $c_2^0 = N_2^0$, the proposition follows by induction on $k$. □

With a proof similar to the one given in the above proposition, we obtain the following result.
2.2 Proposition. Let $X$ be a finite $(q+1)$-regular graph. Then for $k \geq 3$, the following relation holds true

\[ N_k = c_k - (q-1)(c_{k-2} + c_{k-4} + \ldots) \]

the last term being $c_1$ or $c_2$ depending on the parity of $k$.

2.3 The combinatorial Laplacian and heat kernel

Given a $(q+1)$-regular graph $X$ and function $f$ on the vertices $X$, the Laplacian of $f$, written as $\Delta f$, is the function of the vertices of $X$ which is defined by the formula

\[ \Delta f(x) = (q+1)f(x) - \sum_{e \text{ s.t. } o(e)=x} f(t(e)). \]

The Laplacian is a semi-positive, bounded self-adjoint operator on $L^2(VX)$. For a finite graph with $N$ vertices we label the eigenvalues of $\Delta$ as follows: $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{N-1} \leq 2(q+1)$.

The heat kernel $K_X(t,x,y) : \mathbb{R}_\geq 0 \times X \times X \to \mathbb{R}$ on $X$ is the solution of

\[ \begin{align*}
\Delta K_X(t,x_0,x) + \frac{\partial}{\partial t} K_X(t,x_0,x) &= 0 \\
K_X(0,x_0,x) &= \begin{cases} 1 & \text{if } x = x_0 \\
0 & \text{otherwise.} \end{cases}
\end{align*} \tag{1a} \tag{1b} \]

One can show that the heat kernel decays sufficiently rapidly so that the above equalities are not only formal, but indeed are convergent series. We refer to [CJK10, DM06] as well as the heat kernel formula for trees in Proposition 3.1 and the bounds in subsection 2.5.

Consider the numbers $a_n(x)$ defined by

\[ e^{(q+1)t}K_X(t,x) = \sum_{n=0}^{\infty} a_n(x) \frac{t^n}{n!} \]

then it is well-known and simple to see that $a_n(x)$ is the number of paths from $x_0$ to $x$ as defined above.

2.4 The $I$-Bessel function

Classically, the $I$-Bessel function $I_x(t)$ is defined as a certain solution to the differential equation

\[ t^2 \frac{d^2w}{dt^2} + t \frac{dw}{dt} - (t^2 + x^2) = 0. \]

For integer values of $x$, it is immediately shown that $I_x = I_{-x}$ and, for positive integer values of $x$, we have the series representation

\[ I_x(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+x}}{n! \Gamma(n+1+x)} \tag{2} \]
as well as the integral representation

\[ I_x(t) = \frac{1}{\pi} \int_0^\pi e^{t \cos(\theta)} \cos(\theta x) d\theta. \] (3)

The mathematical literature contains a vast number of articles and monographs which study the many fascinating properties and manifestations of the \( I \)-Bessel functions, as well as other Bessel functions. The connection with the discrete heat equation comes from the basic relation

\[ I_{x+1}(t) + I_{x-1}(t) = 2 \frac{d}{dt} I_x(t), \] (4)

which easily can be derived from the integral representation and trigonometric identities. This relation will be used in the proof of Proposition 3.1.

### 2.5 Universal bounds for the \( I \)-Bessel function

We have the following uniform bounds from [CJK10] which used [Pal99]. For any \( t > 0 \) and integer \( x \geq 0 \), we have that

\[ \sqrt{t} \cdot e^{-t} I_x(t) \leq \left( \frac{t}{t+x} \right)^{x/2} = \left( 1 + \frac{x}{t} \right)^{-x/2}. \]

As stated, the above bound is enough to show that the periodization procedure in our setting gives rise to convergent sum expressions for the heat kernel.

### 2.6 An integral transform of I-Bessel

For integers \( n \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) \neq 0 \) we have e.g. from [OB73], that

\[ \int_0^\infty e^{-st} e^{-t} I_n(t) dt = \frac{(s + 1 - \sqrt{s^2 + 2})^n}{\sqrt{s^2 + 2}}. \]

We will consider the transform, essentially the Laplace transform,

\[ Gf(u) = (u^{-2} - q) \int_0^\infty e^{-(qu+1/u)t} e^{(q+1)t} f(t) dt. \]

In view of the above formula, applying the transform to the heat kernel building block, we get

\[ G \left( e^{-(q+1)t} q^{-k/2} I_k(2\sqrt{qt}) \right)(u) = u^{k-1} \]

for \( k \geq 0 \) and \( u > 0 \).

### 3 Heat kernels on regular graphs

#### 3.1 A heat kernel expression for regular trees

Let \( X \) be the \((q+1)\)-regular tree and \( x_0 \in X \) a base point. From its characterizing properties [La] and [B], it is immediate to show that the heat kernel on a graph is invariant with respect to any graph automorphism \( g \):

\[ K(t, gx_0, gx) = K(t, x_0, x). \]

In particular, we have that the heat kernel \( K(t, x_0, x) \) on the tree \( X \) is radial, that is it depends only on \( r = d(x_0, x) \). Therefore we can write the heat kernel as \( K(t, r) \). Expressions for \( K(t, r) \) were established by Bednarchak [Bed97], Chung and Yau [CY99], Cowling, Meda and Setti [CMS00], as well as Horton, Newland and Terras [HNT06]. In the physics literature regular trees are called
Bethe lattices. As stated, one of the main results of this paper is a formula for the heat kernel on \( X \), which we consider to be new since we did not find the expression in either the mathematical or physics literature.

As an example of a known expression, Chung and Yau \cite{CY99} prove that in the radial coordinate \( r \), the heat kernel of the \((q+1)\)-regular tree is given by

\[
K(t, r) = \frac{2e^{-(q+1)t}}{\pi q r^{2q-1}} \int_{0}^{\pi} \exp \left( 2t \sqrt{q} \cos u \right) \frac{\sin u (q \sin (r+1)u - \sin (r-1)u)}{(q+1)^2 - 4q \cos^2 u} du
\]  

for \( r > 0 \), and

\[
K(t, 0) = \frac{2q(q+1)e^{-(q+1)t}}{\pi} \int_{0}^{\pi} \exp \left( 2t \sqrt{q} \cos u \right) \frac{\sin^2 u}{(q+1)^2 - 4q \cos^2 u} du.
\]

The formula which we prove is given in the following proposition.

**Proposition 3.1.** The heat kernel of the \((q+1)\)-regular tree is given in the radial coordinate \( r \) as

\[
K(t, r) = q^{-r/2}e^{-(q+1)t}I_r(2\sqrt{q}t) - (q-1) \sum_{j=1}^{\infty} q^{-(r+2j)/2} e^{-(q+1)t}I_{r+2j}(2\sqrt{q}t),
\]

where \( I \) denotes the I-Bessel function.

**Proof.** It is immediate that when \( t = 0 \), the above series is equal to \( \delta_0(r) \) as required, since \( I_x(0) = 0 \) for \( x \neq 0 \) and \( I_0(0) = 1 \). Since \( K(0, r) = \delta_0(r) \), it remains to show that the above series is equal to \( K(t, 0) \) for \( t > 0 \).

Denote by \( f(r) = K(t, r) \) and \( \dot{f}(r) = \partial K(t, r) / \partial t \). If \( r = 0 \), then the differential equation (1a) for the heat kernel takes the form

\[
(q + 1)f(0) - (q + 1)f(1) + \dot{f}(0) = 0,
\]

and for \( r > 0 \) the differential equation becomes

\[
(q + 1)f(r) - q f(r+1) - f(r-1) + \dot{f}(r) = 0.
\]

Let \( g(r) \) be the above series expansion times \( e^{(q+1)t} \), or, when written out,

\[
g(r) = q^{-r/2}I_r(2\sqrt{q}t) - (q-1) \sum_{j=1}^{\infty} q^{-(r+2j)/2}I_{r+2j}(2\sqrt{q}t).
\]

It is an elementary exercise to show that the series expansion satisfies the characterizing differential equation for the heat kernel if and only if we have the differential equations

\[
- (q + 1)g(1) + \dot{g}(0) = 0, \quad \text{(6)}
\]

and for every \( r > 0 \)

\[
- qg(r+1) - g(r-1) + \dot{g}(r) = 0. \quad \text{(7)}
\]

Let us first verify (6). We begin by writing the

\[\text{Left-hand-side of (6) } = -(q + 1)q^{-(1/2)}I_1(2\sqrt{q}t) + (q + 1)(q-1) \sum_{j=1}^{\infty} q^{-(1+2j)/2}I_{1+2j}(2\sqrt{q}t) + 2\sqrt{q}I_0(2\sqrt{q}t) - (q-1)2\sqrt{q} \sum_{j=1}^{\infty} q^{-2j}I_{2j}(2\sqrt{q}t).\]


Using the basic relation $I_{r-1}(z) + I_{r+1}(z) = 2I_r(z)$, we obtain the expression

Left-hand-side of (8) = $-(q + 1)q^{-1/2}I_1(2\sqrt{q}t) + (q + 1)(q - 1)\sum_{j=1}^{\infty} q^{-(1+2j)/2} I_{1+2j}(2\sqrt{q}t)$

$$+ \sqrt{q}(I_1(2\sqrt{q}t) + I_{-1}(2\sqrt{q}t))$$

$$- (q - 1)\sqrt{q}\sum_{j=1}^{\infty} q^{-j} I_{2j+1}(2\sqrt{q}t) + I_{2j-1}(2\sqrt{q}t)).$$  (8)

Collecting terms, and recalling that $I_{-1} = I_1$, we can evaluate the coefficient of each $I$-Bessel function in (8):

$I_0$ : there are no $I_0$ terms,

$I_1$ : $-(q + 1)q^{-1/2} + 2\sqrt{q} - (q - 1)\sqrt{q}q^{-1} = 0$,

$I_{2j+1}$ : $(q^2 - 1)q^{-(1+2j)/2} - (q - 1)\sqrt{q}q^{-j} - (q - 1)\sqrt{q}q^{-(j+1)}$

$$= (q - 1)q^{-j}((q + 1)q^{-1/2} - q^{1/2} - q^{-1/2}) = 0.$$

In other words, the left-hand-side of (8) is zero, as required.

Let us now check the case when $r > 0$. Again, we begin by writing the

Left-hand-side of (7) = $-q^{1-(r+1)/2}I_{r+1}(2\sqrt{q}t) + q(q - 1)\sum_{j=1}^{\infty} q^{-(r+1+2j)/2} I_{r+1+2j}(2\sqrt{q}t)$  (9)

$$- q^{-(r-1)/2}I_{r-1}(2\sqrt{q}t) + (q - 1)\sum_{j=1}^{\infty} q^{-(r-1+2j)/2} I_{r-1+2j}(2\sqrt{q}t)$$

$$+ 2\sqrt{q}q^{-r/2}I_r'(2\sqrt{q}t) - 2\sqrt{q}(q - 1)\sum_{j=1}^{\infty} q^{-(r+2j)/2} I_{r+2j}(2\sqrt{q}t)$$

$$= -q^{1/2-r/2}I_{r+1}(2\sqrt{q}t) + q(q - 1)\sum_{j=1}^{\infty} q^{-(r+1+2j)/2} I_{r+1+2j}(2\sqrt{q}t)$$

$$- q^{-(r-1)/2}I_{r-1}(2\sqrt{q}t) + (q - 1)\sum_{j=1}^{\infty} q^{-(r-1+2j)/2} I_{r-1+2j}(2\sqrt{q}t)$$

$$+ \sqrt{q}q^{-r/2}(I_{r+1}(2\sqrt{q}t) + I_{r-1}(2\sqrt{q}t))$$

$$- \sqrt{q}(q - 1)\sum_{j=1}^{\infty} q^{-(r+2j)/2}(I_{r+1+2j}(2\sqrt{q}t) + I_{r-1+2j}(2\sqrt{q}t)).$$  (10)

As above, we can evaluate the coefficient of each $I$-Bessel function in (9):

$I_{r-1}$ : $-q^{-(r-1)/2} + \sqrt{q}q^{-r/2} = 0$

$I_r$ : there are no $I_r$ terms,

$I_{r+1}$ : $q^{1/2-r/2} + (q - 1)q^{-(r+1)/2} + \sqrt{q}q^{-r/2} - \sqrt{q}(q - 1)q^{-(r+2)/2} = 0$

$I_{r+2j+1}$ : $(q - 1)q^{-(r+1+2j)/2} + (q - 1)q^{-(r+1+2j)/2}$

$$- \sqrt{q}(q - 1)q^{-(r+2j)/2} - \sqrt{q}(q - 1)q^{-(r+2j+2)/2}$$

$$= (q - 1)q^{-r/2}q^{-j}(q^{1/2} + q^{-1/2} - q^{1/2} - q^{-1/2}) = 0.$$

In other words, the left-hand-side of (9) is zero, as required, which completes the proof of the proposition. □

In the following subsection we indicate another approach to the proof of Proposition 3.1.
3.2 The horospherical transform

Every geodesic ray $\gamma$ in the tree emanating from a fixed base point $x_0$ can be viewed as an “ideal boundary point at infinity”. To each such $\gamma$ there are associated horospheres, one for each integer $n$:

$$\mathcal{H}_n = \{x \in X : \lim_{k \to \infty} [d(\gamma(k), x) - k] = n\}$$

where $d$ is the natural combinatorial distance in the graph.

We fix a geodesic ray $\gamma$ and may then consider the associated horospherical transform of functions $f : X \to \mathbb{R}$ denoted by

$$Hf : \mathbb{Z} \to \mathbb{R}$$

and defined by $Hf(n) = \sum_{x \in \mathcal{H}_n} f(x)$. For a radial function, decaying fast enough, we have the inversion formula

$$f(r) = q^{-r}(Hf)(r) - (q - 1) \sum_{j=1}^{\infty} q^{-(r+2j)}(Hf)(r + 2j)$$

(11)

for $r \geq 0$. This is stated in [HNT06] on pages 7-8 for $f$ of finite support.

If we apply the horospherical transform to the difference-differential equation characterizing the heat kernel, we get

$$(q + 1)HK(t, n) - (qHK(t, n + 1) + HK(t, n - 1)) + \partial_t HK(t, n) = 0,$$

for $n \in \mathbb{Z}$ and with $HK(0, n) = \delta_0(n)$. The solution to this difference-differential equation can be seen to be (cf. section 3.2 in [HNT06] or [KN06])

$$f(t, n) = q^{-n/2}e^{-(q+1)t}I_n(2\sqrt{qt}).$$

As already remarked, the heat kernel on a regular tree is radial, so by inserting the above expression into the inversion formula (11) (here we need to go beyond finitely supported functions for which this formula was stated in [HNT06]) we can get a different proof of Proposition 3.1.

3.3 Heat kernels on regular graphs

Let $q > 0$ be an integer and $X$ a $(q+1)$-regular graph. We fix a base point $x_0 \in VX$ which we will suppress in the notation. The heat kernel on $X$ can be obtained from periodizing the heat kernel $K_{q+1}$ on the universal covering space, the $(q+1)$-regular tree $T_{q+1}$, over the covering group $\Gamma$. Following the remarks in subsection 3.3 we have (with a slight abuse of notation) that

$$K_X(t, x) = \sum_{\gamma \in \Gamma} K_{q+1}(t, \gamma x).$$

Recall that $c_n(x)$ denotes the number of paths without backtracking from the identity $x_0$ to $x$ of length $n$ in $X$. We also use $c_n^0 = c_n(x_0)$ as notation for the number of geodesic loops, i.e. closed paths without backtracking, starting at $x_0$. Note that $c_n(x)$ is equal to the number of elements of the form $\gamma x$ for some $\gamma \in \Gamma$ on the radius $n$ sphere in $T_{q+1}$. We therefore have

$$K_X(t, x) = \sum_{n \geq 0} c_n(x)K_{q+1}(t, n)$$

or more explicitly by inserting the expression from Proposition 3.1 for $K_{q+1}(t, n)$,

$$K_X(t, x) = e^{-(q+1)t} \sum_{n \geq 0} c_n(x) \sum_{j=0}^{\infty} d_q(j)q^{-n/2-j}I_{n+2j}(2\sqrt{qt}),$$

where $d_q(j)$ is 1 if $j = 0$ and $1 - q$ otherwise. A rearrangement of the terms gives

$$K_X(t, x) = e^{-(q+1)t} \sum_{m \geq 0} b_m(x)q^{-m/2}I_m(2\sqrt{qt}),$$

where
where $b_m(x) = c_m(x) - (q-1)(c_{m-2}(x) + c_{m-4}(x) + \ldots)$ where the last term is $c_1(x)$ if $m$ is odd and $c_0(x)$ if $m$ is even. This is also with the understanding that $b_0(x) = c_0(x)$ and $b_1(x) = c_1(x)$. This proves Theorem 1.1.

Now specialize to $x = x_0$. In view of Propositions 2.1 and 3.1 we obtain Corollary 1.2 as well.

### 3.4 Spectral theory

An excellent reference here is that of Mohar and Woess [MW89]. One has that there are spectral measures $\mu_x$ such that (suppressing $x_0$)

$$K_X(t, x) = \int e^{-\lambda t} d\mu_x(\lambda). \quad (12)$$

In particular if $X$ is a finite graph with $n$ vertices, then the Laplacian has eigenvalues $0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_{n-1}$ and corresponding orthonormal eigenfunctions $\phi_j$. The heat kernel may thus be written as

$$K_X(t, x_0, x) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-\lambda_j t} \phi_j(x) \phi_j(x_0). \quad (13)$$

### 4 Ihara formulas

In this final section we describe how the $G$-transform introduced in Section 2.6 applied to the heat kernel gives rise to the Ihara zeta function.

#### 4.1 Zeta functions

Motivated by Selberg’s work, Ihara defined a zeta function for a finite graph $X$, which is now referred to as the Ihara zeta function. The product formula for the Ihara zeta function is

$$\zeta_{Ih}^X(u) = \prod_{[P]} (1 - u^l(P))^{-1}$$

where the product is over equivalence classes of prime geodesics and $l$ the length. (Actually, Ihara worked in a specific group setting, but Serre remarked in the preface of [Ser03] that the definition could be given a simple interpretation in terms of graphs.) By a general calculation, see for example [Ter11, p. 29], one has

$$\log \zeta_{Ih}^X(u) = \sum_{m=1}^{\infty} \frac{N_m}{m} u^m,$$

where $N_m$ is the number of closed geodesics of length $m$. Thus the Ihara function is a zeta type function similar to those appearing in the classical works of Artin, Hasse, and Weil on counting points of varieties in finite fields.

The numbers $N_m$ are not defined for infinite graphs $X$. In the case of transitive graphs, a natural replacement is $N_m^0$, since in the finite case one has $N_m = nN_m^0$, where $n$ is the number of vertices. So we can define a zeta function for any (not necessarily finite) vertex transitive graph via

$$\log \zeta_X(u) = \sum_{m=1}^{\infty} \frac{N_m^0}{m} u^m.$$

More exotically one could define a two variable function $\zeta$ via

$$\log \zeta_X(u, x) = \sum_{m=1}^{\infty} \frac{b_m(x)}{m} u^m.$$

Following Riemann one has another set of zeta functions by instead taking the Mellin transform of the heat kernel. This is a subject for another paper.
4.2 The periodization side

We will apply the transform

$$Gf(u) = (u^{-2} - q) \int_0^\infty e^{-(q+1)/u} t e^{(q+1)t} f(t) dt$$

first to our heat kernel expression

$$K_X(t, x_0, x) = e^{-(q+1)t} \sum_{m=0}^\infty b_m(x) q^{-m/2} I_m(2\sqrt{qt})$$

of Theorem 4.1. Using the basic formula in subsection 2.6, in the case \( x \neq x_0 \), so \( b_0(x) = 0 \) we have that the \( G \)-transform of the heat kernel is equal to

$$\frac{1}{u} \sum_{m=0}^\infty b_m(x) u^m = \frac{\partial}{\partial u} \sum_{m=1}^\infty \frac{b_m(x)}{m} u^m = \frac{\partial}{\partial u} \log \zeta_X(u, x).$$

In the case \( x = x_0 \), the \( G \)-transform of the heat kernel on the diagonal is equal to

$$\frac{1}{u} \sum_{m=0}^\infty b_m u^m = \frac{\partial}{\partial u} \sum_{m=1}^\infty b_m u^m + \frac{\partial}{\partial u} \log u.$$

In the vertex transitive case, for the case \( x = x_0 \) we apply the transform to the expression of Corollary 4.2

$$K_X(t, x_0, x_0) = K_{q+1}(t, x_0) + e^{-(q+1)t} \sum_{m=1}^\infty N_m q^{-m/2} I_m(2\sqrt{qt})$$

which gives (summing the geometric series arising from the first term)

$$1/u - (q - 1) \frac{u}{1-u^2} + \frac{1}{u} \sum_{m=1}^\infty N_m^0 u^m.$$

This can in turn be written as

$$\frac{\partial}{\partial u} \log u + \frac{q - 1}{2} \frac{\partial}{\partial u} \log(1 - u^2) + \frac{\partial}{\partial u} \sum_{m=1}^\infty \frac{N_m^0}{m} u^m$$

$$= \frac{\partial}{\partial u} \log u + \frac{q - 1}{2} \frac{\partial}{\partial u} \log(1 - u^2) + \frac{\partial}{\partial u} \log(\zeta_X(u)).$$

We have proven

**Proposition 4.1.** For \( X \) a \((q+1)\)-regular vertex transitive graph,

$$(GK_X)(\cdot, x_0)(u) = \frac{\partial}{\partial u} \left[ \log u + \frac{q - 1}{2} \log(1 - u^2) + \log \zeta_X(u) \right].$$

In summary, the \( G \)-transform of the heat kernel yields expressions involving the Ihara zeta function together with other trivial terms. In the setting of compact quotients of rank one symmetric spaces, there is a similar change of variables in the Laplace transform so that when applied to the trace of the heat kernel, one obtains the Selberg zeta function. With this in mind, our approach to the Ihara zeta function as an integral transform of the heat kernel is in line with a known approach to the Selberg zeta function in many settings.
4.3 Ihara’s determinantal formula

Now we deduce the classical Ihara determinantal formula. Let $X$ be a $(q+1)$-regular graph of finite vertex cardinality $n$. From (13),

$$K_X(t,x_0) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-\lambda_j t}.$$ \n
The $G$-transform of the righthand side is a simple integration which yields

$$\frac{1}{n} (u-2 - q) \sum_{j=0}^{n-1} qu + 1/u - (q+1-\lambda_j) = -\frac{1}{n} \frac{\partial}{\partial u} \sum_{j=0}^{n-1} \log \frac{1}{u} (1 - (q+1 - \lambda_j)u + qu^2).$$

Comparing this last expression with Proposition 4.1 from the periodization side (and also verifying that integration constants match up) we immediately get Ihara’s formula, namely

$$\zeta_{Ih}^{X}(u) = (1 - u^2)^{n(q-1)/2} \det((1 - (q+1)u + qu^2)I + \Delta u),$$

since, as remarked above,

$$\frac{1}{u} \sum_{m=1}^{\infty} N_m u^m = \frac{\partial}{\partial u} \log \zeta_{Ih}^{X}(u).$$

This formula is also known to hold more generally for non-regular graphs, see the references mentioned in the introduction.

4.4 First extension of Ihara’s formula

Using spectral theory, we obtain a similar formula for infinite transitive graphs, this time using our zeta function instead of Ihara’s. The common point is that both zeta functions are obtained as $G$-transforms of the heat kernel, and the determinantal formula follows from having another expression for the heat kernel, namely that which comes from spectral theory.

We use the notation $\mu = \mu_{x_0}$. Since the functions involved are positive, we may change the order of integration in our integral transforms and arrive at the expression Equating the expression in Proposition 4.1 with the $G$-transform of the spectral expansion of the heat kernel given in (12), we arrive at

$$\frac{\partial}{\partial u} \left[ \log u + \frac{q-1}{2} \log(1 - u^2) + \log \zeta_X(u) \right] = -\frac{\partial}{\partial u} \int \log \frac{1}{u} (1 - (q+1 - \lambda)u + qu^2) \, d\mu(\lambda).$$

Note: since the functions involved are positive, we are justified in interchanging the spectral integral with the $G$-transform integral on the righthand side. We now integrate this equality, noting that at $u = 0$ both sides are 0 to determine the integration constants. We get the formula

$$\log u + \frac{q-1}{2} \log(1 - u^2) + \log \zeta_X(u) = -\int \log \frac{1}{u} (1 - (q+1 - \lambda)u + qu^2) \, d\mu(\lambda),$$

which leads to

$$\zeta_X(u)^{-1} = (1 - u^2)^{(q-1)/2} \exp \left[ \int \log (1 - (q+1 - \lambda)u + qu^2) \, d\mu(\lambda) \right]. \tag{14}$$

This is Theorem 1.3 We further remark that equation (14) clearly generalizes the Ihara determinant formula since for vertex transitive graphs with a finite number $n$ vertices one has $\zeta_{Ih}^{n} = \zeta_{X}^{n}$. 

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4.5 Second extension of Ihara’s formula

Here we do not specialize to $x = x_0$. The resulting identities involve counting geodesics paths, not only closed geodesics paths. Alternatively, as in the most classical situation, our consideration corresponds to computing the Hurwitz zeta function instead of the Riemann zeta function. With the same calculations as above, one gets in the finite graph case, (at one point one uses orthogonality of eigenfunctions) the formula

$$-\log \zeta_X(u, x) = \frac{1}{n} \sum_{j=0}^{n-1} f_j(x) \bar{f}_j(x_0) \log(1 - (q + 1 - \lambda_j)u + qu^2).$$

Thus the eigenfunctions come in to determine the more precise count of geodesics. The lead asymptotic as the length goes to infinity behaves the same as for the closed geodesics since the trivial eigenvalue has the constant function as eigenfunction. From an intuitive viewpoint, this observation is clear: For a fixed $x$ and large length $m$ the geodesics do not look much different from a closed geodesic. In symbols, if $m \gg 1$ then $x \approx x_0$.

We have the analogous formula for infinite regular graphs, namely that

$$-\log \zeta_X(u, x) = \int \log(1 - (q + 1 - \lambda_j)u + qu^2)d\mu_x(\lambda).$$

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