Estimate of dimension of Noether-Lefschetz locus for Beilinson-Hodge cycles on open complete intersections

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0. Introduction

In his lectures in [G1], M. Green gives a lucid explanation how fruitful the infinitesimal method in Hodge theory is in various aspects of algebraic geometry. A significant idea is to use Koszul cohomology for Hodge-theoretic computations. The idea originates from Griffiths work [Gri] where the Poincaré residue representation of the cohomology of a hypersurface played a crucial role in proving the infinitesimal Torelli theorem for hypersurfaces. Since then many important applications of the idea have been made in different geometric problems such as the generic Torelli problem and the Noether-Lefschetz theorem for Hodge cycles and the study of algebraic cycles (see [G1, Lectures 7 and 8]).

In this paper we apply the method to study an analog of the Noether-Lefschetz theorem in the context of Beilinson’s Hodge conjecture. Beilinson’s Hodge conjecture and its Tate variant concern the regulator maps for open varieties (cf. [J1, Conjecture 8.5 and 8.6]).

To be more precise we let $U$ be a smooth variety over a field $k$ of characteristic zero.

(0-1) (Hodge version) When $k = \mathbb{C}$ we have the regulator map from the higher Chow group to the singular cohomology of $U(\mathbb{C})$ (cf. [Bl] and [Sch])

$$reg^q_U : CH^q(U, q) \otimes \mathbb{Q} \rightarrow H^q(U, \mathbb{Q}(q)) \cap F^qH^q(U, \mathbb{C})$$
\[ Q(q) = (2\pi \sqrt{-1})^q \mathbb{Q} \] and \( F^* \) denotes the Hodge filtration of the mixed Hodge structure on the singular cohomology defined by Deligne [D1]. Taking a smooth compactification \( U \subset X \) with \( Z = X \setminus U \), a simple normal crossing divisor on \( X \), we have the following formula for the value of \( \text{reg}_U \) on decomposable elements in \( CH^q(U,q) \):

\[
\text{reg}_U^q(\{g_1, \ldots, g_q\}) = \frac{dg_1}{g_1} \wedge \cdots \wedge \frac{dg_q}{g_q} \in H^0(X, \Omega^q_X(\log Z)) = F^q H^q(U, \mathbb{C}),
\]

where \( \{g_1, \ldots, g_q\} \in CH^q(U,q) \) is the products of \( g_j \in CH^1(U,1) = \Gamma(U, \mathcal{O}_{\text{Zar}}) \).

\[ \text{(0-2)} \] (Tate version) When \( k \) is a finitely generated field over \( \mathbb{Q} \) we have the \( \acute{e} \text{tale} \) regulator map from the higher Chow group to the \( \ell \)-adic \( \acute{e} \text{tale} \) cohomology

\[
\text{reg}^q_{\text{et},U} : CH^q(U, q) \otimes \mathbb{Q}_\ell \to H^q_{\text{et}}(U_{\mathbb{C}_\ell}, \mathbb{Q}_\ell(j))^{\text{Gal}(\overline{k}/k)}
\]

where \( U_{\mathbb{C}_\ell} = U \times_k \overline{k} \) with \( \overline{k} \), an algebraic closure of \( k \).

**Conjecture 0.1** In case \( k = \mathbb{C} \), \( \text{reg}^q_U \) is surjective.

**Conjecture 0.2** In case \( k \) is finitely generated over \( \mathbb{Q} \), \( \text{reg}^q_{\text{et},U} \) is surjective.

We call them Beilinson’s Hodge and Tate conjectures respectively. They are analogs of the Hodge and Tate conjectures for algebraic cycles on smooth projective varieties. The following are some remarks on the conjectures.

(i) The conjectures hold in case \( q = 1 \). They follows from the injectivity of Abel-Jacobi map (and its \( \ell \)-adic variant) for divisors on smooth proper varieties.

(ii) If \( U \) is proper over \( k \), we have

\[
H^q(U, Q(q)) \cap F^q H^q(U, \mathbb{C}) = 0 \quad \text{in case} \ k = \mathbb{C},
\]

\[
H^q_{\text{et}}(U_{\mathbb{C}_\ell}, \mathbb{Q}_\ell(j))^{\text{Gal}(\overline{k}/k)} = 0 \quad \text{in case} \ k \text{ is finitely generated over} \ \mathbb{Q},
\]

by the Hodge symmetry and by the reason of weight respectively. Thus Beilinson’s conjectures are interesting only for non-proper varieties.

(iii) When \( X \) is a projective smooth surface and \( U \subset X \) is the complement of a simple normal crossing divisor \( Z \subset X \), then the surjectivity of \( \text{reg}^2_U \) or \( \text{reg}^2_{\text{et},U} \) has an implication on the injectivity of the regulator maps for \( CH^2(X,1) \), which can be viewed as an analog of the Abel’s theorem for \( K_1 \) of surfaces. The detail will be explained in \( \S 6 \).

(iv) As a naive generalization of Beilinson’s Hodge conjecture, one may expect the surjectivity of the more general regulator map (in case \( k = \mathbb{C} \))

\[
\text{reg}^{p,q}_U : CH^q(U, 2q - p) \otimes \mathbb{Q} \to (2\pi \sqrt{-1})^q W_{2q} H^p(U, \mathbb{Q}) \cap F^q H^p(U, \mathbb{C})
\]

where \( W_* \) denotes the weight filtration. Jannsen ([J1, 9.11]) has shown that the above map in case \( p = 1 \) and \( q \geq 3 \) is odd, is not surjective in general by using a theorem of Mumford [Mu], which implies the Abel-Jacobi for cycles of codimension \( \geq 2 \) on smooth projective varieties is not injective even modulo torsion.
Before introducing the setup of the main result of this paper, we explain its background, the Noether-Lefschetz problem for Hodge cycles. Let $X \subset \mathbb{P}^n$ be a smooth projective variety over $\mathbb{C}$. Recall that the Hodge conjecture predicts that the space of Hodge cycles in codimension $q$ on $X$:

$$F^0H^{2q}(X, \mathbb{Q}(q)) := H^{2q}(X, \mathbb{Q}(q)) \cap F^qH^{2q}(X, \mathbb{C})$$

is generated by classes of algebraic subvarieties on $X$. One defines the space of trivial cycles in codimension $q$ on $X$ to be

$$H^{2q}(X, \mathbb{Q}(q))_{\text{triv}} := \text{Image}(H^{2q}(\mathbb{P}^n, \mathbb{Q}(q)) \to H^{2q}(X, \mathbb{Q}(q))) \subset F^0H^{2q}(X, \mathbb{Q}(q)).$$

It is generated by the class of the section on $X$ of a linear subspace of codimension $q$ in $\mathbb{P}^n$. Now let $S$ be a non-singular quasi-projective variety over $\mathbb{C}$ and assume that we are given $X \subset \mathbb{P}^n_S$, an algebraic family over $S$ of smooth projective varieties. Let $X_t$ be the fiber of $X$ over $t \in S$. Then the Noether-Lefschetz locus for Hodge cycles in codimension $q$ on $X/S$ is defined to be

$$S^q_{NL} = \{ t \in S \mid F^0H^{2q}(X_t, \mathbb{Q}(q)) \neq H^{2q}(X_t, \mathbb{Q}(q))_{\text{triv}} \}.$$

It is the locus of such $t \in S$ that there are non-trivial Hodge cycles in codimension $q$ on $X_t$ and hence that the Hodge conjecture is non-trivial for $X_t$. One can prove $S^q_{NL}$ is the union of countable number of (not necessarily proper) closed algebraic subsets of $S$ (cf. [CDK]).

The celebrated theorem of Noether-Lefschetz affirms that in case $X/S$ is the universal family of smooth surface of degree $\geq 4$ in $\mathbb{P}^3$, $S^q_{NL} \neq S$, which implies the Picard group of general surface of that kind is generated by the class of a hyperplane section.

In this paper we propose the following analog of the above problem in the context of Beilinson’s Hodge conjecture. Assume that we are given schemes over $\mathbb{C}$

$$(0-3) \quad Y = \bigcup_{1 \leq j \leq s} Y_j \hookrightarrow \mathbb{P}^n \leftrightarrow X \leftrightarrow Z = \bigcup_{1 \leq j \leq s} Z_j, \quad V = \mathbb{P}^n \setminus Y, \quad U = X \setminus Z$$

where $X$ is projective smooth, $Y_j \subset \mathbb{P}^n$ is a smooth hypersurface, $Y$ is a simple normal crossing divisor on $\mathbb{P}^n$, $Z_j = X \cap Y_j$ intersecting transversally, and $Z$ is a simple normal crossing divisor on $X$. We are interested in the space of Beilinson-Hodge cycles in degree $q$ on $U$:

$$F^0H^q(U, \mathbb{Q}(q)) := H^q(U, \mathbb{Q}(q)) \cap F^qH^q(U, \mathbb{C})$$

and the regulator map $\text{reg}_q^U : CH^q(U, q) \otimes \mathbb{Q} \to F^0H^q(U, \mathbb{Q}(q))$. Since $U$ is affine, $H^q(U, \mathbb{Q}(q)) = 0$ for $q > \dim(U)$ by the weak Lefschetz theorem. Thus we are interested only in case $1 \leq q \leq \dim(U)$. For a positive integer $q \neq n$ we put

$$H^q(U, \mathbb{Q}(q))_{\text{triv}} = \text{Image}(H^q(V, \mathbb{Q}(q)) \to H^q(U, \mathbb{Q}(q))).$$

We will define a subspace $CH^q(U, q)_{\text{triv}} \subset CH^q(U, q) \otimes \mathbb{Q}$ such that (cf. Definition 2.4)

$$H^q(U, \mathbb{Q}(q))_{\text{triv}} = \text{Image}(\text{reg}_q^U(CH^q(U, q)_{\text{triv}}).$$

and hence that $H^q(U, \mathbb{Q}(q))_{\text{triv}} \subset \text{Image}(\text{reg}_q^U) \subset F^0H^q(U, \mathbb{Q}(q))$. We note that in case $\dim(X) > 1$ we have (cf. Lemma 2.3)

$$CH^q(U, q)_{\text{triv}} = CH^q(U, q)_{\text{dec}}.$$
where $CH^q(U, q)_{dec} \subset CH^q(U, q) \otimes \mathbb{Q}$ is the subspace generated by decomposable elements, namely products of elements in $CH^1(U, 1) = \Gamma(U, \mathcal{O}_S^*)$. Now assume that we are given schemes over $S$, a non-singular quasi-projective variety over $\mathbb{C}$,

\[(0-4) \quad \mathcal{Y} = \bigcup_{1 \leq j \leq s} \mathcal{Y}_j \leftrightarrow \mathbb{P}^n_S \leftrightarrow \mathcal{X} \leftrightarrow \mathcal{Z} = \bigcup_{1 \leq j \leq s} \mathcal{Z}_j, \quad \mathcal{V} = \mathbb{P}^n_S \setminus \mathcal{Y}, \quad U = \mathcal{X} \setminus \mathcal{Z} \]

whose fibers satisfies the same conditions as (0-3). Let $U_t$ be the fiber of $U$ over $t \in S$.

**Definition 0.3** For $1 \leq q \leq \dim(U_t)$ we define the Noether-Lefschetz locus for Beilinson-Hodge cycles in degree $q$ on $U/S$ to be

$$S_{NL}^q = \{ x \in S \mid F^0 H^q(U_t, \mathbb{Q}(q)) \neq H^q(U_t, \mathbb{Q}(q))_{triv} \}.$$

By definition $S_{NL}^q$ is the locus of such $t \in S$ that there are non-trivial Beilinson-Hodge cycles in degree $q$ on $U_t$ and hence that Beilinson’s Hodge conjecture is non-trivial for $U_t$. We note that a standard method in Hodge theory shows that $S_{NL}^q$ is the union of countable number of (not necessarily proper) closed analytic subsets of $S$.

A main purpose of this paper is to study the Noether-Lefschetz locus in case the fibers of $\mathcal{X}/S$ are smooth complete intersection of multi-degree $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$. In this case the Lefschetz theory implies $S_{NL}^q = \emptyset$ unless $q = m := n - r = \dim(U_t)$ (cf. Lemma 2.3). Thus we are interested only in $S_{NL} := S_{NL}^m$. In order state the main result we will introduce an invariant $c_S(\mathcal{X}, \mathcal{Z})$ in §3 that measures the “generality” of the family (0-4), or how many independent parameters $S$ contains (cf. Definition 3.2 and Lemma 3.4). Assume that the fibers of $\mathcal{Y}_j \subset \mathbb{P}^n_S$ for $1 \leq j \leq s$ is a hypersurface of degree $e_j$ and put

$$\delta_{\min} = \min \{ d_i, e_j \mid 1 \leq j \leq s, \ 1 \leq i \leq r \}.$$

**Theorem 0.4** For any irreducible component $E \subset S_{NL}$,

$$\text{codim}_S(E) \geq \delta_{\min}(n - r - 1) + \sum_{1 \leq i \leq r} d_i - c_S(\mathcal{X}, \mathcal{Z}) - n.$$

The original proof of the Noether-Lefschetz theorem was based on the study of monodromy action on cohomology of surfaces (cf. [D2]). The idea of improving the Noether-Lefschetz theorem by means of the infinitesimal method in the theory of variations of Hodge structures was introduced by Carlson, Green, Griffiths and Harris (cf. [CGGH]). By using the method closer analyses have been made on the Noether-Lefschetz locus for Hodge cycles on hypersurfaces, particularly on the problem to give a lower bound of codimensions of its irreducible components and to determine the components of maximal dimension (cf. [G4], [G5], [V] and [Ot]).

The proof of Theorem 0.4 follows the same line of arguments as [G4] by using the variation of mixed Hodge structures arising from cohomology of the fibers of $U/S$. A technical renovation is the results stated in §1 on generalized Jacobian rings, which give an algebraic description of the infinitesimal part of the variation of mixed Hodge structures. It is a natural generalization of the Poincaré residue representation of the cohomology of hypersurfaces in [Gri].
Having a result such as Theorem 0.4, a natural question to ask is if the estimate in the theorem is optimal. In this paper we can give a positive answer to this question only in one case where the fibers of $\mathcal{X}/S$ are plane curves (cf. Theorem 4.1). In case the fibers of $\mathcal{X}/S$ are of dimension $> 1$, the estimate seems far from being optimal. Indeed in a forthcoming paper [AS2] we will show that the optimal estimate of codimensions of the Noether-Lefschetz locus for Beilinson’s Hodge cycles on complements of three hyperplanes in surfaces of degree $d$ in $\mathbb{P}^3$ is given explicitly by a quadratic polynomial in $d$.

Concerning Beilinson’s Tate conjecture, we will show the following results. Let $k$ be a finitely generated field over $\mathbb{Q}$. Let $S$ be a quasi-projective variety over $k$ and assume that we are given schemes over $S$ which satisfy the same condition as (0-4). Assume that the fibers of $\mathcal{X}/S$ are smooth complete intersection of multi-degree $(d_1, \ldots , d_r)$ in $\mathbb{P}^n$. Write $m = n - r$.

**Theorem 0.5** Assume $\sum_{1 \leq i \leq r} d_i \geq n + 1 + c_S(\mathcal{X}, \mathcal{Z})$.

1. Let $K = k(S)$ be the function field of $S$. For any a finite generated field $L$ over $K$ we have

$$H^m_{et}(U, \mathbb{Q}_L(m))^{	ext{Gal}(\overline{L}/L)} = \text{reg}_{et, U_L}^m(\text{CH}^m(U_L, m)_\text{triv} \otimes \mathbb{Q}_\ell),$$

where $\text{reg}_{et, U_L}^m : \text{CH}^m(U_L, m) \otimes \mathbb{Q}_\ell \rightarrow H^m_{et}(U, \mathbb{Q}_L(m))^{	ext{Gal}(\overline{L}/L)}$ is the etale regulator map for $U_L = U \times_S \text{Spec}(L)$ (cf. (0-2)).

2. Assume that $k$ is a finite extension of $\mathbb{Q}$ and $S(k) \neq \emptyset$. Let $\pi : S \rightarrow \mathbb{P}^N_k$ be a dominant quasi-finite morphism. There exist a subset $H \subset \mathbb{P}^N_k(k)$ such that:

   i. For all $x \in S$ such that $\pi(x) \in H$ and for any subgroup $G \subset \text{Gal}(k(\overline{x})/k(x))$ of finite index we have

$$H^m_{et}(U, \mathbb{Q}_L(m))^G = \text{reg}_{et, U_x}^m(\text{CH}^m(U_x, m)_\text{triv} \otimes \mathbb{Q}_\ell),$$

where $\text{reg}_{et, U_x}^m : \text{CH}^m(U_x, m) \otimes \mathbb{Q}_\ell \rightarrow H^m_{et}(U, \mathbb{Q}_L(m))^{	ext{Gal}(k(\overline{x})/k(x))}$ is the etale regulator map for $U_x$, the fiber of $U$ over $x \in S$, with $\overline{x}$, a geometric point over $x$.

   ii. Let $\Sigma$ be any finite set of primes of $k$ and let $k_v$ be the completion of $k$ at $v \in \Sigma$. Then the image of $H$ in $\prod_{v \in \Sigma} \mathbb{P}^N_{k_v}(k_v)$ is dense.

Now we explain how the paper is organized. In §1 we state the fundamental results on the generalized Jacobian rings, the duality theorem and the symmetrizer lemma. The proof is given in another paper [AS1]. It is based on the basic techniques to compute Koszul cohomology developed by M. Green ([G2] and [G3]). In §2 we define the trivial part of cohomology of complements of unions of hypersurface sections in smooth projective varieties, which is necessary to set up the Noether-Lefschetz problem for Beilinson-Hodge cycles. In §3 Hodge theoretic implications of the results in §1 are stated, which plays a crucial role in the proof of Theorem 0.4 given in this section. The case of plane curves is treated in §4, where the estimate in Theorem 0.4 is shown to be optimal. Theorem 0.5 is proven in §5 by using the results in §2. In §6 we explain an implication of the Beilinson’s conjectures on the injectivity of Chern class maps for $K_1$ of surfaces.
The purpose of this section is to introduce Jacobian rings for open complete intersections and state their fundamental properties. Throughout the whole paper, we fix integers \(r, s \geq 0\) with \(r + s \geq 1\), \(n \geq 2\) and \(d_1, \cdots, d_r, e_1, \cdots, e_s \geq 1\). We put
\[
d = \sum_{i=1}^{r} d_i, \quad e = \sum_{j=1}^{s} e_j, \quad \delta_{\min} = \min_{1 \leq i \leq r} \{d_i, e_j\}, \quad d_{\max} = \max_{1 \leq i \leq r} \{d_i\}, \quad e_{\max} = \max_{1 \leq j \leq s} \{e_j\}.
\]
We also fix a field \(k\) of characteristic zero. Let \(P = k[X_0, \ldots, X_n]\) be the polynomial ring over \(k\) in \(n+1\) variables. Denote by \(P^l \subset P\) the subspace of the homogeneous polynomials of degree \(l\). Let \(A\) be a polynomial ring over \(P\) with indeterminants \(\mu_1, \cdots, \mu_r, \lambda_1, \cdots, \lambda_s\). We use the multi-index notation
\[
\mu^a = \mu_1^{a_1} \cdots \mu_r^{a_r}, \quad \lambda^b = \lambda_1^{b_1} \cdots \lambda_s^{b_s}, \quad \text{for} \quad a = (a_1, \cdots, a_r) \in \mathbb{Z}_{\geq 0}^r, \quad b = (b_1, \cdots, b_s) \in \mathbb{Z}_{\geq 0}^s.
\]

For \(q \in \mathbb{Z}\) and \(\ell \in \mathbb{Z}\), we write
\[
A_q(\ell) = \bigoplus_{a+b=q} P^{ad} \cdot \mu^a \lambda^b, \quad (a = \sum_{i=1}^{r} a_i, \quad b = \sum_{j=1}^{s} b_j), \quad ad = \sum_{i=1}^{r} a_i d_i, \quad be = \sum_{j=1}^{s} b_j e_j.
\]

By convention \(A_q(\ell) = 0\) if \(q < 0\).

**Definition 1.1** For \(\underline{F} = (F_1, \cdots, F_r), \quad \underline{G} = (G_1, \cdots, G_s)\) with \(F_i \in P^{d_i}, \quad G_j \in P^{e_j}\), we define the Jacobian ideal \(J(\underline{F}, \underline{G})\) to be the ideal of \(A\) generated by
\[
\sum_{1 \leq i \leq r} \frac{\partial F_i}{\partial X_k} \mu_i + \sum_{1 \leq j \leq s} \frac{\partial G_j}{\partial X_k} \lambda_j, \quad F_i, \quad G_j \lambda_j \quad (1 \leq i \leq r, \ 1 \leq j \leq s, \ 0 \leq k \leq n).
\]
The quotient ring \(B = B(\underline{F}, \underline{G}) = A/J(\underline{F}, \underline{G})\) is called the Jacobian ring of \((\underline{F}, \underline{G})\). We denote
\[
B_q(\ell) = B_q(\ell)(\underline{F}, \underline{G}) = A_q(\ell)/A_q(\ell) \cap J(\underline{F}, \underline{G}).
\]

**Definition 1.2** Suppose \(n \geq r + 1\). Let \(\mathbb{P}^n = \text{Proj} \ P\) be the projective space over \(k\). Let \(X \subset \mathbb{P}^n\) be defined by \(F_1 = \cdots = F_r = 0\) and let \(Z_j \subset X\) be defined by \(G_j = F_1 = \cdots = F_r = 0\) for \(1 \leq j \leq s\). We also call \(B(\underline{F}, \underline{G})\) the Jacobian ring of the pair \((X, Z = \cup_{1 \leq j \leq s} Z_j)\) and denote \(B(\underline{F}, \underline{G}) = B(X, Z)\) and \(J(\underline{F}, \underline{G}) = J(X, Z)\).

In what follows we fix \(\underline{F}\) and \(\underline{G}\) as Definition 1.1 and assume the condition
\[
(1-1) \quad F_i = 0 \ (1 \leq i \leq r) \quad \text{and} \quad G_j = 0 \ (1 \leq j \leq s) \quad \text{intersect transversally in} \ \mathbb{P}^n.
\]

We mention three main theorems. The first main theorem concerns with the geometric meaning of Jacobian rings.

**Theorem 1.3** Suppose \(n \geq r + 1\). Let \(X\) and \(Z\) be as Definition 1.2.
coincides with the ring multiplication up to non-zero scalar. This result was originally invented by P. Griffiths in case of hypersurfaces and generalized to complete intersections (2).

There is a natural map

\[ T : X^1 \to \mathbf{H}^1(X,\Omega_X^p) \]

and the contraction

\[ T : X^1 \to \mathbf{H}^1(X,\Omega_X^p) \]

induced by the cup-product and the contraction

\[ T : X^1 \to \mathbf{H}^1(X,\Omega_X^p) \]

which is an isomorphism if \( \dim(X) \geq 2 \). Here \( T_X(-\log Z) \) is the \( \mathcal{O}_X \)-dual of \( \Omega_X^1(\log Z) \) and the group on the right hand side is defined in Definition 1.4 below.

The following map

\[ H^1(X, T_X(-\log Z)) \otimes H^q(X, \Omega_X^p(\log Z)) \to H^q+1(X, \Omega_X^{p-1}(\log Z)) \]

induced by the cup-product and the contraction \( T_X(-\log Z) \otimes \Omega_X^p(\log Z) \to \Omega_X^{p-1}(\log Z) \) is compatible through \( \psi_{(X,Z)} \) with the ring multiplication up to scalar.

Roughly speaking, the generalized Jacobian rings describe the infinitesimal part of the Hodge structures of open variety \( X \setminus Z \), and the cup-product with Kodaira-Spencer class coincides with the ring multiplication up to non-zero scalar. This result was originally invented by P. Griffiths in case of hypersurfaces and generalized to complete intersections by Konno [K]. Our result is a further generalization.

**Definition 1.4** Let the assumption be as in Theorem 1.3. We define \( H^1(X, T_X(-\log Z))_{alg} \) to be the kernel of the composite map

\[ H^1(X, T_X(-\log Z)) \to H^1(X, T_X) \to H^2(X, \mathcal{O}_X), \]

where the second map is induced by the cup product with the class \( c_1(\mathcal{O}_X(1)) \in H^1(X, \Omega_X^1) \) and the contraction \( T_X \otimes \Omega_X \to \mathcal{O}_X \). It can be seen that

\[ \dim_k(H^1(X, T_X(-\log Z))/H^1(X, T_X(-\log Z))_{alg}) = \begin{cases} 1 & \text{if } X \text{ is a K3 surface}, \\ 0 & \text{otherwise}. \end{cases} \]

The second main theorem is the duality theorem for the generalized Jacobian rings.

**Theorem 1.5** Let the notation be as as above.

1. There is a natural map (called the trace map)

\[ \tau : B_{n-r}(2(d - n - 1) + e) \to k. \]

Let

\[ h_p(\ell) : B_p(d - n - 1 + \ell) \to B_{n-r-p}(d + e - n - 1 - \ell)^* \]

be the map induced by the following pairing induced by the multiplication

\[ B_p(d - n - 1 + \ell) \otimes B_{n-r-p}(d + e - n - 1 - \ell) \to B_{n-r}(2(d - n - 1) + e) \xrightarrow{\tau} k. \]

When \( r > n \) we define \( h_p(\ell) \) to be the zero map by convention.
(2) The map \( h_\ell \) is an isomorphism in either of the following cases.

(i) \( s \geq 1 \) and \( p < n - r \) and \( \ell < e_{\text{max}} \).

(ii) \( s \geq 1 \) and \( 0 \leq \ell \leq e_{\text{max}} \) and \( r + s \leq n \).

(iii) \( s = \ell = 0 \) and either \( n - r \geq 1 \) or \( n - r = p = 0 \).

(3) The map \( h_{n-r} \) is injective if \( s \geq 1 \) and \( \ell < e_{\text{max}} \).

We have the following auxiliary result on the duality.

**Theorem 1.6** Assume \( n - r \geq 1 \) and consider the composite map

\[
\eta_{(X,Z)} : H^0(X, \Omega^n_{X}(\log Z)) \xrightarrow{(\phi_0^X)} B_0(d + e - n - 1) \xrightarrow{h_{n-r}(0)} B_{n-r}(d - n - 1)^*
\]

where the second map is the dual of \( h_{n-r}(0) \). Then \( \eta_{(X,Z)} \) is surjective and we have (cf. Definition 1.7 below)

\[
\text{Ker}(\eta_{(X,Z)}) = \wedge^r_{X}(G_1, \ldots, G_s).
\]

**Definition 1.7** Let \( G_1, \ldots, G_s \) be as in Definition 1.7 and let \( Y_j \subset \mathbb{P}^n \) be the smooth hypersurface defined by \( G_j = 0 \). Let \( X \subset \mathbb{P}^n \) be a smooth projective variety such that \( Y_j \) \((1 \leq j \leq s)\) and \( X \) intersect transversally. Put \( Z_j = X \cap Y_j \). Take an integer \( q \) with \( 0 \leq q \leq s - 1 \). For integers \( 1 \leq j_1 < \cdots < j_{q+1} \leq s \), let

\[
\omega_X(j_1, \ldots, j_{q+1}) \in H^0(X, \Omega^q_{X}(\log Z)) \quad (Z = \sum_{1 \leq j \leq s} Z_j)
\]

be the restriction of

\[
\sum_{\nu = 1}^{q+1} (-1)^{\nu-1} e_{j_\nu} \bigwedge G_{j_1}^{\nu} \cdots \bigwedge G_{j_{\nu}}^{\nu} \bigwedge \cdots \bigwedge G_{j_{q+1}}^{\nu} \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(\log Y))
\]

where \( Y = \sum_{1 \leq j \leq s} Y_j \subset \mathbb{P}^n \). We let

\[
\wedge_{X}^q(G_1, \ldots, G_s) \subset H^0(X, \Omega^q_{X}(\log Z))
\]

be the subspace generated by \( \omega_X(j_1, \ldots, j_{q+1}) \). For \( 1 \leq j_1 < \cdots < j_q \leq s - 1 \) we have

\[
e_1^{q-1} \omega_X(1, j_1, \ldots, j_q) = \frac{dg_{j_1}}{g_{j_1}} \cdots \frac{dg_{j_q}}{g_{j_q}} \quad \text{with} \quad g_j = (G_j^{e_{j_1}} / G_j^{e_{j_q}})|_X \in \Gamma(U, \mathcal{O}_U^*) \quad (U = X \setminus Z)
\]

and \( \omega_X(1, j_1, \ldots, j_q) \) with \( 1 < j_1 < \cdots < j_q \leq s \) form a basis of \( \wedge_{X}^q(G_1, \ldots, G_s) \).

Our last main theorem is the generalization of Donagi’s symmetrizer lemma [Do] (see also [DG], [Na] and [N]) to the case of open complete intersections at higher degrees.

**Theorem 1.8** Assume \( s \geq 1 \). Let \( V \subset B_1(0) \) is a subspace of codimension \( c \geq 0 \). Then the Koszul complex

\[
B_p(\ell) \otimes \wedge^{q+1} V \to B_{p+1}(\ell) \otimes \wedge^q V \to B_{p+2}(\ell) \otimes \wedge^{q-1} V
\]

is exact if one of the following conditions is satisfied.
(i) $p ≥ 0$, $q = 0$ and $δ_{min}p + ℓ ≥ c$.

(ii) $p ≥ 0$, $q = 1$ and $δ_{min}p + ℓ ≥ 1 + c$ and $δ_{min}(p + 1) + ℓ ≥ d_{max} + c$.

(iii) $p ≥ 0$, $δ_{min}(r + p) + ℓ ≥ d + q + c$, $d + e_{max} − n − 1 > ℓ ≥ d − n − 1$ and either $r + s ≤ n + 2$ or $p ≤ n − r − [q/2]$, where $[•]$ denotes the Gaussian symbol.

2. Trivial part of cohomology of complements of hypersurfaces

In this section we introduce some notions necessary to set up the Noether-Lefschetz problem for Beilinson-Hodge cycles on the complement of the union of hypersurface sections in a smooth projective variety. We fix the base field $k$ which is either $C$ or finitely generated over $Q$. Assume that we are given schemes over $k$

(2-1) $Y = \bigcup_{1 ≤ j ≤ s} Y_j \hookrightarrow \mathbb{P}^n \hookleftarrow X \hookrightarrow Z = \bigcup_{1 ≤ j ≤ s} Z_j$, $V = \mathbb{P}^n \setminus Y$, $U = X \setminus Z$

where $X$ is projective smooth, $Y_j \subset \mathbb{P}^n$ is a smooth hypersurface of degree $e_j$, $Y$ is a simple normal crossing divisor on $\mathbb{P}^n$, $Z_j = X \cap Y_j$ intersecting transversally, and $Z$ is a simple normal crossing divisor on $X$. In what follows $CH^i(•, j)$ denotes the Bloch’s higher Chow group (cf. [Bl]).

**Definition 2.1** Assume $s ≥ 2$.

(1) Let $G_j \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e_j))$ be a non-zero element defining $Y_j \subset \mathbb{P}^n$. For $1 ≤ j ≤ s − 1$ put

$$g_j = (G_j^{e_j}/G_s^{e_j})_{|U} \in \Gamma(U, \mathcal{O}_U^*) = CH^1(U, 1).$$

Let $CH^1(U, 1)_{triv} \subset CH^1(U, 1) \otimes Q$ be the subspace generated by $k^*$ and $g_j$ with $1 ≤ j ≤ s − 1$.

(2) For $q ≥ 1$ let $CH^q(U, q)_{dec} \subset CH^q(U, q) \otimes Q$ be the subspace generated by the products of elements in $CH^1(U, 1)$ and let

$$CH^q(U, q)_{triv} \subset CH^q(U, q)_{dec}$$

be the subspace generated by the products of elements in $CH^1(U, 1)_{triv}$.

(3) For $q ≥ 1$ let $H^0(X, \Omega_X^q(\log Z))_{triv} \subset H^0(X, \Omega_X^q(\log Z))$ be the subspace generated by the wedge products of $dg_j/g_j \in H^0(X, \Omega_X^1(\log Z))$ with $1 ≤ j ≤ s − 1$.

(4) Let $H^p(U, q)$ denotes the singular cohomology $H^p(U, Q(q))$ of $U(\mathbb{C})$ in case $k = \mathbb{C}$ (resp. the étale cohomology $H^p_{et}(\overline{U}, Q(q))$ in case $k$ is finitely generated over $Q$, where $\overline{U} = U \times_k \overline{k}$ with $\overline{k}$, an algebraic closure of $k$). For $q ≥ 1$ let

$$H^q(U, q)_{triv} \subset H^q(U, q)$$

be the subspace generated by the image of $CH^q(U, q)_{triv}$ under the regulator map introduced in (0-1) and (0-2).

In case $s = 0, 1$ we put $M_{triv} = 0$ by convention for $M = CH^q(U, q)$, $H^0(X, \Omega_X^q(\log Z))$, $H^q(U, q)$. 

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Lemma 2.3 Assume $s \geq 1$ and $q \geq 1$.

1. If $\dim(X) > 1$, $CH^q(U, q)_{\text{dec}} = CH^q(U, q)_{\text{triv}}$.

2. If $q \neq n$, $H^q(U, q)_{\text{triv}} = \text{Image}(H^q(V, q) \to H^q(U, q))$.

3. If $X \subset \mathbb{P}^n$ is a smooth complete intersection, $H^q(U, q) = H^q(U, q)_{\text{triv}}$ for $q \neq \dim(X)$.

Proof. To show Lemma 2.3(1) it suffices to show $CH^1(U, 1)_{\text{triv}} = CH^1(U, 1) \otimes \mathbb{Q}$, which follows at once from the exact sequence

$$0 \to k^* \to CH^1(U, 1) \to \bigoplus_{1 \leq j \leq s} \mathbb{Z} \to CH^1(X)$$

where the second map is induced by the residue maps along $Z_j$ and the last by the class of $Z_j$ in $CH^1(X)$.

We show Lemma 2.3(3). Since $U$ is affine by the assumption $s \geq 1$, $H^q(U, q) = 0$ for $q > \dim(X)$ by the weak Lefschetz theorem. Thus we may assume that $1 \leq q < \dim(X)$ so that $\dim(X) > 1$. If $s = 1$, $Z = Z_1$ is a smooth complete intersection of dimension $\geq 1$ and we have the exact sequence

$$H^{q-2}(Z, q - 1) \xrightarrow{\alpha} H^q(X, q) \xrightarrow{\beta} H^q(U, q) \to H^{q-1}(Z, q - 1) \xrightarrow{\beta} H^{q+1}(X, q)$$

where $\alpha$ and $\beta$ are the Gysin maps. The assumption $q \neq \dim(X)$ implies $q - 1 \neq \dim(Z)$ and the Lefschetz theory implies that $\alpha$ is surjective and $\beta$ is injective so that $H^q(U, q) = 0$. If $q = 1$ the desired assertion follows from the exact sequence

$$0 \to H^1(U, 1) \to \bigoplus_{1 \leq j \leq s} H^0(Z_j, 0) \to H^2(X, 1),$$

where the last map is the Gysin map and the injectivity of the first map follows from the vanishing of $H^1(X, 1)$ by the assumption on $X$. Now assuming $\dim(X) > 1$, $s > 1$ and $q > 1$, we proceed by the double induction on $\dim(X)$ and $s$. Put

$$U' = X \setminus \bigcup_{j=2}^{s} Z_j, \quad W = U' - U = Z_1 \setminus (Z_1 \cap \bigcup_{j=2}^{s} Z_j).$$

We have the short exact sequence

$$H^q(U', q) \to H^q(U, q) \to H^{q-1}(W, q - 1).$$

By induction $H^q(U', q) = H^q(U', q)_{\text{triv}}$ and $H^{q-1}(W, q - 1) = H^{q-1}(W, q - 1)_{\text{triv}}$ while it is easy to check that the residue map $H^q(U, q)_{\text{triv}} \to H^{q-1}(W, q - 1)_{\text{triv}}$ is surjective. This proves the desired assertion for $H^q(U, q)$.

To show Lemma 2.3(2) let $H^q(V, q)_{\text{triv}} \subset H^q(V, q)$ be defined as $H^q(U, q)_{\text{triv}}$ by taking $X = \mathbb{P}^n$. By definition we have $H^q(U, q)_{\text{triv}} = \text{Image}(H^q(V, q)_{\text{triv}} \to H^q(U, q))$. By Lemma 2.3(3) $H^q(V, q)_{\text{triv}} = H^q(V, q)$ if $q \neq n$. This proves the desired assertion.
In this section we prove Theorem 0.4 by using the results in \S 1. Let the assumption be as in \S 2. We fix a non-singular affine algebraic variety $S$ over $k$ and the following schemes over $S$

\[(3-1) \quad \mathcal{Y} = \bigcup_{1 \leq j \leq s} \mathcal{Y}_j \hookrightarrow \mathbb{P}^n_S \hookleftarrow \mathcal{X} \hookrightarrow \mathcal{Z} = \bigcup_{1 \leq j \leq s} \mathcal{Z}_j, \quad \mathcal{V} = \mathbb{P}^n_S \setminus \mathcal{Y}, \quad \mathcal{U} = \mathcal{X} \setminus \mathcal{Z}\]

whose fibers are as (2-1). We assume that the fibers of $\mathcal{X}/S$ are smooth complete intersections of multi-degree $(d_1, \ldots, d_r)$ and those of $\mathcal{Y}_j \subset \mathbb{P}^n_S$ for $1 \leq j \leq s$ are hypersurfaces of degree $e_j$. For integers $p, q$ we introduce the following sheaf on $S_{zar}$

$$H^{p,q}(U/S) = R^q f_* \Omega^p_{\mathcal{X}/S}(\log Z),$$

where $f : \mathcal{X} \to S$ is the natural morphism and $\Omega^p_{\mathcal{X}/S}(\log Z) = \wedge^p \Omega^1_{\mathcal{X}/S}(\log Z)$ with $\Omega^1_{\mathcal{X}/S}(\log Z)$, the sheaf of relative differentials on $\mathcal{X}$ over $S$ with logarithmic poles along $Z$. In case $s \geq 1$ the Lefschetz theory implies $H^{p,q}(U/S) = 0$ if $p + q \neq n - r$. In case $s = 0$ it implies $H^{p,q}(\mathcal{X}/S)_{prim} = 0$ if $p + q \neq n - r$ where “prim” denotes the primitive part (cf. Theorem 1.3 (1)). The results in \S 1 implies that under an appropriate numerical condition on $d_i$ and $e_j$ we can control the cohomology of the following Koszul complex

$$\Omega^q_S \otimes H^{a+2,b-2}(U/S) \xrightarrow{\nabla} \Omega^q_S \otimes H^{a+1,b-1}(U/S) \xrightarrow{\nabla} \Omega^{q+1}_S \otimes H^{a,b}(U/S).$$

Here $\nabla$ is induced by the Kodaira-Spencer map

\[(3-1) \quad \kappa_{(X,Z)} : \Theta_S \to R^1 f_* T_{\mathcal{X}/S}(- \log Z),\]

with $\Theta_S = \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S)$ and $T_{\mathcal{X}/S}(- \log Z) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}/S}(\log Z), \mathcal{O}_X)$, and the map

$$R^1 f_* T_{\mathcal{X}/S}(- \log Z) \otimes R^{b-1} f_* \Omega^{a+1}_{\mathcal{X}/S}(\log Z) \to R^b f_* \Omega^a_{\mathcal{X}/S}(\log Z)$$

induced by the cup product and $T_{\mathcal{X}/S}(- \log Z) \otimes \Omega^a_{\mathcal{X}/S}(\log Z) \to \Omega^a_{\mathcal{X}/S}(\log Z)$, the contraction. For the application to the Beilinson’s conjectures the kernel of the following map plays a crucial role

$$\nabla^{p,q} : H^{p,q}(U/S) \to \Omega^1_S \otimes_{\mathcal{O}} H^{p-1,q+1}(U/S).$$

In case $s = 0$ we let $\nabla^{p,q}$ denote the primitive part of the above map:

$$H^{p,q}(\mathcal{X}/S)_{prim} \xrightarrow{\nabla} \Omega^1_S \otimes_{\mathcal{O}} H^{p-1,q+1}(\mathcal{X}/S)_{prim}.$$ 

Now we fix a $k$-rational point $0 \in S$ and let $U \subset X \supset Z$ denote the fibers of $U \subset \mathcal{X} \supset \mathcal{Z}$. Taking the fibers of the above maps we get

$$\nabla^{p,q}_0 : H^{p,q}(U) \to \Omega^1_{S,0} \otimes H^{p-1,q+1}(U) \quad (s \geq 1),$$

$$\nabla^{p,q}_0 : H^{p,q}(\mathcal{X})_{prim} \to \Omega^1_{S,0} \otimes H^{p-1,q+1}(\mathcal{X})_{prim} \quad (s = 0).$$

3. Hodge theoretic implication of generalized Jacobian rings
where $H^{p,q}(U) = H^q(X, \Omega^p_X(\log Z))$ and $H^{p,q}(X)_{prim} = H^q(X, \Omega^p_X)_{prim}$. Let $T_0(S)$ be the tangent space of $S$ at 0 and fix a linear subspace $T \subset T_0(S)$. Via the canonical isomorphism $\Omega^1_{S,0} \cong \text{Hom}_k(T_0(S), k)$, the above maps induce

$$\nabla_{0,T}^{\mu,q} : H^{p,q}(U) \to \text{Hom}_k(T, H^{p-1,q+1}(U)) \quad (s \geq 1),$$

$$\nabla_{0,T}^{\mu,0} : H^{p,q}(X)_{prim} \to \text{Hom}_k(T, H^{p-1,q+1}(X)_{prim}) \quad (s = 0).$$

Now the key result is the following. See Definition 3.2 below for the definition of $c_S(\mathcal{X}, Z)_0$. Put

$$d = \sum_{1 \leq i \leq r} d_i, \quad \delta_{\min} = \min\{d_i, e_j | 1 \leq j \leq s, 1 \leq i \leq r\}.$$

**Theorem 3.1** Assume $p + q = m := n - r \geq 1$. Let $c = \text{codim}_{T_0(S)}(T)$.

1. Assuming $1 \leq p \leq m - 1$ and $\delta_{\min}(p - 1) + d \geq n + 1 + c_S(\mathcal{X}, Z)_0 + c$, Ker($\nabla_{0,T}^{\mu,q}$) = 0.

2. Assuming $\delta_{\min}(n - r - 1) + d \geq n + 1 + c_S(\mathcal{X}, Z)_0 + c$, Ker($\nabla_{0,T}^{\mu,0}$) = $H^0(X, \Omega^m_X(\log Z))_{\text{triv}}$ (cf. Definition 2.1).

**Definition 3.2** We define

$$c_S(\mathcal{X}, Z)_0 = \text{dim}_k(\text{Image}(\psi_{(X,Z)})/\text{Image}(\psi_{(X,Z)}) \cap \text{Image}(\kappa_{(X,Z)})).$$

where $\kappa_{(X,Z)} : T_0(S) \to H^1(X, T_X(-\log Z))$ (resp. $\psi_{(X,Z)} : B_1(0) \to H^1(X, T_X(-\log Z))$) is the fiber at 0 of the map (3.2) (resp. the map in Theorem 1.3(2) for $(X, Z)$). We also define

$$c_S(\mathcal{X}, Z_t) = \min_{t \in S}\{c_S(\mathcal{X}, Z)_t\}$$

where $c_S(\mathcal{X}, Z)_t$ is defined by the same way as $c_S(\mathcal{X}, Z)_0$ for the fibers of $U \subset \mathcal{X} \supset Z$ over $t \in S$.

**Remark 3.3** If $n - r \geq 2$ and $X$ is not a K3 surface, $\psi_{(X,Z)}$ is surjective so that $c_S(\mathcal{X}, Z)_0 = \text{dim}_k\text{Coker}(\kappa_{(X,Z)})$.

Let $P = \mathbb{C}[z_0, \ldots, z_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$ and let $P^d \subset P$ be the space of the homogeneous polynomials of degree $d$. The dual projective space

$$\mathbb{P}(P^d) = \mathbb{P}^N_k(n,d) \quad (N(n,d) = \binom{n+d}{d} - 1)$$

parametrizes hypersurfaces $Y \subset \mathbb{P}^n$ of degree $d$ defined over $k$. Let

$$B \subset \prod_{1 \leq i \leq r} \mathbb{P}^N_k(n,d_i) \times \prod_{1 \leq j \leq s} \mathbb{P}^N_k(n,e_j)$$

be the Zariski open subset parametrizing such $((X_i)_{1 \leq i \leq r}, (Y_j)_{1 \leq j \leq s})$ that $X_1 + \cdots + X_r + Y_1 + \cdots + Y_s$ is a simple normal crossing divisor on $\mathbb{P}^n_k$. We consider the family

$$\mathcal{X}_B \leftrightarrow Z_B = \bigcup_{1 \leq j \leq s} Z_{B,j} \quad \text{over } B$$

whose fibers are $X \leftrightarrow Z = \bigcup_{1 \leq j \leq s} Z_j$ with $X = X_1 \cap \cdots \cap X_r$ and $Z_j = X \cap Y_j$. We omit the proof of the following.
Lemma 3.4 Let $E \subset B$ be a non-singular locally closed subvariety of codimension $c \geq 0$ and let $S \to E$ be a dominant map. Assume that the family $(\mathcal{X}, \mathcal{Z})/S$ is the pullback of $(\mathcal{X}_B, \mathcal{Z}_B)/B$ via $S \to B$. Then we have $c_S(\mathcal{X}, \mathcal{Z}) \leq c$.

Now we prove Theorem 3.4. We only show the second assertion and leave the first to the readers. The fact $H^0(X, \Omega^m_{X/S}(\log Z))_{\text{triv}} \subset \text{Ker}(\nabla_{0,T}^{m,0})$ follows from the fact that it lies in the image of

$$H^0(\mathcal{X}, \Omega^m_{\mathcal{X}/k}(\log Z)) \rightarrow \Gamma(S, f_*\Omega^m_{\mathcal{X}/S}(\log Z)) \rightarrow H^0(X, \Omega^m_X(\log Z)),$$

where $\Omega^m_{\mathcal{X}/k}(\log Z)$ is the sheaf of differential forms of $\mathcal{X}$ over $k$ with logarithmic poles along $Z$. Thus we are reduced to show the injectivity of the induced map

$$H^0(X, \Omega^m_X(\log Z))/H^0(X, \Omega^m_X(\log Z))_{\text{triv}} \rightarrow \text{Hom}_k(T, H^1(X, \Omega^m_X(\log Z))).$$

By Theorem 3.5 and Theorem 1.6 in §1 this is reduced to show the surjectivity of

$$W \otimes B_{n-r-1}(d-n-1) \rightarrow B_{n-r}(d-n-1)$$

where $B_1(0) \supset W := \psi^{-1}(\kappa(X,Z)(T))$. By definition $W$ is of codimension $\leq c + c_S(\mathcal{X}, \mathcal{Z})_0$ in $B_1(0)$. Hence the desired assertion follows from Theorem 1.8 in §1.

Having Theorem 3.4, Theorem 0.4 is proven by a standard method in Hodge theory (cf. [G1, p.75]), which we recall in what follows. Let the assumption be as in Theorem 0.4. Fix $0 \in E \subset S_{NL}$ and let $U \subset X \supset Z$ be as before. Choosing a simply connected neighbourhood $\Delta$ of 0 in $S(\mathbb{C})$ we have the natural identification for $\forall t \in \Delta$ via flat translation with respect to the Gauss-Manin connection:

$$H^m(U, \mathbb{Q}(m)) \xrightarrow{\psi} H^m(U_t, \mathbb{Q}(m)); \lambda \rightarrow \lambda_t$$

where $U_t$ is the fiber of $U$ over $t \in S$. By definition of $S_{NL}$ there exists $\lambda \in F^0H^m(U, \mathbb{Q}(m))$ not contained in $H^m(U, \mathbb{Q}(m))_{\text{triv}}$ such that $\lambda_t \in F^mH^m(U_t, \mathbb{C})$ for $\forall t \in E \cap \Delta$. By noting Remark 2.2 it implies that $\lambda \in \text{Ker}(\nabla_{0,T}^{m,0}) \neq H^0(X, \Omega^m_X(\log Z))_{\text{triv}}$, where $T \subset T_0(0)$ is the tangent space of $E$. Hence we have

$$\text{codim}_S(E) \geq \text{codim}_{T_0(S)}(T) \geq \delta_{\min}(n-r-1) + \sum_{1 \leq i \leq s} d_i - n - c_S(\mathcal{X}, \mathcal{Z})$$

where the second inequality follows from Theorem 3.1. This completes the proof of Theorem 0.4.

4. Case of plane curves

One may naturally asks if the estimate in Theorem 0.4 is optimal. In this section we answer the question in case $n = 2$ and $r = 1$, namely in case the fibers of $\mathcal{X}/S$ are plane curves. We work over $\mathbb{C}$. Let $P = \mathbb{C}[z_0, z_1, z_2]$ be the homogeneous coordinate ring of $\mathbb{P}^2$ and let $P^d \subset P$ be the space of the homogeneous polynomials of degree $d$. Let

$$S \subset P^d/\mathbb{C}^* \times \prod_{1 \leq j \leq s} P^{e_j}/\mathbb{C}^* = \mathbb{P}^{N(2,d)} \times \prod_{1 \leq j \leq s} \mathbb{P}^{N(2,e_j)} (N(2, u) = \left(\frac{u + 2}{2}\right) - 1)$$

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be the moduli space of such sets of plane curves \((X, (Y_j))_{1 \leq j \leq s}\) that \(X \cup Y_1 \cup \cdots \cup Y_s\) is a simple normal crossing divisor over \(P^2\). Let \(X \hookrightarrow P^2 \hookrightarrow Y_j\) be the universal families over \(S\) and put \(Z = X \cap \bigcup_{1 \leq j \leq s} Y_j\) and \(U = X \setminus Z\). Let \(U_t \subset X_t \supset Z_t\) denote the fibers of \(U \subset X \supset Z\) over \(t \in S\). We are interested in the Noether-Lefschetz locus for Beilinson-Hodge cycles on \(U_t\);

\[
S_{NL} = \{ t \in S \mid F^0H^1(U_t, \mathbb{Q}(1)) \neq H^1(U_t, \mathbb{Q}(1)) \}.
\]

Noting that \(reg^1_t\) is known to be surjective and that \(\text{Ker}(reg^1_t) = \mathbb{C}^* \otimes \mathbb{Q} \subset CH^1(U_t, 1) \otimes \mathbb{Q}\), we have

\[
S_{NL} = \{ t \in S \mid CH^1(U_t, 1) \otimes \mathbb{Q} \neq CH^1(U_t, 1)_{\text{triv}} \}.
\]

**Theorem 4.1** Let \(E \subset S_{NL}\) be an irreducible component. Then \(\text{codim}_S(E) \geq d - 2\). Moreover there exists \(E\) for which the equality holds.

**Proof.** The inequality follows from Theorem 0.4 together with the fact \(c_S(X, Z) = 0\), which follows from Lemma 3.4. As for the second assertion we consider the subset \(\Sigma \subset S\) of those \(t \in S\) that satisfy the condition: there exists the unique point \(x_t \in Z_t\) such that \(X_t \cap H_t = \{ x_t \}\) set-theoretically, where \(H_t\) is the tangent line of \(X_t\) at \(x_t\). It is easy to check that \(\Sigma\) is locally closed algebraic subset of \(S\). Thus it suffices to show the following.

**Lemma 4.2** \(\Sigma \subset S_{NL}, \Sigma\) is equidimensional, and \(\text{codim}_S(\Sigma) = d - 2\).

**Proof.** Fix any \(t \in \Sigma\) and let \(Y_j\) and \(U\) denote the fibers of \(Y_j\) and \(U\) over \(t\) for simplicity. Choose \(L \in P^1\) defining \(H_t \subset \mathbb{P}^2\) and \(G_j \in P^{e_j}\) defining \(Y_j\) which contains \(x_t\). Then it is easy to see that \((G_j/L^{e_j})|_U \subset CH^1(U, 1)\) while it is not contained in \(CH^1(U, 1)_{\text{triv}}\). This proves the first assertion of Lemma 4.2. In order to show the remaining assertions, let \(M\) be the space of pairs \((x, H)\) of a point \(x \in \mathbb{P}^2\) and a line \(H \subset \mathbb{P}^2\) passing through \(x\). Define the morphism \(\pi : \Sigma \to M\) by \(\pi(t) = (x_t, H_t)\). Considering the action of \(\text{PGL}_3(\mathbb{C})\), the group of linear transformations on \(\mathbb{P}^2\), one easily see that \(\pi\) is surjective. Thus it suffices to show the following.

**Lemma 4.3** Let \(\Sigma_0 \subset \Sigma\) be the fiber of \(\pi\) over \((0, H_0)\) with \(0 := (0 : 0 : 1)\) and \(H_0 \subset \mathbb{P}^n\) defined by \(z_0 \in P^1\). Then \(\Sigma_0\) is equidimensional and \(\text{codim}_S(\Sigma) = d - 2 + \dim(M) = d + 1\).

**Proof.** Let \(T \subset S' := \prod_{1 \leq j \leq s} \mathbb{P}^{N(2,e_j)} \) be the closed subset parametrizing such \((Y_j)_{1 \leq j \leq s}\) that \(0 \in Y_j\) for some \(j\). Clearly \(T\) is equidimensional and of \(\text{codim}_{S'}(T) = 1\). By definition one sees that \(X_t\) for \(t \in E_0\) is defined by an equation of the form \(z_0 A + z_j^{e_j}\) with \(A \in P^{d-1}\). This implies that \(E_0\) is identified with a non empty open subset of \(P^{d-1} \times T\), where \(P^{d-1}\) is viewed as an affine space over \(\mathbb{C}\). This shows that \(E_0\) is equidimensional and that \(\text{codim}_S(E_0) = \left(\frac{d+2}{2}\right) - 1 - \left(\frac{d+1}{2}\right) + \text{codim}_{S'}(T) = d + 1\) as desired.
Remark 4.4 It is interesting to ask if the components $E$ of $\Sigma$ are the only ones satisfying $\text{codim}_S(E) = d - 2$. It is possible to give a positive answer to this question in case $s = 2$ and $e_1 = e_2 = 1$ by using the method in [Gr5] and [V]. The general case remains open.

5. Beilinson’s Tate conjecture

In this section we show Theorem 0.5. Let the assumption be as in the beginning of §3. Write $m = n - r$ and $d = \sum_{1 \leq i \leq r} d_i$. Theorem 0.5(1) is an immediate consequence of Theorem 5.1 and Remark 5.2 below.

**Theorem 5.1** Assume that $k$ is finitely generated over $\mathbb{Q}$ and that $d \geq n + 1 + c_S(X, Z)$. Let $\overline{k}$ be an algebraic closure of $k$ and put $\overline{S} = S \times_k \overline{k}$. Let $\overline{\eta}$ be a geometric generic point of $\overline{S}$ and let $U \subset X$ be the fibers of $U \subset X$ over $\overline{\eta}$.

1. Assuming $s \geq 1$, $H^m_{et}(U, \mathbb{Q}(m))_{\overline{S}} = H^m_{et}(U, \mathbb{Q}(m))_{\overline{S}}$.

2. $H^m_{et}(X, \mathbb{Q}(m))_{\overline{S}} = 0$.

**Remark 5.2** Let $\widetilde{S} \rightarrow S$ be a dominant morphism and $\widetilde{X} \supset \widetilde{Z}$ be the base change. Then $c_S(X, Z) = c_{\overline{S}}(X, Z)$. Hence Theorem 5.1 holds after replacing $X \supset Z$ by the base change. In particular Theorem 5.1 holds if one replaces $\pi_1(\overline{S}, \overline{\eta})$ by any open subgroup of finite index.

**Theorem 5.3** Assume that $k = \mathbb{C}$ and $d \geq n + 1 + c_S(X, Z)$. Let $U \subset X$ be the fibers of $U \subset X$ over a fixed base point $0 \in S(\mathbb{C})$.

1. Assuming $s \geq 1$, $H^m(U, \mathbb{Q}(m))_{\overline{S}} = H^m(U, \mathbb{Q}(m))_{\overline{S}}$.

2. $H^m(U, \mathbb{Q}(m))_{\overline{S}} = 0$.

First we deduce Theorem 5.1 from Theorem 5.3. By the Lefschetz principle we may assume that $k$ is a subfield of $\mathbb{C}$. We fix an embedding $k(\overline{\eta}) \hookrightarrow \mathbb{C}$ and let $0 \in \overline{S}(\mathbb{C})$ be the corresponding $\mathbb{C}$-valued point of $\overline{S}$. Write $S_\mathbb{C} = S \otimes_k \mathbb{C}$ and put $U_\mathbb{C} = X \times_S \text{Spec}(\mathbb{C})$ via $\text{Spec}(\mathbb{C}) \rightarrow \overline{S} \rightarrow S$. We have the comparison isomorphisms ([SGA4, XVI Theorem 4.1])

$$H^m(U_\mathbb{C}, \mathbb{Q}) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^m_{et}(U_\mathbb{C}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^m_{et}(U, \mathbb{Q}_\ell)$$

that are equivariant with respect to the maps of topological and algebraic fundamental groups:

$$\pi_1^{\text{top}}(S_\mathbb{C}, 0) \rightarrow \pi_1^{\text{alg}}(S_\mathbb{C}, 0) \rightarrow \pi_1^{\text{alg}}(\overline{S}, \overline{\eta}).$$

The desired assertion follows at once from this.
In order to show Theorem 5.3, we need a preliminary. We assume $k = \mathbb{C}$. Let $S_{an}$ be the analytic site on $S(\mathbb{C})$. For a coherent sheaf $\mathcal{F}$ on $S_{zar}$ let $\mathcal{F}_{an}$ be the associated analytic sheaf on $S_{an}$. We introduce local systems on $S_{an}$

$$H^q_{\mathcal{F}}(\mathcal{U}/S)(p) = R^q g_* \mathbb{Q}(p) \quad \text{and} \quad H^q_{\mathbb{C}}(\mathcal{U}/S) = R^q g_* \mathbb{C},$$

where $g: \mathcal{U} \to S$ is the natural morphism. Let $H^q_{\mathcal{F}}(\mathcal{U}/S)$ be the sheaf of holomorphic sections of $H^q_{\mathcal{F}}(\mathcal{U}/S)$ and let $F^p H^q_{\mathcal{F}}(\mathcal{U}/S) \subset H^q_{\mathcal{F}}(\mathcal{U}/S)$ be the holomorphic subbundle given by the Hodge filtration on the cohomology of fibers of $\mathcal{U}/S$. We have the analytic Gauss-Manin connection

$$\nabla : H^q_{\mathcal{F}}(\mathcal{U}/S) \to \Omega^1_{S_{an}} \otimes H^q_{\mathcal{F}}(\mathcal{U}/S) \quad \text{with Ker}(\nabla) = H^q_{\mathcal{F}}(\mathcal{U}/S)$$

that satisfies $\nabla(F^p H^q_{\mathcal{F}}(\mathcal{U}/S)) \subset \Omega^1_{S_{an}} \otimes F^{p-1} H^q_{\mathcal{F}}(\mathcal{U}/S)$. The induced map

$$F^p H^{p+q}_{\mathcal{F}}(\mathcal{U}/S)/F^{p+1} \to \Omega^1_{S_{an}} \otimes F^{p-1} H^{p+q}_{\mathcal{F}}(\mathcal{U}/S)/F^p$$

is identified with $(\nabla^{p,q})_{an}$ via the identification $F^p H^{p+q}_{\mathcal{F}}(\mathcal{U}/S)/F^{p+1} = H^{p,q}(\mathcal{U}/S)_{an}$ where

$$\nabla^{p,q} : H^{p,q}(\mathcal{U}/S) \to \Omega^1_S \otimes H^{p-1,q+1}(\mathcal{U}/S)$$

is defined in §3. Therefore Theorem 3.1 implies the following.

**Theorem 5.4** Let the assumption be as in Theorem 5.3.

(1) If $s \geq 1$, $F^1 H^m_{\mathcal{F}}(\mathcal{U}/S) \cap H^m_{\mathcal{C}}(\mathcal{U}/S)$ is generated over $\mathbb{C}$ by the image of

$$H^m(U, \mathbb{Q}(m))_{triv} \hookrightarrow H^m(U, \mathbb{Q}(m))^{\pi_1(S,0)} \simeq \Gamma(S_{an}, H^m_{\mathcal{F}}(\mathcal{U}/S)(m)) \hookrightarrow H^m_{\mathcal{C}}(\mathcal{U}/S).$$

(2) If $s = 0$, $F^1 H^m_{\mathcal{F}}(\mathcal{X}/S) \cap H^m_{\mathcal{C}}(\mathcal{X}/S)_{prim} = 0$, where $H^m_{\mathcal{C}}(\mathcal{X}/S)_{prim}$ is the primitive part of $H^m_{\mathcal{C}}(\mathcal{X}/S)$.

Now we start the proof of Theorem 5.3. First we show Theorem 5.3 (2). Write $H = H^m(X, \mathbb{Q})^{\pi_1(S,0)}$. We have the natural isomorphism

$$H \otimes \mathbb{C} \simeq \Gamma(S_{an}, H^m_{\mathcal{C}}(\mathcal{X}/S)_{prim}).$$

By [D1] $H$ is a sub-Hodge structure of $H^m(X, \mathbb{Q})$ so that we have the Hodge decomposition

$$H \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q} \quad \text{with} \quad H^{p,q} \subset \Gamma(S_{an}, F^p H^m_{\mathcal{F}}(\mathcal{X}/S) \cap H^m_{\mathcal{C}}(\mathcal{X}/S)_{prim}).$$

By Theorem 5.4 this implies $H^{p,q} = 0$ for $p \geq 1$ which implies $H = 0$ by the Hodge symmetry.

Next we show Theorem 5.3 (1). By Lemma 5.3 below we have

$$H^m(U, \mathbb{Q}(m))^{\pi_1(S,0)} \simeq \Gamma(S_{an}, H^m_{\mathcal{F}}(\mathcal{U}/S)(m)) \subset \Gamma(S_{an}, H^m_{\mathcal{C}}(\mathcal{U}/S) \cap F^1 H^m_{\mathcal{F}}(\mathcal{U}/S)).$$

Hence the assertion follows from Theorem 5.4 (1).
Lemma 5.5 Let the assumption be as in Theorem 5.3. Then $H := H^m(U, \mathbb{Q})^{π_1(S,0)}$ is a submixed Hodge structure of $H^m(U, \mathbb{Q})$ and $F^iH_C = H_C$ where $H_C = H \otimes \mathbb{C}$.

Proof. The fact that $H$ is a submixed Hodge structure follows from the theory of mixed Hodge modules [SaM]. We show the second assertion. We recall that the graded subquotients of the weight filtration of $H^m(U, \mathbb{Q})$ is given by

$$Gr^W_{m+p}H^m(U, \mathbb{Q}) = \bigoplus_{1 \leq j_1 < \cdots < j_p \leq s} H^{m-p}(Z_{j_1} \cap \cdots \cap Z_{j_p}, \mathbb{Q}(-p))^{prim}. \quad (0 \leq p \leq m)$$

Therefore it suffices to note that $H^m(X, \mathbb{Q})^{π_1(S,0)} = 0$ by Theorem 5.3(2).

Now we show Theorem 5.3(2). It will follows from Theorem 5.6 below by using the theory of Hilbert set (cf. [La]).

Theorem 5.6 Under the assumption of Theorem 5.3(2), there exists an irreducible variety $\widetilde{S}$ over $k$ with a finite etale covering $φ : \widetilde{S} → S$ for which the following holds: Let $\widetilde{π} = π \circ φ$ and let $H ⊂ \mathbb{P}^N(k)$ be the subset of such points that $\widetilde{π}^{-1}(y)$ is irreducible. For $∀x ∈ S$ such that $π(x) ∈ H$ and for any subgroup $G ⊂ \text{Gal}(k(\overline{x})/k(x))$ of finite index we have

$$H^m_{et}(U_{\overline{x}}, \mathbb{Q}_ℓ(m))^G = H^m_{et}(U_{\overline{x}}, \mathbb{Q}_ℓ(m))^{triv},$$

where $\overline{x}$ is a geometric point over $x$ and $U_{\overline{x}}$ is the fiber of $U/S$ over $\overline{x}$.

Proof. (cf. [T] and [BE]) By choosing a $k$-rational point $0$ of $S$, we get the decomposition

$$π_1(S, \overline{π}) = π_1(\overline{S}, \overline{π}) × \text{Gal}(\overline{k}/k),$$

where the notation is as in Theorem 5.1 and $π_1(\overline{S}, \overline{π})$ is identified with the quotient of $π_1(S, \overline{π})$ which classifies the finite etale coverings of $S$ that completely decompose over $0$. Let

$$Γ = \text{Image}(π_1(\overline{S}, \overline{π}) → GL_{2ℓ}(H^m_{et}(U_{\overline{x}}, \mathbb{Q}_ℓ(m)))).$$ 

From the fact that $Γ$ contains an $ℓ$-adic Lie group as a subgroup of finite index, we have the following fact (cf. [T]): There exists a subgroup $Γ' ⊂ Γ$ of finite index such that a continuous homomorphism of pro-finite group $P → Γ$ is surjective if and only if $P → Γ → Γ/Γ'$ is surjective as a map of sets. Let $φ : \widetilde{S} → S$ be a finite etale covering that corresponds to the inverse image of $Γ'$ in $π_1(\overline{S}, \overline{π})$ and let $H$ be defined as in Theorem 5.6. Fix $x ∈ S$ with $π(x) ∈ H$ and let $\overline{x}$ be a geometric point of $x$. By choosing a “path” $x → \overline{x}$, we get the map $\text{Gal}(k(\overline{x})/k(x)) = π_1(x, \overline{x}) → π_1(S, \overline{π})$. By the definition of $H$ the composite of $i$ with $π_1(S, \overline{π}) → Γ → Γ/Γ'$ is surjective so that $\text{Gal}(k(\overline{x})/k(x))$ surjects onto $Γ$ by the above fact. It implies that for any subgroup $G ⊂ \text{Gal}(k(\overline{x})/k(x))$ of finite index, its image $π$ in $Γ$ is of finite index and hence that we have

$$H^m_{et}(U_{\overline{x}}, \mathbb{Q}_ℓ(m))^G → H^m_{et}(U_{\overline{x}}, \mathbb{Q}_ℓ(m))^{triv} = H^m_{et}(U_{\overline{x}}, \mathbb{Q}_ℓ(m))^{triv},$$

where the last equality follows from Theorem 5.1 and Remark 5.2. This completes the proof of Theorem 5.6.
Let \( k \) be a number field and let \( V \) be an irreducible variety over \( k \). Let \( \pi : V \to \mathbb{P}^N_k \) be an etale morphism and let \( H \subset \mathbb{P}^N_k \) be the subset of such points \( x \) that \( \pi^{-1}(x) \) is irreducible. Let \( \Sigma \) be any finite set of primes of \( k \) and let \( k_v \) be the completion of \( k \) at \( v \). Then the image of \( H \) in \( \prod_{v \in \Sigma} \mathbb{P}^N_{k_v} \) is dense.

6. Implication on injectivity of regulator maps for \( K_1 \) of surfaces

Let \( X \) be a projective smooth surface over a field \( k \). Let \( U \subset X \) be the complement of a simple normal crossing divisor \( Z \subset X \). In this section we discuss an implication of the surjectivity of \( \text{reg}^2_U \) and \( \text{reg}^2_{ct,U} \) on the regulator maps for \( CH^2(X,1) \). Recall that \( CH^2(X,1) \) is by definition the cohomology of the following complex

\[
K_2(k(X)) \xrightarrow{\partial_{\text{tame}}} \bigoplus_{C \subset X} k(C)^* \xrightarrow{\partial_{\text{div}}} \bigoplus_{x \in X} \mathbb{Z},
\]

where the sum on the middle term ranges over all irreducible curves on \( X \) and that on the right hand side over all closed points of \( X \). The map \( \partial_{\text{tame}} \) is the so-called tame symbol and \( \partial_{\text{div}} \) is the sum of divisors of rational functions on curves. Thus an element of \( \text{Ker}(\partial_{\text{div}}) \) is given by a finite sum \( \sum_i (C_i, f_i) \), where \( f_i \) is a non-zero rational function on an irreducible curve \( C_i \subset X \) such that \( \sum_i \text{div}(f_i) = 0 \) on \( X \). We recall that \( K_2(k(X)) \) is generated as an abelian groups by symbols \( \{f,g\} \) for non-zero rational functions \( f, g \) on \( X \) and that

\[
\partial_{\text{tame}}(\{f,g\}) = ((f)_0, g) + ((f)_\infty, 1/g) + ((g)_0, 1/f) + ((g)_\infty, f),
\]

where \( (f)_0 \) (resp. \( (f)_\infty \)) is the zero (resp. pole) divisor of \( f \). An important tool to study \( CH^2(X,1) \) is the regulator map: In case \( k = \mathbb{C} \) it is given by

\[
\text{reg}_{D,X} : CH^2(X,1) \to H^3_D(X,\mathbb{Z}(2)),
\]

where the group on the right hand side is the Deligne cohomology group (cf. [EV] and [J2]). Assuming the first Betti number \( b_1(X) = 0 \), we have the following explicit description of \( \text{reg}_{D,X} \). Take \( \alpha = \sum_i (C_i, f_i) \in \text{Ker}(\partial_{\text{div}}) \). Under the isomorphism (cf. [EV, 2.10])

\[
H^3_D(X,\mathbb{Z}(2)) \cong \frac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z}(2)) + F^2H^2(X,\mathbb{C})} \cong \frac{F^1H^2(X,\mathbb{C})^*}{H_2(X,\mathbb{Z})},
\]

\( \text{reg}_{D,X}(\alpha) \) is identified with a linear function on complex valued \( C^\infty \)-forms \( \omega \) and we have

\[
\text{reg}_{D,X}(\alpha)(\omega) = \frac{1}{2\pi \sqrt{-1}} \sum_i \int_{C_i - \gamma_i} \log(f_i) \omega + \int_{\Gamma} \omega,
\]

where \( \gamma_i := f_i^{-1}(\gamma_0) \) with \( \gamma_0 \), a path on \( \mathbb{P}^1_C \) connecting 0 with \( \infty \) and \( \Gamma \) is a real piecewise smooth 2-chain on \( X \) such that \( \partial \Gamma = \bigcup_i \gamma_i \) which exists due to the assumption \( \alpha \in \text{Ker}(\partial_{\text{div}}) \) and \( b_1(X) = 0 \).

In case \( k \) is finitely generated over \( \mathbb{Q} \) we have the regulator map

\[
\text{reg}_{\text{cont},X} : CH^2(X,1) \otimes \mathbb{Z}_\ell \to H^3_{\text{cont}}(X,\mathbb{Z}_\ell(2)),
\]

where \( H^3_{\text{cont}} \) denotes the continuous etale cohomology of \( X \) (cf. [J3]).
Theorem 6.1 (1) Assume $k = \mathbb{C}$ and that there exists a subspace $\Delta \subset CH^2(U, 2) \otimes \mathbb{Q}$ such that the restriction of $reg^2_{\Delta}$ on $\Delta$ is surjective. Let $\alpha \in CH^1(Z_1, 1)$ and assume $reg_{D,X}(\alpha) = 0$. Then $\alpha \in \partial_Z(\Delta)$ in $CH^1(Z_1, 1) \otimes \mathbb{Q}$. In particular $\iota(\alpha) = 0 \in CH^2(X, 1) \otimes \mathbb{Q}$.

(2) Assume that $k$ is finitely generated over $\mathbb{Q}$. The analogous fact holds for $reg^2_{et,U}$ and $reg_{cont,X}$.

The following result is a direct consequence of Theorem 6.1 and Definition 0.3. Let $Z \subset X_\xi$ be as in the beginning of this section and let $Z_i \subset X_i$ be the fibers over $t \in S$.

Corollary 6.2 Let the assumption be as in Theorem 0.4. For any $t \in S \setminus S_{NL}$, $reg_{D,X}$ restricted on the image of $CH^1(Z_t, 1)$ is injective modulo torsion.

Remark 6.3 In the forthcoming paper [AS2] we study the Noether-Lefschetz locus in case (0-4) is the universal family of smooth surfaces of degree $d$ and three hyperplanes in $\mathbb{P}^3$ intersecting transversally. In this case it is shown that there exist $t \in S \setminus S_{NL}$ such that the image of $CH^1(Z_t, 1)$ in $CH^2(X_t, 1)$ is non-torsion if $d$ is large enough. Thus Corollary 6.2 has indeed non-trivial implication on the injectivity of the regulator map.

Now we start the proof of Theorem 6.1. The idea of the following proof is taken from [J1, 9.8]. We only treat the first assertion. The second is proven by the same way. We have the commutative diagram (cf. [Bl] and [J2, 3.3 and 1.15]).

$$
\begin{array}{cccc}
CH^2(X, 2) & \to & CH^2(U, 2) & \to & CH^1(Z, 1) & \to & CH^2(X, 1) \\
\downarrow ch^{2,2}_{D,X} & & \downarrow ch^{2,2}_{D,U} & & \downarrow ch^{1,1}_{D,Z} & & \downarrow ch^{2,1}_{D,X} \\
H^*_D(X, \mathbb{Z}(2)) & \to & H^*_D(U, \mathbb{Z}(2)) & \to & H^*_D(Z, \mathbb{Z}(2)) & \to & H^*_D(X, \mathbb{Z}(2))
\end{array}
$$

Here the horizontal sequences are the localization sequences for higher Chow groups and Deligne cohomology groups and they are exact. The vertical maps are regulator maps. By a simple diagram chasing it suffices to show that $ch^{1,1}_{D,Z}$ is injective and $ch^{2,2}_{D,U}(\Delta) + \text{Image}(\iota)$ spans $H^2_D(U, \mathbb{Q}(2))$. In order to show the first assertion we note the commutative diagram

$$
\begin{array}{cccc}
0 & \to & \bigoplus_{1 \leq i \leq r} CH^1(Z_i, 1) & \to & CH^1(Z, 1) & \to & \bigoplus_{x \in W} \mathbb{Z} \\
& \downarrow ch^{1,1}_{D,Z_i} & & \downarrow ch^{1,1}_{D,Z} & & \downarrow \simeq ch^{0,0}_{D,x} \\
0 & \to & \bigoplus_{1 \leq i \leq r} H^1_D(Z_i, \mathbb{Z}(1)) & \to & H^1_D(Z, \mathbb{Z}(1)) & \to & \bigoplus_{x \in W} H^1_D(x, \mathbb{Z}(0)) \\
& \downarrow \simeq & & \downarrow = & & \downarrow \simeq & \\
& \bigoplus_{1 \leq i \leq r} H^3_D(Z_i, \mathbb{Z}(2)) & \to & H^3_D(Z, \mathbb{Z}(2)) & \to & \bigoplus_{x \in W} H^4_D(x, \mathbb{Z}(2))
\end{array}
$$

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where $W = \bigsqcup_{1 \leq i \neq j \leq r} Z_i \cap Z_j$. The horizontal sequences come from the Mayer-Vietoris spectral sequence and they are exact. Thus the desired assertion follows from the fact that $ch^{1,1}_{D,Z}$ is an isomorphism (cf. [J2, 3.2]). To show the second assertion we recall the exact sequence (cf. [EV, 2.10])

$$0 \to H^1(U, \mathbb{C})/H^1(U, \mathbb{Z}(2)) \xrightarrow{\gamma} H^2_D(U, \mathbb{Z}(2)) \xrightarrow{\beta} H^2(U, \mathbb{Z}(2)) \cap F^2 H^2(U, \mathbb{C}) \to 0$$

and the same sequence with $U$ replaced by $X$. We have $ch^{2,2}_{B,U} = \beta \cdot ch^{2,2}_{D,U}$ and the commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}^* \otimes CH^1(U, 1) & \rightarrow & CH^1(U, 1) \otimes CH^1(U, 1) \\
\downarrow \log \otimes ch^{1,1}_{B,U} & & \downarrow \text{product} \\
\mathbb{C}/\mathbb{Z}(1) \otimes H^1(U, \mathbb{Z}(1)) & \rightarrow & CH^2(U, 2) \\
\downarrow \cong & & \downarrow ch^{2,2}_{D,U} \\
H^1(U, \mathbb{C})/H^1(U, \mathbb{Z}(2)) & \xrightarrow{\gamma} & H^2_D(U, \mathbb{Z}(2))
\end{array}
\]

Therefore the desired assertion follows from the surjectivity of

$$CH^1(U, 1) \xrightarrow{ch^{1,1}_{B,U}} H^1(U, \mathbb{Z}(1))/H^1(X, \mathbb{Z}(1))$$

which is easily seen.

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