A Class of FRT Quantum Groups
and \( \text{Fun}_q(G_2) \) as a Special Case

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November 1993
revised: December 1993

Abstract

A class of quantum groups that includes a \( G_2 \) type quantum group \( \text{Fun}_q(G_2) \) is considered. After elucidating its connection with the quantized universal enveloping algebras, a concrete construction of \( \text{Fun}_q(G_2) \) is presented.

1 introduction

There are at least two approaches to quantum groups: One is based on the Chevalley basis of Lie algebras and gives, in terms of Hopf algebra, a remarkable deformation — the quantized universal enveloping (QUE) algebra — for every Kac-Moody Lie algebra. The other — known as Faddeev-Reshetikhin-Takhtajan (FRT) approach — is based on matrix relations that employ noncommuting matrix entries and gives a quantization of Lie groups with retaining a formal correspondence to the classical objects.
The former provides a way of constructing $R$-matrices, while the latter needs an $R$-matrix as a starting point. The latter, instead of this cost, seems to have advantages in some directions, especially in investigation on noncommutative geometries as pointed out in Ref. 7.

Since there are $R$-matrices not explained currently in terms of the QUE-algebras, the latter should give more various quantum groups apart from the problem of incorporating an antipode. Nevertheless, as far as the author’s knowledge, quantum groups of the exceptional type so far have been constructed concretely only in the former approach. So we think it natural to ask how the exceptional Lie groups are quantized in the former approach. The preprint provides an answer in part; we present a $G_2$ type quantum group in the FRT approach after giving a consideration on a class of quantum groups with slightly general context and elucidating its connection with the QUE-algebras.

The work was inspired by Okubo’s lecture note.

2 notations and conventions

This section summarizes well-known notions and facts for the definiteness on notations and conventions in the preprint. For the concepts and the foundations of the FRT approach, we refer the reader to Ref. 5.

In the preprint an algebra means an algebra over the complex field $\mathbb{C}$. Any algebra except for Lie algebras is supposed to have the unity element $1$. We do not distinguish the unity element (if exist) of an algebra from that of the base field $\mathbb{C}$ unless otherwise stated.

For any Hopf algebra $A$, the comultiplication, counit and antipode are denoted by $\Delta$, $\epsilon$ and $S$ respectively.

For any Hopf algebra $A$, an $A$-module $V$ or representation $(\rho_V, V)$ of $A$ means merely a complex representation as an algebra, however, the category of $A$-modules acquires extra structures coming from $\Delta$, $\epsilon$ and $S$ for $A$: For any two $A$-modules $V$ and $W$, there is the tensor product module $V \otimes W$, which is a tensor product space endowed with the action $\rho_{V \otimes W} := (\rho_V \otimes \rho_W) \circ \Delta$. There is the trivial module $\mathbb{1}$, which is a 1 dimensional vector space endowed with the action $\rho_{\mathbb{1}} := \epsilon$. For any $A$-module $V$, there is the contragredient module $V^*$, which is a dual vector space endowed with the action $\rho_{V^*}$ defined by

$$\langle \rho_{V^*}(x) \zeta, v \rangle = \langle \zeta, \rho_V(S(x)) v \rangle \quad \forall x \in A, \quad \forall \zeta \in V^*, \quad \forall v \in V,$$

where $\langle , \rangle$ stands for the pairing between $V^*$ and $V$. The symbol for the action is often suppressed in the formula such that the module that the vectors appearing there belong to is clearly identifiable. The situation that $A$-modules $V$ and $W$ are isomorphic is denoted by $V \simeq W$.

For any Hopf algebra $A$, the dual Hopf algebra $A^*$ means a dual space of $A$ endowed with the Hopf algebra structure defined by

$$(l l')(t) := (l \otimes l')(\Delta(t)), \quad 1_{A^*}(t) := \epsilon(t),$$

$$\Delta(l)(t \otimes t') := l(t t'), \quad \epsilon(l) := l(1_A), \quad S(l)(t) := l(S(t)).$$
with \( l, l' \in A^* \) and \( t, t' \in A \). Here \( 1_A, 1_{A^*} \) are the unities of \( A, A^* \) respectively, and the pairing between the tensor space is defined by \((l \otimes l')(t \otimes t') = l(t) \cdot l'(t') \) as usual.

A Hopf algebra \( A \) is called quasitriangular if there exists an invertible element \( R \in A \otimes A \) such that

\[
\Delta'(x) = R \Delta(x) R^{-1} \quad \forall x \in A, \tag{1}
\]

\[
(\Delta \otimes \text{id})(R) = R_{13} R_{23}, \tag{2}
\]

\[
(\text{id} \otimes \Delta)(R) = R_{13} R_{12}, \tag{3}
\]

where \( \Delta' := \sigma \circ \Delta \) is the opposite comultiplication and \( R_{12} := R \otimes 1_A, R_{23} := 1_A \otimes R, R_{13} := (\text{id} \otimes \sigma)(R_{12}) \). In the above, \( \sigma \) stands for the permutation on \( A \otimes A \), namely, \( \sigma: x \otimes y \mapsto y \otimes x \). Such an element \( R \) is referred to as the universal \( R \)-matrix. As a consequence of quasitriangularity, \( R \) satisfies

\[
(e \otimes \text{id})(R) = 1_A = (\text{id} \otimes e)(R), \tag{4}
\]

\[
(S \otimes \text{id})(R) = R^{-1}, \quad (\text{id} \otimes S)(R^{-1}) = R. \tag{5}
\]

More importantly, \( R \) gives a solution to the Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \tag{6}
\]

which is an abstraction of the original form \((26)\). A solution to \((26)\) is called an \( R \)-matrix. A representation of the universal \( R \)-matrix gives an \( R \)-matrix.

The QUE-algebra \( U_q(\mathfrak{g}) \) is a \( q \)-deformation of the universal enveloping algebra of a Lie algebra \( \mathfrak{g} \). In the preprint it is assumed that \( \mathfrak{g} \) is a finite dimensional simple Lie algebra and the deformation parameter \( q \in \mathbb{C} \setminus \{0\} \) is generic (not a root of unity).

Let \( \alpha_i \) be the \( i \)th simple root for \( \mathfrak{g} \). The invariant inner product \((\cdot , \cdot)\) in the root space defines the Cartan integers \( a_{ij} := 2 (\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \) and subsidiary parameters \( q_i := q^{\alpha_i, \alpha_j} \). We follow the convention of Ref. 3 for the QUE-algebras. The QUE-algebra \( U_q(\mathfrak{g}) \) is an associative algebra generated by \( X^\pm_i, K_i^{\pm 1} \) \((i = 1, \ldots, \text{rank} \mathfrak{g})\) satisfying the defining relations

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \]

\[
K_i X^\pm_j K_i^{-1} = q_i^{a_{ij}} X^\pm_j, \quad X^+_i X^-_j - X^-_j X^+_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.
\]

\[
\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \binom{1-a_{ij}}{\nu} q_i \left( X^\pm_j \right)^{1-a_{ij}-\nu} X^\pm_j (X^\pm_i)^\nu = 0 \quad (i \neq j),
\]

where \( \binom{m}{n} \) stands for the \( q \)-binomial coefficient

\[
\binom{m}{n} := \frac{q^m - q^{-m}}{q^n - q^{-n}} - \frac{q^{m-1} - q^{-1-m}}{q^{n-1} - q^{-1-n}} - \cdots - \frac{q^{m-n+1} - q^{-n-m-1}}{q - q^{-1}}
\]

defined for \( n, m \in \mathbb{Z}, \ 0 \leq n \leq m \). The Hopf algebra structure is introduced to the algebra \( U_q(\mathfrak{g}) \) as follows:

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\[ \Delta(X^+_i) = X^+_i \otimes 1 + K_i \otimes X^+_i, \quad \Delta(X^-_i) = X^-_i \otimes K_i^{-1} + 1 \otimes X^-_i, \]
\[ \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \]
\[ \epsilon(X^+_i) = 0, \quad \epsilon(K_i^{\pm 1}) = 1, \]
\[ S(X^+_i) = -K_i^{-1} X^+_i, \quad S(X^-_i) = -X^-_i K_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}. \]

The Hopf algebra \( U_q(g) \) is known to be quasitriangular.

It is easily seen by writing \( K_i = q^H_i \) that the algebra in the limit \( q \to 1 \) reduces to the universal enveloping algebra \( U(g) \); in the same limit \( H_i \) and \( X^\pm_i \) are identified as generators of the Lie algebra \( g \).

3 a class of FRT quantum groups

We consider a class of quantum groups that can be viewed as missing ones in the FRT approach.

We hereafter always suppose that latin indices \( i, j, \ldots \) run over the values 1, 2, \ldots, \( N \) and that repeated indices imply summation over them, where \( N \) is a certain fixed integer (we will set \( N = 7 \) for the case of \( \text{Fun}_q(G_2) \) in Sec. 4).

Let \( A_{Rgf} \) be an associative algebra generated by \( t^i_j \) satisfying the defining relations

\[ R^{ij}_{kl} t^k_m t^l_n = t^l_i t^k_j R^{kl}_{mn}, \quad \text{(7)} \]
\[ g^{ij} = t^j_i t^i_j g^{kl}, \quad \text{(8)} \]
\[ g_{kl} t^k_m t^l_n = g_{mn}, \quad \text{(9)} \]
\[ f^{ij}_{kl} t^k_m t^l_n = t^l_i t^k_j f^{kl}_{mn}, \quad \text{(10)} \]
\[ \tilde{f}^{ij}_{kl} t^k_m t^l_n = t^l_i \tilde{f}^{kl}_{mn}. \quad \text{(11)} \]

Here, \( R^{ij}_{kl}, \ g^{ij}, \ g_{ij}, \ f^{ij}_{kl} \) and \( \tilde{f}^{ij}_{kl} \) belong to \( C \) and are referred to, in the preprint, as the structure constants of \( A_{Rgf} \).

We impose a condition on the structure constants in order to ensure that the algebra \( A_{Rgf} \) admits a Hopf algebra structure

\[ \Delta(t^i_j) = t^i_k \otimes t^k_j, \quad \text{(12)} \]
\[ \epsilon(t^i_j) = \delta^i_j, \quad \text{(13)} \]
\[ S(t^i_j) = g^{jl} t^i_l \tilde{g}_{kj}, \quad \text{(14)} \]

that is to suppose

\[ g^{ik} \tilde{g}_{jk} = \delta^i_j = g^{ki} \tilde{g}_{kj}, \quad \text{(15)} \]
\[ g^{ni} f^{lm}_{n} \tilde{g}_{lj} \tilde{g}_{mk} = \tilde{f}^{ij}_{jk} = g^{jn} f^{lm}_{n} \tilde{g}_{jl} \tilde{g}_{km}, \quad \text{(16)} \]
\[ R^{ij}_{kl} g^{km} g^{ln} = g^{jl} g^{ik} R^{mn}_{kl}. \quad \text{(17)} \]

Using (15)–(17), one can verify the Hopf algebra axioms straightforwardly.
In addition to the conditions (13)–(17), if
\begin{align*}
g^{ij}_{kl} \delta^k_l &= R^{ik}_{np} R^{jp}_{ml} g^{mn}, \quad (18) \\
\delta^i_j \delta_{kl} &= g_{mn} R^{mi}_{kp} R^{np}_{ij}, \quad (19) \\
g^{ij}_{kl} \delta^k_l &= R^{ki}_{pm} R^{pj}_{ln} g^{mn}, \quad (20) \\
\delta^i_j \delta_{kl} &= g_{mn} R^{pl}_{it} R^{pm}_{jk}, \quad (21) \\
f^{ij}_{pk} R^{pk}_{sl} &= R^{ik}_{np} R^{jp}_{ml} f^{mn}_{s}, \quad (22) \\
R^{ij}_{pk} \tilde{f}^{kp}_{pl} &= \tilde{f}^{js}_{mn} R^{mi}_{kp} R^{np}_{ij}, \quad (23) \\
f^{ij}_{pm} R^{pk}_{ls} &= R^{ki}_{pm} R^{pj}_{ln} f^{mn}_{s}, \quad (24) \\
R^{is}_{jp} \tilde{f}^{jp}_{kl} &= \tilde{f}^{is}_{mn} R^{in}_{pl} R^{pm}_{jk} \quad (25)
\end{align*}
and the Yang-Baxter equation
\[ R^{i_1 i_2}_{j_1 j_2} R^{j_1 j_3}_{k_1 k_3} R^{j_2 j_3}_{k_2 k_3} = R^{i_1 i_2}_{j_1 j_3} R^{j_1 j_3}_{k_1 k_3} R^{j_2 j_3}_{k_2 k_2} \quad (26) \]
are satisfied, then the formula
\[ l^{(\pm)k_{i_k}}(t^{i_1}_{j_1} \cdots t^{i_{\nu}}_{j_{\nu}}) := R^{(\pm)k_{i_k}}_{i_1 j_1} \cdots R^{(\pm)k_{i_{\nu}}}_{i_{\nu} j_{\nu}} \quad (27) \]
(for \( \nu = 0 \) in particular, \( l^{(\pm)j}_{i_j} := \delta^{i_j}\)) with
\[ R^{(+)}_{+j} := R^{ij}_{ik}, \]
\[ R^{(-)j}_{-j} := R^{-1 ij}_{ik} \quad \text{inverse as a matrix} \quad (28) \]
is well-defined as a definition of the linear mappings \( l^{(\pm)j}_{i_j} : A_{Rgf} \to C \).

The linear mappings \( l^{(\pm)j}_{i_j} \) by definition can be considered to be in the dual Hopf algebra \( A^*_{Rgf} \). A tedious calculation leads to the relations in \( A^*_{Rgf} \)
\begin{align*}
R^{ij}_{kl} l^{(+)}_{+n} l^{(-)}_{-m} &= l^{(+)}_{+n} l^{(-)}_{-m} R^{kl}_{mn}, \quad (29) \\
R^{ij}_{kl} l^{(-)}_{-n} l^{(+)}_{+m} &= l^{(-)}_{-n} l^{(+)}_{+m} R^{kl}_{mn}, \quad (30) \\
g^{ij}_{kl} l^{(+)}_{+n} l^{(-)}_{-m} &= g^{kl}_{mn}, \quad (31) \\
\tilde{g}_{kl} l^{(+)}_{+n} l^{(-)}_{-m} &= \tilde{g}_{mn}, \quad (32) \\
f^{ij}_{kl} l^{(+)}_{+n} l^{(-)}_{-m} &= l^{(+)k}_{+j} f^{kl}_{mn}, \quad (33) \\
\tilde{f}^{ij}_{kl} l^{(+)}_{+n} l^{(-)}_{-m} &= l^{(-)k}_{+j} \tilde{f}^{kl}_{mn} \quad (34)
\end{align*}
and also leads to the expressions for \( \Delta, \epsilon \) and \( S \) for \( l^{(\pm)j}_{i_j} \in A^*_{Rgf} \)
\begin{align*}
\Delta(l^{(\pm)j}_{i_j}) &= l^{(\pm)j}_{k} \otimes l^{(\pm)j}_{k}, \quad (35) \\
\epsilon(l^{(\pm)j}_{i_j}) &= \delta^{i_j}, \quad (36) \\
S(l^{(\pm)j}_{i_j}) &= g^{ij} l^{(\pm)k}_{k} \tilde{g}_{jk}. \quad (37)
\end{align*}

We would like to mention that these equations excluding those employing \( f^{ij}_{kl}, \tilde{f}^{ij}_{kl} \)
have been considered in Ref. 5 for the construction of \( B_n, \ C_n \) and \( D_n \) type quantum groups. Especially, the set of formulae (7) and (26)–(30) is inherent in the FRT approach.
4 connection with the QUE-algebras

We will look for the structure constants of a quantum group considered in the preceding section in a certain representation of a QUE-algebra of Drinfel’d-Jimbo. Equations (7) and (26) — main relations in the FRT approach — were the very origin of the QUE-algebras; a connection between these relations and the representations of the QUE-algebras was already noticed in the pioneering work of Drinfel’d (see Ref. 9 also). The following proposition describes this well-known fact:

**Proposition 1:** Let $U_q(g)$ be a QUE-algebra and $\mathcal{R}$ be the universal $R$-matrix. Suppose that $V$ is a $U_q(g)$-module, $\dim V = N$. And let us define $R^{ij}_{kl} \in \mathbb{C}$ and $\rho_{ij} \in U_q(g)^*$ by

\begin{align*}
\mathcal{R} v_k \otimes v_l &= v_i \otimes v_j R^{ij}_{kl}, \\
x v_j &= v_i \rho_{ij}(x) \quad \forall x \in U_q(g)
\end{align*}

with $\{v_1, \ldots, v_N\}$ being a basis of $V$. Then $R^{ij}_{kl}$ gives a solution to the Yang-Baxter equation (26). Reading $t^{ij}_{kl}$ in the preceding section as $\rho_{ij}$, one finds (7), (12) and (13) as equations for the dual Hopf algebra $U_q(g)^*$.

**Proof:** The Yang-Baxter equation is obtained as a representation of (6). The others are translations of $U_q(g)$’s some properties into the language of the dual Hopf algebra $U_q(g)^*$: The relation (7) follows from (1). The formula (12) together with (13) merely says that $U_q(g) \ni x \mapsto (\rho_{ij}) \in \text{Mat}(N, \mathbb{C})$ is an algebra homomorphism (i.e., that $V$ is a $U_q(g)$-module).

The setting in the proposition without any more restriction seems inadequate to give a closed expression for $S(\rho_{ij})$, however,

$$S(\rho^i_k) \rho^j_k = \delta^i_j = \rho^i_k S(\rho^k_j)$$

is satisfied quite generally due to the Hopf algebra axioms.

Just like in the proposition, we assume without notice that $t_{ij}^{\prime}$ is read as $\rho_{ij}^{\prime}$ whenever equations in the preceding section are referred in this section. In addition to the setting in the proposition, we from now on assume:

**Assumption 1:** The $U_q(g)$-module $V$ is an $N$ (finite) dimensional vector space over $\mathbb{C}$ and its square has the irreducible decomposition

$$V \otimes V \simeq 1 \oplus V \oplus \cdots,$$

where we mean the right-hand side for expressing appearances of 1 and $V$ with their multiplicity 1 and only 1.

We stress the following: The assumption includes that $V$ is irreducible. Since any finite dimensional $U_q(g)$-module is known to be completely reducible if $q$ is generic, the modules
$V^*$ and $V \otimes V$ in particular are completely reducible ($q$ have been assumed generic).

The following definition of $g^{ij}$ and $f^{ijk}$ makes sense under the assumption:

**Definition 1:** Let $1'$ and $V'$ be the irreducible submodules of $V \otimes V$ such that

$$V \otimes V = 1' \oplus V' \oplus \cdots, \quad 1' \cong 1, \quad V' \cong V.$$

Define $g^{ij}$ and $f^{ijk}$ as Clebsch-Gordan coefficients as follows.

$$1' \ni v'_0 = v_j \otimes v_i g^{ij}, \quad (40)$$

$$V' \ni v'_k = v_j \otimes v_i f^{ijk}, \quad (41)$$

where $\{v'_0\}$ is a basis of the submodule $1'$, namely,

$$\Delta(x) v'_0 = v'_0 \epsilon(x) \quad \forall x \in U_q(\mathfrak{g}) \quad (42)$$

and $\{v'_1, \ldots, v'_N\}$ is a basis of the submodule $V'$ such that $v'_j \mapsto v_j$ defines a $U_q(\mathfrak{g})$-isomorphism $V' \rightarrow V$, namely,

$$\Delta(x) v'_j = v'_i \rho^i_j(x) \quad \forall x \in U_q(\mathfrak{g}) \quad (43)$$

with $\rho^i_j$ appeared in (39).

Then one readily finds (8) and (10).

**Proposition 2:** There is a $U_q(\mathfrak{g})$-isomorphism $\varphi^{(1)}: V^* \rightarrow V$,

$$\varphi^{(1)}(\zeta^j) = v_i g^{ij}, \quad (44)$$

where $\{\zeta^1, \ldots , \zeta^N\}$ is the dual basis of $\{v_1, \ldots , v_N\}$, i.e., $\langle \zeta^i, v_j \rangle = \delta^{ij}$.

**Proof:** In terms of $\zeta^i \in V^*$, the contragredient representation is expressed as

$$x \zeta^i = \zeta^j S(\rho^j)(x) \quad \forall x \in U_q(\mathfrak{g}). \quad (45)$$

Using relation (8), we calculate

$$\varphi^{(1)}(x \zeta^k) = \varphi^{(1)}(\zeta^j) S(\rho^k_j)(x) = v_i g^{ij} S(\rho^k_j)(x) = v_i (S(\rho^k_j) g^{ij})(x) = v_i (S(\rho^k_j) \rho^i_m g^{ml})(x) = v_i \rho^i_m(x) g^{mk} = x v_m g^{mk} = x \varphi^{(1)}(\zeta^k)$$

for $\forall x \in U_q(\mathfrak{g})$, hence $\varphi^{(1)}$ is a $U_q(\mathfrak{g})$-homomorphism. Moreover, $\varphi^{(1)}$ is a nonzero $U_q(\mathfrak{g})$-homomorphism because $v_0 \neq 0$ and so is the matrix $\left( g^{ij} \right)$. Since $V$ is irreducible and $V^*$ is completely reducible, we see that Schur’s lemma asserts existence of a submodule $W$ such that $W \subset V^*$, $W \cong V$. Comparing the dimensions of these modules, we find $V^* \cong V$.

Therefore $\varphi^{(1)}$ is a $U_q(\mathfrak{g})$-isomorphism. \qed
Definition 2: Since \( V^* \) is isomorphic to \( V \), we have

\[
V^* \otimes V^* = \mathbf{1}^\sim \oplus V^\sim \oplus \cdots, \quad \mathbf{1}^\sim \simeq \mathbf{1}, \quad V^\sim \simeq V^* (\simeq V).
\]

Here, as a consequence of the proposition, the submodules isomorphic to \( \mathbf{1} \) or \( V \) are exhausted by those appeared above. So we define \( \tilde{g}_{ij} \) and \( \tilde{f}_{ijk} \) by

\[
\mathbf{1}^\sim \ni \tilde{\zeta}^0 = \tilde{g}_{ij} \zeta^j \otimes \zeta^i, \quad (46)
\]

\[
V^\sim \ni \tilde{\zeta}^k = \tilde{f}_{ijk} \zeta^j \otimes \zeta^i, \quad (47)
\]

where \( \{ \tilde{\zeta}^0 \} \) is a basis of the submodule \( \mathbf{1}^\sim \), namely,

\[
\Delta(x) \tilde{\zeta}^0 = \tilde{\zeta}^0 \epsilon(x) \quad \forall x \in U_q(\mathfrak{g}) \quad (48)
\]

and \( \{ \tilde{\zeta}^1, \ldots, \tilde{\zeta}^N \} \) is a basis of the submodule \( V^\sim \) such that \( \tilde{\zeta}^j \mapsto \zeta^j \) defines a \( U_q(\mathfrak{g}) \)-isomorphism \( V^\sim \to V^* \), namely,

\[
\Delta(x) \tilde{\zeta}^i = \tilde{\zeta}^i S(\rho^i_j)(x) \quad \forall x \in U_q(\mathfrak{g}) \quad (49)
\]

with \( S(\rho^i_j) \) appeared in (45).

Using the inverse of \( S \), one readily finds (4) and (11).

Proposition 3: There is a \( U_q(\mathfrak{g}) \)-isomorphism \( \varphi^{(-1)}: V \to V^* \),

\[
\varphi^{(-1)}(v_i) = \tilde{g}_{ij} \zeta^j, \quad (50)
\]

which, apart from a scalar factor, gives the inverse of \( \varphi^{(1)} \).

**Proof:** Most of the proof goes parallel to the preceding one. The last statement follows from irreducibility of \( V \) and Schur’s lemma.

Now we adjust the scalar factor to fulfill

\[
\varphi^{(1)} \circ \varphi^{(-1)} = \text{id}_V, \quad \varphi^{(-1)} \circ \varphi^{(1)} = \text{id}_{V^*}, \quad (51)
\]

by using scaling ambiguity in the definition of \( v'_0 \) or \( \tilde{\zeta}^0 \). Then (15) is apparent and (14) is easily obtained.

We have already verified (1)–(15) and (26).

Proposition 4: There exists a \( U_q(\mathfrak{g}) \)-isomorphism \( \varphi: (V \otimes V)^* \to V^* \otimes V^* \),

\[
\varphi(\zeta^i \otimes \zeta^j) := R^{ij}_{kl} \zeta^k \otimes \zeta^l. \quad (52)
\]

**Proof:** We think it appropriate here to use the symbol for the action to a module. The modules \( (V \otimes V)^* \) and \( V^* \otimes V^* \) are identical as vector spaces, but the actions are different;

\[
\rho_{V^* \otimes V^*} = (\rho_V^* \otimes \rho_V^*) \circ \Delta, \quad (53)
\]

\[
\rho_{(V \otimes V)^*} = (\rho_V^* \otimes \rho_V^*) \circ \Delta', \quad (54)
\]

The former (53) is by definition. The latter (54) is due to coalgebra anti-homomorphism property of an antipode; \( \Delta \circ S = (S \otimes S) \circ \Delta' \). However, as is well-known, the both modules
are isomorphic because \(1\) gives an isomorphism. Using \(3\) and \(38\), we verify that \(\varphi\) is the isomorphism.

Let \(\{\zeta^1, \ldots, \zeta^N\}\) be the dual basis of \(\{v'_1, \ldots, v'_N\}\), i.e., \(\langle \zeta^i, v'_j \rangle = \delta^i_j\). Then
\[
\Delta'(x) \zeta^i = \zeta^j S(\rho^i_j)(x) \quad \forall x \in U_q(\mathfrak{g})
\]
(55)
follows from \(13\) on the understanding that \(\zeta^i\) belongs to the contragredient module \((V')^*\).

Since \(1^* \simeq 1\) and \((V')^* \simeq V\) (the former is easy and not so crucial; the latter follows from proposition 2), we have
\[
(V \otimes V)^* = (1^*)^* \oplus (V')^* \oplus \cdots, \quad (1^*)^* \simeq 1, \quad (V')^* \simeq V^* (\simeq V).
\]
Here, as a consequence of proposition 4, the irreducible submodules isomorphic to \(1\) or \(V\) are exhausted by those appeared above. Therefore, Schur’s lemma uniquely identify \(\zeta^i\) as
\[
\zeta^i = c^{(0)} \hat{f}^i_{jk} \zeta^j \otimes \zeta^k
\]
(56)
up to the constant \(c^{(0)} \in C\setminus\{0\}\); for, comparing with \(13\), one immediately verify \(33\) with \(50\) substituted. So we arrive at

**Proposition 5:** There is an identity
\[
\hat{f}^i_{kl} f^{lk} = \frac{1}{c^{(0)}} \delta^i_j \quad \exists c^{(0)} \in C\setminus\{0\}.
\]
(57)

**Proof:** This is shown by evaluating \(\langle \zeta^i, v'_j \rangle\).

**Proposition 6:** The linear mappings \(\chi^{(\nu)}: V^* \otimes V^*\), \(\nu = 1, 2\) defined by
\[
\chi^{(1)}(\hat{\zeta}^j) = g^{ij} f^{kl} \hat{g}^{km} \hat{g}^{ln} \zeta^m \otimes \zeta^n, \quad \nu = 1, 2
\]
(58)
are shown to be
\[
\chi^{(1)} = \chi^{(2)} = c \text{id}_{V^*} \quad \exists c \in C\setminus\{0\},
\]
(60)
where, strictly speaking, we mean id_{V^*} as the inclusion mapping \(V^* \to V^* \otimes V^*\).

**Proof:** Similarly to the calculation for \(\varphi^{(1)}\), commutativity of \(\chi^{(\nu)}\) with the action of \(\forall x \in U_q(\mathfrak{g})\) can be verified (for \(\nu = 2\), the verification will be made easier by writing \(x = S^{-1}(y)\)). Hence these mappings are \(U_q(\mathfrak{g})\)-homomorphisms. Moreover, these mappings are nonzero \(U_q(\mathfrak{g})\)-homomorphisms because \((g^{ij})\) and \((\hat{g}_{ij})\) are regular as matrices and some of the constants \(f^{ij}_{kl}\) by definition are nonzero. Therefore Schur’s lemma with the irreducible decomposition of \(V^* \otimes V^*\) states that
\[
\chi^{(\nu)} = c^{(\nu)} \text{id}_{V^*} \quad \exists c^{(\nu)} \in C\setminus\{0\}.
\]
This says
\[ g^{ni} f^{lm} n \tilde{y}_{ij} \tilde{y}_{mk} = c^{(1)} \tilde{f}_{jk}^{i}, \quad (61) \]
\[ g^{jn} f^{lm} n \tilde{y}_{jl} \tilde{y}_{km} = c^{(2)} \tilde{f}_{jk}^{i}. \quad (62) \]

Here, \( c^{(1)} = c^{(2)} \) because
\[
\frac{c^{(1)}}{c^{(0)}} \delta_{ij} = c^{(1)} \tilde{f}_{k}^{ij} f^{lk} = g^{ni} f^{pm} n \tilde{y}_{pk} \tilde{y}_{ml} f^{lk} = g^{ni} \left( \tilde{y}_{pk} \tilde{y}_{ml} f^{lk} \right) f^{pm} n = g^{ni} \left( c^{(2)} \tilde{f}_{mp} \tilde{y}_{kj} \right) f^{pm} n
\]
\[
= \frac{c^{(2)}}{c^{(0)}} \delta_{kn} g^{ni} \tilde{y}_{kj} = \frac{c^{(2)}}{c^{(0)}} \delta_{kj}. \]

So we get the proposition by putting \( c = c^{(1)} = c^{(2)} \).

The value of \( c \) is a matter of convention; using scaling ambiguity in the definition of \( v'_{j} \) or \( \tilde{\zeta}_{j} \), we can adjust \( c = c^{(1)} = c^{(2)} = 1 \).

Then (61) and (62) turn out to be (16).

As is easily verified, (17) follows from (18) and (20). So (18)–(25) are at this stage all remaining equations that we wish to show. These rely mostly on quasi-triangularity of \( U_{q}(\mathfrak{g}) \). The formulae (2) and (3) are represented on the module \( V \otimes V \otimes V \) as follows:
\[
\left( \rho_{i_{1}j_{1}}^{i_{2}j_{2}} \otimes \rho_{i_{3}j_{3}}^{i_{4}j_{4}}(R) \right) = R_{i_{1}i_{2}j_{1}j_{2}}^{i_{3}i_{4}j_{3}j_{4}}, \quad (64)
\]
\[
\left( \rho_{i_{1}j_{1}}^{i_{2}j_{2}} \otimes \rho_{i_{2}j_{2}}^{i_{3}j_{3}}(R) \right) = R_{i_{1}i_{2}j_{1}j_{2}}^{i_{3}i_{2}j_{3}j_{2}}, \quad (65)
\]

Multiplying (64) by \( g^{j_{2}j_{1}}, \tilde{y}_{i_{1}i_{2}}, f^{j_{2}j_{1}} \) or \( \tilde{f}_{i_{1}i_{2}}^{j_{2}j_{1}} \), we get four formulae. Multiplying (65) by \( g^{j_{3}j_{2}}, \tilde{y}_{i_{2}i_{3}}, f^{j_{3}j_{2}} \) or \( \tilde{f}_{i_{2}i_{3}}^{j_{3}j_{2}} \), we get four more formulae. Using (8)–(11) and (4) suitably, we can turn these eight formulae into (18)–(25). This derivation with slightly general setting is found in Ref. 11. One can derive (18)–(21) from (5) also.

Although definition 2 with (51) and (63) gives \( \tilde{y}_{ij} \) and \( \tilde{f}^{i}_{jk} \), it is perhaps more convenient for the purpose of computing these constants to use (15) and (16) instead. In this point of view, we summarize the arguments above as follows:

**Theorem 7:** Suppose that a \( U_{q}(\mathfrak{g}) \)-module \( V \) fulfills the assumption 1. Let \( (R_{ijkl}) \) be the \( R \)-matrix defined by (38) and let \( g^{ij} \) and \( f^{ij} \) be the constants given in definition 1. The remaining constants \( \tilde{y}_{ij} \) and \( \tilde{f}^{i}_{jk} \) are given by the 1st (or equivalently the 2nd) equations in (15) and (16) respectively. Then these constants satisfy all of the equations (15)–(26).

Furthermore, \( \rho_{ij}^{i} \in U_{q}(\mathfrak{g}) \) defined by (33) satisfies (7)–(14) with these constants. In other words, \( \tilde{t}^{i}_{j} \mapsto \rho_{ij}^{i} \) defines a Hopf algebra homomorphism \( A_{Rgf} \rightarrow U_{q}(\mathfrak{g})^{*} \), where \( A_{Rgf} \) is a Hopf algebra considered in the preceding section with these constants as the structure constants.

10
5  construction of $\text{Fun}_q(G_2)$

We notice that the 7 dimensional irreducible module of the $G_2$ type QUE-algebra $U_q(g_2)$ fulfills the assumption $[3]$; the irreducible decomposition

$$\mathbf{7} \otimes \mathbf{7} \simeq \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27}$$

(66)

is valid for generic $q$, where $\mathbf{n}$ stands for the $n$ dimensional irreducible $U_q(g_2)$-module. Therefore this representation induces a Hopf algebra $A_{Rgf}$ considered in Sec. [3] via the prescription for the structure constants described in the theorem $[3]$. This Hopf algebra is denoted by $A(q)$ temporarily.

Let us consider the situation at $q = 1$:

**Proposition 8:** Suppose that $(w_{ij}) \in \text{Mat}(7, \mathbb{C})$ is a solution to the equations (7)–(11) with the structure constants of $A(1)$ provided that one reads $t_{ij}$ in the equations as $w_{ij}$. Let $G$ be the set of all such solutions $(w_{ij})$ and define a multiplication between two elements of $G$ as matrices. Then $G$ constitutes a Lie group isomorphic to $\text{Aut} \mathbb{C}^C$, where $\mathbb{C}^C$ stands for Cayley algebra over $\mathbb{C}$.

Before going into the proof, let us remember the fact that $\text{Aut} \mathbb{C}^C$ defines a $G_2$ type complex Lie group. It is known $[4]$ that $\text{Aut} \mathbb{C}^C$ is simply-connected and has the center of order 1. Theory of Lie groups $[4]$ hence deduces that any connected complex Lie group of $G_2$ type is isomorphic to $\text{Aut} \mathbb{C}^C$.

**Proof:** It is easily shown that $G$ constitute a group; the verification goes parallel to that of the Hopf algebra axioms for $A_{Rgf}$.

We first consider the relation

$$\zeta^i \zeta^j = -\mu g^{ij} + f^{ijk} \zeta^k, \quad \mu = \frac{\hat{f}_{jk} f^{jk}}{42},$$

where the constants $g^{ij}$, $f^{ijk}$, and $\hat{f}_{jk}$ are those for $q = 1$ also. It is known $[4]$ that the relation defines Cayley algebra $\mathbb{C}^C$ with the basis $\{1, \zeta^1, \ldots, \zeta^7\}$. As is readily verified, for arbitrary $(w^i_j) \in G$ the transformation $\zeta^i \rightarrow w^i_j \zeta^j$ preserves the relation above, therefore $G \subset \text{Aut} \mathbb{C}^C$.

Next we use the fact that $\mathbf{7}$ is a faithful representation of the $G_2$ type Lie algebra $g_2$. This fact can be seen with an appearance of the adjoint representation $\mathbf{14}$ (the adjoint representation of a semi-simple Lie algebra is faithful $[3, 13]$) in the irreducible decomposition $[10]$, which holds for $q = 1$. As we have studied in the preceding section, $[3]$, $[11]$ with $\rho^i_j$ in place of $t^i_j$ are right equations in $U(q_2)^\sim$. Hence, considering the pairing between each one of these equations and $x := e^X$ for $X \in g_2 \subset U(g_2)$, and using the equations $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$ satisfied at $q = 1$, we easily find $(\rho^i_j(x)) \in G$. Such elements $(\rho^i_j(x))$ generate a connected Lie group with the Lie algebra $g_2$ because $\mathbf{7}$ is faithful. Hence $\text{Aut} \mathbb{C}^C \subset G$ follows.

So we arrive at $G = \text{Aut} \mathbb{C}^C$. 

\[\square\]
The proposition implies that the Hopf algebra $A(q)$ is considered as a quantization of the $G_2$ type Lie group $\text{Aut} \, \mathfrak{c}^C$; at $q = 1$, $t_{ij}$ can be viewed as the tautological mappings $(w^i_j) \mapsto w^i_j$, which generate $\text{Fun}(G_2) := \text{Fun}(\text{Aut} \, \mathfrak{c}^C)$. Henceforth $A(q)$ is denoted by $\text{Fun}_q(G_2)$.

It is possible in practice to determine the explicit form of the structure constants of $\text{Fun}_q(G_2)$; the $R$-matrix ($R^{ij}_{kl}$) is already known explicitly, while the other constants $g^{ij}$, $\tilde{g}^{ij}$, $f^{ijk}$, $\tilde{f}^{ijk}$ are rather easily determined ($f^{ij}_{\, \, k}$ in particular is also already given explicitly). With the help of Reduce — a computer software for mathematical-formula processing — the author also computed the structure constants of $\text{Fun}_q(G_2)$ with the convention described in Sec. 2 with $a_{11} = 2$, $a_{12} = -1$, $a_{21} = -3$, $a_{22} = 2$, $q_1 = q^3$, $q_2 = q$, and obtained the following decomposition:

$$
\hat{R} := PR = q^{12}P_{12} - q^{-6}P_{14} + q^2 P_{27},
$$

where $P = (P^{ij}_{kl}) := (\delta^i_l \delta^j_k)$ stands for the permutation matrix and $P_n$ stands for the projection matrix that extracts the irreducible component isomorphic to $\mathbb{r}^n$ from the tensor product representation $\mathbb{r}^7 \otimes \mathbb{r}^7$. That the matrix $\hat{R}$ has a decomposition into the projections is an immediate consequence of (1) and (66). The author determined the eigenvalues by virtue of (18)–(25) with using explicit form of $g^{ij}$, $f^{ijk}$ and $P_n$; much more systematic way to derive the decomposition had been given by Reshetikhin (see Ref. [16] also).

Before giving the explicit form of the structure constants, we think it in order to clarifying the basis of $\mathbb{r}$ adopted in the calculation; we use the basis that gives the following representation matrices:

$$
\begin{align*}
(\rho^i_j(K_1)) &= \text{diag}(1, q^3, q^{-3}, 1, q^3, q^{-3}, 1), \\
(\rho^i_j(K_2)) &= \text{diag}(q, q^{-1}, q^2, 1, q^{-2}, q, q^{-1}), \\
(\rho^i_j(X^+_1)) &= (0) \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus (0) \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus (0), \\
(\rho^i_j(X^+_2)) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & [2] \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus (0), \\
(\rho^i_j(X^-_1)) &= (0) \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus (0) \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus (0), \\
(\rho^i_j(X^-_2)) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus (0),
\end{align*}
$$

where $[2] := q + q^{-1}$ and $i$ is a row index whereas $j$ is a column one. In this setting, all the nonzero components among the structure constants of $\text{Fun}_q(G_2)$ are as follows:
There are 112 nonzero entries of the $R$-matrix

\[
\begin{align*}
R_{11}^{11} &= R_{22}^{22} = R_{33}^{33} = R_{55}^{55} = R_{66}^{66} = R_{77}^{77} = q^2, \\
R_{12}^{12} &= R_{33}^{13} = R_{25}^{21} = R_{52}^{25} = R_{31}^{31} = R_{56}^{36} = R_{52}^{52} = R_{57}^{57} \\
&= R_{63}^{63} = R_{67}^{67} = R_{75}^{75} = R_{76}^{76} = q, \\
R_{14}^{14} &= R_{24}^{44} = R_{34}^{41} = R_{42}^{42} = R_{43}^{43} = R_{44}^{44} = R_{45}^{45} \\
&= R_{46}^{46} = R_{47}^{47} = R_{54}^{54} = R_{64}^{64} = R_{74}^{74} = 1, \\
R_{15}^{15} &= R_{16}^{16} = R_{23}^{23} = R_{27}^{27} = R_{32}^{32} = R_{37}^{37} = R_{51}^{51} = R_{56}^{56} \\
&= R_{61}^{61} = R_{65}^{65} = R_{72}^{72} = R_{73}^{73} = q^{-1}, \\
R_{17}^{17} &= R_{26}^{26} = R_{35}^{35} = R_{53}^{53} = R_{62}^{62} = R_{71}^{71} = q^{-2}, \\
R_{21}^{12} &= R_{31}^{13} = R_{52}^{25} = R_{63}^{63} = R_{75}^{75} = R_{76}^{76} = q^2 - 1, \\
R_{23}^{14} &= R_{51}^{54} = R_{56}^{56} = R_{61}^{64} = R_{72}^{74} = R_{73}^{74} = q - q^{-3}, \\
R_{24}^{15} &= R_{34}^{34} = R_{41}^{11} = R_{45}^{45} = R_{47}^{47} = R_{67}^{67} = q - q^{-1}, \\
R_{26}^{17} &= R_{71}^{72} = q^{-1} - q^{-3}, \\
R_{32}^{14} &= R_{65}^{65} = q^{-2} + q^{-6}, \\
R_{32}^{23} &= R_{65}^{66} = q^2 - q^{-4}, \\
R_{35}^{17} &= R_{71}^{73} = q^{-4} + q^{-6}, \\
R_{35}^{26} &= R_{62}^{53} = q - q^{-5}, \\
R_{41}^{14} &= R_{74}^{47} = q^2 - 1 + q^{-4} - q^{-6}, \\
R_{41}^{23} &= R_{44}^{26} = R_{74}^{56} = -q^{-2} + q^{-4}, \\
R_{42}^{15} &= R_{43}^{16} = R_{54}^{27} = R_{64}^{63} = q^{-1} + q^{-3}, \\
R_{42}^{24} &= R_{43}^{34} = R_{54}^{45} = R_{64}^{66} = q^2 - q^{-2}, \\
R_{44}^{17} &= q - 2q^{-1} + 2q^{-3} - q^{-5}, \\
R_{51}^{15} &= R_{61}^{16} = R_{72}^{27} = R_{73}^{37} = q^2 - 1 + q^{-2} - q^{-6}, \\
R_{51}^{24} &= R_{61}^{34} = R_{72}^{45} = R_{73}^{46} = -q^{-1} + q^{-5}, \\
R_{53}^{17} &= R_{71}^{35} = -1 + q^{-4} - q^{-6} + q^{-8}, \\
R_{53}^{26} &= R_{62}^{55} = q^{-5} - q^{-7}, \\
R_{53}^{35} &= q^2 - 1 - q^{-2} + q^{-4}, \\
R_{53}^{44} &= q + q^{-1} - q^{-3} - q^{-5}, \\
R_{62}^{17} &= R_{71}^{26} = q^{-3} - q^{-7} + q^{-9} - q^{-11}, \\
R_{62}^{26} &= q^2 - 1 - q^{-8} + q^{-10}, \\
R_{62}^{44} &= -q^2 - q^{-4} + q^{-6} + q^{-8}, \\
R_{71}^{17} &= q^2 - 1 + q^{-2} - q^{-4} - q^{-6} + q^{-8} - q^{-10} + q^{-12}, \\
R_{71}^{44} &= q - q^{-3} + q^{-5} - q^{-9}.
\end{align*}
\]

13
There are $2 \times (7 + 31)$ nonzero structure constants of $\text{Fun}_{q}(G_2)$ other than the entries of the $R$-matrix

\[
g^{17} = q^5, \quad g^{26} = -q^4, \quad g^{35} = q, \quad g^{44} = -1/(1 + q^{-2}), \quad g^{53} = q^{-1}, \quad g^{62} = -q^{-4}, \quad g^{71} = -q^{-5},
\]

\[
f^{14}_1 = f^{47}_7 = -q^3,
f^{15}_2 = f^{16}_3 = f^{26}_4 = f^{27}_5 = f^{37}_6 = -q^3 - q,
f^{17}_4 = -q^3 - 1, \quad f^{23}_1 = f^{36}_7 = q^2 + 1,
f^{24}_2 = f^{34}_3 = f^{45}_5 = f^{46}_6 = q,
f^{32}_1 = f^{65}_7 = -q^{-1} - q^{-3},
\]

\[
f^{35}_4 = 1 + q^{-2}, \quad f^{41}_1 = f^{74}_7 = q^{-3},
f^{42}_2 = f^{43}_3 = f^{54}_5 = f^{64}_6 = -q^{-1},
f^{44}_4 = q - q^{-1},
f^{51}_2 = f^{61}_3 = f^{71}_4 = f^{72}_5 = f^{73}_6 = q^{-2} + q^{-4},
f^{53}_4 = -1 - q^{-2}, \quad f^{62}_4 = q^{-3} + q^{-5},
\]

\[
\tilde{g}_{17} = q^{-5}, \quad \tilde{g}_{26} = -q^{-4}, \quad \tilde{g}_{35} = q^{-1}, \quad \tilde{g}_{44} = -1 - q^{-2},
\]

\[
\tilde{g}_{53} = q, \quad \tilde{g}_{62} = -q^{-4}, \quad \tilde{g}_{71} = q^5,
\]

which satisfy $\tilde{f}^{ik}_l \ f^{lj}_i = -(1 + q^{-2})(q^4 + q^{-4})(q^2 + 1 + q^{-2}) \delta_{ij}$ (cf. (23)).

The explicit form of the $R$-matrix exhibits

\[
7 (i - 1) + j < 7 (k - 1) + l \quad \Rightarrow \quad R^{ij}_{kl} = 0,
\]

\[
7 (j - 1) + i > 7 (l - 1) + k \quad \Rightarrow \quad R^{ij}_{kl} = 0.
\]

These with (23) state that $L^{(-)} := (l^{(-)i}_j)$ is lower triangular whereas $L^{(+)} := (l^{(+)i}_j)$ is upper triangular in this setting.
acknowledgements

The author is indebted to Prof. R. Sasaki for important information and helpful advices. The author is grateful to Dr. Y. Yasui and Dr. M. Takama for giving him a good opportunity to study quantum groups and for helpful conversations on this work. The author is also grateful to M. Miyajima for his help in computing the constants.

references

1 V. G. Drinfel’d, *Quantum Groups*, in Proc. Internat. Congr. Math., 798–820 (1986).
2 M. Jimbo, Lett. Math. Phys. 10, 63 (1985).
3 M. Jimbo, *Quantum groups and the Yang-Baxter equations* (in Japanese, Springer-Verlag, Tokyo, 1990).
4 L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, in Algebraic Analysis, 129–139 (Academic, Boston, 1989).
5 N. Yu. Reshetikhin, L. A. Takhtajan and L. D. Faddeev, Leningrad Math. J. 1, 193 (1990).
6 S. L. Woronowicz, Commun. Math. Phys. 111, 613 (1987); S. L. Woronowicz, Commun. Math. Phys. 122, 125 (1989).
7 B. Jurčo, Lett. Math. Phys. 22, 177 (1991).
8 S. Okubo, *Introduction to Octonion and Other Nonassociative Algebras in Physics*, UR-1164 (lecture note, Univ. of Rochester, 1990).
9 N. Burroughs, Commun. Math. Phys. 133, 91 (1990).
10 M. Rosso, Commun. Math. Phys. 117, 581 (1988).
11 N. Yu. Reshetikhin, *Quantized Universal Enveloping Algebras, the Yang-Baxter Equation and Invariants of Links*. I, LOMI E-4-87 (preprint, Leningrad, 1988).
12 I. Yokota, *Classical Simple Lie Groups* (in Japanese, Gendai-sugaku-shya, Kyoto, 1990).
13 M. Ise, *Lie Groups* I (in Japanese, Iwanami, Tokyo, 1977).
14 S. Okubo, Alg. Group. Geom. 3, 60 (1986).
15 I. Satake, *Lecture on Lie Algebras* (in Japanese, original title: *Lie-kan no Hanashi*, Nippon-hyoron-shya, Tokyo, 1987).
16 M. D. Gould, Lett. Math. Phys. 24, 183 (1992).