Nonlocal equations with gradient constraints

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Abstract
We prove the existence and $C^{1,\alpha}$ regularity of solutions to nonlocal fully nonlinear elliptic equations with gradient constraints. We do not assume any regularity about the constraints; so the constraints need not be $C^1$ or strictly convex. We also obtain $C^{0,1}$ boundary regularity for these problems. Our approach is to show that these nonlocal equations with gradient constraints are related to some nonlocal double obstacle problems. Then we prove the regularity of the double obstacle problems. In this process, we also employ the monotonicity property for the second derivative of obstacles, which we have obtained in a previous work.

Mathematics Subject Classification 35R35 · 47G20 · 35B65

1 Introduction
In this paper we consider the existence and regularity of solutions to the equation with gradient constraint
\[
\begin{cases}
\max\{-Iu, H(Du)\} = 0 & \text{in } U, \\
u = \varphi & \text{in } \mathbb{R}^n - U.
\end{cases}
\] (1.1)

Here $I$ is a nonlocal elliptic operator, of which a prototypical example is the fractional Laplacian
\[-(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy.
\]

Nonlocal operators appear naturally in the study of discontinuous stochastic processes as the jump part of their infinitesimal generator. These operators have also been studied extensively in recent years from the analytic viewpoint of integro-differential equations. The foundational works of Caffarelli and Silvestre [7–9] paved the way and set the framework for such studies. They provided an appropriate notion of ellipticity for nonlinear nonlocal equations, and obtained their $C^{1,\alpha}$ regularity. They also obtained Evans-Krylov-type $C^{2s+\alpha}$ regularity for

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convex equations. An interesting property of their estimates is their uniformity as $s \uparrow 1$, which provides a new proof for the corresponding classical estimates for local equations.

Free boundary problems involving nonlocal operators have also seen many advancements. Silvestre [47] obtained $C^{1,\alpha}$ regularity of the obstacle problem for fractional Laplacian. Caffarelli et al. [10] proved the optimal $C^{1,s}$ regularity for this problem when the obstacle is smooth enough. Bjorland, Caffarelli, and Figalli [5] studied a double obstacle problem for the infinity fractional Laplacian which appears in the study of a nonlocal version of the tug-of-war game. Korvenpää et al. [24] studied the obstacle problem for operators of fractional $p$-Laplacian type. Petrosyan and Pop [30] considered the obstacle problem for the fractional Laplacian with drift in the subcritical regime $s \in (\frac{1}{2}, 1)$, and Fernández-Real and Ros-Oton [17] studied the critical case $s = \frac{1}{2}$. There has also been some works on other types of nonlocal free boundary problems, like the work of Rodrigues and Santos [32] on nonlocal linear variational inequalities with constraint on the fractional gradient.

A major breakthrough in the study of nonlocal free boundary problems came with the work of Caffarelli et al. [11], in which they obtained the regularity of the solution and of the free boundary of the obstacle problem for a large class of nonlocal elliptic operators. These problems appear naturally when considering optimal stopping problems for Lévy processes with jumps, which arise for example as option pricing models in mathematical finance. We should mention that in their work, the boundary regularity results of Ros-Oton and Serra [33, 34] for nonlocal elliptic equations were also essential. In [41] we proved existence and $C^{1,\alpha}$ regularity of solutions to double obstacle problems for a wide class of nonlocal fully nonlinear operators. We also obtained their boundary regularity. In contrast to [11], we allowed less smooth obstacles, and did not require them to be $C^1$.

In this paper we prove $C^{1,\alpha}$ regularity of the equation with gradient constraint (1.1) for a large class of nonlocal fully nonlinear operators $I$. We do not require the operator to be convex. We also do not require the constraint to be strictly convex or differentiable. Furthermore, we obtain $C^{0,1}$ boundary regularity for these problems, which is more than the expected boundary regularity for nonlocal Dirichlet problems, due to the presence of the constraint. Our estimates in this work are uniform as $s \uparrow 1$; hence they provide a new proof for the corresponding regularity results for local equations with gradient constraints. Nonlocal equations with gradient constraints appear for example in portfolio optimization with transaction costs when prices are governed by Lévy processes with jumps; see [3, 4] and [29, Chapter 8], for more details.

Let us now mention some of the works on local equations with gradient constraints. The study of elliptic equations with gradient constraints was initiated by Evans [16] when he considered the problem

$$\max\{Lu - f, \ |Du| - g\} = 0,$$

where $L$ is a (local) linear elliptic operator of the form

$$Lu = -a_{ij} D_{ij}^2 u + b_i D_i u + cu.$$

Equations of this type stem from dynamic programming in a wide class of singular stochastic control problems. Evans proved $W^{2,p}_{\text{loc}}$ regularity for $u$. He also obtained the optimal $W^{2,\infty}_{\text{loc}}$ regularity under the additional assumption that $a_{ij}$ are constant. Wiegner [50] removed this additional assumption and obtained $W^{2,\infty}_{\text{loc}}$ regularity in general. Later, Ishii and Koike [23] allowed the gradient constraint to be more general, and proved global $W^{2,\infty}$ regularity. We also mention that Shreve and Soner [45, 46] considered similar problems with special structure, and proved the existence of classical solutions.
Yamada [51] allowed the differential operator to be more general, and proved the existence of a solution in $W^{2,\infty}_{\text{loc}}$ to the problem
\[
\max_{1 \leq k \leq N} \{L_k u - f_k, \ |Du| - g\} = 0,
\]
where each $L_k$ is a (local) linear elliptic operator. Recently, there has been new interest in these types of problems. Hynd [20] considered problems with more general gradient constraints of the form
\[
\max \{Lu - f, \ \tilde{H}(Du)\} = 0,
\]
where $\tilde{H}$ is a convex function. He proved $W^{2,\infty}_{\text{loc}}$ regularity when $\tilde{H}$ is strictly convex. Finally, Hynd and Mawi [22] studied (local) fully nonlinear elliptic equations with strictly convex gradient constraints of the form
\[
\max \{F(x, D^2 u) - f, \ \tilde{H}(Du)\} = 0.
\]
Here $F(x, D^2 u)$ is a fully nonlinear elliptic operator. They obtained $W^{2,p}_{\text{loc}}$ regularity in general, and $W^{2,\infty}_{\text{loc}}$ regularity when $F$ does not depend on $x$. Let us also mention that Hynd [19, 21] considered eigenvalue problems for (local) equations with gradient constraints too.

Closely related to these problems are variational problems with gradient constraints. An important example among them is the well-known elastic–plastic torsion problem, which is the problem of minimizing the functional $\int_U \frac{1}{2} |Dv|^2 - v \, dx$ over the set $W_{B_1} := \{v \in W^{1,2}_0(U) : |Dv| \leq 1 \text{ a.e.}\}$. The $W^{2,p}_{\text{loc}}$ regularity for this problem was proved by Brezis and Stampacchia [6], and its optimal $W^{2,\infty}_{\text{loc}}$ regularity was obtained by Caffarelli and Riviére [13]. An interesting property of variational problems with gradient constraints is that under mild conditions they are equivalent to double obstacle problems. For example the minimizer of $\int_U G(Dv) \, dx$ over $W_{B_1}$ also satisfies $-d \leq v \leq d$ and
\[
\begin{align*}
-D_i(D_i G(Dv)) &= 0 \quad \text{in } \{-d < v < d\}, \\
-D_i(D_i G(Dv)) &\leq 0 \quad \text{a.e. on } \{v = d\}, \\
-D_i(D_i G(Dv)) &\geq 0 \quad \text{a.e. on } \{v = -d\},
\end{align*}
\]
where $d$ is the distance to $\partial U$; see for example [37, 40]. This problem can be more compactly written as
\[
\max \{\min \{F(x, D^2 v), v + d\}, v - d\} = 0,
\]
where $F(x, D^2 v) = -D_i(D_i G(Dv)) = -D^2_{ij} G(x) D_{ij}^2 v$.

Variational problems with gradient constraints have also seen new developments in recent years. De Silva and Savin [15] investigated the minimizers of some functionals subject to gradient constraints, arising in the study of random surfaces. In their work, the functional is allowed to have certain kinds of singularities. Also, the constraints are given by convex polygons; so they are not strictly convex. They showed that in two dimensions, the minimizer is $C^1$ away from the obstacles. In [35–37, 40] we have studied the regularity and the free boundary of several classes of variational problems with gradient constraints. Our goal was to understand the behavior of these problems when the constraint is not strictly convex; and we have been able to obtain their optimal $C^{1,1}$ regularity in arbitrary dimensions. This has been partly motivated by the above-mentioned problem about random surfaces.
There has also been similar interests in elliptic equations with gradient constraints which are not strictly convex. These problems emerge in the study of some singular stochastic control problems appearing in financial models with transaction costs; see for example [2, 31]. In [39] we extended the results of [22] and proved the optimal $C^{1,1}$ regularity for (local) fully nonlinear elliptic equations with non-strictly convex gradient constraints. Our approach was to obtain a link between double obstacle problems and elliptic equations with gradient constraints. This link has been well known in the case where the double obstacle problem reduces to an obstacle problem (for example in the elastic–plastic torsion problem). However, we have shown that there is still a connection between the two problems in the general case. In this approach, we also studied (local) fully nonlinear double obstacle problems with singular obstacles.

In this paper we follow the same strategy to prove $C^{1,\alpha}$ regularity for nonlocal fully nonlinear elliptic equations with non-strictly convex gradient constraints. As we have mentioned before, these problems appear in portfolio optimization with transaction costs when prices are governed by Lévy processes with jumps (see [3, 4, 29]). Although the idea of the proof of regularity is similar to the local case given in [39], many technical details had to be adjusted and revamped for nonlocal equations. Notably, the barriers introduced in the proof of Theorem 3 allow us to accommodate for the lack of $C^{1,1}$ regularity. Also, instead of pointwise uniform bounds in the local case, the barriers allow us to obtain the required uniform bounds on neighborhoods (which can intersect the singular sets of the obstacles); see the remark after Proposition 2 in Appendix A.

Now let us introduce the problem in more detail, and recall some of the definitions and conventions about nonlocal operators introduced in [7]. Let

$$\delta u(x, y) := u(x + y) + u(x - y) - 2u(x).$$

A linear nonlocal operator is an operator of the form

$$Lu(x) = \int_{\mathbb{R}^n} \delta u(x, y)a(y) dy,$$

where the kernel $a$ is a positive function which satisfies $a(-y) = a(y)$, and

$$\int_{\mathbb{R}^n} \min\{1, |y|^2\}a(y) dy < \infty.$$

The potential theoretic techniques, which are for example employed in [47] to study the obstacle problem for fractional Laplacian $-(-\Delta)^s$, and are suitable for studying linear nonlocal operators, are inadequate when applied to nonlinear operators. Such nonlinear nonlocal operators can for example arise in optimal control problems and be of the form

$$Iu(x) = \sup_\alpha L_\alpha u(x),$$

where $L_\alpha$ is a family of linear nonlocal operators. More complicated nonlinear operators of the form

$$Iu(x) = \inf_\beta \sup_\alpha L_{\alpha\beta} u(x)$$

can also appear in competitive games. The theory set forth in [7–9] is powerful enough to accommodate these and more complicated nonlinear operators, along with their linear counterparts.

Let us now present the general definition of nonlocal operators and their ellipticity. We say a function $u$ belongs to $C^{1,1}(x)$ if there are quadratic polynomials $P, Q$ such that $P(x) = \ldots$
u(x) = Q(x), and \( P \leq u \leq Q \) on a neighborhood of \( x \). A nonlocal operator \( I \) is an operator for which \( Iu(x) \) is well-defined for bounded functions \( u \in C^{1,1}(x) \), and \( I(\cdot) \) is a continuous function on an open set if \( u \) is \( C^2 \) over that open set. The operator \( I \) is uniformly elliptic with respect to a family of linear operators \( \mathcal{L} \) if for any bounded functions \( u, v \in C^{1,1}(x) \) we have

\[
M^-\mathcal{L}(u - v)(x) \leq I(u) - I(v) \leq M^+\mathcal{L}(u - v)(x),
\]

where the extremal Pucci-type operators \( M^\pm\mathcal{L} \) are defined as

\[
M^\pm\mathcal{L}u(x) = \inf_{L \in \mathcal{L}} Lu(x), \quad M^\pm\mathcal{L}u(x) = \sup_{L \in \mathcal{L}} Lu(x).
\]

Let us also note that \( \pm M^\pm\mathcal{L} \) are subadditive and positively homogeneous.

An important family of linear operators is the class \( \mathcal{L}_0 \) of linear operators whose kernels are comparable with the kernel of fractional Laplacian \(-\Delta)\), i.e.

\[
(1 - s) \frac{\lambda}{|y|^{n+2s}} \leq a(y) \leq (1 - s) \frac{\Lambda}{|y|^{n+2s}},
\]

where \( 0 < s < 1 \) and \( 0 < \lambda \leq \Lambda \). It can be shown that in this case the extremal operators \( M^\pm_{\mathcal{L}_0} \), which we will simply denote by \( M^\pm \), are given by

\[
M^+u = (1 - s) \int_{\mathbb{R}^n} \frac{\Lambda \delta u(x, y) - \lambda \delta u(x, y)}{|y|^{n+2s}} dy,
\]

\[
M^-u = (1 - s) \int_{\mathbb{R}^n} \frac{\lambda \delta u(x, y) - \Lambda \delta u(x, y)}{|y|^{n+2s}} dy,
\]

where \( r^\pm = \max\{\pm r, 0\} \) for a real number \( r \).

We will also only consider “constant coefficient” nonlocal operators, i.e. we assume that \( I \) is translation invariant:

\[
I(\tau_z u) = \tau_z(Iu)
\]

for every \( z \), where \( \tau_z u(x) := u(x - z) \) is the translation operator. In addition, without loss of generality we can assume that \( I(0) = 0 \), i.e. the action of \( I \) on the constant function 0 is 0. Because by translation invariance \( I(0) \) is constant, and we can consider \( I - I(0) \) instead of \( I \).

Next let us recall some concepts from convex analysis. Let \( K \) be a compact convex subset of \( \mathbb{R}^n \) whose interior contains the origin.

**Definition 1** The gauge function of \( K \) is the function

\[
\gamma_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}.
\]

And the polar of \( K \) is the set

\[
K^\circ := \{x : \langle x, y \rangle \leq 1 \text{ for all } y \in K\},
\]

where \( \langle , \rangle \) is the standard inner product on \( \mathbb{R}^n \).

The gauge function \( \gamma_K \) is convex, subadditive, and positively homogeneous; so it looks like a norm on \( \mathbb{R}^n \), except that \( \gamma_K(-x) \) is not necessarily the same as \( \gamma_K(x) \). The polar set \( K^\circ \) is also a compact convex set containing the origin as an interior point. (For more details, and the proofs of these facts, see [44].)
We assume that the exterior data $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a bounded Lipschitz function which satisfies
\[
-\gamma_K(y - x) \leq \varphi(x) - \varphi(y) \leq \gamma_K(x - y),
\]
for all $x, y \in \mathbb{R}^n$. Then by Lemma 2.1 of [49] this property implies that $D\varphi \in K^\circ$ a.e.

Let $U \subset \mathbb{R}^n$ be a bounded open set. Then for $x \in \overline{U}$ the obstacles are defined as
\[
\rho(x) = \rho_{K,\varphi}(x; U) := \min_{y \in \partial U} [\gamma_K(x - y) + \varphi(y)],
\]
\[
\bar{\rho}(x) = \bar{\rho}_{K,\varphi}(x; U) := \min_{y \in \partial U} [\gamma_K(y - x) - \varphi(y)].
\]

It is well known (see [26, Section 5.3]) that $\rho$ is the unique viscosity solution of the Hamilton–Jacobi equation
\[
\begin{cases}
\gamma_{K^\circ}(Dv) = 1 & \text{in } U, \\
v = \varphi & \text{on } \partial U.
\end{cases}
\]

Now, note that $-K$ is also a compact convex set whose interior contains the origin. We also have $\bar{\rho}_{K,\varphi} = \rho_{-K,-\varphi}$, since $\gamma_{-K}(\cdot) = \gamma_K(-\cdot)$. Thus we have a similar characterization for $\bar{\rho}$ too.

**Notation** To simplify the notation, we will use the following conventions
\[
\gamma := \gamma_K, \quad \gamma^\circ := \gamma_{K^\circ}, \quad \check{\gamma} := \gamma_{-K}.
\]
Thus in particular we have $\check{\gamma}(x) = \gamma(-x)$.

In [40] we have shown that $-\bar{\rho} \leq \rho$, and
\[
-\gamma(x - y) \leq \rho(y) - \rho(x) \leq \gamma(y - x).
\]
The above inequality also holds if we replace $\rho, \gamma$ with $\bar{\rho}, \check{\gamma}$. Thus in particular, $\rho, \check{\rho}$ are Lipschitz continuous. We have also shown that $-\bar{\rho} = \varphi = \rho$ on $\partial U$. We extend $\rho, -\bar{\rho}$ to $\mathbb{R}^n$ by setting them equal to $\varphi$ on $\mathbb{R}^n - U$. In other words we set
\[
\rho(x) = \begin{cases}
\min_{y \in \partial U} [\gamma(x - y) + \varphi(y)] & x \in U, \\
\varphi(x) & x \in \mathbb{R}^n - U,
\end{cases}
\]
and similarly for $-\bar{\rho}$. Note that these extensions are continuous functions, and using (1.7) we can easily show that $\rho, -\bar{\rho}$ satisfy (1.10) for all $x, y \in \mathbb{R}^n$.

Motivated by the double obstacle problems arising from variational problems, in [39] we have studied (local) fully nonlinear double obstacle problems of the form
\[
\begin{cases}
\max\{\min[F(D^2u), u + \bar{\rho}], u - \rho\} = 0 & \text{in } U, \\
u = \varphi & \text{on } \partial U,
\end{cases}
\]
and employed them to obtain the regularity of (local) elliptic equations with gradient constraints. And motivated by this result, in [41] we proved the existence and $C^{1,\alpha}$ regularity of solutions to nonlocal double obstacle problems of the form (1.13), in which the obstacles are assumed to be semi-concave/convex functions (see Appendix B for a sketch of proof of this result). As we will see, this regularity result will play a critical role in obtaining the regularity of nonlocal equations with gradient constraints.
Now let us state our main results. We denote the Euclidean distance to $\partial U$ by $d(\cdot, \partial U) := \min_{y \in \partial U} |\cdot - y|$. First let us collect all the assumptions we made so far to facilitate their referencing.

**Assumption 1** We assume that

(a) $I$ is a translation invariant operator which is uniformly elliptic with respect to $L_0$ with ellipticity constants $\lambda, \Lambda$, and $\frac{1}{2} < s_0 < s < 1$. We also assume that $I(0) = 0$.

(b) $U \subset \mathbb{R}^n$ is a bounded open set, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is bounded and satisfies (1.7). Also, $\varphi$ is convex on a neighborhood of $\partial U$.

(c) $K \subset \mathbb{R}^n$ is a compact convex set whose interior contains the origin, and the obstacles $\rho, -\bar{\rho}$ are defined by (1.8) and extended as in (1.11).

(d) The gradient constraint is $H := \gamma \circ -1$.

**Remark** Note that we are not assuming any regularity about $\partial K$ or $\partial K^\circ$. In particular, $\gamma^\circ$, which defines the gradient constraint, need not be $C^1$ or strictly convex. Also, the obstacles can be highly irregular. Furthermore, note that any convex gradient constraint of the general form $\tilde{H}(Du) \leq 0$ for which the set $\{\tilde{H}(\cdot) \leq 0\}$ is bounded, and contains a neighborhood of the origin (which is a natural requirement in these problems), can be written in the form $\gamma^\circ - 1$ with respect to the convex set $K = [\tilde{H}(\cdot) \leq 0]^\circ$. (Note that $[\tilde{H}(\cdot) \leq 0] = K^\circ$, because the double polar of such convex sets are themselves, as shown in Section 1.6 of [44].)

**Theorem 1** Suppose Assumption 1 holds. Also suppose $\partial U$ is $C^2$, and $\varphi$ is $C^2$ with $\gamma^\circ(D\varphi) < 1$. In addition, suppose there is a bounded continuous function $-\bar{\rho} \leq v \leq \rho$ that satisfies $-Iv \leq 0$ in the viscosity sense in $U$. Then the nonlocal elliptic equation with gradient constraint (1.1) has a viscosity solution $u$, and

$$u \in C^{1,\alpha}_{\text{loc}}(U) \cap C^{0,1}(\overline{U})$$

for some $\alpha > 0$ depending only on $n, \lambda, \Lambda, s_0$. And for an open subset $V \subset \subset U$ we have

$$\|u\|_{C^{1,\alpha}(V)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0),$$

(1.12)

where $C$ depends only on $n, \lambda, \Lambda, s_0$, and $d(V, \partial U)$; and $C_0$ depends only on these constants together with $K, \partial U, \varphi$.

**Remark** Note that if the equation with gradient constraint (1.1) has a solution then we must have a subsolution $(-I \cdot \leq 0)$ in $U$ with $\gamma^\circ(D\cdot) \leq 1$, which easily implies that $-\bar{\rho} \leq \cdot \leq \rho$. Thus the existence of $v$ is a natural requirement.

**Remark** Note that our estimate is uniform for $s > s_0$; so we can retrieve the interior estimate for local equations with gradient constraints as $s \to 1$. In addition, note that unlike the results on local fully nonlinear equations with gradient constraints in [22, 39], we do not have any convexity assumption about $I$.

We split the proof of Theorem 1 into two parts. In Theorem 3 we show that there is $u \in C^{1,\alpha}_{\text{loc}}(U)$ that satisfies the nonlocal double obstacle problem (1.13). (Note that this result is stronger than the regularity result in [41], since here we do not have the assumption of semi-concavity/convexity of the obstacles; an assumption which fails to hold when $\partial K$ is not smooth enough.) And in Theorem 2 we prove that $u$ must also satisfy the nonlocal equation with gradient constraint (1.1).
Theorem 2 Suppose Assumption 1 holds. Also suppose there is a bounded continuous function $-\bar{\rho} \leq v \leq \rho$ that satisfies $-Iv \leq 0$ in the viscosity sense in $U$. Let $u \in C^1(U)$ be a viscosity solution of the double obstacle problem

$$\begin{cases}
\max\{\min\{-Iu, u + \bar{\rho}\}, u - \rho\} = 0 & \text{in } U, \\
u = \varphi & \text{in } \mathbb{R}^n - U.
\end{cases}$$

(1.13)

Then $u$ is also a viscosity solution of the equation with gradient constraint (1.1).

Remark It should be noted that not all problems with gradient constraint are equivalent to double obstacle problems, especially when the equation explicitly depends on $x$. This is also the case for local equations; see [42] for a simple counterexample. This is mainly because we need the full power of the maximum principle for $Du$, and mere estimates of $|Du|$ are not sufficient for our purposes. So we essentially need to differentiate the equation, and an explicit $x$-dependence can be problematic; see the proof of Lemma 1.

Theorem 3 Suppose Assumption 1 holds. Also suppose $\partial U$ is $C^2$, and $\varphi$ is $C^2$ with $\gamma^0(D\varphi) < 1$. Then the double obstacle problem (1.13) has a viscosity solution $u$, and $u \in C^{1,\alpha}_{\text{loc}}(U) \cap C^{0,1}(\overline{U})$ for some $\alpha > 0$ depending only on $n, \lambda, \Lambda, s_0$. And for an open subset $V \subset \subset U$ the estimate (1.12) holds for $\|u\|_{C^{1,\alpha}(V)}$.

The paper is organized as follows. In Sect. 2 we prove Theorem 2. Here we use Lemma 1 in which we show that a solution to the double obstacle problem (1.13) must also satisfy the gradient constraint; and Lemma 2 in which we prove that a solution to the double obstacle problem (1.13) must be larger than a solution to the elliptic equation with gradient constraint (1.1). Then we review some well-known facts about the regularity of $K$, and its relation to the regularity of $K^\circ, \gamma, \gamma^0$. After that we consider the function $\rho$ more carefully. We will review the formulas for the derivatives of $\rho$ that we have obtained in [40], especially the novel explicit formula (2.20) for $D^2\rho$. To the best of author's knowledge, formulas of this kind have not appeared in the literature before, except for the simple case where $\rho$ is the Euclidean distance to the boundary. (Although, some special two dimensional cases also appeared in our earlier works [35, 38].)

One of the main applications of the formula (2.20) for $D^2\rho$ is in the relation (2.21) for characterizing the set of singularities of $\rho$. Another important application is in Lemma 4, which implies that $D^2\rho$ attains its maximum on $\partial U$. This interesting property is actually a consequence of a more general property of the solutions to Hamilton–Jacobi equations (remember that $\rho$ is the viscosity solution of the Hamilton–Jacobi equation (1.9)). This little-known monotonicity property is investigated in [40]; but we included a brief account at the end of Sect. 2 for reader's convenience.

In Sect. 3 we prove the regularity result for double obstacle problem (1.13), aka Theorem 3, when the exterior data $\varphi$ is zero. We separated this case to reduce the technicalities so that the main ideas can be followed more easily. The proof of the general case of nonzero exterior data is postponed to Appendix A. The idea of the proof of Theorem 3 is to approximate $K^\circ$ with smoother convex sets. Then we have to find uniform bounds for the various norms of the approximations $u_k$ to $u$. In order to do this, we construct appropriate barriers, which resemble the obstacles $\rho, -\bar{\rho}$. Then, among other estimations, we will use the fact that the second derivative of the barriers attain their maximums on the boundary. A more detailed sketch of proof for Theorem 3 is given at the beginning of its proof (before its Part I).
Finally, for reader’s convenience, in Appendix B we provide a sketch of proof of a regularity result for nonlocal double obstacle problems from [41], which we relied on in the proof of Theorem 3.

## 2 Assumptions and preliminaries

Let us start by defining the notion of viscosity solutions of nonlocal equations. We want to study the nonlocal double obstacle problems and nonlocal equations with gradient constraints. So we state the definition of viscosity solution in the more general case of a nonlocal operator \( I(x, u(x), Du(x), u(\cdot)) \) whose value also depends on the pointwise values of \( x, u(x), Du(x) \) (see [1] for more details). For example, in the case of the double obstacle problem and the equation with gradient constraint we respectively have

\[
-\tilde{I}(x, r, u(\cdot)) = \max\{\min\{-Iu(x), r - \psi^-(x)\}, r - \psi^+(x)\},
\]

\[
-\tilde{I}(x, p, u(\cdot)) = \max\{-Iu(x), H(p)\},
\]

where we replaced \( u(x) \) with \( r \) and \( Du(x) \) with \( p \) to clarify the dependence of \( \tilde{I} \) on its arguments.

**Definition 2** An upper semi-continuous function \( u : \mathbb{R}^n \to \mathbb{R} \) is a viscosity subsolution in \( U \) if whenever \( \phi \) is a bounded \( C^2 \) function and \( u - \phi \) has a maximum over \( \mathbb{R}^n \) at \( x_0 \in U \) we have

\[
-\tilde{I}(x_0, u(x_0), Du(x_0), \phi(\cdot)) \leq 0.
\]

And a lower semi-continuous function \( u \) is a viscosity supersolution in \( U \) if whenever \( \phi \) is a bounded \( C^2 \) function and \( u - \phi \) has a minimum over \( \mathbb{R}^n \) at \( x_0 \in U \) we have

\[
-\tilde{I}(x_0, u(x_0), Du(x_0), \phi(\cdot)) \geq 0.
\]

A continuous function \( u \) is a viscosity solution of \( -\tilde{I} = 0 \) in \( U \) if it is a subsolution and a supersolution in \( U \).

**Remark** We can always add a constant to \( \phi \) to make \( \phi(x_0) = u(x_0) \), but there is no need to make this an assumption in the above definition.

**Lemma 1** Suppose Assumption 1 holds. Let \( u \in C^1(U) \) be a viscosity solution of the double obstacle problem (1.13). Then over \( U \) we have

\[
g^\circ(Du) \leq 1.
\]

**Remark** An immediate consequence of this lemma is that \( u \in C^{0,1}(\overline{U}) \), since \( u \) and its derivative are bounded, and \( \partial U \) is smooth enough.

**Proof** On the open set \( V := \{ u < \rho \} \) we know that \( -Iu \geq 0 \) in the viscosity sense. Also, on the open set \( V_0 := \{ -\bar{\rho} < u \} \) we know that \( -Iu \leq 0 \) in the viscosity sense. Hence due to the translation invariance of \( I \), for any \( h \) we have \( -Iu(\cdot + h) \leq 0 \) in the viscosity sense in \( V_0 - h \). Therefore by Lemma 5.8 of [7] we have

\[
M^+(u(\cdot + h) - u(\cdot)) \geq 0
\]

in the viscosity sense in \( V_1 := V \cap (V_0 - h) \). Thus by Lemma 5.10 of [7] we have

\[
\sup_{V_1}(u(\cdot + h) - u(\cdot)) \leq \sup_{\mathbb{R}^n-V_1}(u(\cdot + h) - u(\cdot)). \tag{2.1}
\]
Let us estimate \( u(+h) - u(\cdot) \) on \( \mathbb{R}^n - V_1 \). Let \( x \in \mathbb{R}^n - V_1 \). Then either \( x \notin V \), or \( x+h \notin V_0 \). Suppose \( x \notin V \). Then \( u(x) = \rho(x) \) (note that outside of \( U \) we also have \( u = \varphi = \rho \)). So
\[
u(x + h) - u(x) \leq \rho(x + h) - \rho(x) \leq \gamma(h).
\]
Next suppose \( x+h \notin V_0 \). Then \( u(x + h) = -\bar{\rho}(x + h) \). Hence we have
\[
u(x + h) - u(x) \leq -(\bar{\rho}(x + h) - \bar{\rho}(x)) = \bar{\rho}(x) - \bar{\rho}(x + h) \leq \bar{\gamma}(-h) = \gamma(h).
\]
Therefore by (2.1) for every \( x \in V_1 \), and hence for every \( x \in \mathbb{R}^n \), we have \( u(x + h) - u(x) \leq \gamma(h) \). Hence for \( t > 0 \) we get
\[
\frac{u(x + th) - u(x)}{t} \leq \frac{1}{t} \gamma(th) = \gamma(h).
\]
Thus at the points of differentiability of \( u \) we have \( \langle Du, h \rangle \leq \gamma(h) \), and consequently \( \gamma^0(Du) \leq 1 \) due to (2.5).

\[\square\]

**Lemma 2** Suppose Assumption 1 holds. Also suppose there is a bounded continuous function \( -\bar{\rho} \leq v \leq \rho \) that satisfies \( -lv \leq 0 \) in the viscosity sense in \( U \). Let \( u \) be a viscosity solution of the double obstacle problem (1.13). Then we have
\[v \leq u.
\]
As a result we get
\[
\max\{-Iu, u - \rho\} = 0 \tag{2.2}
\]
in the viscosity sense in \( U \).

**Remark** In fact, this lemma is still true if we replace \( \rho, -\bar{\rho} \) by any other upper and lower obstacles which agree on \( \mathbb{R}^n - U \). We can also replace \( I \cdot \) by \( I \cdot -f \) for some continuous function \( f \).

**Proof** On the open set \( V := \{u < \rho\} \subset U \) we have \( -Iu \geq 0 \) and \( -lv \leq 0 \) in the sense of viscosity. Also note that on \( \mathbb{R}^n - V \) we have \( u = \rho \geq v \). Hence by Theorem 5.2 of [7] (also see Lemma 6.1 of [28]) we have \( u \geq v \) on \( V \) too, as desired.

Now let us prove (2.2). Suppose \( u - \phi \) has a global maximum at \( x_0 \in U \). Then at \( x_0 \) we have
\[
\max\{\min\{-I\phi(x_0), u + \bar{\rho}\}, u - \rho\} \leq 0.
\]
We know that \( -\bar{\rho}(x_0) \leq u(x_0) \leq \rho(x_0) \). If \( -\bar{\rho}(x_0) < u(x_0) \) then we must have \( -I\phi(x_0) \leq 0 \). And if \( -\bar{\rho}(x_0) = u(x_0) \) then \( v(x_0) = -\bar{\rho}(x_0) = u(x_0) \), since \( -\bar{\rho} \leq v \leq u \). Hence for every \( x \in \mathbb{R}^n \) we have
\[
v(x_0) - \phi(x_0) = u(x_0) - \phi(x_0) \geq u(x) - \phi(x) \geq v(x) - \phi(x).
\]
So \( v - \phi \) has a global maximum at \( x_0 \), and therefore \( -I\phi(x_0) \leq 0 \). Thus in either case we have
\[
\max\{-I\phi(x_0), u - \rho\} \leq 0.
\]

Next suppose \( u - \phi \) has a global minimum at \( x_0 \in U \). Then at \( x_0 \) we have
\[
\max\{\min\{-I\phi(x_0), u + \bar{\rho}\}, u - \rho\} \geq 0.
\]
So if \( u(x_0) < \rho(x_0) \) then we must have \(-I\phi(x_0) \geq 0\), which implies that
\[
\max\{-I\phi(x_0), \ u - \rho\} \geq 0.
\]

\[\Box\]

**Proof of Theorem 2.** Suppose \( u - \phi \) has a global minimum at \( x_0 \in U \). Then Lemma 2 implies that at \( x_0 \) we have
\[
\max\{-I\phi(x_0), \ u - \rho\} \geq 0.
\]

We need to show that
\[
\max\{-I\phi(x_0), \ \gamma^\circ(D\phi(x_0)) - 1\} \geq 0.
\]

If \(-I\phi(x_0) \geq 0\) then we have the desired. Otherwise we must have \( u(x_0) = \rho(x_0) \). Let \( y_0 \in \partial U \) be a \( \rho \)-closest point to \( x_0 \), i.e., \( \rho(x_0) = \gamma(x_0 - y_0) + \varphi(y_0) \). Then by Lemma 3, \( y_0 \) is also a \( \rho \)-closest point on \( \partial U \) to \( x_0 + t(y_0 - x_0) \) for \( t \in [0, 1] \). So we have
\[
\rho(x_0 + t(y_0 - x_0)) = \gamma(x_0 + t(y_0 - x_0) - y_0) + \varphi(y_0) = (1 - t)\gamma(x_0 - y_0) + \varphi(y_0).
\]

So \( u(x_0 + t(y_0 - x_0)) \leq \rho(x_0 + t(y_0 - x_0)) = (1 - t)\gamma(x_0 - y_0) + \varphi(y_0) \). On the other hand, for small negative \( t \) we have \( \rho(x_0 + t(y_0 - x_0)) \leq (1 - t)\gamma(x_0 - y_0) + \varphi(y_0) \), since \( y_0 \) may not be a \( \rho \)-closest point on \( \partial U \) to \( x_0 + t(y_0 - x_0) \). Hence for \( t \) near 0 we have
\[
u(x_0 + t(y_0 - x_0)) \leq (1 - t)\gamma(x_0 - y_0) + \varphi(y_0).
\]
And the equality holds at \( t = 0 \). Thus by differentiating we get \( \langle Du(x_0), x_0 - y_0 \rangle = \gamma'(x_0 - y_0) \). Therefore by (2.5) we get \( \gamma^\circ(D\phi(x_0)) \geq 1 \), as desired.

Next suppose \( u - \phi \) has a global maximum at \( x_0 \in U \). We need to show that
\[
\max\{-I\phi(x_0), \ \gamma^\circ(D\phi(x_0)) - 1\} \leq 0.
\]

By Lemma 2 we know that at \( x_0 \) we have
\[
\max\{-I\phi(x_0), \ u - \rho\} \leq 0.
\]

So \(-I\phi(x_0) \leq 0\); and we only need to show that \( \gamma^\circ(D\phi(x_0)) \leq 1 \). However we know that
\[
D\phi(x_0) = Du(x_0),
\]
since \( u \) is \( C^1 \) in \( U \). Hence we get the desired by Lemma 1. \[\Box\]

**2.1 Regularity of the gauge function**

In this subsection we review some of the properties of \( \gamma, \gamma^\circ \) and \( K, K^\circ \) briefly. For detailed explanations and proofs see [44]. Recall that the gauge function \( \gamma \) satisfies
\[
\gamma(rx) = r\gamma(x),
\]
\[
\gamma(x + y) \leq \gamma(x) + \gamma(y),
\]
for all \( x, y \in \mathbb{R}^n \) and \( r \geq 0 \). Also, note that as \( B_{1/c}(0) \subseteq K \subseteq B_{1/c}(0) \) for some \( C \geq c > 0 \), we have
\[
c|x| \leq \gamma(x) \leq C|x|
\]
(2.3)
for all \( x \in \mathbb{R}^n \). In addition, since \( K \) is closed we have \( K = \{ y \leq 1 \} \), and since \( K \) has nonempty interior we have \( \partial K = \{ y = 1 \} \).

It is well known that for all \( x, y \in \mathbb{R}^n \) we have

\[
\langle x, y \rangle \leq \gamma(x)\gamma^o(y).
\]

(2.4)

In fact, more is true and we have

\[
\gamma^o(y) = \max_{x \neq 0} \frac{\langle x, y \rangle}{\gamma(x)}.
\]

(2.5)

For a proof of this see page 54 of [44]. Also, by homogeneity of \( \gamma \), its strict convexity is equivalent to

\[
\gamma(x + y) < \gamma(x) + \gamma(y)
\]

when \( x \neq cy \) and \( y \neq cx \) for any \( c \geq 0 \).

Suppose that \( \partial K \) is \( C^{k,\alpha} \) \((k \geq 1, 0 \leq \alpha \leq 1) \). Then \( \gamma \) is \( C^{k,\alpha} \) on \( \mathbb{R}^n - \{0\} \) (see for example [40]). Conversely, note that as \( \partial K = \{ y = 1 \} \) and \( D\gamma \neq 0 \) by (2.6), \( \partial K \) is as smooth as \( \gamma \). Suppose in addition that \( K \) is strictly convex. Then, by Remark 1.7.14 and Theorem 2.2.4 of [44], \( K^o \) is also strictly convex and its boundary is \( C^1 \). Therefore \( \gamma^o \) is strictly convex, and it is \( C^1 \) on \( \mathbb{R}^n - \{0\} \). Furthermore, by Corollary 1.7.3 of [44], for \( x \neq 0 \) we have

\[
D\gamma(x) \in \partial K^o, \quad D\gamma^o(x) \in \partial K,
\]

(2.6)

or equivalently

\[
\gamma^o(D\gamma) = 1, \quad \gamma(D\gamma^o) = 1.
\]

In particular \( D\gamma, D\gamma^o \) are nonzero on \( \mathbb{R}^n - \{0\} \).

Now assume that \( k \geq 2 \) and the principal curvatures of \( \partial K \) are positive everywhere. Then \( K \) is strictly convex. We can also show that \( \gamma^o \) is \( C^{k,\alpha} \) on \( \mathbb{R}^n - \{0\} \). To see this let \( n_K : \partial K \to \mathbb{S}^{n-1} \) be the Gauss map, i.e. let \( n_K(y) \) be the outward unit normal to \( \partial K \) at \( y \). Then \( n_K \) is \( C^{k-1,\alpha} \) and its derivative is an isomorphism at the points with positive principal curvatures, i.e. everywhere. Hence \( n_K \) is locally invertible with a \( C^{k-1,\alpha} \) inverse \( n_K^{-1} \), around any point of \( \mathbb{S}^{n-1} \). Now note that as it is well known, \( \gamma^o \) equals the support function of \( K \), i.e.

\[
\gamma^o(x) = \sup\{\langle x, y \rangle : y \in K\}.
\]

Thus as shown in page 115 of [44], for \( x \neq 0 \) we have

\[
D\gamma^o(x) = n_K^{-1}\left(\frac{x}{|x|}\right).
\]

Which gives the desired result. As a consequence, \( \partial K^o \) is \( C^{k,\alpha} \) too. Furthermore, as shown on page 120 of [44], the principal curvatures of \( \partial K^o \) are also positive everywhere.

Let us recall a few more properties of \( \gamma, \gamma^o \). Since they are positively 1-homogeneous, \( D\gamma, D\gamma^o \) are positively 0-homogeneous, and \( D^2\gamma, D^2\gamma^o \) are positively \((-1)\)-homogeneous, i.e.

\[
\gamma(tx) = t\gamma(x), \quad D\gamma(tx) = D\gamma(x), \quad D^2\gamma(tx) = \frac{1}{t}D^2\gamma(x),
\]

\[
\gamma^o(tx) = t\gamma^o(x), \quad D\gamma^o(tx) = D\gamma^o(x), \quad D^2\gamma^o(tx) = \frac{1}{t}D^2\gamma^o(x),
\]

(2.7)
for \( x \neq 0 \) and \( t > 0 \). As a result, using Euler’s theorem on homogeneous functions we get
\[
\begin{align*}
\langle D\gamma(x), x \rangle &= \gamma(x), D^2\gamma(x) x = 0, \\
\langle D\gamma^0(x), x \rangle &= \gamma^0(x), D^2\gamma^0(x) x = 0,
\end{align*}
\] (2.8)
for \( x \neq 0 \). Here \( D^2\gamma(x) x \) is the action of the matrix \( D^2\gamma(x) \) on the vector \( x \).

Finally let us mention that by Corollary 2.5.2 of [44], when \( x \neq 0 \), the eigenvalues of \( D^2\gamma(x) \) are all positive except for one 0. We have a similar characterization of the eigenvalues of \( D^2\gamma^0(x) \).

### 2.2 Regularity of the obstacles

Next let us consider the obstacles \( \rho, -\tilde{\rho} \), and review some of their properties. All the results of this subsection are proved in [40].

**Definition 3** When \( \rho(x) = \gamma(x - y) + \varphi(y) \) for some \( y \in \partial U \), we call \( y \) a \( \rho \)-closest point to \( x \) on \( \partial U \). Similarly, when \( \rho(x) = \gamma(y - x) - \varphi(y) \) for some \( y \in \partial U \), we call \( y \) a \( \rho \)-closest point to \( x \) on \( \partial U \).

Let us also introduce some more notation. For two points \( x, y \in \mathbb{R}^n, [x, y], ]x, y[, ]x, y[ ]x, y] \) will denote the closed, open, and half-open line segments with endpoints \( x, y \), respectively.

**Lemma 3** Suppose \( y \) is one of the \( \rho \)-closest points on \( \partial U \) to \( x \in U \). Then

(a) \( y \) is a \( \rho \)-closest point on \( \partial U \) to every point of \( [x, y] \). Therefore \( \rho \) varies linearly along the line segment \( [x, y] \).

(b) If in addition, for all \( x \neq y \in \mathbb{R}^n \) we have the strict Lipschitz property
\[
-\gamma(y - x) < \varphi(x) - \varphi(y) < \gamma(x - y),
\] (2.9)
then we also have \( ]x, y[ \subset U \).

(c) If in addition \( \gamma \) is strictly convex, and the strict Lipschitz property (2.9) for \( \varphi \) holds, then \( y \) is the unique \( \rho \)-closest point on \( \partial U \) to the points of \( ]x, y[ \).

Next, we generalize the notion of ridge introduced by Ting [48], and Caffarelli and Friedman [12]. Intuitively, the \( \rho \)-ridge is the set of singularities of \( \rho \).

**Definition 4** The \( \rho \)-ridge of \( U \) is the set of all points \( x \in U \) where \( \rho(x) \) is not \( C^{1,1} \) in any neighborhood of \( x \). We denote it by
\[
R_\rho.
\]

We have shown that when \( \gamma \) is strictly convex and the strict Lipschitz property (2.9) for \( \varphi \) holds, the points with more than one \( \rho \)-closest point on \( \partial U \) belong to \( \rho \)-ridge, since \( \rho \) is not differentiable at them. This subset of the \( \rho \)-ridge is denoted by
\[
R_{\rho,0}.
\]

Similarly we define \( R_{\tilde{\rho}}, R_{\tilde{\rho},0} \).

We know that \( \rho, \tilde{\rho} \) are Lipschitz functions. We want to characterize the set over which they are more regular. In order to do that, we need to impose some additional restrictions on \( K, U \) and \( \varphi \).
Assumption 2 Suppose that $k \geq 2$ is an integer, and $0 \leq \alpha \leq 1$. We assume that

(a) $\partial K$ is $C^{k,\alpha}$, and its principal curvatures are positive at every point.
(b) $\partial U$ is $C^{k,\alpha}$.
(c) $\varphi$ is $C^{k,\alpha}$ and $\gamma^\circ(D\varphi) < 1$.

Remark As shown in Sect. 2.1, the above assumption implies that $K$, $\gamma$ are strictly convex. In addition, $K^\circ$, $\gamma^\circ$ are strictly convex, and $\partial K^\circ$, $\gamma^\circ$ are also $C^{k,\alpha}$. Furthermore, the principal curvatures of $\partial K^\circ$ are also positive at every point. Similar conclusions obviously hold for $-K$, $-\varphi$ and $(-K)^\circ = -K^\circ$ too. Hence in the sequel, whenever we state a property for $\rho$, it holds for $\bar{\rho}$ too.

Let $\nu$ be the inward unit normal to $\partial U$. Then, by the implicit function theorem, and using the fact that $\gamma^\circ(D\varphi) < 1$, we can easily show that there is a unique $C^{k-1,\alpha}$ function $\lambda$ of $y \in \partial U$ with $\lambda(y) > 0$ such that

$$\gamma^\circ(D\varphi(y) + \lambda(y)v(y)) = 1.$$  \hfill (2.10)

We set

$$\mu(y) := D\varphi(y) + \lambda(y)v(y).$$  \hfill (2.11)

We also set

$$X := \frac{1}{\langle D\gamma^\circ(\mu), v \rangle} D\gamma^\circ(\mu) \otimes v,$$  \hfill (2.12)

where $a \otimes b$ is the rank 1 matrix whose action on a vector $z$ is $\langle z, b \rangle a$. We also know that

$$\langle D\gamma^\circ(\mu), v \rangle > 0.$$  \hfill (2.13)

Let $x \in U$, and suppose $y$ is one of the $\rho$-closest points to $x$ on $\partial U$. Then we have

$$\frac{x - y}{\gamma(x - y)} = D\gamma^\circ(\mu(y))$$  \hfill (2.14)

(see Fig. 1), or equivalently

$$x = y + (\rho(x) - \varphi(y)) D\gamma^\circ(\mu(y)).$$  \hfill (2.15)

Also, $\rho$ is differentiable at $x$ if and only if $x \in U - R_{\rho,0}$. And in that case we have

$$D\rho(x) = \mu(y),$$  \hfill (2.16)

where $y$ is the unique $\rho$-closest point to $x$ on $\partial U$.

In addition, for every $y \in \partial U$ there is an open ball $B_r(y)$ such that $\rho$ is $C^{k,\alpha}$ on $U \cap B_r(y)$. Furthermore, $y$ is the $\rho$-closest point to some points in $U$, and we have

$$D\rho(y) = \mu(y).$$  \hfill (2.17)

We also have

$$D^2\rho(y) = (I - X^T)(D^2\varphi(y) + \lambda(y)D^2d(y))(I - X),$$  \hfill (2.18)

where $I$ is the identity matrix, $d$ is the Euclidean distance to $\partial U$, and $X$ is given by (2.12).

Remark As a consequence, $R_{\rho}$ has a positive distance from $\partial U$. 

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Let $x \in U - R_\rho$, and let $y$ be the unique $\rho$-closest point to $x$ on $\partial U$. Let

$$
W = W(y) := -D^2 \gamma(y) D^2 \rho(y),
Q = Q(x) := I - (\rho(x) - \varphi(y)) W,
$$

where $I$ is the identity matrix. If $\det Q \neq 0$ then $\rho$ is $C^{k,\alpha}$ on a neighborhood of $x$. In addition we have

$$
D^2 \rho(x) = D^2 \rho(y) Q(x)^{-1}.
$$

We also have

$$
x \in R_\rho \text{ if and only if } \det Q(x) = 0.
$$

**Remark** When $\varphi = 0$, the function $\rho$ is the distance to $\partial U$ with respect to the Minkowski distance defined by $\gamma$. So this case has a geometric interpretation. An interesting fact is that in this case the eigenvalues of $W$ coincide with the notion of curvature of $\partial U$ with respect to some Finsler structure. For the details see [14].

**Lemma 4** Suppose the Assumption 2 holds. Let $x \in U - R_\rho$, and let $y$ be the unique $\rho$-closest point to $x$ on $\partial U$. Then we have

$$
D^2_{\xi \xi} \rho(x) \leq D^2_{\xi \xi} \rho(y)
$$

for every $\xi \in \mathbb{R}^n$.

As we mentioned in the introduction, the above monotonicity property is true because $\rho$ satisfies the Hamilton–Jacobi equation (1.9), and the segment $[x, y]$ is the characteristic curve associated to it. Let us review the general case of the monotonicity property below. Although, note that the assumptions in Lemma 4 are weaker than what we assume in the following calculations. For a proof of Lemma 4 see the proof of Lemma 4 in [40].

**Monotonicity of the second derivative of the solutions to Hamilton–Jacobi equations:** Suppose $v$ satisfies the equation $\tilde{H}(Dv) = 0$, where $\tilde{H}$ is a convex function. Let $x(s)$ be a characteristic curve of the equation. Then we have $\dot{x} = D \tilde{H}$. Let us assume that $v$ is $C^3$ on a neighborhood of the image of $x(s)$. For some vector $\xi$ let

$$
q(s) := D^2_{\xi \xi} v(x(s)) = \xi_i \xi_j D^2_{ij} v.
$$

Here we have used the convention of summing over repeated indices. Then we have

$$
\dot{q} = \xi_i \xi_j D^3_{ijk} v \dot{x}^k = \xi_i \xi_j D^3_{ijk} v D_k \tilde{H}.
$$
On the other hand, if we differentiate the equation we get
\[ D_{ki}^2 \tilde{H} D_{kj}^2 v + D_k \tilde{H} D_{kj}^3 v = 0. \]
Now if we multiply the above expression by \( \xi_i \xi_j \), and sum over \( i, j \), we obtain the following Riccati type equation
\[
\dot{q} = -\xi^T D^2 v D^2 \tilde{H} D^2 v \xi. 
\] (2.22)
So \( \dot{q} \leq 0 \), since \( \tilde{H} \) is convex. Thus we have
\[ D^2 \xi \xi v(x(s)) = q(s) \leq q(0) = D^2 \xi \xi v(x(0)), \]
as desired. This result also holds in the more general case of \( \tilde{H}(x, v, Dv) = 0 \), when \( \tilde{H} \) is a convex function in all of its arguments (see [40]).

3 Proof of Theorem 3

In this section we prove Theorem 3, i.e. we will prove that the double obstacle problem (1.13) has a viscosity solution \( u \in C^{1,\alpha}_{\text{loc}}(U) \), without assuming any regularity about \( K \). Here we only consider the case of zero exterior data. The case of general nonzero exterior data \( \varphi \) is considered in Appendix A.

**Proof of Theorem 3 for zero exterior data**

As it is well known, a compact convex set with nonempty interior can be approximated, in the Hausdorff metric, by a shrinking sequence of compact convex sets with nonempty interior which have smooth boundaries with positive curvature (see for example [43]). We apply this result to \( K \). Thus there is a sequence \( K_{k+1} \) of compact convex sets, that have smooth boundaries with positive curvature, and
\[ K^\circ_k \subset \text{int}(K^\circ_{k+1}), \quad K^\circ = \bigcap K^\circ_k. \]
Notice that we can take the approximations of \( K^\circ \) to be the polar of other convex sets, because the double polar of a compact convex set with 0 in its interior is itself. Also note that \( K_k \)'s are strictly convex compact sets with 0 in their interior, which have smooth boundaries with positive curvature. Furthermore we have \( K = (K^\circ)^{\circ} \supset K_{k+1} \supset K_k \). For the proof of these facts see [44, Sections 1.6, 1.7 and 2.5].

To simplify the notation we use \( \gamma_k, \gamma_k^\circ, \rho_k, \tilde{\rho}_k \) instead of \( \gamma_{K_k}, \gamma_{K_k^\circ}, \rho_{K_k,0}, \tilde{\rho}_{K_k,0} \), respectively. Note that \( K_k, U, \varphi = 0 \) satisfy the Assumption 2. Hence as we have shown in [40], \( \rho_k, \tilde{\rho}_k \) satisfy the assumptions of Theorem 1 of [41] (see Appendix B). Thus there are viscosity solutions \( u_k \in C^{1,\alpha}_{\text{loc}}(U) \) of the double obstacle problem
\[
\begin{cases}
\max\{\min\{-I u_k, u_k + \tilde{\rho}_k\}, u_k - \rho_k\} = 0 & \text{in } U, \\
u_k = 0 & \text{in } \mathbb{R}^n - U.
\end{cases}
\] (3.1)
And \( \alpha > 0 \) depends only on \( n, \lambda, \Lambda, s_0 \).

In addition, we know that
\[ -\tilde{\rho}_1 \leq -\tilde{\rho}_k \leq u_k \leq \rho_k \leq \rho_1. \] (3.2)
Note that \( \rho_k \leq \rho_1 \) and \( \tilde{\rho}_k \leq \tilde{\rho}_1 \), since \( \gamma_k \leq \gamma_1 \) due to \( K_k \supset K_1 \). Let us also show that \( \rho_k \to \rho \) and \( \tilde{\rho}_k \to \tilde{\rho} \) uniformly on \( U \). It is easy to see that for nonzero \( z \) we have \( \frac{z}{\gamma(z)} \in \partial K \). Each
ray emanating from the origin intersects \( \partial K_k, \partial K \) at a pair of points. Let \( \delta_k \) be the maximum distance between these pairs of points. Then we have
\[
\left| \frac{z}{\gamma(z)} - \frac{z}{\gamma_k(z)} \right| \leq \delta_k \implies |\gamma(z) - \gamma_k(z)| \leq \delta_k \frac{\gamma(z)\gamma_k(z)}{|z|} \leq C\delta_k|z|,
\]
where the last estimate is obtained by (2.3). Now let \( x \in U \), and let \( y \in \partial U \) be a \( \rho \)-closest point to \( x \). Then we have
\[
|\rho_k(x) - \rho(x)| \leq |\gamma_k(x - y) - \gamma(x - y)| \leq C\delta_k|x - y|.
\] (3.3)
Since \( \delta_k \to 0 \) and \( U \) is bounded we get the desired.

Furthermore, by Lemma 1 we have \( \gamma_k^0(Du_k) \leq 1 \). Hence we have
\[
Du_k \in K_k^0 \subset K_1 \quad \text{a.e.}
\]
Therefore \( u_k \) is a bounded sequence in \( W^{1,\infty}(U) = C^{0,1}(\overline{U}) \). Hence by the Arzela-Ascoli Theorem a subsequence of \( u_k \), which we still denote by \( u_k \), uniformly converges to a continuous function \( u \in C^0(\overline{U}) \). Note that \( u|_{\partial U} = 0 \), because \( u_k|_{\partial U} = 0 \) for every \( k \). We extend \( u \) to all of \( \mathbb{R}^n \) by setting it equal to 0 in \( \mathbb{R}^n - U \). Note that \( u \) is a continuous function.

We divide the rest of this proof into three parts. In Part I we derive the uniform bound (3.4), i.e. we show that \( Iu_k \) is bounded independently of \( k \) by using suitable barriers. This is possible mainly for two reasons. First we will use the fact that similarly to \( D^2\rho_k \), the second derivative of the barrier attains its maximum on the boundary; so we get a one-way bound for \( Iu_k \). For the other bound, we use the fact that \( u_k \) is a subsolution or a supersolution of \(-I = 0 \) in the regions in which \( u_k \) touches one of the obstacles. In Part II we prove the main properties of the barriers which we employed in Part I. Then in Part III we show that a subsequence of \( u_k \) converges to \( u \) in \( C^{1,\alpha}_{\text{loc}} \), and \( u \) is a viscosity solution of the double obstacle problem (1.13). Here we use the bound (3.4), obtained in Part I, to show that the \( C^{1,\alpha}_{\text{loc}} \) norm of \( u_k \) is uniformly bounded, so that we can extract a convergent subsequence of \( u_k \).

PART I:
Let us show that for every bounded open set \( V \subset\subset U \) and every \( k \) we have
\[
-C_0 \leq Iu_k \leq C_0 \tag{3.4}
\]
in the viscosity sense in \( V \), for some constant \( C_0 \) independent of \( k \). In addition, we can choose \( C_0 \) uniformly for \( s > s_0 \).

Suppose \( \phi \) is a bounded \( C^2 \) function and \( u_k - \phi \) has a minimum over \( \mathbb{R}^n \) at \( x_0 \in V \). We must show that
\[
-I\phi(x_0) \geq -C_0.
\]
We can assume that \( u_k(x_0) - \phi(x_0) = 0 \) without loss of generality, since we can consider \( \phi + c \) instead of \( \phi \) without changing \( I \) (because \( M^+(c) = 0 \)). So we can assume that \( u_k - \phi \geq 0 \), or \( u_k \geq \phi \). We also know that at \( x_0 \) we have
\[
\max\{\min\{-I\phi(x_0), u_k + \rho_k\}, u_k - \rho_k\} \geq 0,
\]
since \( u_k \) is a viscosity solution of (3.1). In addition remember that \(-\rho_k \leq u_k \leq \rho_k \). Now if \( u_k(x_0) < \rho_k(x_0) \) then we must have \(-I\phi(x_0) \geq 0 \). And if \( u_k(x_0) = \rho_k(x_0) \) then \( \phi \) is also touching \( \rho_k \) from below at \( x_0 \), since \( \phi \leq u_k \leq \rho_k \). Now let \( y_0 \in \partial U \) be a \( \rho_k \)-closest point to \( x_0 \). (Note that \( y_0 \) can depend on \( k \).) Let \( B \subset \mathbb{R}^n - \overline{U} \) be an open ball such that
\( \partial B \cap \partial U = \{ y_0 \} \). Note that the radius of \( B \) can be chosen to be independent of \( y_0 \), since \( \partial U \) is \( C^2 \). Now on the domain \( \mathbb{R}^n - \overline{B} \) consider the function
\[
\rho_{k,B^c} := \rho_{K_k,0}(\cdot; \mathbb{R}^n - \overline{B}) = \min_{z \in \partial B} \gamma_k(\cdot - z).
\] (3.5)
The idea is to use \( \rho_{k,B^c} \) as a barrier to obtain the desired bound for \(-I \phi\).

Let \( z \in \partial B \), and let \( y \) be a point on \( \partial U \cap [z, x_0] \). Then we have
\[
\gamma_k(x_0 - y) \leq \gamma_k(x_0 - y) = \gamma_k(x_0 - z) - \gamma_k(y - z) \leq \gamma_k(x_0 - z).
\]

Hence \( y_0 \) is also a \( \rho_{k,B^c} \)-closest point to \( x_0 \) on \( \partial B \).

Consequently we have
\[
\rho_{k,B^c}(x_0) = \gamma_k(x_0 - y_0) = \rho_k(x_0).
\]
Now consider \( x \in U \subset \mathbb{R}^n - \overline{B} \), and let \( z \in \partial B \) be a \( \rho_{k,B^c} \)-closest point to \( x \). Let \( y \) be a point on \( \partial U \cap [z, x] \). Then similarly to the above we can show that
\[
\rho_k(x) \leq \gamma_k(x - y) \leq \gamma_k(x - z) = \rho_{k,B^c}(x).
\]

Similarly to the extension of \( \rho_k \), we extend \( \rho_{k,B^c} \) to all of \( \mathbb{R}^n \) by setting it equal to 0 on \( B \). Then for \( x \notin U \) we have \( \rho_k(x) = 0 \leq \rho_{k,B^c}(x) \). Hence, \( \phi \) is also touching \( \rho_{k,B^c} \) from below at \( x_0 \).

As we will show below in Part II, \( I \rho_{k,B^c}(x) \leq C_0 \) for \( x \in V \). So by ellipticity of \( I \) we must have
\[
I \phi(x_0) \leq I \rho_{k,B^c}(x_0) + M^+(\phi - \rho_{k,B^c})(x_0) \leq C_0,
\]
since it is easy to see that \( M^+(\phi - \rho_{k,B^c})(x_0) \leq 0 \) as \( L(\phi - \rho_{k,B^c})(x_0) \leq 0 \) for any linear operator \( L \). The upper bound for \(-I u_k \) can be shown to hold similarly.

PART II:

Next we want to show that \( I \rho_{k,B^c}(x) \leq C_0 \) for \( x \in V \), independently of \( k, B \). To this end, first we have to show that \( \rho_{k,B^c} \) is a \( C^2 \) function on \( \mathbb{R}^n - \overline{B} \). Let \( x \in \mathbb{R}^n - \overline{B} \). By the results of Sect. 2.2 we need to show that \( x \) has only one \( \rho_{k,B^c} \)-closest point on \( \partial B \), and det \( Q_{k,B^c}(x) \neq 0 \) where \( Q_{k,B^c} \) is given by (3.8). (Note that although \( \mathbb{R}^n - \overline{B} \) is not bounded, the function \( \rho_{k,B^c} \) can be studied as in Sect. 2.2, since \( \partial B \) is compact.) Let \( y \in \partial B \) be a \( \rho_{k,B^c} \)-closest point to \( x \). Then the convex set \( x - \rho_{k,B^c}(x)K_k \) intersects \( \partial B \) at \( y \), and has empty intersection with \( B \). But \( B \) is strictly convex; so \( x - \rho_{k,B^c}(x)K_k \) cannot intersect \( \partial B \) at any other point (see Fig. 1). Thus \( y \) is the unique \( \rho_{k,B^c} \)-closest point on \( \partial B \) to \( x \).

Next note that since the exterior data \( \varphi = 0 \), the equations (2.10)-(2.12) reduce to
\[
\lambda_k(y) = \frac{1}{\gamma_k^2(v)}, \quad \mu_k(y) = \frac{v}{\gamma_k^2(v)}, \quad X_k = \frac{1}{\gamma_k^2(v)}D\gamma_k^2(v) \otimes v,
\] (3.6)
where \( v \) is the normal to \( \partial B \) at \( y \). Here we also used (2.7),(2.8) to simplify the last expression. Hence by (2.18) we have
\[
D^2 \rho_{k,B^c}(y) = \frac{1}{\gamma_k^2(v)}(I - X_k^T)D^2 d_{B^c}(y)(I - X_k),
\] (3.7)
where \( d_{B^c} \) is the Euclidean distance to \( \partial B \) on \( \mathbb{R}^n - B \). But the eigenvalues of \( D^2 d \) are minus the principal curvatures of the boundary, and a zero eigenvalue corresponding to the normal direction (see [18, Section 14.6]). So the nonzero eigenvalues of \( D^2 d_{B^c} \) are \( \frac{1}{r_0} \), where \( r_0 \) is the radius of \( B \) (note that we are in the exterior of \( B \), so the principal curvatures are \( \frac{1}{r_0} \)).
Therefore $D^2 \rho_{k,B^c}(y)$ is a positive semidefinite matrix. Now let us consider the matrices $W, Q$ for $\rho_{k,B^c}$ given by (2.19). The eigenvalues of

$$W_{k,B^c}(y) = -D^2 \gamma_k^\circ(\mu_k(y))D^2 \rho_{k,B^c}(y)$$

must be nonpositive, because $D^2 \gamma_k^\circ(\mu_k)$ is a positive semidefinite matrix. Therefore the eigenvalues of

$$Q_{k,B^c}(x) = I - \rho_{k,B^c}(x)W_{k,B^c}(y) = I - \gamma_k(x - y)W_{k,B^c}(y)$$

are greater than or equal to 1. Hence $\det Q_{k,B^c}(x) > 0$, and we can conclude that $\rho_{k,B^c}$ is $C^2$ on a neighborhood of an arbitrary point $x \in \mathbb{R}^n - \overline{B}$, as desired.

In addition, by Lemma 4 we have

$$D^2 \rho_{k,B^c}(x) \leq D^2 \rho_{k,B^c}(y) = \frac{1}{\gamma_k^\circ(v)}(I - X_k^T)D^2 d_{B^c}(y)(I - X_k).$$

Let us show that $D^2 \rho_{k,B^c}(y)$ is bounded independently of $k, y, B$. Since the radius of $B$ is fixed, $D^2 d_{B^c}(y)$ is bounded independently of $y, B$. So we only need to show that $X_k, 1/\gamma_k^\circ(v)$ are uniformly bounded. And in order to do this, by (3.6), it suffices to show that $D\gamma_k^\circ(v), 1/\gamma_k^\circ(v)$ are uniformly bounded. Note that we have $\gamma_k(D\gamma_k^\circ(v)) = 1$ due to (2.6). Thus $\gamma(D\gamma_k^\circ(v)) \leq 1$ for every $k$, since $\gamma \leq \gamma_k$ due to $K \supset K_k$. So $D\gamma_k^\circ(v)$ is bounded independently of $k$. On the other hand, by (2.3) we have $\gamma_k^\circ(v) \geq \gamma_k^\circ(v) \geq c_1|v| = c_1$ for some $c_1 > 0$. Hence $1/\gamma_k^\circ(v)$ is uniformly bounded too. Therefore $D^2 \rho_{k,B^c}(x)$ has a uniform upper bound, independently of $k, x, B$.

Then it follows that

$$\delta \rho_{k,B^c}(x, h) = \rho_{k,B^c}(x + h) + \rho_{k,B^c}(x - h) - 2\rho_{k,B^c}(x)$$

$$\hat{h} = h/|h| = |h|^2 \int_0^1 \int_{-1}^1 tD^2_h \rho_{k,B^c}(x + sth) d s d t \leq C|h|^2$$

for some constant C independent of $k, x, B$, provided that the segment $[x - h, x + h]$ does not intersect $B$. Now we truncate $\rho_{k,B^c}$ outside of a compact neighborhood of $\overline{U}$ to make it bounded, by redefining $\rho_{k,B^c}$ to be equal to $c$ on the set $\rho_{k,B^c} \geq c$ for some suitable $c$ (note that $\rho_{k,B^c}(x) \to \infty$ as $|x| \to \infty$). Note that, after the truncation, we can choose the bound for $\rho_{k,B^c}$ uniformly, since

$$\rho_{k,B^c}(x) = \gamma_k(x - y) \leq \gamma_1(x - y),$$

and $\gamma_1$ is bounded on the compact set of differences $x - y$ for $x, y$ in the chosen compact neighborhood of $\overline{U}$. Also note that we can still make sure that $\rho_k \leq \rho_{k,B^c}$ after truncation, since $\rho_k = 0$ outside $U$.

Now suppose $x \in V$ and $d(V, \partial U) > \tau$. We can make C larger if necessary so that for $|h| > \tau$ we have $\delta \rho_{k,B^c}(x, h) \leq C\tau^2$, because

$$\delta \rho_{k,B^c}(x, h) = \rho_{k,B^c}(x + h) + \rho_{k,B^c}(x - h) - 2\rho_{k,B^c}(x) \leq 4 \sup_{k,B} \|\rho_{k,B^c}\|_{L^\infty};$$
and thus it suffices to take $C \geq \frac{4}{\tau^2} \sup_{k, B} \| \rho_{k, B^c} \|_{L^\infty}$, noting that by truncation we made $\rho_{k, B^c}$ bounded independently of $k, B$. Then by ellipticity of $I$ we have

$$I \rho_{k, B^c}(x) \leq 10(x) + M^+ \rho_{k, B^c}(x)$$

$$= 0 + (1 - s) \int_{\mathbb{R}^n} \frac{\Lambda \delta \rho_{k, B^c}(x, h) + \lambda \delta \rho_{k, B^c}(x, h)}{|h|^{n+2s}} \, dh$$

$$\leq (1 - s) \int_{\mathbb{R}^n} \frac{(\Lambda + \lambda) C \min\{\tau^2, |h|^2\}}{|h|^{n+2s}} \, dh$$

$$= (1 - s)(\Lambda + \lambda) C \int_{\mathbb{R}^n} \int_0^\infty \frac{\min\{\tau^2, r^2\} r^{n-1} dr dS}{r^{n+2s}}$$

$$= (1 - s) \hat{C} \int_0^\infty \min\{\tau^2, r^2\} r^{-1-2s} dr \leq \frac{\hat{C}(1 + \tau)}{2s_0} =: C_0 < \infty,$$

as desired.

**PART III:**

Let $x_0 \in V$ then $B_\tau(x_0) \subset U$. Thus by the estimate (3.4) and Theorem 4.1 of [25] we have

$$\|u_k\|_{C^{1,\alpha}(B_{\tau/2}(x_0))} \leq \frac{C}{\tau^{1+\alpha}}(\|u_k\|_{L^\infty(\mathbb{R}^n)} + C_0 \tau^{2s}),$$

where $C, \alpha$ depend only on $n, s_0, \lambda, \Lambda$. For simplicity we are assuming that $\tau \leq 1$. (Note that by considering the scaled operator $I_{\tau} v(\cdot) = \tau^{2s} I v(\tau \cdot)$, which has the same ellipticity constants $\lambda, \Lambda$ as $I$, and using the translation invariance of $I$, we have obtained the estimate on the domain $B_{\tau/2}(x_0)$ instead of $B_{1/2}(0)$.) Then we can cover $V \subset U$ with finitely many open balls contained in $U$ and obtain

$$\|u_k\|_{C^{1,\alpha}(\bar{V})} \leq C(\|u_k\|_{L^\infty(\mathbb{R}^n)} + C_0),$$

where $C$ depends only on $n, \lambda, \Lambda, s_0$, and $d(V, \partial U)$. In particular $C$ does not depend on $k$.

Therefore $u_k$ is bounded in $C^{1,\alpha}(\bar{V})$ independently of $k$, because $\|u_k\|_{L^\infty}$ is uniformly bounded by (3.2). Hence there is a subsequence of $u_k$ that is convergent in $C^1$ norm to a function in $C^{1,\alpha}(\bar{V})$. But the limit must be $u$, since $u_k$ converges uniformly to $u$ on $U$. Thus $u \in C^{1,\alpha}_{\text{loc}}(U)$. Also note that $u_k$ converges uniformly to $u$ on $\mathbb{R}^n$, because $u_k, \rho$ are zero outside of $U$. In addition, if we let $k \to \infty$ in (3.9), we obtain the same estimate for $\|u\|_{C^{1,\alpha}(\bar{V})}$.

Finally, let us show that due to the stability of viscosity solutions, $u$ must satisfy the double obstacle problem (1.13) for zero exterior data. Suppose $\phi$ is a bounded $C^2$ function and $u - \phi$ has a maximum over $\mathbb{R}^n$ at $x_0 \in U$. Let us first consider the case where $u - \phi$ has a strict maximum at $x_0$. We must show that at $x_0$ we have

$$\max\{\min\{-I\phi(x_0), u + \tilde{\rho}\}, u - \rho\} \leq 0.$$

Now we know that $u_k - \phi$ takes its global maximum at a point $x_k$ where $x_k \to x_0$; because $u_k$ uniformly converges to $u$ on $\mathbb{R}^n$.

We also know that $-\tilde{\rho} \leq u \leq \rho$. If $-\tilde{\rho}(x_0) = u(x_0)$ then (3.10) holds trivially. So suppose $-\tilde{\rho}(x_0) < u(x_0)$. Then for large $k$ we have $-\tilde{\rho_k}(x_k) < u_k(x_k)$, since $u_k + \tilde{\rho_k}$ locally uniformly converges to $u + \tilde{\rho}$. Hence since $u_k$ is a viscosity solution of the equation (3.1), at $x_k$ we have

$$\max\{\min\{-I\phi(x_k), u_k + \tilde{\rho_k}\}, u_k - \rho_k\} \leq 0.$$
But $u_k + \tilde{\rho}_k > 0$ at $x_k$, so we must have $-I\phi(x_k) \leq 0$. Thus by letting $k \to \infty$ and using the continuity of $I\phi$ we see that (3.10) holds in this case too.

Now if the maximum of $u - \phi$ at $x_0$ is not strict, we can approximate $\phi$ with $\phi_k = \phi + \epsilon \tilde{\phi}$, where $\tilde{\phi}$ is a bounded $C^2$ functions which vanishes at $x_0$ and is positive elsewhere. Then, as we have shown, when $-\tilde{\rho}(x_0) < u(x_0)$ we have $-I\phi_k(x_0) \leq 0$. Hence by the ellipticity of $I$ we get

$$-I\phi(x_0) \leq M^+(\epsilon \tilde{\phi})(x_0) - I\phi_k(x_0) \leq \epsilon M^\prime \tilde{\phi}(x_0) \xrightarrow{\epsilon \to 0} 0,$$

as desired. Similarly, we can show that when $u - \phi$ has a minimum at $x_0 \in U$ we have

$$\max\{\min\{-I\phi(x_0), u + \tilde{\rho}\}, u - \rho\} \geq 0.$$

Therefore $u$ is a viscosity solution of equation (1.13) as desired. At the end note that by Lemma 1 we also have $u \in C^{0,1}(\overline{U})$, since $u$ and its derivative are bounded, and $\partial U$ is smooth enough. \hfill \square

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Appendix A: Proof of Theorem 3 for nonzero exterior data

We do not use the next two propositions directly in the proof of Theorem 3, however, they enhance our understanding of double obstacle problems and equations with gradient constraints. First let us introduce the following terminology for the solutions of the double obstacle problem (1.13). (The notation is motivated by the physical properties of the elastic–plastic torsion problem, in which $E$ stands for the elastic region, and $P$ stands for the plastic region.)

Definition 5 Let

$$P^+ := \{x \in U : u(x) = \rho(x)\}, \quad P^- := \{x \in U : u(x) = -\tilde{\rho}(x)\}.$$ 

Then $P := P^+ \cup P^-$ is called the **coincidence** set; and

$$E := \{x \in U : -\tilde{\rho}(x) < u(x) < \rho(x)\}$$

is called the **non-coincidence** set. We also define the **free boundary** to be $\partial E \cap U$.

Proposition 1 Suppose Assumption 1 holds. Let $u \in C^1(U)$ be a viscosity solution of the double obstacle problem (1.13). If $x \in P^+$, and $y$ is a $p$-closest point on $\partial U$ to $x$ such that $[x, y] \subset U$, then we have $[x, y] \subset P^+$. Similarly, if $x \in P^-$, and $y$ is a $\tilde{\rho}$-closest point on $\partial U$ to $x$ such that $[x, y] \subset U$, then we have $[x, y] \subset P^-$. \hfill \square

Remark If the strict Lipschitz property (2.9) for $\phi$ holds, then by Lemma 3 we automatically have $[x, y] \subset U$.

Proof Suppose $x \in P^-$; the other case is similar. We have

$$u(x) = -\tilde{\rho}(x) = -\gamma(y - x) + \phi(y).$$

Let $\tilde{v} := u - (-\tilde{\rho}) \geq 0$, and $\xi := \frac{y - x}{\gamma(y - x)} = -\frac{x - y}{\gamma(x - y)}$. Then $\tilde{\rho}$ varies linearly along the segment $[x, y]$, since $y$ is a $\tilde{\rho}$-closest point to the points of the segment. So we have $D_{\xi}(-\tilde{\rho}) = D_{-\xi}\tilde{\rho} = 1$ along the segment. Note that we do not assume the differentiability.
of $\tilde{\rho}$; and $D_{-\xi} \tilde{\rho}$ is just the derivative of the restriction of $\tilde{\rho}$ to the segment $[x, y]$. Now by Lemma 1 we get
\[ D_{\xi}u = \langle Du, \xi \rangle \leq \gamma^\circ(Du)\gamma(\xi) \leq 1. \]
So we have $D_{\xi} \bar{v} \leq 0$ along $[x, y]$. Thus as $\bar{v}(x) = \bar{v}(y) = 0$, and $\bar{v}$ is continuous on the closed segment $[x, y]$, we must have $\bar{v} \equiv 0$ on $[x, y]$. Therefore $u = -\tilde{\rho}$ along the segment as desired. \hfill \Box

**Proposition 2** Suppose Assumption 1 holds. Let $u \in C^1(U)$ be a viscosity solution of the double obstacle problem (1.13). Suppose Assumption 2 holds too. Then we have
\[ R_{\rho, 0} \cap P^+ = \emptyset, \quad R_{\tilde{\rho}, 0} \cap P^- = \emptyset. \]

**Proof** Note that due to Assumption 2, the strict Lipschitz property (2.9) for $\phi$ holds, and $\gamma$ is strictly convex. Let us show that $R_{\tilde{\rho}, 0} \cap P^- = \emptyset$; the other case is similar. Suppose to the contrary that $x \in R_{\tilde{\rho}, 0} \cap P^-$. Then there are at least two distinct points $y, z \in \partial U$ such that
\[ \tilde{\rho}(x) = \gamma(y - x) - \varphi(y) = \gamma(z - x) - \varphi(z). \]

Now by Lemma 3 we know that $[x, y], [x, z] \subseteq U$; so by Proposition 1 we get $[x, y], [x, z] \subseteq P^-$. In other words, $u = -\tilde{\rho}$ on both of these segments. Therefore by Lemma 3, $u$ varies linearly on both of these segments. Hence we get
\[ \langle Du(x), \frac{y - x}{\gamma(y - x)} \rangle = 1 = \langle Du(x), \frac{z - x}{\gamma(z - x)} \rangle. \]
However, since $\gamma$ is strictly convex, and by Lemma 1 we know that $\gamma^\circ(Du(x)) \leq 1$, this equality implies that $z, y$ must be on the same ray emanating from $x$. But this contradicts the fact that $[x, y], [x, z] \subseteq U$. \hfill \Box

**Remark** Since here we do not have $C^{1,1}$ regularity for $u$, we cannot imitate the proof given in the local case to show that $P^+, P^-$ do not intersect $R_0, R_{\tilde{\rho}}$. In addition, merely knowing that the solution does not touch the obstacles at their singularities is not enough to obtain uniform bounds for $Iu_k$ in the proof of Theorem 3, because we need to have a uniform positive distance from the ridges too, due to the nonlocal nature of the operator. As we have seen in the proof of Theorem 3, we overcome these limitations by using some suitable barriers.

Next let us review some well-known facts from convex analysis which are needed in the following proof. Consider a compact convex set $K$. Let $x \in \partial K$ and $v \in \mathbb{R}^n - \{0\}$. We say the hyperplane
\[ \Gamma_{x,v} := \{ y \in \mathbb{R}^n : \langle y - x, v \rangle = 0 \} \quad \text{(A.1)} \]
is a **supporting hyperplane** of $K$ at $x$ if $K \subseteq \{ y : \langle y - x, v \rangle \leq 0 \}$. In this case we say $v$ is an **outer normal vector** of $K$ at $x$ (see Fig. 2). The **normal cone** of $K$ at $x$ is the closed convex cone
\[ N(K, x) := \{ 0 \} \cup \{ v \in \mathbb{R}^n - \{0\} : v \text{ is an outer normal vector of } K \text{ at } x \}. \quad \text{(A.2)} \]
It is easy to see that when $\partial K$ is $C^1$ we have
\[ N(K, x) = \{ tD\gamma(x) : t \geq 0 \}. \]
For more details see [44, Sections 1.3 and 2.2].
Proof of Theorem 3 for nonzero exterior data

As before we approximate $K^o$ by a sequence $K^o_k$ of compact convex sets, that have smooth boundaries with positive curvature, and

$$K^o_{k+1} \subset \text{int}(K^o_k), \quad K^o = \bigcap K^o_k.$$ 

Then $K_k$’s are strictly convex compact sets with 0 in their interior, which have smooth boundaries with positive curvature. Furthermore we have $K = (K^o)^o \supset K_{k+1} \supset K_k$. To simplify the notation we use $\gamma_k$, $\gamma_k^o$, $\rho_k$, $\bar{\rho}_k$ instead of $\gamma_{K_k}$, $\gamma_{K^o_k}$, $\rho_{K_k}$, $\bar{\rho}_{K_k}$, respectively. Note that $K_k$, $U$, $\varphi$ satisfy the Assumption 2. In particular we have $\gamma_k^o(D\varphi) < 1$, since $D\varphi \in K^o \subset \text{int}(K^o_k)$. Hence as we have shown in [40], $\rho_k$, $\bar{\rho}_k$ satisfy the assumptions of Theorem 1 of [41] (see Appendix B). Thus there are viscosity solutions $u_k \in C^{1,a}_{\text{loc}}(U)$ of the double obstacle problem

$$\begin{cases}
\max\{\min\{-Iu_k, \ u_k + \bar{\rho}_k\}, u_k - \rho_k\} = 0 & \text{in } U, \\
u_k = \varphi & \text{in } \mathbb{R}^n - U.
\end{cases} \quad (A.3)$$

And $\alpha > 0$ depends only on $n$, $\lambda$, $\Lambda$, $s_0$.

As before, we can easily see that $u_k$’s and their derivatives are uniformly bounded. Hence a subsequence of them converges uniformly to a continuous function $u$, which we extend to all of $\mathbb{R}^n$ by setting it equal to $\varphi$ in $\mathbb{R}^n - U$. In Part I we consider the barrier

$$\rho_{k,B^c} := \rho_{K_k,\varphi}(\cdot; \mathbb{R}^n - \overline{B}) = \min_{z \in \partial B} \gamma_k(\cdot - z) + \varphi(z), \quad (A.4)$$

where $B \subset \mathbb{R}^n - \overline{U}$ is an open ball such that $\partial B \cap \partial U = \{y_0\}$, in which $y_0 \in \partial U$ is a $\rho_k$-closest point to $x_0$.

Let $z \in \partial B$, and let $y$ be a point on $\partial U \cap [z, x_0]$. Then by the Lipschitz property (1.7) for $\varphi$ with respect to $\gamma_k$ (note that $\gamma_k^o(D\varphi) < 1$) we have

$$\gamma_k(x_0 - y_0) + \varphi(y_0) \leq \gamma_k(x_0 - y) + \varphi(y)$$

$$= \gamma_k(x_0 - z) + \varphi(z) \gamma_k(y - z) + \varphi(y) - \varphi(z) \leq \gamma_k(x_0 - z) + \varphi(z).$$

Hence $y_0$ is also a $\rho_{k,B^c}$-closest point to $x_0$ on $\partial B$.

Consequently we have

$$\rho_{k,B^c}(x_0) = \gamma_k(x_0 - y_0) + \varphi(y_0) = \rho_k(x_0).$$

Now consider $x \in U \subset \mathbb{R}^n - \overline{B}$, and let $z \in \partial B$ be a $\rho_{k,B^c}$-closest point to $x$. Let $y$ be a point on $\partial U \cap [z, x]$. Then similarly to the above we can show that

$$\rho_k(x) \leq \gamma_k(x - y) + \varphi(y) \leq \gamma_k(x - z) + \varphi(z) = \rho_{k,B^c}(x).$$

As for $\rho_k$, we extend $\rho_{k,B^c}$ to all of $\mathbb{R}^n$ by setting it equal to $\varphi$ on $B$. Then for $x \notin U$ we either have $\rho_k(x) = \varphi(x) = \rho_{k,B^c}(x)$ when $x \in B$, or

$$\rho_k(x) = \varphi(x) \leq \gamma_k(x - z) + \varphi(z) = \rho_{k,B^c}(x),$$

when $x \notin B$ and $z \in \partial B$ is a $\rho_{k,B^c}$-closest point to $x$. Hence, $\varphi$ is also touching $\rho_{k,B^c}$ from below at $x_0$.

The rest of the proof in Part I, and Part III of the proof, go as before. So we only need to prove the properties of the new barrier $\rho_{k,B^c}$, similarly to the Part II of the proof in the case of zero exterior data. First let us show that $\rho_{k,B^c}$ is $C^2$ on $\mathbb{R}^n - \overline{B}$. Let $x \in \mathbb{R}^n - \overline{B}$. By the results of Sect. 2.2 we need to show that $x$ has only one $\rho_{k,B^c}$-closest point on $\partial B$, and det $Q_{k,B^c}(x) \neq 0$ where $Q_{k,B^c}$ is given by (A.6).
Suppose to the contrary that \( y, \tilde{y} \in \partial B \) are two \( \rho_{k,B_c} \)-closest points to \( x \). Let \( z = \frac{y + \tilde{y}}{2} \in B \). We assume that the radius of \( B \) is small enough so that \( \varphi \) is convex on a neighborhood of it. Then we have

\[
\gamma_k(x - z) + \varphi(z) \leq \frac{1}{2} (\gamma_k(x - y) + \gamma_k(x - \tilde{y}) + \varphi(y) + \varphi(\tilde{y})) = \rho_{k,B_c}(x).
\]

Let \( \tilde{z} \) be the point on \( \partial B \cap [z, x] \). Then by the strict Lipschitz property (2.9) for \( \varphi \) with respect to \( \gamma_k \) (note that \( \gamma_k(D\varphi) < 1 \)) we have

\[
\gamma_k(x - \tilde{z}) + \varphi(\tilde{z}) = \gamma_k(x - z) + \varphi(z) - \gamma_k(\tilde{z} - z) + \varphi(\tilde{z}) - \varphi(z) < \gamma_k(x - z) + \varphi(z) \leq \rho_{k,B_c}(x),
\]

which is a contradiction. So \( x \) must have a unique \( \rho_{k,B_c} \)-closest point \( y \) on \( \partial B \).

Next note that by (2.18) at \( y \in \partial B \) we have

\[
D^2 \rho_{k,B_c}(y) = (I - X_k^T)(D^2 \varphi(y) + \lambda_k(y)D^2 d_{B_c}(y))(I - X_k),
\]

where \( d_{B_c} \) is the Euclidean distance to \( \partial B \) on \( \mathbb{R}^n - B \), and \( \lambda_k, X_k \) are given by (2.10),(2.12) (using \( \gamma_k^0 \) instead of \( \gamma^0 \)). But the eigenvalues of \( D^2 d \) are minus the principal curvatures of the boundary, and a zero eigenvalue corresponding to the normal direction (see [18, Section 14.6]). So the nonzero eigenvalues of \( D^2 d_{B_c} \) are \( \frac{1}{r_0} \), where \( r_0 \) is the radius of \( B \) (note that we are in the exterior of \( B \), so the curvatures are \( -\frac{1}{r_0} \)). Hence \( D^2 d_{B_c} \) is a positive semidefinite matrix. Therefore \( D^2 \rho_{k,B_c}(y) \) is also a positive semidefinite matrix, since \( \varphi \) is convex on a neighborhood of \( \partial U \). (Although the convexity of \( \varphi \) is not really needed here. Because by using \( \gamma^0(D\varphi) < 1 \) we can easily show that \( \lambda_k \) has a uniform positive lower bound independently of \( k, B \). Then by decreasing the radius \( r_0 \) and using the boundedness of \( D^2 \varphi \) we can get the desired.)

Now let us consider the matrices \( W, Q \) for \( \rho_{k,B_c} \) given by (2.19). The eigenvalues of

\[
W_{k,B_c}(y_0) := -D^2 \gamma_k^0(\mu_k(y))D^2 \rho_{k,B_c}(y)
\]

must be nonpositive, because \( D^2 \gamma_k^0(\mu_k) \) is a positive semidefinite matrix. Therefore the eigenvalues of

\[
Q_{k,B_c}(x) := I - (\rho_{k,B_c}(x) - \varphi(y))W_{k,B_c}(y) = I - \gamma_k(x - y)W_{k,B_c}(y) \tag{A.6}
\]

are greater than or equal to 1. Hence \( \det Q_{k,B_c}(x) > 0 \), and we can conclude that \( \rho_{k,B_c} \) is \( C^2 \) on a neighborhood of an arbitrary point \( x \in \mathbb{R}^n - B \), as desired.

In addition, by Lemma 4 we have

\[
D^2 \rho_{k,B_c}(x) \leq D^2 \rho_{k,B_c}(y) = (I - X_k^T)(D^2 \varphi(y) + \lambda_k(y)D^2 d_{B_c}(y))(I - X_k).
\]

Let us show that \( D^2 \rho_{k,B_c}(y) \) is bounded independently of \( k, y, B \). Since the radius of \( B \) is fixed, \( D^2 d_{B_c}(y) \) is bounded independently of \( y, B \). So we only need to show that \( X_k, \lambda_k \) are uniformly bounded. Note that \( \gamma_k^0 \geq \gamma_k^0 \), since \( K_k \subset K_k^0 \). Thus by (2.10) we have

\[
\gamma_k^0(D\varphi + \lambda_k v) \leq \gamma_k^0(D\varphi + \lambda_k v) = 1.
\]

Hence by (2.3) applied to \( \gamma_k^0 \), we have \( |D\varphi + \lambda_k v| \leq C \) for some \( C > 0 \). Therefore we get \( |\lambda_k| = |\lambda_k v| \leq C + |D\varphi| \). Thus \( \lambda_k \) is bounded independently of \( k, y, B \).

Hence we only need to show that the entries of \( X_k = \frac{1}{(D\gamma_k^0(\mu_k) v)} D\gamma_k^0(\mu_k) v \) are bounded. Note that by (2.6) we have \( \gamma_k(D\gamma_k^0(v)) = 1 \). Thus \( \gamma(D\gamma_k^0(v)) \leq 1 \) for every \( k \), since \( \gamma \leq \gamma_k \) due to \( K \supset K_k \). So \( D\gamma_k^0(v) \) is bounded independently of \( k \). Therefore it only remains to show that \( D\gamma_k^0(\mu_k), v \) has a positive lower bound independently of \( k, y, B \). \( \square \)
Note that for every \( k, B \), \( \langle D\gamma^\circ_k(\mu_k), v \rangle \) is a continuous positive function on the compact set \( \partial B \), as explained in Sect. 2.2. Hence there is \( c_{k, B} > 0 \) such that \( \langle D\gamma^\circ_k(\mu_k), v \rangle \geq c_{k, B} \).

Suppose to the contrary that there is a sequence of balls \( B_j \), points \( y_j \in \partial B_j \), and \( k_j \) (which we simply denote by \( j \)) such that
\[
\langle D\gamma^\circ_j(\mu_j(y_j)), v_j \rangle \to 0, \quad (A.7)
\]
where \( v_j \) is the unit normal to \( \partial B_j \) at \( y_j \). By passing to another subsequence, we can assume that \( y_j \to y \), since \( y_j \)'s belong to a compact neighborhood of \( \partial U \). Now remember that
\[
\mu_j(y_j) = D\varphi(y_j) + \lambda_j(y_j)v_j,
\]
where \( \lambda_j > 0 \). As we have shown in the last paragraph, \( \lambda_j \) is bounded independently of \( j, B_j \). Hence by passing to another subsequence, we can assume that \( \lambda_j \to \lambda^* \geq 0 \). Also, \( |v_j| = 1 \). Thus by passing to yet another subsequence we can assume that \( v_j \to v^* \), where \( |v^*| = 1 \). Therefore we have
\[
\mu_j(y_j) \to \mu^* := D\varphi(y) + \lambda^*v^*.
\]
On the other hand we have \( \gamma_j(\mu_j(y_j)) = 1 \). Hence \( \gamma_j(\mu_j(y_j)) \geq 1 \), since \( \gamma_j \leq \gamma^\circ \) due to \( K^o \subset K_j^o \). Thus we get \( \gamma_j(\mu^*) \geq 1 \). However we cannot have \( \gamma_j(\mu^*) > 1 \). Because then \( \mu^* \) will have a positive distance from \( K^o \), and therefore it will have a positive distance from \( K_j^o \) for large enough \( j \). But this contradicts the facts that \( \mu_j(y_j) \to \mu^* \) and \( \mu_j(y_j) \in K_j^o \). Thus we must have \( \gamma_j(\mu^*) = 1 \), i.e. \( \mu^* \in \partial K^o \).

Now note that \( v_j := D\gamma_j^\circ(\mu_j(y_j)) \) belongs to the normal cone \( N(K_j^o, \mu_j(y_j)) \). In addition we have \( \gamma_j(v_j) = 1 \) due to (2.6). Hence we have \( v_j \in K_j \subset K \). Thus by passing to yet another subsequence we can assume that \( v_j \to v \in K \). We also have \( \gamma_1(v_j) \geq 1 \), since \( \gamma_j \leq \gamma_1 \) due to \( K_j \supseteq K_1 \). So we get \( \gamma_1(v) \geq 1 \). In particular \( v \neq 0 \). We claim that \( v \in N(K^o, \mu^*) \). To see this note that we have
\[
K^o \subset K_j^o \subset \{ z : \langle z - \mu_j(y_j), v_j \rangle \leq 0 \}.
\]
Hence for every \( z \in K^o \) we have \( \langle z - \mu_j(y_j), v_j \rangle \leq 0 \). But as \( j \to \infty \) we have \( z - \mu_j(y_j) \to z - \mu^* \). So we get \( \langle z - \mu^*, v \rangle \leq 0 \). Therefore
\[
K^o \subset \{ z : \langle z - \mu^*, v \rangle \leq 0 \},
\]
as desired.
On the other hand, by (A.7) we obtain
\[ \langle v, v^* \rangle = \lim \langle v_j, v_j \rangle = 0. \] (A.8)

Now note that \( D\varphi = \mu^* - \lambda^* v^* \) belongs to the ray passing through \( \mu^* \in \partial K^0 \) in the direction \( -v^* \). However, we know that \( D\varphi \) is in the interior of \( K^0 \), since \( \gamma^5(D\varphi) < 1 \). Hence we must have \( \lambda^* > 0 \) (since \( \gamma^5(\mu^*) = 1 \)). And thus the ray \( t \mapsto \mu^* - tv^* \) for \( t > 0 \) passes through the interior of \( K^0 \). Therefore this ray and \( K^0 \) must lie on the same side of the supporting hyperplane \( \Gamma_{\mu^*} \). In addition, the ray cannot lie on the hyperplane, since it intersects the interior of \( K^0 \). Hence we must have \( \langle v, v^* \rangle = -\langle v, -v^* \rangle > 0 \), which contradicts (A.8). See Fig. 2 for a geometric representation of this argument.

Thus \( \langle D\gamma_k^0(\mu_k), v \rangle \) must have a positive lower bound independently of \( k, y, B \), as desired. Therefore \( D^2\rho_{k,B^c}(x) \) has a uniform upper bound, independently of \( k, x, B \). Then as before it follows that
\[ \delta \rho_{k,B^c}(x, h) = \rho_{k,B^c}(x + h) + \rho_{k,B^c}(x - h) - 2 \rho_{k,B^c}(x) \leq C|h|^2 \]
for some constant \( C \) independent of \( k, x, B \), provided that the segment \([x - h, x + h]\) does not intersect \( B \). Now, as in the case of zero exterior data, we truncate \( \rho_{k,B^c} \) outside of a compact neighborhood of \( \overline{U} \) to make it bounded. Note that, after the truncation, we can choose the bound for \( \rho_{k,B^c} \) uniformly, since
\[ \rho_{k,B^c}(x) = \gamma_k(x - y) + \varphi(y) \leq \gamma_1(x - y) + \psi(y) \leq C + \|\psi\|_{L^\infty}, \]
where \( C \) is an upper bound for \( \gamma_1 \) on the compact set of differences \( x - y \) for \( x, y \) in the chosen compact neighborhood of \( \overline{U} \). Also note that we can still make sure that \( \rho_k \leq \rho_{k,B^c} \) after truncation, since \( \rho_k = \varphi \) outside \( U \), and \( \varphi \) is bounded. And finally we can show that \( I \rho_{k,B^c}(x) \leq C_0 \) for \( x \in V \), similarly to the case of zero exterior data. \( \square \)

**Appendix B: Nonlocal Double Obstacle Problems**

In this appendix, for reader’s convenience, we provide a sketch of proof of the following result from [41], which we relied on in the proof of Theorem 3.

**Theorem** Suppose \( I \) satisfies Assumption 1, and \( \partial U \) is \( C^2 \). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. In addition, suppose \( \psi^\pm, \varphi : \mathbb{R}^n \to \mathbb{R} \) are bounded Lipschitz functions with Lipschitz constant \( C_1 \) which satisfy

(a) \( \psi^\pm = \varphi \) on \( \mathbb{R}^n - U \), and for all \( x \in U \) we have
\[ 0 < \psi^+(x) - \psi^-(x) \leq 2C_1d(x), \] (B.1)
where \( d \) is the Euclidean distance to \( \partial U \).

(b) For every \( x \in U \) and \( |y| \leq d(x) - \epsilon \) we have
\[ \pm \delta \psi^\pm(x, y) \leq C|y|^2, \] (B.2)
where the constant \( C \) depends only on \( \epsilon \). In other words, \( \psi^+, \psi^- \) are respectively semi-concave and semi-convex on compact subsets of \( U \).

Then the double obstacle problem
\[
\begin{cases}
\max \{ \min \{ -Iu - f, u - \psi^+ \}, u - \psi^- \} = 0 & \text{in } U, \\
u = \varphi & \text{in } \mathbb{R}^n - U,
\end{cases}
\] (B.3)
has a viscosity solution \( u \in C^{1,\alpha}_{\text{loc}}(U) \) for some \( \alpha > 0 \) depending only on \( n, \lambda, \Lambda, s_0 \). And for an open subset \( V \subset U \) we have
\[
\|u\|_{C^{1,\alpha}(\bar{V})} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U)} + C_0),
\]
where \( C \) depends only on \( n, \lambda, \Lambda, s_0, \) and \( d(V, \partial U) \); and \( C_0 \) depends only on these constants together with \( \|\psi\|_{L^\infty(\mathbb{R}^n)} \) and the semi-concavity constants of \( \pm \psi \) on \( V \).

**Sketch of Proof** Let \( \eta_\varepsilon \) be a standard mollifier whose support is \( B_\varepsilon(0) \), and consider the mollified functions \( \psi_\varepsilon^\pm := \eta_\varepsilon * \psi^\pm \) and \( \varphi_\varepsilon := \eta_\varepsilon * \varphi \). Let
\[
U_\varepsilon := \{ x \in \mathbb{R}^n : d(x, \overline{U}) < \varepsilon \}.
\]
Then it is easy to see that for \( x \in U_\varepsilon \) we have \( \psi_\varepsilon^-(x) < \psi_\varepsilon^+(x) \), and for \( x \in \mathbb{R}^n - U_\varepsilon \) we have \( \psi_\varepsilon^+(x) = \varphi_\varepsilon(x) \). In addition, \( \partial U_\varepsilon \) is \( C^2 \). Also, similarly to Part II of the proof of Theorem 3, we can show that for every bounded open set \( V \subset U \) and small enough \( \varepsilon \) there is a constant \( C_0 \) such that \( \pm I \psi_\varepsilon^\pm \leq C_0 \).

Now we consider the double obstacle problem
\[
\begin{cases}
\max\{\min\{-Iu_\varepsilon - f, u_\varepsilon - \psi_\varepsilon^-\}, u_\varepsilon - \psi_\varepsilon^+\} = 0 \quad \text{in } U_\varepsilon, \\
u_\varepsilon = \varphi_\varepsilon \quad \text{in } \mathbb{R}^n - U_\varepsilon.
\end{cases}
\tag{B.4}
\]
We show that it has a viscosity solution \( u_\varepsilon \in C^{1,\alpha}_{\text{loc}}(U_\varepsilon) \). To this end, let \( \beta \) be a smooth increasing function that vanishes on \( (-\infty, 0] \) and equals \( \frac{1}{\theta} \) for \( t \geq \theta \). Then the equation
\[
\begin{cases}
-I\tilde{u} - f - \beta(\psi_\varepsilon^- - \tilde{u}) + \beta(\tilde{u} - \psi_\varepsilon^+) = 0 \quad \text{in } U_\varepsilon, \\
\tilde{u} = \varphi_\varepsilon \quad \text{in } \mathbb{R}^n - U_\varepsilon.
\end{cases}
\tag{B.5}
\]
has a viscosity solution \( \tilde{u} = \tilde{u}_\theta \) (see Theorem 5.6 of [27]).

Now let us show that
\[
\|\beta(\pm(\tilde{u} - \psi_\varepsilon^\pm))\|_{L^\infty(U_\varepsilon)} \leq C_2,
\]
where \( C_2 \) is independent of \( \theta \). Note that \( \beta(\pm(\tilde{u} - \psi_\varepsilon^\pm)) \) is zero on \( \mathbb{R}^n - U_\varepsilon \). So assume that \( \beta(\pm(\tilde{u} - \psi_\varepsilon^\pm)) \) attains its positive maximum at \( x_0 \in U_\varepsilon \). Since \( \beta \) is increasing, \( \tilde{u} - \psi_\varepsilon^+ \) has a positive maximum at \( x_0 \) too. Therefore by the definition of viscosity solution we have
\[
-I\psi_\varepsilon^+(x_0) - f(x_0) - \beta(\psi_\varepsilon^-(x_0) - \tilde{u}(x_0)) + \beta(\tilde{u}(x_0) - \psi_\varepsilon^+(x_0)) \leq 0.
\]
So at \( x_0 \) we have
\[
-I\psi_\varepsilon^+(x_0) - f(x_0) \leq \beta(\psi_\varepsilon^- - \tilde{u}) - \beta(\tilde{u} - \psi_\varepsilon^+) = -\beta(\tilde{u} - \psi_\varepsilon^+),
\]
since by our assumption \( \psi_\varepsilon^-(x_0) < \psi_\varepsilon^+(x_0) < \tilde{u}(x_0) \). Thus \( \beta(\tilde{u} - \psi_\varepsilon^+) \leq I\psi_\varepsilon^+ + f \) at \( x_0 \). Therefore \( \beta(\tilde{u} - \psi_\varepsilon^+) \) is bounded independently of \( \theta \), because \( I\psi_\varepsilon^+, f \) are continuous functions.

The bound \( \beta(\pm(\tilde{u} - \psi_\varepsilon^\pm)) \leq C_2 \) and the definition of \( \beta \) imply that
\[
\tilde{u} - \psi_\varepsilon^+ \leq \theta(C_2 + 1), \quad \psi_\varepsilon^- - \tilde{u} \leq \theta(C_2 + 1).
\tag{B.6}
\]
This also shows that \( \tilde{u} \) is uniformly bounded independently of \( \theta \). Then from the equation (B.5) we conclude that \(-3C_2 \leq I\tilde{u} \leq 3C_2 \) in the viscosity sense. Thus by Theorem 4.1 of [25] we obtain
\[
\|\tilde{u}\|_{C^{1,\alpha}(\bar{V})} \leq C(\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)} + 3C_2),
\]
where $V \subset U_{\varepsilon}$. Therefore $\tilde{u}$ is bounded in $C^{1,\alpha}(\bar{V})$ independently of $\theta$, due to its uniform boundedness in $L^\infty$. So, similarly to Part III of the proof of Theorem 3, we can show that as $\theta \to 0$, a subsequence of the $\tilde{u}$’s converges to a solution of (B.4) in $V$. Then by expanding $V$ and a diagonalization argument we can construct a viscosity solution $u_\varepsilon$ of (B.4) in $U_{\varepsilon}$.

Next, similarly to Part I of the proof of Theorem 3, we can show that for every bounded open set $V \subset U$ and every small enough $\varepsilon$ we have

$$\begin{equation}
-C_0 - \|f\|_{L^\infty(U)} \leq Iu_\varepsilon \leq \|f\|_{L^\infty(U)} + C_0, \tag{B.7}
\end{equation}$$

where the constant $C_0$ satisfies $\pm I\psi^\pm_\varepsilon \leq C_0$. Heuristically, on the contact set $\{u_\varepsilon = \psi^\pm_\varepsilon\}$, although a priori we do not have a lower bound for the second derivative of $\psi^\pm_\varepsilon$, and hence an upper bound for $-I\psi^\pm_\varepsilon$, we can obtain the desired bound for $-Iu_\varepsilon$ from the equation.

Thus by Theorem 4.1 of [25], we can show that there is $\alpha$ depending only on $n, \lambda, \Lambda, s_0$, such that for an open subset $V \subset U$ we have

$$\|u_\varepsilon\|_{C^{1,\alpha}(\bar{V})} \leq C(\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(U)} + C_0),$$

where $C$ depends only on $n, \lambda, \Lambda, s_0$, and $d(V, \partial U)$. Therefore $u_\varepsilon$ is bounded in $C^{1,\alpha}(\bar{V})$ independently of $\varepsilon$, because $\|u_\varepsilon\|_{L^\infty}$ is bounded by $\|\psi^\pm_\varepsilon\|_{L^\infty}$, and $\|\psi^\pm_\varepsilon\|_{L^\infty}$ are uniformly bounded by $\|\psi^\pm\|_{L^\infty}$. Now, again by using a diagonalization argument, we can construct a function $u$ in $C^{1,\alpha}_{loc}(U)$. We furthermore extend $u$ to all of $\mathbb{R}^n$ by setting it equal to $\psi$ on $\mathbb{R}^n - U$. Then $u_\varepsilon$ converges uniformly to $u$ on $\mathbb{R}^n$. To see this note that since $u, u_\varepsilon$ are between their corresponding obstacles we get

$$|u_\varepsilon - u| \leq \max\{|\psi^+_\varepsilon - \psi^-|, |\psi^-_\varepsilon - \psi^+|\}$$

$$\leq |\psi^+ - \psi^-| + \max\{|\psi^+_\varepsilon - \psi^+|, |\psi^-_\varepsilon - \psi^-|\} \leq \begin{cases} 2C_1d(\cdot) + C_1\varepsilon & \text{in } U, \\ C_1\varepsilon & \text{in } \mathbb{R}^n - U, \end{cases}$$

because $|\psi^+ - \psi^-| = 0$ outside of $U$. Finally, due to the stability of viscosity solutions, $u$ must satisfy the double obstacle problem (B.3).

\[ \square \]

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