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Equation (A.1) on the page 11 should be read as

$$
\hbar \Delta = - \frac{\hbar c \omega \mathbf{p} \cdot (\mathbf{n} - \mathbf{n}')} {mc^2 \left[ 1 + \frac{\hbar \omega}{mc} (1 - \mathbf{n} \cdot \mathbf{n}') - \frac{\mathbf{p} \cdot \mathbf{v}'} {mc} \right]}.
$$
Photon plasma–wave interaction via Compton scattering

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Abstract
The Kompaneets theory of photon kinetic evolution due to the Compton effect in the absence of absorption and emission is extended to the case of the Vlasov plasma wave oscillations. Under the assumption that the electron distribution function at equilibrium is perturbed by a solution of the linearized Vlasov equation in the long-wavelength limit, a solution of the Kompaneets kinetic equation for the photon distribution function is found and discussed.

Keywords: Compton scattering, Kompaneets equation, Vlasov equation

(Some figures may appear in colour only in the online journal)

1. Introduction

The importance of scattering processes resulting from the interaction of electromagnetic radiation (or photons) with charged particles (here electrons) via the Compton effect was noted for the first time by Kompaneets [1] in the case of a plasma described by a Maxwellian distribution. An equation for the distribution function of the photons in energy space was derived in this case. In [2], Zel’dovich and Levitch exhibited a singular solution in the case of large occupation numbers, which diverges at a finite time. This singular solution corresponds to the accumulation of photons in the state of zero-energy and was referred to as a Bose condensation of photons. The question we address in this article is the persistence of a finite-time singularity for the photon distribution when the plasma is no longer described by a Maxwellian distribution, but instead by a linearized Vlasov equation.

The photon gas is characterized by an arbitrary initial distribution with averaged energy $E_{\text{ph}}$. This may, in particular, be a Planck distribution with a characteristic temperature $T_{\text{ph}}$. We assume that the electron gas is described by a Maxwellian distribution with temperature $T_e$. Furthermore the system is assumed to be homogeneous and closed, which means that no
particles leave or enter the system. To take into account the role of Compton scattering in the convergence to equilibrium between photons and non-relativistic electrons, Kompaneets [1] proposed, for a given electron distribution $f_0$, the following kinetic equation for the photon distribution function $n_t$, $K_0$ in a unit volume:

$$\frac{\partial n_t}{\partial t} = - \int d^3p \int dW n_{K_0}(\hbar \omega, t) \left( 1 + n_{K_0}(\hbar \omega', t) \right) f_0(\varepsilon)$$

$$+ \int d^3p \int dW n_{K_0}(\hbar \omega', t) \left( 1 + n_{K_0}(\hbar \omega, t) \right) f_0(\varepsilon + \hbar(\omega - \omega')),$$

(1)

with $n_{K_0}(\hbar \omega, t) = \int d^3r n_{K_0}(\hbar \omega, r, t)$ and the integrand is the photon space–density for a given energy $\hbar \omega$ and time $t$. Here $f_0(\varepsilon)$ is the Maxwellian distribution function for the free electrons with temperature $T_e$ and energy $\varepsilon = p^2/(2m)$, and $dW$ is the differential probability of transition from a state of energy $\hbar \omega'$ into another state of energy $\hbar \omega$, compatible with the laws of conservation of energy and momentum (for details, see [1]). In equation (1), inelastic emission and absorption processes have not been taken into account, so that the transitions are produced exclusively by Compton scattering processes. Since free electrons do not absorb or emit, but only scatter photons, the total number of photons is conserved, which has to be satisfied by equation (1). In what follows, it is convenient to use dimensionless variables related to the electron temperature $T_e$:

$$\tilde{t} = \frac{k_B T_e c}{mc^2} t = \alpha t, \quad x = \frac{\hbar \omega}{k_B T_e},$$

where $\tilde{t}$ is referred to as a rescaled time, $x > 0$ as the energy of photons which have the distribution function $n_{K_0}(x, \tilde{t})$, and $l$ is the Compton range, determined by the total Thomson scattering cross section. So $l = (\rho \sigma_T)^{-1}$, where $\sigma_T = \frac{8\pi}{3\rho} \left( \frac{e^2}{\hbar c} \right)^2$ in Gaussian units, and $\rho$ is the average electron density in configuration space. In terms of these variables, the condition that the total number of photons $N$ is constant can be written in the form [2]

$$\frac{\partial N}{\partial \tilde{t}} = 0 \quad \text{with} \quad N = \int_0^\infty n_{K_0}(x, \tilde{t}) x^2 dx.$$

Figure 1. $n^x(\xi, \tilde{t})$ for $x = \beta$ given by equation (31) as a function of $\tilde{t}$ for $\mu = 1/2$ and $\omega_0 \alpha^{-1} = 100$. 

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Kompaneets [1] considered the non-relativistic case when \( k_B T \ll mc^2 \), and he assumed that the energy transferred during each scattering process is small compared to \( \hbar \omega \), i.e., \( |\Delta| \ll \omega \). Expanding the integrand of equation (1) in power series in \( \Delta \) up to second order, equation (1) becomes

\[
\frac{\partial n_{Ko}}{\partial t} = \left( \frac{\partial n_{Ko}}{\partial x} + n_{Ko}(1 + n_{Ko}) \right) \frac{\hbar}{k_B T_e} \int d^3 p \int dW_f(\varepsilon) \Delta + \frac{\partial^2 n_{Ko}}{\partial x^2} + 2(1 + n_{Ko}) \frac{\partial n_{Ko}}{\partial x} + n_{Ko}(1 + n_{Ko}) \right) \frac{\hbar^2}{2(k_B T_e)^2} \int d^3 p \int dW_f(\varepsilon) \Delta^2.
\]

The integrals in equation (2) can be explicitly computed (see appendix). As a consequence, Kompaneets [1] obtained the following equation for the photon distribution function \( n_{Ko}(x, t) \):

\[
\frac{\partial n_{Ko}}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n_{Ko}}{\partial x} + n_{Ko} + n_{Ko}^2 \right) \right].
\]

The terms \( n_{Ko} \) and \( n_{Ko}^2 \) in equation (3) correspond to the spontaneous scattering (Compton effect) and the induced scattering, respectively (see [3] and references therein). It is easy to check that in the stationary case \( \partial_t n_{Ko} = 0 \), the solution of equation (3) is the Planck distribution \( n_{Ko}(\hbar \omega) = 1/(\exp(\hbar \omega/k_B T_e) - 1) \).

In [2] Zel’dovich and Levich considered the convergence to equilibrium in a system without absorption and emission. Later they were interested in the expression of the photon distribution function in the case of large occupation numbers. They assumed that the occupation numbers are extremely large, so that \( n_{Ko} \gg 1 \) and \( n_{Ko}^2 \gg |\partial n_{Ko}/\partial x| \). These two approximations lead to simplification of equation (3). It reduces to the inviscid Burgers’ equation:

\[
\frac{\partial n_{ZL}}{\partial t} = \frac{1}{x^2} \frac{\partial}{\partial x} \left( x^4 n_{ZL}^2 \right),
\]

\[
x^2 n_{ZL}(x, 0) = F(x).
\]

Using equation (4), Zel’dovich and Levich [2] evidenced a drastic increase of the photon distribution function \( n_{ZL} \) at \( x = 0 \) in absence of absorption. They refer to this accumulation of photons in the vicinity of the zero energy, as a Bose condensation. This kinetic condensation depends on the form of the initial photon distribution \( F(x) \). For a specific form of initial distribution, a shock-wave profile for \( n_{ZL} \) as the photon energy function occurs. The form of \( n_{ZL} \) is extremely non-uniform across the frequency spectrum and is substantially affected by absorption if it is not negligible. Using the method of characteristics, Zel’dovich and Levich [2] found the following solution of equation (4):

\[
x = F^{-1}(x^2 n_{ZL}) - 2i x^2 n_{ZL}.
\]

Considering the Planck distribution \( n_{ZL}(x, 0) = (\exp(\mu x) - 1)^{-1} \), where \( \mu = T_e/ T_{ph} \), for the photons with temperature \( T_{ph} \) and using the approximation \( n_{ZL}(x, 0) \approx (\mu x)^{-1} \) for \( \mu x \ll 1 \), they have inferred the existence of a Bose condensation in the state \( x = 0 \) (see also [4–6]) for the minimal critical time

\[
\tilde{t}_c = \frac{\mu}{2}.
\]
However, in this special case of the initial Planck distribution the terms, \( n_x ZL \) and \( n ZL^2 \) are of the same order in \( x \) at \( t_0 = 0 \), and its ratio is of order \( \mu \). The inequality \( n_x ZL \gg \left| \frac{\partial n ZL}{\partial x} \right| \) is valid only for \( \tilde{t} \sim \mu/2 \).

For the laboratory plasma this minimal critical time \( \tilde{t}_c \) can be extremely long invalidating the assumption for deriving equation (4). Moreover, in the case of a weakly collisional plasma, the relaxation to a Maxwellian distribution for the electrons can also be extremely long.

In this paper we investigate the Kompaneets equation (1) by taking into account a perturbation of the Maxwellian distribution by plasma waves. We found an exact solution for the photon distribution function. From this solution we deduce a stability of the solution with respect to perturbation by the plasma waves in the long-wavelength limit.

2. Solution of equation (3) in the case of large occupation numbers
\( n_{Ko}(x, \tilde{t}) \gg 1 \)

In general the rescaled critical time \( \tilde{t}_c = \mu/2 \) (see equation (5)) is very large compared with the lifetime of hot and low density laboratory plasmas. For instance, for \( T = 10^9 \) K and \( \rho = 10^{13} \) cm\(^{-3} \), this characteristic time might of the order of two days [7]. It is worth noting that the critical time varies as \( 1/(\rho T) \). The hotter and the more dense the plasma is, the lower is the critical time. Here we are interested in the dynamics corresponding to equation (3) for \( \tilde{t} \lesssim \mu/2 \). The aim is to find a photon distribution function taking into account the interaction of photons with electrons via Compton scattering in the case when the Maxwellian distribution function for the electrons is perturbed by a solution of the linearized Vlasov equation in the long-wavelength limit \( \lambda \to \infty \).

Note that the photon temperature is related to the radiation spectrum of the plasma. Radiation in plasmas have various sources, e.g., cyclotron radiation, bremsstrahlung, etc. At high radiation frequency, i.e., at high photon temperature, a dominant mechanism is the recombination process in which electrons undergo a free-bound transition with the positive ions of the plasma.

![Figure 2.](image-url)
We consider a special case of the initial Planck distribution in the frequency region, which corresponds to large occupation numbers \( n_{Ko} \gg 1 \), and we set \( n_{Ko}(x, 0) = 1/(\exp(\mu x) - 1) \approx (\mu x)^{-1} \) for \( \mu x \ll 1 \). In this limit, which is known as the case with dominating induced scattering regime [3], equation (3) becomes

\[
\frac{\partial n_{Ko}}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n_{Ko}}{\partial x} + n_{Ko}^2 \right) \right], \quad \text{with } n_{Ko}(x, 0) = \frac{1}{\mu x}.
\]  

(6)

To find a solution of equation (6) we introduce the new variables

\[ z = -\ln x, \quad \tau = i, \quad \text{and} \quad \bar{n}(z, \tau) = x n_{Ko}(x(z), \tau(z)). \]

Then one gets for \( \bar{n}(z, \tau) \) the following nonlinear equation with constant coefficients:

\[
\frac{\partial \bar{n}}{\partial \tau} = \frac{\partial^2 \bar{n}}{\partial z^2} - 2\frac{\partial \bar{n}}{\partial z} + 2\bar{n}^2 - \frac{\partial \bar{n}}{\partial z} - 2\bar{n}, \quad \text{with } \bar{n}(z, 0) = 1/\mu.
\]  

(7)

We are looking for a solution of equation (7) in the form:

\[ \bar{n}(z, \tau) = \varphi(\tau) + \psi(\tau)\exp(z). \]  

(8)

Inserting this form of solution (8) into equation (7), we obtain

\[
\left( \varphi' - 2\varphi^2 + 2\varphi \right) = \left( -\psi' + 2\varphi\psi - 2\psi \right)\exp(z).
\]  

(9)

The left-hand side of equation (9) is a function of \( \tau \), contrary to the right-hand side, which is a function of two variables \( \tau \) and \( z \). Therefore the left- and the right-hand sides of equation (9) must be equal to zero. Consequently, we get two ordinary differential equations with initial conditions:

\[
\varphi' - 2\varphi^2 + 2\varphi = 0, \quad 0 \leq \tau < \mu/2, \quad \varphi(0) = 1/\mu,
\]  

(10)

and

\[
\psi' - 2\varphi\psi + 2\psi = 0, \quad 0 \leq \tau < \mu/2, \quad \psi(0) = 0.
\]  

(11)

The solutions of equations (10) and (11) are correspondingly

\[ \varphi(\tau) = \frac{1}{1 - (1 - \mu)\exp(2\tau)}, \quad \psi(\tau) = 0. \]

Consequently, the solution of equation (6) is

\[
n_{Ko}(x, i) = \frac{1}{x 1 - (1 - \mu)\exp(2i)}.
\]  

(12)

which coincides with the invariant solution (28) in [3]. This solution is valid for \( 0 < \mu x \ll 1 \), which implies that \( n_{Ko} \gg 1 \). Here we notice that \( \partial n_{Ko}/\partial x \) is of the same order as \( n_{Ko}^2 \). When \( \mu > 1 \) this solution does not have any singularity, whereas when \( \mu < 1 \), i.e., when \( T_{ph} > T_c \), there is a finite-time singularity at \( \hat{t}_c = -\ln \sqrt{1 - \mu} \). This time is always larger than the minimal critical time \( \hat{t}_c = \mu/2 \) [2]. When \( \mu = 1 \) one obtains the stationary solution of equation (6).
3. Photon distribution extended to the case of Vlasov plasma wave oscillations

In the previous section, it was assumed that the distribution of the electrons \( f_0 \) was Maxwellian. However if the collisionality is small, the relaxation to a Maxwellian distribution is extremely long. For collisionless plasmas, the evolution of the distribution is determined by the Vlasov equation. Here we restrict ourselves to the solutions of the linearized Vlasov equation in the long-wavelength (low-frequency) approximation for the plasma oscillations. Then the zeroth order approximation coincides with the space-homogeneous Maxwellian distribution \( f_v^0(x, \tilde{r}) \) and we consider a perturbation of this distribution. To this aim we denote by

\[
f(\mathbf{r}, \mathbf{v}, \tilde{r}) = f_0(\mathbf{v}) + \sigma f^*(\mathbf{r}, \mathbf{v}, \tilde{r}),
\]

the electron distribution function, where \( f^*(\mathbf{r}, \mathbf{v}, \tilde{r}) \) is a solution of the linearized Vlasov equation. Similarly the photon distribution function is given by

\[
n(x, \tilde{r}) = n_{K\omega}(x, \tilde{r}) + \sigma n^*(x, \tilde{r}).
\]

Here \( n_{K\omega}(x, \tilde{r}) \) is a non-stationary solution of equation (6) given by equation (12). We notice that in the long-wave limit the effective collisions due to photon-plasmon interactions are attenuated and we neglect them (see [8, 9] for more detail).

In a nutshell, our strategy is to obtain an equation for the perturbation \( n^*(x, \tilde{r}) \) by inserting the expressions (13) and (14) into equation (1). Then our goal is to solve this equation in order to estimate the impact of plasma wave oscillations on the background non-stationary photon distribution \( n_{K\omega}(x, \tilde{r}) \).

The linearized Vlasov–Poisson equation has the form [8, 10]

\[
\frac{\partial f^*}{\partial t} + \mathbf{v} \cdot \frac{\partial f^*}{\partial \mathbf{r}} - \frac{eE^*}{m} \cdot \frac{\partial f^*}{\partial \mathbf{v}} = 0,
\]

\[
\nabla \cdot \mathbf{E}^* = -4\pi e\rho \int f^*(\mathbf{r}, \mathbf{v}, t)d^3v,
\]

where

\[
f_0(\mathbf{v}) = \left(\frac{m}{2\pi k_BT_e}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_BT_e}\right),
\]

and \( \rho \) is the average electron density. We consider plasma oscillations propagating in the \( r_1 \)-direction, i.e.,

\[
f^*(\mathbf{r}, \mathbf{v}, t) = f_{K\omega}(\mathbf{v})\exp(-i\omega t + ikr_1),
\]

\[
E_{r_1}^*(\mathbf{r}, t) = E_{K\omega} \exp(-i\omega t + ikr_1),
\]

\[
E_{r_2} = E_{r_3} = 0.
\]

In particular we are interested in the long-wavelength limit \( k \to 0 \). Following Kvasnikov [8] we insert equation (16) into equation (15) to obtain the expressions for the amplitudes \( f_{K\omega} \):

\[
-i\omega f_{K\omega} + ik\nu f_{K\omega} - \frac{e}{m} \frac{\partial f^0}{\partial \nu_1} E_{K\omega} = -\varepsilon f_{K\omega} |_{\omega \to 0},
\]

\[
ike_{K\omega} = -4\pi e\rho \int f_{K\omega}(\mathbf{v})d^3v.
\]
Then from equation (17) follows:

\[
f_{k \omega} = \frac{e^2}{m} \frac{\partial f_0}{\partial v_1} \frac{1}{\omega + i \varepsilon - v_1 k} E_{k \omega}.
\]

(19)

Inserting equation (19) into equation (18), we obtain

\[
\frac{\partial f_0}{\partial v_i} \int_{-\infty}^{\infty} \frac{d^3 v}{\omega - v_i k + i \varepsilon} = 0.
\]

Excluding the trivial solution \( f_{k \omega} = E_{k \omega} = 0 \), which corresponds to the unperturbed case, one gets the dispersion relation \( k E = \omega m \) in the following form:

\[
1 = \frac{\omega_0^2}{\omega} \left( 1 + \frac{3 k_B T_e k^2}{m \omega_0^2} \right) + \frac{m}{k_B T_e} k^4 \left( \frac{\omega_0}{\omega} \right)^2 + i J(k, \omega) = 0,
\]

(20)

where \( \omega_0^2 = 4 \pi e^2 / m \). Calculating the integral in equation (20) we get:

\[
1 = \frac{\omega_0^2}{\omega} \left( 1 + \frac{3 k_B T_e k^2}{m \omega_0^2} \right) + \frac{m}{k_B T_e} k^4 \left( \frac{\omega_0}{\omega} \right)^2 - \frac{3}{2} \Omega^2 k^2 - \frac{1}{2} \gamma, \Omega \approx 1.
\]

(21)

where \( J(k, \omega) \) is

\[
J(k, \omega) = \frac{\pi}{2} \frac{\omega_0^2 m^{3/2}}{k^3 (k_B T_e)^{3/2}} \omega \exp \left( - \frac{m \omega^2}{2 k^2 k_B T_e} \right)
\]

and the bar in equation (21) denotes the mean value with respect to the Maxwellian distribution.

In the long-wavelength approximation, when \( k^2 \ll \frac{m}{k_B T_e} \omega^2 \), we neglect the term proportional to \( k^4 \) in equation (21). Since \( J(k, \omega) \neq 0 \), equation (21) has no real solution. We consider the solutions of equation (21) in the form \( \omega = \Omega - i \gamma \) where \( \gamma / k \ll 1 \). Then

\[
\frac{1}{\omega^2} = \frac{1}{\Omega^2} \approx \frac{1}{\Omega^2} + i \frac{2 \gamma}{\Omega}.
\]

(22)

From equation (22) it follows that

\[
1 = \frac{\omega_0^2}{\Omega^2} \left( 1 + \frac{3 k_B T_e k^2}{m \omega_0^2} \right) - \frac{1}{\Omega^2} - i J(k, \Omega) = 0.
\]

(23)

Considering the real and imaginary parts of equation (23) we finally obtain

\[
\Omega(k) = \omega_0 \left( 1 + \frac{3 k_B T_e k^2}{2 m \omega_0^2} + \cdots \right),
\]

(24)

\[
\gamma(k) \approx \frac{\omega_0^2}{2} J(k, \omega_0) = \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{\omega_0^3}{k^3 (k_B T_e)^{3/2}} \omega \exp \left( - \frac{m \omega^2}{2 k^2 k_B T_e} \right).
\]

(25)

where \( \gamma \) is the Landau damping, which vanishes in the long-wavelength limit \( k \rightarrow 0 \). We notice that for \( k = 0 \) the problem is much more complicated (see [9, 11]). From equation (24) it follows that the first approximation for \( \omega(k) \) is \( \omega(k) = \omega_0 \).
Without loss of generality we assume that $E_{0\omega} = E$. Taking into account equation (19) we get for $f_{0\omega}$ in the long-wavelength limit ($k \to 0$)

$$f_{0\omega} = \frac{eE}{m\omega_0 \partial v_1}.$$ 

In summary, we consider the following solution of the linearized Vlasov equation (15):

$$f^*(\mathbf{r}, \mathbf{v}, \tilde{t}) = \frac{m\omega_0}{eE} \sin \left( \omega_0 \alpha^{-1} \tilde{t} \right) v_1 f_0^{(v)} = \theta(\tilde{t}) v_1 f_0^{(v)},$$

(26)

where $\omega_0 = \sqrt{4\pi e^2 \rho / m}$ is the Langmuir frequency.

Next we insert $n(x, \tilde{t})$ given by equation (14) and $f(\mathbf{r}, \mathbf{v}, t)$ given by equation (13) with $f^*(\mathbf{r}, \mathbf{v}, \tilde{t})$ obtained in equation (26) in equation (1), to get an equation for the unknown function $n^*(x, \tilde{t})$:

$$\frac{\partial n^*}{\partial \tilde{t}} = \alpha^{-1} \left[ \gamma_2 \frac{\partial^2 n^*}{\partial x^2} + \left( \gamma_1 + 2\gamma_2 (n_{K0} + 1) \right) \frac{\partial n^*}{\partial x} \right. \right.$$

$$+ \left. \left. \left( \gamma_1 (1 + 2n_{K0}) + \gamma_2 \left( 1 + 2n_{K0} + 2 \frac{\partial n_{K0}}{\partial x} \right) \right) n^* + \Phi(x, \tilde{t}) \right],$$

(27)

where (see appendix):

$$\gamma_1 = \frac{\hbar}{k_B T_e} \int d^3v \int dW_0^{(v)} \Delta = \alpha x (4 - x),$$

$$\gamma_2 = \frac{1}{2} \left( \frac{\hbar}{k_B T_e} \right)^2 \int d^3v \int dW_0^{(v)} \Delta^2 = \alpha x^2,$$

$$\alpha^{-1} \Phi(x, \tilde{t}) = \left( \frac{\partial n_{K0}}{\partial x} + n_{K0} (1 + n_{K0}) \right) \frac{h^2}{k_B T_e} \left[ h \int d^3p \int dW_0^{(\tilde{t})} v_1 f_0^{(v)} \Delta \right]$$

$$- \left( \frac{\partial n_{K0}}{\partial x} + n_{K0} (1 + n_{K0}) \right) \int \Delta \sin \left( \omega_0 \alpha^{-1} \tilde{t} \right).$$

(28)

Here $\beta = (eE \alpha)/(2\omega_0 k_B T_e)$. Plugging in equation (27) the expressions for $\gamma_1, \gamma_2$ and $\Phi$, and keeping only the dominant terms when $\mu x \ll 1$ and $\tilde{t} < \mu/2$, one gets for $n^*(x, \tilde{t})$ the following equation:

$$\frac{\partial n^*}{\partial \tilde{t}} = \alpha^2 + \alpha x \left( 4 + \frac{2a}{1 - 2(a - 1)\tilde{t}} \right) \frac{\partial n^*}{\partial x} + \frac{6a}{1 - 2(a - 1)\tilde{t}} n^*$$

$$+ \frac{a(a - 1)\beta \mu}{x} \frac{1 + 2\tilde{t}}{1 - 2(a - 1)\tilde{t}} \sin \left( \omega_0 \alpha^{-1} \tilde{t} \right),$$

where $a = 1/\mu$ and with $n^*(x, 0) = 0$. The changes of variables, valid for $1 - 2(a - 1)\tilde{t} > 0$,

$$\eta = \tilde{t},$$

$$\xi = - \ln x + \frac{a}{a - 1} \ln \left[ 1 - 2(a - 1)\tilde{t} \right],$$

$$u(x, \eta) = n^*(\xi, \eta) \left[ 1 - 2(a - 1)\eta \right]^{3a/(a - 1)},$$

(29)
lead to equation for \( u(\xi, \eta) \) with constant coefficients:

\[
\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \xi^2} - 3 \frac{\partial u}{\partial \xi} + a(a - 1) \beta \exp(\xi) \frac{1 + 2 \eta}{(1 - 2(a - 1) \eta)^{1/4}} \sin(\omega_0 \alpha^{-1} \eta),
\]

\( u(\xi, 0) = 0, -\infty < \xi < \infty, \eta > 0. \)

The second change of variables is a well-known trick to eliminate the first derivative:

\[
uu u a\exp \left( \frac{3}{2} \xi - \frac{9}{4} \eta \right) u(\xi, \eta). \]

The equation for \( u(\xi, \eta) \) is then

\[
\frac{\partial u^*}{\partial \eta} = \frac{\partial^2 u^*}{\partial \xi^2} + a(a - 1) \beta \exp \left( -\frac{\xi}{2} + \frac{9 \eta}{4} \right) \frac{1 + 2 \eta}{(1 - 2(a - 1) \eta)^{1/4}} \sin(\omega_0 \alpha^{-1} \eta),
\]

\( u^*(\xi, 0) = 0, -\infty < \xi < \infty, \eta > 0. \) (30)

Using the Fourier transformation one finds the solution of the non-homogeneous linear partial differential equation (30) with constant coefficients:

\[
uu uu a\exp \left( \frac{3}{2} \xi - \frac{9}{4} \eta \right) u(\xi, \eta). \]

\[
\frac{\partial u^*}{\partial \eta} = \frac{\partial^2 u^*}{\partial \xi^2} + a(a - 1) \beta \exp \left( -\frac{\xi}{2} + \frac{9 \eta}{4} \right) \frac{1 + 2 \eta}{(1 - 2(a - 1) \eta)^{1/4}} \sin(\omega_0 \alpha^{-1} \eta),
\]

\( u^*(\xi, 0) = 0, -\infty < \xi < \infty, \eta > 0. \)

Finally, from equation (29) it follows the expression for the photon distribution function \( n^*(x, \tilde{t}) \), which takes into account the effect of the Vlasov plasma wave oscillations:

\[
n^*(x, \tilde{t}) = a \alpha (a - 1) \beta \frac{e^{-2i} \eta}{(1 - 2(a - 1) \tilde{t})^{1/2}} I(\tilde{t}, a).
\]

Then equation (14) leads to

\[
n(x, \tilde{t}) = n_{Ko}(x, \tilde{t}) + \sigma n^*(x, \tilde{t})
\]

\[
= a \frac{1}{x - 1 - 2(a - 1) \tilde{t}} \left[ 1 + \sigma(a - 1) \beta \frac{e^{-2i} \eta}{(1 - 2(a - 1) \tilde{t})^{1/2}} I(\tilde{t}, a) \right].
\]

The integral \( I(\tilde{t}, a) \) in equation (32) can be computed using the incomplete Gamma function [12]. However the explicit expression of these integrals are not very useful for the analysis. In what follows, we discuss the two main cases \( \mu > 1 \) and \( \mu < 1 \) using a combination of numerical computation and series expansions of the integrals.

- If \( \mu < 1 \), i.e., \( T_{ph} > T_e \), the Bose condensation is amplified by the perturbation of the Maxwellian distribution since the exponent \( (a + 1)/(a - 1) \) is positive. Therefore, the
solution for \( n_{Ko}(x, \tilde{t}) \) is unstable. On figure 1, we represent the computation of \( n^g(x, \tilde{t}) \) given by equation (31) for \( \mu = 1/2 \) and \( \omega_0 \alpha^{-1} = 100 \). It is clear that there is a divergence of the perturbed number of photons when \( \tilde{t} \) approaches 1/2. Note that this threshold is different from the one in equation (5). For \( \mu = 1/2 \), a series expansion of the integral given by equation (32) can be obtained when \( \omega_0 \alpha^{-1} \) is large:

\[
I(\tilde{t}, 2) = \frac{1}{\omega_0 \alpha^{-1}} - \frac{\cos(\omega_0 \alpha^{-1} \tilde{t})}{\omega_0 \alpha^{-1}} e^{2(1 - 2\tilde{t})(1 + 2\tilde{t})} + O\left(\frac{1}{(\omega_0 \alpha^{-1})^2}\right),
\]

which leads to an approximation of \( n^g(x, \tilde{t}) \):

\[
n^g(x, \tilde{t}) = \frac{2a}{x} \frac{1}{\omega_0 \alpha^{-1}} \left[ e^{-2\tilde{t}} \left( 1 - 2\tilde{t} \right)^4 - \frac{(1 + 2\tilde{t})}{(1 - 2\tilde{t})^2} \cos(\omega_0 \alpha^{-1} \tilde{t}) \right] + O\left(\frac{1}{(\omega_0 \alpha^{-1})^2}\right).
\]

It should be noted that this expression is a very good approximation of \( n^g(x, \tilde{t}) \) in a wide range of values of \( \omega_0 \alpha^{-1} \). From this expression, we see that initially, the dominant contribution is caused by the plasma wave oscillations. In this regime, we argue that from the measurement of the photon distribution \( n(x, \tilde{t}) \), one can have access to the plasma density variations. Near the singularity, the dominant term is proportional to \( (1 - 2\tilde{t})^{-4} \), and this divergence of \( n^g(x, \tilde{t}) \) is much faster than the unperturbed distribution \( n_{Ko}(x, \tilde{t}) \).

- For \( \mu > 1 \), which corresponds to the case \( T_e > T_{ph} \), the resulting perturbation does not have any singularity. So the solution for the photon distribution is rather stable with respect to perturbations of the Maxwellian distribution for the electrons (see figure 2). It should be noted that the values of \( \sigma \) are restricted to the case, when \( \sigma n^g \) is much smaller than \( n_{Ko} \). So \( \sigma \) has be chosen such that \( \sigma \ll \frac{1}{\beta} \max_{\tilde{t} \in [0, 1/2]} \left[ \left\{ \left[ 1 - 2(\tilde{t} - 1) \right]^{1+\mu/1+\mu} \left[ 1 + \frac{a}{1-\tilde{t}} \right]^{1+\mu/1+\mu} \right\} \right] \).
- If \( \mu = 1 \), i.e., \( T_e = T_{ph} \), the Vlasov plasma wave oscillations have no effect on the distribution of photons. It corresponds to a stationary case.

4. Conclusion

We considered some modifications of the Kompaneets equation for the photon distribution function when plasma wave oscillations perturb the distribution of the electrons. We have shown that the solution of the modified Kompaneets equation in the limit of large occupation numbers \( n(x, \tilde{t}) \gg 1 \) can be unstable for sufficiently large times around the zero energy \( x = 0 \). In the unstable case, Bose condensation occurs. This instability is generated by the diffusion term in the equation for the photon distribution.

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Appendix. Computation of the integrals of equation (2)

Following Kompaneets [1] we calculate the coefficients

\[
\gamma_1 = \frac{\hbar}{k_B T_e} \int d^3v \int dW_{\Omega}(v) \Delta,
\]

\[
\gamma_2 = \frac{1}{2} \left( \frac{\hbar}{k_B T_e} \right)^2 \int d^3v \int dW_{\Omega}(v) \Delta^2.
\]

The laws of momentum and energy conservation in the non-relativistic case are written as:

\[
\frac{\hbar \omega}{c} \mathbf{n} + \mathbf{p} = \frac{\hbar \omega}{c} \mathbf{n'} + \mathbf{p'},
\]

\[
\hbar \omega + \frac{p^2}{2m} = \hbar \omega' + \frac{p'^2}{2m},
\]

where \( \varepsilon = \frac{p^2}{2m} \), and \( \mathbf{p} \) and \( \mathbf{p}' \) are the momenta of the electron before and after the collision, \( \mathbf{n} \) and \( \mathbf{n}' \) are unit vectors in the propagation direction of the photon. Eliminating \( \mathbf{p}' \) from these equations, we obtain an equation which determines \( \mathbf{n}' \) as a function of \( \omega', \mathbf{p} \) and the scattering angles. Taking only into account the linear order in \( \Delta \) one gets

\[
\hbar = - \frac{\hbar \omega}{mc^2} \mathbf{p} \cdot (\mathbf{n} - \mathbf{n'}).
\]  

(A.1)

Taking into account the Kompaneets approximation \( kT \ll mc^2 \), we estimate the terms in the denominator of equation (A.1). Since \( \hbar \omega \sim 10^4 \text{eV} \) and \( mc^2 = 0.5 \times 10^6 \text{eV} \), one gets \( \hbar \omega / mc^2 \ll 1 \). Consequently, \( \hbar \omega(1 - \mathbf{n} \cdot \mathbf{n'})/(mc^2) < 2 \hbar \omega/(mc^2) \ll 1 \) and \( |\mathbf{p} \cdot \mathbf{n'}/(mc)| < pc/(mc^2) = c \sqrt{2mE_{\text{kin}}/(mc^2)} \approx \sqrt{3k_B T_e/(mc^2)} \ll 1 \). The first term in the numerator is of order \( (k_B T_e/mc^2)^{3/2} \) whereas the second term is of order \( (k_B T_e/mc^2)^2 \). Then from equation (A.1) it follows that

\[
\hbar \Delta = - \frac{\hbar \omega}{mc} \mathbf{p} \cdot (\mathbf{n} - \mathbf{n'}). \tag{A.2}
\]

Now we calculate \( \gamma_2 \) using spherical coordinates:

\[
\gamma_2 = \frac{1}{2} \left( \frac{\hbar}{k_B T_e} \right)^2 \int d^3v \int dW_{\Omega}(v) \Delta^2
\]

\[
= \frac{\pi}{(k_B T_e)^2} \left( \frac{\hbar \omega}{mc} \right)^2 \int_0^\infty dp \frac{1}{\left(2 \pi m k_B T_e \right)^{3/2}} \exp \left(- \frac{p^2}{2 mk_B T_e} \right)
\]

\[
\times \int_0^\pi d\theta \cos^2 \theta \sin \theta \int_0^{2\pi} dW |\mathbf{n} - \mathbf{n'}|^2. \tag{A.3}
\]

We replace the Compton cross section in the non relativistic approximation by the Thomson cross section which is symmetric relative to scattering over the angles \( \theta \) and \( \pi - \theta \); so that \( \int_0^\pi dW \mathbf{n} \cdot \mathbf{n'} = 0 \). Here \( dW \) is the differential transition probability, compatible with the energy and momentum conservation laws. The Thomson cross section does not depend on the energy of the photons. Consequently
\[ \gamma_2 = \left( \frac{\hbar \omega}{k_B T_e} \right)^2 \frac{k_B T_e}{mc^2} l = \alpha x^2, \]  
(A.4)

where \( l \) is the Compton range, which is determined by the total cross section \( \frac{8\pi}{3} \left( \frac{e^2}{4\pi\varepsilon_0 mc^2} \right)^2 \).

For the expression for \( \gamma_1 \), we rewrite the kinetic equation for the photons in a conservative form:

\[ \left( \frac{\partial n}{\partial t} \right)_c = -\frac{1}{x^2} \frac{\partial}{\partial x} \left( x^2 J(x, t) \right), \]  
(A.5)

where \( J(x, t) \) is the ‘flow’ of photons in frequency space. It is easy to show that in the state of total equilibrium, i.e., when \( \left( \frac{\partial n}{\partial t} \right)_c = 0 \), the solution of equation (1) is \( \bar{n} = (\exp(x) - 1)^{-1} \), and satisfies \( \partial \bar{n}/\partial x = -\bar{n}(1 + \bar{n}) \). Let us find \( J(x, t) \) in the following form:

\[ J(x, t) = g(x)\left[ \partial \bar{n}/\partial x + n(n + 1) \right], \]

where the function \( g(x) \) must be determined and \( n(x, t) \) is the solution of equation (2). Let \( g(x) = -x^2 \). Substituting equation (A.5) in equation (2) leads to:

\[ \left( \frac{\partial n}{\partial t} \right)_c = \alpha \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n}{\partial x} + n(n + 1) \right) \right], \]

which resembles equation (2) with the right coefficient \( \gamma_2 \). The identification leads to \( \gamma_1 = \alpha x(4 - x) \).

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