Non-local Field Theories and the Non-commutative Torus

Micha Berkooz
Institute for Advanced Study
Princeton, NJ 08540.
and
Institute for Theoretical Physics
University of California
Santa-Barbara, CA, 93106.
berkooz@ias.edu

We argue that by taking a limit of SYM on a non-commutative torus one can obtain a theory on non-compact space with a finite non-locality scale. We also suggest that one can also obtain a similar generalization of the (2,0) field theory in 5+1 dimensions, and that the DLCQ of this theory is known.
1. Introduction

Among the spin-offs of Matrix theory \cite{1} are new kinds, or new formulations, of field theories (or other theories that may not have standard gravity in them \cite{2,3,4,5,6,7,8}). An interesting class of such theories arises in the Matrix description of M-theory compactified on a torus with a non-zero 3-form Wilson line \cite{10,11}. The resulting theories on the world-volume of the Matrix description are Super-Yang-Mills on a non-commuting manifold \cite{12}.

Beyond their application for Matrix theory, these theories are interesting in their own right. One reason is that one loses the notion of exact locality of the theory and it only reappears, in some cases, as an approximate notion for the low-energy (or low wave-length) observer. In particular, the theory is no longer scale invariant in the UV. This is in contrast with more conventional field theories. One of the elements that make “standard” field theory work is the existence of a UV fixed point. Using this fixed point one obtains a theory in the continuum. The theory on the non-commuting torus seems to be defined in the continuum by some other means. Returning to the low energy observer, such an observer may see a renormalization group trajectory, if such a notion is still useful, coming from an unexpected region of field theory parameter space.

In the following we discuss some of the above statements. We argue that it is possible to disentangle the UV from the IR by taking a limit such that the size of the torus goes to infinity, keeping the size of the non-locality fixed. This configuration, as the one in \cite{11} also arises within string theory. To better understand the UV of the theory we note that by a similar procedure we can obtain a non-local version of the (2,0) field theory \cite{13}. A Matrix description of the latter theory exists and may be more useful in the study of the UV of these theories.

As this work was completed we were informed that N. Nekrasov and A. Schwarz have obtained related results \cite{14}.

2. A Limit of Super-Yang-Mills on the Non-commutative 2-Torus

Our goal in this section is to obtain a deformation of SYM on 2+1 non-compact dimensions such that it is characterized by a finite non-locality length. Operationally, we will be interested in a deformation of the SYM theory of the form (x,y,z being appropriate 3-vectors)

\[
\int_{T^2 \times \mathbb{R}} d^3x \phi_1(x) e^{i \frac{\phi_1}{\phi_2} \frac{\partial}{\partial x} \phi_2(y) \phi_3(z)}|_{y=z=x} \tag{2.1}
\]
where $\phi_i$ are some fields in the SYM. The requirement of a finite non-locality length is that as we take the size of the spatial torus to infinity, $\zeta$ (of mass dimension -2) remains finite. It then characterizes a finite non-locality length in the theory on $R^2$. A low-energy observer (below the scale set by $\zeta$) may consider the expansion in $\zeta$ as an expansion in momenta of some effective action.

To obtain such a theory, one uses a variant of the procedure used by Douglas and Hull. Throughout the discussion we will set $\alpha' = 1$. These authors considered a compactification of the IIA string on a torus with

$$ G + B = \begin{pmatrix} R_1^2 & B \\ -B & R_2^2 \end{pmatrix} $$

as $R_1 = R_2 \to 0$ and $B$ held fixed. We will be interested in

$$ G + B = \begin{pmatrix} R_1^2 & \eta R_1 R_2 \\ -\eta R_1 R_2 & R_2^2 \end{pmatrix} \tag{2.2} $$

in the limit in which $R_1 = R_2 = R \to 0$ and $\eta$ is held fixed. In this limit one keeps the value of the B field fixed (in physical coordinates, where $G = I_{2\times2}$) rather than the flux of the B field through the 2-torus.

When we now perform a T-duality in both directions we obtain that the new $G + B$ is

$$ G + B = \frac{1}{R_1^2 R_2^2 (1 + \eta^2)} \begin{pmatrix} R_2^2 & -\eta R_1 R_2 \\ -\eta R_1 R_2 & R_1^2 \end{pmatrix}. \tag{2.3} $$

The T-dual radii go to infinity but so does the $\int_{T^2} B$. In particular, we need to perform different $SL(2, Z)$ transformations in order to map this configuration into the fundamental domain. These $SL(2, Z)$ transformations are weaker than the ones in [11] as they involve only shifts in $\int B$. The trajectory is therefore not ergodic. Still, the $SL(2, Z)$ transformations that are needed are rapidly varying when $R \to 0$, and it is not clear how to take the limit of the entire string theory.

A somewhat clearer picture appears if we continue with the rest of the analysis in [11] (T-dualizing a single circle etc.), adapted to this case. Inserting the current value of $B$ into the action there (equation (2)) one obtains a non-local term of the form,

$$ S_{int} \propto \int_{0<\sigma^1<1} d\sigma^1 d\sigma^2 \phi_1(\sigma^1, \sigma^2) exp\left(\eta R_1 R_2 \frac{\partial}{\partial \sigma^1} \frac{\partial}{\partial \sigma^2} \phi_2(\sigma^1, \sigma^2) \phi_3(\sigma^1', \sigma^2')|_{\sigma^i' = \sigma^i} \right) $$

We can now go back to the physical coordinates $x^i = \frac{\sigma^i}{R_i}$ on the dual torus and obtain a finite non-locality size, in the form of (2.1), while the size of the torus is taken to infinity.
3. Relation to the Non-local (2,0) field theory

3.1. The Limit Procedure for the D4-brane

Let us repeat the same procedure as before but for the D4-brane. For convenience let us divide the 4-torus into a product of 2-tori and turn on a B field separately in each of these. We can now copy the result of the previous section, and obtain a 4-brane on a torus of radii $R_1, \ldots, R_4$ which go to infinity and a finite B field, inducing a finite non-locality length. We will again parameterize the non-locality by a parameter of $\zeta$ of mass dimension $-2$. $\zeta$ is related to $B$ (in the frame where $G = I_{4 \times 4}$) by

$$B = \frac{\zeta}{\alpha'^2}.$$ 

We are interested in lifting this configuration to M-theory. The background $B$ field now becomes a background $C_3$ field, given by

$$B = R_{11} C,$$

and the D4-brane is now to be interpreted as a wrapped M5-brane. As discussed in [15], even though the $C$ field is constant, it is not pure gauge. This is so because of the M5-brane. There is now a gauge invariant quantity, $H - C$, which is not zero in our case. Phrasing it in another way, we can gauge away $C$ and generate a non-zero $H$ field. This is a field strength and is not pure gauge. Using the above relations one finally obtains

$$\zeta = \frac{C}{R_{11} M_p^6}. \quad (3.1)$$

A DLCQ of a non-local deformation of the (2,0) field theory with exactly this non-locality scale was proposed in [15]. One therefore conjectured that the non-local deformation of the (2,0) theory in 5+1 dimensions in non-compact space is a well-defined theory. There is also a concrete quantum mechanical system that describes the DLCQ of this theory.

Upon compactification, this theory becomes the natural definition of the non-local deformation of 4+1 SYM (or 4+1 SYM on the noncommutative torus). By appropriately compactifying the theory one expects to obtain similar non-local SYM on lower dimensional $R^d$. It may be that deformations of the models described in [3] are Matrix models of these SYM.

In the following subsection we will review some facts about the Matrix description of the non-local deformation of the (2,0) field theory. The discussion will include only basic facts that will be useful for the a subsequent brief discussion. For a more complete picture the reader is referred to [15].

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1 The choice of $C$ there was actually more restrictive. This will be discussed later.
3.2. The Non-local \((2,0)\) field theory

Part of the analysis in this subsection, as well as part of the following section, is extracted from [15]. The reader interested in more details and in a more complete picture is referred there. It is brought here for completeness, and in order to set up some details that will be necessary in the discussion later.

The Matrix description of the DLCQ [14,17,18] of the \((2,0)\) field theory, at \(P\)-momentum sector \(N\), is given by quantum mechanics on the moduli space of N-instantons in \(U(k)\) on \(R^4\) [4]. This manifold is described in the ADHM [14] construction by the following HyperKähler quotient.

The initial space is a linear representation of \(U(N)\). It is given by 2 complex \(N*N\) matrices, \(X\) and \(\tilde{X}\), and by two complex \(N*K\) matrices \(Q\) and \(\tilde{Q}\). This space admits an action of \(U(N)\) by

\[
X \rightarrow gXg^{-1}, \quad \tilde{X} \rightarrow g\tilde{X}g^{-1}, \quad Q \rightarrow gQ, \quad \tilde{Q} \rightarrow g^{-1}\tilde{Q}.
\]

The hyperKähler quotient is then given by the

\[
\left(\begin{array}{c}
[X, X^\dagger] - [\tilde{X}, \tilde{X}^\dagger] + QQ^\dagger - \tilde{Q}^*\tilde{Q}^T = 0 \\
[X, \tilde{X}] + Q\tilde{Q}^T = 0
\end{array}\right)/U(N)
\]

Locality in spacetime manifests itself in the quantum mechanics in the following way. The theory in spacetime is in a superconformal fixed point. When we go to the DLCQ the light-like circle breaks most of the conformal invariance. It leaves however an SO(1,2) subgroup, which becomes the conformal symmetry of the quantum mechanics. One of the generators of the latter, which we will denote by \(T\), can be identified as \(T = D + M_{01}\), where \(D\) is the dilatation operator in spacetime and \(M_{01}\) is the generator of Lorentz transformations in the \(X^+ - X^-\) plane. \(T\) acts in the quantum mechanics by contracting the entire sigma model to the origin.

One way to see that the theory is local is the following. Take a state in the quantum mechanics, i.e. some normalizeble wave function on the quantum mechanics, and act on it by \(T\). The support of the wave function shrinks to the origin. As \(T\) contains \(D\) in it, the spacetime interpretation of this operation is that we take some state in spacetime and shrink it to the point. We therefore obtain a local state, or local operator, in spacetime.

The fact that we were not obstructed in this process implies that the theory has local objects in it.
The analysis in [15] included a resolution of the singularities in the space. The resolved space [20] is given by

\[
\left( [X, X^\dagger] - [\tilde{X}, \tilde{X}^\dagger] + QQ^\dagger - \tilde{Q}^* \tilde{Q}^T = \zeta I_{N \times N} \right) / U(N). \tag{3.4}
\]

In [15], this resolution was used primarily to analyze the \((2, 0)_k\) field theory, but it was also noted that the theory is interesting in its own right. The theory is now non-local and does not possess scale invariance. Now we cannot shrink the support of the function to the origin, since the origin is no longer in the manifold. It has been replaced by some manifold of size \(\zeta\). On scales larger the \(\zeta\) we can approximately shrink the support of the function but this procedure fails when we reach the scale set by \(\zeta\). The implications in spacetime are that on larger length scales the theory will appear as a deformation of a local theory, but we will begin to feel the non-locality of the full theory at a scale set by \(\zeta\). This is exactly the situation that we have for SYM on a noncommuting manifold.

Furthermore, thinking about the deformed \((2, 0)_k\) theory as the theory on the M5-brane, it is shown in [15] that the deformation is given by turning on a certain \(C\) field, and that the scale of non-locality is given by

\[
\zeta = \frac{C_{+ij}}{RM^6_p} \tag{3.5}
\]

which is the same relation that we had before.

One more issue remains to be discussed. The world-volume of the D4-brane has an \(SO(4) \sim SU(2)_R \times SU(2)_L\) symmetry of spatial rotation. In the previous subsection, it seemed that we were allowed to turn on a background \(B\) field with arbitrary \(SO(4)\) chirality. That is not the case with the \(\zeta\) parameter in [15]. \(SU(2)_R\) is an R symmetry of the QM which rotates the different constraint equations in the ADHM construction and \(\zeta\) takes value in the \((3, 1)\) of \(SO(4)\). The reason for the restriction is not a fundamental one in the theory, rather it is an artifact of the DLCQ description. We have chosen \(P_- > 0\) momenta in a certain lightlike direction and we define what we call 0-branes accordingly. In the sector with \(P_- > 0\), turning on the \(H\) field with the wrong \(SO(4)\) chirality will break supersymmetry. We have, however, restricted ourselves to deformation that preserve supersymmetry, and therefore we can not describe turning on all the \(SO(4)\) chiralities of \(C\).
4. A Comment on the UV of the Non-local Theories

Some brief comments are due:
1. The fact that we have obtained a Matrix description of the non-local \((2, 0)_{k}\) field theory, and perhaps of SYM on a non-commuting lower dimensional manifold, lends further support to the conjecture that these theories exist as consistent quantum mechanical theories.
2. Although we can not localize a state at a region smaller than \(\zeta\), the theory is still defined in the continuum. If we take a bounded region in space, of some size much larger than \(\zeta\), then the number of states in this region is infinite. To roughly see how many states one has in a bounded region in space, we can focus, in the quantum mechanics, on wave functions that have support in some bounded, much larger than \(\zeta\), region around the origin. In the resolved sigma model the number of such states is infinite.
3. A more interesting point is that we can now also discuss the UV of the theory. The simplest case to analyze is the K=1 case. The theory in the IR contains a free tensor multiplet and we can now probe the UV by a high momentum exchange scattering process.

For concreteness let us discuss the scattering process of two particles carrying each one unit of momentum along the null circle (for the K=1 case, \([21]\)). We are therefore interested in the N=2 moduli space which is \(R^4 \times R^4 / Z_2\). When we resolve the singularity it is replaced by \(R^4 \times ALE\). The first component is the center of mass of the two particles and the second is their relative position. A 2-particle scattering process is described by sending a wave packet on the ALE from infinity towards the origin. The rough picture will be that in momentum scales \(P\) much smaller than the one set by \(\zeta\), the theory will have an expansion in \(\zeta P^2\). At high energy, however, the wave packet will explore the resolved sphere, which suggests a qualitative difference in the result, beyond an expansion by \(\zeta P^2\). One also doubts whether the deformed interaction \(e^{\zeta \partial^2}\) captures the large \(P\) behavior correctly.

This picture suggests that even though the theory is defined in the continuum, the UV is not governed by any scale invariance. In fact, the resolved region of the sigma model, i.e., the one that high energy processes probe, is the one which is the most sensitive to the violation of scale invariance by \(\zeta\). The principal that should replace the UV fixed point is not clear.
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