Dyson’s Brownian-motion model for random matrix theory - revisited

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Abstract. We offer an alternative viewpoint on Dyson’s original paper regarding the application of Brownian motion to random matrix theory (RMT). In particular we show how one may use the same approach in order to study the stochastic motion in the space of matrix traces
\[ t_n = \sum_{\nu=1}^{N} \lambda_{\nu}^{n}, \]
rather than the eigenvalues \( \lambda_{\nu} \). In complete analogy with Dyson we obtain a Fokker-Planck equation that exhibits a stationary solution corresponding to the joint probability density function in the space \( t = (t_1, \ldots, t_n) \), which can in turn be related to the eigenvalues \( \lambda = (\lambda_1, \ldots, \lambda_N) \). As a consequence two interesting combinatorial identities emerge, which are proved algebraically in the appendix. We also offer a number of comments on this version of Dyson’s theory and discuss its potential advantages.

1. Introduction

In his seminal 1962 paper, *A Brownian-Motion Model for the Eigenvalues of a Random Matrix* [1], F. Dyson provided a conceptually novel and practical approach to the theory of random matrices, paving the way for many interesting developments (see e.g. [2–8] and references cited therein.) In it he explains how to introduce a dynamical approach to the theory of random matrices and the traditional Gaussian ensembles in particular. We briefly recapitulate the results here in this introductory section.

Consider a self-adjoint matrix \( M \) of size \( N \times N \), whose entries are of the form
\[ M_{ij} = \sum_{\alpha=0}^{\beta-1} M_{ij;\alpha} e_{\alpha}. \]
The coefficients \( M_{ij;\alpha} \) being real parameters and \( e_{\alpha} \) are the units of the three potential algebras: real (\( \beta = 1 \)), complex (\( \beta = 2 \)) and real-quaternion (\( \beta = 4 \)), satisfying \( e_{\alpha}^2 = 1 \) and \( e_{\alpha}^2 = -1 \) \( \forall \alpha > 0 \). Choosing the real coefficients \( M_{ij;\alpha} \) independently from a Gaussian distribution with zero mean and variance \( \mathbf{E}(M_{ij;\alpha}^2) = (1 + \delta_{ij})/(2\beta) \) we obtain the Gaussian orthogonal, unitary and symplectic ensembles (GOE, GUE and GSE) for \( \beta = 1, 2 \) and 4 respectively. Thus the probability distribution for the matrix \( M \) may be neatly summarised in the following form
\[ P(M) = \kappa^{(N)}_{\beta} e^{-\frac{\beta}{2} \text{tr} MM^\dagger}, \]
with \( \kappa^{(N)}_{\beta} \) a normalization constant.
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Crucially, Dyson realised that the above distribution can be identified as the stationary distribution of a Brownian particle in $N + \beta N(N − 1)/2$ dimensions. More precisely, this means that each independent element $M_{ij,\alpha}$, $1 \leq i \leq j \leq N$, undergoes a 1D Ornstein-Uhlenbeck process, so that in the (fictitious) time $s$ the motion of $M_{ij,\alpha}$ is completely determined by the following moments:

$$E(\delta M_{ij,\alpha}) = -M_{ij,\alpha}\delta s$$  \hspace{1cm} (1.2)

$$E(\delta^2 M_{ij,\alpha}^2) = \frac{1}{\beta}(1 + \delta_{ij})\delta s.$$  \hspace{1cm} (1.3)

The latter implies that $E(|\delta M_{ij}|^2) = (1 + (2/\beta - 1)\delta_{ij})\delta s$ (since the diagonal elements $M_{ii}$ are always real).

Importantly, this stochastic motion is invariant under unitary transformations, meaning the eigenvectors do not play any role in the corresponding motion induced in the $N$ dimensional space of eigenvalues $\lambda = (\lambda_1, \cdots, \lambda_N)$. Therefore one may choose a representation in which $M$ is diagonal, leading to a perturbation of the eigenvalue $\lambda_\mu$ due to a small change in the matrix $\delta M$ of

$$\delta \lambda_\mu = \delta M_{\mu\mu,0} + \sum_{\nu \neq \mu} \frac{|\delta M_{\mu\nu}|^2}{\lambda_\mu - \lambda_\nu}.$$  \hspace{1cm} (1.4)

Obtaining the first two moments of the evolution then follows directly from the expressions (1.2) and (1.3), given by

$$E(\delta \lambda_\mu) = F_\mu(\lambda)\delta s = \left[ \sum_{\nu \neq \mu} \frac{1}{\lambda_\nu - \lambda_\mu} - \lambda_\mu \right] \delta s$$  \hspace{1cm} (1.5)

$$E(\delta^2 \lambda_\mu^2) = \frac{2}{\beta}\delta s.$$  \hspace{1cm} (1.6)

Using these two moments, one obtains a Fokker-Planck equation that describes how the joint probability distribution function (JPDF) $P(\lambda; s)$ evolves in time, given some specific initial distribution $P(\lambda; 0)$:

$$\frac{\partial P}{\partial s} = \sum_{\mu=1}^{N} \left[ -\frac{\partial}{\partial \lambda_\mu}(F_\mu(\lambda)P(\lambda; s)) + \beta^{-1}\frac{\partial^2 P(\lambda; s)}{\partial \lambda_\mu^2} \right].$$  \hspace{1cm} (1.7)

The real advantage, and one might add elegance, of this approach is expressed in the above equation. In general it is not known how to obtain $P(\lambda; s)$ for arbitrary initial conditions and times $s$. However, since we are interested in the stationary distribution, we can reduce the complexity by setting the LHS equal to zero, at which point one solves the equation easily to obtain

$$P(\lambda) = C_\beta^{(N)} \prod_{\mu<\nu} |\lambda_\mu - \lambda_\nu|^\beta \exp \left( -\frac{\beta}{2} \sum_\mu \lambda_\mu^2 \right),$$  \hspace{1cm} (1.8)

with $C_\beta^{(N)}$ a normalisation constant (see e.g. Chapter 3 of [2]). Moreover, since we know that the underlying motion (1.2) and (1.3) in the space of matrices leads to the probability distribution (1.1), the expression (1.8) must be the unique stationary distribution for the process (1.3) and (1.4) and is therefore the JPDF of the eigenvalues in the appropriate Gaussian ensembles.

The key component of (1.8) is the Vandermonde determinant $\prod_{\mu<\nu} |\lambda_\mu - \lambda_\nu|$, which is responsible for the apparent repulsion of neighbouring eigenvalues. This
factor emerges as the Jacobian of the transformation from \(1.1\) to \(1.8\). However, as Dyson highlights, the above approach offers a new insight into its appearance - as it is nothing more than the effect coming from the second order term in the perturbation formula \(1.4\).

Recently, the authors have adapted the above approach to investigate the spectral statistics of Bernoulli matrices \([17]\) (matrices in which the elements come from the set \(\{\pm1\}\)). In this instance higher terms in the perturbation formula had to be accounted for, which meant assumptions regarding the delocalisation of eigenvectors were required. This inevitably led to the following question - can Dyson’s Brownian motion model be used without the requirement of the perturbation formula \(1.4\)?

In this article we demonstrate that the answer is indeed positive. To achieve this we start from a slightly different viewpoint to Dyson: Rather than following the evolution \(P(\lambda; s)\) of the eigenvalues directly, we instead follow \(Q(t; s)\) - the JPDF of the \(N\)-dimensional vector of traces \(t = (t_1, \cdots, t_N)\), where \(t_k = \sum_{\nu=1}^{N} \lambda_{\nu}^k = \text{tr} M^k\). Performing a transformation of variables then allows us to recover the stationary solution \(1.8\) expressed in terms of the \(t\) variables.

To the best of our knowledge, the distribution of the traces (or spectral moments) has not been extensively studied, although there are exceptions for both the Gaussian and circular ensembles (see e.g. \([9–13]\) and references therein). We therefore find it worthwhile to pursue this direction, not only as it sheds new light on Dyson’s approach, but because it may offer different perspectives on such trace distributions. In addition, our method has led to the discovery of two identities (see Proposition \(1\)) that relate the traces \(t_n\) with \(n > N\) to those with \(n \leq N\). We are unaware of the existence of similar identities in the literature and a direct proof of their validity has kindly been supplied by D. Zagier in Appendix A.

It has also been brought to our attention\(^\dagger\) that a similar philosophy has also been undertaken by Bakry and Zani \([14]\). Rather than looking at the traces \(t\) they follow the motion of the secular coefficients (given by \(c_k\) in Section \(2\) ). Their motivation comes from wanting to generalise the probability density functions to Gaussian random matrices with Clifford algebras (rather than real, complex or quaternion entries) and they too note that such approaches have not been utilised before.

The paper is organised as follows: In Section \(2\) we introduce the basic concepts to be discussed, provide some useful relations and outline the identities mentioned above. We also provide explicit formulae for the stationary distribution \(Q_\beta(t)\) for \(\beta = 1, 2, 4\) and arbitrary dimension \(N\). In Section \(3\) we derive the Fokker-Planck equation for \(Q(t; s)\) and give an example of its form in two dimensions in Section \(3.1\). In Section \(3.2\) we analyse the equation in \(N\)-dimensions and show how the aforementioned identities arise from considering the stationary solution \(Q_\beta(t)\). Section \(3.3\) is used briefly to explain how the mean spectral moments also arise naturally in this context. Finally in Section \(4\) and Section \(5\) we provide an application of this method to Bernoulli ensembles and discuss the potential advantages of the whole approach.

2. Definitions and useful relations

The first essential feature to outline is the relationship between the spectral and trace distribution functions \(P(\lambda; s)\) and \(Q(t; s)\). The elements of the Jacobian of

\(\dagger\) For which we would like to thank P. Forrester.
the transformation are given by \( \frac{\partial w}{\partial \lambda_r} = n \lambda_r^{n-1} \), which means that

\[
P(\lambda; s) = \left| \frac{\partial P}{\partial \lambda} \right| Q(t; s) = N! \det(V) Q(t; s). \tag{2.1}
\]

Here \( V \) is the familiar Vandermonde matrix

\[
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_N \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1}
\end{pmatrix}
\]

and so \( \det(V) = \prod_{\mu < \nu} |\lambda_{\mu} - \lambda_{\nu}| \), as seen in (1.8). The mapping \( \lambda \to t \) is one-to-one as long as the Jacobian does not vanish, hence we must restrict the spectral variables to an ordered sector, e.g. \( \lambda_1 < \lambda_2 < \cdots < \lambda_N \). In order to obtain an expression for \( Q(t; s) \) we need to write \( P(\lambda; s) \), and thus the Vandermonde determinant \( \det(V) \), in terms of the traces \( t \). Fortunately this is relatively straightforward, since \( G(t) = \det(V) = \sqrt{\det(V^*V)} \), with

\[
VV^T = \begin{pmatrix}
t_0 & t_1 & \cdots & t_{N-1} \\
t_1 & t_2 & \cdots & t_N \\
t_2 & t_3 & \cdots & t_{N+1} \\
\vdots & \vdots & \ddots & \vdots \\
t_{N-1} & t_N & \cdots & t_{2N-2}
\end{pmatrix}
\]

and \( t_0 = N \) (see [15][16] for examples of this identity in other contexts).

At this point \( G(t) \) is expressed entirely in terms of the traces, as desired, however this includes traces of higher degree than \( N \), which are themselves functions of the traces \( t_n \), \( 1 \leq n \leq N \). The expressions for \( t_{N+r} \) in terms of the first \( N \) \( t_n \), whilst complicated, can be written down explicitly. They originate from the characteristic polynomial \( \Phi(X) := \det(XI - M) = \sum_{k=0}^N c_k X^{N-k} \), with \( c_0 = 1 \). For any eigenvalue \( \lambda_\nu \) we have \( \Phi(\lambda_\nu) = 0 \) and thus it follows

\[
t_{N+r} = -\sum_{k=1}^N c_k t_{N+r-k}. \tag{2.3}
\]

Newton’s identities give the coefficients \( c_k \) in terms of the \( t_n \), \( n \leq k \) via the determinant

\[
c_k = \frac{(-1)^k}{k!} \begin{vmatrix}
t_1 & 1 & 0 & \cdots & 0 \\
t_2 & t_1 & 2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
t_{k-1} & t_{k-2} & \cdots & t_1 & k-1 \\
t_k & t_{k-1} & \cdots & t_2 & t_1
\end{vmatrix}. \tag{2.4}
\]

Therefore, using a combination of relations (2.3) and (2.4) one may write \( G(t) \) explicitly in terms of the first \( N \) traces \( t \). Clearly \( \Delta = G(t)^2 \) is nothing but the discriminant of \( \Phi(X) \) expressed as a function of \( t \).

Using the transformation (2.1) and the stationary distribution for the eigenvalues \( \chi_N(t) \), we can obtain the JPDF for the traces in the three canonical ensembles

\[
Q_\beta(t) = C_\beta^{(N)} G(t)^{\beta-1} \exp \left( -\frac{\beta}{2} t^2 \right) \chi_N(t). \tag{2.5}
\]
\( \chi_N(t) \) is an indicator function for the domain \( T \subset \mathbb{R}^N \) which is the support for \( Q_\beta(t) \). In contrast to the spectrum, which is defined over the entire space \( \mathbb{R}^N \), the trace parameters are restricted to the domain \( T \). This is because the traces are sums of powers of real variables, which must satisfy certain consistency relations: The inverse mapping \( t \rightarrow \lambda \) should yield real spectra. For example, in 2 dimensions we have \( 2t_2 - t_1^2 = (\lambda_2 - \lambda_1)^2 \geq 0 \). Hence, \( T = \{(t_1, t_2) \in \mathbb{R}^2 : 2t_2 - t_1^2 \geq 0\} \). In higher dimensions it becomes increasingly more difficult to write an explicit definition of \( T \), other than stating that it is the image of the mapping \( \lambda \rightarrow t \). It should be emphasized, however, that \( T \) is independent of the ensemble under consideration - one may consider matrices with non-Gaussian elements, or even correlated elements, and \( T \) will remain the same.

We would also like to highlight that the GOE distribution takes a very simple form in this space, i.e. \( Q_1(t) = C_{1}(N) \exp \left(-\frac{1}{2}t_2\right) \). At first sight it might seem strange that the JPDF for all the traces depends only on one parameter \( t_2 \); however, as alluded to above, one must pay very close attention to the domain of integration \( T \). This is exemplified in Section 3.1, in which we calculate expectations values and marginal probabilities.

In the following section we shall derive the Fokker-Planck equation for \( Q(t; \lambda) \). Its stationary solution is known and given explicitly in \( \text{(2.5)} \). As will be shown below, by substituting this solution into the stationary Fokker-Planck equations we obtain two identities which are summarized in the following proposition.

**Proposition 1.** For \( n \geq 0 \) we have

\[
2 \sum_{m=1}^{N} m \frac{\partial t_{n+m}}{\partial t_{m}} = \sum_{i, j \geq 0} t_{i} t_{j} + (n + 1) t_{n} \tag{2.6}
\]

\[
2 \sum_{m=1}^{N} m t_{n+m} \frac{\partial G}{\partial t_{m}} = \sum_{i, j \geq 0} t_{i} t_{j} - (n + 1) t_{n} \tag{2.7}
\]

As mentioned in the introduction, we are unaware of such identities arising before in RMT or any other context and a direct algebraic proof is given by D. Zagier in Appendix A.

### 3. The Fokker-Planck equation

The main reason for studying Dyson’s Brownian motion in the space of traces is that the Fokker-Planck equation for \( Q(t; s) \) can be derived directly, avoiding the use of perturbation theory \( \text{(1.4)} \). The expectation values of the components of \( t \) due to an incremental changes in the matrices will be evaluated directly from the matrix elements statistics. Once \( Q(t; s) \) has been computed, one can then transform back to the spectral representation in order to deduce the eigenvalue statistics though \( P(\lambda, s) \).

We begin by expressing the change in the \( n \)th trace via the change in the matrix \( \delta M \), up to second order (since higher terms in \( \delta M \) will be of orders \( \delta s^2 \) or greater after taking the expectation)

\[
\delta t_{n} = [\text{tr}(M + \delta M)^n) - \text{tr}(M^n)]
= n \text{tr}(M^{n-1} \delta M) + \frac{n}{2} \sum_{x=0}^{n-2} \text{tr}(M^x \delta M M^{n-x-2} \delta M) + \ldots . \tag{3.1}
\]
The simplest way to compute $E(\delta t_n)$ and $E(\delta t_n \delta t_m)$ is to invoke the invariance of the stochastic motion under unitary transformations. We are then free to write the initial matrix $M$ in a diagonal representation of eigenvalues, i.e. $M_{ij} = \lambda_i \delta_{ij}$. Using this and the expressions (3.2) and (3.3) we find

$$E(\delta t_n) = -nt_n \delta s + \frac{n-1}{2} \sum_{x=0}^{n-2} \lambda_i^n \lambda_k^n - x^2 \delta E(\delta M_{ji} \delta M_{ik})$$

$$= \left[-nt_n + \frac{n-1}{2} \sum_{x=0}^{n-2} t_x t_{n-2-x} + \frac{2 - \beta}{\beta} n (n-1) t_{n-2} \right] \delta s, \quad (3.2)$$

where we have used that $E(\delta M_{ij})^2 = (1 + (2/\beta - 1) \delta_{ij}) \delta s$. In particular this means for $n = 1$ and $2$ that we have $E(\delta t_1) = -t_1 \delta s$ and $E(\delta t_2) = (-2t_2 + t_0^2 + (2/\beta - 1)t_0) \delta s$.

For the second order moments, since again we need terms proportional to $\delta s$ and no more, we only require the first term in (3.1). Therefore, for $n, m = 1, \ldots, N$, we get

$$E(\delta t_n \delta t_m) = nm \sum_{ijkl} \lambda_i^{n-1} \lambda_j^{m-1} \delta E(\delta M_{ji} \delta M_{ik})$$

$$= nm \sum_{ik} \lambda_i^{n-1} \lambda_k^{m-1} E(\delta M_{ii} \delta M_{kk}) = \frac{2nm}{\beta} t_{n+m-2} \delta s, \quad (3.3)$$

where we have used $E(\delta M_{ii} \delta M_{kk}) = \frac{2}{\beta} \delta_{kk} \delta s$. Note that in the above equations, and in the following, one should remember that the independent parameters in the present theory are the components $t$ which consist of the first $N$ traces. Whenever there appears $t_x$ with $x > N$, it should be considered as a function of the independent parameters as explained in the previous section. Similarly, one must substitute $t_0 = N$.

We are now in a position to obtain our Fokker-Planck equation for determining the probability distribution $Q_\beta(t; s)$ of the traces. For simplicity we write (3.2) and (3.3) in the form $R_n^{(\beta)} = E(\delta t_n)/\delta s$ and $R_{nm}^{(\beta)} = E(\delta t_n \delta t_m)/\delta s$, so that (see for instance [24])

$$\frac{\partial Q_\beta}{\partial s} = -\sum_n \frac{\partial (R_n^{(\beta)} Q_\beta)}{\partial t_n} + \frac{1}{2} \sum_{n,m} \frac{\partial^2 (R_{nm}^{(\beta)} Q_\beta)}{\partial t_n \partial t_m}. \quad (3.4)$$

Just as $P_\beta(\lambda)$, given in (1.8), is the stationary solution to the Fokker-Planck equation (1.7) for the eigenvalues, so we would like to verify $Q_\beta(t)$, given in (2.4), is the stationary solution of (3.4) above. For this to be the case, $Q_\beta(t)$ must therefore satisfy the following $N$ simultaneous equations

$$R_n^{(\beta)} Q_\beta = \frac{1}{2} \sum_m \frac{\partial (R_{nm}^{(\beta)} Q_\beta)}{\partial t_m}, \quad \forall \ 1 \leq n \leq N. \quad (3.5)$$

These will be discussed shortly for arbitrary matrix dimension $N$ but prior to this we outline, for illustrative purposes, the scenario for $N = 2$.

3.1. Example: $2 \times 2$ Gaussian ensembles

The $N = 2$ case offers the particular advantage that the expressions (3.2) and (3.3) do not contain traces larger than $t_N$ (i.e. $t_2$ in this case), which is not true for $N > 2$. In
order to satisfy (3.8) $Q \equiv Q_\beta(t_1, t_2)$ must be a solution of the simultaneous equations (3.9), which in 2 dimensions are given by

$$0 = t_1 Q + \frac{1}{\beta} \left[ (t_0 \partial Q) \frac{\partial Q}{\partial t_1} + \partial Q \frac{\partial Q}{\partial t_2} \right]$$

$$0 = \left( 2t_2 - t_1^2 - \frac{(2 - \beta)}{\beta} t_0 \right) Q + \frac{2}{\beta} \left[ \frac{\partial Q}{\partial t_1} + 2 \frac{\partial Q}{\partial t_2} \right].$$

One may verify by substitution that the solution is, including the normalisation constant presented in (1.8),

$$Q_\beta(t_1, t_2) = \frac{1}{2} C^{(2)}_\beta (2t_2 - t_1^2)^{\frac{\beta-1}{2}} e^{-\frac{\beta t_1^2}{2}}. \quad (3.6)$$

Written in terms of the eigenvalues, using $G(t)^2 = (2t_2 - t_1^2) = (\lambda_2 - \lambda_1)^2$, this yields

$$P_\beta(\lambda_1, \lambda_2) = C^{(2)}_\beta |\lambda_2 - \lambda_1| Q(\lambda_1, \lambda_2) = C^{(2)}_\beta |\lambda_2 - \lambda_1| e^{-\beta(\lambda_1^2 + \lambda_2^2)/2},$$

which is the expected result for the JPDF.

From (3.6) we can immediately calculate the marginal probability distributions for the traces. Importantly, the limits of integration are defined by the domain $T$. For 2 dimensions this was outlined in Section 2

$$q_\beta(t_1) = \int_{t_1^{1/2}}^{\infty} dt_2 Q_\beta(t_1, t_2) = \frac{1}{2} C^{(2)}_\beta r_\beta e^{-\beta t_1^2/4} \quad (3.7)$$

$$q_\beta(t_2) = \int_{-\sqrt{2\beta t_2}}^{\sqrt{2\beta t_2}} dt_1 Q_\beta(t_1, t_2) = \frac{1}{2} C^{(2)}_\beta s_\beta t_2^{3/2} e^{-\beta t_2/2}, \quad (3.8)$$

where $(C^{(2)}_\beta)^{-1} = 4\sqrt{\pi}, \pi, 3\pi/8$, $r_\beta = 2, \sqrt{\pi}/2, 3\sqrt{\pi}/8$ and $s_\beta = 2^{3/2}, \pi, 3\pi/2$ for $\beta = 1, 2, 4$ respectively. The expected value of $t_2$ is therefore $\langle t_2 \rangle = \int_0^\infty dt_2 t_2 q_\beta(t_2) = 3, 2, 3/2$ in the three cases.

### 3.2. Stationary solution

Finding the stationary solution in $N$ dimensions requires solving the $N$ simultaneous equations given by (3.9). Therefore, substituting in the expressions (3.2) and (3.3) we get for each $n$

$$\left(-nt_n + \frac{n-2}{2} \sum_{x=0}^{n-2} t_x t_{n-2-x} + \frac{2 - \beta}{\beta} (n-1)t_{n-2} \right) Q_\beta = \frac{n}{\beta} \sum_{m=1}^{N} m \frac{\partial Q_{\beta}(n+m-2)}{Q_{\beta}(n+m-2)} \frac{\partial Q}{\partial t_m}. \quad (3.9)$$

The derivative in the RHS can be expanded using the chain rule to obtain

$$\frac{\partial Q_{\beta}(n+m-2)e^{-\beta t_2/2}}{\partial t_m} = \left( \frac{\partial Q_{\beta}(n+m-2)}{\partial t_m} + t_{n+m-2} \frac{\beta - 1}{G} \frac{\partial G}{\partial t_m} \frac{\beta}{2} t_{n+m-2} \right) Q_\beta.$$

Therefore, after some algebra in which we divide through by a factor $n Q_\beta(2\beta)$ and cancel the term involving $-nt_n$ on both sides, we arrive at the following relationship between the traces

$$\beta \sum_{x=0}^{n-2} t_x t_{n-2-x} + (2 - \beta)(n-1)t_{n-2} = 2 \sum_{m=1}^{N} m \left( \frac{\partial Q_{\beta}(n+m-2)}{\partial t_m} + t_{n+m-2} \frac{\beta - 1}{G} \frac{\partial G}{\partial t_m} \right). \quad (3.10)$$
In the particular case $\beta = 1$ there is no dependence on the Vandermonde determinant $G(t)$ and we get

$$2 \sum_{m=1}^{N} m \frac{\partial t_{n+m-2}}{\partial t_m} = \sum_{x=0}^{n-2} t_x t_{n-2-x} + (n-1)t_{n-2}. \quad (3.11)$$

Replacing $n-2$ by $n$ thus gives the identity (2.6). If we then rearrange (3.10) in terms of $\beta$ we find

$$\beta \left( \sum_{x=0}^{n-2} t_x t_{n-2-x} - (n-1)t_{n-2} - 2 \sum_{m=1}^{N} mt_{n+m-2} \frac{1}{G} \frac{\partial G}{\partial t_m} \right)$$

$$= 2 \left( \sum_{m=1}^{N} m \left[ \frac{\partial t_{n+m-2}}{\partial t_m} - t_{n+m-2} \frac{1}{G} \frac{\partial G}{\partial t_m} \right] - (n-1)t_{n-2} \right). \quad (3.12)$$

The above must be fulfilled simultaneously for both $\beta = 2, 4$, which only occurs if the expressions in large brackets on the two sides of (3.12) vanish. Therefore, by using the substitution (3.11) we arrive at the second identity (2.7)

$$2 \sum_{m=1}^{N} mt_{n+m-2} \frac{1}{G} \frac{\partial G}{\partial t_m} = \sum_{x=0}^{n-2} t_x t_{n-2-x} - (n-1)t_{n-2},$$

where again we must replace $n-2$ by $n$. Since we know that the expression (2.6) must be our stationary solution the method above constitutes a proof of the identities (2.6) and (2.7). However, a direct proof of these is given by D. Zagier in Appendix A, which therefore implies that (2.6) must be our stationary JPDF, without the need for any transformation of variables.

### 3.3. The mean values $\langle t_n \rangle$

Computations of expected values of any function of $t$ involve integrating over the domain $\chi_N(t)$, which is not explicitly defined for any $N > 2$. However, one can use a simple heuristic reasoning in order to identify the mean values $\langle t_n \rangle$ as the coordinates of the vector $t$ for which the drift force (3.2) vanishes, i.e.

$$\langle t_n \rangle = \frac{1}{2} \sum_{x=0}^{n-2} \langle t_x \rangle \langle t_{n-2-x} \rangle + \frac{2-\beta}{2\beta}(n-1)\langle t_{n-2} \rangle. \quad (3.13)$$

It is natural, and customary, to scale the matrices $M$ by $1/\sqrt{N}$ and the resulting traces by $1/N$, so that we may define $\tau_n = N^{-\frac{\beta}{2}-1}t_n$. Thus

$$\langle \tau_n \rangle = \frac{1}{2} \sum_{x=0}^{n-2} \langle \tau_x \rangle \langle \tau_{n-2-x} \rangle + \frac{2-\beta}{2\beta N}(n-1)\langle \tau_{n-2} \rangle. \quad (3.14)$$

If we take $\beta = 2$, with initial conditions $\tau_0 = 1$ and $\tau_1 = 0$, then (3.14) implies that $\langle \tau_{n+1} \rangle = 0$ and $\langle \tau_k \rangle = \frac{1}{k}C_k$ where $C_k$ are the Catalan numbers. This is the well known result obtained by computing the moments using the semi-circle spectral distribution function (see e.g. [4, 9, 18]). For other $\beta$ the last term is of order $O(1/N)$ smaller than the rest and thus its effect vanishes in the limit of large $N$. This is consistent with the fact that the spectral distribution of the three canonical ensembles converge to the semi-circle distribution for $N \to \infty$. Moreover for $N = 2$, (3.13) returns $\langle t_2 \rangle = 3, 2, 3/2$ for $\beta = 1, 2, 4$ respectively, which is exactly the result obtained in Section 3.1.
4. Application to Bernoulli ensembles

Recently, the authors have used a discrete analogue of Dyson’s Brownian motion model to investigate the spectral statistics of Bernoulli ensembles [17]. Here we provide a brief illustration of how this can be adapted to the traces setting and discuss why this offers certain advantages. Our Bernoulli ensemble \( \mathfrak{B}_N \) is given by the set of \( N \times N \) symmetric matrices with 0 on the diagonal and off-diagonal entries chosen randomly and independently from the set \( \{ \pm a \} \) with equal probability (in the following we shall choose, without loss of generality, \( a = 1/\sqrt{2} \) in order to match the variance of the GOE defined in Section 1). The spectral properties of \( \mathfrak{B}_N \) were first analysed by E. Wigner in 1955, who showed the empirical spectral density converges to the semicircle distribution in the limit of large \( N \) [18]. Recent works have gone much further, establishing that local eigenvalue correlations do indeed converge to the corresponding Gaussian expressions as \( N \) increases [19–22].

In [17] the random walk is defined on \( \mathfrak{B}_N \) such that at each single time-step, one of the \( d_N = \frac{1}{2}N(N-1) \) off-diagonal matrix entries \( B_{pq} \) is chosen at random and its sign is flipped (together with \( B_{qp} \)). This leads to a change in the matrix \( B \) of

\[
\delta B^{pq} = -2B_{pq}|p\rangle\langle q| + |q\rangle\langle p|, \tag{4.1}
\]

where \( |p\rangle \) is a vector whose elements are all zero but for 1 in the position \( p \), and \( |p\rangle \) is its transposed. This perturbation in turn induces a change in the eigenvalue \( \lambda_\mu \) of

\[
\delta \lambda_\mu = \langle \mu | \delta B^{pq} | \mu \rangle + \sum_{\nu \neq \mu} \frac{|\langle \nu | \delta B^{pq} | \mu \rangle|^2}{\lambda_\mu - \lambda_\nu} + \cdots, \tag{4.2}
\]

in a similar manner to (1.4). In order to construct the coefficients in the Fokker-Planck equation one has to average \( \delta \lambda_\mu \) over the entire neighbourhood of matrices that can be reached in a single step. In particular, \( \mathbb{E}(\langle \mu | \delta B | \mu \rangle) = -2\lambda_\mu/d_N \) and

\[
\mathbb{E}(|\langle \nu | \delta B | \mu \rangle|^2) = \frac{1}{d_N} \sum_{p<q} |\langle \nu | \delta B^{pq} | \mu \rangle|^2 = \frac{2}{d_N} \left( 1 + \delta_{\nu\mu} - 2 \sum_{p=1}^N \nu_p^2 \mu_p^2 \right). \tag{4.3}
\]

Here, in contrast to [17], there is an additional term \( \sum_{p=1}^N \nu_p^2 \mu_p^2 \) that cannot be written purely in terms of the eigenvalues, meaning the motion is not autonomous.

Collating the above expressions allows one to derive a Fokker-Planck equation which describes the motion of a suitable observable, up to an error that depends on \( N \). This error comes from a combinations of factors such as higher moments \( \mathbb{E}(\delta \lambda_\mu^k) \) and higher terms in the perturbation formula (4.2). This is because, ultimately, our process in discrete and, unlike Dyson’s Brownian motion, one cannot assume that the change of the matrix due to a single step can be made arbitrarily small. These errors, together with the correction to the second moment from the additional term in (4.3), all depend on the eigenvectors and can only be assumed to become negligible in the large \( N \) limit if they are sufficiently delocalised. For the present ensemble, it has been proved this is correct with high probability (see [19–22] and references therein) but for Bernoulli ensembles with correlated matrix entries there are no rigorous results thus far in this direction. Moreover, perturbation theory only converges when \( |\langle \mu | \delta B | \mu \rangle| \) is small relative to \( \lambda_\mu - \lambda_{\mu \pm 1} \). In ensembles such as random regular graphs, this is not the case, even though the eigenvectors are delocalised, due to the growth rate (or lack thereof) of the mean level spacing. These observations therefore motivate the search for another approach.
Dyson’s Brownian-motion model for random matrix theory - revisited

In complete analogy to Section 3 we can also study the random walk in the space of traces. In fact we shall find it more amenable to use the rescaled traces \( \tau_n = N^{-n/2-1}t_n \), as used in Section 3.3. In this basis all the variables are \( \mathcal{O}(1) \) in \( N \) and thus it becomes transparent as to which terms can be neglected. To facilitate this transition let us therefore scale the original matrices by \( \bar{B} = B / \sqrt{N} \). Applying this to (3.1) we have

\[
\delta \tau_n = \frac{1}{N} \left( \text{Tr}(\bar{B} + \delta \bar{B})^n - \text{Tr}(\bar{B}^n) \right) = \frac{1}{N} \left[ n \text{Tr}(\bar{B}^{n-1} \delta \bar{B}) + \frac{n-2}{2} \sum_{x=0}^{n-2} \text{Tr}(\bar{B}^x \delta \bar{B} \bar{B}^{n-2-x} \delta \bar{B}) + \ldots \right]. \tag{4.4}
\]

Although we shall eventually seek to neglect those higher terms, as in Section 3 the whole expansion is finite for fixed \( n \) and thus exact. It means this formalism offers a distinct advantage over the perturbation formula (4.2), which has no such guarantees. Moreover, in this way the change in the variables can be expressed directly in terms of the matrix elements, which is not the case for the eigenvalue representation, since it relies on the appearance of the eigenvectors.

Proceeding in a similar manner, the expected change of \( \delta \tau_n \) in one time step may be calculated as follows:

\[
\mathbb{E}(\text{Tr}(\bar{B}^{n-1} \delta \bar{B})) = -\frac{2}{d_N} \sum_{p<q} \bar{B}_{pq} \text{Tr} \left( \bar{B}^{n-1} ([|p\rangle \langle q|] + [\langle q| \langle p|]) \right) = -\frac{4}{d_N} \sum_{p<q} \bar{B}_{pq} \bar{B}_{pq}^{n-1} = -\frac{2}{d_N} N^{-n/2} t_n = -\frac{2}{d_N} N \tau_n \tag{4.5}
\]

and

\[
\mathbb{E}\left(\text{Tr}(\bar{B}^x \delta \bar{B} \bar{B}^{n-2-x} \delta \bar{B})\right) = \frac{2}{d_N} \sum_{p<q} \bar{B}_{pq}^2 \text{Tr} \left( \bar{B}^x ([|p\rangle \langle q|] + [\langle q| \langle p|]) \right) = \frac{2}{d_N} \frac{1}{N} \sum_{p \neq q} \left( \bar{B}_{pq}^x \bar{B}_{qp}^{n-2-x} + \bar{B}_{pp}^x \bar{B}_{qq}^{n-2-x} \right) = \frac{2}{d_N} \left[ \frac{1}{N} \sum_{p, q} \left( \bar{B}_{pq}^x \bar{B}_{qp}^{n-2-x} + \bar{B}_{pp}^x \bar{B}_{qq}^{n-2-x} \right) - \frac{2}{N} \sum_p \bar{B}_{pp}^x \bar{B}_{pp}^{n-2-x} \right] = \frac{2}{d_N} \left[ \tau_n - 2 \tau_x \tau_{n-2-x} - 2 \zeta(x, n - 2 - x) \right], \tag{4.6}
\]

where

\[
\zeta(r, s) = \frac{1}{N} \sum_p \bar{B}_{pp}^r \bar{B}_{pp}^s.
\]

The most striking difference between (4.6) and the Gaussian equivalent (4.2) is the appearance of the term \( \zeta(x, n - 2 - x) \), which cannot be expressed in terms of the variables \( t \). Writing \( \tau_s \tau_r - \zeta(r, s) = \frac{1}{N} \left( \sum_p \bar{B}_{pp}^r \left[ \frac{1}{N} \sum_q \bar{B}_{qq}^s - \bar{B}_{pp}^s \right] \right) \) we see that \( \zeta(r, s) \) is very close to \( \tau_s \tau_r \) if the diagonal elements \( \bar{B}_{pp}^s \) are close to their average over the whole diagonal \( \sum_q \bar{B}_{qq}^s \). Using Wigner’s combinatorial method of counting Dyck paths (see e.g. [1][18]) one can show that by averaging over \( \mathcal{B}_N \) we have for fixed \( r \) and \( s \)

§ We use the convention that \( B_{pq}^n \) denotes the \( p,q \)-th element of the matrix \( B^n \) and \( (B_{pq})^n \) is the matrix element \( B_{pq} \) raised to the \( n \)-th power.
that \(\langle \tau_r \tau_s - \zeta(r,s) \rangle_{\mathfrak{B}_N}\) tends to 0 as \(N \to \infty\). Moreover, using the same technique one finds \(\text{Var}_{\mathfrak{B}_N}(\tau_r \tau_s - \zeta(r,s)) = \mathcal{O}(N^{-2})\). Hence with high probability \(\zeta(r,s)\) is \(\mathcal{O}(1)\). This shows that (4.10) is dominated by the term \(N \tau_r \tau_{r-2} - x\). In addition we can also estimate those higher terms in the expectation \(E(\delta \tau_n)\) coming from the expansion (4.12). For example, we have \(E(\text{Tr}(\delta B^n B^{n-3})) = 4N^{-n/2}E(\text{Tr}(\delta B^n B^{n-3})) = -8N^{-n/2}t_{n-2}/d_N = -8t_{n-2}/d_N\). This again is of an order in \(N\) less than the dominant term in (4.10). Therefore, in the large \(N\) limit we find that \(E(\delta \tau_n)/\delta s\) (taking \(\delta s = 2/d_N\)) tends to the expression (4.2) calculated for the GOE.

Similarly, for the second moment we find

\[
E(\delta \tau_n \delta \tau_m) = \frac{2}{d_N} \frac{2m n}{N^2} (\tau_{n+m-2} - \zeta(n-1, m-1)) + \ldots \tag{4.7}
\]

The difference in comparison to the first moment is that, by the arguments above, the additional term \(\zeta(n-1, m-1)\) is of the same order in \(N\) as the supposed leading term. This is also in contrast to the outcome for the second order term in the eigenvalue representation (4.3), where the effect of removing the matrix diagonal leaves only a \(1/N\) correction. Nevertheless we present arguments that allow for it to be neglected. Let us continue by inserting the expressions (4.5), (4.6), and (4.7) into the appropriately scaled version of the \(n\) simultaneous equations (3.3), which determine the stationary solution \(Q\) (the method for calculating the error terms in the analogous eigenvalue representation approach is discussed at length in [17] and thus we refrain from details here). Therefore, for large \(N\), the stationary solution \(Q\) for \(\mathfrak{B}_N\) approximately satisfies

\[
\left[ -\tau_n + \sum_{x=0}^{n-2} \left( \tau_x \tau_{n-x} + \frac{\tau_{n-2}}{N} \right) \right] Q = \sum_m \frac{2m}{N^2} \frac{\partial}{\partial \tau_m} \{ (\tau_{n+m-2} - \zeta(n-1, m-1))Q \}.
\]

To estimate the contribution of \(\zeta(n-1, m-1)\) we replace the exact value with its mean, i.e. \(N^{-1} \sum_p B^{n-1}B^{m-1} \approx \tau_{n-1} \tau_{m-1}\). For all matrices \(B \in \mathfrak{B}_N\) we have \(\tau_1 = 0\) and \(\tau_2 = N(N-1)/N^2 = 1 - 1/N\), meaning our space of variables is reduced to \(\tau_n\) for \(n = 3, \ldots, N\). Assuming then, that in all the remaining directions our JPDF \(Q\) is constant (as is the case in the GOE expression (2.24)) we find for \(n \geq 3\)

\[
\sum_m \frac{2m}{N^2} \frac{\partial}{\partial \tau_m} (\tau_{n-1} \tau_{m-1} Q) = \frac{2}{N^2} (n-1) \tau_{n-2} Q,
\]

where we have used that \(\partial \tau_{m-1}/\partial \tau_m = 0\) and \(\partial \tau_{m-1}/\partial \tau_m = \delta_{m,n-1}\) for all \(n, m \leq N\). This results in a term which is of order \(1/N\) less than the corresponding term on the LHS and a full order \(1/N^2\) less than the leading term.

5. Discussion

The efforts invested in developing the formalism presented above were motivated by our initial observations regarding random regular graphs. Dyson’s original model could not be transcribed to this matrix ensemble as the perturbation formula is effectively useless (a consequence of small separation between eigenvalues) in this context (see [24]). Here we offer a method which does away with the requirement of the perturbation formula and therefore offers a potential method for circumventing such problems. We have demonstrated this method in the standard Gaussian setting and also illustrated how this can be used for Bernoulli matrices. The former case leads immediately to two previously unseen identities regarding symmetric functions, which
are proved directly below. Finally we also note the relation with those studies [9–13] regarding the distributions of traces. Except for [11], these works did not consider any dynamical aspects and so what we have outlined here may offer alternative ways for studying traces distributions. For instance one should be able to apply the same techniques to the circular ensembles.

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Appendix A. Proof of Proposition 1 by Don Zagier

Following the notation of the paper, we let \( \lambda_\alpha (\alpha = 1, \dots, N) \) be independent variables and let \( c_i \) \((0 \leq i \leq N)\), \( t_n \) \((n = 0, 1, \dots)\) and \( \Delta \) (discriminant) be the elements of the algebra \( S = \mathbb{C}[\lambda_1, \dots, \lambda_N]^{S_N} \) of symmetric polynomials in the \( \lambda_\alpha \) defined by

\[
\Phi(X) := \prod_{\alpha=1}^{N} (X - \lambda_\alpha) = \sum_{i=0}^{N} c_i X^i ,
\]

\[
t_n = \sum_{\alpha=1}^{N} \lambda^n_\alpha , \quad \Delta = \text{disc}(\Phi) = \prod_{1 \leq \alpha < \beta \leq N} (\lambda_\alpha - \lambda_\beta)^2 .
\]

For \( n < 0 \) we set \( t_n = 0 \). We have \( c_N = 1 \) and \( t_0 = N \), while both \( (c_1, \dots, c_N) \) and \( (t_1, \dots, t_N) \) generate the algebra \( S \). In particular, if we take the latter as coordinates on \( S \), then we can ask for the values of \( \partial t_n / \partial t_m \) and \( \partial \Delta / \partial t_m \) for \( n \geq 0 \) and \( 1 \leq m \leq N \).

The identities (2.6) and (2.7) were proved in the body of this paper using an indirect proof coming from random matrix theory. Here we give a purely algebraic verification of both of these identities, and some small generalizations. For the reader’s convenience we repeat these identities here, expressing the second one in terms of the polynomial invariant \( \Delta \) rather than its square-root \( G \).

**Proposition.** For \( n \geq 0 \) we have

\[
2 \sum_{m=1}^{N} m \frac{\partial t_{n+m}}{\partial t_m} = t_i t_j + (n + 1) t_n , \quad (A.1)
\]

\[
\frac{1}{\Delta} \sum_{m=1}^{N} m t_{n+m} \frac{\partial \Delta}{\partial t_m} = t_i t_j - (n + 1) t_n . \quad (A.2)
\]

We use that the logarithmic derivative of \( \Phi(X) \) is a generating series for the \( t_n \), i.e.,

\[
T(X) := \frac{\Phi'(X)}{\Phi(X)} = \sum_{\alpha=1}^{N} \frac{1}{X - \lambda_\alpha} = \sum_{n=0}^{\infty} \frac{t_n}{X^{n+1}} ,
\]
where the last expression can be taken either as a formal power series in $S[[1/X]]$ or as a holomorphic function in the annulus $|X| > \max_{\alpha} |\lambda_\alpha|$ if the $\lambda_\alpha$ are complex numbers.

Dividing (A.1) and (A.2) by $X \leq 0$ and summing over $n \geq -m$ (or equivalently $n \geq 0$, since $\partial t_{n+m}/\partial t_m$ vanishes for $-m \leq n < 0$), we can rewrite these two identities as

$$2 \sum_{m=1}^{N} m \frac{\partial T(X)}{\partial t_m} X^{m-1} = T(X)^2 - T'(X) \quad (A.3)$$

and

$$\frac{T(X)}{\Delta} \sum_{m=1}^{N} m \frac{\partial \Delta}{\partial t_m} X^{m-1} = T(X)^2 + T'(X). \quad (A.4)$$

For the proof, we define polynomials $\Phi_\alpha(X)$ and coefficients $c_{\alpha,n}$ for $1 \leq \alpha \leq N$ and $0 \leq n \leq N - 1$ by

$$\Phi_\alpha(X) = \prod_{\beta \neq \alpha} \frac{X - \lambda_\beta}{\lambda_\alpha - \lambda_\beta} = \frac{1}{\Phi'(\lambda_\alpha)} \frac{\Phi(X)}{X - \lambda_\alpha} = \sum_{n=0}^{N-1} c_{\alpha,n} X^n.$$

Then $\Phi_\alpha(\lambda_\beta) = \delta_{\alpha,\beta}$, so $(c_{\alpha,n})$ is the inverse of the Vandermonde matrix $(\lambda_\alpha^m)_{n,\alpha}$. On the other hand, we have $\frac{1}{m} \frac{\partial}{\partial \lambda_\alpha} = \lambda_\alpha^{m-1}$, so $c_{\alpha,m-1} = m \frac{\partial}{\partial \lambda_\alpha}$ for $1 \leq m \leq N$. Hence

$$m \frac{\partial T(X)}{\partial t_m} = \sum_{\alpha=1}^{N} c_{\alpha,m-1} \frac{\partial T(X)}{\partial \lambda_\alpha} = \sum_{\alpha=1}^{N} \frac{c_{\alpha,m-1}}{(X - \lambda_\alpha)^2}, \quad (A.5)$$

so each term $\frac{\partial T(X)}{\partial t_m}$ is a rational function of the form $P_m(X)/\Phi(X)^2$ where $P_m(X)$ is a polynomial of degree $\leq 2n - 2$. Multiplying (A.5) by $X^{m-1}$ and summing over $m = 1, \ldots, N$ gives

$$\sum_{m=1}^{N} m \frac{\partial T(X)}{\partial t_m} X^{m-1} = \sum_{\alpha=1}^{N} \frac{\Phi_\alpha(X)}{(X - \lambda_\alpha)^2}$$

$$= \Phi(X) \sum_{\alpha=1}^{N} \frac{1}{\Phi'(\lambda_\alpha)} \frac{1}{(X - \lambda_\alpha)^3}$$

$$= \Phi(X) \sum_{\alpha=1}^{N} \text{Res}_{z=\lambda_\alpha} \left( \frac{1}{\Phi(z)} \frac{dz}{(X - z)^2} \right)$$

$$= \Phi(X) \text{Res}_{z=X} \left( \frac{dz}{(z - X)^2 \Phi(z)} \right)$$

$$= \frac{\Phi(X)}{2} \frac{d^2}{dX^2} \frac{1}{\Phi(X)} = -\frac{1}{2} \frac{\Phi''(X)}{\Phi(X)} + \frac{\Phi'(X)^2}{\Phi(X)^2}$$

$$= -\frac{T'(X)}{2} + \frac{T(X)^2}{2},$$

where in the fourth line we have used the residue theorem. This prove the first identity (A.3). The calculation for (A.4) is similar. We have

$$\frac{m}{2\Delta} \frac{\partial \Delta}{\partial t_m} = \frac{1}{2} \sum_{\alpha=1}^{N} c_{\alpha,m-1} \frac{\partial \log \Delta}{\partial \lambda_\alpha} = \sum_{\alpha=1}^{N} c_{\alpha,m-1} \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta} = \sum_{\alpha=1}^{N} c_{\alpha,m-1} \frac{\Phi'_\alpha(\lambda_\alpha)}{\Phi_\alpha(\lambda_\alpha)}.$$
Substituting into this the identity
\[
\frac{\Phi'_n(\lambda_\alpha)}{\Phi(\lambda_\alpha)} = \left. \left( \frac{\Phi(t)}{\Phi(t)} - \frac{1}{t - \lambda_\alpha} \right) \right|_{t=\lambda_\alpha} = \left. \left( \frac{\Phi'(\lambda_\alpha + \varepsilon) - 1}{\varepsilon} \right) \right|_{\varepsilon=0} = \left. \left( \frac{\Phi'(\lambda_\alpha) + \Phi''(\lambda_\alpha) \varepsilon + \cdots - \frac{1}{2} \Phi''(\lambda_\alpha) \varepsilon^2 + \cdots}{\Phi'(\lambda_\alpha) \varepsilon + \frac{1}{2} \Phi''(\lambda_\alpha) \varepsilon^2 + \cdots} - \frac{1}{\varepsilon} \right) \right|_{\varepsilon=0} = \left. \frac{1}{2} \Phi''(\lambda_\alpha), \right.
\]

multiplying by \( X^{m-1} \) and summing over \( m \), we obtain the second identity (A.4):
\[
\frac{1}{\Delta} \sum_{m=1}^{N} m \frac{\partial \Delta}{\partial t_m} X^{m-1} = \sum_{\alpha=1}^{N} \frac{\Phi''(\lambda_\alpha)}{\Phi'(\lambda_\alpha)} \Phi(\lambda_\alpha) = \sum_{\alpha=1}^{N} \frac{\Phi''(\lambda_\alpha)}{\Phi'(\lambda_\alpha)^2} \Phi(X) X - \lambda_\alpha = \Phi(X) \sum_{\alpha=1}^{N} \text{Res}_{z=\lambda_\alpha} \left( \frac{\Phi''(z) dz}{(X - z) \Phi(z)} \right) = \Phi''(X) \Phi(X) = T(X) + \frac{T'(X)}{T(X)}. \quad \Box
\]

We mention that one can use the same method of calculation to obtain other identities of this type. For instance,
\[
\sum_{m=1}^{N} m(m-1) \frac{\partial T(X)}{\partial t_m} X^{m-2} = \sum_{\alpha=1}^{N} \frac{\Phi'_n(\lambda_\alpha)}{(X - \lambda_\alpha)^2} = \sum_{\alpha=1}^{N} \frac{1}{2} \Phi'(\lambda_\alpha) \left( \frac{\Phi'(X) (z - \lambda_\alpha)^3 - \Phi(X) (z - \lambda_\alpha)}{(z - X)^4 \Phi(z)} \right) = \text{Res}_{z=X} \left[ \frac{\Phi'(X) (1 - \Phi(X) (z - X)^3 \Phi(z))}{2 \Phi'(X)^2 (z - X)^3 (z - X)^3 + 6 (z - X)^6 \Phi(X)^3} \right] = \frac{1}{3} T(X)^3 - \frac{1}{6} T''(X)
\]
and hence, in analogy with (A.1),
\[
3 \sum_{m=1}^{N} m(m-1) \frac{\partial t_{n+m}}{\partial t_m} = \sum_{i+j+k=n, i+j+k=0}^{n} t_i t_j t_k - \frac{(n+1)(n+2)}{2} t_n.
\]

Identities with polynomials of higher degree in \( m \) on the left could be proved in the same way.

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