ON THE BIRATIONALITY OF THE ADJUNCTION MAPPING
OF PROJECTIVE VARIETIES

ANDREAS LEOPOLD KNUTSEN

Abstract. Let $X$ be a smooth projective $n$-fold such that $q(X) = 0$ and $L$ a globally generated, big line bundle on $X$ such that $h^0(K_X + (n-2)L) > 0$. We give necessary and sufficient conditions for the adjoint systems $|K_X + kL|$ to be birational for $k \geq n - 1$. In particular, for Calabi-Yau $n$-folds we generalize and prove parts of a conjecture of Gallego and Purnaprajna.

1. Introduction and main results

Let $X$ be a smooth $n$-dimensional complex variety with canonical divisor (or bundle) $K_X$ and let $L$ be a line bundle on $X$ satisfying some positivity properties, e.g. $L$ is globally generated, big and nef, ample, very ample, etc. A lot of attention in algebraic geometry has been devoted to studying properties of adjoint bundles $K_X + kL$, where $k$ is a positive integer (or even a rational number in some cases), like nefness, global generation, very ampleness, and the properties $N_p$. We refer to the book [1] and references therein for an overview of results in adjunction theory.

A particularly nice case is when $k = n - 1$ and $L$ is globally generated and big. Then, by adjunction, any smooth curve $C$ obtained by intersecting $n - 1$ general members of $|L|$ has the property that $K_C = (K_X + (n-1)L)|_C$. Therefore, one may expect that properties of $K_X + (n-1)L$ can be described in terms of properties of (a sufficiently general) $C$. Moreover, since $L$ is globally generated, $|L|$ defines a morphism of $X$ to some projective space and there are several instances where properties of $K_X + (n-1)L$ are closely related to properties of the morphism defined by $|L|$.

The picture is particularly nice for $n = 2$ (resp. 3) and $k = n - 1$ or $n$ when $K_X = 0$ and $h^1(O_X) = 0$, that is, for $K3$ surfaces (resp. Calabi-Yau threefolds). We start by recalling the following two results of Saint-Donat [12]:

Theorem 1.1. (Saint-Donat [12]) Let $X$ be a smooth $K3$ surface and $L$ a globally generated, big line bundle on $X$.

The following conditions are equivalent:

(i) $L$ is birationally very ample (meaning that $|L|$ defines a birational morphism onto its image);
(ii) the morphism determined by $|L|$ does not map $X$ generically $2 : 1$ onto a surface of minimal degree;
(iii) $|L|$ contains a smooth, irreducible, nonhyperelliptic curve;
(iv) all smooth irreducible curves in $|L|$ are nonhyperelliptic.

Furthermore, if these conditions are satisfied, then $L$ is also normally generated.

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Theorem 1.2. (Saint-Donat [12]) Let $X$ be a smooth $K3$ surface and $L$ a globally generated, big line bundle on $X$. Then $2L$ is birationally very ample if and only if the morphism defined by $|L|$ does not map $X$ generically $2:1$ onto $\mathbb{P}^2$.

Furthermore, $2L$ is also normally generated in these cases.

The Calabi-Yau threefold case has been treated by Gallego and Purnaprajna [6], who also posed an interesting conjecture:

Theorem 1.3. (Gallego and Purnaprajna [6, Thm. 1]) Let $X$ be a smooth Calabi-Yau threefold and $L$ a globally generated, ample line bundle on $X$. Then $3L$ is very ample and the morphism it defines embeds $X$ as a projectively normal variety if and only if the morphism defined by $|L|$ does not map $X$ onto a variety of minimal degree other than $\mathbb{P}^3$.

Conjecture 1.5. (Gallego and Purnaprajna [6, Conj. 1.9]) Let $X$ be a smooth Calabi-Yau threefold and $L$ a globally generated, ample line bundle on $X$. Then $2L$ is very ample and the morphism it defines embeds $X$ as a projectively normal variety if and only if there is a smooth nonhyperelliptic curve $C \in |L \otimes \mathcal{O}_S|$ for some $S \in |L|$.

In this note we generalize the parts of the above results not concerning normal generation. In particular, we generalize and prove part of Conjecture [15].

We make the following definition:

Definition 1.6. Let $X$ be a smooth projective $n$-fold and $L$ a line bundle on $X$. We say that the curve $C$ is a curve section of the pair $(X,L)$ if there are $n-1$ distinct members $H_1,H_2,\ldots,H_{n-1}$ of $|L|$ such that $C = H_1 \cap \cdots \cap H_{n-1}$.

Our two main results, which will be proved in Section 2, are the following:

Theorem 1.7. Let $X$ be a smooth projective $n$-fold such that $q(X) = 0$ and $L$ a globally generated, big line bundle on $X$ such that $h^0(K_X + (n-2)L) > 0$.

The following conditions are equivalent:

(i) the rational map defined by $|K_X + (n-1)L|$ is not birational;
(ii) the rational map defined by $|K_X + (n-1)L|$ has generic degree two;
(iii) $(X,L)$ has a smooth, irreducible, hyperelliptic curve section;
(iv) all smooth irreducible curve sections of $(X,L)$ are hyperelliptic;
(v) the morphism $\psi$ defined by $|L|$ is of generic degree two onto a variety of minimal degree.

Theorem 1.8. Let $X$ be a smooth projective $n$-fold such that $q(X) = 0$ and $L$ a globally generated, big line bundle on $X$ such that $h^0(K_X + (n-2)L) > 0$. Then the rational map defined by $|K_X + nL|$ is birational. Furthermore, the following conditions are equivalent:

(i) the rational map defined by $|K_X + nL|$ is not birational;
(ii) the rational map defined by $|K_X + nL|$ has generic degree two;
(iii) the morphism $\psi$ defined by $|L|$ is of generic degree two onto $\mathbb{P}^n$. 
Similarly, in Theorem 1.7 will be satisfied (with a hyperelliptic curve section. These have been classified in [11, Thm. (3.1)]. Then (iii) in Theorem 1.7 will be satisfied (with $L$ the hyperplane bundle), but not (v). Similarly, $X = \mathbb{P}^n$ with $L = \mathcal{O}(1)$ will satisfy (i) but not (iii) in Theorem 1.8 and $K_X + (n+1)L$ is trivial, so it does not define a birational map.

We refer to [5, 7, 3, 8] for more interesting results on hyperelliptic sections in the case of surfaces.

We also remark that the equivalence between (iii) and (iv) in Theorem 1.7 is of independent interest: it says that a $g^2$ on one smooth curve section necessarily propagates to all smooth curve sections. (Again this does not hold if one drops the assumption that $h^0(K_X + (n-2)L) > 0$, see the remark at the end of [11].) In general, questions of propagations of linear series have attracted a lot of attention in the case of surfaces (see for instance [4, 7, 3, 8] for the case of $K3$ surfaces), but we do not know of similar results on higher dimensional varieties.

Remark 1.9. In Theorems 1.7 and 1.8, if one adds the assumption that $L$ be ample and $K_X + (n-2)L$ be globally generated (e.g., $K_X$ globally generated), then the rational maps in questions are in fact morphisms and they are birational if and only if they are embeddings.

Remark 1.10. If $X$ is a smooth Calabi-Yau projective $n$-fold with $n \geq 2$ (that is, $K_X = 0$ and $h^i(\mathcal{O}_X) = 0$ for $i = 1, \ldots, n-1$), and $L$ is a globally generated line bundle on $X$, then the condition that $h^0(K_X + (n-2)L) > 0$ is automatically satisfied. Hence Theorems 1.7 and 1.8 give necessary and sufficient conditions for $kL$ to be birationally very ample, for $k \geq n-1$. In particular, $(n-1)L$ is birationally very ample if and only if there is a smooth nonhyperelliptic curve section of $(X, L)$. This (together with Remark 1.9) proves Conjecture 1.5 except for the projective normality part and generalizes it to higher dimensions and the nonample case.

2. Proofs of Theorems 1.7 and 1.8

We remark that by adjunction, any irreducible curve section $C$ of a pair $(X, L)$, where $X$ is a smooth projective $n$-fold and $L$ a line bundle on $X$, satisfies

\begin{equation}
(K_X + (n-1)L)|_C \simeq K_C,
\end{equation}

and that

\begin{equation}
2g(L) - 2 = 2p_a(C) - 2 = (K_X + (n-1)L) \cdot L^{n-1},
\end{equation}

where $g(L)$ is the sectional genus of $L$ and $p_a(C)$ is the arithmetic genus of $C$.

We will in the rest of this note use the following:

Notation 2.1. If $C$ is a curve section and $C = H_1 \cap \cdots \cap H_{n-1}$ with $H_i \in |L|$ for $i = 1, \ldots, n$, we set $X_n := X$ and $X_{n-i} := H_1 \cap \cdots \cap H_i$, for $i = 1, \ldots, n-1$, so that, in particular, $C = X_1$, and $\dim X_i = i$. Note that neither the $H_i$ nor $X_i$ are uniquely determined, nor need they be smooth. However, they can be chosen smooth if $L$ is globally generated and $C$ is a general curve section.

We will also often set $L_i := L|_{X_i}$.

We remark the following:
Lemma 2.2. Let $L$ be a globally generated, big line bundle on a smooth projective $n$-fold $X$ and $A$ any line bundle on $X$ such that $H^0(A) \neq 0$. Then no curve section of $L$ can lie in the base locus of $|A|$.

Proof. Let $C$ and $X_i$ be as in Notation 2.1 $i = 1, \ldots, n$. Consider, for each $i$, the short exact sequence

$$0 \longrightarrow A|_{X_i} - L_i \longrightarrow A|_{X_i} \overset{\alpha_i}{\longrightarrow} A|_{X_{i-1}} \longrightarrow 0$$

Now $(L_i)^i = L^n > 0$, as $L$ is big. Hence $L_i$ is globally generated and nontrivial. Therefore, $H^0(A|_{X_i} - L_i) \neq H^0(A|_{X_i})$, so that the restriction map of sections $H^0(\alpha_i)$ is nonzero. Therefore, the restriction map $H^0(A) \to H^0(A|_C)$ is also nonzero, so that $H^0(A \otimes I_C) \neq H^0(A)$, proving the lemma. 

Lemma 2.3. Let $L$ be a globally generated, big line bundle on a smooth projective $n$-fold $X$ with $q(X) = h^1(O_X) = 0$. Let $C$ be a smooth curve section of $(X, L)$ and $k \geq n - 1$.

The natural restriction maps $H^0(L) \to H^0(L|_C)$ and $H^0(K_X + kL) \to H^0(K_C + (k - n + 1)L|_C)$ are surjective and $h^0(L) = h^0(L|_C) + n - 1$.

Proof. Let again $C$ and $X_i$ be as in Notation 2.1 $i = 1, \ldots, n$. Note that we cannot use Kawamata-Viehweg vanishing on each $X_i$, since we do not know whether $X_i$ is smooth, except $C = X_1$. Instead we claim that

$$(2.3) \quad H^j(X_i, (K_X + kL)|_{X_i}) = 0 \quad \text{for all} \quad j \geq 1, \quad k \geq n - i + 1, \quad 2 \leq i \leq n$$

and for $j = i - 1, k = n - i$.

Indeed, this holds for $i = n$ by Kawamata-Viehweg vanishing and the assumption that $h^1(O_X) = 0$ together with Serre duality. If $(2.3)$ holds for $i = i_0 \geq 3$, then it also holds for $i = i_0 - 1$, by the short exact sequences

$$0 \longrightarrow (K_X + (k - 1)L)|_{X_i} \longrightarrow (K_X + kL)|_{X_i} \longrightarrow (K_X + kL)|_{X_{i-1}} \longrightarrow 0.$$ 

In particular, if $k \geq n - 1$, each of the restriction maps $H^0(X_i, (K_X + kL)|_{X_i}) \to H^0(X_{i-1}, (K_X + kL)|_{X_{i-1}})$ is surjective, for $2 \leq i \leq n$, whence also the restriction map $H^0(K_X + kL) \to H^0(K_C + (k - n + 1)L|_C)$ is surjective.

Similarly, by the short exact sequences

$$0 \longrightarrow (-k - 1)L_i \longrightarrow -kL_i \longrightarrow -kL_{i-1} \longrightarrow 0.$$ 

we obtain that

$$H^j(X_i, -kL_i) = 0 \quad \text{for all} \quad k > 0, \quad 0 \leq j \leq i - 1, \quad 2 \leq i \leq n,$$

so that $H^1(O_X) = 0$ for all $2 \leq i \leq n$, as $H^1(O_X) = 0$. It follows that each of the restriction maps $H^0(X_i, L_i) \to H^0(X_{i-1}, L_{i-1})$ is surjective with kernel $H^0(O_{X_{i-1}}) \simeq C$, for $2 \leq i \leq n$, whence also the restriction map $H^0(L) \to H^0(L|_C)$ is surjective and $h^0(L) = h^0(L|_C) + n - 1$. 

We can now give the proofs of the two main theorems stated in the introduction.

Proof of Theorem 1.7. Let

$$\varphi : X \dashrightarrow \mathbb{P}(H^0(K_X + (n - 1)L))$$

be the rational map defined by $|K_X + (n - 1)L|$ and let $U$ be the dense, open subset of $X$ where it is a morphism. As $L$ is globally generated and $h^0(K_X + (n - 2)L) > 0$, 


the complement of $U$ is contained in the base locus of $|K_X + (n - 2)L|$. Moreover, as $L$ is big and nef, we have that $\dim \varphi(U) = \dim X = n$.

Let $Y$ be the projective closure of $\varphi(U)$ and let

$$
\begin{array}{c}
\bar{X} \\
\pi \\
X - \varphi \\
\downarrow \\
\varphi \\
\downarrow \\
Y
\end{array}
$$

be the resolution of indeterminacies of $\varphi$. Then $\tilde{\varphi}$ is the morphism defined by the complete linear system associated to the line bundle $H := \pi^*(K_X + (n - 1)L) - E$, for some effective divisor $E$ on $\bar{X}$. The fact that the complement of $U$ is contained in the base locus of $|K_X + (n - 2)L|$ implies that $H - \pi^*L = \pi^*(K_X + (n - 2)L) - E$ is effective. By abuse of notation, we will consider $U$ to be a subset of $\bar{X}$.

Let $C$ be any smooth curve section of $(X, L)$. By Lemma 2.8, the natural restriction map $H^0(X, K_X + (n - 1)L) \to H^0(C, \omega_C)$ is surjective. If $g(C) = 0$, then $L^{-1}$ $(K_X + (n - 1)L) = C \cdot (K_X + (n - 1)L) = 2g(C) - 2 = -2$ by (2.2), contradicting the fact that $L$ is nef, as $h^0(K_X + (n - 1)L) > 0$ by assumption. Hence $g(C) > 0$. In particular, $C \subseteq U$ and $\varphi_C$ is the canonical morphism of $C$. Again by abuse of notation, we will often consider $C$ as a curve in $\bar{X}$.

Let

$$
\tilde{\varphi} : \bar{X} \xrightarrow{f} X' \xrightarrow{g} Y.
$$

be the Stein factorization of $\varphi$.

We now prove that (iii) implies (i). Assume therefore that $C$ is hyperelliptic. Then $\varphi_C$ is a $2 : 1$ map. If $x \in C$ is general, then there is a point $y \neq x$ on $C$ such that $\tilde{\varphi}(x) = \tilde{\varphi}(y)$.

If $\deg g = 1$, then there has to exist a curve $\Gamma \subset \bar{X}$ passing through $x$ and $y$ and contracted to a point by $f$. In particular, $\Gamma \cdot H = 0$. Since $x \in U$, we have $\pi(\Gamma) \subseteq \text{BS}[(K_X + (n - 1)L) \otimes I_x]$. Moreover, as $C$ is a curve section of $L$ passing through $x$, we have that $\pi(\Gamma) \nsubseteq \text{BS}[L \otimes I_x]$, so that $\pi(\Gamma) \subseteq \text{BS}[(K_X + (n - 2)L)]$. In particular, $x \in \text{BS}[(K_X + (n - 2)L)]$. Since this holds for a general $x \in C$, we must in fact have $C \subseteq \text{BS}[(K_X + (n - 2)L)]$, contradicting Lemma 2.2. Hence $\deg g > 1$ and $\varphi$ is not birational, as desired.

We next prove that (i) implies (ii), (iv) and (v). Let $\ell := \deg g > 1$.

To prove (iv), it suffices to prove that the general curve section of $(X, L)$ is hyperelliptic. So pick a general point $x \in U$ and a smooth curve section $C$ containing $x$. (Recall that $C \subseteq U$ and that we can therefore consider $C$ as a curve in $\bar{X}$.) Since $x$ is general, the fiber over $\tilde{\varphi}(x)$ of $\tilde{\varphi}$ consists of $\ell$ distinct points $x = x_1, \ldots, x_\ell$. Moreover, $\tilde{\varphi}(x) \notin \tilde{\varphi}(\text{BS}[H - \pi^*L])$, as $x$ is general. Therefore, all the points $x = x_1, \ldots, x_\ell$ lie outside the base locus of $|H - \pi^*L|$. As $\pi^*L$ is globally generated, it follows that also the morphism $\psi$ defined by $|\pi^*L|$ must identify the points $x = x_1, \ldots, x_\ell$. Hence $|\pi^*L \otimes I_x| = |\pi^*L \otimes I_{x_1} \otimes \cdots \otimes I_{x_\ell}|$. In particular, since $C$ is a curve section of $(\bar{X}, \pi^*L)$, all the points $x_1, \ldots, x_\ell \in C$. Therefore, $\tilde{\varphi}_C = \varphi_C$ is not an embedding, but a morphism of degree $\ell$. Since it is the canonical morphism, $C$ must be hyperelliptic and $\ell = 2$. This proves (iv) and also (ii). To prove (v), note that $L|_C$ is a special, globally generated line bundle on $C$. Hence, as is well-known, $|L|_C$ must be a multiple of the $g_2^1$ on $C$. Consequently, the morphism $\psi$ defined by $|L|$ is $2 : 1$ on every smooth curve section, whence it
has generic degree two. Now \(|L_{C}|\) is a \(g_{2}^{r}\), where \(r := \frac{1}{2}L^{n}\). By Lemma 2.3 we have \(\dim |L| = \dim |L_{C}| + (n - 1) = r + n - 1\). Therefore, \(\psi(X)\) has degree \(r\) in \(\mathbb{P}^{r + n - 1}\) and is therefore a variety of minimal degree. This proves (v).

Now clearly (ii) implies (i) and (iv) implies (iii). Finally, (v) implies (iv), as every smooth curve section of \(|L|\) is mapped \(2 : 1\) onto a curve section of \(\psi(X)\), which is a rational normal curve.

Proof of Theorem 1.8. First we note that by Lemma 2.3 the rational map defined by \(|K_{X} + nL|\) restricted to any smooth curve section \(C\) of \((X, L)\) is the morphism defined by

\[
(K_{X} + nL)_{C} \cong K_{C} + L_{C}.
\]

Also recall that we showed that \(C\) is not rational in the proof of Theorem 1.7.

We first prove the equivalence of (i)-(iii).

Clearly (ii) implies (i). Assume now that the rational map defined by \(|K_{X} + nL|\) is not birational. Then clearly the same holds for the rational map defined by \(|K_{X} + (n - 1)L|\), so that (ii) follows from Theorem 1.7. Furthermore, each smooth curve section \(C\) of \((X, L)\) is hyperelliptic, and, arguing as in the proof of Theorem 1.7, also the morphism defined by \(K_{C} + L_{C}\) is not an embedding. Therefore, \(|L_{C}|\) must be the \(g_{2}^{0}\) on \(C\), so that \(L^{2} = 2\) and \(h^{0}(L) = h^{0}(L_{C}) + n - 1 = n + 1\) by Lemma 2.3. Thus, the morphism \(\psi\) defined by \(|L|\) is of generic degree two onto \(\mathbb{P}^{n}\), proving (iii). Finally, assume (iii). Then any smooth curve section \(C\) of \((X, L)\) is hyperelliptic and \(|L_{C}|\) is the \(g_{2}^{0}\) on \(C\), so that the morphism defined by \(|K_{C} + L_{C}|\) is of degree two, so that (i) follows.

Finally, to prove the first statement, assume that the rational map defined by \(|K_{X} + (n + 1)L|\) were not birational. By Lemma 2.3 this map is the morphism on any smooth curve section \(C\) of \((X, L)\) defined by \((K_{X} + (n + 1)L)_{C} \cong K_{C} + 2L_{C}\), which is very ample, as \(L \cdot C \geq 2\), since \(C\) is not rational. However, arguing as in the proof of Theorem 1.7, one easily proves that this map cannot be an isomorphism on the general curve section, a contradiction.

\[\square\]

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Andreas Leopold Knutsen, Department of Mathematics, University of Bergen, Johannes Brunsgate 12, 5008 Bergen, Norway.

E-mail address: andreas.knutsen@math.uib.no