ON THE SETS OF LENGTHS OF PUISEUX MONOIDS GENERATED BY MULTIPLE GEOMETRIC SEQUENCES

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Abstract. It is well known that a Puiseux monoid $M$ is atomic if and only if $M$ contains a minimal set of generators. This characterization, whose verification involves a great deal of effort given that Puiseux monoids are not, in general, finitely generated, motivated a series of papers studying the atomicity of Puiseux monoids generated by structured sets such as monotone and geometric sequences. In this paper, we focus on the study of rational multicyclic monoids, that is, Puiseux monoids generated by multiple geometric sequences. In particular, we provide a complete description of the rational multicyclic monoids that are hereditarily atomic. Additionally, we show that the sets of lengths of certain family of atomic rational multicyclic monoids are finite unions of infinite multidimensional arithmetical progressions, a result we use to realize infinite arithmetic progressions as delta sets of some Puiseux monoids.

1. Introduction

The systematic study of additive submonoids of the nonnegative cone of $\mathbb{Q}$, also known as Puiseux monoids, started just a few years ago in [15]. However, Puiseux monoids have been used as a natural source of crucial examples in the realms of commutative ring theory due to the intrinsic complexity of its atomicity. For instance, A. Grams utilized Puiseux monoids to refute P. Cohn’s conjecture that every atomic integral domain satisfies the ACCP [22]. More recently, J. Coykendall and F. Gotti used Puiseux monoids to provide insight about a question posed by R. Gilmer in the 1980s [6].

In [15], the author provides a characterization of the atomicity of Puiseux monoids in terms of its generating sets. This characterization, whose verification involves a great deal of effort given that Puiseux monoids are not, in general, finitely generated, motivated a series of papers studying the atomicity of Puiseux monoids generated by structured sets; for example, Puiseux monoids generated by monotone and geometric sequences have been study in [20] and [3], respectively. This paper introduces a family of Puiseux monoids that constitutes a generalization of rational cyclic semirings, i.e., Puiseux monoids generated by the elements of a geometric progression.
Systems of sets of lengths have been studied in the context of Krull monoids \[14\], C-monoids \[8\], affine monoids \[10\], and submonoids of \(\mathbb{N}^d\) \[19\] (see also \[12, 7\] for some recent work). Given that Puiseux monoids started receiving attention just a few years ago, little is known about its system of sets of lengths (for recent progress see \[16\, 21\]). In \[3, \text{Theorem 3.3}\], Chapman et al. prove that the system of sets of lengths of a rational cyclic semiring is an arithmetic sequence. We extend this result to a family of rational multicyclic monoids that encompasses rational cyclic semirings. Moreover, we show that the sets of lengths of canonical rational multicyclic monoids are finite unions of infinite multidimensional arithmetical progressions, a result we use to realize infinite arithmetic progressions as delta sets of some Puiseux monoids.

This paper is organized as follows. In Section 2, we establish the notation we will be using throughout this paper. In Section 3, we provide a complete description of the rational multicyclic monoids that are hereditarily atomic. We conclude by characterizing, in Section 4, the sets of lengths of canonical rational multicyclic monoids.

2. Background

Throughout this paper, we let \(\mathbb{N}\) and \(\mathbb{N}_0\) denote the set of positive and nonnegative integers, respectively. Moreover, we denote by \(\mathbb{P}\) the set of all prime numbers. For \(n, m \in \mathbb{N}_0\) we let \([n, m]\) denote the set of nonnegative integers between \(n\) and \(m\), i.e.,

\([n, m] := \{k \in \mathbb{N}_0 \mid n \leq k \leq m\}\).

In addition, if \(S \subseteq \mathbb{Q}\) then \(S_{\geq t} := \{s \in S \mid s \geq t\}\); in the same spirit we define \(S_{>t}\) and \(S_{<t}\). For every positive rational number \(q = \frac{n}{m}\), we assume that \(\gcd(n, m) = 1\), and we set \(n(q) = n\) and \(d(q) = m\). We say that a positive fraction \(n/m\) is proper if \(n < m\); otherwise, we call \(n/m\) improper.

**Definition 2.1.** For \(d \in \mathbb{N}\) and \(l \in \mathbb{N}_0 \cup \{\infty\}\) we set \(P_l(d) = d\mathbb{Z} \cap [0, ld]\). If \(r \in \mathbb{N}\), then a nonempty subset \(S \subseteq \mathbb{N}_0\) is called an infinite \(r\)-dimensional arithmetical progression (with differences \(d_1, \ldots, d_r \in \mathbb{N}\)) if

\[ S = \min S + \sum_{i=1}^{r} P_{l_i}(d_i) \]

for some \(l_1, \ldots, l_r \in \mathbb{N}_0 \cup \{\infty\}\). We say that \(S\) is an infinite multidimensional arithmetical progression (or IMAP) if \(S\) is an infinite \(r\)-dimensional arithmetical progression for some \(r \in \mathbb{N}\).

Let \(M\) be a commutative, cancellative, and reduced monoid (which is written additively). We say that \(a \in M^* := M \setminus \{0\}\) is an atom provided that whenever \(a = x + y\) for some \(x, y \in M\), either \(x = 0\) or \(y = 0\). Let \(\mathcal{A}(M)\) represent the set of all atoms of \(M\). For a subset \(S\) of \(M\), we denote by \(\langle S \rangle\) the minimal submonoid of \(M\) containing \(S\), and if \(M = \langle S \rangle\), then we say that \(S\) is a generating set of \(M\). \(M\) is atomic if
$M = \langle A(M) \rangle$. In addition, $M$ is called hereditarily atomic if all submonoids of $M$ are atomic.

A subset $I$ of $M$ is an ideal of $M$ provided that $I + M \subseteq I$; if $I = x + M$ for some $x \in M$, then $I$ is principal. We say that $M$ satisfies the ascending chain condition on principal ideals (or ACCP) if there is no infinite strictly increasing sequence of principal ideals of $M$. It is not hard to argue that an ACCP monoid is atomic (see [13, Proposition 1.1.4]).

**Definition 2.2.** A Puiseux monoid is an additive submonoid of $\mathbb{Q}_{\geq 0}$.

Puiseux monoids have a fascinating atomic structure. While some Puiseux monoids have no atoms at all, for instance, $\langle 1/2^n \mid n \in \mathbb{N}_0 \rangle$, others have exactly $m$ atoms for each $m \in \mathbb{N}$ ([15, Proposition 5.4]). The atomicity of Puiseux monoids have received considerable attention lately (see [4] and references therein). One particularly interesting family of Puiseux monoids is that one consisting of all rational cyclic semirings.

**Definition 2.3.** The rational cyclic semiring over $r \in \mathbb{Q}_{>0}$ is the Puiseux monoid generated by the nonnegative powers of $r$, i.e., $M_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$.

Surprisingly, the atomicity of rational cyclic semirings is quite simple. Consider the following theorem.

**Theorem 2.4.** [20, Theorem 6.2] Let $r \in \mathbb{Q}_{>0}$ and $M = \langle r^n \mid n \in \mathbb{N}_0 \rangle$. The following statements hold:

1. if $d(r) = 1$, then $M$ is atomic with $A(M) = \{1\}$;
2. if $d(r) > 1$ and $n(r) = 1$, then $A(M) = \emptyset$;
3. if $d(r) > 1$ and $n(r) > 1$, then $A(M) = \{r^n \mid n \in \mathbb{N}_0 \}$.

The factorization monoid of $M$, denoted by $Z(M)$, is the free monoid on $A(M)$. The elements of $Z(M)$ are called factorizations, and if $z = a_1 + \cdots + a_n \in Z(M)$ for $a_1, \ldots, a_n \in A(M)$ then we say that $|z|$, the length of $z$, is $n$. The unique monoid homomorphism $\pi: Z(M) \to M$ satisfying $\pi(a) = a$ for all $a \in A(M)$ is called the factorization homomorphism of $M$. For each $x \in M$, there are two important sets associated to $x$: $Z_M(x) := \pi^{-1}(x) \subseteq Z(M)$, which is called the set of factorizations of $x$, and $L_M(x) := \{|z| : z \in Z_M(x)\}$, the set of lengths of $x$; we omit subscripts when $M$ is clear from the context. In addition, the collection $L(M) := \{L(x) \mid x \in M\}$ is called the system of sets of lengths of $M$. See [11] for a friendly survey about sets of lengths and the role they play in factorization theory. We say that $M$ satisfies the finite factorization property if $Z(x)$ is finite for all $x \in M$. In this case we also say
that $M$ is an \textit{FF-monoid}. If $L(x)$ is finite for all $x \in M$, then we say that $M$ is a \textit{BF-monoid}. It is well known that BF-monoids satisfy the ACCP.

Now we proceed to introduce a factorization invariant that is closely related to sets of lengths: \textit{delta set}. For a nonzero element $x$ of $M$ we say that $d$ is a \textit{distance} of $x$ if $L(x) \cap [l, l + d] = \{l, l + d\}$ for some $l \in L(x)$. The \textit{delta set} of $x$, denoted by $\Delta(x)$, is the set consisting of all the distances of $x$. In addition, the set

$$\Delta(M) = \bigcup_{x \in M^*} \Delta(x)$$

is called the \textit{delta set} of $M$. In general, determining the delta set of a given monoid is no simple task, and just a few specific calculations are known (see, for example, [2], [5], [9]). In [1], Bowles et al. prove that every set of the form $\{d, 2d, \ldots, nd\}$ occurs as the delta set of some numerical monoid for $n, d \in \mathbb{N}$. As we mentioned before, we prove that infinite arithmetic progressions can be realized as delta sets of some Puiseux monoids.

### 3. Rational Multicyclic Monoids

In this section, we initiate the study of the atomic properties of Puiseux monoids generated by multiple geometric sequences, but first we make a definition to avoid long descriptions.

**Definition 3.1.** Let $B$ be a finite subset of $\mathbb{Q}$. The \textit{rational multicyclic monoid over $B$} is the monoid $M_B := \langle b^n \mid b \in B, n \in \mathbb{N}_0 \rangle$. We always assume that $B$ is minimal, that is, if $B' \subset B$ then $M_{B'} \subset M_B$.

We say that the elements of $B$ are the \textit{primitive generators} of $M_B$. Note that $(\mathbb{N}_0, +)$ is a rational multicyclic monoid. Furthermore, $\mathbb{N}_0 \subseteq M$ for all rational multicyclic monoids $M$ whose set of primitive generators is nonempty. Especial cases of rational multicyclic monoids have been studied before.

**Example 3.2.** For a fixed $r \in \mathbb{Q}_{>0}$, consider the monoid $M = \langle r^n \mid n \in \mathbb{N}_0 \rangle$. These monoids are called rational cyclic semirings, and they were introduced by F. Gotti and M. Gotti in [20] and deeper studied by S. T. Chapman et al. in [3]. It is well known that rational cyclic semirings are always atomic unless $r$ is a unit fraction, i.e., $r = n^{-1}$ for some $n \in \mathbb{N}_{>1}$ (Theorem 2.4).

**Example 3.3.** Let $b \in \mathbb{Q}_{>0}$ such that $n(b), d(b) > 1$, and consider the rational multicyclic monoid $M_B$ with $B = \{b, b^{-1}\}$. In [18, Proposition 3.5], it is proved not only that $M_B$ is atomic but also that $\text{Aut}(M_B) \cong \mathbb{Z}$.

Note that if a rational multicyclic monoid $M_B$ has a unit fraction $1/d$ as primitive generator, then $M_B$ is not atomic. Indeed, since $M_B$ is minimally generated by $B$, not all elements of the form $1/d^n$ with $n \in \mathbb{N}_0$ are generated by $B \setminus \{1/d\}$. This, together with the fact that $(1/d)^n = d(1/d)^{n+1}$ for all $n \in \mathbb{N}_0$, implies that $M_B$ is not atomic.
For the rest of the section, we assume that primitive generators of rational multicyclic monoids are never unit fractions.

The next theorem gives evidence of the complexity of the atomic structure of rational multicyclic monoids, but first let us collect two lemmas.

**Lemma 3.4.** Let \((\alpha_n)\) be an unbounded increasing sequence of nonnegative rational numbers, and let \(M = \langle \alpha_n \mid n \in \mathbb{N} \rangle\). Then \(M\) contains no strictly decreasing sequence of elements.

**Proof.** In virtue of [17, Theorem 6.3], \(M\) is atomic, and there is no loss in assuming that \(\mathcal{A}(M) = \{\alpha_n \mid n \in \mathbb{N}\}\). Now let
\[
S = \{s \in \mathbb{R}_{\geq 0} \mid (\gamma_n) \to s, (\gamma_n)\text{ is a strictly decreasing sequence of elements of } M\}.
\]
Clearly, \(0 \notin S\). Assume, by contradiction, that \(S \neq \emptyset\). Note that \(S\) has a minimal element since \(\inf(S) \in S\). Let \((\gamma_n)\) be a strictly decreasing sequence of elements of \(M\) that converges to \(\inf(S)\). Note that for any \(m \in \mathbb{N}\) there exists \(K_m \in \mathbb{N}\) such that \(\alpha_n \nmid \gamma_n\) for \(n > K_m\); otherwise, \((\inf(S) - \alpha_m) \in S\), which contradicts the minimality of \(\inf(S)\). Consequently, there are arbitrarily large atoms of \(M\) dividing some \(\gamma_n\), but this is impossible as \((\gamma_n)\) is a bounded sequence. Therefore, \(S = \emptyset\). \(\square\)

**Lemma 3.5.** Let \(M_B\) and \(M_{B'}\) be rational multicyclic monoids, where \(B' = B \cap (0, 1)\). Then \(M_B\) is atomic if and only if \(M_{B'}\) is atomic.

**Proof.** Without loss of generality we can assume that \(B' \neq \emptyset\). Now since \(M_B\) is reduced, \(\mathcal{A}(M_{B'}) \supseteq \mathcal{A}(M_B) \cap M_{B'}\). This, along with the fact that \(B' = B \cap (0, 1)\), implies that \(\mathcal{A}(M_{B'}) = \mathcal{A}(M_B) \cap (0, 1]\). Now assume that \(M_B\) is atomic. For \(b \in B'\) and \(n \in \mathbb{N}_0\), we can write \(b^n\) as the sum of atoms of \(M_B\) with values less than or equal to 1, which implies that \(b^n \in \langle \mathcal{A}(M_{B'})\rangle\). Then \(M_{B'}\) is atomic.

Conversely, assume that \(M_{B'}\) is atomic, and let \(x\) be a nonzero element of \(M_B\). Assume, by contradiction, that \(x \notin \langle \mathcal{A}(M_B)\rangle\). Then \(x = x_1 + x_2\), where \(x_1, x_2 \neq 0\). There is no loss in assuming that \(x_1 \notin \langle \mathcal{A}(M_B)\rangle\). Then \(x_1 \notin \langle \mathcal{A}(M_{B'})\rangle = M_{B'}\), which implies that there exists \(y_1 \in M_{B' \backslash B'}\), such that \(y_1 \mid x_1\) and \(y_1 \notin \langle \mathcal{A}(M_B)\rangle\). Note that \(x > y_1\). Repeating the same reasoning for \(y_1 \notin \langle \mathcal{A}(M_B)\rangle\) (as it was the case for \(x\)), we obtain an element \(y_2 \in M_{B' \backslash B'}\) such that \(y_2 \notin \langle \mathcal{A}(M_B)\rangle\) and \(y_1 > y_2\). We can repeat this infinitely many times, but this contradicts Lemma 3.4 since the generators of \(M_{B' \backslash B'}\) form an unbounded increasing sequence of nonnegative rational numbers. Therefore, \(x \in \langle \mathcal{A}(M_B)\rangle\), which concludes our proof. \(\square\)

The previous lemma tells us that, when analyzing the atomicity of rational multicyclic monoids, there is no loss in assuming that the primitive generators are all proper fractions. Now we proceed to prove the main result of this section.

**Theorem 3.6.** Let \(M_B\) be a rational multicyclic monoid. The following statements hold:
(1) $M_B$ is hereditarily atomic if and only if $n(b) \geq d(b)$ for all $b \in B$;

(2) there are infinitely many non-atomic and non-isomorphic rational multicyclic monoids with exactly $n$ proper fractions as primitive generators if and only if $n \geq 2$.

Proof. For the reverse implication of (1), it is not hard to see that 0 is not a limit point of $M_B$, which implies that $M_B$ is a BF-monoid by [4, Theorem 5.8]. Since $M_B$ satisfies the ACCP, it follows that $M_B$ is hereditarily atomic. As for the direct implication, we know that $M_B$ is hereditarily atomic and, a fortiori, atomic. Assume, by way of contradiction, that not all primitive generators of $M_B$ are improper fractions. Then there exists $a/b \in \mathcal{A}(M_B)$ such that $1 < a < b/2$. For $n \in \mathbb{N}$,

\[(3.1) \quad b \left( \frac{a}{b} \right)^n = b \left( \frac{a}{b} \right)^{n+1} + (b-a) \left( \frac{a}{b} \right)^n ,\]

as the reader can check. Now let $M' = \langle (b(a/b)^n, (b-a)(a/b)^n \mid n \in \mathbb{N} \rangle$. Clearly, $M'$ is a submonoid of $M_B$, which implies that $M'$ is atomic. In virtue of Equation (3.1) $b(a/b)^n \notin \mathcal{A}(M')$ for any $n \in \mathbb{N}$. Consequently, $\mathcal{A}(M') \subseteq \{(b-a)(a/b)^n \mid n \in \mathbb{N} \}$. As $M'$ is atomic,

\[(3.2) \quad b \left( \frac{a}{b} \right)^n = \sum_{i=1}^{k} c_i (b-a) \left( \frac{a}{b} \right)^{m_i} ,\]

where $k, c_i, m_i \in \mathbb{N}$ for every $i \in [1,k]$ and $n \in \mathbb{N}_{>2}$. Without loss of generality, we can assume that $m_1 < \cdots < m_k$. Given that $2a < b$, it is not hard to check that $b(a/b)^n < (b-a)(a/b)^{n-2}$ for $n \geq 2$, which implies that $m_1 \geq n-1$. After multiplying both sides of Equation (3.2) by $(b-a)^{-1}(a/b)^{n-1}$, we obtain that

\[(3.3) \quad a^nb^{m_k-n+1} = \sum_{i=1}^{k} c_i (b-a)a^{m_i}b^{m_k-m_i} ,\]

where both sides of Equation (3.3) represent integers as $n-1 \leq m_1 < \cdots < m_k$. Now take $p \in \mathbb{P}$ such that $p \mid b-a$. In virtue of Equation (3.3) either $p \mid a$ or $p \mid b$. This contradiction proves that our hypothesis is untenable. Therefore, all primitive generators of $M_B$ are improper fractions, and (1) follows.

The direct implication of (2) follows after (1), Lemma 3.5 and Theorem 2.4. As for the reverse implication, let $p_0, \ldots, p_n$ be a finite sequence of prime numbers such that $p_0 < p_1 < p_0 \cdot p_1 + 1 < p_2 < \cdots < p_n$; it is easy to see that there are infinitely many sequences of prime numbers of length $n$ satisfying this property. Now consider the rational multicyclic monoid $M_B$, where

\[\mathcal{B} = \left\{ p_0, p_1, p_0p_1, \frac{p_0p_1}{p_2}, \frac{p_0p_1}{p_2}, \ldots, \frac{p_0p_1}{p_n} \right\} .\]
First, it is not hard to see that $B$ contains a minimal set of primitive generators of $M_B$. Indeed, we cannot generate $p_0/p_2$ (resp., $p_1/p_2$) using the rest of the generators since their numerators share a prime number, namely, $p_0$ (resp., $p_0$). Note that $p_0p_1/p_2$ is the only element of $B$ that can be generated by the rest of the primitive generators. Second, $(p_0/p_2)^m \notin \mathcal{A}(M_B)$ for any $m \in \mathbb{N}$: note that for a fixed $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $(p_2/p_0)^m < (p_0/p_0p_1)^N$ and $m < N$. Thus,

$$p_0^N p_1^N - p_0^N - p_1^N < (p_0p_1)^N < p_0^m p_2^{N-m}.$$ 

Then there exist $\alpha, \beta \in \mathbb{N}_0$ such that $\alpha p_0^N + \beta p_1^N = p_0^m p_2^{N-m}$, which implies that $\alpha \cdot \beta \neq 0$ and $\alpha(p_0/p_2)^N + \beta(p_1/p_2)^N = (p_0/p_2)^m$. Hence $M_B$ is not atomic, and the proof follows after [18, Proposition 3.2].

**Corollary 3.7.** Let $M_B$ be a rational multicyclic monoid. If $M_B$ satisfies the ACCP then $n(b) \geq d(b)$ for all $b \in B$.

**Remark 3.8.** Corollary 3.7 was first proved in [1] Corollary 5.5.

We already showed that not all rational multicyclic monoids with more than one proper fraction as primitive generators are atomic. Our next proposition, which is a generalization of [18, Proposition 3.5], establishes that the atomicity of rational multicyclic monoids is not constant in the sense that, for a given $n \in \mathbb{N}_{\geq 2}$, there are atomic and non-atomic rational multicyclic monoids with exactly $n$ proper fractions as primitive generators.

**Proposition 3.9.** Let $b_0 \in \mathcal{B}$ with $\mathcal{B} \subseteq \mathbb{Q} \setminus \mathbb{N}$, and let $M = \langle b^n \mid b \in \mathcal{B}, n \in \mathbb{N}_0 \rangle$. If for all $b \in \mathcal{B} \setminus \{b_0\}$ we have that $\gcd(d(b_0), d(b)) = 1$ then $b_0^n \in \mathcal{A}(M)$ for all $n \in \mathbb{N}$.

**Proof.** Assume, by contradiction, that $b_0^k \notin \mathcal{A}(M)$ for some $k \in \mathbb{N}$. Thus,

$$(3.4) \quad b_0^k = c_0 b_0^{m+k} + \cdots + c_{m+k-1} b_0 + c_1 b_1^{e_1} + \cdots + c_l b_l^{e_l},$$

where $b_i \in \mathcal{B} \setminus \{b_0\}$, $c_j, c_1, e_i \in \mathbb{N}_0$ for every $j \in [0, m + k - 1]$ and every $i \in [1, l]$. Now if $b_0 < 1$ then $c_m = \cdots = c_{m+k-1} = 0$. There is no loss in assuming that $c_0 \neq 0$ and $c_j < d(b_0)$ for each $j \in [0, m + k - 1]$; otherwise, we can transform the right-hand side of Equation (3.4) by using the transformation $(d(b) + r)b^k = n(b)b^{k-1} + rb^k$ finitely many times. It is not hard to see that $c_0 + \cdots + c_{m+k-1} > 0$. After multiplying Equation (3.3) by $N = d(b_1)^{e_1} \cdots d(b_l)^{e_l}$, it is easy to see that $b_0$ is a rational root of the polynomial $p(x) = c_0 N x^{m+k} + \cdots - N x^k + K$ for some $K \in \mathbb{N}$. In virtue of the Rational Root Theorem, $d(b_0) \mid c_0$, which is a contradiction. We proceed similarly for the case $b_0 > 1$: note that $c_0 = \cdots = c_m = 0$, which implies that $b_0$ is a rational root of the polynomial $q(x) = -N x^k + \cdots + K$. Again, applying the Rational Root Theorem we obtain that $d(b_0) \mid N$, which is a contradiction. Therefore, our result follows. \hfill \square

The examples of rational multicyclic monoids that we have seen so far are multiplicatively closed. However, this is not always the case as the next example illustrates.
Example 3.10. Let $p_1$ and $p_2$ be two prime numbers such that $2 < p_1 < p_2$. Consider the rational multicyclic monoid $M_B$ with $B = \{(p_2 - p_1)/p_1, (p_1/p_2)\}$. Note that if a rational multicyclic monoid $M$ is multiplicatively closed, then $A(M) = \emptyset$ if and only if $1 \notin A(M)$. Since $M_B$ is atomic by Proposition 3.9, $M_B$ is not multiplicatively closed.

We conclude this section by introducing an atomic subfamily of rational multicyclic monoids that will play an important role in the next section.

Definition 3.11. Let $M_B$ be a rational multicyclic monoid. We say that $M_B$ is canonical if $M_B \ne (\mathbb{N}_0, +)$ and $\gcd(d(b), d(b')) = 1$ for all $b, b' \in B$ with $b \ne b'$.

Note that, in virtue of Proposition 3.9, a canonical rational multicyclic monoid $M_B$ is always atomic and $A(M) = \{b^n \mid b \in B, n \in \mathbb{N}_0\}$.

4. Sets of Lengths of Canonical Rational Multicyclic Monoids

In a rational cyclic semiring, the set of lengths of an element is an arithmetic sequence ([3, Theorem 3.3]). The proof of this result relies on the fact that each element has exactly one factorization of minimum length ([3, Lemma 3.1] and [3, Lemma 3.2]). Unfortunately, this is not true for (canonical) rational multicyclic monoids in general. Consider the following example.

Example 4.1. Let $M_B$ be a rational multicyclic monoid with $B = \{7/4, 8/5\}$. In virtue of Proposition 3.9, $A(M_B) = \{(7/4)^n, (8/5)^n \mid n \in \mathbb{N}_0\}$. It is not hard to check that $|z| \ge 6$ for all $z \in \mathbb{Z}(9)$. Hence $z_1 = 4(7/4) + 2$ and $z_2 = 5(8/5) + 1$ are two different factorizations of $9$ of minimum length.

In this section, we extend [3, Theorem 3.3] to a family of rational multicyclic monoids that encompasses all rational cyclic semirings. Moreover, we prove that, in a canonical rational multicyclic monoid, the set of lengths of an element is the union of finitely many IMAPs. For the rest of the section, whenever we refer to an expression of the form $z = \sum_{k=0}^N c_k b_k^{e_k}$ we assume that $b_i^{e_i} \ne b_j^{e_j}$ for $i \ne j$, $e_k = 0$ if and only if $k = 0$, and if $b_i = b_j$ then $e_i < e_j$ if and only if $i < j$.

Definition 4.2. Let $M_B$ be a canonical rational multicyclic monoid, and let $x$ be an element of $M_B$. If $z = \sum_{k=0}^N c_k b_k^{e_k} \in \mathbb{Z}(x)$, where $N, c_k, e_k \in \mathbb{N}_0$ and $b_k \in B$ for every $k \in [0, N]$, then we say that $z$ is a hub factorization of $x$ if $c_k < d(b_k)$ for all $k \in [1, N]$.

Lemma 4.3. Let $M_B$ be a canonical rational multicyclic monoid. Then every element of $M_B$ has exactly one hub factorization.

Proof. Let $x \in M_B$. First, we prove the existence of a hub factorization of $x$. Let $z = \sum_{k=0}^N c_k b_k^{e_k} \in \mathbb{Z}(x)$, where $N, c_k, e_k \in \mathbb{N}_0$ and $b_k \in B$ for every $k \in [0, N]$. If $c_i \ge d(b_i)$ for some $i \in [1, N]$, then $c_i = q_i d(b_i) + r_i$, where $q_i \in \mathbb{N}$ and $r_i \in [0, d(b_i) - 1]$. Then we can transform $c_i b_i^{e_i}$ as follows:

$$c_i b_i^{e_i} = [q_i d(b_i) + r_i] b_i^{e_i} = r_i b_i^{e_i} + q_i d(b_i) b_i^{e_i} = r_i b_i^{e_i} + q_i n(b_i) b_i^{e_i - 1}.$$
This transformation reduces the exponent of the summand $cb^e$ violating the condition $c < d(b)$, where $e > 0$, which means that we cannot carry out this transformation infinitely many times. Once this process stops, it is not hard to see that we obtain a hub factorization of $x$.

Now let $z_h = \sum_{k=0}^N c_k b_k^{e_k}$ and $z'_h = \sum_{k=0}^N d_k b_k^{e_k}$ be two hub factorizations of $x$, where $N, c_k, e_k, d_k \in \mathbb{N}_0$ and $b_k \in \mathcal{B}$ for every $k \in \llbracket 0, N \rrbracket$. Notice that there is no loss in assuming that $z_h$ and $z'_h$ share the same atoms. For the sake of a contradiction, assume that $z_h \neq z'_h$. Then there exists $j \in \llbracket 1, N \rrbracket$ such that $c_j \neq d_j$. Assuming that $j$ is as large as possible, we have

$$(c_j - d_j)b_j^{e_j} = \sum_{k=0}^{j-1} (d_k - c_k)b_k^{e_k}.$$ 

Since $e_j \in \mathbb{N}$ and $M_\mathcal{B}$ is canonical, the previous equality implies that $d(b_j)$ divides $(c_j - d_j)$, but this is impossible as $c_j, d_j < d(b_j)$. Therefore, $z_h = z'_h$. \hfill \Box

Note that a hub factorization of an element not only minimizes the number of atoms that are proper fractions but also maximizes the number of atoms that are improper fractions.

**Definition 4.4.** Let $x$ be an element of a canonical rational multicyclic monoid $M_\mathcal{B}$ and $z = \sum_{k=0}^N c_k b_k^{e_k} \in \mathbb{Z}(x)$, where $N, c_k, e_k \in \mathbb{N}_0$ and $b_k \in \mathcal{B}$ for each $k \in \llbracket 0, N \rrbracket$. We say that $z$ is a minimalist factorization of $x$ if $c_k < \max(n(b_k), d(b_k))$ for every $k \in \llbracket 1, N \rrbracket$ and $c_0 < n(b)$ for all $b \in \mathcal{B} \cap (1, \infty)$.

Note that a factorization of minimum length of an element is minimalist; consequently, it is possible for an element of a canonical rational multicyclic monoid to have multiple minimalist factorizations (Example 4.1). However, for certain canonical rational multicyclic monoids, there exists a close relation among the lengths of the different minimalist factorizations of an element.

**Lemma 4.5.** Let $\mathcal{B}$ be a finite subset of $\mathbb{Q}$ such that $|n(b) - d(b)| = |n(b') - d(b')|$ for all $b, b' \in \mathcal{B}$, and let $M_\mathcal{B}$ be a canonical rational multicyclic monoid. If $z$ and $z'$ are two minimalist factorizations of a nonzero element $x \in M_\mathcal{B}$, then $||z| - |z'|| = k \cdot |n(b) - d(b)|$ for some $k \in \mathbb{N}_0$ and $b \in \mathcal{B}$.

*Proof.* If all the elements of $\mathcal{B}$ are proper fractions, then all minimalist factorizations of $x$ have the same length since there is only one, namely, the hub factorization (Lemma 4.3). Then we can assume that there exists $b \in \mathcal{B}$ such that $b > 1$. In the first part of the proof of Lemma 4.3, we already established that we can transform $z$ into the hub factorization of $x$, and this process increases the length of $z$ in a multiple of $|n(b) - d(b)|$ with $b \in \mathcal{B}$. Consequently, $|n(b) - d(b)|$ divides $|z_h| - |z|$, where $z_h$ represents the hub factorization of $x$. Similarly, $|n(b) - d(b)|$ divides $|z_h| - |z'|$. Thus,

$$||z| - |z'|| = ||z_h| - l \cdot |n(b) - d(b)| - |z_h| + r \cdot |n(b) - d(b)|| = k \cdot |n(b) - d(b)|,$$
where $k = |r - l|$, which concludes our proof.

We left the proof of the following lemma to the reader as it mimicks that one of [3, Lemma 3.1].

**Lemma 4.6.** Let $B \subseteq \mathbb{Q}_{<1}$ such that $\gcd(d(b), d(b')) = 1$ for all $b, b' \in B$, and consider the Puiseux monoid $M = (\mathbb{Q}^n \mid b \in B, n \in \mathbb{N}_0)$. Let $x \in M$ and $z = \sum_{k=0}^{N} c_k b_k^e \in \mathbb{Z}(x)$, where $N, c_k, e_k \in \mathbb{N}_0$ and $b_k \in B$ for each $k \in [0, N]$. The following statements hold:

1. $\min L(x) = |z|$ if and only if $c_k < d(b_k)$ for all $k \in [1, N]$;
2. there exists exactly one factorization in $\mathbb{Z}(x)$ of minimum length.

Now we are in a position to prove the main result of this section.

**Theorem 4.7.** Let $M_B$ be a canonical rational multicyclic monoid, and let $x$ be an element of $M_B$. Then the following statements hold:

1. $L(x)$ is the union of finitely many IMAPs;
2. if $|n(b) - d(b)| = |n(b') - d(b')|$ for all $b, b' \in B$, then $L(x)$ is an arithmetic sequence.

**Proof.** To tackle (1), we first analyze the case where all elements of $B$ are proper fractions. Let $z_h = \sum_{k=0}^{N} c_k b_k^e \in \mathbb{Z}(x)$, where $N, c_k, e_k \in \mathbb{N}_0$ and $b_k \in B$ for each $k \in [0, N]$, be the hub factorization of $x$ (Lemma 4.3). Since the elements of $B$ are proper fractions, $z_h$ is the factorization of minimum length of $x$ by Lemma 4.6. Now let

$$V = \{b_k \in B \mid c_k \geq n(b_k), k \in [1, N]\}, \quad U = \left\{U \subseteq B \mid \sum_{b \in U} n(b) \leq c_0 \right\},$$

and let $W = \{V \cup U \mid U \subseteq U\}$. Note that $\emptyset \in U$, which implies that $W$ is nonempty. For each $W = \{b_1, \ldots, b_m\} \in W$, let

$$L_W = \left\{|z_h| + \sum_{i=1}^{m} P_\infty(d(b_i) - n(b_i))\right\};$$

on the other hand, if $\emptyset \in W$ then $L_\emptyset = \{z_0\}$. We shall prove that $L(x) = \bigcup_{W \in W} L_W$.

Let $z = \sum_{k=0}^{N} c'_k b_k^e \in \mathbb{Z}(x)$, where $N, c'_k, e'_k \in \mathbb{N}_0$ and $b_k \in B$ for each $k \in [0, N]$, such that $z \neq z_h$. Since $z_h$ is the only minimalist factorization of $x$, $c'_k \geq d(b_k)$ for some $k \in [1, N]$. As before, the transformation

$$c'_k b_k^e = (d(b_k) + r)b_k^e = n(b_k)b_k^{e-1} + rb_k^e$$

decreases the value of the coefficient of $b_k^e$, which means that we cannot apply this transformation, over the same term, infinitely many times. Carrying out this transformation as many times as necessary we obtain a sequence $z_1, \ldots, z_n \in \mathbb{Z}(x)$ such that $z_1 = z$, $z_n = z_h$, and $|z_i - z_{i+1}| = |n(b_{k_i}) - d(b_{k_i})|$ for some $k_i \in [1, N]$ and $i \in [1, n - 1]$. Moreover, note that if $c'_k \geq d(b_k)$ for some $k \in [1, N]$, then either...
Consider the canonical rational multicyclic monoids $M$. Theorem 6.3, and it is not hard to see that 

\[
\{ \text{trivial Puiseux monoid (} M\text{submonoids of } x \text{in } \mathbb{N}\}
\]

Note that $1 \leq |D|$ where $L = \infinitely many times. Consequently, it follows that $L \subseteq L(x)$, which implies that $U_{W \in W} L_W \subseteq L(x)$.

Now we proceed to prove (1) in the general case. Let $B' = \{b \in B \mid b < 1\}$. Consider the canonical rational multicyclic monoids $M_{B'}$ and $M_{B \setminus B'}$, which are clearly submonoids of $M_B$. There is no loss in assuming that neither $M_{B'}$ nor $M_{B \setminus B'}$ is the trivial Puiseux monoid ($\{0\}, +$). Moreover, $M_{B \setminus B'}$ is an FF-monoid in virtue of [17, Theorem 6.3], and it is not hard to see that $\mathcal{L}(M_{B'}) \cup \mathcal{L}(M_{B \setminus B'}) \subseteq \mathcal{L}(M_B)$. Now let

\[
\mathcal{D}(x) := \{ (y, y') \mid x = y + y', y \in M_{B'}, \text{and } y' \in M_{B \setminus B'}\}.
\]

Note that $1 \leq |\mathcal{D}(x)| < \infty$ since there are only finitely many elements of $M_{B \setminus B'}$ dividing $x$ in $M_B$. Thus,

\[
L(x) = L_{M_B}(x) = \bigcup_{(y, y') \in \mathcal{D}(x)} \left[ L_{M_{B'}}(y) + L_{M_{B \setminus B'}}(y') \right] = \bigcup_{(y, y') \in \mathcal{D}(x)} \left( \bigcup_{l \in \mathcal{L}(y')} (l + L_{M_{B'}}(y)) \right) = \bigcup_{(y, y') \in \mathcal{D}(x)} \bigcup_{l \in \mathcal{L}(y')} (l + L_{M_{B'}}(y)),
\]

where $\mathcal{L}(y') = L_{M_{B \setminus B'}}(y')$, and (1) follows readily.

To prove (2), let $z_0 \in Z(x)$ be a factorization of minimum length, and set $l = |z_0|$. Let $z = \sum_{k=0}^N c_k b_k^{e_k} \in Z(x)$, where $N, c_k, e_k \in \mathbb{N}_0$ and $b_k \in B$ for every $k \in \mathbb{N}$. Now take $b \in B$. If $z$ is a minimalist factorization of $x$, then $|z| - |z_0| = m \cdot |n(b) - d(b)|$ for some $m \in \mathbb{N}$ by Lemma 4.3. Consequently, $|z| = l + m \cdot |n(b) - d(b)|$. On the other hand, if $z$ is not minimalist, then we shall describe an algorithm to transform $z$ into a minimalist factorization of $x$.

Since $z$ is not minimalist, either $c_0 \geq n(b')$ for some $b' \in B \cap (1, \infty)$ or $c_k \geq n(b_k), d(b_k)$ for some $k \in \mathbb{N}$. Our algorithm starts by applying, as many times as necessary, the transformation

\[
c_k b_k^{e_k} = (d(b_k) + r)b_k^{e_k} = n(b_k)b_k^{e_k-1} + r b_k^{e_k}
\]
over all terms for which \( k > 0 \) and \( c_k \geq d(b_k) = \max(n(b_k), d(b_k)) \). Clearly, we cannot apply this transformation infinitely many times. Next, if \( c_0 \geq n(b') \) for some \( b' \in B \cap (1, \infty) \), then we use the transformation \( c_0 b^0 = (n(b') + r)b^0 = d(b')b' + rb^0 \) and, recursively, we repeat this step for \( rb^0 \). Notice that this transformation does not create new instances of the first issue solved by our algorithm, and again it is not possible to apply this step infinitely many times. Finally, we apply as many times as necessary the transformation

\[
c_k b^k_k = (n(b_k) + r)b^k_k = d(b_k)b^k_{k+1} + rb^k_k
\]

to all terms \( c_k b^k_k \) for which \( k > 0 \) and \( c_k \geq n(b_k) = \max(n(b_k), d(b_k)) \). It is not hard to see that our algorithm stops, and it yields a sequence \( z_1, \ldots, z_t \in \mathbb{Z}(x) \) such that \( z_1 = z \), \( z_t \) is minimal, and \( |z_i| - |z_{i+1}| = |n(b) - d(b)| \) for each \( i \in [1, t-1] \). Hence \( |z| - |z_0| = t \cdot |n(b) - d(b)| \). In virtue of Lemma 4.8, \( |z_i| - |z_0| = m \cdot |n(b) - d(b)| \) for some \( m \in \mathbb{N} \). Thus,

\[
L(x) \subseteq \{ l + n \cdot |n(b) - d(b)| : n \in \mathbb{N} \}.
\]

Now assume, by contradiction, that \( |\Delta(x)| > 1 \). In this case, it is not hard to see that \( |n(b) - d(b)| \in \Delta(x) \), which implies that \( t \cdot |n(b) - d(b)| \in \Delta(x) \) for some \( t \in \mathbb{N}_{>1} \). Let \( z, z' \in \mathbb{Z}(x) \) such that \( |z| - |z'| = t \cdot |n(b) - d(b)| \) and either \( |z_3| \leq |z'| \) or \( |z| \leq |z_3| \) for all \( z_3 \in \mathbb{Z}(x) \). There is no loss in assuming that \( |z_h| \leq |z'| \), where \( z_h \) is the hub factorization of \( x \). We already established that there exists a sequence \( z_1, \ldots, z_m \in \mathbb{Z}(x) \) such that \( z_1 = z \), \( z_m = z_h \), and \( |z_i| - |z_{i+1}| = |n(b) - d(b)| \) for each \( i \in [1, m-1] \). This implies that \( |z'| \leq |z_2| \leq |z| \), which is a contradiction. Therefore, \( |\Delta(x)| \leq 1 \), which concludes our proof.

Note that our previous result does not hold for all Puiseux monoids. The following example exhibits an atomic Puiseux monoid with an element whose set of lengths is not the union of finitely many IMAPs.

**Example 4.8.** Let \( M = \langle 1/p \mid p \in \mathbb{P} \rangle \). It is not hard to see that \( M \) is atomic and \( \mathcal{A}(M) = \{ 1/p \mid p \in \mathbb{P} \} \). Moreover, \( L(1) = \mathbb{P} \). Since arbitrarily large prime gaps exist, \( L(1) \) is not the union of finitely many IMAPs.

As we mentioned before, determining the delta set of a given monoid is, in general, a difficult task. In \[1\], Bowles et al. prove that every set of the form \( \{ d, 2d, \ldots, nd \} \) occurs as the delta set of some numerical monoid for \( n, d \in \mathbb{N} \). We conclude this paper by proving that, for \( d \in \mathbb{N} \), every set of the form \( \{ nd \mid n \in \mathbb{N} \} \) occurs as the delta set of some Puiseux monoid.

**Proposition 4.9.** Let \( d \) be a positive integer. Then there exists a Puiseux monoid \( M \) such that \( \Delta(M) = \{ md \mid m \in \mathbb{N} \} \).
Proof. Let $p_2, p_3, \ldots$ be a sequence of prime numbers such that $\{p_n - dn\}_{n \in \mathbb{N}_{\geq 2}}$ is an increasing sequence. There is no loss in assuming that $p_n > dn + 1$ for all $n \in \mathbb{N}_{\geq 2}$. Now consider the Puiseux monoid

$$M = \left\{ \left( \frac{p_{2k} - 2dk}{p_{2k}} \right)^r, \left( \frac{p_{2k+1} - 2dk + d}{p_{2k+1}} \right)^s \mid r, s \in \mathbb{N}_0 \text{ and } k \in \mathbb{N} \right\}.$$ 

First, $M$ is atomic in virtue of Proposition \[3,9\]. Second, all elements of $M$ have a unique factorization of minimum length by Lemma \[4,6\]. Third, for all $x \in M$, $\mathbb{Z}(x)$ uses finitely many primitive generators as $\{p_n - dn\}_{n \in \mathbb{N}_{\geq 2}}$ is an increasing sequence. This last property implies that, when analyzing $L(x)$ for a fixed $x \in M$, there is no loss in assuming that $M$ is a canonical rational multicyclic monoid. Now let

$$z = (p_{2k} - 2dk)\left( \frac{p_{2k} - 2dk}{p_{2k}} \right)^2 + (p_{2k+1} - 2dk + d)\left( \frac{p_{2k+1} - 2dk + d}{p_{2k+1}} \right)^2 \in \mathbb{Z}(x_{2k}),$$

where $k \in \mathbb{N}$. In virtue of Lemma \[4,6\] $z$ is the factorization of minimum length of $x_{2k}$. Moreover, $L(x_{2k}) = \{ |z| + k_1(2dk) + k_2d(2k - 1) \mid k_1, k_2 \in \mathbb{N}_0 \}$ by the argument used in the first part of the proof of Theorem \[4,7\]. Clearly, $d(2k - 1) \in \Delta(x_{2k})$. Now let $h \in [1, 2k - 2]$, and take $z_1, z_2 \in \mathbb{Z}(x_{2k})$ such that $|z_1| = |z| + h(2dk)$ and $|z_2| = |z| + d(h + 1)(2k - 1)$. Assume, by way of contradiction, that there exists $z_3 \in \mathbb{Z}(x_{2k})$ such that $|z_1| < |z_3| < |z_2|$. Thus, $h(2dk) < t_0(2dk) + s_0d(2k - 1) < d(h + 1)(2k - 1)$ for some nonnegative integers $t_0$ and $s_0$. After some computations, the first inequality yields that $h < t_0 + s_0$, and from the second inequality we obtain that $t_0 + s_0 < h + 1$. This contradiction proves that our hypothesis is untenable. Hence $d(2k - h - 1) \in \Delta(x_{2k})$ for $h \in [1, 2k - 2]$. In other words, $\{d, 2d, \ldots, (2k-1)d\} \subseteq \Delta(x_{2k})$, where $k \in \mathbb{N}$. Consequently, $\{md \mid m \in \mathbb{N} \} \subseteq \Delta(M)$. Conversely, it is not hard to see that an element of $\Delta(x)$ is divisible by $d$ for all $x \in M$, which implies that $\Delta(x) \subseteq \{md \mid m \in \mathbb{N} \}$. Therefore, our result follows.

\[ \square \]

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