On the interior regularity criterion and the number of singular points to the Navier-Stokes equations

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January 3, 2012

Abstract

We establish some interior regularity criterions of suitable weak solutions for the 3-D Navier-Stokes equations, which allow the vertical part of the velocity to be large under the local scaling invariant norm. As an application, we improve Ladyzhenskaya-Prodi-Serrin’s criterion and Escauriza-Seregin-Šverák’s criterion. We also show that if weak solution \( u \) satisfies

\[ \|u(\cdot, t)\|_{L^p} \leq C(-t)^{-\frac{3-p}{2p}} \]

for some \( 3 < p < \infty \), then the number of singular points is finite.

1 Introduction

We consider the three dimensional incompressible Navier-Stokes equations

\[
\begin{align*}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\
\text{div} u &= 0,
\end{align*}
\]

where \( u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \) denotes the unknown velocity of the fluid, and the scalar function \( \pi(x,t) \) denotes the unknown pressure.

In a seminal paper [12], Leray proved the global existence of weak solution with finite energy. It is well known that weak solution is unique and regular in two spatial dimensions. In three dimensions, however, the question of regularity and uniqueness of weak solution is an outstanding open problem in mathematical fluid mechanics.

In a fundamental paper [1], Caffarelli-Kohn-Nirenberg proved that one-dimensional Hausdorff measure of the possible singular points of suitable weak solution \( u \) is zero (see also [13] [22] [14] [23]). The proof is based on the following \( \varepsilon \)-regularity criterion: there exists an \( \varepsilon > 0 \) such that if \( u \) satisfies

\[ \limsup_{r \to 0} r^{-1} \int_{Q_r(z_0)} |\nabla u(y,s)|^2 dyds \leq \varepsilon, \]

then \( u \) is regular at \( z_0 \). The same result remains true if (2) is replaced by

\[ \limsup_{r \to 0} r^{-2} \int_{Q_r(z_0)} |u(y,s)|^3 dyds \leq \varepsilon. \]

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The quantities on the left hand side of (2) and (3) are scaling invariant. More general interior regularity criterions were obtained by Gustafson-Kang-Tsai \cite{7} in terms of scaling invariant quantities (see Proposition 2.5). In the first part of this paper, we will establish some interior regularity criterions, which allow the vertical part of the velocity to be large under the local scaling invariant norm. The proof is based on the blow-up argument and an observation that if the horizontal part of the velocity is small, then the blow-up limit satisfies $u_h = 0$, hence $\partial_t u_3 = \Delta u_3 = \partial_3 \pi = 0$.

Using new interior regularity criterions, we improve Ladyzhenskaya-Prodi-Serrin regularity criterions, which state if the weak solution $u$ satisfies

$$u \in L^q(0,T;L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p \geq 3,$$

then it is regular in $(0,T) \times \mathbb{R}^3$, see \cite{19, 5, 21, 4}. It should be pointed out that the regularity in the class $L^\infty(0,T;L^3(\mathbb{R}^3))$ is highly nontrivial, since it does not fall in the framework of small energy regularity. This case was solved by Escauriza-Seregin-Šverák \cite{4} by using blow-up analysis and the backward uniqueness for the parabolic equation.

In Leary’s paper \cite{12}, he also proved that if $[0^+, T)$ is the maximal existence interval of smooth solution, then for $p > 3$, there exists $c_p > 0$ such that

$$\|u(\cdot,t)\|_{L^p} \geq c_p(-t)^{\frac{3-p}{2p}}.$$  

In general, if $u$ satisfies

$$\|u(\cdot,t)\|_{L^p} \leq C(-t)^{\frac{3-p}{2p}},$$

the regularity of the solution at $t = 0$ remains unknown except $p = 3$. Recently, for the axisymmetric Navier-Stokes equations, important progress has been made by Chen-Strain-Yau-Tsai \cite{2, 3} and Koch-Nadirashvili-Segegin-Šverák \cite{10}, where they showed that the solution does not develop Type I singularity (i.e, $\|u(\cdot,t)\|_{L^\infty} \leq C(-t)^{-\frac{1}{2}}$) by using De-Giorgi-Nash method and Liouville theorem respectively. However, the case without the axisymmetric assumption is still open. The second part of this paper will be devoted to show that the number of singular points is finite if the solution satisfies (4) for $3 < p < \infty$. The proof is based on an improved $\varepsilon$-regularity criterion: if the suitable weak solution $(u, \pi)$ satisfies

$$\sup_{t \in [-1 + t_0, t_0]} \int_{B_1(x_0)} |u(x,t)|^2 \, dx + \int_{-1 + t_0}^{t_0} \left( \int_{B_1(x_0)} |u(x,t)|^4 \, dx \right)^{\frac{3}{2}} \, dt$$

$$+ \int_{-1 + t_0}^{t_0} \left( \int_{B_1(x_0)} |\pi(x,t)|^2 \, dx \right)^{\frac{1}{2}} \, dt \leq \varepsilon_6,$$

then $u$ is regular in $Q_{1+}(z_0)$, see Proposition 5.1.

This paper is organized as follows. In section 2, we introduce some definitions and notations. In section 3, we establish some new interior regularity criterions of suitable weak solutions. In section 4, we apply them to improve Ladyzhenskaya-Prodi-Serrin’s criterion and Escauriza-Seregin-Šverák’s criterion. Section 4 is devoted to the proof of the number of singular points under the condition (4). In the appendix, we present the estimates of the pressure and some scaling invariant quantities.
2 Definitions and notations

Let us first introduce the definition of weak solution.

**Definition 2.1** Let \( \Omega \subset \mathbb{R}^3 \) and \( T > 0 \). We say that \( u \) is a Leray-Hopf weak solution of (1) in \( \Omega_T = \Omega \times (-T,0) \) if

1. \( u \in L^\infty(-T,0;L^2(\Omega)) \cap L^2(-T,0;H^1(\Omega)) \);
2. \( u \) satisfies (1) in the sense of distribution;
3. \( u \) satisfies the energy inequality: for a.e. \( t \in [-T,0] \),
   \[
   \int_{\Omega} |u(x,t)|^2 dx + 2 \int_{-T}^t \int_{\Omega} |\nabla u|^2 dxds \leq \int_{\Omega} |u(x,-T)|^2 dx.
   \]

Furthermore, the pair \((u,\pi)\) is called a suitable weak solution if \( \pi \in L^{3/2}(\Omega_T) \) and the energy inequality is replaced by the following local energy inequality: for any nonnegative \( \phi \in C_\infty_c(\mathbb{R}^3 \times \mathbb{R}) \) vanishing in a neighborhood of the parabolic boundary of \( \Omega_T \),

\[
\int_{\Omega} |u(x,t)|^2 \phi dx + 2 \int_{-T}^t \int_{\Omega} |\nabla u|^2 \phi dxds \\
\leq \int_{-T}^t \int_{\Omega} |u|^2 (\partial_s \phi + \Delta \phi) + u \cdot \nabla \phi (|u|^2 + 2\pi) dxds, \quad \text{for a.e. } t \in [-T,0].
\]

**Remark 2.2** In general, we don’t know whether a Leray-Hopf weak solution is a suitable weak solution. However, if \( u \) is a Leray-Hopf weak solution and \( u \in L^4(\Omega_T) \), then it is also a suitable weak solution, which can be verified by using a standard mollification procedure.

Let \((u,\pi)\) be a solution of (1) and introduce the following scaling

\[
\begin{align*}
  u^\lambda(x,t) &= \lambda u(\lambda x, \lambda^2 t), \\
  \pi^\lambda(x,t) &= \lambda^2 \pi(\lambda x, \lambda^2 t),
\end{align*}
\]

for any \( \lambda > 0 \), then the family \((u^\lambda,\pi^\lambda)\) is also a solution of (1). We introduce some invariant quantities under the scaling (5):

\[
\begin{align*}
  A(u, r, z_0) &= \sup_{-r^2 + t_0 \leq t < t_0} r^{-1} \int_{B_r(z_0)} |u(y,t)|^2 dy, \\
  C(u, r, z_0) &= r^{-2} \int_{Q_r(z_0)} |u(y,s)|^3 dyds, \\
  E(u, r, z_0) &= r^{-1} \int_{Q_r(z_0)} |\nabla u(y,s)|^2 dyds, \\
  D(\pi, r, z_0) &= r^{-2} \int_{Q_r(z_0)} |\pi(y,s)|^\frac{3}{2} dyds.
\end{align*}
\]
where $z_0 = (x_0, t), Q_r(z_0) = (-r^2 + t_0, t_0) \times B_r(x_0)$, and $B_r(x_0)$ is a ball of radius $r$ centered at $x_0$. We also denote $Q_r$ by $Q_r(0)$ and $B_r$ by $B_r(0)$. We also denote

$$G(f, p, q; r, z_0) = r^{-\frac{\ell}{p} - \frac{2}{q}} \|f\|_{L_p,q(Q_r(z_0))},$$

$$H(f, p, q; r, z_0) = r^{2 - \frac{\ell}{p} - \frac{2}{q}} \|f\|_{L_p,q(Q_r(z_0))},$$

$$\tilde{G}(f, p, q; r, z_0) = r^{-\frac{\ell}{p} - \frac{2}{q}} \|f - (f)_{B_r(x_0)}\|_{L_p,q(Q_r(z_0))},$$

$$\tilde{H}(f, p, q; r, z_0) = r^{2 - \frac{\ell}{p} - \frac{2}{q}} \|f - (f)_{B_r(x_0)}\|_{L_p,q(Q_r(z_0))},$$

where the mixed space-time norm $\|\cdot\|_{L_p,q(Q_r(z_0))}$ is defined by

$$\|f\|_{L_p,q(Q_r(z_0))} = \left( \int_{t_0-r^2}^{t_0} \left( \int_{B_r(x_0)} |f(x, t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

and $(f)_{B_r(x_0)}$ is the average of $f$ in the ball $B_r(x_0)$. For the simplicity of notations, we denote

$$A(u, r, (0, 0)) = A(u, r), \quad \tilde{C}(u, r) = C(u - (u)_{B_r}, r), \quad G(f, p, q; r, (0, 0)) = G(f, p, q; r)$$

and so on. These scaling invariant quantities will play an important role in the interior regularity theory.

Now we recall the definitions of Lorentz space and BMO space [6].

**Definition 2.3** Let $\Omega \subset \mathbb{R}^n$ and $1 \leq p, \ell \leq \infty$. We say that a measurable function $f \in L^{p,\ell}(\Omega)$ if $\|f\|_{L^{p,\ell}(\Omega)} < +\infty$, where

$$\|f\|_{L^{p,\ell}(\Omega)} \overset{\text{def}}{=} \begin{cases} \left( \int_0^\infty \sigma^{\ell-1} |\{x \in \Omega; |f| > \sigma\}|^p \sigma d\sigma \right)^{\frac{1}{p}} & \text{for } \ell < +\infty, \\ \sup_{\sigma > 0} \sigma |\{x \in \Omega; |f| > \sigma\}|^{\frac{1}{\ell}} & \text{for } \ell = +\infty. \end{cases}$$

Moreover, $f(x, t) \in L^{\ell,\infty}(-T, 0; L^{p,\ell}(\Omega))$ if $\|f(\cdot, t)\|_{L^{p,\ell}(\Omega)} \in L^{\ell,\infty}(-T, 0)$.

The following facts will be used frequently: for any $R > 0$,

$$\|f\|_{L^{p,\ell}} \leq \|f\|_{L^{p,2}}, \quad \text{if} \quad \ell_1 \geq \ell_2;$$

$$\|f\|_{L^{p,1}(\Omega)} \leq C(R^p|\Omega| + R^{p_1-\ell} \|f\|_{L^{p,\infty}(\Omega)}^{p_1}), \quad \text{if} \quad p > p_1. \quad (7)$$

Recall that a local integrable function $f \in BMO(\mathbb{R}^n)$ if it satisfies

$$\sup_{R > 0, x_0 \in \mathbb{R}^n} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| dx < \infty.$$ 

Moreover, $f(x) \in VMO(\mathbb{R}^n)$ if $f(x) \in BMO(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$, 

$$\limsup_{R \downarrow 0} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}| dx = 0.$$ 

We say that a function $u \in BMO^{-1}(\mathbb{R}^n)$ if there exist $U_j \in BMO(\mathbb{R}^n)$ such that $u = \sum_{j=1}^n \partial_j U_j$. $VMO^{-1}(\mathbb{R}^n)$ is defined similarly. A remarkable property of BMO function is

$$\sup_{R > 0, x_0 \in \mathbb{R}^n} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x) - f_{B_R(x_0)}|^p dx < \infty.$$
for any $1 \leq q < \infty$.

Let us conclude this section by recalling the following $\varepsilon$-regularity results. Here and what follows, we define a solution $u$ to be regular at $z_0 = (x_0, t_0)$ if $u \in L^\infty(Q_r(z_0))$ for some $r > 0$.

**Proposition 2.4** [1, 14] Let $(u, \pi)$ be a suitable weak solution of (1) in $Q_1(z_0)$. There exists an $\varepsilon_0 > 0$ such that if
\[
\int_{Q_1(z_0)} |u(x,t)|^3 + |\pi(x,t)|^{3/2} dxdt \leq \varepsilon_0,
\]
then $u$ is regular in $Q_{1/2}(z_0)$. Moreover, $\pi$ can be replaced by $\pi - (\pi)_B$, in the integral.

**Proposition 2.5** [7] Let $(u, \pi)$ be a suitable weak solution of (1) in $Q_1(z_0)$ and $w = \nabla \times u$. There exists an $\varepsilon_1 > 0$ such that if one of the following two conditions holds,

1. $G(u,p,q;r,z_0) \leq \varepsilon_1$ for any $0 < r < \frac{1}{2}$, where $1 \leq \frac{2}{p} + \frac{2}{q} \leq 2$;

2. $H(w,p,q;r,z_0) \leq \varepsilon_1$ for any $0 < r < \frac{1}{2}$, where $2 \leq \frac{3}{p} + \frac{2}{q} \leq 3$ and $(p,q) \neq (1, \infty)$;

then $u$ is regular at $z_0$.

3 Interior regularity criterions of suitable weak solution

The purpose of this section is to establish some interior regularity criterions, which allow the vertical part of the velocity to be large under the local scaling invariant norm. These results improve some classical results and Gustafson-Kang-Tsai’s result (Proposition 2.5).

Set $u = (u_h, u_3)$. Let us state our main results.

**Theorem 3.1** Let $(u, \pi)$ be a suitable weak solution of (1) in $Q_1$ and satisfy
\[
C(u,1) + D(\pi,1) \leq M.
\]

Then there exists a positive constant $\varepsilon_2$ depending on $M$ such that if
\[
C(u_h,1) \leq \varepsilon_2,
\]
then $u$ is regular at $(0, 0)$.

**Theorem 3.2** Let $(u, \pi)$ be a suitable weak solution of (1) in $Q_1$ and satisfy
\[
G(u,p,q;r) \leq M \quad \text{for any } 0 < r < 1,
\]
where $1 \leq \frac{3}{p} + \frac{2}{q} < 2$, $1 < q \leq \infty$. There exists a positive constant $\varepsilon_3$ depending on $p, q, M$ such that $(0, 0)$ is a regular point if
\[
G(u_h,p,q;r^*) \leq \varepsilon_3
\]
for some $r^*$ with $0 < r^* < \min\{\frac{1}{2}, (C(u,1) + D(\pi,1))^{-2}\}$.
Theorem 3.3 Let \((u, \pi)\) be a suitable weak solution of (1) in \(Q_1\) and satisfy
\[
H(\nabla u, p, q; r) \leq M \quad \text{for any } 0 < r < 1,
\]
where \(2 \leq \frac{3}{p} + \frac{2}{q} < 3, 1 < p \leq \infty\). There exists a positive constant \(\varepsilon_4\) depending on \(p, q, M\) such that \((0, 0)\) is a regular point if
\[
H(\nabla u_h, p, q; r^*) \leq \varepsilon_4 \tag{8}
\]
for some \(r^*\) with \(0 < r^* < \min\{\frac{1}{2}, (C(u, 1) + D(p, 1))^{-2}\} \).

Remark 3.4 As a special case of Theorem 3.2, it follows that \(u\) is regular if
\[
|u_3| \leq \frac{M}{\sqrt{T - t}}, \quad |u_h| \leq \frac{\varepsilon_3}{\sqrt{T - t}}.
\]
which improves Leray’s result [12]. And from Theorem 3.3, it follows that \(u\) is regular at \((0, 0)\) if for any \(0 < r < 1\),
\[
r^{-1} \int_{Q_r} |\nabla u_3|^2 dxdt \leq M^2, \quad r^{-1} \int_{Q_r} |\nabla u_h|^2 dxdt \leq \varepsilon_4^2,
\]
which improves Caffarelli-Kohn-Nirenberg’s result [1].

The proof of Theorem 3.1 is based on compactness argument and the following lemma.

Lemma 3.5 Let \((u, \pi)\) be a suitable weak solution of (1) in \(Q_1\) and \(\bar{D}(\pi, 1) \leq M\). Then \(u\) is regular at \((0, 0)\) if
\[
C(u, r_0) \leq c\varepsilon_0^{9/5} r_0^{8/5} \quad \text{for some } 0 < r_0 \leq 1.
\]
Here \(c\) is a small constant depending on \(M\).

Proof. By (27) and Hölder inequality, for \(0 < r < r_0/4\) we have
\[
C(u, r) + \bar{D}(\pi, r) \leq \frac{r_0^2}{r^2} C(u, r_0) + C\left(\frac{r}{r_0}\right)^{5/2} \bar{D}(\pi, r_0) + C\frac{r_0^2}{r^2} C(u, r_0)
\]
\[
\leq CM \frac{r_0^{5/2}}{r^{9/2}} + C\frac{r_0^2}{r^2} C(u, r_0).
\]
Choosing \(r = (\frac{r_0}{2C M})^{2/5} r_0^{9/5}\) and by assumption, we infer that
\[
C(u, r) + \bar{D}(\pi, r) < \varepsilon_0,
\]
which implies that \((0, 0)\) is a regular point by Proposition 2.4. \(\square\)
Now let us turn to the proof of Theorem 3.1.

Proof of Theorem 3.1 Assume that the statement of the proposition is false, then there exist a constant $M$ and a sequence $(u^k, \pi^k)$, which are suitable weak solutions of (11) in $Q_1$ and singular at $(0,0)$, and satisfies

$$C(u^k,1) + D(\pi^k,1) \leq M, \quad C(u^k_h,1) \leq \frac{1}{k}.$$  

Then by the local energy inequality, it is easy to get

$$A(u^k,3/4) + E(u^k,3/4) \leq C(M),$$

hence by using Lions-Aubin’s lemma, there exists a suitable weak solution $(v,\pi')$ of (11) such that (at most up to subsequence),

$$u^k \to v, \quad u^k \to 0 \text{ in } L^3(Q_{1/2}), \quad \pi^k \to \pi' \text{ in } L^3(Q_{1/2}),$$

as $k \to +\infty$. That is, $v_b = 0$, which gives $\partial_3 v_3 = 0$ by $\nabla \cdot v = 0$, and hence,

$$\partial_t v_3 + \partial_3 \pi' - \Delta v_3 = 0, \quad -\Delta \pi' = 0, \quad \text{or} \quad \partial_t v + \nabla \pi' - \Delta v = 0,$$

which implies that $|v| \leq C(M)$ in $Q_{1/4}$ by the classical result of linear Stokes equation (see [2] for example). However, $(0,0)$ is a singular point of $u^k$, hence by Lemma 3.5, for any $0 < r < 1/4$,

$$c \leq C(v,r) \leq C(M)r^3,$$

which is a contradiction by letting $r \to 0$. \hfill $\Box$

The proof of Theorem 3.2 is motivated by [13] and based on the blow-up argument.

Proof of Theorem 3.2 Assume that the statement of the proposition is false, then there exist constants $p,q,M$ and a sequence $(u^k,\pi^k)$, which are suitable weak solutions of (11) in $Q_1$ and singular at $(0,0)$, and satisfy

$$G(u^k,p,q;r) \leq M \quad \text{for all} \quad 0 < r < 1,$$

$$G(u^k_h,p,q;r_k) \leq \frac{1}{k},$$

where $0 < r_k < \min\{\frac{1}{2},(C(u^k,1) + D(\pi^k,1))^{-2}\}$. Then it follows from Lemma 5.2 that

$$A(u^k,r) + E(u^k,r) + D(\pi^k,r) \leq C(M,p,q)$$

for any $0 < r < r_k$.

Set $u^k(x,t) = r_k u^k(r_k x, r_k^2 t), q^k(x,t) = r_k^2 \pi^k(r_k x, r_k^2 t)$. Then

$$A(v^k,r) + E(v^k,r) + D(q^k,r) \leq C(M,p,q)$$

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for any $0 < r < 1$. Lions-Aubin’s lemma ensures that there exists a suitable weak solution $(\bar{v}, \bar{\pi})$ of (1) such that (at most up to subsequence),

$$v^k \rightarrow \bar{v} \quad \text{in} \quad L^3(Q_{\frac{3}{4}}), \quad q^k \rightarrow \bar{q} \quad \text{in} \quad L^\frac{3}{2}(Q_{\frac{3}{4}}),$$

$$v^k_h \rightarrow 0 \quad \text{in} \quad L^3((-\frac{1}{4},0);L^p(B_{\frac{3}{4}})),$$

as $k \rightarrow +\infty$. Then we have $\bar{v}_h = 0$ and

$$\partial_t \bar{v}_3 + \partial_3 \bar{q} - \Delta \bar{v}_3 = 0,$$

which implies that $|\bar{v}_3| \leq C(M)$ in $Q_{\frac{3}{4}}$. However, $(0,0)$ is a singular point of $v^k$, hence by Proposition 2.4 and (26), for any $0 < r < r^*$, without loss of generality, let us assume that

$$\frac{8}{3} < \frac{3}{p} + \frac{2}{q} < 3.$$

The other case can be reduced to it by Hölder inequality. By Lemma 6.2, we have

$$A(u,r) + E(u,r) + D(\pi, r) \leq C(M) \left( r^{1/2} (C(u,1) + D(\pi,1)) + 1 \right) \leq C(M),$$

for any $0 < r \leq r_1 \doteq \min \left\{ \frac{1}{2}, (C(u,1) + D(p,1))^{-2} \right\}$. This together with interpolation inequality gives

$$C(u,r) \leq C(M) \quad \text{for any} \quad 0 < r \leq r_1. \quad (9)$$

We get by Poincaré inequality that

$$\bar{G}(u_h, p_1, q_1; r) \leq C\bar{H}(\nabla u_h, p, q; r),$$

where $p_1 = \frac{3p}{4-p}, q_1 = q$, hence it follows from (28) and (8) that

$$\bar{C}(u_h, r^*) \leq C(M) \left( A(u_h, r^*) + E(u_h, r^*) \right)^{\frac{1-\delta}{1-25}} \bar{G}(u_h, p_1, q_1; r^*)^{\frac{1}{1-25}},$$

$$\leq C(M)\bar{G}(u_h, p_1, q_1; r^*)^{\frac{1}{1-25}} \leq C(M)\varepsilon_4^{\frac{1}{1-25}},$$

where $\delta = 2 - \frac{3}{p_1} - \frac{2}{q_1} \in (0, \frac{1}{2})$, hence by (21) for $0 < r < r^*$

$$C(u_h, r) \leq C\left( \frac{r}{r^*} \right) C(u_h, r^*) + C\left( \frac{r}{r^*} \right)^2 \bar{C}(u_h, r^*) \leq C(M) \left( \frac{r}{r^*} + \left( \frac{r}{r^*} \right)^2 \varepsilon_4^{\frac{1}{1-25}} \right).$$

Taking $r$ small enough, and then $\varepsilon_4$ small enough such that

$$C(u_h, r) \leq \varepsilon_3^3,$$

Then the result follows from Theorem 3.2. \hfill \Box
4 Applications of interior regularity criterions

4.1 Ladyzhenskaya-Prodi-Serrin’s criterion

Using the interior regularity criterions established in Section 3, we present Ladyzhenskaya-Prodi-Serrin’s type criterions in Lorentz spaces.

Theorem 4.1 Let u be a Leray-Hopf weak solution of (1) in \( \mathbb{R}^3 \times (-1, 0) \). Assume that u satisfies

\[
\|u\|_{L^\infty((-1,0);L^p,\infty(\mathbb{R}^3))} \leq M, \quad \|u_h\|_{L^\infty((-1,0);L^{p,\infty}(\mathbb{R}^3))} < \infty,
\]

where \( \frac{3}{p} + \frac{2}{q} = 1 \), \( 3 < p \leq \infty \), and \( 1 \leq \ell < \infty \). Then u is regular in \( \mathbb{R}^3 \times (-1, 0) \). For \( \ell = \infty \) or \( p = 3 \), the same result holds if the second condition of (10) is replaced by

\[
\|u_h\|_{L^\infty((-1,0);L^p,\infty(\mathbb{R}^3))} \leq \varepsilon_5,
\]

where \( \varepsilon_5 \) is a small constant depending on M.

Remark 4.2 For \( \ell = \infty \), we improve Kim-Kozono’s result [9] and He-Wang’s result [8], where the smallness of all components of the velocity is imposed. In general case, we improve Sohr’s result [20] by allowing the vertical part of the velocity to fall in weak \( L^p \) space.

Remark 4.3 Under the condition (10), it can be verified that Leray-Hopf weak solution is suitable weak solution. We left it to the interested readers.

The proof is based on the following lemma.

Lemma 4.4 Assume that u satisfies

\[
\|u\|_{L^\infty((-1,0);L^p,\infty(\mathbb{R}^3))} \leq m,
\]

where \( \frac{3}{p} + \frac{2}{q} = 1 \), \( 3 \leq p \leq \infty \). Then for any \( 0 < r < 1 \) and \( 0 < \epsilon < 1 \), there hold

\[
G(u, \frac{9}{10}p, \frac{4}{5}q, r) \leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q, \quad 3 < p < \infty,
\]

\[
A(u, r) \leq C\epsilon^2 + C\epsilon^{-1}m^3, \quad p = 3,
\]

\[
G(u, \infty, \frac{3}{2}; r) \leq C\epsilon^{3/2} + C\epsilon^{-1/2}m^2, \quad p = \infty.
\]

Proof. First we consider the case of \( 3 < p < \infty \). Using the definition of Lorentz space, we infer that

\[
R^{\frac{4}{5} - \frac{8}{5}p} q^{-2} \int_{-r^2}^0 \left( \int_{B_r} \frac{u^q}{|x|^p} dx \right)^{\frac{8q}{5p}} dt
\]

\[
\leq Cr^{\frac{4}{5} - \frac{8}{5}p} q^{-2} \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]

\[
\leq C\epsilon^{\frac{4a}{5}} + C\epsilon^{-\frac{2}{5}}m^q + \int_{-r^2}^0 \left( \int_0^\infty \sigma^{\frac{q}{p}-1} |\{x \in B_r; |u(x,t)| > \sigma\}|d\sigma \right)^{\frac{8q}{5p}} dt
\]
where we take $R = \varepsilon r^{-1}$ and the estimate of $I(r)$ is given by

$$I(r) \equiv r^{(1-\frac{2}{p})\frac{8q}{q-2}} \int_{-r^2}^{0} \|u(\cdot, t)\|_{L^{p,\infty}(B_1)}^{\frac{8q}{q-2}} dt$$

$$\leq Cr^{(1-\frac{3}{p})\frac{8q}{q-2}} \int_{-r^2}^{0} |\{t \in (-r^2, 0); \|u(\cdot, t)\|_{L^{p,\infty}} > \sigma\}| d\sigma$$

$$\leq Cr^{(1-\frac{3}{p})\frac{8q}{q-2}} \left( R^{\frac{8q}{q-2}} r + \int_{-r^2}^{0} \|u(\cdot, t)\|_{L^{p,\infty}(-1,0;L^{q,\infty}(B_1))}^{\frac{8q}{q-2}} dt \right)$$

$$\leq Cr^{(1-\frac{3}{p})\frac{8q}{q-2}} \left( R^{\frac{8q}{q-2}} r + R^{-\frac{q}{2}} m^q \right)$$

$$\leq Cr^{\frac{8q}{q-2}} + C\varepsilon^{-\frac{q}{2}} m^q \quad (R = \varepsilon r^{-1}).$$

This gives the first inequality. For $p = 3$, we consider

$$\sup_{-r^2 < t < 0} r^{-1} \int_{B_r} |u|^2 dx \leq C \sup_{-r^2 < t < 0} r^{-1} \int_{0}^{\infty} \sigma |\{x \in B_r; |u(x, t)| > \sigma\}| d\sigma$$

$$\leq C \sup_{-r^2 < t < 0} r^{-1} \left( R^2 r^3 + \int_{-r^2}^{0} \sigma |\{x \in B_r; |u(x, t)| > \sigma\}| d\sigma \right)$$

$$\leq C \sup_{-r^2 < t < 0} r^{-1} \left( R^2 r^3 + R^{-1} \|u(\cdot, t)\|_{L^{3,\infty}}^3 \right),$$

which gives the second inequality by taking $R = \varepsilon r$. Let $g(t) = \|u(\cdot, t)\|_{L^\infty(B_1)}$. Then we have

$$r^{-1/2} \int_{-r^2}^{0} g(t)^{3/2} dt \leq C r^{-1/2} \int_{0}^{\infty} \sigma^{\frac{1}{2}} |\{t \in (-r^2, 0); |g(t)| > \sigma\}| d\sigma$$

$$\leq C r^{-1/2} \left( R^{\frac{3}{2}} r^2 + \int_{-r^2}^{0} \sigma^{\frac{1}{2}} |\{t \in (-r^2, 0); |g(t)| > \sigma\}| d\sigma \right)$$

$$\leq Cr^{-\frac{1}{2}} \left( R^{\frac{3}{2}} r^2 + R^{-\frac{1}{2}} m^2 \right),$$

which gives the third inequality by taking $R = \varepsilon r$.  

**Proof of Theorem 4.1** By translation invariance and Theorem 3.2 it suffices to show that

$$G(u, p_1, q_1; r) \leq M, \quad G(u_h, p_1, q_1; r) \leq \varepsilon_3, \quad (11)$$

for any $0 < r < 1/2$ and some $(p_1, q_1)$ with $1 \leq \frac{3}{p_1} + \frac{2}{q_1} < 2$. For $3 < p < \infty$, let $p_1 = \frac{9}{10}p$ and $q_1 = \frac{4}{5}q$, then $\frac{3}{p_1} + \frac{2}{q_1} < \frac{3}{2} < 2$. For $\ell < \infty$, we have $\|u_h\|_{L^\infty((-r^2, 0;L^{p,\infty}(B_1))} \rightarrow 0$ as $r \rightarrow 0$. Hence by Lemma 4.3 the condition (11) holds if we take $\varepsilon$ small enough, and then take $r$ small enough. The proof of the other cases is similar. We omit the details.

### 4.2 Escauriza-Seregin-Šverák’s criterion

The following theorem improves Escauriza-Seregin-Šverák’s criterion by noting the inclusion

$$L^3(\mathbb{R}^3) \subset L^{3,\ell}(\mathbb{R}^3) \quad \text{for } \ell > 3 \quad \text{and} \quad L^3(\mathbb{R}^3) \subset VMO^{-1}(\mathbb{R}^3).$$
Theorem 4.5 Let \((u, \pi)\) be a suitable weak solution of (4) in \(\mathbb{R}^3 \times (-1, 0)\). If
\[
\|u_h\|_{L^\infty((-1,0);L^3,\ell(\mathbb{R}^3))} + \|u_3\|_{L^\infty((-1,0);BMO^{-1}(\mathbb{R}^3))} = M < \infty,
\]
for some \(\ell < \infty\), and \(u_3(x,t) \in VMO^{-1}(\mathbb{R}^3)\) for \(t \in (-1, 0]\), then \(u\) is regular in \(\mathbb{R}^3 \times (-1, 0]\).

We need the following lemma, which gives a bound of local scaling invariant energy.

Lemma 4.6 Under the assumptions of Theorem 4.5, there holds
\[
A(u, r) + E(u, r) + D(\pi, r) \leq C(M, C(u, 1), D(\pi, 1)) \quad \text{for any } 0 < r < 1/2.
\]

Proof. Let \(\zeta(x,t)\) be a smooth function with \(\zeta \equiv 1\) in \(Q_r\) and \(\zeta = 0\) in \(Q_{2r}^c\). Since \(u_3 \in L^\infty(-1,0;BMO^{-1}(\mathbb{R}^3))\), there exists \(U(x,t) \in L^\infty(-1,0;BMO(\mathbb{R}^3))\) such that \(u_3 = \nabla \cdot U\). We have by Hölder inequality that
\[
r^{-2} \int_{Q_{2r}} |u_3|^3 \zeta^2 dxdt
= r^{-2} \int_{Q_{2r}} \sum_{j=1}^3 \partial_j U_j \cdot u_3 |u_3| \zeta^2 dxdt
\]
\[
\leq 6r^{-2} \int_{Q_{2r}} |U - U_{B_{2r}}| (|\nabla u_3| |u_3| + |u_3|^2 |\nabla \zeta|) dxdt
\]
\[
\leq 6r^{-2} \left( \int_{Q_{2r}} |U - U_{B_{2r}}|^6 dxdt \right)^{1/6} \left( \int_{Q_{2r}} |\nabla u_3|^2 dxdt \right)^{1/2} \left( \int_{Q_{2r}} |u_3|^3 dxdt \right)^{1/3}
+ 12r^{-3} \left( \int_{Q_{2r}} |U - U_{B_{2r}}|^3 dxdt \right)^{1/3} \left( \int_{Q_{2r}} |u_3|^3 dxdt \right)^{2/3},
\]
which implies that
\[
C(u_3, r) \leq C(M) (E(u, 2r)^{1/2} C(u, 2r)^{1/3} + C(u, 2r)^{2/3}).
\]
(12)

On the other hand, we have by Lemma 4.3 that
\[
A(u_h, r) \leq C(M) \quad \text{for any } 0 < r < 1,
\]
which along with the interpolation inequality gives
\[
C(u_h, r) \leq A(u, r)^{3/4} (E(u, r) + A(u, r))^{3/4} \leq C(M)(E(u, r) + A(u, r))^{3/4}.
\]
(13)

We infer from (12) and (13) that
\[
C(u, r) \leq C(M)(E(u, 2r) + A(u, 2r) + C(u, 2r))^{5/6}.
\]

With this, following the proof of Lemma 6.2, we conclude the result.
Proof of Theorem 4.5. Following [4], the proof is based on the blow-up analysis and unique continuation theorem. Without loss of generality, assume that \((0,0)\) is a singular point. Then by Theorem 3.2, there exists a sequence of \(r_k \downarrow 0\) such that
\[
\int_{Q_{r_k}} |u_h|^3 \, dx \, dt \geq \varepsilon_1. \tag{14}
\]
Let \(u^k(x,t) = r_k u(r_k x, r_k^2 t)\) and \(\pi^k(x,t) = r_k^2 \pi(r_k x, r_k^2 t)\). Then for any \(a > 0\) and \(k\) large enough, it follows from Lemma 4.6 that
\[
A(u^k, a) + E(u^k, a) + C(u^k, a) + D(\pi^k, a) \leq C(M, D(\pi, 1)). \tag{16}
\]
Using Lions-Aubin lemma, there exists \((v, \pi')\) such that for any \(a, T > 0\) (up to subsequence)
\[
u^k \to v \quad \text{in} \quad L^3(B_a \times (-T, 0)),
\pi^k \to \pi' \quad \text{in} \quad L^3(-T, 0; L^\infty(B_a)),
\]
as \(k \to +\infty\) (see the proof of Theorem 4.1 in [24] for the details). Furthermore, there hold
\[
\|v_h\|_{L^\infty(-a^2, 0; L^3,\ell(\mathbb{R}^3))} \leq \sup_k \|u_h^k\|_{L^\infty(-a^2, 0; L^3,\ell(\mathbb{R}^3))} \leq M, \tag{15}
\]
and for any \(z_0 = (x_0, t_0) \in (-T + 1, 0) \times \mathbb{R}^3\),
\[
A(v, 1; z_0) + E(v, 1; z_0) + C(v, 1; z_0) + D(\pi', 1; z_0) \leq C(M, D(p, 1)). \tag{16}
\]
Due to (15) and (16), we infer that
\[
\int_{Q_1(z_0)} |v_h|^2 \, dx \, dt \to 0, \quad \text{as} \ z_0 \to \infty,
\]
which along with (16) implies that
\[
\int_{Q_1(z_0)} |v_h|^3 \, dx \, dt \to 0, \quad \text{as} \ z_0 \to \infty.
\]
Hence by Theorem 3.1, there exists \(R > 0\) such that
\[
|v(x,t)| + |\nabla v(x,t)| \leq C, \quad (t,x) \in (-T + 1, 0) \times \mathbb{R}^3 \setminus B_R.
\]
Due to \(u_h(x,0) \in L^{3,\ell}\), we infer that
\[
\int_{B_a} |v_h(x,0)| \, dx \leq \int_{B_a} |v_h(x,0) - u_h^k(x,0)| \, dx + \int_{B_a} |u_h^k(x,0)| \, dx \leq \int_{B_a} |v_h(x,0) - u_h^k(x,0)| \, dx + r_k^{-2} \int_{B_{ar_k}} |u_h(y,0)| \, dy \leq \int_{B_a} |v_h(x,0) - u_h^k(x,0)| \, dx + C \|u_h(0)\|_{L^{3,\ell}(B_{ar_k})} \to 0, \quad \text{as} \ k \to \infty,
\]

which implies \( v_h(x,0) = 0 \) a.e. \( \mathbb{R}^3 \). And due to \( u_3(x,0) \in VMO^{-1}(\mathbb{R}^3) \), we have \( v_3(x,0) = 0 \) (see Theorem 4.1 in [24]).

Let \( w = \nabla \times v \), then \( w(x,0) = 0 \) and

\[
|\partial_t w - \Delta w| \leq C(|w| + |\nabla w|), \quad (-T + 1,0) \times \mathbb{R}^3 \setminus B_R.
\]

By the backward uniqueness property of parabolic operator [4], we have \( w = 0 \) in \((-T + 1,0) \times \mathbb{R}^3 \setminus B_R \).

Similar arguments as in [4], using spatial unique continuation we have \( w \equiv 0 \) in \((-T + 1,0) \times \mathbb{R}^3 \), which implies \( \Delta v \equiv 0 \) in \((-T + 1,0) \times \mathbb{R}^3 \), hence \( v_h \equiv 0 \) in \((-T + 1,0) \times \mathbb{R}^3 \), since \( v_h(\cdot,t) \in L^3, \ell \). This is a contradiction to (14). □

5 The number of singular points

5.1 An improved \( \varepsilon \)-regularity criterion

We need the following improved version, which may be independent of interest.

**Proposition 5.1** Let \((u,\pi)\) be a suitable weak solution of (1) in \(Q_{1/2}(z_0)\). There exists an \( \varepsilon_6 > 0 \) such that if

\[
\sup_{t \in [-1+\tau,\tau]} \int_{B_1(x_0)} |u(x,t)|^2 dx + \int_{-1+\tau}^{\tau} \left( \int_{B_1(x_0)} |u(x,t)|^4 dx \right)^{3/4} dt
\]

\[
+ \int_{-1+\tau}^{\tau} \left( \int_{B_1(x_0)} |\pi(x,t)|^2 dx \right)^{3/4} dt \leq \varepsilon_6,
\]

then \( u \) is regular in \( Q_{1/2}(z_0) \).

**Remark 5.2** Due to Lemma 6.1, the above norm of the pressure can be replaced by \( L^1(Q_{1/2}(z_0)) \) norm. A slightly different version of Proposition 5.1 was obtained by Vasseur [23], who used the De Giorgi iterative method.

**Proof.** By Proposition 2.5 and translation invariance, it suffices to prove that

\[
A(u,r) + E(u,r) \leq \varepsilon_6^{3/4} \leq \varepsilon_1^2
\]

for any \( 0 < r < 1/2 \). Set \( r_n = 2^{-n} \), where \( n = 1, 2, \ldots \). First of all, (17) holds for \( r = r_1 \) by local energy inequality. Suppose that (17) holds for \( r_k \) with \( k \leq n - 1 \). We need to show that

\[
A(u,r_n) + E(u,r_n) \leq \varepsilon_6^{3/4}.
\]

Let \( \phi_n = \chi \psi_n \), where \( \chi \) is a cutoff function which equals 1 in \( Q_{1/4} \) and vanishes outside of \( Q_{1/3} \), and \( \psi_n \) is as follows:

\[
\psi_n = (r_n^2 - t)^{-3/2} e^{-\frac{|x|^2}{4(r_n^2 - t)}}.
\]

Direct computations show that \( \phi_n \geq 0 \) and

\[
(\partial_t + \Delta)\phi_n = 0 \quad \text{in} \quad Q_{1/4},
\]
\[ |(\partial_t + \Delta)\phi_n| \leq C_1 \quad \text{in} \quad Q_{1/3}, \]
\[ C_1^{-1}r_n^{-3} \leq \phi_n \leq C_1 r_n^{-3}, \quad |\nabla \phi_n| \leq C_1 r_n^{-4} \quad \text{on} \quad Q_{r_n} \quad n \geq 2, \]
\[ \phi_n \leq C_1 r_k^{-3}, \quad |\nabla \phi_n| \leq C_1 r_k^{-4} \quad \text{on} \quad Q_{r_{k-1}/r_k} \quad 1 < k \leq n. \]

Using \( \phi_n \) as a test function in the local energy inequality, we get
\[
\sup_{-r_n^2 < t < 0} r_n^{-1} \int_{B_r} |u(x, t)|^2 dx + r_n^{-1} \int_{Q_{r_n}} |\nabla u|^2 dx dt \\
\leq C_1^2 r_n^2 \int_{Q_1} |u|^2 dx dt + C_1 r_n^2 \int_{Q_1} |u|^3 |\nabla \phi_n| dx dt + C_1 r_n^2 \int_{Q_1} |\nabla \phi_n| dx dt \\
def \quad I_1 + I_2 + I_3.
\]

Firstly, we have by assumption that
\[ I_1 \leq C_1 T r_n^{-2} \varepsilon_6. \]

Recall that the following well-known interpolation inequality from [1]: for \( \rho \geq r > 0 \)
\[ C(u, r) \leq C\left( \frac{\rho}{r} \right)^3 A(u, \rho)^{3/4} E(u, \rho)^{3/4} + C\left( \frac{\rho}{r} \right)^3 A(u, \rho)^{3/2}, \]
from which and the induction assumption, it follows that
\[ I_2 \leq C_1^2 r_n^2 \sum_{k=1}^n r_k^{-4} \int_{Q_{r_k}} |u|^3 dx dt \\
\leq Cr_n^2 \sum_{k=1}^n r_k^{-2} \varepsilon_6^{3/4} \leq C \varepsilon_6^{3/4}. \]

To estimate \( I_3 \), we choose \( \chi_k \) to be a cutoff function, which vanishes outside of \( Q_{r_k} \) and equals 1 in \( Q_{7/8r_k} \), and \( |\nabla \chi_k| \leq Cr_k^{-1} \). We have by the induction assumption that
\[ I_3 \leq C_1 r_n^2 \sum_{k=1}^{n-1} \left| \int_{Q_1} \pi(u \cdot \nabla ((\chi_k - \chi_{k+1})\phi_n)) dx dt \right| + C_1 r_n^2 \int_{Q_1} \pi u \cdot \nabla (\chi_n \phi_n) dx dt \\
\leq C_1 r_n^2 \sum_{k=1}^{n-1} \left| \int_{Q_1} (\pi - (\pi)_{B_k}) u \cdot \nabla ((\chi_k - \chi_{k+1})\phi_n) dx dt \right| \\
+ C_1 r_n^2 \left| \int_{Q_1} (\pi - (\pi)_{B_n}) u \cdot \nabla (\chi_n \phi_n) dx dt \right| \\
\leq C_1 r_n^2 \sum_{k=3}^{n} r_k^{-4} \int_{Q_{r_k}} |(\pi - (\pi)_{B_k}) u| dx dt + C_1 r_n^2 \int_{Q_1} |u| |\pi| dx dt \\
\leq C_1 r_n^2 \sum_{k=3}^{n} r_k^{-2} \varepsilon_6^{1/4} H(\pi, 2, 1; r_k) + C \varepsilon_6^{3/2},
\]

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and by Lemma 6.1 and interpolation inequality, we get
\[
\tilde{H}(\pi, 2, 1; \theta^j) \leq C\theta \tilde{H}(\pi, 1, 1; \theta^{j-1}) + C\theta^{-\frac{3}{2}}G(u, 4, 2; \theta^{j-1})^2
\]
\[
\leq (C\theta)^j \tilde{H}(\pi, 1, 1; 1) + C\theta^{-\frac{3}{2}} \sum_{\ell=1}^{j} (C\theta)^{-1} G(u, 4, 2; \theta^{j-1})^2
\]
\[
\leq (C\theta)^j \varepsilon_6 + C\theta^{-\frac{3}{2}} \sum_{\ell=1}^{j} (C\theta)^{-1} \varepsilon_6^2
\]
\[
\leq C\varepsilon_6^2,
\]
where we take \( \theta \) such that \( C\theta < \frac{1}{2} \) and \( j \) satisfies \( \theta^j \geq r_n \). This gives
\[
I_3 \leq C\varepsilon_6^{3/4}.
\]
Summing up the estimates for \( I_1 - I_3 \) and taking \( \varepsilon_6 \) small enough, we conclude (18).

5.2 The number of singular points

**Theorem 5.3** Let \( u \) be a Leray-Hopf weak solution in \( \mathbb{R}^3 \times (-1, 0) \) and satisfy
\[
\|u\|_{L^q, \infty((-1, 0), L^p(\mathbb{R}^3))} = M < \infty,
\]
where \( \frac{3}{p} + \frac{2}{q} = 1 \), \( 3 < p < \infty \). Then the number of singular points of \( u \) is finite at any time \( t \in (-1, 0] \), and the number depends on \( M \).

**Remark 5.4** The case of \( (p, q) = (3, \infty) \) has been proved by Neustupa [15] and Seregin [16]. In fact, the solution is regular in this case [4]. A special case satisfying (19) is
\[
\|u(t)\|_{L^p(\mathbb{R}^3)} \leq M(-t)^{-\frac{3}{2p}}.
\]
Note that the solution is regular if \( M \) is small, which was proved by Leray [12].

**Lemma 5.5** Let \( (u, \pi) \) be a suitable weak solution of (1) in \( Q_1 \) and satisfy
\[
\|u\|_{L^q, \infty((-1, 0), L^p(B_1))} < M,
\]
where \( \frac{3}{p} + \frac{2}{q} = 1 \) and \( 3 < p < \infty \). There exists \( \varepsilon_7 > 0 \) depending on \( C(u, 1), D(\pi, 1) \) such that \( u \) is regular at \( (0, 0) \) if
\[
\|u\|_{L^{q_0/2} L^p(Q_1)} + \|\pi\|_{L^{q_0/2} L^p(Q_1)} \leq \varepsilon_7.
\]
where \( q_0 = 3 \) for \( 3 < p < 9 \) and \( q_0 = \frac{9+2}{2} \) for \( p \geq 9 \).

**Proof.** For \( p \in (3, 9) \), the result follows from Proposition 2.4 and 21. Now we assume \( p \geq 9 \). Similar to the proof of Lemma 4.4, we can infer from (20) that
\[
G(u, p, q_0; r) \leq C(M) \quad \text{for any } 0 < r < 1,
\]

which along with Lemma 6.2 gives
\[ A(u, r) + E(u, r) + D(\pi, r) \leq C(M, C(u, 1), D(\pi, 1)) \quad \text{for any } 0 < r < 1/2. \tag{23} \]

Due to \( p \geq 9 \), hence \( q_0 > 2 \), Hölder inequality gives
\[ G(u, 4, 2; r) \leq CG(u, p, q_0; r). \tag{24} \]

Let \( \zeta \) be a cutoff function, which vanishes outside of \( Q_\rho \) and equals 1 in \( Q_{\rho/2} \), and satisfies
\[ |\nabla \zeta| \leq C_1 \rho^{-1}, \quad |\partial_t \zeta|, |\Delta \zeta| \leq C_1 \rho^{-2}. \]

Define the backward heat kernel as
\[ \Gamma(x, t) = \frac{1}{4\pi(r^2 - t)^{3/2}} e^{-\frac{|x|^2}{4(r^2 - t)}}. \]

Taking the test function \( \phi = \Gamma \zeta \) in the local energy inequality, and noting \((\partial_t + \Delta)\Gamma = 0\), we obtain
\[ \sup_t \int_{B_\rho} |u|^2 \phi dx + \int_{Q_\rho} |\nabla u|^2 \phi dx dt \leq \int_{Q_\rho} (|u|^2(\Delta \phi + \partial_t \phi) + u \cdot \nabla \phi(|u|^2 + \pi)) dx dt. \]

This implies that
\[ A(u, r) + E(u, r) \leq C \left( \frac{r}{\rho} \right)^2 \rho^{-3} \int_{Q_\rho} |u|^2 dx dt + C(u, \rho) + \rho^{-2} |u|\pi - (\pi)_{B_\rho} |dx dt). \]

While by (22) and (24), we have
\[ C(u, \rho) \leq A(u, \rho)^{1/2} G(u, 4, 2; \rho)^2 \leq C(M) A(u, \rho)^{1/2}. \]

And we get by Lemma 6.1 that
\[ \tilde{H}(\pi, 2, 1; r) \leq C \left( \frac{\rho}{r} \right)^{3/2} G(u, 4, 2; \rho)^2 + C \left( \frac{\rho}{\rho} \right) \tilde{H}(\pi, 1, 1; \rho) \]
\[ \leq C(M) \left( \frac{\rho}{r} \right)^{3/2} + C \left( \frac{\rho}{\rho} \right) \tilde{H}(\pi, 1, 1; \rho), \]

which gives by a standard iteration that
\[ \tilde{H}(\pi, 2, 1; r) \leq C(M) \quad \text{for } 0 < r < 1/2. \]

Hence, we have
\[ \rho^{-2} \int_{Q_\rho} |u|\pi - \pi_{B_\rho} |dx dt \leq CA(u, \rho)^{1/2} \tilde{H}(\pi, 2, 1; \rho) \leq C(M) A(u, \rho)^{1/2}. \]

Let \( F(r) = A(u, r) + E(u, r) + \tilde{H}(\pi, 2, 1; r)^2 \). Then we conclude
\[ F(r) \leq C \left( \frac{r}{\rho} \right)^2 F(\rho) + C(M) \left( \frac{r}{\rho} \right)^2 + C \left( \frac{\rho}{r} \right)^{3} G(u, 4, 2; \rho)^4. \tag{25} \]

Letting \( \rho = 1 \), and taking \( r \) small and then \( \varepsilon > 0 \), small, we infer from (25), (23) and Proposition 5.1 that \((0, 0)\) is a regular point. The proof is completed. \( \Box \)
Now we are in position to prove Theorem 5.3.

Proof of Theorem 5.3. We denote \( z_1 = (x_1, t_0), \ldots, z_K = (x_K, t_0) \) by the singular points of the solution at \( t = t_0 \). Then Lemma 5.5 implies that at every singular point we have

\[
G(u, p, q_0; r, z_l)^2 + H(\pi, p/2, q_0/2; r, z_l) > \varepsilon_7, \quad \text{for any } 0 < r < 1,
\]

where \( l = 1, \ldots, K \). We choose \( r_0 > 0 \) small such that \( B_r(x_i) \cap B_r(x_j) = \emptyset \) for \( i \neq j \) and all \( 0 < r \leq r_0 \). Taking \( r = \theta^k r_0 \) and \( \rho = \theta^{-k} r_0 \) in \((26)\), we find

\[
H(\pi, p/2, q_0/2; \theta r_0)^{q_0/2}
\leq C \theta^{q_0-2} H(\pi, p/2, q_0/2; \theta^{k-1} r_0)^{q_0/2} + C \theta^{-2} \frac{3q_0}{p} + q_0 G(u, p, q_0; \theta^{k-1} r_0)^{q_0}
\leq (C \theta^{q_0-2})^k H(\pi, p/2, q_0/2; r_0)^{q_0/2} + C \theta^{-2} \frac{3q_0}{p} + q_0 \sum_{i=0}^{k-1} (C \theta^{q_0-2})^{k-i-1} G(u, p, q_0; \theta^i r_0)^{q_0}.
\]

Now, for \( \frac{q_0}{p} \geq 1 \), noting that

\[
\sum_{l=1}^{K} a_l^{q_0} \leq \left( \sum_{l=1}^{K} a_l \right)^{q_0/p}, \quad a_l \geq 0,
\]

we deduce by \((7)\) with \( R = r^{-2/q} \|u\|_{L^{q,\infty}(-1,0;L^p(\mathbb{R}^3))} \) that

\[
eq \left( \varepsilon_7 \right)^{q_0} k \leq C \sum_{l=1}^{K} \left( G(u, p, q_0; r, z_l)^{q_0} + H(\pi, p/2, q_0/2; r, z_l)^{q_0/2} \right)
\leq C r^\alpha \int_{t_0-r^2}^{t_0} \left( \int_{\Omega} |u|^p \, dx \right)^{q_0/p} \, dt + C (C \theta^{q_0-2})^k r^\beta \int_{t_0-r^2}^{t_0} \left( \int_{\Omega} |\pi|^p \, dx \right)^{q_0/p} \, dt
\leq + C \theta^{-2} \frac{3q_0}{p} + q_0 \sum_{i=0}^{k-1} (C \theta^{q_0-2})^{k-i-1} r^\alpha \int_{t_0-r^2}^{t_0} \left( \int_{\Omega} |u|^p \, dx \right)^{q_0/p} \, dt
\leq C \|u\|_{L^{q,\infty}(-1,0;L^p(\mathbb{R}^3))}^{q_0/2} + C (C \theta^{q_0-2})^k \|\pi\|_{L^{q,\infty}(-1,0;L^p(\mathbb{R}^3))}^{q_0/2},
\]

where \( \Omega = \bigcup_{l=1}^{K} B_r(x_l), \alpha = (1 - \frac{3}{p} - \frac{3}{q_0}) q_0, \beta = (2 - \frac{3}{p} - \frac{2}{q_0}) \), and choose \( \theta \) such that \( C \theta^{q_0-2} < \frac{1}{2} \). Letting \( k \to \infty \), we infer that

\[
K \leq C \varepsilon_7^{-q_0} \|u\|_{L^{q,\infty}(-1,0;L^p(\mathbb{R}^3))}^{q_0/2},
\]

Similarly, for \( \frac{q_0}{p} < 1 \), noting that

\[
\sum_{l=1}^{K} a_l^{q_0} \leq K^{1-\frac{q_0}{p}} \left( \sum_{l=1}^{K} a_l \right)^{\frac{q_0}{p}}, \quad a_l \geq 0,
\]

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we infer that
\[
\begin{align*}
\varepsilon_7^{\frac{q_0}{p}} K & \leq C \sum_{i=1}^{K} \left( r^\alpha \int_{t_0}^{t_0-r} \left( \int_{B_r(x_i)} |u|^p dx \right)^{q_0/p} dt + (C_\theta^{q_0-2})^{r} \int_{t_0}^{t_0} \left( \int_{B_r(x_i)} |\pi|^p dx \right)^{q_0/p} dt 
\right. \\
& \quad + C_\theta^{q_0-2} \sum_{i=1}^{K} \left( C_\theta^{q_0-2})^{r} \int_{t_0}^{t_0-r} \left( \int_{B_r(x_i)} |u|^p dx \right)^{q_0/p} dt 
\right) \\
& \leq C_\theta^{q_0-2} \left( C_\theta^{q_0-2} \int_{t_0}^{t_0} \left( \int_{\Omega} |u|^p dx \right)^{q_0/p} dt + (C_\theta^{q_0-2})^{r} \int_{t_0}^{t_0} \left( \int_{\Omega} |\pi|^p dx \right)^{q_0/p} dt 
\right) \\
& \leq C_\theta^{q_0} \left( \|u\|_{L^{q_0,\infty}(-1,0;L^p(\mathbb{R}^3))} + (C_\theta^{q_0})^{r} \|\pi\|_{L^{q_0,\infty}(-1,0;L^p(\mathbb{R}^3))} \right).
\end{align*}
\]

Letting \( k \to \infty \), we get
\[
K \leq C_\theta^{q_0} \|u\|_{L^{q_0,\infty}(-1,0;L^p(\mathbb{R}^3))}.
\]
The proof is completed. \( \Box \)

6 Appendix

We first present some estimates of the pressure in terms of some scaling invariant quantities.

Lemma 6.1 Let \((u, \pi)\) be a suitable weak solution of \((\mathcal{D})\) in \( Q_1 \). Then there hold
\[
\begin{align*}
H(\pi, 2, 1; r) & \leq C \left( \frac{\rho}{r} \right)^{\frac{2}{3}} G(u, 4, 2; \rho)^2 + C \cdot H(\pi, 1, 1; \rho), \\
\bar{H}(\pi, 2, 1; r) & \leq C \left( \frac{\rho}{r} \right)^{\frac{2}{3}} G(u, 4, 2; \rho)^2 + C \left( \frac{\rho}{\rho} \right) \bar{H}(\pi, 1, 1; \rho),
\end{align*}
\]
for any \( 0 < 4r < \rho < 1 \). Here \( C \) is a constant independent of \( r, \rho \).

Proof. We write \( \pi = \pi_1 + \pi_2 \) with \( \pi_1 \) satisfying
\[
\Delta \pi_1 = -\partial_i \partial_j (u_i u_j \zeta),
\]
where \( \zeta \) is a cut-off function, which equals 1 in \( B_{\rho/2} \) and vanishes outside of \( B_\rho \). Hence,
\[
\Delta \pi_2 = 0 \quad \text{in} \quad B_{\rho/2}.
\]
By Calderón-Zygmund inequality, we have
\[
\int_{B_\rho} |\pi_1|^2 dx \leq C \int_{B_\rho} |u|^4 dx,
\]
and using the properties of harmonic function, for \( r < \rho/4 \)
\[
\begin{align*}
\sup_{x \in B_r} |\pi_2| & \leq C \rho^{-3} \int_{B_{\rho/4}} |\pi_2| dx, \\
\sup_{x \in B_r} |\pi_2 - (\pi_2)_{B_r}| & \leq C \rho \sup_{x \in B_{\rho/4}} |\nabla \pi_2| \leq C \left( \frac{\rho}{\rho} \right)^{\frac{2}{3}} \int_{B_\rho} |\pi_2 - (\pi_2)_{B_\rho}| dx.
\end{align*}
\]
Then it follows that for $0 < r < \rho/4$,
\[
\int_{B_r} |\pi|^2 dx \leq \int_{B_r} |\pi_1|^2 dx + \int_{B_r} |\pi_2|^2 dx \\
\leq C \int_{B_\rho} |u|^4 dx + Cr^3 \rho^{-6} \left( \int_{B_\rho} |\pi| dx \right)^2,
\]
and
\[
\int_{B_r} |\pi - (\pi)_{B_r}|^2 dx \leq \int_{B_r} |\pi_1 - (\pi_1)_{B_r}|^2 dx + \int_{B_r} |\pi_2 - (\pi_2)_{B_r}|^2 dx \\
\leq C \int_{B_\rho} |u|^4 dx + Cr^5 \rho^{-8} \left( \int_{B_\rho} |\pi - (\pi)_{B_\rho}| dx \right)^2.
\]
Integrating with respect to $t$, we get
\[
\int_0^{-r^2} \left( \int_{B_r} |\pi|^2 dx \right)^{\frac{1}{2}} dt \leq C \int_0^{-\rho^2} \left( \int_{B_\rho} |u|^4 dx \right)^{\frac{1}{2}} dt + Cr^4 \rho^{-3} \int_0^{-\rho^2} \int_{B_\rho} |\pi| dx dt,
\]
and
\[
\int_0^{-r^2} \left( \int_{B_r} |\pi - (\pi)_{B_r}|^2 dx \right)^{\frac{1}{2}} dt \\
\leq C \int_0^{-\rho^2} \left( \int_{B_\rho} |u|^4 dx \right)^{\frac{1}{2}} dt + r^4 \rho^{-4} \int_0^{-\rho^2} \int_{B_\rho} |\pi - (\pi)_{B_\rho}| dx dt.
\]
The proof is completed. 

The same proof also yields that for any $0 < 4r < \rho < 1$,
\[
H(\pi, p/2, q/2; r) \leq C \left( \frac{\rho}{r} \right)^{\frac{4q}{p} + \frac{6}{q} - 2} G(u, p, q; \rho)^2 + C \left( \frac{\rho}{r} \right)^{2 - \frac{4}{q}} H(\pi, 1, q/2; \rho),
\]
where $p > 2, q \geq 2$. Similarly, one can show that (see also [17])
\[
\tilde{D}(\pi, r) \leq C \left( \frac{\rho}{r} \right)^{5/2} \tilde{D}(\pi, \rho) + \left( \frac{\rho}{r} \right)^2 C(u, \rho),
\]
for any $0 < 4r < \rho < 1$.

The following lemma gives a bound of local scaling invariant energy, see also [7] and [25].

**Lemma 6.2** Let $(u, \pi)$ be a suitable weak solution of (1) in $Q_1$. If
\[
G(u, p, q; r) \leq M \quad \text{with} \quad 1 \leq \frac{3}{p} + \frac{2}{q} < 2, 1 < q \leq \infty \quad \text{or} \quad \text{and}
\]
\[
H(\nabla u, p, q; r) \leq M \quad \text{with} \quad 2 \leq \frac{3}{p} + \frac{2}{q} < 3, 1 < p \leq \infty,
\]
for any $0 < r < 1$, then there holds for $0 < r < 1/2$
\[
A(u, r) + E(u, r) + D(\pi, r) \leq C(p, q, M) \left( r^{1/2} (C(u, 1) + D(\pi, 1)) + 1 \right).
\]
Proof. First of all, we assume $G(u, p, q; r) \leq M$ and moreover,

$$
\frac{3}{2} < \frac{3}{p} + \frac{2}{q} < 2, \quad \frac{3}{p} + \frac{3}{q} \geq 2, \quad \frac{4}{p} + \frac{2}{q} \geq 2, \quad p, q < \infty.
$$

Otherwise, we can choose $(p_1, q_1)$ satisfying the above condition and by Hölder inequality,

$$
G(u, p_1, q_1; r) \leq CG(u, p, q; r).
$$

By Hölder inequality and Sobolev inequality, we get

$$
\int_{B_r} |u|^3 \, dx = \int_{B_r} |u|^{3\alpha + 3\beta + 3 - 3\alpha - 3\beta} \, dx
\leq \left( \int_{B_r} |u|^2 \, dx \right)^{3\alpha/2} \left( \int_{B_r} |u|^6 \, dx \right)^{\beta/2} \left( \int_{B_r} |u|^p \, dx \right)^{(3 - 3\alpha - 3\beta)/p}
\leq C \left( \int_{B_r} |u|^2 \, dx \right)^{3\alpha/2} \left( \int_{B_r} \nabla u^2 + |u|^2 \, dx \right)^{3\beta/2} \left( \int_{B_r} |u|^p \, dx \right)^{(3 - 3\alpha - 3\beta)/p},
$$

where $\alpha, \beta \geq 0$ are chosen such that

$$
\frac{1}{3} = \frac{\alpha}{2} + \frac{\beta}{6} + \frac{1 - \alpha - \beta}{p}, \quad 1 = \frac{3\beta}{2} + \frac{3 - 3\alpha - 3\beta}{q}.
$$

That is,

$$
\alpha = \frac{2 (\frac{3}{p} + \frac{3}{q} - 2)}{3 (\frac{3}{p} + \frac{3}{q} - 3)}, \quad \beta = \frac{\frac{4}{p} + \frac{2}{q} - 2}{\frac{2}{p} + \frac{1}{q} - 3}.
$$

Integrating with respect to time, we get

$$
\int_{Q_r} |u|^3 \, dx \, dt \leq C \left( \sup_{-r^2 < t < 0} \int_{B_r} |u|^2 \, dx \right)^{\frac{3\alpha}{2}} \left( \int_{Q_r} \nabla u^2 + |u|^2 \, dx \, dt \right)^{\frac{\beta}{2}} \times \left( \int_{-r^2}^0 \left( \int_{B_r} |u|^p \, dx \right)^{\frac{3}{p}} \, dt \right)^{\frac{3 - 3\alpha - 3\beta}{q}},
$$

this means that

$$
C(u, r) \leq C \left( A(u, r) + E(u, r) \right)^{\frac{3\alpha + 3\beta}{2}} \left( G(u, p, q; r) \right)^{3 - 3\alpha - 3\beta}.
$$

Set $\frac{3}{p} + \frac{3}{q} = 2 - \delta$ with $0 \leq \delta < 1/2$. Then $\frac{3\alpha + 3\beta}{2} = \frac{3}{2} - \frac{\delta}{2 (\frac{3}{p} + \frac{1}{q} - 3)} = \frac{1 - 3\delta}{2 - 2\delta}$ and

$$
C(u, r) \leq C \left( A(u, r) + E(u, r) \right)^{\frac{1 - 3\delta}{2 - 2\delta}} \left( G(u, p, q; r) \right)^{\frac{1}{1 - 2\delta}}. \tag{28}
$$

By the assumption, we get

$$
C(u, r) \leq C(p, q, M) \left( A(u, r) + E(u, r) \right)^{\frac{1 - 3\delta}{2 - 2\delta}}.
$$

Using the local energy inequality and (25), we deduce that

$$
A(u, r) + E(u, r) \leq C \left( C(u, 2r)^{2/3} + C(u, 2r) + C(u, 2r)^{1/3} D(\pi, 2r)^{2/3} \right),
$$

$$
D(\pi, r) \leq C \left( \frac{r}{\rho} \right) D(\pi, \rho) + \left( \frac{r}{\rho} \right)^2 C(u, \rho) \quad \text{for} \quad 0 < 4r < \rho < 1.
$$

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Set $F(r) = A(u, r) + E(u, r) + D(\pi, r)$. It follows from the above three inequalities that

$$F(r) \leq C(1 + C(u, 2r) + D(\pi, 2r))$$

$$\leq C + C\left(\frac{r}{\rho}\right)F(\rho) + C(p, q, M)\left(\left(\frac{\rho}{r}\right)^2 + \left(\frac{\rho}{r}\right)^{\frac{4}{2+q}}\right)(A(u, \rho) + E(u, \rho))^{\frac{1-3\delta}{2}}$$

$$\leq C + C\left(\frac{r}{\rho}\right)F(\rho) + C(p, q, M, \frac{\rho}{r})$$

for $0 < 8r < \rho < 1$. By the standard iteration and local energy inequality, we deduce that

$$F(r) \leq C(p, q, M)(r^{1/2}(A(u, 1/2) + E(u, 1/2) + D(\pi, 1)) + 1)$$

$$\leq C(p, q, M)(r^{1/2}(C(u, 1) + D(\pi, 1)) + 1).$$

Now let us assume that $H(\nabla u, p, q; r) \leq M$ and $\frac{3}{p} < \frac{2}{q} < 3, p < 3$. General case can be reduced to this case as above. Similarly, we have

$$\int_{Q_r} |u - u_{B_r}|^3 dxdt$$

$$\leq C\left(\sup_{-r^2 < t < 0} \int_{B_r} |u|^2 dx \right)^{\frac{3p}{2}} \left(\int_{Q_r} |\nabla u|^2 dxdt\right)^{\frac{3p}{2}}$$

$$\times \left(\frac{\int_0^2}{r^2} \left(\int_{B_r} |u - u_{B_r}|^{\frac{3p}{2}} (\int_{Q_r} |\nabla u|^2 dxdt)^{\frac{3p}{2}} \left(\frac{\int_0^2}{r^2} |\nabla u|^p dx\right)^{\frac{2}{p}} dt\right)^{\frac{3-3\alpha-3\beta}{3-3\alpha-3\beta}},$$

where $\alpha + \beta = 1 - \frac{1}{2} - \frac{1}{2} = 3 - \delta_0$ with $0 \leq \delta_0 < \frac{1}{2}$, then

$$\tilde{C}(u, r) \leq C(A(u, r) + E(u, r))^{\frac{1-3\delta}{2+3\delta}} H(\nabla u, p, q; r)^{\frac{1-3\delta}{2+3\delta}}$$

$$\leq C(p, q, M)(A(u, r) + E(u, r))^{\frac{1-3\delta}{2+3\delta}}.$$

Note that

$$C(u, r) \leq C\left(\left(\frac{r}{\rho}\right)C(u, \rho) + \left(\frac{\rho}{r}\right)^2 \tilde{C}(u, \rho)\right),$$

and

$$A(u, r) + E(u, r) \leq C\left(C(u, 2r)^{2/3} + C(u, 2r) + C(u, 2r)^{1/3} D(\pi, 2r)^{2/3}\right),$$

$$D(\pi, r) \leq C\left(\left(\frac{r}{\rho}\right)D(\pi, \rho) + \left(\frac{\rho}{r}\right)^2 \tilde{C}(u, \rho)\right),$$

for $0 < 4r < \rho < 1$. Let $F(r) = A(u, r) + E(u, r) + C(u, r) + D(\pi, r)$. Then we have

$$F(r) \leq C\left(1 + C(u, 2r) + D(\pi, 2r)\right)$$

$$\leq C\left(1 + \left(\frac{r}{\rho}\right)C(u, \rho) + \left(\frac{\rho}{r}\right)^2 \tilde{C}(u, \rho) + \left(\frac{r}{\rho}\right)D(\pi, \rho)\right)$$

$$\leq C\left(1 + \left(\frac{r}{\rho}\right)F(\rho) + \left(\frac{\rho}{r}\right)^2 F(\rho)^{\frac{1-3\delta}{2+3\delta}}\right)$$

$$\leq C + C\left(\frac{r}{\rho}\right)F(\rho) + C(p, q, M, \frac{\rho}{r}),$$

which implies the required result. 

\[\square\]
Acknowledgments. Both authors thank helpful discussions with Professors Gang Tian and Liqun Zhang. Zhifei Zhang is partly supported by NSF of China under Grant 10990013 and 11071007.

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