THE GENERIC ISOGENY DECOMPOSITION OF THE PRYM VARIETY OF A CYCLIC BRANCHED COVERING

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Abstract. Let \( f: S' \to S \) be a cyclic branched covering of smooth projective surfaces over \( \mathbb{C} \) whose branch locus \( \Delta \subset S \) is a smooth ample divisor. Pick a very ample complete linear system \( |H| \) on \( S \), such that the polarized surface \((S, |H|)\) is not a scroll nor has rational hyperplane sections. For the general member \([C] \in |H|\) consider the \( \mu_n \)-equivariant isogeny decomposition of the Prym variety \( \text{Prym}(C'/C) \) of the induced covering \( f: C' := f^{-1}(C) \to C \):

\[
\text{Prym}(C'/C) \sim \prod_{d|n, d \neq 1} \mathcal{P}_d(C'/C).
\]

We show that for the very general member \([C] \in |H|\) the isogeny component \( \mathcal{P}_d(C'/C) \) is \( \mu_d \)-simple with \( \text{End}_{\mu_d}(\mathcal{P}_d(C'/C)) \cong \mathbb{Z}[\zeta_d] \). In addition, for the non-ample case we reformulate the result by considering the identity component of the kernel of the map \( \mathcal{P}_d(C'/C) \subset \text{Jac}(C') \to \text{Alb}(S') \).

1. Introduction

The main result of this paper is the following:

**Theorem 1.1.** Let \( S \) be a smooth projective surface over \( \mathbb{C} \) with an ample line bundle \( \mathcal{L} \). Assume \( \Delta \in |\mathcal{L}^\otimes n| \) is smooth and consider the \( n \)-fold cyclic covering \( f: S' \to S \) branched along the divisor \( \Delta \). Given a very ample complete linear system \( |H| \) on \( S \), such that \((S, |H|)\) is not a scroll nor has rational hyperplane sections. Then, for the very general member \([C] \in |H|\) we have that

\[
\text{Prym}(C'/C) \sim \prod_{d|n, d \neq 1} \mathcal{P}_d(C'/C),
\]

with \( \text{End}_{\mu_d}(\mathcal{P}_d(C'/C)) \cong \mathbb{Z}[\zeta_d] \). Especially, each \( \mathcal{P}_d(C'/C) \) is a \( \mu_d \)-simple abelian variety.

If we restrict to the case of double coverings, we note that the involution \( \sigma \) of the covering \( f \) acts as \(-\text{id}\) on \( \mathcal{P}_2(C'/C) = \text{Prym}(C'/C) \) and thus, \( \text{End}_{\mu_2}(\text{Prym}(C'/C)) = \text{End}(\text{Prym}(C'/C)) \). In particular, (1.1) can be stated as follows:

**Corollary 1.2.** Let \( S \) be a smooth projective surface over \( \mathbb{C} \) with an ample line bundle \( \mathcal{L} \). Assume \( \Delta \in |\mathcal{L}^\otimes 2| \) is smooth and consider the double covering \( f: S' \to S \) branched along the divisor \( \Delta \). Given a very ample complete linear system \( |H| \) on \( S \), such that \((S, |H|)\) is not...
a scroll nor has rational hyperplane sections. Then, for the very general member \([C] \in |\mathcal{H}|\) we have that
\[
\text{End}(\text{Prym}(C'/C)) \cong \mathbb{Z}.
\]

The condition the line bundle \(\mathcal{L}\) is ample in (1.1) implies that \(\text{Alb}(f): \text{Alb}(S') \rightarrow \text{Alb}(S)\) is an isomorphism cf. page 10 and therefore the map \(\mathcal{P}_d(C'/C) \rightarrow \text{Alb}(S')\) is trivial. For the general situation one needs to consider the abelian subvariety
\[
\mathcal{R}_d(C', C, S') := \ker^0(\mathcal{P}_d(C'/C) \rightarrow \text{Alb}(S')).
\]

Then, the result can be reformulated as follows:

**Theorem 1.3.** Let \(S\) be a smooth projective surface over \(\mathbb{C}\) with a line bundle \(\mathcal{L}\). Assume \(\Delta \in |\mathcal{L}^\otimes n|\) is smooth and consider the \(n\)-fold cyclic covering \(f: S' \rightarrow S\) branched along the divisor \(\Delta\). Given a very ample complete linear system \(|\mathcal{H}|\) on \(S\), such that \((S, |\mathcal{H}|)\) is not a scroll nor has rational hyperplane sections. Then, exactly one of the following assertions holds true:

(i) For the general member \([C] \in |\mathcal{H}|\) we have that \(\mathcal{R}_d(C', C, S') = 0\).

(ii) For the very general member \([C] \in |\mathcal{H}|\) we have that \(\text{End}_{\mu_d}(\mathcal{R}_d(C', C, S')) \cong \mathbb{Z}[\zeta_d]\).

In this paper we present a complete proof for the above results, inspired by Ciliberto and Van der Geer’s approach in [3]. We note that this method does not capture the étale situation, cf. (3.2), (3.3) and (3.4). In addition, if we rephrase the statement for \(n > 2\) by requiring simplicity instead of \(\mu_d\)-simplicity to the isogeny components, we observe that this method cannot be adopted. Namely, the abelian variety \(B\) in (3.4) cannot be chosen in general to be \(\mu_d\)-invariant and for this reason the last combinatorial argument in (3.4) fails. Lastly, a result due to Ortega and Lange, cf. [6] may be used to find counter-example for the case the covering \(f\) is étale of degree 7.

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2. Preliminaries

In this section, we state some well-known results, which are needed later.
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Proposition 2.1. Let \( \pi: A \to S \) be a projective abelian scheme over a Noetherian base \( S \). Then, the endomorphism functor of \( A \) over \( S \) is representable by an \( S \)-scheme \( \text{End}_{A/S} \), which is a disjoint union of projective and unramified \( S \)-schemes.

Proof. This is well-known, cf. [4, pp. 133]. \( \Box \)

The following proposition relates the correspondences on \( C \times C \) with the endomorphisms of the Jacobian \( \text{Jac}(C) \).

Proposition 2.2. Let \( \pi: X \to S \) be a projective smooth morphism over a Noetherian base \( S \), whose fibres are geometrically integral curves. Furthermore, assume that the morphism \( \pi \) admits a section, i.e. \( X(S) \neq \emptyset \). Then, there is a natural and functorial isomorphism

\[
\text{Corr}_S(X) := \text{Pic}(X \times_S X) / (\text{pr}_1)^* \text{Pic}(X) \otimes (\text{pr}_2)^* \text{Pic}(X) \cong \text{End}_S(\text{Pic}^0_X/S).
\]

Proof. Consider the commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Pic}(X)/\pi^* \text{Pic}(S) & \xrightarrow{(\text{pr}_1)^*} & \text{Pic}(X \times_S X)/\text{pr}_2^* \text{Pic}(X) & \xrightarrow{q} & \text{Corr}_S(X) & \longrightarrow & 0 \\
& & \cong & & \cong & & \cong & & \\
0 & \longrightarrow & \text{Pic}_{X/S}(S) & \xrightarrow{c=\circ \pi} & \text{Pic}_{X/S}(X) & \xrightarrow{d} & \text{End}_S(\text{Pic}^0_{X/S}) & \longrightarrow & 0
\end{array}
\]

The first row is clearly exact: Indeed, the relative Picard functor is an fppf-sheaf, cf. [5, Thm. 2.5] and thus, the restriction map \( (\text{pr}_1)^* \) is injective. Furthermore, the map \( q \) is just the cokernel of \( (\text{pr}_1)^* \). Next, we give the definition of the map \( d \). Fix \( x \in X(S) \) and let \( \phi: X \to \text{Pic}_{X/S} \) be any \( S \)-morphism. Then, \( d\phi \) is the unique endomorphism of \( \text{Pic}^0_{X/S} \), making the diagram below commutative.

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\text{can}} & \text{Alb}_{X/S} \cong \text{Pic}^0_{X/S} \\
\phi \circ \phi \circ \pi & \downarrow & \downarrow d\phi \\
\text{Pic}_{X/S} & \Longleftarrow & \text{Pic}^0_{X/S}
\end{array}
\]

Note that under our assumptions the Albanese map \( \text{can}: \mathcal{X} \to \text{Alb}_{X/S} \) exists and has the desired universal property, cf. [1, Thm. 2.17], [1, Rem. 2.19] and [8, Thm. 10.2]. Moreover, the construction of the map \( d \) indicates that \( d \) is surjective and also that the second row in the diagram above is exact at the middle. Now, the existence of \( g \) and the fact that it is an isomorphism are clear. \( \Box \)

The following proposition is well-known.
Proposition 2.3. Suppose that the polarized surface \((S, |H|)\) is not a scroll nor has rational hyperplane sections. Then, the following assertions hold true:

(i) The discriminant divisor \(D\) is irreducible and has codimension one in \(|H|\), i.e. \(D\) is a prime divisor of \(|H|\).

(ii) The general curve \([C] \in D\) is irreducible and has a single ordinary double point as its only singularity.

Proof. Cf. [3, Lem. 3.1].

We close this section by introducing the \(\mu_n\)-equivariant isogeny decomposition in (1.1). Let \(f: C' \rightarrow C\) be a cyclic branched covering of smooth complex projective curves with \(\deg(f) = n\) and let \(\sigma\) stand for a generator of the Galois group of \(f\). The \(\mu_n\)-action on \(C'\) induces an action on \(\text{Jac}(C')\) and thus, it defines a \(\mathbb{Q}\)-algebra homomorphism

\[ \rho: \mathbb{Q}[\mu_n] \cong \mathbb{Q}[T]/(T^n - 1) \rightarrow \text{End}^0(\text{Jac}(C')), \ T \mapsto \sigma^* . \]

For any divisor \(d|n\), we define \(\mathcal{P}_d(C'/C') := \ker^0(\Psi_d(\sigma^*))\), where \(\Psi_d(T) \in \mathbb{Z}[T]\) is the \(d\)-th cyclotomic polynomial. Then, the addition map

\[ \mu: \prod_{d|n} \mathcal{P}_d(C'/C') \rightarrow \text{Jac}(C') \]

is a \(\mu_n\)-equivariant isogeny. Lange and Recillas [7] have stated and proved the relation between \(\mathbb{Q}\)-representations and the \(G\)-equivariant isogeny decomposition of an abelian variety with \(G\)-action, in terms of the finite group \(G\) involved, cf. [7, Thm. 2.2]. The \(\mu_n\)-equivariant isogeny decomposition of \(\text{Jac}(C')\) given above is in fact identical with the one introduced by Lange and Recillas [7]. This can be seen for example by using [2, Rem. 5.5] and [2, Cor. 5.7]. Moreover, we also note that the isogeny components \(\mathcal{P}_d(C'/C')\) are non-trivial as long as the genus \(g(C) \geq 1\), cf. [7, Thm. 3.1], [11, Thm. 5.12] and [11, Thm. 5.13].

3. Reduction to the generic fibre

Let \(S\) be a smooth projective surface over \(\mathbb{C}\) with an ample line bundle \(\mathcal{L}\). Assume \(\Delta \in |\mathcal{L}^\otimes n|\) is smooth and consider the \(n\)-fold cyclic covering \(f: S' \rightarrow S\) branched along the divisor \(\Delta\). Furthermore, fix a very ample complete linear system \(|H|\) on \(S\), such that the polarized surface \((S, |H|)\) is not a scroll nor has rational hyperplane sections. In this section we reduce the proof of Theorem 1.1 to showing that \(\mathcal{P}_d(C_\eta/C_\eta)\) is a \(\mu_d\)-simple abelian variety, where \([C_\eta]\) is the generic member of \(|H|\).
Let \( x \in S \) be a closed point of \( S \). We denote by \(|H|_x\) the linear system of hyperplane sections in \(|H|\) passing through \( x \). In the following we impose restrictions on the point \( x \), i.e. \( x \in S \) will be taken from some appropriate non-empty open subset of \( S \).

Let \( g: \mathcal{X} \subset S \times |H|_x \longrightarrow |H|_x \) denote the universal family of hyperplane sections and \( h: \mathcal{Y} \subset S' \times |H|_x \longrightarrow |H|_x \) its pullback to \( S' \), i.e. \( \mathcal{Y} := \mathcal{X} \times_S S' \). Note that over the non-empty open subset \( U \subset |H|_x \) of smooth curves which intersect the branch locus \( \Delta \) transversally both \( g \) and \( h \) are smooth families of curves having a section. The latter allows us to consider their families of Jacobians over \( U \), which we denote by \( p: \text{Pic}^0_{\mathcal{X}/U} \longrightarrow U \) and \( q: \text{Pic}^0_{\mathcal{Y}/U} \longrightarrow U \), respectively.

A generator \( \sigma: S' \longrightarrow S' \) of the Galois group of the covering \( f \) induces an automorphism of \( \mathcal{Y} \) over \( U \) and thus, an automorphism \( \sigma^*: \text{Pic}^0_{\mathcal{Y}/U} \longrightarrow \text{Pic}^0_{\mathcal{Y}/U} \). We define

\[
P_d := \ker^0(\Psi_d(\sigma^*)) \quad \text{for any divisor } d|n.
\]

Then, \( \varphi_d: P_d \longrightarrow U \) is an abelian fibration with fibres \((P_d)_{[C]} = P_d(C'/C)\) for \([C] \in U\).

As a first step we use the representability of the endomorphism functor of abelian schemes cf. (2.1) to reduce the proof of Theorem 1.1 to showing that \( \text{End}_{\mu_d}((P_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d] \), where \( \bar{\eta} \) is a fixed geometric generic point of \(|H|_x\). The proof of this is standard and so we omit it.

**Lemma 3.1.** Assume that \( \text{End}_{\mu_d}((P_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d] \). Then, for the very general member \([C] \in U\), one has that \( \text{End}_{\mu_d}((P_d)_{[C]}) \cong \mathbb{Z}[\zeta_d] \).

Let \([C] \in |H|_x\) be an irreducible member with a single ordinary double point as its only singularity and intersecting the branch locus \( \Delta \) transversally. Then, \( C' := f^{-1}(C) \) is irreducible and has \( n \) ordinary double points as its only singularities. In this case the group variety \( P_d(C'/C) \) is semi-abelian. In particular, the result is the following:

**Lemma 3.2.** For an irreducible member \([C] \in |H|_x\) with a single ordinary double point as its only singularity and intersecting the branch locus \( \Delta \) transversally, there is an exact sequence:

\[
0 \longrightarrow \mathbb{G}_m^{\varphi(d)} \longleftarrow \mathcal{P}_d(C'/C) \longrightarrow \mathcal{P}_d(\tilde{C}'/\tilde{C}) \longrightarrow 0,
\]

where \( \nu: \tilde{C} \longrightarrow C \) is the normalisation map and \( \varphi(d) \) is the Euler’s totient function.

**Proof.** We have a commutative diagram

\[
\begin{array}{ccc}
\tilde{C}' & \xrightarrow{\nu'} & C' \\
\downarrow f & & \downarrow f \\
\tilde{C} & \xrightarrow{\nu} & C,
\end{array}
\]
where \( \tilde{f} \) is the cyclic covering branched along the divisor \( \nu^* \Delta_{\tilde{C}} \in |\nu^* \mathcal{L}|^\otimes n \) and \( \nu' \) is the normalisation of \( C' \). Fix a generator \( \sigma \) of \( \text{Aut}(C'/C) \) and let \( \tilde{\sigma} \) be the corresponding generator of \( \text{Aut}(\tilde{C}'/\tilde{C}) \), i.e. the one for which the diagram below commutes

\[
\begin{array}{ccc}
\tilde{C}' & \xrightarrow{\nu'} & C' \\
\downarrow{\tilde{\sigma}} & & \downarrow{\sigma} \\
\tilde{C}' & \xrightarrow{\nu'} & C'.
\end{array}
\]

Let \( \{y, \sigma(y), \sigma^2(y), \ldots, \sigma^{n-1}(y)\} \) be the set of ordinary double points of \( C' \). Then, we find a commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\alpha) & \xleftarrow{\gamma} & \Psi_d(\sigma^*) \text{Pic}^0(C') & \xrightarrow{\alpha} & \Psi_d(\tilde{\sigma}^*) \text{Pic}^0(\tilde{C}') & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{C}^*_y \times \cdots \times \mathbb{C}^*_{\sigma^{n-1}(y)} & \xrightarrow{\nu^*} & \text{Pic}^0(C') & \xrightarrow{\nu^*} & \text{Pic}^0(\tilde{C}') & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{C}^*_y \times \cdots \times \mathbb{C}^*_{\sigma^{(d-1)}(y)} & \longrightarrow & \ker(\Psi_d(\sigma^*)) & \xrightarrow{\beta} & \ker(\Psi_d(\tilde{\sigma}^*)) & \longrightarrow & 0.
\end{array}
\]

We show that \( \beta \) induces a surjection \( \mathcal{P}_d(C'/C) = \ker^0(\Psi_d(\sigma^*)) \twoheadrightarrow \mathcal{P}_d(\tilde{C}'/\tilde{C}) = \ker^0(\Psi_d(\tilde{\sigma}^*)) \).

Indeed, by Snake lemma we have the exact sequence

\[ \ker(\Psi_d(\sigma^*)) \longrightarrow \ker(\Psi_d(\tilde{\sigma}^*)) \longrightarrow \text{coker}(\gamma) \longrightarrow 0. \]

Note that coker(\( \gamma \)) is an affine algebraic group, as it is the quotient of a commutative affine algebraic group by an algebraic subgroup. Especially, by [13, Cor. 12.67] and the exactness of the above sequence, we find that \( \dim \text{coker}(\gamma) = 0 \). The latter provides the surjectivity of the map \( \ker^0(\Psi_d(\sigma^*)) \twoheadrightarrow \mathcal{P}_d(\tilde{C}'/\tilde{C}) = \ker^0(\Psi_d(\tilde{\sigma}^*)) \), as claimed. \( \square \)

We are now in the position to prove the following:

**Proposition 3.3.** The abelian variety \( (\mathcal{P}_d)_{\bar{\eta}} \) is \( \mu_d \)-simple if and only if \( \text{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d] \).

**Proof.** The one direction is clear: Indeed, if \( \text{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d] \), then every non-zero \( \mu_d \)-equivariant endomorphism of \( (\mathcal{P}_d)_{\bar{\eta}} \) is an isogeny and thus, \( (\mathcal{P}_d)_{\bar{\eta}} \) is a \( \mu_d \)-simple abelian variety. Conversely, assume that \( (\mathcal{P}_d)_{\bar{\eta}} \) is \( \mu_d \)-simple. We divide the proof into steps.

**Step 1.** There is a closed subscheme \( \text{End}_{\mu_d}^\text{ad}_{\mathcal{P}_d/U}(0) \subset \text{End}_{\mu_d}^\text{ad}_{\mathcal{P}_d/U} \) whose points parametrise the \( \mu_d \)-equivariant endomorphisms of \( \mathcal{P}_d \), which are not isogenies, i.e. the ones, which are of degree 0.
Proof of Step 1. Observe that the functor of $\mu_d$-equivariant endomorphisms of $P_d$ denoted by $\text{End}^{\mu_d}_{P_d/U}$ is representable by a closed subscheme of $\text{End}_{P_d/U}$, since the equivariant condition is closed. It follows that we have a universal endomorphism $\alpha$, such that every other $\mu_d$-equivariant endomorphism of $P_d$ over some scheme $T$ is obtained by pulling-back $\alpha$ along a morphism $T \longrightarrow \text{End}^{\mu_d}_{P_d/U}$. By [13, Prop. 12.93] the set $V := \{x \in \text{End}^{\mu_d}_{P_d/U} \mid \alpha x := \alpha \times \text{id}_{\kappa(x)} \text{ is an isogeny}\}$ is open. Therefore, $\text{End}^{\mu_d}_{P_d/U}(0) := \text{End}^{\mu_d}_{P_d/U} \setminus V$ with the reduced induced closed subscheme structure has the desired property. □

Step 2. The fibre $(P_d)_C$ for the very general member $[C] \in |H|_x$ is a $\mu_d$-absolutely simple abelian variety.

Proof of Step 2. Since $(P_d)_\eta$ is a $\mu_d$-simple abelian variety, we can determine countably many non-empty open subsets $U_i \subset U$, such that the $U$-scheme $\text{End}^{\mu_d}_{P_d/U}(0)$ has (geometrically) connected fibres for all points lying in the intersection of the $U_i$’s, cf. [Stacks, Tag 055C]. □

Pick a Lefschetz pencil $(C_t)_{t \in \mathbb{P}^1} \subset |H|_x$. We may assume that all its singular members are irreducible and intersect the branch locus $\Delta$ transversally, cf. (2.3).

Step 3. Given a Lefschetz pencil $(C_t)_{t \in \mathbb{P}^1}$ as above, we construct a homomorphism:

$$\rho: \text{End}_{\mu_d}(P_d)_\bar{\mu} \longrightarrow \text{End}(\mathbb{G}_{m}^{(d)}),$$

where $\bar{\mu}$ is a fixed geometric generic point of $\mathbb{P}^1$.

Proof of Step 3. Since the endomorphism ring of any abelian variety is finitely generated, cf. [9, Thm. 12.5], we find a finite field extension $L \supset \kappa(\mu)$, such that every endomorphism of $P_d$ over $\kappa(\bar{\mu})$ is defined over $L$, i.e. $\text{End}((P_d)_{\bar{\mu}}) = \text{End}((P_d)_L)$. Consider the smooth projective model $E$ of $L$ together with the morphism $E \longrightarrow \mathbb{P}^1$ induced by this field extension and fix a closed point $y \in E$ lying over a point of the pencil that corresponds to a nodal curve. The map $\rho: \text{End}_{\mu_d}(P_d)_\bar{\mu} \longrightarrow \text{End}(\mathbb{G}_{m}^{(d)})$ is constructed as follows: Let $f \in \text{End}_{\mu_d}(P_d)_L$. Then, $f$ extends to an endomorphism over the local ring $R$ of $E$ at the point $y$, cf. [12, Prop. 7.4.3]. The restriction of the first projection of $(P_d) \times_R P_d$ to the graph of $f$ is an isomorphism. We set $\alpha := \text{pr}_1 |_{(P_d)_y}$. By pulling back $\alpha$ along $\mathbb{G}_{m}^{(d)} \hookrightarrow (P_d)_y$, we get an isomorphism $\alpha: \alpha^{-1}(\mathbb{G}_{m}^{(d)}) \longrightarrow \mathbb{G}_{m}^{(d)}$. We claim that $\alpha^{-1}$ is the graph of a homomorphism $\mathbb{G}_{m}^{(d)} \longrightarrow \mathbb{G}_{m}^{(d)}$. Indeed, it suffices to show that $\text{pr}_2(\alpha^{-1}(\mathbb{G}_{m}^{(d)})) \subset \mathbb{G}_{m}^{(d)}$. To see this,
observe that the composite
\[
\mathbb{G}_{m}^{\varphi(d)} \xrightarrow{\cong} \alpha^{-1}(\mathbb{G}_{m}^{\varphi(d)}) \subset (\Gamma_f)_y \xrightarrow{\text{pr}_2} (\mathcal{P}_d)_y \longrightarrow \mathcal{P}_d(\mathcal{C}_y'/\mathcal{C}_y)
\]
is the zero map by [9, Cor. 3.9] and hence, \( \text{pr}_2 |_{\mathbb{G}_{m}^{\varphi(d)}} \) factors through the kernel of \((\mathcal{P}_d)_y \rightarrow \mathcal{P}_d(\mathcal{C}_y'/\mathcal{C}_y) \) which is \( \mathbb{G}_{m}^{\varphi(d)} \). Finally, we define \( \rho(f) \) to be this endomorphism of \( \mathbb{G}_{m}^{\varphi(d)} \). One checks that \( \rho \) is a homomorphism of rings.

\[\square\]

**Conclusion.** Eventually, we are in the position to complete the proof. Suppose \( \text{End}_{\mu_d}((\mathcal{P}_d)_\eta) \neq \mathbb{Z}[\zeta_d] \) and choose a \( \mu_d \)-equivariant endomorphism \( f \) not in \( \mathbb{Z}[\zeta_d] \). The endomorphism \( f \) can be described as a \( \kappa(\bar{\eta}) \)-point of \( \text{End}_{\mu_d}^{\mathbb{Z}}(\mathcal{P}_d)_U \) and we let \( Z \subset \text{End}_{\mu_d}(\mathcal{P}_d)_U \) be the irreducible component containing this point. Then, the generic point \( \theta \in Z \) corresponds to a \( \mu_d \)-equivariant endomorphism not in \( \mathbb{Z}[\zeta_d] \). Consider the finite set
\[\Gamma := \{ n := (n_0, n_1, \ldots, n_{\varphi(d)-1}) \in \mathbb{Z}^{\varphi(d)} \mid \text{im}([n]) \cap Z \neq \emptyset \}.\]
Each \( \text{im}([n]) \cap Z \) is a proper closed subset of \( Z \). Setting
\[Z_n := \pi(\text{im}([n]) \cap Z),\]
for \( n \in \Gamma \), we get finitely many nowhere dense closed subsets of \( U \), such that for every point \( u \in U \setminus \bigcup_{n \in \Gamma} Z_n \) the fibre \( \pi^{-1}(u) \) contains a point, which is not in \( \mathbb{Z}[\zeta_d] \). We can choose a Lefschetz pencil as above, such that \( (\mathcal{P}_d)_{\bar{\mu}} \) is \( \mu_d \)-simple, cf. Step 2 and \( \text{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \neq \mathbb{Z}[\zeta_d] \).
By Step 3 this leads to a contradiction.

\[\square\]

The next lemma consists of the final reduction step.

**Lemma 3.4.** The abelian variety \((\mathcal{P}_d)_\eta\) is \( \mu_d \)-simple if and only if it is \( \mu_d \)-absolutely simple.

**Proof.** Clearly, if \((\mathcal{P}_d)_\eta\) is \( \mu_d \)-absolutely simple, then it is \( \mu_d \)-simple. Conversely, assume that \((\mathcal{P}_d)_\eta\) is \( \mu_d \)-simple but not \( \mu_d \)-absolutely simple. Then, there is a finite field extension \( L \supset \kappa(\eta) \) and a non-zero and proper \( \mu_d \)-simple abelian subvariety \( B \) of \((\mathcal{P}_d)_L\), such that \((\mathcal{P}_d)_L\) can be written up to isogeny as a product \( \prod B^\tau \), where \( B^\tau \) stands for a Galois conjugate of \( B \) and \( \tau \) runs through a finite subset \( J \subset \text{Gal}(L/\kappa(\eta)) \) of cardinality greater equal to 2.
The field extension \( L \supset \kappa(\eta) \) gives rise to a morphism \( g: U' \longrightarrow U \), which we may assume is étale. For \( \tau \in J \), we let \( \varphi_\tau \) be the endomorphism of \((\mathcal{P}_d)_L\) whose image is \( B^\tau \). More
explicitly, $\varphi_\tau$ is given by

$$(\mathcal{P}_d)_L \overset{\sim}{\longrightarrow} \prod B^\tau \overset{pr_\tau}{\longrightarrow} B^\tau \subset (\mathcal{P}_d)_L.$$ 

Pick a Lefschetz pencil $\{C_t\}_{t \in \mathbb{P}^1}$, such that its singular members are irreducible and intersect the branch locus $\Delta$ transversally. Let $X$ be any irreducible component of $g^{-1}(\mathbb{P}^1 \cap U)$. Then, $X$ dominates $\mathbb{P}^1 \cap U$ and if $\theta \in X$ is its generic point, then each $\varphi_\tau$ determines an endomorphism of $\mathcal{P}_d$ over $\theta$, e.g. using the Néron mapping property, such that if $B^\tau := \text{im}(\varphi_\tau)$, then $\prod B^\tau \sim (\mathcal{P}_d)_\theta$. Let $\tilde{X}$ be a smooth compactification of $X$ and $\tilde{X} \longrightarrow \mathbb{P}^1$ the extension of $g: X \longrightarrow \mathbb{P}^1 \cap U$. Fix a point $y \in \tilde{X}$ lying over a point of the pencil which corresponds to a nodal curve and consider the local ring $R$ of $\tilde{X}$ at $y$. Since $\mathcal{P}_d$ admits a semi-abelian reduction over $R$, cf. (3.2) the same is true for all $B^\tau$, cf. [12, Cor. 7.1.6].

We denote by $\tilde{\mathcal{B}}^\tau$ the identity component of the Néron model of $B^\tau$. Then, the isogeny of the generic fibre extends to an isogeny $\prod \tilde{\mathcal{B}}^\tau \sim \mathcal{P}_d$ over $R$, cf. [12, Prop. 7.3.6]. Since $(\mathcal{P}_d)_y$ is an extension of an abelian variety by a torus of rank $\varphi(d)$, cf. (3.2), it follows that the toric part of $\tilde{B}^\tau_y$ has rank $\delta$, $1 \leq \delta \leq \varphi(d)$, such that $\delta | J = \varphi(d)$. As in Step 3, one constructs a homomorphism $\rho_\tau: \text{End}_{\mu_d}(\tilde{B}^\tau) \longrightarrow \text{End}(\mathbb{G}_m^\delta)$. Since the restriction of $\psi \circ \rho_\tau$ to $\mathbb{Z}[\zeta_d] \subset \text{End}_{\mu_d}(\tilde{B}^\tau)$ is injective, where $\psi := \text{pr}_1 \circ \vdash: \text{End}(\mathbb{G}_m^\delta) \longrightarrow \text{Hom}(\mathbb{G}_m^\delta, \mathbb{G}_m)$ we find that $\delta = \varphi(d)$ and $|J| = 1$, which is absurd. 

□

4. The Proof of Theorem 1.1

According to the results of Section 3, our task to prove Theorem 1.1 is reduced to showing $(\mathcal{P}_d)_\eta$ is a $\mu_d$-simple abelian variety. Recall, that we have an isogeny

$$\text{Jac}(C'_\eta) \sim \text{Jac}(C_\eta) \times \prod_{d|n, d \neq 1} (\mathcal{P}_d)_\eta.$$ 

Given a non-zero endomorphism $\varepsilon \in \text{End}_{\mu_d}((\mathcal{P}_d)_\eta)$. Then, by considering the composite

$$\varepsilon': \text{Jac}(C'_\eta) \overset{\sim}{\longrightarrow} \text{Jac}(C'_\eta) \times \prod_{d|n, d \neq 1} (\mathcal{P}_d)_\eta \overset{pr_\eta}{\longrightarrow} (\mathcal{P}_d)_\eta \overset{\varepsilon}{\longrightarrow} (\mathcal{P}_d)_\eta \hookrightarrow \text{Jac}(C'_\eta),$$

we get a $\mu_d$-equivariant endomorphism of $\text{Jac}(C'_\eta)$ whose restriction to $(\mathcal{P}_d)_\eta$ is simply $\varepsilon \circ [n]$. Hence, it suffices to show that the restriction of $\varepsilon'$ to $(\mathcal{P}_d)_\eta$ lies in $\mathbb{Z}[\zeta_d]$. Recall, that abelian schemes satisfy a stronger Néron mapping property, cf. [10, Sec. 3.1.5]. Thus, the endomorphism $\varepsilon'$ extends to an endomorphism

$$\varepsilon': \text{Pic}^0_{Y/U} \longrightarrow \mathcal{P}_d \subset \text{Pic}^0_{Y/U}.$$
Let \([T] \in \text{Corr}_U(\mathcal{Y})\) be the class of a correspondence \(T\) on \(\mathcal{Y} \times_U \mathcal{Y}\) associated to the endomorphism \(\varepsilon'\), cf. (2.2). We write \(T = \sum n_i T_i\), where \(T_i\) are prime divisors. Let \(\Sigma\) be a general two dimensional linear system in \(|\mathcal{H}|_x\), i.e. the general member of \(\Sigma\) is smooth and intersects the branch locus \(\Delta\) transversally. Then, the correspondences \(T_i\) are all defined over a non-empty open subset of \(\Sigma\) and we can construct a rational map \(\phi_{\Sigma,T} : S' \longrightarrow \text{Div}^+(S')\), \(y \mapsto \Gamma_y^i\), cf. [3, pp. 38]. Especially, we get a rational map
\[
\phi_{\Sigma,T} : S' \longrightarrow \text{Pic}(S'),\ y \mapsto [\Gamma_y] := \sum n_i [\Gamma_y^i].
\]

Let \([C] \in |\mathcal{H}|_x\) be a general member and choose a general two-dimensional linear system \(\Sigma\) containing \([C]\). Consider the rational map \(\phi_{\Sigma,T}\). Then, for a general point \(y \in C'\) we get a divisor \(\Gamma_y = \phi_{\Sigma,T}(y)\) on \(S'\). Set \(w = f(y) \in C\), \(f^{-1}(w) = \{y, \sigma(y), \ldots, \sigma^{n-1}(y)\}\) and \(f^{-1}(x) = \{z, \sigma(z), \ldots, \sigma^{n-1}(z)\}\), where \(\sigma\) is a generator of the Galois group of the covering \(f\). The following lemma computes the divisor \(E_y\) in \(C'\) corresponding to the intersection of \(C'\) with \(\Gamma_y\).

**Lemma 4.1.** We have that \(E_y = \alpha_0 z + \alpha_1 \sigma(z) + \ldots + \alpha_{n-1} \sigma^{n-1}(z) + \beta_0 y + \beta_1 \sigma(y) + \ldots + \beta_{n-1} \sigma^{n-1}(y) + \gamma B'_{x,w} + T_{C'}(y)\), where \(\alpha_i, \beta_i, \gamma \in \mathbb{Z}\) and \(B'_{x,w}\) is the pull-back of the divisor of base points different from \(x\) and \(w\) of \(\Sigma_w\) under the covering \(f\).

**Proof.** Cf. [3, Lem. 3.6].

1. **Regular case.** The branched locus \(\Delta\) of the covering \(f\) is a smooth ample divisor and thus, the canonical map \(\text{Alb}(f) : \text{Alb}(S') \longrightarrow \text{Alb}(S)\) induced by \(f\) is an isomorphism. Indeed, since \(f_\ast \mathcal{O}_{S'} \cong \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}\), the Kodaira Vanishing theorem gives \(H^1(\mathcal{O}_{S'}) = H^1(\mathcal{O}_S)\) and hence, \(\text{Alb}(f)\) is an isogeny. From this one immediately sees that the induced action on \(\text{Alb}(S')\) is trivial, i.e. \(\text{Alb}(\sigma) = \text{id}\). Consider the Albanese map \(\text{Alb}_{\xi_0} : S' \longrightarrow \text{Alb}(S')\), where the point \(\xi_0 \in S'\) lies over a point of the branch locus \(\Delta \subset S\) and observe that the map is invariant under the \(\mu_n\)-action. Therefore, we find a homomorphism \(\text{Alb}(S) \longrightarrow \text{Alb}(S')\) that is inverse to \(\text{Alb}(f)\), proving the claim. In particular, we deduce that \(q(S) = q(S')\). Here, we give the proof for the case \(S\) is regular, i.e. \(q(S) = 0\).

**Proof of Theorem 1.1 for the regular case.** If \(S\) is regular, then \(\text{Pic}(S')\) is discrete and thus, the rational map \(\phi_{\Sigma,T}\) is constant. Hence, for a general point \(y \in C'\), the curves \(\Gamma_y\) and \(\Gamma_{\sigma(y)}\) are linearly equivalent. It follows that \(E_y\) and \(E_{\sigma(y)}\) are also linearly equivalent and so,
\[
E_y - E_{\sigma(y)} = \beta_0 (y - \sigma(y)) + \beta_1 \sigma(y - \sigma(y)) + \ldots + \beta_{n-1} \sigma^{n-1}(y - \sigma(y)) + T_{C'}(y - \sigma(y)) \sim 0.
\]
Since \(\text{Prym}(C'/C) = \text{im}(\text{id} - \sigma^*)\), the latter forces \(T_{C'}(y) = (-\beta_0)y + \ldots + (-\beta_{n-1}) \sigma^{n-1}(y)\) for all \(y \in \text{Prym}(C'/C)\). Eventually, we see that the restriction of \(T_{C'}\) to \(\mathcal{P}_d(C'/C)\) takes the desired form. This yields that the restriction of \(\varepsilon'\) to \((\mathcal{P}_d)_\eta\) lies in \(\mathbb{Z}[\xi_d]\), as claimed. \(\square\)
2. Irregular case. We shall use the following lemma.

Lemma 4.2. Let \( a : \text{Jac}(C') \longrightarrow \mathcal{P}_d(C'/C) \subset \text{Jac}(C') \) be a homomorphism and let \( T_a \) be a correspondence associated to it, cf. (2.2). Assume that there exist \( \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z} \), such that for general \( y \in C' \) the divisor class \( T_a(y-\sigma(y)) + \alpha_0(y-\sigma(y)) + \ldots + \alpha_{n-1}\sigma^{n-1}(y-\sigma(y)) \) lies in \( \text{Pic}^0(S') \). Then, the restriction of \( a \) to \( \mathcal{P}_d(C'/C) \) lies in \( \mathbb{Z}[\zeta_d] \subset \text{End}(\mathcal{P}_d(C'/C)) \).

Proof. As \( \text{Prym}(C'/C) = \text{im}(\text{id} - \sigma^*) \), the assumption clearly implies that \( a(y) + \alpha_0 y + \ldots + \alpha_{n-1}\sigma^{n-1}(y) \in \text{im}(i^*) \) for all \( y \in \text{Prym}(C'/C) \), where \( i^* : \text{Pic}^0(S') \longrightarrow \text{Pic}^0(C'') = \text{Jac}(C') \) is the natural pull-back induced by \( C' \rightarrow S' \). If we show that the intersection \( \text{im}(i^*) \cap \text{Prym}(C'/C) \) is finite, then the result follows immediately. Indeed, consider the commutative square:

\[
\begin{array}{ccc}
\text{Pic}^0(S) & \overset{i^*}{\longrightarrow} & \text{Pic}^0(C) \\
\downarrow f^* & & \downarrow f^* \\
\text{Pic}^0(S') & \overset{i^*}{\longrightarrow} & \text{Pic}^0(C').
\end{array}
\]

The canonical map \( \text{Alb}(S') \longrightarrow \text{Alb}(S) \) induced by \( f \) is an isomorphism and so, is its dual, which is \( f^* \). Hence, the latter yields that \( \text{im}(i^*) : \text{Pic}^0(S') \longrightarrow \text{Pic}^0(C'') \subset f^*(\text{Pic}^0(C)) \). By the definition of \( \text{Prym}(C'/C) \), we know that \( f^*(\text{Pic}^0(C)) \cap \text{Prym}(C'/C) \) is finite and so, is the intersection \( \text{im}(i^*) \cap \text{Prym}(C'/C) \), proving the claim. \( \square \)

Proof of Theorem 1.1 for the irregular case. Using the curves \( \Gamma_y \) we find that \( E_y - E_{\sigma(y)} \) lies in the image of \( \text{Pic}(S') \longrightarrow \text{Pic}(C') \). Therefore, we have that \( T_{C'}(y-\sigma(y)) + \beta_0(y-\sigma(y)) + \beta_1\sigma(y-\sigma(y)) + \ldots + \beta_{n-1}\sigma^{n-1}(y-\sigma(y)) \in \text{im}(i^* : \text{Pic}(S') \longrightarrow \text{Pic}(C')) \) for general \( y \in C' \).

It follows that \( \varepsilon' \in \mathbb{Z}[\zeta_d] \subset \text{End}(\mathcal{P}_d) \), cf. (4.2). \( \square \)

5. The proof of Theorem 1.3

The proof is similar to the case of (1.1). First, we need to replace our earlier family \( \varphi_d : \mathcal{P}_d \longrightarrow U \). In particular, we consider the abelian fibration

\[ \mathcal{R}_d := \ker^0(\mathcal{P}_d \longrightarrow \text{Alb}(S') \times U). \]

Assume that the abelian fibration \( \varphi_d : \mathcal{R}_d \longrightarrow U \) is non-zero, i.e. \( \mathcal{R}_{[C]} \neq 0 \) for \( [C] \in U \). Then, we show that for the very general member \( [C] \in U \), we have that \( \text{End}_{\mu_d}(\mathcal{R}_d_{[C]}) \cong \mathbb{Z}[\zeta_d] \). One checks that the results (3.3) and (3.4) still hold true for the family \( \varphi_d : \mathcal{R}_d \longrightarrow U \).

We proceed as in the proof of Theorem 1.1. A non-zero endomorphism \( \varepsilon \in \text{End}_{\mu_d}(\mathcal{R}_d_{[C]}) \) gives rise to a \( \mu_d \)-equivariant endomorphism \( \varepsilon' \in \text{End}(\text{Jac}(C')) \) and it is enough to check that the restriction of \( \varepsilon' \) to \( (\mathcal{R}_d)_{[C]} \) lies in \( \mathbb{Z}[\zeta_d] \). The following lemma is needed.
Lemma 5.1. Let $a: \Jac(C') \to \mathcal{R}_d(C', C, S') \subset \Jac(C')$ be a homomorphism and let $T_a$ be a correspondence associated to it, cf. (2.2). Assume that there exist $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z}$, such that for general $y \in C'$ the divisor class $T_a(y - \sigma(y)) + \alpha_0(y - \sigma(y)) + \ldots + \alpha_{n-1}y^{n-1}(y - \sigma(y))$ lies in $\Pic^0(S')$. Then, the restriction of $a$ to $\mathcal{R}_d(C', C, S')$ lies in $\mathbb{Z}[\zeta_d] \subset \End(\mathcal{R}_d(C', C, S'))$.

Proof. Clearly, we have that $a(y) + \alpha_0y + \ldots + \alpha_{n-1}y^{n-1} \in \im(i^*)$ for all $y \in \Prym(C'/C)$, where $i^* : \Pic^0(S') \to \Pic^0(C') = \Jac(C')$ is the pull-back induced by $C' \to S'$. Let $K(C', S') := \ker(\Jac(C') \to \Alb(S'))$ and observe that the intersection $\im(i^*) \cap K(C', S')$ is finite. Since $\mathcal{R}_d(C', C, S') \subset K(C', S')$, we find that $a(y) + \alpha_0y + \ldots + \alpha_{n-1}y^{n-1}(y - \sigma(y)) = 0$ for all $y \in \mathcal{R}_d(C', C, S')$. Therefore, the restriction of $a$ to $\mathcal{R}_d(C', C, S')$ belongs to $\mathbb{Z}[\zeta_d]$, as claimed.

Proof of Theorem 1.3. Using the curves $\Gamma_y$ one sees that $E_y - E_{\sigma(y)}$ lies in the image of $\Pic(S') \to \Pic(C')$. It follows that $T_{C'}(y - \sigma(y)) + \beta_0(y - \sigma(y)) + \beta_1y(y - \sigma(y)) + \ldots + \beta_{n-1}y^{n-1}(y - \sigma(y)) \in \im(i^* : \Pic(S') \to \Pic(C'))$. Now, the result is an immediate consequence of (5.1).

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