An anisotropic geometrical approach for non-relativistic extended dynamics

M. Neagu*, A. Oană* and V.M. Red’kov†

Abstract

In this paper we present the distinguished (d-) Riemannian geometry (in the sense of nonlinear connection, Cartan canonical linear connection, together with its d-torsions and d-curvatures) for a possible Lagrangian inspired by optics in non-uniform media. The corresponding equations of motion are also exposed, and some particular solutions are given. For instance, we obtain as geodesic trajectories some circular helices (depending on an angular velocity $\omega$), certain circles situated in some planes (ones are parallel with $xOy$, and other ones are orthogonal on $xOy$), or some straight lines which are parallel with the axis $Oz$. All these geometrical geodesics are very specific because they are completely determined by the non-constant index of refraction $n(x)$.

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1 Introduction

In the geometrical optics [3], a central role is played by the Synge-Beil metric [1, 6, 7]

$$g_{\alpha\beta}(x, y) = \varphi_{\alpha\beta}(x) + \gamma^2 y_\alpha y_\beta, \quad (1.1)$$

where $\gamma(x) \geq 0$ is a positive smooth function on the space-time $M^4$, and $\varphi_{\alpha\beta}(x)$ is a pseudo-Riemannian metric on $M^4$. One assumes that the four-dimensional manifold $M^4$ (which is connected and simply connected) is endowed with the local coordinates

$$(x^\alpha)_{\alpha=0,3} = (x^0 = t, x^1, x^2, x^3);$$

for simplicity we use the system of units where the light velocity is $c = 1$. Obviously, the following rule holds good: $y_\alpha = \varphi_{\alpha\beta} y^\beta$. Because the components

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*Department of Mathematics and Informatics, University Transilvania of Brașov, 50 Iuliu Maniu Blvd., 500091 Brașov, Romania, E-mails: mircea.neagu@unitbv.ro, alexandru.oana@unitbv.ro

†Institute of Physics, National Academy of Sciences of Belarus, Minsk, Belarus, E-mail: redkov@dragon.bas-net.by
of $\varphi_{\alpha\beta}(x)$ are dimensionless, the same are $\gamma y_\alpha$:

$[\varphi_{\alpha\beta}(x)] = 1, \ [\gamma y_\alpha] = 1.$

In such a context, let us restrict our geometric-physical study to the Euclidean manifold $\mathcal{E}^3 = (\Sigma^3, \delta_{ij})$ which has the local coordinates $(x) := (x^i)_{i=1,3}$. It follows that the corresponding tangent bundle $T\Sigma^3$ has the dimension equal to six, and its local coordinates are

$$(x, y) := (x^i, y^i)_{i=1,3} = \left(\begin{array}{c} x^1, x^2, x^3 \\ y^1, y^2, y^3 \end{array}\right).$$

Let us introduce a metric on $T\Sigma^3$ inspired by optics in a non-uniform medium (see formula (1.1)):

$$g_{ij}(x, y) = \delta_{ij} + \gamma^2(x) y_i y_j,$$

where $\delta = (\delta_{ij}) = \text{diag}(1, 1, 1)$ is the Euclidean metric, and $y_i = \delta_{ir} y^r$. Usually, we have $\gamma^2(x) = n^2(x) - 1$, where $n = n(x)$ is the refractive index of the non-uniform medium (see [1, 6, 7]). Using this spatial metric, below we will examine the special case of a possible anisotropic non-relativistic dynamical model, which is governed by the Lagrangian (in this model one considers that the particle has the mass $m = 1$)

$$L(x, y) = \frac{1}{2} g_{ij}(x, y) y^i y^j = \frac{1}{2} \left[\delta_{ij} + \gamma^2 y_i y_j\right] y^i y^j = \frac{1}{2} \delta_{ij} y^i y^j + \frac{\gamma^2}{2} ||y||^4,$$

where $||y||^2 = (y^1)^2 + (y^2)^2 + (y^3)^2 = \delta_{ij} y^i y^j$.

**Remark 1.1** Because the Euclidean metric $\delta_{ij}$ is invariant with respect to the linear transformations of coordinates induced by the Lie group of orthogonal transformations

$$O(3) = \{ A \in M_3(\mathbb{R}) \mid A^T \cdot A = I_3 \},$$

it immediately follows that the Lagrangian (1.2) has a global geometrical character with respect to these orthogonal transformations.

Following as a pattern the geometrical ideas from Lagrangian geometry of tangent bundles [5] or jet bundles [2], in what follows we construct the Riemann-Lagrange geometrical objects (the canonical nonlinear connection, the Cartan canonical linear connection, together with its d-torsions and d-curvatures) produced by the Lagrangian (1.2).

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In this paper the Latin letters $i, j, k, ...$ run from 1 to 3. The Einstein convention of summation is adopted all over this work.
2 Geometrical objects in the non-relativistic extended dynamics

The fundamental metrical distinguished tensor induced by the Lagrangian (1.2) is given by

\[ g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} = \sigma(x, y) \delta_{ij} + 2\gamma^2(x) y_i y_j, \]

where \( \sigma(x, y) = \frac{1}{2} + \gamma^2(x) \|y\|^2 > 0. \)

**Remark 2.1** Because the quadratic form \( q(\xi) = (y_i y_j) \xi^i \xi^j \) is degenerate and has the signature \((2, 1, 0)\), while the Euclidean metric \( \delta(\xi) = \delta_{ij} \xi^i \xi^j \) is non-degenerate and has the signature \((0, 3, 0)\), we easily deduce that the quadratic form \( g(\xi) = g_{ij}(x, y) \xi^i \xi^j \) has the constant signature \((0, 3, 0)\). It follows that it is invariant under orthogonal linear transformation of coordinates. Consequently, all the subsequent geometrical objects constructed in this paper will have the same form in any chart of coordinates induced by a linear transformation of coordinates produced by the orthogonal group \( O(3) \).

The inverse matrix \( g^{-1} = (g^{jk}) \) has the entries

\[ g^{jk}(x, y) = \frac{1}{\sigma(x, y)} \delta^{jk} - \frac{2\gamma^2(x)}{\sigma(x, y) \cdot \tau(x, y)} y^k, \]

where \( \delta^{jk} = \delta_{jk} \) and \( \tau(x, y) = (1/2) + 3\gamma^2(x) \|y\|^2 = \sigma(x, y) + 2\gamma^2(x) \|y\|^2. \)

**Proposition 2.2** For the anisotropic Lagrangian (1.2), the action

\[ E(x(t)) = \int_a^b L(x(t), y(t)) dt, \]

where \( y = dx/dt \), produces on the tangent bundle \( T\Sigma^3 \) the **canonical nonlinear connection** \( N = (N^i_j) \), whose components are

\[
N^i_j = \frac{2\gamma}{\sigma} y^i y_j (\gamma_s y^s) + \\
+ \frac{\gamma \|y\|^2}{\sigma} \left[ \delta_j^i (\gamma_s y^s) + y^i \gamma_j - \gamma^i y_j - \frac{2\gamma^2}{\sigma} y^i y_j (\gamma_s y^s) - \\
- \frac{6\gamma^2}{\tau} y^i y_j (\gamma_s y^s) \right] + \\
+ \frac{\gamma^3 \|y\|^4}{2\sigma} \left[ \frac{1}{\sigma} \gamma^i y_j - \frac{3}{\tau} y^i \gamma_j - \frac{3}{\tau} \delta_j^i (\gamma_s y^s) + \\
+ \frac{6\gamma^2}{\sigma \tau^2} (\tau + 3\sigma) y^i y_j (\gamma_s y^s) \right],
\]

where \( \gamma_s = \partial \gamma / \partial x^s \) and \( \gamma^i = \delta^i \gamma = \gamma_i. \)
Proof. For the energy action functional $E$, the associated Euler-Lagrange equations can be written in the equivalent form (see [5, 2])

$$\frac{d^2 x^i}{dt^2} + 2G^i(x^k(t), y^k(t)) = 0, \quad \forall \, i = 1, 3,$$

(2.2)

where the local components

$$G^i \overset{\text{def}}{=} \frac{g^{ir}}{4} \left[ \frac{\partial^2 L}{\partial y^r \partial x^s} y^s - \frac{\partial L}{\partial x^r} \right] = \frac{\gamma}{\sigma} ||y||^2 y^i \gamma_s y^r - \frac{\gamma}{4\sigma} ||y||^4 \gamma^i - \frac{3\gamma^3}{2\sigma \tau} ||y||^4 y^i (\gamma_s y^r)$$

represent, from a geometrical point of view, a semispray on the tangent vector bundle $T\Sigma^3$. The canonical nonlinear connection associated to this semispray has the components (see [5])

$$N^i_j \overset{\text{def}}{=} \frac{\partial G^i}{\partial y^j}.$$

In conclusion, by direct computations, we find the expression (2.1). \qed

Remark 2.3 In an uniform medium with the constant refractive index

$$n(x) = n \in [1, \infty),$$

we have $\gamma_s = 0$. Consequently, in this case we obtain $G^i = 0$ and $N^i_j = 0$.

The nonlinear connection (2.1) produces the dual adapted bases of d-vector fields

$$\left\{ \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r} \right\} \subseteq \mathcal{X}(T\Sigma^3)$$

and d-covector fields

$$\left\{ dx^i ; \delta y^j = dy^j + N^i_j dx^r \right\} \subseteq \mathcal{X}^*(T\Sigma^3).$$

The naturalness of the geometrical adapted bases (2.3) and (2.4) is coming from the fact that, via a general transformation of coordinates, their elements transform as tensors on $\Sigma^3$. Therefore, the description of all subsequent geometrical objects on the tangent space $T\Sigma^3$ (e.g., the Cartan canonical linear connection, its torsion and curvature) will be done in local adapted components.

For instance, using the notations

$$N_{ij} := N^r_i \delta_{rj}, \quad N_{ir} := N_{ir} y^r, \quad N_{0j} := N_{rj} y^r, \quad N_{00} := N_{ij} y^i y^j,$$

by direct local computations, we obtain the following geometrical result:

Proposition 2.4 The Cartan canonical $N$-linear connection produced by the anisotropic Lagrangian (1.2) has the adapted local components

$$C\Gamma(N) = (L^i_{jk}, C^i_{jk}),$$

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where
where
\[ L^i_{jk} = -\frac{\gamma}{\sigma} \left[ (\delta_i^j N_{k0} + \delta_i^k N_{j0} - \delta_{jk} \delta^{ir} N_{r0}) + \|y\|^2 \left( \delta_{jk} \gamma^i - \delta_i^j \gamma_k - \delta_i^k \gamma_j \right) + \gamma \left\{ (N_{jk} + N_{kj}) y^i + \left( N_i^j - \delta^{ir} N_{rk} \right) y_j + \left( N_i^k - \delta^{ir} N_{rj} \right) y_k \right\} + 2 \left( \gamma y_j y_k - y^i y_j \gamma_k - y^i y_k \gamma_j \right) \right] + \frac{2\gamma^2 y_i}{\sigma^2} \left[ \gamma (y_j N_{k0} + y_k N_{j0} - \delta_{jk} N_{00}) + \|y\|^2 (\delta_{jk} \gamma_i y^r - y^i \gamma_k y^j - y^j \gamma_k y^i - y^i \gamma_j y^j) + 2 (y_j y_k \gamma_i y^r - y^i \gamma_k y^j - y^j \gamma_i y^j) \right] + \gamma \left\{ (N_{jk} + N_{kj}) \|y\|^2 + (N_{k0} - N_{0k}) y_j + (N_{j0} - N_{0j}) y_k \right\} , \]

\[ C^i_{jk} = \frac{\gamma^2}{\sigma} \left( \delta_i^j y_k + \delta_i^k y_j + \delta_{jk} y^i \right) - \frac{2\gamma^2}{\sigma^2} \left( \|y\|^2 \delta_{jk} + 2y_j y_k \right) y^i . \]

**Proof.** The adapted components of the Cartan canonical connection are given by the general formulas (see [5])

\[ L^i_{jk} \overset{\text{def}}{=} \frac{g^{ir}}{2} \left( \delta g^r jk + \delta g^r kr - \delta g^r jk \right) , \]

\[ C^i_{jk} \overset{\text{def}}{=} \frac{g^{ir}}{2} \left( \delta g^r jk + \delta g^r kr - \delta g^r jk \right) = \frac{g^{ir}}{2} \delta g^r jk . \]

Using the derivative operators (2.3), the direct calculations lead us to the required results. ■

The Cartan canonical N-linear connection produced by the anisotropic Lagrangian (12) is characterized by three effective local torsion d-tensors, namely

\[ R^i_{jk} \overset{\text{def}}{=} \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} , \quad P^i_{jk} \overset{\text{def}}{=} \frac{\partial N^i_j}{\partial y^k} - L^i_{kj} - C^i_{jk} , \]

and three effective local curvature d-tensors:

\[ R^i_{jkl} \overset{\text{def}}{=} \frac{\delta L^i_{jk}}{\delta x^l} - \frac{\delta L^i_{jl}}{\delta x^k} + L^i_{jk} L^j_{rl} - L^j_{jl} L^i_{rk} + C^i_{jkl} , \]

\[ P^i_{jkl} \overset{\text{def}}{=} \frac{\partial L^i_{jk}}{\partial y^l} - L^i_{jkl} + C^i_{jkl} P^i_{kl} , \]

\[ S^i_{jkl} \overset{\text{def}}{=} \frac{\partial C^i_{jk}}{\partial y^l} - C^i_{jkl} + C^r_{jkl} C^i_{rl} - C^r_{jrl} C^i_{rk} , \]

where

\[ C^i_{jkl} \overset{\text{def}}{=} \frac{\delta C^i_{jk}}{\delta x^l} + C^i_{jkl} L^i_{jk} - C^i_{jrl} L^i_{jk} - C^i_{jrl} L^i_{jk} . \]

At the end of this Section, we would like to point out that in our anisotropic Lagrangian geometrical theory for non-relativistic extended dynamics, the classical Riemannian Levi-Civita connection (which in our case would be attached
even to the Euclidean metric $\delta_{ij}$ is replaced by Cartan canonical connection (associated to the perturbed Lagrangian $L^\prime$). It is well known that in Finsler-Lagrange geometrical framework (see [5]) there are a lot of important linear distinguished (d-) connections (Cartan, Berwald, Chern-Rund or Hashiguchi, for instance). However, the use of the Cartan canonical connection is preferred because it is the single linear d-connection which is a metrical connection, like the Levi-Civita connection in the classical Riemannian framework. An important difference between these two connections is that the Cartan connection is only partial torsion-free because the Poisson brackets of the distinguished vector fields $\delta/\delta x^i$ are not generally equal to zero.

3 The equations of motion in the anisotropic non-relativistic dynamics

The Euler-Lagrange equations (2.2), for

$$(y^1, y^2, y^3) = (V^1, V^2, V^3) := V, \quad V^i = \frac{dx^i}{dt},$$

$$v^2 = |V|^2 = (V^1)^2 + (V^2)^2 + (V^3)^2,$$

lead us to the following anisotropic equations of motion for a particle:

$$\frac{dV^i}{dt} + \frac{4\gamma v^2 (1 + 3\gamma^2 v^2)}{(1 + 2\gamma^2 v^2)(1 + 6\gamma^2 v^2)} (\gamma_i V^s V^s) - \frac{\gamma v^4}{1 + 2\gamma^2 v^2} \gamma^i = 0, \quad (3.1)$$

where $i \in \{1, 2, 3\}$.

Remark 3.1 In a uniform medium with the constant refractive index $n(x) = n_0 \in [1, \infty)$, the above anisotropic equations of motion simplify as

$$\frac{dV^i}{dt} = 0 \iff V = (V^1, V^2, V^3) = \text{constant} \iff$$

$$\frac{dx^i}{dt} = V^i \iff x(t) = (V^1 t + x^1_0, V^2 t + x^2_0, V^3 t + x^3_0),$$

where $(x^1_0, x^2_0, x^3_0) = \text{constant}$. It follows that, in this case, the particles are moving only on straight lines.

3.1 Particular solutions with cylindrical symmetry

Let us investigate now the case of a non-uniform medium with cylindrical symmetry. This means that we have $n^2 = 1 + f^2(\rho) \iff \gamma = f(\rho)$, where $f : (0, \infty) \to (0, \infty)$ is an arbitrary non-constant smooth function, and

$$\rho^2 = (x^1)^2 + (x^2)^2 > 0.$$
In such a special context, let us search for solutions of (3.1) in cylindrical coordinates:

\[
x_1(t) = \rho(t) \cos \phi(t), \quad x_2(t) = \rho(t) \sin \phi(t), \quad x_3(t) = \zeta(t),
\]

where \( \phi \in [0, 2\pi] \) and \( \zeta \in \mathbb{R} \). By direct computations, we deduce that the equations (3.1) rewrite as

\[
\begin{align*}
\cos \phi (\ddot{\rho}) - 2 \sin \phi \left( \dot{\rho} \dot{\phi} \right) - \rho \cos \phi \left( \dot{\phi} \right)^2 - \rho \sin \phi \left( \ddot{\phi} \right) + \\
+ \frac{4ff'v^2}{(1 + 2f^2v^2)^2} \left( \cos \phi (\ddot{\rho}) - \rho \sin \phi (\dot{\rho} \dot{\phi}) \right) - \\
- \frac{ff'v^4}{1 + 2f^2v^2} \cos \phi = 0,
\end{align*}
\]

\[
\begin{align*}
\sin \phi (\ddot{\rho}) + 2 \cos \phi \left( \dot{\rho} \dot{\phi} \right) - \rho \sin \phi \left( \dot{\phi} \right)^2 + \rho \cos \phi \left( \ddot{\phi} \right) + \\
+ \frac{4ff'v^2}{(1 + 2f^2v^2)^2} \left( \sin \phi (\ddot{\rho}) + \rho \cos \phi \left( \dot{\rho} \dot{\phi} \right) \right) - \\
- \frac{ff'v^4}{1 + 2f^2v^2} \sin \phi = 0,
\end{align*}
\]

\[
\ddot{\zeta} + \frac{4ff'v^2}{(1 + 2f^2v^2)^2} \left( \dot{\rho} \dot{\zeta} \right) = 0,
\]

where

\[
v^2 = (\dot{\rho})^2 + \rho^2 (\dot{\phi})^2 + (\dot{\zeta})^2.
\]

In what follows, we will look for some particular solutions of the system of differential equations (3.2 - 3.4).

**Case 1:** Let us consider that \( \dot{\rho} = 0 \) and \( v \neq 0 \). We will look for solutions of the form

\[
c(\zeta) = (\rho = \text{constant}, \ \phi = \phi(\zeta), \ \zeta \in \mathbb{R}) \Rightarrow \dot{c}(\zeta) = \left( \dot{\rho} = 0, \ \dot{\phi} = \frac{d\phi}{d\zeta}, \ \dot{\zeta} = 1 \right).
\]

These conditions imply

\[
v^2 = \rho^2 \left( \frac{d\phi}{d\zeta} \right)^2 + 1.
\]

In this case the anisotropic equations of motion reduce to

\[
\begin{cases}
-\rho \cos \phi \left( \frac{d\phi}{d\zeta} \right)^2 - \rho \sin \phi \left( \frac{d^2\phi}{d\zeta^2} \right) - \frac{ff'v^4}{1 + 2f^2v^2} \cos \phi = 0 \\
-\rho \sin \phi \left( \frac{d\phi}{d\zeta} \right)^2 + \rho \cos \phi \left( \frac{d^2\phi}{d\zeta^2} \right) - \frac{ff'v^4}{1 + 2f^2v^2} \sin \phi = 0.
\end{cases}
\]

\[\text{(3.5)}\]
Multiplying the first equation of (3.5) by \((- \sin \phi)\), and the second by \((\cos \phi)\), by summing we get the equation:

\[
\rho \cdot \left( \frac{d^2 \phi}{d \zeta^2} \right) = 0 \Leftrightarrow \frac{d^2 \phi}{d \zeta^2} = 0 \Leftrightarrow \phi(\zeta) = \omega \zeta + \phi_0, \tag{3.6}
\]

where \(\omega, \phi_0 \in \mathbb{R}\) are arbitrary constants (here \(\omega\) has the physical meaning of angular velocity). Consequently, the function (3.6) is a solution for the equations of motion (3.5) if and only if

\[
\begin{cases}
(\rho \cos \phi) (\omega^2) + \frac{f f' v^4}{1 + 2 f^2 v^2} \cos \phi = 0 \\
(\rho \sin \phi) (\omega^2) + \frac{f f' v^4}{1 + 2 f^2 v^2} \sin \phi = 0
\end{cases} \Leftrightarrow \rho \omega^2 + \frac{f f' v^4}{1 + 2 f^2 v^2} = 0, \tag{3.7}
\]

where \(v^2 = \rho^2 \omega^2 + 1\).

1. If \(2f^2 - 1 < 0\) and

\[2f + \rho f' \in \left[-\frac{(2f^2 - 1)^2}{4f}, 0\right) \cup (0, 2f),\]

then the solutions of the equation (3.7) are

\[
\omega_0 = \pm \frac{1}{\rho} \sqrt{\frac{2f^2 - 1 \pm \sqrt{\Delta}}{2f (2f + \rho f')}} - 1,
\]

where \(\Delta = (1 - 2f^2)^2 + 4f (2f + \rho f')\).

2. If \(2f^2 - 1 > 0\) and \(2f + \rho f' \in (0, 2f)\), then the solutions of (3.7) are

\[
\omega_0 = \pm \frac{1}{\rho} \sqrt{\frac{2f^2 - 1 + \sqrt{\Delta}}{2f (2f + \rho f')}} - 1.
\]

In conclusion, the solutions of equations of motion (3.5) are

\[
c(t) = (\rho = \rho_0 = \text{constant}, \ \phi = \omega_0 t + \phi_0, \ \zeta = t) \Rightarrow \]

\[x^1(t) = \rho_0 \cos \phi(t), \quad x^2(t) = \rho_0 \sin \phi(t), \quad x^3(t) = t.\]

It follows that in this case the particles move on some circular helices. Note that these circular helices are some very specific trajectories (non-evident in advance) because the values of the angular velocity \(\omega_0\) are completely determined by the initial function \(f(\rho)\). Solutions corresponding to the sign plus and
minus, which are intimately related to the orientation of rotation (right-handed and left-handed respectively).

**Case 2:** Let it be $\dot{\phi} = 0$ and $v \neq 0$. We will look for solutions of the form

$$c(\zeta) = (\rho = \rho(\zeta), \phi = \text{constant}, \zeta \in \mathbb{R}) \Rightarrow \dot{c}(\zeta) = \left( \dot{\rho} = \frac{d\rho}{d\zeta}, \dot{\phi} = 0, \dot{\zeta} = 1 \right).$$

These conditions imply

$$v^2 = \left( \frac{d\rho}{d\zeta} \right)^2 + 1.$$ 

In this case the anisotropic equations of motion reduce to

$$\begin{align*}
\cos \phi \left( \frac{d^2 \rho}{d\zeta^2} \right) + \frac{4ff'v^2 \left( 1 + 3f^2v^2 \right)}{(1 + 2f^2v^2)(1 + 6f^2v^2)} \cos \phi \left( \frac{d\rho}{d\zeta} \right)^2 &- \frac{ff'v^4}{1 + 2f^2v^2} \cos \phi = 0 \\
\sin \phi \left( \frac{d^2 \rho}{d\zeta^2} \right) + \frac{4ff'v^2 \left( 1 + 3f^2v^2 \right)}{(1 + 2f^2v^2)(1 + 6f^2v^2)} \sin \phi \left( \frac{d\rho}{d\zeta} \right)^2 &- \frac{ff'v^4}{1 + 2f^2v^2} \sin \phi = 0 \\
left( \frac{d\rho}{d\zeta} \right) f' &= 0.
\end{align*} \tag{3.8}$$

The last equation of (3.8) implies $d\rho/d\zeta = 0$ or $f' = 0$. In both situations the solutions of the equations of motion are

$$c(t) = (\rho = \rho_0 = \text{constant}, \phi = \phi_0 \in [0, 2\pi], \zeta = t \in \mathbb{R}) \Rightarrow$$

$$\Rightarrow x^1(t) = \rho_0 \cos \phi_0, \quad x^2(t) = \rho_0 \sin \phi_0, \quad x^3(t) = t.$$ 

Consequently, in this case the trajectories are the generators of the right circular cylinders

$$(x^1)^2 + (x^2)^2 = \rho_0^2,$$

where $\rho_0 > 0$ is a solution of the equation $f'(\rho) = 0$.

**Case 3:** Let us consider that $\dot{\zeta} = 0$ and $v \neq 0$. We will look for solutions of the form

$$c(\phi) = (\rho = \rho(\phi), \phi := t \in [0, 2\pi], \zeta = \text{constant}) \Rightarrow$$

$$\Rightarrow \dot{c}(\phi) = \left( \dot{\rho} = \frac{d\rho}{d\phi}, \dot{\phi} = 1, \dot{\zeta} = 0 \right).$$

These conditions imply

$$v^2 = \left( \frac{d\rho}{dt} \right)^2 + \rho^2.$$
In this case the anisotropic equations of motion reduce to

\[
\begin{aligned}
\cos t \left( \frac{d^2 \rho}{dt^2} \right) - 2 \sin t \left( \frac{d\rho}{dt} \right) - \rho \cos t - \frac{f f' v^4}{1 + 2 f^2 v^2} \cos t + \\
\frac{4f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} \left( \cos t \left( \frac{d\rho}{dt} \right) \right)^2 - \rho \sin t \left( \frac{d\rho}{dt} \right) = 0 \\
\sin t \left( \frac{d^2 \rho}{dt^2} \right) + 2 \cos t \left( \frac{d\rho}{dt} \right) - \rho \sin t - \frac{f f' v^4}{1 + 2 f^2 v^2} \sin t + \\
\frac{4f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} \left( \sin t \left( \frac{d\rho}{dt} \right) \right)^2 + \rho \cos t \left( \frac{d\rho}{dt} \right) = 0.
\end{aligned}
\]  

(3.9)

Multiplying the first equation of (3.9) by \((- \sin \phi)\), and the second by \((\cos \phi)\), by summing we get the equation:

\[
\left( \frac{d\rho}{dt} \right) \left( 1 + \frac{2 \rho f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} \right) = 0.
\]

1. If \(d\rho/dt = 0\), then the equations of motion (3.9) become

\[
\begin{aligned}
\rho \cos t + \frac{f f' v^4}{1 + 2 f^2 v^2} \cos t = 0 \\
\rho \sin t + \frac{f f' v^4}{1 + 2 f^2 v^2} \sin t = 0.
\end{aligned}
\]

\[
\Leftrightarrow 1 + 2 f^2 \rho^2 + 4 f' \rho^3 = 0,
\]

(3.10)

where \(v^2 = \rho^2\). The corresponding solutions are

\[
c(t) = (\rho = \rho_0 > 0, \phi = t \in [0,2\pi], \zeta = \zeta_0 \in \mathbb{R}) \Rightarrow \\
x^1(t) = \rho_0 \cos t, \quad x^2(t) = \rho_0 \sin t, \quad x^3(t) = \zeta_0.
\]

It follows that in this case the trajectories are circles situated in planes parallel with the plane \(xOy\), having the centers situated on the axis \(Oz\), and the radii \(\rho_0 > 0\), where \(\rho_0\) is a solution of (3.10).

2. If \(d\rho/dt \neq 0\), then we get

\[
1 + \frac{2 \rho f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} = 0.
\]

(3.11)

The preceding equation has solution if and only if

\[
2f + \rho f' \in \left( \frac{1 \pm 4 \rho^2 f^2}{2 \rho^2 f (1 + 3 \rho^2 f^2)}, 0 \right).
\]
In this case, the solutions of the equation (3.11) are

$$\frac{d\rho}{dt} = \pm \sqrt{-4f^2 - \rho f f' - \sqrt{\Delta'}} \cdot \frac{6f^2 (2f + \rho f')}{\Delta' 6f^3} - \rho^2 := F(\rho),$$

(3.12)

where $\Delta' = 4f^4 + \rho^2 f^2 (f')^2 + 2\rho f^3 f'$. Imposing as a solution of (3.12) to be also a solution for the system (3.9), we get

$$FF' - \rho - \frac{2F^2}{\rho} = \frac{f f' (F^2 + \rho^2)^2}{1 + 2f^2 (F^2 + \rho^2)} \Rightarrow$$

$$\Rightarrow \rho = \text{constant (contradiction!)}. $$

So, the case $d\rho/dt \neq 0$ does not provide us any solution.

3.2 Particular solutions with spherical symmetry

We investigate now the case of a non-uniform medium with spherical symmetry. This means that we have $n^2 = 1 + f^2(r) \iff \gamma = f(r)$, where $f : (0, \infty) \rightarrow (0, \infty)$ is an arbitrary non-constant smooth function, and

$$r^2 = |x|^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 > 0.$$  

Example 3.2 In the study of mirages from heat above warm surfaces, one works with indices of refraction which have spherical symmetry (see [4, 8]):

$$n(x) = 1 + \varepsilon e^{-r^2/\sigma^2} \iff f(r) = \sqrt{2\varepsilon e^{-r^2/\sigma^2} + \varepsilon^2 e^{-2r^2/\sigma^2}},$$

where $\varepsilon > 0$ and $\sigma > 0$ are some given constants.

In such a special context, let us search for solutions of (3.11) in spherical coordinates:

$$x_1(t) = r(t) \sin \theta(t) \cos \phi(t), \quad x_2(t) = r(t) \sin \theta(t) \sin \phi(t), \quad x_3(t) = r(t) \cos \theta(t),$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. By direct computations, we deduce that the equations (3.11) rewrite as

$$\sin \theta \cos \phi (\ddot{r}) + r \cos \theta \cos \phi (\dot{\theta}) - r \sin \theta \sin \phi (\dot{\phi}) +$$

$$+ 2 \cos \theta \cos \phi (\dot{r} \dot{\theta}) - 2 \sin \theta \sin \phi (\dot{r} \dot{\phi}) - 2r \cos \theta \sin \phi (\dot{\theta} \dot{\phi}) -$$

$$- r \sin \theta \cos \phi \left( (\dot{\theta})^2 + (\dot{\phi})^2 \right) + \frac{4ff'v^2 (1 + 3f^2v^2)}{(1 + 2f^2v^2)(1 + 6f^2v^2)}$$

$$\cdot \left( \sin \theta \cos \phi (\ddot{r}) + r \cos \theta \cos \phi (\dot{r} \dot{\theta}) - r \sin \theta \sin \phi (\dot{r} \dot{\phi}) -$$

$$- \frac{ff'v^4}{1 + 2f^2v^2} \sin \theta \cos \phi = 0,$$
These conditions imply

\[
\sin \theta \sin \phi (\ddot{r}) + r \cos \theta \sin \phi (\ddot{\theta}) + r \sin \theta \cos \phi (\ddot{\phi}) +
+ 2 \cos \theta \sin \phi (\dot{r} \ddot{\theta}) + 2 \sin \theta \cos \phi (\dot{r} \dot{\phi}) + 2r \cos \theta \cos \phi (\theta \ddot{\phi}) -
- r \sin \theta \sin \phi \left( (\dot{\theta})^2 + (\dot{\phi})^2 \right) + \frac{4ff'v^2(1 + 3f^2v^2)}{(1 + 2f^2v^2)(1 + 6f^2v^2)} \cdot \left( \sin \theta \sin \phi (\dot{r})^2 + r \cos \theta \sin \phi (\dot{r} \dot{\theta}) + r \sin \theta \cos \phi (\dot{r} \dot{\phi}) \right) -
\frac{ff'v^4}{1 + 2f^2v^2} \sin \theta \sin \phi = 0,
\cos \theta (\ddot{r}) - r \sin \theta (\dot{r} \ddot{\theta}) - 2 \sin \theta (\dot{r} \ddot{\theta} - r \cos \theta (\ddot{\theta})^2 +
+ \frac{4ff'v^2(1 + 3f^2v^2)}{(1 + 2f^2v^2)(1 + 6f^2v^2)} \left( \cos \theta (\dot{r})^2 - r \sin \theta (\dot{r} \dot{\theta}) \right) -
\frac{ff'v^4}{1 + 2f^2v^2} \cos \theta = 0,
\]

where

\[ v^2 = (\dot{r})^2 + r^2 (\dot{\theta})^2 + r^2 \sin^2 \theta (\dot{\phi})^2. \]

Let us look for some particular solutions of the DEs system \((3.13 - 3.15)\).

**Case 1:** Let it be \( \dot{r} = 0 \) and \( v \neq 0 \). We will look for solutions of the form

\[ c(\phi) = (r = \text{constant}, \ \theta = \theta(\phi), \ \phi := t \in [0, 2\pi]) \Rightarrow \]

\[ \Rightarrow \hat{c}(\phi) = \left( \dot{r} = 0, \ \dot{\theta} = \frac{d\theta}{d\phi}, \ \dot{\phi} = 1 \right). \]

These conditions imply

\[ v^2 = r^2 \left( \left( \frac{d\theta}{dt} \right)^2 + \sin^2 \theta \right). \]

In this case the anisotropic equations of motion reduce to

\[
\begin{cases}
  r \cos \theta \cos t \left( \frac{d^2\theta}{dt^2} \right) - 2r \cos \theta \sin t \left( \frac{d\theta}{dt} \right) - \\
  -r \sin \theta \cos t \left( \left( \frac{d\theta}{dt} \right)^2 + 1 \right) - \frac{ff'v^4}{1 + 2f^2v^2} \sin \theta \cos t = 0 \\
  r \cos \theta \sin t \left( \frac{d^2\theta}{dt^2} \right) + 2r \cos \theta \cos t \left( \frac{d\theta}{dt} \right) - \\
  -r \sin \theta \sin t \left( \left( \frac{d\theta}{dt} \right)^2 + 1 \right) - \frac{ff'v^4}{1 + 2f^2v^2} \sin \theta \sin t = 0 \\
  r \sin \theta \left( \frac{d^2\theta}{dt^2} \right) + r \cos \theta \left( \frac{d\theta}{dt} \right)^2 + \frac{ff'v^4}{1 + 2f^2v^2} \cos \theta = 0.
\end{cases}
\]
Multiplying the first equation of (3.16) by \((-\sin t)\), and the second by \((\cos t)\), by summing we get the equation:

\[ 2r \cos \theta \left( \frac{d\theta}{dt} \right) = 0. \]

1. If \(\cos \theta = 0 \Leftrightarrow \theta = \pi/2\), then the equations of motion (3.16) become

\[
\begin{align*}
 r \sin \theta \cos t + & \frac{f'v^4}{1 + 2f^2v^2} \sin \theta \cos t = 0 \\
 r \sin \theta \sin t + & \frac{f'v^4}{1 + 2f^2v^2} \sin \theta \sin t = 0,
\end{align*}
\]

\[ \Leftrightarrow 1 + \frac{f'v^3}{1 + 2f^2v^2} = 0, \quad (3.17) \]

where \(v^2 = r^2\). It follows that in this case the solutions of the anisotropic equations of motion are

\[ c(t) = \left( r = r_0 = \text{constant}, \quad \theta = \frac{\pi}{2}, \quad \phi = t \right) \Rightarrow \]

\[ x^1(t) = r_0 \cos t, \quad x^2(t) = r_0 \sin t, \quad x^3(t) = 0, \]

where \(r_0 > 0\) is a solution of the equation (3.17). It follows that in this case the particles move on the \textit{circles situated in the plane} \(xOy\), \textit{which have the centers in origin and the radii equal to the roots of the equation} (3.17). We recall that in the case of cylinder solutions we obtained some \textit{circles situated in planes parallel with the plane} \(xOy\), \textit{having the centers situated on the axis} \(Oz\), and the radii \(\rho_0 > 0\), where \(\rho_0\) is a solution of the same equation.

2. If we have \(d\theta/dt = 0\), then the anisotropic equations of motion become

\[
\begin{align*}
 r \sin \theta \cos t + & \frac{f'v^4}{1 + 2f^2v^2} \sin \theta \cos t = 0 \\
 r \sin \theta \sin t + & \frac{f'v^4}{1 + 2f^2v^2} \sin \theta \sin t = 0, \\
 f'v^4 - & \frac{1}{1 + 2f^2v^2} \cos \theta = 0.
\end{align*}
\]

This system implies \(\theta = \pi/2\) (we recover then the above Case 1.) or \(f' = 0\). So, the new corresponding solutions are

\[ c(t) = (r = r_0 > 0, \quad \theta = \theta_0 \in [0, \pi] \setminus \{\pi/2\}, \quad \phi = t) \Rightarrow \]

\[ x^1(t) = (r_0 \sin \theta_0) \cos t, \quad x^2(t) = (r_0 \sin \theta_0) \sin t, \quad x^3(t) = r_0 \cos \theta_0. \]

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These trajectories are circles situated in planes parallel with the plane \( xOy \), having the centers situated on the axis \( Oz \), and the radii \( r_0 \sin \theta_0 \), where \( r_0 > 0 \) is a solution of the equation \( f'(r) = 0 \).

By comparing, we recall that in the case of cylinder solutions we obtained the generators of the right circular cylinders

\[
(x^1)^2 + (x^2)^2 = \rho_0^2,
\]

where \( \rho_0 > 0 \) is a solution of the same equation.

**Case 2:** Let us consider that \( \dot{\theta} = 0 \) and \( v \neq 0 \). We will look for solutions of the form

\[
\begin{align*}
& c(\phi) = (r = r(\phi), \theta = \text{constant}, \phi := t \in [0, 2\pi]) \Rightarrow \\
& \Rightarrow \dot{c}(\phi) = \left( \dot{r} = \frac{dr}{d\phi}, \dot{\theta} = 0, \dot{\phi} = 1 \right).
\end{align*}
\]

These conditions imply

\[
v^2 = \left( \frac{dr}{dt} \right)^2 + r^2 \sin^2 \theta.
\]

In this case the anisotropic equations of motion reduce to

\[
\sin \theta \cos t \left( \frac{d^2 r}{dt^2} \right) - 2 \sin \theta \sin t \left( \frac{dr}{dt} \right) - r \sin \theta \cos t +
\]

\[
+ \frac{4 f' \nu^2 (1 + 3 f^2 \nu^2)}{(1 + 2 f^2 \nu^2) (1 + 6 f^2 \nu^2)} \left( \sin \theta \cos t \left( \frac{dr}{dt} \right)^2 - r \sin \theta \sin t \left( \frac{dr}{dt} \right) \right) -
\]

\[
- \frac{f' \nu^4}{1 + 2 f^2 \nu^2} \sin \theta \cos t = 0,
\]

\[
\sin \theta \sin t \left( \frac{d^2 r}{dt^2} \right) + 2 \sin \theta \cos t \left( \frac{dr}{dt} \right) - r \sin \theta \sin t +
\]

\[
+ \frac{4 f' \nu^2 (1 + 3 f^2 \nu^2)}{(1 + 2 f^2 \nu^2) (1 + 6 f^2 \nu^2)} \left( \sin \theta \sin t \left( \frac{dr}{dt} \right)^2 + r \sin \theta \cos t \left( \frac{dr}{dt} \right) \right) -
\]

\[
- \frac{f' \nu^4}{1 + 2 f^2 \nu^2} \sin \theta \sin t = 0,
\]

\[
\cos \theta \left( \frac{d^2 r}{dt^2} \right) + \frac{4 f' \nu^2 (1 + 3 f^2 \nu^2)}{(1 + 2 f^2 \nu^2) (1 + 6 f^2 \nu^2)} \cos \theta \left( \frac{dr}{dt} \right)^2 -
\]

\[
- \frac{f' \nu^4}{1 + 2 f^2 \nu^2} \cos \theta = 0.
\]
1. If \( \cos \theta = 0 \iff \theta = \pi/2 \), then the equations of motion become

\[
\cos t \left( \frac{d^2r}{dt^2} \right) - 2 \sin t \left( \frac{dr}{dt} \right) - r \cos t - \frac{f f' v^4}{1 + 2 f^2 v^2} \cos t + \\
\frac{4 f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} \cdot \left( \cos t \left( \frac{dr}{dt} \right)^2 - r \sin t \left( \frac{dr}{dt} \right) \right) = 0,
\]

(3.18)

\[
\sin t \left( \frac{d^2r}{dt^2} \right) + 2 \cos t \left( \frac{dr}{dt} \right) - r \sin t - \frac{f f' v^4}{1 + 2 f^2 v^2} \sin t + \\
\frac{4 f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} \cdot \left( \sin t \left( \frac{dr}{dt} \right)^2 + r \cos t \left( \frac{dr}{dt} \right) \right) = 0,
\]

where

\[ v^2 = \left( \frac{dr}{dt} \right)^2 + r^2. \]

Multiplying the first equation by \((- \sin t)\), and the second by \((\cos t)\), by summing we get the equation:

\[
\left( \frac{dr}{dt} \right) \left( 1 + \frac{2 r f f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2)(1 + 6 f^2 v^2)} \right) = 0.
\]

(3.20)

(a) If \( dr/dt = 0 \), then the equations of motion become

\[
\begin{cases}
  r \cos t + \frac{f f' v^4}{1 + 2 f^2 v^2} \cos t = 0 \\
  r \sin t + \frac{f f' v^4}{1 + 2 f^2 v^2} \sin t = 0 \\
  
\end{cases}
\]

\[ \iff r + \frac{f f' v^4}{1 + 2 f^2 v^2} = 0 \iff \]

\[ 1 + 2 f^2 v^2 + f f' r^3 = 0, \]

(3.21)

where \( v^2 = r^2 \). The corresponding solutions are

\[ c(t) = \left( r = r_0 > 0, \theta = \frac{\pi}{2}, \phi = t \right) \Rightarrow \]

\[ x^1(t) = r_0 \cos t, \quad x^2(t) = r_0 \sin t, \quad x^3(t) = 0. \]

It follows that in this case the particles move on circles situated in the plane \( xOy \), having the centers in origin, and the radii...
\( \rho_0 > 0, \) where \( \rho_0 \) is a solution of the equation \((3.21)\). Note that these trajectories also appear in the case of cylindric solutions (see again the preceding Case 3. // Subcase 1.).

(b) If \( dr/dt \neq 0 \), then we get

\[
1 + \frac{2rf f' v^2 (1 + 3f^2 v^2)}{(1 + 2f^2 v^2) (1 + 6f^2 v^2)} = 0.
\]

The preceding equation has solution if and only if

\[
2f + r f' \in \left( -\frac{1 + 4r^2 f^2}{2r^2 f (1 + 3f^2 v^2)}, 0 \right),
\]

and the solutions are

\[
\frac{dr}{dt} = \pm \sqrt{- \frac{4f^2 - r f' - \sqrt{\Delta'}}{6f^3 (2f + r f')}} - r^2 := F(r), \quad (3.22)
\]

where \( \Delta' = 4f^4 + r^2 f^2 (f')^2 + 2rf^3 f' \). Imposing as a solution of \((3.22)\) to be also a solution for equations \((3.18)\) and \((3.19)\), we get

\[
FF' - r - \frac{2F^2}{r} = \frac{ff' (F^2 + r^2)^2}{1 + 2f^2 (F^2 + r^2)} \Rightarrow
\]

\[
\Rightarrow r = \text{constant (contradiction!).}
\]

So, the subcase \( dr/dt \neq 0 \) does not provide us any solution.

2. If \( \cos \theta \neq 0 \) and \( \sin \theta \neq 0 \), then the equations of motion reduce to the equations \((3.18)\), \((3.19)\) and

\[
\left( \frac{d^2r}{dt^2} \right) + \frac{4f f' v^2 (1 + 3f^2 v^2)}{(1 + 2f^2 v^2) (1 + 6f^2 v^2)} \left( \frac{dr}{dt} \right)^2 - \frac{f f' v^4}{1 + 2f^2 v^2} = 0.
\]

Using the equation \((3.22)\), we deduce that the equations \((3.18)\) and \((3.19)\) simplify as

\[
-2 \sin t \left( \frac{dr}{dt} \right) - r \cos t - \frac{4rf f' v^2 (1 + 3f^2 v^2)}{(1 + 2f^2 v^2) (1 + 6f^2 v^2)} \sin t \left( \frac{dr}{dt} \right) = 0,
\]

\[
2 \cos t \left( \frac{dr}{dt} \right) - r \sin t + \frac{4rf f' v^2 (1 + 3f^2 v^2)}{(1 + 2f^2 v^2) (1 + 6f^2 v^2)} \cos t \left( \frac{dr}{dt} \right) = 0.
\]

Multiplying the first equation from above by \( \cos t \), and the second by \( \sin t \), we deduce that this system of equations has no solution.
3. If \( \sin \theta = 0 \), then the equations of motion are equivalent with differential equation (3.23), where

\[
v^2 = \left( \frac{dr}{dt} \right)^2.
\]

To integrate the equation (3.23) seems to be very difficult but it is obvious that the solutions of the equations of motion are segments of the axis \( Oz \), whose lengths are determined by the images of the solutions \( r(t) \) of the equation (3.23).

**Case 3:** Let us consider that \( \dot{\phi} = 0 \) and \( v \neq 0 \). We will look for solutions of the form

\[
c(\theta) = (r = r(\theta), \, \theta := t \in [0, \pi], \, \phi = \text{constant}) \Rightarrow \\
\Rightarrow \dot{c}(\theta) = \left( \dot{r} = \frac{dr}{dt}, \, \dot{\theta} = 1, \, \dot{\phi} = 0 \right).
\]

These conditions imply

\[
v^2 = \left( \frac{dr}{dt} \right)^2 + r^2.
\]

In this case the anisotropic equations of motion reduce to

\[
\begin{align*}
\sin t \cos \phi \left( \frac{d^2r}{dt^2} \right) + 2 \cos t \cos \phi \left( \frac{dr}{dt} \right) - r \sin t \cos \phi + \\
+ \frac{4 f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2) (1 + 6 f^2 v^2)} \left( \sin t \cos \phi \left( \frac{dr}{dt} \right)^2 + r \cos t \cos \phi \left( \frac{dr}{dt} \right) \right) - \\
- \frac{f f' v^4}{1 + 2 f^2 v^2} \sin t \cos \phi = 0,
\end{align*}
\]

\[
\begin{align*}
\sin t \sin \phi \left( \frac{d^2r}{dt^2} \right) + 2 \cos t \sin \phi \left( \frac{dr}{dt} \right) - r \sin t \sin \phi + \\
+ \frac{4 f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2) (1 + 6 f^2 v^2)} \left( \sin t \sin \phi \left( \frac{dr}{dt} \right)^2 + r \cos t \sin \phi \left( \frac{dr}{dt} \right) \right) - \\
- \frac{f f' v^4}{1 + 2 f^2 v^2} \sin t \sin \phi = 0,
\end{align*}
\]

\[
\begin{align*}
\cos t \left( \frac{d^2r}{dt^2} \right) - 2 \sin t \left( \frac{dr}{dt} \right) - r \cos t+ \\
+ \frac{4 f' v^2 (1 + 3 f^2 v^2)}{(1 + 2 f^2 v^2) (1 + 6 f^2 v^2)} \left( \cos t \left( \frac{dr}{dt} \right)^2 - r \sin t \left( \frac{dr}{dt} \right) \right) - \\
- \frac{f f' v^4}{1 + 2 f^2 v^2} \cos t = 0.
\end{align*}
\]
Multiplying the first equation by \((\cos \phi \cos t)\), the second by \((\sin \phi \cos t)\), and the third by \((-\sin t)\), by summing we get again the equation (3.20). But, this case was previously treated, and the corresponding solutions are

\[
e(t) = (r = r_0 > 0, \theta = t, \phi = \phi_0 \in [0, 2\pi]) \Rightarrow
\]

\[
x_1(t) = (r_0 \cos \phi_0) \sin t, \quad x_2(t) = (r_0 \sin \phi_0) \sin t, \quad x_3(t) = r_0 \cos t,
\]

where \(r_0 > 0\) is a solution of the equation (3.21). It follows that in this case the trajectories are the following circles:

\[
\begin{cases}
(x^1)^2 + (x^2)^2 + (x^3)^2 = r_0^2 \\
(sin \phi_0) x^1 - (cos \phi_0) x^2 = 0.
\end{cases}
\]

4 Conclusion

At the end of this paper, we consider it is important to underline some geometrical differences between the classical non-relativistic dynamics (determined only by the Euclidean metric \(\delta_{ij}\)) and the present anisotropic non-relativistic extended dynamics (determined by the perturbed Lagrangian (1.2)). For instance, we recall that in the first case we have only a Riemannian curvature (determined by the Levi-Civita connection), while in the second case we have more Lagrangian curvatures (determined by the more complicated Cartan connection). Moreover, in the first case we live in a flat space (its Riemannian curvature is zero), while in the second case we work in a curved spaces (its Lagrangian curvatures being non-zero). At the same time, note that the corresponding geodesics are in these different cases:

1. any straight lines in space, in the first case;

2. only certain circular helices or circles situated in some specific planes, together with certain specific straight lines which are parallel with the axis \(Oz\), in the second case. It is also important to note that, in this second anisotropic situation, the preceding geometrical geodesics do not represent the general solutions of the geodesic equations, but only some particular cases of them. To find the general solution of these geodesic equations can be considered as an open problem for a research study in this geometric-physical domain.

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