Abstract. Building on recent work of Jaikin-Zapirain, we provide a homological criterion for a ring to be a pseudo-Sylvester domain, that is, to admit a non-commutative field of fractions over which all stably full matrices become invertible. We use the criterion to study skew Laurent polynomial rings over free ideal rings (firs).

As an application of our methods, we prove that crossed products of division rings with free-by-cyclic and surface groups are pseudo-Sylvester domains unconditionally and Sylvester domains if and only if they admit stably free cancellation. This relies on the recent proof of the Farrell–Jones conjecture for poly-free groups and extends previous results of Linnell–Lück and Jaikin-Zapirain on universal localizations and universal fields of fractions of such crossed products.

Introduction

Given a domain \( R \), i.e., a not necessarily commutative ring without non-trivial zero divisors, it is natural to ask whether there exists a division ring \( D \) in which \( R \) can be embedded. In the commutative world, the existence of the field of fractions of \( R \) settles the question, but in the non-commutative setting, a division ring with the desired property may not exist in general (\[\text{Ma37}\]).

It was P.M. Cohn who realized that, in the same way that we can obtain a field from a commutative ring by localizing at a prime ideal (and then taking the
residue field), we can obtain a division ring $D$ from any ring $R$ by means of universal localization at prime matrix ideals (cf. [Coh06]). Similarly to the commutative case, the division ring obtained in this way is generated as a division ring by the image of $R$ under the corresponding map $R \to D$. The pair given by $D$ and the map $R \to D$, or sometimes just $D$ if the map is clear from the context, is usually referred to as epic division $R$-ring.

Adopting the previous terminology, recall that a homomorphism from a commutative ring $R$ to an epic field $K$ is completely characterized by its kernel, which is a prime ideal of $R$, in the sense that $K$ can be recovered as mentioned above, i.e., by localizing at the kernel and taking the residue field. This is equivalent to saying that such a homomorphism is determined by the set of elements that become invertible in $K$, the ones outside the kernel. In the very same spirit, P.M. Cohn showed that a prescribed epic division $R$-ring is completely characterized by its singular kernel, which is a prime matrix ideal, or equivalently by the set $\Sigma$ of matrices becoming invertible under the homomorphism. The latter point of view is particularly useful since the map will be injective if and only if $\Sigma$ contains every non-zero element of $R$.

Assume that we are given an embedding $R \hookrightarrow D$ of the domain $R$ into the division ring $D$. Then, a natural necessary condition for an $n \times n$ matrix $A$ over $R$ to become invertible over $D$ is that it cannot be expressed as a product $A = BC$ for some matrices $B, C$ of sizes $n \times m$ and $m \times n$, respectively, where $m < n$. Otherwise, the usual rank $\text{rk}_D(A)$ of $A$ over $D$ would be less or equal than $m$, and hence $A$ would not be invertible. A matrix satisfying this necessary condition is called full. Therefore, one may wonder whether, among the division rings in which $R$ can be embedded, there exists one in which we can invert every full matrix. The rings for which this is possible, originally studied by W. Dicks and E. Sontag ([DS78]) as those satisfying the law of nullity with respect to the inner rank function, comprise the family of Sylvester domains. The first examples of Sylvester domains were the free ideal rings (firs) (cf. [Coh06, Section 5.5]).

In addition, observe that if the matrix $A$ is to become invertible in a division ring, then the same holds true for $A \oplus I_m$, the block diagonal matrix with blocks $A$ and $I_m$, where $I_m$ denotes the $m \times m$ identity matrix. Thus, $A \oplus I_m$ must in fact be full for every non-negative integer $m$. A matrix with this property is called stably full and, of course, in a Sylvester domain it is the case that every full matrix is stably full. Nevertheless, in general there may be full matrices that are not stably full, and hence, the question of whether there exists a division ring $D$ in which $R$ embeds and in which we can invert every stably full matrix over $R$ is interesting in itself. The rings with this property are the pseudo-Sylvester domains, originally defined as the family of stably finite rings satisfying the law of nullity with respect to the stable rank function (cf. [Coh06 Section 5.6]). Notice that if such a division ring $D$ exists, then it is necessarily universal in the sense of P.M. Cohn (see Section 1.2), meaning that if a matrix $A$ over $R$ becomes invertible over some division ring, then it is also invertible over $D$.

Recently, in [Jai19c], A. Jaikin-Zapirain introduced a new homological criterion for a ring to be a Sylvester domain. In the current paper, we provide a similar recognition principle for pseudo-Sylvester domains and use it to prove the following result:

**Theorem A.** Let $\mathcal{F}$ be a fir with universal division $\mathcal{F}$-ring of fractions $\mathcal{D}_\mathcal{F}$, and consider a crossed product ring $S = \mathcal{F} \ast \mathbb{Z}$. Then, the following holds:

a) $S$ is a pseudo-Sylvester domain if and only if every finitely generated projective $S$-module is stably free.

b) $S$ is a Sylvester domain if and only if it is projective-free.
In any of the previous situations, $D_S = \text{Ore}(D_S \ast \mathbb{Z})$ is the universal localization of $S \ast \mathbb{Z}$ with respect to the set of all stably full (resp. full) matrices. In particular, it is the universal division $S$-ring of fractions.

As a particular application of Theorem A we obtain the next result through the recent advances on the Farrell–Jones conjecture by Bestvina–Fujiwara–Wigglesworth and Brück–Kielak–Wu:

**Theorem B.** Let $E$ be a division ring and $G$ a group arising as an extension

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where $F$ is a free group. Then any crossed product $E \ast G$ is a pseudo-Sylvester domain and $D_{E \ast G} = \text{Ore}(D_{E \ast F} \ast \mathbb{Z})$ is its universal localization with respect to the set of all stably full matrices. In particular, it is the universal division $E \ast G$-ring of fractions. Moreover, $E \ast G$ is a Sylvester domain if and only if it has stably free cancellation.

Some examples of groups as in Theorem B with and without stably free cancellation are discussed in Section 3.2.

Note that Jaikin-Zapirain already showed in [Jai19b, Theorem 1.1] that $E \ast G$ has a universal division ring of fractions. With Theorem B we provide an independent proof of this fact as well as a description of the matrices that become invertible over $D_{E \ast G}$. Furthermore, in [LL18, Theorem 2.17], it has already been shown that $K_G$ where $K$ is a subfield of $\mathbb{C}$, admits a universal localization that is a division ring.

The universal division $E \ast G$-rings shown to exist above can actually be given explicit realizations. First, observe that the groups $G$ considered are locally indicable, so we can define a Conradian left order $\leq$ on $G$, and hence construct the space of Malcev-Neumann series $K((G, \leq))$, which is the $K$-vector space consisting of formal power series on $G$ with coefficients in $K$ and well-ordered support with respect to $\leq$. Second, for the particular case of a classical group ring $K_G$ with a subfield $K$ of $\mathbb{C}$, let $U(G)$ denote the algebra of unbounded affiliated operators. We prove the following:

**Theorem C.** Let $G$ be as in Theorem B and let $K$ be a field. Then $D_{K_G}$ can be realized as the Dubrovin division ring, i.e., the division closure $D_{\leq}(K_G)$ of $K_G$ inside $\text{End}(K((G, \leq)))$, where $\leq$ is a Conradian left order in $G$. If $K$ is a subfield of $\mathbb{C}$, then $D_{K_G}$ can also be realized as the Linnell division ring $D(G; K)$, namely, the division closure of $K_G$ inside $U(G)$.

The paper is organized as follows. In Section 1 we recall the major notions that are going to play a role in the proof of our main result. We recall in Section 1.2 and Section 1.4 the basics on localization, stably freeness and stably finiteness, three notions needed to introduce properly (pseudo-)Sylvester domains in Section 1.5. In Section 1.3 we introduce the main homological tools that we are going to work with.

Section 2 is devoted to prove Theorem A. We first state the criteria for Sylvester and pseudo-Sylvester domains in Section 2.1 and we explore the case we are interested in through Section 2.2.

In Section 3 we prove Theorem B as an application of Theorem A and the recent proof of the Farrell–Jones conjecture for the family of groups considered, and Theorem C through the notion of Hughes-freeness. In Section 3.2 we list some examples for which our results apply.

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1. Definitions and background

All rings are assumed to be associative with unit, but not necessarily commutative. If not otherwise specified, modules are taken to be left modules. If $R$ is a ring, then the group of units of $R$ is denote by $R^\times$.

1.1. Crossed products. Let $R$ be a ring and $G$ a group. A crossed product $R \ast G$ is a ring that as a left $R$-module is free on a copy of $G$ usually denoted by $\tilde{G} = \{ \tilde{g} \mid g \in G \}$ and such that the ring multiplication is determined by the following two properties:

- There is a map of sets $\alpha: G \times G \to R^\times$, called the twisting, such that $\tilde{g} \cdot \tilde{h} = \alpha(g, h) \cdot gh$ for every $g, h \in G$.
- There is a map of sets $\sigma: G \to \text{Aut}(R)$, called the action, such that $\tilde{g} \cdot r = \sigma(g)(r) \cdot \tilde{g}$ for every $r \in R$ and $g \in G$.

We will additionally assume that $\sigma(e) = \text{id}_R$ and $\alpha(g, e) = \alpha(e, g) = 1$, for every $g \in G$ and $e \in G$ the neutral element, which makes $\tilde{e}$ the unit of the crossed product. The map $r \mapsto r \cdot \tilde{e}$ is then an embedding of $R$ into $R \ast G$. For any given crossed product together with a choice of basis and structure maps, this can always be arranged by a diagonal change of basis and modifications to the twisting and action, but without changing the ring.

We are mainly interested in crossed products of the form $R \ast \mathbb{Z}$. Moreover, in order to simplify our proofs, we often consider skew Laurent polynomial rings $R[t^{\pm 1}; \tau]$, where $\tau$ is an automorphism of the ring $R$ and $tx = \tau(x)t$ for all $x \in R$. These rings are particular instances of crossed products of the previous form and, in fact, every crossed product $R \ast \mathbb{Z}$ is isomorphic to such a skew Laurent polynomial ring for some choice of $\tau$ (cf. [San08, Remark 4.6] and [Haz16, 1.1.4]).

For a detailed treatment of crossed products and their properties, we refer the reader to [Pas89].

1.2. Ore and universal localizations. Recall that whenever we have a commutative domain $R$, we can consider its field of fractions, a field in which every element can be expressed as a fraction of the form $rs^{-1}$ for some $r, s \in R$ where $s$ is non-zero. When $R$ is a non-commutative domain, a division ring with such a description may not exist in general, since we have no way in principle to ensure that sums and products of elements of the form $rs^{-1}$ admit a similar expression. The condition to ensure the feasibility of this procedure is the so-called Ore condition.

Assume that we are given a ring $R$, non necessarily a domain for the moment, and let $T$ be a multiplicative set of non-zero-divisors in $R$. We say that $T$ is right Ore if, for every $r \in R, t \in T$, there exist $s \in R, u \in T$ such that $ru = ts$. Under this condition, one can construct the right Ore localization $\text{Ore}_{r, T}(R) = RT^{-1}$, a ring whose elements can be written as $rt^{-1}$ for $r \in R, t \in T$ (cf. [GW04, Theorem 6.2]). Moreover, the ring $\text{Ore}_{r, T}(R)$ is flat as a left $R$-module (cf. [GW04, Corollary 10.13]), i.e., the tensor $\square \otimes_R \text{Ore}_{r, T}(R)$ preserve short exact sequences of right $R$-modules.
Similarly we can define left Ore sets, and if

T

is both left and right Ore, then

Ore\textsubscript{l,T}(R) = Ore\textsubscript{r,T}(R) (cf. [GW04, Proposition 6.5]).

Thus, observe that if R is a right Ore domain, i.e., if it is a domain and the set

T = R\backslash\{0\}

of all non-zero elements of R is right Ore, the right Ore localization

Ore\textsubscript{r}(R) := Ore\textsubscript{r,T}(R)

is a division ring whose elements are fractions of the previous form (cf. [GW04, Theorem 6.8]). If R is a right and left Ore domain, we just say that it is an Ore domain, and denote its Ore localization by Ore(R). For instance, this is the case of a skew Laurent polynomial ring of the form R[t\pm 1; τ] where R is both a right and a left Noetherian domain and τ is an automorphism of R (cf. [GW04, Corollary 1.15 & Corollary 6.7]).

Going a step further, one can consider the general question of whether a given non-commutative domain R can be embedded at all into a division ring. In this full generality, it can be treated by means of P. M. Cohn’s theory of epic division R-rings (cf. [Coh06, Chapter 7]), which relies on the existence of prime matrix ideals (for the definition of this notion, we refer the reader to [Coh06, Chapter 7, Section 3]), and universal localizations.

Definition 1.1. Given a set Σ of (square) matrices over R, and a homomorphism of rings \( \varphi : R \to S \), we say that the map \( \varphi \) is Σ-inverting if every element of Σ becomes invertible over S. We say that \( \varphi \) is universal Σ-inverting if any other Σ-inverting homomorphism factors uniquely through \( \varphi \). In this latter case, we denote \( S = R_\Sigma \) and we call \( R_\Sigma \) the universal localization of R with respect to Σ.

If we allow \( R_\Sigma \) to be the zero ring, the existence of the universal localization can always be proved by taking a presentation of R as a ring and formally adding the necessary generators and relations. Moreover, the universal Σ-inverting homomorphism will be injective if and only if there exists a Σ-inverting embedding to some ring (Coh06, Theorem 7.2.4).

To briefly explain P.M. Cohn’s main result on the topic, we need to introduce the notion of epic division R-ring.

Definition 1.2. Given a ring R, an epic division R-ring is a division ring D together with a ring homomorphism \( R \to D \) such that D is generated, as a division ring, by the image of R.

The “epic” terminology is justified through the fact that D being generated by the image of R is equivalent to the homomorphism being an epimorphism in the category of rings ([Coh06, Corollary 7.2.2]). With this, P.M. Cohn proved that epic division R-rings are completely characterized (up to R-isomorphism) by the set Σ of matrices over R that become invertible in the division ring, and that they always arise as residue fields of a universal localization \( R_\Sigma \) ([Coh06, Theorem 7.2.5 & Theorem 7.2.7]). In addition, the sets \( \Sigma \) for which the localization \( R_\Sigma \) is a non-zero local ring are precisely the complements, in the set of square matrices over R, of prime matrix ideals \( P \) ([Coh06, Theorem 7.4.3]). Thus, we would have an embedding of R into a division ring if we can construct such a set Σ including all non-zero elements in R.

Finally, if among all the possible epic division R-rings in which we can embed R, there exists one in which we can invert “the most” (relative to R) matrices possible, we call it the universal division R-ring of fractions. More precisely:

Definition 1.3. The epic division R-ring D is called the universal division R-ring of fractions if R embeds into D and, for any other epic division R-ring \( D' \), the set \( \Sigma' \) of matrices that become invertible over \( D' \) is contained in the set \( \Sigma \) of matrices that become invertible over D.
In Section 1.5 we will introduce two families of rings, namely Sylvester and pseudo-Sylvester domains, for which there exists a universal division ring of fractions and for which the set $\Sigma$ of matrices becoming invertible under the embedding can be characterized in a natural way only depending on $R$. Our main result will be to build on a homological criterion for a ring to belong to one of these families, which is why we need to introduce parts of the dimension theory of (non-commutative) rings in the following.

1.3. Weak and global dimensions. Recall that a module $N$ over a ring $R$ has projective dimension at most $n$ (abbreviated $\text{pd}(N) \leq n$) if $N$ admits a resolution

$$0 \to P_n \to \ldots \to P_0 \to N \to 0$$

of projective $R$-modules. In particular, observe that $M$ is projective if and only if $\text{pd}(N) = 0$. The supremum among the projective dimensions of all left (resp. right) $R$-modules is called the left (resp. right) global dimension of $R$, and it is not left-right symmetric in general. This concept is deeply related to Ext functors.

Lemma 1.4 ([Rot09, Proposition 8.6]). The following are equivalent for a left $R$-module $N$:

1. The projective dimension of $N$ is at most $n$.
2. $\text{Ext}^i_R(N, N') = 0$ for all left $R$-modules $N'$ and $i > n$.
3. $\text{Ext}^{n+1}_R(N, N') = 0$ for all left $R$-modules $N'$.
4. If $0 \to I \to P_{n-1} \to \ldots P_0 \to N \to 0$ is an exact sequence where every $P_i$ is projective, then $I$ is projective.

Analogously, we say that the flat dimension of $N$ is at most $n$, and we write $\text{fd}(N) \leq n$, if it admits a resolution of flat $R$-modules

$$0 \to Q_n \to \ldots \to Q_0 \to N \to 0,$$

and define the left (resp. right) weak dimension of $R$ as the supremum of the flat dimensions of all left (resp. right) $R$-modules. It turns out that this notion is always left-right symmetric ([Rot09, Theorem 8.19]) and hence we can just talk about the weak dimension of $R$. As it happens with $\text{pd}(N)$ and $\text{Ext}^*_R(N, \Box)$, the flat dimension of $N$ (resp. of a right $R$-module $M$) can be characterized in terms of $\text{Tor}^R_*(\Box, N)$ (resp. $\text{Tor}^R_*(M, \Box)$). Observe though that, unlike the previous case, here we need to change the argument while considering left or right modules.

Lemma 1.5 ([Rot09, Proposition 8.17]). The following are equivalent for a left $R$-module $N$:

1. The flat dimension of $N$ is at most $n$.
2. $\text{Tor}^i_R(M, N) = 0$ for all right $R$-modules $M$ and $i > n$.
3. $\text{Tor}^{n+1}_R(M, N) = 0$ for all right $R$-modules $M$.
4. If $0 \to J \to Q_{n-1} \to \ldots Q_0 \to N \to 0$ is an exact sequence where every $Q_i$ is flat, then $J$ is flat.

We finish this section with the following result regarding Tor, sometimes referred to as Shapiro’s Lemma.

Lemma 1.6. Let $R$ be a subring of $S$ such that $S$ is flat as a left $R$-module. Then, for any right $R$-module $M$, for any left $S$-module $N$ and for any $n \geq 0$, we have

$$\text{Tor}^R_n(M, R N) \cong \text{Tor}^S_n(M \otimes_R S, N)$$

where $rN$ denotes $N$ considered as a left $R$-module.
Proof. Assume that we have a projective resolution for $M$
\[ \ldots \to P_k \to \ldots \to P_0 \to M \to 0. \]
Since $S$ is a flat left $R$-module, the following sequence is also exact of projective right $S$-modules, i.e., a projective resolution for $M \otimes_R S$
\[ \ldots \to P_k \otimes_R S \to \ldots \to P_0 \otimes_R S \to M \otimes_R S \to 0. \]
Now, just observe that computing $\text{Tor}_*^R(M, R_N)$ amounts to computing the homology of the chain
\[ \ldots \to P_k \otimes_R N \to \ldots \to P_0 \otimes_R N \to 0 \]
and that computing $\text{Tor}_*^R(M \otimes_R S, N)$ amounts to computing the homology of
\[ \ldots \to P_k \otimes_R S \otimes_S N \to \ldots \to P_0 \otimes_R S \otimes_S N \to 0 \]
Since $S \otimes_R N \cong N$, the result follows. \qed

1.4. Stably freeness and stably finite rings. The criteria we are going to introduce in Section 2 rely on proving that certain submodules are finitely generated free or stably free, respectively. Therefore, we need to deal with the latter concept and its relation with the notion of stably finite rings.

Definition 1.7. A module $M$ over a ring $R$ is called stably free if there exists $n \geq 0$ such that $M \oplus R^n$ is a free $R$-module.

By a result of Gabel, a proof of which is given in \cite[Lam78, Proposition 4.2]{Lam78}, any stably free module that is not finitely generated is already free. For this reason, we will restrict our attention to finitely generated stably free modules in the following.

If $M$ is a finitely generated stably free $R$-module and $M \oplus R^n$ is free, then this free module is necessarily finitely generated and hence isomorphic to some $R^m$. In general, the difference $m - n$ need neither be positive nor uniquely determined by $M$. It is here where the stably finite property enters the scene.

Recall that a ring $R$ is said to be stably finite (or weakly finite) if whenever $A$ and $B$ are two $n \times n$-matrices over $R$ such that $AB = I_n$, then also $BA = I_n$. For example, every division ring is stably finite, as can be deduced from the characterization in the next paragraph using a dimension argument. Also, if $K$ is a field of characteristic 0 and $G$ is any group, or if $K$ has positive characteristic and $G$ is socf, the group ring $KG$ is stably finite (cf. \cite[Jai19a, Corollary 13.7]{Jai19a}). Furthermore, any subring of a stably finite ring is clearly again stably finite.

Observe that being stably finite can be reformulated in terms of modules by saying that if $R^n \oplus K \cong R^n$, then $K = 0$. Indeed, for instance, consider the projection onto the first summand $R^n \cong R^n \oplus K \to R^n$, which will be given by right multiplication by some $n \times n$ matrix $B$. Since $R^n$ is free, this splits and there exists a homomorphism $R^n \to R^n$, defined by a matrix $A$, such that $AB = I_n$. By stably finiteness, $BA = I_n$, so the projection is actually an isomorphism and hence $K = 0$.

Conversely, if $AB = I_n$ then, in particular, the homomorphism given by right multiplication by $B$ is surjective. Thus, the sequence $0 \to \ker(B) \to R^n \xrightarrow{B} R^n \to 0$ is exact, and splits because $R^n$ is free. Now, the hypothesis tells us that $\ker(B) = 0$. Hence, $B$ is invertible and therefore $BA = BAB^{-1} = BB^{-1} = I_n$.

Thus, if $M$ is a module over a stably finite ring $R$ and $M \oplus R^n \cong R^n$, then the difference $m - n$ is positive and constant among all such representations. We call this positive number the stably free rank of $M$ and denote it by $\text{rk}_{sf}(M)$.

To finish this subsection, let $P$ be a finitely generated projective module over $R$. We will recall in the next subsection that if $R$ is a Sylvester domain then $P$ is necessarily free, while if $R$ is just a pseudo-Sylvester domain, we can only deduce
Remark 1.9. Let $R$ be a ring. If $M$ is a left (right) $R$-module, then $M^* := \text{Hom}_R(M, R)$, called the dual of $M$, is naturally a right (left) $R$-module. For every ring $R$, the functor $P \mapsto P^*$ defines an equivalence between the category of finitely generated projective left $R$-modules and the opposite of the category of finitely generated projective right $R$-modules, with the inverse functor given in the same way. To see that $P \cong P^{**}$, note that taking the dual commutes with direct sums and the claim thus needs to be checked only for $R$ itself viewed as an $R$-module, where it is clear. The equivalence defined in this way restricts to equivalences of the respective subcategories of finitely generated stably free and finitely generated free modules.

As a consequence, every property of rings that can be expressed in terms of these categories in a way that is invariant under passing to an equivalent or opposite category will hold for left modules if and only if it holds for right modules. In particular, whether or not any of the classes of finitely generated projective, stably full, stably free or free modules coincide for a ring does not depend on whether left or right modules are considered.

1.5. (Pseudo-)Sylvester domains. In this section we introduce the main families of rings we are going to deal with throughout the paper, namely, Sylvester domains and pseudo-Sylvester domains. We are going to define them in terms of inner and stable rank of matrices, what a priori may seem to be unrelated to the topics discussed in the previous subsections, but we will see how they interact.

Let $R$ be a ring, and $A$ an $m \times n$ matrix over $R$. Recall that the inner rank $\rho(A)$ is defined as the least $k$ such that $A$ admits a decomposition $A = B_{m \times k}C_{k \times n}$. We say that a square matrix $A$ of size $n \times n$ is full if $\rho(A) = n$. Recall also that the stable rank $\rho^*(A)$ is given by

$$\rho^*(A) = \lim_{s \to \infty} \left[ \rho(A \oplus I_s) - s \right],$$

whenever the limit exists, where $A \oplus I_s$ denotes the block diagonal matrix with blocks $A$ and $I_s$. We analogously say that a square matrix is stably full if it has maximum stable rank. When $R$ is stably finite, $\rho^*(A)$ is well-defined and non-negative, and it is positive if $A$ is a non-zero matrix ([Coh06 Proposition 0.1.3]). For this reason, in the following we restrict our attention to stably finite rings.

Observe that from the definition of the inner rank it follows that the sequence in the limit is always non-increasing and bounded above by $\rho(A)$. In particular, for an $n \times n$ matrix $A$ we obtain that $\rho^*(A) \leq \rho(A) \leq n$ and that $\rho^*(A) = n$ if and only if the sequence is constantly $n$. Thus, $A$ is stably full if and only if $\rho(A \oplus I_s) = n + s$ for every $s \geq 0$.

We summarize useful properties of the stable rank over stably finite rings.

Lemma 1.10. Let $R$ be a stably finite ring. Then, for any matrix $A$ over $R$,

1. For every $k \geq 0$, $\rho^*(A \oplus I_k) = \rho^*(A) + k$.
2. There exists $N \geq 0$ such that for every $l \geq N$, $\rho^*(A \oplus I_l) = \rho(A \oplus I_l)$.
3. $0 \leq \rho^*(A) \leq \rho(A)$. 

Proof. Since $R$ is stably finite, we know that $\rho^*(A) = r \geq 0$. This means that there exists $N \geq 0$ such that for any $l \geq N$ we have $\rho(A \oplus I_l) - l = r$. Thus, for $k \geq 0$,
\[
\rho^*(A \oplus I_k) = \lim_{s \to \infty} (\rho(A \oplus I_k \oplus I_s) - (s + k) + k) = r + k = \rho^*(A) + k.
\]
From here, we also deduce that for $l \geq N$ one has
\[
\rho(A \oplus I_l) = l + r = l + \rho^*(A) = \rho^*(A \oplus I_l).
\]
The last statement has already been observed above.

We can now introduce the main notions of the subsection. Let us define first the notion of Sylvester domain, together with the main examples and properties.

**Definition 1.11.** A non-zero ring $R$ is a Sylvester domain if $R$ is stably finite and satisfies the law of nullity with respect to the inner rank, i.e., if $A \in \text{Mat}_{m \times n}(R)$ and $B \in \text{Mat}_{n \times k}(R)$ are such that $AB = 0$, then
\[
\rho(A) + \rho(B) \leq n.
\]

In fact, it can be shown that the condition that $R$ is stably finite is redundant here, but we keep it as a requirement to show the symmetry with the upcoming definition of pseudo-Sylvester domain. The following rings serve as the most prominent examples of Sylvester domains ([Coh06, Proposition 5.5.1]):

**Definition 1.12.** A free ideal ring (fir) is a ring in which every left and every right ideal is free of unique rank (as a module).

As a consequence, in a fir every submodule of a free module is again free (cf. [Coh06, Corollary 2.1.2] and note that every submodule of a free $R$-module of rank $\kappa$ is $\max([R],\kappa)$-generated). For instance, a division ring $D$ is a fir, and the inner rank over $D$ is just its usual rank, which will be denoted by $\text{rk}_D$. An important example is the group ring $KF$, where $K$ is a field and $F$ is a free group, is a fir. This result was originally proved by P.M. Cohn, and we refer the reader to [Lew69, Theorem 1] for a concise treatment. More generally, for any division ring $E$ and free group $F$, the crossed product $E \ast F$ is a fir. This is a consequence of Bergman’s coproduct theorem (cf. [San08, Theorem 4.22 (i)]).

The following property of a ring is intimately related to Sylvester domains.

**Definition 1.13.** A ring $R$ is called projective-free if every finitely generated projective $R$-module is free of unique rank.

By Remark 1.9, this notion is left-right symmetric.

Note, for instance, that if $K$ is a field, then the polynomial ring $K[t_1, \ldots, t_n]$ in $n$ indeterminates is projective-free, a result known as the Quillen-Suslin theorem.

Every Sylvester domain is projective-free and has weak dimension at most 2 (cf. [DS75, Theorem 6] and the subsequent discussion). The question whether projective-freeness is equivalent to being a Sylvester domain for a ring of weak dimension 2 is a very delicate one. We will see in the next section that this is actually the case for crossed products $E \ast G$ in which $E$ is a division ring and $G$ has a normal free subgroup $F$ with infinite cyclic quotient $G/F$.

In the same way that Sylvester domains are defined in terms of inner rank, pseudo-Sylvester domains are defined in terms of stable rank.

**Definition 1.14.** A non-zero ring $R$ is a pseudo-Sylvester domain if $R$ is stably finite and satisfies the law of nullity with respect to the stable rank, i.e., if $A \in \text{Mat}_{m \times n}(R)$ and $B \in \text{Mat}_{n \times k}(R)$ are such that $AB = 0$, then
\[
\rho^*(A) + \rho^*(B) \leq n.
\]
In analogy to the case of Sylvester domains, any finitely generated projective module over a pseudo-Sylvester domain is stably free \cite[Proposition 5.6.2]{Coh06}. Thus, for a pseudo-Sylvester domain to be a Sylvester domain it is necessary that the ring enjoys the stably free cancellation property.

Several characterizations of Sylvester and pseudo-Sylvester domains can be found in \cite[Theorem 7.5.13]{Coh06} and \cite[Theorem 7.5.18]{Coh06}, respectively. In particular, they can be defined in terms of universal localizations and universal division rings of fractions. In this flavour, observe that for an \( n \times n \) matrix \( A \) to become invertible over a division ring \( D \), we need \( A \) to be stably full, since otherwise there would exists \( s \geq 0 \) such that \( \rho(A \oplus I_s) < n + s \) and hence \( A \oplus I_s \) would not be invertible over \( D \). Thus, one can wonder whether there exists a division ring in which every stably full matrix can be inverted. The family of rings for which this is possible is precisely the family of pseudo-Sylvester domains.

For a Sylvester domain, the inner rank is additive, in the sense that \( \rho(A \oplus B) = \rho(A) + \rho(B) \) holds for any matrices \( A \) and \( B \) (cf. \cite[Lemma 5.5.3]{Coh06}), and thus the inner and stable rank coincide. Indeed, if \( \rho'(A) = r \), then by Lemma \ref{additivity} (2) there exists \( s \geq 0 \) such that \( \rho(A \oplus I_s) = \rho'(A \oplus I_s) \), from where Lemma \ref{additivity} (1) and additivity tell us that \( \rho'(A) = \rho(A) \). As a consequence, every full matrix is actually stably full, and hence Sylvester domains will form the family of rings embeddable into a division ring in which we can invert all full matrices.

We record this in the following proposition and, although its content is implicit in the proofs of \cite[Theorem 7.5.13]{Coh06} and \cite[Theorem 7.5.18]{Coh06}, the absence of the “rank preserving” property may make them look weaker, so we add a few lines to clarify this point. Throughout the proof, given a division \( R \)-ring \( R \to D \) and a matrix \( A \) over \( R \), \( \rho_D(A) \) will denote the usual rank over \( D \) of the image of \( A \) (i.e. its inner rank as a matrix over \( D \)). Note that, from the definition of the inner rank and the properties of \( \rho_D \), we clearly have \( \rho_D(A) \leq \rho(A) \), and hence, from Lemma \ref{additivity} (1) and (2), we also deduce \( \rho_D(A) \leq \rho'(A) \).

**Proposition 1.15.** For a non-zero ring \( R \), the following are equivalent:

1. \( R \) is a Sylvester (resp. pseudo-Sylvester) domain.
2. There exists a division ring \( D \) and an epic embedding \( R \to D \) such that every full (resp. stably full) matrix over \( R \) becomes invertible over \( D \).

Moreover, if \( R \) satisfies one, and hence each of the previous properties, \( D \) is the universal \( R \)-division ring of fractions, and it is isomorphic to the universal localization of \( R \) with respect to the set of all full (resp. stably full) matrices over \( R \).

**Proof.** Assume (1). Since for a Sylvester domain we have \( \rho = \rho' \), as discussed earlier, it suffices to work with \( \rho' \). Now, \cite[Theorem 7.5.13 (f)]{Coh06} (resp. \cite[Theorem 7.5.18 (d)]{Coh06}) tells us that there exists a stable rank preserving homomorphism \( R \to D \), where \( D \) is a division ring. Thus, any \( n \times n \) stably full matrix over \( R \) has maximum \( D \)-rank, and hence becomes invertible in \( D \). Since \( R \) is stably finite, we have \( \rho'(x) = 1 \) for every non-zero \( x \in R \), and thus the homomorphism is injective.

Assume (2), and note that \( R \), as a subring of a division ring, is stably finite. Notice also that the product of two \( n \times n \) (stably) full matrices \( A, B \) over \( R \) is again (stably) full, since using (2) we would deduce that \( n = \rho_D(AB) \leq \rho'(AB) \). Similarly, the block diagonal matrix \( A \oplus B \) is (stably) full.

Thus, let \( A \) be any \( m \times n \) matrix over \( R \) with \( \rho_d(A) = r \). Invoking \cite[Theorem 5.4.9]{Coh06} (resp. \cite[Proposition 5.6.4]{Coh06}), there exists an \( r \times r \) (stably) full submatrix \( A' \) of \( A \), which is then invertible over \( D \). Hence, \( \rho_D(A) \geq \rho_D(A') = r = \rho'(A) \), and therefore \( \rho_D(A) = \rho'(A) \). We have just proved that the embedding is rank preserving, and the law of nullity follows from the properties of \( \rho_D \).
For the last statement, it follows from [Coh06 Theorem 7.5.13 (e)] (resp. [Coh06 Theorem 7.5.18 (c)]) that \( R\), the universal localization of \( R \) with respect to the stably full matrices, is a division ring, and by its universal property we have an \( R \)-homomorphism (in particular non-trivial) \( R\to D \). Since \( R\) is a division ring, the homomorphism is injective, and since \( R\to D \) is epic, it is also surjective. Since being stably full is a necessary prerequisite for a matrix to become invertible, \( D \) is clearly the universal \( R \)-division ring of fractions.

□

2. Towards Theorem A

This section is devoted to prove Theorem A by verifying the conditions of Theorems 2.3 and 2.5, both of which will be stated in Section 2.1. The former is a particular case of a homological criterion introduced by Jaikin-Zapirain in [Jai96] to determine when a ring with a prescribed embedding into a division ring is a Sylvester domain. The latter is the analogous recognition principle adapted to pseudo-Sylvester domains.

Throughout this section, \( \mathcal{S} \) will always denote a fir with universal division \( \mathcal{S} \)-ring of fractions \( \mathcal{D}\), and we will consider any crossed product ring \( S = \mathcal{S} \ast \mathbb{Z} \).

The following lemma tells us in particular that the crossed product structure \( \mathcal{S} = \mathcal{S} \ast \mathbb{Z} \) can always be extended to a crossed product structure \( \mathcal{D}\ast \mathbb{Z} \), and that this ring is an Ore domain.

**Lemma 2.1.** Let \( R \) be a (pseudo-)Sylvester domain with universal division \( R \)-ring of fractions \( \mathcal{D}\). Then any crossed product structure \( R \ast \mathbb{Z} \) extends to a crossed product \( \mathcal{D}\ast \mathbb{Z} \). Moreover, \( \mathcal{D}\ast \mathbb{Z} \) is an Ore domain and Ore(\( \mathcal{D}\ast \mathbb{Z} \)) is an epic division \( R \ast \mathbb{Z} \)-ring.

**Proof.** First, we are going to see that every automorphism \( \varphi \) of \( R \) extends uniquely to an automorphism of \( \mathcal{D}\). Indeed, let \( \Sigma \) denote the set of (stably) full matrices over \( R \) and notice that \( \varphi \) preserves \( \Sigma \) (i.e., \( \varphi(\Sigma) = \Sigma \)). Thus, the composition \( R \xrightarrow{\varphi} R \leftarrow \mathcal{D}\) is a \( \Sigma \)-inverting embedding, and hence the universal property of universal localization gives us a unique injective map \( \varphi : R\Sigma = \mathcal{D}\to \mathcal{D}\) such that the diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\varphi} & \mathcal{D}\ \\
\downarrow & & \downarrow \\
R & \xrightarrow{\varphi} & \mathcal{D}\ 
\end{array}
\]

commutes. Since \( \mathcal{D}\) is generated by \( R \) as a division ring, \( \varphi \) is also surjective, and hence an automorphism of \( \mathcal{D}\).

Therefore, given a crossed product structure \( R \ast \mathbb{Z} \) with action \( \sigma : \mathbb{Z} \to \text{Aut}(R) \) and twisting \( \alpha : \mathbb{Z} \times \mathbb{Z} \to R^\times \), we can extend \( \sigma \) to a map \( \sigma : \mathbb{Z} \to \text{Aut}(\mathcal{D}) \) by the previous reasoning, and consider the composition \( \alpha : \mathbb{Z} \times \mathbb{Z} \to R^\times \leftarrow \mathcal{D}\). In fact, given abstract action and twisting maps, it is possible to construct a crossed product, which will be associative if and only if certain compatibility conditions on the two maps are satisfied (cf. [Pas89 Lemma 1.1]). In our particular situation, these conditions on the two maps \( \alpha \) and \( \sigma \) are easily seen to be preserved when passing from \( R \to \mathcal{D}\) since automorphisms of \( R \) extend uniquely to \( \mathcal{D}\). As a result, we obtain a crossed product ring \( \mathcal{D}\ast \mathbb{Z} \) such that \( R \ast \mathbb{Z} \to \mathcal{D}\ast \mathbb{Z} \) via the obvious map.

Now, as mentioned at the beginning of the section, we can find an automorphism \( \tau \) of \( \mathcal{D}\) such that \( \mathcal{D}\ast \mathbb{Z} \cong \mathcal{D}\ast [t^{\pm 1} ; \tau] \). Since \( \mathcal{D}\) is a division ring, \( \mathcal{D}\ast [t^{\pm 1} ; \tau] \) is a left and right Ore domain, as mentioned in Section 1.2. Hence, \( \mathcal{D}\ast \mathbb{Z} \) is an Ore domain and we can consider Ore(\( \mathcal{D}\ast \mathbb{Z} \)), a division ring.
Finally, we have a commutative diagram

\[
\begin{array}{ccc}
R & \longrightarrow & \mathcal{D}_R \\
\downarrow & & \downarrow \\
R \ast \mathbb{Z} & \longrightarrow & \mathcal{D}_R \ast \mathbb{Z}.
\end{array}
\]

Let \( S \) be any ring and \( f, g : \mathcal{D}_R \ast \mathbb{Z} \to S \) ring homomorphisms that agree on \( R \ast \mathbb{Z} \). They induce ring homomorphisms \( \mathcal{D}_R \to S \) that coincide on \( R \), and hence, since the embedding \( R \hookrightarrow \mathcal{D}_R \) is epic, \( f \) and \( g \) agree on \( \mathcal{D}_R \). Since by assumption they also coincide on the basis of \( R \ast \mathbb{Z} \), we deduce that \( f = g \). Therefore, \( R \ast \mathbb{Z} \hookrightarrow \mathcal{D}_R \ast \mathbb{Z} \) is epic. On the other hand, the embedding \( \mathcal{D} \ast \mathbb{Z} \hookrightarrow \text{Ore}(\mathcal{D} \ast \mathbb{Z}) \) is also epic, and hence the composition \( R \ast \mathbb{Z} \hookrightarrow \text{Ore}(\mathcal{D} \ast \mathbb{Z}) \) is epic. \( \square \)

We are interested in the homological properties of \( \mathcal{D} \ast \mathbb{Z} = \text{Ore}(\mathcal{D} \ast \mathbb{Z}) \), to which we will dedicate Section 2.2. For this reason, we collect in the following two lemmas basic structural results on crossed products and Ore domains which will prove useful later.

We first explore the \( S \)-module structure of the crossed product \( \mathcal{D} \ast \mathbb{Z} \) constructed in Lemma 2.1.

**Lemma 2.2.** Let \( R \) be a ring and fix any crossed product \( R \ast \mathbb{Z} \). If \( T \) is an overring of \( R \) such that the crossed product structure on \( R \ast \mathbb{Z} \) extends to a crossed product \( T \ast \mathbb{Z} \), then the left \( R \ast \mathbb{Z} \)-modules \( T \ast \mathbb{Z} \) and \((R \ast \mathbb{Z}) \otimes_R T \) are isomorphic. Similarly, the right \( R \ast \mathbb{Z} \)-modules \( T \ast \mathbb{Z} \) and \( T \otimes_R (R \ast \mathbb{Z}) \) are isomorphic.

**Proof.** As at the beginning of the section, every crossed product with \( \mathbb{Z} \) is isomorphic to a skew Laurent polynomial ring for some choice of an automorphism. Since the crossed product structure of \( T \ast \mathbb{Z} \) extends that of \( R \ast \mathbb{Z} \), we can actually choose an automorphism \( \tau \) of \( T \) restricting to an automorphism of \( R \), also denoted \( \tau \), such that \( R \ast \mathbb{Z} \cong (R \ast \mathbb{Z}) \otimes_R T \) and \( T \ast \mathbb{Z} \cong T \otimes_R (R \ast \mathbb{Z}) \) as rings, respectively.

The left \( R[t^{\pm 1}; \tau] \otimes_R T \to T[t^{\pm 1}; \tau] \)

\[
t^n \otimes \lambda \mapsto \tau^n(\lambda)t^n
\]

is an isomorphism since it is also right \( T \)-linear and maps the basis \( \{t^n \otimes 1 \mid n \in \mathbb{Z}\} \) to the basis \( \{t^n \mid n \in \mathbb{Z}\} \). This proves the first statement; the second statement is proved analogously. \( \square \)

The second lemma, applied to the case \( R := \mathcal{D} \ast \mathbb{Z}, \mathcal{O} := \mathcal{D} \), and \( S := \mathcal{S} \), will allow us later to restrict our attention to \( S \)-submodules of \( \mathcal{D} \ast \mathbb{Z} \).

**Lemma 2.3.** Let \( R \) be a right Ore domain with Ore localization \( \mathcal{O} \) and \( S \) a subring of \( R \). Then every finitely generated \( S \)-submodule \( M \) of the left \( S \)-module \( \mathcal{O} \) is isomorphic to a finitely generated \( S \)-submodule of \( R \).

**Proof.** Let \( M \) be generated as a left \( S \)-module by \( x_1, \ldots, x_m \in \mathcal{O} \). We find \( p_i, q_i \in R \) such that \( x_i = p_iq_i^{-1} \) for \( i = 1, \ldots, m \). If \( m \geq 2 \) we can use the Ore condition to find non-zero \( a, b \in R \) such that \( q_1a = q_2b \), and hence \( x_1 = (p_1a)(q_1a)^{-1} \) and \( x_2 = (p_2b)(q_2b)^{-1} \) can be expressed as fractions with common denominators. By repeatedly applying this procedure we produce \( p_i', q_i \in R, q \neq 0 \) such that \( x_i = p_i'q_i^{-1} \) for all \( i \).

We now consider the left \( S \)-submodule \( M' \) of \( R \) generated by \( x_1q, \ldots, x_mq \). The map \( f : M \to M' \) given by \( y \mapsto yq \) is \( S \)-linear since \( \mathcal{O} \) is associative and surjective since its image contains the generators. Finally, it is injective, since \( \mathcal{O} \) is a division ring and hence \( zq \neq 0 \) for every \( z \neq 0 \). We conclude that \( f \) is an \( S \)-linear isomorphism. \( \square \)
2.1. Homological recognition principles for (pseudo-)Sylvester domains.

As mentioned above, we are going to use the next two theorems to prove Theorem 2.5 provided by Jaikin-Zapirain for a ring to be a Sylvester domain.

**Theorem 2.4.** Let \( R \rightarrow D \) be an epic division \( R \)-ring. Assume that

1. \( \text{Tor}^1_R(D, D) = 0 \)
2. for any finitely generated left or right \( R \)-submodule \( M \) of \( D \) and any exact sequence \( 0 \rightarrow J \rightarrow R^n \rightarrow M \rightarrow 0 \), the \( R \)-module \( J \) is free of finite rank.

Then \( R \) is a Sylvester domain and \( D \) is the universal localization of \( R \) with respect to all full matrices.

The second theorem is an analogue for pseudo-Sylvester domains, involving stably free modules instead of free modules. The proof proceeds similarly, but we include it here for the sake of completeness. Given an embedding \( R \hookrightarrow D \) of \( R \) into a division ring and a matrix \( A \) over \( R \), we will denote by \( \text{rk}_D(A) \) the usual \( D \)-rank of \( A \) considered as a matrix over \( D \). Similarly, if \( M \) is a left \( R \)-module, we take \( \dim_{D}(M) \) to denote the \( D \)-dimension \( \dim_{D}(D \otimes_{R} M) \) of the left \( D \)-module \( D \otimes_{R} M \).

**Theorem 2.5.** Let \( R \rightarrow D \) be an epic division \( R \)-ring. Assume that

1. \( \text{Tor}^1_R(D, D) = 0 \)
2. for any finitely generated left or right \( R \)-submodule \( M \) of \( D \) and any exact sequence \( 0 \rightarrow J \rightarrow R^n \rightarrow M \rightarrow 0 \), the \( R \)-module \( J \) is finitely generated stably free.

Then \( R \) is a pseudo-Sylvester domain and \( D \) is the universal localization of \( R \) with respect to all stably full matrices.

**Proof.** Notice that by Proposition 1.15 it suffices to show that every stably full matrix over \( R \) becomes invertible over \( D \). Thus, let \( A \) be an \( n \times n \) matrix over \( R \) with \( \rho^*(A) = n \), and assume that \( A \) is not invertible over \( D \), i.e., \( \text{rk}_D(A) < n \). Since \( R \) is a subring of a division ring, it is necessarily stably finite.

Let \( N \) be the left \( R \)-module \( N = R^n / R^sA \). Then \( A \) is also the presentation matrix of \( D \otimes_{R} N \), and therefore \( \dim_{D}(N) = n - \text{rk}_{D}(A) \), which is finite and positive. This implies that \( D \otimes_{R} N \cong D^k \) as \( D \)-modules for some \( k \geq 1 \) and, thus, composing the \( R \)-homomorphism \( N \rightarrow D \otimes_{R} N \) given by \( x \rightarrow 1 \otimes x \) with an appropriate projection, we obtain a non-trivial \( R \)-homomorphism \( N \rightarrow D \). Therefore, if \( M \) is the image of this map, the surjection \( N \rightarrow M \) gives us a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & R^nA & \rightarrow & R^n & \rightarrow & N & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & J & \rightarrow & R^n & \rightarrow & M & \rightarrow & 0.
\end{array}
\]

Here, \( J \) is the kernel of the map \( R^n \rightarrow M \) and the dotted arrow is such that the left square commutes (cf. [Rot09], Proposition 2.71) and therefore injective. Moreover, notice that \( D \otimes_{R} M \) is non-trivial since the multiplication map to \( D \) is non-trivial. We conclude that \( \dim_{D}(M) > 0 \).

Now we have by (2) that \( J \) is stably free, i.e., there exists \( l \geq 0 \) such that \( J \oplus R^s \) is free for all \( s \geq l \). Moreover, since \( J \) is finitely generated and \( R \), as a subring of a division ring, is stably finite, we conclude that \( J \oplus R^s \cong R^{k_{s,l}}(J)^{s} \) for every \( s \geq l \). In fact, we obtain that \( \text{rk}_{D}(J) = \dim_{D}(J) \) by applying \( D \otimes_{R} \). Notice also that the previous diagram remains exact and commutative if we add \( 0 \rightarrow R^s \rightarrow R^s \rightarrow 0 \rightarrow 0 \) to both rows. Thus, setting \( t = \dim_{D}(J) \) and taking any \( s \geq l \), the situation can
be summarized in the following commutative diagram:

\[
\begin{array}{ccc}
R^{n+s} & \xrightarrow{r_{A\oplus I_s}} & R^n A \oplus R^s \\
\downarrow & & \downarrow \\
R^t A \oplus R^s & \xrightarrow{r_{A\oplus I_s}} & R^{n+s} \\
\end{array}
\]

Here, \(r_{A\oplus I_s}\) denotes the homomorphism given by right multiplication by \(A \oplus I_s\), so that all maps except the isomorphism behave identically on the \(R^s\) summand.

In terms of matrices, this factorization of \(r_{A\oplus I_s}\) allows us to express \(A \oplus I_s\) as a product of two matrices of dimensions \((n+s) \times (t+s)\) and \((t+s) \times (n+s)\), respectively. Thus, \(\rho(A \oplus I_s) \leq t + s\) right by definition. Since \(A\) is stably full, we have \(\rho(A \oplus I_s) = n + s\) for every \(s\), so we conclude that \(n \leq t\).

We are going to show on the other hand that \(t < n\), a contradiction. Observe first that the condition (2) tells us in particular that the flat (in fact, projective) dimension of any finitely generated right \(R\)-submodule of \(D\) is at most 1. Hence, using Lemma [1.5] and the fact that \(\text{Tor}\) commutes with directed colimits (cf. [Rot09, Proposition 7.8]), we obtain that for any left \(R\)-module \(Q\),

\[
\text{Tor}_2^R(D, Q) = \text{Tor}_2^R \left( \lim_{\to} L, Q \right) \cong \lim_{\to} \text{Tor}_2^R(L, Q) = 0,
\]

where we express the right \(R\)-module \(D\) as the direct union of its finitely generated right \(R\)-submodules. Again by Lemma [1.5] this means that \(D\) itself has flat dimension at most 1 as a right \(R\)-module.

Now, since \(M\) is an \(R\)-submodule of \(D\), we have an exact sequence of left \(R\)-modules

\[
0 \to M \to D \to Q \to 0
\]

for some left \(R\)-module \(Q\), and hence, applying \(D \otimes_R \square\) we can construct a long exact sequence containing the following exact part:

\[
\cdots \to \text{Tor}_2^R(D, Q) \to \text{Tor}_1^R(D, M) \to \text{Tor}_1^R(D, D) \to \cdots
\]

The first term is trivial by the previous argument, while the third term is trivial because of (1). Thus, we deduce that \(\text{Tor}_1^R(D, M) = 0\). From here, it follows that applying \(D \otimes_R \square\) to the exact sequence \(0 \to J \to R^n \to M \to 0\) returns an exact sequence of left \(D\)-modules

\[
0 \to D \otimes_R J \to D^n \to D \otimes_R M \to 0,
\]

from which we obtain

\[
t = \dim_D(J) = n - \dim_D(M) < n
\]

This is the desired contradiction, which shows that necessarily \(\text{rk}_D(A) = n\). □

In the case we are interested in, namely \(R = \mathcal{S} * \mathbb{Z}\), the role of \(D\) will be played by the Ore division ring of fractions \(D_S = \text{Ore}(D_S * \mathbb{Z})\).

2.2. The homological properties of \(D_S\) and the proof of Theorem A. We will now study the homological properties of the \(S\)-module \(D_S\) and its submodules. In particular, we will derive vanishing results for \(\text{Tor}\) and \(\text{Ext}\), which will allow us to verify condition (1) and a weak version of condition (2) of Theorems 2.4 and 2.5.

From this, we will finally derive Theorem A.

The following theorem, which combines Theorem 4.7 and 4.8 of [Sch85], will be very useful in verifying condition (1):

\[
\text{Theorem A}
\]
Theorem 2.6. Let $R \to S$ be an epic ring homomorphism. Then the following are equivalent:

1. $\text{Tor}_1^S(S, S) = 0$.
2. $\text{Tor}_1^S(M, N) = \text{Tor}_1^S(M, N)$ for every right $S$-module $M$ and every left $S$-module $N$.
3. $\text{Ext}_1^S(M, M') = \text{Ext}_1^S(M, M')$ for all right $S$-modules $M$ and $M'$.
4. $\text{Ext}_1^S(N, N') = \text{Ext}_1^S(N, N')$ for all left $S$-modules $N$ and $N'$.

If $S = R\mathbb{Z}$ is a universal localization of $R$, then all of these properties are satisfied.

The importance of this theorem in our paper is given by the fact that, since firs are Sylvester domains, the universal division $\mathfrak{S}$-ring of fractions $D_{\mathfrak{S}}$ is precisely the universal localization of $\mathfrak{S}$ with respect to the set of all full matrices. Therefore, each of the statements in Theorem 2.6 holds for the epic embedding $\mathfrak{S} \hookrightarrow D_{\mathfrak{S}}$, and this will serve as the starting point for the proof of the main result. The other crucial property in our setting is the following.

Lemma 2.7. Let $R$ be a ring of right (resp. left) global dimension 1. Then any crossed product $R * \mathbb{Z}$ has right (resp. left) global dimension at most 2. In particular, if $\mathfrak{S}$ is a fir, then $\mathfrak{S} * \mathbb{Z}$ has right and left global dimension at most 2.

Proof. We choose an automorphism $\tau$ of $R$ such that $R * \mathbb{Z} \cong R[t^{\pm 1}, \tau]$. Now [MR01] Theorem 7.5.3 applies (notice though some notational changes, since their polynomials are defined to be of the form $\sum_k t^k a_k$) to show that the right global dimension of $R[t^{\pm 1}, \tau]$ is at most 2, and, via an entirely symmetrical argument, that its left global dimension is also at most 2. The second statement follows because firs are particular examples of rings with right and left global dimension equal to 1. □

We are now ready to study the homological properties of $D_{\mathfrak{S}}$ and its submodules.

Lemma 2.8.

1. $\text{Ext}_1^S(M, M') = 0$ for all left (resp. right) $\mathfrak{S}$-modules $M$ and $M'$.
2. $D_{\mathfrak{S}} * \mathbb{Z}$ has projective dimension at most 1 as a left and right $\mathfrak{S}$-module.
3. Every left or right $\mathfrak{S}$-submodule of $D_{\mathfrak{S}} * \mathbb{Z}$ has projective dimension at most 1.
4. Every finitely generated left or right $\mathfrak{S}$-submodule of $D_{\mathfrak{S}}$ has projective dimension at most 1.

Proof. (1) Since $\mathfrak{S}$ has global dimension at most 2 by Lemma 2.7, this is a consequence of Lemma 1.4.

(2) Since $\mathfrak{S}$ has global dimension 1, the left $\mathfrak{S}$-module $D_{\mathfrak{S}}$ admits a resolution $0 \to P_1 \to P_0 \to D_{\mathfrak{S}} \to 0$ with $P_1$ and $P_0$ projective left $\mathfrak{S}$-modules. We now apply the functor $S \otimes_\mathfrak{S} -$ to this short exact sequence, where we view $S$ as an $S$-$\mathfrak{S}$-bimodule. Since $S$ is a free right $\mathfrak{S}$-module, the resulting sequence is a projective resolution of the left $\mathfrak{S}$-module $S \otimes_\mathfrak{S} D_{\mathfrak{S}}$, and thus the projective dimension of this module is at most 1. This finishes the proof, since the left $\mathfrak{S}$-modules $S \otimes_\mathfrak{S} D_{\mathfrak{S}}$ and $D_{\mathfrak{S}} * \mathbb{Z}$ are isomorphic by Lemma 2.2. The corresponding statement for the right $\mathfrak{S}$-module $D_{\mathfrak{S}} * \mathbb{Z}$ follows analogously.

(3) For every left (resp. right) $\mathfrak{S}$-module $M'$, the Ext long exact sequence obtained by applying the functor $\text{Hom}_{\mathfrak{S}}(\square, M')$ to the short exact sequence $0 \to M \to D_{\mathfrak{S}} * \mathbb{Z} \to Q \to 0$ for an appropriate $\mathfrak{S}$-module $Q$ contains the following exact part:

$$\ldots \to \text{Ext}_1^S(D_{\mathfrak{S}} * \mathbb{Z}, M') \to \text{Ext}_1^S(M, M') \to \text{Ext}_1^S(Q, M') \to \ldots$$

Here, the first term vanishes by (2) and Lemma 1.4 and the third term vanishes by property (1). By exactness, we conclude that the term in the middle also vanishes. Thus, the claim follows from Lemma 1.4.
(4) This follows directly from (3) and Lemma 2.3.

Lemma 2.9.

(1) Tor^1_\delta(D_{\mathfrak{g}}, D_{\mathfrak{g}}) = 0.
(2) Tor^2_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) = 0 for every left \( S \)-module \( N \).
(3) Tor^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) = 0 for every left \( D_{\mathfrak{g}} * \mathbb{Z} \)-module \( N \).
(4) Tor^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) = 0 for every left \( S \)-submodule \( N \) \( \leq \) \( D_S \).
(5) Tor^1_\delta(D_S, N) = 0 for every left \( S \)-submodule \( N \) \( \leq \) \( D_S \).
(6) Tor^2_\delta(N, D_S) = 0 for every right \( S \)-submodule \( N \) \( \leq \) \( D_S \).
(7) Tor^1_\delta(D_S, D_S) = 0.

Proof. (1) Since \( \mathfrak{g} \) is a fir, we know that \( D_{\mathfrak{g}} \) is the universal localization of \( \mathfrak{g} \) with respect to the set of all full matrices, so this follows from Theorem 2.6.
(2) The flat dimension of a module is at most its projective dimension, so this follows from Lemma 2.8 (2) and Lemma 1.5.
(3) Observe that \( D_{\mathfrak{g}} * \mathbb{Z} \) is isomorphic to \( D_{\mathfrak{g}} \otimes_{\mathfrak{g}} S \) as a right \( S \)-module by Lemma 2.2 and that \( S \) is a free left \( \mathfrak{g} \)-module (in particular flat). Thus, Lemma 1.6 together with (1) and Theorem 2.6 (2), tells us that

\[
\text{Tor}^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) = \text{Tor}^1_\delta(D_{\mathfrak{g}}, N) = \text{Tor}^1_\delta(D_{\mathfrak{g}}, N) = 0.
\]

(4) We have a short exact sequence \( 0 \to N \to D_S \to Q \to 0 \) for some left \( S \)-module \( Q \). Applying \( D_{\mathfrak{g}} * \mathbb{Z} \otimes \mathfrak{g} \) \( \square \) to this sequence, we obtain a long exact sequence that contains the following subsequence:

\[
\ldots \to \text{Tor}^2_\delta(D_{\mathfrak{g}} * \mathbb{Z}, Q) \to \text{Tor}^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) \to \text{Tor}^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, D_S) \to \ldots
\]

Since the first and third term vanish by (2) and (3), respectively, we obtain the result.
(5) Let

\[
\ldots \to P_k \to \ldots \to P_0 \to N \to 0
\]

be a projective resolution of \( N \). We can compute \( \text{Tor}^1_\delta(D_S, N) \) as the first homology group of the \( S \)-chain complex

\[
\ldots \to D_S \otimes_S P_k \to \ldots \to D_S \otimes_S P_0 \to 0.
\]

Since \( D_S \otimes \mathfrak{g} \) \( \cong \) \( D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} \) \( D_{\mathfrak{g}} * \mathbb{Z} \otimes \mathfrak{g} \), this complex is \( S \)-isomorphic to:

\[
C_*: \ldots \to D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} \to D_{\mathfrak{g}} * \mathbb{Z} \otimes \mathfrak{g} P_k \to \ldots \to D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} \otimes \mathfrak{g} P_0 \to 0.
\]

Using that \( D_{\mathfrak{g}} \) is the Ore localization of \( D_{\mathfrak{g}} * \mathbb{Z} \), which implies that the functor \( D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} \mathfrak{g} \) is exact, we obtain that \( H_*(C_*) \cong D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} H_*(D_*) \), where

\[
D_*: \ldots \to D_{\mathfrak{g}} \otimes \mathfrak{g} P_k \to \ldots \to D_{\mathfrak{g}} * \mathbb{Z} \otimes \mathfrak{g} P_0 \to 0.
\]

But the homology of this complex computes \( \text{Tor}^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) \), and thus

\[
\text{Tor}^1_\delta(D_S, N) \cong H_1(C_*) \cong D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} H_1(D_*) \cong D_S \otimes_{D_{\mathfrak{g}}} \mathbb{Z} \text{Tor}^1_\delta(D_{\mathfrak{g}} * \mathbb{Z}, N) \cong 0.
\]

(6) Every step in the proof of (5) can be adapted for right modules since \( S \) is also a free right \( \mathfrak{g} \)-module, and we can apply Lemma 2.8, Lemma 2.2 and the corresponding version of Lemma 1.6 for right modules.
(7) This is a special case of (5). \qed

We obtain from the previous results a weaker version of conditions (2) of Theorem 2.4 and Theorem 2.5.

Proposition 2.10. For every finitely generated left or right \( S \)-submodule \( M \) of \( D_S \) and every exact sequence \( 0 \to J \to S^m \to M \to 0 \), the \( S \)-module \( J \) is finitely generated projective.
Proof. Since $M$ has projective dimension at most 1 by Lemma 2.8 (4) and $S^n$ is projective, we conclude from Lemma 1.4 that $J$ is projective.

If $M$ is a left $S$-module and we apply the functor $D_S \otimes_S -$ to the short exact sequence defining $J$, the sequence remains exact by Lemma 2.9 (5). In particular, $D_S \otimes_S J$ is isomorphic to a $D_S$-submodule of the finitely generated $D_S$-module $(D_S)^n$. But $D_S$ is a division ring, thus $D_S \otimes_S J$ is itself finitely generated. Since $J$ is projective, [LLS03, Lemma 4] applies and we obtain that $J$ is finitely generated.

We finally have all the necessary ingredients for the proof of Theorem A.

Proof of Theorem A. By Lemma 2.9 (7), the conditions (1) of Theorem 2.5 and Theorem 2.4 are satisfied for $S \rightarrow D_S$, while we obtain from Proposition 2.10 that the module $I$ appearing in the conditions (2) is finitely generated and projective. Therefore, if every finitely generated projective $S$-module is stably free (resp. free), we deduce that $S$ is a pseudo-Sylvester domain (resp. Sylvester domain). Conversely, over a pseudo-Sylvester domain every finitely generated projective module is stably free (cf. [Coh06, Proposition 5.6.2]), while Sylvester domains are always projective-free (cf. [Coh06, Proposition 5.5.7]).

In any of the previous cases, we conclude from the criteria that $D_S = \text{Ore}(D_S \otimes \mathbb{Z})$ is the universal localization of $\mathfrak{s} \ast \mathbb{Z}$ with respect to the set of all stably full (resp. full) matrices, and hence its universal division ring of fractions. □

3. Applications: Theorem B and Theorem C

The aim of this section is to prove Theorem B and Theorem C. Thus, throughout this section the main object of study will be a crossed product $E \ast G$, where $E$ is a skew field and $G$ denotes a group that fits into a short exact sequence

$$1 \rightarrow F \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1,$$

with $F$ a non-necessarily finitely generated free group. Since $\mathbb{Z}$ is a free group, any such extension splits and $G$ arises as a semi-direct product $F \rtimes \mathbb{Z}$.

The crossed product $E \ast G$ can be expressed as an iterated crossed product $(E \ast F) \ast \mathbb{Z}$ by [Pas89, Lemma 1.3], using that the free subgroup $F$ is normal in $G$. Since $E \ast F$ is a fir, we are in the situation of Theorem A with $\mathfrak{s} = E \ast F$ and $S = E \ast G$.

In Section 3.1 we use the Farrell–Jones conjecture in algebraic K-theory to show that $E \ast G$ is always a pseudo-Sylvester domain. Whether this ring is even a Sylvester domain is a much more delicate question and not much can be said in general. In Section 3.2 we give examples of group rings for which this question has a known answer. Finally, in Section 3.3 we introduce locally indicable groups and Hughes-free division rings to prove Theorem C.

In this section, we use $D_{E \ast F}$ to denote the universal division $E \ast F$-ring of fractions and set $D_{E \ast G} = \text{Ore}(D_{E \ast F} \ast \mathbb{Z})$.

3.1. The Farrell–Jones conjecture and stably freeness. In this subsection we use recent results on the Farrell–Jones conjecture to prove that the finitely generated projective $E \ast G$-module $J$ that appears in condition (2) of Theorem 2.5 is actually stably free, which will conclude the first part of the proof of Theorem B. The following piece of the algebraic K-theory of a ring is needed to phrase the results:

Definition 3.1. Let $R$ be a ring. Then we denote by $K_0(R)$ the abelian group generated by the isomorphism classes $[P]$ of finitely generated projective $R$-modules together with the relations

$$[P \oplus Q] - [P] - [Q] = 0$$

for all finitely generated projective $R$-modules $P$ and $Q$. 

Every element of $K_0(R)$ is of the form $[P] - [P']$ for finitely generated projective $R$-modules $P$ and $P'$. The identity $[P] = [P'] \in K_0(R)$ holds for two finitely generated projective $R$-modules $P$ and $P'$ if and only if there is a finitely generated projective $R$-module $Q$ such that $P \oplus Q \cong P' \oplus Q$, where $Q$ can even be taken to be free.

If $f : R \to S$ is a ring homomorphism and $P$ is a finitely generated projective $R$-module, then $S \otimes_R P$ is a finitely generated projective $S$-module. In this way, $K_0(\square)$ becomes a functor from rings to abelian groups.

The conditions of Remark 1.9 are satisfied for $K_0(\square)$ and thus it does not depend on whether we use left or right modules in its definition.

The Farrell–Jones conjecture makes far-reaching claims about the K-theory and L-theory of group rings or, more generally, additive categories with group actions, in particular for torsion-free groups. It is known for many classes of groups and satisfies a number of useful inheritance properties. For a full statement of the Farrell–Jones conjecture and an overview of the groups for which it is known, we refer the reader to the surveys [BLR08] and [RV18], and also to [Lüc10, Lüc19].

We will need the following consequence of the Farrell–Jones conjecture which is certainly well-known, but has not been made explicit in the literature.

Proposition 3.2. Let $E$ be a division ring, $\Gamma$ a torsion-free group and $E \ast \Gamma$ a crossed product. If the $K$-theoretic Farrell–Jones conjecture with coefficients in an additive category holds for $\Gamma$, then the embedding $E \hookrightarrow E \ast \Gamma$ induces an isomorphism

$$K_0(E) \xrightarrow{\cong} K_0(E \ast \Gamma).$$

In particular, since $K_0(E) = \{n[E] \mid n \in \mathbb{Z}\}$, every finitely generated projective $E \ast \Gamma$-module is stably free.

Proof. For a given crossed product $E \ast \Gamma$, we will denote the additive category defined in [BR07, Corollary 6.17] by $\mathcal{A}_{E \ast \Gamma}$. We will freely use the terminology and notation of that paper. Furthermore, we will denote the family of virtually cyclic subgroups of a given group by $\text{VCyc}$ and the family consisting just of the trivial subgroup by $\text{Tr}$. The K-theoretic Farrell–Jones conjecture for the group $\Gamma$ with coefficients in the additive category $\mathcal{A}_{E \ast \Gamma}$ arises as an instance of the more general meta-isomorphism conjecture [Lüc10, Conjecture 13.2] for the $\Gamma$-homology theory $H_\mathcal{F}^\Gamma(\square; \mathcal{K}_{\text{VCyc}})$ introduced in [BR07] and the family $\mathcal{F} = \text{VCyc}$. It states that the assembly map

$$H_\mathcal{F}^\Gamma(\mathcal{E}_{\text{VCyc}}(\Gamma); \mathcal{K}_{\text{VCyc}}) \to H_\mathcal{F}^\Gamma(\text{pt}; \mathcal{K}_{\text{VCyc}})$$

is an isomorphism, where the right-hand side is isomorphic to $K_*(E \ast \Gamma)$ by [BR07, Corollary 6.17].

In order to arrive at the desired conclusion, we need to reduce the family from $\text{VCyc}$ to $\text{Tr}$. Since $\Gamma$ is assumed to be torsion-free and hence all its virtually cyclic subgroups are infinite cyclic, we can arrange for this via the transitivity principle of [Lüc19, Theorem 13.13 (i)] if the meta-isomorphism conjecture with the $\mathbb{Z}$-homology theory $H_\mathcal{F}^\Gamma(\square; \mathcal{K}_{\text{VCyc}})$ and the family $\mathcal{F} = \text{Tr}$ holds. A model for the classifying space $E_{\mathcal{F}}(\mathbb{Z})$ is given by $S^1$ and we may again assume that the crossed product $E \ast \mathbb{Z}$ is a skew Laurent polynomial ring $E[t^{\pm 1}, \tau]$. In this situation, since $E$ is regular, the assembly map coincides with the map provided by the analogue of the Fundamental Theorem of algebraic $K$-theory for skew Laurent polynomial rings, which is an isomorphism (cf. [BL20, Theorems 6.8 & 9.1] or [Gra88, for a more classical treatment]).

Since the K-theoretic Farrell–Jones conjecture with coefficients in an additive category is assumed to hold for $G$, we now obtain from the transitivity principle
that the assembly map
\[ H_*^c(\mathcal{E}_T(\Gamma); K_{A_{E\Gamma}}) \to H_*^c(\text{pt}; K_{A_{E\Gamma}}) \cong K_*(E \ast \Gamma) \]

is an isomorphism. The space \( \mathcal{E}_T \) is a free \( \Gamma \)-space and the value at the coset \( \Gamma/(1) \) of the Ort(\( \Gamma \))-spectrum \( K_{A_{E\Gamma}} \) is \( K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1)) \). We can thus simplify the left-hand side of the assembly map as follows:

\[ H_*^c(\mathcal{E}_T(\Gamma); K_{A_{E\Gamma}}) \cong H_*^c(\mathcal{E}(\Gamma); K_{A_{E\Gamma}}) \cong H_*(B\Gamma; K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1))). \]

Here, \( B\Gamma \) denotes the standard classifying space of the group \( \Gamma \) and homology is again taken with local coefficients. Since \( K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1)) \) is weakly equivalent to \( K^\infty(E) \), which is connective by \[\text{Lüc19}\] Theorem 3.6 since \( E \) is a regular ring. In particular, the Atiyah–Hirzebruch spectral sequence provides the following natural isomorphism:

\[ H_0(B\Gamma; K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1))) \cong H_0(B\Gamma; \pi_0(K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1)))), \]

where homology is again taken with local coefficients. Since \( \pi_0(K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1))) \cong K_0(\mathcal{A}_{E\Gamma} \ast \Gamma/(1)) \) and the \( \Gamma \)-action on \( \mathcal{A}_{E\Gamma} \ast \Gamma/(1) \), which is induced from that on the \( \Gamma \)-space \( \Gamma/(1) \), preserves isomorphism types, the local coefficients are in fact constant. We conclude that

\[ H_0(B\Gamma; K^\infty(\mathcal{A}_{E\Gamma} \ast \Gamma/(1))) \cong H_0(B\Gamma; K_0(E)), \]

and thus the assembly map in degree 0 simplifies to

\[ K_0(E) \cong H_0(B\Gamma; K_0(E)) \cong K_0(E \ast \Gamma). \]

This proves the first statement.

The second statement is now a consequence since every finitely generated projective \( E \ast \Gamma \)-module \( P \) represents an element \( n[E \ast \Gamma] \) in \( K_0(E \ast \Gamma) \) for \( n \geq 0 \), and thus there exists a finitely generated free \( E \ast \Gamma \)-module \( Q \) such that \( P \oplus Q \cong (E \ast \Gamma)^n \oplus Q \), which is free.

The following is the K-theoretic part of \[\text{[BFW19]}\] Theorem 1.1 in the case of a finitely generated free group \( F \) and \[\text{[BKW19]}\] Theorem A in the general case:

**Theorem 3.3.** The K-theoretic Farrell–Jones conjecture with coefficients in an additive category holds for every group that arises as an extension

\[ 1 \to F \to G \to \mathbb{Z} \to 1 \]

with \( F \) a (not necessarily finitely generated) free group.

The previous result provides the final step in the proof of Theorem [3]

**Proof of Theorem [3].** Since \( G \) satisfies the K-theoretic Farrell–Jones conjecture with coefficients in additive categories by Theorem 3.3, we obtain from Proposition 3.2 that every finitely generated projective \( E \ast G \)-module is stably free. Therefore, the statement follows from Theorem [3].

**3.2. Examples and non-examples.** The main examples of groups of the form \( 1 \to F \to G \to \mathbb{Z} \to 1 \) are the free-by-cyclic groups (terminology usually reserved in the literature for the case where \( F \) is finitely generated) and fundamental groups of connected closed surfaces with genus \( g \geq 1 \) other than the projective plane, which has to be excluded since its fundamental group has torsion. In the latter family, we have to distinguish the fundamental groups \( S_g \) of orientable closed surfaces of genus \( g \geq 1 \), which admit the presentations

\[ S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot \ldots \cdot [a_g, b_g] \rangle. \]
and the fundamental groups of non-orientable closed surfaces of genus \( g \geq 2 \), which admit the presentations
\[
\mathfrak{G}_g = \langle a_1, \ldots, a_g \mid a_1^2 \cdots a_g^2 \rangle.
\]
That these groups contain a normal free subgroup \( F \) such that \( G/F \) is infinite cyclic is a consequence of the fact that their infinite index subgroups are free (cf. [HKLS72]) and that they all admit surjections onto the infinite cyclic group with generator \( t \).

For \( S_g \) one can consider the map sending \( a_1 \) and \( b_1 \) to \( t \) and any other generator to \( 1 \), and for \( \mathfrak{G}_g \) one can send \( a_1 \) to \( t \), \( a_2 \) to \( t^{-1} \) and every other generator to \( 1 \).

Within these families, there are some cases of group rings for which it is known whether they admit stably free cancellation. In the following examples, \( K \) is any field of characteristic 0.

- Examples of group rings with stably free cancellation are \( K[\mathbb{Z}^2] = K[S_1] \) (c.f. [Swa78]) and \( K[F_2 \times \mathbb{Z}] \) (c.f. [Bas68 IV.6.4]).
- Examples of group rings which do admit non-free stably free modules are given by \( K[\mathbb{Z} \times \mathbb{Z}] = K[\mathcal{G}_2] \) (c.f. [Sta85 Theorem 2.12]) and \( \mathbb{Q}[\langle x, y \mid x^3 = y^7 \rangle] = \mathbb{Q}[F_2 \times \mathbb{Z}] \) (c.f. [Lew82]).

Here, the latter example is the rational group ring of the fundamental group of the complement of the trefoil knot, which fibers over the circle and hence admits a free-by-cyclic fundamental group (c.f. [BZH13 Corollary 4.12]). Both group rings serve as examples of pseudo-Sylvester domains that are not Sylvester domains.

To the best of the authors’ knowledge, it is an open question whether \( \mathbb{C}[S_g] \) for \( g \geq 2 \) and \( \mathbb{C}[\mathfrak{G}_g] \) for \( g \geq 3 \) have stably free cancellation.

3.3. Locally indicable groups and Hughes-freeness: identifying \( \mathcal{D}_{E \ast G} \). In order to prove Theorem [C], we need to introduce the following definitions. Recall that a group \( \Gamma \) is locally indicable if it is either trivial or every non-trivial finitely generated subgroup \( H \) of \( \Gamma \) admits a surjection onto \( \mathbb{Z} \). Observe that in this case, if \( H \) is non-trivial finitely generated, \( N \) is the kernel of such a surjection, and \( t \in H \) generates the quotient \( H/N \), then left conjugation by \( t \) induces an automorphism \( \tau \) of \( E \ast N \) and we can identify \( E \ast H \) with the skew Laurent polynomial ring \( E \ast N[t^\pm 1; \tau] \).

In other words, the powers of \( t \) in \( E \ast H \) are \( E \ast N \)-linearly independent.

We will say that an injective epic division \( E \ast \Gamma \)-ring \( \mathcal{D} \) is Hughes-free if this linear independence, for every \( H \) and \( N \) as before, is also reflected in \( \mathcal{D} \). To state this properly, recall that the division closure of a subring \( R \) of a ring \( S \) is the smallest subring \( D(R \subset S) \) of \( S \) that contains \( R \) and is division closed, i.e., such that for any element \( d \) of \( D(R \subset S) \) which is invertible in \( S \) we have \( d^{-1} \in D(R \subset S) \). If \( \mathcal{D} \) is an injective division \( E \ast \Gamma \)-ring, then, for any subgroup \( H \) of \( \Gamma \), let \( \mathcal{D}_H \) denote the division closure of \( E \ast H \) in \( \mathcal{D} \).

**Definition 3.4.** Given an epic embedding \( E \ast \Gamma \hookrightarrow \mathcal{D} \) of \( E \ast \Gamma \) into a division ring \( \mathcal{D} \), we say that \( \mathcal{D} \) is a Hughes-free epic division \( E \ast \Gamma \)-ring if, for every non-trivial finitely generated subgroup \( H \) of \( \Gamma \), every normal subgroup \( N \trianglelefteq H \) such that \( H/N \) is infinite cyclic and every \( t \in H \) projecting to a generator of the quotient, the powers of \( t \) are \( \mathcal{D}_H \)-linearly independent, i.e., there exists no non-trivial expression \( \sum d_i t^i = 0 \) with \( d_i \in \mathcal{D}_N \).

This notion was introduced by Hughes in ([Hug70]), where he also proved that, if there exists one Hughes-free division ring for \( E \ast \Gamma \), then it is unique up to \( E \ast \Gamma \)-isomorphism (see also [San08 Hughes Theorem I] for a detailed proof of this result).

In ([JL20]) the existence of a Hughes-free division ring is settled for group rings \( \Gamma \) when \( K \) has characteristic zero, but in fact, for the groups \( G \) under consideration in this paper, which as extensions of locally indicable by locally indicable groups
are locally indicable, this problem was already solved in full generality for crossed product group rings. More precisely, we have the following:

**Theorem 3.5.** Let $G$ be a group obtained as an extension

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where $F$ is a free group. Then, for every division ring $E$ and any crossed product $E \ast G$, there exists a Hughes-free epic division $E \ast G$-ring $D$. Moreover, if there exists a universal division $E \ast G$-ring, then it is isomorphic to $D$.

**Proof.** Every crossed product of a division ring with a free group admits a Hughes-free division ring (cf. [Lew74, Proposition 6], [Sán08, Example 5.6(e)]). Therefore, we obtain by a result of Hughes (cf. [Sán08, Hughes Theorem II]) that $E \ast G$ admits a Hughes-free division ring. The final statement can be proved either applying [Sán08, Example 6.19 & Proposition 6.23] to the subnormal series $1 \leq F \leq G$, or using [JL20, Corollary 8.2].

Thus, in our particular setting, $D_{E \ast G}$ is Hughes-free, and we are going to use the uniqueness of the Hughes-free division ring to describe concrete realizations of $D_{E \ast G}$.

The first one has to do with the space of Malcev-Neumann series for left orderable groups. Recall that a group $\Gamma$ is left orderable if it admits a left order, i.e., a total order $\leq$ compatible with left multiplication by elements of $\Gamma$. Brodskii ([Bro84]) proved that the family of locally indicable groups coincides with the family of left orderable groups admitting a Conradian order, i.e., a left order with the property that for any $1 \leq g, h \in \Gamma$, we have $h \leq gh^2$ (cf. [Sán08, Proposition 2.31]).

The second one, available for group rings $K \Gamma$ with $K$ a subfield of $\mathbb{C}$, has an analytical nature and is intimately related to the strong Atiyah conjecture for torsion-free groups.

### 3.3.1. $D_{E \ast G}$ and the Malcev-Neumann construction.

Let $\Gamma$ be a left orderable group, and let $\leq$ be a left order on $\Gamma$. Then, one can consider the set $E((\Gamma, \leq))$ of formal power series

$$x = \sum_{g \in \Gamma} \tilde{g}\lambda_g, \text{ with } \lambda_g \in E,$$

whose support $\text{supp}(x) = \{g \in \Gamma : \lambda_g \neq 0\}$ is well-ordered with respect to $\leq$.

Malcev ([Mal48]) and Neumann ([Neu49]) proved independently that, if $\leq$ is also compatible with the right multiplication (i.e., $\leq$ is a bi-order), then the natural sum and product of series are well-defined, and $K((\Gamma, \leq))$ for a field $K$ is a division ring in which $K\Gamma$ embeds. In general, $E((\Gamma, \leq))$ is not a ring, but it is still a right $E$-vector space. Now, if $x \in E((\Gamma, \leq))$ and $\mu h$ is an element of $E \ast \Gamma$, we can define $\mu h \cdot x$ by just extending the product defined in $E \ast \Gamma$. In this way, the support of the element obtained is $\{h g : g \in \text{supp}(x)\}$, which by left compatibility of $\leq$ is well-ordered with least element $h_0$. Thus, left multiplication by $\mu h$ defines an element of $\text{End}(E((\Gamma, \leq)))$, and this can be linearly extended to any element in the crossed product $E \ast \Gamma$ since subsets and finite unions of well-ordered sets are again well-ordered. By construction, this identification is compatible with the product in $E \ast \Gamma$, and therefore we can see $E \ast \Gamma$ as a subring of $\text{End}(E((\Gamma, \leq)))$ (cf. [Grä19] Section 7 for a detailed explanation and further properties of this embedding).

The following is a combination of [Grä19] Theorem 8.1 & Corollary 8.3 and will give us the first half of Theorem [Grä19].

**Theorem 3.6.** Let $E$ be a division ring and $\Gamma$ a locally indicable group. If there exists a Hughes-free epic division $E \ast \Gamma$-ring, then it is isomorphic to the division
closure $\mathcal{D}_< (E \star \Gamma)$ of $E \star \Gamma$ inside $\text{End}(E((\Gamma, \leq)))$, where $\leq$ is any Conradian left order on $\Gamma$.

3.3.2. The Atiyah conjecture and the characteristic zero case. Let $\Gamma$ be a countable group and denote by $\ell^2(\Gamma)$ the $\ell^2$-Hilbert space with orthonormal basis $\Gamma$, i.e., the space whose elements are square summable series $\sum_{g \in \Gamma} \lambda_g g$ with complex coefficients and which is equipped with the standard $\ell^2$-scalar product. The group von Neumann algebra $\mathcal{N}(\Gamma)$ is the algebra of bounded $\Gamma$-equivariant operators $T: \ell^2(\Gamma) \to \ell^2(\Gamma)$, where $\Gamma$ acts on $\ell^2(\Gamma)$ by left multiplication. We can consider $\mathcal{K} \Gamma$ as a subring of the group von Neumann algebra $\mathcal{N}(\Gamma)$ by identifying any element $a$ with the bounded $\Gamma$-equivariant operator $r_a: \ell^2(\Gamma) \to \ell^2(\Gamma)$ given by right multiplication by $a$.

Over $\mathcal{N}(\Gamma)$, there exists a well-defined notion of dimension $\dim_{\mathcal{N}(\Gamma)}$ for $\mathcal{N}(\Gamma)$-modules, the so-called von Neumann dimension (cf. (Lüc02, Theorems 6.5 & 6.7)). Moreover, the set $T$ of all non-zero-divisors in $\mathcal{N}(\Gamma)$ is (right and left) Ore (cf. (Rei92, Proposition 2.8)), and its Ore localization, the algebra of unbounded affiliated operators $\mathcal{U}(\Gamma)$, is a von Neumann regular ring to which the dimension function can be extended (cf. (Lüc02, Theorem 8.22 and Section 8.3)).

The equivalence of results presented in the following proposition can be found, for example, in [Rei98, Conjecture 5.3 & Proposition 8.30].

**Proposition 3.7.** Let $\Gamma$ be a torsion-free group and $K$ a subfield of $\mathbb{C}$. Then, the following are equivalent:

1. For every finitely presented left $\mathcal{K} \Gamma$-module $M$,
   $$\dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathcal{K} \Gamma} M) \in \mathbb{Z}.$$ 

2. The division closure $\mathcal{D}(\Gamma; K)$ of $\mathcal{K} \Gamma$ in $\mathcal{U}(\Gamma)$ is a division ring.

**Definition 3.8.** A torsion-free group $\Gamma$ is said to satisfy the strong Atiyah conjecture over the subfield $K$ of $\mathbb{C}$ if any (and hence each) of the statements in Proposition 3.7 is satisfied.

There is no known example of a torsion-free group that does not satisfy the strong Atiyah conjecture. It is known to hold for locally indicable groups ([JL07, Theorem 1.1]), and the first proof for the groups under consideration goes back to Linnell ([Lin93], see also (Lüc02, Chapter 10]), since they all lie in Linnell’s class $\mathcal{C}$.

In Linnell’s proof, Hughes-freeness was already used to identify the $\mathcal{C}F$-rings $\mathcal{D}(F; \mathbb{C})$ and $\mathcal{D}_{CF}$, and the same arguments apply for any subfield $K$ of $\mathbb{C}$. Using this, one can directly exhibit $\mathcal{D}(G; K)$ as the Ore field of fractions of $\mathcal{D}_{KF} \ast \mathbb{Z}$. Indeed, this crossed product can be built as a subring of $\mathcal{U}(G)$, and hence, inasmuch as $\mathcal{D}(G; K)$ is a division ring containing $\mathcal{D}_{KF} \ast \mathbb{Z}$, the universal property of the Ore localization tells us that it also contains the ring $\text{Ore}(\mathcal{D}_{KF} \ast \mathbb{Z})$. Since the latter is a sub-division ring containing $KG$, necessarily $\mathcal{D}(G; K) = \text{Ore}(\mathcal{D}_{KF} \ast \mathbb{Z})$.

This gives us a proof of the second half of Theorem C. An alternative proof can be given by directly applying the Hughes-freeness of $\mathcal{D}(G; K)$ when $G$ is locally indicable ([JL07, Corollary 6.3]) and the last assertion of Theorem 3.5.

**Proof of Theorem C.** Using Theorem B, the first half of the theorem follows from Theorem 3.5 and Theorem 3.6, while the second half follows from the previous discussion.

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