Transformation of the Mean Value of Integral On Fourier Series Expansion

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Abstract. Approximation of sigma is a damping factor which is obtained through transformation of mean value of integral to the functionality expanded via Fourier series. Where the result of the transformation is in the form of oscillation function, so as to form a modified partial sums of Fourier series. Through modification of the partial sums of Fourier series, the leap (overshoot) near discontinuity points of the oscillations function can be suppressed.

1. Introduction

Gibbs phenomenon appears near discontinuity points when a periodic function of \( f(x) \) is approximated by the partial sums of Fourier series or through a Fourier expansion. Near the discontinuity points, the approximation of periodic function occurred leap (overshoot) are not lost to \( N \to \infty \) with smaller area, but the overshoot near discontinuity points is still maintained. It should be according to the convergence Fourier theorem, a function that is approximated by the Fourier expansion oscillate around the numerical values, so that the limit of an infinite of partial sum equal to the numerical value of a given function.

There has been some effort made to minimize the overshoot that appears near discontinuity points. One of them is the method of Fejer, by determining the arithmetic mean of a finite number of terms partial sum ([4] p. 15).

In this paper, it will be introduced a mathematical modification of the Fourier series which is called approximation of sigma (\( \sigma \)). The approximation of \( \sigma \) method is done by determining the mean value of the function is approximated by Fourier series on the interval \( \left[ t - \frac{\pi}{N}, t + \frac{\pi}{N} \right] \) so that the partial sums of Fourier series load \( \sigma \) factors. The influence of these factors may dampened the overshoot that appears near discontinuity points.

2. Fourier Series

The general form of Fourier series is expressed as follows, (see [5])

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi x}{L} \right) + b_n \sin \left( \frac{n \pi x}{L} \right)
\]  (2.1)

According to Franklin (1964) and Churchill (1972), a periodic function of \( f(x) \) is approximated by the partial sums of Fourier series \( S_N(x) \) will always converge, such that

\[
\lim_{N \to \infty} S_N(x) = f(x)
\]

The condition for determining when a Fourier series for a periodic function converges, has been investigated by Dirichlet (Folland, 1992). In this case, what is the condition of a function \( f(x) \) if approximated by Fourier expansion in order to converges, particularly if \( f(x) \) is a smooth piecewise
function at a given interval. The fact shows that convergence of Fourier expansion at the discontinuity points always lies on the mean of the left and right limit function \( f(x) \), as shown in the Figure 1 below.

![Figure 1. \( S_N \rightarrow f(x) \), for continuity points](image)

According Wibraham (1898), a phenomenon has occurred in the Fourier series expansion around at discontinuity points. That is overshoot around at discontinuity points for periodic function approximation. Furthermore, by Josiah Willard Gibbs (1839-1903) investigated that if the function is approximated by a Fourier series, then the overshoot of approximation numerical values around the discontinuity points will not have lost up to \( N \rightarrow \infty \) with smaller area, but the overshoot near the discontinuity point is still maintained as if contrary to the convergence theorem of Fourier series. This phenomenon is called Gibbs phenomenon. The convergence theorem of Fourier series is expressed as follows.

**Theorem 2.1**

Suppose \( S_N \) partial sum of the Fourier series \((1)\) for the function \( f \). If \( f \in C_{2\pi} \), then for \( n \rightarrow \infty \), \( \| S_N - f \| \rightarrow 0 \)

Gibbs phenomenon can be overcome by averaging the partial sums of Fourier series, as had been done by Cesaro (Folland, 1992) and Fejer (Churchill, 2001). The average of Cesaro and Fejer summation show analytically that convergence of the Fourier expansion, but explorative convergence obtained from both can be achieved for the terms partial sum that is not relatively little, so that the time to achieve convergence is relatively long, although the overshoot around the discontinuity point can be eliminated.

### 3. Gibbs Phenomenon

As an illustration of view of the following functions,

\[
f(x) = \begin{cases} 
1, & 0 \leq x \leq \pi \\
-1, & \pi \leq x \leq 2\pi 
\end{cases}
\]

Where \( f(x + 2\pi) = f(x) \). Because this function is odd, then obtained by Fourier coefficients \( a_n = 0 \) for \( n = 1, 2, \ldots \) and \( a_0 = 0 \). While the coefficient \( b_n \) is

\[
b_n = \frac{1}{\pi} \left( \int_0^\pi x \sin nx \, dx - \int_{\pi}^{2\pi} \sin nx \, dx \right) = \begin{cases} 
\frac{4}{n\pi}, & \text{if } n \text{ odd} \\
0, & \text{if } n \text{ even} 
\end{cases}
\]

for \( n = 1, 2, \ldots \). The partial sums of Fourier series \( 2n - 1 \) the first term for the function \( f(x) \) is

\[
S_{2n-1}(x) = \frac{4}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \ldots + \frac{\sin(2n-1)x}{(2n-1)} \right)
\]  \hspace{1cm} (3.1)
which converges to \( f(x) \) except at the point \( x = k\pi \) for \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \) which are points of discontinuity of \( f(x) \). The graph of the Fourier partial sum for the function \( f(x) \) can be seen in Figure 2 on the interval \([0, 2\pi]\).

\[
\text{Figure 2. Gibbs Phenomenon}
\]

In the picture, it appears that at the discontinuity points of oscillation function, the approximations of the partial sum occurred Gibbs phenomenon. View of discontinuity point of \( x = 0 \). It appears that, when \( x \) is close to the point of discontinuity \( 0 \), the approximation of the oscillation function is having overshoot, in which the overshoot is always maintained for sufficiently large \( N \), as shown in Figure 3.

\[
\text{Figure 3. Fourier Series converge to } f(x)
\]

In this paper will do a calculation to explain this phenomenon. The first derivative \( S_{2n-1} \) of view, to determine the critical points so that \( S_{2n-1} \) maximum.

\[
S_{2n-1}(x) = \frac{4}{\pi} \left( \frac{\sin x}{3} + \frac{\sin 3x}{3} + \ldots + \frac{\sin (2n-1)x}{(2n-1)} \right)
\]

By using trigonometric identities, we get \( \pi \sin(x) S'_{2n-1}(x) = 2\sin(2nx) \). So obtained critical points of \( S_{2n-1} \), that is \( 2nx = \pm \pi, \pm 2\pi, \ldots, \pm (2n-1)\pi \). Since the function is odd, view of the interval \([0, \pi]\) and we will only consider the nature of the right of the 0. The critical point that is closest to 0 from the right is \( \frac{\pi}{2n} \).

The result is

\[
S_{2n-1}(\frac{\pi}{2n}) = \frac{4}{\pi} \left( \frac{\sin \left( \frac{\pi}{2n} \right)}{3} + \frac{\sin \left( \frac{3\pi}{2n} \right)}{3} + \ldots + \frac{\sin \left( \frac{(2n-1)\pi}{2n} \right)}{(2n-1)} \right)
\]
When \( n \) is large enough, we use the Riemann sum. Of view of function \( F(x) = \frac{\sin x}{x} \) on interval \([0, \pi]\), and partition \( \left\{ \frac{k\pi}{n}, k \in [1, n] \right\} \) of \([0, \pi]\). So the Riemann sum

\[
\frac{\pi}{n} \left( \sin \left( \frac{\pi}{2n} \right) + \sin \left( \frac{2n-1}{2n} \pi \right) \right)
\]

converges to \( \int_{0}^{\pi} F(x)dx \) where the Riemann will equal to the Riemann \( \frac{\pi}{2} S_{2n+1} \left( \frac{\pi}{2n} \right) \) which consequently

\[
\lim_{n \to \infty} S_{2n+1} \left( \frac{\pi}{2n} \right) = 2 \int_{0}^{\pi} \frac{\sin x}{x} dx = 1.1789797...
\]

where \( \int_{0}^{\pi} \frac{\sin x}{x} dx = \int_{0}^{\pi} \frac{\sin x}{x} dx = \text{Si}(\pi) = 1.851937052 \ldots \) (see [21]). Thus obtained

\[
\int_{0}^{\pi} \frac{\sin x}{x} dx = \frac{\pi}{2} 1.1789797...
\]

this phenomenon has been observed by Willbraham (1848) and Gibbs (1899), where the value of the quantities 1.1789797... declared as constants Gibbs, as stated in theorem 3.2. Before the theorem 3.2 proved, note the theorem 3.1 this below

**Theorem 3.1.**

\[
\sum_{k=1}^{n} \frac{\sin kx}{x} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi
\]

**Proof**

For \( 0 < x < 2\pi \) and for any \( n \in \mathbb{N} \), belonged to Dirichlet kernel to \( n \) or \( D_n(x) \), ([2] p. 88).

\[
\frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{\sin \left( \frac{2n+1}{2} x \right)}{2 \sin \left( \frac{x}{2} \right)}, \quad 0 < x < 2\pi
\]

As a result of partial sum can be written

\[
S_n = \frac{\sum_{k=1}^{n} \sin kx}{k} = \frac{1}{2} \int_{0}^{\pi} \sum_{k=1}^{n} \cos(kt) dt
\]

\[
= \int_{0}^{\pi} \left( -1 + \frac{\sin \left( \frac{2n+1}{2} t \right)}{2 \sin \left( \frac{t}{2} \right)} \right) dt = \frac{\pi - x}{2} + \frac{1}{2n+1} \left( -\cos \left( \frac{2n+1}{2} t \right) \right) \left[ \frac{t}{\sin \left( \frac{t}{2} \right)} \right]_{0}^{\pi} - \frac{1}{2} \int_{0}^{\pi} \frac{\cos \left( \frac{t}{2} \right)}{\sin ^{3} \left( \frac{t}{2} \right)} \cos \left( \frac{2n+1}{2} t \right) dt
\]
Because \( \frac{\lambda \cos\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)} \) \( dt \) \( = \frac{2}{\sin\left(\frac{x}{2}\right)} - 2 \), such that for \( n \to \infty \) we get

\[
\sum_{k=1}^{\infty} \frac{\sin kx}{k} - \frac{\pi - x}{2} \leq \frac{1}{2n+1} \left( \frac{2}{\sin\left(\frac{x}{2}\right)} - 1 \right) \to 0
\]

**Theorem 3.2. (Gibbs Phenomenon)**

Let \( x_n = \frac{2\pi}{2n+1} \) for \( n = 1, 2, \ldots \)

then

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{\sin \left(\frac{k\pi}{2n+1}\right)}{k} \right) = \int_0^\pi \frac{\sin x}{x} dx
\]

**Proof**

by replacing partial sum \( S_n = \sum_{k=1}^{n} \frac{\sin kx}{k} \) with integral, i.e.

\[
S_n(x_n) + \frac{1}{2} x_n = \frac{\lambda}{\pi} \int_0^\pi \frac{\sin \left(\frac{2n+1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)} dt = \frac{\pi}{2n+1} \int_0^\pi \frac{\sin t}{\sin\left(\frac{\pi}{2n+1}\right)(2n+1)} \to \int_0^\pi \frac{\sin t}{t} dt (n \to \infty)
\]

And then, it can be determined the critical point of \( S_n(x) \) in order to achieve maximum value, obtained at the time of \( S_n'(x)=0 \), i.e. occurs at points \( x_n = \frac{2\pi}{2n+1} \), for \( n = 1, 2, \ldots \)

Such that we have

\[
S_n(x_n) + \frac{1}{2} x_n = \frac{\lambda}{\pi} \int_0^\pi \frac{\sin \left(\frac{2n+1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)} dt = \frac{\pi}{2n+1} \int_0^\pi \frac{\sin t}{\sin\left(\frac{\pi}{2n+1}\right)(2n+1)} \to \int_0^\pi \frac{\sin t}{t} dt (n \to \infty)
\]

and

\[
\int_0^\pi \frac{\sin x}{x} dx = \frac{\pi}{2} \cdot 1.1789797...
\]

So that it appears from the explanation that a overshoot by the approximation of 17.9% (see Figure 4) always maintained when \( n \to \infty \) (at smaller interval around the \( x = 0 \))

![Figure 4. overshoot by the approximation of 17.9%](image-url)
4. Approximation of Fejer

Fejer approximation is based on the fact that if the sequence \( \{a_n\} \) converges to \( a \), then the average of \( k \) the first term of the sequence, that is \( \frac{1}{k} \sum_{i=1}^{k} a_n \) which also converges to \( a \) for \( k \to \infty \), but the average is probably convergent when the own sequence is not. For example, the sequence 1,0,1,0,1,0, ... is a sequence that is divergent, but the average of its \( k \) first term is \( \frac{(k+1)}{2k} \) or \( \frac{1}{2} \) in accordance with the value of \( k \) odd or even, and an average of the sequence will converge to \( \frac{1}{2} \) for \( k \to \infty \). Now given a series \( \sum_{n=0}^{\infty} b_n \) with a partial sum \( \sum_{n=0}^{N} b_n \), the average of the \( (k+1) \) the first term of that partial sum is

\[
\frac{1}{k+1} (S_0 + S_1 + ... + S_k)
\]

which is called the average of Cesaro and its series is called the sum of Cesaro. It is equal to \( s \) if the average of Cesaro converges to \( s \). Based on the average of these Cesaro, Fejer generalize on a series of functions. He stated that if we have a function of \( f \) is periodic with period \( 2 \pi \) and piecewise continuous on \( P \), then the Fourier series for the function \( f \) is the sum of Cesaro which converge to \( \frac{1}{2} [f(x-) + f(x+)] \) for every \( x \). Furthermore, it is said that if \( f \) is continuous everywhere, the average of Cesaro of the series converges uniform to \( f \). Therefore, in general, we have the following Fejer theorem

**Theorem 2.2**

Suppose \( \sigma_N = \frac{1}{N+1} (S_0 + S_1 + ... + S_N) \) where \( S_N \) partial sums of the Fourier series for the function \( f \). If \( f \in C_{2\pi} \) then for \( n \to \infty \), \( \| \sigma_N - f \| \to 0 \)

Several research to improve the overshoot (or undershoot) of Gibbs phenomenon that occurs has been done by Helmberg, G., and Wagner, P (1997) which has been published in the Journal of Approximation. They can fix it through interpolation. Numerical calculations done by manipulating the phenomenon through interpolation method. However, aspects of the numerical calculations were done too complicated and not simple. While, by using the method of Fejer we will plot the convergence of the results of calculations for the function \( f \) given in the illustration above.

![Figure 5. Plotting the convergence of the results of Fejer](image)

From Figure 5 the average number of partial to \( N = 1 \), \( N = 5 \) and \( N = 10 \) each curve yellow, blue and red. It appears that the overshoot near points of discontinuity is lost, but the time required for partial sum finite
$N$ is long enough at the point of discontinuity. This means the required $N$ large enough to produce a partial sum at near point of discontinuity.

5. Approximation of sigma

Based on research of the introduction whatever do Gunawan G (2005), obtained that mean value theorem of integral have the behavior analytic geometry which may averaging function $f$ on the closed interval. The basic idea of this research is to transform function is approximated by a Fourier series using the mean value theorem of integral to the area definition $\left[ t - \frac{\pi}{N}, t + \frac{\pi}{N} \right]$ as a domain of function $f$. The Mean Value Theorem of Integral expressed as follows.

Theorem 5.1
If $f$ is continuous on a closed interval $[a, b]$, then there is a point $c$ in $[a, b]$ such that

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c)$$

In this paper, it will be introduced a better method than Fejer method in tackling the overshoot near points of discontinuity. In this case, it is proposed a mathematical modification against the Fourier series which is called approximation of sigma which is symbolized by $\sigma$. That is obtained from transformation of partial sum of Fourier series $S_N(t)$ through the operation of mean value of integral, namely

$$\overline{S_N(t)} = \frac{N}{2\pi} \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} S_N(x) dx$$

(5.1)

with the function $\overline{S_N(t)}$ is flattening of $S_N(t)$ function centered at a point $t$, such that the modifications of Fourier series $S_N(t)$ of partial sum formed. Now, view of partial sum of the Fourier series is stated by equation (2.1), using equation (5.1) it will be obtained

$$\overline{S_N(t)} = \frac{N}{2\pi} \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} S_N(x) dx$$

$$= \frac{N}{2\pi} \left[ a_0 \frac{2\pi}{2N} + \sum_{k=1}^{\frac{N}{2}} \left( a_k \frac{\sin kx}{k} - b_k \frac{\cos kx}{k} \right) \right]_{-\frac{\pi}{N}}^{\frac{\pi}{N}}$$

$$= \frac{N}{2\pi} \left[ a_0 \frac{\pi}{N} a_0 + \sum_{k=1}^{\frac{N}{2}} \left( a_k \frac{\sin k \left( t + \frac{\pi}{N} \right) - \sin k \left( t - \frac{\pi}{N} \right)}{k} - b_k \frac{\cos k \left( t + \frac{\pi}{N} \right) - \cos \left( t - \frac{\pi}{N} \right)}{k} \right) \right]$$

$$= \frac{2a_0}{2} + \sum_{k=1}^{\frac{N}{2}} \left( \frac{\pi k}{n} \left( a_k \frac{\sin \left( \pi k \frac{n}{N} \right) \cos \left( \frac{\pi k}{n} \right) + 2b_k \frac{\sin \left( \frac{\pi k}{n} \right) \sin \left( \frac{k}{n} \right)}{N} \right) \right)$$

$$= \frac{2a_0}{2} + \sum_{k=1}^{\frac{N}{2}} \left( a_k \cos kt + b_k \sin kt \right)$$
So we get

\[
S_N(t) = \frac{a_0}{2} + \sum_{k=1}^{N} \sigma_k \left( a_k \cos(kt) + b_k \sin(kt) \right)
\]

with \( \sigma_k = \frac{\sin(\pi k N)}{\pi k N} \).

Note that \( \sigma_k \) inserted in the respective coefficients of Fourier series, so the term of \( \sin(kt) \) and \( \cos(kt) \) are both getting the factor. Through modification of the partial sum of Fourier series in equation (5.2), the overshoot near discontinuity points of function oscillation can be muted. So by using the equation (5.2) it will be obtained the form of \( \sigma \) approximation to the function given in the illustration above, namely

\[
S_N(t) = 4 \sum_{k=1}^{N} \frac{1}{\pi} \frac{\sin((2k+1)\frac{\pi}{N})}{(2k+1)} \sin((2k+1)t)
\]

Plot graphs of equations (5.3) is shown for \( N = 10 \) in Figure 6 below. Comparison of the method of \( \sigma \) approximation and Fejer for the same value of \( N \), i.e. \( N = 10 \) are shown in Figure 7.

The figures above show that the overshoot can be dampened. It is caused by influence of \( \sigma_k \) factor in Fourier series. Through this method, it takes the time more short when the function close to discontinuity point than Fejer method for the same partial sums. It means that the function of \( f \) can be closed by \( \sigma \) approximation with the short time.

6. Concluding Remarks

Qualitatively, to achieve convergence to the discontinuity points of a function which is approximated, approximation of sigma is better than Fejer method. However, in overcoming the overshoot near discontinuity points, Fejer method tend to be eliminating. Whereas the approximation method of sigma is muffled and for number of terms a partial sum, function of \( f \) can be closed in a short time.
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