ON THE GROUP OF SYMPLECTIC AUTOMORPHISMS OF \( \mathbb{C}P^m \times \mathbb{C}P^n \)

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1. Introduction

Let \( \omega_n \) be the standard symplectic form on \( \mathbb{C}P^n \), normalized in such a way that \([\omega_n] \) is Poincaré dual to a hyperplane. We denote the product \((\mathbb{C}P^m, \omega_m) \times (\mathbb{C}P^n, \omega_n)\), for \( m, n \geq 1 \), by \((\mathbb{P}^{mn}, \eta^{mn})\). Let \( \text{Diff}(\mathbb{P}^{mn}) \) be the group of diffeomorphisms with the \( C^\infty \)-topology, \( \text{Aut}(\mathbb{P}^{mn}, \eta^{mn}) \) the subgroup of symplectic automorphisms, and

\[
\beta_k : \pi_k(\text{Aut}(\mathbb{P}^{mn}, \eta^{mn})) \to \pi_k(\text{Diff}(\mathbb{P}^{mn}))
\]

the homomorphisms induced by inclusion.

**Theorem 1.1.** Let \( k \) be odd and \( \leq \max\{2m - 1, 2n - 1\} \). Then \( \beta_k \) is not surjective. In fact

\[
\text{rank}(\text{coker } \beta_k) \geq b_{2m+1-k}(\mathbb{P}^{mn}) - b_{2m+1-k}(\mathbb{C}P^m) + b_{2n+1-k}(\mathbb{P}^{mn}) - b_{2n+1-k}(\mathbb{C}P^n) > 0,
\]

where \( b_* \) are the Betti numbers.

For \( m = n = 1 \) the Theorem says that

(1.1) \hspace{1cm} \text{rank}(\text{coker } \beta_1) \geq 2.

This can be derived from a result of Gromov [5, 0.3.C] which says that the group of Kähler isometries of \( P_{11} \) (with respect to the standard metric) is a deformation retract of \( \text{Aut}(P_{11}, \eta_{11}) \). Since the isometry group is an extension of \( \mathbb{Z}/2 \) by \( SO(3) \times SO(3) \), it follows that \( \pi_1(\text{Aut}(P_{11}, \eta_{11})) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). The topology of \( \text{Diff}(P_{11}) \) is unknown, but by looking at its image in the space of continuous self-maps of \( P_{11} \) one can show that \( \pi_1(\text{Diff}(P_{11})) \) is a group of rank \( \geq 2 \), which implies (1.1).

Gromov’s theorem on \( \text{Aut}(P_{11}, \eta_1) \) and its cousin for \( (\mathbb{C}P^2, \omega_2) \) were the first non-trivial results about the topology of symplectic automorphism groups. More recently Abreu [1] and Abreu-McDuff [2] have studied the group of automorphisms of the symplectic structures \( \eta^{(\lambda)}_{11} = \lambda(\omega_1 \times 1) + 1 \times \omega_1 \) on \( P_{11} \) for \( \lambda \neq 1 \), using an approach suggested by Gromov [3, 2.4.C2]. The automorphism groups of blowups of \( \mathbb{C}P^2 \) are closely related to the spaces of...
symplectic embeddings of balls, which have been studied by Biran [3] and McDuff [10]. These results rely on specific properties of rational or ruled symplectic four-manifolds, and little is known outside this class (although see [14] for some information about \(\pi_0\) of the automorphism groups of symplectic four-manifolds). It seems that Theorem 1.1 provides the first examples of symplectic manifolds of dimension > 4 for which the map \(\pi_1(\text{Aut}) \to \pi_1(\text{Diff})\) is known not to be surjective. The fact that no such examples were known was pointed out to the author by F. Lalonde. Lalonde also suggested that it may be possible to construct such examples by exploiting the ‘rigidity theorem’ of [7] instead of the methods used here (any such examples would have positive first Betti number).

By definition, the symplectic automorphism group is the structure group of symplectic fibre bundles. The study of pseudo-holomorphic curves in each fibre of such a fibre bundle \(E\) yields ‘parametrized Gromov-Witten invariants’ which are multilinear maps on \(H^*(E)\). These invariants have been considered by Lê [8] and others. They satisfy (at least in principle) axioms similar to those of Kontsevich and Manin [6]. In this paper we argue that in certain cases the existence of such a system of invariants restricts the possibilities for what \(H^*(E)\) can be. To simplify the technical issues, we do not consider all Gromov-Witten invariants but just a single particularly simple one.

Although our argument relies on Gromov-Witten invariants, it does not suppose any knowledge of what the actual value of the invariants is. In this respect it resembles the recent ‘rigidity theorem’ of Lalonde-McDuff-Polterovich [7]. Note that the proof of the ‘rigidity theorem’ involves pseudo-holomorphic curves of a different kind, namely, pseudo-holomorphic sections of a fibre bundle whose base is a Riemann surface (in that case, \(S^2\)). It is possible that a combination of the two approaches would yield more information about symplectic automorphism groups.

This paper is structured as follows: the next section reviews some basic facts about the cohomology rings of fibrations. In section 3 we apply these considerations to the case where the fibre is \(P_{mn}\). Up to this point the argument is purely topological. The relevant Gromov-Witten invariant is introduced in sections 4 and 5. Section 6 contains the main computation. In the final section we derive Theorem 1.1 and discuss a related result and some possible further developments.

Acknowledgements. The author is indebted to François Lalonde, Dusa McDuff, and Richard Thomas for helpful discussions.

2. The cohomology rings of fibrations over spheres

Fix a field \(F\). By a graded \(F\)-algebra \(R = \bigoplus_{i \in \mathbb{Z}} R^i\) we mean one which is finite-dimensional, commutative (in the graded sense), and has a unit. All homomorphisms between graded algebras preserve units.
Definition 2.1. Let $R$ be a graded $\mathbb{F}$-algebra. A deformation of $R$ of dimension $d > 0$ consists of

1. a graded $\mathbb{F}$-algebra $\tilde{R}$,
2. an element $t \in \tilde{R}^d$ with $t^2 = 0$, such that the sequence

$$0 \rightarrow \tilde{R}/t\tilde{R} \rightarrow \tilde{R} \rightarrow \tilde{R}/t\tilde{R} \rightarrow 0$$

is exact, and
3. a homomorphism of graded algebras $j : \tilde{R} \rightarrow R$ which is surjective with kernel $t\tilde{R}$.

To be precise, these objects should be called ‘first order infinitesimal deformations’; we use the shorter name for brevity’s sake. The exactness of the sequence $\tilde{R}/t\tilde{R} \rightarrow \tilde{R} \rightarrow \tilde{R}/t\tilde{R}$ is equivalent to the flatness of $\tilde{R}$ as a module over $\mathbb{F}[\epsilon]/\epsilon^2$, where $\epsilon$ acts by multiplication with $t$.

Given two graded algebras $R_1, R_2$ and a homomorphism $f : R_1 \rightarrow R_2$, one defines a morphism over $f$ from a $d$-dimensional deformation $(\tilde{R}_1, t_1, j_1)$ of $R_1$ to a $d$-dimensional deformation $(\tilde{R}_2, t_2, j_2)$ of $R_2$ to be a homomorphism of graded algebras $\tilde{f} : \tilde{R}_1 \rightarrow \tilde{R}_2$ such that $\tilde{f}(t_1) = t_2$ and $j_2\tilde{f} = f j_1$.

In the special case $R_1 = R_2 = R$ and $f = \text{id}$, $\tilde{f}$ is called a morphism of deformations of $R$. All morphisms of deformations of $R$ are isomorphisms.

The set $\text{Def}_d(R)$ of isomorphism classes of $d$-dimensional deformations of $R$ carries a natural structure of an $\mathbb{F}$-vector space. The sum of two $d$-dimensional deformations $(\tilde{R}_i, t_i, j_i)$, $i = 1, 2$, is the deformation $(\tilde{R}, t, j)$ defined as follows: let $\Delta \subset \tilde{R}_1 \oplus \tilde{R}_2$ be the subalgebra of those $(x_1, x_2)$ such that $j_1(x_1) = j_2(x_2)$. Then $\tilde{R} = \Delta/(t_1, -t_2)\Delta$, $t = [t_1, 0] = [0, t_2] \in \tilde{R}$, and $j : \tilde{R} \rightarrow R$ is given by $j([x_1, x_2]) = j_1(x_1) = j_2(x_2)$.

Remark 2.2. There is a second and more explicit definition of $\text{Def}_d(R)$. Let $Z_d(R)$ be the space of $\mathbb{F}$-bilinear maps $\psi : R \times R \rightarrow R$ of degree $-d$ which are graded symmetric and satisfy

$$(-1)^{d \deg(x)} x \psi(y, z) - \psi(xy, z) + \psi(x, yz) - \psi(x, y)z = 0,$$

and $B_d(R) \subset Z_d(R)$ the subset of maps of the form

$$\psi(x, y) = (-1)^{d \deg(x)} x \xi(y) - \xi(xy) + \xi(x)y$$

where $\xi : R \rightarrow R$ is some linear map of degree $-d$. Then $\text{Def}_d(R) = Z_d(R)/B_d(R)$. The equivalence of this definition with the previous one is proved by choosing, for a deformation $(\tilde{R}, t, j)$ of $R$, an isomorphism of $\tilde{R}$ with $R[\epsilon]/\epsilon^2$ as an $\mathbb{F}[\epsilon]/\epsilon^2$-module. Then the product on $\tilde{R}$ has the form

$$(x_0 + \epsilon x_1)(y_0 + \epsilon y_1) = x_0y_0 + \epsilon((-1)^{d \deg(x_0)} x_0y_1 + x_1y_0 + \psi(x_0, y_0))$$

for some $\psi \in Z_d(R)$. The equivalence class of $\psi$ in $Z_d(R)/B_d(R)$ is independent of the choice of isomorphism. This alternative description shows that if $R$ is concentrated in even dimensions then $\text{Def}_d(R) = 0$ for all odd $d$. 
In the deformation theory of algebras (see [1] for a survey) $\text{Def}_*(R)$ is called the second Harrison cohomology group of the graded algebra $R$.

Let $M$ be a compact manifold. A smooth fibre bundle over a sphere $S^d$ with fibre $M$ consists of a manifold $E$, a submersion $\pi : E \rightarrow S^d$, and a diffeomorphism $i : M \rightarrow E_{b_0}$ for some fixed $b_0 \in S^d$. The cohomology of $E$ (unless otherwise stated, all cohomology groups have $\mathbb{F}$-coefficients) sits in a Wang sequence

$$
\cdots \xrightarrow{\delta} H^*(M) \xrightarrow{i_!} H^{*+d}(E) \xrightarrow{i^*} H^{*+d}(M) \xrightarrow{\delta} H^{*+1}(M) \xrightarrow{\cdots}
$$

where $i_!$ is the cohomology transfer or pushforward. Let $\epsilon$ be the standard generator of $H^d(S^d)$, and $t = \pi^*(\epsilon) \in H^d(E)$.

**Lemma 2.3.** If $i^*$ is onto then $(H^*(E), t, i^*)$ is a $d$-dimensional deformation of $H^*(M)$.

**Proof.** Because $t$ is Poincaré dual to the fibre of $E$, $i_!(i^*(x)) = xt$ for all $x \in H^*(E)$. Hence $\ker(i^*) = \text{im}(i_!) = \text{im}(ii^*) = tH^*(E)$. It remains to show that any $x$ with $tx = 0$ lies in $tH^*(E)$. Because $i_!$ is injective, $tx = vi^*(x) = 0$ implies that $i^*(x) = 0$, and we have already seen that $\ker(i^*) = tH^*(E)$. \qed

The isomorphism classes of smooth fibre bundles over $S^d$ with fibre $M$ form a group under the operation of fibre connected sum. This group is isomorphic to $\pi_{d-1}(\text{Diff}(M))$; the isomorphism is given by a clutching construction which associates to a map $\phi : (S^{d-1}, b_0) \rightarrow (\text{Diff}(M), \text{id})$ a fibre bundle $(E_{\phi}, \pi_{\phi}, i_{\phi})$. On the level of cohomology $\phi$ and $E_\phi$ are related in the following way: since $\phi$ can be written as a map $M \times S^{d-1} \rightarrow M$, it induces a homomorphism $\delta_{\phi} : H^{*+d-1}(M) \rightarrow H^*(M)$, and this homomorphism is the connecting map in the Wang sequence for $E_{\phi}$. In particular, the subgroup $G_{d-1} \subset \pi_{d-1}(\text{Diff}(M))$ of classes $[\phi]$ such that $\delta_{\phi} = 0$ corresponds to the isomorphism classes of fibre bundles $(E, \pi, i)$ for which $i^*$ is onto. Lemma 2.3 associates to any such fibre bundle an element of $\text{Def}_d(H^*(M))$. Since the sum of deformations imitates the behaviour of cohomology under fibre connected sum, this defines a homomorphism

$$
\alpha_{d-1} : G_{d-1} \rightarrow \text{Def}_d(H^*(M)).
$$

Although we have used smooth fibre bundles throughout, the construction does not really depend on the smooth structure of $M$. In fact the group of diffeomorphisms can be replaced by the bigger semigroup of homotopy equivalences $M \rightarrow M$. To take into account the smooth structure of $M$ would mean to consider deformations with certain distinguished elements (the Pontryagin classes). For $\mathbb{F} = \mathbb{Q}$ these questions can be treated in a much more satisfactory way in the framework of rational homotopy theory; see [13], pp. 313–314, 322–326.

**Example 2.4.** Let $R = \mathbb{F}[u]/u^{n+1}$ where $u$ has dimension 2. Since $R$ is concentrated in even dimensions, $\text{Def}_d(R) = 0$ for all odd $d$. Now fix an
even $d$. To any element $a \in R^{2n+2-d}$ one can associate a $d$-dimensional deformation $(\tilde{R}_a, t, j)$ of $R$. This deformation is defined in the following way: $a = \alpha u^{n+1-d/2}$ for some $\alpha \in \mathbb{F}$. Then

$$\tilde{R}_a = \mathbb{F}[\tilde{u}, t]/(\tilde{u}^{n+1} + \alpha t\tilde{u}^{n+1-d/2}, t^2),$$

and $j : \tilde{R}_a \to R$ is the algebra homomorphism with $j(\tilde{u}) = u$ and $j(t) = 0$. The sum of $\tilde{R}_a$ and $\tilde{R}_b$ is isomorphic to $\tilde{R}_{a+b}$.

Every $d$-dimensional deformation $(\tilde{R}, t, j)$ of $R$ is isomorphic to $\tilde{R}_a$ for some $a$. This is proved as follows: choose a $\tilde{u} \in \tilde{R}^2$ with $j(\tilde{u}) = u$. Since $u^{n+1} = 0$, $\tilde{u}^{n+1} = -t\tilde{a}$ for some $\tilde{a}$. At this point it is easy to construct a morphism of deformations of $R$ from $\tilde{R}_a$, where $a = j(\tilde{a})$, to $\tilde{R}$; recall that any such morphism is an isomorphism.

If $d \neq 2$ then $\tilde{R}_a$ is isomorphic to $\tilde{R}_b$ iff $a = b$. This can be proved either directly or by observing that for an arbitrary deformation, the element $\tilde{u}$ is unique and hence $a = j(\tilde{a})$ is an isomorphism invariant of the deformation. For $d = 2$, $\tilde{R}_a$ and $\tilde{R}_b$ are isomorphic iff $b - a$ is a multiple of $(n+1)u^n$. Therefore

$$\text{Def}_d(R) \cong \begin{cases} H^{2n+2-d}(R) & d \neq 2, \\ H^{2n}(R)/(n+1)H^{2n}(R) & d = 2. \end{cases}$$

We will now interpret this result geometrically. $R = H^*(\mathbb{C}P^n)$. Because the cohomology is concentrated in even dimensions, any smooth fibre bundle $(E, \pi, i)$ with fibre $\mathbb{C}P^n$ over $S^d$ satisfies the condition that $i^*$ is onto if $d$ is even. In other words $G_{d-1} = \pi_{d-1}(\text{Diff}(\mathbb{C}P^n))$ for all even $d$. Hence one has homomorphisms

$$\alpha_{d-1} : \pi_{d-1}(\text{Diff}(\mathbb{C}P^n)) \to \text{Def}_d(R) \cong \begin{cases} \mathbb{F} & d = 4, \ldots, 2n+2, \\ \mathbb{F}/(n+1)\mathbb{F} & d = 2. \end{cases}$$

If we restrict them to the subgroup $PU(n+1) \subset \text{Diff}(\mathbb{C}P^n)$, these homomorphisms are the Chern classes $c_2, \ldots, c_{n+1}$ and the first Chern class mod $n+1$. This follows from the standard formula for the cohomology ring of the projective bundle associated to a vector bundle.

The argument above is an instance of a general way of computing the deformation spaces of an algebra given by generators and relations. We will use this method again in the next section.

### 3. Deformations of $H^*(P_{mn})$

Let $R = H^*(P_{mn}) = \mathbb{F}[u, v]/(u^{m+1}, v^{n+1})$. Fix an even $d$. One can associate to any pair $(a, b) \in R^{2m+2-d} \oplus R^{2n+2-d}$ a $d$-dimensional deformation of $R$ in the following way: write $a = \sum_i a_i u^{i-d/2} v^{m+1-i}$ and $b = \sum_i b_i u^{i-d/2} v^{n+1-i}$
with \(a_i, b_i \in \mathbb{F}\). Let \(\tilde{R}_{a,b}\) be the algebra with generators \(\tilde{u}, \tilde{v}, t\) of degrees 2, 2, \(d\) and relations \(t^2 = 0\),

\[
\tilde{u}^{m+1} + \sum_i a_i t \tilde{u}^{i-d/2} \tilde{v}^{m+1-i} = 0, \quad \tilde{v}^{n+1} + \sum_i b_i t \tilde{u}^{i-d/2} \tilde{v}^{n+1-i} = 0.
\]

\(\tilde{R}_{a,b}, t,\) and the homomorphism \(j : \tilde{R}_{a,b} \to R\) with \(j(\tilde{u}) = u, j(\tilde{v}) = v,\) and \(j(t) = 0,\) define a deformation of \(R\).

Using the same idea as in Example 2.4, one can check that for \(d \neq 2\) these examples form a complete list, with no repetitions, of the \(d\)-dimensional deformations of \(R\). For \(d = 2\) there are again some isomorphisms between the \(\tilde{R}_{a,b}\) for different \((a, b)\). In this way one obtains isomorphisms

\[
\text{Def}_d(R) \cong \begin{cases} R^{2m+2-d} \oplus R^{2n+2-d} & d \neq 2, \\
R^{2m}/\mathbb{F}(m+1)u^m \oplus R^{2n}/\mathbb{F}(n+1)v^n & d = 2. \end{cases}
\]

As in the case of \(\mathbb{C}P^n\), the homomorphisms \(\alpha_{d-1}\) are defined on all of \(\pi_{d-1}(\text{Diff}(P_{mn}))\) if \(d\) is even. For the rest of this section let \(\mathbb{F} = \mathbb{Q}\).

**Proposition 3.1.** For any even \(d\) the homomorphism

\[
\alpha_{d-1} \otimes \text{id}_\mathbb{Q} : \pi_{d-1}(\text{Diff}(P_{mn})) \otimes \mathbb{Q} \to \text{Def}_d(R)
\]

is surjective.

Clearly the proof consists in finding sufficiently many examples of smooth fibre bundles. We will use the following construction: let \(\xi : \mathbb{C}P^n \times S^d\) be a complex vector bundle of rank \(n + 1\), such that \(\xi|\mathbb{C}P^n \times \{b_0\}\) is trivial, and \(\mathbb{P}(\xi)\) the associated bundle of projective spaces. The map \(\mathbb{P}(\xi) \to \mathbb{C}P^m \times S^d \to S^d\) makes \(\mathbb{P}(\xi)\) into a smooth fibre bundle with fibre \(\mathbb{C}P^m \times \mathbb{C}P^n\). \(H^*(\mathbb{P}(\xi))\) is a module over \(H^*(\mathbb{C}P^m \times S^d)\) with one two-dimensional generator \(v\) and one relation

\[
v^{n+1} + \sum_{i=0}^n c_i(\xi)v^{n+1-i} = 0.
\]

Since \(\xi|\mathbb{C}P^m \times \{b_0\}\) is trivial, the Chern classes can be written as \(c_i(\xi) = \gamma_i(u^{i-d/2} \otimes \epsilon),\) where \(\gamma_i \in \mathbb{Z},\) \(u\) is the generator of \(H^2(\mathbb{C}P^m)\), and \(\epsilon\) is the generator of \(H^2(S^d)\). It follows that the deformation of \(R\) determined by \(\mathbb{P}(\xi)\) is isomorphic to \(\tilde{R}_{a,b}\) with \(b = \sum \gamma_i u^{i-d/2} v^{n+1-i}.\) The Chern classes which can be nonzero are those with \(d/2 \leq i \leq \nu = \min\{n+1, m+d/2\}\). Now assume that for every such \(i\) there is a vector bundle \(\xi_i\) such that \(c_i(\xi_i) \neq 0\) and \(c_j(\xi_i) = 0\) for all \(j > i.\) Since \(R^{2n+2-d} = \bigoplus_{i=d/2}^{\nu} \mathbb{Q} u^{i-d/2} v^{n+1-i},\) it would follow that \(0 \oplus R^{2n+2-d} \subset \text{Def}_d(R)\) lies in the image of \(\alpha_{d-1} \otimes \text{id}_{\mathbb{Q}}.\) By exchanging \(m\) and \(n,\) one would obtain the same result for the complementary subspace \(R^{2m+2-d} \oplus 0,\) and this would prove Proposition 3.1. Hence it remains to construct the \(\xi_i.\) In order to fulfil \(c_j(\xi_i) = 0\) for \(j > i,\) we will take \(\xi_i\) to be the sum of a vector bundle \(\eta_i\) of rank \(i\) and a trivial bundle. All we need to prove is that
Lemma 3.2. For any integer $i$ with $d/2 \leq i \leq m + d/2$ there is a complex vector bundle $\eta_i : \mathbb{C}P^k \times S^d \rightarrow \mathbb{C}P^m \times S^d$ of rank $i$ such that $\eta_i|\mathbb{C}P^m \times \{b_0\}$ is trivial and $c_i(\eta_i) \neq 0$.

Proof. Vector bundles $\eta : \mathbb{C}P^k \times S^d \rightarrow \mathbb{C}P^m \times S^d$ of rank $i$ equipped with a trivialization of $\eta|\mathbb{C}P^k \times \{b_0\}$ are classified by the homotopy classes of (based) maps

$$\mathbb{C}P^k \rightarrow \Omega^{d-1}U(i) \simeq \Omega^dBU(i).$$

Since $\Omega^{d-1}U(i)$ is a topological group, two such maps can be multiplied. This multiplication associates to two vector bundles a third one, whose $i$-th Chern class is the sum of the $i$-th Chern classes of the original ones.

The obstruction to extending a map (3.2) from $\mathbb{C}P^k$ to $\mathbb{C}P^{k+1}$ is an element of the finite group $\pi_{2k+1}(\Omega^{d-1}U(i)) = \pi_{2k+d}(U(i))$. Hence the extension can always be carried out after replacing the original map by a positive multiple.

Start with a map $\mathbb{C}P^{i-d/2} \rightarrow \Omega^{d-1}U(i)$ which collapses everything except the top-dimensional cell to a point and represents a nontrivial element of $\pi_{2i-d}(\Omega^{d-1}U(i)) \otimes \mathbb{Q} \cong \pi_{2i-1}(\Omega^{d-1}U(i)) \otimes \mathbb{Q} \cong \mathbb{Q}$. This map corresponds to a vector bundle over $\mathbb{C}P^{i-d/2} \times S^d$ with nonzero $i$-th Chern class. After passing to a suitable multiple if necessary, one can extend the map to $\mathbb{C}P^m$.

## 4. Symplectic fibre bundles

In this section and the next one, $(M, \omega)$ is a compact symplectic manifold, $R = H^*(M)$, and $A \in H_2(M; \mathbb{Z}) = \text{im}(\pi_2(M) \rightarrow H_2(M; \mathbb{Z}))$ is a spherical homology class such that $\omega(A)$ is positive and generates the period group $\omega(H_2(M; \mathbb{Z})) \subset \mathbb{R}$. In other words, we assume that there is no $A' \in H_2(M; \mathbb{Z})$ with $0 < \omega(A') < \omega(A)$. Of course, such a class can only exist if the period group is discrete.

The Gromov-Witten invariant which counts rational pseudo-holomorphic $A$-curves with three marked points can be written as a bilinear map $\psi_A : R \times R \rightarrow R$. This map is (graded) symmetric, of degree $-2\langle c_1(M, \omega), A \rangle$, and has the following properties:

1. $x\psi_A(y, z) = \psi_A(xy, z) + \psi_A(x, yz) - \psi_A(x, y)z = 0$,
2. $\psi_A(1, x) = 0$ for all $x$, and
3. if $u \in R^2$ satisfies $\langle u, A \rangle = 0$ then $\psi_A(u, x) = 0$ for all $x$.

Define a product $*_A$ on $R \otimes \mathbb{F}[q]/q^2$ by

$$(x_0 + x_1q)_A(y_0 + y_1q) = (x_0y_0) + (x_0y_1 + x_1y_0 + \psi_A(x_0, y_0))q.$$ 

Property [1] is equivalent to the associativity of $*_A$. Property [2] says that $1 \in R$ is a unit for $*_A$, and property [3] can be written as $u*_A(x_0 + x_1q) = ux_0 + (ux_1)q$. 
Definition 4.1. Let \((\tilde{R}, t, j)\) be a deformation of \(R\). An extension of \(\psi_A\) to \(\tilde{R}\) is a bilinear map \(\tilde{\psi}_A : \tilde{R} \times \tilde{R} \to \tilde{R}\) of degree \(-2\langle c_1(M, \omega), A\rangle\) which is (graded) symmetric and has the following properties:

1. \(\tilde{\psi}_A(\tilde{x}, \tilde{y}) - \tilde{\psi}_A(\tilde{y}, \tilde{x}) = 0;\)
2. \(\tilde{\psi}_A(1, \tilde{x}) = \tilde{\psi}_A(\tilde{t}, \tilde{x}) = 0\) for all \(\tilde{x}\);
3. if \(u \in \tilde{R}^2\) satisfies \(\langle j(u), A\rangle = 0\) then \(\tilde{\psi}_A(u, \tilde{x}) = 0\) for all \(\tilde{x}\);
4. \(j(\tilde{\psi}_A(\tilde{x}, \tilde{y})) = \psi_A(j(\tilde{x}), j(\tilde{y}))\).

These properties can again be interpreted in terms of a suitably defined product \(\tilde{\ast}_A\) on \(\tilde{R} \otimes F[q]/q^2\). (4') says that \(j \otimes \text{id}_{F[q]/q^2} : (\tilde{R} \otimes F[q]/q^2, \tilde{\ast}_A) \to (R \otimes F[q]/q^2, \ast_A)\) is a ring homomorphism. The deformations of \(R\) which admit an extension of \(\psi_A\) form a subset \(\text{Def}_d(R, \psi_A) \subset \text{Def}_d(R)\). We leave it to the reader to verify that this is actually a linear subspace.

Let \((E, \pi, i)\) be a smooth fibre bundle over \(S^d\) with fibre \(M\). A fibrewise symplectic structure on \(E\) is a smooth family \(\Omega = (\Omega_b)\) of symplectic forms on the fibres \(E_b\) such that the cohomology class \([\Omega_b] \in H^2(E_b; \mathbb{R})\) is locally constant in \(b\). If in addition \(i^*(\Omega_b) = \omega\), we call \((E, \pi, i, \Omega)\) a symplectic fibre bundle with fibre \((M, \omega)\). The isomorphism classes of such bundles form a group under fibre connected sum, and this group is isomorphic to \(\pi_{d-1}(\text{Aut}(M, \omega))\). Forgetting \(\Omega\) corresponds to passing from \(\text{Aut}(M, \omega)\) to \(\text{Diff}(M)\).

Proposition 4.2. Let \((E, \pi, i, \Omega)\) be a symplectic fibre bundle over \(S^d\) \((d \geq 2)\) with fibre \((M, \omega)\), such that \(i^*\) is onto. Then the deformation of \(R\) determined by \(H^*(E)\) admits an extension of \(\psi_A\).

In the case when \(G_{d-1} = \pi_{d-1}(\text{Diff}(M))\), the situation can be summarized in the commutative diagram

\[
\begin{array}{ccc}
\pi_{d-1}(\text{Aut}(M, \omega)) & \longrightarrow & \pi_{d-1}(\text{Diff}(M)) \\
\downarrow & & \downarrow \alpha_{d-1} \\
\text{Def}_d(R, \psi_A) & \longrightarrow & \text{Def}_d(R).
\end{array}
\]

5. Proof of Proposition 4.2

We begin by recalling the definition of the Gromov-Witten invariant \(\psi_A\). Let \(J\) be an \(\omega\)-compatible almost complex structure on \(M\), \(M(A, J)\) the moduli space of \(J\)-holomorphic maps \(u : \mathbb{C}P^1 \to M\) which represent \(A\), and

\[
M_r(A, J) = M(A, J) \times_{PSL_2(\mathbb{C})} (\mathbb{C}P^1)^r
\]
(r = 0, 1, 2, . . . ) the associated moduli spaces of marked J-holomorphic curves. Our assumptions on A imply that any \( u \in \mathcal{M}(A, J) \) is simple (not multiply covered) and that A cannot be represented by a J-holomorphic cusp curve. It follows from the standard theory of pseudo-holomorphic curves (see [12] for an exposition) that for generic J the spaces \( \mathcal{M}_r(A, J) \) are compact oriented smooth manifolds. Let \( e_r : \mathcal{M}_r(A, J) \to M^r \) be the r-fold evaluation map. \( \psi_A \) is defined by

\[
\langle \psi_A(x, y)z, [M] \rangle = \langle x \times y \times z, (e_3)_*[\mathcal{M}_3(A, J)] \rangle
\]

for \( x, y, z \in H^*(M) \) and generic J. A cobordism argument [12, Theorem 3.1.3] shows that \( \psi_A \) is independent of the choice of J.

Now let \( (E, \pi, i, \Omega) \) be a symplectic fibre bundle over \( S^d \) (\( d \geq 2 \)) with fibre \( (M, \omega) \). The assumption that \( d \geq 2 \) implies that the homology of any fibre \( E_b \) can be canonically identified with the homology of \( E_{b_0} \) and hence (using i) with that of M. Therefore it makes sense to say that a map \( u : \mathbb{C}P^1 \to E_b \) represents \( A \in H_3^J(M; \mathbb{Z}) \). Let \( J = (J_b)_{b \in S^d} \) be a family of almost complex structures on the fibres \( E_b \) such that \( J_b \) is \( \Omega_b \)-compatible for all \( b \). Equivalently, one can think of \( J \) as an almost complex structure on the subbundle \( TE^v = \ker(T\pi) \subset TE \), with the property that \( \Omega(\cdot, J \cdot) \) is a metric on \( TE^v \). Let \( \mathcal{M}^p(A, J) \) be the space of pairs \( (b, u) \) where \( b \in S^d \) and \( u : \mathbb{C}P^1 \to E_b \) is a \( J_b \)-holomorphic map which represents \( A \) in the sense explained above. \( \mathcal{M}^p(A, J) \) is called the parametrized moduli space of rational J-holomorphic A-curves on \( E \). As before, we define

\[
\mathcal{M}^p_r(A, J) = \mathcal{M}^p(A, J) \times_{PSL_2(\mathbb{C})} (\mathbb{C}P^1)^r.
\]

The basic theory of pseudo-holomorphic curves carries over to the parametrized situation (this is familiar from the cobordism argument which we have mentioned above, in which one considers one-parameter families of almost complex structures on \( M \)). In particular, for generic \( J \) the spaces \( \mathcal{M}^p_r(A, J) \) are again compact smooth oriented manifolds; their dimension is

\[
\dim \mathcal{M}^p_r(A, J) = \dim E + 2\langle c_1(M, \omega), A \rangle + 2(r - 3).
\]

\( \mathcal{M}^p_r(A, J) \) comes with two canonical maps: the projection \( P_r : \mathcal{M}^p_r(A, J) \to S^d \) and the r-fold evaluation map \( e^p_r : \mathcal{M}^p_r(A, J) \to E^r \). Note that in general \( P_r \) is not a smooth fibration. We define the parametrized Gromov-Witten invariant \( \tilde{\psi}_A : H^*(E) \times H^*(E) \to H^*(E) \) by

\[
\langle \tilde{\psi}_A(x, y)z, [E] \rangle = \langle x \times y \times z, (e^p_3)_*[\mathcal{M}^p_3(A, J)] \rangle
\]

for generic J. Here \( [E] \) is the orientation induced by the symplectic orientation of \( M \) and the standard orientation of \( S^d \). An argument similar to that for \( \psi_A \) proves that \( \psi_A \) is independent of \( J \).

The three properties of \( \psi_A \) listed in section 3 are special cases of properties which are known to hold for far more general Gromov-Witten invariants. Nevertheless, we will outline proofs of them, making use of the special features of our case to simplify the argument.
(1) Consider the cycles $Z, Z' \subset M_4(A, J)$ consisting of those $[u, z_1, z_2, z_3, z_4]$ such that $z_1 = z_2$ or $z_3 = z_4$ (for $Z$) resp. $z_2 = z_3$ or $z_1 = z_4$ (for $Z'$). One can prove easily, using the cross-ratio $(z_1, z_2, z_3, z_4) \mapsto \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$, that $Z$ and $Z'$ are homologous. On the other hand
\[
\langle x \times y \times z \times w, (e_4)_*[Z] \rangle = \langle xy \times z \times w + x \times y \times zw, (e_3)_*[M_3(A, J)] \rangle
\]
and similarly
\[
\langle x \times y \times z \times w, (e_4)_*[Z'] \rangle = \langle \psi_A(xy, z)w + \psi_A(x, y)zw, [M] \rangle.
\]

(2) Let $\phi : M_3(A, J) \to M_2(A, J)$ be the map which forgets the first marked point. There is a commutative diagram
\[
\begin{array}{ccc}
M_3(A, J) & \xrightarrow{e_3} & M^3 \\
\downarrow{\phi} & & \downarrow{p} \\
M_2(A, J) & \xrightarrow{e_2} & M^2
\end{array}
\]
where $p$ is the projection. Because $\dim M_2(A, J) = \dim M_3(A, J) - 2$, $\phi_*[M_3(A, J)] = 0$. This implies that
\[
\langle 1 \times y \times z, (e_3)_*[M_3(A, J)] \rangle = \langle y \times z, p_*(e_3)_*[M_3(A, J)] \rangle = 0.
\]

(3) The forgetful map $\phi$ is a fibration with fibre $CP^1$. Let $\phi_!$ be the cohomology transfer.
\[
\langle \psi_A(u, y, z), [M] \rangle = \langle e_3^*(u \times y \times z), [M_3(A, J)] \rangle
\]
\[
= \langle e_3^*(u \times 1 \times 1)\phi^*(e_2^*(y \times z)), [M_3(A, J)] \rangle
\]
\[
= \langle \phi_!(e_3^*(u \times 1 \times 1))e_2^*(y \times z), [M_2(A, J)] \rangle
\]
for all $u, y, z \in H^*(M)$. Now assume that $u$ is two-dimensional. Since $e_3$ maps the homology class of the fibre of $\phi$ to $A \times [\text{point}] \times [\text{point}] \in H_2(M^3)$, one has
\[
\phi_! e_3^*(u \times 1 \times 1) = \langle u, A \rangle 1 \in H^0(M_2(A, J)).
\]
Hence $\psi_A(u, y)$ vanishes if $\langle u, A \rangle = 0$.

To prove Proposition 4.2 we have to show that $\tilde{\psi}_A$ is an extension of $\psi_A$ in the sense of Definition 11. The first three properties listed there are analogues of the properties of $\psi_A$, and the proofs given above can be easily adapted to the parametrized case. The remaining property is
\[
i^* (\tilde{\psi}_A(x, y)) = \psi_A(i^*(x), i^*(y)).
\]
As usual, let $t \in H^d(E)$ be the pullback of the fundamental class of $S^d$. $1 \times 1 \times t \in H^d(E^3)$ is Poincaré dual to the submanifold $C = E \times E \times E_0 \subset E^3$. 
Therefore
\[\langle i^*(\tilde{\psi}_A(x,y))i^*(z), [M] \rangle = \langle \tilde{\psi}_A(x,y)z, [E_0] \rangle\]
\[= \langle \tilde{\psi}_A(x,y)zt, [E] \rangle\]
\[= \langle x \times y \times zt, (e_3^p)_*[\mathcal{M}_3^p(A,J)] \rangle\]
\[= \langle x \times y \times z, (e_3^p)_*[\mathcal{M}_3^p(A,J)] \circ [C] \rangle\]
for all \(x, y, z \in H^*(E)\). Here \(\circ\) denotes the intersection product on \(H^*(E^3)\).

Now, if we assume that
\[
(e_3^p)_*[\mathcal{M}_3^p(A,J)] \circ [C] = (i \times i \times i)_*(e_3)_*[\mathcal{M}_3(A,J)],
\]
we obtain
\[
\langle i^*(\tilde{\psi}_A(x,y))i^*(z), [M] \rangle = \langle i^*(x) \times i^*(y) \times i^*(z), (e_3)_*[\mathcal{M}_3(A,J)] \rangle = \langle \psi_A(i^*(x), i^*(y))i^*(z), [M] \rangle.
\]
Since \(i^*(z)\) can be any element of \(H^*(M)\) (we have assumed in Proposition 1.2 that \(i^*\) is onto), this implies (5.1).

Equation (5.2) expresses the (rather obvious) fact that for suitable choices of \(J\) and \(\mathcal{J}\), the unparametrized moduli space \(\mathcal{M}_3(A,J)\) can be identified with a fibre of the projection \(P_3 : \mathcal{M}_3^p(A,J) \rightarrow S^d\). A more precise formulation of the argument goes as follows: choose \(J\) and \(\mathcal{J}\) in such a way that the moduli spaces \(\mathcal{M}_3(A,J)\) and \(\mathcal{M}_3^p(A,J)\) are smooth and \(i_*(J) = J_0\); this is possible. Then \(e_3^p\) is transverse to \(C\), and \((e_3^p)^{-1}(C) \subset \mathcal{M}_3^p(A,J)\), which consists of those elements \([b,u,z_1,z_2,z_3]\) with \(b = b_0\), can be identified with \(\mathcal{M}_3(A,J)\) in such a way that the diagram
\[
\begin{array}{ccc}
\mathcal{M}_3(A,J) & \xrightarrow{e_3} & M^3 \\
\downarrow & & \downarrow i \times i \times i \\
(e_3^p)^{-1}(C) & \xrightarrow{e_3^p} & E^3
\end{array}
\]
commutes. This proves (5.2) and hence completes the proof of Proposition 1.2.

6. The main computation

Let \(R_1, R_2\) be a pair of graded \(\mathbb{F}\)-algebras, and \(R = R_1 \otimes R_2\) their graded tensor product. Given a deformation \((\tilde{R}_1, t_1, j_1)\) of \(R_1\) and a deformation \((\tilde{R}_2, t_2, j_2)\) of \(R_2\) of the same dimension, one can define a deformation \((\tilde{R}, t, j)\) of \(R\) by \(\tilde{R} = \tilde{R}_1 \otimes_{\mathbb{F}[\epsilon]/\epsilon^2} \tilde{R}_2\) (where \(\epsilon\) acts on \(\tilde{R}_i\) by multiplication with \(t_i\)), \(t = t_1 \otimes 1 = 1 \otimes t_2\), and \(j = j_1 \otimes j_2\). This operation, which we call the exterior product of deformations, defines homomorphisms \(\times_d : \text{Def}_d(R_1) \otimes \text{Def}_d(R_2) \rightarrow \text{Def}_d(R)\). We call a deformation of \(R\) split if it is isomorphic to the exterior product of deformations of \(R_i\), that is, if its isomorphism class lies in the image of \(\times_d\).
Remark 6.1. In the alternative picture of Remark 2.2, the exterior product is defined by assigning to a pair of bilinear maps $\psi_i : R_i \otimes R_i \to R_i$ the product

$$\psi(x_1 \otimes x_2, y_1 \otimes y_2) = (-1)^{\deg(x_2) \deg(y_1)}(\psi_1(x_1, y_1) \otimes x_2 y_2 +$$

$$+ (-1)^{\deg(x_1 y_1)} x_1 y_1 \otimes \psi_2(x_2, y_2)).$$

The main result of this section is

**Proposition 6.2.** Let $(E, \pi, i, \Omega)$ be a symplectic fibre bundle over $S^d$, for some even $d$, with fibre $(Pmn, \eta mn)$. Then the deformation of $H^*(Pmn) = H^*(\mathbb{CP}^m) \otimes H^*(\mathbb{CP}^n)$ determined by $H^*(E)$ is split.

We will obtain this as a consequence of a more general result. To state that result we need to introduce one more algebraic notion. Let $f_i : R_i \to \tilde{R}$ be the obvious inclusions.

**Definition 6.3.** Let $(\tilde{R}, t, j)$ be a deformation of $R$. We say that $\tilde{R}$ is semi-split with respect to $R_i$ if there is a deformation $(\tilde{R}_i, t, j_i)$ of $R_i$ and a morphism $\tilde{f}_i : \tilde{R}_i \to \tilde{R}$ which lies over $f_i$.

A split deformation is clearly semi-split with respect to both $R_1$ and $R_2$. The converse it also true: a deformation of $R$ which is semi-split with respect to both $R_1$ and $R_2$ is split. Later, we will use the following elementary criterion:

**Lemma 6.4.** Let $(\tilde{R}, t, j)$ be a deformation of $R$. If $\tilde{R}$ has a subalgebra $\tilde{R}_1$ with $t \in \tilde{R}_1$, $j(\tilde{R}_1) = \text{im}(f_1)$ and $\dim(\tilde{R}_1 \cap t \tilde{R}) \leq \dim R_1$ then it is semi-split with respect to $R_1$. \hfill $\Box$

Let $M'$ be a compact manifold whose cohomology ring is generated by $H^2(M')$, and $\omega'$ a symplectic form on $M'$ such that $\omega'(H^2_2(M'; \mathbb{Z}) \subset \mathbb{Z}$. We consider the product $(M, \omega) = (M', \omega') \times (\mathbb{CP}^n, \omega_n)$ for some $n$. Let $A \in H^2_2(M; \mathbb{Z})$ be the homology class of a line in $\mathbb{CP}^n$. $A$ satisfies the conditions of section 3, and the corresponding Gromov-Witten invariant is well-known:

$$\psi_A(x \otimes u^i, y \otimes u^j) = \begin{cases} xy \otimes u^{i+j-n-1} & i + j \geq n + 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in H^*(M')$ and $0 \leq i, j \leq n$ (as usual, $u$ denotes the standard generator of $H^2(\mathbb{CP}^n)$).

**Proposition 6.5.** Let $(E, \pi, i, \Omega)$ be a symplectic fibre bundle with fibre $(M, \omega)$ over $S^d$ for some even $d$. Then the deformation of $R = H^*(M)$ given by $H^*(E)$ is semi-split with respect to $H^*(M')$.

Setting $(M', \omega') = (\mathbb{CP}^m, \omega_m)$, it follows that a symplectic fibre bundle over $S^d$ with fibre $(Pmn, \eta mn)$ determines a deformation of $H^*(Pmn)$ which
is semi-split with respect to $H^*(\mathbb{CP}^m)$. Since the situation is symmetric with respect to $m$ and $n$, it follows that the deformation is also semi-split with respect to $H^*(\mathbb{CP}^n)$, hence split. Therefore Proposition 6.3 implies Proposition 6.2.

**Proof of Proposition 6.3.** Note that $i^*$ is onto because $d$ is even and $H^*(M)$ is concentrated in even dimensions. Let $(R, t, j)$ be the deformation of $R$ determined by $H^*(E)$. Proposition 4.2 ensures the existence of an extension $\tilde{\psi}_A$ of $\psi_A$ to $\tilde{R}$. Choose a basis $\xi_1, \ldots, \xi_g$ of $H^2(M')$ and elements $\tilde{\xi}_1, \ldots, \tilde{\xi}_g, \tilde{u} \in \tilde{R}^2$ such that $j(\tilde{\xi}_i) = \xi_i \otimes 1$ and $j(\tilde{u}) = 1 \otimes u$. Let $\tilde{R}_1 \subset \tilde{R}$ be the subalgebra generated by $t, \tilde{\xi}_1, \ldots, \tilde{\xi}_g$. Clearly $j(\tilde{R}_1) = H^*(M') \otimes 1$. We want to apply Lemma 6.4 to this subalgebra. What remains to be shown is that the dimension of $\tilde{R}_1 \cap tR$ is not greater than the dimension of $H^*(M')$. This is a consequence of the following

**Assertion.** Every $x \in \tilde{R}_1 \cap tR$ is of the form $x = ty$ for some $y$ such that $j(y) \in H^*(M') \otimes 1$.

The Assertion is proved in two steps:

**Step 1:** Every $x \in \tilde{R}_1$ satisfies $\tilde{\psi}_A(x, \tilde{u}^n) = 0$. Since $(j(\tilde{\xi}_i), A) = (\xi_i, A) = 0$, $\tilde{\psi}_A(\xi_i, x) = 0$ for all $x$ by property (3') of $\tilde{\psi}_A$. Therefore the equation

$$\tilde{\xi}_i \tilde{\psi}_A(y, \tilde{u}^n) - \tilde{\psi}_A(\tilde{\xi}_i y, \tilde{u}^n) + \tilde{\psi}_A(\tilde{\xi}_i, y \tilde{u}^n) - \tilde{\psi}_A(\tilde{\xi}_i, y) \tilde{u}^n = 0,$$

which is a special case of property (1) of $\tilde{\psi}_A$, simplifies to

$$(6.2) \quad \tilde{\psi}_A(\tilde{\xi}_i, y, \tilde{u}^n) = \tilde{\xi}_i \tilde{\psi}_A(y, \tilde{u}^n).$$

Note that by property (2) $\tilde{\psi}_A(1, \tilde{u}^n) = \tilde{\psi}_A(t, \tilde{u}^n) = 0$. Arguing inductively using (6.2) it follows that $\tilde{\psi}_A(x, \tilde{u}^n) = 0$ for all $x \in \tilde{R}_1$.

**Step 2.** If $x = ty \in tR$ satisfies $\tilde{\psi}_A(x, \tilde{u}^n) = 0$, then $j(y) \in H^*(M') \otimes 1$. Since $\tilde{\psi}_A(t, w) = 0$ for all $w$, the equation $t\tilde{\psi}_A(y, \tilde{u}^n) - \tilde{\psi}_A(x, \tilde{u}^n) + \tilde{\psi}_A(t, y \tilde{u}^n) - \tilde{\psi}_A(t, y) \tilde{u}^n = 0$ reduces to

$$(6.3) \quad \tilde{\psi}_A(x, \tilde{u}^n) = t \tilde{\psi}_A(y, \tilde{u}^n).$$

Now write $j(y) = \sum_{i=0}^n c_i \otimes u^i$ with $c_i \in H^*(M')$, and choose $\tilde{c}_0, \ldots, \tilde{c}_n \in \tilde{R}$ with $j(\tilde{c}_i) = c_i$. Using property (4) of $\tilde{\psi}_A$ and the formula (6.1) for $\tilde{\psi}_A$ one sees that

$$j(\tilde{\psi}_A(y, \tilde{u}^n)) = \psi_A(j(y), u^n) = \sum_{i=1}^n c_i \otimes u^{i-1}.$$

Therefore $\tilde{\psi}_A(y, \tilde{u}^n) = \sum_{i=1}^n \tilde{c}_i \tilde{u}^{i-1} + tz$ for some $z \in \tilde{R}$. Using (6.3) it follows that

$$\tilde{\psi}_A(x, \tilde{u}^n) = t \left( \sum_{i=1}^n \tilde{c}_i \tilde{u}^{i-1} \right).$$
If this vanishes then $t\tilde{c}_i = 0$ and hence $c_i = 0$ for all $i > 0$, which means that $j(y) \in H^*(M') \otimes 1$.

7. Conclusion

We can now prove Theorem 1.1. Set $F = \mathbb{Q}$, and let $R = H^*(P_{mn})$. Fix an even $d$, and let $\text{Def}_2(R) \subset \text{Def}_d(R)$ be the subspace of split deformations. According to Proposition 6.2, there is a commutative diagram

$$
\pi_{d-1}(\text{Aut}(P_{mn}, \eta_{mn})) \otimes \mathbb{Q} \xrightarrow{\beta_{d-1} \otimes \text{id}_\mathbb{Q}} \pi_{d-1}(\text{Diff}(M)) \otimes \mathbb{Q}
$$

$$
\text{Def}_2(R) \xrightarrow{\alpha_{d-1} \otimes \text{id}_\mathbb{Q}} \text{Def}_d(R)
$$

Proposition 3.1 shows that $\alpha_{d-1} \otimes \text{id}_\mathbb{Q}$ is onto. Therefore

$$
\text{rank(coker } \beta_{d-1}) \geq \dim \mathbb{Q} \text{Def}_d(R) - \dim \mathbb{Q} \text{Def}_2(R).
$$

Assume first that $d = 2$. Using Example 2.4 and the results of section 3, one can identify the subspace $\text{Def}_2(R) \subset \text{Def}_d(R)$ with $H^{2m+2-d}(\mathbb{CP}^m) \otimes H^{2n+2-d}(\mathbb{CP}^n) \subset R^{2m+2-d} \otimes R^{2n+2-d}$. Therefore

$$
\text{rank(coker } \beta_{d-1}) \geq b_{2m+2-d}(P_{mn}) - b_{2m+2-d}(\mathbb{CP}^m) + b_{2n+2-d}(P_{mn}) - b_{2n+2-d}(\mathbb{CP}^n).
$$

This is positive as long as $d \leq \max\{2m, 2n\}$. For $d = 2$ one has $\text{Def}_2(R) = 0$ (by Example 2.4) and hence $\text{rank(coker } \beta_1) \geq \dim \mathbb{Q} \text{Def}_2(R) = \dim \mathbb{Q} R^{2m} + \dim \mathbb{Q} R^{2n} - 2 = b_{2m}(P_{mn}) - b_{2m}(\mathbb{CP}^m) + b_{2n}(P_{mn}) - b_{2n}(\mathbb{CP}^n).

One can ask what happens if one considers $P_{mn}$ with a weighted symplectic form $\eta_{mn} = \lambda(\omega_n \times 1) + 1 \times \omega_n$ where $\lambda > 1$. Proposition 6.2 is no longer true for symplectic fibre bundles with fibre $(P_{mn}, \eta_{mn}^{(\lambda)})$, at least if $\lambda$ is sufficiently large. Indeed, all the examples constructed in Proposition 1.1, which were of the form $E = \mathbb{P}(\xi)$ for some $\xi \to \mathbb{CP}^m \times S^d$, admit a fibrewise symplectic structure modelled on $\eta_{mn}^{(\lambda)}$ for some large $\lambda$, while the corresponding deformations are not always split. If $\lambda$ is integral, we can still use Proposition 6.5 to obtain some information about symplectic fibre bundles with fibre $(P_{mn}, \eta_{mn}^{(\lambda)})$. This leads to the following

**Theorem 7.1.** Let $\lambda \geq 2$ be an integer. Then the homomorphism

$$
\beta_k^{(\lambda)} : \pi_k(\text{Aut}(P_{mn}, \eta_{mn}^{(\lambda)})) \to \pi_k(\text{Diff}(P_{mn}))
$$

induced by inclusion is not surjective for any odd $k$ with $1 \leq k \leq 2m - 1$. In fact $\text{rank(coker } \beta_k^{(\lambda)}) \geq b_{2m+1-k}(P_{mn}) - b_{2m+1-k}(\mathbb{CP}^m) > 0$.

We omit the proof, which is similar to that of Theorem 1.1. For $\lambda \notin \mathbb{N}$ it becomes possible that the class $A \in H_2^{(\lambda)}(P_{mn}; \mathbb{Z})$ coming from the generator of $H_2(\mathbb{CP}^n; \mathbb{Z})$ can be represented by a rational pseudo-holomorphic
cusp-curve. However, our argument would still work if one could prove that these cusp-curves can be removed by perturbing the almost complex structure. More precisely, one needs to show that no such cusp-curves occur in a generic family of compatible almost complex structures depending on a certain number of parameters.

**Example 7.2.** Let \((E, \pi, i, \Omega)\) be a symplectic fibre bundle over \(S^2\) with fibre \((P_{21}, \eta_{21})\). Choose a family \(J = (J_b)_{b \in S^2}\) of compatible almost complex structures on its fibres. If \(A\) is represented by a rational \(J_b\)-holomorphic cusp-curve for some \(b\), there must be \(A_1, A_2 \in H^2_c(P_{21}; \mathbb{Z})\) with \(A_1 + A_2 = A\), such that \(\omega(A_1), \omega(A_2) > 0\) and \(\mathcal{M}(A_1, J), \mathcal{M}(A_2, J) \neq \emptyset\). The virtual dimensions of these moduli spaces are \(v \dim \mathcal{M}(A_1, J) = 8 + 2\langle c_1, A_1 \rangle\), \(v \dim \mathcal{M}(A_2, J) = 8 + 2\langle c_1, A_2 \rangle\). A pair \((A_1, A_2)\) such that both dimensions are nonnegative exists iff there are integers \(k, l\) with \(1 > \lambda k + l > 0\) and \(6 \geq 3k + 2l \geq -4\). An elementary argument shows that this is possible only if \(\lambda < 3\). Therefore it is plausible that if \(\lambda \geq 3\) then the deformation of \(H^*(P_{21})\) determined by \(H^*(E)\) is semi-split with respect to \(H^*(\mathbb{C}P^2)\). To make this argument rigorous, one has to deal with the problem of multiply-covered pseudo-holomorphic maps of negative Chern number, in the manner of [9], [13], [15].

It is interesting to compare our approach with the geometric methods used by Gromov and others in the four-dimensional case. For instance, consider the following result, which is a version of the main step in the proof of Gromov’s theorem on \(\text{Aut}(P_{11}, \eta_{11})\).

**Proposition 7.3** (Gromov). Any symplectic fibre bundle over \(S^d\) \((d \geq 2)\) with fibre \((P_{11}, \eta_{11})\) is the fibre product of two symplectic fibre bundles with fibre \((\mathbb{C}P^1, \omega_1)\).

Proposition 6.2 is a higher-dimensional analogue of this in which the splitting takes place only on the level of cohomology rings. The reason for the stronger nature of the four-dimensional result is that the geometric behaviour of pseudo-holomorphic curves on a symplectic four-manifold is tightly controlled by the positive intersection theorem and the adjunction inequality [13], both of which are used in the proof of Proposition 7.3. We end this discussion with a general conjecture suggested by Proposition 6.3.

**Conjecture 7.4.** Let \((M, \omega)\) and \((N, \eta)\) be compact symplectic manifolds. Let \((E, \pi, i)\) be a smooth fibre bundle over \(S^d\) with fibre \(M \times N\), such that \(i^* : H^*(E) \to H^*(M \times N)\) is surjective. Assume that for all sufficiently large \(\lambda\) there is a fibrewise symplectic structure \(\Omega^{(\lambda)}\) on \(E\) such that \(i^*(\Omega^{(\lambda)}) = \lambda(\omega \times 1) + 1 \times \eta\). Then the deformation of \(H^*(M \times N)\) determined by \(H^*(E)\) is semi-split with respect to \(H^*(M)\).

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