SIMPLE SURFACE SINGULARITIES

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ABSTRACT. By the famous ADE classification rational double points are simple. Rational triple points are also simple. We conjecture that the simple normal surface singularities are exactly those rational singularities, whose resolution graph can be obtained from the graph of a rational double point or rational triple point by making (some) vertex weights more negative. For rational singularities we show one direction in general, and the other direction (simplesness) within the special classes of rational quadruple points and of sandwiched singularities.

INTRODUCTION

Simple hypersurface singularities were classified by Arnol’d in the famous ADE list [1]. In the surface case these are exactly the rational double points. In Giusti’s list [6] of simple isolated complete intersection singularities no surface singularities occur. They do appear in the classification of simple determinantal codimension two singularities by Frühbis-Krüger and Neumer [4]; the simple surface singularities are the rational triple points.

In this paper we address the question:

Question. What are the simple normal surface singularities?

Simple means here that there occur only finitely many isomorphism classes in the versal deformation. The known cases suggest that simple singularities are rational. There exist certainly simple singularities of higher embedding dimension, as all two-dimensional quotient singularities are simple (they deform only in other quotients [3]). But not all rational singularities are simple: already for rational quadruple points there can be a cross ratio involved in the exceptional divisor. To exclude this we want the singularity to be taut. Furthermore it is natural to expect that the simple ones are quasi-homogeneous. The quasi-homogeneous taut singularities make up the parts I, II and III in Laufer’s list [12]. Their graphs have at most one vertex of valency three and no higher valencies. There is a simple characterisation in terms of rational double point and triple point graphs. Using it we can formulate the conjectural answer to our question as follows.

Conjecture. Simple normal surface singularities are exactly those rational singularities, whose resolution graphs can be obtained from the graphs of rational double points and rational triple points by making (some) vertex weights more negative.
A rigid singularity, one having no nontrivial deformations at all, is certainly simple. It is an old unsolved question whether rigid normal surface singularities (or rigid reduced curve singularities) exist. If they do, they are rather special. Our conjecture says not only that they do not exist, but even more, that there are no singularities with nontrivial infinitesimal deformations, which however all are obstructed.

Without the condition of normality there are rigid singularities. In fact, the standard example of a nonnormal isolated singularity, two planes in 4-space meeting transversally in one point, is rigid and therefore simple.

The problem in studying rational singularities of multiplicity at least four is that their deformation space has (in general) many components, and for only one, the Artin component, one has good methods to study adjacencies: it suffices to look at deformations of the resolution. In the case of almost reduced fundamental cycle there is even a complete description of the adjacencies [13]. For special classes we know more, mainly by the work of De Jong and Van Straten. This includes rational quadruple points [8] and sandwiched singularities [9].

Our results on the conjecture are restricted to rational singularities. We prove one direction of the conjecture for rational singularities in general, that those not covered by the conjecture are not simple. We show that they deform on the Artin component to a singularity with a cross ratio on the exceptional set. Proving simpleness is more difficult. We succeed in the cases where there are good methods to study deformations.

Our result is:

**Proposition.** For the following classes of rational singularities the conjecture is true, that is, the simple ones are those obtained from rational double and triple points:

- quotient singularities,
- rational quadruple points,
- sandwiched singularities.

As singularities of type Laufer’s III.5 (see Table 1) are deformations of those of type III.6, and type III.7 and III.8 deformations of type III.9, it remains to prove that singularities with graph of type III.9, of type III.6 and those of type III.4, which are not sandwiched, are simple.

The contents of this paper is as follows. We first recall the concepts of simpleness and modality. Then we discuss and state the conjecture, in particular why rationality is to be expected. We use the results of Laufer [13] and Wahl [19] to determine adjacencies on the Artin component, giving one direction of the conjecture for rational singularities. Thereafter we show that the conjecture holds for rational quadruple points. The last section proves the result for sandwiched singularities.
1. MODALITY

Modality was introduced by Arnol’d in connection with the classification of singularities of functions under right equivalence. It has been generalised to arbitrary actions of algebraic groups by Vinberg [18]. Wall [20] described two possible generalisations for use in other classification problems in singularity theory.

The first one is developed in detail by Greuel and Nguyen [7]. Let an algebraic group $G$ act on an algebraic variety $X$. Consider a Rosenlicht stratification of $X$, that is a stratification with locally closed $G$-invariant subvarieties $X_i$, for which a geometric quotient $X_i/G$ exists. The modality $\text{mod}(X,G)$ of the action is the maximal dimension of the quotients $X_i/G$. For an open subset $U$ the modality $\text{mod}(U,G)$ is the maximal dimension of the images of $U \cap X_i$ in $X_i/G$. Finally, the modality $\text{mod}((X,x),G)$ of a point $x \in X$ is the minimum of $\text{mod}(U,G)$ over open neighbourhoods $U$ of $x$. The modality is independent of the Rosenlicht stratification used. One obtains right and contact modality of a germ by passing to the space of $k$-jets, for sufficiently large $k$.

It is not obvious how to use this approach for modality of general singularities, which are not complete intersections. Wall’s second description starts from the versal deformation $X_\Sigma \to S$ of a given singularity $X_0$. The modality is the largest number $m$ such that in any neighbourhood of $0 \in S$ there exists an $m$-dimensional analytic subset such that at most finitely many points represent analytically isomorphic singularities.

The modality considered here is sometimes referred to as outer modality. For right equivalence Gabrielov [5] has shown that outer modality and inner modality coincide, where inner or proper modality is the dimension of the $\mu$-constant stratum in the semi-universal unfolding. For contact equivalence of hypersurface germs the inner modality is in general smaller than the outer modality. For general deformations one can use as inner modality the dimension of the modular stratum, as introduced by Palamodov [16]: the maximal subgerm of the base of the versal deformation, over which the deformation is universal.

A singularity is simple if it is 0-modal. By the above discussion this means the following.

**Definition 1.1.** A singularity is *simple* if it only can deform into finitely many different isomorphism classes of singularities.

A singularity is not simple if it deforms into a singularity with positive inner modality, that is, a singularity with moduli.

2. THE CONJECTURE

The simple 2-dimensional hypersurface singularities are the rational double points. No complete intersection surface singularity, which is not a hypersurface, is simple: such a singularity deforms always into the intersection of two quadrics, so in a simple elliptic singularity of type $\tilde{D}_5$. In the classification of simple
determinantal codimension two singularities by Frühbis-Krüger and Neumer [4] the surface singularities are the rational triple points; in particular non-rational singularities of this type are not simple.

Minimally elliptic singularities deform into simple elliptic singularities of the same multiplicity [10], and are therefore never simple. A general elliptic singularity has on the resolution a minimally elliptic cycle, such that its complement consists of rational cycles. We expect that a deformation exists to a singularity with an elliptic curve as minimally elliptic cycle and the same complementary rational cycles. This would imply that elliptic singularities are never simple. From Laufer’s classification of taut and pseudotaut singularities [12] it follows that a ‘general’ non-rational singularity has moduli. To explain what ‘general’ means we first recall the definition of (pseudo)taut.

**Definition 2.1.** A normal surface singularity is **taut**, if every other singularity with the same resolution graph, is isomorphic to it. A singularity is **pseudotaut** if there are only finitely many isomorphism classes of singularities with the same resolution graph.

The only non-rational (pseudo)taut singularities are the minimally elliptic singularities whose graph is a Kodaira graph, and those are not simple.

Actually, Laufer defines a singularity as pseudotaut if there are countably many isomorphism classes with the same graph, but he proves that there are then only finitely many, as result of a more precise description. Given a pseudotaut graph, all analytic types occur in a certain versal deformation: construct by the standard plumbing construction a manifold $M$ with exceptional divisor $E$, and consider deformations of the pair $(M, E)$, which are locally trivial deformations of $E$. For the general fibre all such deformations are in fact trivial. Conversely, any singularity whose resolution can be deformed in this way into one with only trivial such deformations, is pseudotaut. For all other singularities the resolution has moduli. This does not imply that the singularity itself has moduli, as deformations of the resolution blow down to deformations of the singularity if and only if the geometric genus $p_g$ is constant. So ‘general’ has to mean that $p_g$ has the lowest possible value.

We conjecture that simple singularities are rational. As such singularities are always smoothable, the conjecture says in particular that no rigid normal surface singularities exist. Rigid singularities, that is singularities with no deformations at all, not even infinitesimally, are trivially simple. In fact, the (non)-existence of rigid normal surface singularities is an old open problem.

But not all rational singularities are simple. Already for quadruple points we find singularities with a modulus in the resolution. An example is the $n$-star singularity of [8], which has a star-shaped graph with a central $(-4)$-vertex and four arms of $(-2)$’s of equal length $n - 1$. On the other hand, there exist simple singularities of arbitrary multiplicity, as quotient singularities are simple: they deform only into other quotient singularities [3].
What quotient singularities and rational triple points have in common, is that they are taut. For rational singularities deformations of the resolution with the same graph blow down to equisingular deformations of the singularity. Therefore pseudotautness is a necessary condition for simplicity. Without using the classification we show below (Proposition 3.6) that pseudotaut but not taut rational singularities are not simple. From the same Proposition it follows that simple rational surface singularities are quasi-homogeneous.

Conjecture 2.2. A normal surface singularity is simple if and only if it is taut and quasi-homogeneous.

Remark 2.3. Quasi-homogeneity is a feature of many lists of simple objects, but in each case it is the result of the classification. In fact, the list of simple map germs \((C,0) \to (C^2,0)\) [2] contains also germs which are not quasi-homogeneous.

The quasi-homogeneous taut singularities make up the parts I, II and III in Laufer’s list of graphs [12]. We give the list in Table 1. It is organised as to have no duplicates. The meaning of the symbols is the following. A dot \(\bullet\) denotes a vertex of any weight \(-b \leq -2\), a dot \(\bullet -2\) has weight exactly \(-2\), whereas a square \(\square\) is a vertex of any weight \(-b \leq -3\), less than \((-2)\). A chain \(\cdots \bullet \cdots \bullet\) is a chain of vertices of any length \(k \geq 0\). So the first entry of the Table gives exactly the cyclic quotient singularities, with the cone over a rational normal curve being the case \(k = 0\). The second entry gives all other quasi-homogeneous taut singularities with reduced fundamental cycle.

Using the characterisation of the graphs in the list, already observed by Laufer, we get the equivalent form of our conjecture, which was stated in the Introduction.

Conjecture 2.4. Simple normal surface singularities are exactly those rational singularities, whose resolution graphs can be obtained from the graphs of rational double points and rational triple points by making (some, or none) vertex weights more negative.

We will say shortly that the singularities are obtainable from rational double and triple points. As the singularities under consideration are taut, their analytic structure is determined by the graph. For general singularities we get a relation between analytic structures by the following construction.

Definition 2.5. A singularity \(Y\) is obtainable from a singularity \(X\), if the exceptional set on its minimal resolution \(\tilde{Y}\) is the strict transform of the exceptional set \(E\) of the minimal resolution \(\tilde{X} \to X\) under a blow up \(\tilde{X}' \to \tilde{X}\) in smooth points of \(E\).

In this construction we allow that points are infinitely near, that is, we blow up successively in smooth points of the strict transform of \(E\). In the terminology of [14] the singularity \(Y\) is a sandwiched singularity relative to \(X\), but of a special form, in that \(X\) and \(Y\) have the same underlying graph.
Table 1. Quasi-homogeneous taut surface singularities
The deformation space of a rational surface singularity has a special component, of largest dimension, over which a simultaneous resolution exists (after base change). Therefore the adjacencies on this component can be found from deformations of the resolution. These were studied by Laufer and Wahl [13, 19].

Let \( X_T \to T \) be a one-parameter deformation with simultaneous resolution \( \tilde{X}_T \to T \). Consider an irreducible component \( E_T \to T \) of the exceptional set, mapping surjectively onto \( T \). Then \( E_T \) is flat and proper over \( T \). The fibre \( E_t \) for \( t \neq 0 \) is irreducible [13, Thm 2.1]; in fact, if there are several components, there is monodromy and we find two components, which are homologous in \( \tilde{X}_T \), contradicting the negative definiteness of the intersection form on \( X_0 \). We say that \( D \) lifts to the deformation \( \tilde{X}_T \to T \). Laufer and Wahl give sufficient conditions on cycles \( D \) for the existence of a deformation to which \( D \) lifts. To formulate the result we need some definitions.

**Definition 3.1** ([19]). A cycle \( D > 0 \) on the minimal resolution of a rational singularity is a **positive root** if \( p_a(D) = 0 \).

For the ADE singularities the positive roots correspond exactly to the positive roots of the root system of the same type A, D or E under the identification of the resolution graph with the Dynkin diagram.

**Lemma 3.2** ([13, Lemma 3.1]). A cycle \( D > 0 \) is a positive root if and only if \( D \) is part of a computation sequence.

*Proof*. In a computation sequence the genus cannot go down: let \( Z_{j+1} = Z_j + E_{i(j)} \) be a step in the sequence, where \( Z_j \cdot E_{i(j)} > 0 \), then

\[
p_a(Z_{j+1}) = p_a(Z_j) + p_a(E_{i(j)}) + Z_j \cdot E_{i(j)} - 1 \geq p_a(Z_j).
\]

So for a rational singularity \( p_a(Z_j) = 0 \) in every step. Conversely, let \( C \leq D \) be a cycle, which is part of a computation sequence and is maximal with respect to this property. This implies that \( C \cdot E_i \leq 0 \) for every \( E_i \) in the the support of \( D - C \). For every cycle \( A \) one has \( p_a(A) \leq 0 \). If the support of \( D - C \) is not empty, then

\[
0 = p_a(D) = p_a(C) + p_a(D - C) + C \cdot (D - C) - 1 < 0.
\]

This contradiction shows that \( C = D \). \( \square \)

**Definition 3.3.** A cycle \( D = \sum d_i E_i \) on the exceptional set of the minimal resolution of a normal surface singularity is **almost reduced**, if it is reduced at the non-\((-2)\)'s, i.e., \( d_i = 1 \) if \( E_i^2 < -2 \).
This condition is important, as it implies vanishing of the obstructions to deform $D$, which lie in $H^1(T_D)$, because $h^1(T_D) = h^1(\mathcal{O}_{D,D_{\text{red}}}(D)) = (D - D_{\text{red}}) \cdot K$

[19, (2.14)].

**Definition 3.4** ([13, Definition 3.5]). A collection of cycles $D_1, \ldots, D_m$ on the exceptional set the minimal resolution of a rational surface singularity is **integrally minimal**, if each $D_i$ is a positive root, the cycle $D = D_1 + \ldots + D_m$ is almost reduced, and no other collection $C_1, \ldots, C_m$ of $m$ positive roots can generate the $D_i$ with non-negative integral coefficients.

**Theorem 3.5.** Let $D_1, \ldots, D_m$ be an integrally minimal collection of cycles on the minimal resolution $\bar{X}_0$ of a rational surface singularity with $D = \sum D_i$ itself a positive root. Then there exists a 1-parameter deformation $\bar{X}_T \to T$ such that the exceptional set $E_t$ of $\bar{X}_t$, $t \neq 0$, has a decomposition in $m$ irreducible components $E_{t,i}$ with $E_{t,i}$ homologous to $D_i$ in $\bar{X}_T$.

If the fundamental cycle $Z$ of $\bar{X}_0$ is itself almost reduced, then every adjacency arises this way.

**Proof.** The result is contained in [13, Thm 3.12]. Note that Laufer has almost reduced fundamental cycle as assumption, but his proof shows that the existence of an integrally minimal collection is sufficient for the existence of a deformation.

Some adjacencies, which Laufer calls *reduced*, are particularly simple. In terms of the resolution graph they are the following. First of all, one can take a (connected) subgraph of a given graph. One can also replace two intersecting curves $E_a$ and $E_b$ of self-intersection $-a$ and $-b$, with one curve with same self-intersection as $E_a + E_b$, that is $-(a + b - 2)$. This means smoothing the double point of the exceptional divisor at the intersection point of $E_a$ and $E_b$.

**Proposition 3.6.** A rational singularity, which is not obtainable from a rational double or triple point, is not simple. It deforms into a singularity with a modulus in the exceptional divisor, with (unweighted) graph of the form:

![Graph](image)

**Proof.** For the purpose of this proof we call a graph as in the statement a *star*. If the graph of a rational singularity has a vertex of valency at least four, then it has a star as subgraph. If there are two vertices $E_a$ and $E_b$ of valency three, then we can smooth all double points of the exceptional divisor on the chain between $E_a$ and $E_b$. In terms of the graph: we combine all vertices on the chain, including $E_a$ and $E_b$, into one new vertex. The new graph has a star as subgraph.
Table 2. Confining non-simple singularities

We are left with graphs with exactly one vertex of valency three. We claim that every such graph, which is not obtained from a double or triple point graph, has as subgraph a rational graph of form given in Table 2. The meaning of the symbols is the same as in Table 1. If all vertex weights are \((-2)\), then the graph is \(\tilde{E}_6\), \(\tilde{E}_7\) or \(\tilde{E}_8\) (also known as the Kodaira graphs IV\(^*\), III\(^*\) or II\(^*\)). One has to make appropriate vertex weights more negative to obtain rational graphs. We refer to the resulting graphs as graphs of type \(\tilde{E}_k\). To prove the claim one has only to carefully inspect Table 1, and note which graphs are not there. First of all, the central curve has to be a \((-2)\)-curve. If the three arms all have length at least three (counting from the central vertex), there is a subgraph of type \(\tilde{E}_6\). Otherwise, if there is one arm of length two, and the other two have length at least four, then the two vertices on the long arms next to the central vertex have to be \((-2)\)-vertices, and there is a subgraph of type \(\tilde{E}_7\). If not, the second arm has to have length 3, and the third arm at least 6, and there are at least 6 \((-2)\)-curves, so there is a subgraph of type \(\tilde{E}_8\). The Proposition follows from the following Lemma.

**Lemma 3.7.** Rational singularities with a graph of type \(\tilde{E}_k\) deform into a star.

**Proof.** We need some notation. We denote the central vertex by \(E_0\). There are three arms, of length \(p\), \(q\) and \(r\) (counted from the central vertex). Here \((p, q, r) = (3, 3, 3), (2, 4, 4)\) or \((2, 3, 6)\). Referring to Table 2 this means that \(p\) is the length of the arm pointing downwards, and \(r\) the length of the right arm. The arms are \(E_{1,1} + \cdots + E_{1,p-1}, E_{2,1} + \cdots + E_{2,q-1}\) and \(E_{3,1} + \cdots + E_{3,r-1}\), with \(E_{i,j}^2 = -b_{i,j}\), and \(E_{i,1}\) intersecting \(E_0\).

In each of the three cases we define a collection \(\{D_0, \ldots, D_4\}\) of integrally minimal cycles, such that the graph of the collection is a star with \(D_0\) as central
vertex, that is, \( D_0 \cdot D_i = 1 \) and \( D_i \cdot D_j = 0 \) for \( 1 \leq i < j \leq 4 \). Theorem 3.5 then gives the existence of a deformation into a star with the given graph. For graphs of type \( 
abla_6 \) we take
\[
D_0 = E_0 + E_{1,1} + E_{2,1} + E_{3,1}, \quad D_0^2 = -(b_{1,1} + b_{2,1} + b_{3,1} - 4),
\]
\[
D_i = E_{i,2}, \quad 1 \leq i \leq 3, \quad D_i^2 = -b_{i,2},
\]
\[
D_4 = E_0, \quad D_4^2 = -2,
\]
for \( 
abla_7 \)
\[
D_0 = E_{1,1} + E_{2,2} + E_{2,1} + E_0 + E_{3,1} + E_{3,2}, \quad D_0^2 = -(b_{1,1} + b_{2,2} + b_{3,2} - 4),
\]
\[
D_1 = E_0 + E_{2,1}, \quad D_1^2 = -2,
\]
\[
D_2 = E_0 + E_{3,1}, \quad D_2^2 = -2,
\]
\[
D_3 = E_{2,3}, \quad D_3^2 = -b_{2,3},
\]
\[
D_4 = E_{3,3}, \quad D_4^2 = -b_{3,3}
\]
and for \( 
abla_8 \)
\[
D_0 = E_{2,2} + E_{2,1} + E_0 + E_{3,1} + E_{3,2} + E_{3,3} + E_{3,4}, \quad D_0^2 = -(b_{2,2} + b_{3,4} - 2),
\]
\[
D_1 = E_{1,1} + E_0 + E_{2,1}, \quad D_1^2 = -2,
\]
\[
D_2 = E_{1,1} + E_0 + E_{3,1} + E_{3,2} + E_{3,3}, \quad D_2^2 = -2,
\]
\[
D_3 = E_{1,1} + E_{2,1} + 2E_0 + 2E_{3,1} + E_{3,2}, \quad D_3^2 = -2,
\]
\[
D_4 = E_{3,5} \quad D_4^2 = -b_{3,5}
\]
\[
\square
\]

**Remark 3.8.** In the special cases that \( -b_{1,p-1} = -b_{2,q-1} = -b_{3,r-1} = -2 \) for \( \nabla_6 \), \( -b_{3,q-1} = -b_{3,r-1} = -2 \) for \( \nabla_7 \) and \( -b_{3,r-1} = -2 \) for \( \nabla_8 \), the deformation has a quotient construction. Then \(-2K\) is an integral cycle, whose coefficients are just the familiar multiplicities of the \( \nabla_k \)-diagram. The canonical cover is a minimally elliptic singularity and the quotient of a deformation to a simple elliptic singularity gives a deformation to a star. For \( \nabla_8 \) the multiplicities are
\[
2 \quad 4 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1
\]
and the double cover has a minimally elliptic graph of type \( \nabla_6 \) with \( E_{1,2}^2 = E_{2,2}^2 = -b_{2,2} \) and \( E_{3,2}^2 = -(2b_{3,4} - 2) \).

**Corollary 3.9.** Any non-simple rational singularity deforms into a star.

**Proof.** If a singularity obtainable from a rational double or triple point only deforms into such singularities (necessarily only finitely many), then it is simple. So if it is not simple, then it deforms to a rational singularity, not of this type, and therefore also into a star. \( \square \)
Next we study the adjacencies on the Artin component of quasi-homogeneous taut singularities. If the singularity is obtainable from a double point and the multiplicity is at least at least four, then it can also be obtained from a triple point. Note that it can deform into double points. We start by determining the positive roots.

**Lemma 3.10.** Let \( X' \) be obtained from the rational triple point \( X \), with the irreducible components \( E_i' \) of the exceptional divisor of \( X' \) corresponding to the components \( E_i \) of \( X \). A cycle \( D' = \sum d_i E_i' \) is a positive root of \( X' \) if and only if \( D = \sum d_i E_i \) is a positive root of \( X \) with \( d_i = 1 \) for all \( i \) with \( (E_i')^2 < E_i^2 \).

**Proof.** We first remark that an \( E_i' \) with \( (E_i')^2 = E_i^2 - \beta_i < E_i^2 \) has coefficient 1 in the fundamental cycle. Indeed, by construction the exceptional divisor \( E' \) is a subset of the exceptional set of a non-minimal resolution of \( X \); the fundamental cycle on this resolution can be computed by first computing the fundamental cycle on \( E' \), and by rationality each \((-1)\)-curve intersects this cycle with multiplicity 1. Therefore the condition on \( d_i \) is necessary.

Let \( D' \) be a positive root, so \( p_\alpha(D') = 0 \). We compute \( p_\alpha(D) = 1 + \frac{1}{2} D \cdot (D + K) \).

If \( d_i > 1 \), then \( E_i \cdot D = d_i E_i' + \sum_{E_j, E_j > 0} d_j = d_i (E_i')^2 + \sum_{E_j', E_j > 0} d_j = E_i' \cdot D' \) and \( E_i \cdot K = E_i' \cdot K' \). If \( d_i = 1 \) and \( (E_i')^2 = E_i^2 - \beta_i \), then \( E_i \cdot (D + K) = E_i' \cdot D' + \beta_i + E_i' \cdot K' - \beta_i = E_i' \cdot (D' + K') \). So also \( p_\alpha(D) = 0 \). The same computation shows that \( p_\alpha(D) \) implies \( p_\alpha(D') = 0 \), if \( d_i = 1 \) whenever \( \beta_i > 0 \). \( \square \)

**Proposition 3.11.** A singularity, obtainable from a rational triple point, deforms on the Artin component only into singularities, obtainable from triple and double points.

**Proof.** Let \( X' \) deform into a surface with several singularities. By openness of versality we can smooth all but one of them, so we may as well assume that there is only one singularity. As \( X' \) has almost reduced fundamental cycle, the deformation \( X_T' \) can be described by an integrally minimal collection \( D_1', \ldots, D_m' \) of positive roots. Lemma 3.10 gives an integrally minimal collection \( D_1, \ldots, D_m \), determining a deformation \( X_T \) of the triple point \( X \). As triple points deform only into triple or double points, the graph of \( D \) is a double or triple point graph. An \( E_i' \) with \( (E_i')^2 = E_i^2 - \beta_i < E_i^2 \) can occur in at most one \( D_j' \), as its coefficient in \( D' \) is one. Therefore \( (D_j')^2 = D_j^2 - \sum \beta_i(j) \), where the sum runs over all \( i \) such that \( E_i' \) is contained in the support of \( D_j \). So the graph of \( D' \) is obtainable from that of \( D \). \( \square \)

4. **RATIONAL QUADRUPLE POINTS**

The deformation theory of rational quadruple points was studied by De Jong and Van Straten [8]. The base space of the versal deformation is up to a smooth factor isomorphic to a explicitly described space \( B(n) \) with \( n + 1 \) irreducible components. The integer \( n \) can be found from the resolution graph of the rational
quadruple points, as it is the number of virtual quadruple points, or in other words the number of quadruple points in the resolution process.

If the base space has only two components, then a deformation to any other quadruple point has to occur over the intersection of the two components. So in particular it is a deformation on the Artin component.

**Proposition 4.1.** A rational quadruple point, obtained from a triple point, is simple.

**Proof.** By Proposition 3.11 any other quadruple point on the Artin component is obtainable from a triple point. Therefore simpleness follows if the base space is $B(1)$, with exactly two components.

We show that $n = 1$, that is, there is no quadruple point on the first blow up. Let $E_0$ be the $(-3)$-curve of the triple point $X$, and $E_m$ the unique curve of the quadruple point $X'$ with $E'_m \cdot E'_m < E_m \cdot E_m$ (possibly $E_m = E_0$). There is a quadruple point on the first blow up of $X'$ if and only if $E'_i \cdot Z' = 0$ for every curve $E'_i$ on the chain from $E'_0$ to $E'_m$.

The multiplicity of $E'_0$ in the fundamental cycle $Z'$ is 1. If $E'_0 \cdot Z' = 0$ on $\tilde{X}'$, then the neighbour $E'_1$ of $E'_0$ on the chain has the same multiplicity in $Z'$ as $E_1$ in $Z$, and if $E'_1 \cdot Z' = 0$, its neighbour has also the same multiplicity, and so on. This process stops with an $E'_i \cdot Z' < 0$, or reaches $E'_m$, and $E'_m$ has the same multiplicity in $Z'$ as $E_m$ in $Z$. But then $E'_m \cdot Z' < 0$. $\square$

**Remark 4.2.** We can characterise the simple rational quadruple points as those with almost reduced fundamental cycle and without quadruple points on the first blow up. The easiest way to see this is to check the classification of quadruple points [17, Prop. 4], as it is not always immediately obvious which of the two $(-3)$-curves one has to make into a $(-2)$ to get a triple point.

5. Sandwiched singularities

Sandwiched singularities are normal surface singularities, which admit a birational map to $(\mathbb{C}^2,0)$. Following De Jong and Van Straten [9] we describe them and their deformations in terms of decorated curves.

**Definition 5.1.** Let $C = \bigcup_{i \in B} C_i$ be a plane curve singularity. The number $m(i)$ is the sum of the multiplicities of the branch $C_i$ in the multiplicity sequence of the minimal embedded resolution of $C$.

**Definition 5.2.** A decorated curve is a curve singularity together with a function $l: B \rightarrow \mathbb{N}$ on the set of branches, with the property that $l(i) \geq m(i)$. A decorated curve $(C, l)$ is non-singular if $C$ consists of one smooth branch, and $l(1) = 0$.

**Definition 5.3.** Let $(C, l)$ be a singular decorated curve on a smooth surface $(Z, p)$ and let $m_i$ be the multiplicity of the $i$-th branch $C_i$. Consider the blow up $Bl_p Z \rightarrow Z$ of the singular point. The strict transform $(\overline{C}, \overline{l})$ of $(C, l)$ is
the decorated curve, consisting of the strict transform $\overline{C}$ of $C$ with decoration $\overline{l}(i) = l(i) - m_i$.

Note that we do not allow blow-ups in non-singular decorated curves, for then $\overline{l}$ would become a negative function. Therefore the embedded resolution of a decorated curve is the unique (minimal) composition of point blow-ups, such that the strict transform of the decorated curve is non-singular. It is obtained from the minimal embedded resolution of $C$ by $l(i) - m(i)$ consecutive point blow-ups in each branch $C_i$.

**Definition 5.4.** Let $(C, l) \subset (Z, p)$ be a decorated curve with embedded resolution $(\overline{C}, 0) \subset \overline{Z}(C, l)$. The analytic space $X(C, l)$ is obtained from $\overline{Z}(C, l) - \overline{C}$ by blowing down the maximal compact subset, that is, all exceptional divisors, not intersecting the strict transform $\overline{C}$.

The space $X(C, l)$ can be smooth, or it may have several singularities. Each singularity is a sandwiched singularity. Given a sandwiched singularity, it is always possible to find a decorated curve $(C, l)$ such that the sandwiched singularity is the only singularity of the space $X(C, l)$. Even then the representation of a sandwiched singularity as $X(C, l)$ is not unique.

**Proposition 5.5.** A singularity is not sandwiched, if its graph contains $D_4$ as subgraph or has the following subgraph:

$$
\begin{array}{cccc}
-2 & -2 & -2 & -2 \\
\bullet & \bullet & \bullet & \bullet \\
& & & -3 \\
\end{array}
$$

*Proof.* A sandwiched singularity has at least one exceptional curve $E_i$ with $E_i \cdot Z < 0$ and multiplicity one in the fundamental cycle $Z$. Indeed, the strict transform of the first blown-up curve in the construction has this property: the compact part of the divisor of a general linear function is an upper bound for $Z$. This criterion excludes $D_4$ and the graph shown. \qed

It follows that the singularities of type III.5 and higher in Table 1 are not sandwiched. Most of the other ones are not excluded by the above criterion, and in fact they are sandwiched.

The numbers $l(i)$ determine a divisor on the normalisation $\overline{C}$ of $C$. This interpretation allows a more global point of view, in which a decorated curve $(C, l)$ is a curve $C$ on a smooth surface $Z$ together with a divisor $l$ on the normalisation $\overline{C}$. We form again the embedded resolution $\overline{Z}(C, l)$ and blow down the maximal compact subset of $\overline{Z}(C, l) - \overline{C}$ to obtain the space $X(C, l)$.

We now consider deformations. Recall that a deformation of a plane curve singularity admits a simultaneous normalisation if and only if it is $\delta$-constant.

**Definition 5.6.** A 1-parameter deformation $(C_S, l_S)$ of a decorated curve $(C, l)$ over a germ $S$ of a smooth curve is a deformation $C_S \to S$ of $C$ with simultaneous
normalisation $\overline{C}_S \to S$, together with a deformation of the divisor $l$, such that each fibre $(C_s, l_s)$ is a decorated curve.

**Theorem 5.7** ([9, Thm. 4.4]). The 1-parameter deformations of a sandwiched singularity $X(C, l)$ are exactly deformations $X(C_S, l_S)$ for 1-parameter deformations $(C_S, l_S)$ of the decorated curve $(C, l)$.

A simplified proof can be found in the thesis of Konrad Möhring [15], who constructs the deformation $X(C_S, l_S)$ directly from $(C_S, l_S)$ by blowing up a family of complete ideals.

To study deformations of taut sandwiched singularities we use a specific representation with a decorated curve. We start with singularities with reduced fundamental cycle, see also [9, Ex. 1.5 (4)]. We construct the resolution graph of a decorated curve. This is an embedded resolution graph for the curve $C$, with as usual arrows for the strict transforms $\overline{C}_i$, and decorations $\overline{l}(i) = 0$, which we omit. We choose one end of the resolution graph of the singularity, whose vertex $E_0$ will correspond the strict transform of the first curve blown up. To a vertex $E_i$ we connect $-Z \cdot E_i (-1)$-vertices with an arrow attached to it, except for $E_0$, to which we connect one vertex and arrow less: only $-Z \cdot E_i - 1$ ones.

**Example 5.8.** The cyclic quotient singularity $X_{37,11}$. We start at the left end. The resulting graph is

![Graph](image)

It blows down to the following decorated curve.

![Decorated Curve](image)

For cyclic quotient singularities this representation with smooth branches has the property that $\min\{l(i), l(j)\} = C_i \cdot C_j + 1$ for each pair of branches. It was observed by Möhring [15] that this property can be used to give a new proof of Riemenschneider’s conjecture that cyclic quotients deform only into cyclic quotients. Kollár and Shepherd-Barron [11] derive it from the stronger result that in a deformation of a rational singularity with reduced fundamental cycle the number of ends of the graph cannot go up. Their result can also be obtained with the present methods. We first treat the cyclic quotient case, where the argument is more transparent.
Lemma 5.9. Let \((C, l)\) be the germ of a decorated curve with smooth branches with the property that for each pair of branches
\[ C_i \cdot C_j \leq \min\{l(i), l(j)\} \leq C_i \cdot C_j + 1. \]
Then the only singularities of the space \(X(C, l)\) are cyclic quotients.

Proof. Let \(C_r\) be a branch such that \(l(i) \leq l(r)\) for all branches \(C_i\). We construct the embedded resolution of \((C, l)\) in two steps. We first consecutively blow up \(l(r)\) times in the origin of the strict transform of the branch \(C_r\). This introduces a chain \(E_1, \ldots, E_{l(r)}\) of exceptional curves. If \(l(i) = C_i \cdot C_r + 1\), we blow up once in the intersection point of \(E_{l(i)}\) and the strict transform of \(C_i\), and we do not blow up further in \(C_i\). The newly introduced \((-1)\)-curve intersects the strict transform of \(C_i\) on the minimal resolution and is therefore not part of the exceptional set for \(X(C, l)\). If \(l(i) = C_i \cdot C_r\), we do not blow up in \(C_i\) and the curve \(E_{l(i)}\) does not belong to the exceptional set. This set is thus a subset of the chain \(E_1, \ldots, E_{l(r)-1}\), which may consist of several connected components.

Proposition 5.10. Cyclic quotient singularities deform only into cyclic quotients.

Proof. Choose, as above, a representation \(X(C, l)\) with \((C, l)\) a decorated curve with smooth branches and the property that \(\min\{l(i), l(j)\} = C_i \cdot C_j + 1\) for each pair of branches. Let \(X(\tilde{C}, \tilde{l})\) be a general fibre of a 1-parameter deformation. Consider a pair of branches. Suppose that \(C_i \cdot C_j = n\) and that \(l(i) = n + 1\). Then \(\tilde{C}_i \cdot \tilde{C}_j = \sum_p n_p = n\), where the sum runs over the intersection points. The support of \(\tilde{l}(i)\) on \(\tilde{C}_i\) may contain other points. Now
\[
\sum_{p \in \tilde{C}_i \cap \tilde{C}_j} \tilde{l}_p(i) + \sum_{q \notin \tilde{C}_j} l_q(i) = l = n + 1 = 1 + \sum_{p \in \tilde{C}_i \cap \tilde{C}_j} n_p.
\]
Because always \(\tilde{l}_p(i) \geq n_p\) we see that for at most one point \(\tilde{l}_p(i) = n_p + 1\) while for the others \(\tilde{l}_p(i) = n_p\). So for each singularity \(p\) of \((\tilde{C}, \tilde{l})\) the property \((\tilde{C}_i \cdot \tilde{C}_j)_p \leq \min\{\tilde{l}_p(i), \tilde{l}_p(j)\} \leq (\tilde{C}_i \cdot \tilde{C}_j)_p + 1\) holds.

Lemma 5.11. Let \((C, l)\) be the germ of a decorated curve with smooth branches. Suppose that the set of branches \(B\) can be written as the (not necessarily disjoint) union \(B_1 \cup \cdots \cup B_k\) such that for all \(1 \leq m \leq k\) the property
\[ C_i \cdot C_j \leq \min\{l(i), l(j)\} \leq C_i \cdot C_j + 1 \]
holds for all pairs \((i, j) \in B_m \times B_m\). Then the number of ends of the singularities of the space \(X(C, l)\) is at most \(k + 1\).

Proof. Again we construct the embedded resolution of \((C, l)\) in two steps. For each subset \(B_m\) we choose a branch \(C_m\) with \(l(m)\) maximal. The first step is to resolve the curve \(\cup_m C_m\). As this curve has \(k\) branches, the resulting embedded
resolution graph has (at most) \( k + 1 \) ends. The exceptional curves of the additional blow-ups needed to resolve \((C, l)\) are not exceptional for \(X(C, l)\). □

**Proposition 5.12.** In a deformation of a rational singularity with reduced fundamental cycle the number of ends cannot increase.

**Proof.** We choose a representation with a decorated curve with smooth branches. For each end of the graph of the singularity (except the root) we choose a curve \( C_m \), whose strict transform is connected to this end by a \((-1)\)-curve. The set \( B_m \) contains all branches, which are connected by \((-1)\)-curves to the chain from the root to \( C_m \). Then \( \min \{l(i), l(j)\} = C_i \cdot C_j + 1 \) for all \( i, j \in B_m \). As before, we deduce that \( \tilde{C}_i \cdot \tilde{C}_j \leq \min \{l_p(i), l_p(j)\} \leq (\tilde{C}_i \cdot \tilde{C}_j) + 1 \) for each singular point of the deformed curve \((\tilde{C}, \tilde{l})\), through which branches in the set \( B_m \) pass. Therefore we have a division in at most \( k + 1 \) ends. □

**Remark 5.13.** For non-reduced fundamental cycle the number of ends can increase. Examples are provided by the deformations of Lemma 3.7. We give a sandwich description of the deformation for the case of a surface singularity of type \( \tilde{E}_6 \) with \(-b_{1,1} = -4\) and all other curves \(-2\). It has a sandwiched representation with decorated curve \((E_{12}, 12)\), where \( E_{12} \) is the curve \( x^3 + y^7 + axy^5 = 0 \). It deforms into \((\tilde{E}_7, (4, 4, 4))\), giving a 2-star.

Next we study sandwiched singularities in the classes III.2 and III.3. The singularities of type III.2 whose graph contains a \( D_4 \) subgraph, are not sandwiched, but they are simple as they are dihedral quotients. The sandwiched ones can be seen as special case of the type III.3, if we allow the arms to be shorter. This means that we are looking at graphs of the form:

\[
\begin{array}{cccccccc}
\bullet & \cdots & \begin{array}{c}
\text{\(-2\)}
\end{array} & \bullet & \cdots & \bullet
\end{array}
\]

**Proposition 5.14.** Sandwiched singularities with graph as above deform only to singularities of the same type or to singularities with reduced fundamental cycle and at most three ends.

**Proof.** We start by describing a decorated curve \((C, l)\). We need some notation. The left arm of the graph is \( E_{1,1} + \cdots + E_{1,k} \), with \( E_{1,k} \) the \((-3)\). The right arm is \( E_{2,1} + \cdots + E_{2,s+1} \) and the short arm consists only of \( E_{3,1} \). First look at the triple point graph of this form, having exactly one \((-3)\) and further only \((-2)\)'s. A decorated curve giving this graph is \((A_{2k}, 2k + 4 + s)\). We call the curve \( C_0 \). To make the self-intersections more negative we add branches. They come in three types, one for each arm. On the left arm, if a \((-1)\) intersects \( E_{1,m} \), then we have a smooth branch \( C_i \) with \( C_0 \cdot C_i = 2m \) and \( l(C_i) = m + 1 \). We make the short arm \( E_{3,1} \) more negative with a smooth branch \( C_i \) with \( C_0 \cdot C_i = 2k + 1 \) and
$l(C_1) = k + 2$. Finally, if a $(-1)$ intersects $E_{2,n}$ on the right arm, then we have a branch $C_i$ of type $A_{2k}$ with $C_0 \cdot C_i = 4k + 2 + n$ and $l(C_i) = 2k + 3 + n$.

We use induction on the number of branches of $(C, l)$. If the curve consists only of $(C_0, l(0))$ the claim is true as an $A_{2k}$ deforms with $\delta$-const into one $A_{2l}$, $0 \leq l < k$ and some $A_{2m-1}$ with $k = l + \sum m_i$. The $(A_{2l}, 2l + 1 + t)$ gives at most a smaller graph of the same type, while $X(A_{2m-1}, (m + t_1, m + t_2))$ has reduced fundamental cycle and at most three ends.

Now consider a deformation $X(\tilde{C}, \tilde{l})$ of $X(C, l)$ where $C$ has several branches. Let $C_i$ be a branch different from $C_0$. We wish to compare the singularities of $X(\tilde{C}, \tilde{l})$ and $X(\tilde{C} \setminus \tilde{C_i}, \tilde{l})$. The Proposition follows from the following claim.

**Claim.** The graph of the singularities of $X(\tilde{C}, \tilde{l})$ is a subgraph of the graph of $X(\tilde{C} \setminus \tilde{C_i}, \tilde{l})$ or it is obtainable from a subgraph by making some self-intersections more negative.

We first resolve the decorated curve $(\tilde{C} \setminus \tilde{C_i}, \tilde{l})$. If this resolution also resolves $(\tilde{C}, \tilde{l})$, i.e., no further blow ups on the strict transform of $\tilde{C_i}$ are needed, then the exceptional curves intersected by the strict transform of $\tilde{C_i}$ are not exceptional for $X(\tilde{C}, \tilde{l})$, and the graph of the singularities of $X(\tilde{C}, \tilde{l})$ is a subgraph of the graph of $X(\tilde{C} \setminus \tilde{C_i}, \tilde{l})$.

We shall show that if this is not the case, then there is exactly one extra blow-up needed. If the center of the blow up does not lie on an exceptional divisor, we get just one $(-1)$-curve and no new singularity. If the center is a smooth point on an irreducible component $E_a$ of the exceptional divisor, then the self intersection $E_a \cdot E_a$ is made more negative. The graph of the singularity in question is therefore obtainable from a subgraph of the graph of $X(\tilde{C} \setminus \tilde{C_i}, \tilde{l})$. Finally, if the center is an intersection point of two divisors $E_a$ and $E_b$, then the newly introduced $(-1)$, being not exceptional for $X(\tilde{C}, \tilde{l})$, breaks up the graph; the self-intersections of $E_a$ and $E_b$ become more negative. Therefore the graphs for $X(\tilde{C}, \tilde{l})$ are obtainable from subgraphs of the graph of $X(\tilde{C} \setminus \tilde{C_i}, \tilde{l})$.

We look only at the intersection of a branch $\tilde{C_i}$ with $\tilde{C_0}$. If $\tilde{C_i}$ is a smooth branch with $C_0 \cdot C_i = 2m$ and $l(C_i) = m + 1$, then one has to blow up in $m + 1$ points to resolve $(\tilde{C_i}, m + 1)$. Let $m_{0,p}$ be the multiplicity of $\tilde{C_0}$ in such a point $p$. As $\tilde{C_0} \cdot \tilde{C_i} = \sum_p m_{0,p} = 2m$, it follows that either $m_{0,p} = 2$ for $m$ points and one point does not lie on $\tilde{C_0}$, or all $m + 1$ points lie on $\tilde{C_0}$ and for two of them $m_{0,p} = 1$. A similar argument settles the case of a smooth branch $C_i$ with $C_0 \cdot C_i = 2k + 1$ and $l(C_i) = k + 2$.

Finally we look at a branch $C_i$ of type $A_{2k}$ with $C_0 \cdot C_i = 4k + 2 + n$ and $l(C_i) = 2k + 3 + n$. To resolve $(\tilde{C_i}, l(\tilde{C_i}))$ we need $k + 3 + n$ blow ups, and $\tilde{C_i}$ has multiplicity 2 in $k$ of them. In these points the multiplicity of $\tilde{C_0}$ can be 2, 1 or 0. As intersection multiplicity $\tilde{C_0} \cdot \tilde{C_i}$ is $4k + 2 + n$, there are only two possibilities. The first one is $k \times 4 + (n + 1) \times 1 + 0$, the same numbers as for
$C_0 \cdot C_i$, or in one multiplicity 2 point of $\tilde{C}_i$ the multiplicity of $\tilde{C}_0$ is 1. Then $4k + 2 + n = (k - 1) \times 4 + 2 + (n + 2) \times 1$.

Therefore in all cases at most one point blown up does not lie on the strict transform of $\tilde{C}_0$. \hfill \Box

Singularities with a graph of type III.4 are not sandwiched if the graph has the graph of Proposition 5.5 as subgraph. To be sandwiched some vertex weights on this subgraph have to be more negative. We distinguish three types, with different sandwich representation. Let the left arm be $E_{1,2} + E_{1,1}$, where always $-b_{1,1} = -2$, the short arm $E_{3,1}$ and the right arm $E_{2,1} + E_{2,2} + \cdots + E_{2,k}$. For each of the three types the singularity of lowest multiplicity is a quadruple point, realisable as $X(C_0, l)$ with $C_0$ irreducible.

For the first type $-b_{3,1} = -4$ (for the quadruple point). We take as decorated curve $(E_6, k + 7)$. To make the self-intersection of $E_{3,1}$ (the first blown up curve) more negative we add one or more smooth branches with $l = 2$, for $E_{1,2}$ smooth branches with $l = 3$, and finally for $E_{2,t}$ on the right arm one or more branches $(E_6, t + 7)$, intersecting $C_0$ with multiplicity $12 + t$.

The second type has always $-b_{3,1} = -3$. If also $-b_{1,2} = -3$, we get the quadruple point with the irreducible curve $(E_8, k + 8)$. To make $E_{1,2} \cdot E_{1,2}$ more negative we add one or more smooth branches with $l = 2$ and for $E_{2,t}$ on the right arm one or more branches $(E_8, t + 8)$, intersecting $C_0$ with multiplicity $15 + t$.

For the last type always $-b_{3,1} = -3$ and $-b_{1,2} = -2$. In this case $E_{2,k}$ is the first blown up curve. For the quadruple point the only other ($-3$)-curve besides $E_{3,1}$ is $E_{2,2}$. We take a curve equisingular with $(x^3 + y^{3k-1}, 4 + 3k)$; here we may assume that $k > 2$, as $k = 2$ is dealt with in the previous case. The only curves, which can be made more negative, are on the right arm, and we use smooth branches for this purpose.

**Proposition 5.15.** Sandwiched singularities with graph of type III.4 deform only to singularities of type I, II or III.1 up till III.4.

*Proof.* We use again induction on the number of branches. For the induction start we have to describe $\delta$-const deformations of $(C_0, l)$. For $E_6$ and $E_8$ this is not difficult, but for the third type it is not so easy. Therefore we use a different argument. By Proposition 4.1 quadruple points, obtainable from triple points, are simple. As sandwiched singularities deform only into sandwiched singularities, we get the statement of the Proposition.

For the induction step we use the same claim as in the proof of Proposition 5.14. As before the result follows if we can prove that either the resolution $(\tilde{C} \setminus \tilde{C}_i, \tilde{l})$ is also the resolution of $(\tilde{C}, \tilde{l})$, or that only one extra blow up in a smooth point of the strict transform of $\tilde{C}_i$ is needed. For smooth branches this is the same argument as before. It remains to look at branches of type $E_6$ and $E_8$. Both cases being similar, we do here only the last one.
If \( E_8 \) deforms \( \delta \)-const to a collection of \( A_k \) singularities, then the multiplicities of the (infinitely near) points in the minimal embedded resolution are \((2,2,2,2)\), whereas they are \((3,2)\) if there is a triple point. This can be checked from the list of possible combinations, but it is all the information we need here. Therefore the multiplicities in the resolution of \((E_8, t + 8)\) are \((3,2,1^t+3)\) or \((2^4,1^t)\). The intersection multiplicity with \( \tilde{C}_0 \) has to be \(15 + t\). If the multiplicities in the minimal resolution of \( \tilde{C}_0 \) are \((3,2)\) then we can get \(15 + t\) as \(9 + 4 + (t + 2) + 0\), or \(9 + 2 + 2 + (t + 2)\), but not from a \( \tilde{C}_i \) with multiplicities \((2^4,1^t)\), as \(6 + 4 + 2 + 2 + t = 14 + t\). If \( \tilde{C}_0 \) is of type \((2^4)\), then we can get \(6 + 4 + 2 + 2 + (t + 1)\), or \(4 \times 4 + (t - 1) + 0\) or \(3 \times 4 + 2 + 2 + (t - 1)\). So there is at most one smooth point of the strict transform of \( \tilde{C}_i \) on the resolution of \((\tilde{C}_0, \tilde{I})\), which still has to be blown up. □

**Corollary 5.16.** Sandwiched singularities are simple if and only if they are taut and quasi-homogeneous.

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