Quantum measurement in the charge representation

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Counting statistics of charge transfers in a point contact interacting with an arbitrary quantum system is studied. The theory for the charge specific density matrix is developed, allowing the evaluation of the probability of the outcome of any joint measurement of the state of the quantum system and the transferred charge. Applying the method of charge projectors, the master equation for the charge specific density matrix is derived in the tunneling Hamiltonian model of the point contact. As an example, the theory is applied to a quantum measurement of a two-state system: The evolution of the charge specific density matrix in the presence of Nyquist or Schottky noise is studied and the conditions for the realization of a projective measurement are established.

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I. INTRODUCTION

In recent years, interest in quantum measurement has emerged in the context of solid state devices. This is in part forced by practical issues in connection with experiments involving nanostructures, and general questions of decoherence and issues arising in nanomechanics, for example in probing the quantum electro-mechanical behavior of a nano-resonator, and is also crucial for the read-out of a quantum computational process. Solid state nanodevices such as quantum dots are candidates for implementation of spin qubits and charge qubits and superconducting nanodevices containing Josephson junctions have been studied in detail and are being tested for their potential as charge qubits. Nanomechanics and monitoring qubits thus confront us with the practical details of a quantum measurement. In a quantum measurement one must, for the system purported to function as a detector, identify a collective variable which behaves classically. Such a variable is not a priori provided by quantum theory, but in an electronic device the candidate is ultimately related to the charge flow. It is therefore of importance to have a quantum description of the charge transfer statistics, and demonstrate that a charge measurement can be a measurement of the state of a quantum object coupled to it. The purpose of the paper is to base such a description directly on the density matrix of the many-body system.

The renewed interest in the quantum measurement process has in particular focussed on a two-level system coupled to a quantum point contact in the limit where the point contact can be modelled as a low transparency tunnel junction. The realization of the two-level system could be two coherently coupled quantum dots between which an excess electronic charge can tunnel. The two dots are electrostatically coupled to the tunnel junction, or quantum point contact, thereby making the tunnelling amplitude depend on the state of the two-level system. This model has recently attracted much attention. Gurvitz derived for the zero temperature case, a Markovian master equation for the density matrix of the two-level system keeping track of the charge transferred through the junction. Goan et al. considered the Bloch-Redfield equation for the spin dynamics, i.e., the electron degrees of freedom of the tunnel junction are traced out. To account for individual tunnelling events they employed the quantum jump approach often used in quantum optics, and obtained from the stochastic master equation the spin evolution for specific realizations of tunnelling events. To account for specific realizations of tunnelling events, Korotkov similarly employed a stochastic treatment and showed by numerical simulation how an initially mixed spin state can evolve into a pure state. Ruskov and Korotkov considered the Markovian master equation for the density matrix for the two-level system keeping track of the charge transferred through the junction at finite temperatures and calculated the noise spectrum. Recent achievements in counting statistics of charge transfers were by Shnirman et al. who derived a master equation for the spin density matrix, keeping track of the charges passing through the tunnel junction by a counting field as practised in counting statistics, and calculated the noise spectrum of a quantum point contact, with the same result as obtained by Bulaevskii et al. Recently, a quantum oscillator coupled to the junction has been studied, but by a method only valid at zero temperature. The witnessed variety of techniques, calls for a standard method to attack these types of problems.

We shall in the following develop a regular method for a joint description of the charge kinetics of a tunnel junction and a quantum system coupled to it. Essential to the approach is that for an arbitrary many-body system we show how to treat the number of particles in a given spatial region or piece of material as a degree of freedom. This is achieved by employing suitably constructed charge projection operators introduced previously in the context of counting statistics. We shall refer to this reduced description, where at any moment in time the probability distribution for the number of particles or charges in a chosen region is specified, as the charge representation. The key construct of the theory
is the charge specific density matrix where the degrees of freedom of the environment are partially traced out. The charge specific density matrix allows one to evaluate the probability of the outcome of any joint measurement of the system and the charge state. Applying the charge projectors, allows us to use a standard kinetic approach to obtain the equation of motion for the density matrix in the charge representation.

Tunnel junctions and quantum point contacts are ubiquitous in nanodevices such as for example those utilizing two-dimensional electron gases in semiconductor inversion layers combined with split gate technique. The developed charge specific density matrix description of the dynamics of a quantum object we believe is the optimal description of such nanodevices since in electrical measurements any information beyond the charge distribution is irrelevant. This sets the stage for considering questions regarding quantum measurements, for example whether a measurement of the charge can provide a measurement of the state of a quantum object coupled to the junction. Indeed we shall demonstrate that the junction is able to perform a projective von Neumann measurement. The model allows us to study analytically a quantum measurement in the proper language of the density matrix. The intrinsic quantum bound on the measurement time necessary for the tunnel junction to operate as a measuring device is established. Quantum theory and measurement has a long and controversial history, and the model provides another simple illustration that a quantum measurement can be described in standard quantum mechanical terms, in fact in full detail for a realistic model of a nanodevice. The measuring scheme illustrates that there is no need to postulate the classicality of any variable, the charge state of the junction being directly accessible, nor to invoke “wave function collapse.” The transmission of the electrical noise in the tunnel junction to the quantum object turns out to be sufficient for the density matrix to decohere in the pointer basis. Amplification from the quantum to the classical level, the emergence of a projective measurement, can thus be followed in a realistic model of a nanodevice.

The paper is organized as follows: in section II we construct the charge representation. In section III, the coupled object-junction Hamiltonian is introduced, and in section IV the equation of motion in the object and charge variables is obtained. In section V, we briefly discuss an isolated tunnel junction in the charge representation. The method we present is quite general, but for illustration we shall in this paper apply it to a low transparency tunnel junction coupled to a two-level system or for short a spin, and thereby obtain a master equation in terms of the relevant variables, charge and spin. In section VI, a short time measurement of the two-level system is considered and an analytic solution of the master equation is obtained. Expressions for the characteristic times for decoherence and spin-charge separation are obtained. In section VII, the temporal progression of a quantum measurement is studied and the emergence of a projective measurement is seen to be different depending on the relationship between voltage and temperature. In section VIII, we estimate the intrinsic quantum bounds on the measurement time. Finally, we summarize and conclude. Details of calculations are presented in appendices.

II. CHARGE REPRESENTATION

We start by showing that the number of particles in a given spatial region can be treated as a degree of freedom. We construct the probability distribution for the number of particles in a specified region for an arbitrary many-body system. This is accomplished by using charge projection operators originally introduced in Ref. 22. Consider an N-particle quantum system in a volume \( V \) and in state \( \Psi \). According to quantum mechanics, the probability, \( p_n^A \), that \( n \) particles are in a sub-volume \( A \) is

\[
p_n^A = \int_V dr_1 \ldots dr_N |\Psi(r_1, \ldots, r_N)|^2 \Theta^A_{n,N}(r_1, \ldots, r_N)
\]

where the indicator function \( \Theta^A_{n,N}(r_1, \ldots, r_N) \) is equal to 1 if exactly \( n \) of its \( N \) spatial arguments belong to the region \( A \), and is zero otherwise. The volume \( A \) and its supplement volume \( B \) partitions the total volume into two non-overlapping parts, and the probability that there are \( N-n \) particles in region \( B \) equals the probability for having \( n \) particles in region \( A \).

The probability \( p_n^A \) can be written as the expectation value

\[
p_n^A = \langle \Psi | \mathcal{P}_n^A | \Psi \rangle ,
\]

of the Hermitian operator \( \mathcal{P}_n^A \) which acts in the position representation on the wave function according to

\[
\mathcal{P}_n^A \Psi(r_1, \ldots, r_N) = \Theta^A_{n,N}(r_1, \ldots, r_N) \Psi(r_1, \ldots, r_N).
\]

From their definition, one observes that the introduced operators, \( \mathcal{P}_n^A \), have the properties

\[
\mathcal{P}_n^A \mathcal{P}_n^A = \delta_{nn'} \mathcal{P}_n^A \quad , \quad \sum_n \mathcal{P}_n^A = 1
\]

and therefore constitute a complete set of Hermitian projectors. The projected state \( |\Psi_n\rangle = \mathcal{P}_n^A |\Psi\rangle \) is the component of \( |\Psi\rangle \) for which exactly \( n \) of the particles are in region \( A \).

The many-particle operator \( \mathcal{P}_n^A \) can be expressed in a simple way through single particle operators. We introduce the gauge transformation operators

\[
U^A_\lambda = \exp \left[ i \lambda \sum_{k=1}^N \theta^A(r_k) \right],
\]

where \( \lambda \) is a real parameter, and \( \theta^A(r_k) \) equals 1 if \( r_k \) belongs to the region \( A \) and is zero otherwise. By virtue
of the identity,
\[
\int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-i\lambda n} U^{A}_\lambda = \Theta^{A}_{n,N} \tag{2.6}
\]
the projection operator can be presented in any representation as
\[
P^{A}_n = \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-in\lambda} U^{A}_\lambda \tag{2.7}
\]
thus presenting the many-particle operator \(P^{A}_n\) in terms of the product of the single particle gauge transformations in \(U^{A}_\lambda\). This result can also be obtained by observing that the projected state, \(|\Psi_n\rangle\), is an eigenstate of \(U^{A}_\lambda\), \(U^{A}_\lambda |\Psi_n\rangle = e^{i\lambda n} |\Psi_n\rangle\). If the region \(A\), where the particles are counted, occupies the whole volume \(V\), the charge projector coincides with the one used by P.W. Anderson in the theory of superconductivity for projecting the BCS-wave function onto the state with a fixed particle number.

A. The charge specific density matrix

We shall quite generally consider a system consisting of two parts: a quantum object to be measured and the measuring device, which is taken to be a tunnel junction connecting two electron reservoirs.

For the sake of the derivation, we assume that the object together with the electron reservoirs constitute a closed quantum system. The system is described by its full density matrix \(\rho(\xi, R_{el}; \xi', R'_{el})\), where \(\xi\) is the coordinate of the object and \(R_{el} = r_1, \ldots, r_N\) comprises the electron coordinates, \(N\) being the total number of electrons relevant for the functioning of the measuring device. With the understanding that the only accessible information about the electrons is their charge distribution, we introduce the charge specific density matrix of the measured system
\[
\hat{\rho}_n = Tr_{el} (P_n \rho) \tag{2.8}
\]
where the trace is with respect to the degrees of freedom of the quantum object, i.e., \(\xi\). It follows from Eq. (2.6) that the probability distribution is normalized, \(\sum_n p_n = 1\). Apart from proper normalization, \(\hat{\rho}_n\) is the density matrix for the quantum object after a charge measurement.

The charge projection operators define what we shall refer to as the charge representation. The degree of freedom in the charge representation is the variable \(n\): the number of charges in a specified spatial region or equivalently in a piece of material. All other information about the charges is traced out. The charge representation is a strongly reduced representation, the description of the environment is reduced to one variable, but if interest is in the currents in a system, information beyond the charge representation is irrelevant.

We now consider the charge specific dynamics, i.e., the equation of motion of the quantum object for specified charge. The dynamics of the combined system is governed by the equation for the density matrix
\[
i\dot{\rho} = [H, \rho] \tag{2.10}
\]
and the equation of motion for the charge specific density matrix is thus
\[
i\dot{\rho}_n = Tr_{el} (P_n [H, \rho]) \tag{2.11}
\]
The charge specific density matrix describes the quantum dynamics of the coupled object conditioned on the charge variable being specified. Since the charge specific description is a reduced description, generally a hierarchy of equations is generated. In the following we shall study a situation where \(\hat{\rho}_n\) can be obtained explicitly.

III. MODEL HAMILTONIAN

In this section we discuss the Hamiltonian for an arbitrary quantum system coupled to a tunnel junction
\[
\hat{H} = \hat{H}_S + H_I + H_r + \hat{H}_T. \tag{3.1}
\]
Here, \(\hat{H}_S\) is the Hamiltonian for the isolated quantum object, the system to be measured, and a hat marks terms which are operators with respect to the degrees of freedom of the quantum object. The Hamiltonians \(H_I, H_r\) specify the isolated left and right electrodes of the junction
\[
H_I = \sum_l \varepsilon_l c_l^\dagger c_l \quad , \quad H_r = \sum_r \varepsilon_r c_r^\dagger c_r \tag{3.2}
\]
where \(l, r\) labels the quantum numbers of the single particle energy eigenstates in the left and right electrodes, respectively, with corresponding energies \(\varepsilon_{l, r}\) and annihilation and creation operators. The operator \(\hat{H}_T\) describes tunnelling,
\[
\hat{H}_T = \hat{T} + \hat{T}^\dagger \tag{3.3}
\]
where
\[
\hat{T} = \sum_{i,r} \hat{T}_{i} c_i^r c_r, \quad \hat{T}^\dagger = \sum_{i,r} \hat{T}_{i} c_r^i c_t
\]
(3.4)
and \(\hat{T}_{i}\) are operators acting on the coordinates, \(\xi\), of the quantum object. The Hermitian property of the Hamiltonian requires that \(\hat{T}_{i} = \hat{T}_{i}^\dagger\). Compared to an isolated tunnel junction, the additional feature of the model is that the tunnelling amplitude, and thereby the conductance of the tunnel junction, depends on the state of the measured system.

In the next section we shall study the dynamics of the system when the probability distribution for the number of charge transfers through the junction is specified at all times. This is achieved by using the charge projectors introduced in section II. For a tunnel junction, the spatial region of interest for counting charges is either of the two electrodes, say we choose the left one. Of importance are in view of Eq. (2.11) the commutation relations of the charge projectors and the Hamiltonian. The terms in the Hamiltonian commute with the charge projectors except for the tunnelling term. The discrete charge dynamics of the tunnel junction is specified by the charge projection operators according to
\[
P_n c_i^r c_r = c_i^r c_r P_{n-1}, \quad P_n c_r^i c_t = c_r^i c_t P_{n+1}.
\]
(3.5)
In terms of the tunnelling operators, Eq. (3.4), the identities read
\[
P_n \hat{T} = \hat{T} P_{n-1}, \quad P_n \hat{T}^\dagger = \hat{T}^\dagger P_{n+1}.
\]
(3.6)
These identities are used repeatedly in the derivation of the equation of motion in the charge representation.

IV. CHARGE SPECIFIC DYNAMICS

In this section we shall obtain the equation of motion for the charge specific density matrix, \(\hat{\rho}_n\), to lowest order in the tunnelling.

Taking advantage of the relations in Eq. (3.5), the equation of motion for the charge specific density matrix, Eq. (2.11), can be written in the form
\[
\dot{\hat{\rho}}_n(t) + i[H_S, \hat{\rho}_n(t)] = \sum_{i,r} \left( \hat{T}_{i} \hat{A}_{i}^{(n)} \right) (t) + \hat{T}_{r} \hat{B}_{r}^{(n)} (t) + h.c.
\]
(4.1)
where here and in the following h.c. represents the hermitian conjugate term, and the time dependent system operators \(\hat{A}\) and \(\hat{B}\) are given by
\[
\hat{A}_{i}^{(n)} (t) = \frac{1}{i} \text{Tr}_{el} \left( c_i^r c_r \rho(t) P_n \right), \quad \hat{B}_{r}^{(n)} (t) = \frac{1}{i} \text{Tr}_{el} \left( c_r^i c_t \rho(t) P_n \right).
\]
(4.2, 4.3)

One sees from Eq. (4.1), that the time evolution of the charge diagonal component of the density matrix, \(\hat{\rho}_n\), is determined (in addition to the internal dynamics of the system) by the charge off-diagonal components \(\rho_{n \pm 1} \rho \mathcal{P}\) that control the matrices \(\hat{A}\) and \(\hat{B}\). The latter two are small, being generated by the rare tunnelling events, and can be expressed in terms of the diagonal elements using perturbation theory. To obtain, e.g. \(\hat{A}\), one uses the equation of motion, Eq. (2.10), to obtain
\[
\dot{\hat{A}}_{i}^{(n)}(t) = -i \omega_{tr} \hat{\rho}_{i}^{(n)}(t) + i[\hat{H}_S, \hat{\rho}_{i}^{(n)}(t)] = -\text{Tr}_{el} \left( c_i^r c_r [H_T, \rho](t) P_n \right)
\]
(4.4)
where \(\omega_{tr} \equiv \varepsilon_l - \varepsilon_r\). To lowest order in tunnelling, one evaluates the source term on the r.h.s. retaining only the charge diagonal \(\mathcal{P}_m \rho \mathcal{P}_m\) components \((m = n\) and \(m = n - 1\)) of the full density matrix \(\rho\); the off-diagonal components, \(\mathcal{P}_{n \pm 1} \rho \mathcal{P}_{n \pm 1}\), give higher order corrections which are neglected (for details we refer to appendix A). The diagonal terms can be expressed in terms of the single particle distribution functions for the electrodes, \(f_{l,r}\), which, since the electrodes act as particle reservoirs, can be taken independent of the total charge number of the particles in the left electrode, \(n\). As shown in appendix A, to lowest order in the tunnelling the inhomogeneous term on the right in Eq. (4.4) can be expressed in terms of the charge specific density matrix and an explicit solution for \(\hat{A}\) (and similarly for \(\hat{B}\)) can be obtained. The resulting equation is a non-Markovian master equation for the charge specific density matrix for the system, Eq. (A15). In the limit where the part of the tunnelling matrix element in Eq. (6.1) which depends on the coupling to the system is small, the temporal non-locality of the kernels can be neglected and the Markovian equation is obtained. The master equation for the charge specific density matrix \(\hat{\rho}_n\) has the form:
\[
\dot{\hat{\rho}}_n(t) = \frac{1}{i} \left[ [\hat{H}_S + \hat{\mathcal{M}}, \hat{\rho}_n(t)] + \mathcal{L} \{ \hat{\rho}_n(t) \} + \mathcal{D} \{ \hat{\rho}_n(t) \} + \mathcal{J} \{ \hat{\rho}_n(t) \} \right]
\]
(4.5)
where \(\hat{\rho}_n'\) and \(\hat{\rho}_n''\) denote “discrete derivatives”:
\[
\hat{\rho}_n' = \frac{1}{2} (\hat{\rho}_{n+1} - \hat{\rho}_{n-1}), \quad \hat{\rho}_n'' = \hat{\rho}_{n+1} + \hat{\rho}_{n-1} - 2\hat{\rho}_n.
\]
(4.6, 4.7)
The Hamiltonian \(\hat{H}_S\) describes the dynamics of the isolated system, and the rest of the terms on the r.h.s. of Eq. (4.5) describe the effect of the tunnelling events; in the formulae below, the subscript \(\pm\) marks the contribution of the processes when an electron tunnels from the left (right) electrode to the right (left) one. The “magnetization” term, \(\hat{\mathcal{M}} = \hat{\mathcal{M}}_+ + \hat{\mathcal{M}}_-\),
\[
\hat{\mathcal{M}}_+ = \frac{1}{2i} \sum_{i,r} f_l (1 - f_r) \left( \hat{T}_{i} \left[ \hat{T}_{i}^\dagger \right] - h.c. \right),
\]
(4.8)
\[
\hat{\mathcal{M}}_- = \frac{1}{2i} \sum_{i,r} f_r (1 - f_l) \left( \hat{T}_{r} \left[ \hat{T}_{r}^\dagger \right] - h.c. \right)
\]
(4.9)
renormalizes the system Hamiltonian $\hat{H}_S$. The operator

$$\mathcal{L}_+ \{ \hat{\rho} \} = \sum_{l,r} f_l (1 - f_r) \left( \hat{T}_{l,r}^{\dagger} \hat{\rho} \hat{T}_{l,r} - \frac{1}{2} \left[ \left( \hat{\rho} \hat{T}_{l,r}^{\dagger} \right)_+ + \text{h.c.} \right] \right)$$

(4.10)

and

$$\mathcal{L}_- \{ \hat{\rho} \} = \sum_{l,r} f_r (1 - f_l) \left( \hat{T}_{l,r} \hat{\rho} \hat{T}_{l,r}^{\dagger} - \frac{1}{2} \left[ \left( \hat{\rho} \hat{T}_{l,r} \right)_+ + \text{h.c.} \right] \right).$$

(4.11)

The diffusion, $\mathcal{D}$, and drift, $\mathcal{J}$, operators are

$$\mathcal{D} \{ \hat{\rho} \} = \frac{1}{2} \left( \mathcal{L}_+ \{ \hat{\rho} \} + \mathcal{L}_- \{ \hat{\rho} \} \right),$$

$$\mathcal{J} \{ \hat{\rho} \} = \mathcal{L}_+ \{ \hat{\rho} \} - \mathcal{L}_- \{ \hat{\rho} \},$$

(4.12)

where

$$\mathcal{D}_+ \{ \hat{\rho} \} = \sum_{l,r} f_l (1 - f_r) \left[ \hat{T}_{l,r}^{\dagger} \hat{\rho} \hat{T}_{l,r} + \text{h.c.} \right]$$

(4.13)

and

$$\mathcal{D}_- \{ \hat{\rho} \} = \sum_{l,r} f_r (1 - f_l) \hat{T}_{l,r} \hat{\rho} \hat{T}_{l,r}^{\dagger} + \text{h.c.}$$

(4.14)

We have introduced the notation (but suppressed the time dependence in the above formulas)

$$\left[ \hat{T}_{l,r}^{\dagger} \right] (\tau) = \int_0^\tau d\tau' e^{i\omega_{l,r} \tau + i \int_0^{\tau'} \frac{dV(t')}{e}} e^{-i\hat{H}_S \tau} \hat{T}_{l,r}^{\dagger} e^{i\hat{H}_S \tau}$$

(4.15)

and assumed a voltage $U$ applied to the junction, $V(t) = eU(t)$, $e$ the electron charge.

The master equation, Eq. (15), requires an initial condition. We assume that the left and right electrodes of the junction are disconnected at times preceding the initial moment $t = 0$, so that the electrodes are in definite charge states, say with $n^{(0)}$ electrons in the left electrode. Using the convention that $n$ is counted from $n^{(0)}$, the variable $n$ becomes the number of electrons transferred from the right to the left electrode. The initial condition reads

$$\hat{\rho}_n (t = 0) = \delta_{n,0} \hat{\rho}^{(0)}$$

(4.16)

where $\hat{\rho}^{(0)}$ is the initial density matrix of the measured system.

The kernel $\mathcal{D}$ describes diffusion in charge space, and the average charge current from the left to the right electrode, $-e \sum_n n \text{Tr} \hat{\rho}_n(t)$, is given by the drift term $\mathcal{J}$

$$I(t) = e \text{Tr} \mathcal{J} \{ \hat{\rho}(t) \}$$

(4.17)

where $\hat{\rho}$ is the (unconditional) density matrix

$$\hat{\rho} = \sum_{n} \hat{\rho}_n(t).$$

(4.18)

We stress that the form of the derived master equation for the charge specific density matrix is valid for any quantum object coupled to the junction. In the present paper, we shall consider the electrons coupled to a two-level system or for short a spin, a system considered in numerous papers as discussed in the introduction and $\hat{\rho}_n$ will be a $2 \times 2$ matrix. A remaining task is thus to unravel the complex spin structure of the master equation presently hidden in the bracket operation. However, first we consider the isolated tunnel junction.

V. ISOLATED TUNNEL JUNCTION

To familiarize with the charge representation we pause to consider the case where the system is decoupled from the junction. In this case, the only degree of freedom of interest is the charge $n$, and the equation for the probability, $p_n$, for the transfer of $n$ charges

$$\dot{p}_n = D p''_n + I p'_n$$

(5.1)

is obtained by taking trace of the master equation Eq. (4.15). Here,

$$I = 2\pi \sum_{l,r} |T_{l,r}|^2 (f_l - f_r) \delta (\epsilon_l + V - \epsilon_r).$$

(5.2)

and

$$D = \pi \sum_{l,r} |T_{l,r}|^2 (f_l + f_r - 2 f_l f_r) \delta (\epsilon_l + V - \epsilon_r).$$

(5.3)

For simplicity, we assume the bias $V$ is time independent.

The physical meaning of the parameters $I$ and $D$ in Eq. (5.1) can be deduced from the structure of the equation. Multiplying Eq. (5.1) by $n$ and performing summation with respect to $n$, one readily sees that $I$ equals the dc current through the junction, i.e., the average rate
of charge transfers from the left to the right electrode, \( I = -\frac{1}{e\tau} \sum_{n} np_{n}(t) \). The value of the \( dc \) current \( I = I(V) \) is given by Eq. (5.2). For the time evolution of the variance of the charge distribution, \( \Delta n(t) = \langle n^{2}(t) \rangle - \langle n(t) \rangle^{2} \), \( t \), where \( \langle n^{k}(t) \rangle = \sum_{n} n^{k}p_{n}(t) \), one obtains from Eq. (5.1)
\[ \frac{d\Delta n}{dt} = 2D \] (5.5)
thus \( D \) has the meaning of the charge diffusion coefficient, characterizing the randomness of the charge transfers.

To connect with the standard noise discussion in terms of current fluctuations, we express the transferred charge, \( n(t) \) as the time integral of the current \( i(t) \), \( n(t) = \int_{0}^{t} dt'i(t') \), and the variance takes the form
\[ \Delta n(t) = 2 \int_{0}^{t} dt' \int_{0}^{t'} d\tau S(\tau) \] (5.6)

where \( S(\tau) = S(-\tau) \) is the current-current correlator, \( S(\tau) = (\delta i(t)\delta i(t + \tau)) \), \( \delta i(t) = i(t) - \langle i \rangle \). (5.7)

At times much larger than the current-current correlation time, one obtains from Eq. (5.5)
\[ \frac{d\Delta n}{dt} = 2 \int_{0}^{\infty} d\tau S(\tau) \] (5.8)

Recalling Eq. (5.5), one establishes the relation,
\[ 4D = S_{\omega=0} \] (5.9)

between the charge diffusion coefficient \( D \) and the power spectrum of the current noise, \( S_{\omega} = 2 \int_{0}^{\infty} d\tau S(\tau) \cos(\omega \tau) \), at zero frequency.

In quasi-equilibrium, when \( f_{l,r} \) are Fermi functions corresponding to the temperature \( T \), one readily obtains from Eqs. (5.5) and (5.9) that the charge diffusion constant \( D \) is related to the \( dc \) current \( I \) as \( 2D = I \coth V/2T \). In view of Eq. (5.2), this corresponds to the well-known result \( 23 \)
\[ S_{\omega=0} = 2I(V) \coth \frac{V}{2T} \] (5.10)

which expresses the fluctuation-dissipation theorem (and holds for arbitrary voltages for the case of a tunnel junction.) \( 24 \)

Given the initial charge distribution, Eq. (5.1) can be solved introducing the Fourier transform \( \chi(\lambda; t) = \sum_{n} p_{n}(t)e^{i\lambda n} \). For the initial condition \( p_{n}(t=0) = \delta_{n,0} \), the solution reads
\[ \chi(\lambda; t) = \exp[2Dt(\cos \lambda - 1) - iIt \sin \lambda] \] (5.11)

Not surprisingly, this expression reproduces the long time (Markovian) limit of the generating function of the full counting statistics for a tunneling junction first derived in Ref. 26. Inverting the Fourier transform (see appendix B.1 for details), the probability of \( n \)-charge transfers in the time span \( t \) reads
\[ p_{n}(t) = I_{n} \left( \frac{\langle n \rangle}{\sinh \nu} \right) \exp[vn - \langle n \rangle \coth \nu] \] (5.12)

where \( I_{n} \) is the modified Bessel function. \( \langle n \rangle = I(V)t \) is the average charge transfer for the bias \( V \), and \( v = V/2T \). The moments of the distribution, \( \langle n^{k} \rangle \) and \( \langle n - \langle n \rangle \rangle^{k} \), can be evaluated with the help of the generating functions Eq. (B.9) in appendix B.1.

VI. SHORT TIME MEASUREMENT

The charge specific density matrix can be used to describe a quantum measurement in a two-fold way: (i) measuring the charge transferred through the junction provides information about the state of the system coupled to the junction, or (ii) a measurement of the coupled system can reveal information about the charge state of the junction. In this paper we shall consider the case where the system coupled to the junction is a two-level system or for short a spin, whose internal dynamics is described by \( H_{S} = \Omega \sigma_{x} \). We are interested in the read-out of the state of the spin, and we analyze how measuring the charge can be a projective measurement of the spin.

For definiteness, we chose to measure the \( z \)-projection of the spin, and for the measurement in this basis the coupling to the spin is taken to be
\[ \hat{T}_{lr} = v_{lr} + \sigma_{z} w_{lr} \] (6.1)
so that the spin up and down states correspond to extreme values in tunnelling strength. It is convenient to introduce the following quantities:
\[ \left\{ \begin{array}{l} G_{W} \\ G_{1} \\ G_{2} \end{array} \right\} = 2\pi \sum_{lr} \left\{ \begin{array}{l} V_{lr} \\ W_{lr} \\ U_{1lr} \\ U_{2lr} \end{array} \right\} \left( -\frac{\partial f(\varepsilon_{l})}{\partial \varepsilon_{l}} \right) \delta(\varepsilon_{l} - \varepsilon_{r}) \] (6.2)

where \( V_{lr} = |v_{lr}|^{2} \), \( W_{lr} = |w_{lr}|^{2} \), \( U_{1lr} = \Re w_{lr}^{*} v_{lr} \), and \( U_{2lr} = 3v_{lr}^{*} w_{lr} \). For the up and down spin orientation, the tunneling conductance (in the units \( e^{2}/\hbar \)) is given by \( G_{W} + G_{1} \pm 2G_{2} \), respectively.

Following von Neumann\( 23 \) we aim at an effectivley instantaneous measurement of the spin quantum state, which means that the time of the measurement must be short compared to the intrinsic spin precession time. To describe the measurement process, we are therefore interested in solving the master equation for times much shorter than the inverse of the Rabi frequency \( \Omega \). In that case, \( H_{S} \) in Eq. (4.15) and Eq. (4.16) can be set to zero, and the spin structure of the master equation, Eq. (4.15), reduces considerably.
In this limit, the renormalization term \( \hat{M} = \hat{M}_+ + \hat{M}_- \), defined by Eqs. (4.8), and (4.9) is \( \hat{M} = M_\sigma \sigma \) where 

\[
M_\sigma = 2 \sum_{\text{tr}} \Re v_{\text{tr}} w_{\text{tr}} \frac{f_l - f_r}{\varepsilon_l + V - \varepsilon_r} \tag{6.3}
\]

The physical mechanism behind the renormalization of the spin “Zeeman” energy given by \( M_\sigma \) is the sensitivity of the total electron energy to the spin orientation. Indeed, in second order perturbation theory with respect to tunneling, the single electron levels acquire a shift controlled by the square modulus of the tunneling matrix element which is either \( |v_{\text{tr}} + w_{\text{tr}}|^2 \), when the spin is up, or \( |v_{\text{tr}} - w_{\text{tr}}|^2 \), when the spin is down. The interference term \( 2 Re v_{\text{tr}} w_{\text{tr}} \) gives rise to the spin dependence of the energy and, therefore, \( M_\sigma \) in Eq. (6.3). The sign of the combination \( Re v_{\text{tr}} w_{\text{tr}} \) is not fixed by any physical requirement. In a mesoscopic system, we expect it to be a random function of the quantum numbers \( l \) and \( r \), fluctuating both in absolute value and sign. In view of this, the contributions to \( M_\sigma \) from the states in the wide energy range of the order of the Fermi energy, have the tendency for cancellation. As an estimate, \( M_\sigma \sim G_1 \delta E \) where \( \delta E \) is a narrow energy interval within which \( Re v_{\text{tr}} w_{\text{tr}} \) are correlated. The renormalization may therefore be a small correction, which we neglect below.

The master equation, Eq. (4.15), becomes 

\[
\dot{\hat{\rho}}_n = \frac{G_2 V}{i} [\hat{\sigma}_z, \hat{\rho}_n] + 2G_2 W V \coth \frac{V}{2T} (\hat{\sigma}_z \hat{\rho}_n \hat{\sigma}_z - \hat{\rho}_n) \\
+ \frac{1}{2} V \coth \frac{V}{2T} (G_V \hat{\rho}_n'' + G_1 \hat{\rho}_n', \hat{\sigma}_z) + G_W \hat{\sigma}_z \hat{\rho}_n'' \hat{\sigma}_z \\
+ V (G_V \hat{\rho}_n'' + G_1 \hat{\rho}_n', \hat{\sigma}_z) + G_W \hat{\sigma}_z \hat{\rho}_n'' \hat{\sigma}_z \\
+ iG_2 V [\hat{\rho}_n', \hat{\sigma}_z] + iG_2 V \coth \frac{V}{2T} [\hat{\rho}_n', \hat{\sigma}_z], \tag{6.4}
\]

where \( G_{V,W,1,2} \) are defined in Eq. (6.2).

At short times, the components of the density matrix, 

\[
\hat{\rho}_n = \left( \begin{array}{c}
u_n \\ \alpha_n^* \\ \alpha_n \\ d_n \end{array} \right), \tag{6.5}
\]

decoupled, and solving Eq. (6.4) amounts to solving the following three equations:

\[
\dot{u}_n = \frac{1}{2} V \coth \frac{V}{2T} (G_V + G_W + 2G_1) u_n'' \\
+ V (G_V + G_W + 2G_1) u_n', \tag{6.6}
\]

\[
\dot{d}_n = \frac{1}{2} V \coth \frac{V}{2T} (G_V + G_W - 2G_1) d_n'' \\
+ V (G_V + G_W - 2G_1) d_n', \tag{6.7}
\]

\[
\dot{\alpha}_n = -2iG_2 V \alpha_n - 2G_2 W V \coth \frac{V}{2T} \alpha_n \\
+ \frac{1}{2} \left( V \coth \frac{V}{2T} (G_V - G_W) - 2iG_2 V \right) \alpha_n'' \\
+ \left( V (G_V - G_W) - 2iG_2 V \coth \frac{V}{2T} \right) \alpha_n'. \tag{6.8}
\]

Solving the master equation, Eq. (6.4), is thus achieved once the equation with the structure

\[
\dot{x}_n = D_x x_n'' + J_x x_n' \tag{6.9}
\]

is solved. According to appendix B, the solution is specified in terms of the modified Bessel function

\[
x_n(t) = x_0 \left( \frac{D_x}{D_{x+}} \right)^{n/2} e^{-2D_x t} I_n(2t \sqrt{D_{x+} D_{x-}}) \tag{6.10}
\]

where \( D_{x\pm} = D_x \pm \frac{1}{2} J_x \). All moments of the stochastic process \( x_n(t) \) can be expressed in terms of the parameters \( D_x \) and \( J_x \) which for the original problem becomes in terms of voltage, temperature and time.

### A. Characteristic times

Let us now analyze the time dependence of the charge specific density matrix. It is found solving master equation with the initial condition Eq. (4.10), where \( \dot{\rho}^{(0)} \) is the initial spin state (to be measured),

\[
\dot{\rho}^{(0)} = \left( \begin{array}{c} u_0 \\ \alpha_0 \\ d_0 \end{array} \right). \tag{6.11}
\]

First, we consider the unconditional spin density matrix \( \rho \), Eq. (4.18), which gives information about the spin state irrespective of the outcome of the charge measurement \( \text{i.e.} \), discarding the results of a charge measurement. The equation for \( \dot{\rho} \),

\[
\dot{\rho} = \frac{G_2 V}{i} [\hat{\sigma}_z, \rho] + 2G_2 W V \coth \frac{V}{2T} (\hat{\sigma}_z \hat{\rho} \hat{\sigma}_z - \rho), \tag{6.12}
\]

is obtained by summation with respect to \( n \) in Eq. (6.14). Its solution reads,

\[
\dot{\rho}(t) = \left( \begin{array}{c} u_0 \\ \alpha_0 e^{2iG_2 V t e^{-t/\tau_d}} \\ d_0 \end{array} \right), \tag{6.13}
\]

where the parameter \( \tau_d \),

\[
\tau_d^{-1} = 2G_2 W V \coth \frac{V}{2T}, \tag{6.14}
\]

is the decoherence time, which gives the decay rate of the charge unconditional off-diagonal elements of the density matrix \( \sum_n \alpha_n \).

As time passes, the probability distributions for being in the spin up or down states, \( u_n(t) \) and \( d_n(t) \), will according to Eq. (6.10) drift and broaden in charge space. To investigate the different drift in the two distributions we consider their overlap in charge space, \( \sum_n u_n(t) d_n(t) \) and we get according to Eq. (B4)

\[
\sum_n u_n(t) d_n(t) = u_0 d_0 e^{-2V t \coth \frac{V}{2T} (G_V + G_W)} I_0(t) \tag{6.15}
\]
where \( \tilde{t} = 2Vt \coth \frac{V}{2T} \sqrt{(G_V + G_W)^2 - 4G_1^2 \tanh^2 \frac{V}{2T}} \).

At large times, \( \tilde{t} \gg 1 \), the overlap decays exponentially, \( e^{-t/\tau_z} \), with the characteristic rate

\[
\tau_z^{-1} = 8V \tanh \frac{V}{2T} \frac{G_1^2}{G_V + G_W} \\
\times \left(1 + \frac{4G_1^2 \tanh^2 \frac{V}{2T}}{(G_V + G_W)^2}\right)^{-1}.
\]

After the time span \( \tau_z \), the probability distributions for spin up and down have separated in charge space.

A similar calculation for the off-diagonal elements of the charge specific spin density matrix gives

\[
\sum_n |\alpha_n(t)|^2 = |\alpha_n(0)|^2 e^{-2i(G_V + G_W)\coth \frac{V}{2T} t} I_0(t^*),
\]

where \( t^* = 2tV \coth \frac{V}{2T} |G_V - G_W + 2G_2| \), and at large times the off-diagonal elements have decayed exponentially at the characteristic decoherence rate

\[
\tau_\perp^{-1} = 2V \coth \frac{V}{2T} (G_V + G_W - |G_V - G_W + 2G_2|).
\]

After a time span \( \tau_\perp \), the coupling to the current has thus reduced the spin state to a mixture, the charge specific density matrix being diagonal in the measurement basis. We note that there is no simple relation between \( \tau_\perp \) and the decoherence time \( \tau_d \) in Eq. (6.19). Generally, \( \alpha_n \) is more robust relative to decoherence than \( \sum_n \alpha_n \).

Although the characteristic times in Eqs. (6.16), (6.17), and (6.18) depend on different combinations of parameters of the model, their ratio seems to be rather universal and we expect quite generally that

\[
\frac{\tau_\perp}{\tau_z} = c \tanh^2 \frac{V}{2T}.
\]

where \( c \) is a constant of order unity.\(^{28}\)

We have identified the characteristic time scales in the problem, and found their dependences on voltage and temperature. The three time scales describe the coherent dynamics of the spin under different experimental conditions. When the charge state of the junction is left unobserved, the dynamics of the spin is given by the unconditional spin density matrix, Eq. (6.13), which decohers on the time scale, \( \tau_d \), given in Eq. (6.14). When the spin evolution is conditioned on the charge state, the dynamics of the spin is given by the charge specific density matrix. In the shot noise regime, \( V \gg T \), spin-charge separation and the decoherence of the charge specific density matrix happens on the same time scale, \( \tau_\perp \sim \tau_z \), and the tunnel junction provides a projective measurement of the spin after a time span of this order. For small voltages, \( V \ll T \), when \( \tau_\perp \ll \tau_z \), the tunnel junction provides a projective measurement of the spin only after the long time span \( \tau_2 \).

To study the time evolution of the charge specific spin density matrix in detail we turn to analyze the obtained analytical results.

\[\text{VII. "WATCHING" A QUANTUM MEASUREMENT}\]

We shall now consider the time evolution of the measurement of the state of the spin performed by the tunnel junction. The charge specific density matrix is presented in the Pauli basis

\[
\hat{\rho}_n(t) = \frac{p_n(t)}{2} \left(\mathbb{1} + s_n(t) \cdot \hat{\sigma}\right),
\]

where \( p_n(t) \) is the probability that \( n \) electrons have passed through the junction at time \( t \) and the polarization or Bloch vector \( s_n(t) \) indicates a point within the unit Bloch sphere.

The coupling between the spin and the tunnel junction is chosen to have the spin structure given in Eq. (6.14), and the parameters \( \tau_r \) and \( \tau_w \) are in the following taken to be real constants. In that case \( G_2 \) vanishes; the ratio of the conductivities is taken to be \( G_W = 0.01G_V \).

Initially the junction is in the definite charge state, \( p_n(0) = \delta_{n,0}, \) as depicted in figure (d). For definiteness, we choose the spin initially in the pure state corresponding the Bloch vector \( s_0(0) = (-1, 0, 0) \). The further evolution of Bloch vectors takes place only in the xz-plane, and the charge specific spin density matrix can therefore be represented by a line rising vertically from the point indicating the location of the Bloch vector \( s_n(t) \). Its height (ending in a black dot for visual clarity) represents the probability for the transfer of \( n \) charges, \( p_n(t) \).

The initial state is thus represented as depicted in figure (a). At a later time \( t \), many vertical lines are present, representing the probabilities for various charge transfers and their corresponding spin states as depicted in figure (b). In fact, many Bloch vectors \( s_n(t) \) will appear close to each other, and placing a square grid over the x-z plane cross section several Bloch vectors, differing from each other in charge number by units of one, can lie in the same square. For visual clarity, however, only one vertical line in the figures will be associated with a square, and its height is the sum of the individual probabilities \( p_n \) belonging to the Bloch vectors in the square.

We now turn to study the time evolution of the charge specific spin density matrix, and start by considering the shot noise regime where the junction is biased by a high voltage, \( V \gg T \).

\[\text{A. Schottky limit}\]

The evolution of the Bloch spin density for the case where the voltage is larger than the temperature, \( V = 20T \), is presented in figure (b) and (c). At time \( t = \tau_z/2 \approx \tau_\perp/2 \), the Bloch spin density is spread along the unit circle as depicted in figure (b) (the corresponding charge numbers increasing in the clockwise direction). The visualization shows that during the evolution, the charge specific density matrix stays pure. This is characteristic of the shot noise regime, where the charge trans-
fer statistics is Poissonian, though not universal since to some extent dependent on the choice of constant tunnelling matrix elements. At this time, $t = \tau_z/2$, where on the average roughly ten electrons have tunnelled, the charge probability distribution, $p_n = u_n + d_n$, depicted in figure 1(e), is built equally of the charge conditioned probabilities for the spin to be in state up or down. At a larger time, $t = 5\tau_z$, the Bloch spin density distribution, depicted in figure 1(c), has split and is located near either of the Bloch vectors $s_1 = (0, 0, 1)$ or $s_1 = (0, 0, -1)$, corresponding to the spin up and down states, respectively. At this time, where on the average roughly one hundred electrons have tunnelled, the charge probability peak built solely by the probabilities for the spin to be in state up and a charge probability peak built by the probabilities for the spin to be in state down, and has the shape depicted in figure 1(f). The total probability in the two peaks both equal one half, the probabilities initially for the spin to be in state up or down. The probability distributions for charge and spin have thus through interaction come in one-to-one correspondence, and measuring the charge state of the junction at times larger than $\tau_z$ is thus a measurement of the state of the spin prior to interaction with the tunnel junction. The relative frequency with which a charge state in either of the two peaks is realized is equal to the probability for the corresponding spin state at the start of the measurement. The junction thus functions as a projective measuring device of the spin. To be effective, we see from figure 1(f), that on the average roughly one thousand electrons have tunnelled.

### B. Nyquist limit

Next we study the evolution of the charge specific spin density matrix for the case where the voltage is smaller than the temperature, $V = 0.2T$. The initial spin and charge states are chosen as in the high voltage case depicted in figure 1(a) and (d), so initially the Bloch spin density is concentrated near the initial Bloch vector, $s = (1, 0, 0)$. However at larger times, the evolution of the Bloch spin density is quite different from the high voltage case proceeding in two steps as noticed from figure 2(a). At time $\tau_z$, the $x$-component of the Bloch vectors, $s_n(\tau_z)$, have started to decay, and the charge specific spin density matrix is no longer pure. At this time on the average only roughly five electrons have tunnelled according to figure 2(d). At the much larger time $t = \tau_z/2$, $\tau_z \simeq 100\tau_1$, it is seen from figure 2(b), that the Bloch vectors are concentrated on the $z$-axis, the charge specific spin density matrix has evolved into a mixture. At this time, where on the average roughly one hundred electrons have tunnelled, the charge specific spin up and down distributions, $u_n$ and $d_n$, have started to charge separate, but not enough to significantly distort the total charge distribution as evident from figure 2(e). At the later time $t = 5\tau_z$, the Bloch spin density is again separated and located at the sites corresponding to the spin up and down states as depicted in figure 2(c), and the charge probability distribution is again spin separated, and the total probability in the two peaks equals one half, the probabilities initially for the spin to be in state up or down. A projective spin measurement has again been performed by the tunnel junction. To be effective, we see from figure 2(f), that on the average roughly one thousand electrons must have tunnelled.

### VIII. Measurement Time

In the measuring scheme under consideration, information about the spin state is obtained from a charge measurement. In the long time limit, the charge probability distribution acquires two distinct peaks (as seen in Figures 1(f) and 2(f)), and the outcome of a charge measurement is almost always in the vicinity of $Q_u$ or $Q_d$, $Q_u \perp Q_d$ being the positions of the peaks. Ideally, one reads off the $z$-projection of the spin, $\uparrow$ or $\downarrow$, from the charge outcome $Q_u$ or $Q_d$. At any finite time, this scheme may produce errors. The intrinsic error mechanism is electrical noise due to which the peaks have a finite width and, therefore, partially overlap. Additionally, the spin state is not pure at finite times, spin purification occurring only asymptotically. Also, errors may occur due to a finite resolution $\Delta Q_D$ of the charge detector. The time span, $t_n$, needed to make a reliable measurement, is defined by the condition that the information gained from the charge measurement suffices to identify the $z$-projection of the spin after the measurement.

#### a. An ideal charge detector

Generates certain discrete values, $n$, as the outcome of the charge measurement with the probability $p_n = Tr \hat{\rho}_n$, and leaves the spin in the state described by the density matrix proportional to $\rho_n$. With $\hat{\rho}_n$ of the form in Eq. (6.4), the probability $p_n$ equals $p_n = u_n + d_n$, and

$$ p_n^\uparrow = \frac{u_n}{u_n + d_n}, \quad p_n^\downarrow = \frac{d_n}{u_n + d_n} \quad (8.1) $$

are the probabilities for the spin to be in the up or down states for a given observed $n$.

Asymptotically, the distributions $u_n(t)$ and $d_n(t)$ do not overlap in accordance with Eq. (6.13). Therefore, only one of the diagonal elements, $u_n$ or $d_n$, remains finite.
To quantify the efficiency of the measurement at finite times, we take an approach well known in the context of information theory (see e.g. ref. [30]). Considering the outcomes of the joint measurement of the charge $n$ and spin $\sigma = \uparrow, \downarrow$, as random variables, the probability for the outcome $(n, \uparrow)$ (or $(n, \downarrow)$) is given by $u_n$ (or $d_n$). The joint entropy, $H(\sigma, n)$, equals

$$
H(\sigma, n) = - \sum_n (u_n \log_2 u_n + d_n \log_2 d_n), \quad (8.2)
$$

and the entropy $H(n)$ of the charge variable $n$ is,

$$
H(n) = - \sum_n p_n \log_2 p_n. \quad (8.3)
$$

The conditional entropy, $H(\sigma|n) = H(\sigma, n) - H(n)$, is a measure of the information gained about the spin from a charge measurement. Combining Eq. (8.2) and Eq. (8.3), we obtain

$$
H(\sigma|n) = - \sum_n p_n \left( p^\uparrow_n \log_2 p^\uparrow_n + p^\downarrow_n \log_2 p^\downarrow_n \right). \quad (8.4)
$$

This quantity changes from the initial value, $-(u_0 \log_2 u_0 + d_0 \log_2 d_0)$, to zero at large times.
FIG. 2: (a) Spin-charge evolution in the low voltage case, \( V = 0.2T \). (a) At time \( t = \tau_\perp \), the x-component of the Bloch vectors have started to decay, and the charge specific spin density matrix is no longer pure. (b) At the much larger time \( t = \tau_z / 2 \approx 50\tau_\perp \), the Bloch vectors are concentrated on the z-axis, and the charge specific spin density matrix has evolved into a mixture. (c) At time \( t = 5\tau_z \), the Bloch spin density is located at the pure states \( |\uparrow\rangle \) and \( |\downarrow\rangle \). (d) At time \( t = \tau_\perp \), only an average of hundred charges has tunneled, and the charge distribution, \( p_n = u_n + d_n \), is built equally of the charge conditioned probabilities for the spin to be in state up or down. (e) At time \( t = \tau_z / 2 \approx 50\tau_\perp \), the charge specific spin up and down distributions, \( u_n \) and \( d_n \), have started to charge separate, but not enough to significantly distort the total charge distribution. (f) At time \( t = 5\tau_z \), the charge distribution is spin separated into a peak built solely by the charge conditioned probabilities for the spin to be in state up and a charge probability peak built by the probabilities for the spin to be in state down. A projective spin measurement has been performed by the tunnel junction after an average of roughly one thousand electrons have tunneled.

where \( p_n^{\uparrow \downarrow} \) is either zero or 1. At intermediate times it is a measure of the error in the predicted spin state after the charge measurement. The time dependence of the conditional entropy for the initial state of the spin in the \( x \)-direction is shown in Fig. 3. The curve shows exponential-like decrease of the conditional entropy from the initial value, one bit, to the value zero.

For the case of an ideal charge detector, we define the measurement time \( t_m \) as the time when the spin entropy decreases below a chosen threshold determined by the required fidelity of the measurement. In Fig. 4 we show the measurement time for various parameters with the entropy threshold (arbitrarily) chosen as 0.01. The main conclusion is that the intrinsic measurement time is of order \( \tau_z \) as given in Eq. (6.10), \( t_m \sim \tau_z \).

The most favorable condition for the accuracy of the measurement is the shot noise regime where the charge noise is relatively small. As a rough estimate of the measuring time, \( t_m \sim \tau_z \), in the shot noise regime we obtain from Eq. (6.10)

\[
\frac{t_m^{-1}}{\hbar} \sim \frac{\gamma eU}{\hbar} , \quad \gamma = \frac{8G_v^2}{G_V + G_W}
\]  

(8.5)

where \( U \) is the voltage applied to the junction and \( \gamma \) is a dimensionless constant. For the tunnelling Hamiltonian approach and the Markovian approximation to be justified, the constant \( \gamma \) should be small. If the conducting modes are fully spin effective, i.e., \( v_{\uparrow \downarrow} \sim w_{\uparrow \downarrow} \) in Eq. (6.1), the constant \( \gamma \) is of the order of the conductivity of the junction in units of \( e^2 / \hbar \). Although smallness of \( \gamma \) is needed for the derivation, the theory is expected to be qualitatively applicable even for \( \gamma \sim 1 \). From this one concludes that the intrinsic quantum bound on the measurement time is given by the inequality, \( t_m \gtrsim \hbar / eU \).
for any voltage (in the shot noise regime).

The separation, $\Delta Q = |Q_\uparrow - Q_\downarrow|$, of the two spin orientations in charge space by the time of measurement, can be evaluated as $\Delta Q \sim |G_1|U t_m$. Using the estimate in Eq. (8.6),

$$\frac{\Delta Q}{e} \sim \frac{G_V + G_W}{4|G_1|}$$

for any voltage (in the shot noise regime).

b. A real charge detector may have a resolution lower than $\Delta Q$ given by Eq. (8.6). In this case, the time of measurement is estimated from the condition that the distance in the charge space between the peaks corresponding to opposite spin orientations, exceeds the detector resolution, i.e., $\Delta Q(t_m) > \Delta Q_D$. From this, we get for the lower bound on the measurement time

$$t_m \geq \frac{\Delta Q_D}{e} \frac{e^2}{2\hbar|G_1| eU} .$$

In this case the time of measurement essentially depends on the resolution of the charge measuring device.

In recent experiments involving quantum point contacts coupled to two-level systems, realized by coupled quantum dots, the typical voltage bias was in the $mV$-range and the Rabi frequency of the order of $10^{10}Hz$.\textsuperscript{6,7}

In accordance with the estimate of the measurement time, the scheme studied in the present paper can be realized in currently studied nanostructures.

**IX. SUMMARY AND CONCLUSION**

We have derived the equation of motion, Eq. (1.3), for the charge specific density matrix for a quantum system interacting with an environment, the role of which is played by a tunnel junction connecting two electron reservoirs. In our approach we use the density matrix technique with the environment degrees of freedom partially traced out: We keep track of the charge distribution between the reservoirs and derived the master equation for the charge specific density matrix of the system, $\dot{\rho}_n$, $n$ being the charge variable. The interaction with the environment is through the dependence of the tunnelling amplitudes on the state of the system. This dependence being specified, the derived master equation, Eq. (4.3), is general and can be applied to any system. The charge distribution is a collective variable of the environment, and we have shown that to lowest order in the tunnelling it can be treated classically.

In our analysis of the measurement of a spin (two-level system), the charge variable plays the role of the pointer coordinate in von Neumann's general theory of quantum measurement.\textsuperscript{22} At the conceptual level, the measurement scheme works as follows: One first disconnects the junction, preparing thereby the environment in a certain initial charge state, say $n = 0$. To measure the spin wave function $|\psi_0\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle$ at time $t = 0$, the spin sensitive tunnelling, Eq. (6.1), is switched on at $t = 0$ together with the voltage $U$ and kept till the instant $t = t_m$, the measurement time discussed in Section VIII. Then, the junction is again disconnected, and the transferred charge $Q = en$, the change in the charge of one of the electrodes, is examined. The transferred charge $Q$ is random due to noise, both classical and quantum. However, its probability distribution at the instant $t = t_m$ is concentrated in two well-separated peaks located at $Q_\uparrow$ and $Q_\downarrow$ (as seen in Figures 1(f) and 2(f)). The frequency of occurrence of transferred charge $Q$ in the vicinity of the corresponding peaks, gives the probabilities $|\alpha|^2$ and $|\beta|^2$. The measurement is projective: An observation of the charge state in a state around $Q_\uparrow$ (or $Q_\downarrow$) ensures that the spin after the measurement is in the pure state $|\uparrow\rangle$ (or $|\downarrow\rangle$).
In the shot noise regime, which is the most favorable for performing the measurement, the measurement time and the voltage $U$ are related as $eU t_m \sim \hbar / \gamma$ where $\gamma$ (see Eq. (3.5)) is the effective coupling constant. For our theory to be valid, the coupling constant must be small. However, one may expect that the estimate holds even for $\gamma \sim 1$, giving the ultimate quantum bound, $eU t_m \gtrsim \hbar$, for the time duration of the interaction needed in order for the tunnel junction to function as a measuring device. On the other hand, the duration must be short on the scale of the inverse Rabi frequency $\Omega$ for the measurement to be “instantaneous”. For the typical Rabi frequencies $\Omega \sim 10^{10} \text{Hz}$ the scale of the inverse Rabi frequency $\Omega$ for the measurement becomes “instantaneous”, that is $\Omega t_m \ll 1$, provided the amplitude of the voltage pulse $U$ is above the $mV$ level.

The application of the charge projection method to a tunnel junction, a semi-realistic model of a nanodevice, allows us to study questions regarding quantum measurement, and we have considered the emergence of a projective spin measurement. We have obtained explicit expressions for the characteristic times for decoherence and spin state purification, and visualized the emergence of a projective spin measurement. The model allowed a detailed study of the purification of the charge specific density matrix. The spin measurement by the tunnel junction is quite analogous to the spin measurement of any variable and no “wave function collapse” needs to be invoked. The transmission of electrical noise from the tunnel junction to the quantum object accomplishes the projective measurement. Amplification to the classical level and the emergence of a projective measurement, have thus been illustrated using a model of a nanodevice.

The use of charge projectors to study charge kinetics originated in the context of counting statistics. The derived master equation for the charge specific density matrix shows that the method of charge projectors leads to a useful application of counting statistics. Finally, we note that the obtained master equation allows the study of the charge specific dynamics of an arbitrary quantum object coupled to a tunnel junction. The presented method is therefore suitable for attacking problems in a variety of fields, such as for example quantum electro-mechanical effects in nanostructures.

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**APPENDIX A: DERIVATION OF EQUATION OF MOTION**

In this appendix, the equation of motion for the charge specific density matrix, $\hat{\rho}_n$, is derived for the case where tunneling events are rare, i.e., to lowest order in the tunnelling. It is convenient to use the Heisenberg picture

$$\rho' = e^{-iH_0 t} \rho e^{-iH_0 t}$$

where the density matrix evolves only due to the dynamics of the electrons

$$i\dot{\rho}' = [H', \rho']$$

and the Hamiltonian in the Heisenberg picture is

$$H'(t) = H_I + H_r + H_T'(t)$$

where $H_T'(t) = T(t) + T^\dagger(t)$ and

$$\dot{T}(t) = \sum_{l,r} \hat{T}_{lr}(t) c_l^\dagger c_r,$$

$$\hat{T}_{lr}(t) = e^{iH_0 t} \hat{T}_{lr} e^{-iH_0 t}.$$ (A4)

The equation of motion in the Heisenberg picture for the charge specific density matrix of the system coupled to the junction can be written on the form

$$\dot{\hat{\rho}}_n(t) = \sum_{lr} \left( \hat{T}_{lr}(t) \hat{A}^{(n)}_{lr}(t) + \hat{T}_{rl}(t) \hat{B}^{(n)}_{rl}(t) \right) + h.c. \quad \text{(A5)}$$

where the time dependent operators $\hat{A}$ and $\hat{B}$ are

$$\hat{A}^{(n)}_{lr}(t) = \frac{1}{t} \text{Tr}_{el} \left( c_{l}^\dagger c_{r} \rho'(t) |P\rangle \langle P| \right),$$

$$\hat{B}^{(n)}_{rl}(t) = \frac{1}{t} \text{Tr}_{el} \left( c_{l}^\dagger c_{r} \rho'(t) |P\rangle \langle P| \right). \quad \text{(A6)}$$

A hat indicates operators with respect to the degrees of freedom of the system coupled to the junction.

To obtain, e.g. $\hat{A}$, one uses the equation of motion, Eq. (A2), to obtain

$$\dot{\hat{A}}_{lr}(t) - i\omega_{lr} \hat{A}_{lr}(t) = -\text{Tr}_{el} \left( c_{l}^\dagger c_{r} [T(t), \rho'(t)] |P\rangle \langle P| \right) \quad \text{(A7)}$$

where the term containing $\omega_{lr} \equiv \varepsilon_{l} - \varepsilon_{r}$ originates from the commutator with the electrode Hamiltonian, $H_0 = H_I + H_r$. We must therefore consider the inhomogeneous term on the right of Eq. (A7). According to the relations Eq. (3.5) and Eq. (3.6), it can be rewritten on the form

$$\text{Tr}_{el} \left( c_{l}^\dagger c_{r} \left( \tau^{\dagger} + T \right) \rho' |P\rangle \langle P| \right) = \text{Tr}_{el} \left( c_{l}^\dagger c_{r} \tau T^\dagger P_{n-2} \rho' |P\rangle \langle P| \right)$$

$$-\text{Tr}_{el} \left( c_{l}^\dagger c_{r} P_{n-1} \rho' |P\rangle \langle P| \right) + \text{Tr}_{el} \left( c_{l}^\dagger c_{r} \tau T \rho' |P\rangle \langle P| \right)$$

$$-\text{Tr}_{el} \left( c_{l}^\dagger c_{r} P_{n-1} \rho' |P\rangle \langle P| \right) \quad \text{(A8)}$$

where the time argument has been suppressed. The first two terms are charge “far” off-diagonal components of
the density matrix like $\mathcal{P}_m\rho\mathcal{P}_{m+2}$, and these terms can be neglected since they are of higher order in the tunnelling matrix element and an expression involving only the two charge diagonal components of the density matrix results. These charge diagonal terms, which have the form

$$\text{Tr}_c(\hat{c}_r^\dagger\hat{c}_r T^l\mathcal{P}_n\rho^l\mathcal{P}_n) = \sum_{l'l'} \hat{T}_{rl} T_{l'l'} \text{Tr}_c(\hat{c}_r^\dagger\hat{c}_{l'}^\dagger\rho_{l'}^l\mathcal{P}_n),$$

(A9)

can be approximated by the expressions

$$\text{Tr}_c(\hat{c}_r^\dagger\hat{c}_r T^l\mathcal{P}_n\rho^l\mathcal{P}_n) = \delta_{ll'} T_{rl} \rho_n^l,$$

(A10)

where the $f$'s are the single particle energy distribution functions for the electrodes. The distribution functions are taken independent of the number of charges on the left electrode $n$, since the electrodes are inexhaustible particle reservoirs. The diagonality in the electrode quantum numbers are justified by the fact that when tunnelling is rare, superpositions of different single particle states do not occur.

Approximating similarly the second charge diagonal term, Eq. (A8), makes Eq. (A7) acquire the form

$$\dot{\rho}_n = \text{Tr}_c(\mathcal{P}_n\rho^l\mathcal{P}_n)$$

(A11)

In the following we shall take as initial condition that tunnelling first starts at time $t = 0$, i.e., the junction is disconnected at earlier times. The solution of Eq. (A11) thus becomes

$$\dot{\rho}_n(t) = -f_r(1-f_r) \left[ \hat{T}_{rl}(t) \rho_n^l(t) \right] + f_r(1-f_r) \left[ \hat{T}_{rl}(t) \rho_{n-1}^l(t) \right]$$

(A12)

where the notation

$$X(t) = \int_0^t dt' e^{i\omega_{rl}(t-t')} \int_0^{t'} dt'' V(t'') X(t'')$$

(A13)

has been used, and the feature that the junction may be biased by a time dependent voltage, $V(t)$, is included.

The evaluation of $B$ in Eq. (A6) is analogous and we obtain (of course in the same approximation and for the same initial condition)

$$\dot{\rho}_n(t) = -f_r(1-f_r) \left[ \hat{T}_{rl}(t) \rho_n^l(t) \right] + f_r(1-f_r) \left[ \hat{T}_{rl}(t) \rho_{n+1}^l(t) \right]$$

(A14)

where the dagger indicates hermitian conjugation of the system operators.

Collecting the results, we obtain a non-Markovian master equation for the charge specific density matrix:

$$\dot{\rho}_n(t) = \Lambda' \{ \rho_n^l(t) \} + D_+ \{ \rho_{n+1}^l(t) - \rho_n^l(t) \} + D_- \{ \rho_{n-1}^l(t) - \rho_n^l(t) \}$$

(A15)

where the kernels are

$$\Lambda' \{ \rho \} = \sum_{l'r} f_r(1-f_r) \left[ \hat{T}_{rl}(t) \hat{T}_{rr'} - \hat{T}_{rl}(t) \hat{T}_{lr'}^\dagger \right]$$

(A16)

denoted by temperature and voltage. In the limit where $\text{tunnelling}$ can be neglected on this time scale, the master equation for the spin dynamics becomes Markovian since $\rho'$ can be taken outside the bracket operation [...] in Eq. (A15). We therefore finally arrive at the Markovian master equation, Eq. (156), however there displayed in the Schrödinger picture.

**APPENDIX B: ONE-COMPONENT MASTER EQUATION**

The equations Eqs. (5.1) and (6.6-6.8), have the structure

$$\dot{x}_n = D_x x_n'' + J_x x_n', \quad x_n(t=0) = x_0 \delta_{n,0}.$$  (B1)

The equation can be solved by Fourier transform,

$$x_n(t) = x_0 \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i2D_x(\cos \varphi - 1) + iJ_x \sin \varphi t + i\varphi n}$$  (B2)

giving

$$x_n(t) = x_0 \left( \frac{D_x-}{D_x+} \right)^n e^{-2D_x t} I_n(2\sqrt{D_x+D_x-})$$  (B3)

where $D_x = D_x \pm \frac{1}{2} J_x$, and $I_n$ is the Bessel function.

Let $y_n(t)$ be another variable obeying Eq. (B1) with corresponding coefficients $D_y$ and $J_y$ Then, the “overlap” of the variables, $\sum_n x_n(t)y_n(t)$, can be calculated as

$$\sum_n x_n(t)y_n(t) = x_0 y_0 e^{-2(D_x+D_x)t} I_0(t')$$  (B4)

where $t' = 2t\sqrt{(D_x+)(D_x+D_x-)}$. This expression is used to derive Eqs. (6.15), and (6.17).
1. Charge transfer probability distribution

The solution to the master equation, Eq. (5.1), for the probability of \( n \)-charge transfers, \( p_n \), which has the structure of Eq. (B1), can be read off Eq. (B3) giving

\[
p_n(\tau) = e^{-\tau + \mu n} I_n \left( \frac{\tau}{\cosh \nu} \right) \tag{B5}
\]

in the dimensionless variables

\[
\tau = 2Dt \quad , \quad \nu = \sqrt{\frac{D + \frac{1}{2}I}{D - \frac{1}{2}I}}. \tag{B6}
\]

With \( D \) and \( I \) evaluated from Eqs. (5.2), and (5.3), the parameters \( D \) and \( I \) have the relationship \( I/2D = \tanh(V/2T) \), as required by the fluctuation-dissipation theorem, and the parameter \( \nu \) has the meaning of the dimensionless bias, \( V/2T \).

The charge expectation values, \( \langle n^k \rangle = \sum_n n^k p_n \) and the moments \( \langle (\Delta n)^k \rangle = \), where \( \Delta n = n - \langle n \rangle \), can be obtained from the generating functions, \( F_u \) and \( F_u \),

\[
F_u(\tau) = \sum_{n=-\infty}^{\infty} p_n(\tau) e^{nu}, \quad F_u = \sum_{n=-\infty}^{\infty} p_n(\tau) e^{u(n-\langle n \rangle)} \tag{B7}
\]
as

\[
\langle n^k \rangle = \left. \frac{\partial^k F_u}{\partial u^k} \right|_{u=0}, \quad \langle (\Delta n)^k \rangle = \left. \frac{\partial^k F_u}{\partial u^k} \right|_{u=0}. \tag{B8}
\]

After summation Eq. (B7) with \( p_n \) from Eq. (B5), we get

\[
F_u(\tau) = \exp \left[ \tau \frac{\cosh(v + u)}{\cosh v} - \tau \right], \quad F_u = e^{-u\tau \tanh v} F_u(\tau) \tag{B9}
\]

For the first five moments one gets: \( \langle n \rangle = \tau \tanh v \), \( \langle (\Delta n)^2 \rangle = \tau \), \( \langle (\Delta n)^3 \rangle = \tau \tanh v \), \( \langle (\Delta n)^4 \rangle = \tau + 3\tau^2 \), \( \langle (\Delta n)^5 \rangle = \tau(1 + 10\tau) \tanh v \). In particular, the third moment satisfies the relation

\[
\langle (\Delta n)^3 \rangle = \langle n \rangle \tag{B10}
\]
in agreement with Ref. 24.

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In the simplest model where $\hat{T}_{lr} = v + w\tau$ is a constant, the “constant” $c$ in Eq. (6.19) equals $c = \left(1 + \frac{2|\hat{T}_{u,d}|}{|\hat{T}_{u}|^2 + |\hat{T}_{d}|^2}\right) \left(1 + \sqrt{1 - \frac{(|\hat{T}_{u}|^2 - |\hat{T}_{d}|^2 \tanh \frac{\psi}{2T}}{T_{u,d}^2}}\right)^{-1}$, where $T_{u,d} = v \pm w$. For any value of the parameters, $\frac{1}{2} < c < 1$.

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