A NEW APPROACH TO GAPS BETWEEN ZETA ZEROS

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ABSTRACT. We study the value-distribution of Dirichlet polynomials on the critical line, under the assumption of the so-called alternative hypothesis on the distribution of zeroes of the Riemann zeta function. We also consider some related problems. As an application, we introduce a new approach to the problem of gaps between the zeta zeroes.

1. Introduction.

Studying the zeros of the Riemann zeta function is one of the central themes in analytic number theory. The Riemann hypothesis, which remains unsolved for more than a century, predicts that all of the non-trivial zeros of the zeta function lie on the line \( \text{Re}(s) = \frac{1}{2} \), known as the critical line in the complex plane. Having all the zeros on the critical line leads us to questions about distribution of gaps between zeta zeroes.

In this paper we study the value-distribution of Dirichlet polynomials with respect to the size of gaps between the zeros of the Riemann zeta function. Our main objective is to introduce a new approach that we believe has a more reasonable chance to resolve the problem of “alternative hypothesis” (AH) on the distribution of zeros of the zeta function. AH, formulated by Farmer, Gonek and Lee in [7], models the distribution that the Riemann zeta zeros would have if Landau-Siegel zeros were to exist. According to Farmer, Gonek and Lee “AH is obviously absurd, but it has not been disproved. A sufficiently strong disproof would show that there are no Landau-Siegel zeros”.

To explain these terms and motivate our investigation, we begin with the class number of imaginary quadratic fields. The class number, in a way, would measure the failure of unique factorization in the ring of integers of a number field. For example in \( \mathbb{Z}[\sqrt{-5}] \), which is the ring of integers of \( \mathbb{Q}[\sqrt{-5}] \), we have that 6 can be written as a product of irreducible in two different ways: \( 2 \times 3 \) and \( (1 + \sqrt{-5}) \times (1 - \sqrt{-5}) \). The class number of this field is two.

An important question is how big the class number can get. The answer is very much dependant on the possible existence of so called Landau-Siegel zeros for Dirichlet \( L \)-functions. These zeros are possible counterexamples to the generalized Riemann hypothesis which are on the real line. More precisely a Dirichlet \( L \)-function attached to a real character \( \chi \) of conductor
q might have a real zero within a distance $\log^{-1} q$ from $s = 1$.

The possibility of Landau-Siegel zeros has an unfortunate effect in the class number formula. They can force $L(1, \chi)$, and hence the class number, to be very small. It took a significant effort by Goldfeld, Gross and Zagier [9, 10] to show that the class number gets arbitrary large. This was achieved by introducing the $L$-functions attached to elliptic curves into the problem. In light of this, there is a significant incentive to eliminate this possibility and there has been much effort in number theory in this direction.

To explain the connection with zeros of the Riemann zeta function, it was noticed by Montgomery [13] that small class numbers would imply that zeros of the Riemann zeta function are rigidly spaced. Conrey and Iwaniec in [6] provided a detailed analysis of this. They show that existence of many pairs of consecutive zeros of the zeta function with gap smaller than 0.5 times the average gap would imply that there are no Landau-Siegel zeros.

Over the years this problem attracted much attention and resulted in considerable progress in the theory of $L$-function, including Montgomery’s pair correlation conjecture. If we normalize zeros in a way that their average gap equals 1, Montgomery’s work, under the assumption of the Riemann hypothesis, would imply that there are infinitely many gaps smaller than 0.68. His conjecture, though predicts that the size of the gaps between consecutive zeros of the zeta function can be arbitrary small and arbitrary large. With another method Montgomery and Odlyzko [14] showed that there are normalized gaps smaller than 0.5179. The caveat with this method is that it is unable to prove existence of normalized gaps smaller than 0.5.

Montgomery and Odlyzko’s result was improved to 0.5172 by Conrey, Ghosh and Gonek [5] and subsequently reduced to to 0.5155 by Bui, Mondayovich and Ng [4], all assuming the Riemann hypothesis. Further improvements was achieved in [20, 8] by Feng and Wu and Preobrazhenskiï. Recently, Goldston and Turnage-Butterbaugh [12] using generalized Wu Weighs with a number of new ideas, proved that gaps between the zeros of the zeta function are smaller than 0.50412 infinitely often.

Returning to the problem of the effect of Landau-Siegel zeros on the distribution of zeros of the Riemann zeta function, from the work of Conrey and Iwaniec [6] one can deduce that the existence of these zeros would imply the normalized gap between zeros of the zeta function are close to being half integers. This was further explored by Farmer, Gonek and Lee [7] and they proposed an alternative hypothesis (to the pair correlation conjecture) as follows: Let $\gamma$ be an imaginary part of the a zero of the zeta function. The number of zeros up to the height $T$ is about $(2\pi)^{-1} T \log T$, which makes
the average gap about $2\pi \log^{-1} T$. We normalize them by setting
\[ \tilde{\gamma} = \frac{1}{2\pi} \gamma \log \left( \frac{\gamma}{2\pi} \right), \]
and by $\gamma^+$ and $\gamma^-$ we mean the zero after and before $\gamma$ respectively.

**The Alternative Hypothesis (AH).** There exists a real number $T_0$ such that if $\gamma > T_0$, then
\[ \tilde{\gamma}^+ - \tilde{\gamma} \in \frac{1}{2}\mathbb{Z}. \]
That is, almost all the normalized neighbor spacings are an integer or half-integer.

Let $g_\sigma$ denote the proportion of normalized gaps between consecutive zeros that are equal to $\sigma$. Farmer, Gonek and Lee [7], assuming the AH, showed that $g_{0.5} \approx 0.297$ and $0.405 \leq g_1 \leq 0.5$. (Also see [21] for a similar result under a milder formulation of AH.) Recently, Tao, Rogers, and Lagarias [23, 22] constructed a sequence in $\frac{1}{2}\mathbb{Z}$ that is compatible with what is known about the pair correlation function of zeta zeros. Their result is a good indication as to why disproving AH seems to be a hard problem. We will say more about this in our subsequent paper [2].

1.1. **A new approach to gaps between zeta zeros.** Let us begin by explaining the method of Montgomery and Odlyzko [13] for detecting small gaps between zeros of the zeta function. For $a, b > 0$, consider the probability measure
\[
\mu_A((a, b]) := \frac{\int_a^b \omega(\frac{1}{2} + it)|A(\frac{1}{2} + it)|^2}{\int \omega(\frac{1}{2} + it)|A(\frac{1}{2} + it)|^2},
\]
where
\[
A(s) = \sum_{n<T^1} a(n)n^{-s}
\]
is a Dirichlet polynomial and $\omega$ a cut-off weight, centered around $T$, with $\| \omega \|_1 = 1$. For $A(s) = 1$, the above gives the Lebesgue measure.

Using analytic methods one can show that
\[
\sum_{\zeta(1/2+it) = \gamma} \mu_A\left( \tilde{\gamma} - \frac{c}{2}, \tilde{\gamma} + \frac{c}{2} \right) \approx c - 2 \sum_{mp<T} \frac{a(m)a(mp)}{mp} \sin \left( \pi \frac{\log p}{\log T} \right).\]

To proceed, let us assume that we do not have any (normalized) gap smaller than $c$. Based on this assumption intervals in (1.3) are disjoint, therefore the sum equals the measure of the union of these intervals. By the definition, $\mu$ is a probability measure and the LHS of (1.3) must be smaller than 1. Choosing $a(n) = \lambda(n)$ would flip the negative sign in (1.3) and by the prime number theorem we have the RHS is about $2c - 0.0276\pi^2c^3$ which
by setting \( c = 0.5192 \), is greater than 1.

The improvements \([5, 4, 8, 20, 12]\) that we mentioned are results of various efforts to maximize the quantity

\[
(1.4) \sum_{mp<T} \frac{a(m)a(mp)}{mp} \sin \left( \frac{\pi \log p}{2 \log T} \right) \frac{\alpha}{m}.
\]

The disadvantage of their method is that no matter the choice of \( a(\cdot) \), we have that (1.4) is always smaller than 0.5, which means that the method is unable to prove existence of normalized gaps smaller than 0.5.

In this paper we develop a general tool to study the value-distribution of Dirichlet polynomials given information about distribution of gaps between the zeros of the zeta function. Our main objective is to introduce a refinement of the previous method which seems to have a more reasonable chance to resolve AH.

Our first observation is that in order to reject AH, we need to maximize

\[
(1.5) \frac{2 \sum_{mp<T} a(m)a(mp) \log p}{\log T \sum_{m<T} |a(m)|^2} \frac{1 - \log p}{\log T},
\]

while minimizing (1.4). The logic behind it is that (1.5) is the expectation of a certain test function with respect to the probability measure induced by \( A\left( \frac{1}{2} + it \right) \) i.e.

\[
\int C(t) d\mu_A,
\]

while assuming AH, the measure of the region where \( C(t) \) is positive is around 0.5 + (1.4). Let us give a precise definition of our test function.

**Definition.** Let \( \alpha \in \mathbb{R}^+ \) and let \( \gamma \) be an imaginary part of a zero of the zeta function. Define

\[
(1.7) C_\alpha(t) := \sum_\gamma \left( \frac{\sin \left( \frac{\alpha}{2}(\gamma - t) \log T \right)}{\frac{\alpha}{2}(\gamma - t) \log T} \right)^2 - \alpha^{-1}.
\]

This is a function with large values where we have an accumulation of zeros and it is negative around the large gaps between the zeros. If we assume AH, for \( \alpha \geq 1 \) we have that the test function is always smaller than one.

1.2. **How to contradict the alternative hypothesis.** From the last two paragraphs and assuming AH, we see that the \( \mu_A \)-measure of the region where \( C_1(t) \) is positive is about 0.5 + (1.4) and is bounded above by (1.5). Thus, if we could prove that

\[
(1.5) \geq 0.5 + (1.4),
\]
that would reject AH. In practice, assuming AH would determine, to some extent, the distribution \( \mu_A \). Therefore we can employ more complicated arguments that allows us to get a contradiction with even less severe inequality. For example the quantity

\[
\sum_{mp<T} \frac{a(m)a(mp)}{mp} \left( \sin \left( \frac{2\pi \log p}{\log T} \right) - \sin \left( \frac{\pi \log p}{\log T} \right) \right)
\]

(1.8)
gives the \( \mu_A \)-probability of \( C_\alpha \) attain large values (\( \geq 0.8 \)). Therefore minimizing it will result to an easier path to contradict AH.

1.3. **Advantages over the previous method.** Here we list some major advantages of the refinements compared to the previous method.

- There is no obvious reason the refined method cannot contradict the AH, unlike the Montgomery-Odlyzko method.
- Using the Liouville function is not necessary. In the Montgomery-Odlyzko method to get to small gaps, it is essential to define \( a(n) \) as \( \lambda(n) \) times a positive function. In our method having or not having \( \lambda \) involved, does not make a big difference.
- Since we could avoid using the Liouville function, it would be much easier to incorporate longer Dirichlet polynomials. Considering them requires off-diagonal estimation. If the Liouville function was used then this would require assuming, at a minimum, the strong version of Chowla’s conjecture.

In light of recent work of Tao, Rogers, and Lagarias [23, 22], I could see a possibility of a result of the following type: No matter the choice of \( a(\cdot) \), with support in \([1, T^{1-\epsilon}]\), the distribution of \( \mu_A (\cdot) \) is consistent with AH. However proving such result requires work to be done and I am not sure how hard it would be.

2. **Statement of Results**

As our major goal, we are seeking to maximize (1.5) while minimizing the measure of the region in which the test function is large. In this direction the first question we answer is, what would the best bound (for Dirichlet polynomials of length \( \leq T^{1-\epsilon} \)) we can hope for? The “trivial” bound is

\[
(1.5) = \int C_1(t)d\mu_A \leq 1.
\]

We prove a stronger result:

**Theorem 2.1.** Let \( A(s) \) be as (1.2). We have that

\[
(2.1) \quad \int C_1(t)d\mu_A < 0.79371.
\]

In other words the theorem states that our test function is expected to be smaller than 0.79371, for any probability measure that we make using
Dirichlet polynomials of length smaller than $T$.

In terms of lower bound the best we have so far is

$$\text{(1.3)} \geq 0.74097 \cdots$$

by using $a(n) = \lambda(n) d_{1,8}(n) \left(1 - \frac{\log(n)}{\log T}\right)^{0.4}$. For this choice (1.3) with $c = 0.25$ is about 0.94787, which means that the measure of the region for which the test function is positive is about 0.95.

To make a comparison, if we choose $a(n) = 1$, it would give that (1.3) is 0.6666 $\cdots$ and (1.5) is 0.66 $\cdots$. Recently, Goldston and Turnage-Butterbaugh [12] using generalized Wu weighs with a number of new ideas proved that gaps between the zeros of the zeta function is smaller than 0.50412 infinitely often. Interestingly, their choice is less suitable for my method than the trivial $a(n) = 1$, because it makes (1.5) about 0.63343 and the measure of region where the test function is positive is 0.99254. In a way it maximizes (1.4) and minimizes (1.5). However, it may be possible to manipulate their weight to reach our purpose. The interesting feature of their choice is that the support of the sequence is sparse.

We obtain the above results from the following general theorem.

**Theorem 2.2.** Define $\lambda_{\beta_1, \beta_2}$ to be a completely multiplicative function with $\lambda_{\beta_1, \beta_2}(p) = -1$ for $T^{\beta_1} < p \leq T^{\beta_2}$ and $\lambda_{\beta_1, \beta_2}(p) = 1$ otherwise. Furthermore, let $\mu_{A_{\beta_1, \beta_2}, r, \eta}$ be the measure we build, as in (1.2), using

$$(2.2) \quad a(n) = \lambda_{\beta_1, \beta_2}(n) d_r(n) \left(1 - \frac{\log(n)}{\log T}\right)^\eta,$$

where $d_r$ is the generalized divisor function. Assuming the Riemann hypothesis we have

$$\int C_{\alpha} \left(t + \frac{2\pi d}{\log T}\right) d\mu_{A_{\beta_1, \beta_2}, r, \eta} =$$

$$2r \int_{0}^{\min(\alpha, 1)} \int_{0}^{1-u} \lambda_{\beta_1, \beta_2}(u) \cos(2\pi du) v^{r-1} (1 - u/\alpha)(1 - v)^\eta (1 - u - v)^\eta \frac{\alpha \int_{0}^{1} v^{r-1} (1 - v)^{2\eta}}$$

where $\lambda_{\beta_1, \beta_2}(u) = -1$ for $\beta_1 < u \leq \beta_2$ and equals 1 otherwise.

2.1. **The Maximize vs Minimize Problem.** It is known that AH determines the pair correlation of zeta zeros, but it is not clear if it completely determine the triple or higher correlations. This means that for example under AH we can show that the Lebesgue measure of gaps of length 0.5 is about 0.15. However, for the Lebesgue measure of gaps of length 1 we can just say that it is between 0.4 and 0.5. The reason is that there are two ways to cover intervals of length 1, with two consecutive 0.5-gaps or just one gap of length 1. Therefore we need to know the Lebesgue measure of two
consecutive gaps of length 0.5 for which we need the triple correlation function. In [22, 23] Tao, Rogers, and Lagarias proposed a conjectural model (AGUE) for all the correlation functions assuming the truth of AH.

Similar to the Lebesgue measure AH has a somewhat deterministic effect on every measure we get from Dirichlet polynomials. For example quantity (1.8), under AH, would give the measure of intervals of length 0.5 that comes after a 0.5-gap. This is important in our analysis since it would give an upper bound for the measure of region that $C_\alpha$ is large. We will examine the effect of AH on $\mu_A$ more precisely in Section 4.

An important point of the last paragraph is that assuming AH we can get an upper bound for the expectation of $C_\alpha$ with respect to $\mu_A$. Therefore our strategy to contradict it is to make a measure $\mu_A$ such that the expectation of $C_\alpha d\mu_A$ under AH is smaller than its expectation from Theorem 2.2. This explains why we seek to minimize (1.8) and maximize (1.5).

So far I only tried the conventional choices for $a(\cdot)$. The best choice I found is

$$a(n) = d_{1.4}(n) \left(1 - \frac{\log(n)}{\log T}\right)^{0.2},$$

for which the expectation of the test function is about 0.73. Assuming AH, about 96% of the measure is where the test function is positive and at most 58% of the measure can be where the test function attains large values (\geq 0.8). With a minor calculation we have that in order to have it consistent with AH we must have $C(t) > 0.9$ for almost all of the 58% of the measure on which the test function can be large. Note that for $C(t)$ to be bigger than 0.9, $t$ should vary inside at least four consecutive 0.5-gaps.

I believe that the best results may come from non conventional choices of $a(\cdot)$. For example consider the multiplicative function

$$f(p) := \begin{cases} 1.4, & \text{if } p < T^{0.4} \text{ and } T^{0.6} < p < T \\ 10, & \text{if } T^{0.6} \leq p \leq T^{0.6}. \end{cases}$$

The point is having large values for primes around $\sqrt{T}$ would decrease the measure of the region on which $C(t)$ is large. We conclude this section with some remarks.

**Remark 1.** If the Riemann hypothesis fails to hold we have to amend the definition of (1.7) and state our theorem with

$$C_\alpha(t) = \sum_{\zeta(\rho) = 0} \left(\frac{\sin(\frac{\alpha}{2}(\rho - (\frac{1}{2} + it)) \log T)}{\frac{\alpha}{2}(\rho - (\frac{1}{2} + it)) \log T}\right)^2,$$

where we consider Sine as a complex valued function. In this case $C_\alpha(t)$ can get very large for $t$ that are close to zeros off the critical line. However, for
a prospective application to the Landau-Siegel zero problem, because of the
Deuring-Heilbronn phenomenon the assumption of the Riemann hypothesis
in the theorem may be relaxed a bit.

Remark 2. For \( \alpha \geq 1 \), the alternative hypothesis forces \( C_{\alpha}(t) \) to be smaller
than 1, while we know that \( C_{\alpha}(t) \) is bigger than one near to consecutive gaps
of small size. So one might hope that the moments

\[
(2.5) \quad \int (C_{\alpha}(t) - \frac{1}{\alpha})^k d\mu_A
\]

are bounded away from zero, whereas under AH they tend to zero as \( k \to \infty \). Estimating these moments seems difficult, as there will be complicated
off-diagonal terms. We will say more about higher moments in \([2]\).

3. Discussion and Questions

There are similarities between maximizing (1.5) and maximizing

\[
(3.1) \quad \sum_{mk<T} \frac{a(m)a(mk)}{mk} \sum_{m<T} \frac{|a(m)|^2}{m},
\]

which is the essence of the resonance method for finding large values of
the Riemann zeta function. Soundararajan in \([18]\) found the optimal choice
for Dirichlet polynomials of length \( < T^{1-\epsilon} \). Interesting work was done by
Bondarenko and Seip \([19]\). They used Dirichlet polynomials of length bigger
than \( T \) and dealt with the off-diagonal terms in a following way: by using
an appropriate weight they made sure that contributions of off-diagonal
terms are positive. Then they could throw them out and have a lower
bound which improved Soundararajan’s result. Theoretically, one can use
a similar method here, however we have to to avoid using the Liouville
function and therefore our measure would likely be concentrated on large
gaps. This brings me to the following question:

**Question 3.1.** Is it possible to construct a Dirichlet polynomial, \( A \), of
arbitrary length such that

\[
\int C_1(t) d\mu_A \to 1,
\]
as the length goes to infinity?

Proving such a result may not immediately contradict AH, however it
would settle the large gap conjecture i.e. gaps between zeros of the zeta
function get arbitrary large.

Now consider a Dirichlet polynomial, \( A \), such that in its support we can
find many pairs \( m, m' \) that \( m = m'p \) for some \( p < T \). From (1.5) and our
discussion in previous sections it is clear that \( \mu_A \) correlates with the size of
gaps between zeta zeros. Note that in all of our examples in Theorem 2.2
we used sequences that have full support among integers less than \( T^{1-\epsilon} \).
This brings us to the choice of Goldston and Turnage-Butterbaugh \([12]\). An
interesting feature of their choice is that it is supported on integers with
less than four prime factors, which is sparse in the set of integers. This tells us, using their method, we may be able to take the length of the Dirichlet polynomial slightly bigger than $T$ and still not have too much trouble form off-diagonal terms.

From Theorem 2.1 we can conclude that if the length of $A$ is smaller than $T^{1-\epsilon}$, then $\mu_A$ is unable to detect the microscopic behaviour of zeros. For example the theorem shows that we cannot find $A$ such that $\mu_A$ has more then 50% of its support concentrated on two consecutive gaps of length 0.25.

Now let $A$ be a Dirichlet polynomial of arbitrary length such that in calculation of
\[
\int C_1(t)d\mu_A
\]the contribution of off-diagonal terms are negligible. We call $A$ a sparse Dirichlet polynomial. We conclude this section with a question that we whether or not we can build a measure using sparse Dirichlet polynomials that beats the upper bound in Theorem 2.1

**Question 3.2.** Is it possible to find a sparse Dirichlet polynomial, $A$, of arbitrary length such that
\[
\int C_1(t)d\mu_A > 79371.
\]

4. **Interpretation of $\mu_A$ Under the Alternative Hypothesis.**

AH would approximately determine the distribution of the measures $\mu_A$ defined in (1.1). In this section we will use the following lemmas as a tool to investigate our measures.

**Lemma 4.1.** Let $A(s)$ be as (1.2). Assuming the Riemann hypothesis
\[
\sum_{\gamma} \mu_A[(\gamma+\alpha\frac{2\pi}{\log T}, \gamma + \beta\frac{2\pi}{\log T}] = \beta - \alpha
\]
(4.1) \[-\frac{1}{\pi} \sum a(mp)a(m)\frac{1}{mp} \left( \sin(2\pi\beta\frac{\log p}{\log T}) - \sin(2\pi\alpha\frac{\log p}{\log T}) \right) + O(\frac{1}{\log T}).\]

The next lemma is about our specific choice of $a(n)$ in (2.2).

**Lemma 4.2.** Let $\mu_{A_{\beta_1,\beta_2,r,\eta}}$ be the measure we build, as in (1.2), using
\[
a(n) = \lambda_{\beta_1,\beta_2}(n)d_r(n)(1 - \frac{\log(n)}{\log T})^\eta.
\]Assuming the Riemann hypothesis
\[
\sum_{\gamma} \mu_{A_{\beta_1,\beta_2,r,\eta}}[(\gamma+\alpha\frac{2\pi}{\log T}, \gamma + \beta\frac{2\pi}{\log T}] = \beta - \alpha
\]
\[-r \int_0^1 \int_0^{1-u} \frac{\lambda_{\beta_1,\beta_2}(u)}{u} \left( \sin(2\pi\beta u) - \sin(2\pi\alpha u) \right) v^{r-1}(1-v)^\eta(1-u-v)^\eta \pi \int_0^1 v^{r-1}(1-v)^{2\eta}.
\]
To proceed, we will use the following notations:

- By $\sigma$-gap we mean a normalized gap of length equal to $\sigma$.
- By $\mu_{\alpha_2,\alpha_1,\sigma,\tilde{\beta}_1, x \in I}$ we mean the measure of sub-interval $I$ inside a $\sigma$-gap that is followed by, respectively, an $\alpha_1$ and an $\alpha_2$-gaps. The next gap after $\sigma$ equals $\tilde{\beta}_1$. If we put a dot in any of these places we mean that we have no restriction on that gap or interval.

4.1. Distribution of $\mu_A$ Under AH. We continue with providing a manual as how we can get information about the distribution of $\mu_A$ under the assumption of AH.

1. Measure of region around each zeros. Using Lemma 4.1 we can find a precise estimate for

$$\mu_{A,\pm c} := \mu_A \left( \cup [\tilde{\gamma} - c, \tilde{\gamma} + c] \right)$$

We define $\mu_{A,\text{Out}} := 1 - \mu_{A,\pm 0.25}$.

2. Measure of 0.5-gaps. To estimate the measure of 0.5-gaps we first need to calculate

$$\mu_A \left( \cup \tilde{\gamma} \pm [0.25, 0.5] \right).$$

This will cover all 0.5-gaps plus some middle parts of gaps $\geq 1$, which is smaller than $\mu_{A,\text{Out}}$. Therefore we get that

$$\mu_{A,\pm c} - \mu_{A,\text{Out}} \leq \mu_A(0.5 - \text{gaps}) \leq \mu_{A,\pm c}.$$

This would yield a good estimates if $\mu_{A,\text{Out}}$ is small. For example for the choice Goldston and Turnage-Butterbaugh we get that the measure of 0.5-gaps is about 0.77 with error smaller than 0.01.

3. Measure of region that the test function attain high values. The region where $C_1(t)$ is large often come after (or before) a 0.5-gap. For example two or three consecutive gaps or length 0.5. Consider the following events $E_1 : \cup \tilde{\gamma} + (0, 0.5)$ and $E_2 : \cup \tilde{\gamma} + (0.5, 1)$. We have that $E_1 \cup E_2$ contains all the intervals except parts of gaps $\geq 1.5$. On the other hand we have $E_1 \cap E_2$ contains intervals of length 0.5 that comes after a 0.5-gap. By using $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$ we can get an estimate of measure 0.5-intervals that comes after a 0.5-gap.

4. Measure of isolated 0.5-gaps. Let $E_1 : \cup \tilde{\gamma} + (0.5, 1)$ and $E_2 : \cup \tilde{\gamma} - (0.5, 1)$. We have that $E_1 \cup E_2$ contains all of the 0.5-gaps except the ones that comes before and after $\geq 1$-gaps. On the other hand we have $E_1 \cap E_2$ contains 0.5-gaps that comes after and before 0.5-gaps. Again by using $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$ we can get an estimate on the measure of isolated 0.5-gaps. More precisely using the notation above, we get

$$\mu_{0,5,0.5,0.5, \cdot} + \mu_{0,5,1, \cdot, x < 0.5} + \mu_{\cdot, 1,0.5, x > 0.5} + \mu_{\cdot, 1,5, \cdot, 0.5 < x < 1} = |E_1| + |E_2| - 1 + 2\mu_{\cdot, \geq 1, \cdot, 1.5, x < 0.5} + \mu_{\cdot, \geq 1, 0.5, \cdot, 1}.$$
Remark 3. We will quickly summarize what we get if we apply the above to the basic choice $a(n) = \lambda(n)$. Roughly speaking, we get that the measure of the 0.5-gaps is about 0.77 and the measure of $\geq 1$-gaps about 0.23. Around 40\% of the measure is distributed in 0.5-gaps that are followed by another 0.5-gap. There is a direct correlation between the measure of the 0.5-gaps with next and previous gaps equal to 0.5, and the 0.5-gaps such that next and previous gaps are $\geq 1$.

5. Approximation and Numerical Analysis

In this section we explain how we can test AH using a computer program. First we need to approximate the integral in Theorem 2.2 with a sum and our problem turns to a linear programming problem. To proceed let us define some new notations that we will need in our approximation.

Definition (Approximation of $\mu_A$). For $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{0.5, 1, 1.5, 2\}$, we define

$$\gamma_{\alpha_2, \alpha_1, \beta_1, \beta_2}$$

to be the set of zeros $\gamma$ with the property that $\gamma^+ - \gamma = \beta_1$, $(\gamma^+)^n - \gamma^+ = \beta_2$, $\gamma - \gamma^- = \alpha_1$, and $(\gamma^-)^n - \gamma^- = \alpha_2$, where $\gamma^+$ is the next zero after $\gamma$ and $\gamma^-$ is the zero before $\gamma$. Using this we define:

$$\mu_{\alpha_2, \alpha_1, \beta_1, \beta_2, i} := \sum_{\gamma \in \gamma_{\alpha_2, \alpha_1, \beta_1, \beta_2}} \mu_A((\gamma + \frac{i - 1}{20}, \gamma + \frac{i + 1}{20})).$$

Definition (Approximation of $C_\alpha$). Let $G = \{0, \alpha_1, -\beta_1, \alpha_1 + \alpha_2, -\beta_1 - \beta_2, \alpha_1 + \alpha_2 + m, -\beta_1 - \beta_2 - m : m \in \mathbb{N}\}$, we also define

$$C_{\alpha, \alpha_2, \alpha_1, \beta_1, \beta_2, i} := \sum_{d \in G} \left(\frac{\sin\left(\frac{\alpha}{2}(d + i/20) \log T\right)}{\alpha/2(d + i/20) \log T}\right)^2 - \frac{1}{\alpha}.$$

Employing the notations above and applying Theorem 2.2 with $a(n) = \lambda(n)$ we get he following set of linear equations

$$\sum_{\alpha_i, \beta_i} \sum_{i=1}^{20\beta_1} \mu_{\alpha_2, \alpha_1, \beta_1, \beta_2, i} C_{\alpha, \alpha_2, \alpha_1, \beta_1, \beta_2, i} = \begin{cases} 1 - \frac{\alpha}{3}, & \alpha < 1, \\ \sqrt{1 - \frac{1}{\alpha}}, & \alpha \geq 1, \end{cases}$$

with variables $\mu_{\alpha_2, \alpha_1, \beta_1, \beta_2, i}$ and coefficients $C_{\alpha, \alpha_2, \alpha_1, \beta_1, \beta_2, i}$.

The choice of $i/20$ is just for numerical convenience and it refers to partitioning an interval of length one into 20 equal segments. Also we limited the values of $\alpha_i, \beta_i$ to $\{0.5, 1, 1.5, 2\}$, again just for the sake of approximation. In order to increase the precision we need to insert the effect of more of the neighboring gaps in our variables, considering variables of the form $\mu_{\alpha_m, \cdots, \alpha_1, \beta_1, \cdots, \beta_m, i}$. Doing so would increase both the precision and the number of variables, of which the latter makes it more difficult for the numerical analysis of the linear system.
To solve this system we used the Pulp package in Python that is designed to solve linear system of equalities and inequalities. We applied (5.1) with $\alpha \in \{1, 1.1, 1.2, 1.5, 1.8, 2, 2.5\}$ and we use the shift parameter $\tilde{d} \in \{0, 0.25, 0.45, 0.6, 0.75\}$, and also the information in Remark 3. We allowed an error of the size $0.02\alpha^{-2}$ for each equation we derived from Theorem 2.2. The program we wrote concludes that the system is infeasible, hence the AH is false. There are two caveat here, first, is that we approximate $C_\alpha$ with just four term in its series expansion, therefore there is an error depends on $\alpha$. Second, is that the fact that system is infeasible means that the solver we used was not able to find a solution, and it is not a mathematical proof of of there is no solution to the system. Using other choices than the Liouville’s function may simplifies the calculation to some extent but problem of approximation of $C_\alpha$ remains.

6. Proof of Lemmas and Theorems

We begin giving the proof of of Lemma 4.1. The proof of Lemma 4.2 goes similarly to proof of Theorem 2.2.

Proof of Lemma 4.1. We need to calculate the following

$$
\sum_\gamma \omega\left(\frac{1}{2} + i\gamma\right) \int_{2\alpha\pi \log^{-1} T}^{2\beta\pi \log^{-1} T} |A\left(\frac{1}{2} + i\gamma + ix\right)|^2 \, dx
$$

for $A(s) = a(n)n^{-s}\mathbb{1}_{[n \leq T\log^{-2} T]}$, we expand the square in the numerator and (6.1) comes down to calculating

$$
\sum_\gamma \omega\left(\frac{1}{2} + i\gamma\right) \sum_{m,n} a(m)\overline{a(n)} \left(\frac{n}{m}\right)^{\frac{1}{2} + i\gamma} \int_{2\alpha\pi \log^{-1} T}^{2\beta\pi \log^{-1} T} \left(\frac{n}{m}\right)^{ix} \, dx.
$$

For the $m = n$ we get $(\beta - \alpha) \sum |a(m)|^2 m^{-1}$. For $m \neq n$ we apply the smooth version of the Landau-Gonek formula, [1][Lemma 2.1], and we get

$$
\frac{1}{\pi} \sum_{m,p} \frac{a(mp)a(m)}{mp} \left(\sin \left(2\pi \beta \frac{\log p}{\log T}\right) - \sin \left(2\pi \alpha \frac{\log p}{\log T}\right)\right),
$$

plus a negligible error term. This completes the proof of the lemma. \qed

Proof of Theorem 2.2. We begin by writing the numerator of (1.4) as

$$
\frac{2}{\log T} \sum_{mp < T} \left(\frac{a(m)\sqrt{\log p}}{\sqrt{mp}}\left(1 - \frac{\log p}{\log T}\right)\right)\left(\frac{a(mp)\sqrt{\log p}}{\sqrt{mp}}\right).
$$
By Cauchy-Schwarz inequality we have (6.2) is smaller than
\[ \frac{2}{\log T} \left( \sum_{mp<T} |a(m)|^2 \log p \left( 1 - \log p / \log T \right)^2 \right)^{\frac{1}{2}} \left( \sum_{mp<T} |a(mp)|^2 \log p \right)^{\frac{1}{2}} \]
\[ = \frac{2}{\log T} \left( \sum_{m<T} |a(m)|^2 \sum_{p<T/m} \log p \left( 1 - \log p / \log T \right)^2 \right)^{\frac{1}{2}} \left( \sum_{m<T} |a(m)|^2 \log m \right)^{\frac{1}{2}} \]
\[ = \frac{2}{\sqrt{3}} \left( \sum_{m<T} |a(m)|^2 \left( 1 - \log m / \log T \right)^3 \right)^{\frac{1}{2}} \left( \sum_{m<T} |a(m)|^2 \left( \log m / \log T \right) \right)^{\frac{1}{2}}. \]
(6.3)

To continue for \(0 \leq i \leq k - 1\), we set
\[ \sum_{\frac{m}{k} < m < T^{(i+1)/k}} \frac{|a(m)|^2 m^{-1}}{\sum_{m<T} |a(m)|^2 m^{-1}} = A_i \]
(6.4)

Using this partition, we have that (6.3) is smaller than
\[ \frac{2}{\sqrt{3}} \sum_{i=1}^{k} \left( A_i \left( 1 - \frac{i}{k} \right)^3 \right)^{\frac{1}{2}} \left( A_i \frac{i + 1}{k} \right)^{\frac{1}{2}} = \frac{2}{\sqrt{3}} \sum_{i=1}^{k} A_i \sqrt{1 - \frac{i}{k} \frac{i + 1}{k}}. \]

To finish the proof we use the fact that \(\sum A_i = 1\) and note that
\[ \max \frac{2}{\sqrt{3}} \sqrt{1 - x^3} x = 0.79370 \ldots . \]

Now we will give the proof of Theorem 2.2. The proof of Lemma 4.2 goes similarly applying Lemma 4.1. We begin by considering the series expansion of \(C_\alpha(t)\) that we defined in (1.7). We have that (see in [16, proof of Lemma 1])
\[ C_\alpha(t) = -\frac{1}{\alpha \log T} \sum_{n<T^\alpha} \left( \frac{\Lambda(n)}{n^{\frac{1}{2} + it}} + \frac{\Lambda(n)}{n^{\frac{1}{2} - it}} \right) \left( 1 - \frac{\log n}{\alpha \log T} \right) \]
\[ + 2 \Re \frac{T^{\alpha \frac{1}{2} - it}}{(\frac{1}{2} - it)^2 \alpha^2 \log^2 T} + O \left( \frac{1}{\log T} \right). \]
(6.5)

For detailed proof of (6.5) see [1]. Let \(T_0 = T \log^{-2} T\) and first let us consider the effect of the poles of the Riemann zeta function in the above. We have that
\[ \int \omega(\frac{1}{2} + it) \frac{T^{\alpha \frac{1}{2} - it}}{(\frac{1}{2} - it)^2 \alpha^2 \log^2 T} \left[ \sum_{n<T_0} \left| \frac{a(n)}{n^{\frac{1}{2} + it}} \right|^2 \right]. \]
For $\alpha < 2$ the above is $O(T^{-1})$. For $\alpha > 2$ it comes down to considering

$$
(6.6) \quad \int \omega\left(\frac{1}{2} + it\right) \frac{1}{(\frac{1}{2} - it)^2} \left(\frac{s}{r T^\alpha}\right)^{\frac{1}{2} + it}.
$$

We can look at the above as the Fourier transform of $\omega(\frac{1}{2} + it)(\frac{1}{2} - it)^{-2}$ at

$\log s - \log r T^\alpha$. Since $|\log s - \log r T^\alpha| \geq \log T$ and the function is smooth

we have that (6.6) is very small. With the above explanation and using (6.5) we have that the expectation of $C_\alpha$ with respect to $\mu_A$ equals

$$
- \int \frac{\omega\left(\frac{1}{2} + it\right)}{\alpha \log T} \sum_{r,s \leq T^\alpha} \frac{a(r)a(s)}{s} \left(\frac{s}{r}\right)^{\frac{1}{2} + it} \sum_{n \leq T^\alpha} \left(\frac{\Lambda(n)}{n^{\frac{1}{2} + it}} + \frac{\Lambda(n)}{n^{\frac{1}{2} - it}}\right) \left(1 - \frac{\log n}{\alpha \log T}\right) 
\quad \cdot \left(1 - \frac{\log n}{\alpha \log T}\right) \int \omega\left(\frac{1}{2} + it\right) \left(\frac{s}{r}\right)^{\frac{1}{2} + it}
$$

$$
(6.7) \quad = -\frac{2}{\alpha \log T} \sum_{\substack{r,s \leq T^\alpha \\ n \leq T^\alpha}} \frac{a(r)a(s)\Lambda(n)}{ns} \left(1 - \frac{\log n}{\alpha \log T}\right) \int \omega\left(\frac{1}{2} + it\right) \left(\frac{s}{r}\right)^{\frac{1}{2} + it},
$$

plus a small error term. This basically establishes (1.3) for $\alpha = 1$. For the proof of Theorem (2.2) we consider

$$
a(n) = \lambda_{\beta_1, \beta_2}(n)d_\nu(n)\left(1 - \frac{\log (n)}{\log T}\right)^\nu,
$$
as in (2.2). Substituting this in (6.7) and considering the shift by $d$ we need to estimate

$$
r \sum_{p \leq T} \frac{\lambda_{\beta_1, \beta_2}(p)\log p}{p^{1-id}} \left(1 - \frac{\log p}{\alpha \log T}\right) \sum_{m < T_0/p} \frac{d_r^2(m)}{m} \left(1 - \frac{\log m}{\log T}\right) \nu \left(1 - \frac{\log m}{\log T}\right) \frac{\log p}{\log T}.
$$

We use the following on the sum of the generalized divisor function [4]

$$
(6.8) \quad \sum_{m < x} \frac{d_r^2(m)}{m} = A_r(\log x)^r + O((\log T)^{r^2 - 1}),
$$

and the prime number theorem to get

$$
r \int_1^T \frac{\lambda_{\beta_1, \beta_2}(x)}{x^{1-id}} \left(1 - \frac{\log x}{\alpha \log T}\right) \int_1^{T/x} A_r \frac{r^2(\log y)^r y^{r^2 - 1}}{y} \left(1 - \frac{\log y}{\log T}\right) \nu \left(1 - \frac{\log y}{\log T}\right) \left(1 - \frac{\log x}{\log T}\right) \left(1 - \frac{\log y}{\log T}\right) dx dy.
$$

By changing variable $u = \log x/\log T$ and $v = \log y/\log T$, we get

$$
r^3 A_r(\log T)^{r^2 + 1} \int_0^1 \int_0^{1-u} \lambda_{\beta_1, \beta_2}(u)e^{2\pi i du} v^{r^2 - 1}(1 - u/\alpha)(1 - v)^\nu (1 - u - v)^\eta.
$$

Following a similar argument we have

$$
\sum_{m < T} \frac{|a(m)|^2}{m} = r^2 A_r(\log T)^r \int_0^1 v^{r^2 - 1}(1 - v)^{2\eta} dv.
$$

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