Late comment on Astumian’s paradox

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In 2001 Astumian \(^{1}\) published a very simple game which can be described by a Markov chain with absorbing initial and final states. In August 2004 Piotrowski and Sladowski \(^{2}\) asserted that Astumian’s analysis was flawed. However, as was shown by Astumian \(^{3}\), this statement was wrong. Since the analysis of the problem in a slightly more general frame than it was done earlier could be a good exercise for graduate students, we came to the conclusion that it might be useful to publish our elementary considerations about the properties of Markov chains corresponding to Astumian type games.

For entirely didactic reasons, in Sections II and III we present a brief summary of definitions and statements which are needed for the analysis of the Astumian type Markov chains. In Section IV we analyze the properties of such chains and determine the probabilities of losing and winning. Conclusions are made in Section V.

I. INTRODUCTION

The present note was initiated by the revisited Astumian’s paradox. In August 2004 Piotrowski and Sladowski \(^{2}\) asserted that Astumian’s analysis was flawed. However, as shown by Astumian \(^{3}\), this statement was wrong. Since the analysis of the problem in a slightly more general frame than it was done earlier could be a good exercise for graduate students, we came to the conclusion that it might be useful to publish our elementary considerations about the properties of Markov chains corresponding to Astumian type games.

II. PRELIMINARIES

Let \(\mathcal{N} = \{1, 2, \ldots, N\}\) be a finite set of positive integers, and \(\mathcal{Z} = \{0, 1, \ldots\}\) be a set of non-negative integers. Denote by \(\xi_n, n \in \mathcal{Z}\) the random variable which assumes the elements of \(\mathcal{N}\). We say that the sequence \(\{\xi_n\}\) forms a Markov chain if for all \(n \in \mathcal{Z}\) and for all possible values of random variables the equation

\[
P\{\xi_n = j|\xi_0 = i_0, \xi_1 = i_1, \ldots, \xi_{n-1} = i_{n-1}\} = P\{\xi_n = j|\xi_{n-1} = i_{n-1}\}
\]

is fulfilled. If \(\xi_n = j\) then the process is said to be in state \(S_j\) at the \(n\)th (discrete time instant) step. The states \(S_1, S_2, \ldots, S_N\) define the space of states of the process. The probability distribution \(P\{\xi_0 = i\}, i \in \mathcal{N}\) of the random variable \(\xi_0\) is called the initial distribution and the conditional probabilities \(P\{\xi_n = j|\xi_{n-1} = i\}\) are called transition probabilities. If \(\xi_{n-1} = i\) and \(\xi_n = j\), then we say that the process made a transition \(S_i \rightarrow S_j\) at the \(n\)th step. The Markov chain is homogeneous if the transition probabilities are independent of \(n \in \mathcal{Z}\). In this case we may write

\[
P\{\xi_n = j|\xi_{n-1} = i\} = w_{ij}(1) = w_{ij},
\]

and it obviously holds that

\[
\sum_{j=1}^{N} w_{ij} = 1, \quad \forall i \in \mathcal{N}.
\]

In what follows we shall consider only homogeneous Markov chains. We would like to emphasize that the transition probability matrix

\[
w = \begin{pmatrix}
w_{11} & w_{12} & \cdots & w_{1N} \\
w_{21} & w_{22} & \cdots & w_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
w_{N1} & w_{N2} & \cdots & w_{NN}
\end{pmatrix},
\]

which is a stochastic matrix, and the initial distribution \(p_i = P\{\xi_0 = i\}, i \in \mathcal{N}\) determine the random process uniquely. For the sake of simplicity, we assume that the process is a random walk of an abstract object, called particle on the space of states \(S_1, S_2, \ldots, S_N\). The \(n\)th step transition probability

\[
P\{\xi_{m+n} = j|\xi_m = i\} = w_{ij}(n)
\]
satisfies the following equation:

\[ w_{ij}(n) = \sum_{k=1}^{N} w_{ik}(r) \cdot w_{kj}(s), \quad (5) \]

where

\[ r + s = n. \]

It is to note that \( w_{ij}(n) \) is the probability that at the \( n \)th step the particle is in the state \( S_j \) provided that at \( n = 0 \) it was in the state \( S_i \). From Eq. (5) we obtain that

\[ w_{ij}(n) = \sum_{k=1}^{N} w_{ik}(n-1) \cdot w_{kj}, \]

and by using the rules of matrix multiplication we arrive at

\[ w(n) = w \cdot w(n-1) = w(n-1) \cdot w = w^n, \quad (6) \]

where

\[
w(n) = \begin{pmatrix}
w_{11}(n) & w_{12}(n) & \cdots & w_{1N}(n) \\
w_{21}(n) & w_{22}(n) & \cdots & w_{2N}(n) \\
\vdots & \vdots & \ddots & \vdots \\
w_{N1}(n) & w_{N2}(n) & \cdots & w_{NN}(n)
\end{pmatrix},
\]

(7)

and

\[
w(0) = \begin{pmatrix}1 & 0 & \cdots & 0 \\0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

is the \( N \times N \) unit matrix.

Making use of the total probability theorem we can determine the absolute probabilities \( p_j(n) \) as follows:

\[ p_j(n) = \sum_{i=1}^{N} p_i \cdot w_{ij}(n), \quad j \in \mathcal{N}, \quad (8) \]

where \( p_i = \mathbb{P} \{ \xi_0 = i \} \) is the initial probability. Clearly, \( p_j(n) \) is the probability that the particle is in the state \( S_j \) at the \( n \)th step. Introducing the row vector

\[ \vec{p} = \{p_1, p_2, \ldots, p_N\}, \quad (9) \]

Eq. (8) can be rewritten in the form:

\[ \vec{p}(n) = \vec{p} \cdot w(n) = w^{(T)}(n) \cdot \vec{p}^{(T)}, \quad (10) \]

where the upper index \( T \) indicates the transpose of matrix \( w(n) \) and vector \( \vec{p} \) defined by (7) and (9), respectively. If the process starts from the state \( S_i \), then

\[ \vec{p} = \{0_1, 0_2, \ldots, 1_i, \ldots, 0_N\}, \]

and

\[ \vec{p}_i(n) = \{w_{i1}(n), w_{i2}(n), \ldots, w_{iN}(n)\}. \]
III. TYPES OF STATES AND ASYMPTOTIC BEHAVIOR

A. Basic definitions

In order to use clear notions, we introduce several well-known definitions. If there is an integer \( n \geq 0 \) such that \( w_{jk}(n) > 0 \), then we say the state \( S_k \) can be reached from the state \( S_j \). If \( S_k \) can be reached from \( S_j \) and \( S_j \) can be reached from \( S_k \), then \( S_j \) and \( S_k \) are connected states. Obviously, if \( S_j \) and \( S_k \) are not connected, then either \( w_{jk}(n) = 0 \), or \( w_{kj}(n) = 0 \). The set of states which are connected forms a class of equivalence. A Markov chain is called irreducible if every state can be reached from every state i.e., the entire state space consists of only one class of equivalence. In other words, the Markov chain is irreducible when all of the states are connected.

The probability \( f_{ij}(n) \) of passage from \( S_i \) to \( S_j \) in exactly \( n \) steps, that is, without passing through \( S_j \) before the \( n \)th step, is given by

\[
f_{ij}(n) = \sum_{j_1 \neq j, j_2 \neq j, \ldots, j_{n-1} \neq j} w_{ij_1} w_{j_1 j_2} \cdots w_{j_{n-1} j}, \tag{11}
\]

There exists an important relationship between the probabilities \( w_{ij}(n) \) and \( f_{ij}(n) \) which is easy to prove. The relationship is given by

\[
w_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k) w_{jj}(n-k), \quad \forall \ n \in \mathbb{Z}. \tag{12}
\]

One has to note that the expressions \( w_{jj}(0) = 1 \) are the diagonal elements of the unit matrix \( w(0) \).

The proof of (12) is immediate upon applying the total probability rule. The particle passes from \( S_i \) to \( S_j \) in \( n \) steps if, and only if, it passes from \( S_i \) to \( S_j \) for the first time in exactly \( k \) steps, \( k = 1, 2, \ldots, n \), and then passes from \( S_j \) to \( S_j \) in the remaining \( n-k \) steps. These “paths” are disjoint events, and their probabilities are given by \( f_{ij}(k) w_{jj}(n-k) \). Summing over \( k \) one obtains the equation (12).

Let us introduce the generating functions

\[
\varphi_{ij}(z) = \sum_{n=0}^{\infty} f_{ij}(n) z^n \quad \text{and} \quad \omega_{ij}(z) = \sum_{n=1}^{\infty} w_{ij}(n) z^n. \tag{13}
\]

Taking into account that \( w_{jj}(0) = 1 \), from Eq. (12) we obtain

\[
\omega_{ij}(z) = \varphi_{ij}(z) \left[ 1 + \omega_{jj}(z) \right], \tag{14}
\]

and from this

\[
\varphi_{ij}(z) = \frac{\omega_{ij}(z)}{1 + \omega_{jj}(z)}, \tag{15}
\]

so we have

\[
F_{ij} = \sum_{n=1}^{\infty} f_{ij}(n) = \frac{\sum_{n=1}^{\infty} w_{ij}(n)}{1 + \sum_{n=1}^{\infty} w_{jj}(n)}, \tag{16}
\]

and in particular

\[
\sum_{n=1}^{\infty} w_{jj}(n) = \frac{\sum_{n=1}^{\infty} f_{jj}(n)}{1 - \sum_{n=1}^{\infty} f_{jj}(n)}. \tag{17}
\]

\( F_{ij} \) defined by (16) is the probability that a particle starting its walk from \( S_i \) passes through the state \( S_j \) at least once. Clearly, \( F_{ii} = F_i \) is the probability of returning to \( S_i \) at least once.

More generally, the probability \( F_{ij}(k) \) that a particle starting its walk from \( S_i \) passes through \( S_j \) at least \( k \) times is given by

\[
F_{ij}(k) = \left[ \sum_{n=1}^{\infty} f_{ij}(n) \right] F_{jj}(k-1) = F_{ij} F_{jj}(k-1). 
\]
In particular, the probability of returning to $S_i$ at least $k$ times is given by $F_{ij}(k) = (F_{ii})^k$. Its limit

$$R_{ij} = \lim_{k \to \infty} (F_{ii})^k = \begin{cases} 0, & \text{if } F_{ii} < 1, \\ 1, & \text{if } F_{ii} = 1 \end{cases}$$

is the probability of returning to $S_i$ infinitely often. It follows from the previous relationship that the probability that a particle starting its walk from $S_i$ passes through $S_j$ infinitely many times is

$$R_{ij} = \lim_{k \to \infty} F_{ij}(k) = F_{ij} R_{jj},$$

so that

$$R_{ij} = \begin{cases} 0, & \text{if } F_{ii} < 1, \\ F_{ij}, & \text{if } F_{ii} = 1. \end{cases}$$

We say that $S_i$ is a return state or a nonreturn state according as $F_i > 0$ or $F_i = 0$. As a further definition, we say that $S_i$ is a recurrent state or a nonrecurrent state according as $F_i = 1$ or $0 \leq F_i < 1$. A nonrecurrent state is often called a transient state.

The state $S_i$ is called periodic with period $\ell$ if a return to $S_i$ can occur only at steps $\ell, 2\ell, 3\ell, \ldots$ and $\ell > 1$ is the greatest integer with this property. If $n$ is not divisible by $\ell$, then $w_{ij}(n) = 0$. If the period of each state is equal to 1, i.e., if $\ell = 1$, then the Markov chain is called aperiodic. In the sequel we are dealing with aperiodic Markov chains.

A set $C$ of states in a Markov chain is closed if it is impossible to move out from any state of $C$ to any state outside $C$ by one-step transitions, i.e., $w_{ij}(1) = w_{ij} = 0$ if $S_i \in C$ and $S_j \notin C$. In this case $w_{ij}(n) = 0$ obviously holds for every $n \in \mathbb{Z}$. If a single state $S_i$ forms a closed set, then we call this an absorbing state, and we have $w_{ij} = 1$.

The states of a closed set $C$ are recurrent states since the return probability $F_i$ for any state $S_i \in C$ is equal to 1. Therefore, the set of recurrent states is denoted by $C$. The set of states having return probabilities $F_i < 1$ is the set of transient states and it is denoted by $T$. Obviously, if $S_i \in T$ and $S_j \in C$, i.e., if $S_j$ is an absorbing state, then $F_{ij}$ is the probability that a particle starting at $S_i$ is finally absorbed at $S_j$.

Let $\nu_{ij}$ be the passage time of a particle from the state $S_i$ to the state $S_j$, taking values $m = 1, 2, \ldots$, with probabilities $f_{ij}(m)$. If

$$\sum_{m=1}^{\infty} f_{ij}(m) = F_{i,j} = 1,$$

then the expected passage time $\tau_{ij} = \mathbb{E}\{\nu_{ij}\}$ from $S_i$ to $S_j$ is defined by

$$\tau_{ij} = \sum_{m=1}^{\infty} m f_{ij}(m) = \left[ \frac{d\varphi_{ij}(z)}{dz} \right]_{z=1},$$

while if $F_{ij} < 1$, one says that $\nu_{ij} = \infty$ with probability $1 - F_{ij}$, i.e., if $F_{ij} < 1$, then the expected passage time $\tau_{ij} = \infty$. If the state $S_j = S_i$ and it is recurrent, i.e., if $F_{ii} = F_i = 1$, then the expectation

$$\mathbb{E}\{\nu_{ii}\} = \sum_{m=1}^{\infty} m f_{ii}(m) = \left[ \frac{d\varphi_{ii}(z)}{dz} \right]_{z=1} = \tau_{ii} = \mu_i$$

(18)

is called mean recurrent time. If $\mu_i = \infty$, then we say that $S_i$ is a recurrent null-state, whereas if $\mu_i < \infty$, then we say that $S_i$ is a recurrent non-null-state. If $F_i < 1$, i.e., the state $S_i$ is transient, then $1 - F_i$ is the probability that the recurrence time is infinitely long, and so $\mu_i = \infty$.

We say that the recurrent state $S_i$ is ergodic, if it is not a null-state and is aperiodic, that is, if $F_i = 1$, $\mu_i < \infty$ and $\ell = 1$.

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1 The set $C$ can be decomposed into mutually disjoint closed sets $C_1, C_2, \ldots, C_r$ such that from any state of a given set all states of that set and no others can be reached. States $C_1, C_2, \ldots, C_r$ can be reached from $T$, but not conversely.
B. Asymptotic behavior

The first statement is very simple, hence it is given without proof. If $S_j$ is a transient or a recurrent null-state, then for any arbitrary $S_i$

$$\lim_{n \to \infty} w_{ij}(n) = 0$$  \hspace{1cm} (19)

holds.

If $S_i$ and $S_j$ are recurrent aperiodic states due to the same closed set, then

$$\lim_{n \to \infty} w_{ij}(n) = \frac{1}{\mu_j}$$  \hspace{1cm} (20)

irrespective of $S_i$. $^2$

If $i = j$, then we have from Eq. (14) the formula

$$\omega_{jj}(z) = \frac{\varphi_{jj}(z)}{1 - \varphi_{jj}(z)}.$$  \hspace{1cm} (21)

Substituting this into (14) we obtain the following expression:

$$\omega_{ij}(z) = \varphi_{ij}(z) \left( 1 + \frac{\varphi_{jj}(z)}{1 - \varphi_{jj}(z)} \right) = \frac{\varphi_{ij}(z)}{1 - \varphi_{jj}(z)}.$$  \hspace{1cm} (22)

By using Tauber’s Theorem we can state that

$$\lim_{z \uparrow 1} (1 - z) \frac{\varphi_{ij}(z)}{1 - \varphi_{jj}(z)} = \lim_{n \to \infty} w_{ij}(n).$$  \hspace{1cm} (23)

Since $S_i$ and $S_j$ are aperiodic recurrent states due to the same closed set,

$$\lim_{z \uparrow 1} \varphi_{ij}(z) = \lim_{z \uparrow 1} \varphi_{jj}(z) = 1,$$

i.e., the limit value we have to determine

$$\lim_{z \uparrow 1} \frac{1 - z}{1 - \varphi_{jj}(z)}.$$

Applying L’Hospital’s rule we find that

$$\lim_{z \uparrow 1} \frac{1 - z}{1 - \varphi_{jj}(z)} = \frac{1}{\varphi'(1)} = \frac{1}{\mu_j},$$

and thus we obtain (20). This completes the proof.

As a generalization we would like to consider the case when $S_i$ is a transient state ($S_i \in T$) and $S_j$ is an aperiodic recurrent state due to the closed set $C$. It can be shown that

$$\lim_{n \to \infty} w_{ij}(n) = \frac{F_{ij}}{\mu_j},$$  \hspace{1cm} (24)

where $F_{ij}$ is the probability that a particle starting from $S_i$ will ultimately reach and stay in the state $S_j \in C$. In other words, $F_{ij}$ is the absorption probability that satisfies the following system of equations:

$$F_{ij} = w_{ij} + \sum_{S_k \in T} w_{ik} F_{kj}, \quad \forall S_i \in T.$$  \hspace{1cm} (25)

$^2$ In order to prove the limit relationship (20) Tauber’s Theorem is used instead of the lemma by Erdős-Feller-Kac.
Clearly, if $T \cup C$ contains all of the possible states of the particle, then
\[
\sum_{S_j \in C} F_{ij} = 1.
\]  
(26)

The proof of (24) follows immediately from (22). Since
\[
\lim_{z \uparrow 1} \varphi_{ij}(z) \frac{1 - z}{1 - \varphi_{jj}(z)} = \sum_{n=1}^{\infty} f_{ij}(n) \frac{1}{\mu_j} = \frac{F_{ij}}{\mu_j},
\]
we obtain the limit relationship (24).

Finally, we would like to present a brief classification of Markov chains.

- A Markov chain is called **irreducible** if and only if all its states form a closed set and there is no other closed set contained in it.

- A Markov chain is called **ergodic** if the probability distributions
\[
p_j(n) = \sum_{k=1}^{N} p_k(0) w_{kj}(n), \quad j \in N
\]
always converge to a limiting distribution $p_j$ which is independent of the initial distribution $p_j(0)$, that is, when $\lim_{n \to \infty} p_j(n) = p_j, \quad \forall j \in N$. All states of a finite, aperiodic irreducible Markov chain are ergodic.

- The probability distribution $p_i^{(st)}$ is a **stationary** distribution of a Markov chain if, when we choose it as an initial distribution all the distributions $p_i(n)$ will coincide with $p_i^{(st)}$. Every stationary distribution of a Markov chain satisfies the following system of linear equations:
\[
p_j^{(st)} = \sum_i p_i^{(st)} w_{ij} \quad \text{and} \quad \sum_j p_j^{(st)} = 1,
\]
and conversely, each solution $p_j^{(st)}$ of this system is a stationary distribution of the Markov chain, if it is a probability distribution.

It is to mention that some parts of this short summary is based on the small but excellent book by Takács [4].

**IV. MARKOV CHAINS WITH ABSORBING STATES**

In this section we are going to deal with Markov chains containing two absorbing states $S_1$ and $S_N$, and $N - 2$ transient states. In this case, the Markov chain is **reducible** and **aperiodic**. The set of its states is the union of two closed sets $C_1 = \{S_1\}$ and $C_2 = \{S_N\}$, and of the set of transient states $T = \{S_2, S_3, \ldots, S_{N-1}\}$ The states $S_1$ and $S_N$ can be reached from each state of $T$ but the converse doesn’t hold, no state of $T$ can be reached from the states $S_1$ and $S_N$. The states of $T$ are **non-recurrent** since the particle leaves the set never to return to it. In contrary, the states of $C_1$ and $C_2$ are ergodic.

**A. Chains of five states**

Let us assume that the transition matrix $w$ has the following form:
\[
w = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & w_{21} & w_{22} & w_{23} & 0 \\
0 & w_{31} & w_{33} & w_{34} & 0 \\
0 & 0 & w_{43} & w_{44} & w_{45} \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]  
(27)

where
\[
\sum_j w_{ij} = 1.
\]
The particle, which starts his walk from one of the states \( S \), \( i = 2, 3, 4 \), is captured when it enters the states \( S_1 \) or \( S_5 \). By using the foregoing formulae for \( F_{ij} \) and \( F_{ij} \), we can immediately obtain the capture probabilities by the absorbing states \( S_1 \) and \( S_5 \), respectively. In order to have a direct insight into the nature of the process, we derive the backward equations for the probabilities \( w_{ij}(n) \). Clearly,

\[
\begin{align*}
    w_{1j}(n) &= \delta_{1j}, \\
    w_{2j}(n) &= w_{21} w_{1j}(n-1) + w_{22} w_{2j}(n-1) + w_{23} w_{3j}(n-1), \\
    w_{3j}(n) &= w_{32} w_{2j}(n-1) + w_{33} w_{3j}(n-1) + w_{34} w_{4j}(n-1), \\
    w_{4j}(n) &= w_{43} w_{3j}(n-1) + w_{44} w_{4j}(n-1) + w_{45} w_{5j}(n-1), \\
    w_{5j}(n) &= \delta_{5j},
\end{align*}
\]

and by introducing the generating function

\[
    g_{ij}(z) = \delta_{ij} + \sum_{n=1}^{\infty} w_{ij}(n) z^n = \delta_{ij} + \omega_{ij}(z), \quad |z| < 1,
\]

we obtain the following system of equations:

\[
\begin{align*}
    g_{1j}(z) &= \delta_{1j} \frac{1}{1-z}, \\
    g_{2j}(z) &= \delta_{2j} + zw_{21} g_{1j}(z) + zw_{22} g_{2j}(z) + zw_{23} g_{3j}(z), \\
    g_{3j}(z) &= \delta_{3j} + zw_{32} g_{2j}(z) + zw_{33} g_{3j}(z) + zw_{34} g_{4j}(z), \\
    g_{4j}(z) &= \delta_{4j} + zw_{43} g_{3j}(z) + zw_{44} g_{4j}(z) + zw_{45} g_{5j}(z), \\
    g_{5j}(z) &= \delta_{5j} \frac{1}{1-z}.
\end{align*}
\]

This can be simplified and rewritten in the form:

\[
\begin{align*}
    (1-zw_{22}) g_{2j}(z) - zw_{23} g_{3j}(z) &= \delta_{2j} + w_{21} \frac{z}{1-z} \delta_{1j}, \\
    -zw_{32} g_{2j}(z) + (1-zw_{33}) g_{3j}(z) - zw_{34} g_{4j}(z) &= \delta_{3j}, \\
    -zw_{43} g_{3j}(z) + (1-w_{44}) g_{4j}(z) &= \delta_{4j} + w_{45} \frac{z}{1-z} \delta_{5j}.
\end{align*}
\]

After elementary algebra, we can determine all the generating functions \( g_{ij}(z) \), \( i, j = 1, 2, 3, 4, 5 \), nevertheless we are now interested only in those functions which correspond to processes starting from the state \( S_5 \). In this case we have

\[
\begin{align*}
    g_{31}(z) &= \frac{z^2}{1-z} \frac{w_{32} w_{21} (1-w_{44} z)}{D(z)}, \\
    g_{32}(z) &= \frac{w_{32} z (1-w_{44} z) z}{D(z)}, \\
    g_{33}(z) &= \frac{(1-w_{22} z)(1-w_{44} z)}{D(z)}, \\
    g_{34}(z) &= \frac{w_{43} (1-w_{22} z)}{D(z)}, \\
    g_{35}(z) &= \frac{z^2}{1-z} \frac{w_{34} w_{45} (1-w_{22} z)}{D(z)},
\end{align*}
\]

where

\[
D(z) = (1-w_{44} z) \left[ (1-w_{22} z)(1-w_{31} z) - w_{23} w_{32} z^2 \right] - (1-w_{22} z) w_{34} w_{43} z^2.
\]

Applying Tauber’s Theorem we obtain that

\[
\lim_{n \to \infty} w_{31}(n) = \lim_{z \to 1} (1-z) g_{31}(z) = \frac{w_{32} w_{21} (1-w_{44})}{D(1)},
\]
\[ \lim_{n \to \infty} w_{3j}(n) = \lim_{z \to 1} (1 - z) g_{3j}(z) = 0, \quad j = 2, 3, 4, \quad (44) \]

and

\[ \lim_{n \to \infty} w_{35}(n) = \lim_{z \to 1} (1 - z) g_{35}(z) = \frac{w_{34}w_{45}(1 - w_{22})}{D(1)}. \quad (45) \]

Performing the substitutions

\[ w_{22} = 1 - w_{21} - w_{23}, \quad w_{33} = 1 - w_{32} - w_{34} \quad w_{44} = 1 - w_{43} - w_{45}, \]

we have

\[ w_{31}(\infty) = \frac{w_{32}w_{21}(w_{43} + w_{45})}{w_{34}w_{45}(w_{21} + w_{23}) + w_{32}w_{21}(w_{43} + w_{45})}, \quad (46) \]

and

\[ w_{35}(\infty) = \frac{w_{34}w_{45}(w_{21} + w_{23})}{w_{34}w_{45}(w_{21} + w_{23}) + w_{32}w_{21}(w_{43} + w_{45})}. \quad (47) \]

It is elementary to show that

\[ F_{31} = \varphi_{31}(1) = w_{31}(\infty) \quad \text{and} \quad F_{35} = \varphi_{35}(1) = w_{35}(\infty). \quad (48) \]

In order to prove these equations, let us take into account relationship (15) and write

\[ \varphi_{31}(z) = \omega_{31}(z) = \frac{g_{31}(z)}{g_{11}(z)}, \]

and

\[ \varphi_{35}(z) = \frac{g_{35}(z)}{g_{55}(z)} = \frac{1}{1 - z}. \]

Since

\[ g_{11}(z) = g_{55}(z) = \frac{1}{1 - z}, \]

we have

\[ \varphi_{31}(z) = (1 - z)g_{31}(z) \quad \text{and} \quad \varphi_{35}(z) = (1 - z)g_{35}(z). \quad (49) \]

Comparing (43) and (46) with (49) we see that Eqs. (48) are true.

It is convenient to write the absorption probabilities \( F_{31} \) and \( F_{35} \) in the form:

\[ F_{31} = \frac{1}{1 + r}, \quad \text{and} \quad F_{35} = \frac{r}{1 + r}, \quad (50) \]

where

\[ r = \frac{w_{34}w_{45}(w_{21} + w_{23})}{w_{32}w_{21}(w_{43} + w_{45})}. \quad (51) \]

and we see immediately that \( F_{31} + F_{35} = 1 \), as expected.

It seems to be worthwhile to study the history of a particle starting its random walk from the state \( S_3 \). Let us consider a trap containing a special ladder with 5 rungs. Each rung corresponds to a given state of the Markov chain under investigation. The process starts when a particle enters (say,) on the third rung of the ladder, i.e., in the state \( S_3 \). Once the particle has entered, it is free to move up and down the rungs randomly. Fig. 1 illustrates this random walk. If the particle reaches the states either \( S_1 \) or \( S_5 \), it is absorbed. (If the random walk is considered as a game,
then the absorption state with probability smaller than 1/2 is the “winning” state.) Having chosen the transition matrix

\[
\mathbf{w} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
4/36 & 24/36 & 8/36 & 0 & 0 \\
0 & 5/36 & 29/36 & 2/36 & 0 \\
0 & 0 & 4/36 & 24/36 & 8/36 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

we calculated the dependencies of probabilities \(w_{31}(n), w_{33}(n)\) and \(w_{35}(n)\) on the number of steps \(n\). The results of calculation are shown in Fig. 2. We see that the probability to find the particle after \(n \approx 100\) steps in the transient state \(S_3\) is practically zero. The same holds for the transient states \(S_2\) and \(S_4\). After \(n \approx 100\) steps the particle is absorbed either in \(S_1\) with probability \(w_{31}(100) \approx F_{31} = 5/9\) or in \(S_5\) with probability \(w_{35}(100) \approx F_{35} = 4/9\).

It is instructive to determine also the probabilities \(F_{32}, F_{33}\) and \(F_{34}\). As a reminder, we note that \(F_{3j}\) is the probability that a particle starting from \(S_3\) passes through \(S_j\), \((j = 2, 3, 4)\) at least once. By using the transition matrix \((52)\) we obtain the following values: \(F_{32} = 15/19\), \(F_{33} = 11/12\) and \(F_{34} = 6/11\). Fig. 3 shows the histogram of these probabilities. It is evident that passing through either \(S_1\) or \(S_5\) at least once means that the particle is absorbed.
FIG. 3: Probabilities that a particle passes through the state $S_j$, $(j = 1, 2, 3, 4, 5)$ at least once provided that it started from $S_3$.

As expected in the present case, the probability $F_{33}$ that the particle starting from $S_3$ returns to $S_3$ at least once, is nearly 1. It is to mention that the two absorbing states $S_1$ and $S_5$ are recurrent since $F_{11} = F_{55} = 1$.

In what follows we would like to deal with the determination of the absorption time probability. Denote by $\tau_i$ the number of steps leading to the absorption of a particle starting its random walk from the state $S_i$. By definition, $f_{i1}(n)$ and $f_{i5}(n)$ are the probabilities that the particle starting from the state $S_i$, $(i = 2, 3, 4)$ is absorbed exactly at the $n$th step in $S_1$ or in $S_5$, respectively. Hence we can write that

$$P\{\tau_i = n\} = T_i(n) = f_{i1}(n) + f_{i5}(n), \quad i = 2, 3, 4. \quad (53)$$

It is easy to prove that

$$T_i(n) = w_{i1}(n) - w_{i1}(n - 1) + w_{i5}(n) - w_{i5}(n - 1), \quad \forall \, n \geq 1. \quad (54)$$

From (12) one obtains

$$w_{i1}(n) = \sum_{k=1}^{n} f_{i1}(k) w_{11}(n - k),$$
$$w_{i5}(n) = \sum_{k=1}^{n} f_{i5}(k) w_{55}(n - k),$$

and by taking into account that

$$w_{11}(\ell) = w_{55}(\ell) = 1, \quad \forall \, \ell \geq 0,$$

one has

$$w_{i1}(n) = \sum_{k=1}^{n} f_{i1}(k) \quad \text{and} \quad w_{i5}(n) = \sum_{k=1}^{n} f_{i5}(k).$$

It follows immediately from these equations that

$$f_{i1}(n) = w_{i1}(n) - w_{i1}(n - 1) \quad \text{and} \quad f_{i5}(n) = w_{i5}(n) - w_{i5}(n - 1),$$

and this completes the proof. The absorption time probabilities $T_i(n)$, $(i = 2, 3, 4)$ can be determined by the “forward” equations:

$$f_{i1}(n) = \sum_{\ell=2}^{4} w_{i\ell}(n - 1) w_{\ell1}$$
and

\[ f_{15}(n) = \sum_{\ell=2}^{4} w_{i\ell}(n - 1) w_{\ell5}. \]

By using these expressions one can write

\[ T_i(n) = \sum_{\ell=2}^{4} w_{i\ell}(n - 1) \left[ w_{\ell1} + w_{\ell5} \right], \quad (55) \]

which in the case of \( w \) defined by (24) has the following form:

\[ T_i(n) = w_{i2}(n - 1) w_{21} + w_{i4}(n - 1) w_{45}. \quad (56) \]

For the sake of completeness, we would like to show that

\[ \sum_{n=1}^{\infty} T_i(n) = 1. \quad (57) \]

In the case of Eq. (53) we see that

\[ \sum_{n=1}^{\infty} T_i(n) = F_{i1} + F_{i5}, \]

and by using the expression (26) we find (57). In the case of Eq. (55)

\[ \sum_{n=1}^{\infty} T_i(n) = 4 \sum_{\ell=2}^{4} \left[ \sum_{n=1}^{\infty} w_{i\ell}(n - 1) \right] \left[ w_{\ell1} + w_{\ell5} \right] = \]

\[ \sum_{\ell=2}^{4} \left[ \delta_{i\ell} + \omega_{i\ell}(1) \right] \left[ w_{\ell1} + w_{\ell5} \right] = 4 \sum_{\ell=2}^{4} g_{i\ell}(1) \left[ w_{\ell1} + w_{\ell5} \right] = F_{i1} + F_{i5} = 1. \]

FIG. 4: Absorption time probabilities of a particle starting its random walk from the state \( S_i \), \( i = 2, 3, 4 \)

Using the transition matrix \( w \) given by (52), we calculated the dependence of the probability \( T_i(n) \) on the number of steps \( n \). The results are seen in Fig. 4. As expected, if the starting state is \( S_3 \), then the probability \( T_3(n) \) varies
differently with the step number as the probabilities \( T_2(n) \) and \( T_4(n) \). It is characteristic the probabilities have a rather long tail. Since \( T_i(n) \) is the probability that a particle starting from \( S_i \) is absorbed exactly in the \( n \)th step, the expectation and the standard deviation of the absorption time \( \tau_i \) are given by

\[
E\{\tau_i\} = \sum_{n=1}^{\infty} n T_i(n) = \langle \tau_i \rangle, \tag{58}
\]

and

\[
D\{\tau_i\} = \left[ \sum_{n=1}^{\infty} (n - \langle \tau_i \rangle)^2 T_i(n) \right]^{1/2}. \tag{59}
\]

For a transition matrix of the form \( \text{[52]} \) these values are presented in the Table I.

| \( S_2 \) | \( S_3 \) | \( S_4 \) |
|---|---|---|
| \( E\{\tau_i\} \) | 15.7 | 19.0 | 9.3 |
| \( D\{\tau_i\} \) | 16.3 | 16.4 | 13.3 |

### B. Properties of the absorption probability \( F_{31} \)

As it has been shown, \( F_{31} \) is the probability that a particle starting its random walk from the state \( S_3 \) is finally absorbed in the state \( S_1 \). If \( F_{31} > 1/2 \), then \( S_1 \) is called a “losing” state, while if \( F_{31} < 1/2 \), then it is a “winning” state. The game is “fair” when \( F_{31} = 1/2 \), i.e. when the equation

\[
w_{32}w_{21} (w_{43} + w_{45}) = w_{34}w_{45} (w_{21} + w_{23}) \tag{60}
\]

is fulfilled as it follows from Eq. \( \text{[51]} \).

Astumian \( \text{[1]} \) proposed two transition matrices, namely

\[
w_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
4/36 & 24/36 & 8/36 & 0 & 0 \\
0 & 5/36 & 29/36 & 2/36 & 0 \\
0 & 0 & 4/36 & 24/36 & 8/36 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
w_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
4/36 & 24/36 & 8/36 & 0 & 0 \\
0 & 5/36 & 29/36 & 2/36 & 0 \\
0 & 0 & 4/36 & 24/36 & 8/36 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

resulting in the absorption probability \( F_{31} = 5/9 > 1/2 \) and showed that the arithmetic mean of these two matrices

\[
w = \frac{1}{2} (w_1 + w_1) =
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
9/72 & 53/72 & 10/72 & 0 & 0 \\
0 & 9/72 & 53/72 & 10/72 & 0 \\
0 & 0 & 9/72 & 53/36 & 10/72 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[\text{[3]} \] There is no need to deal separately with the absorption probability \( F_{35} \) since \( F_{31} + F_{35} = 1 \).
brings about the probability $F_{31} = 9/19 < 1/2$, i.e., in this case the state $S_1$ becomes “winning” state. This property of the transition matrix (61) is general if the diagonal entries of the matrix are different from zero. By using a simple example we would like to demonstrate this statement.

Let us choose the transition matrix in the following form:

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 - a - b & b & 0 & 0 \\ 0 & b & 1 - a - b - x & a + x & 0 \\ 0 & 0 & a + x & 1 - a - b & b - x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (61)$$

One obtains immediately that

$$F_{31} = H(x) = \frac{ab}{ab + (a + x)(b - x)}, \quad (62)$$

where

$$0 < a < 1, \quad 0 < b < 1, \quad 0 < a + b < 1 \quad \text{and} \quad -a < x < \min(b, 1 - a - b).$$

If $x = 0$ or $x = b - a$, then the game is “fair”, i.e., $F_{31} = 1/2$. The function $H(x)$ assumes its minimal value at

$$x = x_{\text{min}} = \frac{1}{2}(b - a),$$

and this value is

$$H(x_{\text{min}}) = \begin{cases} 1/2, & \text{if } a = b, \\ \frac{4ab}{4ab + (a + b)^2} < 1/2, & \text{if } a \neq b. \end{cases}$$

Introducing the notation $x = x_{\text{min}} + y$ one has

$$H(x_{\text{min}} + y) = J(y) = \frac{4ab}{4ab + (a + b)^2 - 4y^2}.$$ 

Choosing $y$ according to the inequalities

$$x_1 = x_{\text{min}} + y > 0 \quad \text{and} \quad x_2 = x_{\text{min}} - y < b - a,$$

i.e., $y > |x_{\text{min}}|$ and $a \neq b$ one finds that

$$H(x_1) = H(x_2) > \frac{1}{2} \quad \text{and} \quad H\left(\frac{x_1 + x_2}{2}\right) < \frac{1}{2}.$$ 

Evidently, there are infinitely many pairs of transition matrices which result in probabilities of losing in the state $S_1$ but the arithmetic means of corresponding pairs bring about probabilities of winning in the state $S_1$.

For the sake of illustration in Fig. 5 the probability $F_{31} = H(x)$ vs. $x$ curve is plotted by the values $a = 1/4$ and $b = 1/8$. The black points $c_1$, $c_2$ and $c$ correspond to the probabilities

$$H(x_1 = 1/16) = H(x_2 = -3/16) = 8/13 \quad \text{and} \quad H\left(\frac{1}{2}(x_1 + x_2) = -1/16\right) = 8/17,$$

respectively. It seems to be not superfluous to write down the corresponding transition matrices:

$$w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/16 & 10/16 & 2/16 & 0 & 0 \\ 0 & 2/16 & 9/16 & 5/16 & 0 \\ 0 & 0 & 5/16 & 10/16 & 1/16 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4/16 & 10/16 & 2/16 & 0 & 0 \\ 0 & 2/16 & 13/16 & 1/16 & 0 \\ 0 & 0 & 1/16 & 10/16 & 5/16 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (63)$$
Let us now define a Markov chain with transition matrix $w$ randomly chosen from $w_1$ and $w_2$ defined by (63). In
this case
\[ w(n) = [pw_1 + (1-p)w_2] \times w(n-1), \]
i.e.,
\[ w(n) = [pw_1 + (1-p)w_2]^n. \]  \hspace{1cm} (65)

In Fig. 6 the dependencies of the absorption probabilities \( w_{31}(n) \) on the number of steps \( n \) are shown when the transition matrices are \( w_1, w_2 \) and \( w \), respectively. The last one corresponds to the random selection of the entries from \( w_1 \) and \( w_2 \) with probability \( p = 1/2 \). Obviously, not all values of \( p \in [0, 1] \) bring about a “winning” game, i.e., an absorption probability less than \( 1/2 \).

Taking into account the transition matrices \( w_1 \) and \( w_2 \) defined by (63), we determined the dependence of \( F_{31} \) on \( p \). As seen in Fig. 7, there is a well defined subinterval \([p_1, p_2] \in [0, 1]\) containing the \( p \) values which result in absorption probabilities \( F_{31} \) smaller than \( 1/2 \). In the present case we obtained that \( p_1 = 0.25 \) and \( p_2 = 0.75 \).

V. CONCLUSIONS

It has been shown that the random walk of a particle defined by the stochastic transition matrix of a Markov chain is equivalent to an Astumian type game if the diagonal entries of the matrix are different from zero and the first \((w_{11})\) as well as the last \((w_{NN})\) entries are equal to 1. By using a simple example, we have proved that there are infinitely many pairs of transition matrices which result in absorption probabilities in the state \( S_1 \) larger than \( 1/2 \) but the arithmetic means of the corresponding pairs lead to probabilities smaller than \( 1/2 \).

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4 As seen before, \( w_{31}(n) \) is the first entry of the third row of the matrix \( w(n) \).