Adaptive Quadratic Finite Element Method for the Unilateral Contact Problem

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Abstract
In this paper, we present and analyze a posteriori error estimates in the energy norm of a quadratic finite element method for the frictionless unilateral contact problem. The reliability and the efficiency of a posteriori error estimator is discussed. The suitable decomposition of the discrete space $V_h^0$ and a discrete space $Q_h$, where the discrete counterpart of the contact force density is defined, play crucial role in deriving a posteriori error estimates. Numerical results are presented exhibiting the reliability and the efficiency of the proposed error estimator.

Keywords Signorini problem · Quadratic finite elements · A posteriori error analysis · Variational inequalities

1 Introduction

Numerical analysis of the non-linear problems arising from unilateral contact problems using finite element methods exhibits technical adversity both in approximating the continuous problem and numerical modeling of contact conditions on a part of the boundary. The Signorini contact model typically is a prototype model for the class of unilateral contact problems [32]. The Signorini contact problem can be recasted as an elliptic variational inequality of the first kind [22] where the inequality constraint arises due to non linearity condition on the contact boundary. Later, the location of the free boundary (the part of the boundary where

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it touches the given obstacle) is not a priori known, and therefore, it forms a part of the numerical approximation. Hence, it is quite challenging both in the theory and computation to analyze finite element approximation of the Signorini problem using quadratic elements.

Adaptive finite element methods (AFEM) [2, 39] are considered as an essential tool in boosting the precision of the numerical approximation of the non-linear problems. AFEM is mainly based on the reliable and (locally) efficient error estimators, referred as a posteriori error estimators. These error estimators are computable quantities which depends on the given data and the discrete solution.

Subsequently, there has been a tremendous work on the analysis and development of finite element methods for variational inequalities. We refer to articles [4, 11, 22, 35] for convergence analysis of the Signorini problem using linear finite element method, whereas in [5, 26] a priori analysis of quadratic finite element method has been derived for the contact problem. An extensive study of convergence analysis of discontinuous Galerkin (DG) methods for simplified Signorini problem has been carried out in [42]. In the monograph [40] several DG methods have been discussed for the Signorini problem and therein a priori error analysis has been established. Adaptive conforming finite element method for the Signorini problem has been discussed in [3, 27, 28, 34, 44]. The articles [23, 43] analyze a posteriori error analysis of discontinuous Galerkin finite element methods for the Signorini problem. Note that, the solution of the Signorini problem don’t lie in $H^3$ because of the presence of the free boundary. Thereby, the uniform refinement does not yield the optimal convergence using quadratic finite element approximation (see [26]) but adaptive refinement gives the optimal convergence. In this paper, we derive the residual based a posteriori error estimates for the quadratic conforming finite element method for the unilateral contact problem. To the best of the knowledge of the authors, quadratic AFEM for the Signorini problem has not been discussed so far. One of the key ingredient of our analysis is the appropriate construction of the discrete counterpart of the continuous contact force density which helps in proving the main results of this article.

The outline of this article is as follows. In Sect. 2, we introduce continuous contact force density and some notations which are used in later analysis. Therein, we also present the continuous (strong and weak) formulation of the Signorini problem and discuss some preliminary results. Section 3 is devoted to the introduction of discrete spaces on which discrete problem and discrete contact force density are defined followed by introducing the discrete Lagrange multiplier and deriving its basic properties. Further, in Sect. 4, quasi discrete contact force density is introduced to imitate the property of continuous Lagrange multiplier [3, 8] but computed using discrete contact force density. In Sect. 5, we propose and analyze a posteriori error estimator, therein the reliability and efficiency of the error estimator is discussed. Finally, numerical experiments illustrating the convergence behavior of proposed a posteriori error estimator using quadratic finite elements are depicted in Sect. 6.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, polygonal elastic body with Lipschitz boundary $\partial \Omega = \Gamma$ which is partitioned into three non overlapping, relatively open parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ with $\text{meas}(\Gamma_D) > 0$ and $\Gamma_C \subseteq \Gamma \setminus \Gamma_D$. Here, $\text{meas}(A)$ denotes the measure of any set $A \subset \overline{\Omega}$ and let $\{e_i : i = 1, 2\}$ denotes the standard ordered basis functions of $\mathbb{R}^2$.

### 2 Basic Preliminaries and Definitions

We recall here some basic notations associated with the finite element setting which are required in the subsequent sections:
- $T_h$ is a family of regular triangulation of $\Omega$,
- $\mathcal{E}_h$ denotes the set of all edges of $T_h$,
- $\mathcal{E}_h^i$ denotes the set of all interior edges of $T_h$,
- $\mathcal{E}_h^b$ denotes the set of all boundary edges of $T_h$,
- $\mathcal{E}_h^{N}$ denotes the set of all boundary edges lying on $\Gamma_N$,
- $\mathcal{E}_h^{C}$ denotes the set of all boundary edges lying on $\Gamma_C$,
- $V_h$ denotes the set of all the vertices of $T_h$,
- $M_h$ denotes the set of all the midpoints of edges of $T_h$,
- $V_e$ denotes the set of vertices lying on edge $e$,
- $M_e$ refers to the midpoint of the edge $e$,
- $\gamma_p$ refers to the set of all elements sharing the node $p$,
- $h_{\gamma_p}$ refers to the maximum of the set $\{h_T : T \in \omega_p\}$,
- $\gamma_{p,D}$ denotes the set of vertices of $T_h$ lying on $\Gamma_D$,
- $\gamma_{p,N}$ denotes the set of vertices of $T_h$ lying on $\Gamma_N$,
- $\gamma_{p,C}$ denotes the set of vertices of $T_h$ lying on $\Gamma_C$,
- $\gamma_{p,I}$ refers to all interior edges in $\omega_p$,
- $\gamma_{p,T}$ is the diameter of $T$ where $T \in T_h$,
- $h$ refers to the maximum of the set $\{h_T : T \in T_h\}$,
- $h_e$ is the length of an edge $e$,
- $\omega_p$ refers to the set of all elements sharing the node $p$,
- $h_p$ refers to the maximum of the set $\{h_T : T \in \omega_p\}$,
- $\gamma_{p,D}$ is the set of all elements sharing the node $p$,
- $\gamma_{p,N}$ is the set of all elements sharing the node $p$,
- $\gamma_{p,C}$ is the set of all elements sharing the node $p$,
- $\gamma_{p,I}$ is the set of all elements sharing the node $p$,
- $\gamma_{p,T}$ is the set of all elements sharing the node $p$.

Next, we define the following differential operators and preliminary definitions for the further use:

- For any Banach space $X$, let $X^*$ denotes the dual space of $(X, \| \cdot \|_X)$ with the dual norm $\| \cdot \|_*$ defined by
  \[ \| L \|_* \ := \ \sup_{v \in X, \ v \neq 0} \frac{L(v)}{\| v \|_X}, \quad \forall \ L \in X^*, \]

- $\nabla v$ is a $2 \times 2$ gradient matrix of a vector $v \in \mathbb{R}^2$,
- For any matrix $M = (m_{ij}) \in \mathbb{R}^{2 \times 2}$, the divergence of $M$ is defined as
  \[ \text{div}(M) := \sum_{j=1}^{2} \frac{\partial}{\partial x_j} (m_{ij}), \quad i = 1, 2. \]

- $\epsilon(v)$ is the linearized strain tensor defined by $\frac{1}{2}(\nabla v + \nabla v^T)$,
- $A$ is the fourth-order elasticity tensor of the material,
- $\sigma(v)$ is the linearized stress tensor defined by $A \epsilon(v)$,
$H^m(\Omega)$ denotes the usual Sobolev space \cite{10} of square integrable functions whose weak derivative up to order $m$ is also square integrable with the corresponding norm $\| \cdot \|_{H^m(\Omega)}$ and seminorm $| \cdot |_{H^m(\Omega)}$.

- For a non integer positive number $s = m + k$, where $m$ is an integer and $0 < k < 1$, the fractional ordered subspace $H^s(\Omega)$ is defined as

$$H^s(\Omega) := \{ v \in H^m(\Omega) : \frac{|v(x) - v(y)|}{|x - y|^{k+1}} \in L^p(\Omega \times \Omega) \}.$$ 

- For any vector $v = (v_1, v_2) \in H^m(\Omega) = H^m(\Omega)^2$, we define the product norm on the domain as $\|v\|_{H^m(\Omega)} := \left( \sum_{i=1}^2 \|v_i\|^2_{H^m(\Omega)} \right)^{\frac{1}{2}}$ and seminorm $|v|_{H^m(\Omega)} := \left( \sum_{i=1}^2 |v_i|^2_{H^m(\Omega)} \right)^{\frac{1}{2}}$.

- $\langle \cdot , \cdot \rangle_{-1,1}$ denotes the duality pairing between $H^1(\Omega)$ and $H^{-1}(\Omega)$.
- For any $\Omega \subset \Omega, \langle \cdot , \cdot \rangle_{-1,1,\Omega}$ denotes the duality pairing between $H^1(\Omega')$ and $H^{-1}(\Omega')$.
- $\langle \cdot , \cdot \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C}$ denotes the duality pairing between $H^{\frac{1}{2}}(\Gamma_C)$ and $H^{-\frac{1}{2}}(\Gamma_C)$.
- For any $v \in H^1(\Omega)$, we denote $v^+ = \max\{v, 0\}$ to be the positive part of the function.

Throughout this article, we assume that $C$ is a positive generic constant independent of the mesh parameter $h$. Further, the notation $x \lesssim y$ denotes that there is a generic constant $C$ such that $x \leq Cy$.

Next, we define the broken Sobolev space $[H^1(\Omega, T_h)]^2$, with the aim of defining the jump and averages of discontinuous functions efficiently as

$$[H^1(\Omega, T_h)]^2 := \{ v \in [L^2(\Omega)]^2 : v|_T \in [H^1(T)]^2 \ \forall \ T \in T_h \}.$$ 

Let $e \in E^*_h$ be an interior edge and let $T^+$ and $T^-$ be the neighbouring elements s.t. $e \in \partial T^+ \cup \partial T^-$ and let $n^+$ be the unit outward normal vector on $e$ pointing from $T^+$ to $T^-$ and $n^- = -n^+$. For a vector valued function $v \in [H^1(\Omega, T_h)]^2$ and a matrix valued function $\Phi \in [H^1(\Omega, T_h)]^{2 \times 2}$, averages $\{[\cdot]\}$ and jumps $\|\cdot\|$ across the edge $e$ are defined as follows:

$$\{[v]\} := \frac{1}{2}(v^+ + v^-) \quad \text{and} \quad \|v\| := v^+ \otimes n^+ + v^- \otimes n^-,$$

$$\{[\Phi]\} := \frac{1}{2}(\Phi^+ + \Phi^-) \quad \text{and} \quad \|\Phi\| := \Phi^+ n^+ + \Phi^- n^-,$$

where $v^\pm|_T = v^\pm|_{T^\pm}$, $\Phi^\pm = \Phi|_{T^\pm}$.

For any $e \in E^b_h$, it is clear that there is a triangle $T \in T_h$ such that $e \in \partial T \cap \partial \Omega$. Let $n_e$ be the unit normal of $e$ that points outside $T$. Then, the averages $\{[\cdot]\}$ and jumps $\|\cdot\|$ of vector valued function $v \in [H^1(\Omega, T_h)]^2$ and a matrix valued function $\Phi \in [H^1(\Omega, T_h)]^{2 \times 2}$ are defined as follows:

$$\{[v]\} := v \quad \text{and} \quad \|v\| := v \otimes n_e,$$

$$\{[\Phi]\} := \Phi \quad \text{and} \quad \|\Phi\| := \Phi n_e.$$ 

In the above definitions $v \otimes n$ is a $2 \times 2$ matrix with $v_i n_j$ as its $(i, j)^{th}$ entry.

For any displacement field $v$, we adopt the notation $v_m = v \cdot m$ and $v_T = v - v_m m$, respectively, as its normal and tangential component on the boundary where $m$ is the outward unit normal vector to $\Gamma$. Similarly, for a tensor-valued function $\Phi$, the normal and tangential
components are defined as \( \Phi_m = \Phi m \cdot m \) and \( \Phi_T = \Phi m - \Phi m m \), respectively. Further, we have the following decomposition formula

\[
(\Phi m) \cdot v = \Phi_m v_m + \Phi_T \cdot v_T.
\]

In the further analysis, for any tensor-valued function \( \Phi \) the term \( \Phi(v) \) denotes the boundary contact stresses in the direction of the normal at the potential contact boundary and is equal to \( \Phi(v)n \) where \( n \) is the outward unit normal on \( \Gamma_C \). In this article, we assume that the outward unit normal vector \( n \) to \( \Gamma_C \) is constant and for the sake of simplicity, we define \( n = e_1 \).

In this article, we assume that our elastic body is homogeneous and isotropic, as a result

\[
\sigma(v) = A \varepsilon(v) := \chi tr(\varepsilon(v))I + 2\mu \varepsilon(v). 
\]  

(2.1)

where, \( \chi > 0 \) and \( \mu > 0 \) denote the Lamé coefficients. Here \( I \) denotes the identity matrix of size \( 2 \times 2 \). In order to define the continuous problem, we define the space \( V \) of admissible displacements as

\[
V := \{ v \in [H^1(\Omega)]^2 : v = 0 \text{ on } \Gamma_D \},
\]

and a non empty, closed and convex subset of \( V \) is defined as

\[
K := \{ v \in V : v_n = v_1 \leq 0 \text{ a.e on } \Gamma_C \}.
\]

Given \( f \in [L^2(\Omega)]^2, g \in [L^2(\Gamma_N)]^2 \), the weak formulation of unilateral contact problem is to find \( u \in K \) such that

\[
a(u, v - u) \geq L(v - u) \quad \forall \ v \in K, 
\]  

(2.2)

where, the bilinear form \( a(\cdot, \cdot) \) and the linear functional \( L(\cdot) \) are defined by

\[
a(w, v) = \int_{\Omega} \sigma(w) : \varepsilon(v) \ dx, 
\]

\[
L(v) = \int_{\Omega} f \cdot v \ dx + \int_{\Gamma_N} g \cdot v \ ds \quad \forall \ w, v \in V. 
\]

The strong form associated to the variational inequality of the first kind (2.2) is to find the displacement vector \( u : \Omega \rightarrow \mathbb{R}^2 \) such that the following holds:

\[-div \ \sigma(u) = f \quad \text{in } \Omega, \]

\[u = 0 \quad \text{on } \Gamma_D, \]

\[\hat{\sigma}(u) = g \quad \text{on } \Gamma_N, \]

\[u_n \leq 0, \ \sigma_n(u) \leq 0, \ u_n \sigma_n(u) = 0 \text{ and } \sigma_T(u) = 0 \text{ on } \Gamma_C, \]

where \( \hat{\sigma}(u) = (\hat{\sigma}_1(u), \hat{\sigma}_2(u)) := \sigma(u)n \). The existence and uniqueness of the solution of problem (2.2) is well known from the theory of variational inequalities [22].

Next, we define the continuous contact force density \( \lambda \in V^* \) as

\[
(\lambda, v)_{-1,1} := L(v) - a(u, v) \forall \ v \in V. 
\]  

(2.3)

In the next lemma, we collect some important properties corresponding to continuous contact force density \( \lambda \).

**Lemma 2.1** The following holds

\[
(\lambda, v - u)_{-1,1} \leq 0 \quad \forall \ v \in K. 
\]  

(2.4)

\[
(\lambda, \phi)_{-1,1} \geq 0 \quad \forall \ 0 \leq \phi \in V. 
\]  

(2.5)
**Proof** The relation (2.4) can be realized directly from (2.2) and (2.3).

To prove (2.5), let \( \theta \leq \phi \in V \) and substitute \( v = u - \phi \in K \) in the variational inequality (2.2) to get \( (\lambda, \phi)_{-1,1} \geq 0 \). \( \square \)

In order to realize another representation to continuous contact force density \( \lambda \), we further define an intermediate space \( V_0 \) as

\[
V_0 := \{ v = (v_1, v_2) \in V, \ v_1 = 0 \ \text{on} \ \Gamma_C \}.
\]

Since \( v = u \pm \bar{v} \in K \forall \bar{v} \in V_0 \), therefore inequality (2.2) reduces to

\[
a(u, \bar{v}) = L(\bar{v}) \ \forall \bar{v} \in V_0.
\]

(2.6)

**Remark 2.2** For each \( v \in V \), we have \( v = (v_1, v_2) = w_1 + w_2 \) where \( w_1 = (v_1, 0) \) and \( w_2 = (0, v_2) \). In view of equation (2.3), we can rewrite \( (\lambda, v)_{-1,1} = (\lambda_1, v_1)_{-1,1} + (\lambda_2, v_2)_{-1,1} \) where

\[
(\lambda_1, v_1)_{-1,1} = L(w_1) - a(u, w_1),
\]

\[
(\lambda_2, v_2)_{-1,1} = L(w_2) - a(u, w_2).
\]

As \( w_2 \in V_0 \), using Eq. (2.6), we obtain

\[
(\lambda_2, v_2)_{-1,1} = 0 \ \forall v = (v_1, v_2) \in V.
\]

Thus, we have the representation

\[
(\lambda, v)_{-1,1} = (\lambda_1, v_1)_{-1,1} \ \forall v = (v_1, v_2) \in V.
\]

(2.7)

An application of Green’s theorem [40] yields

\[
(\lambda_1, v_1)_{-1,1} = -\langle \hat{\sigma}_1(u), v_1 \rangle_{-1/2, \Gamma_C},
\]

\[
(\lambda_2, v_2)_{-1,1} = -\langle \hat{\sigma}_2(u), v_2 \rangle_{-1/2, \Gamma_C} = 0.
\]

### 3 Discrete Problem

In this article, we consider the approximation of the problem (2.2) by quadratic finite element method. To this end, we define the finite element space \( V^h \) as the space of continuous piecewise quadratic finite element functions over \( T_h \) i.e.

\[
V^h := \{ v \in [C(\bar{\Omega})]^2 : v|_T \in [P_2(T)]^2 \ \forall \ T \in T_h \}.
\]

Further, we set

\[
V^h_0 := \{ v \in V^h : v = 0 \ \text{on} \ \Gamma_D \}.
\]

For concreteness, we state the following discrete trace inequality and inverse inequality which will be used in the subsequent analysis [10].

**Lemma 3.1** Let \( v \in [H^1(T)]^2 \). Then,

\[
\|v\|_{L^2(e)}^2 \leq h_T^{-1} \|v\|_{L^2(T)}^2 + h_T |v|_{H^1(T)}^2,
\]

where \( T \in T_h \) and \( e \) is an edge of \( T \).
Lemma 3.2 Let $T \in \mathcal{T}_h$ and $e$ be an edge of $T$. For $v \in V_h$, the following estimates hold
\[
\|v\|_{L^2(e)} \lesssim h_e^{-\frac{1}{2}} \|v\|_{L^2(T)},
\]
\[
|v|_{H^1(T)} \lesssim h_T^{-1} \|v\|_{L^2(T)}.
\]
Next, we define the discrete formulation of the continuous problem (2.2). Further, we construct an auxiliary discrete space $Q_h$, where the discrete counterpart of the contact force density is defined. It will play the crucial role in forthcoming a posteriori error analysis.

Let $(\psi_z e_i, z \in V_h \cup M_h, i = 1, 2)$ represents the canonical nodal Lagrange basis for the space $V_h$, i.e., for $z \in V_h \cup M_h$
\[
\psi_z(p) = \begin{cases} 
1 & \text{if } z = p \\
0 & \text{if } z \neq p
\end{cases} \quad \forall \ p \in V_h \cup M_h.
\]
Analogously, $[\psi_z e_i, z \in V_h \cup M_h, i = 1, 2]$ denote the basis functions for the discrete space $V_0^h$. Note that, for any $v^h = (v_1^h, v_2^h) \in V_0^h$, we have the following representation
\[
v^h = \sum_{p \in V_h^0 \cup M_h^0} \sum_{i=1}^2 v_i^h(p) \psi_p e_i. \tag{3.1}
\]
We define the two discrete subspaces $W_1$ and $W_2$ of $V_0^h$ as
\[
W_1 := \text{Span}\{\psi_z e_i : z \in (V_h^0 \cup M_h^0) \setminus (V_h^C \cup M_h^C), i = 1, 2\},
\]
\[
W_2 := \text{Span}\{\psi_z e_i : z \in V_h^C \cup M_h^C, i = 1, 2\}.
\]
Then, clearly $V_0^h = W_1 \oplus W_2$. It can be observed that the subspace $W_2$ of $V_0^h$ is orthogonal to $W_1$ with respect to inner product:
\[
\langle v^h, w^h \rangle_{V_0^h} := \frac{1}{3} \sum_{T \in \mathcal{T}_h} \left( \sum_{z \in \partial T} v^h(z) w^h(z) + \sum_{z \in \mathcal{M}_T} v^h(z) w^h(z) \right)
\]
where $\partial T$ and $\mathcal{M}_T$ refers to vertices and midpoints of the element $T$, respectively.

Further, we introduce the discrete set $K_h^1$ of admissible displacements by
\[
K_h^1 := \{v^h = (v_1^h, v_2^h) \in V_0^h \text{ s.t. } v_1^h(z) \leq 0 \ \forall \ z \in V_h^C \cup M_h^C\}.
\]
The quadratic finite element approximation of (2.2) is to find $u^h \in K_h^1$ such that
\[
a(u^h, v^h - u^h) \geq L(v^h - u^h) \quad \forall \ v^h \in K_h^1. \tag{3.2}
\]
It can be observed that the non-empty, closed and convex set $K_h^1 \subset K$ in general. For all $z \in (V_h \cup M_h) \setminus (V_h^C \cup M_h^C)$, observe that $v^h = u^h + \psi_z e_i \in K_h^1$ since
\[
v_i^h(p) = u_i^h(p) \leq 0 \ \forall \ p \in V_h^C \cup M_h^C.
\]
Therefore, we find
\[
a(u^h, \psi_z e_i) = L(\psi_z e_i) \quad \forall \ z \in (V_h \cup M_h) \setminus (V_h^C \cup M_h^C). \tag{3.3}
\]
Henceforth,
\[
a(u^h, v^h) = L(v^h) \quad \forall \ v^h \in W_1. \tag{3.4}
\]
Further, for \( z \in (\mathcal{V}_h^C \cup \mathcal{M}_h^C) \), we observe that \( \mathbf{v}^h = \mathbf{u}^h - \psi_z \mathbf{e}_1 \in \mathcal{K}^h \) as
\[
\mathbf{v}_i^h(p) = \begin{cases} 
\frac{u_i^h(p)}{u_i^h(p)} - 1 & \text{if } p \neq z \\
0 & \text{if } p = z
\end{cases} \quad \forall \ p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C.
\]

On the similar lines, one can verify \( \mathbf{v}^h = \mathbf{u}^h \pm \psi_z \mathbf{e}_2 \in \mathcal{K}^h \) \( \forall \ z \in (\mathcal{V}_h^C \cup \mathcal{M}_h^C) \). Thus,
\[
a(\mathbf{u}^h, \psi_z \mathbf{e}_1) \leq L(\psi_z \mathbf{e}_1) \quad \forall \ z \in (\mathcal{V}_h^C \cup \mathcal{M}_h^C), \\
a(\mathbf{u}^h, \psi_z \mathbf{e}_2) = L(\psi_z \mathbf{e}_2) \quad \forall \ z \in (\mathcal{V}_h^C \cup \mathcal{M}_h^C). \quad (3.5)
\]

We now proceed to introduce a discrete space where we can define discrete counterpart of the contact force density \( \lambda \). The construction of the discrete space requires the introduction of some more notations related to the contact zone. Let \( \mathcal{T}_h^C \) denotes the mesh formed by the edges of \( \mathcal{T}_h \) on \( \Gamma_C \) and characterized by the subdivision of \( (x_i^e)_{0 \leq i \leq n} \) where \( (x_i^e)_{0 \leq i \leq n} \in \mathcal{V}_h^C \).

Let \( t_i = [x_{i-1}^e, x_i^e]_{0 \leq i \leq n-1} \) denotes the element on \( \Gamma_C \) with the midpoint \( m_i^e \). Hence, we can write each element \( t_i \) as union of two subintervals \( q_i^1 \cup q_i^2 \) where \( q_i^1 = [x_i^e, m_i^e] \) and \( q_i^2 = [m_i^e, x_{i+1}^e] \). Thus, we can rewrite
\[
\Gamma_C = \bigcup_{0 \leq i \leq n-1} q_i^1 \cup q_i^2.
\]

Now, with the following notations we define the discrete space \( \mathcal{Q}_h \) as
\[
\mathcal{Q}_h := \{ \mathbf{v}^h \in [C(\Gamma_C)]^2 : \mathbf{v}^h_{q_i^j} \in [P_1(q_i^j)]^2, \ 1 \leq i \leq n - 1, \ j = 1, 2 \}. \quad (3.6)
\]

We observe that the dimension of the space \( \mathcal{Q}_h \) is \( 2|\mathcal{V}_h^C \cup \mathcal{M}_h^C| \). Let \( \{\phi_z \mathbf{e}_i : z \in \mathcal{V}_h^C \cup \mathcal{M}_h^C\} \) be the canonical nodal Lagrange basis for \( \mathcal{Q}_h \), i.e., for \( z \in \mathcal{V}_h^C \cup \mathcal{M}_h^C \)
\[
\phi_z(p) = \begin{cases} 
1 & \text{if } z = p \\
0 & \text{if } z \neq p
\end{cases} \quad \forall \ p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C.
\]

Define a linear map \( \pi_h : \mathcal{Q}_h \rightarrow W_2 \) by
\[
\pi_h \mathbf{v}^h := \sum_{z \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \sum_{i=1}^{2} v_i^h(z) \phi_z \mathbf{e}_i \quad \forall \ \mathbf{v}^h = (v_1^h, v_2^h) \in \mathcal{Q}_h. \quad (3.7)
\]

Clearly, the map \( \pi_h \) is well defined and one-one. Since the dimension of spaces \( \mathcal{Q}_h \) and \( W_2 \) are equal, therefore the map \( \pi_h \) is bijective and hence \( \pi_h^{-1} : W_2 \rightarrow \mathcal{Q}_h \) exists and is given by
\[
\pi_h^{-1} \mathbf{v}^h = \sum_{z \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \sum_{i=1}^{2} v_i^h(z) \phi_z \mathbf{e}_i \quad \forall \ \mathbf{v}^h = (v_1^h, v_2^h) \in W_2.
\]

It can be observed that
\[
\pi_h^{-1} \mathbf{v}^h(z) = v_i^h(z) \quad \forall \ z \in \mathcal{V}_h^C \cup \mathcal{M}_h^C \quad \forall \ \mathbf{v}^h \in W_2. \quad (3.8)
\]

Now, we turn our attention to introduce the discrete contact force density \( \lambda^h \in \mathcal{Q}_h \) which is defined as
\[
(\lambda^h, \mathbf{v}^h) = L(\pi_h \mathbf{v}^h) - a(C, \pi_h \mathbf{v}^h) \quad \forall \ \mathbf{v}^h \in \mathcal{Q}_h. \quad (3.9)
\]
where, the inner product $\langle \cdot, \cdot \rangle_h$ on the space $Q_h$ is defined as
\[
\langle w^h, v^h \rangle_h := \sum_{z \in V_h^C \cup M_h^C} w^h(z) \cdot v^h(z) \int_{\gamma_{z,c}} \phi_z \, ds.
\]

Note that, $\lambda^h$ is well-defined since $\langle \cdot, \cdot \rangle_h$ defines an inner product on $Q_h$. In the following lemma, we will establish the properties of discrete contact force density $\lambda^h \in Q_h$.

**Lemma 3.3** The discrete contact force density $\lambda^h = (\lambda_1^h, \lambda_2^h) \in Q_h$ satisfies the following sign properties.

\[
\lambda_1^h(p) \geq 0 \quad \forall \ p \in V_h^C \cup M_h^C,
\]

\[
\lambda_2^h(p) = 0 \quad \forall \ p \in V_h^C \cup M_h^C.
\]

**Proof** The proof of this lemma follows by suitable construction of a test function $v^h \in Q_h$. Let $p \in V_h^C \cup M_h^C$ be an arbitrary node. We choose a test function $v^h$ as follows
\[
v^h(z) = \begin{cases} 
(1, 0) & \text{if } z = p, \\
(0, 0) & \text{if } z \neq p,
\end{cases} \quad \forall \ z \in V_h^C \cup M_h^C.
\]

Further, using the definition of $\pi_h$, we have
\[
\pi_h v^h = \sum_{z \in V_h^C \cup M_h^C} (v_1^h(z) \psi_z, v_2^h(z) \psi_z) = (\psi_p, 0) = \psi_p e_1.
\]

Thus, the use of (3.5) yields
\[
\langle \lambda^h, v^h \rangle_h = L(\pi_h v^h) - a(u^h, \pi_h v^h)
\]
\[
= L(\psi_p e_1) - a(u^h, \psi_p e_1)
\]
\[
\geq 0,
\]
\[
(3.10)
\]

whereas, using the definition of $\langle \cdot, \cdot \rangle_h$, we find
\[
\langle \lambda^h, v^h \rangle_h = \sum_{z \in V_h^C \cup M_h^C} \lambda^h(z) \cdot v^h(z) \int_{\gamma_{z,c}} \phi_z \, ds
\]
\[
= \lambda^h(p) \cdot v^h(p) \int_{\gamma_{p,c}} \phi_p \, ds
\]
\[
= \lambda_1^h(p) \int_{\gamma_{p,c}} \phi_p \, ds.
\]
\[
(3.11)
\]

Combining (3.10), (3.11) and taking into account $\int_{\gamma_{p,c}} \phi_p \, ds > 0$, we find $\lambda_1^h(p) \geq 0$. Since $p \in V_h^C \cup M_h^C$ is arbitrary, it follows that $\lambda_1^h(p) \geq 0 \ \forall \ p \in V_h^C \cup M_h^C$. Analogously for any $p \in V_h^C \cup M_h^C$, we define $v^h \in Q_h$ such that
\[
v^h(z) = \begin{cases} 
(0, 1) & \text{if } z = p, \\
(0, 0) & \text{if } z \neq p
\end{cases} \quad \forall \ z \in V_h^C \cup M_h^C.
\]

In this case, we have $\pi_h v^h = \psi_p e_2$. Therefore, using (3.5) we have
\[
\langle \lambda^h, v^h \rangle_h = L(\pi_h v^h) - a(u^h, \pi_h v^h)
\]
Using equation (3.12) and (3.7), we have

\[
\langle \lambda^h, v^h \rangle_h = \sum_{z \in \mathcal{V}_h \cup \mathcal{M}_h} \lambda^h(z) \cdot v^h(z) \int_{\gamma_{c}} \phi_z \ ds
\]

\[= \lambda^h(p) \cdot v^h(p) \int_{\gamma_{p,c}} \phi_p \ ds
\]

\[= \lambda_2^h(p) \int_{\gamma_{p,c}} \phi_p \ ds. \tag{3.13}
\]

Using equation (3.12) and \( \int_{\gamma_{p,c}} \phi_p > 0 \), it follows \( \lambda_2^h(p) = 0 \). Consequently, it follows \( \lambda_2^h(p) = 0 \ \forall \ p \in \mathcal{V}_h \cup \mathcal{M}_h \).

In order to carry out further analysis, we define the linear residual \( R^{lin} \in V^* \) as

\[\langle R^{lin}, \phi \rangle_{-1,1} := L(\phi) - a(u^h, \phi) \ \forall \phi \in V. \tag{3.14}\]

For any \( \phi = (\phi_1, \phi_2) \in V \), the linear residual can be represented as

\[\langle R^{lin}, \phi \rangle_{-1,1} = \sum_{i=1}^{2} \langle R^{lin}_i, \phi_i \rangle_{-1,1}, \]

where,

\[\langle R^{lin}_1, \phi_1 \rangle_{-1,1} := L((\phi_1, 0)) - a(u^h, (\phi_1, 0)), \]

\[\langle R^{lin}_2, \phi_2 \rangle_{-1,1} := L((0, \phi_2)) - a(u^h, (0, \phi_2)). \]

Further, for any \( \phi^h \in V^h \), we have

\[\langle R^{lin}, \phi^h \rangle_{-1,1} = L(\phi^h) - a(u^h, \phi^h) \ \forall \phi^h \in V^h. \tag{3.15}\]

In particular, we assume \( \phi^h = \psi_z e_i \) for \( z \in \mathcal{V}_h \cup \mathcal{M}_h \) to derive

\[\langle R^{lin}, \psi_z e_i \rangle_{-1,1} = L(\psi_z e_i) - a(u^h, \psi_z e_i). \tag{3.16}\]

Using equation (3.7), we have \( \pi_h(\phi_z e_i) = \psi_z e_i \) for \( i = 1, 2 \). Finally, using the Eqs. (3.9) and (3.16), we have the following relation for any \( z \in \mathcal{V}_h \cup \mathcal{M}_h \)

\[\langle R^{lin}, \psi_z e_i \rangle_{-1,1} = L(\pi_h(\phi_z e_i)) - a(u^h, \pi_h(\phi_z e_i)).
\]

\[= \langle \lambda^h, \phi_z e_i \rangle_h. \tag{3.17}\]

The above relation between \( \lambda^h \) and \( R^{lin} \) plays a key role in later analysis. Let \( v^h = (v_1^h, v_2^h) \in V^h \), using integration by parts and Eq. (3.1), we find

\[\langle R^{lin}, v^h \rangle_{-1,1} = \sum_{i=1}^{2} \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \left[ L(v_i^h(p) \psi_p e_i) - a(u^h, v_i^h(p) \psi_p e_i) \right]
\]

\[= \sum_{i=1}^{2} \int_{\omega_p} (f + div(u^h)) \cdot v_i^h(p) \psi_p e_i \ dx \]
Using the Eqs. (3.3) and (3.5), we derive important characterizations for $R^{lin}$

$$\langle R^{lin}, \psi_z e_i \rangle_{-1,1} = 0 \quad \forall \ z \in (V_h \cup M_h) \setminus (V_h^C \cup M_h^C), \ i = 1, 2,$$

$$\langle R^{lin}, \psi_z e_2 \rangle_{-1,1} = 0 \quad \forall \ z \in V_h^C \cup M_h^C. \quad (3.19)$$

In the subsequent analysis, for the ease of the presentation we abbreviate the interior residual as $r(u^h) = f + div (u^h).$ Further, the jump terms which are either the difference between the contact stresses of two neighboring elements or the difference between Neumann data and boundary stress at Neumann boundary or the boundary stresses at contact boundary are abbreviated as

- For $e \in E_h^i$
  \[ J^i(u^h) := [\sigma(u^h)], \]

- For $e \in E_h^N$
  \[ J^N(u^h) := g - \sigma(u^h)n, \]

- For $e \in E_h^C$
  \[ J^C_{tan}(u^h) := \hat{\sigma}(u^h). \]

### 4 Quasi Discrete Contact Force Density

In this section, we introduce the quasi discrete contact force density which imitates the properties of continuous contact force density $\lambda.$ It is further computed using the discrete solution and discrete contact force density. For any $p \in V_h^C \cup M_h^C,$ we take the node values of a discrete contact force density obtained by lumping the boundary mass matrix and define

$$s_p = (s^1_p, s^2_p), \quad (4.1)$$

where $s^1_p := \langle \lambda^h \cdot e_1 \rangle_{h, p} \phi_{p} = \lambda^h_1(p)$ and $s^2_p := \lambda^h_2(p) = 0.$

Next, we categorize actual contact nodes $p \in V_h^C \cup M_h^C$ (i.e., $u^h_1(p) = 0$) in two different categories.

1. Full contact nodes $\mathcal{N}_h^{FC} := \{ p \in V_h^C \cup M_h^C \mid u^h_1(p) = 0 \}$ on $\gamma_{p,c}.$
2. The remaining actual contact nodes are called semi contact nodes and denoted by $\mathcal{N}_h^{SC}.$

Denote $\mathcal{N}_h^{NC}$ as the set of no actual contact nodes, i.e., for $p \in \mathcal{N}_h^{NC},$ $u^h_1(p) \neq 0.$
Further, for $\mathbf{v} = (v_1, v_2) \in V$, we will introduce the constants $c_p(v_i), p \in \mathcal{V}_h \cup \mathcal{M}_h$ which are key ingredients in defining quasi discrete contact force density. These constants are defined such that they fulfill $L^2$ approximation properties. We set $c_p(v_i) = 0$ for all the Dirichlet nodes. For all the non-contact nodes and all contact nodes with $i \neq 1$, we define the constants

$$c_p(v_i) := \frac{\int_{\omega_p} v_i \psi_p \, dx}{\int_{\omega_p} \psi_p \, dx} \quad (4.2)$$

and for semi contact and full contact nodes, the constants $c_p(v_1)$ are chosen such that

$$c_p(v_1) := \frac{\int_{\tilde{\gamma}_{p,c}} v_1 \phi_p \, ds}{\int_{\tilde{\gamma}_{p,c}} \phi_p \, ds} \quad (4.3)$$

where $\tilde{\gamma}_{p,c}$ is a proper subset of $\gamma_{p,c}$ such that it contains $p$ and for any two different nodes $p_1$ and $p_2$ in $\gamma_{p,c} \setminus \tilde{\gamma}_{p_1,c} \cap \tilde{\gamma}_{p_2,c} = \emptyset$. These constants are helpful in deriving the lower bound of the error estimator. Also, we have the following approximation properties [34].

**Lemma 4.1** Let $\phi = (\phi_1, \phi_2)$ be an arbitrary function in $V$ and $c_p(\phi_i)$ be one of the mean values defined in (4.2) and (4.3). Then, we have the following $L^2$ approximation properties.

$$\|\phi_i - c_p(\phi_i)\|_{L^2(\omega_p)} \lesssim h_p \|\nabla \phi_i\|_{L^2(\omega_p)},$$

$$\|\phi_i - c_p(\phi_i)\|_{L^2(\gamma_p)} \lesssim h_p^{1/2} \|\nabla \phi_i\|_{L^2(\omega_p)},$$

where $\gamma_p \in \{\gamma_{p,1}, \gamma_{p,2}, \gamma_{p,3}, \gamma_{p,4}\}$.

We remark here that the estimates of Lemma 4.1 also hold for $p \in \mathcal{V}_h^{D} \cup \mathcal{M}_h^{D}$ by taking into account Poincaré-Fredrichs inequality and trace inequality.

Now with the help of Eqs. (4.2), (4.3) and the property of nodal basis functions $\sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \psi_p = 1$, we introduce the quasi discrete contact force density $\tilde{\lambda}^h \in V^*$ in the following way

$$\langle \tilde{\lambda}^h, \mathbf{v} \rangle_{-1,1} := \sum_{i=1}^{2} \langle \tilde{\lambda}^h_i, v_i \rangle_{-1,1} \quad (4.4)$$

where, for $i = 1, 2$

$$\langle \tilde{\lambda}^h_i, v_i \rangle_{-1,1} := \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \langle \tilde{\lambda}^h_i, v_i \psi_p \rangle_{-1,1} := \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \langle R^{lin}, \psi_p e_i \rangle_{-1,1} c_p(v_i) \quad \forall \mathbf{v} = (v_1, v_2) \in V. \quad (4.5)$$

Next, we derive a useful preliminary result and the sign property for $\tilde{\lambda}^h$ in the next two lemmas, respectively.

**Lemma 4.2** The following holds

$$\langle \tilde{\lambda}^h, \mathbf{v} \rangle_{-1,1} = \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle \tilde{\lambda}^h, v_1 \psi_p \rangle_{-1,1}, \quad (4.6)$$

where $\mathbf{v} = (v_1, v_2) \in V$.  

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Proof Using the definition of quasi discrete contact force density (4.5), we have
\[
\langle \tilde{\lambda}^h, v \rangle_{-1,1} = \sum_{i=1}^{2} \sum_{p \in V_h \cup M_h} \langle \tilde{\lambda}^h_i, v_1 \psi_p \rangle_{-1,1}
\]
\[
= \sum_{i=1}^{2} \sum_{p \in V_h \cup M_h} \langle R^{lin}, \psi_p e_i \rangle_{-1,1} c_p(v_i).
\]
(4.7)
Using (3.19) and (3.20), the last relation (4.7) reduces to
\[
\langle \tilde{\lambda}^h, v \rangle_{-1,1} = \sum_{p \in V_h \cup M_h} \langle R^{lin}, \psi_p e_1 \rangle_{-1,1} c_p(v_1)
\]
\[
= \sum_{p \in V_h \cup M_h} \langle \tilde{\lambda}^h_1, v_1 \psi_p \rangle_{-1,1}.
\]

Lemma 4.3 It holds that
\[
\langle \tilde{\lambda}^h, v \rangle_{-1,1} = \langle \tilde{\lambda}^h_1, v_1 \rangle_{-1,1} \geq 0 \quad \text{whenever} \quad v_1 \geq 0,
\]
where \( v = (v_1, v_2) \in V \).

Proof With the help of Eqs. (4.5), (3.19) and (3.20), we find
\[
\langle \tilde{\lambda}^h_1, v_1 \rangle_{-1,1} = \sum_{p \in V_h \cup M_h} \langle \tilde{\lambda}^h_1, v_1 \psi_p \rangle_{-1,1} = \sum_{p \in V_h \cup M_h} \langle R^{lin}, \psi_p e_1 \rangle_{-1,1} c_p(v_1)
\]
\[
= \sum_{p \in V_h \cup M_h} \langle R^{lin}, \psi_p e_1 \rangle_{-1,1} c_p(v_1)
\]
\[
= \sum_{p \in V_h \cup M_h} \langle \tilde{\lambda}^h_1, v_1 \psi_p \rangle_{-1,1}.
\]
Therefore, in view of the last equation and Lemma 4.2, we have
\[
\langle \tilde{\lambda}^h, v \rangle_{-1,1} = \langle \tilde{\lambda}^h_1, v_1 \rangle_{-1,1} = \sum_{p \in V_h \cup M_h} \langle \tilde{\lambda}^h_1, v_1 \psi_p \rangle_{-1,1}.
\]
Now, a use of relation (3.17) and the definition of constant \( s_p \) yields
\[
\langle \tilde{\lambda}^h, v \rangle_{-1,1} = \sum_{p \in V_h \cup M_h} \langle R^{lin}, \psi_p e_1 \rangle_{-1,1} c_p(v_1)
\]
\[
= \sum_{p \in V_h \cup M_h} \langle \lambda^h, \phi_p e_1 \rangle_{h} c_p(v_1)
\]
\[
= \sum_{p \in V_h \cup M_h} s^1_p c_p(v_1) \int_{\gamma_p,C} \phi_p \, ds.
\]
(4.9)
Using Lemma 3.3, we deduce that \( s^1_p = \lambda^h_1(p) \geq 0 \). Hence
\[
\langle \tilde{\lambda}^h_1, v_1 \rangle_{-1,1} = \sum_{p \in V_h \cup M_h} s^1_p c_p(v_1) \int_{\gamma_p,C} \phi_p \, ds \geq 0
\]
(4.10)
Next, we derive an important property of $\tilde{\lambda}$ with the help of upcoming lemma.

**Lemma 4.4** It holds that

$$\langle \lambda_h, \phi_p e_1 \rangle_h = 0 \quad \forall \ p \in N_h^{NC}. \quad (4.11)$$

**Proof** Let $p$ be any non-actual contact node, i.e., we have $u^h(p) < 0$. We note for sufficiently small $\kappa > 0$ such that $0 < \kappa < -u^h(p)$, we have $v^h = u^h + \kappa \psi_p e_1 \in K^h$. Using (3.2), we conclude

$$a(u^h, \psi_p e_1) \geq \mu(u^h, \psi_p e_1). \quad (4.12)$$

Finally, we have $\langle \lambda_h, \phi_p e_1 \rangle_h = 0$ in view of (3.17) and (3.5). \hfill \Box

**Remark 4.5** From the last lemma, we deduce that $s_1^p := \langle \lambda_h, \phi_p e_1 \rangle_h \int_{\gamma_p, C} \phi_p ds = 0 \quad \forall \ p \in N_h^{NC}$. Thus, using relation (4.9) we obtain,

$$\langle \tilde{\lambda}_1^h, v_1 \rangle_{-1, 1} = \sum_{p \in V_h^C \cup M_h^C} s_1^p c_p(v_1) \int_{\gamma_{p, c}} \phi_p ds = \sum_{p \in N_h^{FC} \cup N_h^{SC}} s_1^p c_p(v_1) \int_{\gamma_{p, c}} \phi_p ds$$

for any $v = (v_1, v_2) \in V$.

### 5 A posteriori Error Analysis

In this section, we turn our attention to analyze reliability and efficiency of a posteriori error estimator. We begin by introducing the following contributions of the error estimator

$$\eta_1 := \left( \sum_{p \in V_h \cup M_h} \eta^2_{1, p} \right)^{\frac{1}{2}}, \quad \eta_{1, p} := h_p \| \text{div} \sigma(u^h) \|_{L^2(\omega_p)},$$

$$\eta_2 := \left( \sum_{p \in V_h \cup M_h} \eta^2_{2, p} \right)^{\frac{1}{2}}, \quad \eta_{2, p} := h_p \| \| \sigma(u^h) \| \|_{L^2(\gamma_{p, 1})},$$

$$\eta_3 := \left( \sum_{p \in N_h^N \cup M_h^N} \eta^2_{3, p} \right)^{\frac{1}{2}}, \quad \eta_{3, p} := h_p \| \sigma(u^h) n - g \|_{L^2(\gamma_{p, N})},$$

$$\eta_4 := \left( \sum_{p \in V_h^C \cup M_h^C} \eta^2_{4, p} \right)^{\frac{1}{2}}, \quad \eta_{4, p} := h_p \| \hat{\sigma}_2(u^h) \|_{L^2(\gamma_{p, C})},$$

$$\eta_5 := \left( \sum_{p \in N_h^{FC} \cup N_h^{SC}} \eta^2_{5, p} \right)^{\frac{1}{2}}, \quad \eta_{5, p} := h_p \| \hat{\sigma}_1(u^h) \|_{L^2(\gamma_{p, C})},$$

$$\eta_6 := \left( \sum_{p \in N_h^{SC}} \eta^2_{6, p} \right)^{\frac{1}{2}}, \quad \eta_{6, p} := (s_p d_p)^{\frac{1}{2}},$$

$$\eta_7 := \| (u_1^h)^+ \|_{H^\frac{1}{2}(\Gamma_C)}.$$
where \( d_p := \int_{\gamma_{p,c}} (-u_1^h)^+ \phi_p \, ds \) and \( \eta_h \) denotes the total residual estimator defined by

\[
\eta_h^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 + \eta_5^2 + \eta_6^2 + \eta_7^2.
\] (5.1)

**Remark 5.1** The estimator \( \eta_1 \) refers to the volume residual and \( \eta_i, 2 \leq i \leq 5 \) are the jump residuals corresponding to interior, Neumann and contact edges of the triangulation. In fact, \( \eta_i, 1 \leq i \leq 5 \) correspond to the contributions of the residual type error estimator for unconstrained problems in linear elasticity. The estimator \( \eta_6 \) is localized to semi contact nodes and it measures the violation of the complementarity condition (on \( \Gamma_C \)) at the discrete level. Lastly, \( \eta_7 \) measures the error resulting from non-penetration condition incorporated in the set \( \mathcal{K}_h \) in the sense that it measures the violation of the constraint \( u_1^h \leq 0 \) on \( \Gamma_C \).

The following subsection guarantees the reliability of the error estimator \( \eta_h \).

### 5.1 Reliability of the Error Estimator

Define the Galerkin functional \( G_h : V \rightarrow \mathbb{R} \) by

\[
G_h(v) := a(u - u^h, v) + \langle \lambda - \tilde{\lambda}^h, v \rangle_{-1,1} \forall v \in V.
\] (5.2)

**Theorem 5.2** Let \((u, \lambda)\) satisfy (2.2)–(2.3) and \((u^h, \tilde{\lambda}^h)\) satisfy (3.2) and (4.4), respectively. Then, the following reliability estimate holds

\[
\|u - u^h\|^2_{H^1(\Omega)} + \|\lambda - \tilde{\lambda}^h\|_{V^*} \lesssim \eta_h^2.
\]

We need to establish the following lemmas to prove Theorem 5.2.

**Lemma 5.3** It holds that

\[
\|u - u^h\|^2_{H^1(\Omega)} + \|\lambda - \tilde{\lambda}^h\|_{V^*} \leq C_1 \|G_h\|^2_{V^*} + C_2 \langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1},
\]

where \( C_1 \) and \( C_2 \) are positive generic constants.

**Proof** Using the \( V^{-} \) ellipticity of the bilinear form \( a(\cdot, \cdot) \) and Eq. (5.2), we find

\[
\alpha \|u - u^h\|^2_{H^1(\Omega)} \leq a(u - u^h, u - u^h)
\]

\[
= G_h(u - u^h) + \langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1}
\]

\[
\lesssim \|G_h\|_{V^*} \|u - u^h\|_{H^1(\Omega)} + \langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1}.
\]

A use of Young’s inequality in the last equation yields

\[
\alpha \|u - u^h\|^2_{H^1(\Omega)} \lesssim \frac{1}{2\alpha} \|G_h\|^2_{V^*} + \frac{\alpha}{2} \|u - u^h\|^2_{H^1(\Omega)} + \langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1},
\] (5.3)

for some positive constant \( \alpha \). As a result, we obtain a bound on \( \|u - u^h\|^2_{H^1(\Omega)} \). Further, using

\[
\|\lambda - \tilde{\lambda}^h\|_{V^*} := \sup_{\phi \in V, \phi \neq 0} \frac{G_h(\phi) - a(u - u^h, \phi)}{\|\phi\|_V}.
\]
together with the bound on $\|u - u^h\|_{H^1(\Omega)}$ given in (5.3) and the continuity of the bilinear form, we obtain the bound for $\|\lambda - \tilde{\lambda}^h\|_{V^*}$. □

In the next lemma, we infer the relation between the functional $G_h$ and estimator $\eta_h$.

**Lemma 5.4** It holds that

$$\|G_h\|_{V^*} \lesssim \eta_h.$$  

**Proof** For any $v \in V$, we have

$$G_h(v) = a(u - u^h, v) + \langle \lambda - \tilde{\lambda}^h, v \rangle_{-1,1} \quad \text{(using equation (5.2))}$$

$$= L(v) - a(u^h, v) - \langle \tilde{\lambda}^h, v \rangle_{-1,1} \quad \text{(using equation (2.3))}$$

$$= (R_{lin}^{1}, v)_{-1,1} - \langle \tilde{\lambda}^h, v \rangle_{-1,1}. \quad \text{(using equation (3.14))}$$

Using the property of nodal basis functions and Lemma 4.3, we find

$$G_h(v) = \sum_{i=1}^{2} \sum_{p \in V_h \cup M_h} \langle R_{lin}^{1}, v_i \psi_p \rangle_{-1,1} - \langle \tilde{\lambda}^h, v_i \rangle_{-1,1} - \sum_{p \in V_h \cup M_h} \langle R_{lin}^{2}, v_2 \psi_p \rangle_{-1,1}$$

$$= \sum_{i=1}^{2} \sum_{p \in (V_h \cup M_h) \setminus (V^C_h \cup M^C_h)} \langle R_{lin}^{1}, v_i \psi_p \rangle_{-1,1} + \sum_{p \in V^C_h \cup M^C_h} \langle R_{lin}^{2}, v_2 \psi_p \rangle_{-1,1}$$

$$+ \sum_{p \in V^C_h \cup M^C_h} \langle R_{lin}^{1}, v_1 \psi_p \rangle_{-1,1} - \langle \tilde{\lambda}^h, v_1 \rangle_{-1,1}. \quad (5.4)$$

Using the constants $c_p(v_i)$ introduced in the Sect. 4 together with Eqs. (3.19) and (3.20), for $i = 1, 2$, we derive

$$c_p(v_i)(R_{lin}^{1}, \psi_p e_i)_{-1,1} = 0 \quad \forall \ p \in (V_h \cup M_h) \setminus (V^C_h \cup M^C_h), \quad (5.5)$$

$$c_p(v_2)(R_{lin}^{1}, \psi_p e_2)_{-1,1} = 0 \quad \forall \ p \in V^C_h \cup M^C_h. \quad (5.6)$$

Next, we subtract Eqs. (5.5) and (5.6) from Eq. (5.4) to get

$$G_h(v) = \sum_{i=1}^{2} \sum_{p \in (V_h \cup M_h) \setminus (V^C_h \cup M^C_h)} \langle R_{lin}^{1}, (v_i - c_p(v_i)) \psi_p \rangle_{-1,1}$$

$$+ \sum_{p \in V^C_h \cup M^C_h} \langle R_{lin}^{2}, (v_2 - c_p(v_2)) \psi_p \rangle_{-1,1}$$

$$+ \sum_{p \in V^C_h \cup M^C_h} \langle R_{lin}^{1}, v_1 \psi_p \rangle_{-1,1} - \langle \tilde{\lambda}^h, v_1 \rangle_{-1,1}. \quad (5.7)$$

Now, using Lemma 4.2, we obtain the following equation

$$G_h(v) = \sum_{i=1}^{2} \sum_{p \in (V_h \cup M_h) \setminus (V^C_h \cup M^C_h)} \langle R_{lin}^{1}, (v_i - c_p(v_i)) \psi_p \rangle_{-1,1}$$

$$+ \sum_{p \in V^C_h \cup M^C_h} \langle R_{lin}^{2}, (v_2 - c_p(v_2)) \psi_p \rangle_{-1,1}$$

$\square$
Finally, using Eq. (3.18), Hölder’s inequality and Lemma 4.1, we find

\[
G_h(v) = 2 \sum_{i=1}^2 \sum_{p \in V_h} \left( \int_{\omega_p} r_i(u^h)(v_i - c_p(v_i)) \psi_p \, dx + \int_{\gamma_p,l} J_{i,l}^l(u^h)(v_i - c_p(v_i)) \psi_p \, ds \right) + 2 \sum_{i=1}^2 \sum_{p \in V_h} \int_{\gamma_p,N} J_{i,N}^N(u^h)(v_i - c_p(v_i)) \psi_p \, ds - \sum_{p \in V_h} \int_{\gamma_{p,C}} \hat{\sigma}_2(u^h)(v_2 - c_p(v_2)) \psi_p \, ds - \sum_{p \in V_h} \int_{\gamma_{p,C}} \hat{\sigma}_1(u^h)(v_1 - c_p(v_1)) \psi_p \, ds
\]

\[
\lesssim \left( \sum_{i=1}^2 \sum_{p \in V_h} h_p^2 \| r_i(u^h) \|^2_{L^2(\omega_p)} \right)^{1/2} \left( \sum_{i=1}^2 \sum_{p \in V_h} \| \nabla v_i \|^2_{L^2(\omega_p)} \right)^{1/2} + \left( \sum_{i=1}^2 \sum_{p \in V_h} h_p \| J_{i,l}^l(u^h) \|^2_{L^2(\gamma_p,l)} \right)^{1/2} \left( \sum_{i=1}^2 \sum_{p \in V_h} \| \nabla v_i \|^2_{L^2(\omega_p)} \right)^{1/2} + \left( \sum_{p \in V_h} h_p \| \hat{\sigma}_2(u^h) \|^2_{L^2(\gamma_{p,C})} \right)^{1/2} \left( \sum_{p \in V_h} \| \nabla v_2 \|^2_{L^2(\omega_p)} \right)^{1/2} + \left( \sum_{p \in V_h} h_p \| \hat{\sigma}_1(u^h) \|^2_{L^2(\gamma_{p,C})} \right)^{1/2} \left( \sum_{p \in V_h} \| \nabla v_1 \|^2_{L^2(\omega_p)} \right)^{1/2} \lesssim \left( \sum_{i=1}^2 \eta_i^2 \right)^{1/2} \| \nu \|.
\]

This completes the proof of this lemma. \( \square \)

Next, we find an upper bound on the term \( \langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1} \).

**Lemma 5.5** It holds that

\[
\langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1} \lesssim \frac{1}{2} \| \lambda_1 - \tilde{\lambda}_1 \|^2_{H^{-1}(\Omega)} + \eta_0^2 + \eta_1^2.
\]

**Proof** Let \( \tilde{u}_1^h := \min\{u_1^h|_{\Gamma_C}, 0\} \in H^{1/2}(\Gamma_C) \). Let \( \tilde{w} \) be the harmonic extension of \( w = u_1^h - \tilde{u}_1^h \in H^{1/2}(\Gamma_C) \) such that \( \| \tilde{w} \|_{H^1(\Omega)} \lesssim \| w \|_{H^{1/2}(\Gamma_C)} \) [36]. A use of (2.7) and Lemma 4.3 yields

\[
\langle \tilde{\lambda}^h - \lambda, u - u^h \rangle_{-1,1} = \langle \tilde{\lambda}^h, u - u^h \rangle_{-1,1} + \langle \lambda, u^h - u \rangle_{-1,1} = \langle \tilde{\lambda}^h, u_1 - u_1^h \rangle_{-1,1} + \langle \lambda, u_1^h - u_1 \rangle_{-1,1}.
\]

(5.7)
Employing the relation (2.7), we deal with the second term on the right hand side of the last Eq. (5.7) as follows

\[ \langle \lambda_1, u_1^h - u_1 \rangle_{\Gamma_C} = \langle \lambda_1, u_1^h - \tilde{u}_1^h \rangle_{\Gamma_C} + \langle \lambda_1, \tilde{u}_1^h - u_1 \rangle_{\Gamma_C}. \]

By definition of \( \tilde{u}_1^h \), we have \( \tilde{u}_1^h \leq 0 \) on \( \Gamma_C \), therefore by Lemma 2.1 we have

\[ \langle \lambda_1, u_1^h - u_1 \rangle_{\Gamma_C} \leq \langle \lambda_1, u_1^h - \tilde{u}_1^h \rangle_{\Gamma_C} + \langle \tilde{\lambda}_1^h, u_1^h - \tilde{u}_1^h \rangle_{\Gamma_C}. \]

Using the Young’s inequality and the stability estimate for the harmonic extension, we find

\[ \langle \lambda_1, u_1^h - u_1 \rangle_{\Gamma_C} \leq \frac{1}{2} \| \lambda_1 - \tilde{\lambda}_1^h \|_H^{-1}(\Omega) + \frac{1}{2} \| u_1^h - \tilde{u}_1^h \|_H^{-1}(\Omega) + \langle \tilde{\lambda}_1^h, u_1^h - \tilde{u}_1^h \rangle_{\Gamma_C}. \]

Thus, using Eq. (5.7), we get

\[ \langle \tilde{\lambda}_1^h, u_1^h - u_1 \rangle_{\Gamma_C} \leq \frac{1}{2} \| \lambda_1 - \tilde{\lambda}_1^h \|_H^{-1}(\Omega) + \frac{1}{2} \| u_1^h - \tilde{u}_1^h \|_H^{-1}(\Omega) + \langle \tilde{\lambda}_1^h, u_1^h - \tilde{u}_1^h \rangle_{\Gamma_C}. \] (5.8)

Note that on \( \Gamma_C \), we have

\[ (u_1^h - \tilde{u}_1^h) + u_1 - u_1^h = (u_1^h)^+ + u_1 - u_1^h \]

\[ \leq (u_1^h)^+ - u_1^h \]

\[ = (-u_1^h)^+. \]

With this realization and in view of Lemma 4.2, we handle the third term on the right hand side of Eq. (5.8) as follows

\[ \langle \tilde{\lambda}_1^h, u_1^h - \tilde{u}_1^h - u_1 + u_1 \rangle_{\Gamma_C} = \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \langle \tilde{\lambda}_1^h, (u_1^h - \tilde{u}_1^h + u_1)\psi_p \rangle_{\Gamma_C}. \]

\[ \leq \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \langle \tilde{\lambda}_1^h, (-u_1^h)^+ \psi_p \rangle_{\Gamma_C}. \]

Using the Lemma 4.4, Eq. (4.9) together with the Remark 4.5, we have

\[ \langle \tilde{\lambda}_1^h, u_1^h - \tilde{u}_1^h - u_1 + u_1 \rangle_{\Gamma_C} = \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} s_p^1 c_p ((-u_1^h)^+) \int_{\gamma_{p,C}} \phi_p ds \]

\[ + \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} s_p^1 c_p ((-u_1^h)^+) \int_{\gamma_{p,C}} \phi_p ds. \] (5.9)

Since for any full contact node \( p \), i.e., \( p \in \mathcal{N}_h^{FC} \) we have \( u_1^h = 0 \) on \( \gamma_{p,C} \), thus \( (-u_1^h)^+ = 0 \). Therefore, the Eq. (5.9) reduces to

\[ \langle \tilde{\lambda}_1^h, u_1^h - \tilde{u}_1^h - u_1 + u_1 \rangle_{\Gamma_C} = \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} s_p^1 c_p ((-u_1^h)^+) \int_{\gamma_{p,C}} \phi_p ds \]

\[ = \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} s_p^1 \left( \int_{\gamma_{p,C}} \phi_p \frac{f_{\gamma_{p,C}} ((-u_1^h)^+ \phi_p ds)}{f_{\gamma_{p,C}} \phi_p ds} \right), \]

\[ \square \]
where we have used the definition of constants \( c_p((-u_1^h)^+) \) on semi contact nodes in the last step. Note that, \( \tilde{\gamma}_{p,c} \) is a fixed portion of \( \gamma_{p,c} \) in the sense that \( \frac{|\gamma_{p,c}|}{|\tilde{\gamma}_{p,c}|} \) is a fixed constant (independent of mesh parameter). To be specific, for \( p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C \), the choice for \( \tilde{\gamma}_{p,c} \) is described in Figs. 1 and 2. Thus, \( \tilde{\gamma}_{p,c} \) results to be a strict subset of \( \gamma_{p,c} \). Thereby, \( \frac{\int_{\gamma_{p,c}} \phi_p \, ds}{\int_{\tilde{\gamma}_{p,c}} \phi_p \, ds} \) is a mesh-independent constant taking into account that \( \phi_p \) is the Lagrange basis function.

We refer the article [34, Section 5] for more details. Therefore,

\[
\eta_{k,p} \lesssim \|u - u^h\|_{H^1(\omega_p)} + \|\lambda - \tilde{\lambda}_h\|_{H^{-1}(\omega_p)} + \text{Osc}(f) + \text{Osc}(g),
\]

where the oscillation terms are defined as

\[
\text{Osc}(f)^2 := \sum_{T \in \omega_p} h_T^2 \|f - \tilde{f}\|^2_{L^2(T)},
\]

\[
\text{Osc}(g)^2 := \sum_{e \in \gamma_{p,N}} h_e \|g - \tilde{g}\|^2_{L^2(e)},
\]

with \( \tilde{v} \) representing \( L^2 \) projection of \( v \) onto the space of piecewise constant functions.

**Proof (i) (Local bound for \( \eta_1 \))** To this end, we choose an arbitrary triangle \( T \in \omega_p \). Let \( b_T \in P_3(T) \) be the interior bubble function which is zero on \( \partial T \) and admits unit value at the barycenter of \( T \). Set \( \beta_T = b_T(\tilde{f} + d_i v_0 (u^h)) \) on \( T \), where \( \tilde{f} \) is the function with the piecewise constant approximation of \( f \) with the components \( \tilde{f}_i \). Extend \( \beta_T \) to \( \Omega \) by defining it to be zero on \( \Omega \setminus \bar{T} \), call it \( \beta \). It is evident that \( \beta \in [H^1_0(\Omega)]^2 \). Using the equivalence of
norms in finite dimensional normed spaces on a reference triangle and scaling arguments [17], we find  
\[ \| \tilde{f} + \text{div} \sigma(u^h) \|_{L^2(T)}^2 \lesssim \int_T \beta \cdot (\tilde{f} + \text{div} \sigma(u^h)) \, dx \]
\[ = \int_T \beta \cdot (\tilde{f} - f) \, dx + \int_T r(u^h) \cdot \beta \, dx. \]  
(5.11)  

Using integration by parts, (2.3) together with the definition of functional \( G_h \) in (5.11), we find  
\[ \| \tilde{f} + \text{div} \sigma(u^h) \|_{L^2(T)}^2 \lesssim \int_T \beta \cdot (\tilde{f} - f) \, dx + \langle G_h, \beta \rangle_{-1,1,T}. \]

Henceforth using Cauchy-Schwarz inequality and standard inverse estimates, we obtain  
\[ \| \tilde{f} + \text{div} \sigma(u^h) \|_{L^2(T)}^2 \lesssim \left( \| \tilde{f} - f \|_{L^2(T)} + h_T^{-1} \| G_h \|_{H^{-1}(T)} \right) \| \beta \|_{L^2(T)} \]
\[ \lesssim \left( \| \tilde{f} - f \|_{L^2(T)} + h_T^{-1} \| G_h \|_{H^{-1}(T)} \right) \| \tilde{f} + \text{div} \sigma(u^h) \|_{L^2(T)}. \]

Thus, we find  
\[ h_T^2 \| \tilde{f} + \text{div} \sigma(u^h) \|_{L^2(T)}^2 \lesssim h_T^2 \| \tilde{f} - f \|_{L^2(T)}^2 + \| G_h \|^2_{H^{-1}(T)}. \]

Using the definition of Galerkin functional in the last equation, it follows that  
\[ h_T^2 \| \tilde{f} + \text{div} \sigma(u^h) \|_{L^2(T)}^2 \lesssim h_T^2 \| \tilde{f} - f \|_{L^2(T)}^2 + \| u - u_h \|^2_{H^1(\omega_p)} + \| \lambda - \tilde{\lambda}^h \|^2_{H^{-1}(\omega_p)}. \]

Summing the above equation, over all the triangles in \( \omega_p \) and using triangle inequality, we get  
\[ \eta_{1,p}^2 \lesssim \sum_{T \in \omega_p} h_T^2 \| \tilde{f} - f \|_{L^2(T)}^2 + \| u - u_h \|^2_{H^1(\omega_p)} + \| \lambda - \tilde{\lambda}^h \|^2_{H^{-1}(\omega_p)}. \]

(ii) (Local bound for \( \eta_2 \)) In order to prove this, let \( e \in \gamma_{p,l} \) be an arbitrary interior edge sharing the elements \( T^+ \) and \( T^- \) and let \( n_e \) be the outward unit vector normal vector to \( e \) heading from the triangle \( T^- \) to \( T^+ \). To this end, we will construct an edge bubble function and exploit its properties as follows: Define \( b_e \in P_4(T^- \cup T^+) \) to be the polynomial such that it takes value 1 at the midpoint of \( e \) and zero on the boundary of polygon \( T^- \cup T^+ \). Further, we define \( \xi \in \{ P_1(T^- \cup T^+) \}^2 \) to be the polynomial such that \( \xi = [\sigma(u^h)] \) on edge \( e \). Set \( \beta = b_e \xi \) on \( T^- \cup T^+ \) and zero outside the polygon \( T^- \cup T^+ \) yielding \( \beta \in \left[ H^1_0(\Omega) \right]^2 \). A use of equivalence of norms on a reference element in finite dimensional spaces together with scaling arguments yields  
\[ \| \xi \|^2_{L^2(e)} \lesssim \int_e \beta \cdot \xi \, ds. \]  
(5.12)  

Using integration by parts together with the definition of \( G_h \) we find  
\[ \int_e \xi \cdot \beta \, ds = \int_{T^- \cup T^+} \sigma(u^h) : \epsilon(\beta) \, dx + \int_{T^- \cup T^+} \text{div} \sigma(u^h) \cdot \beta \, dx \]
\[ = \int_{T^- \cup T^+} \sigma(u^h) : \epsilon(\beta) \, dx + \int_{T^- \cup T^+} r(u^h) \cdot \beta \, dx - \int_{T^- \cup T^+} f \cdot \beta \, dx \]
Thus, combining (5.12) and (5.13), we find that the edge \( e \) using integration by parts and (2.3), we have which is the desired estimate.

In view of the definition of Galerkin functional \( G_h \) together with the estimate in \( (i) \), we find

\[
\eta \in [0, \lambda - \tilde{\lambda} h] \quad \text{and} \quad \lambda \in [\lambda, \beta].
\]

Further a use of Cauchy-Schwarz inequality and standard inverse estimate yields

\[
\frac{1}{h^2} \left\| \sigma(u^h) \right\|_{L^2(e)} \lesssim \sum_{T \in \{T^+, T^-\}} \left( \| G_h \|_{H^{-1}(T)} + h_T \left\| r(u^h) \right\|_{L^2(T)} \right).
\]

In view of the definition of Galerkin functional \( G_h \) together with the estimate in \( (i) \), we find

\[
\int_T \zeta \cdot \beta \, ds \lesssim \sum_{T \in \{T^+, T^-\}} \left( \| G_h \|_{H^{-1}(T)} + h_T \left\| r(u^h) \right\|_{L^2(T)} \right).
\]

Thus, combining (5.12) and (5.13), we find

\[
\int_T \zeta \cdot \beta \, ds \lesssim \sum_{T \in \{T^+, T^-\}} \left( \| G_h \|_{H^{-1}(T)} + h_T \left\| r(u^h) \right\|_{L^2(T)} \right).
\]

In view of the definition of Galerkin functional \( G_h \) together with the estimate in \( (i) \), we find

\[
\int_T \zeta \cdot \beta \, ds \lesssim \sum_{T \in \{T^+, T^-\}} \left( \| G_h \|_{H^{-1}(T)} + h_T \left\| r(u^h) \right\|_{L^2(T)} \right).
\]

In view of the definition (2.7), we have

\[
\frac{1}{h^2} \left\| \sigma(u^h) \right\|_{L^2(e)} \lesssim \int_T \zeta \cdot \beta \, ds.
\]

Using integration by parts and (2.3), we have

\[
\int_T \zeta \cdot \beta \, ds = \int_T \sigma(u^h) : \epsilon(\beta) \, dx + \int_T \text{div} \sigma(u^h) \cdot \beta \, dx
\]

\[
= \int_T \sigma(u^h) : \epsilon(\beta) \, dx + \int_T r(u^h) \cdot \beta \, dx - \int_T f \cdot \beta \, dx
\]

\[
= \int_T \sigma(u^h) : \epsilon(\beta) \, dx + \int_T r(u^h) \cdot \beta \, dx - a(u, \beta) - \langle \lambda, \beta \rangle_{-1,1}.
\]

In view of the definition (2.7), we have \( \langle \lambda, \beta \rangle_{-1,1} = 0 \). Therefore, the continuity of bilinear form \( a(\cdot, \cdot) \) together with Cauchy-Schwarz inequality and standard inverse estimate yields

\[
\int_T \zeta \cdot \beta \, ds = a(u^h - u, \beta) + \int_T r(u^h) \cdot \beta \, dx
\]
In order to get rid of the last term, we construct a suitable function \( \theta \).

Now, if \( p \) is a non actual contact node, then using Lemma 4.4 it holds that \( \tilde{\lambda}^h = 0 \), thus using the Eq. (5.17) we can proceed similar to the proof of lower bound of \( \eta_4 \). We consider the case when the node \( p \) is a full contact node or semi contact node then the equation (5.17) reduces to

\[
G_h(\zeta_e e_1) = \sum_{p \in (\mathcal{V}_h \cup \mathcal{M}_h) \setminus (\mathcal{V}_h^C \cup \mathcal{M}_h^C)} \langle R_{1}^{lin}, \zeta_e \psi_p \rangle_{1,1} + \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle R_{1}^{lin}, \zeta_e \psi_p \rangle_{1,1}
\]

\[
- \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle \tilde{\lambda}^h, \zeta_e \psi_p \rangle_{1,1}
\]

\[
= \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \int_{\omega_p} r_1 \zeta_e \psi_p \, dx - \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \int_{\gamma_{p,c}} \hat{\sigma}_1(u^h) \zeta_e \psi_p \, ds
\]

\[
- \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle \tilde{\lambda}^h, \zeta_e \psi_p \rangle_{1,1}.
\]  

Combining (5.15) and (5.16) together with the estimate in (i), we obtain

\[
h_e \| \hat{\sigma}_2(u^h) \|_{L^2(e)}^2 \lesssim h_e^2 \| \vec{f} - f \|_{L^2(T)}^2 + \| u - u^h \|_{H^1(T)}^2 + \| \lambda - \tilde{\lambda}^h \|_{H^{-1}(T)}^2.
\]

Further, summing over all the \( e \in \gamma_{p,c} \), we find

\[
\eta_{\lambda,p}^2 \lesssim \sum_{e \in \omega_p} h_e^2 \| \vec{f} - f \|_{L^2(T)}^2 + \| u - u^h \|_{H^1(\omega_p)}^2 + \| \lambda - \tilde{\lambda}^h \|_{H^{-1}(\omega_p)}^2.
\]

(v) (Local bound for \( \eta_5 \)) Let \( p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C \) be arbitrary. Let \( e \in \gamma_{p,c} \) be arbitrary. On the similar lines as in the proof of (iii) we construct an edge bubble function \( \zeta_e \in P_2(T) \).

Using the definition of Galerkin functional \( G_h \), we have

\[
G_h(\zeta_e e_1) = \sum_{p \in (\mathcal{V}_h \cup \mathcal{M}_h) \setminus (\mathcal{V}_h^C \cup \mathcal{M}_h^C)} \langle R_{1}^{lin}, \zeta_e \psi_p \rangle_{1,1} + \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle R_{1}^{lin}, \zeta_e \psi_p \rangle_{1,1}
\]

\[
- \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle \tilde{\lambda}^h, \zeta_e \psi_p \rangle_{1,1}
\]

\[
= \sum_{p \in \mathcal{V}_h \cup \mathcal{M}_h} \int_{\omega_p} r_1 \zeta_e \psi_p \, dx - \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \int_{\gamma_{p,c}} \hat{\sigma}_1(u^h) \zeta_e \psi_p \, ds
\]

\[
- \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle \tilde{\lambda}^h, \zeta_e \psi_p \rangle_{1,1}.
\]  

Now, if \( p \) is a non actual contact node, then using Lemma 4.4 it holds that \( \tilde{\lambda}^h = 0 \), thus using the Eq. (5.17) we can proceed similar to the proof of lower bound of \( \eta_4 \). We consider the case when the node \( p \) is a full contact node or semi contact node then the equation (5.17) reduces to

\[
G_h(\zeta_e e_1) = \sum_{p \in (\mathcal{V}_h \cup \mathcal{M}_h) \setminus (\mathcal{V}_h^C \cup \mathcal{M}_h^C)} \int_{\omega_p} r_1 \zeta_e \psi_p \, dx - \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \int_{\gamma_{p,c}} \hat{\sigma}_1(u^h) \zeta_e \psi_p \, ds
\]

\[
- \sum_{p \in \mathcal{V}_h^C \cup \mathcal{M}_h^C} \langle \tilde{\lambda}^h, \zeta_e \psi_p \rangle_{1,1}.
\]  

In order to get rid of the last term, we construct a suitable function \( \theta_e \) such that \( c_p(\theta_e) = 0 \). To this end, we will exploit the definition of \( c_p(\cdot) \) which depends on \( \gamma_{p,c} \). For this we have the following two cases:

- If \( p \) is an interior vertex of \( \mathcal{V}_h^C \), then \( \gamma_{p,c} \) consists of two intervals. In that case, we set \( \gamma_{p,c} \) as inner third of \( \gamma_{p,c} \) containing \( p \) (see Fig. 1).
- If \( p \) is a midpoint in \( \mathcal{M}_h^C \), then \( \gamma_{p,c} \) consist of one interval and again we set \( \gamma_{p,c} \) as inner third of \( \gamma_{p,c} \) containing midpoint \( p \) (see Fig. 2).

Let \( e_i \) be the sides of subgrid containing \( p_i \) where \( p_i \in \mathcal{V}_h^C \) and \( e_M \) be the part of the subgrid containing the midpoint of \( [p_i, p_{i+1}] \) (see Figs. 1 and 2). Now, we will use the above construction to define the function \( \theta_e \) in the following way.
Fig. 1 Subgrid of $\gamma_p, C$ when $p \in \mathcal{V}_h^C$.

Fig. 2 Subgrid of $\gamma_p, C$ when $p \in \mathcal{M}_h^C$.

$$
\theta_e = \sum_{i=1}^{2} \alpha_i \psi_i + \alpha_M \psi_M, \quad (5.19)
$$

where $\psi_M$ and $\psi_i$ are the edge bubble functions corresponding to $e_M$ and $e_i$ respectively. The coefficients $\alpha_i$ and $\alpha_M$ are determined such that the following holds

1. $\int_e \hat{\sigma}_1(u^h) \theta_e \phi_p = 0 \forall$ semi contact and full contact nodes lying on edge $e$.

Inserting the expression of $\theta_e$ from (5.19) in the aforementioned conditions results into a solvable system of three equations with the unknown coefficients $\alpha_i$, $i = 1, 2$ and $\alpha_M$. In addition, the uniqueness of $\alpha_i$ and $\alpha_M$ is ensured as the resulting coefficient matrix is non-singular. Thus, we have $c_p(\hat{\sigma}_1(u^h) \theta_e) = 0$. Further, using the equivalence of norms in finite dimensional spaces, Hölder’s inequality and the construction of $\theta_e$, we have

$$
\|\hat{\sigma}_1(u^h)\|_{L^2(e)}^2 \lesssim \int_e \hat{\sigma}_1(u^h)\hat{\sigma}_1(u^h) \theta_e \psi_p \, ds
$$

$$
\lesssim -\langle G_h, \hat{\sigma}_1(u^h) \theta_e e_1 \rangle_{-1,1, \omega_p} + \int_{\omega_p} r_1(u^h) \hat{\sigma}_1(u^h) \theta_e \psi_p \, dx
$$

$$
\lesssim \|G_h\|_{H^{-1}(\omega_p)} \|\hat{\sigma}_1(u^h) \theta_e\|_{H^1(\omega_p)} + \|r(u^h)\|_{L^2(\omega_p)} \|\hat{\sigma}_1(u^h) \theta_e\|_{L^2(\omega_p)}
$$

$$
\lesssim \|G_h\|_{H^{-1}(\omega_p)} h_e^{-1} \|\hat{\sigma}_1(u^h) \theta_e\|_{L^2(\omega_p)} + \|r(u^h)\|_{L^2(\omega_p)} \|\hat{\sigma}_1(u^h) \theta_e\|_{L^2(\omega_p)}
$$

$$
\lesssim \left( \|G_h\|_{H^{-1}(\omega_p)} + h_e \|r(u^h)\|_{L^2(\omega_p)} \right) h_e^{-\frac{1}{2}} \|\hat{\sigma}_1(u^h)\|_{L^2(e)}
$$

Thus,

$$
h_e \|\hat{\sigma}_1(u^h)\|_{L^2(e)}^2 \lesssim \left( \|G_h\|_{H^{-1}(\omega_p)}^2 + h_T^{-2} \|r(u^h)\|_{L^2(\omega_p)}^2 \right).
$$

We conclude the proof using the upper bound of $G_h$ and the estimate in (i).

\[ \square \]

**Remark 5.8** We would like to remark here that the efficiency of the estimator terms $\eta_6$ and $\eta_7$ involving positive part of $u^h$ is still not clear theoretically due to quadratic nature of the discrete solution and this will be pursued in future.
Table 1  Convergence of error and estimator on uniform mesh for Example 6.1

| $h$   | $\|u - u^h\|_{H^1(\Omega)}$ | Order of Conv. | $h$   | $\eta_h$ | Order of Conv. |
|-------|----------------------------|----------------|-------|----------|----------------|
| 1/2   | 0.1763                     | -              | 1/2   | 1.6770   | -              |
| 1/4   | 0.0448                     | 1.9680         | 1/4   | 0.4504   | 1.9195         |
| 1/8   | 0.0112                     | 1.9879         | 1/8   | 0.1210   | 1.8353         |
| 1/16  | 0.0028                     | 1.9944         | 1/16  | 0.0340   | 1.8411         |
| 1/32  | 0.0007                     | 1.9972         | 1/32  | 0.0101   | 1.8805         |
| 1/64  | 0.0001                     | 1.9986         | 1/64  | 0.0032   | 1.9233         |

Table 2  Convergence of error and estimator on adaptive mesh for Example 6.1

| $h$    | $\|u - u^h\|_{H^1(\Omega)}$ | Order of Conv. | $h$    | $\eta_h$ | Order of Conv. |
|--------|----------------------------|----------------|--------|----------|----------------|
| 0.03181| 0.0097                     | 2.0392         | 0.03181| 0.1112   | 1.53953        |
| 0.01177| 0.0013                     | 2.1722         | 0.01177| 0.0146   | 1.78039        |
| 0.00580| 0.0003                     | 2.2653         | 0.00580| 0.0033   | 2.43540        |
| 0.00389| 0.00009                    | 2.1432         | 0.00389| 0.0014   | 1.57421        |
| 0.00326| 0.00006                    | 2.0485         | 0.00326| 0.0009   | 2.56407        |
| 0.00266| 0.00006                    | 2.1069         | 0.00266| 0.0006   | 2.19607        |

Fig. 3  Convergence of error, estimator and efficiency index for Example 6.1

Remark 5.9 The analysis carried out in this article can be extended to the case when obstacle is a non-zero affine function, say $w$. The following two terms in the error estimator $\eta_h$ will correspondingly change to $d_p = \int_{\Gamma_C} (w - u^h_1)^+ \phi_p \, ds$ and $\eta_7 = \|(u^1_1 - w)^+\|_{H^1/2(\Gamma_C)}$.

6 Numerical Results

The aim of the given section is to numerically illustrate the theoretical results derived in Sects. 3 and 5, respectively. Therein, the numerical experiments are performed on two model problems using MATLAB(version R2020b). The first model problem is constructed in such
a way that the exact solution \( u \) is a priori known. Henceforth, the exact error is computed and the results are compared with the convergence of a posteriori error estimator \( \eta_h \). In the second model problem, the exact solution is unknown and therefore we compute the iterative \( H^1 \) error between the solutions of two consecutive meshes and thus focus on the convergence of the computed error and the estimator \( \eta_h \) therein. The discrete variational inequality (3.2) is solved using the primal dual active set strategy [29]. The tests are being carried out on both uniform and adaptive meshes. For the adaptive algorithm, we make use of the following paradigm

\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}
\]

The step \text{SOLVE} comprises of computing the discrete solution \( u_h \) by solving the discrete variational inequality (3.2) with the help of primal-dual active set strategy. Thereafter, in the next step \text{ESTIMATE}, the error estimator \( \eta_h \) discussed in Section 4 is computed elementwise and further making the use of Dörfler’s marking strategy [18] with the parameter \( \theta = 0.4 \), we mark the elements of the triangulation in the step \text{MARK}. Followed by that in the step \text{REFINE}, the marked elements are refined using the newest vertex bisection algorithm [16, 21] to obtain the new mesh and the algorithm is repeated. Note that when \( \Gamma_C \) lies on the \( x \)-axis, we have \( \mathbf{n} = (0, -1) \) on \( \Gamma_C \). Therefore in this case, the estimator given in Eq. (5.1) will have modified estimator contributions

\[
\eta_{6,p} = (s_p d_p)^{1/2}, \quad \text{where} \quad d_p = \int_{\gamma_p,C} (u_2^h)^2 + \phi_p \, ds
\]

and

\[
\eta_{7} = \|(-u_2^h)^{1/2}\|_{H^1(\Gamma_C)}.
\]

For the given examples, the Lamé’s parameter \( \mu \) and \( \chi \) are computed as follows

\[
\mu = \frac{E}{2(1+v)}, \quad \chi = \frac{E\nu}{(1-2\nu)(1+v)},
\]

where \( E \) and \( \nu \) represents the Young’s modulus and Poisson ratio [32], respectively.

**Example 6.1** We assume the unit square \( \Omega = (0, 1)^2 \) to be the domain \( \Omega \) under consideration. The displacement fields vanishes on the top of the square i.e. zero Dirichlet boundary condition is applied on \( (0, 1) \times \{1\} \). The Neumann force \( g \) is acting on the left and right hand side of the square namely \( \{0, 1\} \times (0, 1) \). The bottom of the unit square is in contact with the rigid
Fig. 5  Physical setting of Example 6.2

Fig. 6  Convergence of error, estimator and efficiency index for Example 6.2

Fig. 7  Adaptive mesh and estimator contributions for Example 6.2
foundation and hence represent the contact boundary $\Gamma_C$. The Lamé’s parameter $\mu$ and $\chi$ are set to be 1. The source term $f$ and Neumann data $g$ are computed in such a way that exact solution takes the form $u = (y^2(y - 1), (x - 2)y(1 - y)e^y)$.

Since the exact solution in Example 6.1 is sufficiently regular, it can be observed from Table 1 that the energy norm error in the displacement vector and the residual estimator $\eta_h$ converge with the optimal rate (1/NDF), where NDF is number of degrees of freedom. Further, Table 2 depicts the order of convergence of $H^1$ error and the residual estimator $\eta_h$ on the adaptive mesh. Note that, in Table 2, the mesh parameter $h$ is taken to be $1/\sqrt{\text{NDF}}$. Figure 3A illustrates the convergence behavior of error and estimator with the increase in the number of degrees of freedom (NDF) on uniform and adaptive meshes. We observe that both the error and estimator converge with the optimal rate, thus ensuring the reliability of the error estimator $\eta_h$. The efficiency index which is calculated as ratio of the estimator and the error is depicted in Fig. 3B. It can be observed that the efficiency index is both bounded above and below by a generic constant, which indicates that the error estimator $\eta_h$ is efficient. Figure 4 ensures the convergence of each estimator contributions $\eta_i$, $1 \leq i \leq 5$ versus the degrees of freedom. It is to be noted that the estimators $\eta_6, \eta_7$ vanishes for the given example as the entire contact boundary forms the active set.

**Example 6.2** In this example (motivated from [43]), we simulate the deformation of unit elastic square which is displaced in $x$-direction towards the non zero obstacle $w(y) = -0.2 + 0.5|y - 0.5|$. The obstacle can be described as a wedge inclined at an angle $\alpha = 63^\circ$ (see

![Plot of first component $u_1^h$ (left) and second component $u_2^h$ (right) of discrete solution $u^h$ for Example 6.2](image-url)
Fig. 5). The Dirichlet boundary is set to be on the left side of elastic square at \( x = 0 \) with the non-homogeneous condition \( u = (0.1, 0) \) on \( \Gamma_D \) while the force density \( f \) and the Neumann forces \( g \) are set to zero. The Poisson ratio is \( \nu = 0.3 \) and the Young’s modulus is \( E = 500 \). In view of Remark 5.9, the following two terms in the error estimator \( \eta_h \) will correspondingly change to 

\[
d_p = \int_{\gamma_{p,c}} (w - u_h^1)^+ \phi_p \, ds \\
\eta_7 = \| (u_h^1 - w)^+ \|_{H^1/2(\Gamma_C)}.
\]

Figure 6A illustrates the convergence behavior of iterative error and residual estimator \( \eta_h \) on the adaptive mesh with the increase in degrees of freedom. It is evident that the error and estimator converges optimally. The efficiency of the estimator can be observed through the efficiency index shown in Fig. 6B. Figure 7A depicts the adaptive mesh refinement at certain level. The high mesh refinement is observed near the intersection of disjoint Neumann and Dirichlet boundaries and also near the free boundary region. The rate of convergence of the individual estimator contributions \( \eta_i, 1 \leq i \leq 7 \) is illustrated in Fig. 7B. The plot of the discrete solution \( u_h = (u_h^1, u_h^2) \) at a certain refinement level is shown in Fig. 8.

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**Declarations**

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**References**

1. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, 140, Pure and Applied Mathematics, a subsidiary of Harcourt Brace Jovanovich. Academic Press, New York (2003)
2. Ainsworth, M., Oden, J.T.: A Posteriori Error Estimation in Finite Element Analysis. Pure and Applied Mathematics. Wiley-Interscience, Wiley, New York (2000)
3. Banz, L., Schröder, A.: A posteriori error control for variational inequalities with linear constraints in an abstract framework. J. Appl. Numer. Optim. 3, 333–359 (2021)
4. Belgacem, F.B.: Numerical Simulation of Some Variational Inequalities Arisen from Unilateral Contact Problems by the Finite Element Methods. SIAM J. Num. Ana. 37(4), 1198–1216 (2000)
5. Belhachmi, Z., Belgacem, F.B.: Quadratic finite element approximation of the Signorini problem. Math. Comp. 72(241), 83–104 (2003)
6. Bostan, V., Han, W.: Recovery-based error estimation and adaptive solution of elliptic variational inequalities of the second kind. Commun. Math. Sci. 2, 1–18 (2004)
7. Bostan, V., Han, W., Reddy, B.: A posteriori error estimation and adaptive solution of elliptic variational inequalities of the second kind. Appl. Numer. Math. 52, 13–38 (2004)
8. Braess, D.: A posteriori error estimators for obstacle problems-another look. Numer. Math. 101, 415–421 (2005)
9. Brenner, S.C.: Two-level additive Schwarz preconditioners for nonconforming finite element methods. Math. Comp. 65, 897–921 (1996)
10. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 3rd edn. Springer, New York (2008)
11. Brezzi, F., Hager, W.W., Raviart, P.A.: Error estimates for the finite element solution of variational inequalities. Num. Math. 28, 431–443 (1977)
12. Brezzi, F., Hager, W.W., Raviart, P.A.: Error estimates for the finite element solution of variational inequalities. Part I: Primal theory. Numer. Math. 28, 431–443 (1977)
13. Bustinza, R., Sayas, F.J.: Error estimates for an LDG method applied to a Signorini type problems. J. Sci. Comput. 52, 322–339 (2012)
14. Bürg, M., Schröder, A.: A posteriori error control of hp-finite elements for variational inequalities of the first and second kind. Comput. Math. Appl. 70, 2783–2802 (2015)
15. Castillo, P., Cockburn, B., Perugia, I., Schötzau, D.: An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. SIAM J. Numer. Anal. 38, 1676–1706 (2000)
16. Chen, L., Zhang, C.: A coarsening algorithm on adaptive grids by newest vertex bisection and its applications. J. Comput. Math. 28(6), 767–789 (2010)
17. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
18. Dörfler, W.: A convergent adaptive algorithm for Poisson’s equation. SIAM J. Numer. Anal. 33, 1106–1124 (1996)
19. Duvaux, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)
20. Falk, R.S.: Error estimates for the approximation of a class of variational inequalities. Math. Comp. 28, 963–971 (1974)
21. Funken, S., Praetorius, D., Wissgott, P.: Efficient implementation of adaptive P1-FEM in Matlab. Comp. Meth. Appl. Math. 11, 460–490 (2011)
22. Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer, Berlin (2008)
23. Gudi, T., Porwal, K.: A posteriori error estimator for a class of discontinuous Galerkin methods for the Signorini problem. J. Comp. Appl. Math. 292, 257–278 (2016)
24. Hage, D., Klein, N., Suttmeier, F.T.: Adaptive finite elements for a certain class of variational inequalities of the second kind. Calcolo 48, 293–305 (2011)
25. Hesthaven, J.S., Warburton, T.: Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications. Springer, New York (2007)
26. Hild, P., Laborde, P.: Quadratic finite element methods for unilateral contact problems. Appl. Num. Math. 41, 401–422 (2000)
27. Hild, P., Nicaise, S.: A posteriori error estimations of residual type for Signorini’s problem. Numer. Math. 101, 523–549 (2005)
28. Hild, P., Nicaise, S.: Residual a posteriori error estimators for contact problems in elasticity. ESAIM:M2AN 41, 897–923 (2007)
29. Hüeber, S., Mair, M., Wohlmuth, B.I.: A priori error estimates and an inexact primal-dual active set strategy for linear and quadratic finite elements applied to multibody contact problems. Appl. Num. Math. 54, 555–576 (2005)
30. Karakashian, O.A., Pascal, F.: A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. SIAM J. Numer. Anal. 41, 2374–2399 (2003)
31. Kesavan, S.: Topics in Functional Analysis and Applications. Wiley, Hoboken (1989)
32. Kikuchi, N., Oden, J.T.: Contact Problem in Elasticity. SIAM, Philadelphia (1988)
33. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. SIAM, Philadelphia (2000)
34. Krause, R., Veeser, A., Walloth, M.: An efficient and reliable residual-type a posteriori error estimator for the Signorini problem. Num. Math. 130, 151–197 (2015)
35. Scarpini, F., Vivaldi, M.A.: Error estimates for the approximation of some unilateral problems. Ana. Numér. 11, 197–208 (1977)
36. Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems. Springer, New York (2008)
37. Veeser, A.: Efficient and reliable a posteriori error estimators for elliptic obstacle problems. SIAM J. Numer. Anal. 39, 146–167 (2001)
38. Verfürth, R.: A posteriori error estimation and adaptive mesh-refinement techniques. In: Proceedings of the 5th International Congress on Computational and Applied Mathematics (Leuven, 1992), vol. 50, pp. 67–83 (1994)
39. Verfürth, R.: A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Wiley-Teubner, Chichester (1995)
40. Wang, F., Han, W., Cheng, X.: Discontinuous Galerkin methods for solving the Signorini problem. IMA J. Numer. Anal. 31, 1754–1772 (2011)
41. Wang, F., Han, W., Eichholz, J., Cheng, X.: A posteriori error estimates for discontinuous Galerkin methods of obstacle problems. Nonlinear Anal. Real World Appl. 22, 664–679 (2015)
42. Wang, F., Han, W., Cheng, X.: Discontinuous Galerkin methods for elliptic variational inequalities. SIAM J. Numer. Anal. 48, 708–733 (2010)
43. Walloth, M.: A reliable, efficient and localized error estimator for a discontinuous Galerkin method for the Signorini problem. Appl. Num. Math. 135, 276–296 (2019)
44. Weiss, A., Wohlmuth, B.: A posteriori error estimator and error control for contact problems. Math. Comp. 78, 1237–1267 (2004)
45. Wohlmuth, B.I., Popp, A., Gee, M.W., Wall, W.A.: An abstract framework for a priori estimates for contact problems in 3D with quadratic finite elements. Comput. Mech. 49(6), 735–747 (2012)

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