THE PENROSE TRANSFORM AND SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

AHMED SEBBAR, DANIELE STRUPPA & OUMAR WONE

Abstract. We study the origins of twistor theory and of the Penrose twistor, from the point of view of partial differential equations. We show how the fundamental ideas require the ability to correctly calculate the dimensions of spaces associated to the varieties of zeros of the symbols of those differential equations. This brings to the center of the analysis several classical results from algebraic geometry, including the Cayley-Bacharach theorem and some of its variants as Serret’s theorem, and the Brill-Noether Restsatz theorem.

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1. INTRODUCTION

The Penrose transform is originally an integral geometric method of interpreting various analytic cohomology groups on open subsets of projective complex 3-space as solutions of linear partial differential equations on the Grassmannian of 2-planes in 4-space.

The original motivation for its introduction arises from physics where it was used to interpret the Grassmannian as the complexification of the conformal compactification of

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Minkowski space, the differential equations being the massless field equations with various helicities.

In this paper we trace the beginnings of the Penrose transform and associated twistor theory. We find that the theory was already present in its infancy in Bateman [1], where he showed how to give an integral representation for the solutions of the Laplace equation. Since the argument is essentially a counting argument, it is not surprising that in order to explain the connection between the Penrose transform and the solution of suitable differential equations, we are forced to delve into the classical theory of algebraic curves, the Riemann-Roch theorem, the Cayley-Bacharach theorem, Serret’s theorem, and more generally those theorems that give us the dimensions of suitable cohomology groups, such as the Brill-Noether Restsatz theorem. The plan of the paper is as follows: Section 2 is devoted exactly to the origin of twistor theory and its connection to Bateman’s results. In the process we will explore the Plucker-Klein correspondence and its relation to the twistor correspondence. In Section 3 we generalize these ideas, recover the results of [18], and finally extend this approach to more general partial differential equations, whose symbols are homogeneous polynomials. We finally come back to the modern incarnation of the theory i.e. the twistor transform and finally the Penrose transform, which we explain in detail in Section 4. The paper concludes with two appendices of abstract materials that is necessary to fully formalize our approach, but that we have isolated in appendices to facilitate the reading of the rest of the article.

The novelty of our presentation lies in the fact that throughout the paper we will always emphasize the discussion from the point of view of partial differential equations, so that the tools of twistor theory are seen as finalized to finding the holomorphic or real-analytic solutions of constant coefficients partial differential equations. We should conclude this introduction to point out that our own interest in this topic stems from our desire to better understand the interconnections between the theory of twistors and the fundamental principle of Ehrenpreis-Palamodov-Malgrange. This interest was stimulated by our reading of Ehrenpreis’ [7], [8] and we are currently investigating further how to clarify those interconnections.

2. Motivational examples for twistor theory

It has been known for some time that problems in real differential geometry can often be simplified by using complex coordinates. For example in the plane \( \mathbb{R}^2 \) we can write \( z = x + iy \) and thereby identify \( \mathbb{R}^2 \cong \mathbb{C} \). We then discover that a \( C^2 \) function \( f : \mathbb{R}^2 \to \mathbb{R} \) is harmonic if and only if we can write it as

\[
f = \psi + \overline{\psi}
\]

where \( \psi : \mathbb{C} \to \mathbb{C} \) is a holomorphic (complex-analytic) function. This is because a \( C^2 \) real-valued function \( f \) is harmonic if and only if its laplacian is zero and the Laplace operator \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is proportional to the operator \( \frac{\partial^2}{\partial z \partial \overline{z}} \). This shows us how to connect harmonic real-valued functions-an object from real differential geometry in the plane- to holomorphic functions of one complex variable- the natural objects of complex analysis. If we try the same technique in \( \mathbb{R}^3 \) we have to accept the fact that odd dimensional spaces cannot be identified with complex spaces \( \mathbb{C}^n \), for any integer \( n \). We can however form another space
closely associated to the geometry of $T = \mathbb{R}^3$ that is intrinsically complex, and this is the fundamental idea behind twistor theory. Consider therefore the space $M$ of all oriented lines of $\mathbb{R}^3$. The generic element of the space $M$ is the oriented line $L(u, v)$ given by
\[ L(u, v) = \{ v + tu, t \in \mathbb{R} \} \]
where $\|u\| = 1$ and $u, v \in \mathbb{R}^3$. Consider now the tangent bundle of the 2-sphere $S^2$ defined by
\[ TS^2 = \{ (u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|u\| = 1, (u, v) = 0 \} \]
with $(u, v)$ denoting the Euclidean scalar product of $u$ and $v$. We can now define a bijection
\[ M \rightarrow TS^2 \]
\[ L(u, v) \mapsto (u, v - (v, u)u) \]
where the second component is the point on $L(u, v)$ closest to the origin of $T = \mathbb{R}^3$. Remark that the map is indeed $TS^2$-valued and is clearly surjective. It is injective because if $(u, v - (v, u)u) = (u_1, v_1 - (v_1, u_1)u_1)$ then $u = u_1$ and $v - v_1 = (v - v_1, u)u$ which gives $L(u, v) = L(u_1, v_1)$. The mapping and its inverse mapping $(u, v) \in TS^2 \mapsto L(u, v) \in M$ are smooth, a fact that shows that $M$ and $S^2$ are at least diffeomorphic.

To get to the next stage, we recall that the unit sphere $S^2$ can be endowed with a structure of complex manifold by choosing a covering atlas $\{U_0, U_1\}$, where $U_0 = S^2 \setminus \{(0, 0, 1)\}$ and $U_1 = S^2 \setminus \{(0, 0, -1)\}$. We define complex coordinates on $U_0$ by
\[ \xi_0(x, y, z) = \frac{x + iy}{1 - z} \]
which is the stereographic projection of the point $(x, y, z)$ from the north pole and on $U_1$ by
\[ \xi_1(x, y, z) = \frac{x - iy}{1 + z} \]
which is the stereographic projection of the point $(x, y, z)$ from the south pole. We have by construction
\[ \xi_0(x, y, z) = \frac{1}{\xi_1(x, y, z)} = F(\xi_1(x, y, z)) \]
where $F(w) = \frac{1}{w}$, on $U_0 \cap U_1$. This defines a complex structure on $S^2$. To define a complex structure on $TS^2$ we use standard constructions in differential geometry; for example a chart on $TS^2$ corresponding to the chart $\xi_0$ is given in local coordinates $(u, v)$ where $u \in U_0$ and $v \in \mathbb{R}^3$ by
\[ (\xi(u, v), \eta(u, v)) = \left( \frac{u_1 + iu_2}{1 - u_3}, \frac{v_1 + iv_2}{1 - u_3} + \frac{(u_1 + iu_2)v_3}{(1 - u_3)^2} \right) \]
By definition the points of $M$ are oriented lines on $\mathbb{R}^3$. Moreover any point $p$ in $\mathbb{R}^3$ defines a 2-sphere of lines, namely all oriented lines going through that point. Specifically the set of all lines through $p$ is the set of all $(u, v) \in S^2 \times \mathbb{R}^3$ satisfying
\[ v = p - (p, u)u. \]
We call this a real section of $M$ and denote it by $X_p$. Let us explore in more detail the geometry of these real sections.
First we observe that these \( X_p \) are called sections because the map

\[
\rho_p : S^2 \to M \\
u \mapsto (u, p - (p, u)u)
\]

defines a section of the projection \( \pi : M \to S^2 \) (namely \( \pi(\rho_p(u)) = u \), for all \( u \in S^2 \)), and, with some abuse of notation, the image of this section is \( X_p \). To understand why we called the sections \( X_p \) real sections, we need to define a real structure on \( M \), through a map

\[
\tau : M \to M
\]
called a real structure. This map is defined as the involution that sends an oriented line to the same line with opposite orientation, i.e.

\[
\tau(u, v) = (-u, v).
\]

This real structure fixes the set \( X_p \) because

\[
\tau(u, p - (p, u)u) = (-u, p - (p, u)u) = (-u, p - (p, -u)(-u)),
\]

and this explains why \( X_p \) is called a real section.

If \( p = (x, y, z) \) is a point of \( \mathbb{R}^3 \) then \( X_p \) is the set of all \( (u, v) \) that correspond to lines through \( p \):

\[
X_p = \{(u, p - (p, u)u), u \in S^2\}.
\]

If we substitute \( v = p - (p, u)u \) into equation (2.1) and simplify, we see that the equation of \( X_p \) as a subset of \( M \) is

\[
\eta = \frac{1}{2}((x + iy) + 2z\xi - (x - iy)\xi^2)
\]

when we identify \( X_p \) with its image by the local chart given in equation (2.1). Hence under a similar identification, in coordinates, \( \rho_p \) is given by

\[
\rho_p(\xi) = (\xi, \frac{1}{2}((x + iy) + 2z\xi - (x - iy)\xi^2)).
\]

We will call any section that can be written in this way a holomorphic section. It is then possible to show that all holomorphic sections \( S^2 \to M \) take the form

\[
\xi \mapsto (\xi, a + b\xi + c\xi^2), a, b, c \in \mathbb{C}
\]
in local coordinates (this is because the holomorphic line \( T\mathbb{C}P_1 \) is the line bundle \( \Theta(2) \) whose holomorphic sections are given by degree two homogeneous complex polynomials in two variables, hence by a second degree trinomial in non-homogeneous coordinates). With our choice of coordinates and the definition of a real structure one can show that if \( (\xi, \eta) \) are the coordinates of a point \( m \in M \), then \( \left(\frac{-1}{\xi}, \frac{-\eta}{\xi}\right) \) are the coordinates of \( \tau(m) \). So \( \tau \) is anti-holomorphic. Therefore a section is real (i.e. invariant under the anti-holomorphic involution \( \tau \)) if and only if the equation

\[
\eta = a + b\xi + c\xi^2
\]
defines the same subset of $M$ as

$$-rac{\eta}{\xi^2} = a + b\left(-\frac{1}{\xi}\right) + c\frac{1}{\xi^2}.$$  

This immediately implies that $a = -c$ and that $b$ is real. Hence the real sections defined by points of $\mathbb{R}^3$ as in equation (2.3) are precisely all the real sections of $M$. Thus we have a surjection between points of $\mathbb{R}^3$ and real sections of $M$. The correspondence we have now established between $\mathbb{R}^3$ and real sections of $M$ is completely symmetric: points in $\mathbb{R}^3$ define special subsets (oriented lines) in $\mathbb{R}^3$ and points in $\mathbb{R}^3$ define special subsets (holomorphic real sections) in $M$.

Set $\omega = g(\xi, \eta)d\xi$ a differential one form on $M$. If

$$\phi(x, y, z) = \int g(\xi, \frac{1}{2}((x + iy) + 2z\xi - (x - iy)\xi^2))d\xi$$

and we differentiate under the integral sign we have

$$\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} = 0$$

that is $\phi$ is harmonic.

Consider now the ultrahyperbolic equation in $\mathbb{R}^4$ given by

$$\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial s \partial z} = 0.$$  

Let $T = \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be an arbitrary element of the Schwartz space $S(\mathbb{R}^3)$ and identify locally $M$ with $\mathbb{R}^4$. Choose local coordinates $(s, x, y, z)$ for $M$, on the open set where the third coordinate of $v$ does not vanish or again on the open set of lines which do not lie on the planes of constant $x_3$. A typical line in this open set is given by

$$L = \{(s + ty, x + tz, t), t \in \mathbb{R}\}.$$  

Define a function $\varphi$ on $M$ by

$$\varphi(L) = \int_L f$$

which gives in coordinates

$$\varphi(s, x, y, z) = \int_{-\infty}^{+\infty} f(s + ty, x + tz, t)dt.$$  

Now there are four variables $s, x, y, z$ and $f$ is defined on $\mathbb{R}^3$ so we expect a differential condition on $\varphi$ (constraint). Indeed differentiating under the integral sign one has

$$\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial s \partial z} = \int_{-\infty}^{+\infty} t(\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial x \partial y})f(s + ty, x + tz, t)dt = 0.$$  

It is natural to ask if this procedure, which goes under the name of John transform, can be inverted. This the case as shown by John in [12].

This example illustrates the defining philosophy of "twistor" theory. Namely, an unconstrained function on "twistor" space $T$ yields the solution to a differential equation on
Minkowski space $M$, by means of an integral transform. We also have a simple geometric correspondence, another characteristic feature of twistor methods. More precisely we see

$$T \leftrightarrow M$$

{point in T} $\rightarrow$ {oriented lines through point}

{line in T} $\leftarrow$ {point in M}.

**Remark 2.1.** We recall that the tautological line bundle $H$ over $\mathbb{CP}_1$, also denoted by $\mathcal{O}$, is the holomorphic line bundle whose fibre over a point $[z] = [z_0 : z_1]$ is given by the line $[z]$. This can be written also as $H = \{(z, w) \mid w = \lambda z, \lambda \in \mathbb{C} - \{0\}\} \subset \mathbb{CP}_1 \times \mathbb{C}^2$. The projection map $\pi : H \to \mathbb{CP}_1$ is the restriction of the projection from $\mathbb{CP}_1 \times \mathbb{C}^2$, that is $([z], w) \mapsto [z]$. By covering $\mathbb{CP}_1$ with the two open sets $U_0 = \{[z] = [z_0 : z_1], z_0 \neq 0\}$ and $U_1 = \{[z] = [z_0 : z_1], z_1 \neq 0\}$ we see that the transition function for the line bundle $H$ is given by

$$g_{01} : U_0 \cap U_1 \to \mathbb{C}^\times,
\quad [z] \mapsto \frac{z_1}{z_0}.$$  

This follows from the fact that one can define local sections $\psi_i : U_i \to H$, $i \in \{0, 1\}$ by

$$\psi_0([z]) = ([z], (1, \frac{z_1}{z_0}))$$

and

$$\psi_1([z]) = ([z], (\frac{z_0}{z_1}, 1)).$$

And one sees that $\psi_0([z]) = \frac{z_1}{z_0} \psi_1([z])$. By dualizing the line bundle $H$ one obtains the line bundle $\mathcal{O}(1)$ with transition function $g_{01}^*([z]) = \frac{z_0}{z_1}$ and taking the tensor product of $\mathcal{O}(1)$ with itself one gets the line bundle $\mathcal{O}(2)$ with transition function given by $[z] \mapsto \left(\frac{z_0}{z_1}\right)^2$.

Set $\xi = \frac{z_0}{z_1}$ on $U_1$ and $w = \frac{z_1}{z_0}$ on $U_0$, the two coordinates associated to $U_1$ and $U_0$ respectively. We have $\xi = \frac{1}{w}$ on $U_0 \cap U_1$. This gives $d\xi = -\frac{1}{w^2} dw$, i.e. $dw = -\xi^{-2} d\xi$, and therefore $\partial_w = -\xi^2 \partial_\xi$. This shows that the line bundles $\mathcal{O}(2)$ and $TCP_1$ on $\mathbb{CP}_1$ have the same transition functions and as a consequence they are isomorphic.

**2.1. The Plücker-Klein correspondence.** Given a four dimensional complex vector space $V$, consider the projective space $PV$ is the set of one dimensional linear subspaces of $V$. Recall that a line in $PV$ is the projective space associated to a two dimensional subspace of $V$, and Plücker has shown that any line in $PV$ could be represented by six scalars. To understand this process, consider the line $L$ is spanned by two non-collinear vectors of $V$, $u = (u_0, u_1, u_2, u_3)$ and $v = (v_0, v_1, v_2, v_3)$. Form now the $2 \times 4$ matrix $M_{u,v}$ by placing one vector above the other

$$M_{u,v} := \begin{bmatrix} u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \end{bmatrix}.$$
Since $u$ and $v$ are non-collinear, at least one of its $2 \times 2$ minors has non-zero determinant. Define $p_{ij} = u_i v_j - v_i u_j$. Then the 6-tuple

$$p_L := (p_{01} : p_{02} : p_{03} : p_{12} : p_{31} : p_{23})$$

defines (in homogeneous coordinates) a point in the projective space $\mathbb{CP}_5$. The coordinates of $p_L$ are called the Plücker coordinate of $L$. If we choose another representation of $L$, by taking another two linearly independent vectors on it: $u' = (u'_0, u'_1, u'_2, u'_3)$ and $v' = (v'_0, v'_1, v'_2, v'_3)$, then we can write the corresponding matrix $M_{u', v'}$ as $X M_{u, v}$ for a $2 \times 2$ invertible matrix $X$. Then the new Plücker coordinates are

$$\det(X)(p_{01} : p_{02} : p_{03} : p_{12} : p_{31} : p_{23}).$$

Therefore the Plücker coordinates of $L$ give a unique representation of $L$ if we regard them as homogeneous coordinates of a point of $\mathbb{CP}_5$. Let us consider now the matrix

$$
\begin{bmatrix}
  u_0 & u_1 & u_2 & u_3 \\
  v_0 & v_1 & v_2 & v_3 \\
  u_0 & u_1 & u_2 & u_3 \\
  v_0 & v_1 & v_2 & v_3
\end{bmatrix}.
$$

This matrix has rank 2 hence its determinant is zero and this is equivalent to the relation

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.$$

Therefore the Plücker coordinates of a line yield a point of $\mathbb{CP}_5$ that is a zero of the following quadratic form

$$Q(x_0, x_1, x_2, x_3, x_4, x_5) := x_0x_5 + x_1x_4 + x_2x_3.$$

The zeros of this quadratic form define a non-singular quadric called the Klein quadric. We have therefore shows that the map defined by taking the Plücker coordinates defines a bijection between lines in $PV \cong \mathbb{CP}_5$ and points of the Klein quadric.

This argument can be formalized in a more precise way. Let $G_{2,4}(V)$ be the Grassmannian of 2-planes in $V$; it turns out that it is also the manifold of lines in $\mathbb{CP}_5$. Define the mapping

$$pl : G_{2,4}(V) \to P(\Lambda^2 V)$$

by

$$(2.4) \quad pl([Z, W]) = [Z \wedge W] \in P(\Lambda^2 V),$$

where $Z$ and $W$ are linearly independent vectors in $V$, and $[Z, W]$ and $[Z \wedge W]$ denote $\text{span}(Z, W)$ and $\text{span}(Z \wedge W)$, respectively. Choosing a basis $\{e_0, \ldots, e_3\}$ for $V$ and letting $(p_{ij})$ be the coordinates of $\Lambda^2 V$, then we have the following theorem

**Theorem 2.2.** The mapping $pl$ in (2.4) is an embedding and the image $Q_4 = pl(G_{2,4}(V))$ is the projective algebraic hypersurface of degree two in $\mathbb{CP}_5$ given by

$$Q_4 = \{p_{ij} : p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0\}.$$

To give a proof of this theorem we first need the following lemma [10]

**Lemma 2.3.** Let $V$ be a finite dimensional complex vector space. Let $\omega \in \Lambda^2 V$; then $\omega$ is decomposable as $\omega = Z \wedge W$, for $Z, W \in V$, if and only if $\omega \wedge \omega = 0$. 

Let us now prove the theorem.

\textit{Proof.} We have already seen in the discussion above that the image of $G_{2,4}(V)$ under $pl$ lies on the Klein quadric $Q_4$. Now given $\omega = \sum p_{ij}e_i \wedge e_j \in \wedge^2 V$, the lemma gives us the characterization of those $\omega$ which are decomposable and a computation gives the result. \hfill \square

To close this subsection, let us expand more on the Plucker-Klein correspondence. Our starting point is to consider a complex vector space $T$ of dimension $n$ and to introduce its flag manifold

$$F_{d_1 \ldots d_m}(T) = \{(S_1, \ldots, S_m)\mid S_i \subset T, \dim S_i = d_i, S_1 \subset S_2 \subset \ldots \subset S_m\}$$

where the $S_i$ are complex subspaces of $T$. Typical examples of such flag manifolds are the projective space $F_1 = \mathbb{CP}_{n-1}$ and the grassmannian $G_{k,n}$. Let now $\mathcal{V}^4$ be a complex four-dimensional vector space that we will call the twistor space. Then we have a natural double fibration in terms of flag manifolds

$$F_{12}(\mathcal{V}^4) \xrightarrow{\pi_1} F_1(\mathcal{V}^4) \xleftarrow{\pi_2} F_2(\mathcal{V}^4)$$

(2.5)

together with canonical projections

$$\pi_i(S_1, S_2) = S_i, \ i = 1, 2.$$ 

Following [17] we define $F_1(\mathcal{V}^4) = \mathbb{CP}_3$, $M^4 = F_2(\mathcal{V}^4) = G_{2,4}(\mathcal{V}^4)$ and $F^5 = F_{12}(\mathcal{V}^4)$ and we call them projective twistor space, compactified Euclidean four dimensional space and correspondence space, respectively. The following lemma tells us how basic geometric data is translated from $M^4$ to $\mathbb{CP}_3$ and vice versa.

\textbf{Lemma 2.4.} We have the following geometric twistor correspondence

\begin{enumerate}
  \item Point in $\mathbb{CP}_3 \leftrightarrow \mathbb{CP}_2 \subset M^4$
  \item $\mathbb{CP}_1 \subset \mathbb{CP}_3 \leftrightarrow$ point in $M^4$.
\end{enumerate}

\textit{Proof.} Let us start with (1). By definition, a point in $\mathbb{CP}_3$ is a one-dimensional subspace $S_1^0$ of $\mathcal{V}^4$. Thus,

$$\pi_2 \circ \pi^{-1}(S_1^0) = \{S_2 \subset \mathcal{V}^4 \mid \dim_C S_2 = 2, S_1^0 \subset S_2\}.$$ 

Let $e_0 \in S_1^0$ be a non-zero vector and choose a basis for $\mathcal{V}^4$ of the form $\{e_0, e_1, e_2, e_4\}$.

The correspondence we are interested in is defined by mapping $[w^1, w^2, w^3]$ to $S_2^w = \text{span}\{e_0, w^1e_1 + w^2e_2 + w^3e_3\}$. Note that $S_1^0 \subset S_2^w$. In fact, all subspaces arise in this manner and thus we have established a complex-analytic isomorphism.

To prove (2), we consider a fixed two dimensional subspace of $\mathcal{V}^4$ and denote it by $S_2^0$. We have

$$\pi_1 \circ \pi_2^{-1}(S_2^0) = \{S_1 \subset \mathcal{V}^4 \mid \dim_C S_1 = 1, S_1 \subset S_2^0\}.$$ 

But $S_2^0 \simeq \mathbb{C}^2$ and hence $\pi_1 \circ \pi_2^{-1}(S_2^0)$ is isomorphic to the set of one dimensional subspaces of $\mathbb{C}^2$, i.e., $\mathbb{CP}_1$. \hfill \square
3. The solution of partial differential equations by means of definite integrals

We want to begin this section with what Atiyah regarded as the beginning of twistor theory, intended as the representation of solutions of linear homogeneous (as a polynomial in the partial derivatives) partial differential equations with constant coefficients on \( \mathbb{R}^n \) or \( \mathbb{C}^n \) by means of definite integrals. We will soon need some results from classical algebraic geometry, but we begin here with a relatively simple example where all the calculations can be made explicit.

3.1. The Laplace equation and its general solutions. Consider two points \( P = (a, b, c) \) and \( M = (x, y, z) \) in the usual Euclidean space \( \mathbb{R}^3 \), and assume they are subjected to Newtonian attraction, with \( P \) being the attracting point, and \( M \) the attracted one. By a suitable normalization we have that the force exerted by \( P \) on \( M \) is \( \vec{F} = \frac{P \vec{M}}{||P \vec{M}||^3} \). We set \( r = ||P \vec{M}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \). Then the components of the force \( \vec{F} \) are

\[
X = -\frac{x-a}{r^3}, \quad Y = -\frac{y-b}{r^3}, \quad Z = -\frac{z-c}{r^3},
\]

and this attraction derives from the potential

\[
U(x, y, z) = \frac{1}{r},
\]

since

\[
\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial x} = -\frac{x-a}{r^3}
\]

(the same calculation holds for the other partial derivatives of \( U \)).

If instead of an attracting point \( P \), one has a finite attracting volume \( V \), then the potential \( U \) is given for points \((x, y, z)\) lying outside the volume \( V \) by

\[
U(x, y, z) = \int \int \int_V \frac{dadbdc}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}.
\]

By computing now the derivatives of \( U(x, y, z) \) we obtain

\[
\frac{\partial U}{\partial x} = -\int \int \int_V \frac{(x-a)dadbdc}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^\frac{3}{2}}
\]

and

\[
\frac{\partial^2 U}{\partial x^2} = -\int \int \int_V \frac{dadbdc}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^\frac{5}{2}}
\]

\[+ 3 \int \int \int_V \frac{(x-a)^2 dadbdc}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^\frac{7}{2}}.\]

This gives

\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0
\]
and so
\[ \Delta U = 0. \]

In the introduction we made reference to how Whittaker, [18], found a way to write the general solution to this Laplace equation by means of definite integrals. We will now review in detail how that can be achieved.

The first observation is the fact that the Laplace differential operator \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) is elliptic. This means that there is no non-zero element of \( \mathbb{R}^3 \) which satisfies \( x^2 + y^2 + z^2 = 0 \). Hence by the elliptic regularity theorem any solution to the Laplace partial differential equation is real analytic.

Let then \( U(x, y, z) \) be a solution to the Laplace differential equation expressed as a convergent power series with respect to the three variables \( x, y, z \), in the neighborhood of a given point \( x_0, y_0, z_0 \), and set
\[ x = x_0 + X, \quad y = y_0 + Y, \quad z = z_0 + Z. \]

The series
\[ U = a_0 + a_1 X + b_1 Y + c_1 Z + a_2 X^2 + b_2 Y^2 + c_2 Z^2 + 2d_2 YZ + 2e_2 ZX + 2f_2 XY + \ldots \]
is therefore convergent for \( X^2 + Y^2 + Z^2 \) sufficiently small. To determine the coefficients \( a_0, a_1, \ldots \), we will calculate the second order partial derivatives of \( U \) with respect to \( X, Y, Z \), and put the resulting expressions in the equation
\[ \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} = 0. \]

By identification, this will give us linear relations from which we can deduce the values of the coefficients. Now note that if we consider, in the series of \( U \), the homogenous part \( U_n \) of degree \( n \) in \( X, Y, Z \), the number of its coefficients is \( \frac{(n+1)(n+2)}{2} \), because this is the dimension of the space of homogeneous polynomials of degree \( n \) in three variables. As the laplacian is of second order, when \( n \geq 2 \), its action on \( U_n \) will give a homogeneous polynomial of degree \( n - 2 \). This term has to vanish identically so we have \( \frac{n(n-1)}{2} \) (dimension of the space of homogeneous polynomials of degree \( n - 2 \) in three variables) linear relations among the coefficients of \( U_n \). Therefore in the terms of degree \( n \) of \( U_n \), there will be
\[ \frac{(n+1)(n+2)}{2} - \frac{n(n-1)}{2} = 2n + 1 \]
arbitrary coefficients when \( n \geq 2 \). Note that for \( n = 0, 1 \), \( \frac{(n+1)(n+2)}{2} = 2n + 1 \), and so there are \( 2n + 1 \) arbitrary coefficients in \( U_n \) regardless of the value of \( n \). By superposition these terms will be linear combinations of \( 2n + 1 \) particular solutions, of degree \( n \), to the Laplace equation. Let us look for such solutions.

For that let us start with the expression
\[ E_n := (Z + iX \cos(u) + iY \sin(u))^n, \quad u \in \mathbb{R}, \]
which is clearly a solution to the Laplace equation of degree \( n \). We can develop \( E_n \) into a Fourier series because it is smooth and \( 2\pi \)-periodic in \( u \). This gives

\[
\sum_{0}^{\infty} g_m(X,Y,Z) \cos(mu) + \sum_{0}^{\infty} h_j(X,Y,Z) \sin(ju);
\]

with coefficients \( g_m \) and \( h_j \) linearly independent.

However the development in Fourier series of \( E_n \) contains only a finite number of terms. This follows by computing \( E_n \) via the binomial formula, by linearizing the various powers of \( \cos(u) \), \( \sin(u) \) and by uniqueness of the Fourier expansion of a continuous \( 2\pi \)-periodic function. Therefore one can write

\[
E_n = \sum_{0}^{n} g_m(X,Y,Z) \cos(mu) + \sum_{1}^{n} h_j(X,Y,Z) \sin(ju)
\]

where by Fourier one has

\[
\pi g_m(X,Y,Z) = \int_{-\pi}^{\pi} (Z + iX \cos u + iY \sin u)^n \cos(mu) du
\]

\[
\pi h_j(X,Y,Z) = \int_{-\pi}^{\pi} (Z + iX \cos u + iY \sin u)^n \sin(ju) du.
\]

To show that \( E_n \) may be written in such a form one can use an induction based on the classical formulas, valid for \( a, b \in \mathbb{R} \)

\[
\cos(a) \cos(b) = (\cos(a - b) + \cos(a + b))/2, \quad \sin(a) \sin(b) = (\cos(a - b) - \cos(a + b))/2
\]

\[
\cos(a) \sin(b) = (\sin(a + b) - \sin(a - b))/2.
\]

We remark that the \( g_m \) are even in \( Y \) and that the \( h_j \) are odd in \( Y \). For instance one has by definition

\[
\pi g_m(X,-Y,Z) = \int_{-\pi}^{\pi} (Z + iX \cos u - iY \sin u)^n \cos(mu) du;
\]

by setting \( u = -v \), we obtain

\[
\pi g_m(X,Y,Z) = \int_{-\pi}^{\pi} (Z + iX \cos v + iY \sin v)^n \cos(mu)(-dv) = \pi g_m(X,Y,Z).
\]

Also the highest power of \( Z \) present in \( g_m \) or \( h_j \) is \( n - m \) (respectively \( n - j \)). To see this one may use an induction based on the formula

\[
E_n = \sum_{0}^{n} g_m(X,Y,Z) \cos(mu) + \sum_{1}^{n} h_j(X,Y,Z) \sin(ju)
\]

and the fact that

\[
E_{n+1} = (Z + iX \cos(u) + iY \sin(u))E_n.
\]

Now we can use these properties of \( g_m \) and \( h_j \) to show that they are linearly independent. Let \( \lambda_0, \lambda_1, \ldots, \lambda_n \) and \( \mu_1, \mu_2, \ldots, \mu_n \) be scalars such that

\[
\lambda_0g_0 + \ldots + \lambda_ng_n + \mu_1h_1 + \ldots \mu_nh_n = 0.
\]
Then since the $g_m$ are even and the $h_j$ are odd, we have separately
\[ \lambda_0 g_0 + \ldots + \lambda_n g_n = 0 \]
and
\[ \mu_1 h_1 + \ldots + \mu_n h_n = 0. \]
Therefore from the fact that $g_m$ and $h_m$ are of degree $n - m$, one deduces immediately that all the coefficients $\lambda_m$ and $\mu_m$ are zero.

This being said every linear combination of the independent $2n + 1$ solutions can then be put in the form
\[ \int_{-\pi}^{\pi} (Z + iX \cos u + iY \sin u)^n f_n(u) du. \]
Hence we shall have
\[ U(x, y, z) = \sum_{0}^{\infty} \int_{-\pi}^{\pi} (Z + iX \cos u + iY \sin u)^n f_n(u) du \equiv \int_{-\pi}^{\pi} F(Z + iX \cos u + iY, u) du, \]
for $F$ a suitable function in two variables. We have therefore obtained the desired integral representation of the general solution of the Laplace equation in $\mathbb{R}^3$.

3.2. The kernel of the partial differential operator $F \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$. In this subsection we will show how to generalize the result of the previous subsection to the case in which the Laplacian is replaced by another partial differential operator, whose symbol is still a homogeneous polynomial in three variables. As we will see, it is not so easy to calculate the dimensions of the spaces of coefficients in the series expansion of the solution, and therefore we have to resort to some pretty significant results from classical algebraic geometry.

To this end, we introduce some classical terminology before being able to prove the theorem stated later in this section. Let $C$ be a smooth projective plane algebraic curve and let $\tilde{C}$ be its associated compact Riemann surface. A divisor $D$ on $C$ is a formal sum $D = \sum_{p \in C} n_p p$ with $n_p$ an element of the set of integers and all but a finite number of the $n_p$’s are equal to zero. The degree of a divisor $D$ is defined by $\deg D := \sum_{p \in C} n_p$. A divisor $D = \sum_{p \in C} n_p p$ is called effective, and we will write $D \geq 0$, when $n_p \geq 0$ for all $p \in C$. Two divisors $D$ and $D'$ are linearly equivalent if there exists a rational function $f$ on $C$ such that $D - D' = (f)$, where $(f)$ stands for $f^{-1}(0) - f^{-1}(\infty)$. Equivalence between divisors is easily seen to be an equivalence relation. One important example of a divisor class on a smooth algebraic curve is the class of the divisor of any given rational differential on the curve. It will be called the canonical divisor class and will be denoted $K$. It is the. Let $\mathcal{K}(C)$ be the field of rational functions on $C$. Let us introduce now the following vector space associated to a given divisor $D$

\[
L(D) = \{0\} \cup \{f, f \in \mathcal{K}(C), (f) + D \geq 0\}.
\]

The dimension of $L(D)$ is denoted by $l(D)$. Note that $l(D)$ only depends on the divisor class of $D$. The fundamental theorem which enables one to compute $l(D)$ in general is the Riemann-Roch theorem.
Theorem 3.1 (R-R, [9]). Let \( C \) be a smooth projective plane algebraic curve of degree \( d \), \( D \) a divisor on \( C \), \( K \) the canonical divisor class of \( C \) and \( g \) the genus of \( C \). Then
\[
l(D) = \deg D + 1 - g + l(K - D).
\]

The following result is a simple consequence of the Riemann-Roch theorem.

Theorem 3.2. Any rational function \( f \) on a nonsingular curve plane \( C \) can be written as the quotient of two homogeneous polynomials in three variables and of the same degree, restricted to \( C \):
\[
f = \frac{Q}{R}|_C
\]

Let \( L \) be a line on \( \mathbb{CP}^2 \) not containing \( C \); we set \( H := \sum_{p \in L \cap C} I_p(C, L)p \), where \( I_p(C, L) \) is the intersection multiplicity at \( p \) between \( C \) and \( L \). More generally, let \( X \) be any plane curve intersecting \( C \) only in isolated points; then we define the divisor cut on \( C \) by \( X \), denoted by \( C \cdot X \), by the formula
\[
C \cdot X = \sum_{p \in X \cap C} I_p(C, X)p.
\]
Remark that any two such divisors \( C \cdot X \) and \( C \cdot X' \) associated to different such curves \( X \) and \( X' \) of the same degree, are linearly equivalent. This is because if \( X = \{ F = 0 \} \) and \( X' = \{ F' = 0 \} \) then
\[
C \cdot X - C \cdot X' = \left( \frac{F}{F'}|_C \right).
\]

Let us now recall the Bézout’s theorem

Theorem 3.3 (Bézout). If \( C \) is a smooth plane curve of degree \( d \). If \( X \) is a plane curve of degree \( e \) not containing \( C \), then the degree of the divisor \( C \cdot X \) cut by \( X \) on \( C \) is \( d \cdot e \).

It follows from Bézout’s theorem [9, p. 86] that the degree of \( H \) is equal to \( n \). We also have

Theorem 3.4 (L=AF+BG). Let \( C = \{ F = 0 \} \) be a smooth curve and \( X = \{ G = 0 \} \) a curve not containing \( C \). Then if a curve \( Y = \{ L = 0 \} \) contains \( C \cdot X \) one can write \( L = AF + BG \), with \( A \) and \( B \) homogeneous polynomials of degrees, respectively, \( \deg(L) - \deg(F) \) and \( \deg(L) - \deg(G) \).

One of the central results in the classical theory of plane algebraic curves is the following theorem of Brill and Noether

Theorem 3.5 (Brill-Noether Restsatz). Let \( C \) be a non-singular plane curve, and let \( X \) be any plane curve not containing \( C \). Then for any divisor linearly equivalent to \( C \cdot X \), there is a plane curve \( X' \) not containing \( C \), and such that \( C \cdot X' = D \).

An important important consequence of the Brill-Noether theorem [6, cor. 6] is

Corollary 3.6. Let \( C \) be a smooth plane curve of degree \( d \), and let \( \Lambda \) be a subset of \( \lambda \) distinct points of \( C \) considered as an effective divisor on \( C \). Then the dimension of the space of homogeneous polynomials of degree \( m \) in three variables modulo those vanishing on \( \Lambda \) is equal to
\[
l(mH) - l(mH - \Lambda).
\]

Proof. We apply the Restsatz to the curve \( C \) and to the \( m \)-th power of a generic line. The vector space of homogeneous polynomials of degree \( m \) cuts out on \( C \) the family of divisors
linearly equivalent to \( mH \). This family denoted \(|mH|\) is isomorphic to the projective space \( P(L(mH)) \). This set has the same dimension as the projective space associated to the vector space of homogeneous polynomials of degree \( m \) not containing \( C \) or, in other words, the projective space of the vector space of homogeneous polynomials of degree \( m \) modulo those homogeneous polynomials vanishing on \( C \).

To see this, take a divisor \( D \) linearly equivalent to \( mH \). We have \( D = C \cdot X \) for some \( X \) of degree \( m \) not containing \( C \). Therefore the map

\[
P(V) \to |mH| \\
X \mapsto C \cdot X = D
\]

is surjective. Suppose that \( D = C \cdot X' \) for another curve \( X' \). We show that the defining polynomials of \( X \) and \( X' \) are proportional. The fact that \( X \) and \( X' \) do not contain \( C \) is equivalent to the fact that their defining polynomials \( F \) and \( F_1 \) (resp) are not divisible by the polynomial \( P \) which defines \( C \). From Theorem 3.4 we can write \( F = A_1 F_1 + B_1 P \) and \( F_1 = A_2 F + B_2 P \) with \( A_1 \) and \( A_2 \) complex numbers. Moreover \( F(1 - A_1 A_2) = (A_1 B_2 + B_1 P \) and \( F_1(1 - A_1 A_2) = (A_2 B_1 + B_2 P \). So we have a contradiction unless \( B_1 = B_2 = 0 \) and \( F_1 = \lambda_0 F \). Hence the map

\[
P(V) \to |mH| \\
X \mapsto C \cdot X = D
\]

is injective.

We have thus shown that \( P(V) \simeq |mH| \simeq P(L(mH)) \). This gives \( \dim V = l(mH) = \dim(L(mH)) \). Likewise one shows that \( l(mH - \Lambda) = \dim L(mH - \Lambda) \) is the dimension of \( V' \), the vector space of homogeneous polynomials of degree \( m \) passing through the points of \( \Lambda \) modulo those vanishing on \( C \). This follows from the fact that \( L(mH - \lambda) \simeq \Gamma(C, [mH - \Lambda]) \), where \([mH - \Lambda] \) is the line bundle, [10, chap. 1], associated to the divisor \( mH - \Lambda \). This latter line bundle is \( \mathcal{O}(m) \otimes \mathcal{O}_C([-\Lambda]) \) and this concludes the proof. \( \square \)

**Lemma 3.7.** Given a divisor \( D \) on a nonsingular projective algebraic curve \( C \), the set

\[
|D| := \{D' \sim D, D' \geq 0 \} \simeq P(L(D)).
\]

**Proof.** For every \( D' \in |D| \), there exists \( f \in \mathcal{K}(C) \) such that \( D' = (f) + D \). And any two such \( f \in \mathcal{K}(C) \) differ by a non-zero constant. Indeed if \( (f) + D = (g) + D \) then \( (f) = (g) \) so \( (f/g) = 0 \). Let us take two representatives of \( f \) and \( g \), still denoted \( f \) and \( g \) \( (f \neq 0 \) and \( g \neq 0) \). One sees then that \( f/g \) has no zeros or poles (so it is in particular holomorphic on \( \hat{C} \)) therefore it is an element of \( \mathbb{C}^\times \) because \( \mathcal{O}_C(C) = \mathbb{C} \). Therefore we have a bijective map

\[
P(L(D)) \to |D| \\
f \mapsto (f) + D
\]

\( \square \)

We will need the following version of Cayley-Bacharach’s theorem [6, th. CB4]

**Theorem 3.8 (C-B1).** Let \( X_1, X_2 \) be plane curves of degrees \( m \) and \( n \) respectively, with \( X_1 \) smooth and meeting in a collection of \( mn \) distinct points \( \Gamma = \{p_1, \ldots, p_{mn}\} \). If \( X \subset \mathbb{CP}_2 \)
is any plane curve of degree \( m + n - 3 \) containing all but one point of \( \Gamma \), then \( X \) contains all of \( \Gamma \).

**Corollary 3.9** (Chasles' theorem). Let \( X_1, X_2 \subseteq \mathbb{CP}_2 \) be cubic plane curves, with \( X_1 \) smooth, meeting in nine points \( P_1, P_2, \ldots, P_9 \). If \( X \subseteq \mathbb{CP}_2 \) is any cubic plane curve containing 8 among them, then \( X \) contains the remaining point as well.

Let us introduce the following theorem of Serret [16, p. 99]

**Theorem 3.10** (Serret). Let \( p = [a, b, c] \) be a point of \( \mathbb{CP}_2 \) and associate to it the linear form \( l_p(x, y, z) = ax + by + cz \). Then the necessary and sufficient condition for any given curve \( C^r \), of given degree \( r \), which passes through \( q - 1 \) of \( q \) given points of \( \mathbb{CP}_2 \) to pass through them all, is that there should be a linear relation (or syzygyy) connecting the \( r \)th powers of the linear forms (or, by abuse of language, tangential equation) associated to each given point.

**Proof.** If one has a relation of the form

\[
\sum_{i=1}^{q} \lambda_i (l_{p_i})^r = 0
\]

and the equation of the curve \( C^r \) is given by \( h \), then \( h \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (l_{p_i})^r = r! h(p_i) \). So if \( q - 1 \) of the points lie on \( C^r \) the remaining one also does. The converse is shown along similar lines. \( \square \)

**Corollary 3.11.** When \( q = \binom{m + 2}{m} \) Serret’s theorem gives the necessary and sufficient condition that \( q \) points should lie on a curve of degree \( m \).

**Proof.** Indeed it follows from the Riemann-Roch theorem (see below the proof of Theorem 3.12) that there always exists a curve of degree \( m \) which passes through any \( q - 1 = \frac{1}{2} m(m + 3) \) given points of the fixed smooth curve \( C \). So if there is linear relation between the \( m \)th powers of the linear terms associated to the \( q \) points, then necessarily all the \( q \) points lie on a curve of degree \( m \). The converse is the necessity statement of Serret’s theorem. \( \square \)

We finally prove the following generalized Cayley-Bacharach theorem inspired by [16] and which easily follows from Serret’s theorem and The Riemann-Roch theorem.

**Theorem 3.12** (C-B2). Let \( X_1 \) and \( X_2 \) be plane curves of degrees \( m \) and \( n \) respectively, with \( X_1 \) smooth and meeting \( X_2 \) in a collection of \( mn \) distinct points \( \Gamma = \{p_1, \ldots, p_{mn}\} \). Every curve \( C^{m+n-\gamma} \) (\( \gamma > 3 \)) of degree \( m + n - \gamma \) which passes through \( mn - \frac{1}{2} (\gamma - 1)(\gamma - 2) \) of the points \( X_1 \cap X_2 \) passes through the remainder except when these remaining \( \frac{1}{2} (\gamma - 1)(\gamma - 2) \) points lie on a curve \( C^{\gamma - 3} \) of degree \( \gamma - 3 \).

**Proof.** Denote by \([a_s, b_s, c_s] \) the points of \( X_1 \cap X_2 \). From the Cayley-Bacharach Theorem 3.8 every curve \( C^{m+n-3} \) which passes through all but one point of \( X_1 \cap X_2 \) necessarily passes
through the remaining point. Therefore from Serret’s theorem we have a syzygy

\[ \sum_{s=1}^{mn} k_s (a_s x + b_s y + c_s z)^{m+n-3} = 0, k_s \in \mathbb{C}. \]

Since this last equation is an identity in \( x, y, z \), it can be differentiated repeatedly with respect to the variables \( x, y, z \). Then, if \( F \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \) is a homogeneous polynomial of degree \( \nu \) in the operators, we evidently have

\[ \sum_{s=1}^{mn} k_s F(a_s, b_s, c_s)(a_s x + b_s y + c_s z)^{m+n-3-\nu} = 0. \]

In particular, taking \( \nu = m + n - \gamma \)

\[ \sum_{s=1}^{mn} k_s F(a_s, b_s, c_s)(a_s x + b_s y + c_s z)^{\gamma-3} = 0. \]

Thus if \( F \) is the equation of the curve \( C^{m+n-\gamma} \) provided by the hypothesis of the theorem, we obtain an identity involving only \( \frac{1}{2}(\gamma-1)(\gamma-2) \) of the points. But from the Riemann-Roch theorem we know that there always exists a curve \( C^{\gamma-3} \) passing through any given \( \frac{1}{2}\gamma(\gamma-3) \) given points of \( X_1 \). Indeed if \( \Lambda \) is that set of points considered as an effective divisor

\[ l((\gamma-3)H) - l((\gamma-3)H - \Lambda) = \frac{1}{2}\gamma(\gamma-3) + l(K - (\gamma - 3)H) - l(K - (\gamma - 3)H + \Lambda), \]

with \( K \) and \( H \) defined with respect to \( X_1 \). Because \( \Lambda \geq 0 \) it follows that \( l(K - (\gamma - 3)H) - l(K - (\gamma - 3)H + \Lambda) \leq 0 \), by definition. Thus \( l((\gamma-3)H) - l((\gamma-3)H - \Lambda) \leq \frac{1}{2}\gamma(\gamma-3) \)

which is strictly less than the dimension \( \frac{1}{2}(\gamma-1)(\gamma-2) \) of the vector space of homogeneous polynomials of degree \( \gamma-3 \) in three variables. Given the interpretation we gave of the quantity \( l((\gamma-3)H) - l((\gamma-3)H - \Lambda) \), we conclude that there exists a non-trivial polynomial of degree \( \gamma-3 \) vanishing on \( \Lambda \). Hence if the remaining \( \frac{1}{2}(\gamma-1)(\gamma-2) \) points do not lie on a \( C^{\gamma-3} \), they must all lie on the given curve of degree \( m + n - \gamma \), by Serret’s theorem.

\[ \square \]

We are finally ready for the main result of this work:

**Theorem 3.13.** Let \( C \) be a smooth degree \( n \) projective plane curve of equation \( F(\xi, \eta, \zeta) = 0 \). Let \([\xi, \eta, \zeta]\) be the coordinates of its points expressed as functions of a uniformizing parameter \( t \). Then any analytic solution \( V \) of the equation

\[ F \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \phi(x, y, z) = 0 \]

on a sufficiently small open set can be put in the form

\[ V(x, y, z) = \int \Phi(\xi x + \eta y + \zeta z, t) dt, \]
for a suitable path of integration.

Proof. Without loss of generality we assume $V$ to be analytic near the origin as a function of $x$, $y$, $z$. Expand $V$ as an absolutely and uniformly convergent power series in $x$, $y$, $z$ near the origin itself.

We now apply $F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ to this series, and we set to zero the coefficients of the various powers. Let’s look at the homogeneous part of $F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V$ of degree $m$, and when $m \geq n$ we find

$$\frac{1}{2}(m - n + 1)(m - n + 2)$$

relations among the coefficients since the action of $F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ on $V(x, y, z)$ will leave a polynomial of degree $m - n$ and $\frac{1}{2}(m - n + 1)(m - n + 2)$ represents the dimension of the space of homogeneous polynomials of degree $m - n$ in three variables. But when $m < n$ no such relations arise, because the operation of $F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ on the homogeneous parts of $V(x, y, z)$ of degree $m < n$ will cancel them entirely.

We now recall that the dimension of the space of homogeneous polynomials of degree $m$ in three variables is $\left(\frac{m + 2}{m}\right)$. Therefore the terms of order $m$ are a linear combination of

$$\frac{1}{2}(m + 1)(m + 2) - \frac{1}{2}(m - n + 1)(m - n + 2) = mn + 1 - \frac{1}{2}(n - 1)(n - 2)$$

linearly independent elements when $m \geq n$, and of $\frac{1}{2}(m + 1)(m + 2)$ independent terms when $m < n$.

In order to express the terms of order $m$ in the form (3.4) we proceed as follows. If we take $mn + 1 - \frac{1}{2}(n - 1)(n - 2)$ arbitrary points on the curve $F(\xi, \eta, \zeta) = 0$, then the corresponding quantities $(\xi x + \eta y + \zeta z)^m$ will in general be linearly independent.

For, if not, there would be a linear relation between them, of the form

$$\sum_{i=1}^{M} \lambda_i (\xi_i x + \eta_i y + \zeta_i z)^m = 0$$

where $M = mn + 1 - \frac{1}{2}(n - 1)(n - 2)$. Leaving out one of the points, say $(\xi_1, \eta_1, \zeta_1)$, we can draw a curve of the $m$-th order $f(\xi, \eta, \zeta) = 0$ through the remainder. Indeed with the notations introduced above we have

$$l(mH - \Lambda) = \deg(mH - \Lambda) + 1 - g + l(K - mH + \Lambda),$$

where $\Lambda$ is the set of the $mn - \frac{1}{2}(n - 1)(n - 2)$ chosen points, considered as an effective divisor. From this last equality we deduce that $l(mH - \Lambda) \geq 1$. Because $L(mH - \Lambda)$ is interpreted [6] as the vector space of homogeneous polynomials of degree $m$ vanishing on
A modulo those of degree $m$ vanishing on $C$, we can ascertain that there exists a curve $f$ of degree $m$ passing through the $mn - \frac{1}{2}(n-1)(n-2)$ points. We now operate on the equation (3.5) with $f \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$. Then the terms corresponding to the points disappear on account of the relation

$$f \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)(\xi x + \eta y + \zeta z)^m = m!f(\xi, \eta, \zeta)$$

and we are left with the equation

$$\lambda_1 f(\xi_1, \eta_1, \zeta_1) = 0.$$ 

Therefore either $f(\xi_1, \eta_1, \zeta_1) = 0$, in which case all the points lie on a curve of the $m$-th degree and they would not have been chosen arbitrarily; or $\lambda_1 = 0$. But $\lambda_1$ can be taken to be anyone of the coefficients; hence all the coefficients are zero and the syzygy or linear relation (3.5) does not exist.

This shows that we have $M = mn + 1 - \frac{1}{2}(n-1)(n-2)$ independent solutions (or tangential equation) of the equation $F(\xi, \eta, \zeta)$; therefore since $(\xi x + \eta y + \zeta z)^m$ (tangential equation associated to the point $[\xi_l, \eta_l, \zeta_l]$ of the curve $F(\xi, \eta, \zeta) = 0$), is a solution of the equation, it can be expressed as a linear combination of these solutions: in other words, there exists a linear relation between the $m$-th powers of the tangential equations of any $mn + 2 - \frac{1}{2}(n-1)(n-2)$ points on the curve, unless all but one of them lie on a curve of the $m$-th degree, in which case the syzygy does not contain a point corresponding to the last point.

More generally we may obtain the linear relation between the $m$th powers of the tangential equations of any $mn + 2 - \frac{1}{2}(n-1)(n-2)$ points on the curve as follows: draw a curve $C^{m+1}$, $\chi(x, y, z) = 0$ of degree $m+1$ through the points (this exists by a similar Riemann-Roch argument); this curve will cut the curve again in $\frac{1}{2}(n+1)(n-2)$ points by Bezout’s theorem. Through these we may draw a curve $C^{m-2}$ of degree $n-2$, say $G(x, y, z) = 0$. Now by the Cayley-Bacharach Theorem 3.8 together with the Serret’s theorem, we may deduce that there is a linear relation between the $(m + n - 2)^{th}$ powers of the tangential equations of the points of intersection of $F = 0$ and $\chi = 0$: this relation is

$$\sum_{r=1}^{n(m+1)} k_r (x\xi_r + y\eta_r + z\zeta_r)^{m+n-2} = 0.$$ 

Operating on this syzygy with $G \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$, the terms corresponding to $G = 0$ disappear, and we are left with the equation

$$\sum_{r=1}^{mn+2-\frac{1}{2}(n-1)(n-2)} k_r G(\xi_r, \eta_r, \zeta_r)(x\xi_r + y\eta_r + z\zeta_r)^m = 0.$$
Similarly, when \( m < n \) we may take \( \binom{m+2}{m} \) points on the curve which do not lie on a curve of the \( m^{th} \) degree. This is possible by Corollary 3.11 and the fact that \( m < n \).

Having so chosen the \( \binom{m+2}{m} \) points on \( C \) we observe that the corresponding tangential equations are linearly independent. This follows from the fact that one can always draw a curve \( C^m \) of degree \( m \) through any \( \frac{1}{2}m(m+3) \) given points and from Serret’s theorem.

Again we also obtain that when \( m < n \) there is a linear relation between the \( m \)th powers of the tangential equations of any \( \frac{1}{2}(m+2)(m+1)+1 \) points on the curve, with the \( \binom{m+2}{m} \) points we started with included and which satisfy the equation (3.3) (from the theorem of Serret).

So the conclusion of what we have said so far is that we can find a basis of the space of homogeneous polynomials in the form \((\xi x + \eta y + \zeta z)^m\), \(1 \leq i \leq r\) for \( r \) well-chosen points on \( C \) with \( r = mn + 1 - \frac{1}{2}(n-1)(n-2) \) in case \( m \geq n \) and \( r = \binom{m+2}{2} \) when \( m < n \).

Let us take the two cases together, and denote by \( r \) the number of independent solutions, we see that

\[
(\xi x + \eta y + \zeta z)^m = \sum_{i=1}^{r} (\xi_i x + \eta_i y + \zeta_i z)^m \mu_i
\]

is a solution of the partial differential equation (3.3) with \([\xi, \eta, \zeta] \in C\), expressed as a function of the uniformizing parameter \( t \).

Indeed when \( m < n \) we have \( F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(\xi x + \eta y + \zeta z)^m = 0 \) for degree reasons.

And when \( m \geq n \) we have

\[
F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(\xi_i x + \eta_i y + \zeta_i z)^m = F(\xi_i, \eta_i, \zeta_i)(\xi_i x + \eta_i y + \zeta_i z)^{m-n}.
\]

Let us take the \( \mu_i \) as above; we remark that for some \( t = t_i \), \((\xi x + \eta y + \zeta z)^m = (\xi_i x + \eta_i y + \zeta_i z)^m\). This implies that the \( \mu_i \) are linearly independent as functions of \( t \) because otherwise we would have a syzygy between the linear forms of any \( r \) points on the curve and \((\xi x + \eta y + \zeta z)^m\) would be expressible as a linear combination of the tangential equations of \( r - 1 \) points on the curve and this is not possible.

We have

\[
\int (\xi x + \eta y + \zeta z)^m f_s(t) dt = \sum_{i=1}^{r} (\xi_i x + \eta_i y + \zeta_i z)^m \int \mu_i f_s(t) dt.
\]

and the vectors \((\theta_{s,i})_{1 \leq i \leq r} := (\int \mu_i f_s(t) dt)_{1 \leq i \leq r}, s = 1, \ldots, r\) will, in general, be linearly independent.
For example choosing \( r \) linearly independent meromorphic functions \( f_s \), the determinant of the quantities \( \theta_{s,i} \) will not, in general be zero; accordingly we may choose \( r \) constants \( \lambda_j \), so that the expressions

\[
\lambda_1 \theta_{1,i} + \lambda_2 \theta_{2,i} + \ldots + \lambda_r \theta_{r,i} \quad (i = 1, 2, \ldots, r)
\]

take any \( r \) assigned values \( p_1, \ldots, p_r \), and we have

\[
\int (\xi x + \eta y + \zeta z)^m \sum_1^r \lambda_s f_s(t) dt = \sum_1^r p_i (\xi_i x + \eta_i y + \zeta_i z)^m.
\]

But any homogeneous part of the \( m^{th} \) order can be expressed in the form

\[
\sum_1^r p_i (\xi_i x + \eta_i y + \zeta_i z)^m,
\]

and can therefore be put in the form

\[
\int (\xi x + \eta y + \zeta z)^m f(t) dt.
\]

This being done for all values of \( m \), our series for \( V \) takes the required form

\[
V = \int \Phi(\xi x + \eta y + \zeta z, t) dt.
\]

\[\Box\]

4. The twistor correspondence and solutions of differential equations

Let \( f(z_0, z_1, \ldots, z_n) \) be a complex homogeneous polynomial of degree \( k > 1 \) in the indicated variables with \( n > 1 \). In this section we will show how to use a general twistor correspondence to describe all the solutions of

\[
D_f \phi := f \left( \frac{\partial}{\partial z_0}, \ldots, \frac{\partial}{\partial z_n} \right) \phi(z_0, \ldots, z_n) = 0.
\]

We will do so by showing that there is a twistor space \( Z \), a vector space \( H^{n-1}(Z, \mathcal{O}(-n - 1 + k)) \) of Dolbeault cohomology classes and a twistor correspondence

\[
T : H^{n-1}(Z, \mathcal{O}(-n - 1 + k)) \to H^0(\mathbb{C}^{n+1}, \mathcal{O}),
\]

whose image is the space of solutions of 4.1. Here and in what follows we will denote by \( \mathcal{O} \) is the sheaf of holomorphic functions on \( \mathbb{C}^m \) for some \( m \in \mathbb{N}^\times \), and we will show that \( T \) is an injective map, obtained by integrating the cohomology class against "a cycle", and onto the space of analytic functions in the kernel of \( D_f \) for all \( k > 1 \).

To begin with, we recall that the complex projective space \( \mathbb{C}P_n \) is the set of all lines through the origin in \( \mathbb{C}^{n+1} \), namely the set of all \([z], z \in \mathbb{C}^{n+1} - \{0\}\) with

\[
[z] := [z_0 : z_1 : \ldots : z_n] = \{\lambda z, \lambda \in \mathbb{C}^\times\}.
\]
It is a compact complex manifold with a covering family of charts given by the open sets $U_i$ where the $i$-th homogeneous coordinate is non-zero.

$$U_i \to \mathbb{C}^n$$

$$[z_0 : z_1 : \ldots : z_n] \mapsto \left( \frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right).$$

As in the previous sections let $H$ be the tautological line bundle over $\mathbb{C}P_n$ whose fibre over a point $[z] \in \mathbb{C}P_n$ is just the line $[z]$, i.e.

(4.3) $$H = \{(z, w) : w = \lambda z, \lambda \in \mathbb{C}^* \} \subset \mathbb{C}P_n \times \mathbb{C}^{n+1}.$$ We define local sections $\psi_i : U_i \to H$ for each $i = 0, \ldots, n$ by

(4.4) $$\psi_i([z]) = \left( [z], \left( \frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right) \right).$$

Note that $\psi_i$ is $H$-valued because

$$\left( \frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right) = \frac{1}{z_i} (z_0 : z_1 : \ldots : z_n).$$

From the definition of the $\psi_i$ we have that

$$\psi_i([z]) = \frac{z_j}{z_i} \psi_j([z])$$

and hence the transition functions of $H$ are

$$g_{ij} = \frac{z_j}{z_i}.$$ We set $O(-1) := H$. For $p > 0$ we define $O(p) = \bigotimes_{1}^{p} H^*$, with $H^*$ the line bundle dual to $H$. We also define $O(p) = \bigotimes_{1}^{-p} H$, for $p < 0$ and $O(0)$ as the trivial line bundle on $\mathbb{C}P_n$.

For a given sheaf $S$ on $\mathbb{C}P_n$, we denote by $H^p(\mathbb{C}P_n, S)$ its $p$-th cohomology group $p \geq 0$. The dimension of the vector space $H^p(\mathbb{C}P_n, S)$ (which is finite by Hodge theory) is denoted by $h^p(\mathbb{C}P_n, S)$. We have the following formulas of Bott [15, p. 4]

(4.5) $$h^q(\mathbb{C}P_n, O(k)) = \begin{cases} \binom{k+n}{k}, & k \geq 0, q = 0 \\ \binom{-k}{-k-n}, & q = n, k \leq -n - 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have [10, p. 165] $Sym^d((\mathbb{C}^{n+1})^*) \simeq H^0(\mathbb{C}P_n, O(d))$

here $Sym^d((\mathbb{C}^{n+1})^*)$ is the space of homogeneous polynomials of degree $d$ in the variables $z_0, \ldots, z_n$. Moreover from [10, p. 135] one knows that to any global section in $H^0(\mathbb{C}P_n, O(d))$, one can associate an effective divisor which is precisely where the considered section vanishes.
Let us now go back to the homogeneous polynomial $f(z_0, z_1, \ldots, z_n)$ of degree $k$ we started with. From what we have explained we can associate to it a section $f(\xi_0, \ldots, \xi_n)$ of $\mathcal{O}(k)$ which vanishes exactly on an effective divisor denoted $X$.

We then have the exact sequence of sheaves

\[(4.6)\quad 0 \longrightarrow \mathcal{O}(-k) \longrightarrow \mathcal{O}_{\mathbb{C}P^n} \longrightarrow \mathcal{O}_X \longrightarrow 0.\]

By taking the tensor product of the sequence (4.6) with the locally free sheaf $\mathcal{O}(p)$ and using the Bott’s formulas we get (by the long exact sequence in cohomology) the short exact sequences

\[(4.7)\quad 0 \longrightarrow H^0(\mathbb{C}P^n, \mathcal{O}(p-k)) \longrightarrow H^0(\mathbb{C}P^n, \mathcal{O}(p)) \longrightarrow H^0(X, \mathcal{O}(p)) \longrightarrow 0.\]

and

\[(4.8)\quad 0 \longrightarrow H^{n-1}(\mathbb{C}P^n, \mathcal{O}(p)) \longrightarrow H^n(\mathbb{C}P^n, \mathcal{O}(p-k)) \longrightarrow H^n(\mathbb{C}P^n, \mathcal{O}(p)) \longrightarrow 0.\]

Set $p = -n - 1 = k$ in (4.8). Then since $k > 0$ we have $-n - 1 + k > -n - 1$ and from formulas (4.5) we get

\[H^{n-1}(\mathbb{C}P^n, \mathcal{O}(-n - 1 + k)) \simeq H^n(\mathbb{C}P^n, \mathcal{O}(-n - 1)) \simeq \mathbb{C}\]

where the last isomorphism also follows from (4.5).

If we assume $X$ to be smooth, this isomorphism can be realized by integrating smooth $(0, n - 1)$-forms with values in $\mathcal{O}(-n - 1 + k)$.

\[(4.9)\quad 0 \longrightarrow \mathcal{O}(-n - 1 + k) \longrightarrow \xi_0^\bullet \otimes \mathcal{O}(-n - 1 + k).\]

Recall now that $H^0(\mathbb{C}P^n, \mathcal{O}(1))$ is the space of homogeneous polynomials of degree 1 in $\mathbb{C}^{n+1}$, and let $\xi_0, \ldots, \xi_n$ be a basis of $H^0(\mathbb{C}P^n, \mathcal{O}(1))$. We then have an isomorphism

\[\mathbb{C}^{n+1} \rightarrow H^0(\mathbb{C}P^n, \mathcal{O}(1))
\]

\[(x_0, \ldots, x_n) \mapsto \sum_{i=0}^n x_i \xi_i.\]

Moreover the restriction map

\[(4.11)\quad H^0(\mathbb{C}P^n, \mathcal{O}(1)) \rightarrow H^0(X, \mathcal{O}(1))\]

is an isomorphism by Lefschetz’s hyperplane theorem, and therefore if $X$ is the divisor defined by $f$ we obtain

\[(4.12)\quad \mathbb{C}^{n+1} = H^0(\mathbb{C}P^n, \mathcal{O}(1)) = H^0(X, \mathcal{O}(1)).\]
Remark 4.2. If $E$ is a vector bundle over a manifold $M$ then there is a sequence of induced jet bundles $J^pE$, $p = 0, 1, \ldots$. The fibre of $J^pE$ at $m \in M$ is the vector space of all $p$-jets at $m$ or of equivalence classes of local sections about $m$ where two sections are equivalent if they have the same Taylor series up to order $p$. This equivalence relation is independent of the charts necessary to define it and the fibers fit together to form smooth vector bundles. There is a natural projection from $J^pE$ to $J^{p-1}E$ which "forgets" the $p$-th term in the Taylor series. Associated with the jet bundles one has the following exact sequence

$$0 \longrightarrow S^pT^*M \boxtimes E \longrightarrow J^pE \longrightarrow J^{p-1}E \longrightarrow 0. \tag{4.13}$$

In this invariant language a $p$th order differential operator acting on sections of $E$ is just a bundle map

$$D : J^pE \rightarrow E. \tag{4.14}$$

When $E = \mathbb{C}^{n+1} \times \mathbb{C}$, $J^pE$ will be denoted $J^p\mathbb{C}$ and (4.13) becomes

$$0 \longrightarrow (\mathbb{C}^{n+1} \times S^p\mathbb{C}^{n+1}) \longrightarrow J^p\mathbb{C} \longrightarrow J^{p-1}\mathbb{C} \longrightarrow 0, \tag{4.15}$$

where $\mathbb{C}^{n+1}$ is identified with its dual. If $p = k$, then the natural flat connection on $\mathbb{C}^{n+1}$, which is just the exterior derivative, defines a splitting $J^k\mathbb{C} \rightarrow (\mathbb{C}^{n+1} \times S^k\mathbb{C}^{n+1})$ and composing with the projection on $\mathbb{C}^{n+1}$ followed by the composition with $f$ defines the homogeneous polynomial differential operator

$$D_f : J^k\mathbb{C} \rightarrow \mathbb{C}. \tag{4.16}$$

This is how one should interpret the differential operator given in equation (4.1).

Now let us define the twistor space $Z$ as the total space of the line bundle $O(1)$ on $\mathbb{C}P_n$ restricted (or pulled-back) to $X$.

The relation between $Z$ and $\mathbb{C}^{n+1}$ revolves around the following double fibration where we identify $\mathbb{C}^{n+1}$ with $H^0(X, O(1))$ via the isomorphism (4.12).

$$\begin{array}{c}
\mathbb{C}^{n+1} \times X \\
\mu \\
\downarrow \\
\mathbb{C}^{n+1} \\
\nu \\
\downarrow \\
Z \\
\end{array} \tag{4.17}$$

If $\left(\sum_{0}^{n} x_i \xi_i, z\right) \in \mathbb{C}^{n+1} \times X$, then the left hand map sends it to $(x_1, \ldots, x_n)$ and the right hand arrow sends it to $\sum_{0}^{n} \xi_i(z)$. If we fix a point $x \in \mathbb{C}^{n+1}$ then the image of the section

$$\sum_{0}^{n} x_i \xi_i : X \rightarrow Z \tag{4.18}$$

is a subvariety $X_x$ of $Z$ which is identified with $X$ by the projection map $\pi : Z \rightarrow X$. We will need various line bundles on $Z$. We use the mapping $\pi$ to pull-back line bundles from $X$. That is if $L$ is a line bundle on $X$, we define a line bundle $\pi^*L$ on $Z$ by $(\pi^*L)_z = L_{\pi(z)}$ (the fiber of $(\pi^*L)$ over $z$ is by definition given by $L_{\pi(z)}$). Notice that a peculiar thing happens for the bundle $O(1)$. Here if $z$ is an element of $Z$ then because $Z$ is itself the line
bundle $\mathcal{O}(1)$, $z \in \mathcal{O}(1)_{\pi(z)}$, that is $z \in (\pi^*\mathcal{O}(1))_z$. We denote this section of $\mathcal{O}(1)$, the one that sends $z$ to $z$, by $\eta$. It is the section whose divisor is the zero section $Z_0$ of the line bundle $Z \to X$. Notice that the subvariety $X_x$ in this notation is the subset of $Z$ where

$$ \eta = \sum_{i=0}^{n} \xi_i x_i. $$

Let $\omega_l$, $l = 0, \ldots, m$ be $(0, n - 1)$ forms on $X$ with values in $\mathcal{O}(-n - 1 + k - l)$. Consider now the $(0, n - 1)$ form on $Z$ with values in $\mathcal{O}(-n - 1 + k)$ defined by

$$ \omega = \sum_{l=0}^{m} \pi^*(\omega_l) \eta^l. $$

Because each $X_x$ is a copy of $X$ we can restrict $\omega$ to $X_x$ and integrate. We obtain

$$ \int_{X_x = \pi^{-1}(X)} \omega = \int_{X_x} \sum_{l=0}^{m} \pi^*(\omega_l) \eta^l $$

$$ = \int_{X_x} \sum_{l=0}^{m} \pi^*(\omega_l)(\sum_{i=0}^{n} \xi_i x_i)^l $$

$$ = \int_{X} \sum_{l=0}^{m} \omega_l(\sum_{i=0}^{n} \xi_i x_i)^l =: \phi(x). $$

This $\phi$ is clearly in $\ker D_f$, because if we apply a monomial differential operator

$$ \left( \frac{\partial}{\partial x_0} \right)^{i_0} \left( \frac{\partial}{\partial x_1} \right)^{i_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{i_n} $$

to $\phi(x)$ we obtain

$$ \int_{X} \sum_{l=0}^{m} \omega_l \xi_0 \xi_1 \cdots \xi_l \xi_{l+1} \cdots \xi_k. $$

Hence if we apply $D_f$ to $\phi(x)$ we obtain

$$ \int_{X} \sum_{l=0}^{m} \omega_l \xi_0 \xi_1 \cdots \xi_l \xi_{l+1} \cdots \xi_k f(\xi^0, \xi^1, \ldots, \xi^k) $$

which vanishes because $f$ vanishes on $X$.

In [14] it is shown that more generally one should integrate over $X_x$ elements belonging to $H^{n-1}(Z, \pi^*\mathcal{O}(-n - 1 + k))$.

Finally, as we indicated at the beginning of this section, one has in full generality the twistor transform

$$ T : H^{n-1}(Z, \pi^*\mathcal{O}(-n - 1 + k)) \to H^0(C^{n+1}, \mathcal{O}) $$

$$ T(\omega)(x) = \int_{X_x} \omega, $$

which is a bijection onto the kernel of $D_f$ for $k \leq n$. 
Remark 4.3. $H^{n-1}(Z, \pi^*\mathcal{O}(-n - 1 + k))$ is an infinite dimensional vector space because $Z$ is non-compact and $-n - 1 + k < 0$ as $k \leq n$. Also the appearance of $\mathcal{O}(-n - 1 + k)$ is due to the fact that the canonical bundle of $X$ is exactly $\mathcal{O}(-n - 1 + k)$. Moreover the fact that one should integrate $(0, n - 1)$-forms is a consequence of the Dolbeault resolution which implies that $H^{n-1}(X, \mathcal{O}(-n - 1 + k)) \simeq H^{n-1}(\Gamma(X, \mathcal{E}^{0,n}(\mathcal{O}(-n - 1 + k))))$. Finally if $f(x^0, x^1, \ldots, x^n)$ is a homogenous polynomial then $f(\xi) = f(\xi^0, \xi^1, \ldots, \xi^n)$ defines a section of the line bundle $\mathcal{O}(k)$ which vanishes precisely on $X$, namely $\{f = 0\}$.

Let us now see from what we have said in this section how the classical contour integral formula, for the Laplacian in three dimensions given in [18], arises. First, from [18], the general solution of the Laplacian in three dimensions is

\begin{equation}
\phi(x, y, z) = \int_{-\pi}^{\pi} f(z + ix \cos u + iy \sin u, u) du
\end{equation}

for an arbitrary real analytic function $f$. In three dimensions the manifold $X$ is the quadric $Q$ which is the vanishing set in $\mathbb{C}P_2$ of $z_0^2 + z_1^2 + z_2^2$. We have $\mathbb{C}P_2 \supseteq Q \simeq \mathbb{C}P_1$. From equation (4.21) the general solution is

\begin{equation}
\phi(x, y, z) = \int_0^\infty \sum_{0}^{\infty} f_p(\xi)(x\xi_1 + y\xi_2 + z\xi_3)^p d\xi.
\end{equation}

Indeed from (4.21), and with the notations used before, $\sum_0^m \omega_p(\sum_0^n \xi_i x_i)^p$ is a $(0,1)$-form with values in $\mathcal{O}(-1)$. A crucial remark is that for any complex projective hypersurface $M = \{g = 0\}$, of degree $k$ of $\mathbb{C}P^n$, its canonical bundle with sheaf of sections given by the holomorphic one forms on $M$, is isomorphic to $\mathcal{O}(-n - 1 + k)$. Therefore in the case at hand $\mathcal{O}(-1)$ is the canonical bundle of $Q$, and $\sum_0^m \omega_p(\sum_0^n \xi_i x_i)^p$ is a $(1, 1)$-form on $Q$ (where we interpret $d\xi$ as a $(1,1)$-form on $Q$).

But $f(\xi_0, \xi_1, \xi_2) = 0$ precisely on $X$. Thus we can identify $(\xi_0, \xi_1, \xi_2)$ as a variable point (function of $p \in Q$) which lies on $Q$. The embedding of $\mathbb{C}$ in the quadric which will be used is

$u \mapsto [i(e^{2iu} + 1), (e^{2iu} - 1), 2e^{iu}], \quad u \in [-\pi, \pi]
$

where square brackets denote the point of $\mathbb{C}P_2$ which is the line through $(i(e^{2iu} + 1), (e^{2iu} - 1), 2e^{iu})$. So $(x\xi_1 + y\xi_2 + z\xi_3)$ becomes

$2e^{iu}(xi \cos u + yi \sin u + z).
$

Therefore we have

\begin{equation}
\phi(x, y, z) = \int_{-\pi}^{\pi} \sum_{p} f_p(u)2^p e^{ipu}(z + ix \cos u + iy \sin u)^p h(u) du
\end{equation}

\begin{equation}
= \int_{-\pi}^{\pi} f(z + ix \cos u + iy \sin u, u) du
\end{equation}

by change of variables and this recovers the result (4.23).
We now note that the formula in [18] for a solution of the wave equation is

\begin{equation}
\phi(x, y, z, t) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t + x \sin u \cos v + y \sin u \sin v + z \cos u, u, v) du dv.
\end{equation}

Again the hypersurface \( X \subseteq \mathbb{CP}_3 \) is the quadric \( z_0^2 + z_1^2 + z_2^2 - z_3^2 = 0 \) which is simply \( \mathbb{CP}_1 \times \mathbb{CP}_1 \), which accounts for the double contour integral in (4.26) and \( Z = O(1)|_X \). To obtain (4.26) one uses the 2-1 ramified map given by

\[(\xi, \xi') \mapsto [(1 + \xi^2)(-1 + \xi'^2), -i(-1 + \xi^2)(-1 + \xi'^2), 2i(1 + \xi^2)\xi', 4i\xi\xi'] .\]

Setting \( \xi = e^{iu}, \xi' = e^{iv} \) and using formula (4.21), we obtain

\begin{equation}
\phi(x, y, z, t) = \int_{\xi} \int_{\xi'} \sum_p f_p(\xi, \xi')(2i(1 + \xi^2)\xi' z + x(1 + \xi^2)(-1 + \xi'^2)

- i(-1 + \xi^2)(-1 + \xi'^2)y + 4i\xi\xi' t)^p d\xi d\xi'

= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_p f_p(u, v)(t + x \sin u \cos v + y \sin u \sin v + z \cos u)^p du dv

= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t + x \sin u \cos v + y \sin u \sin v + z \cos u, u, v) du dv.
\end{equation}

5. Correspondences

In this final appendix we follow [5], and [13], to collect the fundamental abstract machinery that provides the background for the treatment of the Penrose transform.

Suppose that \( X, Y, Z \) are complex manifolds and suppose given the diagram of mappings

\begin{equation}
\begin{array}{ccc}
Y & \mu \downarrow & X \\
\nu \uparrow & & \\
Z & \downarrow & \\
\end{array}
\end{equation}

where \( \mu \) and \( \nu \) are surjective holomorphic mappings of maximal rank (hence submersions), and the pair \((\mu, \nu)\) embeds \( Y \) as a submanifold of \( Z \times X \). Such a diagram of mappings with those properties is called a correspondence or double fibration. For \( z \in Z \), let \( \tilde{z} \) denote the submanifold \( \nu(\mu^{-1}(z)) \) of \( X \) and for \( x \in X \), let \( L_x \) denote the submanifold \( \mu(\nu^{-1}(x)) \) of \( Z \). The transforms with which we are concerned will start with data on \( Z \) and end up with solutions of differential equations on \( X \). The data in question will be analytic cohomology classes.

In all cases the procedure will be in two stages.

1. Pull-back to \( Y \). Roughly speaking the data on \( Z \) will be transferred to data on \( Y \) constant along the fibers of \( \mu \), this constancy being interpreted via differential equations along the fibers of \( \mu \). If suitable topological conditions to be explained later are satisfied, then this transfer will be an isomorphism.
(2) Push-down to $X$. The mapping $\nu$ will generally be taken to have compact fibers and this second stage of the transform will be effected by taking direct images of the data obtained from (1) and plugging into the Leray spectral sequence. In this way the differential equations along the fibers of $\mu$ manifest themselves on $X$ and an isomorphism is established.

As a simple example we can consider a form $\omega$ on $Z$ of degree equal to the dimension of the fibers of $\nu$ and suppose that the fibers of $\nu$ are compact. Then the pullback of $\omega$ to $Y$ is $\mu^*(\omega)$; this data when descended to $X$ gives a function, which comes from integrating $\mu^*(\omega)$ along the fibers of $\nu$.

5.1. **Topological considerations.** In this subsection we have collected together few properties from sheaf theory, necessary for the understanding and the illustration of twistor theory.

5.1.1. **The pullback process.** This subsection concerns the mapping $\mu : Y \to Z$. Suppose that $V$ is a homomorphic vector bundle on $Z$ with sheaf of holomorphic sections $\mathcal{O}(V)$. This bundle can be pulled back to $\mu^*V$ on $Y$, characterized by the fact that its sheaf of holomorphic sections is

$$\mathcal{O}(\mu^*V) = \mu^*\mathcal{O}(V) := \mu^{-1}\mathcal{O}(V) \otimes_{\mu^{-1}\mathcal{O}_Z} \mathcal{O}_Y.$$  

But conversely to recover $V$ from $\mu^*V$ extra structure is needed. In order to make this precise let us introduce the sheaf of relative 1-forms $\Omega^1_\mu$ on $Y$ by the exact sequence

$$0 \longrightarrow \mu^*\Omega^1_Z \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_\mu \longrightarrow 0$$

and $\Omega^p_\mu := \wedge^p \Omega^1_\mu$. There is an induced relative exterior differential on $Y$

$$d_{\mu} : \mathcal{O}_Y \to \Omega^1_\mu$$

$$\mathcal{O}_Y(U) \ni s \mapsto d_{\mu}|_U(s) \mod (\mu^*\Omega^1_Z)(U),$$

$U \subset Y$, open.

When $\mu$ is of maximal rank, we have the following long exact sequence

$$0 \longrightarrow \mu^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_Y \longrightarrow \Omega^1_\mu \longrightarrow \Omega^2_\mu \longrightarrow \ldots$$

with the obvious mappings. The kernel of the mapping $d_{\mu}$ is $\mu^{-1}\mathcal{O}_Z$. Indeed the stalk of $\Omega^1_\mu$ at the point $x$ is $\mathcal{O}_{\mathcal{Y},\mu}/\mathcal{O}_{\mathcal{Z},\mu(y)}$, the module of Kähler differentials for the $\mathcal{O}_{\mathcal{Z},\mu(y)}$-module $\mathcal{O}_{\mathcal{Y},\mu}$ and the kernel of the differential $d_{\mathcal{O}_{\mathcal{Y},\mu}/\mathcal{O}_{\mathcal{Z},\mu(y)}}$ is $\mathcal{O}_{\mathcal{Z},\mu(y)}$ the stalk of $\mu^{-1}\mathcal{O}_Z$ at the point $y \in Y$. The exactness of the sequence for the other maps follows from the usual Poincaré lemma.

For a holomorphic vector bundle $W$ on $Y$, a relative connection on $W$ (relative to $\mu$) is a homomorphism $\nabla$ of abelian sheaves

$$\nabla : W \to \Omega^1_\mu \otimes_{\mathcal{O}_Y} W$$

such that

$$\nabla(ge) = g\nabla(e) + d_{\mu}(g) \otimes e$$

where $g$ and $e$ are sections of $\mathcal{O}_Y$ and $W$ resp. over an open set of $Y$. The kernel of $\nabla$, noted $W^\nabla$, is the sheaf of germs of horizontal sections of $(W, \nabla)$.
A relative connection $\nabla$ may be extended to a homomorphism of abelian sheaves
$$\nabla_i : \Omega^i_\mu \otimes \mathcal{O}_Y W \to \Omega^{i+1}_\mu \otimes \mathcal{O}_Y W$$
by
$$\nabla_i(\omega \otimes e) = d_\mu(\omega) \otimes e + (-1)^i \omega \wedge \nabla(e)$$
where $\omega$ and $e$ are sections of $\Omega^i_\mu$ and $W$ resp. over an open set of $Y$, and where $\omega \wedge \nabla(e)$ denotes the image of $\omega \otimes \nabla(e)$ under the canonical map
$$\Omega^i_\mu \otimes \mathcal{O}_Y \to \Omega^{i+1}_\mu \otimes \mathcal{O}_Y$$
$$\omega \otimes \tau \otimes e \mapsto (\omega \wedge \tau) \otimes e.$$

The curvature $K = K(W, \nabla)$ of the connection is the $\mathcal{O}_Y$-linear map
$$K = \nabla_1 \circ \nabla : W \to \Omega^2_\mu \otimes \mathcal{O}_Y W.$$
One has
$$\nabla_{i+1} \circ \nabla_i(\omega \otimes e) = \omega \wedge K(e)$$
with $\omega$ and $e$ sections of $\Omega^i_\mu$ and $W$ resp. over an open set of $Y$. The relative connection $\nabla$ is called integrable or flat when $K = 0$. An integrable relative connection, thus gives rise to a complex (the de Rham complex of $(W, \nabla)$)
$$0 \to W \to \Omega^1_\mu \otimes \mathcal{O}_Y W \to \Omega^2_\mu \otimes \mathcal{O}_Y W \to \cdots$$
which we denote by $\Omega^\bullet_\mu \otimes W$ when the integrable connection is understood.

For $\mu$ as above, let $x \in X$ and assume further that the fibers $\mu^{-1}(x)$ for $x \in X$ are connected and simply connected. Consider the following commutative diagram of morphisms

$$\begin{array}{ccc}
\mu^{-1}(x) & \longrightarrow & Y \\
\mu_{\mu^{-1}(x)} & \downarrow & \mu \\
x & \longleftarrow & X
\end{array}$$

The pull-back vector bundle over $\mu^{-1}(x)$ together with its canonical relative connection is flat and moreover holomorphically trivial since $\mu^{-1}(x)$ is connected and simply-connected and the collection of fibers of such trivial bundles for varying $x \in X$ glue to give a holomorphic vector bundle over $X$. Besides if $V$ is a holomorphic vector bundle on $Z$, $\mu^*V$ can be given by transition functions constant along the fibers of $\mu$ and consequently annihilated by $d_\mu$ (the transition functions for $\mu^*V$ are the compositions with $\mu$ of those for $V$). Thus the value of $d_\mu$ is independent of the choice of a trivialization (two sections are related by a transition function). So $\mu^*V$ comes equipped with a flat relative connection which is just $d_\mu \otimes \text{Id}$. Thus we can conclude that

**Proposition 5.1.** Suppose $\mu : Y \to Z$ has connected and simply-connected fibers. Then there is a 1-1 correspondence between holomorphic vector bundles on $Z$ and holomorphic vector bundles on $Y$ with a flat relative connection.
Remark 5.2. Let $V$ be holomorphic vector bundle on a complex manifold $Z$, $(V, \nabla)$ a vector bundle with connection on $Z$ and $f : Y \to Z$ a holomorphic mapping. Then there is an induced connection on the vector bundle $f^*V$. First of all the map $f$ induces a mapping
\[ df : TY \to f^*(TZ) \]
which by taking the dual gives a map
\[ f^*\Omega^1_Z \to \Omega^1_Y \]
still denoted $df$. The connection $f^*\nabla$ on $f^*V$ is uniquely characterized by
\[ (f^*\nabla)(f^*s) = (df \otimes \text{Id}_{f^*V})(f^*(\nabla s)). \]
Here for a complex manifold $X$, $TX$ stands for its holomorphic tangent bundle. The map $df : TY \to f^*(TZ)$ is given by the well-known formula in differential geometry
\[ df(X)(g) = X_x(g \circ f) \]
where $x \in Y$, $X$ is a holomorphic vector field on an open set containing $x$ and $g$ is a germ of holomorphic function at $f(x)$. An analogous construction exists in the relative situation. Indeed consider a commutative diagram of morphisms of complex manifolds, which for simplicity are assumed to be holomorphic and such that $\pi$ and $\pi'$ are of maximal rank:

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Y' & \xrightarrow{h} & Y \\
\end{array}
\]

We suppose given a vector bundle with connection $(\mathcal{E}, \nabla)$ relative (to $\pi$) on $X$. We have
\[ \nabla : \mathcal{E} \to \Omega^1_{\pi} \otimes \mathcal{E} \]
which by pullback gives a map
\[ g^*(\mathcal{E}) \to g^*\Omega^1_{\pi} \otimes g^*(\mathcal{E}) = (g, h)^*\Omega^1_{\pi} \otimes g^*(\mathcal{E}) \]
because the construction depends on $h$. But given such a commutative diagram we have a canonical map
\[ \eta : (g, h)^*\Omega^1_{\pi} \to \Omega^1_{\pi'} \]
which by taking the tensor product with $\text{Id}_{g^*(\mathcal{E})}$ gives a mapping
\[ (g, h)^*\Omega^1_{\pi} \otimes g^*(\mathcal{E}) \to \Omega^1_{\pi'} \otimes g^*(\mathcal{E}). \]
By composition one gets a mapping
\[ g^*(\mathcal{E}) \to \Omega^1_{\pi'} \otimes g^*(\mathcal{E}) \]
which is by definition the pullback connection $(g^*(\mathcal{E}), (g, h)^*(\nabla))$. It is characterized uniquely by
\[ (g, h)^*(\nabla)(g^*(s)) = (\eta \otimes \text{Id}_{g^*(\mathcal{E})})(g^*(\nabla(s))). \]
5.1.2. Effect of a continuous map on sheaf Cohomology. We consider as before a holomorphic mapping \( \mu : Y \to Z \) surjective and of maximal rank. We remark that for any holomorphic vector bundle \( V \) on \( Z \) there is an induced map on cohomology

\[
\mu^{-1} : H^r(Z, \mathcal{O}(V)) \to H^r(Y, \mu^{-1} \mathcal{O}(V)).
\]

Let us recall that an abelian sheaf \( \mathcal{F} \) is called injective if for any monomorphism \( f : \mathcal{F} \to \mathcal{G} \) and any \( g : \mathcal{G} \to \mathcal{H} \), there exists \( h : \mathcal{G} \to \mathcal{I} \) such that \( g = hf \).

\[
\begin{array}{c}
0 \longrightarrow \mathcal{F} \overset{f}{\longrightarrow} \mathcal{G} \\
\downarrow{g} \hspace{0.5cm} h \downarrow \mathcal{G}
\end{array}
\]

**Theorem 5.3** ([11], p. 40). Any bounded below complex of sheaves \( \mathcal{F}^* \) admits a quasi-isomorphism (isomorphism in cohomology) into a bounded below complex of injective objects \( \mathcal{I}^* \) (an injective resolution of \( \mathcal{F} \)).

Assume quite generally that \( f \) is continuous map between two topological spaces \( X \) and \( Y \) and that \( \mathcal{G} \) is a sheaf of abelian groups on \( Y \). We shall introduce a natural map \( f^{-1} : H^r(Y, \mathcal{G}) \to H^r(X, f^{-1} \mathcal{G}) \). Choose an injective resolution \( \mathcal{G} \to \mathcal{I}^* \), that is an exact sequence

\[
0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}_2 \longrightarrow \ldots
\]

and an injective resolution \( f^{-1} \mathcal{I}^* \to \mathcal{I}^* \) on \( X \) (by the previous theorem). Since the functor \( f^{-1} \) is exact, this makes \( \mathcal{I}^* \) an injective resolution of \( f^{-1} \mathcal{G} \). By adjunction one has a map \( a : \mathcal{I} \to f_*f^{-1} \mathcal{G} \), for every sheaf \( \mathcal{G} \) of abelian groups on \( Y \). Therefore applying the global section functor on \( Y \), which is left exact, we have morphisms of complexes

\[
\Gamma(Y, \mathcal{I}^*) \longrightarrow \Gamma(Y, f_*f^{-1} \mathcal{I}^*) = \Gamma(X, f^{-1} \mathcal{I}^*) \longrightarrow \Gamma(X, \mathcal{I}^*).
\]

By passing to cohomology one gets the induced by \( f \) map on cohomology \( f^{-1} : H^r(Y, \mathcal{G}) \to H^r(X, f^{-1} \mathcal{G}) \).

Recall that given a complex manifold \( Z \) and a holomorphic vector bundle \( V \) on \( Z \) that one has the Dolbeault resolution

\[
0 \longrightarrow \mathcal{O}(V) \longrightarrow \xi^0(V)
\]

where \( \xi^{0,q}(V) \) is the sheaf of \( C^\infty \) \( (0, q) \)-forms with values on \( V \), \( q \in \mathbb{N} \). This resolution gives rise since the functor \( \mu^{-1} \) is exact to a corresponding resolution of \( \mu^{-1} \mathcal{O}(V) \)

\[
0 \longrightarrow \mu^{-1}\mathcal{O}(V) \longrightarrow \mu^{-1}\xi^0(V).
\]

If the resolution \( \mu^{-1}\xi^0(V) \) is suitably acyclic, namely

\[
H^q(Y, \mu^{-1}\xi^0(V)) = 0, \ 1 \leq q \leq r
\]

then the map \( \mu^{-1} : H^r(Z, \mathcal{O}(V)) \to H^r(Y, \mu^{-1}\mathcal{O}(V)) \) is an isomorphism. The proof of this assertion is based on a theorem of Buchdahl [3] which states that for any smooth vector bundle \( E \) on \( Z \) with sheaf of smooth sections \( \xi(E) \)

\[
H^q(Y, \mu^{-1}\xi(E)) = 0
\]

provided that \( H^q(\mu^{-1}(z), \mathbb{C}) = H^{q-1}(\mu^{-1}(z), \mathbb{C}) \), for all \( z \in Z \) (this last condition being replaced by the connectedness of \( \mu^{-1}(z) \), for all \( z \in Z \), when \( q = 1 \)).
Suppose that \( \mu \) has connected fibers and that \( H^q(\mu^{-1}(z), \mathbb{C}) = 0, 1 \leq q \leq r \) and all \( z \in Z \). Then \( \mu^{-1} : H^q(Z, \mathcal{O}(V)) \to H^q(Y, \mu^{-1}\mathcal{O}(V)) \) is an isomorphism for \( q = 0, 1, \ldots, r \) for any holomorphic vector bundle \( V \) on \( Z \).

5.1.3. Introduction to spectral sequences. In order to explain the proof of the Proposition 5.4 we give a brief introduction to spectral sequences. Let \( \mathcal{C} \) be the category of \( \mathcal{O}_X \)-modules (to fix ideas), where \( (X, \mathcal{O}_X) \) is fixed ringed space. Suppose that we have a complex \((\mathcal{K}^\bullet, d)\) of objects in \( \mathcal{C} \), together with a filtration \( F_p \mathcal{K}^\bullet = (F_p \mathcal{K})_{p \in \mathbb{Z}} \). This means that we have a non-increasing sequence of subcomplexes of \( \mathcal{K}^\bullet \) (given by monomorphisms of complexes):

\[
\mathcal{K}^\bullet \supseteq \ldots \supseteq F_p \mathcal{K}^\bullet \supseteq F_{p+1} \mathcal{K}^\bullet \supseteq \ldots
\]

The filtration induces a filtration on the cohomology given by

\[
F_p H^n(\mathcal{K}^\bullet) = \text{Im}(H^n(F_p \mathcal{K}^\bullet) \to H^n(\mathcal{K}^\bullet)).
\]

Instead of describing the cohomology sheaves \( H^n(\mathcal{K}^\bullet) \) we will only describe the graded pieces with respect to the above filtration, that is,

\[
\text{gr}_p H^n(\mathcal{K}^\bullet) = F_p H^n(\mathcal{K}^\bullet)/F_{p+1} H^n(\mathcal{K}^\bullet).
\]

The main idea is to describe these graded pieces by using a sequence of approximations built out the complexes \( F_p \mathcal{K}^\bullet \). This is achieved by the spectral sequence associated to the filtration on \( \mathcal{K}^\bullet \).

For every \( r \geq 0 \) and every \( p, q \in \mathbb{Z} \), we will define an object \( E_r^{p,q} \) in \( \mathcal{C} \). For each \( r \), the \((E_r^{p,q})_{p,q \in \mathbb{Z}}\) form the \( r \)th page of the spectral sequence.

For every \( r \in \mathbb{Z} \) and \( p, q \in \mathbb{Z} \), we put

\[
Z_r^{p,q} = \{ u \in F_p \mathcal{K}^{p+q} \mid d(u) \in F_{p+r} \mathcal{K}^{p+q+1}\}.
\]

Note that for \( r \leq 0 \), we have \( Z_r^{p,q} = F_p \mathcal{K}^{p+q} \). It is clear that we have

\[
Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2}) \subseteq Z_r^{p,q}
\]

and for \( r \geq 0 \), we put

\[
E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})}.
\]

It is instructive to look at the first two pages. For \( r = 0 \), we have

\[
E_0^{p,q} = \frac{F_p \mathcal{K}^{p+q}}{F_{p+1} \mathcal{K}^{p+q} + d(F_p \mathcal{K}^{p+q-1})}.
\]

Note that \( d \) induces morphisms \( d_0 : E_0^{p,q} \to E_0^{p,q+1} \) such that \( d_0 \circ d_0 = 0 \) and

\[
E_1^{p,q} = \frac{\ker(E_0^{p,q} \to E_0^{p,q+1})}{\text{Im}(E_0^{p,q-1} \to E_0^{p,q})}.
\]

A similar picture holds also for higher \( r \). For every \( r \geq 0 \), \( d \) induces a map \( d_r \) of bidegree \((r, 1-r)\), that is

\[
d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}
\]

for every \( p, q \). Indeed it is clear that \( d(Z_r^{p,q}) \subseteq Z_r^{p+r,q-r+1} \) and

\[
d(Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})) = d(Z_{r-1}^{p+1,q-1}),
\]
which implies the claim. Since \( d \circ d = 0 \), it follows that \( d_r \circ d_r = 0 \). The sequence \(((Z^{p,q}_r),d_r)_{r \geq 0}\) is the spectral sequence associated to the given filtered complex.

**Proposition 5.5.** With the above notation, for every \( r \geq 0 \), we have \( E_{r+1} = H(E_r,d_r) \), in the sense that for every \( p, q \in \mathbb{Z} \), we have a canonical isomorphism

\[
E_{r+1}^{p,q} = \frac{\ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\text{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})}
\]

**Definition 5.6.** The filtration on \( \mathcal{K}^* \) is pointwise finite if for every \( n \in \mathbb{Z} \), we have \( F_p \mathcal{K}^n = \mathcal{K}^n \) for \( p \ll 0 \) and \( F_p \mathcal{K}^n = 0 \) for \( p \gg 0 \). Note that in this case a similar property holds (by definition) for the filtration on cohomology: for every \( n \in \mathbb{Z} \), we have \( F_p H^n(\mathcal{K}^*) = H^n(\mathcal{K}^*) \) for \( p \ll 0 \) and \( F_p H^n(\mathcal{K}^*) = 0 \) for \( p \gg 0 \).

**Proposition 5.7.** Given a pointwise finite filtration \((F_p \mathcal{K}^*)_{p \in \mathbb{Z}}\) on the complex \( \mathcal{K}^* \), for every \( p, q \), there is \( r(p,q) \) such that \( E^{p,q}_r = E^{p,q}_{r+1} \) for all \( r \geq r(p,q) \). If we denote this stable value by \( E^{p,q}_\infty \), then for every \( p \) and \( q \) we have a canonical isomorphism

\[
E^{p,q}_\infty \simeq gr_p H^{p+q}(\mathcal{K}^*)
\]

where

\[
E^{p,q}_\infty = \frac{\{ u \in F_p \mathcal{K}^{p+q} | d(u) = 0 \}}{\{ u \in F_{p+1} \mathcal{K}^{p+q} | d(u) = 0 \} + \{ F_p \mathcal{K}^{p+q} \cap d(\mathcal{K}^{p+q-1}) \}}.
\]

We will refer to the conclusion of the above proposition by saying that the spectral sequence converges or abuts to \( H^{p+q}(\mathcal{K}^*) \), and this is written

\[
E_r^{p,q} \Longrightarrow_p H^{p+q}(\mathcal{K}^*).
\]

**Definition 5.8.** Suppose that the filtration on \( \mathcal{K}^* \) is pointwise finite. We say that the spectral sequence \((E_r^{p,q})_{r \geq 0}\) degenerates at level \( r_0 \geq 0 \) if \( d_r = 0 \) for all \( r \geq r_0 \) and all \( p, q \). In this case we have \( E_r^{p,q} = E_{r+1}^{p,q} \) for all \( r \geq r_0 \), by definition of \( E_r^{p,q} \). It follows then from the Proposition 5.7 that

\[
E_r^{p,q} \simeq gr_p H^{p+q}(\mathcal{K}^*), p, q \in \mathbb{Z}.
\]

**Remark 5.9.** There are two important cases in which the spectral sequence degenerates at level \( r_0 \). Suppose for example, that \( r_0 \geq 1 \) and there is \( a \in \mathbb{Z} \) such that \( E_{r_0}^{p,q} = 0 \) unless \( p = a \). This clearly implies, using Proposition 5.5 that \( E_r^{p,q} = 0 \) for \( r \geq r_0 \) if \( p \neq a \). Since \( d_r \) has bidegree \((r,1-r)\) we see that \( d_r = 0 \) for all \( r \geq r_0 \). Therefore from the Definition 5.8 the spectral sequence degenerates at level \( r_0 \). Moreover in this case one shows that \( gr_p H^{p+q}(\mathcal{K}^*) = 0 \), unless \( p = a \). From this one may conclude that

\[
H^n(\mathcal{K}^*) \simeq E_{r_0}^{n,a-a}, n \in \mathbb{Z}.
\]

Likewise if \( r_0 \geq 2 \) and there is \( b \in \mathbb{Z} \) such that \( E_{r_0}^{p,q} = 0 \), unless \( q = b \). Again we see that \( d_r = 0, r \geq r_0 \), and we have

\[
H^n(\mathcal{K}^*) \simeq E_{r_0}^{n-b,b}, n \in \mathbb{Z}.
\]
5.1.4. The spectral sequence of a double complex.

Definition 5.10. A double complex $A^{\bullet, \bullet}$ in $\mathcal{C}$ is given by a family $(A^{p,q})_{p,q \in \mathbb{Z}}$ of objects in $\mathcal{C}$, together with morphisms $d_1^{p,q}: A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q}: A^{p,q} \to A^{p,q+1}$, such that the following conditions hold

\[
d_1d_1 = 0, \quad d_2d_2 = 0 \quad \text{and} \quad d_1d_2 = d_2d_1.
\]

Note that, in this case, for every $p$, we have a complex $(A^{p,\bullet}, d_1)$ and similarly for every $q$, we have a complex $(A^{\bullet,q}, d_2)$. The total complex of a double complex $A^{\bullet, \bullet}$ is the complex $\mathcal{K}^{\bullet} = \text{Tot}(A^{\bullet, \bullet})$, $\mathcal{K}^{n} = \oplus_{i+j=n} A^{i,j}$ and with map $d : \mathcal{K}^{n} \to \mathcal{K}^{n+1}$ defined on $A^{i,j}$ by

\[
d_1^{i,j} + (-1)^i d_2^{i,j}.
\]

This makes $\mathcal{K}^{\bullet}$ into a complex. Indeed, let $x^{i,j} \in A^{i,j}$ then

\[
d \circ d(x^{i,j}) = (d_1^{i+1,j} + (-1)^{i+1} d_2^{i+1,j})(d_1^{j}(x^{i,j})) + (d_1^{i,j+1} + (-1)^{i} d_2^{i,j+1})(d_2^{j}(x^{i,j})) = 0.
\]

The complex $\mathcal{K}^{\bullet}$ admits two natural filtrations, as follows. The first filtration is given by

\[
F_p^{\mathcal{K}^{n}} = \oplus_{i \geq p} A^{i,n-i};
\]

note that indeed, we have $d(F_p^{\mathcal{K}^{n}}) \subseteq F_p^{\mathcal{K}^{n}+1}$. This follows easily from the remarks and the fact that if $i \geq p$, so does $i+1$ and that $n-i = n+1 - (i+1)$.

The second filtration is given by

\[
F_q^{\mathcal{K}^{n}} = \oplus_{j \geq q} A^{n-j,j};
\]

again, we have $d(F_q^{\mathcal{K}^{n}}) \subseteq F_q^{\mathcal{K}^{n}+1}$. Let us compute the first terms of the spectral sequences associated to the filtrations. We do the computations for the first filtration, the other follows by symmetry. We denote the terms of this spectral sequence by $\ 'E_1^{p,q}$. By the general procedure of computation of the 0-th page, we see that

\[
\ 'E_1^{0,q} = F_p^{\mathcal{K}^{p+q}}/F_{p+1}^{\mathcal{K}^{p+q}} = A^{p,q}, \quad p,q \in \mathbb{Z}.
\]

Moreover the induced map by $d$ on $F_p^{\mathcal{K}^{p+q}}/F_{p+1}^{\mathcal{K}^{p+q}}$ is equal to the map induced by $(-1)^p d_2^{p,q}$. Hence

\[
\ 'E_1^{p,q} = H_{d_2}^q(A^{p,\bullet}).
\]

Note that for every $p$, the map $d_1$ induces a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$, and thus for every $q$, it induces a morphism in cohomology $H^q(A^{p,\bullet}) \to H^q(A^{p+1,\bullet})$. One shows that this map gets identified with the morphism $\ 'E_1^{p,q} \to \ 'E_1^{p+1,q}$. Thus from Proposition 5.5 we deduce that

\[
\ 'E_2^{p,q} = H_{d_2}^p(H_d^q(A^{\bullet,\bullet}))
\]

where the right-hand side stands for the cohomology of the complex

\[
H_{d_1}^q(A^{\bullet,p-1}) \to H_{d_1}^q(A^{\bullet,p}) \to H_{d_1}^q(A^{\bullet,p+1})
\]

with the maps induced by $d_1$.

Similarly, for the other filtration, we obtain the first terms of the spectral sequence given by

\[
\ ''E_0^{p,q} = A^{q,p}, \quad \ ''E_1^{p,q} = H^q(A^{\bullet,p})
\]

\[
\ 'E_2^{p,q} = H_{d_2}^p(H_{d_1}^q(A^{\bullet,\bullet}))
\]
Definition 5.15. Let $H$ be a bounded below complex of sheaves on a topological space $X$. The hypercohomology is induced by the morphism of sheaves $d_2 : H^q_d(A^{p-1}\cdot) \to H^q_d(A^p\cdot) \to H^q_d(A^{p+1}\cdot)$ with the maps induced by $d_2$.

Remark 5.11. Often one is interested in the case when the double complex satisfies the following extra condition: for every $n$, there are only finitely many pairs $(p,q)$ with $i+j = n$ and with $A^{p,q} \neq 0$. In this case, both filtrations on $\text{Tot}(A^{\bullet\bullet})$ are pointwise finite, hence by Proposition 5.7, they converge.

Definition 5.12. A first-quadrant double complex $A^{\bullet\bullet}$ is a double complex such that $A^{p,q} = 0$ for $p < 0$ or $q < 0$.

5.1.5. The spectral sequence of a composition of functors. We consider two left exact functors $G : C_1 \to C_2$ and $F : C_2 \to C_3$, where $C_1$, $C_2$, $C_3$ are subcategories of the category $C$ of $\mathcal{O}_X$-modules, with $(X, \mathcal{O}_X)$ a fixed ringed space.

Theorem 5.13. With the above notation, suppose that for every injective object $\mathcal{I}$ in $C_1$, the object $G(\mathcal{I})$ in $C_2$ is $F$-acyclic. In this case, for every object $\mathcal{A}$ in $C_1$, we have a spectral sequence

$$E_2^{p,q} = R^pF(R^qG(\mathcal{A})) \Longrightarrow_p R^{p+q}(F \circ G)(\mathcal{A}),$$

where $R^qG(\mathcal{A})$ stands for the right derived functor of the left-exact functor $G$ applied to $\mathcal{A}$. This is called the Grothendieck spectral sequence.

5.1.6. Spectral sequence associated to a differential complex of sheaves. In this part we use [2, p. 20-21].

Proposition 5.14. Let $(\mathcal{K}^{\bullet}, d_{\infty})$ be a bounded below complex of sheaves (there is a $k \in \mathbb{Z}$ such that $\mathcal{K}^n = 0$ for all $n < k$) on a topological space $X$. There exists a double complex $(\mathcal{J}^{\bullet\bullet}, \delta, d)$, with $\mathcal{J}^{p,q} = 0$, for $p < 0$, and a morphism of complexes $u : \mathcal{K}^{\bullet} \to (\mathcal{J}^{p\bullet}, d)$ such that

1. For each $q \in \mathbb{Z}$, the complex of sheaves (with differential $\delta$) is an injective resolution of $\mathcal{J}^q$.
2. For each $q \in \mathbb{Z}$, the complex of sheaves $d(\mathcal{J}^{p,q-1}) \subseteq \mathcal{J}^{\bullet,q}$ is an injective resolution of $d_{\infty}(\mathcal{J}^{p,q-1})$.
3. For each $q \in \mathbb{Z}$, the complex of sheaves $\ker(d) \subseteq \mathcal{J}^{\bullet,q}$ is an injective resolution of $\ker(d_{\infty} : \mathcal{K}^{q-1} \to \mathcal{K}^q)$.
4. For each $q \in \mathbb{Z}$, the complex of sheaves $H^q_H(\mathcal{J}^{p\bullet})$ (horizontal cohomology) is an injective resolution of $H^q(\mathcal{K}^{\bullet})$.

Definition 5.15. Let $\mathcal{K}^{\bullet}$ be a bounded below complex of sheaves on a topological space $X$. The hypercohomology $H^p(X, \mathcal{K}^{\bullet})$ is the $p$-th cohomology of the total complex associated to the double complex $(\Gamma(X, \mathcal{J}^{p,q}))_{p,q \in \mathbb{Z}}$, where $\mathcal{J}^{p\bullet}$ is as in Proposition 5.14. The definition is independent of the choice of $\mathcal{J}^{p\bullet}$.

Proposition 5.16. Let $\mathcal{K}^{\bullet}$ be a bounded below complex of sheaves on a topological space $X$. Then there is a convergent spectral sequence with $E_1^{p,q} = H^q(X, \mathcal{K}^p)$ and $E_\infty^{p,q} = \text{gr}_F^p(H^{p+q}(X, \mathcal{K}^{\bullet}))$ for some filtration $F$. The differential $d_1 : H^q(X, \mathcal{K}^p) \to H^q(X, \mathcal{K}^{p+1})$ is induced by the morphism of sheaves $d_{\infty} : \mathcal{K}^p \to \mathcal{K}^{p+1}$. We have $E_2^{p,q} = H^q_{d_1}(H^q(X, \mathcal{K}^{\bullet}))$. 

Proposition 5.17. For any bounded below complex of sheaves $\mathcal{K}^\bullet$ on a topological space $X$, there is a convergent spectral sequence $\lim^p E_2^{p,q} = H^p(X, H^q(\mathcal{K}^\bullet)) \Longrightarrow_p \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet)$ for some filtration $F$ and where $H^q(\mathcal{K}^\bullet)$ is the sheaf associated to the presheaf

$$U \mapsto \frac{\ker(d^1_{\mathcal{K}}|_U)}{\text{Im}(d^2_{\mathcal{K}}|_U)}.$$ 

To make clear the ideas let us see some examples. We suppose that $\mathcal{K}^\bullet$ is a resolution of the sheaf $\mathcal{S}$:

$$0 \to \mathcal{S} \to \mathcal{K}^\bullet$$

is exact. Then $H^q(\mathcal{K}^\bullet) = 0$ when $q > 0$ and $H^0(\mathcal{K}^\bullet) = \mathcal{S}$.

Thus $\lim^1 E_2^{0,0} = H^0(X, \mathcal{S})$ and $\lim^1 E_2^{0, q} = 0$, $q > 0$. Hence from Remark 5.9 we have

$$\mathbb{H}^p(X, \mathcal{K}^\bullet) = \lim^1 E_2^{0,0} = H^p(X, \mathcal{S}).$$

Similarly, suppose now that instead of being a resolution, the differential sheaf $\mathcal{K}^\bullet$ is acyclic for the global section functor:

$$H^q(X, \mathcal{K}^p) = 0, \quad q > 0, \quad p \geq 0.$$ 

This will be the case for instance if $\mathcal{K}^\bullet$ consists of fine sheaves. Then we see that $\lim^\infty E_2^{p,q} = 0$ for $q > 0$ and $\lim^\infty E_2^{p,0} = H^p(\Gamma(X, \mathcal{K}^\bullet))$. From this one sees, using again Remark 5.9, that

$$\mathbb{H}^p(X, \mathcal{K}^\bullet) = \lim^\infty E_2^{p,0} = H^p(\Gamma(X, \mathcal{K}^\bullet)).$$

If the differential sheaf $\mathcal{K}^\bullet$ is both a resolution and acyclic we therefore have

$$H^p(X, \mathcal{S}) = H^p(\Gamma(X, \mathcal{K}^\bullet)).$$

Let us now summarize the two principal conclusions which we can draw from the above discussions. If $0 \to \mathcal{S} \to \mathcal{K}^\bullet$ is an acyclic differential sheaf then we have a spectral sequence

$$\lim^1 E_2^{p,q} = \lim^1 E_2^{0,0} = H^p(X, H^q(\mathcal{K}^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet) = H^{p+q}(\Gamma(X, \mathcal{K}^\bullet)).$$

On the other hand if the differential sheaf $\mathcal{K}^\bullet$ is a resolution which is not necessarily acyclic, then we have a spectral sequence

$$\lim^p E_2^{p,q} = \lim^p E_2^{0,0} = H^p(X, H^q(\mathcal{K}^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(X, \mathcal{K}^\bullet) = H^{p+q}(X, \mathcal{S}).$$

Proof of the Proposition 5.4. By hypothesis $\mu$ has connected fibers, hence $H^0(Z, \mathcal{O}(V)) = H^0(Y, \mu^{-1}\mathcal{O}(V))$, where $V$ is a holomorphic vector bundle on $Z$. Also we have the Dolbeault resolution

$$0 \to \mathcal{O}(V) \to \xi^0(\mathcal{V}).$$

By exactness of the functor $\mu^{-1}$, we have a corresponding resolution

$$0 \to \mu^{-1}\mathcal{O}(V) \to \mu^{-1}\xi^0(\mathcal{V}).$$

The hypotheses imply by Buchdahl's theorem [3, prop. 1] that

$$H^q(Y, \mu^{-1}\xi^0(\mathcal{V})) = 0, \quad q = 1, \ldots, r.$$ 

The spectral sequence of the resolution $\mu^{-1}\xi^0(\mathcal{V})$ of $\mu^{-1}\mathcal{O}(V)$ has the form

$$E_2^{p,q} = H^p_{d_1}(H^q(Y, \mu^{-1}\xi^0(\mathcal{V}))) \Longrightarrow \mathbb{H}^{p+q}(Y, \mu^{-1}\mathcal{O}(V)).$$
Now remark that the spectral sequence arising from a resolution of a sheaf is a first quadrant spectral sequence and that because $H^q(Y, \mu^{-1}\xi^0(\bullet)(V))) = 0$, $q = 1, \ldots, r$ we have $E^{p,q}_2 = 0$, $0 < q < r + 1$.

It then follows from [4, p. 325] that

$$E_2^{p,0} = H^p(\Gamma(Y, \mu^{-1}\xi^0(\bullet)(V))) \simeq H^p(Y, \mu^{-1}\mathcal{O}(V)), p \leq r$$

and that there is a monomorphism

$$E_2^{p,0} = H^p(\Gamma(Y, \mu^{-1}\xi^0(\bullet)(V))) \hookrightarrow H^p(Y, \mu^{-1}\mathcal{O}(V)).$$

Now $H^p(Z, \mathcal{O}(V)) \simeq H^p(\Gamma(Z, \xi^0(\bullet)(V)))$ because the resolution $\xi^0(\bullet)(V)$ of $\mathcal{O}(V)$ is acyclic. Also as $\mu$ has connected fibers one has the isomorphism

$$H^p(\Gamma(Y, \xi^0(\bullet)(V))) \simeq H^p(\Gamma(Y, \mu^{-1}\xi^0(\bullet)(V))).$$

Putting all these facts together we get the conclusion of the proposition. □

5.2. The pushforward process. This subsection concerns the mapping $\nu$. Recall that one has the following exact sequence where $\mu$ satisfies the hypothesis given above:

$$0 \longrightarrow \mu^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_{\nu Y} \longrightarrow \Omega^1_{\mu} \longrightarrow \Omega^2_{\mu} \longrightarrow \ldots$$

Since $\mathcal{O}(\mu^*V)$ is the sheaf of sections of the holomorphic vector bundle $\mu^*(V)$, it is locally free, hence flat over $\mathcal{O}_{\nu Y}$. Therefore the sequence

$$0 \longrightarrow \mu^{-1}\mathcal{O}_Z \otimes_{\mathcal{O}_{\nu Y}} \mu^*(V) \longrightarrow \Omega^1_{\mu} \otimes_{\mathcal{O}_{\nu Y}} \mu^*(V) \longrightarrow \Omega^2_{\mu} \longrightarrow \ldots$$

i.e.

$$0 \longrightarrow \mu^{-1}\mathcal{O}(V) \longrightarrow \Omega^0_{\mu}(V) \longrightarrow \Omega^1_{\mu}(V) \longrightarrow \Omega^2_{\mu}(V) \longrightarrow \ldots \longrightarrow \Omega^p_{\mu}(V) \longrightarrow \ldots$$

is exact, and where $\nabla_{\mu} = d_{\mu} \otimes \text{Id}_{\mu^*(V)}$ and $\Omega_{\mu}(V) = \Omega^0_{\mu} \otimes \mu^*(V)$.

Let us introduce now the Leray spectral sequence. Let $f : Y \rightarrow X$ be a continuous map of topological spaces, which in particular gives a functor $f_*$ from sheaves of abelian groups on $Y$, $\text{Sh}(Y)$ to sheaves of abelian groups on $X$, $\text{Sh}(X)$:

$$\text{Sh}(Y) \xrightarrow{f_*} \text{Sh}(X) \xrightarrow{\Gamma(X, \bullet)} \text{Ab}$$

where $\text{Ab}$ is the category of abelian groups. Composing $f_*$ with taking global sections of sheaves in $\text{Sh}(X)$ is the same as taking the global sections of the initial sheaves on $Y$, by definition of the direct image functor. Recall that for $f$ as above and $\mathcal{I}$ an injective sheaf
in \( Sh(Y) \), \( f_* (\mathcal{J}) \) is also injective, hence \( \Gamma(X, \mathbb{1}) \) acyclic. This follows from the adjunction isomorphism
\[
\text{Hom}(f^{-1} \mathcal{G}, \mathcal{J}) \simeq \text{Hom}(\mathcal{G}, f_* \mathcal{J});
\]
because \( \mathcal{J} \) is injective, the functor
\[
\mathcal{G} \ni Sh(X) \to \text{Hom}(f^{-1} \mathcal{G}, \mathcal{I})
\]
is also injective, hence \( \Gamma(X, \mathbb{1}) \) acyclic. This follows from the adjunction isomorphism
\[
\text{Hom}(f^{-1} \mathcal{G}, \mathcal{J}) \simeq \text{Hom}(\mathcal{G}, f_* \mathcal{J});
\]
because \( \mathcal{I} \) is injective, the functor
\[
\mathcal{G} \ni \text{Sh}(X) \to \text{Hom}(f^{-1} \mathcal{G}, \mathcal{I})
\]
is exact. Hence \( f_* (\mathcal{J}) \) is injective by definition.

Therefore we can build the Grothendieck spectral sequence for the composition \( \Gamma(X, \mathbb{1}) \circ f_* = \Gamma(Y, \mathbb{1}) \), which is called the Leray spectral sequence: for every sheaf \( \mathcal{F} \in Sh(Y) \) one has a spectral sequence whose \( E_2 \) term is given by
\[
E_{p,q}^2 = H^p(X, R^q f_* (\mathcal{F})) \implies H^{p+q}(Y, \mathcal{F}).
\]
We return now to our situation with \( f = \nu : Y \to X \) holomorphic and proper and \( \mathcal{F} = \Omega^s \mu(\mathcal{V}) \). By Grauert's direct image theorem, the higher direct images of a coherent sheaf under a proper holomorphic map are coherent (an analytic sheaf \( \mathcal{S} \) on a complex manifold \( X \) is called coherent if every \( x \in X \) has an open neighborhood \( U \) for which there is an exact sequence
\[
\mathcal{O}_X^p|_U \to \mathcal{O}_X^q|_U \to \mathcal{S}|_U \to 0, \ p, q > 0).
\]
Assume further that \( X \) is a Stein manifold, then because \( R^q \nu_* (\Omega^s \mu) \) is coherent we have \( H^p(X, R^q \nu_* (\Omega^s \mu)) = 0 \), for \( p \geq 1 \). Thus the Leray spectral sequence degenerates at \( E_2 \) and gives (see Remark 5.9)
\[
H^q(Y, \Omega^s \mu(V)) \simeq E_{0,q}^2 = \Gamma(X, R^q \nu_* (\Omega^s \mu))).
\]
Setting \( s = p \) we get
\[
H^q(Y, \Omega^p \mu(V)) = \Gamma(X, R^q \nu_* (\Omega^p \mu))).
\]
Recall that we have the exact sequence
\[
0 \to \mu^{-1} \mathcal{O}(V) \to \Omega^0_\mu(V) \to \Omega^1_\mu(V) \to \Omega^2_\mu(V) \to \ldots
\]
From it one can build as previously explained a spectral sequence with \( E_1 \) term given by
\[
E_{1,p}^q = H^q(Y, \Omega^p_\mu(V))
\]
with differentials induced by \( \nabla_\mu \) and converging to \( H^{p+q}(Y, \mu^{-1} \mathcal{O}(V)) \). Combining these observations and the Remark 5.9 gives the

**Theorem 5.18.** Suppose that \( X \) is a Stein manifold and that for all \( z \in Z \), \( \hat{z} = \mu^{-1}(z) \) is connected and that \( H^s(\hat{z}, \mathbb{C}) = 0 \) for \( 1 \leq s \leq p + q \). Then there is a spectral sequence
\[
E_{1,p}^q = \Gamma(X, R^q \nu_* (\Omega^p_\mu(V)))
\]
with differential induced by \( \nabla_\mu \) and converging to \( H^{p+q}(Z, \mathcal{O}(V)) \) for any holomorphic vector bundle \( V \) on \( Z \).
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