Quantum gravity effects on Hořava-Lifshitz black hole

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Abstract: In this paper, we would like to obtain quantum gravity effects by using Hořava-Lifshitz black hole. We consider logarithmic corrected thermodynamics quantities and investigate the effects of logarithmic correction term. Logarithmic correction comes from thermal fluctuation and may be interpreted as quantum loop corrections. As black hole is a gravitational system, hence we can investigate quantum gravity effect. We find such effects on the black hole stability and obtain domain of correction coefficient.

Keywords: Hořava-Lifshitz black hole, Quantum gravity, Thermodynamics.
1 Overview and motivation

As we know, black holes are related to the usual thermodynamics laws and are objects with maximum entropy \[ 1 – 3 \]. In fact, the study of black hole thermodynamics is a subject of considerable physical importance in the development of quantum field theory in curved space-time. It is found that the black hole entropy \( S_0 \) depends on the black hole event horizon area \( A \) rather than the black hole volume \( V \) with initial formula,

\[
S_0 = \frac{A}{4} \quad (1.1)
\]

The black hole thermodynamics plays an important role in, for example, the very early universe. Black hole thermodynamics provides a real connection between gravity and quantum mechanics \([4]\). In this connection, there are several kinds of black holes have been studied and black holes with lower dimensions are really been interesting. For instance, 2D black holes with vanishing horizon area are discussed in Refs. \([5–9]\). Thermodynamics of different black holes with hyperscaling violation are studied in the Refs. \([10, 11]\). Recently, the thermodynamics, stability and Hawking-Page phase transition of the massive Banados, Teitelboim, and Zanelli (BTZ) black holes with energy dependent space-time are studied \([12]\). A full discussion on getting the thermodynamic and statistical mechanical properties of black holes is given in \([13]\). A brief review on black hole thermodynamics can be found in Ref. \([14]\), where the Unruh effect and Hawking radiation, showing how quantum fields on black hole backgrounds behave thermally are discussed. Here, a discussion of entropy from the Euclidean path-integral point of view and the understanding of black hole thermodynamics in AdS/CFT can also be found.

Despite knowing the fact that the black holes much larger than the Planck scale have entropy proportional to its horizon area, it is important to investigate the leading order corrections to entropy, as one reduces the size of the black hole. These corrections interpreted
as quantum effect, due to quantum fluctuations, in turn modify the holographic principle [15, 16]. Kaul and Majumdar [17] derived the lowest order corrections to the Bekenstein-Hawking entropy in a particular formulation of the quantum geometry program of Ashtekar et al and found that the leading correction is logarithmic, with

$$S = S_0 + \alpha \log A + \cdots$$  \hspace{1cm} (1.2)

where coefficient $\alpha$ depends on the details of the model and dots denotes higher order corrections. The leading order correction to the geometry of large AdS black holes in a four-dimensional Einstein gravity with negative cosmological constant is discussed [18] and found that the Hawking temperature grows without bound with increasing black hole mass. It may be noted that corrections to the thermodynamics of black holes can be studied using the non-perturbative quantum general relativity. It is also possible to study the corrected thermodynamics of a black hole with the help of matter fields in neighborhood of a black hole [19–21].

The study of logarithmic corrections in various contexts are subject of current interests [22–25]. For example, it has been studied in the contexts of Gödel black hole [23], Schwarzschild-Beltrami-de Sitter black hole [26] and massive black hole in AdS space [27]. The corrected thermodynamics of a dilatonic black hole has also been studied [28] to find the same universal form of correction term. The partition function of a black hole is used to study the corrected thermodynamics of a black hole [29]. The universality of the correction can be understood in the Jacobson formalism, where the Einstein equations are famous thermodynamics identities [30, 31]. As a result, the quantum correction to the structure of space-time would produce thermal fluctuations in the black holes thermodynamics. Such corrected thermodynamics has same universal shape as expected from the quantum gravitational effects [32–34]. Therefore, the equation (1.2) can be extended to the following forms,

$$S = S_0 + \alpha \log[S_0T_H^2] + \cdots = S_0 + \alpha \log[C_0T_H^2] + \cdots,$$  \hspace{1cm} (1.3)

where $C_0$ is uncorrected specific heat. In that case quantum corrected charged dilaton 2d black holes and BTZ black holes has been studied already [35–37]. Thermodynamics of a small singly spinning Kerr-AdS black hole under the effects of thermal fluctuations has been studied by the Ref. [38] and concluded that this form of corrections (logarithmic correction) becomes important when the size of the black hole is sufficient small. Such corrections may affect the critical behaviors of black object, for example in the Ref. [39] a dyonic charged AdS black hole (holographic dual of a van der Waals fluid) considered and logarithm-corrected thermodynamics investigated to show that holographic picture is still valid.

On the other hand, Horava-Lifshitz (HL) gravity theory, a new approach to deal the quantum gravity, is based on the idea of the breaking of the Lorentz invariance by equipping the space-time with additional geometric structure [40]. The HL gravity has been studied widely [41–44], and has been considered in several works of particle physics and cosmology [45, 46]. The static spherically symmetric black hole solutions in HL theory is investigated in Ref. [47]. The generalized second law of thermodynamics in Hořava-Lifshitz cosmology
investigated by [48]. The thermodynamics of HL black holes has been studied in [49–52], which has certain instabilities. The thermal fluctuation (quantum gravity effects) for HL black hole is still to explore. We would like this opportunity to explore the quantum gravity effects on the thermodynamics of HL gravity due to the first order correction.

In this paper, we consider HL black hole with Lu-Mei-Pop (LMP) solution and discuss the effects of thermal fluctuations to the entropy of a black hole which gets first-order (logarithmic) correction, and it is interpreted as quantum effects because these are important as the size of black hole reduces due to the Hawking radiation. We discuss these thermal fluctuations for the curvatures corresponding to spherical, flat and hyperbolic horizon. We find that the first order-corrected equations of states also satisfy the first-law of thermodynamics for all three curvatures. In order to see the effects of thermal fluctuations, we do comparative analysis of corrected thermal quantities with uncorrected ones. Here, in case of spherical space, we find that the pressure with negative correction coefficient shows opposite behavior when horizon radius tending to zero and pressure takes positive value for large value of radius horizon. However, in case of flat space, the pressure is negative for small black hole only when the higher positive values of correction parameter. In case of hyperbolic space, for both the cases of corrected and uncorrected one, the pressure is increasing function and takes positive value for finite size of black hole. However, the pressure is decreasing function for very small horizon radius in both the uncorrected and corrected with positive coefficients and corrected pressure becomes asymptotically negative when horizon radius tends to zero. The Helmholtz free energy for spherical space takes negative asymptotic value for negative correction parameter only when horizon radius tends to zero. There exists a critical radius of horizon for such black hole. With higher positive value of correction parameter the Helmholtz free energy falls faster till critical horizon radius and then falls bit slower. In flat space, the Helmholtz free energy for black hole is an increasing function always. In hyperbolic space, the behavior of the Helmholtz free energy is negative to that of pressure. In spherical space, there exists a critical radius for HL black hole and the large black hole without thermal fluctuations is in completely stable phase, however the small black hole is in completely unstable phase. For hyperbolic case, the situation is completely opposite to spherical space. However, in flat space, there exists only stable black holes. Unlike to flat and hyperbolic spaces, there is not any first-order correction for internal energy in spherical space. In spherical space, the Gibbs free energy takes positive value only for smaller black hole. On the other hand, in flat and hyperbolic spaces, the Gibbs free energy takes negative values only.

Rest of the paper is organized as follows. Next, in the section 2, we recapitulate the basics of HL black hole and their solutions. In section 3, we study the effects of thermal fluctuations on the thermodynamics and stability/instability of HL black hole with LMP solution. Here, the effects of thermal fluctuation are discussed for three types of space, namely, spherical space, flat space and hyperbolic space. Finally, in the last section, we summarize our results with brief discussion.
2 Hořava-Lifshitz Black Hole

Here, we first recapitulate the HL black holes and discuss the special solutions. We start by writing the four-dimensional gravity action of HL theory as follow \[ S_{HL} = \int d^4x \sqrt{gN} \left( \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3 \Lambda_W^2)}{8(1 - 3\lambda)} + \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 ight)^{\frac{1}{2}} - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} - \frac{\kappa^2 \mu^2}{2\omega^2} \epsilon^{ijk} R_i \nabla_j R_k + \frac{\kappa^2}{2\omega^4} C_{ij} C^{ij}, \] (2.1)

where \( \kappa^2, \lambda, \omega, \Lambda_W \) and \( \mu \) are constant parameters, \( C_{ij} = \epsilon^{ikl} \nabla_k (R^l_j - \frac{1}{4} R \delta^l_j) \) is cotton tensor, \( K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \) is extrinsic curvature written in terms of shift function \( N_i \) and lapse function \( N \). The cosmological constant \( \Lambda \) is related to constant parameter \( \Lambda_W \) as following \[ \Lambda = \frac{3}{2} \Lambda_W. \] (2.2)

In order to study the static and spherically symmetric solution, we assume the following metric ansatz \[ ds^2 = f(r) dt^2 - f^{-1}(r) dr^2 - r^2 d\Omega^2, \] (2.3)

where metric of a two-dimensional symmetric space is given by

\[
d\Omega^2 \equiv \begin{cases} 
  d\theta^2 + \sin^2 \theta d\phi^2 & (k = 1) \\
  d\theta^2 + \theta^2 d\phi^2 & (k = 0) \\
  d\theta^2 + \sinh^2 \theta d\phi^2 & (k = -1) 
\end{cases}
\]
(2.4)

and \( k \) is the curvature corresponding to spherical, flat or hyperbolic horizon, respectively.

The function \( f(r) \) has the following expression for \( \lambda = 1 \) case:

\[
f(r) = k + (\omega - \Lambda_W) r^2 - \sqrt{(r(\omega(\omega - 2\Lambda_W)r^2 + \beta))},
\]
(2.5)

here \( \beta \) refers to an integration constant. The solution (2.5) is obtained from the equations of motion of the action (2.1) with metric (2.3). There exist different solutions with different cases. For example, in the case of \( \Lambda_W = 0 \) and \( \beta = 4\omega M \), we have Kehagias-Sfetsos (KS) solution [54],

\[
f(r) = k + \omega r^2 - \omega r^2 \sqrt{1 + \frac{4M}{\omega r^2}},
\]
(2.6)

while in the case of \( \omega = 0 \) and \( \beta = -\frac{\lambda^2}{\Lambda_W^2} \), we have LMP solution [42, 47],

\[
f(r) = k - \Lambda_W r^2 - \gamma \sqrt{\frac{r}{-\Lambda_W}},
\]
(2.7)

which was given by Ref. [42] for any \( k \), and Ref. [47] for \( k = 1 \). Here, parameter \( \gamma (= aM) \) is related to the black hole mass \( M \).

The black hole thermodynamics in KS solution of HL Gravity have been studied by the Ref. [55], and thermodynamical quantities of LMP solution of HL black hole for three different cases of spherical, flat and hyperbolic spaces are discussed in Ref. [49]. Next, we analyze the effects of thermal fluctuations on the thermodynamics of the system.
3 Effects of thermal fluctuations

In this section, we discuss the effects of thermal fluctuation on the thermodynamics of the system. Due to the thermal fluctuations the entropy of a black hole gets logarithmic correction, and it is interpreted as quantum effects because these are important as the size of black hole reduces due to the Hawking radiation. It is also possible to relate the black hole microscopic degrees of freedom to a conformal field theory. In this case the modular invariance of the partition function can constrain the entropy of the HL black hole.

The partition function for the HL black hole considered as the statistical mechanics of $N$ particles with energy spectrum $E$ is given by [56, 57],

\[ Z = \int_0^\infty dE \rho(E)e^{-\beta E}, \quad (3.1) \]

where $\beta$ is inverse of Hawking temperature $T_H$ in the units of Boltzmann constant and $\rho(E)$ is the canonical density of the system with energy average $E$ defined as

\[ \rho(E) = \frac{1}{2\pi i} \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta e^{S(\beta)}. \quad (3.2) \]

Here entropy, $S = \beta E + \ln Z$, is measured around the equilibrium $\beta_0$, and all thermal fluctuations are neglected. However, it is possible to consider mentioned thermal fluctuations and expand $S(\beta)$ around the equilibrium temperature $\beta_0$ to the first order,

\[ S = S_0 + \frac{1}{2}(\beta - \beta_0)^2 \left( \frac{\partial^2 S(\beta)}{\partial \beta^2} \right)_{\beta=\beta_0}, \quad (3.3) \]

where $S_0$ is uncorrected entropy given by the equation (1.1) and the higher order corrections of the entropy neglected. Following the Ref. [32], one can write the corrected form of the entropy as

\[ S = S_0 + \alpha \ln(S_0T_H^2). \quad (3.4) \]

The case of $\alpha = -\frac{1}{2}$ gives the equation (1.3) hence it is generalization of the logarithmic corrected entropy.

The first-law of thermodynamics is given by

\[ dM = T_HdS + VdP, \quad (3.5) \]

where $M, T_H, V$ and $P$ denote the mass, Hawking temperature, volume and pressure of the HL black hole. The Hawking temperature for the black hole can be obtained from following relation:

\[ T_H = \frac{1}{4\pi} \left( \frac{\partial f}{\partial r} \right)_{r=r_h}, \quad (3.6) \]

where $r_h$ is the black hole horizon radius obtained from $f(r) = 0$. The zeroth-order entropy of the black hole is given by

\[ S_0 = \int \frac{1}{T_H} \frac{\partial H}{\partial r_h} dr_h, \quad (3.7) \]
where $H$ denotes the enthalpy and interpreted as the black hole mass ($H = M$) \[58\]. It is easy to check that $T = T_H$. Thermodynamical quantities of the HL black hole for LMP solution are different for spherical space ($k = 1$), flat space ($k = 0$), and hyperbolic space ($k = -1$).

Now, we can investigate of such correction on the thermodynamics quantities.

### 3.1 Spherical space

For the spherical space ($k = 1$), the black hole mass is given by,

$$M = a^{-1} \sqrt{-\frac{\Lambda_W}{r_h}} (1 - \Lambda_W r_h^2), \quad (3.8)$$

here condition $f(r)|_{r=r_h} = 0$ from (2.7) is utilized. Exploiting relation (3.6) together with (2.7) and (3.8) yields the expression for Hawking temperature,

$$T_H = \frac{1}{8\pi r_h} [-1 - 3\Lambda_W r_h^2]. \quad (3.9)$$

It is evident here that the black hole temperature depends on the cosmological constant.

With the help of relations (3.7), (3.8) and (3.9), the zeroth-order entropy of the black hole in spherical space is computed as

$$S_0 = \frac{8\pi}{a} \sqrt{-\Lambda_W r_h}. \quad (3.10)$$

Now, from relation (3.4), it is easy to compute the first-order corrected entropy due to thermal fluctuations as

$$S = \frac{8\pi}{a} \sqrt{-\Lambda_W r_h} + \alpha \log \left[ \frac{1}{8\pi ar_h} \sqrt{-\frac{\Lambda_W}{r_h}} (1 + 3\Lambda_W r_h^2)^2 \right]. \quad (3.11)$$

The expression for first-order corrected pressure for HL black hole ($P = \frac{1}{2} T_H S$ \[59, 60\]) is given by

$$P = -\frac{1}{2ar_h} \sqrt{-\Lambda_W r_h (1 + 3\Lambda_W r_h^2)}$$

$$- \frac{\alpha}{16\pi r_h} (1 + 3\Lambda_W r_h^2) \log \left[ \frac{1}{8\pi ar_h} \sqrt{-\frac{\Lambda_W}{r_h}} (1 + 3\Lambda_W r_h^2)^2 \right]. \quad (3.12)$$

For $\Lambda_W = -2$, this further reduces to

$$P = -\frac{1}{a\sqrt{2r_h}} (1 - 6r_h^2) + P_1(\alpha), \quad (3.13)$$

where,

$$P_1(\alpha) = -\frac{\alpha}{16\pi r_h} (1 - 6r_h^2) \log \left[ \frac{1}{8\pi ar_h} \sqrt{2} r_h (1 - 6r_h^2)^2 \right]. \quad (3.14)$$

In Fig. 1, we can see the effect of thermal fluctuation on the pressure of HL black hole

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Figure 1. Pressure in spherical space in terms of the black hole horizon for $a = 1$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.

in spherical space. We notice that the pressure is negative when horizon radius tends to smaller value for both without correction term and with positive value of correction parameter $\alpha$. However, we see that the negative value of $\alpha$ compensates the negative pressure and makes it positive even for the smaller horizon radius. We found that the pressure takes asymptotic values for point-like back hole. After a critical value, the pressure increases with horizon radius and increases more sharply with higher values of $\alpha$.

The corrected volume of black hole thus is given by,

$$V = \left( \frac{\partial H}{\partial P} \right)_S = \left( \frac{\partial M}{\partial r_h} \frac{\partial P}{\partial P} \right)_S,$$

$$= \frac{4\pi r^2 (3\Lambda r^2 + 1)}{\Lambda (3\Lambda r^2 - 1)} \left( -\frac{\Lambda}{r} \right)^{3/2} \left( 8\pi r \sqrt{-\frac{\Lambda}{r}} + a \alpha \log \left( \frac{\sqrt{-\frac{\Lambda}{r}} (3\Lambda r^2 + 1)^2}{8\pi ar} \right) \right)^{-1}.$$

This further simplifies to

$$V = \frac{1}{2} \left( 1 + 3\Lambda_W r_h^2 \right) - \frac{a \alpha}{16\pi \sqrt{-\Lambda_W} r_h} \left( 1 + 3\Lambda_W r_h^2 \right) \log \left( \frac{\sqrt{-\Lambda} (3\Lambda r^2 + 1)^2}{8\pi ar^{3/2}} \right).$$

The first-order corrected HL black hole Helmholtz free energy is derived as

$$F = -\int SdT_H,$$

$$= \frac{2}{a} \sqrt{-\frac{\Lambda_W}{r_h}} (1 + \Lambda_W r_h^2) - \frac{3a}{16\pi r_h} (1 + 5\Lambda_W r_h^2)$$

$$+ a \frac{(1 + 3\Lambda_W r_h^2)^2}{8\pi r_h} \log \left[ \frac{1}{8\pi ar_h} \sqrt{-\frac{\Lambda_W}{r_h}} (1 + 3\Lambda_W r_h^2)^2 \right].$$

Now, by setting $\Lambda_W = -2$, then the Helmholtz free energy reduces to

$$F = \frac{2}{a} \sqrt{\frac{2}{r_h}} (1 - 2r_h^2) + F_1(\alpha) + F_2(\alpha).$$
where

\[ F_1(\alpha) = -\frac{3\alpha}{16\pi r_h} (1 - 10r_h^2), \]
\[ F_2(\alpha) = \frac{\alpha}{8\pi r_h} \log \left[ \frac{1}{8\pi \alpha r_h} \sqrt{\frac{2}{r_h}} (1 - 6r_h^2)^2 \right]. \] (3.19)

In the plot 2, the change in behavior of the Helmholtz free energy due to the logarithmic correction is shown. We examined both positive and negative value of correction coefficient.

Figure 2. Helmholtz free energy in spherical space in terms of the black hole horizon for \( a = 1 \). Here, \( \alpha = 0 \) denoted by blue line, \( \alpha = -0.5 \) denoted by green line, \( \alpha = 3 \) denoted by red line, and \( \alpha = 10 \) denoted by black line.

In the case of negative \( \alpha \), the Helmholtz free energy takes negative asymptotic value when horizon radius tends to zero. However, uncorrected and corrected Helmholtz free energy with positive \( \alpha \) do not fall asymptotically near the vanishing small \( r_h \). In comparison to smaller \( \alpha \), the higher \( \alpha \) effects Helmholtz free energy differently and the positive span of Helmholtz free energy for higher \( \alpha \) is more. For the very large \( r_h \), there is no main differences in corrected and uncorrected energy. There exists a critical radius of horizon for which the Helmholtz free energy is constant irrespective to all different cases.

Now, in order to study the critical points and stability, we calculate specific heat as

\[ C = T_H \frac{\partial S}{\partial T_H} = -\frac{4\pi}{a} \sqrt{-\Lambda_W r_h} \left( \frac{1 + 3\Lambda_W r_h^2}{1 - 3\Lambda_W r_h^2} \right) + \frac{3\alpha}{2} \left( \frac{1 - 5\Lambda_W r_h^2}{1 + 3\Lambda_W r_h^2} \right). \] (3.20)

Fig. 3 discusses some important aspects about the stability/instability of HL black hole in spherical space. We find that there exists a critical radius \( r_{hc} \) for HL black hole and the large black hole \( (r_h > r_{hc}) \) without thermal fluctuations is found in completely stable phase, however the small black hole \( (r_h < r_{hc}) \) exists in completely unstable phase. Remarkably, we notice that there exists phase transition at critical radius due to the thermal fluctuations. Also, with the positive coefficient \( \alpha \), we can obtain some stable regions for the small HL black hole in spherical space. The correction with the positive \( \alpha \) also causes some instabilities for larger black hole as well.
Finally the internal energy is obtained as,

$$E = M - PV = a^{-1} \sqrt{\frac{2}{r_h}}(1 + 2r_h^2) + a^{-1} \frac{1}{4} \sqrt{\frac{2}{r_h}} \frac{(1 - 6r_h^2)^2}{(1 + 6r_h^2)},$$ \hspace{1cm} (3.21)

From the expression, it is evident that there is no first-order correction on internal energy in spherical space. However, still there is possibility of higher-order corrections on the internal energy. The Gibbs free energy using the relation

$$G = M - T_H S,$$

$$= \frac{2}{a} \sqrt{\frac{2}{r_h}}(1 - 2r_h^2) + \frac{\alpha}{8\pi r_h}(1 - 6r_h^2) \log \left[ \frac{1}{4\sqrt{2}\pi a r_h^{3/2}}(1 - 6r_h^2)^2 \right],$$ \hspace{1cm} (3.22)

where $\Lambda_W = -2$ is set. In Fig. 4, we see that both uncorrected and corrected Gibbs free energy take positive values for smaller black hole. However, corrected Gibbs free energy

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**Figure 3.** Specific heat in spherical space in terms of the black hole horizon for $\Lambda_W = -2$ and $a = 1$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.

**Figure 4.** Gibbs free energy in spherical space in terms of the black hole horizon for $a = 1$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.
with negative $\alpha$ takes negative asymptotic value for point like back holes. There exists two critical points and Gibbs free energy is positive in between there two points. However, for large sized black hole, there is not much effect of thermal fluctuation on Gibbs free energy and it becomes more negative with larger radius.

### 3.2 Flat space

In order to discuss the first-order corrected thermodynamics quantities for flat space $(k = 0)$, the LMP solution is given by

$$f(r) = -\Lambda_W r^2 - aM \sqrt{-\Lambda_W}, \quad (3.23)$$

which leads to following expression for mass:

$$M = \frac{1}{a}(-\Lambda_W r_h)^{\frac{3}{2}}. \quad (3.24)$$

Here we see that this is an increasing function of $r_h$. Now, utilizing relations $(3.6)$, $(3.23)$ and $(3.24)$, we are able to calculate the Hawking temperature

$$T_H = \frac{3}{8\pi} \Lambda_W r_h. \quad (3.25)$$

Here, we see that the magnitude of temperature of black hole increases linearly with the cosmological constant.

Now, the zeroth-order entropy for the black hole is calculated as

$$S_0 = \frac{8\pi}{a} \sqrt{-\Lambda_W r_h}, \quad (3.26)$$

where $(3.7)$, $(3.24)$ and $(3.25)$ are utilized. With the help of $(3.4)$, it matter of calculation only to write the first-order corrected entropy due to thermal fluctuations as

$$S = \frac{8\pi}{a} \sqrt{-\Lambda_W r_h} + \alpha \log \left[ \frac{9}{8\pi a}(-\Lambda_W r_h)^{5/2} \right]. \quad (3.27)$$

Following the procedure of above section, the first-order corrected pressure is given by

$$P = \frac{3}{2a}(-\Lambda_W r_h)^{3/2} - \frac{3\alpha}{16\pi} \Lambda_W r_h \log \left[ \frac{9}{8\pi a}(-\Lambda_W r_h)^{5/2} \right]. \quad (3.28)$$

From the plot of Fig. 5, we see that for smaller black holes in flat space the pressure becomes more negative when the value of correction parameter $\alpha$ takes large positive values. However, without correction or with correction with negative $\alpha$, the pressure is positive always. But after the critical value, pressure increases more sharply with size of black holes in case of positive $\alpha$.

With the help of expression for pressure, one can obtain the corrected volume in case of flat space as,

$$V = \frac{2}{3} - \frac{\alpha a}{36\pi \sqrt{-\Lambda_W r_h}} \left( 5 + \log \left[ \frac{9}{8\pi a}(-\Lambda_W r_h)^{5/2} \right] \right). \quad (3.29)$$
Figure 5. Pressure in flat space in terms of the black hole horizon for $a = 1$ and $\Lambda_W = -2$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.

In this case, the first-order corrected black hole Helmholtz free energy is obtained using the relations (3.25) and (3.27), as following:

$$F = \frac{2}{a}(-\Lambda_W r_h)^{3/2} - \frac{15\alpha}{16\pi} \Lambda_W r_h + \frac{3\alpha}{8\pi} \Lambda_W r_h \log \left[ \frac{9}{8\pi a}(-\Lambda_W r_h)^{5/2} \right].$$

It is clear that the uncorrected Helmholtz free energy is twice to mass $M$ (3.24). From Fig.

Figure 6. Helmholtz free energy in flat space in terms of the black hole horizon for $a = 1$ and $\Lambda_W = -2$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.

it is evident that the Helmholtz free energy for black hole in flat space is an increasing function with $r_h$. We observe that for smaller value of $\alpha$ the corrected Helmholtz free energy behaves more or less like uncorrected one. However, for rather higher value of $\alpha$, the Helmholtz free energy starts behaving differently and increases with lesser slope.

The heat capacity is an important parameter to study stability of the black hole. So, for flat space, we found the following expression:

$$C = \frac{4\pi}{a} \sqrt{-\Lambda_W r_h} + \frac{5}{2} \alpha,$$

(3.31)
We can see that uncorrected heat capacity \( C_0 = \frac{S_0}{2} \) and this justifies the corrected

\[
C = \frac{S}{2}.
\]

\( r_h \)

*Figure 7.* Heat capacity in flat space in terms of the black hole horizon for \( a = 1 \) and \( \Lambda_W = -2 \). Here, \( \alpha = 0 \) denoted by blue line, \( \alpha = -0.5 \) denoted by green line, \( \alpha = 3 \) denoted by red line, and \( \alpha = 10 \) denoted by black line.

relation (1.3). From the both expression (3.31) and Fig. 2.5, we see that the heat capacity in flat space is always positive with the negative cosmological constant. Therefore, there exists stable black holes only.

Now, the internal energy \( (E = M - PV) \) is obtained as follow,

\[
E = -\frac{5a}{24\pi} \Lambda_W r_h + \frac{\alpha}{12\pi} \Lambda_W r_h \log \left[ \frac{9}{8\pi a} (-\Lambda_W r_h)^{5/2} \right].
\]

(3.32)

Here we see that the internal energy for black hole depends completely on logarithmic correction. There is no zeroth-order internal energy for black hole in flat space. In Fig. 8,

*Figure 8.* Internal energy in flat space in terms of the black hole horizon for \( a = 1 \) and \( \Lambda_W = -2 \). Here, \( \alpha = -0.5 \) denoted by blue line, \( \alpha = 3 \) denoted by green line, \( \alpha = 10 \) denoted by red line.

for negative \( \alpha \), the internal energy is negative and positive for small black hole (smaller than critical radius) and large black hole (larger than critical radius), respectively. However, for positive \( \alpha \), the internal energy is positive and negative for small black hole and large black hole, respectively.
Finally, we derive the corrected Gibbs free energy for LMP black hole in flat space as,

$$G = -\frac{2}{a}(-\Lambda_W r_h)^{3/2} + \frac{3\alpha}{8\pi} \Lambda_W r_h \log \left[ \frac{9}{8\pi a}(-\Lambda_W r_h)^{5/2} \right]. \quad (3.33)$$

Here we observe that the zeroth-order Gibbs free energy is negative to zeroth-order black Helmholtz free energy. In Fig. 9, we see that the Gibbs free energy in flat space is negative in all cases. Although, the higher positive values make the Gibbs free energy more negative, the behavior of Gibbs free energy does not change on the correction parameter.

### 3.3 Hyperbolic space

In this subsection, we study the effects of quantum gravity effects on LMP black hole in hyperbolic space ($k = -1$). In this case, the LMP solution is given by

$$f(r) = -1 - \Lambda_W r^2 - a M \sqrt{-\frac{r}{\Lambda_W}}. \quad (3.34)$$

The black hole mass at horizon $f(r)|_{r=r_h} = 0$ is calculated by,

$$M = H = \frac{1}{a} \sqrt{-\frac{\Lambda_W}{r_h}} [1 - \Lambda_W r_h^2]. \quad (3.35)$$

It can be seen that the magnitude of the cosmological constant increases the black hole mass.

Exploiting relation (3.6), the Hawking temperature in this case has following expression:

$$T_H = \frac{1}{8\pi r_h} (1 - 3\Lambda_W r_h^2). \quad (3.36)$$

The expression (3.7), (3.35) and (3.36) induce the following zeroth-order entropy of the black hole in hyperbolic space:

$$S_0 = \frac{8\pi}{a} \sqrt{-\Lambda_W r_h}, \quad (3.37)$$
here we notice that the zeroth-order entropy for both the spherical and hyperbolic spaces is same. Utilizing relation (3.4), the first-order corrected entropy in hyperbolic space is given by

\[
S = \frac{8\pi}{a} \sqrt{-\Lambda W r_h} + \alpha \log \left[ \frac{1}{8\pi a r_h} \sqrt{-\Lambda W} (1 - 3\Lambda W r_h^2) \right]. \tag{3.38}
\]

It is clear that correction terms are different for the cases of the spherical and hyperbolic spaces. Now, one can find pressure as follow,

\[
P = \frac{1}{2} T_H S = \frac{1}{2 a r_h} \sqrt{-\Lambda W r_h (1 - 3\Lambda W r_h^2)} + \frac{\alpha}{16 \pi a r_h} (1 - 3\Lambda W r_h^2) \log \left[ \frac{1}{8\pi a r_h} \sqrt{-\Lambda W} (1 - 3\Lambda W r_h^2) \right], \tag{3.39}
\]

which is positive for all values of negative cosmological constant. In Fig. 10, we can see the
dayeffect of thermal fluctuation on the pressure of HL black hole in hyperbolic space. We find that for both the cases of corrected and uncorrected one, the pressure is increasing function and takes positive value for finite size of black hole. However, the pressure is decreasing function for the uncorrected and corrected with positive \(\alpha\) for vanishingly small \(r_h\). In case of negative \(\alpha\), the pressure becomes asymptotically negative for \(r_h\) tending to zero.

Once expression for pressure is know, it is easy to compute volume with the help of relation \(V = (\frac{\partial M}{\partial P})_S\) as

\[
V = 2 \left( \frac{1 - 3\Lambda W r_h^2}{1 - 9\Lambda W r_h^2} \right) + \frac{2\alpha}{3\pi \sqrt{-\Lambda W r_h}} \frac{(1 + 3\Lambda W r_h^2)(1 + 5\Lambda W r_h^2)}{(1 - 9\Lambda W r_h^2)^2} + \frac{4\alpha}{9\pi \sqrt{-\Lambda W r_h}} \left( \frac{1 + 3\Lambda W r_h^2}{1 - 9\Lambda W r_h^2} \right)^2 \log \left[ \frac{1}{8\pi a r_h} \sqrt{-\Lambda W} (1 - 3\Lambda W r_h^2) \right]. \tag{3.40}
\]

**Figure 10.** Pressure in hyperbolic space in terms of the black hole horizon for \(a = 1\) and \(\Lambda W = -2\). Here, \(\alpha = 0\) denoted by blue line, \(\alpha = -0.5\) denoted by green line, \(\alpha = 3\) denoted by red line, and \(\alpha = 10\) denoted by black line.
In the case of hyperbolic space \((k = -1)\), the black hole Helmholtz free energy obtained using the relation, \(F = -\int SdT_H\), as

\[
F = -\frac{2}{a} \sqrt{\frac{-\Lambda_W}{r}} (1 - \Lambda_W r_h^2) + F_1(\alpha) + F_2(\alpha),
\]

where

\[
F_1(\alpha) = \frac{3\alpha}{16\pi r_h} (1 - 5\Lambda_W^2),
\]
\[
F_2(\alpha) = -\frac{\alpha}{8\pi r_h} (1 - 3\Lambda_W r_h^2) \log \left[ \frac{1}{8\pi a r_h} \sqrt{\frac{-\Lambda_W}{r_h}} (1 - 3\Lambda_W r_h^2)^2 \right].
\]

If we set \(\Lambda_W = -2\), this reduces to

\[
F = -\frac{2\sqrt{2}}{a\sqrt{r_h}} (1 + 2r_h^2) + \frac{3\alpha(1 + 10r_h^2)}{16\pi r_h} - \frac{\alpha(1 + 6r_h^2)}{2\sqrt{2\pi a r_h^{3/2}}} (1 + 6r_h^2)^2).
\]

In Fig. 11, we notice that the Helmholtz free energy for finite sized HL black hole in hyperbolic space is negative. However, there exists a critical point after which the behavior of Helmholtz free energy changes. For point-like HL black hole, the corrected Helmholtz free energy for negative \(\alpha\) takes asymptotically positive value only. However, for the same size of black hole the corrected Helmholtz free energy takes negative value very fast analogous to uncorrected case.

The logarithmic corrected specific heat is given by

\[
C = -\frac{4\pi}{a} \sqrt{-\Lambda_W r_h} \left( \frac{1 - 3\Lambda_W r_h^2}{1 + 3\Lambda_W r_h^2} \right) + \frac{3\alpha}{2} \left( \frac{1 + 5\Lambda_W r_h^2}{1 + 3\Lambda_W r_h^2} \right).
\]

From the plot of Fig. 12, we observe that in hyperbolic space the phase transition occurs for both corrected and uncorrected cases of specific heat. The phase transition occurs at
critical radius ($r_{hc}$). Interestingly, we find that the large HL black hole ($r_h > r_{hc}$) is in stable phase, while small black hole ($r_h < r_{hc}$) is completely at unstable phase. In absence of quantum effects, there are some instabilities for the small radius. Moreover, in presence of thermal fluctuations with positive parameter, the black hole can have stable regions for the small HL black hole also.

Now, we calculate the first-order corrected internal energy as follow,

$$E = -\frac{2}{a} \sqrt{\frac{-\Lambda_W}{r_h}} \left( \frac{1 - 7 \Lambda_W r_h^2}{1 - 9 \Lambda_W r_h^2} \right) + \frac{\alpha}{3 \pi r_h} \frac{(1 - 9 \Lambda_W r_h^4)(1 + 5 \Lambda_W r_h^2)}{(1 - 9 \Lambda_W r_h^2)^2}$$

$$+ \frac{\alpha (1 - 3 \Lambda_W r_h^2)(25 - 12 \Lambda_W r_h^2 + 387 \Lambda_W^2 r_h^4)}{72 \pi r_h(1 - 9 \Lambda_W r_h^2)^2} \log \left[ \frac{\sqrt{-\Lambda_W}}{8 \pi a r_h^{3/2}} (1 - 3 \Lambda_W r_h^2)^2 \right]. \quad (3.45)$$

Due to thermal fluctuation, the internal energy shows different behavior for small black

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**Figure 12.** Specific heat in hyperbolic space in terms of the black hole horizon for $a = 1$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.

**Figure 13.** Internal energy in hyperbolic space in terms of the black hole horizon for $a = 1$. Here, $\alpha = 0$ denoted by blue line, $\alpha = -0.5$ denoted by green line, $\alpha = 3$ denoted by red line, and $\alpha = 10$ denoted by black line.
of positive correction coefficient. However, the corrected internal energy with negative correction coefficient follows similar behavior to that of without correction.

Finally, the corrected Gibbs free energy for hyperbolic space is calculated by

\[
G = -\frac{2}{a} \sqrt{-\Lambda_W r_h} (1 - \Lambda_W r_h^2) - \frac{\alpha}{8\pi r_h} (1 - 3\Lambda_W r_h^2) \log \left[ \frac{\Lambda_W}{8\pi a r_h^{3/2}} (1 - 3\Lambda_W r_h^2)^2 \right].
\]  

(3.46)

For fixed value of \(\Lambda_W = -2\), this turns to following:

\[
G = -\frac{2}{a} \sqrt{2} r_h (1 + 2r_h^2) - \frac{\alpha}{8\pi r_h} (1 + 6r_h^2) \log \left[ \frac{1}{4\sqrt{2\pi a r_h^{3/2}}} (1 + 6r_h^2)^2 \right].
\]  

(3.47)

In Fig. 14, we see that there exists two critical radius for the Gibbs free energy of black hole with finite size in hyperbolic space. The corrected Gibbs free energy takes positive value (asymptotically) only with negative correction coefficient. However, for small black hole, the corrected Gibbs free energy with positive correction coefficients show similar behavior to that of uncorrected case.

4 Concluding remarks

In this paper, we considered the LMP solution of HL black hole in flat, spherical and hyperbolic spaces and studied the effects of quantum correction on the black hole thermodynamics. Quantum corrections appear due to the statistical thermal fluctuations and have logarithmic shape. Here, we introduced the origin of logarithmic correction and finally discussed the effects of such correction on the thermodynamics quantities.

The first order-corrected equations of state are calculated so that they satisfy the first-law of thermodynamics for all kind of curvatures. The quantum effects on thermodynamical quantities is shown in the various plots above. For example, in case of spherical space, the pressure is found negative when horizon radius is small for both without correction and with

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{Gibbs free energy in hyperbolic space in terms of the black hole horizon for \(a = 1\). Here, \(\alpha = 0\) denoted by blue line, \(\alpha = -0.5\) denoted by green line, \(\alpha = 3\) denoted by red line, and \(\alpha = 10\) denoted by black line.}
\end{figure}
positive correction coefficients. The negative value of correction coefficient compensates the negative pressure and makes it positive even for the smaller horizon radius. However, in case of flat space, the pressure with large positive coefficient is found negative for small black hole only. In case of hyperbolic space, the pressure is an increasing function and takes positive value for finite size of black hole. The corrected pressure is asymptotically negative when horizon radius tends to zero. We have observed that the Helmholtz free energy for spherical space takes negative asymptotic value for negative correction parameter only when horizon radius tends to zero. On the other hand, in flat space, the Helmholtz free energy is an increasing function always. The behavior of the Helmholtz free energy is found opposite to the pressure in case of hyperbolic space.

In spherical space, we have found the critical radius \( r_{hc} \) for HL black hole and have observed that the large black hole (with horizon radius \( r_h \) greater than critical radius) without thermal fluctuations is in completely stable phase, however the small black hole \( (r_h < r_{hc}) \) is in completely unstable phase. Remarkably, the thermal fluctuations causes a phase transition at critical radius. Even some stable phase exists for the small HL black hole with the positive coefficient in spherical space. However, the correction with the positive correction parameter is also responsible for some instabilities for larger black hole. In flat space, the black hole is always stable and thermal fluctuations do not affect the stability of black hole. Moreover, in hyperbolic space, the phase transition occurs for both with and without thermal fluctuations. The phase transition occurs at critical radius. Incidentally, we have found that the large HL black hole is in stable phase, while instabilities occur for small black holes. The thermal fluctuations with positive parameter removes instabilities of the small HL black hole also.

Apart from flat and hyperbolic cases, we have not found any correction for internal energy to the first-order in spherical space. However, the higher-order corrections may still be present. Further, we have studied quantum effects on the Gibbs free energy. We have found that in spherical space, the Gibbs free energy takes only positive value for the smaller black hole. On the other hand, in flat and hyperbolic spaces, the Gibbs free energy for black hole with finite horizon radius takes negative values only. In hyperbolic space, the first-order corrected Gibbs free energy with negative correction coefficient takes asymptotic positive value when the horizon radius tends to zero limit. However, for small black holes, the corrected Gibbs free energy with positive correction coefficients shows similar behavior to that of without thermal fluctuations.

References

[1] Simon F. Ross, “Black hole thermodynamics”, arXiv: hep-th/0502195.
[2] S. Vongehr, “Black hole thermodynamics in semi-classical and superstring theory”, arXiv: hep-th/9709172.
[3] John Preskill, “Do Black Holes Destroy Information?”, arXiv: hep-th/9209058.
[4] D. N. Page, “Hawking Radiation and Black Hole Thermodynamics”, New J. Phys. 7, 203 (2005).
[5] J. Sadeghi, M. R. Setare and B. Pourhassan, “Two Dimensional Black Hole Entropy”, Eur. Phys. J. C 53, 95 (2008).
[6] M. Cadoni and P. Carta, “2D black holes, conformal vacua and CFTs on the cylinder”, Phys. Lett. B 522, 126 (2001).
[7] J. Sadeghi, M. R. Setare and B. Pourhassan, “Entropy Of Extremal Black Holes In Two Dimensions”, Acta Phys. Pol. B 40, 251 (2009).
[8] J. Sadeghi and B. Pourhassan, “Entropy Function in the Liouville Theory”, Int. J. Theor. Phys. 48, 3526 (2009).
[9] M. Cadoni and M. Cavaglia, “Two-dimensional black holes as open strings: a new realization of the ADS/CFT correspondence”, Phys. Lett. B 499, 315 (2001).
[10] J. Sadeghi, B. Pourhassan and F. Pourasadollah, “Thermodynamics of Schrodinger black holes with hyperscaling violation”, Phys. Lett. B 720, 244 (2013).
[11] J. Sadeghi, B. Pourhassan and A. Asadi, “Thermodynamics of string black hole with hyperscaling violation”, Eur. Phys. J. C 74, 2680 (2014).
[12] S. H. Hendi, S. Panahiyan, S. Upadhyay and B. E. Panah, “Charged BTZ black holes in the context of massive gravity’s rainbow”, Phys. Rev. D 95, 084036 (2017).
[13] S. Carlip, “Black Hole Thermodynamics”, Int. J. Mod. Phys. D23, 1430023 (2014).
[14] S. F. Ross, “Black hole thermodynamics”, arXiv:hep-th/0502195.
[15] D. Bak and S. J. Rey, “Holographic Principle and String Cosmology”, Class. Quant. Grav. 17, L1 (2000).
[16] S. K. Rama, “Holographic principle in the closed universe: a resolution with negative pressure matter”, Phys. Lett. B 457, 268 (1999).
[17] R. K. Kaul and P. Majumdar, “Logarithmic Correction to the Bekenstein-Hawking Entropy”, Phys. Rev. Lett. 84 (2000) 5255.
[18] S. Hemming and L. Thorlacius, “Thermodynamics of Large AdS Black Holes”, JHEP 0711, 086 (2007)
[19] R. B. Mann and S. N. Solodukhin, “Universality of quantum entropy for extreme black holes”, Nucl. Phys. B523, 293 (1998).
[20] A. J. M. Medved and G. Kunstatter, “Quantum corrections to the thermodynamics of charged 2D black holes”, Phys. Rev. D60, 104029 (1999).
[21] A. J. M. Medved and G. Kunstatter, “One-loop corrected thermodynamics of the extremal and nonextremal spinning Banados-Teitelboim-Zanelli black hole”, Phys. Rev. D63, 104005 (2001).
[22] S. Upadhyay, “Thermodynamics and galactic clustering with a modified gravitational potential”, Phys. Rev. D 95, 043008 (2017).
[23] A. Pourdarvish, J. Sadeghi, H. Farahani, and B. Pourhassan, Int. J. Theor. Phys. 52, 3560 (2013).
[24] B. Pourhassan and M. Faizal, “Thermal Fluctuations in a Charged AdS Black Hole”, Europhys. Lett. 111, 40006 (2015).
[25] M. Faizal and B. Pourhassan, “Corrections Terms for the Thermodynamics of a Black Saturn”, Phys. Lett. B 751, 487 (2015).
[26] B. Pourhassan, S. Upadhyay and H. Farahani, “Thermodynamics of Higher Order Entropy Corrected Schwarzschild-Beltrami-de Sitter Black Hole”, arXiv:1701.08650.

[27] S. Upadhyay, B. Pourhassan and H. Farahani, “P-V criticality of first-order entropy corrected AdS black holes in massive gravity”, arXiv:1704.01016.

[28] J. Jing and M. L Yan, “Statistical Entropy of a Stationary Dilaton Black Hole from Cardy Formula”, Phys. Rev. D63, 024003 (2001).

[29] D. Birmingham and S. Sen, “Exact black hole entropy bound in conformal field theory”, Phys. Rev. D63, 047501 (2001).

[30] T. Jacobson, “Thermodynamics of Spacetime: The Einstein Equation of State”, Phys. Rev. Lett. 75, 1260 (1995).

[31] R. G. Cai and S. P. Kim, “First Law of Thermodynamics and Friedmann Equations of Friedmann-Robertson-Walker Universe”, JHEP 0502, 050 (2005).

[32] S. Das, P. Majumdar and R. K. Bhaduri, “General Logarithmic Corrections to Black Hole Entropy”, Class. Quant. Grav. 19, 2355 (2002).

[33] J. Sadeghi, B. Pourhassan, and F. Rahimi, “Logarithmic corrections to charged hairy black hole in (2+1) dimensions”, Can. J. Phys. 92, 1638 (2014).

[34] S. S. More, “Higher Order Corrections to Black Hole Entropy”, Class. Quant. Grav. 22, 4129 (2005).

[35] S. Nojiri, Sergei D. Odintsov, ”Quantum (in)stability of 2-D charged dilaton black holes and 3-D rotating black holes”, Phys.Rev. D59, 044003 (1999)

[36] Andrei A. Bytsenko, S. Nojiri, Sergei D. Odintsov, ”Quantum generation of Schwarzschild-de Sitter (Nariai) black holes in effective Dilaton - Maxwell gravity”, Phys.Lett. B443, 121 (1998)

[37] S. Nojiri, Sergei D. Odintsov, ”Thermodynamics of Schwarzschild-(anti) de Sitter black holes with account of quantum corrections”, Int. J. Mod. Phys. A15, 989 (2000)

[38] B. Pourhassan and M. Faizal, “Thermodynamics of a Sufficient Small Singly Spinning Kerr-AdS Black Hole”, Nucl. Phys. B 913, 834 (2016).

[39] J. Sadeghi, B. Pourhassan and M. Rostami, “P-V criticality of logarithm-corrected dyonic charged AdS black holes”, Phys. Rev. D 94, 064006 (2016).

[40] P. Hořava, “Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point”, Phys. Rev. D 79, 084008 (2009).

[41] G. Calcagni, “Cosmology of the Lifshitz universe”, JHEP 0909, 112 (2009).

[42] R. G. Cai, L. M. Cao and N. Ohta, “Topological Black Holes in Horava-Lifshitz Gravity”, Phys. Rev. D 80 (2009) 024003.

[43] R. G. Cai, Y. Liu and Y. W. Sun, “On the z=4 Horava-Lifshitz Gravity”, JHEP 0906, 010 (2009).

[44] B. R. Majhi, “Hawking radiation and black hole spectroscopy in Horava-Lifshitz gravity”, Phys. Lett. B 686 (2010) 49.

[45] B. Pourhassan, “Extended Chaplygin Gas in Horava-Lifshitz Gravity”, Physics of the Dark Universe 13, 132 (2016).

[46] J. Sadeghi, B. Pourhassan, K. Jafarzadeh, E. Reisi and M. Rostami, “Massless Fermion Quasinormal Modes in the Horava-Lifshitz Background”, Can. J. Phys. 91, 251 (2013).
[47] H. Lu, J. Mei and C.N. Pope, “Solutions to Horava Gravity”, Phys. Rev. Lett. 103 (2009) 091301.

[48] M. Jamil, E. N. Saridakis, M. R. Setare, "The generalized second law of thermodynamics in Horava-Lifshitz cosmology”, JCAP 1011 (2010) 032.

[49] J. Sadeghi, K. Jafarzade, and B. Pourhassan, “Thermodynamical Quantities of Horava-Lifshitz Black Hole”, Int. J. Theor. Phys. 51, 3891 (2012).

[50] Y. S. Myung and Y. W. Kim, “Thermodynamics of Horava-Lifshitz black holes”, EPJC 68, 265 (2010).

[51] S.-Wei Zhou and W.-Biao Liu, “Black Hole Thermodynamics of Horava-Lifshitz and IR Modified Horava-Lifshitz Gravity Theory”, Int. J. Theor. Phys. 50, 1776 (2011).

[52] H. Quevedo, A. Sanchez, S. Taj and Alejandro Vazquez, “Geometrothermodynamics in Horava-Lifshitz gravity”, J. Phys. A 45 (2012) 055211.

[53] M.-In Park, “The Black Hole and Cosmological Solutions in IR modified Horava Gravity”, JHEP 0909 (2009) 123.

[54] A. Kehagias and K. Sfetsos, “The black hole and FRW geometries of non-relativistic gravity”, Phys. Lett. B 678 (2009) 123.

[55] R. Biswas, and S. Chakraborty, “Black Hole Thermodynamics in Horava Lifshitz Gravity and the Related Geometry”, Astrophys. Space Sci. 332 (2011) 193.

[56] G. W. Gibbons and S. W. Hawking, “Action integrals and partition functions in quantum gravity”, Phys. Rev. D. 15 (1977) 2752.

[57] V. Iyer and R. M. Wald, “Comparison of the Noether charge and Euclidean methods for computing the entropy of stationary black holes”, Phys. Rev. D. 52 (1995) 4430.

[58] B. P. Dolan, “The cosmological constant and black hole thermodynamic potentials”, Class. Quant. Grav. 28 (2011) 125020.

[59] I. Amado and A. Faedo, “Lifshitz black holes in string theory”, JHEP 1107, 004 (2011).

[60] K. Balasubramanian and J. Mc Greevy, “An analytic Lifshitz black hole”, Phys. Rev. D 80, 104039 (2009).