RMPC for Uncertain Nonlinear Systems With Non-Additive Dynamic Disturbances and Noisy Measurements

SAMI EL-FERIK
Systems Engineering Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
e-mail: selferik@kfupm.edu.sa
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ABSTRACT In this paper, we present a robust, model-predictive control scheme for the general class of uncertain and constrained discrete-time nonlinear systems subject to noisy measurements. The relationships between the system’s dynamics, uncertainties, disturbances and the measurement noise are nonlinear and not necessarily additive. In particular, the disturbance is the output of an uncertain system with an unknown input. This study serves the threefold ultimate objective of ensuring robust satisfaction of the state constraints, recursive feasibility and stability. To satisfy state constraints, the proposed algorithms adopt a constraints tightening approach using the restricted constraint sets computed online. Several bounds on the prediction level and rate are derived and the size of the terminal region is maximized using polytopic linear differential inclusions (PLDI). An explicit bound on the maximum allowable disturbance for recursive feasibility is also derived based on optimization of the one-step ahead controllable set to the terminal region. The disturbance and uncertainties are non-vanishing and therefore only Input-to-state practical stability (ISpS) can be ensured. A simulation example demonstrates the efficacy of the mathematical framework and algorithms developed in this work.

INDEX TERMS Nonlinear MPC, robust MPC, constraint tightening, convex optimization.

I. INTRODUCTION
Model predictive control (MPC) is a moving horizon control approach that has been recognized as the most used control strategy for systems under inputs and state constraints. MPC is a model-based control and can be called either linear or nonlinear depending on the nature of the model used in the prediction of the system’s dynamics. In both linear MPC and nonlinear MPC (NMPC) schemes, the closed loop system is nonlinear due to the presence of constraints. Linear MPC has become a preferred control strategy in many industrial applications [24]. It is worthwhile noting that despite the fact that industrial systems are inherently nonlinear, several features, including the ability to handle constraints and balance competing control objectives, contributed to the widespread use of MPC strategy. However, there is a growing need to operate the systems closer to the admissible operating conditions for maximum efficiency and profit. Linear models are therefore no longer adequate to satisfy requirements that are more rigorous and the use of more representative nonlinear system models is necessary. In practice, due to modeling errors all real-world systems suffer from system uncertainties, exogenous disturbances and measurement noise.

The idea behind the model predictive control strategy is very well known, mature, and can be summarized as follows. Based on the system’s information collected at time $t$, MPC solves an online open-loop constrained optimal control problem over a predetermined finite-horizon at every sampling instant. At each instant, the controller uses the system’s model to predict the dynamic of the real system under a set of control actions up to a predetermined horizon $N_p$. The control is optimized over a horizon $N_c$ and only the first optimal control action is implemented. These steps in the control strategy repeat themselves over time at every single sampling period and because the prediction and optimization are done open loop, the stability and performance of the closed-loop behavior can differ from what is expected. To overcome this weakness, MPC applies the optimal control law obtained for only a single sampling period and repeat the open loop optimization.
The third technique is Tube-based MPC. It is a closed loop approach [26] used to ensure that the deviation of the actual state from the nominal state is smaller than the one obtained by open-loop nominal MPC. In Tube-MPC, the controller is composed of two terms:

- the nominal control input, which is the online solution of nominal finite horizon control problem when applying nominal MPC and
- an additive state feedback law which is computed off-line and guarantees that the real trajectory of the closed-loop system will be within an envelope for all possible trajectories called hyper-tube [27].

The center of the envelope is the nominal trajectory [29]. Several successful studies using Tube-MPC have been reported in various literature (see [6], [12], [32], [33], [9], [13], [21]). More recently, [38] addressed RMPC for continuous systems. In [1] and [41], tube-based RMPC for time varying system is presented. There are 508 publications since 2018, some with successful application of tube-based MPC (see [21], [36], [22]). Tube-based MPC includes constraint tightening in the MPC optimization problem and a simple constraint tightening formulation has been proposed in [12]. Most of the available literature either for nominal-MPC or Tube-based MPC have considered different constraint tightening approaches for systems with additive uncertainties ([12], [17]). In [30], the study addressed uncertainties which decay as function of the system states. In [28], an output feedback model predictive controller (MPC) with the integration of an extended state observer (ESO) is proposed for hydraulic systems. The experimental application shows an enhancement of the robustness.

### A. CONTRIBUTION

In this paper, we extend the work initially developed in [35] and consider the system to be affected by an external dynamic disturbance having its own exo-system dynamic model. Both states and disturbance measurements needed for the control action and also to initialize the prediction phase are considered imperfect and noisy. The robust MPC design takes into account the bounds of the disturbance, the measurement noise, and the dynamic uncertainty. Therefore, in the case of this work, the tightened constraints sets are more general. Asymptotic stability for MPC strategies can be proven in case of additive and vanishing disturbance (decaying with state) (see [23]), however, only ISpS can be guaranteed in case the uncertainties are non-vanishing [18]. Likewise, in this work, the uncertainties are also not decaying with the state and hence only ISpS can be considered. Therefore, Input-to-state practical stability (ISpS) framework is used to prove the stability of the proposed algorithm. Moreover, stability can also be guaranteed by imposing a terminal cost and terminal constraint set [24], and the stability margin of the system is established depending on the size of this terminal set [3]. Several approaches can be used to maximize the size of the terminal set (see for example [3], [42]). To determine recursive feasibility, one needs to find the size.
of the l-step controllable sets to the terminal region, \( C_l(X_f) \). A conservative set-based estimation of \( C_l(X_f) \) by iteratively computing convex inner approximations of \( C_l(X_f) \) is presented in [29], however, the sensitivity of the size of \( C_l(\cdot) \) with respect to the disturbance has not been taken into account. Another set-based approach [40] considers Minkowski differences of a collection of polytopes. This approach is applied to determine the one-step ahead robust invariant sets in the case of polytopic additive uncertainties. A similar approach solved the disturbance. In addition, the paper provides the following contributions to the literature.

i. When using Lipschitz bounds, the literature mainly considers either dynamic uncertainty assuming perfect measurements [30] or measurement noise with perfect model [8]. The present work is an attempt to develop a unified framework to address both types of uncertainties.

The non-additive disturbance is generated by an uncertainty set containing either the error level or its growth rate along the prediction horizon. In addition, we utilize the bounds to tighten the constraints. Then, conditions on recursive feasibility (Section and practical stability III-C) are derived. The terminal constraint set and its associated control law are optimized using PLDIs in Section III-D. The robust one-step controllability set to the terminal region is calculated in Section III-D, which is extended in Section III-E to find the robust output feasible set. Finally, simulation studies are presented in Section IV. The paper is concluded in Section V with recommendations for future work.

### II. PROBLEM FORMULATION

#### A. NOTATION

Let \( \mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}, \mathbb{Z}_{\geq 0} \) denote real, non-negative real, integer and non-negative integer sets of numbers, respectively. For a set \( A \subseteq \mathbb{R}^n \), the point to set distance from \( \zeta \in \mathbb{R} \) to \( A \) is denoted by dist \((\zeta, A) \equiv \inf \{ |\eta - \zeta|, \eta \in A \} \), and if \( A \) is a closed set, its boundary is denoted by \( \partial A \). The difference between two sets \( A, B \subseteq \mathbb{R}^n \) is denoted by \( A \setminus B \equiv \{ x \in A, x \notin B \} \). The Pontryagin (or Minkowski) difference between these sets is defined as \( A \sim B \equiv \{ x + y | x \in A, y \in B \} \), while the Minkowski sum of these two sets is defined as \( A \oplus B \equiv \{ x + y | x \in A, y \in B \} \). Let \( L_2 \) Euclidean norm be denoted by \( \| \cdot \| \). For a discrete-time series \( \phi = [\phi_0, \phi_1, \ldots, \phi_T]^T \), we define \( \| \phi \| \equiv \sup_{t \geq 0} (|\phi_t|) \) and \( \| \phi_0 \| \equiv \sup_{0 \leq t \leq T} (|\phi_t|) \). We also use class \( K, K_{\infty} \) and \( KL \) comparison functions [37].

#### B. SYSTEM NOTATION AND PRELIMINARIES

Consider the discrete-time nonlinear system

\[
x_{t+1} = f(x_t, u_t)
\]

with \( f(0, 0) = 0 \), where \( x_t \in \mathbb{R}^n \) is the state, and \( u_t \in \mathbb{R}^p \) is the control input. Given a signal \( x \), let \( x_{t,t+N} \) denote the discrete-time realization of \( x_t \) within the period \( t \) to \( t+N \). \( T : \mathbb{R} \rightarrow \mathbb{R} \) is the identity function, \( \gamma_1 \) and \( \gamma_2 \) by \( \gamma_1 \circ \gamma_2 \) is functional composition of \( \gamma_1 \) and \( \gamma_2 \), \( \alpha^{-1} \) is the inverse function of \( \alpha \).

**Definition 1**: If \( x_t \in \Xi, \forall t \geq t_0 \) whenever \( x_{t_0} \in \Xi \) and bounded input \( w_t \in W \), then \( \Xi \) is called a Robust Positively Invariant (RPI) set. Moreover, if \( \Xi \) is compact, \( \Xi \) and contains the origin as an interior point, the system (1) is said to be regionally Input-to-State practically Stable (ISpS) in \( \Xi \) for \( x_0 \in \Xi \) and \( w \in W \), if there exists \( KL \)-function \( \beta \), \( K \)-function \( \gamma \) and constant \( c > 0 \) such that

\[
|x_t| \leq \beta (|x_0|, t) + \gamma (|w|) + c
\]

where \( c \equiv 0 \), then the system is said to be regionally Input-to-State Stable (ISS) in \( \Xi \) [37].

**Definition 3**: Function \( V : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty) \) is an ISpS Lyapunov function in \( \Xi \), if for suitable functions \( \alpha_1, 2, 3, \sigma_3 \in K_{\infty}, \sigma_1, 2 \in K \) and constants \( \tilde{c}, \tilde{c} > 0 \), there exists a compact and RPI set \( \Xi \) and another set \( \Omega \subset \Xi \) with origin as an interior-point (\( \Omega \) is also RPI), such that the following conditions hold,

\[
V(x_t, w_t) \geq \alpha_1 (|x_t|), \quad \forall x_t \in \Xi
\]

\[
V(f(x_t, w_t), w_{t+1}) - V(x_t, w_t) \leq -\alpha_2 (|x_t|) + \sigma_1 (|w_t|) + \sigma_2 (|w_{t+1}|) + \tilde{c}, \quad \forall x_t \in \Xi
\]

\[
V(x_t, w_t) \leq \alpha_3 (|x_t|) + \sigma_3 (|w_t|) + \tilde{c}, \quad \forall x_t \in \Omega
\]

ISS implies ISpS, but converse is not true, since an ISS system with 0-input, i.e. \( w_k = 0, \forall k \geq 0 \) implies...
asymptotic stability to the origin, while for an ISpS system, 0—input implies asymptotic stability to a compact set (ball of radius $c$) containing the origin. In this paper, the stability analysis will demonstrate that according to the proposed control approach, closed-loop dynamics is ISpS, not ISS, due to uncertainty resulting from data compression. We also state an important result in regional input-to-state practical stability.

Theorem 1 (Ref. [5]): If system $x_{t+1} = f(x_t, u_t, w_t)$ admits an ISpS Lyapunov function in $\mathcal{Z}$, then it is regionally ISpS and satisfies condition (2), with $\beta(r, s) \triangleq \alpha_1^{-1}(3\beta(3\alpha_3(r), s))$, $\gamma(s) \triangleq \alpha_1^{-1}(3\beta(\sum_{i=1}^3 \sigma_i(s)) + \beta(3\alpha_3(s), 0)))$ and $c \triangleq \alpha_1^{-1}(3\beta(3\bar{c} + d), 0) + \alpha_1^{-1}\gamma(\mu(3\tilde{c})) + \alpha_1^{-1}\gamma(3\tilde{c})$, where $\mu, \gamma \in \mathcal{K}_{\infty}$ while $\beta \in \mathcal{KL}$ for some $d > 0$.

III. ROBUST MODEL PREDICTIVE CONTROL

A. SETUP

The paper addresses NMPC of the following nonlinear discrete-time system

$$ x_{t+1} = f(x_t, u_t, w_t), $$

$$ w_{t+1} = g(w_t, \phi_t), $$

where states $x_t$, controls $u_t$ and disturbance $w_t$ belong to the following constrained convex sets: $x_t \in X \subset \mathbb{R}^n$, $u_t \in U \subset \mathbb{R}^m$, $w_t \in W \subset \mathbb{R}^p$, bounded by some maximum and minimum values. The dynamic of the system $f(\ldots)$ and the dynamic of the disturbance are both uncertain. In addition, the states and disturbance are assumed to have both inaccurate estimates $\hat{x}_t$ and $\hat{w}_t$, respectively, such that

$$ \hat{x}_t = x_t + \eta_{x_t}, \quad \hat{x}_{t+1} = x_{t+1} + \eta_{x_{t+1}}, $$

$$ \hat{w}_t = w_t + \eta_{w_t}, \quad \hat{w}_{t+1} = w_{t+1} + \eta_{w_{t+1}}, $$

where $\eta_{x_t} \in H_x \subset \mathbb{R}^n$ and $\eta_{w_t} \in H_w \subset \mathbb{R}^p$ are bounded random noise affecting the measurements of $x_t$ and $w_t$ respectively. $\phi$ is an unknown input vector. Nominal states and disturbance are denoted by $\tilde{x}_t$ and $\tilde{w}_t$, for which we have the following nominal models used for prediction.

$$ \tilde{x}_{t+1} = \tilde{f}(\tilde{x}_t, u_t, \tilde{w}_t), \quad \tilde{w}_{t+1} = \tilde{g}(\tilde{w}_t), $$

Let

$$ \Delta_x(x, u, w) = f(x_t, u_t, w_t) - \tilde{f}(\tilde{x}_t, u_t, \tilde{w}_t), $$

$$ \Delta_w(w) = g(w_t, \phi_t) - \tilde{g}(\tilde{w}_t), $$

Therefore, equation 6 can be rewritten as

$$ \hat{x}_{t+1} = \tilde{f}(\tilde{x}_t, u_t, \tilde{w}_t) + \Delta_x(x, u, w) + \eta_{x_{t+1}}, $$

$$ \hat{w}_{t+1} = \tilde{g}(\tilde{w}_t) + \Delta_w(w) + \eta_{w_{t+1}}. $$

In the remaining of the paper we will use $\hat{x}_t$ and $x_t$ as well as $\hat{w}_t$ and $w_t$ interchangeably as $x$ and $w$ are accessible only through measurements. The cost function $J_t(x, u, w, N_c, N_p, k_f)$ is defined by

$$ J_t(x, u, w, N_c, N_p, k_f) = \sum_{l=t+1}^{t+N_c-1} [h_l(x_l, u_l) + q_l(x_l, w_l)] $$

$$ + \sum_{l=t+N_c}^{t+N_p-1} [h_l(x_l, k_f(x_l))] + h_f(x_{t+N_c}), $$

where $N_p$ and $N_c$ are prediction and control horizons. Cost function (12) consists of transition cost $h_l$ terminal cost $h_f$ and robustness cost $q_l$. Control sequence $u_{t:t+N_p}$ consists of two parts, $u_{t:t+N_c-1}$ and $u_{t+N_c:t+N_p-1}$. The latter part is generated by a terminal control law $u_l = k_f(x_l)$ for $l \geq t + N_c$, while the former is finite horizon optimal control $u_{t:t+N_c}$ which is the solution of the OCP 1.

Problem 1: At every instant $t \geq 0$, given prediction and control horizons $N_p, N_c \in \mathbb{Z}_{>0}$, auxiliary control $k_f(x)$ : $\mathbb{R}^n \to \mathbb{R}$, state $\tilde{x}_t$ and disturbance $\tilde{w}_t$, measurements, find the optimal control sequence $u_{t:t+N_c-1}$, which minimizes the finite horizon cost (12)

$$ u_{t:t+N_c-1}^0 = \arg\min_{u_{t:t+N_c-1}} J_t(\tilde{x}_{t:t+N_p}, \tilde{w}_{t:t+N_p}, u_{t:t+N_c-1}, N_c), $$

subject to nominal plant model and nominal exo-system model (8), tightened constraints $\tilde{x} \in \tilde{X}_{t:t+N_p}$, and terminal state $\tilde{x}_{t+N_c}$ is constrained to an invariant terminal set $X_f \in \tilde{X}_{t:t+N_c}$, i.e.

$$ \tilde{x}_{t+l} \in X_f, \quad \forall l = N_c, \ldots, N_p $$

The loop is closed by implementing only the first element of $u_{t:t+N_c-1}^0$ at each instant, such that the NLMPC implicit control law becomes

$$ \Theta_t(\tilde{x}, \tilde{w}) = u_{t}^{0}(\tilde{x}_{t:t+N_p}, \tilde{w}_{t:N_p}, N_p, N_c) $$

In the remaining of this paper,

- section III-B will go through all the details mentioned in Step 2 in Algorithm 1.
- Guarantees for recursive feasibility will be detailed in section III-C
- Determination of $C_1(X_f)$ in Step 4 will be covered in section III-D.
- Determination of $X_{\text{MPC}}$ in Step 5 will be covered in section III-E.
- Robust Stability of Algorithm 1 will be discussed in I.

B. CONSTRAINT TIGHTENING

The system under consideration contains several possible sources of uncertainty which is more representative to the real word than those in the literature (e.g. [17] and [30]). To ensure stability, an envelop similar to a growing tube will be determined. The dynamic bounds take into accounts all
these type of uncertainties and unlike [12], they do not decay exponentially. For that, a bound on the growth on uncertainty will be determined first. Due to uncertainty, the constraint sets $x$ and $w$ are 'larger' than the constraint sets for $\tilde{x}$ and $\tilde{w}$, such that $\tilde{x}_t \in X_t \subset X$, $\tilde{w}_t \in \tilde{W}_t$. Moreover, the tightened constraint sets have the contractive property such that $\tilde{X}_{t+1} \subseteq \tilde{X}_{t+1}$ for $l \geq 2$. Since uncertainty grows with horizon length, one needs to assume that measurements and transition uncertainties are bounded (see Assumption 1). The standard MPC assumptions of initial feasibility [29] is also assumed.

**Assumption 1:**

i. Each prediction starts from the present state $\tilde{x}_l = \tilde{x}_t = x_t + \eta_x$ and $\tilde{w}_l = \tilde{w}_t = w_t + \eta_w$, where $\eta_x \triangleq |x_t - \tilde{x}_t| \leq \tilde{n}_x$ and $\eta_w \triangleq |w_t - \tilde{w}_t| \leq \tilde{n}_w$.

ii. The transition uncertainties $|\Delta_x(x, u, w)| \triangleq |f(x, u, w) - f(\tilde{x}_t, \tilde{u}_t, \tilde{w}_t)| \leq \Delta_x$ and $|\Delta_u(w)| \triangleq |\tilde{g}(w) - g(w, \phi)| \leq \Delta_u(w)$.

iii. The nominal maps are locally Lipschitz with constants $L_{fx}, L_{fu}, L_{gw}$ and $L_{gw}$ with respect to $x$, $u$, and $w$, and $\tilde{f}(0, 0, 0) = 0, \tilde{g}(0) = 0$.

iv. A compact robust output feasible set $X_{MPC} \subseteq X_t$ defined as the set of initial states for which optimal control problem (Problem 1) is feasible, exists.

**Lemma 1:** Under Assumption 1, the $l$-step ahead prediction errors $\tilde{\rho}_{x_l+1} \triangleq |x_l+1 - \tilde{x}_l+1|$ and $\tilde{\rho}_{w_l+1} \triangleq |w_l+1 - \tilde{w}_l+1|$; $l = 1, \ldots, N_P$ are bounded by

$$\tilde{\rho}_{w_{l+1}} = \tilde{n}_w \frac{L_{gw}^l - 1}{(L_{gw} - 1)} + \Delta_w \frac{L_{gw}^l - 1}{(L_{gw} - 1)}$$

$$\tilde{\rho}_{x_{l+1}} = \tilde{n}_x \frac{L_{fx}^l}{L_{fx} - 1} + \Delta_x \frac{L_{fx}^l - 1}{L_{fx} - 1}$$

$$+ \eta_x \frac{L_{gw}^l}{L_{gw} - 1} \left( \frac{L_{gw}^l - L_{gw}^l}{L_{fx} - L_{gw}} - L_{gw}^l \right)$$

$$+ \Delta_w \frac{L_{gw}^l}{L_{gw} - 1} \left( \frac{L_{gw}^l - L_{gw}^l}{L_{fx} - L_{gw}} - L_{gw}^l \right),$$

for $l = 0, \ldots, N_P$, and $L_{fx}, L_{gw} \neq 1$ and $L_{fx} \neq L_{gw}$.

**Proof:** From Assumption 1, for $l = 1$

$$|w_{t+1} - \tilde{w}_{t+1}| = |\tilde{g}(w_t) + \Delta_w(w) + \eta_{w_{t+1}} - \tilde{g}(\tilde{w}_{t+1}),$$

$$\leq |\tilde{g}(w_t) - \tilde{g}(\tilde{w}_{t+1})| + \Delta_w + \tilde{n}_w,$$

$$\leq |w_t - \tilde{w}_{t+1}| + \Delta_w + \tilde{n}_w,$$

$$\leq (L_{gw} + 1) \tilde{n}_w + \Delta_w.$$  

Using similar derivation, $l$-step ahead prediction error can be written as

$$|x_{t+1} - \tilde{x}_{t+1}| \leq \tilde{n}_x \sum_{k=0}^{k} L_{fx}^l + \Delta_x \sum_{k=0}^{k} L_{fx}^l$$

$$+ \eta_x \sum_{k=0}^{k} \left( \sum_{j=0}^{j} L_{gw}^j \right)$$

$$+ \Delta_w \sum_{k=0}^{k} \left( \sum_{j=0}^{j} L_{gw}^j \right),$$

for $l = 0, \ldots, N_P$, and $L_{fx}, L_{gw} \neq 1$ and $L_{fx} \neq L_{gw}$.

Using nested geometric series’ sum will lead to the second part of (16).

In Lemma 1, $L_{fx}$ and $L_{gw}$ are assumed $\neq 1$ and $L_{fx} \neq L_{gw}$. However, the result for the case where $L_{fx}, L_{gw} = 1$ and $L_{fx} = L_{gw}$ is straightforward. Bounds (16) are conservative due to the use of the Lipschitz constants and methods suggested in [17] to reduce this conservatism can be used. A lower bound on rate of convergence of the prediction error can be established in the following claim.

**Claim 1:** There exists a terminal control $u_t = k_f(\tilde{x}_t) \in U$, $l = N_C, \ldots, N_P$ such that the solution $X_f$ of the nominal plant dynamic given in (8) is robust positively invariant (RPI) i.e. $x_t \in X_f$ and $\tilde{x}_t \in X_f$, $W_t = t + N_C + \ldots, \ldots, t + N_P$ for any $x_t$, $\forall x_t \in X_f$, such that,

i. The rate of convergence $\delta_{\rho_{l+1}} \triangleq |\tilde{f}(\tilde{x}_t_{l+1}, k_f(\tilde{x}_t_{l+1}, 0)) - \tilde{x}_{l+1}|$ of nominal state under control $k_f(\tilde{x})$ is
lower bounded by
\[
\delta_{\rho_{t+1}} \geq \tilde{\delta}_{\rho_{t+1}} = \eta_k L_{f_k}^{l+1} + L_{f_k}^{l} \left( \tilde{\Delta}_x + (\eta_w + \tilde{\Delta}_w) \frac{L_{fw}}{L_{gw} - 1} \right) + \left( \tilde{\eta}_w L_{gw} + \tilde{\Delta}_w \right) \frac{L_{fw}}{L_{gw} - 1} \times \left( \frac{L_{f_k}^{l}(L_{f_k} - 1) - L_{fw}^{l}(L_{fw} - 1)}{L_{f_k} - L_{fw}} \right)
\] (18)
for \( l = N_c - 1 \ldots N_p - 2 \).

ii. There exists \( a \in \mathbb{Z}_{>0} \) and \( \delta \leq Q_f^l \in \mathbb{R}^{n \times n} \) such that
\[
\tilde{x}^T Q_f^l \tilde{x} \leq a, \quad \forall \tilde{x} \in X_f
\] (19)

Proof:

i. Using the definition of the prediction error \( \hat{\rho}_{t+1} = x_{t+1} - \tilde{x}_{t+1} \) and \( \delta_{\rho_{t+1}} \) can be written as \( \delta_{\rho_{t+1}} \geq |x_{t+1} - \tilde{x}_{t+1}| \). Adding and subtracting \( x_{t+1} \) leads to \( \delta_{\rho_{t+1}} \geq \tilde{\rho}_{\rho_{t+1}+1} - \tilde{\rho}_{\rho_{t+1}+1} \). Using Lemma 1, the proof is complete.

ii. The proof for part ii) and the existence of \( k_f(\tilde{x}_f) \in U, l = N_c, \ldots, N_p - 1 \) is typical and based on linear theory analysis using a linearized version of the nonlinear model. Similar claim have been made in the literature (see for example \cite{29} or \cite{35}). It is omitted in this paper for the benefit of the reader and a better management of the paper length.

By taking into account the established prediction uncertainty bounds and study their impact on the constrained FHOCP, it is possible to guarantee the admissibility of the state/output dynamic behavior of the actual system.

Theorem 2: With nominal constraints \( X \) and \( W \) on the system (6), let the tightened constraints for nominal model (8) be given by
\[
\tilde{x}_{t+1} \triangleq X - \text{dist}(\tilde{x}_{t+1} | X_{t+1}),
\] (20)
for \( l = 0, \ldots, N_p, \) and \( \tilde{\rho}_x \) and \( \tilde{\rho}_w \) are prediction error bounds defined in (16). \( \text{dist}(\tilde{x}_{t+1} | X_{t+1}) \) is an \( n \)-ball centered at \( \tilde{x}_{t+1} \) and with a radius \( \tilde{\rho}_x \). Then, any admissible control sequence \( u_{t,t+N_c-1}, k_f(\tilde{x}_{t+1}, u_t, w_t) \) which is feasible with respect to tightened constraints (20), guarantees the satisfaction of original constraints, i.e. \( x_{t+1} \in X \) for \( w_{t+1} \in W \), \( l = 0, \ldots, N_p \) and \( x_0 \in X_{t+1} \).

Proof: To prove the theorem, one needs to proceed in two steps:

1) Let \( \eta'_x < \tilde{\eta}_x, \eta'_w < \tilde{\eta}_w \), \( \Delta'_x(x, u, w) \leq \Delta'_w < \Delta_x \) and \( \Delta'_w(x, u, w) \leq \Delta'_w < \Delta_w \). Then, following the same procedure as in proof of Lemma 1, one can find that \( |x_{t+1} - \tilde{x}_{t+1}| \leq \rho'_x \tilde{\rho}_x \tilde{x}_{t+1} \leq \tilde{\rho}_x \frac{L_{fw}}{L_{gw} - 1} \left( \tilde{\eta}_x L_{f_k}^{l+1} - L_{f_k}^{l} \right) + \Delta'_w \frac{L_{fw}}{L_{gw} - 1} \)

2) Comparing (16) and using (17), \( \rho'_x \tilde{\rho}_x < \tilde{\rho}_x \tilde{\rho}_x \) and hence \( B^0(\rho'_x \tilde{\rho}_x) \subset B^0(\tilde{\rho}_x \tilde{\rho}_x) \). This leads to \( x_{t+1} \in \tilde{x}_{t+1} \). On the other hand, based on (20), one can state that \( X = \tilde{x}_{t+1} \). Then the proof is complete.

2) This part is similar to what was reported in the literature (see for example \cite{29}). The sketch of the proof is as follows: Consider the prediction sequence \( \tilde{x}_{t+1}, \tilde{x}_{t+1} \) made using the control sequence \( u_{t,t+N_c-1}, k_f(\tilde{x}_{t+1}, \tilde{x}_{t+1}) \) respectively and with the initial conditions \( \tilde{x}_{t+1} = \tilde{x}_{t+1} + \tilde{x}_{t+1} \). Let \( \hat{X}_{t+1} \in X \sim B^0(\tilde{\rho}_x \tilde{\rho}_x) \). Let \( \tilde{\rho}_x \tilde{\rho}_x = \tilde{x}_{t+1} + \tilde{x}_{t+1} + w_{t+1} \). This leads to \( x_{t+1} \in X \sim B^0(\tilde{\rho}_x \tilde{\rho}_x) \). □

C. RECURSIVE FEASIBILITY OF NMPC ALGORITHM

This section addresses the important issue of recursive feasibility. Let the one-step controllability set towards the terminal constraint set \( X_f \) be defined as \( C_1(X_f, \tilde{x}_{t+1}) \triangleq \{ \tilde{x}_t \in \tilde{x}_{t+1} : \tilde{x}_t \in X_f, u_t \in U, \} \). Let us also define the minimum size of this set as \( \hat{d} \triangleq \text{dist}(\tilde{x}_{t+1} | X_f, \tilde{x}_{t+1}) \).

Theorem 3: Under Assumption 1, terminal control (Claim 1) and tightened constraints (Theorem 2), given the feasibility of initial state \( \tilde{x}_t \) in X_{t+1}, Problem 1 is recursively feasible, if the minimum size of \( C_1(X_f) \) upper bounds the uncertainties as follows
\[
\left( L_{f_k}^{l} + 1 \right) \left( \tilde{\eta}_x L_{fw} \tilde{\rho}_x + \tilde{\Delta}_w \right) \geq \hat{d}
\] (21)

Proof: The sketch of the proof is presented here. First, one needs to show that starting at a time instant \( t \) from an initially feasible state \( \tilde{x}_t \in X_{t+1} \) and a feasible control input \( u_t \), a feasible control input solution of the FHOCP (1) for \( t + 1 \) can be found. This will lead to the determination of the control input sequence for the entire horizon. Assume that at \( t, \) a feasible control \( u_{t+1}^{0_{t,t+N_c-1}} \) exists. Let \( u_{t+1,t+N_c-1} \) be a possible feasible control sequence for \( t+1, \ldots, t+N_c \). The objective is to find (a) a feasible \( u_{t+1,t+1}^{0_{t,t+N_c-1}} \) for the progressive tightened constraints (20) for \( t+1, \ldots, t+N_c - 1 \), and

(b) \( u_{t+1,t+N_c-1} \) at \( t+1 \).

(a) Given \( \tilde{x}_{t+1} \) and \( w_{t+1} \) at \( t \), assume the existence of a feasible control \( u_{t+1,t+1}^{0_{t,t+N_c-1}} \) with state and disturbance predictions \( \tilde{x}_{t+1}^{0_{t,t+N_c-1}} \) and \( w_{t+1}^{0_{t,t+N_c-1}} \) respectively. At \( t+1 \), new predictions generated using \( u_{t+1,t+1}^{0_{t,t+N_c-1}} \) are \( \tilde{x}_{t+1}^{0_{t,t+N_c-1}} \) and \( w_{t+1}^{0_{t,t+N_c-1}} \). Using Assumption 1, \( \tilde{x}_{t+1}^{0_{t,t+N_c-1}} - \tilde{x}_{t+1}^{0_{t,t+N_c-1}} \leq L_{fw} |w_{t+1}^{0_{t,t+N_c-1}} + \tilde{w}_{t+1}^{0_{t,t+N_c-1}} |. \) By induction,
it follows that
\[
\tilde{x}_{t+1|t} = \tilde{x}_{t|t} + L_{fx}^{-1} \left| \tilde{x}_{t+1|t} - \tilde{x}_{t|t} \right| + L_{fw} \begin{bmatrix} f_{x} & I \\ L_{fx} & L_{fw} - L_{gx} \end{bmatrix} \begin{bmatrix} \tilde{w}_{t+1|t} \\ \tilde{w}_{t+1|t} \end{bmatrix},
\]
for \( l = 1, \ldots, N_c - 1 \). Let \( \xi_1 \triangleq |\tilde{x}_{t+1|t} - \tilde{x}_{t|t}| \) and \( \delta_{\tilde{w}_i} \triangleq |\tilde{w}_{t+1|t} - \tilde{w}_{t|t}| \). Now, based on the assumption that \( \tilde{x}_{t+1|t} \in \tilde{X}_{t+1|t} \triangleq X \sim B^n(\tilde{\rho}_{\tilde{x}_{t+1|t}}) \) and using (22), one gets \( \tilde{x}_{t+1|t} \in X \sim B^n(\tilde{\rho}_{\tilde{x}_{t+1|t}}) \). On the other hand, \( \tilde{x}_{t+1|t} \in \tilde{X}_{t+1|t} \) requires that \( \tilde{\rho}_{\tilde{x}_{t+1|t}} + L_{fx}^{-1} \left| \delta_{\tilde{x}_1} \right| + L_{fw} \begin{bmatrix} f_{x} & I \\ L_{fx} & L_{fw} - L_{gx} \end{bmatrix} \begin{bmatrix} \tilde{w}_{t+1|t} \\ \tilde{w}_{t+1|t} \end{bmatrix} \leq \tilde{x}_{t+1|t} \). In view of \( \tilde{\rho}_{\tilde{x}_{t+1|t}} = \tilde{\rho}_{\tilde{x}_{t+1|t}} \), it turns out that \( \tilde{\rho}_{\tilde{x}_{t+1|t}} \geq L_{fx}^{-1} \left| \delta_{\tilde{x}_1} \right| + L_{fw} \begin{bmatrix} f_{x} & I \\ L_{fx} & L_{fw} - L_{gx} \end{bmatrix} \begin{bmatrix} \tilde{w}_{t+1|t} \\ \tilde{w}_{t+1|t} \end{bmatrix} \). Using triangle inequality leads to \( \tilde{\rho}_{\tilde{x}_{t+1|t}} \geq L_{fx}^{-1} \left| \delta_{\tilde{x}_1} \right| + L_{fw} \begin{bmatrix} f_{x} & I \\ L_{fx} & L_{fw} - L_{gx} \end{bmatrix} \begin{bmatrix} \tilde{w}_{t+1|t} \\ \tilde{w}_{t+1|t} \end{bmatrix} \). Knowing that \( 0 \leq |\tilde{x}_{t+1|t} - x_{t+1}| \) and \( 0 \leq |\tilde{w}_{t+1|t} - w_{t+1}| \), one gets \( \tilde{\rho}_{x_{t+1|t}} \geq \tilde{\rho}_{\tilde{x}_{t+1|t}} \) and hence, \( \tilde{x}_{t+1|t} \in X \sim B^n(\tilde{\rho}_{\tilde{x}_{t+1|t}}) \), \( \tilde{x}_{t+1|t} \) for \( l = 1, \ldots, N_c - 1 \).

(b) To prove \( \tilde{x}_{t+1+N_c|t+1} \in X_f \subseteq \tilde{X}_{t+1+N_c|t+1|+1}, \) inequality (22) is written as \( \tilde{x}_{t+1+N_c|t+1} - \tilde{x}_{t+1+N_c|t} \leq L_{fx}^{-1} \left( \eta_{t+1} + \eta_{t+1} \right) + L_{fw} \begin{bmatrix} f_{x} & I \\ L_{fx} & L_{fw} - L_{gx} \end{bmatrix} \left( \eta_{t+1} + \eta_{t+1} \right). \) The upper bounds on the accumulated prediction errors are given by (16), therefore one has \( \tilde{x}_{t+1+N_c|t+1} - \tilde{x}_{t+1+N_c|t} \leq L_{fx}^{-1} \left( \left(f_{x} + 1\right)\eta_{t+1} + L_{fw} \tilde{w}_{t+1} + \Delta x \right) + L_{fw} \begin{bmatrix} f_{x} & I \\ L_{fx} & L_{fw} - L_{gx} \end{bmatrix} \left( \tilde{w}_{t+1} + \tilde{w}_{t+1} \right). \) In view of (21), it follows that \( \tilde{x}_{t+1+N_c|t+1} - \tilde{x}_{t+1+N_c|t} \leq \tilde{d}. \) Using \( \tilde{x}_{t+1+N_c|t} \in X_f \), leads to \( \tilde{x}_{t+1+N_c|t+1} \in X_f \sim B^n(\tilde{\rho}_{x_{t+1+N_c|t+1}}) \triangleq C_1(X_f, X). \)

D. DETERMINATION OF C_1(X_f)

In this section, a method is presented to determine the minimum size \( \tilde{d} \) of \( C_1(X_f) \) as given by (21). To find \( \tilde{d} \), the topology of \( C_1(X_f) \) should be known. The proposed approach uses min-max optimization to find the estimate of \( C_1(X_f) \).

Problem 2: Given target set \( X_f \) and tightened constraints (20), if the boundary of \( X_f \) is discretized appropriately into \( \tilde{N} \) point \( \tilde{x}_i \) \( \in \partial(X_f) \) for \( i = 1, \ldots, \tilde{N}, \) then \( C_1(X_f) \) is the solution of the following \( \tilde{N} \) min-max OCPs

\[
\tilde{x}_i = \max_u \left\{ \min_{\tilde{w}} \left( -\log \tilde{x}_i Q f \tilde{x}_i \right) \right\}
\]

for \( i = 1, \ldots, \tilde{N} \), subject to

\[
\tilde{x}_i = \tilde{f}(\tilde{x}_i, u, \tilde{w})
\]

Let \( \partial(C_1(X_f)) \) denote the boundary of \( C_1(X_f) \), it follows that \( \partial(C_1(X_f)) = \{ \tilde{x}_i \}, \forall i = 1, \ldots, \tilde{N} \).

Remark 1:
- The cost functional (23) is convex by design. However, the overall OCP may not be convex due to the presence of the nonlinear constraints (24)(25).

To overcome this challenge and mainly avoid local minima, a priori knowledge of an adequate initial guess is required to converge to a feasible solution. (25) ensures that is outside \( X_f \).
- while computationally expensive, the determination of \( X_f \) runs offline, therefore the complexity in terms of time and memory is not of great concern.

E. DETERMINATION OF X_{MPC}

In this section, the entire feasibility region of the MPC algorithm will be presented. The following propositions are needed and some of them will be stated without proof as they are straightforward.

**Proposition 1:** \( X_f \subseteq C_1(X_f) \).

Proof: \( X_f \) is RI if and only if \( X_f \subseteq C_1(X_f) \) (see for example [19]). and since \( X_f \) is RI, therefore Proposition 1 follows.

**Proposition 2:** \( C_1(X_f) \) is contained in the robust output feasible set \( X_{MPC} \), i.e. \( C_1(X_f) \subseteq C_1(X_{MPC}) \).

Proof: According to [10], \( \Omega \subseteq \bar{\Omega} \Rightarrow C_1(\Omega) \subseteq C_1(\bar{\Omega}) \). It follows that \( X_f \subseteq X_{MPC} \), and hence \( C_1(X_f) \subseteq C_1(X_{MPC}) \).

**Proposition 3:** \( C_1(X_f) \) is a finite union of polyhedra.

Proof: Given \( X_f \) is convex (RI) and \( X_f \subseteq C_1(X_f) \) according to Proposition 1, \( C_1(X_f) \) is compact and is the union of intersecting polyhedra.

**Proposition 4:** \( C_1(X_f) \) contains the boundary of the terminal set, \( \partial(X_f) \), i.e. \( \partial(X_f) \subseteq C_1(X_f) \).

The following main theorem can now be stated

**Theorem 4:** Given the terminal set \( X_f \), the tightened constraints (20) and the control \( u \in U \), \( X_{MPC} \) is obtained by recursively applying the one-step controllable set operator \( C_1(\cdot) \) to solve the OCP 2 as follows

\[
X_{MPC} = \bigcup_{l=1}^{l=N_c} \bigcup_{l=1}^{l=2} C_1(C_{l-1}(X_f)) \subseteq C_1(X_f) \cup X_f
\]
F. ROBUST STABILITY

In this work, the uncertainties $\eta_t$ and $\eta_u$ are non-vanishing and only ISpS can be proven as explained previously. In order to state the main theorem characterizing ISpS of the proposed control strategy, the following assumptions and properties are introduced.

Assumption 2 (Cost Lipschitz Continuity): i. The cost functions $f_t(\cdot)$, $h(\cdot)$, $q(\cdot)$ and $h_f(\cdot)$ are nonlinear, continuous and locally Lipschitz with constants $L_{f_t}$, $L_{hx}$, $L_{hu}$, $L_{qw}$, $L_{gwx}$ and $L_{gwd}$.

ii. $|k_f(\bar{x})| \leq L_{f_k}|\bar{x}|$, for $\bar{x} \in \bar{X}_f$.

iii. $|h(\bar{x}, u)| \leq L_{hx}|\bar{x}| + L_{hu}|u|$, for $\bar{x} \in \bar{X}_f$ and $u \in U$.

iv. $0 \leq |q(\bar{x}, \bar{w})| \leq L_{qw}|\bar{x}| + L_{qw}|\bar{w}|$, for $\bar{x} \in \bar{X}_f$ and $\bar{w} \in \bar{W}_f$.

v. $|h_f(\bar{x})| \leq L_{hf}|\bar{x}|$, for $\bar{x} \in \bar{X}_f$.

vi. $\alpha_t(\bar{x}(\cdot)) \leq h_f(\bar{x}(\cdot)) \leq \alpha_{t+1}(\bar{x}(\cdot))$, for all $\bar{x} \in \bar{X}_f$.

Assumption 3 (Technical): Based on the definition of ISpS Lyapunov functions given in Section II-A, the following holds,

(i). $\alpha_1(s) = \alpha_2(s) \equiv \min h(s, 0)$

(ii). $\alpha_3(s) \equiv \alpha_{L_{f_k}^2s} + s$ 1 s

(iii). $\alpha_3(s) \equiv (c_2 + c_3 + c_4) L_{fu} L_{gws}$

(iv). $\alpha_3(s) \equiv \alpha_1(s) L_{fu} + \psi(s)$

(v). $\alpha_3(s) \equiv L_{gwp}(L_{gwp} - 1)^{-1} f_{gwp}(N_p - 1) s$

(vii). $\tilde{c} \equiv \tilde{c}_L + \tilde{L}_{hu}|u_{\text{max}} - u_{\text{min}}|$

where, $c_1 = (L_{hx} + L_{hu} L_{kf} + L_{gwx}) (L_{gwp} - 1) s$, $c_2 = L_{gwp}^2 L_{gwp}^2 (L_{gwp} - 1) s$, $c_3 = \frac{L_{hx} + L_{hu} L_{kf} + L_{gwx}}{L_{fu}}$, $c_4 = \frac{L_{hx} + L_{hu} L_{kf} + L_{gwx}}{L_{fu}}$, $c_5 = \frac{L_{hx} + L_{hu} L_{kf} + L_{gwx}}{L_{fu}}$.

In view of Lemma 3. In view of Theorem 3, using the optimal control $u^0_{t+1,Nc-1|t}$, it turns out that at least one feasible control $u^*_{t+1,Nc-1|t}$ is $[u^*_{t+1,Nc-1|t}] = [u^0_{t+1,Nc-1|t}]$. Summing the geometric series, we obtain $V_{t|t} \leq \alpha_1(x_{t|t}) + \alpha_1(w_{t|t}) + \tilde{c}$, with $\alpha_3, \sigma_3$ and $\tilde{c}$ defined in Lemma 3. In view of Theorem 3, the input sequence proceeds as follows. From (22) for $t \geq 0$, $l \geq 1$, $N_c \geq 0$, the input sequence $u^0_{t+1,Nc-1|t}$...
and finally, $h_f(\tilde{x}_{t+N|f+1}) - h_f(\tilde{x}_{t+N|f}) \leq c_6 L_f^{Np-1} + L_{fu}^{Np-1} \frac{L_{fu}}{L_{fu}-L_{gjm}} (L_{gjm} (\|\tilde{w}_{t+1|f+1}\|) + \|\tilde{w}_{t+1|f+1}\|)$.
Using (29), and in view of (28), $V_{t+1|f+1} - V_{t|f}$ is written as $V_{t+1|f+1} - V_{t|f} \leq -\alpha_2(x_t) + \sigma_1(\|w_t\|) + \sigma_2(\|\tilde{w}_{t+1}\|) + C$ for $\tilde{x}_t \in X_{\text{MPC}}, \forall \tilde{w}_t \in \tilde{W}_t, \tilde{w}_{t+1} \in \tilde{W}_{t+1}$, with $\alpha_2, \sigma_1, \sigma_2$, and $\tilde{C}$ are given in Lemma 3. Therefore, according to Theorem 1

$$|\tilde{x}_{t+1|f+1}| \leq \tilde{P} (|\tilde{x}_{t|f}|, I) + \tilde{y} (\|\tilde{w}_{t+1|f+1}\|) + \tilde{C} \quad (30)$$

Stability of the uncertain system is always a concern. The following theorem establishes the conditions that allow to conclude about the stability of the uncertain system based on the ISpS of the nominal model.

**Theorem 6:** If the nominal system (8) with tightened constraint set (20) is ISpS under the RH control law (15), then the uncertain system (6) is also ISpS.

**Proof:** In view of Assumption 1, $|x_{t+1} - \tilde{x}_{t+1|f+1}| \leq \tilde{n}_x$, hence $\tilde{x}_{t+1|f+1} \geq |x_{t+1} - \tilde{n}_x|$. Using (30), one gets $|x_{t+1}| \leq \tilde{P} (|x_{t|f}|, I) + \tilde{y} (\|\tilde{w}_{t+1|f+1}\|) + \tilde{C} + \tilde{n}_x$. Similarly, $\tilde{x}_{t|f} \leq |x_{t+1}| + \tilde{n}_x$, therefore from the proof of Theorem 5, $|x_{t+1}| \leq \tilde{P} (|x_{t|f}| + \tilde{n}_x, I) + \tilde{y} (\|\tilde{w}_{t+1|f+1}\|) + \tilde{C} + \tilde{n}_x \leq \tilde{P} (|x_{t|f}|, I) + \tilde{y} (\|\tilde{w}_{t|f+1}| + \tilde{n}_x, I) + \tilde{C} + \tilde{n}_x + \tilde{n}_x$. On the other hand $\tilde{w}_{t+1|f+1} \leq \tilde{P} (|x_{t|f}| + \tilde{n}_x, I) + \tilde{y} (\|\tilde{w}_{t|f+1}| + \tilde{n}_x, I) + \tilde{C} + \tilde{n}_x + \tilde{n}_x + \tilde{n}_x$. Consequently, using (31), one can conclude that the perturbed system (6) under RH control law (15) is ISpS. □

The following algorithm 1 summarizes the robust nonlinear model predictive control approach presented in this paper. Two classes of optimization problems are solved: offline and online.

**IV. SIMULATION EXAMPLE**

The numerical example of a simple nonlinear oscillator considered in this paper has been studied in the literature and will be used here with an important modification to illustrate the main concepts proposed in this paper. The simulation work used the semi-definite programming packages SDPT3-4.0 [39] and PENBMI [14], running on the optimization toolbox YALMIP [20]. The second order uncertain nonlinear oscillator system dynamic of interest has been modified to include a non-additive nonlinear disturbance and is given by

$$x_{1,t+1} = x_{1,t} + a_1[-x_{2,t} + a_2 (1 + x_{1,t}) u_t] (1 + w_{1,t}) + w_{1,t}$$
$$x_{2,t+1} = x_{2,t} + a_3[x_{1,t} + a_4 (1 - 4x_{2,t}) u_t] (1 + w_{2,t}) + w_{2,t} \quad (32)$$

where the uncertain parameters $a_1 = 0.051, a_2 = 0.4902, a_3 = 0.049$ and $a_4 = 0.5102$. The external input $w$ is driven by the following exosystem

$$w_{1,t+1} = -10^{-3} w_{1,t} + \phi_{1,t}$$
$$w_{2,t+1} = -10^{-3} w_{2,t} + \phi_{2,t}$$

where $|\phi_t| \leq 10^{-4}$ is a random noise. The constraints are $U = \{ u \in \mathbb{R} : -2 \leq u \leq 2 \}$, $X = \{ x \in \mathbb{R} : -0.125 \leq x_1 \leq 0.125, -0.125 \leq x_2 \leq 0.125 \}$ and $W = \{ w_{1,2} \in \mathbb{R} : -5 \times 10^{-3} \leq w_{1,2} \leq 5 \times 10^{-3} \}$. The nominal model is given as

$$\tilde{x}_{1,t+1} = \tilde{x}_{1,t} + \tilde{a}_1 [-\tilde{x}_{2,t} + \tilde{a}_2 (1 + \tilde{x}_{1,t}) u_t] (1 + \tilde{w}_{1,t}) + \tilde{w}_{1,t}$$
$$\tilde{x}_{2,t+1} = \tilde{x}_{2,t} + \tilde{a}_3 [\tilde{x}_{1,t} + \tilde{a}_4 (1 - 4\tilde{x}_{2,t}) u_t] (1 + \tilde{w}_{2,t}) + \tilde{w}_{2,t} \quad (33)$$

with the nominal parameters $\tilde{a}_1 = 0.05, \tilde{a}_2 = 0.5, \tilde{a}_3 = 0.05$ and $\tilde{a}_4 = 0.5$. On the other hand, the nominal dynamic model of the exosystem causing the disturbance is

$$\tilde{w}_{1,t+1} = 1.0110^{-3} \tilde{w}_{1,t}$$
$$\tilde{w}_{2,t+1} = 1.0110^{-3} \tilde{w}_{2,t}$$

The random measurement uncertainties are not known but bounded by $|x_t - \tilde{x}_t| \leq \tilde{n}_x = 10^{-3}$ and $|w_t - \tilde{w}_t| \leq \tilde{n}_w = 10^{-5}$. Let the cost functional be given by (??), with $Q = 0.1 \times I_2, R = 1, S = 10^{-3} I_2, N_p = 12$ and $N_c = 5$. The lipschitz constants are $L_f = 1.189, L_{fu} = 0.8011, L_{fu} = 1.4142, L_{gjm} = 1.01 \times 10^{-3}, \alpha_1(\|r\|) = \alpha_2(\|r\|) = 0.1|\|r\|, L_{hi5} = 0.06325, L_{hu} = 2, L_{gjm} = 6.325 \times 10^{-4}$, $L_{gjm} = 1.414 \times 10^{-5}, \Delta_x = 6.237 \times 10^{-4}$ and $\Delta_w = 5 \times 10^{-8}$. The shrink constraint sets are determined using with (20). The exponential growth of the uncertainty, leads to tighter constraint sets which may lead in turn to unfeasible solution of OCP. Therefore, as it is expected and known in the literature, the maximum horizon which can be selected and still ensures feasibility is limited. To determine the set $M \subset Z$ within which the OCP for maximum terminal region is feasible, the algorithm starts from $Z$ and makes progressive

**Algorithm 1 Robust NMPC**

1. Define nominal model $\tilde{f}(\tilde{x}, u, w)$, the constraint sets $X, U$ and $W$, nominal cost (12) and error bounds as in Assumption 1.
2. Determine the tightened constraint sets using Algorithm.
3. Compute optimal terminal set $X_f$ and terminal control $k_f$.
4. Compute One-step controllability set $C_t(X_f)$.  
5. Compute Robust output feasibility set $X_{MPC}$ using Algorithm.
6. Initialization at $t, l = 0$.
7. While target state is not reached do.
8. Compute state $\tilde{x}_{t+l}$ and disturbance $\tilde{w}_{t+l}$.
9. Determine $u^0_{t+l,1}, t+l+Nc$ solution to the FHOCP 1.
10. Check feasibility of the optimal solution.
11. Implement first the control action, $u^0_t$.
12. End while.
Determination of the constraints. Therefore, set $M \subset Z$ is selected as as $-2 \leq u \leq 2$, $-0.1125 \leq \tilde{x}_1 \leq 0.1125$ and $-0.1125 \leq \tilde{x}_2 \leq 0.1125$. For that, the normalized set value is ($\hat{v} = 6$)

$$M = \left\{ \left[ \tilde{x}_1 \tilde{x}_2 u \right]^T \in \mathbb{R}^{n+m} : \tilde{c}_i \tilde{x} + \tilde{d}_i u \leq 1 \right\}$$

with $\tilde{c}_1 = \tilde{c}_3 = [-8.89, 0]$, $\tilde{c}_2 = \tilde{c}_4 = [0, 8.89]$, $\tilde{c}_5 = c_6 = [0, 0]$, $\tilde{d}_{1,2,3,4} = 0$, $\tilde{d}_5 = -0.5$ and $\tilde{d}_5 = 0.5$. The linearized nominal system is $\tilde{x}_{i+1} = A_i \tilde{x}_i + B_i u$, with $A_i = \begin{bmatrix} 1 + 0.025t_i & -0.05 \\ 0.05 & 1 - 0.1t_i \end{bmatrix}$ and $B_i = 0.025 [1 + \tilde{x}_{1,i} - 4\tilde{x}_{2,i} ]^T$. There are $\hat{v} = 8$ vertices and $\hat{S}$ has to be selected such that $-q(\tilde{x}, \tilde{w}) + \psi(\tilde{w}) \leq \tilde{x}^T \hat{S} \tilde{x}$. So, let $\hat{S} = 10 \cdot \hat{S}$. Figure 4 illustrates the results where $Q_f^\infty = \begin{bmatrix} 342.04 & 69.16 \\ 69.16 & 670.08 \end{bmatrix}$ and $K^\infty = \begin{bmatrix} -6.52 & -10.45 \end{bmatrix}^T$.

Starting from the warm start results, the optimum terminal region is computed for $a = 1$ as indicated in figure. Warm starting generally helps to improve the determination of a larger optimum terminal region. Figure 2 shows the evolution of the tightened constraints sets during the prediction horizon. The exponential growth rate of the uncertainty depends on the size of the bounds relative to the disturbance, dynamic uncertainty and noise level.

$$Q_f = \begin{bmatrix} 6222.7 & 631.1 \\ 631.1 & 3709.7 \end{bmatrix}, \quad K = \begin{bmatrix} -11.66 & -36.33 \end{bmatrix}$$

Determination of $C_1(X_f)$ leads to $\tilde{x}_i \in \partial \left( C_1(X_f) \right)$ for $i = 1, \ldots, \tilde{N}$. The minimum size of $C_1(X_f)$ is $\bar{d} = \min \left( \left| \tilde{x}^i - \tilde{x}_j \right| \right)$, $\forall i = 1, \ldots, \tilde{N} = 6.18 \times 10^{-2}$. The left side of inequality (21) gives $1.34 \times 10^{-3} \leq \bar{d}$. Figure 3 displays the shape and evolution of the feasibility sets within the state space based on the initial feasible solution in $X_{MPC}$. Recursive feasibility of OCP 1 is guaranteed. In Figure 4, one can also observe that boundary of $X_{MPC}$ starts from the boundaries of the tightened state constraint set $\delta \tilde{x}_{T+Nc}$. This is
because the uncertainty bound is set at $\tilde{p}_{1+N_c}$. The generated optimal control signal can be seen in Figure 5. The online part of Algorithm 1 uses fmincon package of Matlab [2]. Several initial conditions has been tried as can be seen in Figure 7 shows state trajectories to the origin. Using the terminal set size one can observe the equivalent Tube-based dynamic

**V. CONCLUSION**

This paper addressed the problem of robust model predictive control of uncertain nonlinear system. Several sources of uncertainties have been considered and algorithms that can facilitate the implementation has been presented. Some of these algorithms are offline, while others are online. They can be used as a guideline to solve constraint tightening, terminal region optimization, feasibility determination and online optimization. The analysis confirmed several observations seen in the literature especially those related to tightened constraints and uncertainty growth over the prediction horizon. To maximize the terminal region and increase robustness, polytopic linear difference inclusions (PLDI) based on LMI formulation has been used. Only practical stability (ISpS) can be ensured for this type of system and the limit of disturbances’ levels that can be handled by the controller are bounded by the size of the one-step controllability set to the terminal constraint condition. Simulations results confirmed the theoretical analysis and showed performance of the proposed approach.

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Sami El-Ferik received the B.Sc. degree in electrical engineering from Laval University, Quebec, QC, Canada, and the M.S. and Ph.D. degrees in electrical and computer engineering from Ecole Polytechnique, University of Montreal, Montreal, QC.

His Ph.D. work, on flexible manufacturing systems modeling and control, was co-supervised with mechanical engineering. After completion of his Ph.D. and postdoctoral positions, he was worked as a Staff Control Analyst with the Research and Development Center of Systems, Controls, and Accessories, Pratt and Whitney Canada. He is currently an Associate Professor in control and instrumentation with the Department of Systems Engineering, King Fahd University of Petroleum and Minerals (KFUPM). His research interests are in sensing, monitoring, and control with strong multidisciplinary research and applications. His research contributions are in control of drug administration, process control and control loop performance-monitoring, control of systems with delays, modeling and control of stochastic systems, analysis of network stability, condition monitoring, and condition-based maintenance.