Inverse problem in the calculus of variations - functional exterior calculus approach

Radosław Antoni Kycia$^{1,2,a}$

$^1$Masaryk University
Department of Mathematics and Statistics
Kotlářská 267/2, 611 37 Brno, The Czech Republic

$^2$Cracow University of Technology
Department of Computer Science and Telecommunications
Warszawska 24, Kraków, 31-155, Poland

$^akycia.radoslaw@gmail.com$

Abstract

The well-known Helmholtz conditions are commonly used in the Inverse Problem of the Calculus of Variations (IPCV), including multiplier problems. However, their generality and enormous complexity for higher-order Lagrangians make demands for alternative and more straightforward methods. We present the functional version of the exterior differential forms that prove to be easier in calculations and understanding. Rigorous introduction to Functional Exterior Calculus (FEC) and its application to the multiplier problem in the calculus of variations is given. Many examples, including applied mathematics and physics, are presented. Using FEC we also formulate, for the first time, the problem for nonlinear transformation of equations to make them variational, called here nonlinear transformations problem. This is complicated, if even possible, to achieve using standard approach to IPCV using the Helmholtz conditions.

Keywords: functional exterior calculus; Euler-Lagrange equation; inverse problem in the calculus of variations; multipliers problem; nonlinear transformations; variationality

Mathematical Subject Classification: 49-02, 49N99

1 Introduction

The calculus of variations is the unique bridge between various mathematics disciplines as differential geometry and functional analysis. It also has an unprecedented role in physics, being a universal language for Field Theory and various
problems in other areas of science. Due to its importance, there have been developed many approaches to this subject, including the traditional way in terms of functional analysis [11, 12] and purely geometrical approach in terms of jet spaces [4, 13, 14].

From one side, for a given functional of suitable differentiability class, the derivation of the Euler-Lagrange equations is algorithmic. The most crucial step is integration by parts made under the assumption of a variation that vanishes at the boundary. The integration by parts is also responsible for complications of the jet approach to the calculus of variation. From the other side, this crucial step is the most problematic in the so-called inverse problems in the calculus of variations [24, 5, 6], where one has to check if a given set of differential equations comes from some Lagrangian. In the classical approach, we assume that we do not modify equations - we take them 'as they stand'. In this approach, a set of Helmholtz conditions was derived to check the variationality. In the more relaxed version of this problem, one initially allows a linear transformation of the equations to make them variational if there are not variational as they stand [5].

The inverse problem of the calculus of variation has a long history originating from the pioneering papers of Helmholtz through Veinberg [22] and the more recent approach of Anderson [5]. The general summary from the viewpoint of the geometrical (jet bundle) approach is in [24].

Most approaches to the Inverse Problem of the Calculus of Variations (IPCV) are based on Helmholtz conditions (HC) which complexity grows enormously with the degree of Lagrangian. The other approach that relates variation operation properties to exterior derivative, proposed by Aldrovandi and Kraenkel in [2, 3] leads to the same results in a usually more straightforward way than Helmholtz conditions. We will call it in this paper Variational Functional Exterior Calculus (VFEC).

In this paper we provide extension and generalization of VFEC, defining Functional Exterior Derivative (FED), which is general exterior derivative on functionals and in special cases is variational derivative. This simplifies considerations, has more profound insight into IPCV, and solves economically inverse problems, including the multiplier problem. Moreover, it allows the formulation of a nonlinear transformations approach to the Inverse Problem, which is impossible using HC.

The paper is organized as follows: In the next section we formulate FEC and provide standard results from the Inverse Problem of the Calculus of Variations obtained in this way. This should convince the reader about the effectiveness of this approach. The following section describes the multiplier problem and its clean formulation in terms of functional exterior calculus. Finally, we provide a discussion of the nonlinear transformation formulation of the Inverse Problem. FEC bases on variations of the ‘shape’ of the function. In the appendix we present what variation in the general case is.

2
2 Functional Exterior Calculus

This section defines the FEC that combines the geometric theory of the calculus of variations and the functional analytic approach.

2.1 Introduction

First, we fix the notation by restoring the classical results of the calculus of variations.

Consider a fiber bundle \( \pi : E \to M \) with typical fiber \( W \). The sections \( \Gamma(E) \) are the functions of our interests, e.g., physical fields.

Since our considerations will be local, we restrict the whole \( M \) to an open bounded subset, and therefore locally we will consider a space a product space of fiber and base open subset.

Then we construct a jet bundle \( J_k(\pi) \), \( k > 0 \) over \( E \), see [19]. When \( (U, u) \) is a coordinate system on \( E \) with \( u = (x^i, u) \). Then the coordinates on \( J_k(\pi) \) is given by \( (U_k, u_k) \) defined by

\[
x^i(j^k\phi) = x^i(p),
\]

\[
u(j^k\phi) = u(\phi(p)),
\]

\[
u_I(j^k\phi) = \frac{\partial^{|I|} \phi}{\partial x^{|I|}}|_p, \quad |I| \leq k.
\]

(1)

Next, we can extend it to the infinite jet bundle \( J^\infty(\pi) \) as an equivalence class of the sections of \( \pi \) with the contact of all degrees. We also have the usual projections \( \pi_{\infty,k} : J^\infty(\pi) \to J_k(\pi) \).

We can now define a Lagrange density function/Lagrangian defined on \( J_k(\pi) \) treated as projection of the element of \( J^\infty(\pi) \),

\[
j^k\phi \to L(j^k\phi)dx : J_k(\pi) \to \Lambda^{dim(M)}(M),
\]

(2)

where \( dx \) is the volume element on \( M \). In view of applications, in the calculus of variations it is assumed that \( L(\cdot)dx \in C^\infty(J^k(\pi)) \otimes \Lambda^{dim(M)}(M) \), i.e., \( L(\cdot) \) is a smooth function of all its variables.

Finally, we can define the basic object of the calculus of variation, the (action) functional

\[
(j^k\phi)^*F = F[j^k\phi] := \int_{\pi U} L(j^k\phi)dx,
\]

(3)

where the integration is carried over an open bounded set \( \pi U \). It is usually assumed that \( \partial \pi U \) is a regular submanifold.

The (Gateaux) directional derivative \( 1 \) of the functional \( F \) is

\[
DF[\phi] \cdot \eta = \lim_{\epsilon \to 0} \frac{F[\phi + \epsilon\eta] - F[\phi]}{\epsilon},
\]

(4)

\( \text{IIf Gateaux derivative is continuous then there exists differential (Frechet derivative) \[1\].} \)
for some variation $\eta \in C^\infty(U \times J^k(\pi))$ usually with additional assumptions explained below. The variation is a function of $E$ and $j^k\phi \in J^k(\pi)$.

We therefore have

$$DF[\phi] \cdot \eta := \int_{\pi U} \left( \frac{\partial f}{\partial u}(\phi) \eta(\phi) + \ldots + \frac{\partial f}{\partial u_K}(\phi) D_{x_K} \eta(\phi) \right) dx,$$

for $|K| = k$, where $D_{x_K}$ is the total derivative.

Integrating by parts one gets

$$DF[\phi] \cdot \eta = \int_{\pi U} \left( \frac{\partial L}{\partial u}(\phi) \eta + \ldots + (-1)^{|K|} D_{x_K} \frac{\partial L}{\partial u_K}(\phi) \right) \eta(\phi) dx + \int_{\partial \pi U} d\sigma \left( \frac{\partial L}{\partial u_i}(\phi) \eta(\phi) \right),$$

where the second integral is a boundary term and $d\sigma = \partial_i \cdot dx$. The two typical conditions for vanishing the boundary term are:

- Dirichlet boundary conditions - $\eta(\phi)|_{\partial \pi U} = 0$ for a function with assigned boundary value $\phi|_{\partial \pi U}$;
- Neumann boundary conditions - $\frac{\partial L}{\partial u_i}(\phi)|_{\partial \pi U} = 0$ and $\eta$ is unassigned/arbitrary at the boundary of $\pi U$.

See [8] for analogous definitions for jet bundle approach.

Since $\eta$, apart of additional conditions on vanishing boundary term, is arbitrary, it leads to the Euler-Lagrange equation for an extremum of $F$ and the definition of variational derivative $\frac{\delta F}{\delta \phi}$:

$$\left( j^k \phi \right)^* \left( \frac{\delta F}{\delta \phi} \right) = \left( j^k \phi \right)^* \left( \frac{\partial L}{\partial u} - \partial_x \frac{\partial L}{\partial u_i} + \ldots \right) = 0.$$  \hspace{1cm} (7)

Using the pairing we can rewrite this expression, and this will be done in the following subsection. Recalling from Supplement 2.4C of [1].

**Definition 1.** For a two Banach spaces $E$ and $F$ a bilinear functional $< \cdot, \cdot >$: $E \times F \rightarrow \mathbb{R}$ is $E$-nondegenerate if from $< x, y > = 0$ for all $y \in F$ it results that $x = 0$. Similarly $F$-nondegeneracy is defined.

If the mappings $E \rightarrow E^*$ ($E^*$ is dual of $E$) defined by $x \rightarrow < x, \cdot >$ is an isomorphism, then the functional is called $E$-strongly nondegenerate. A similar definition occurs for $F$-strong nondegeneracy.

If the functional $<,>$ is nondegenerate, then it is called pairing and $E$ and $F$ are in duality. For strongly nondegenerate pairing, the duality is called strong.

**Example 1.** In the calculus of variations, $E = C_0^k(\pi U) = F$, $k > 0$ and the pairing is $L^2$ pairing

$$<,>: C_0^k(\pi U) \times C_0^k(\pi U) \rightarrow \mathbb{R}, \quad < f, g > = \int_D f(x)g(x)dx.$$  \hspace{1cm} (8)

Then the first variation can be written as

$$DF[\phi] \cdot \eta = \left< \frac{\delta F}{\delta \phi}, \eta \right> = \int_U \frac{\delta F}{\delta \phi} \eta dx.$$  \hspace{1cm} (9)
The above pairing, due to the use of integral and vanishing of the the boundary term have the property
\[ < f, \partial^\alpha g > = (-1)^{|\alpha|} < \partial^\alpha f, g >, \tag{10} \]
for $|\alpha| \leq k$. This will be used in what follows.

2.2 Development

Returning to the formula (5) we reinterpret it as a functional version of exterior derivative.

First we remind [8, 9] definition of variation of the section $\phi \in \Gamma(E)$.

**Definition 2.** For a vertical vector field $v \in \Gamma(TE)$, i.e., $\pi_* v = 0$ or in local coordinates $v(x^i) = 0, \forall i \in \{1, \ldots, \dim(M)\}$ we have
\[ v_\eta = \eta(x,u) \partial_u. \tag{11} \]
where $\eta \in C^\infty(E)$ is called variation.

The variational vector field is the prolongation of such vertical vector field, that we will denote here by the same symbol, and which in coordinates has a form
\[ v_\eta = \eta(x,u) \partial_u + \sum_{|I| \leq k} (D_I \eta) \partial_{u_I}, \tag{12} \]
where $k < \infty$ is a number adjusted to applications, and where $D_I$ is the total derivative for multiindex $I$, see e.g., [18, 19]. The variational vector field induces a vertical flow on $E$ and its prolongation on $J^k(\pi)$.

In the Appendix A we will present general variations that also involve change/flows on the base manifold $M$.

We now rewrite (5) using the following pairing
\[ (j^\infty_\phi)^* \left\langle \left[ \frac{\partial L}{\partial u}, \ldots, \frac{\partial L}{\partial u_K} \right], v \right\rangle = \int_{\pi U} \left( \frac{\partial L}{\partial u}(\phi) \eta(\phi) + \ldots + \frac{\partial L}{\partial u_K}(\phi) \eta_K(\phi) \right) dx, \tag{13} \]
Since this expression is linear in the second slot, we can define a (functional) 1-form $\rho_F$
\[ \rho_F := \sum_{|I| \leq k} \frac{\partial L}{\partial u_I} \delta u_I, \quad |K| = k, \tag{14} \]

\[ ^2\text{In this paper we consider for simplicity only } C^\infty \text{ mappings or their subclasses. However, since the pairing is the integral, we can extend the maps classes as in the distributional calculus. The simplest case is to complete smooth functions to suitable Sobolev spaces [10, 20].} \]
where integration sign is not written and where 1-forms fulfil

\[ v_\delta u_I = D_I \eta, \quad |I| \leq k, \quad (15) \]

and zero otherwise. The integration in \( \rho F \) is performed only after inserting variational vector field and evaluating functional on some section \( \Gamma(E) \). From this viewpoint it must be considered as a formal expression. Yet, its properties can be easily checked by evaluating it on variational vector field and some section of \( E \).

We have

\[ \delta u_K = D_K \delta u, \quad (16) \]

that results from considering insertion of some \( v_\eta \).

We can now define

**Definition 3.** The functional exterior derivative is defined as

\[ \rho := \sum_{|K| \leq d} \delta u_K \wedge \frac{\partial}{\partial u_K} \quad (17) \]

This notation involves only vertical coordinates on \( J^\infty(\pi) \) and hides dependence on \( x \) coordinates of base manifold. In [18] there is also a notion of ‘functional forms’ calculus, however it differs to those presented here, since the forms of [18] are also base manifold forms, i.e., it mix \( dx \) and \( du_K \) coordinates.

We can then define functional exterior vector space \( \Lambda_F \) that contains the forms of all degrees with only a finite terms non-zero. It is a projection of a multilinear integral form on \( F \in J^\infty(\pi U) \times T^J^\infty(\pi U) \)

\[ F = \int_{\pi U} \sum_{k_1, \ldots, k_l} f^{k_1 \cdots k_l}(u) \delta u_{k_1} \otimes \cdots \otimes \delta u_{k_l}, \quad (18) \]

to antisymmetric tensors

\[ F = \int_{\pi U} \sum_{k_1, \ldots, k_l} f^{k_1 \cdots k_l}(u) \delta u_{k_1} \wedge \cdots \wedge \delta u_{k_l}. \quad (19) \]

In this vector space the FED makes a complex.

For example,

\[ \delta u \wedge \delta u_x, \quad (20) \]

can be seen as an antisymmetrization of

\[ \delta u \otimes \delta u_x := \int_{\pi U} \delta u \otimes \delta u_x dx. \quad (21) \]

The difference between variational derivative \( \delta \) and functional exterior derivative \( \rho \) defined above results from integration by parts and neglecting boundary
term by choosing variational vector field that fulfills suitable boundary conditions. We have the Euler(-Lagrange) functional 1-form \( \mathcal{E} = E\delta u \) from the following:

\[
\delta F = E\delta u = \frac{\delta F}{\delta u} \delta u := \int_{\pi U} dx \sum_{|I| \leq k} (-1)^{|I|} D_{x_I} \frac{\partial L}{\partial u_K} \delta u.
\]  

(22)

Since the pairing is the integration, we have

**Corollary 1.** When variation is taken so that the boundary term vanishes, then one Euler form is equivalent to many functional differential forms of various degrees. Each such form is obtained by performing integration by parts.

This non-uniqueness is kept in our notation by distinguishing symbols \( \rho \) (functorial exterior derivative) and \( \delta \) (variational exterior derivative).

As a simple example consider a form

\[
u_{xx}\delta u = \{u_{xx}\delta u; \ -u_x\delta u_x; \ u\delta u_{xx}\}.
\]

(23)

Moreover, applying FED one gets

\[
\rho u_{xx}\delta u = \{\rho u_{xx} \wedge \delta u; \ -\delta u_x \wedge \delta u_x; \ \delta u \wedge \delta u_{xx}\},
\]

(24)

which are all related by integration by parts.

In what follows we will be considering variations that nullify boundary terms. We will present general situations in the Appendix A.

### 2.3 Inverse problem of the calculus of variations

The inverse problem of the calculus of variations can now be stated as follows: start from the given system of equations \( \{E_\alpha\}_{\alpha} \) and construct Euler functional 1-form

\[
\mathcal{E} = E_\alpha \delta u^\alpha.
\]

(25)

from the equations \( E_\alpha[\phi] \) by passing to \( J^\infty(\pi) \) and then multiply by a functional 1-form \( \delta u^\alpha \). The central question is if the Euler functional form is an exterior functional derivative of some functional, i.e., if there exists \( F \) such that

\[
\rho F \equiv E_\alpha \delta u^\alpha,
\]

(26)

modulo boundary terms that are assumed to vanish when suitable variation is selected.

As it was pointed out in [2][3], the true power of mimic of exterior calculus for functionals - the variational functional exterior derivative - is the antisymmetry and its combination with the functional version of the Poincaré lemma. We have for functional exterior calculus the analog of Poincaré lemma:
Theorem 1. If the functional $1$-form $F$ is closed ($\rho F = 0$) in a starshaped neighbourhood of a $u_0$ section of the bundle, then it is (locally) a functional exterior derivative of a functional, i.e., (functionally) exact. The functional is given by the formula

$$G = G[u_0] + HF,$$

where $G[u_0]$ is a constant, and functional homotopy operator is defined by

$$HF = \int_0^1 K \wedge F[u_0 + t(u - u_0)]dt,$$

where the functional Euler vector field is

$$K = \sum_{|\kappa| = 0}^{[F]} \partial^K(u - u_0)\partial_0u.$$  

Proof. The proof is a simple consequence of symmetry of the second derivative, which is automatically provided by the wedge product in functional exterior calculus defined above. This is the claim of the following

Theorem 2. ([22], Theorem 5.1) Suppose that the following conditions are fulfilled:

- $A$ is operator from $E$ to $E^*$, where $E$ is a real Banach space with the norm $||\cdot||$,
- $A$ has linear Gateaux differential $DA(u, h)$ at every point of the ball $D$: $||u - u_0|| < r$ for some $r \in \mathbb{R}^+$,
- The functional $(DA(u, h_1), h_2)$ is continuous in $x$ at every point of $D$.

Then, in order that the operator $A$ be potential (is a differential) in the ball $D$, it is necessary and sufficient that the bilinear functional $(DA(u, h_1), h_2)$ be symmetric for every $u \in D$, that

$$(DA(u, h_1), h_2) = (DA(u, h_2), h_1),$$

for every $h_1, h_2 \in E$ and every $u \in D$.

In the course of the proof of this theorem [22] the following homotopy formula was introduced

$$a(u) = a(u_0) + \int_0^1 (A(u_0 + t(u - u_0)), u - u_0)dt,$$

where $u_0$ is a center of linear homotopy: $[0; 1] \ni t \to u_0 + t(u - u_0)$.

3 Usually called in the context of the calculus of variation the Helmholtz-Infeld theorem, however, it can also be seen as a functional analysis version of the Poincare lemma.
From Theorem 2, it results that the second functional exterior derivative must vanish in order for a functional 1-form to be an exact functional form, i.e., to the existence of a Lagrangian. This is a simple consequence of contracting symmetric second derivative and antisymmetric 2-form. It is not important which of the representative one finds out to vanish making integrations by parts, since if one representative vanishes then all of them vanishes. This proves equivalence of the various Helmholtz-type conditions obtained by different ways to make integration by parts.

The homotopy formula (31) was rewritten to (28) to resemble those of non-functional formulas [8, 9, 17, 21, 23, 14, 16]. Moreover, by the integration by parts, and then applying the homotopy formula, one can choose such a representative that yields a Lagrangian of a lower degree than the degree of the equations. 

The homotopy formula can be extended to functional form $F$ of any degree by [8, 15]

$$HF = \int_{0}^{1} K \cdot_{\alpha} F[u_0 + t(u - u_0)]t^{|F|-1}dt,$$

with the homotopy invariance formula

$$(I - s_{u_0})F = H \rho F + \rho HF,$$

where $s_{u_0}$ is evaluation of a functional at $u_0$ and 0 for $|F| > 0$.

Therefore we have the fundamental theorem of the inverse problem of the calculus of variations in terms of functional differential forms

**Corollary 2.** The system $\{E_{\alpha}\}_{\alpha}$ is variational as it stands if the Euler form

$$\mathcal{E} := E_{\alpha}\delta u^{\alpha},$$

is closed, when pulled-back on the set $\{E_{\alpha}\}$, i.e.,

$$\rho \mathcal{E}|\{E_{\alpha}\}_{\alpha} = 0.$$ 

In contrast to our presentation, the symmetry of the derivative idea from Theorem 2 in the form of the adjointness of $DA$ was further explored in the jet spaces approach to the calculus of variations in [18], Chapter 5.

### 2.4 Center of homotopy and antiexact forms

Relating above theory to the ideas from [8, 9, 15, 16], the Lagrangian (functional 0-form) is an example of antiexact functional forms defined by

$$\mathcal{A} = \{F \mid F[u_0] = 0, \ K \cdot_{\alpha} F = 0\},$$

where $u_0$ is the center of homotopy and for zero forms the second condition ($K \cdot_{\alpha} F = 0$) is trivially fulfilled. The first condition ($F[u_0] = 0$) for functionals
in the image of homotopy operator $H$ is obvious from (27). Complementary, the exact functional 0-forms $\mathcal{E}^0$ are constant functionals with respect to functional exterior derivative. Then in analogous way as for finite-dimensional calculus we can decompose functional 1-forms into direct product

$$\mathcal{E}^0 \oplus A^0.$$  \hfill (37)

According to the homotopy invariance formula (33) we have that the operator $H\rho$ is the projector onto $A^0$.

Moreover, one can construct the structure of a homotopic harmonic oscillator that is based on operators $\rho$ and $H$ as annihilation and creation operators [15, 16].

From the physicist’s point of view, the introduction of antiexact forms instantiates the fact that the Lagrangian can be defined up to a term which is the Lagrangian evaluated at the arbitrary center of a homotopy $u_0$. It was pointed in [2, 3] that not always the center should be zero, e.g., for a field theory when a nonzero metric is a fiber coordinate. This is precisely the same situation as for the relation between forces and their potentials.

### 2.5 Examples

This section presents how FEC can easily solve the inverse problem of the calculus of variations. Some of the examples are trivial, however they were selected to make the introduction to FEC smooth for non-mathematicians.

#### 2.5.1 Harmonic oscillator

The first example is a harmonic oscillator equation with Euler form

$$\mathcal{E} = (u_{tt} + u)\delta u,$$ \hfill (38)

with all equivalent forms by integration by parts

$$(u_{tt} + u)\delta u, \quad -u_t\delta u_t + u\delta u, \quad u\delta u_{tt} + +u\delta u$$ \hfill (39)

If one assumes that the Lagrangian is of the form $L(u, u_t)$ then one have to choose only the middle form since other ones will generate $u_{tt}$ terms in Lagrangian. Checking all combinations introduces combinatorial complexity to this approach, which results from nonuniqueness between transition between functional exterior derivative and variational functional derivative using integration by parts. We are left with the form

$$\mathcal{E} = -u_t\delta u_t + u\delta u.$$ \hfill (40)

Its exterior derivative is

$$\rho\mathcal{E} = -\delta u_t \wedge \delta u_t + u\delta u \wedge \delta u = 0.$$ \hfill (41)
Therefore using homotopy formula, we have

\[ F[u] = \int_U dx \left( -\frac{u_t^2}{2} + \frac{u_x^2}{2} \right), \]

which differ by the sign from the usual action for the harmonic oscillator.

The same results one gets when one uses variational derivative and integration by parts to nullify the results \([2, 3]\), i.e.,

\[ \delta((u_{tt} + u)\delta u) = \delta u_{tt} \wedge \delta u + \delta u \wedge \delta u = -\delta u_t \wedge \delta u_t = 0. \]  

In this case it is a more economical approach, yet, at the end of computations, to nullify the result (if possible), one must make a combinatorial check of all possible integrations by parts.

### 2.5.2 Dumped harmonic oscillator

For dumped harmonic oscillator the Euler form is

\[ \mathcal{E} = (u_{tt} + u_t + u)\delta u. \]

The equivalent forms by integration by parts are

\[ (u_{tt} + u_t + u)\delta u, \quad -u_t\delta u_t + (u_t + u)\delta u, \quad -u_t\delta u_t - u\delta u_t + u\delta u, \]
\[ u\delta u_{tt} + (u_t + u)\delta u, \quad u\delta u_{tt} - u\delta u_t + u\delta u. \]  

We now assume that the possible Lagrangian is of the form \(L(u, u_t)\), which leaves the following elements in the equivalence class

\[ -u_t\delta u_t + (u_t + u)\delta u, \quad -u_t\delta u_t - u\delta u_t + u\delta u. \]  

The functional exterior derivative of any of these elements is nonzero, i.e.,

\[ \pm \delta u_t \wedge \delta u \neq 0, \]  

and cannot be nullified by integrations by parts. This means that the equation has no Lagrangian. Later it will be shown that there is a multiplier making this equation the Euler-Lagrange equation.

### 2.5.3 Heat equation

The next example is the heat equation with the Euler form

\[ (u_t - u_{xx})\delta u, \]

which represents equivalent functional 1-forms

\[ (u_t - u_{xx})\delta u, \quad -u_t\delta u_t - u_{xx}\delta u, \quad u_t\delta u_t + u_x\delta u_x, \]
\[ -u\delta u_t + u_x\delta u_x, \quad u_t\delta u - u\delta u_{xx}, \quad -u\delta u_t - u\delta u_{xx}. \]  

11
For the Lagrangian $L(u, u_t, u_x)$ one picks

$$u_t \delta u + u_x \delta u_x, -u \delta u_t + u_x \delta u_x,$$

which after taking functional exterior derivative lead to

$$\pm \delta u_t \wedge \delta u.$$

(51)

As it was pointed out in [2, 3] (using variational exterior derivative) the above approach shows which term spoils variationality. In the case of the heat equation it is $u_t$ term.

Similarly, this term is the obstruction for the equation $E[x] = \dot{x} - f(x)$ to be variational, i.e., the first order autonomous ODE is nonvariational.

### 2.5.4 Helmholtz conditions for the second order ODEs

Now we will derive Helmholtz conditions for the equations $\epsilon_i(x, \dot{x}, \ddot{x}) = 0$. We start from the Euler form

$$E = \epsilon_i \delta x^i.$$  

(52)

Computing functional exterior derivative we get

$$\rho E = \frac{\partial \epsilon_i}{\partial x^j} \delta x^j \wedge x^i + \left( \frac{\partial \epsilon_i}{\partial \dot{x}^j} - \frac{d}{dt} \left( \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \right) \right) \delta \dot{x}^j \wedge \delta x^i - \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \delta \ddot{x}^j \wedge \delta \dot{x}^i.$$

(53)

The vanishing (antisymmetrized) coefficients of 2-forms provide Helmholtz conditions, i.e.,

$$\left( \frac{\partial \epsilon_i}{\partial x^j} - \frac{d}{dt} \left( \frac{\partial \epsilon_i}{\partial \dot{x}^j} \right) \right) = 0,$$

$$\left( \frac{\partial \epsilon_i}{\partial \dot{x}^j} + \frac{\partial \epsilon_j}{\partial x^i} - \frac{d^2}{dt^2} \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \right) \delta \dot{x}^j \wedge \delta \dot{x}^i = 0,$$

$$\frac{\partial \epsilon_i}{\partial \ddot{x}^j} = 0,$$

(54)

where $[i, j]$ denotes antisymmetrization of $i, j$ indices.

However, in order to obtain those HC from Chapter 2 of [24] one has to provide quite complicated symmetrization procedure that involves splitting some of the terms in half and perform integration by parts. The resulting form is equivalent (modulo boundary term) to the one above. Due to this, one can also say that the resulting HC are equivalent modulo boundary terms. We provide such calculation.

Continuing transformation of (53) we get

$$\rho E = \frac{\partial \epsilon_i}{\partial x^j} \delta x^j \wedge x^i + \left( \frac{\partial \epsilon_i}{\partial \dot{x}^j} - \frac{d}{dt} \left( \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \right) \right) \delta \dot{x}^j \wedge \delta x^i +$$

$$\left( \frac{1}{2} \left( \frac{\partial \epsilon_i}{\partial x^j} + \frac{\partial \epsilon_i}{\partial x^j} \right) - \frac{d}{dt} \left( \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \right) - \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \delta \ddot{x}^j \wedge \delta \dot{x}^i \right) =$$

$$\left( \frac{\partial \epsilon_i}{\partial x^j} - \frac{1}{2} \frac{d}{dt} \frac{\partial \epsilon_i}{\partial \ddot{x}^j} + \frac{1}{2} \frac{d^2}{dt^2} \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \right) \delta \dot{x}^j \wedge \delta \dot{x}^i +$$

$$\frac{1}{2} \left( \frac{\partial \epsilon_i}{\partial x^j} + \frac{\partial \epsilon_i}{\partial x^j} - \frac{d}{dt} \left( \frac{\partial \epsilon_i}{\partial \ddot{x}^j} + \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \right) \right) \delta \ddot{x}^j \wedge \delta \dot{x}^i - \frac{\partial \epsilon_i}{\partial \ddot{x}^j} \delta \ddot{x}^j \wedge \delta \dot{x}^i,$$

(55)
which, by demanding that $E$ is closed, gives
\begin{align*}
\frac{\partial e_1}{\partial x} + \frac{\partial e_1}{\partial y} - \frac{d}{dt} \left( \frac{\partial e_1}{\partial x} + \frac{\partial e_1}{\partial y} \right) &= 0, \\
\frac{\partial e_1}{\partial x} - \frac{\partial e_1}{\partial y} - \frac{1}{2} \frac{d}{dt} \left( \frac{\partial e_1}{\partial y} - \frac{\partial e_1}{\partial x} \right) &= 0.
\end{align*}  
\eqref{eq:56}

That is the system (2.15)-(2.17) of Chapter 2 of \cite{24}. We also have

**Corollary 3.** Functional 1-form being exact is equivalent (modulo boundary terms) to that the Helmholtz conditions are fulfilled.

### 2.5.5 Circles in the plane

Consider the system
\begin{align*}
\frac{d}{dt}(\dot{x} - f_1(x, y)) &= 0 \\
\frac{d}{dt}(\dot{y} - f_2(x, y)) &= 0.
\end{align*}  
\eqref{eq:57}

The Euler form is
\[ E = (\dot{x} - f_1(x, y))\delta \dot{x} + (\dot{y} - f_2(x, y))\delta y. \]
\eqref{eq:58}

Its functional exterior derivative gives
\[ \rho E = -\frac{\partial f_1}{\partial x} \delta x \wedge \delta \dot{x} - \frac{\partial f_1}{\partial y} \delta y \wedge \delta \dot{x} - \frac{\partial f_2}{\partial x} \delta x \wedge \delta \dot{y} - \frac{\partial f_2}{\partial y} \delta y \wedge \delta \dot{y}. \]
\eqref{eq:59}

$E$ is closed if
\[ \frac{\partial f_1}{\partial x} = 0, \quad \frac{\partial f_2}{\partial y} = 0, \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}. \]
\eqref{eq:60}

We have the only solution $f_1(y) = y$ and $f_2(x) = -x$. We therefore ended with
\[ E = (\dot{x} - y)\delta \dot{x} + (\dot{y} + x)\delta \dot{y}, \]
\eqref{eq:61}

that after applying homotopy formula gives (electro)magnetic action functional
\[ F[x, y] = \int dxdy (\dot{x}^2 + \dot{y}^2 + x\dot{y} - y\dot{x}). \]
\eqref{eq:62}

The solution of the Euler-Lagrange system gives circles in the plane.

### 3 The multiplier problem in the functional exterior calculus

The general multiplier problem in variational calculus is formulated as follows, having the set of equations/Euler forms coefficients \( \{E_\alpha\} \) which are not variational
as they stand find (if possible) a linear transformation/matrix \( A = [a^\beta_\alpha] \) such that the new set of equations
\[
E'_\alpha := a^\beta_\alpha E_\beta,
\]
is variational.

The order of variables is also not fixed in classical formulation of the problem, meaning, that the term, say, \( \ddot{x} \) can come from the Lagrangian \( \ddot{y} \), i.e., from \( \ddot{x}\delta y \sim \delta y \) and not the Lagrangian \( \dot{x}^2 \), i.e., the term \( \dot{x}\delta \dot{x} \sim \delta x \) by integration by parts.

In the FEC presented above the complete solution of the multiplier problem can be reformulated as follows:

**Theorem 3.** For a system of equations \( \{E_\alpha\}_{\alpha=1}^r \) for variables \( \{u^1, \ldots, u^r\} \) find a matrix \( a^\beta_\alpha \) such that the expression
\[
\mathcal{E} = [\delta u^1, \ldots, \delta u^r] \begin{bmatrix} a^1_1 & \ldots & a^1_r \\ \vdots & \ddots & \vdots \\ a^r_1 & \ldots & a^r_r \end{bmatrix} \begin{bmatrix} E_1 \\ \vdots \\ E_r \end{bmatrix}, \tag{64}
\]
is variational, i.e.,
\[
\rho \mathcal{E}|_{\{E_\alpha\}_\alpha} = 0. \tag{65}
\]

Note that in this formulation the order of variables \( u^i \) is irrelevant since the multiplier matrix \( A \) also mix the order of coordinates in the vector \( [\delta u^1, \ldots, \delta u^r] \). Therefore the functional reformulation resolves this issue. The check of the variationality of (65) is a straightforward yet sometimes tedious calculation, which is no harder and always easier than applying the general version of HC.

The examples will be provided in the following subsection.

### 3.1 Examples

#### 3.1.1 Dumped harmonic oscillator

We will find multiplier \( \lambda(t) \) for a dumped harmonic oscillator. Consider the Euler form with multiplier
\[
\mathcal{E} = \lambda(u_{tt} + bu_t + u)\delta u. \tag{66}
\]
We integrate by parts the first term since we assume the Lagrangian in the form \( L(u, u_t) \), we get
\[
\mathcal{E} = -u_t(\lambda_t \delta u + \lambda \delta u_t) + \lambda_b u_t \delta u + \lambda u \delta u. \tag{67}
\]
The closedness condition is
\[
\rho \mathcal{E} = -\lambda_t \delta u_t \wedge \delta u + b \lambda \delta u_t \wedge \delta u = 0, \tag{68}
\]
which gives equation for \( \lambda \)
\[
\lambda_t = b \lambda, \tag{69}
\]
that is solved by

\[ \lambda(t) = Ce^{bt}, \]

where \( C \) is a constant. Substituting \( \lambda \) to (67) we get

\[ \mathcal{E} = \lambda(-u_t \delta u_t + u \delta u), \]

which gives

\[ \int_{\pi U} dx \lambda \left( -\frac{u_t^2}{2} + \frac{u^2}{2} \right), \]

which is the standard form of the action for dumped harmonic oscillator. In this example one can note that the multiplier is responsible for deleting the term that spoils closedness.

### 3.1.2 Sonin problem

As pointed out in Chapter 2 of [24] the Sonin problem is to find a multiplier function \( g(t, x, \dot{x}) \) for the equation \( E = \ddot{x} - F(t, x, \dot{x}) \). The Euler 1-form is

\[ \mathcal{E} = g(\ddot{x} - F) \delta x = -\dot{x} \frac{d}{dt}(g \delta x) - g F \delta x. \]  

Computing functional exterior derivative one gets

\[ \rho \mathcal{E}|_{E=0} = -\delta \dot{x} \wedge \frac{d}{dt}(g \delta x) - \dot{x} \frac{d}{dt}(\rho g \wedge \delta x) - \frac{\partial g F}{\partial \dot{x}} \delta \dot{x} \wedge \delta x = \]

\[ \left( \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \dot{x} + \frac{\partial g F}{\partial x} \right) \delta x \wedge \delta \dot{x}, \]

so \( \mathcal{E} \) is closed on solutions when

\[ \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \dot{x} + \frac{\partial g F}{\partial x} = 0, \]

and coincides with equation (2.61) of Chapter 2 of [24].

The extension of this problem is a Sonin-Douglas problem when one deals with many equations and the multiplier matrix, see [24] for the solution using Helmholtz conditions.

### 3.1.3 Multiplier problem for the first order ODE

For the first order ODE \( E = \dot{x} - f(t, x) \) we have the Euler form with multiplier:

\[ \mathcal{E} = g(\dot{x} - f) \delta x. \]

Computing functional exterior derivative, we get

\[ \rho \mathcal{E}|_{E=0} = g \delta \dot{x} \wedge \delta x. \]

Therefore \( \mathcal{E} \) is closed only when \( g = 0 \), that is, there is no multiplier for this problem, which is a well-known classical result [5][6].
3.1.4 Point in $\mathbb{R}^2$

Consider the system

$$\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= 0.
\end{align*}$$

(78)

The multiplier problem is related to the closedness of the following functional 1-form

$$E = [\delta x, \delta y] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = (a\dot{x} + b\dot{y}) \delta x + (c\dot{x} + d\dot{y}) \delta y,$$

(79)

where matrix elements $a, b, c, d$ are assumed to be numbers. Applying functional exterior derivative we get

$$\rho E = a\delta \dot{x} \wedge \delta x + b\delta \dot{y} \wedge \delta x + c\delta \dot{x} \wedge \delta y + d\delta \dot{y} \wedge \delta y.$$

(80)

The form $E$ is closed if $a = 0 = d$ and $b = -c$, so

$$E = b(\dot{y}\delta x - \dot{x}\delta y).$$

(81)

The functional is therefore

$$F[x, y] = b \int dt (\dot{y}x - \dot{x}y).$$

(82)

This example shows that for a two-dimensional system, there is a multiplier, in contrast to one dimensional ODE above.

3.1.5 Straight line in $\mathbb{R}^2$

Consider now a slight generalization of the system from the previous example

$$\begin{align*}
\dot{x} &= x \\
\dot{y} &= y.
\end{align*}$$

(83)

The multiplier problem is related to the closedness of the following functional 1-form

$$E = [\delta x, \delta y] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \dot{x} - x \\ \dot{y} - y \end{bmatrix},$$

(84)

where now we assume that the elements $a, b, c, d$ are functions of $t$. The functional exterior derivative gives

$$\rho E = a\delta \dot{x} \wedge \delta x + b\delta \dot{y} \wedge \delta x + c\delta \dot{x} \wedge \delta y + (b - c + b)\delta \dot{x} \wedge \delta y + d\delta \dot{y} \wedge \delta y,$$

(85)

which vanish when $a = 0 = d$ and $b = -c$, that gives $2b + \dot{b} = 0$ with solution

$$b(t) = Ae^{-2t} = -c(t),$$

(86)

for some nonzero constant $A$. We therefore have

$$E = Ae^{-2t}((\dot{y} - y)\delta x - (\dot{x} - x)\delta y)$$

(87)

which gives functional

$$F = A \int dx e^{-2t}(xy - y\dot{x}).$$

(88)
4 Nonlinear transformations

Using the framework of functional exterior calculus, we can formulate a nonlinear version of the inverse problem of the calculus of variations. We want to stress here that for the approach using Helmholtz conditions, to our best knowledge, there is even no attempt to formulate the nonlinear transformations method due to the complexity of HC. This shows the economy and efficiency of functional exterior forms.

We have at once

**Theorem 4. (Nonlinear Transformations Problem)** Consider a system \( \{ E_\alpha \}_\alpha \) of differential equations for variables \( \{ u^\alpha \}_\alpha \). For some (possibly nonlinear) smooth functions \( \{ F_\alpha \}_\alpha \) construct the functional nonlinear Euler 1-form

\[
E = F_\beta \{ [E_\alpha]_\alpha \} \delta u^\beta. \tag{89}
\]

Then there exists a functional for this system iff

\[
\rho E|_{E_\alpha} = 0. \tag{90}
\]

This equation provides a system of equations for unknown functions \( \{ F_\alpha \}_\alpha \).

A particular case of this theorem is the multiplier problem described above.

In the following subsection we provide an example that shows how the nonlinear transformation method can be used.

4.1 Example

Consider again the system

\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= 0.
\end{align*} \tag{91}
\]

We will construct an Euler functional form using nonlinear transformations. Assume that there exists two \( C^\infty(\mathbb{R}^2) \) functions \( F \) and \( G \) such that the Euler form is

\[
E = F(\dot{x}, \dot{y}) \delta x + G(\dot{x}, \dot{y}) \delta y. \tag{92}
\]

Evaluating functional exterior derivative and equate it to zero we get

\[
0 = \rho E = \frac{\partial F}{\partial \dot{x}} \delta \dot{x} \land \delta x + \frac{\partial F}{\partial \dot{y}} \delta \dot{y} \land \delta x + \frac{\partial G}{\partial \dot{x}} \delta \dot{x} \land \delta y + \frac{\partial G}{\partial \dot{y}} \delta \dot{y} \land \delta y. \tag{93}
\]

At once we have that

\[
\frac{\partial F}{\partial \dot{x}} = \frac{\partial G}{\partial \dot{y}}, \tag{94}
\]

i.e., \( F = F(\dot{y}) \), \( G = G(\dot{x}) \). Next, integrating by parts we get

\[
0 = \rho E = - \left( \frac{\partial^2 F}{\partial \dot{y}^2} \dot{y} \delta x + \frac{\partial F}{\partial \dot{x}} \delta \dot{x} \right) \land \delta y + \frac{\partial G}{\partial \dot{x}} \delta \dot{x} \land \delta y, \tag{95}
\]
from which results that
\[ \frac{\partial^2 F}{\partial \dot{y}^2} \rightarrow F(\dot{y}) = a\dot{y}, \]  \hspace{1cm} (96)
for some constant \( a \). The second condition is
\[ \frac{\partial F}{\partial \dot{y}} = -\frac{\partial G}{\partial \dot{x}}, \]  \hspace{1cm} (97)
that is, \( G(\dot{x}) = -a\dot{x} \). As a result, the Euler form is
\[ \mathcal{E} = a(\dot{y}\delta x - \dot{x}\delta y), \]  \hspace{1cm} (98)
which gives, as in multiplier problem, the functional in the form
\[ F[x, y] = a \int dt(\dot{x}y - \dot{y}x). \]  \hspace{1cm} (99)

We therefore have

**Corollary 4.** For the system (91), the general nonlinear method for inverse problem gives the same solution as a linear multiplier problem. In this case this is the only solution when assuming the following dependence \( F(\dot{x}, \dot{y}), G(\dot{x}, \dot{y}) \) of \( F \) and \( G \) on equations.

## 5 Conclusions

In this paper we defined Functional Exterior Calculus and showed that it is a flexible tool for computations in variational calculus. Using this framework one does not have to derive or remember the Helmholtz conditions to check the variationality of a given system of differential equations. This enormously easy computation for higher-order Lagrangians where the complexity of the Helmholtz conditions grows.

Application of the Functional Exterior Calculus to the multiplier problem was given. Main advantage comparing to the classical approach is that this approach removes the problem of the ordering of equations present in the approach using Helmholtz conditions.

Finally, the nonlinear transformations method for the inverse problem of the calculus of variations was formulated and easily solved using FEC. This, if at all possible, cannot be done easily using solely Helmholtz conditions.

We hope this new tool for calculations will become the first choice in applications in mathematics and sciences as physics, and engineering.
Acknowledgments

This research was supported by the GACR grant GA19-06357S, the grant 8J20DE004 of Ministry of Education, Youth and Sports of the CR, and Masaryk University grant MUNI/A/0885/2019. RK also thank the SyMat COST Action (CA18223) for partial support.

I would like to thanks prof. Vladimir Matveev and all of his group in Jena for the invitation and inspiring discussions. I also want to thank Josef Šilhan for continuous support.

A Arbitrary variations and boundary functional forms

We will now consider general variations for completeness. The results contained here are classical and provided here for comparison with the existing literature.

In this section we assume that the section $\phi$ fulfills Euler-Lagrange equations, and therefore only boundary term remains from the whole variational derivative.

The first case consists of variations involving the transformation of base manifold. Consider a one parameter group of diffeomorphisms $x' = \Phi_t(x)$ with the generator $X = X^i \partial_{x^i}$, where $X^i = \frac{d\Phi^i}{dt}|_{t=0}$. For a compact oriented submanifold $V$ we define $V(t) = \Phi_t V$, and form the functional

$$F(t) = \int_V L dx. \quad (100)$$

Differentiating with respect to the parameter $t$ we get

$$\dot{F}|_{t=0} = \int_V \mathcal{L}_X (L dx) = \int_{\partial V} L(X_i dx) = \int_{\partial V} L X^i d\sigma_i, \quad (101)$$

where $d\sigma_i = \partial_{x^i} dx$. We can rewrite this result by introducing $\delta x^i$ such that

$$X_i \delta x^i = X^i. \quad (102)$$

Then we get

$$\dot{F}|_{t=0} = \partial_{X^i} \int_{\partial V} d\sigma_i L \delta x^i. \quad (103)$$

The total variation $\eta_T$ of a section $\phi$ contains its change with respect to the flow generated by $X$, i.e., $X_\phi = (j^\infty \phi)^*(X^i u_i)$ and the change under variational field $v = \eta \partial_u + \sum_{|I|<k} D_I \eta \partial_{u_I}$, which gives

$$\eta_T(x, \phi) = X_\phi + \eta(x, \phi). \quad (104)$$
Assuming that \( \phi \) fulfils Euler-Lagrange equations, the boundary term of \( \eta \) variation is

\[
v \gamma \rho F = \int_{\partial V} d\sigma \left( \frac{\partial L}{\partial u^i} (\phi) \eta (\phi) \right) = \int_{\partial V} d\sigma \left[ \frac{\partial L}{\partial u^i} (\phi) (\eta_T (x, \phi) - X \phi) \right] = (j^\infty \phi)^* \left( X \gamma v_T \int_{\partial V} d\sigma \frac{\partial L}{\partial u^i} (\delta_T u - u_i \delta x^i) \right),
\]

(105)

where \( v_T = \eta_T \partial_a + \sum_{|I| \leq k} D_I \eta_T \partial_{u_I} \), and \( v_T \gamma \delta_T u_I = D_I \eta_T \).

Combining base variation (103) and vertical variation (105) we get the total variation formula

\[
\delta F = \int_{\partial V} d\sigma \left( \left( L - \frac{\partial L}{\partial u^i} u_i \right) \delta x^i + \frac{\partial L}{\partial u^i} \delta_T u \right),
\]

(106)

which agrees with the standard formula from the physics books, e.g., Eq. (163) from [7].

References

[1] R. Abraham, J.E. Marsden, T.Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd edition, Springer, 1988

[2] R. Aldrovandi, R.A. Kraenkel, *On exterior variational calculus* J. Phys. A: Math. Gen. 21 1329 (1988); DOI: 10.1088/0305-4470/21/6/010

[3] R. Aldrovandi, J.G. Pereira, *An Introduction to Geometrical Physics*, 2nd edition, World Scientific 2016; Chapter 20

[4] I.M Anderson, *Variational Bicomplex*, unpublished script

[5] I. Anderson, G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations*, 98, 473, Memoirs of the American Mathematical Society, 1992

[6] M. Dafinger, *Existence of a Variational Principle for PDEs with Symmetries and Current Conservation*, [arXiv:1906.10976]

[7] F. Dyson, *Advanced Quantum Mechanics*, World Scientific, 2007

[8] D.G.B. Edelen, *Applied Exterior Calculus*, Dover Publications, Revised edition, 2011

[9] D.G.B. Edelen, *Isovector Methods for Equations of Balance*, Springer, 1980

[10] F.G. Friedlander, M. Joshi, *Introduction to the Theory of Distributions*, 2nd edition, Cambridge University Press 1999

[11] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Dover Publications, 2000
[12] M. Giaquinta, S. Hildebrandt, *Calculus of Variations*, 2 vols. Springer 2010
[13] D. Krupka, *Introduction to Global Variational Geometry*, Atlantis Press, 2015
[14] O. Krupkova, *The Geometry of Ordinary Variational Equations*, Springer 1997
[15] R.A. Kycia, *The Poincare Lemma, Antieexact Forms, and Fermionic Quantum Harmonic Oscillator*, Results Math 75, 122 (2020); DOI: 10.1007/s00025-020-01247-8
[16] R.A. Kycia, *The Poincare lemma for codifferential, anticoexact forms, and applications to physics*, [arXiv:2009.08542] [math.DG]
[17] J. Lee, *Introduction to Smooth Manifolds*, Springer, 2nd edition, 2012
[18] P.J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd edition, Springer, 2000
[19] D. J. Saunders, *The Geometry of Jet Bundles*, Cambridge 1989
[20] R.E. Showalter, *Hilbert Space Methods in Partial Differential Equations*, Dover 2010
[21] L.W. Tu, *An Introduction to Manifolds*, Springer, 2nd edition, 2010
[22] M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day 1964
[23] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer, 1983
[24] D. Zenkov, *The Inverse Problem of the Calculus of Variations*, Atlantis Press, 2015