Integrable Four-Fermi Models with a Boundary and Boson-Fermion Duality

Takeo Inami, Hitoshi Konno and Yao-Zhong Zhang

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606, Japan

Abstract

Construction of integrable field theories in space with a boundary is extended to fermionic models. We obtain general forms of boundary interactions consistent with integrability of the massive Thirring model and study the duality equivalence of the MT model and the sine-Gordon model with boundary terms. We find a variety of integrable boundary interactions in the $O(3)$ Gross-Neveu model from the boundary supersymmetric sine-Gordon theory by using boson-fermion duality.
1 Introduction

A physical system can often be described in two apparently different languages, which are dual to each other. One of the examples, which has recently attracted much attention, is the conjecture of electric-magnetic duality in supersymmetric Yang-Mills theories in four dimensions [1]. In two dimensional quantum field theories (QFTs), the duality relation between the bosonic and fermionic descriptions can be proven in a large class of integrable models, e.g. the sine-Gordon (SG) and massive Thirring (MT) equivalence [2] and the WZNW and massless Thirring equivalence [3]. Studies of the duality relation have in the past deepened our understanding of the non-perturbative nature of QFTs.

Integrable QFTs including SG and supersymmetric sine-Gordon (SSG) theories have recently been extended to cases in which the space has a boundary [4, 5, 6, 7, 8, 9, 10]. Integrable boundary QFT can only be constructed if one can find boundary interactions which satisfy suitable conditions. Integrable boundary QFTs have many applications in physical systems of low dimensionality.

A large class of integrable QFTs consisting of fermions is known. It is believed that such models bear a resemblance, in some respects such as chiral symmetry breaking, to the fermions in gauge theories in four dimensions.

In this paper we extend the construction of integrable boundary QFT to fermionic models and study boson-fermion duality in models with a boundary potential. We consider the two simplest cases, MT and \(O(3)\) Gross-Neveu (GN) models [11]. In the bulk theory, the MT and the \(O(3)\) GN models are known to be bosonized to give the SG and the SSG theories [2, 12], respectively. We will find a few different classes of integrable boundary interactions, given a fermionic model in the bulk, each of them defining a distinct boundary QFT.

2 Four-Fermi Interaction Models and Their Dual Theories

Let us recall in this section some known facts about the four-fermi interaction models in the bulk and their duals.

The MT model is defined by

\[
\mathcal{L}_{0,MT} = \frac{1}{2} \bar{\psi} i \gamma^\mu \partial_\mu \psi - \frac{1}{2} \partial_\mu \bar{\psi} i \gamma^\mu \psi - im \bar{\psi} \psi - \frac{g}{2} (\bar{\psi} \gamma_\mu \psi)^2, \tag{2.1}
\]

where \(\psi\) is a complex spinor (Dirac fermion), \(m\) a mass parameter and \(g\) a coupling constant.

The MT model is known to be equivalent to the sine-Gordon theory [2], which is a model of

\[\bar{\psi} = \psi^\dagger \gamma^0, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[\gamma_\mu = \eta_{\mu\nu} \gamma^\nu, \quad \eta_{\mu\nu} = diag(-1, 1).\]
a single real scalar field $\phi(x)$ defined by

$$\mathcal{L}_{0SG} = \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{m_0^2}{\sigma^2} \cos \sigma \phi. \quad (2.2)$$

The coupling constants in the two theories are connected by

$$g/\pi = 4\pi/\sigma^2 - 1. \quad (2.3)$$

This equivalence between the SG theory and the MT model has been shown to be a duality relation [13].

The $O(3)$ GN model is a model of a triplet of real spinors (Majorana fermions) $\chi_i(x)$ and is defined by

$$\mathcal{L}_{0GN} = -\frac{1}{2} \sum_{i=1}^{3} \bar{\chi}_i i\gamma^\mu \partial_\mu \chi_i - g \left( \sum_{i=1}^{3} \bar{\chi}_i \chi_i \right)^2. \quad (2.4)$$

This model was shown by Witten [12] to be equivalent to the SSG model which is defined by the lagrangian

$$\mathcal{L}_{0SSG} = -\frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} \bar{\chi}_i \gamma^\mu \partial_\mu \chi_i - 2A^2 \cos^2 B\phi - iAB \bar{\chi}_1 \cos B\phi, \quad (2.5)$$

where $\chi$ is a Majorana spinor and $A$, $B$ are coupling constants. The point is to group the Majorana fermions in pairs. Given two Majorana spinors $\chi_1$ and $\chi_2$ (two Majorana fermions being equivalent to a Dirac fermion), one has the following fermion-boson correspondence [12]

$$\frac{1}{2} (\bar{\chi}_1 i\gamma^\mu \partial_\mu \chi_1 + \bar{\chi}_2 i\gamma^\mu \partial_\mu \chi_2) = \frac{1}{2} (\partial_{\mu} \phi)^2, \quad (2.6)$$

$$i\bar{\chi}_1 \chi_1 + i\bar{\chi}_2 \chi_2 =: \cos \sqrt{4\pi} \phi :,$$

$$\bar{\chi}_1 \gamma^\mu \chi_2 = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial_\nu \phi,$$

together with the following important relation

$$: \cos \sqrt{4\pi} \phi :^2 = -\frac{2}{\pi} (\partial_{\mu} \phi)^2 + c \text{ number}, \quad (2.7)$$

where :: stands for normal ordering. The last relation can be deduced from (2.6) and has no classical counterpart.

Using these fermion-boson relations, one gets [12]

$$\mathcal{L}_{0GN} = -\frac{1}{2} (\partial_{\mu} \varphi)^2 + \frac{1}{2} 2\bar{\chi}_3 i\gamma^\mu \partial_\mu \chi_3 - 2A^2 : \cos \rho \varphi :^2 - iA\rho \bar{\chi}_3 \chi_3 : \cos \rho \varphi :, \quad (2.8)$$

where

$$A = \sqrt{C/2}, \quad \rho = 2g/A, \quad \varphi = \sqrt{\pi} \frac{A}{g} \phi,$$

$$C = 2g^2 (1 - 4g/\pi) / (\pi - 8g^2 / \pi). \quad (2.9)$$

The coupling constant $A$ is connected to the coupling constant $g$ by the above relations (2.9) and $B$ to $g$ by

$$B = 2g/A. \quad (2.10)$$

The equivalence of the two descriptions (2.4) and (2.8) may be interpreted as a duality relation, in the same way as the MT–SG equivalence.
3 Integrable MT Model on a Half-Line

We use the light-cone notation: \( z = \frac{1}{2}(t - x), \ z = \frac{1}{2}(t + x) \), which implies \( \partial_z = \partial_t - \partial_x, \ \partial_{\bar{z}} = \partial_t + \partial_x \). The Dirac fermion \( \psi \) appearing in (2.1) can be expressed as \( \psi = (\psi_1, \psi_2)^T \).

The theory on a half-line \( x \in (-\infty, 0] \) is defined by adding the boundary term \( S_b \) to the bulk part \( S_0 \) of the action. The total action \( S \) can be written as

\[
S = S_0 + S_b \equiv \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \mathcal{L}_0(\psi_1, \psi_2, \partial_z \psi_1, \partial_z \psi_2) + \int_{-\infty}^{\infty} dt \mathcal{B}(\psi_1, \psi_2), \tag{3.1}
\]

where the boundary potential \( \mathcal{B} \) is assumed to be a functional of the fields at \( x = 0 \) but not of their derivatives. In addition to the bulk field equations:

\[
\frac{\delta \mathcal{L}_0}{\delta \psi_1} - \partial_z \frac{\delta \mathcal{L}_0}{\delta \partial_z \psi_1} = 0, \quad \frac{\delta \mathcal{L}_0}{\delta \psi_2} - \partial_z \frac{\delta \mathcal{L}_0}{\delta \partial_z \psi_2} = 0,
\]

we have equations of motion at the boundary \( x = 0 \):

\[
-\frac{\delta \mathcal{L}_0}{\delta \psi_1} + \frac{\partial \mathcal{B}}{\partial \psi_1} = 0, \quad \frac{\delta \mathcal{L}_0}{\delta \psi_2} + \frac{\partial \mathcal{B}}{\partial \psi_2} = 0.
\]

In the case of MT model, if one makes the substitution (rescaling):

\[
\frac{1}{m} \partial_z \rightarrow \partial_\pm, \quad \sqrt{\frac{2g}{m}} \psi_1 \rightarrow \psi_1, \quad \sqrt{\frac{2g}{m}} \psi_2 \rightarrow i\psi_2, \tag{3.2}
\]

then from (2.1) one has the following lagrangian for the \( \mathcal{L}_0 \)'s in (3.1):

\[
\mathcal{L}_{0MT} = -\frac{i}{2} \psi_1^\dagger \partial_z \psi_1 + \frac{i}{2} \partial_z \psi_1^\dagger \psi_1 - \frac{i}{2} \psi_2^\dagger \partial_z \psi_2 + \frac{i}{2} \partial_z \psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2. \tag{3.3}
\]

The equations of motion read in the bulk

\[
i\partial_z \psi_1 = \psi_2 + \psi_1 \psi_2^\dagger \psi_1, \quad i\partial_z \psi_2 = \psi_1 + \psi_2 \psi_1^\dagger \psi_1, \tag{3.4}
\]

and at the boundary \( x = 0 \):

\[
-\psi_1 + \frac{\partial \mathcal{B}}{\partial \psi_1^\dagger} = 0, \quad i\psi_2 + \frac{\partial \mathcal{B}}{\partial \psi_2^\dagger} = 0. \tag{3.5}
\]

In the bulk theory there is an infinite number of conserved charges \([14]\) constructed from densities \( T_{s+1}, \ \bar{T}_{s+1}, \ \Theta_{s-1} \) and \( \bar{\Theta}_{s-1} \) with \( s = 1, 3, 5, \cdots \). These densities satisfy the following continuity equations

\[
\partial_z T_{s+1} = \partial_x \Theta_{s-1}, \quad \partial_r \bar{T}_{s+1} = \partial_x \bar{\Theta}_{s-1}. \tag{3.6}
\]

The densities for \( s = 1 \) are given by the energy-momentum tensor,

\[
T_2 = i\psi_1^\dagger \partial_z \psi_1 - i\partial_z \psi_1^\dagger \psi_1, \quad T_2 = i\psi_2^\dagger \partial_z \psi_2 - i\partial_z \psi_2^\dagger \psi_2, \quad \Theta_0 = \bar{\Theta}_0 = -\psi_2^\dagger \psi_1 - \psi_1^\dagger \psi_2. \tag{3.7}
\]

The \( s = 3 \) densities are

\[
T_4 = -i\psi_1^\dagger \partial_z^3 \psi_1 + i\partial_z^3 \psi_1^\dagger \psi_1 + 6\partial_z \psi_1^\dagger \psi_1 \partial_x \psi_1, \quad T_4 = -i\psi_2^\dagger \partial_z^3 \psi_2 + i\partial_z^3 \psi_2^\dagger \psi_2 + 6\partial_z \psi_2^\dagger \psi_2 \partial_x \psi_2, \quad \Theta_2 = \psi_2^\dagger \partial_z^2 \psi_1 + \partial_z^2 \psi_1^\dagger \psi_2 - i\psi_2^\dagger \psi_1^\dagger \psi_1 \partial_x \psi_1 + i\partial_z \psi_1^\dagger \psi_1^\dagger \psi_1 \psi_2, \quad \Theta_2 = \psi_1^\dagger \partial_z^2 \psi_2 + \partial_z^2 \psi_2^\dagger \psi_1 - i\psi_1^\dagger \psi_2^\dagger \psi_2 \partial_x \psi_2 + i\partial_z \psi_2^\dagger \psi_2^\dagger \psi_2 \psi_1. \tag{3.8}
\]
These densities can be checked to satisfy the continuity equations, using only the field equations (3.4) and the anticommutation relations for the $\psi$’s and their derivatives.

Suppose that one can choose the boundary potential $\mathcal{B}$ such that at $x = 0$

$$T_4 - \bar{T}_4 + \Theta_2 - \bar{\Theta}_2 = \frac{d}{dt}\Sigma_3(t), \tag{3.9}$$

where $\Sigma_3(t)$ is some functional of boundary fields $\psi_1(t)$, $\psi_2(t)$. Then the charge $P_3$, given by

$$P_3 = \int_{-\infty}^{0} dx \left( T_4 + \bar{T}_4 - \Theta_2 - \bar{\Theta}_2 \right) - \Sigma_3(t) \tag{3.10}$$

is a non-trivial integral of motion.

We now examine in what circumstances $T_4 - \bar{T}_4 + \Theta_2 - \bar{\Theta}_2$ may be written as a total $t$-derivative. By using the field equations (3.4) and the anticommutation property of the $\psi$’s, it can be shown after a tedious computation that

$$T_4 - \bar{T}_4 + \Theta_2 - \bar{\Theta}_2 = \partial_t(\text{something}) - 24(i \psi_1^{\dagger} \psi_1 + \psi_2^{\dagger} \psi_2)(\psi_1^{\dagger} \psi_1^{\dagger} - \psi_2^{\dagger} \psi_2^{\dagger}) - 24i \psi_1^{\dagger} \psi_1(\psi_1^{\dagger} \psi_2 - \psi_2^{\dagger} \psi_1^{\dagger}) - 24i \psi_2^{\dagger} \psi_2(\psi_1^{\dagger} \psi_2^{\dagger} - \psi_2^{\dagger} \psi_1^{\dagger}) - 16i(\psi_1^{\dagger} \psi_1^{\dagger} \psi_2 - \psi_2^{\dagger} \psi_1^{\dagger} \psi_2^{\dagger}) - 12i(\psi_1^{\dagger} \psi_1^{\dagger} - \psi_2^{\dagger} \psi_2^{\dagger}). \tag{3.11}$$

We look for the boundary potential $\mathcal{B}(\psi_1, \psi_2)$ which will make the r.h.s of (3.11) be a total $t$-derivative. The most general form of $\mathcal{B}(\psi_1, \psi_2)$ satisfying hermiticity must have the form $^3$

$$\mathcal{B}(\psi_1, \psi_2) = a_1 \psi_1^{\dagger} \psi_1 + a_2 \psi_2^{\dagger} \psi_2 + ia_3 \left( \psi_1^{\dagger} \psi_2^{\dagger} - \psi_2 \psi_1 \right) + ia_4 \left( e^{ib_1} \psi_1^{\dagger} \psi_2 - e^{-ib_1} \psi_2^{\dagger} \psi_1 \right) + a_5 \left( e^{ib_2} \psi_2^{\dagger} \psi_2 - e^{-ib_2} \psi_2 \psi_1 \right) + a_6 \left( e^{ib_3} \psi_1^{\dagger} \psi_1 - e^{-ib_3} \psi_2 \psi_2 \right) + ia_7 \partial_1 \epsilon_1 + ia_8 \partial_2 \epsilon_2 + ia_9 \partial_3 \epsilon_3, \tag{3.12}$$

where $a_l$ ($l = 1, \ldots, 9$), $b_1$, $b_2$ are real constant bosonic parameters, and $\epsilon_1(t)$, $\epsilon_2(t)$ are fermionic boundary operators $\hat{\mathcal{B}}$. Then, the $\psi_1$, $\psi_2$ field equations at the boundary $x = 0$, eqs.(3.3), become

$$(1 + ia_1)\psi_1 - a_4 e^{ib_1} \psi_2 - a_3 \psi_2^{\dagger} = -ia_5 \epsilon_1,$$

$$(1 - ia_2)\psi_2 + a_4 e^{-ib_1} \psi_1 - a_3 \psi_1^{\dagger} = ia_6 e^{-ib_2} \epsilon_2. \tag{3.13}$$

It turns out that in order for solutions of (3.13) to make the r.h.s of (3.11) a total $t$-derivative one has to set $\epsilon_1(t) = \epsilon_2(t) \equiv \epsilon(t)$. The constraints for the other constant parameters can also be determined. We find the following four classes of boundary potential which preserve the integrability of the bulk MT model:

(i) $\mathcal{B}(\psi_1, \psi_2) = \frac{\sin \alpha_0}{\cos \alpha_0} \left( \psi_1^{\dagger} \psi_1 - \psi_2^{\dagger} \psi_2 \right) + \frac{i}{\cos \alpha_0} \left( \psi_1^{\dagger} \psi_2^{\dagger} - \psi_2 \psi_1 \right)$, \hspace{1cm} $\alpha_0 \neq \frac{\pi}{2} \text{ mod } \pi; \tag{3.14}$

$^3$We have excluded terms such as $\psi_1^{\dagger} \psi_2^{\dagger} \psi_2$, $\psi_1^{\dagger} \psi_1 \psi_2^{\dagger}$ etc, since they do not have the right dimensions.
\[ B(\psi_1, \psi_2) = \frac{\sin(\beta - \beta_0)}{\cos(\beta - \beta_0)} \left( \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 \right) + \frac{i}{\cos(\beta - \beta_0)} \left( e^{i\beta} \psi_1^\dagger \psi_2 - e^{-i\beta} \psi_2^\dagger \psi_1 \right) \]
\[ \beta - \beta_0 \neq \frac{\pi}{2} \mod \pi; \quad (3.15) \]
\[ B(\psi_1, \psi_2) = \alpha (\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2) + i f \left( \psi_1^\dagger \psi_2^\dagger - \psi_2 \psi_1 \right) + h \left( e^{i\alpha} \psi_1^\dagger \psi_2 - e^{-i\alpha} \psi_2^\dagger \psi_1 \right) + ib \partial_t \epsilon, \]
\[ f \neq \pm 1, \text{ if } a = 0, \text{ and } a \neq 0 \text{ if } f = \pm 1; \quad (3.16) \]
\[ B(\psi_1, \psi_2) = r (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) + ic \left( e^{i\alpha} \psi_1^\dagger \psi_2 - e^{-i\alpha} \psi_2^\dagger \psi_1 \right) + id \partial_t \epsilon, \]
\[ c \neq \pm 1, \text{ if } r = 0, \text{ and } r \neq 0 \text{ if } c = \pm 1, \quad (3.17) \]

where \( \alpha_0, \beta_0, \beta, \alpha, \gamma, \lambda, \) and \( a, b \) etc, are free real constant parameters. It is remarkable that in the cases (iii) and (iv), five and six free parameters are allowed in the integrable boundary potential, respectively.

These boundary potentials have been derived by examining the first non-trivial conserved charge \( P_3 \) and our argument is restricted to the classical case. We believe that the above computation can be extended to the quantum theory, and that our analysis will be completed by showing that all conserved charges of higher spin give the same results. To this end one may use the prescription suggested in [15].

The boundary conditions for the \( \psi \) fields can be derived from the above boundary potentials. The boundary term (3.14) gives rise to the free boundary condition,

\[ (i) \quad \psi_1 = e^{i\alpha_0} \psi_2^\dagger; \quad (3.18) \]

and (3.15) to the fixed boundary condition,

\[ (ii) \quad \psi_1 = e^{i\beta_0} \psi_2. \quad (3.19) \]

In the last two cases one needs to eliminate the “external field” \( \epsilon(t) \) by using the \( \epsilon \) field equation at the boundary \( x = 0 \), which takes the form for the case (iii), \( ib \partial_t \epsilon + h(\psi_1 + e^{i\gamma} \psi_2) = 0 \), and \( id \partial_t \epsilon + s(\psi_1 + e^{i\lambda} \psi_2) = 0 \) for the case (iv). The results are

\[ (iii) \quad f \psi_2^\dagger - (1 + ia) \psi_1 - e^{i\gamma} \left( (1 + ia) \psi_2 - f \psi_2^\dagger \right) = 0, \]
\[ f \partial_t \left( \psi_2^\dagger - (1 + ia) \psi_1 \right) + h^2 \left( \psi_1 + e^{i\gamma} \psi_2 \right) = 0; \quad (3.20) \]
\[ (iv) \quad ce^{i\alpha} \psi_2 - (1 + ir) \psi_1 - e^{i\lambda} \left( (1 + ir) \psi_2 - ce^{-i\alpha} \psi_1 \right) = 0, \]
\[ d \partial_t \left( ce^{i\alpha} \psi_2 - (1 + ir) \psi_1 \right) + s^2 \left( \psi_1 + e^{i\lambda} \psi_2 \right) = 0, \quad (3.21) \]

plus the corresponding equations obtained by hermitian conjugation.

Let us examine whether the integrable boundary MT models obtained above have a dual description in terms of bosons. Consider the special case of (iii) (the argument below applies...
to the case (iv) as well),
\[ B(\psi_1, \psi_2) = h \left( \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 + e^{-i\gamma} \psi_2^\dagger \psi_2^\dagger \right) \] (3.22)
which can be rewritten as, after a redefinition of the field \( \epsilon = \epsilon e^{-i\delta} \),
\[ B(\psi_1, \psi_2) = h \left( e^{i\delta} \psi_1^\dagger \psi_1 + e^{-i\delta} \psi_2^\dagger \psi_2 + e^{i(\gamma+\delta)} \psi_2^\dagger \psi_2^\dagger + e^{-i(\gamma+\delta)} \psi_2^\dagger \psi_2^\dagger \right). \] (3.23)

With the help of a result obtained in [16], one may write:
\[ \cos \sigma \left( \frac{\phi - \phi_0}{2} \right) := e^{i\sigma \phi_0/2} (\psi_1^\dagger \psi_1 + \epsilon \psi_2^\dagger \psi_2) + e^{-i\sigma \phi_0/2} (\epsilon^\dagger \psi_2 + \psi_2^\dagger \epsilon). \] (3.24)

So, the boundary MT model with (3.23) as its boundary potential is equivalent to the boundary SG theory [6] which is defined by the lagrangian,
\[ L_{0SG} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{\sigma^2} : \cos \phi : + \delta(x) \Lambda : \cos \left( \frac{\phi - \phi_0}{2} \right), \] (3.25)
provided the coupling constants in the two theories are connected by
\[ h = \Lambda, \quad \delta = \frac{\sigma \phi_0}{2}, \quad \gamma = \pi - \sigma \phi_0. \] (3.26)

Other types of boundary potential in the MT model do not appear to give rise to integrable boundary potentials in the SG model by means of bosonization. The reason why bosonization formula, which holds in the bulk theory, may not hold in some cases with a boundary is not very clear to us. The following may be a possible explanation. The bosonization involves an exponential of integral from \(-\infty \) (\(+\infty \)) to \(x\). The presence of a boundary (at \(x = 0\)) may be an obstruction to this non-local expression and may cause the breakdown of boson-fermion duality in boundary integrable field theories.

4 Integrable \(O(3)\) GN Model on a Half-Line via Boson-Fermion Duality

The boundary potential compatible with integrability of the SSG has been investigated previously [9]. Two different forms of potentials are obtained:

1) \[ B(\phi, \chi, \bar{\chi}) = \Lambda : \cos B(\phi - \phi_0) : + \frac{i}{4} M \bar{\chi} \chi, \quad (M \neq \pm 1); \] (4.1)

2) \[ B(\phi, \chi, \bar{\chi}) = \pm \left( \frac{2A}{B} : \cos B \phi : + \frac{i}{4} \bar{\chi} \chi \right), \] (4.2)

where \(\Lambda, M\) and \(\phi_0\) are bosonic constants, and \(A, B\) are the same constants as in (2.3). Only the boundary potential of the form 2) preserves supersymmetry.

In this section we determine the integrable boundary \(O(3)\) GN model which is equivalent to the integrable boundary SSG theory. An analysis of finding the general form of integrable
boundary potential will be reported elsewhere. This can be done by use of the boson-fermion
duality relations (2.6) and (2.7). Experience suggests us to make the following ansatz for the
general form of the boundary GN model lagrangian,

\[ L_{bGN} = -\frac{1}{2} \sum_{i=1}^{3} \bar{\chi}_i \gamma^\mu \partial_\mu \chi_i - g \left( \sum_{i=1}^{3} \bar{\chi}_i \chi_i \right)^2 + i\delta(x) \left( \frac{g'}{4} \bar{\chi}_3 \chi_3 + g''(\bar{\chi}_1 \chi_1 + \bar{\chi}_2 \chi_2) \right). \] (4.3)

We determine the constants \( g' \) and \( g'' \) by demanding that the boundary potential be inte-
grable. Following the same prescription as in [12], one obtains from (4.3)

\[ L_{bGN} = -\frac{1}{2} \left( \partial_\mu \varphi \right)^2 - \frac{1}{2} \bar{\chi}_3 \gamma^\mu \partial_\mu \chi_3 - 2A^2 : \cos \rho \varphi :^2 - iA \rho \bar{\chi}_3 \chi_3 : \cos \rho \varphi : + i\delta(x) \left( \frac{g'}{4} \bar{\chi}_3 \chi_3 + g'' : \cos \rho \varphi : \right), \] (4.4)

where \( \varphi, \rho \) and \( A \) are given by (2.9).

Comparing with the integrable boundary SSG theories, one sees that in order for the bound-
dary GN to be integrable the constants \( g' \) and \( g'' \) should take either of the two values:

1) \( g' = M, \ g'' = \Lambda \);

2) \( g' = \pm 1, \ g'' = \pm \frac{2A}{B} = \pm \frac{g(1 - 4g/\pi)}{\pi - 8g^2/\pi}. \) (4.5)

Thus one has two alternative classes of integrable boundary GN model which are defined by

1) \[ L_{bGN} = -\frac{1}{2} \sum_{i=1}^{3} \bar{\chi}_i \gamma^\mu \partial_\mu \chi_i - g \left( \sum_{i=1}^{3} \bar{\chi}_i \chi_i \right)^2 + i\delta(x) \left( M \bar{\chi}_3 \chi_3 + \Lambda (\bar{\chi}_1 \chi_1 + \bar{\chi}_2 \chi_2) \right), \] \( M(\neq \pm 1), \ \Lambda \) arbitrary; \quad (4.6)

2) \[ L_{bGN} = -\frac{1}{2} \sum_{i=1}^{3} \bar{\chi}_i \gamma^\mu \partial_\mu \chi_i - g \left( \sum_{i=1}^{3} \bar{\chi}_i \chi_i \right)^2 \pm i\delta(x) \left( \frac{1}{4} \bar{\chi}_3 \chi_3 + \frac{g(1 - 4g/\pi)}{\pi - 8g^2/\pi} (\bar{\chi}_1 \chi_1 + \bar{\chi}_2 \chi_2) \right). \] (4.7)

For the special case \( M = 4\Lambda \) the model 1) is reduced to

1') \[ L_{bGN} = -\frac{1}{2} \sum_{i=1}^{3} \bar{\chi}_i \gamma^\mu \partial_\mu \chi_i - g \left( \sum_{i=1}^{3} \bar{\chi}_i \chi_i \right)^2 + i\delta(x) \Lambda \sum_{i=1}^{3} \bar{\chi}_i \chi_i \] (4.8)

This model preserves the \( O(3) \) symmetry of the bulk theory. On the other hand, the \( O(3) \)
symmetry is always broken in the model 2).

5 Discussions

We have derived the general form of boundary interactions \( B(\psi) \) by finding all solutions of (3.11)
in the MT model. We are then able to discuss the boson-fermion duality between the boundary
MT model and the boundary SG model. It is very curious that the integrable boundary potential
for the MT model contains a larger number of free parameters than that for the SG model.
To study the duality between the few classes of boundary SSG model and those of the $O(3)$ GN model in details, we have to go beyond the present investigation in section 4 and to carry out an analysis similar to the one in section 3. It is of interest to do the same for the $SU(2)$-invariant Thirring model. We wish to return to these problems in the future.

The SG/MT model is obtained by taking the continuum limit of the spin-$\frac{1}{2}$ XYZ chain with appropriate choice of the coupling constants in the latter model [17]. It is very interesting to see whether the boundary SG/MT model can be obtained from the boundary XYZ spin chain [18] by taking the continuum limit. This problem has been discussed in [19] and is under investigation.

The authors are grateful to Ed Corrigan for a careful reading of the manuscript. T.I. wishes to thank Choonkyu Lee for a discussion on the boson-fermion duality while he was visiting Seoul supported by the Japan-Korea exchange program of JSPS. This work was partially supported by the Grant-in-Aid for Scientific Research, priority area 231, from the Ministry of Education, Science and Culture of Japan. H.K. is supported by the Yukawa Memorial Foundation. Y.Z.Z. is supported by the Japan Society for the Promotion of Science (JSPS).

When writing this paper, we saw a recent preprint [20] in which a boundary MT model corresponding to the special case of one of our solutions, (3.16), is derived via fermionization of the boundary SG theory by using the same method as in [16].

References

[1] C. Montonen, D. Olive, *Phys.Lett.* B72 (1977) 117;
    P. Goddard, J. Nuyts, D. Olive, *Nucl.Phys.* B125 (1977) 1;
    N. Seiberg, E. Witten, *Nucl.Phys.* B426 (1994) 19.

[2] S. Coleman, *Phys.Rev.* D11 (1975) 2088;
    S. Mandelstam, *Phys.Rev.* D11 (1975) 3026.

[3] E. Witten, *Commun.Math.Phys.* 92 (1984) 455;
    M.B. Halpern, *Phys.Rev.* D12 (1975) 1684, and D13 (1976) 337.

[4] I. Cherednik, *Theor.Math.Phys.* 61 (1984) 35.

[5] J. Cardy, *Nucl.Phys.* B324 (1989) 581.

[6] S. Ghoshal, A.B. Zamolodchikov, *Int.J.Mod.Phys.* A9 (1994) 3841.

[7] E. Corrigan, P.E. Dorey, R.H. Rietdijk, R. Sasaki, *Phys.Lett.* B333 (1994) 83.

[8] H. Saleur, S. Skorik, *J.Phys.* A: Math.Gen. 28 (1995) 6605.

[9] T. Inami, S. Odake, Y.-Z. Zhang, *Phys.Lett.* B359 (1995) 118.
[10] C. Ahn, W.-M. Koo, preprint hep-th/9509056.

[11] A.B. Zamolodchikov, Al.B. Zamolodchikov, Ann.Phys. 120 (1979) 253.

[12] E. Witten, Nucl.Phys. B142 (1978) 285;
    H. Aratyn, P.H. Damgaard, Nucl.Phys. B241 (1984) 253.

[13] C.P. Burgess, F. Quevedo, Nucl.Phys. B421 (1994) 373;
    P.H. Damgaard, H.B. Nielsen, R. Sollacher, Phys.Lett. B296 (1992) 132.

[14] B. Berg, M. Karowski, H.J. Thun, Phys.Lett. 64B (1976) 286;
    P.P. Kulish, E.R. Nissinov, JETP Lett. 24 (1976) 247;
    R. Flume, P.K. Mitter, N. Papanicolaou, Phys.Lett. 64B (1976) 289.

[15] P. Bowcock, E. Corrigan, P.E. Dorey, R.H. Rietdijk, Nucl.Phys. B445 (1995) 469.

[16] M. Ameduri, R. Konik, A. LeClair, Phys.Lett. B354 (1995) 376.

[17] A. Luther, Phys.Rev. B14 (1976) 2153.

[18] B.-Y. Hou, R.-H. Yue, Phys.Lett. A183 (1993) 169;
    T. Inami, H. Konno, J.Phys. A: Math.Gen. 27 (1994) L913.

[19] P. Fendley, H. Saleur, Nucl.Phys. B428 [FS] (1994) 681;
    M.T. Grisaru, L. Mezincescu, R.I. Nepomechie, J.Phys. A: Math.Gen. 28 (1995) 1027.

[20] H.-B. Gao, Z.-M. Sheng, preprint hep-th/9512011.