On the rainbow matching conjecture for 3-uniform hypergraphs

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Abstract Aharoni and Howard and, independently, Huang et al. (2012) proposed the following rainbow version of the Erdős matching conjecture: For positive integers \( n, k \) and \( m \) with \( n \geq km \), if each of the families \( F_1, \ldots, F_m \subseteq \left(\begin{array}{l} n \\ k \end{array}\right) \) has size more than \( \max\left\{ \left(\begin{array}{l} n \\ k \end{array}\right) - \left(\begin{array}{l} n-m+1 \\ k \end{array}\right), \left(\begin{array}{l} km-1 \\ k \end{array}\right) \right\} \), then there exist pairwise disjoint subsets \( e_1, \ldots, e_m \) such that \( e_i \in F_i \) for all \( i \in [m] \). We prove that there exists an absolute constant \( n_0 \) such that this rainbow version holds for \( k = 3 \) and \( n \geq n_0 \). We convert this rainbow matching problem to a matching problem on a special hypergraph \( H \). We then combine several existing techniques on matchings in uniform hypergraphs: find an absorbing matching \( M \) in \( H \); use a randomization process of Alon et al. (2012) to find an almost regular subgraph of \( H - V(M) \); find an almost perfect matching in \( H - V(M) \). To complete the process, we also need to prove a new result on matchings in 3-uniform hypergraphs, which can be viewed as a stability version of a result of Luczak and Mieczkowska (2014) and might be of independent interest.

Keywords rainbow matching conjecture, Erdős matching conjecture, stability

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1 Introduction

For a positive integer \( k \) and a set \( V \), let \([k] := \{1, \ldots, k\} \) and

\[
\left(\begin{array}{l} V \\ k \end{array}\right) := \{A \subseteq V : |A| = k\}.
\]

A hypergraph \( H \) consists of a vertex set \( V(H) \) and an edge set \( E(H) \subseteq 2^{V(H)} \). A hypergraph \( H \) is \( k \)-uniform if all its edges have size \( k \) and we call it a \( k \)-graph for short. Throughout this paper, we often identify \( E(H) \) with \( H \) when there is no confusion and, in particular, denote by \( |H| \) the number of edges in \( H \). Given a set \( T \) of edges in \( H \), we use \( V(T) \) to define \( \bigcup_{e \in T} e \). Given a vertex subset \( S \subseteq V(H) \) in \( H \), we use \( H[S] \) to denote the subgraph of \( H \) induced by \( S \), and let \( H - S \equiv H[V(H) \setminus S] \).

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A matching in a hypergraph $H$ is a set of pairwise disjoint edges in $H$. We use $\nu(H)$ to define the maximum size of a matching in $H$. Let $F = \{F_1,\ldots,F_m\}$ be a family of hypergraphs on the same vertex set. A set of $m$ pairwise disjoint edges is called a rainbow matching for $F$ if each edge is from a different $F_i$. If such a matching exists, then we also say that $F$ admits a rainbow matching.

A classical problem in extremal set theory asks for the maximum number of edges in $n$-vertex $k$-graphs $H$ with $\nu(H) < m$. Let $n$, $k$ and $m$ be positive integers with $n \geq km$. The $k$-graphs

$$S(n,m,k) := \binom{n}{k} \setminus \binom{n}{k} \cdot \frac{m-1}{k}$$

and $D(n,m,k) := \binom{km-1}{k}$ on the same vertex set $[n]$ do not have matchings of size $m$. Erdős [6] conjectured in 1965 that among all the $k$-graphs with no matching of size $m$, $S(n,m,k)$ or $D(n,m,k)$ has the maximum number of edges: Any $n$-vertex $k$-graph $H$ with $\nu(H) < m$ contains at most

$$f(n,m,k) := \max \left\{ \binom{n}{k} - \binom{n-m+1}{k}, \binom{km-1}{k} \right\}$$

edges. This is often referred to as the Erdős matching conjecture in the literature, and there has been extensive research on this conjecture (see, for example, [3, 5, 8–11, 13, 22]). In particular, the special case for $k = 3$ was settled for large $n$ by Lučzak and Mieczkowska [22] and completely resolved by Frankl [9].

The following analogous conjecture, known as the rainbow matching conjecture, was made by Aharoni and Howard [1] and, independently, by Huang et al. [15]. For related topics on rainbow type problems, we refer the interested readers to [16, 18, 20, 23].

**Conjecture 1.1** (See [1, 15]). Let $n$, $k$ and $m$ be positive integers with $n \geq km$. Let $F = \{F_1,\ldots,F_m\}$ be a family of $k$-graphs on the same vertex set $[n]$ such that $|F_i| > f(n,m,k)$ for all $i \in [m]$. Then $F$ admits a rainbow matching.

The case $k = 2$ of this conjecture is in fact a direct consequence of an earlier result of Akiyama and Frankl [2] (which was restated in [7]). The following was obtained by Huang et al. [15].

**Theorem 1.2** (See [15, Theorem 3.3]). Conjecture 1.1 holds when $n > 3k^2 m$.

Keller and Lifshitz [17] proved that Conjecture 1.1 holds when $n \geq f(m) k$ for some large constant $f(m)$ which only depends on $m$, and this was further improved to $n = \Omega(m \log m)$ by Frankl and Kupavskii [12]. Both proofs use the junta method. Very recently, Lu et al. [19] showed that Conjecture 1.1 holds when $n \geq 2km$ and $n$ is sufficiently large.

The following is our main result, which proves Conjecture 1.1 for $k = 3$ and sufficiently large $n$.

**Theorem 1.3.** There exists an absolute constant $n_0$ such that the following holds for all $n \geq n_0$. For any positive integers $n$ and $m$ with $n \geq 3m$, let $F = \{F_1,\ldots,F_m\}$ be a family of $3$-graphs on the same vertex set $[n]$ such that $|F_i| > f(n,m,3)$ for all $i \in [m]$. Then $F$ admits a rainbow matching.

Our proof of Theorem 1.3 uses some new ideas and combines different techniques from Alon et al. [3], Lučzak and Mieczkowska [22], and Lu et al. [21]. (For a high level description of our proof, we refer the readers to Section 2 and/or Section 7.) In the process, we prove a stability result on 3-graphs (see Lemma 4.2) that plays a crucial role in our proof and might be of independent interest: If the number of edges in an $n$-vertex 3-graph $H$ with $\nu(H) < m$ is close to $f(n,m,3)$, then $H$ must be close to $S(n,m,3)$ or $D(n,m,3)$.

The rest of the paper is organized as follows. In Section 2, we introduce additional notation, and state and/or prove a few lemmas for later use. In Section 3, we deal with the families $F$ in which most 3-graphs are close to the same 3-graph, i.e., $S(n,m,3)$ or $D(n,m,3)$. To deal with the remaining families, we need the above mentioned stability result for matchings in 3-graphs, which is done in Section 4. In Section 5, we show that there exists an absolute constant $c > 0$ such that Theorem 1.3 holds for $m > (1-c)n/3$. The proof of Theorem 1.3 for $m \leq (1-c)n/3$ is completed in Section 6. Finally, we complete the proof of Theorem 1.3 in Section 7.
2 Previous results and lemmas

In this section, we define saturated families and stable hypergraphs, and state several lemmas that we will use frequently. We begin with some notation. Suppose that \( H \) is a hypergraph and \( U \) and \( T \) are subsets of \( V(H) \). Let
\[
N_H(T) := \{ A : A \subseteq V(H) \setminus T \text{ and } A \cup T \in E(H) \}
\]
be the neighborhood of \( T \) in \( H \), and let \( d_H(T) := |N_H(T)| \). We write \( d_H(v) \) for \( d_H(\{v\}) \). Let
\[
\Delta(H) := \max_{v \in V(H)} d_H(v) \quad \text{and} \quad \Delta_2(H) := \max_{F \in (\binom{V(H)}{2})} d_H(T).
\]

In the case \( T \subseteq U \), we often identify \( d_{H[U]}(T) \) with \( d_U(T) \) when there is no confusion.

It will be helpful to consider “maximal” counterexamples to Conjecture 1.1. Let \( n, k \) and \( m \) be positive integers with \( n \geq km \) and let \( \mathcal{F} = \{ F_1, \ldots, F_m \} \) be a family of \( k \)-graphs on the same vertex set \([n]\). We say that \( \mathcal{F} \) is saturated, if \( \mathcal{F} \) does not admit a rainbow matching, but for every \( F \in \mathcal{F} \) and \( e \notin F \), the new family \( \mathcal{F}(e, F) := (\mathcal{F} \setminus \{ F \}) \cup \{ F \cup \{ e \} \} \) admits a rainbow matching. The following lemma says that the vertex degrees of every \( k \)-graph in a saturated family are typically small.

**Lemma 2.1.** Let \( n, k \) and \( m \) be positive integers with \( n \geq km \). Let \( \mathcal{F} = \{ F_1, \ldots, F_m \} \) be a saturated family of \( k \)-graphs on the same vertex set \([n]\). Then for each \( v \in [n] \) and each \( i \in [m] \),
\[
d_{F_i}(v) \leq \left( \frac{n-1}{k-1} \right) - \left( \frac{n-1-k(m-1)}{k-1} \right) \quad \text{or} \quad d_{F_i}(v) = \left( \frac{n-1}{k-1} \right).
\]

**Proof.** Suppose \( d_{F_i}(v) < \left( \frac{n-1}{k-1} \right) \), where \( v \in [n] \) and \( i \in [m] \). Then there exists \( e \in \binom{[n]}{k} \setminus F_i \) such that \( v \in e \). Since \( \mathcal{F} \) is saturated, the family \( \mathcal{F}(e, F_i) \) admits a rainbow matching, say \( M \cup \{ e \} \), with \( M \) being a rainbow matching for the family \( \mathcal{F} \setminus \{ F_i \} \).

If
\[
d_{F_i}(v) > \left( \frac{n-1}{k-1} \right) - \left( \frac{n-1-k(m-1)}{k-1} \right),
\]
then there exists an edge \( f \in F_i \) such that \( v \in f \) and \( f \cap V(M) = \emptyset \). Now \( M \cup \{ f \} \) is a rainbow matching for \( \mathcal{F} \), which leads to a contradiction. So \( d_{F_i}(v) \leq \left( \frac{n-1}{k-1} \right) - \left( \frac{n-1-k(m-1)}{k-1} \right) \).

We will remove vertices of degree \( \left( \frac{n-1}{k-1} \right) \) and use Lemma 2.1 to produce the saturated family \( \mathcal{F} = \{ F_1, \ldots, F_m \} \) of \( k \)-graphs such that for each \( v \in V(F_i) \) and each \( i \in [m] \),
\[
d_{F_i}(v) \leq \left( \frac{n-1}{k-1} \right) - \left( \frac{n-1-k(m-1)}{k-1} \right).
\]

Next, we define stable hypergraphs. Let \( n \) and \( k \) be positive integers with \( n \geq k \). Let \( e = \{a_1, \ldots, a_k\} \) and \( f = \{b_1, \ldots, b_k\} \) be members of \( \binom{[n]}{k} \) with \( a_1 < a_2 < \cdots < a_k \) and \( b_1 < b_2 < \cdots < b_k \). We write \( e \leq f \) if \( a_i \leq b_i \) for all \( 1 \leq i \leq k \), and \( e < f \) if \( e \not\leq f \) and \( e \not< f \).

A \( k \)-graph \( F \subseteq \binom{[n]}{k} \) is said to be stable if \( e < f \) implies \( e \in F \). A family \( \mathcal{F} \) of \( k \)-graphs on the same vertex set \([n]\) is stable if each \( k \)-graph in \( \mathcal{F} \) is stable.

The following result of Huang et al. [15] will be used frequently, which enables us to work with stable families when proving Conjecture 1.1.

**Lemma 2.2** (See [15, Lemma 2.1]). Let \( n, k \) and \( m \) be positive integers with \( n \geq km \). If the family \( \{ F_1, \ldots, F_m \} \) of \( k \)-graphs with \( V(F_i) = [n] \) for all \( i \in [m] \) does not admit a rainbow matching, then there exists a stable family \( \{ F_1, \ldots, F'_m \} \) of \( k \)-graphs with \( |F_i| = |F'_i| \) and \( V(F'_i) = [n] \) for all \( i \in [m] \) which still preserves this property.

**Corollary 2.3.** Let \( n, k \) and \( m \) be positive integers with \( n \geq km \). Let \( \mathcal{F} = \{ F_1, \ldots, F_m \} \) be a family of \( k \)-graphs on the vertex set \([n]\) that does not admit a rainbow matching. Then there exists a family \( \mathcal{F}' = \{ F'_1, \ldots, F'_m \} \) of \( k \)-graphs on the same vertex set \([n]\) such that \( \mathcal{F}' \) is both stable and saturated and \( |F'_i| \geq |F_i| \) for \( i \in [m] \).
Proof. Let $F^* = \{F_1', \ldots, F_m'\}$ be a family of $k$-graphs on the same vertex set $[n]$ such that $F^*$ admits no rainbow matching, $|F_i'| > |F_i|$ for $i \in [m]$, and subject to these, $\sum_{i \in [m]} |F_i'|$ is maximum.

Then $F^*$ is saturated. Now applying Lemma 2.2 to $F^*$, we obtain a stable family $F' = \{F_1', \ldots, F_m'\}$ of $k$-graphs on the vertex set $[n]$ such that $F'$ admits no rainbow matching, and $|F_i'| = |F_i|$ for $i \in [m]$. By the choice of $F^*$, we see that $F'$ is also saturated. \qed

We now describe an operation that converts a rainbow matching problem to a matching problem on a single hypergraph. Let $n$, $k$, $m$ and $r$ be non-negative integers with $r = \lfloor n/k \rfloor - m$ and $m \geq 1$. Let $F = \{F_1, \ldots, F_m\}$ be a family of $k$-graphs on the same vertex set $[n]$, and let $V = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_r\}$ be two disjoint sets such that $(V \cup U) \cap [n] = \emptyset$. We use $H(F)$ to define the $(k+1)$-graph with the vertex set $[n] \cup V$ and the edge set $\bigcup_{i=1}^m \{v \cup \{v_i\} : v \in F_i\}$, and use $H^*(F)$ to define the $(k+1)$-graph with the vertex set $[n] \cup V \cup U$ and the edge set

$$E(H(F)) \cup \bigcup_{i=1}^r \{e \cup \{u_i\} : e \in \binom{[n]}{k}\}.$$

If $F_1 = \cdots = F_m = S(n, m, k)$ (resp. $F_1 = \cdots = F_m = D(n, m, k)$), then we write $H(F)$ as $H_S(n, m, k)$ (resp. $H_D(n, m, k)$).

It is easy to see that $F$ admits a rainbow matching if and only if $H(F)$ has a matching of size $m$, which is also if and only if $H^*(F)$ has a matching of size $m + r$. This allows us to access existing approaches and tools invented for matching problems. For example, we take the approach by considering whether or not the hypergraphs $H(F)$ in question are close to the extremal configurations $H_S(n, m, k)$ and $H_D(n, m, k)$. We will see in Section 3 that if $H(F)$ is close to $H_D(n, m, k)$ and $F$ is stable, then $F$ admits a rainbow matching.

Here, we give an easy lemma concerning a case where $H(F)$ is not close to $H_S(n, m, k)$, which will be used along with Lemma 2.1. Let $H_1$ and $H_2$ be two $k$-graphs on the same vertex set $V$ and let $\epsilon$ be some positive real number; we say that $H_2$ is $\epsilon$-close to $H_1$ if $|E(H_1) \setminus E(H_2)| \leq \epsilon |V|^k$.

Lemma 2.4. For any given integer $k \geq 3$, let $\epsilon$ and $c$ be real numbers such that $0 < \epsilon \ll c \leq 1$). Let $n$ and $m$ be integers such that $n/3k^2 \leq m \leq (1 - c)n/k$. Let $F = \{F_1, \ldots, F_m\}$ be a family of $k$-graphs on the vertex set $[n]$. If for every $i \in [m]$ and $v \in [n]$,

$$d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1},$$

then $H(F)$ is not $\epsilon$-close to $H_S(n, m, k)$.

Proof. We note that $S(n, m, k)$ has $m - 1$ vertices of degree $\binom{n-1}{k-1}$. Since for every $i \in [m]$ and $v \in [n]$,

$$d_{F_i}(v) \leq \binom{n-1}{k-1} - \binom{n-k(m-1)-1}{k-1},$$

we have

$$|E(H_S(n, m, k)) \setminus E(H(F))| \geq m \cdot (m - 1) \cdot \binom{n-k(m-1)-1}{k-1} \cdot \frac{1}{k} > \frac{n^2}{10k^3} \binom{cn}{k-1} > \epsilon (n + m)^{k+1},$$

where the second inequality is due to $n/3k^2 \leq m \leq (1 - c)n/k$ and the third inequality follows from $\epsilon \ll c$. This shows that $H(F)$ is not $\epsilon$-close to $H_S(n, m, k)$.

To deal with the case where $H(F)$ is not close to $H_D(n, m, 3)$, we first find a small matching $M$ in $H^*(F)$ such that $M$ can “absorb” all small vertex sets and $H^*(F) - V(M)$ has an almost perfect matching. When $F$ is stable, the matching $M$ can be found very easily by the following lemma and its proof.

1) Here and throughout the rest of the paper, the notation $a \ll b$ means that $a$ is sufficiently small compared with $b$ which need satisfy finitely many inequalities in the proof.
Lemma 2.5. Let $k$ be a fixed positive integer and let $0 < \gamma' < \gamma < c < 1$ be real numbers. Let $n$ and $m$ be positive integers with $n/3k^2 \leq m \leq (1-c)n/k$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a stable family of $k$-graphs such that $V(F_i) = [n]$ and $|F_i| > \gamma n$ for all $i \in [m]$. Then for sufficiently large $n$, $H^*(\mathcal{F})$ has a matching $M$ with $|M| \leq \gamma n$ such that for any set $S \subseteq V(H^*(\mathcal{F})) \setminus V(M)$ with $|S| \leq \gamma' n$ and $k|S \cap (V \cup U)| = |S \cap [n]|$, $H^*(\mathcal{F})[V(M) \cup S]$ has a perfect matching.

Proof. Recall that $V = \{v_1, \ldots, v_n\}$ and $U = \{u_1, \ldots, u_r\}$, where $r = |n/k| - m$. Fix an integer $t$ satisfying $\gamma' n < t < \gamma n$. Then $t < \gamma n \leq |cn/k| \leq |n/k| = m = r$. Let $s = \lceil n/3k^2 \rceil - 1$.

By Theorem 1.2 (viewing all the $k$-graphs as the same $k$-graph), since $|F_i| > \gamma n$, $\gamma n > f(n, m, k)$ for all $i \in [m]$, every $F_i$ has a matching of size $s$. Since $F_i$ is stable, $F_i[[s]]$ is a complete $k$-graph. Hence, if for any $i_1, i_2, \ldots, i_k \leq kt < k\gamma n < s$ and $j \in [m]$, we have $\{v_j, i_1, i_2, \ldots, i_k\} \in H^*(\mathcal{F})$.

From the definition of $H^*(\mathcal{F})$, we have

(i) for any $i_1, i_2, \ldots, i_k \in [n]$ and $j \in [r]$, $\{u_j, i_1, i_2, \ldots, i_k\} \in H^*(\mathcal{F})$.

(ii) since $t < r$, we may choose a matching $M$ of size $t$ in $H^*(\mathcal{F})$ with $V(M) = \{u_1, \ldots, u_t\} \cup \{kt\}$. Note that $|M| = t \leq \gamma n$. We claim that this $M$ is the desired matching. To see this, consider any subset $S$ with $S \cap (V \cup U) = \emptyset$, $|S| \leq \gamma' n$ and $k|S \cap (V \cup U)| = |S \cap [n]|$. Let $t' = |S \cap (V \cup U)|$. So $t' \leq \gamma' n < t$. Then by (i) and (ii), there is a perfect matching $M_1$ in $H^*(\mathcal{F})[S \cap (V \cup U) \cup \{kt\}]$. By (ii), there exists a perfect matching $M_2$ in $H^*(\mathcal{F})[(V(M) \cup S) \setminus V(M_1)]$. So $M_1 \cup M_2$ is a perfect matching in $H^*(\mathcal{F})[V(M) \cup S]$.

For the “absorbing” matching $M$ in $H^*(\mathcal{F})$ in Lemma 2.5, we also want $H^*(\mathcal{F}) - V(M)$ to have an almost perfect matching. For this, we need to use the following result of Frankl and Rödl [14].

Theorem 2.6 (See [14]). For every integer $k \geq 2$ and any real number $\sigma > 0$, there exist $\tau = \tau(k, \sigma)$ and $d_0 = d_0(k, \sigma)$ such that for every integer $n \geq D \geq d_0$ the following holds: Every $n$-vertex $k$-graph $H$ with

$$(1 - \tau)D < \Delta_1(H) < (1 + \tau)D$$

and $\Delta_2(H) < \tau D$ contains a matching covering all but at most $\sigma n$ vertices.

In order to obtain a $k$-graph $H$ satisfying Theorem 2.6, we use the approach from [3] by conducting two rounds of randomization on $H^*(\mathcal{F}) - V(M)$. We summarize part of the proof in [3] (more precisely, their proof of Claim 4.1) as a lemma. A fractional matching in a $k$-graph $H$ is a function $w : E(H) \to [0, 1]$ such that for any $v \in V(H)$, $\sum_{e \in E(H), w(e) > 0} w(e) \leq 1$. A fractional matching is called perfect if $\sum_{e \in E(H)} w(e) = |V(H)|/k$.

Lemma 2.7 (See [3], retained from their proof of Claim 4.1). Let $k \geq 3$ and $H$ be a $k$-graph on at most $2n$ vertices. Suppose that there are subsets $R^i \subseteq V(H)$ for $i = 1, \ldots, n^{1.1}$ satisfying the following:

(a) every vertex $v \in V(H)$ satisfies that $|\{i : v \in R^i\}| \leq (1 + o(1))n^{0.2}$,

(b) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets $R^i$,

(c) every edge $e \in H$ is contained in at most one set $R^i$, and

(d) for every $i = 1, \ldots, n^{1.1}$, $R^i$ has a perfect fractional matching $w^i$.

Then $H$ has a spanning subgraph $H'$ such that

$$d_{H'}(v) = (1 + o(1))n^{0.2}$$

for all $v \in V(H')$ and $\Delta_2(H') \leq n^{0.1}$. We will also need to control the independence number of random subgraphs of $H^*(\mathcal{F}) - V(M)$. The intuition is that when $H(\mathcal{F})$ is not close to $H_D(n, m, k)$ or $H_S(n, m, k)$, $H^*(\mathcal{F}) - V(M)$ does not have very large independence number. The following lemma in [21] was proved by Lu et al. using the container method.

Lemma 2.8 (See [21, Lemma 5.4]). Let $d$, $\varepsilon'$ and $\alpha$ be positive real numbers and let $k$ and $n$ be positive integers. Let $H$ be an $n$-vertex $k$-graph such that $d(H) \geq dn^k$ and $d(H[S]) \geq \varepsilon' \varepsilon(H)$ for all $S \subseteq V(H)$ with $|S| > \alpha n$. Let $R \subseteq V(H)$ be obtained by taking each vertex of $H$ uniformly at random with probability $n^{-0.9}$. Then for any positive real number $\gamma \ll \alpha$, the size of maximum independent sets in $H[R]$ is at most $(\alpha + \gamma)n^{0.1}$ with probability at least $1 - (n^{O(1)})e^{-\Omega(n^{0.1})}$.
Lemma 2.10 (See [9, Proposition 5.1]). Let $n$, $m$ and $k$ be positive integers with $n \geq km - 1$. Then
\[ f(n, m, k) \geq f(n - 1, m - 1, k) + \binom{n - 1}{k - 1}. \]

We conclude this section with the well-known Chernoff inequality.

Lemma 2.10 (Chernoff inequality [4]). Suppose that $X_1, \ldots, X_n$ are independent random variables taking values in $\{0, 1\}$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}(X)$. Then for any $0 < \delta \leq 1$,
\[ \mathbb{P}[X \geq (1 + \delta)\mu] \leq e^{-\delta^2\mu/3} \quad \text{and} \quad \mathbb{P}[X \leq (1 - \delta)\mu] \leq e^{-\delta^2\mu/3}. \]
In particular, if $X \sim \text{Bin}(n, p)$ and $\lambda < \frac{3}{2}np$, then
\[ \mathbb{P}(|X - np| \geq \lambda) \leq e^{-\Omega(\lambda^2/(np))}. \]  

3 Extremal configuration $H_D(n, m, 3)$

From Lemmas 2.1 and 2.4, we see that if $\mathcal{F}$ is a saturated family of $k$-graphs on the vertex set $[n]$ and $H(\mathcal{F})$ is close to the extremal configuration $H_S(n, m, k)$, then there exist $F \in \mathcal{F}$ and $v \in [n]$ such that $d_F(v) = \binom{n-1}{k-1}$. Such vertices $v$ can be removed from all the $k$-graphs in $\mathcal{F} \setminus \{F\}$ to obtain a smaller family $\mathcal{F}'$, so that if $\mathcal{F}'$ admits a rainbow matching, then $\mathcal{F}$ admits a rainbow matching.

In this section, we consider the case where $H(\mathcal{F})$ is close to $H_D(n, m, 3)$ and $\mathcal{F}$ is stable.

Lemma 3.1. Let $\epsilon$ and $c$ be real numbers such that $0 < \epsilon \ll c \ll 1$. Let $n$ and $m$ be positive integers such that $n/27 \leq m \leq (1 - \epsilon)n/3$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a stable family of 3-graphs on the vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for all $i \in [m]$. If $H(\mathcal{F})$ is $\epsilon$-close to $H_D(n, m, 3)$, then $\mathcal{F}$ admits a rainbow matching.

Proof. Let $b = 6\epsilon^{1/6}n$. If $F_i$ is $\sqrt{\epsilon}$-close to $D(n, m, 3)$, then $F_i$ contains a complete subgraph of size $3m - b$; for otherwise, as $F_i$ is stable, we have
\[ |E(D(n, m, 3)) \setminus E(F_i)| \geq \binom{b}{3} > \sqrt{\epsilon}n^3, \]
which leads to a contradiction.

We claim that for any $i \in [m]$ and $j \in \{0, \ldots, b\}$, $\{2j + 1, 2j + 2, 3m - j\} \subset F_i$. To prove this claim we fix $i \in [m]$. Suppose for a contradiction that there exists an integer $t$ with $0 \leq t \leq b$ such that $\{2t + 1, 2t + 2, 3m - t\} \notin F_i$. Since $|F_i| > \binom{3m - 1}{3}$ and $F_i$ is stable, we have $\{1, 2, 3m\} \subset F_i$. So $t \geq 1$. We now count the edges in $F_i$: Let $q_1$ be the number of edges of $F_i$ in $[3m - 1]$, and $q_2$ be the number of edges of $F_i$ not contained in $[3m - 1]$. Since $F_i$ is stable and $\{2t + 1, 2t + 2, 3m - t\} \notin F_i$, we see that $\{a, b, c\} \notin F_i$ when $2t + 2 \leq a < b < 3m - t \leq c \leq 3m - 1$. So
\[ q_1 \leq \binom{3m - 1}{3} - t\binom{3m - 3t - 3}{2}. \]

Since $\{2t + 1, 2t + 2, 3m - t\} \notin F_i$, we have that for any $e \in F_i$ with $e \cap ([n] \setminus [3m - 1]) \neq \emptyset$, $e \cap [2t] \neq \emptyset$. This shows $q_2 \leq 2t(n - 3m + 1)n$. First suppose that $n \leq 7m/2$. Then we have
\[ |F_i| \leq \binom{3m - 1}{3} - t\binom{3m - 3t - 3}{2} + 2tn(n - 3m + 1) \leq \binom{3m - 1}{3} - t\left[\binom{3m - 3t - 3}{2} - 7m(m/2 + 1)\right] < \binom{3m - 1}{3}. \]
where the second inequality holds since \( n \leq 7m/2 \), and the last inequality holds since \( t \leq b = 6e^{1/6}n \ll m \), which leads to a contradiction. So we may assume \( n > 7m/2 \). Let \( m = an \). Then \( 1/27 \leq \alpha < 2/7 \). We assert that
\[
\binom{n}{3} - \binom{n-m+1}{3} > \binom{3m-1}{3} + 2tn^2.
\]
To see this, let \( f(x) = 1 - (1 - x)^3 - (3x)^3 \), and then
\[
\frac{6}{n^3} \left( \binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3} \right) = f(\alpha) + o(1).
\]
Since \( f'(x) = 3(1 - 2x - 26x^2) \) is decreasing in \([1/27, 2/7]\) with \( f'(1/27) > 0 \) and \( f'(2/7) < 0 \), we have
\[
f(\alpha) \geq \min\{f(1/27), f(2/7)\} = f(2/7) = \frac{2}{343}
\]
for \( 1/27 \leq \alpha < 2/7 \). This shows that
\[
\binom{n}{3} - \binom{n-m+1}{3} - \binom{3m-1}{3} = \frac{f(\alpha)}{6}n^3 + o(n^3) \geq 2tn^2,
\]
as asserted. Then it follows that
\[
|F_i| \leq \binom{3m-1}{3} - t\binom{3m-3t-3}{2} + 2tn(n - 3m + 1) < \binom{3m-1}{3} + 2tn^2 < \binom{n}{3} - \binom{n-m+1}{3},
\]
which leads to a contradiction as \(|F_i| > f(n, m, 3)\). This finishes the proof of the claim.

Recall \( V = \{v_1, \ldots, v_m\} \) from the definition of \( H(F) \). By the above claim,
\[
M_1 := \{\{v_i, 2i - 1, 2i, 3m - i + 1\} : i \in [b]\}
\]
is a matching in \( H(F) \). Without loss of generality, let \( F_1, \ldots, F_a \) be all the \( k \)-graphs in \( F \) which are not \( \sqrt{\epsilon} \)-close to \( D(n, m, 3) \). Since \( H(F) \) is \( \epsilon \)-close to \( D(n, m, 3) \), we have \( a \leq \sqrt{\epsilon}n < b \). Then for any \( j \in [m]\backslash[b] \), since \( F_j \) is \( \sqrt{\epsilon} \)-close to \( D(n, m, 3) \), \( F_j \) contains a complete subgraph with size at least \( 3m - b \). Hence we have \( \{2j - 1, 2j, 3m - j + 1\} \in F_j \). So \( M_2 := \{\{v_j, 2j - 1, 2j, 3m - j + 1\} : b < j \leq m\} \) is a matching in \( H(F) \) which is disjoint from \( M_1 \). Then \( M_1 \cup M_2 \) forms a matching of size \( m \) in \( H(F) \). So \( F \) admits a rainbow matching, completing the proof of Lemma 3.1. \( \square \)

4 A stability lemma

In this section, we prove a result for stable 3-graphs, which may be viewed as a stability version of the following result of Luczak and Mieczkowska proved in [22].

**Theorem 4.1** (See [22]). There exists a positive integer \( n_1 \) such that for integers \( m \) and \( n \) with \( n \geq n_1 \) and \( 1 \leq m \leq n/3 \), if \( H \) is an \( n \)-vertex 3-graph with \( e(H) > f(n, m, 3) \), then \( \nu(H) \geq m \).

Building on the proof in [22], we prove the following lemma.

**Lemma 4.2.** For any real number \( \epsilon > 0 \), there exists a positive integer \( n_1(\epsilon) \) such that the following holds. Let \( m \) and \( n \) be integers with \( n \geq n_1(\epsilon) \) and \( 1 \leq m \leq n/3 \), and let \( H \) be a stable 3-graph on the vertex set \([n]\). If \( e(H) > f(n, m, 3) - \epsilon^4n^3 \) and \( \nu(H) < m \), then \( H \) is \( \epsilon \)-close to \( S(n, m, 3) \) or \( D(n, m, 3) \).

**Proof.** Suppose that \( e(H) > f(n, m, 3) - \epsilon^4n^3 \) and \( s := \nu(H) < m \). Let \( M = \{(i, j, k) : \ell \in [s]\} \) be a largest matching in \( H \) and the partition \( V(M) = I \cup J \cup K \) such that every edge \((i, j, k) \in E(M)\) with \( i < j < k \) satisfies \( i \in I, j \in J \) and \( k \in K \). Since \( H \) is stable, we may choose \( V(M) \) to be \([3s]\).

Let \( V' = [n]\backslash[3s] \). For \( x \in [3s] \), let \( e(x) \) denote the edge in \( M \) containing \( x \). Let \( F_1 = \{v \in [3s]_1 : d_{V'}(v) \geq 20n\} \), \( F_2 = \{v, w \in [3s]_2 : e(v) \neq e(w) \) and \( d_{V'}(v, w) \geq 20\} \).
Let $H^* = ([3s], F)$ be the hypergraph with the vertex set $[3s]$ and the edge set $F = M \cup F_1 \cup F_2 \cup F_3$.

Call an edge $e \in H$ traceable if $e \cap [3s] \in F$, and untraceable otherwise. Since $M$ is a maximum matching in $H$, $V'$ is independent in $H$. So the number of untraceable edges of $H$ is bounded from above by

\[
\binom{3s}{1} \cdot 20n + \left( \binom{s}{2} \binom{3}{1} \binom{3}{1} \right) \times 19 + \left( \binom{s}{1} \binom{3}{2} n \right) + \left( \binom{s}{1} \binom{3}{2} \right) \left( 3s - 3 \right) \leq 32n^2 = o(n^3),
\]

where we use $s < m \leq n/3$. We point out that those edges (there being $o(n^3)$ of them) will be negligible in the following proof.

Let $T$ be a triple of edges from $M$. We say that $T$ is bad if $V(T)$ contains three pairwise disjoint edges of $H^*$ whose union intersects $I$ in at most 2 vertices, and good otherwise. For each $i \in [3]$, let $f_i(T)$ denote the number of edges of $F_i$ contained in $V(T)$. Note that $f_3(T) \leq 27$. The following two claims are explicit in [22].

**Claim 1.** There exist no three pairwise disjoint bad triples (of edges in $M$). Hence, there exist at most six edges in $M$ such that each bad triple contains one of these edges.

**Claim 2.** Let $T$ be a good triple.

(i) If $f_3(T) \geq 24$, then $f_1(T) = f_2(T) = 0$.

(ii) If $f_3(T) = 20$, then $f_1(T) \leq 1$ and $f_2(T) \leq 12$.

(iii) If $f_3(T) \leq 19$, then $f_1(T) \leq 3$ and $f_2(T) \leq 15$. Moreover, the only triples $T$ for which $f_3(T) = 19$, $f_2(T) = 15$ and $f_1(T) = 3$ are those in which each edge of $H^*$ contained in $V(T)$ intersects $I$.

(iv) If $f_3(T) = 21$, then $f_1(T) \leq 1$ and $f_2(T) \leq 10$.

(v) If $22 \leq f_3(T) \leq 23$, then $f_1(T) = 0$ and $f_2(T) \leq 7$.

We remove exactly six edges from $M$ such that the resulting matching $M'$ only contains good triples. Since $H$ has at most $18n^2$ edges intersecting $V(M \setminus M')$ and $32n^2$ untraceable edges, we have

\[
e(H) \leq |F_1| \left( \frac{n - 3s}{2} \right) + |F_2| (n - 3s) + |F_3| + 50n^2.
\]

To bound $|F_i|$, let us consider the summation of $f_i(T)$ over all $T \in (M')_3$. Since each edge from $F_i$ is counted exactly $\binom{(s-6) - i}{3 - i}$ times in this sum, we have

\[
|F_i| \binom{(s-6) - i}{3 - i} = \sum_{T \in (M')_3} f_i(T).
\]

Therefore,

\[
e(H) \leq \sum_{T \in (M')_3} \left( f_1(T) \frac{n - 3s}{2} + f_2(T) \frac{n - 3s}{s - 8} + f_3(T) \right) + 50n^2
\]

\[
\leq \sum_{T \in (M')_3} \left( f_1(T) \frac{(n - 3s)^2}{s^2} + f_2(T) \frac{n - 3s}{s} + f_3(T) \right) + O(n^2).
\]

Here, the last inequality is trivial when $s \leq 15$, and it holds when $s > 15$ because the difference between the above two summations is at most

\[
\sum_{T \in (M')_3} \left( f_1(T) \frac{15(n - 3s)^2}{s(s^2 - 15s)} + f_2(T) \frac{8(n - 3s)}{s(s - 8)} \right) \leq \left( \frac{s - 6}{3} \right) \left( \frac{45(n - 3s)^2}{s(s^2 - 15s)} + \frac{120(n - 3s)}{s(s - 8)} \right) = O(n^2),
\]

where $3s < n$, $f_1(T) \leq 3$ and $f_2(T) \leq 15$ (from Claim 2).
To further bound $e(H)$, we partition good triples $T$ depending on $f_3(T)$ and $f_1(T)$. Let
\[ T_i = \left\{ T \in \binom{M'}{3} : f_3(T) = i \right\} \]
for $i \in [27]$ and
\[ X = \left\{ T \in \binom{M'}{3} : f_1(T) = 3 \right\}. \]
Consider any $T \in X$, and then $T$ is a good triple\(^2\). Since $f_1(T) = 3$, the three edges of $F_i$ contained in $V(T)$ are precisely the three vertices in $V(T) \cap I$, and each edge of $H^*$ contained in $V(T)$ intersects $I$. Since $H$ is stable and $V(M) = [3s]$, by using the definition of $F_i$, it is not hard to see that $X \subseteq T_{19}$. Define
\[ x_1 = \sum_{i=1}^{18} |T_i| + |T_{19} \setminus X|, \quad x_2 = |T_{20}|, \quad x_3 = |T_{21}|, \quad x_4 = |T_{22}| + |T_{23}|, \quad x_5 = \sum_{i=24}^{26} |T_i|, \quad x = |X| \]
and $y = |T_{27}|$. So
\[ \sum_{i=1}^{5} x_i + x + y = \binom{s-6}{3}. \]
From now on, we let $t = (n - 3s)/s$. By Claim 2 and the fact $X \subseteq T_{19}$, we can derive from the above upper bound on $e(H)$ that
\[ e(H) \leq (3x + 2x_1 + x_2 + x_3)t^2 + (15x + 15x_1 + 12x_2 + 10x_3 + 7x_4)t \\
+ (19x + 19x_1 + 20x_2 + 21x_3 + 23x_4 + 26x_5 + 27y) + O(n^2). \]
For convenience, we write
\[ f_t(x_1, x_2, x_3, x_4, x_5, x, y) = \sum_{i=1}^{5} \alpha_i(t) \cdot x_i + \beta_1(t) \cdot x + \beta_2(t) \cdot y, \]
where
\[ \alpha_1(t) = 2t^2 + 15t + 19, \quad \alpha_2(t) = t^2 + 12t + 20, \quad \alpha_3(t) = t^2 + 10t + 21, \]
\[ \alpha_4(t) = 7t + 23, \quad \alpha_5(t) = 26, \quad \beta_1(t) = 3t^2 + 15t + 19 \quad \text{and} \quad \beta_2(t) = 27. \]
Then it follows that
\[ e(H) \leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2). \]
Next, we derive properties of the functions $\alpha_i(t)$ and $\beta_j(t)$.

**Claim 3.** For any $t \geq 0$, $\max\{\beta_1(t), \beta_2(t)\} \geq \max\{\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t)\} + 0.2$.

**Proof.** We have $\beta_2(t) = 27$. It is easy to see that for each $i \in [5]$, the functions $\alpha_i(t)$, $\beta_1(t) - \alpha_i(t)$ and $\beta_2(t)$ are increasing for $t \geq 0$. Note that $\beta_1(0.5) = 27.25$, $\alpha_2(0.5) = 26.25$, $\alpha_3(0.5) = 26.25$ and $\alpha_4(0.5) = 26.5$, and then $\max\{\beta_1(t), 27\} \geq \alpha_i(t) + 0.2$ for $t \geq 0$ and $i = 2, 3, 4$. Since $\beta_1(t) - \alpha_i(t) = t^2$ and $\alpha_1(\sqrt{0.2}) < 27 - 0.2$, we see $\max\{\beta_1(t), 27\} \geq \alpha_1(t) + 0.2$ for all $t \geq 0$. \qed

Since
\[ \beta_1(t) \binom{s-6}{3} = \frac{1}{2} (n - 3s)^2 s + \frac{5}{2} (n - 3s) s^2 + \frac{19}{6} s^3 = \frac{1}{6} n^3 - \frac{1}{6} (n-s)^3, \]
we see
\[ \max\{\beta_1(t), \beta_2(t)\} \binom{s-6}{3} \leq \max \left\{ \binom{n}{3} - \binom{n-s+1}{3}, \binom{3s-1}{3} \right\} + O(n^2) = f(n, s, 3) + O(n^2). \]

\(^2\) Since $T$ is good, the union of any three disjoint edges of $H^*$ in $V(T)$ must contain the three vertices in $V(T) \cap I$. 
By Claim 3 and the fact that $\sum_{i=1}^{5} x_i + x + y = \left(\frac{s-6}{3}\right)$, we have
\[
f_t(x_1, x_2, x_3, x_4, x_5, x, y) \\
\leq (\max\{\beta_1(t), \beta_2(t)\} - 0.2) \sum_{i=1}^{5} x_i + \beta_1(t)x + \beta_2(t)y \\
\leq \max\{\beta_1(t), \beta_2(t)\}\left(\frac{s-6}{3}\right) - 0.2 \sum_{i=1}^{5} x_i \leq f(n, s, 3) - 0.2 \sum_{i=1}^{5} x_i + O(n^2). \quad (4.1)
\]

Let $\cup X$ (resp. $\cup T_{27}$) denote the set of edges each of which belongs to some triple in $X$ (resp. in $T_{27}$). Now we show the following claim.

**Claim 4.** \( s > m - cn/4 \), and $x > \left(\frac{s-6}{3}\right) - 10\epsilon^4 n^3 - \left(\frac{en/24}{3}\right)$ or $y > \left(\frac{s-6}{3}\right) - 10\epsilon^4 n^3 - \left(\frac{en/12}{3}\right)$.

**Proof.** If $s \leq m - cn/4$, then by (4.1) we have
\[
e(H) \leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leq f(n, s, 3) + O(n^2) \\
\leq f(n, m, 3) - \left(\frac{\epsilon/4n}{3}\right) + O(n^2) \leq f(n, m, 3) - \epsilon^4 n^3,
\]
which leads to a contradiction. So $s > m - cn/4$. First we see that $x + y > \left(\frac{s-6}{3}\right) - 10\epsilon^4 n^3$; for, otherwise, $\sum_{i=1}^{5} x_i \geq 10\epsilon^4 n^3$, which together with (4.1) implies
\[
e(H) \leq f_t(x_1, x_2, x_3, x_4, x_5, x, y) + O(n^2) \leq f(n, m, 3) - 2\epsilon^4 n^3 + O(n^2) \leq f(n, m, 3) - \epsilon^4 n^3,
\]
which leads to a contradiction. Now suppose that $x > \left(\frac{en/12}{3}\right)$ and $y > \left(\frac{en/24}{3}\right)$. Then $|\cup X| > \epsilon n/12$ and $|\cup T_{27}| > \epsilon n/24$. For any edge $e = (i, j, k) \in \cup X$ with $i < j < k$, by the previous discussion, we have $i \notin F_1$. For any edge $e = (i, j, k) \in \cup T_{27}$ with $i < j < k$, by Claim 2 we see $i \notin F_1$. Thus $\cup X \cap \cup T_{27} = \emptyset$. The triples $T = \{e_1, e_2, e_3\}$ with $e_1 \in \cup X$ and $e_2, e_3 \in \cup T_{27}$ cannot satisfy both $f_3(T) = 27$ and $f_1(T) = 3$. This shows
\[
x + y < \left(\frac{s-6}{3}\right) - |\cup X|\left(\frac{|\cup T_{27}|}{2}\right) \leq \left(\frac{s-6}{3}\right) - \epsilon n/12 \left(\frac{en/24}{2}\right),
\]
contradicting that $x + y > \left(\frac{s-6}{3}\right) - 10\epsilon^4 n^3$. Hence, we have that either $x \leq \left(\frac{en/12}{3}\right)$ or $y \leq \left(\frac{en/24}{3}\right)$. $\square$

Suppose
\[
x > \left(\frac{s-6}{3}\right) - 10\epsilon^4 n^3 - \left(\frac{en/24}{3}\right).
\]
So $x > \left(\frac{s-6}{3}\right) - \left(\frac{en/12}{3}\right)$ and thus $|\cup X| > s - 6 - \epsilon n/12$. Recall that for any $T \in X$, $T$ is a good triple, and hence each edge of $H^*$ contained in $V(T)$ intersects $I$. Hence any traceable edge which intersects $V(\cup X)$ must also intersect $I$. Thus, the number of edges of $H$ not intersecting $I$ is at most
\[
|V(M') \setminus V(\cup X)| \left(\frac{n}{2}\right) + 50n^2 \leq \epsilon n \left(\frac{n}{4}\right) + 50n^2 \leq \frac{\epsilon}{4} n^3.
\]
As $|I| = s \leq m - 1$,
\[
|E(S(n, m, 3)) \setminus E(H)| = |E(H) \setminus E(S(n, m, 3))| + e(S(n,m,3)) - e(H) \leq \frac{\epsilon}{4} n^3 + \epsilon^4 n^3 < \epsilon n^3.
\]
So in this case, we see that $H$ is $\epsilon$-close to $S(n, m, 3)$.

By Claim 4, it remains to consider
\[
y > \left(\frac{s-6}{3}\right) - 10\epsilon^4 n^3 - \left(\frac{en/12}{3}\right).
\]
We claim that there exists a complete 3-graph $K$ on more than $3m - 3en/2$ vertices and $V(K) \subseteq V(M')$. Suppose to the contrary that $V(M')$ does not contain such a complete 3-graph $K$. Since 

$$|V(M')| = (3m - 3en/2) = 3(s - 6) - 3m + 3en/2 > \frac{en}{2}$$

and $H$ is stable, $V(M')$ contains an independent set of size $\frac{en}{2}$, say $A$. Note that if $T = \{e_1, e_2, e_3\}$ with $e_i \cap A \neq \emptyset$ for all $i \in [3]$, then $f_3(T) < 27$. Since there are at least $|A|/3 \geq cn/6$ edges in $M'$ which intersect with $A$, we see that $y \leq \left(\frac{s}{3}\right) - \binom{cn}{3}$, which leads to a contradiction.

Then 

$$E(D(n, m, 3)) \subseteq E(H) \leq \frac{3}{2} \frac{n!}{(\frac{n}{2})!} < cn^3,$$

i.e., $H$ is $c$-close to $D(n, m, 3)$. This finishes the proof of Lemma 4.2. \hfill \square

5 Almost perfect rainbow matchings

In this section, we prove a lemma about almost perfect rainbow matchings that we will need. In fact, this result holds for families of $k$-graphs, for any $k \geq 3$.

**Lemma 5.1.** For any given integer $k \geq 3$, there exist positive real numbers $c$ and $n_2$ such that the following holds. Let $n$ and $m$ be integers with $n \geq km$ and $n \geq n_2$, and let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a stable family of $k$-graphs on the same vertex set $[n]$ such that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$. If $m > (1-c)n/k$, then $\mathcal{F}$ admits a rainbow matching.

**Proof.** We choose $c' = c'(k)$ and $c = c(k)$ small enough such that $0 < c \ll c' \ll 1$. Let $n$ be sufficiently large and $n/k \geq m > (1-c)n/k$. Suppose to the contrary that $|F_i| > \binom{km-1}{k}$ for each $i \in [m]$ and $\mathcal{F}$ does not admit a rainbow matching.

By Corollary 2.3, we may additionally assume that $\mathcal{F}$ is saturated. Let $U_i$ be the vertex set of a largest complete $k$-graph in $F_i$ for $i \in [m]$. Since $F_i$ is stable, we may choose $U = \bigcup U_i$ such that $[n] \setminus U_i$ is an independent set in $F_i$. For each $i \in [m]$, we have $|U_i| > (1-c')km$; for, otherwise, we have the following contradiction for some $i \in [m]$:

$$|F_i| \leq \binom{n}{k} - c' \binom{km}{k} \leq \binom{n}{k} - (cn + 1) \binom{n - 1}{k - 1} \leq \binom{n}{k} - (n - km + 1) \binom{n - 1}{k - 1} < \binom{km - 1}{k},$$

where the second inequality holds since $c \ll c' \ll 1$ and $m > (1-c)n/k$, the third inequality holds since $n - km < cn$, and the last inequality holds since

$$\binom{n}{k} - \binom{km - 1}{k} = \sum_{i=1}^{n-km+1} \binom{n-i}{k-1} < (n - km + 1) \binom{n - 1}{k - 1}.$$

Let $U = \bigcap_{i=1}^{m} U_i$. By the above paragraph, we see that $|U| \geq (1-c')km$. If $|U| \geq km$, then it is clear that $\mathcal{F}$ admits a rainbow matching. So we may assume that $U_m = U \subseteq [km - 1]$. Because $U_m$ is the vertex set of a largest complete $k$-subgraph of $F_m$ and since $F_m$ is stable and $|F_m| > \binom{km-1}{k}$, there exists some $k$-set $e \in F_m$ such that $|e \cap U| = k - 1$ and $km \in e$. Since $\mathcal{F}$ is saturated, there exists a rainbow matching $M$ in $\mathcal{F} \setminus F_m$ such that $M \cup \{e\}$ is a rainbow matching in $\mathcal{F}(e, F_m)$. Since $F_i$ is stable for each $i \in [m]$, we may assume that $V(M) \cup e = [km]$. Let $M' = \{e' \in M : e' \not\subseteq U\}$.

**Claim 5.**

(a) $|M'| < c' km$,

(b) each edge of $F_m$ is contained in $U$ or intersects an edge of $M'$, and

(c) for any $v \in V(M) \setminus U$, $d_{F_m}(U)(v) \leq c' k^2 m_2(U)$.  

**Proof.** To prove (a), just observe that $|M'| \leq |V(M) \setminus U| = (km - 1) - |U| < c' km$.

Suppose that (b) fails, i.e., there exists an edge $f \in F_m$ such that $f \cap U = \emptyset$ and $f \cap V(M') = \emptyset$. Note that $f \cap (U \setminus V(M')) \neq \emptyset$, as $[n] \setminus U$ is independent in $F_m$. In particular, $|f \cap (U \setminus V(M'))| \leq k - 1$. Let $|M'| = m - t$ for some $t \geq 1$. Recall that $U \cup V(M') = V(M) = [km - 1]$. Hence $|U \setminus V(M')| = kt - 1$,
and thus $U \setminus (V(M') \cup f)$ induces a common complete $k$-graph of size at least $k(t - 1)$ in all $F_i$. Then we see that $M' \cup \{f\}$ together with a matching of size $t - 1$ in $U \setminus (V(M') \cup f)$ form a rainbow matching for $F$. So (b) holds.

Now we prove (c). For any $v \in V(M) \setminus U \subseteq [km]$, by the maximality of $U$, there exists $f \in \binom{[n]}{k} \setminus F_m$ such that $v \notin f$ and $|f \cap U| = k - 1$. So there exists a rainbow matching $N$ in $F \setminus F_m$ such that $N \cup \{f\}$ is a rainbow matching in $F'(f, F_m)$. Since $F_i$ is stable for $i \in [m]$, we may assume that $V(N) \cup f = [km]$. Let $N' = \{v' \in N : v' \notin U\}$. By applying (b) to $N'$, every edge of $F_m$ containing $v$ intersects $V(N')$. Since

$$V(N') \leq k|N'| \leq k(km - |U|) \leq c'k^2m,$$

there are at most $c'k^2m\binom{|U|}{k - 2}$ edges $e'$ in $F_m$ containing $v$ such that $e' \subseteq U \cup \{v\}$. Hence (c) holds. This proves the claim.

Note that $|e \cap U| = k - 1$ and $V(M) \cup U = [km]$. Let $q_1$ be the number of edges of $F_m$ contained in $[km - 1]$, and $q_2$ be the number of edges of $F_m$ with at least one vertex in $[n] \setminus [km - 1]$. By (c), we have

$$q_1 \leq \binom{km - 1}{k} - |V(M) \setminus U| \left(\binom{|U|}{k - 1} + c'k^2m\binom{|U|}{k - 2}\right) + |V(M')|(n - km + 1)\binom{n - 2}{k - 2}.$$ 

By (b), we see $q_2 \leq |V(M')| \cdot (n - km + 1)\binom{n - 2}{k - 2}$. So we have

$$|F_m| \leq \binom{km - 1}{k} - |V(M) \setminus U| \left(\binom{|U|}{k - 1} + c'k^2m\binom{|U|}{k - 2}\right) + |V(M')|(n - km + 1)\binom{n - 2}{k - 2},$$

where the second inequality holds since $n - km < cn$ and $|M'| \leq |V(M) \setminus U|$, and the last inequality holds since $c'$ and $c$ are small enough and $|U| > (1 - c')km > (1 - c')(1 - c)n$. This is a contradiction, finishing the proof of Lemma 5.1.

## 6 Non-extremal configurations

Note that if there exist $F \in \mathcal{F}$ and $v \in [n]$ such that $d_F(v) = \binom{n - 1}{k - 1}$, then $v$ can be removed from all the $k$-graphs in $\mathcal{F} \setminus \{F\}$ to obtain a smaller family $\mathcal{F}'$ so that $\mathcal{F}'$ admits a rainbow matching if and only if $\mathcal{F}$ admits a rainbow matching. Hence, if such vertex does not exist in a saturated family $\mathcal{F}$, then from Lemma 2.1, we see that $d_F(v) \leq \binom{n - 1}{k - 1} - \binom{n - k(m - 1) - 1}{k - 1}$ for all $v \in F$ and $F \in \mathcal{F}$. This leads us to the following result.

**Lemma 6.1.** Given real numbers $0 < \epsilon \ll c \ll 1$, let $n \geq n(\epsilon, c)$ be a sufficiently large integer and $m$ be an integer such that $n/27 < m < (1 - c)n/3$. Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a stable family of $3$-graphs on the vertex set $[n]$ such that for every $i \in [m]$, $|F_i| > f(n, m, 3)$ and

$$d_{F_i}(v) \leq \binom{n - 1}{2} - \binom{n - 3(m - 1) - 1}{2}$$

for each $v \in [n]$. If $H(\mathcal{F})$ is $\epsilon$-close to neither $H_S(n, m, 3)$ nor $H_D(n, m, 3)$, then $\mathcal{F}$ admits a rainbow matching.

**Proof.** Given $0 < \epsilon \ll c \ll 1$, let $n'$ and $m'$ be integers such that $n'$ is sufficiently large and $n'/27 < m' < (1 - c)n'/3$. Let $\mathcal{F} = \{F_1, \ldots, F_{m'}\}$ be a family of $3$-graphs on the vertex set $[n']$ such that $|F_i| > f(n', m', 3)$ and

$$d_{F_i}(v) \leq \binom{n' - 1}{2} - \binom{n' - 1 - 3(m' - 1)}{2}$$

for each $v \in [n']$. If $H(\mathcal{F})$ is $\epsilon$-close to neither $H_S(n, m, 3)$ nor $H_D(n, m, 3)$, then $\mathcal{F}$ admits a rainbow matching.
for $i \in [m']$ and $v \in [n']$. Suppose that $H(F)$ is not $\epsilon$-close to $H_S(n', m', 3)$ or $H_D(n', m', 3)$. Our ultimate goal is to find a rainbow matching in $F$.

Let $n' = 3m' + 3r' + s$, where $0 \leq s < 3$. Recall the definitions of $H(F)$ and $H^*(F)$ such that $V(H(F)) = [n'] \cup V'$ and $V(H^*(F)) = [n'] \cup V' \cup U'$, where $|V'| = m'$ and $|U'| = r'$. By Lemma 2.5, for $0 < \gamma' < \gamma < \epsilon < c < 1$, there exists a matching $M_a$ in $H^*(F)$ with $|M_a| \leq \gamma'n'$ such that for any $S \subseteq V(H^*) \cup V(M_a)$ with $|S| \leq \gamma'n'$ and $3|S \cap (V' \cup U')| = |S \cap [n']|$, $H^*(F)[V(M_a) \cup S]$ has a perfect matching. In the rest of the proof, without loss of generality, we use the following notation:

$$
H = H^*(F) - V(M_a), \quad [n] = [n'] \setminus V(M_a),
$$

$$
V = V' \setminus V(M_a) = \{v_1, \ldots, v_m\}, \quad U = U' \setminus V(M_a) = \{u_1, \ldots, u_r\}.
$$

Then $n = 3m + 3r + s$. By using the above property of the matching $M_a$, it now suffices for us to find an almost perfect matching in $H$. To find this almost perfect matching, our plan is to show that there exists an almost regular subgraph of $H$ with bounded maximum co-degree so that Theorem 2.6 can be applied.

To that end, in what follows we will use the two-round randomization technique developed in [3].

Let $R$ be chosen from $V(H)$ by taking each vertex independently of probability $n^{-0.9}$. We take $n^{1.1}$ independent copies of $R$ and denote them by $R^i$ for $1 \leq i \leq n^{1.1}$. For $S \subseteq V(H)$, denote $Y_S = \{|i : S \subseteq R^i\}$. First we have the following claim.

**Claim A.** With probability $1 - o(1)$, the following hold:

(i) for every $v \in V(H)$, $Y_{[v]} = (1 + o(1))n^{0.2}$,

(ii) every pair $\{u, v\} \subseteq V(H)$ is contained in at most two sets $R^i$, and

(iii) every edge $e \in H$ is contained in at most one set $R^i$.

**Proof.** Note that $Y_S \sim \text{Bin}(n^{1.1}, n^{-0.9}|S|)$ for any $S \subseteq V(H)$. Thus, $\mathbb{E}[Y_{[v]}] = n^{0.2}$ for every $v \in V(H)$. By (2.2) in Lemma 2.10, we have $\mathbb{P}(|Y_{[v]} - n^{0.2}| > n^{0.15}) \leq e^{-\Omega(n^{0.1})}$. By the union bound, we see that (i) holds. To prove (ii) and (iii), let

$$
Z_2 = \left|\left\{\{u, v\} \in \binom{V(H)}{2} : Y_{\{u, v\}} \geq 3\right\}\right| \quad \text{and} \quad Z_3 = \left|\left\{S \in \binom{V(H)}{3} : Y_S \geq 2\right\}\right|.
$$

Then

$$
\mathbb{E}[Z_2] = \binom{|V(H)|}{2}(n^{0.2})^3 \leq \frac{n}{2}(n^{1.1})^3(n^{-1.8})^3 \leq 4n^{-0.1}
$$

and

$$
\mathbb{E}[Z_3] \leq \frac{n}{3}(n^{1.1})^2(n^{-2.7})^2 \leq 8n^{-0.2}.
$$

By Markov’s inequality, we have

$$
\mathbb{P}(Z_2 = 0) > 1 - 4n^{-0.1} \quad \text{and} \quad \mathbb{P}(Z_3 = 0) > 1 - 8n^{-0.2}.
$$

That implies that (ii) and (iii) hold with probability at least $1 - 4n^{-0.1}$ and $1 - 8n^{-0.2}$, respectively.

Next, we want to prove that there exists a perfect (or, rather, maximum) fractional matching in each $H[R^i]$. To do so, we define a maximal subset $R^i \subseteq R^i$ that satisfies $R^i \cap [n] = 3|R^i \cap (V \cup U)|$ as follows. If $|R^i \cap [n]| \geq 3|R^i \cap (V \cup U)|$, we take a subset of $R^i$ denoted by $R^i$, which is chosen from $R^i$ by deleting $|R^i \cap [n] - 3|R^i \cap (V \cup U)||$ vertices in $R^i \cap [n]$ independently and uniformly at random. Otherwise $|R^i \cap [n]| < 3|R^i \cap (V \cup U)|$, we take a subset of $R^i$ denoted by $R^i$ by the following step: First we delete at most 3 vertices (chosen independently and uniformly at random) in $R^i \cap [n]$ so that the number $\ell$ of the remaining vertices is a multiple of 3. Then we delete $|R^i \cap (V \cup U)| - \ell/3$ vertices in $R^i \cap (V \cup U)$ independently and uniformly at random.

For $S \subseteq V(H)$, define $Y'_S = \{|i : S \subseteq R^i|\}$. Note that $\mathbb{E}(|R^i \cap [n]|) = n^{0.1}$, $\mathbb{E}(|R^i \cap (V \cup U)|) = n^{0.1}/3$ and $\mathbb{E}(|R^i \cap V|) = n^{-0.9}m$. For each $i$, let $A_i$ be the event $|R^i \cap [n]| - n^{-0.1} < n^{0.095}$, $B_i$ be the event $|R^i \cap (V \cup U)| - n^{-0.1}/3 < n^{0.095}$, and $C_i$ be the event $|R^i \cap V| - n^{-0.9}m < n^{0.095}$.

**Claim B.** With probability $1 - o(1)$, the following hold:
(i) \( \bigwedge_i (A_i \land B_i \land C_i) \) holds,
(ii) for every \( v \in V(H) \), \( Y'_{v} = (1 + o(1))n^{0.2} \),
(iii) every pair \( \{u, v\} \subseteq V(H) \) is contained in at most two sets \( R^i \), and
(iv) every edge \( e \in H \) is contained in at most one set \( R^i \).

**Proof.** Since \( R^i \subseteq R \), it is clear from Claim A that (iii) and (iv) hold with probability \( 1 - o(1) \). Next, we consider (i). By (2.2) in Lemma 2.10 (with \( \lambda = \nu^{0.095} \)), for each \( 1 \leq i \leq n^{1.1} \), we have
\[
\mathbb{P}(A_i) \leq e^{-\Omega(n^{0.09})}, \quad \mathbb{P}(B_i) \leq e^{-\Omega(3n^{0.09})} = e^{-\Omega(n^{0.09})} \quad \text{and} \quad \mathbb{P}(C_i) \leq e^{-\Omega(\nu^{0.09})} = e^{-\Omega(n^{0.09})}.
\]
Thus by the union bound, \( \mathbb{P}(\bigwedge (A_i \land B_i \land C_i)) = 1 - o(1) \), proving (i).

Assuming \( A_i \land B_i \land C_i \), we see
\[
|R^i \setminus R^\circ| < \max\{|R^i \cap [n]| - 3|R^i \cap (V \cup U)|, |R^i \cap (V \cup U)| - \frac{|R^i \cap [n]|}{3} + 3\} \leq 4n^{0.095}.
\]
Then by the choice of \( R^i \), for all \( v \in V(H) \), the probability \( \mathbb{P}(\{v \in R^i \setminus R^\circ \mid (A_i \land B_i \land C_i) \land (v \in R^i)\}) \) is at most
\[
\max \left\{ \frac{|R^i \setminus R^\circ|}{|R^i \cap [n]|}, \frac{|R^i \setminus R^\circ|}{|R^i \cap (V \cup U)|} \right\} \leq \frac{|R^i \setminus R^\circ|}{|R^i |} < \frac{4n^{0.095}}{n^{0.1/3} - n^{0.095}} \leq 13n^{-0.095}.
\]
Using coupling and applying (2.2) in Lemma 2.10 to Bin(\{Y_v, 13n^{-0.095}\} with \( \lambda = 3n^{0.195} \), we have
\[
\mathbb{P}\left(\left|Y'_{v} - Y'_{v'} \right| > 16n^{0.195} \bigg| \bigwedge_i (A_i \land B_i \land C_i) \land (Y_{v} = (1 + o(1))n^{0.2})\right) \leq e^{-\Omega(n^{0.195})}.
\]
Note that with probability \( 1 - o(1) \), \( \bigwedge_i (A_i \land B_i \land C_i) \) and \( Y_{v} = (1 + o(1))n^{0.2} \) hold for all \( v \in V(H) \). By the union bound, we can derive that \( 0 \leq Y_{v} - Y'_{v} \leq 16n^{0.195} = o(n^{0.2}) \) for all \( v \in V(H) \) with probability \( 1 - o(1) \). Hence (ii) holds with probability \( 1 - o(1) \). This proves Claim B. \( \Box \)

Let \( n_i = |R^i \cap [n]| \) and \( m_i = |R^i \cap V| \). Using Claim B(i), we see that with probability \( 1 - o(1) \),
\[ m_i = (1 + o(1))mn^{-0.9} = \Theta(n^{0.1}) = \Theta(n_i) \text{ for all } 1 \leq i \leq n^{1.1}. \]

**Claim C.** With probability \( 1 - o(1) \), the following hold for all \( 1 \leq i \leq n^{1.1}:
(a) \( H[R^i \setminus U] \) is not \( \epsilon^2/4 \)-close to \( H_2(n_i, m_i, 3) \) or \( H_3(n_i, m_i, 3) \), and
(b) there exists a perfect fractional matching in \( H[R^i] \).

**Proof.** For each \( T \in \binom{V(H)}{2} \), let
\[
\text{Deg}^i(T) := \left| N_H(T) \cap \left( R^i \setminus \frac{T}{4} \right) \right|.
\]
By the definition of \( H \), we have that
- for any \( v_j \in V \), \( d_H(v_j) \geq f(n', m', 3) - (\gamma n')^{(\eta')/2} \geq f(n, m, 3) - \gamma n^3 \), and
- for any \( T = \{v_j, u\} \) with \( v_j \in V \) and \( u \in [n] \),
\[
d_H(T) = d_{F_j}(u) \leq \left( \frac{n' - 1}{2} - \left( \frac{n' - 1 - 3(m' - 1)}{2} \right) \right) \leq \left( \frac{n - 1}{2} - \left( \frac{n - 1 - 3(m - 1)}{2} \right) + \gamma n^2 \right).
\]
Assume that \( \bigwedge_i (A_i \land B_i \land C_i) \) holds. Then \( n_i = (1 + o(1))n^{0.1} \) and \( m_i = (1 + o(1))mn^{-0.9} \). Since \( R^i \setminus R^\circ = o(n_i) \), for each \( T \in \binom{V(R^i)}{2} \) with \( t \in [2] \), we have
\[
E[\text{Deg}^i(T)] = (1 + o(1))d_H(T)(n^{-0.9})^{t - 1}.
\]
Thus for any \( v \in V \cap R^i \),
\[
E[\text{Deg}^i(v)] \geq (1 + o(1))(f(n, m, 3) - \gamma n^3)(n^{-0.9})^3 \geq f(n_i, m_i, 3) - 2\gamma n^3.
\]
and for any $T = \{u, v\}$ with $v \in \mathcal{V}$ and $u \in [n]$, $\mathbb{E}[\text{Deg}(T)]$ is at most

$$(1 + o(1)) \left[ \binom{n - 1}{2} - \binom{n - 1 - 3(m - 1)}{2} + \gamma n^2 \right] (n - 0.9)^2 \leq \binom{n - 1}{2} - \binom{n - 1 - 3(m - 1)}{2} + 2\gamma n^2.$$ 

We apply Janson’s inequality (see [4, Theorem 8.7.2]) to bound the deviation of $\text{Deg}(T)$ for $|T| \leq 2$. Write $\text{Deg}(T) = \sum_{e \in \mathcal{N}(T)} X_e$, where $X_e = 1$ if $e \subseteq R^n$ and $X_e = 0$ otherwise. Let $t = |T| \in \{1, 2\}$ and $p = n^{-0.9}$. Then

$$\Delta^* = \sum_{e_i \cap e_j \neq \emptyset, e_i, e_j \in \binom{[n]}{4-t}} \mathbb{P}(X_{e_i} = X_{e_j} = 1) \leq 4^{-t} \cdot \sum_{t=1}^{4-t} p^{2(4-t)} \left( \frac{n - t}{4 - t} \right) \left( \frac{4 - t}{t} \right)^{(n - 4)} (4 - t - t) = O(n^{0.1(2(4-t)-1)}).$$

By Janson’s inequality, for $v \in \mathcal{V} \cap R^n$,

$$\mathbb{P}(\text{Deg}(v) \leq (1 - \gamma)\mathbb{E}[\text{Deg}(v)]) \leq e^{-\gamma^2 \mathbb{E}[\text{Deg}(v)]/(2 + \Delta^*/\mathbb{E}[\text{Deg}(v)])} \leq e^{-\mathbb{E}[\text{Deg}(v)]/(2 + \Delta^*/\mathbb{E}[\text{Deg}(v)])} = e^{-\Omega(n^{0.1})},$$

and for the pair $\{v, u\}$ with $v \in \mathcal{V}$ and $u \in [n]$ (by considering the complement of $H$), we can have

$$\mathbb{P}(\text{Deg}(\{v, u\}) \geq (1 + \gamma)\mathbb{E}[\text{Deg}(\{v, u\})]) \leq e^{-\Omega(n^{0.1})}.$$ 

By the union bound, with probability $1 - o(1)$ we derive from above that for all $1 \leq i \leq n^{1 - \epsilon}$, (1) for any $v \in \mathcal{V} \cap R^n$, $\text{Deg}(v) \geq (1 - \gamma)\mathbb{E}[\text{Deg}(v)] \geq f(n, m, 3) - 3\gamma n^3$, and (2) for any pair $\{v, u\} \subseteq R^n$ with $v \in \mathcal{V}$ and $u \in [n]$,

$$\text{Deg}(\{v, u\}) \leq \left( \binom{n - 1}{2} - \binom{n - 1 - 3(m - 1)}{2} + 3\gamma n^2 \right) \leq \left( \binom{n - 1}{2} - \Omega(n^2) \right),$$

which implies that $F_j[R^n \cap [n]]$ is not $e^j/2$-close to $S(n_i, m_i, 3)$, since $m_i = (1 + o(1))mn^{0.9}$ and $m < (1 - \epsilon)n/3$.

This shows that $H[R^n \setminus U]$ is not $e^j/3$-close to $H_S(n_i, m_i, 3)$, where $\gamma < \epsilon$.

Let $\mathcal{V}_0 := \{v \in \mathcal{V} : F_j[[n]]$ is not $c\epsilon$-close to $D(n, m, 3)\}$. We claim that $|\mathcal{V}_0| < cn$. Otherwise $|\mathcal{V}_0| \leq cn$, we have

$$|E(H_D(n', m', 3)) \setminus E(H(F))| \leq cn \left( \binom{n}{3} + (m - cn)\epsilon n^3 + \gamma(n')^4 \right) \leq \epsilon(n')^4,$$

which leads to a contradiction as $H(F)$ is not $c\epsilon$-close to $H_D(n', m', 3)$. As $|\mathcal{V}_0| > cn$, with probability $1 - o(1)$ we have (by using Lemma 2.10) that

(3) $|R^n \cap \mathcal{V}_0| \geq \frac{cn}{2}$ for all $1 \leq i \leq n^{1 - \epsilon}$.

For $v_i \in R^n \cap \mathcal{V}_0$, we consider $F_j[[n]]$. Let $G$ be the complement of $F_j[[n]]$. Then for any $S \subseteq \mathcal{V}(G)$ with $|S| > 3m - cn$, we have $e(G[S]) \geq e(G)$. Otherwise,

$$|E(D(n, m, 3)) \setminus E(F_j[[n]])| \leq cn \left( \binom{n}{2} + \epsilon e(G) < \epsilon n^3,$$

contradicting $v_j \in \mathcal{V}_0$. By Lemma 2.8, the maximum size of the complete 3-graph in $F_j[R^n \cap [n]]$ is no more than $(3m/n - \epsilon + \gamma)n^{0.1} \leq 3m_i - cn_i/2$ with probability at least $1 - (n^{0.1}/e^{-\Omega(n^{0.1})})$. By assuming $\bigwedge_i(A_i \land B_i \land C_i)$, this implies that $F_j[R^n \cap [n]]$ is not $e^j/2$-close to $D(n_i, m_i, 3)$. By the union bound, with probability $1 - o(1)$ we have

(4) for all $1 \leq i \leq n^{1 - \epsilon}$ and $v_i \in R^n \cap \mathcal{V}_0$, $F_j[R^n \cap [n]]$ is not $e^j/2$-close to $D(n_i, m_i, 3)$.

By (3) and (4), we see that with probability $1 - o(1)$, $H[R^n \setminus U]$ is not $e^j/4$-close to $H_D(n, m, 3)$, proving Claim C(a).

It remains to show Claim C(b), i.e., to construct a perfect fractional matching $w_i$ in $H[R^n]$ for each $1 \leq i \leq n^{1 - \epsilon}$. Our main tool is the stability result, Lemma 4.2.
Fix some $1 \leq i \leq n^{1.1}$. We write $R^i \cap [n] = \{x_1^i, \ldots, x_n^i\}$ with $x_1^i < x_2^i < \cdots < x_n^i$ and define $[d]^i := \{x_1^i, x_2^i, \ldots, x_d^i\}$ for any integer $d$. We now state two simple inequalities for later use:

$$f(x, y, 3) \geq f(x, y - a, 3) + \left(\frac{a}{3}\right) \quad \text{and} \quad f(x, y, 3) \geq f(x, y + a, 3) - 3ax^2$$  \hspace{1cm} (6.1)

hold for any positive integers $x, y$ and $a$ with $a < y$.

To construct a perfect fractional matching $w_i$ in $H[R^i]$, first we consider $v_j \in R^i \cap \mathcal{V}_0$ and assign weights to the edges of $H[R^i]$ containing $v_j$. Using (1), by (6.1) and the fact that $\gamma \ll \epsilon \ll 1$, we have

$$|F_j[R^i \cap [n]]| = \deg^i(v_j) \geq f(n_i, m_i, 3) - 3\gamma n_i^3 \geq f(n_i, m_i + 20n_i, 3) - \epsilon^6 n_i^3.$$  

By (2) and (4), $F_j[R^i \cap [n]]$ is not $\epsilon^4/2$-close to $S(n_i, m_i, 3)$ or $D(n_i, m_i, 3)$. Since $|E(S(n_i, m_i + 20n_i, 3)) \setminus E(S(n_i, m_i, 3))| \leq \epsilon^2 n_i^3$ and $|E(D(n_i, m_i + 20n_i, 3)) \setminus E(D(n_i, m_i, 3))| \leq 3\epsilon^2 n_i^3$, we see that $F_j[R^i \cap [n]]$ is not $\epsilon^4$-close to $S(n_i, m_i + 20n_i, 3)$ or $D(n_i, m_i + 20n_i, 3)$. Then by Lemma 4.2 and the fact that $F_j$ is stable, $F_j[R^i \cap [n]]$ contains a matching $M_j$ with $V(M_j) = [3m_i + 20n_i]$. Now we assign weights $w_i(e)$ to all the edges $e$ of $H[R^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i - 6\gamma n_i}$, and otherwise let $w_i(e) = 0$.

Next, we consider $v_j \in R^i \cap (V \setminus \mathcal{V}_0)$. By (1) and (6.1), we have

$$|F_j[R^i \cap [n]]| \geq f(n_i, m_i, 3) - 3\gamma n_i^3 \geq f(n_i, m_i - 6\gamma^2 n_i, 3).$$

By Theorem 4.1 and the fact that $F_j$ is stable, $F_j[R^i \cap [n]]$ contains a matching $M_j$ with $V(M_j) = [3m_i - 18\gamma^2 n_i]$. Then we assign weights $w_i(e)$ to all the edges $e$ of $H[R^i]$ with $v_j \in e$ as follows: If $e \setminus v_j \in M_j$, then let $w_i(e) = \frac{1}{m_i - 6\gamma n_i}$, and otherwise let $w_i(e) = 0$.

Note that for every $v_j \in R^i \cap V$, we have defined weights $w_i(e)$ for all the edges $e \in H[R^i]$ with $v_j \in e$, whose total weights equal one. In the remaining proof, we want to extend this function $w_i$ to the entire $H[R^i]$ to form a perfect fractional matching. We complete this in two steps.

First, we define a perfect fractional matching $w$ (as the projection of $w_i$) in the complete 3-graph $K$ on the vertex set $R^i \cap [n]$. Note that a function $w : E(K) \to [0, 1]$ is a perfect fractional matching if and only if $w(v) := \sum_{e \in f} w(f) = 1$ holds for every $v \in V(K)$. Initially, we define a function $w' : E(K) \to [0, 1]$ such that for each $f \in E(K)$, $w'(f) := \sum_{e \in f} w_i(e)$ over all the edges $e \in H[R^i]$ with $f \subseteq e$ and $|e \cap \mathcal{V}| = 1$.

Since $|\mathcal{V}_0| > cn$ and $\gamma \ll \epsilon$, it follows from the above definitions on $w_i$ that for any $e \in R^i \cap [n]$,

$$w'(v) := \sum_{v \in f} w'(f) \leq \frac{|\mathcal{V}_0|}{m_i + \epsilon^2 n_i} + \frac{m_i - |\mathcal{V}_0|}{m_i + 6\gamma n_i} \leq \frac{\epsilon n_i}{m_i + \epsilon^2 n_i} + \frac{m_i - \epsilon n_i}{m_i - 6\gamma n_i} < 1.$$  

Since $\epsilon \ll 1$, we have $3m_i + 3\epsilon^2 n_i < n_i - 4$. So there exists a vertex set $\{a_1, a_2, a_3, a_4\}$ in $K$ such that $w'(a_i) = 0$ for $i \in [4]$. Let $K'$ be the 3-graph obtained from $K$ by deleting vertices $a_1, a_2, a_3$ and $a_4$.

Starting with $w := w'$, let us follow the iterating steps: (i) pick a vertex $v$ in $V(K')$ with the maximum $w(v)$; (ii) pick any edge $f \in K'$ containing $v$ and update $w(f) \leftarrow w(f) + 1 - w(v)$; (iii) delete all the vertices $u \in V(K')$ with $w(u) = 1$ (which must include the vertex $v$) from $K'$; (iv) if $|V(K')| \leq 2$, then terminate; otherwise go to (i) again. This must terminate in finitely many iterations and when it terminates, we obtain a fractional matching $w$ in $K$ such that $w(a_i) = 0$ for $i \in [4]$ and $|V(K')| \leq 2$. So there exist two vertices $b_1$ and $b_2$ in $V(K) \setminus \{a_1, a_2, a_3, a_4\}$ such that for any vertex $v$ in $V(K) \setminus \{a_1, a_2, a_3, a_4, b_1, b_2\}$, $w(v) = 1$. We may suppose $1 > w(b_1) \geq w(b_2)$. Let

$$w(a_1, a_2, b_1) = 1 - w(b_1), \quad w(a_1, a_2, b_2) = \frac{w(b_1) - w(b_2)}{2}, \quad w(a_3, a_4, b_2) = 1 - w(b_1) + \frac{w(b_1) - w(b_2)}{2}$$

and

$$w(a_1, a_2, a_3) = w(a_1, a_2, a_4) = w(a_1, a_3, a_4) = w(a_2, a_3, a_4) = \frac{w(b_1) + w(b_2)}{6}.$$

\[3\] Note that this maximum $w(v)$ is strictly less than 1.
It is easy to check that $w$ is a perfect fractional matching in $K$.

Now we notice that

$$\sum_{f \in K} w'(f) = \sum_{\{e \in H[R^i]; e \cap V = 1\}} w_i(e) = |R^i \cap V|$$

and

$$\sum_{f \in K} w(f) = \frac{|R^i \cap [n]|}{3} = |R^i \cap (V \cup U)|.$$ 

Moreover, the neighborhood of any $u_j \in R^i \cap U$ in $H[R^i]$ is the complete 3-graph $K$. So we can partition the total weight $\sum_{f \in K}(w(f) - w'(f)) = |R^i \cap U|$ into $|R^i \cap U|$ copies of 1’s (say each is represented by a set $E_j$ of edges in $K$), and then for each $u_j \in R^i \cap U$, we assign the weight of each $f \in E_j$ to be $w_i(f \cup \{u_j\})$. One can easily check that we obtain a perfect fractional matching $w_i$ in $H[R^i]$. This completes the proof of Claim C. \hfill \Box

From Claims B and C, we see that the sets $R^i$ for $1 \leq i \leq n^{1.1}$ satisfy (a)–(d) in Lemma 2.7. Then by Lemma 2.7, there exists a spanning subgraph $H'$ of $H$ such that for each $v \in V(H)$, $d_H(v) = (1+o(1))n^{0.2}$ and $\Delta_2(H') \leq n^{0.1}$. By Theorem 2.6, $H$ contains a matching $M_b$ such that $S = V(H) \setminus V(M_b)$ contains at most $\gamma' n^2$ vertices. Since $|S \cup M_a \cup M_b| = n^2 = 3n^2 + 3m' + s$ where $0 \leq s \leq 2$, we can delete at most $s$ elements from $S$ to get a subset $S'$ such that $3|S'| \leq |V(H')| - |S'|$. By the setting at the beginning of the proof, Lemma 2.5 assures that $H^*(F)[V(M_a) \cup S']$ has a perfect matching, which together with $M_b$ forms a matching in $H^*(F)$ of size $r' + m'$. Equivalently, this says that $F$ admits a rainbow matching, finishing the proof of Lemma 6.1. \hfill \Box

7 Proof of Theorem 1.3

Let $n$ be a sufficiently large integer. Let $m$ be a positive integer with $n \geq 3m$ and let $F = \{F_1, \ldots, F_m\}$ be a family of 3-graphs on the same vertex set $[n]$ such that $|F_i| > f(n, m, 3)$ for each $i \in [m]$. Suppose to the contrary that $F$ does not admit a rainbow matching. In view of Lemma 2.2, we may assume that $F$ is stable. Then by Lemma 5.1, there exists an absolute constant $c = c(3) > 0$ such that $m \leq (1-c)n/3$. By Theorem 1.2, $m \geq n/27$. Hence,

$$n/27 \leq m \leq (1-c)n/3. \quad (7.1)$$

We now apply the following algorithm. Initially, let $F_0 = F$, $n_0 = n$ and $m_0 = m$. We repeat the following iterations. Suppose that we have defined $F_i$, which contains $m_i$ 3-graphs on the same vertex set $[n_i]$.

Step 1. Applying Corollary 2.3 to $F_i$, we obtain a family $F_{i+1}$ of 3-graphs on the vertex set $[n_i]$ that is both stable and saturated, and set $n_{i+1} = n_i$ and $m_{i+1} = m_i$.

Step 2. If for any $F \in F_{i+1}$ and any $v \in [n_{i+1}]$, $d_F(v) < \left(\begin{array}{c} n_{i+1} - 1 \\ 2 \end{array}\right)$, then set $t := i + 1$ and output $F_i$, $n_t$ and $m_t$.

Step 3. If there exist $F \in F_{i+1}$ and $v \in [n_{i+1}]$ such that $d_F(v) = \left(\begin{array}{c} n_{i+1} - 1 \\ 3 \end{array}\right)$, then set $n_{i+1} = n_{i+1} - 1$, $m_{i+1}' = m_{i+1} - 1$ and $F_{i+1}' := \{F' \in F_{i+1} \setminus \{F\}\}$. Relabel the vertices if necessary so that all the 3-graphs in $F_{i+1}'$ have the same vertex set $[n_{i+1}']$. Set $F_i := F_{i+1}'$, $n_i := n_{i+1}'$, $m_i := m_{i+1}'$ and go to Step 1.

Let $F_t$ be the resulting family of 3-graphs, which contains $m_t$ 3-graphs on the same vertex set $[n_t]$ and admits no rainbow matching. By (7.1), we see that $n_t \geq n - m > cn$ is sufficiently large. We also see from Lemma 2.9 that $|F| > f(n_t, m_t, 3)$ holds for any $F \in F_t$.

By definition, we see that $F_t$ is stable and saturated such that for any $F \in F_t$ and $v \in V_t$, $d_F(v) < \left(\begin{array}{c} n_t - 1 \\ 2 \end{array}\right)$ for any $F \in F_t$ and $v \in V_t$. On the other hand, by Lemma 2.1, it further holds that

$$d_F(v) \leq \left(\begin{array}{c} n_t - 3(m_t - 1) \\ 2 \end{array}\right) \quad \text{for any } F \in F_t \text{ and } v \in V_t.$$
Since $n_t$ is sufficiently large, using Lemma 5.1 and Theorem 1.2 again, we may assume that
\[ nt/27 \leq m_t \leq (1 - c)n_t/3. \]

Now we choose $0 < \epsilon \ll c$. Since $F_t$ satisfies the above properties, by applying Lemmas 2.4, 3.1 and 6.1, we can conclude that $F_t$ admits a rainbow matching. This is a contradiction, completing the proof of Theorem 1.3.

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