On the interplay between CPE metrics and $\sigma_2$-singular spaces.

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Abstract

We call CPE metrics the critical points of the total scalar curvature functional restricted to the space of metrics with constant scalar curvature of unitary volume. In this short note, we give a necessary and sufficient condition for a CPE metric to be Einstein in terms of $\sigma_2$-singular spaces introduced in [7]. Such a result improves our understanding about CPE metrics and Besse's conjecture (see [2]) with a new geometric point of view.

Keywords: Total scalar curvature, Critical point metric, Einstein metric, $\sigma_2$-curvature.

MSC: 53C20, 53C21, 53C24, 53C25.

1 Introduction

Scalar curvature appears in the study of Einstein metrics, since this is a type of curvature closed related with the Ricci tensor. Recall that a compact Riemannian manifold is said to be Einstein if the Ricci tensor is multiple of the metric $g$, i.e., $\text{Ric}_g = \lambda g$, where $\lambda : M \to \mathbb{R}$. In others words, $(M^n, g)$ is Einstein if its traceless tensor $\text{Ric}_g - \frac{R_g}{n} g$ is identically zero, where $\text{Ric}_g$ and $R_g$ are Ricci and scalar curvatures, respectively.

Let $(M^n, g)$ be an $n$-dimensional closed (compact without boundary) oriented manifold with $n \geq 3$, $\mathcal{M}$ be the Riemannian metric space and $S_2(M)$ be the space of symmetric 2-tensors on $M$. Fischer and Marsden [3] consider the scalar curvature map $R : \mathcal{M} \to C^\infty$ which associates to each metric $g \in \mathcal{M}$ its scalar curvature. If $\gamma_g$ is the linearization of the map $R$ and $\gamma^*_g$ is its $L^2$-formal adjoint, then they proved that

$$\gamma_g h = -\Delta_g \text{tr}_g h + \delta^2 g h - \langle \text{Ric}_g, h \rangle$$

and

$$\gamma^*_g f = \nabla^2_g f - (\Delta_g f) g - f \text{Ric}_g$$

where $\delta_g = -\text{div}_g$, $h \in S_2(M)$ and $\nabla^2_g$ is the Hessian form on $M^n$.

The Einstein-Hilbert functional $S : \mathcal{M} \to \mathbb{R}$ is defined by

$$S(g) = \int_M R_g dv_g,$$

It is well known that the solution of the Yamabe problem shows that any compact Manifold $M^n$ admits a Riemannian metric with constant scalar curvature. In particular, the set $\mathcal{C} = \{ g \in \mathcal{M} ; R_g \text{ is constant} \} \neq \emptyset$. Thus, we can consider $\mathcal{M}_1 = \{ g \in \mathcal{M} ; R_g \in \mathcal{C} \text{ and } \text{vol}_g(M) = 1 \} \neq \emptyset$.

Besse [2] conjectured that the critical points of the total scalar curvature functional $S$ restricted to $\mathcal{M}_1$ are Einstein. More precisely, the Euler-Lagrangian equation of Hilbert-Einstein action restricted to $\mathcal{M}_1$ may be written as the following critical point equation (CPE)

$$\gamma^*_g f = \nabla^2_g f - (\Delta_g f) g - f \text{Ric}_g = \text{Ric}_g. \tag{1}$$

In this setting, it is interesting to study the critical points of the restriction of the Einstein-Hilbert functional to $\mathcal{M}_1$. Following the notations developed in [1] and [6] we consider the following definition.

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Definition 1.1. A CPE metric is a triple \((M^n, g, f)\), where \((M^n, g)\) is a compact oriented Riemannian manifold of dimension \(n \geq 3\) with constant scalar curvature and volume 1 and \(f\) is a smooth potential satisfying equation (1).

The Besse’s conjecture (or CPE conjecture) can be rewritten as

Conjecture 1.2 (2). A CPE metric is always Einstein.

Note that the equation (1) is equivalent to

\[
\hat{\text{Ric}}_g = \nabla^2 g f - \left(\text{Ric}_g - \frac{R_g}{n-1}\right)f
\]

and

\[
(1+f)\hat{\text{Ric}}_g = \nabla^2 g f + \frac{R_g f}{n(n-1)} g
\]

Observe that if \(f\) is a constant function and satisfies the equation (3) then \(f = 0\) and this implies that \((M^n, g)\) is Einstein. Moreover, if \((M^n, g, f)\) is a CPE metric with \(f\) non-constant function, then the set

\[
B = \{x \in M^n / f(x) = -1\}
\]

has zero \(n\)-dimensional measure (see [4] and [6]). Thus, to prove that CPE metric is Einstein is equivalent to show that \((g, f)\) satisfies the equation

\[
\nabla^2 g f + \frac{R_g f}{n(n-1)} g = 0
\]

where \(f\) is not a constant function. This implies that

\[
\Delta_g f = -\frac{R_g}{n-1} f.
\]

Now, multiplying the equation (5) by \(f\) and integrating over \(M\), we obtain

\[
\frac{R_g}{n-1} \int_M f^2 dv_g = -\int_M f \Delta_g f dv_g = \int_M |\nabla_g f|^2 dv_g > 0
\]

since \(f\) is not a constant function, we get that \(R_g > 0\).

If \(f\) satisfies the equation (4), then \((M^n, g)\) is isometric to \(S^n(r)\) where \(r = \left(\frac{R_g}{n(n-1)}\right)^{1/2}\) (see [8]). In this way, the Conjecture [1] can be rewritten changing Einstein property by the manifold being the canonical sphere.

Hwang, [4], has proved the Besse’s conjecture when \(n = 3\) and under hypothesis that \(\ker(\gamma^*_g) \neq \{0\}\). Barros and Ribeiro Jr., [1], proved that the Besse’s conjecture is true for 4-dimensional half locally conformally flat manifolds. More recently, Neto (see [6]) proved a necessary and sufficient condition for a CPE metric to be Einstein in terms of the potential function.

In this note, we give a necessary and sufficient condition for a CPE metric to be Einstein for \(n \geq 3\), improving the understanding about CPE metrics and Besse’s conjecture with a different geometric point of view. More precisely, we prove the following result.

Theorem 1.3. Let \((M^n, g, f)\) be an \(n\)-dimensional CPE metric, \(n \geq 3\). \((M^n, g)\) is Einstein if and only if \(\ker \Lambda^*_g \neq \{0\}\), where \(\Lambda_g : S^2(M) \to C^\infty(M)\) is the linearization of the \(\sigma_2\)-curvature and \(\Lambda^*_g\) is the \(L^2\)-formal adjoint of the operator \(\Lambda_g\).

An immediate consequence is the following

Corollary 1.1. Let \((M^n, g)\) be a compact, oriented \(n\)-dimensional manifold, with \(n \geq 3\). Let \((g, f)\) be a non-trivial solution of (1). If \(\ker \Lambda_g^* \neq \{0\}\), then \((M^n, g)\) is isometric to the standard sphere \(S^n\).

2 Background and proofs

We start this section recalling some important results about \(\sigma_2\)-curvature. Then we prove the main result.
Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold, \(n \geq 3\). The \(\sigma_2\)-curvature, which we will denote by \(\sigma_2(g)\), is as a nonlinear map \(\sigma_2 : M \to C^\infty(M)\), defined as the second elementary symmetric function of the eigenvalue of the Schouten tensor \(\Lambda_g = \text{Ric}_g - \frac{R_g}{2(n-1)}g\). In this case, we obtain that

\[
\sigma_2(g) = -\frac{1}{2}|\text{Ric}_g|^2 + \frac{n}{8(n - 1)}R_g^2.
\]

Motivated by works of Fischer and Marsden (3) and Lin and Yuan (5), in [7] was proved that the linearization of the \(\sigma_2\)-curvature at the metric \(g\),

\[
\Lambda_g : S_2(M) \to C^\infty(M),
\]

is given by

\[
\Lambda_g(h) = \frac{1}{2} \left\{ (\text{Ric}_g, \Delta_g h + \nabla^2 \text{tr}_g h + 2\delta^*_g \delta_g h + 2\tilde{R}(h)) - \frac{n}{4(n - 1)}R_g (\Delta_g \text{tr}_g h - \delta^*_g h + \langle \text{Ric}, h \rangle) \right\},
\]

where \(\delta^*_g\) is the \(L^2\)-formal adjoint of \(\delta_g\) and \(\tilde{R}(h)_{ij} = g^{kl}g^{st}R_{kijh}.\)

Thus, its \(L^2\)-formal adjoint, \(\Lambda^*_g : C^\infty(M) \to S_2(M)\), is

\[
\Lambda^*_g(f) = \frac{1}{2} \Delta_g (f \text{Ric}_g) + \frac{1}{2} \delta^*_g (f \text{Ric}_g) - \frac{n}{4(n - 1)}(\Delta_g (f R_g) g - \nabla^2 (f R_g) + f R_g \text{Ric}_g). \tag{7}
\]

This implies that

\[
\text{tr}_g \Lambda^*_g(f) = \frac{2 - n}{4} R_g \Delta_g f + \frac{n - 2}{2} \langle \nabla^2 f, \text{Ric}_g \rangle - 2\sigma_2(g)f. \tag{8}
\]

Note that,

\[
\Lambda^*_g(1) = \frac{1}{2} \Delta_g \text{Ric}_g - \frac{1}{4(n - 1)}(\Delta_g R_g) g + \frac{2 - n}{4(n - 1)} \nabla^2 R_g + \tilde{R}(\text{Ric}_g) - \frac{n}{4(n - 1)} R_g \text{Ric}_g \tag{9}
\]

Then, by (7) and (8) we obtain

\[
\text{tr}_g \Lambda^*_g(1) = -2\sigma_2(g) \tag{10}
\]

and

\[
\text{div}_g \Lambda^*_g(1) = -\frac{1}{2} \sigma_2(g). \tag{11}
\]

The relations (9) and (10) are similar to the relations between the Ricci tensor and the scalar curvature, namely \(R_g = \text{tr}_g \text{Ric}_g\) and \(\text{div}_g \text{Ric}_g = \frac{1}{2} dR_g\).

In [7] was introduced the notion of \(\sigma_2\)-singular space, which has the \(L^2\)-formal adjoint of the linearization of the \(\sigma_2\)-curvature map with nontrivial kernel, and under certain hypotheses it was proved rigidity and others results. More precisely,

**Definition 2.1.** A complete Riemannian manifold \((M, g)\) is \(\sigma_2\)-singular if

\[
\ker \Lambda^*_g \neq \{0\},
\]

where \(\Lambda^*_g : C^\infty(M) \to S_2(M)\) is the \(L^2\)-formal adjoint of \(\Lambda_g\). We will call the triple \((M, g, f)\) as a \(\sigma_2\)-singular space if \(f\) is a nontrivial function in \(\ker \Lambda^*_g\).

**Theorem 2.2** [7]. Let \((M^n, g, f)\) be a closed \(\sigma_2\)-singular Einstein manifold with positive \(\sigma_2\)-curvature. Then \((M^n, g)\) is isometric to the round sphere with radius \(r = \left(\frac{n(n-1)}{2R_g}\right)^{\frac{1}{2}}\) and \(f\) is an eigenfunction of the Laplacian associated to the first eigenvalue \(\frac{R_g}{n-1}\) on \(S^n(r)\). Hence \(\dim \ker \Lambda^*_g = n + 1\) and \(\int_M f dv_g = 0\).

The next Lemma is crucial for our result.
Lemma 2.3. Let \((M^n, g, f)\) be an \(n\)-dimensional CPE metric, then
\[
\text{tr} \Lambda_g^*(f) = \left(\frac{n-2+nf}{2}\right)|\text{Ric}|^2.
\]

Proof. Since \((M^n, g, f)\) is an \(n\)-dimensional CPE metric, then \(f\) satisfies the equation (11)
\[
\nabla_g^2 f = \text{Ric}_g - \left(\text{Ric}_g - \frac{R_g}{n-1}\right) f.
\]
This, by equations (5), (6) and (11), we get
\[
\text{tr}_g \Lambda_g^*(f) = \frac{2-n}{4} R_g\left(\frac{-R_g}{n-1}\right) f + \frac{n-2}{2} \left< \text{Ric}_g - \left(\text{Ric}_g - \frac{R_g}{n-1}\right) f, \text{Ric}_g \right>
\]
\[
+ \left(\text{Ric}_g^2 - \frac{n}{4(n-1)} R_g^2\right) f
\]
\[
= \frac{n-2}{2} |\text{Ric}_g|^2 + \frac{n}{2} \left(\text{Ric}_g^2 - \frac{R_g^2}{n}\right) f
\]
\[
= \left(\frac{n-2+nf}{2}\right)|\text{Ric}_g|^2.
\]
This prove the result. \(\Box\)

Proof. (Theorem 1.3) First, we assume that \((M^n, g, f)\) is a CPE metric. Now, we suppose that \((M^n, g)\) is Einstein, then by Lemma 2 in [7] we obtain that
\[
\Lambda_g^*(f) = \frac{R_g(n-2)^2}{4n(n-1)} \left< \nabla^2 f - (\Delta_g f) g - \frac{R_g}{n} f g \right>.
\]
So, by equation (5), we get
\[
\Lambda_g^*(f) = \frac{R_g(n-2)^2}{4n(n-1)} \left< \nabla^2 f + \frac{R_g}{n} f g \right>.
\]
By hypothesis \((M^n, g)\) is Einstein, then using this in the equation (11) we obtain that the equation (11) is satisfied. Thus, by expression in (12) we conclude that \(f \in \ker \Lambda_g^*\).

Conversely, we assume that \((M^n, g, f)\) is a \(\sigma_2\)-singular space, i.e, \(\Lambda_g^*(f) = 0\), in particular \(\text{tr} \Lambda_g^*(f) = 0\). Since \(f\) is not a constant function and \(n \geq 3\), Lemma 2.3 implies that \(|\text{Ric}_g|^2 = 0\), thus \((M^n, g)\) is Einstein. \(\Box\)

Proof. (Corollary 1.1) Let \((M^n, g, f)\) be an \(n\)-dimensional CPE metric. If \((M^n, g)\) is a \(\sigma_2\)-singular space, then by Theorem 1.3 we obtain that \((M^n, g)\) is Einstein, and in this case \(\sigma_2 = \frac{(n-2)^2-R_g^2}{8n(n-1)} > 0\). Thus, by Theorem 2.2, \((M^n, g)\) is isometric to the round sphere with radius \(r = \left(\frac{n(n-1)}{R_g}\right)^{\frac{1}{2}}\) and \(f\) is an eigenfunction of the Laplacian associated to the first eigenvalue \(\frac{R_g}{n-1}\) on \(S^n(r)\) with \(\text{dim ker } \Lambda_g^* = n+1\) and \(\int_M f dv_g = 0\). \(\Box\)

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