ON THE ENDMORPHISM MONOID OF A PROFINITE SEMIGROUP

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Abstract. Necessary and sufficient conditions are given for the endomorphism monoid of a profinite semigroup to be profinite. A similar result is established for the automorphism group.

1. Introduction

A classical result in profinite group theory says that if $G$ is a profinite group with a fundamental system of neighborhoods of the identity consisting of open characteristic subgroups, then the group Aut($G$) of continuous automorphisms of $G$ is profinite with respect to the compact-open topology [5]. This applies in particular to finitely generated profinite groups. Hunter proved that the monoid of continuous endomorphisms End($S$) of a finitely generated profinite semigroup $S$ is profinite in the compact-open topology [3]. This result was later rediscovered by Almeida [2], who was unaware of Hunter’s result. Almeida was the first to use to good effect that End($S$) is profinite. In this note I give necessary and sufficient conditions for Aut($S$) and End($S$) to be profinite for a profinite semigroup $S$. This came out of trying to find an easier proof than Almeida’s for the finitely generated case. This led me unawares to exactly Hunter’s proof, which I afterwards discovered via a Google search. Like Hunter [3] and Ribes and Zalesskii [5], I give an explicit description of End($S$) and Aut($S$) as inverse limits in the case they are profinite. I also deduce the Hopfian property for $S$ in this case. Recall that a topological semigroup $S$ is Hopfian if each surjective continuous endomorphism of $S$ is an automorphism.

2. The main result

My approach, like that of Hunter [3], but unlike that of Almeida [2] and Ribes and Zalesskii [5], relies on the uniform structure on a profinite semigroup and Ascoli’s theorem. Recall that a congruence $\rho$ on a profinite semigroup $S$ is called open if it is an open subset of $S \times S$. It is easy to see that open congruences are precisely the kernels of continuous surjections from $S$ to finite semigroups [4, Chapter 3]. A congruence $\rho$ on $S$ is called fully invariant if, for all continuous endomorphisms $f: S \to S$, one has $(x, y) \in \rho$

Date: March 22, 2010.
The author was supported in part by NSERC.
implies \((f(x), f(y)) \in \rho\). Equivalently, \(\rho\) is fully invariant if and only if \(\rho \subseteq (f \times f)^{-1}(\rho)\) for all \(f \in \text{End}(S)\). In group theory, it is common to call a subgroup invariant under all automorphisms ‘characteristic.’ As I do not know of any terminology in vogue for the corresponding notion for congruences, it seems reasonable to define a congruence \(\rho\) on \(S\) to be characteristic if \((x, y) \in \rho\) implies \((f(x), f(y)) \in \rho\) for all continuous automorphisms \(f\) of \(S\). Again, this amounts to \(\rho \subseteq (f \times f)^{-1}(\rho)\) for all \(f \in \text{Aut}(S)\). Clearly, any fully invariant congruence is characteristic.

Let \((X, U)\) and \((Y, V)\) be uniform spaces. Recall that a family \(F\) of functions from \(X\) to \(Y\) is said to be uniformly equicontinuous if, for all entourages \(R \in V\), one has \(\bigcap_{f \in F} (f \times f)^{-1}(R) \in U\). Of course, a uniformly equicontinuous family consists of uniformly continuous functions. It is clearly enough to have this condition satisfied for all \(R\) running over a fundamental system of entourages for the uniformity \(V\).

Every compact Hausdorff space \(X\) has a unique uniformity compatible with its topology, namely the collection of all neighborhoods (not necessarily open) of the diagonal in \(X \times X\). The following theorem is a special case of the Ascoli theorem for uniform spaces.

**Theorem 1** (Ascoli). Let \(X, Y\) be compact Hausdorff spaces equipped with their unique uniform structures and let \(C(X, Y)\) be the space of continuous map from \(X\) to \(Y\) equipped with the compact-open topology. Then, for a family \(\mathcal{F} \subseteq C(X, Y)\), the following are equivalent:

1. \(\mathcal{F}\) is compact in the compact-open topology;
2. \(\mathcal{F}\) is closed and (uniformly) equicontinuous.

In the case of a profinite semigroup \(S\) the uniform structure is given by taking the open congruences as a fundamental system of entourages. The multiplication on \(S\) is uniformly continuous and all continuous endomorphisms of \(S\) are uniformly continuous. The following result gives a sufficient condition for a profinite semigroup to have a fundamental system of entourages consisting of open fully invariant congruences. The index of an open congruence \(\rho\) on a profinite semigroup \(S\) is the cardinality of \(S/\rho\).

**Proposition 2.** Let \(S\) be a profinite semigroup admitting only finitely many open congruences of index \(n\) for each \(n \geq 1\). Then \(S\) has a fundamental system of open fully invariant congruences. This applies in particular if \(S\) is finitely generated.

**Proof.** Let \(\mathcal{F}_n\) be the set of open congruences on \(S\) of index at most \(n\) and let \(\rho_n = \bigcap \mathcal{F}_n\); it is open because \(\mathcal{F}_n\) is finite. Clearly, the family \(\{\rho_n \mid n \geq 1\}\) is a fundamental system of entourages for the uniformity. I claim \(\rho_n\) is fully invariant. Indeed, let \(f \in \text{End}(S)\) and \(\sigma \in \mathcal{F}_n\). Then if \(p: S \to S/\sigma\) is the quotient map, one has \((f \times f)^{-1}(\sigma) = \ker pf\) and hence is of index at most \(n\). Thus

\[
(f \times f)^{-1}(\rho_n) = (f \times f)^{-1}\left(\bigcap_{\sigma \in \mathcal{F}_n} \sigma\right) = \bigcap_{\sigma \in \mathcal{F}_n} (f \times f)^{-1}(\sigma) = \bigcap_{\sigma \in \mathcal{F}_n} \mathcal{F}_n = \rho_n
\]
as required.

The final statement follows since if $X$ is a finite generating set for $S$, then any congruence of index $n$ on $S$ is determined by its restriction to the finite set $X \times X$.

\[ \square \]

Remark 3. There are non-finitely generated profinite groups that satisfy the hypothesis of Proposition 2. For example, one can take the direct product of all finite simple groups (one copy per isomorphism class); see [5, Exercise 4.4.5].

It is well known that, for any locally compact Hausdorff space $X$, the compact-open topology turns $C(X,X)$ into a topological monoid with respect to the operation of composition. The main result of this note is:

**Theorem 4.** Let $S$ be a profinite semigroup. Then $\text{End}(S)$ (respectively, $\text{Aut}(S)$) is compact in the compact-open topology if and only if $S$ admits a fundamental system of open fully invariant (respectively, characteristic) congruences. Moreover, if $\text{End}(S)$ (respectively, $\text{Aut}(S)$) is compact, then it is profinite and the compact-open topology coincides with the topology of pointwise convergence.

**Proof.** I just handle the case of $\text{End}(S)$ as the corresponding result for $\text{Aut}(S)$ is obtained by simply replacing the words ‘fully invariant’ by ‘characteristic’ and ‘endomorphism’ by ‘automorphism’.

First observe that $\text{End}(S)$ is closed in $C(S,S)$. Indeed, suppose that $f: S \to S$ is a continuous map that is not a homomorphism. Then there are elements $s,t \in S$ such that $f(st) \neq f(s)f(t)$. Choose disjoint open neighborhoods $U,V$ of $f(st)$ and $f(s)f(t)$ respectively. By continuity of multiplication one can find open neighborhoods $W,W'$ of $f(s)$ and $f(t)$ so that $W \cdot W' \subseteq V$. Then let $N$ be the set of all continuous functions $g: S \to S$ such that $g(st) \subseteq U$, $g(s) \subseteq W$ and $g(t) \subseteq W'$. Then $f \in N$ and $N$ is open in the compact-open topology. Clearly, if $g \in N$, then $g(g)g(t) \in W \cdot W' \subseteq V$ and $g(st) \in U$, whence $g(st) \neq g(s)g(t)$. Thus $\text{End}(S)$ is closed.

Assume that $\text{End}(S)$ is compact. By Ascoli’s theorem, it is uniformly equicontinuous. Let $\rho$ be an open congruence on $S$. Then uniform equicontinuity implies that

$$\sigma = \bigcap_{f \in \text{End}(S)} (f \times f)^{-1}(\rho)$$

is an entourage of the uniformity on $S$. Evidently, $\sigma$ is a congruence. It must contain an open congruence by definition of the uniformity on $S$ and so $\sigma$ is an open congruence (the open congruences being a filter in the lattice of congruences on $S$). Since the identity belongs to $\text{End}(S)$, trivially $\sigma \subseteq \rho$. It remains to observe that $\sigma$ is fully invariant. Indeed, if $g \in \text{End}(S)$, then

$$(g \times g)^{-1}(\sigma) = \bigcap_{f \in \text{End}(S)} (fg \times fg)^{-1}(\rho) \supseteq \bigcap_{h \in \text{End}(S)} (h \times h)^{-1}(\rho) = \sigma$$
establishing that $\sigma$ is fully invariant. Thus $S$ has a fundamental system of open fully invariant congruences.

Conversely, suppose that $S$ has a fundamental system of open fully invariant congruences. Uniform equicontinuity follows because if $\rho$ is an open fully invariant congruence, then for any $f \in \text{End}(S)$, one has $(f \times f)^{-1}(\rho) \supseteq \rho$ and hence $\bigcap_{f \in \text{End}(S)} (f \times f)^{-1}(\rho) \supseteq \rho$. Since the set of entourages is a filter, it follows that $\bigcap_{f \in \text{End}(S)} (f \times f)^{-1}(\rho)$ is an entourage. Because the open fully invariant congruences form a fundamental system of entourages for the uniformity on $S$, this shows that $\text{End}(S)$ is uniformly equicontinuous.

Compactness of $\text{End}(S)$ is now direct from Ascoli’s theorem. Let us equip $S^S$ with the topology of pointwise convergence. Since the compact-open topology is finer than the topology of pointwise convergence, the natural inclusion $i : \text{End}(S) \to S^S$ is continuous. As $\text{End}(S)$ and $S^S$ are compact Hausdorff, it follows that $i$ is a topological embedding and hence the compact-open topology on $\text{End}(S)$ coincides with the topology of pointwise convergence. Also $\text{End}(S)$ is totally disconnected being a subspace of $S^S$. Thus $\text{End}(S)$ is profinite.

In light of Proposition 2, Hunter’s result for finitely generated profinite semigroups (and the corresponding well-known result for automorphism groups of finitely generated profinite groups) is immediate.

**Corollary 5.** If $S$ is a finitely generated profinite semigroup, then $\text{End}(S)$ is a profinite monoid and $\text{Aut}(S)$ is a profinite group in the compact-open topology, which coincides with the topology of pointwise convergence.

Theorem 4 also implies the converse of [5, Proposition 4.4.3]: a profinite group $G$ has profinite automorphism group if and only if it has a fundamental system of neighborhoods of the identity consisting of open characteristic subgroups.

**Remark 6.** If $S$ is a profinite semigroup generated by a finite set $X$, then we have the composition of continuous maps $\text{End}(S) \to S^S \to S^X$ where the last map is induced by restriction. Moreover, this composition is injective. Since $\text{End}(S)$ is compact, it follows that $\text{End}(S)$ is homeomorphic to the closed space of all maps $X \to S$ that extend to an endomorphism of $S$ equipped with the topology of pointwise convergence. In the case $S$ is a relatively free profinite semigroup on $X$, we in fact have $\text{End}(S)$ is homeomorphic to $S^X$. Under this assumption, if $T$ is the abstract subsemigroup generated by $X$ (which is relatively free in some variety of semigroups), then it easily follows that $T^X$ is dense in $S^X$ and so $\text{End}(T)$ is dense in $\text{End}(S)$.

A corollary is the well-known fact that finitely generated profinite semigroups are Hopfian. In fact, there is the following stronger result.

**Corollary 7.** Let $S$ be a profinite semigroup admitting a fundamental system of open fully invariant congruences, e.g., if $S$ is finitely generated. Then $S$ is Hopfian.
Proof. Suppose that $f : S \to S$ is a surjective continuous endomorphism that is not an automorphism and let $f(x) = f(y)$ with $x \neq y \in S$. Then there is an open fully invariant congruence $\rho$ so that $(x, y) \notin \rho$. Since $\rho$ is fully invariant, there is an induced endomorphism $f' : S/\rho \to S/\rho$, which evidently is surjective. Thus $f'$ is an automorphism by finiteness. But if $[x], [y]$ are the classes of $x, y$ respectively, then $f'([x]) = f'([y])$ but $[x] \neq [y]$. This contradiction shows that $S$ is Hopfian.

Remark 8. In fact a more general result is true. Let $X$ be a compact Hausdorff space and let $M$ be a compact monoid of continuous maps on $X$ with respect to the compact-open topology. Then every surjective element of $M$ is invertible cf. [1]. The proof goes like this. First one shows that the surjective elements of $M$ form a closed subsemigroup $S$ (its complement is the union over all points $x \in X$ of the open sets $\mathcal{N}(X, X \setminus \{x\})$ of maps $f$ with $f(X) \subseteq X \setminus \{x\}$). Clearly, the identity is the only idempotent of $S$. But a compact Hausdorff monoid with a unique idempotent is a compact group so every element of $S$ is invertible. Consequently, any compact Hausdorff semigroup whose endomorphism monoid is compact must be Hopfian.

Not all profinite semigroups have a fundamental system of open fully invariant congruences. For instance, if $S$ is the Cantor set $\{a, b\}^\omega$ equipped with the left zero multiplication, then $S$ is a profinite semigroup and every continuous map on $S$ is an endomorphism. In particular, the shift map $\sigma$ that erases the first letter of an infinite word is a surjective continuous endomorphism, which is not an automorphism. Thus $S$ is not Hopfian and so $S$ does not have a fundamental system of open fully invariant congruences by Corollary [7]. As another example, let $F$ be a free profinite group on a countable set of generators $X = \{x_1, x_2, \ldots\}$ converging to 1 [5]. Let $\sigma : F \to F$ be the continuous endomorphism induced by the shift $x_1 \mapsto 1$ and $x_i \mapsto x_{i-1}$ for $i \geq 2$. Then $\sigma$ is surjective but not injective and so $\text{End}(F)$ is not profinite.

As is the case for automorphism groups of profinite groups [5, Proposition 4.4.3], $\text{End}(S)$ can be explicitly realized as a projective limit of finite monoids given a fundamental system of open fully invariant congruences on $S$. For finitely generated profinite semigroups, this was observed by Hunter [3]. It was pointed out to me by Luis Ribes that the realizations as a projective limit in the above sources, and in a previous version of this note, are slightly wrong. The statement and the proof in the next theorem are based on a modification suggested by him that appears in the second edition of [5].

Theorem 9. Let $S$ be a profinite semigroup and suppose that $\mathcal{F}$ is a fundamental system of entourages for $S$ consisting of open fully invariant congruences. If $\rho \in \mathcal{F}$, then there is a natural continuous projection $r_\rho : \text{End}(S) \to \text{End}(S/\rho)$. Let $\hat{\rho}$ be the corresponding open congruence on
End(S). Let \( \widehat{\mathcal{F}} = \{ \hat{\rho} \mid \rho \in \mathcal{F} \} \). Then
\[
\text{End}(S) \cong \lim_{\rightarrow} \text{End}(S)/\widehat{\rho}.
\]
(1)

The analogous result holds for \( \text{Aut}(S) \) if there exists a fundamental system of open characteristic congruences for \( S \).

Proof. First we must show that \( r_\rho \) is continuous so that \( \hat{\rho} \) is indeed an open congruence. Indeed, if \( f \in \text{End}(S) \), then \( r_\rho^{-1}r_\rho(f) \) consists of those endomorphisms \( g \in \text{End}(S) \) that take each block \( B \) of \( \rho \) into the block of \( \rho \) containing \( f(B) \). But since each block of \( \rho \) is compact and open, and there are only finitely many blocks, it follows that \( r_\rho^{-1}r_\rho(f) \) is an open set in the compact-open topology on \( \text{End}(S) \). Thus \( \hat{\rho} \) is an open congruence.

Since the open fully invariant congruences on \( S \) are closed under intersection, the set \( \widehat{\mathcal{F}} \) is closed under intersection and so it makes sense to form the projective limit in (1). Since the canonical homomorphism from \( \text{End}(S) \) to the inverse limit on the right hand side of (1) is surjective, to prove that it is an isomorphism it suffices to show that \( \widehat{\mathcal{F}} \) separates points. If \( f, g \) are distinct endomorphisms of \( S \), we can find \( s \in S \) so that \( f(s) \neq g(s) \). Then since \( \mathcal{F} \) is a fundamental system of entourages, there exists \( \rho \in \mathcal{F} \) such that \((f(s), g(s)) \notin \rho \). It follows that \( r_\rho(f) \neq r_\rho(g) \).

\[\square\]

Acknowledgments

I would like to thank Luis Ribes for pointing out an error in the original version of Theorem 9.

References

[1] E. Akin, J. Auslander and E. Glasner. The topological dynamics of Ellis actions. Mem. Amer. Math. Soc., 195(913), 2008.
[2] J. Almeida. Profinite semigroups and applications. In V. B. Kudryavtsev and I. G. Rosenberg, editors, Structural Theory of Automata, Semigroups and Universal Algebra, pages 1–45, New York, 2005. Springer.
[3] R. P. Hunter. Some remarks on subgroups defined by the Bohr compactification. Semigroup Forum, 26(1-2):125–137, 1983.
[4] J. Rhodes and B. Steinberg. The \( q \)-theory of finite semigroups. Springer Monographs in Mathematics. Springer, New York, 2009.
[5] L. Ribes and P. Zalesskii. Profinite groups, volume 40 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 2000.