ON THE CRITICAL MASS PATLAK-KELLER-SEGEL SYSTEM FOR MULTI-SPECIES POPULATIONS: GLOBAL EXISTENCE AND INFINITE TIME AGGREGATION

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Abstract. We obtained the global in time existence of solutions to the parabolic-elliptic Patlak-Keller-Segel system for the multi-species populations in the whole Euclidean space \( \mathbb{R}^2 \) at the borderline case of critical mass. We prove that there is a well-defined notion of criticality which separates the dichotomy between the global existence and the finite-time blow-up. Moreover, we showed that as time \( t \) approaches to infinity, all the components of the solutions concentrate in the form of a Dirac measure at a common point. Our approach utilizes the gradient flow structure in Wasserstein space in the spirit of De Giorgi's minimizing movement or the JKO-schemes. Due to the critical mass, the minimization problem in JKO-schemes may not admit a solution in general. We found a necessary and sufficient criterion for which the minimizers do exist.

1. Introduction

In this article, we study the global in time existence of solutions to the parabolic-elliptic Patlak-Keller-Segel system (henceforth abbreviated PKS-system) for the multi-species populations at the critical mass regime (to be defined in a moment). Multi-species PKS-system models the evolution of cells (a phenomenon called chemotaxis in biology) interacting via a self-produced sensitivity agent (called chemoattractant) and their natural habitat domain is the two-dimensional Euclidean space \( \mathbb{R}^2 \). Both the cells and the sensitivity agents are also subject to independent diffusive fluctuations. The \( n \)-component multi-species PKS-system governed by the following system of equations:

\[
\begin{aligned}
\partial_t \rho_i(x, t) &= \Delta_x \rho_i(x, t) - \sum_{j=1}^{n} a_{ij} \nabla_x \cdot (\rho_i(x, t) \nabla_x u_j(x, t)), \quad \text{in} \ \mathbb{R}^2 \times (0, \infty), \\
-\Delta_x u_i(x, t) &= \rho_i(x, t), \quad \text{in} \ \mathbb{R}^2 \times (0, \infty), \\
\rho_i(x, 0) &= \rho_i^0, \quad i = 1, \ldots, n,
\end{aligned}
\]

1. INTRODUCTION

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where $\rho_i(x, t)$ denotes the cell density of the $i$-th population, $u_i(x, t)$ denotes the concentration of the chemoattractant, produced by the $i$-th population and $\rho_i^0$ is the initial cell distribution of the $i$-th population. The constants $a_{ij}$ measures the sensitivity of the $i$-th population towards the chemical gradient produced by the $j$-th population. If $a_{ij} > 0$ (respectively, $a_{ij} < 0$) then $i$-th population is attracted (respectively, repelled) by the $j$-th population known as positive (respectively, negative) chemotaxis. In this article, we assume the sensitivity matrix $(a_{ij})$ is symmetric with non-negative entries $a_{ij} \geq 0$, for all $i, j$.

Since the solutions to the Poisson equation $-\Delta u = \rho$ is unique up to a harmonic function, we define concentration of the chemoattractant $u_i$ by the Newtonian potential of $\rho_i$

$$u_i(x, t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \rho_i(y, t) \, dy, \quad i = 1, \ldots, n. \quad (1.2)$$

There have been several prototype models for chemotaxis in the literature. The first of its kind has been proposed by Patlak [Pat53] in 1953 and Keller and Segel [KS70] in 1970. The original model of [KS70] consists of a coupled parabolic-parabolic equation and comprehends the single species chemotaxis $n = 1$. The parabolic-elliptic model is the quasi-equilibrium state of the chemoattractant and when the time scale of observation is a lot smaller compared to the speed at which the chemoattractant degrades. Over the last four decades PKS-system (1.1) for $n = 1$ and $a_{11} = a > 0$ has been widely studied in the literature, see [CP81, NS08, SS02, SS04, Suz05, BKLN06a, BKLN06b, BDP06, BCM08, BDEF10, BCC12, CD14, FM16] and the references therein.

One of the main reasons for so many interests in the mathematical community is that the system adores a critical mass $\beta := \int_{\mathbb{R}^2} \rho^0(x) \, dx$. In other words, the parameter $\beta$ solely determines the dichotomy between the global in time existence and the chemotactic collapse (or finite time blow up): if the initial number of bacteria is smaller than the critical threshold $\beta \leq 8\pi/a$, then there exists a global in time solution $[BDP06, BCM08, BKLN06b]$. However, if it crosses the critical threshold, i.e. $\beta > 8\pi/a$, then all the solutions blow-up in finite time. Thus completing the whole picture of the existence vs non-existence expedition. The above mentioned existence vs non-existence phenomena renders us to define the sub-critical regime i.e. $a\beta < 8\pi$, and the critical regime i.e. $a\beta = 8\pi$. Moreover, in the critical case, if the second moment of the initial data is finite, then the solutions do blow up in the form of a Dirac delta measure as time $t$ goes to infinity $[BCM08]$.

Chemotaxis for the multi-species population is not unnatural. The model (1.1) has been proposed by the second author in [Wol02] and subsequently further expanded in [Hor03, Hor11]. The existence vs non-existence phenomena is also quite expected in the multi-species systems, because of the two opposing forces of equal order are competing against each other. The smoothing effect diffusion term $\Delta \rho_i$ and the weighted cumulative drift induced by the chemical gradients $\sum_{j=1}^{n} a_{ij} \nabla u_j$, which is assisting the cells to accumulate are competing. However, the notion of sub-critical and critical mass is a lot more involved quantities (see Definition 1.2). In [KW19] we obtained global in time solutions in the sub-critical regime (see also [IT19]). Moreover, if the mass crosses the critical zone, then the chemotactic collapse is inevitable. We refer the readers to [EASV09, EASV10, CEV11, BG12, EVC13] for related works in 2-species model.

The main focus of this article is to study the existence of a global solution at the critical mass. The set of all critical mass contained in a $(n - 1)$-dimensional ellipsoidal domain, which separates the dichotomy between the global existence and non-existence of solutions. Also, quite surprisingly we found that all the components of the solutions concentrate at a common point in the form of a Dirac delta measure when the time approaches infinity.
1.1. **Mathematical analysis of PKS-system.** A solution \( \rho := (\rho_1, \ldots, \rho_n) \) to the PKS-system (1.1), at least formally, satisfies the following identities:

- Conservation of mass:
  \[
  \int_{\mathbb{R}^2} \rho(x, t) \, dx = \int_{\mathbb{R}^2} \rho^0(x) \, dx = \beta, \quad \text{for all } t > 0. \tag{1.3}
  \]

- Conservation of the center of mass:
  \[
  \sum_{i=1}^{n} \int_{\mathbb{R}^2} x \rho_i(x, t) \, dx = \sum_{i=1}^{n} \int_{\mathbb{R}^2} x \rho_i^0(x) \, dx. \tag{1.4}
  \]

- Free energy dissipation or the free energy identity:
  \[
  \mathcal{F}(\rho(\cdot, t)) + \int_0^t \mathcal{D}_\mathcal{F}(\rho(\cdot, s)) \, ds = \mathcal{F}(\rho^0), \tag{1.5}
  \]
  where the free energy \( \mathcal{F} \) is defined by
  \[
  \mathcal{F}(\rho) = \sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho_i(x) \ln \rho_i(x) \, dx + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i(x) \ln |x - y| \rho_j(y) \, dxdy, \tag{1.6}
  \]
  and the dissipation of free energy \( \mathcal{D}_\mathcal{F} \) is defined by
  \[
  \mathcal{D}_\mathcal{F}(\rho) = \sum_{i=1}^{n} \int_{\mathbb{R}^2} \left| \nabla \rho_i(x) - \frac{\sum_{j=1}^{n} a_{ij} \nabla u_j(x)}{\rho_i(x)} \right|^2 \rho_i(x) \, dx. \tag{1.7}
  \]

- If the second moment of the initial condition \( M_2(\rho^0) := \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \rho_i^0(x) \, dx \) is finite then formally
  \[
  M_2(\rho(\cdot, t)) = \frac{\Lambda_I(\beta)}{2\pi} t + M_2(\rho^0), \tag{1.8}
  \]
  where \( I = \{1, \ldots, n\} \) and \( \Lambda_I(\beta) \) is a quadratic polynomial in \( \beta \) defined by
  \[
  \Lambda_I(\beta) := \sum_{i \in J} \beta_i \left( 8\pi - \sum_{j \in J} a_{ij} \beta_j \right), \quad \text{for all } \emptyset \neq J \subset I. \tag{1.9}
  \]

As a consequence, if \( \Lambda_I(\beta) < 0 \), then a solution can not exists globally. If \( T^* \) is the maximal time of existence then necessarily \( T^* \leq -\frac{2\pi M_2(\rho^0)}{\Lambda_I(\beta)} \). On the other hand, if \( \Lambda_I(\beta) = 0 \), then the second moment is preserved throughout the time.

- Moreover, We observe formally that the system (1.1) can be written as
  \[
  \partial_t \rho_i = \nabla \cdot \left( \rho_i \nabla \frac{\delta \mathcal{F}}{\delta \rho_i} (\rho) \right), \quad i = 1, \ldots, n. \tag{1.10}
  \]
  As a consequence, the dissipation of energy (1.7) can be written as
  \[
  \mathcal{D}_\mathcal{F}(\rho(\cdot, t)) = \sum_{i=1}^{n} \int_{\mathbb{R}^2} \left| \nabla \frac{\delta \mathcal{F}(\rho)}{\delta \rho_i} (\rho_i(x, t)) \right|^2 \rho_i(x, t) \, dx.
  \]

Equation (1.10) is the formal structure of a gradient flow of the free energy \( \mathcal{F} \) in the space \( \mathcal{P}^\beta_2(\mathbb{R}^2)^n = \prod_{i=1}^{n} \mathcal{P}^\beta_2(\mathbb{R}^2) \) equipped with the 2-Wasserstein distance \( d_w \) (see section 2 for definition), where \( \mathcal{P}^\beta_2(\mathbb{R}^2) \) denotes the space of non-negative Borel measures on \( \mathbb{R}^2 \) with total mass \( \beta_i \) and finite second moment and \( \frac{\delta \mathcal{F}}{\delta \rho_i} \) denotes the first variation of
the functional $\mathcal{F}$ with respect to the variable $\rho_i$. The functional $\mathcal{F}$ on the product space $\mathcal{P}_2^\beta_1(\mathbb{R}^2) \times \cdots \times \mathcal{P}_2^\beta_n(\mathbb{R}^2)$ is defined by $\mathcal{F}(\rho)$ if $\rho \in \Gamma^\beta$, where

$$
\Gamma^\beta = \left\{ \rho = (\rho_i)_{i=1}^n \mid \rho_i \in L^1_+(\mathbb{R}^2), \int_{\mathbb{R}^2} \rho_i(x) \ln \rho_i(x) \, dx < +\infty, \int_{\mathbb{R}^2} \rho_i(x) \, dx = \beta_i, \int_{\mathbb{R}^2} \rho_i(x) \ln(1 + |x|^2) \, dx < +\infty \right\}
$$

and $+\infty$ elsewhere.

Before proceeding further let us first introduce the appropriate notion of a weak solution to the PKS-system (1.1). Throughout this article, we use the notation $\mathcal{H}(\rho) := \sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i \ln \rho_i$ to denote the entropy of the solutions and $\mathcal{H}_+(\rho) := \sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i \ln \rho_i^+$ is the positive part of the entropy.

**Definition 1.1.** For any initial data $\rho^0 \in \Gamma^\beta \cap \{ \rho \mid M_2(\rho) < \infty \}$ and $T^* > 0$ we say that a non-negative vector valued function $\rho \in (C([0,T^*); \mathcal{D}'(\mathbb{R}^2)))^n$ satisfying

$$
\mathcal{A}_T(\rho) := \sup_{t \in [0,T]} \left( \mathcal{H}(\rho(t)) + M_2(\rho(t)) \right) + \int_0^T \mathcal{D}(\rho(t)) \, dt < +\infty, \forall \, T \in (0,T^*)
$$

is a weak solution to the PKS-system (1.1) on the time interval $(0,T^*)$ associated to the initial condition $\rho^0$ if $\rho$ satisfies (1.3) and

$$
\int_0^{T^*} \int_{\mathbb{R}^2} \frac{\partial_t \xi(x,t) \rho_i(x,t)}{\rho_i(x,t)} \, dxdt + \int_{\mathbb{R}^2} \xi(x,0) \rho_i^0(x) \, dx \\
- \int_0^{T^*} \int_{\mathbb{R}^2} \rho_i(x,t) \left( \frac{\nabla_i \rho_i(x,t)}{\rho_i(x,t)} - \sum_{j=1}^n a_{ij} \nabla_x u_j(x,t) \right) \cdot \nabla_x \xi(x,t) \, dxdt = 0
$$

for all $\xi \in C^2([0,T^*] \times \mathbb{R}^2)$ and for all $i = 1, \ldots, n$. If $T^* = +\infty$ we say $\rho$ is a global weak solution of the system.

Note that thanks to the finite dissipation assumption and Cauchy-Schwartz inequality, all the terms in the weak formulation makes sense.

For the convenience let us write down our assumptions:

**Assumption 1.** The initial condition satisfies $\rho^0 \in \Gamma^\beta_2$ where

$$
\Gamma^\beta_2 := \left\{ \rho \in \Gamma^\beta \mid M_2(\rho) := \sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \rho_i \, dx < +\infty, \sum_{i=1}^n \int_{\mathbb{R}^2} x \rho_i(x) \, dx = 0 \right\},
$$

Note that the equation is translation invariant so the center of mass is conserved throughout the time.

**Assumption 2.** We also assume

- $A = (a_{ij})_{n \times n}$ symmetric and non-negative matrix satisfying $a_{ii} > 0$ for all $i \in I := \{1, \ldots, n\}$.

- The initial mass $\beta$ is critical i.e.,

$$
\Lambda_I(\beta) = 0, \, \Lambda_J(\beta) > 0, \text{ for all } \emptyset \neq J \subsetneq I.
$$

**1.2. Our approach and major difficulties.** There is an illuminating theory devoted to the gradient flows in Wasserstein-space in their book by Ambrosio, Gigli and Savaré [AGS05]. However, the functional $\mathcal{F}$ fails to satisfy the necessary convexity assumption in [AGS05] to have a complete well-posed theory. On the positive side, we can rely upon the PDE based approach of Wasserstein gradient flow. In precise, we could utilize De
Giorgi’s generalized minimizing movements [DG93], to study the PDE (1.1). Such connections first discovered by Otto [Ott98, Ott01] and subsequently, Jordan, Kinderlehrer and Otto [BCC08] implemented this idea for the class of Fokker-Plank equation and the heat equation. In the literature, this approach now referred to minimizing movement scheme or the JKO-scheme: for a time step \( \tau > 0 \), we define recursively

\[
\rho^k \in \arg \min_{\rho \in \Gamma^\beta} \left( F(\rho) + \frac{1}{2\tau} d_w^2(\rho, \rho^{k-1}) \right), \quad k \geq 1
\]

with \( \rho^0 = \rho^0 \), provided all the minimizers exist. The goal is to show that an appropriate interpolation of the minimizers converge to a solution in the sense of Definition 1.1.

The sharp conditions under which the functional involved are bounded below have been well studied in the literature. Indeed, it follows from the results of [CSW97, SW05] that the bound from below of \( F \) and, in particular, the functional \( \rho \mapsto G_{\eta}(\rho) := F(\rho) + \frac{1}{2\tau} d_w^2(\rho, \eta) \) depends on the the following relations of \( \beta \) and the interaction matrix \( A \):

\[
\begin{cases}
\Lambda_J(\beta) \geq 0, & \text{for all } \emptyset \neq J \subset \{1, \ldots, n\}, \\
\text{if for some } J, \Lambda_J(\beta) = 0, & \text{then } a_{ii} + \Lambda_{J \setminus \{i\}}(\beta) > 0, \forall i \in J
\end{cases}
\]

where \( \Lambda_J(\beta) \) is defined by (1.9). In particular, it is shown in [CSW97, SW05] that \( \Lambda_I(\beta) = 0 \) and (1.13) is necessary and sufficient condition for the bound from below of \( F \) over \( \Gamma^\beta \).

Needless to say, if \( a_{ii} > 0 \) for all \( i \in I \), then \( \Lambda_I(\beta) = 0 \) and \( \Lambda_J(\beta) \geq 0 \) for \( J \neq I \) is necessary and sufficient condition for the bound from below for \( F \). In addition, in [SW05] the authors showed that \( F \) admits a minimizer in \( \Gamma^\beta \) if and only if (1.11) is satisfied, which allows us to define the notion of critical mass.

**Definition 1.2.** Given a symmetric non-negative matrix \( A \).

- \( \beta \) is said to be sub-critical if
  \[
  \Lambda_J(\beta) > 0, \quad \text{for all } \emptyset \neq J \subset I.
  \]

- \( \beta \) is said to be critical if
  \[
  \Lambda_I(\beta) = 0, \quad \text{and } \Lambda_J(\beta) > 0, \quad \text{for all } \emptyset \neq J \subsetneq I.
  \]

The criterion for existence of minimizers in (1.12) drastically differs from that of \( F \). Indeed, for sub-critical mass \( \beta \) there always exists a minimizer in (1.12). And the global existence of solutions to (1.1) has also been dealt with in [KW19]. In the sub-critical case, the sharp condition on \( \beta \) for the bound from below of \( F \) gives us the uniform entropy bound on any minimizing sequence (the readers can consult [BCC08, KW18]). However, the arguments of the sub-critical regime do not work in the critical case, and apparently, one can not rule out the possibilities of concentration of minimizing sequences.

There are two major difficulties: first of all, it is not clear that the MM-scheme (1.12) is well defined for critical \( \beta \). In section 5 we study this delicate point and obtain a necessary and sufficient criterion for the existence of minimizers in the MM-scheme for this case. In particular, we showed that, given an initial datum \( \rho^0 \), there exists a \( \tau^* \in (0, 1) \) such that for every \( \tau \in (0, \tau^*) \), the MM-scheme is well defined.

The next hurdle is to obtain uniform estimates on the minimizers obtained in (1.12). In particular, obtaining uniform entropy estimates is the next big challenge, and we have dealt with it in section 6. First, we showed that if along a subsequence the entropy goes to +\( \infty \), then up to a further subsequence all the components concentrates at the origin in the form of Dirac delta measures. This result follows from the existence theorem established in section 5. Next, we used de la Vellé Poussin’s (see Lemma 6.2) the gain of integrability result to obtain higher moment estimates, and conclude that the second moment converges to 0. Because of conservation of the second moment, that is impossible. However, there
is a technical issue in obtaining higher entropy bound using de la Vellée Poussin lemma. To stress on this fact let us remark that even if the initial condition $\rho^0$ has finite $2k$-th moment $\sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^{2k} \rho^0_i(x) \, dx < \infty$, for some $k > 1$, it is not at all clear that $\rho^1_t$ also have finite $2k$-th moment. It is not clear to us either, one can get such finiteness results. To bypass such issue, we applied de la Vellée Poussin lemma on a suitably constructed measure (see section 6), to obtain uniform integrability of the second moments. We refer the reader to section 6 for more details.

1.3. Main results. The main results of this article are as follows:

**Theorem 1.3.** Assume $\rho^0 \in \Gamma_2^\beta$ and $\beta$ is critical. Further assume that the interaction matrix $(a_{ij})$ have strictly positive diagonal entries. Then the PKS-system (1.1) admits a global weak solution $\rho$ in the sense of Definition 1.1 with initial data $\rho^0$. Moreover, $\rho$ satisfies for every $T > 0$

(a) $\rho \in \left( L^2((0,T) \times \mathbb{R}^2) \right)^n \cap \left( L^1(0,T; W^{1,1}(\mathbb{R}^2)) \right)^n$, and Fischer information bound:

$$\sum_{i=1}^{n} \int_0^T \int_{\mathbb{R}^2} \left| \frac{\nabla \rho_i(x,t)}{\rho_i(x,t)} \right|^2 \rho_i(x,t) \, dx \, dt < +\infty.$$

(b) Free energy inequality:

$$\sum_{i=1}^{n} \int_0^T \int_{\mathbb{R}^2} \left| \frac{\nabla \rho_i(x,t)}{\rho_i(x,t)} - \sum_{j=1}^{n} a_{ij} \nabla u_j(x,t) \right|^2 \rho_i(x,t) \, dx \, dt + \mathcal{F}(\rho(T)) \leq \mathcal{F}(\rho^0).$$

A solution to (1.1) satisfying (a) and (b) of Theorem 1.3 is called free energy solution. By now it is well known that free energy solutions are of class $C^\infty$ in both space and in time away from $t=0$. Moreover, such solutions found to be unique. The proof of smoothness and uniqueness relies on the a posteriori estimates and depends on the novel ideas of Diperna and Lions renormalized solutions. We refer the interested readers to [FM16] for more details.

Our second main result deals with the asymptotic behaviour of the solutions as time $t \to \infty$.

**Theorem 1.4.** Assume $A$ and $\beta$ satisfies the assumptions of Theorem 1.3 and $\rho^0 \in \Gamma_2^\beta$. Then for any free energy solution $\rho$ to (1.1)

$$\lim_{t \to \infty} \rho(\cdot,t) = \beta \delta_0,$$

in the weak* convergence of measures.

The organizations of this article are as follows: in section 2, we recalled necessary definitions and known results, that are essential in this article. The topic of section 3 focuses on the description of the MM-scheme and their regularity results. Section 4 devoted to the Euler-Lagrange equations for the minimizers obtained through MM-scheme, and as a consequence, we derived several moment estimates. We proved, in section 5, the necessary and sufficient criterion for the existence of a minimizer for the functionals $\mathcal{G}_\eta(\rho)$ at the critical mass regime, which ensures the well-definedness of the MM-scheme. In section 6, we established the a priori estimates of the minimizers obtained by MM-scheme. Section 7 is devoted to the proof of convergence of the scheme and hence provides the existence of a global weak solution satisfying free energy inequality. Finally, in section 8, we proved the aggregation to the Dirac measure as time $t$ approaches to infinity.
2. Notations and Preliminaries

In this section we have listed the main notations used in this article and also some of the well known results about the Wasserstein distance and the free energy functional $\mathcal{F}$.

Any bold letters will be used to denote $n$-vectors or $n$-vector valued functions. For example $\rho = (\rho_1, \ldots, \rho_n) \in (L^1_+(\mathbb{R}^2))^n$, $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{R}^+)^n$ and so on. The entropy of a scalar function $\rho \in L^1_+(\mathbb{R}^2)$ will be denoted by $\mathcal{H}(\rho) := \int_{\mathbb{R}^2} \rho(x) \ln \rho(x) \, dx$. For the vector valued functions $\rho \in (L^1_+(\mathbb{R}^2))^n$ the entropy is also denoted by $\mathcal{H}(\rho)$ and is defined by

$$\mathcal{H}(\rho) = \sum_{i=1}^n \mathcal{H}(\rho_i).$$

We will use similar definitions for the second moment:

$$M_2(\rho) = \sum_{i=1}^n M_2(\rho_i) := \sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \rho_i(x) \, dx.$$

Let $\mathcal{P}(\mathbb{R}^2)$ be the space of all Borel probability measures on $\mathbb{R}^2$, $\mathcal{P}_2(\mathbb{R}^2)$ denotes the subset of $\mathcal{P}(\mathbb{R}^2)$ having finite second moments and $\mathcal{P}_{ac,2}(\mathbb{R}^2)$ denotes the subset of $\mathcal{P}_2(\mathbb{R}^2)$ which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.

Given two elements $\mu, \nu$ of $\mathcal{P}(\mathbb{R}^2)$ and a Borel map $T : \mathbb{R}^2 \to \mathbb{R}^2$, we say $T$ pushes forward $\mu$ to $\nu$, denoted by $T \# \mu = \nu$, if for every Borel measurable subset $U$ of $\mathbb{R}^2$, $\nu(U) = \mu(T^{-1}(U))$. Equivalently,

$$\int_{\mathbb{R}^2} \psi(x) \, d\nu(x) = \int_{\mathbb{R}^2} \psi(T(x)) \, d\mu(x), \text{ for every } \psi \in L^1(\mathbb{R}^2, d\nu). \quad (2.1)$$

2.1. Wasserstein distances. Our main sources for the optimal transportation problems and Wasserstein distances are the book of Villani [Vil03], the bible on the subject of gradient flows on the Wasserstein spaces written by Ambrosio, Gigli and Savaré [AGS05] and a recent book by Santambrogio [San15]. The results we are going to recall in this subsection are very classical and can be found in any of these text books.

2.1.1. $2$-Wasserstein distance. On $\mathcal{P}_2(\mathbb{R}^2)$ we can define a distance $d_w$, using the Monge-Kantorovich transportation problem with quadratic cost function $c(x, y) = |x - y|^2$. More precisely, given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^2)$ define

$$d^2_w(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y|^2 \, d\pi(x, y) \right], \quad (2.2)$$

where

$$\Pi(\mu, \nu) := \{ \pi \in \mathcal{P}(\mathbb{R}^2 \times \mathbb{R}^2) \mid (P_1) \# \pi = \mu, \ (P_2) \# \pi = \nu \},$$

is the set of transport plans and $P_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ denotes the canonical projections on the $i$-th factor.

A well known theorem of Brenier [Bre91] asserts that: if $\mu \in \mathcal{P}_{ac,2}(\mathbb{R}^2)$ then there exists a unique (up to additive constants) convex, lower semi continuous function $\varphi$ such that $\nabla \varphi \# \mu = \nu$ and the optimal transference plan $\tilde{\pi}$ on the right hand side of (2.2) is given by $\tilde{\pi} = (Id, \nabla \varphi) \# \mu$, where $Id : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity mapping (see [McC95], [Vil03, Theorem 2.12]). As a consequence, we have

$$d^2_w(\mu, \nu) = \int_{\mathbb{R}^2} |x - \nabla \varphi(x)|^2 \, d\mu(x), \quad \text{where } \nabla \varphi \# \mu = \nu. \quad (2.3)$$
If $\mu, \nu$ are two non-negative measures on $\mathbb{R}^2$ (not necessarily probability measures) satisfying the total mass compatibility condition $\mu(\mathbb{R}^2) = \nu(\mathbb{R}^2) (= \beta > 0)$, then we define the Wasserstein 2-distance between them as follows:

$$d_w(\mu, \nu) = \beta^{\frac{1}{2}}d_w \left( \frac{\mu}{\beta}, \frac{\nu}{\beta} \right). \quad (2.4)$$

We will denote by $P^\beta(\mathbb{R}^2)$ the space of non-negative Borel measures with total mass $\beta$ and $P^\beta_{ac}(\mathbb{R}^2)$ and $P^\beta_{uc,2}(\mathbb{R}^2)$ are defined analogously. We will also use the bold $\beta$ notation in $P^\beta(\mathbb{R}^2)$ to denote the product space $P^\beta(\mathbb{R}^2) \times \cdots \times P^\beta(n)(\mathbb{R}^2)$.

One advantage of defining the Wasserstein distance on $P^\beta(\mathbb{R}^2)$ by (2.4) is that, if $\mu$ is absolutely continuous with respect to the Lebesgue measure and if $\nabla \varphi$ is the gradient of a convex function pushing $\mu/\beta$ forward to $\nu/\beta$ then $\nabla \varphi \# \mu = \nu$ and

$$d^2_w(\mu, \nu) = \int_{\mathbb{R}^2} |x - \nabla \varphi(x)|^2 d\mu(x),$$

where note that $d_w(\mu, \nu)$ is defined by (2.4).

2.1.2. **Kantorovich-Rubinstein distance.** If the cost function $c(x, y)$ is the usual Euclidean distance $|x - y|$, then

$$d_{w1}(\mu, \nu) = \beta^{\frac{1}{2}}d_{w1} \left( \frac{\mu}{\beta}, \frac{\nu}{\beta} \right) = \beta^{\frac{1}{2}} \inf_{\pi \in \Pi(\frac{\mu}{\beta}, \frac{\nu}{\beta})} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |x - y| \pi(dx, dy), \quad (2.5)$$

also defines a distance on $P^\beta_{ac}(\mathbb{R}^2)$ and is called the 1-Wasserstein distance. The constant $\beta^{\frac{1}{2}}$ on the right hand side of (2.5) does not really matter for this article. We defined it in order to be consistent with our definition of 2-Wasserstein distance on $P^\beta_{uc,2}(\mathbb{R}^2)$. On unbounded domains, such as our case $\mathbb{R}^2$, $d_{w1}$ is weaker than $d_w$ in the sense that $d_{w1}(\cdot, \cdot) \leq d_w(\cdot, \cdot)$. There is however a sharp distinctive feature of the 1-Wasserstein distance, it is the pinning property: $d_{w1}(\mu, \nu)$ depends only on the difference $\mu - \nu$. This is manifested by the alternative dual formulation of $d_{w1}$:

$$d_{w1}(\mu, \nu) = \beta^{-\frac{1}{2}} \sup \left\{ \int_{\mathbb{R}^2} \phi(x)d(\mu - \nu)(x) \mid ||\phi||_{Lip} \leq 1 \right\} \quad (2.6)$$

where $||\phi||_{Lip} := \sup_{x \neq y}(|\phi(x) - \phi(y)|/|x - y|)$. Under fairly reasonable assumptions on $\mu, \nu$, such as both $\mu$ and $\nu$ are absolutely continuous with respect to the Lebesgue measure, one can impose additional assumption that $\phi \in C^1_{c}$ in the supremum (2.6).

We define the Wasserstein distance for vector valued functions $\rho, \eta \in P^\beta(\mathbb{R}^2)$ as follows:

$$d_w(\rho, \eta) := \left( \sum_{i=1}^{n} d^2_w(\rho_i, \eta_i) \right)^{\frac{1}{2}}; \quad d_{w1}(\rho, \eta) := \sum_{i=1}^{n} d_{w1}(\rho_i, \eta_i).$$

We end this subsection with a result regarding the weak* lower semi-continuity of the 2-Wasserstein distance: a sequence of measures $\mu^m \in P^\beta(\mathbb{R}^2)$ is said to converge to $\mu \in P^\beta(\mathbb{R}^2)$ in the weak* topology of measures if

$$\lim_{m \to \infty} \int_{\mathbb{R}^2} g(x) \ d\mu^m(x) = \int_{\mathbb{R}^2} g(x) \ d\mu(x)$$

for every $g \in C_b(\mathbb{R}^2)$, where $C_b(\mathbb{R}^2)$ denotes the space of all bounded continuous functions in $\mathbb{R}^2$.

**Lemma 2.1** (Weak* lower semi-continuity of $d_w$). Let $\{\mu^m\}, \{\nu^m\}$ be a sequence in $P^\beta_{ac}(\mathbb{R}^2)$ converging to $\mu, \nu \in P^\beta_{uc,2}(\mathbb{R}^2)$, respectively, in the weak* topology of measures. Further
assume that the second moments \( M_2(\mu^m), M_2(\nu^m) \) are uniformly bounded by some constant \( C < \infty \) independent of \( m \). Then
\[
d_m(\mu, \nu) \le \liminf_{m \to \infty} d_m(\mu^m, \nu^m).
\]

2.2. Properties of the free energy functional. We recall a few properties of the free energy functional \( F \) whose proof can be found in [SW05, KW18].

**Proposition 2.2.** The followings hold:

(a) \( F \) is bounded from below on \( \Gamma^\beta_2 \) if and only if \( \beta \) satisfies
\[
\Lambda_f(\beta) = 0 \quad \text{and (1.13)}.
\]

(b) For any \( n \)-numbers \( \alpha_i > 0 \), the functional
\[
F_\alpha(\rho) := F(\rho) + \sum_{i=1}^n \alpha_i M_2(\rho_i)
\]
is bounded from below on \( \Gamma^\beta_2 \) if and only if \( \beta \) satisfies (1.13).

(c) The functionals \( F \) and \( F_\alpha \) are sequentially lower semi-continuous with respect to the weak topology of \( L^1(\mathbb{R}^2) \).

(d) If \( \beta \) is sub-critical and \( \min_{i \in I} \alpha_i > 0 \), then all the sub-level sets \( \{ F_\alpha \le C \} \) are sequentially precompact with respect to the weak topology of \( L^1(\mathbb{R}^2) \).

In this sequel we will be using a slight variant of the functional \( F \) and the sharp conditions for bound from below as stated in Proposition 2.2(a). Naturally, it follows form a simple scaling argument. Since we assumed \( a_{ii} > 0 \) for all \( i \in I \), let us only state a weaker version of it, which is enough for our purpose. Let \( A = (a_{ij}) \) be as before satisfying \( a_{ii} > 0 \) for all \( i \in I \), then for any \( n \)-numbers \( b_i > 0 \) the functional
\[
\sum_{i=1}^n b_i \int_{\mathbb{R}^2} \rho_i \ln \rho_i + \sum_{i=1}^n \sum_{j=1}^n \frac{a_{ij}}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i(x) \ln |x-y| \rho_j(y) \tag{2.7}
\]
is bounded from below over \( \Gamma^\beta_2 \) if and only if \( \Lambda_f(\beta; b) = 0 \) and \( \Lambda_f(\beta; b) \ge 0 \) for all \( \emptyset \neq J \subseteq I \), where \( \Lambda_f(\beta; b) \) is defined by
\[
\Lambda_f(\beta; b) := 8\pi \sum_{i \in J} b_i \beta_i - \sum_{i \in J} \sum_{j \in J} a_{ij} \beta_i \beta_j, \quad \text{for all } \emptyset \neq J \subset I. \tag{2.8}
\]

**Remark 2.3.** The same conclusions of Proposition 2.2 hold true for the functional (2.7) as well, of course, one needs to frame the conditions in terms of \( \Lambda_f(\beta; b) \). The smallest lower bound of the functional \( F \) or (2.7), whenever it is finite, will always be denoted by \( C_{\text{LHLS}}(\beta) \).

**Proposition 2.4.**

(a) Assume \( \beta \) satisfies (1.13) and fix \( \eta \in \Gamma^\beta_2 \) and \( \tau > 0 \). Then the functional \( G_\eta : \Gamma^\beta_2 \to \mathbb{R} \) defined by
\[
G_\eta(\rho) := F(\rho) + \frac{1}{2\tau} d_m^2(\rho, \eta)
\]
is bounded from below on \( \Gamma^\beta_2 \). Moreover, \( G_\eta \) is sequentially lower semicontinuous with respect to the weak topology of \( L^1(\mathbb{R}^2) \).

(b) If \( \beta \) is sub-critical then all the sub-level sets \( \{ G_\eta \le C \} \) are sequentially precompact with respect to the weak topology of \( L^1(\mathbb{R}^2) \). In particular, the minimization problem
\[
\inf_{\rho \in \Gamma^\beta_2} G_\eta(\rho)
\]
ads a solution.
Remark 2.5. The proof of Proposition 2.4(a) (b) follows from Proposition 2.2, the weak* lower semi-continuity of $d_w$ and the inequality
\[ M_2(\rho) \leq 2d_w^2(\rho, \eta) + 2M_2(\eta). \] (2.9)

The conclusion (c) is just a restatement of (a) phrased in a different way and follows from Dunford-Pettis theorem. From the entropy and second moment bound it follows that the measures $\rho_m$ are tight and converges to $\tilde{\rho}$ in the weak* convergence of measures having a density with respect to the Lebesgue measure. This is enough to improve the weak* convergence of measures to the weak convergence in $L^1(\mathbb{R}^2)$. We refer to [KW18] for a proof in this framework.

3. Minimizing Movement Scheme

Given two elements $\rho, \eta \in P^\beta_2(\mathbb{R}^2)$, recall that the Wasserstein distance between them is defined by
\[ d_w(\rho, \eta) = \left[ \sum_{i=1}^n d_w^2(\rho_i, \eta_i) \right]^{\frac{1}{2}}, \]
where $d_w$ is defined as in (2.4). Further recall that we are interested in the case $\beta$-critical.

3.1. Minimizing Movement Scheme (MM-scheme): Given $\rho^0 \in \Gamma^\beta_2$ and a time step $\tau \in (0, 1)$ sufficiently small, we set $\rho_0^2 = \rho^0$ and define recursively
\[ \rho^k_\tau \in \arg\min_{\rho \in \Gamma^\beta_2} \left\{ \mathcal{F}(\rho) + \frac{1}{2\tau} d_w^2(\rho, \rho^{k-1}_\tau) \right\}, \quad k \in \mathbb{N}. \] (3.1)

It is not yet clear that the scheme is well defined. We will prove in section 5 (Theorem 5.2) a necessary and sufficient condition for the existence of minimizers. In particular, if the initial datum $\rho^0 = \rho^0_\tau$ satisfies
\[ \mathcal{F}(\rho^0_\tau) < \inf_{\rho \in \Gamma^\beta_2} \mathcal{F}(\rho) + \frac{1}{2\tau} M_2(\rho^0_\tau), \] (3.2)
then there exists a solution $\rho^1_\tau$ to the minimization problem $\min_{\rho \in \Gamma^\beta_2} \mathcal{F}(\rho) + \frac{1}{2\tau} d_w^2(\rho, \rho^0_\tau)$ and moreover, $\rho^1_\tau$ satisfies
\[ \mathcal{F}(\rho^1_\tau) < \inf_{\rho \in \Gamma^\beta_2} \mathcal{F}(\rho) + \frac{1}{2\tau} M_2(\rho^1_\tau). \] (3.3)

Evidently, (3.2) is satisfied if $\tau$ is small enough. For the rest of this article we will always assume $\tau \in (0, \tau^*)$, where $\tau^*$ is small enough such that the initial condition $\rho^0$ satisfies (3.2) with $\tau$ replaced by $\tau^*$.

Similarly, as a consequence of (3.3) and Theorem 5.2, $\rho^2_\tau$ exists and satisfies (3.3) with $\rho^1_\tau$ replaced by $\rho^2_\tau$. By induction, we conclude that the minimizing movement scheme (3.1) is well defined for every $\tau \in (0, \tau^*)$.

In section 7, we will prove that a suitable interpolation of these minimizers converge to a weak solution to (1.1). We have mentioned in the introduction that there are two major difficulties to obtain global existence: (a) well definedness of the MM-scheme and (b) uniform estimates on the entropy and the second moment. If we trace back carefully
the existing literature on the gradient flow approach to the PKS-system, one can easily understand that the remaining analysis leading to the global existence depends only on these two issues.

The first one of these results (Lemma 3.1 below) deals with the regularity of the mini-

mizers. These regularity results have been proved rigorously in our earlier article [KW19], Lemma 4.3, which relies on the flow interchange technique introduced by Matthes-McCann and Savaré [MMS09].

3.2. Regularity of the minimizers.

Lemma 3.1. Let \( \tau \in (0, \tau^*) \) and let \( \rho^k \) be a sequence obtained using the MM-scheme (3.1) satisfying

\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \rho^k_{r,i} |\mu^k_{r,i}| \, dx + M_2(\rho^k_r) \leq \Theta,
\]

for some constant \( \Theta > 0 \). Then \( \rho^k_{r,i} \in W_{\text{loc}}^{1,1}(\mathbb{R}^2), \frac{\nabla \rho^k_{r,i}}{\rho^k_{r,i}} \in L^2(\mathbb{R}^2, \rho^k_{r,i}) \) for all \( i \in I \).

Moreover, there exists a constant \( C(\Theta) \) such that

\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \frac{\nabla \rho^k_{r,i}(x)}{\rho^k_{r,i}(x)}^2 \rho^k_{r,i}(x) \, dx \leq \frac{2}{\tau} \left[ \mathcal{H}(\rho^{k-1}) - \mathcal{H}(\rho^k) \right] + C(\Theta).
\]

These regularity results, in particular, the \( W_{\text{loc}}^{1,1} \)-regularity is necessary in persuasion of the Euler-Lagrange equations (see section 4) satisfied by the minimizers.

4. The Euler-Lagrange equations and moment estimates

In this section we will derive several auxiliary results, assuming that a minimizer exists. Our main focus will be on the situation of critical \( \beta \). However, most of these results hold true irrespective of \( \beta \), as long as a minimizer exists. We begin by recalling the Euler-Lagrange equation satisfied by a minimizer.

Lemma 4.1. Let \( \eta \in \Gamma^\beta \) and \( \tau > 0 \) be given and assume that \( \varrho \) be a minimizer of \( \inf_{\rho \in \Gamma^\beta} G_\eta(\rho) \). Let \( \nabla \varphi_i \) denotes the map transporting \( \eta_i \) to \( \varrho_i \) then

(a) The Euler-Lagrange equation:

\[
\frac{1}{\tau} \int_{\mathbb{R}^2} (\nabla \varphi_i(x) - x) \cdot \nabla(\nabla \varphi_i(x)) \eta_i(x) \, dx = - \int_{\mathbb{R}^2} \zeta(x) \cdot \nabla \varrho_i(x) \, dx + \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} \zeta(x) \cdot \nabla u_j(x) \varrho_i(x) \, dx,
\]

holds for all \( i = 1, \ldots, n \) and any \( \zeta \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) \), where

\[
u_j(x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \varrho_i(y) \, dy.
\]

(b) Free energy production term: the following identity holds

\[
\frac{1}{\tau^2} d_{\varrho_i}(\varrho_i, \eta_i) = \int_{\mathbb{R}^2} \frac{|\nabla \varrho_i(x)|}{\varrho_i(x)} - \sum_{j=1}^{n} a_{ij} \nabla u_j(x) \varrho_i(x) \, dx.
\]

(c) The following approximate weak solution is satisfied

\[
\left| \int_{\mathbb{R}^2} \psi(x) (\varrho_i(x) - \eta_i(x)) + \tau \int_{\mathbb{R}^2} \nabla \psi(x) \cdot \nabla \varrho_i \right|
\]
\[-\tau \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} \nabla \psi(x) \cdot \nabla u_j(x) \varphi_i(x) dx = O \left( \|D^2 \psi\|_{L^\infty} \right) d^2_{\omega}(\varphi_i, \eta_i), \]  

for all \( \psi \in C_c^\infty(\mathbb{R}^2) \).

(d) In particular, for all \( \psi \in C_c^\infty(\mathbb{R}^2) \)

\[
\frac{1}{\tau} \sum_{i=1}^{n} \int_{\mathbb{R}^2} \left( \nabla \varphi_i(x) - \nabla \varphi_i(x) \right) \eta_i(x) dx = \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Delta \psi(x) \varphi_i(x) dx
\]

\[= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\nabla \psi(x) - \nabla \psi(y)) \cdot (x-y)}{|x-y|^2} \varphi_i(x) \varphi_j(y) dx. \]

A proof of Lemma 4.1 can be found in [KW19, Lemma 5.1]. The proof is obtained by following the main ideas of the seminal work by Jordan-Kinderlehrer and Otto [JKO98] with necessary modifications. See also [BCC08, Theorem 3.4] for the Euler-Lagrange equation related to the PKS-system of single population. Note that by the regularity results of Lemma 3.1, \( \varphi_i \in W^{1,1}(\mathbb{R}^2) \) and hence all the terms in (4.1) makes sense. The conclusion (d) is a particular case of (a) and follows by plugging \( \zeta = \nabla \psi \) and \( \nabla u_i(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \varphi_i(y) dy \) in (4.1) and summing over all \( i = 1, \ldots, n \).

4.1. Consequences of the Euler Lagrange equation. We introduce the following cut off function: Let \( \Psi \) be a non-negative smooth function such that \( \tilde{\varphi} \equiv 1 \) in \( B(0,1) \) and vanishes outside \( B(0,2) \). We set \( \Psi_R(x) := \Psi(\frac{x}{R}) \) where \( R > 0 \). Then

\[|\nabla \Psi_R(x)| = O \left( \frac{1}{R} \right), \quad |D^2 \Psi_R(x)| = O \left( \frac{1}{R^2} \right) \]  

uniformly in \( x \in \mathbb{R}^2 \).

4.1.1. Center of mass preservation and second moment estimate.

Lemma 4.2. Let \( \eta \in \Gamma^g_2 \) and \( \tau > 0 \) be given and assume that \( \varrho \) be a minimizer of \( \inf_{\rho \in \Gamma^g_2} G_\eta(\rho) \). Then

\[\sum_{i=1}^{n} \int_{\mathbb{R}^2} x \varphi_i(x) dx = \sum_{i=1}^{n} \int_{\mathbb{R}^2} x \eta_i(x) dx = 0.\]

Proof. Fix \( v \in \mathbb{R}^2 \). We will approximate the linear functional \( x \cdot v \) by smooth compactly supported functions. We use \( \psi_R(x) := (x \cdot v) \Psi_R(x) \) as a test function in the Euler-Lagrange equation Lemma 4.1(d). Let us estimate one by one:

\[\int_{\mathbb{R}^2} (\nabla \varphi_i(x) - x) \cdot \nabla \psi_R(\nabla \varphi_i(x)) \eta_i(x) dx
\]

\[= \int_{\mathbb{R}^2} x \cdot \nabla \psi_R(x) \varphi_i(x) dx - \int_{\mathbb{R}^2} x \cdot \nabla \psi_R(\nabla \varphi(x)) \eta_i(x) dx
\]

\[= \int_{\mathbb{R}^2} (x \cdot v) \Psi_R(x) \varphi_i(x) dx - \int_{\mathbb{R}^2} (x \cdot v) \Psi_R(\nabla \varphi(x)) \eta_i(x) dx + O \left( \frac{1}{R} \right)
\]

\[= \int_{\mathbb{R}^2} (x \cdot v) \varphi_i(x) dx - \int_{\mathbb{R}^2} (x \cdot v) \eta_i(x) dx + o(1) \quad \text{as} \ R \to \infty. \]

On the other hand it is easy to see that

\[|\Delta \psi_R(x)| = O \left( \frac{|x|}{R} \right) + O \left( \frac{1}{R^2} \right). \]
\[
\frac{(\nabla \psi_R(x) - \nabla \psi_R(y)) \cdot (x - y)}{|x-y|^2} = \frac{1}{|x-y|^2} \left[ v \cdot (x-y)(\Psi_R(x) - \Psi_R(y)) + ((x-y) \cdot v)((x-y) \cdot \nabla \psi_R(x)) + (y \cdot v)(\nabla \psi_R(x) - \nabla \psi_R(y)) \cdot (x-y) \right]
\]

\[
= O \left( \frac{1}{R} \right),
\]

Plugging \(\psi_R\) in Lemma 4.1(d) and letting \(R \to \infty\) we obtain

\[
\left( \sum_{i=1}^{n} \int_{\mathbb{R}^2} x \varrho_i(x) \, dx - \sum_{i=1}^{n} \int_{\mathbb{R}^2} x \eta_i(x) \, dx \right) \cdot v = 0
\]

for any \(v \in \mathbb{R}^2\).

\[\square\]

**Lemma 4.3.** Let \(\eta \in \Gamma_{\beta}^\theta\) and \(\tau > 0\) be given and assume that \(\varrho\) be a minimizer of \(\inf_{\rho \in \Gamma_{\beta}^\theta} \mathcal{G}_{\eta}(\rho)\). If \(\beta\) is critical then

\[
d^2_{\alpha}(\varrho, \eta) = M_2(\eta) - M_2(\varrho).
\]

**Proof.** Here we will approximate \(|x|^2\) by smooth compactly supported functions. For that matter we choose \(\psi_R(x) = |x|^2 \Psi_R(x)\) in Lemma 4.1(d). A straight forward computation gives

\[
\Delta \psi_R(x) = 4 \Psi_R(x) + O \left( \frac{|x|}{R} + \frac{|x|^2}{R^2} \right),
\]

\[
\frac{(\nabla \psi_R(x) - \nabla \psi_R(y)) \cdot (x - y)}{|x-y|^2} = \frac{1}{|x-y|^2} \left[ 2|x-y|^2 \Psi_R(x) - 2y \cdot (x-y)(\Psi_R(x) - \Psi_R(y)) + |x|^2 - |y|^2 \right] (x-y) \cdot \nabla \psi_R(y)
\]

\[
= 2 \Psi_R(x) + O \left( \frac{|y|}{R} \right) + O \left( \frac{|x|^2}{R^2} \right) + O \left( \frac{|x| + |y|}{R} \right).
\]

As a consequence, we see that

\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \Delta \psi_R(x) \varrho_i(x) \, dx - \sum_{i=1}^{n} a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\nabla \psi_R(x) - \nabla \psi_R(y)) \cdot (x-y) \varrho_i(x) \varrho_j(y)}{|x-y|^2} \, dx \, dy
\]

\[
= 4 \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Psi_R(x) \varrho_i(x) - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{1}{2 \pi} \int_{\mathbb{R}^2} \Psi_R(x) \varrho_i(x) \varrho_j(y) + O \left( \frac{1}{R} \right)
\]

\[
= \frac{\Lambda_{I}(\beta)}{2\pi} + o(1), \text{ as } R \to \infty.
\]

On the other hand

\[
\int_{\mathbb{R}^2} (\nabla \varphi_i(x) - x) \cdot \nabla \psi_R(\nabla \varphi_i(x)) \eta_i(x) \, dx
\]

\[
= 2 \int_{\mathbb{R}^2} |x|^2 \Psi_R(x) \varrho_i(x) \, dx + \int_{\mathbb{R}^2} x \cdot \nabla \Psi_R(x) |x|^2 \varrho_i(x) \, dx
\]

\[
- 2 \int_{\mathbb{R}^2} x \cdot \nabla \varphi_i(x) \Psi_R(\nabla \varphi_i(x)) \eta_i(x) \, dx - \int_{\mathbb{R}^2} x \cdot \nabla \Psi_R(\nabla \varphi_i(x)) |\nabla \varphi_i(x)|^2 \eta_i(x) \, dx.
\]

Note that as \(\Psi_R\) vanishes outside \(B(0,2R)\), the quantity \(|y \nabla \Psi_R(y)|\) remains uniformly bounded for all \(y \in \mathbb{R}^2\) and converges point wise to 0 as \(R \to \infty\). It follows that the second and forth terms of the above expression decay to zero as \(R \to \infty\). Indeed, \(|x|^2 \varrho_i\)
is integrable by assumption so the second term vanish as $R \to \infty$ by the dominated convergence theorem. As for the forth term, we may estimate
\[
\int_{\mathbb{R}^2} x \cdot \nabla \Psi_R(\nabla \varphi_i) |\nabla \varphi_i|^2 \eta(x) dx \leq C \int_{\mathbb{R}^2} |x| |\nabla \varphi_i(x)| \eta(x) dx.
\]
\[
\leq \frac{C}{2} \left( \int_{\mathbb{R}^2} |x|^2 \eta(x) dx + \int_{\mathbb{R}^2} |y|^2 \theta_i(y) dy \right).
\]

By the assumed criticality of $\beta$ and Lemma 4.4-(d) we conclude
\[
\sum_{i=1}^n \int_{\mathbb{R}^2} |x|^2 \theta_i(x) dx = \sum_{i=1}^n \int_{\mathbb{R}^2} x \cdot \nabla \varphi_i(x) \eta_i(x) dx.
\]
Using (4.5) and the definition of the Wasserstein distance we get
\[
d_{W}^2(\varrho, \eta) = \sum_{i=1}^n \int_{\mathbb{R}^2} |x - \nabla \varphi_i(x)|^2 \eta_i(x) dx
\]
\[
= M_2(\eta) + M_2(\varrho) - 2 \sum_{i=1}^n \int_{\mathbb{R}^2} x \cdot \nabla \varphi_i(x) \eta_i(x) dx
\]
\[
= M_2(\eta) - M_2(\varrho).
\]

4.1.2. Higher moment estimates. For the next Lemma, we introduce the de la Vallée Poussin convex function. Let $\Upsilon \in C^\infty([0, \infty); [0, \infty))$ be a convex function satisfying the growth conditions:
\[
\begin{aligned}
\Upsilon(0) &= \Upsilon'(0) = 0, \quad r \mapsto \Upsilon'(r) \text{ is concave}, \\
\Upsilon(r) &\leq r \Upsilon'(r) \leq 2 \Upsilon(r) \text{ for } r > 0, \\
r \mapsto r^{-1} \Upsilon(r) \text{ is concave}, \quad \lim_{r \to \infty} \frac{\Upsilon(r)}{r} = \lim_{r \to \infty} \Upsilon'(r) = \infty.
\end{aligned}
\]
(4.6)

We like to give a special emphasize to the convex function $\Upsilon$ and all its properties. It is one of the key step in obtaining uniform bound on the entropy of the iterates obtained through minimizing movement scheme (see section 3 for the definition).

Lemma 4.4. Let $\eta \in C^{\beta} \mathbb{R}^2$ and assume that $\sum_{i=1}^n \Upsilon(|x|^2) \eta_i \in L^1(\mathbb{R}^2)$ where $\Upsilon$ is the de la Vallée Poussin convex function introduced in (4.6). Then there exists a $\tau_0 > 0$ such that for any $\tau \in (0, \tau_0)$ if the minimization problem $\inf_{\rho \in C^{\beta} \mathbb{R}^2} G_\eta(\rho)$ admits a solution and the minimizer $\rho$ satisfies $\sum_{i=1}^n \Upsilon(|x|^2) \theta_i \in L^1(\mathbb{R}^2)$ then
\[
\sum_{i=1}^n \int_{\mathbb{R}^2} \Upsilon(|x|^2) \theta_i(x) dx \leq (1 + C_\Omega \tau) \sum_{i=1}^n \int_{\mathbb{R}^2} \Upsilon(|x|^2) \eta_i(x) dx + C_\Omega \tau,
\]
where $C_\Omega$ is a constant depending only on $\Omega, \beta$ and $\tau_0$.

The proof of Lemma 4.4 is rather technically cumbersome. For the moment the validity of the lemma is taken for granted. We will give a proof in the appendix. We emphasize on the fact that the constant $C_\Omega$ does not depend on the $L^1$ norm of $\sum_{i=1}^n \Upsilon(|x|^2)(\theta_i + \eta_i)$.

5. The critical case: Existence of minimizers and blow-up behaviour

At this present section, we are going to establish a necessary and sufficient criterion for the existence of minimizers of $G_\eta(\rho)$ at the critical mass regime. We found that there are only two possibilities for a minimizing sequence. Either they converge to a minimizer, or, if they were to blow-up, they must concentrate in the form of a Dirac delta measure at a common point.
Our starting point is the following functional inequality, which turns out to be very crucial in the analysis to come.

**Lemma 5.1.** Assume $\beta$ is critical, then

$$\inf_{\rho \in \Gamma_2^\beta} G_n(\rho) \leq \inf_{\rho \in \Gamma_2^\beta} F(\rho) + \frac{1}{2\tau} M_2(\eta). \tag{5.1}$$

**Proof.** Let $\{\rho^m\} \subset \Gamma_2^\beta$ be a minimizing sequence for $\inf_{\rho \in \Gamma_2^\beta} F(\rho)$. Choose a sequence $R_m \to \infty$ and let $\tilde{\rho}^m(x) := (R_m)^2 \rho^m(R_m x)$. It follows that $M_2(\tilde{\rho}^m) \to 0$. Since $\beta$ is critical, $F(\tilde{\rho}^m) = F(\rho^m)$, so is also a minimizing sequence.

The proof follows from the following

$$\inf_{\rho \in \Gamma_2^\beta} G_n(\rho) \leq G_n(\tilde{\rho}^m) = F(\tilde{\rho}^m) + \frac{1}{2\tau} d_w^2(\tilde{\rho}^m, \eta) = \inf_{\rho \in \Gamma_2^\beta} F(\rho) + \frac{1}{2\tau} M_2(\eta) + o(1), \text{ as } m \to \infty. \quad \square$$

Next we state the main result of this section.

**Theorem 5.2.** Assume $A$ and $\beta$ satisfies Assumption 2 and $\eta \in \Gamma_2^\beta$. Then either one of the following alternative holds:

(a) equality holds in (5.1),

(b) the minimization problem $\inf_{\rho \in \Gamma_2^\beta} G_n(\rho)$ admits a solution.

In particular, if $F(\eta) < \inf_{\rho \in \Gamma_2^\beta} F(\rho) + \frac{1}{2\tau} M_2(\eta)$ then the minimization problem $\inf_{\rho \in \Gamma_2^\beta} G_n(\rho)$ admits a solution $\rho \in \Gamma_2^\beta$. Moreover, the minimizer $\rho \in \Gamma_2^\beta$ satisfies

$$F(\rho) < \inf_{\rho \in \Gamma_2^\beta} F(\rho) + \frac{1}{2\tau} M_2(\rho).$$

**Proof.** The proof of the theorem is divided into several steps.

**Step 1:** Let $\{\rho^m\}$ be a minimizing sequence for $\inf_{\rho \in \Gamma_2^\beta} G_n(\rho)$. By Proposition 2.2(b) and (2.9), the second moment of $\rho^m : M_2(\rho^m)$ remains uniformly bounded and hence the measures $\rho^m$ are tight. By Prohorov’s theorem, $\rho^m \xrightarrow{\ast} \rho^*$ in the weak * topology of measures. In the following we will prove that either all the components of $\rho^m$ concentrates in the form of a Dirac delta measure at a common point $v_0$ or, the entropy $\mathcal{H}(\rho^m)$ remains uniformly bounded. In other words, either $\rho_i^* = \beta_i \delta_{v_i}$ for every $i \in I$, or, $\mathcal{H}(\rho^m) \leq C$ for some constant independent of $m$. The concentration phenomena of $\rho^m$ will give the conclusion (a) and on the other hand part (b) of the theorem will follow from the uniform entropy bound.

**Step 2:** If at least one component does not concentrate then no component concentrate at all.

With out loss of generality we can assume $\rho_i^*$ is not a Dirac mass and $0 \in \text{supp}(\rho^*_i)$ for $0 \notin \text{supp}(\rho^*_i)$ we can work with any point within the support of $\rho^*_i$ and the same argument can be carried over. We define

$$\alpha^*_n(r) := \int_{B_r} d\rho^*_i. \tag{5.2}$$

Then $\lim_{r \to \infty} \alpha^*_n(r) = \beta_n$ and $\rho^*_i$ is not a Dirac mass is equivalent to saying $\lim_{r \to 0^+} \alpha^*_n(r) < \beta_n$. Since $\alpha^*_n$ is monotone, the points of discontinuities of $\alpha^*_n$ is at most countable. We can
choose a point \( r_1 \) small enough such that \( 0 < \alpha_n^*(r_1) < \beta_n \) and \( r_1 \) is a point of continuity of \( \alpha_n^* \).

We choose \( \delta > 0 \) small so that \( 0 < \alpha_n^*(r_1) - \delta < \alpha_n^*(r_1) + \delta < \beta_n \). Later we will make further smallness assumption on \( \delta \). By continuity of \( \alpha_n^* \) at \( r_1 \) and monotonicity, we can choose \( r_0 < r_1 < r_2 \) such that

\[
0 < \alpha_n^*(r_1) - \delta < \alpha_n^*(r_0) \leq \alpha_n^*(r_1) \leq \alpha_n^*(r_2) < \alpha_n^*(r_1) + \delta < \beta_n. \tag{5.3}
\]

Since \( \rho_n^m \overset{\text{a}}{\to} \rho_n^* \), for large enough \( m \) we have

\[
\alpha_n^*(r_1) - \delta \leq \int_{B_{r_1}} \rho_n^m(x) \, dx, \quad \beta_n - \alpha_n^*(r_1) - \delta \leq \int_{B_{r_2}} \rho_n^m(x) \, dx,
\]

\[
\int_{B_{r_2}\setminus B_{r_0}} \rho_n^m(x) \, dx \leq 2\delta.
\]

Applying HLS inequality in three regions \( B_{r_0+\epsilon}, B_{r_2-\epsilon} \), and \( B_{r_2}\setminus B_{r_0} \) and proceeding as in [BCM08, Lemma 3.1] we conclude that

\[
\kappa_n^m \int_{\mathbb{R}^2} \rho_n^m(x) \ln \rho_n^m(x) + \frac{2}{3} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \rho_n^m(x) \ln |x-y| \rho_n^m(y) \, dx dy \right) \geq C, \tag{5.4}
\]

where \( \kappa_n^m = \max \{a_1^m + a_2^m, a_2^m + a_3^m, \} \), \( \epsilon = \frac{1}{3}(r_2 - r_0) \) and

\[
a_1^m = \int_{B_{r_0+\epsilon}} \rho_n^m(x) \, dx, \quad a_2^m = \int_{B_{r_2}\setminus B_{r_0}} \rho_n^m(x) \, dx, \quad a_3^m = \int_{B_{r_2-\epsilon}} \rho_n^m(x) \, dx.
\]

By choosing \( \delta < \frac{1}{6} \min \{\alpha_n^*(r_1), \beta_n - \alpha_n^*(r_1)\} \) we see that \( \kappa_n^m < \beta_n - \delta \) for sufficiently large \( m \).

**Step 3:** Uniform boundedness on the entropy.

Since \( \rho_m \) is a minimizing sequence for sufficiently large \( m \)

\[
\sum_{i=1}^n \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i=1}^n \sum_{j=1}^n \frac{\alpha_{ij}}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x-y| \rho_i^m(y) \, dx dy \leq F(\eta) + 1. \tag{5.5}
\]

Multiplying (5.4) by \( \frac{\alpha_{ii}}{8\pi} \) and subtracting from (5.5) we get

\[
\sum_{i=1}^n b_i \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i=1}^n \sum_{j=1}^n \frac{\tilde{\alpha}_{ij}}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x-y| \rho_i^m(y) \, dx dy \leq C, \tag{5.6}
\]

where

\[
b_i = \begin{cases} 1, & \text{if } i \neq n, \\ \left(1 - \frac{\alpha_{ii}}{8\pi}\right), & \text{if } i = n, \end{cases}
\]

\[
\tilde{\alpha}_{ij} = \begin{cases} 0, & \text{if } i = j = n, \\ \alpha_{ij}, & \text{otherwise}. \end{cases}
\]

Now we claim that the mass \( \beta \) is sub-critical with respect to \( b \) and the interaction matrix \( \tilde{\alpha}_{ij} \), i.e., \( \Lambda_J(\beta; b) \geq \bar{\varepsilon} > 0 \) for all \( J \subset I \) and for some \( \bar{\varepsilon} \). Let \( J \subset I \) be any subset. If \( n \notin J \) then there is nothing to prove, because, in that case \( \Lambda_J(\beta; b) = \Lambda_J(\beta) \). If \( n \in J \) then

\[
\Lambda_J(\beta; b) = 8\pi \sum_{i \in J} b_i \beta_i - \sum_{i \in J \setminus \{n\}} \alpha_{ij} \beta_i \beta_j
\]

\[
= \Lambda_{J \setminus \{n\}}(\beta) - 2 \sum_{i \in J \setminus \{n\}} a_{in} \beta_i \beta_n + 8\pi b_n \beta_n. \tag{5.7}
\]
By our assumption $\Lambda_J(\beta) > 0$ if $J \neq I$ and equal to zero if $J = I$. In either case the condition $\Lambda_J(\beta) \geq 0$ can be rewritten as

$$\Lambda_J(n)(\beta) - 2 \sum_{i \in J \setminus \{n\}} a_{nn} \beta_i \beta_n \geq a_{nn} \beta_n^2 - 8\pi \beta_n.$$  \hspace{1cm} (5.8)

Using (5.8) in (5.7) we get

$$\Lambda_J(\beta, b) \geq a_{nn} \beta_n^2 - 8\pi \beta_n + 8\pi \left(1 - \frac{a_{nn} \kappa_m}{8\pi}\right) \beta_n = a_{nn} \beta_n(\beta_n - \kappa_n^m) \geq a_{nn} \delta \beta_n > 0$$

provided $a_{nn} > 0$ which is valid owing to our assumption. According to Proposition 2.2(d) and Remark 2.3 and the uniform second moment bound we deduce that the entropy $\mathcal{H}(\rho^m)$ is uniformly bounded.

At this stage we know that if at least one component does not concentrate then we have uniform bound on the entropy. As a consequence of Proposition 2.4(c), the existence of a minimizer follows.

Next we will deal with the case when concentration does happen. Assume $\rho_i = \beta_i \delta_{v_i}$ for some points $v_i \in \mathbb{R}^2$.

**Step 4:** $v_1 = \cdots = v_n = 0$.

Let $I_1 \subset I$ be the set of all indices such that $v_i = v_1$ for all $i \in I_1$. It is enough to show that $I_1 = I$. We set $I_2 = I \setminus I_1$. Let us take $i \in I_1$ and $j \in I_2$. We claim that for any $\epsilon > 0$ small, we can estimate

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \leq -\alpha \int_{\mathbb{R}^2} \rho_i^m \ln \rho_j^m - \alpha \int_{\mathbb{R}^2} \rho_j^m \ln \rho_i^m - C,$$  \hspace{1cm} (5.9)

for $m$ large enough and where $\max\{\alpha_i^m, \alpha_j^m\} < \epsilon$. To do so, we set $\epsilon = \min\{\frac{1}{10}|v_i - v_j|, 1\}$ and decompose the integral $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy$ into three pieces $A, B$ and $C$:

$$A := \int \int_{\{|x-y|<\epsilon\}} \cdot, \quad B := \int \int_{\{\epsilon \leq |x-y|<R\}} \cdot, \quad C := \int \int_{\{|x-y|>R\}} \cdot$$

and $R > 1$ is large. The integral $B$ is easy to estimate by the $L^1$ bound of $\rho_i^m, \rho_j^m$. The integral $C$ is estimated by the bound on $M_2(\rho^m)$.

To estimate the integral $A$ we see that the region $\{|x-y|<\epsilon\}$ is contained in the union of $\{|x-y|<\epsilon\} \cap \{x \in B_r(v_1)\}$ and $\{|x-y|<\epsilon\} \cap \{y \in B_r(v_j)\}$. Indeed, if $x \in B_r(v_1)$ and $y \in B_r(v_j)$ then $|x-y| \geq |v_i-v_j| - |x-v_i| - |x-v_j| \geq 8\epsilon$. We only estimate the region $\{|x-y|<\epsilon\} \cap \{y \in B_r(v_j)\}$, the other being verbatim copy with the role of $i$ and $j$ interchanged. Choose $\alpha < 1$, then using the inequality $ab \leq b \ln b + \epsilon^\alpha$, we deduce

$$\left| \int \int_{\{|x-y|<\epsilon\} \cap \{y \in B_r(v_j)\}} \rho_i^m(x) \ln |x-y| \rho_j^m(y) \, dx \, dy \right|$$

$$= \int \int_{\{|x-y|<\epsilon\} \cap \{y \in B_r(v_j)\}} \left(\alpha^{-1} \rho_i^m(x)\right) \left(\alpha \ln \frac{1}{|x-y|}\right) \rho_j^m(y) \, dx \, dy$$

$$\leq \int \int_{\{|x-y|<\epsilon\} \cap \{y \in B_r(v_j)\}} \rho_j^m(y) \left[\alpha^{-1} \rho_i^m(x) \ln \rho_i^m(x) \frac{|x-v_j|}{|x-y|^\alpha} \right] \, dx \, dy$$

$$\leq \alpha^{-1} \left(\int_{B_r(v_j)} \rho_j^m(x) \, dx\right) \int_{\mathbb{R}^2} \rho_i^m(x) \ln \rho_i^m(x) \, dx + C$$
\[ \leq \alpha^{-1} \left( \int_{B_r(v)} \rho_j^m(x) \, dx \right) \int_{\mathbb{R}^2} \rho_i^m(x) \ln \rho_i^m(x) + C, \]

where in the last line we have used the inequality [BDP06, Lemma 2.6]:
\[ \int_{\mathbb{R}^2} \rho(x) \ln \rho(x) \, dx \leq \int_{\mathbb{R}^2} \rho(x) \ln \rho(x) + M_2(\rho) + 2 \ln(2\pi) \int_{\mathbb{R}^2} \rho(x) \, dx + \frac{2}{\epsilon}. \]

Taking into account the integrals A, B and C we conclude that (5.9) holds with \( \alpha_l^m = 1/\epsilon \). Since \( \rho_i^m \to \beta \delta_{v_0} \), for large enough \( m \) we can make \( \alpha_l^m < \epsilon \).

We can rewrite the upper bound condition \( \mathcal{F}(\rho^m) \leq C \) in the following way:
\[ \sum_{i \in I_1} \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i \in I_1, j \in I_1} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \]
\[ + \sum_{i \in I_2} \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i \in I_2, j \in I_2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \]
\[ + \sum_{i \in I_1, j \in I_2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \leq C. \] (5.10)

Using (5.9) in (5.10) we conclude that
\[ \left[ \sum_{i \in I_1} (1 - b_i^m) \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i \in I_1, j \in I_1} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \right] \]
\[ + \left[ \sum_{i \in I_2} (1 - b_i^m) \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i \in I_2, j \in I_2} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \right] \leq C. \]

where \( b_i^m = \sum_{j \in I_2} \frac{a_{ij} \alpha_l^m}{2\pi} \) if \( i \in I_1 \) and \( b_i^m = \sum_{j \in I_1} \frac{a_{ij} \alpha_l^m}{2\pi} \). At least one of the quantities within the brackets must be bounded above. We assume
\[ \sum_{i \in I_1} (1 - b_i^m) \int_{\mathbb{R}^2} \rho_i^m \ln \rho_i^m \, dx + \sum_{i \in I_1, j \in I_1} \int_{\mathbb{R}^2} \rho_i^m(x) \ln |x - y| \rho_j^m(y) \, dx \, dy \leq C. \]

Since \( \Lambda(J(\beta)) > 0 \), for all \( J \subset I_1 \) and \( b_i^m \) can be made as small as one desires, by Remark 2.3 we obtain uniform bound on the entropy of \( \rho_i^m \) for all \( i \in I_1 \). This contradicts that \( \rho^m \) is a Dirac measure. Hence \( I = I_1 \) and we denote the common blow-up point by \( v_0 \). By Lemma 4.2 and the second moment bound
\[ (\sum_{i=1}^n \beta_i) v_0 = \lim_{m \to \infty} \sum_{i=1}^n \int_{\mathbb{R}^2} x \rho_i^m(x) \, dx = 0. \]

**Step 5:** Concentration implies equality holds in the functional inequality.

Since \( \rho_i^m \to \beta \delta_0 \) for all \( i \in I \), by lower semi-continuity Lemma 2.1 we conclude that
\[ \inf_{\rho \in L^2} \mathcal{G}_\eta(\rho) = \lim_{m \to \infty} \left( \mathcal{F}(\rho^m) + \frac{1}{2\tau} d_w^2(\rho^m, \eta) \right) \]
\[ \geq \inf_{\rho \in L^2} \mathcal{F}(\rho) + \frac{1}{2\tau} \lim_{m \to \infty} d_w^2(\rho^m, \eta) \]
\[ \geq \inf_{\rho \in L^2} \mathcal{F}(\rho) + \frac{1}{2\tau} d_w^2(\beta \delta_0, \eta) \]
\[ = \inf_{\rho \in \Gamma^2} \mathcal{F}(\rho) + \frac{1}{2\tau} M_2(\eta). \]

Using Lemma 5.1 we obtain the equality in the functional inequality.

**Step 6:** Concluding the proof of the theorem.

In view of the first part of the theorem, if \( \eta \) satisfies \( \mathcal{F}(\eta) < \inf_{\rho \in \Gamma^2} \mathcal{F}(\rho) + \frac{1}{2\tau} M_2(\eta) \) then there exists a minimizer of \( \inf_{\rho \in \Gamma^2} \mathcal{G}_\eta(\rho) \), and let us denote the minimizer by \( \varrho \).

Utilizing Lemma 4.3, the obtained minimizer satisfies
\[
\mathcal{F}(\varrho) = \mathcal{G}_\eta(\varrho) = \frac{1}{2\tau} d_w^2(\varrho, \eta) \leq \inf_{\rho \in \Gamma^2} \mathcal{F}(\rho) + \frac{1}{2\tau} M_2(\eta) - \frac{1}{2\tau} \left( \frac{1}{2} M_2(\eta) - M_2(\varrho) \right) = \inf_{\rho \in \Gamma^2} \mathcal{F}(\rho) + \frac{1}{2\tau} M_2(\eta).
\]

This completes the proof of the theorem. \( \square \)

**Remark 5.3.** A careful tracing back of the proof of Theorem 5.2, in particular, step 2 to step 4 confirms that: suppose we have a sequence \( \{\rho_m^k\} \subset \Gamma^2 \) such that the energy \( \mathcal{F}(\rho_m^k) \) and the second moment \( M_2(\rho_m^k) \) are uniformly bounded above then one of the following alternative holds

- either the entropy is uniformly bounded, or
- all the components of \( \rho_m^k \) concentrates at the origin in the form of a Dirac delta measure.

This observation will be useful in the next section while trying to obtain uniform entropy estimates.

### 6. A Priori Estimates

In this section we will obtain uniform estimates on the interpolates \( \rho^k_\tau, k \in \mathbb{N} \) obtained by the MM-scheme (3.1). Recall that if \( \tau \in (0, \tau^*) \) then MM-scheme is well defined.

#### 6.1. Uniform bound on the second moment and the Wasserstein distance.

**Lemma 6.1.** For any \( \tau \in (0, \tau^*) \) and any positive integer \( k \) there holds
\[
M_2(\rho^k_\tau) + \frac{1}{2\tau} \sum_{l=1}^k d^2_w(\rho^l_\tau, \rho^{l-1}_\tau) \leq \mathcal{F}(\rho^0_\tau) - C_{LHLS}(\beta) + M_2(\rho^0). \tag{6.1}
\]

**Proof.** For every \( l \in \{1, \ldots, k\} \), the minimizing property of \( \rho^l_\tau \) gives
\[
\mathcal{F}(\rho^l_\tau) + \frac{1}{2\tau} d^2_w(\rho^l_\tau, \rho^{l-1}_\tau) \leq \mathcal{F}(\rho^{l-1}_\tau).
\]

Summing over \( l \in \{1, \ldots, k\} \) we obtain
\[
\mathcal{F}(\rho^k_\tau) + \frac{1}{2\tau} \sum_{l=1}^k d^2_w(\rho^l_\tau, \rho^{l-1}_\tau) \leq \mathcal{F}(\rho^0_\tau). \tag{6.2}
\]

Since \( \beta \) is critical, \( \mathcal{F}(\rho^k_\tau) \geq C_{LHLS}(\beta) \), and hence
\[
\frac{1}{2\tau} \sum_{l=1}^k d^2_w(\rho^l_\tau, \rho^{l-1}_\tau) \leq \mathcal{F}(\rho^0_\tau) - C_{LHLS}(\beta). \tag{6.3}
\]
On the other hand, by Lemma 4.3
\[ M_2(\rho^{k}_\tau) \leq M_2(\rho^{k-1}_\tau) \leq \cdots \leq M_2(\rho^0_\tau) = M_2(\rho^0), \]
completing the proof. \qed

6.2. Uniform bound on the entropy. To obtain uniform bound on the entropy, we are going to use a refined version of de la Vellé Poussin’s lemma on the gain on integrability [DLVP15, Le77, GTW95, LM02, Lau15]. Let us first recall the lemma

Lemma 6.2 (de la Vellé Poussin). Let \( \mu \) be a non-negative measure on \( \mathbb{R}^2 \) and \( Z \subset L^1(\mathbb{R}^2;\mu) \). The set \( Z \) is uniformly integrable in \( L^1(\mathbb{R}^2;\mu) \) if and only if \( Z \) is uniformly bounded in \( L^1(\mathbb{R}^2;\mu) \) and there exists a convex function \( \Upsilon \in C^\infty([0,\infty);[0,\infty)) \) satisfying all the properties listed in (4.6) and

\[ \sup_{g \in Z} \int_{\mathbb{R}^2} \Upsilon(g(x)) \, d\mu(x) < +\infty. \]

We are going to apply de la Vellé Poussin lemma for a suitably constructed measure \( \mu \) and to the family \( Z \) containing only one element \( g(x) = |x|^2 \). We remark that it is, in general, impossible to iterate the higher integrability information on the initial data to the sequence \( \rho^k_\tau \) obtained by the MM-scheme. In particular, if we know \( \sum_{i=1}^n \Upsilon(|\cdot|^2)\rho^0_{\tau,i}(\cdot) \in L^1(\mathbb{R}^2) \), then it is not evident that \( \sum_{i=1}^n \Upsilon(|\cdot|^2)\rho^1_{\tau,i}(\cdot) \) will also be uniformly integrable.

Lemma 6.3. For every \( T \in (0,\infty) \) there exists a constant \( C_{ap}(T) \) such that for each \( \tau \in (0,\tau^*) \) and positive integers \( k \) satisfying \( k\tau \leq T \) there holds

\[ \sum_{i=1}^n \int_{\mathbb{R}^2} \rho^k_{\tau,i} |\ln \rho^k_{\tau,i}| \, dx \leq C_{ap}(T). \] \hspace{1cm} (6.4)

Proof. Fix \( T \), and assume that the claim is false. Then there exists a sequence \( \tau_m \to 0^+ \) as \( m \to \infty \), and \( k_m \in \mathbb{N} \) such that \( k_m \tau_m \leq T \) and \( \mathcal{H}(\rho^{k_m}_{\tau_m}) \to 0 \). By (6.2) we know that

\[ \mathcal{F}(\rho^{k_m}_{\tau_m}) \leq \mathcal{F}(\rho^0), \hspace{0.5cm} \text{for all } m. \]

Moreover, recall that by Lemma 4.2 each \( \rho^{k_m}_{\tau_m} \) satisfies zero center mass condition i.e., \( \sum_{i=1}^n \int_{\mathbb{R}^2} x \rho^{k_m}_{\tau_m,i}(x) \, dx = 0 \). Proceeding as in the proof of Theorem 5.2 (and heeding Remark 5.3) we conclude that \( \rho^{k_m}_{\tau_m,i} \xrightarrow{\text{ap}} \beta_i \delta_0 \) in the weak*-topology of measures. In the following we are going to obtain uniform integrability of the second moment of \( \rho^{k_m}_{\tau_m} \). We set

\[ \mu = \sum_{m=1}^\infty \frac{1}{(k_m+1)2^{m+1}} \sum_{l=0}^{k_m} \sum_{i=1}^n \rho^l_{\tau_m,i}. \]

Then taking into account Lemma 6.1, we obtain

\[ M_2(\mu) = \sum_{m=1}^\infty \frac{1}{(k_m+1)2^{m+1}} \sum_{l=0}^{k_m} M_2(\rho^l_{\tau_m}) \leq 2M_2(\rho^0). \] \hspace{1cm} (6.5)

For the above choice of the measure \( \mu \), we apply de la Vellé Poussin lemma to \( g(x) = |x|^2 \). There exists a smooth convex function \( \Upsilon \) satisfying all the properties enumerated in (4.6) such that \( \int_{\mathbb{R}^2} \Upsilon(|x|^2) \, d\mu(x) = C_0 < +\infty \). As a consequence we obtain

\[ \sum_{i=1}^n \int_{\mathbb{R}^2} \Upsilon(|x|^2)\rho^l_{\tau_m,i} \, dx \leq (k_m+1)2^{m+1}C_0, \hspace{0.5cm} \text{for all } l \in \{0,\ldots,k_m\}. \] \hspace{1cm} (6.6)
Now we use Lemma 4.4 to obtain uniform estimates on \( \sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^k_{\tau,m,i}(x) \, dx \). Applying Lemma 4.4 with \( \varrho = \rho^k_{\tau,m} \) and \( \eta = \rho^{k,-1}_{\tau,m} \) we see that
\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^k_{\tau,m,i}(x) \, dx \leq (1 + C_0 \tau_m) \sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^{k,-1}_{\tau,m,i}(x) \, dx + C_0 \tau_m.
\]

Iterating this process \( k_m \) times we obtain
\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^{k_m}_{\tau,m,i}(x) \, dx \leq (1 + C_0 \tau_m)^{k_m} \sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^0_{\tau,m,i}(x) \, dx + C_0 \tau_m (1 + C_0 \tau_m)^{k_m}. \tag{6.7}
\]

Since \( k_m \tau_m \leq T \), and \( \rho^0_{\tau,m,i} = \rho_i^0 \) we deduce from (6.7)
\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^{k_m}_{\tau,m,i}(x) \, dx \leq e^{C_0 T} \sum_{i=1}^{n} \int_{\mathbb{R}^2} \mathcal{Y}(|x|^2) \rho^0_{\tau,m,i}(x) \, dx + C_0 T e^{C_0 T}. \tag{6.8}
\]

Taking into account (6.8) and the weak* convergence to the Dirac measure we conclude \( M_2(\rho^k_{\tau,m}) \to 0 \) as \( m \to \infty \). On the other hand, by Lemma 4.3
\[
M_2(\rho^{k_m,-1}_{\tau,m}) - M_2(\rho^{k_m}_{\tau,m}) = d^2_m(\rho^{k_m}_{\tau,m}, \rho^{k_m,-1}_{\tau,m})
\]
\[
\vdots
\]
\[
M_2(\rho^0_{\tau,m}) - M_2(\rho^1_{\tau,m}) = d^2_m(\rho^1_{\tau,m}, \rho^0_{\tau,m}). \tag{6.9}
\]
Adding all the terms in (6.9) and using \( \rho^0_{\tau,m} = \rho^0 \) and Lemma 6.1 we get
\[
M_2(\rho^0) = M_2(\rho^{k_m}_{\tau,m}) + \sum_{l=1}^{k_m} d^2_m(\rho^l_{\tau,m}, \rho^{l-1}_{\tau,m}) = M_2(\rho^{k_m}_{\tau,m}) + O(\tau_m) \to 0
\]
as \( m \to 0 \), contradicting \( \rho^0 \in \Gamma^2 \). This contradiction appeared because we assumed the entropy is not uniformly bounded. Hence our assumption was wrong and the lemma is proved. \( \square \)

6.3. A priori estimates on the time interpolation.

6.3.1. Time interpolation. We define the piece wise constant time dependent interpolation
\[
\rho_r(t) = \rho^k_r, \quad \text{if} \ t \in ((k-1)\tau, k\tau], \ k \geq 1.
\]
Recall that the Newtonian potential of \( \rho^k_r \) is defined by
\[
u^k_r(i)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - y| \rho^k_r(y) \, dy, \quad \text{for} \ i = 1, \ldots, n. \tag{6.10}
\]
By definition of \( \rho_r(t) \), we have \( u_{r,i}(t) = u^k_r(i)(t) \) for all \( t \in ((k-1)\tau, k\tau] \), \( k \geq 1 \).

As a consequence of the a priori estimates obtained in Lemma 6.1, Lemma 6.3 and the regularity estimates stated in Lemma 3.1 we obtain the following:

Lemma 6.4. For every \( T > 0 \), there exists a finite constant \( C(T) = C(T, \rho^0) > 0 \), depending only on the time \( T \) and the initial data \( \rho^0 \), such that for every \( \tau \in (0, \tau^*) \) we have

(a) uniform entropy and second moment bound
\[
\sup_{t \in [0,T], \tau \in (0,\tau^*)} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^2} \rho_{r,i}(t) \ln \rho_{r,i}(t) \, dx + M_2(\rho_{r,i}(t)) \right) \leq C(T).
\]
applied to the measures $\rho_m$. Moreover, since $v\rho$ is continuous in time with respect to $d$, from the $L^2$ estimate:}

$$\lim_{m \to \infty} \int_{\mathbb{R}^2} \zeta \cdot v_m \rho_{\tau_m,i} \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} \zeta \cdot v_\rho \, dx \, dt, \quad \text{for all } \zeta \in C_c^\infty((0,T) \times \mathbb{R}^2; \mathbb{R}^2).$$

Moreover, since $v_\rho \in L^1((0,T) \times \mathbb{R}^2; \mathbb{R}^2)$ we can identify $v_i = \frac{\nabla_\rho}{\rho_i}$ through the following identity

$$\lim_{m \to \infty} \int_0^T \int_{\mathbb{R}^2} \zeta \cdot v_m \rho_{\tau_m,i} = \lim_{m \to \infty} \int_0^T \int_{\mathbb{R}^2} (\nabla_x \cdot \zeta) \rho_{\tau_m,i} = -\int_0^T \int_{\mathbb{R}^2} (\nabla_x \cdot \zeta) \rho_i,$$
which also ensures that $\rho_i \in L^1((0,T);W^{1,1}(\mathbb{R}^2))$. By lower semicontinuity (Lemma A equation (9.9)) and Lemma 6.4(b)

$$\int_0^T \int_{\mathbb{R}^2} \left| \nabla \rho_i(t) \rho_i(t) \right|^2 dx dt < \infty,$$

(proving that $\nabla \rho_i \in L^2((0,T) \times \mathbb{R}^2, \rho_i ; \mathbb{R}^2)$ for all $i \in \{1, \ldots, n\}$.

In addition, using the $H^1_{loc}$ estimates of Lemma 6.4(d), and proceeding as in [KW19, Lemma 7.1] we get up to a subsequence $u_{\tau,i}(t) \to u_i(t)$ strongly in $L^2((0,T) ; L^2_{loc}(\mathbb{R}^2))$, where $u_i$ is the Newtonian potential of the obtained solution $\rho_i$. This together with the $L^4 - H^2_{loc}$ estimate of Lemma 6.4(d) and Simon’s compactness results [Sim87, Lemma 9] ensures

$$u_{\tau,i} \to u_i \text{ strongly in } L^2((0,T) ; H^1_{loc}(\mathbb{R}^2)).$$

7.2. Global in time existence of solutions. For any time dependent test function $\xi \in C_c^\infty((0,T) \times \mathbb{R}^2)$, we apply Lemma 4.1(c) with the following choices: $\varphi_i = \rho_{\tau,i}, \eta_i = \rho_{\eta,i}^{k_{\tau,i}}, \psi(\cdot) = \xi(\cdot, (k-1)\tau)$, where $k$ is such that $k \tau \leq T$. Summing over all $k$ satisfying $k \tau \leq T$ we obtain the following time discrete formulation of the system (1.1)

$$- \int_0^T \int_{\mathbb{R}^2} \partial_t \xi \rho_{\tau,i} - \int_{\mathbb{R}^2} \xi(0) \rho_i^0 + \int_0^T \int_{\mathbb{R}^2} \nabla \xi \cdot \nabla \rho_{\tau,i}$$

$$- \sum_{j=1}^n a_{ij} \int_0^T \int_{\mathbb{R}^2} \nabla \xi \cdot \nabla u_{\tau,j} \rho_{\tau,i} \rho_{\tau,j} dx dt = O(\tau^{\frac{1}{2}}),$$

for all $i = 1, \ldots, n$. Now it is easy to pass the limit in (7.2) using the convergence results (C1) - (C4). For the first term we use (C2), for the third term we use (C3) and finally for the last term we use (C2), (C4) and the duality relation. The limiting equation is the weak formulation of (1.1) stated in Definition 1.1. Since $T$ is arbitrary, we conclude the proof of global existence.

So far we have proved Theorem 1.3 (a). The proof of (b) is the content of the next subsection.

7.3. The free energy inequality. As in the sub-critical case, we show using De Giorgi variational interpolation, that the obtained solution satisfies the free energy inequality. Define for $\tau \in (0, \tau^*)$

$$\tilde{\rho}_{\tau}(t) := \arg \min_{\rho \in [1/2]} \left\{ \mathcal{F}(\rho) + \frac{1}{2} \frac{\rho^{k_{\tau}}} {2(t - (k-1)\tau)^2} \right\}, \quad t \in ((k - 1)\tau, k\tau].$$

With out loss of generality we can assume that $\tilde{\rho}_{\tau}(k\tau) = \rho_{\tau}^{k_{\tau}}$. Note that since $0 < t - (k-1)\tau \leq \tau^* \text{ and } \rho_{\tau}^{k_{\tau} - 1}$ satisfies the inequality

$$\mathcal{F}(\rho_{\tau}^{k_{\tau} - 1}) < \inf_{\rho \in [1/2]} \mathcal{F}(\rho) + \frac{1}{2} M_2(\rho_{\tau}^{k_{\tau} - 1}) \leq \inf_{\rho \in [1/2]} \mathcal{F}(\rho) + \frac{1}{2(t - (k-1)\tau)} M_2(\rho_{\tau}^{k_{\tau} - 1}),$$

the De Giorgi’s interpolation (7.3) is well defined. Moreover, combining [AGS05, Theorem 3.1.4 and Lemma 3.3.2] together with Lemma 4.1(b) we get the following discrete energy identity:

Lemma 7.1 (Discrete energy identity). For every $k \in \mathbb{N}$ and $\tau \in (0, \tau^*)$ the De-Giorgi interpolation defined by (7.3) satisfies the following energy identity:

$$\sum_{i=1}^n \frac{1}{2} \int_0^{k\tau} \int_{\mathbb{R}^2} \left| \nabla \rho_{\tau,i} \right|^2 dx dt - \sum_{j=1}^n a_{ij} \int_0^{k\tau} \int_{\mathbb{R}^2} \nabla u_{\tau,j} dx dt$$
where $\tilde{u}_{\tau,j}$ is the Newtonian potential associated to $\tilde{\rho}_{\tau,j}$.

Furthermore, for every $T > 0$ there exists a constant $C(T) > 0$ such that
\[
\mathbf{d}_w(\rho_\tau(t), \tilde{\rho}_\tau(t)) \leq C(T) \tau, \quad \text{for all } t \in [0, T].
\] (7.4)

As a consequence of (7.4) we infer that $\rho_{\tau,m}$ and $\tilde{\rho}_{\tau,m}$ enjoys the same convergence properties stated in the previous section (C1)-(C4), provided we establish the entropy, second moment and the Fisher information bound (Lemma 6.4(a,b)). Here $\tau_m$ is the same monotone decreasing sequence used in the proof of global in time existence. Moreover, both $\rho_{\tau,m}$ and $\tilde{\rho}_{\tau,m}$ converge to the same limit $\rho$. The finite Fisher information bound follows from Lemma 7.1 and the inequality
\[
\frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^2} \frac{|\nabla \tilde{\rho}_{\tau,i}(t)|^2}{\tilde{\rho}_{\tau,i}(t)} \, dx \leq \sum_{i=1}^{n} \int_{\mathbb{R}^2} \frac{|\nabla \tilde{\rho}_{\tau,i}(t)|^2}{\tilde{\rho}_{\tau,i}(t)} - \sum_{j=1}^{n} a_{ij} \nabla \tilde{u}_{\tau,j}(t) \tilde{\rho}_{\tau,j}(t) \, dx
\]
\[
+ 4n(\max_{i,j} a_{ij}) C_0 \sum_{i=1}^{n} \beta_i, \quad \text{for all } t \in [0, T],
\] (7.5)

where $C_0$ is a constant independent of the interpolates. A proof of this inequality can be found in [FM16, Lemma 2.2] (see also [KW19, Lemma 8.3] in this setting). It remains to show that
\[
\sup_{m} \sup_{t \in [0,T]} \left[ \sum_{i=1}^{n} \int_{\mathbb{R}^2} \tilde{\rho}_{\tau,m,i}(t) |\ln \tilde{\rho}_{\tau,m,i}(t)| \, dx + M_2(\tilde{\rho}_{\tau,m}(t)) \right] < \infty.
\] (7.6)

Recall that we have all the estimates on the interpolates $\rho_{\tau,m}$ given by Lemma 6.4. In this sequel, any uniform constant will be denoted by $C$.

Second moment bound: Let $t \in [0, T]$ and $k \in \mathbb{N}$ be such that $t \in ((k-1)\tau_m, k\tau_m)$. Then by (7.4) and Lemma 6.1 we obtain the required bound:
\[
\mathbf{d}_w^2(\tilde{\rho}_{\tau,m}(t), \rho_{\tau,m}^{k-1}) \leq C \tau_m, \quad M_2(\tilde{\rho}_{\tau,m}(t)) \leq C.
\] (7.7)

Entropy bound: Assume by contradiction, there exists a sequence $t_m \in [0, T]$ such that $\mathcal{H}(\tilde{\rho}_{\tau,m}(t_m)) \to \infty$. Let $k_m \in \mathbb{N}$ be such that $t_m \in ((k_m-1)\tau_m, k_m\tau_m)$. Since the energy of the interpolates $\mathcal{F}(\tilde{\rho}_{\tau,m}(t_m)) \leq \mathcal{F}(\rho_{\tau,m}^{k_m-1})$ are uniformly bounded, by Theorem 5.2 and Remark 5.3 we have $\lim_{m \to \infty} \tilde{\rho}_{\tau,m}(t_m) = \beta \delta_0$ in the weak* topology of measures. In view of Lemma 4.3 we get
\[
M_2(\rho_{\tau,m}^{k_m-1}) - M_2(\tilde{\rho}_{\tau,m}(t_m)) = \mathbf{d}_w^2(\tilde{\rho}_{\tau,m}(t_m), \rho_{\tau,m}^{k_m-1})
\]
\[
M_2(\rho_{\tau,m}^{k_m-2}) - M_2(\rho_{\tau,m}^{k_m-1}) = \mathbf{d}_w^2(\rho_{\tau,m}^{k_m-1}, \rho_{\tau,m}^{k_m-2})
\]
\[
\vdots
\]
\[
M_2(\rho_{\tau,m}^0) - M_2(\rho_{\tau,m}^1) = \mathbf{d}_w^2(\rho_{\tau,m}^1, \rho_{\tau,m}^0).
\]

Adding all the terms and using (7.7), Lemma 6.1 we see that
\[
M_2(\rho^0) = M_2(\tilde{\rho}_{\tau,m}(t_m)) + O(\tau_m).
\] (7.8)

On the other hand, invoking Lemma 6.2 to the measure
\[
\mu = \sum_{i=1}^{n} \tilde{\rho}_{\tau,m,i}(t_m) + \sum_{m=1}^{\infty} \frac{1}{k_m 2^{m+1}} \sum_{l=0}^{k_m-1} \sum_{i=1}^{n} \tilde{\rho}_{\tau,m,i}^l
\]
we obtain the existence of a convex function $\Upsilon$ satisfying the all the properties (4.6) and such that $\int_{\mathbb{R}^2} \Upsilon(\|x\|^2) d\mu < \infty$. Using Lemma 4.4 and iterating the process to the interpolates $\tilde{\rho}_{\tau_m}(t_m), \rho_{\tau_m}^{k-1}, \rho_{\tau_m}^{k-2}, \ldots, \rho_0$ we get

$$\sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(\|x\|^2) \tilde{\rho}_{\tau_m,i}(t_m) \, dx \leq c \epsilon^{-2} T \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(\|x\|^2) \rho_i^0 \, dx + C_0 T \epsilon^{-2} T. \quad (7.9)$$

In view of (7.8), (7.9) and the convergence to the Dirac mass we get a contradiction. This establishes the uniform entropy bound.

One more application of Lemma A with the measures $\mu_m = (T \beta_i)^{-1} \rho_{\tau_m,i} \cdot dx$ and the vector fields $(v_m, 1)$ where

$$v_{m,i} = \nabla \rho_{\tau_m,i} \frac{\rho_{\tau_m,i}}{\rho_{\tau_m,i}},$$

and using the semi-continuity (9.9) we obtain

$$\int_0^T \int_{\mathbb{R}^2} \left( \nabla \rho_i(t) \frac{\rho_i(t)}{\rho_{\tau_m,i}} - \sum_{j=1}^{n} a_{ij} \nabla u_j(t) - \sum_{j=1}^{n} a_{ij} \nabla u_{\tau_m,j}(t) \right) \rho_i(t) \, dx \, dt \leq \liminf_{m \to +\infty} \int_0^T \int_{\mathbb{R}^2} \left( \nabla \rho_{\tau_m,i}(t) \frac{\rho_{\tau_m,i}(t)}{\rho_{\tau_m,i}(t)} - \sum_{j=1}^{n} a_{ij} \nabla u_{\tau_m,j}(t) \right) \rho_{\tau_m,i}(t) \, dx \, dt. \quad (7.10)$$

$\tilde{\rho}_{\tau_m}$ being converge to the same limit as $\rho_{\tau_m}$, the inequality (7.10) holds for $\tilde{\rho}_{\tau_m}$ as well. Passing to the limit in the discrete energy identity (Lemma 7.1) and using the lower semi-continuity of $F$ with respect to the weak $L^1$ convergence we get

$$\sum_{i=1}^{n} \int_0^T \int_{\mathbb{R}^2} \left( \nabla \rho_i(t) \frac{\rho_i(t)}{\rho_{\tau_m,i}} - \sum_{j=1}^{n} a_{ij} \nabla u_j(t) \right) \rho_i(t) \, dx \, dt + F(\rho(T)) \leq F(\rho^0).$$

This completes the proof of Theorem 1.3(b).

8. CONCENTRATION DOES OCCUR AS TIME $T \nearrow \infty$

This section divulges the behavior of the obtained solution $\rho(\cdot, t)$ as time $T \nearrow \infty$. Recall that $\rho(\cdot, t)$ satisfies the free energy inequality

$$F(\rho(\cdot, t)) + \int_0^t D_F(\rho(\cdot, s)) \, ds \leq F(\rho^0),$$

for all $t \in [0, \infty)$ and consequently, the energy $F(\rho(\cdot, t))$ is bounded above by $F(\rho^0)$ for all $t \in [0, \infty)$. Moreover, it follows from the weak formulation (Definition 1.1) that the second moment is conserved in time. Notice that the weak $L^1$-convergence and the a priori estimates of Lemma 6.1 provide the second moment bound: $M_2(\rho(t)) \leq M_2(\rho^0)$. Indeed, applying de la Velleé Poussin’s lemma, we confirm that $\|x\|^2 \rho_{\tau_m}(t)$ are equi-integrable, and hence the second moment is conserved: $M_2(\rho(t)) = M_2(\rho^0)$. By Theorem 5.2 (see remark 5.3), as $t_m \nearrow \infty$ either $\rho(\cdot, t_m)$ converges to $\beta \delta_0$ or the entropy $\mathcal{H}(\rho(\cdot, t_m))$ remains uniformly bounded. In the following, we will show that the later situation is inconceivable.

**Lemma 8.1.** Assume $A$ and $\beta$ satisfies the Assumption 2 and $\rho^0 \in \Gamma_{\beta_+}$. Given any free energy solution $\rho$ to (1.1) we have

$$\lim_{t \to \infty} \rho(\cdot, t) = \beta \delta_0,$$

where the convergence is in the sense of weak* convergence of measures.
Proof. Assume by contradiction that there exists a sequence $t_m \uparrow \infty$ as $m \to \infty$ such that

$$\sup_{m \in \mathbb{N}} \mathcal{H}(\rho(\cdot, t_m)) < \infty. \quad (8.1)$$

Passing to a subsequence we may assume that $t_{m+1} - t_m > 1$ for all $m$. In the following, we may have to pass to a further subsequence quite often, and for the simplicity of notations, we will not distinguish between the original sequence and its subsequences. Since in the critical case the second moment is conserved in time, the measures $\rho(\cdot, t_m)$ are tight. Moreover, since the entropy is also uniformly bounded, by Dunford-Pettis theorem there exists $\rho^\infty \in \Gamma_2^\beta$ such that $\rho(\cdot, t_m) \Rightarrow \rho^\infty(\cdot)$ weakly in $(L^1(\mathbb{R}^2))^n$. In addition, we have

$$0 < M_2(\rho^\infty) \leq \liminf_{m \to \infty} M_2(\rho(\cdot, t_m)) = M_2(\rho^0) < \infty. \quad (8.2)$$

Let us set

$$\rho^m(x, t) = \rho(x, t_m + t), \text{ for } x \in \mathbb{R}^2 \text{ and } t \in [0, 1].$$

In the following we will show that $\rho^m$ converges, in some sense, to a steady state of (1.1) having finite second moment. To do that we need to obtain uniform estimates on entropy and the Fisher information of $\rho^m$ all over again. Because all the estimates obtained earlier are local in time. Most importantly, we need to obtain uniform Hölder estimates in time. We know that for the solution obtained in subsection 7.2, $t \mapsto \rho(\cdot, t)$ is a Hölder continuous with respect to the 2-Wasserstein distance. Since we are considering any free energy solution (and we did not prove the uniqueness) we can not avail that information. However, in the next few steps, we will show that if we replace the 2-Wasserstein distance by 1-Wasserstein distance then we have a global Hölder estimate. We divide the proof into several steps. Any universal constant independent of $m$ will be denoted by $C$.

**Step 1:** Uniform $L^2$ and Fisher information bound:

$$\sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^2} (\rho_i^m(x, t))^2 \, dx \, dt + \sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^2} \frac{\nabla \rho_i^m(x, t)^2}{\rho_i^m(x, t)} \, dx \, dt \leq C. \quad (8.3)$$

Since $\beta$ is critical $\mathcal{F}(\rho(\cdot, t))$ is bounded from below uniformly with respect to $t$. Using the free energy inequality we conclude that the dissipation of the free energy is integrable, i.e.,

$$\lim_{t \to \infty} \int_0^t \mathcal{D}_\mathcal{F}(\rho(\cdot, s)) \, ds \leq \mathcal{F}(\rho^0) - \liminf_{t \to \infty} \mathcal{F}(\rho(\cdot, t)) \leq C. \quad (8.4)$$

As a consequence of (8.4) and the assumption $t_{m+1} - t_m > 1$ for all $m$ we deduce

$$\lim_{m \to \infty} \int_{t_m}^{t_{m+1}} \mathcal{D}_\mathcal{F}(\rho(\cdot, s)) \, ds = \lim_{m \to \infty} \int_0^1 \mathcal{D}_\mathcal{F}(\rho^m(\cdot, s)) \, ds = 0. \quad (8.5)$$

The inequality (8.5) together with (7.5) establishes the Fisher information bound

$$\sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^2} \frac{\nabla \rho_i^m(x, t)^2}{\rho_i^m(x, t)} \, dx \, dt \leq C.$$

By [FM16, Lemma 2.1] any $L^p$-norm can be controlled by the Fisher information and in particular

$$\sum_{i=1}^n \int_0^1 \int_{\mathbb{R}^2} (\rho_i^m(x, t))^2 \, dx \, dt \leq C.$$

**Step 2.** Uniform Hölder estimate: $d_{\text{w}1}(\rho(\cdot, t_0), \rho(\cdot, t_1)) \leq C|t_0 - t_1|^\frac{1}{2}$, for all $t_0, t_1 \in [0, \infty)$. 


Fix $t_0 < t_1$. For simplicity of presentation we abbreviate the equation (1.1) as $\partial_t \rho_i = \nabla \cdot (\rho_i \nabla F(\rho_i(t)))$. So that the weak formulation can be reformulated as

$$\int_{t_0}^{t_1} \int_{\mathbb{R}^2} \partial_t \xi \rho_i = \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \nabla_x \xi \cdot \left( \nabla_x \frac{\partial F(\rho_i(t))}{\partial \rho_i} \right) \rho_i. \quad (8.6)$$

for every $\xi \in C_0^1((t_0, t_1) \times \mathbb{R}^2)$. We apply test functions of the form $\xi(x, t) = h(t)\psi(x)$ in (8.6), where $h \in C^1_c(t_0, t_1)$ and $\psi \in C^1_c(\mathbb{R}^2)$ and set $f_i(t) = \int_{\mathbb{R}^2} \psi(x) \rho_i(x, t) \, dx$. Then we can rewrite (8.6)

$$\int_{t_0}^{t_1} h'(t) f_i(t) \, dt = \int_{t_0}^{t_1} h(t) \int_{\mathbb{R}^2} \nabla_x \psi \cdot \left( \nabla_x \frac{\partial F(\rho_i(t))}{\partial \rho_i} \right) \rho_i. \quad (8.6)$$

Hence $f_i'(t) = -\int_{\mathbb{R}^2} \nabla_x \psi \cdot \left( \nabla_x \frac{\partial F(\rho_i(t))}{\partial \rho_i} \right) \rho_i$ and moreover, $L^2$-norm of $f_i'$ can be estimated as

$$\int_{t_0}^{t_1} |f_i'(t)|^2 \, dt = \int_{t_0}^{t_1} \left( \int_{\mathbb{R}^2} \nabla_x \psi \cdot \left( \nabla_x \frac{\partial F(\rho_i(t))}{\partial \rho_i} \right) \rho_i \right)^2 \, dt$$

$$\leq ||\nabla \psi||_L^2 \int_{t_0}^{t_1} \left( \int_{\mathbb{R}^2} \left| \nabla \frac{\partial F(\rho_i(t))}{\partial \rho_i} \right|^2 \rho_i \right) \left( \int_{\mathbb{R}^2} \rho_i \right) \, dt$$

$$\leq \beta_i ||\nabla \psi||_{L^2}^2 \left( \int_0^\infty D_F(\rho_i, t) \, dt \right)$$

$$\leq C ||\nabla \psi||_{L^2}^2.$$ 

Since $\rho_i \in L^2_{\text{loc}}((0, \infty); L^2(\mathbb{R}^2))$, we also deduce that $f_i \in L^2(t_0, t_1)$ and hence $f_i \in W^{1,2}(t_0, t_1)$. As a consequence, $f_i$ are absolutely continuous and by fundamental theorem of calculus

$$|f_i(t_0) - f_i(t_1)| \leq \int_{t_0}^{t_1} |f_i'(t)| \, dt \leq ||f_i'||_{L^2(t_0, t_1)} |t_0 - t_1|^{\frac{1}{2}} \leq C ||\nabla \psi||_{L^\infty} |t_0 - t_1|^{\frac{1}{2}}. \quad (8.7)$$

Taking the supremum over all $\psi \in C^1_c(\mathbb{R}^2)$ satisfying $||\nabla \psi||_{L^\infty} \leq 1$ and using Kantorovich duality (2.6) we get the desired result:

$$d_{w1}(\rho_i(\cdot, t_0), \rho_i(\cdot, t_1)) = \beta_i^{\frac{1}{2}} \sup_{||\nabla \psi||_{L^\infty} \leq 1} |f_i(t_0) - f_i(t_1)| \leq C |t_0 - t_1|^{\frac{1}{2}}.$$

Applying step 2 to the sequence $\rho_i^m$ we obtain $d_{w1}(\rho^m(\cdot, t), \rho^m(\cdot, s)) \leq C |t - s|^{\frac{1}{2}}$ for all $s, t \in [0, 1]$.

**Step 3:** Uniform entropy bound: $\sup_{t \in [0, 1]} H(\rho^m(\cdot, t)) \leq C$.

We evaluate the time derivative of the entropy

$$\frac{dH(\rho^m(\cdot, t))}{dt} = \int_{\mathbb{R}^2} (1 + \ln \rho_i^m) \partial_t \rho_i^m \, dx$$

$$= -\int_{\mathbb{R}^2} \nabla \rho_i^m \cdot \left( \nabla \frac{\partial F(\rho_i^m(t))}{\partial \rho_i} \right) \rho_i^m \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^2} \frac{\nabla \rho_i^m \rho_i^m}{\rho_i^m} + \frac{1}{2} \int_{\mathbb{R}^2} \left| \nabla \frac{\partial F(\rho_i^m(t))}{\partial \rho_i} \right|^2 \rho_i^m \, dx$$

For $t \in [0, 1]$ integrating the above inequality from 0 to $t$ we get

$$H(\rho_i^m(\cdot, t)) \leq H(\rho_i^m(\cdot, 0)) + \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \frac{\nabla \rho_i^m \rho_i^m}{\rho_i^m} \, dx \, ds + \frac{1}{2} \int_0^\infty D_F(\rho_i(t), s) \, ds \leq C.$$
In the last inequality we have used the hypothesis (8.1) and step 1.

In view of step 1 and step 3 we also have

**Step 4:** For any \( R > 0 \) the Newtonian potential \( u_i^m(x, t) := -\frac{1}{2\pi} \ln |x| \ast \rho_i^m(\cdot, t) \) satisfies

\[
\sup_{t \in [0, 1]} ||u_i^m(\cdot, t)||_{H^1(B_R)} + \int_0^1 ||u_i^m(\cdot, t)||_{H^2(B_R)}^2 dt \leq C, \text{ for all } i \in I. \tag{8.8}
\]

It is essentially Lemma 4.4 and Lemma 6.1 of [KW19]. We skip the proof and refer the reader to [KW19] for details.

**Step 5:** Identifying the limit.

We define \( \mu_i^m = \beta_i^{-1} \rho_i^m(x, t) dt \) and \( v_i^m(x, t) = \frac{\nabla \rho_i^m(x, t)}{\rho_i^m} - \sum_{j=1}^n a_{ij} \nabla u_j^m(x, t) \) for \( x \in \mathbb{R}^2, t \in [0, 1] \) and \( \tilde{v}_i^m = (v_i^m, 1) \). By step 1, \( \sup_m ||\tilde{v}_i^m||_{L^2((0, 1) \times \mathbb{R}^2, \mu_i^m; \mathbb{R}^3)} < \infty \) and moreover, using (8.5) we see that

\[
||v_i^m||_{L^2((0, 1) \times \mathbb{R}^2, \mu_i^m; \mathbb{R}^2)} \to 0, \text{ as } m \to \infty. \tag{8.9}
\]

In view of step 1 - step 4, we can invoke refined Arzelà-Ascoli’s lemma [AGS05, Proposition 3.3.1] and the arguments used in section 7, to conclude that \( \rho_i^m(\cdot, t) \to \rho_i^\infty(\cdot, t) \) weakly in \( L^1(\mathbb{R}^2) \) for every \( t \in [0, 1], \rho_i^m \to \rho_i^\infty \) weakly in \( L^2((0, 1) \times \mathbb{R}^2) \) and \( u_i^m \to u_i^\infty \) strongly in \( L^2((0, 1); H^1_{loc}(\mathbb{R}^2)) \). Applying Lemma A to \( \mu_i^m \) and \( v_i^m \) we find the existence of a vector field \( v_i^\infty \in L^2((0, 1) \times \mathbb{R}^2, \rho_i^\infty; \mathbb{R}^2) \) such that

\[
\int_0^1 \int_{\mathbb{R}^2} \zeta \cdot v_i^m \rho_i^m \, dx dt \to \int_0^1 \int_{\mathbb{R}^2} \zeta \cdot v_i^{\infty} \rho_i^{\infty} \, dx dt, \text{ for all } \zeta \in C_0^\infty((0, 1) \times \mathbb{R}^2).
\]

Taking into account the convergence results mentioned above, we can identify the vector fields \( v_i^{\infty} \) through

\[
\int_0^1 \int_{\mathbb{R}^2} \zeta \cdot v_i^{\infty} \rho_i^{\infty} = \lim_{m \to +\infty} \int_0^1 \int_{\mathbb{R}^2} \zeta \cdot v_i^m \rho_i^m = \int_0^1 \int_{\mathbb{R}^2} \zeta \cdot \left( \nabla \rho_i^{\infty} - \sum_{j=1}^n a_{ij} \nabla u_j^\infty \rho_i^\infty \right) \tag{8.10}
\]

On the other hand, as a consequence of (8.9) \( v_i^\infty \equiv 0 \) and hence from (8.10) we conclude

\[
\int_0^1 \int_{\mathbb{R}^2} \zeta \cdot \left( \nabla \rho_i^{\infty} - \sum_{j=1}^n a_{ij} \nabla u_j^\infty \rho_i^\infty \right) = 0, \text{ for all } \zeta \in C_0^\infty((0, 1) \times \mathbb{R}^2). \tag{8.11}
\]

**Step 6:** Concluding the proof.

The equation (8.11) is equivalent to saying \( \rho_i^{\infty}(x, t) = \frac{\beta_i e^{\sum_{j=1}^n a_{ij} u_j^\infty(x, t)}}{\int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij} u_j^\infty(x, t)} \, dz} \) and \( u_i^\infty \) satisfies the Liouville system

\[
-\Delta u_i^\infty(x, t) = \frac{\beta_i e^{\sum_{j=1}^n a_{ij} u_j^\infty(x, t)}}{\int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij} u_j^\infty(x, t)} \, dz} \text{ in } \mathbb{R}^2.
\]

It follows from a result of Chipot, Shafrir and the second author [CSW97, Lemma 3.1, Proposition 3.1] that \( u_i^\infty \) has the asymptotic behaviour

\[
\left| \sum_{j=1}^n a_{ij} u_j^\infty(x, t) + \frac{1}{2\pi} \sum_{j=1}^n a_{ij} \beta_j \ln |x| \right| = O(1) \quad \text{if } |x| > R \text{ is large.}
\]
As a result
\[
\int_{\mathbb{R}^2} |x|^2 \rho_1^\infty(x, 0) \, dx = \int_{\mathbb{R}^2} |x|^2 \frac{\beta_1 e^{\sum_{j=1}^n a_{ij} u_j^\infty(x, 0)}}{\sum_{j=1}^n a_{ij} u_j^\infty(x, 0)} \, dx \\
\geq c_0(1) \int_{|x| \geq R} |x|^{(2 - \frac{2}{\pi} \sum_{j=1}^n a_{ij} \beta_j)} \, dx.
\]

Since, by (8.2), the second moment is finite, we must have \(2 - \frac{1}{2\pi} \sum_{j=1}^n a_{ij} \beta_j < -2\) i.e., \(\sum_{j=1}^n a_{ij} \beta_j > 8\pi\) for all \(i \in I\). But according to our assumption
\[
0 = \Lambda_I(\beta) = \sum_{i=1}^n \beta_i (8\pi - \sum_{j=1}^n a_{ij} \beta_j) < 0.
\]

This contradiction assures (8.1) is not possible. In view of the energy bound and Theorem 5.2, remark 5.3, \(\rho^n\) must concentrate, and hence the proof of the theorem is completed. □

9. Appendix

The last section is devoted to the proof of Lemma 4.4, and further we recall a compactness result which has been used frequently in this article. Recall that \(\Psi_R\) is the smooth cut off function introduced in (4.4) and \(\Upsilon\) is the de la Vallée Poussin convex function satisfying all the properties in (4.6). Furthermore, we have the following estimates: the concavity of \(\Upsilon'\) and \(\Upsilon'(0) = 0\) implies
\[
r \Upsilon''(r) \leq \frac{\Upsilon(r)}{r}.
\]
By smoothness of \(\Upsilon\), it is not difficult to see that
\[
\Upsilon'(r) \leq 2 \frac{\Upsilon(r)}{r} \leq 2 (c_1 \Upsilon(r) + c_2).
\]
In the proof we are going to use these estimates frequently. Any universal constant will be denoted by \(C_1, C_2\).

Proof of Lemma 4.4:

Proof. As before the central idea is to use the test function \(\psi_R(x) = \Upsilon(|x|^2) \Psi_R(x)\) in the Euler-Lagrange equation Lemma 4.1(d). We estimate one by one:

\[
\nabla \psi_R(x) = 2x \Upsilon'(|x|^2) \Psi_R(x) + \Upsilon(|x|^2) \nabla \Psi_R(x),
\]
\[
\Delta \psi_R(x) = 4 \Upsilon'(|x|^2) \Psi_R(x) + 4|x|^2 \Upsilon''(|x|^2) \Psi_R(x) \\
+ 4 \Upsilon'(|x|^2) (x \cdot \nabla \Psi_R(x)) + \Upsilon(|x|^2) \Delta \Psi_R(x).
\]

It follows from the properties of \(\Upsilon\) mentioned above and that \(\sup_x |x \nabla \Psi_R(x)| = O(1)\)
\[|\Delta \psi_R(x)| \leq C_1 \Upsilon(|x|^2) + C_2.
\]

As a result
\[
\left| \sum_{i=1}^n \int_{\mathbb{R}^2} \Delta \psi_R(x) \theta_i(x) \right| \leq C_1 \sum_{i=1}^n \int_{\mathbb{R}^2} \Upsilon(|x|^2) \theta_i + C_2.
\]

(9.1)

Now we can write
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\nabla \psi_R(x) - \nabla \psi_R(y)) \cdot (x - y)}{|x - y|^2} \theta_i(x) \theta_j(x) \, dx \, dy = I_1 + I_2 + I_3 + I_4
\]
where

\[
I_1 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{2(x \cdot y) - y \cdot |y|^2}{|x-y|^2} \Psi_R(x) \phi_i(x) \phi_j(y) \, dx \, dy,
\]

\[
I_2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{2y \cdot (x - y)^2}{|x-y|^2} \nabla \Psi_R(x) - \Psi_R(y)) \phi_i(x) \phi_j(y) \, dx \, dy,
\]

\[
I_3 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\nabla \Psi_R(x) - \nabla \Psi_R(y)) \cdot (x - y)}{|x-y|^2} \nabla \Psi_R(x) \phi_i(x) \phi_j(y) \, dx \, dy,
\]

\[
I_4 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(x - y) \cdot \nabla \Psi_R(y)}{|x-y|^2} \nabla \Psi_R(y) \phi_i(x) \phi_j(y) \, dx \, dy.
\]

The integrals \(I_2\) and \(I_3\) are easy to estimate:

\[
|I_2 + I_3| \leq C_1 \sum_{i=1}^n \int_{\mathbb{R}^2} \nabla \Psi_R(x) \phi_i(x) + C_2.
\]  

(9.2)

To estimate \(I_4\) we use the following

\[
|\nabla \Psi_R(x) - \Psi_R(y)| \leq \frac{1}{|x-y|^2} \int_0^1 \frac{d}{ds} \nabla \Psi_R(y) \phi_i(x) + \nabla \Psi_R(x) \phi_j(y) \, ds.
\]

As a result

\[
|I_4| \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_0^1 |s \cdot x + (1-s)y\nabla \Psi_R(y)| \phi_i(x) \phi_j(y) \, ds \, dx \, dy
\]

\[
\leq 2 \int_0^1 \int_{\mathbb{R}^2} \nabla \Psi_R(y)| \phi_i(x) \phi_j(y) \, dx \, dyds
\]

\[
+ 4 \int_0^1 \int_{\mathbb{R}^2} \nabla \Psi_R(y)| \phi_i(x) \phi_j(y) \, dx \, dyds
\]

\[
\leq C_1 \sum_{i=1}^n \int_{\mathbb{R}^2} \nabla \Psi_R(x) \phi_i(x) \, dx + C_2.
\]  

(9.3)

In the third inequality we have used \(s \nabla \Psi_R(s^2) \leq 2s \frac{\Psi(s^2)}{s^2} \leq 2 \Psi(s^2)\) provided \(s > 1\). The last inequality follows from the continuity of \(|x|^2\) and \(\Psi\) and the monotonicity of \(\Psi\). Finally, to estimate \(I_1\) we use the following

\[
(x \cdot \nabla \Psi_R(x) - y \cdot \nabla \Psi_R(y)) \cdot (x - y) = \frac{1}{2} |x-y|^2 (\nabla \Psi_R(x) \phi_i(x) + \nabla \Psi_R(y) \phi_j(y))
\]

\[
+ \frac{1}{2} (|x|^2 - |y|^2) (\nabla \Psi_R(x) \phi_i(x) - \nabla \Psi_R(y) \phi_j(y))
\]

\[
\geq \frac{1}{2} |x-y|^2 (\nabla \Psi_R(x) \phi_i(x) + \nabla \Psi_R(y) \phi_j(y)),
\]
where in the second line we used the convexity of \( \Upsilon \). Using \( \psi_R \) as a test function, the right hand side of the Euler-Lagrange equation Lemma 4.1(d) can be estimated as follows:

\[
\sum_{i=1}^{n} \int_{\mathbb{R}^2} \Delta \psi_R(x) \varphi_i(x) dx - \frac{n}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\nabla \psi_R(x) - \nabla \psi_R(y) \cdot (x-y)}{|x-y|^2} \varphi_i(x) \varphi_j(y) dxdy
\leq C_1 \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \varphi_i(x) dx + C_2
\]

\[
- \frac{n}{4\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{2} (\Upsilon'(|x|^2) + \Upsilon'(|y|^2)) \psi_R(x) \varphi_i(x) \varphi_j(y) dxdy
\leq C_1 \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \varphi_i(x) dx + C_2
\]

Meanwhile the left hand side of Lemma 4.1(d) can be written as

\[
\frac{1}{\tau} \sum_{i=1}^{n} \int_{\mathbb{R}^2} ((\nabla \varphi_i(x) - x) \cdot \nabla \psi_R(\nabla \varphi_i(x))) \eta_i(x) dx
= \frac{1}{\tau} \left[ \sum_{i=1}^{n} \int_{\mathbb{R}^2} (x \cdot \nabla \psi_R(x)) \varphi_i(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^2} (x \cdot \nabla \psi_R(\nabla \varphi_i(x))) \eta_i(x) dx \right]
= \frac{2}{\tau} \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(|x|^2) \psi_R(x) \varphi_i(x) dx
- \sum_{i=1}^{n} \int_{\mathbb{R}^2} (x \cdot \nabla \psi_R(x)) \Upsilon'(\nabla \varphi_i(x) |^2) \psi_R(\nabla \varphi_i(x)) \eta_i(x) dx
+ \frac{1}{\tau} \left[ \sum_{i=1}^{n} \int_{\mathbb{R}^2} (x \cdot \nabla \psi_R(\nabla \varphi_i(x))) \Upsilon(|\nabla \varphi_i(x)|^2) \eta_i(x) dx \right]
= \frac{2}{\tau} \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(|x|^2) \varphi_i(x) dx - \sum_{i=1}^{n} \int_{\mathbb{R}^2} (x \cdot \nabla \varphi_i(x)) \Upsilon'(|\nabla \varphi_i(x)|^2) \eta_i(x) dx + o(1)
\]

(9.4)

As \( R \to \infty \). In the last line we have used the integrability assumption \( \sum_{i=1}^{n} \Upsilon(|\cdot|^2)(\eta_i + \varphi_i) \in L^1(\mathbb{R}^2) \) and the dominated convergence theorem. The justification of passing to the limit will be clear in a moment (see (9.6), (9.7) below). For the time being note that

\[
|x \cdot \nabla \varphi_i(x)) \Upsilon'(|\nabla \varphi_i(x)|^2) \psi_R(\nabla \varphi_i(x))| \leq |x||\nabla \varphi_i(x)||\nabla \psi_R(\nabla \varphi_i(x))| \frac{\Upsilon(|\nabla \varphi_i(x)|^2)}{|\nabla \varphi_i(x)|^2} = O(|x||\nabla \varphi_i(x)| \Upsilon'(|\nabla \varphi_i(x)|^2))
\]

As a consequence we deduce from the Euler-Lagrange equation with \( \psi_R \) as a test function

\[
2 \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(|x|^2) \varphi_i(x) dx \leq 2 \int_{\mathbb{R}^2} (x \cdot \nabla \varphi_i(x)) \Upsilon'(|\nabla \varphi_i(x)|^2) \eta_i(x) dx
+ \tau \left( C_1 \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \varphi_i(x) dx + C_2 \right)
\]

(9.5)
Using the inequality $2a \cdot b \leq |a|^2 + |b|^2$ we can estimate the first term on the right hand side of (9.5) as follows:

$$2 \int_{\mathbb{R}^2} (x \cdot \nabla \varphi_i(x)) \Upsilon'(\|
abla \varphi_i(x)\|^2) \eta_i(x) \, dx \leq \sum_{i=1}^{n} \int_{\mathbb{R}^2} |\nabla \varphi_i(x)|^2 \Upsilon'(\|
abla \varphi_i(x)\|^2) \eta_i(x) \, dx$$

$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(\|
abla \varphi_i(x)\|^2) \eta_i(x) \, dx.$$

$$= \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(\|
abla \varphi_i(x)\|^2) \eta_i(x) \, dx$$

$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(\|
abla \varphi_i(x)\|^2) \eta_i(x) \, dx. \quad (9.6)$$

Let $\Upsilon^*$ be the conjugate convex function to $\Upsilon$. Then the last term in (9.6) can be bound from above by

$$\sum_{i=1}^{n} \int_{\mathbb{R}^2} |x|^2 \Upsilon'(\|
abla \varphi_i(x)\|^2) \eta_i(x) \, dx$$

$$\leq \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \eta_i(x) \, dx + \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon^*(\Upsilon'(\|
abla \varphi_i(x)\|^2)) \eta_i(x) \, dx$$

$$= \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \eta_i(x) \, dx + \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon^*(\Upsilon'(\|
abla \varphi_i(x)\|^2)) \eta_i(x) \, dx \quad (9.7)$$

The properties mentioned in (4.6) ensures that $\Upsilon^*(\Upsilon'(r)) \leq \Upsilon(r)$ for every $r > 0$ (see [LM02, Lemma B.1]). As a result, all the terms in (9.7) are finite, which also justifies the passing to the limit in (9.4). Plugging (9.6) and (9.7) into (9.5) we get

$$\sum_{i=1}^{n} \int_{\mathbb{R}^2} [|x|^2 \Upsilon'(\|
abla \varphi_i(x)\|^2) - \Upsilon^*(\Upsilon'(\|
abla \varphi_i(x)\|^2))] \varphi_i(x) \, dx \leq \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \eta_i(x) \, dx$$

$$+ \tau \left( C_1 \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \varphi_i(x) \, dx + C_2 \right) \quad (9.8)$$

Using the properties of $\Upsilon$ mentioned in (4.6) it can also be shown that $\tau \Upsilon'(r) - \Upsilon^*(\Upsilon'(r)) = \Upsilon(r)$ (see [LM02, Lemma B.1]). Thus we obtain

$$(1 - C_1 \tau) \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \varphi_i(x) \, dx \leq \sum_{i=1}^{n} \int_{\mathbb{R}^2} \Upsilon(|x|^2) \eta_i(x) \, dx + C_2 \tau,$$

which is equivalent to the result claimed in the lemma. \qed

We have used the following compactness of vector fields result whose proof can be found in [AGS05, Theorem 5.4.4]:

**Lemma A (Compactness of vector fields).** Let $\Omega$ be an open set in $\mathbb{R}^N$. If $\{\mu_m\}_m$ is a sequence of probability measures in $\Omega$ narrowly converging to $\mu$ (in duality with $C_b(\mathbb{R}^N)$, continuous bounded functions) and $\{v_m\}_m$ is a sequence of vector fields in $L^2(\Omega, \mu_m; \mathbb{R}^N)$ satisfying

$$\sup_m \|v_m\|_{L^2(\Omega, \mu_m; \mathbb{R}^N)} < +\infty,$$
then there exists a vector field \( v \in L^2(\Omega; \mathbb{R}^N) \) such that

\[
\lim_{m \to \infty} \int_{\Omega} \zeta \cdot v_m \, d\mu_m = \int_{\Omega} \zeta \cdot v \, d\mu, \quad \text{for all } \zeta \in C_c^\infty(\Omega; \mathbb{R}^N)
\]

and satisfy

\[
||v||_{L^2(\Omega, \mu; \mathbb{R}^N)} \leq \liminf_{m \to \infty} ||v_m||_{L^2(\Omega, \mu_m; \mathbb{R}^N)}.
\]

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