ALGEBRAIC $K$-THEORY AND ALGEBRAIC COBORDISM OF ALMOST MATHEMATICS

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Abstract. Faltings [Fal88]; Gabber and Ramero [GR03] introduced almost mathematics. In another way, almost mathematics can be characterized bilocalization abelian category of modules mentioned in Quillen’s unpublished note [Qui96]. Applying the concept of Quillen’s bilocalization to Gabber and Ramero’s work, this paper establishes the almost version of algebraic $K$-theory and cobordism. As a result of almost $K$-theory, we prove that, in the case an almost algebra containing a field, the almost $K$-theory of the almost algebra is a direct factor of the $K$-theory of the field, implying that almost $K$-theory holds the Gersten property. We clarify that an almost $K$-theory is a $K$-theory spectrum of non-unital firm algebras in the sense of Quillen [Qui96]. Furthermore, we obtain that almost algebraic cobordism holds tilting equivalence on the category of zero-section stable integral perfectoid algebras with finite syntomic topology.

1. Introduction

Faltings [Fal88] first introduced almost mathematics, proving almost purity. More concisely, Gabber and Ramero established [GR03] almost ring theory in their textbook: theories of almost modules, almost algebras, and almost homotopical algebra. While almost mathematics has various applications to arithmetic geometry, for example, Scholze’s work perfectoid geometry [Sch12], Quillen [Qui96] mentioned linear algebra over non-unital rings which is the same as almost mathematics. Quillen’s work is more conceptional in the sense of using categorical language: in his work, almost mathematics is characterized as bilocalization of an abelian category of modules.

In almost mathematics, the theory of derived categories is a bit complicated. For instance, exact sequences in the almost-world are not exact in the corresponding module category in general. By considering the derived category of a bilocalization of the category of modules, this paper provides algebraic $K$-theory and cobordism of almost mathematics. More precisely, we define the derived category (or stable $\infty$-category) of almost perfect complexes, enabling us to obtain the $K$-theory of almost modules. In this paper, we technically introduce a new $K$-theory $K^+(A)$, for any almost algebra $A$, which decomposes into almost $K$-theory of $A$ and the $K$-theory of almost acyclic parts. Using the properties of $K^+(A)$, we prove the almost Gersten

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property (Theorem 3.17) and an equivalence of almost $K$-theories between almost algebras and the corresponding non-unital algebras (Theorem 3.21).

Furthermore, defining almost finite syntomic algebras, we can get the defining algebraic cobordism in almost mathematics similar to Elmanto, Hoyois, Khan, Sosnilo, and Yakerson’s work [EHK+20b] via framed correspondences.

This paper’s another result is the tilting equivalence between the almost algebraic cobordism of zero-section stable integral perfectoid algebras by a similar argument to the author’s previous work [Kat22] about the algebraic cobordism of non-unital integral perfectoid algebras (Theorem 4.23). This result lets us expect some analogy between non-unital and almost algebra. Here the condition zero-section stability is defined as follows: Let $i : \{0\} \rightarrow \Delta^\infty = \text{colim} \Delta^n$ denote the canonical injection and $\pi : \Delta^\infty \rightarrow \Delta^0$ the projection. One has a Quillen adjunction

$$
\pi_* \circ i_* : (\text{Set}_\Delta)_{/\Delta^0} \rightleftarrows (\text{Set}_\Delta)_{/\Delta^0} : \pi^* \circ i^*
$$

of simplicial model categories. A simplicial set $X$ is zero-section stable if the morphism $\pi \circ i \circ \eta : X \rightarrow \pi_*(i_*(X))$ induced by the unit map $\eta : X \rightarrow (i^* \circ \pi^*)(\pi_* \circ i_*(X))$ is a weak equivalence. Let $\mathbb{A}_S^*$ denote the cosimplicial motivic space defined by $\mathbb{A}_S^*(n) = \mathbb{A}_S^n$ for each $n \geq 0$. Then the singular functor $\text{Hom}_S(\mathbb{A}_S^*, -) : \text{MS} \rightarrow \text{MS}$ admits a left adjoint. We say that $X$ is locally zero-section stable if the simplicial sheaf $\text{Hom}_S(\mathbb{A}_S^*, X)$ is stalk-wise zero-section stable. Let $\text{Msp}^0$ denote the full subcategory of zero-section motivic spectra. Then the inclusion functor: $\text{Msp}^0 \rightarrow \text{Msp}$ admits a left adjoint $Z_0 : \text{Msp} \rightarrow \text{Msp}^0$ which is defined to be the homotopy colimit

$$Z_0(X) = \lim_{\longrightarrow} (\pi_* \circ i_*) \circ \cdots \circ (\pi_* \circ i_*)(X).$$

We call $Z_0(X)$ the zero-section stabilization of $X$.

This paper is organized as follows: In Section 2, we provide a brief review of Quillen’s work [Qui96] bilocalization by the Serre subcategory of almost zero modules. Section 3 defines almost perfect complexes by using derived functors of categorical equivalences between full subcategories spanned by local objects and spanned by colocal objects. Via the equivalence, we prove the comparison theorem of almost $K$-theories (Theorem 3.14). In Section 4 on the viewpoint of non-unital algebra, we study the left adjoint of the localization functor from the category of algebras to one of almost algebras. We define almost finite syntomic algebras, enabling us to establish the almost version of algebraic cobordism by following [EHK+20b]. In the final part of this paper, localizing the stable $\infty$-category of motivic spectra by zero-section stable finite syntomic morphisms, we prove the tilting equivalence of the almost algebraic cobordism for zero-section stable integral perfectoid algebras.

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In this paper, we fix an unital commutative ring $V$. In this section; we review Quillen’s unpublished note [Qui96] in the almost mathematics language, giving the proof of the propositions.

Let $m$ be an idempotent ideal of $V$. A $V$-module $M$ is said to be almost zero if $mM = 0$. A $V$-homomorphism $f : M \to N$ of $V$-modules is called an almost isomorphism if both the kernel and the cokernel of $f$ are almost zero.

**Lemma 2.1** ([Qui96]). Let $V$ be an unital commutative ring and $m$ an idempotent ideal. Then $m \otimes_V M = 0$ if and only if $M$ is almost zero.

**proof.** The only if direction is clear. Assume $mM = 0$. For $a \otimes x \in m \otimes_V M$ $(a \in m, x \in M)$, there exists $(a_i, b_i) \in m^2$ such that $a = \sum_i a_i b_i$. Hence one has $a \otimes x = (\sum_i a_i b_i) \otimes x = \sum_i a_i b_i \otimes x = 0$. □

**Proposition 2.2** ([Qui96]). Let $V$ be an unital ring, $m$ an idempotent ideal of $V$, and $M$ a $V$-module. Then the canonical morphisms $\mu : m \otimes_V M \to M$ and $\mu' : M \to \text{Hom}_V(m, M)$ are both almost isomorphisms.

**proof.** Since the abelian category $\text{Mod}_V$ is both enough projective and injective, the kernels and the cokernels; $\text{Tor}_i^V(V/m, M)$ and $\text{Ext}_i^V(V/m, M)$ for $i = 0, 1$, are killed by $m$. □

**Definition 2.3.** In the situation of Proposition 2.2, we say that $M$ is firm if $\mu$ is an isomorphism and $M$ is closed if $\mu'$ is an isomorphism.

In particular, an injective $V$-module $Q$ is closed if and only if it is $m$-torsion free (i.e. $\text{Hom}_V(V/m, Q) = 0$).

**Corollary 2.4** ([Qui96]). Let $m$ be an idempotent ideal of an unital ring $V$. Then the map $\mu_m : m \otimes_V m \to m$ is an almost isomorphism and $\mu_m \otimes m : m \otimes_V m \otimes_V m \to m \otimes_V m$ already an isomorphism.

**proof.** This follows from Lemma 2.1 and Proposition 2.2. □

**Corollary 2.5** ([Qui96] Proposition 4.1 and Proposition 5.3). Let $V$ be an unital ring and $m$ an idempotent ideal of $V$. Write $\tilde{m} = m \otimes_V m$. For any $V$-module $M$, $\tilde{m} \otimes_V M$ is firm and $\text{Hom}(\tilde{m}, M)$ closed.

**proof.** By Corollary 2.4, the counit $\mu_{\tilde{m} \otimes_V M} : m \otimes_V \tilde{m} \otimes_V M \to \tilde{m} \otimes_V M$ is an isomorphism, inducing canonical isomorphisms $\text{Hom}(\tilde{m}, M) \simeq \text{Hom}(m \otimes_V \tilde{m}, M) \simeq \text{Hom}(m, \text{Hom}(\tilde{m}, M))$. □

Let $M$ be a closed module and an injective hall $M \to Q$. Then, by the right exactness of $\text{Hom}_V(m, -)$, one has an injection $M \to \text{Hom}_V(\tilde{m}, Q)$. Then $\text{Ext}_1^V(N, \text{Hom}_V(\tilde{m}, Q)) = \text{Ext}_1^V(N, \text{Hom}_V(m, \text{Hom}(\tilde{m}, M))) = \text{Ext}_1^V(N, \text{Hom}_V(\tilde{m}, Q))$.
$\text{Ext}^i_V(N, \mathbb{R}\text{Hom}_V(\tilde{m}, Q)) \approx \text{Ext}^i_V(N \otimes_V \tilde{m}, Q) = 0$ implies that $\text{Hom}_V(\tilde{m}, Q)$ is injective. Hence, a $V$-module is closed if and only if it is a kernel of a homomorphism of closed injective modules.

Let $S$ denote the Serre subcategory of $\text{Mod}_V$ spanned by almost zero modules. We say that a $V$-module $M$ is $S$-colocal (resp. $S$-local) if for any almost isomorphism $f : N_1 \to N_2$, the induced map

$$f_* : \text{Hom}_V(M, N_1) \to \text{Hom}_V(M, N_2) \quad (\text{resp. } f^* : \text{Hom}_V(N_2, M) \to \text{Hom}_V(N_1, M))$$

is an isomorphism. The following well-known proposition is crucial for almost mathematics. We explain the proof:

**Proposition 2.6** ([Qui96] Proposition 4.1 and Proposition 5.3). *Let $V$ be an unital ring, $m$ an idempotent ideal of $V$ and $M$ a $V$-module. Let $S$ denote the Serre subcategory of almost zero modules of $\text{Mod}_V$. Then the following conditions are equivalent:

1. The $V$-module $M$ is $S$-colocal.
2. The $V$-module $M$ is firm.
3. The $V$-module $M$ satisfies that $\text{Ext}^i_V(M, N) = \text{Ext}^i_V(M, M) = 0$ for any almost zero module $N$.

Similarly, the following conditions are equivalent:

1. The $V$-module $M$ is $S$-local.
2. The $V$-module $M$ is firm.
3. The $V$-module $M$ satisfies that $\text{Ext}^i_V(M, M) = \text{Ext}^i_V(M, N) = 0$ for any almost zero module $N$.

**proof.** Assume that $M$ is $S$-local. By Proposition 2.6, the induced map

$$\mu_* : \text{Hom}(M, m \otimes_V M) \to \text{Hom}(M, M)$$

is bijective. Hence $\mu : m \otimes_V M \to M$ is surjective, in particular $M = mM$. Since $M$ is a direct factor of $m \otimes_V M$, the composition $M \to m \otimes_V M \to \tilde{m} \otimes_V M$ is split. A direct factor $M$ of the firm module $\tilde{m} \otimes_V M$ is also firm.

Assume that $M$ is firm. Let $N$ be an almost zero module. We prove that $\text{Ext}^i_V(M, N) = 0$ for $i = 0, 1$. The condition $mN = 0$ implies that $N$ is also $V/m$-module. The adjunction between the derived categories:

$$(V/m) \otimes^L_V - : D(V) \rightleftarrows D(V/m) : \mathbb{R}\text{Hom}_V(V/m, -)$$

induces a canonical equivalence $\mathbb{R}\text{Hom}_V((V/m) \otimes^L_V M, N) \approx \mathbb{R}\text{Hom}_V(M, N)$ and the spectral sequence

$$E_2^{pq} = \text{Ext}^p_{V/m}(\text{Tor}^V_q(V/m, M), N) \implies \text{Ext}^{p+q}_V(M, N)$$

where $E_2^{pq} = 0$ for $q = 0, 1$. Hence one has $\text{Ext}^0_V(M, N) = \text{Ext}^1_V(M, N) = 0.$
Assume that condition (3). Let \( f : N_1 \to N_2 \) be a morphism of \( V \)-modules with almost zero modules \( N' = \text{Ker} \, f \) and \( N'' = \text{Coker} \, f \). Write \( N = \text{Im} \, f \) and consider an exact sequences:

\[
0 \to N' \to N_1 \to N \to 0 \quad \text{and} \quad 0 \to N \to N_2 \to N'' \to 0.
\]

We obtain that the induced map \( f_* : \text{Hom}_V(M, N_1) \to \text{Hom}_V(M, N) \to \text{Hom}_V(M, N_2) \) is composition of isomorphisms by the assumption (3).

Next, assume that \( M \) is \( \mathcal{S} \)-colocal. Then the almost isomorphism : \( \mathfrak{m} \to V \) induces an isomorphism \( M \to \text{Hom}_V(\mathfrak{m}, M) \). Therefore \( M \) is closed.

If \( M \) is closed, there exist an injective resolution \( 0 \to M \to Q^0 \to Q^1 \to Q^2 \to \cdots \), where \( Q^i \) is closed for \( i = 0, 1 \). By the canonical isomorphism \( \text{Hom}_V(N, \text{Hom}_V(\mathfrak{m}, M)) \cong \text{Hom}_V(N \otimes_V \mathfrak{m}, M) \), for any almost zero module \( N \), \( \text{Hom}_V(N, Q^i) \cong \text{Hom}_V(N \otimes_V \mathfrak{m}, Q^i) = 0 \) (i = 0, 1).

Therefore we obtain \( \text{Ext}^i_V(N, M) = 0 \) for \( i = 0, 1 \).

By the similar argument of the above (3) to (1), the implication (3)' to (1)' holds.

\[\square\]

**Theorem 2.7** ([Qui96] Theorem 4.5 and Theorem 5.6). Let \( V \) be an unital ring and \( \mathfrak{m} \) an idempotent ideal of \( V \). Let \( \mathcal{S} \) denote the Serre subcategory of almost zero modules of \( \text{Mod}_V \) and \( \text{alMod}_V \) the localization of \( \text{Mod}_V \) by \( \mathcal{S} \). Then the localization functor \( (-)^\mathfrak{m} : \text{Mod}_V \to \text{alMod}_V \) is also a colocalization. Furthermore, the left adjoint is isomorphic to the functor \( \mathfrak{m} \otimes_V - : \text{alMod}_V \to \text{Mod}_V \) whose essential image is the full subcategory of film modules, and the right adjoint is isomorphic to \( \text{Hom}_V(\mathfrak{m}, -) : \text{alMod}_V \to \text{Mod}_V \) whose essential image is the full subcategory of closed modules.

\[\square\]

Let \( V \) be an unital ring and \( \mathfrak{m} \) an idempotent ideal of \( V \). An almost \( V \)-module is an object of \( \text{alMod}_V \) in Theorem 2.7 and an almost \( V \)-algebra a commutative algebra object object of \( \text{alMod}_V \). Let \( A \) be a unital \( V \)-algebra and \( (-)^\mathfrak{m} : \text{Mod}_A \to \text{alMod}_A \) denote the bilocalization of \( \text{Mod}_A \) by the Serre subcategory of almost zero \( A \)-modules.

**Corollary 2.8.** Let \( V \) be an unital ring and \( \mathfrak{m} \) an idempotent ideal of \( V \) and \( A \) a unital \( V \)-algebra. Let \( \text{Mod}_A^\mathfrak{m} \) denote the full subcategory of \( \text{Mod}_A \) spanned by firm \( A \)-modules and \( \text{Mod}_A^\mathfrak{cl} \) by closed \( A \)-modules. Then those adjunctions

\[
\mathfrak{m} \otimes_V (-) : \text{alMod}_A \rightleftarrows \text{Mod}_A^\mathfrak{m} : (-)^\mathfrak{m}
\]

\[
(-)^\mathfrak{m} : \text{Mod}_A^\mathfrak{cl} \rightleftarrows \text{alMod}_A : \text{Hom}_V(\mathfrak{m}, -)
\]

are categorical equivalences. Assume that \( \mathfrak{m} \) is a flat \( V \)-module. The functor \( \mathfrak{m} \otimes_V (-) \) induces categorical equivalences of the derived categories:

\[
\mathfrak{m} \otimes_V (-) : D(\text{alMod}_A) \rightleftarrows D(\text{Mod}_A^\mathfrak{m}) : (-)^\mathfrak{m}
\]

\[
\mathfrak{m} \otimes_V (-) : D(\text{Mod}_A^\mathfrak{cl}) \rightleftarrows D(\text{Mod}_A^\mathfrak{m}) : \mathbb{R}\text{Hom}_V(\mathfrak{m}, -).
\]

\[\square\]
Write \((-): \tilde{m} \otimes_V \text{Hom}_V(\tilde{m}, (-)) : \text{alMod}_A \to \text{Mod}_A^\otimes\). Since \(\tilde{m}\) is flat, the functor \((-):\) is left exact. Furthermore, if \(f : M \to N\) is almost surjective, one has \((\text{Coker} f)_*: = 0\), being \((-):\) is right exact. Hence \((-):\) is an exact functor.

3. Almost algebraic \(K\)-theory

3.1. Almost perfect complexes. In this section, we always assume that \(\tilde{m}\) is a flat \(V\)-module.

Definition 3.1. An \(A\)-module \(M\) is almost finitely generated (resp. almost finitely presented) if for any filtered inductive system \((N_\alpha)\) of firm \(A\)-module, the canonical map

\[
\lim \rightarrow \text{Hom}_{\text{Mod}_A^\otimes}(\tilde{m} \otimes_V M, N_\alpha) \to \text{Hom}_{\text{Mod}_A^\otimes}(\tilde{m} \otimes_V M, \lim \rightarrow N_\alpha)
\]

is almost injective (resp. an almost isomorphism).

Definition 3.2. An \(A\)-module \(P\) is almost projective if the functor \(\text{Hom}_{\text{alMod}_A}(P, -) : \text{alMod}_A \to \text{alMod}_A\) is exact, equivalently, \(\tilde{m} \otimes_V \text{Hom}_{\text{Mod}_A^\otimes}(\tilde{m} \otimes_V P, -) : \text{Mod}_A^\otimes \to \text{Mod}_A^\otimes\) is exact.

Proposition 3.3. Let \(P\) be a finitely generated projective \(A\)-module. Then \(\tilde{m} \otimes_V P\) is an almost finitely presented projective \(A\)-module.

proof. Since \(P\) is a direct factor of finitely generated free \(A\)-module, it is sufficient to check that \(\tilde{m} \otimes_V A\) is an almost finitely presented projective \(A\)-module. Then the source of the canonical morphism (3.1) is isomorphic to \(\lim \rightarrow (N_\alpha)_*\), and the target \((\lim \rightarrow N_\alpha)_*\). Therefore, (3.1) is an isomorphism after tensoring with \(\tilde{m}\), implying that \(\tilde{m} \otimes_V A\) is almost finitely presented. For any firm module \(M\), both the morphisms of the diagram \(M \leftarrow \tilde{m} \otimes_V M \to \tilde{m} \otimes_V (M_*):\) are isomorphisms. Hence the functor \(\text{Hom}_{\text{Mod}_A^\otimes}(\tilde{m} \otimes_V A, -) = \tilde{m} \otimes (-)_*\) is clearly exact. □

Let \(\text{alPMod}_A\) denote the full subcategory of \(\text{Mod}_A\) spanned by those objects \(\tilde{m} \otimes_V P\) where \(P\) is a finitely generated projective \(A\)-module.

Proposition 3.4. The firm module \(\tilde{m} \otimes_V A\) is a compact object of \(D(\text{Mod}_A^\otimes)\). □

We define the \(K\)-theory of almost mathematics by using Thomason–Trobaugh’s [TT90] method: For any algebra \(A\), \(\text{Perf}(A)\) denotes the full triangulated subcategory of \(D(\text{Mod}_A^\otimes)\) generated by an object \(A\), and an object of \(\text{Perf}(A)\) is called a perfect complex. Any full triangulated subcategory of \(D(\text{Mod}_A^\otimes)\) generated by some objects is assumed to be idempotent complete throughout this section. We define an almost version of perfect complexes:

Definition 3.5. Let \(\text{APerf}(A)\) denote the full triangulated subcategory of \(D(\text{Mod}_A^\otimes)\) generated by \(\tilde{m} \otimes_V A\). An almost perfect complex \(E\) is an object of \(D(\text{alMod}_A)\) satisfying \(\tilde{m} \otimes_V E \in \text{APerf}(A)\). That is, The category \(\text{Perf}^{\text{al}}(A)\) of almost perfect \(A\)-complexes is defined to be the pullback \(D(\text{alMod}_A) \times_{D(\text{Mod}_A^\otimes)} \text{APerf}(A)\) of triangulated categories.
Definition 3.6. Let $A$ be an almost $V$-algebra and $K^{al}(A)$ denote the $K$-theory spectrum of the triangulated category of $A$Perf($A$). We call $K^{al}(A)$ the almost $K$-theory spectrum of $A$.

We will characterize almost $K$-theory.

Proposition 3.7. The exact functor $\tilde{m} \otimes_V (-) : \text{Mod}_A \to \text{Mod}_A^{al}$ induces an essential surjective functor form the triangulated category of perfect $A$-complexes to the one of almost perfect $A$-complexes. □

Lemma 3.8. Let $P$ be a finitely generated projective $A_{\ast}$-module. Then $P$ is closed.

proof. Clearly $A_{\ast}^n$ is closed for any integer $n \geq 0$. By the assumption, $P$ is a direct factor of some $A_{\ast}^n$, implying that $P$ is closed. □

By Proposition 2.6, $\tilde{m} \otimes_V (-)$ and $\text{Hom}_V(\tilde{m}, -)$ invert almost isomorphisms, implying that one has $\tilde{m} \otimes_V (\text{Hom}_V(\tilde{m}, A)) \cong \tilde{m} \otimes_V A$ and $\text{Hom}_V(\tilde{m}, \tilde{m} \otimes_V A) \cong \text{Hom}_V(\tilde{m}, A)$.

Remark 3.9. While the triangulated category $D(\text{Perf}(A_{\ast}))$ is contained in $D(\text{Mod}_A^{cl})$, which is distinct to $D(\text{Mod}_A^{al})$. An $A_{\ast}$-module $\tilde{m} \otimes_V A_{\ast}$ is not closed in general. Indeed, if $A_{\ast}$ is an unital ring, $\text{Hom}_V(\tilde{m}, \tilde{m} \otimes_V A_{\ast}) \cong A_{\ast} \neq \tilde{m} \otimes_V A_{\ast} \neq 1$.

3.2. Localization by almost quasi-isomorphisms. Let $f : E \to E'$ be a morphism of $A$-complexes. Then $f$ is said to be an almost quasi-isomorphism if the induced morphism $\tilde{m} \otimes_V f : \tilde{m} \otimes_V E \to \tilde{m} \otimes_V E'$ is a quasi-isomorphism.

Proposition 3.10. Let $f : E \to E'$ be a morphism of $A$-complexes. Then the following conditions are equivalent:

(1) The morphism $f$ is an almost quasi-isomorphism.
(2) The firm complex $\tilde{m} \otimes_V \text{cone} f$ is acyclic.

proof. The proof is straightforward. □

Let $A$ be an almost $V$-algebra and Perf($A$) denote the triangulated category of perfect complexes of $A$-modules. Choosing classes of weak equivalences on Perf($A$) defines the following $K$-theory spectra:

Definition 3.11. We define $K$-theory spectra as the following:

- The almost Thomason–Trobaugh $K$-theory $K^{aTT}(A)$ is the Waldhausen $K$-theory of Perf($A$) whose weak-equivalences are almost quasi-isomorphisms.
- The Thomason–Trobaugh $K$-theory $K^{TT}(A_{m})$ is the Waldhausen $K$-theory of the category Perf($A_{m}$) spanned by almost acyclic perfect $A$-complexes.

To compare almost $K$-theories mentioned above, we recall the following the approximation theorem:
Theorem 3.12 ([Wal85], Theorem 1.6.7 (Approximation theorem)). Let $C$ and $D$ be small Waldhausen categories whose weak equivalences satisfy the 2-out-of-3 property and $F : C \to D$ an exact functor. If the following conditions hold:

1. The category $C$ is closed under cylinder objects.
2. A morphism in $f$ is a weak equivalence in $C$ if and only if $F(f)$ is a weak equivalence in $D$.
3. The functor $F$ is a coCartesian fibration. That is, for any object $c$ of $C$ and any map $x : F(c) \to d$ in $D$, there exists a map $a : c \to c'$ and a weak equivalence $x' : F(c') \to d$ such that $x = x' \circ F(a)$.

Then the induced morphism $K(F) : K(C) \to K(D)$ is a weak equivalence. □

Lemma 3.13. Those triangulated categories in Definition 3.11 admit cylinder objects.

proof. Since $m \otimes_V -$ and $- \otimes_V m$ are exact and commute with shift and the mapping cone functor, the all of $\text{Perf}(A)$, $\text{APerf}(A)$, and $\text{Perf}^{\text{al}}(A)$, admit the usual mapping cylinder complexes $\text{cyl}(f : E \to F) = \text{cone}(g : \text{cone} f[-1] \to E)$ satisfying the cylinder axioms. □

The following theorem holds without the flatness of $m$:

Theorem 3.14. Assume that $\mathfrak{m}$ is a flat $V$-module. Then the functors $\text{Perf}(A_+) \to \text{Perf}^{\text{al}}(A) \to \text{APerf}(A)$ induces weak equivalences

$$K(\mathfrak{m} \otimes_V (-)) : K^{\text{alTT}}(A_+) \to K(\text{Perf}^{\text{al}}(A)) \to K^{\text{al}}(A)$$

of $K$-theory spectra.

proof. Clearly, $\mathfrak{m} \otimes_V (-)$ is exact and sends all almost quasi-isomorphisms to quasi-isomorphism of almost perfect complexes. By definition of almost quasi-isomorphisms, the condition (2) in Theorem 3.12 is satisfied. By the definition of $\text{Perf}^{\text{al}}(A)$ and $\text{APerf}(A)$, the condition (3) in Theorem 3.12 holds, implying that the canonical morphism $K(\text{Perf}^{\text{al}}(A)) \to K^{\text{al}}(A)$ is an equivalence. Let $E$ be a perfect $A$-complex and $F$ be a firm almost perfect complex. Given an morphism $f : \mathfrak{m} \otimes_V E \to F \simeq \mathfrak{m} \otimes_V F$, we obtain a morphism $\text{Hom}(\mathfrak{m}, f) : \text{Hom}_V(\mathfrak{m}, E) \to \text{Hom}_V(\mathfrak{m}, F)$. Then $\text{Hom}_V(\mathfrak{m}, E)$ and $\text{Hom}_V(\mathfrak{m}, F)$ are perfect complexes of $A_+$-modules. Therefore, $K^{\text{alTT}}(A_+) \to K^{\text{al}}(A)$ is also an equivalence. □

3.3. Splitting of almost $K$-theory. In most cases, $m \otimes V A$ or $A/m A$ is not finitely presented over $A$, in particular, being not an object of $\text{Perf}(A)$. Let $\text{Perf}^+ (A)$ denote the stable full subcategory of the triangulated category $D(\text{Mod}_A)$ generated by $A$ and $\mathfrak{m} \otimes V A$ and $K^+ (A)$ denote the $K$-theory spectrum of $\text{Perf}^+ (A)$. Then the functor $\mathfrak{m} \otimes_V (-) : \text{Perf}^+ (A) \to \text{Perf}^+ (A)$ is categorically idempotent, inducing a homotopy projector $K(\mathfrak{m} \otimes_V (-)) : K^+ (A) \to K^+ (A)$. Furthermore, the essential image of $\mathfrak{m} \otimes_V (-) : \text{Perf}^+ (A) \to \text{Perf}^+ (A)$ is equivalent to $\text{APerf}(A)$. Thus, the canonical map $K^{\text{al}}(A) \to K^+ (A)$ is homotopically split. We obtain the following:
Theorem 3.15. Let $A$ be an almost $V$-algebra. Assume that $\tilde{m}$ is a flat $V$-module. Then one has the following weak equivalences:

$$K^+(A) \simeq K^{al}(A) \oplus K(\text{Perf}^+(A)^m).$$

\[\square\]

Corollary 3.16. The projection $K^+(A) \to K(\text{Perf}^+(A)^m)$ is induced by the functor

$$\varphi : E \mapsto \text{cone}(\tilde{m} \otimes_V E \to E).$$

proof. For any complex $E$, one has $\tilde{m} \otimes \varphi(E) \simeq \text{cone}(\tilde{m} \otimes_V E \to \tilde{m} \otimes_V E)$, being acyclic. If $E$ is almost acyclic, one has a chain of quasi-isomorphisms $\varphi(E) \simeq \text{cone}(0 \to E) \simeq E$. Therefore $\varphi$ induces a homotopic identity on $K(\text{Perf}^+(A)^m)$ and one has $\varphi(\varphi(E)) \simeq \varphi(E)$. Furthermore, $\varphi(\tilde{m} \otimes_V E)$ are quasi-isomorphic to an acyclic complex $\text{cone}(\tilde{m} \otimes_V E \to \tilde{m} \otimes_V E)$, the functor $(\tilde{m} \otimes_V (-)) \oplus \varphi$ induces a homotopic identity map. \[\square\]

Using Theorem 3.15, we can obtain the almost Gersten property of almost $K$-theory:

Theorem 3.17. Assume that there is a field $F$ containing $V$. For any almost $V$-algebra $A$ and $A \otimes_V F$-algebra $C$, let $\overline{K^m}(C)$ denote the cofiber of the composition $K^{al}(A) \to K^+(A) \to K(C)$ induced by the functor $(-) \otimes_A C : \text{Mod}_A \to \text{Mod}_C$. Then we have a homotopical decomposition

$$K(C) \simeq K^{al}(A) \amalg \overline{K^m}(C)$$

of $K$-theory spectra. In particular, the pullback $K(- \otimes_V F) : K^{al}(A) \to K(A \otimes_V F)$ is homotopically split injective.

proof. Since the canonical morphism $\tilde{m} \otimes_V C \to C$ is an isomorphism, the $\text{Perf}(C) \to \text{Perf}^+(C)$ functor is a categorical equivalence of triangulated categories. By definition of $\overline{K^m}(C)$, the $K$-theory spectrum $\overline{K^m}(C)$ is equivalent to the homotopy coCartesian product $K(C) \amalg K^+(A) \amalg K^{al}(A)$. Therefore, by the formal argument, we have a chain of equivalences:

$$\overline{K^m}(C) \amalg K^{al}(A) \simeq (K(C) \amalg K^+(A) \amalg K^{al}(A)) \amalg K^{al}(A)$$

$$\simeq K(C) \amalg K^+(A) \amalg K^{al}(A) \simeq K(C).$$

\[\square\]

3.4. The left adjoint of the localization $(-)^a : \text{CAlg}(\text{Mod}_A) \to \text{CAlg}(\text{alMod}_A)$ from the non-unital algebra viewpoint. We will review almost mathematics from the theory of non-unital algebra. We assume that $1 \notin m$. Let $A$ be an almost $V$-algebra and $B$ an $A$-algebra. Then $B_1 := \tilde{m} \otimes_V \text{Hom}_A(\tilde{m}, B)$ is a (non-unital) $B$-algebra. Consider the unitalization functor

$$V \oplus (-) : \text{CAlg}^{nu}(\text{Mod}_A) \to \text{CAlg}(\text{Mod}_A)$$
Then the direct sum $V \oplus B_1$ has a canonical unital ring structure defined by

$$(v, b) \cdot (v', b') = (vv', vb' + v'b + bb')$$

for $v, v' \in V$ and $b, b' \in B$. Note the functor $V \oplus - : \text{CAlg}^{\text{nu}}(\text{Mod}_A) \to \text{CAlg}(\text{Mod}_A)$ induces a categorical equivalence between $\text{CAlg}^{\text{nu}}(\text{Mod}_A)$ and the category of augmented commutative rings $\text{CAlg}(\text{Mod}_A)_V$. The right adjoint is the augmented ideal functor $\text{Ker}(- \to V) : \text{CAlg}(\text{Mod}_A)_V \to \text{CAlg}^{\text{nu}}(\text{Mod}_A)$, becoming the quasi-inverse of $V \oplus -$.

Let $B_!!$ denote the coCartesian product $V \oplus \tilde{m}B_!$. Then one has an exact sequence

$$\tilde{m} \to V \oplus B_1 \to B_!! \to 0,$$

where the left map is almost injective. After tensoring with $\tilde{m}$, indeed, one has a trivial split exact sequence:

$$0 \to \tilde{m} \to \tilde{m} \oplus (\tilde{m} \otimes_V B_1) \to \tilde{m} \otimes_V B_!! \to 0,$$

implying that $\tilde{m} \otimes_V B_1 \to \tilde{m} \otimes_V B_!!$ is an isomorphism. Therefore the functor $(\cdot)_!! : \text{CAlg}(\text{Mod}_A) \to \text{CAlg}(\text{alMod}_A)$ is left adjoint to the localization $(\cdot)_! : \text{CAlg}(\text{Mod}_A)_V \to \text{CAlg}(\text{alMod}_A)$. If $\tilde{m} \to V \oplus (\tilde{m} \otimes_V B_1)$ is exactly injective, $B$ is called an exact almost $V$-algebra \cite[Definition 2.2.27]{GR}. In the case $\tilde{m} \simeq m$, in particular, that $m$ is flat, $B$ is always exact \cite[Remark 2.2.28]{GR}.

**Remark 3.18.** In Hovey’s Smith ideal theory \cite{HoveySmith}, a $V$-homomophism $j : \tilde{m} \to V$ is regarded a Smith ideal of $V$. For any $A$-algebra $B$, the push-forward product

$$j \square (\eta_B : V \to B) = (V \oplus (\tilde{m} \otimes_V B)) \to B$$

is the canonical ring homomorphism $B!! \to B$. Since $j$ induces an isomorphism $\text{id}_{\tilde{m}} \square j \simeq \text{id}_{\tilde{m}}$, the induced homomorphism

$$(\tilde{m} \otimes_V B_!! \to \tilde{m} \otimes_V B) = \text{id}_{\tilde{m}} \square (V \to B) \simeq \text{id}_{\tilde{m}} \square (V \to B) = (\tilde{m} \otimes_V B \to \tilde{m} \otimes_V B)$$

is an isomorphism, entailing that the canonical ring homomorphism $B!! \to B$ is an almost isomorphism of $A$-algebras.

**Lemma 3.19.** Let $E$ be an almost perfect $A!!$-complex. Then $E$ is also an almost perfect $V \oplus \tilde{m} \otimes_V A$-complex.

**proof.** Write $B = V \oplus \tilde{m} \otimes_V A$. Then $\tilde{m} \otimes_V B = \tilde{m} \oplus \tilde{m} \otimes_V A$. Therefore $\tilde{m} \otimes_V A \simeq \tilde{m} \otimes_V A!!$ is a direct factor of $\tilde{m} \otimes_V B$. Since $\text{APerf}(B)$ is closed under retracts, $\tilde{m} \otimes_V A!!$ is an object of $\text{APerf}(B)$. □

**Corollary 3.20.** For any almost $V$-algebra $A$, one has an decomposition:

$$K^{\text{al}}(V \oplus \tilde{m} \otimes_V A) \simeq K^{\text{al}}(V) \oplus K^{\text{al}}(A).$$

□
Write $B = V \oplus \tilde{m} \otimes V A$. Let $K^+ (\tilde{m} \otimes V A)$ denote the homotopy fiber of the map $K^+ (- \otimes_B V : K^+ (B) \to K^+ (V)$. For any $V$-complex $E$, one has an isomorphism $E \to E \otimes_V B \otimes_B V$ induced by the identity composition $V \to B \to V$. Therefore the canonical map $K^+ (B) \to K^+ (V)$ has a homotopically section. Therefore we have a homotopically decomposition:

$$K^+ (B) \simeq K^+ (V) \oplus K^+ (\tilde{m} \otimes V A).$$

By Corollary 3.20, one has a weak equivalence $K^\text{al} (\tilde{m} \otimes V A) \simeq K^\text{al} (A)$. Let $\text{Perf}^+ (B, \tilde{m} \otimes V A)$ denote the full subcategory of $\text{Perf} (B)$ spanned by $V$-acyclic complexes. That is. A $B$-complex $E$ is $V$-acyclic if $E \otimes_B V$ is acyclic. Note that for any $B$-complex $E$, $E \otimes_B (\tilde{m} \otimes V A)$ is already firm, implying any $V$-acyclic complex is quasi-isomorphic to a firm complex. Therefore one has $K^+ (B, \tilde{m} \otimes V A) = K^\text{al} (B, \tilde{m} \otimes V A)$. Furthermore, one has a canonical isomorphism: $E \otimes_B (\tilde{m} \otimes V A) = \tilde{m} \otimes_V (E \otimes_B A)$, implying there is a canonical functor $(-) \otimes_B A : \text{Perf}^+ (B, \tilde{m} \otimes V A) \to \text{APerf} (A)$. Further, we call a morphism $f : E \to E'$ of $B$-complexes $- \otimes_B V$-quasi-isomorphism if the induced morphism $f \otimes_B V : E \otimes_B V \to E' \otimes_B V$ is a quasi-isomorphism.

The main result of almost $K$-theory is that the almost $K$-theory of $A$ is represented by the $K^+$-theory of the non-unital algebra $\tilde{m} \otimes V A$:

**Theorem 3.21.** Let $A$ be an almost $V$-algebra. Let $K^+ (\tilde{m} \otimes V A)$ denote the homotopy fiber of the canonical map $K^+ (V \oplus \tilde{m} \otimes V A) \to K^+ (V)$. Then the $K$-theory spectrum $K^+ (\tilde{m} \otimes V A)$ is canonically equivalent to the almost $K$-theory $K^\text{al} (A)$.

**Proof.** Let $\text{Perf}^+ (B, V)$ denote the triangulated category whose objects are the same of $\text{Perf}^+ (B)$ and weak equivalences are $- \otimes_B V$-quasi-isomorphisms. By applying Theorem 3.12 to the exact functor $- \otimes_B V : \text{Perf}^+ (V) \to \text{Perf}^+ (B, V)$, the induced map $K^+ (B, V) \to K^+ (V)$ is a weak equivalence, entailing that the induced map $K^+ (\tilde{m} \otimes V A) \to K^+ (B, \tilde{m} \otimes V A)$ is also a weak equivalence: For any object $E'$ of $\text{Perf}^+ (V)$, indeed, the extension $E' \otimes_V B$ contains $E'$ as a direct factor. Therefore, $E'$ is also an object of $\text{Perf}^+ (B)$ and the inclusion $E' \to E' \otimes_V B \simeq E' \oplus (E' \otimes_B \tilde{m} \otimes V A)$ a $- \otimes_B V$-quasi-isomorphism.

Furthermore, for any $B$-complex, one has an equivalence $E \simeq (E \otimes_B V) \oplus (E \otimes_B (\tilde{m} \otimes V A))$, implying that any $V$-acyclic complex is quasi-isomorphic to a firm complexes. Hence, we obtain a chain of equivalences: $K^+ (\tilde{m} \otimes V A) \simeq K^+ (B, \tilde{m} \otimes V A) \simeq K^\text{al} (B, \tilde{m} \otimes V A) \simeq K^\text{al} (\tilde{m} \otimes V A)$. By Corollary 3.20 we get a weak equivalence: $K^\text{al} (A) \simeq K^+ (\tilde{m} \otimes V A)$. \qed

4. **Finite syntomic algebraic cobordism of almost algebras**

Fix a base ring $V$ which is unital and commutative and an almost $V$-algebra $A$. Let $\tilde{m}$ be an idempotent ideal of $V$. Following the previous section, this section assumes that $\tilde{m}$ is a flat $V$-module.
4.1. Definition of finite syntomic motivic model structure of the category of simplicial presheaves. Let \( \mathcal{X} \) be a Grothendieck site with an interval object \( I \). We assume that \( \mathcal{X} \) has enough points: That is, a morphism \( f : X \to Y \) in \( \mathcal{X} \) is an isomorphism if \( f_x : X_x \to Y_x \) is an isomorphism of sets for any point \( x : * \to \mathcal{X} \) where the functor \((−)_x : \mathcal{X} \to \text{Sets}\) denotes the right adjoint of the induced functor \( x_* : \text{Sets} \to \mathcal{X} \). A simplicial object \( U_* : \Delta^{\text{op}} \to \mathcal{X} \) with an augmentation \( \pi : U_* \to X \in \mathcal{X} \) is a hypercover of \( X \) if the following conditions are hold:

- For any \( n \geq 0 \), \( U_*(\{n\}) \) is a coproduct of compact objects represented by small objects of \( \mathcal{X} \).
- The augmentation \( \pi : U_* \to X \) is a stalk-wise trivial Kan fibration: That is, \( \pi_x : U_x, \bullet \to * \) is a trivial Kan fibration for any point \( x : * \to X \).

The category \( \text{Set}_A \) of simplicial set has a proper combinatorial simplicial model structure, called Kan–Quillen model structure. Then the injective model structure of the category \( \text{Set}^{\text{op}}_A \) of simplicial presheaves on \( \mathcal{X} \) is also proper combinatorial. Let \( \text{MS}^A_X \) denote the Bousfield localization of \( \text{Set}^{\text{op}}_A \) defined as follows: A simplicial presheaf \( F \) is motivic local if \( F \) satisfies the following conditions:

- The presheaf \( F \) is stalk-wise fibrant.
- For any hypercover \( \pi : U_* \to X \) of \( X \in \mathcal{X} \), the induced map \( F(f) : F(X) \to F(U) \) is a weak equivalence. Here the functor \(|−| : \text{Fun}(\Delta^{\text{op}}, \mathcal{X}) \to \mathcal{X} \) denotes the geometric realization of simplicial objects.
- The canonical map \( 1 : I \to * \) induces a weak equivalence \( F(U) \to F(U \times I) \) for any \( U \in \mathcal{X} \).

A map \( f : F \to G \) of simplicial presheaves on \( \mathcal{X} \) is a motivic equivalence if the induced map

\[
f^* : \text{Hom}(G, Z) \to \text{Hom}(F, Z)
\]

is a weak homotopy equivalence of simplicial sets for each motivic local presheaf \( Z \). By [Bar10, p.56, Corollary 4.55], the injective model structure of \( \text{Set}^{\text{op}}_A \) is proper combinatorial and symmetric monoidal. Therefore the Bousfield localization \( \text{MS}^A_X \) of \( \text{sSet}^{\text{op}}_A \) is also proper.

We apply the finite syntomic site on the category \( \text{Sch}^{\text{op}}_V \) of finitely presented schemes over \( \text{Spec} V \), and the interval object \( \mathbb{A}^1_V \). We let \( \text{MS}^\Delta_{\text{FSyn}} \) denote the \( \infty \)-category determined by the simplicial model category \( \text{MS}^\Delta_{\text{Sch}^{\text{op}}_V} \) and we call objects of \( \text{MS}^\Delta_{\text{FSyn}} \) (finite syntomic) motivic spaces. Moreover, \( \text{MS}^\Delta_{\text{FSyn}} \) denotes the stable \( \infty \)-category of (finite syntomic) motivic spectra defined by

\[
\text{MS}^\Delta_{\text{FSyn}} = \lim_{\leftarrow} (\text{MS}^\Delta_{\text{FSyn}})_* \overset{\Omega_{\mathbb{P}^1}}{\longrightarrow} (\text{MS}^\Delta_{\text{FSyn}})_*,
\]

where \( \text{MS}^\Delta_{\text{FSyn}} \), the \( \infty \)-category of pointed motivic spaces and \( \Omega_{\mathbb{P}^1}(-) = \text{Map}(\text{MS}^\Delta_{\text{FSyn}}, (\mathbb{P}^1_+ , -)) \) is the pointed \( \mathbb{P}^1 \)-loop functor. The stable \( \infty \)-category \( \text{MS}^\Delta_{\text{FSyn}} \) is the full subcategory of the model category \( \text{Sp}_{\mathbb{P}^1} (\text{MS}^\Delta_{\text{FSyn}}) \) spanned by \( \mathbb{P}^1_+ \)-stable fibrant objects. (See [Jar15, Section 10.2]).
Theorem 4.1 ([Kat22] Corollary 3.9). The localization : $L_{FSyn} : \text{MSp} \rightarrow \text{MSp}$ by the family of zero-section stable finite syntomic surjective morphisms induces a categorical equivalence of stable $\infty$-categories : $L_{FSyn}(\text{MSp}) \rightarrow \text{MSp}^{FSyn}$. □

4.2. Some of almost homotopical algebra. We will recall and prove some result of almost homotopical algebra.

Definition 4.2. Let $A$ be an almost $V$-algebra. The Jacobson radical is defined to be $\text{Jac}(A) = \text{Jac}(A_\ast)^{\ast} \subset A$.

Clearly $I \rightarrow \text{Jac}(A)$ is almost injective if and only if $\tilde{m} \otimes_V I \subset \text{Jac}(A_\ast)$.

Definition 4.3. Let $A$ be an almost $V$-algebra and $I$ an ideal of $A$. We say that $I$ is tight if there exists a finitely generated ideal $m_0 \subset m$ and an integer $n \geq 0$ such that $I^n \subset m_0 A$.

Proposition 4.4 ([GR03] Corollary 5.1.17.). Let $A$ be an almost $V$-algebra, $I$ a tight ideal of $A$ contained in the Jacobson radical of $A$. Then $I$, is contained in the Jacobson radical of $A_\ast$. □

Lemma 4.5 ([GR03] Lemma 5.1.7.). Let $A$ be an almost $V$-algebra, $I$ a tight ideal of $A$ which is contained in the Jacobson radical of $A$. If $M$ is an almost finitely generated $A$-module satisfying $IM = M$, then exactly $M = 0$. □

Corollary 4.6 ([GR03] Corollary 5.1.8.). Let $A$ be an almost $V$-algebra, $I$ a tight ideal of $A$ which is contained in the Jacobson radical of $A$. Let $f : M \rightarrow N$ be a homomorphism of almost finitely generated projective $A$-modules. If $f \otimes_A A/I : M/IM \rightarrow N/IN$ is an isomorphism. Then $f$ is also an isomorphism. □

Theorem 4.7 ([GR03] Theorem 5.3.24.). Let $A$ be an almost $V$-algebra, $I$ a tight ideal of $A$ which is contained in the Jacobson radical of $A$. Write $A_n = A/I^n$ for each $n \geq 1$, and $A_\infty = \varprojlim A_n$. Then

$$\varprojlim : 2 - \varprojlim \text{alPMod}_{A_\ast} \rightarrow \text{alPMod}_{A_\infty}$$

is a categorical equivalence. □

proof. Let $(P_n)$ be an inverse system of finitely generated projective $A_n$-modules. Since the canonical map : $A'_n \rightarrow (\varprojlim A'_n) \otimes_{A_\infty} A_n$ is an isomorphism for each $n \geq 1$ and $r \geq 0$, $P_n \rightarrow (\varprojlim P_n) \otimes_{A_\infty} A_n$ is an isomorphism for each $n \geq 1$. Then $(P_n) \rightarrow ((\varprojlim P_n) \otimes_{A_\infty} A_n)$ is an isomorphism of inverse systems. Then one has a chain of isomorphisms $\text{Hom}_V(\tilde{m}, P_n) \simeq \text{Hom}_V(\tilde{m}, (\varprojlim P_n) \otimes_{A_\infty} A_n) \simeq \text{Hom}_V(\tilde{m}, \text{Hom}_{A_\ast}((\varprojlim P_n)^\vee, A_n)) \simeq \text{Hom}_V((\varprojlim P_n)^\vee, \text{Hom}_V(\tilde{m}, A_n))$. Therefore $\text{Hom}_V(\tilde{m}, P_n) \simeq \text{Hom}_V(\tilde{m}, (\varprojlim P_n) \otimes_V A_n)$.

Conversely, let $P_\infty$ be a finitely generated projective $A_\infty$-module. Then we show that $P_\infty \rightarrow \varprojlim P_\infty \otimes_{A_\infty} A_n$ is an isomorphism. Since $P_\infty$ is canonically isomorphic to the second dual $P_\infty^{c\vee}$.
\[ P_\infty \otimes_{A_\infty} A_n \cong \text{Hom}_{A_\infty}(P^\vee_\infty, A_n). \] Therefore one has \[ \lim P_\infty \otimes_{A_\infty} A_n \cong \lim \text{Hom}_{A_\infty}(P^\vee_\infty, A_n) \cong \text{Hom}_{A_\infty}(P^\vee_\infty, A_\infty) \cong P_\infty. \] We obtain
\[
\text{Hom}_V(\hat{m}, P_\infty) \cong \text{Hom}_V(\hat{m}, \lim P_\infty \otimes_{A_\infty} A_n) \cong \lim \text{Hom}_V(\hat{m}, P_\infty \otimes_{A_\infty} A_n)
\]

\[ \square \]

**Theorem 4.8.** Let \( A \) be an almost \( V \)-algebra and \( I \) a tight ideal which is contained in the Jacobson radical of \( A \). For \( n \geq 1 \), set \( A_n = A/I^n \), and \( A_\infty = \varprojlim A_n \). Then the canonical adjunction \((- \otimes_A^I A_n) : D(A_\infty) \rightleftarrows 2 - \lim D(A_n) : \mathbb{R} \varprojlim \) induces a categorical equivalence
\[
\mathbb{R} \varprojlim : 2 - \lim \text{APerf}(A_n) \to \text{APerf}(A_\infty)
\]

between stable \( \infty \)-categories.

**proof.** Let \( E \) be an almost perfect \( A_\infty \)-complex. Since \( E \) is a dualizable object of the derived category \( D(A_\infty) \), \( E \otimes_{A_\infty}^I (\_ \_ ) \) preserves all small limits. Therefore the canonical morphism : \( E_n \to \mathbb{R} \varprojlim E \otimes_{A_\infty}^I A_n \) is a quasi-isomorphism.

Conversely, let \( (E_n) \) be an inverse system of complex of almost finitely generated projective \( A_n \)-modules. Since \( (A_n)_{\infty} \to A_n/(I_n)^n \) is surjective for each \( n \geq 1 \), \( \lim E_n \to E_n \) is also surjective for each \( n \geq 1 \). Therefore, clearly, the canonical morphism \( (\lim E_n) \otimes_{A_\infty}^I A_n \to E_n \) is a quasi-isomorphism for each \( n \geq 1 \). Hence the morphism \( ((\lim E_n) \otimes_{A_\infty}^I A_n) \to (E_n) \) is an inverse system of quasi-isomorphisms. \( \square \)

### 4.3. Almost finite syntomic morphisms.

We consider the restriction of the functor \( \hat{m} \otimes_V (\_ \_ ) : \text{Mod}_V \to \text{Mod}_V^\hat{m} \) to the subcategory \( \text{CAlg}_V \) of commutative unital \( V \)-algebras. Since the functor \( \hat{m} \otimes_V (\_ \_ ) \) is (strong) symmetric monoidal, the essential image of the restriction \( \hat{m} \otimes_V (\_ \_ )|_{\text{CAlg}_V} \) is equivalent to the full subcategory of the category of non-unital commutative \( V \)-algebras \( \text{CAlg}^\text{nu}_V \).

Let \( \text{CAlg}_{\hat{m}} \) denote the full subcategory \( \text{CAlg}^\text{nu}_V \) spanned by those objects of the essential image of \( \hat{m} \otimes_V (\_ \_ )|_{\text{CAlg}_V} \).

**Theorem 4.9.** The functor \( \hat{m} \otimes_V (\_ \_ ) : \text{CAlg}_V \to \text{CAlg}^\text{nu}_V \) induces a categorical equivalence
\[
\hat{m} \otimes_V (\_ \_ ) : \text{CAlg}_{\hat{m}}^\text{al} \to \text{CAlg}_{\hat{m}}
\]
whose quasi-inverse is equivalent to the functor \( (V \oplus_{\hat{m}} (\_ \_ ))^a : \text{CAlg}_{\hat{m}} \to \text{CAlg}_{V}^\text{al} \), where \( V \oplus_{\hat{m}} (\_ \_ ) \) denotes the cokernel of the diagonal: \( \hat{m} \to V \oplus (\hat{m} \otimes_V (\_ \_ )). \)

**proof.** Let \( \mu_A : A \otimes_V A \to A \) and \( \mu_{\hat{m}} : \hat{m} \otimes_V \hat{m} \to \hat{m} \) denote the multiplication maps. By the idempotention of \( \hat{m} \), the map \( \mu_{\hat{m}} : \hat{m} \otimes_V \hat{m} \to \hat{m} \) is an isomorphism. Therefore, in the
algebras satisfying the property \( P \). Let \( \text{CAlg} \) be fully faithful: By definition of the functor \( (\cdot)_! \), the counit \( (\cdot)_!(\mathfrak{m} \otimes_V (-)) \rightarrow \text{Id}_{\text{CAlg}^\text{al}_V} \) is an equivalence of functors. Therefore \( \mathfrak{m} \otimes_V (-) \) is fully faithful, entailing a categorical equivalence. □

By Theorem 4.9 we can formulate theories of almost algebras from the non-unital algebras viewpoint. Let \( A \) be an almost \( V \)-algebra and \( B \) an almost \( A \)-algebra. Then the canonical morphism \( \mathfrak{m} \otimes_V B \rightarrow L_0(B)_! \) is a quasi-isomorphism. Hence, both \( \mathfrak{m} \otimes_V (-) \) is fully faithful, entailing a categorical equivalence.

By Gabber–Ramero [GR03, Proposition 2.5.43 and Remark 2.2.28], the canonical morphism \( \mathfrak{m} \otimes_V B \rightarrow L_0(B)_! \) is a quasi-isomorphism. Therefore, \( L_B/A \) can be an object of \( D(\text{Mod}_{B!!}) \).

**Proposition 4.11.** Let \( A \) be an almost \( V \)-algebra and \( B \) an almost \( A \)-algebra. Then the canonical morphism \( L_B/A \rightarrow L_{B!!}/A_{!!} \) induced by the projection \( V \times B \rightarrow B \) is an almost quasi-isomorphism.

**proof.** Since the projection \( (V \times A))_! \rightarrow A!! \) is flat and surjective, by Stacks project [Sta22, 08QY, Lemma 91.8.4], the relative cotangent complex \( L_{A_{!!}/(V \times A)_!!} \) is acyclic. Note that \( B \simeq (V^a \times B)/V^a \simeq (V^a \times B) \otimes_{V^a \times A} ((V^a \times A)/V^a) \). Therefore the colimit preserving functor \( L_{(-)_!/(V \times A)_!!} \) induces a quasi-isomorphism \( L_{(V^a \times B)/V^a}/(V^a \times A)_!! \rightarrow L_{B!!}/(V^a \times A)_!! \). By the distinguish triangle

\[
B!! \otimes_{(V^a \times A)_!!} L_{A_{!!}/(V \times A)_!!} \rightarrow L_{B!//(V \times A)_!!} \rightarrow L_{B!!}/A_{!!} \rightarrow B!! \otimes_{(V^a \times A)_!!} L_{A_{!!}/(V \times A)_!!}[1]
\]

induced by the sequence \( (V^a \times A))_! \rightarrow A!! \rightarrow B!! \) of \( V \)-algebras, we obtain that the canonical morphism \( L_{B!!}/(V^a \times A)_!! \rightarrow L_{B!!}/A_{!!} \) is a quasi-isomorphism. Hence, both \( \mathfrak{m} \otimes_V L_{B!!}/A_{!!} \) and \( L_{B!!}/A_{!!} \) are quasi-isomorphic to the same object of \( D(\text{Mod}_{B!!}^\text{al}) \). □

**Definition 4.12.** A morphism \( f : A \rightarrow B \) of almost \( V \)-algebras is almost finite syntomic if the following conditions are satisfied:

\[
\begin{align*}
\mathfrak{m} \otimes_V A \otimes_V \mathfrak{m} \otimes_V A^\mu \twoheadrightarrow \mathfrak{m} \otimes_V A \otimes_V A, \\
\mathfrak{m} \otimes_V \mathfrak{m} \otimes_V A \otimes_V A^\mu \twoheadrightarrow \mathfrak{m} \otimes_V A \otimes_V A,
\end{align*}
\]

both horizontal maps are isomorphisms. We prove that the \( \mathfrak{m} \otimes_V (-) : \text{CAlg} \rightarrow \text{CAlg}^\text{al}_V \) functor is fully faithful: By definition of the functor \((\cdot)_! : \text{CAlg}^\text{al}_V \rightarrow \text{CAlg}_V \), the counit \( (V \otimes \mathfrak{m} \otimes_V (-))_a \rightarrow \text{Id}_{\text{CAlg}^\text{al}_V} \) is an equivalence of functors. Therefore \( \mathfrak{m} \otimes_V (-) \) is fully faithful, entailing a categorical equivalence. □
(1) An $A$-algebra $B$ is an almost finitely projective $A$-module.

(2) The almost relative cotangent complex $L_{B/A}$ is almost perfect and tor-amplitude in $[-1, 0]$ in the category $D(\text{Mod}_{B_1})$.

**Theorem 4.13.** Let $\text{Alg}^{\text{FSyn}}_{\mathbb{m}\mathbb{m}\mathbb{n}V}$ denote the full subcategory of $\text{CAlg}_{\mathbb{m}\mathbb{m}\mathbb{n}V}$ spanned by those objects $\mathbb{n} \otimes V B$ satisfying $(V \oplus \mathbb{n} \otimes V) A$ is almost finite (V \oplus \mathbb{n} \otimes V) A$-algebra. Then the categorical equivalence $(V \oplus \mathbb{n} \otimes V (-))^a : \text{CAlg}_{\mathbb{m}\mathbb{m}\mathbb{n}V} \to \text{CAlg}_{\mathbb{m}\mathbb{m}\mathbb{n}V}^a$ induces a categorical equivalence between $\text{CAlg}_{\mathbb{m}\mathbb{m}\mathbb{n}V}^{\text{FSyn}}$ and the full subcategory $\text{Alg}_{\text{alFSyn}} A$ of $\text{CAlg}_{\mathbb{m}\mathbb{m}\mathbb{n}V}^a$ spanned by almost finite syntomic $A$-algebras.

**proof.** By the definition of almost finitely generated projective modules and Proposition 4.11 the category $\text{Alg}^{\text{FSyn}}_{\mathbb{m}\mathbb{m}\mathbb{n}V}$ is equivalent to the pullback of $\text{Alg}^{\text{alFSyn}}_{\mathbb{m}\mathbb{m}\mathbb{n}V}$ along the inclusion $(\mathbb{m}\mathbb{m}\mathbb{n}V \otimes _\mathbb{m} \mathbb{m}\mathbb{n}V)$ $\mathbb{m}\mathbb{m}\mathbb{n}V$ is an almost finitely projective $A$-algebra. Let $\mathbb{n} \otimes V A$ be an almost $V$-algebra. We show that the relative cotangent sequence $\mathbb{n} \otimes V A$ is almost perfect and tor-amplitude in $[\mathbb{n} \otimes V A, 0]$.

Fix a regular cardinal $\kappa$. We write $\text{MS} = \text{MS}_{\text{FSyn}}$ and let $\text{MS}^a$ denote full subcategory spanned by all $\kappa$-small objects of $\text{MS}$. Referring the results \[\text{EHK}+20b\] Theorem 3.4.1 and Lemma 3.5.1 and \[\text{EHK}+21\] Lemma 5.1.3, we define the almost version of algebraic cobordism:

**Definition 4.14.** Let $A$ be an almost $V$-algebra. Let $\text{alFSyn}(A)$ denote the largest groupoid of the category $\text{Alg}_{\mathbb{m}\mathbb{m}\mathbb{n}V}^{\text{FSyn}}$. and set $a\text{MGL} = \Sigma_\kappa^a \text{alFSyn}^a(-)$, where $\text{alFSyn}^a$ is the right Kan extension of the restriction $\text{alFSyn}_{(\text{MS}^a, \mathbb{n})}$ along the inclusion $(\text{MS}^a)^{op} \to \text{MS}^{op}$. We say that the motivic spectrum $a\text{MGL}$ is the *almost algebraic cobordism*.

For an almost $V$-algebra $A$, let $\text{Alg}^{\text{alFSyn}}_{A}$ denote the category of almost finite syntomic $A$-algebras. The results in Section 4.2 provides us the following:

**Proposition 4.15.** Let $A$ be an almost $V$-algebra and $I$ a tight ideal which is contained in the Jacobson radical of $A$. For $n \geq 1$, set $A_n = A/I^n$, and $A_\infty = \lim_{n \to \infty} A_n$. Let $(A_n)_{n \geq 1}$ be an inverse system of commutative rings and write $A = \lim_{n \to \infty} A_n$. The functor $(- \otimes A_n)_{n \geq 1} : \text{Alg}_{A_n}^{\text{alFSyn}} \to 2 - \lim_{n \to \infty} \text{Alg}_{A_n}^{\text{alFSyn}}$ is a categorical equivalence.

**proof.** Let $(B_n)$ be an inverse system of almost finite syntomic $A_n$-algebras. Then, by the assumption, $B = \lim_{n \to \infty} A_n$ is almost finite, projective $A$-algebra. We show that the relative cotangent complex $L_{B/A}$ is perfect and tor-amplitude in $[-1, 0]$. Since $B \to B_n$ is unramified, the sequence $A \to B \to B_n$ induces a quasi-isomorphism $\mathbb{n} \otimes V L_{B/A} \otimes B_n \to \mathbb{n} \otimes V L_{B_n/A}$. Similarly, the sequence $A \to A_n \to B_n$ induces a quasi-isomorphism $\mathbb{n} \otimes V L_{B_n/A} \to \mathbb{n} \otimes V L_{B_{n/A}}$. Therefore $\mathbb{n} \otimes V L_{B/A} \otimes B_n \to \mathbb{n} \otimes V L_{B_{n/A}}$ is a quasi-isomorphism. Since each $B_n$ is an almost finitely generated projective $A$-module, $\mathbb{n} \otimes V B \otimes A_n \to \mathbb{n} \otimes V B_n$ is an isomorphism for each $n$. Then $\mathbb{n} \otimes V L_{B/A} \otimes B_n \to \mathbb{n} \otimes V L_{B_{n/A}} (B \otimes A_n) \to \mathbb{n} \otimes V L_{B/A} \otimes B_n$ is a composition of quasi-isomorphisms for each $n$. By Theorem 4.13 $\mathbb{n} \otimes V L_{B/A} \to \lim_{n \to \infty} \mathbb{n} \otimes V L_{B_{n/A}}$ is a quasi-isomorphism.
and they are almost perfect complexes of $B$-modules. By the Milnor exact sequence

$$0 \rightarrow \lim_{n \rightarrow \infty} H_{*+1}(\tilde{m} \otimes V L_{B_n/A_n}) \rightarrow H_{*}(\tilde{m} \otimes V L_{B/A}) \rightarrow \lim_{n \rightarrow \infty} H_{*}(\tilde{m} \otimes V L_{B_n/A_n}) \rightarrow 0,$$

the projective limit $\tilde{m} \otimes V L_{B_n/A_n}$ is tor-amplitude in $[-1, 0]$ in $D(\text{Mod}_B)$.

**Corollary 4.16.** Let $A$ be an almost $V$-algebra and $I$ a tight ideal which is contained in the Jacobson radical of $A$. Let $(A_n)_{n \geq 1}$ be an inverse system of commutative rings and write $A = \lim_{n \rightarrow \infty} A_n$. The induced map $\text{alFSyn}^\ast(\text{Spec}A) \rightarrow \text{alFSyn}^\ast(\lim_{n \rightarrow \infty} \text{Spec}A_n)$ is a weak equivalence of spaces.

**proof.** We may assume that the functor $\text{Spec}A$ is small. By Proposition 4.15, one has a chain of weak equivalences:

$$\text{alFSyn}(\text{Spec}A) \cong \text{Map}_{\text{Cat}}(\Delta^0, \text{alFSyn}_A) \cong \text{Map}_{\text{Cat}}(\Delta^0, 2 - \lim_{n \rightarrow \infty} \text{alFSyn}_A_n) \cong \lim_{n \rightarrow \infty} \text{Map}_{\text{Cat}}(\Delta^0, \text{alFSyn}_A_n) \cong \lim_{n \rightarrow \infty} \text{alFSyn}(\text{Spec}A_n),$$

where $\text{Cat}$ denotes the large category of small categories that mapping spaces are the largest groupoids of the functor categories. Therefore, one has an equivalences: $\text{alFSyn}^\ast(\text{Spec}A) \cong \lim_{n \rightarrow \infty} \text{alFSyn}^\ast(\text{Spec}A_n) \cong \text{alFSyn}^\ast(\lim_{n \rightarrow \infty} \text{Spec}A_n)$.

**Remark 4.17.** Proposition 4.15 and its corollary holds under the only the assumption that $\lim_{n \rightarrow \infty} 2 - \lim_{n \rightarrow \infty} \text{alPMod}_A_n \rightarrow \text{alPMod}_A$ is a categorical equivalence.

Let $V$ be a commutative ring and $\omega \in V$ a non-zero divisor. For $n \geq 0$, set $A_n = A/\omega^{n+1}A$. Write $\overline{A_n} = \omega^n A / \omega^{n+1} A$. Since $\overline{A_n}$ is square-zero non-unital ring, the $A$-linear inclusion: $\overline{A_n} \rightarrow A_n$ is a ring homomorphism as non-unital rings, inducing a ring homomorphism:

$$(4.1) \quad \varphi_n : (\overline{A_n})!! \rightarrow (A_n)!!$$

Hence $(A_n)!!$ has an $(\overline{A_n})!!$-algebra structure.

By the canonical isomorphism $\overline{A_1} \rightarrow \overline{A_n}$, the following holds.

**Lemma 4.18.** The linear map $(\omega^{n-1} \cdot \text{Id})!! : (A_n)!! \rightarrow (A_n)!!$ induces an isomorphism: $(\overline{A_1})!! \rightarrow (\overline{A_n})!!$ of $A!!$-algebras.

### 4.4. The almost algebraic cobordism of perfectoid algebras

In this section, we consider the case that $\mathfrak{m}$ is a flat $V$-module, entailing the canonical map $\mu_{\mathfrak{m}} : \tilde{m} \rightarrow \mathfrak{m}$ is bijective. We recall the definition of perfectoid algebra.

**Definition 4.19.** Let $K$ be a complete non-Archimedean non-discrete valuation field of rank 1, and $K^\circ$ denote the subring of powerbounded elements. We say that $K$ is a **perfectoid field** if the Frobenius $\Phi : K^\circ / p \rightarrow K^\circ / p$ is surjective, where $p$ is a positive prime integer which is equal to the characteristic of the residue field of $K^\circ$. 

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In this section, we fix a perfectoid field $K$ whose valuation ring $K^o$ is mixed characteristic $(0, p)$. We put $V = K^p$ and $m = K^oo$, where $K^oo = \{x \in K \mid |x| < 1\}$ is the maximal ideal of $K^o$. Then $m$ is the set of topologically nilpotent elements, being an idempotent ideal. We fix a pseudouniformizer $\omega \in V$ with $|p| \leq |\omega| < 1$. In this case, the ideal $m = \lim_{\to} \omega^{1/p^n} V$, which is a filtered colimit of free $V$-modules, is flat by Lazard’s theorem.

**Definition 4.20.** An integral perfectoid $V$-algebra is an $\omega$-adic complete flat $V$-algebra $A$ on which Frobenius induces an isomorphism $\Phi : A/\omega A \to A/\omega A$.

For any $V$-algebra $B$, let $B^@$ denote the tilting algebra $\lim \leftarrow_{\omega^{1/p^n}} B/\omega B$ of $B$. The tilting ideal $m^@ \subset V^@$ is a flat $V^@$-module as $m$ is. Therefore, we can apply almost mathematics to perfectoid algebras.

**Proposition 4.21.** Let $K$ be a perfectoid field with the valuation ring $V$ whose residue field is of characteristic $p > 0$. Let $\omega$ be a pseudouniformizer and $m = \lim_{\to} \omega^{1/p^n} V$. Let $A$ be an integral perfectoid $V$-algebra and write $A_n = A/\omega^n A$. Then $(A_n)^!!$ is an ind-finite syntomic $(A_1)^!!$-algebra for $n \geq 1$.

**proof.** We may assume $A_\ast = A$ and $A_1 = m \otimes_V A$. The injection $(A_1)^!! \to (A_n)^!!$ is induced by the inductive system of (non-unital) ring homomorphisms:

$$\omega^{n+1/ p} A/\omega^{n+1+1/ p} A \to \omega^{1/ p} A/\omega^{n+1+1/ p} A,$$

which induces a finite syntomic homomorphism

$$\varphi_{n,m} : V \oplus (\omega^{1/ p})^{n+1} A/\omega^{n+1+1/ p} A \to V \oplus \omega^{1/ p} A/\omega^{n+1+1/ p} A$$

for each $m \geq 1$. Since the functor $(-)^!!$ preserves all small colimit, by definition of almost relative cotangent complex, $(A_n)^!!$ is ind-finite syntomic over $(A_1)^!!$ by Lemma 4.18 for each $n \geq 1$. 

**Lemma 4.22.** On the condition of Proposition 4.21 let $A_\ast^+$ denote the cokernel of

$$m/\omega^n m \to V/\omega^n \oplus (m \otimes_V A/\omega^n A)$$

for each $n$. Then there exists a canonical isomorphism $(A_n)^!! \to A_\ast^+$ for each $n \geq 1$.

**proof.** Consider the following commutative diagram:

$$\begin{array}{ccc}
m & \to & m/\omega^n m \to m \otimes_V A_n \\
\downarrow & & \downarrow \\
V/\omega^n & \to & A_\ast^+,
\end{array}$$

where both squares are coCartesian. Therefore the morphism $(A_n)^!! \to A_\ast^+$ is canonically an isomorphism. 

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Set $\text{alFSyn}^b(A) = \text{alFSyn}(A^b)$ for any perfectoid algebra $A$, and write $\text{alMGL}^b = \Sigma^\infty_+ \text{alFSyn}^b$. Finally, we prove the tilting equivalence of almost algebraic cobordism:

**Theorem 4.23.** Let $V$ be a mixed characteristic perfectoid valuation ring with unit and $V^b$ denote the tilting. Then these spectra $\text{alMGL}$ and $\text{alMGL}^b$ are equivalent on zero-section stable affine integral perfectoid schemes over $V$.

**proof.** By Lemma 4.22, there exists a zig-zag of isomorphisms $(A_1)_{!!} \to (A^+_1)_{!!} \leftrightarrow (A^+_1)_{!!} \leftrightarrow (A^+_1)_{!!}$ of commutative rings, where the middle map is a canonical isomorphism. By Proposition 4.21 for each $n \geq 1$, $\text{alMGL}(Z_0(A_n))$ (resp. $\text{alMGL}(Z_0(A^+_n))$) is weakly equivalent to the same $\text{alMGL}(A_1)$ (resp. $\text{alMGL}(A^+_1))$, implying that $\mathbb{R} \lim \text{alMGL}(Z_0(A_n))$ and $\mathbb{R} \lim \text{alMGL}(Z_0(A^+_n))$ are weakly equivalent, where $Z$ denotes the zero-section stabilization. Note that the zero-section stabilization preserves small colimits. The zero-section stability of $\lim \text{Spec}A_n$ implies that one has a chain of equivalences $\mathbb{L} \lim \text{Spec}A_n \simeq Z_{0}(\mathbb{L} \lim \text{Spec}A_n) \simeq \mathbb{L} \lim Z_0(\text{Spec}A_n)$. Similarly, one has $\mathbb{L} \lim \text{Spec}^b_n \simeq \mathbb{L} \lim Z_0(\text{Spec}A_n^b)$. Consider the diagram of fiber sequences:

\[
\begin{align*}
&\text{alMGL}(\mathbb{L} \lim \text{Spec}A_n) \longrightarrow \text{alMGL}(\oplus_n Z_0(\text{Spec}A_n)) \longrightarrow \text{alMGL}(\oplus_n Z_0(\text{Spec}A_n)) \\
&\mathbb{R} \lim \text{alMGL}(Z_0(A_n)) \longrightarrow \prod_n \text{alMGL}(Z_0(A_n)) \longrightarrow \prod_n \text{alMGL}(Z_0(A_n)),
\end{align*}
\]

where the middle and right vertical arrows are weak equivalences. Since each $\text{Spec}A_n$ is cofibrant and closed immersion $t^*_n : \text{Spec}A_n \to \text{Spec}A_{n+1}$ a cofibration with respect to the motivic model structure, the inductive system $(\text{Spec}A_n)_{n \geq 1}$ is injectively cofibrant. Therefore $\lim \text{Spec}A_n \to \mathbb{L} \lim \text{Spec}A_n$ is a motivic equivalence. Furthermore, by Proposition 4.15 one has weak equivalences of spectra $\text{alFSyn}^*(\text{Spec}A) \simeq \text{alFSyn}^*(\lim \text{Spec}A_n)$ and $\text{alFSyn}^*(\text{Spec}^b) \simeq \text{alFSyn}^*(\lim \text{Spec}A^b_n)$. Hence $\text{alMGL}$ and $\text{alMGL}^b$ are weakly equivalent on the category of integral perfectoid $V$-algebras.

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