Response of Quasi-Integrable and Non-Resonant Hamiltonian Systems to Fractional Gaussian Noise

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ABSTRACT The main difficulty of analyzing the response of nonlinear dynamical systems to fractional Gaussian noise (fGn) is the non-Markov property and non-usefulness of diffusion process theory. Currently, only numerical simulation can be applied to obtain the response of nonlinear systems to fGn. In the present paper, noting the rather flat property of the fGn power spectral density (PSD) in most part of frequency band, the stochastic averaging method for quasi-integrable Hamiltonian systems under wide-band noise excitation is applied to predict the response of quasi-integrable and non-resonant Hamiltonian systems to fGn. By using this method, the averaged Itô stochastic differential equations (SDEs) are established and the probability density function (PDF) of system response can be obtained from solving the corresponding Fokker-Planck-Kolmogorov (FPK) equation. All of the statistics of system response are then obtained from the PDF analytically and verified through the comparison with the simulation results.

INDEX TERMS Fractional Gaussian noise; quasi-integrable and non-resonant Hamiltonian systems; stochastic averaging method.

I. INTRODUCTION
In the past half century, great achievements on nonlinear stochastic dynamics have been made [1]–[5], mainly due to the wide applications of the diffusion process theory. Recently, the fractional Gaussian noise (fGn) has been introduced to stochastic dynamics. FGN is a proper mathematical model of some real noises with long-range (or long-memory), strongly spatial and/or temporal correlation, and has been used in finance [6], signal processing [7], communication network [8] and turbulence [9], etc. It is very difficult to predict the response of nonlinear dynamical systems to fGn due to non-Markov property and non-usefulness of diffusion process theory. The exact solutions of dynamical systems excited by fGn have been obtained only for one degree-of-freedom (DOF) linear oscillator [10]. As for multi DOF linear systems and nonlinear systems excited by fGn, some approximate methods, such as the stochastic averaging method, have been developed recently.

The stochastic averaging methods, including the stochastic averaging methods for quasi-Hamiltonian systems, are the powerful approximate analytical methods that have been widely used in nonlinear stochastic dynamics [4], [11], [12]. Recently, the stochastic averaging method for quasi-Hamiltonian systems excited by fGn [13], [14] has been developed based on averaging principal [15], [16]. The dimension of the averaged fractional stochastic differential equation (FSDE) is less than that of original system while the dynamical characteristics of averaged system keep the same as the original system. However, due to the non-Markov property of the system response, numerical simulation has to be used for obtaining the response statistics. To develop an approximate analytical method for predicting the response of nonlinear dynamical systems to fGn then becomes an interesting topic, and that is the motivation of the present paper.

In the present paper, the power spectral density (PSD) of fGn are firstly introduced. It is pointed out that the PSD of fGn varies slowly as frequency changes in most frequency band, which indicates fGn is a wide-band process in this
frequency band. Based on this observation, the stochastic averaging method for quasi-integrable Hamiltonian systems excited by wide-band noise is applied to study the response of quasi-integrable Hamiltonian systems under fGn excitation. After stochastic averaging, the response of the system is approximated as diffusion process and the transition probability density of the system is governed by an averaged Fokker-Planck-Kolmogorov (FPK) equation. All statistics of system response are then obtained from the solution of FPK equation. Finally, the obtained analytical results are verified by comparison with the results from the simulations of original and averaged FSDEs.

II. POWER SPECTRAL DENSITY OF FGN

Similar to Gaussian white noise, fGn $W^H(t)$ is the formal derivative of fractional Brown motion (fBm) $B^H(t)$, which can be written as

$$W^H(t) = dB^H(t)/dt. \quad (1)$$

Mandelbrot and van Ness [17] have given the definition of fBm $B^H(t)$, an integral expansion of classical Brownian motion (Bm) $B(t)$, as follows

$$B^H(t) = C_H \left\{ \int_{-\infty}^{0} [(t - s)^{H-1/2} - (-s)^{H-1/2}]d-B(s) + \int_{0}^{t} (t - s)^{H-1/2}d-B(s) \right\}. \quad (2)$$

where parameter $H$ is called Hurst index and $0 \leq H \leq 1$. The coefficient $C_H$ in Eq. (2) reads [10]

$$C_H = \left[ \frac{\Gamma(H+1)\sin(\pi H)}{\sqrt{\pi}\Gamma(H+1/2)} \right]^{1/2}. \quad (3)$$

It can be seen from Eq. (2) that the fBm is more general than Bm and will reduce to the standard Bm in the case of $H = 1/2$. There are some useful properties of a unit fBm:

(i) $B^H(t) = 0$, $t < 0$,
(ii) $E[B^H(t)] = 0$, $t \geq 0$,
(iii) $E[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, $t, s \geq 0$. \quad (4)

The fGn is a stationary Gaussian process with following auto-correlation function (ACF) [18]

$$R(\tau) = E[W^H(t + \tau)W^H(t)] = H(2H - 1)|\tau|^{2H-2} + 2H|\tau|^{2H-1}\delta(\tau). \quad (5)$$

Note that when $H = 1/2$, $R(\tau)$ reduces to Dirac function $\delta(\tau)$, denoting fGn for $H = 1/2$ is a Gaussian white noise.

It is pointed out that the fGn with $0 \leq H < 1/2$ is not proper for modeling physical noise since its ACF is negative and there is no PSD in the sense of traditional definition. In the present paper, only $1/2 \leq H < 1$ is considered. In the case of $1/2 \leq H < 1$, we can obtain the following PSD for fGn from Eq. (5) according to the Wiener-Khintchine relation

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-j\omega \tau} d\tau = \frac{H\Gamma(2H)\sin(\pi H)}{\pi} |\omega|^{1-2H}. \quad (6)$$

where $\Gamma(\cdot)$ is the gamma function. Note that when $H = 1/2$, the PSD $S(\omega)$ reduces to $1/2\pi$, constant PSD of Gaussian white noise.

Based on Eq. (6), the PSD of fGn for several Hurst indexes $H$ are shown in Fig. 1. We can see from Fig. 1 that the PSD for larger frequencies, e.g. larger than 0.6, varies very slowly or flat as frequency changes. And changing $H$ hardly change this flat feature of fGn’s PSD. Thus, in this frequency band, fGn can be approximately regarded as a wide-band noise. And the stochastic averaging method for quasi-integrable Hamiltonian systems under wide-band noise excitation [19] can be applied to predict the response of dynamic systems driven by fGn.

III. PROBLEM FORMULATION AND THE MAIN METHOD

The equations of a quasi-Hamiltonian system excited by fGn is of the form

$$\dot{Q}_j = \frac{\partial H}{\partial P_j},$$

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} - \varepsilon^2 c_{ij}(Q, P)\frac{\partial H}{\partial P_j} + \varepsilon f_{ik}(Q, P)W^H_k(t),$$

$$i, j = 1, 2, \ldots, n; \quad k = 1, \ldots, 2, m. \quad (7)$$

where $Q_i$ and $P_i$ are generalized displacements and momenta, respectively; $\varepsilon$ is a small parameter; $H = H(Q, P)$ is twice differentiable Hamiltonian; $\varepsilon^2 c_{ij}(Q, P)$ mean the coefficients of light quasi-linear dampings; $W^H_k(t)$ are stationary fGn with Hurst index $H$, ACF $R_k(\tau) = E[W^H_k(t)W^H_k(t + \tau)]$ and PSD $S_k(\omega)$ (see Eq. (6)).

Assume system (7) is quasi-integrable, then Hamiltonian of system (7) is

$$H(Q, P) = \sum_{i=1}^{n} H_i(Q_i, P_i)$$

$$H_i(Q_i, P_i) = P_i^2/2 + U_i(Q_i) \quad (8)$$

where

$$U_i(Q_i) = \int_{0}^{Q_i} g_i(u)du \quad (9)$$

FIGURE 1. The PSD of fGn with different Hurst index (see Eq. (6)).
With certain conditions [19], system (7) has following randomly periodic solution

\[ Q(t) = A_i \cos \Phi_i(t) + B_i, \quad \dot{Q}_i(t) = -A_i V_j(A_j, \Phi_j) \sin \Phi_i(t), \]
\[ \Phi_i(t) = \Psi_i(t) + \Theta_i(t) \]  

(10) and

\[ V_i(A_j, \Phi_j) = \sqrt{2[U_i(A_i + B_i) - U_i(A_j \cos \Phi_i + B_i)]} \]

(11) where \( A, \Phi, \Psi, \Theta \) are all random processes, \( A_i \) are amplitude of displacements, \( V_i(A_j, \Phi_j) \) are the instantaneous frequency of the oscillators. Regard Eq. (10) as a transformation from \( Q, \dot{Q} \) to \( A_i, \Phi_i \). System (7) is transformed into

\[ \dot{A}_i = \varepsilon^2 F_i^A(A, F) + \varepsilon G_i^A(A, F) \xi_i(t), \]
\[ \dot{\Phi}_i = \varepsilon^2 F_i^\Phi(A, F) + \varepsilon G_i^\Phi(A, F) \xi_i(t) \]

(12) where

\[ F_i^A = \frac{A_i}{g_i(A_i + B_i)(1 + h_i)} \left[ c_{ij}(A, \Phi) A_j V_j(A_j, \Phi_j) \sin \Phi_j \right] \]
\[ F_i^\Phi = \frac{1}{g_i(A_i + B_i)(1 + h_i)} \left[ c_{ij}(A, \Phi) A_j V_j(A_j, \Phi_j) \sin \Phi_j \right] \]
\[ G_i^A = \frac{A_i}{g_i(A_i + B_i)(1 + h_i)} f_{ik}(A, \Phi) v_i(A_i, \Phi_i) \sin \Phi_i \]
\[ G_i^\Phi = \frac{1}{g_i(A_i + B_i)(1 + h_i)} f_{ik}(A, \Phi) v_i(A_i, \Phi_i)(\cos \Phi_i + h_i) \]

(13)

By substituting \( A \) and \( B \) into Eq. (10), the \( h_i \) in Eq. (13) can be written as the following form

\[ h_i = \frac{g_i(-A_i + B_i) + g_i(A_i + B_i)}{g_i(-A_i + B_i) - g_i(A_i + B_i)} \]

(14)

In this paper, fGn \( W^H(t) \) is approximately regarded as stationary wide band process. In non-resonant case, according to Stratonovich-Khasminskii limit theorem, the processes \( A_i(t) \) in Eq. (12) converge weakly to an \( n \)-dimensional Markov diffusion processes as \( \varepsilon \to 0 \) in a finite time interval and is described by following averaged Itô SDEs

\[ dA_i = m_i(A) dt + \sigma_i(A) dB_k(t), \]
\[ i = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m \]  

(15) where

\[ m_i(A) = \varepsilon \int_{-\infty}^{0} \left( \frac{\partial G_{ik}}{\partial A_j} \right) \left|_{\tau} \right| + (G^A_{ik}/\Phi_i) \left|_{\tau} \right| R_k(\tau) d\tau \]
\[ b_{ij}(A) = \varepsilon \int_{-\infty}^{0} \left( G^A_{ik} \right) \left|_{\tau} \right| R_k(\tau) d\tau \]

\[ \langle \cdot \rangle_i = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \langle \cdot \rangle dt \]

(16)

Usually it is more useful to convert Eq. (15) into that for \( H_i \). By using Itô differential rule and noting \( H_i = U_i(A_i + B_i) \), the equations for \( H_i \) are

\[ dH_i = \tilde{m}_i(H) dt + \tilde{\sigma}_{ik}(H) dB_k(t), \]
\[ i = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m \]  

(17) where

\[ \tilde{m}_i(H) = \left[ \frac{g_i(A_i + B_i)(1 + h_i) m_i(A)}{A_i - U_i^{-1}(H_i) - B_i} \right]_{A_i = U_i^{-1}(H_i) - B_i} \]
\[ \tilde{b}_{ij}(H) = \left[ \frac{g_i(A_i + B_i) g_j(A_j + B_j)}{A_i - U_i^{-1}(H_i) - B_i} \right] \]

(18)

The FPK equation associated with Eq. (17) is

\[ \frac{\partial p}{\partial t} = -\frac{\partial}{\partial H_s}[\tilde{m}_r(H)p] + \frac{1}{2} \frac{\partial^2}{\partial H_s \partial H_r}[\tilde{b}_{rs}(H)p], \]
\[ r, s = 1, 2, \ldots, n \]  

(19) where \( p = p(H, t | H_0) \) is the transition probability density of energy vector process \( H(t) \). The initial condition of Eq. (19) is

\[ p(H, 0 | H_0) = \delta(H - H_0) \]

(20)

For quasi-integrable and non-resonant Hamiltonian system, the stationary probability density can be calculated as [4]

\[ p(q, p) = p(H(t)/T(H)|H_0 = H(t), q, p) \]

(21) Thus, the probability density for generalized displacements and generalized momenta of response of quasi-integrable and non-resonant Hamiltonian system excited by fGn is obtained analytically. The following example is carrying out to show the effective of proposed method and will be compared with numerical method in the Ref. [14].

IV. EXAMPLE

Consider the following two strong nonlinear Duffing oscillators with external and parametric excitations of fGn

\[ \ddot{X}_1 + \beta_{11} \dot{X}_1 + \beta_{12} \dot{X}_2 + \omega^2 X_1 + \alpha_1 X_1^3 \]
\[ = \sqrt{2D_1 X_1 W^H(t)} + \sqrt{2D_2 W^2(t)}, \]
\[ \ddot{X}_2 + \beta_{21} \dot{X}_1 + \beta_{22} \dot{X}_2 + \omega^2 X_2 + \alpha_2 X_2^3 \]
\[ = \sqrt{2D_3 X_2 W^2(t)} + \sqrt{2D_4 W^4(t)}. \]

(22) where \( \beta_{ij}, \omega, \alpha_i, (i = 1, 2) \) are constants; \( W^H(t), W^2(t) \) are independent unit fGn with Hurst index \( H \) and with PSD in Eq. (6); \( 2D_1, 2D_2, 2D_3, 2D_4 \) play the role of modulating the excitation intensity for fGns. Letting \( X_1 = q_1, X_1 = p_1, X_2 = q_2, \dot{X}_2 = p_2 \), the system (22) can be transformed to the form of quasi-Hamiltonian system (7). The associated Hamiltonian is

\[ H = H_1 + H_2, H_i = \frac{1}{2} p_i^2 + U_i(q_i), \]
\[ U_i(q_i) = \frac{1}{2} \alpha_i q_i^2 + \frac{1}{4} \alpha_i q_i^4. \]

(23)
Using the method in Ref. [19], the following averaged FSDEs governing Hamiltonian $H_1(t)$, $H_2(t)$ can be obtained

$$
\begin{align*}
dH_1 &= a_1(H_1, H_2)dt + \sigma_{1k}(H_1, H_2)d^{-B_k^H}(t), \\
dH_2 &= a_2(H_1, H_2)dt + \sigma_{2k}(H_1, H_2)d^{-B_k^H}(t),
\end{align*}
$$

where

$$
\begin{align*}
a_1 &= \langle -\beta_{11}p_1^2 \rangle, \\
a_2 &= \langle -\beta_{22}p_2^2 \rangle, \\
b_{11} &= \sigma_{1k}\sigma_{1k} = \langle 2D_2p_1^2 + 2D_1q_1^2p_1^2 \rangle, \\
b_{22} &= \sigma_{2k}\sigma_{2k} = \langle 2D_4p_2^2 + 2D_3q_2^2p_2^2 \rangle, \\
b_{12} &= b_{21} = \sigma_{1k}\sigma_{2k} = 0.
\end{align*}
$$

Then, treating $fGn$ as wide-band noise and applying the method described in the previous section, the following averaged Itô SDEs are obtained

$$
dA_i = m_i(A)dt + \sigma_{ik}(A)dB_k(t), \quad i = 1, 2; \quad k = 1, 2, 3, 4.
$$

where

$$
m_i(A) = \frac{-A_i^2}{8g_i}\beta_{ii}\left(4\omega_i^2 + \frac{5}{2}a_iA_i^2\right) + \frac{\pi A_i}{32g_i} \times \sum_{n=2}^{\infty} \left\{ A_i \left( c_{i,n-2} - c_{i,n+2} \right) \right\} S_{2i-1}(n\omega_i)
$$

\[ + 4 \left( c_{i,n-2} - c_{i,n} \right) \left( \frac{\sqrt{A_i^2 - 2U_1(q_1)}}{g_i} \right) \left( \frac{-1}{g_i} \right) S_{2i}(n-1\omega_i) \] (25)
$b_{ii}(A) = \sum_{n=2}^{\infty} \left\{ A_i^2 (c_i, n-2 - c_i, n+2)^2 S_{2i-1}(n\bar{\omega}_i) + 4(c_i, n-2 - c_i, n)^2 S_{2i}(n-1)\bar{\omega}_i) \right\}$

$\bar{\omega}_i = c_{i,0}$. In terms of the following relations between $A_i$ and $H_i$

$A_i = U_i^{-1}(H_i) = \sqrt{\left(\sqrt{\omega_i^2 + 4\alpha_i H_i - \omega_i^2}\right)/\alpha_i} \quad (28)$

the averaged Itô SDEs for $H_i$ are then

$dH_i = \tilde{m}_i(H)dt + \tilde{\sigma}_{1i}(H)dB_{1i}(t) + \tilde{\sigma}_{2i}(H)dB_{2i}(t) \quad (29)$
FIGURE 6. Mean value $E[H_1], E[H_2]$ and mean-square $E[Q_1^2], E[P_1^2]$ with varying natural frequencies $\omega_1, \omega_2$. $\mathcal{H} = 0.7$.

FIGURE 7. Stationary marginal PDF $p(q_1, p_1)$ of system (22), $\omega_1 = 1.414$, $\omega_2 = 1$ and $\mathcal{H} = 0.7$.

Thus, the stationary FPK equation associated with Eq. (29) is

$$-rac{\partial}{\partial H_i} [\bar{m}_i(H)p] + \frac{1}{2} \frac{\partial^2}{\partial H_i^2} [\bar{b}_{ii}(H)p] = 0, \quad i = 1, 2. \quad (31)$$

The boundary condition is

$$p(H_1, H_2 \to \infty) = p(H_1 \to \infty, H_2) = 0 \quad \int_0^\infty \int_0^\infty p(H_1, H_2)dH_2dH_1 = 1. \quad (32)$$

where

$$\bar{m}_i(H) = \left[ (\omega_i^2A_i + \alpha_i A_i^3)m_i(A) + \frac{1}{2}(\omega_i^2A_i^3) \right] \bigg|_{A_i = U_i^{-1}(H_i)} + 3\omega_i A_i^2 b_{ii}(A) \bigg|_{A_i = U_i^{-1}(H_i)}$$

$$\bar{b}_{ii}(H) = \left[ (\omega_i^2A_i + \alpha_i A_i^3)^2 b_{ii}^2(A) \right] \bigg|_{A_i = U_i^{-1}(H_i)}$$

$$\bar{b}_{12}(H) = \bar{b}_{21}(H) = 0 \quad (30)$$
We can get the stationary PDF $p(H_1, H_2)$ by solving Eq. (31) using Peaceman-Rachford scheme with conditions (32). And the stationary PDF $p(q_1, q_2, p_1, p_2)$ is then obtained as follows [20]

$$p(q_1, q_2, p_1, p_2) = \frac{\bar{\omega}_1 \bar{\omega}_2}{4\pi^2} p(H_1, H_2) \bigg|_{H_i = \frac{1}{4} p_i^2 + U(q_i)}. \tag{33}$$

The other statistics, such as the marginal stationary PDFs $p(H_1), p(q_1), p(q_1, p_1)$ and the momenta $E[H_1], E[Q_1^2]$, can be calculated as follows

$$p(H_1) = \int_0^{H_2} p(H_1, H_2) dH_2, \quad E[H_1] = \int_0^{H_2} H_1 p(H_1) dH_1,$$

$$p(q_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(q_1, q_2, p_1, p_2) dp_1 dp_2 dq_2.$$
\[ E[Q^2_1] = \int_{-\infty}^{\infty} q^2_1 p(q_1) dq_1, \]
\[ p(q_1, p_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(q_1, q_2, p_1, p_2) dq_2 dp_2 \]  
(34)

Some numerical calculations are performed, including simulation from original system (22), simulation of averaged FSDE (24) and calculation from averaged expressions in Eqs. (33)-(34). The system parameters in all figures are \( \beta_{11} = 0.1, \beta_{12} = 0.05, \beta_{21} = 0.05, \beta_{22} = 0.2, \alpha_1 = 1, \alpha_2 = 0.6, \]
\( D_1 = 0.04, D_2 = 0.1, D_3 = 0.04 \) and \( D_4 = 0.1 \).

Summary, two kinds of averaged SDEs have been built to govern the Hamiltonian processes \( H_1(t), H_2(t) \). One is FSDEs in Eq. (24) that is obtained by using the stochastic averaging method for quasi-Hamiltonian systems excited by fGn [13], [14]. Another is Itô SDEs in Eq. (29) and its corresponding formula (33) and (34) that are obtained by using the method in Sec. III. The response in Eq. (24) is not Markov process and the probability density and statistics of the response have to be obtained using Monte Carlo simulation while the response in Eq. (29) is Markov diffusion process and the analytical expressions for the probability density and statistics of the response can be obtained via solving the averaged FPK equation. So, the stochastic averaging leading to Eq. (29) is more advantageous than that leading to Eq. (24).

The stationary PDFs \( p(H_1), p(H_2) \) and mean values \( E[H_1], E[H_2] \) shown in Figs. 2-3 are obtained from simulation of original system (22), simulation of averaged SDEs and analytical expressions (33), (34), respectively. Due to the Euler integration scheme in the simulation of averaged SDEs (24), the results from simulation of averaged SDEs (24) are more scattering than those from simulation of original system (22) The results from analytical expressions (33), (34) are in better agreement with those from simulation of original system (22).

The results shown in Figs. 4-8 further indicate the validity of analytical expressions (33), (34) in wide range of Hurst index \( H \) provided the natural frequencies \( \omega_1, \omega_2 \) of the system are larger than some values, e.g., larger than 0.6, since 0.6 is roughly a boundary value between steep range and flat range of PSD (see Fig. 1). Accordingly, Fig. 6 show that the response prediction is fairly effective in flat range while ineffective in steep range. The reason, from the physical view, the response amplitude mainly depends on the system bandwidth and together with those part of noise bandwidth that fall into the range of system bandwidth.

V. CONCLUDING REMARKS

The stochastic averaging method for quasi-Hamiltonian systems under fGn excitation previously developed by us can reduce the dimension and simplify the system equation. However, the derived averaged FSDEs can only be simulated to obtain the numerical results of the response due to its non-Markov property. In the present paper, based on the observation that the PSD of fGn is quite flat for large frequency, the stochastic averaging method for quasi-integrable Hamiltonian systems under wide-band random excitation is applied to quasi-integrable and non-resonant Hamiltonian systems under fGn excitation. The response of averaged system is an approximate diffusion process and averaged FPK equation can be established and solved to yield the approximately analytical PDF and statistics of the response of original system. It has been shown via an example that the stochastic averaging method for quasi-integrable Hamiltonian systems under wide-band random excitation is applicable to quasi-integrable and non-resonant Hamiltonian systems under fGn excitation for wide range of Hurst index provided the natural frequencies are larger than some value, e.g., larger than 0.6 for studied example.

The novelty and main contribution of this paper are getting an analytical prediction for the response of quasi Hamiltonian system driven by fGn by applying the stochastic averaging method for quasi Hamiltonian system under wide-band noise for the first time. This application provide some effective parameter conditions, and more important, avoid handling the non-Markov problem of system response.

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