On the local well-posedness of a Benjamin-Ono-Boussinesq system

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Abstract
Consider a Benjamin-Ono-Boussinesq system
\[
\begin{align*}
\eta_t + u_x + a u_{xxx} + (u\eta)_x &= 0, \\
u_t + \eta_x + u u_x + c \eta_{xxx} - d u_{xxt} &= 0,
\end{align*}
\]
where \(a, c\) and \(d\) are constants satisfying
\[
a = c > 0, \quad d > 0 \quad \text{or} \quad a < 0, \quad c < 0, \quad d > 0. 
\]
We prove that this system is locally well-posed in Sobolev space \(H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})\) with \(s > \frac{1}{4}\).

Keywords: Boussinesq equation, local well-posedness.
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1 Introduction and main results
We consider the Cauchy problem for a Benjamin-Ono-Boussinesq system
\[
\begin{align*}
\eta_t + u_x + a u_{xxx} + (u\eta)_x &= 0, & t > 0, \quad x \in \mathbb{R}, \\
u_t + \eta_x + u u_x + c \eta_{xxx} - d u_{xxt} &= 0, & t > 0, \quad x \in \mathbb{R}, \\
\eta|_{t=0} = f(x), \quad u|_{t=0} = g(x),
\end{align*}
\]
where \(a, c\) and \(d\) are constants satisfying
\[
a = c > 0, \quad d > 0 \quad \text{or} \quad a < 0, \quad c < 0, \quad d > 0. 
\]
The system is called a Benjamin-Ono-Boussinesq system because it can be reduced to a pair of equations whose linearization uncouples to a pair of linear Benjamin-Ono equations.

Equations of type (1.1) are a class of essential model equations appearing in physics and fluid mechanics. To describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel, Boussinesq in 1871 derived from the Euler equation the classical Boussinesq system
\[
\begin{align*}
\eta_t + u_x + (u\eta)_x &= 0, & t > 0, \quad x \in \mathbb{R}, \\
u_t + \eta_x + u u_x + \frac{\eta}{3} \eta_{xxt} &= 0, & t > 0, \quad x \in \mathbb{R}.
\end{align*}
\]
In [1], Bona, Chen and Saut derived by considering first-order approximations to the Euler equation the following alternative ( a four-parameter Boussinesq system ) to the classical Boussinesq system
\[
\begin{align*}
\eta_t + u_x + a u_{xxx} + (u\eta)_x - b \eta_{xxt} &= 0, \\
u_t + \eta_x + u u_x + c \eta_{xxx} - d u_{xxt} &= 0,
\end{align*}
\]
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where the constants obey the relations
\[ a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad a + b + c + d = \frac{1}{3}, \]
with \( \theta \in [0, 1] \). The system (1.1) is one of the four-parameter systems associated with \( b = 0 \). When \( b = 0 \), Bona, Chen and Saut in [1] determined exactly that the four-parameter systems are linearly well posed if and only if \( a, c \) and \( d \) satisfy the relation (1.2). The local well-posedness of the nonlinear system (1.1) is considered in [2]. They prove that the system (1.1) associated with (1.2) is locally well-posed in the Sobolev space \( H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \) with \( s \geq 1 \). In this work we shall give some local well-posedness for the Cauchy problem (1.1) in the Sobolev spaces \( H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \) with \( s > \frac{1}{3} \) by using the so-called \( L^p - L^q \) smoothing effect of the Strichartz type.

Denote by \( J \) the Fourier multiplier with symbol \( (1 + \xi^2)^{1/2} \), and denote by \( \mathcal{H} \) the usual Hilbert transform. Our result is

**Theorem 1.1** Fix \( s > \frac{1}{4} \). Then for every \( (f(x), g(x)) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \), there exist \( T > 0 \) depending only on \( \|f(x)\|_{H^s} + \|g(x)\|_{H^{s+1}} \) and a unique solution of (1.1) on the time interval \([0, T]\) satisfying

\[
(J^{-1}\mathcal{H}\eta, u) \in C([0, T], L^2(\mathbb{R}) \times L^2(\mathbb{R})), \quad (J^{-1}\mathcal{H}\eta_{t}, u_{x}) \in C([0, T], L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})).
\]

Moreover, for any \( R > 0 \), there exists \( T \) depending on \( R \) such that the nonlinear map \( (f(x), g(x)) \rightarrow (\eta, u) \) is continuous from the ball of radius \( R \) of \( H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \) to \( C([0, T]; H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})) \).

In the sequel, we say the pair \((p, q) \in \mathbb{R}^2\) admissible if they satisfy

\[
\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad p > 4.
\]

We denote by \( J \) the Fourier multiplier with symbol \((1 + \xi^2)^{1/2}\), denote by \( \mathcal{H} \) the usual Hilbert transform and denote by \( m(D) \) the Fourier multiplier associated with symbol \( m(\xi) \). We also denote the dyadic integers \( 2^k, k \geq 0 \), by \( \lambda \) or \( \mu \). Whenever a summation over \( \lambda \) or \( \mu \) appears it means that we sum over the dyadic integers. The notation \( A \lesssim B \) (resp., \( A \gtrsim B \)) means that there exists a harmless positive constant \( C \) such that \( A \leq CB \) (resp., \( A \geq CB \)). We denote by \( L^p_X \) (resp., \( L^p_Y \)) the space of \( X \)-valued measurable and \( p \)-integrable functions defined on \([0, T]\) (resp., \( I \)), equipped with the natural norm. We also use the notation \( \|u_1, u_2, \ldots, u_k\|_X = \|u_1\|_X + \cdots + \|u_k\|_X \).

The rest of this paper is organized as follows. In section 2 we prove some Strichartz type estimates for smooth solutions of (1.1). In section 3 we give the proof of the local well-posedness of the Cauchy problem (1.1).

2 Some estimates

In this section we give some smoothing effects for the equation (1.1). These estimate will be the main ingredient in the proof of local well-posedness of the Cauchy problem (1.1). Consider the following linear system

\[
\begin{aligned}
\eta_t + u_x + au_{xxx} &= F(x, t), \quad t > 0, \quad x \in \mathbb{R}, \\
u_t + \eta_x + c\eta_{xx} - du_{xxt} &= G(x, t), \quad t > 0, \quad x \in \mathbb{R}.
\end{aligned}
\] (2.1)

Let

\[
\sigma(\xi) = \left[ \frac{(a\xi^2 - 1)(c\xi^2 - 1)}{d\xi^2 + 1} \right]^{1/2}, \quad h(\xi) = \left[ \frac{(a\xi^2 - 1)(c\xi^2 + 1)}{c\xi^2 - 1} \right]^{1/2}.
\] (2.2)
Consider the change of variables
\[ \eta = h(D)(v + w), \ u = v - w \text{ and } \tilde{\eta} = h^{-1}(D)\eta = v + w, \] (2.3)
where \( h(D) \) (resp. \( h^{-1}(D) \)) is the Fourier multiplier with the symbol \( h(\xi) \) (resp. \( h^{-1}(\xi) \)). Then we have
\[
\begin{aligned}
v_t + \sigma(D)\partial_x v &= \frac{1}{2}h^{-1}(D)F + \frac{1}{2}(1 + dD^2)^{-1}G, \\
w_t - \sigma(D)\partial_x w &= \frac{1}{2}h^{-1}(D)F - \frac{1}{2}(1 + dD^2)^{-1}G.
\end{aligned}
\]

Rewrite \( \sigma(\xi) = (ac/d)^{1/2}|\xi| + \gamma(\xi) \) with
\[ \gamma(\xi) = \frac{d - (ac + ad + dc)\xi^2}{d[(a\xi^2 - 1)(d\xi^2 + 1)(c\xi^2 - 1)]^{1/2} + (acd)^{1/2}|\xi|(d\xi^2 + 1)} \]

Let \( \Gamma(D) \) be the Fourier multiplier with the symbol \( i\xi\gamma(\xi) \), which is a skew-adjoint operator in \( L^2(\mathbb{R}) \) and is bounded from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \) for \( 1 < p < +\infty \). Then
\[
\begin{aligned}
v_t + (ac/d)^{1/2}\partial_x v &= \Gamma(D)v + \frac{1}{2}h^{-1}(D)F + \frac{1}{2}(1 + dD^2)^{-1}G, \\
w_t - (ac/d)^{1/2}\partial_x w &= -\Gamma(D)w + \frac{1}{2}h^{-1}(D)F - \frac{1}{2}(1 + dD^2)^{-1}G.
\end{aligned}
\] (2.4)

Using the Strichartz inequality for the constant coefficients Benjamin-Ono equation to the system (2.4) we deduce, for admissible pair \( (p, q) \),
\[
\|v\|_{L^p_t L^q_x} \lesssim \|v\|_{L^p_t L^q_x} + \|\Gamma(D)v + \frac{1}{2}h^{-1}(D)F + \frac{1}{2}(1 + dD^2)^{-1}G\|_{L^1_t L^2_x}
\]
\[
\lesssim \|v\|_{L^p_t L^q_x} + \|v\|_{L^2_t L^2_x} + \|h^{-1}(D)F + (1 + dD^2)^{-1}G\|_{L^1_t L^2_x}
\]
\[
\lesssim (1 + |I|)\|v\|_{L^p_t L^q_x} + \|h^{-1}(D)F\|_{L^1_t L^2_x} + \|(1 + dD^2)^{-1}G\|_{L^1_t L^2_x},
\]
and similarly
\[
\|w\|_{L^2_t L^2_x} \lesssim (1 + |I|)\|w\|_{L^2_t L^2_x} + \|h^{-1}(D)F\|_{L^1_t L^2_x} + \|(1 + dD^2)^{-1}G\|_{L^1_t L^2_x}.
\]
Thus by (2.3),
\[
\|(\tilde{\eta}, u)\|_{L^p_t L^q_x} \lesssim (1 + |I|)\|u\|_{L^p_t L^q_x} + \|h^{-1}(D)F(x, t)\|_{L^1_t L^2_x} + \|(1 + dD^2)^{-1}G(x, t)\|_{L^1_t L^2_x}.
\] (2.5)

Consider a standard Littlewood-Paley decomposition
\[
u = \sum_{\lambda} u_{\lambda}, \ \eta = \sum_{\lambda} \eta_{\lambda}, \ \tilde{\eta} = \sum_{\lambda} \tilde{\eta}_{\lambda},
\]
where
\[
u_{\lambda} = \Delta_{\lambda}u, \ \eta_{\lambda} = \Delta_{\lambda}\eta \text{ and } \tilde{\eta}_{\lambda} = \Delta_{\lambda}\tilde{\eta},
\]
\(\Delta_{\lambda}\) are the Fourier multipliers with symbols \( \phi(\xi/\lambda) \) when \( \lambda = 2^k \) with \( k \geq 1 \) and \( \chi(\xi) \) when \( \lambda = 1 \), and where the nonnegative functions \( \chi \in C_0^\infty(\mathbb{R}) \) and \( \phi \in C_0^\infty(\mathbb{R}) \) satisfy
\[
\chi(\xi) + \sum_{\lambda} \phi(\xi/\lambda) = 1
\]
and
\[
\phi(\xi) = \begin{cases} 
0, & \text{if } |\xi| \leq \frac{\delta}{2} \text{ or } |\xi| \geq 2, \\
1, & \text{if } 1 \leq |\xi| \leq \frac{\delta}{2}.
\end{cases}
\]
For a dyadic integer \( \lambda \) we set
\[
\tilde{\Delta}_{\lambda} = \begin{cases} 
\Delta_{\lambda^2} + \Delta_{\lambda} + \Delta_{2\lambda}, & \text{if } \lambda > 1, \\
\Delta_{1} + \Delta_{2}, & \text{if } \lambda = 1.
\end{cases}
\]
Lemma 2.1 Let \( \{a_\lambda\} \), \( \{d_\lambda\} \) and \( \{\delta_\lambda\} \) be three sequences indexed on positive dyadic integers \( \lambda \). Assume that there exist two positive constants \( 1 < \kappa_1 < \kappa_2 \) such that \( \kappa_1 \delta_\lambda \leq \delta_{2^\lambda} \leq \kappa_2 \delta_\lambda \). Then
\[
\sum_\lambda \delta_\lambda \sum_{\mu \geq \lambda / 8} a_\mu d_\lambda \lesssim \left( \sum_\lambda \delta_\lambda^2 a_\lambda^2 \right)^{1/2} \left( \sum_\lambda d_\lambda^2 \right)^{1/2},
\]
and hence by duality
\[
\sum_\lambda \delta_\lambda^2 \left( \sum_{\mu \geq \lambda / 8} a_\mu \right)^2 \lesssim \sum_\lambda \delta_\lambda^2 a_\lambda^2.
\]

Proof. Notice that \( \kappa_1 \delta_\lambda \leq \delta_{2^\lambda} \leq \kappa_2 \delta_\lambda \) implies \( \delta_\lambda / \delta_{2^\lambda} \leq \kappa_1^{-k} \). Using the Cauchy-Schwarz inequality we get
\[
\sum_\lambda \delta_\lambda \sum_{\mu \geq \lambda / 8} a_\mu d_\lambda = \sum_\lambda \delta_\lambda \sum_{k=-3}^{\infty} \sum_{\lambda \geq 2^{k}} a_{2^k \lambda} d_\lambda = \sum_{k=-3}^{\infty} \kappa_1^{-k} \sum_{\lambda \geq 2^{-k}} \delta_{2^\lambda} a_{2^\lambda \lambda} d_\lambda \leq \sum_{k=-3}^{\infty} \kappa_1^{-k} \left( \sum_{\lambda \geq 2^{-k}} \delta_\lambda^2 a_{2^\lambda \lambda}^2 \right)^{1/2} \left( \sum_{\lambda \geq 2^{-k}} d_\lambda^2 \right)^{1/2} \lesssim \left( \sum_\lambda \delta_\lambda^2 a_\lambda^2 \right)^{1/2} \left( \sum_\lambda d_\lambda^2 \right)^{1/2}.
\]

Lemma 2.2 Fix \( T > 0 \) and \( \sigma > 1/2 \). Let \((\eta, u)\) be a smooth solution of the system \((1.1)\). Then for every admissible pair \((p, q)\),
\[
\left\{ \lambda^{2\sigma} \left\| (H\bar{\eta}_\lambda, u_\lambda) \right\|_{L^2_p L^\infty}^2 \right\}^{1/2} \lesssim (1 + T)^{1/p} \left( 1 + \| J^\sigma u \|_{L^p L^2} \right) \times \left( 1 + \| (u, u_x, H\bar{\eta}, \partial_x H\bar{\eta}, \bar{\eta}) \|_{L^1_T L^\infty} \right) \left( \sum_\lambda \lambda^{2/p + 2\sigma} \left\| (H\bar{\eta}_\lambda, u_\lambda) \right\|_{L^2_p L^2}^2 \right)^{1/2},
\]
where \( \bar{\eta} = h^{-1}(D)\eta \), \( \bar{\eta}_\lambda = h^{-1}(D)\Delta_\lambda \eta \), \( h^{-1}(D) \) is the Fourier multiplier with the symbol \( h^{-1}(\xi) \) defined in \((2.2)\).

Proof. Let \((\eta, u)\) be a smooth solution of the system \((1.1)\). Then \((\eta_\lambda, u_\lambda)\) satisfies the following system
\[
\begin{cases}
(\eta_\lambda)_t + (u_\lambda)_x + a(u_\lambda)_{xxx} = -\left( u h(D) \bar{\eta}_\lambda + [\Delta_\lambda, u h(D)] \bar{\eta}_\lambda \right)_x, & t > 0, x \in \mathbb{R}, \\
(u_\lambda)_t + (\eta_\lambda)_x + c \eta_\lambda_{xxx} - d(u_\lambda)_{xx} = -\left( u_\lambda(\lambda)_x + [\Delta_\lambda, u \partial_x] u \right)_x, & t > 0, x \in \mathbb{R}.
\end{cases}
\]
Using \((2.5)\) to the system \((2.6)\) and choosing the interval \( I \) satisfying \( |I| \leq 1 \) and \( |I| \leq \frac{1}{\kappa_1} \), we get
\[
\begin{align*}
\| (H\bar{\eta}_\lambda, u_\lambda) \|_{L^\infty_T L^\infty} & \lesssim \| (\bar{\eta}_\lambda, u_\lambda) \|_{L^\infty_T L^\infty} \\
& \lesssim (1 + |I|) \| (\bar{\eta}_\lambda, u_\lambda) \|_{L^\infty_T L^2} + \| \partial_x h^{-1}(D) (u h(D) \bar{\eta}_\lambda + [\Delta_\lambda, u h(D)] \bar{\eta}_\lambda) \|_{L^1_T L^2} \\
& \quad + \| (1 + dD^2)^{-1} (u_\lambda (\lambda)_x + [\Delta_\lambda, u \partial_x] u) \|_{L^1_T L^2} \\
& \lesssim (1 + |I|) \| (\bar{\eta}_\lambda, u_\lambda) \|_{L^\infty_T L^2} + \| u \|_{L^\infty_T L^\infty} \| h(D) \bar{\eta}_\lambda \|_{L^1_T L^2} \\
& \quad + \| [\Delta_\lambda, u h(D)] \bar{\eta}_\lambda \|_{L^1_T L^2} + \| u_\lambda (\lambda)_x \|_{L^1_T L^2} + \| [\Delta_\lambda, u \partial_x] u \|_{L^1_T L^2} \\
& \lesssim (1 + |I|) \| (\bar{\eta}_\lambda, u_\lambda) \|_{L^\infty_T L^2} + \| u \|_{L^\infty_T L^\infty} \| h(D) \bar{\eta}_\lambda \|_{L^1_T L^2} \\
& \quad + \| [\Delta_\lambda, u h(D)] \bar{\eta}_\lambda \|_{L^1_T L^2} + \| \bar{\eta}_\lambda \|_{L^1_T L^\infty} + \| [\Delta_\lambda, u \partial_x] u \|_{L^1_T L^2}.
\end{align*}
\]
By the Sobolev embedding, $|I| \leq 1$ and $|I| \leq 1/\lambda$,  
\[ \|u\|_{L_t^\infty L_x^\infty} \lesssim \|J^\sigma u\|_{L_t^\infty L_x^2} \]  
with $\sigma > 1/2$, and  
\[ \|h(D)\tilde{\eta}_\lambda\|_{L_t^1 L_x^2} \lesssim \|\tilde{\eta}_\lambda\|_{L_t^1 L_x^2} + \|\partial_\xi \tilde{\eta}_\lambda\|_{L_t^1 L_x^2} \lesssim |I| \|\tilde{\eta}_\lambda\|_{L_t^\infty L_x^\infty} + |I| |\lambda| \|\tilde{\eta}_\lambda\|_{L_t^\infty L_x^\infty} \lesssim \|\tilde{\eta}_\lambda\|_{L_t^\infty L_x^2}, \]  
and  
\[ \|\partial_\xi u\lambda\|_{L_t^1 L_x^2} \lesssim |I| |\lambda| \|\tilde{\eta}_\lambda\|_{L_t^\infty L_x^\infty} \lesssim \|u\lambda\|_{L_t^\infty L_x^2}. \]  
Then, we deduce from (2.7),  
\[ \|(H\tilde{\eta}_\lambda, u\lambda)\|_{L_t^p L_x^q} \lesssim (1 + \|J^\sigma u\|_{L_t^\infty L_x^2})\|(H\tilde{\eta}_\lambda, u\lambda)\|_{L_t^p L_x^q} + \||\Delta_\lambda, uh(D)\tilde{\eta}_\lambda\|_{L_t^1 L_x^2} + \||\Delta_\lambda, u\partial_\xi|u\|_{L_t^1 L_x^2}. \]  
(2.8)  
Partition $[0, T] = \cup_k I_k$, where each interval $I_k$ is of size $\lesssim \min\{1/\lambda, 1\}$. We can choose $I_k$ such that their number is bounded by $(1 + T)\lambda$. Therefore by (2.8) we obtain  
\[ \|(H\tilde{\eta}_\lambda, u\lambda)\|_{L_t^p L_x^q} \lesssim (1 + T)^{1/p} \lambda^{1/p}(1 + \|J^\sigma u\|_{L_t^\infty L_x^2}) \times \left( \|(H\tilde{\eta}_\lambda, u\lambda)\|_{L_t^p L_x^q} + \||\Delta_\lambda, uh(D)\tilde{\eta}_\lambda\|_{L_t^1 L_x^2} + \||\Delta_\lambda, u\partial_\xi|u\|_{L_t^1 L_x^2} \right). \]  
(2.9)  
To estimate the terms $\||\Delta_\lambda, uh(D)\tilde{\eta}_\lambda\|_{L_t^1 L_x^2}$ and $\||\Delta_\lambda, u\partial_\xi|u\|_{L_t^1 L_x^2}$, we introduce the following estimates which come from Lemma 4.2 and Lemma 4.3 in [3]:  
\[ \||\Delta_\lambda, v\partial_\xi|w\|_{L_x^2} \lesssim \|v\|_{L_t^\infty} \|w\|_{L_x^2}, \||\Delta_\lambda, v|w\|_{L_x^2} \lesssim \|v\|_{L_t^\infty} \|w\|_{L_x^2}. \]  
(2.10)  
Then, for $\lambda \geq 4$,  
\[ \||\Delta_\lambda, uh(D)\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_t^2} \leq \||\Delta_\lambda, u\partial_\xi\|_{L_t^2} \left( \frac{h(D)}{iD} \tilde{\Delta}_\lambda\tilde{\eta} \right)_{L_x^2} \lesssim \|u\|_{L_t^\infty} \|\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_x^2}, \]  
and for $\lambda = 1, 2$,  
\[ \||\Delta_\lambda, uh(D)\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_x^2} = \||\Delta_\lambda, uh(D)\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_x^2} \leq \|u\|_{L_t^\infty} \|h(D)\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_x^2} \lesssim \|u\|_{L_t^\infty} \|\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_x^2}, \]  
so we get  
\[ \||\Delta_\lambda, uh(D)\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_t^1 L_x^2} \lesssim \left( \|u\|_{L_t^1 L_x^\infty} + \|u\|_{L_t^1 L_x^2} \right) \|\tilde{\Delta}_\lambda\tilde{\eta}\|_{L_t^\infty L_x^2}. \]  
(2.11)  
Now we consider the term $\||\Delta_\lambda, uh(D)\|_{(Id - \tilde{\Delta}_\lambda)\tilde{\eta}}$, that is $\|\Delta_\lambda \left( uh(D)(Id - \tilde{\Delta}_\lambda)\tilde{\eta} \right)\|_{L_x^2}$. Notice that the frequencies of order $\lambda \leq \lambda/8$ in the Littlewood-Paley decomposition of $u$ do not contribute, therefore  
\[ \|\Delta_\lambda \left( uh(D)(Id - \tilde{\Delta}_\lambda)\tilde{\eta} \right)\|_{L_t^1 L_x^2} = \left| \|\Delta_\lambda \sum_{\mu \geq \lambda/8} u\mu h(D)(Id - \tilde{\Delta}_\lambda)\tilde{\eta} \right|_{L_t^1 L_x^2} \lesssim \sum_{\mu \geq \lambda/8} \|\Delta_\lambda h(D)(Id - \tilde{\Delta}_\lambda)\tilde{\eta}\|_{L_t^\infty L_x^\infty} \|u\|_{L_t^\infty L_x^2}. \]  
(2.12)
Denote by
\[
A(\xi) = h(\xi) - \left(\frac{ad}{c}\right)^{1/2} |\xi|
\]
\[
= \frac{(ad + ac - cd)\xi^2 - c}{c[(a\xi^2 - 1)(c\xi^2 - 1)(d\xi^2 + 1)]^{1/2} + (acd)^{1/2}\xi|\xi^2 - 1|}.
\] (2.13)

Let \(\psi(\xi) \in C^\infty(\mathbb{R})\) be a nonnegative function satisfying \(\psi(\xi) = 1\) for \(|\xi| > 2\) and \(\psi(\xi) = 0\) for \(|\xi| \leq 1\). It is obvious that \(\psi(\xi)A(\xi)/|\xi| \in S^{-1}\), that is, \(\psi(\xi)A(\xi)|\xi| \in C^\infty(\mathbb{R})\) satisfies
\[
|\partial_\xi^\alpha (\psi(\xi)A(\xi)|\xi|/|\xi|)| \lesssim (1 + \xi^2)^{-\frac{1-\gamma}{2}}
\]
for any nonnegative integers \(\alpha\). Let \(\Lambda_\gamma\) be the Lipschitz space defined by
\[
\Lambda_\gamma = \{ f : \text{there exists a positive constant } A \text{ such that } \|f\|_{L^\infty} \leq A, \|\Delta f\|_{L^\infty} \leq A\lambda^{-\gamma}\}.
\]

Proposition 6 in [5] (Ch.6 \S 5.3) shows that \(\psi(D)A(D)\mathcal{H}\) is a bounded mapping from \(\Lambda_{1/2}\) to \(\Lambda_{3/2}\). We have
\[
\|\Delta_\lambda h(D)(Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty} \lesssim \|\psi(D)\partial_\xi (Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty}
\]
\[
\lesssim \|\psi(D)\partial_\xi (Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty} + \|A(D)\psi(D)\mathcal{H}(Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty}
\]
\[
\lesssim \|\partial_\xi (\mathcal{H}\vec{\eta})\|_{L^1_\lambda L^\infty} + \|A(D)\psi(D)\mathcal{H}\vec{\eta}\|_{L^1_\lambda L^\infty} + \|\Delta h(D)(Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty},
\] (2.14)
and
\[
\|\Delta_\lambda h(D)(Id - \psi(D))(Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty} \lesssim \|h(D)(Id - \psi(D))\vec{\eta}\|_{L^1_\lambda L^\infty} \lesssim \|\vec{\eta}\|_{L^1_\lambda L^\infty},
\]
because of \(h(\xi)(1 - \psi(\xi)) \in C^\infty(\mathbb{R})\). Hence we get
\[
\|\Delta_\lambda h(D)(Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty} \lesssim \|\Delta_\lambda h(D)(Id - \vec{\Delta}_\lambda)\vec{\eta}\|_{L^1_\lambda L^\infty}
\]
\[
\lesssim \|\partial_\xi (\mathcal{H}\vec{\eta})\|_{L^1_\lambda L^\infty} + \|\mathcal{H}\vec{\eta}\|_{L^1_\lambda L^\infty} + \|\vec{\eta}\|_{L^1_\lambda L^\infty}.
\] (2.15)

A combination (2.11) with (2.12) and (2.15) yields
\[
\|\Delta_\lambda \left(uh(D)(Id - \vec{\Delta}_\lambda)\vec{\eta}\right)\|_{L^1_\lambda L^\infty} \lesssim \left(\|\partial_\xi \mathcal{H}\vec{\eta}\|_{L^1_\lambda L^\infty} + \|\mathcal{H}\vec{\eta}\|_{L^1_\lambda L^\infty} + \|\vec{\eta}\|_{L^1_\lambda L^\infty}\right) \sum_{\mu \geq \lambda/8} \|u_\mu\|_{L^\infty T^L_2},
\]
and so
\[
\|\|\Delta_\lambda, uh(D)\|_{\mathcal{F}_\lambda} \|_{L^1_\lambda L^\infty} \lesssim \left(\|u\|_{L^1_\lambda L^\infty} + \|u_\lambda\|_{L^1_\lambda L^\infty}\right) \|\Delta_\lambda \mathcal{H}\vec{\eta}\|_{L^1_\lambda L^\infty} + \left(\|\partial_\xi (\mathcal{H}\vec{\eta})\|_{L^1_\lambda L^\infty} + \|\mathcal{H}\vec{\eta}\|_{L^1_\lambda L^\infty} + \|\vec{\eta}\|_{L^1_\lambda L^\infty}\right) \sum_{\mu \geq \lambda/8} \|u_\mu\|_{L^\infty T^L_2}.
\] (2.16)

Similarly,
\[
\|\|\Delta_\lambda, u\partial_\lambda u\|_{L^1_\lambda L^\infty} \lesssim \|u_\lambda\|_{L^1_\lambda L^\infty} \|\Delta_\lambda u\|_{L^1_\lambda L^\infty} + \|u_\lambda\|_{L^1_\lambda L^\infty} \sum_{\mu \geq \lambda/8} \|u_\mu\|_{L^\infty T^L_2}.
\] (2.17)
It follows from (2.9), (2.16) and (2.17) that

\[ \| (\mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^q} \leq (1 + T)^{1/p} \lambda^{2/3}(1 + \| J^\sigma u \|_{L^p_T L^2}) \times \left( \| (\mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2} + \| (\Delta \lambda, u\nu(D)\tilde{h}, \nu, \lambda) \|_{L^p_T L^2} \right) \]

\[ \leq (1 + T)^{1/p} \lambda^{2/3}(1 + \| J^\sigma u \|_{L^p_T L^2}) \times \left( \| (\lambda, \nu, \lambda) \|_{L^p_T L^2} + \| (\Delta \lambda, \partial x \mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2} \right) \]

Hence,

\[ \sum_{\lambda} \lambda^2 \| (\mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^q}^2 \]

\[ \leq \sum_{\lambda} (1 + T)^{2/p} \lambda^{2/3}(1 + \| J^\sigma u \|_{L^p_T L^2})^2 \times \left( \| (\lambda, \nu, \lambda) \|_{L^p_T L^2}^2 + \| (\Delta \lambda, \partial x \mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2}^2 \right) \]

\[ \leq (1 + T)^{2/p} (1 + \| J^\sigma u \|_{L^p_T L^2}) \left( 1 + \| (\lambda, \nu, \lambda) \|_{L^p_T L^2}^2 + \| (\Delta \lambda, \partial x \mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2}^2 \right) \]

where we have used the inequality due to Lemma 2.1

\[ \sum_{\lambda} \lambda^{2/p+2\sigma} \left( \sum_{\mu \geq 2/8} \| u_{\mu} \|_{L^p_T L^2} \right)^2 \leq \sum_{\lambda} \lambda^{2/p+2\sigma} \| u_{\lambda \nu} \|_{L^p_T L^2}^2, \]

and the inequality

\[ \sum_{\lambda} \lambda^{2/p+2\sigma} \| (\Delta \lambda, \mathcal{H}\tilde{h}, \partial x \mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2} \leq \sum_{\lambda} \lambda^{2/p+2\sigma} \| (\mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2}^2. \]

Thus we get

\[ \sum_{\lambda} \lambda^{2\sigma} \| (\mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^q}^2 \leq (1 + T)^{2/p} (1 + \| J^\sigma u \|_{L^p_T L^2})^2 \times \left( 1 + \| (\lambda, \nu, \lambda) \|_{L^p_T L^2}^2 + \| (\Delta \lambda, \partial x \mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2}^2 \right) \sum_{\lambda} \lambda^{2/p+2\sigma} \| (\mathcal{H}\tilde{h}, \nu, \lambda) \|_{L^p_T L^2}^2. \quad (2.18) \]

We complete the proof. \hfill \Box

**Lemma 2.3** Let $1 \leq \kappa_1 \leq \kappa_2$, and let $\{\delta_{\lambda}\}$ be the dyadic sequence of positive numbers satisfying $\kappa_1 \delta_{\lambda} \leq \delta_{2\lambda} \leq \kappa_2 \delta_{\lambda}$ and $\lambda \leq \delta_{\lambda} \leq \lambda^2$ for all dyadic integers $\lambda$. Then for all $\tau, t \in I$, the smooth solution $(\nu, u)$ of (1.1) satisfies

\[ \sum_{\lambda} \delta_{\lambda}^2 \| (\nu(t), u(t)) \|_{L^2}^2 \leq \sum_{\lambda} \delta_{\lambda}^2 \| (\tilde{\nu}_{\lambda}(\tau), u_{\lambda}(\tau)) \|_{L^2}^2 \exp \left( 2 \| (\nu, u, \mathcal{H}\tilde{h}, \partial x \mathcal{H}\tilde{h}) \|_{L^1_T L^\infty} \right). \]
Proof. Let $F = -(u\eta)_x$ and $G = -uw_x$ in (2.4), and without loss of generality assume $\tau < t$. Multiplying (2.4) by $(\Delta_\lambda^2 v_\lambda, \Delta_\lambda w_\lambda)$ and integrating by parts we get

$$
\frac{1}{2} \frac{d}{dt} \left( \frac{\|v_\lambda(t)\|_{L^2}^2}{\|w_\lambda(t)\|_{L^2}^2} \right) = \text{Re} \frac{d}{dt} \left( \int \overline{w_\lambda} \overline{w_\lambda} d\xi \right) = \text{Re} \left( -\int [h^{(-1)}(D)(u\eta)_x)_\lambda + (1 + dD^2)^{-1}(uw_x)_\lambda] v_\lambda dx - \int [h^{(-1)}(D)(u\eta)_x)_\lambda - (1 + dD^2)^{-1}(uw_x)_\lambda] w_\lambda dx \right),
$$

where we denote by $(\tilde{\nu}_\lambda, \tilde{w}_\lambda) = F_x(v_\lambda, v_\lambda)$. Using $(h^{(-1)}(D)\eta, u) = (v + w, v - w) = (\tilde{\eta}, u)$, we get

$$
\frac{d}{dt} \left( \frac{\|v_\lambda(t)\|_{L^2}^2}{\|w_\lambda(t)\|_{L^2}^2} \right) = 2\text{Re} \left( -\int h^{(-1)}(D) (u\eta)_x)_\lambda \tilde{\eta}_\lambda dx - \int (1 + dD^2)^{-1}(uw_x)_\lambda u_\lambda dx \right),
$$

and so for the dyadic sequence $\{\delta_\lambda\}$ we get

$$
\sum_\lambda \delta_\lambda^2 \|\tilde{\eta}_\lambda(t), u_\lambda(t)\|_{L^2}^2 \lesssim \sum_\lambda \delta_\lambda^2 \left( \|v_\lambda(t)\|_{L^2}^2 + \|w_\lambda(t)\|_{L^2}^2 \right)
$$

$$
\lesssim \sum_\lambda \delta_\lambda^2 \left( \|v_\lambda(t)\|_{L^2}^2 + \|w_\lambda(t)\|_{L^2}^2 \right) + \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 h^{(-1)}(D)(u(\sigma)\eta(\sigma))_\lambda \tilde{\eta}(\sigma) dx \right| d\sigma
$$

$$
+ \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 (1 + dD^2)^{-1}(u(\sigma)w_\lambda(\sigma))_\lambda \eta(\sigma) dx \right| d\sigma
$$

$$
\lesssim \sum_\lambda \delta_\lambda^2 \|\tilde{\eta}_\lambda(t), u_\lambda(t)\|_{L^2}^2 + I + II,
$$

(2.19)

with

$$
I = \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 h^{(-1)}(D)(u\eta)_x)_\lambda \tilde{\eta}_\lambda dx \right| d\sigma
$$

and

$$
II = \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 (1 + dD^2)^{-1}(uw_x)_\lambda u_\lambda dx \right| d\sigma.
$$

The estimate of (II). Using $\lambda \leq \delta_\lambda \leq \lambda^2$ we have

$$
II \lesssim \sum_\lambda \int_\tau^t \|\delta_\lambda^2 u_\lambda\|_{L^2} \|\delta_\lambda^2 J^{-2}(uw_x)\|_{L^2} d\sigma
$$

$$
\lesssim \int_\tau^t \left( \sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \left( \sum_\lambda \delta_\lambda^2 \|J^{-2}(uw_x)\|_{L^2}^2 \right)^{1/2} d\sigma
$$

$$
\lesssim \int_\tau^t \left( \sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \left( \sum_\lambda \lambda^4 \|J^{-2}(uw_x)\|_{L^2}^2 \right)^{1/2} d\sigma
$$

$$
\lesssim \int_\tau^t \left( \sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \|uw_x\|_{L^2} d\sigma \lesssim \int_\tau^t \left( \sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \|u\|_{L^2} \|uw_x\|_{L^2} d\sigma
$$

$$
\lesssim \int_\tau^t \|u_x\|_{L^2} \sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 d\sigma.
$$

(2.20)
The estimate of (I). To estimate the term \( I \), we introduce

\[ B(\xi) = h^{-1}(\xi) - (c/ad)^{1/2} 1/|\xi| \]  \hspace{1cm} (2.21)

with

\[ B(\xi) = \frac{(cd - ac - ad)\xi^2 + c}{(ad|\xi|[(a\xi^2 - 1)(\xi^2 - 1)]^{1/2} + (acd)^{1/2}(a\xi^2 - 1)(\xi^2 + 1))|\xi|}. \]

For \( \lambda > 2 \),

\[ \left| \sum_{\lambda > 2} \int \delta_\lambda^2 h^{-1}(D) ((u\eta)_x)_\lambda \tilde{\eta}_\lambda dx \right| \leq \sum_{\lambda > 2} \left| \int \delta_\lambda^2 B(D) ((u\eta)_x)_\lambda \tilde{\eta}_\lambda dx \right| + \sum_{\lambda > 2} \left| \int \delta_\lambda^2 B(D) ((u\eta)_x)_\lambda \tilde{\eta}_\lambda dx \right|. \]

Using \( |\phi(\xi/\lambda)B(\xi)| \leq \lambda^{-3} \) for \( \lambda > 2 \) we have

\[ \left| \sum_{\lambda > 2} \int \delta_\lambda^2 B(D) ((u\eta)_x)_\lambda \tilde{\eta}_\lambda dx \right| \leq \sum_{\lambda > 2} \delta_\lambda^2 \| B(D) ((u\eta)_x)_\lambda \|_{L^2} \| \tilde{\eta}_\lambda \|_{L^2} \]

\[ \leq \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| B(D) ((u\eta)_x)_\lambda \|_{L^2}^2 \right\}^{1/2} \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2} \]

\[ \leq \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2} \left\{ \sum_{\lambda > 2} \lambda^{-2} \| (u\eta)_x \|_{L^2}^2 \right\}^{1/2} \]

\[ \leq \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\} \| u\eta \|_{L^2}. \]  \hspace{1cm} (2.23)

By (2.13),

\[ \| u\eta \|_{L^2} = \| u \cdot h(D)\tilde{\eta} \|_{L^2} = \| u \cdot (A(D) + (ad/c)^{1/2}|D|)\tilde{\eta} \|_{L^2} \]

\[ \leq \| uA(D)\tilde{\eta} \|_{L^2} + \| u(D\tilde{\eta}) \|_{L^2} \leq \| u\|_{L^\infty} \| A(D)\tilde{\eta} \|_{L^2} + \| u(D\tilde{\eta}) \|_{L^2} \]

\[ \leq \| u\|_{L^\infty} \| \tilde{\eta} \|_{L^2} + \| u \cdot (H\tilde{\eta}) \|_{L^2} + \| u \cdot (H\tilde{\eta}) \|_{L^2} \leq \| u\|_{L^\infty} \| \tilde{\eta} \|_{L^2} + \| u \cdot (H\tilde{\eta}) \|_{L^2} \]

\[ \leq \left( \| u\|_{L^\infty} + \| u_x \|_{L^\infty} \right) \left\{ \lambda \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2}. \]  \hspace{1cm} (2.24)

A combination of (2.23) with (2.24) yields

\[ \left| \sum_{\lambda > 2} \int \delta_\lambda^2 B(D) ((u\eta)_x)_\lambda \tilde{\eta}_\lambda dx \right| \leq \left( \| u\|_{L^\infty} + \| u_x \|_{L^\infty} \right) \sum_{\lambda} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2. \]  \hspace{1cm} (2.25)

Using (2.13) again we get

\[ \left| \sum_{\lambda > 2} \int \frac{1}{|D|} ((u\eta)_x)_\lambda \cdot \tilde{\eta}_\lambda dx \right| = \sum_{\lambda > 2} \frac{1}{|D|} \int \mathcal{H} \left( u(A(D) + (ad/c)^{1/2}|D|)\tilde{\eta} \right) \cdot \tilde{\eta}_\lambda dx \]

\[ \leq \sum_{\lambda > 2} \mathcal{H} \left( u(D\tilde{\eta}) \right) \cdot \tilde{\eta}_\lambda dx + \sum_{\lambda > 2} \delta_\lambda^2 \int \mathcal{H} (uA(D)\tilde{\eta}) \cdot \tilde{\eta}_\lambda dx. \]  \hspace{1cm} (2.26)
\( A(\xi) \in S^{-1} \) and the inequality \( \lambda \leq \delta_\lambda \leq \lambda^2 \) imply

\[
\left| \sum_{\lambda > 2} \delta_\lambda^2 \int \mathcal{H}(uA(D)\tilde{\eta})_\lambda \cdot \tilde{\eta}_\lambda dx \right| \\
\leq \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \mathcal{H}(uA(D)\tilde{\eta})_\lambda \|_{L^2}^2 \right\}^{1/2} \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2} \lesssim \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \left\| \tilde{\eta}_\lambda \right\|_{L^2}^2 \right\}^{1/2} \|uA(D)\tilde{\eta}\|_{H^2} \\
\lesssim \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2} \left( \|uA(D)\tilde{\eta}\|_{L^2} + \|\partial_2 (u \cdot A(D)\tilde{\eta})\|_{H^1} \right) \\
\lesssim \left\{ \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2} \left( \|u\|_{L^\infty} \|A(D)\tilde{\eta}\|_{L^2} + \|u_x\|_{L^\infty} \|A(D)\tilde{\eta}\|_{H^1} + \|u\|_{L^\infty} \|\partial_2 (A(D)\tilde{\eta})\|_{H^1} \right) \\
\lesssim (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \sum_{\lambda} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 \tag{2.27}
\]

and

\[
\left| \sum_{\lambda > 2} \delta_\lambda^2 \int \mathcal{H}(u[D(\tilde{\eta})]_\lambda \cdot \tilde{\eta}_\lambda dx \right| = \sum_{\lambda > 2} \delta_\lambda^2 \int (u\partial_2 \mathcal{H}\tilde{\eta})_\lambda \cdot \mathcal{H}\tilde{\eta}_\lambda dx \\
\leq \sum_{\lambda > 2} \delta_\lambda^2 \int u\partial_2 \mathcal{H}\tilde{\eta}_\lambda \cdot \mathcal{H}\tilde{\eta}_\lambda dx + \sum_{\lambda > 2} \delta_\lambda^2 \int [\Delta_\lambda, u\partial_2] \mathcal{H}\tilde{\eta}_\lambda \cdot \mathcal{H}\tilde{\eta}_\lambda dx \\
\lesssim \sum_{\lambda > 2} \delta_\lambda^2 \left( \mathcal{H}(\tilde{\eta})_\lambda \right)^2 \cdot u_x dx + \sum_{\lambda > 2} \delta_\lambda^2 \|\Delta_\lambda, u\partial_2 [\mathcal{H}\tilde{\eta}]_\lambda \|_{L^2} \cdot \|\mathcal{H}\tilde{\eta}_\lambda \|_{L^2} \\
\lesssim \|u_x\|_{L^\infty} \sum_{\lambda > 2} \delta_\lambda^2 \| \mathcal{H}\tilde{\eta}_\lambda \|_{L^2}^2 + \sum_{\lambda > 2} \delta_\lambda^2 \|\Delta_\lambda, u\partial_2 [\mathcal{H}\tilde{\eta}]_\lambda \|_{L^2} \cdot \|\mathcal{H}\tilde{\eta}_\lambda \|_{L^2}. \tag{2.28}
\]

Using (2.10) we obtain

\[
\|\Delta_\lambda, u\partial_2 \tilde{\Delta}_\lambda \mathcal{H}\tilde{\eta}\|_{L^2} \lesssim \|\Delta_\lambda, u\partial_2 \tilde{\Delta}_\lambda \mathcal{H}\tilde{\eta}\|_{L^2} + \|\Delta_\lambda, u\partial_2 \| \left( I_d - \tilde{\Delta}_\lambda \right) \mathcal{H}\tilde{\eta}\|_{L^2},
\]

and

\[
\|\Delta_\lambda, u\partial_2 \tilde{\Delta}_\lambda \mathcal{H}\tilde{\eta}\|_{L^2} \lesssim \|u_x\|_{L^\infty} \|\tilde{\Delta}_\lambda \mathcal{H}\tilde{\eta}\|_{L^2}.
\]

Then we get

\[
\left| \sum_{\lambda > 2} \delta_\lambda^2 \|\Delta_\lambda, u\partial_2 \| \tilde{\Delta}_\lambda \mathcal{H}\tilde{\eta}\|_{L^2} \|\mathcal{H}\tilde{\eta}_\lambda \|_{L^2} \right| \lesssim \|u_x\|_{L^\infty} \sum_{\lambda > 2} \delta_\lambda^2 \| \tilde{\eta}_\lambda \|_{L^2}^2 . \tag{2.29}
\]

Now we estimate the term \( \|\Delta_\lambda, u\partial_2 \| \left( I_d - \tilde{\Delta}_\lambda \right) \mathcal{H}\tilde{\eta}\|_{L^2} \). Let \( u = \sum_{\mu} u_\mu \). We have

\[
\|\Delta_\lambda, u\partial_2 \| \left( I_d - \tilde{\Delta}_\lambda \right) \mathcal{H}\tilde{\eta}\|_{L^2} = \|\Delta_\lambda (u\partial_2 (I_d - \tilde{\Delta}_\lambda)) \mathcal{H}\tilde{\eta}\|_{L^2} \\
\lesssim \sum_{\mu \geq \lambda/8} \|u_\mu \partial_2 (I_d - \tilde{\Delta}) \mathcal{H}\tilde{\eta}\|_{L^2} \lesssim \sum_{\mu \geq \lambda/8} \|\partial_2 \mathcal{H}\tilde{\eta}\|_{L^\infty} \|u_\mu\|_{L^2}.
\]
and then by using Lemma 2.1

\[ \sum_{\lambda > 2} \delta_\lambda^2 \|\Delta_{\lambda}, u\partial_x \| (I_d - \tilde{\Delta}_{\lambda}) \mathcal{H}\tilde{\eta} \|_{L^2} \|\mathcal{H}\tilde{\eta}_\lambda \|_{L^2} \]

\[ \lesssim \|\partial_x \mathcal{H}\tilde{\eta} \|_{L^\infty} \sum_{\lambda > 2} \delta_\lambda^2 \|\mathcal{H}\tilde{\eta}_\lambda \|_{L^2} \sum_{\mu \geq \lambda / 8} \|u_\mu\|_{L^2} \]

\[ \lesssim \|\partial_x \mathcal{H}\tilde{\eta} \|_{L^\infty} \left\{ \sum_{\lambda} \delta_\lambda^2 \|\mathcal{H}\tilde{\eta}_\lambda \|_{L^2}^2 \right\}^{1/2} \left\{ \sum_{\lambda} \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right\}^{1/2} \]

\[ \lesssim \|\partial_x \mathcal{H}\tilde{\eta} \|_{L^\infty} \left( \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}_\lambda \|_{H^1}^2 + \sum_{\lambda} \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right). \] (2.30)

It follows from (2.28), (2.29) and (2.30) that

\[ \sum_{\lambda > 2} \delta_\lambda^2 \int \mathcal{H}u((ad/c)^{1/2}|D|\tilde{\eta})_\lambda \cdot \tilde{\eta}_\lambda dx \]

\[ \lesssim (\|u_x\|_{L^\infty} + \|\partial_x \mathcal{H}\tilde{\eta} \|_{L^\infty})(\sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}_\lambda \|_{H^1}^2 + \sum_{\lambda} \delta_\lambda^2 \|u_\lambda\|_{L^2}^2). \] (2.31)

What remains is to estimate the term \( \|\partial_x \tilde{\eta}\|_{L^2} \) with \( \lambda = 1 \) or 2. Let \( \chi(x) \in C^\infty_0(\mathbb{R}) \) satisfy \( \chi(x) = 1 \) for \( |x| < 10 \) and \( \chi(x) = 0 \) for \( |x| > 20 \). Since \( \lambda = 1 \) or 2 implies \( \|\partial_x \tilde{\eta}\|_{L^2} \lesssim \|\tilde{\eta}\|_{L^2} \), we get

\[ \left| \delta_\lambda^2 \int h^{-1}(D)((u\eta)_x)_\lambda \cdot \tilde{\eta}_\lambda dx \right| \]

\[ \lesssim \int h^{-1}(D)\partial_x \chi(D) \|u\|_{L^\infty} \|\eta\|_{L^1} \]

\[ \lesssim \|\tilde{\eta}\|_{L^2}^2 + \|u_x\|_{L^\infty} \|\partial_x \tilde{\eta}\|_{L^1} \]

\[ \lesssim (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \|\tilde{\eta}\|_{L^2}^2 \lesssim (\|u\|_{L^\infty} + \|u_x\|_{L^\infty}) \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}_\lambda, u_\lambda\|_{L^2}^2. \] (2.32)

A combination of (2.25), (2.31) and (2.32) with (2.22) yields

\[ \left| \sum_{\lambda} \delta_\lambda^2 \int h^{-1}(D)((u\eta)_x)_\lambda \cdot \tilde{\eta}_\lambda dx \right| \lesssim \|u_x, \mathcal{H}\tilde{\eta}\|_{L^\infty} \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}_\lambda, u_\lambda\|_{L^2}^2, \]

and so

\[ I \lesssim \int_t^1 \|u_x, \mathcal{H}\tilde{\eta}\|_{L^\infty} \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}_\lambda, u_\lambda\|_{L^2}^2 d\sigma. \] (2.33)

Hence, by (2.19), (2.20) and (2.33)

\[ \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}(t), u_\lambda(t)\|_{L^2}^2 \]

\[ \lesssim \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}(\tau), u_\lambda(\tau)\|_{L^2}^2 + \int_\tau^1 \|u_x, \mathcal{H}\tilde{\eta}\|_{L^\infty} \sum_{\lambda} \delta_\lambda^2 \|\tilde{\eta}_\lambda, u_\lambda\|_{L^2}^2 d\sigma. \] (2.34)
The Gronwall inequality implies
\[ \sum_\lambda \delta^2_\lambda \|(\tilde{\eta}(t), u_\lambda(t))\|_{L^2}^2 \lesssim \sum_\lambda \delta^2_\lambda \|(\tilde{\eta}(\tau), u_\lambda(\tau))\|_{L^2}^2 \exp \left( 2\|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^1_t L^\infty} \right). \]

We complete the proof. \( \square \)

**Theorem 2.1** Fix \( T > 0 \) and \( 1 < \sigma \leq 2 - 1/p \). Let \((\eta, u)\) be a smooth solution of (1.1). Then for every admissible pair \((p, q)\),
\[ \|J^\sigma(\tilde{\eta}, u)\|_{L^p_x L^q_t} \lesssim (1 + T)^{1/p} (1 + \|J^\sigma u\|_{L^p_x L^2_t}) \|J^{\sigma+1/p}(\tilde{\eta}, u)\|_{L^2_x L^\infty_t} \times (1 + \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^1_t L^\infty}^2), \]

**Proof.** If \((p, q)\) is an admissible pair, then both \( p \) and \( q \) are greater than or equal to two and different from infinity. Therefore the Minkowski inequality, the Littlewood-Paley square function theorem and the Mikhlin-Hörmander theorem show
\[ \|J^\sigma(\tilde{\eta}, u)\|_{L^p_x L^q_t}^2 \lesssim \sum_\lambda \|J^\sigma(\tilde{\eta}_\lambda, u_\lambda)\|_{L^p_x L^q_t}^2 \lesssim \sum_\lambda \lambda^{2\sigma} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^p_x L^q_t}^2. \]  
(2.35)

By Lemma 2.2,
\[ \sum_\lambda \lambda^{2\sigma} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^p_x L^q_t}^2 \lesssim (1 + T)^2 (1 + \|J^\sigma u\|_{L^p_x L^2_t})^2 (1 + \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^1_t L^\infty}^2)^2 \times \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^p_x L^q_t}^2. \]  
(2.36)

Choosing \( \tau = 0 \), \( \delta_\lambda = \lambda^{\sigma+1/p} \) and \( I = [0, T] \) in (2.34) we get
\[ \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^p_x L^q_t}^2 \lesssim \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda(0), u_\lambda(0))\|_{L^2}^2 + \int_0^T \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^\infty} \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda(\tau), u_\lambda(\tau))\|_{L^2}^2 d\tau \lesssim (1 + \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^1_t L^\infty}^2) \|J^{\sigma+1/p}(\tilde{\eta}, u)\|_{L^p_x L^2_t}. \]  
(2.37)

Theorem 2.1 follows from (2.35), (2.36) and (2.37). \( \square \)

3 The local well-posedness

3.1 Uniqueness

Let \((\eta_1, u_1)\) and \((\eta_2, u_2)\) be two solutions of the system (1.1). Let \( \eta_j = h(D)(v_j + w_j) \) and \( u_j = v_j - w_j \ (j = 1, 2) \). Then \((v_1 - v_2, w_1 - w_2)\) satisfies the system (2.4) associated with \( F = -(u_1 \eta_1 - u_2 \eta_2)_x \) and \( G = -(u_1 \partial_x u_1 - u_2 \partial_x u_2) \). Multiplying the equation satisfied by \( v_1 - v_2 \)
Moreover, by (3.11) we have
\[
\frac{1}{2}\frac{d}{dt} \|(v_1 - v_2, w_1 - w_2)\|_{L^2}^2
= -\int \Gamma(D)(v_1 - v_2) \cdot (v_1 - v_2)dx + \int \Gamma(D)(w_1 - w_2) \cdot (w_1 - w_2)dx
- \frac{1}{2} \int [h^{-1}(D)(u_1 \eta_1 - u_2 \eta_2)x] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2)dx
- \frac{1}{2} \int \left[ \frac{1}{1 + DD^2}(u_1 \partial_x u_1 - u_2 \partial_x u_2) \right] \cdot (u_1 - u_2)dx
\lesssim \| (\tilde{\eta}_1 - \tilde{\eta}_2, u_1 - u_2)\|_{L^2}^2 + III + IV,
\] (3.1)

with
\[
III = \frac{1}{2} \int \partial_x h^{-1}(D)[(u_1 - u_2)\eta_1 + u_2(\eta_1 - \eta_2)] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2)dx
= \frac{1}{2} \int (u_1 - u_2)\eta_1 \cdot \partial_x h^{-1}(D)(\tilde{\eta}_1 - \tilde{\eta}_2)dx - \frac{1}{2} \int \partial_x h^{-1}(D)(u_2(\eta_1 - \eta_2)) \cdot (\tilde{\eta}_1 - \tilde{\eta}_2)dx,
\]
and
\[
IV = \frac{1}{2} \int \frac{1}{1 + DD^2}[(u_1 - u_2)\partial_x u_1 + u_2(\partial_x u_1 - \partial_x u_2)] \cdot (u_1 - u_2)dx.
\]

It is obvious that
\[
|IV| \lesssim \| (\partial_x u_1, \partial_x u_2)\|_{L^\infty} \| u_1 - u_2 \|_{L^2}^2.
\] (3.2)

By (2.13) we have
\[
h(\xi) = (ad/c)^{1/2} |\xi| + |(\xi| + 1) A(\xi) \frac{A(\xi)}{1 + |\xi|}.
\]

Moreover, \( \frac{A(\xi)}{1 + |\xi|} \in H^1(\mathbb{R}) \) and \( \frac{A(\xi)\text{sgn}(\xi)}{1 + |\xi|} \in H^1(\mathbb{R}) \) decay like \( |\xi|^{-2} \) at infinity, and so they define some \( L^\infty \)-multipliers. Thus,
\[
\| \eta_1 \|_{L^\infty} = \| (ad/c)^{1/2} \partial_x + \frac{A(D)}{1 + |D|} \partial_x - \frac{A(D)\mathcal{H}}{1 + |D|} \mathcal{H}\eta_1 \|_{L^\infty}
\lesssim \| \mathcal{H}\eta_1 \|_{L^\infty} + \| \partial_x \mathcal{H}\eta_1 \|_{L^\infty},
\]
and so
\[
\left| \int (u_1 - u_2)\eta_1 \cdot \partial_x h^{-1}(D)(\tilde{\eta}_1 - \tilde{\eta}_2)dx \right| \lesssim \| \eta_1 \|_{L^\infty} \| (\tilde{\eta}_1 - \tilde{\eta}_2, u_1 - u_2)\|_{L^2}^2
\lesssim \| \mathcal{H}\eta_1 \|_{L^\infty} + \| \partial_x \mathcal{H}\eta_1 \|_{L^\infty} \| (\tilde{\eta}_1 - \tilde{\eta}_2, u_1 - u_2)\|_{L^2}.
\] (3.3)
To bound the term \( \int \partial_x h^{-1}(D)(u_2(\eta_1 - \eta_2)) \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \), we use (2.13) again and get

\[
\left| \int \partial_x h^{-1}(D)(u_2(\eta_1 - \eta_2)) \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \right|
= \left| \int \partial_x h^{-1}(D)[u_2((ad/c)^{1/2}D + A(D))(\tilde{\eta}_1 - \tilde{\eta}_2)] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \right|
= \left| \left[ \left( \frac{c}{ac} \right)^{1/2} \mathcal{H} + \partial_x B(D) [u_2(ac)^{1/2}D(\tilde{\eta}_1 - \tilde{\eta}_2)] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \right] + \left[ \int \partial_x h^{-1}(D)[u_2 A(D)(\tilde{\eta}_1 - \tilde{\eta}_2)] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \right] \right|
\leq \left[ \left| \int [u_2 \partial_x \mathcal{H}(\tilde{\eta}_1 - \tilde{\eta}_2)] \cdot \mathcal{H}(\tilde{\eta}_1 - \tilde{\eta}_2) dx \right| + \left| \int \partial_x B(D) \mathcal{H}[u_2 \partial_x \mathcal{H}(\tilde{\eta}_1 - \tilde{\eta}_2)] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \right| 
+ \int \partial_x u_2^2 \mathcal{H}[\partial_x u_2 \mathcal{H}(\tilde{\eta}_1 - \tilde{\eta}_2)] \cdot (\tilde{\eta}_1 - \tilde{\eta}_2) dx \right| + \|u_2\|_{L^\infty} \|\tilde{\eta}_1 - \tilde{\eta}_2\|_{L^2}^2
\leq \|\partial_x u_2\|_{L^\infty} + \|u_2\|_{L^\infty} \|\tilde{\eta}_1 - \tilde{\eta}_2\|_{L^2}^2.
\]

(3.4)

It follows from (3.1)–(3.4) that

\[
\frac{d}{dt} \|(v_1 - v_2, u_1 - u_2)\|_{L^2}^2 \leq [1 + \|(\mathcal{H}\tilde{\eta}_1, \partial_x \mathcal{H}\tilde{\eta}_1, u_2, \partial_x u_2)\|_{L^\infty}] \|(\tilde{\eta}_1 - \tilde{\eta}_2, u_1 - u_2)\|_{L^2}^2.
\]

By the Gronwall lemma,

\[
\|(\tilde{\eta}_1 - \tilde{\eta}_2, u_1 - u_2)(t)\|_{L^2} \leq \|(\tilde{\eta}_1(0) - \tilde{\eta}_2(0), u_1(0) - u_2(0))\|_{L^2} \exp \left( 1 + \|(\mathcal{H}\tilde{\eta}_1, \partial_x \mathcal{H}\tilde{\eta}_1, u_2, \partial_x u_2)\|_{L^1_t L^\infty} \right),
\]

(3.5)

which clearly implies the uniqueness.

### 3.2 Existence

Without loss of generality we assume \(1/4 < s < 1\). Let \((\eta, u)\) be a smooth solution of the system (1.1). Setting \(\sigma = s + 3/4\), \(\delta_\lambda = \lambda^{s+1}\) and \(I = [0, T]\) in Lemma 2.3, we deduce

\[
\| J^{s+1}(\tilde{\eta})(t), u(t) \|_{L^\infty_t L^2}^2 \leq \| J^{s+1}(\tilde{\eta}(0), u(0)) \|_{L^2}^2 \exp \left( 2 \|(u, x_\lambda, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^1_t L^\infty} \right).
\]

(3.6)

If \((p, q)\) is an admissible pair, then \(s + 1/p < s + 1/4 = s + 1 < 2\). Therefore using Theorem 2.1 and (3.5) we get that for every admissible pair \((p, q)\) and every \(T > 0\),

\[
\| J^s(\tilde{\eta}, u) \|_{L^p_t L^q} \leq (1 + T)^{1/p} \left( \| J^s u \|_{L^p_t L^q} \left( 1 + \|(u, x_\lambda, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta}, \tilde{\eta})\|_{L^1_t L^\infty} \right) \right) \times \left( \left( \| J^{s+1}(\tilde{\eta}(0), u(0)) \|_{L^p_t L^q} \right) \right) \exp \left( 2 \|(u, x_\lambda, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta}, \tilde{\eta})\|_{L^1_t L^\infty} \right).
\]

(3.7)

Using the Sobolev embedding in the spatial variable together with the H"older inequality in time variable we can choose an admissible pair \((p, q)\) such that

\[
\|(u, x_\lambda, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta}, \tilde{\eta})\|_{L^1_t L^\infty} \lesssim T^{1-1/p} \| J^s(\tilde{\eta}, u) \|_{L^p_t L^q}.
\]

(3.8)
A combination of (3.8) with (3.7) yields
\[
\begin{align*}
&\|(u, u_x, H\tilde{u}, \partial_t H\tilde{u}, \tilde{u})\|_{L^1_t L^\infty_x} \\
&\leq T^{1-1/p}(1 + T)^{1/p}(1 + \|J^s u\|_{L^p_x L^2}(1 + \|\{(u, u_x, H\tilde{u}, \partial_x H\tilde{u}, \tilde{u})\|_{L^1_t L^\infty_x}^2) \\
&\times \|J^{s+1} (\tilde{u}(0), u(0))\|_{L^2_x} \exp \left(2\|(u, u_x, H\tilde{u}, \partial_x H\tilde{u}, \tilde{u})\|_{L^1_t L^\infty_x} \right). \tag{3.9}
\end{align*}
\]
Choosing \(\delta_\lambda = \lambda^{s+1}, \tau = 0\) and \(t = T\) in (2.34) we deduce
\[
\begin{align*}
&\|J^{s+1} (\tilde{u}(t), u(t))\|_{L^p_x L^2} \leq \|J^{s+1} (\tilde{u}(0), u(0))\|_{L^2_x} \\
&+ \|J^{s+1} (\tilde{u}(t), u(t))\|_{L^p_x L^2} \|(u, u_x, H\tilde{u}, \partial_x H\tilde{u})\|_{L^1_t L^\infty_x}. \tag{3.10}
\end{align*}
\]
Then there exists a positive constant \(C_0\) so small that
\[
\|J^{s+1} (\tilde{u}(t), u(t))\|_{L^p_x L^2} \leq \frac{1}{1 - C_0} \|J^{s+1} (\tilde{u}(0), u(0))\|_{L^2_x} \text{ for } t \in [0, T] \tag{3.11}
\]
providing \(\|(u, u_x, H\tilde{u}, \partial_x H\tilde{u}, \tilde{u})\|_{L^1_t L^\infty_x} \leq C_0\). Let \(H(T) = \|(u, u_x, H\tilde{u}, \partial_x H\tilde{u}, \tilde{u})\|_{L^1_t L^\infty_x}\). (3.9) and (3.11) imply that
\[
H(T) \leq (1 + \frac{1 - C_0}{C_0}) \|J^{s+1} (\tilde{u}(0), u(0))\|_{L^2_x} \exp(T) \tag{3.12}
\]
providing \(H(T) \leq C_0\). For every \(R > 0\) we choose a positive constant \(T_R\) such that, for all \(T \in [0, T_R]\),
\[
(1 + \frac{1}{C_0}) \|J^{s+1} (\tilde{u}(0), u(0))\|_{L^2_x} \exp(T_R) \leq C_0/2.
\]
Then for \(\|(\tilde{u}(0), u(0))\|_{H^{s+1}} \leq R\) we deduce from (3.12) that, for all \(T \in [0, T_R]\)
\[
H(T) \leq C_0 \text{ implies } H(T) \leq \frac{1}{2} C_0.
\]
Notice that \(H(0) = 0\). A straightforward continuity argument shows that \(H(T) \leq C_0/2\) for all \(T \leq T_R \leq \|(\tilde{u}(0), u(0))\|_{H^{s+1}}^{-1}\). Using (3.9) we obtain that if \((\eta, u)\) be a smooth solution of the system (1.1) then it satisfies
\[
\|(u, u_x, H\tilde{u}, \partial_x H\tilde{u})\|_{L^1_t L^\infty_x} \leq C \text{ for all } T \leq T_R \tag{3.13}
\]
and
\[
\|J(u, u_x)\|_{L^p_x H^{s+1}} \leq \|J(\tilde{u}(0), u(0))\|_{H^{s+1}} \text{ for all } T \leq T_R. \tag{3.14}
\]
The bounds (3.13) and (3.14) enable us to perform a standard compactness argument. More precisely, consider that the smooth sequence \(\{f_n(x), g_n(x)\}\) satisfying \(\|(f_n, g_n)\|_{H^{s+1}} \leq R\) for some positive constant \(R\), which converges to \((f(x), g(x))\) in \(H^s(\mathbb{R}) \times H^{s+1} (\mathbb{R})\), where we denote by \(f_n = h^{-1}(D)f_n\). Let \(\{\eta_n, u_n\}\) be the solution of the system (1.1) with data \((f_n(x), g_n(x))\) which exists globally in time due to Theorem 3.5 in [2]. We shall prove that \(\{\eta_n, u_n\}\) converges and the limit object is a solution of the system (1.1) with data \((f(x), g(x))\). Indeed, (3.14) implies that \(\{\eta_n, u_n\}\) converges in weak-topology of \(L^\infty([0, T_R]: H^s \times H^{s+1})\) to some limit \((\eta, u)\). Using (3.5) we deduce that \(\{\eta_n, u_n\}\) converges strongly to \((\eta, u)\) in \(L^\infty([0, T_R]: H^{-1} \times L^2)\) and therefore \(\|u_n u_n\|_{L^2} \text{ and } u_n u_n\) converge to \(u u_n, u u_n\), respectively, in a distributional sense. This prove that the limit \((\eta, u)\) satisfies the system (1.1) in a distributional sense. The map \([0, T_R] \ni t \mapsto (\eta(t), u(t))\in H^s(\mathbb{R}) \times H^{s+1} (\mathbb{R})\) is weakly continuous. Lemma 2.3 with \(\delta_\lambda = \lambda^{s+1}\) implies that the map \([0, T_R] \ni t \mapsto \|J(\eta(t), u(t))\|_{H^{s+1}}\) is continuous because exp \(C\|(u, u_x, H\tilde{u}, \partial_x H\tilde{u})\|_{L^1_t L^\infty_x}\) tends to one as \(\tau\) tends to \(t\) if \(I = [\tau, t]\). Hence \([0, T_R] \ni t \mapsto (\eta(t), u(t))\in H^s(\mathbb{R}) \times H^{s+1} (\mathbb{R})\) is continuous.
3.3 Continuous dependence on the data

We present a proof of continuous dependence on the data based on Lemma 2.3.

**Lemma 3.1** Fix $s \in [0, 1)$. Suppose that $(\nu^n, \gamma^n) \to (\nu, \gamma)$ in $H^{s+1}$. Then there exists a sequence $\{\delta_\lambda\}$ of positive numbers which satisfying $2^{s+1}\delta_\lambda \leq \delta_2\lambda \leq 4\delta_\lambda$, $\lambda \leq \delta_\lambda \leq \lambda^2$ and $\delta_\lambda / \lambda^{s+1} \to +\infty$ such that $\sup_n \sum \delta_\lambda^2 \| (\Delta \lambda \nu^n, \Delta \lambda \gamma^n) \|^2_{L^2} < +\infty$.

**Proof.** For $\lambda = 2^j$ set $a_j^n = \lambda^{2(s+1)} \| (\nu^n, \gamma^n) \|^2_{L^2}$, $a_j = \lambda^{2(s+1)} \| (\nu, \gamma) \|^2_{L^2}$. The assumptions implies that $\{a_j^n\}_{j \in \mathbb{N}} \to \{a_j\}_{j \in \mathbb{N}}$ in $\ell^1(\mathbb{N})$. Then for all $k \in \mathbb{N}$ there exists $N_k$ such that

$$N_k \geq k \quad \text{and} \quad \sup_n \sum_{j=N_k}^{\infty} a_j^n < 2^{-2k}.$$ 

For a fixed $j \in \mathbb{N}$, there exists a unique $k \in \mathbb{N}$ such that $N_{k-1} \leq j < N_k$. We set $\mu_j = 2^{(1-s)}$ and $\delta_\lambda = \lambda^{s+1} \mu_j$ for $\lambda = 2^j$. Obviously $2^{s+1}\delta_\lambda \leq \delta_2\lambda \leq 4\delta_\lambda$ and $\delta_\lambda / \lambda^{s+1} \to +\infty$, and

$$\sup_n \sum \delta_\lambda^2 \| (\Delta \lambda \nu^n, \Delta \lambda \gamma^n) \|^2_{L^2} \leq \sum_{j=1}^{\infty} \mu_j a_j^n \leq \sum_{k=0}^{N_{k+1}} \sum_{j=N_k}^{\infty} \mu_j a_j^n \leq \sum_{k=0}^{N_{k+1}} \sum_{j=N_k}^{\infty} 2^{(1-s)}2^{-2k} < +\infty.$$

□

Let $\{(\eta^n, u^n)\}$ be a sequence of solutions in $C([0, T]; H^{s+1} \times H^s)$ with $(\eta^n(0), u^n(0)) \to (\eta(0), u(0))$ in $H^s \times H^{s+1}$. As in the proof of the existence of solutions, we have

$$(\tilde{\eta}^n(t), u^n(t)) \to (\tilde{\eta}(t), u(t)) \text{ in } C([0, T]; L^2 \times L^2). \quad (3.15)$$

Using Lemma 3.1 and Lemma 2.3 we deduce

$$\sup_n \sup_{0 \leq t \leq T} \sum \delta_\lambda^2 \| (\tilde{\eta}^n(t), u^n(t)) \|^2_{L^2} < +\infty. \quad (3.16)$$

Set $\tilde{\eta}_\Lambda = \sum_{\lambda \leq \Lambda} \tilde{\eta}_\lambda$, $u_\Lambda = \sum_{\lambda \leq \Lambda} u_\lambda$. Fix $\epsilon > 0$, there exists by (3.16) a $\Lambda$ such that for every $t \in [0, T]$,

$$\sup_n \{\| (\tilde{\eta}^n(t) - \tilde{\eta}_\Lambda(t), u^n(t) - u_\Lambda(t)) \|_{H^{s+1}} + \| (\tilde{\eta}_\Lambda(t) - \tilde{\eta}(t), u_\Lambda(t) - u(t)) \|_{H^{s+1}} \} < \epsilon/2. \quad (3.17)$$

By (3.15) there exists $n_0$ such that for $n \geq n_0$ and $0 \leq t \leq T$,

$$\| (\tilde{\eta}^n(t) - \tilde{\eta}_\Lambda(t), u^n(t) - u_\Lambda(t)) \|_{H^{s+1}} \leq (2\Lambda)^{s+1} \| (\tilde{\eta}^n(t) - \tilde{\eta}_\Lambda(t), u^n(t) - u_\Lambda(t)) \|_{L^2} < \epsilon/2. \quad (3.18)$$

Therefore, we get for $n \geq n_0$ and $0 \leq t \leq T$,

$$\| (\tilde{\eta}^n(t) - \tilde{\eta}(t), u^n(t) - u(t)) \|_{L^2} \leq \| (\tilde{\eta}^n(t) - \tilde{\eta}_\Lambda(t), u^n(t) - u_\Lambda(t)) \|_{L^2} + \| (\tilde{\eta}_\Lambda(t) - \tilde{\eta}(t), u_\Lambda(t) - u(t)) \|_{H^{s+1}} < \epsilon.$$
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