Calabi-Yau manifolds constructed by Borcea-Voisin method

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ABSTRACT

We construct Calabi-Yau manifolds and their mirrors from K3 surfaces. This method was first developed by Borcea and Voisin. We examined their properties torically and checked mirror symmetry for Calabi-Yau 4-fold case. From Borcea-Voisin 3-fold or 4-fold examples, it may be possible to probe the S-duality of Seiberg-Witten.
1 Introduction

In recent development of string duality, the mathematical properties of the underlying manifold, on which theories are compactified, are playing significant role [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In this paper, Calabi-Yau manifolds and their mirrors are constructed using the method developed by Borcea and Voisin [11, 12]. Gross and Wilson [13] showed that Calabi-Yau 3-fold by Borcea-Voisin method is a special Lagrangian three tori fibered when $\text{Pic}(K3) = U$ by using degenerate Calabi-Yau metrics. The special Lagrangian fibration also exist on one family of Borcea-Voisin three fold with respect to non-degenerate Calabi-Yau metrics [14]. From Borcea-Voisin 3-fold or 4-fold examples, it may be possible to probe the S-duality of Seiberg-Witten [15] by using compact manifolds.

In the next section, we give the list of some mirror pairs of K3 as the reflexive pyramids. In section 3, we give the list of the mirror pairs of Calabi-Yau 3-folds constructed by Borcea-Voisin method in weighted projective manifolds. In section 4, we give the list of the mirror pairs of Calabi-Yau 4-folds constructed by Borcea-Voisin method. Section 5, is devoted to Discussions. In Appendix A, we review Borcea-Voisin method by using some polynomials of Calabi-Yau manifolds. In Appendix B, we show a way of the mirror check of Calabi-Yau 4-folds. In Appendix C, we examine properties of two dual polyhedrons of Calabi-Yau 3-fold and Calabi-Yau 4-fold constructed by Borcea-Voisin. Especially, we present a dual polyhedron and polyhedron of Calabi-Yau 3-fold with $\text{Pic}(K3)=U$ case.

\footnote{It has been pointed out that T-duality on the special tori causes local mirror transformation [16], which may relate the S-duality of Seiberg-Witten.}
2 K3 surface

We start by looking at a definition of K3 surface [17]. A K3 surface is defined as a compact manifold of complex dimension two with trivial canonical bundle such that $h^{0,1}(K3) = 0$. $h^{p,q}(K3)$ denote the Hodge number of a K3 surface.

For this paper, we will consider algebraic K3 surfaces defined by a set of algebraic equations in a $N$ dimensional complex (weighted) projective space, $\mathbb{CP}^N$. The reason why we consider them is that weighted projective space is easy to describe torically. Such surfaces have been classified by Reid and Yonemura [18]. The equations of K3 surface which we will discuss are given by table 1. The numbers in parenthesis in the second column denote weights.

| surface | K3 or K3* | Equation of K3 |
|---------|-----------|----------------|
| (1)     | $\mathbb{CP}^3(6, 4, 1, 1)[12]$ | $w^2 = s^3 + t^{12} + u^{12}$ |
| (1)*    | $\mathbb{CP}^3(33, 22, 6, 5)[66]$ | $w^2 = s^4 + t^{11} + tu^{12}$ |
| (2), (2)*, (3) | $\mathbb{CP}^3(21, 14, 6, 1)[42]$ | $w^2 = s^3 + t^7 + u^{42}$ |
| (3)*    | $\mathbb{CP}^4(18, 12, 5, 1)[36]$ | $w^2 = s^5 + t^7u + u^{36}$ |
| (4)     | $\mathbb{CP}^4(5, 2, 2, 1)[10]$ | $w^2 = s^5 + t^{10} + u^{10}$ |
| (5)*    | $\mathbb{CP}^3(10, 5, 4, 1)[20]$ | $w^2 = s^4 + t^5 + u^{20}$ |
| (5)     | $\mathbb{CP}^3(15, 7, 6, 2)[30]$ | $w^2 = s^4u + t^5 + u^{15}$ |
| (6), (6)* | $\mathbb{CP}^3(15, 10, 3, 2)[24]$ | $w^2 = s^5 + t^{15} + u^{10}$ |
| (7)     | $\mathbb{CP}^3(12, 8, 3, 1)[30]$ | $w^2 = s^4 + t^5 + u^{24}$ |
| (7)*    | $\mathbb{CP}^3(21, 14, 5, 2)[42]$ | $w^2 = s^5 + t^7u + u^{21}$ |
| (8), (9) | $\mathbb{CP}^3(9, 6, 2, 1)[18]$ | $w^2 = s^4 + t^9 + u^{18}$ |
| (8)*    | $\mathbb{CP}^3(18, 11, 4, 3)[36]$ | $w^2 = s^3u + t^9 + u^{12}$ |
| (9)*    | $\mathbb{CP}^3(24, 16, 5, 3)[48]$ | $w^2 = s^3 + t^9u + u^{16}$ |

Table 1: Equation of K3 surfaces of K3 surfaces.
The superscript * denotes the mirror of the corresponding K3 surface. We construct Calabi-Yau 3- and 4-fold, using the method of Borcea and Voisin. These manifolds have some nice properties. One of them is that some fibers of mirror pairs are known. Borcea used these K3 surfaces which allow involution to construct Calabi-Yau 3-folds (the way of involution $\sigma$ is reviewed in Appendix A). The K3 and K3* surfaces in table 1 satisfy

$$\text{Table 2: K3 surface and Picard lattice.}$$

| surface | Pic(K3) | $\rho$ | $(r, a, \delta)$ | $\text{quoti. - sing.}$ |
|---------|---------|--------|-----------------|----------------------|
| (1)     | $U$     | 2      | $(2, 0, 0)$     | $A_1$                |
| (1)*    | $U \oplus E_8^2$ | 18    | $(18, 0, 0)$    | $A_1 + A_2 + A_4 + A_{10}$ |
| (2), (2)*, (3) | $U \oplus E_8$ | 10    | $(10, 0, 0)$    | $A_1 + A_2 + A_6$    |
| (3)*    | $U \oplus E_8$ | 10    | $(10, 0, 0)$    | $A_4 + A_5$          |
| (4)     | $U(2) \oplus D_4$ | 6     | $(6, 4, 0)$     | $5A_1$               |
| (4)*    | $E_8 \oplus T_{2,5,5}$ | 18    | $(14, 4, 0)$    | $A_3 + 2A_4 + A_6$   |
| (5)     | $T_{2,5,5}$ | 10    | $(6, 4, 0)$     | $A_1 + 2A_4$         |
| (5)*    | $D_4 \oplus D_8 \oplus U$ | 14    | $(14, 4, 0)$    | $5A_1 + A_2 + A_6$   |
| (6), (6)* | $D_4 \oplus E_6 \oplus U$ | 12    | $(10, 4, 0)$    | $3A_1 + 2A_2 + A_4$  |
| (7)     | $U \oplus E_6$ | 8      | $(6, 2, 0)$     | $2A_2 + A_3$         |
| (7)*    | $E_8 \oplus D_4 \oplus U$ | 14    | $(14, 2, 0)$    | $3A_1 + A_4 + A_6$   |
| (8), (9) | $U \oplus D_4$ | 6      | $(6, 2, 0)$     | $3A_1 + A_2$         |
| (8)*    | $E_6 \oplus E_8 \oplus U$ | 16    | $(14, 2, 0)$    | $A_1 + 2A_2 + A_{10}$ |
| (9)*    | $E_6 \oplus E_8 \oplus U$ | 16    | $(14, 2, 0)$    | $2A_2 + A_4 + A_7$   |

the "reflexive pyramid" property. This is a sufficient condition for a pair of Calabi-Yau 3-folds constructed by Borcea method to be a mirror pair. This condition is stronger than that of K3 and K3* being a mirror pair.

The properties of our K3 surface are summed in table 2. The second column is the Picard lattice and the third column is rank of Pic(K3) denoted

\(^2\)There are some definitions of mirror symmetry for K3 case\cite{19, 20, 21}. The relation between them and the extension of weight duality on K3 case are given in ref. \cite{22}. Their physical applications were discussed in \cite{23}.

\(^3\)We used the results of \cite{25} for Picard lattice and $\rho$. 

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by $\rho$. Note that all K3 surfaces in table 2 are all elliptic fibered \cite{25}. The fourth column is the lattice invariants $(r, a, \delta)$. Nikulin used these lattice invariant to characterize the fixed part $L^\sigma$ of $\sigma$ on the K3 lattice, up to the lattice isomorphism \cite{24}. $H_2(K3) = L = U^3 \oplus (-E_8)^2$. $a$ is defined as $(L^\sigma)^*/L^\sigma \simeq (\mathbb{Z}/2\mathbb{Z})^a$. $(L^\sigma)^*$ is the dual of $L^\sigma$, namely $(L^\sigma)^* = \text{Hom}(L^\sigma, \mathbb{Z})$. $r = \text{rank}(L^\sigma)$. $\delta$ denotes the genus of the lattice, that is, $\delta = 0$ if $(x^*)^2 \in \mathbb{Z}$ for any $x^* \in (L^\sigma)^*$, otherwise $\delta = 1$. The fifth column is the quotient singularity \cite{18}.
Table 3: Calabi-Yau 3-fold 1.

| Surface | (\(\mathbb{CP}^4\) base, K3 fiber) | CY3 | \(h^{1,1}, h^{1,2}\) | \(\chi\) |
|---------|-----------------------------------|-----|-----------------|-----|
| (2)     | \((\mathbb{CP}^4(1,1), \mathbb{CP}^3(21,14,6,1)[42])\) | \(\mathbb{CP}^4(21,21,2(14,6,1))[84]\) | 35, 35 | 0 |
| (4)     | \((\mathbb{CP}^4(1,2), \mathbb{CP}^3(5,2,2,1)[10])\) | \(\mathbb{CP}^4(5,10,3(2,2,1))[30]\) | 15, 39 | -48 |
| (4*)    | \((\mathbb{CP}^4(1,2), \mathbb{CP}^3(20,8,7,5)[40])\) | \(\mathbb{CP}^4(20,40,3(8,7,5))[120]\) | 39, 15 | 48 |
| (6)     | \((\mathbb{CP}^4(1,1), \mathbb{CP}^3(15,10,3,2)[30])\) | \(\mathbb{CP}^4(15,15,2(10,3,2))[60]\) | 27, 27 | 0 |

3 Calabi-Yau 3-fold

3.1 K3 fibered Calabi-Yau 3-fold

Let us now consider Calabi-Yau manifolds with base, \(\mathbb{CP}^1 = \mathbb{CP}^1(1,k)\) and fiber, \(K3 = \mathbb{CP}^3(u_1, u_2, u_3, u_4)[d]\) [30]. They are represented as hypersurface in weighted projective 4-space, \(\mathbb{CP}^4\),

\[
\text{CY}_3 = \mathbb{CP}^4(u_1, ku_1, (k + 1)u_2, (k + 1)u_3, (k + 1)u_4)[(k + 1)d],
\]

where \(d = \sum_{i=1}^{4} u_i\).

There are some mirror pairs of Calabi-Yau 3-folds, which have mirror pair of K3 surfaces as fiber. See, for instance, the self-mirror Calabi-Yau 3-folds (2), (6) in table 5 [1]. These manifolds have self-mirror K3 surfaces \(\mathbb{CP}^3(21,14,6,1)[42]\) and \(\mathbb{CP}^3(15,10,3,2)[30]\) as K3 fiber. Another example of a mirror pair is (4) and (4)*, which can be obtained by Borcea-Voisin method [4].

The word ”self-mirror manifold” contains a deformed manifold from the strict self-mirror one whose faces are lattice equivalent to vertices. Therefore,

4 The equivalence for type IIA string on \(\text{CY}_3\) with \(k \geq 2\) dual to Heterotic string on \(K3 \times T^2\) is not clear yet. However, some extension of the duality to \(k \geq 2\) and \(u_1 = 1\) may be possible. For example, \(\text{CY}_3 = \mathbb{CP}^4(1,k,(k + 1)(1,4,6))[12(6k + 1)]\) with \(K3 = \mathbb{CP}^3(1,1,4,6)[12]\) fiber relate to the terminal A-chain with shrinking \(E_8\) instantons [26] by exchanging of base under the elliptic fibration, that is, \(F_0\) blown-up and \(F_2\) blown-up [27]. They are the same manifolds with double K3 fibrations.

5 These Calabi-Yau 3-folds were already investigated and listed in [29].

6 All Calabi-Yau 3-folds given in ref. [21] are smooth. However, by picking up appropriate terms, we obtain Calabi-Yau 3-fold which we are treating.
in case (2) and case (6), their faces are not lattice equivalent to vertices.

### 3.2 Borcea-Voisin construction

We use the K3 surfaces listed in table 2. In the above equation, \((-1)\) denotes involution acting on \(T^2\) and \(\sigma\) on K3 surface. For K3 surface, the involution changes the sign of one of the coordinates describing the torus in elliptic fiber.

\[
\text{CY}_3 = \frac{T^2 \times K3}{(-1) \times \sigma}, \quad \text{CY}_3^* = \frac{T^2 \times K3^*}{(-1) \times \tilde{\sigma}}.
\] (2)

For the details of this construction, see ref. [11] and appendix A\(^7\). Because of the condition \(\gcd(u_0, v_0) = 1\) (see appendix A), we obtain only three mirror pairs as the hypersurfaces in \(\mathbb{CP}^4\) with weight representations. The results of the construction of mirror pairs of Calabi-Yau 3-folds are given by table 4. However, it is possible to obtain other mirror pairs of Calabi-Yau 3-folds by using all K3 surfaces in table 1 and their toric data. This is because one side of Calabi-Yau 3-folds can be represented by the hypersurface representation in terms of weight by using either \(T^2=\mathbb{CP}^2(1,1,2)[4]\) or \(T^2=\mathbb{CP}^2(1,2,3)[6]\) at least. For example, toric data of a Calabi-Yau 3-fold constructed from \(K3=\mathbb{CP}^3(1,1,4,6)[12]\) with \(\text{Pic}(K3)=U\) in case (1) of table 1 can be obtained (in Appendix C). This is the case with special three tori fibered [[13]].

| surfaces | \(T^2\) fiber | K3 fiber | \(CY_3\) |
|----------|----------------|----------|---------|
| (2)      | \(\mathbb{CP}^2(2, 1, 1)[4]\) | \(\mathbb{CP}^4(1, 6, 14, 21)[42]\) | \(\mathbb{CP}^4(21, 21, 28, 12, 2)[84]\) |
| (4)      | \(\mathbb{CP}^2(3, 2, 1)[6]\) | \(\mathbb{CP}^4(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^4(5, 10, 6, 6, 3)[30]\) |
| (4*)     | \(\mathbb{CP}^2(3, 2, 1)[6]\) | \(\mathbb{CP}^4(20, 8, 7, 5)[40]\) | \(\mathbb{CP}^4(20, 40, 24, 21, 15)[120]\) |
| (6)      | \(\mathbb{CP}^2(2, 1, 1)[4]\) | \(\mathbb{CP}^4(15, 10, 3, 2)[30]\) | \(\mathbb{CP}^4(15, 15, 20, 6, 4)[60]\) |

Table 4: Calabi-Yau 3-fold 2.

Equations of these manifolds are listed in table 5. The Hodge numbers given

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\(^7\) In ref. [31, 32], Calabi-Yau manifolds are constructed by the extended way of the Borcea method using K3 surfaces listed in [37].
The number of the tensor multiplets in D=6 and N=1 are given by \( r-2 \) (or \( 20-r-2 \)) for Calabi-Yau 3-fold (or the mirror of Calabi-Yau 3-fold) \(^6\) when \( T^2 \) fiber from the direct product part of \( T^2 \) is used in the compactification. They lead to the same numbers of the U(1) vector multiplets for D=4 and N=2 case from type IIA side, which might enhance.
(We examine the possibility of existing a elliptic fibration coming from K3 fiber side in the Appendix C.) 

If Calabi-Yau 3-fold by Borcea-Voisin is double K3 fibered, then case (4) may relate to the Heterotic and type IIA string duality in C-chain whose gauge symmetries is $U(1)^2 \times D_5$ with $H^{1,1} = 15$ and $H^{1,2} = 39$. 

\footnote{If Calabi-Yau 3-fold by Borcea-Voisin is double K3 fibered, then case (4) may relate to the Heterotic and type IIA string duality in C-chain whose gauge symmetries is $U(1)^2 \times D_5$ with $H^{1,1} = 15$ and $H^{1,2} = 39$.}
4 Calabi-Yau 4-fold

In this section, we consider three types of Calabi-Yau 4-folds.

The first basic examples are composed with two K3 surfaces with involution given by \([\pit]\).

\[
\text{CY}_4 = \frac{\mathbb{K}_3 \times \bar{\mathbb{K}_3}}{\sigma \times \sigma'}, \quad \text{CY}^*_4 = \frac{\mathbb{K}_3^* \times \bar{\mathbb{K}_3}^*}{\sigma \times \sigma'}, \tag{5}
\]

The involution \(\sigma\) acts on \(\mathbb{K}_3\) and \(\sigma'\) on \(\mathbb{K}_3^*\). Note that the condition \(\gcd(u_0, v_0) = 1\) restricts the choice of K3 pairs. Table 6 contains all mirror pairs with weight representations constructed from K3 surfaces listed in table 2. We can prove that these pairs are mirror by using their polyhedra derived from the weight representation in Appendix B.

| surface | K3 fiber | K3 fiber | CY_4 |
|---------|----------|----------|------|
| (1)     | \(\mathbb{CP}^3(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^3(6, 4, 1, 1)[12]\) | \(\mathbb{CP}^3(20, 5, 5, 12, 12, 6)[60]\) |
| (1)*    | \(\mathbb{CP}^3(20, 8, 7, 5)[40]\) | \(\mathbb{CP}^3(33, 22, 6, 5)[66]\) | \(\mathbb{CP}^3(440, 120, 100, 264, 231, 165)[1320]\) |
| (2)     | \(\mathbb{CP}^3(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^3(21, 14, 6, 1)[42]\) | \(\mathbb{CP}^3(70, 30, 5, 42, 42, 21)[210]\) |
| (2)*, (3)| \(\mathbb{CP}^3(20, 8, 7, 5)[40]\) | \(\mathbb{CP}^3(21, 14, 6, 1)[42]\) | \(\mathbb{CP}^3(280, 120, 20, 168, 147, 105)[840]\) |
| (3)*    | \(\mathbb{CP}^3(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^3(18, 12, 5, 1)[36]\) | \(\mathbb{CP}^3(60, 25, 5, 36, 36, 18)[180]\) |
| (7)     | \(\mathbb{CP}^3(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^3(12, 8, 3, 1)[24]\) | \(\mathbb{CP}^3(40, 15, 5, 24, 24, 12)[180]\) |
| (7)*    | \(\mathbb{CP}^3(20, 8, 7, 5)[40]\) | \(\mathbb{CP}^3(21, 14, 5, 2)[42]\) | \(\mathbb{CP}^3(280, 100, 40, 168, 147, 105)[840]\) |
| (8)*    | \(\mathbb{CP}^3(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^3(18, 11, 4, 3)[36]\) | \(\mathbb{CP}^3(55, 20, 15, 36, 36, 18)[180]\) |
| (8), (9)| \(\mathbb{CP}^3(20, 8, 7, 5)[40]\) | \(\mathbb{CP}^3(9, 6, 2, 1)[18]\) | \(\mathbb{CP}^3(120, 40, 20, 72, 63, 45)[360]\) |
| (9)*    | \(\mathbb{CP}^3(5, 2, 2, 1)[10]\) | \(\mathbb{CP}^3(24, 16, 5, 3)[48]\) | \(\mathbb{CP}^3(80, 25, 15, 48, 48, 24)[240]\) |

Table 6: Calabi-Yau 4-fold

The Hodge numbers of Calabi-Yau 4-folds given by \([\pit]\) are

\[
\begin{align*}
    h_1^{1,1} &= r_1 + r_2 + f_1 f_2, \\
    h_1^{3,1} &= (20 - r_1) + (20 - r_2) + g_1 g_2, \\
    h_2^{2,1} &= f_1 g_2 + f_2 g_1, \\
    h_2^{2,2} &= 2[102 + (r_1 - 10) \cdot (r_2 - 10) + f_1 f_2 + g_1 g_2],
\end{align*}
\]
where \( f_i \) and \( g_i \) are given by eq. (13). The suffix \( i \) specifies K3 or K3\(^*\) in eq.(14). The Euler number of Calabi-Yau 4-fold is
\[
\chi = 4 + 2(h^{1,1} + h^{1,3}) + h^{2,2} - 4h^{1,2}.
\]
(7)

Equations and the Euler number are given by table 7.

| surface | Equation of CY\(_4\) | \( h^{1,1} \) | \( h^{2,1} \) | \( h^{3,1} \) | \( h^{2,2} \) | \( \chi \) |
|---------|----------------------|----------------|----------------|----------------|----------------|--------|
| (1)     | \( y^3 + z^4 + w^{12} + s^5 + t^5 + u^{10} = 0 \) | 12  | 32  | 92  | 396  | 480  |
| (1)*    | \( y^3 + z^4 + z^w^{12} + s^5 + t^5u + u^8 = 0 \) | 92  | 32  | 12  | 396  | 480  |
| (2)     | \( y^3 + z^4 + w^{42} + s^5 + t^5 + u^{10} = 0 \) | 28  | 48  | 60  | 300  | 464  |
| (2)*, (3) | \( y^3 + z^4 + w^{42} + s^5 + t^5u + u^8 = 0 \) | 60  | 48  | 28  | 300  | 464  |
| (3)*    | \( y^3 + z^4 + w^{36} + s^5 + t^5 + u^{10} = 0 \) | 28  | 48  | 60  | 300  | 464  |
| (7)     | \( y^3 + z^8 + w^{24} + s^5 + t^5 + u^{10} = 0 \) | 14  | 44  | 102 | 420  | 480  |
| (7)*    | \( y^3 + z^8 + w^{42} + s^5 + t^5u + u^8 = 0 \) | 102 | 44  | 14  | 420  | 480  |
| (8)     | \( y^3w + z^9 + w^{12} + s^5 + t^5 + u^{10} = 0 \) | 34  | 48  | 38  | 236  | 192  |
| (8), (9)| \( y^3 + z^9 + w^{18} + s^5 + t^5u + u^8 = 0 \) | 38  | 48  | 34  | 236  | 192  |
| (9)*    | \( y^3 + z^9 + w^{16} + s^5 + t^5 + u^{10} = 0 \) | 34  | 48  | 38  | 236  | 192  |

Table 7: Equation of CY\(_4\)

When both K3 surfaces are elliptically fibered, the dual theory of F-theory compactified on Calabi-Yau 4-fold will be type I' theory compactified on \( T^2 \times K3/Z_2 \). The enhanced gauge symmetries will be the related to the singular elliptic fibers or the quotient singularities of a Calabi-Yau 4-fold relating to K3 singularities.

The second type,
\[
CY_4 = \frac{T^2 \times CY^3}{(-1) \times \sigma},
\]
(8)
can be obtained from Borcea-Voisin method (here, Calabi-Yau 3-fold has an
involution) [11, 31, 32]. We can construct the following example. By using
\( \text{CY}_3=\mathbb{C}P^4(1,42,258,602,903)[1806] \) and \( T_2=\mathbb{C}P^2(1,1,2)[4] \), we obtain
\[ \text{CY}_4=\mathbb{C}P^5(2,84,516,903,903,1204)[3612] \] Hodge numbers are
\[ h^{1,1} = h^{3,1} = 500, \ h^{1,2} = 0 \text{ and } \chi = 6048. \] (9)

In this case, Calabi-Yau 3-fold is K3 fibered Weierstrass type one such as Calabi-Yau 3-fold and K3 fiber have the same \( T^2 \) fiber. K3 fiber is
\( \mathbb{C}P^3(1,6,14,21)[42] \) with \( E_8 \) type elliptic singularity. It may be self-mirror though their faces and vertices are not \( \text{SL}(5,\mathbb{Z}) \) equivalent [3]. It has \( h^{1,1} = h^{2,1} = 251 \) and \( k = 42 \) in eq.(1). CY 3-fold satisfies the condition of being reflexive pyramid which is extended to the higher dimensions. \( k = 42 \) is the only case when \( (k+1)903 \) is coprime with 2 or 3 among \( k = \{2,3,7,41,42\} \) in the list of [30].

For the third type, the manifold may be obtained by the extending Borcea-Voisin method,
\[ \text{CY}_4 = \frac{T^2 \times T^2 \times K^3}{(-1) \times \sigma}. \] (10)
Holonomy group of covering space of this manifold is \( SU(2) \). It would be interesting to consider the application of this manifold to verify string duality.

\[ ^{10} \text{It is also in the list of Calabi-Yau 4-fold [29].} \]
\[ ^{11} \text{For self-mirror families, vertices and faces are not \( \text{SL}(5,\mathbb{Z}) \) equivalent and they are only \( \text{GL}(5,\mathbb{Z}) \) equivalent. This condition may be a necessary one for self-mirror cases.} \]
5 Discussion

Many Calabi-Yau manifolds were constructed by Borcea-Voisin method. Furthermore, many mirror pairs are possible without weight representations in addition to our list with weight representations\footnote{The reason why we made mirror pairs of Calabi-Yau manifolds with weight representations is that we wanted to examine their properties torically.}.

These manifolds have some nice properties. For example, in Calabi-Yau 3-fold case:
1) Mirror pair can be constructed easily,
2) Mirror pair manifolds have the same $T^2$ fiber,
3) $K3$ fibers of mirror pair manifolds have the same base $T^2/(-1)$,
4) The number of the rational curves are known. Therefore, the superpotentials are obtained in the exact forms \cite{12} in Type IIA and Type IIB side. The correspondence of type IIA and type IIB is apparent.

It should be possible to use these CY4-folds to see the relation between F-theory on Calbi-Yau 4-folds and type I' theory on Calbi-Yau 3-folds or $T^2 \times K3/Z_2$. For example, Sen \cite{5} used Calabi-Yau 3-folds constructed by Borcea-Voisin method, \((CY_3 = (T^2, F_m), m = 0, 1, 4)\) and showed the relation among F-theory compactified on Calabi-Yau 3-fold and type I' theory on K3 surface.

Studying the duality of supersymmetric field theory with brane will require the clarification of a relation among T-duality, mirror symmetry and Fourier-Mukai transformation \cite{10, 15, 41, 42}. The manifolds constructed in the present paper will be applied to investigate this relation in future work.
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Appendix

A  Borcea-Voisin method

We start with the manifolds with weights, $u = (u_0, u_1, \ldots, u_n)$ and $v = (v_0, v_1, \ldots, v_m)$, where, $u_0 = \sum_{i=1}^{n} u_i$ and $v_0 = \sum_{i=1}^{m} v_i$. We assume the following form to the equations which describes these manifolds,

$$x_0^2 = f(x_1, x_2, \ldots, x_n) \text{ and } y_0^2 = f(y_1, y_2, \ldots, y_m).$$  \hfill (11)

Then the Calabi-Yau hypersurfaces $X_n^{(u)}$ and $Y_m^{(v)}$ of degree $2u_0$ and $2v_0$, respectively are defined,

$$\mathbb{C}P(u) = \mathbb{C}P(u_0, u_1, \ldots, u_n)\text{ and } \mathbb{C}P(v) = \mathbb{C}P(v_0, v_1, \ldots, v_m).$$  \hfill (12)

Furthermore, assume that $\gcd(u_0, v_0) = 1$ to obtain the rational map

$$\mathbb{C}P(u) \times \mathbb{C}P(v) = \mathbb{C}P(v_0u_1, \ldots, v_0u_n, u_0v_1, \ldots, u_0v_m),$$  \hfill (13)

defined by

$$(x_0, \ldots, x_n) \times (y_0, \ldots, y_m) \to (x_1y_0^{u_1/u_0}, \ldots, x_ny_0^{u_n/u_0}, y_1x_0^{v_1/v_0}, \ldots, y_mx_0^{v_m/v_0})$$  \hfill (14)

where all the fractional powers use the same determination for $y_0^{1/u_0}$ and $x_0^{1/v_0}$ respectively. This is a Calabi-Yau hypersurface $X_{n+m}^{u \times v}$ of total degree $2u_0v_0$.

We used two types of tori for the construction of $CY_3$,

$$\mathbb{C}P^2(1,1,2)[4] : \quad y_1^2 = y_2^4 + y_3^4,$$  \hfill (15)

and

$$\mathbb{C}P^2(1,2,3)[6] : \quad y_1^2 = y_2^3 + y_3^6.$$  \hfill (16)
B  The mirror check of Calabi-Yau 4-folds

Here, we will show a simple way to check mirror using a following pair of Calabi-Yau 4-folds. We will use (2) and (2)$^*$ in the table 8 as an example. All of their weight are not equal one.

(2) $\mathbb{CP}^5(70, 30, 5, 42, 21)$ [210]  
(2)$^*$ $\mathbb{CP}^5(280, 120, 20, 168, 147, 105)$ [840].

The coordinate transformation of the polyhedra in the lattice is as follows:

$$(x_1, \cdots, x_6) \in \mathbb{Z}^6 \rightarrow (m_1, \cdots, m_5) \in \mathbb{Z}^5.$$  

For case (2), the basic equation which they must satisfy is

$$5(14x_1 + 6x_2 + x_3) + 21(2x_4 + 2x_5 + x_6)$$

$$\equiv 5 \times (-21m_3) + 21 \times 5m_3 = 0.$$  \hfill (17)

A solution is given by

$$x_1 = m_1, \ x_2 = m_2, \ x_3 = -14m_1 - 6m_2 - 21m_3, \ x_4 = m_4, \ x_5 = m_5, \ x_6 = 5m_3 - 2m_4 - 2m_5, \ \forall x_i \geq -1.$$  \hfill (18)

Some similar deformations are possible for (2)$^*$,

$$x_1 = m_1, \ x_2 = m_2, \ x_3 = -14m_1 - 6m_2 - 21m_3, \ \forall x_i \geq -1$$

$$x_4 = 6m_3 - 7m_4 + 2m_5, \ x_5 = -4m_3 + 8m_4 - 3m_5, \ x_6 = m_5.$$  \hfill (19)

This comes from the relation between weights and degrees.

Using a program of analyzing polytopes and polyhedra, the faces and the vertices of polyhedra are obtained. For example, the faces of (2) and the vertices of (2)$^*$ are given by

$$\vec{f}_1 = (14, 6, 21, 0, 0), \quad \vec{v}_1 = (-1, -1, -1, -1),$$

$$\vec{f}_2 = (-1, 0, 0, 0, 0), \quad \vec{v}_2 = (2, -1, -1, -1),$$

$$\vec{f}_3 = (0, -1, 0, 0, 0), \quad \vec{v}_3 = (-1, 6, -1, -1),$$

$$\vec{f}_4 = (0, 0, 0, -1, 0), \quad \vec{v}_4 = (-1, -1, 1, 1),$$

$$\vec{f}_5 = (0, 0, 0, 0, -1), \quad \vec{v}_5 = (-1, -1, 1, 0),$$

$$\vec{f}_6 = (0, 0, -5, 2, 2), \quad \vec{v}_6 = (-1, -1, 1, 3, 7).$$  \hfill (20)

\footnote{We owe this way to Mohri.}
They are lattice isomorphic, i.e., $\tilde{z}A \in SL(5, \mathbb{Z})$, $A\bar{f}_i = \bar{v}_i$ for $i = 1 \cdots 6$.

$$A = \begin{pmatrix}
-2, & 1, & 1, & 1, & 1 \\
1, & -6, & 1, & 1, & 1 \\
1, & 1, & -1, & -1, & -1 \\
1, & 1, & -1, & -1, & 0 \\
1, & 1, & -1, & 0, & 1
\end{pmatrix}$$

Thus, (2) and (2)* are a mirror pair.
C Dual Polyhedra of Calabi-Yau 3-fold in Borcea-Voisin construction

There are some works about singularities in algebraic manifolds including K3 surfaces [33, 34, 35, 36, 37, 38]. It is difficult to know all elliptic fibrations and all degeneration which Calabi-Yau manifolds have even if they are K3 surfaces. Some ways of obtaining them in terms of toric varieties are proposed [26, 29]. They use a dual polyhedron of Calabi-Yau manifolds to find fibrations and their singularities. For example, in finding singularities, top points and the bottom points in the edge of the dual polyhedron of K3 denote the extended Dynkin diagrams of the singularities. Their method is simple and visible. However, there are some ambiguities because they have the lattice equivalence and some vanishing points. Therefore, we think that the sufficient condition to identify the final step corresponding to the boundary of Kahler cone may be useful in using their method. The boundary of the Kahler cone is where we can find all elliptic fiberations and their degenerations which Calabi-Yau manifolds have. It corresponds to the most smooth Calabi-Yau manifold. Vinberg gave an algorism to obtain the boundary of the Kahler cones of K3. He derived the sufficient condition to identify the last stage of the algorism for the signature (1,n) case of the Picard lattice [39]. If we can translate this condition in terms of dual polyhedron, we can apply this condition to the method of refs in [26, 29] for these cases. From Picard lattice, it may be possible to obtain the boundary of the Kahler cone by the following method [26]. The dual graph of Kahler cone (secondary polytope) is

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14 For K3 cases, Belcastro gave some more improved ways of calculating Pic (K3), finding elliptic fibrations [25]. Her ways are composite ones of using toric varieties and some theorems of algebraic geometry. Her way to calculate the intersection number of two curves is to look at the graph associated to the incidence matrix associated to the desingularization of the polytope. If the vertices representing the curve have an edge between them, then thier intersection multiplicity is the multiplicity of the edge (otherwise 0). She gave the list of Pic (K3) and the elliptic fibers for 95 K3 surfaces and types of generations [25].

15 Kahler cone can be obtained as the fundamental region of the Weyl transformation of the Picard lattice for K3 case.
obtained by the triangulation of dual polyhedra. After choosing the case which has the most triangulations which corresponds to the most smooth K3, we can identify it as the dual of the boundary of the Kahler cone. If we can get the dual graph of the secondary polytope then we will be able to see the boundary of the Kahler cone constructed by CP\(^1\). If we can make the Gram matrix whose element is represented by the intersection number of each CP\(^1\) in the boundary of the Kahler cone then it contains all the elliptic curves which are resolved and forming the extended Dynkin diagrams.

We applied the methods of the references of \([26, 29]\) to investigate fibers in Calabi-Yau 3-folds in table 5. Their method is summarized as below. We will follow the notation of the ref. of \([26]\). The upper prefix in \(\nabla\) denotes the dimension of the lattice of the polyhedron or dual polyhedron. 

\[\{(x_1, x_2, x_3, x_4)\} \in 4\nabla\ \text{form integral points in the four-dimensional dual polyhedron of Calabi-Yau 3-fold up to the points in codimension one face.}\]

\[\{(x_1, x_2)\} \in 2\nabla \subset 4\nabla\ \text{represent the integral points in the dual polyhedron of a base under the elliptic fibration of Calabi-Yau 3-folds.}\]

\[\{(x_2, x_3, x_4)\} \mid_{x_1=0} 3\nabla \subset 4\nabla\ \text{denote the integral points in the three-dimensional dual polyhedron of K3 fiber of Calabi-Yau 3-fold.}\]

\[\{(x_3, x_4)\} \mid_{x_1=x_2=0} 2\nabla \subset 4\nabla\ \text{form the integral points in the two-dimensional dual polyhedron of a common elliptic fiber in Calabi-Yau 3-folds and in K3 fiber.}\]

We confirmed that most Calabi-Yau 3-folds and Calabi-Yau 4-folds in table 8 have the dual polyhedrons of K3 and of elliptic curve which satisfy the above conditions. More precisely, for case (2), the dual polyhedron contains two dual sub-polyhedra of T\(^2\) and a dual polyhedron of K3 satisfying the above conditions. They are CP\(^2\)(1,1,2)[4] and CP\(^2\)(1,2,3)[6]. CP\(^2\)(1,2,3)[6] comes from the elliptic fiber of the K3 fiber. If the above condition for having elliptic fibration is sufficient, then we can conclude that the two kinds of elliptic fibrations coming from both sides of the direct product of the Borcea-Voisin construction are possible for Calabi-Yau 3-folds. One comes from the T\(^2\) side and the other comes from the elliptic fibration of K3. However, this

\[16\] There are some softwares to get secondary polytopes. Unfortunately, even for K3 cases, if \(\rho \geq 8\), then they will not work unless one uses some symmetries.

\[17\] It may be possible to link the intersection number of the singularities and the secondary polytope by using \([43]\).
will not be not sufficient condition for having elliptic fibration and needs farther conditions to be sufficient. $T^2 = \mathbb{CP}^1(1,2,3)[6]$ in Calabi-Yau 3-fold in case (2) does not satisfy the necessary condition of having elliptic fibration about canonical bundles which is quoted in the reference of \cite{2}.

$$K_{CY3} = \pi^*(K_B + \sum a_i E_i) + \text{error terms} = 0$$ \hspace{1cm} (21)

The dual polyhedron of table 8 leads to $n_T = 1 = h^{1,1}(B) - 1$. This is not consistent with the above equation when we suppose that $T^2 = \mathbb{CP}^1(1,2,3)[6]$ fibered with $E_8$ type degeneration and $B = F_0$ based for case (2).

The conclusion of the property of Calabi-Yau 3-folds by Borcea-Voisin methods is that they do not have the elliptic fibration coming from K3 fiber side. Even if they have them as fibers, then there may be no sections. Therefore, they cannot be represented in the Weierstrass form as the extension of this elliptic curve with the singularity. The degenerations of elliptic fibers of Calabi-Yau 3-folds do not relate to the degenerations of K3 fibers.

For the dual polyhedron of Calabi-Yau 4-folds of in table 6 which constructed by the method in appendix B. It has two dual sub polyhedra of K3 as two fibers. Table 9 and Table 10 are a polyhedron and a dual polyhedron of Calabi-Yau 3-fold (1) constructed by $K3 = \mathbb{CP}^3(1,1,4,6)[12]$ and $T^2 = \mathbb{CP}^2(1,1,2)[12]$. \{x$_2$, x$_3$\} and \{x$_4$\} may relate polyhedron and dual polyhedron of three tori.

\footnote{18 We follow the notations of this reference \cite{2}.}
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\[
\begin{array}{ccccccc}
\text{Table 8: Dual polyhedron of CY3-fold(2) and their sub dual polyhedra}
\end{array}
\]
Table 9: a polyhedron of CY3-fold(1) = \( CP^1(1, 1, 2) \times CP^3(1, 1, 4, 6)/(-1) \times \sigma \) and their sub polyhedra

| \( CP^2(1, 2, 3)[6] \) | \( CP^2(1, 1, 2)[4] \) | \( CP^3(1, 1, 4, 6)[42] \) | CY3-fold(1) |
|------------------------|------------------------|------------------------|--------------|
| \( x_2 \) \( x_3 \) | \( x_2 \) \( x_4 \) | \( x_1 \) \( x_2 \) \( x_3 \) | \( x_1 \) \( x_2 \) \( x_3 \) \( x_4 \) |
| 1 -1 | 1 -1 | -1 1 -1 | -1 1 -1 -1 |
| -1 -1 | -1 -1 | 0 -1 -1 | 0 -1 -1 -1 |
| -1 -1 | -1 0 | 0 -1 -1 | 0 -1 -1 0 |
| -1 -1 | -1 1 | 0 -1 -1 | 0 -1 -1 1 |
| -1 -1 | -1 2 | 0 -1 -1 | 0 -1 -1 2 |
| -1 -1 | -1 -3 | 0 -1 -1 | 0 -1 -1 3 |
| 0 -1 | 0 -1 | 0 0 -1 | 0 0 -1 -1 |
| 0 -1 | 0 0 | 0 0 -1 | 0 0 -1 0 |
| 0 -1 | 0 1 | 0 0 -1 | 0 0 -1 1 |
| 0 0 | 0 -1 | 0 0 0 | 0 0 0 -1 |
| 0 0 | 0 0 | 0 0 0 | 0 0 0 0 |
| 0 0 | 0 1 | 0 0 0 | 0 0 0 1 |
| 1 -1 | 1 -1 | 0 1 -1 | 0 1 -1 -1 |
| 1 0 | 1 -1 | 0 1 0 | 0 1 0 -1 |
| 1 1 | 1 -1 | 0 1 1 | 0 1 1 -1 |
| 1 2 | 1 -1 | 0 1 2 | 0 1 2 -1 |
| 1 -1 | 1 -1 | 1 1 -1 | 1 1 -1 -1 |
| $CP^2(1, 2, 3)[6]$ | $CP^2(1, 1, 2)[4]$ | $CP^3(5, 6, 22, 33)[66]$ | CY3-fold(1)* |
|---|---|---|---|
| $x_2$ $x_3$ | $x_3$ $x_4$ | $x_1$ $x_2$ $x_3$ | $x_1$ $x_2$ $x_3$ $x_4$ |
| -6 3 -2 | -6 3 -2 0 | -6 3 -2 0 |
| 5 3 -2 | -5 3 -2 0 |
| -4 2 -1 | -4 2 -1 0 |
| -4 3 -2 | -4 3 -2 0 |
| -3 1 -1 | -3 1 -1 0 |
| -3 2 -1 | -3 2 -1 0 |
| -3 3 -2 | -3 3 -2 0 |
| 0 0 -1 | -2 1 -1 0 |
| -2 1 0 | -2 1 0 0 |
| -2 2 -1 | -2 2 -1 0 |
| -2 3 -2 | -2 3 -2 0 |
| -1 1 -1 | -1 1 -1 0 |
| -1 1 -1 | -1 1 -1 0 |
| -1 1 0 | -1 1 0 0 |
| -1 2 -1 | -1 2 -1 0 |
| 1 3 -2 | -1 3 -2 0 |
| -2 -1 | 0 -2 0 -1 |
| -1 0 | -1 0 0 |
| 0 0 | 0 0 0 0 |
| 0 0 | 0 0 0 0 |
| 0 1 | 0 0 0 1 |
| 0 1 | 0 0 1 0 |
| 1 -1 | 0 1 -1 0 |
| 1 0 | 0 1 -1 0 |
| 0 1 0 | 0 1 0 0 |
| 0 2 -1 | 0 2 0 0 0 |
| 2 -1 | 0 2 -1 0 |
| 3 -2 | 0 3 -2 0 |
| 3 -2 | 0 3 -2 0 |
| 1 0 0 | 1 0 0 0 |
| 1 1 -1 | 1 1 -1 0 |
| 1 1 0 | 1 1 0 0 |
| 1 3 -2 | 1 2 -1 0 |
| 1 2 -1 | 1 2 -1 0 |
| 1 2 -1 | 1 2 -1 0 |
| 1 3 -2 | 1 3 -2 0 |
| 2 1 -1 | 2 1 -1 0 |
| 2 1 0 | 2 1 0 0 |
| 2 2 -1 | 2 2 -1 0 |
| 2 3 -2 | 2 3 -2 0 |
| 3 1 -1 | 3 1 -1 0 |
| 3 1 -1 | 3 1 -1 0 |
| 3 2 -1 | 3 2 -1 0 |
| 3 3 -2 | 3 3 -2 0 |
| 4 2 -1 | 4 2 -1 0 |
| 4 3 -2 | 4 3 -2 0 |
| 5 3 -2 | 5 3 -2 0 |
| 6 3 -2 | 6 3 -2 0 |

Table 10: a dual polyhedron of CY3-fold(1) and their sub dual polyhedra