ON THE ALMOST SURE RUNNING MAXIMA OF SOLUTIONS OF AFFINE NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper studies the large fluctuations of solutions of finite–dimensional affine stochastic neutral functional differential equations with finite memory, as well as related nonlinear equations. We find conditions under which the exact almost sure growth rate of the running maximum of each component of the system can be determined, both for affine and nonlinear equations. The proofs exploit the fact that an exponentially decaying fundamental solution of the underlying deterministic equation is sufficient to ensure that the solution of the affine equation converges to a stationary Gaussian process.

1. Introduction

In the last decade, a large number of papers have been written about the stability of solutions of stochastic neutral functional differential equations (SNFDEs). Asymptotic (usually exponential) stability has been studied by Mao [25, 26, 27], Liao and Mao [28], Liu and Xia [20], Luo [19], Luo, Mao and Shen [22], Shen and Liao [27] and Jankovic, Randjelovic, and Jovanovic [35, 34]. Equations with Markovian switching have also been studied in Mao, Shen and Yuan [30]. Monographs which consider at least in part the theory of neutral SFDEs (including stability theory) have been written by Kolmanovskii, Nosov and Myshkis [16, 17, 18] and Mao [23, 24]. The last book in particular contains results on asymptotic behaviour which do not necessarily appear in journals. The stability theory has even been extended to stochastic neutral partial equations; in this regard, we refer the reader to Luo [21] and Govindan [10], for example.

Despite this surge in activity, it appear that much less work has been done on determining the asymptotic behaviour of SNFDEs whose solutions are not asymptotically stable. In part this stems from the interest in neutral equations in control engineering, in which pathwise or moment stability is of great importance. However, there are good reasons, both in terms of mathematical interest, and applications to consider SNFDEs whose solutions are not asymptotically stable. One paper in this direction, which considers stability in distribution is Frank [9], which studies conditions under which affine stochastic neutral delay differential equations possess unique solutions. Since the solution of such an equation is Gaussian, and the limiting distribution is stationary, it seems that the solution cannot be bounded. In a

Date: 11 July 2013.

1991 Mathematics Subject Classification. Primary: 34K06, 34K25, 34K50, 34K60, 60F15, 60F10.

Key words and phrases. stochastic neutral functional differential equation, neutral differential equation, Gaussian process, stationary process, differential resolvent, running maxima, almost sure asymptotic estimation, finite delay.

We gratefully acknowledge the support of this work by Science Foundation Ireland (SFI) under the Research Frontiers Programme grant RFP/MAT/0018 “Stochastic Functional Differential Equations with Long Memory”. JA also thanks SFI for the support of this research under the Mathematics Initiative 2007 grant 07/MI/008 “Edgeworth Centre for Financial Mathematics".
finite dimensional setting therefore, we might expect solutions to obey
\[ \lim_{t \to \infty} \max_{0 \leq s \leq t} |X(s)| = \infty, \quad \text{a.s.} \]

The scalar process \( t \mapsto X^*(t) := \max_{0 \leq s \leq t} |X(s)| \) is called the running maximum.

Therefore, it is natural to ask, at what rate does the running maximum tend to infinity, or, more precisely to find a deterministic function \( \rho \) with \( \rho(t) \to \infty \) as \( t \to \infty \) such that
\begin{equation}
\lim_{t \to \infty} \frac{X^*(t)}{\rho(t)} = 1, \quad \text{a.s.}
\end{equation}

We call such a function \( \rho \) the essential growth rate of the running maxima of \( X \). In applications this is important, as the size of the large fluctuations may represent the largest bubble or crash in a financial market, the largest epidemic in a disease model, or a population explosion in an ecological model.

To date there is comparatively little literature regarding the size of such large fluctuations for SNFDEs, and to the best of our knowledge, no comprehensive theory for affine stochastic neutral functional differential equations. In this paper, we determine the essential growth rate of the running maximum for affine SNFDEs. The class of equations covered includes equations with both point and distributed delay by using measures in the delay. The results exploit the fact that given an exponentially decaying differential resolvent, the finite delay in the equation forces the limiting autocovariance function to decay exponentially fast, so that the solution of the linear equation is an asymptotically stationary Gaussian process. The results apply to both scalar and finite-dimensional equations and can moreover be extended to equations with a weak nonlinearity at infinity.

The paper bears many similarities to results proved in a recent paper of the authors [2] which considers the large fluctuations of affine non-neutral stochastic functional differential equations. Indeed the main results here are all analogues of those in [2]. However, the proofs of both main results differ because the differential resolvent of the neutral equation is not guaranteed to be differentiable, while that of the non-neutral functional differential equation is differentiable. In the proofs of the main results in [2], this differentiability plays a crucial role in controlling the behaviour of the process between discrete mesh points at which the process is sampled. This is a key point of the proof, because at these mesh points a sharp almost sure upper bound on the growth rate of the process is known. In this paper however, due to the uncertainty of the differentiability of the resolvent of the SNFDE, we cannot apply the same analysis as in [2]. However, part of the strategy of our proof involves writing the solution of the affine SNFDE in terms of the solution of an affine SFDE which does have an underlying deterministic differential resolvent which is continuously differentiable, enabling some of the techniques and estimates of [2] to be employed once more.

More precisely, we study the asymptotic behaviour of the finite-dimensional process which satisfies
\begin{align}
(1.2a) & \quad X(t) - D(X_t) = \phi(0) - D(\phi_0) + \int_0^t L(X_s) \, ds + \int_0^t \Sigma \, dB(s), \quad t \geq 0, \\
(1.2b) & \quad X(t) = \phi(t), \quad t \in [-\tau, 0],
\end{align}

where \( B \) is an \( m \)-dimensional standard Brownian motion, \( \Sigma \) is a \( d \times m \)-matrix with real entries, and \( D, L : C[-\tau, 0] \to \mathbb{R}^d \) are linear functional with \( \tau \geq 0 \) and
\[ L(\phi) = \int_{[-\tau,0]} \nu(d)s) \phi(s), \quad D(\phi) = \int_{[-\tau,0]} \mu(d)s) \phi(s), \quad \phi \in C([-\tau,0]; \mathbb{R}^d). \]
The asymptotic behaviour of (1.2) is determined in the case when the resolvent $\rho$ of the deterministic equation
\begin{equation}
\frac{d}{dt} (x(t) - D(x_i)) = L(x_i), \quad t \geq 0
\end{equation}
obey $\rho \in L^1([0, \infty); \mathbb{R}^{d \times d})$ and the integral resolvent of $-\mu_+ \in M([0, \infty), \mathbb{R}^d)$ is a finite measure, where $\mu_+(E) = \mu(-E)$ for every Borel subset $E$ of $[0, \infty)$, and $\mu(E) = 0$ for all Borel sets $E \subset (-\infty, -\tau)$. In particular, we show that the running maxima of each component grows according to
\begin{equation}
\limsup_{t \to \infty} \frac{\langle X(t), \eta_i \rangle}{\sqrt{2 \log t}} = \sigma_i, \quad \liminf_{t \to \infty} \frac{\langle X(t), \eta_i \rangle}{\sqrt{2 \log t}} = -\sigma_i, \quad \text{a.s.}
\end{equation}
where $\sigma_i > 0$ depends on $\Sigma$ and the resolvent $\rho$. Moreover
\begin{equation}
\limsup_{t \to \infty} \frac{|X(t)|_\infty}{\sqrt{2 \log t}} = \max_{i=1, \ldots, d} \sigma_i, \quad \text{a.s.}
\end{equation}
We can also subject (1.2) to a general nonlinear perturbation to get the equation
\begin{equation}
d(X(t) - N_1(t, X_i)) = (L(X_i) + N_2(t, X_i)) dt + \Sigma dB(t), \quad t \geq 0,
\end{equation}
and still retain the asymptotic behaviour of (1.2). More specifically, if the nonlinear functionals $N_1, N_2 : [0, \infty) \times C[-\tau, 0] \to \mathbb{R}^d$ is of smaller than linear order as $\|\varphi\|_2 := \sup_{-\tau \leq s \leq 0} |\varphi(s)|_2 \to \infty$ in the sense that
\begin{equation}
\lim_{\|\varphi\|_2 \to \infty} \frac{|N_i(t, \varphi_i)|_2}{\|\varphi\|_2} = 0 \text{ uniformly in } t \geq 0,
\end{equation}
then (1.4) and (1.5) still hold.

It should be remarked that we establish a stochastic variation of parameters formula for solutions of (1.2). To the best of our knowledge, such a formula does not appear in the literature to date. If $\rho$ is the differential resolvent of (1.3) and $x$ is the solution of (1.3), then the solution $X$ of (1.2) obeys
\begin{equation}
X(t) = x(t) + \int_0^t \rho(t-s)\Sigma dB(s), \quad t \geq 0,
\end{equation}
One interesting aspect of the proof of (1.8) is that it can be applied to equations with non-constant diffusion coefficient. We intend to give some applications of this result to characterise the asymptotic behaviour of SNFDEs in later work.

Neutral delay differential equations have been used to describe various processes in physics and engineering sciences [13, 38]. For example, transmission lines involving nonlinear boundary conditions [13, 39] cell growth dynamics [3], propagating pulses in cardiac tissue [4] and drill-string vibrations [4] have been described by means of neutral delay differential equations. Reliable simulation of such equations in applications in which stochastic perturbations are present is facilitated by Euler–Maruyama methods for SNFDEs developed by Mao and Wu [31].

2. Preliminaries

Let $d, m$ be some positive integers and $\mathbb{R}^{d \times m}$ denote the space of all $d \times m$ matrices with real entries. We equip $\mathbb{R}^{d \times m}$ with a norm $|\cdot|$ and write $\mathbb{R}^d$ if $m = 1$ and $\mathbb{R}$ if $d = m = 1$. We denote by $\mathbb{R}^+$ the half-line $[0, \infty)$. The complex plane is denoted by $\mathbb{C}$.

The total variation of a measure $\nu$ in $M([0, \infty), \mathbb{R}^{d \times d})$ on a Borel set $B \subseteq [-\tau, 0]$ is defined by
\[
|\nu|(B) := \sup_{i=1}^N \sum_{i=1}^N |\nu(E_i)|,
\]
where \((E_i)_{i=1}^N\) is a partition of \(B\) and the supremum is taken over all partitions. The total variation defines a positive scalar measure \(|\nu|\) in \(M([[-\tau,0], \mathbb{R})\). If one specifies temporarily the norm \(|\cdot|\) as the \(\ell^1\)-norm on the space of real-valued sequences and identifies \(\mathbb{R}^{d \times d}\) by \(\mathbb{R}^d\) one can easily establish for the measure \(\nu = (\nu_{i,j})_{i,j=1}^d\) the inequality

\[
|\nu|(B) \leq C \sum_{i=1}^d \sum_{j=1}^d |\nu_{i,j}|(B) \quad \text{for every Borel set } B \subseteq [-\tau,0]
\]

with \(C = 1\). Then, by the equivalence of every norm on finite-dimensional spaces, the inequality (2.1) holds true for the arbitrary norms \(|\cdot|\) and some constant \(C > 0\). Moreover, as in the scalar case we have the fundamental estimate

\[
\left| \int_{[-\tau,0]} \nu(ds) f(s) \right| \leq \int_{[-\tau,0]} |f(s)| \nu(ds)
\]

for every function \(f : [-\tau,0] \rightarrow \mathbb{R}^{d \times d'}\) which is \(|\nu|\)-integrable.

We first turn our attention to the deterministic delay equation underlying the stochastic differential equation (1.2). For a fixed constant \(\tau \geq 0\) we consider the deterministic linear delay differential equation

\[
\frac{d}{dt} \left( x(t) - \int_{[-\tau,0]} \mu(ds)x(t+s) \right) = \int_{[-\tau,0]} \nu(ds)x(t+s), \quad t \geq 0,
\]

\[
x(t) = \phi(t) \quad \text{for } t \in [-\tau,0],
\]

for measures \(\nu \in M([-\tau,0], \mathbb{R}^{d \times d}), \mu \in M([-\tau,0], \mathbb{R}^{d \times d}).\) The initial function \(\phi\) is assumed to be in the space \(C[-\tau,0] := \{ \phi : [-\tau,0] \rightarrow \mathbb{R}^d : \text{continuous} \}\). A function \(x : [-\tau,\infty) \rightarrow \mathbb{R}^d\) is called a solution of (2.2) if \(x\) is continuous on \([-\tau,\infty)\) and \(x\) satisfies the first and second identity of (2.2) for all \(t \geq 0\) and \(t \in [-\tau,0]\), respectively. It is well-known that for every \(\phi \in C[-\tau,0]\) the problem (2.2) admits a unique solution \(x = x(\cdot, \phi)\) provided that \(\det(I_d - \mu(\{0\})) \neq 0\), where \(I_d\) is the \(d \times d\) identity matrix, and \(\det(A)\) signifies the determinant of a \(d \times d\) matrix \(A\). This condition on \(\mu\) is equivalent to the notion of uniform non-atomicity at \(0\) of the functional \(D : C[-\tau,0] \rightarrow \mathbb{R}^d\) given by

\[
D(\psi) = \int_{[-\tau,0]} \mu(ds)\psi(s), \quad \psi \in C([-\tau,0]; \mathbb{R}^d).
\]

Results on the existence of deterministic neutral equations, including a definition of uniform non-atomicity of \(D\), may be found in Chukwu [5], Chukwu and Simpson [6], Hale [12] and Hale and Cruz [15].

The fundamental solution or resolvent of (2.2) is the unique continuous function \(\rho : [0, \infty) \rightarrow \mathbb{R}\) which satisfies

\[
\frac{d}{dt} \left( \rho(t) - \int_{[-\tau,0]} \mu(ds)\rho(t+s) \right) = \int_{[-\tau,0]} \nu(ds)\rho(t+s), \quad t \geq 0;
\]

\[
\rho(t) = 0, \quad t \in [-\tau,0); \quad \rho(0) = I_d.
\]

It plays a role which is analogous to the fundamental system in linear ordinary differential equations and the Green function in partial differential equations.

For a function \(x : [-\tau,\infty) \rightarrow \mathbb{R}^d\) we denote the segment of \(x\) at time \(t \geq 0\) by the function

\[
x_t : [-\tau,0] \rightarrow \mathbb{R}, \quad x_t(s) := x(t+s).
\]

If we equip the space \(C([-\tau,0])\) of continuous functions with the supremum norm \(\text{Riesz}\)' representation theorem guarantees that every continuous functional \(D : \mathbb{R}^d\) to \(\mathbb{R}^d\) is of the form

\[
D(\psi) = \int_{[-\tau,0]} \mu(ds)\psi(s), \quad \psi \in C([-\tau,0]; \mathbb{R}^d).
\]
\[ C[-\tau, 0] \to \mathbb{R}^{d \times d} \] is of the form
\[ D(\psi) = \int_{[-\tau, 0]} \mu(ds) \psi(s), \]
for a measure \( \mu \in M([-\tau, 0]; \mathbb{R}^d) \). Hence, we will write (2.2) in the form
\[ \frac{d}{dt}(x(t) - D(x_t)) = L(x_t) \quad \text{for } t \geq 0, \quad x_0 = \phi \]
where
\[ L(\psi) = \int_{[-\tau, 0]} \nu(ds) \psi(s), \]
\( \nu \) is a measure in \( \mu \in M([-\tau, 0]; \mathbb{R}^d) \), and assume \( D \) and \( L \) to be continuous and linear functionals on \( C([-\tau, 0]; \mathbb{R}^d) \).

Fix a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \( \{\mathcal{F}(t)\}_{t \geq 0} \) satisfying the usual conditions and let \((B(t) : t \geq 0)\) be a standard \( m\)–dimensional Brownian motion on this space. Equation (2.2) can be written as
\[ d[X(t) - D(X_t)] = L(X_t) dt + \Sigma dB(t) \quad \text{for } t \geq 0, \]
\[ X(t) = \phi(t) \quad \text{for } t \in [-\tau, 0], \]
where \( D \) and \( L \) are as previously defined, and \( \Sigma \in \mathbb{R}^{d \times m} \).

In [1], we discussed the conditions under which (2.3) has a unique solution on \([-\tau, T]\) for any \( T > 0 \). In addition to the Lipschitz continuity of the linear functional \( L \), we require that the neutral functional \( D \) is uniformly nonatomic at zero. In the scalar case, it can be shown that if \( \mu(\{0\}) = 1 \), \( D \) does not obey the uniformly nonatomic condition. For \( \mu(\{0\}) \in \mathbb{R}^\ast \{1\} \), (2.3) can be rescaled, so that a unique solution exists. In the general finite–dimensional case, if \( I_d - \mu(\{0\}) \) is invertible, we can rescale the equation so that the neutral functional \( \tilde{D} \) is given by
\[ \tilde{D}(\phi) = (I_d - \mu(\{0\}))^{-1} \int_{[-\tau, 0]} \mu(ds)\phi(s) =: \int_{[-\tau, 0]} \tilde{\mu}(ds)\phi(s), \quad \phi \in C([-\tau, 0]; \mathbb{R}^d), \]
where \( \tilde{\mu} \in M \). Then \( \tilde{D} \) is uniformly non–atomic at zero, and indeed \( \tilde{\mu}(\{0\}) = 0 \). Hence without loss of generality, we can assume that
\[ \mu(\{0\}) = 0. \]

The dependence of the solutions on the initial condition \( \phi \) is neglected in our notation in what follows; that is, we will write \( x(t) = x(t, \phi) \) and \( X(t) = X(t, \phi) \) for the solutions of (2.3) and (2.4) respectively.

We also constrain ourselves with the condition
\[ \det \left( I_d - \int_{[-\tau, 0]} e^{\lambda s} \mu(ds) \right) \neq 0 \quad \text{for every } \lambda \in \mathbb{C} \text{ with } \Re \lambda \geq 0. \]

Define the function \( h_{\mu, \nu} : \mathbb{C} \to \mathbb{C} \) by
\[ h_{\mu, \nu}(\lambda) = \det \left( \lambda (I_d - \int_{[-\tau, 0]} e^{\lambda s} \mu(ds)) - \int_{[-\tau, 0]} e^{\lambda s} \nu(ds) \right). \]
The asymptotic behaviour of \( \rho \) relies on the value of
\[ v_0(\mu, \nu) := \sup \left\{ \Re(\lambda) : \lambda \in \mathbb{C}, h_{\mu, \nu}(\lambda) = 0 \right\}. \]

We summarize some conditions on the asymptotic behaviour of \( \rho \) in the following lemma:

**Lemma 1.** Let \( \rho \) satisfy (2.3), and \( v_0(\mu, \nu) \) be defined as (2.7). If (2.6) holds, then the following statements are equivalent:
(a) $v_0(\mu, \nu) < 0$.
(b) $\rho$ decays to zero exponentially.
(c) $\rho(t) \to 0$ as $t \to \infty$.
(d) $\rho \in L^1(\mathbb{R}^+; \mathbb{R}^{d \times d})$.
(e) $\rho \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d})$.

Since (2.4) may be viewed as a perturbation of the non–stochastic equation (2.2), it is natural to expect that its solution can be written in terms of $x$ and the differential resolvent $\rho$ via a variation of constants formula. We show below that the solution of (2.4) has the following representation

**Theorem 1.** Suppose that $L$ and $D$ are linear functionals and that $\mu$ obeys (2.5). If $x$ is the solution of (2.2) and $\rho$ is the continuous solution of (2.3), then the unique continuous adapted process $X$ which satisfies (2.4) obeys

\[
X(t) = x(t) + \int_0^t \rho(t-s)\Sigma dB(s), \quad t \geq 0,
\]

and $X(t) = \phi(t)$ for $t \in [-\tau, 0]$.

In the paper, we let $(\cdot, \cdot)$ stand for the standard inner product on $\mathbb{R}^d$, and $|\cdot|_2$ for the standard Euclidean norm induced from it. We also let $|\cdot|_{\infty}$ stand for the infinity norm on $\mathbb{R}^d$, and if $\phi \in C([-\tau, 0]; \mathbb{R}^d)$, we define $||\phi||_2 = \sup_{-\tau \leq s \leq 0} |\phi(s)|_2$. By way of clarification, we note that $|\cdot|_{\infty}$ stands here for a vector norm rather than a norm on a space of continuous functions. For $i = 1, \ldots, d$, the $i$-th standard basis vector in $\mathbb{R}^d$ is denoted $e_i$. If $X$ and $Y$ are two random variables, then we denote the correlation and the covariance between $X$ and $Y$ by $\text{Corr}(X, Y)$ and $\text{Cov}(X, Y)$ respectively.

### 3. Statement and Discussion of Main Results

In this section the main asymptotic results of the paper are stated. We start by stating our main result for the solution of the finite–dimensional affine equation (2.2).

**Theorem 2.** Suppose that $\rho$ is the solution of (2.2) and that $\mu$ satisfies (2.0). Moreover suppose that $v_0(\mu, \nu) < 0$, where $v_0(\mu, \nu)$ is defined by (2.7). Let $X$ be the unique continuous adapted $d$-dimensional process which obeys (2.2). Then for each $1 \leq i \leq d$,

\[
\limsup_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = \sigma_i \quad \text{and} \quad \liminf_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = -\sigma_i, \quad \text{a.s.}
\]

where $\sigma_i$ is given by

\[
\sigma_i^2 = \int_0^\infty \sum_{k=1}^m \theta_{ik}(s)^2 \, ds
\]

where $\theta(t) = \rho(t)\Sigma \in \mathbb{R}^{d \times m}$. Moreover

\[
\limsup_{t \to \infty} \frac{|X(t)|_\infty}{\sqrt{2 \log t}} = \max_{1 \leq i \leq d} \sigma_i, \quad \text{a.s.}
\]

The results of Theorem 2 are very similar to those of Theorem 2 in [2] which considers the affine functional differential equation

\[
dX(t) = L(X_t) \, dt + \Sigma \, dB(t), \quad t \geq 0; \quad X_0 = \phi.
\]

We will refer to this result throughout the paper, so it is stated shortly for convenience. To do so we need some auxiliary deterministic functions. Define the differential resolvent $r$ by

\[
r'(t) = L(r_t), \quad t \geq 0; \quad r(0) = I_d, \quad r(t) = 0, \quad t \in [-\tau, 0).
\]
We define \( h_\nu : \mathbb{C} \to \mathbb{C} \) by
\[
 h_\nu(\lambda) = \text{det} \left( \lambda I_d - \int_{[\tau, 0]} e^{\lambda s} \nu(ds) \right),
\]
and suppose that
\[
(3.6) \quad v_0(\nu) := \sup \{ \text{Re} (\lambda) : h_\nu(\lambda) = 0 \} < 0.
\]

Theorem 2 of [2] is as follows.

**Theorem 3.** Suppose that \( r \) is the solution of (3.5) and that \( v_0(\nu) < 0 \), where \( v_0(\nu) \) is defined as (3.6). Let \( X \) be the unique continuous adapted \( d \)-dimensional process which obeys (3.4). Then for each \( 1 \leq i \leq d \),
\[
(3.7) \quad \limsup_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = \sigma_i \quad \text{and} \quad \liminf_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = -\sigma_i, \quad \text{a.s.}
\]
where
\[
(3.8) \quad \sigma_i = \sqrt{\sum_{k=1}^{m} \int_{0}^{\infty} \theta^2_{ik}(s) \, ds}
\]
and \( \theta(t) = r(t) \Sigma \in \mathbb{R}^{d \times m} \). Moreover
\[
(3.9) \quad \limsup_{t \to \infty} \frac{|X(t)|_{\infty}}{\sqrt{2 \log t}} = \max_{i=1, \ldots, d} \sigma_i, \quad \text{a.s.}
\]

In other words, the solution \( X \) of (3.5) obeys all the conclusions of Theorem 2 above with \( r \) in place of \( \rho \).

The proof of Theorem 3 depends on two key properties of the differential resolvent \( r \) satisfying (3.5). The first is that \( r \) decays exponentially fast because \( v_0(\nu) < 0 \). This is in common with the condition \( v_0(\mu, \nu) < 0 \) in Theorem 2. The second is that \( r \) is in \( C^1((0, \infty); \mathbb{R}^d) \), which plays a crucial role in the proof of Theorem 3 in controlling the behaviour of the process between mesh points. In contrast with the differentiability of \( r \), the neutral differential resolvent \( \rho \) may not be differentiable everywhere on \( (0, \infty) \). Therefore the proof of Theorem 3 departs from that of Theorem 2 in controlling the behaviour of the process between mesh points.

Our other main result shows that (2.4) can be perturbed by nonlinear functionals \( N_1 \) and \( N_2 \) in the neutral term and drift respectively (which is of lower than linear order at infinity) without changing the asymptotic behaviour of the underlying affine stochastic neutral functional differential equation. To make this claim more precise, we characterize the perturbing nonlinear functionals \( N_1 \) and \( N_2 \) as follows: suppose \( \{N_i\}_{i=1, 2} : [0, \infty) \times C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d \) obey

For all \( n \in \mathbb{N} \) there exists a \( K_n > 0 \) such that if \( \varphi, \psi \in C([-\tau, 0]; \mathbb{R}^d) \)
\[
(3.10) \quad \text{obey } \|\varphi\|_2 \vee \|\psi\|_2 \leq n, \text{ then } |N_i(t, \varphi, \psi)|_2 \leq K_n \|\varphi - \psi\|_2,
\]
and \( N_i \) is continuous in its first argument for \( i = 1, 2 \);

\[
(3.11) \quad \lim_{\|\varphi\|_2 \to \infty} \frac{|N_i(t, \varphi)|_2}{\|\varphi\|_2} = 0 \quad \text{uniformly in } t \text{ for } i = 1, 2,
\]
and
\[
(3.12) \quad t \mapsto |N_i(t, 0)|_2 \quad \text{is bounded on } [0, \infty) \text{ for } i = 1, 2.
\]

Before stating our main result, we examine the hypotheses (3.10)–(3.12) and prove an important estimate deriving therefrom. By the hypothesis (3.11) we mean that
for every $\varepsilon > 0$ there is a $\Phi = \Phi(\varepsilon) > 0$ such that if $\varphi \in C([-\tau, 0]; \mathbb{R}^d)$ obeys $\|\varphi\|_2 \geq \Phi(\varepsilon)$, we then have
\[|N_i(t, \varphi)|_2 \leq \varepsilon \|\varphi\|_2, \quad \text{for all } t \geq 0 \text{ and } i = 1, 2.\]

By (3.12), we have that there is a $\bar{n} \geq 0$ such that $|N_i(t, 0)|_2 \leq \bar{n}$ for all $t \geq 0$. Also by (3.11) for all $\varphi$ such that $\|\varphi\|_2 \leq \Phi(\varepsilon)$ (where $\lfloor x \rfloor$ denotes the smallest integer greater than or equal to $x \geq 0$), we have that there is a $K(\varepsilon) = K(\Phi(\varepsilon))$ such that $|N_i(t, \varphi)|_2 \leq |N_i(t, \varphi) - N_i(t, 0)|_2 + |N_i(t, 0)|_2 \leq K(\varepsilon)\|\varphi\|_2 + \bar{n} \leq K(\varepsilon)\Phi(\varepsilon) + \bar{n}$. Therefore with $L(\varepsilon) := K(\varepsilon)\Phi(\varepsilon) + \bar{n}$ we have
\[|N_i(t, \varphi)|_2 \leq L(\varepsilon), \quad \text{for all } t \geq 0 \text{ and all } \|\varphi\|_2 \leq \Phi(\varepsilon).\]

Hence for every $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that
\[(3.13) \quad |N_i(t, \varphi)|_2 \leq L(\varepsilon) + \varepsilon \|\varphi\|_2, \quad \text{for all } t \geq 0 \text{ and all } \varphi \in C([-\tau, 0]; \mathbb{R}^d).\]

The hypothesis (3.12) ensures that the functional $N_i$ is (in some sense) close to being an autonomous functional, or is bounded by an autonomous functional.

We study the following nonlinear stochastic differential equation with time delay:
\[(3.14) \quad d[X(t) - D(X_t) - N_1(t, X_t)] = [L(X_t) + N_2(t, X_t)] dt + \Sigma dB(t) \quad \text{for } t \geq 0,\]
\[X(t) = \phi(t) \quad \text{for } t \in [-\tau, 0],\]
where $D$ and $L$ are continuous and linear functionals on $C([-\tau, 0]; \mathbb{R})$ as defined in the preliminaries.

The following theorem is a consequence of the affine finite–dimensional result Theorem 2.

**Theorem 4.** Suppose that $N_1$ and $N_2$ obey (5.10), (5.11) and (5.12) and that $N_1$ is uniformly nonatomic at 0. Also suppose that $\rho$ is the solution of (2.3) and that $\mu$ satisfies (2.6). Moreover suppose that $\nu_0(\mu, \nu) < 0$, where $\nu_0(\mu, \nu)$ is defined by (2.7). Let $X$ be the unique continuous adapted $d$-dimensional process which obeys (3.13). Then for each $1 \leq i \leq d$,
\[(3.15) \quad \lim_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = \sigma_i, \quad \text{and} \quad \lim_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = -\sigma_i, \quad \text{a.s.}\]
where $\sigma_i$ is given (6.2). Moreover
\[(3.16) \quad \lim_{t \to \infty} \frac{|X(t)|}{\sqrt{2 \log t}} = \max_{1 \leq i \leq d} \sigma_i, \quad \text{a.s.}\]

Since in general, it is not possible to obtain a representation that is analogous to (2.8) for non-linear equations such as (5.11), the proof cannot directly rely on Gaussianity of the process. Instead, by using a *comparison* argument, we conclude that if the non-linear term in the drift is smaller than linear order at infinity (cf. assumption (5.11)), the size of the large fluctuations of a Gaussian stationary process is retained.

### 4. Proofs from Section 3

#### 4.1. Proof of Lemma 1
We extend the measures $\mu$ and $\nu$ to $M((\infty, 0]; \mathbb{R}^{d \times d})$ by assuming
\[(4.1) \quad \mu(E) = \nu(E) = 0 \quad \text{for every Borel set } E \subseteq (\infty, -\tau).\]

For any Borel set $E \subseteq \mathbb{R}$ we use the notation
\[-E := \{x \in \mathbb{R}; -x \in E\}.\]
to define the reflected Borel set \((-E)\). Introduce the measures \(\mu_+\) and \(\nu_+\) in \(M([0, \infty); \mathbb{R}^{d\times d})\), related to \(\mu\) and \(\nu\) in \(M((-\infty, 0]; \mathbb{R}^{d\times d})\) by

\[
\mu_+(E) := \mu(-E), \quad \nu_+(E) := \nu(-E).
\]

Then for \(t \geq 0\),

\[
\int_{(-\tau, 0]} \nu(ds) = \int_{[0, \tau]} \mu_+(ds) = \int_{(0, \tau)} \mu_+(ds) - \int_{(\tau, \infty)} \mu_+(ds) = \int_{[0, \tau]} \mu_+(ds) = \int_{[0, \tau]} \mu_+(ds).
\]

The last step is obtained by the fact that \(\rho(t) = 0\) for \(t \in [-\tau, 0)\) and \(\mu([0]) = 0\). Similarly

\[
\int_{(-\tau, 0]} \nu(ds) = \int_{[0, \tau]} \nu_+(ds), \quad t \geq 0.
\]

Define

\[
\kappa(t) := \begin{cases} \rho(t) - \int_{[-\tau, 0]} \mu_+(ds), & t \geq 0, \\ 0, & t \in [-\tau, 0). \end{cases}
\]

Also since \(\rho(0) = I_d\) and \(\mu([0]) = 0\), \(\kappa(0) = I_d\). Moreover, by (4.3) and (4.4), we have

\[
\kappa = \rho - \mu_+ * \rho,
\]

and \(\kappa \in C^1((0, \infty); \mathbb{R}^{d\times d})\) with

\[
k' = \int_{[0, \tau]} \nu_+(ds) \rho(t - s), \quad t \geq 0.
\]

Since \(\mu_+ \in M([0, \infty); \mathbb{R}^{d\times d})\) and \(\mu_+([0]) = \mu([0]) = 0\), we may define \( \rho_0 \in M_{\text{loc}}([0, \infty); \mathbb{R}^{d\times d})\) to be the integral resolvent of \((-\mu_+)\), i.e.,

\[
\rho_0 - \mu_+ * \rho_0 = -\mu_+.
\]

Then by Theorem 4.1.7 in [11],

\[
\rho = \kappa - \rho_0 * \kappa.
\]

Therefore, if we define \(\beta \in M_{\text{loc}}([0, \infty); \mathbb{R}^{d\times d})\) by

\[
\beta := \nu_+ - \nu_+ * \rho_0,
\]

we have that

\[
k' = (\beta * \kappa)(t), \quad t \geq 0; \quad \kappa(0) = I_d.
\]

By condition (2.6) and by Theorem 4.1.5 (half line Paley–Wiener theorem) in [11], we have that \(\rho_0\) defined by (4.7) obeys \(\rho_0 \in M([0, \infty]; \mathbb{R}^{d\times d})\). Moreover, it is even true that condition (2.6) implies that \(\rho_0\) decays exponentially, so

\[
\int_{[0, \infty)} e^{\alpha t}\rho_0(dt) < \infty.
\]
Therefore by using Theorem 3.6.1 in [11], (4.15), and (11.11) we have the following equivalences

\begin{align}
(4.12) \quad \lim_{t \to \infty} \kappa(t) &= 0 \iff \lim_{t \to \infty} \rho(t) = 0; \\
(4.13) \quad \kappa \text{ decays to zero exponentially} &\iff \rho \text{ decays to zero exponentially}; \\
(4.14) \quad \kappa \in L^1(\mathbb{R}^+; \mathbb{R}^{d \times d}) &\iff \rho \in L^1(\mathbb{R}^+; \mathbb{R}^{d \times d}); \\
(4.15) \quad \kappa \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d}) &\iff \rho \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d}).
\end{align}

Since \( \rho_0 \in M([0, \infty); \mathbb{R}^{d \times d}) \), we have that \( \beta \) defined by (4.19) is in \( M(\mathbb{R}^+; \mathbb{R}^{d \times d}) \). Now by Theorem 3.3.17 from [11], if \( \beta \) has a fortiori a finite first moment, i.e., \( \int_{[0, \infty)} t|\beta|(dt) < \infty \), where \( \beta := \nu_+ - \nu_+ * \rho_0 \), then

\begin{equation}
(4.16) \quad \lim_{t \to \infty} \kappa(t) = 0 \iff \kappa \in L^1(\mathbb{R}^+; \mathbb{R}^{d \times d}).
\end{equation}

We now show that \( \beta \) has a finite first moment. Note that

\[
\int_{[0, \infty)} t|\beta|(dt) \leq \int_{[0, \infty)} t|\nu_+| + \int_{[0, \infty)} t|\nu_+ * \rho_0|(dt)
\]

\[
= \int_{[0, \infty)} t|\nu_+|(dt) + \int_{[0, \infty)} t|\nu_+ * \rho_0|(dt).
\]

Thus by Young’s inequality,

\[
\int_{[0, \infty)} t|\nu_+ * \rho_0|(dt) \leq \frac{1}{\alpha} \int_{[0, \infty)} e^{\alpha t}|\nu_+ * \rho_0|(dt)
\]

\[
\leq \frac{1}{\alpha} \int_{[0, \infty)} e^{\alpha t}|\nu_+|(dt) \int_{[0, \infty)} e^{\alpha t}|\rho_0|(dt)
\]

\[
= \frac{1}{\alpha} \int_{[0, \infty)} e^{\alpha t}|\nu_+|(dt) \int_{[0, \infty)} e^{\alpha t}|\rho_0|(dt)
\]

\[
< \infty.
\]

So \( \beta \) has finite first moment. Therefore (4.16) holds. Moreover,

\begin{equation}
(4.17) \quad \int_{[0, \infty)} e^{\alpha t}|\beta|(dt) < \infty.
\end{equation}

So by (4.12), (4.14) and (4.16), statements (c) and (d) are equivalent. Now if \( \kappa \in L^1(\mathbb{R}^+; \mathbb{R}^{d \times d}) \), due to (4.17), we have that \( \kappa \) decays to zero exponentially, which by (4.13) implies that \( \rho \) decays to zero exponentially. Hence (b) and (c) are equivalent. If \( \lim_{t \to \infty} \rho(t) = 0 \), then \( \rho \) decays to zero exponentially, which implies \( \rho \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d}) \). On the other hand, if \( \rho \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d}) \), then \( \kappa \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d}) \). Also \( \kappa' \in L^2(\mathbb{R}^+; \mathbb{R}^{d \times d}) \). Let \( f := |\kappa|^2 = \sum_{i=1}^{d} \sum_{j=1}^{d} \kappa_{ij}^2 \). Then

\[
|f'| = \sum_{i=1}^{d} \sum_{j=1}^{d} 2\kappa_{ij}'(t)\kappa_{ij}(t) \leq \sum_{i=1}^{d} \sum_{j=1}^{d} 2|\kappa_{ij}'(t)|\kappa_{ij}(t)\]

\[
\leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left( |\kappa_{ij}'(t)|^2 + |\kappa_{ij}(t)|^2 \right).
\]

Therefore \( |f'| \leq |\kappa|^2 + |\kappa'|^2 \), so \( f' \in L^1(\mathbb{R}^+; \mathbb{R}) \). Therefore as \( f \in L^1(\mathbb{R}^+; \mathbb{R}) \), we have \( \lim_{t \to \infty} \kappa(t) = 0 \) and consequently \( \lim_{t \to \infty} \rho(t) = 0 \). Hence (b)–(d) are equivalent. For part (a), suppose \( \rho \in L^1([0, \infty); \mathbb{R}^{d \times d}) \), which holds if and only if \( \kappa \in L^1([0, \infty); \mathbb{R}^{d \times d}) \), which in turn is equivalent to

\begin{equation}
(4.18) \quad \det(\lambda M - \beta(\lambda)) \neq 0, \quad \text{Re}(\lambda) \geq 0.
\end{equation}
Now
\[ \hat{\rho}_0(\lambda) = -(I_d - \hat{\mu}_+(\lambda))^{-1}\hat{\mu}_+(\lambda) = -\hat{\mu}_+(\lambda)(I_d - \hat{\mu}_+(\lambda))^{-1} \]
for all \( \Re \lambda \geq 0 \), because \( \det(I_d - \hat{\mu}_+(\lambda)) \neq 0 \) for all \( \Re \lambda \geq 0 \) due to (2.6). We have, for \( \Re \lambda \geq 0 \),
\[
\lambda I_d - \hat{\beta}(\lambda) = \lambda I_d - \hat{\mu}_+(\lambda) + \hat{v}_+(\lambda)\hat{\rho}_0(\lambda)
\]
\[
= \lambda I_d - \hat{\mu}_+(\lambda) - \hat{v}_+(\lambda)\hat{\mu}_+(\lambda)(I_d - \hat{\mu}_+(\lambda))^{-1}
\]
\[
= \left[ \lambda(I_d - \hat{\mu}_+(\lambda)) - \hat{v}_+(\lambda)(I_d - \hat{\mu}_+(\lambda)) - \hat{\nu}_+(\lambda)\hat{\mu}_+(\lambda) \right](I_d - \hat{\mu}_+(\lambda))^{-1}
\]
\[
= \left[ \lambda I_d - \hat{\mu}_+(\lambda) - \hat{\nu}_+(\lambda) \right](I_d - \hat{\mu}_+(\lambda))^{-1}
\]
\[
= \left[ \lambda \left( I_d - \int_{[-\tau,0]} e^{\lambda s}\mu(ds) \right) - \int_{[-\tau,0]} e^{\lambda s}\nu(ds) \right](I_d - \hat{\mu}_+(\lambda))^{-1}
\]
Clearly, under (2.6), (4.19) holds if and only if
\[
\det \left( \lambda \left( I_d - \int_{[-\tau,0]} e^{\lambda s}\mu(ds) \right) - \int_{[-\tau,0]} e^{\lambda s}\nu(ds) \right) \neq 0, \text{ for all } \Re \lambda \geq 0
\]
which is true if and only if \( v_0(\mu, \nu) < 0 \). Hence statements (a)–(e) are all equivalent.

4.2. Proof of Theorem 1. We first show that the solution of (2.4) can be represented in the form of (2.8). Let \( \mu_+ \) and \( \nu_+ \) be as in (4.2). Then for \( t \geq 0 \), the solution \( X \) of (2.4) satisfies
\[
d \left( X(t) - \int_{[0,t]} \mu_+(ds)X(t - s) \right) = \left( \int_{[0,t]} \nu_+(ds)X(t - s) \right) dt + \Sigma dB(t),
\]
with \( X(t) = \phi(t) \) for \( t \in [-\tau,0] \). Let \( x \) be the solution of (2.2) with \( x(t) = \phi(t) \) for \( t \in [-\tau,0] \). By (4.3) and (4.4) we have that the fundamental solution \( \rho \) of (2.3) satisfies
\[
(4.19) \quad \frac{d}{dt} \left( \rho(t) - \int_{[0,t]} \mu_+(ds)\rho(t - s) \right) = \int_{[0,t]} \nu_+(ds)\rho(t - s), \quad t \geq 0,
\]
with \( \rho(t) = 0 \) for \( t \in [-\tau,0] \) and \( \rho(0) = I_d \). Define \( W(t) := X(t) - x(t) \) for \( t \geq -\tau \). Then \( W \) obeys
\[
(4.20) \quad d \left( W(t) - \int_{[0,t]} \mu_+(ds)W(t - s) \right) = \int_{[0,t]} \nu_+(ds)W(t - s) dt + \Sigma dB(t), \quad t \geq 0;
\]
\[ W(t) = 0, \quad t \in [-\tau,0], \]
and is the unique solution of the above equation. With \( \kappa \) defined by (4.1) we have (4.15) with \( \kappa(0) = I_d \) and \( \kappa(t) = 0 \) for all \( t < 0 \). Let
\[
(4.21) \quad Z(t) := W(t) - \int_{[0,t]} \mu_+(ds)W(t - s), \quad t \in \mathbb{R}.
\]
Then \( Z(0) = W(0) = 0 \), and we may write \( Z = W - \mu_+ * W \). Clearly \( Z \) is continuous. Let \( \rho_0 \) be the integral resolvent of \( -\mu_+ \) defined by (4.7). Then by Theorem 4.1.7 in [11] we have \( W = Z - \rho_0 * Z \). Therefore by this, the definition of \( \beta \) from (4.10), (4.20), and (4.21) we get
\[
dZ(t) = (\nu_+ * W)(t) dt + \Sigma dB(t) = \left[ \nu_+ \left( Z - \rho_0 * Z \right) \right](t) dt + \Sigma dB(t)
\]
\[ = (\beta * Z)(t) dt + \Sigma dB(t) \]
Now by (4.10), and using the fact that $Z(0) = 0$, we have by the variation of constants formula (cf. [36]) that $Z$ obeys
\begin{equation}
Z(t) = \int_0^t \kappa(t-s)\Sigma dB(s), \quad t \geq 0.
\end{equation}
Therefore as $W = Z - \rho_0 * Z$ we have
\begin{equation}
W(t) = \int_0^t \kappa(t-s)\Sigma dB(s) - \int_0^t \int_0^{t-s} \rho_0(ds)\kappa(t-s-u)\Sigma dB(u).
\end{equation}
Hence by a stochastic Fubini theorem (cf., e.g., [33, Ch. IV.6, Theorem 64]) and (4.8), we have for all $t \geq 0$,
\begin{align*}
W(t) &= \int_0^t \kappa(t-s)\Sigma dB(s) - \int_0^t \int_0^{t-s} \rho_0(ds)\kappa(t-s-u)\Sigma dB(u) \\
&= \int_0^t \kappa(t-s)\Sigma dB(s) - \int_0^t (\rho_0 * \kappa)(t-s)\Sigma dB(s).
\end{align*}
Since $X(t) = x(t) + W(t)$ we have (2.8) as required.

5. Proof of Theorem 2

To prove this result, we need a result about the rate of growth of the running maxima of a sequence of standard normal random variables which have an exponentially decaying autocovariance function. It is Lemma 3 in [2].

Lemma 2. Suppose $(X_n)_{n=1}^\infty$ is a sequence of jointly normal standard random variables satisfying
\begin{equation}
|\text{Cov}(X_i, X_j)| \leq \lambda |i-j|
\end{equation}
for some $\lambda \in (0, 1)$. Then
\begin{equation}
\lim_{n \to \infty} \frac{\max_{1 \leq j \leq n} X_j}{\sqrt{2 \log n}} = 1, \quad \text{a.s.}
\end{equation}
This result will be required in the proof of the following lemma.

Lemma 3. Let $B$ be an $m$-dimensional standard Brownian motion. Suppose that for each $j = 1, \ldots, m$, $\gamma_j$ is a deterministic function such that $\gamma_j \in C([0, \infty); \mathbb{R}) \cap L^2([0, \infty); \mathbb{R}^{d \times d})$. Define
\begin{equation}
U(t) = \sum_{j=1}^m \int_0^t \gamma_j(t-s) dB_j(s), \quad t \geq 0.
\end{equation}
Then
(a) For every $\theta \in (0, 1)$, there is an a.s. event $\Omega_\theta$ such that
\begin{equation}
\limsup_{n \to \infty} \frac{|U(n^\theta)|}{\sqrt{2 \log n}} \leq \left( \frac{1}{\theta} \sum_{j=1}^m \int_0^\infty \gamma_j^2(s) \, ds \right)^{1/2}, \quad \text{a.s. on } \Omega_\theta.
\end{equation}
(b) If there exists $c > 0$ and $\alpha > 0$ such that $|\gamma_j(t)| \leq ce^{-\alpha t}$ for all $t \geq 0$ and $j = 1, \ldots, m$ then
\begin{equation}
\limsup_{t \to \infty} \frac{|U(t)|}{\sqrt{2 \log t}} \geq \left( \sum_{j=1}^m \int_0^\infty \gamma_j^2(s) \, ds \right)^{1/2}, \quad \text{a.s.}
\end{equation}
Furthermore we have

\[ \limsup_{t \to \infty} \frac{U(t)}{\sqrt{2 \log t}} \geq \left( \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds \right)^{1/2}, \quad \text{a.s.} \]  

(5.2)

\[ \liminf_{t \to \infty} \frac{U(t)}{\sqrt{2 \log t}} \leq -\left( \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds \right)^{1/2}, \quad \text{a.s.} \]  

(5.3)

Proof. Note that with \( v \) defined by

\[ v^{2}(t) := \int_{0}^{t} \sum_{j=1}^{m} \gamma_{j}^{2}(s) \, ds, \quad \text{for all } t \geq 0. \]  

(5.4)

we have that \( U(t) \) is normally distributed with mean zero and variance \( v(t) \).

We first prove part (a). In the case when \( \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds = 0 \) the result is trivial, because we have \( \gamma_{j}(t) = 0 \) for all \( t \geq 0 \) and each \( j = 1, \ldots, m \), in which case \( U(t) = 0 \) for all \( t \geq 0 \) a.s.

Suppose that \( \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds > 0 \). Since \( \gamma \) is square integrable there exists \( T_{0} > 0 \) such that

\[ v^{2}(t) \geq \frac{1}{2} \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds > 0, \quad \text{for all } t \geq T_{0}. \]  

(5.5)

Let \( N_{0} \) be the minimal integer greater that \( T_{0} \). Let \( \theta \in (0, 1) \). Note that for \( n \geq N_{0} \) we have that \( X_{n}(\theta) := U(n^{\theta})/v(n^{\theta}) \) is a standard normal random variable. Let \( \epsilon > 0 \) and \( n \geq N_{0} \). Then

\[ \Pr[|X_{n}(\theta)| \geq \sqrt{1 + \epsilon \sqrt{2 \log n}}] \leq \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{1 + \epsilon \sqrt{2 \log n}}} e^{-(1+\epsilon)\log n} \]

\[ = \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{1 + \epsilon \sqrt{2 \log n}}} \frac{1}{n^{1+\epsilon}}. \]

By the Borel–Cantelli lemma there exists an a.s. event \( \Omega_{\epsilon, \theta} \) such that

\[ \limsup_{n \to \infty} \frac{|X_{n}(\theta)|}{\sqrt{2 \log n}} \leq \sqrt{1 + \epsilon}, \quad \text{a.s. on } \Omega_{\epsilon, \theta}. \]

Let \( \Omega_{\theta} = \cap_{\epsilon \in (0, 1) \cap \mathbb{Q}} \Omega_{\epsilon, \theta} \). Then \( \Omega_{\theta} \) is an a.s. event. Then

\[ \limsup_{n \to \infty} \frac{|U(n^{\theta})/v(n^{\theta})|}{\sqrt{2 \log n}} = \limsup_{n \to \infty} \frac{|X_{n}(\theta)|}{\sqrt{2 \log n}} \leq 1, \quad \text{a.s. on } \Omega_{\theta}. \]

Since \( v^{2}(n^{\theta}) \to \int_{0}^{\infty} \sum_{j=1}^{m} \gamma_{j}^{2}(s) \, ds \), we have proven part (a).

We now prove part (b). In the case when \( \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds = 0 \) the result is trivial. Suppose instead that \( \sum_{j=1}^{m} \int_{0}^{\infty} \gamma_{j}^{2}(s) \, ds > 0 \). Note that

\[ \text{Cov}(U(t), U(t+h)) = \int_{0}^{t} \sum_{j=1}^{m} \gamma_{j}(s) \gamma_{j}(s+h) \, ds, \quad t \geq 0, h \geq 0. \]
With $T_0 > 0$ defined in (5.5), we let $t \geq T_0$, so that $v(t) > 0$ and $v(t+h) \geq v(t) > 0$. Then by (5.5)

\[
|\text{Corr}(U(t), U(t+h))| = \frac{1}{v(t)v(t+h)} \left| \int_0^t \sum_{j=1}^m \gamma_j(s)\gamma_j(s+h) \, ds \right|
\]

\[
\leq \frac{1}{v(t)^2} \int_0^t \sum_{j=1}^m |\gamma_j(s)||\gamma_j(s+h)| \, ds
\]

\[
\leq \frac{1}{2} \sum_{j=1}^m \int_0^\infty \gamma_j^2(s) \, ds \int_0^t \sum_{j=1}^m |\gamma_j(s)|ce^{-\alpha(s+h)} \, ds
\]

\[
\leq e^{-\alpha h} \sum_{j=1}^m \int_0^\infty \gamma_j^2(s) \, ds \int_0^t \sum_{j=1}^m |\gamma_j(s)|e^{-\alpha s} \, ds.
\]

Since $|\gamma_j(t)| \leq ce^{-\alpha t}$, the righthand side is finite, and moreover for all $t \geq T_0$ and $h \geq 0$ there is a $c_1 > 0$ independent of $t$ and $h$ such that

\[
|\text{Corr}(U(t), U(t+h))| \leq c_1 e^{-\alpha h}.
\]

If $c_1 \in (0, 1]$, we can define the process $(X_n)_{n \geq 0}$ by $X_n = U(T_0 + n)/v(T_0 + n)$ for $n \geq 0$. Define $\lambda := e^{-\alpha} \in (0, 1)$. Clearly $X_n$ is a standard normal random variable for each $n$. Furthermore for all $h \in \mathbb{N}$ we have

\[
|\text{Cov}(X_n, X_{n+h})| = |\text{Cov}(U(T_0 + n)/v(T_0 + n), U(T_0 + n+h)/v(T_0 + n+h))|
\]

\[
= |\text{Corr}(U(T_0 + n), U(T_0 + n+h))| \leq c_1 e^{-\alpha h} \leq e^{-ah} = \lambda^h.
\]

Therefore by Lemma 2 we have

\[
\lim_{n \to \infty} \max_{1 \leq j \leq n} \frac{U(T_0 + j)/v(T_0 + j)}{\sqrt{2 \log n}} = 1, \quad \text{a.s.}
\]

Next we have

\[
\limsup_{n \to \infty} \frac{|U(T_0 + n)|/v(T_0 + n)}{\sqrt{2 \log n}} = \limsup_{n \to \infty} \frac{\max_{1 \leq j \leq n} |U(T_0 + j)|/v(T_0 + j)}{\sqrt{2 \log n}}
\]

combining these relations gives

\[
\limsup_{n \to \infty} \frac{|U(T_0 + n)|/v(T_0 + n)}{\sqrt{2 \log n}} \geq 1, \quad \text{a.s.}
\]

Define $\Gamma^2 = \int_0^\infty \sum_{j=1}^m \gamma_j^2(s) \, ds$. Then $v(t) \to \Gamma > 0$ as $t \to \infty$. Therefore

\[
\limsup_{t \to \infty} \frac{|U(t)|}{\Gamma \sqrt{2 \log t}} = \limsup_{t \to \infty} \frac{|U(t)|/v(t)}{\sqrt{2 \log t}}
\]

\[
\geq \limsup_{n \to \infty} \frac{|U(T_0 + n)|/v(T_0 + n)}{\sqrt{2 \log(T_0 + n)}}
\]

\[
= \limsup_{n \to \infty} \frac{|U(T_0 + n)|/v(T_0 + n)}{\sqrt{2 \log n}} \cdot \frac{\sqrt{2 \log n}}{\sqrt{2 \log(T_0 + n)}}
\]

Therefore we have

\[
\limsup_{t \to \infty} \frac{|U(t)|}{\Gamma \sqrt{2 \log t}} \geq 1, \quad \text{a.s.}
\]

proving part (b) in the case where $c_1 \in [0, 1]$.

Suppose to the contrary that $c_1 > 1$. Choose $N \in \mathbb{N}$ so that $c_1 e^{-\alpha N} < 1$. We can define the process $(X_n)_{n \geq 0}$ by $X_n = U(T_0 + Nn)/v(T_0 + Nn)$ for $n \geq 0$. Define
\( \lambda := c_1 e^{-\alpha N} \in (0, 1) \). Clearly \( X_n \) is a standard normal random variable for each \( n \). Furthermore for all \( h \in \mathbb{N} \) we have
\[
|\text{Cov}(X_n, X_{n+h})| = |\text{Cov} \left( \frac{U(T_0 + Nn)}{v(T_0 + Nn)}, \frac{U(T_0 + Nn + Nh)}{v(T_0 + Nn + Nh)} \right)|
\]
\[
= |\text{Corr}(U(T_0 + Nn), U(T_0 + Nn + Nh))| \leq c_1 e^{-\alpha Nh}
\]
\[
\leq c_1 e^{-\alpha Nh} = \lambda^h,
\]
because \( c_1 > 1 \). Therefore by Lemma \[2\] we have
\[
\lim_{n \to \infty} \max_{1 \leq j \leq n} \frac{U(T_0 + Nj)/v(T_0 + Nj)}{\sqrt{2 \log n}} = 1, \text{ a.s.}
\]
Next we have
\[
\lim_{n \to \infty} \frac{|U(T_0 + Nn)|/v(T_0 + Nn)}{\sqrt{2 \log n}} = \lim_{n \to \infty} \sup_{1 \leq j \leq n} \frac{|U(T_0 + Nj)|/v(T_0 + Nj)}{\sqrt{2 \log n}},
\]
so combining these relations gives
\[
\lim_{n \to \infty} \frac{|U(T_0 + Nn)|/v(T_0 + Nn)}{\sqrt{2 \log n}} \geq 1, \text{ a.s.}
\]
Therefore we have
\[
\lim_{t \to \infty} \frac{|U(t)|}{\sqrt{2 \log t}} \geq 1, \text{ a.s.}
\]
proving part (b) in the case where \( c_1 > 1 \).

In the case when \( c_1 \in (0, 1) \) or when \( c_1 > 1 \) we have that there exist \( T_0 > 0 \) and \( N \geq 1 \) such that \( X_n := U(T_0 + Nn)/v(T_0 + Nn) \) defines a sequence of standard zero mean normal random variables for which
\[
\lim_{n \to \infty} \max_{1 \leq j \leq n} \frac{U(T_0 + jN)}{\sqrt{2 \log n}} = 1, \text{ a.s.}
\]
Therefore a.s. we have
\[
\lim_{t \to \infty} \frac{U(t)}{\sqrt{2 \log t}} \geq \lim_{n \to \infty} \frac{U(T_0 + nN)}{\sqrt{2 \log(T_0 + nN)}} = \lim_{n \to \infty} \frac{X_n v(T_0 + nN)}{\sqrt{2 \log(T_0 + nN)}}
\]
\[
= \lim_{n \to \infty} \frac{X_n}{\sqrt{2 \log n}} \cdot \Gamma = \lim_{n \to \infty} \frac{\max_{1 \leq j \leq n} X_j}{\sqrt{2 \log n}} \cdot \Gamma
\]
proving (5.2). The above argument for part (b) can be applied equally to \( -U \), so that we get
\[
\lim_{t \to \infty} \frac{-U(t)}{\sqrt{2 \log t}} \geq \Gamma, \text{ a.s.,}
\]
from which (5.3) can be deduced. \( \square \)

We need one other estimate on the asymptotic behaviour of a Gaussian process.
Lemma 4. Suppose $B$ is an $m$-dimensional Brownian motion. For $j = 1, \ldots, m$ suppose that $\kappa_j$ is in $C^1((0, \infty); \mathbb{R})$ in such a way that $\kappa_j \in L^2((0, \infty), \mathbb{R})$ and $\kappa_j' \in L^p([0, \infty), \mathbb{R})$. and define $Z = \{Z(t) : t \geq 0\}$ by

$$Z(t) = \sum_{j=1}^{m} \int_{0}^{t} \kappa_j(t-s) dB_j(s), \quad t \geq 0.$$ 

Suppose that $\theta \in (0, 1)$. Then

$$\lim_{n \to \infty} \sup_{n^\theta \leq t \leq (n+1)^\theta} |Z(t) - Z(n^\theta)| = 0, \quad \text{a.s.}$$

Proof. We note that

$$Z(t) = \sum_{j=1}^{m} \int_{0}^{t} \kappa_j(0) + \int_{0}^{t-s} \kappa_j'(u) du \, dB_j(s)$$

$$= \sum_{j=1}^{m} \kappa_j(0) B_j(t) + \sum_{j=1}^{m} \int_{0}^{t} \int_{0}^{u} \kappa_j'(u-s) dB_j(s) du.$$ 

Hence for $t \in [n^\theta, (n+1)^\theta]$, we get

$$Z(t) - Z(n^\theta) = \sum_{j=1}^{m} \kappa_j(0) (B_j(t) - B_j(n^\theta)) + \sum_{j=1}^{m} \int_{n^\theta}^{t} \int_{n^\theta}^{u} \kappa_j'(u-s) dB_j(s) du,$$

which implies

$$\sup_{n^\theta \leq t \leq (n+1)^\theta} |Z(t) - Z(n^\theta)| \leq \sum_{j=1}^{m} |\kappa_j(0)| \sup_{n^\theta \leq t \leq (n+1)^\theta} |B_j(t) - B_j(n^\theta)| + \sum_{j=1}^{m} U_j(n),$$

where we have defined

$$U_j(n) = \sup_{n^\theta \leq t \leq (n+1)^\theta} \left| \int_{n^\theta}^{t} \int_{n^\theta}^{u} \kappa_j'(u-s) dB_j(s) du \right|.$$ 

Now if $V$ is a normal random variable with zero mean, and $p \geq 2$, there is a constant $c(p) > 0$ such that $\mathbb{E}[|V|^p] = c(p) \mathbb{E}[V^2]^{p/2}$. Let $p \geq 2/(1-\theta) > 2$. Therefore we have

$$\mathbb{E}[U_j(n)^p] = \mathbb{E} \left[ \sup_{n^\theta \leq t \leq (n+1)^\theta} \left| \int_{n^\theta}^{t} \int_{n^\theta}^{u} \kappa_j'(u-s) dB_j(s) du \right|^p \right]$$

$$\leq \mathbb{E} \left[ \sup_{n^\theta \leq t \leq (n+1)^\theta} \left( \int_{n^\theta}^{t} \left| \int_{n^\theta}^{u} \kappa_j'(u-s) dB_j(s) \right| du \right)^p \right]$$

$$\leq \mathbb{E} \left\{ \sup_{n^\theta \leq t \leq (n+1)^\theta} (t-n^\theta)^{p-1} \int_{n^\theta}^{t} \left| \int_{n^\theta}^{u} \kappa_j'(u-s) dB_j(s) \right|^p du \right\}$$

$$= ((n+1)^\theta - n^\theta)^{p-1} \int_{n^\theta}^{(n+1)^\theta} \mathbb{E} \left( \int_{n^\theta}^{u} \kappa_j'(u-s) dB_j(s) \right)^p du$$

$$= ((n+1)^\theta - n^\theta)^{p-1} \int_{n^\theta}^{(n+1)^\theta} c(p) \left( \int_{n^\theta}^{u} (\kappa_j')^2(s) ds \right)^{p/2} du.$$ 

Since $\kappa_j'$ is in $L^2([0, \infty), \mathbb{R})$ we have

$$\mathbb{E}[U_j(n)^p] \leq c(p) \left( \int_{0}^{\infty} (\kappa_j')^2(s) ds \right)^{p/2} ((n+1)^\theta - n^\theta)^p.$$
Since \( p \geq 2/(1 - \theta) \), we have that \( \sum_{j=1}^{\infty} \mathbb{E}[U_j(t)^p] < +\infty \), and therefore by the Borel–Cantelli lemma it follows that

\[
\lim_{n \to \infty} U_j(n) = 0, \quad \text{a.s.}
\]

Let \( \epsilon > 0 \). Then by the properties of a standard Brownian motion, we have

\[
P \left[ \sup_{n^\theta \leq t \leq (n+1)^\theta} |B_j(t) - B_j(n^\theta)| > \epsilon \right] \leq 2P \left[ \sup_{0 \leq t \leq (n+1)^\theta - n^\theta} B_j(t) > \epsilon \right] = 2P(|B_j((n+1)^\theta - n^\theta)| > \epsilon) = 4P \left[ Z > \frac{\epsilon}{\sqrt{(n+1)^\theta - n^\theta}} \right],
\]

where \( Z \) is a standard normal random variable. Since \( \{(n+1)^\theta - n^\theta\}/n^{\theta-1} \to \theta \) as \( n \to \infty \), by Mill’s estimate and the Borel–Cantelli lemma, there exists \( N(\omega, \epsilon) \in \mathbb{N} \), such that for all \( n > N(\epsilon) \)

\[
\sup_{n^\theta \leq t \leq (n+1)^\theta} |B_j(t) - B_j(n^\theta)| \leq \epsilon, \quad \text{a.s. on } \Omega_\epsilon
\]

Define \( \Omega^* = \cap_{\epsilon \in (0,1)} \Omega_\epsilon \). Then \( \Omega^* \) is an a.s. event and we have

\[
\lim_{n \to \infty} \sup_{n^\theta \leq t \leq (n+1)^\theta} |B_j(t) - B_j(n^\theta)| = 0, \quad \text{a.s. on } \Omega^*.
\]

Taking the limit as \( n \to \infty \) across both sides of (5.7) and using (5.8) and (5.9) we obtain (5.6) as required.

5.1. Proof of Theorem 2 Let \( x \) be the solution of (2.2) and \( X \) the solution of (2.3). Then by Theorem 1 for \( t \geq 0 \) we have \( X(t) = x(t) + W(t) \). If \( Z \) is defined by (1.24), the argument of Theorem 1 tells us that \( W = Z - \rho_0 * Z \) and \( Z \) obeys (1.22). Therefore

\[
X(t) = x(t) + W(t) = x(t) + Z(t) - \int_{[0,t]} \rho_0(ds)Z(t-s), \quad t \geq 0.
\]

Let \( \theta \in (0,1) \). Let \( t \geq 0 \) and \( n \) be an integer such that \( t \in [n^\theta, (n+1)^\theta) \). Then

\[
X(t) - X(n^\theta) = x(t) - x(n^\theta) + Z(t) - Z(n^\theta)
\]

\[
= x(t) - x(n^\theta) + \left( \int_{[0,t]} \rho_0(ds)Z(t-s) - \int_{[0,n^\theta]} \rho_0(ds)Z(n^\theta - s) \right)
\]

\[
= x(t) - x(n^\theta) + Z(t) - Z(n^\theta)
\]

\[
= x(t) - x(n^\theta) + \left( \int_{[0,t]} \rho_0(ds)Z(t-s) - \int_{[0,n^\theta]} \rho_0(ds)Z(t-s) \right)
\]

\[
+ \left( \int_{[0,t]} \rho_0(ds)Z(t-s) - \int_{[0,n^\theta]} \rho_0(ds)Z(t-s) \right)
\]

\[
= x(t) - x(n^\theta) + Z(t) - Z(n^\theta) - \int_{[n^\theta,t]} \rho_0(ds)Z(t-s) - \int_{[n^\theta,t]} \rho_0(ds)Z(t-s)
\]

\[
- \int_{[0,n^\theta]} \rho_0(ds) \left( Z(t-s) - Z(n^\theta - s) \right).
\]
Therefore

\begin{equation}
(5.10) \quad \sup_{n^\theta \leq t \leq (n+1)^\theta} |X(t) - X(n^\theta)| \\
\leq \sup_{n^\theta \leq t \leq (n+1)^\theta} |x(t) - x(n^\theta)| + \sup_{n^\theta \leq t \leq (n+1)^\theta} |Z(t) - Z(n^\theta)| \\
+ \sup_{n^\theta \leq t \leq (n+1)^\theta} \left| \int_{[n^\theta, t]} \rho_0(ds) Z(t - s) \right| \\
+ \sup_{n^\theta \leq t \leq (n+1)^\theta} \left| \int_{[0, n^\theta]} \rho_0(ds) (Z(t) - Z(n^\theta - s)) \right|.
\end{equation}

We now consider each of the four terms on the right-hand side of (5.10) in turn. It is easy to see that

\[
\lim_{n \to \infty} \sup_{n^\theta \leq t \leq (n+1)^\theta} |x(t) - x(n^\theta)| = 0.
\]

By applying Lemma 4 component by component it follows that

\begin{equation}
(5.11) \quad \lim_{n \to \infty} \sup_{n^\theta \leq t \leq (n+1)^\theta} |Z(t) - Z(n^\theta)| = 0, \quad \text{a.s.}
\end{equation}

For the third term,

\[
\sup_{n^\theta \leq t \leq (n+1)^\theta} \left| \int_{[n^\theta, t]} \rho_0(ds) Z(t - s) \right| \\
\leq \sup_{n^\theta \leq t \leq (n+1)^\theta} \int_{[n^\theta, t]} |\rho_0|(ds) |Z(t - s)| \\
\leq \sup_{n^\theta \leq t \leq (n+1)^\theta} \sup_{n^\theta \leq s \leq t} |Z(t - s)| \cdot \int_{[n^\theta, \infty)} |\rho_0|(ds) \\
= \sup_{n^\theta \leq t \leq (n+1)^\theta} |Z(t)| \cdot \int_{[0, \infty)} |\rho_0|(ds) \\
= \sup_{0 \leq u \leq (n+1)^\theta - n^\theta} |Z(u)| \cdot \int_{[0, \infty)} |\rho_0|(ds),
\]

which implies

\begin{equation}
(5.12) \quad \lim \sup_{n \to \infty} \sup_{n^\theta \leq t \leq (n+1)^\theta} \left| \int_{[n^\theta, t]} \rho_0(ds) Z(t - s) \right| \\
\leq \lim \sup_{n \to \infty} \sup_{0 \leq u \leq (n+1)^\theta - n^\theta} |Z(u)| \cdot \int_{[0, \infty)} |\rho_0|(ds) = |Z(0)| \cdot \int_{[0, \infty)} |\rho_0|(ds) = 0, \quad \text{a.s.}
\end{equation}

For the last term on the right-hand side of (5.10), we note that for \( t \geq 0 \), by (1.22)

\[
Z(t) = \int_0^t (\kappa(0) + \int_0^{t-s} \kappa'(v) \, dv) \Sigma dB(s) \\
= \kappa(0) \Sigma B(t) + \int_0^t \int_0^{t-s} \kappa'(v) \, dv \Sigma dB(s) \\
= \Sigma B(t) + \int_0^t \int_0^s \kappa'(u - s) \Sigma du \, dB(s) \\
= \Sigma B(t) + \int_0^t \int_0^u \kappa'(u - s) \Sigma dB(s) \, du.
\]

So for \( n^\theta \leq t \leq (n+1)^\theta \),

\[
Z(t) - Z(n^\theta - s) = \Sigma \left( B(t) - B((n^\theta - s)) \right) + \int_{n^\theta - s}^t \int_0^u \kappa'(u - v) \Sigma dB(v) \, du.
\]
Hence

\[ (5.13) \sup_{n^\theta \leq (n+1)^{\theta}} \left| \int_{[0,n^\theta]} \rho_0(ds)(Z(t - s) - Z(n^\theta - s)) \right| \]

\[ \leq |\Sigma| \sup_{n^\theta \leq (n+1)^{\theta}} \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)| \]

\[ + \sup_{n^\theta \leq (n+1)^{\theta}} \int_{[0,n^\theta]} |\rho_0|(ds) \int_{n^\theta - s}^{u} \kappa'(u - v) dB(v) du. \]

For the first term on the right-hand side of (5.13), for some \( p_\theta > 1 \) and \( q_\theta > 1 \) such that \( 1/p_\theta + 1/q_\theta = 1 \),

\[ E \left[ \left( \sup_{n^\theta \leq (n+1)^{\theta}} \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)| \right)^{p_\theta} \right] \]

\[ \leq E \left[ \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{q_\theta} \left( \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)|^{p_\theta} \right) \right] \]

\[ \leq \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{q_\theta} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_\theta} \left( \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)|^{p_\theta} \right) \]

\[ \leq \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{q_\theta} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_\theta} \left( \int_{[0,n^\theta]} \sup_{n^\theta - s \leq u \leq (n+1)^{\theta} - s} |B(u) - B(n^\theta - s)|^{p_\theta} \int_{n^\theta - s}^{u} dB(v) \right)^{p_\theta} \]

\[ \leq \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_\theta} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_\theta} \left[ (n + 1)^{\theta} - n^\theta \right]^{p_\theta}, \]

where we have used the Hölder inequality and Burkholder-Davis-Gundy inequality in the second and penultimate lines respectively. Hence by the Chebyshev inequality

\[ P \left[ \sup_{n^\theta \leq (n+1)^{\theta}} \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)| > 1 \right] \]

\[ \leq E \left[ \left( \sup_{n^\theta \leq (n+1)^{\theta}} \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)| \right)^{p_\theta} \right] \]

\[ \leq \left( \frac{32}{p_\theta} \right)^{\frac{p_\theta}{q_\theta}} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{\frac{p_\theta q_\theta}{q_\theta}} \left[ (n + 1)^{\theta} - n^\theta \right]^{p_\theta}. \]

Now since \( \lim_{n \to \infty} [(n + 1)^{\theta} - n^\theta]^{1/(\theta - 1)} = 0 \), if we choose \( p_\theta = 4/(1 - \theta) > 1 \), then by the Borel-Cantelli lemma, we get

\[ (5.14) \limsup_{n \to \infty} \sup_{n^\theta \leq (n+1)^{\theta}} \int_{[0,n^\theta]} |\rho_0|(ds)|B(t - s) - B(n^\theta - s)| \leq 1 \quad \text{a.s.} \]
Therefore there exists \( K \), define \( I(u) := \int_0^u \kappa'(u - v) \Sigma dB(v) \) and \( H_n(s) := \int_{n^\theta - s}^{(n+1)^\theta - s} |I(u)| \, du \). Then

\[
A_n := \sup_{n^\theta \leq t \leq (n+1)^\theta} \int_{[0,n^\theta]} |\rho_0|(ds) \int_{n^\theta - s}^u \kappa'(u - v) \Sigma dB(v) \, du
\]

\[
\leq \sup_{n^\theta \leq t \leq (n+1)^\theta} \int_{[0,n^\theta]} |\rho_0|(ds) \int_{n^\theta - s}^u \kappa'(u - v) \Sigma dB(v) \, du
\]

\[
\leq \int_{[0,n^\theta]} |\rho_0|(ds) H_n(s).
\]

Therefore if \( p_0 > 1 \) and \( q_0 > 1 \) are such that \( 1/p_0 + 1/q_0 = 1 \), then by Hölder’s inequality we have

\[
A_n^{p_0} \leq \left( \int_{[0,n^\theta]} |\rho_0|(ds) H_n(s) \right)^{p_0}
\]

\[
\leq \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0 \over q_0} \left( \int_{[0,n^\theta]} |\rho_0|(ds) H_n(s) \right)^{p_0}
\]

\[
\leq \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0 \over q_0} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0}
\]

\[
\times \int_{[0,n^\theta]} |\rho_0|(ds) \left( (n+1)^\theta - n^\theta \right)^{p_0 - 1} \int_{n^\theta - s}^{(n+1)^\theta - s} |I(u)| \, du.
\]

Now as \( \kappa' \in L^2 \), we have that

\[
\mathbb{E}[|I(u)|^2] = \int_0^u |\kappa'(s)\Sigma|^2 dF \, ds, \quad u \geq 0.
\]

Therefore there exists \( K(p) > 0 \) such that \( \mathbb{E}[|I(u)|^p] \leq K(p) \) for all \( u \geq 0 \). Therefore

\[
\mathbb{E}[A_n^{p_0}] \leq \left( (n+1)^\theta - n^\theta \right)^{p_0 - 1} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0 \over q_0}
\]

\[
\times \int_{[0,n^\theta]} |\rho_0|(ds) \left( (n+1)^\theta - n^\theta \right)^{p_0 - 1} \mathbb{E}[|I(u)|^p] \, du
\]

\[
\leq \left( (n+1)^\theta - n^\theta \right)^{p_0 - 1} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0 \over q_0}
\]

\[
\times \int_{[0,n^\theta]} |\rho_0|(ds) \left( (n+1)^\theta - n^\theta \right)^{p_0} \mathbb{E}[|I(u)|^p]
\]

\[
\leq K(p_0) \left( (n+1)^\theta - n^\theta \right)^{p_0} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0 \over q_0 + 1}.
\]

Therefore we have

\[
\mathbb{P}[A_n > 1] \leq \mathbb{E}[A_n^{p_0}] \leq K(p_0) \left( (n+1)^\theta - n^\theta \right)^{p_0} \left( \int_{[0,n^\theta]} |\rho_0|(ds) \right)^{p_0 \over q_0 + 1}.
\]

Let \( p_0 = 2/(1-\theta) \); then \( p_0 > 2 \). Then the righthand side of (5.15) is summable in \( n \), because \( \{(n+1)^\theta - n^\theta\} / n^{\theta - 1} \to \theta \) as \( n \to \infty \), so by the Borel–Cantelli Lemma
we have that
\[
\limsup_{n \to \infty} \sup_{n^\kappa \leq t \leq (n+1)^\kappa} \int_{[0,n^\kappa]} |\rho_0(ds)| \int^{t-s}_{n^\kappa-s} \int_{0}^{u} \kappa'(u-v) \Sigma dB(v) \, du = \limsup_{n \to \infty} A_n \leq 1, \text{ a.s.}
\]

Combining (5.13), (5.14) and (5.10), it follows that
\[
\limsup_{n \to \infty} \sup_{n^\kappa \leq t \leq (n+1)^\kappa} \left| \int_{[0,n^\kappa]} \rho_0(ds)\left( Z(t-s) - Z(n^\theta - s) \right) \right| = 0, \text{ a.s.}
\]

Gathering the results (5.10), (5.11), (5.12) and (5.17), we obtain
\[
\limsup_{n \to \infty} \sup_{n^\kappa \leq t \leq (n+1)^\kappa} |X(t) - X(n^\theta)| = 0, \text{ a.s.}
\]

Next we consider each component. We have
\[
W(t) := \int_{0}^{t} \rho(t-s) \Sigma dB(s), \quad t \geq 0.
\]

Notice that $W(t) \in \mathbb{R}^d$ for each $t \geq 0$. Also $W(t) = \int_{0}^{t} \theta(t-s) dB(s), \quad t \geq 0$, where $\theta(t) = \rho(t)\Sigma$ is a $d \times m$-matrix valued function in which each entry must obey $|\theta_{ij}(t)| \leq C e^{-\alpha(t,\kappa)t/2}, \quad t \geq 0$ for some $C > 0$. Hence $W_i(t) := \langle W(t), e_i \rangle$ obeys
\[
W_i(t) = \sum_{j=1}^{m} \int_{0}^{t} \theta_{ij}(t-s) dB_j(s), \quad t \geq 0.
\]

Define $\theta_i(t) \geq 0$ with $\theta^2_i(t) = \sum_{j=1}^{m} \theta^2_{ij}(t), \quad t \geq 0$. Then $W_i(t)$ is normally distributed with mean zero and variance $\nu_i(t) = \int_{0}^{\infty} \theta^2(t) \, ds$. Since $\theta_i \in L^2(0, \infty)$, we have that $\nu_i(t) \to 0$ as $t \to \infty$, and moreover $|\theta_i(t)| \leq C m e^{-\alpha(t,\kappa)t/2}, \quad t \geq 0$. By part (b) of Lemma 3 we have
\[
\limsup_{t \to \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} \geq \sigma_i, \quad \limsup_{t \to \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} \geq \sigma_i, \quad \liminf_{t \to \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} \leq -\sigma_i, \text{ a.s.}
\]

We now wish to prove
\[
\limsup_{t \to \infty} \frac{W_i(t)}{\sqrt{2 \log t}} \leq \limsup_{t \to \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} \leq \sigma_i, \text{ a.s.}
\]

We first note for each $\theta > 0$ that part (a) of Lemma 5 yields the estimate
\[
\limsup_{n \to \infty} \frac{|W_i(n^\theta)|}{\sqrt{2 \log(n^\theta)}} \leq \sqrt{\frac{\sigma_i^2}{\theta}}, \text{ a.s.}
\]

Define $X_i(t) = \langle X(t), e_i \rangle$ for $t \geq 0$. Using (5.21), the fact that $x(t) \to 0$ as $t \to \infty$, and the fact that
\[
W_i(t) = W_i(n^\theta) + x_i(n^\theta) - x_i(t) + X_i(t) - X_i(n^\theta)
\]

we may use (5.18) to obtain
\[
\limsup_{n \to \infty} \sup_{n^\kappa \leq t \leq (n+1)^\kappa} \frac{|W_i(t)|}{\sqrt{2 \log t}} \leq \sqrt{\frac{\sigma_i^2}{\theta}}, \text{ a.s.,}
\]

which implies
\[
\limsup_{t \to \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} \leq \sqrt{\frac{\sigma_i^2}{\theta}}, \text{ a.s.}
\]
Letting \( \theta \rightarrow 1 \) through the rational numbers implies (5.20). Therefore by (5.19) and (5.20) we have

\[
\limsup_{t \rightarrow \infty} \frac{W_i(t)}{\sqrt{2 \log t}} = \sigma_i, \quad \limsup_{t \rightarrow \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} = \sigma_i, \quad \text{a.s.}
\]

It is a consequence of (5.20) that

\[
-\liminf_{t \rightarrow \infty} \frac{W_i(t)}{\sqrt{2 \log t}} = \limsup_{t \rightarrow \infty} \frac{-W_i(t)}{\sqrt{2 \log t}} \leq \limsup_{t \rightarrow \infty} \frac{|W_i(t)|}{\sqrt{2 \log t}} \leq \sigma_i, \quad \text{a.s.}
\]

Combining this with the third inequality in (5.19) we obtain

\[
\liminf_{t \rightarrow \infty} \frac{W_i(t)}{\sqrt{2 \log t}} = -\sigma_i, \quad \text{a.s.}
\]

From all these estimates, and recalling that \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \), we obtain (3.1) as required.

To prove (3.3), note that there is an \( i^* \in \{1, \ldots, d\} \) such that \( \sigma_{i^*} = \max_{1 \leq i \leq d} \sigma_i \), Next, note for each \( t \geq 0 \) that

\[
\max_{1 \leq i \leq d} |X_i(t)| = \max(|X_1(t)|, |X_2(t)|, \ldots, |X_{i^*}(t)|, \ldots, |X_d(t)|) \geq |X_{i^*}(t)|.
\]

Hence

\[
\limsup_{t \rightarrow \infty} \frac{\max_{1 \leq i \leq d} |X_i(t)|}{\sqrt{2 \log t}} \geq \limsup_{t \rightarrow \infty} \frac{|X_{i^*}(t)|}{\sqrt{2 \log t}} = \sigma_{i^*} = \max_{i=1, \ldots, d} \sigma_i, \quad \text{a.s.}
\]

Let \( p \) be an integer greater than unity. Note that \( \max_{1 \leq i \leq d} |x_i| \leq (\sum_{i=1}^d |x_i|^p)^{1/p} \), so we have

\[
\left( \limsup_{t \rightarrow \infty} \frac{\max_{1 \leq i \leq d} |X_i(t)|}{\sqrt{2 \log t}} \right)^p = \limsup_{t \rightarrow \infty} \frac{(\max_{1 \leq i \leq d} |X_i(t)|)^p}{(\sqrt{2 \log t})^p} \leq \limsup_{t \rightarrow \infty} \frac{\sum_{i=1}^d |X_i(t)|^p}{(\sqrt{2 \log t})^p} \leq \sum_{i=1}^d \limsup_{t \rightarrow \infty} \frac{|X_i(t)|^p}{(\sqrt{2 \log t})^p} = \sum_{i=1}^d \left( \limsup_{t \rightarrow \infty} \frac{|X_i(t)|}{\sqrt{2 \log t}} \right)^p = \sum_{i=1}^d \sigma_i^p.
\]

Hence

\[
\limsup_{t \rightarrow \infty} \frac{\max_{1 \leq i \leq d} |X_i(t)|}{\sqrt{2 \log t}} \leq \left( \sum_{i=1}^d \sigma_i^p \right)^{1/p}, \quad \text{a.s.}
\]

Letting \( p \rightarrow \infty \) through the natural numbers yields

\[
(5.23) \quad \limsup_{t \rightarrow \infty} \frac{\max_{1 \leq i \leq d} |X_i(t)|}{\sqrt{2 \log t}} \leq \max_{1 \leq i \leq d} \sigma_i, \quad \text{a.s.,}
\]

since \( \left( \sum_{i=1}^d \sigma_i^p \right)^{1/p} \rightarrow \max_{1 \leq i \leq d} \sigma_i \) as \( p \rightarrow \infty \). Combining (5.22) and (5.23) yields (3.3).
Let $X$ be the solution of \((3.14)\). Suppose that $Y$ obeys
\begin{equation}
(6.1) \quad d(Y(t) - D(Y_t)) = L(Y_t) dt + \Sigma dB(t), \quad t \geq 0; \quad Y_0 = \phi.
\end{equation}
Define for $t \geq -\tau$ the process $Z(t) := X(t) - Y(t)$. Since $D$ and $L$ are linear, $Z$ obeys
\begin{equation}
(6.2) \quad \frac{d}{dt} (Z(t) - D(Z_t) - N_1(t, Z_t + Y_t)) = L(Z_t) + N_2(t, Z_t + Y_t), \quad t > 0; \quad Z_0 = 0.
\end{equation}
Define $U = \{U(t) : t \geq 0\}$ by
\begin{equation}
(6.3) \quad U(t) = Z(t) - D(Z_t) - N_1(t, Z_t + Y_t), \quad t \geq 0.
\end{equation}
By the definition of $\mu_+$ and the fact that $Z(t) = 0$ for all $t \in [-\tau, 0]$, we have
\begin{equation}
U(t) = Z(t) - \int_{[0,t]} \mu_+(ds) Z(t-s) - N_1(t, Z_t + Y_t), \quad t \geq 0,
\end{equation}
so $Z - \mu_+ * Z = U + N_1$. Recall that $\rho_0$ defined by \((4.7)\) is in $M([0,\infty); \mathbb{R}^{d \times d})$ because $\mu$ obeys \((2.6)\). Therefore $Z = N_1 + U - \rho_0 * (N_1 + U)$ or
\begin{equation}
(6.4) \quad Z(t) = N_1(t, Z_t + Y_t) + U(t) - \int_{[0,t]} \rho_0(ds) (N_1(t-s, Y_{t-s} + Z_{t-s})) + U(t-s), \quad t \geq 0.
\end{equation}
By \((6.2)\), \((6.3)\) we have $U'(t) = L(Z_t) + N_2(t, Z_t + Y_t)$ for $t \geq 0$, so by the definition of $\nu_+$ we have $U'(t) = \int_{[0,t]} \nu_+(ds) Z(t-s) + N_2(t, Z_t + Y_t)$. Therefore
\begin{equation}
(6.5) \quad U'(t) = (\nu_+ * Z)(t) + N_2(t, Z_t + Y_t), \quad t > 0.
\end{equation}
By \((2.5)\) we have $U(0) = -N_1(0, Y_0) = -N_1(0, \phi)$. Thus by \((6.4)\) and the definition of $\beta$ from \((4.10)\), we get
\begin{equation}
U'(t) = \left\{ \nu_+ * [N_1 + U - \rho_0 * (N_1 + U)] \right\}(t) + N_2(t, Z_t + Y_t)
\end{equation}
(6.6)
\begin{equation}
= [\nu_+ - \nu_+ * \rho_0] * N_1(t) + N_2(t, Z_t + Y_t) + [\beta * U](t).
\end{equation}
Recall that $\kappa$ is the differential resolvent defined by \((4.10)\). Therefore, we also have
\begin{equation}
(6.7) \quad \kappa'(t) = (\kappa * \beta)(t), \quad t > 0.
\end{equation}
Now by \((6.6)\) and \((6.10)\) and the fact that $U(0) = -N_1(0, \phi)$ we have
\begin{equation}
U(t) = -\kappa(t) N_1(0, \phi) + \kappa * [\nu_+ - \nu_+ * \rho_0] * N_1](t)
\end{equation}
(6.8)
\begin{equation}
+ \int_0^t \kappa(t-s) N_2(s, Z_s + Y_s) ds, \quad t \geq 0.
\end{equation}
By \((6.7)\) this implies
\begin{equation}
U(t) = -\kappa(t) N_1(0, \phi) + (\kappa' * N_1)(t) + \int_0^t \kappa(t-s) N_2(s, Z_s + Y_s) ds, \quad t \geq 0.
\end{equation}
Let $\kappa_1 := \kappa'$. Then $\kappa_1 = \nu_+ * \rho$. Since $\rho \in L^1([0,\infty); \mathbb{R}^{d \times d}), \nu_+ \in M([0,\infty); \mathbb{R}^{d \times d})$, we have that $\kappa_1 \in L^1([0,\infty); \mathbb{R}^{d \times d})$. Therefore with $N_2(t) := N_2(t, Z_t + Y_t)$ we have
\begin{equation}
U(t) = -\kappa(t) N_1(0, \phi) + (\kappa_1 * N_1)(t) + (\kappa * N_2)(t), \quad t \geq 0.
\end{equation}
Inserting this into (6.4) we get

\[ Z(t) = N_1(t) - \kappa(t)N_1(0, \phi) + (\kappa_1*N_1)(t) + (\kappa*N_2)(t) - (\rho_0*N_1)(t) \]
\[ - \{\rho_0*[-\kappa N_1(0, \phi) + \kappa_1 N_1 + \kappa*N_2]\}(t), \quad t \geq 0. \]

This gives

\[ Z(t) = N_1(t) - \{\kappa(t) - (\rho_0*\kappa)(t)\}N_1(0, \phi) + \{(\kappa - \rho_0*\kappa)*N_2\}(t) \]
\[ + \{(\kappa_1*N_1)(t) - (\rho_0*N_1)(t) - (\rho_0*\kappa_1*N_1)(t), \quad t \geq 0. \]

By (6.8) we have \( \rho = \kappa - \rho_0*\kappa \), so if we define \( \kappa_2 \in M([0, \infty); \mathbb{R}^{d \times d}) \)

we have

\[ (6.8) \quad Z(t) = N_1(t) - \rho(t)N_1(0, \phi) + (\rho*N_2)(t) + (\kappa_2*N_1)(t), \quad t \geq 0. \]

\( \kappa_2 \) is guaranteed to be in \( M([0, \infty); \mathbb{R}^{d \times d}) \) because \( \kappa_1 \in L^1([0, \infty); \mathbb{R}^{d \times d}) \) and \( \rho_0 \in M([0, \infty); \mathbb{R}^{d \times d}) \). Since \( N_1 \) and \( N_2 \) are bounded by maximum functionals of \( Z + Y \), the asymptotic behaviour of \( Y \) is known, and the convolution “kernels” \( \kappa_2 \) and \( \rho \) are finite on the right hand side, we may treat (6.8) pathwise as a Volterra integral equation.

By (6.13), for any \( \phi \in C([-\tau, 0]; \mathbb{R}^d) \) and \( \varepsilon > 0 \), there exists \( L(\varepsilon) > 0 \) such that

\[ |N_1(t, \phi_t)| \leq L(\varepsilon) + \varepsilon \sup_{t - \tau \leq u \leq t} |\phi(s)|, \quad \text{for all } (t, \phi) \in (0, \infty) \times C([-\tau, \infty); \mathbb{R}^d). \]

Define

\[ (6.9) \quad c = 1 + \int_0^\infty |\rho(s)|_2 \, ds + \int_{[0, \infty)} |\kappa_2|_2 (ds) \]

Choose \( \varepsilon > 0 \) so small that \( \varepsilon c < 1/2 \). Therefore by (6.8) for \( t \geq 0 \)

\[ |Z(t)|_2 \leq L(\varepsilon) + \varepsilon \sup_{t - \tau \leq u \leq t} |Y(u) + Z(u)|_2 + |\rho(t)||N_1(0, \phi)| \]
\[ + \int_0^t |\rho(t - s)|_2 \left\{ L(\varepsilon) + \varepsilon \sup_{t - \tau \leq u \leq t} |Y(u) + Z(u)|_2 \right\} \, ds \]
\[ + \int_{[0, t]} |\kappa_2|_2 (ds) \left\{ L(\varepsilon) + \varepsilon \sup_{t - \tau \leq u \leq t - \tau} |Y(u) + Z(u)|_2 \right\}. \]

By Lemma\[ \text{1} \] since \( \rho \in L^1([0, \infty); \mathbb{R}^{d \times d}) \) we have that \( |\rho(t)| \to 0 \) as \( t \to \infty \). Therefore there exists \( T_1 > 0 \) such that \( |\rho(t)| \leq 1 \) for all \( t \geq T_1 \). Hence for \( t \geq T_1 \), using (6.9), we have

\[ |Z(t)|_2 \leq |N_1(0, \phi)| + L(\varepsilon)c + \varepsilon \sup_{t - \tau \leq u \leq t} (|Y(u)|_2 + |Z(u)|_2) \]
\[ + \varepsilon \int_0^t |\rho(t - s)|_2 \sup_{t - \tau \leq u \leq t - \tau} (|Y(u)|_2 + |Z(u)|_2) \, ds \]
\[ + \varepsilon \sup_{t - \tau \leq u \leq t - \tau} (|Y(u)|_2 + |Z(u)|_2). \]
Define $L_2(\varepsilon) := |N_1(0, \phi)| + L(\varepsilon)c$ and

\[(6.10) \quad f_\varepsilon(t) = L_2(\varepsilon) + \varepsilon \int_0^t |\rho(t-s)|_2 \sup_{s-t \leq u \leq s} |Y(u)|_2 \]
\[+ \varepsilon \int_{[0,t]} |k_2|(ds) \sup_{t-s \leq u \leq t-s} |Y(u)|_2 \sup_{t \leq u \leq t} |Y(u)|_2.\]

Hence for $t \geq T_1$ we have

\[|Z(t)|_2 \leq f_\varepsilon(t) + \varepsilon \int_0^t |\rho(t-s)|_2 \sup_{s-t \leq u \leq s} |Z(u)|_2 ds \]
\[+ \varepsilon \int_{[0,t]} |k_2|(ds) \sup_{t-s \leq u \leq t-s} |Z(u)|_2 + \varepsilon \sup_{t \leq u \leq t} |Z(u)|_2.\]

Now by (6.9) we have

\[|Z(t)|_2 \leq f_\varepsilon(t) + \varepsilon c \sup_{-\tau \leq u \leq t} |Z(u)|_2, \quad t \geq T_1.\]

By (6.9) and (6.10) we have

\[f_\varepsilon(t) \leq L_2(\varepsilon) + \varepsilon c \sup_{-\tau \leq u \leq t} |Y(u)|_2.\]

Since $Z(u) = 0$ for all $u \in [-\tau, 0]$, and $Y(u) = \phi(u)$ for all $u \in [-\tau, 0]$, for $t \geq T_1$ we have

\[|Z(t)|_2 \leq L_2(\varepsilon) + \varepsilon c \sup_{-\tau \leq u \leq 0} |\phi(u)|_2 + \varepsilon \sup_{0 \leq u \leq t} |Y(u)|_2 + \varepsilon \sup_{0 \leq u \leq t} |Z(u)|_2.\]

Define $L_3(\varepsilon) = L_2(\varepsilon) + \varepsilon c \sup_{-\tau \leq u \leq 0} |\phi(u)|_2$. Then

\[|Z(t)|_2 \leq L_3(\varepsilon) + \varepsilon c \sup_{0 \leq u \leq t} |Y(u)|_2 + \varepsilon \sup_{0 \leq u \leq t} |Z(u)|_2, \quad t \geq T_1.\]

Now for $\omega \in \Omega$ define $L_4(\varepsilon, \omega) = L_3(\varepsilon) \vee \max_{0 \leq u \leq T_1} |Z(s, \omega)|_2$. Then for $t \in [0, T_1]$ we have $|Z(t, \omega)|_2 \leq L_4(\varepsilon, \omega)$ and so

\[|Z(t, \omega)|_2 \leq L_4(\varepsilon, \omega) + \varepsilon c \sup_{0 \leq u \leq t} |Y(u, \omega)|_2 + \varepsilon \sup_{0 \leq u \leq t} |Z(u, \omega)|_2, \quad t \geq 0.\]

Define $Z^*(T) = \max_{0 \leq s \leq T} |Z(s)|_2$ for any $T \geq 0$. Therefore

\[Z^*(T, \omega) \leq L_4(\varepsilon, \omega) + \varepsilon c \sup_{0 \leq u \leq T} |Y(u, \omega)|_2 + \varepsilon Z^*(T, \omega), \quad T \geq 0.\]

Since $\varepsilon c < 1/2$ we have

\[(6.11) \quad Z^*(T, \omega) \leq 2L_4(\varepsilon, \omega) + 2\varepsilon c \sup_{0 \leq u \leq T} |Y(u, \omega)|_2, \quad T \geq 0.\]

Define $Y^*(T) = \sup_{0 \leq u \leq T} |Y(u)|_2$ for $T \geq 0$. Next, we have that

\[\limsup_{t \to \infty} \frac{Y^*(t)}{\sqrt{2 \log t}} = \limsup_{t \to \infty} \frac{|Y(t)|_2}{\sqrt{2 \log t}}.\]

We already know from Theorem 2 that there is a $c_0 > 0$ such that

\[\lim sup_{t \to \infty} \frac{|Y(t)|_2}{\sqrt{2 \log t}} = c_0, \quad a.s.\]

so by norm-equivalence there is a deterministic $c_1 > 0$ such that

\[\limsup_{t \to \infty} \frac{|Y(t)|_2}{\sqrt{2 \log t}} \leq c_1, \quad a.s.\]

Hence

\[\limsup_{t \to \infty} \frac{Y^*(t)}{\sqrt{2 \log t}} \leq c_1, \quad a.s.\]
Let $\Omega^*$ be the event for which this holds. Then by (6.11), for each $\omega \in \Omega^*$ we have

$$\limsup_{T \to \infty} \frac{Z^*(T, \omega)}{\sqrt{2 \log T}} \leq 2 \varepsilon \limsup_{T \to \infty} \frac{Y^*(T)}{\sqrt{2 \log T}} \leq 2c_1 \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{T \to \infty} \frac{Z^*(T, \omega)}{\sqrt{2 \log T}} = 0,$$

for each $\omega \in \Omega^*$.

Since $\Omega^*$ is an almost sure event we have

$$\limsup_{T \to \infty} \frac{Z^*(T)}{\sqrt{2 \log T}} = 0,$$

a.s.

That is

$$\lim_{t \to \infty} \frac{|X(t) - Y(t)|}{\sqrt{2 \log t}} = 0, \quad a.s.,$$

and moreover

(6.12) $$\lim_{t \to \infty} \frac{|X_i(t) - Y_i(t)|}{\sqrt{2 \log t}} = 0, \quad a.s.$$

Now, it is known from Theorem 2 that

$$\limsup_{t \to \infty} \frac{|Y_i(t)|}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.$$ 

where $\sigma_i$ is given by (3.2). Thus

$$\limsup_{t \to \infty} \frac{|X_i(t)|}{\sqrt{2 \log t}} \leq \limsup_{t \to \infty} \frac{|Y_i(t)|}{\sqrt{2 \log t}} + \limsup_{t \to \infty} \frac{|X_i(t) - Y_i(t)|}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.$$

Similarly

$$\limsup_{t \to \infty} \frac{|X_i(t)|}{\sqrt{2 \log t}} \geq \limsup_{t \to \infty} \left( \frac{|Y_i(t)|}{\sqrt{2 \log t}} - \frac{|X_i(t) - Y_i(t)|}{\sqrt{2 \log t}} \right) = \limsup_{t \to \infty} \frac{Y_i(t)}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.$$

Combining these inequalities, we get

$$\limsup_{t \to \infty} \frac{|X_i(t)|}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.$$

Write $X_i(t) = X_i(t) - Y_i(t) + Y_i(t)$. By Theorem 2 we have

$$\limsup_{t \to \infty} \frac{Y_i(t)}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.$$

Therefore, taking this in conjunction with (6.12) we get

$$\limsup_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = \limsup_{t \to \infty} \frac{X_i(t) - Y_i(t)}{\sqrt{2 \log t}} + \limsup_{t \to \infty} \frac{Y_i(t)}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.,$$

which is the first part of (3.15).

Similarly, since Theorem 2 implies

$$\limsup_{t \to \infty} \frac{-Y_i(t)}{\sqrt{2 \log t}} = \sigma_i, \quad a.s.$$

by using this in conjunction with (6.12) we have

$$\limsup_{t \to \infty} \frac{-X_i(t)}{\sqrt{2 \log t}} = \limsup_{t \to \infty} \left( \frac{-Y_i(t)}{\sqrt{2 \log t}} + \frac{Y_i(t) - X_i(t)}{\sqrt{2 \log t}} \right) = \sigma_i, \quad a.s.$$

This implies

$$\liminf_{t \to \infty} \frac{X_i(t)}{\sqrt{2 \log t}} = -\sigma_i, \quad a.s.,$$
which is the second part of (3.15). We may proceed as in the proof of Theorem 2 to show that these limits imply
\[
\limsup_{t \to \infty} \frac{|X(t)|}{\sqrt{2 \log t}} = \max_{1 \leq i \leq d} \sigma_i, \text{ a.s.,}
\]
proving the result.

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