LONG-TIME BEHAVIOR OF A CLASS OF NONLOCAL PARTIAL DIFFERENTIAL EQUATIONS

CHANG ZHANG
School of Mathematics and Physics, Jiangsu University of Technology
Changzhou 213001, China

FANG LI
School of Mathematics and Statistics, Xidian University
Xi’an 710126, China

JINQIAO DUAN
Department of Applied Mathematics, Illinois Institute of Technology
Chicago, IL 60616, USA

Abstract. This work is devoted to investigate the well-posedness and long-time behavior of solutions for the following nonlocal nonlinear partial differential equations in a bounded domain

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (-\Delta)^{\sigma/2} u + f(u) &= g, & \text{in } \Omega \times \mathbb{R}^+, \\
&= 0, & \text{on } \mathbb{R}^n \setminus \Omega \times \mathbb{R}^+, \\
&= u_0, & \text{in } \mathbb{R}^n,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) with a sufficiently smooth boundary, \( u_0 \in L^2_0(\Omega) \), \( g(x) \in L^2_0(\Omega) \), and \( 0 < \sigma < 2 \). We assume that the nonlinearity \( f \) satisfies a dissipativity condition

\[
f \in C^1(\mathbb{R}), f'(s) \geq -C_1, f(s) \geq -C_2
\]

1. Introduction. We consider the existence of global attractors for the following nonlocal nonlinear reaction-diffusion equations:

\[
\begin{aligned}
u_t + (-\Delta)^{\sigma/2} u + f(u) &= g(x), & \text{in } \Omega \times \mathbb{R}^+, \\
u(x, t) &= 0, & \text{on } \mathbb{R}^n \setminus \Omega \times \mathbb{R}^+, \\
u(x, 0) &= u_0, & \text{in } \mathbb{R}^n,
\end{aligned}
\]

2010 Mathematics Subject Classification. Primary: 35R11, 35D30, 35K61; Secondary: 35A01, 35B41.

Key words and phrases. Fractional Laplacian, long-time behavior, weak solution, nonlocal partial differential equation, bounded domain.

Zhang was supported by NSFC Grant (11701230).

*Corresponding author.
for some positive constants $C_1, C_2$. Recently, a lot of interest is devoted to the study of the fractional Laplacian (or nonlocal) operator $(-\Delta)^{\sigma/2}$, also known as the Laplacian of order $\frac{\sigma}{2}$. It is defined for every function $g$ in the Schwartz class through the Fourier transform:

$$(-\Delta)^{\sigma/2}g(\xi) = |\xi|^\sigma \hat{g}(\xi).$$

It can also be represented by a singular integral,

$$(-\Delta)^{\sigma/2}g(x) := CP.V. \int_{\mathbb{R}^n} g(x) - g(z) \frac{1}{|x-z|^{n+\sigma}} dz,$$

where $C = \frac{2^{\sigma-1} \Gamma((n+\sigma)/2)}{\pi^{n/2} \Gamma(1-\sigma/2)}$ is a normalization constant (see [21]). Furthermore, the fractional Laplacian can further be defined by a $\sigma-$harmonic extension which was introduced by Caffarelli and Silvestre [6] in the whole space. This extension is commonly used in the recent literature since it allows nonlocal problems to be written in a local way, which enables the use of variational techniques for these kind of problems (see [2, 5, 6, 22, 23, 27, 31]).

The fractional Laplacian operator arises in several areas such as physics, probability and finance (see [1, 4, 9, 7, 14]). In particular, it can be understood as the infinitesimal generator of a stable Lévy process (see [1, 4, 26]).

We will consider partial differential equations (PDEs) with the fractional Laplacian operators. It arises in the Fokker-Planck equations (see [10]) for stochastic differential equations with non-Gaussian $\sigma$-stable Lévy motion $L^\sigma_t$, for $\sigma \in (0, 2)$. This type of diffusion is nowadays intensively studied both from theoretical and experimental point of views, since it conveniently explains a large number of phenomena in physics, finance, biology, ecology, geophysics, and others. The fractional partial differential equations also appear in the modeling of various complex systems, such as heat transfer processes in fractal and disordered media, and fluid flows and acoustic propagation in porous media (see [19, 20, 22, 23, 30]).

Some authors have investigated important properties of fractional PDEs. For example, Ros-Oton and collaborators (see [11, 24]) have studied the global regularity of solutions to the fractional elliptic PDEs and heat equations. Lu et. al. (see [18, 17]) have obtained the existence of a random attractor for fractional Ginzburg-Landau equation with multiplicative noise. Vázquez et. al. (see [22, 23, 28]) have investigated the well-posedness and the asymptotic behavior, speed of propagation and many other properties for the fractional porous medium equations.

The main purpose of the present paper is to study the long-time dynamical behavior of solutions for the fractional reaction-diffusion equations (1). We first prove the well-posedness of weak solutions by a Galerkin method. Because of the absence of an upper growth restriction of $f$, it is impossible to estimate the boundedness of $f(u_m)$ (per Galerkin sequence $u_m$) to determine its weak limit. In order to overcome this difficulty, we apply the weak compactness theorem in an Orlicz space (see [15]) as it is used in [13]. Then, we examine the existence of absorbing sets in $L^2_0(\Omega)$ and $H^{\sigma/2}_0(\Omega)$ for the semigroup $\{S(t)\}_{t \geq 0}$ corresponding to the fractional reaction-diffusion equations (1) and the existence of a global attractor in $L^2_0(\Omega)$. Finally, we will obtain the $(L^2_0(\Omega), H^{\sigma/2}_0(\Omega))$-asymptotical compactness of the solution semigroup. Utilizing the norm-to-weak continuous semigroup method in [32], we prove the existence of a global attractor in $H^{\sigma/2}_0(\Omega))$. Our main results are stated below.
Theorem 1.1. Assume that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $f$ satisfies (2) and $0 < \sigma < 2$. Then for any initial date $u_0 \in L^2_0(\Omega)$ and any $T > 0$, there exists a unique solution and $u(t)$ is continuous on $L^2_0(\Omega)$.

Combining Theorem 1.1 with Lemma 2.6 (next section), we can define a solution semigroup $\{S(t)\}_{t \geq 0}$ in $L^2_0(\Omega)$ as

$$S(\cdot) : L^2_0(\Omega) \times \mathbb{R}^+ \rightarrow L^2_0(\Omega),$$

which is $(L^2_0(\Omega), L^2_0(\Omega))$-continuous and $(L^2_0(\Omega), (H^{\sigma/2}_0(\Omega))_w)$-continuous.

Theorem 1.2. The semigroup $\{S(t)\}_{t \geq 0}$ generated by (1) has a $(L^2_0(\Omega), L^2_0(\Omega))$-global attractor $A_1$.

Theorem 1.3. The semigroup $\{S(t)\}_{t \geq 0}$ generated by (1) has a $(L^2_0(\Omega), H^{\sigma/2}_0(\Omega))$-global attractor $A_2$.

Notations in these theorems will be explained in the next section.

This paper is organized as follows. In the next section, we present some definitions and lemmas used in the sequel. In Section 3, we will prove the well-posedness of weak solutions to the fractional reaction-diffusion equations (1). Finally, in Section 4 we prove the existence of global attractors in $L^2_0(\Omega)$ and in $H^{\sigma/2}_0(\Omega)$ for the nonlocal system (1).

2. Preliminaries. In this section, we introduce several function spaces and recall basic concepts about the global attractors. See [3, 29] for more details. We recall

$$L^p_0(\Omega) := \{u \in L^p(\mathbb{R}^n) : u = 0 \ \text{a.e. on} \ \mathbb{R}^n \setminus \Omega\}, \quad 1 \leq p \leq \infty,$$

and

$$H^{\sigma/2}(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\sigma}} \, dx \, dy < \infty \right\},$$

endowed with the natural norm

$$\|u\|_{H^{\sigma/2}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |u|^2 \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\sigma}} \, dx \, dy \right)^{\frac{1}{2}},$$

where the term

$$[u]_{H^{\sigma/2}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\sigma}} \, dx \, dy \right)^{\frac{1}{2}}$$

is the so-called Gagliardo (semi)norm of $u$, and q.e. is the abbreviation for quasi-everywhere with respect to Riesz capacity (see [8, 12, 26]). We also introduce the space

$$H^{\sigma/2}_0(\Omega) := \left\{ u \in H^{\sigma/2}(\mathbb{R}^n) : u = 0 \ \text{q.e. on} \ \mathbb{R}^n \setminus \Omega \right\},$$

equipped with the norm inherited from $H^{\sigma/2}(\mathbb{R}^n)$.

Denote by $H^{-\sigma/2}(\Omega)$ the dual space of $H^{\sigma/2}_0(\Omega)$ and let the space

$$W = H^{\sigma/2}_0(\Omega) \cap L^\infty(\Omega)$$

be endowed with the norm

$$\|u\|_W := \|u\|_{H^{\sigma/2}(\mathbb{R}^n)} + \|u\|_{L^\infty(\Omega)}. $$
Moreover, we define a linear operator $A : H^{\sigma/2}_0(\Omega) \to H^{-\sigma/2}(\Omega)$ by

$$
(Au, v) = \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4}u(-\Delta)^{\sigma/4}v\, dx, \quad \forall \ v \in H^{\sigma/2}_0(\Omega).
$$

**Remark 1.** If $u$ belong to $H^{\sigma/2}_0(\Omega)$, then $u = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$.

**Remark 2.** (See [8]) The embedding $H^{\sigma/2}_0(\Omega) \subset W^{\sigma/2,2}_0(\Omega)$ is continuous. Particularly, $H^{\sigma/2}_0(\Omega) = W^{\sigma/2,2}_0(\Omega)$ for every $\sigma \in (0, 1) \cup (1, 2)$, where $W^{\sigma/2,2}_0(\Omega)$ is the classical fractional Sobolev spaces.

**Remark 3.** (See [21]) For every $\sigma \in (0, 2)$ and $u \in H^{\sigma/2}(\mathbb{R}^n)$, then $|u|_{H^{\sigma/2}(\mathbb{R}^n)}^2 = C\|(-\Delta)^{\sigma/4}u\|_{L^2(\mathbb{R}^n)}^2$.

**Remark 4.** (See [8]) The space $H^{\sigma/2}_0(\Omega)$ can also be characterized as the $[\cdot]_{H^{\sigma/2}(\mathbb{R}^n)}$-closure of $C_c^\infty(\Omega)$.

**Definition 2.1.** For every $T > 0$, the function $u \in C([0, T]; L^2_0(\Omega)) \cap L^2(0, T; H^{\sigma/2}_0(\Omega)) \cap W^{1,2}_{loc}(0, T; L^2(\Omega))$ is called weak solutions of Problem (1), if it satisfies

$$
\int_0^T \int_\Omega f(u(x, t))u(x, t) < \infty, \quad u(x, 0) = u_0(x)
$$

and the equations

$$
\int_\Omega u_t \phi dx + \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4}u(-\Delta)^{\sigma/4}\phi dx + \int_\Omega f(u)\phi dx = \int_\Omega g\phi dx \text{ a.e. on } (0, T),
$$

for every $\phi \in W$. For any $Y$ and $X$, let $\{S(t)\}_{t \geq 0}$ be a semigroup from $Y$ to $X$, that is, $S(t)Y \subset X$ for every $t > 0$ and

1. $S(0) = \text{Id}$;
2. $S(t)S(s) = S(t+s)$.

We say that $\{S(t)\}_{t \geq 0}$ is a norm-to weak continuous semigroup on $(Y, X)$, if $\{S(t)\}_{t \geq 0}$ satisfies that (1), (2) and

3. $S(t_n)y_n \to S(t)y$ in $X$, if $t_n \to t$ and $y_n \to y$ in $Y$.

We denote the condition (3) as $(Y, X_w)$-continuous.

**Definition 2.3.** Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach spaces $Y$ and $X$ be a metric space. A set $A \subset X \cap Y$, which is invariant, closed in $Y$, compact in $X$ and attracts bounded subsets of $Y$ in the topology of $X$ is called a $(Y, X)$-global attractor.

**Definition 2.4.** Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach space $Y$. A bounded subset $B_0$ of $X$ is called a $(Y, X)$-bounded absorbing set, if for any bounded subset $B \subset Y$, there is $T = T(B)$, such that $S(t)B \subset B_0$ for any $t \geq T$.

**Definition 2.5.** Let $\{S(t)\}_{t \geq 0}$ be a semigroup on Banach space $Y$. $\{S(t)\}_{t \geq 0}$ is called $(Y, X)$-asymptotically compact, if for any bounded $(in Y)$ sequence $\{y_n\}_{n=1}^\infty \subset Y$ and $t_n \geq 0, t_n \to \infty$ as $n \to \infty$, $\{S(t_n)y_n\}_{n=1}^\infty$ has a convergent subsequence with respect to the topology of $X$. 
Let $X,Y$ be two Banach spaces and $X^*,Y^*$ be their dual spaces, respectively. Assume that $X$ is a dense subspace of $Y$, the injection $i : X \hookrightarrow Y$ is continuous and its adjoint $i^* : Y^* \hookrightarrow X^*$ is densely injective. Under these assumptions, the following results hold.

**Lemma 2.6.** (See [32]) Let $X,Y$ be two Banach spaces satisfy the assumptions just above, $\{S(t)\}_{t\geq 0}$ be a semigroup on $X$ and $Y$, respectively, and assume furthermore that $\{S(t)\}_{t\geq 0}$ is continuous or weak continuous on $Y$. Then $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous semigroup on $X$ if and only if $\{S(t)\}_{t\geq 0}$ maps compact subsets of $X \times \mathbb{R}^+$ into bounded sets of $X$.

**Lemma 2.7.** (See [32]) Let $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous semigroup on $(Y,X)$. Then $\{S(t)\}_{t\geq 0}$ has a $\{Y,X\}$-global attractor, if and only if

1. $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $X$,
2. $\{S(t)\}_{t\geq 0}$ is $(Y,X)$-asymptotically compact.

**Lemma 2.8.** For every $\sigma \in (0,2)$, there exists a positive constant $\lambda_1$ such that for every $u \in H_0^{\sigma/2}(\Omega)$, we have

$$
\lambda_1 \|u\|_{L_0^2(\Omega)}^2 \leq \|(-\Delta)^{\sigma/4}u\|_{L_2(\mathbb{R}^n)}^2.
$$

(3)

**Remark 5.** We can verify Lemma 2.8 by Theorem 6.5 in [21] for $n \geq 2$.

3. **Well-posedness.** We start with the discussion of existence and uniqueness of solutions by a Galerkin method. Define

$$
\tilde{e}_i(x) = \begin{cases} 
  e_i(x), & x \in \Omega, \\
  0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
$$

where $e_i(x)$ is eigenfunctions of $-\Delta$ with Dirichlet boundary, and $\{e_i\}_{i=0}^{\infty}$ denote an orthonormal basis of $L^2(\Omega)$. Hence, it is easy to check that $\{\tilde{e}_i\}_{i=0}^{\infty}$ is an orthonormal basis of $L_0^2(\Omega)$, and $\tilde{e}_i \in H_0^{\sigma/2}(\Omega)$.

We will take $f(0) = 0$ to simplify the argument in the rest of the paper, but this is not, in fact, necessary.

**Proof of Theorem 1.1.** Let us consider the approximate solutions $\{u_m(t)\}_{m=1}^{\infty}$ in the form

$$
u_m(t) = \sum_{k=1}^{m} u_{mk}(t) \tilde{e}_k,
$$

where $\{u_{mk}(t)\}_{k=1}^{m}$ are the solutions of the following problem:

$$
\begin{align*}
&\left( \sum_{k=1}^{m} u_{mk}' \tilde{e}_k, \tilde{e}_j \right) + \left( (-\Delta)^{\sigma/4} \sum_{k=1}^{m} u_{mk} \tilde{e}_k, (-\Delta)^{\sigma/4} \tilde{e}_j \right) + \left( f(\sum_{k=1}^{m} u_{mk} \tilde{e}_k), \tilde{e}_j \right) \\
= & \left( g(x), \tilde{e}_j \right), \quad j = 1, \ldots, m, \\
\sum_{k=1}^{m} u_{mk}(0) \tilde{e}_k \to u_0 & \text{ strongly in } L_0^2(\Omega) \text{ as } m \to \infty.
\end{align*}
$$

(4)

Since the nonlinearity in (4) is locally Lipschitz, there exists a unique solution to (4). Multiplying the equations (4) by the function $u_{mj}(t)$ for each $j$ and adding
these equalities for $j = 1, \ldots, m$, we conclude that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_{m}^{2}(x, t) dx + \int_{\mathbb{R}^{n}} [(-\Delta)^{\sigma/4} u_{m}(x, t)]^{2} dx + \int_{\Omega} f(u_{m}(x, t)) u_{m}(x, t) dx = \int_{\Omega} g(x) u_{m}(x, t) dx.
\] (5)

Now we can use lemma 2.8 and Cauchy inequality to get
\[
\frac{d}{dt} \int_{\Omega} u_{m}^{2}(t) dx + \lambda_{1} \int_{\mathbb{R}^{n}} u_{m}^{2} dx \leq \frac{1}{\lambda_{1}} \|g(x)\|_{L_{x}^{2}(\Omega)}^{2} + 2C_{2} |\Omega|.
\] (6)

By Gronwall inequality, we obtain that
\[
\|u_{m}(t)\|_{L_{x}^{2}(\Omega)}^{2} \leq\frac{\lambda_{1}}{\lambda_{1}} [\|g(x)\|_{L_{x}^{2}(\Omega)}^{2} + 2C_{2} |\Omega|] + \|u_{m}(0)\|_{L_{x}^{2}(\Omega)}^{2} e^{-\lambda_{1} t}.
\] (7)

Therefore
\[
\|u_{m}\|_{L^{\infty}(0, T; L_{x}^{2}(\Omega))} \leq C(\|u_{0}\|_{L_{x}^{2}(\Omega)}, \|g\|_{L_{x}^{2}(\Omega)}).
\] (8)

Define functions $f(s) = f(s) + C_{1} s$. Because of (5) and (8), we infer that
\[
\|u_{m}\|_{L^{\infty}(0, T; L_{x}^{2}(\Omega))} + \|u_{m}\|_{L^{2}(0, T; H_{0}^{\sigma/2}(\Omega))} + \int_{0}^{T} \int_{\Omega} \tilde{f}(u_{m}) u_{m} dx dt \leq C(\|u_{0}\|_{L_{x}^{2}(\Omega)}, \|g\|_{L_{x}^{2}(\Omega)}).
\] (9)

Now, multiplying equation (4), by the function $u'_{mj}(t)$, for each $j$, adding these relations for $j = 1, \ldots, m$, we deduce that
\[
\|u_{mt}(t)\|^{2} + \frac{1}{2} \frac{d}{dt} \|u_{m}(t)\|_{H_{0}^{\sigma/2}(\Omega)}^{2} + \int_{\Omega} F(u_{m}(x, t)) dx - \int_{\Omega} g(x) u_{m}(x, t) dx = 0,
\] (10)

where, $F(u) = \int_{0}^{u} f(s) ds$. Integrating the last inequality over $(s, T)$ with respect to the variable $t$, we conclude that
\[
\frac{1}{2} \|u_{m}(T)\|_{H_{0}^{\sigma/2}(\Omega)}^{2} + \int_{\Omega} F(u_{m}(x, T)) dx + \int_{s}^{T} \|u_{mt}(t)\|_{L_{x}^{2}(\Omega)}^{2} dt
\]
\[
= \frac{1}{2} \|u_{m}(s)\|_{H_{0}^{\sigma/2}(\Omega)}^{2} + \int_{\Omega} F(u_{m}(x, s)) dx + \int_{\Omega} g(x) (u_{m}(x, T) - u_{m}(x, s)) dx
\]
\[
\leq \frac{1}{2} \|u_{m}(s)\|_{H_{0}^{\sigma/2}(\Omega)}^{2} + \int_{\Omega} F(u_{m}(x, s)) dx + \|g\|_{L_{x}^{2}(\Omega)} (\|u_{m}(T)\|_{L_{x}^{2}(\Omega)} + \|u_{m}(s)\|_{L_{x}^{2}(\Omega)}).
\] (11)

Again integrating the last inequality over $(0, T)$ with respect to the variable $s$, we obtain that
\[
\frac{T}{2} \|u_{m}(T)\|_{H_{0}^{\sigma/2}(\Omega)}^{2} + T \int_{\Omega} F(u_{m}(x, T)) dx + \int_{0}^{T} \int_{s}^{T} \|u_{mt}(t)\|_{L_{x}^{2}(\Omega)}^{2} dt ds
\]
\[
\leq \frac{1}{2} \int_{0}^{T} \|u_{m}(s)\|_{H_{0}^{\sigma/2}(\Omega)}^{2} ds + \int_{0}^{T} \int_{\Omega} F(u_{m}(x, s)) dx ds + \|g\|_{L_{x}^{2}(\Omega)} (T \|u_{m}(T)\|_{L_{x}^{2}(\Omega)}
\]
\[
+ \int_{0}^{T} \|u_{m}(s)\|_{L_{x}^{2}(\Omega)} ds).
\]
Noting that
\[- \frac{C_1}{2} s^2 \leq F(s) = \int_0^s f(\tau) d\tau = \int_0^s \hat{f}(\tau) - C_1 \tau d\tau \leq \hat{f}(s) s - \frac{C_1}{2} s^2 \leq \hat{f}(s)s,\]
taking into account (8) and (9), we conclude that
\[
\int_0^T t \|u_m(t)\|_{L_0(\Omega)}^2 dt + T \|u_m(T)\|_{H_0^{\sigma/2}(\Omega)}^2 + T \int_\Omega F(u_m(x,T)) dx 
\leq C(\|u_0\|_{L_0(\Omega)},\|g\|_{L_0(\Omega)},T).
\]
Hence, for every \( \varepsilon \in (0, T) \), we deduce that
\[
\|u_m\|_{L_0(\Omega)} \leq C_\varepsilon, \quad (12)
\]
\[
\|u_m\|_{L_0^{(\varepsilon,T,H_0^{\sigma/2}(\Omega))}} \leq C_\varepsilon. \quad (13)
\]
Thanks to Remark 2 and Aubin-Lions Lemma [25], there exists a subsequence, which we still denote by \( u_m \), such that
\[
u_m \to u \text{ in } C([\varepsilon,T];L_0^2(\Omega)). \quad (14)
\]
Therefore, applying the diagonalization procedure, we infer that
\[
u_m \to u \text{ a.e. in } \Omega \times (0,T) \quad (15)
\]
as \( m \to \infty \).

Hence, we obtain that
\[
\begin{cases}
u_m \to u & \text{in } L_0^\infty(0,T;L_0^2(\Omega)), \\
u_m t \to u & \text{in } L_0^2(\varepsilon,T;L_0^2(\Omega)), \quad \forall \varepsilon \in (0,T), \\
u_m \to u & \text{in } L_0^2(0,T;H_0^{\sigma/2}(\Omega)), \\
u_m \to Au & \text{in } L_0^2(0,T;H^{-\sigma/2}(\Omega)).
\end{cases} \quad (16)
\]

We now show that \( \hat{f}(u_m) \in L_0^1(0,T;L_0^1(\Omega)) \). It is easy to check that \( \int_0^T \int_\Omega \hat{f}(u_m) u_m dx dt \leq C \), where \( C \sim m \). Let \( \chi_{\Omega_1} \) and \( \chi_{\Omega_2} \) be the characteristic functions of the sets \( \Omega_1 = \{(x,t) \in \Omega \times (0,T) : |u_m| > 1\} \), \( \Omega_2 = \{(x,t) \in \Omega \times (0,T) : |u_m| \leq 1\} \).

Since
\[
\int_0^T \int_\Omega |\hat{f}(u_m) \chi_{\Omega_1}| dx dt = \int_{\Omega_1} |\hat{f}(u_m)| dx dt \leq \int_{\Omega_1} \hat{f}(u_m) u_m dx dt 
\leq \int_0^T \int_\Omega \hat{f}(u_m) u_m dx dt \leq C,
\]
we deduce that \( \hat{f}(u_m) \in L_0^1(0,T;L_0^1(\Omega)) \). Furthermore, since
\[
\int_0^T \int_{\Omega_2} |\hat{f}(u_m) \chi_{\Omega_2}|^2 dx dt = \int_{\Omega_2} |\hat{f}(u_m)|^2 dx dt = \int_{\Omega_2} |\hat{f}(u_m) - \hat{f}(0)|^2 dx dt
\]
by the mean value theorem we infer that
\[
\int_0^T \int_{\Omega_2} |\hat{f}(u_m) \chi_{\Omega_2}|^2 dx dt \leq C \int_{\Omega_2} |u_m|^2 dx dt \leq C \int_0^T \int_{\Omega_2} |u_m|^2 dx dt \leq C.
\]
Thus we get \( \hat{f}(u_m) \chi_{\Omega_2} \in L_0^2(0,T;L_0^2(\Omega)) \). Hence, \( \hat{f}(u_m) \in L_0^1(0,T;L_0^1(\Omega)) + L_0^2(0,T;L_0^2(\Omega)) \subseteq L_0^1(0,T;L_0^2(\Omega)) \).

We need to show that \( \int_0^T \int_\Omega \hat{f}(u_m) \varphi dx dt \to \int_0^T \int_\Omega f(u) \varphi dx dt \), for test function \( \varphi \in C_0(\Omega \times (0,T)) \), by the methods in Orlicz space [13, 15]. Since the function
\(\hat{f}(s)\) is continuous, increasing, positive for \(s > 0\) and \(\hat{f}(0) = 0\). It is easy to check that
\[
\int_0^T \int_\Omega \hat{f}((u^+_m(x, \tau)))(u^+_m(x, \tau))dx\,d\tau \leq C,
\]
\[
\int_0^T \int_\Omega \hat{f}((u^-_m(x, \tau)))(u^-_m(x, \tau))dx\,d\tau \leq C,
\]
where \(u^+_m = \max\{u_m, 0\}\) and \(u^-_m = \min\{u_m, 0\}\). Define an N-function
\[
\hat{F}(x) = \int_0^{|x|} \hat{f}(s)ds,
\]
which has a complementary N-function \(\hat{G}\) as follows,
\[
\hat{G}(y) = \int_0^{|y|} \hat{f}^{-1}(\tau)d\tau.
\]
By the definition and (17), we obtain that
\[
\int_0^T \int_\Omega \hat{G}((\hat{f}(u^+_m(x, \tau))))dx\,d\tau \leq \int_0^T \int_\Omega \hat{f}((u^+_m(x, \tau)))(u^+_m(x, \tau))dx\,d\tau \leq C.
\]
Hence
\[
\|\hat{f}(u^+_m)\|_{L^\infty_{\hat{G}}(\Omega \times [0,T])} \leq C,
\]
where \(L^\infty_{\hat{G}}(\Omega \times [0,T])\) is the Orlicz space ([15]). Taking into account \(u_m \to u\) for almost every \((x, t) \in \Omega \times [0,T]\), continuity of \(\hat{f}(\cdot)\) and the functions \(\max\{s, 0\}\) and \(\min\{s, 0\}\), it can be inferred that \(\hat{f}(u^+_m) \to \hat{f}(u^+)\) in measure on \(\Omega \times [0,T]\). Hence, we deduce that
\[
\int_0^T \int_\Omega \hat{f}(u^+_m(x, t))v(x, t)dx\,dt \to \int_0^T \int_\Omega \hat{f}(u^+(x, t))v(x, t)dx\,dt, \quad \forall v \in E_{\hat{F}}.
\]
where \(E_{\hat{F}}\) is the closures of the set of bounded functions in the spaces \(L^\infty_{\hat{F}}(\Omega \times [0,T])\). We obtain that
\[
\int_0^T \int_\Omega \hat{f}(u^+_m(x, t))w(x, t)dx\,dt \to \int_0^T \int_\Omega \hat{f}(u^+(x, t))w(x, t)dx\,dt, \quad \text{for every } w \in L^\infty(\Omega \times (0, T)).
\]
Similarly, we can infer that
\[
\int_0^T \int_\Omega \hat{f}(u^-_m(x, t))w(x, t)dx\,dt \to \int_0^T \int_\Omega \hat{f}(u^-(x, t))w(x, t)dx\,dt, \quad \text{for every } w \in L^\infty(\Omega \times (0, T)).
\]
Since
\[
\hat{f}(u_m(x, t)) = \hat{f}(u^+_m(x, t)) + \hat{f}(u^-_m(x, t)),
\]
we conclude that
\[
\int_0^T \int_\Omega \hat{f}(u_m(x, t))w(x, t)dx\,dt \to \int_0^T \int_\Omega \hat{h}(u(x, t))w(x, t)dx\,dt \quad \text{for every } w \in L^\infty(\Omega \times (0, T)).
\]
Because \(u_m \to u\) for almost every \((x, t) \in \Omega \times [0,T]\), we obtain that \(\int_0^T \int_\Omega u_mwdx\,dt \to \int_0^T \int_\Omega uwdx\,dt\).
Therefore,
\[ \int_0^T \int_{\Omega} f(u_m) \varphi dx \, dt \rightarrow \int_0^T \int_{\Omega} f(u) \varphi dx \, dt \]  
(18)
for test functions \( \varphi \in C^\infty_c(\mathbb{R}^n) \), \( \varphi = 0 \) a.e. in \( \mathbb{R}^n \setminus \Omega \).

As a result, we have that
\[ \int_{\Omega} u_t \varphi dx + \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u(-\Delta)^{\sigma/4} \varphi dx + \int_{\Omega} f(u) \varphi = \int_{\Omega} g \varphi dx \quad \text{in} \quad \mathcal{D}'(0,T). \]  
(19)

Moreover, by (9) and Fatou Lemma, we deduce that
\[
\begin{align*}
\int_0^T \int_{\Omega} |f(u(x,t))u(x,t)| \, dx \, dt &= \int_0^T \int_{\Omega} |\tilde{f}(u(x,t))u(x,t) - C_1|u(x,t)|^2| \, dx \, dt \\
&\leq \int_0^T \int_{\Omega} \tilde{f}(u(x,t))u(x,t) + C_1|u(x,t)|^2 \, dx \, dt \\
&\leq \liminf_{m \to \infty} \int_0^T \int_{\Omega} \tilde{f}(u_m(x,t))u_m(x,t) + C_1|u_m(x,t)|^2 \, dx \, dt \\
&\leq C.
\end{align*}
\]

The last inequality tells us that
\[ f(u) \in L^1(0,T; L^1_0(\Omega)). \]

The equation (19) is satisfied a.e. in \((0,T)\) and by the dense property we conclude that
\[ \int_{\Omega} u_t \varphi dx + \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u(-\Delta)^{\sigma/4} \varphi dx + \int_{\Omega} f(u) \varphi = \int_{\Omega} g \varphi dx \quad \text{a.e. on} \quad (0,T), \]
where \( \varphi \in W \).

To prove the continuity of the solution \( u(t) \) from \([0,T]\) to \( L^2_0(\Omega) \). From the last equality it follows that \( u_t \in L^1(0,T; L^1_0(\Omega) + H^{-\sigma/2}(\Omega)) \). Combining with \( u \in L^\infty(0,T; L^2_0(\Omega)) \to L^1(0,T; L^1_0(\Omega) + H^{-\sigma/2}(\Omega)) \), we obtain that \( u \in C([0,T]; L^1_0(\Omega) + H^{-\sigma/2}(\Omega)) \) and consequently \( u \in C_s([0,T]; L^1_0(\Omega) + H^{-\sigma/2}(\Omega)) \). By the Lemma 8.1 in [16],
\[ L^\infty(0,T; L^2_0(\Omega)) \cap C_s(0,T; L^1_0(\Omega) + H^{-\sigma/2}(\Omega)) = C_s([0,T]; L^2_0(\Omega)). \]

Therefore
\[ u \in C_s([0,T]; L^2_0(\Omega)). \]  
(20)

Multiplying both sides of (4) by test function \( \eta \in C^1[0,T], \eta(T) = 0 \), we infer that
\[
\begin{align*}
&- \int_{\Omega} u_m(0) \phi(x) dx \eta(0) - \int_0^T \int_{\Omega} u_m(t) \phi(x) \eta'(t) dx dt \\
&+ \int_0^T \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u_m(-\Delta)^{\sigma/4} \phi(x) \eta(t) dx dt + \int_0^T \int_{\Omega} f(u_m) \phi(x) \eta(t) dx dt \\
&= \int_0^T \int_{\Omega} g(x) \phi(x) \eta(t) dx dt,
\end{align*}
\]
and, for every $\varepsilon > 0$,
\[
- \int_{\Omega} u_m(\varepsilon) \phi(x) dx \eta(\varepsilon) - \int_{\varepsilon}^{T} \int_{\Omega} u_m(t) \phi(x) \eta'(t) dx dt \\
+ \int_{\varepsilon}^{T} \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u_m(-\Delta)^{\sigma/4} \phi(x) \eta(t) dx dt + \int_{\varepsilon}^{T} \int_{\Omega} f(u_m) \phi(x) \eta(t) dx dt \\
= \int_{\varepsilon}^{T} g(x) \phi(x) \eta(t) dx dt.
\]
Taking into account (16), (18) and passing to the limit in the last two equations when $m \to \infty$, we obtain that
\[
- \int_{\Omega} u_0 \phi(x) dx \eta(0) - \int_{0}^{T} \int_{\Omega} u(t) \phi(x) \eta'(t) dx dt \\
+ \int_{0}^{T} \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u(-\Delta)^{\sigma/4} \phi(x) \eta(t) dx dt + \int_{0}^{T} \int_{\Omega} f(u) \phi(x) \eta(t) dx dt \\
= \int_{0}^{T} \int_{\Omega} g(x) \phi(x) \eta(t) dx dt,
\]
and
\[
- \int_{\Omega} u(\varepsilon) \phi(x) dx \eta(\varepsilon) - \int_{\varepsilon}^{T} \int_{\Omega} u(t) \phi(x) \eta'(t) dx dt \\
+ \int_{\varepsilon}^{T} \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u(-\Delta)^{\sigma/4} \phi(x) \eta(t) dx dt + \int_{\varepsilon}^{T} \int_{\Omega} f(u) \phi(x) \eta(t) dx dt \\
= \int_{\varepsilon}^{T} g(x) \phi(x) \eta(t) dx dt.
\]
Passing to the limit in (22) when $\varepsilon \to 0^+$ and taking into account (20), we infer that
\[
- \int_{\Omega} u(0) \phi(x) dx \eta(0) - \int_{0}^{T} \int_{\Omega} u(t) \phi(x) \eta'(t) dx dt \\
+ \int_{0}^{T} \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} u(-\Delta)^{\sigma/4} \phi(x) \eta(t) dx dt + \int_{0}^{T} \int_{\Omega} f(u) \phi(x) \eta(t) dx dt \\
= \int_{0}^{T} \int_{\Omega} g(x) \phi(x) \eta(t) dx dt.
\]
Combining with (21) and (23), we deduce that
\[
u(0) = u_0.
\]
Integrating (5) over $[0, T]$ with respect to the variable $t$, we have
\[
\|u_m(t)\|^2_{L^2_\delta(\Omega)} \leq \int_{0}^{t} \int_{\Omega} g(x) u_m(x, t) dx dt + \|u_m(0)\|^2_{L^2_\delta(\Omega)} + 2tC_2|\Omega|.
\]
Passing to the limit in the last inequality when $m \to \infty$, we obtain that
\[
\|u(t)\|^2_{L^2_\delta(\Omega)} \leq \int_{0}^{t} \int_{\Omega} g(x) u(x, t) dx dt + \|u(0)\|^2_{L^2_\delta(\Omega)} + 2tC_2|\Omega|.
\]
Therefore,
\[
\limsup_{t \to 0} \|u(t)\|^2_{L^2_\delta(\Omega)} \leq \|u(0)\|^2_{L^2_\delta(\Omega)}.
\]
Combining with (20), we conclude that
\[ u \in C([0, T]; L^2_0(\Omega)). \]

It remains to show that uniqueness and continuous dependence. Let \( u_0 \) and \( v_0 \) be in \( L^2_0(\Omega) \) and consider \( w(t) = u(t) - v(t) \). Then
\[
\begin{cases}
  w_t + Aw - C_1 + \hat{f}(v + w) - \hat{f}(v) = 0, \\
  w(0) = u_0 - v_0.
\end{cases}
\]

Define a truncated function
\[
\psi_k(s) = \begin{cases}
  k, & s > k, \\
  s, & -k \leq s \leq k, \\
  -k, & s < -k.
\end{cases}
\]

It is easy to check that \( \psi_k(w) \in H^{\sigma/2}_0(\Omega) \cap L^\infty_0(\Omega) \). Multiplying the first equation in (24) by \( \psi_k(w) \) integrating over \( \Omega \times (0, T) \), we obtain that
\[
\int_{\Omega} w(x, T) \psi_k(w(x, T)) dx - \frac{1}{2} \left\| \psi_k(w(T)) \right\|^2_{L^2_0(\Omega)} + \int_{\Omega} w(x, \varepsilon) \psi_k(w(x, \varepsilon)) dx - \frac{1}{2} \left\| \psi_k(w(\varepsilon)) \right\|^2_{L^2_0(\Omega)} + C_1 \int_{\varepsilon}^T \int_{\Omega} w(x, t) \psi_k(w(x, t)) dx dt.
\]

According to the definition of \( \psi_k(\cdot) \) and \( \sigma \)-harmonic extension (see [6, 23]), we deduce that
\[
\int_{\varepsilon}^T \int_{\mathbb{R}^n} (-\Delta)^{\sigma/4} w(x, t)(-\Delta)^{\sigma/4} \psi_k(w(x, t)) dx dt
\geq 0.
\]
The last equation follows from Lemma 3.1 in [23].

Hence, we infer that
\[
\int_{\Omega} w(x, T) \psi_k(w(x, T)) dx - \frac{1}{2} \left\| \psi_k(w(T)) \right\|^2_{L^2_0(\Omega)} + C_1 \int_{\varepsilon}^T \int_{\Omega} w(x, t) \psi_k(w(x, t)) dx dt.
\]

Passing to the limit in the last inequality when \( k \to \infty \) and \( \varepsilon \to 0 \), we obtain that
\[
\left\| w(T) \right\|^2_{L^2_0(\Omega)} \leq \left\| u_0 - v_0 \right\|^2_{L^2_0(\Omega)} + 2C_1 \int_{0}^{T} \left\| w(t) \right\|^2_{L^2_0(\Omega)} dt.
\]
The Gronwall Lemma gives
\[ \|w(T)\|_{L^2(\Omega)}^2 \leq \|u_0 - v_0\|_{L^2(\Omega)}^2 \exp(2C_1 T). \]
This completes the proof. \( \square \)

4. **Global attractors.** We begin with the existence of the absorbing set for the solution semigroup \( \{S(t)\}_{t \geq 0} \).

**Lemma 4.1.** \( \{S(t)\}_{t \geq 0} \) has a \((L^2_0(\Omega), L^2_0(\Omega))\)-bounded absorbing set, that is, there is a positive constant \( \rho_H \), so that for every bounded subset \( B \) in \( L^2_0(\Omega) \), there exists a positive constant \( T \) which depends only on the \( L^2 \)-norm of \( B \) such that
\[ \|S(t)u_0\|_{L^2_0(\Omega)}^2 \leq \rho_H \text{ for every } t \geq T \text{ and } u_0 \in B. \]

**Proof.** By the inequality (7), we conclude that
\[
\|u(t)\|_{L^2_0(\Omega)}^2 \leq \liminf_{m \to \infty} \|u_m(t)\|_{L^2_0(\Omega)}^2 \\
\leq \exp(-\lambda_1 t) \liminf_{m \to \infty} \|u_m(0)\|_{L^2_0(\Omega)}^2 + \frac{1}{\lambda_1} \left( \frac{1}{\lambda_1} \|g\|_{L^2_0(\Omega)}^2 + 2C_2|\Omega| \right) \\
= \exp(-\lambda_1 t) \|u_0\|_{L^2_0(\Omega)}^2 + \frac{1}{\lambda_1} \left( \frac{1}{\lambda_1} \|g\|_{L^2_0(\Omega)}^2 + 2C_2|\Omega| \right),
\]
Hence,
\[ B_0 = \left\{ u \in L^2_0(\Omega) \leq \rho_H = \frac{1}{\lambda_1} \left( \frac{1}{\lambda_1} \|g\|_{L^2_0(\Omega)}^2 + 2C_2|\Omega| \right) + 1 \right\} \]
is an \( L^2_0(\Omega) \)-bounded absorbing set. \( \square \)

We now prove the existence of \((L^2_0(\Omega), H^{\sigma/2}_0(\Omega))\)-bounded absorbing set.

**Lemma 4.2.** \( \{S(t)\}_{t \geq 0} \) has a \((L^2_0(\Omega), H^{\sigma/2}_0(\Omega))\)-bounded absorbing set, that is, there is a positive constant \( C \), such that for any bounded subset \( B \) in \( L^2_0(\Omega) \), there exists a positive constant \( T \) which depends only on the \( L^2_0 \)-norm of \( B \) such that
\[ \|u\|_{H^{\sigma/2}_0(\Omega)} < C \text{ for every } t > T \text{ and } u_0 \in B. \]

**Proof.** By Poincaré inequality, Hölder inequality and the equality (5), we deduce that
\[
\frac{d}{dt} \|u_m(t)\|_{L^2_0(\Omega)}^2 + \alpha \|u_m(t)\|_{L^2_0(\Omega)}^2 + \alpha \|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 \\
+ 2 \int_{\Omega} f(u_m(x,t))u_m(x,t)dx \leq C,
\]
for some \( \alpha > 0 \). Applying the Gronwall inequality, we obtain that
\[ \|u_m(t)\|^2 + \int_s^t e^{\alpha(t-t)} \left( \alpha \|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} f(u_m(x,\tau))u_m(x,\tau)dx \right) d\tau \leq \|u_m(s)\|^2 e^{\alpha(s-t)} + C. \]
By Lemma 4.1, there exists a positive constant \( T = T(B) \), such that for every \( t \geq T \),
\[ \|u_m(t)\|^2 \leq C, \]
and
\[ \int_{t-1}^t \left( \alpha \|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} f(u_m(x,\tau))u_m(x,\tau)dx \right) d\tau \leq C. \]
Integrating the equation (10) over \((s, t)\) with respect to the variable \(t\) leads to
\[
\frac{1}{2}\|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} F(u_m(x, t))dx - \int_{\Omega} g(x)u_m(x, t)dx
\leq \frac{1}{2}\|(-\Delta)^{\sigma/4} u_m(s)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} F(u_m(x, s))dx - \int_{\Omega} g(x)u_m(x, s)dx.
\]
Integrating the last inequality over \((t-1, t)\) with respect to the variable \(s\), we obtain that
\[
\frac{1}{2}\|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} F(u_m(x, t))dx - \int_{\Omega} g(x)u_m(x, t)dx
\leq \int_{t-1}^t \left(\frac{1}{2}\|(-\Delta)^{\sigma/4} u_m(s)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} F(u_m(x, s))dx - \int_{\Omega} g(x)u_m(x, s)dx\right)ds.
\]
Hence, for every \(t \geq T\)
\[
\frac{1}{2}\|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \int_{\Omega} F(u_m(x, t))dx - \int_{\Omega} g(x)u_m(x, t)dx \leq C.
\]
Therefore
\[
\|u_m(t)\|_{H^\sigma/2(\Omega)}^2 = \|(-\Delta)^{\sigma/4} u_m(t)\|_{L^2(\mathbb{R}^n)}^2 \leq C.
\]
Taking into account \(u_m \rightharpoonup u \in L^\infty(T, \infty; H^{\sigma/2}_{0}(\Omega))\) and passing to the limits when \(m \to \infty\), we infer that
\[
\|u(t)\|_{H^\sigma/2(\Omega)}^2 \leq C, \quad \text{for all} \quad t \geq T.
\]

Thus, taking into account Lemma 4.1 and Lemma 4.2, we have proved Theorem 1.2.

Now, let us prove the \((L^2_0(\Omega), H^{\sigma/2}_{0}(\Omega))-\)asymptotical compactness for the solution semigroup.

**Lemma 4.3.** Assume that \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^n\), \(f\) satisfies (2) and \(g \in L^2(\Omega)\). Then for every bounded subset \(B\) in \(L^2_0(\Omega)\), there exists a positive constant \(T_B = T(B)\) such that
\[
\|u_t(s)\|_{L^2_0(\Omega)}^2 \leq C \quad \text{for every} \quad u_0 \in B \quad \text{and} \quad s \geq T_B,
\]
where \(C\) independent of \(B\).

**Proof.** By differentiating (4) in time, denoting \(v_m = u_{mt}\) and multiplying the new equation by the function \(u^\prime_{m(t)}\), we infer that
\[
\frac{1}{2} \frac{d}{dt} \|v_m\|_{L^2_0(\Omega)}^2 + \|v_m\|_{H^{\sigma/2}_{0}(\Omega)}^2 \leq C_1 \|v_m\|_{L^2_0(\Omega)}^2.
\]
Integrating the inequality (11) over \((T, T + 2)\) with respect to \(s\) and using (8), we conclude that
\[
\int_{T+1}^{T+2} \|v_m(s)\|_{L^2_0(\Omega)}^2 ds \leq C
\]
as \(T\) large enough, where \(C\) is independent of \(T\). Combining (26) with (27), and using the uniform Gronwall lemma, we deduce that
\[
\int_{T}^{T+2} |u_{mt}(s)|^2 dx \leq C.
\]
as $s$ large enough. Therefore, there exists $T_B$ such that
\[
\int_\Omega |u_t(s)|dx \leq \liminf_{m \to \infty} \int_\Omega |u_{n_1}(s)|^2dx \leq C \quad \text{for every} \quad s \geq T_B. \tag{28}
\]
Lemma 4.3 follows from (28) immediately. \hfill \Box

**Theorem 4.4.** Assume that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $f$ satisfies (2) and $g \in L^0_0(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ generated by (1) with initial data $u_0 \in L^0_0(\Omega)$ is $(L^0_0(\Omega), H^{\sigma/2}_0(\Omega))$-asymptotically compact.

**Proof.** Let $B_0$ be a $(L^0_0(\Omega), H^{\sigma/2}_0(\Omega))$-bounded absorbing set obtained in Lemma 4.2. We only need to show that
\[
\text{for any} \{u_{0n}\} \subset B_0 \text{ and } t_n \to \infty, \{u_{nt}(t_n)\}_{n=1}^\infty \text{ is precompact in } H^{\sigma/2}_0(\Omega), \tag{29}
\]
where $u_{nt}(t_n) = S(t_n)u_{0n}$.

Thanks to Theorem 1.2, we know that $\{u_{nt}(t_n)\}_{n=1}^\infty$ is precompact in $L^0_0(\Omega)$. Without loss of generality, we assume that $\{u_{nk}(t_{nk})\}$ is a Cauchy sequence in $L^0_0(\Omega)$. Hence, combining with Lemma 4.3, we deduce that
\[
\|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\|^2_{H^{\sigma/2}_0(\Omega)} = \langle Au_{nk}(t_{nk}) - Au_{nj}(t_{nj}), u_{nk}(t_{nk}) - u_{nj}(t_{nj}) \rangle
\]
\[
= \left\langle \frac{d}{dt}u_{nk}(t_{nk}) - f(u_{nk}(t_{nk})) + \frac{d}{dt}u_{nj}(t_{nj}) + f(u_{nj}(t_{nj})), u_{nk}(t_{nk}) - u_{nj}(t_{nj}) \right\rangle
\]
\[
\leq \|\frac{d}{dt}u_{nk}(t_{nk}) - \frac{d}{dt}u_{nj}(t_{nj})\|_{L^0_0(\Omega)}\|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\|_{L^0_0(\Omega)}
\]
\[
+ C_1\|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\|^2_{L^0_0(\Omega)},
\]
\[
\leq C\|u_{nk}(t_{nk}) - u_{nj}(t_{nj})\|_{L^0_0(\Omega)},
\]
which yields (29) immediately. \hfill \Box

Thanks to Lemma 2.7, we then obtain Theorem 1.3.

**Acknowledgments.** We would like to express our sincere thanks to the anonymous referee for his/her valuable comments and suggestions which led to an important improvement of our original manuscript.

**REFERENCES**

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Second edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009.

[2] I. Athanasopoulos and L. A. Caffarelli, *Continuity of the temperature in boundary heat control problems*, Adv. Math., **224** (2010), 293–315.

[3] A. Babin and M. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[4] J. Bertoin, *Lévy Processes*, Cambridge Tracts in Mathematics, 121. Cambridge University Press, Cambridge, 1996.

[5] C. Brändle, E. Colorado, A. de Pablo and U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Math. Roy. Soc. Edinb., **143** (2013), 39–71.

[6] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Diff. Eq., **32** (2007), 1245–1260.

[7] L. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. Math., **171** (2010), 1903–1930.

[8] Z. Chen and R. Song, *Hardy inequality for censored stable processes*, Tohoku Math. J., **55** (2003), 439–450.
[9] R. Cont and P. Tankov, *Financial Modelling With Jump Processes*, Boca Raton, FL: Chapman Hall/CRC, 2004.
[10] J. Duan, *An Introduction to Stochastic Dynamics*, Cambridge University Press, New York, 2015.
[11] X. Fernández-Real and X. Ros-Oton, Boundary regularity for the fractional heat equation, *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM*, 10 (2016), 49–64.
[12] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Second revised and extended edition. De Gruyter Studies in Mathematics, 19. Walter de Gruyter & Co., Berlin, 2011.
[13] P. Geredeli and A. Khanmamedov, Long-time dynamics of the parabolic p-Laplacian equation, *Commun. Pure Appl. Anal.*, 12 (2013), 735–754.
[14] A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, *Invent. Math.*, 167 (2007), 445–453.
[15] M. Krasnoselskii and Y. Rutickii, *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen, 1961.
[16] J. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, New York: Springer-Verlag, Vol I, 1973.
[17] H. Lu, P. Bates, S. Lü and M. Zhang, Dynamics of the 3-D fractional complex Ginzburg-Landau equation, *J. Differ. Equ.*, 259 (2015), 5276–5301.
[18] H. Lu, P. Bates, S. Lü and M. Zhang, Dynamics of the 3D fractional Ginzburg-Landau equation with multiplicative noise on an unbounded domain, *Commun. Math. Sci.*, 14 (2016), 273–295.
[19] J. Mercado, E. Guido, A. Sánchez-Sesma, M. Íñiguez and A. González, Analysis of the Blasius Formula and the Navier-Stokes Fractional Equation, Chapter Fluid Dynamics in Physics, Engineering and Environmental Applications Part of the series Environmental Science and Engineering, (2012), 475–480.
[20] R. Metzler and J. Klafter, The restaurant at the end of the random walk: Recent developments in the description of anomalous transport by fractional dynamics, *J. Phys. A: Mathematical and General*, 37 (2004), 161–208.
[21] E. Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. sci. math.*, 136 (2012), 521–573.
[22] A. de Pablo, F. Quirós, A. Rodríguez and J. Vázquez, A fractional porous medium equation, *Adv. Math.*, 226 (2011), 1378–1409.
[23] A. de Pablo, F. Quirós, A. Rodríguez and J. Vázquez, A general fractional porous medium equation, *Comm. Pure Applied Math.*, 65 (2012), 1242–1284.
[24] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary, *J. Math. Pures Appl.*, 101 (2014), 275–302.
[25] J. Simon, Compact sets in the space $L^p(O,T;B)$, Annali di Matematica Pura ed Applicata, 146 (1987), 65–96.
[26] R. Song and Z. Vondraček, Potential theory of subordinate killed Brownian motion in a domain, *Probab. Theory Relat. Fields*, 125 (2003), 578–592.
[27] P. Stinga and J. Torrea, Extension problem and Harnack’s inequality for some fractional operators, *Commun. Partial Differ. Equ.*, 35 (2010), 2092–2122.
[28] J. Vázquez, Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators, *Discrete Contin. Dyn. Syst. Ser. S*, 7 (2014), 857–885.
[29] M. Yang, C. Sun and C. Zhong, Global attractors for p-Laplacian equation, *J. Math. Anal. Appl.*, 327 (2007), 1130–1142.
[30] X. Zhang, Stochastic Lagrangian particle approach to fractal Navier-Stokes equations, *Commun. Math. Phys.*, 311 (2012), 133–155.
[31] C. Zhong, J. Zhang and C. Zhong, Existence of weak solutions for fractional porous medium equations with nonlinear term, *Appl. Math. Lett.*, 61 (2016), 95–101.
[32] C. Zhong, M. Yang and C. Sun, The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations, *J. Differ. Equ.*, 223 (2006), 367–399.

Received October 2016; revised October 2017.
E-mail address: chzhnju@126.com
E-mail address: fli@xidian.edu.cn
E-mail address: duan@iit.edu