THE CRITICAL ORDER OF CERTAIN HECKE
L-FUNCTIONS OF IMAGINARY QUADRATIC FIELDS

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Abstract. Let \(-D < -4\) denote a fundamental discriminant which is either odd or divisible by 8, so that the canonical Hecke character of \(\mathbb{Q}(\sqrt{-D})\) exists. Let \(d\) be a fundamental discriminant prime to \(D\). Let \(2k - 1\) be an odd natural integer prime to the class number of \(\mathbb{Q}(\sqrt{-D})\). Let \(\chi\) be the twist of the \((2k - 1)\)th power of a canonical Hecke character of \(\mathbb{Q}(\sqrt{-D})\) by the Kronecker’s symbol \(n \mapsto \left(\frac{d}{n}\right)\). It is proved that the order of the Hecke \(L\)-function \(L(s, \chi)\) at its central point \(s = k\) is determined by its root number when \(|d| \leq c(\varepsilon)D^{1/12 - \varepsilon}\) or, when \(|d| \leq c(\varepsilon)D^{1/24 - \varepsilon}\) and \(k \geq 2\), where \(\varepsilon > 0\) and \(c(\varepsilon)\) is a constant depending only on \(\varepsilon\).

0. Introduction

Let \(K\) be an imaginary quadratic field with discriminant \(-D < -4\) and class number \(h\). Suppose that \(D\) is either odd or divisible by 8. Then, according to Rohrlich [Re], there are exactly gcd(2, \(D\)) \(h\) Hecke characters \(\chi_{can}\) of \(K\) satisfying

1. The conductor of \(\chi_{can}\) is \((2\sqrt{-D}, D)\), where \((2\sqrt{-D}, D)\) denotes the ideal generated by \(2\sqrt{-D}\) and \(D\).
2. \(\chi_{can}(\alpha) = \pm\alpha\) if \((\alpha)\) is a principle ideal prime to \((2\sqrt{-D}, D)\).
3. \(\chi_{can}(n) = \left(\frac{-D}{n}\right)n\) if \(n\) is a positive integer prime to \((2\sqrt{-D}, D)\).

We call such a character \(\chi_{can}\) canonical. Let \(d\) be a fundamental discriminant prime to \(D\). Let \(2k - 1\) be an odd positive integer prime to the class number \(h\). Let \(\chi\) be the product of the \((2k - 1)\)th power of a canonical Hecke character of \(K\) and the lifting of the Kronecker’s symbol \(n \mapsto \left(\frac{d}{n}\right)\). Let \(L(s, \chi)\) be the Hecke \(L\)-function attached to \(\chi\). Then

\[\Lambda(s, \chi) = (D^*|d|)^*(2\pi)^{-s}\Gamma(s)L(s, \chi) = W(\chi)\Lambda(2k - s, \chi),\]

where \(D^* = D\gcd(2, D)\) and \(W(\chi) = \pm 1\) is the root number. It is well known that the Hecke \(L\)-function \(L(s, \chi)\) is the \(L\)-function of a newform \(f\) of level

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\((D^*|d|)^2\) and weight \(2k\) with coefficients in \(\mathbb{Q}\). Let \(M\) be the Grothendieck motive over \(\mathbb{Q}\) attached to \(f\) by U. Jannsen and A. J. Scholl. According to a conjecture of Beilinson and Bloch and a result of S. Zhang, the order of the Hecke \(L\)-function \(L(s, \chi)\) at its central point \(s = k\) is closely related to the arithmetic of \(M\). Because of their arithmetic nature, these \(L\)-functions have been extensively studied by, among other authors, Gross [Gr], Rohrlich, Rodriguez-Villegas and Yang. Throughout this paper \(\varepsilon > 0\) is arbitrary small and the constants \(c(\varepsilon), c_1(\varepsilon)\) and \(c_2(\varepsilon)\), and those implied in the symbols \(<, >\) and \(O\) depend at most on \(\varepsilon\) and \(k\). We list the following update results.

- If \(k = 1\) and \(W(\chi) = 1\), Rohrlich [Rb] proved that \(L(1, \chi)\) doesn’t vanish when \(|d| \ll D^{1/39-\varepsilon}\).
- If \(k = 1\) and \(W(\chi) = -1\), Miller-Yang [MY] proved that \(L'(1, \chi)\) doesn’t vanish when \(|d| \ll D^{1/35-\varepsilon}\).
- If \(W(\chi) = 1\), Yang [Ya] proved that \(L(k, \chi)\) doesn’t vanish when \(D \mod 7, d \mod 4\) is positive and \(\sqrt{D} > d^2(\frac{12}{\pi} \ln d + M(k))\), where \(M(k)\) is a constant depending only on \(k\).

In this paper, we shall prove the following two theorems.

**Theorem 1.** If \(W(\chi) = 1\), then \(L(k, \chi)\) does not vanish when \(|d| \ll D^{1/24-\varepsilon}\).

**Theorem 2.** If \(W(\chi) = -1\), then \(L'(k, \chi)\) doesn’t vanish when \(|d| \ll D^{1/24-\varepsilon}\) or, when \(|d| \ll D^{1/24-\varepsilon}\) and \(k \geq 2\).

From the theorems one sees that

**Corollary 1.** The order of the Hecke \(L\)-function \(L(s, \chi)\) at its central point \(s = k\) is determined by its root number if \(|d| \ll D^{1/24-\varepsilon}\) or, if \(|d| \ll D^{1/24-\varepsilon}\) and \(k \geq 2\).

Assume that \(k = 1\). Then \(\chi\) lifts to a Hecke character \(\psi\) of the Hilbert class field \(H\) of \(K\) commuting with the action of the Galois group of \(H\) over \(\mathbb{Q}\). And

\[
L(s, \psi) = \prod_{\phi} L(s, \chi \phi),
\]

where \(\phi\) runs over all characters of the class group of \(K\). Let \(j\) be the \(j\)-invariant of an elliptic curve over \(H\) with complex conjugation by the ring of integers of \(K\). According to Gross [Gr], there is a unique elliptic curve \(A\) over \(H\) with \(j\)-invariant \(j\) which is isogenous to all its Galois conjugates and whose \(L\)-function is

\[
L(s, A/H) = L(s, \psi)^2.
\]

\(A\) descends to two isogenous elliptic curves over \(\mathbb{Q}(j)\) with \(L\)-function \(L(s, \psi)\). Let \(A_d\) be one of them. By results of Kolyvagin-Logachev [KL] and Gross-Zagier [GZ], the theorems imply the following arithmetic consequences.
Corollary 2. If $W(\chi) = 1$ and $|d| \ll D^{1/2} - \varepsilon$, then the Mordell-Weil group and the Shafarevich-Tate group of $A_d/\mathbb{Q}(j)$ are finite.

Corollary 3. If $W(\chi) = -1$ and $|d| \ll D^{1/2} - \varepsilon$, then $A_d/\mathbb{Q}(j)$ has a finite Shafarevich-Tate group and a Mordell-Weil group of rank $h$.

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1. $L$-functions with root number 1

In this section we shall prove Theorem 1. Put

$$L(s, \chi, p) = \sum_{(\alpha) \in p} \chi((\alpha))(N(\alpha))^{-s}, \quad \Re s > 3/2,$$

where $p$ is the set of all principal integral ideals of $K$. According to Rohrlich [Rb], for all ideal class characters $\varphi$ of $K$, the $L$-functions $L(s, \chi \varphi)$ satisfy the same functional equation as $L(s, \chi)$ does. So

$$\Lambda(s, \chi, p) = (D^*|d|)^s (2\pi)^{-s} \Gamma(s)L(s, \chi, p) = W(\chi)\Lambda(2k - s, \chi, p).$$

Note that

$$L(s, \chi, p) = L_D(2s - 2k + 1) + L(s),$$

where

$$L_D(s) = \sum_{n \geq 1, (n,d)=1} (\frac{-D}{n}) n^{-s},$$

and

$$L(s) = \sum_{(\alpha) \in p'} \chi((\alpha))N(\alpha)^{-s}$$

with $p'$ denoting the set of all principal integral ideals not generated by rational numbers. It is easy to see that $L_D(s)$ is the Dirichlet $L$-function attached to the Dirichlet character:

$$(\mathbb{Z}/(D|d|)^\times \rightarrow \mathbb{C}\times, \quad n \mapsto (\frac{-D}{n})).$$

We now suppose that $W(\chi) = 1$ and proceed to prove Theorem 1. According to Shimura [Sh] and Rohrlich [Rb], $L(k, \chi) = 0$ implies $L(k, \chi \varphi^{2k-1}) = 0$ for all ideal class characters $\varphi$ of $K$. So it also implies $L(k, \chi, p) = 0$ as

$$hL(s, \chi, p) = \sum_\varphi L(s, \chi \varphi) = \sum_\varphi L(s, \chi \varphi^{2k-1}),$$
where $\varphi$ runs over all ideal class characters of $K$. Hence, to prove $L(k, \chi) \neq 0$, it suffices to prove that $L(k, \chi, p) \neq 0$. It follows from the functional equation and a formula of Cauchy that

$$\frac{1}{2} L(k, \chi, p) = \frac{1}{2\pi i} \int_{(2k)} (2\pi)^k \Lambda(s, \chi, p) \frac{ds}{(D^* |d|)^k \Gamma(k) s - k}.$$ 

As

$$L(s, \chi, p) = L_D(2s - 2k + 1) + L(s),$$

we get the following approximation to the central value:

$$\frac{1}{2} L(k, \chi, p) = I_1 + I_2,$$

where

$$I_1 = \frac{1}{2\pi i} \int_{(2k)} (D^* |d|)^{s-k} (2\pi)^{k-s} \frac{\Gamma(s)}{\Gamma(k)} L_D(2s + 1 - 2k) \frac{ds}{s - k}$$

and

$$I_2 = \frac{1}{2\pi i} \int_{(2k)} (D^* |d|)^{s-k} (2\pi)^{k-s} \frac{\Gamma(s)}{\Gamma(k)} L(s) \frac{ds}{s - k}.$$

Theorem 1 now follows from the estimate

$$I_2 \ll D^{-\frac{1}{16} + \varepsilon} |d|^{\frac{3}{4} + \varepsilon},$$

which will be proved in the next section, and the estimate

$$I_1 \geq c_1(\varepsilon) (|D| |d|)^{-\varepsilon} - c_2(\varepsilon) (|D| |d|)^{-\frac{1}{16} + \varepsilon},$$

which we are going to prove. Shifting the line of integration in $I_1$ to $\Re s = k - 1/4$, we get

$$I_1 = L_D(1) + \frac{1}{2\pi i} \int_{(k-1/4)} (D^* |d|)^{s-k} (2\pi)^{k-s} \frac{\Gamma(s)}{\Gamma(k)} L_D(2s + 1 - 2k) \frac{ds}{s - k}.$$

Applying Burgess’ estimate [Bu]

$$L_D \left( \frac{1}{2} + it \right) \ll (D |d|)^{\frac{1}{16} + \varepsilon} (|t| + 1),$$

we get

$$I_1 = L_D(1) + O((D |d|)^{-\frac{1}{16} + \varepsilon}).$$

The required estimate for $I_1$ now follows from Siegel’s estimate

$$L_D(1) \gg (D |d|)^{-\varepsilon}.$$
2. **The complex part of the approximation to the central value**

In this section, we shall prove that

$$I_2 \ll D^{-\frac{1}{16} + \varepsilon} |d|^\frac{3}{4} + \varepsilon.$$  

Recall that

$$I_2 = \frac{1}{2\pi i} \int_{(2k)} (D^*|d|)^{s-k} (2\pi)^{k-s} \frac{\Gamma(s)}{\Gamma(k)} L(s) \frac{ds}{s-k}.$$  

Applying Mellin’s inversion and writing

$$\chi((\alpha)) = \epsilon(\alpha)\alpha^{2k-1},$$

where $\epsilon$ is a quadratic character with conductor $d(2\sqrt{-D}, D)$ on the subgroup of $K^\times$ consisting of elements prime to $d(2\sqrt{-D}, D)$, we get

$$I_2 = \frac{1}{\Gamma(k)} \sum_{(\alpha) \in \mathcal{D}'} \epsilon(\alpha)\alpha^{2k-1} N(\alpha)^{-k} \int_{\frac{2\pi N(\alpha)}{D^*|d|}}^{\infty} e^{-\xi \epsilon k \frac{d\xi}{\xi}}.$$  

The contribution from the terms with $\Re \alpha$ or $|\Im \alpha| \geq (D|d|)^{\frac{3}{4}} \log(D|d|)$ is bounded by

$$\sum_{n \geq D^*|d| \log(D|d|)}^{2} \epsilon(\alpha)\alpha^{2k-1} \frac{d\xi}{\xi} \ll (D^*|d|)^{-1}.  $$

The subsum from the terms of $I_2$ with $0 < \Re \alpha$, $|\Im \alpha| < (D|d|)^{\frac{3}{4}} \log(D|d|)$ equals

$$\sum_{u,v} \frac{2(u + \sqrt{-D}v) \epsilon(u + \sqrt{-D}v)}{\Gamma(k)(u^2 + Dv^2)^k} e^{-\xi \epsilon k \frac{d\xi}{\xi}},$$

where $(u, v)$ runs over pairs of integers satisfying

$$0 < u, \sqrt{|D|}v < 2(D|d|)^{\frac{1}{2}} \log(D|d|), \quad 4|(u^2 + Dv^2).$$

Conjugate terms grouped together, it becomes

$$\sum_{u,v} a(u,v) \epsilon(u + \sqrt{-D}v) \int_{\frac{2\pi N(\alpha)}{2D^*|d|}}^{\infty} e^{-\xi \epsilon k \frac{d\xi}{\xi}},$$

where $(u, v)$ runs over pairs of integers satisfying

$$0 < u, \sqrt{|D|}v < 2(D|d|)^{\frac{1}{2}} \log(D|d|), \quad 4|(u^2 + Dv^2).$$
and
\[
\frac{1}{2}a(u, v) = \frac{(u + \sqrt{-Dv})^{2k-1} + (u - \sqrt{-Dv})^{2k-1}}{\Gamma(k)(u^2 + Dv^2)^k}.
\]

It splits dyadically into at most \(4 \log_2(2D|d|)\) sums of the form

\[
\sum_{N \leq v < N'} \sum_{M \leq u < M', 4|(u^2+Dv^2)} a(u, v)e\left(\frac{u+\sqrt{-Dv}}{2}\right) \int_0^\infty \frac{e^{-\xi k} d\xi}{\sqrt{(u^2+Dv^2)}} \xi,
\]

where \(0 < M', \sqrt{DN} < (D|d|)^{1/2} \log(D|d|), N' \leq 2N, \) and \(M' \leq 2M.\) By Abel’s summation formula, the inner sum is bounded by

\[
O(\log^{2k}(D|d|)) \min(M^{-1}, D^{-1/2}N^{-1}) \max_{M < w \leq 2M} |S_v(w)|,
\]

where

\[
S_v(w) = \sum_{M \leq u < w, 4|(u^2+Dv^2)} e\left(\frac{u+\sqrt{-Dv}}{2}\right).
\]

We claim that

\[
S_v(w) \ll |d| + M^{1/2}D^{3/16+\varepsilon}|d|^{1/2}, \quad w < 2M,
\]

from which the estimate for \(I_2,\) which is stated at the beginning of this section, follows. Write \(\epsilon = \epsilon_0\epsilon_1,\) where \(\epsilon_0\) and \(\epsilon_1\) have conductors \(\sqrt{-D}/(\sqrt{-D},4)\) and \(d(\sqrt{-D}, 4)\) respectively. Let \(k_0\) and \(k_1/2\) be the least positive integers in \(\sqrt{-D}/(\sqrt{-D},4)\) and \(d(\sqrt{-D}, 4)\) respectively. Then

\[
S_v(w) = \sum_{1 \leq j < k_1, 4|(j^2+Dv^2)} \sum_{M \leq u < w, u \equiv k_0 j(k_1)} \epsilon_0(u/2)\epsilon_1\left(\frac{u+\sqrt{-Dv}}{2}\right).
\]

The inner sum equals

\[
\epsilon_0(k_1/2)\epsilon_1\left(\frac{k_0 j + \sqrt{-Dv}}{2}\right) \sum_{M-k_0 l < l < w-k_0 j} \epsilon_0(l),
\]

which, according to Burgess [Bu], is bounded by \(O(1 + (M/|d|)^{1/2}D^{3/16+\varepsilon}).\) So

\[
S_v(w) \ll |d| + M^{1/2}D^{3/16+\varepsilon}|d|^{1/2}
\]

as claimed.
3. Twists of root number $-1$

In this section we shall prove Theorem 2. Similarly, it suffices to show that \( L'(k, \chi, p) \neq 0 \) under the condition of Theorem 2 (see [GZ] or [MY]). Suppose that \( W(\chi) = -1 \). It follows from this functional equation and a formula of Cauchy that

\[
\frac{1}{2} L'(k, \chi, p) = \frac{1}{2\pi i} \int_{(2k)} (2\pi)^k \Lambda(s, \chi, p) \frac{ds}{(D^*|d|)^k \Gamma(k) (s-k)^2}.
\]

As

\[
L(s, \chi, p) = L_D(2s - 2k + 1) + L(s),
\]

we get the following approximation to the central derivative:

\[
\frac{1}{2} L'(k, \chi, p) = R_k + C,
\]

where

\[
R_k = \frac{1}{2\pi i} \int_{(2k)} \frac{(D^*|d|)^{s-k} \Gamma(s)}{\Gamma(k)} L_D(2s - 2k + 1) \frac{ds}{(s-k)^2}
\]

and

\[
C = \frac{1}{2\pi i} \int_{(2k)} (D^*|d|)^{s-k} (2\pi)^k \frac{\Gamma(s)}{\Gamma(k)} L(s) \frac{ds}{(s-k)^2}.
\]

Theorem 2 now follows from the estimate

\[
R_k \geq .0351 - c(\varepsilon)(D|d|)^{-\frac{1}{16} + \varepsilon},
\]

which will be proved in the next section, and the estimate

\[
C \ll \begin{cases} D^{-\frac{1}{16} + \varepsilon}|d|^{\frac{1}{4} + \varepsilon}, & k = 1, \\ D^{-\frac{1}{16} + \varepsilon}|d|^{\frac{1}{4} + \varepsilon}, & k > 1, \end{cases}
\]

which we are going to prove. Applying Mellin’s inversion we get

\[
C = \sum_{(\alpha) \in p'} \chi((\alpha)) N(\alpha)^{-k} \int_{\frac{2\pi n N(\alpha)}{D^*|d|}}^{\infty} e^{-\xi} \xi^k (\log \xi - \log \frac{2\pi n N(\alpha)}{D^*|d|}) \frac{d\xi}{\xi}.
\]

The contribution from the terms with \( \Re \alpha \) or \( |\Im \alpha| \geq (D^*|d|)^{\frac{1}{4}} \log(D|d|) \) is bounded by

\[
\sum_{n \geq (D^*|d|)^{\frac{1}{4}} \log(D|d|)}^{\infty} e^{-\xi} \xi^k (\log \xi - \log \frac{2\pi n}{D^*|d|}) \frac{d\xi}{\xi} \ll (D^*|d|)^{-1}.
\]
The subsum from the terms of \( C \) with \( 0 < \Re \alpha, |\Im \alpha| < (D|d|)^{\frac{3}{4}} \log(D|d|) \) equals

\[
\sum_{u,v} \frac{2(u + \sqrt{-Dv})^{2k-1}}{(u^2 + Dv^2)^k} f(u, v) \epsilon \left( \frac{u + \sqrt{-Dv}}{2} \right),
\]

where \((u, v)\) runs over pairs of integers satisfying

\[
0 < u, \sqrt{D} |v| < 2(D|d|)^{\frac{3}{4}} \log(D|d|), \quad 4|(u^2 + Dv^2)|
\]

and

\[
f(u, v) = \int_{\frac{\pi(u^2 + Dv^2)}{2D^2|d|}}^{\infty} e^{-\xi} \xi^k (\log \xi - \log \frac{\pi(u^2 + Dv^2)}{2D^2|d|}) d\xi.
\]

Conjugate terms grouped together, it becomes

\[
\sum_{u,v} a(u, v) f(u, v) \epsilon \left( \frac{u + \sqrt{-Dv}}{2} \right),
\]

where \((u, v)\) runs over pairs of integers satisfying

\[
0 < u, \sqrt{D} v < 2(D|d|)^{\frac{3}{4}} \log(D|d|), \quad 4|(u^2 + Dv^2)|
\]

It splits dyadically into at most \(4 \log^2(2D|d|)\) sums of the form

\[
\sum_{N \leq v < N'} \sum_{M \leq u < M', 4|(u^2 + Dv^2)} a(u, v) f(u, v) \epsilon \left( \frac{u + \sqrt{-Dv}}{2} \right),
\]

where \(0 < M, \sqrt{D} N < (D|d|)^{\frac{3}{4}} \log(D|d|), \quad N' \leq 2N, \quad \text{and} \quad M' \leq 2M\). By Abel’s summation formula, the inner sum is bounded by

\[
O(D|d| \log^2(D|d|)) \min(M^{-3}, D^{-3/2} N^{-3}) \max_{M \leq w \leq 2M} |S_v(w)|,
\]

if \(k = 1\), and by

\[
O(\log^{2k}(D|d|)) \min(M^{-1}, D^{-1/2} N^{-1}) \max_{M \leq w \leq 2M} |S_v(w)|,
\]

if \(k > 1\), where

\[
S_v(w) = \sum_{M \leq u < w, 4|(u^2 + Dv^2)} \epsilon \left( \frac{u + \sqrt{-Dv}}{2} \right).
\]

In §2, we have proved that

\[
S_v(w) \ll |d| + M^{\frac{3}{4}} D^{\frac{3}{4} + \epsilon} |d|^{\frac{3}{4}}, \quad w < 2M.
\]

The desired estimate for \(C\) now follows.
4. The rational part of the approximation to the central derivative

In this section we shall prove that

\[ R_k \geq 0.0351 - c(\varepsilon)(D|d|)^{-\frac{1}{16} + \varepsilon}. \]

Recall that

\[ R_k = \frac{1}{2\pi i} \int_{(2k)} \left( \frac{D^*|d|}{2\pi} \right)^{s-k} \frac{\Gamma(s)}{\Gamma(k)} L_D(2s - 2k + 1) \frac{ds}{(s-k)^2}. \]

A change of variable yields

\[ R_k = \frac{1}{\pi i} \int_{(k+1)} \left( \frac{D^*|d|}{2\pi} \right)^{s-1} \frac{\Gamma(s + k - 1)}{\Gamma(k)} L_D(2s - 1) \frac{ds}{(s-1)^2}. \]

Shifting the line of integration to \( \Re s = 3/4 \) and applying Burgess’ estimate, we get

\[ R_k = \Lambda_k'(1) + O(16^{-1/16} + \varepsilon), \]

where

\[ \Lambda_k(s) = \left( \frac{D^*|d|}{2\pi} \right)^{s-1} \frac{\Gamma(s + k - 1)}{\Gamma(k)} L_D(2s - 1). \]

So

\[ R_k = R_1 + \Lambda_k'(1) - \Lambda_1'(1) + O((D|d|)^{-\frac{1}{16} + \varepsilon}). \]

As

\[ \Lambda_k'(1) = \frac{\Gamma'(k)}{\Gamma(k)} L_D(1) + \log \frac{D^*|d|}{2\pi} L_D(1) + 2L_D'(1) \geq \Lambda_1'(1), \]

we claim that

\[ R_1 \geq 0.0351, \]

from which the estimate for \( R_k \), which is stated at the beginning of this section, follows. Write

\[ \frac{\zeta(s)L_D(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} a_n n^{-s} \]

with \( a_1 = 1 \), and \( a_n \geq 0 \). Then

\[ R_1 = \sum_n a_n n^{-1} I(\frac{D^*|d|}{2\pi n^2}), \]

where

\[ I(x) = \frac{1}{2\pi i} \int_{(2)} x^{s-1} \frac{\Gamma(s)\zeta(4s - 2)}{\zeta(2s - 1)} \frac{ds}{(s-1)^2}. \]

Miller-Yang [MY] proved that \( I(x) > 0 \) if \( x > 0 \) and that \( I(x) > 0.0351 \) if \( x \geq 4 \). So we have \( R_1 \geq 0.0351 \) as claimed.
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