A new criterion for finite non-cyclic groups

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Abstract

Let $H$ be a subgroup of a group $G$. We say that $H$ satisfies the power condition with respect to $G$, or $H$ is a power subgroup of $G$, if there exists a non-negative integer $m$ such that $H = G^m = \langle g^m | g \in G \rangle$. In this note, the following theorem is proved: Let $G$ be a group and $k$ the number of non-power subgroups of $G$. Then (1) $k = 0$ if and only if $G$ is a cyclic group (theorem of F. Szász); (2) $0 < k < \infty$ if and only if $G$ is a finite non-cyclic group; (3) $k = \infty$ if and only if $G$ is an infinite non-cyclic group. Thus we get a new criterion for the finite non-cyclic groups.

Keywords: power subgroup, cyclic group, Dedekind group.

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1. Introduction

Let $H$ be a subgroup of group $G$. We say that $H$ satisfies the power condition with respect to $G$, or $H$ is a power subgroup of $G$ if there exists a non-negative integer $m$ such
that $H = G^m = \langle g^m \mid g \in G \rangle$. On the other hand, if $H \neq G^m$ for all $m$, we say that $H$ is a non-power subgroup of $G$. If $H$ is a power subgroup of $G$, the power exponent is the least non-negative integer $m$ such that $H = G^m$.

It is obvious that a nontrivial group $G$ has at least two trivial power subgroups: $\{1\}$ and $G$ itself, and the power exponents of the two subgroups are 0 and 1, respectively.

The power subgroups have some properties. For example, if $H$ is a power subgroup of group $G$, then $H$ is a full-invariant subgroup, in particular $H$ is normal in $G$; and if $H$ is a non-power subgroup of $G$, then a conjugate subgroup $H^x(x \in G)$ is also a non-power subgroup of $G$.

The number of non-trivial power subgroups affects the structure of the group. In [2], F.Szász proved that $G$ is a cyclic group if and only if all subgroups of $G$ are power subgroups. That is, a non-cyclic group contains at least one non-power subgroup. In this paper, we generalize the above result of F. Szász and prove the following theorem:

**Theorem.** Let $G$ be a group and $k$ the number of non-power subgroups of $G$. Then

(1) $k = 0$ if and only if $G$ is a cyclic group;

(2) $0 < k < \infty$ if and only if $G$ is a finite non-cyclic group;

(3) $k = \infty$ if and only if $G$ is an infinite non-cyclic group.

**Remark 1.** The conclusion (1) of this theorem is the theorem of F. Szász in [2].

**Remark 2.** Note that for some $k$ in the theorem, the group $G$ may not exist. In [3],[4] the author proved that the case of $k = 1$ or $k = 2$ do not occur. But there are groups having exactly 3 non-power subgroups: For example the quaternion group $Q_8$ of order 8, and $Z_2 \times Z_2$ are groups possessing just 3 non-power subgroups.

From the theorem the following corollary gives a new criterion for finite non-cyclic groups.

**Corollary.** Suppose $G$ is a non-cyclic group. Then $G$ is finite if and only if $G$ contains only finitely many non-power subgroups.

**Problem.** For any integer $k(k \geq 3)$, does there exist groups possessing just $k$ non-power subgroups?

In the proof of the Theorem, we use the structure of the Dedekind group. A group is called Dedekind group if all its subgroups are normal. From Theorem 5.3.7 in [1], we know that $G$ is a Dedekind group if and only if $G$ is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order. Notation is standard and may be found for instance in[1]. In particular, we denote $k$ the number of of non-power subgroups of $G$. 

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2. Some lemmas

**Lemma 1.** Let $A$ be an abelian power subgroup in a group $G$. Then the set of power subgroups of $G$ contained in $A$ coincides with the set of power subgroups of $A$.

**Proof:** If $G = A$, then there it is nothing to prove. Let $G \neq A$ and let $l$ be the power exponent of $A$. Then $l > 1$ and $A = G^l$. Suppose that $G^m \leq A$ for $m \geq 1$ and let $d$ be the greatest common divisor of $l, m$. Then $d = lr + ms$ for some integers $r$ and $s$, and $g^d = (g^l)^r (g^m)^s \in A$ for every $g \in G$, so $G^d \leq A$. It is obvious that $A = G^l \leq G^d$, so $d = l$ and $l$ divides $m$. Hence $m = ls_0$ for some $s_0 \geq 0$. Note that $A = G^l = < g^l | g \in G >$ is abelian. We get $A^{s_0} = (G^l)^{s_0} = < g^l | g \in G >^{s_0} = < g^{ls_0} | g \in G > = G^{ms_0}$. On the other hand, if $A^n$ is a power subgroup of $A$, then $A^n = G^{ln}$ is a power subgroup of $G$. The lemma is proved.

**Lemma 2.** If $A/N$ is a non-power subgroup of a factor-group $G/N$, then $A$ is a non-power subgroup of $G$.

**Proof:** Suppose the result is false. That is, $A$ is a power subgroup of $G$. Let $A = G^m = < g_i^m | g_i \in G, i = 1, 2, \cdots, >$. Since $g_1^m \cdots g_n^m N = (g_1 N)^m \cdots (g_n N)^m$, $A/N = (G/N)^m$, a contradiction. Hence the lemma is proved.

Now we need a lemma about the structure of the cyclic group.

**Lemma 3.** If $G$ is an infinite abelian group all of whose proper quotient groups are finite, then $G$ is infinite cyclic.

**Proof:** Suppose $G$ is not infinite cyclic. We choose an element $a_0 \in G$ and $a_0 \neq 1$. $< a_0 >$ is a proper subgroup of $G$ and $< a_0 > \neq 1$. Then $G/ < a_0 >$ is finite. So $G/ < a_0 >$ is finitely generated and $G$ is finitely generated. Then $G$ is a direct product of finitely many cyclic groups of infinite or prime-power orders. Let $G = < g_1 > \times < g_2 > \cdots \times < g_n >$, and $n > 1$ by the assumption. If there exists an $i$, such that $|g_i| = \infty$. Then $G/ < g_1 > \times \cdots \times < g_{i-1} > \times < g_{i+1} > \cdots \times < g_n > \cong < g_i >$ is finite by the condition. That is impossible. So $|g_i| < \infty, \forall i$, which makes $G$ be a finite group, a contradiction. We prove the lemma.

**Lemma 4.** If the number of cyclic subgroups of a group $G$ is finite, then $G$ is finite.

**Proof:** Firstly, we have $G$ is a torsion group, otherwise suppose $g \in G$ and $|g| = \infty$, then $< g^n >$ will be different cyclic subgroups in $G$ with infinitely many different $n$, a contradiction.

For any element $g$ in $G$, it can generate a cyclic subgroup $< g >$. Suppose $G$ have $n$ cyclic subgroups and $g_i (i = 1, \cdots, n)$ be the generated elements of all the cyclic subgroups. $G = \cup_{i=1}^{n} < g_i >$. So $|G| \leq \sum_{i=1}^{n} |g_i| < \infty$, that is to say $G$ is finite.

3. Proof of the theorem

**Proof of the theorem:** We need only to prove the necessity of (2).
Case 1. $G$ is abelian.

i) Suppose $G$ is a torsion abelian group, we prove that $G$ is a finite group.

Let $\pi(G)$ be the set of all primes dividing the orders of elements of $G$. For every prime $p$, let $G_p$ be the set of all $p$-elements in $G$. Then $G_p$ is a subgroup of $G$, that is, the $p$-component of $G$. We claim $|\pi(G)|$ is finite. If $|\pi(G)| = \infty$, then there must exist a prime $p \in \pi(G)$ such that $G_p$ is a power subgroup for there are finitely many non-power subgroups. Hence there exists a positive integer $m$ such that $G_p = G^m$, and then $G/G_p = G/G^m$, which means the exponent of $G/G_p$ is finite. But $G/G_p \cong \prod_{q \in \pi(G) - \{p\}} G_q$. So the exponent of $G/G_p$ can not be finite since $|\pi(G)|$ is infinite and then we get the contradiction.

If $G_p$ is infinite for some $p$, then it contains infinitely many finite cyclic subgroups $C_i$ and hence we may assume that $C_i$ is a power subgroup in $G$ and so $G/C_i$ is of finite exponent, hence $G$ is of finite exponent and $G$ is a direct product of cyclic subgroups of finite order. In particular, $G_p$ is of finite exponent and since $G_p \cong G/N$ for some subgroup $N$, $G_p$ has only finitely many non-power subgroups in $G_p$ by Lemma 2. Since $G_p$ has only finitely many power subgroups, i.e., $G_p, G_p^p, \ldots, G_p^p$, where $p^i$ is the power exponent of $G_p$, join with its finitely many non-power subgroups, $G_p$ has only finitely many subgroups and so $G_p$ is finite. Then for every $p \in \pi(G)$, $G_p$ is finite. So $G$ is finite.

ii) Suppose $G$ is a non-torsion abelian group. Then $G$ is an infinite abelian group with some elements of infinite order. We prove $G$ is cyclic, which is contrary to $k > 0 (k$ is the number of non-power subgroups of $G$).

Firstly, we prove that the subgroup $T$ of $G$ consisting of all elements of finite orders in $G$ is finite. Evidently, no subgroup of $T$ can be a power subgroup of $G$, otherwise $G$ would be torsion and so, by i), $G$ is finite, a contradiction. Hence, since has only finitely many non-power subgroups, we have that $T$ contains only finitely many subgroups. Thus $T$ have only finitely many cyclic subgroups and $T$ is finite by Lemma 4.

Now $T \neq G$ and $\overline{G} = G/T$ is an infinite torsion-free group. By Lemma 2, $\overline{G}$ has only finitely many non-power subgroups. We prove $\overline{G}$ is cyclic.

We claim that $\overline{G}/\overline{H}$ is finite for all non-trivial subgroup $\overline{H}$ of $\overline{G}$. In fact, let $\overline{H}$ be a nontrivial subgroup of $\overline{G}$. If $\overline{H}$ is a power subgroup of $\overline{G}$, then there exists a non-negative integer $m$ such that $\overline{H} = \overline{G}^m$. Now $\overline{G}/\overline{H}$ is torsion, and it has only finitely many non-power subgroups by Lemma 2. Then $\overline{G}/\overline{H}$ is finite by i). On the other hand, suppose $\overline{H}$ is a non-power subgroup. Since $\overline{G}$ is torsion-free with finitely many non-power subgroups, there must exist $1 \neq \overline{H}_1 < \overline{H}$, with $\overline{H}_1$ being a power subgroup of $\overline{G}$. Then $\overline{G}/\overline{H}_1$ is finite and then $\overline{G}/\overline{H}$ is finite.

By Lemma 3, $\overline{G}$ is cyclic. Hence $G = T \times < z >$ where $z$ is of infinite order. It is obvious
that \( G^n = T^n \times < z^n > \), so if \( s = |T| > 1 \) then \( T \times < z^n > \) is a non-power subgroup of \( G \) for each \( n = 2, 3, \cdots \). Thus \( T = 1 \) and so \( G \) is cyclic, a contradiction.

Case 2. \( G \) is non-abelian.

If \( G \) contains no non-power subgroups, then \( G \) is cyclic and every subgroup of a cyclic group is a power subgroup. So, suppose that \( G \) contains some non-power subgroups, and let \( H_1, \cdots, H_s \) be all of those. Since every conjugate of \( H_i(i = 1, \cdots, s) \) is also a non-power subgroup of \( G \), so \( H_i \) has only finitely many conjugate subgroups. Then the normalizer \( N_i \) of \( H_i \) has a finite index in \( G \) and hence the subgroup \( K_i = \cap_{g \in G} g^{-1}N_ig \) is normal in \( G \) and has a finite index in \( G \). Let \( K = \cap_{i=1}^s K_i \), then \( K \) is a normal subgroup of finite index in \( G \) and so it normalizes every \( H_i \). Observe that every subgroup of \( K \) is normal in \( K \). In fact, if \( L \leq K \) and \( L \) is a power subgroup of \( G \), then \( L \leq G \) and hence \( L \leq K \); if \( L \) is a non-power subgroup of \( G \), then \( L = H_i \) for some \( i = 1, \cdots, s \), and by construction of \( K \), \( H_i \leq K \). Thus \( K \) is a Dedekind group. So \( K \) is either abelian or a direct product of an abelian group and the quaternion group of order 8. Hence the center \( Z = Z(K) \) of \( K \) is of finite index in \( K \) and evidently \( Z \triangleleft G \). Thus \( Z \) is an abelian normal subgroup of finite index, say \( m \), in \( G \). In particular, \( x^m \in Z \) for every \( x \in G \). Let \( l \) be the smallest natural number such that \( x^l \in Z \) for every \( x \in G \). Then \( G^l \leq Z \).

Observe that \( G^l \) is a subgroup of finite index in \( Z \). Otherwise, then \( Z/G^l \) is infinite and contains infinitely many proper non-trivial subgroups \( R_i/G^l, i = 1, 2, \cdots \). By assumption, there exists \( t \) such that \( R_t \) is a power subgroup of \( G \), so \( R_t = G^r \) for some \( r \neq l \). It is easy to get \( r < l \), but this contradicts the choice of \( l \).

Thus the index of \( G^l \) in \( Z \) is finite, so we may replace \( Z \) by \( G^l \) and assume that \( Z \) is an abelian power subgroup of finite index in \( G \), \( Z = G^l \).

Observe that, by Lemma 1, every power subgroup of \( G \), being contained in \( Z \), is a power subgroup of \( Z \) and every power subgroup of \( Z \) is a power subgroup of \( G \). In particular, \( Z \) contains only finitely many non-power subgroups, and hence \( Z \) is finite or an infinite cyclic group. If \( Z \) is finite, then \( G \) is finite and so the result is proved. Thus we suppose \( Z \) is an infinite cyclic group, that is, \( Z = < x > \) with \( |x| = \infty \). If \( C_G(Z) \neq G \), then there exists \( y \in G \) such that \( x^y = y^{-1}xy \neq x \in < x > \). Obviously \( Z^y = Z \). Since \( x \) and \( x^{-1} \) are the only generators of \( < x > \), we have \( x^y = x^{-1} \).

Let \( Y_i = < x^{-1}yx^i >, i = 1, 2, \cdots \). We claim there are infinite many different \( Y_i \). Suppose it is false. Let \( Y_{i_1}, \cdots, Y_{i_s} \) be all the different such type subgroups. Then there are infinite subgroups of this type equal to \( Y_{it}, 1 \leq t \leq s \). Without any loss, we suppose \( Y_{i_1} = Y_{m_1} = \cdots = Y_{m_r} = \cdots \). If \( |y| = \infty \), by the generator properties of infinite cyclic group we have \( y^{x^{m_1}} = (y^{x^{m_r}})^{\varepsilon_r}, r = 2, 3, \cdots \), where \( \varepsilon_r = 1 \) or \(-1 \). Hence there must exist \( r_0, r_1 \) such that...
$\varepsilon_{r_0} = \varepsilon_{r_1}$ and $r_0 \neq r_1$. And then $y^{\varepsilon_{r_0}} = y^{\varepsilon_{r_1}}$. Suppose $m_{r_0} < m_{r_1}$ without any loss. Then we have $y = y^{x^{m_{r_1} - m_{r_0}}}$. But since $x^y = x^{-1}$, we get $y^{x^{m_{r_1} - m_{r_0}}} = x^{-2(m_{r_1} - m_{r_0})}y$. Therefore $y = y^{x^{m_{r_1} - m_{r_0}}} = x^{-2(m_{r_1} - m_{r_0})}y$, and then $x^{-2(m_{r_1} - m_{r_0})} = 1$. We get contradiction for $|x| = \infty$. On the other hand if $|y| = n$. Similarly there are infinite subgroups such that $Y_{m_1} = \cdots = Y_{m_r} = \cdots$. By the generator properties of cyclic group we have $y^{x_{m_1}} = (y^{x_{m_r}})^{\varepsilon_r}, r = 2, 3, \cdots$, where $1 \leq \varepsilon_r \leq n$ is coprime to $n$. Hence there must exist $r_0, r_1$ such that $\varepsilon_{r_0} = \varepsilon_{r_1}$ and $r_0 \neq r_1$. And then $y^{x_{m_{r_0}}} = y^{x_{m_{r_1}}}$. We get a contradiction in the same way as before.

So there are infinite many different $Y_i$ which are conjugate with each other. In particular, these infinite many $Y_i$ are non-normal in $G$ and non-power subgroups of $G$, which is contrary to the assumption. Thus $C_G(Z) = G$ and hence $Z \leq Z(G)$. So the center of $G$ has a finite index in $G$. By Schur Theorem ([1] Theorem 10.1.3), the commutator subgroup $C = [G, G]$ is finite. If $C = 1$ then $G$ is abelian contrary to the assumption. So let $y$ be an element of prime order $p$ in $C$. Obviously $Z \cap < y >= 1$ and $< y > < Z, y >$ by $Z \leq Z(G)$, so $< Z, y >= Z \times < y >$ and all subgroups $Z^r \times < y >$ are distinct for $r = 1, 2, \cdots$. Thus there exists $r$ such that $A = Z^r \times < y >$ is a power subgroup of $G$. $A$ contains, by Lemma 1, only a finitely many non-power subgroups of $A$ and hence $A$ is finite or cyclic. But this is not true. The theorem is proved.

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