New solutions to the complex Ginzburg-Landau equations

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(Dated: July 03, 2022; revised August 26, 2022, accepted August 31, 2022)

The various régimes observed in the one-dimensional complex Ginzburg-Landau equation result from the interaction of a very small number of elementary patterns such as pulses, fronts, shocks, holes, sinks. We provide here three exact such patterns observed in numerical calculations but never found analytically. One is a quintic case localized homoclinic defect, observed by Popp et alii, the two others are bound states of two quintic dark solitons, observed by Afanasyev et alii.

Keywords: cubic and quintic complex Ginzburg-Landau equation, traveling waves, patterns, defect-mediated turbulence, dark solitons, closed-form solutions.

I. INTRODUCTION

Slowly varying amplitudes of numerous physical phenomena evolve in time according to the ubiquitous one-dimensional complex Ginzburg-Landau equation (CGL) equation

\[ iA_t + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0, \]

in which the constants \( p, q, r \) are complex and \( \gamma \) is real.

These phenomena include pattern formation, spatio-temporal intermittency, superconductivity, nonlinear optics, Bose-Einstein condensation, etc, see the reviews [1,2].

In the cubic case (\( r = 0 \)), it describes for instance the formation of patterns near a Hopf bifurcation (\( \gamma \) being the distance from criticality) to an oscillatory state.

The régimes are extremely rich and, depending on the parameters, they range from chaotic (turbulent) to regular (laminar), see the phase diagrams in the plane (Re(\( p \))/Im(\( p \)), Re(\( q \))/Im(\( q \))) [3, Fig. 1] [4, Fig. 1a]. The observed patterns have been classified [5, Fig. 1] according to both their homoclinic (equal values of \( \lim_{x \to -\infty} |A| \) and \( \lim_{x \to +\infty} |A| \)) or heteroclinic (unequal values) nature, and their topology: pulses, fronts, shocks, holes, sinks. For instance, the holes (characterized by the existence of a minimum of \( |A| \)) can be either heteroclinic (like the analytic solution of Bekki and Nozaki [6]), or homoclinic (as displayed by numerical simulations of van Hecke [4]). Of particular interest is the case when \( |A| \) can vanish; since the phase arg \( A \) is then undefined, it can undergo discontinuities, a feature which creates topological defects. This “defect-mediated turbulence” [3,8] is a major mechanism [9] of transition to a turbulent state, in addition to the mechanism of phase turbulence.

The situation is similar in the complex quintic case (CGL5, \( r/p \) not real) [5], and the importance of these coherent structures is their role of separators between different régimes, cf. [5, Figs. 5,6]. All these coherent structures are indeed observed both in physical phenomena and in numerical simulations [2,4].

In Taylor-Couette flows between rotating or counter-rotating cylinders [10], when a parameter varies, one first observes the expected Benjamin-Feir instability, followed by the occurrence of spatio-temporal defects (i.e. a vanishing of \( |A| \) which allows a discontinuity in the phase arg \( A \)) and a large variation of the amplitude. A similar behaviour is also observed in Rayleigh-Bénard convection (fluid between two conducting plates, heated from below) [11].
In nonlinear optics where $t$ is a coordinate along the fiber and $x$ a transverse coordinate, the goal is to carefully select the initial signal (for instance one of the coherent structures) and to tune the parameters of the CGL equation in order to minimize the attenuation during the propagation of this signal. Various kinds of bright or dark solitons, or more general “dissipative” solitons can be used for this purpose [12, 13, 14].

There also exist other optical devices described by a system [15], Eqs. (1)–(3)] involving a CGL-like equation for the electric field. In these broad area vertical cavity surface-emitting lasers (VCSELs) with a single longitudinal mode, two transverse coordinates (among them our $x$) are necessary to describe the broad area, and one observes for instance various patterns similar to those in spatiotemporal intermittency [16, 17].

From a more general point of view, there is a strong evidence [18] that coherent structures govern the transition between different scenarios of chaos.

Of special interest to our purpose is the fact that some of the observed patterns have been represented by exact, closed form, analytic expressions, such as the CGL5 heteroclinic hole 4, a traveling wave with arbitrary velocity. Conversely, it is reasonable to believe that every elementary observed pattern could be associated to some analytic solution, to be found, this is the motivation of the present work.

In this Letter, we consider the most challenging situation, i.e. the so-called “complex cases” in which $r/p$ is not real and, if $r$ vanishes, $q/p$ is not real, respectively denoted CGL5 and CGL3. This excludes the nonlinear Schrödinger (NLS) limit ($p, q$ real, $\gamma$ zero), which does not display any chaotic régime. Our interest is to look for exact traveling wave solutions ($c$ and $\omega$ real),

$$A = \sqrt{M(\xi)e^{i(\omega t + \varphi(\xi))}}, \quad \xi = x - ct.$$  

Such traveling waves are characterized by a third order nonlinear ordinary differential equation (ODE), see [3] hereafter.

The current list of exact traveling waves is very short, it comprises only six solutions [19]: for CGL3, a homoclinic pulse [20], a heteroclinic front [21], a heteroclinic source/hole [6, 22, Fig. 5]; for CGL5, a homoclinic pulse [23], a heteroclinic front [24], a homoclinic source/sink [23, 24]. Other elementary patterns exist, but they have only been observed experimentally, an important one being a CGL3 homoclinic hole [4, 25, Fig. 1b].

A lot of effort has been devoted to the search for exact solutions of CGL5/5, by essentially three methods, which we now outline. A slight modification of the method of Hirota allowed Bekki and Nozaki [21] to uncover two of the three above mentioned traveling waves of CGL3. By making heuristic assumptions among the components of a three-dimensional dynamical system equivalent to [3], van Saarloos and Hohenberg [5, §3.3] succeeded to obtain two remarkable solutions of CGL5: the pulse and the front solutions. In the third method, building on previous work [23], Marcu et alii [24] made an assumption for the complex amplitude $A$ matching the structure of singularities [27] and thus found the full CGL5 source/sink.

In the present work, by combining two mathematical methods, we present new traveling waves, outlined in [27], and moreover we prove that these are the only ones whose square modulus $M(\xi)$ admits only poles as singularities in the complex plane of $\xi$ (in short, is meromorphic on $\mathbb{C}$). Each such meromorphic solution is characterized by a first order nonlinear ODE for $M(\xi)$.

Only three of these new solutions are bounded, they are all homoclinic, decrease exponentially fast to a constant value at infinity and only exist for CGL5. One represents a topological defect, previously observed experimentally [28, Fig. 3b] [29, Fig. 4] for a set of parameters compatible with ours. The two other traveling waves are bound states made of two CGL5 dark solitons; while in CGL3 the presence of sources inhibits the formation of bound states of dark solitons [30], such bound states have been numerically observed in CGL5 by Afanasjev et alii [31, Fig. 4] and more recently in [31, Fig. 3d].

The three other solutions, outlined in [27] (one doubly periodic for CGL3, one doubly periodic and one rational for CGL5) are unbounded, but, under a small perturbation which moves the poles outside the real axis $x - ct$, their analytic expression becomes bounded, and the corresponding periodic patterns could be good approximations to a variety of periodic patterns observed experimentally, this is currently under investigation.

In the next section, we simply outline the mathematics involved. Then, for each bounded solution, the amplitude $A$ is presented as a product of complex powers of entire functions.

II. THE METHOD

It arises from a simple remark: in all presently known exact traveling waves (six, recalled above), the only singularities of the square modulus $|A|^2 = M$ in the complex plane of $\xi$ are poles. Conversely, let us make the only assumption that $M(\xi)$ is meromorphic on $\mathbb{C}$.

First, using Nevalinna theory [32], it has been proven [33] that, for all values of the CGL parameters $p, q, r$ (complex), $\gamma$ (real) and of the traveling waves parameters $c, \omega$ (real), for both CGL3 and CGL5, all solutions $M(\xi)$ meromorphic on $\mathbb{C}$ are elliptic or degenerate elliptic and therefore obey a nonlinear ODE of first order.

As a consequence, in order to find all meromorphic solutions of both CGL3 and CGL5, only a finite number of possibilities need to be examined. This is done by two methods. The first method (Hermite decomposition [34]) represents $M$ as a finite sum of derivatives of “simple elements” admitting only one pole of residue unity (Weierstrass’ $\zeta(\xi)$ [35] or its degeneracies $k \coth(k\xi)$ and $1/\xi$), while the second one (subequation method [36, 37]) builds the first order ODE obeyed by $M(\xi)$ then integrates it. Full details can be found in [27].
The modulus $M(\xi)$ obeys a third order ODE \[ (G' - 2k_i G)^2 - 4GM'^2 + d_i M - g_i r)^2 = 0, \]

\[ G = \frac{MM''}{2} - \frac{M'^2}{4} - \frac{k_i}{2} MM' + g_i M + d_i M^3 + e_i M^4, \]

\[ \varphi' = \frac{k_i}{2} + \frac{M'^2}{2M(g_r - d_i M - e_i M^2)}, \tag{3} \]

in which the real parameters $d_i, e_i, \kappa_i, \kappa_i, g_i, r_i$ are

\[ \frac{q}{p} = d_r + i d_i, \quad \frac{r}{p} = e_r + i e_i, \quad \frac{c}{p} = \kappa_r - i \kappa_i, \]

\[ \gamma = \frac{1}{2} \kappa_r, \quad \omega = \frac{1}{4} \kappa_i^2. \tag{4} \]

This ODE is the key to obtain all meromorphic solutions $M(\xi)$ and, by the quadrature \[ \frac{d}{d \xi} \int \frac{2}{2 sinh (k \xi) + 2 sinh (k \xi)}, \]

This new analytic pattern decreases exponentially fast at infinity, has the topology of a double pulse (Fig. 1) and is the first exact representation of a defect in CGL. The study of its stability under small perturbations has not been performed in this Letter, this will be investigated later.

Although defect-mediated turbulence is mainly observed in two-dimensional CGL3 where this is a major mechanism of turbulence \[ 30, 22, \] it has also been reported in CGL5 \[ 28, \] and Fig. 4 \[ 22, \] p. 278, where, for a destabilizing CGL5 term (negative $\delta$ in the notation of Ref. \[ 28, \] compatible with the present numerical values \[ 4, \] one observes a succession of phase slips (every time $\dot{M}$ vanishes), which create hole-shock collisions, ending in a process of as many annihilations as creations. Such a process is known as “hole-mediated turbulence”. We must also mention that a pattern topologically identical to the present defect has been numerically observed in a system of two amplitude equations coupled quadratically \[ 40, \] Fig. 2.

This exact pattern should be quite useful for determining the range of parameters for which topologically similar patterns are stable or metastable. Indeed, choosing as initial data an analytic expression similar to \[ 11, \] for
appropriate sets of parameters of CGL5 should considerably shorten the duration of the transient régime and accordingly increase the convergence towards a topologically similar pattern.

FIG. 1. CGL5. Homoclinic defect in dimensionless units \( M/(d_i/e_i) \) vs. \( k\xi/2 \).

B. CGL5, bound state of two dark solitons

When \( e_r/e_i \) is one of the four real roots of

\[
1089 - 81327\lambda^2 + 323512\lambda^4 + 456976\lambda^6 = 0,
\]

there also exists a four-pole subequation (see Appendix for its coefficients) without any free parameter. To each of these four values of \( \lambda = e_r/e_i \) correspond two values \( \alpha_1, \alpha_2 \) of the exponent \( \alpha \) defined in (5), whose product is \( \alpha_1\alpha_2 = -3/4 \),

\[
\lambda = \frac{e_r}{e_i} - \frac{\alpha}{2} = \frac{3}{8\alpha},
\]

\[
\lambda = \pm 0.1192, \alpha = (\mp 0.7550, \pm 0.9934),
\]

\[
\lambda = \pm 0.4300, \alpha = (\mp 0.5369, \pm 1.397).
\]

The derivation of \( M \) then follows exactly the same logic as for the defect solution. The complex amplitude \( A \) is the product of powers of six sinh functions

\[
A = e^{-i\omega t + i k\xi/2} \left( \sinh \frac{k(\xi - \xi_N)}{2} \sinh \frac{k(\xi + \xi_N)}{2} \right) \left( \sinh \frac{k(\xi - \xi_A)}{2} \sinh \frac{k(\xi + \xi_A)}{2} \right)^{-1/2 + i\alpha_1} \left( \sinh \frac{k(\xi - \xi_B)}{2} \sinh \frac{k(\xi + \xi_B)}{2} \right)^{-1/2 + i\alpha_2}.
\]

The zeroes \( (\pm \xi_A, \pm \xi_B) \) have their squares real and are the poles of \( M \), the other zeroes \( (\pm \xi_N) \) and their complex conjugates \( \pm \xi_N \) are the four zeroes of \( M \),

\[
M = M_0 + \frac{d_i}{e_i^2} \left( K_1 \coth^2 \frac{k\xi}{2} + K_2 (\coth^2 \frac{k\xi}{2} - 1) \right) \coth \frac{k\xi}{2} + D_1 \coth^2 \frac{k\xi}{2} + D_0.
\]

This defines two similar-looking (but different) homoclinic patterns in the shape of a double well, whose aspect ratios \( \min(M)/\lim_{\xi \to \pm \infty} M : \max(M) \) are \( (1:7.46:11.5) \) (Fig. 2) and \( (1:1.09:1.23) \) (Fig. 3).

Like for the defect pattern, these two patterns are stationary if \( p \) is not real and they move with an arbitrary velocity if \( p \) is real. They compare, at least qualitatively, quite well with the bound state of two CGL5 dark solitons, as reported apparently for the first time in [30, Fig. 4].

FIG. 2. CGL5. The homoclinic bound state in dimensionless units \( M/(d_i/e_i) \) vs. \( k\xi/2 \), for \( \lambda = \pm 0.4300 \), with aspect ratio \( (1:7.46:11.5) \).

IV. CONCLUSION

On the numerical side, these exact patterns can be used as building blocks to study the interaction of defects and dark solitons with various other patterns. On the analytic side, more exact traveling waves (with less constraints on \( p, q, r, \gamma, c, \omega \)) would necessarily be non-meromorphic, this question will be addressed in future work.

V. ACKNOWLEDGMENTS

The authors warmly thank Joceline Lega for sharing her expertise. The first two authors thank CIRM, Marseille (grant 2311) and IHES, Bures-sur-Yvette for their
The homoclinic bound state in dimensionless units $M/(d_i/e_i)$ vs $k_i/2$, for $\lambda = \pm 0.1192$, with aspect ratio (1:1.09:1.23).

\begin{align}
    d_r &= d_i \lambda \frac{828038745921 + 7649070764998 \lambda^2 + 9025535790856 \lambda^4}{138644084775}, \\
    g_r &= g_i = \frac{119473478956925997 - 1651180178784048644 \lambda^2 + 1243896610551884848 \lambda^4}{178728931719095040}, \\
    g_i &= g_i = \frac{119473478956925997 - 1651180178784048644 \lambda^2 + 1243896610551884848 \lambda^4}{178728931719095040}, \\
    k^2 &= k_i = \frac{470534592586268997 + 16800138410952093932 \lambda^4 + 15744055491100758536 \lambda^4}{2010700481839819200}, \\
    M_0 &= \frac{\alpha_i}{\alpha_i} - 344373082347 + 2958053216864 \lambda^2 + 3382994698928 \lambda^4, \\
    \coth_A \coth_B &= \frac{2i \sqrt{3} \lambda}{6800175} - 493029007920 - 41722436 \lambda^2 - 37472032 \lambda^4,
\end{align}

but $\coth_A = \coth(\xi_A/2)$, $\coth_B = \coth(\xi_B/2)$ and the four parameters $K_1, K_2, D_0, D_1$ of (15) are algebraic functions of $\lambda$. They are obtained by equating the rational functional (15) and the Hermite decomposition (sum of four simple poles),

\begin{align}
    K_1 &= -\frac{ke_i}{d_i}(a_2 \coth_A + b_2 \coth_B), \\
    K_2 &= \frac{ke_i}{d_i}(a_2/\coth_A + b_2/\coth_B)(\coth_A \coth_B)^2, \\
    D_0 &= \coth_A^2 \coth_B^2, \\
    D_1 &= -(\coth_A^2 + \coth_B^2).
\end{align}

All numerical values characterizing the two bound state solutions are listed in Table I.[1] [2] [3]

APPENDIX. Details on CGL5 two homoclinic bound states

The subequation has the structure

\begin{equation}
    \left( M^2 + c_6 P_{2a}(M) P_{2b}(M) \right)^2 - c_7 P_1(M)^2 P_{2a}(M)^3 = 0, \\
    \kappa_i = 0, P_1(M) = M + c_1, \\
    P_{2a}(M) = M^2 + c_2 M + c_3, P_{2b}(M) = M^2 + c_4 M + c_5,
\end{equation}

with $P_1, P_{2a}, P_{2b}$ polynomials and $c_j$ constants.

Let us choose $e_i$ and $d_i$ as scaling parameters. The fixed parameters, as well as three movable constants, are polynomials of $e_i/e_i$ (denoted $\lambda$),

[1] I.S. Aranson and L. Kramer, The world of the complex Ginzburg-Landau equation, Rev. Math. Phys. 74 (2002) 99–143. https://doi.org/10.1142/1922. http://arXiv.org/abs/cond-mat/0106115

[2] W. van Saarloos, Front propagation into unstable states, Physics reports 386 29–222 (2003). https://doi.org/10.1016/j.physrep.2003.08.001

[3] H. Chaté, Spatiotemporal intermittency regimes of the one-dimensional complex Ginzburg-Landau equation, Nonlinearity 7 (1994) 185–204. https://doi.org/10.1088/0951-7715/7/1/007

[4] M. van Hecke, Building blocks of spatiotemporal intermittency, Phys. Rev. Lett. 80 (1998) 1896–1899. https://doi.org/10.1103/PhysRevLett.80.1896

[5] W. van Saarloos and P.C. Hohenberg, Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations. The third author was partially supported by the RGC grant 17307420. The last author was supported by the National Natural Science Foundation of China (grant 11701382).
Landau equations, Physica D 56 (1992) 303–367. Erratum 69 (1993) 209. https://doi.org/10.1016/0167-2789(92)0175-M

[6] N. Bekki and K. Nozaki, Formations of spatial patterns and holes in the generalized Ginzburg-Landau equation, Phys. Lett. A 110 (1985) 133–135. https://doi.org/10.1016/0375-9601(85)90759-5

[7] P. Coullet, L. Gil, and J. Lega, Defect-mediated turbulence, Phys. Rev. Lett. 62 (1989) 1619–1622. https://doi.org/10.1103/PhysRevLett.62.1619

[8] M. van Hecke and Martin Howard, Ordered and self-disordered dynamics of holes and defects in the one-dimensional complex Ginzburg-Landau equation, Phys. Rev. Lett. 86 (2001) 2018–2021. 10.1103/PhysRevLett.86.2018 https://arxiv.org/abs/niln/0002031

[9] B.I. Shraiman, A. Pumir, W. van Saarloos, P.C. Hohenberg, H. Chaté, M. Holen, Spatiotemporal chaos in the one-dimensional complex Ginzburg-Landau equation, Physica D 57:3–4 (1992) 241–248. https://doi.org/10.1016/0167-2789(92)90004-4

[10] N. Latrache, N. Abcha, O. Crumeyrolle, I. Mutabazi, Defect-mediated turbulence in ribbons of viscoelastic Taylor-Couette flow, Physical Review E 93 (2016) 043126 (10pp). https://doi.org/10.1103/PhysRevE.93.043126

[11] P. Manneville, Dissipative structures and weak turbulence (Academic Press, Boston, 1990). ISBN 978-0-08-092445-8. French adaptation: Structures dissipatives, chaos et turbulence (Aléa-Saclay, Gif-sur-Yvette, 1991).

[12] Y.S. Kivshar and G.P. Agrawal, Optical solitons: from fibers to photonics crystals (Academic Press, San Diego, 2003). ISBN 9780080538099

[13] J.V. Moloney and A.C. Newell, Nonlinear optics (CRC press, Boca Raton, 2004). https://doi.org/10.1201/9780429052842

[14] N. Akhmediev and A. Ankiewicz (eds.), Dissipative solitons: from optics to biology and medicine, Lecture notes in physics 751 (Springer, NY, 2008). https://doi.org/10.1007/978-3-540-78217-9

[15] X. Hachair, F. Pedaci, E. Caboche, S. Barland, M. Giuliani, J.R. Tredicce, F. Prati, G. Tissoni, R. Kheradmand, L.A. Lugliato, I. Protosenko and M. Brambilla, Cavity solitons in a driven VCSEL above threshold, IEEE Journal of selected topics in quantum electronics 12 (2006) 339–351. https://doi.org/10.1109/JSTQE.2006.872711

[16] S. Coulbaly, M.G. Clerc, F. Selmi and S. Barbay, Extreme events following bifurcation to spatiotemporal chaos in a spatially extended microwavely laser, Phys. Rev. A 95 (2017) 023816 (12pp). https://doi.org/10.1103/PhysRevA.95.023816

[17] C. Rimoldi, M. Eslami, F. Prati and G. Tissoni, Extreme events in a broad-area semiconductor laser with coherent injection, Phys. Rev. A 105 (2022) 023525 (9pp). https://doi.org/10.1103/PhysRevA.105.023525

[18] Lutz Brusch, Martin G. Zimmermann, Martin van Hecke, Markus Baer, Alessandro Torcini, Modulated amplitude waves and the transition from phase to defect chaos, Phys. Rev. Lett. 85 (2000) 86–89. https://doi.org/10.1103/PhysRevLett.85.86 https://arxiv.org/abs/niln/0001068

[19] R. Conte and M. Musette, Solitary waves of nonlinear nonintegrable equations, Dissipative solitons, eds. N. Akhmediev and A. Ankiewicz, Lecture notes in physics 661 (2005) 375–406. https://doi.org/10.1007/978-3-540-78208-15 http://arXiv.org/abs/niln/0407026

[20] L.M. Hocking and K. Stewartson, On the nonlinear response of a marginally unstable plane parallel flow to a two-dimensional disturbance, Proc. Roy. Soc. London A 326 (1972) 289–313. https://doi.org/10.1098/rspa.1972.0010

[21] K. Nozaki and N. Bekki, Exact solutions of the generalized Ginzburg-Landau equation, J. Phys. Soc. Japan 53 (1984) 1581–1582. https://doi.org/10.1143/JPSJ.53.1581

[22] J. Lega, Traveling hole solutions of the complex Ginzburg-Landau equation: a review, Physica D 152–153 (2001) 269–287. https://doi.org/10.1016/S0167-2789(01)00174-9

[23] J.D. Moores, On the Ginzburg-Landau laser mode-locking model with fifth-order saturable absorber term, Optics communications 96 (1993) 65–70. https://doi.org/10.1016/0030-4018(93)90524-9

[24] P. Marcq, H. Chaté and R. Conte, Exact solutions of the one-dimensional quintic complex Ginzburg-Landau equation, Physica D 73 (1994) 305–317. https://doi.org/10.1016/0167-2789(94)90102-3 http://arXiv.org/abs/patt-sol/9310004

[25] Yusuke Uchiyama, Hidetoshi Kanno, Velocity fluctuation of hole in 1D defect turbulence, 2014 (2014) 356–360. Proceedings of the 45th ISCIE international symposium on stochastic systems theory and its applications, Okinawa, Nov 1–2, 2013. https://doi.org/10.5687/sss.2014.356

[26] R. Conte and M. Musette, Linearity inside nonlinearity: exact solutions to the complex Ginzburg-Landau equation, Physica D 69 (1993) 1–17. https://doi.org/10.1016/0167-2789(93)90177-3

[27] R. Conte and M. Musette, The Painlevé handbook, Mathematical physics studies, xxxi+389 pages (Springer Nature, Switzerland, 2020). https://doi.org/10.1007/978-3-030-53340-3

[28] S. Popp, O. Stiller, I. Aranson, A. Weber and
L. Kramer, Localized hole solutions and spatiotemporal chaos in the 1D complex Ginzburg-Landau equation, Phys. Rev. Letters **70**:25 (1993) 3880–3883. https://doi.org/10.1103/PhysRevLett.70.3880

[29] S. Popp, O. Stiller, I. Aranson and L. Kramer, Hole solutions in the 1d complex Ginzburg-Landau equation, Physica D **84** (1995) 398–423. https://doi.org/10.1016/0167-2789(95)00070-K

[30] V.V. Afanasjev, P.L. Chu and B.A. Malomed, Bound states of dark solitons in the quintic Ginzburg-Landau equation, Phys. Rev. E **57** (1998) 1088–1091. https://doi.org/10.1103/PhysRevE.57.1088

[31] Marco A. Viscarra and Deterlino Urzagasti, Dark soliton solutions of the cubic-quintic complex Ginzburg-Landau equation with high-order terms and potential barriers in normal-dispersion fiber lasers, Journal of nonlinear optical physics and materials **31**:1 (2022) 2250003 (17 pp). https://doi.org/10.1142/S0218863522500035

[32] I. Laine, Nevanlinna theory and complex differential equations, (de Gruyter, Berlin and New York, 1992). https://doi.org/10.1515/9783110134223

[33] R. Conte and T.W. Ng, Meromorphic traveling wave solutions of the complex cubic-quintic Ginzburg-Landau equation, Acta applicandae mathematicae **122** 153–166 (2012). https://doi.org/10.1007/s10440-012-9734-y [Corrigenda: change d, to ad, e, to ad, in (15), (16).] http://arxiv.org/abs/1204.3032

[34] C. Hermite, Remarques sur la décomposition en éléments simples des fonctions doublement périodiques, Annales de la faculté des sciences de Toulouse II (1888) C1–C12. Oeuvres d’Hermite, vol IV, pp 262–273. http://www.numdam.org/item/AFST_1888_1_2__C1_0/

[35] M. Abramowitz, I. Stegun, Handbook of mathematical functions, Tenth printing (Dover, New York, 1972). ISBN 978-0486612720

[36] M. Musette and R. Conte, Analytic solitary waves of nonintegrable equations, Physica D **181** (2003) 70–79. [http://dx.doi.org/10.1016/S0167-2789(03)00069-1] http://arXiv.org/abs/nlin.PS/0302051

[37] R. Conte and M. Musette, Elliptic general analytic solutions, Studies in applied mathematics **123** (2009) 63–81. https://doi.org/10.1111/j.1467-9590.2009.00447.x http://arxiv.org/abs/0903.2009

[38] F. Cariello and M. Tabor, Painlevé expansions for nonintegrable evolution equations, Physica D **39** (1989) 77–94. https://doi.org/10.1016/0167-2789(89)90040-7

[39] J. Lega, Défauts topologiques associés à la brisure de l’invariance de translation dans le temps, Thèse (Université de Nice, 28 mars 1989).

[40] S. Darmanyan, L. Crasovan and F. Lederer, Double-hump solitary waves in quadratically nonlinear media without loss and gain, Phys. Rev. E **61** (2000) 3267–3269. https://doi.org/10.1103/PhysRevE.61.3267