Excited D-brane decay in Cubic String Field Theory 
and in Bosonic String Theory

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ABSTRACT

In the cubic string field theory, using the gauge invariant operators corresponding to the on-shell closed string vertex operators, we have explicitly evaluated the decay amplitudes of two open string tachyons or gauge fields to one closed string tachyon or graviton up to level two. We then evaluated the same amplitudes in the bosonic string theory, and shown that the amplitudes in both theories have exactly the same pole structure. We have also expanded the decay amplitudes in the bosonic string theory around the Mandelstam variable $s=0$, and shown that their leading contact terms are fully consistent with a tachyonic Dirac-Born-Infeld action which includes both open string and closed string tachyon.

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1 Introduction

Open string field theory, containing in an elegant way a kinetic term and one cubic interaction for string field, is invariant under a non-trivial gauge transformation\([1]\). In terms of components of string field, this action includes kinetic terms for open string tachyon, for massless gauge field and for infinite number of massive fields, and cubic interaction among them. Being non-perturbative and having off-shell tachyon, it is believed that this theory might provide a direct approach to study the physics of unstable D-brane\([2]\), in particular, appearance of closed string fields at the stable point of the tachyon potential.

Similar to the ordinary gauge theory one can, in principle, evaluate, for instance, the tree level S-matrix elements of external massless fields. However, in contrast to the ordinary gauge theory, one finds a tower of massive poles in the amplitude indicating the coupling between massless and massive fields in the cubic string theory\([1]\). Even more, one loop calculation shows that off-shell closed string states appear in the S-matrix elements\([3]\). Unitarity then implies that the closed string states must also appear as on-shell states in the theory. Recently, such states introduced to the theory as some sort of gauge invariant operators which include on-shell closed string vertex operators\([4, 5]\).

It has been suggested in \([4]\) that the correlation function of these gauge invariant operators could be interpreted as the on-shell S-matrix elements of their corresponding closed string states. Using the truncated method, the tree level correlation function of two such operators corresponding to closed string tachyon or graviton has been explicitly evaluated in \([3]\) up to level two. It has been shown in that paper that up to some contact terms they are exactly the same as S-matrix elements of two closed string tachyons or gravitons in perturbative bosonic string theory. An interesting observation in \([3]\) is that the tree level correlation of two gauge invariant operators may contain off-shell closed string states upon adding back the infinite tower of truncated open string states.

In the present paper we would like to extend the calculation in \([3]\) to the case of correlation functions of one gauge invariant operator corresponding to closed string tachyon or graviton, and two on-shell open string tachyons or massless states. To compare with the corresponding S-matrix elements in bosonic string theory, we also evaluate, explicitly, the S-matrix elements of one closed string tachyon or graviton and two open string tachyons or massless vertex operators. Expanding the latter amplitudes in terms of tachyon, massless and infinite number of massive poles, and keeping only poles up to level two, we find exact agreement between the amplitudes in the two theories up to some contact terms. We expect that the discrepancy in the contact terms of the amplitudes in the two theories to be related to the fact that in string field theory side we have only level truncated results. Adding all contact terms resulting from all poles of amplitude in string field theory, one should find all contact terms of the corresponding amplitude in the bosonic string theory side as well.

The decay amplitudes in bosonic string theory have all infinite number of poles in a single beta function with one unconstrained Mandelstam variable \(s = \alpha'(k_1 + k_2)^2/2\) where \(k\)'s are external open string momenta. When open string states are massless, the low energy limit\((\alpha' \to 0)\) is \(s \to 0\). We shall show that the leading contact terms of the decay
amplitude in this limit are reproduced exactly by DBI action, as expected. This calculation shows that the open string tachyon potential does not have linear term. When the open string states are tachyon, at the top of tachyon potential one should not in general take the low energy limit of the amplitude. Indeed, similar amplitude in the superstring theory has been analysed in [8] where it has been shown that the leading contact terms of the decay amplitude at low energy limit \( s \to 1/2 \) (\( \alpha' \to 0 \)) are reproduced exactly by BSFT effective action while in the limit \( s \to 0 \) (at the top of the tachyon potential) they are reproduced exactly by tachyonic DBI action. We shall show in the present paper that the leading contact terms of the decay amplitude in bosonic string theory in the limit \( s \to 0 \) (not \( \alpha' \to 0 \) when open string states are tachyon) are reproduced exactly by tachyonic DBI action [9, 10, 8].

An outline of the paper is as follows. In the next section, using the string field truncated up to level two, we evaluate the decay amplitudes of two open string tachyons or gauge fields to one closed string tachyon or graviton in the cubic string field theory. In section 3, we evaluate the same amplitudes in the bosonic string theory. All infinite tower of massive states appears as off-shell poles of the beta function in these amplitudes. Using an expansion for the beta function, and keeping only poles up to level two, we show that up to some contact terms the amputated amplitudes are exactly the same as the amplitudes evaluated in the cubic string field theory. In section 4, we expand the exact form of the amplitudes in the bosonic string theory at \( s \to 0 \) and show that their leading contact terms are exactly reproduced by the tachyonic DBI action. The last section is devoted to a short discussion.

## 2 Cubic String Field Theory

The cubic open string field theory action is given by [1]

\[
S(\Psi) = -\frac{1}{2\alpha'} \int \Psi \star Q \Psi - \frac{g_0}{3\alpha'} \int \Psi \star \Psi \star \Psi ,
\]

(1)

where \( g_0 \) is the open string coupling, \( Q \) is the BRST charge with ghost number one, and the string field, \( \Psi \), is a ghost number one state in the Hilbert space of the first-quantized string theory. This field can be expanded using the Fock space basis as [2]

\[
|\Psi\rangle = \int d^{p+1}k \ (\phi + A_\mu \alpha^{-1}_\mu + i\alpha b_{-1}c_0 + \frac{i}{\sqrt{2}} B_\mu \alpha^{-1}_\mu + \frac{1}{\sqrt{2}} B_{\mu\nu} \alpha^{-1}_\mu \alpha^{-1}_\nu + \beta_0 b_{-2}c_0 + \beta b_{-1}c_0 + i k_\mu \alpha^{-1}_\mu b_1 c_0 + \cdots) c_1 |k\rangle .
\]

\(^1\)Note that our convention for \( s \) here is minus the \( s \) in [8].

\(^2\)Here, we use the convention fixed in [11] that uses the V and N matrices for projecting a space-time field to its component in the world-volume and transverse spaces, respectively. So in this convention \( \mu, \nu = 0, 1, 2, ..., 25 \), and \( A_\mu \alpha^{-1}_\mu = A \cdot V \cdot \alpha^{-1} + A \cdot N \cdot \alpha^{-1} \). Our conventions also set \( \alpha' = 2 \).
The SFT action (1) is invariant under the gauge transformation, $\delta \Psi = Q \Lambda + g_c \Psi \Lambda - g_c \Lambda \Psi$. By choosing so-called Feynman-Siegel gauge $b_\gamma |\Psi\rangle = 0$ we will carry out gauge-fixing. In this gauge the truncated field up to level two reads

$$|\Psi\rangle = \int d^{p+1}k \left( \phi(k) + A_\mu(k) \alpha^\mu_{-1} + \frac{i}{\sqrt{2}} B_\mu(k) \alpha^\mu_{-2} + \frac{1}{\sqrt{2}} B_{\mu\nu}(k) \alpha^\mu_{-1} \alpha^\nu_{-1} + \beta(k) b_{-1} c_{-1} \right) c_{1} |k\rangle .$$

The corresponding string vertex is given by

$$\Psi(0) = \int d^{p+1}k \left[ \phi(k) c(0) + i A_\mu(k) c \partial X^\mu(0) - \frac{1}{\sqrt{2}} B_\mu(k) c \partial^2 X^\mu(0) - \frac{1}{2} \beta(k) \partial^2 c(0) \right] e^{2ik \cdot X(0)} .$$

(2)

In writing the above vertex, we have used the doubling trick [11]. Hence, the world-sheet field $X^\mu(z)$ in above equation is only holomorphic part of $X^\mu(z, \bar{z})$.

The gauge invariant operators in string field theory have been constructed in [4, 5]. The general form of these operators are given by

$$O = g_c \int V \Psi,$$

where $g_c$ is the closed string coupling and $V$ is an on-shell closed string vertex operator with ghost number two. In order to be gauge invariant, the closed string vertex operator has to be inserted at the midpoint of open string.

To make sense out of the abstract form of the open string field theory action, one can use CFT method. In this method the $\star$-product between string fields transforms to the correlation function of string vertexes on a disk or upper-half plane [12, 13]. In the CFT language the action (1) and the gauge invariant operator (3) are given by

$$S = -\frac{1}{4} \left\{ f_{2}^{(2)} \circ \Psi(0) f_{1}^{(2)} \circ (Q \Psi(0)) + \frac{2g_c}{3} f_{1}^{(3)} \circ \Psi(0) f_{2}^{(3)} \circ \Psi(0) f_{3}^{(3)} \circ \Psi(0) \right\},$$

$$O = g_c \left\{ V(i) \bar{V}(-i) f_{1}^{(1)} \circ \Psi(0) \right\} ,$$

(3)

where $f_{k}^{(n)} \circ \Psi(0)$ denotes the conformal transformation of the vertex operator $\Psi(0)$ by the conformal map $f_{k}^{(n)}$. In above equations, $V(\bar{V})$ denotes the holomorphic(antiholomorphic) part of the closed string vertex operators.\footnote{We assume that there is a normal order sign between fields at different points in the correlation functions.\footnote{Note that when vertexes are mapped on the boundary of a unit complex disk the midpoint of strings are mapped to center of disk. The conformal map that transforms the disk to upper-half plane maps the center of disk to $i$.} 3\footnote{We assume that there is a normal order sign between fields at different points in the correlation functions.} 4\footnote{Note that when vertexes are mapped on the boundary of a unit complex disk the midpoint of strings are mapped to center of disk. The conformal map that transforms the disk to upper-half plane maps the center of disk to $i$.}
half plane and the conformal map \( f_k^{(n)} \) is

\[
f_k^{(n)}(z_k) = g \left( e^{\frac{2\pi i (k-2)}{n}} \left( \frac{1 + iz_k}{1 - iz_k} \right)^\frac{2}{n} \right), \quad 1 \leq k \leq n,
\]

where \( g(\zeta) = -i\frac{\zeta - 1}{\zeta + 1} \). The three point interaction in (3) has twist symmetry [14] which vanishes the interaction of three states that their level numbers add up to an odd number.

2.1 Decay amplitude in Cubic String Field Theory

S-matrix elements of open and closed string states can be evaluated in the cubic string field theory by evaluating explicitly the correlators in equation (3) and using the standard Feynman rules. In performing these correlators one should find the conformal transformation of the string vertex (2). These have been done in [6] for vertex up to level two. Then one ends up with some elementary correlators that can be done using the Wick theorem and the world sheet propagator

\[
\langle X_\mu(z) X_\nu(w) \rangle = -\eta_{\mu\nu} \ln(z - w).
\]

In this way one finds momentum space propagators for tachyon, massless and massive states and their three points vertexes. Evaluation of the kinetic term in (3) gives the propagators of fields which for states, up to level two, are (see for example [15]),

\[
\begin{align*}
\tilde{G}_T &= \frac{-i}{k^2 - \frac{1}{2}}, \\
\tilde{G}_A &= \frac{i}{k^2 + \frac{1}{2}}, \\
(\tilde{G}_B^{(1)})^{\mu\nu} &= \frac{-i\eta^{\mu\nu} + \eta_{\mu\rho}\eta_{\nu\lambda}}{k^2 + \frac{1}{2}}, \\
(\tilde{G}_B^{(2)})^{\mu\nu\lambda\rho} &= -\frac{i}{2} \frac{\eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda}}{k^2 + \frac{1}{2}}.
\end{align*}
\] (4)

Evaluation of the correlator in the gauge invariant operator \( \mathcal{O} \) for off-shell open string states up to level two has been done in [3],

\[
\mathcal{O}_\tau(p) = \frac{ig_c}{8} e^{4\ln(2)p \cdot V \cdot p} \left( T + 4i(p \cdot N \cdot A) + 2i\sqrt{2}(p \cdot V \cdot B^{(1)}) - 8\sqrt{2} \left[ (p \cdot N \cdot B^{(2)} \cdot N \cdot p) - \frac{1}{16} \text{Tr}(B^{(2)}) \right] - \beta \right),
\]

\[
\mathcal{O}_\h(p, \varepsilon) = \frac{ig_c}{8} e^{4\ln(2)p \cdot V \cdot p} \left( T a + 4iA_{\mu}b^{\mu} + 2i\sqrt{2}B_{\mu\nu}c^{\mu} - 8\sqrt{2}B_{\mu\nu}d^{\mu\nu} - \beta a \right),
\]

where \( \tau \) and \( h \) stand for on-shell closed string tachyon and graviton, respectively. The tachyon \( \tau \) has on-shell condition \( p_{\mu}p^{\mu} = 4/\alpha' = 2 \), and the graviton with polarization \( \varepsilon^{\mu\nu} = \varepsilon_{\nu\mu} \) has the on-shell conditions \( p_{\mu}\varepsilon^{\mu\nu} = 0 = p_{\mu}b^{\mu} \). We have also dropped in above and subsequent amplitudes the conservation of momentum factor, e.g., \((2\pi)^{p+1}\delta(p \cdot V + k)\) in above equations. The factors \( a, b^{\mu}, c^{\mu}, d^{\mu\nu} \) are

\[
a = \text{Tr}(\varepsilon \cdot D) - p \cdot D \cdot \varepsilon \cdot D \cdot p.
\]

4
\[
\begin{align*}
    b^\mu &= a p \cdot N^\mu + p \cdot D \cdot \varepsilon \cdot D^\mu - \varepsilon^\mu \cdot D \cdot p , \\
    c^\mu &= a p \cdot V^\mu - 4 p \cdot D \cdot \varepsilon \cdot D^\mu - 4 \varepsilon^\mu \cdot D \cdot k , \\
    d^{\mu \nu} &= a(p \cdot N^\mu p \cdot N^\nu - \frac{1}{16} \eta^{\mu \nu}) + 2(\varepsilon \cdot D)^{\mu \nu} + 2 p \cdot D \cdot \varepsilon \cdot D(p \cdot N^\nu - 2 \varepsilon(\mu \cdot D \cdot p \cdot N^\nu).
\end{align*}
\]

Now using the results in (3), one can easily read the vertex function for on-shell closed string and off-shell open string states up to level two. They are

\[
\begin{align*}
    \bar{V}_{\tau_T} &= \frac{ig_c}{8} 2^{4p} V^p \quad ; \quad (\bar{V}_{\tau_A})^\mu = \frac{ig_c}{8} (4i)(p \cdot N)^\mu 2^{4p} V^p \\
    (\bar{V}_{\tau_B^{(1)})}^{\mu \nu} &= \frac{ig_c}{8} (2i\sqrt{2})(p \cdot V)^{\mu \nu} 2^{4p} V^p \quad ; \quad \bar{V}_{\tau_B}(k) = \frac{ig_c}{8} (-1) 2^{4p} V^p \\
    (\bar{V}_{\tau_B^{(2)})}^{\mu \nu} &= \frac{ig_c}{8} (-8\sqrt{2}) (p \cdot N)^\mu (p \cdot N)^\nu - \frac{1}{16} \eta^{\mu \nu} 2^{4p} V^p \quad ; \quad \bar{V}_{\tau_T} = \frac{ig_c}{8} (a) 2^{4p} V^p \\
    (\bar{V}_{\tau_B^{(2)})}^{\mu \nu} &= \frac{ig_c}{8} (-8\sqrt{2}) (d^{\mu \nu}) 2^{4p} V^p \quad ; \quad \bar{V}_{\tau_B} = \frac{ig_c}{8} (-a) 2^{4p} V^p
\end{align*}
\]

Using above vertexes and propagators (4), one can evaluate truncated S-matrix elements of any closed string tachyon or graviton. For example scattering amplitude of two tachyons or two gravitons from D-brane are

\[
\begin{align*}
    A(\tau_1, \tau_2) &= \sum_\phi \bar{V}_{\tau_1 \phi} \bar{G}_\phi \bar{V}_{\phi \tau_2}, \\
    A(h_1, h_2) &= \sum_\phi \bar{V}_{h_1 \phi} \bar{G}_\phi \bar{V}_{\phi h_2},
\end{align*}
\]

where the summation is over off-shell fields \( \phi \in \{ T, A, B^{(1)}, B^{(2)}, \beta, \ldots \} \) where dots represents the off-shell open starting states belonging to the levels more than two where we are not considering in our calculations. In [6], it was shown that the above scattering amplitudes produce exactly the same pole structure as the amplitudes in the bosonic string theory.

Now to evaluate decay amplitude of excited D-branes, i.e., S-matrix elements of open and closed string states, one has to find the vertex function for three open string states as well. These can be done by performing the correlator in the second terms of action (3). Since we are interested in the tree level amplitudes for decaying two on-shell open string states to one closed string state, the vertex functions need to have only one off-shell leg.

### 2.1.1 Two tachyons decay

The vertex functions with two on-shell tachyons and one off-shell open string states up to level two are

\[
\bar{V}_{T_1 T_2 T} = \frac{ig_c}{96 \gamma} (27)^4 \gamma^{4(1+k_1 \cdot k_2)}.
\]
\[
(\tilde{V}_{T_1T_2B(1)})^\mu = \frac{ig_0}{96\gamma}(6i\sqrt{2})(k_1^\mu + k_2^\mu)\gamma^{4(1+k_1-k_2)},
\]
\[
(\tilde{V}_{T_1T_2B(2)})^{\mu\nu} = \frac{ig_0}{96\gamma}(8\sqrt{2}) \left[ (k_2^\mu - k_1^\mu)(k_2^\nu - k_1^\nu) - \frac{5}{16}g_{\mu\nu} \right] \gamma^{4(1+k_1-k_2)},
\]
\[
\tilde{V}_{T_1T_2\beta} = \frac{ig_0}{96\gamma}(-11)\gamma^{4(1+k_1-k_2)},
\]
and \((\tilde{V}_{T_1T_2A})^\mu = 0\). The vanishing of \((\tilde{V}_{T_1T_2A})^\mu\) can also be understood from twist symmetry \([14]\). According to this symmetry the vertex function of three open string states that their level numbers add up to an odd number is zero, e.g., level number of tachyon is zero and of gauge field is one, hence \((\tilde{V}_{T_1T_2A})^\mu = 0\). In above equation \(\gamma = 4/(3\sqrt{3})\) and we have used the on-shell condition \(k_1^2 = 1/\alpha' = 1/2 = k_2^2\) for two tachyons. Now with above vertexes, the vertex functions in (7) and propagators (4), one can evaluate the tree level decay amplitude of two tachyons to one graviton becomes

\[
A(T_1, T_2, \tau_3) = \sum \tilde{V}_{T_1T_2\phi}\tilde{G}_\phi\tilde{V}_{\phi\tau_3}
\]
\[
= \frac{ig_0g_c}{6} \left\{ e^{(2s-1)\ln(4\gamma)} \frac{2s-1}{2s+1} + e^{(2s+1)\ln(4\gamma)} \left[ -\frac{3}{32}(s + \frac{1}{2}) + \frac{1}{8} \right] + \cdots \right\},
\]

where \(s = p_3\cdot V\cdot p_3\) and dots represent poles of more massive fields. In the above expression, those terms in each pole that have a factor which is a positive power of its denominator give only contact terms. Structure of these contact terms highly affected by individual contact terms of more massive poles, whereas, pole structure in each level is independent of poles of other levels. To compare level truncated scattering amplitudes in string field theory and in bosonic string theory, we keep tract of only pole structure of amplitudes, that is

\[
A(T_1, T_2, \tau_3) = \frac{ig_0g_c}{6} \left\{ \frac{1}{2s-1} + \frac{1/8}{2s+1} + \cdots \right\},
\]

where dots represent some contact terms as well more massive poles. Similarly, the decay amplitude of two tachyons to one graviton becomes

\[
A(T_1, T_2, h_3) = \sum \tilde{V}_{T_1T_2\phi}\tilde{G}_\phi\tilde{V}_{\phi h_3}
\]
\[
= \frac{ig_0g_c}{6} \left\{ e^{(2s-1)\ln(4\gamma)} \frac{2s-1}{2s+1} a + e^{(2s+1)\ln(4\gamma)} \left( -\frac{3}{32}(s + \frac{1}{2})a \right.ight.
\]
\[
+ \frac{1}{8} \left[ \text{Tr}(\varepsilon_3\cdot D) + p_3\cdot D\cdot \varepsilon_3\cdot D\cdot p_3 - 8(k_1 - k_2)\cdot \varepsilon_3\cdot (k_1 - k_2) \right] \bigg\} + \cdots \bigg\},
\]
\[
= \frac{ig_0g_c}{6} \left\{ \frac{a}{2s-1} \right. 
\]
\[
+ \frac{1}{8} \left[ \text{Tr}(\varepsilon_3\cdot D) + p_3\cdot D\cdot \varepsilon_3\cdot D\cdot p_3 - 8(k_1 - k_2)\cdot \varepsilon_3\cdot (k_1 - k_2) \right] \bigg\} + \cdots \bigg\},
\]
where $a$ is given in (6), and in the last equality we have again dropped some contact terms. We shall compare these amplitudes with the corresponding decay amplitudes in the bosonic string theory in section 3.

2.1.2 One tachyon and one gauge field decay

The only non-zero vertex function, allowed by the twist symmetry, for one on-shell tachyon, one on-shell gauge field and one off-shell state up to level two is

\begin{equation}
(\tilde{V}_{T_1A_2})^\mu = \frac{ig_\phi}{6\gamma}(-16)[(k_2^\mu - k_1^\mu)(-2k_1\cdot \zeta_2) - \zeta_2^\mu] \gamma^{4(1/2+k_1\cdot k_2)},
\end{equation}

where we have used the on-shell conditions $k_1^2 = 1/2$ and $k_2^2 = 0 = k_2 \cdot \zeta_2$ where $\zeta_2$ is the polarization of the external gauge field. With this vertex and (7), (4) one can evaluate the following decay amplitude:

\begin{equation}
A(T_1, A_2, \tau_3) = \sum_\phi \tilde{V}_{T_1A_2} \tilde{G}_\phi \tilde{V}_{\phi\tau_3} = \frac{ig_\phi g_c}{6} \left\{ \frac{e^{2s \ln(4\gamma)}}{2s} (p_3 \cdot N \cdot \zeta_2) + \cdots \right\}.
\end{equation}

In order to keep only massless pole, one should replace exponential factor by one, i.e., $e^{2s \ln(4\gamma)} \rightarrow 1$. Similarly decay to graviton is

\begin{equation}
A(T_1, A_2, h_3) = \sum_\phi \tilde{V}_{T_1A_2} \tilde{G}_\phi \tilde{V}_{\phi h_3} = \frac{ig_\phi g_c}{6} \left\{ \frac{e^{2s \ln(4\gamma)}}{2s} [a(p_3 \cdot N \cdot \zeta_2) - 2p_3 \cdot D - \varepsilon_3 \cdot N \cdot \zeta_2] + \cdots \right\},
\end{equation}

where $a$ is given in (6) and again to keep only pole one should replace the exponential factor by one. In section 3, we shall compare these amplitudes with their corresponding amplitudes in bosonic string theory.

2.1.3 Two gauge fields decay

The vertex functions for two on-shell gauge fields and one off-shell state up to level two extracted from the action (3) are

\begin{align*}
\tilde{V}_{A_1A_2} & = \frac{ig_\phi}{6\gamma}(-16) [(2k_2\cdot \zeta_1)(2k_1\cdot \zeta_2) - \zeta_1\cdot \zeta_2] \gamma^{4(k_1\cdot k_2)}, \\
(\tilde{V}_{A_1A_2B^1})^\mu & = \frac{ig_\phi}{6\gamma} \left( \frac{2i\sqrt{2}}{27} \right) \gamma^{4(k_1k_2)} \left( 3[\zeta_1\cdot \zeta_2 - 4(k_1\cdot \zeta_2)(k_2\cdot \zeta_1)](k_1 + k_2)^\mu \\
& + 8[(k_1\cdot \zeta_2)\zeta_2^\mu + \zeta_2^\mu (k_2\cdot \zeta_1)] \right),
\end{align*}

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\((\tilde{V}_{A1A_2\beta(2)})^{\mu\nu} = \frac{ig_\phi}{6\gamma} \frac{2\sqrt{2}}{27} \gamma^{4(k_1k_2)} \left( 4(\zeta_1^\mu \zeta_2^\nu + \zeta_1^\nu \zeta_2^\mu) \right) \right)

\[= \frac{ig_\phi}{6\gamma} \frac{-11}{27} \gamma^{4(k_1k_2)} \left( \zeta_1^\mu \zeta_2^\nu + \zeta_1^\nu \zeta_2^\mu \right) \right) ,

\]

and \((\tilde{V}_{A1A_2A})^\mu = 0\). Here again we have used the on-shell condition \(k_i^2 = 0 = k_1 \cdot \zeta_i\) for \(i = 1, 2\). Now decay amplitude of these two gauge fields to closed string tachyon becomes

\[
A(A_1, A_2, \tau_3) = \sum_{\phi} \tilde{V}_{A1A_2\phi} \tilde{G}_{\phi} \tilde{V}_{\phi\tau_3} = \frac{ig_\phi g_\epsilon}{6} \left\{ \frac{e^{(2s-1)\ln(4\gamma)}}{2s - 1} \left[ \zeta_1 \cdot \zeta_2 - 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) \right] \right. \\
+ \left. \frac{e^{(2s+1)\ln(4\gamma)}}{2s + 1} \left( -\frac{3}{32}(s + \frac{1}{2})[\zeta_1 \cdot \zeta_2 - 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1)] \right) \\
+ \frac{1}{8}[\zeta_1 \cdot \zeta_2 + 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) - 8(p_3 \cdot N \cdot \zeta_1)(p_3 \cdot N \cdot \zeta_2)] + \cdots \right\} .
\]

In this case to retain only poles, one should replace each exponential by one and drop also the term in the third line above. Similarly the decay to graviton becomes

\[
A(A_1, A_2, h_3) = \sum_{\phi} \tilde{V}_{A1A_2\phi} \tilde{G}_{\phi} \tilde{V}_{\phi h_3} = \frac{ig_\phi g_\epsilon}{6} \left\{ \frac{e^{(2s-1)\ln(4\gamma)}}{2s - 1} a[\zeta_1 \cdot \zeta_2 - 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1)] \right. \\
+ \left. \frac{e^{(2s+1)\ln(4\gamma)}}{2s + 1} \left( -\frac{3}{32}(s + \frac{1}{2})a[\zeta_1 \cdot \zeta_2 - 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1)] \right) \\
- [\zeta_2 \cdot \varepsilon_3 \cdot D \cdot \zeta_1 + \zeta_1 \cdot \varepsilon_3 \cdot D \cdot \zeta_2] - \frac{1}{8}[\zeta_1 \cdot \zeta_2 - 4(k_1 \cdot V \cdot \zeta_2)(k_2 \cdot \zeta_1)] \right) \\
\times \left[ \text{Tr}(\varepsilon_3 \cdot D) + p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 - 8(k_1 - k_2) \cdot \varepsilon_3 \cdot (k_1 - k_2) \right] \\
+ a[(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) - (p_3 \cdot N \cdot \zeta_2)(p_3 \cdot N \cdot \zeta_1)] \\
+ 2(k_1 \cdot \zeta_2)[p_3 \cdot D \cdot \varepsilon_3 \cdot V \cdot \zeta_1 + 2\zeta_1 \cdot V \cdot \varepsilon_3 \cdot (k_1 - k_2)] \\
+ 2(k_2 \cdot \zeta_1)[p_3 \cdot D \cdot \varepsilon_3 \cdot V \cdot \zeta_2 - 2\zeta_2 \cdot V \cdot \varepsilon_3 \cdot (k_1 - k_2)] \\
+ 2[(p_3 \cdot N \cdot \zeta_2)p_3 \cdot D \cdot \varepsilon_3 \cdot N \cdot \zeta_1 + (p_3 \cdot N \cdot \zeta_1)p_3 \cdot D \cdot \varepsilon_3 \cdot N \cdot \zeta_2] + \cdots \right\} ,
\]

where \(a\) is given by (11). Here again one should replace the exponential factors by one and drop the term in third line above to retain only poles. In the next section we shall evaluate the same decay amplitudes in the bosonic string theory and compare with the above results.
3 Decay amplitude in Bosonic String Theory

The decay amplitude of excited stable and unstable D-branes in superstring theory have been studied in [16, 9]. In this section we shall study decay amplitudes of excited unstable D-brane in the bosonic string theory. These amplitudes are given by world-sheet correlation function of some open and closed string vertex operators, that is

\[ A \sim \langle \prod_n V_o \prod_m V_c \rangle, \]

where the open string tachyon and gauge field vertex operators, and closed string tachyon and graviton vertex operators are given as

\[ V_T = \int dx e^{2i k \cdot X(x)} ; k^2 = 1/2, \]
\[ V_A = \zeta_\mu \int dx \partial X^\mu e^{2i k \cdot X(x)} ; k^2 = 0, k \cdot \zeta = 0, \]
\[ V_\tau = \int d^2 z e^{ip \cdot X(z)} e^{ip \cdot D \cdot X(\bar{z})} ; p^2 = 2, \]
\[ V_h = (\varepsilon \cdot D)_{\mu\nu} \int d^2 z \partial X^\mu e^{ip \cdot X(z)} \partial X^\nu e^{ip \cdot D \cdot X(\bar{z})} ; p^2 = 0, p \cdot \varepsilon = 0 = \varepsilon \cdot p, \]

where in our convention we have set the normalization of all vertices to one, and restore them by normalizing properly the scattering amplitudes. We are interested in the world sheet tree level S-matrix elements of two open string states and one closed string state. They can be evaluated by using the world sheet propagator \( <X^\mu(z)X^\nu(w)> = -\eta^{\mu\nu} \ln(z - w) \) and Wick theorem. After performing the correlators, one ends up with a multi integral whose integrand has SL(2,R) symmetry reflecting the conformal symmetry that maps upper-half plane to itself. This symmetry is usually gauged away by fixing world sheet position of one open string vertex operators at infinity and the closed string vertex operator at \( z = i \). Then the remaining integral over position of the other vertex operator gives a beta function. We begin with presenting in some details the result for decay amplitude of two open string tachyons to one closed string tachyon.

3.1 Two tachyons decay

The decay amplitude of two open string tachyons and one closed string tachyon in bosonic string theory is given by the following correlation:

\[ A(T_1, T_2, \tau_3) \sim \langle V_{T_1}V_{T_2}V_{\tau_3} \rangle \]
\[ = \left( \frac{ig_olc}{12} \right) 2^{2s-1} \int_{-\infty}^{\infty} dx (x - i)^{-s} (x + i)^{-s} \]
\[ = \left( \frac{ig_olc}{12} \right) 2^{2s-1} B(s - \frac{1}{2}, \frac{1}{2}) \]
\[ = \left( \frac{ig_olc}{12} \right) 2\pi \frac{\Gamma(2s - 1)}{\Gamma(s)\Gamma(s)} \]

(17)
where in the last equality we have used the identity \( \sqrt{\pi} \Gamma(s+1) = 2^s \Gamma(s/2+1) \Gamma(s/2+1/2) \), to show that the result can be written in a form similar to the case of super string theory \([16, 9]\).

We have also normalized the amplitude here and subsequent amplitudes in this section by factor \( \left( \frac{i g_0 g_c}{12} \right) \), i.e., the same factor as in amplitudes in the string fields theory side. By looking at the poles of gamma function, one realizes that the amplitude has simple pole only when mediators are in even levels, i.e., their on-shell masses are \( -s = m^2 = -1/2, 1/2, 3/2, \cdots \). This is consistent with the twist symmetry in the cubic string field theory. According to it when two legs of three point vertex function are tachyons, i.e., level zero, the other leg must be in even levels, i.e., \( m^2 = -1/2, 1/2, 3/2, \cdots \). Now to compare in more details the decay amplitude here with the corresponding amplitude in string field theory, we use the following pole expansion of the beta function:

\[
B(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{1}{\alpha + n} \frac{(-1)^n}{n!} (\beta - 1) \cdots (\beta - n). \tag{18}
\]

By making use of this pole expansion for beta function, the decay amplitude \([17]\) becomes

\[
A(T_1, T_2, \tau_3) = 2 \left( \frac{i g_0 g_c}{12} \right) \left( \frac{e^{(2s-1) \ln(2)}}{2s-1} + \frac{1}{8} \frac{\ln^2(2)}{2s+1} + \cdots \right)
= 2 \left( \frac{i g_0 g_c}{12} \right) \left( \frac{1}{2s-1} + \frac{1}{8} \frac{\ln^2(2)}{2s+1} + \cdots \right),
\]

where in the last line above we dropped the contact terms resulting from each pole in the first line. Now comparing the first line above with the corresponding amplitude in \([4]\), one realizes that the two amplitudes are not quite the same. The difference is in their contact terms, i.e., the second line above is exactly the same as the amplitude in \([10]\). Similarly, the decay amplitude of two tachyons to one graviton or dilaton becomes

\[
A(T_1, T_2, h_3) \sim \langle V_{T_1} V_{T_2} V_{h_3} \rangle
= \left( \frac{i g_0 g_c}{12} \right)^{2s-1} \left[ \Tr(\varepsilon_3 \cdot D) + p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 \right] B(s - \frac{1}{2}, \frac{1}{2})
- 4(k_1 - k_2) \cdot \varepsilon_3 \cdot (k_1 - k_2) B(s + \frac{1}{2}, \frac{1}{2})
- p_3 \cdot \varepsilon_3 \cdot D \cdot p_3 [B(s + \frac{1}{2}, \frac{1}{2}) + 2B(s - \frac{1}{2}, \frac{3}{2})] \tag{19}
\]

By looking at the poles of beta functions, one finds that the amplitude has no simple pole corresponding to states in odd levels which is consistent with the twist symmetry in string field theory side. Using the expansion \([18]\) for the beta functions in above amplitude, one finds the following pole structure:

\[
A(T_1, T_2, h_3) = 2 \left( \frac{i g_0 g_c}{12} \right) \left( \frac{1}{2s-1} \langle \Tr(\varepsilon_3 \cdot D) - p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 \rangle \right)
+ \frac{1}{8} \left[ \langle \Tr(\varepsilon_3 \cdot D) + p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 - 8(k_1 - k_2) \cdot \varepsilon_3 \cdot (k_1 - k_2) \rangle \right] + \cdots
\]
where we have dropped some contact terms. This is exactly equal to the corresponding amplitude in string field theory side in the last equality in equation (11).

For later reference, we evaluate the amplitude (19) for dilaton by replacing $\epsilon_{\mu \nu} = (\eta_{\mu \nu} - p_\mu l_\nu - l_\mu p_\nu) / \sqrt{24}$ where $p \cdot l = 1$, that is,

$$A(T_1, T_2, h_3) = \left(\frac{ig_0 g_c}{12 \sqrt{24}}\right) 2^{2s-1} \left( \text{Tr}(D)B(s - \frac{1}{2}, \frac{1}{2}) - 4(2 - s)B(s + \frac{1}{2}, \frac{1}{2}) \right),$$

as expected the auxiliary vector $l_\mu$ does not appears in the amplitude. In reaching to this result we have used the on-shell condition $k_1^2 = 1/2 = k_2^2$.

### 3.2 One tachyon and one gauge field decay

The amplitude for decaying one open string tachyon and one gauge field to one closed string tachyon is

$$A(T_1, A_2, \tau_3) \sim \langle V_{T_1} V_{A_2} V_{\tau_3} \rangle = \left(\frac{ig_0 g_c}{12}\right) 2^{2s} B(s, \frac{1}{2}) (p_3 \cdot N \cdot \zeta_2)$$

where in the last equality we have used the expansion (18) and dropped some contact terms. It is easy to see that this is equal to the pole of the decay amplitude in string field theory side (12). The amplitude in the second line above has simple poles only when the mediator is in any odd level, i.e., $m^2 = 0, 1, 2, \cdots$. This is again consistent with twist symmetry in string field theory side, i.e., the tachyon is in zero level, the gauge field is in level one hence the third leg must be in odd level. Similarly the decay amplitude to graviton or dilaton is

$$A \sim \langle V_{T_1} V_{A_2} V_{\tau_3} \rangle = \left(\frac{ig_0 g_c}{12}\right) 2^{2s} \left( \text{Tr}(\epsilon_3 \cdot D)(p_3 \cdot N \cdot \zeta_2) - p_3 \cdot D \cdot \epsilon_3 \cdot N \cdot \zeta_2 \right) B(s, \frac{1}{2})$$

In the last line we just retain pole structure up to level two. Here again one can see that the amplitude in last line above is exactly equal to the corresponding amplitude in string theory side (13), and check that the simple pole of the beta functions in the amplitude are consistent with the twist symmetry in string field theory side. Note that when the
external massless vertex operator has world-volume polarization $\zeta_2^\nu$, the whole amplitudes above vanish. This predicts for the string field theory that the off-shell states in the 3-point vertex with two other states being on-shell tachyon and gauge field, does not couple with closed string tachyon or graviton in the gauge invariant operators $\mathcal{F}$.

### 3.3 Two gauge fields decay

The amplitude for decaying an unstable D-brane excited with two gauge fields to an unstable D-brane with no open string excited and one closed string tachyon is

$$A(A_1, A_2, \tau_3) \sim \langle V_{A_1} V_{A_2} V_{\tau_3} \rangle$$

$$= \left( \frac{i\alpha_0 g_e}{12} \right) 2^{2s-1} \left( \zeta_1 \cdot \zeta_2 B(s - \frac{1}{2}, \frac{1}{2}) - 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) B(s - \frac{1}{2}, \frac{3}{2}) - 4(p_3 \cdot N \cdot \zeta_1)(p_3 \cdot N \cdot \zeta_2) B(s + \frac{1}{2}, \frac{1}{2}) \right)$$

$$= \left( \frac{i\alpha_0 g_e}{12} \right) \frac{2}{2s-1} \left( \zeta_1 \cdot \zeta_2 - 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) \right)$$

$$+ \frac{1}{4} \frac{1}{2s+1} \left( \zeta_1 \cdot \zeta_2 + 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) - 8(p_3 \cdot N \cdot \zeta_1)(p_3 \cdot N \cdot \zeta_2) \right) + \cdots,$$

where in the last equality again some contact terms are dropped. As a check of our calculation, one may replace each gauge field polarization with its momentum and find that the result is zero. In this case also the pole structure in the last equality above is exactly equal to the poles of the decay amplitude $\mathcal{F}$ in string field theory side, and the infinite number of simple poles of beta functions above are consistent with the twist symmetry. Similarly, the decay amplitude to graviton or dilaton is

$$A \sim \langle V_{A_1} V_{A_2} V_{\tau_3} \rangle$$

$$= \left( \frac{i\alpha_0 g_e}{12} \right) 2^{2s-2} \left( - \left[ \zeta_1 \cdot \zeta_2 p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 + 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) \text{Tr}(\varepsilon_3 \cdot D) \right] B(s - \frac{1}{2}, \frac{3}{2}) + \zeta_1 \cdot \zeta_2 \text{Tr}(\varepsilon_3 \cdot D) B(s + \frac{5}{2}, \frac{1}{2}) + 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1) p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 B(s + \frac{5}{2}, \frac{1}{2}) - \left[ 4(k_1 \cdot \zeta_2)(k_1 - k_2) \cdot \varepsilon_3 \cdot (k_1 - k_2) + 4(p_3 \cdot N \cdot \zeta_1)(p_3 \cdot N \cdot \zeta_2) \text{Tr}(\varepsilon_3 \cdot D) \right] B(s + \frac{1}{2}, \frac{1}{2})$$

$$+ 16 \left[ (k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1)(k_1 - k_2) \cdot \varepsilon_3 \cdot (k_1 - k_2) + \frac{1}{4}(p_3 \cdot N \cdot \zeta_1)(p_3 \cdot N \cdot \zeta_2) p_3 \cdot D \cdot \varepsilon_3 \cdot D \cdot p_3 + (p_3 \cdot N \cdot \zeta_1) \zeta_2 \cdot N \cdot D \cdot p_3 + (k_1 \cdot \zeta_2) \zeta_1 \cdot V \cdot \varepsilon_3 \cdot (k_1 - 3k_2) - \frac{1}{2} \zeta_1 \cdot \varepsilon_3 \cdot D \cdot \zeta_2 \right] B(s + \frac{1}{2}, \frac{3}{2}) + 16(p_3 \cdot N \cdot \zeta_1)(p_3 \cdot N \cdot \zeta_2)(k_1 - k_2) \cdot \varepsilon_3 \cdot (k_1 - k_2)$$

$$+ 32(p_3 \cdot N \cdot \zeta_1) \zeta_2 \cdot N \cdot \varepsilon_3 \cdot (k_1 - k_2) + 8 \zeta_1 \cdot \varepsilon_3 \cdot D \cdot \zeta_2 \right] B(s + \frac{3}{2}, \frac{1}{2}) + (1 \leftrightarrow 2).$$

As a check of our calculations, we have replaced the gauge field polarization with its momentum and found that the amplitude vanishes. Now if one replaces the expansion $\mathcal{F}$
for the beta functions in above amplitude and drops the unwanted contact terms, one will find exactly the same pole structure as in the string field theory case (16). Also the simple poles of the above amplitude is consistent with twist symmetry in string field theory side.

4 Effective action

As pointed out by Sen in [17], the general structure of unstable D-brane action should be consistent with disk amplitude in string theory. In this section we would like to compare the disk amplitudes found in previous section with a tachyonic DBI action, i.e., an action which is consistent with the leading order terms of the amplitudes expanded for massless open string fields around $\alpha' \to 0 (s \to 0)$, and for open string tachyon around the top of the tachyon potential ($s \to 0$). We propose the following tachyonic DBI action in the bosonic string theory which includes both open string and closed string tachyon:

$$ S = -T_p \int d^{p+1}x f(\tau) V(T) e^{-\Phi} \sqrt{-\det (P[g_{ab} + b_{ab}] + 2\pi \alpha' F_{ab} + 2\pi \alpha' \partial_a T \partial_b T)} , \quad (20) $$

where $V(T) = 1 - \pi T^2 + O(T^3)$ is the open string tachyon potential expanded around its maximum, $f(\tau) = 1 + \tau + O(\tau^2)$ is the closed string tachyon coupling to the D-brane, and $g_{ab}$ is flat space metric $\eta_{ab}$ plus its graviton fluctuation. Here $b_{ab}$, $\Phi$, $A_a$ and $T$ are the antisymmetric Kalb-Ramond tensor, dilaton, gauge field and the tachyon fluctuations, respectively. In above action $P[\cdots]$ is also the pull-back of the closed string fields. For example, $P[h_{ab}] = b_{ab} + 2h_{ai} \partial_b X^i + h_{ij} \partial_a X^i \partial_b X^j$ in the static gauge. Now it is straightforward to expand (20) to find different couplings involving two open string states and one closed string state. When two open string states are massless, the couplings are

$$ \mathcal{L}(X, X, \tau) = -T_p \left( \tau \partial_a X^i \partial^a \partial^i X_1 + \frac{1}{2} X^i X^j \partial_i \partial_j \tau \right) , $$
$$ \mathcal{L}(X, X, h) = -T_p \left( \frac{1}{4} h^a_a \partial_b X^i \partial^a \partial^i X_1 - \frac{1}{2} h^{ab} \partial_b X^i \partial_a X_i + \frac{1}{2} \partial_a X^i \partial^a X^j \partial_{ij} \right) , $$
$$ \mathcal{L}(A, A, \tau) = -T_p \tau \left( -\frac{(2\pi \alpha')^2}{4} F_{ab} F^{ab} \right) , $$
$$ \mathcal{L}(A, A, h) = -T_p \left( -\frac{(2\pi \alpha')^2}{8} h^a_a F_{bc} F^{cb} + \frac{(2\pi \alpha')^2}{2} h_{ab} F^{bc} F_{ca} \right) , $$

and $\mathcal{L}(X, A, \tau) = 0 = \mathcal{L}(X, A, h)$.

Now we would like to expand the decay amplitudes in section 3.1 at low energy limit, i.e., $\alpha' \to 0$ or $s = \alpha' k_1 k_2 \to 0$, and compare with the above couplings. Expanding the beta

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*We explicitly restore $\alpha'$ in this section.*
functions in these amplitude around \(s = 0\), one finds the following leading order terms:

\[
A(A_1, A_2, \tau_3) = \left(\frac{ig_0g_c}{12}\right) \left( (\zeta_1 \cdot \zeta_2)(-\pi s) + 4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1)\frac{\pi}{2} + \cdots \right),
\]

\[
A(X_1, X_2, \tau_3) = \left(\frac{ig_0g_c}{12}\right) \left( (\zeta_1 \cdot \zeta_2)(-\pi s) - 4(p_3 \cdot \zeta_1)(p_3 \cdot \zeta_2)\frac{\pi}{2} + \cdots \right),
\]

\[
A(A_1, A_2, h_3) = \left(\frac{ig_0g_c}{12}\right) \left( [ (\zeta_1 \cdot \zeta_2)\text{Tr}(\varepsilon_3 \cdot D) - 4\zeta_1 \cdot \varepsilon_3 \cdot \zeta_2 ](-\pi s) 
+ [4(k_1 \cdot \zeta_2)(k_2 \cdot \zeta_1)\text{Tr}(\varepsilon_3 \cdot D) + 16(\zeta_1 \cdot \zeta_2)k_1 \cdot \varepsilon_3 \cdot k_2 
- 16(k_1 \cdot \zeta_2)\zeta_1 \cdot \varepsilon_3 \cdot k_2 - 16(k_2 \cdot \zeta_1)\zeta_2 \cdot \varepsilon_3 \cdot k_1]\frac{\pi}{2} + \cdots \right),
\]

\[
A(X_1, X_2, h_3) = \left(\frac{ig_0g_c}{12}\right) \left( [ (\zeta_1 \cdot \zeta_2)\text{Tr}(\varepsilon_3 \cdot D) + 4\zeta_1 \cdot \varepsilon_3 \cdot \zeta_2 ](-\pi s) 
+ [-4(p_3 \cdot \zeta_1)(p_3 \cdot \zeta_2)\text{Tr}(\varepsilon_3 \cdot D) + 16(\zeta_1 \cdot \zeta_2)k_1 \cdot \varepsilon_3 \cdot k_2 
- 16(p_3 \cdot \zeta_1)\zeta_2 \cdot \varepsilon_3 \cdot k_2 - 16(p_3 \cdot \zeta_2)\zeta_1 \cdot \varepsilon_3 \cdot k_1]\frac{\pi}{2} + \cdots \right),
\]

and \(A(A_1, X_2, \tau_3) = 0 = A(A_1, X_2, h_3)\). The dots above represents terms with more than two open string momenta, which are related to the higher derivative terms in the effective action. Now it is a simple exercise to check that above leading terms of the decay amplitudes are reproduced exactly by the corresponding terms in (21) as expected. To have overall factors agreement as well, one should appropriately normalize fields in (20). This agreements insures us that the open string tachyon potential in (20) dose not have a linear terms. Having such a term , the field theory action would produce for example an amplitude for decaying two massless scalars to one graviton, i.e., \(h^{a} \alpha_{a} \zeta_1 \cdot \zeta_2 s/(1 - 2s) \sim h^{a} \alpha_{a} \zeta_1 \cdot \zeta_2 s\) which must be subtracted from string theory amplitude \(A(X_1, X_2, h_3)\). Consequently the result would not be consistent with \(\mathcal{L}(X, X, h)\) in (21). Similarly, above agreement predicts that the field theory does not have coupling \(iX^iX_i\). It is important to note that the string field theory in its present field variables has this coupling (14), however using field redefinition \(^6\), in another field variables this coupling can be disappeared (20).

The decay amplitudes in section 3.2 have simple pole at \(s \rightarrow 0\), hence one may expect that field theory should reproduce them. However, as above analysis predicts the tachyonic DBI action (20) does not have couplings \(T \partial_a X^i \partial^a X_i\) or \(T X^i X_i\), hence the field theory (20) does not produce the simple massless poles of the string theory amplitude. We expect that these massless simple poles are reproduced by the coupling \(X^i \partial^a \tau\) or \(X^i \partial_a h^{a} \alpha\) which are in (20) and higher derivative couplings like \(\partial^a \partial_a T \partial_b X^i \partial^b X_i\) (or \(T \partial_a \partial_a X^i \partial^a \partial^b X_i\)) which are not included in the tachyonic DBI action\(^1\). This means that the on-shell amplitudes in section 3.2 should have a factor like \(1 = (2k_0^2)^2\) that in off-shell case \(2k_0^2 \neq 1\) produces higher derivative couplings in the field theory. Similarly the contact terms of the amplitudes around \(s \rightarrow 0\) also describe the higher derivative terms which lead to these on-shell contact terms.

\(^6\)A similar argument is presented in [18] for consistency between sigma model approach effective action and S-matrix elements.
Finally, the decay amplitudes in section 3.1 are smooth around the top of tachyon potential $s \to 0$. Their expansion around this point are

\[
A(T_1, T_2, \tau_3) = \left( \frac{ig_0 g_c}{12} \right) \left( -\pi s \right) + \cdots,
\]

\[
A(T_1, T_2, h_3) = \left( \frac{ig_0 g_c}{12} \right) \left( \text{Tr}(\varepsilon_3 \cdot D)(-\pi s) + 16(k_1 \cdot \varepsilon_3 \cdot k_2)\left( \frac{\pi}{2} \right) + \cdots \right),
\]

\[
A(T_1, T_2, \Phi_3) = \left( \frac{ig_0 g_c}{12 \sqrt{24}} \right) \left( \text{Tr}(D)(-\pi s) - 4(2 - s)\left( \frac{\pi}{2} \right) + \cdots \right),
\]

where dots represent terms that we expect them to be related to higher derivative terms. Using the fact that in this case $s = 1 + 2k_1 \cdot k_2$, one can easily verifies that these contact terms are reproduced exactly by the following couplings extracted from (20):

\[
\mathcal{L}(T, T, \tau) = -\pi T_T T \left( T^2 + \alpha' \partial_a T \partial^a T \right),
\]

\[
\mathcal{L}(T, T, h) = -\pi T_T \left( -\frac{1}{2} h^a \partial_a T^2 + \frac{\alpha'}{2} h^a \partial_a T \partial^b T - \alpha' h^{ab} \partial_a T \partial_b T \right),
\]

\[
\mathcal{L}(T, T, \Phi) = -\pi T_T \Phi \left( \frac{(p - 11)}{12} T^2 + \frac{(p - 13)}{12} \alpha' \partial_a T \partial^a T \right),
\]

where in writing the last line we have used the fact that vertex operators in the string amplitudes correspond to Einstein frame metric, whereas, the metric in the tachyonic DBI action is string frame metric. The relation between them is $g_S = e^{\phi/6} g_E$. The couplings in the first line above indicates that the tachyon has mass $m^2 = -1/\alpha'$, as expected. The couplings in the second line indicates that the graviton couples to tachyon by making covariant the free action of tachyon. However, the couplings in the last line have a unique signature that imply the kinetic term of tachyon should appear under square root of a determinant as in (20). This ends our illustration of the complete consistency between S-matrix elements at $s \to 0$ limit and tachyonic DBI action.

5 Discussion

In this paper, using gauge invariant operators corresponding to on-shell closed string states, 3-point interaction of open string field theory and standard Feynman rules, we have explicitly evaluated the amplitude for decaying two on-shell open string states in level zero or one and one closed string tachyon or graviton. In our calculation, the off-shell states propagating between the on-shell closed string state and two open string states in level more than two are truncated. We have then evaluated explicitly the same amplitudes in the bosonic string theory. Using an expansion for the beta function in these amplitudes, we have expand the amplitudes in terms of a tower of simple poles reflecting the off-shell states propagating between one on-shell closed string and two open string states, like in string field theory case. Truncating poles with more than level two, we have found that
the results are exactly the same as in string field theory side up to some contact terms. To compare the contact terms of the amplitude in two theories as well, one needs the amplitude in string field theory side to have all infinite number of massive poles. This because individual contact terms of each massive pole has contribution to contact terms of the whole amplitude. It would be interesting then to find an exact tree level amplitude in string field theory side and compare with the exact form of the amplitude in the bosonic string theory found in section 3. Similar comparison has been done for the scattering amplitude of four open string tachyons in \[21\]. Lastly, we have expanded the beta functions in the exact disk level decay amplitudes in the bosonic string theory case around \(s \to 0\) and found that their leading order terms can be fully described by the tachyonic DBI action (20).

The tachyonic DBI action produces only the leading terms of S-matrix elements expanded around top of the tachyon potential (not \(\alpha' \to 0\) limit). Hence one may expect that the other terms of expansion have significant effect. So the field theory effective action of string theory at the top of the tachyon potential should have those contact terms as well, \(i.e.,\) the action should have higher derivative terms as well. Interestingly, Sen has shown that some exact result of string theory, like production of a pressure-less gas with non-zero energy density at the late time of the tachyon condensation \[19\], can be derived also from the tachyonic DBI action \[1, 14\] at minimum of the tachyon potential. Hence one may conclude that, as an effective action of string theory, although the higher derivative terms have important impact at the top of the tachyon potential, they have little effects at the minimum of the potential. Therefore, the tachyonic DBI action derived by studying S-matrix elements at the top of potential may be, in fact, effective action of string theory at the minimum of the tachyon potential.

In our calculation of decay amplitude we have ignored recoil of D-brane which allows us to ignore momentum conservation in the transverse directions and hence keep \(k_1 \cdot k_2\) arbitrary. This procedure of relaxing momentum conservation is in fact a particular way of off-shell extension of the S-matrix elements. It would be interesting also to extend the on-shell conditions \(k_1^2 = 1/2 = k_2^2\) to off-shell values as well. This might be possible by comparing the S-matrix elements in bosonic string theory with the corresponding off-shell amplitudes in the string field theory side. In this way one may be able to find higher derivative corrections to the action (20). Alternatively, one may start from beginning with an off-shell approach to effective action like sigma model approach \[18\]. In this approach one directly reads effective action from a sigma model partition function. It would be interesting then to see if the tachyonic DBI action (21) and its higher derivative corrections can be extracted in this way. We hope to answer these questions in our future works.

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