A plane stress yield function described by multi-segment spline curves and its application

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Abstract. A plane stress yield function which is described by multi-segment spline curve is proposed. This model is able to consider an arbitrary number of multi-axial stress points and its normal directions on the yield surface. To show the applicability of the model, some evaluation examples are shown and its accuracy is discussed.

1. Introduction
Precise prediction of the material anisotropy is an important factor of an accurate sheet metal forming simulation. In order to describe the material anisotropy properly, many yield functions have been proposed. For example, Hill’48\(^1\) quadratic yield function is widely used, because of its simplicity. By considering r-values in three tension axis directions from rolling direction, its anisotropic parameters are determined. On the other hand, it is well known that Hill’48 quadratic model has a serious problem of its poor prediction of equi-biaxial stress state. Barlat et al.\(^2\) introduced Yld2000-2d, which has a formulation using two linear transformations to represent the yield function. Yld2000-2d considers r-values and stress in three directions, equi-biaxial stress and its strain increment ratio. Vegter et al.\(^3\) proposed a plane stress yield function based on the interpolation of multi-axial stress states by second order Bézier curves. This model considers equi-biaxial, plane strain, uniaxial, and pure shear stress states as reference points of the yield surface. Tsutamori et al.\(^4, 5\) have proposed a plane stress yield function using third order Bézier curves, which connects uniaxial stress state and equi-biaxial stress. In that model, by tuning variables of control points of third order Bézier curve, yield surface is described flexibly, however the experimental data of multi-axial stress without equi-biaxial is not considered as reference points of Bézier curves. In this study, yield function which is able to consider arbitrary number of multi-axial state is proposed.

2. Descriptions of yield function
In the two-dimensional principal stress space, a stress point is represented by the vector \(\{\sigma_1, \sigma_2\}^T\). Let \(\theta\ (0^\circ \leq \theta \leq 90^\circ)\) be the angle between the first principle axis and the rolling direction. In this study, only orthotropic materials are considered. Divide the angle from \(0^\circ\) to \(90^\circ\) into \(n\) sections, and set \(0 \leq i \leq n\).
Let $A_i$, $D_i$, $E_i$ and $C_i$ be the components of third order Bézier curves as shown in figure 1. Points $A_i$ and $C_i$ are arbitrary reference stress state, and assume its tangential directions are given. Point $B_i$ is defined as a point of intersection of the two tangents. Point $D_i$ is the point of interior division ratio $(1 - p:p)$ of line $A_iB_i$, where $p$ is a material parameter, and point $E_i$ is the point of interior division ratio $(1 - p:p)$ of line $C_iB_i$. By the interpolation based on third order Bézier curve yield locus located between $A_i$ and $C_i$ is described by function $R_i$.

$$R_i(t) = (1 - t)^3A_i + 3(1 - t)^2tD_i + 3(1 - t)t^2E_i + t^3C_i, \quad 0 \leq t \leq 1$$ (1)

If there are $N$ reference stress points are considered as yield locus in the area $\sigma_1 \geq \sigma_2 \geq 0$, its yield surface described by $N$-1 segments. For example, if uniaxial stress and equi-biaxial stress are referred, the yield surface in the area $\sigma_1 \geq \sigma_2 \geq 0$ described by one segment interpolation of Bézier curve, if the points $\sigma_1: \sigma_2$=4:1, 2:1 and 4:3 are made addition to uniaxial and equi-biaxial stress as reference data specifically, yield surface in the area $\sigma_1 \geq \sigma_2 \geq 0$ described by four segments.

For planar anisotropic sheet metal, the yield locus and the reference strain ratio depend on the angle between the first principle axis and the rolling direction. The angular dependency of the reference points and strain ratio can be described by using cubic spline basis function $f_i$. If $n=7$, spline basis functions $f_i$ are determined as in figure 2. By using spline function $f_i$, the function which describes yield locus $R$ is written as following.

$$R(\theta, t) = \sum_{i=1}^{n} f_i(\theta)(1 - t)^3A_i + 3(1 - t)^2tD_i + 3(1 - t)t^2E_i + t^3C_i, \quad 0 \leq t \leq 1$$ (2)

![Figure 1](image1.png)  Reference and control points

![Figure 2](image2.png)  Cubic spline function basis (Case: $n=7$)

3. Implementation to FEM cord

To implement the proposed yield function into FEM cord of dynamic explicit, a summary of the formulation is shown as follows:

Set the normalized yield locus $R(\theta, t) = (R_1(\theta, t), R_2(\theta, t))$. Then the yield locus is represented by using effective stress $\sigma_{eq}$ as follows:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \sigma_{eq} \begin{bmatrix} R_1(\theta, t) \\ R_2(\theta, t) \end{bmatrix}$$ (3)

Let $\phi$ be a plastic potential

$$\phi(\sigma) = \sigma_{eq}(\sigma)$$ (4)

The associate flow rule is employed. Set $X = (\sigma_1, \sigma_2, \theta), \sigma = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$, then description of plastic strain increment is given by
\[ \frac{\partial \phi}{\partial \sigma} = \left( \frac{\partial X}{\partial \sigma} \right)^T \frac{\partial \phi}{\partial X} \]  \hspace{1cm} (5)

Each component of \( \frac{\partial \phi}{\partial \sigma} \) is derived by calculations of following equations.

\[ \sigma_1 = \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} + \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2} \right) \]  \hspace{1cm} (6)

\[ \sigma_2 = \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} - \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2} \right) \]  \hspace{1cm} (7)

\[ \cos 2\theta = \frac{\sigma_{xx} - \sigma_{yy}}{\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2}} = \frac{\sigma_{xx} - \sigma_{yy}}{\sigma_1 - \sigma_2} \]  \hspace{1cm} (8)

The consistency condition during plastic deformation is written as follows:

\[ \frac{\partial \phi}{\partial \sigma} \cdot d\sigma + \frac{\partial \phi}{\partial \varepsilon} d\varepsilon = 0 \]  \hspace{1cm} (9)

where \( \varepsilon \) implies the effective plastic strain. \( \frac{\partial \phi}{\partial \varepsilon} \) is equivalent to \( d\sigma_{eq}/d\varepsilon \), which corresponds to the slope of a hardening rule. Let \( C \) be an elastic tensor, and \( \varepsilon = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}) \) be a total strain vector. Then the stress increment is given by

\[ d\sigma = C \left( d\varepsilon - d\lambda \frac{\partial \phi}{\partial \sigma} \right) \]  \hspace{1cm} (10)

where \( d\lambda \) is a plastic multiplier. Set the effective plastic strain increment as follows:

\[ d\varepsilon = \left( \frac{2}{3} d\lambda \frac{\partial \phi}{\partial \sigma} \cdot d\lambda \frac{\partial \phi}{\partial \sigma} \right)^{1/2} \]  \hspace{1cm} (11)

By equation (9), equation (10) and equation (11), the plastic multiplier is determined as follows:

\[ d\lambda = \frac{\frac{\partial \phi}{\partial \sigma} \cdot C \frac{\partial \phi}{\partial \sigma} - \frac{\partial \phi}{\partial \sigma} \left( \frac{2}{3} \frac{\partial \phi}{\partial \sigma} \right)^{1/2}}{\frac{\partial \phi}{\partial \sigma} \cdot C \frac{\partial \phi}{\partial \sigma} - \frac{\partial \phi}{\partial \sigma} \left( \frac{2}{3} \frac{\partial \phi}{\partial \sigma} \right)^{1/2}} \]  \hspace{1cm} (12)

Now the plastic strain increment \( d\varepsilon^p = d\lambda \left( \frac{\partial \phi}{\partial \sigma} \right) \) is computed explicitly.

4. Application

To demonstrate the applicability of the proposed yield function, some evaluations of a dual phase steel sheet with a tensile strength of 980 MPa are shown. The material properties are being opened to the public for the benchmarks problem of NUMISHEET 2018 [6]. The material properties consists of uniaxial tension tests conducted every 15 degree from the rolling direction provide seven yield stresses and r-values. Moreover, the balanced biaxial tensile tests of seven linear stress paths, \( \sigma_x: \sigma_y = 4:1, 2:1, 4:3, 1:1, 3:4, 1:2, \) and \( 1:4 \) are included.

Figure 3 and figure 4 show the prediction of the anisotropy of the uniaxial tension tests. For comparison, evaluation results of Hill’48 and Yld2000-2d are also shown. Parameters of Hill’48 are determined by r-value in three directions. Parameters of Yld2000-2d consider uniaxial stress and r-value in three directions, equi-biaxial stress and its strain ratio. The proposed model describes the planar anisotropy by the interpolation based on cubic spline, thus all experiment data in seven directions are predicted.

Figure 5 shows the prediction of the yield surface. In the figure, the difference between the prediction of Yld2000-2d and that of the proposed model is slight. The proposed model describes the yield surface by the interpolation of multi-segment based on Bézier curve. Yield locus from uniaxial tension stress in rolling direction to equi-biaxial stress is described by four segments. Likewise, yield
locus from equi-biaxial stress to uniaxial stress in transverse direction is described by four segments. Thus all experiment data of multi-axial stress is considered in the prediction of the proposed model.

Figure 3. Prediction of r-value

Figure 4. Prediction of normalized uniaxial stress

Figure 5. Prediction of normalized yield surface

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