ON CONSTRAINED MARKOV-NIKOLSKII TYPE INEQUALITY FOR \( k \)-ABSOLUTELY MONOTONE POLYNOMIALS

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Abstract. We consider the classical problem of estimating norms of higher order derivatives of algebraic polynomial via the norms of polynomial itself. The corresponding extremal problem for general polynomials in uniform norm was solved by A. A. Markov. In 1926, Bernstein found the exact constant in the Markov inequality for monotone polynomials. It was shown in [3] that the order of the constants in constrained Markov-Nikolskii inequality for \( k \)-absolutely monotone polynomials is the same as in the classical one in case \( 0 < p \leq q \leq \infty \). In this paper, we find the exact order for all values of \( 0 < p, q \leq \infty \). It turned out that for the case \( q < p \) constrained Markov-Nikolskii inequality can be significantly improved.

1. Introduction

For \( n \geq k \geq 0 \), we denote

\[
M_{q,p}(n, k) := \sup_{P_n \in \mathbb{P}_n} \frac{\|P_n^{(k)}\|_{L^q[-1,1]}}{\|P_n\|_{L^p[-1,1]}}.
\]

In paper [4], complete information about the orders of \( M_{q,p}(n, k) \) for all values \( p > 0, q > 0 \) is given.

Theorem 1.1. For \( 0 < p, q \leq \infty \) and \( P_n \in \mathbb{P}_n \) we have:

\[
M_{q,p}(n, k) \asymp \begin{cases} 
  n^{2k+2/p-2/q}, & \text{if } k > 2/q - 2/p, \\
  n^k(\log n)^{1/q-1/p}, & \text{if } k = 2/q - 2/p, \\
  n^k, & \text{if } k < 2/q - 2/p.
\end{cases}
\]

By \( \Delta_n \) we denote the set of all monotone polynomials of degree \( n \) on \([-1,1]\). In 1926, S. Bernstein [1] pointed out that Markov’s inequality for monotone polynomials is not essentially better than for all polynomials, in the sense, that the order of \( \sup_{P_n \in \Delta_n} \|P_n\|/\|P_n\| \) is \( n^2 \). He proved his result only for odd \( n \). In 2001, Qazi [6] extended Bernstein’s idea to include polynomials of even degree. Next theorem contains their results:

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Theorem 1.2 (Bernstein [1], Qazi [6]).

\[
\sup_{P_n \in \Delta_n} \frac{\|P_n\|}{\|P_n\|_p} = \begin{cases} 
\frac{(n+1)^2}{4n(n+2)}, & \text{if } n = 2k + 1, \\
\frac{n(n+2)}{4}, & \text{if } n = 2k.
\end{cases}
\]

Definition 1.3. The function \( f : [a, b] \to \mathbb{R} \) is absolutely monotone of order \( k \) if, for all \( x \in [a, b] \),

\[
f^{(m)}(x) \geq 0,
\]

for all \( 0 \leq m \leq k \), and denote by \( \Delta_n^{(k)} \) the set of all absolutely monotone polynomials of order \( k \) on \([-1, 1]\).

For example, absolutely monotone functions of order 0 are just nonnegative functions on \([a, b]\), and \( \Delta_n^{(0)} = \Delta_n \) is the set of all nonnegative monotone polynomials on \([-1, 1]\).

A natural modification of \( M_{q,p}^{(n,k)}(n, m) \) for \( \Delta_n^{(k)} \) is

\[
M_{p,q}^{(k)}(n, m) = \sup_{P_n \in \Delta_n^{(k)}} \frac{\|P_n\|_q}{\|P_n\|_p},
\]

for \( 0 \leq m \leq n \), \( 0 \leq k \leq n \).

In 2009, J. Szabados and A. Kroo [5] found the exact constants for Markov-Nikolskii inequalities in \( L_1 \) and \( L_\infty \). Note, that J. Szabados and A. Kroo referred to absolutely monotone polynomials of order \( k \) as “\( k \)-monotone polynomials.”

The next theorem contains theirs results:

Theorem 1.4 (Kroo and Szabados [5], 2009). For \( 2 \leq k \leq n \), \( m = \left\lfloor \frac{n-k}{2} \right\rfloor + 1 \), \( \beta = \frac{1-(n-k)^{n-k}}{2} \):

\[
M_{\infty,\infty}^{(k)}(n, 1) = \frac{k - 1}{1 - x_{1,m}^{(k-2,\beta)}},
\]

\[
M_{1,1}^{(k)}(n, 1) = M_{\infty,\infty}^{(k+1)}(n + 1, 1),
\]

where \( x_{1,m}^{(k-2,\beta)} \) is the largest zero of the Jacobi polynomial \( J_m^{(k-2,\beta)} \), associated with the weight \( (1-x)^{k-2}(1+x)^\beta \).

T. Erdelyi [3] found the order of \( M_{q,p}^{(k)}(n, m) \) in the case \( q \geq p \). He was interested in how this order depends on \( k \).

Theorem 1.5 (Erdelyi [3], 2009). For \( 0 \leq m \leq k/2 \), \( 1 \leq k \leq n \), \( 0 < p \leq q \leq \infty \), we have

\[
M_{q,p}^{(k)}(n, m) \asymp \left( \frac{n^2}{k} \right)^{m+1/p-1/q} \asymp M_{q,p}(n, m).
\]

Second asymptotic is taken when \( k \) is fixed.
it follows from Theorem 1.5 that whenever \( q \geq p \) the order of constants in constrained Markov-Nikolskii inequality remains the same as in the classical case. In this paper, we find exact order for all values of \( 0 < p, q \leq \infty \). In particular, the results imply that the order can be significantly improved whence \( q < p \). Our main result is:

**Theorem 1.6.** For \( 0 < p, q \leq \infty \) and \( p \neq \infty \),

\[
M^{(l)}_{q,p}(n, k) \simeq \begin{cases} 
  n^{2l+2/p-2/q}, & \text{if } l > 1/q - 1/p, \\
  \log^l n, & \text{if } l = 1/q - 1/p, \\
  1, & \text{if } l < 1/q - 1/p.
\end{cases}
\]

If \( p = \infty \), \( 0 < q \leq \infty \) and \( 0 \leq k \leq n \), then

\[
M^{(l)}_{q,\infty}(n, k) \leq C(k, q) \begin{cases} 
  n^{2l-2/q}, & \text{if } q > l, \\
  \log^{l-1} n, & \text{if } l = q, \\
  1, & \text{if } q < l.
\end{cases}
\]

### 2. Proof of the Main Result

The following lemma is well-known (see, for example [2]), however we will need particular estimates for the constants. Thus, the proof is included.

**Theorem 2.1.** If \( q > 0 \), then for every \( n \in \mathbb{N} \) and an interval \( [a, b] \subset [-1, 1] \) such that \( 0 < b - a \leq (cn^2)^{-\max(q,1)} \), the following inequality holds for all \( P_n \in \mathbb{P}_n \):

\[
\|P_n\|_{L_q[a,b]} \leq \frac{C(q)}{c} \|P_n\|_{L_q[-1,1]}.
\]

**Proof.** For every \( P_n \in \mathbb{P}_n \) and \(-1 \leq a < b \leq 1\) we have

\[
\left( \int_a^b |P_n(x)|^q \, dx \right)^{1/q} \leq (b-a)^{1/q} \|P_n\|_{C[a,b]} \leq (b-a)^{1/q} \|P_n\|.
\]

Now, if \( G_n(x) = \int_a^x P_n(t) \, dt \), then

\[
|G_n(x)| = \left| \int_{-1}^x P_n(t) \, dt \right| \leq \int_{-1}^1 |P_n(t)| \, dt \leq \int_{-1}^1 |P_n(t)| \, dt,
\]

and Markov’s inequality implies

\[
(b-a)^{-1/q} \left( \int_a^b |P_n(x)|^q \, dx \right)^{1/q} \leq \|P_n\| = \|G_n'\| \leq (n+1)^2 \|G_n\| \leq (n+1)^2 \int_{-1}^1 |P_n(t)| \, dt.
\]
If \( q \geq 1 \), take \( b - a \leq (cn^2)^{-q} \). Applying Hölder’s inequality we get
\[
\|P_n\|_{L^q[a,b]} = \left( \int_a^b |P_n(x)|^q dx \right)^{1/q} \leq \frac{(n + 1)^2}{cn^2} \int_{-1}^1 |P_n(x)| dx
\]
\[
< \frac{10}{c} \|P_n\|_{L^1[-1,1]}
\]
\[
\leq \frac{20}{c} \|P_n\|_{L^q[-1,1]}.\]

If \( q < 1 \), take \( b - a \leq \frac{1}{cn^2} \) and apply Nikolskii inequality. We get
\[
\|P_n\|_{L^q[a,b]} = \left( \int_a^b |P_n(x)|^q dx \right)^{1/q} \leq (n + 1)^2 \int_{-1}^1 |P_n(x)| dx
\]
\[
\leq C_1(q)(1/cn^2)^{1/q}n^{2/q-2}(n + 1)^2\|P_n\|_{L^q[-1,1]}
\]
\[
\leq \frac{1}{c} \cdot C_1(q)\|P_n\|_{L^q[-1,1]}\]

We start with an upper bound of Theorem 1.6.

**Theorem 2.2.** For \( 0 < p, q \leq \infty \), \( 0 \leq l \leq k \leq n \) and \( p \neq \infty \),
\[
M^{(l)}_{q,p}(n,k) \leq C(k,p,q) \begin{cases} n^{2l+2/p-2/q}, & \text{if } l > 1/q - 1/p, \\ \log^l n, & \text{if } l = 1/q - 1/p, \\ 1, & \text{if } l < 1/q - 1/p. \end{cases}
\]

**Proof.** Consider the case \( k = 1 \). We distinguish between two cases.

**Case 1.** \( q \geq 1 \). Clearly, \( \frac{1}{q} - \frac{1}{p} \leq 1 \). WLOG, we can assume that \( P_n(-1) = 0 \). Note, that for each \( P_n \in \Delta_n^{(1)} \) we have \( \|P'_n\|_{L^1[-1,1]} = \|P_n\| \). By Nikolskii inequality
\[
\|P'_n\|_{L^q[-1,1]} \leq C_1(q)n^{2 - \frac{2}{q}}\|P'_n\|_{L^1[-1,1]},
\]
and
\[
\|P'_n\|_{L^1[-1,1]} \leq \|P_n\| \leq C_1(p)n^{\frac{2}{p}}\|P_n\|_{L^p[-1,1]},
\]
so
\[
\|P'_n\|_{L^1[-1,1]} \leq C_2(q,p)n^{2 - \frac{2}{q} + \frac{2}{p}}\|P_n\|_{L^p[-1,1]}.
\]

**Case 2.** Let \( q < 1 \). We first prove, that for all \( P_n \in \Delta_n^1 \), \( P_n(-1) = 0 \) the following inequality holds:
\[
\int_{-1}^1 P_n^q(x)dx \leq \frac{1}{q} \int_{-1}^1 \frac{P_n^q(x)}{(1-x)^q}dx.
\]
Indeed, integration by parts yields
\[ S = \int_{-1}^{1} \frac{P_n^q(x)}{(1 - x)^q} dx = \frac{1}{q - 1} P_n^q(x)(1 - x)^{1 - q}|_{-1}^{1} + \frac{q}{1 - q} \int_{-1}^{1} P_n'(x) P_n^{q-1}(x)(1 - x)^{1 - q} dx. \]
Since \( P_n(-1) = 0 \), we have
\[ S_1 = \frac{1 - q}{q} S = \int_{-1}^{1} P_n' P_n^{q-1}(x)(1 - x)^{1 - q} dx. \]
We now estimate \( S_1 + S \) to get the result:
\[ S_1 + S = \frac{1}{q} S = \int_{-1}^{1} \left[ \frac{P_n^q(x)}{(1 - x)^q} + P_n'(x) P_n^{q-1}(x)(1 - x)^{1 - q} \right] dx \geq \int_{-1}^{1} P_n^q(x) dx \]
since
\[ \frac{P_n^q(x)}{(1 - x)^q} + P_n'(x) P_n^{q-1}(x)(1 - x)^{1 - q} \geq P_n^q(x) \]
pointwise. Indeed, if
\[ \frac{P_n^q(x)}{(1 - x)^q} \geq P_n^q(x) \]
the inequality clearly holds. In the other case, if
\[ \frac{P_n^q(x)}{(1 - x)^q} < P_n^q(x), \]
then
\[ P_n^{q-1}(x) < (1 - x)^{1 - q} P_n^{q-1}(x) \]
and second term dominates RHS.

Next we show that it is possible to stay bounded away from the endpoints of the interval in the sense, that
\[ \int_{-1}^{1} P_n^q(x) dx \leq C_3(q) \int_{-1}^{1-c/n^2} \frac{P_n^q(x)}{(1 - x)^q} dx. \]
To prove the last inequality, we estimate
\[ \int_{1-c/n^2}^{1} \frac{P_n^q(x)}{(1 - x)^q} dx \leq P_n^q(1) \int_{1-c/n^2}^{1} (1 - x)^{-q} dx = \frac{1}{1 - q} c^{1 - q} P_n^q \| P_n \|_{L_1[-1,1]} \]
\[ \leq c^{1 - q} C(q) \| P_n \|_{L_q[-1,1]}, \]
where the constant \( C_1(q) \) comes from the classical Nikoskii inequality for polynomial \( P_n \) and spaces \( L_1[-1,1] \) and \( L_q[-1,1] \) respectively. Taking \( c \) to be sufficiently small, we can make \( c^{1 - q} C(q) \leq \frac{q}{2} \). Combining this with Lemma 2.1 we can choose such \( c_0 = c_0(p, q) \), that
\[ \int_{-1}^{1} P_n^q(x) dx \leq \frac{2}{q} \int_{1-c/n^2}^{1} \frac{P_n^q(x)}{(1 - x)^q} dx \]
and
\[ \| P_n \|_{L_p[-1,1]} \leq 2 \| P_n \|_{L_p[-1,1-c/n^2]} . \]
We are ready to prove bounds from above for \( k = 1 \). For \( q \leq p \) the result follows from the classical Markov-Nikolskii inequality. Let \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \) and \( r > 0 \). Combining \( \square \) with Young’s inequality we get

\[
\| P_n \|_{L_q[-1,1]} \leq \frac{2}{q} \| P_n(x)(1-x)^{-1} \|_{L_q[-1,1-c/n^2]} \| P_n \|_{L_p[-1,1-c/n^2]} \leq \frac{2}{q} \| (1-x)^{-1} \|_{L_r[-1,1-c/n^2]}.
\]

The only thing left is to observe that

\[
\| (1-x)^{-1} \|_{L_r[-1,1-c/n^2]} \geq \begin{cases} n^{2+2/p-2/q}, & \text{if } 1 > 1/q - 1/p, \\
\log n, & \text{if } 1 = 1/q - 1/p, \\
1, & \text{if } 1 < 1/q - 1/p.
\end{cases}
\]

We prove an upper bound of the theorem for all \( k \) by induction. The base case has been proved above. Let us assume that for each \( P_n \in \triangle_{n}^{k-1} \), \( k \geq 2 \), \( 1 \leq l \leq k-1 \) we have

\[
\| P_n \|_{L_q[-1,1]} \leq C(k - 1, q, p) \begin{cases} n^{2(l-1)+2/p-2/q}, & \text{if } l-1 > 1/q - 1/p, \\
\log^{l-1} n, & \text{if } l-1 = 1/q - 1/p, \\
1, & \text{if } l-1 < 1/q - 1/p.
\end{cases}
\]

Take \( P_n \in \triangle_{n}^{k} \). If \( \frac{1}{q} - \frac{1}{p} = l \), then \( \frac{1}{q} - \frac{1}{p/p+1} = l - 1 \). Using induction hypothesis for \( Q_n = P'_n \in \triangle_{n}^{k-1} \), we get

\[
\| P_n \|_{L_q[-1,1]} \leq C(k - 1, q, p/p + 1) \log^{l-1} n \| P_n \|_{L_p[-1,1]} \leq C(q, p, k) \log^{l} n.
\]

Following the same lines, if \( \frac{1}{q} - \frac{1}{p} > l \), take \( r < \frac{p}{p+1} \) such that \( \frac{1}{q} - \frac{1}{r} > l - 1 \) and use induction hypothesis to arrive at

\[
\| P_n \|_{L_q[-1,1]} \leq C(k - 1, q, r) \| P_n \|_{L_q[-1,1]} \leq C(q, p, k).
\]

If \( \frac{1}{q} - \frac{1}{p} < l \), take \( r > \frac{p}{p+1} \) such that \( \frac{1}{q} - \frac{1}{r} < l - 1 \) and use induction hypothesis to get

\[
\| P_n \|_{L_q[-1,1]} \leq C(k - 1, q, r)n^{2(l-1)+2/p-2/q} \| P_n \|_{L_q[-1,1]} \leq C(q, p, k)n^{2l+2/p-2/q}.
\]

The proof of an upper bound is now complete. \( \square \)

We treat the case \( p = \infty \) separately.
Lemma 2.3. For $0 < q \leq \infty$, $0 \leq l \leq k \leq n$ and $p = \infty$,

$$
M_{q,\infty}^{(l)}(n, k) \leq C(k, q) \begin{cases} 
    n^{2l-2/q}, & \text{if } q > l, \\
    \log^{l-1} n, & \text{if } l = q, \\
    1, & \text{if } q < l.
\end{cases}
$$

Proof. Since $\|P_n\| \geq \|P_n\|_{L_1[-1,1]}$ the result immediately follows from Theorem 2.2 \hfill \square

To prove lower bounds we begin with the following two lemmas:

Lemma 2.4. Consider

$$
Q_n(x) = \sum_{i=1}^{n} \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!} x^k
$$

for $\alpha > 0$. Then

$$
\int_0^1 Q_n^\frac{1}{\alpha}(x)dx \geq C(\alpha) \log n.
$$

Proof. For $\alpha = 1$ the result immediately follows from direct integration. Let $\alpha \neq 1$. We first note, that

$$
\frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!} \sim k^{\alpha-1}
$$

and therefore, $Q_n(1) \sim n^\alpha$. Let us prove that

$$
Q_{n+1}^\frac{1}{\alpha}(x) - Q_n^\frac{1}{\alpha}(x) \geq C(\alpha) x^{n+1},
$$

for all $x \in [0,1]$. Indeed, by the Mean Value Theorem

$$
Q_{n+1}^\frac{1}{\alpha}(x) - Q_n^\frac{1}{\alpha}(x) = \frac{1}{\alpha} (Q_{n+1}(x) - Q_n(x)) \theta^\frac{1}{\alpha-1},
$$

for some $\theta \in [Q_n(x), Q_{n+1}(x)]$. If $\alpha > 1$, then

$$
Q_{n+1}^\frac{1}{\alpha}(x) - Q_n^\frac{1}{\alpha}(x) \geq C_1(\alpha) n^{\alpha-1} x^{n+1} Q_{n+1}^\frac{1}{\alpha}(x)
\geq C_1(\alpha) x^{n+1} n^{\alpha-1} Q_{n+1}^\frac{1}{\alpha-1}(1)
\geq C(\alpha) x^{n+1} n^{\alpha-1} (n^{\alpha})^\frac{1}{\alpha-1}
= C(\alpha) x^{n+1}.
$$

If $\alpha < 1$, we strengthen the result and prove that

$$
\int_{1-1/n}^1 Q_n^\frac{1}{\alpha}(x) dx \geq C_0(\alpha) \log n.
$$
By the Mean Value Theorem
\[ Q_{n+1}^\frac{1}{n}(x) - Q_n^\frac{1}{n}(x) \geq C_1(\alpha) x^{n+1} n^{\alpha-1} Q_n^{\frac{1}{n}}(x) \]
\[ \geq C_1(\alpha) x^{n+1} n^{\alpha-1} \frac{Q_n^1(x)}{Q_n(1)} \]
\[ \geq C_3(\alpha) \frac{1}{n} x^{n+1} Q_n^{\frac{1}{n}}(x) \]
and therefore
\[ Q_{n+1}^\frac{1}{n}(x) \geq Q_n^\frac{1}{n}(x) + C_4(\alpha) \frac{1}{n} Q_n^{\frac{1}{n}}(x), \]
where we again used \( Q_n(1) \sim n^\alpha \).

Now integrating the last inequality over the interval \([1 - 1/n, 1]\) and using that \( x^{n+1} \geq 1/20e \) there, we arrive at
\[ I_{n+1}(\alpha) \geq I_n(\alpha) + C(\alpha) \frac{1}{n} I_n(\alpha), \]
where
\[ I_k(\alpha) = \int_{1-1/n}^1 Q_k^{\frac{1}{n}}(x) dx. \]

Iterating \( \Box \) and using \( 1 + \frac{C(\alpha)}{k} \sim e^{\frac{C(\alpha)}{k}} \) we get the result.

**Lemma 2.5.** Let \( n = 2l, \)
\[ Q_{n,\alpha} = \sum_{i=1}^{n} \frac{\alpha(\alpha + 1)...(\alpha + k - 1)}{k!} x^k, \]
and \( \alpha \leq 1. \) Then
\[ |Q_{n,\alpha}(x)| \leq C(\alpha), \]
for all \( x \in [-1, 0]. \)

**Proof.** Observe, that from Taylor’s formula for \( x \in (-1, 1) \)
\[ (1 - x)^{-\alpha} - Q_{n,\alpha}(x) = \frac{\alpha(\alpha + 1)...(\alpha + n)}{(n+1)!} \int_0^x (1 - y)^{-\alpha-n-1}(x - y)^n dy \]
it follows that in the case \( n = 2l \) the remainder is nonpositive for \( x \in [-1, 0] \) and therefore
\[ 0 < Q_{n,\alpha}(x) = (1 - x)^{-\alpha} - \frac{\alpha(\alpha + 1)...(\alpha + n)}{(n+1)!} \int_0^x (1 - y)^{-\alpha-n-1}(x - y)^n dy \]
\[ \leq 1 + C_1(\alpha) n^{\alpha-1} \leq C(\alpha). \]

We now prove bound from below.

**Theorem 2.6.** For \( 0 < p, q \leq \infty, \ p \neq \infty \) \( 0 \leq l \leq k \leq n \)
\[ M^{(l)}_{q,p}(n,k) \geq C(k, p, q) \begin{cases} n^{2l+2/p-2/q}, & \text{if } l > 1/q - 1/p, \\
\log^l n, & \text{if } l = 1/q - 1/p, \\
1, & \text{if } l < 1/q - 1/p. \end{cases} \]
Proof. Note, that in the case when our order is \( n^{2l+2/p-2/q} \) the polynomial was constructed by Erdelyi, more precisely, the construction in Theorem 1.5 is valid for all \( 0 < p, q \leq \infty \). So of interest is to construct a polynomial \( P_n \in \Delta_n^k \) such that for all \( 0 < l \leq k \)

\[
M_{q,p}^l(n, k) \asymp \log^l n.
\]

Take

\[
Q_n(x) = \sum_{i=1}^{n} \frac{1}{2mq} \frac{1}{2mq} + 1 \ldots \frac{1}{2mq} + k - 1 \frac{x^k}{k!}
\]

and consider

\[
P_n(y) = P_{2mn+k}(y) = \int_{-1}^{y} Q_{2m}^m(x)(y-x)^{k-1} dx.
\]

Clearly, \( \nu = 2mn + k \) and \( \deg P_n = 2mn + k \). It is easy to see, that \( P_n \in \Delta_n^k \) and \( P_n^{(k)}(x) = Q_{2m}^n(x) \). Lemma 2.2 implies \( \| P_n^{(k)} \|_{L_p[-1,1]} \geq C(k, q) \log^{1/q} n \). Thus, we are left to prove that \( \| P_n \|_{L_p[-1,1]} \leq C(k, p) \log^{1/p} n \). Since \( P_n \in \Delta_n^k \) Lemma 2.2 implies that for sufficiently small \( c = c(p) \)

\[
\| P_n \|_{L_p[-1,1]} \leq 2 \int_0^{1-c/n^2} P_n^p(x) dx \leq C(p) \int_0^{1-c/n^2} P_n^p(x) dx.
\]

Now, for \( y > 0 \) using that \( |a+b|^p \leq C(p)(|a|^p + |b|^p) \), \( y-x \leq 1-x \) and \( 0 < Q_n(x) \leq (1-x)^{-1/2mq} \) for \( x \geq 0 \) we can estimate

\[
\int_0^{1-c/n^2} P_n^p(x) dx = \int_0^{1-c/n^2} \left( \int_0^y Q_{2m}^m(x)(y-x)^{k-1} dx \right)^p dy
\]

\[
\leq C(p) \int_0^{1-c/n^2} \left( \int_{-1}^{y} Q_{2m}^n(x)(y-x)^{k-1} dx \right)^p dy
\]

\[
+ C(p) \int_0^{1-c/n^2} \left( \int_{-1}^{0} Q_{2m}^n(x)(y-x)^{k-1} dx \right)^p dy
\]

\[
\leq C(p) \int_0^{1-c/n^2} \left( \int_{-1}^{y} Q_{2m}^n(x)(y-x)^{k-1} dx \right)^p dy
\]

\[
+ 2^k C(p) \left( \int_{-1}^{y} Q_{2m}^n(x)(y-x)^{k-1} dx \right)^p
\]

\[
\leq \int_{-1}^{1-c/n^2} \left( \int_{-1}^{y} (1-x)^{-1/2mq}(1-x)^{k-1} dx \right)^p dy + C_1(p, k, m)
\]

\[
\leq C_2(p) \log n.
\]

It is now straightforward to get a sharp result for all intermediate derivatives of \( k \)-absolutely monotone polynomials by using Theorem 2.2 and Theorem 2.6 \( \Box \)

The result for the case \( p = \infty \) immediately follows from the construction in the 2.6 and the fact \( \| P_n \| = \| P_n^p \|_{L_1[-1,1]} \).
References

[1] Serge Bernstein. Sur l’extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes. *Math. Ann.*, 97(1):1–59, 1927.

[2] Peter Borwein and Tamás Erdélyi. *Polynomials and polynomial inequalities*, volume 161 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[3] Tamás Erdélyi. A Markov Nikolskii type inequality for absolutely monotone polynomials of order $K$. *J. Anal. Math.*, 112:369–381, 2010.

[4] P. Yu. Glazyrina. The sharp Markov-Nikolskii inequality for algebraic polynomials in the spaces $L_q$ and $L_0$ on an interval. *Mat. Zametki*, 84(1):3–22, 2008.

[5] A. Kroó and J. Szabados. On the exact Markov inequality for $k$-monotone polynomials in uniform and $L_1$-norms. *Acta Math. Hungar.*, 125(1-2):99–112, 2009.

[6] Mohammed A. Qazi. On polynomials monotonic on the unit interval. *Analysis (Munich)*, 21(2):129–134, 2001.

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