Abstract

We prove the existence of maximal (and minimal) solution for one-dimensional generalized doubly reflected backward stochastic differential equation (RBSDE for short) with irregular barriers and stochastic quadratic growth, for which the solution $Y$ has to remain between two rcll barriers $L$ and $U$ on $[0, T]$, and its left limit $Y_-$ has to stay respectively above and below two predictable barriers $l$ and $u$ on $[0, T]$. This is done without assuming any $P$-integrability conditions and under weaker assumptions on the input data. In particular, we construct a maximal solution for such a RBSDE when the terminal condition $\xi$ is only $\mathcal{F}_T$-measurable and the driver $f$ is continuous with general growth with respect to the variable $y$ and stochastic quadratic growth with respect to the variable $z$.

Our result is based on a (generalized) penalization method. This method allow us find an equivalent form to our original RBSDE where its solution has to remain between two new rcll reflecting barriers $\bar{Y}$ and $\bar{Y}$ which are, roughly speaking, the limit of the penalizing equations driven by the dominating conditions assumed on the coefficients.

A standard and equivalent form to our initial RBSDE as well as a characterization of the solution $Y$ as a generalized Snell envelope of some given predictable process $l$ are also given.

Key Words: Doubly reflected backward stochastic differential equation, Stochastic quadratic growth, Comparison theorem, Penalization method, Snell envelope.

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1 Introduction

The notion of backward stochastic differential equations with two reflecting barriers has been first introduced by Civitanić and Karatzas [3]. A solution for such equation, associated with a coefficient $f$, a terminal value $\xi$ and two barriers $L$ and $U$, is a quadruple of processes $(Y, Z, K^+, K^-)$ with
values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ satisfying:

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T \left[ f(s, Y_s, Z_s)ds + dK^+_s - dK^-_s - Z_sdB_s \right], \ t \leq T, \\
(ii) & \quad L_t \leq Y_t \leq U_t, \ \forall t \leq T, \ \text{a.s.,} \\
(iii) & \quad \int_0^T (Y_t - L_t) dK^+_t = \int_0^T (U_t - Y_t) dK^-_t = 0, \ \text{a.s.,} \\
(iv) & \quad K^+, K^- \text{ are continuous non-decreasing processes with } K^+_0 = K^-_0 = 0,
\end{align*}
\]

where $(B_t)_{t \leq T}$ is the standard Brownian motion. Cvitanic and Karatzas \cite{CVK1} have proved the existence and uniqueness of solutions if, on the one hand, the coefficient $f$ is uniformly Lipschitz and, on the other hand, the barriers $L$ and $U$ are either regular or satisfy Mokobodski’s condition. This later condition essentially postulates the existence of a quasimartingale between the barriers $L$ and $U$. It has also been shown in \cite{CVK1} that the solution coincides with the value of a stochastic Dynkin game. The link between PDEs with obstacles and RBSDEs has been given in Hamadène and Hassani \cite{HH1}. More studies on RBSDEs can be found in \cite{AVV1, AVV2, AVV3, AVV4, AVV5, AVV6, AVV7, AVV8} and the references therein.

The problem of existence of solutions for generalized doubly reflected backward stochastic differential equation (RBSDE for short), which involves an integral with respect to a continuous increasing process, under weaker assumptions on the input data has been studied in \cite{AVV9} (see also \cite{AVV10, AVV11} for the non-reflecting case and \cite{AVV12} for discontinuous barriers). In \cite{AVV9}, the authors have proved the existence of a maximal solution when the terminal condition $\xi$ is $\mathcal{F}_T$-measurable, the coefficient $f$ is continuous with a general growth with respect to the variable $y$ and a stochastic quadratic growth with respect to the variable $z$. The reflecting barriers $L$ and $U$ are assumed to be continuous. The result has been proved without assuming any $P$-integrability conditions on the terminal data. It should be noted that the integral with respect to the continuous increasing process appears naturally when the authors tried to eliminate the quadratic term by using an exponential transformation. An application of the above result to the Dynkin game problem as well as to the American game option is studied in \cite{AVV13}.

The above result on RBSDE has been generalized by Essaky, Hassani and Ouknine \cite{AVV14} to the case of a RBSDE with two rcll barriers which involves a term of the form $\sum_{t<s \leq T} h(s, Y_{s-}, Y_s)$. The authors have proved the existence of solutions by establishing first a correspondence between the initial RBSDE and another RBSDE whose coefficients are more tractable. They have shown that the existence of solutions for the initial RBSDE is equivalent to the existence of solutions for the auxiliary RBSDE. Since the integrability conditions on parameters are weaker, they have made use of approximations and truncations to establish the existence result for the auxiliary RBSDE.

It should be pointed out that, in the case of continuous or rcll solutions, all the previous papers on RBSDEs were developed in the framework of continuous or rcll obstacles. It is then natural to ask the following question: is there any weaker conditions on the data under which the RBSDE has a solution?

In another context, we recall that, when the barriers is $L^2$-process, Peng and Xu \cite{PENG2} have proved the existence and uniqueness of solutions for RBSDE under Lipschitz condition on the generator and square integrable data where the following condition has been considered instead of condition (iii) in Equation (1.1):

\[
\forall (L^*, U^*) \in \mathcal{D} \times \mathcal{D} \text{ satisfying } L_t \leq L^*_t \leq Y_t \leq U^*_t \leq U_t \text{ a.e a.s.} \text{ we have}
\]

\[
\int_0^T (Y_t - L^*_t) dK^+_t = \int_0^T (U^*_t - Y_t) dK^-_t = 0, \ \text{a.s.}
\]

In this paper, we are concerned with the study of the following RBSDE with irregular barriers, for which the solution $Y$ has to remain between two rcll barriers $L$ and $U$ on $[0, T]$, and its left limit
such that:

\[ Y_t = \xi + \int_0^T \left[ f(s, Y_s, Z_s)ds + g(s, Y_s, Z_s)dA_s + dK^+_t - dK^-_t - Z_sdB_s \right] \]

and

\[ \forall t \in [0, T], \; L_t \leq Y_t \leq U_t, \]

on \([0, T]\), \(Y_{t-} \leq u_t, \; d\alpha_t \text{ a.e.} \; \; l_t \leq Y_{t-}, \; d\delta_t \text{ a.e.} \)

we have \(\int_0^T (Y_{t-} - L_{t-})dK^+_t = \int_0^T (U_{t+} - Y_{t-})dK^-_t = 0, \; \text{a.s.}\)

\[ Y \in \mathcal{D}, \; K^+, K^- \in \mathcal{K}, \; Z \in L^{2,d}, \]

Here, \(\mathcal{D}\) is the set of measurable and right continuous with left limits \((rcll\; \text{for short})\) processes with values in \(\mathbb{R}\), the notation \(dK^+ \perp dK^-\) means that \(K^+\) and \(K^-\) are singular, \(\alpha, \delta\) and \(A\) are predictable, right continuous and nondecreasing processes, the generators \(f\) and \(g\) are assumed to be continuous with general growth with respect to the variable \(y\) and \(f\) satisfies further the following so-called stochastic quadratic growth condition:

\[ \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \; |f(t, \omega, L_t(\omega) \vee y \wedge U_t(\omega), z)| \leq \eta(\omega) + C(\omega)|z|^2, \; dtP(d\omega) \text{ a.e.} \]

Without assuming any \(P\)-integrability conditions on the data, we prove the existence of maximal and minimal solutions to Equation (1.2) by constructing two new \(rcll\) reflecting barriers \(\overline{Y}\) and \(\underline{Y}\) which are in fact the limit of the penalizing equations driven by the dominating conditions assumed on the data. With these new \(rcll\) barriers a new RBSDE is considered. Our new idea consists to deduce the solvability of the original RBSDE (1.2) from the solvability of the new one.

More precisely, we construct two \(rcll\) reflecting barriers \(\overline{Y}\) and \(\underline{Y}\) such that: \((Y, Z, K^+, K^-)\) is a solution of the above RBSDE (1.2) if and only if it is a solution of the following RBSDE with the two reflecting \(rcll\) barriers \(\overline{Y}\) and \(\underline{Y}\):

\[ \begin{align*}
\left\{ \begin{array}{l}
(i) \quad Y_t = \xi + \int_0^T \left[ f(s, Y_s, Z_s)ds + g(s, Y_s, Z_s)dA_s + dK^+_t - dK^-_t - Z_sdB_s \right], \\
(ii) \quad \forall t \in [0, T], \; \underline{Y}_t \leq Y_t \leq \overline{Y}_t, \\
(iii) \quad \int_0^T (\overline{Y}_{t-} - \underline{Y}_{t-})dK^+_t = \int_0^T (\overline{Y}_{t+} - \underline{Y}_{t-})dK^-_t = 0, \; \text{a.s.}, \\
(iv) \quad Y \in \mathcal{D}, \; K^+, K^- \in \mathcal{K}, \; Z \in L^{2,d}, \\
v) \quad dK^+ \perp dK^-.
\end{array} \right. \]

It should be noted that, since the processes \(\overline{Y}\) and \(\underline{Y}\) are \(rcll\), the existence of solutions for this last equation is ensured by the work [9, Theorem 2.1.].

Our second goal is to write the RBSDE (1.2) in a standard form without introducing the test barriers \(L^*\) and \(U^*\). To this purpose, we construct two predictable processes \(l^*\delta(t)\) and \((-u)^*\alpha(t)\), associated to \(l\) and \(u\), by the following formulas (see Section 3 and the Appendix for more details)

\[ l^*\delta(t) = \inf_n \left\{ -nt + \inf \left\{ a \in \mathbb{R} : \int_0^t \left[ l_s(\omega) + ns - a \right] \; d\delta_s(\omega) = 0 \right\} \right\}, \]

(1.3)

\[ (-u)^*\alpha(t) = \inf_n \left\{ -nt + \inf \left\{ a \in \mathbb{R} : \int_0^t \left[ -u_s(\omega) + ns - a \right] \; d\alpha_s(\omega) = 0 \right\} \right\}. \]

(1.4)
and then we prove that our RBSDE (1.2) can be written in a standard form as follows

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T \left[ f(s, Y_s, Z_s) \, ds + g(s, Y_{s-}, Y_s) \, dA_s + dK^+_s - dK^-_s - Z_s \, dB_s \right], \\
(ii) & \quad \forall t \in [0, T], \quad L_{t-} \lor t^\ast(t) \leq Y_{t-} \leq -(u)^\ast(t) \land U_{t-}, \\
(iii) & \quad \int_0^T (Y_{t-} - [L_{t-} \lor t^\ast(t)]) \, dK^+_t = \int_0^T ([U_{t-} \land -(u)^\ast(t)] - Y_{t-}) \, dK^-_t = 0, \ a.s., \\
(iv) & \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^+ \perp dK^-.
\end{align*}
\]

This is done by proving the following characterization:

\[
(\ell(\omega) \leq Y_{t-}(\omega)) \, d\delta_t(\omega) \, P(d\omega)-a.e \quad \text{if and only if} \quad \left( \forall t \in [0, T], \quad t^\ast(t, \omega) \leq Y_{t-}(\omega) \right) \, P-\text{a.s.}
\]

A particular form of such RBSDE is related to the notion of generalized Snell envelope of a predictable process \( l := (l_t)_{0 \leq t \leq T} \) introduced in [10]. Roughly speaking, let \( l := (l_t)_{0 \leq t \leq T} \) be an \( \mathcal{F}_t \)-adapted right continuous with left limits process with values in \( \mathbb{R} \) of class \( D[0, T] \), that is the family \( (l_\nu)_{\nu \in \mathcal{T}} \) is uniformly integrable, where \( \mathcal{T} \) is the set of all \( \mathcal{F}_t \)-stopping times \( \nu \), such that \( 0 \leq \nu \leq T \). The Snell envelope \( \mathcal{S}_t(l) \) of \( l := (l_t)_{0 \leq t \leq T} \) is defined as

\[
\mathcal{S}_t(l) = \text{ess sup}_{\nu \in \mathcal{T}} \mathbb{E} \left[ l_\nu | \mathcal{F}_t \right],
\]

where \( \mathcal{T}_t \) is the set of all stopping times valued between \( t \) and \( T \). According to the work of Mertens (see [3]), \( \mathcal{S} \) is the smallest \( rcll \)-supermartingale of class \( D[0, T] \) which dominates the process \( l \), i.e.,

\[
\forall t \leq T, \quad l_t \leq \mathcal{S}_t(l), \quad P-\text{a.s.}
\]

Suppose now that the process \( l \) is neither of class \( D[0; T] \) nor a \( rcll \) process but just a predictable process. Let \( L \in \mathcal{D} \) and \( \delta \in \mathcal{K} \) and assume that there exists a local martingale \( M_t = M_0 + \int_0^t \kappa_s \, dB_s \) such that \( P-\text{a.s.} \)

\[
L_t \leq M_t \quad \text{on} \quad [0, T] \quad \text{and} \quad l_t \leq M_t \, d\delta_t - a.e. \quad \text{on} \quad [0, T] \quad \text{and} \quad l_T \leq M_T.
\]

Then, we prove that the minimal solution \( Y \) of the following RBSDE with lower barriers \( L \) and \( l \),

\[
\begin{align*}
(i) & \quad Y_t = L_T + \int_t^T \left[ dK^+_s - Z_s \, dB_s \right], \\
(ii) & \quad \forall t \in [0, T], \quad L_{t-} \lor t^\ast(t) \leq Y_{t-}, \\
(iii) & \quad \int_0^T (Y_{t-} - [L_{t-} \lor t^\ast(t)]) \, dK^+_t = 0, \ a.s., \\
(iv) & \quad Y \in \mathcal{D}, \quad K^+ \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}.
\end{align*}
\]

is the smallest \( rcll \) local supermartingale satisfying

\[
\forall t \in [0, T], \quad L_t \leq Y_t, \quad l_t \leq Y_{t-} - d\delta - a.e., \quad \text{on} \quad [0, T] \quad \text{and} \quad l_T \leq Y_T.
\]

The process \( Y \) (denoted by \( \mathcal{S}(L, l d\delta) \)) is called the generalized Snell envelope (see Section 4 for more details). As an example, if we assume that \( \delta = \lambda \) the Lebesgue measure, then \( Y = \mathcal{S}(L, l 1_{(t<T)} + \)
ξ1_{(t=T)}, l(dλ)) the solution of the following RBSDE
\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T \left[ dK_t^+ - Z_s dB_s \right], \ t \leq T, \\
(ii) & \quad l_t^\lambda \leq Y_t^- \text{ on } [0, T], \\
(iii) & \quad \mathbb{E} \int_0^T (Y_t - l_t^\lambda) dK_t^+ = 0, \\
(iv) & \quad Y \in \mathcal{D}, \ K^+ \in \mathcal{K}, \ Z \in \mathcal{L}^{2,d},
\end{align*}
\]
is the smallest local super-martingale dominating the predictable process \( l_t \), i.e.
\[ l_t \leq Y_t, \ d\lambda - a.e \text{ and } \xi \leq Y_T. \]
A second example in the case of \( \delta_t = 1_{(T'_T \leq \xi)} \), where \( T' \) is a stopping time on \([0, T]\) is also given.

Let us describe our plan. In Section 2, we introduce the definition of our RBSDE with irregular barriers. Remarks on assumptions and main result of the paper is introduced in Section 3. In Section 4, we prove the existence of maximal (and minimal) solution of the RBSDE. Section 5 is devoted to find an equivalent and standard form to our initial RBSDE. An application to the notion of generalized Snell envelope is given in Section 6. Finally, most of the material needed in Sections 5 and 6 is given, in a more general setting, in the appendix.

2 Definition of a solution for RBSDEs

2.1 Notations

Let \((B_t)_{t \leq T}\) be a Brownian motion defined on some probability space \((\Omega, \mathcal{F}, P)\) and let \((\mathcal{F}_t)_{t \leq T}\) be the standard augmentation of the filtration generated by \((B_t)_{t \leq T}\).

For simplicity, we omit sometimes dependence on \( \omega \) of some processes or random functions.

The following sets will be frequently used in the sequel.
- \( \mathcal{P} \) is the sigma algebra of \((\mathcal{F}_t)_{t \leq T}\)-predictable sets on \([0, T] \times \Omega \).
- \( \mathcal{D} \) is the set of \( \mathcal{P} \)-measurable and right continuous with left limits (rdl for short) processes \((Y_t)_{t \leq T}\) with values in \( \mathbb{R}\).
- For a given process \( Y \in \mathcal{D} \), we denote: \( Y_{t-} = \lim_{\lambda \downarrow t} Y_s, t \leq T \) (\( Y_{0-} = Y_0 \)), and \( \Delta_s Y = Y_s - Y_{s-} \) the size of its jump at time \( s \).
- \( \mathcal{K} := \{ K \in \mathcal{D} : K \text{ is nondecreasing and } K_0 = 0 \} \).

For a given process \( V \in \mathcal{K} - \mathcal{K} \) and for each \( \omega \in \Omega \), \( dV_t(\omega) \) denotes the signed measure on \([0, T], B_{[0, T]}\) associated to \( V_t(\omega) \) where \( B_{[0, T]} \) is the Borel sigma-algebra on \([0, T]\) and
\[
\int_a^b dV_s = V_b - V_a = \int_{[a, b]} dV_s.
\]
- \( \mathcal{L}^{2,d} \) the set of \( \mathbb{R}^d \)-valued and \( \mathcal{P} \)-measurable processes \((Z_t)_{t \leq T}\) such that
\[
\int_0^T |Z_t|^2 ds < \infty, P - a.s.
\]

The following notations are also needed:
- For a set \( B \), \( 1_B \) denotes the indicator of \( B \).
- For each \((a, b) \in \mathbb{R}^2 \), \( a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}, a^+ = a \vee 0 \) and \( a^- = (-a) \vee 0 \).
- For all \((a, b, c) \in \mathbb{R}^3 \) such that \( a \leq c, a \vee b \wedge c = (a \vee b) \wedge c = a \vee (b \wedge c) \).

\[
\int_0^T |Z_t|^2 ds < \infty, P - a.s.
\]
2.2 The data
Throughout the paper we need the following data.

1. **Terminal data** : \(\xi\) is a \(\mathcal{F}_T\)-measurable one dimensional random variable.

2. **Lower Barriers** :
   a. \(l := \{l_t, 0 \leq t \leq T\}\) is a \(\mathcal{P}\)-measurable process.
   b. \(L := \{L_t, 0 \leq t \leq T\}\) is a process which belong to \(\mathcal{D}\).

3. **Upper Barriers** :
   a. \(u := \{u_t, 0 \leq t \leq T\}\) is a \(\mathcal{P}\)-measurable processes.
   b. \(U := \{U_t, 0 \leq t \leq T\}\) is a process which belong to \(\mathcal{D}\).

4. **Drivers (or generators)** :
   a. \(f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}\) is a function such that for every \((y, z) \in \mathbb{R} \times \mathbb{R}^d\),
      \((t, \omega) \mapsto f(t, \omega, L_t(\omega) \lor y \land U_t(\omega), z)\) is \(\mathcal{P}\)-measurable.
   b. \(g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) is a function such that for any \((x, y) \in \mathbb{R} \times \mathbb{R}\),
      \((t, \omega) \mapsto g(t, \omega, L_t(\omega) - (\omega) \lor x \land U_t(\omega), L_t(\omega) \lor y \land U_t(\omega))\) is \(\mathcal{P}\)-measurable.

5. **Processes** : \(\alpha, \delta\) and \(A\) are processes in \(\mathcal{K}\).

2.3 Definition of a solution
Before giving the definition of our RBSDE, let us first give the following definition.

**Definition 2.1.** Let \(K^1\) and \(K^2\) be two processes in \(\mathcal{K}\). We say that :

1. \(K^1\) and \(K^2\) are singular if and only if there exists a set \(D \in \mathcal{P}\) such that
   \[
   \mathbb{E} \int_0^T 1_D(s, \omega)dK^1_s(\omega) = \mathbb{E} \int_0^T 1_D(s, \omega)dK^2_s(\omega) = 0.
   \]
   This is denoted by \(dK^1 \perp dK^2\).

2. \(dK^1 \leq dK^2\) if and only if for each set \(B \in \mathcal{P}\)
   \[
   \mathbb{E} \int_0^T 1_B(s, \omega)dK^1_s(\omega) \leq \mathbb{E} \int_0^T 1_B(s, \omega)dK^2_s(\omega), \text{ i.e. } K^1_t - K^1_s \leq K^2_t - K^2_s, \forall s \leq t \text{ a.s.}
   \]

Let us now introduce the definition of our RBSDE for which the solution is constrained to stay between given rcll processes \(L\) and \(U\) and \(\mathcal{P}\)-measurable processes \(l\) and \(u\) (conditions \((ii)\) and \((iii)\)).

Two nondecreasing processes \(K^+\) and \(K^-\) are introduced in order to push the solution \(Y\) to stay between the barriers in a minimal way. This minimality property on \(K^+\) and \(K^-\) is ensured by the generalized Skorohod conditions (condition \((iv)\)) together with the additional constraint \(dK^+ \perp dK^-\) (condition \((vi)\)). It should be noted that this orthogonality condition is introduced for the first time in the domain of BSDE in the paper [6].
Definition 2.2. 1. A quadruple \((Y, Z, K^+, K^-) \in D \times \mathcal{L}^{2.d} \times \in K \times K\), is a solution of the RBSDE, associated with the data \((\xi, f dt + gdA, t d\bar{A} + L dt, u d\alpha + Ud\bar{t})\), if (i), (ii), (iii), (iv) and (vi) of Equation (2.3) are satisfied.

2. We say that the RBSDE (1.2) has a maximal (resp. minimal) solution \((Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}\) if for any other solution \((Y'_t, Z'_t, K^+_t, K^-_t)_{t \leq T}\) we have for all \(t \leq T, Y'_t \leq Y_t\), \(P\text{-a.s.}\) (resp. \(Y'_t \geq Y_t, P\text{-a.s.}\)).

In order to rewrite equation (1.2) to a more tractable form we introduce the following set:

\[
\text{Dom} = \left\{ Y \in D : \mathbb{E} \int_0^T (L_{t^-} - Y_{t^-})^+ dt + (l_t - Y_{t^-})^+ d\delta_t + (U_{t^-} - Y_{t^-})^- dt + (u_t - Y_{t^-})^- d\alpha_t = 0 \right\}.
\]

Remark 2.1. It should be pointed out that, for every processes \(Y, Y'\) and \(Y''\) in \(D\), if \(Y, Y' \in \text{Dom}\) and for all \(t \in [0,T], Y_t \land Y'_t \leq Y''_t \leq Y_t \lor Y'_t\) then \(Y'' \in \text{Dom}\).

Remark 2.2. Using the set \(\text{Dom}\), the RBSDE (1.2) can be written as follows:

\[
\begin{cases}
(i) & Y_t = \xi + \int_t^T \left[ f(s, Y_s, Z_s) ds + g(s, Y_s, Z_s) dA_s + dK^+_s - dK^-_s - Z_s dB_s \right], \\
(ii) & \forall L^* \in \text{Dom}, \quad \mathbb{E} \int_0^T (Y_{t^-} - L^*_{t^-})^+ dK^+_t + (U^{*^-}_{t^-} - Y_{t^-}) dK^-_t = 0.
\end{cases}
\]

3 Main result

We shall need the following assumptions:

1. Assumption (A.1) on \(f\):
   There exist two processes \(\eta \in L^0(\Omega, L^1([0,T], dt, \mathbb{R}_+))\) and \(C \in \mathcal{D}\) such that the driver \(f\) satisfies the following conditions:
   (a) for all \((y, z) \in \mathbb{R} \times \mathbb{R}^d, |f(t, \omega, L_t(\omega) \lor y \land U_t(\omega), z)| \leq \eta_t(\omega) + C_t(\omega)|z|^2, dtP(d\omega)\text{-a.e.}\)
   (b) \(dtP(d\omega)\text{-a.e.}, the function \((y, z) \mapsto f(t, \omega, y, z)\) is continuous.

2. Assumption (A.2) on \(g\):
   There exists \(\beta \in L^0(\Omega, L^1([0,T], A(dt), \mathbb{R}_+))\) such that:
   (a) \(A(dt)P(d\omega)\text{-a.e.}, for every \((x, y) \in \mathbb{R} \times \mathbb{R}, |g(t, \omega, L_{t^-}(\omega) \lor x \land U_{t^-}(\omega), L_t(\omega) \lor y \land U_t(\omega))| \leq \beta_t(\omega),\)
   (b) \(A(dt)P(d\omega)\text{-a.e.} the function \((x, y) \mapsto g(t, \omega, x, y)\) is continuous.
   (c) \(P\text{-a.s.}, for every \((t, x) \in ]0,T[ \times \mathbb{R}, the function \(y \mapsto y + g(t, \omega, x, y)\) is nondecreasing.\)
3. Assumption (A.3):

There exists a semimartingale \( S = S_0 + V^- - V^+ + \int_0^\gamma dB_s \), \( S_0 \in \mathbb{R}, V^+ \in \mathcal{K} \) and \( \gamma \in \mathcal{L}^{2,d}, \) such that \( S \in \text{Dom}. \)

The following theorem constitutes the main result of the paper whose proof is postponed to the next section.

**Theorem 3.1.** If assumptions (A.1)-(A.3) hold then the RBSDE (2.3) has a maximal and minimal solution.

Let us give the following remarks on the assumptions.

**Remark 3.1.**

1. By taking \( L_t 1_{[0,T]}(t) + \xi_1(t) \), \( U_t 1_{[0,T]}(t) + \xi_1(t) \) and \( S_t 1_{[0,T]}(t) + \xi_1(t) \), instead of \( L_t \), \( U_t \) and \( S_t \) respectively, we can assume without loss of generality, that 
   \[ L_T = U_T = S_T = \xi. \]

2. It should be pointed out that conditions (A.1)(a) and (A.2)(a) hold if the functions \( f \) and \( g \) satisfy the following: \( \forall (s,\omega), \forall x,y \in \mathbb{R}, \forall z \in \mathbb{R}^d, \)
   \[ |f(s,\omega, y, z)| \leq \varphi(|y|) \left( \tilde{\eta}(\omega) + \tilde{C}(\omega)|z|^2 \right), \]
   \[ |g(s,\omega, x, y)| \leq \varphi(|x| + |y|) \tilde{\eta}(\omega), \]
   where \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a nondecreasing function, \( \tilde{\eta} \in L^0(\Omega, L^1([0,T], ds, \mathbb{R}^d)), \tilde{C} \in \mathcal{D} \) and \( \tilde{\gamma} \in L^0(\Omega, L^1([0,T], dA_s, \mathbb{R}_+)). \)
   Indeed, in conditions (A.1)(a) and (A.2)(a), we have just to take the processes \( \eta, C \) and \( \beta \) as follows:
   \[ \eta_t(\omega) = \varphi(D_t(\omega)) \tilde{\eta}(\omega), \]
   \[ C_t(\omega) = \varphi(D_t(\omega)) \tilde{C}_t(\omega) \]
   and \( \beta_t(\omega) = \varphi(D_t(\omega)) \tilde{\gamma}(\omega), \)
   where
   \[ D_t = 2 \sup_{s \leq t} \left( U_s^+ + L_s^- \right). \]
   This means that the functions \( f \) and \( g \) can have, in particular, a general growth with respect to \( (x, y) \) and stochastic quadratic growth with respect to \( z \). In this respect, assumptions (A.1)(a) and (A.2)(a) are not restrictive.

3. Suppose that there exist two processes \( \overline{\mathcal{L}}, \overline{\mathcal{U}} \in \text{Dom} \) completely separated, i.e. the processes \( \overline{\mathcal{L}} \)

and \( \overline{\mathcal{U}} \) are such that:

\[ \overline{\mathcal{L}}_t < \overline{\mathcal{U}}_t \quad \text{on} \quad [0, T] \quad \text{and} \quad \overline{\mathcal{L}}_{t-} < \overline{\mathcal{U}}_{t-} \quad \text{on} \quad [0, T], \]

then assumption (A.3) holds true. Indeed by setting

\[ k_t = 1 + \sup_{s \leq t} (|\overline{\mathcal{L}}_s| + |\overline{\mathcal{U}}_s|), \]

\[ L_t' = \frac{\overline{\mathcal{L}}_t}{k_t} 1_{(t<T)} + \frac{\overline{\mathcal{U}}_T - \overline{\mathcal{L}}_t}{k_t} 1_{(t=T)}, \quad \text{and} \quad U_t' = \frac{\overline{\mathcal{U}}_t}{k_t} 1_{(t<T)} + \frac{\overline{\mathcal{L}}_T - \overline{\mathcal{U}}_t}{k_t} 1_{(t=T)}, \]

we have for all \( t \in [0, T] \)

\[ -1 \leq L_t' < U_t' \leq 1 \quad \text{and} \quad L_{t-}' < U_{t-}'. \]

It follows then from [Theorem 4.1., [10]] that there is a semimartingale \( S' \) such that for all \( t \in [0, T], \)

\[ L_t' \leq S_t' < U_t', \]

and then the semimartingale \( S_t'k_t 1_{(t<T)} \) is between \( \overline{\mathcal{L}}_t 1_{(t<T)} \) and \( \overline{\mathcal{U}}_t 1_{(t<T)} \). Hence \( S_t'k_t \in \text{Dom}. \)
4 Proof of the main result

This section is devoted to the proof of the existence of maximal solution to Equation (4.5) by using a penalization method. This method allows us to construct two reflexive barriers which are in fact the limit of penalized equations driven by the dominating conditions assumed on \( f \) and \( g \). However, our approach consists of deducing the solvability of a RBSDE (4.5) from a suitable RBSDE with two reflexive barriers which is equivalent to our initial RBSDE and its solvability is ensured by [9].

### 4.1 Comparison theorem for maximal solutions

Let us now give the following comparison theorem which plays a crucial role in the proof of our main result. For this reason, suppose that assumptions (A.1)–(A.2) hold and \((Y, Z, K^+, K^-)\) is the maximal solution for the following RBSDE

\[
\begin{cases}
(i) & Y_t = \xi + \int_t^T [f(s, Y_s, Z_s)ds + g(s, Y_s, Z_s)dB_s + dK^+_s - dK^-_s - Z_sdB_s], \\
(ii) & \forall t \in [0, T], \quad L_t \leq Y_t \leq U_t, \\
(iii) & \int_0^T (Y_{t-} - L_{t-})dK^+_t = \int_0^T (U_{t-} - Y_{t-})dK^-_t = 0, \quad \text{a.s.,} \\
(iv) & \forall \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2, d}, \\
(v) & dK^+ \perp dK^-.
\end{cases}
\]

Let \((Y', Z', K'^+, K'^-)\) be a solution for the following RBSDE

\[
\begin{cases}
(i) & Y'_t = \xi' + \int_t^T \left[ dA'_t + dK'^+_t - dK'^-_t - Z'_tdB_s \right], \\
(ii) & \forall t \in [0, T], \quad L'_t \leq Y'_t \leq U'_t, \\
(iii) & \int_0^T (Y'_{t-} - L'_{t-})dK'^+_t = \int_0^T (U'_{t-} - Y'_{t-})dK'^-_t = 0, \quad \text{a.s.,} \\
(iv) & \forall \in \mathcal{D}, \quad K'^+, K'^- \in \mathcal{K}, \quad Z' \in \mathcal{L}^{2, d}, \\
(v) & dK'^+ \perp dK'^-.
\end{cases}
\]

where \( R \in \mathcal{K}, A' \in \mathcal{K} - \mathcal{K}, L' \) and \( U' \) are two barriers which belong to \( \mathcal{D} \).

To derive a comparison theorem, we assume the following assumption.

**Assumption (H):**

(H.1) \( \xi' \leq \xi, \quad Y'_t \leq U_t, \quad L'_t \leq Y_t, \quad \forall t \in [0, T] \).

(H.2) \( dA'_s \leq f(s, Y'_s, Z'_s)ds + g(s, Y'_s, Z'_s)dB_s \) on \([0, T] \).

Suppose that assumptions (A.1)–(A.2) are in force. From Theorem 6.1 in the paper [9], we have the following comparison theorem for maximal solutions.

**Theorem 4.1.** [Comparison theorem for maximal solutions] Under hypothesis (H), we get

1. \( Y'_t \leq Y_t \), for every \( t \in [0, T] \), \( P \)-a.s.

2. \( 1_{\{U'_t = U_t\}}dK'^-_t \leq dK^-_t \) and \( 1_{\{L'_t = L_t\}}dK'^+_t \leq dK'^+_t \).

**Proof.** In order to prove Theorem 4.1 we should only verify that assumptions of Theorem 6.1 in [9] are satisfied. To begin with, set 

\[
b_t = R_t + |A'_t|.
\]
Let \( a \in \mathcal{K}, \alpha \in L^0(\Omega, L^1([0,T], dt)) \) such that
\[
db_t = \alpha_t \, dt + \, da_t, \quad \, da_t \perp dt.
\]

Put
\[
\begin{align*}
(i) & \quad \bar{f}(s,y,z) = f(s,y,z) + \alpha_s g(s,y,y) \frac{dR_s}{db_s} \\
(ii) & \quad \bar{g}(s,y) = g(s,y,y) \frac{dR_s}{db_s} \\
(iii) & \quad \bar{h}(s,x,y) = h(s,x,y) \frac{dR_s}{db_s} \Delta a_s \\
(iv) & \quad \bar{f}(s) = \alpha_s \frac{dA'_s}{db_s} \\
(v) & \quad \bar{g}(s) = \frac{dA'_s}{db_s} \\
(v) & \quad \bar{h}(s) = \frac{dA'_s}{db_s} \Delta a_s.
\end{align*}
\]

where \( \frac{dR}{db} \) (respectively \( \frac{dA'}{db} \)) is the Radon-Nikodym derivative of the measure \( dR \) (respectively \( dA' \)) by the measure \( db \).

Hence Equations (4.6)-(4.7) can be written respectively as follows:
\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T \left[ f(s,Y_s,Z_s) \, ds + \bar{g}(s,Y_s) \, da_s + dK_t^+ - dK_t^- - Z_s \, dB_s \right] + \sum_{t<s\leq T} \Delta Y_{s-}, \\
(ii) & \quad \forall t \in [0,T], \quad L_t \leq Y_t \leq U_t, \\
(iii) & \quad \int_0^T (Y_{t-} - L_{t-}) \, dK_t^+ = \int_0^T (U_{t-} - Y_{t-}) \, dK_t^- = 0, \text{ a.s.,} \\
(iv) & \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^+ \perp dK^-.
\end{align*}
\]

and
\[
\begin{align*}
(i) & \quad Y'_t = \xi' + \int_t^T \left[ \bar{f}(s) \, ds + \bar{g}(s) \, da_s + dK_t'^+ - dK_t'^- - Z'_s \, dB_s \right] + \sum_{t<s\leq T} \Delta Y'_{s-}, \\
(ii) & \quad \forall t \in [0,T], \quad L'_t \leq Y'_t \leq U'_t, \\
(iii) & \quad \int_0^T (Y'_{t-} - L'_{t-}) \, dK_t'^+ = \int_0^T (U'_{t-} - Y'_{t-}) \, dK_t'^- = 0, \text{ a.s.,} \\
(iv) & \quad Y'_0 \in \mathcal{D}, \quad K'^+, K'^- \in \mathcal{K}, \quad Z' \in \mathcal{L}^{2,d}, \\
(v) & \quad dK'^+ \perp dK'^-.
\end{align*}
\]

where \( a^c_s = a_s - \sum_{0<\tau\leq s} a_{\tau-} \), is the continuous part of the process \( a \). By assumptions (H.1) – (H.2) we get
\[
\frac{dA'_s}{db_s}(\alpha_s \, ds + da_s) \leq f(s,Y'_s,Z'_s) \, ds + g(s,Y'_s,Y'_s) \frac{dR_s}{db_s}(\alpha_s \, ds + da_s).
\]

Since \( da_s \perp ds \), we obtain from the above inequality that
\[
\begin{align*}
(i) & \quad \frac{dA'_s}{db_s} \alpha_s \, ds \leq \left[ f(s,Y'_s,Z'_s) \, ds + \alpha_s g(s,Y'_s,Y'_s) \frac{dR_s}{db_s} \right] \, ds \\
(ii) & \quad \frac{dA'_s}{db_s} \, ds \leq g(s,Y'_s,Y'_s) \frac{dR_s}{db_s} \, ds \\
(iii) & \quad \frac{dA'_s}{db_s} \Delta a_s \leq g(s,Y'_s,Y'_s) \frac{dR_s}{db_s} \Delta a_s.
\end{align*}
\]
which is equivalent to

\[
\begin{align*}
(i) & \quad \overline{f}(s) ds \leq \overline{f}(s, Y_s', Z_s') ds \\
(ii) & \quad \underline{g}(s) da_s^c \leq \underline{g}(s, Y_s') da_s^c \\
(iii) & \quad \underline{h}(s) \leq \underline{h}(s, Y_s', Z_s').
\end{align*}
\]

(4.11)

It follows that assumptions of Theorem 6.1 in [9] are satisfied for equations (1.8) and (1.9). Therefore

1. \( Y'_t \leq Y_t \), for every \( t \in [0, T] \), \( P \)-a.s.
2. \( 1_{\{u_t = u_t^-\}} dK_t^- \leq dK_t^- \) and \( 1_{\{u_t = u_t^+\}} dK_t^+ \leq dK_t^+ \).

\[\]

4.2 Penalization method

To begin, we set for every \( s \in [0, T] \),

\[ m_s = 1 + 8 \sup_{r \leq s} \left| C_r \right|. \]

We consider \( (Y^n, Z^n, K^{n+}, K^{n-}) \) the minimal solution of the following penalized RBSDE where the driver is derived from the conditions assumed on \( f \) and \( g \) and the semimartingale \( S \):

\[
\begin{align*}
(i) & \quad Y_s^n = \xi - \int_t^T \left[ (\eta_s + 4C_s|\gamma_s|^2 + \frac{1}{2} m_s|Z_s^n - \gamma_s|^2) ds - (dV_s^+ + dV_s^- + \beta_s dA_s) \\
& \quad + n(\xi - Y_s^n) + d\delta_s + dK_s^{n+} - dK_s^{n-} - Z_s^n dB_s \right], t \leq T, \\
(ii) & \quad \forall t \in [0, T[, L_t \leq Y_t^n \leq S_t, \\
(iii) & \quad \int_0^T (Y_{t-}^n - L_{t-}) d\delta_t^+ = \int_0^T (S_{t-} - Y_{t-}^n) dK_t^{n-} = 0, P \text{- a.s.}, \\
(iv) & \quad Y_t^n \in D, \quad K^{n+}, K^{n-} \in K, \quad Z^n \in L^2, \\
(v) & \quad dK^{n+} \perp dK^{n-}.
\end{align*}
\]

where \( S, V^+, V^- \) and \( \gamma \) are the processes appeared in Assumption (A.3).

Let also \( (\overline{Y}^n, \overline{Z}^n, \overline{K}^{n+}, \overline{K}^{n-}) \) be the maximal solution of the following penalized RBSDE:

\[
\begin{align*}
(i) & \quad \overline{Y}_s^n = \xi - \int_t^T \left[ (\eta_s + 4C_s|\gamma_s|^2 + \frac{1}{2} m_s|Z_s^n - \gamma_s|^2) ds - (dV_s^+ + dV_s^- + \beta_s dA_s) \\
& \quad - n(\overline{Y}_s^n - \overline{Y}_s) + d\delta_s + d\overline{K}_s^{n+} - d\overline{K}_s^{n-} - \overline{Z}_s^n dB_s \right], t \leq T, \\
(ii) & \quad \forall t \in [0, T[, S_t \leq \overline{Y}_{t-}^n \leq U_t, \\
(iii) & \quad \int_0^T (\overline{Y}_{t-}^n - S_{t-}) d\delta_t^+ = \int_0^T (U_{t-} - \overline{Y}_{t-}) d\overline{K}_t^{n-} = 0, P \text{- a.s.}, \\
(iv) & \quad \overline{Y}_t^n \in D, \quad \overline{K}^{n+}, \overline{K}^{n-} \in K, \quad \overline{Z}^n \in L^2, \\
(v) & \quad d\overline{K}^{n+} \perp d\overline{K}^{n-}.
\end{align*}
\]

We should point out here that, since the barriers are \( rell \) and the drivers are of stochastic quadratic growth, the existence of minimal (resp. maximal) solution to (4.12) (resp. (4.13)) is ensured by the work [9, Theorem 2.1].

\[\]
4.3 Study of the penalized equations (4.12)-(4.13)

In this subsection, we will prove that limiting processes $\overline{Y}$ and $\underline{Y}$ of $Y_n$ and $\overline{Y}^n$ respectively are in Dom. To begin with, we recall that the semimartingale $S$ is given by

$$S_t = \xi - \int_0^T dV_t^- + \int_0^T dV_t^+ - \int_0^T \gamma_s dB_s.$$

and consider the solution $Y_n$ of Equation 4.13. Then assumptions (H.1) and (H.2) of Theorem 4.1 are satisfied by taking $\xi' = \xi, L' = S, U' = U, A' = V^+ - V^-$

and

$$Y' = S, Z' = \gamma, dK_+ = dK_- = 0,$$

and

$$f(s, y, z) = \eta_s + 4C_s|\gamma_s|^2 + \frac{1}{2}m_s|z - \gamma_s|^2, \quad dR_s = dV_s^+ + dV_s^- + \beta_s dA_s + d\alpha_s,$$

$$g(s, x, y) = -n(x - u_s) \frac{d\alpha_s}{dR_s} + \frac{dV_s^+ + dV_s^- + \beta_s dA_s}{d\alpha_s}, \quad L = S,$$

$$Y = \overline{Y}, K^{\pm} = \overline{K}^{n\pm}, Z = \overline{Z}.$$

Applying comparison theorem (Theorem 4.1) to $Y = \overline{Y}$ and $Y' = S$, it follows that

$$dK_n^- + \int_0^T 1 \{L_s = L_s'\} d\overline{K}_s^{n+} \leq dK_s' = 0.$$

Henceforth

$$dK_s^{n+} = 0.$$

By a symmetric argument, it follows also that for every $n \in \mathbb{N},$

$$dK_s^{n-} = 0.$$

Again, by using comparison theorem (Theorems 4.1) we get also that

$$L_t \leq Y_t^n \leq Y_t^{n+} \leq S_t \leq Y_t^{n+1} \leq \overline{Y}_t \leq Y_t \leq U_t.$$

Set

$$\overline{Y}_t = \inf_n \overline{Y}_t^n, \quad \underline{Y}_t = \inf_n \underline{Y}_t^n,$$

$$\overline{Y}_t = \sup_n \overline{Y}_t^n, \quad \underline{Y}_t = \sup_n \underline{Y}_t^n.$$

By letting $n$ to infinity in (4.14) and using assumption (A.3) we get that the semimartingale $S$ is between $\underline{Y}$ and $\overline{Y}$. More precisely, we have the following.

**Proposition 4.1.** For every $t \in [0, T], we get

$$L_t \leq \underline{Y}_t \leq S_t \leq \overline{Y}_t \leq U_t \quad \text{and} \quad L_t \leq \underline{Y}_t \leq S_t \leq \overline{Y}_t \leq U_t.$$ 

**Proposition 4.2.** The processes $\overline{Y}$ and $\underline{Y}$ defined by (4.15) are in Dom. In particular, $\overline{Y}$ and $\underline{Y}$ are rcll.
Proof. Let \( R_t = \int_0^t \left[ \left( \eta_s + 4C_s \gamma_s \right)^2 \right] ds + 2dV^- + \beta_s dA_s \). We have

\[
\bar{Y}_t^n - S_t = \int_t^T dR_s + \int_t^T \frac{m_s}{2} \left( \bar{Z}_s^n - \gamma_s \right)^2 ds - n \int_t^T \left( \bar{Y}_s^n - u_s \right) \, d\alpha_s - \int_t^T m_s \, dK^{-}_s - \int_t^T \left( \bar{Z}_s^n - \gamma_s \right) dB_s.
\]

Then

\[
m_t(\bar{Y}_t^n - S_t) = \int_t^T m_{s-}dR_s + \frac{1}{2} \int_t^T m_s(\bar{Z}_s^n - \gamma_s)^2 \, ds - n \int_t^T m_{s-}(\bar{Y}_s^n - u_s)^+ \, d\alpha_s - \int_t^T m_{s-} dK^{-}_s
\]

\[
- \int_t^T \psi(m_s(\bar{Y}_s^n - \gamma_s)) \, dB_s - \int_t^T \psi(m_s(\bar{Y}_s^n - \gamma_s)) \, dm_s
\]

\[
- \frac{1}{2} \int_t^T \psi''(m_s(\bar{Y}_s^n - \gamma_s)) \left| m_s(\bar{Z}_s^n - \gamma_s) \right|^2 \, ds
\]

\[
- \sum_{t<s \leq T} \left[ \psi(m_s(\bar{Y}_s^n - \gamma_s)) - \psi(m_s(\bar{Y}_s^n - \gamma_s)) - \psi(m_s(\bar{Y}_s^n - \gamma_s)) \Delta m_s \right].
\]

Hence

\[
\psi(m_t(\bar{Y}_t^n - S_t)) = 1 + \int_t^T \psi(m_{s-}(U_{s-} - S_{s-})) \, m_{s-} dR_s
\]

\[
- \int_t^T dV^n_s - \int_t^T \psi(m_s(\bar{Y}_s^n - S_s)) m_s(\bar{Z}_s^n - \gamma_s) dB_s,
\]

(4.16)

where \( V^n_s \) is the process in \( K \) given by

\[
V^n_t = \int_0^t \left[ \psi(m_{s-}(U_{s-} - S_{s-})) - \psi(m_{s-}(\bar{Y}_{s-}^n - S_{s-})) \right] m_{s-} dR_s
\]

\[
+ n \int_0^t \psi(m_{s-}(\bar{Y}_{s-}^n - S_{s-})) (\bar{Y}_{s-}^n - u_s)^+ \, d\alpha_s + \int_0^t \psi(m_{s-}(\bar{Y}_{s-}^n - S_{s-})) m_{s-} dK^{-}_s
\]

\[
+ \int_0^t \psi(m_{s-}(\bar{Y}_{s-}^n - S_{s-})) (\bar{Y}_{s-}^n - S_s) \, dm_s
\]

\[
+ \sum_{0<s \leq t} \psi(m_{s-}(\bar{Y}_{s-}^n - S_{s-})) \left[ \psi(\Delta m_s(\bar{Y}_s^n - S)) - 1 - \Delta m_s(\bar{Y}_s^n - S) \right].
\]

(4.17)
Let (\tau_i) be the sequence of stopping times defined by
\begin{align*}
\tau_i = \inf \left\{ s \geq 0 : -D_s \geq i \right\} \land T.
\end{align*}
We should note here that the family (\tau_i) satisfies the following property
\begin{align*}
P \left[ \bigcup_{i=1}^{\infty} (\tau_i = T) \right] = 1.
\end{align*}
We say that (\tau_i) is a stationary sequence of stopping times. Now, set
\begin{align*}
^i M^n_t &= M^n_0 1_{\{ t < \tau_i \}} + M^n_{\tau_i} 1_{\{ t \geq \tau_i \}}, \\
^i V^n_t &= V^n_0 1_{\{ t < \tau_i \}} + V^n_{\tau_i} 1_{\{ t \geq \tau_i \}}, \\
^i \hat{Z}^n_t &= I_{\{ s < \tau_i \}} Z^n_s.
\end{align*}
Then we have
\begin{align*}
^i M^n_t &= M^n_0 - ^i V^n_t - \int_0^t ^i \hat{Z}^n_s dB_s. 
\end{align*}
It follows from this last equation that:
1. \( Y \) and \( \hat{Y} \) are \( \text{rrl} \): In fact, we have that \((^i M^n_t)\) is a \( \text{rrl} \) supermartingale satisfying
\begin{align*}
-i \leq ^i M^n_t \leq ^i M^{n+1}_t \leq 0.
\end{align*}
It follows then from Dellacherie and Meyer [Theorem 18 Chapter 5 page 79, 4] that \( \sup_n \, ^i M^n_t \) is also a \( \text{rrl} \) process (supermartingale). Then \( \psi(m_t(Y_t - S_t)) \) is \( \text{rrl} \) on \([0, \tau_i]\), but \((\tau_i)_{i \geq 0}\) is a stationary sequence of stopping times, then \( \hat{Y} \) is \( \text{rrl} \) on \([0, T]\).
By the same way, we obtain that \( \hat{Y} \) is \( \text{rrl} \).
2. \( Y_{s-} \leq u_s \) \( \text{a.e. on } [0, T] \): Indeed, since all terms in equation 4.17 are positive it follows that
\begin{align*}
\eta E \int_0^{\tau_i} m_{s-} \psi(m_{s-}(Y^n_{s-} - S^n_{s-})) \left( Y^n_{s-} - u_s \right)^+ ds \leq \eta E V^n_{\tau_i}.
\end{align*}
By using a localization procedure and taking the expectation in equation 4.19 we have
\begin{align*}
E(Y^n_T) = E(M^n_0 - ^i M^n_T).
\end{align*}
Hence

$$\mathbb{E}(V^n_{\tau_\ell^-}) = \mathbb{E}(\mathcal{M}^n_{\ell} - \mathcal{M}^n_{\tau_\ell^-})$$

Since $\mathcal{M}^n_{\ell} \leq 0$, it follows from inequality (4.15) and the definition of $\tau_\ell$ that

$$\mathbb{E}V^n_{\tau_\ell^-} = \mathbb{E}(\mathcal{M}^n_{\ell} - \mathcal{M}^n_{\tau_\ell^-}) \leq \mathbb{E}(-D_{\tau_\ell^-}) \leq \delta.$$ 

Therefore

$$\mathbb{E}\int_0^{\tau_\ell^-} m_s \psi \left(m_s \left(\overline{Y}^n_{s^-} - Y_{s^-}\right) \left(\overline{Y}^n_{s^+} - Y_{s^+}\right)^+ \right) \, ds \leq \frac{\delta}{n}.$$ 

Fatou’s lemma gives

$$\mathbb{E}\int_0^{\tau_\ell^-} m_s \psi \left(m_s \left(\overline{Y}^n_{s^-} - Y_{s^-}\right) \left(\overline{Y}^n_{s^+} - Y_{s^+}\right)^+ \right) \, ds = 0.$$ 

Hence

$$\overline{Y}^n_{s^-} \leq Y_{s^-} \text{ a.e. on } [0, T].$$

But for every $s \in [0, T]$ and $n \in \mathbb{N}$, $\overline{Y}^n_{s^-} \leq \overline{Y}^n_{s^-}$ then $Y_{s^-} \leq \overline{Y}^n_{s^-}$. Consequently $\overline{Y}^n_{s^-} \leq Y_{s^-} \text{ a.e. on } [0, T].$

Assume now that $\overline{Y}^-_{T^+} > u_T$ and $\Delta_T \sigma > 0$. It follows from [9, Lemma 3.1.] that

$$\overline{Y}^n_{T^-} = S_{T^-} \vee \left[ \xi + \Delta_T V^+ + \Delta_T V^- + \beta_T \Delta_T A_n \left(\overline{Y}^n_{T^-} - u_T\right)^+ \Delta_T \sigma \right] \land U_{T^-}.$$

Since $-n(\overline{Y}^n_{T^-} - u_T)^+$ converges to $-\infty$ if $n$ goes to $+\infty$, we get $\overline{Y}^-_{T^-} = S_{T^-}$. Now since $S_{T^-} \leq \overline{Y}^-_{T^-} \leq \overline{Y}^-_{T^-}$ we have $\overline{Y}^-_{T^-} = S_{T^-}$. Hence $S_{T^-} > u_T$ which is absurd since $\Delta_T \sigma > 0$ and $S_{T^-} \leq u_T$, $\delta = 0$, $a.e.$ on $[0, T]$. Consequently

$$\overline{Y}^n_{s^-} \leq Y_{s^-} \text{ a.e. on } [0, T].$$

3. By the same method as in the previous step, we get also that

$$l_t \leq \overline{Y}^-_{s^-} \text{ a.e. on } [0, T].$$

Hence $Y$ and $\overline{Y}$ are in Dom. The proof of Proposition 4.2 is finished.

4.4 Proof of Theorem 3.1

With the help of processes $Y$ and $\overline{Y}$, another RBSDE is considered which is equivalent to our original RBSDE. More precisely, we have the following proposition.

**Proposition 4.3.** $(Y, Z, K^+, K^-)$ is a solution of RBSDE (2.5) if and only if it is a solution of the following RBSDE

\[
\begin{align*}
(i) \quad Y_t &= \xi + \int_t^T [f(s, Y_s, Z_s)ds + g(s, Y_s, Z_s)dA_s + dK^+_s - dK^-_s - Z_s dB_s], t \leq T, \\
(ii) \quad \forall t \in [0, T], \quad Y_t \leq Y_t, \\
(iii) \quad \int_0^T (Y_t - \overline{Y}^-_{s^-})dK^+_t = \int_0^T (\overline{Y}^n_{T^-} - Y_{s^-})dK^-_t = 0, a.s., \\
(v) \quad Y \in \mathcal{D}, \quad K^+, K^- \in K, \quad Z \in \mathcal{L}^{2d}, \\
(vi) \quad dK^+ \perp dK^-.
\end{align*}
\]
Proof. Let \((Y, Z, K^+, K^-)\) be a solution of RBSDE \((2.5)\). Put

\[ U_t^* = Y_t \vee \bar{Y}_t \quad \text{and} \quad L_t^* = Y_t \wedge \bar{Y}_t. \]

It is obvious that \(L_t^* \leq Y_t \leq U_t^*\). By Remark \((2.1)\), it follows that \(U^*, L^*\) are in \(\text{Dom}\). Then \((Y, Z, K^+, K^-)\) is a solution of the following RBSDE

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T \left[ f(s, Y_s, Z_s)ds + g(s, Y_{s-}, Y_s) dA_s + dK_s^+ - dK_s^- - Z_s dB_s \right], t \leq T; \\
(ii) & \quad \forall t \in [0, T], L_t^* \leq Y_t \leq U_t^*, \\
(iii) & \quad \int_0^T (Y_{s-} - L_{s-}) dK_s^+ = \int_0^T (U_{s-} - Y_{s-}) dK_s^- = 0, \ a.s., \\
(iv) & \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2, d}, \\
(vi) & \quad dK^+ \perp dK^-.
\end{align*}
\]

Since \(Y_s \leq U_s\) and \(L_s^* \leq Y_s \leq Z_s \leq \bar{Y}_s\),

\[
\begin{align*}
& \begin{align*}
& (a) \quad f(s, Y_s, Z_s)ds + g(s, Y_{s-}, Y_s) dA_s \\
& \quad \leq (\eta_s + C_s |Z_s|^2)ds + \beta_s dA_s \\
& \quad \leq (\eta_s + 4C_s |\gamma_s|^2 + \frac{m_s}{2} |Z_s - \gamma_s|^2)ds + \beta_s dA_s + dV_s^+ + dV_s^- - n(u_s - Y_s-)^- d\alpha_s, \\
& \quad = 0
\end{align*}
\end{align*}
\]

then it follows from comparison theorem (Theorem \((4.1)\)) applied to \(Y\) and \(\bar{Y}^n\), that for all \(n \in \mathbb{N}\),

\[ Y_t \leq \bar{Y}_t^n, \]

and then \(Y_t \leq \bar{Y}_t\). Hence \(U_t^* = \bar{Y}_t\). By a symmetric argument we get also that \(L_t^* = Y_t\). Therefore \((Y, Z, K^+, K^-)\) is a solution to RBSDE \((4.21)\).

Conversely, suppose now that \((Y, Z, K^+, K^-)\) is a solution of \((4.21)\). In order to prove that \((Y, Z, K^+, K^-)\) is a solution of RBSDE \((2.5)\), we just need to prove \((iii)\) of RBSDE \((2.5)\). Let \(L^* \in \text{Dom}\) and consider \((Y^*, Z^*, K^{*,+}, K^{*,-})\) the minimal solution of the following RBSDE

\[
\begin{align*}
(i) & \quad Y_t^* = \xi + \int_t^T \left[ f(s, Y_s, Z_s) - \frac{m_s}{2} |Z_s^* - Z_s|^2 \right] ds + g(s, Y_{s-}, Y_s) dA_s + dK_s^{*,+} - dK_s^{*-} - Z_s dB_s, \\
(ii) & \quad \forall t \in [0, T], Y_t \leq Y_t^* \leq U_t \cup Y_t, \\
(iii) & \quad \int_0^T (Y_{s-} - L_{s-}) dK_s^{*,+} = \int_0^T (U_{s-} - Y_{s-}) dK_s^{*,-} = 0, \ a.s., \\
(iv) & \quad Y^* \in \mathcal{D}, \quad K^{*,+}, K^{*,-} \in \mathcal{K}, \quad Z^* \in \mathcal{L}^{2, d}, \\
(vi) & \quad dK^{*,+} \perp dK^{*,-}.
\end{align*}
\]

We note here that this minimal solution exists according to \((9)\). On the other hand we have

\[
\begin{align*}
& \begin{align*}
& (a) \quad Y_s \leq \bar{Y}_s \leq \bar{Y}_s^* \quad \text{and} \quad Y_s^* \leq L_s^* \vee Y_s \leq L_s^* \vee \bar{Y}_s \leq U_s, \\
& (b) \quad \left[ f(s, Y_s, Z_s) - \frac{m_s}{2} |Z_s^* - Z_s|^2 \right] ds + g(s, Y_{s-}, Y_s) dA_s \\
& \quad \leq (\eta_s + C_s |Z_s|^2 - \frac{m_s}{2} |Z_s^* - Z_s|^2) ds + \beta_s dA_s \\
& \quad \leq (\eta_s + 4C_s |\gamma_s|^2 + \frac{m_s}{2} |Z_s^* - \gamma_s|^2) ds + \beta_s dA_s + dV_s^+ + dV_s^- - n(u_s - Y_s^-)^- d\alpha_s, \\
& \quad = 0
\end{align*}
\end{align*}
\]

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Applying the comparison theorem (Theorem 5.1) to \( Y^* \) and \( \overline{Y} \), we get \( \forall n \in \mathbb{N}, Y^* \leq \overline{Y} \). Letting \( n \to \infty \) we obtain \( Y^* \leq \overline{Y} \). Applying again comparison theorem (Theorem 5.1) to \(-Y^*\) and \(-Y\) it follows that \( Y^* \leq Y \). Then

\[
Y^* = Y, \quad Z = Z^* \quad \text{and} \quad dK^{*-} = dK^-.
\]

Henceforth

\[
(Y_s - L^+_s)^- dK^- = (L^+_s \vee Y_s - Y_s^-) dK^{*-} = 0.
\]

By symmetric argument we get also,

\[
(Y_s - L^-_s)^+ dK^+ = 0.
\]

Consequently \((Y, Z, K^+, K^-)\) is a solution of (2.5). \hfill \qed

**Corollary 4.1.** \((Y, Z, K^+, K^-)\) is a maximal (resp. minimal) solution of RBSDE (2.5) if and only if \((Y, Z, K^+, K^-)\) is a maximal (resp. minimal) solution of RBSDE (4.27).

**Proof of Theorem 3.1** According to [Theorem 2.1., [9]], there exists \((Y, Z, K^+, K^-)\) a maximal (resp. minimal) solution of RBSDE (2.5) and then by Corollary 4.1 \((Y, Z, K^+, K^-)\) is a maximal (resp. minimal) solution of RBSDE (2.5). \hfill \qed

5 Further study: standard form of RBSDE

In this section, we want to find an equivalent and standard form to our initial RBSDE (2.5) by giving another characterization of the Dom without introducing the test barriers \( L^* \) and \( U^* \).

First, from the appendix, we have the following characterization of Dom.

**Proposition 5.1.** \( \text{Dom} = \left\{ Y \in \mathcal{D} : \left[ \forall t \in [0, T], \ L_{t-} \lor \ t^* \leq Y_{t-} \leq U_{t-} \land \ (-u)^* \right] \ P\text{-a.s.} \right\} \), where \( l^*(t) \) and \((-u)^*(t)\) are defined respectively by (see the appendix for more details)

\[
l^*(t) = \inf_n \left\{ -nt + \inf \left\{ a \in \mathbb{R} : \int_0^t \left[ l_s + ns - a \right]^+ d\delta_s = 0 \right\} \right\},
\]

\[
(-u)^*(t) = \inf_n \left\{ -nt + \inf \left\{ a \in \mathbb{R} : \int_0^t \left[ -u_s + ns - a \right]^+ d\alpha_s = 0 \right\} \right\}.
\]

The following theorem proves that our original RBSDE can be written in a standard form.

**Theorem 5.1.** \((Y, Z, K^+, K^-)\) is a solution of RBSDE (2.5) if and only if \((Y, Z, K^+, K^-)\) satisfies

\[
\begin{align*}
(i) \quad Y_t = \xi + \int_t^T \left[ f(s, Y_s, Z_s) ds + g(s, Y_s, Y_s) dA_s + dK^+_s - dK^-_s - Z_s dB_s \right], \\
(ii) \quad \forall t \in [0, T], \ L_{t-} \lor l^*(t) \leq Y_{t-} \leq (-u)^*- (\cdot)^* \land U_{t-}, \\
(iii) \quad \int_0^T (Y_{t-} - [L_{t-} \lor l^*(t)]) dK^+_t = \int_0^T ([U_{t-} \land (-u)^*] - Y_{t-}) dK^-_t = 0, \ a.s., \\
(iv) \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{1, d}, \\
(v) \quad dK^+ \perp dK^-.
\end{align*}
\]
Proof.
Let \((Y,Z,K^+,K^-)\) be a solution of RBSDE (2.5), then for all processes \(L^*\) and \(U^*\) in \(Dom\) such that for all \(t \in [0,T]\), \(L^*_t \leq Y_t \leq U^*_t\) we have \((Y,Z,K^+,K^-)\) is also a solution of the following RBSDE

\[
\begin{align*}
(i) \quad & Y_t = \xi + \int_0^T [f(s,Y_s,Z_s)ds + g(s,Y_{s-},Z_s)dA_s + dK^*_s - dK^-_s] \quad , \\
(ii) \quad & \forall t \in [0,T], \ L^*_t \leq Y_t \leq U^*_t, \\
(iii) \quad & \int_0^T (Y_{t-} - L^*_{t-})dK^+_t = \int_0^T (U^*_t - Y_{t-})dK^-_t = 0, \ \text{a.s.,} \\
(iv) \quad & Y \in \mathcal{D}, \ K^+, K^- \in \mathcal{K}, \ Z \in \mathcal{L}^{2,d}, \\
(v) \quad & dK^+ \bot dK^-.
\end{align*}
\]

But for each integers \(n\) and \(m\),

\[
L^*_m(\omega) = L^*_n(\omega) \lor I^{m,\delta}(t,\omega) \land Y_t(\omega) \quad \text{and} \quad U^*_m(\omega) = Y_t(\omega) \lor (-u)^{m,\alpha}(t,\omega) \land U_t(\omega),
\]

are in \(Dom\) and \(L^*_n \leq Y_t \leq U^*_n\), where

\[
l^{m,\delta}(t,\omega) = -nt + \text{esssup}_{s \leq t} [l(s,\omega) + ns] \quad \text{and} \quad (-u)^{m,\alpha}(t,\omega) = -mt + \text{esssup}_{s \leq t} [-u(s,\omega) + ms].
\]

We deduce that, for each integers \(n\) and \(m\),

\[
\int_0^T (Y_{t-} - L^*_{t-})dK^+_t = \int_0^T (U^*_t - Y_{t-})dK^-_t = 0.
\]

It follows from Appendix that

\[
l^{\cdot,\delta}(t) \leq l^{\cdot,\delta}(t) \leq Y_{t-} \leq (-u)^{\cdot,\alpha}(t) \leq (-u)^{\cdot,\alpha}(t).
\]

Then passing in limit and using monotone convergence theorem we get

\[
\int_0^T \left( Y_{t-} - [L_{t-} \lor l^{\cdot,\delta}(t)] \right) dK^+_t = \int_0^T \left( (U^*_t - Y_{t-}) - (-u)^{\cdot,\alpha}(t) \right) dK^-_t = 0.
\]

This gives the necessary implication.

Let us now show the reverse. Suppose that \((Y,Z,K^+,K^-)\) is a solution of RBSDE (5.22). In order to prove that \((Y,Z,K^+,K^-)\) is a solution of RBSDE (2.5), it remains to prove (iii) and \(Y \in \mathcal{D}\). Let \(L^*_t \in \mathcal{D}\), it follows that

\[
\left( l_t(\omega) \leq L^*_t(\omega) \right) \quad \text{d}d_t(\omega)\text{P}(d\omega)\text{-a.e and } L_{t-} \leq L^*_t\quad \text{d}t\text{P}(d\omega)\text{-a.e.}
\]

Since \(L\) and \(L^*_t\) are rcll, by using Corollary 7.1 we have

\[
\left( \forall t \in [0,T], \ l^{\cdot,\delta}_t \leq L^*_t \land L_t \leq L^*_t \right) \quad \text{P-as}
\]

Henceforth

\[
\left( \forall t \in [0,T], \ l^{\cdot,\delta}_t \lor L_{t-} \leq L^*_t \right) \quad \text{P-as}
\]

Using the same method as above we have also

\[
\left( \forall t \in [0,T], \ L^*_t \leq (-u)^{\cdot,\alpha}(t) \land U_{t-} \right) \quad \text{P-as}
\]
Henceforth
\[
\mathbb{E} \int_0^T (Y_t - L^*_t)^+ dK_t^+ + (Y_t - L^*_t)^- dK_t^- 
\leq \mathbb{E} \int_0^T (Y_t - [L_t \lor l^*\delta(t)])dK_t^+ + \mathbb{E} \int_0^T ([U_t \land -(-u)^{*-\alpha}(t)] - Y_t^-)dK_t^- = 0.
\]
Then (iii) is satisfied.

Let us now show that \( Y \in \text{Dom} \). It follows from (ii) and Corollary 7.1 that
\[
\mathbb{E} \int_0^T (L_t - Y_t^-)^+ dt + (l_t - Y_t^-)^+ d\delta_t = 0.
\]
Now by taking \(-Y\) and \(-u\), in Corollary 7.1 instead of \( Y \) and \( l \) we have also
\[
\mathbb{E} \int_0^T (U_t - Y_t^-)^- dt + (u_t - Y_t^-)^- d\alpha_t = 0.
\]
This gives the result. \( \square \)

**Remark 5.1.** From the above proof, we have
\[
\int_0^T \left([L_t \lor l^*\delta(t)] - [L_t \lor l^*\delta(t)]\right)dK_t^+ = \int_0^T \left([U_t \land -(-u)^{*-\alpha}(t)] - [U_t \land -(-u)^{*-\alpha}(t)]\right)dK_t^- = 0
\]
which is equivalent to
if \( l_t > [L_t \lor l^*\delta(t)] \) and \( \Delta_t \delta > 0 \) then \( \Delta_t K^+ = 0 \),
and
if \( u_t < [U_t \land -(-u)^{*-\alpha}(t)] \) and \( \Delta_t \alpha > 0 \) then \( \Delta_t K^- = 0 \).

## 6 Particular case: Generalized Snell envelope

Let \( l \) be a predictable process, \( L \in \mathcal{D} \) and \( \delta \in \mathcal{K} \) satisfying the following hypothesis:

**A** There exists a local martingale \( M_t = M_0 + \int_0^t \kappa_s dB_s \) such that \( P\)-a.s.,
\[
L_t \leq M_t \quad \text{on } [0, T] \quad \text{and} \quad l_t \leq M_t \quad \text{d}\delta\text{-a.e. on } [0, T].
\]

Let \( U_t = U_0 - V_t + \int_0^t \chi_s dB_s \), where \( V \in \mathcal{K} \) and \( \chi \in \mathcal{L}^{2,d} \), be a rcll local supermartingale such that \( P\)-a.s.,
\[
L_t \leq U_t \quad \text{on } [0, T] \quad \text{and} \quad l_t \leq U_t \quad \text{d}\delta\text{-a.e. on } [0, T].
\]

According to our main result, let \( (Y, Z, K^+, K^-) \) be the minimal solution of the following RBSDE

\[
\begin{align*}
(i) & \quad Y_T = L_T + \int_T^T [dK^+_s - dK^-_s - Z_s dB_s], \\
(ii) & \quad \forall t \in [0, T], \quad L_t \lor l^*\delta(t) \leq Y_t^- \leq U_t^-, \\
(iii) & \quad \int_0^T (Y_t^- - [L_t \lor l^*\delta(t)])dK^+_t = \int_0^T (U_t^- - Y_t^-)dK^-_t = 0, \quad \text{a.s.,} \\
(iv) & \quad Y \in \mathcal{D}, \quad K^+, K^- \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}, \\
(v) & \quad dK^+ \perp dK^-.
\end{align*}
\]
Applying comparison theorem (Theorem 4.1) to the processes $U_t$ and $Y_t$, we get $dK^- = 0$. Then $Y$ is a local supermartingale minimal solution of the following RBSDE

\[
\begin{aligned}
  (i) & \quad Y_t = L_T + \int_t^T \left[ dK^+_s - Z_s dB_s \right], \\
  (ii) & \quad \forall t \in [0,T], \; L_{t-} \lor t^\Delta(t) \leq Y_{t-}, \\
  (iii) & \quad \int_0^T (Y_{t-} - [L_{t-} \lor t^\Delta(t)]) dK^+_t = 0, \; \text{a.s.}, \\
  (iv) & \quad Y \in \mathcal{D}, \; K^+ \in \mathcal{K}, \; Z \in L^{2,d}.
\end{aligned}
\] (6.23)

Henceforth, we obtain the following result.

**Theorem 6.1.** Suppose that (A) hold. Then the minimal solution $Y$ of (6.23) is the smallest rcll local super-martingale satisfying $P$-a.s.,

\[ L_t \leq Y_t \text{ on } [0,T] \quad \text{and} \quad l_t \leq Y_{t-} - d\delta_t\text{-a.e. on } [0,T]. \]

We say that $Y$ is the generalized Snell envelope associated to $L$ and $ld\delta$. We denote it by $S(L, ld\delta)$.

**Remark 6.1.** We know that if $L$ is of class $D$ then $L$ satisfies assumption (A) (see Dellacherie-Meyer [4], Theorem 24 page 419). In this case our generalized Snell envelope $S(L) = S(L, 0d\delta)$ coincides with the usual Snell envelope $\text{esssup}_{\tau \in T} E[L_T | F_\tau]$, where $T$ is the set of all stopping times valued between $t$ and $T$, as presented in [Dellacherie-Meyer [4], page 416] and studied by several authors.

Let us give the following two examples in the case where $\delta_t = \lambda$ the Lebesgue measure and $\delta_t = 1_{\{T^* \leq t\}}$, where $T^*$ is a stopping time with values in $[0,T]$.

**Example 6.1.** Let $l$ be a predictable process and $\xi$ a $\mathcal{F}_T$-measurable random variable such that there exist $L \in \mathcal{D}$ and $M$ a local martingale such that $L_t \leq l_t \leq M_t$ d$\lambda$-a.e and $\xi \leq M_T$ (where $\lambda$ denotes the Lebesgue measure). Let $(Y, Z, K^+, K^-)$ be the minimal solution of the following RBSDE

\[
\begin{aligned}
  (i) & \quad Y_t = \xi + \int_t^T \left[ dK^+_s - Z_s dB_s \right], \; t \leq T, \\
  (ii) & \quad l_t^\lambda \leq Y_{t-} \text{ on } [0,T], \\
  (iii) & \quad \mathbb{E} \int_0^T (Y_{t-} - l_t^\lambda) dK^+_t = 0, \\
  (iv) & \quad Y \in \mathcal{D}, \; K^+ \in \mathcal{K}, \; Z \in L^{2,d}.
\end{aligned}
\]

Then $Y \left( = S\left(L_t 1_{\{t \leq T\}} + \xi 1_{\{t = T\}}, ld\lambda\right) \right)$ is the smallest local supermartingale such that $l_t \leq Y_t$, d$\lambda$ - a.e and $\xi \leq Y_T$.

**Example 6.2.** Let $T^*$ be a stopping time with values in $[0,T]$, $\xi'$ a $\mathcal{F}_T$-measurable random variable, $\xi$ a $\mathcal{F}_T$-measurable random variable and $L \in \mathcal{D}$ such that there exists a local martingale $M$ such that $L_t \leq M_t$ on $[0,T]$, $\xi' \leq M_{T^*}$ and $\xi \leq M_T$. Let $(Y, Z, K^+, K^-)$ be the minimal solution of the following RBSDE

\[
\begin{aligned}
  (i) & \quad Y_t = \xi + \int_t^T \left[ dK^+_s - Z_s dB_s \right], \; t \leq T, \\
  (ii) & \quad L_{t-} \lor l_t^\Delta(t) \leq Y_{t-} \text{ on } [0,T], \\
  (iii) & \quad \mathbb{E} \int_0^T (Y_{t-} - L_{t-} \lor l_t^\Delta(t)) dK^+_t = 0, \\
  (iv) & \quad Y \in \mathcal{D}, \; K^+ \in \mathcal{K}, \; Z \in L^{2,d}.
\end{aligned}
\]
where $\delta_t = 1_{\{T^i \leq t\}}$ and $L_t = \xi_t 1_{\{t = T^i\}}$ hence $L_t^\delta = -\infty 1_{\{t \neq T^i\}} + \xi_t 1_{\{t = T^i\}}$. Then $Y = S(L_t 1_{\{t < T\}} + \xi_1 1_{\{t = T\}}, \lfloor d\delta\rfloor)$ is the smallest local super-martingale such that

$$L_t \leq Y_t, \text{ on } [0, T], \quad \xi_t \leq Y_{T^i} - \text{ and } \xi \leq Y_T.$$ 

## 7 Appendix

In this appendix we give, in particular, the following characterization: for $Y \in \mathcal{D}$ we have the following

$$\{g(t, \omega) \leq Y_t - (\omega)\} \ \text{ dP}(\omega)\text{-a.e. if and only if } \forall t \in [0, T], \ g^{n, \rho}(t, \omega) \leq Y_t - (\omega) \ \text{ P-as},$$

where $\rho$ is a process in $\mathcal{K}$. This characterization allow us to write our RBSDE in standard form. Let $g : [0, T] \times \Omega \to \mathbb{R}$ be a progressively measurable function and $\rho$ be a process in $\mathcal{K}$.

Note for $(t, \omega) \in [0, T] \times \Omega$ and $n \in \mathbb{N}$

$$g^{n, \rho}(t, \omega) = -nt + \operatorname{esssup}_{s \leq t} \left[ g(s, \omega) + ns \right]$$

$$= -nt + \inf \left\{ \alpha \in \mathbb{R} : \int_0^t \left[ g(s, \omega) + ns - \alpha \right]^+ d\rho_s(\omega) = 0 \right\}. \quad (7.24)$$

We have the following.

**Proposition 7.1.** For every $n \in \mathbb{N}^\ast$, $g^{n, \rho}$ is a predictable process satisfying:

1. For each $\omega \in \Omega$, $t \mapsto nt + g^{n, \rho}(t, \omega)$ is a non-decreasing function and then $t \mapsto g^{n, \rho}(t, \omega)$ is lsgld with $g^{n, \rho}(t-, \omega) \leq g^{n, \rho}(t, \omega) \leq g^{n, \rho}(t+, \omega)$ and such that
   
   (a) $g^{n, \rho}(t-, \omega) = -nt + \inf \left\{ \alpha \in \mathbb{R} : \int_0^{t-} \left[ g(s, \omega) + ns - \alpha \right]^+ d\rho_s(\omega) = 0 \right\}$.
   
   (b) If $\Delta_t \rho(\omega) > 0$, then $g^{n, \rho}(t, \omega) = g^{n, \rho}(t-, \omega) \vee g(t, \omega)$.
   
   (c) If $\Delta_t \rho(\omega) = 0$, then $g^{n, \rho}(t, \omega) = g^{n, \rho}(t-, \omega)$.
   
2. For each $(t, \omega) \in [0, T] \times \Omega$, $-\infty \leq g^{n+1, \rho}(t, \omega) \leq g^{n, \rho}(t, \omega)$.
   
3. If $g \leq h \ \text{ dP}(\omega)\text{-a.e. then } P\text{-a.e. } \omega \in \Omega$ for every $t \in [0, T]$ and $n \in \mathbb{N}$, we have

$$g^{n, \rho}(t, \omega) \leq h^{n, \rho}(t, \omega) \leq \sup_{s \leq t} \left[ h(s, \omega) - n(t - s) \right].$$

**Proof.** Properties 1, 2 and 3 are obvious. The predictability follows from the fact that

$$\left\{ (t, \omega) : nt + g^{n, \rho}(t, \omega) \leq a \right\} = \left\{ (t, \omega) : \int_0^t \left[ g(s, \omega) + ns - a \right]^+ d\rho_s(\omega) = 0 \right\}. \quad \blacksquare$$
Let us now define
\[ g^{*}(t, \omega) = \inf_{n} g^{n}(t, \omega), \quad g^{-}(t, \omega) = \inf_{n} g^{n}(t-, \omega) \quad \text{and} \quad g^{+}(t, \omega) = \inf_{n} g^{n}(t+, \omega). \]

**Remark 7.1.** We have the following:

1. \( g^{*}(t, \omega) \leq g^{*}(t, \omega) \leq g^{+}(t, \omega). \)
2. If \( \Delta \rho(\omega) > 0 \), then \( g^{*}(t, \omega) = g^{-}(t, \omega) \vee g(t, \omega). \)
3. If \( \Delta \rho(\omega) = 0 \) then \( g^{*}(t, \omega) = g^{+}(t, \omega). \)

In particular, for every \( \omega \in \Omega \)
\[ \left\{ t \in [0, T] : g^{*}(t, \omega) > g^{+}(t, \omega) \right\} \subset \left\{ t \in [0, T] : \Delta \rho(\omega) > 0 \right\} \]
and then \( \left\{ t \in [0, T] : g^{*}(t, \omega) > g^{+}(t, \omega) \right\} \) is a countable set.

**Proposition 7.2.** For every \( \omega \in \Omega \),
\[ g(t, \omega) \leq g^{*}(t, \omega) \text{ d} \rho(\omega) \text{ a.e. on } [0, T]. \]

**Proof.** From the definition of \( g^{n, \rho} \), we have that for every \( (\omega, t, n) \in \Omega \times [0, T] \times \mathbb{N} \), there exists a negligible Borel set \( N(\omega, t, n) \) with respect to the measure \( d\rho(\omega) \) such that for every \( s \in N_{(\omega, t, n)}^{c} \)
\[ 1_{\{s \leq t\}} g(s, \omega) \leq \left( g^{n}(t, \omega) + n(t - s) \right) 1_{\{s \leq t\}}. \]

For \( \omega \in \Omega \), let
\[ I_{\omega} = \bigcup_{n \in \mathbb{N}} \left\{ t \in [0, T] : g^{n}(t-, \omega) < g^{n}(t+, \omega) \right\} \bigcup \left[ \mathbb{Q} \cap [0, T] \right] \bigcup \{ T \}, \]
which is countable and dense in \([0, T]\).

Define the following negligible Borel set with respect to the measure \( d\rho(\omega) \)
\[ N_{\omega} = \bigcup_{t \in I_{\omega}} \bigcup_{n \in \mathbb{N}} N(\omega, t, n). \]
It follows that for every \( \omega \in \Omega, s \in N_{\omega}^{c}, n \in \mathbb{N} \) and \( t \in I_{\omega} \), we have
\[ 1_{\{s \leq t\}} g(s, \omega) \leq \left( g^{n}(t, \omega) + n(t - s) \right) 1_{\{s \leq t\}}. \]

Let \( \omega \in \Omega, s \in N_{\omega}^{c} \). If \( s \in I_{\omega} \), we take \( t = s \) and by letting \( n \) to infinity it follows that \( g(s, \omega) \leq g^{*}(s, \omega) \). Now, if \( s \notin I_{\omega} \), then there exists a sequence \( (t_{p})_{p} \in I_{\omega} \) such that \( t_{p} \downarrow s \). Then
\[ g(s, \omega) \leq \left[ g^{n}(t_{p}, \omega) + n(t_{p} - s) \right]. \]
Letting \( p \) to infinity we have
\[ g(s, \omega) \leq g^{n}(s+, \omega) = g^{n}(s, \omega). \]
The result follows by letting \( n \) to infinity. \( \blacksquare \)
We have the following characterization.

**Corollary 7.1.** Let \( Y \in D \)

\[
\left( g(t, \omega) \leq Y_{t-}(\omega) \right) dp_t(\omega) P(d\omega) - a.e \quad \text{if and only if} \quad \left( \forall t \in [0,T], \ g^*(t, \omega) \leq Y_{t-}(\omega) \right) P-as
\]

**Proof**

We use the left continuity of \( Y_{t-} \) to prove that \( \sup_{s \leq t} \left[ Y_{s-} - n(t-s) \right] \) converges to \( Y_{t-} \) as \( n \) goes to infinity. \( \blacksquare \)

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