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Exact Controllability Results for Sobolev-Type Hilfer Fractional Neutral Delay Volterra-Fredholm Integro-Differential Systems

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Abstract: This manuscript mainly focuses on the exact controllability of Sobolev-type Hilfer fractional neutral delay Volterra-Fredholm integro-differential systems. The principal findings of this discussion are established by using the theories on fractional calculus, the measure of noncompactness and Mönch fixed point technique. Initially, the exact controllability of the system is presented and then we improve the discussion to the system with nonlocal conditions. Finally, abstract and filter systems are provided for the illustration.

Keywords: controllability; Hilfer fractional system; neutral system; Volterra-Fredholm integro-differential equations; nonlocal conditions; measure of noncompactness

1. Introduction

In many physical processes, fractional differential equations incorporating not only one fractional derivative but also several fractional derivatives are heavily concentrated. The meaning of fractional systems has recently attracted a lot of attention due to its astonishing applications in showcasing the wonders of science and engineering. The use of fractional order differential equations allows for the management of a wide range of issues in a variety of fields, including fluid flow, electrical systems, visco-elasticity, electro-chemistry, and so on. The monographs [1–9] and the research articles [8–13] show the interlinking in the same way that the separation between classical and fractional differential representations seems to. Applications of the differential systems can be found in [14–16]. Neutral structures with delays or without delays, in particular, serve as a summary association of a large number of partial neutral structures that appear in problems involving heat flow in substances, visco-elasticity, and a variety of natural processes. Neutral systems appear in many areas of applied mathematics; as a result, the most successful neutral structures have gotten a lot of attention in the current generation; readers can look at [12,13,17–21].

Recently, in [22,23], the author initiated another kind of derivative of fractional order, that including Riemann-Liouville and Caputo fractional derivative. In [24], the authors proved the existence of mild solution for evolution equation with Hilfer fractional derivative which generalized the famous Riemann-Liouville fractional derivative by using the semigroup theory, measure of noncompactness and fixed point approach. In [25], the authors proved the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems by using fractional calculus and Bohenblust-Karlin’s theorem. In [18,26,27], the authors proved the existence and controllability of various extensions related with Hilfer fractional derivative by using semigroup theory, measure of noncompactness and various fixed point theorems.

In [28], the authors discussed the approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system by using Bohenblust-Karlin’s fixed point theorem. In [29], the authors proved the existence of nonlocal functional
integro-differential equations via Hilfer fractional derivative by using Mönch fixed point theorem. In [13], the authors discussed the existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay by using Mönch fixed point theorem. In [30], the authors proved the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions by using Bohenblust–Karlin’s fixed point theorem. The existence and exact controllability described in our paper have still to be investigated, and it is the motivation of this article.

Motivated by the monograph, nowadays, several authors focus on this Hilfer fractional derivative, and we refer to [13,18,22,24–27,29–32]. The potential of controllability is an important part of engineering and mathematical control theory. Finding a suitable control function to the point where one may guide the considered dynamic system to a final state is the controllability problem. As a result, many researchers have investigated the controllability of a variety of nonlinear systems in recent years, and the articles are available for viewing. For instance, refs. [12,13,18,19,30,32–45] and references therein.

Assume that the Hilfer fractional neutral delay Volterra-Fredholm integro-differential system of Sobolev-type has the following form

\[ D_{0+}^{\alpha} [Ju(t) - F_1(t, u_t)] = Au(t) + Bx(t) \]
\[ + F_2(t, u_t, \int_0^t e(t, \tau, u_\tau)d\tau, \int_0^t f(t, \tau, u_\tau)d\tau), \quad t \in I = (0, c), \]
\[ I_{0+}^{(1-\nu)(1-\xi)} u(0) = \phi \in \mathcal{R}_I, \]

where \( D_{0+}^{\alpha} \) stands for the Hilfer fractional derivative, \( 0 \leq \nu \leq 1, \frac{1}{2} < \xi < 1 \) and \( x(\cdot) \) takes value in Banach Space \( X \) with \( \| \cdot \| \). The histories \( u_0 : (-\infty, 0] \to \mathcal{R}_I, u_0(s) = u(q+s), \) \( s \leq 0 \) with phase space \( \mathcal{R}_I, F_1 : I \times \mathcal{R}_I \to X, F_2 : I \times \mathcal{R}_I \times X \times X \to X, e : I \times I \times \mathcal{R}_I \to X \) and \( f : I \times I \times \mathcal{R}_I \to X \) are appropriate functions. \( u(\cdot) \) is a control function and \( B \) is a bounded linear operator from \( \mathcal{V} \to X \).

The rest of the paper is organized as follows:
1. Section 2: Theoretical notions linked to fractional calculus and the measure of non-compactness are presented.
2. Section 3: This section is focused on exact controllability results of the fractional system (1) and (2).
3. Section 4: Our findings are expanded to include the concept of nonlocal situations.
4. Section 5: Finally, abstract and filter systems are provided for the illustration of the obtained theory.

2. Preliminaries

We now provide some fundamental theories, lemmas, and facts to discuss our main results. \( C(I, X) \) of all continuous functions. Let us take \( \eta = \nu + \xi - v \xi, \) then \( (1-\eta) = (1-\nu)(1-\xi). \) Assume \( C_{1-\eta}(I, X) = \{ u : t^{1-\eta}u(t) \in C(I, X) \} \) with \( \| \cdot \|_\eta \) by

\[ \| u \|_\eta = \sup \{ t^{1-\eta} \| u(t) \|, t \in I, \eta = (\nu + \xi - v \xi) \}. \]

Clearly, \( C_{1-\eta}(I, X) \) is a Banach space. Define \( H \) with \( \| H \|_{L^p(I, \mathbb{R}^+)} \) if \( H \in L^p(I, \mathbb{R}^+) \) for any \( p \) with \( 1 \leq p \leq \infty. \) \( A^\alpha, 0 < \alpha \leq 1, \) a closed linear operator on \( D(A^\alpha) \) with inverse \( A^{-\alpha}, \) see [46].

**Definition 1** ([46]).

(i) \( D(A^\alpha), \) a Banach space with \( \| u \|_\alpha = \| A^\alpha u \| \) for \( u \in D(A^\alpha). \)
(ii) \( T(t) : X \to X_\alpha, \) for \( t \geq 0. \)
(iii) \( A^\alpha T(t)u = T(t)A^\alpha u, \) for \( u \in D(A^\alpha) \) and \( t \geq 0. \)
(iv) \( A^\alpha T(t) \) is bounded on \( X \) and there exists \( M_\alpha > 0 \) such that

\[ \| A^\alpha T(t) \| \leq \frac{M_\alpha}{t^\alpha}. \]
Definition 2 ([47]). The operators define $A : D(A) \subset X \to X$ and $J : D(A) \subset X \to X$ satisfy the following:

1. $A$ and $J$ are closed linear operators.
2. $D(J) \subset D(A)$ and $J$ is bijective.
3. $J^{-1} : X \to D(J)$ is continuous.

Additionally, because of (1) and (2) $J^{-1}$ is closed, by (3) and by referring closed graph theorem, we obtain the boundedness of $A J^{-1} : X \to X$. Define $\|J^{-1}\| = I_m$ and $\|I\| = I_m$.

Definition 3 ([4]). The left sided Riemann-Liouville fractional integral of order $\zeta$ having lower limit $c$ for $F : [c, +\infty) \to \mathbb{R}$ is presented as

\[
I_c^\zeta F(q) = \frac{1}{\Gamma(\zeta)} \int_c^q \frac{F(\tau)}{(q - \tau)^{1-\zeta}} d\tau, \quad q > c; \; \zeta > 0,
\]

if the right side is pointwise determined on $[c, +\infty)$, where $\Gamma(\cdot)$ denotes gamma function.

Definition 4 ([4]). The left-sided Riemann-Liouville fractional derivative of order $\zeta \in [k-1, k)$, $k \in X$ for $F : [c, +\infty) \to \mathbb{R}$ is given by

\[
D_c^\zeta F(q) = \frac{d^k}{dq^k} I_c^{(1-k)\zeta} F(q), \quad q > c, \; k - 1 < \zeta < k.
\]

Definition 5 ([4]). The left-sided Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \zeta < 1$ function of $F(q)$ is given by

\[
D_c^\nu\zeta F(q) = (I_c^{\nu(1-k)\zeta} D(\eta^{(1-k)\zeta}) F)(q).
\]

Remark 1 ([23]).

(i) Given $\zeta = 0, 0 < \nu < 1$ also $c = 0$, the Hilfer fractional derivative identical with standard Riemann-Liouville fractional derivative:

\[
D_c^\nu F(q) = \frac{d}{dq} I_0^{\nu-\zeta} F(q) = l D_c^\zeta F(q).
\]

(ii) Given $\zeta = 1, 0 < \nu < 1$ also $c = 0$, the Hilfer fractional derivative identical with standard Caputo derivative:

\[
D_c^\nu F(q) = I_0^{\nu-\zeta} \frac{d}{dq} F(q) = C D_c^\zeta F(q).
\]

We define the abstract phase space $R_t$ by referring [33]. Consider $l : (-\infty, 0] \to (0, +\infty)$ is continuous along with $j = \int_{-\infty}^0 l(\gamma) d\gamma < +\infty$. Now for every $a > 0$, we define

\[
R_t = \{ \eta : [-a, 0] \to X \text{ such that } \eta(\gamma) \text{ is bounded and measurable} \},
\]

and

\[
\|\eta\|_{[-a,0]} = \sup_{\zeta \in [-a,0]} \|\eta(\zeta)\|, \text{ for all } \eta \in \mathcal{R}.
\]

Now, we define

\[
\mathcal{R}_t = \left\{ \eta : (-\infty, 0] \to X \text{ such that for some } c > 0, \eta|_{[-c,0]} \in \mathcal{R} \text{ and } \int_{-\infty}^0 l(\zeta) \|\eta\|_{[\zeta,0]} d\zeta < +\infty \right\}.
\]
and
\[ \| \eta \|_{R_i} = \int_{-\infty}^{0} j(\zeta) \| \eta \|_{l_\infty[\zeta,0]} d\zeta, \quad \forall \eta \in R_i, \]
therefore $\langle R_i, \| \cdot \|_{R_i} \rangle$ is a Banach space.
Consider
\[ R'_i = \{ u : (-\infty, c] \to X \text{ such that } u |_{V} \in C(I, X), u(0) = \phi(0) \in R_i \}. \]
Set $\| \cdot \|_c$ be a seminorm in $R'_i$ which is defined by
\[ \| u \|_c = \| \phi(0) \|_{B_0} + \sup \{ |y(\zeta)| : \zeta \in [0, c] \}, \quad u \in R'_i. \]

**Lemma 1** ([48]). If $u \in R'_i$, then for $\gamma \in I$, $u_\gamma \in R_i$. Furthermore,
\[ j\| u(\gamma) \| \leq \| u_\gamma \|_{R_i} \leq \| \phi(0) \|_{R_i} + I \sup_{\zeta \in [0, \gamma]} |u(\zeta)|, \]
where $j = \int_{-\infty}^{0} l(\gamma) d\gamma < +\infty$.

**Lemma 2.** The function $u : [0, c] \to X$ is said to be an integral solution of the fractional system (1) and (2) if it satisfies the following:
- $u : [0, c] \to X$ is continuous.
- $I_{0+}^{1-\zeta} u(t) \in D(A)$, for $t \in I$.
- The fractional system (1)–(2) is similar with
\[
\begin{align*}
 u(t) = & \frac{J^{-1}[\phi(0) - F_1(0, \phi(0))]}{\Gamma(\nu(1-\zeta) + \zeta)} J^{(\nu-1)(1-\zeta)} + J^{-1} F_1(t, u_t) \\
 & + \frac{1}{\Gamma(\zeta)} \int_{0}^{t} (t-s)^{\zeta-1} J^{\zeta} \left[ A(s) + Bx(s) \\
 & + F_2 \left( s, u_s, \int_{0}^{s} e(s, \tau, u_\tau) d\tau, \int_{0}^{c} f(s, \tau, u_\tau) d\tau \right) \right] ds, \quad t \in I.
\end{align*}
\]

**Remark 2.** By referring Wright function $M_{\zeta}(s)$, we present the mild solution of the fractional system (1)–(2) as follows:
\[ M_{\zeta}(s) = \sum_{k=1}^{\infty} \frac{(-s)^{k-1}}{(k-1)! \Gamma(1-k\zeta)}, \quad 0 < \zeta < 1, \quad s \in \mathbb{C}, \]
and satisfies
\[ \int_{0}^{\infty} s^{\zeta} M_{\zeta}(s) ds = \frac{\Gamma(1+s)}{\Gamma(1+\zeta s)}, \quad \text{for } s \geq 0. \]

**Lemma 3.** If the fractional system (1)–(2) are satisfied, then there exists $F_1 : I \times R_i \to X$ and $F_2 : I \times R_i \times X \times X \to X$ such that
\[
\begin{align*}
 u(t) = & J^{-1} \mathcal{P}_{\nu,\zeta}(t) \phi(0) - F_1(0, \phi(0)) + F_1(t, u_t) + \int_{0}^{t} J^{-1} \mathcal{H}_{\zeta}(t) A F_1(s, u_s) ds \\
 & + \int_{0}^{t} J^{-1} \mathcal{H}_{\zeta}(t) F_2 \left( s, u_s, \int_{0}^{s} e(s, \tau, u_\tau) d\tau, \int_{0}^{c} f(s, \tau, u_\tau) d\tau \right) ds \\
 & + \int_{0}^{t} J^{-1} \mathcal{H}_{\zeta}(t) Bx(s) ds, \quad t \in I,
\end{align*}
\]
where
\[ \mathcal{P}_{v,\xi}(t) = I_0^{(1-\xi)}(t)^{\xi-1}\mathcal{R}_v(t); \quad \mathcal{R}_v(t) = t^{\xi-1}\mathcal{F}_v(t); \quad \mathcal{F}_v(t) = \int_0^\infty \xi \omega \mathcal{M}_v(\omega)S(t^\xi \omega)\,d\omega. \]

**Definition 6** ([20]). A function \( u : (-\infty, c] \to X \) is said to be a mild solution of the fractional system (1)–(2) provided that \( u_0 = \phi(0) \in \mathcal{R}_1 \) on \( (-\infty, 0] \) and satisfies the following integral equation
\[
\begin{align*}
u(t) &= \nu(0) f_1(\nu(0)) + \int_0^t (t-s)^{\xi-1} \mathcal{R}_v(t-s)A\nu_1(s)\,ds \\
&\quad + \int_0^t (t-s)^{\xi-1} \mathcal{R}_v(t-s)F(s, \mu(s))\,ds \\
&\quad + \int_0^t (t-s)^{\xi-1} \mathcal{F}_v(t-s)Bx(s)\,ds, \quad t \in I_1,
\end{align*}
\]

where
\[
\mathcal{P}_{v,\xi}(t) = \int_0^\infty \xi \omega \mathcal{M}_v(\omega)M(t^\xi \omega)\,d\omega, \quad \mathcal{R}_v = \int_0^\infty \omega \xi \omega M(\omega)\,d\omega,
\]
and for \( \omega \in (0, \infty) \)
\[
\xi(\omega) = \frac{1}{\xi} \omega^{-1-\xi} \mathcal{M}_v(\omega^{-1}) \geq 0, \quad \mathcal{M}_v(\omega) = \frac{1}{\xi} \sum_{n=1}^\infty (-1)^n n^{\xi-1} \Gamma(n\xi + 1) \sin(n\pi\xi),
\]
where \( \xi \) is a probability density function defined on \( (0, \infty) \), that is,
\[
\xi(\omega) \geq 0, \quad \omega \in (0, \infty) \quad \text{and} \quad \int_0^\infty \xi(\omega)\,d\omega = 1.
\]

**Lemma 4** ([20]). The operators \( \mathcal{P}_{v,\xi} \) and \( \mathcal{R}_v \) satisfies the following:

(i) For \( t \geq 0 \), \( \mathcal{P}_{v,\xi} \) and \( \mathcal{R}_v \) are linear and bounded, that is, for every \( u \in X \),
\[
\| \mathcal{P}_{v,\xi}(t)u \| \leq \frac{M^\mu-1}{\Gamma(\mu(1-\xi) + \xi)} \| u \| \quad \text{and} \quad \| \mathcal{R}_v(t)u \| \leq \frac{M}{\Gamma(\xi)} \| u \|,
\]
where \( \mathcal{P}_{v,\xi}(t) = I_0^{(1-\xi)}(t)^{\xi-1}\mathcal{R}_v(t) \), \( \mathcal{R}_v(t) = t^{\xi-1}\mathcal{F}_v(t) \).

(ii) The operators \( \{ \mathcal{P}_{v,\xi}(t) \}_{t \geq 0} \) and \( \{ \mathcal{R}_v(t) \}_{t \geq 0} \) are strongly continuous.

(iii) For every \( u \in X \), \( \mu, \xi \in (0, 1] \), we have
\[
A^\mu \mathcal{F}_v(t)u = A^{1-\mu} \mathcal{F}_v(t)A^\mu u, \quad 0 \leq t \leq c;
\]
\[
\| A^\mu \mathcal{F}_v(t) \| \leq \frac{\xi C_\mu \Gamma(2-\mu)}{t^\mu \Gamma(1+\xi(1-\mu))}, \quad 0 < t \leq c.
\]

**Lemma 5.** The operators \( \{ \mathcal{F}_v(t) \}_{t \geq 0} \) and \( \{ \mathcal{P}_{v,\xi}(t) \}_{t \geq 0} \) are strongly continuous, that is, \( 0 < t' < t'' \leq c \),
\[
\| (t')^{\xi-1}\mathcal{F}_v(t')u - (t'')^{\xi-1}\mathcal{F}_v(t'')u \| \to 0 \quad \text{and} \quad \| \mathcal{P}_{v,\xi}(t')u - \mathcal{P}_{v,\xi}(t'')u \| \to 0 \quad \text{as} \quad t'' \to t'.
\]

**Definition 7** ([49,50]). The Measure of noncompactness of Hausdorff \( \mathcal{F}(\cdot) \) determined on every bounded subset \( q \) of \( X \) by \( \mathcal{F}(q) = \inf \{ e > 0 : q \) can be covered by a finite number of balls of radii lesser than \( e \} \).
Definition 8 ([35]). Let $F^+$ be the positive cone of an order Banach space $(F, \leq)$. The value $E$ of $F^+$ is said to be measure of noncompactness on $X$ of $\mathcal{D}$ determined on the set of all bounded subsets of $X$ if and only if $E(\overline{\mathcal{D}q}) = E(q)$ for all bounded subsets $q \subseteq X$, where $\overline{\mathcal{D}q}$ is a closed convex hull of $q$.

Definition 9 ([49,51]). For every bounded subsets $q, q_1, q_2$ of $X$.

(i) Monotone if and only if for all bounded subsets $q, q_1, q_2$ of $X$ we get: $(q_1 \subseteq q_2) \Rightarrow (\mathcal{D}(q_1) \leq \mathcal{D}(q_2))$;

(ii) Non singular if and only if $\mathcal{D}(\{a\} \cup q) = \mathcal{D}(q)$ for each $a \in X, q \subseteq U$;

(iii) Regular if and only if $\mathcal{D}(q) = 0$ if and only if $q$ is relatively compact in $X$;

(iv) $\mathcal{D}(q_1 + q_2) \leq \mathcal{D}(q_1) + \mathcal{D}(q_2)$, where $q_1 + q_2 = \{x_1 + x_2 : x_1 \in q_1, x_2 \in q_2\}$;

(v) $\mathcal{D}(q_1 \cup q_2) \leq \max\{\mathcal{D}(q_1), \mathcal{D}(q_2)\}$;

(vi) $\mathcal{D}(\gamma q) \leq |\gamma| \mathcal{D}(q)$, for all $\gamma \in \mathbb{R}$;

(vii) If $Q : D(Q) \subseteq X \to Y$ is a Lipschitz continuous function with $k > 0$, then $\mathcal{F}_k(Qq) \leq k\mathcal{F}(q)$, for $q \subseteq D(Q)$ and $Y$ is a Banach space.

Lemma 6 ([49]). Assume that $\mathcal{X} \subset C([a, b], X)$ is bounded and equicontinuous, then $\mathcal{T}(\mathcal{X})$ is continuous for any $t \in [a, b]$,

$$\mathcal{T}(\mathcal{X}) = \sup_{t \in \mathcal{N}} \{\mathcal{T}(\mathcal{X}(t)), t \in [a, b]\},$$

whereby $\mathcal{X}(t) = \{u(t) : u \in \mathcal{X}\} \subseteq X$.

Theorem 1 ([45,52]). Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence of Bochner integrable functions from $I \to X$ with $\|u_n(t)\| \leq \varepsilon(t)$, for all $t \in I$ and every $n \geq 1$, where $\varepsilon \in L^1(1, \mathbb{R})$, then $q(t) = \mathcal{T}(\{u_n(t) : n \geq 1\}) \in L^1(I, \mathbb{R})$ and satisfies $\mathcal{T}\{\int_0^t \varepsilon(\tau)d\tau : n \geq 1\} \leq 2\int_0^t \varepsilon(\tau)d\tau$.

Lemma 7 ([53]). Assume $F$ be closed convex subset of $X$ and $0 \in F$, $K : F \to X$ is continuous and that satisfies Mönch’s condition, that is, $(P \subseteq F$ is countable, $P \subseteq \overline{\mathcal{D}(\{0 \cup K(P) \Rightarrow \overline{P}$ is compact $)$}. Then $K$ has a fixed point in $F$.

3. Existence

The reason for this part is to examine the existence of the fractional system (1)–(2).

(H0) If $F \subset X$ and $w \in F$, then

$$\|T(t_2^t q)w - T(t_1^t q)w\| \to 0, \text{ when } t_2 \to t_1,$$

for each fixed $q \in (0, \infty)$.

(H1) The function $F_1 : I \times \mathcal{R}_t \to X$ is continuous and there exists $\mu \in (0, 1)$ such that

$F_1 \in D(A^\mu)$ for every $w \in X, \mu \in I, A^\mu J^{-1} F_1(, w)$ is strongly measurable, there exists $M_\delta > 0, M_\delta > 0$ such that $r, s \in X, A^\mu F_1(t, r)$ satisfies the following

$$\|A^\mu J^{-1} F_1(t, r(t)) - A^\mu J^{-1} F_1(t, s(t))\| \leq M_\delta t^{1-\eta} \|r(t) - s(t)\|_{\mathcal{R}_1},$$

$$\|A^\mu J^{-1} F_1(t, w(t))\| \leq M_\delta (1 + t^{1-\eta}\|w\|_{\mathcal{R}_1}).$$

(H2) The function $F_2 : I \times \mathcal{R}_t \times X \times X \to X$ satisfies the following:

(i) The function $F_2(, \phi, y, z)$ is measurable for all $\phi, y, z \in \mathcal{R}_t \times X$ and $F_2(t, r, s, \cdot)$ is continuous for a.e. $\mu \in I, u \in \mathcal{R}_t, F_2(t, \cdot, r, s) : I \to X$ is strongly measurable.

(ii) There exists $q_0 \in (0,q)$ and $\epsilon_1 \in L^\infty(I, \mathbb{R}^+)$ and the integrable function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|E(t, \phi, y, z)\| \leq \epsilon_1(t)\|\|\phi\|_{\mathcal{R}_1} + t^{1-\eta}\|y\| + t^{1-\eta}\|z\||$,

for all $t, \phi, y, z \in N \times \mathcal{R}_t \times X \times X$, where $k$ satisfies $\lim_{n \to \infty} \frac{\epsilon_1(t)}{n} = 0.$
(iii) There exists $q_2 \in (0, q)$ and $\varepsilon_2 \in L^\frac{1}{p}(I, \mathbb{R}^+)$ such that for any bounded subset $D_1 \subset X$ and $G_1 \subset \mathcal{R}_I$,  
\[
\mathcal{T}(F_2(t, G_1, D_1, D_2)) \leq \varepsilon_2(t) \left[ \sup_{-\infty < \varphi \leq 0} \mathcal{T}(G_1(\varphi)) + \mathcal{T}(D_1) + \mathcal{T}(D_2) \right],
\]
for a.e. $t \in I$, $G_1(\varphi) = \{ v(\varphi) : v \in G_1 \}$ and $\mathcal{T}$ is the Hausdorff MNC.

(H$_3$) The function $e : I \times I \times \mathcal{R}_I \to X$ satisfies the following:

(i) $e(\cdot, \phi, z)$ is measurable for all $(\phi, z) \in \mathcal{R}_I \times X$, $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in I$.

(ii) There exists $E_0 \geq 0$ such that $\|e(t, \tau, \varphi)\| \leq E_0(1 + \|\varphi\|_{\mathcal{R}_I})$, for all $t \in I$, $\tau \in X$, $\varphi \in \mathcal{R}_I$.

(iii) There exists $q_3 \in (0, q)$ and $\varepsilon_3 \in L^\frac{1}{p}(I, \mathbb{R}^+)$ such that for any bounded subset $D_3 \subset X$ 
\[
\mathcal{T}(F_2(t, \tau, D_3)) \leq \varepsilon_3(t, \tau) \left[ \sup_{-\infty < \varphi \leq 0} \mathcal{T}(D_3(\varphi)) \right] \text{ for a.e. } t \in I,
\]
with $\varepsilon_3^* = \sup_{t \in I} \int_0^t \varepsilon_3(t, \tau) d\tau < \infty$.

(H$_4$) The function $f : I \times I \times \mathcal{R}_I \to X$ satisfies the following:

(i) $f(\cdot, \phi, z)$ is measurable for all $(\phi, z) \in \mathcal{R}_I \times X$, $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in I$.

(ii) There exists $E_1 \geq 0$ such that $\|f(t, \tau, \varphi)\| \leq E_1(1 + \|\varphi\|_{\mathcal{R}_I})$, for all $t \in I$, $\tau \in X$, $\varphi \in \mathcal{R}_I$.

(iii) There exists $q_4 \in (0, q)$ and $\varepsilon_4 \in L^\frac{1}{p}(I, \mathbb{R}^+)$ such that for any bounded subset $D_4 \subset X$, 
\[
\mathcal{T}(F_2(t, \tau, D_4)) \leq \varepsilon_4(t, \tau) \left[ \sup_{-\infty < \varphi \leq 0} \mathcal{T}(D_4(\varphi)) \right] \text{ for a.e. } t \in I,
\]
with $\varepsilon_4^* = \sup_{t \in I} \int_0^t \varepsilon_4(t, \tau) d\tau < \infty$.

(H$_5$) The operator $B : L^2(I, \mathcal{Y}) \to L^1(I, \mathcal{Y})$ is bounded and $W : L^2(I, \mathcal{Y}) \to L^1(I, \mathcal{Y})$ is defined by
\[
Wx = \int_0^c (c - s)^{\frac{p-1}{2}} t^{-1} \mathcal{T}^b(t - s) Bx(s) ds,
\]
which satisfies the following:

(i) $W$ have an inverse $W^{-1}$ acquires the value in $L^2(I, \mathcal{Y})/\text{Ker} W$, there exists $\varepsilon_0 > 0, \varepsilon_\infty > 0$ such that $\|B\| \leq \varepsilon_0$ and $\|W^{-1}\| \leq \varepsilon_\infty$.

(ii) For $q_5 \in (0, q)$ and for every bounded subset $D \subset X$, there exists $\varepsilon_4 \in L^\frac{1}{p}(I, \mathbb{R}^+)$ such that $\mathcal{T}(W^{-1}(D)(t)) \leq \varepsilon_5(t) \mathcal{T}(D)$. Here $\varepsilon_\infty \in L^\frac{1}{p}(I, \mathbb{R}^+)$.

(iii) For $q_6 \in (0, q)$ and $M_{\mathcal{Y}} \in L^\frac{1}{q_6}(I, \mathbb{R}^+)$ such that for any $S \subset X$, $\mathcal{T}(W^{-1}(S)(t)) \leq M_{\mathcal{Y}}(t) \mathcal{T}(S), q_n \in (0, q), n = 0, 1, 2, 3$.

We present the following for our convenience:

\[
K_1 = k_1\|e_1\|_{L^\frac{1}{p}(I, \mathbb{R}^+)}, \quad K_2 = k_2\|e_2\|_{L^\frac{1}{p}(I, \mathbb{R}^+)}, \quad K_3 = k_3\|e_3\|_{L^\frac{1}{p}(I, \mathbb{R}^+)}, \quad \|A^{-\mu}\| = M_0, \quad K_4 = k_4\|e_4\|_{L^\frac{1}{p}(I, \mathbb{R}^+)}, \quad K_5 = k_5\|e_5\|_{L^\frac{1}{q_5}(I, \mathbb{R}^+)}, \quad K_0 = k_0\|M_{\mathcal{Y}}\|_{L^\frac{1}{q_6}(I, \mathbb{R}^+)}, \quad K_n = \left( \frac{1 - q_n}{q_n} \right)^{\frac{1}{p_0}} c^{\frac{1}{p_0}} - q_n, \quad n = 0, 1, 2, 3, 4, \quad K = \frac{\zeta - 1}{1 - \rho}, \quad K^* = \frac{c^{(1+K)(1-\rho)}}{(1+K)^{1-\rho}}.
\]
Theorem 2. Suppose that the hypotheses \((H_0)-(H_5)\) are satisfied, then the fractional system (1) and (2) is controllable if

\[
L^* = \frac{2M \int_0 \zeta (1 + 2\epsilon + 2\epsilon^2) \Gamma (1 - \eta)}{\Gamma (\zeta)} \left[ 1 + \frac{2M \int_0 \zeta (1 + 2\epsilon + 2\epsilon^2) \Gamma (1 - \eta)}{\Gamma (\zeta)} \right], \text{ for some } \frac{1}{2} < \zeta < 1. \tag{4}\]

Proof. We now define the operator \(\Phi : \mathcal{R}_\iota \rightarrow \mathcal{R}_{\iota}^*\) by

\[
\Phi u(t) = \begin{cases} 
\phi(t), & t \in (-\infty, 0], \\
\int_0^t (-s)^{-1} (s - t) J_1 J_2 (s, u_s + \hat{\beta}_s) ds + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds \\
\int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds \\
(\times) \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds.
\end{cases}
\]

For \(\phi \in \mathcal{R}_\iota\), we present \(\hat{\beta}\) as follows:

\[
\hat{\beta}(t) = \begin{cases} 
\phi(t), \quad t \in (-\infty, 0], \\
\mathcal{R}_\iota (t) \phi(0), \quad t \in \iota,
\end{cases}
\]

then \(\hat{\beta} \in \mathcal{R}_\iota^*\). Let \(u(t) = p(t) + \hat{\beta}(t)\), \(-\infty < t \leq d\). Clearly \(u\) satisfies (3), if and only if \(p\) satisfies \(p_0 = 0\) and

\[
p(t) = -\int_0^t (-s)^{-1} \mathcal{R}_\iota (t) \phi(0) + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s + \hat{\beta}_s) ds.
\]

where

\[
x_p(t) = W^{-1} \left[ f_0 \int_0^t (-s)^{-1} \mathcal{R}_\iota (c) \phi(0) - F_1 (0, \phi(0)) \right] - \int_0^t (-s)^{-1} u \mathcal{R}_\iota (c) \phi(0) - F_1 (0, \phi(0)) + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s) ds + \int_0^t (-s)^{-1} J_1 J_2 (s, u_s) ds.
\]

Take \(\mathcal{R}_\iota^{p} = \{ p \in \mathcal{R}_\iota : p_0 = 0 \in \mathcal{R}_\iota \}\). For every \(p \in \mathcal{R}_\iota^{p}\),

\[
\|p\|_c = \|p_0\|_{\mathcal{R}_\iota} + \sup \{ \|p(s)\| : 0 \leq s \leq c\},
\]

\[
= \sup \{ \|p(s)\| : 0 \leq s \leq c\}.
\]

Hence \((\mathcal{R}_\iota^{p}, \|\cdot\|_c)\) is a Banach space. For \(r > 0\), fix \(F_r = \{ p \in \mathcal{R}_\iota^{p} : \|w\|_c \leq r\}\), thus \(F_r \subseteq \mathcal{R}_\iota^{p}\) is uniformly bounded, and for \(p \in F_r\), by referring Lemma 1, we have

\[
\|p_t + \hat{\beta}_t\|_{\mathcal{R}_\iota} \leq \|p_t\|_{\mathcal{R}_\iota} + \|\hat{\beta}_t\|_{\mathcal{R}_\iota} \leq j \left( r + \frac{M}{\Gamma (v (1 - \zeta)) \phi(0)} \right) + \|\phi\|_{\mathcal{R}_\iota} = r' \tag{5}\]

where

\[
\phi(t) = \begin{cases} 
\phi(t), \quad t \in (-\infty, 0], \\
\mathcal{R}_\iota (t) \phi(0), \quad t \in \iota,
\end{cases}
\]

and

\[
\hat{\beta}(t) = \begin{cases} 
\phi(t), \quad t \in (-\infty, 0], \\
\mathcal{R}_\iota (t) \phi(0), \quad t \in \iota,
\end{cases}
\]
Introduce $\tilde{Y} : \mathcal{R}_r'' \to \mathcal{R}_r''$ by

$$\tilde{Y}p(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -\mathcal{P}_{\mathcal{U}}(t)F_1(0, \phi) + J^{-1}F_1(t, pt + \tilde{\beta}_t) \\ + \int_0^t (t-s)^{\gamma-1}A_0\mathcal{U}_1(t-s)J^{-1}F_1(s, p_s + \tilde{\beta}_s)ds \\ + \int_0^t (t-s)^{\gamma-1}\mathcal{U}_1(t-s)Bx_{p_s}(s)ds \\ + \int_0^t (t-s)^{\gamma-1}\mathcal{U}_1(t-s) \\ \times \left[ F_2(s, p_s + \tilde{\beta}_s, \int_0^s e(s, \tau, pt + \tilde{\beta}_r)d\tau, \int_0^s f(s, \tau, pt + \tilde{\beta}_r)d\tau \right] ds. \end{cases}$$

Clearly, $Y$ having a fixed point and which is similar to $\tilde{Y}$. To prove $\tilde{Y}$ having a fixed point, we subdivide the whole proof as follows:

**Step 1:** We state that there exists $r > 0$ such that $\tilde{Y}(F_r) \subseteq F_r$.

If it is not correct, then there exists $p' \in F_r$ and $t \in I$ such that $\|\tilde{Y}(p')(t)\| > r$, i.e., $\tilde{Y}(p') \notin F_r$.

Fix $r > 0$, and assume $\{F_r = u \in C : \|u\| \leq r\}$. Clearly, $F_r$ is a closed, bounded and convex set of $C$.

Now, we need to check there exists $r > 0$ such that $\phi(F_r) \subseteq F_r$. If it fails, then $p' \in F_r$. However, $\phi(u') \notin F_r$. Hence,

$$\|\tilde{Y}(u')\| \equiv \sup \{t^{1-\eta}||s'||_c : s' \in \phi(u') > r\}.$$

Using assumptions (H$_2$) and (H$_3$) and Lemma 4, we have

$$r < \sup_{t \in I} t^{1-\eta}\|\tilde{Y}(p')(t)\|$$

$$\leq c^{1-\eta}\left(\| - J^{-1}\mathcal{P}_{\mathcal{U}}(t)F_1(0, \phi(0))\| + \| J^{-1}F_1(t, pt + \tilde{\beta}_t)\| \\ + \| \int_0^t (t-s)^{\gamma-1}A_0\mathcal{U}_1(t-s)J^{-1}F_1(s, p_s + \tilde{\beta}_s)ds \| \\ + \| \int_0^t (t-s)^{\gamma-1}\mathcal{U}_1(t-s)Bx_{p_s}(s)ds \| \\ + \| \int_0^t (t-s)^{\gamma-1}\mathcal{U}_1(t-s) \\ \times \left[ F_2(s, p_s + \tilde{\beta}_s, \int_0^s e(s, \tau, pt + \tilde{\beta}_r)d\tau, \int_0^s f(s, \tau, pt + \tilde{\beta}_r)d\tau \right] ds \| \right)$$

$$= \sum_{i=1}^5 V_i,$$

where

$$V_1 = c^{1-\eta}\| J^{-1}\mathcal{P}_{\mathcal{U}}(t)F_1(0, \phi(0))\|$$

$$\leq \frac{M_0M_fM_m}{1(\rho(1-\xi) + \xi)}M'_p,$$

$$V_2 = c^{1-\eta}\| J^{-1}F_1(t, pt + \tilde{\beta}_t)\|$$

$$\leq c^{1-\eta}M_0M'_m(1 + r').$$

By referring the Hölder’s inequality and Lemma 4, we have

$$V_3 = c^{1-\eta} \int_0^t \| (t-s)^{\gamma-1}A_0^{1-\xi}\mathcal{U}_1(t-s)A_0^{\xi}J^{-1}F_1(s, p_s + \tilde{\beta}_s)\| ds$$

$$\leq c^{1-\eta}M_0M'_m(1 + r').$$
By referring the Hypotheses (H₁)–(H₅), we get

\[
V_4 = c^{1-\eta}\| \int_0^t (t-s)^{\xi-1}J^{-1} \mathcal{X}_\xi(t-s)
\]

\[
\times F_2\left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_\tau + \hat{\beta}_\tau) d\tau, \int_0^c f(s, \tau, p_\tau + \hat{\beta}_\tau) d\tau \right) ds \|,
\]

\[
\leq c^{1-\eta} \bar{J}_m \int_0^t (t-s)^{\xi-1} \mathcal{X}_\xi(t-s) c_1(s) Y(r' + cE_0(1+r') + cE_1(1+r')) ds \|
\]

\[
\leq c^{1-\eta} \frac{M_\bar{J}_m K_1}{\Gamma(\xi)} Y(r' + cE_0(1+r') + cE_1(1+r')) ,
\]

and

\[
V_5 = c^{1-\eta} \| \int_0^t (t-s)^{\xi-1} J^{-1} \mathcal{X}_\xi(t-s) Bx_p(s) ds \|,
\]

\[
\leq c^{1-\eta} \| \int_0^t (t-s)^{\xi-1} J^{-1} \mathcal{X}_\xi(t-s) BW^{-1} \left[ u^1 - J^{-1} \mathcal{X}_\xi(c) J[\phi(0) - F_1(0, \phi)] + J^{-1} F_1(c, p_c + \hat{\beta}_c) + \int_0^c (c-s)^{\xi-1} J^{-1} A \mathcal{X}_\xi(c-s) J^{-1} F_1(s, p_s + \hat{\beta}_s) ds + \int_0^c (c-s)^{\xi-1} J^{-1} A \mathcal{X}_\xi(c-s) \right] ds \|
\]

\[
\leq c^{1-\eta+\xi} \frac{M_\bar{J}_m e_\beta e_w}{\Gamma(\eta)} \left[ u^1 + \frac{M_\bar{J}_m J_m c^{\eta-1}}{\Gamma(s(1-\eta) + \eta)} M_0(\|\phi(0)\| + K_0) + M_0 \bar{J}_m M_\delta'(1+r') + c^{1-\eta+\xi} \frac{C_1-\mu \Gamma(1+\mu)}{\mu \Gamma(1+\mu)} \bar{J}_m M_\delta'(1+r') + \frac{M_\bar{J}_m}{\Gamma(\xi)} K_1 Y(r' + cE_0(1+r') + cE_1(1+r')) \right].
\]

Combining all the above results \( V_1 - V_5 \), we get

\[
r \leq \frac{M_0 \bar{J}_m J_m}{\Gamma(v(1-\xi) + \xi)} M_\delta' + c^{1-\eta+\xi} M_0 \bar{J}_m M_\delta'(1+r') + c^{1-\eta+\xi} \frac{C_1-\mu \Gamma(1+\mu)}{\mu \Gamma(1+\mu)} \bar{J}_m M_\delta'(1+r') + c^{1-\eta+\xi} \frac{M_\bar{J}_m J_m c^{\eta-1}}{\Gamma(s(1-\eta) + \eta)} M_0(\|\phi(0)\| + K_0) + M_0 \bar{J}_m M_\delta'(1+r') + c^{1-\eta+\xi} \frac{C_1-\mu \Gamma(1+\mu)}{\mu \Gamma(1+\mu)} \bar{J}_m M_\delta'(1+r') + \frac{M_\bar{J}_m}{\Gamma(\xi)} K_1 Y(r' + cE_0(1+r') + cE_1(1+r')) \]

\[
\leq c^{1-\eta+\xi} \frac{M_\bar{J}_m e_\beta e_w}{\Gamma(\eta)} \left[ u^1 + \frac{M_\bar{J}_m J_m c^{\eta-1}}{\Gamma(s(1-\eta) + \eta)} M_0(\|\phi(0)\|) + \frac{M_\bar{J}_m}{\Gamma(\xi)} K_1 Y(r' + cE_0(1+r') + cE_1(1+r')) \right] + \left( 1 + c^{1-\eta+\xi} \frac{M_\bar{J}_m e_\beta e_w}{\Gamma(\xi)} \right) \frac{M_0 \bar{J}_m J_m}{\Gamma(v(1-\xi) + \xi)} M_\delta' + c^{1-\eta} M_0 \bar{J}_m M_\delta'(1+r')
\]
\[ F_n(s) = 2 \left( t, u_n(t), \int_{0}^{t} e(t, \tau, t^{1-\eta} u_n(\tau)) d\tau, \int_{0}^{c} f(t, \tau, \tau^{1-\eta} u_n(\tau)) d\tau \right) \]

\[ F(s) = 2 \left( t, u(t), \int_{0}^{t} e(t, \tau, t^{1-\eta} u(\tau)) d\tau, \int_{0}^{c} f(t, \tau, \tau^{1-\eta} u(\tau)) d\tau \right), \]

where

\[ \mathcal{F}_n(s) = F_n(s), \quad \mathcal{F}(s) = F(s). \]

By referring to (H1) and Lebesgue’s dominated convergence theorem, we have

\[ \int_{0}^{t} (t-s)^{x-1} \| J^{-1} \mathcal{F}_n(s) - J^{-1} \mathcal{F}(s) \| ds \to 0 \text{ as } n \to \infty, \quad t \in I. \]

In view of (H1),

\[ \| \tilde{Y}p^n - \tilde{Y}p\|_{L^p} \leq \frac{M_{K_1}}{1(\xi)} \int_{0}^{t} (t-s)^{x-1} \| \mathcal{F}_n(s) - \mathcal{F}(s) \| ds + \frac{M_{K_1}}{1(\xi)} \| x^n - x \|_{L^p}, \]
By using (7) and (8), we have
\[ \| \bar{Y} p^n - \bar{Y} p \|_{L^p} \to 0 \text{ when } n \to \infty. \]

Therefore \( \bar{Y} \) is continuous on \( F_r \).

**Step 3:** For \( p \in F_r \), assume \( p(t) = t^{\eta - 1} u(t) \), \( \bar{Y} \) sends bounded sets into equicontinuous sets of \( C \), for all \( z \in F_r \), there exists \( z \in \bar{Y}(z) \) such that \( \| z(t_2) - z(t_1) \| \to 0 \) when \( t_2 \to t_1 \).

\[ z(t) = r_{1-\eta}^{-1} P_{\nu, \xi}(t)[\phi(0) - F_1(0, \phi(0))] + \int_0^t r_{1-\eta}^{-1} P_{\nu, \xi}(t_1) \int_0^{t_1} (\tau, s, t_1 + \hat{\beta}_s) d\tau \int_0^{t_1} f(s, \tau, p_\tau + \hat{\beta}_\tau) d\tau ds \]

Assume \( 0 < \delta < t \) and \( 0 < t_1 < t_2 < c \). Then \( \bar{Y}(F_r) \) is equicontinuous on \( I \).
+ M_0 \| t_2^{1-\eta} A^\mu F_1(t_2, p_{t_2} + \hat{\beta}_{t_2}) - t_1^{1-\eta} A^\mu F_1(t_1, p_{t_1} + \hat{\beta}_{t_1}) \| \\
+ t_2^{1-\eta} \int_0^{t_2} (t_2 - s)^{1-\nu} A^{1-\mu} \mathcal{J}_\xi(t_2 - s) A^{1-\mu} f_1(s, p_s + \hat{\beta}_s) ds \\
+ \int_0^{t_2} \left[ t_2^{1-\eta} (t_2 - s)^{1-\nu} - t_1^{1-\eta} (t_1 - s)^{1-\nu} \right] A^{1-\mu} \mathcal{J}_\xi(t_2 - s) \\
\left( \times A^\mu \int_0^{t_1} f_1(s, p_s + \hat{\beta}_s) ds \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times [ A^{1-\mu} \mathcal{J}_\xi(t_2 - s) - A^{1-\mu} \mathcal{J}_\xi(t_1 - s) ] A^{1-\mu} \int_0^{t_1} f_1(s, p_s + \hat{\beta}_s) ds \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times A^\mu \int_0^{t_1} f_1(s, p_s + \hat{\beta}_s) ds \right) \\
+ t_2^{1-\eta} \int_0^{t_2} (t_2 - s)^{1-\nu} \mathcal{J}_\xi(t_2 - s) F_2(s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau) ds \\
+ \int_0^{t_2} \left[ t_2^{1-\eta} (t_2 - s)^{1-\nu} - t_1^{1-\eta} (t_1 - s)^{1-\nu} \right] \mathcal{J}_\xi(t_2 - s) \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_2^{1-\eta} \int_0^{t_2} (t_2 - s)^{1-\nu} \mathcal{J}_\xi(t_2 - s) B x_p(s) ds \\
+ \int_0^{t_2} \left[ t_2^{1-\eta} (t_2 - s)^{1-\nu} - t_1^{1-\eta} (t_1 - s)^{1-\nu} \right] \mathcal{J}_\xi(t_2 - s) B x_p(s) ds \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
+ t_1^{1-\eta} \int_0^{t_1} (t_1 - s)^{1-\nu} \| \\
\left( \times F_2 \left( s, p_s + \hat{\beta}_s, \int_0^s e(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau, \int_0^c f(s, \tau, p_{\tau} + \hat{\beta}_{\tau}) d\tau \right) \right) \\
\leq \sum_{i=1}^{16} T_i ,

where

\begin{align*}
T_1 &= \frac{M_0 M_1 m}{\Gamma(\nu(1-\xi))} \int_0^{t_1} t_2^{1-\eta} (t_2 - s)^{\nu(1-\xi) - 1} \| [\phi(0) - F_1(0, \phi(0))] \| ds , \\
T_2 &= t_2^{1-\eta} \frac{M_0 M_1 m}{\Gamma(\xi)} \int_0^{t_1} (t_2 - s)^{1-\nu} - (t_1 - s)^{1-\nu} \| [\phi(0) - F_1(0, \phi(0))] \| ds , \\
T_3 &= (t_1^{1-\eta} - t_2^{1-\eta}) \frac{M_0 M_1 m}{\Gamma(\xi)} \int_0^{t_1} (t_1 - s)^{1-\nu} \| [\phi(0) - F_1(0, \phi(0))] \| ds , \\
T_4 &= M_0 m t_2^{1-\eta} - t_1^{1-\eta} M_1 M_2 \epsilon^{1+ \epsilon} , \\
T_5 &= t_1^{1-\eta} \frac{K_{1-\mu} \Gamma(1+\mu)}{\Gamma(1+\xi \mu)} \| [(t_2 - t_1)^{1-\nu}] M_1 m \| ,
\end{align*}
Now, we need to prove that the Mönch’s condition holds.

**Step 4:** Now, we need to prove that the Mönch’s condition holds.

Consider \((p^0 + \hat{\beta})(t) = t^{1-\eta} \mathcal{A}_F(t)\phi(0),\) for all \(t \in I\) and \((p^{m+1} + \hat{\beta}) = \tilde{Y}(p^m + \hat{\beta}), m = 0, 1, 2, 3, \ldots\) and \(\tilde{Y}\) is relatively compact.

Consider \(H \subseteq F\) is countable and \(H \subseteq \text{conv}\{0\} \cup \tilde{Y}(H)\). We need to verify that \(\mathcal{T}(H) = 0\), where \(\mathcal{T}\) is the Hausdorff measure of noncompactness. Consider \(H = \{(p^m + \hat{\beta})\}_{m=1}^\infty\). Presently, we have to verify that \(\tilde{Y}(H)\) is relatively compact in \(X\), for all \(t \in I\). By referring Theorem 1,

\[
\mathcal{T}(H) = \mathcal{T}(\{(p^m + \hat{\beta})(t)\}_{m=0}^\infty) = \mathcal{T}(\{(p^0 + \hat{\beta})(t)\} \cup \{(p^m + \hat{\beta})(t)\}_{m=1}^\infty)
\]

and

\[
\mathcal{T}(\{(\tilde{Y}p^m)(t)\}_{m=1}^\infty) = \mathcal{T}(\{t^{1-\eta} \int_0^t (t-s)^{\frac{1}{\xi}-1} \mathcal{A}_F(t-s)\xi_m(s) + Bx_{\xi}(s)\}_{m=1}^\infty)
\leq \frac{2M\Gamma(1-\eta)}{\Gamma(1)\Gamma(\xi)} \int_0^1 \int_0^t (t-s)^{\frac{1}{\xi}-1} \mathcal{A}_F(t-s)\xi_m(s) + Bx_{\xi}(s)_{m=1}^\infty)ds
\]

The right-hand side of \(T_1\) to \(T_{12}\) tends to '0' as \(\delta \to 0\). On implementing the absolute continuity of the Lebesgue integral dominance convergence theorem for inequality, we conclude that \(T_1\) to \(T_{12}\) gives '0' when \(t_2 - t_1 \to 0\). Therefore, \(\tilde{Y}(F_t)\) is equicontinuous on \(I\).
$= U_1 + U_2$.

Now,

$$U_1 \leq \frac{2 M \tilde{m} c^{1-\eta}}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1} \mathcal{T} \left( \left\{ \mathcal{E}_m(p^{m}_s + \hat{\beta}_s) \right\}_{m=1}^{\infty} \right) ds$$

$$\leq \frac{2 M \tilde{m} c^{1-\eta}}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1} \left[ \mathcal{T} \left( \left\{ p^m(\tau + \phi) + \hat{\beta}(\tau + \phi) \right\}_{m=1}^{\infty} \right) + \mathcal{T} \left( \left\{ \int_0^\tau c(\tau, \epsilon, p^{m}_\epsilon + \hat{\beta}_\epsilon) d\epsilon, \int_0^b f(\tau, \epsilon, p^{m}_\epsilon + \hat{\beta}_\epsilon) d\epsilon \right\}_{m=1}^{\infty} \right) \right] ds$$

$$\leq \frac{2 M \tilde{m} c^{1-\eta}}{\Gamma(\xi)} K_2(1 + 2c^*_3 + 2c^*_4) \sup_{-\infty < \phi \leq 0} \mathcal{T}(\mathcal{X}(\phi)),$$

$$U_2 = \mathcal{T} \left( \{ c^{1-\eta} \int_0^t (t-s)^{\eta-\frac{1}{2}} J^{-1} \mathcal{J}_\xi(c-s) B w^{m}_\phi(s) ds \}_{m=1}^{\infty} \right)$$

$$\leq \frac{2 M \tilde{m} e c^{1-\eta}}{\Gamma(\xi)} \int_0^t (t-s)^{\eta-\frac{1}{2}} \mathcal{T} \left( \{ w^{m}_\phi(s) \}_{m=1}^{\infty} \right) ds$$

$$\leq \frac{2 M \tilde{m} e c^{1-\eta}}{\Gamma(\xi)^2} \int_0^t (t-s)^{\eta-\frac{1}{2}} \mathcal{T} \left[ W^{-1} \left( \int_0^t (c-s)^{\eta-\frac{1}{2}} J^{-1} \mathcal{J}_\xi(c-s) \mathcal{T}(E_m(p^{m}_s + \hat{\beta}_s))_{m=1}^{\infty} ds \right) \right] ds$$

$$\leq \frac{2 M \tilde{m} e c^{1-\eta}}{\Gamma(\xi)} \int_0^t (t-s)^{\eta-\frac{1}{2}} M w^{m}_\phi(s) ds \frac{2 M \tilde{m} K_2(1 + 2c^*_3 + 2c^*_4)}{\Gamma(\xi)} \sup_{-\infty < \phi \leq 0} \mathcal{T}(\mathcal{X}(\phi))$$

$$\leq \frac{4 M^2 \tilde{m} e c^{1-\eta}}{\Gamma(\xi)^2} K_0 K_2 \tilde{m}(1 + 2c^*_3 + 2c^*_4) \sup_{-\infty < \phi \leq 0} \mathcal{T}(\mathcal{X}(\phi)).$$

Now

$$U_1 + U_2 \leq \frac{2 M \tilde{m} K_2(1 + 2c^*_3 + 2c^*_4) c^{1-\eta}}{\Gamma(\xi)} \left[ 1 + \frac{2 M \tilde{m} K_0 \Gamma(\xi)}{\Gamma(\xi)} \right] \sup_{-\infty < \phi \leq 0} \mathcal{T}(\mathcal{X}(\phi)).$$

By referring Lemma 5,

$$\mathcal{T}(\tilde{Y}(H)) \leq L^* \mathcal{T}(H).$$

Hence by using Mönch’s condition, we have

$$\mathcal{T}(Y) \leq \mathcal{T}(\text{conv}(\{0\} \cup \tilde{Y}(H))) = \mathcal{T}(\tilde{Y}(H)) \leq L^* \mathcal{T}(H),$$

this implies $\mathcal{T}(H) = 0$. Therefore, $\tilde{Y}$ has a fixed point $y \in F_r$, in view of Lemma 6. Next, $u = p + \hat{\beta}$ is a mild solution of the fractional system (1)-(2) satisfying $u(c) = u^1$, then the fractional system (1)-(2) is controllable on $X$. □

**4. Nonlocal Conditions**

Physical problems prompted the development of evolution equations with nonlocal conditions. In [54, 55], the authors explored nonlocal issues for the first time in 1990, obtaining the existence and uniqueness of mild solutions for nonlocal differential equations of integer order. For more details on the systems with integer of fractional orders, one can refer [29, 30, 32, 35, 45, 54, 55]. Assume that nonlocal Hilfer fractional delay Volterra-Fredholm integro-differential system has the following form

$$D_0^{c,\xi} [Ju(t) - F_1(t, u_t)] = Au(t) + Bx(t)$$
\[ F_2 \left( t, u(t), \int_0^t e(t, \tau, u(\tau)) d\tau, \int_0^t f(t, \tau, u(\tau)) d\tau \right), \quad t \in (0, c], \] (9)

\[ I_{0^+}^{(1-\nu)(1-\zeta)} u(0) = \phi + g(u_{1, t}, u_{1, t}, \ldots, u_{1, t}) \in \mathcal{R}_I, \] (10)

where \( 0 < t_1 < t_2 < t_3 < \ldots < t_n \leq d, g : \mathcal{R}_I^n \to \mathcal{R}_I \) and satisfies the following:

\((\mathcal{H}_6)\)  \( g : \mathcal{R}_I^n \to \mathcal{R} \) is continuous, there exists \( K_1(g) > 0 \) such that

\[ \| g(u_{1, t}, u_{2, \ldots, u_{n}}) - g(w_{1, t}, w_{2, \ldots, w_{n}}) \| \leq \sum_{i=1}^{n} K_1(g) \| u_i - w_i \| \text{ for all } u_i, w_i \in \mathcal{R}_I \] and consider \( K_g = \sup \{ \| g(u_{1, t}, u_{2, \ldots, u_{n}}) \| : u_i \in \mathcal{R}_I \} \).

**Definition 10.** A function \( u : (-\infty, c] \to X \) is said to be a mild solution of (9) and (10) if \( u_0 = \phi + g(u_{1, t}, u_{1, t}, \ldots, u_{1, t})(0) \in \mathcal{R}_I \) on \(( -\infty, 0) \) and

\[ u(t) = I_{0^+}^{(1-\nu)(1-\zeta)} \left[ \left( \int_0^t e(t, \tau, u(\tau)) d\tau, \int_0^t f(t, \tau, u(\tau)) d\tau \right) \right] + F_2 \left( t, u(t), \int_0^t e(t, \tau, u(\tau)) d\tau, \int_0^t f(t, \tau, u(\tau)) d\tau \right), \quad t \in (0, c], \] (9)

is satisfied.

**Theorem 3.** Assume that the hypotheses \((\mathcal{H}_6)-(\mathcal{H}_9)\) are satisfied, then the fractional system (9) and (10) is controllable if

\[ L^* = \frac{2M_{11} K_2 (1 + 2\epsilon^*_1 + 2\epsilon^*_4) c^{1 - \eta}}{\Gamma(\zeta)} \left[ 1 + \frac{2M_{11} K_0 c_{1b}}{\Gamma(\zeta)} \right], \] for some \( \frac{1}{2} < \zeta < 1. \]

**5. Examples**

5.1. Abstract System

Assume that the Hilfer fractional differential system with control of the following form

\[ D_{0^+}^{\nu, \zeta} \left[ u(s, h) - \frac{\partial^2}{\partial h^2} u(s, h) + \int_0^h \zeta(z, h) u(s, z) dz \right] = \frac{\partial^2}{\partial h^2} u(s, h) \]

\[ + E \left( s, \int_0^h \zeta_1(\epsilon - s) u(\epsilon, h) d\epsilon, \int_0^h \int_{-\infty}^0 \zeta_2(\tau, x, \epsilon - \tau) u(\epsilon, h) d\epsilon d\tau, \right. \]

\[ \left. \int_0^h \int_{-\infty}^0 \zeta_3(\tau, x, \epsilon - \tau) u(\epsilon, h) d\epsilon d\tau \right] + \mathcal{V}(s, h), \quad \tau \in [0, \pi], \]

\[ t_{(1-\nu)(1-\zeta)} [u(s, h)]|_{h=0} = u_0(h), \quad h \in [0, \pi], \]

\[ u(s, 0) = u(s, \pi) = 0, \quad s \geq 0, \]

\[ u(0, \mu) = s(\mu), \quad 0 \leq \mu \leq \pi, \]

(11)

where \( D_{0^+}^{\nu, \zeta} \) denotes the Hilfer fractional derivative of order \( (\zeta = \frac{\zeta}{2}) \) and type \( \nu, \mathcal{V} : I \times [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous.

To transform the fractional system (11)–(14) to abstract form, assume \( X = L^2(0, \pi) \) and \( A : D(A) \subset X \to X, J : D(J) \subset X \to X \) be defined by \( Aw = w^0, \) and \( Pw = w - A \) where \( D(A) \) and \( D(J) \) is given by \( \{ w \in U : w, w^0 \) are absolutely continuous, \( w(0) = w(\pi) = 0 \}. \)

Then, \( A \) and \( J \) are presented as \( Aw = \sum_{m=1}^{\infty} h^2 \langle w, z_m \rangle z_m, \) \( w \in D(A), \)

After applying the conditions and transformations, the abstract form of the system can be solved.

To conclude, the abstract form allows for a more general framework to analyze the fractional differential equations with control. The controllability condition (15) provides a criterion for determining whether the system can be steered to a desired state from a given initial state within a finite time.
where $z_m(x) = \sqrt{\frac{2}{\pi}} \sin(mx)$, $m = 1, 2, 3, \cdots$ are the orthonormal vectors of $A$. Then, for $z \in U$, we have
\[
P^{-1} z = \sum_{m=1}^{\infty} \frac{1}{1 + m^2} \langle z, z_m \rangle z_m,
\]
\[
AJ^{-1} z = \sum_{m=1}^{\infty} \frac{m^2}{(1 + m^2)} \langle z, z_m \rangle z_m,
\]
and
\[
Q_p(\theta) z = \sum_{m=1}^{\infty} \exp \left( \frac{m^2 \theta}{1 + m^2} \right) \langle z, z_m \rangle z_m.
\]

A is defined by $T(\theta)z(\tau) = z(\theta + \tau)$ for $z \in X$, $T(\theta)$ is not compact on $X$ with $\mathcal{F}(T(\theta)D) \leq \mathcal{F}(D)$, $\mathcal{F}$ is the Hausdorff measure of noncompactness.

Here $A$ is an infinitesimal generator of a semigroup $\{T(s), s \geq 0\}$ in $X$ and which is presented as $T(s)w(\varepsilon) = w(s + \varepsilon)$, for $w \in U$, $T(s)$ is not compact on $X$ with $\mathcal{F}(T(s)D) \leq \mathcal{F}(D)$. Furthermore, $s \rightarrow w(s^2 + \varepsilon)u$ is equicontinuous ([45]), where $s \geq 0$ and $\mu \in (0, \infty)$.

We assume $F_1: [0, \pi] \times X \times X \rightarrow X$ and $F_2: [0, \pi] \times X \times X \rightarrow X$ by
\[
e(\theta, s) = \int_{-\infty}^{\theta} \zeta_2(\theta, \tau, \varepsilon)s(\varepsilon)d\varepsilon,
\]
\[
f(\theta, s) = \int_{-\infty}^{\varepsilon} \zeta_3(\theta, \tau, \varepsilon)s(\varepsilon)d\varepsilon,
\]
\[
F_1(u)(\varepsilon) = \int_{0}^{\pi} \zeta(z, \varepsilon)u(s, \varepsilon)d\varepsilon,
\]
\[
F_2(\theta, s, \varepsilon, s, \varepsilon, \varepsilon, \varepsilon) = E(s, \int_{-\infty}^{\pi} v_{1}(\varepsilon - s)u(\varepsilon, \zeta)d\varepsilon, \int_{0}^{\varepsilon} e(\varepsilon, s)(\tau)d\varepsilon, \int_{0}^{\varepsilon} f(\varepsilon, s)(\tau)d\varepsilon),
\]
and $D_{\mu}^{\frac{3}{2}}(u)(\varepsilon) \mu = \frac{3}{2}\mu u(s, \mu), u(s)(\varepsilon) = u(s, \varepsilon)$.

Let $B: X \rightarrow X$ be given by $(Bx)(s)(h) = \mathcal{F}x(s, h)$, $0 < s < 1$. For $h \in (0, \pi)$, $W$ is given by
\[
Wx(h) = \int_{0}^{1} (1 - s) h_{\frac{1}{2}}^{1-\frac{1}{2}} \mathcal{F}_{1}(1 - s) \mathcal{F}x(s, h)ds,
\]
where
\[
\mathcal{F}_{1}^{\frac{3}{2}} = \frac{2}{3} \int_{0}^{\infty} h_{\frac{3}{2}}^{n-1} x_{\frac{3}{2}}(h \frac{3}{2}) dh,
\]
and for $h \in (0, \infty)$
\[
\mathcal{F}_{1}^{\frac{3}{2}}(h) = \frac{3}{2} h^{-1} - \frac{2}{3} \mathcal{F}_{1}(h \frac{3}{2}),
\]
\[
\mathcal{F}_{1}^{\frac{3}{2}}(h) = \frac{1}{n} \sum_{n=1}^{\infty} (-1)^{n-1} h^{-\frac{3}{2}n-1} \frac{\Gamma\left(\frac{3}{2}n + 1\right)}{n!} \sin\left(\frac{2n\pi}{3}\right),
\]
where $\zeta_{\frac{3}{2}}$ is determined on $(0, \infty)$, i.e.,
\[
\zeta_{\frac{3}{2}}(h) \geq 0, \quad h \in (0, \infty) \quad \text{and} \quad \int_{0}^{\infty} \zeta_{\frac{3}{2}}(h)dh = 1.
\]
Finally $F_1$, $F_2$ and $\mathcal{V}$ satisfy the hypotheses $(H_1)-(H_5)$, thus all the requirements of the Theorem 2 are satisfied, then the fractional system (11)–(14) is controllable on $I$.

5.2. Filter System

An advanced filter is a framework that performs mathematical operations on an inspected, digitized sign to decrease or upgrade certain highlights of the prepared signal. Propelled by the plans examined in [13,19,29,56,57], we presented a filter design for our framework which is shown in Figure 1. Figure 1 portrays the rough pattern of block diagram which helps to improve the viability of arrangement with least measure of sources of input and which is presented as follows.

![Filter System Diagram](image)

**Figure 1.** Filter System.

Product modulator (PM)-1 receives inputs $u_t$ and $F_1$ generates the output as $F_1(t, u_t)$. PM-2 receives $A$ and $F_1(t, u_t)$ generates $AF_1(t, u_t)$. PM-4 receives $u_\tau$ and $f$ generates $f(t, \tau, u_\tau)$. PM-5 receives $u_\tau$ and $e$ generates $e(t, \tau, u_\tau)$. PM-6 receives $F_2(t, u_t)$, $e(t, \tau, u_\tau)$ and $f(t, \tau, u_\tau)$ generates

$$F_2 \left( t, u_t, \int_0^t e(t, \tau, u_\tau)d\tau, \int_0^c f(t, \tau, u_\tau)d\tau \right).$$

PM-7 receives $x(t)$ and $B$ generates $Bx(t)$. PM-8 receives $[\phi(0) - F_1(0, \phi(0))]$ and $\mathcal{P}_{\nu, \xi}(t)$ at time $t = 0$, generates $\mathcal{P}_{\nu, \xi}(t)$. The integrators execute the integral of

$$\mathcal{P}_{\nu, \xi}(t) \left[ A\mathcal{P}_{\nu, \xi}(t)F_1(t, u_t) + F_2 \left( t, u_t, \int_0^t e(t, \tau, u_\tau)d\tau, \int_0^c f(t, \tau, u_\tau)d\tau \right) + Bx(t) \right],$$

over $I$.

Additionally, Inputs $\mathcal{P}_{\nu, \xi}(t)$, $AF_1(t, u_t)$ are joined and multiplying with the output on the interval $(0, t)$. $\mathcal{P}_{\nu, \xi}(t)$, $F_2 \left( t, u_t, \int_0^t e(t, \tau, u_\tau)d\tau, \int_0^c f(t, \tau, u_\tau)d\tau \right)$ are joined and multiplying with the output on the interval $(0, t)$. $\mathcal{P}_{\nu, \xi}(t)$, $Bx(t)$ are joined and multiplying with the output on the interval $(0, t)$.

Finally, if we shift all the outputs from the integrators to summer network, then, the output of $u(t)$ is achieved, which is bounded and controllable.
6. Conclusions

The exact controllability of Sobolev-type Hilfer fractional neutral integro-differential systems via measure of noncompactness is the topic of our article. The main conclusions of our paper are based on theoretical ideas such as fractional calculus, the measure of noncompactness, and the fixed-point approach. First, we looked at the exact controllability of mild solutions for fractional evolution systems. Then we expanded on our findings to consider the system in nonlocal conditions. Finally, we presented theoretical and practical applications to aid in the efficacy of the discussion.

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