MINI REVIEW

Some subgroup embeddings in finite groups: A mini review

A. Ballester-Bolinches a,*, J.C. Beidleman b, R. Esteban-Romero c,1, M.F. Ragland d

a Departament d’Àlgebra, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain
b Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA
c Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, 46022 València, Spain
d Department of Mathematics, Auburn University at Montgomery, P.O. Box 244023, Montgomery, AL 36124-4023, USA

ARTICLE INFO

Article history:
Received 3 April 2014
Received in revised form 15 April 2014
Accepted 18 April 2014
Available online 26 April 2014

Keywords:
Finite group
Permutability
S-permutability
Semipermutability
Primitive subgroup
Quasipermutable subgroup

ABSTRACT

In this survey paper several subgroup embedding properties related to some types of permutability are introduced and studied.

Introduction

All groups in the paper are finite.

The purpose of this survey paper is to show how the embedding of certain types of subgroups of a finite group $G$ can determine the structure of $G$. The types of subgroup embedding properties we consider include: S-permutability, S-semipermutability, semipermutability, primitivity, and quasipermutability.

A subgroup $H$ of a group $G$ is said to permute with a subgroup $K$ of $G$ if $HK$ is a subgroup of $G$. $H$ is said to be permutable in $G$ if $H$ permutes with all subgroups of $G$. A less restrictive subgroup embedding property is the S-permutability introduced by Kegel and defined in the following way:

Definition 1. A subgroup $H$ of $G$ is said to be $S$-permutable in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ for every prime $p$.

In recent years there has been widespread interest in the transitivity of normality, permutability and S-permutability.
Definition 2.

1. A group $G$ is a $T$-group if normality is a transitive relation in $G$, that is, if every subnormal subgroup of $G$ is normal in $G$.
2. A group $G$ is a $PT$-group if permutability is a transitive relation in $G$, that is, if $H$ is permutable in $K$ and $K$ is permutable in $G$, then $H$ is permutable in $G$.
3. A group $G$ is a $PST$-group if $S$-permutability is a transitive relation in $G$, that is, if $H$ is $S$-permutable in $K$ and $K$ is $S$-permutable in $G$, then $H$ is $S$-permutable in $G$.

If $H$ is $S$-permutable in $G$, it is known that $H$ must be subnormal in $G$ ([1, Theorem 1.2.14(3)]). Therefore, a group $G$ is a $PST$-group (respectively a $PT$-group) if and only if every subnormal subgroup is $S$-permutable (respectively permutable) in $G$.

Note that $T$ implies $PT$ and $PT$ implies $PST$. On the other hand, $PT$ does not imply $T$ (non-Dedekind modular $p$-groups) and $PST$ does not imply $PT$ (non-modular $p$-groups). The reader is referred to [1, Chapter 2] for basic results about these classes of groups. Other characterisations based on subgroup embedding properties can be found in [2].

Agrawal ([1, 2.1.8]) characterised soluble $PST$-groups. He proved that a soluble group $G$ is a $PST$-group if and only if the nilpotent residual in $G$ is an abelian Hall subgroup of $G$ on which $G$ acts by conjugation as power automorphisms. In particular, the class of soluble $PST$-groups is subgroup-closed.

Let $G$ be a soluble $PST$-group with nilpotent residual $L$. Then $G$ is a $PT$-group (respectively $T$-group) if and only if $G/L$ is a modular (respectively Dedekind) group ([1, 2.1.11]).

Definition 3 [3]. A subgroup $H$ of a group $G$ is said to be semipermutable (respectively, $S$-sempermutable) provided that it permutes with every subgroup (respectively, Sylow subgroup) $K$ of $G$ such that $\gcd(|H|, |K|) = 1$.

An $S$-sempermutable subgroup of a group need not be subnormal. For example, a Sylow 2-subgroup of the nonabelian group of order 6 is semipermutable and $S$-sempermutable, but not subnormal.

Definition 4 (see [4]). A group $G$ is called a $BT$-group if semipermutability is a transitive relation in $G$.

L. Wang, Y. Li, and Y. Wang proved the following theorem which showed that soluble $BT$-groups are a subclass of $PST$-groups:

Theorem 5 [4]. Let $G$ be a group with nilpotent residual $L$. The following statements are equivalent:

1. $G$ is a soluble $BT$-group.
2. Every subgroup of $G$ of prime power order is $S$-sempermutable.
3. Every subgroup of $G$ of prime power order is semipermutable.
4. Every subgroup of $G$ is semipermutable.
5. $G$ is a soluble $PST$-group and if $p$ and $q$ are distinct primes not dividing the order of $L$ with $G_p$ a Sylow $p$-subgroup of $G$ and $G_q$ a Sylow $q$-subgroup of $G$, then $[G_p, G_q] = 1$.

Research papers on BT-groups include [4-7].

We next present an example of a soluble $PST$-group which is not a $BT$-group.

Example 6. Let $L$ be a cyclic group of order 7 and $A = C_3 \times C_2$ be the automorphism group of $L$. Here $C_3$ (respectively, $C_2$) is the cyclic group of order 3 (respectively, 2). Let $G = [L, A]$ be the semidirect product of $L$ by $A$. Let $L = \langle x \rangle$, $C_3 = \langle y \rangle$ and $C_2 = \langle z \rangle$ and note that $[\langle y \rangle^2, \langle z \rangle] \neq 1$. Now $G$ is a $PST$-group by Agrawal’s theorem, but $G$ is not a $BT$-group by Theorem 5.

A subclass of the class of soluble BT-groups is the class of soluble SST-groups, which has been introduced in [8].

Definition 7 (see [9]). A subgroup $H$ of a group $G$ is said to be $SS$-permutable (or $SS$-quasinormal) in $G$ if $H$ has a supplement $K$ in $G$ such that $H$ permutes with every Sylow subgroup of $K$.

Definition 8 (see [8]). We say that a group $G$ is an $SST$-group if $SS$-permutability is a transitive relation.

$SS$-permutability can be used to obtain a characterisation of soluble $PST$-groups.

Theorem 9 [8]. Let $G$ be a group. Then the following statements are equivalent:

1. $G$ is soluble and every subnormal subgroup of $G$ is SS-permutable in $G$.
2. $G$ is a soluble $PST$-group.

Theorem 10 [8]. A soluble $SST$-group $G$ is a $BT$-group.

The following example shows that a soluble BT-group is not necessarily an SST-group.

Example 11 [8]. Let $G = \langle x, y \mid x^3 = y^4 = 1, x^y = x^{-1} \rangle$. The nilpotent residual of $G$ is the Sylow 5-subgroup $\langle x \rangle$. By Theorem 5, $G$ is a soluble BT-group. Let $H = \langle x \rangle$ and $M = \langle y^2 \rangle$. Suppose that $M$ is SS-permutable in $G$. Then $G$ is the unique supplement of $M$ in $G$. It follows that $M$ is S-permutable in $G$, and thus $M \leq O_2(G)$. This implies that either $O_2(G) = H$ or $O_2(G) = M$. Since $y^2 = yx^{-1}$ and $(y^2)^2 = y^2x^2$, neither $H$ nor $M$ are normal subgroups of $G$. This contradiction shows that $M$ is not SS-permutable in $G$. Since $M$ is SS-permutable in $\langle x, y^2 \rangle$ and this subgroup is SS-permutable in $G$, we obtain that the soluble group $G$ cannot be an SST-group.

A less restrictive class of groups is the class of $T_0$-groups which has been studied in [5,7,10-12].

Definition 12. A group $G$ is called a $T_0$-group if the Frattini factor group $G/\Phi(G)$ is a $T$-group.

Theorem 13 [11]. Let $L$ be the nilpotent residual of the soluble $T_0$-group. Then:

1. $G$ is supersoluble;
2. $L$ is a nilpotent Hall subgroup of $G$. 
Theorem 14 [10]. Let $G$ be a soluble $T_0$-group. If all the subgroups of $G$ are $T_0$-groups, then $G$ is a PST-group.

A group $G$ is called an MS-group if the maximal subgroups of all the Sylow subgroups of $G$ are S-semipermutable.

Theorem 15 [13]. If $G$ is an MS-group, then $G$ is supersoluble.

Theorem 16 [7]. Let $L$ be the nilpotent residual of an MS-group $G$. Then:

1. $L$ is a nilpotent Hall subgroup of $G$;
2. $G$ is a soluble $T_0$-group.

We now provide three examples which illustrate several properties and differences of some of the classes presented in this paper. These examples are from [6,7].

Example 17. Let $C = \langle x \rangle$ be a cyclic group of order 7 and let $A = \langle y \rangle \times \langle z \rangle$ be a cyclic group of order 6 with $y$ an element of order 3 and $z$ an element of order 2. Then $A = \text{Aut}(C)$. Let $G = [C, A]$ be the semidirect product of $C$ by $A$. Then $\langle [y]^2, z \rangle \neq 1$ and $G$ is not a soluble BT-group. However, $G$ is an MS-group.

Example 18 shows that the classes of MS- and $T_0$-groups are not subgroup closed.

Example 18. Let $H = \langle x, y \mid x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1 \rangle$ be an extraspecial group of order 27 and exponent 3. Then $H$ has an automorphism $a$ of order 2 given by $x^a = x^{-1}$, $y^a = y^{-1}$ and $[x, y]^a = [x, y]$. Put $G = \langle H(a) \rangle$, the semidirect product of $H$ by $\langle a \rangle$. Let $z = \langle x, y \rangle$. Then $\Phi(G) = \Phi(H) = \langle z \rangle = Z(G) = Z(H)$. Note that $G/\Phi(G)$ is a T-group so that $G$ is a $T_0$-group. The maximal subgroups of $H$ are normal in $G$ and it follows that $G$ is an MS-group. Let $K = \langle x, z, a \rangle$. Then $\langle xz \rangle$ is a maximal subgroup of $\langle x, z \rangle$, the Sylow 3-subgroup of $K$. However, $\langle xz \rangle$ does not permute with $\langle a \rangle$ and hence $\langle xz \rangle$ is not an S-semipermutable subgroup of $K$. Therefore, $K$ is not an MS-subgroup of $G$. Also note that $\Phi(K) = 1$ and so $K$ is not a T-subgroup of $G$ and $K$ is not a T-$0$-subgroup of $G$. Hence the class of soluble $T_0$-groups is not closed under taking subgroups. Note that $G$ is not a soluble PST-group.

Example 19 presents an example of a soluble PST-group which is not an MS-group.

Example 19. Let $C = \langle x \rangle$ be a cyclic group of order 19$^2$, $D = \langle y \rangle$ a cyclic group of order 3$^2$, and $E = \langle z \rangle$ is a cyclic group of order 2 such that $D \times E \leq \text{Aut}(C)$. Then $G = [C, (D \times E)]$ is a soluble PST-group and $G$ is not an MS-group since $\langle [y]^2, z \rangle \neq 1$.

The following notation is needed in the presentation of the next theorem which characterises MS-groups. Let $G$ be a group whose nilpotent residual $L$ is a Hall subgroup of $G$. Let $\pi = \pi(L)$ and let $\theta = \pi^c$, the complement of $\pi$ in the set of all prime numbers. Let $\theta_P$ denote the set of all primes $p$ in $\theta$ such that if $P$ is a Sylow $p$-subgroup of $G$, then $P$ has at least two maximal subgroups. Further, let $\theta_q$ denote the set of all primes $q$ in $\theta$ such that if $Q$ is a Sylow $q$-subgroup of $G$, then $Q$ has only one maximal subgroup or, equivalently, $Q$ is cyclic.

Theorem 20 [6]. Let $G$ be a group with nilpotent residual $L$. Then $G$ is an MS-group if and only if $G$ satisfies the following:

1. $G$ is a $T_0$-group.
2. $L$ is a nilpotent Hall subgroup of $G$.
3. If $p \in \pi$ and $P \in \text{Syl}_p(G)$, then a maximal subgroup of $P$ is normal in $G$.
4. Let $p$ and $q$ be distinct primes with $p \in \theta_P$ and $q \in \theta$. If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$, then $[P, Q] = 1$.
5. Let $p$ and $q$ be distinct primes with $p \in \theta_q$ and $q \in \theta$. If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ and $M$ is the maximal subgroup of $P$, then $QM = MQ$ is a nilpotent subgroup of $G$.

Theorem 21 [6]. Let $G$ be a soluble PST-group. Then $G$ is an MS-group if and only if $G$ satisfies 4 and 5 of Theorem 20.

Theorem 22 [6]. Let $G$ be a soluble PST-group which is also an MS-group. If $\theta_5$ is the empty set, then $G$ is a BT-group.

Definition 23 [14]. A subgroup $H$ of a group $G$ is called primitive if it is a proper subgroup in the intersection of all subgroups containing $H$ as a proper subgroup.

All maximal subgroups of $G$ are primitive. Some basic properties of primitive subgroups include:

Proposition 24.

1. Every proper subgroup of $G$ is the intersection of a set of primitive subgroups of $G$.
2. If $X$ is a primitive subgroup of a subgroup $T$ of $G$, then there exists a primitive subgroup $Y$ of $G$ such that $X = Y \cap T$.

Johnson [14] proved that a group $G$ is supersoluble if every primitive subgroup of $G$ has prime power index in $G$.

The next results on primitive subgroups of a group $G$ indicate how such subgroups give information about the structure of $G$.

Theorem 25 [15]. Let $G$ be a group. The following statements are equivalent:

1. Every primitive subgroup of $G$ containing $\Phi(G)$ has prime power index.
2. $G/\Phi(G)$ is a soluble PST-group.

Theorem 26 [16]. Let $G$ be a group. The following statements are equivalent:

1. Every primitive subgroup of $G$ has prime power index.
2. $G = [L, M]$ is a supersoluble group, where $L$ and $M$ are nilpotent Hall subgroups of $G$. $L$ is the nilpotent residual of $G$ and $G = LN_G(L \cap X)$ for every primitive subgroup $X$ of $G$. In particular, every maximal subgroup of $L$ is normal in $G$.

Let $X$ denote the class of groups $G$ such that the primitive subgroups of $G$ have prime power index. By Proposition 24 (1), it is clear that $X$ consists of those groups whose subgroups are intersections of subgroups of prime power indices.
The next example shows that the class $\mathcal{X}$ is not subgroup closed.

**Example 27.** Let $P = \langle x, y | x^5 = y^5 = [x, y]^5 = 1 \rangle$ be an extraspecial group of order 125 and exponent 5. Let $z = [x, y]$ and note that $Z(P) = \Phi(P) = \langle z \rangle$. Then $P$ has an automorphism $a$ of order four given by $x^a = x^2$, $y^a = y^2$, and $z^a = z^{-1}$. Put $G = [P] \langle a \rangle$ and note that $Z(G) = 1$, $\Phi(G) = \langle z \rangle$, and $G/\Phi(G)$ is a T-group. Thus $G$ is a soluble $T_0$-group. Let $H = \langle y, z, a \rangle$ and notice that $\Phi(H) = 1$. Then $H$ is not a $T$-group since the nilpotent residual $L$ of $H$ is $\langle y, z \rangle$ and $a$ does not act on $L$ as a power automorphism. Thus $H$ is not a $T_0$-group, and hence not a soluble PST-group. By Theorem 25, $G$ is an $X$-group and $H$ is not an $X$-group.

**Theorem 28 [17].** Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PST-group.
2. Every subgroup of $G$ is an $X$-group.

We bring the paper to a close with the quasipermutable embedding which is defined in the following way.

**Definition 29.** A subgroup $H$ is called *quasipermutable* in $G$ provided there is a subgroup $B$ of $G$ such that $G = N_G(H)B$ and $H$ permutes with $B$ and with every subgroup (respectively, with every Sylow subgroup) $A$ of $B$ such that $\gcd(|H|, |A|) = 1$.

Theorem 30 contains new characterisations of soluble PST-groups with certain Hall subgroups.

**Theorem 30 [18].** Let $D = G^N$ be the nilpotent residual of the group $G$ and let $\pi = \pi(D)$. Then the following statements are equivalent:

1. $D$ is a Hall subgroup of $G$ and every Hall subgroup of $G$ is quasipermutable in $G$.
2. $G$ is a soluble PST-group.
3. Every subgroup of $G$ is quasipermutable in $G$.
4. Every $\pi$-subgroup of $G$ and some minimal supplement of $D$ in $G$ are quasipermutable in $G$.

**Acknowledgements**

The work of the first and the third authors has been supported by the Grant MTM2010-19938-C03-03 from the Ministerio de Economía y Competitividad, Spain. The first author has also been supported by the Grant 11271085 from the National Natural Science Foundation of China.

**References**

[1] Ballester-Bolinches A, Esteban-Romero R, Asaad M. Products of finite groups. Vol. 53 of de Gruyter Expositions in Mathematics. Berlin: Walter de Gruyter; 2010. http://dx.doi.org/10.1515/9783110220612.

[2] Beidleman JC. Weakly normal subgroups and classes of finite groups. Note Mat 2012;32(2):115–21.

[3] Chen ZM. On a theorem of Srinivasan. Southwest Normal Univ Nat Sci 1987;12(1):1–4.

[4] Wang L, Li Y, Wang Y. Finite groups in which (S-) semipermutability is a transitive relation. Int J Algebra 2008;2(1–4):143–52, Corrigendum in Int J Algebra 2012;6(13–16):727–8.

[5] Al-Sharo KA, Beidleman JC, Heineken H, Ragland MF. Some characterizations of finite groups in which semipermutability is a transitive relation. Forum Math 2010;22(5):855–62, Corrigendum in Forum Math 2012;24(7):1333–34.

[6] Ballester-Bolinches A, Beidleman JC, Esteban-Romero R, Ragland MF. On a class of supersoluble groups. Bull Austral Math Soc 2014:90:220–26.

[7] Beidleman JC, Ragland MF. Groups with maximal subgroups of Sylow subgroups satisfying certain permutability conditions. Southeast Asian Bull Math (in press).

[8] Chen XY, Guo WB. Finite groups in which SS-permutability is a transitive relation. Acta Math Hungar 2014;143(2):446–79.

[9] Li SR, Shen ZC, Liu JJ, Liu XC. The influence of SS-quasinormality of some subgroups on the structure of finite groups. J Algebra 2008;319:4275–87.

[10] Ballester-Bolinches A, Esteban-Romero R, Pedraza-Aguilera MC. On a class of p-supersoluble groups. Algebra Colloq 2005;12(2):263–7.

[11] Ragland MF. Generalizations of groups in which normality is transitive. Commun Algebra 2007;35(10):3242–52.

[12] van der Waal RW, Fransman A. On products of groups for which normality is a transitive relation on their Frattini factor groups. Quaestiones Math 1996;19:59–82.

[13] Ren YC. Notes on $\pi$-quasi-normal subgroups in finite groups. Proc Am Math Soc 1993;117:631–6.

[14] Johnson DL. A note on supersoluble groups. Can J Math 1971;23:562–4.

[15] He X, Qiao S, Wang Y. A note on primitive subgroups of finite groups. Commun Korean Math Soc 2013;28(1):55–62.

[16] Guo W, Shum KP, Skiba AN. On primitive subgroups of finite groups. Indian J Pure Appl Math 2006;37(6):369–76.

[17] [17] Ballester-Bolinches A, Beidleman JC, Esteban-Romero R. Primitive subgroups and PST-groups. Bull Austral Math Soc 2014;89(3):373–8.

[18] Yi X, Skiba AN. Some new characterizations of PST-groups. J Algebra 2014;399:39–54.