COMBINATORICS OF DOUBLE GROTHENDIECK POLYNOMIALS

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Abstract. We give a generalized Cauchy identity for double Grothendieck polynomials, a combinatorial interpretation of the stable double Grothendieck polynomials in terms of triples of tableaux, and an interpolation between the stable double Grothendieck polynomial and the weak stable double Grothendieck polynomial. This so-called half weak stable double Grothendieck polynomial evaluated at $x = y$ generalizes the type $B$ Stanley symmetric function of Billey and Haiman and is $Q$-Schur positive by degree.

1. Introduction

1.1. Background. Grothendieck polynomials [LS82b],[LS83], are a non-homogeneous generalization of Schubert polynomials, the latter a family of polynomials indexed by permutations which are studied among other things in relation to the combinatorics of Coxeter groups. In particular, the lowest degree term of a Grothendieck polynomial is a Schubert polynomial. Combinatorially, Grothendieck polynomials replace the notion of the symmetric group with that of the 0-Hecke monoid [BKS+08]. Double Schubert polynomials are considered in [LS82a] to generalize Schubert polynomials by extending them to two sets of variables in such a way that setting the second variable set to zero returns a regular Schubert polynomial. In turn, double Grothendieck polynomials generalize Grothendieck polynomials by extending them to two sets of variables in such a way that setting the second variable set to zero returns a regular Grothendieck polynomial.

Double Grothendieck polynomials themselves are generally not symmetric in either set of variables. However, there exists a way to derive a (doubly) symmetric function from a double Grothendieck polynomial through a process of letting the number of variables go to infinity and then setting all but finitely many of them to zero. The limit is stable, meaning that once the number of variables is high enough the result of the process is the same for all higher numbers of variables. In fact, what this number is, is not difficult to specify, so to define these so-called stable double Grothendieck polynomials requires no real appeal to any notion of the infinite.

An important special class of stable double Grothendieck polynomials is that which is composed of the polynomials indexed by Grassmannian permutations. In fact, all stable double Grothendieck polynomials expand in terms of these so-called symmetric double Grothendieck polynomials. (The relation between these polynomials and factorial Grothendieck polynomials is given in [McN06]). Moreover, they are Schur positive ([Len00]) and have a very nice combinatorial interpretation in

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terms of set valued tableaux [Buc02]. We also mention there exist weak versions of the double stable and double symmetric Grothendieck polynomials. In particular the weak symmetric Grothendieck polynomials have a combinatorial interpretation in terms of multiset valued tableaux ([LP07]).

1.2. Contributions and Organization. We explain what this paper contributes and how it is structured simultaneously. In section 2 we recall various constructions and results pertaining to Grothendieck polynomials appearing elsewhere in the literature that will be needed for the rest of the paper.

Section 3 deals with the most general version of double Grothendieck polynomials. The main result of section 3 is Theorem 3.6 which gives three formulae for the double Grothendieck polynomial. The first two are combinatorial expressions in terms of certain factorizations of Hecke words. Such interpretations are more useful for our purposes than pipe dream formulations as they are amenable to the Hecke insertion of [BKS+08]. Moreover, the relation between our two models helps to explicate the relationship between double Grothendieck and single Grothendieck polynomials. This relation is made explicit in the proof of the third formula, which is a generalization of the Cauchy identity for Schubert polynomials given in [FK94]. Somewhat surprisingly, some 25 years after the original formula was published this generalization has been independently discovered by the author and Brubaker et al. simultaneously (see [BFH+20]).

Section 4 deals with stable double Grothendieck polynomials, symmetric double Grothendieck polynomials and their relation. The main result of section 4 is Theorem 4.19 which gives a formula for the stable double Grothendieck polynomial in terms of triples of tableaux (see remark 4.20) as well as a similar formula for the weak double Grothendieck polynomial. If it were not for the fact that Hecke insertion lacks a certain property (see remark 2.9) such an expression would be an easy corollary of the work of [BKS+08]. Indeed in almost all imaginable analogous cases (i.e., for choices of the parameters single vs double, type A vs other types, standard vs k-theoretic) that have been defined a similar expression follows directly from the relevant insertion algorithm. However, in the absence of this property of Hecke insertion (we leave it as an open problem to amend Hecke insertion so that it does have this property) some additional work must be done. This work comprises the majority of section 4.

In section 5 we investigate the relation between double stable Grothendieck polynomials (of type A) and a potential definition of single stable Grothendieck polynomials of type B/C. The latter definition is given by first taking an interpolation of the stable double Grothendieck polynomial and the weak stable double Grothendieck polynomial and then evaluating the result when the two sets of variables are set equal to each other (i.e., at $x = y$). As we will see, the combinatorial definition of this “half weak” double Grothendieck polynomial is more amenable to Hecke insertion than either the weak or non-weak versions. Moreover, unlike the others, it is $Q$-Schur positive at $x = y$. This is the main result of section 5, which is stated as Theorem 5.20 which also gives a combinatorial interpretation of the coefficients in the $Q$-Schur expansion. The half weak double Grothendieck polynomial (evaluated at $x = y$) generalizes nicely the type B Stanley symmetric function of [BH95]: It agrees on the lowest degree terms and is $Q$-Schur positive by degree.
2. Single Grothendieck Polynomials

2.1. Operator definition.

**Definition 2.1.** Let \( f \in \mathbb{Z}[x_1, \ldots, x_{n+1}] \). For each \( 1 \leq i \leq n \) define the divided difference operator \( \delta_i \) by \( \delta_i(f) = \frac{f-s_i f}{x_i-x_{i+1}} \) where \( s_i \) acts by interchanging the variables \( x_i \) and \( x_{i+1} \). Define \( \pi_i \) by the formula \( \pi_i(f) = \delta_i(f) + \delta_i(x_{i+1} f) \).

**Lemma 2.2.** The divided difference operators satisfy the following relations:

1. If \( |i-j| > 1 \) then \( \delta_i \delta_j = \delta_j \delta_i \) and \( \pi_i \pi_j = \pi_j \pi_i \).
2. If \( i = j + 1 \) then \( \delta_i \delta_j = \delta_j \delta_i \) and \( \pi_i \pi_j = \pi_j \pi_i \).
3. \( \delta_i^2 = 0 \).
4. \( \pi_i^2 = -\pi_i \).

**Proof.**

1. Computing \( \delta_i \delta_j(f) \) and \( \delta_j \delta_i(f) \) shows that the second is obtained from the first by replacing \( s_i s_j f \) with \( s_j s_i f \). However, since \( |i-j| > 1 \), \( s_i \) and \( s_j \) commute so this difference is trivial. From this and the fact that \( s_i \) commutes with multiplication by \( x_{j+1} \) and vice versa we see \( \pi_i \pi_j = \pi_j \pi_i \).

2. Since the operators are clearly linear it suffices to prove the statement for monomials. Doing this for \( \pi \) will imply the same result for \( \delta \) by comparing lowest degree terms. To prove the statement for \( \pi \) on a monomial it suffices without loss of generality to show that \( \pi_1 \pi_2 \pi_3 x_1^{a_1} x_2^{a_2} x_3^{a_3} \) is the same as \( \pi_2 \pi_1 \pi_3 x_1^{a_1} x_2^{a_2} x_3^{a_3} \). Computing both of these directly yields:

\[
\sum_{\sigma \in S_3} \frac{x_{\sigma_1}^{a_1} x_{\sigma_2}^{a_2} x_{\sigma_3}^{a_3} (1 + x_{\sigma_2} + 2 x_{\sigma_2} x_{\sigma_3} + x_{\sigma_2}^2 + x_{\sigma_2} x_{\sigma_3}^2)}{(x_{\sigma_1} - x_{\sigma_2})(x_{\sigma_1} - x_{\sigma_3})(x_{\sigma_2} - x_{\sigma_3})}
\]

3. \( \delta_i(f) \) is a symmetric function with respect to \( x_i \) and \( x_{i+1} \). But \( \delta_i(f) \) vanishes on any function with such symmetry.

4. We have

\[
\pi_i^2(f) = \pi_i(\delta_i f + \delta_i x_{i+1} f) = \delta_i^2 f + \delta_i^2 x_{i+1} f + \delta_i x_{i+1} \delta_i f + \delta_i x_{i+1} \delta_i x_{i+1} f
\]

\[
= \delta_i x_{i+1} (\delta_i f + \delta_i x_{i+1} f) - x_i s_i (\delta_i f + \delta_i x_{i+1} f)
\]

\[
= \frac{x_{i+1} (\delta_i f + \delta_i x_{i+1} f) - x_i (\delta_i f + \delta_i x_{i+1} f)}{x_i - x_{i+1}} = - (\delta_i f + \delta_i x_{i+1} f) = -\pi_i(f)
\]

Given a permutation \( \omega \in S_n \) one can write down (non-uniquely in general) \( \omega \) as sequence of adjacent transpositions, i.e.: \( \omega = s_{i_1} \cdots s_{i_\ell} \) where \( \ell \) is the inversion number of the permutation. We can then define \( \delta_\omega \) by \( \delta_{i_1} \cdots \delta_{i_\ell} \) and \( \pi_\omega \) to be \( \pi_{i_1} \cdots \pi_{i_\ell} \). By parts 1 and 2 of Lemma 2.2 this procedure is well defined, i.e., the definition of \( \delta_\omega \) and \( \pi_\omega \) does not depend on the chosen reduced word.

**Definition 2.3.** Fix \( \omega \in S_n \) and let \( \omega_0 \) refer to the element of \( S_n \) with maximal inversion number. Define the Grothendieck polynomial for \( \omega \) by:

\[
G_\omega = \pi_{(\omega \cdot \omega_0)}(x_1^{n} x_2^{n-1} \cdots x_n^{1} x_{n+1}^{0}).
\]

See [LS82b] and [LS83] for original formulations.
2.2. Hecke Insertion. Consider a new operator \( \bar{s}_i \) acting on permutations of the set \( \{1, 2, 3, \ldots, n, n+1\} \) where the operation \( \bar{s}_i \) is given by interchanging \( i \) and \( i+1 \) if \( i \) lies to the left of \( i+1 \) and by doing nothing otherwise (In particular \( \bar{s}_i^2 = \bar{s}_i \) whereas \( s_i^2 = e \)). In this setting, a Hecke word for \( \omega \) is a sequence \( \bar{s}_{i_1} \cdots \bar{s}_{i_h} \) such that applying this sequence (right to left) to the starting arrangement \((1, 2, 3, \ldots, n, n+1)\) gives the permutation \( \omega \).

We will give an overview of a simple insertion algorithm [BKS+08] for Hecke words which will be necessary at various stages. First we need to define two types of tableaux:

**Definition 2.4.** A standard set-valued tableau of shape \( \lambda \) is a filling of a Young diagram of shape \( \lambda \) with exactly one of each of the letters \( \{1, \ldots, N\} \) for some integer \( N \geq |\lambda| \) such that each box contains at least one entry and such that all entries in a given box are smaller than all the entries in the box below and smaller than all the entries in the box to the right.

**Definition 2.5.** Given a permutation \( \omega \), a Hecke tableau for \( \omega \) of shape \( \lambda \) or element of \( HT_\omega(\lambda) \) is a tableau where each box of \( \lambda \) is filled with exactly one of the symbols \( \{\bar{s}_1, \ldots, \bar{s}_n\} \) in such a way that reading the boxes by rows, moving left to right within the rows and moving bottom to top amongst the rows gives a Hecke word for \( \omega \), and, such that the rows and columns are strictly increasing in the order \( \bar{s}_1 < \cdots < \bar{s}_n \). In actual examples, the boxes of a Hecke tableau will be written as, for instance \([3]\) instead of \([\bar{s}_3]\).

To define Hecke insertion, we first show how to insert some \( a \in \{\bar{s}_1, \ldots, \bar{s}_n\} \) into some row of a Hecke tableau, say \( Y = (y_1, y_2, \ldots, y_j) \) read from left to right. Suppose that the row above \( Y \) (if it exists) is \( X = (x_1, x_2, \ldots, x_k) \) and the row below it is \( Z = (z_1, z_2, \ldots, z_l) \). Here all \( x, y, z \in \{\bar{s}_1, \ldots, \bar{s}_n\} \). We assume that \( a \in [x_h, y_h) \) in the order \( \bar{s}_1 < \cdots < \bar{s}_n \) for some \( h \) (where possibly \( h > j \) and \( y_h \) is taken, by convention to be \( \infty \)) and that \( a > x_1 \). There are no restrictions on \( a \) if \( X \) does not exist that is, if \( Y \) is the first row of the tableau.

\[
\begin{array}{ccccccc}
  x_1 & x_2 & \cdots & x_i & \cdots & x_j & \cdots & x_k \\
y_1 & y_2 & \cdots & y_i & \cdots & y_j & \\
z_1 & z_2 & \cdots & z_l & \\
\end{array}
\]

\( \leftarrow a \)

We insert \( a \) into \( Y \) as follows:

1. If \( a \geq y_j \) and:
   a. \( a > y_j \) and \( a > x_{j+1} \). Then \( a \) is appended to the right of \( y_j \).
   b. \( a = y_j \) or \( a = x_{j+1} \). Then \( a \) simply disappears.

2. If \( a < y_j \). Let \( h \) be minimal such that \( a \leq y_h \).
   a. \( a = y_h \). Then \( Y \) stays the same and \( y_{h+1} \) is inserted into \( Z \).
   b. \( a < y_h \) and \( a > x_h \) then \( a \) replaces \( y_h \) and \( y_h \) is inserted into row \( Z \).
   c. \( a < y_h \) and \( a = x_h \) then \( Y \) is unchanged and \( y_h \) is inserted into row \( Z \).

Note that the result is strictly decreasing down columns by construction and that our assumption on \( a \) guarantees one of the situations above must occur. Moreover, the assumption is maintained moving on to the next insertion. That is, the element
We now describe complete Hecke insertion. Given a Hecke word say \( w_1 \cdots w_m \) create a sequence of pairs of tableaux of the same shapes \((P_0, Q_0), (P_1, Q_1), \ldots, (P_m, Q_m)\) by setting \( P_0 = \emptyset = Q_0 \) and creating \((P_{i+1}, Q_{i+1})\) from \((P_i, Q_i)\) as follows. Insert \( w_{i+1} \) into \( P_i \) by inserting it into the first row of \( P_i \). As long as there is an output, insert the output into the next row. The algorithm stops when either an element is appended to the end of a row or disappears. The resulting Hecke tableau is \( P_{i+1} \). If the algorithm ends by appending an element, add a box to the corresponding position of \( Q_i \) and fill it with the number \( i + 1 \) to form \( Q_{i+1} \). If the algorithm stops by an element disappearing, take the row where the last insertion occurred and caused this element to disappear and consider its rightmost box \( b \). Now find the lowest box in the same column as \( b \), call it \( b' \). Add an \( i + 1 \) to the position corresponding to \( b' \) in \( Q_i \) to form \( Q_{i+1} \). (Of course, it is possible \( b' = b \).)

\[
\begin{array}{cccc}
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & \\
\ & \ & \ & b \\
\ & \ & \ & \\
\ & \ & \ & b'
\end{array}
\]

\[\text{\leftarrow \text{disappearing element}}\]

**Example 2.6.** Suppose that we have

\[
P_{16} = \begin{array}{cccc}
1 & 2 & 4 & 5 \\
2 & 4 & 6 & 8 \\
3 & 5 & 7 & \\
4 & 7 & \\
6 & 8 & \\
9 & \\
\end{array} \quad Q_{16} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & \\
12 & 13 & \\
14 & 15 & \\
16 & \\
\end{array} \quad \omega_{17} = 3
\]

Then \( P_{17} \) and \( Q_{17} \) are computed as follows. First the 3 is inserted into row one of \( P_{16} \). 3 replaces 4 in this row and 4 is sent to be inserted into row two. When 4 is inserted into row two, since a 4 already appears in row two this row does not change and the number to the right of the 4 in row two, which is 6, will be inserted into row three. When 6 is inserted into row three it does not replace the 7 with itself because the number above this 7 in row two is not less than 6 (it is 6). Thus row three remains unchanged and the number 7 is inserted into row four. Row four ends in 7 itself so the inserted 7 is disappeared. Finally the 17 is added to the recording tableau not in the box, \( b \), at the end of row four but to the box \( b' \) at the bottom of the column containing \( b \). All in all the only changes are in the first row.
of the insertion tableau and the fifth row of the recording tableau and the result is:

\[
P_{17} = \begin{array}{cccc}
1 & 2 & 3 & 5 \\
2 & 4 & 6 & 8 \\
3 & 5 & 7 & \\
4 & 7 & \\
6 & 8 & \\
9 & 
\end{array} \quad Q_{17} = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & \\
12 & 13 & \\
14 & 15, 17 & \\
16 & 
\end{array}
\]

**Proposition 2.7 ([BKS+08]).** Fix \( \omega \in S_{n+1} \). Hecke insertion is a bijection from Hecke words for \( \omega \) to pairs \((P,Q)\) of tableaux of the same shape where \( P \) is a Hecke tableau for \( \omega \) and \( Q \) is a standard set-valued tableau.

Additionally, Hecke insertion also has the following convenient property:

**Lemma 2.8 ([BKS+08]).** If the word \( w_1 \cdots w_m \) maps to \((P,Q)\). Then \( w_i > w_{i+1} \) if and only if \( i+1 \) shows up in a row strictly below \( i \) in \( Q \).

**Remark 2.9.** Unfortunately (this fact will cause mild consternation later) it is not true that if \( w_1 \cdots w_m \) maps to \((P,Q)\) then \( w_i < w_{i+1} \) if and only if \( i+1 \) shows up in a column to the right of \( i \) in \( Q \). For instance applying Hecke insertion to \( \bar{s}_3 \bar{s}_2 \bar{s}_1 \) through abuse of notation as 1322 results in:

\[
\begin{cases}
\{1, 1\} \\
\{1, 3, 1, 2\} \\
\{1, 2, 1, 24\}
\end{cases}
\]

showing that although 4 = 3 + 1 shows up in a column to the right of 3 in \( Q \), it is not true that \( w_3 = 2 < 2 = w_4 \). As another example applying Hecke insertion to 1312 results in:

\[
\begin{cases}
\{1, 1\} \\
\{1, 3, 1, 2\} \\
\{1, 2, 1, 2\} \\
\{1, 2, 1, 2\}
\end{cases}
\]

showing that although \( w_3 = 1 < 2 = w_4 \) it is not true that 4 = 3 + 1 shows up in a column to the right of 3 in \( Q \).

**Definition 2.10.** A Hecke word that has been partitioned into groups of transpositions with decreasing indices is known as a **Hecke factorization**. For instance \((\bar{s}_3 \bar{s}_2)(\bar{s}_3 \bar{s}_2 \bar{s}_1)()((\bar{s}_1)\) is a Hecke factorization with four factors for the permutation \((4, 1, 3, 2) \in S_4 \). If \( f \) is a Hecke factorization we denote by \( wt(f) \) the vector whose \( i^{th} \) coordinate records the number of entries in the \( i^{th} \) factor (from left to right) of \( f \).

**Definition 2.11.** A Hecke factorization of an element in \( S_{n+1} \) with \( n+1 \) factors that has only entries with indices of at least \( i \) in the \( i^{th} \) subdivision is known as a **bounded Hecke factorization**. For instance, \((\bar{s}_3 \bar{s}_2 \bar{s}_1)()((\bar{s}_3)\), or \((321)(2)(3)\) is a
bounded Hecke factorization in $S_4$. (Note that the definition implies the last factor is always empty.)

- Let $F_{\omega}$ denote the set of all (unbounded) Hecke factorizations of $\omega$ into $m + 1$ parts.
- Let $\mathfrak{F}_{\omega}$ denote the set of all bounded circled Hecke factorizations of $\omega$ into $n + 1$ parts.

**Definition 2.12.** A (semistandard) set-valued tableau or SVT of shape $\lambda$ is a filling of a Young diagram of shape $\lambda$ with the letters $\{1, \ldots, m, m+1\}$ (repetition allowed) for some integer $m$ such that each box contains a nonempty set of numbers and

- If box $b$ lies left of box $b'$ then $\max(b) < \max(b')$.
- If box $b$ lies above box $b'$ then $\max(b) \leq \max(b')$.

A skew SVT of shape $\lambda/\mu$ is defined similarly.

**Definition 2.13.** Semistandard Hecke insertion is the following algorithm. Starting with $f \in F_{\omega}$ first consider the underlying Hecke word, $w$, given by erasing the parentheses in $f$. Apply regular Hecke insertion to obtain a pair of tableaux $(P, Q)$. Now form $Q'$ from $Q$ as follows. Wherever $j$ appears in $Q$ replace the $j$ with an $i$ where $i$ is chosen such that the $j^{th}$ entry in $f$ appears in the $i^{th}$ factor of $f$. The result of the algorithm is the pair $(P, Q')$.

We end this section with a few results that will be useful later.

**Lemma 2.14** ([BKS+08]). Semistandard Hecke insertion is a weight preserving bijection from $F_{\omega}$ to pairs $(P, Q)$ where $P \in HT_{\omega}$ and $Q \in SVT$ have the same shape.

**Theorem 2.15** ([Las90]). We have that

$$\Phi_{\omega} = \sum_{f \in F_{\omega}} x^{\omega(f)}$$

**Claim 2.16.** We have:

$$\sum_{Q \in SVT(\ell_1, \ell_2)} x_{r}^{(\#1)} x_{r+1}^{(\#2)} = \pi_s(\ell_1+1, \ell_2+1)$$

where the left hand sum is over all SVT with columns of lengths $\ell_1 \geq \ell_2$ in the letters $\{1, 2\}$ and $(\#1)$ and $(\#2)$ is the number of $1s$ and $2s$ respectively in $Q$.

**Proof.** Counting the elements in the left hand sum and doing the required algebra on the right hand side shows both are equal to:

$$x_r^{\ell_1} x_{r+1}^{\ell_2} + x_r^{\ell_1-1} x_{r+1}^{\ell_2+1} + \cdots + x_r^{\ell_2+1} x_{r+1}^{\ell_1-1} + x_r^{\ell_2} x_{r+1}^{\ell_1} + x_r^{\ell_1} x_{r+1}^{\ell_2+1} + x_r^{\ell_1-1} x_{r+1}^{\ell_2+2} + \cdots + x_r^{\ell_2+2} x_{r+1}^{\ell_1-1} + x_r^{\ell_2+1} x_{r+1}^{\ell_1}$$

\[\square\]

### 3. Double Grothendieck Polynomials

Consider two sets of variables $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_{n+1})$. We extend the action of $\pi_s$ linearly over $\mathbb{Z}[y]$ to get an action on $\mathbb{Z}[x,y]$. 

Definition 3.1. A circled Hecke factorization is a factorization of a Hecke word into factors, where some of the elements have been circled. Moreover, each factor must be decreasing in the order $1 < 2 < \cdots < n$. For instance, $(32\,\overline{2})(\overline{3}\,21\overline{1})(1\overline{1})$ is a circled Hecke factorization for the permutation $(4,1,3,2) \in S_4$.

Definition 3.2. A bounded circled Hecke factorization is a circled Hecke factorization with $n+1$ factors such that all the elements in the $i^{th}$ factor are $\geq 1$. For instance, $(\overline{3}3\overline{2}1)(3\overline{3})(43\overline{3})(4)(1)$ is a bounded circled Hecke factorization for the permutation $(5,1,4,3,2) \in S_5$. The $x$-weight of a bounded circled Hecke factorization is the vector whose $i^{th}$ entry records the number of uncircled elements in its $i^{th}$ factor. The $x$-weight of the example above is $(2,1,2,1,0)$. The $y$-weight of such a factorization is the vector whose $i^{th}$ entry records the number of circled entries that have some value $j$ and appear in some factor $k$ such that $j-k+1=i$. The $y$-weight of the example above is $(1,2,0,1,0)$.

Definition 3.3. A double Hecke factorization is a factorization into an even number of factors where the first half of the factors are increasing in the order $1 < \cdots < n$ and the last half of the factors are decreasing in the order $1 < \cdots < n$. For example, $(12)(13)(21)(32)$ is a double Hecke factorization for $(4,3,2,1) \in S_4$, where we have drawn a “|” between the left half and the right half of the factors for viewing convenience.

Definition 3.4. A bounded double Hecke factorization is such a factorization into $2n+2$ factors where all elements in the $i^{th}$ factor to the right of center are $\geq i$ and all elements in the $i^{th}$ factor to the left of center are $\geq i$. For instance, $(1\overline{3}3\overline{2}1)(\overline{3}3)(\overline{4}3\overline{3})(\overline{4})(1)$ is a bounded double Hecke factorization for the permutation $(4,3,2,1) \in S_4$. The $x$-weight of a bounded or unbounded double Hecke factorization is the vector whose $i^{th}$ entry records the number of elements in the $i^{th}$ factor to the right of center. The $x$-weight of the example above is $(2,2,1,0)$. The $y$-weight of a bounded or unbounded double Hecke factorization is the vector whose $i^{th}$ entry records the number of entries in the $i^{th}$ factor to the left of center. The $y$-weight of the example above is $(2,1,1,0)$.

We will use the following notation:

- Let $\mathcal{F}^\omega$ denote the set of all (unbounded) circled Hecke factorizations of $\omega$ into $m+1$ parts.
- Let $\mathcal{F}_c^\omega$ denote the set of all bounded circled Hecke factorizations of $\omega$ into $n+1$ parts.
- Let $\mathcal{F}_d^\omega$ denote the set of all (unbounded) double Hecke factorizations of $\omega$ into $2m+2$ parts.
- Let $\mathcal{F}_d^\omega$ denote the set of all bounded double Hecke factorizations of $\omega$ into $2n+2$ parts.

If $f$ is one of the factorizations above we write $(x,y)^{wt(f)}$ to mean the monomial $x^{x\text{-weight}(f)}y^{y\text{-weight}(f)}$. If two Hecke words represent the same permutation, we denote this by writing a “$\sim$” between them. Moreover, if $\mu$ is any permutation, let $\bar{\mu}$ denote an arbitrary Hecke word for $\mu$. Finally, if $\mu$ is any permutation let $X_\mu$ be the set of all pairs of permutations $(u,v)$ such that the concatenation $\overline{uv}$ represents the permutation $\mu$. Finally, we need one more definition before we can state the main result:
**Definition 3.5.** The double Grothendieck polynomial for $\omega$ [Las85]:

\[
G_\omega(x, y) = \pi(\omega^{-1} \omega_0) \left( \prod_{i+j \leq n+1} x_i + y_j + x_i y_j \right)
\]

The rest of this section will be devoted to proving that:

**Theorem 3.6.** We have:

(3.1) \[ G_\omega(x, y) = \sum_{f \in \mathcal{F}/\omega} (x, y)^{wt(f)} \]

(3.2) \[ G_\omega(x, y) = \sum_{f \in \mathcal{F}^\square/\omega} (x, y)^{wt(f)} \]

(3.3) \[ G_\omega(x, y) = \sum_{(u,v) \in X_\omega} G_{u^{-1}}(y) G_v(x) \]

**Remark 3.7.** Equation 3.3 is a generalization of that given in [FK94] for double Schubert polynomials. It has been independently discovered by Brubaker et al. (see Theorem 7.1 of [BFH+20]).

To do this we define:

**Definition 3.8.**

\[ G_\omega^\square(x, y) = \sum_{f \in \mathcal{F}^\square/\omega} (x, y)^{wt(f)} \]

**Definition 3.9.**

\[ G_\omega^\bullet(x, y) = \sum_{f \in \mathcal{F}^\bullet/\omega} (x, y)^{wt(f)} \]

**Definition 3.10.**

\[ G_\omega^\bullet(x, y) = \sum_{(u,v) \in X_\omega} G_{u^{-1}}(y) \cdot G_v(x) \]

**Lemma 3.11.** We have $G_\omega^\square(x, y) = G_\omega^\bullet(x, y)$

**Proof.** We have:

\[ \mathcal{F}_\omega^\square = \bigcup_{(u,v) \in X_\omega} \mathcal{F}^\square_{(u,v)} \]

where $\mathcal{F}^\square_{(u,v)}$ is the subset of $\mathcal{F}^\square/\omega$ such that the first $n+1$ factors give a Hecke word for the permutation $u$ and the second $n+1$ factors give a Hecke word for the permutation $v$. However, this is precisely the set of all of factorizations such that reading the first half of the factors from right to left gives a bounded Hecke factorization, $f_l$, for $u^{-1}$ and reading the second half of the factors from left to right gives a bounded Hecke factorization, $f_r$, for $v$. Moreover, the $y$-weight of this factorization is the weight of $f_l$ and the $x$-weight of this factorization is the weight of $f_r$. From this it follows that,

\[ \sum_{f \in \mathcal{F}^\square_{(u,v)}} (x, y)^{wt(f)} = G_{u^{-1}}(y) G_v(x) \]

which completes the proof. \hfill \square
Claim 3.12. There is a bijection from $\wedge(\mu) \rightarrow \vee(\mu)$ such that, denoting word length by $|\cdot|$, if $(b, c) \rightarrow (a, d)$ then $|a| = |c|$ and $|b| = |d|$.

Proof. Denote by $W^k(\mu)$ the set of all quadruples of Hecke words $(a, b, c, d)$ such that the concatenation $abcd$ is a Hecke word for $\mu$ and such that

- $a$ and $c$ are strictly decreasing.
- $b$ and $d$ are strictly increasing.
- $b$ and $c$ only contain elements from the set $\{1, 2, \ldots, k\}$.
- $a$ and $d$ only contain elements from the set $\{k+1, \ldots, n\}$.

It suffices to find a bijection $W^{k+1}(\mu) \rightarrow W^k(\mu)$, such that if $(a, b, c, d) \rightarrow (a', b', c', d')$ then $|a| + |c| = |a'| + |c'|$ and $|b| + |d| = |b'| + |d'|$. The bijection is given by the identity in the case that no Hecke word for $\mu$ contains a $k+1$. If some Hecke word for $\mu$ does contain a $k+1$ then set $K = k + 1$. In this case the bijection fixes all entries which are not equal to $k$ or $K$ and changes the entries equal to $k$ or $K$ as follows:

\[
(* \cdots *)(* \cdots k K)(K k \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots k)(k \cdots *)(K \cdots *)
\]
\[
(* \cdots *)(* \cdots k K)(k \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots k)(* \cdots )(K \cdots *)
\]
\[
(* \cdots *)(* \cdots k K)(Kk \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots *)(* \cdots )(K \cdots *)
\]
\[
(* \cdots *)(* \cdots k)(K \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots *)(* \cdots *)(* \cdots K)
\]
\[
(* \cdots*)(* \cdots K)(K \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots K)(* \cdots *)(* \cdots *)
\]
\[
(* \cdots *)(* \cdots k)(K \cdots *)(* \cdots ) \rightarrow (* \cdots *)(* \cdots k)(* \cdots *)(* \cdots K)
\]
\[
(* \cdots *)(* \cdots k K)(* \cdots *)(* \cdots ) \rightarrow (* \cdots *)(* \cdots k)(* \cdots *)(* \cdots )\]
\[
(* \cdots *)(* \cdots k K)(* \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots *)(* \cdots K)(* \cdots *)
\]
\[
(* \cdots *)(* \cdots K)(K \cdots *)(* \cdots ) \rightarrow (* \cdots K)(* \cdots K)(* \cdots *)(* \cdots *)
\]

It is easy to see that this defines a bijection with the desired properties. \qed

We denote the map $\wedge(\mu) \rightarrow \vee(\mu)$ by $\downarrow$ and its inverse by $\uparrow$. 
Example 3.13. Let \((123568)(8752) \in \wedge(\mu)\). Thus:

\[\begin{align*}
W_8 &= ()(123568)(8752)() \\
W_7 &= (8)(123567)(752)() \\
W_6 &= (8)(12356)(652)(7) \\
W_5 &= (86)(1235)(52)(67) \\
W_4 &= (865)(123)(2)(567) \\
W_3 &= (865)(123)(2)(567) \\
W_2 &= (8653)(12)(())(3567) \\
W_1 &= (8653)(1)()(23567) \\
W_0 &= (8653)()()()()()
\end{align*}\]

so that \(\downarrow(123568)(8752) = (8653)(123567) \in \vee(\mu)\).

Lemma 3.14. We have \(G^\circ_\omega(x, y) = G^\Box_\omega(x, y)\)

Proof. We need to find a bijection from \(F^\circ_\omega\) to \(F^\Box_\omega\) that preserves the \(x\)-weight and the \(y\)-weight. The arguments are quite technical and we give an example here that the reader is welcome to follow along through the proof.

Example 3.15. Let \(\omega = (4, 3, 2, 1) \in S_3\). The following sequence of factorizations would be computed under the bijection of this proof to get from an element of \(F^\circ_\omega\) to an element of \(F^\Box_\omega\). The sets \(F^\omega_{jk}\) will be defined inside the proof.

\[
(3\overline{3}2\overline{1}1)(\overline{3}2)(\overline{3}3)(())(3)() \in \mathfrak{F}^\circ_\omega
\]

\[
(()|)(3\overline{3}2\overline{1}1)(\overline{3}2)(\overline{3}3)(())(3)() \in \mathfrak{F}^{13}_\omega
\]

\[
(()|)(2\overline{3}2\overline{1}1)(\overline{3}2)(\overline{3}3)(())(3)() \in \mathfrak{F}^{22}_\omega
\]

\[
(()|)(3\overline{2}1\overline{1})(\overline{3})(\overline{3})(3)(())() \in \mathfrak{F}^{12}_\omega
\]

\[
(()|)(23)(21\overline{1})(\overline{2})(\overline{3})(3)(())() \in \mathfrak{F}^{10}_\omega
\]

\[
(()|)(3)(23)(21\overline{1})(\overline{2})(\overline{3})(3)(())() \in \mathfrak{F}^{31}_\omega
\]

\[
(()|)(3)(23)(321\overline{1})(2)(\overline{3})(3)(())() \in \mathfrak{F}^{21}_\omega
\]

\[
(()|)(3)(23)(321\overline{1})(2)(3)(3)(())() \in \mathfrak{F}^{11}_\omega
\]

\[
(()|)(3)(23)(1)(3)(3)(3)(())() \in \mathfrak{F}^{01}_\omega
\]

\[
(()|)(3)(23)(12)(3)(3)(3)(())() \in \mathfrak{F}^\Box_\omega
\]

For each \(k \in \{1, 2, \ldots, n\}\) and \(j \in \{0, 1, \ldots, n - k + 1\}\) we define the set \(\mathfrak{F}^{jk}_\omega\) to be the set of factorizations \(\circ\) such that:

- \(\circ\) contains \(n + 1 - k\) left factors which we denote \(f_{-(n+1)}, f_{-(n)}, \ldots, f_{-(k+1)}\).
- \(\circ\) contains \(n + 1\) right factors which we denote \(f_1, f_2, \ldots, f_{n+1}\).
- \(\circ\) contains 1 extra factor which we denote \(f_{xx}\). (Red in the example.)
- The Hecke word \(f_{-(n+1)} \cdots f_{-(k+1)} f_1 \cdots f_j f_{xx} f_{j+1} \cdots f_{n+1}\) represents \(\omega\).
- For each \(i\), the left factor \(f_{-(i)}\) contains elements from \(\{i, \ldots, n\}\).
- For each \(i\), the right factor \(f_i\) contains uncircled elements from \(\{i, \ldots, n\}\).
- For \(i \leq j\), \(f_i\) contains circled elements from \(\{\overline{1}, \ldots, \overline{s}\}: s = k + i - 1\).
For \( i > j \), \( f_i \) contains circled elements from \( \{1, \ldots, i\} : t = k + i - 2 \).

- The extra factor \( f_{ex} \) contains elements from \( \{j + k, \ldots, n\} \).
- The left factors and \( f_{ex} \) are strictly increasing in the order \( 1 < \cdots < n \).
- The right factors strictly decrease in the order \( \overline{1} < \cdots < \overline{n} < n \).

If \( f \in \mathfrak{F}^\omega \) we define \((x, y)^{\omega f(1)}\) to be the monomial such that the power of \( x_i \) is the number of uncircled elements in \( f_i \). For \( i > k \) the power of \( y_i \) is the number of elements in \( f_{-i} \). For \( i < k \) the power of \( y_i \) is the number of times some \( \overline{m} \) appears in some factor \( f_{\ell} \) such that \( m - \ell + 1 = i \). The power of \( y_k \) is the number of times some \( \overline{m} \) appears in some factor \( f_{\ell} \) such that \( m - \ell + 1 = k \) plus the number of elements in \( f_{ex} \). Essentially what we want to do now is show that we can take a factorization of the form \( f_{-(n+1)} \cdots f_{-(k+1)} f_{1} \cdots f_{j} f_{ex} f_{j+1} \cdots f_{n+1} \) and move the extra factor \( f_{ex} \) from the right of \( f_j \) to its left via some process. If we repeat this process eventually we can pull \( f_{ex} \) all the way to the left of \( f_1 \cdots f_{n+1} \) and make it \( f_{-k} \). If this in turn can be done for each \( k \) it means that if we start with a factorization of the form \( f_1 \cdots f_n \in \mathfrak{F}^\omega \) we can get one of the form \( f_{-(n+1)} \cdots f_{-(j+1)} f_{j+1} \cdots f_{n} \). We will then want to show the latter lies in \( \mathfrak{H}^\omega \).

We begin by noting that \( \mathfrak{H}^{\omega 1} = \mathfrak{F}^{\overline{1}} \) and that \( \mathfrak{H}^{01} = \mathfrak{F}^{\overline{1}} \). Moreover, we have that \( \mathfrak{H}^{0k} = \mathfrak{H}^{\omega^{n-k+1}(k-1)} \) for \( k > 1 \). Hence it suffices to find an \( x \)-weight and \( y \)-weight preserving bijection from \( \mathfrak{F}^{jk} \) to \( \mathfrak{F}^{\omega^{j-1}k} \) for \( j \in \{1, \ldots, n-k+1\} \) and \( k \in \{1, \ldots, n\} \).

To do the latter it suffices to find a bijection, \( \Psi_{jk} \) between pairs \((f_j, f_{ex})\) such that:

- \( f_j \) contains uncircled elements from \( \{j, \ldots, n\} \).
- \( f_j \) contains circled elements from \( \{\overline{1}, \ldots, \overline{k}\} : s = j + k - 1 \).
- \( f_{ex} \) contains elements from \( \{j + k, \ldots, n\} \).
- \( f_{ex} \) is strictly increasing in the order \( 1 < \cdots < n \).
- \( f_j \) is strictly decreasing in the order \( \overline{1} < \cdots < \overline{n} < n \).

To pairs \((f'_{ex}, f'_j)\) such that:

- \( f'_{ex} \) contains uncircled elements from \( \{j, \ldots, n\} \).
- \( f'_{ex} \) contains circled elements from \( \{\overline{1}, \ldots, \overline{k}\} : t = j + k - 2 \).
- \( f'_{ex} \) contains elements from \( \{j + k - 1, \ldots, n\} \).
- \( f'_{ex} \) is strictly increasing in the order \( 1 < \cdots < n \).
- \( f'_j \) is strictly decreasing in the order \( \overline{1} < \cdots < \overline{n} < n \).

with the property that if \((f_j, f_{ex}) \to (f'_{ex}, f'_j)\) then the Hecke words \( f_j f_{ex} \) and \( f'_{ex} f'_j \) represent the same permutation and whenever \((f_j)(f_{ex})\) appears in an element of \( \mathfrak{H}^{jk} \) (in the expected position) these two factors make the same contribution to the \( x \)-weight and \( y \)-weight as the pair \((f'_{ex})(f'_j)\) when it appears in an element of \( \mathfrak{H}^{\omega^{j-1}k} \) (in the expected position).

To do this write \( f_j = f_j'^+ f_j'^- f_j'^\) where \( f_j'^+ \), \( f_j'^- \), \( f_j'^\) are the parts of \( f_j \) composed respectively of elements greater than, equal to, or less than \( \overline{q} \) (where \( s = j + k - 1 \)) in the order \( \overline{1} < \cdots < \overline{q} < \overline{q+k} \).

If \( f_j'^+ \) is nonempty then append \( s \) to the left of \( f_{ex} \) to form \( f_{ex}^+ \). Otherwise set \( f_{ex}^+ = f_{ex} \). Then let \((g_1, g_2) = \uparrow (f_j'^+, f_{ex}^+)\). We define \( \Psi_{jk}(f_j, f_{ex}) = (f'_{ex}, f'_j) \) where \( f'_{ex} = g_1 \) and \( f'_j = g_2 f_j'^- \).

On the other hand given a pair \((f'_{ex}, f'_j)\) write \( f'_j = f'_j'^+ f'_j'^- f'_j'^\) where \( f'_j'^+ \) and \( f'_j'^- \) are the parts of \( f'_j \) composed respectively of elements greater than or less than \( \overline{q} \) in the order \( \overline{1} < \cdots < \overline{q} < n \). Next set \((h_1, h_2) = \downarrow (f'_{ex}, f'_j'^+)\). Now write \( h_2 = h_2^+ h_2^- \) where \( h_2^+ \) and \( h_2^- \) are the parts of \( h_2 \) composed respectively of elements equal to or
greater than $s$ in the order $1 < \cdots < n$. We define $\Psi_{jk}^{-1}(f'_{ex}, f''_j) = (f_j, f_{ex})$ where $f_j = h_1 f'_j$ if $h_0^j$ is empty and $f_j = h_1 \otimes f'_j$ otherwise and $f_{ex} = h_2^j$.

The fact that $\uparrow$ and $\downarrow$ preserve the permutation represented along with the commutation of nonadjacent transpositions implies that $\Psi$ and $\Psi^{-1}$ do not change the permutation represented. Moreover the constructions of $\Psi$ and $\Psi^{-1}$ make it clear that they map into the proper images. One can easily check that $\Psi^{-1} \circ \Psi$ is the identity by considering the two cases where either $f_j$ contains a $\otimes$ or does not. Similarly, one can check that $\Psi \circ \Psi^{-1}$ is the identity by considering the two cases where either the $h_2$ of the construction of $\Psi^{-1}$ contains an $s$ or does not.

Finally, if $(f_j, f_{ex}) \rightarrow (f'_{ex}, f'_j)$, we need to check these two pairs make the same contributions to the $x$-weight and $y$-weight of the factorization they are part of. For the first pair, the contribution to the $x$-weight is simply to add $r$ to the $j^{th}$ coordinate of the $x$-weight where $r$ is the number of uncircled elements in $f_j$. For the second pair, the contribution to the $x$-weight is simply to add $r'$ to the $j^{th}$ coordinate of the $x$-weight where $r'$ is the number of uncircled elements in $f'_j$. Clearly the construction of $\Psi$ implies that $r = r'$. Now the circled elements of $f_j$ and $f'_j$ other than $\otimes$ (which only affects the $k^{th}$ coordinate of the $y$-weight since $s - (j - 1) = k$) are the same and $f_{ex}$ and $f'_{ex}$ only affect the $k^{th}$ coordinate of the $y$-weight. Thus it suffices just to check that $(f_j)(f_{ex})$ and $(f'_{ex})(f'_j)$ make the same contribution to the $k^{th}$ coordinate of the $y$-weight. If $f_j$ does not contain a $\otimes$ then $f_{ex}$ and $f'_{ex}$ have the same length $\ell$ and the contribution to the $k^{th}$ coordinate of the $y$-weight is just $\ell + 0$ in either case since neither $f_j$ nor $f'_j$ contain a $\otimes$. If $f_j$ does contain a $\otimes$ then if $f_{ex}$ has length $\ell$ then $f'_{ex}$ has length $\ell + 1$. The contribution to the $k^{th}$ coordinate of the $y$-weight from $(f_j)(f_{ex})$ is $(1) + (\ell)$ since $f_j$ contains one $\otimes$. The contribution to the $k^{th}$ coordinate of the $y$-weight from $(f'_{ex})(f'_j)$ is $(\ell + 1) + (0)$ in since $f'_j$ contains no $\otimes$.

**Example 3.16.** Set $n = 9$ and $k = 3$ and $j = 2$. Suppose that $f_j = (9764(3)2(2))$ and $f_{ex} = (5689)$. Then as in the construction of $\Psi_{jk}$ we set $s = j + k - 1 = 4$ and $f_j^\uparrow$ becomes (9764) while $f_j^\downarrow$ becomes (3) and $f_j' \downarrow$ becomes (322). To compute $f_{ex}^\uparrow$ we append a 4 to $f_{ex}$, and so $f_{ex}^\uparrow$ becomes (45689). Next we set $(g_1, g_2) = \uparrow(9764)(45689)$. To evaluate this we compute:

\[ W^3 = (9764)(45689) \]
\[ W^4 = (976)(4)(4)(5689) \]
\[ W^5 = (976)(45)(5)(689) \]
\[ W^6 = (97)(456)(65)(89) \]
\[ W^7 = (9)(457)(765)(89) \]
\[ W^8 = (9)(4578)(865)(9) \]
\[ W^9 = (45789)(9865)(1) \]

and see that $g_1 = (45789)$ and $g_2 = (9865)$. Therefore we get that $f_{ex}' = g_1 = (45789)$ and $f_j' = g_2 f_j^\downarrow = (9865(3)2(2))$. All in all, we see that $\Psi_{23}$ sends

\[ (9764(3)2(2))(5689) \rightarrow (45789)(9865(3)2(2)) \]

**Lemma 3.17.** We have $\Theta^\circ_\omega(x, y) = \Theta^*_\omega(x, y)$

**Proof.** We proceed by induction on the number of inversions of $\omega^{-1} \omega_0$. First suppose that this number is 0. That is, $\omega = \omega_0$. First we compute $\Theta^\circ_\omega(x, y)$: For
any \( f \in \mathfrak{F}^0_{\omega_0} \) the boundedness condition implies that for each \( i \), the \( i^{th} \) factor of \( f \) contains a subset of \( \{ \mathfrak{o}, i, \ldots, \mathfrak{c}, n \} \). Letting \( \ell_i \) denote the number of distinct numerical values that appear (uncircled, circled, or both) in the \( i^{th} \) factor of \( f \), it is clear that the inversion number of the permutation represented by \( f \) is bounded by \( \sum \ell_i \), which, in turn is bounded by \( n + (n - 1) + \cdots + 1 + 0 = \binom{n+1}{2} \). But \( \binom{n+1}{2} \) actually is the inversion number of \( \omega_0 \in S_{n+1} \). Thus \( \sum \ell_i = \binom{n+1}{2} \), which means that \( \ell_i = n + 1 - i \) for each \( i \) or that all the numerical values \( \{ i, i + 1, \ldots, n \} \) show up in the \( i^{th} \) factor of \( f \) (uncircled, circled, or both). This means that there are precisely \( \binom{n+1}{2} \) factorizations in \( \mathfrak{F}^0_{\omega_0} \). Each factorization, \( f \), is specified by choosing, for each \( i \in \{ 1, n + 1 \} \) and \( j \in \{ 1, n + 1 - i \} \) whether the value \( (i + j - 1) \) appears in the \( i^{th} \) factor as circled, uncircled, or both. The value of \( (x, y)_{\text{wt}}(f) \) is computed by starting with 1 and, for each \( i \in \{ 1, n + 1 \} \) and \( j \in \{ 1, n + 1 - i \} \), multiplying by \( x_i, y_j \) or \( x_i y_j \) depending on whether the value \( (i + j - 1) \) appears in the \( i^{th} \) factor as circled, uncircled, or both.

It follows that:

\[
\theta^\circ_{\omega_0}(x, y) = \sum_{f \in \mathfrak{F}^0_{\omega_0}} (x, y)_{\text{wt}}(f) = \prod_{i + j \leq n + 1} (x_i + y_j + x_i y_j) = \theta_{\omega_0}(x, y)
\]

Combining this with Lemmas 3.14 and 3.11 we see that

\[
\theta_{\omega_0}(x, y) = \theta^\circ_{\omega_0}(x, y) = \theta^\circ_{\omega_0}(x, y) = \theta_{\omega_0}(x, y)
\]

which completes the base step of induction. Now suppose that \( \omega^{-1}\omega_0 \) has at least one inversion. Choose \( s_r \) such that \( \omega s_r \) has more inversions than \( \omega \). By definition we have \( \pi_{s_r} \theta_{\omega s_r}(x, y) = \theta_{\omega}(x, y) \) and by induction we have \( \theta_{\omega s_r}(x, y) = \theta_{\omega s_r}(x, y) \) so that \( \pi_{s_r} \theta_{\omega s_r}(x, y) = \theta_{\omega}(x, y) \). Therefore it suffices to show that \( \pi_{s_r} \theta_{\omega s_r}(x, y) = \theta_{\omega}(x, y) \). That is, we must show that:

\[
(3.4) \quad \pi_{s_r} \sum_{(u, v) \in X_{\omega s_r}} \theta_{\omega s_r-1}(y) \theta_{s_r}(x) = \sum_{(u, v) \in X_{\omega s_r}} \theta_{\omega s_r-1}(y) \theta_{s_r}(x)
\]

First we write \( X_{\omega s_r} = X_{\omega s_r}(1) \cup X_{\omega s_r}(2) \cup X_{\omega s_r}(3) \) where:

1. \( X_{\omega s_r}(1) = \{ (u, v) \in X_{\omega s_r} : \text{no reduced word for} \ v \ \text{ends in} \ s_r \} \).
2. \( X_{\omega s_r}(2) = \{ (u, v) \in X_{\omega s_r} : \exists \nu \text{ s.t.} \ \nu \sim \bar{\nu} \bar{s}_r, \ \bar{\nu} \neq \bar{\nu} \bar{s}_r, \ \bar{\nu} \bar{\nu} \sim \bar{\nu} \bar{s}_r \} \).
3. \( X_{\omega s_r}(3) = \{ (u, v) \in X_{\omega s_r} : \exists \nu \text{ s.t.} \ \nu \sim \bar{\nu} \bar{s}_r, \ \bar{\nu} \neq \bar{\nu} \bar{s}_r, \ \bar{\nu} \bar{\nu} \sim \bar{\nu} \bar{s}_r \} \).

Claim 3.18. There exist bijections

1. \( X_{\omega s_r}(2) \rightarrow X_{\omega s_r}(1) \)
2. \( X_{\omega s_r}(3) \rightarrow X_{\omega} \)

**Proof.** We describe a single map and its inverse that actually gives both the bijections. First let \( (u, v) \in X_{\omega s_r}(2) \) or \( (u, v) \in X_{\omega s_r}(3) \). In either case we may write \( v = \nu s_r \) for a (unique) permutation \( \nu \) with \( \bar{\nu} \neq \bar{\nu} \bar{s}_r \). We now simply send \( (u, v) \) to \( (u, \nu) \).

If \( (u, v) \in X_{\omega s_r}(2) \) then \( \bar{\nu} \bar{\nu} \sim \bar{\nu} \bar{s}_r \bar{s}_r \) so that \( (u, \nu) \in X_{\omega s_r} \). Moreover, \( \bar{\nu} \neq \bar{\nu} \bar{s}_r \) implies that \( v \) has no reduced word ending in \( s_r \), so \( (u, \nu) \in X_{\omega s_r}(1) \). If \( (u, v) \in X_{\omega s_r}(3) \) then \( \bar{\nu} \bar{\nu} \neq \bar{\nu} \bar{s}_r \bar{s}_r \) implies that \( (u, \nu) \in X_{\omega} \).

Next let \( (u, v) \in X_{\omega s_r}(1) \) or \( (u, v) \in X_{\omega} \). Then the inverse map sends \( (u, v) \) to \( (u, \nu s_r) \). First, suppose that \( (u, v) \in X_{\omega s_r}(1) \). Then the permutation that the Hecke word \( \bar{\nu} \bar{\nu} \bar{\nu} \bar{\nu} \) represents is \( \omega s_r \) which obviously has a reduced word ending in \( s_r \). Thus the permutation represented by \( \bar{\nu} \bar{s}_r \) is also \( \omega s_r \). Further, in the definition of
it suffices to show if $(\overline{v}s_r) \sim \overline{v}s_r$ (clearly), $\overline{v} \not\sim \overline{v}s_r$ (because $v$ has no reduced word ending in $s_r$), and $\overline{u} \overline{w} \sim \overline{u} \overline{w}s_r$ (by the previous sentence). Thus $(u, vs_r) \in X_{\omega s_r}(2)$. On the other hand if $(u, v) \in X_\omega$ then the fact that $\overline{u} \overline{w}$ represents the permutation $\omega$ implies that $\overline{u} \overline{w}s_r$ represents the permutation $\omega s_r$. Moreover, in the definition of $X_{\omega s_r}(3)$, replacing $v$ with $vs_r$ and $v$ with $v$ satisfies the three conditions: $(\overline{v}s_r) \sim \overline{v}s_r$ (clearly), $\overline{v} \not\sim \overline{v}s_r$ (because the fact $uv$ has no reduced word ending in $s_r$ implies the same for $v$), and $\overline{u} \overline{w} \not\sim \overline{u} \overline{w}s_r$ (because $\overline{u} \overline{w} \sim \overline{w}$ and $\omega$ has no reduced word ending in $s_r$). Finally, it is clear the given maps are mutual inverses. □

Claim 3.19.

\[
\pi_{s_r} \sum_{(u,v) \in X_{\omega s_r}(2)} \mathcal{G}_{u^{-1}}(y) \mathcal{G}_{v}(x) = -\pi_{s_r} \sum_{(u,v) \in X_{\omega s_r}(1)} \mathcal{G}_{u^{-1}}(y) \mathcal{G}_{v}(x) \tag{5.5}
\]

\[
\pi_{s_r} \sum_{(u,v) \in X_{\omega s_r}(3)} \mathcal{G}_{u^{-1}}(y) \mathcal{G}_{v}(x) = \sum_{(u,v) \in X_\omega} \mathcal{G}_{u^{-1}}(y) \mathcal{G}_{v}(x) \tag{5.6}
\]

Proof. To prove 3.5 it suffices to show if $(u, vs_r) \rightarrow (u, v)$ under bijection (i) then,

\[\pi_{s_r} \mathcal{G}_{u s_r}(x) = -\pi_{s_r} \mathcal{G}_{v}(x)\]

To prove 3.6 it suffices to show if $(u, vs_r) \rightarrow (u, v)$ under bijection (ii) then,

\[\pi_{s_r} \mathcal{G}_{u s_r}(x) = \mathcal{G}_{v}(x)\]

In either case if $\mu = \mu_1 \cdots \mu_l$ is a reduced word for the permutation $(\nu s_r)^{-1} \omega_0$, then $s_r \mu_1 \cdots \mu_l$ is a reduced word for $\nu^{-1} \omega_0$. So by the divided difference operator definition, the equations become:

\[\pi_{s_r}(\pi_{\mu_1} \cdots \pi_{\mu_l})(x^0_1 \cdots x^0_{n+1}) = -\pi_{s_r}(\pi_{s_r \mu_1} \cdots \pi_{\mu_l})(x^0_1 \cdots x^0_{n+1})\]

and

\[\pi_{s_r}(\pi_{\mu_1} \cdots \pi_{\mu_l})(x^0_1 \cdots x^0_{n+1}) = (\pi_{s_r \mu_1} \cdots \pi_{\mu_l})(x^0_1 \cdots x^0_{n+1})\]

The first follows by part (4) of Lemma 2.2 and the second is immediate. □

Combining equations 3.5 and 3.6 gives equation 3.4 thereby completing the induction step and finishing the proof. □

Proof of Theorem 3.6. Combine Lemmas 3.11, 3.14, and 3.17. □

4. Stable Grothendieck Polynomials

In this section we discuss certain limits of Grothendieck polynomials as well as the doubled versions of these limits. The main result of this section is a formula for the double stable Grothendieck polynomials in terms of set-value tableaux.

We will use the following notation in this section: Let $\omega \in S_{k+1}$ and choose some $m \geq 0$. Let $\bar{\omega} \in S_{m+k+1}$ be the permutation of $[1, \ldots, m, (m+1), \ldots, (m+k+1)]$ that fixes the first $m$ entries and applies the permutation $\omega$ to the last $k+1$ entries. In other words, $s_{i_1} \cdots s_{i_k}$ is a reduced word for $\omega$ if and only if $s_{i_1+m} \cdots s_{i_k+m}$ is a reduced word for $\bar{\omega}$. Let $\bar{x} = (x_1, \ldots, x_{m+1})$ and $\bar{\bar{x}} = (x_{m+2}, \ldots, x_{m+k+1})$. Similarly let $\bar{y} = (y_1, \ldots, y_{m+1})$ and $\bar{\bar{y}} = (y_{m+2}, \ldots, y_{m+k+1})$. Write $x = (\bar{x}, \bar{\bar{x}})$ and $y = (\bar{y}, \bar{\bar{y}})$.
4.1. Single Stable and Single Symmetric Grothendieck polynomials. We quickly review the situation for single Grothendieck polynomials:

**Definition 4.1.** The stable Grothendieck polynomial for $\omega$ is
$$G_\omega(\hat{x}) = G_\omega(x)|_{\hat{x} = 0}$$

**Lemma 4.2 ([Las90]).** We have
$$G_\omega(\hat{x}) = \sum_{f \in F_\omega} (\hat{x})^{wt(f)}$$

**Proof.** By Theorem 2.15 we have that
$$G_\omega(\hat{x})|_{\hat{x} = 0} = \sum_{f \in F_\omega} (\hat{x})^{wt(f)}$$
where $F_\omega(m+1)$ is the subset of $F_\omega$ where all but the first $m+1$ factors are empty. Every Hecke word for $\hat{\omega}$ only contains elements from the set of $\{(m+1), \ldots, (m+k)\}$ and the boundedness condition on the first $m+1$ factors of a factorization of $F_\omega(m+1)$ only requires elements to be greater than $i$ for some $i \leq m+1$. Therefore no factorization of $F_\omega$ fails to lie inside of $F_\omega(m+1)$ (after changing each $s_i$ to $s_i + m$), that is, $F_\omega = \tilde{F}_\omega(m+1)$ (after changing each $s_i$ to $s_i + m$). \qed

**Definition 4.3.** The symmetric Grothendieck polynomial for a partition $\lambda$ is
$$G_\lambda(\hat{x}) = \sum_{T \in SVT(\lambda)} (\hat{x})^{wt(T)}$$
The symmetric Grothendieck polynomial is defined for skew shapes similarly.

**Proposition 4.4 ([BKS+08]).** We have
$$G_\omega(\hat{x}) = \sum_{T \in HT_\omega(\lambda)} G_\lambda(\hat{x})$$

**Proof.** This follows immediately from Lemma 2.14. \qed

4.2. Stable Double Grothendieck Polynomials.

**Definition 4.5.** The stable double Grothendieck polynomial for $\omega$ is given by:
$$G_\omega(\hat{x}, \hat{y}) = \Theta_\omega(x, y)|_{\hat{x} = 0 = \hat{y}}$$

**Proposition 4.6.**
$$G_\omega(\hat{x}, \hat{y}) = \sum_{f \in F_\omega^{\square}} (\hat{x}, \hat{y})^{wt(f)}$$

**Proof.** We have that:
$$\Theta_\omega(x, y) = \Theta_\omega(x, y) = \sum_{f \in \tilde{F}_\omega^{\square}} (x, y)^{wt(f)}$$
Therefore we have that
$$\Theta_\omega(x, y)|_{\hat{x} = 0 = \hat{y}} = \sum_{f \in \tilde{F}_\omega^{\square}(2m+2)} (\hat{x}, \hat{y})^{wt(f)}$$
where $\tilde{F}_\omega^{\square}(2m+2)$ is the subset of $\tilde{F}_\omega^{\square}$ where all but the middle $2m+2$ factors are empty. But every Hecke word for $\hat{\omega}$ only contains elements from the set of
\{(m+1), \ldots, (m+k)\} and the boundedness condition on the central 2m+2 factors of a factorization of $$\mathcal{F}_\omega(2m+2)$$ only requires elements to be greater \(\geq i\) for some \(i \leq m+1\). Therefore no factorization of $$\mathcal{F}_\omega$$ fails to lie inside of $$\mathcal{F}_\omega(2m+2)$$ (after changing each \(s_i\) to \(s_{i+m}\)), that is, $$\mathcal{F}_\omega = \mathcal{F}_\omega(2m+2)$$ (after changing each \(s_i\) to \(s_{i+m}\)). \(\square\)

4.3. Symmetric Double Grothendieck Polynomials.

**Definition 4.7.** Consider the orderdered alphabet \(\{1' < 2' < \cdots < (m+1)' < 1 < 2 < \cdots < m+1\}\). A *primed set valued tableau* of shape \(\lambda\), or an element of \(P_{SVT}(\lambda)\), is a filling of a Young diagram of shape \(\lambda\) such that each box is nonempty and contains a set from this alphabet such that

- All of the entries in a box are less than or equal to all of the entries in the box to its right.
- All of the entries in a box are less than or equal to all of the entries in the box below it.
- \(i\) appears in at most one box in each row.
- \(i'\) appears in at most one box in each column.

The \(x\)-weight of such a tableau is the vector whose \(i\)th coordinate records the number of times \(i\) appears in the tableau. The \(y\)-weight is the vector whose \(i'\)th coordinate records the number of times \(i'\) appears in the tableau.

**Example 4.8.** The following is a \(P_{SVT}\) with \(x\)-weight \((3, 3, 3, 2)\) and \(y\)-weight \((1, 2, 2, 0)\)

\[
\begin{array}{cccc}
1'2' & 2'3' & 123 \\
3'1 & 23 & 4 \\
12 & 34 \\
\end{array}
\]

**Definition 4.9.** We define the symmetric double Grothendieck polynomial by:

\[
G_\lambda(\bar{x}, \bar{y}) = \sum_{T \in P_{SVT}(\lambda)} (\bar{x}, \bar{y})^{wt(T)}
\]

4.4. Relationship between Stable Double Grothendieck Polynomials and Symmetric Double Grothendieck Polynomials. We are interested now in the relationship between \(G_\omega(\bar{x}, \bar{y})\) and \(G_\lambda(\bar{x}, \bar{y})\).

**Proposition 4.10.** There is an \(x\)-weight and \(y\)-weight preserving bijection from \(\mathcal{F}_\omega\) to pairs \((P, Q)\) where \(P \in HT_\omega\) and \(Q \in PSVT\) have the same shape.

**Proof.** Let \(f \in \mathcal{F}_\omega\) and let \(f_L\) represent the leftmost \(m+1\) factors of \(f\) and \(f_R\) represent the rightmost \(m+1\) factors of \(f\). Suppose that \(f_L\) represents the permutation \(\mu\) and denote by \(\overleftarrow{f_L}\) the factorization given by reversing the order of the factors of \(f_L\) and reversing the order of the letters within each factor. Note that if \(f_L\) is a Hecke for \(\mu\) then \(\overleftarrow{f_L}\) is a Hecke factorization of \(\mu^{-1}\). Apply semistandard Hecke insertion to \(\overleftarrow{f_L}\) to obtain a pair \((P_\ell, Q_\ell)\) where \(P_\ell \in HT_{\mu^{-1}}(\lambda_\ell)\) and \(Q_\ell \in SVT(\lambda_\ell)\) for some \(\lambda_\ell\). Now prime all entries of \(Q\) and transpose both tableaux to get a pair \((P'_\ell, Q'_\ell)\) of shape \(\lambda'_\ell\). Now, proceed with semistandard Hecke insertion as if the current insertion tableau were \(P'_\ell\) and the current recording tableau were \(Q'_\ell\) and exactly
the factors of \( f_r \) remained to be inserted. The only ambiguity to starting in the middle of Hecke insertion like this is not knowing what entry to add to the recording tableau during insertion of the \( i \)th factor of \( f_r \): Use the entry \( i \). Denote the final insertion tableau and recording tableau as \( P \) and \( Q \) respectively. We can now define the bijection: \( \Phi(f) = (P, Q) \).

**Example 4.11.** Let \( f = (124)(13)(432)(3) \in \mathcal{F}_{\omega} \). we have \( f_{\ell} = (124)(13) \) and \( f_r = (432)(3) \). First apply semistandard Hecke insertion to \( f_{\ell} = (31)(421) \) to find that

\[
P_{\ell} = \begin{array}{|c|c|}
1 & 2 \\
2 & 4 \\
\hline
3 & \end{array}, \quad Q_{\ell} = \begin{array}{|c|c|}
1 & 2 \\
2 & 4 \\
\hline
3 & \end{array}, \quad P_{\ell}^{\ell} = \begin{array}{|c|c|c|}
1 & 2 & 3 \\
2 & 4 & \end{array}, \quad Q_{\ell}^{\ell} = \begin{array}{|c|c|c|}
1' & 1' & 2' \\
2' & 2' & \end{array}
\]

Now apply semistandard Hecke insertion of \( f_r = (432)(3) \) to starting pair \((P_{\ell}^{\ell}, Q_{\ell}^{\ell})\)

\[
P = P_{\ell}^{\ell} \leftarrow (432)(3) = \begin{array}{|c|c|c|}
1 & 2 & 3 \\
2 & 4 & \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|c|}
1' & 1' & 2' & 1 \\
2' & 2' & 1 & \end{array}
\]

There is much to prove:

- \( P \) is a Hecke tableau and it represents the permutation \( \omega \): Suppose \( f_{\ell} \) is a Hecke word for some permutation \( \mu \). Now, \( P_{\ell} \) was formed by applying Hecke insertion to \( f_{\ell} \) and so is a Hecke tableau whose rows read left to right, from bottom row to top row form a Hecke word for \( \mu^{-1} \). Since the only requirement for being a Hecke tableau is that the rows and columns are strictly increasing, (which is clearly preserved under transposition) it is also true that \( P_{\ell}^{\ell} \) is a Hecke tableau. Next, the columns of \( P_{\ell}^{\ell} \) read from top to bottom from rightmost column to leftmost column give a Hecke word for \( \mu^{-1} \). Therefore the columns of \( P_{\ell}^{\ell} \) read from bottom to top from leftmost column to rightmost column give a Hecke word for \( \mu \). However:

**Claim 4.12.** The column reading word and row reading word of a Hecke tableau, \( H \), represent the same permutation.

*Proof.* Let \( \omega_k(H) \) be the permutation represented by reading the leftmost \( k \) columns of \( H \) bottom to top, leftmost column to rightmost column and then, ignoring the first \( k \) columns of \( H \), reading rows left to right, bottom row to top row. It suffices to show that \( \omega_k(H) = \omega_{k+1}(H) \). Without loss of generality we may assume \( k = 0 \). Now let \( \omega^j(H) \) be the permutation represented by reading the lowest \( j \) entries of the leftmost column of \( H \) from bottom to top and then reading the remaining entries of \( H \) by rows, left to right, bottom to top. To show that \( \omega_k(H) = \omega_{k+1}(H) \) for \( k = 0 \) it
suffices to show that \( \omega^j(H) = \omega^{j+1}(H) \). If \( a \) is the entry in the leftmost column of \( H \) in the \( j + 1 \)st row from the bottom and \( b \) is any entry in \( H \) in the \( j \)th row from the bottom or lower not in the first column of \( H \) then \( a < b - 1 \). Therefore \( a \) commutes with all such \( b \) which shows that \( \omega^j(H) = \omega^{j+1}(H) \). \( \square \)

Therefore reading the rows of \( P^\ell \) left to right, bottom to top also gives a Hecke word for \( \mu \). Since \( f_r \) gives a Hecke word for some permutation \( \nu \) such that \( \bar{\mu} \bar{\nu} \sim \bar{\omega} \) the properties of Hecke insertion imply that the Hecke word formed by reading the rows of \( P \) from left to right, bottom to top also represents \( \omega \). All this shows that \( P \) is a Hecke tableau and it represents the permutation \( \omega \).

- \( Q \in PSVT(\lambda) \) where \( \lambda \) is the shape of \( P \): First, \( Q_\ell \in SVT(\lambda_\ell) \) by Lemma 2.14 so it follows that \( Q_\ell^\ell \in PSVT(\lambda_\ell^\ell) \) (and has no unprimed entries). On the other hand it also follows from Lemma 2.14 that the unprimed entries from \( Q \) will give an element of \( SVT(\lambda/\rho) \) for some \( \rho \subseteq \lambda_\ell^\ell \) such that \( \lambda_\ell^\ell \setminus \rho \) contains no more than one box in any row or column. The fact that the primed and unprimed entries give such tableaux along with the fact that \( i' < j \) for any \( i \) and \( j \) imply that \( Q \in PSVT(\lambda) \).

- \( \Phi \) is injective. Let \( f, f^\times \in \mathcal{F}_\omega \). Suppose that \( \Phi(f) = \Phi(f^\times) \) with \( f \neq f^\times \). We use the same notation as in the construction of \( \Phi(f) \) and also set \( p_\ell \) equal to the Hecke factorization given by reading the columns of \( P^\ell \) bottom to top from left column to right column. Use the same notation for corresponding objects associated to \( f^\times \) but with a \( \times \).

  If \( f_\ell \neq f^\times_\ell \) then by Lemma 2.14 \( (P_\ell, Q_\ell) \neq (P_\ell^\times, Q_\ell^\times) \). But \( Q_\ell \neq Q_\ell^\times \) would force \( Q \neq Q^\times \) so we must have \( P_\ell \neq P_\ell^\times \) and so \( P_\ell^\ell \neq (P_\ell^\times)^\ell \). Thus either \( f_\ell \neq f^\times_\ell \) in which case \( p_\ell \neq p_\ell^\times \) or else \( f_\ell \neq f^\times_\ell \). Either way, \( p_\ell f_\ell \neq p_\ell^\times f^\times_\ell \). But it is easy to see that the insertion tableau of \( p_\ell \) is just \( P^\ell_\ell \) and the insertion tableau of \( p_\ell^\times \) is just \( P^\ell_\ell^\times \). Meanwhile the recording tableaux of \( p_\ell \) and \( p_\ell^\times \) are the same. Thus the Hecke factorizations \( p_\ell f_\ell \) and \( p_\ell^\times f^\times_\ell \) would be two distinct elements mapping to the same insertion and recording tableaux under the bijection of Lemma 2.14 which is a contradiction.

- \( \Phi \) is surjective. Suppose we are given \( (P, Q) \) of the same shape \( \lambda \) where \( P \in HT_\omega \) and \( Q \in PSVT \). Let \( Q_{out} \) denote the skew tableau formed by only taking the unprimed entries of \( Q \). Let \( Q_{in} \) denote the tableau formed by taking only the primed entries of \( Q \) and then erasing all their prime marks. Take \( j \) sufficiently large, (for example more than the number of primed entries in \( Q \)) and let \( Q_{can} \) be any \( SVT \) such that erasing all integers less than or equal to \( j \) and subtracting \( j \) from the rest gives \( Q_{out} \) and such that removing all entries greater than \( j \) gives a tableau of the same shape as \( Q_{in} \).

  Now use Lemma 2.14 to find a Hecke factorization \( f_{\ell} \) mapping to \( (P, Q_{can}) \). Write \( \bar{f} = \bar{f}_r f_{\ell} \) where \( \bar{f}_r \) represents the first \( j \) factors of \( \bar{f} \). Suppose the insertion tableau of \( f_{\ell} \) is \( T \). Use Lemma 2.14 to find a Hecke factorization \( f^r_{\ell} \) mapping to \( (T^r, Q^r_{can}) \). Let \( \bar{f}_r \) represent the result of reversing the order of the factors of \( f_{\ell} \) and reversing the order of the entries within each factor. Then we have that \( \Phi(\bar{f}_r f_{\ell}) = (P, Q) \). Now \( \bar{f}_r f_{\ell} \in \mathcal{F}_\omega \) for some \( \omega' \) just by construction. But by the first bullet point we have \( \omega' = \omega \).
• $\Phi$ preserves the $x$-weight and the $y$-weight: Suppose $\Phi(f) = (P, Q)$ where $f = f_{\ell} f_r$. The $y$-weight of $f$ is the vector whose $i^{th}$ coordinate records the number of entries in the $i^{th}$ factor of $f$ which is the number of times $i$ appears in $Q_\ell$ or equivalently the number of times $i'$ appears in $Q'_r$ or equivalently in $Q$. This is the definition of the $y$-weight of $Q$. The $x$-weight of $f$ is the vector whose $i^{th}$ coordinate records the number of entries in the $i^{th}$ factor of $f_r$ which is the number of times $i$ appears in $Q$. This is the definition of the $x$-weight of $Q$.

$\square$

**Remark 4.13.** If it were not for the unfortunate fact mentioned in Remark 2.9 the whole process of reversing the left side of the factorization and then inserting and then transposing would not be necessary and the proposition could be proved through just inserting the factors directly. We leave it as an open problem to find a way of altering Hecke insertion so it has the additional properties needed for this simpler proof.

**Corollary 4.14.** Given a permutation $\mu$ we say that $\rho \subseteq \cdot \mu$ if $\rho \subseteq \mu$ and $\mu/\rho$ contains no two boxes in the same row and no two boxes in the same column. For a tableau, $T$, let $T_s$ denote the shape of $T$. We have:

$G_\omega (\hat{x}, \hat{y}) = \sum_{T \in HT_\omega} \sum_{\rho \subseteq \cdot \mu \subseteq T_s} G_{T_s/\rho}(\hat{x}) G_{\mu'}(\hat{y})$

(4.1)

**Proof.** It follows from proposition 4.10 that we have:

$G_\omega (\hat{x}, \hat{y}) = \sum_{T \in HT_\omega} G_{T_s}(\hat{x}, \hat{y})$

Next, note that there is a canonical bijection from $PSVT(\lambda)$ to pairs of tableaux $(P, Q)$ where $P$ is a skew $SVT$ and $Q^t$ is a straight shape $SVT$ such that $P_s \cap Q_s$ contains no two boxes in the same row or column and where $P_s \cup Q_s = \lambda$. Since the bijection sends $x$-weight to the weight of $P$ and $y$-weight to the weight of $Q$ the theorem follows from the formula above. $\square$

### 4.5. Double Grothendieck functions

We will now be interested in Grothendieck polynomials over infinite set(s) of variables. To distinguish when we are talking about such polynomials will refer to them as functions from now on. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be an infinite lists of variables. Let $\Omega_x$ be the $\mathbb{Z}[y]$ linear involution on functions symmetric with respect to $x$ in $\mathbb{Z}[x, y]$ which sends $s_\lambda(x) \to s_{\lambda^+}(x)$. Let $\Omega_y$ be the $\mathbb{Z}[x]$ linear involution on functions symmetric with respect to $y$ in $\mathbb{Z}[x, y]$ which sends $s_\lambda(y) \to s_{\lambda^+}(y)$.

**Definition 4.15.** Define the symmetric Grothendieck function and the stable Grothendieck function, respectively, by:

$G_\lambda(x) = \lim_{m \to \infty} G_\lambda(\hat{x})$

$G_\omega(x) = \lim_{m \to \infty} G_\omega(\hat{x})$
Definition 4.16. Define the weak symmetric Grothendieck function and the weak stable Grothendieck function, respectively, by:
\[ ^{*}\mathcal{G}_\lambda(x) = \Omega_x(G_\lambda(x)) \]
\[ ^{*}\mathcal{G}_\omega(x) = \Omega_x(G_\omega(x)) \]

Definition 4.17. Define the symmetric double Grothendieck function and the stable double Grothendieck function, respectively, by:
\[ G_\lambda(x, y) = \lim_{m \to \infty} G_\lambda(\check{x}, \check{y}) \]
\[ \mathcal{G}_\omega(x, y) = \lim_{m \to \infty} \mathcal{G}_\omega(\check{x}, \check{y}) \]

Definition 4.18. Define the weak symmetric double Grothendieck function and the weak stable double Grothendieck function, respectively, by:
\[ ^{*}\mathcal{G}_\lambda(x, y) = \Omega_x \Omega_y(G_\lambda(x, y)) \]
\[ ^{*}\mathcal{G}_\omega(x, y) = \Omega_x \Omega_y(G_\omega(x, y)) \]

This allows us to state the main theorem of this section:

Theorem 4.19. Given a permutation \( \mu \) we say that \( \rho \subseteq \mu \) if \( \rho \subseteq \mu \) and \( \mu / \rho \) contains no two boxes in the same row and no two boxes in the same column. For a tableau, \( T \), let \( T_s \) denote the shape of \( T \). We have:
\[ \mathcal{G}_\omega(x, y) = \sum_{T \in HT_{\omega}} \sum_{\rho \subseteq \mu \subseteq T_s} G_{T_{s/\rho}}(x) G_{\mu'}(y) \]
(4.2)
\[ ^{*}\mathcal{G}_\omega(x, y) = \sum_{T \in HT_{\omega}} \sum_{\rho \subseteq \mu \subseteq T_s} ^{*}G_{T_{s/\rho}}(x) ^{*}G_{\mu'}(y) \]
(4.3)

Proof. Equation 4.2 follows from taking the limit of the equation 4.1. Equation 4.3 follows by applying \( \Omega_x \circ \Omega_y \) to equation 4.2. \( \square \)

Remark 4.20. Since \( G_\lambda(x) \) has a combinatorial interpretation in terms of set-valued tableaux ([Buc02]), equation 4.2 gives us a way to express \( \mathcal{G}_\omega(x, y) \) in terms of triples of tableaux, \( (T, P, Q) \) where \( T \) is a Hecke tableau and \( P \) and \( Q \) are (possibly skew) set-valued tableaux. Similarly, since \( ^{*}G_\lambda(x) \) has a combinatorial interpretation in terms of multiset valued tableaux ([LP07]), equation 4.3 gives us a way to express \( ^{*}\mathcal{G}_\omega(x, y) \) in terms of triples of tableaux, \( (T, P, Q) \) where \( T \) is a Hecke tableau and \( P \) and \( Q \) are (possibly skew) multiset valued tableaux.

5. Half weak double Grothendieck functions

5.1. Motivation. We begin by noting that there is a large degree of flexibility in the combinatorial models we have chosen in this paper.

Remark 5.1. Note that \( G_\omega(x, y) \) may be expressed combinatorially in terms of primed set valued tableaux if we alter the definition to allow the tableaux to be filled with any positive integers (and their primed versions). Similarly, it may be shown that \( ^{*}G_\omega(x, y) \) may be expressed in terms of “primed multiset valued tableaux” where boxes can be filled with multisets from \{1′, 2′, . . . , 1, 2, . . . \} and where the roles of the primed and nonprimed entries are reversed from their roles in primed set valued tableaux.

Further, the ordering of the alphabet \{1′, 2′, . . . , 1, 2, . . . \} is irrelevant as long as it is fixed. We can for instance assume the order 1′ < 1 < 2′ < 2 < ··· in
the definition of primed set valued tableaux or in the definition of primed multiset
valued tableaux. (We note that for primed set valued tableaux this particular choice
recovers the definition of (a nonshifted version of) the shifted set valued tableaux
of [IN13].) However, we still must require that all unprimed entries contribute to
the \( x \)-weight and all unprimed entries contribute to the \( y \)-weight.

These results can be proven using a generalization of the maps \( \nearrow \) and \( \searrow \) from
Lemma 2.2 of [Haw22] to the set valued and multiset valued cases respectively.

**Remark 5.2.** Note also that \( \omega_w(x, y) \) may be expressed combinatorially in terms
of (unbounded) double Hecke factorizations if we change the definition to allow an
infinite number of factors to the left and to the right. Similarly, it may be shown
that \( \omega_w(x, y) \) has such a combinatorial interpretation if we change the strictly
increasing and strictly decreasing requirements on the left and right hand factors,
respectively, to weakly decreasing and weakly increasing, respectively.

Moreover the order in which the factors appear in an (unbounded) double Hecke
factorization is irrelevant as long as it is consistent among all factorizations. For
example, instead of requiring the increasing factors of an (unbounded) double Hecke
factorization to all appear to the left of the decreasing factors, we may instead
require they appear to the right of the decreasing factors, or, we could require
that increasing and decreasing factors alternate, or, any other fixed arrangement
of increasing and decreasing factors. However, the decreasing factors must always
contribute to the \( x \)-weight and the increasing factors to the \( y \)-weight. Similar
alterations may be made in the weak case as long as weakly increasing factors
always contribute to the \( x \)-weight and weakly decreasing factors to the \( y \)-weight.

The statements above can be proven using the maps \( \uparrow \) and \( \downarrow \) of section 3 along
with an adaptation of these maps to the weak case.

If we were to choose to define (unbounded) double Hecke factorizations using
the requirement that we must alternate between (strictly) decreasing and (strictly)
increasing factors we would in essence end up with a definition of in terms of “strict
hook factors” where the decreasing part of each factor contributes to the \( x \)-weight
and the increasing part to the \( y \)-weight. Comparing this with the combinatorial
definition of type \( B \) Stanley symmetric functions in [BH95] shows that \( \omega_w(x, x) \)
agrees with the type \( B \) Stanley symmetric function on terms of lowest degree.

However, since \( \omega_w(x, y) \) can be interpreted combinatorially using “weak” (un-
bounded) double Hecke factorizations with the requirement that we must alternate
between (weakly) decreasing and (weakly) increasing factors we see that \( \omega_w(x, y) \)
has a similar combinatorial interpretation in terms of “weak hook factorizations.”

Now, any weak hook factorization corresponding to a reduced word is automatically
also a strict hook factorization (and vice versa). Thus \( \omega_w(x, x) \) also agrees with
the type \( B \) Stanley symmetric function on terms of lowest degree.

Finally, we could also define a new function \( \omega_w(x, y) \) using “half weak” hook
factorizations that are composed of factors that are strictly decreasing and then
weakly increasing. By similar reasoning as above we see that \( \omega_w(x, x) \) would also
generalize the type \( B \) Stanley symmetric function in the sense that its lowest degree
term would return the latter. Moreover, there is a natural interpolation of \( \omega_w(x, y) \)
and \( \omega_w(x, y) \) which we denote \( \omega_w(x, y) \) such that the relation between \( \omega_w(x, y) \)
and \( \omega_w(x, y) \) parallels the relation between of \( \omega_w(x, y) \) and \( \omega_w(x, y) \) as well as the
relation between \( \omega_w(x, y) \) and \( \omega_w(x, y) \). In fact the relation between \( \omega_w(x, y) \)
and \( \omega_w(x, y) \) appears more naturally in the sense that it can be proven directly
using Hecke insertion (remark 2.9 does not cause issues in this case). However, the most compelling evidence that $^\times G_\omega(x, x)$ is a more natural generalization of the type $B$ Stanley symmetric function than $G_\omega(x, x)$ or $^* G_\omega(x, x)$ is that, like the type $B$ Stanley symmetric function, $^\times G_\omega(x, x)$ is $Q$-Schur positive whereas neither $G_\omega(x, x)$ nor $^* G_\omega(x, x)$ is.

For these reasons it seems like $^* G_\omega(x, x)$ is a suitable candidate for a $k$-theoretic version of the type $B$ Stanley symmetric function, or, as it were, a type $B$ stable Grothendieck function. Of course, our definition is incomplete in the sense that it is only defined for (unsigned) permutations whereas we would ideally like something defined for all signed permutations. This is equivalent to adding a rule as to how the special generator, $s_0$, of the type $B$ Weyl group should be incorporated into the definition of hook Hecke factorization found in the next subsection. We leave this as an open problem.

5.2. Results. We need to introduce a number of definitions:

Definition 5.3. A hook Hecke factorization of $\omega$ is a factorization into hook factors. Each hook factor contains a subset of $\{1, 2, \ldots\}$ and a multiset from $\{1, 2, \ldots\}$ arranged so that all circled factors lie to the left of all uncircled factors and such that the circled elements are strictly decreasing left to right and the uncircled elements are weakly increasing left to right. Moreover, erasing the circles and parentheses should give a Hecke word for $\omega$. For instance, $((3\,2\,3\,3))((1\,2\,2))((3\,2\,1\,1\,3\,3))$ is a hook Hecke factorization for the permutation $(4, 3, 2, 1) \in S_4$. The $x$-weight of a hook Hecke factorization is the vector whose $i$th entry records the number of uncircled elements in its $i$th factor. The $x$-weight of the example above is $(3, 2, 4)$. The $y$-weight of such a factorization is the vector whose $i$th entry records the number of circled entries in the $i$th factor. The $y$-weight of the example above is $(2, 1, 2)$.

Definition 5.4. Denote the set of all hook Hecke factorizations of $\omega$ with an infinite number of factors by $^\times F_\omega$. Define the half weak stable double Grothendieck function by

$$^\times G_\omega(x, y) = \sum_{f \in ^\times F_\omega} (x, y)^{wt(f)}$$

Definition 5.5. Consider the ordered alphabet $1' < 1 < 2' < 2 < \cdots$. A primed set multiset tableau of shape $\lambda$, or an element of $PSMT(\lambda)$, is a filling of a Young diagram of shape $\lambda$ such that each box is nonempty and contains a multiset from this alphabet such that

- All of the entries in a box are less than or equal to all of the entries in the box to its right.
- All of the entries in a box are less than or equal to all of the entries in the box below it.
- $i$ appears in at most one box in each column.
- $i'$ appears in at most one box in each row.
- Each box contains at most one $i'$.

Definition 5.6. If $T \in PSMT(\lambda)$ then the $x$-weight of $T$ is vector whose $i$th entry records the number of instances of $i$ in $T$ and the $y$-weight of $T$ is vector whose $i$th entry records the number of instances of $i'$ in $T$. Define the half weak symmetric
double Grothendieck function by
\[
\times G_\omega(x, y) = \sum_{T \in \text{PSMT}(\lambda)} (x, y)^{wt(T)}
\]

**Example 5.7.** A primed set multiset tableau, \(Q \in \text{PSMT}(3, 3, 2)\), with \(x\)-weight of \((3, 2, 3, 0, 0, \ldots)\) and \(y\)-weight of \((1, 3, 3, 0, 0, \ldots)\) is shown below.

\[
Q = \begin{array}{ccc}
1'11 & 12' & 23' \\
2' & 2 & 3'33 \\
2'3' & 3
\end{array}
\]

The following lemma and its corollary relates these definitions.

**Lemma 5.8.** There is an \(x\)-weight and \(y\)-weight preserving bijection from \(\times F_\omega\) to pairs \((P, Q)\) where \(P \in \text{HT}_\omega\) and \(Q \in \text{PSMT}\) have the same shape. Here the \(x\)-weight and \(y\)-weight of \((P, Q)\) are defined as the \(x\)-weight and \(y\)-weight of \(Q\).

**Proof.** A hook Hecke factorization is just a Hecke word, \(w\), along with an ordered set partition of \(\{1, 2, \ldots, \text{length}(w)\}\) into parts such that
- Each part contains consecutive numbers and parts with smaller numbers precede parts with larger numbers.
- If \(a\) and \(a + 1\) occur in the same one of one of parts number 1, 3, 5, \ldots then, \(w_a > w_{a+1}\).
- If \(a\) and \(a + 1\) occur in the same one of one of parts number 2, 4, 6, \ldots then, \(w_a \leq w_{a+1}\).

On the other hand using a standardization argument we see that a \(\text{PSMT}\) is just a standard set valued tableau, \(T\), along with a set partition of \(\{1, 2, \ldots, \text{max}(T)\}\) into parts such that
- Each part contains consecutive numbers and parts with smaller numbers precede parts with larger numbers.
- If \(a\) and \(a + 1\) occur in the same one of one of parts number 1, 3, 5, \ldots then \(a + 1\) lies strictly below \(a\) in \(T\).
- If \(a\) and \(a + 1\) occur in the same one of one of parts number 2, 4, 6, \ldots then \(a + 1\) lies either in the same box as \(a\) or strictly right of \(a\) in \(T\).

Lemma 2.8 implies that if \(w \rightarrow (P, Q)\) under the bijection of Proposition 2.7 then a given set partition of \(\{1, 2, \ldots, \text{length}(w)\}\) turns \(w\) into a hooked Hecke factorization if and only if the same set partition turns \(Q\) into a \(\text{PSMT}\). Thus combining the bijection of Proposition 2.7 with the identity on set partitions induces the weight preserving bijection of the lemma. \(\square\)

**Corollary 5.9.** Letting \(H_\rho^\rho\) denote the number of Hecke tableaux for \(\omega\) with shape \(\rho\) we have

\[
\times G_\omega(x, y) = \sum_\rho (H_\rho^\rho) \times G_\rho(x, y)
\]

**Proof.** This follows from the lemma above. \(\square\)

Next, we need to define another couple of types of tableaux:
**Definition 5.10.** Let $\mu \subseteq \lambda$ be partitions with an equal number of rows. An over flagged tableau of shape $\lambda/\mu$, or an element of $OFT(\lambda/\mu)$, is a filling of a Young diagram of shape $\lambda/\mu$ using the alphabet $1 < 2 < \cdots$ such that:

- Each box in row $i$ of $\lambda/\mu$ contains one element from $\{1, 2, \ldots, \mu_i\}$.
- The rows are weakly decreasing from left to right.
- The columns are strictly decreasing from top to bottom.

On the other hand, if $\lambda$ contains more rows than $\mu$, we define $OFT(\lambda/\mu) = \emptyset$.

**Example 5.11.** An over flagged of shape $(6, 6, 5, 4)/(4, 3, 2, 1)$ with the inner shape shown filled with *s is shown below.

\[
\begin{array}{cccc}
* & * & * & 4 & 2 \\
* & * & 3 & 2 & 1 \\
* & 2 & 2 & 1 \\
1 & 1 & 1
\end{array}
\]

Note that the definition requires that the maximum number in each row is no greater than the number of stars in that row.

**Definition 5.12.** A primed tableau of shape $\lambda$, or element of $PT(\lambda)$ is an element of $PSMT(\lambda)$ with exactly one entry in each box.

**Definition 5.13.** The double $Q$-Schur function is defined as:

\[
R_\lambda(x, y) = \sum_{T \in PT(\lambda)} (x, y)^{wt(T)}
\]

The relationship between these definitions and the earlier ones is given by part (1) of Lemma 1.9 of [Haw22] along with a corollary of this result:

**Lemma 5.14 ([Haw22]).** There is an $x$-weight and $y$-weight preserving bijection from $PSMT(\mu)$ to pairs of tableaux $(P, Q)$ where $P \in PT(\lambda)$ and $Q \in OFT(\lambda/\mu)$ for some $\lambda \supseteq \mu$. Here the $x$-weight and $y$-weight of $(P, Q)$ are defined as the $x$-weight and $y$-weight of $P$.

**Corollary 5.15.** Letting $K^\mu_\rho$ be the number of overfull tableaux of shape $\mu/\rho$ we have

\[
\chi G_\omega(x, y) = \sum_{\rho \subseteq \mu} H_\omega^\rho K^\mu_\rho R_\rho(x, y)
\]

**Proof.** This follows from the lemma above along with corollary 5.9. \hfill \Box

Finally, we need to identify certain special elements of $PT(\lambda)$. To do this we make some more definitions:

**Definition 5.16.** Let $T \in PT(\lambda)$. Start with your finger in the leftmost box of the lowest row of $T$. Now move your finger left to right across rows moving from bottom row to top row until your finger lies over an instance of $i$ or $i'$ for the first time. If it lies over an $i$ we say that $T$ has the $i$ starting property. If $T$ has no $i$ or $i'$ we also say that $T$ has the $i$-starting property.

**Definition 5.17.** Let $T \in PT(\lambda)$. 
• Place your finger in the rightmost box of the top row of T. Drag your finger right to left moving down a row each time you get to the leftmost box of a row until you reach the bottom left box of T. While you are scanning, any time your finger lies over an i place a tally above T. Any time your finger lies over an i − 1 place a tally under T. If there are ever more tallies above T than below T, break your finger. If there are ever an equal number of tallies above T and below T and your finger lies on an i′ also break your finger.

• Next (do NOT erase the tallies from the last step), start with your finger in the leftmost box of the lowest row of T. Now move your finger left to right across rows moving from bottom row to top row. While you are scanning, any time your finger lies over an i′ place a tally above T. Any time your finger lies over an (i − 1)′ place a tally under T. If there are ever more tallies above T than below T, break your finger. If there are ever an equal number of tallies above T and below T and your finger lies on an i − 1 also break your finger.

We say that T has the i-lattice property if at the end of this process you have no broken fingers.

Example 5.18. An element \( P \in PT(6, 4, 4, 4, 3) \) is shown below.

\[
\begin{array}{cccccc}
1' & 1 & 1 & 1 & 1 & 1 \\
1 & 2' & 2 & 2 & & \\
2' & 2 & 3' & 3 & & \\
2 & 3' & 3 & 4 & & \\
3 & 4' & 4 & & & \\
\end{array}
\]

Note that:

• \( P \) has the 1-starting property.
• \( P \) has the 2-starting property.
• \( P \) has the 3-starting property.
• \( P \) does not have the 4-starting property.

These definitions allow us to state a particular case of Theorem 8.3 of [Ste89]:

Proposition 5.19 ([Ste89]). We have:

\[
R_\mu(x, x) = \sum_\lambda F_\mu^\lambda Q_\lambda(x)
\]

where \( F_\mu^\lambda \) is the number of elements \( T \in PT(\mu) \) that have i-lattice property and the i-starting property for all \( i \) and such that the sum of the x-weight and y-weight of \( T \) is equal to \( \lambda \).

Combining everything from this subsection will now give us the \( Q \)-Schur expansion of \( ^xG_\omega(x, x) \):
**Theorem 5.20.** The function $\times G_\omega$ evaluated at $x = y$ is $Q$-Schur positive and:

$$\times G_\omega(x, x) = \sum_{\rho \subseteq \mu, \lambda} H_{\rho_{\mu}}^\mu K_{\rho_{\mu}}^\lambda F_{\lambda}^\mu Q_{\lambda}(x)$$

**Example 5.21.** Consider the permutation $\omega = (3, 1, 2, 5, 4) \in S_5$ and suppose that we are interested in computing the degree 4 part of $\times G_{(3, 1, 2, 5, 4)}(x, x)$.

To do this we first compute the three elements of $HT_\omega$ shown in the top of the diagram below. For each such tableau, $H$, we compute all elements of $OFT(\mu/H_s)$ for all possible partitions $\mu \supseteq H_s$ that have 4 boxes (since we are concerned with the degree 4 part). These are shown in the middle of the diagram below. Finally, for each such tableau, $O$, we compute all elements of $PT(O_s)$ that have the $i$-starting property and the $i$-lattice property for all $i$. The resulting tableaux are shown in the last two lines of the diagram.

After counting the tableaux appearing in the last two lines of the diagram above and computing their weights we see by Theorem 5.20 that the degree 4 part of $\times G_{(3, 1, 2, 5, 4)}(x, x)$ is equal to $6Q_{(4,0)}(x) + 4Q_{(3,1)}(x)$. In particular, the coefficient of $x_1^4 x_2^0 x_3^0 \cdots$ in this expression is 12 since this coefficient is 2 in $Q_{(4,0)}(x)$ and 0 in $Q_{(3,1)}(x)$. This implies that there ought to be exactly 12 hook Hecke factorizations of $\omega$ which are composed of only one factor which has length 4. Indeed the following are all such factorizations:

$$(1124), (1224), (1244), (3112), (3122), (4124), (1124), (11224), (1224), (3112), (3122), (31124)$$

6. **Open Problems**

We remind the reader that we have left them with two open problems. The first open problem is to reformulate Hecke insertion so that it does not suffer from the inconvenient flaw mentioned in remark 2.9 (while still maintaining the properties of Lemma 2.8). Experimentally this appears possible on a case by case basis. That is, given a fixed permutation there seems always to be a way to define a “reasonable” insertion algorithm that has the desired properties. It is not obvious a priori that
such an algorithm (reasonable or not) should even be possible for every arbitrary permutation even if it is allowed to depend on the permutation. Therefore, the fact that it appears possible is encouraging evidence that a globally defined algorithm with the desired properties may exist.

The second open problem is to generalize our definition of hook Hecke factorization to include signed permutations that have one or more instance of the generator $s_0$ in a reduced word for said signed permutation. This should be done in such a way that the resulting generating function is symmetric, $Q$-Schur positive, and agrees with the type $B$ Stanley symmetric function on terms of lowest degree. We have tried a few of the more obvious ways of doing this, and so far none have been successful.

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