On hidden broken nonlinear superconformal symmetry of conformal mechanics and nature of double nonlinear superconformal symmetry

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Abstract

We show that for positive integer values \( l \) of the parameter in the conformal mechanics model the system possesses a hidden nonlinear superconformal symmetry, in which reflection plays a role of the grading operator. In addition to the even \( so(1, 2) \oplus u(1) \)-generators, the superalgebra includes \( 2l + 1 \) odd integrals, which form the pair of spin-\((l + \frac{1}{2})\) representations of the bosonic subalgebra and anticommute for order \( 2l + 1 \) polynomials of the even generators. This hidden symmetry, however, is broken at the level of the states in such a way that the action of the odd generators violates the boundary condition at the origin. In the earlier observed double nonlinear superconformal symmetry, arising in the superconformal mechanics for certain values of the boson-fermion coupling constant, the higher order symmetry is of the same, broken nature.

1 Introduction

The conformal mechanics model of De Alfaro, Fubini and Furlan\cite{1, 2} is the simplest nontrivial (0+1)-dimensional conformal field theory. The interest to it and to its supersymmetric extension\cite{3, 4, 5} has revived recently in the context of the black hole physics, AdS/CFT correspondence and integrable Calogero-Moser type systems\cite{6–12}. This model provides an example of quantum mechanical system, in which the problem of a self-adjointness of the Hamiltonian arises in certain region of its parameter values\cite{13, 14, 11, 15}. The latter aspect is related to the problem of existence of bound quantum states and spontaneous breaking of scale symmetry, that also finds applications in the physics of black holes\cite{16}.

Recently, it was observed that a certain change of the boson-fermion coupling constant in the supersymmetric conformal mechanics model gives rise to a radical change of symmetry: instead of the \( osp(2|2) \) superconformal symmetry, the modified system is characterized by its nonlinear generalization\cite{17} \cite{18}. It was also found that when the boson-fermion coupling constant takes integer values, the system is described simultaneously by the two nonlinear superconformal symmetries of the orders relatively shifted in odd number\cite{17}. In particular, such nonlinear superconformal symmetry arises in the original superconformal mechanics model\cite{3, 4} in addition to the \( osp(2|2) \). However, the nature of this double superconformal symmetry left to be mysterious.

In the present Letter we show that when the parameter \( \alpha \) of the purely bosonic conformal mechanics model\cite{8.1} takes integer values, in addition to the \( so(1, 2) \) conformal symmetry the system possesses a set of the integrals of motion which are odd differential operators. Identifying the nonlocal reflection operator as a grading operator, we find that these additional integrals extend

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the \( so(1,2) \) to the nonlinear superconformal symmetry discussed in [17] [13]. However, this hidden nonlinear superconformal symmetry of the conformal mechanics model is broken at the level of the states. Similarly to the usual spontaneous supersymmetry breaking mechanism, where a zero energy state loses the normalizability due to a violation of the boundary condition at infinity, here the odd generators acting on the Hamiltonian eigenstates (being the non-normalizable, scattering states) produce its other eigenstates, which violate the boundary condition at the origin. Then, we show that in the case of the double superconformal symmetry observed in ref. [17], the symmetry of higher order has the same, broken nature.

The Letter is organized as follows. In Section 2 we show that a free particle on a line possesses a hidden \( \text{osp}(2|2) \) superconformal symmetry in the exact, unbroken phase, in which reflection plays a role of a grading operator. Section 3 is devoted to the discussion of a hidden broken nonlinear superconformal symmetry of the conformal mechanics model. In Section 4 we analyse the double nonlinear superconformal symmetry of superconformal mechanics. In Section 5 we discuss some problems to be interesting for further investigation.

## 2 Hidden superconformal symmetry of a free particle on a line

Let us start with a free unit mass particle on a line, which is given by the Hamiltonian

\[
H = -\frac{1}{2} \frac{d^2}{dx^2}. \tag{2.1}
\]

A linear momentum \( p = -i\frac{d}{dx} \) is an integral of motion satisfying the relation

\[
p^2 = 2H. \tag{2.2}
\]

As a consequence, every energy level with \( E > 0 \) is doubly degenerated: the two non-normalizable eigenstates \( \psi_{E,+}(x) = C_+ \cos \sqrt{2E} x \) and \( \psi_{E,-}(x) = C_- \sin \sqrt{2E} x \) belong to it, while a nondegenerate state \( \psi_{0,+} = C_0 = \text{const} \) corresponds to \( E = 0 \). Due to the energy levels structure, this elementary pure bosonic system possesses a hidden \( N = 1 \) supersymmetry in exact, not spontaneously broken, phase. Indeed, take the reflection operator \( R, R\psi(x) = \psi(-x) \), satisfying the relations \( \{ R, x \} = \{ R, p \} = 0 \), \( R^2 = 1 \), where \( \{ ., . \} \) is an anticommutator. Then the operators \( Q_a, a = 1, 2 \),

\[
Q_1 = \frac{1}{\sqrt{2}} p, \quad Q_2 = iRQ_1, \tag{2.3}
\]

can be identified as Hermitian supercharges,

\[
\{ Q_a, Q_b \} = 2\delta_{ab} H, \quad [Q_a, H] = 0. \tag{2.4}
\]

Their linear combinations, \( Q_{\pm} = \frac{1}{\sqrt{2}} (Q_1 \pm iQ_2) = -i\frac{d}{dx} \cdot \frac{1}{2} (1 \pm R) \), provide us with Hermitian conjugate nilpotent supercharges, \( Q_+^2 = Q_-^2 = 0, \{ Q_+, Q_- \} = 2H \), which mutually transform the even, \( \psi_{E,+} \), and the odd, \( \psi_{E,-} \), eigenstates with \( E > 0 \), and annihilate the ground state \( \psi_{0,+} \). In this construction the reflection plays a role of a grading operator, which identifies the \( H \) as an even generator and the \( Q \)'s as odd generators of the \( N = 1 \) superalgebra \( 2.4 \).

The described hidden ("bosonized" [13] [20] [21]) \( N = 1 \) supersymmetry can be extended to the \( \text{osp}(2|2) \) superconformal symmetry by supplying the set of the integrals \( H \) and \( Q_a \) with the dynamical odd,

\[
S_1 = \frac{1}{\sqrt{2}} X, \quad S_2 = iRS_1,
\]

and even, \( D = \frac{1}{4} \{ X, p \}, \quad K = \frac{1}{2} X^2 \), integrals, where \( X := x - tp \). The \( D \) and \( K \) are presented equivalently as

\[
D = \frac{1}{4} \{ x, p \} - tH, \quad K = \frac{1}{2} x^2 - 2t D - t^2 H. \tag{2.5}
\]

The dynamical integrals of motion satisfy the equation of the form

\[
\frac{d}{dt} I = \frac{\partial I}{\partial t} - i[I, H] = 0. \tag{2.6}
\]
The $Q_a$, $S_a$, $H$, $D$, $K$ and the operator

$$\Sigma = -\frac{1}{2} \hat{R}$$

(2.7)

satisfy the $osp(2|2)$ superalgebra given by the nontrivial (anti)commutation relations

$$[H, K] = -2iD, \quad [D, H] = iH, \quad [D, K] = -iK,$$

(2.8)

$$\{Q_a, Q_b\} = 2\delta_{ab}H, \quad \{S_a, S_b\} = 2\delta_{ab}K, \quad \{S_a, Q_b\} = 2\delta_{ab}D - \epsilon_{ab}\Sigma,$$

(2.9)

$$[H, S_a] = -iQ_a, \quad [K, Q_a] = iS_a, \quad [D, Q_a] = \frac{i}{2}Q_a, \quad [D, S_a] = -\frac{i}{2}S_a,$$

(2.10)

$$[\Sigma, Q_a] = i\epsilon_{ab}Q_b, \quad [\Sigma, S_a] = i\epsilon_{ab}S_b,$$

(2.11)

in which the even generators form the bosonic $so(1, 2) \oplus u(1)$ subalgebra, while the odd generators form a pair of its spin+$\frac{1}{2}$ representations.

### 3 Hidden superconformal symmetry of conformal mechanics

Let us turn now to the conformal mechanics model [1] described by the Hamiltonian

$$H_\alpha = \frac{1}{2} \left( -\frac{d^2}{dx^2} + \frac{g(\alpha)}{x^2} \right), \quad \text{where} \quad g(\alpha) := \alpha(\alpha + 1),$$

(3.1)

$0 < x < \infty$, and it is assumed that the wave functions are subjected to the boundary conditions

$$\psi(x) \to 0, \quad \psi'(x) \to 0 \quad \text{for} \quad x \to 0,$$

that we write in the form

$$\psi(0) = 0, \quad \psi'(0) = 0.$$

(3.2)

Formally, for $g = 0$ the model is reduced to a free particle on a half-line. However, for $-\frac{1}{4} \leq g < \frac{3}{4}$ the operator (3.1) is not essentially self-adjoint (and therefore cannot play the role of a Hamiltonian, for the details see refs. [13, 14, 11, 15, 16]). On the other hand, Hamiltonian (3.1) with $g \geq \frac{3}{4}$ is essentially self-adjoint, and in what follows we shall assume that the parameter $\alpha$ takes the values corresponding to the latter case.

We shall show that analogously to the model of the free particle on the line, for integer values of the parameter $\alpha = l$, $l = 1, 2, \ldots$, (or, equivalently, for $-\alpha = 2, 3, \ldots$) the conformal mechanics model is described by the nonlinear superconformal symmetry $osp(2|2)_{2l+1}$. It generalizes the hidden superconformal symmetry of the free particle, and is produced by the even $so(1, 2) \oplus u(1)$ generators $H_l$, $D_l$, $K_l$ and $\Sigma$, and by the set of odd operators $S_{n,m}^+, S_{n,m}^-, n = 2l+1, m = 0, 1, \ldots, n-1$. The odd generators constitute the pair of spin-($l + \frac{1}{2}$) representations of the $so(1, 2) \oplus u(1)$, and anticommute for order $2l + 1$ polynomials of the even generators. As we shall see, it is this hidden supersymmetry of the purely bosonic conformal mechanics model [3.1], [3.2] that explains the origin and nature of the double superconformal symmetry in the superconformal mechanics at certain values of the boson-fermion coupling parameter [17]. On the other hand, it will be shown that unlike the case of the model (3.1), the hidden superconformal nonlinear symmetry of the conformal mechanics model is of the broken nature.

System (3.1) has two dynamical integrals of motion, $D_\alpha$ and $K_\alpha$, given by Eq. (2.8) with $H$ changed for (3.1). Together with $H_\alpha$, they form the $so(1, 2)$ algebra (2.8). Define the operator

$$\nabla_\gamma = \frac{d}{dx} + \frac{\gamma}{x}, \quad \nabla_\gamma^{\dagger} = -\nabla_{-\gamma},$$

(3.3)

where $\gamma$ is a real parameter, and write down the relation

$$-2H_\alpha = \nabla_{\alpha+1} \nabla_{-(\alpha+1)} = \nabla_{-\alpha} \nabla_\alpha = -2H_{-(\alpha+1)},$$

(3.4)
which is valid for Hamiltonian \(3.1\) and is based on the elementary equality \(g(\alpha) = g(-\alpha - 1)\). In terms of operator \(3.3\), construct the order \(n\) differential operator

\[
P_{\gamma,n} := (-i)^n \nabla_{\gamma-n+1} \nabla_{\gamma-n+2} \ldots \nabla_{\gamma-1} \nabla_\gamma. \tag{3.5}
\]

Due to definition \(3.3\), operator \(3.5\) is Hermitian in the case

\[
\gamma = \frac{1}{2} (n - 1), \tag{3.6}
\]

i.e. when parameter \(\gamma\) takes integer or half-integer values. When \(n\) takes an even value, \(n = 2l\), \(l = 1, 2, \ldots\), and in accordance with \(3.6\) \(\gamma\) is half-integer, \(3.5\) takes the form

\[
P_{l-\frac{1}{2},2l} = (-1)^l (\nabla_{-(l-\frac{1}{2})} \nabla_{-(l-\frac{1}{2})} \ldots \nabla_{l-\frac{1}{2}}). \tag{3.7}
\]

Making use of relation \(3.4\) starting from the center, we obtain the chain of equalities

\[
\ldots \nabla_{-\frac{3}{2}} \ldots \nabla_{\frac{3}{2}} \ldots = \ldots \nabla_{-\frac{3}{2}} \ldots \nabla_{\frac{3}{2}} \ldots = \ldots \nabla_{\frac{5}{2}} \ldots \nabla_{\frac{5}{2}} \ldots = \ldots,
\]

and find that operator \(3.7\) is reduced to the \(l\)-th order of the operator \(H_{l-\frac{1}{2}}\):

\[
P_{l-\frac{1}{2},2l} = (2H_{l-\frac{1}{2}})^l. \tag{3.8}
\]

Therefore, the Hermitian operator \(3.7\) is an integral of motion for the system \(3.1\) with \(\alpha = l - \frac{1}{2}\), but it is reduced to the \(l\)-th order of the Hamiltonian itself.

Consider now the case of the odd \(n = 2l+1\) and integer \(\gamma = l\). Then, we have the order \((2l+1)\) differential operator \(3.5\) of the form

\[
P_{l,2l+1} := (-i)^{2l+1} \nabla_{-l} \nabla_{-l+1} \ldots \nabla_0 \ldots \nabla_{l-1} \nabla_l. \tag{3.9}
\]

Let us show that it is an integral of motion for system \(3.1\) with \(\alpha = l\). First, this is so for \(l = 0\) when Hamiltonian \(3.1\) is reduced to the formal free particle Hamiltonian \(2.1\) (see the comment on self-adjointness above, which, however, is not important at the moment), and first order operator \(3.9\) is reduced to the momentum operator. In a generic case, with taking into account Eq. \(3.4\) we have

\[
[P_{l,2l+1}, H_l] = -\frac{1}{2} [P_{l,2l+1}, \nabla_{-l} \nabla_l] = \frac{1}{2} \nabla_{-l} [P_{l-1,2l-1}, \nabla_l \nabla_l] \nabla_l = -\nabla_{l} [P_{l-1,2l-1}, H_{l-1}] \nabla_l. \tag{3.10}
\]

Since \([P_{0,1}, H_0] = 0\), by induction we conclude that \([P_{l,2l+1}, H_l] = 0\) for any \(l = 0, 1, 2, \ldots\). A simple algebraic calculation with repeated application of relation \(3.4\) shows that the integral \(3.9\) satisfies the relation

\[
(P_{l,2l+1})^2 = (2H_l)^{2l+1}, \tag{3.10}
\]

cf. the free particle relation \(2.2\) corresponding to \(l = 0\).

In what follows, remembering the problem of self-adjointness for operator \(3.1\), we shall assume that

\[
\alpha = l, \quad l = 1, 2, \ldots. \tag{3.11}
\]

Let us denote \(n = 2l + 1\) and \(S_{n,0} := P_{l,2l+1}\). The commutator of any two dynamical integrals is also a dynamical integral. Then we find that the subsequent commutation of the integral \(S_{n,0}\) with dynamical integral \(K_l\) produces a set of the new \(n-1\) dynamical integrals in accordance with relation

\[
[K_l, S_{n,m}] = -(n-m) S_{n,m+1}, \quad m = 0, \ldots, n - 1. \tag{3.12}
\]

The \(n\)-times repeated commutator of \(K_l\) with \(S_{n,0}\) produces finally zero and we obtain the finite chain of dynamical integrals \(S_{n,m}, m = 1, \ldots, n - 1\) in addition to the integral \(S_{n,0}\). These additional
integrals, like $S_{n,0}$, are also the order $n = 2l+1$ differential operators, and all together they explicitly can be presented in the form

$$S_{n,m} = (-i)^n (x + it\nabla_{-l})(x + it\nabla_{-l+1}) \cdots (x + it\nabla_{-l+(m-1)})\nabla_{-l+m} \cdots \nabla_{l}, \quad m = 0, \ldots, n - 1.$$  

(3.13)

To get this explicit form, we have used, in particular, the equality $\nabla_\gamma x = x\nabla_{\gamma+1}$. The $S_{n,m}$ satisfies the relations $S_{n,m}^\dagger = (-1)^m S_{n,m}$ and

$$\frac{\partial S_{n,m}}{\partial t} = i m S_{n,m-1}.$$  

Therefore, being the dynamical integral of motion, the $S_{n,m}$ satisfies the commutation relation

$$[H_l, S_{n,m}] = -m S_{n,m-1}. \quad (3.14)$$

We have also the relation

$$[D_l, S_{n,m}] = i \left( \frac{n}{2} - m \right) S_{n,m}. \quad (3.15)$$

According to (3.12), (3.14) and (3.15), the set of the operators $S_{n,m}$ forms the $so(1,2)$ spin-$\frac{m}{2}$ representation. All the $S_{n,m}$ are the order $n = 2l+1$ differential operators in $x$, and so, anticommute with the reflection operator $R$, while the $so(1,2)$ generators are even operators commuting with $R$. Then, as in the free particle case, one can extend the set of odd operators $S_{n,m}$ with the set of odd operators $iRS_{n,m}$, which satisfy the commutation relations with the $so(1,2)$ generators exactly of the same form as $S_{n,m}$. To distinguish these two sets of odd operators, we, again, add the $u(1)$ generator $2\gamma_l$ to the set of even operators. All this set of integrals forms the nonlinear superconformal algebra $osp(2|2)_{2l+1}$ given by nontrivial relations of the form (2.8) and

$$[\Sigma, S_{n,m}^\pm] = \pm S_{n,m}^\pm, \quad [D_l, S_{n,m}^\pm] = i \left( \frac{n}{2} - m \right) S_{n,m}^\pm, \quad (3.16)$$

$$[H_l, S_{n,m}^\pm] = \mp m S_{n,m-1}^\pm, \quad [K_l, S_{n,m}^\pm] = \mp (n - m) S_{n,m+1}^\pm, \quad (3.17)$$

$$\{S_{n,m}^+, S_{n,m}^-\} = P_{2l+1}^{\,m,m'} (H_l, K_l, D_l, \Sigma). \quad (3.18)$$

where $S_{n,m}^+ = \frac{1}{2} (1-R) S_{n,m}$, $S_{n,m}^- = (-1)^m \frac{1}{2} (1+R) S_{n,m} = (S_{n,m}^+)\dagger$, and $P_{2l+1}^{\,m,m'}$ is an order $n = 2l+1$ polynomial of its arguments, whose explicit form is not important for us here (see ref. 18).

We have found the odd integrals of motion not taking into account the boundary condition (3.2). To clarify this aspect, we turn to the spectral problem for Hamiltonian (3.1). Having in mind that for $g \geq \frac{3}{2}$ the system has no states with $E \leq 0$, we introduce the notations

$$k = \sqrt{2E}, \quad z = kx, \quad \psi(x) = \sqrt{z} u(z), \quad \nu = \alpha + \frac{1}{2}.$$  

(3.19)

implying that $E > 0$. Then the spectral equation $(H_\alpha - E) \psi(x) = 0$ is reduced to the Bessel equation

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2) u = 0.$$  

(3.20)

For non-integer values of $\nu$ (in this case $\alpha$ is not half-integer, that includes (3.11)), the general solution of Eq. (3.20) is $u(z) = AJ_\nu(z) + BJ_{-\nu}(z)$, where $J_\nu(z)$ is Bessel function, and $A, B$ are some constants. Bessel functions satisfy the recursive differential relations

$$\nabla_{-\nu} J_\nu(z) = -J_{\nu+1}(z), \quad \nabla_\nu J_\nu(z) = J_{\nu-1}(z), \quad (3.21)$$

where $\nabla_\nu$ is the first order differential operator given by Eq. (2.8) with $x$ changed for $z$. Using repeatedly the second relation, we get

$$\nabla_{\nu,n+1} J_\nu(z) = J_{\nu-(n+1)}(z), \quad \nabla_{\nu,n+1} := \nabla_{\nu-n} \nabla_{\nu-n+1} \cdots \nabla_\nu. \quad (3.22)$$
When \( \nu = l + \frac{1}{2} \) and \( n = 2l \), the operator \( \hat{P}_{l+\frac{1}{2},2l+1} \) transforms the solution \( J_{l+\frac{1}{2}}(z) \) into independent solution \( J_{-(l+\frac{1}{2})}(z) \) of the same equation \( (3.20) \). Since \( J_{\nu}(z) \sim z^{\nu} \) for \( z \sim 0 \), from the two solutions \( J_{l+\frac{1}{2}}(z) \) and \( J_{-(l+\frac{1}{2})}(z) \) of the same equation \( (3.20) \) with \( \nu^2 = (l + \frac{1}{2})^2 \) only the first one satisfies the boundary condition \( (3.2) \) corresponding to the conformal mechanics model. Remembering relations \( (3.19) \), we find that the action of the operator \( \hat{P}_{l+\frac{1}{2},2l+1} \) on the function \( u(z) \) up to a numerical coefficient is reduced to the action of the supercharge \( S_{n,0} \) on the corresponding function \( \psi(x) \). This means that when the integral \( S_{n,0} \) acts on a physical state, which is an eigenstate of the Hamiltonian \( H_l \) satisfying boundary condition \( (3.2) \), it produces a state which formally still is an eigenstate of the same differential operator \( H_l \) but does not satisfy the boundary condition at the origin. This picture is somewhat reminiscent to the the spontaneously broken supersymmetry, in which a zero energy state being non-normalisable does not satisfy the boundary condition at infinity. On the other hand, here, unlike the usual spontaneously broken supersymmetry, we have no pairing of the states even in a part of the spectrum. Because of this reason the hidden symmetry of the conformal mechanics can be called a virtual superconformal symmetry.

Note also that taking in \( (3.22) \) \( n = 2l + 1 \), \( \nu = l + 1 \), \( l = 0, 1, \ldots \), and using the earlier observed relation \( (3.8) \), we reproduce the well known identity

\[
J_{-l}(z) = (-1)^l J_l(z). \tag{3.23}
\]

### 4 Double superconformal symmetry of superconformal mechanics

The Hamiltonian of the system possessing superconformal symmetry of order \( n \) \cite{17, 18} can be presented in the form

\[
H_{n,\alpha} = \begin{pmatrix} H_{\alpha-n} & 0 \\ 0 & H_{\alpha} \end{pmatrix}, \tag{4.1}
\]

where the upper and lower Hamiltonian operators are of the form \( (3.1) \), and the upper, \( \psi^+ \), and lower, \( \psi^- \), components of the state \( \Psi^T = (\psi^+, \psi^-) \), are subjected to the boundary condition \( (3.2) \).

For the sake of definiteness we shall assume that the parameter \( \alpha \) is non-negative. The set of odd generators of nonlinear superconformal symmetry associated with Hamiltonian \( (4.1) \) has the structure similar to \( (3.13) \)

\[
S_{n,m;\alpha}^\pm := (x + it \nabla_{\alpha-n+1})(x + it \nabla_{\alpha-n+2}) \cdots (x + it \nabla_{\alpha-n+l})\mathcal{P}_{\alpha,n-l} \sigma_+, \tag{4.2}
\]

where \( \sigma_+ = \frac{1}{2}(\sigma_1 + i \sigma_2) \), and operator \( \mathcal{P}_{\alpha,n-l} \) is defined by Eq. \( (3.5) \). These odd supercharges together with conjugate operators anticommute for order \( n \) polynomials in even generators \( H_{n,\alpha} \), \( K_{n,\alpha} \), \( D_{n,\alpha} \) and \( \Sigma = \frac{1}{2} \sigma_3 \) (for the details see refs. \cite{17, 18}). In the case \( n = 1 \) the system \( (4.1) \) corresponds to the superconformal mechanics model \( (2.8) \) possessing the \( osp(2|2) \) superconformal symmetry of the form \( (2.8) - (2.11) \).

Before we pass over to the discussion of the double superconformal symmetry, let us note that applying the results of the previous section, we find that when the parameters \( \alpha \) and \( n \) satisfy the relation \( \alpha > n \), we have the order \( n \) unbroken superconformal symmetry. In this case, in particular, when the supercharges \( S_{n,0;\alpha}^\pm \) commuting with the Hamiltonian act on the eigenstates of \( (4.4) \) satisfying boundary condition \( (3.2) \), they mutually transform these eigenstates. For \( \alpha = n \), there appears the problem with self-adjointness of the supersymmetric Hamiltonian \( (4.1) \). Finally, when the parameters satisfy the relation \( 0 < \alpha < n \) and \( \alpha \neq l + \frac{1}{2}, l = 0, 1, \ldots, n-1 \), the order \( n \) superconformal symmetry is broken in the same way as the hidden symmetry discussed in the previous section. Acting on the Hamiltonian eigenstates being proportional to \( J_{\alpha+\frac{1}{2}} \) or \( J_{n-\alpha-\frac{1}{2}} \), the \( S_{n,0;\alpha}^\pm \) will produce the states violating boundary condition \( (3.2) \) except for the case when \( \alpha = l + \frac{1}{2}, l = 0, 1, \ldots, n-1 \). In the last case due to relation of the form \( (3.28) \) we have the unbroken nonlinear superconformal symmetry \( osp(2|2)_n \).

Now we turn to the double superconformal symmetry. In ref. \cite{17} it was observed that when the parameter \( \alpha \) takes an integer value, the system \( (4.1) \) can simultaneously be characterized by the two nonlinear superconformal symmetries of the orders shifted in the odd number.
specifically, when \( \alpha = n + p \), \( p = 1, 2, \ldots \), the system (4.1) is characterized also, in addition to the nonlinear superconformal symmetry of the order \( n \), by the superconformal symmetry of the order \( n' = n + 2p + 1 \). Let us show that this second supersymmetry really is of the broken, virtual nature. Indeed, the supercharge \( S^+_{n+2p+1,0;n+p} \) constructed in accordance with Eq. (4.2), commutes with Hamiltonian (4.1) and anticommutes with its conjugate operator for the operator \( (H_{n,n+p})^{n+2p+1} \). It can be presented in the form

\[
S^+_{n+2p+1,0;n+p} = (\nabla - p \cdots \nabla p) S^+_{n,0;n+p} = i^{2p+1} P_{p,2p+1} S^+_{n,0;n+p}.
\]

(4.3)

The operator \( P_{p,2p+1} \), as we have seen, commutes with the conformal mechanics model Hamiltonian \( H_p \), but acting on an eigenstate of the latter which satisfies boundary condition (3.2), it produces a state violating (3.2). So, we conclude that the higher order symmetry of the system with double superconformal symmetry (like in the case \( 0 < \alpha < n \), \( \alpha \neq l + \frac{1}{2} \), mentioned above) has the same broken, virtual nature as a hidden nonlinear superconformal symmetry of the conformal mechanics model with integer parameter (3.11).

For the sake of completeness, we note that the double nonlinear superconformal symmetry with supersymmetry orders shifted in an even number is trivial \[17\]: in this case the parameter \( \alpha \) takes a half-integer value and the structure of the higher order odd generators is reduced to the corresponding lower order odd generators multiplied by monomials in bosonic generators due to relations of the form (3.8).

5 Discussion and outlook

We have seen that the breaking of hidden superconformal symmetry in conformal mechanics model is complete in the sense that no energy eigenstates pairing remains in the system. It would be interesting to clarify whether there exist some systems in which the supersymmetry would be broken via violation of a boundary condition at the finite extremum only for the part of the Hamiltonian eigenstates. If so, it could provide us with the supersymmetry breaking mechanism different from the usual mechanism of spontaneous supersymmetry breaking associated with the lost of normalisability of the ground state via violation of boundary condition at infinity.

We have shown that the higher order symmetry of the double nonlinear superconformal symmetry of superconformal mechanics has the same, broken nature as a hidden superconformal symmetry of the conformal mechanics model. This is because the corresponding higher order supercharges have the structure of the lower order superconformal symmetry generators multiplied by the differential operators generating the described hidden superconformal symmetry of the corresponding bosonic subsystem. As a result, the higher order superconformal symmetry odd generators acting on the Hamiltonian eigenstates produce the states which violate the boundary condition at the origin.

Other systems with double nonlinear supersymmetry were found outside the context of conformal symmetry \[22, 23\]. However, potentials of such systems reveal a structure similar to the potential of the conformal mechanics: they are singular at the origin. Moreover under appropriately taken parameter limit, they are reduced to the \( 1/x^2 \) potential. Therefore, other known systems with double supersymmetry belong to the class of the systems to which the conformal and superconformal mechanics model do belong: they are the systems formulated on a half-line, and so, include the boundary condition at the origin as a defining ingredient. By analogy, one could expect that the higher order supersymmetry in such systems should also be of the broken, virtual nature. Note here that independently from the nonlinear supersymmetry context \[24, 20, 23, 26\], the supersymmetry breaking in the systems with singular potentials was discussed recently in \[27\]. We are going to present the results of investigation of such systems elsewhere.

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