BOUNDEDNESS FOR A CLASS OF FRACTIONAL CARLESON TYPE MAXIMAL OPERATOR

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Abstract. In this paper, the authors study the fractional Carleson type maximal operators $T^*_\beta$ which is defined by

$$T^*_\beta f(x) = \sup_{\lambda} \left| \int_{\mathbb{R}^n} e^{i\lambda \cdot y} \frac{\Omega(y)}{|y|^{n-\beta}} f(x-y) dy \right|,$$

where $0 < \beta < n$ and $\Omega$ satisfies the $L^q$-Dini conditions with $1 < q < \infty$. The authors prove the $L^p \to L^p$ boundedness of $T^*_\beta$ under certain conditions.

1. Introduction

In 1966, Carleson [2] studied the following Carleson type maximal operator $C^*$ as

$$C^* f(x) = \sup_{\lambda \in \mathbb{R}} \left| \int_{-\pi}^{\pi} e^{-i\lambda t} f(t) dt \right|,$$  \hfill (1.1)

where $f \in L^2([-\pi, \pi])$ and $x \in [-\pi, \pi]$. Carleson [2] proved the almost everywhere convergence of the Fourier series of the functions in $L^2([-\pi, \pi])$ by using the weak type $(2,2)$ of $C^*$. Later, Hunt [9] improved Carleson’s results to $L^p([-\pi, \pi])$ with $1 < p < \infty$.

In 1970, Sjölin [12] studied another type of following Carleson type operator $\mathcal{J}^*$ on $\mathbb{R}^n$, that is

$$\mathcal{J}^* (f)(x) = \sup_{\lambda \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{-i\lambda \cdot y} K(x-y)f(y) dy \right|,$$  \hfill (1.2)

where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ and $K$ is an appropriate Calderón-Zygmund kernel. Sjölin [12] proved the following theorem.

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THEOREM A. ([12]) If $K$ satisfies the following conditions:

1. $K(tx) = t^{-n}K(x)$, for $t > 0$;
2. $\int_{S^{n-1}} K(x')d\sigma(x') = 0$;
3. $K \in \mathcal{C}^{n+1}(\mathbb{R}^n \setminus \{0\})$.

Then $\|\mathcal{J}^*(f)\|_{L^p} \leq C_p\|f\|_{L^p}$ for $1 < p < \infty$.

In 2001, Stein and Wainger [13] extended Theorem A to a broader context. That is, the authors in [13] replace the linear phase $\lambda \cdot y$ in the definition of $\mathcal{J}^*$ by a more general phase with a fixed degree. Now, let us state the main results of [13].

Define

$$T_\lambda (f)(x) = \int_{\mathbb{R}^n} e^{iP_\lambda(y)} K(y) f(x-y) dy,$$

where $P_\lambda(x) = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$ is the polynomial in $\mathbb{R}^n$ with real coefficients $\lambda := (\lambda_\alpha)_{1 \leq |\alpha| \leq d}$.

Then, the definition of the Carleson type maximal operator $\mathcal{T}^*$ is

$$\mathcal{T}^* f(x) = \sup_\lambda |T_\lambda (f)(x)|,$$

(1.1)

where the supremum is taken over all the real coefficients $\lambda$ of $P_\lambda$. Stein and Wainger proved the following result.

THEOREM B. ([13]) Suppose that $P_\lambda(x) = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$ and $K$ satisfies the following conditions:

1. $K$ is a tempered distribution and agrees with a $C^1$ function $K(x)$ for $x \neq 0$;
2. $\hat{K} \in L^\infty$;
3. $|\partial^\gamma K(x)| \leq A|x|^{-n-|\gamma|}$ for $0 \leq |\gamma| \leq 1$.

Then $\|\mathcal{T}^*(f)\|_{L^p} \leq C_p\|f\|_{L^p}$ for $1 < p < \infty$.

Obviously, Theorem B is a essential extension of Theorem A. Recently, Ding and Liu [5] gave a weighted variant version of Theorem B under weak conditions. Before giving the main results of [5], we introduce some definitions.

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ ($n \geq 2$), equipped with the usual Lebesgue measure $d\sigma$. Suppose that $\Omega$ is a homogeneous of degree zero and measurable function on $\mathbb{R}^n \setminus \{0\}$. Furthermore, we assume that $\Omega$ satisfies the following conditions:

$$\Omega \in L^1(S^{n-1}), \quad \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$  

(1.3)

DEFINITION 1.1. ([1]) Suppose that $\Omega \in L^q(S^{n-1})$ for some $1 \leq q \leq \infty$. Then a function $\Omega$ is said to satisfy the $L^q$-Dini condition if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty,$$
where $\omega_q(\delta) (0 < \delta \leq 1)$ is called the integral continuous modulus of $\Omega$ of degree $q$, which is defined by

$$\omega_q(\delta) = \sup_{\|\rho\| < \delta} \left( \int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q d\sigma(x') \right)^{1/q} \quad \text{for} \quad 1 \leq q < \infty$$

and

$$\omega_\infty(\delta) = \sup_{\|\rho\| < \delta} |\Omega(\rho x') - \Omega(x')|,$$

where $\rho$ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup \{|\rho x' - x'| : x' \in S^{n-1}\}$.

Then the Carleson type maximal operator with a rough kernel on $\mathbb{R}^n$ studied by Ding and Liu in [5] can be written as

$$T^*_f(x) := \sup_{\lambda} |T_{\lambda} f(x)| = \sup_{\lambda} \left| \int_{\mathbb{R}^n} e^{iP_\lambda(y)} K(y) f(x - y) dy \right|,$$

where $K(y) = \frac{\Omega(y)}{|y|^n}$. In [5], Ding and Liu proved the following theorem.

**Theorem C.** ([5]) Suppose that $P_\lambda(x) = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha x^\alpha$ and $K(x) = \Omega(x)|x|^{-n}$, where $\Omega$ satisfies (1.3). If $\Omega$ satisfies the $L^q$-Dini condition for some $1 < q \leq \infty$, then for $1 \leq q' < p < \infty$ and $w \in A_{p/q'}$, the Carleson type maximal operator $T^*_f$ is a bounded operator on the weighted space $L^p(w)$. That is, there exists a constant $C > 0$ such that for all $f \in L^p(w)$

$$\|T^*_f\|_{L^p(w)} \leq C \|f\|_{L^p(w)},$$

(1.4)

where $A_{p/q'}$ denotes the classical Muckenhoupt class (see [8] or [10]).

By the way, we would like to point out that Ding and Liu [4] also proved that if $\Omega \in H^1(S^{n-1})$, then $T^*$ is bounded on $L^p$ for $1 < p < \infty$. Here $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere $S^{n-1}$ and one may see [3] for more details. Noting the following fact

$$C^1(S^{n-1}) \subset \text{Lip}_1(S^{n-1}) \subset L^q(S^{n-1}) (1 < q \leq \infty) \subset H^1(S^{n-1}) \subset L^1(S^{n-1}),$$

we find that Ding and Liu’s results in [4, 5] are improvements of the main results of [13].

On the other hand, the fractional integral was also studied a lot by many authors. Especially in [6], Ding and Lu studied the fractional integral with a rough kernel defined by

$$T_{\Omega, \beta} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n-\beta}} f(y) dy,$$

where $0 < \beta < n$ and $\Omega \in L^1(S^{n-1})$. Ding and Lu [6] proved the following results.
THEOREM D. ([6]) Let $0 < \beta < n, s < p < n/\alpha$ and $1/q = 1/p - \beta/n$. If $\Omega \in L^q(S^{n-1})$ and $\omega(x)^{s'} \in A(p/s', q/s')$, then there exists a constant $C$ independent of $f$, such that
\[
\left( \int_{\mathbb{R}^n} [T_{\Omega, \beta} f(x) \omega(x)]^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx \right)^{1/p}
\]
where $A(p/s', q/s')$ denotes the fractional type Muckenhoupt-Wheeden class (see [11]).

In this paper, we will study the following fractional Carleson type maximal operators $\mathcal{T}_{\beta}^*$ with the following definition,
\[
\mathcal{T}_{\beta}^* f(x) := \sup_{\lambda} |T_{\Lambda, \beta} f(x)| = \sup_{\lambda} \left| \int_{\mathbb{R}^n} e^{iP_{\Lambda}} K_{\beta}(y) f(x - y) dy \right|,
\]
where $K_{\beta}(y) = \frac{\Omega(y)}{|y|^{n-\beta}}$ with $0 < \beta < n$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$.

Furthermore, for any $c > 0$ and vector $(\lambda_{\alpha})_{2 \leq |\alpha| \leq d}$, the set $\Lambda$ is defined by
\[
\Lambda = \left\{ \lambda = (\lambda_{\alpha})_{2 \leq |\alpha| \leq d} : |\lambda| = \sum_{2 \leq |\alpha| \leq d} |\lambda_{\alpha}| \geq c \right\}.
\]

Our results can be stated as follows.

**Theorem 1.2.** Suppose that $P_{\lambda}(x) = \sum_{2 \leq |\alpha| \leq d} \lambda_{\alpha} x^\alpha$ with $\lambda \in \Lambda$ and $\Omega$ satisfies the $L^q$-Dini condition for some $1 < q \leq \infty$. Then for $q' < p < \infty$ and $0 < \beta < \frac{\delta \gamma(p)}{n-1+\gamma(p)}$, with $\gamma(p) = \min\{1/p, 1/p'\}$ and $\delta = \frac{1}{c}$, we have
\[
\| \mathcal{T}_{\beta}^* f \|_{L^p} \leq C \| f \|_{L^p},
\]
where the constant $C$ is dependent on $1/c$ and $\alpha$ but independent of $f$.

**Remark 1.3.** By a simple computation, we have $\mathcal{T}_{\beta}^* (f)(x) \leq T_{\Omega, \beta}(|f|)(x)$ with $0 < \beta < n$. Thus by Theorem D, we can easily get the $L^p_{\omega p} \to L^q_{\omega q}$ boundedness of $\mathcal{T}_{\beta}^*$ under the conditions of Theorem D.

**2. Proof of Theorem 1.2**

In this section, we will give the proof of Theorem 1.2. Some basic ideas and techniques of this proof comes from [4, 5].

First, we will introduce a variant version of the Hardy-Littlewood maximal function which will be very useful in the proof of Theorem 1.2 (See [13]).

Let $B_3 = \{ x \in \mathbb{R}^n : |x| \leq 3 \}$. For a measurable set $E \subset B_3$, $\chi_E$ denotes the characteristic function of $E$. Then for any $\epsilon > 0$, the maximal operator $M_{\epsilon}$ is defined by
\[
M_{\epsilon}(f)(x) = \sup_{\epsilon > 0} \left| f \right| \ast (\chi_E)_{\epsilon}(x),
\]
where $(\chi_E)_{\epsilon}(x) = a^{-n} \chi_E(x/a)$.
Lemma 2.1. ([13]) For $1 < p < \infty$, there exists a constant $C > 0$, independent of $\varepsilon$, such that

$$\|M_{\varepsilon}(f)\|_{L^p} \leq C \varepsilon^{1-1/p} \|f\|_{L^p}.$$  

Proof of Theorem 1.2. As $\lambda \in \Lambda$, we have $|\lambda| = \sum_{2 \leq |\alpha| \leq d} |\lambda_{\alpha}| \geq c$. Thus, it suffices to consider the case for $\mathcal{T}_\beta^{\varepsilon} f(x) := \sup_{\lambda > 0} |T_{\lambda, \beta} f(x)|$.

Let $\lambda(x)$ be the nonzero vector $(\lambda_{\alpha}(x))_{2 \leq |\alpha| \leq d}$ satisfying

$$|T_{\lambda(x), \beta}(f)(x)| \geq \frac{1}{2} \sup_{\lambda} |T_{\lambda, \beta}(f)(x)|,$$

where $f \in L^p$ and $x \in \mathbb{R}^n$. Thus by the above estimates, to prove Theorem 1.2, it suffices to show that there exists a constant $C$, such that

$$\|T_{\lambda(\cdot), \beta}(f)\|_{L^p} \leq C \|f\|_{L^p},$$

where the constant $C$ is independent of the choice of $\lambda(\cdot)$.

For any $x \in \mathbb{R}^n$, we define $N(\lambda(x)) = \sum_{2 \leq |\alpha| \leq d} |\lambda_{\alpha}(x)| \frac{1}{|\alpha|}$. By the definition of $\lambda(\cdot)$ and the condition $\lambda \in \Lambda$, we have $\sum_{2 \leq |\alpha| \leq d} |\lambda_{\alpha}(x)| \geq c$. As $\frac{1}{|\alpha|} < 1$, we obtain

$$N(\lambda(x)) = \sum_{2 \leq |\alpha| \leq d} |\lambda_{\alpha}(x)| \frac{1}{|\alpha|} \geq \left( \sum_{2 \leq |\alpha| \leq d} |\lambda_{\alpha}(x)| \right) \frac{1}{|\alpha|} \geq C_{\alpha}, \quad (2.1)$$

where the constant $C_{\alpha}$ is only dependent on $c$ and $\alpha$ but independent of $x$.

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function with supp$(\psi) \subseteq \{ \frac{1}{4} < |y| \leq 1 \}$. Furthermore, we assume that $\psi$ satisfies

$$\sum_{j=-\infty}^{\infty} \psi_j(y) = 1 \quad \text{for} \quad y \neq 0,$$

where $\psi_j(y) = \psi(2^{-j}y)$. Then, it is easy to get that

$$\sum_{j=-\infty}^{\infty} \psi_j(N(\lambda(x))y) = 1 \quad \text{for} \quad y \neq 0.$$

Now we may decompose $K_\beta$ as

$$K_\beta(y) = \sum_{j=0}^{\infty} K_{\beta; j}(y),$$
where $K_{\beta,0}(y) = \sum_{j=0}^{\infty} \psi_j(y) K_\beta(y)$, $K_{\beta,j}(y) = \psi_j(y) K_\beta(y)$ and $\psi_{j,\lambda}(y) = \psi_j(N(\lambda(x))y)$ for $j \geq 1$ and $x \in \mathbb{R}^n$. Thus, we have

$$\left| T_{\lambda(x),\beta}^0(f)(x) \right| \leq \left| \int_{\mathbb{R}^n} e^{i\lambda(x)(y)} K_{\beta,0}(y) f(x-y) \, dy \right| + \sum_{j=1}^{\infty} \left| \int_{\mathbb{R}^n} e^{i\lambda(x)(y)} K_{\beta,j}(y) f(x-y) \, dy \right|$$

$$:= T_{\lambda(x),\beta}^0(f)(x) + \sum_{j=1}^{\infty} T_{\lambda(x),\beta}^j(f)(x). \quad (2.2)$$

The estimates of $T_{\lambda(x),\beta}^0(f)(x)$

By the fact supp $(K_{\beta,0}) \subset \{ |y| \leq \frac{1}{N(\lambda(x))} \}$ and (2.1), we have

$$\left| T_{\lambda(x),\beta}^0(f)(x) \right| = \left| \int_{\mathbb{R}^n} e^{i\lambda(x)(y)} K_{\beta,0} f(x-y) \, dy \right|$$

$$\leq \sum_{j=-\infty}^{0} \int_{\mathbb{R}^n} e^{i\lambda(x)(y)} K_{\beta}(y) \psi_{j,\lambda}(y) f(x-y) \, dy$$

$$\leq C \sum_{j=-\infty}^{0} \left( \frac{2^{j-2}}{N(\lambda(x))} \right)^{\beta-n} \int_{|y| \leq \frac{2^j}{N(\lambda(x))}} |\Omega(y) f(x-y)| \, dy$$

$$\leq C M_\Omega f(x),$$

where $M_\Omega$ is the usual Hardy-Littlewood maximal function with a rough kernel (see [7]). Thus, by the $L^p$ boundedness of $M_\Omega$ (see [7]), we have

$$\| T_{\lambda(x),\beta}^0 f \|_{L^p} \leq C \| M_\Omega f \|_{L^p} \leq C \| f \|_{L^p}. \quad (2.3)$$

The estimates of $T_{\lambda(x),\beta}^j(f)(x)$

We use some basic ideas from [5]. Choose a nonnegative function $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfying supp $(\phi) \subset \{ |y| \leq 2^{-6} \}$ and $\| \phi \|_{L^1} = 1$. For any $a > 0$, we denote $\phi_a(x) = a^{-n} \phi(x/a)$.

For a positive real number $\sigma$ which satisfies $\beta < \sigma$ and will be chosen later, we may denote

$$L_{j,\lambda(x),\beta}(y) = K_{\beta,j} * \phi_{j(1-\sigma)}(y) \quad \text{and} \quad R_{j,\lambda(x),\beta}(y) = K_{\beta,j}(y) - L_{j,\lambda(x),\beta}(y), j \in \mathbb{N}.$$ 

Thus, it is easy to see

$$\left| T_{\lambda(x),\beta}^j(f)(x) \right| \leq \left| \mathcal{L}_{\lambda(x),\beta}^j(f)(x) \right| + \left| \mathcal{R}_{\lambda(x),\beta}^j(f)(x) \right|,$$
where \( \mathcal{L}_{\lambda}^{j}(\cdot, \beta) \) and \( \mathcal{R}_{\lambda}^{j}(\cdot, \beta) \) are defined by

\[
\mathcal{L}_{\lambda}^{j}(\cdot, \beta)(f)(x) = \int_{\mathbb{R}^{n}} e^{i \lambda_{\beta}(y)} L_{j, \lambda}(x, \beta)(y) f(x - y) dy
\]

and

\[
\mathcal{R}_{\lambda}^{j}(\cdot, \beta)(f)(x) = \int_{\mathbb{R}^{n}} e^{i \lambda_{\beta}(y)} R_{j, \lambda}(x, \beta)(y) f(x - y) dy.
\]

Next, we will give the estimates of \( \mathcal{L}_{\lambda}^{j}(\cdot, \beta) \) and \( \mathcal{R}_{\lambda}^{j}(\cdot, \beta) \) respectively.

**The estimates of \( \mathcal{L}_{\lambda}^{j}(\cdot, \beta) \)**

First, we give the estimate of \( L_{j, \lambda}(x, \beta) \). By the definition of \( L_{j, \lambda}(x, \beta) \), we have

\[
\text{supp } (L_{j, \lambda}(\cdot, \beta)) \subseteq \left\{ \frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))} \right\}.
\] (2.4)

Define

\[
L_{j, \beta}(y) = \int_{\mathbb{R}^{n}} \psi(y - z) K_{\beta}(y - z) 2^{jnq} \phi(2^{j} z) dz.
\]

Then, it is easy to see that

\[
L_{j, \lambda}(x, \beta)\left(\frac{2^{j} y}{N(\lambda(x))}\right) = \left(\frac{2^{j}}{N(\lambda(x))}\right)^{\beta - n} L_{j, \beta}(y).
\] (2.5)

By the definition of \( L_{j, \lambda}(\cdot, \beta) \) and the Hölder inequality, we have the following estimates for \( L_{j, \lambda}(\cdot, \beta) \).

\[
|L_{j, \lambda}(x, \beta)(y)| = (2^{-jN(\lambda(x))})^{n} 2^{jnq} \left| \int_{\mathbb{R}^{n}} K_{\beta, j}(y) \phi \left( \frac{|x - y|}{2^{j(1 - \sigma)}} \right) dy \right|
\]

\[
\leq (2^{-jN(\lambda(x))})^{n} 2^{jnq} \int_{\frac{1}{4} \leq |y| \leq 1} |\psi_{j, \lambda}(y)| |\Omega(y)| dy
\]

\[
\leq (2^{-jN(\lambda(x))})^{n} 2^{jnq} \left( \int_{\frac{1}{4} \leq |y| \leq 1} |\Omega(y)|^{q} dy \right)^{1/q} \left( \int_{\frac{1}{4} \leq |y| \leq 1} |y|^{(\beta - n)q} dy \right)^{1/q'}
\]

\[
\leq C(2^{-jN(\lambda(x))})^{n} 2^{jnq} \left( \frac{2^{j}}{N(\lambda(x))} \right)^{n/q} \left( \frac{2^{j}}{N(\lambda(x))} \right)^{\beta - n} \left( \frac{2^{j}}{N(\lambda(x))} \right)^{n/q'}
\]

\[
= C(2^{-jN(\lambda(x))})^{n} 2^{jnq} \frac{2^{j\beta}}{(N(\lambda(x)))^{\beta}}.
\]

From (2.1), we get

\[
|L_{j, \lambda}(x, \beta)(y)| \leq C(2^{-jN(\lambda(x))})^{n} 2^{jnq} \frac{2^{j\beta}}{(N(\lambda(x)))^{\beta}} \leq C(2^{-jN(\lambda(x))})^{n} 2^{jnq} 2^{j\beta},
\] (2.6)
which implies
\[ |\mathcal{L}_\lambda^j(\lambda(x), \beta \cdot f(x))| \leq C \int_{\frac{2^n-1}{N(\lambda(x))} \leq |y| \leq \frac{2^n+1}{N(\lambda(x))}} (2^{-jnN(\lambda(x))})^n 2^{jn} 2^{j\beta} |f(x-y)|dy \leq C 2^{jn+\beta}|f(x)|. \]  

(2.7)

Next, we adopt some notations from [5]. Recall that \( \lambda(x) = (\lambda_\alpha(x))_{2 \leq |\alpha| \leq d}. \) For any \( j \in \mathbb{Z}^+ \), we denote
\[ A_{j,\lambda} \circ \lambda = \left( \left( \frac{2^j}{N(\lambda(x))} \right)^{|\alpha|} \lambda_\alpha(x) \right)_{2 \leq |\alpha| \leq d} \]
for convenience. From [5, p.2744], there is
\[ P_{\lambda(x)}(y) = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha(x)y^\alpha = \sum_{2 \leq |\alpha| \leq d} \lambda_\alpha(x) \left( \frac{2^j}{N(\lambda(x))} \right)^{|\alpha|} (2^{-jnN(\lambda(x))})^n \alpha \]
(2.8)

Thus, by (2.5), we obtain
\[ \mathcal{L}_\lambda^j(\lambda(x), \beta \cdot f(x)) = \int_{\mathbb{R}^n} e^{iP_{A_{j,\lambda} \circ \lambda} (\lambda(x), y)} (2^{-jnN(\lambda(x))} f(x-y)) dy. \]

From now on, we denote \( \mathcal{L}_\lambda^j(\lambda(x), \beta \cdot f(x)) \) by \( \mathcal{L}_{j,\beta} \cdot f(x) \) for simplicity.

From [5, p.2745], we know that there exists a constant \( c_0 > 0 \) such that \( N(v) \leq c_0 |v| \) for any vector \( v \) satisfying \( N(v) \geq 1 \). Moreover, there is
\[ |A_{j,\lambda} \circ \lambda| \geq 2^j/c_0 \quad \text{for all} \quad \lambda(x), x \in \mathbb{R}^n. \]

(2.9)

For \( r \geq 2^j/c_0 \), we define
\[ U_{j,r} = \{ x : r \leq |A_{j,\lambda} \circ \lambda| < 2r \} \]
and
\[ \mathcal{L}_{j,r,\beta}(f)(x) = \mathcal{L}_{j,\beta}(f)(x) \chi_{U_{j,r}}(x). \]

Let \( \mathcal{L}_{j,r,\beta}^* \) be the adjoint operator of \( \mathcal{L}_{j,r,\beta} \). Then, it is easy to check that
\[ \mathcal{L}_{j,r,\beta}^*(g)(y) = \int_{\mathbb{R}^n} e^{-iP_{A_{j,\lambda} \circ \lambda}(x-y)} L_{j,x,y} \beta(x-y) g(x) \chi_{U_{j,r}}(x) dx \]
and
\[ (\mathcal{L}_{j,r,\beta} \mathcal{L}_{j,r,\beta}^*) (f)(x) = \int_{\mathbb{R}^n} \mathcal{K}_\beta(x,z) f(z) dz, \]
where
\[ \mathcal{K}_\beta(x,z) = \int_{\mathbb{R}^n} e^{iP_{A_{j,\lambda} \circ \lambda}(z-x+y)} L_{j,x,y} \beta(y) L_{j,y,z} \beta(z-x+y) dy \chi_{U_{j,r}}(x) \chi_{U_{j,r}}(z) \]
First, we have the following estimates.

\[
\nu \text{ where over, we can divide it into two cases:}
\]

Let \( h \)

\[
\text{De}
\]

Now, we give the proof of (2.10) according to [5, p.2747-p.2748].

Following [5], we are going to prove that for \( r > 2j/c_0 \) and fixed \( x, z \in U_{j,r} \), the following inequality holds.

\[
\left| \mathcal{X}_{\beta}(x,z) \right| \leq C \left( 2^{-j} N (\lambda(z)) \right)^n + 2^{j+n+4} \left[ r^{-2\delta} \mathcal{X}_{B_3} \left( 2^{-j} N (\lambda(z)) (x-z) \right) + C \left( 2^{-j} N (\lambda(x)) \right)^n + 2^{j+n+4} \left[ r^{-2\delta} \mathcal{X}_{B_3} \left( 2^{-j} N (\lambda(x)) (x-z) \right) + C \left( 2^{-j} N (\lambda(x)) \right)^n \right] \right]
\]  

(2.10)

where the sets \( E_j^{\lambda(x)} \), \( E_j^{\lambda(z)} \subset B_3 = \{ |y| \leq 3 \} \) satisfying \( |E_j^{\lambda(x)}|, |E_j^{\lambda(z)}| \leq r^{-4\delta} \) with \( \delta = (6d)^{-1} \).

Now, we give the proof of (2.10) according to [5, p.2747-p.2748].

Define \( \mathcal{F}_{\mu,\nu} \) by

\[
\mathcal{F}_{j,\beta} (\mu,\nu) (u) = \left( e^{iP_{\nu}(\cdot)} L_{j,\nu,\beta} (\cdot) \right) * \left( e^{-iP_{\nu}(\cdot)} L_{j,\mu,\beta} (\cdot) \right) (u),
\]

where \( \nu = (\nu_{\alpha})_{2 \leq |\alpha| \leq d} \) and \( \mu = (\mu_{\alpha})_{2 \leq |\alpha| \leq d} \) to satisfy

\[
\nu \leq |A_{j,\nu} \circ \nu|, |A_{j,\mu} \circ \mu| < 2r.
\]

Let \( h = \frac{N(\mu)}{N(\nu)} \) and we may assume that \( h \leq 1 \). Hence, by (2.5) and (2.8), we have

\[
\mathcal{F}_{j,\beta} \left( \frac{2i}{N(\mu)} \right) = \int_{\mathbb{R}^n} e^{i \left[ P_{\nu,\circ \nu}(y) - P_{\nu,\mu}(y) - u + hy \right]} L_{j,\beta} (y) \left( \frac{2i}{N(\mu)} \right)^{-n+\beta} L_{j,\beta} (hy - u) dy.
\]

(2.11)

To estimate \( \mathcal{F}_{j,\beta} \left( \frac{2i}{N(\mu)} \right) \), we adopt some basic ideas and estimates from [5, 13]. Moreover, we can divide it into two cases: \( h \) is near the origin and away from the origin.

**Case 1.** \( 0 < h \ll 1 \), where \( \eta \) will be chosen later. Note that

\[
\supp(L_{j,\beta}) \subseteq \{ 1/8 < |y| \leq 3/2 \} \quad \text{and} \quad |u| \leq |hy - u| + h|y| \leq 3.
\]

First, we have the following estimates.

\[
|L_{j,\beta} (y)| \leq C \int_{2j^{-2} \leq |y-z| \leq 2j} \left| \Omega (y-z) \right| \left( \frac{2j^{-2} \leq |y| \leq 2j} |y-z|^{-n+\beta} \right) d\sigma d\nu
\]

\[
\leq C 2^{j/n+\sigma} \left( \int_{2j^{-2} \leq |y| \leq 2j} \left| \Omega (y) \right|^q dy \right)^{1/q} \left( \int_{2j^{-2} \leq |y| \leq 2j} |y|^{(\beta-n)q'} dy \right)^{1/q'}
\]

\[
\leq C 2^{j/n+2\beta}.
\]

(2.13)

Similarly, there is

\[
|\nabla \mathcal{L}_{j,\beta} (y)| \leq C 2^{j/n+2\beta}.
\]

(2.14)
From [5, p.2747], we know that if we choose $\eta$ small enough, then

$$\sum_{2 \leq |\alpha| \leq d} \left| \left( (A_{j,v} \circ v)_{\alpha} + O(h|A_{j,\mu} \circ \mu|) \right) \right| \geq \sum_{2 \leq |\alpha| \leq d} \left| (A_{j,v} \circ v)_{\alpha} \right| - C\eta |A_{j,\mu} \circ \mu| \geq Cr.$$ 

Thus, using (2.13), (2.14) and the van der Corput lemma in $n$-dimensional (see Proposition 2.1 in [13, p. 791]), we obtain

$$\left| \mathcal{F}_{j,\beta}^{\mu,v} \left( \frac{2ju}{N(\mu)} \right) \right| \leq C(2^{-j}N(\mu))^n 2^{2jns_\sigma+4j\beta} r^{-1/d} \chi_{B_3}(u). \quad (2.15)$$

**Case 2.** $\eta < h \leq 1$. From the assumption on polynomial, we know that there is no first order term in $y$ of $P_{A_j,v}(y)$. Let $e_k = (0, \ldots, 1, 0, \ldots)$ with 1 in the $k^{th}$ component. From [5, p.2748], we know that the first order term in $y$ in $P_{A_j,v}(y) - P_{A_j,\mu}(y + h)$ can be written as

$$-h \sum_{k=1}^n P^{(k)}_{A_j,\mu}(u) y_k = -h \sum_{k=1}^n \sum_{2 \leq |\alpha| \leq d} \alpha_k \left( \frac{2j}{N(\mu)} \right)^{|\alpha|} \mu_\alpha u^{\alpha - e_k} y_k.$$ 

Now applying (2.13), (2.14) and Proposition 2.1 in [13] again, we have

$$\left| \mathcal{F}_{j,\beta}^{\mu,v} \left( \frac{2ju}{N(\mu)} \right) \right| \leq C(2^{-j}N(\mu))^n 2^{2jns_\sigma+4j\beta} \left( \sum_{k=1}^n |P^{(k)}_{A_j,\mu}(u)| \right)^{-1/d} \chi_{B_3}(u).$$ 

For $\rho > 0$, we define $E^j_{\mu} = \left\{ u \in B_3 : \sum_{k=1}^n |P^{(k)}_{A_j,\mu}(u)| \leq \rho \right\}$. Thus, we get

$$\left| \mathcal{F}_{j,\beta}^{\mu,v} \left( \frac{2ju}{N(\mu)} \right) \right| \leq C(2^{-j}N(\mu))^n 2^{2jns_\sigma+4j\beta} \rho^{-1/d} \chi_{B_3}(u), \quad (2.16)$$

for $u \in (E^j_{\mu})^c$. From Proposition 2.2 in [13, p. 791], we have

$$|E^j_{\mu}| \leq C_{n,d} \left( \sum_{k=1}^n \sum_{2 \leq |\alpha| \leq d} \alpha_k \left( \frac{2j}{N(\mu)} \right)^{|\alpha|} |\mu_\alpha| \right)^{-1/d} \rho^{1/d}.$$ 

According to [5, p.2748], there is

$$\sum_{k=1}^n \sum_{2 \leq |\alpha| \leq d} \alpha_k \left( \frac{2j}{N(\mu)} \right)^{|\alpha|} |\mu_\alpha| \geq \sum_{2 \leq |\alpha| \leq d} \left( \frac{2j}{N(\mu)} \right)^{|\alpha|} |\mu_\alpha| = |A_{j,\mu} \circ \mu| \geq r.$$ 

Let $\rho = (C_{n,d})^{-d} r^{1/3}$ and $\delta = \frac{1}{6d}$. Then, for $u \in E^j_{\mu}$, we have

$$\left| \mathcal{F}_{j,\beta}^{\mu,v} \left( \frac{2ju}{N(\mu)} \right) \right| \leq C(2^{-j}N(\mu))^n 2^{2jns_\sigma+4j\beta} \chi_{E^j_{\mu}}(u) \quad (2.17)$$
with $|E_{j}^{\mu}| \leq C_{n,d}(\rho/r)^{1/d} \leq r^{-4\delta}$. As $r \geq 2^{j}/c_{0}$, then using (2.15)-(2.17), we get

$$
\left| \mathcal{B}_{j,\beta}^{\mu,v} \left( \frac{2^{j}u}{N(\mu)} \right) \right| \leq C \left( 2^{-j}N(\mu) \right)^{n} 2^{jn+4j} \left[ r^{-2\delta} \chi_{B_{3}}(u) + \chi_{E_{j}^{\mu}}(u) \right].
$$

Now, we conclude that for $\mu$, $\nu$ with $r \leq |A_{j,\nu} \circ \nu| \leq 2r, r \leq |A_{j}\circ \mu| \leq 2r$ and $h \leq 1$, there is

$$
\left| \mathcal{B}_{j,\beta}^{\mu,v} (u) \right| \leq C \left( 2^{-j}N(\mu) \right)^{n} 2^{jn+4j} \left[ r^{-2\delta} \chi_{B_{3}}(u) + \chi_{E_{j}^{\mu}}(u) \right].
$$

(2.18)

For fixed $x,z \in U_{j,r}$, let $\nu = \lambda(x)$, $\mu = \lambda(z)$. $u = x - z$. By the symmetry of $\mu$, $\nu$, and (2.18), we finish the proof of (2.10).

Now, we return to the estimates of $\mathcal{L}_{\lambda(\cdot),\beta}^{j}$. Recall the definition of $\mathcal{M}_{\varepsilon}(f)(x)$ and Lemma 2.1. Denoting $\varepsilon = r^{-4\delta}$ and using (2.10), we obtain

$$
\left| (\mathcal{L}_{j}\mathcal{L}_{j}^{\epsilon} f, g) \right| \leq \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} |\mathcal{K}_{\beta}(x,z)||f(z)||g(x)|dzdx
$$

$$
\leq C r^{-2\delta} 2^{jn+4j} \int_{\mathbb{R}^{n}} |f(z)|(2^{-j}N(\lambda(z)))^{n} \int_{|x-z| \leq \frac{32j}{N(\lambda(z))}} |g(x)|dzdx
$$

$$
+ C 2^{jn+4j} \int_{\mathbb{R}^{n}} |f(z)|(2^{-j}N(\lambda(z)))^{n} \int_{\mathbb{R}^{n}} \chi_{E_{j}^{\lambda(x)}}(2^{-j}N(\lambda(z))(x-z)) |g(x)|dzdx
$$

$$
+ C r^{-2\delta} 2^{jn+4j} \int_{\mathbb{R}^{n}} |g(x)|(2^{-j}N(\lambda(x)))^{n} \int_{|x-z| \leq \frac{32j}{N(\lambda(x))}} |f(z)|dzdx
$$

$$
+ C 2^{jn+4j} \int_{\mathbb{R}^{n}} |g(x)|(2^{-j}N(\lambda(x)))^{n} \int_{\mathbb{R}^{n}} \chi_{E_{j}^{\lambda(x)}}(2^{-j}N(\lambda(x))(x-z)) |f(z)|dzdx
$$

$$
\leq C r^{-2\delta} 2^{jn+4j} \int_{\mathbb{R}^{n}} |f(z)||M(g)(z)|dz + C 2^{jn+4j} \int_{\mathbb{R}^{n}} |f(z)||\mathcal{M}_{\varepsilon}(g)(z)|dz
$$

$$
+ C r^{-2\delta} 2^{jn+4j} \int_{\mathbb{R}^{n}} |g(x)||M(f)(x)|dx + C 2^{jn+4j} \int_{\mathbb{R}^{n}} |g(x)||\mathcal{M}_{\varepsilon}(f)(x)|dx.
$$

Using the Hölder inequality, the $L^{2}$ boundedness of $M$ (see [8]) and Lemma 2.1, we obtain that

$$
\left| (\mathcal{L}_{j}\mathcal{L}_{j}^{\epsilon} f, g) \right| \leq C r^{-2\delta} 2^{jn+4j} \|f\|_{L^{2}}\|g\|_{L^{2}}.
$$

(2.19)

As

$$
\left\{ x \in \mathbb{R}^{n} : |A_{j,\lambda} \circ \lambda| \geq \frac{2^{j}}{c_{0}} \right\} = \bigcup_{k=0}^{\infty} \left\{ x : \frac{2^{j+k}}{c_{0}} \leq |A_{j,\lambda} \circ \lambda| < \frac{2^{j+k+1}}{c_{0}} \right\},
$$

we may choose $r = 2^{j+k}/c_{0}$ for $k = 0,1,\cdots$, and denote $\mathcal{L}_{j,\beta}^{(k)} := \mathcal{L}_{j,\beta}^{(k)}$. Thus, we obtain $\mathcal{L}_{j,\beta}(f)(x) = \sum_{k=0}^{\infty} \mathcal{L}_{j,\beta}^{(k)}(f)(x)$. Using (2.19), we get

$$
\|\mathcal{L}_{j,\beta}\|_{L^{2} \rightarrow L^{2}} \leq \sum_{k=0}^{\infty} \left\| \mathcal{L}_{j,\beta}^{(k)} \right\|_{L^{2} \rightarrow L^{2}} \leq C 2^{jn+4j} \sum_{k=0}^{\infty} \left( \frac{2^{j+k}}{c_{0}} \right)^{-\delta} \leq C 2^{jn+4j} 2^{-j\delta}.
$$

(2.20)
By the fact $|\mathcal{L}_{j,\beta}f(x)| \leq C2^{jn\sigma+j\beta}Mf(x)$ (see (2.7)), we have

$$\|\mathcal{L}_{j,\beta}f\|_{L^s} \leq C2^{jn\sigma+j\beta}\|f\|_{L^s},$$

for $1 < s < \infty$.

Thus, by the Riesz-Thörin interpolation theorem, we obtain

$$\|\mathcal{L}_{j,\beta}f\|_{L^p} \leq C2^{jn\sigma+j\beta}2^{-j(\delta-\beta)\gamma(p)}\|f\|_{L^p},$$

where $\gamma(p) = \min\{1/p, 1/p'\}$ and $q' < p < \infty$. As $0 < \beta < \frac{\delta\gamma(p)}{n-1+\gamma(p)}$, we may choose $\sigma$ satisfying $\beta < \sigma < \frac{\beta+(\delta-\beta)\gamma(p)}{n}$. Moreover, we denote $\theta = (\delta-\beta)\gamma(p) - n\sigma - \beta > 0$, which implies

$$\|\mathcal{L}^j_{\lambda(\cdot),\beta}f\|_{L^p} = \|\mathcal{L}_{j,\beta}f\|_{L^p} \leq C2^{-j\theta}\|f\|_{L^p}. \quad (2.21)$$

The estimates of $\mathcal{R}^j_{\lambda(\cdot),\beta}$

Before giving the estimates of $\mathcal{R}^j_{\lambda(\cdot),\beta}$, we recall the definition of $\mathcal{R}^j_{\lambda(\cdot),\beta}$ as

$$\mathcal{R}^j_{\lambda(x),\beta}(f)(x) = \int_{\mathbb{R}^n} e^{iP_\lambda(x)}(y)R_{j,\lambda,\beta}(y)f(x-y)dy,$$

where $R_{j,\lambda,\beta}(y)$ is defined by

$$R_{j,\lambda,\beta}(y) = \int_{\mathbb{R}^n} \left[K_{\beta}(y)\psi_{j,\lambda}(y) - K_{\beta}(y-z)\psi_{j,\lambda}(y-z)\right]\phi_{2j(1-\sigma)}(z)dz.$$

As

$$\text{supp} \ (R_{j,\lambda,\beta}) \subseteq \left\{ \frac{2^{j-3}}{N(\lambda(x))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(x))} \right\} \quad (2.22)$$

and $|z| \leq \frac{2j(1-\sigma)-5}{N(\lambda(x))}$, we get $|y-z| \sim |y|$. Thus, we have

$$|R_{j,\lambda,\beta}(y)| \leq \int_{\mathbb{R}^n} |\psi_{j,\lambda}(y-z)||K_{\beta}(y) - K_{\beta}(y-z)||\phi_{2j(1-\sigma)}(z)|dz$$

$$+ \int_{\mathbb{R}^n} |K_{\beta}(y)||\psi_{j,\lambda}(y) - \psi_{j,\lambda}(y-z)||\phi_{2j(1-\sigma)}(z)|dz$$

$$\leq C \frac{\Omega(y)}{|y|^{n+1-\beta}} \int_{\mathbb{R}^n} |z||\phi_{2j(1-\sigma)}(z)|dz$$

$$+ \frac{C}{|y|^{n-\beta}} \int_{\mathbb{R}^n} \Omega(y-z) - \Omega(y)||\phi_{2j(1-\sigma)}(z)|dz$$

$$+ C \frac{\Omega(y)}{|y|^{n-\beta}} \int_{\mathbb{R}^n} 2^{-j}N(\lambda(x))z||\phi_{2j(1-\sigma)}(z)|dz$$

$$\leq C2^{-j\sigma+j\beta}\frac{\Omega(y)}{|y|^n} + \frac{C}{|y|^{n-\beta}} \int_{\mathbb{R}^n} \Omega(y-z) - \Omega(y)||\phi_{2j(1-\sigma)}(z)|dz.$$
Now, we denote $R^j_{\lambda}(\cdot, \beta)(f)$ by $R_{j, \beta}(f)$ for simplicity. Then, from (2.22) and the Hölder inequality, we have

$$|R_{j, \beta}(f)(x)| \leq 2^{-j\sigma + j\beta} M_\Omega(f)(x) + \int_{\mathbb{R}^n} \phi_j(z) \times \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\beta}} |f(x-y)| dydz \leq 2^{-j\sigma + j\beta} M_\Omega(f)(x) + \int_{\mathbb{R}^n} \phi_j(z) \times \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\beta}} |f(x-y)|^q dydz \leq 2^{-j\sigma + j\beta} M_\Omega(f)(x) + \int_{\mathbb{R}^n} \phi_j(z) \times \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\beta}} |f(x-y)|^{q'} dydz.$$ 

From [5, p.2750], we know that for $\alpha = \frac{q'}{n}$, there is

$$\int_{\mathbb{S}^{n-1}} \Omega \left( \frac{y' - \alpha}{|y'|} \right) - \Omega(y') \left| \frac{q}{q'} \right| d\sigma(y') \leq C \omega_n^q(|\alpha|).$$

As $|z| \leq \frac{2^{(j-\sigma-5)}}{N(\lambda(\cdot))}$, we get

$$\left( \int_{\frac{2^{j-3}}{N(\lambda(\cdot))} \leq |y| \leq \frac{2^{j+1}}{N(\lambda(\cdot))}} \frac{|\Omega(y - z) - \Omega(y)|^q}{|y|^{n-\beta}} dy \right)^{1/q} \leq C 2^{jB/q} \omega_q(2^{-j\sigma-2}).$$

Thus, we have

$$|R_{j, \beta}(f)(x)| \leq C \left[ 2^{-j\sigma + j\beta} M_\Omega(f)(x) + 2^{j\beta} \omega_q(2^{-j\sigma-2}) M_{q'}(f)(x) \right],$$

where $C$ is independent of the choice of $\lambda(\cdot)$. Since $p > q'$, using the $L^p$ boundedness of $M_\Omega$ and $M_{q'}$ again, we obtain

$$\left\| R^j_{\lambda}(\cdot, \beta)(f) \right\|_{L^p} = \left\| R_{j, \beta}(f) \right\|_{L^p} \leq C \left( 2^{-j\sigma + j\beta} + 2^{j\beta} \omega_q(2^{-j\sigma-2}) \right) \|f\|_{L^p}. \quad (2.23)$$

**Proof of Theorem 1.2**

From (2.21) and (2.23), we have the following estimates.

$$\left\| T^i_{\lambda}(\cdot, \beta)(f) \right\|_{L^p} \leq \left\| T^j_{\lambda}(\cdot, \beta)(f) \right\|_{L^p} + \left\| R^j_{\lambda}(\cdot, \beta)(f) \right\|_{L^p} \leq C \left( 2^{-j\theta} \|f\|_{L^p} + 2^{-j\sigma + j\beta} \|f\|_{L^p} + 2^{j\beta} \omega_q(2^{-j\sigma-2}) \|f\|_{L^p} \right).$$

By the fact that $\theta < 0$ and $\beta < \sigma$, we can easily get

$$\sum_{j \geq 1} 2^{-j\theta} \|f\|_{L^p} \leq C \|f\|_{L^p}, \quad (2.24)$$
and
\[ \sum_{j \geq 1} 2^{-j\sigma + j\beta} \| f \|_{L^p} \leq C \| f \|_{L^p}. \]  
\hfill (2.25)

Moreover, we have
\[ \sum_{j \geq 1} 2^{j\beta} \omega_q(2^{-j\sigma - 2}) \| f \|_{L^p} \]
\[ = (\ln 2)^{-1} \sum_{j \geq 1} \omega_q(2^{-j\sigma - 2}) 2^{j\beta} \int_{2^{-j\sigma - 2}}^{2^{-j\sigma - 1}} \frac{d\delta}{\delta} \| f \|_{L^p} \]
\[ \leq C \sum_{j \geq 1} 2^{j\beta} \int_{2^{-j\sigma - 2}}^{2^{-j\sigma - 1}} \frac{\omega_q(\delta)}{\delta} d\delta \| f \|_{L^p} \]
\leq C \sum_{j \geq 1} 2^{j\beta} 2^{-j\sigma} \int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta \| f \|_{L^p} \leq C \| f \|_{L^p}. \]  
\hfill (2.26)

Combining (2.2)-(2.3), (2.24)-(2.26), we finish the proof of Theorem 1.2.

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