Fault-Tolerant Quantum Computation via Exchange interactions

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Quantum computation can be performed by encoding logical qubits into the states of two or more physical qubits, and controlling a single effective exchange interaction and possibly a global magnetic field. This “encoded universality” paradigm offers potential simplifications in quantum computer design since it does away with the need to perform single-qubit rotations. Here we show that encoded universality schemes can be combined with quantum error correction. In particular, we show explicitly how to perform fault-tolerant leakage correction, thus overcoming the main obstacle to fault-tolerant encoded universality.

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In the “standard paradigm” of quantum computing (QC) a universal set of quantum logic gates is encoded via the application of a complete set of single-qubit gates, along with a non-trivial (entangling) two-qubit gate $[1]$. It is in this context that the theory of fault tolerant quantum error correction (QEC) (e.g., $[2]$), and the well-known associated threshold results (e.g., $[3]$), have been developed. These results are of crucial importance since they establish the in-principle viability of QC, despite the adverse effects of decoherence and inherently inaccurate controls. However, some of the assumptions underpinning the standard paradigm may translate into severe technical difficulties in the laboratory implementation of QC, in particular in solid-state devices. Any quantum system comes equipped with a set of “naturally available” interactions, i.e., interactions which are inherent to the system and are determined by its symmetries, and are most easily controllable. For example, the symmetries of the Coulomb interaction dictate the special scalar form of the Heisenberg exchange interaction, which features in a number of promising solid-state QC proposals $[4]$. The introduction of single-spin operations requires a departure from this symmetry, and typically leads to complications, such as highly localized magnetic fields $[5]$, powerful microwave radiation that can cause excessive heating, or $g$-tensor engineering/modulation $[6]$. For these reasons the “encoded universality” (EU) alternative to the standard paradigm has been developed (e.g., $[7]$). In EU, single-qubit interactions with external control fields are replaced by “encoded” single-qubit operations, implemented on logical qubits via control of exchange interactions between their constituent physical qubits. It has been shown that such an exchange-only approach is also capable of universal QC, on the (decoherence-free) subspace spanned by the encoded qubits $[8]$. Explicit pulse sequences have been worked out for the implementation of encoded logic gates in the case when only the exchange interaction is available $[9, 10]$, which can be simplified by assuming the controllability of a global, time-dependent magnetic field $[11, 12]$.

The issue of the robustness of encoded universal QC in the presence of decoherence has been addressed in a number of publications, mostly using a combination of decoherence-free subspaces (DFSs) and dynamical decoupling methods $[10, 13]$. However, in contrast to the case of the standard paradigm, so far a theory of fault tolerant QEC has not yet been developed for encoded universal QC. The difficulty originates from the fact that EU constructions use only a subspace of the full system Hilbert space, and hence are subject to leakage errors to the orthogonal subspace. Standard QEC theory then breaks down under the restriction of using only a limited set of interactions, since these interactions are not universal over the orthogonal subspace, and cannot, using pre-established methods, be used to fix the leakage problem. Here we show for the first time how to extend the theory of fault tolerant QEC so as to encompass encoded universal QC. This establishes also the fault tolerance of a class of DFSs, for which prior fault tolerance results were of a heuristic nature $[14]$.

Encoded Universality.— We first briefly review the concept of EU in the context of a particularly simple encoding of one logical qubit into the states of two neighboring physical qubits: $|0_L\rangle_i = |0_{2i-1}\rangle \otimes |1_{2i}\rangle$, $|1_L\rangle_i = |1_{2i-1}\rangle \otimes |0_{2i}\rangle$, where $|0\rangle$ ($|1\rangle$) is the +1 (−1) eigenstate of the Pauli matrix $\sigma_z$. We shall refer to this encoding as a “two-qubit universal code” (2QUC), and more generally to EU encodings involving $n$ qubits per logical qubit as “$n$QUC”. In Ref. $[11]$ it was shown how to construct a universal set of encoded quantum logic gates for the 2QUC, generated from the widely applicable class of (effective or real) exchange Hamiltonian of the form $H_{ex} \equiv \sum_{i<j} H_{ij}$, where

$$H_{ij} = \sum_{i<j} J_{ij} (X_i X_j + Y_i Y_j) + J_{ij}^z Z_i Z_j,$$

(1)
Here \( X_i, Y_i, Z_i \) represent the Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \) acting on the \( i \)th physical qubit. The Heisenberg interaction is the case \( J_{ij} = J_{ij}^x \) (e.g., electron and nuclear spin qudits, [4]), while the XXZ and XY models are, respectively, the cases \( J_{ij} = J_{ij}^x \neq 0 \) (e.g., electrons on helium, [15]) and \( J_{ij} = J_{ij}^y \neq 0 \) (e.g., quantum dots in cavities, [10]). In essentially all pertinent QC proposals one can control the \( J_{ij} \) for \( |i - j| \leq 2 \), though not independently from \( J_{ij} \). As usual in the EU discussion we do not assume that the technically challenging single-qubit external operations of the form \( \sum C_i^n (t) X_i + f_i^n (t) Y_i \) are available. We do assume that a (global) free Hamiltonian \( H_0 = \sum_i \frac{1}{2} \omega_i Z_i \) with non-degenerate \( \omega_i \)'s can be exploited for QC in the sense that the \( \omega_i \) are collectively controllable, e.g., via the application of a global magnetic field. Note that \( X_{2i-1,2i} \) and \( Z_{2i-1,2i} \), where \( X_i \equiv \frac{1}{2} (X_iX_i + Y_i Y_i) \), \( Z_i \equiv \frac{1}{2} (Z_i - Z_i^\dagger) \), generate an su(2) algebra on the \( i \)th 2QC, while \( Z_{2i+1,2i+2} \) generates a controlled-phase (CP) gate between the \( i, i + 1 \)th 2QUCs. Here bars denote logical operations on the 2QC, so that, e.g., \( |0_{1k}⟩ \rangle \equiv \sum_{s=0}^1 X_s \otimes |0⟩ \rangle \otimes |1⟩ \rangle \). Given only the ability to control the \( J_{ij} \), explicit encoded logic gates can be derived using the identity

\[
C_i^0 \equiv \exp(-i\phi I_k) \equiv \exp(-i\phi I_k) \exp(i\phi I_k) = \exp[i\theta(I_i \cos \phi + I_j \sin \phi),]
\]

valid for su(2) generators satisfying the commutation relations \([I_i, I_j] = i_k \) (and cyclic permutations). E.g., an encoded CNOT gate over control (subscript \( C \), qubits 1, 2) and target (subscript \( T \), qubits 3, 4) 2QUCs can be constructed as follows for the XY model: \( C\text{NOT} = C_{\text{YY}} C^\dagger_{\text{XY}} C_{\text{YY}}, \) where \( C_{\text{YY}} = e^{-\frac{1}{2} e^{-i\frac{\pi}{4}}} e^{-\frac{1}{2} e^{-i\frac{\pi}{4}}} e^{-\frac{1}{2} e^{-i\frac{\pi}{4}}} \) is the encoded Hadamard gate,

\[
C_T = \frac{1}{2} \left( C_{\text{YY}} \circ C_{\text{YY}} \circ e^{-i\frac{\pi}{2} X_{12}} \right) e^{-i\frac{\pi}{2} X_{12}} e^{-i\frac{\pi}{2} X_{12}}
\]

is the encoded controlled-phase gate. For the Heisenberg and XXZ models one has

\[
e^{-itH_{2i-1,2i+1}} \equiv e^{-itH_{2i-1,2i+1}} C_{\text{XXZ}}(Z_{2i-1,2i+1}) \circ e^{-itH_{2i-1,2i+1}}
\]

which is equivalent to the \( C_T \) gate when \( t_{J_{2i-1,2i+1}} = \pi/4 \). Importantly, in all these cases universal encoded QC is possible via relaxed control assumptions, namely control of only the parameters \( J_{i,i+1} \) and a global magnetic field.

**Hybrid 2QUC-Stabilizer codes.**— Our solution for fault tolerant EU involves a concatenation of 2QUC and the method of stabilizer codes of QEC theory [2]. We define a hybrid \( n \)QUC-Stabilizer code (henceforth, “\( S-n \)QUC”) as the stabilizer code in which each physical qubit state \( |\psi⟩ = \alpha |0⟩ + \beta |1⟩ \) is replaced by the \( n \)QUC qubit state \( |\psi⟩ = \alpha |0⟩ + \beta |1⟩ \). With this replacement \( X_i \) for physical qubit \( i \) must be replaced by its encoded version \( \overline{X}_i \), and similarly for \( Y_i \) and \( Z_i \). Thus, physical-level operations on the stabilizer code are replaced by encoded-level operations on the 2QUC. This replacement rule also applies to give the new stabilizer for the \( S-n \)QUC. For example, suppose we concatenate the 2QUC with the three-qubit phase-flip code \(|+⟩^3, |−⟩^3\), where \(|±⟩ = (|0⟩ ± |1⟩)/\sqrt{2} \). The stabilizer of the latter is generated by \( X_1X_2, X_2X_3 \). Then the stabilizer for the hybrid S-2QUC \( |00⟩ = \frac{1}{\sqrt{2}} (|0⟩ + |1⟩)/\sqrt{2} \), \( |1⟩^3 = \frac{1}{\sqrt{2}} (|0⟩ - |1⟩)^3 \) is just \( S = \{ X_1X_2, X_2X_3 \} \), with \( \overline{X}_i = X_{2i-1,2i} \).

We will assume that it is possible to make measurements directly in the 2QUC basis. This involves, e.g., distinguishing a singlet \((|01⟩ + |10⟩)/\sqrt{2} \) or performing a non-demolition measurement of the first qubit in each 2QUC logical qubit; these tasks are currently under active investigation, e.g., [17]. In conjunction with the encoded universal gate set, it is then evidently possible to perform the entire repertoire of quantum operations needed to compute fault tolerantly on the 2QUC, using standard stabilizer-QEC methods [2]. Note that because the stabilizer code is, in our case, built from 2QUC qubits, it is a priori not designed to fix errors on the physical qubits. Thus, our next task is to consider these physical-level errors.

**Physical phase flips.**— Let \( C \) be a stabilizer code that can correct a single phase flip error, \( Z_i \), on any of the physical qubits. Therefore at least one of the generators of its stabilizer anticommutes with the error \( Z_i \). This implies that there is at least one stabilizer generator which includes the operator \( X_i \) or \( Y_i \). Consider the hybrid code \( C' \) resulting from concatenating \( C \) and an \( n \)QUC. The stabilizer of \( C' \) is found by replacing \( X_i, Y_i \) or \( Z_i \) by \( \overline{X}_i, \overline{Y}_i \) or \( \overline{Z}_i \), respectively. Therefore at least one of the generators of the stabilizer of \( C' \) includes the operator \( \overline{X}_i \) or \( \overline{Y}_i \), for all \( i \). In the case of a 2QUC we have \( \overline{X}_i = X_{2i-1,2i} \) and \( \overline{Y}_i = Y_{2i-1,2i} \), both of which anti-commute with \( Z_{2i-1,2i} \). Moreover, one readily verifies that arbitrary products of error operators anti-commute with at least one stabilizer generator, or have trivial effect. Therefore the corrigibility condition of errors on stabilizer codes [12] are satisfied, and hence a phase flip error on any physical qubit in a hybrid S-2QUC is always correctible.

**Physical bit flip.**— In contrast to physical-level phase flips, bit flips, \( \{ X_{2i-1,2i} = Y_{2i-1,2i,} X_{2i,2i} Y_{2i} \} \), cause leakage from the 2QUC subspace via transitions to the orthogonal, “leakage” subspace spanned by \( \{ |02i-1,02i⟩, |12i-1,12i⟩ \} \). The generators of the encoded su(2) on a 2QUC qubit, \( X_{2i-1,2i}, X_{2i-1,2i} \), annihilate this subspace, and hence will fail to produce the desired effect if used to implement standard QEC operations.

**Two-physical-qubit errors.**— Lastly we need to consider the case of two physical-level errors affecting two qubits of the same 2QC block (the case of two errors on two qubits in different 2QC blocks is already covered by the considerations above). Listing all possible such errors we find that (i) \( \{ XX = \overline{X}, XY = -\overline{Y}, YX = \overline{X}, YY = \overline{Y} \} \).
\( \{XY, YZ = -\mathbb{T}\} \) act as single encoded-qubit errors, and thus are correctible by the stabilizer QEC, and (ii) \( \{XZ, YZ, ZX, ZY\} \) all act as leakage errors. We conclude that our task is to find a way to solve the leakage problem by using only the available interactions. We do this in two steps: first we construct a "leakage correction unit" (LCU) assuming perfect pulses, then we consider fault tolerance in the presence of imperfections in the LCU and computational operations.

**Leakage correction unit.**— We assume that we can reliably prepare a 2QUC ancilla qubit in the state \( |0_L\rangle \). We now define an LCU as the unitary operator \( L \) whose action (up to phase) is:

\[
\begin{align*}
L |0_L\rangle |0_L\rangle & = |0_L\rangle |0_L\rangle \\
L |0_1 0_2\rangle |0_L\rangle & = |0_3 0_4\rangle \\
L |1_L\rangle |0_L\rangle & = |1_L\rangle |0_L\rangle \\
L |1_{12}\rangle |0_L\rangle & = |0_L\rangle |1_{14}\rangle
\end{align*}
\]

(5)

Here the first (second) qubit is the data (ancilla) qubit, and the action of \( L \) on the remaining 12 basis states is completely arbitrary. The LCU thus conditionally swaps a leaked data qubit with the ancilla, resetting the data qubit to \( |0_L\rangle \); this corresponds to a logical error on the data qubit, which can be fixed by the stabilizer code. Note that \( L \) entangles the data and ancilla qubits, which means that we can determine with certainty if a leakage correction has occurred or not by measuring the state of ancilla. We next show how to construct the transformation \( L \) from the available interactions. We decompose \( L \) in general as follows:

\[
L = \sqrt{SWAP} \times \sqrt{SWAP},
\]

where

\[
\sqrt{SWAP} = \exp[-i \pi/4 (X_{13} + X_{24})]
\]

(6)

\[
\sqrt{SWAP^\prime} = \exp[-i \pi/4 (X_{13}Z_{24} + X_{24}Z_{13})]
\]

(7)

and \( \exp[-i \pi/4 X_{ij}] \) is just the square-root of swap gate between physical qubits \( i \) and \( j \). The gate \( \sqrt{SWAP} \) applies this operation on qubits 1, 3 and 2, 4 in parallel. Depending on whether the eigenvalues of \( Z_2Z_4 \) and \( Z_1Z_3 \) are +1 or -1 on the four basis states of Eq. (5), the gates \( \sqrt{SWAP} \) and \( \sqrt{SWAP^\prime} \) multiply constructively (destructively) to generate a full swap identity.

**Circuits for the LCU.**— Eq. (7) involves four-body spin interactions. We next show how to construct these from available two-body interactions. For systems with XY-type of exchange interactions [16] the \( \sqrt{SWAP} \) gate consumes a single pulse. A circuit for performing \( \sqrt{SWAP} \) is given in Fig. 1.

For the class of Heisenberg systems [4], and for XXZ-type systems [13], we refocus the Ising term \( J_{ij}Z_iZ_j \), and use the following identity:

\[
\sqrt{SWAP^\prime} = \{C_{Z_2Z_3}^{\pi/4} \circ C_{X_{12}}^{\pi/2} \circ \exp[-i \pi X_{14}/4]\} \times \{C_{Z_1Z_4}^{\pi/4} \circ C_{X_{12}}^{\pi/2} \circ \exp[-i \pi X_{23}/4]\}
\]

(8)

To generate \( X_{ij} \) and \( Z_iZ_j \) we use [recall Eq. (1)]

\[
e^{-i t H_{ij}/2} C_{Z_i}^{\pi/2} \circ e^{i \pi t H_{ij}/2} = e^{-i t J_{ij}} \overline{X_{ij}} \rangle = e^{-i t J_{ij} Z_i Z_j} \langle (c)
\]

(9)

which is an example of recoupling [11]. The \( Z_i \) pulses required for this can, in turn, be generated as follows:

\[
e^{i t \sum_i \frac{1}{2} \omega_i Z_i} C_{Z_i}^{\pi/4} \circ e^{-i t \sum_i \frac{1}{2} \omega_i Z_i} = e^{i \frac{\pi}{4} t \Delta_{ik}} Z_k e^{i \frac{\pi}{4} t \Delta_{ik} Z_i}
\]

(10)

where \( i, k \in \{1, ..., 6\} \) and \( \Delta_{ik} = \omega_i - \omega_k \). By adjusting the time so that \( t \Delta_{ik} \equiv \pi \) and inserting Eq. (10) into Eq. (9), we generate the pulses \( \exp[i \pi Z_i/2] \) required in the conjugation step of Eq. (9), since the action on qubit \( k \) cancels out. Note that all spins not participating in the exchange interaction are unaffected by the procedure of Eq. (10). For all types of exchange interactions we have checked that the \( \sqrt{SWAP} \) can be also performed using only the two physical ancilla qubits 3, 4, with the same number of physical pulses, by sacrificing to some degree the possibility of parallel operations within each LCU. In all cases the time required for realizing the LCU is, to within a factor of two, equal to that for performing a single \( CNOT \). We note that a non-unitary QEC leakage detection circuit was described in Ref. 2.

Unfortunately, this method is not in general applicable to \( n \)QUCs, since the required logic gates operate over the full system Hilbert space. Constrains for unitary leakage-correcting operations, similar to our LCU, were derived in Ref. 10 for the 3QUC and Heisenberg-only computation, but no explicit circuit was given there.

**Fault-tolerant computation on the S-2QUC.**— So far we have assumed perfect gates. We now relax this assumption. Fault-tolerant computation is defined as a procedure in which if any component of a circuit fails to operate, at most one error appears in each encoded-block qubit [1, 2]. For a specific component to be fault-tolerant, the probability of error per operation should be below a certain threshold [3]. Transversal quantum
operations, such as the normalizer elements CNOT, phase, and Hadamard (W), are those which can be implemented in pairwise fashion over physical qubits. This ensures that an error from an encoded block of qubits cannot spread into more than one physical qubit in another encoded block of qubit \[^1 \, 2\]. Transversal operations become automatically fault-tolerant. In order to construct a universal fault-tolerant set of gates we should in addition be able to implement, e.g., a fault-tolerant encoded π/8 gate; although this gate is not transversal it can be realized by performing fault-tolerant measurements \[^1\]. Let us denote by a double bar encoded gates that act on the S-nQUC. It is easy to see that \(\overline{\text{CNOT}}\) and \(\overline{\text{T}}\) can be implemented transversally using EU operations as above. Moreover, by inspection of Ref. \[^1\] it is easy to see that all operations needed to construct the π/8 gate, in particular fault tolerant measurements and cat state preparation, can be done in the 2QUC basis, without any modification, as long as one can measure directly in the 2QUC basis (as discussed above). Hence, with respect to logical errors on the 2QUC qubits, the hybrid S-nQUC preserves all the required fault-tolerance properties.

This leaves physical-level phase and bit flip errors during encoded logic gates. We already showed that phase flip errors act as logical errors that the stabilizer QEC can correct. Bit flip errors are more problematic: a single leakage error invalidates the stabilizer code block in which it occurs, since the QEC procedures are ineffective in the leakage subspace. Hence if such errors were to propagate during a logic operation such as \(\overline{\text{CNOT}}\), they would – if left uncorrected – overwhelm the stabilizer level and result in catastrophic failure. We have verified that leakage errors propagate as either: (i) single physical-level leakage errors, remaining localized on the same qubit, in the case of an error taking place before or between the unitary transformations that make up an encoded logic gate \[^1 \, 3\]: (ii) as two-qubit leakage errors if a single-qubit leakage error happened during the latter transformations. In any case, the solution is to invoke the LCU after each logic operation, and before the QEC circuitry. The LCU turns a leakage error into a logical error, after which multilevel concatenated QEC \[^1 \, 2\] can correct these errors with arbitrary accuracy. However, uncontrolled leakage error propagation during QEC syndrome measurements must be avoided by inserting LCUs on each 2QUC after the cat-state preparation and before the verification step.

The final possibility we must contend with are leakage errors taking place during the operation of the LCU itself. Such a faulty LCU could incorrectly change the state of the ancilla qubit in Eq. \[^4\]. Therefore finding the ancilla in either |00\> or |11\> is an inconclusive result. Now let \(p_s\) be the probability of success of the LCU operation in one trail (this depends on accurate gating of the interaction Hamiltonian, etc.). Let \(\omega = \text{Tr}(\rho f |0_L\> \langle 0_L|)\) be the probability of finding the ancilla-qubit in the final state |0_L\>, where \(\rho f\) represents the final entangled state of data-qubit and ancilla (\(\omega\) critically depends on the quantum channel error model). The probability, \(p_c\), of achieving conclusive and correct information about the state of the data-qubit (being in the logical subspace) is \(p_c = (\omega \wedge p_s)/\omega\). This is the conditional probability of LCU success when we already know that the ancilla is in state |0_L\>. Then \(1 - p_c\) is the probability of achieving a conclusive but wrong result. We can arbitrarily boost the success probability of the LCU+measurement, \(1 - (1 - p_c)^n\), to be higher than some constant \(c_o\), by repeating this procedure until we obtain \(n \geq \log_{1 - p_c}(1 - c_o)\) consecutive no-leakage events.

**Conclusion.**— We have presented a theory of fault-tolerant QC for systems governed by XY, XXZ or Heisenberg exchange interactions, operated without single-qubit gates. In doing so, the theories of QEC and EU were reconciled for the first time by introducing a type of hybrid EU-stabilizer code. Leakage out of the EU code space was identified as the key problem and solved here using a fully constructive approach, within the EU framework of utilizing only the system’s intrinsic interactions. Many elements of this theory can be directly generalized to other quantum systems with a known set of experimentally available Hamiltonians. These results confirm the viability of the EU paradigm, with its associated advantages of reduced quantum control constraints and improved experimental compatibility to the interactions that are naturally available in a given quantum system. Moreover, by constructing error correction operations from a Hamiltonian formulation, rather than from gates as the elementary building blocks, a more accurate and reliable calculation of the fault-tolerance threshold is possible than in previous approaches. This will be undertaken in a future publication.

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[18] To see this consider a generic unitary transformation
\[ G_{ij} \in \{ H_{ij} = J_{ij} (X_i X_j + Y_i Y_j) + J_{ij}^z Z_i Z_j, Z_{ij} = (Z_i - Z_j)/2 \}, \]
and a single qubit errors \( E_i \in \{ X_i, Z_i \} \). Then,
using \( U \exp(A)U^\dagger = \exp(UAU^\dagger) \) for unitary \( U \) we can commute \( E_i \) to the left while flipping signs in \( G_{ij} \) appropriately [e.g., \( H_{ij} X_i = X_i (J_{ij} (X_i X_j - Y_i Y_j) - J_{ij}^z Z_i Z_j) \)].
The transformations with flipped sign combine to give a faulty logic gate on the 2QUC qubits, which is followed by the same \( E_i \) error.