Doubly transitive lines I: Higman pairs and roux

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Abstract

We study lines through the origin of finite-dimensional complex vector spaces that enjoy a doubly transitive automorphism group. In doing so, we make fundamental connections with both discrete geometry and algebraic combinatorics. In particular, we show that doubly transitive lines are necessarily optimal packings in complex projective space, and we introduce a fruitful generalization of abelian distance-regular antipodal covers of the complete graph.

1 Introduction

Given a sequence \( \mathcal{L} \) of lines through the origin of \( \mathbb{C}^d \), we consider all unitary operators that permute these lines, and we refer to such permutations as automorphisms of \( \mathcal{L} \). We are interested in doubly transitive lines, that is, lines that enjoy a doubly transitive automorphism group. (Recall that a group action \( G \leq \text{Sym}(X) \) is doubly transitive if for every \( x_1, x_2, y_1, y_2 \in X \) with \( x_1 \neq x_2 \) and \( y_1 \neq y_2 \), there exists \( g \in G \) such that \( g \cdot x_1 = y_1 \) and \( g \cdot x_2 = y_2 \).) This paper is the first in a series that studies doubly transitive lines. One of the key observations in this paper is a surprising connection with a fundamental problem in discrete geometry. Consider the task of packing lines through the origin so that the minimum distance between any two is as large as possible. Given unit-norm representatives \( \varphi_i \in \ell_i \) for \( i \in [n] := \{1, \ldots, n\} \) of lines \( \mathcal{L} = \{\ell_i\}_{i \in [n]} \), then the coherence of the sequence \( \Phi = \{\varphi_i\}_{i \in [n]} \) is defined by

\[
\mu(\Phi) := \max_{i,j \in [n]} |\langle \varphi_i, \varphi_j \rangle|.
\]

Sequences of unit vectors that minimize coherence find applications in compressed sensing [2], multiple description coding [50], digital fingerprinting [44], and quantum state tomography [46]. One popular lower bound on the coherence is the Welch bound [58], given by

\[
\mu(\Phi) \geq \sqrt{\frac{n - d}{d(n - 1)}}. \tag{1}
\]

Coherence achieves equality in the Welch bound precisely when the sequence of vectors form an equiangular tight frame (ETF) [50], meaning there exist \( \alpha > 0 \) and \( \beta \geq 0 \) such that

\[
\sum_{i \in [n]} \varphi_i \varphi_i^* = \alpha I, \quad |\langle \varphi_i, \varphi_j \rangle| = \beta \quad \forall i, j \in [n], \ i \neq j.
\]

Equivalently, the Gram matrix \( (\langle \varphi_j, \varphi_i \rangle)_{ij} \) is a scalar multiple of an orthogonal projection matrix whose off-diagonal entries all have the same modulus. In light of their applications, ETFs have

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received considerable attention over the last decade, resulting in constructions involving difference
sets \[50, 59, 14\], regular graphs \[57, 11, 18\], block designs \[22, 39, 17, 16, 15\], discrete geometry \[21,
19\], association schemes \[37, 36, 35\], and various computational methods \[51, 24, 30, 1\]; see \[20\]
for a living survey. In the next section, we prove that unit-norm representatives of
\(n > d\) doubly transitive lines that span \(\mathbb{C}^d\) necessarily form an ETF.

For real ETFs, the phases in the Gram matrix are discrete, lying in \(\{\pm 1\}\) instead of the entire
complex unit circle \(\mathbb{T}\), and this feature suggests a combinatorial description. In order to elaborate,
we need a few definitions: We say ETFs \(\{\varphi_i\}_{i \in [n]}\) and \(\{\psi_i\}_{i \in [n]}\) are switching equivalent if there
exists \(Q \in \mathrm{U}(d)\) and \(\{\omega_i\}_{i \in [n]}\) in \(\mathbb{T}\) such that \(\psi_i = \omega_i Q \varphi_i\) for every \(i \in [n]\). Switching equivalence classes of real ETFs are in one-to-one correspondence with combinatorial objects known as regular
two-graphs \[48\].

To facilitate the study of doubly transitive lines, the present paper introduces a complex ana-
logue of regular two-graphs called roux. The connection we draw between symmetry and roux
enjoys a historical precedent. Taylor \[53\] and Seidel \[49\] report that G. Higman introduced regular
two-graphs in 1970 while studying doubly transitive permutation groups, in particular the
action of \(\mathrm{Co}_3\) on 276 points. As a group theorist, Higman may have been motivated by Burnside’s
theorem \[6\], which indicates a correspondence between certain doubly transitive actions and finite
simple groups. Evidently, Higman suspected that regular two-graphs could provide a setting for
such groups, acting as their automorphisms. In a sequel paper, we will prove that doubly transi-
tive two-graphs correspond to doubly transitive real lines \[38\]. In this sense, Higman was actually
studying the automorphism groups of doubly transitive real lines when he naturally uncovered real
ETFs, in their guise as regular two-graphs.

In direct analogy, the authors discovered roux while studying doubly transitive lines in complex
space. In fact, roux generalize doubly transitive lines in the following sense: Any sequence of doubly
transitive lines has unit-norm line representatives whose Gram matrix carries a Schurian association
scheme satisfying a few axioms. Conversely, any Schurian association scheme satisfying these axioms
produces doubly transitive lines through its primitive idempotents. By generalizing these axioms to
non-Schurian schemes, we produce the notion of roux. Even in the non-Schurian case, the primitive
idempotents describe equiangular lines (indeed, ETFs) in complex space. The resulting class of
roux lines includes all doubly transitive lines, all lines arising from abelian DRACKNS \[28, 11\], and
all real lines corresponding to ETFs. See Figure 1 for an illustration of these relationships.

In the following section, we discuss our main results with illustrative examples, and we review
the preliminaries needed to read the remainder of the paper. In particular, this section contains
the definitions of Higman pairs and roux. Sections 3 and 4 then study the association schemes and
lines that arise from roux, respectively. Section 5 establishes how roux generalize abelian DRACKNS,
and Section 6 leverages this generalization to resolve open questions about abelian DRACKNS. We
conclude in Section 7 with the proofs of our main results.

2 Preliminaries and main results

We start with an example of four lines in \(\mathbb{C}^2\). Recall that lines in \(\mathbb{C}^2\) correspond to points in complex
projective space \(\mathbb{CP}^1\), which is isometrically isomorphic to the unit sphere \(S^2\) (this correspondence
is known as the Bloch sphere \[45\] in quantum mechanics). As such, we expect symmetric collections
of lines through the origin to correspond to symmetric collections of points in the sphere. Since we
want four lines in \(\mathbb{C}^2\), we are naturally drawn to the vertices of a regular tetrahedron circumscribed
by \(S^2\). In fact, these lines are doubly transitive: The action of \(\mathrm{U}(2)\) on \(\mathbb{CP}^1\) corresponds to \(\mathrm{SO}(3)\)
acting on \(S^2\), and it is easy to convince oneself that \(\mathrm{SO}(3)\) acts doubly transitively on these vertices
Figure 1: Venn diagram of the primary objects in this paper. While equiangular tight frames (ETFs) are not lines, they correspond to lines in a natural way. Many of these containments are nontrivial, corresponding to results in this paper.

(evenly with the help of a four-sided die). Explicitly, the isomorphism we are leveraging is given by \( f: \varphi \mapsto \sqrt{2}(\varphi \varphi^* - \frac{1}{2}I) \), which maps unit vectors in \( \mathbb{C}^2 \) into the 3-dimensional real vector space of \( 2 \times 2 \) self-adjoint matrices with zero trace. This mapping interacts nicely with inner products:

\[
\langle f(\varphi), f(\psi)\rangle_{\text{HS}} = 2|\langle \varphi, \psi \rangle|^2 - 1.
\]

Consider unit-norm representatives of our doubly transitive lines, that is, \( \{\varphi_i\}_{i \in [4]} \) in \( \mathbb{C}^2 \) so that \( \{f(\varphi_i)\}_{i \in [4]} \) form the vertices of a regular tetrahedron. We can use the mapping \( f \) to show that \( \{\varphi_i\}_{i \in [4]} \) forms an equiangular tight frame for \( \mathbb{C}^2 \). First, the vertices sum to zero, and so

\[
\sum_{i \in [4]} \varphi_i \varphi_i^* = \sum_{i \in [4]} \left( \frac{1}{\sqrt{2}}f(\varphi_i) + \frac{1}{2}I \right) = 2I.
\]

Next, when \( i \neq j \), we have \( \langle f(\varphi_i), f(\varphi_j)\rangle_{\text{HS}} = -1/3 \) and so

\[
|\langle \varphi_i, \varphi_j \rangle|^2 = \frac{1}{2} \left( \langle f(\varphi_i), f(\varphi_j)\rangle_{\text{HS}} + 1 \right) = \frac{1}{3}.
\]

The fact that an ETF arose from highly symmetric lines is no coincidence (see also [12]):

**Lemma 2.1.** Given \( n \) doubly transitive lines with span \( \mathbb{C}^d \), select unit-norm representatives \( \{\varphi_i\}_{i \in [n]} \).

(a) There exists \( \beta \) such that \( |\langle \varphi_i, \varphi_j \rangle|^2 = \beta \) for every \( i, j \in [n] \) with \( i \neq j \).

(b) If \( n > d \), then there exists \( \alpha \) such that \( \sum_{i \in [n]} \varphi_i \varphi_i^* = \alpha I \).

We will prove this lemma shortly. First, we note that part (a) does not require \( n \) to be finite, and in fact, part (a) implies that \( n \) is finite; indeed, Gerzon’s bound [14] gives that \( n \) lines are equiangular only if \( n \leq d^2 \). For part (b), the requirement \( n > d \) is important: If \( n = d \), then given
an orthonormal basis \( \{e_i\}_{i \in [d]} \) for \( \mathbb{C}^d \), define \( s = \sum_{j \in [d]} e_j \) and \( \varphi_i = e_i + s \) for every \( i \in [n] \); the lines spanned by \( \{\varphi_i\}_{i \in [n]} \) are doubly transitive (in fact, the automorphism group is all of \( S_n \)), but

\[
\sum_{i \in [n]} \varphi_i \varphi_i^* = \sum_{i \in [n]} (e_i + s)(e_i + s)^* = \sum_{i \in [n]} e_i e_i^* + 3ss^* = I + 3ss^*,
\]

which has two distinct eigenvalues, unlike \( \alpha I \). Overall, we have that doubly transitive lines with \( n > d \) necessarily produce ETFs. (Recall that the previous example had \( n = 4 > 2 = d \).)

For the proof of Lemma 2.1, it is convenient to pass the notion of double transitivity to the unit-norm representatives. To this end, the **projective symmetry group** of \( \{\varphi_i\}_{i \in [n]} \) is the group of permutations \( \sigma \in S_n \) for which there exists \( Q \in U(d) \) and phases \( \{\omega_i\}_{i \in [n]} \) such that \( Q \varphi_i = \omega_i \varphi_{\sigma(i)} \) for every \( i \in [n] \). The automorphism group of a sequence of lines is identical to the projective symmetry group of any choice of unit-norm representatives.

**Proof of Lemma 2.1.** For (a), take \( a, b, a', b' \in [n] \) with \( a \neq b \) and \( a' \neq b' \). Then by double transitivity, there exists \( \sigma \) in the projective symmetry group of \( \{\varphi_i\}_{i \in [n]} \) that maps \( a \mapsto a' \) and \( b \mapsto b' \). Letting \( Q \) and \( \{\omega_i\}_{i \in [n]} \) denote the corresponding unitary and phases, this in turn implies

\[
|\langle \varphi_a, \varphi_b \rangle|^2 = |\langle Q \varphi_a, Q \varphi_b \rangle|^2 = |\langle \omega_a \varphi_{a'}, \omega_b \varphi_{b'} \rangle|^2 = |\langle \varphi_{a'}, \varphi_{b'} \rangle|^2.
\]

Since our choice for \( a, b, a', b' \in [n] \) was arbitrary, we may conclude equiangularity.

For (b), let \( G \) denote the Gram matrix of \( \{\varphi_i\}_{i \in [n]} \), whose \( (i, j) \)th entry is given by \( \langle \varphi_j, \varphi_i \rangle \). Then borrowing notation from the proof of (a), we have

\[
G_{ab} = \langle \varphi_b, \varphi_a \rangle = \langle Q \varphi_b, Q \varphi_a \rangle = \langle \omega_b \varphi_{a'}, \omega_a \varphi_{a'} \rangle = \overline{\omega_a} \omega_b \langle \varphi_{a'}, \varphi_{a'} \rangle = \overline{\omega_a} \omega_b G_{a'a'}
\]

and furthermore

\[
(G^2)_{ab} = \sum_{i \in [n]} \langle \varphi_i, \varphi_a \rangle \langle \varphi_b, \varphi_i \rangle = \sum_{i \in [n]} \langle Q \varphi_i, Q \varphi_a \rangle \langle Q \varphi_b, Q \varphi_i \rangle
\]

\[
= \sum_{i \in [n]} (\omega_i \varphi_{\sigma(i)}, \omega_a \varphi_{a'}) \langle \omega_b \varphi_{b'}, \omega_i \varphi_{\sigma(i)} \rangle = \overline{\omega_a} \omega_b \sum_{i \in [n]} \langle \varphi_{\sigma(i)}, \varphi_{a'} \rangle \langle \varphi_{b'}, \varphi_{\sigma(i)} \rangle = \overline{\omega_a} \omega_b (G^2)_{a'a'}.
\]

As such, the off-diagonal of \( G^2 \) is a constant multiple of the off-diagonal of \( G \). Moreover,

\[
(G^2)_{aa} = \sum_{i \in [n]} |\langle \varphi_i, \varphi_a \rangle|^2 = 1 + (n - 1)\beta = (1 + (n - 1)\beta) G_{aa}.
\]

Overall, \( G^2 = c_1 G + c_2 I \) for some \( c_1, c_2 \in \mathbb{C} \), and so every eigenvalue \( \lambda \) of \( G \) satisfies \( \lambda^2 = c_1 \lambda + c_2 \). Since \( n > d \) by assumption, \( G \) is rank-deficient, meaning \( \lambda = 0 \) is an eigenvalue of \( G \), and so \( c_2 = 0 \). As such, \( G \) is a scalar multiple of an orthogonal projection matrix, which gives the result.

The above proof exploits how ETFs are easily characterized in terms of the Gram matrix, i.e., it is equivalent for the Gram matrix to be a scalar multiple of an orthogonal projection matrix whose off-diagonal entries all have the same modulus. This characterization interacts nicely with the theory of association schemes \([30, 35]\), which we now review (see \([4, 8]\) for a complete treatment). An **association scheme** is a sequence \( \{A_i\}_{i \in [k]} \) in \( \mathbb{C}^{n \times n} \) with entries in \( \{0, 1\} \) such that

(A1) \( A_1 = I \),

(A2) \( \sum_{i \in [k]} A_i = J \) (the matrix of all ones), and
We refer to $\mathcal{A}$ as the adjacency algebra of $\{A_i\}_{i \in [k]}$. We say two association schemes are isomorphic if there exists a permutation matrix $P$ such that conjugating the adjacency matrices from one scheme by $P$ produces the adjacency matrices of the other scheme. An association scheme is said to be commutative if its adjacency algebra is commutative. In this case, the spectral theorem affords $\mathcal{A}$ with an alternative orthogonal basis of primitive idempotents, which can be combined to produce every orthogonal projection matrix in $\mathcal{A}$. As such, if a commutative association scheme’s $k$-dimensional matrix algebra contains the Gram matrix of an ETF, then it can be obtained by searching through all $2^k$ combinations of the primitive idempotents. This correspondence between association schemes and desirable Gram matrices dates back to Delsarte, Goethals and Seidel [13], who coined the following phrase: We say a matrix $M$ carries the association scheme $\{A_i\}_{i \in [k]}$ if $M = \sum_{i \in [k]} c_i A_i$ with $\{c_i\}_{i \in [k]}$ distinct (in words, the $A_i$’s indicate “level sets” of $M$).

A scheme is called thin if all of its adjacency matrices are permutation matrices, in which case the scheme is a permutation representation of a group $G$, and its adjacency algebra is isomorphic to the group ring $\mathbb{C}[G]$. For example, the Cayley representation of the cyclic group $C_n$ produces a commutative association scheme of translation matrices whose adjacency algebra is the set of $n \times n$ circulant matrices. For any association scheme, the adjacency matrices that are permutation matrices form a group known as the thin radical.

Since we are interested in doubly transitive lines, we expect unit-norm representatives of these lines to have a Gram matrix that exhibits additional algebraic structure. Given a group $G$ acting transitively on a set $X$, we may consider the $\ast$-algebra of $G$-stable matrices, that is, matrices $M \in \mathbb{C}^{X \times X}$ satisfying $M_{g,x,y} = M_{x,y}$ for every $x,y \in X$ and $g \in G$. Almost every member of this algebra carries an underlying association scheme, known as a Schurian scheme. To express the scheme’s adjacency matrices, fix a point $x_0 \in X$ and let $H$ denote the stabilizer of $x_0$ in $G$. Since $G$ acts transitively on $X$, we may identify $X$ with $G/H$, as $g \cdot x_0$ corresponds to $gH$. The group $G$ can be partitioned into double cosets in $H \backslash G/H$, defined by

$$HaH := \{hah' : h, h' \in H\},$$

and each double coset can be further partitioned into left cosets. These double cosets determine the adjacency matrices for the adjacency algebra $\mathcal{A}(G,H)$ of $G$-stable matrices with indices in $G/H$:

$$(A_{HaH})_{xH,yH} = \begin{cases} 1 & \text{if } y^{-1}x \in HaH; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We say $(G,H)$ is a Gelfand pair if the $\ast$-algebra $\mathcal{A}(G,H)$ is commutative.

Throughout, it will be convenient to exploit other algebras that are isomorphic to $\mathcal{A}(G,H)$. For example, consider the space

$$L^2(H \backslash G/H) := \{f : G \to \mathbb{C} : f(gh) = f(hg) = f(g) \text{ for every } g \in G, h \in H\}$$

of bi-$H$-invariant functions on $G$. Equivalently, these are complex-valued functions over $G$ that are constant on double cosets of $H$. This vector space is a $\ast$-algebra with convolution and involution:

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1}g), \quad f^\ast(g) = \overline{f(g^{-1})}, \quad (g \in G).$$

Furthermore, the mapping $\phi : \mathcal{A}(G,H) \to L^2(H \backslash G/H)$ defined by $(\phi(M))(g) = \frac{1}{|H|} M_{gH,H}$ is a $\ast$-algebra anti-isomorphism; here, “anti” indicates that the mapping switches the order of multiplication: $\phi(AB) = \phi(B) * \phi(A)$. Next, $L^2(H \backslash G/H)$ embeds into the group ring by $\theta : L^2(H \backslash G/H) \to$
\[\mathbb{C}[G] \text{ defined by } \theta(f) = \sum_{g \in G} f(g)g. \text{ The range of } \theta \text{ is }\]
\[\mathbb{C}[H\backslash G/H] := \left\{ \sum_{S \in H\backslash G/H} \sum_{g \in S} c_{g}Sg : c_{S} \in \mathbb{C} \text{ for every } S \in H\backslash G/H \right\}.\]

In particular, \(\mathbb{C}[H\backslash G/H]\) is also a \(*\)-algebra with the usual group ring multiplication and with involution defined by
\[\left( \sum_{S \in H\backslash G/H} \sum_{g \in S} c_{g}Sg \right)^* = \sum_{S \in H\backslash G/H} \sum_{g \in S} c_{S}g^{-1}.\]

As such, \(\theta\) is a \(*\)-algebra isomorphism. To summarize, we have two \(*\)-algebra (anti-) isomorphisms available for our use:
\[\mathcal{A}(G,H) \xrightarrow{\phi} L^{2}(H\backslash G/H) \xrightarrow{\theta} \mathbb{C}[H\backslash G/H]. \quad (3)\]

Now that we have reviewed the role that groups play in certain association schemes, we introduce pairs of groups that will be particularly relevant to the study of doubly transitive lines. We name the following object after a pair of mathematicians, namely, Graham Higman and Donald G. Higman, who are known for their contributions to the theory of groups, two-graphs, and association schemes \cite{Higman10, Higman3}.

**Definition 2.2.** Given a finite group \(G\) and a proper subgroup \(H \leq G\), let \(K = N_{G}(H)\) be the normalizer of \(H\) in \(G\). We say \((G,H)\) is a **Higman pair** if there exists a key \(b \in G \setminus K\) such that

- (H1) \(G\) acts doubly transitively on \(G/K\),
- (H2) \(K/H\) is abelian,
- (H3) \(HbH = Hb^{-1}H\),
- (H4) \(aba^{-1} \in HbH\) for every \(a \in K\), and
- (H5) \(a \in K\) satisfies \(ab \in HbH\) only if \(a \in H\).

As an example, consider the isomorphism \(\beta : \mathbb{F}_{3}^{4} \rightarrow C_{2}\) and take
\[G := \text{SL}(2,3) \times C_{4}, \quad H := \{([x\ y], \beta(x)) : x, y \in \mathbb{F}_{3}, x \neq 0\}\].

It turns out that \((G,H)\) is a Higman pair with
\[K = \{([x\ 0], z) : x, y \in \mathbb{F}_{3}, x \neq 0, z \in C_{4}\}, \quad b = ([0\ 1\ 1\ 0], i)\].

(Here and throughout, we view \(C_{r}\) as the subgroup of \(\mathbb{C}\) comprised of \(r\)th roots of unity, and we denote \(i = \sqrt{-1}\).) We will only verify (H1) here, as the proofs of \(K = N_{G}(H)\) and (H2)–(H5) are short and unenlightening. Since \(\text{SL}(2,3)\) permutes the set \(X\) of one-dimensional subspaces of \(\mathbb{F}_{3}^{2}\), we may let \(G\) act on \(X\) by setting \((g, z) \cdot x = g \cdot x\). Then since \(\text{SL}(2,3)\) acts doubly transitively, \(G\) does, as well. Now observe that \(K\) is the stabilizer of the line through \([1,0,0]^{T}\), meaning the action of \(G\) on \(G/K\) is equivalent to that on \(X\). This gives (H1).

With the help of GAP \cite{GAP3, GAP4}, one can show that the algebra \(\mathcal{A}(G,H)\) has a basis of eight \(16 \times 16\) adjacency matrices: four of the form \(D^{j}\) and four of the form \(D^{j}A\), where
\[
D = \begin{bmatrix}
T & \cdots & . \\
. & T & \cdots \\
\vdots & \ddots & \ddots \\
. & . & T
\end{bmatrix},
A = \begin{bmatrix}
T^{-1} & T & T \\
T & T^{-1} & T \\
T^{-1} & T & T^{-1}
\end{bmatrix},
T = \begin{bmatrix}
. & . & . & 1 \\
1 & . & . & . \\
. & 1 & . & . \\
. & . & 1 & .
\end{bmatrix}.
\]
Here, dots denote zeros and $T$ is the Cayley representation of $i \in C_4$. It is straightforward to verify that $(G, H)$ is a Gelfand pair, and so $\mathcal{A}(G, H)$ contains eight primitive idempotents. While this determines $2^8 = 256$ different orthogonal projection matrices, it turns out that in this case, the primitive idempotents already yield interesting Gram matrices. Of these, two have rank 1, four have rank 2, and the remaining two have rank 3. One of the rank-2 idempotents is given below:

$$P = \frac{1}{8\sqrt{3}} \begin{pmatrix}
\sqrt{3} & -\frac{1}{\sqrt{3}} & \sqrt{3} & -\frac{1}{\sqrt{3}} & 1 & 1 & -1 & -1 \\
-\sqrt{3} & \frac{1}{\sqrt{3}} & -\sqrt{3} & \frac{1}{\sqrt{3}} & -1 & 1 & 1 & -1 \\
\sqrt{3} & -\frac{1}{\sqrt{3}} & \sqrt{3} & -\frac{1}{\sqrt{3}} & -1 & 1 & 1 & -1 \\
-\sqrt{3} & \frac{1}{\sqrt{3}} & -\sqrt{3} & \frac{1}{\sqrt{3}} & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1
\end{pmatrix}.$$

(We will make use of the highlighted entries later.) Multiplying $P$ by 8 gives the Gram matrix of four unit-norm representatives from each of four lines, and furthermore, any vectors $\varphi_i$ and $\varphi_j$ representing different lines satisfy $|\langle \varphi_i, \varphi_j \rangle|^2 = 1/3$. Indeed, selecting a single representative from each line produces an ETF of four vectors in $\mathbb{C}^2$, i.e., the ETF discussed at the beginning of this section. In particular, these lines are doubly transitive. This correspondence between Higman pairs and doubly transitive lines occurs in general:

**Theorem 2.3 (Higman Pair Theorem).**

(a) Given $n > d$ doubly transitive lines that span $\mathbb{C}^d$, there exists $r$ such that one may select $r$ equal-norm representatives from each of the $n$ lines that together carry the association scheme of a Higman pair $(G, H)$ with $r = [N_G(H) : H]$ and $n = [G : N_G(H)]$. Moreover, their Gram matrix is a primitive idempotent for this scheme.

(b) Every Higman pair $(G, H)$ is a Gelfand pair. Every primitive idempotent of its association scheme is the Gram matrix of $r = [N_G(H) : H]$ equal-norm representatives from each of $n = [G : N_G(H)]$ doubly transitive lines that span $\mathbb{C}^d$ with $d < n$, and the phase of each entry is an $r$th root of unity.

As in the case of four lines in $\mathbb{C}^2$, doubly transitive lines always exhibit the remarkable feature that, for some finite $r$, one may select $r$ unit-norm representatives from each line in such a way that the phase of every inner product is an $r$th root of unity. Next, we note that the block form of $D$ and $A$ above suggest that we embed $\mathcal{A}(G, H)$ as a subalgebra of $\mathbb{C}[C_4]^{4 \times 4}$. Under this mapping, $\{D^j\}_{j \in [4]}$ is sent to $\{\delta g I\}_{g \in C_4}$ and $\{D^j A\}_{j \in [4]}$ is sent to $\{\delta g B\}_{g \in C_4}$, where

$$B = \begin{bmatrix}
0 & \delta_1 & \delta_i & \delta_i \\
\delta_i & 0 & \delta_i & \delta_i \\
\delta_i & \delta_i & 0 & \delta_i \\
\delta_i & \delta_i & \delta_i & 0
\end{bmatrix}. \tag{4}$$

(Usually, we will identify a group element $g$ with the corresponding member $\delta g$ of the group ring, but since we think of $C_4$ as a subgroup of $\mathbb{C}$, we use delta notation here to make clear that $B$ lies in $\mathbb{C}[C_4]^{4 \times 4}$ instead of $\mathbb{C}^{4 \times 4}$.) Note that the embedding $\mathcal{A}(G, H) \rightarrow \mathbb{C}[C_4]^{4 \times 4}$ can be inverted by applying the Cayley representation (or more precisely, its linear extension to the group ring) to each matrix entry. It is convenient to formalize the role that $B$ plays here:
Definition 2.4. A roux for an abelian group $\Gamma$ is an $n \times n$ matrix $B$ with entries in $\mathbb{C}[\Gamma]$ such that each of the following holds simultaneously:

(R1) $B_{ii} = 0$ for every $i \in [n]$.

(R2) $B_{ij} \in \Gamma$ for every $i, j \in [n], i \neq j$.

(R3) $B_{ji} = (B_{ij})^{-1}$ for every $i, j \in [n], i \neq j$.

(R4) The matrices $\{gI\}_{g \in \Gamma}$ and $\{gB\}_{g \in \Gamma}$ span an algebra $\mathcal{A}(B)$.

Note that $\mathcal{A}(B)$ is necessarily a commutative $*$-algebra since

$$(gI)^* = g^{-1}I \in \mathcal{A}(B), \quad (gB)^* = g^{-1}B \in \mathcal{A}(B),$$

and the $gI$’s and $gB$’s all commute. Later, we will explain how roux generalize the theory of abelian distance-regular antipodal covers of the complete graph (abelian DRACKNs), as studied by Godsil and Hensel in [28]. Moreover, $n \times n$ roux for $\Gamma$ are in one-to-one correspondence with what D. G. Higman calls regular weights $w : [n] \times [n] \rightarrow \Gamma$ in [33] (the reader may check the details, which follow very quickly from Lemma 3.3 in the next section).

Given a roux for $\Gamma$, we may evaluate the roux at a character $\alpha$ of $\Gamma$. Specifically, $\alpha$ extends linearly to $\mathbb{C}[\Gamma]$, and its entrywise application amounts to a $*$-algebra homomorphism $\mathbb{C}[\Gamma]^{n \times n} \rightarrow \mathbb{C}^{n \times n}$. Evaluating a roux at a character produces a signature matrix, that is, a self-adjoint matrix with zeros on the diagonal and entries of unit modulus on the off-diagonal. For example, evaluating the above roux for $C_4$ at the character $\alpha$ defined by $\alpha(z) = z$ gives

$$S = \begin{bmatrix} 0 & i & i & i \\ -i & 0 & i & -i \\ -i & -i & 0 & i \\ -i & i & -i & 0 \end{bmatrix}.$$  \hspace{1cm} (5)

Every signature matrix $S$ is nonzero with zero trace, and so its minimum eigenvalue $\lambda$ is negative. Thus, $S - \lambda I$ is the Gram matrix of equal-norm vectors that span equiangular lines. Performing this move in our case gives $S + \sqrt{3}I$, which happens to be a principal submatrix of $P$ (if we ignore the factor of $1/(8\sqrt{3})$ in $P$), namely, the highlighted submatrix indexed by $\{1, 5, 9, 13\}$. Conversely, given equiangular lines, then for any choice of unit-norm representatives, the resulting Gram matrix has the form $I + \mu S$ for some signature matrix $S$. This correspondence between signature matrices and equiangular lines dates back to van Lint and Seidel [55]. We define roux lines to be any sequence of lines for which there exist equal-norm representatives whose signature matrix can be obtained by evaluating a roux at a character.

Theorem 2.5.

(a) All doubly transitive lines are roux.

(b) Every roux line sequence has unit-norm representatives that form an equiangular tight frame for their span.

This gives an alternate explanation for Lemma 2.1 (the proof factors through the combinatorial notion of roux). Overall, our main results are Theorems 2.3 and 2.5. The proofs of Theorem 2.3(b) and Theorem 2.5 appear in Section 7. We save the proof of Theorem 2.3(a) for our sequel paper [38], as the techniques in this proof are very different from the themes in this paper, and furthermore, the proof presents our method for classifying doubly transitive lines, which we also perform in [38].
3 Roux schemes

Given a group $\Gamma$ of order $r$, let $\cdot : \mathbb{C}[\Gamma]^{n \times n} \to \mathbb{C}^{m \times m}$ denote the injective $*$-algebra homomorphism that applies the Cayley representation of $\Gamma$ (extended linearly to $\mathbb{C}[\Gamma]$) to each entry of the input matrix. Given a roux $B$ for an abelian group $\Gamma$, then $\{[gI]\}_{g \in \Gamma}$ and $\{[gB]\}_{g \in \Gamma}$ form the adjacency matrices of a commutative association scheme whose adjacency algebra is isomorphic to $\mathcal{A}(B)$; we refer to this as the corresponding roux scheme. (This is akin to the correspondence between regular weights and coherent configurations given in [17].) We say a scheme is roux if the points have labels in $[n] \times \Gamma$ such that there exists an $n \times n$ roux for $\Gamma$ that produces the scheme through this process. Roux schemes are important to the study of doubly transitive lines because they provide a combinatorial generalization of Higman pairs:

Theorem 3.1. Let $G$ be a finite group, and pick $H \leq G$. The Schurian scheme of $(G,H)$ is isomorphic to a roux scheme if and only if $(G,H)$ is a Higman pair.

We will prove Theorem 3.1 later. The following lemma indicates the origin of the name roux: it “thickens up” an otherwise thin scheme, imitating the role of roux in the culinary arts.

Lemma 3.2. An association scheme is isomorphic to a roux scheme if and only if it is commutative and its thin radical acts regularly (by multiplication) on the other adjacency matrices, at least one of which is symmetric.

Proof. ($\Rightarrow$) Suppose $B$ is a roux for $\Gamma$. Then $[B]$ is symmetric by (R3). Furthermore, the thin radical in the roux scheme is $\{[gI]\}_{g \in \Gamma}$, which acts regularly on the other matrices $\{[gB]\}_{g \in \Gamma}$ since $[gI][hB] = [ghB]$ for $g, h \in \Gamma$. The scheme is commutative since $\Gamma$ is abelian.

($\Leftarrow$) Let $m$ denote the size of the adjacency matrices, and pick $\Gamma \leq S_m$ so that the thin radical is given by the matrix representations $\{P_\sigma\}_{\sigma \in \Gamma}$. Select a symmetric adjacency matrix $A$ outside the thin radical. Since $\Gamma$ acts regularly on these other matrices, they are given by the orbit $\{P_\sigma A\}_{\sigma \in \Gamma}$. Since the scheme is commutative, $\Gamma$ is abelian. Next, $\Gamma$ acts on $[m]$, and we claim that the stabilizer of each point is trivial. To see this, fix $\sigma \in \Gamma$ with $\sigma \neq 1$ and $i \in [m]$. Then we may pick $j \in [m]$ so that $A_{ij} = 1$, in which case $A_{\sigma^{-1}(i),j} = (P_\sigma A)_{ij} = 0$, meaning $\sigma^{-1}(i) \neq i$. As such, the orbits of $[m]$ under $\Gamma$ all have size $r := |\Gamma|$. Put $n := m/r$.

For each orbit $O_i$, arbitrarily label one of the points $p \in O_i$ with $\ell(p) = 1 \in \Gamma$ and label the other points $q \in O_i$ with $\ell(q) = \sigma \in \Gamma$ such that $\sigma(p) = q$. Then we may rearrange $[m]$ by ordering the orbits $O_1, \ldots, O_n$, and ordering points within each orbit according to $\Gamma$. Conjugating $\{P_\sigma\}_{\sigma \in \Gamma} \cup \{P_\sigma A\}_{\sigma \in \Gamma}$ by this ordering produces matrices $\{[\sigma I]\}_{\sigma \in \Gamma} \cup \{[\sigma I]A'\}_{\sigma \in \Gamma}$. Finally, $A' = [\sigma I]A'[\sigma I]^{-1}$, and so $A'_{ij} = A'_{\sigma(i),\sigma(j)}$, meaning each $r \times r$ block of $A'$ is $\Gamma$-circulant, i.e., $A' = [B]$ for some $B \in \mathbb{C}[\Gamma]^{n \times n}$. By counting, each off-diagonal block of $A'$ is a permutation matrix, and so each off-diagonal entry of $B$ lies in $\Gamma$. Overall, the adjacency matrices are $\{[\sigma I]\}_{\sigma \in \Gamma} \cup \{[\sigma B]\}_{\sigma \in \Gamma}$, and since $B$ satisfies (R1)–(R4), we may conclude that the scheme is roux.

Next, we offer an alternative to (R4) that is often easier to work with in practice.

Lemma 3.3. Suppose $B \in \mathbb{C}[\Gamma]^{n \times n}$ satisfies (R1)–(R3). Then $B$ is a roux for $\Gamma$ if and only if

$$B^2 = (n - 1)I + \sum_{g \in \Gamma} c_g gB$$

for some complex numbers $\{c_g\}_{g \in \Gamma}$. In this case, we necessarily have that $\{c_g\}_{g \in \Gamma}$ are nonnegative integers that sum to $n - 2$, with $c_{g^{-1}} = c_g$ for every $g \in \Gamma$. 


Proof. (⇐) It suffices to demonstrate (R4). Note that \( \{gI\}_{g \in \Gamma} \) and \( \{gB\}_{g \in \Gamma} \) commute, and the assumption on \( B^2 \) gives that their span contains their pairwise products:

\[
(gI)(hI) = ghI, \quad (gI)(hB) = ghB, \quad (gB)(hB) = ghB^2. 
\]

Thus, their span is an algebra.

(⇒) The diagonal entries of \( B^2 \) are given by

\[
(B^2)_{ii} = \sum_{j \in [n]} B_{ij}B_{ji} = \sum_{\substack{j \in [n] \\ j \neq i}} B_{ij}(B_{ij})^{-1} = n - 1. \tag{6}
\]

As such, the diagonal component of \( B^2 \) is \((n - 1)I\). The off-diagonal component lies in the span of \( \{gB\}_{g \in \Gamma} \). Thus, we may write

\[
B^2 = (n - 1)I + \sum_{g \in \Gamma} c_g gB.
\]

For the final claim, we consider an off-diagonal entry of \( B^2 \):

\[
\sum_{h \in \Gamma} c_h(B_{ij})^{-1}h = \sum_{g \in \Gamma} c_g(gB)_{ij} = (B^2)_{ij} = \sum_{k \in [n]} B_{ik}B_{kj} = \sum_{\substack{k \in [n] \\ i \neq k \neq j}} B_{ik}B_{kj},
\]

The right-hand side is a sum of \( n - 2 \) (not necessarily distinct) members of \( \Gamma \), and so we conclude that \( \{c_g\}_{g \in \Gamma} \) are nonnegative integers that sum to \( n - 2 \). Furthermore, \( c_{g^{-1}} = c_g \) for every \( g \in \Gamma \) since \( B^2 \) is self-adjoint.

We will see that \( \{c_g\}_{g \in \Gamma} \) serve as fundamental parameters to the study of roux, and we refer to them as roux parameters. For example, the following result generalizes \( [4] \), but requires a definition: A conference matrix is an \( n \times n \) matrix \( M \) with zero diagonal and off-diagonal entries in \( \{\pm 1\} \) such that \( MM^\top = (n - 1)I \).

**Lemma 3.4.** Given an antisymmetric conference matrix \( M \), define \( B \in \mathbb{C}[C_4]^{n \times n} \) by

\[
B_{ij} = \begin{cases} 
0 & \text{if } M_{ij} = 0; \\
\delta_i & \text{if } M_{ij} = 1; \\
\delta_{-i} & \text{if } M_{ij} = -1.
\end{cases}
\]

Then \( B \) is a roux for \( C_4 \) with parameters \( c_{\pm 1} = 0 \) and \( c_{\pm i} = n/2 - 1 \).

**Proof.** First, (R1)–(R3) hold by definition. For (R4), we leverage Lemma 3.3. To this end, we first note that the diagonal of \( B^2 \) satisfies \( [5] \). Next, take any \( i \neq j \). Then

\[
0 = -(MM^\top)_{ij} = (M^2)_{ij} = \sum_{k \in [n] \setminus \{i,j\}} M_{ik}M_{kj},
\]

and since each term on the right-hand side lies in \( \{\pm 1\} \), we conclude that half of these terms (i.e., \( n/2 - 1 \) of them) are \( +1 \) and the other half are \( -1 \). As such,

\[
(B^2)_{ij} = \sum_{k \in [n] \setminus \{i,j\}} \delta_i M_{ik} \delta_i M_{kj} = \sum_{k \in [n] \setminus \{i,j\}} \delta_{-i M_{ik}} M_{kj} = \left(\frac{n}{2} - 1\right) (\delta_i + \delta_{-i}) B_{ij},
\]

implying \( B^2 = (n - 1)I + (n/2 - 1)\delta_i B + (n/2 - 1)\delta_{-i}B \), as desired. \( \square \)
**Lemma 3.5** (Basic roux transformations). Take any $n \times n$ roux $B$ with parameters $\{c_{ij}\}_{i,j \in \Gamma}$.

(a) Given a diagonal matrix $D \in \mathbb{C}[\Gamma]^{n \times n}$ with $D_{ii} \in \Gamma$ for every $i \in [n]$, then $DBD^{-1}$ is a roux for $\Gamma$ with parameters $\{c_{ij}\}_{i,j \in \Gamma}$.

(b) Given $h \in \Gamma$, then $hB$ is a roux for $\Gamma$ if and only if $h^2 = 1$. In that case, its roux parameters are $\{c_{ij}\}_{i,j \in \Gamma}$.

(c) Given a homomorphism $\varphi: \Gamma \to \Lambda$, extend $\varphi$ linearly to the group ring and apply entrywise to get $\tilde{\varphi}: \mathbb{C}[\Gamma]^{n \times n} \to \mathbb{C}[\Lambda]^{n \times n}$. Then $\tilde{\varphi}(B)$ is a roux for $\Lambda$ with parameters $\{\sum_{g \in \varphi^{-1}(\lambda)} c_{ij}\}_{i,j \in \Lambda}$, where the empty sum is taken to be zero.

(d) Given a group $\Lambda \geq \Gamma$, then $B$ is a roux for $\Lambda$ with parameters $\{c_{ij}\}_{i,j \in \Lambda}$, where $c_{ij} = c_{ij}^\prime$ if $\lambda \in \Gamma$ and $c_{ij}^\prime = 0$ otherwise.

These roux transformations suggest various invariants. Part (a) establishes “switching equivalence classes” of roux. Notice that a reasonable representative of each class takes $B_{i,i}$ and $B_{1,i}$ to be the identity element of $\Gamma$ for every $i \neq 1$. For (b), we note that $[B]$ and $[hB]$ are adjacency matrices in the same roux scheme, meaning each roux generates the same scheme, though with relations re-indexed. Given a roux scheme with a fixed indexing of the vertices, the $hB$’s in part (b) are the only roux that produce this scheme. While (d) explains how to view $B$ as a roux for a supergroup, Lemma 3.5 in the next section shows how to view $B$ as a roux for a subgroup (provided the roux parameters are zero on the complement of the subgroup).

**Proof of Lemma 3.5.** First (a) and (c) are straightforward, as is $(\Leftarrow)$ in (b). For $(\Rightarrow)$, pick $i$ and $j$ such that $i \neq j$. Then (R3) implies

$$h B_{ij} = A_{ij} = (A_{ji})^{-1} = (h B_{ji})^{-1} = h^{-1} (B_{ji})^{-1} = h^{-1} B_{ij}.$$  

Multiplying both sides by $h (B_{ij})^{-1}$ then gives $h^2 = 1$. Finally, (d) follows from (c) since the natural injection $\Gamma \to \Lambda$ is a homomorphism.

In order to prove $(\Leftarrow)$ in Theorem 3.1 we need a technical lemma regarding the structure of Higman pairs:

**Lemma 3.6.** Given a Higman pair $(G, H)$, denote $K = N_G(H)$, $n = [G : K]$ and $r = [K : H]$, and select any key $b \in G \setminus K$. Then

(a) $H$ has $2r$ double cosets in $G$: $r$ of the form $aH$, and $r$ of the form $HabH$ for some $a \in K$;

(b) for every $a \in K$, we have $HabH = HbaH$; and

(c) for every $a \in K$, we have $|HabH| = (n-1)|H|$.

**Proof.** For (a), (H1) implies that $G$ is a disjoint union of $K$ and $KbK$. Next, $K$ is covered by left cosets of $H$ in $K/H$ (these are double cosets of $H$ in $G$ since $K$ normalizes $H$), while $KbK$ is covered by sets of the form

$$(aH)b(a'H) = aa'(a'H) = aa'H(a')^{-1}bH \subseteq aa'HbH = Haa'bH \quad (a,a' \in K);$$

here, $(\ast)$ applies (H4). Since $G$ can be partitioned into double cosets of $H$, we conclude that every double coset of $H$ has the form $aH$ or $HabH$ for some $a \in K$. To count these double cosets, consider the action of $K/H$ on the double cosets defined by $aH \cdot HxH = HaxH$. There are two
orbits under this action: those of the form $aH$, and those of the form $HabH$. In particular, $H$ is the stabilizer of $H$, and (H5) gives that $H$ is also the stabilizer of $HbH$, meaning $K/H$ acts regularly on both orbits. This gives (a).

Next, we apply (H4) to get

$$HbaH = Haa^{-1}baH = aHa^{-1}baH \subseteq aHbH = HabH.$$ 

We obtain equality by counting: $|HbaH| = |HbHa| = |HbH| = |aHbH| = |HabH|$. This gives (b).

For (c), note that our proof of (b) gives that $|HabH| = |HbH|$ for every $a \in K$. It suffices to show that $|HbH| = (n - 1)|H|$. Recall that the double cosets of the form $aH$ cover $K$, whereas the double cosets of the form $HabH$ cover $KbK = G \setminus K$. By (a), we therefore have

$$r|HbH| = |KbK| = |G| - |K| = (n - 1)|K| = r(n - 1)|H|,$$

and division by $r$ gives the result. □

Lemma 3.7 (Roux from Higman pairs). Given a Higman pair $(G, H)$, denote $K = N_G(H)$ and $n = [G : K]$, and select any key $b \in G \setminus K$. Choose left coset representatives $\{x_j\}_{j \in [n]}$ for $K$ in $G$ and choose coset representatives $\{a_g\}_{g \in K/H}$ for $H$ in $K$. Define $B \in \mathbb{C}[K/H]^{n \times n}$ entrywise as follows: Given $i \neq j$, let $B_{ij}$ be the unique $g \in K/H$ for which $x_i^{-1}x_j \in Ha_gbH$, and set $B_{ii} = 0$. Then $B$ is a roux for $K/H$ with roux parameters $\{c_g\}_{g \in K/H}$ given by

$$c_g = \frac{n - 1}{|H|} \cdot |bHb^{-1} \cap Ha_gbH|.$$ 

Furthermore, the roux scheme generated by $B$ is isomorphic to the Schurian scheme of $(G, H)$.

Notice that a different choice of $x_j$'s produces a switching equivalent roux (as in Lemma 3.5(a)), whereas a different choice of $a_g$'s makes no change to $B$.

Proof of Lemma 3.7. We start by checking (R1)–(R4). First, $B$ satisfies (R1) and (R2) by definition. For (R3), pick $i \neq j$ and put $g = B_{ij}$. Then

$$x_j^{-1}x_i = (x_i^{-1}x_j)^{-1} \in (a_gHbH)^{-1} = HbH^{-1}Ha_g^{-1} = HbHa_g^{-1} = Hba_g^{-1}H = Ha_g^{-1}bH,$$

where (*) applies (H3) and (†) follows from Lemma 3.6(b). As such, we have $B_{ij} = g^{-1}$, implying (R3). To verify (R4), first observe that $[n] \times K/H$ is in bijection with $G/H$ through the mapping $(i, h) \mapsto x_i a_h H$. Let $\psi: \mathbb{C}^{([n] \times K/H) \times ([n] \times K/H)} \to \mathbb{C}^{G/H \times G/H}$ be the corresponding $*$-algebra isomorphism

$$\psi(M)_{x_i a_h H, x_j a_k H} = M_{(i, h), (j, k)} \quad (M \in \mathbb{C}^{([n] \times K/H) \times ([n] \times K/H)}, i, j \in [n], h, k \in K/H).$$

Pre-composing with $[-] : \mathbb{C}[K/H]^{n \times n} \to \mathbb{C}^{([n] \times K/H) \times ([n] \times K/H)}$ gives an injective $*$-algebra homomorphism of $\mathbb{C}[K/H]^{n \times n}$ into $\mathbb{C}^{G/H \times G/H}$. It is straightforward to verify that $\psi([gI]) = A_{a_gH}$ and $\psi([gB]) = A_{Ha_gbH}$ for all $g \in K/H$. Considering Lemma 3.6(a), these images span a $*$-subalgebra $\mathcal{A}(G, H)$ of $\mathbb{C}^{G/H \times G/H}$, and so $\{gI\}_{g \in K/H}$ and $\{gB\}_{g \in K/H}$ span a $*$-algebra in $\mathbb{C}[K/H]^{n \times n}$. This is (R4). We conclude that $B$ is a roux, and that the scheme it generates is isomorphic to the Schurian scheme of $(G, H)$.

It remains to compute the roux parameters $\{c_g\}_{g \in K/H}$. We accomplish this by computing $B^2$. To this end, we denote $\iota: \mathcal{A}^*(B) \to \mathcal{A}^*(G, H)$ for the $*$-algebra isomorphism

$$\iota(gI) = A_{a_gH}, \quad \iota(gB) = A_{Ha_gbH} \quad (g \in K/H),$$

and
as above. Recalling \ref{3}, it is convenient to perform much of this computation in an anti-isomorphic domain:

\[(\theta \circ \phi \circ \iota)(B) = \frac{1}{|H|} \sum_{x \in HbH} x \in \mathbb{C}[H \setminus G/H].\]

We have

\[(\theta \circ \phi \circ \iota)(B^2) = [(\theta \circ \phi \circ \iota)(B)]^2 = \frac{1}{|H|^2} \sum_{x_1 \in HbH} \sum_{x_2 \in HbH} x_1 \cdot x_2 = \frac{1}{|H|^2} \sum_{x_1 \in HbH} \sum_{x_2 \in Hb^{-1}H} x_1 \cdot x_2,

where the last step follows from (H3). Next, consider the action of \(H \times H\) on \(G\) defined by \((h_1, h_2) \cdot x = h_1 x h_2^{-1}\). The orbits of this action are the double cosets of \(H\) in \(G\). As such, we may continue with the help of the orbit-stabilizer theorem:

\[(\theta \circ \phi \circ \iota)(B^2) = \frac{|HbH|^2}{|H|^6} \sum_{h_1, h_2 \in H} h_1 b h_2 \sum_{h_3, h_4 \in H} h_3 b^{-1} h_4 = \frac{|HbH|^2}{|H|^5} \sum_{h_1, h_2, h_3 \in H} h_1 b h_2 b^{-1} h_3,

where the last step changes variables \((h_2, h_3, h_4) \mapsto (h_2, h_3)\). At this point, we observe that \(h_1 b h_2 b^{-1} h_3 = x\) precisely when \(h_1^{-1} x h_3^{-1} = b h_2^{-1}\), and so

\[(\theta \circ \phi \circ \iota)(B^2) = \frac{|HbH|^2}{|H|^5} \sum_{x \in G} \left| \left\{(h_1, h_2, h_3) \in H^3 : h_1 b h_2 b^{-1} h_3 = x \right\} \right| x

= \frac{|HbH|^2}{|H|^5} \sum_{x \in G} \left| \left\{(h_1, h_3) \in H^2 : h_1^{-1} x h_3^{-1} \in b H b^{-1} \right\} \right| x

= \frac{|HbH|^2}{|H|^5} \sum_{x \in G} \frac{|H|^2}{|HxH|} \cdot |b H b^{-1} \cap H x H| x,

where the normalization in the last step follows from the orbit-stabilizer theorem, as before. We now apply \((\theta \circ \phi \circ \iota)^{-1}\) to both sides to get

\[B^2 = \frac{|HbH|^2}{|H|^4} \left( \sum_{g \in K/H} \frac{|H|^2}{|a g H|} \cdot |b H b^{-1} \cap a g H| g + \sum_{g \in K/H} \frac{|H|^2}{|H a g b H|} \cdot |b H b^{-1} \cap H a g b H| g B \right)

= (n - 1) I + \sum_{g \in K/H} \frac{n - 1}{|H|} \cdot |b H b^{-1} \cap H a g b H| g B,

where the final simplification follows from \ref{3} and Lemma \ref{36}(c).

\textit{Proof of Theorem \ref{7,7}.} It suffices to prove \((\Rightarrow)\), since \((\Leftarrow)\) follows immediately from Lemma \ref{3.7}. Suppose the Schurian scheme of \((G, H)\) is isomorphic to a rox scheme. By Lemma \ref{3.2}, we equivalently have that the scheme is commutative and its thin radical acts regularly on the other adjacency matrices, at least one of which is symmetric.

First, it is straightforward to show that the thin radical is comprised of the adjacency matrices \(\{A_a H\}_{a \in K/H}\) defined in \ref{2}, where \(K = N_G(H)\). Before proceeding, we make a general observation:

\[A_a H \cdot A_{x H} = A_{a x H} \text{ for every } a \in K, \; x \in G. \tag{7}\]

To see this, note that \(\phi(A_a H) = \frac{1}{|H|} \cdot 1_{a H}\) and \(\phi(A_{x H}) = \frac{1}{|H|} \cdot 1_{H x H}\), and so

\[\phi(A_a H \cdot A_{x H}))(y) = \frac{1}{|H|^2} (1_{H x H} \star 1_{a H})(y) = \frac{1}{|H|^2} \sum_{z \in G} 1_{H x H}(z) 1_{a H}(z^{-1} y).\]
Next, change variables \( u = y^{-1}az \), and observe that since \( a \in K \), we have \( u \in H \) if and only if \( z^{-1}y = z^{-1}yu^{-1} = au^{-1} = aH \), if and only if \( 1_{aH}(z^{-1}y) = 1 \):

\[
(\phi(A_{aH}A_{HxH}))(y) = \frac{1}{|H|^2} \sum_{u \in H} 1_{HxH}(a^{-1}yu) = \frac{1}{|H|} \cdot 1_{HxH}(a^{-1}y) = (\phi(A_{HaH}))(y).
\]

By considering the case \( x \in K \) in (7), we have that the thin radical is isomorphic to \( K/H \). Since the scheme is commutative by assumption, this group is abelian, and so we have (H2). We also have that one of the adjacency matrices \( A_{HbH} \) is symmetric by assumption, which is equivalent to \( HbH = Hb^{-1}H \), namely, (H3). Next, take any \( a \in K \). Then (7) and commutativity together give

\[
A_{Ha}A_{bH} = A_{a}A_{b} = (A_{a^{-1}H}A_{b^{-1}H})^\top = A_{H^{-1}a^{-1}b^{-1}H} = A_{HaH}.
\]

As such, \( HabH = HbaH \), with which we establish (H4). For (H5), take any \( a \in K \) such that \( ab \in HbH \). Then (7) gives

\[
A_{Ha}A_{bH} = A_{HaH} = A_{HbH}.
\]

Recall that by assumption, the thin radical acts regularly on the other adjacency matrices, meaning the stabilizer of \( A_{HbH} \) under this action is trivial. As such, we have \( aH = H \), meaning \( a \in H \). Finally, for (H1), take any \( x \in G \) and suppose for the moment that \( x \notin K \). Then \( A_{HxH} \) is not thin, and so there exists \( a \in K \) such that \( A_{aH}A_{HbH} = A_{HxH} \) since the thin radical acts transitively on the non-thin adjacency matrices. By (7), this in turn implies \( HxH = HabH \subseteq KbH \subseteq KbK \). Overall, \( x \in G \) either belongs to \( K \) or \( KbK \), meaning \( K \) has two double cosets, and therefore \( G \) acts doubly transitively on \( G/K \).

Given a commutative association scheme, the primitive idempotents provide an alternative orthogonal basis for the adjacency algebra. As detailed in Section 2, this basis is particularly important to our pursuit of Gram matrices. In the case of roux schemes, all of the primitive idempotents can be expressed in terms of the characters \( \alpha \in \hat{\Gamma} \) of the abelian group \( \Gamma \) and the underlying roux for \( \Gamma \):

**Theorem 3.8.** Given an \( n \times n \) roux \( B \) for \( \Gamma \), the primitive idempotents for the corresponding roux scheme are scalar multiples of

\[
G^\epsilon_\alpha := \sum_{g \in \Gamma} \alpha(g)[gI] + \mu^\epsilon_\alpha \sum_{g \in \Gamma} \alpha(g)[gB], \quad (\alpha \in \hat{\Gamma}, \ \epsilon \in \{+, -\}),
\]

where \( \mu^\epsilon_\alpha \) is defined in terms of the Fourier transform \( \hat{c}_\alpha := \sum_{h \in \Gamma} c_h \alpha(h) \) as follows:

\[
\mu^\epsilon_\alpha = \frac{\hat{c}_\alpha + \epsilon \sqrt{(\hat{c}_\alpha)^2 + 4(n-1)}}{2(n-1)}.
\]

Furthermore,

\[
d^\epsilon_\alpha := \text{rank}(G^\epsilon_\alpha) = \frac{n}{1 + (n-1)(\mu^\epsilon_\alpha)^2}.
\]
Notice that \( G^*_\alpha \) is the Gram matrix of \(|\Gamma|\) phased versions of all \( n \) vectors of an ETF in \( \mathbb{C}^{d_{\alpha}} \) with coherence \( |\mu^\epsilon_{\alpha}| \). Expanding \( d^\epsilon_{\alpha} \) in terms of the definition of \( \mu^\epsilon_{\alpha} \) gives
\[
d^\epsilon_{\alpha} = \frac{2n(n-1)}{(c^\alpha)^2 + 4(n-1) + \epsilon c^\alpha \sqrt{(c^\alpha)^2 + 4(n-1)}}.
\]
By appearances, it seems that \( d^\epsilon_{\alpha} \in \mathbb{Z} \) is a strong necessary condition for the existence of roux.

**Proof of Theorem 3.8.** First, we establish that each \( G^*_\alpha \) is a scalar multiple of an orthogonal projection matrix. To this end, it is helpful to write
\[
G^*_\alpha = \left( \sum_{g \in \Gamma} \alpha(g)[gI] \right) \left( [I] + \mu^\epsilon_{\alpha}[B] \right) =: M_1 M_2.
\]
Notice that \( \mu^\epsilon_{\alpha} \) is real since \( c_{g^{-1}} = c_g \) for every \( g \in \Gamma \) (see Lemma 3.3), and \([B]\) is symmetric since \( B \) is self-adjoint, and so \( M_2 = M_2^\top = M_2 \). Also, a change of variables gives that \( M_1 = M_1 \), and so \((G^*_\alpha)^* = G^*_\alpha \). Next, if we put \( r = |\Gamma| \), then
\[
M_1^2 = r \sum_{g \in \Gamma} \alpha(g)[gI], \quad M_2^2 = \left( 1 + (n-1)(\mu^\epsilon_{\alpha})^2 \right) [I] + 2 \mu^\epsilon_{\alpha}[B] + (\mu^\epsilon_{\alpha})^2 \left( \sum_{g \in \Gamma} c_g[gI] \right) [B].
\]
Indeed, \( M_1^2 \) is computed by a change of variables, whereas \( M_2^2 \) is computed by applying the formula for \( B^2 \) in terms of the roux parameters. With this, we may compute \((G^*_\alpha)^2 = M_1^2 M_2^2\), the third term of which is \( r(\mu^\epsilon_{\alpha})^2 M_3 [B] \), where
\[
M_3 = \left( \sum_{g \in \Gamma} \alpha(g)[gI] \right) \left( \sum_{g \in \Gamma} c_g[gI] \right) = \sum_{g,h \in \Gamma} c_g \alpha(h)[ghI] = \sum_{g,h \in \Gamma} c_h \alpha(g^{-1}h)[gI] = \hat{c}_\alpha \sum_{g \in \Gamma} \alpha(g)[gI].
\]
Putting everything together, we have
\[
(G^*_\alpha)^2 = r \left( 1 + (n-1)(\mu^\epsilon_{\alpha})^2 \right) \sum_{g \in \Gamma} \alpha(g)[gI] + r \left( 2 + \hat{c}_\alpha \mu^\epsilon_{\alpha} \right) \cdot \mu^\epsilon_{\alpha} \sum_{g \in \Gamma} \alpha(g)[gB].
\]
Considering \( \mu^\epsilon_{\alpha} \) is a solution to \( 1 + (n-1)(\mu^\epsilon_{\alpha})^2 = 2 + \hat{c}_\alpha \mu^\epsilon_{\alpha} \), we therefore have that \((G^*_\alpha)^2\) is a scalar multiple of \( G^*_\alpha \). Combined with the fact that \( G^*_\alpha \) is self-adjoint, this then implies that \( G^*_\alpha \) is a scalar multiple of an orthogonal projection matrix.

Next, we show that \( G^*_\alpha \) is a scalar multiple of a *primitive* idempotent. Since the dimension of \( \mathcal{A}(B) \) is \( 2r \) and there are \( 2r \) different \( G^*_\alpha \)'s, it suffices to show that \{\( G^*_\alpha \)\} are linearly independent. We will prove something much stronger: that every pair multiplies to the zero matrix. To this end, take \( \alpha, \beta \in \hat{\Gamma} \) and \( \epsilon, \delta \in \{+,-\} \). We will proceed in two cases. First, suppose \( \alpha \neq \beta \). Then
\[
\sum_{g,h \in \Gamma} \alpha(g) \beta(h)[gI][hI] = \sum_{g,k \in \Gamma} \alpha(g) \beta(g^{-1}k)[kI] = \left( \sum_{g \in \Gamma} \alpha(g) \overline{\beta(g)} \right) \left( \sum_{k \in \Gamma} \beta(k)[kI] \right) = 0,
\]
where the last step is by the orthogonality of characters. As such,
\[
G^*_\alpha G^*_\beta = \left( \sum_{g \in \Gamma} \alpha(g)[gI] \right) \left( [I] + \mu^\epsilon_{\alpha}[B] \right) \left( \sum_{h \in \Gamma} \beta(h)[hI] \right) \left( [I] + \mu^\delta_{\beta}[B] \right) = 0,
\]
where the last step follows from exploiting commutativity to multiply the first and third factors and then applying (R3). This completes the first case. It remains to show that $G^+\alpha G^-\alpha = 0$ for every $\alpha \in \hat{\Gamma}$. To this end, fix $\alpha \in \hat{\Gamma}$ and note that

$$\mu^+_\alpha + \mu^-_\alpha = -\frac{\hat{c}_\alpha}{n-1}, \hspace{1cm} \mu^+_\alpha \mu^-_\alpha = -\frac{1}{n-1}.$$

Combining this with the expression for $B^2$ then gives

$$\left([I] + \mu^+_\alpha [B]\right)\left([I] + \mu^-_\alpha [B]\right) = [I] + \frac{\hat{c}_\alpha}{n-1}[B] - \frac{1}{n-1}[B]^2 = \frac{1}{n-1}[B]\left(\hat{c}_\alpha[I] - \sum_{g \in \Gamma} c_g[gI]\right).$$

With this, we compute the desired product:

$$G^+\alpha G^-\alpha = \frac{1}{n-1}[B]\left(\sum_{g \in \Gamma} \alpha(g)[gI]\right)^2\left(\hat{c}_\alpha[I] - \sum_{g \in \Gamma} c_g[gI]\right)$$

$$= \frac{1}{n-1}[B]\left(\sum_{g \in \Gamma} \alpha(g)[gI]\right)\left(\sum_{g,h \in \Gamma} \alpha(gh^{-1})c_h[gI] - \sum_{g,h \in \Gamma} \alpha(g)c_h[ghI]\right) = 0,$$

where the last step follows from a change of variables.

At this point, we have that the primitive idempotents of $\mathcal{A}(B)$ are given by

$$\frac{1}{r(1 + (n-1)(\mu^\alpha)^2)} \cdot G^\epsilon\alpha, \hspace{1cm} (\alpha \in \hat{\Gamma}, \epsilon \in \{+,-\}).$$

For the last claim, we need to compute the ranks of these idempotents, which amounts to a trace calculation. To this end, we isolate the diagonal contribution to $G^\epsilon\alpha$ to get $\text{tr}(G^\epsilon\alpha) = \text{tr}([I]) = rn$, from which the formula for $d^\epsilon\alpha$ follows.

4 Roux lines

Given unit-norm representatives of equiangular lines, the Gram matrix of these vectors has the form $I + \mu S$ for some $\mu \geq 0$ and signature matrix $S$. As discussed in Section 2, we call the equiangular lines roux if there exist unit-norm representatives such that the signature matrix $S$ can be obtained by evaluating some roux at a character. This operation of evaluating at a character requires some notation: Every $\alpha \in \hat{\Gamma}$ extends to a $*$-algebra homomorphism $\hat{\alpha} : \mathbb{C}[\Gamma] \to \mathbb{C}$, which in turn extends to a $*$-algebra homomorphism $\hat{\alpha} : \mathbb{C}[\Gamma]^{n \times n} \to \mathbb{C}^{n \times n}$ given by applying $\hat{\alpha}$ entrywise.

**Theorem 4.1.** Suppose $B \in \mathbb{C}[\Gamma]^{n \times n}$ satisfies (R1)–(R3). Then $B$ is a roux if and only if for every $\alpha \in \hat{\Gamma}$, $\hat{\alpha}(B)$ is the signature matrix of an equiangular tight frame.

See Theorem 5.2 for the combinatorial significance of (R1)–(R3).

**Proof of Theorem 4.1** $(\Rightarrow)$ Put $S = \hat{\alpha}(B)$. Then $S = S^*$ and Lemma 3.3 gives

$$S^2 = (n-1)I + \left(\sum_{g \in \Gamma} c_g\alpha(g)\right)S.$$  \hspace{1cm} (9)

As such, $S$ has at most two eigenvalues $\lambda_1 \geq \lambda_2$, and since $\text{tr}(S) = 0$, we have $\lambda_1 > 0 > \lambda_2$. Then $(S - \lambda_2I)/(\lambda_1 - \lambda_2)$ is an orthogonal projection with off-diagonal of constant modulus.
By Lemma 3.3 it suffices to compute $[B]^2$. Since

$$[B_{ij}]_{g,h} = \begin{cases} 1 & \text{if } B_{ij} = g \\ 0 & \text{otherwise} \end{cases},$$

we may decompose $[B]$ as follows:

$$[B] = \sum_{\alpha \in \hat{\Gamma}} \hat{\alpha}(B) \otimes v_{\alpha} v^*_{\alpha}, \quad (v_{\alpha})_g := \frac{1}{\sqrt{|\Gamma|}} \cdot \alpha(g).$$

This decomposition provides a useful expression for $[B]^2$:

$$[B]^2 = \sum_{\alpha, \beta \in \hat{\Gamma}} \hat{\alpha}(B) \hat{\beta}(B) \otimes v_{\alpha} v^*_\alpha v_{\beta} v^*_\beta = \sum_{\alpha \in \hat{\Gamma}} \hat{\alpha}(B)^2 \otimes v_{\alpha} v^*_\alpha. \quad (10)$$

Next, since $\hat{\alpha}(B)$ is the signature matrix of an ETF by assumption, it necessarily has exactly two eigenvalues. Furthermore, (R1)–(R3) together imply that every diagonal entry of $(\hat{\alpha}(B))^2$ is $n - 1$, and so we may write

$$(\hat{\alpha}(B))^2 = (n - 1)I + C_{\alpha} \cdot \hat{\alpha}(B), \quad (\alpha \in \hat{\Gamma}), \quad (11)$$

for some sequence $\{C_{\alpha}\}_{\alpha \in \hat{\Gamma}}$ in $\C$. Consider the sequence $\{c_g\}_{g \in \Gamma}$ whose Fourier transform is given by $\hat{c}_\alpha = C_{\alpha^{-1}}$ for $\alpha \in \hat{\Gamma}$. We combine this with (11) to continue (10):

$$[B]^2 = (n - 1)[I] + \sum_{\alpha \in \hat{\Gamma}} \hat{c}_{\alpha^{-1}} \cdot \hat{\alpha}(B) \otimes v_{\alpha} v^*_\alpha =: (n - 1)[I] + M, \quad (12)$$

where the first term follows from the fact that $\sum_{\alpha \in \hat{\Gamma}} v_{\alpha} v^*_\alpha = I$. By (R1), we have that $M_{(i,g),(j,h)} = 0$ whenever $i = j$. For $i \neq j$, we have

$$M_{(i,g),(j,h)} = \sum_{\alpha \in \hat{\Gamma}} \hat{c}_{\alpha^{-1}} \cdot \hat{\alpha}(B_{ij}) (v_{\alpha})_g (v_{\alpha})_h = \frac{1}{|\Gamma|} \sum_{\alpha \in \hat{\Gamma}} \hat{c}_{\alpha} \alpha(B_{ij}^{-1}gh^{-1}) = c_{B_{ij}^{-1}gh^{-1}},$$

which matches the desired sum:

$$\left(\sum_{k \in \Gamma} c_k [kB]_{(i,g),(j,h)}\right)_{(i,g),(j,h)} = \sum_{k \in \Gamma} c_k [kB]_{ij,h} = \sum_{k \in \Gamma} c_k \begin{cases} 1 & \text{if } kB_{ij} = g \\ 0 & \text{otherwise} \end{cases} = c_{B_{ij}^{-1}gh^{-1}}.$$

Overall, $M = \sum_{g \in \Gamma} c_g [gB]$, and so (12) and Lemma 3.3 together give that $B$ is a roux with parameters $\{c_g\}_{g \in \Gamma}$. □

Recalling the primitive idempotents in Theorem 3.8, we see that $(G^{\mu}_{\alpha^{-1}}(i,1)(j,1)) = \mu_{\alpha^{-1}} \cdot \alpha(B_{ij})$ whenever $i \neq j$. This implies a fundamental relationship:

**Lemma 4.2.** For each $\alpha \in \hat{\Gamma}$, the signature matrix $\hat{\alpha}(B)$ from Theorem 4.1 and the Gram matrix $G^{\mu}_{\alpha^{-1}}$ from Theorem 3.8 describe the same lines (each line implicated by the former is represented $|\Gamma|$ times in the latter).

In fact, this relationship can be used to characterize roux lines (see Corollary 4.6 for a more characterizing):

**Theorem 4.3.** Let $\mathcal{L}$ be a sequence of $n$ complex lines. Then $\mathcal{L}$ is roux if and only if both of the following occur simultaneously:

$$\begin{aligned}
\end{aligned}$$
(a) there exist unit-norm representatives \( \{ \varphi_i \}_{i \in [n]} \) of \( \mathcal{L} \) whose signature matrix is comprised of \( r \)th roots of unity for some \( r \), and

(b) the Gram matrix of \( \{ g\varphi_i \}_{i \in [n], g \in C_r} \) carries an association scheme.

In this case, \( \{ \varphi_i \}_{i \in [n]} \) is an equiangular tight frame for its span.

Proof. We start with a general observation. Suppose the Gram matrix of \( \{ \varphi_i \}_{i \in [n]} \) has the form \( I + \mu S \), where \( \mu > 0 \) and \( S \) has entries in \( C_r \), and define \( \tilde{B} \in \mathbb{C}[C_r]^{n \times n} \) to have entries \( \tilde{B}_{ii} = 0 \) and \( \tilde{B}_{ij} = \delta_{S_{ij}} \) for \( i \neq j \). Then the Gram matrix \( G \) of \( \{ g\varphi_i \}_{i \in [n], g \in C_r} \) can be expressed as

\[
G = \sum_{g \in C_r} g^{-1} [\delta_g I] + \mu \sum_{g \in C_r} g^{-1} [\delta_g \tilde{B}].
\]

(13)

Note that this expression leverages our convention that \( C_r \) lies in \( \mathbb{C} \).

With this, we first show \((\Rightarrow)\). By (a), we may define \( \tilde{B} \) as above, which satisfies (R1)–(R3) by definition. Then by (13), the Gram matrix \( G \) of \( \{ g\varphi_i \}_{i \in [n], g \in C_r} \) carries \( \{ [\delta_g I] \}_{g \in C_r} \) and \( \{ [\delta_g \tilde{B}] \}_{g \in C_r} \). By (b), these matrices form an association scheme, and so they span an algebra that is isomorphic to \( \alpha'(\tilde{B}) \), implying (R4). As such, \( B \) is a roux, and so \( \mathcal{L} \) is roux.

For \((\Leftarrow)\), there exists an \( n \times n \) roux \( B \) for some \( \Gamma \), and the lines \( \mathcal{L} \) have signature matrix \( S = \tilde{\alpha}(B) \) for some \( \alpha \in \widetilde{\Gamma} \). That is, there exist unit-norm representatives \( \{ \varphi_i \}_{i \in [n]} \) of \( \mathcal{L} \) whose Gram matrix is \( I + \mu S \) for some \( \mu > 0 \). Furthermore, the off-diagonal entries of \( S \) lie in the image of \( \alpha \), which equals \( C_r \) for some \( r \). This gives (a). Next, we may define \( \tilde{B} \) as above, which by Lemma 3.5(c), equals the roux \( \tilde{\alpha}(B) \) for \( C_r \). Then (13) shows that the Gram matrix \( G \) of \( \{ g\varphi_i \}_{i \in [n], g \in C_r} \) carries the corresponding roux scheme, implying (b).

Finally, Theorem 4.1 gives that the unit-norm representatives \( \{ \varphi_i \}_{i \in [n]} \) of roux lines form an equiangular tight frame for their span.

Signature matrices of unit-norm representatives of roux lines are necessarily comprised of roots of unity, and this feature leads to a necessary integrality condition for the existence of roux lines:

**Corollary 4.4.** Suppose there exist \( n > d \) roux lines for \( \Gamma \) spanning \( \mathbb{C}^d \), and put

\[
q = \frac{(n - 2d)^2(n - 1)}{d(n - d)}.
\]

Then \( q \in \mathbb{Z} \) and \( \sqrt{q} \in \mathbb{Z}[\omega] \), where \( \omega \) is a primitive \( r \)th root of unity with \( r = |\Gamma| \).

Proof. By assumption, there exists an \( n \times n \) roux \( B \) for \( \Gamma \) such that the given lines have unit-norm representatives with signature matrix \( S = \tilde{\alpha}(B) \) for some \( \alpha \in \tilde{\Gamma} \). By Theorem 4.1, \( S \) is the signature matrix of an ETF. Since ETFs achieve equality in the Welch bound (1), the Gram matrix of this ETF is given by

\[
G = I + \sqrt{\frac{n - d}{d(n - 1)}} \cdot \mathcal{S},
\]

and tightness implies \( G^2 = (n/d)G \). We express this quadratic in terms of \( \mathcal{S} \) and isolate \( \mathcal{S}^2 \):

\[
\mathcal{S}^2 = (n - 1)I + \text{sign}(n - 2d) \cdot \sqrt{q} \cdot \mathcal{S}.
\]

Comparing with (19), we note that each \( c_g \) is an integer and each \( \alpha(g) \) is an \( r \)th root of unity, and so \( \sqrt{q} \in \mathbb{Z}[\omega] \). This further implies that \( \sqrt{q} \) and \( q \) are algebraic integers. Since \( q \) is also rational, it must be an integer.

\(\square\)
Recall that Lemma 3.5 provides a few basic roux transformations. We now discuss how some of these interact with evaluating a roux at a character. We say two roux $B, \tilde{B}$ for $\Gamma$ are switching equivalent, denoted $B \sim \tilde{B}$, if there exists a diagonal matrix $D$ as in Lemma 3.5(a) such that $\tilde{B} = DBD^{-1}$. This echoes the more classical notion of switching equivalence between signature matrices, in which the diagonal entries of $D$ are required to be complex with unit modulus. Note that $B \sim \tilde{B}$ implies that $\hat{\alpha}(B)$ and $\hat{\alpha}(\tilde{B})$ are switching equivalent. (The converse fails to hold by taking $\alpha$ to be defined by $\alpha(z) = 1$, for example.) It is convenient to define the normalization of a roux $B$ for $\Gamma$ to be the unique $\tilde{B} \sim B$ with $\tilde{B}_{i,1} = \tilde{B}_{1,i} = 1$ (the identity element of $\Gamma$) for every $i \neq 1$. Regarding Lemma 3.5(d), we note that if $B$ is a roux for $\Gamma \leq \Lambda$, then $\{\hat{\alpha}(B) : \alpha \in \hat{\Gamma}\} = \{\hat{\beta}(B) : \beta \in \hat{\Lambda}\}$ since each $\beta \in \hat{\Lambda}$ restricts to a character $\alpha \in \hat{\Gamma}$. In particular, the additional characters in $\hat{\Lambda}$ fail to produce new roux lines. The following result reverses the transformation in Lemma 3.5(d), and the proof leverages the notion of roux lines:

**Lemma 4.5.** Take any $n \times n$ roux $B$ with parameters $\{c_g\}_{g \in \Gamma}$, and put $\Lambda = \langle g : c_g \neq 0 \rangle$. Then the normalization of $B$ lies in $\mathbb{C}[\Lambda]^{n \times n} \subseteq \mathbb{C}[\Gamma]^{n \times n}$, and is a roux for $\Lambda$. Furthermore, if $\tilde{B} \sim B$ is a roux for $\hat{\Lambda} \leq \Gamma$, then $\Lambda \leq \hat{\Lambda}$.

**Proof.** Define $\Pi := \langle B_{ij}B_{jk}B_{ki} : i \neq j \neq k \neq i \rangle$, and take any $\alpha \in \hat{\Gamma}$. Then since $\tilde{B}_{i,1} = \tilde{B}_{1,i} = 1$ for every $i \neq 1$ and $\tilde{B}_{ij}\tilde{B}_{jk}\tilde{B}_{ki} = B_{ij}B_{jk}B_{ki}$ for every $i,j,k \in [n]$, we have $\tilde{B} \in \mathbb{C}[\Pi]^{n \times n}$. We claim that $\Pi \leq \ker \alpha$ if and only if $\Lambda \leq \ker \alpha$. This in turn would imply $\Pi = \Lambda$ since a subgroup is determined by its annihilator, and so $\tilde{B} \in \mathbb{C}[\Lambda]^{n \times n}$.

We prove our claim by identifying a sequence of equivalent statements. First, $\Pi \leq \ker \alpha$ if and only if $\hat{\alpha}(B)_{ij}\hat{\alpha}(B)_{jk}\hat{\alpha}(B)_{ki} = 1$ whenever $i \neq j \neq k \neq i$. By Theorem 2.2 in [9], this is equivalent to $\hat{\alpha}(B)$ being switching equivalent to $J - I$, where $J$ denotes the $n \times n$ matrix of all ones. Equivalently, $\hat{\alpha}(B)$ is the signature matrix of a 1-dimensional ETF, that is, by Lemma 1.2 we equivalently have $d_{\alpha}^+ = 1$. By Theorem 3.3 and Lemma 3.3, this is equivalent to having $\hat{c}_\alpha = \hat{c}_{\alpha^{-1}} = n - 2$. Since $\Re \alpha(g) \leq 1$ for every $g \in \Gamma$, Lemma 3.3 gives

$$\Re \hat{c}_\alpha = \sum_{g \in \Gamma} c_g \Re \alpha(g) \leq \sum_{g \in \Gamma} c_g = n - 2,$$

with equality only if $\Re \alpha(g) = 1$ for every $g$ with $c_g \neq 0$, implying $\Lambda \leq \ker \alpha$. Conversely, $\Lambda \leq \ker \alpha$ implies $\hat{c}_\alpha = \sum_{g \in \Gamma} c_g = n - 2$. Overall, $\hat{c}_\alpha = n - 2$ if and only if $\Lambda \leq \ker \alpha$, completing the proof of our intermediate claim.

Now that we have $\tilde{B} \in \mathbb{C}[\Lambda]^{n \times n}$, we verify that $\tilde{B}$ is a roux for $\Lambda$. To this end, (R1)–(R3) are immediate, while (R4) follows from Lemma 3.5(a) and Lemma 3.3:

$$\tilde{B}^2 = (n - 1)I + \sum_{g \in \Gamma} c_g g\tilde{B} = (n - 1)I + \sum_{g \in \Lambda} c_g g\tilde{B}.$$

Overall, $\tilde{B}$ is a roux for $\Lambda$.

For the last claim, the previous argument shows that normalizing $\tilde{B}$ produces a roux for $\Pi = \langle \hat{B}_{ij}\hat{B}_{jk}\hat{B}_{ki} : i \neq j \neq k \neq i \rangle$. However, $\tilde{B}_{ij}\tilde{B}_{jk}\tilde{B}_{ki} = B_{ij}B_{jk}B_{ki}$ for every $i,j,k \in [n]$, and so $\Lambda = \Pi = \Pi \leq \Lambda$, as desired. \(\square\)

In what follows and throughout, we let $\circ$ denote the **Hadamard product** defined by $(A \circ B)_{ij} = A_{ij}B_{ij}$, and we let $A^{\circ k}$ denote the $k$th **Hadamard power** of $A$, defined by $(A^{\circ k})_{ij} = (A_{ij})^k$.

**Corollary 4.6** (Roux lines detector). Given a signature matrix $\mathcal{S}$, normalize the first row and column to get $\tilde{\mathcal{S}}$. Then $\tilde{\mathcal{S}}$ is the signature matrix of unit-norm representatives of roux lines if and only if the following occur simultaneously:
(a) The entries of $\tilde{S}$ are all roots of unity.

(b) Every Hadamard power of $\tilde{S}$ has exactly two eigenvalues.

Proof. ($\Rightarrow$) Suppose there exists an $n \times n$ roux $B$ for some $\Gamma$, pick $\alpha \in \hat{\Gamma}$ and let $S$ be switching equivalent to $\hat{\alpha}(B)$. Then $\tilde{S}$ is the normalization of $\hat{\alpha}(B)$. Since the off-diagonal entries of $\hat{\alpha}(B)$ are roots of unity, the same holds for its normalization, implying (a). Take $D$ such that $\tilde{S} = D\hat{\alpha}(B)D^{-1}$, and put $v = \text{diag}(D) \in \mathbb{T}^n$. Then the $k$th Hadamard power of $\tilde{S}$ is given by

$$\tilde{S}^k = (D\hat{\alpha}(B)D^{-1})^k = (\hat{\alpha}(B) \circ vv^*)^k = \hat{\alpha}^k(B) \circ (v^{\circ k})(v^{\circ k})^* = D^k\hat{\alpha}^k(B)(D^k)^{-1}$$

That is, $\tilde{S}^k$ is switching equivalent to $\hat{\alpha}^k(B)$. Theorem 4.1 then implies (b).

($\Leftarrow$) Given an $n \times n$ signature matrix $S$ satisfying (a) and (b), pick any $r$ such that the off-diagonal entries of $\tilde{S}$ lie in $C_r$. Define $B \in \mathbb{C}[C_r]^{n \times n}$ so that $B_{ii} = 0$ for every $i \in [n]$ and $B_{ij} = \delta_{S_{ij}}$ whenever $i \neq j$. We claim that $B$ is a roux, which would imply the result since evaluating $B$ at the character $\alpha$ defined by $\alpha(z) = z$ recovers $\tilde{S}$. First, $B$ satisfies (R1)–(R3) by definition. Next, the following holds for every $k$:

$$\hat{\alpha}^k(B) = (\hat{\alpha}(B))^k = S^k.$$

As such, (b) implies that evaluating $B$ at every character of the form $\alpha^k$ produces the signature matrix of an ETF. Since $\alpha$ generates $\hat{\Gamma}$, we may then conclude (R4) by Theorem 4.1.

We say a sequence of lines is real if their normalized signature matrix is real. For example, letting $\omega$ denote a primitive cube root of unity, then the lines spanned by $(1, 1), (1, \omega), (1, \omega^2) \in \mathbb{C}^2$ are real (even though the Gram matrix of these vectors is not real).

**Lemma 4.7** (Real lines detector). An $n \times n$ signature matrix $S$ is a signature matrix of real lines if and only if the eigenvalues of $S^{\circ 2}$ are $n - 1$ and $-1$.

**Proof.** ($\Rightarrow$) Suppose $S$ is a signature matrix of real lines. Then the off-diagonal entries of its normalization $\tilde{S} = D^{-1}SD$ lie in $\{\pm 1\}$. Put $v = \text{diag}(D)$. Then

$$S^{\circ 2} = (D\tilde{S}D^{-1})^{\circ 2} = (\tilde{S} \circ vv^*)^{\circ 2} = (J - I) \circ (v^{\circ 2})(v^{\circ 2})^* = D^2(J - I)(D^2)^{-1},$$

i.e., $S^{\circ 2}$ has the same eigenvalues as $J - I$, where $J$ is the matrix of all ones.

($\Leftarrow$) Since $S^{\circ 2}$ has zero trace, the eigenvalues $n - 1$ and $-1$ have multiplicities $1$ and $n - 1$, respectively. Thus, $I + S^{\circ 2}$ has rank $1$ with maximum eigenvalue $n$, and so we may write $I + S^{\circ 2} = uu^*$ for some $u \in \mathbb{T}^n$. This in turn implies that $I + S$ is a solution to $X^{\circ 2} = uu^*$. Pick any $v \in \mathbb{T}^n$ such that $v^{\circ 2} = u$. Then every solution has the form $X = vv^* \circ R$, where $R$ has entries in $\{\pm 1\}$. As such, we have $I + S = vv^* \circ R$ for some symmetric $R \in \{\pm 1\}^{n \times n}$ satisfying $R_{ii} = 1$ for every $i \in [n]$. Put $D = \text{diag}(v)$. Then isolating $S$ gives

$$S = vv^* \circ R - I = D(R - I)D^{-1}.$$  

Since $R - I$ is the signature matrix of real lines, we are done.

**Corollary 4.8** (Real roux lines detector). Let $B$ be a roux for $\Gamma$ and pick $\alpha \in \hat{\Gamma}$. Then $\hat{\alpha}(B)$ is a signature matrix of real lines if and only if $\alpha$ is real for every $g \in \Gamma$ such that $c_g \neq 0$.  

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Proof. By Lemma 4.7, \( \hat{\alpha}(B) \) is a signature matrix of real lines if and only if \( \hat{\alpha}(B)^{\circ 2} \) has minimal polynomial \( x^2 - (n - 2)x - (n - 1) \). By Lemma 3.3, evaluating \( B \) at any character \( \beta \in \hat{\Gamma} \) gives

\[
(\hat{\beta}(B))^2 = (n - 1)I + \sum_{g \in \Gamma} c_g \beta(g) \hat{\beta}(B) = (n - 1)I + \hat{c}_{\beta} \hat{\beta}(B) = (n - 1)I + \hat{c}_{\beta} \hat{\beta}(B)
\]

As such, \( \hat{\alpha}(B)^{\circ 2} = \hat{\alpha}^2(B) \) has minimal polynomial \( x^2 - \hat{c}_{\alpha^2}x - (n - 1) \); the minimal polynomial does not have degree 1 since \( \hat{\alpha}(B)^{\circ 2} \) is nonzero with zero trace. Overall, \( \hat{\alpha}(B) \) is a signature matrix of real lines if and only if \( \hat{c}_{\alpha^2} = n - 2 \). Finally, the argument in \([14]\) gives that \( \hat{c}_{\alpha^2} = n - 2 \) if and only if \( \alpha^2(g) = 1 \) for every \( g \in \Gamma \) such that \( c_g \neq 0 \).

\[\square\]

Corollary 4.9. Let \( B \) be an \( n \times n \) roux for \( \Gamma \) with parameters \( \{c_g\}_{g \in \Gamma} \). Assume that \( \Gamma \) has odd order, and that \( \langle g : c_g \neq 0 \rangle = \Gamma \). Then for every nontrivial \( \alpha \in \hat{\Gamma} \), \( \hat{\alpha}(B) \) is the signature matrix of non-real lines spanning \( \mathbb{C}^d \) for some \( d \notin \{1, n - 1\} \).

Proof. To begin, denote \( A = \{g \in \Gamma : c_g \neq 0\} \), and observe that any character \( \beta \in \hat{\Gamma} \) with \( \beta(g) = 1 \) for all \( g \in A \) is in fact trivial. Now let \( \alpha \in \hat{\Gamma} \) be nontrivial, and take \( \beta = \alpha^2 \). Since \( \Gamma \) has odd order, \( \beta \) is necessarily nontrivial, and so \( \alpha^2(g) \neq 1 \) for some \( g \in A \). It follows by Corollary 4.8 that \( \hat{\alpha}(B) \) is not the signature matrix of real lines. In particular, any lines with signature matrix \( \hat{\alpha}(B) \) span a space of dimension \( d \notin \{1, n - 1\} \).

\[\square\]

5 Roux graphs

In this section, we identify graph-theoretic properties that are associated with roux. We start with a review of certain concepts in graph theory (the reader is encouraged to reference \([3, 28, 29]\) for further information). The graphs in this paper will be assumed to be simple without mention, i.e., they will contain neither loops nor multiple edges. A graph is said to be \textbf{distance-regular} if for every ordered pair of vertices \((u, v)\), the number of vertices that are simultaneously at distance \( i \) from \( u \) and distance \( j \) from \( v \) is determined by \( i, j \), and the distance between \( u \) and \( v \). We say a connected graph is \textbf{antipodal} if the vertices can be partitioned into fibres such that the distance between two vertices is the diameter of the graph if and only if they belong to the same fibre. Finally, given graphs \( \mathcal{G} \) and \( \mathcal{H} \), we say \( \mathcal{G} \) is a \textbf{cover} of \( \mathcal{H} \) if there exists a surjective map \( \pi: V(\mathcal{G}) \to V(\mathcal{H}) \) such that for every \( u, v \in V(\mathcal{H}) \), the induced subgraph of \( \pi^{-1}(u) \) and \( \pi^{-1}(v) \) in \( \mathcal{G} \) is a perfect matching if \( u \) and \( v \) are adjacent in \( \mathcal{H} \), and is otherwise empty. In particular, if \( \mathcal{H} \) is connected, then the fibres \( \pi^{-1}(v) \) are necessarily independent sets in \( \mathcal{G} \) of the same size. In what follows, we consider distance-regular antipodal covers of the complete graph (DRACKN)

Proposition 5.1 (Lemma 3.1 in \([28]\)). For every distance-regular antipodal cover \( \mathcal{G} \) of the complete graph, there exist constants \((n, r, c)\) such that \( \mathcal{G} \) is a connected graph on \( r \) \( n \) vertices such that

\[(D1) \text{ every pair of vertices at distance } 2 \text{ has } c \text{ common neighbors,}\]

\[(D2) \text{ the vertices can be partitioned into fibres of size } r \text{ such that the distance between two vertices is the diameter of the graph if and only if they belong to the same fibre, and}\]

\[(D3) \text{ the induced subgraph between any two fibres is a perfect matching.}\]

Conversely, any connected graph on \( r \) \( n \) vertices satisfying \((D1)-(D3)\) for some \((n, r, c)\) is a distance-regular antipodal cover of the complete graph.
Given a DRACKN, we refer to the corresponding constants \((n, r, c)\) above as its parameters. The fibres suggest a block-matrix expression for the \(rn \times rn\) adjacency matrix of a given DRACKN. In particular, the \(r \times r\) blocks are zero on the diagonal and permutation matrices on the off-diagonal. If the off-diagonal permutation matrices generate an abelian group \(\Gamma\), we say the DRACKN is abelian. In \([11]\), Coutinho, Godsil, Shirazi and Zhan establish that evaluating these blocks at any character of \(\Gamma\) produces the \(n \times n\) signature matrix of an ETF (see also \([19]\)). This behavior of abelian DRACKNS should be compared with Theorem 4.1. In a roux scheme, the adjacency matrix \([B]\) is symmetric by (R3), and therefore describes a graph we call a roux graph. The following result identifies the relationship between roux graphs and DRACKNS. The result requires another definition: Given a cover \(G\) of the complete graph, if the group of automorphisms of \(G\) that fix its fibres acts regularly on the fibres, then we say \(G\) is a regular cover.

**Theorem 5.2.**

(a) Every abelian \((n, r, c)\)-DRACKN is a roux graph with parameters

\[
c_g = \begin{cases} 
  n - c(r - 1) - 2 & \text{if } g = \text{id}; \\
  c & \text{otherwise,} 
\end{cases} \quad c > 0. \tag{15}
\]

(b) Given a finite abelian group \(\Gamma\), then \(B \in \mathbb{C}[\Gamma]^{n \times n}\) satisfies (R1)–(R3) if and only if \([B]\) is the adjacency matrix of an abelian cover of the complete graph.

(c) A roux graph has diameter 3 if and only if its parameters satisfy \(c_g > 0\) for every \(g \in \Gamma\), in which case the graph is an antipodal regular cover of the complete graph.

(d) Every roux graph with parameters \((15)\) is an abelian \((n, r, c)\)-DRACKN.

**Proof.** For (a), we may normalize the adjacency matrix (i.e., conjugate with a block-diagonal matrix of permutations in \(\Gamma\) that make the first row and column of off-diagonal blocks equal the \(r \times r\) identity matrix) since the graph is not affected by switching. Translating notation from Corollary 7.5 in \([28]\), we have

\[
B_{ij} = f(i, j), \quad B = A^f, \quad \text{and} \quad B^2 = (n - 1)I + (a_1 - c)B + c \sum_{g \in \Gamma} g (J - I) = (n - 1)I + \sum_{g \in \Gamma} c_g gB
\]

with \(c_{id} = a_1 = n - c(r - 1) - 2\) and \(c_g = c\) otherwise. By Lemma 3.3, \(B\) is a roux. Next, (b) is immediate. For (c), we first apply Lemma 3.3 to get

\[
[B]^2 = (n - 1)[I] + \sum_{g \in \Gamma} c_g [gB], \tag{16}
\]

\[
[B]^3 = (n - 1)[B] + \sum_{g \in \Gamma} c_g [gI] + \sum_{g, h \in \Gamma} c_g c_h [ghB]. \tag{17}
\]

Then (R1) and (16) give that distinct vertices in a common fibre have distance at least 3. Furthermore, (17) implies that all such vertices have distance 3 precisely when every \(c_g\) is strictly positive. This proves \((\Rightarrow)\). For \((\Leftarrow)\), it remains to show that points in different fibres have distance at most 3 when every \(c_g\) is strictly positive. In fact, (16) gives that all such vertices have distance at most 2. This stronger conclusion implies that the roux graph is antipodal. Furthermore, Lemma 7.2 in \([28]\) and our Lemma 4.5 combine to show that we have a regular cover. Finally, for (d), we again normalize without loss of generality and put \(f(i, j) = B_{ij}\). Since we have a regular cover of \(K_n\) by (b) and (c), the result follows from Corollary 7.5 in \([28]\). \(\Box\)
At this point, we identify an example that demonstrates that the theory of roux extends beyond DRACKNS and Higman pairs. In particular, note that the roux constructed from antisymmetric conference matrices in Lemma 3.4 do not arise from DRACKNS since the $c_g$'s for non-identity $g \in C_4$ are not all equal. Consider the following iterative construction of antisymmetric conference matrices (based on Theorem 14 in [42]):

$$M_1 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad M_{k+1} := \begin{bmatrix} M_k & M_k + I \\ M_k - I & -M_k \end{bmatrix}.$$ 

One may verify in GAP [26, 32] that the roux corresponding to $M_4$ is not Schurian. By Theorem 3.1, the roux scheme does not arise from a Higman pair.

**Theorem 5.3** (Abelian DRACKNS from roux). Given a connected $\Gamma$-roux graph of order $n|\Gamma|$, then for every odd prime $p$ dividing $|\Gamma|$, there exists an abelian $(n,p,c)$-DRACKN for some $c$.

The proof of this theorem leverages the spectrum of roux graphs.

**Theorem 5.4.** A roux graph over an abelian group $\Gamma$ on $n|\Gamma|$ vertices with parameters $c = \{c_g\}_{g \in \Gamma}$ has eigenvalues $\lambda^c_\alpha$ given by

$$\lambda^c_\alpha = \hat{c}_\alpha + \epsilon \sqrt{\hat{c}_\alpha^2 + 4(n-1)}, \quad (\alpha \in \hat{\Gamma}, \epsilon \in \{+, -\}).$$

**Proof.** By definition, a roux graph has adjacency matrix $[B]$ for some roux $B$. Furthermore, projection onto any eigenspace of $[B]$ is an idempotent of the corresponding roux scheme. Considering Theorem 3.8, the eigenvalues of $[B]$ are therefore the scalars $\lambda^c_{\hat{\alpha}}$ such that $[B]G^c_\alpha = \lambda^c_{\hat{\alpha}}G^c_\alpha$. The definition of $G^c_\alpha$ and Lemma 5.3 together give $\text{tr}([B]G^c_\alpha) = \#(\alpha \in \hat{\Gamma}) \mu^c_\alpha = (n-1)\mu^c_\alpha \text{tr}(G^c_\alpha)$, and so we must have $\lambda^c_{\alpha} = (n-1)\mu^c_\alpha$. The definition of $\mu^c_\alpha$ in Theorem 3.8 then gives the result. In words, the above theorem gives that the spectrum of a roux graph is a nonlinear function of the Fourier spectrum of the roux parameters. As is standard in spectral graph theory, the spectrum can be leveraged to identify combinatorial structure in the graph.

**Corollary 5.5.** Let $B$ be a roux for $\Gamma$ with parameters $\{c_g\}_{g \in \Gamma}$, and let $\Lambda = \langle g : c_g \neq 0 \rangle$. Then the corresponding roux graph has exactly $[\Gamma : \Lambda]$ connected components. Consequently, the following are equivalent:

(a) The roux graph of $B$ is connected.

(b) $\Lambda = \Gamma$.

(c) $B$ is not switching equivalent to a roux for any proper subgroup of $\Gamma$.

**Proof.** Denote $n$ for the size of $B$ and $G$ for the corresponding roux graph. We begin by verifying the formula for the number of connected components. Since $G$ is an $(n-1)$-regular graph, it suffices to compute the multiplicity of the eigenvalue $n-1$. Using the notation of Theorem 5.4, it is straightforward to verify that $\lambda^c_\alpha = n-1$ if and only if $\epsilon = +$ and $\hat{c}_\alpha = n-2$. As in [14], the latter happens if and only if $\alpha(g) = 1$ whenever $c_g \neq 0$, if and only if $\alpha$ lies in the annihilator $\Lambda^* \leq \hat{\Gamma}$. Consequently, the multiplicity of $n-1$ as an eigenvalue of $G$ equals $|\Lambda^*| = [\Gamma : \Lambda]$. This gives the desired formula for the number of connected components. Everything else follows immediately from Lemma 4.5.
Theorem 5.6. An antipodal roux graph is distance regular if and only if it is connected with exactly four eigenvalues.

Proof. (⇒) By Theorem 5.2, the graph is an abelian DRACKN (which is connected by definition) with parameters (15). Taking the Fourier transform gives that there exists $t$ such that

$$\hat{c}_\alpha = \begin{cases} n - 2 & \text{if } \alpha = 1; \\ t & \text{otherwise}. \end{cases}$$

(18)

Since $\hat{c}_\alpha$ has only two values, Theorem 5.4 then gives that the roux graph has only four eigenvalues.

(⇐) Any roux graph on $n|\Gamma|$ vertices is $(n-1)$-regular, and so $n-1$ is an eigenvalue with all ones eigenvector. In fact, $\lambda_1^+ = n - 1$ since $\hat{c}_1 = n - 2$. Our graph is connected, and so the multiplicity of $n - 1$ is 1, and all other eigenvalues $\lambda$ satisfy $|\lambda| < n - 1$. Since $\hat{c}_\alpha \mapsto \lambda_\alpha^+$ in Theorem 5.4 is strictly increasing, we therefore have $\hat{c}_\alpha < n - 2$ for every $\alpha \neq 1$. Furthermore, by Theorem 5.4, every value of $\hat{c}_\alpha$ produces two distinct eigenvalues. Since our roux graph has exactly four eigenvalues, it must come from a roux with two distinct values of $\hat{c}_\alpha$, that is, there exists $t$ such that (18) holds. Applying the inverse Fourier transform then produces the roux parameters of a DRACKN, and so we are done by Theorem 5.2(d).

Proof of Theorem 5.3. Let $B$ denote the underlying roux for $\Gamma$. As a consequence of the classification of finitely generated abelian groups, there exists a surjection $\varphi : \Gamma \rightarrow C_p$. By Lemma 3.5(c), we have that $\varphi(B)$ is a roux for $C_p$ with parameters $\hat{c}_\lambda = \sum_{g \in \varphi^{-1}(\lambda)} c_g$ for $\lambda \in C_p$. Furthermore, we must have $\hat{c}_\lambda \neq 0$ for some $\lambda \neq 0$ since otherwise $\langle c_g : g \neq 0 \rangle \leq \ker \varphi \leq \Gamma$, which contradicts connectedness by Corollary 5.5. Since $p$ is odd, Corollary 4.9 implies that any nontrivial $\beta \in \hat{C}_p$ produces a signature matrix $S = \hat{\beta}(\varphi(B))$ of an ETF in $\mathbb{C}^d$ for some $d \neq 1$. Finally, the off-diagonal entries of $S$ are $p$th roots of unity, and so Theorem 5.1 in [11] gives the result.

6 Applications

In this section, we identify consequences of our theory for abelian covers of the complete graph and maximal equiangular tight frames.

6.1 Consequences for abelian covers of the complete graph

We start by observing that abelian DRACKNs satisfy a stronger version of Theorem 4.1.

Theorem 6.1. Suppose $B \in \mathbb{C}[\Gamma]^{n \times n}$ satisfies (R1)–(R3). Then $B$ is the adjacency matrix of an abelian DRACKN if and only if there exists $d \neq 1$ such that for every $\alpha \in \Gamma$, $\hat{\alpha}(B)$ is the signature matrix of an equiangular tight frame for $\mathbb{C}^d$.

Note that (⇒) corresponds to Theorem 4.1 in [11], but our theory allows for a quick proof of both directions simultaneously.

Proof of Theorem 6.1. By Theorem 5.2, $B$ is the adjacency matrix of an abelian DRACKN if and only if $B$ has roux parameters of the form

$$c_g = \begin{cases} n - c(r - 1) - 2 & \text{if } g = 1; \\ c & \text{otherwise}, \end{cases}$$

$c > 0$.

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Corollary 6.3. Suppose there exist $n > d$ DRACKN lines for $\Gamma$ spanning $\mathbb{C}^d$, and put

$$q = \frac{(n - 2d)^2(n - 1)}{d(n - d)}.$$ 

Then $\sqrt{q} \in \mathbb{Z}$. 

if and only if $\hat{c}$ has the form

$$\hat{c}_\alpha = \left\{ \begin{array}{ll} n - 2 & \text{if } \alpha = 1; \\ t & \text{otherwise}, \end{array} \right. \quad t < n - 2,$$

if and only if $B$ is a roux with constant $d_\alpha^+ \neq 1$ for all nontrivial $\alpha \in \hat{\Gamma}$ (by Theorem 3.3), and if only if there exists $d \neq 1$ such that for every $\alpha \in \hat{\Gamma}$, $\hat{\alpha}(B)$ is the signature matrix of an equiangular tight frame for $\mathbb{C}^d$ (by Theorem 4.1 and Lemma 4.2). 

Much like roux lines, we say lines are DRACKN if their signature matrix can be obtained by evaluating an abelian DRACKN’s roux at a character (as suggested by the previous theorem).

Corollary 6.2 (DRACKN lines detector). Let $\mathcal{L}$ be a sequence of lines having normalized signature matrix $S$. Then $\mathcal{L}$ forms DRACKN lines if and only if

(a) every off-diagonal entry of $S$ is an $r$th root of unity for some minimal $r > 0$, and

(b) there exist $\theta$ and $\tau$ such that for every $k \in \{1, \ldots, r - 1\}$, $S^{rk}$ has spectrum $\{\theta, \tau\}$.

Proof. ($\Rightarrow$) Let $B$ denote the underlying roux for $\Gamma$. Notice that $S = \hat{\alpha}(B)$ for some character $\alpha \in \hat{\Gamma}$, and $S^{rk} = \hat{\alpha}^k(B)$ for every $k$. By Corollary 4.6, it suffices to show that $\hat{\beta}(B)$ has the same minimal polynomial for every nontrivial character $\beta \in \hat{\Gamma}$. Since $\hat{c}_\beta$ has the form (18), then Lemma 3.3 gives $(\hat{\beta}(B))^2 = (n - 1)I + t\hat{\beta}(B)$ for every nontrivial $\beta$. Since each $\hat{\beta}(B)$ is nonzero with zero trace, we then have that the minimal polynomial of $\hat{\beta}(B)$ is $x^2 - tx - (n - 1)$ for every nontrivial $\alpha$, as desired.

($\Leftarrow$) By Corollary 4.6, $\mathcal{L}$ forms roux lines for the group $C_r$ with roux $B \in \mathbb{C}[C_r]^{n \times n}$ defined by $B_{ii} = 0$ for $i \in [n]$ and $B_{ij} = \delta_{ij}$ for $i, j \in [n]$ with $i \neq j$. The proof of Theorem 1.1 (specifically (11)) gives that $(\hat{\alpha}(B))^2 = (n - 1)I + \hat{c}_\alpha \hat{\alpha}(B)$ for every character $\alpha \in \hat{\Gamma}$. Since each $\hat{\alpha}(B)$ is nonzero with zero trace, we then have that the minimal polynomial of $\hat{\alpha}(B)$ is $x^2 - \hat{c}_\alpha x - (n - 1)$ for every $\alpha \in \hat{\Gamma}$. By assumption, this minimal polynomial is the same for every nontrivial $\alpha$, and so $\hat{c}_\alpha$ has the form (18). Applying the inverse Fourier transform then produces the roux parameters of a DRACKN, and so $B$ defines an abelian DRACKN by Theorem 5.2(d). 

Any signature matrix $S$ comprised of prime roots of unity that has a quadratic minimal polynomial $p \in \mathbb{Q}[x]$ necessarily satisfies (b) above, and therefore corresponds to an abelian DRACKN. Indeed, in this case, taking the $k$th Hadamard power of $S$ is equivalent to applying a field automorphism of $\mathbb{Q}((e^{2\pi i/r}))$ entrywise, which fixes $p$. While signature matrices from DRACKNs necessarily have a quadratic minimal polynomial in $\mathbb{Q}[x]$, namely $x^2 - (n - rc - 2)x - (n - 1)$, one may remove the polynomial’s rationality from the hypothesis here (see Theorem 5.1 in [11]). Comparing Corollary 6.2 with Corollary 4.6, we see that DRACKN lines are the roux lines for which Hadamard powers of the normalized signature matrix correspond to ETFs in a common dimension. In this sense, this completes the picture of “lines from covers” and “covers from lines” introduced in [11]. Next, we provide a stronger version of Corollary 4.4 for DRACKN lines:

Corollary 6.3. Suppose there exist $n > d$ DRACKN lines for $\Gamma$ spanning $\mathbb{C}^d$, and put

Then $\sqrt{q} \in \mathbb{Z}$. 

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Proof. Let $B$ be a roux for $\Gamma$ whose roux graph is an abelian DRACKN, and let $d \neq 1$ be the constant dimension for which $\hat{\alpha}(B)$ is the signature matrix of an ETF in $\mathbb{C}^d$ for every nontrivial $\alpha \in \hat{\Gamma}$ (such $d$ exists by Theorem 6.1). Following the proof of Corollary 4.4, we have

$$\left(\hat{\alpha}(B)\right)^2 = (n - 1)I + \text{sign}(n - 2d) \cdot \sqrt{q} \cdot \hat{\alpha}(B).$$

Comparing with (9) then gives

$$\hat{c}_\alpha = \begin{cases} n = 2 & \text{if } \alpha = 1; \\ \text{sign}(n - 2d) \sqrt{q} & \text{otherwise}. \end{cases}$$

We invert the Fourier transform to get

$$c_g = \frac{1}{|\Gamma|} \sum_{\alpha \in \Gamma} \hat{c}_\alpha \alpha(g) = \frac{1}{|\Gamma|} \left(n - 2 - \text{sign}(n - 2d) \sqrt{q}\right), \quad (g \neq 1).$$

Since $c_g$ is integer by Lemma 3.3, we are done. \qed

Corollary 6.3 rules out the existence of many abelian DRACKNS. For example, Table I lists abelian DRACKN parameters that meet the necessary conditions with $n \leq 500$ and $r$ an odd prime. For the sake of reproducibility, we provide our methodology for constructing this table:

1. Find pairs $(d, n)$ with $n \leq 500$ satisfying Corollary 6.3. By Corollary 4.9, we may ignore $d \in \{1, n - 1\}$. Following Gerzon’s bound [43], we also require $n \leq \min\{d^2, (n - d)^2\}$.

2. Find odd primes $r$ dividing $n$ for which there exists $c \in \mathbb{Z}$ corresponding to $d$. The fact that $r$ necessarily divides $n$ is given by Theorem 9.2 in [28].

3. Check additional constraints from [28], as summarized by Theorem 3.1 in [11].

We note that Theorem 5.3 establishes how Table I can be used to preclude the existence of abelian DRACKNS (and more generally, connected roux graphs) over groups of odd order. For example, since $n = 64$ does not appear in Table I, any connected roux graph with $n = 64$ must necessarily be over a group $\Gamma$ whose order is a power of 2. (In fact, the next subsection constructs such a roux with $\Gamma = C_4$.) As another perspective, Table I and Theorem 5.3 together indicate several directions for future research. Indeed, if there is a connected roux graph with $n \leq 500$ for an abelian group $\Gamma$ of order other than a power of 2, then $n$ must appear in a row of Table I with every odd prime $r$ dividing $|\Gamma|$. As such, repeated values of $n$ suggest the possible existence of roux for groups of composite order.

6.2 Consequences for maximal equiangular tight frames

Gerzon’s bound implies that an equiangular tight frame in $\mathbb{C}^d$ necessarily has $n \leq d^2$ vectors [43]. For this reason, ETFs that saturate this bound are known as maximal ETFs in the frame theory community. It turns out that maximal ETFs find applications in quantum information theory, where they are known as symmetric, informationally complete positive operator–valued measures [25]. Interestingly, Hadamard powers appear naturally in the context of maximal ETFs:

Proposition 6.4 (Corollary 19 in [56]). Given a maximal equiangular tight frame $\{\varphi_i\}_{i \in [d^2]}$ for $\mathbb{C}^d$ with signature matrix $S$, then $\{\varphi_i^{\otimes 2}\}_{i \in [d^2]}$ forms an equiangular tight frame for its $(\frac{d+1}{2})$-dimensional span with signature matrix $S^{\otimes 2}$. 

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Table 1: Abelian DRACKN parameters \((n, r, c)\) that meet the necessary conditions with \(n \leq 500\) and \(r\) an odd prime. Any such abelian DRACKN would necessarily produce equiangular tight frames of \(n\) vectors in \(\mathbb{C}^d\). Next, \(\delta = n - rc - 2\) is a parameter of interest defined in [28]. If such an abelian DRACKN is known by the authors to exist, its construction can be found in the reference(s) cited in the column labeled “Existence.”

| \(d\) | \(n\) | \(r\) | \(c\) | \(\delta\) | Existence |
|-------|-------|-------|-------|----------|-----------|
| 6     | 9     | 3     | 3     | -2       | \([19, 27, 41, 54]\) |
| 15    | 25    | 5     | 5     | -2       | \([19, 27, 41]\) |
| 11    | 33    | 3     | 9     | 4        |           |
| 21    | 36    | 3     | 12    | -2       | \([41]\) |
| 12    | 45    | 3     | 12    | 7        | \([41]\) |
| 33    | 45    | 5     | 10    | -7       |           |
| 28    | 49    | 7     | 7     | -2       | \([19, 27, 41]\) |
| 34    | 51    | 3     | 18    | -5       |           |
| 22    | 55    | 5     | 10    | 3        | \([17]\) |
| 55    | 100   | 5     | 20    | -2       | \([41]\) |
| 14    | 105   | 3     | 27    | 22       |           |
| 40    | 105   | 7     | 14    | 5        | \([17]\) |
| 65    | 105   | 3     | 36    | -5       | \([19, 41]\) |
| 35    | 120   | 3     | 36    | 10       |           |
| 66    | 121   | 11    | 11    | -2       | \([19, 27, 41]\) |
| 105   | 126   | 3     | 48    | -20      | \([19, 54]\) |
| 86    | 129   | 3     | 45    | -8       |           |
| 78    | 144   | 3     | 48    | -2       | \([41]\) |
| 29    | 145   | 5     | 25    | 18       |           |
| 46    | 161   | 7     | 21    | 12       |           |
| 91    | 169   | 13    | 13    | -2       | \([19, 27, 41]\) |
| 30    | 175   | 5     | 30    | 23       |           |
| 145   | 175   | 7     | 28    | -23      |           |
| 133   | 190   | 5     | 40    | -12      |           |
| 105   | 196   | 7     | 28    | -2       | \([41]\) |
| 67    | 201   | 3     | 63    | 10       |           |
| 77    | 210   | 5     | 40    | 8        |           |
| 133   | 210   | 3     | 72    | -8       |           |
| 186   | 217   | 7     | 35    | -30      |           |
| 120   | 225   | 3     | 75    | -2       |           |
| 120   | 225   | 5     | 45    | -2       |           |
| 175   | 225   | 3     | 81    | -20      |           |
| 70    | 231   | 3     | 72    | 13       |           |
| 161   | 231   | 11    | 22    | -13      |           |

| \(d\) | \(n\) | \(r\) | \(c\) | \(\delta\) | Existence |
|-------|-------|-------|-------|----------|-----------|
| 162   | 243   | 3     | 84    | -11      |           |
| 41    | 246   | 3     | 72    | 28       |           |
| 92    | 253   | 11    | 22    | 9        | \([17]\) |
| 52    | 273   | 7     | 35    | 26       |           |
| 221   | 273   | 3     | 99    | -26      |           |
| 217   | 280   | 5     | 60    | -22      |           |
| 42    | 288   | 3     | 84    | 34       |           |
| 153   | 289   | 17    | 17    | -2       | \([19, 27, 41]\) |
| 177   | 295   | 5     | 60    | -7       |           |
| 129   | 301   | 7     | 42    | 5        |           |
| 88    | 320   | 5     | 60    | 18       |           |
| 171   | 324   | 3     | 108   | -2       | \([41]\) |
| 225   | 325   | 13    | 26    | -15      |           |
| 260   | 325   | 5     | 70    | -27      |           |
| 113   | 339   | 3     | 108   | 13       |           |
| 78    | 351   | 3     | 108   | 25       |           |
| 126   | 351   | 13    | 26    | 11       | \([17]\) |
| 225   | 351   | 3     | 120   | -11      |           |
| 190   | 361   | 19    | 19    | -2       | \([19, 27, 41]\) |
| 117   | 378   | 3     | 120   | 16       |           |
| 261   | 378   | 7     | 56    | -16      |           |
| 33    | 385   | 5     | 65    | 58       |           |
| 55    | 385   | 7     | 49    | 40       |           |
| 105   | 385   | 11    | 33    | 20       |           |
| 154   | 385   | 5     | 75    | 8        |           |
| 262   | 393   | 3     | 135   | -14      |           |
| 210   | 400   | 5     | 80    | -2       | \([41]\) |
| 145   | 406   | 7     | 56    | 12       |           |
| 56    | 441   | 7     | 56    | 47       |           |
| 231   | 441   | 3     | 147   | -2       |           |
| 231   | 441   | 7     | 63    | -2       |           |
| 385   | 441   | 3     | 162   | -47      |           |
| 369   | 451   | 11    | 44    | -35      |           |
| 391   | 460   | 5     | 100   | -42      |           |
| 370   | 481   | 13    | 39    | -28      |           |
| 253   | 484   | 11    | 44    | -2       | \([41]\) |
| 97    | 485   | 5     | 90    | 33       |           |
| 209   | 495   | 3     | 162   | 7        |           |
| 286   | 495   | 5     | 100   | -7       |           |
It is widely believed that maximal ETFs exist in every dimension \[23, 24\]. This is the subject of Zauner’s conjecture \[60\]. The following result establishes the extent to which maximal ETFs arise as DRACKN lines.

**Corollary 6.5.** There do not exist \(d^2\) DRACKN lines spanning \(\mathbb{C}^d\). There exist \(d^2\) DRACKN lines spanning \(\mathbb{C}^{d^2-d}\) only if \(d = 3\).

The second part above is Corollary 6.7 in \[11\], whereas the first part answers an open problem posed at the end of Section 6 in the same paper. In particular, our result implies that none of the abelian DRACKNs satisfying case (II.a) of Theorem 6.5 in \[11\] exist. Our proof of both parts of Corollary 6.5 uses the same technique, namely, Corollary 6.2.

**Proof of Corollary 6.2.** Given a maximal ETF, then the Gram matrix is \(G = I + (1/\sqrt{d+1})S\) by equality in the Welch bound \[1\]. Furthermore, the eigenvalues of \(G\) are 0 and \(d\), and so the eigenvalues of \(S\) are given by

\[
\sigma(S) = \left\{-\sqrt{d+1}, (d-1)\sqrt{d+1}\right\}.
\]

By Proposition 6.4, the eigenvalues of \(G^{\circ 2} = I + (1/(d+1))S^{\circ 2}\) are 0 and \(2d/(d+1)\), and so

\[
\sigma(S^{\circ 2}) = \left\{- (d+1), d-1\right\}.
\]

We claim that \(S\) and \(-S\) (and therefore their normalized versions) fail to satisfy Corollary 6.2(b) with one exception. Indeed, the positive eigenvalues of \(S\) and \(S^{\circ 2}\) are equal only if \(d = 0\), whereas the positive eigenvalues of \(-S\) and \((-S)^{\circ 2} = S^{\circ 2}\) are equal only if \(d \in \{0, 3\}\).

Overall, DRACKN lines are too restrictive to produce maximal ETFs beyond \(d = 3\). However, roux lines appear to be a fruitful relaxation in this regard. For example, Theorem 2.5(a) gives that all three of the doubly transitive maximal ETFs classified in \[61\] (namely, those in \(\mathbb{C}^3\), the Hesse ETF in \(\mathbb{C}^4\) \[60\], and Hoggar’s lines in \(\mathbb{C}^8\) \[31\]) span roux lines. The following result provides another indication that roux lines may interact nicely with maximal ETFs:

**Corollary 6.6.** Every maximal equiangular tight frame whose signature matrix consists of \(4\)th roots of unity is roux.

**Proof.** We will check (a) and (b) in Corollary 4.6. Since the signature matrix already consists of roots of unity, its normalized version will as well, and so we have (a). For (b), note that the second Hadamard power has two eigenvalues by Proposition 6.4. Also, the third Hadamard power is equivalent to applying the complex conjugate entrywise, which fixes the (real) minimal polynomial of the signature matrix.

The following provides an explicit example other than \[5\]:

**Example 6.7.** Take \(h \in L^2(\mathbb{Z}_2^3)\) defined by \(h(0) = -1 + 2i\) and \(h(j) = 1\) for \(j \neq 0\) (as given in \[40, 52\]), and let \(T^a\) and \(M^b\) denote translation and modulation operators over \(L^2(\mathbb{Z}_2^3)\):

\[
(T^a)f(x) = f(x + a), \quad (M^b)f(x) = (-1)^{b \cdot x}f(x).
\]

Then \(\{T^aM^bh\}_{a,b \in \mathbb{Z}_2^3}\) is a maximal ETF (namely, Hoggar’s lines) in which every off-diagonal entry of its signature matrix \(S_H\) lies in \(C_4\). By Corollary 6.6, the vectors in this ETF span roux lines. To see this, consider the matrix \(B_k \in \mathbb{C}[C_4]^{22k \times 22k}\) with indices in \((\mathbb{Z}_2^k)^2\) defined by

\[
(B_k)_{(a,b),(c,d)} := \begin{cases} 
0 & \text{if } (a,b) = (c,d); \\
\delta_{\text{gray}^{-1}}(d, (a+c), b, (a+c)) & \text{else if } a = c \text{ or } b = d; \\
\delta_{\text{gray}^{-1}}(d, (a+c), b, (a+c)) & \text{otherwise}.
\end{cases}
\]
Here, \( \text{GRAY} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2^2 \) maps \( j \in \mathbb{Z}_4 \) to the \( j \)th Gray codeword \( \mathbb{Z}_4 \{ 0,1 \} \), i.e.,

\[
\text{GRAY}: \quad 0 \mapsto (0,0); \quad 1 \mapsto (0,1); \quad 2 \mapsto (1,1); \quad 3 \mapsto (1,0).
\]

Then \( B_k \) is a roux for \( C_4 \) when \( k \in \{1,3\} \). In particular, evaluating \( B_1 \) at the character \( \alpha \) defined by \( \alpha(z) = z \) gives the signature matrix of a maximal ETF in \( \mathbb{C}^2 \), whereas doing the same for \( B_3 \) produces the signature matrix \( S_H \) of Hoggar’s lines. The roux parameters of \( B_1 \) are \( c_1 = c_{-1} = 0 \) and \( c_3 = c_{-3} = 1 \) (cf. Lemma 3.4), whereas the roux parameters of \( B_3 \) are

\[
c_1 = 24, \quad c_3 = c_{-3} = 16, \quad c_{-1} = 6.
\]

Interestingly, the roux parameters of \( B_3 \) generalize to an infinite family, leaving open the possibility of an infinite family of maximal ETFs that arise from roux lines. In particular, for any positive integer \( j \), consider the parameters

\[
c_1 = 4j^4 + 12j^3 + 10j^2 - 2, \quad c_3 = c_{-3} = 4j^4 + 8j^3 + 4j^2, \quad c_{-1} = 4j^4 + 4j^3 - 2j^2.
\]

\( B_3 \) exhibits these parameters with \( j = 1 \). Summing these parameters (and adding 2) gives \( n = 16j^2(1+j)^2 \). Take \( \alpha \in \hat{C}_4 \) defined by \( \alpha(z) = z \). Then \( d := d_0^+ = 4j(j+1) = \sqrt{n} \), meaning that for any roux with these parameters, evaluating at \( \alpha \) produces the signature matrix of a maximal ETF in \( \mathbb{C}^d \). Furthermore, the integrality condition in Corollary 4.4 is satisfied with \( \sqrt{q} \in \mathbb{Z} \). In addition, \( d_{2n}^+ = \left( \frac{d+1}{2} \right) \), matching the necessary condition in Proposition 6.4. Finally, a real ETF of this size exists if and only if there exists a regular symmetric Hadamard matrix of constant diagonal, and such matrices necessarily exist whenever there is a Hadamard matrix of order \( d = 4j(j+1) \) [20]; the Hadamard conjecture implies that such a matrix exists for every \( j \).

### 7 Summary and proofs of main results

As demonstrated by Theorem 2.3, roux naturally arise in the study of doubly transitive lines. Indeed, the technology we have developed here plays a prominent role in the sequel paper [38], which classifies all doubly transitive lines with almost simple symmetries. Overall, the notion of roux simultaneously generalizes doubly transitive lines, abelian DRACKNs, and regular two-graphs. These multifaceted objects can be studied variously from the perspectives of group theory (roux proper), algebraic combinatorics (roux schemes), discrete geometry (roux lines), and graph theory (roux graphs). As we have seen, the theory favors a rich interplay between these perspectives. Due in part to their applications in equiangular lines and abelian DRACKNs, roux are worthy of study in their own right. The next steps are to find more constructions. Example 6.7 indicates one promising direction which, if fruitful, would give a combinatorial approach to Zauner’s conjecture [60]. In addition, Table 1 points out many open problems regarding the existence of roux.

What follow are the proofs of our main results, reported in Section 2. We save the proof of Theorem 2.3(a) for the sequel paper [38].

**Proof of Theorem 2.3(b).** Given a Higman pair \( (G,H) \), then the Schurian scheme of \( (G,H) \) is isomorphic to a roux scheme by Lemma 3.7. In particular, this scheme arises from an \( n \times n \) roux for \( \Gamma = K/H \), where \( K = N_G(H) \). Since the roux scheme’s adjacency algebra is necessarily commutative, we may conclude that \( (G,H) \) is a Gelfand pair. Next, Theorem 3.8 provides the primitive idempotents of the roux scheme, each of which is the Gram matrix of \( r \) equal-norm representatives from each of \( n \) lines, where the rank is strictly smaller than \( n \) and the phase of each entry is an \( r \)th root of unity. It remains to show that the lines are doubly transitive.
To this end, fix some $G = G'_G$ with rank $d = d'_G$ from Theorem 3.8. Since $G$ lies in $\mathcal{A}(G, H)$, there exists a unitary representation $\pi: G \to U(d)$ and a vector $v \in \mathbb{C}^d$ such that $\pi(h)v = v$ for every $h \in H$ and $\{\pi(x_ja_g)v\}_{j \in [n], g \in K/H}$ has Gram matrix $G$ by Theorem 3.2 in [35] (here, we follow Lemma 3.7 in selecting left coset representatives $\{x_j\}_{j \in [n]}$ for $K$ in $G$ and coset representatives $\{a_g\}_{g \in K/H}$ for $H$ in $K$). By Theorem 3.8, the entries of $G$ that have modulus 1 appear in the $r \times r$ diagonal blocks; these are the entries $G(i,g)(i,h)$ for $i \in [n]$ and $g, h \in \Gamma = K/H$. As such, we may conclude that $|\langle \pi(x_i a_g)v, \pi(x_j a_h)v \rangle| = 1$ if and only if $i = j$, which in turn means $|\langle \pi(x)v, \pi(y)v \rangle| = 1$ for $x, y \in G$ if and only if $x^{-1}y \in K$.

Now we consider how $G$ acts on the lines spanned by $\{\pi(x_ja_g)v\}_{j \in [n], g \in K/H}$ under the action $g \cdot [\pi(x)v] = [\pi(g)\pi(x)v]$. This action is transitive, and furthermore, since $g \cdot [v] = [v]$ if and only if $|\langle v, \pi(g)v \rangle| = 1$, if and only if $g \in K$, we have that $K$ is the stabilizer of $[v]$ under this action. By the orbit–stabilizer theorem, this action is equivalent to the action of $G$ on $G/K$, which is doubly transitive by (H1).

Proof of Theorem 2.5. Part (a) of the Higman Pair Theorem (see [35]) gives that doubly transitive lines arise from primitive idempotents of the corresponding Schurian scheme, which is a roux scheme by Theorem 3.1 and so the primitive idempotents produce roux lines by Lemma 4.2. This gives (a), while (b) follows immediately from Theorem 4.1.

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