Research Article

Fekete–Szegő Inequality for Bi-Univalent Functions Subordinate to Horadam Polynomials

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1. Preliminaries

Let \( \mathbb{R} \) and \( \mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0\setminus\{0\} \) be the sets of real numbers and positive integers, respectively. Let \( \mathbb{C} \) be the set of all complex numbers, and let \( \mathfrak{D} \) denote the disc \( \{z \in \mathbb{C} : |z| < 1\} \). We denote by \( \mathcal{A} \), the set of all regular functions in \( \mathfrak{D} \) that has the series of the form

\[
g(z) = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1}
\]

and \( \mathcal{S} \) be the set of all members of \( \mathcal{A} \) that are univalent in \( \mathfrak{D} \). According to the well-known Koebe theorem (see [1]), every univalent function \( g \) has an inverse defined by

\[
g^{-1}(\omega) = f(\omega) = \omega - d_2 \omega^2 + \frac{(2d_2^2 - d_3)\omega^3}{2d_2^3 - 5d_2d_3 + d_4} \omega^4 + \cdots, \tag{2}
\]

satisfying \( z = g^{-1}(g(z)) \) and \( \omega = g(g^{-1}(\omega)) \), \( |\omega| < r_0(g) \), \( r_0(g) \geq 1/4 \), \( z, \omega \in \mathfrak{D} \).

A function \( g \) of \( \mathcal{A} \) is said to be bi-univalent (or bi-schlicht) in \( \mathfrak{D} \) if both \( g \) and \( g^{-1} \) are univalent in \( \mathfrak{D} \). Let \( \Sigma \) stand for the set of bi-univalent functions having form (1).

Lewin [2] investigated the family \( \Sigma \) and proved that \( |d_2| < 1.51 \). Brannan and Clunie [3] claimed that \( |d_2| < \sqrt{2} \).

Later, Tan [4] obtained initial coefficient estimates for bi-univalent functions. Subsequently, Brannan and Taha [5] examined certain well-known subfamilies of \( \Sigma \) in \( \mathfrak{D} \). The momentum on the study of bi-univalent function family was gained recently, which is due to the work of Srivastava et al. [6]. This article has revived the topic apparently, and many researchers have investigated several interesting special families of \( \Sigma \) (see [7–10]).

Recently, Hörçum and Koçer [11] (see also Horadam and Mahon [12]) examined the Horadam polynomials \( \mathcal{H}_j(x) \) (or \( \mathcal{H}_j(x, a, b; p, q) \)), which is defined by the recurrence relation

\[
\mathcal{H}_j(x) = px\mathcal{H}_{j-1}(x) + q\mathcal{H}_{j-2}(x),
\]

\[
\mathcal{H}_1(x) = a,
\]

\[
\mathcal{H}_2(x) = bx,
\]

where \( j \in \mathbb{N}\setminus\{1, 2\} \), \( x \in \mathbb{R} \), \( p, q, a, \) and \( b \) are real constants. It is seen from (3) that \( \mathcal{H}_3(x) = ppx^3 + qa \). The generating function of the sequence \( \mathcal{H}_j(x) \), \( j \in \mathbb{N} \), is as follows (see [11]):
\[ G(x, z) = \sum_{j=1}^{\infty} H_j(x)z^{j-1} = \frac{(b - ap)xz + a}{1 - pxz - qz^2}, \quad (4) \]

where \( z \in \mathbb{C} \) is such that \( R(z) \neq x, x \in \mathbb{R} \).

Few particular cases of \( H_j(x, a, b; p, q) \) are

1. The Fibonacci polynomials, \( F_j(x) = H_j(x, 1, 1, 1) \)
2. The second type Chebyshev polynomials, \( U_j(x) = H_j(x, 1, 2, 2, -1) \)
3. The first type Chebyshev polynomials, \( T_j(x) = H_j(x, 1, 1, 2, -1) \)
4. The Lucas polynomials, \( L_j(x) = H_j(x, 2, 1, 1) \)
5. The Pell – Lucas polynomials, \( Q_j(x) = H_j(x, 2, 2, 2, 1) \)
6. The Pell polynomials, \( P_j(x) = H_j(x, 1, 2, 2, 1) \)

The estimates on \( |d_2| \) and \( |d_4| \) and the very popular Fekete–Szegő functional were determined for bi-univalent functions linked with certain polynomials like Lucas polynomials, Fibonacci polynomials, Chebyshev polynomials, Horadam polynomials, and Gegenbauer polynomials. It is well-known that these polynomials play a potentially important role in architecture, approximation theory, physics, statistics, mathematical, and engineering sciences.

The recent research trend is the study of functions in \( \Sigma \) linked with any of the abovementioned polynomials. Generally, interest was shown to obtain the initial coefficient bounds and the celebrated inequality of Fekete–Szegő for the special subfamilies of \( \Sigma \). Recently, the Horadam polynomial was used by Abirami et al. [13] to find coefficient estimates for the families of bi-Bazilevic and \( \lambda \)-bi-starlike function, Frasin et al. [14] obtained coefficient estimates and Fekete–Szegő inequalities for certain subfamilies of Al-Oboudi-type bi-univalent functions related to \( k \)-Fibonacci numbers involving modified activation function, initial coefficient bounds for certain subsets of bi-univalent functions family subordinate to Horadam polynomials were obtained in [15, 16], Shaba and Wanas [17] obtained coefficient bounds which are sharp, for a family of bi-univalent functions using \( (U, V) \)-Lucas polynomials, Srivastava et al. [18] have proposed a methodology to estimate coefficient bounds and Fekete–Szegő problem for certain subsets of bi-univalent function family linked with Horadam polynomials, and Swamy [19] and Swamy et al. [20, 21] have initiated the study of some subfamilies of bi-univalent function family subordinate to Horadam polynomials involving modified activation function. Swamy and Sailaja [22] have used Horadam polynomials to investigate coefficient estimates for two families of bi-univalent functions, Swamy et al. [23] have introduced some subfamilies of Sălăgean type bi-univalent functions subordinate to \( (m, n) \)-Lucas polynomials and found initial coefficients, and Wanas and Alina [24] have fixed the Fekete–Szegő problem for Bazilevic bi-univalent function class linked with Horadam polynomials.

For functions \( g \) and \( f \) holomorphic in \( \mathcal{D} \), \( g \) is said to subordinate \( f \), if there is a Schwarz function \( \psi \) in \( \mathcal{D} \), such that \( \psi(0) = 0, |\psi(z)| < 1, \) and \( g(z) = f(\psi(z)), z \in \mathcal{D} \). This subordination is indicated as \( g < f \). In particular, if \( f \in \mathcal{S} \), then \( g(z) < f(z) \) is equivalent to \( g(0) = f(0) \) and \( g(\mathcal{D}) \subset f(\mathcal{D}) \).

Inspired by the article [25] and the recent trends on functions in \( \Sigma \), we present a comprehensive family of \( \Sigma \) associated with Horadam polynomials \( H_j(x) \) as in (3) having the generating function (4).

Throughout this paper, the inverse function \( g^{-1}(\omega) = f(\omega) \) as in (2) and \( G(x, z) \) is as in (4).

**Definition 1.** A function \( g \) in \( \Sigma \) having the power series (1) is said to be in the set \( \mathcal{S}\mathcal{E}_x^2(\chi, \gamma, \mu), \) is as (4) and \( G(x, z) \) is as in (4).

\[ z(g'(z))^\gamma + \mu z^2 g''(z) < 1 - a + G(x, z), \quad z \in \mathcal{D}, \]
\[ \frac{\omega(f'(\omega))}{\gamma f(\omega)} + \mu \omega^2 f''(\omega) < 1 - a + G(x, \omega), \quad \omega \in \mathcal{D}. \]

(5)

The family \( \mathcal{S}\mathcal{E}_x^2(\chi, \gamma, \mu) \) is of interest as it contains many existing as well as new subfamilies of \( \Sigma \) for particular choices of \( \gamma, \tau, \) and \( \mu, \) as illustrated as follows:

1. \( \mathcal{J}_1^x(\chi, \mu, \mu, 0) \equiv \mathcal{S}\mathcal{E}_x^2(\chi, 0, \mu, \mu, 0) \) and \( x \in \mathbb{R} \), is the collection of functions \( g \in \Sigma \) satisfying
\[ z(g'(z))^\gamma + \mu z^2 g''(z) < 1 - a + G(x, z), \quad z \in \mathcal{D}, \]
\[ (6) \]
2. \( \mathcal{J}_2^x(\chi, \mu, \mu, 0) \equiv \mathcal{S}\mathcal{E}_x^2(\chi, 1, \mu, \mu, 0) \) and \( x \in \mathbb{R} \), is the collection of functions \( g \in \Sigma \) satisfying
\[ z(g'(z))^\gamma + \mu z^2 g''(z) < 1 - a + G(x, z), \quad z \in \mathcal{D}, \]
\[ (7) \]
3. \( \mathcal{J}_3^x(\gamma, \mu, \mu, 0) \equiv \mathcal{S}\mathcal{E}_x^2(\chi, \gamma, 1, \mu, \mu, 0) \) and \( x \in \mathbb{R} \), is the collection of functions \( g \in \Sigma \) satisfying
\[ z(g'(z))^\gamma + \mu z^2 g''(z) < 1 - a + G(x, z), \quad z \in \mathcal{D}, \]
\[ (8) \]
4. The function classes \( \mathcal{S}\mathcal{E}_x^2(\chi, \gamma, \mu) \) \( (0 \leq \gamma \leq 1, \mu \geq 0) \) and \( x \in \mathbb{R} \) were investigated by the author in [19].

**Remark 1.** We note that

(i) \( \mathcal{J}_1^x(\chi, 1, \mu, \mu, 0) \equiv \mathcal{S}\mathcal{E}_x^2(\chi, 0, 1, \mu, \mu, 0) \) and \( x \in \mathbb{R} \)
(ii) \( \mathcal{J}_2^x(\chi, 1, \mu, \mu, 0) \equiv \mathcal{S}\mathcal{E}_x^2(\chi, 1, 1, \mu, \mu, 0) \) and \( x \in \mathbb{R} \)
Remark 2.

(i) For $\tau = 1$, the family $S_1^l(x, \mu)$ was investigated by Swamy and Sailaja [22]
(ii) $S_2^l(x, 1, 0) \equiv S_2(x, 0)$ was due to Abirami et al. [13]
(iii) $S_3^l(x, 1, \mu) \equiv R_2(x, \mu)$ was introduced by Magesh et al. [16]

Remark 3.

(i) For $\mu = 0$ and $\tau = 1$, the class $S_1^l(x, 0) \equiv S_2(x)$ was studied by Alamush [15]
(ii) For $S_1^l(x, 0) \equiv S_3(x)$ and $\tau = 1$, the family $R_2(x, 0) \equiv S_3(x)$ was introduced by Srivastava et al. [6]

In Section 2, we derive the estimates for $|d_2|$ and $|d_3|$ and the inequality of Fekete and Szegö [26] for functions in the class $S_2^l(x, \gamma, \mu)$. In Section 3, relevant connections to the existing results and few interesting consequences of the main result are presented.

2. Bi-Univalent Function Class $S_2^l(x, \gamma, \mu)$

We determine the initial coefficient bounds and the inequality of Fekete–Szegö for functions in $S_2^l(x, \gamma, \mu)$, in the following theorem.

Theorem 1. Let the function $g(z)$ defined by (1) be in the family $S_2^l(x, \gamma, \mu)$ and let $0 \leq \gamma \leq 1, \mu \geq 0, \tau \geq 1$ and $x \in \mathbb{R}$. Then,

\begin{equation}
|d_2| \leq \frac{|bx| \sqrt{|b|}}{\sqrt{\left( y^2 + (\tau - y)(2\tau + 1) + 2\mu(3 - y) \right) \left( bx \right)^2 - (2(\mu + \tau) - y)\left( pbx^2 + qa \right) }}
\end{equation}

\begin{equation}
|d_3| \leq \frac{(bx)^2}{(2(\mu + \tau) - y)z + \frac{|bx|}{(2\mu + \tau) - y}}
\end{equation}

and for $\delta \in \mathbb{R}$,

\begin{equation}
|d_3 - \delta d_2^2| \leq \begin{cases} 
\frac{|bx|}{3(2\mu + \tau) - y} & |1 - \delta| \leq J, \\
\frac{|bx|^3 |1 - \delta|}{\left[ y^2 + (\tau - y)(2\tau + 1) + 2\mu(3 - y) \right] (bx)^2 - (2(\mu + \tau) - y)\left( pbx^2 + qa \right) ^2} & |1 - \delta| \geq J,
\end{cases}
\end{equation}

where

\begin{equation}
J = \frac{1}{(3(2\mu + \tau) - y)} \left[ y^2 + (\tau - y)(2\tau + 1) + 2\mu(3 - y) \right] (bx)^2 - (2(\mu + \tau) - y)\left( \frac{pbx^2 + qa}{b^2 x^2} \right)
\end{equation}

Proof. Let $g \in S_2^l(x, \gamma, \mu)$. Then, on account of Definition 1, we get

\begin{equation}
\frac{z (g'(z))^2 + \mu z^2 g''(z)}{yg(z) + (1 - y)z} = 1 - a + G(x, \mathfrak{M}(z)),
\end{equation}

\begin{equation}
\frac{\omega (f'(\omega))^2 + \mu \omega^2 f''(\omega)}{yf(\omega) + (1 - y)\omega} = 1 - a + G(x, \mathfrak{N}(\omega)),
\end{equation}

are some regular functions in $\mathfrak{D}$ with $\mathfrak{M}(0) = 0, |\mathfrak{M}(z)| < 1, \mathfrak{N}(0) = 0$, and $|\mathfrak{M}(\omega)| < 1, z, \omega \in \mathfrak{D}$. It follows from (13)–(16) with (4) that
\[
\frac{z(g'(z))^2 + \mu z^2 g''(z)}{y g(z) + (1 - y)z} = 1 - a + \mathcal{H}_1(x) + \mathcal{H}_2(x)m(z) + \mathcal{H}_3(x)m^2(z) + \cdots,
\]

\[
\frac{\omega(f'(\omega))^2 + \mu \omega^2 f''(\omega)}{y f(\omega) + (1 - y)\omega} = 1 - a + \mathcal{H}_1(x) + \mathcal{H}_2(x)n(\omega) + \mathcal{H}_3(x)n^2(\omega) + \cdots.
\]

From (17) and (18), in view of (3), we find
\[
\frac{z(g'(z))^2 + \mu z^2 g''(z)}{y g(z) + (1 - y)z} = 1 + \mathcal{H}_2(x)m_1z + \left[\mathcal{H}_2(x)m_2 + \mathcal{H}_3(x)m_1^2\right]z^2 + \cdots,
\]

\[
\frac{\omega(f'(\omega))^2 + \mu \omega^2 f''(\omega)}{y f(\omega) + (1 - y)\omega} = 1 + \mathcal{H}_2(x)n_1\omega + \left[\mathcal{H}_2(x)n_2 + \mathcal{H}_3(x)n_1^2\right]\omega^2 + \cdots.
\]

It is known that if \( |\Re(z)| = |m_1z + m_2z^2 + m_3z^3 + \cdots| < 1, \ z \in \mathcal{D}, \) then
\[
|m_i| \leq 1, \quad (i \in \mathbb{N}).\]  

(20)

Similarly, if \( |\Re(\omega)| = |n_1\omega + n_2\omega^2 + n_3\omega^3 + \cdots| < 1, \ \omega \in \mathcal{D}, \) then

\[
\mathcal{H}_2(x)(m_2 + n_2) + \mathcal{H}_3(x)(m_1^2 + n_1^2),
\]

which yields (10) on using (21) and (22).

After subtracting (26) from (24) and then using (27), we obtain
\[
d_3 = d_2^2 + \frac{\mathcal{H}_2(x)(m_2 - n_2)}{2(3(\mu + \tau) - \gamma)}.
\]

(31)

Then, in view of (28), (31) becomes
\[
|d_3 - \delta d_2^2| = |\mathcal{H}_2(x)|\left(\mathcal{B}(\delta, x) + \frac{1}{2(3(\mu + \tau) - \gamma)}m_2 + \left(\mathcal{B}(\delta, x) - \frac{1}{2(3(\mu + \tau) - \gamma)}\right)n_2\right),
\]

where
\[
\mathcal{B}(\delta, x) = \frac{(1 - \delta)\mathcal{H}_2(x)}{2\left[\left(y^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma)\mathcal{H}_2(x) - (2(\mu + \tau) - \gamma)^2\mathcal{H}_3(x)\right]\right],
\]

(34)
Clearly,

\[
|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|\mathcal{H}_2(x)|}{3(2\mu + \tau) - \gamma}, \\
2|\mathcal{H}_2(x)| |\mathcal{B}(\delta, x)|,
\end{cases}
\]

from which we conclude (9) with \( J \) as in (12). Thus, Theorem 1 is proved.

Remark 4. By taking \( \tau = 1 \) in Theorem 1, we get a result of the author (Corollary 2.4 [19]), and by letting \( \mu = 0 \) in Theorem 1, we obtain another result of the author (Corollary 3.3 in [19]).

3. Outcome of the Main Result

Theorem 1 would yield the following outcome when \( \gamma = 0 \).

\[
|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau)}, \\
|bx|^3|1 - \delta| \\
\frac{|bx|}{\sqrt{(\tau(2\tau + 1) + 6\mu)(bx)^2 - 4(\mu + \tau)^2(pb^2x^2 + qa)}} \\
|1 - \delta| \leq J_1,
\end{cases}
\]

where

\[
J_1 = \frac{1}{3(2\mu + \tau)} \left| \tau(2\tau + 1) + 6\mu - 4(\mu + \tau)^2\left(\frac{pbx^2 + qa}{b^2x^2}\right) \right|
\]

(35)

Corollary 1. Let the function \( g(z) \) defined by (1) be in the family \( \mathcal{S}_2^t(x, \mu) \equiv \mathcal{S}_2^t(x, 0, \mu) \), and let \( \mu \geq 0, \tau \geq 1 \) and \( x \in \mathbb{R} \). Then,

\[
|d_3| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left(\tau(2\tau + 1) + 6\mu)(bx)^2 - 4(\mu + \tau)^2(pb^2x^2 + qa)\right)}} \\
|d_3| \leq \frac{b^2x^2}{4(\mu + \tau)^2} + \frac{|bx|}{3(2\mu + \tau)}, \quad (36)
\]

and for some \( \delta \in \mathbb{R} \).

Corollary 2. Let the function \( g(z) \) defined by (1) be in the set \( \mathcal{S}_2^t(x, \mu) \equiv \mathcal{S}_2^t(x, 1, \mu) \), and let \( \mu \geq 0, \tau \geq 1 \) and \( x \in \mathbb{R} \). Then,

\[
|d_3| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left(\tau(2\tau - 1) + 4\mu)(bx)^2 - 2(\mu + \tau - 1)^2(pb^2x^2 + qa)\right)}} \\
|d_3| \leq \frac{(bx)^2}{(2(\mu + \tau) - 1)^2} + \frac{|bx|}{3(2\mu + \tau - 1)}, \quad (39)
\]

and for \( \delta \in \mathbb{R} \).

Remark 5. For \( \tau = 1 \), Corollary 1 reduces to Corollary 2.1 of Swamy and Sailaja [22]. Also, we obtain Theorem 2.2 of Alamoush [15] from Corollary 1 when \( \mu = 0 \) and \( \tau = 1 \).

Allowing \( \gamma = 1 \) in Theorem 1, we obtain the following.

\[
|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau) - 1}, \\
|bx|^3|1 - \delta| \\
\frac{|bx|}{\sqrt{\left(\tau(2\tau + 1) + 4\mu)(bx)^2 - 2(\mu + \tau - 1)^2(pb^2x^2 + qa)\right)}} \\
|1 - \delta| \leq J_2,
\end{cases}
\]

(38)

(37)}
where

\[ J_2 = \frac{1}{(3(2\mu + \tau) - 1)} \tau(2\tau - 1) + 4\mu - (2(\mu + \tau) - 1)^2 \left( \frac{pbx^2 + qa}{b^2x^2} \right) \]  

Remark 6.
(i) Corollary 2 coincides with Theorem 2.1 of Abirami et al. [13] when \( \mu = 0 \)
(ii) Corollary 2 further coincides with Theorem 2.1 of Magesh et al. [16] when \( \tau = 1 \)
(iii) We obtain Corollary 1 and Corollary 3 of Srivastava et al. [18] from Corollary 1 when \( \mu = 0 \) and \( \tau = 1 \).

Setting \( \mu = 1 \) in Theorem 1, we have the following.

Corollary 3. Let the function \( g(z) \) defined by (1) be in the class \( \mathcal{K}(x, y, 1) = S\mathcal{T}_x(x, y, 1) \) and let \( 0 \leq y \leq 1, \tau \geq 1, \) and \( x \in \mathbb{R} \). Then,

\[ |d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{(1 - \gamma)^2 + (\tau - \gamma)(2\tau + 1) + 5 - \gamma}\left(bx\right)^2 - (2(1 + \tau) - \gamma)^2\left(\frac{pbx^2 + qa}{b^2x^2}\right)} \]  

\[ |d_3| \leq \frac{(bx)^2}{2(1 + \tau) - \gamma^2} + \frac{|bx|}{3(2 + \tau) - \gamma} \]

and for \( \delta \in \mathbb{R} \),

\[ |d_3 - \delta d_2| \leq \begin{cases} \frac{|bx|}{3(2\mu + \tau) - \gamma} & |1 - \delta| \leq J_3, \\ \frac{|bx|}{(1 - \gamma)^2 + (\tau - \gamma)(2\tau + 1) + 5 - \gamma}\left(bx\right)^2 - (2(1 + \tau) - \gamma)^2\left(\frac{pbx^2 + qa}{b^2x^2}\right) & |1 - \delta| \geq J_3, \end{cases} \]

where

\[ J_3 = \frac{1}{(3(2 + \tau) - \gamma)} \left( (1 - \gamma)^2 + (\tau - \gamma)(2\tau + 1) + 5 - \gamma \right) - (2(1 + \tau) - \gamma)^2 \left( \frac{pbx^2 + qa}{b^2x^2} \right). \]

4. Conclusion

A subfamily of bi-univalent (or bi-schlicht) functions is examined by using Horadam polynomial. Bounds of the first two coefficients \(|d_2|\) and \(|d_3|\) and the Fekete–Szegö functional have been fixed for this subfamily. We have presented relevant connections to the existing results and few interesting consequences of the main result.

A subfamily examined in this article could inspire researchers to focus on other aspects such as a family bi-univalent functions using \( q \)-derivative operator [27, 28], a family using \( q \)-integral operator [29], meromorphic bi-univalent function family based on Al-Oboudi differential operator [30], regular bi-univalent function family based on Frasin operator [31], and a family using integro-differential operator [32].
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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