Spectral portraits of the Orr–Sommerfeld operator with large Reynolds numbers.

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Abstract. A model problem of the form

\[-i\varepsilon y'' + q(x)y = \lambda y\]
\[y(-1) = y(1) = 0\]

is associated with well-known in hydrodynamics Orr–Sommerfeld operator. Here \(\lambda\) is the spectral parameter, \(\varepsilon\) is the small parameter which is proportional to the viscosity of the liquid and to the reciprocal of the Reynolds number, and \(q(x)\) is the velocity of the stationary flow of the liquid in the channel \(|x| \leq 1\). We study the behaviour of the spectrum of the corresponding model operator as \(\varepsilon \to 0\) with linear, quadratic and monotonous analytic functions. We show that the sets of the accumulation points of the spectra (the limit spectral graphs) of the model and the corresponding Orr–Sommerfeld operators coincide as well as the main terms of the counting eigenvalue functions along the curves of the graphs.

The well-known in hydrodynamics Orr–Sommerfeld equation is obtained by linearization of the Navier–Stokes equation in the infinite three-dimension spatial layer \((x, \xi, \eta) \in \mathbb{R}^3\), where \(|x| \leq 1\) and \((\xi, \eta) \in \mathbb{R}^2\), assuming that a stationary unperturbed solution for the velocity profile is of the form \((q(x), 0, 0)\). This equation with respect to a function \(y = y(x)\) has the form (see details in the book of Drazin and Reid [2], for example)

\[(0.1) \quad (D^2 - \alpha^2)^2 y - i\alpha R \left[q(x)(D^2 - \alpha^2) - q''(x)\right] y = -i\alpha R\lambda(D^2 - \alpha^2)y.\]

Here \(D = d/dx\), \(\alpha\) is a wave number \((\alpha \neq 0)\) appearing after the separation of the variables \((\xi, \eta) \in \mathbb{R}^2\), \(R\) is the Reynolds number characterizing the viscosity of the liquid and \(\lambda\) is the spectral parameter. Usually the boundary conditions

\[(0.2) \quad y(\pm 1) = y'(\pm 1) = 0\]

are associated with equation (0.1).

The main goal of this paper is to describe qualitatively the limit behaviour of the spectrum of problem (0.1), (0.2) as \(R \to \infty\). The Reynolds number \(R\) is proportional to the reciprocal of the viscosity. Therefore, our problem is devoted to the description of the spectrum of the Orr–Sommerfeld problem for the liquid which is almost ideal. During long time it was supposed that the spectrum of the Rayleigh problem

\[(0.3) \quad q(x)(D^2 - \alpha^2)y - q''(x)y = \lambda(D^2 - \alpha^2)y,\]
\[y(-1) = y(1) = 0,\]

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plays an important role for the solution of the problem which is raised above. The Rayleigh problem is obtained (after dividing of equation (0.1) by \(-i\alpha R\)) by formal passing to the limit as \(R \to \infty\) and by eliminating of the ”superfluous” boundary conditions. A vast literature is devoted to the study of the spectrum of the problem (0.3). The papers of Lin \([6]\) and the cited book \([2]\) can be recommended as a guide in this subject. Actually, the spectrum of the Rayleigh problem plays no essential role in the description of the spectral portraits of the Orr–Sommerfeld problem as \(R \to \infty\). This will be seen in the sequel.

It is known \([2]\), that the spectrum of the problem (0.3) consists of the segment \([m, M]\), where \(m\) and \(M\) are minimum and maximum of the function \(q(x)\) (it is supposed that \(q(x)\) is continuous), and possibly, some isolated eigenvalues of finite multiplicity lying outside this segment. Perhaps, Heisenberg was the first who noticed that there is no continuity between the spectra of the Orr–Sommerfeld problem with large \(R\) and the spectrum of the Rayleigh problem. It may happen that there is a domain containing the interval \((m, M)\) which is free of the spectra of problem (0.1), (0.2) for all sufficiently large \(R\). This phenomenon was named as a ”Heisenberg tongue”. In 1924 Heisenberg proved the existence of a system of fundamental solutions for equation (0.1) having special representation (see \([2]\)). This result is very essential for explanation of such a phenomenon. However, we are not aware of Heisenberg’ papers containing ideas to explain the phenomenon. It was Morawetz \([7]\), who proved that in the case of the Couette profile \(q(x) = x\) the spectrum of the problem (0.1), (0.2) is concentrated in a \(\delta\)-neighbourhoods of the ray \([-i/\sqrt{3}, -i\infty]\), of two segments \([-1, -i/\sqrt{3}]\) and the isolated eigenvalues \(\{\mu_k\}\) of the Rayleigh problem for \(q(x) = x\). Actually, the Rayleigh problem for \(q(x) = x\) has no isolated eigenvalues (see \([2]\)), therefore, the last reservation can be dropped. Moreover, Morawetz made an attempt to prove a similar result for more general functions. She assumed (this assumption is quite essential in her method) that \(q(x)\) is an entire function with real values on the real axis and maps bijectively the whole complex plane \(\mathbb{C}\) into itself. Probably, it was not noticed that all the function possessing these properties have the representation \(q(x) = \Theta x + \theta\), where \(0 \neq \Theta, \theta \in \mathbb{R}\). There is another important problem which was left beyond the paper \([7]\). What is the set of the accumulation points of the eigenvalues as \(R \to \infty\)? Morawetz emphasized in \([7]\) that her method did not allow to get the information if the eigenvalues exists as \(R \to \infty\) near each point of the segments \([-1, -i/\sqrt{3}]\).

In 90-ies there appeared papers (see \([17], [10]\), for example), where a simpler problem of the form

\[
-\ii \varepsilon z'' + q(x)z = \lambda z,
\]

\[
z(-1) = z(1) = 0,
\]

was associated with (0.1), (0.2). Here \(\varepsilon\) and \(\lambda\) are the small and spectral parameters, respectively. One can consider this problem as a simplified model for (0.1), (0.2). The following arguments can confirm this point.

Let us make in equation (0.1) the substitution \(z = (D^2 - \alpha^2)y\). From this equation taking into account the conditions \(y(-1) = y'(-1) = 0\), we find

\[
y(x) = \frac{1}{\alpha^2} \int_{-1}^{x} \sinh \alpha(x - \xi)z(\xi)\, d\xi.
\]

Then equation (0.1) takes the form

\[
-\ii \varepsilon (D^2 - \alpha^2)z + q(x)z + Kz = \lambda z,
\]
where

\[ Kz = \int_{-1}^{x} \sinh(2(x - \xi)q''(\xi)z(\xi)) d\xi, \quad \varepsilon = \frac{1}{\alpha R}. \]

It follows from (0.6) that \( y(-1) = y'(-1) = 0 \). Therefore, one can rewrite boundary conditions (0.2) in the form

\[ (0.8) \quad \int_{-1}^{1} z(\xi) \sinh(\alpha(1 - \xi)) d\xi = 0, \quad \int_{-1}^{1} z(\xi) \cosh(\alpha(1 - \xi)) d\xi = 0. \]

Hence, problem (0.1), (0.2) is equivalent to problem (0.7), (0.8). This reduction was realized by Orr in 1915. Now, if we neglect the influence of the integral operator \( K \) (notice, that \( K = 0 \) in the case \( q(x) = x \)) and assume that boundary conditions do not change essentially the spectral portraits as \( \varepsilon \to 0 \), we come to model problem (0.4), (0.5) up to the shift of the spectral parameter by \( i\varepsilon \alpha^2 \). Of course, these arguments are heuristic. However, we shall show below that the similarity of the spectral portraits of the model and the Orr–Sommerfeld problem as \( \varepsilon \to 0 \) can be proved rigorously.

If we put \( \varepsilon > 0 \) in equation (0.4) instead of \( i\varepsilon \), we get a selfadjoint problem with small parameter. Such a problem is well investigated long ago (see [12], for example). Its spectrum is real, it condenses as \( \varepsilon \to 0 \) and the formulae for the eigenvalue localization can be written out explicitly. They are known as the Bohr–Sommerfeld quantization formulae.

The replacement of \( \varepsilon \) by \( i\varepsilon \) changes the problem dramatically. In 1997 the author [13] described the spectral portraits of model problem (0.4), (0.5) as \( \varepsilon \to 0 \) in the case \( q(x) = x \) and paid attention to the fact that for analytic functions \( q(x) \) the spectrum is concentrated along some curves which are determined by the geometry of the Stokes lines of equation (0.4).

It is useful here to recall the definitions. The zeros of the equation \( q(z) - \lambda = 0 \) in the complex \( z \)-plane are called turning points and the lines

\[ \gamma_{\xi_\lambda} = \left\{ z \in \mathbb{C} \left| \Re \int_{\xi_\lambda}^{z} \sqrt{i(q(\xi) - \lambda)} d\xi = 0 \right. \right\}, \]

outgoing from a fixed turning point \( \xi_\lambda \) are called the Stokes lines of equation (0.4). These lines come either to the boundary of a domain \( G \), where \( q(z) \) is holomorphic (in particular, go to \( \infty \) if \( q(z) \) is an entire function), or end in other turning points. The maximal connected set consisting of the Stokes lines and containing a turning point \( \xi_\lambda \) is called the Stokes complex corresponding to \( \xi_\lambda \). The Stokes complex may contain several turning points but not necessary all of them. The union of Stokes complexes corresponding to all turning points is called the Stokes graph.

As we have mentioned before, for analytic (or piece-wise analytic) functions \( q(x) \) the spectrum of problem (0.4), (0.5) is concentrated near some curves in the \( \lambda \)-plane as \( \varepsilon \to 0 \). We call them limit spectral curves. The union of the limit spectral curves we call limit spectral graph. One should not mix the limit spectral curves or the limit spectral graph with the Stokes lines and the Stokes graph which lie in \( z \)-plane. Concrete forms of the limit spectral curves can be found comparatively easy only for the linear function \( q(x) = x \). For nonlinear functions \( q(x) \) these curves take a complicated form. Therefore, attempting to describe the spectral portraits for non-selfadjoint problems with small or large parameter one meets serious difficulties even in the case of model problem (0.4), (0.5). Now we can formulate our goals more concretely.
To find functions \( q(x) \) of particular and general form for which the spectral portraits of
problem (0.4), (0.5) as \( \varepsilon \to 0 \) can be described completely. Of course, special attention
should be paid to profiles which correspond to stationary solutions of the Navier–Stokes
equation, in particular to the Couette profile \( q(x) = x \), Poiseuille profile \( q(x) = 1 - x^2 \)
and Couette–Poiseuille profile \( q(x) = ax^2 + bx + c, \ a, b, c \in \mathbb{R} \);
(2) To find the formulae for the eigenvalues near the limit spectral curves as \( \varepsilon \to 0 \),
provided that these curves are already determined;
(3) To solve the same problems for the original Orr–Sommerfeld problem with the same
functions \( q(x) \).

At the present moment the problems, which we have raised up, are solved only partially. In
this paper we present the obtained results based on the research of the author and his students
A. V. Dyachenko, S. N. Tumanov and M. I. Neiman-zade. We should also mention recent papers
of Chapman [1], Redparth [11] and Stepin [16] which are closely connected with the problems
in question.

1. Model problem: the case \( q(x) = x \).

Let us consider the spectral problem
\[
\begin{align*}
-i\varepsilon y'' & = (x - \lambda)y, \\
y(-1) & = y(1) = 0,
\end{align*}
\]
where \( \varepsilon \) is a small parameter. As we have mentioned, computer calculations give a striking
result: the eigenvalues of this problem for sufficiently small \( \varepsilon \to 0 \) are concentrated on the
ray \( \gamma_\infty = [-i/\sqrt{3}, -i\infty) \) and near the segments \( \gamma_\pm = [\pm 1, -i/\sqrt{3}] \) (see Figure 1). Of course,
the density of the eigenvalues is increased as \( \varepsilon \to 0 \). Based on the method of the paper of
Morawetz [7] (although problem (1.1), (1.2) was not treated in [7]) one can show, that given
\( \delta > 0 \) the spectrum of this problem is contained in the \( \delta \)-neighbourhood of the limit spectral
graph
\[ \Gamma = \gamma_+ \cup \gamma_- \cup \gamma_\infty, \]
provided that \( \varepsilon < \varepsilon_0(\delta) \). The author was not acquainted with the paper [7] when he started
to study this subject. In [13] he described the spectral portrait of problem (1.1), (1.2) as
\( \varepsilon \to 0 \) based on the properties of fundamental solution of the Airy equation. Moreover, the
explicit formulae for the eigenvalues near \( \gamma_+ \) and \( \gamma_- \) were written down. Later on, these results
were completed and sharpened in the papers of Djachenko and Shkalikov [3], [4]. Here we will
formulate the last results concerning problem (1.1), (1.2) and sketch basic ideas.

First, let us note about the following simple fact.

**Lemma 1.1.** For any \( \varepsilon > 0 \) the spectrum of problem (1.1), (1.2) lies in the closure of the
semi-strip
\[ \Pi = \{ \lambda \mid \text{Im} \lambda < 0, -1 < \text{Re} \lambda < 1 \}. \]

**Proof.** Let us consider the operator \( Ly = i\varepsilon y'' + xy \) generated by boundary conditions (1.2). Its spectrum coincide with the spectrum of the problem. Obviously, the values
of the quadratic form
\[ (Ly, y) = -i\varepsilon (y', y') + (xy, y) \]
lie in \( \Pi \), provided that \( ||y|| = 1 \) and \( y \) belongs to the domain \( \mathcal{D}(L) \) of the operator. The values
of the quadratic form form the numerical range of the operator \( L \). Now, the assertion follows
The substitution $\xi = (-i\varepsilon)^{-1/3}(x - \lambda)$ reduces problem (1.1), (1.2) to the following one:

$$
z''(\xi) = \xi z(\xi), \\
z(\xi_1) = z(\xi_2) = 0,
$$

$$
\xi_1 = e^{i\pi/6} \varepsilon^{-1/3}(-1 - \lambda), \quad \xi_2 = e^{i\pi/6} \varepsilon^{-1/3}(1 - \lambda).
$$

Thus, we have the spectral problem for the Airy equation. This problem is uncommon, since the spectral parameter is involved only in the boundary conditions, and the end points change depending on the spectral parameter. However, the eigenvalues of this problem can be found by standard means. Namely, the eigenvalues are determined by the equation

$$
\Delta(\lambda) = \begin{vmatrix} v(\xi_1) & v(\xi_2) \\ w(\xi_1) & w(\xi_2) \end{vmatrix} = 0,
$$

where $v(\xi)$ and $w(\xi)$ are linear independent solutions of the Airy equation. Let us take the Airy function $v(\xi)$ subject the following asymptotic equality

$$
v(\xi) = \frac{1}{2\sqrt{\pi} \xi^{1/4}} e^{-\frac{2}{3} \xi^{3/2}} \left(1 + O\left(|\xi|^{-3/2}\right)\right), \quad \xi \in \Lambda_{\pi-\delta}, \quad \xi \to \infty,
$$

where $\Lambda_{\pi-\delta} = \{\xi \mid \arg \xi < \pi - \delta\}$, $\delta > 0$, and the main branches of $\xi^{1/4}$ and $\xi^{3/2}$ are chosen in $\Lambda_{\pi-\delta}$. A domain, where representation (1.3) is valid, can be enlarged. We take advantage from the following result obtained in [3].

**Lemma 1.2.** The Airy function $v(\xi)$ preserves asymptotic representation (1.3) in the domain

$$
\Lambda = \left\{\xi \mid |\xi| > 1, \ |\arg \xi| < \pi - \frac{3}{4}|\xi|^{-3/2} \ln |\xi|\right\},
$$

i. e. (1.3) is valid for all $\xi \in \Lambda$, and the remainder is majorated by $C|\xi|^{-3/2}$ uniformly for $\xi \in \Lambda$. 

**Figure 1.**

from the well-known fact: the spectrum of any operator lies in the closure of its numerical range. □
Proof. We make use of the well-known representation [9]

\[ v(\xi) = e^{-\pi i/3} v\left(e^{2\pi i/3} \xi\right) + e^{\pi i/3} v\left(e^{-2\pi i/3} \xi\right). \]

Notice, that in the intersection of \( \Lambda \) with the second quadrant of the complex plane the estimate

\[ |v\left(e^{-2\pi i/3} \xi\right)| \leq C|\xi|^{-3/2} |v\left(e^{2\pi i/3} \xi\right)| \]

is valid. Moreover, the first summand in (1.4) has representation (1.3) in this intersection. Using the symmetry principle we obtain the assertion of Lemma. \( \square \)

Let us consider the function

\[ f(\lambda) = \int_{-1}^{1} e^{-\pi i/4} \sqrt{x - \lambda} dx = \frac{2}{3} e^{-\pi i/4} \cdot \left[(1 - \lambda)^{3/2} - (-1 - \lambda)^{3/2}\right], \]

which will be used in the formulation of the main theorem of this section. The function \( f(\lambda) \) is holomorphic in the domain \( \Pi \) where the spectrum is located. We choose the branch by the condition \( f(-i/\sqrt{3}) > 0 \).

Denote

\[ \Lambda_\alpha = \{\lambda \in \mathbb{C} \mid \arg \lambda < \alpha\}. \]

Fix an arbitrary number \( \sigma > 2^{-2/3}3^{-1/4} \) and consider the domain \( D_\sigma \) (see Figure 2) bounded by the lines \( \text{Re} \lambda = \pm 1 \) from the right and from the left, and by the lines passing through the points \( \pm 1 \) and the point

\[ d_\sigma = -i \left(\frac{1}{\sqrt{3}} + \sigma \varepsilon^{1/2} |\ln \varepsilon|\right) \]

from the above. The constant \( 2^{-2/3}3^{-1/4} \) for the choice of \( \sigma \) is explicit in our calculations, we can not make it smaller.

Lemma 1.3. The function \( f(\lambda) \) has the following properties.

1. \( f(\lambda) \) is holomorphic in the lower half-plane and takes real values on the negative imaginary axis;
(2) for all \( \lambda \in D_\sigma \) the values of \( f(\lambda) \) and \(-i f'(\lambda)\) belong to the sector \( \Lambda_{\pi/6} \), and the values of \( f'''(\lambda) \) belong to the lower half-plane;
(3) \( f(\lambda) \) increases monotonously and tends to \( \infty \) as \( \lambda \rightarrow -i \infty \) along the negative imaginary axis;
(4) \( f(\lambda) = 2 \sqrt{i \lambda} + O\left(|\lambda|^{-3/2}\right), \ f'(\lambda) = i / \sqrt{i \lambda} + O\left(|\lambda|^{-3/2}\right) \) as \( \lambda \rightarrow \infty, \ \lambda \in D_\sigma \);
(5) \( f(D_\sigma) < \Re f(\lambda) \) for \( \lambda \in D_\sigma \);
(6) \( |f(\lambda_1) - f(\lambda_2)| \geq c \frac{|\lambda_1 - \lambda_2|}{\sqrt{|\lambda_1|+|\lambda_2|}} \) for \( \lambda_1, \lambda_2 \in D_\sigma \), where the constant \( c \) does not depend on \( \lambda_1, \lambda_2 \);
(7) the function \( f(\lambda) \) takes real values in \( D_\sigma \) only at the imaginary axis.

**Proof.** All these assertions follow easily from representation (1.5). The details can be found in [4]. \( \square \)

Let \( \{r_k\}_{k=1}^\infty \) be the increasing sequence of the zeros of the function \( v(-r) \), where \( v(\xi) \) is the Airy function with representation (1.3). Then \( r_k > 0 \), and the asymptotic equalities

\[
(1.7) \quad r_k = \left( \frac{3\pi}{2} \left( k - \frac{1}{4} \right) \right)^{2/3} + O\left( \frac{1}{k^{4/3}} \right), \quad k = 1, 2, \ldots
\]

are valid. It is not obvious, that the numeration in these formulas starts from \( k = 1 \). The explanations are given in [4].

Set

\[
\delta_\sigma = \delta_\sigma(\varepsilon) = \sigma \varepsilon |\ln \varepsilon|, \quad \sigma > 2^{-3/2}3^{-1/4}.
\]

Denote

\[
(1.8) \quad \mu^-_k = -1 + e^{-\pi i/6} \varepsilon^{1/3} r_k, \quad k = 1, 2, \ldots, k_1, k_1 + 1
\]
\[
\mu^+_k = 1 - e^{\pi i/6} \varepsilon^{1/3} r_k, \quad k = 1, 2, \ldots, k_1, k_1 + 1,
\]

where the number \( k = k_1(\varepsilon) \) is the largest integer such that the inequality \( \varepsilon^{1/3} r_k < 2/\sqrt{3} - \delta_\sigma \) holds. The numbers \( \mu^-_k \) and \( \mu^+_k \) lie on the segments \( \gamma_- = [1, -i/\sqrt{3}] \) and \( \gamma_+ = [-1, -i/\sqrt{3}] \) respectively. They come up to the \( \delta_\sigma \)-neighbourhood of the knot-point \(-i/\sqrt{3}\), but only the last ones get inside this neighbourhood.

Next, let us construct a sequence of numbers on the imaginary axis. By virtue of Lemma [1.3], \( f(-i\rho) \) is the increasing function for \( \rho \in \mathbb{R}^+ \). Moreover,

\[
f(d_\sigma) = f(-i(1/\sqrt{3} + \delta_-)) > 0.
\]

Given \( \varepsilon > 0 \) choose the smallest integer \( k_0 = k_0(\varepsilon) \) such that \( f(d_\sigma) < \pi k_0 \varepsilon^{1/2} \). Then, each equation

\[
f(-i\rho) - \pi k \varepsilon^{1/2} = 0, \quad k = k_0 - 1, k_0, k_0 + 1, \ldots,
\]

has the only solution \( \rho = \rho_k > -i d_\sigma \) for \( k \geq k_0 \) and \( \rho_{k_0-1} < -i d_\sigma \). The sequence \( \{\rho_k\}_{k=0}^\infty \) is monotonous and \( \rho_k \rightarrow \infty \) as \( k \rightarrow \infty \).

Now, we are ready to formulate the main result on the eigenvalue portrait of model problem (1.1), (1.2).

**Theorem 1.1.** Let \( \sigma > 2^{-3/2}3^{-3/4} \) be any fixed number and \( \delta_\sigma = \sigma \varepsilon |\ln \varepsilon| \). Set

\[
\varphi(t) = \frac{4}{3} \Re (2e^{\pi i/6} - t)^{3/2}, \quad t \in [0, 2/\sqrt{3}].
\]
Denote by $U_k^\pm$ the neighbourhoods of the points $\mu_k^\pm$, $1 \leq k \leq k_1$, of radius $\gamma_k = C \exp \left( -\varepsilon^{-1/2} \varphi (\varepsilon^{1/2} r_k) \right)$, by $U_0^\infty$ the neighbourhood of the points $-i\rho_k$, $k \geq k_0 + 1$, of radius $C\rho_k^{-1}\varepsilon$, and by $U_0$ the $\delta_\varepsilon$-neighbourhood of the knot-point $-i/\sqrt{3}$.

There are numbers $C = C(\sigma)$ and $\varepsilon_0 = \varepsilon_0(\sigma)$ not depending on $\varepsilon$ and $k$ such that for all $\varepsilon < \varepsilon_0$ the circles $\{U_k^\pm\}_{k=1}^{k_0+1}$ and $U_0^\infty$ contain the only eigenvalue of problem (1.1), (1.2). Moreover, the eigenvalues in $U_0^\infty$ are pure imaginary. All the other eigenvalues lie in the circle $U_0$.

In other words, the pure imaginary eigenvalues in the domain $D_\sigma$ have asymptotics

$$\lambda_k = -i \left( \rho_k + \varepsilon \rho_k^{-1} O(1) \right), \quad k = k_0 - 1, k_0, k_0 + 1, \ldots,$$

the eigenvalues near the segments $\gamma_\pm$ have asymptotics

$$\lambda_k^\pm = \mu_k^\pm + \exp \left( -\varepsilon^{-1/2} \varphi (\varepsilon^{1/2} r_k) \right) O(1), \quad k = 1, 2, \ldots, k_1, k_1 + 1,$$

and there are

$$\sigma^{2^{-1/2} 3^{3/4}} |\ln \varepsilon| + O(1)$$

eigenvalues inside the circle $U_0$. In all these formulae the quantity $|O(1)|$ is estimated by a constant $C$ depending only on $\sigma$.

**Proof.** Here we outline only the main ideas, the details can be found in [4].

Let us construct the characteristic determinant of the problem using the Airy solutions $v(\xi)$ and $w(\xi) = v \left( e^{2\pi i/3} \xi \right)$. We have

$$\Delta(\xi) = v(\xi_1) v \left( e^{-2\pi i/3} \xi_2 \right) - v \left( e^{-2\pi i/3} \xi_1 \right) v(\xi_2).$$

**Step 1.** First, we investigate the zeros of $\Delta(\lambda)$ in the domain $D_\sigma$. Notice that the variables $\xi_1$ and $e^{-2\pi i/3} \xi_j$, $j = 1, 2$, lie in the domain $\Lambda$ of Lemma 1.3 provided that $\lambda \in D_\sigma$. Therefore, for all $\lambda \in D_\sigma$, we can use asymptotic representation (1.3) for the functions $v(\xi_j)$ and $v \left( e^{-2\pi i/3} \xi_j \right)$. After simple transformations we get the equation

$$e^{4 \left( \xi_1^{3/2} - \xi_2^{3/2} \right)} = e^{2\pi i - 1/2 f(\lambda)} = 1 + \lambda^{-3/2} \varepsilon^{1/2} O(1),$$

which is equivalent in the domain $D_\sigma$ to the equation $\Delta(\lambda) = 0$. Taking the logarithm of both sides, we come to the equations

$$f(\lambda) - \pi k \varepsilon^{1/2} = \lambda^{-3/2} \varepsilon^{1/2} O(1), \quad k \in \mathbb{Z}, \quad \lambda \in D_\sigma.$$

Now, recall the definition of the number $k_0$ and the estimate $|f'(\lambda)| > C|\lambda|^{-1/2}$, $\lambda \in D_\sigma$, which follows from Lemma 1.3. Then, using the Rouche theorem, we find that the equation (1.11) has the only root $\lambda_k$ inside the circle $U_0^\infty$, provided that $k \geq k_0 + 1$. This root is necessarily pure imaginary, since the spectrum of the problem is symmetric with respect to the imaginary axis.

Actually, we can prove the same for the indices $k_0$ and $k_0 - 1$ if we consider a larger domain $D_{\sigma'}$ with $2^{-2/3} 3^{-1/4} < \sigma' < \sigma$. It is easily seen due to properties of $f(\lambda)$ proved in Lemma 1.3 that there are no roots in the domain $D_\sigma$ outside the circles $\{U_k^\infty\}_{k=0}^{k_0-1}$.

**Step 2.** While investigating the zeros of $\Delta(\lambda)$ near the segment $[-1, -i/\sqrt{3}]$ it is convenient to rewrite the equation $\Delta(\lambda) = 0$ in the form

$$\frac{v(\xi_1)}{v \left( e^{-2\pi i/3} \xi_1 \right)} = \frac{v(\xi_2)}{v \left( e^{-2\pi i/3} \xi_2 \right)}.$$
Let us consider the trapezium $\Omega_\sigma \subset \Pi$, which is bounded by the lines $\Re \lambda = -1$, $\Re \lambda = 0$, the real axis and the line passing through the points 1 and $\beta_\sigma = -1 + e^{-\pi i/6}(2/\sqrt{3} - \delta_\sigma)$. The point $\beta_\sigma$ is the intersection of the segment $\gamma_-$ and the circumference of radius $\delta_\sigma$ centered at the knot-point $-i/\sqrt{3}$. The variables $\xi_1$, $\xi_2$, $e^{-2\pi i/3}\xi_1$, $e^{-2\pi i/3}\xi_2$ take the values in the domain $\Lambda$ of Lemma 1.2 if $\lambda \in \Omega_\sigma$. Hence, we can use asymptotics (1.11) for the functions involved in (1.12). It is easy to show that the right hand side of (1.12) is majorated by $C \left| \exp \left( -\frac{4}{3} \xi_{3/2}^2 \right) \right|$. Introducing the new variable $r = (\lambda + 1)e^{i\pi/6}$ we get from (1.12) the relation

(1.13) \[ V(r) := \frac{v \left( -e^{-1/3}r \right)}{v \left( -e^{-2\pi i/3}r \right)} = O(1) \exp \left( -\varepsilon^{-1/2} \varphi(r) \right), \quad r = (\lambda + 1)e^{i\pi/6}. \]

The function $V(r)$ has the simple zeros $\varepsilon^{1/3}r_k$, $k = 1, 2, \ldots$, where $r_k$ have asymptotics (1.11). Using the Rouche theorem (here technicalities are omitted), we obtain that equation (1.13) has simple roots in the neighbourhoods of the points $\varepsilon^{1/3}r_k$ of radius $\gamma_k = C \exp \left( -\varepsilon^{-1/2} \varphi \left( \varepsilon^{1/3}r_k \right) \right)$, provided that $1 \leq k \leq k_1 + 1$. Coming back to the variable $\lambda$, we obtain the assertion of the theorem about the eigenvalues near the segment $\gamma_-$. It is easily seen from (1.13) that there are no other zeros in $\Omega_\sigma$ except the zeros lying in the circles $\{ U_k \}_{k=1}^{k_1+1}$. We pay attention that $\lambda_k^-$ lie in exponentially small neighbourhoods of the points $\mu_k^-$ provided that $|\mu_k^- - i/\sqrt{3}| > c > 0$. However, the radii $\beta_k$ of the circles $U_k$ increase as $|\mu_k^-|$, since $\varphi(t) \to 0$ as $t \to 2/\sqrt{3}$. It is easy to calculate that $\beta_k < C\varepsilon^\alpha$ with some $\alpha > 1/2$ (depending on $\sigma$) for all $k \leq k_1 + 1$.

**Step 3.** We have to show that there are no eigenvalues in the domain $\Pi^- \setminus (\Omega_\sigma \cup D_\sigma)$, where $\Pi^-$ is the intersection of the semistrip $\Pi$ and the left half-plane. Using representation (1.8) it is easy to show that for $\lambda$ belonging to this domain the inequality

\[ \left| v(\xi_1) v \left( e^{-2\pi i/3}\xi_2 \right) \right| > \left| v \left( e^{-2\pi i/3}\xi_1 \right) v(\xi_2) \right| \]

holds.

**Step 4.** Denote by $N(\rho, \varepsilon)$ the number of the eigenvalues of the problem (1.1), (1.2) lying above the line $\Im \lambda = -\rho$. Suppose that $\rho > 1/\sqrt{3} + \delta$, $\delta > 0$. Fix numbers $\varepsilon_0$ and $\varepsilon$ ($\varepsilon < \varepsilon_0$) and define numbers $k_0$ and $k$ as the largest integers satisfying the conditions

\[ \pi \varepsilon_0^{1/2}k_0 < f(-i\rho), \quad \pi \varepsilon^{1/2}k < f(-i\rho). \]

Then

\[ |N(\rho, \varepsilon) - N(\rho, \varepsilon_0) - (k - k_0)| \leq 2. \]

Therefore,

\[ N(\rho, \varepsilon) = k + O(1) = \frac{1}{\pi \varepsilon^{1/2}} f(-i\rho) + O(1). \]

It is easily seen that $N(\rho, \varepsilon_0) = k_0 = 0$ for sufficiently large $\varepsilon_0$ and $O(1)$ in the last formula takes the values 0 or $\pm 1$.

Now, using asymptotics (1.10) we can calculate the number of the eigenvalues in the circle $U_0$:

\[ N_{\sigma} := \frac{1}{\pi \varepsilon^{1/2}} \left( f \left( -\frac{i}{\sqrt{3}} - i\delta_{\sigma} \right) - \frac{4}{3} \left( \frac{2}{\sqrt{3}} - \delta_{\sigma} \right)^{3/2} \right) + O(1), \]

where $|O(1)| \leq 3$. After simple transformations we obtain

\[ N_{\sigma} = \frac{2^{1/2}3^{3/4}}{\pi} \ln \varepsilon + O(1) + o(1), \]

where $o(1) \to 0$ as $\varepsilon \to 0$ and $|O(1)| \leq 3$. This completes the proof. \(\square\)
In addition to this theorem we shall say some words about the movement of the \( \lambda_k(\varepsilon) \) as \( \varepsilon \to 0 \). Since the function \( f(-i\rho) \) is monotonous at the semiaxis \( \mathbb{R}^+ \), it follows from the proof of Theorem 1.1 that all the eigenvalues \( \lambda_k(\varepsilon) \) lying on the imaginary axis are simple and move up as \( \varepsilon \to 0 \). Due to the symmetry they can jump off the imaginary axis only in the case when \( \lambda_k \) gets into the circle \( U_0 \) and catches up the preceding eigenvalue \( \lambda_{k-1} \). After the collision they leave the imaginary axis. We have no information about their behaviour in the circle \( U_0 \).

However, if we proceed to diminish \( \varepsilon \), the eigenvalues go out from the small circle \( U_0 \) and move along the segments \( \gamma_- \) and \( \gamma_+ \) coming to these segments exponentially close.

2. Model problem: the case of a monotonous analytic function \( q(x) \).

Here we shall show that the spectral portrait of the explicitly solvable model problem (1.1), (1.2) is not accidental; similar phenomena are observed for a wide class of monotonous analytic functions \( q(x) \). Certainly, another method will be used for the study of general problem. Namely, we shall make use of the phase integral method or the so-called WKBJ-method. The results of this section were announced by the author [14].

We consider the problem

\[
\begin{align*}
(2.1) & \quad i\varepsilon^2 y'' + q(x) y = \lambda y, \\
(2.2) & \quad y(-1) = y(1) = 0.
\end{align*}
\]

We pay attention that here for convenience the small parameter is denoted by \( \varepsilon^2 \) instead of \( \varepsilon \) in equation (1.1). It might be assumed for the simplicity that \( q(x) \) admits an analytic continuation in the whole complex plane, although only the analyticity in a neighbourhood of the segment \([-1, 1]\) is required in the sequel in addition to the conditions formulated below.

Let \([a, b]\) be the range of the function \( q(x) \) defined for \( x \in [-1, 1] \). Denote by \( L(\varepsilon) \) the operator corresponding to problem (2.1), (2.2). Obviously, the values of the quadratic form \( (L(\varepsilon)y, y) \) for \( y \in \mathcal{D}(L), \|y\| = 1 \), lie in the semistrip

\[
\Pi = \{\lambda \mid \text{Im} \lambda < 0, a < \text{Re} \lambda < b\}.
\]

Hence, for any \( \varepsilon > 0 \) the eigenvalues of problem (1.1), (1.2) lie in this semistrip (see Lemma 1.2).

Let us formulate the main assumptions for \( q(x) \).

(i) The function \( q(x) \) is real for \( x \in [-1, 1] \) and there is a domain \( G \subset \mathbb{C} \) such that \( q(z) \) is analytic in \( G \) and maps \( \overline{G} \) bijectively onto \( \overline{\Pi} \) (here the overline implies the closure of the domains).

(ii) For any \( c \in (a, b) \) the preimage of the ray \( r_c = \{\lambda \mid \lambda = c - it, \ 0 \leq t < \infty\} \) is a function with respect to the imaginary axis, i.e. any line \( \text{Im} \lambda = \text{const} \) either intersects the preimage of the ray \( r_c \) only once, or has no intersection points.

Condition (i) implies that \( q(x) \) is strictly monotonous on the segment \([-1, 1]\). Without loss of generality we assume that \( q(x) \) is increasing. In this case the domain \( G \) belongs entirely to the lower half-plane (otherwise, we get a contradiction with condition (ii)). By virtue of the symmetry principle, the function \( q(z) \) maps bijectively \( G \cup G^* \cup (-1, 1) \) onto the strip \( a < \text{Re} \lambda < b \), where \( G^* \) is symmetrical to \( G \) with respect to the real axis. In particular, \( q'(x) > 0 \) for all \( x \in (-1, 1) \). The above conditions are satisfied, for example, for the functions \( q(x) = \sin(\pi x/2), q(x) = (x + 1)^2 \) etc.
Let us consider the following functions in the closure of the domain Π:

\[ Q(\lambda) = \int_{-1}^{1} \sqrt{i(q(x) - \lambda)} \, dx, \quad \lambda \in \Pi, \]

\[ Q^{\pm}(\lambda) = \pm \int_{\xi_{\lambda}}^{\pm 1} \sqrt{i(q(\xi) - \lambda)} \, d\xi, \quad \lambda \in \Pi, \]

where \( \xi_{\lambda} \) is the only root of the equation \( q(\xi) - \lambda = 0 \) lying in the domain \( G \). The branches of these functions can be chosen arbitrary. To be certain, fix the branches by the conditions

\[ Q(a) = \int_{-1}^{1} \sqrt{i(q(x) - a)} \, dx = e^{i\pi/4} \alpha, \quad \alpha > 0, \]

\[ Q^{+}(a) = Q(a), \quad Q^{+}(\lambda) + Q^{-}(\lambda) = Q(\lambda). \]

Certainly, the functions \( Q, Q^{+} \) and \( Q^{-} \) are analytic in \( \Pi \) and continuous in \( \overline{\Pi} \).

Define the following curves in the semistrip \( \overline{\Pi} \):

\[ \tilde{\gamma}_{\infty} = \left\{ \lambda \in \overline{\Pi} \mid \text{Re} \, Q(\lambda) = 0 \right\}, \]

\[ \tilde{\gamma}_{\pm} = \left\{ \lambda \in \overline{\Pi} \mid \text{Re} \, Q^{\pm}(\lambda) = 0 \right\}. \]

These curves are depicted in Figure 3 for the case \( q(x) = (x + 1)^2/4 \). Some parts of these curves are drawn by dotted lines. We shall show below that these parts play no role in the description of the spectral portrait. The remaining parts form the limit spectral graph, and each point of this graph accumulates the eigenvalues of the problem as \( \varepsilon \to 0 \).

First, we prove some important properties of the curves \( \tilde{\gamma}_{\pm} \) and \( \tilde{\gamma}_{\infty} \).

**Lemma 2.1.** The curve \( \tilde{\gamma}_{\pm} \) (\( \tilde{\gamma}_{-} \)) passes through the point \( b \) (point \( a \)) of the semistrip \( \overline{\Pi} \). Both curves are the functions with respect to the real axis.
PROOF. Consider the curve $\tilde{\gamma}_+$, the proof for $\tilde{\gamma}_-$ is similar. Fix a number $c \in (a, b)$. By assumption (i) there is the point $\xi_c \in (-1, 1)$ such that $q(\xi_c) - c = 0$. Let $\lambda = c - it$, $t \geq 0$ and $\xi_{\lambda}$ be the root of the equation $q(\xi) - \lambda = 0$, $\xi_{\lambda} \in G$. We have

$$Q^+(\lambda) = \int_{\xi_{\lambda}}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \left( \int_{\xi_{\lambda}}^{\xi_c} + \int_{\xi_c}^{1} \right) \sqrt{i(q(\xi) - \lambda)} \, d\xi =: F_1(\lambda) + F_2(\lambda).$$

The branch of the function $Q^+(\lambda)$ is defined by condition (2.3). Since $Q^+(\lambda)$ is continuous in $\Pi$, we have

$$Q^+(c) = \int_{\xi_c}^{1} \sqrt{i(q(\xi) - c)} \, d\xi = e^{\pi i/4} \alpha_c, \quad \alpha_c > 0.$$ 

Notice, that

$$\text{Re} \, F'_2(\lambda) = -\frac{1}{2} \int_{\xi_c}^{1} \frac{e^{\pi i/4}}{\sqrt{(q(\xi) - c) + it}} \, d\xi < 0, \quad \lambda = c - it,$$

since $0 < \arg [(q(\xi) - c) + it] < \pi/2$. Hence, the function $\text{Re} \, F_2(c - it)$ decreases monotonously, moreover, $\text{Re} \, F_2(c - it) \to 0$ as $t \to \infty$. Further, denote by $\xi(\mu)$ the preimage of the segment $\lambda = c - i\mu$, $0 \leq \mu \leq t$, under the map $q(z)$. Due to assumption (ii) we can parameterize this curve as follows $\xi(s) = -is + r(s)$, $0 \leq s = s(\mu) \leq s(t)$, $s'(\mu) \geq 0$.

While calculating the function $F_1(\lambda)$ we can take the integral along the curve $\xi(\mu)$. Then,

$$F_1(\lambda) = \int_{\xi_{\lambda}}^{\xi_c} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \int_{s(t)}^{0} \sqrt{i(c - i\mu - \lambda)} \, d\xi(s) = \int_{t}^{0} \sqrt{\mu - t} \, d(-is(\mu) + r(s(\mu))).$$

Bearing in mind the choice of the branch, we find that the function

$$\text{Re} \, F_1(c - it) = -\int_{0}^{t} \sqrt{t - \mu} \, ds(\mu)$$

decreases monotonously as $t \to +\infty$. Therefore, the function $\text{Re} \, Q^+(c - it)$ decreases monotonously as $t$ changes from 0 to $+\infty$. Moreover, this function is positive at zero and negative in a neighbourhood of $+\infty$. Hence, the equation $\text{Re} \, Q^+(c - it) = 0$ has the only root $t_c < 0$, and the curve $\tilde{\gamma}_+$ is a function with respect to the real axis. Since $Q^+(\lambda) \to 0$ as $\lambda \to b$, this curve ends in the point $b$. Lemma is proved.

**Lemma 2.2.** The curve $\tilde{\gamma}_\infty$ is a function with respect to the imaginary axis.

**Proof.** The branch of $Q(\lambda)$ in $\Pi$ is defined by condition (2.3). Since $a < q(x) < b$ for $x \in (-1, 1)$, we find that for all $t > 0$

$$\text{Re} \, Q(a - it) = \text{Re} \int_{-1}^{1} \sqrt{i(q(x) - a + it)} \, dx > 0,$$

$$\text{Re} \, Q(b - it) = \text{Re} \int_{-1}^{1} \sqrt{i(q(x) - b + it)} \, dx < 0.$$
Further,
\begin{equation}
\text{Re} \frac{Q'(\lambda)}{Q(\lambda)} = -\frac{1}{2} \int_{-1}^{1} \frac{e^{\pi i/4}}{\sqrt{q(x) - \lambda}} \, dx < 0,
\end{equation}

since \(\text{Im}(q(x) - \lambda) > 0\) for \(x \in (-1, 1)\) and \(\lambda \in \Pi\). This implies that for any fixed \(t > 0\) the function \(Q(c - it)\) of the variable \(c \in [-1, 1]\) vanishes in the only point. Lemma is proved. \(\square\)

**Lemma 2.3.** The functions \(Q^+(\lambda)\), \(Q^- (\lambda)\), \(Q(\lambda)\) are univalent in the semistrip \(\Pi\). In particular, the functions \(\text{Im} Q^+(\lambda)\), \(\text{Im} Q^- (\lambda)\), \(\text{Im} Q(\lambda)\) are strictly monotonous along the curves \(\tilde{\gamma}_+, \tilde{\gamma}_-\) and \(\tilde{\gamma}_\infty\), respectively.

**Proof.** It was shown in Lemma 2.1 that \(\text{Re} \frac{d}{dt} Q^+(c - it) < 0\) for \(t > 0\) and any fixed \(c \in (a, b)\). Consequently, \(\text{Re} \frac{d}{d\lambda} Q^+(\lambda) > 0\) for \(\lambda \in \Pi\). This implies the univalence of \(Q^+(\lambda)\) in \(\Pi\). Since \(\text{Re} Q^+(\lambda) = 0\) along the curve \(\tilde{\gamma}_+\), the derivative of the function \(\text{Im} Q^+(\lambda)\) along this curve does not vanish and preserves the sign. The same assertion is true for the function \(Q^- (\lambda)\). The univalence of the function \(Q(\lambda)\) in \(\Pi\) follows from inequality (2.5). Then, the function \(\text{Im} Q(\lambda)\) is strictly monotonous along the curve \(\tilde{\gamma}_\infty\). Lemma is proved. \(\square\)

**Lemma 2.4.** \(\text{Re} Q^+(\lambda) > 0\) if \(\lambda\) lies above the curve \(\tilde{\gamma}_+\) and \(\text{Re} Q^+(\lambda) < 0\) if \(\lambda\) lies below this curve. Similarly, \(\text{Re} Q^- (\lambda) < 0\) (> 0) if \(\lambda\) lies above (below) the curve \(\tilde{\gamma}_-\).

**Proof.** This assertion follows from the proof of Lemma 2.1 and representation (2.4). \(\square\)

**Lemma 2.5.** The curves \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\) have the only intersection point in \(\Pi\).

**Proof.** We can view the curves \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\) as the graphs of negative continuous function at the interval \((a, b)\). These functions vanish at the point \(b\) and \(a\) respectively, hence they have at least one intersection point \(\lambda_0\). Suppose, that there is another intersection point \(\lambda_1\). We can assume that there are no other intersection points at \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\) between \(\lambda_0\) and \(\lambda_1\). The parts of the curves \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\) between \(\lambda_0\) and \(\lambda_1\) form a closed Jordan curve \(\tilde{\gamma}\), and the interior of \(\tilde{\gamma}\) is a simply connected domain. Consider the function
\[
Q_0(\lambda) = \frac{\int_{\xi_0}^{\xi_1} \sqrt{i(q(x) - \lambda)} \, dx}{\xi_0 - \xi_0},
\]
which is analytic in \(\Pi\). It follows from the definition of \(\tilde{\gamma}_\pm\) that the harmonic function \(\text{Re} Q_0(\lambda)\) vanishes at both curves \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\). In particular, \(\text{Re} Q_0(\lambda) = 0\) for \(\lambda \in \tilde{\gamma}\). By virtue of the maximum principle \(\text{Re} Q_0(\lambda) \equiv 0\) in the interior of \(\tilde{\gamma}\). This implies \(Q_0(\lambda) = \text{const} in \Pi\), although \(Q(\lambda) \neq \text{const}\). This contradiction ends the proof. \(\square\)

Let \(\lambda_0\) be the intersection of \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\). It follows from (2.4) that the curve \(\tilde{\gamma}_\infty\) passes through the point \(\lambda_0\). By Lemma 2.5, \(\tilde{\gamma}_\infty\) has no other intersection points with \(\tilde{\gamma}_+\) and \(\tilde{\gamma}_-\). Denote by \(\gamma_+, \gamma_-\) and \(\gamma_\infty\) the parts of \(\tilde{\gamma}_+\), \(\tilde{\gamma}_-\) and \(\tilde{\gamma}_\infty\) between the knot-point \(\lambda_0\) and the points \(b, a\) and \(-i\infty\), respectively. Set
\begin{equation}
\Gamma = \gamma_+ \cup \gamma_- \cup \gamma_\infty.
\end{equation}

We shall show that \(\Gamma\) is the limit spectral graph of problem (2.1), (2.2). This implies the following: any point \(\lambda \in \Gamma\) is an accumulation point for the eigenvalues, while the complementary points \(\lambda \in \mathbb{C} \setminus \Gamma\) do not possess this property. Moreover, in the end of this section we shall find explicit formulae for the eigenvalue distribution along the curves \(\gamma_\pm\) and \(\gamma_\infty\).
Theorem 2.1. Given \( \tau > 0 \) there exists an \( \varepsilon_0 > 0 \) such that all the eigenvalues of problem (2.1), (2.2) lie in the \( \tau \)-neighbourhood of \( \Gamma \), provided that \( \varepsilon < \varepsilon_0 \).

Proof. We shall use in the sequel the results of asymptotic theory for ordinary differential equation. A comprehensive exposition of this theory can be found in books \( [9, 5] \).

The function
\[
S(z, \lambda) = \int_{\xi_\lambda}^{z} \sqrt{i(q(\xi) - \lambda)} \, d\xi = 0
\]
plays an important role in this theory. As before, here \( \xi_\lambda \) is the root of the equation \( q(\xi) - \lambda = 0 \) from the domain \( G \). By assumption the function \( q(z) \) is analytic in a neighbourhood of the segment \([-1, 1]\) and admits an analytic continuation in \( G \). Hence, there is a number \( \delta > 0 \) such that \( q(z) \) is analytic in the domain \( \Omega = G \cup U_\delta[-1, 1] \), where \( U_\delta[-1, 1] \) is the \( \delta \)-neighbourhood of \([-1, 1]\) (see Figure 4). Obviously, the function \( S(z, \lambda) \) is analytic with respect to the variable \( \lambda \in \Pi \) and continuous for \( \lambda \in \overline{\Pi} \), and locally analytic with respect to the variable \( z \in G \) with the branching point \( \xi_\lambda \).

Fix a number \( \lambda = c \in (-1, 1) \) and chose the branch of this function such that \( \text{Re} \, S(1, c) > 0 \). For other values \( \lambda \in \overline{\Pi} \) the branch of \( S(z, \lambda) \) is chosen by the following condition: the values \( S(1, c) \) and \( S(1, \lambda) \) are analytically connected.

For a fixed \( \lambda \in \Pi \) the set
\[
\{ z \mid \text{Re} \, S(z, \lambda) = 0 \},
\]
determines the lines in the \( z \)-plane which are called the Stokes lines. The point \( \xi_\lambda \) belongs to this set and three lines come out from this point. Conventionally, we will use the terms the right, the left and the lower Stokes lines. Although these terms are not defined rigorously, the identification will be clear from the context. To clarify the situation, we remark that for a fixed \( \lambda \in \Pi \) the domain \( G \) may contain some other Stokes lines, besides the Stokes complex with the knot-point \( \xi_\lambda \) (such a line is depicted near the point \(-1\) in Figure 4).

Lemma 2.6. Let \( \lambda \in \Pi \setminus \{ \gamma_+ \cup \gamma_- \} \), where \( \gamma_\pm \) are defined in (2.6). Consider all the cases (see Figure 5):

(1) The point \( \lambda \) lies above both curves \( \tilde{\gamma}_+ \), \( \tilde{\gamma}_- \);
(2) $\lambda$ lies under the curve $\tilde{\gamma}_-$, but above the curve $\tilde{\gamma}_+$;
(3) $\lambda \in \tilde{\gamma}_+ \setminus \gamma_+$, i.e. $\lambda$ belongs to $\tilde{\gamma}_+$, but lies under the curve $\tilde{\gamma}_-$;
(4) $\lambda$ lies under the curve $\tilde{\gamma}_+$, but above $\tilde{\gamma}_-$;
(5) $\lambda \in \tilde{\gamma}_- \setminus \gamma_-$, i.e. $\lambda$ belongs to $\tilde{\gamma}_-$ and lies under $\tilde{\gamma}_+$;
(6) $\lambda$ lies under both curves $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$.

Then, in the first case the left and the right Stokes lines intersect the interval $(−1, 1)$ in points $c^- = c^- (\lambda)$ and $c^+ = c^+ (\lambda)$, and no other Stokes line intersect the segment $[−1, 1]$. In the second case only the right Stokes line intersect the interval $(−1, 1)$ in a point $c^+ = c^+ (\lambda)$ and no other Stokes lines intersect the segment $[−1, 1]$. In the third case the right Stokes line intersect the point $1$ and no other Stokes lines intersect $[−1, 1]$. The fourth and the fifth cases are similar to the second and the third ones, only the roles of the points $−1$ and $1$ are changed. Finally, in the sixth case no Stokes lines intersect $[−1, 1]$.

**Proof.** The branch of the function $S(z, \lambda)$ is chosen by the condition $\text{Re} S(1, c) > 0$ for some (and hence for all) $c \in (−1, 1)$. Viewing $S(z, c)$ as an analytic function on $z \in G \cup [−1, c) \cup (c, 1]$, we find that $\text{Re} S(z, c) > 0$ for $c < z \leq 1$ and $\text{Re} S(z, c) < 0$ for $−1 \leq z < c$. It is proved in Lemma 2.4 that the function $\text{Re} S(1, −it) = \text{Re} Q^+(−it)$ decreases monotonously as $t$ runs from $0$ to $+\infty$ and vanishes in the only point. The Stokes lines continuously depend on $\lambda$, hence, for small values $t \in [0, t_0]$ these lines intersect the interval $(−1, 1)$ in points $c^−(t), c^+(t)$ which are close to the point $c$ (see Figure 4). For all $z \in (c^+, 1]$ we have

$$\text{Re} S(z, c − it) = \text{Re} \left( \int_{c^+}^{c^−} \int_c^z \sqrt{i(q(x) − λ)} \, dx \right) = \text{Re} \int_c^z \sqrt{i(q(x) − λ)} \, dx > 0$$

since the point $c^+$ lies on the Stokes line and the values of $q(x) − λ$ belong to the first quadrant of the complex plane for $\lambda = c − it$ and $x \geq c$. This inequality shows that no Stokes lines intersect the set $(c^+, 1]$. Similarly, no Stokes lines intersect the set $[−1, c^-)$, and the interval $(c^−, c^+)$.

Further, the function

$$\text{Re} Q^+(c − it) = \text{Re} S(1, c − it) = \text{Re} \int_{c^+}^1 \sqrt{i(q(x) − λ)} \, dx$$

decreases monotonously as $t$ grows from $0$. Therefore, the point $c^+ = c^+(t)$ moves to the right and reaches the point $1$ as $\lambda = c − it$ comes to the curve $\tilde{\gamma}_+$. Similarly, the point $c^−(t)$ moves to the left and reaches to the point $−1$ as $\lambda = c − it$ comes the curve $\tilde{\gamma}_−$. This analysis makes obvious all other assertions of Lemma 2.6. □

Now, let us recall an important concept of canonical domain for equation (2.1). A domain $\Omega_{\lambda}$ in the $z$-plane is called canonical if the function $S(z, \lambda)$ is univalent in $\Omega_{\lambda}$ (we do not define here maximal canonical domains). It follows easily from the definition that any domain not containing the points of the Stokes graph is canonical. Moreover, $\text{Re} S(z, \lambda)$ preserves the sign for $z$ belonging to such a domain. This fact implies that any domain containing only one Stokes line remains to be canonical.

**Lemma 2.7.** Given $\lambda \in \Pi \setminus (\gamma_+ \cup \gamma_-)$ there exist a path connecting the points $\pm 1$ and a canonical domain $\Omega_{\lambda}$ which entirely contains this path.
PROOF. Consider, for instance, the case when the point \( \lambda \) lies above both curves \( \tilde{\gamma}_+ \) and \( \tilde{\gamma}_- \). Recall the notation \( \Omega = G \cup U_{\delta}[-1, 1] \) and consider the domain \( \Omega \setminus \Omega_\lambda^+ \), where \( \Omega_\lambda^+ \) is the domain in \( \Omega \) bounded by the left and the right Stokes lines and containing the interval \((c^-, c^+) \subset (-1, 1)\) (see Figure 4). The domain \( \Omega \setminus \Omega_\lambda^+ \) contains the lower Stokes line outgoing from \( \xi_\lambda \) and, probably, some other Stokes lines. However, these other lines do not intersect the lines outgoing from the point \( \xi_\lambda \) (this is a general property), and do not intersect the sets \([-1, c^-] \) and \((c^+, 1]\) by virtue of Lemma 2.6. Hence, there is a path in \( \Omega \setminus \Omega_\lambda^+ \) connecting the points \( \pm 1 \) which intersects only the lower Stokes line, and there exists a neighbourhood of this path containing no Stokes lines but a part of the lower one. Such a domain is canonical (as we have noted before). All the other cases of disposition of a point \( \lambda \) (see Lemma 2.6) can be treated analogously. Lemma is proved.

The following well-known fact will be essentially used in the sequel.

**Lemma 2.8.** Given fixed \( \lambda \in \Pi \) equation (2.1) possesses two linear independent solution of the form

\[
v_{\pm}(z, \lambda) = \frac{1}{i(q(z) - \lambda)^{1/4}} e^{\pm \varepsilon^{-1} S(z, \lambda)} (1 + O_\pm(\varepsilon)),
\]

where the function \( O_\pm \) are subject the estimate

\[
|O_\pm(\varepsilon)| \leq C \varepsilon
\]

with a constant \( C \) not depending on \( z \), as \( z \) varies on a compact \( K \) belonging to a canonical domain \( \Omega_\lambda \). Moreover, if \( K \subset \Omega_\lambda \) as \( \lambda \) varies on a compact \( K' \) in the \( \lambda \)-plane, then the estimate holds with a constant \( C \) not depending on \( z \in K \) and \( \lambda \in K' \).

**Proof.** See [9, 5].

Now, we start a direct proof of Theorem 2.1. Fix an arbitrary number \( \tau > 0 \) and denote by \( \Gamma_\tau \) the \( \tau \)-neighbourhood of the limit spectral graph \( \Gamma \). We have to consider six different cases mentioned in Lemma 2.6 for the disposition of the point \( \lambda \) with respect to the curves \( \tilde{\gamma}_+ \) and \( \tilde{\gamma}_- \). Consider, for instance the first case, when \( \lambda \) lies under both curves \( \tilde{\gamma}_+ \) and \( \tilde{\gamma}_- \). The characteristic determinant of problem (2.1), (2.2) compiled from fundamental solutions (2.7) has the representation

\[
\Delta(\lambda) = \begin{vmatrix}
    v^+(1, \lambda) & v^+(1, \lambda) \\
    v^-(1, \lambda) & v^-(1, \lambda)
\end{vmatrix} = T(\lambda) \left( e^{\varepsilon^{-1}(S(1, \lambda) - S(-1, \lambda))} [1] - e^{-\varepsilon^{-1}(S(1, \lambda) - S(-1, \lambda))} [1] \right),
\]

where the function

\[
T(\lambda) = (i(q(1) - \lambda)^{-1/4} (i(q(-1) - \lambda)^{-1/4}
\]

does not vanish in \( \Pi \). For abreviation we use the Birkhoff notation \[1 \] = 1 + \( O(\varepsilon) \). It follows from representation (2.8) that \( \Delta(\lambda) \neq 0 \) if

\[
\varepsilon < \varepsilon_0 = \varepsilon_0(\lambda).
\]

Lemma 2.8 asserts that the obtained asymptotic representation for \( \Delta(\lambda) \) is valid, if the points \( \pm 1 \) can be connected by a path lying in a canonical domain of equation (2.1). Lemma 2.7 guarantees the existence of such a path \( \gamma_\lambda \) and a canonical domain \( \Omega_\lambda \). Moreover, the points \( +1 \) and \( -1 \) is this canonical domain are separated by a lower Stokes line. Recall that the function \( \text{Re} S(z, \lambda) \) changes the sign passing through the Stokes line. We fixed the branch of the function \( S(z, \lambda) \) by the condition \( \text{Re} S(1, \varepsilon) > 0 \) for \( \varepsilon \in (-1, 1) \), and this implies \( \text{Re} S(1, \lambda) > 0 \) and \( \text{Re} S(1, -\lambda) < 0 \).
if $\lambda$ lies under the curves $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$. Therefore, $\text{Re}S(-1,\lambda) < 0$ and inequality (2.8) holds. This inequality remains valid in a neighbourhood $U_\lambda$ of the point $\lambda$, by continuity. The neighbourhood $U_\lambda$ can be chosen such small that a path $\gamma$ connected the points 1 and $-1$ does not intersect the Stokes lines of the Stokes graph $\Gamma_\mu$ for all $\mu \in U_\lambda$ with exception of the lower lines. Therefore, the canonical domain $\Omega_\lambda$ containing $\gamma$ can be chosen such small that it remains to be canonical for all $\mu \in U_\lambda$, i.e. we can set $\Omega_\mu = \Omega_\lambda$. By virtue of Lemma 2.8 asymptotic representation (2.8) for $\Delta(\lambda)$ remains to be valid in the whole neighbourhood $U_\lambda$ of the point $\lambda$, and the remainders in the representations $[1] = 1 + O(\varepsilon)$ in (2.8) can be estimated by $C\varepsilon$ with a constant $C$ not depending on $\mu \in U_\lambda$.

Fix a number $R \gg 1$ and denote by $\Pi_R$ the intersection of the semistrip $\Pi$ with the closed circle of radius $R$ with center at the origin. For sufficiently small $\tau$ the set $\Pi_R \setminus \Gamma_\tau$ consists of three compacts, say, the upper, the left and the right ones. With each point $\lambda$ belonging to the upper compact $K^+$ we associate a neighbourhood $U_\lambda$, that was constructed above, and from the cover of $K^+$ by these neighbourhoods take a finite subcover. This procedure allows to obtain representation (2.8) for all $\lambda \in K^+$, moreover,

$$\text{Re}(S(1,\lambda) - S(-1,\lambda)) > c > 0, \quad \lambda \in K^+,$$

with a constant $c$ dependent only on $\tau$, and the estimates $|O(\varepsilon)| < C\varepsilon$ hold for the remainders in the representations $[1] = 1 + O(\varepsilon)$ with a constant $C$ dependent also only on $\lambda$. Then, we obtain $\Delta(\lambda) \neq 0$ for all $\lambda \in K^+$ if $\varepsilon < \varepsilon_0$ and $\varepsilon_0 = \varepsilon_0(\tau)$ is sufficiently small.

Analogously, we can prove the absence of the zeros of $\Delta(\lambda)$ in the left and the right compacts. It is left to show that the choice of a $\tau$-neighbourhood of the curve $\gamma_\infty$ can be realized independently on $R$, i.e. the eigenvalues with large moduli do not leave the $\tau$-neighbourhood of the curve $\gamma_\infty$, vice versa, asymptotically they lie more close to this curve. This fact can be proved in the same way as in Lemma 4.4 of paper [18] (see Remark 4.1 in [18]). Theorem is proved. 

Next, we shall obtain an additional information on the eigenvalue behaviour near the curves of the limit spectral graph. For this purposes we need the concept of counting function of the zeros of holomorphic functions along the curves.

Let $\gamma = \gamma(t), t \in [0, 1]$, be an oriented smooth curve in the complex plane $\mathbb{C}$ with end points $z_0$ and $z_1$ (the value $\infty$ for these points is admitted). One can order points on the curve as follows: $\lambda_1 < \lambda_2$ if $\lambda_j = \gamma(t_j)$ and $t_1 < t_2$. Denote by $\gamma_\tau(\lambda_1, \lambda_2)$ a curvilinear strip of width $2\tau$ containing $\gamma$ as the middle line and having, as the lateral sides, the segments perpendicular to $\gamma$ at the points $\lambda_1$ and $\lambda_2$. Let a function $F(z)$ be holomorphic in $\gamma_\tau(z_0, z_1)$. Fix a point $\lambda_1 \in \gamma$ and denote by $n(\lambda_1, \lambda)$ the number of zeros of the function $F(z)$ in $\gamma_\tau(\lambda_1, \lambda)$, if $\lambda_1 < \lambda$. For $\lambda < \lambda_1$ define $n(\lambda_1, \lambda) = -n(\lambda, \lambda_1)$. Now, define $N(\lambda) = n(\lambda_1, \lambda) + C$, where $C$ is an arbitrary constant, as the zero counting function of $F(z)$ in a $\tau$-neighbourhood of the curve $\gamma$ (or along the curve $\gamma$). Since the eigenvalues of the problem in question are the zeros of the entire function $\Delta(\lambda)$, we may speak in the same context on the eigenvalue counting functions along the curves.
Theorem 2.2. Fix a small number \( \delta > 0 \) and denote by \( \mu_k^+ \), \( \mu_k^- \) and \( \mu_k \) the solutions of the equations

\[
\begin{align*}
  i \int_{\xi}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi &= \varepsilon \pi (k - 1/4), & k \in \mathbb{Z}, \\
  i \int_{-1}^{\xi} \sqrt{i(q(\xi) - \lambda)} \, d\xi &= \varepsilon \pi (k - 1/4), & k \in \mathbb{Z}, \\
  i \int_{-1}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi &= \varepsilon \pi k, & k \in \mathbb{Z},
\end{align*}
\]

lying on the curves \( \gamma_+, \gamma_- \) and \( \gamma_{\infty} \) (note, that left hand-sides take real values along the corresponding curves). Choose the indices \( p_\pm, m_\pm \) and \( s_0 \) such that \( \{\mu_k^+\}_{p_+}, \{\mu_k^-\}_{p_-} \) and \( \{\mu_k\}_{s_0} \) are all solutions of these equations on the curves \( \gamma_+, \gamma_- \) and \( \gamma_{\infty} \), respectively, lying outside the \( \delta \)-neighbourhoods of the points \( a, b \) and the knot-point \( \lambda_0 \). Then, there exists a number \( C \) depending only on \( \delta \), such that all the eigenvalues lie in the \( \delta \)-neighbourhoods \( U_\delta(a) \), \( U_\delta(b) \) and \( U_\delta(\lambda_0) \), and in the circles of the radius \( C\varepsilon^2 \) centered at the points \( \{\mu_k^+\}_{p+,1}, \{\mu_k^-\}_{p-,1} \) and \( \{\mu_k\}_{s_0-1} \). All these circles contain only one eigenvalue. The eigenvalue counting functions along the curves \( \gamma_+, \gamma_- \) and \( \gamma_{\infty} \) have representations

\[
\begin{align*}
  N(\lambda) &= \frac{1}{i \pi \varepsilon} Q^+(\lambda) + O(1), & \text{if } \lambda \in \gamma_+, \\
  N(\lambda) &= \frac{1}{i \pi \varepsilon} Q^-(\lambda) + O(1), & \text{if } \lambda \in \gamma_-, \\
  N(\lambda) &= \frac{1}{i \pi \varepsilon} Q(\lambda) + O(1), & \text{if } \lambda \in \gamma_{\infty}.
\end{align*}
\]

The remainders in these formulae are estimated by a constant not dependent on \( \lambda \) and \( \varepsilon \), if \( \lambda \) lies outside the neighbourhoods \( U_\delta(a) \), \( U_\delta(b) \) and \( U_\delta(\lambda_0) \).

Proof. Consider, for example, the curve \( \gamma_+ \). We shall make use of the transmission formulas for asymptotic representations of solutions in neighbouring canonical domains. Below we formulate the corresponding result in a convenient form, viewing in mind our concrete problem.

Let \( \lambda \) lie in a small \( \tau \)-neighbourhood of the curve \( \gamma_+ \) outside some fixed neighbourhoods of the end points. Let \( C_\lambda \) be the Stokes complex corresponding to the point \( \lambda \), i.e. \( \xi_\lambda \) is the
knot-point of $C_\lambda$ (see Figure 5). One of the Stokes lines, say $l_1$, passes near the point 1 ($l_1$ passes through 1 if $\lambda \in \gamma_+$). As before, denote by $\Omega$ the union of $G$ and a $\delta$-neighbourhood of $[-1, 1]$. Consider the canonical domain $\Omega_1$ lying in $\Omega$ between the lines $l_2$ and $l_3$ and containing the line $l_1$, and the canonical domain $\Omega_2$ in $\Omega$ lying between the lines $l_3$ and $l_1$ and containing $l_2$. Let $v^+_j(z, \lambda)$, $v^-_j(z, \lambda)$ be the pairs of solutions having in canonical domains $\Omega_j$, $j = 1, 2$ asymptotics (2.7). Asymptotics (2.7) for the pair of solutions $v^+_2$, $v^-_2$ in the domain $\Omega_2$, does not remain valid in the domain $\Omega_1$. However, there is a connection formula for the solutions.

**Lemma 2.9.** The following transmission formula is valid: for $z \in \Omega_1$ one has the representation

\[(2.10) \quad \left( \begin{array}{c} v^+_2(z, \lambda) \\ v^-_2(z, \lambda) \end{array} \right) = e^{i\pi/6} \left( \begin{array}{cc} -i[1] & [1] \\ 0 & -i[1] \end{array} \right) \left( \begin{array}{c} v^+_1(z, \lambda) \\ v^-_1(z, \lambda) \end{array} \right), \]

where $[1] = 1 + O(\varepsilon)$ and $|O(\varepsilon)| < C\varepsilon$ with a constant $C$ depending on $z$ and $\lambda$. However, given any compact $K$ in $\Omega_1 = \Omega_1(\lambda)$ there exists a neighbourhood $U_\lambda$ of the point $\lambda$ such that the last estimate holds for all $z \in K$ and all $\lambda$ belonging to $U_\lambda$ with a constant $C$ depending only on $K$ and $U_\lambda$.

**Proof.** See the monographs [9] and [5].

Now, we can complete the proof of the theorem. Consider the characteristic determinant

\[\Delta(\lambda) = \begin{vmatrix} v^+_2(-1, \lambda) & v^+_2(1, \lambda) \\ v^-_2(-1, \lambda) & v^-_2(1, \lambda) \end{vmatrix}.\]

For $\lambda$ lying near the curve $\gamma_+$ the points $-1$ and $+1$ belong to the domains $\Omega_2$ and $\Omega_1$, respectively. Using asymptotics (2.7) and (2.10) and abbreviating by the term $e^{-i\pi/6}T(\lambda)$ (see the proof of Theorem 2.1), we find

\[\Delta(\lambda) = \begin{vmatrix} [1] \exp(-1S(-1, \lambda)) & -i[1] \exp(-1S(1, \lambda)) + [1] \exp(-\varepsilon^{-1}S(1, \lambda)) \\ [1] \exp(-\varepsilon^{-1}S(-1, \lambda)) & [1] \exp(-\varepsilon^{-1}S(1, \lambda)) \end{vmatrix}.\]

The branch of the function $S(z, \lambda)$ is defined by the condition $\Re S(-1, \lambda) =: \alpha(\lambda) < 0$, if $\lambda$ is located in the $\tau$-neighbourhood of $\gamma_+$ outside $U_\delta(b)$ and $U_\delta(\lambda_0)$. Notice, that $\Re S(1, \lambda) \to 0$ as $\tau \to 0$. Hence, we can choose a number $\tau > 0$ such small that $\Re S(1, \lambda) < \alpha(\lambda)/2$. In this case the term $\exp(-1(S(-1, \lambda) + S(1, \lambda)))$ decays exponentially, while the term $\exp(-\varepsilon^{-1}S(-1, \lambda))$ grows exponentially as $\varepsilon \to 0$. So, the equation $\Delta(\lambda) = 0$ is equivalent up to exponentially small terms to the equation

\[\begin{vmatrix} [1]e^{-\varepsilon^{-1}S(1, \lambda)} - i[1]e^{-\varepsilon^{-1}S(1, \lambda)} \end{vmatrix} = 0,
\]

or

\[(2.11) \quad \cos \left( \frac{1}{i\varepsilon} \int_{\xi_\lambda}^{1} \sqrt{i(q(\xi) - \lambda)} d\xi - \frac{\pi}{4} \right) = O(\varepsilon).\]

Assume that the term $O(\varepsilon)$ in this equation equals zero. Then, the roots near the curve $\gamma_+$ are determined explicitly by the equations

\[(2.12) \quad -iQ^+(\lambda) = \varepsilon(k\pi - \pi/4), \quad k \in \mathbb{Z}.\]

The function $-iQ^+(\lambda)$ is real along the curve $\gamma_+$ and monotonous (see Lemma 2.3). Hence, there exist integers $p_+$ and $m_+$, such that for all $p_+ \leq k \leq m_+$ equations (2.12) have solutions $\mu_k$ lying on $\gamma_+$ outside the $\delta$-neighbourhoods of the end points $b$ and $\lambda_0$. The existence of
simple roots of perturbed equation (2.11) in $C\varepsilon^2$-neighbourhoods of the points $\{\mu_k\}_{m^+}$ can be proved by standard means using the Rouché theorem (see details in Theorem 5.1 of [18]). The representation for the counting eigenvalue function $N(\lambda)$ along the curve $\gamma_+$ can be obtained from formulae (2.12). This can be done in the same way as in Theorem 5.2 of [18].

Certainly, the same assertions are valid for the eigenvalues near the curve $\gamma_-$. The proof of the formulae for the eigenvalues near the curve $\gamma_\infty$ can be realized simpler. Namely, there is no need to use the transmission formulae to get asymptotic representation of the characteristic determinant for all $\lambda$ lying below both curved $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$. For such values of $\lambda$ the segment $[-1, 1]$ lies between the left and the right Stokes lines, and we can use asymptotics (2.10) simultaneously in both points $-1$ and $+1$. After simple calculations we obtain that for $\lambda$ lying below the curves $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$ the equation $\Delta(\lambda) = 0$ is equivalent to the following one

$$\sin \frac{1}{i\varepsilon} Q(\lambda) = O(\varepsilon).$$

Using standard arguments we get the localization formulae for the eigenvalues and the representation for the eigenvalue counting function along $\gamma_\infty$. Actually, a sharper analysis can be carried out. Namely, the formulae

$$\lambda_k = \mu_k + \varepsilon^2 \mu_{k}^{-1} O(1), \quad k = s, s + 1, \ldots,$$

can be obtained for the eigenvalues near $\gamma_\infty$ (here $|O(1)|$ is estimated by a constant $C$ not dependent on $k$ and $\varepsilon$). To get these formulae, one has to use an analogue of Lemma 4.3 from paper [18]. Here we omit details. Theorem is proved.

3. The case of Couette–Poiseuille profile

Profiles of the form $q(x) = \alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$, correspond to stationary solutions of the Navier–Stokes equation, therefore, they have special interest in connection with Orr–Sommerfeld problem (0.1), (0.2). Consequently, the study of model problem (0.4), (0.5) for
functions \( q(x) \) of this form is quite important. Since
\[
\alpha x^2 + \beta x + \gamma = \alpha \left( x + \frac{\beta}{2\alpha} \right)^2 + \gamma - \frac{\beta^2}{4\alpha^2},
\]
the substitution of the spectral and small parameters
\[
\varepsilon = \sqrt{\alpha} \varepsilon', \quad \lambda = \sqrt{\alpha} (\lambda' - \gamma + \frac{\beta^2}{4\alpha^2})
\]
leads to model problem (0.4), (0.5) with the function \( q(x) = (x - \beta')^2, \beta' = \beta/2\alpha \). Further we write \( \beta \) instead of \( \beta' \).

The cases \( \beta = 0 \) and \( \beta \neq 0 \) are different and they have to be treated separately. In the first case (i.e. for \( q(x) = x^2 \)) the eigenfunctions are either even or odd (see [2], for example). Consequently, the spectrum of problem (0.4), (0.5) with the function \( q(x) = x^2 \) consists of the spectra of two problems on the segment \([0, 1]\):

\[
i \varepsilon^2 y'' + x^2 y'' = \lambda y \quad \text{and} \quad i \varepsilon^2 y'' + x^2 y'' = \lambda y
\]
\[
y(0) = y(1) = 0, \quad y'(0) = y(1) = 0.
\]
The first of these problems is solved in §1; the second one can be solved in the same way: the replacement of the boundary condition \( y(0) = 0 \) by the condition \( y'(0) = 0 \) does not change the essence of the matter. The method for the solution remains the same. However, the problem

\[
i \varepsilon y'' + x^2 y = \lambda y,
\]
\[
y'(-1) = y(1) = 0
\]
with non-symmetrical boundary conditions can not be reduced to the study of similar problems on the segment \([0, 1]\) with the monotonous function \( x^2 \). The solution of (3.2) is essentially more difficult. The method for the study of this problem is proposed in [18] (formally, the Dirichlet boundary conditions are treated in [18], however, the problem is studied globally on \([-1, 1]\), and the proposed method can be used without changes for arbitrary separated boundary conditions). It is easy to see analyzing this method, that the form of the limit spectral graph \( \Gamma \) for the function \( q(x) = x^2 \) and the formulae for the counting eigenvalue functions along the curves do not depend on boundary conditions. Hence, both problems in (3.1) and problem (3.2) have the same limit spectral graph (see Figure 5) and the same eigenvalue counting functions up to multiplication by the coefficient 2.

Actually, the method of [18] works to solve the problem

\[
i \varepsilon^2 y'' + (x - \beta)^2 y = \lambda y,
\]
\[
y'(-1) = y(1) = 0,
\]
with \( \beta \neq 0 \). However, the limit spectral graph for this problem takes a more complicated form. The same is true for the eigenvalue formulae and the counting eigenvalue functions along the curves of the limit graph.

Let us formulate the main results concerning problem (3.3). Without loss of generality we assume, that \( \beta \in (0, 1) \). Set
\[
a = (-1 - \beta)^2, \quad b = (1 - \beta)^2.
\]
Repeating simple arguments of Lemma [2], we obtain, that the spectrum of problem (3.3) lies in the semistrip
\[
\Pi = \{ \lambda \mid \Im \lambda < 0, \ 0 < \Re \lambda < b \}.
\]
Consider the following curves in $\Pi$:

\[ \tilde{\gamma}_0 = \{ \lambda \in \Pi \mid \arg \lambda = -\pi/4 \}, \]

\[ \tilde{\gamma}_b = \left\{ \lambda \in \Pi \mid \Re \int_{\sqrt{\lambda}+\beta}^{1} \sqrt{i(q(\xi)-\lambda)} \, d\xi = 0 \right\}, \]

\[ \tilde{\gamma}_a = \left\{ \lambda \in \Pi \mid \Re \int_{-\sqrt{\lambda}+\beta}^{-1} \sqrt{i(q(\xi)-\lambda)} \, d\xi = 0 \right\}, \]

\[ \tilde{\gamma}_- = \left\{ \lambda \in \Pi \mid \Re \int_{\sqrt{\lambda}+\beta}^{-1} \sqrt{i(q(\xi)-\lambda)} \, d\xi = 0 \right\}, \]

\[ \tilde{\gamma}_\infty = \left\{ \lambda \in \Pi \mid \Re \int_{-1}^{1} \sqrt{i(q(\xi)-\lambda)} \, d\xi = 0 \right\}, \]

with $q(\xi) = (\xi - \beta)^2$. It is easily seen that all these curves have no self-intersections (see Lemma 2.5).

**Lemma 3.1.** The curves $\tilde{\gamma}_0$, $\tilde{\gamma}_b$ and $\tilde{\gamma}_-$ have the only intersection point $\lambda_1 \in \Pi$. The curves $\tilde{\gamma}_-\,\tilde{\gamma}_a$ and $\tilde{\gamma}_\infty$ intersect in a point $\lambda_2 \in \Pi$. There are no other intersection points of all these curves.

**Proof.** It can be carried out using the ideas from Lemma 2.5. □

Denote by $\gamma_0$ the part of $\tilde{\gamma}_0$ between 0 and $\lambda_1$ (i.e. $\gamma_0 = [0, \lambda_1]$), by $\gamma_b$ the part of $\tilde{\gamma}_b$ between the points $b$ and $\lambda_1$, by $\gamma_\gamma$ the part of $\tilde{\gamma}_\gamma$ between the points $\lambda_2$ and $a$, and by $\gamma_\infty$ the part of $\tilde{\gamma}_\infty$ between $\lambda_2$ and $-i\infty$. 

**Figure 7.**
THEOREM 3.1. Given small $\tau > 0$ there is a number $\varepsilon_0 = \varepsilon_0(\tau)$ such that for all $\varepsilon < \varepsilon_0$ the eigenvalues of problem (3.3) lie inside the $\tau$-neighbourhood of the set

$$\Gamma = \gamma_0 \cup \gamma_a \cup \gamma_- \cup \gamma_b \cup \gamma_{\infty}.$$  

PROOF. The first step in proving of this theorem is to clarify the geometry of the Stokes graphs of the Weber equation (3.3). Actually, this work was carried out in [18]. There are three reasons for points $\lambda \in \Pi$ to be accumulation points for the eigenvalues as $\varepsilon \to 0$. First, at least one point $+1$ or $-1$ belongs to the Stokes graph $C_\lambda$ of equation (3.3). The set of such points form the lines which we call singular. In our case the lines $\tilde{\gamma}_a$, $\tilde{\gamma}_-$ and $\tilde{\gamma}_b$ are singular. However, not all points of a singular line $\tilde{\gamma}$ belong to the limit spectral graph. We have to exclude the points $\mu \in \tilde{\gamma}$ possessing the following property: there is a path in the $z$-plane connecting the points $-1$ and $+1$ such that it intersect only one Stokes line of all Stokes complexes $C_\lambda$ for all $\lambda \in U_\delta(\mu)$ with sufficiently small $\delta > 0$. In our case after elimination of such points we get the curves $\gamma_a$, $\gamma_-$ and $\gamma_b$. Second, there are points $\mu$ in the $\lambda$-plane such that the geometry of the Stokes graphs is not preserved in any sufficiently small neighbourhood $U_\delta(\mu)$ (see details in [18]). The points of such kind form lines which we call critical. In our case there is the only critical line $\tilde{\gamma}_0$ (for $\mu \in \tilde{\gamma}_0$ the Stokes graph consists of one complex, while for $\mu \notin \tilde{\gamma}_0$ it consists of two complexes). Again, we have to exclude the points of critical lines which possess the property that we described before. Then, in our case we get the curve $\gamma_0$. Finally, the curve

$$\tilde{\gamma}_\infty = \left\{ \lambda \left| \operatorname{Re} \int_{-1}^{1} \sqrt{i(q(x) - \lambda)} \, dx = 0 \right. \right\}$$

we call the main line. The part of this line between $-i\infty$ and the first intersection point with singular or critical lines has to be included in the limit spectral graph.

This introduction gives an understanding, how to prove Theorem 3.1. Here a complete proof is omitted. The details can be found in [20]. \[\square\]

THEOREM 3.2. The set (3.4) is the limit spectral graph of problem (3.3), i.e. the points $\lambda \in \Gamma$ and only these points are the accumulation points of the eigenvalues as $\varepsilon \to 0$. Given $\delta > 0$ there are numbers $\varepsilon_0 = \varepsilon_0(\delta)$ and $C = C(\delta)$ such that for all $\varepsilon < \varepsilon_0$ the eigenvalues of problem (3.3) lie in the set

$$U_\delta = U_\delta(0) \cup U_\delta(a) \cup U_\delta(b) \cup U_\delta(\lambda_1) \cup U_\delta(\lambda_2)$$

and in the $C\varepsilon^2$-neighbourhoods of the points $\mu_k \in \Gamma$ which are determined by the equations

$$i \int_{\sqrt{\lambda} + \beta}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \varepsilon(\pi k - \pi/4), \quad \mu_k \in \gamma_b, \quad k \in \mathbb{Z},$$

$$i \int_{-\sqrt{\lambda} + \beta}^{-1} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \varepsilon(\pi k - \pi/4), \quad \mu_k \in \gamma_a, \quad k \in \mathbb{Z},$$

$$i \int_{\sqrt{\lambda} + \beta}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \varepsilon(\pi k - \pi/4), \quad \mu_k \in \gamma_-, \quad k \in \mathbb{Z},$$

$$i \int_{-1}^{0} \sqrt{i(q(\xi) - \lambda)} \, d\xi = \varepsilon \pi k, \quad \mu_k \in \gamma_{\infty}, \quad k \in \mathbb{Z},$$

$$\lambda_k^0 = (2k + 1) \varepsilon e^{-i\pi/4}, \quad \mu_k \in \gamma_0, \quad k \in \mathbb{Z}.$$
The $C\varepsilon^2$-neighbourhoods of all points $\mu_k \in \Gamma \setminus U_\delta$ contain only one simple eigenvalue. The counting eigenvalue functions along the curves of the graph $\Gamma$ have representations

$$N(\lambda) = \frac{1}{i\pi\varepsilon} \int_{\sqrt{\lambda} + \beta}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi + O(1), \quad \lambda \in \gamma_b,$$

$$N(\lambda) = \frac{1}{i\pi\varepsilon} \int_{-\sqrt{\lambda} + \beta}^{-1} \sqrt{i(q(\xi) - \lambda)} \, d\xi + O(1), \quad \lambda \in \gamma_a,$$

$$N(\lambda) = \frac{1}{i\pi\varepsilon} \int_{\sqrt{\lambda} + \beta}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi + O(1), \quad \lambda \in \gamma_-,$$

$$N(\lambda) = \frac{1}{i\pi\varepsilon} \int_{-1}^{1} \sqrt{i(q(\xi) - \lambda)} \, d\xi + O(1), \quad \lambda \in \gamma_\infty,$$

$$N(\lambda) = \frac{1}{2\varepsilon} e^{i\pi/4} \lambda, \quad \lambda \in \gamma_0.$$

**Proof.** The details of the proof of this theorem can be found in [20]. The spectral portrait of problem (3.3) for the function $q(x) = (7x - 1)^2/64, \varepsilon^2 = 5000$ is depicted in Figure 7.

4. The Orr–Sommerfeld problem

The description of global behaviour of the spectrum of the Orr–Sommerfeld problem as $R \to \infty$ have been carried out only for the functions $q(x) = x$ (Couette profile) and $q(x) = x^2$ or $q(x) = 1 - x^2$ (Poiseuille profile). Here we point out the papers [1], [4], [7] and [19]. Actually, the methods developed in [19] and [20] can be used without essential changes to
treat the Couette–Poiseuille profiles $q(x) = \alpha x^2 + \beta x + \gamma$. Here we shall only formulate the result obtained in [4] and the generalization of the result [19]; the detailed proof requires a serious work. Figure 8 (it is borrowed from the paper [8]) gives an illustration of the first theorem of this section. Here the spectrum is calculated for $\alpha = 1$ and $R = 4000$.

It was shown in Introduction, that in the case of the Couette profile $q(x) = x$ the Orr–Sommerfeld problem takes the form

$$
-i\varepsilon z'' + (i\varepsilon\alpha^2 - x)z = \lambda z, \quad \varepsilon = 1/\alpha R,
$$

(4.1)

$$
\int_{-1}^{1} \sinh[\alpha(1 - t)] z(t) \, dt = \int_{-1}^{1} \cosh[\alpha(1 - t)] z(t) \, dt = 0.
$$

To describe the eigenvalue behaviour of this problem as $\varepsilon \to 0$, we consider the rectangular coordinate system $\{t, \gamma\}$ in the $\lambda$-complex plane taking the point $-1$ as the origin and directing the axis $t$ along the segment $[-1, -i/\sqrt{3}]$ (see Figure 9).

Set

$$
c(t) = 2\sqrt{\pi} \left| \frac{\sinh (\alpha (2 - e^{-i\pi/4} t))}{\sinh 2\alpha} \right|, \quad \varphi(t) = \frac{1}{2\pi} \arg \sinh (\alpha (2 - e^{-i\pi/4} t)),
$$

where the main branch of the argument is chosen.

Consider in the $\{t, \gamma\}$ plane the curves

$$
\gamma_{\pm}(t) = \pm \frac{1}{(2\alpha)^{1/2}} \ln \frac{c(t)t^{3/4}}{\varepsilon^{1/4}}, \quad t > 0,
$$

and fix at these curves the points

$$
\mu_{k} = \{t_{k}^{\pm}, \gamma_{\pm}(t_{k})\},
$$

where

$$
t_{k}^{\pm} = \varepsilon^{1/3} (3\pi [k - 1/4 \mp \varphi((3\pi\varepsilon^{1/2}k)^{2/3})])^{2/3}, \quad k_{0} \leq k \leq k_{1},
$$
and the indices $k_0, k_1$ are chosen in such a way that

$$\varepsilon^{1/3} |\ln \varepsilon| \leq t_k \leq 2/\sqrt{3} - \frac{2}{3} \left( \frac{3}{4} \right)^{3/4} \varepsilon^{1/2} |\ln \varepsilon|, \quad k_0 \leq k \leq k_1.$$ 

**Theorem 4.1.** Denote by $U^\pm_k$ the neighbourhoods of the points $\mu^\pm_k$ of radius $\delta_k = C \varepsilon^{3/4} t_k^{-5/4}$ and by $\hat{U}^\pm_k$ the symmetrical reflections of $U^\pm_k$ with respect to the imaginary axis. Denote by $U^\pm_0$ the $\varepsilon^{1/3}|\ln \varepsilon|$-neighbourhoods of the points $\pm 1$ and by $U_0$ the $2/3 \cdot (3/4)^{3/4} \varepsilon^{-1/2} |\ln \varepsilon|$-neighbourhood of the knot-point $-i/\sqrt{3}$. Then, there are numbers $C > 0$ and $\varepsilon_0 > 0$ such that all the eigenvalues of problem (4.1) located near the segments $[\pm 1, -i/\sqrt{3}]$ lie inside the circles $U^\pm_0$, $U_0$ and $\{U^\pm_k\}_{k_0-1}^{k_1+1}$, $\{\hat{U}^\pm_k\}_{k_0-1}^{k_1+1}$, provided that $\varepsilon \leq \varepsilon_0$. The circles $U^\pm_k$ and $\hat{U}^\pm_k$ contain only one simply eigenvalue. All the other eigenvalues lie at the imaginary axis below the point $-i/\sqrt{3}$ and have representation

$$\lambda_k = -i(\rho_k + \varepsilon O(1)), \quad k = k_0, k_0 + 1, \ldots,$$

where the numbers $\rho_k$ and $k_0$ are the same as in Theorem 1.1.

**Proof.** As in Theorem 1.1, special properties of the Airy functions are explored in the proof. Details can be found in [41].

**Theorem 4.2.** Given $\tau > 0$ there is $\varepsilon = \varepsilon_0(\tau)$, such that for $\varepsilon < \varepsilon_0$ all the eigenvalues of the Orr–Sommerfeld problem (0.1), (0.2) with the Couette–Poiseuille profile $q(x) = (x - \beta)^2$, $\beta \in (-1, 1)$, lie in the $\tau$-neighbourhood of the limit spectral graph $\Gamma$ of the corresponding model problem. The main terms of the counting eigenvalue functions along the curves of the graph $\Gamma$ have the representations given in Theorem 3.4.

**Proof.** In the case $\beta = 0$ the proof is obtained in [20]. For $\beta \neq 0$ the proof remains essentially the same, provided that the analysis of the model problem with $q(x) = (x - \beta)^2$ is carried out (see [20]). The proof of the results about the counting eigenvalue functions uses the
tauberian technique developed in [14]. Figure 10 shows the spectrum of the Orr–Sommerfeld problem for \( q(x) = x^2, \alpha = 1 \) and \( R = 3000 \). □

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