A Convex Hull Based Approach for MIMO Radar Waveform Design with Quantized Phases

Weichao Pi\textsuperscript{1}, Chenglin Ren\textsuperscript{1}, Jianming Zhou\textsuperscript{1}
1. School of Information and Electronics, Beijing institute of technology, Beijing, China
peao_kelvin@163.com, rencl3@bit.edu.cn, zhoujm@bit.edu.cn

Abstract—In this letter, we focus on designing constant-modulus waveform with discrete phases for the multi-input multi-output (MIMO) radar system, where the signal-to-interference-plus-noise ratio (SINR) is maximized in the presence of both the signal independent clutter and the noise. Because of the NP-hardness of the above formulated problem, we take advantage of successive convex hulls of the discrete feasible region to relax the original hard optimization problem to a sequence of several continuous quadratic programming (QP) subproblems. Compared with the conventional SDR technique, the proposed method yields approximated solutions with lower computational costs. In the final part, several numerical simulations were taken to prove the effectiveness of the proposed method.

Keywords—waveform design, discrete phases, MIMO radar, convex hull

I. INTRODUCTION

According to the configuration of the antennas, the MIMO radar is classified and studied in two types: distributed MIMO radar \cite{1, 2} and colocated MIMO radar \cite{3, 4}. Compared with traditional phased array radar, MIMO radar offers extra degrees of freedom by allowing independent waveforms to be transmitted at each antenna, and therefore achieves enhanced capabilities \cite{5, 6, 7}. In addition, a well-designed waveform can significantly improve the SINR \cite{8, 9} and probability of detection \cite{10}.

The problem of waveform design for MIMO radar has attracted considerable attentions in recent years \cite{11, 12, 13, 14, 15, 16, 17}. In practical scenarios, waveforms with constant modulus or called low peak-to-average-power ratio (PAPR) properties are needed to avoid signal distortions, as the non-linear power amplifiers are typically operated at the saturation region. Meanwhile, the similarity constraint (SC) is also enforced, which employs a reference waveform as the benchmark and allows the optimized waveform to share the ambiguity properties of the reference waveform. Relying on the semi-definite relaxation (SDR), the authors of \cite{14} introduce the sequential optimization procedures to maximize the SINR accounting for the constant modulus constraint (CMC) and the SC for the case of continuous signal phase. In \cite{16}, a novel successive QCQP refinement (SQR) algorithm is developed for the same optimization problem in \cite{14}, which involves solving a sequence of convex quadratically constrained quadratic programming (QCQP) subproblems. Furthermore, by customizing the projection procedure for the convex QCQP subproblems, the Accelerated Gradient Projection (AGP) is proposed in our previous work \cite{17}, which shares a comparable performance with the SQR method while notably reducing the computational costs. However, the works aforementioned only address the continuous phase case. Given the extensive use of digital phase shifters in many working radar systems, in this letter we consider the MIMO radar waveform design for the case of quantized phases. Pioneered by \cite{18}, the cases of both continuous and discrete phases are considered. Based on the method of semi-definite program (SDP) relaxation and randomization technique, the approximated solutions to the quadratic optimization problem subject to the CMC and the SC are presented. In \cite{19}, the optimization problem under the PAPR constraint and energy constraint is resolved through the same technique.

In this letter, we optimize the SINR of the radar under both CMC and SC, where the phases of the designed signals are drawn from a discrete alphabet. By introducing the convex hull of the finite feasible points, a novel Continuous Approximation Method (CAM) is proposed to relax the original problem as a sequence of convex QP subproblems, which can be solved via standard numerical tools. By doing so, the solution of the original discrete problem can be obtained by a simple quantization procedure. In the last part, Several numerical simulations were taken to prove our method and results show that the proposed approach outperforms the SDR technique used in \cite{18}.

II. SYSTEM MODEL

A. The model of receiving signal

We consider a co-located narrow band MIMO radar system equipped with \(N_t\) transmit antennas and \(N_r\) receive antennas, where each antenna emits or receives N samples. The receive waveform is given as \cite{14}

\[
\tilde{r} = \alpha_0 A(\theta_0) s + \sum_{m=1}^{M} \alpha_m A(\theta_m) s + w
\]  

(1)

Where \(\tilde{r} = [r_1^T, \ldots, r_N^T]^T \in C^{N_t \times N_r \times 1}\) with \(r_n \in C^{N_t \times 1}\), \(n = 1, \ldots, N\) being the n-th snapshot across the NR receive antennas, stands for the transmit waveform with \(s_n\), the n-th snapshot across the \(N_t\) transmit antennas, \(w \in C^{N_t \times 1}\) denotes circular white gaussian noise, \(\alpha_n\) and \(\alpha_m\) represents the complex amplitudes of the target and the m-th interference source, \(\theta_0\) and \(\theta_m\) represents the angles of the target and the m-th interference source, respectively, and \(A(\theta)\) stands for
the steering matrix of a Uniform Linear Array (ULA) with
half-wavelength separation between the antennas, which is
given as
\[ A(\theta) = I_N \otimes [a_t(\theta)a_r(\theta)^T] \]  
(2)

Where \( I_N \) is the \( N \times N \) identity matrix , \( \otimes \) denotes the
kronecker product, \( a_t(\theta) \) and \( a_r(\theta) \) stand for the transmit and the receive steering vectors respectively.

Aiming to maximize the output SINR of the radar, we
jointly design the transmit waveform and the receive filter. Without loss of generality, we assume that a linear Finite
Impulse Response (FIR) filter \( f \in \mathbb{C}^{N_F \times N} \) is employed to
process the receive echo wave. The output of the filter \( r \) is
given as
\[ r = f^H \hat{r} = \alpha_0 f^H A(\theta_0)s + \sum_{m=1}^{M} \alpha_m f^H A(\theta_m)s + f^H w \]  
(3)

Where operation \((\cdot)^H\) denotes the Hermitian transpose.

As a consequence, the output SINR can be expressed as follow
\[ \text{SINR} = \frac{\sigma^2 \left| f^H A(\theta_0)s \right|^2}{\|f^H T(s)f + f^H f\|} \]  
(4)

Where \( \sigma = E\left[ |\alpha_0|^2 \right] / \sigma_n^2 \) with \( E[\cdot] \) denoting the
statistical expectation and the \( \bar{T}(s) \) is expressed as
\[ \bar{T}(s) = \sum_{m=1}^{M} I_m A(\theta_m)ss^H A^H(\theta_m) \]  
(5)

Where \( I_m = E\left[ |\alpha_m|^2 \right] / \sigma_n^2 \).

B. Optimization Problem Formulation

Based on the above system model analyzed above, the
waveform design with continuous phase is firstly considered, where the CMC and the SC aforementioned have been imposed. Thus, the maximization of the SINR in (4) can be formulated as follow
\[ \max_{f \neq 0, s} \frac{\sigma^2 \left| f^H A(\theta_0)s \right|^2}{\|f^H \bar{T}(s)f + f^H f\|} \]  
\[ \text{s.t.} \quad \left| s(k) \right| = 1 / \sqrt{N_r N} \]  
\[ \|s - s_0\|_\infty \leq \varepsilon \]  
(6)

Where \( s(k) \) stands for the k-th entry of \( s \), \( k = 1, \ldots, N_r N \), operation \( \|\cdot\|_\infty \) denote the infinity norm , \( s_0 \) represents the reference waveform and \( \varepsilon (0 \leq \varepsilon \leq 2) \) is a
preset parameter to control the similarity degree between wave \( s \) and \( s_0 \). In addition, by taking into account the CMC, the SC can be rewritten as
\[ \arg s(k) = [w_k, w_k + \delta] \]  
(7)

Where, \( w_k = \arg s_0(k) - \arccos(1 - \varepsilon^2 / 2) \),
\( \delta = 2 \arccos(1 - \varepsilon^2 / 2) \), and \( s_0(k) \) is the k-th entry of
\( s_0 \). Noting that there is no constraint on \( f \) as analyzed in
[14], the maximization problem of (6) can be equivalent to follow,
\[ \max_s s^H Y(s)s \]  
\[ \text{s.t.} \quad \left| s(k) \right| = 1 / \sqrt{N_r N} \]  
\[ \arg s(k) = [w_k, w_k + \delta] \]  
(8)

Where \( Y(s) \) is a positive-semi definite matrix ,which is
given as
\[ Y(s) = A^H(\theta_0) \left[ \bar{T}(s) + I \right]^{-1} A(\theta_0) \]  
(9)

According to [20], we can obtain a suboptimal SINR by
assuming \( Y = Y(s) \) with a fixed \( s \) and optimizing \( s \) with the
new \( Y \) iteratively, resulting in a sequence of subproblems as follows
\[ \max_s s^H Ys \]  
\[ \text{s.t.} \quad \left| s(k) \right| = 1 / \sqrt{N_r N} \]  
\[ \arg s(k) = [w_k, w_k + \delta] \]  
(10)

We consider the subproblem for each iteration in the sequel. In addition, let \( q \in \mathbb{C}^{N_r N} \) be the quantized phase transmit
waveform, and the \( q(k) \) be the k-th entry of waveform \( q \). As
a consequence, the maximization problem of the SINR in (10)
for the continuous phase case can be quantized as,
\[ \max_q q^H Yq \]  
\[ \text{s.t.} \quad q \in Q \]  
(11)

where \( Q \) is the discrete phase alphabet which is discussed
with detailed in the next section. Because of quantity constraint
the optimization problem of (11) is non-convex and NP-hard in
general, whose optimal solution cannot be found in polynomial
time for general cases. To tackle this problem, we first consider
the convex hull of the feasible points for each dimension, and
relax (11) to a continuous QP subproblem. In the next section, we
introduce a novel algorithm-Continuous Approximation Method (CAM) to approximate to the nearest feasible vector
\( q_{\text{opt}} \) for the optimization problem of (11).
III. THE PROPOSED ALGORITHM FOR DISCERTED PHASED WAVEFORM PROBLEM

To further describe $Q$ in (11), we suppose an extreme situation of (7) with $\delta = 2\pi$, the similarity parameter $\varepsilon = 2$, where the SC vanishes and only the CMC is in effect.

![Fig. 1](image)

Fig. 1. (a) The constellation points with $\Omega = 16$ for k-th dimension; (b) The convex hull of with $\Omega = 16$ and the convex hull of $\Gamma_\eta(k)$ with $\eta = 6$ for k-th dimension.

As shown in Fig. 1 (a), the relaxed feasible region of $s(k)$ is a full circle with a constant radius on the complex plane $\mathbb{C}$. In the discrete phase case, we construct a constellation with points on the circle, i.e., $q_1, \ldots, q_\Omega$ as an example with $\Omega = 16$ shown in Fig. 1 (a), where the radius between any two adjacent points is given as

$$\tau = \frac{2\pi}{\Omega}$$

(12)

The constellation points are distributed over the whole continuous feasible region for each dimension. Thus, each sample of the continuous transmit waveform can be quantized to the finite feasible points which can be given as

$$q_s = \exp(j\rho\tau)/\sqrt{N_T N}$$

(13)

where $\rho^\Omega = 1, \ldots, \Omega$. For notational convenience let

$$\Gamma_\Omega = \{ q_s = \exp(j\rho\tau)/\sqrt{N_T N} \mid \rho = 1, \ldots, \Omega \}$$

represent the discrete phase alphabet for each dimension.

As shown in Fig. 1 (b), by connecting all the feasible points, a regular polygon is formulated in the tint area, which is the convex hull of $\Gamma_\Omega$. Furthermore, let $\Lambda_\Omega$ be the discrete phase alphabet for all dimensions, which is given as

$$\Lambda_\Omega = \Gamma_\Omega \odot \ldots \odot \Gamma_\Omega$$

(14)

where $\odot$ denotes the Cartesian product. Obviously,

$$q \in \Lambda_\Omega, q(k) \in \Gamma_\Omega$$

(15)

However, for the general situation of (7) with $\delta$ far less than $2\pi$, the CMC and the SC are both involved. We introduce a positive even integer parameter $\eta$ to indicate the similarity between $q$ and $q_0$, where $q_0$ represents the quantized reference waveform, e.g., linear frequency modulation wave. Using $\eta$, we rewrite (7) as follow

$$\gamma_k = \arg(q_0(k)) - \eta \tau / 2$$

$$\phi = \eta \tau$$

(16)

Where $q_0(k)$ is the k-th entry of $q_0$. For each dimension, $\eta$ stands for the number of feasible points around $q_0(k)$, which is less than $\Omega$. According to the equation $\delta = 2 \arccos(1 - \varepsilon^2 / 2)$, the actual similarity tolerance $\varepsilon$ between $q$ and $q_0$ can be expressed as

$$\varepsilon = \sqrt{2} \left(1 - \cos\left(\phi/2\right)\right)$$

(17)

$$\Gamma_\eta(k) = \{ p(k), \ldots, p_{\eta+1}(k) \}$$

(18)

where the angle of $p_1(k)$ is $\gamma_k$, $p_{\eta+1}(k)$ is $q_0(k)$, and $p_{\eta+1}(k)$ is the mirror constellation point of $p_1(k)$ with respect to $q_0(k)$. The convex hull of $\Gamma_\eta(k)$ is shown as a darker polygon marked by $\Theta(k)$. Noting that $\Gamma_\eta(k) \subseteq \Gamma_\Omega$, any given $p_\mu(k), \mu = 1, \ldots, \eta + 1$ can be expressed as

$$p_\mu(k) = \exp\left\{ j \left[ \gamma_k + (\mu - 1) \tau \right] \right\} / \sqrt{N_T N}$$

(19)

Furthermore, let $\Lambda_\mu$ represent the feasible vector of $q$ for all dimensions.

$$\Lambda_\mu = \Gamma_\eta(1) \odot \ldots \odot \Gamma_\eta(N_T N)$$

(20)

We focus on the feasible points for each dimension, where we relax the finite feasible points set $\Gamma_\eta(k)$ to the convex hull itself, resulting in the following QP problem

$$\max_s \ s^H Y_s$$

s.t. \ $s \in \Delta$

(21)

where $\Delta$ is the Cartesian product of the convex hull for all dimensions, which can be expressed as

$$\Delta = \Theta(1) \odot \ldots \odot \Theta(N_T N)$$

(22)

Obviously, the optimization problem of (21) is a convex problem, which can be solved via numerical solvers, e.g., the CVX toolbox. As shown in Fig. 1 (b), $s_{opt}(k)$ is the k-th entry of the optimal solution $s_{opt}$ of (21).
The convex hulls $\Theta(k)$ for each dimension in the sequel. For notational convenience, we omit the dimension number $k$. As shown in Fig. 1 (b), the convex hull has $\eta+1$ edges which are highlighted in red, i.e., $L_\mu : p_\mu p_{\mu+1}$, $\ldots$, $L_\eta : p_\eta p_{\eta+1}$ and $L_{\eta+1} : p_{\eta+1} p_1$, then define the middle point of $p_\mu p_{\mu+1}$ as

$$m_\mu = \frac{p_\mu + p_{\mu+1}}{2}, \mu = 1, \ldots, \eta$$  \hspace{1cm} (23)

According to the basic plane analytic geometry, given any $s \in C$, the line $L_\mu$ can be expressed as

$$L_\mu : f_\mu(s) = \text{Re}\left(m_\mu^*(s - m_\mu)\right) = 0$$  \hspace{1cm} (24)

In addition, for the $(\eta+1)$-th edge, the middle point $m_{\eta+1}$ is as follow

$$m_{\eta+1} = \frac{p_{\eta+1} + p_1}{2}$$  \hspace{1cm} (25)

And corresponding line $L_{\eta+1}$ is given as

$$L_{\eta+1} : f_{\eta+1}(s) = \text{Re}\left(m_{\eta+1}^*(s - m_{\eta+1})\right) = 0$$  \hspace{1cm} (26)

As a consequence, the $k$-th convex hull for each dimension can be separately formulated in two cases. For the first one, when $0 \leq \phi \leq \pi$, namely $0 \leq \eta \leq \Omega / 2$, $\Theta(k)$ is given by

$$\begin{cases} f_\mu(s) \leq 0, \mu = 1, \ldots, \eta \\ f_{\eta+1}(s) \geq 0 \end{cases}$$  \hspace{1cm} (27)

For the case of $\pi \leq \phi \leq 2\pi$, namely $\Omega / 2 \leq \eta \leq \Omega$, $\Theta(k)$ is given by

$$\begin{cases} f_{\eta+1}(s) \leq 0, \mu = 1, \ldots, \eta \\ f_{\mu}(s) \geq 0 \end{cases}$$  \hspace{1cm} (28)

Finally, by comparing the Euclidean distance between $s_{\text{opt}}(k)$ and the feasible points in $\Gamma_\eta(k)$, the feasible point with the minimal distance is checked for each dimension. Thus, we obtain the optimal feasible vector $q_{\text{opt}}$.

IV. NUMERICAL RESULTS

In this section, we provide numerical results to evaluate the performance in terms of the SINR and the beampattern between the CAM and the SDR [18] for the discrete phase case, and use chirp waveform as our benchmark. For the continuous phase case, the reference waveform $s_0 \in C^{N_t \times N_R}$ can be obtained by stacking the columns of $s_0$, which we choose as the following orthogonal chirp waveform matrix without loss of generality

$$s_\eta(k, n) = \frac{\exp\left[j 2\pi k(n - 1) / N\right]\exp\left[j \pi (n - 1)^2 / N\right]}{\sqrt{N_t N}}$$

where $k = 1, \ldots, N_t, n = 1, \ldots, N$. For the discrete phase case, the reference waveform $q_0$ is obtained by quantizing $s_0$. The numbers of the transmit and the receive antennas are $N_t = 4$ and $N_R = 8$, respectively. In addition, we consider a scenario with three fixed signal-dependent clutters and additive white Gaussian disturbance with variance $\sigma_n = 0 \text{ dB}$. The target is located at an angle $\theta_t = 15^\circ$ with a reflecting power of $|\alpha_0|^2 = 10 \text{ dB}$ and three fixed interference sources located at $\theta_1 = -50^\circ$, $\theta_2 = -10^\circ$, and $\theta_3 = 40^\circ$ reflecting a power parameter of $|\alpha_1|^2 = |\alpha_2|^2 = |\alpha_3|^2 = 30 \text{ dB}$.

Figure 2: (a) A simulation example of the k-th entry of the optimal solution $s_{\text{opt}}$ for the CAM approach; (b) SINR in each iteration; (c) Beampattern.

In order to transform the discrete optimization problem of (11) into the continuous counterpart, we construct the convex hull which is the minimum convex set that contains all the feasible points, approximating to a tractable QP problem. In
Fig. 2 (a), we take a simulation example of $S_{opt}(k)$ for the CAM approach with $N=8$, $\Omega=16$ and $\eta=6$. The red stars indicate the feasible points on the complex plane $\mathbb{C}$, and the radius for the arc equals $1/\sqrt{N/TN} \approx 0.1768$. The blue circle is $S_{opt}(k)$, which obviously locates in the convex hull as constructed in Fig. 1 (b). For each dimension, the optimal solution always locates on the edge of the convex hull, which means that the constraint of the OP problem (21) works, and furthermore, it also suggests that this is a reasonable approximation.

As shown in Fig. 2 (b), we firstly compare the SINR in each iteration between the CAM and the SDR with $N=8$, $\Omega=16$ and $\eta=6$. According to (14), (16) and (17), we have $q=3\pi/4$ and $\varepsilon \approx 1.1$. In addition, for the SDR method, the number of randomization trials $L=20000$. The numerical results indicate that the SINR resulting from the CAM converges very fast (i.e., after 2-3 iterations). On the other hand, the SINR resulting from the SDR cannot converge because of the randomization, and is obviously lower than the proposed CAM approach. Fig. 2 (c) shows the beampattern of the optimal waveform for both methods, using the same parameters as in Fig. 2 (b). The simulation resulting from the CAM exhibits much better suppression performance when compared with the SDR.

V. CONCLUSION

In this letter, we propose a novel approach for MIMO radar waveform design with constant modulus and discrete phases. By constructing the polygon convex hull of the feasible constellation points, we develop a CAM algorithm to relax the discrete optimization problem as sequential convex OP subproblems. The optimal solution is then obtained by quantizing the solution of OP to its nearest neighbor. Compared with the conventional SDR methods, the computational complexity has been reduced, and numerical results reveal that the performance of the proposed approach is superior in terms of the SINR and the beampattern.

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