Sum-over-histories representation for the causal Green function of free scalar field theory

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ABSTRACT

A set of Green functions $G_\alpha(x - y), \alpha \in [0, 2\pi[$, for free scalar field theory is introduced, varying between the Hadamard Green function $\Delta_1(x - y) \equiv \langle 0 | \{ \varphi(x), \varphi(y) \} | 0 \rangle$ and the causal Green function $G_\pi(x - y) = i\Delta(x - y) \equiv [\varphi(x), \varphi(y)]$. For every $\alpha \in [0, 2\pi[$ a path-integral representation for $G_\alpha$ is obtained both in the configuration space and in the phase space of the classical relativistic particle. Especially setting $\alpha = \pi$ a sum-over-histories representation for the causal Green function is obtained. Furthermore using BRST theory an alternative path-integral representation for $G_\alpha$ is presented. From these path integral representations the composition laws for the $G_\alpha$'s are derived using a modified path decomposition expansion.

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1 Introduction

Relativistic quantum mechanics of a spinless point particle is a simple example of a system with constraints and is often viewed as a toy-model for more complicated field theories possessing gauge symmetry or reparametrization invariance, e.g. quantum cosmology. With the aim of understanding better the sum-over-histories formulation of quantum cosmology Halliwell and Ortiz considered in an interesting recent paper [1] sum-over-histories/path-integral representations of various Green functions of the Klein-Gordon equation, constructed canonical representations and investigated the relationship between these representations and the existence of composition laws for these Green functions. A canonical representation of a Green function is a representation as an inner product $G(x''|x') = \langle x''|x' \rangle$, where $|x\rangle$ denotes a complete set of configuration space states for any fixed $x^0$. A sum-over-histories representation is a representation of $G$ of the form $G(x''|x') = \sum \exp(iS)$, where the sum runs over all possible paths from $x'$ to $x''$ and $S$ denotes the action of each path. However Halliwell and Ortiz failed to obtain a path integral and sum-over-histories representation for the causal Green function. The purpose of this paper is to fill this gap. This paper deals exclusively with the case of a spinless relativistic point particle. The reader interested in the connection to quantum cosmology is referred to the References, e.g. [1, 2, 3].

The paper is organized as follows: In the Introduction some results of certain Green functions in free scalar field theory, relativistic classical mechanics of a point particle and relativistic quantum mechanics of a point particle are shortly reviewed for convenience. For complete details the reader is once again referred to the References. A set of Green functions $G_\alpha(x - y)$, $\alpha \in [0, 2\pi]$, varying between the Hadamard Green Function and the causal Green function is introduced. Several path integral representations in phase and in configuration space for the $G_\alpha$'s are presented in the second section. In the third section a modified path decomposition expansion is presented and used to derive the relativistic composition laws for the $G_\alpha$'s from their sum-over-histories representations.

Green functions for free scalar field theory

The causal Green function of free scalar field theory is defined by [1]

$$i\Delta(x - y) = [\varphi(x), \varphi(y)].$$

(1)

$\Delta$ is Lorentz invariant and vanishes outside the light-cone. The following identities are well known

$$i\Delta(x - y) = \frac{1}{(2\pi)^3} \int d^4k \epsilon(k_0) \delta(k^2 - m^2)e^{-ik(x - y)}$$  (2)

$$\Delta(x - y) = \frac{-1}{(2\pi)^3} \int \frac{d^3k}{\omega_k} \sin(\omega_k(x^0 - y^0))e^{ik(x - y)},$$  (3)

where

$$\epsilon(k_0) = \left\{ \begin{array}{ll} 1 & : k_0 > 0 \\ -1 & : k_0 < 0 \end{array} \right.$$  (4)

and

$$\omega_k \equiv + \sqrt{m^2 + \sum_{i=1}^{3} k_i^2}.$$  (5)

The Hadamard Green function of free scalar field theory is defined as

$$\Delta_1(x - y) = \langle 0|\{\varphi(x), \varphi(y)\}|0\rangle.$$  (6)

1The notations in this work differ from those in [1].
The Hadamard Green function may be written
\[ \Delta_1(x - y) = \frac{1}{(2\pi)^3} \int d^4k \delta(k^2 - m^2) e^{-ik\cdot(x-y)} \]
\[ = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty}dT \int d^4ke^{-ik\cdot(x-y)+iT(k^2-m^2)} \]
\[ = \frac{1}{(2\pi)^3} \int d^3k \cos(\omega_k(x^0 - y^0)) e^{ik\cdot(x-y)}. \]
Both the causal and the Hadamard Green function are solutions of the Klein-Gordon equation \((\Box + m^2)\varphi = 0\).

**Relativistic point particle**

A point particle in Minkowski spacetime follows a worldline \(x(\tau)\), which is parametrized by a variable \(\tau\) - not necessarily time -, which is monotonically increasing along the wordline, whether the worldline is proceeding forward or backward in time. The classical action is given by
\[ S_0[x(\tau)] = \int_{\tau}^{\tau''} d\tau L, \]
where
\[ L = -m[\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu]^\frac{1}{2} = -m\sqrt{\dot{x}^2}. \]

Introducing the momenta conjugate to \(x^\mu(\tau)\)
\[ p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -m \frac{\dot{x}^\mu}{\sqrt{\dot{x}^2}}, \]
one finds immediately the well known first-class primary constraint
\[ \mathcal{H}_0 \equiv p_\mu p^\mu - m^2 = 0. \]
One also finds that the canonical Hamiltonian \(H = p_\mu \dot{x}^\mu - L\) vanishes, which is typical for reparametrization invariant systems. \[ ^2 \]

The constraint (13) can incorporated into the Hamiltonian action via a Lagrange multiplier \(N(\tau)\) by writing
\[ S = \int_{\tau}^{\tau''} d\tau (p_\mu \dot{x}^\mu - N\mathcal{H}_0). \]
The classical equations of motion are obtained by extremizing \(S\) under independent variations of \(x, p\) and \(N\) subject to the conditions \(x(\tau') = x'\) and \(x(\tau'') = x''\). The action (14) is still invariant under the following gauge transformations generated by \(\mathcal{H}_0\)
\[ \delta x = \epsilon(\tau)\{x, \mathcal{H}_0\}, \quad \delta p = \epsilon(\tau)\{p, \mathcal{H}_0\}, \quad \delta N = \dot{\epsilon}(\tau), \]
where \(\epsilon(\tau)\) is an arbitrary infinitesimal function satisfying \(\epsilon(\tau') = \epsilon(\tau'') = 0\) and \{\cdot, \cdot\} denotes the Poisson-bracket.

In order to fix the reparametrization invariance (13) one has to impose a further restriction (gauge-fixing condition). In \[ ^3 \] it has been shown that the simplest gauge-fixing condition, which fully fixes the gauge and does not remove physical modes from the system is
\[ \dot{N} = 0. \]

\[ ^2 \]Actually the following is true: if coordinates and momenta for a generally covariant system transform under reparametrisations as scalars, then the Hamiltonian vanishes. \[ ^3 \]
The momentum eigenstates can therefore be labeled by the three-momentum path-integral representations for $\Delta^+$ where $\Theta$ denotes the Heaviside step function, one obtains immediately from the equations (17) and (18) induced path-integral representation for $\Delta^+$. An expression of the form (18) is formally derived in [2] using BRST symmetry.

As usual the path integrals (17),(18) are thought to be defined by a time slicing procedure with the Liouville measure on each time-slice and the following boundary conditions at the end points

$$x(\tau') = x'; x(\tau'') = x''.$$

In eq. (39) below the time slice definition for a path integral is explicitly given. An expression of the form (18) is formally derived in [2] using BRST symmetry.

Recalling that

$$\Delta_1(x - y) = \langle 0|\varphi(x)\varphi(y)|0\rangle + \langle 0|\varphi(y)\varphi(x)|0\rangle, \quad (20)$$

$$i\Delta_F(x - y) = \Theta(x^0 - y^0)\langle 0|\varphi(x), \varphi(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle, \quad (21)$$

where $\Theta$ denotes the Heaviside step function, one obtains immediately from the equations (17) and (18) induced path-integral representations for $\Delta_+(x - y) \equiv \langle 0|\varphi(x)\varphi(y)|0\rangle$ and $\Delta_-(x - y) \equiv \langle 0|\varphi(y)\varphi(x)|0\rangle$. This yields an induced path-integral representation for $\Delta$

$$i\Delta(x - y) = \epsilon(y^0 - x^0) \left[ \int_{0}^{\infty} dN(\tau'' - \tau') - \int_{-\infty}^{0} dN(\tau'' - \tau') \right] \int \mathcal{D}p \mathcal{D}x \exp \left( i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right),$$

where the function $\epsilon$ is given by eq. (19). Below more intrinsic path-integral representations will be derived.

Upon canonical quantization, one introduces abstract momentum eigenstates $|\hat{p}\rangle$ by

$$\hat{p}_\mu |\hat{p}\rangle = p_\mu |\hat{p}\rangle, \quad (22)$$

where the constraint eq. (13) is imposed as a condition on the states

$$(p^2 - m^2) |\hat{p}\rangle = 0. \quad (23)$$

The momentum eigenstates can therefore be labeled by the three-momentum $\mathbf{p}$ and the sign of $p_0$

$$\hat{p}_i |\mathbf{p}\pm\rangle = p_i |\mathbf{p}\pm\rangle, \quad \text{for } i \in \{1, 2, 3\}, \quad (24)$$

$$\hat{p}_0 |\mathbf{p}\pm\rangle = \pm (p^2 + m^2)^{1/2} |\mathbf{p}\pm\rangle. \quad (25)$$

Physical states are obtained by a superposition of momentum eigenstates. By Fourier transformation one defines Lorentz-invariant configuration states [4]

$$|x\rangle \equiv |x+\rangle + |x-\rangle, \quad (26)$$

This so-called proper time gauge is adopted throughout this paper. More generally gauge conditions of the form $\dot{N} = \chi(p, x, N)$ are also admissible. Canonical gauge conditions of the form $C(p, x) = 0$ for all $\tau$ would in general violate eq. (19). In [3] it has also been pointed out that in the gauge (16) the transition amplitude from $x'$ to $x''$ is given by

$$\Delta_F(x'' | x') = -i \int_{-\infty}^{0} dN(\tau'' - \tau') \int \mathcal{D}p \mathcal{D}x \exp \left( i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right), \quad (17)$$

where $\Delta_F(x'' | x') \equiv -i\langle 0|T(\varphi(x')\varphi(x'))|0\rangle$ denotes the Feynman propagator of free scalar field theory. On the other hand in [2] it has been shown that eq. (17) yields $\Delta_F$ provided the range of $N(\tau'' - \tau')$ is taken to be $-\infty$ to $\infty$

$$\Delta_1(x'' | x') = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int \mathcal{D}p \mathcal{D}x \exp \left( i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right). \quad (18)$$

In eq. (13) below the time slice definition for a path integral is explicitly given. An expression of the form (18) is formally derived in [2] using BRST symmetry.
Eq. (29) implies the completeness relation

\[ |x+\rangle \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{p_0=\omega_p} d^{3}p e^{ipx} |p+\rangle, \]  
\[ |x-\rangle \equiv \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{p_0=-\omega_p} d^{3}p e^{ipx} |p-\rangle. \]  

(27)  
(28)

The states of positive and negative energy decouple and may be considered separately. For the abstract momentum eigenstates with positive energy \(|p+\rangle\) an inner product may be defined by

\[ \langle p|p'\rangle = 2\omega_p \delta(p - p'). \]  

(29)

Eq. (29) implies the completeness relation

\[ 1 = \int \frac{d^{3}p}{2\omega_p} |p+\rangle \langle p+| \]  
\[ = i \int x^0=const. d^3x \left[ |x+\rangle \hat{\partial}_0 \langle x+| - |x-\rangle \hat{\partial}_0 \langle x-| \right]. \]  

(30)  
(31)

To include the states with negative energies the most natural choice seems to be

\[ \langle p|i'j'\rangle = 2\omega_p \delta_{ij} \delta(p - p'), \text{ where } i, j \in \{+,-\}, \]  

(32)

\[ 1 = \int \frac{d^{3}p}{2\omega_p} [|p+\rangle \langle p+| + |p-\rangle \langle p-|], \]  

however as it is shown in [1] eq. (31) is replaced by the unusual formula

\[ 1 = i \int x^0=const. d^3x \left[ |x+\rangle \hat{\partial}_0 \langle x+| + |x-\rangle \hat{\partial}_0 \langle x-| \right]. \]  

If one wants to stick to the usual relativistic completeness relation

\[ 1 = i \int x^0=const. d^3x \left[ |x+\rangle \hat{\partial}_0 \langle x+| + |x-\rangle \hat{\partial}_0 \langle x-| \right], \]  

one has to abandon the positive definiteness of the inner product eq. (29) and instead has to set

\[ \langle p|i'j'\rangle = 2\omega_p \delta_{i+} \delta_{j+} \delta(p - p') - 2\omega_p \delta_{i-} \delta_{j-} \delta(p - p'), \text{ where } i, j \in \{+,-\}. \]  

(33)

Now one has abandoned the positive definiteness of the inner product eq. (29), there is no reason to keep \( \langle p|p'\rangle \) real. Indeed, once the absolute values of \( |p|\) are fixed, there remains a phase factor free to choose

\[ \langle p|i'j'\rangle = 2\omega_p \delta_{i+} \delta_{j+} \delta(p - p') + 2e^{i\alpha} \omega_p \delta_{i-} \delta_{j-} \delta(p - p'), \text{ where } i, j \in \{+,-\}. \]  

(34)

Setting \( \alpha = 0 \) (resp. \( \alpha = \pi \)) eq. (32) (resp. eq. (33)) is readily obtained. The following completeness relations are straightforward

\[ 1 = \int \frac{d^{3}p}{2\omega_p} [|p+\rangle \langle p+| + e^{-i\alpha} |p-\rangle \langle p-|], \]  
\[ = i \int x^0=const. d^3x \left[ |x+\rangle \hat{\partial}_0 \langle x+| - e^{-i\alpha} |x-\rangle \hat{\partial}_0 \langle x-| \right]. \]  

The relativistic propagator turns out to be

\[ G_{\alpha}(x' - x) \equiv \langle x'|x \rangle = \frac{1}{2} \Delta_1(x' - x) + i\Delta(x' - x) + \frac{e^{i\alpha}}{2}(\Delta_1(x' - x) - i\Delta(x' - x)) \]  
\[ = \langle 0|\varphi(x')\varphi(x)|0 \rangle + e^{i\alpha} \langle 0|\varphi(x)\varphi(x')|0 \rangle. \]  

(35)  
(36)

One finds \( G_0 = \Delta_1 \) and \( G_{\pi} = i\Delta \). \( G_{\alpha} \) is a solution of the Klein-Gordon equation for every \( \alpha \in [0,2\pi[ \).
2 Path-Integral representation for $G_\alpha$

At the end of the last section it was shown that for special values of $\alpha$ the Green function $G_\alpha$ is equal to the Hadamard Green function resp. to the causal Green function. As remarked above $\Delta_1$ possesses a path-integral representation eq. (13). Therefore the suggestion in [1] that possible there is no sum-over-histories representation for $\Delta$ seems to be rather unnatural. On the contrary one would expect that after introducing a suitable phase factor in eq. (13) a path-integral representation for $G_\alpha$ could straightforwardly be obtained. The aim of this section is to show that this is indeed the case. First the result is stated and then checked.

The path-integral representation for $G_\alpha$ is

$$G_\alpha(x'' - x') = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int Dp \, Dx \left[ \prod_{\tau'} \Theta(p_0) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - N\mathcal{H}_0) \right) \right. $$

$$\left. + e^{i\alpha} \prod_{\tau'} \Theta(-p_0) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - N\mathcal{H}_0) \right) \right].$$

(37)

The meaning of the infinite product will become clearer in eq. (39) below. Setting $\alpha = \pi$ one obtains the following path-integral representations for the causal Green function:

$$\Delta(x'' - x') = -i \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int Dp \, Dx \left( \prod_{\tau'} \Theta(p_0) \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - N\mathcal{H}_0) \right)$$

$$= -i \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int Dp \, Dx \left( \prod_{\tau'} \Theta(p_0) \exp \left( -ip\dot{x} + iN\mathcal{H}_0 \right) \right).$$

(38)

Setting $\alpha = 0$ yields again eq. (13). To prove eq. (37) one uses the well-known constructive definition of path-integrals by a time slicing procedure. The measure at each time slice is taken to be the Liouville measure. Suppressing the integration over $N(\tau'' - \tau')$, the first term in eq. (37) gives with $\delta = \frac{\tau'' - \tau'}{n+1}$

$$\int Dp \, Dx \left( \prod_{\tau'} \Theta(p_0) \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - N\mathcal{H}_0) \right)$$

$$\equiv \lim_{n \to \infty} \int \prod_{k=1}^{n} d^4x_k \frac{d^4p_k}{(2\pi)^4} \prod_{h=0}^{n} \Theta(p_h^0) \exp \left( -ip_h(x_{h+1} - x_h) + i\delta N(p_h^2 - m^2) \right).$$

(39)

In eq. (33) the $x_k$’s and $p_k$’s are thought to define a skeletonized path in phase space. The $x_k$’s are viewed as the values of $x(\tau)$ at parameter-time $\tau_k = \tau' + k\delta$, whereas the $p_k$’s are thought of as the values of $p(\tau)$ at parameter-time $\frac{1}{2}(\tau_{k+1} + \tau_k) = \tau' + (k + \frac{1}{2})\delta$.

Setting $n = 0$ in (39) one obtains a short-time propagator. It is well known that the short-time evolution of any positive energy solution of the Klein-Gordon equation is given by this short-time propagator. The reverse is also true. Namely, given any wave function, whose short-time evolution is determined by this short-time propagator, one can show that under reasonable circumstances this wave function is a solution of the positive square root of the Klein-Gordon equation. The details are given in the appendix.

Performing the $x_i$-integrations in (39) yields $\delta$-functions of the form $\delta(p_i - p_{i-1})$. Therefore all but one of the $p$-integrations may be performed to yield

$$\int Dp \, Dx \left( \prod_{\tau'} \Theta(p_0) \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - N\mathcal{H}_0) \right)$$

$$= \int \frac{d^4p}{(2\pi)^4} \Theta(p_0) \exp \left( -ip(x'' - x') + iN(\tau'' - \tau')(p^2 - m^2) \right).$$
The second term of eq. (37) may be manipulated in the same way. Recalling

$$(2\pi)\delta(p^2 - m^2) = \int_{-\infty}^{\infty} dTe^{iE(p^2 - m^2)},$$

and setting $T \equiv N(\tau'' - \tau')$, comparison with eqs. (27), (26) and (2) - (3) proves eq. (37). 

The appearance of the Heaviside functions in the path-integral (37) prevents the $p$-integrations from being straightforwardly performed. However a little trick makes it possible to perform the $p$-integrations without affecting the $x$-integrations. Inserting the identity

$$\Theta(x) = \lim_{\kappa \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega}{\omega - i\kappa} e^{i\omega x}$$

into the first term in eq. (37) gives (again suppressing the integration over $N(\tau'' - \tau')$),

$$\int DpDx \left( \prod_{\tau'}^{\tau''} \Theta(p^0) \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right)$$

$$= \lim_{n \to \infty} \frac{1}{(2\pi)^n} \int \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^2} \frac{dp_k}{(2\pi)^2} \frac{dp_0}{(2\pi)^2} \left( \prod_{h=0}^{n} \Theta(p^0_h) \exp \left( -ip_h(x_{h+1} - x_h) + i\delta N(p^2_h - m^2) \right) \right)$$

$$= \lim_{n \to \infty} \frac{1}{(2\pi)^n} \int \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^2} \frac{dp_k}{(2\pi)^2} \frac{dp_0}{(2\pi)^2} \left[ \prod_{h=0}^{n} \lim_{\kappa_h \to 0} \frac{d\omega_h}{2\pi i(\omega_h - i\kappa_h)} \exp \left( ip^0_h\omega_h - ip_h(x_{h+1} - x_h) + i\delta N(p^2_h - m^2) \right) \right].$$

After interchanging the integration over $p_h$ and the limit $\kappa_h \to 0$, the Gaussian integral over $p_h$ is straightforward. Performing all $p$-integrations gives the result

$$\int DpDx \left( \prod_{\tau'}^{\tau''} \Theta(p^0) \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right)$$

$$= \frac{1}{(2\pi)^n} \lim_{n \to \infty} \int \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^2} \prod_{h=0}^{n} \lim_{\kappa_h \to 0} \frac{d\omega_h}{2\pi i(\omega_h - i\kappa_h)} i\epsilon(N) \left( \frac{\pi}{\delta N} \right)^2$$

$$\times \exp \left( \frac{-i}{4\delta N} \left( (x_{h+1} - x_h)^2 - 2(x_{h+1}^0 - x_h^0)\omega_h + \omega^2_h + 4\delta^2 N^2 m^2 \right) \right),$$

where $\epsilon$ was defined in (3). Application of Cauchy’s integral formula and taking $\kappa_h \to 0$ yields

$$\int DpDx \left( \prod_{\tau'}^{\tau''} \Theta(p^0) \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right)$$

$$= \frac{1}{(2\pi)^n} \lim_{n \to \infty} \int \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^2} \prod_{h=0}^{n} \left[ i\epsilon(N) \left( \frac{\pi}{\delta N} \right)^2 \Theta \left( \frac{x_{h+1}^0 - x_h^0}{\delta N} \right) \exp \left( -i\frac{(x_{h+1} - x_h)^2}{4\delta N} - i\delta N m^2 \right) \right]$$

$$= \int Dx \Theta \left( \frac{x_0^0}{N} \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau \frac{x^2}{4N} + Nm^2 \right),$$

3 By the way notice that $2 \int [T] = 2 \int [N] (\tau'' - \tau')$ is identical with the so-called ‘fifth parameter’, which was first introduced by Fock. Feynman used the fifth parameter to bring the Klein-Gordon-equation in a form analogous to the Schrödinger-equation. Furthermore Feynman stated a representation of the Feynman propagator $\Delta_F$ as an integral over the fifth parameter, which can also be obtained readily from eq. (40) performing the $DpDx$ integrations.

4 To make the integrations well defined, one has to add to $N$ a suitable imaginary part: for the $p_0$-integration: $N \to N + i\alpha^2$ and for the $p_j$-integration ($j = 1, 2, 3$): $N \to N - i\alpha^2$. After the integration $\alpha^2$ is send to zero.
where the symbolic measure $\tilde{D}x$ is defined as follows
\[ \tilde{D}x = \frac{1}{(2\pi)^4} \lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{dx_k}{(2\pi)^2} \right) \left( \frac{\pi}{\delta N} \right)^{2n+2} (i\epsilon(N))^{n+1}. \] (43)

For notational simplicity we suppress the (formal) infinite product over all times in front of the Heaviside function in eq. [12]. Remarkably enough after performing the $p_0$-integration the factor $\Theta(p_0^2)$ turns into a factor $\Theta(\frac{2N+2}{\delta N})$. Thus if $N > 0$ a particle with positive energy is traveling forward in time and if $N < 0$ a particle with positive energy is traveling backward in time.

For the second term in eq. [13] the $p$-integrations may be performed in the same way as for the first term. The factor $\Theta(-p_0^2)$ turns into $\Theta(-\frac{2N+2}{\delta N})$. Therefore a particle with negative energy is traveling forward (resp. backward) in time, if $N < 0$ (resp. $N > 0$). Therefore all trajectories of physical interest are included if one considers the case $N > 0$ only. These results have been obtained already in [4] by completely different means considering the classical equations of motion of the relativistic particle.

The final result for the path-integral representation of $G_\alpha$ in configuration space is
\[ G_\alpha(x'' - x') = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \left\{ \int \tilde{D}x \left[ \Theta \left( \frac{\dot{x}_0}{N} \right) + e^{i\alpha} \Theta \left( -\frac{\dot{x}_0}{N} \right) \right] \exp \left( -i \int_{\tau'}^{\tau''} d\tau \left( \frac{\dot{x}_0^2}{4N} + Nm^2 \right) \right) \right\}. \] (44)

Especially for $\Delta$ and $\Delta_1$ one obtains
\[ i\Delta(x'' - x') = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int \tilde{D}x \left( \frac{\dot{x}_0}{N} \right) \exp \left( -i \int_{\tau'}^{\tau''} d\tau \left( \frac{\dot{x}_0^2}{4N} + Nm^2 \right) \right), \] (45)
\[ \Delta_1(x'' - x') = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int \tilde{D}x \exp \left( -i \int_{\tau'}^{\tau''} d\tau \left( \frac{\dot{x}_0^2}{4N} + Nm^2 \right) \right). \] (46)

**BRST Path integral representation for $G_\alpha$**

There is another path integral representation in phase space for the Green functions $G_\alpha$, which uses techniques from BRST theory. An introduction to BRST symmetry as needed here can be found in [1].

The idea behind what follows is to derive a path integral representation for the first and second term in eq. [36a] separately. Eq. [13b] induces a path integral representation for $\Delta_+(x' - x) = \langle 0|\varphi(x')\varphi(x)|0 \rangle$, which can also be written as
\[ \Delta_+(x'' - x') = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int DpDx \left( \int_{\tau'}^{\tau''} \Theta(-p_0) \right) \exp \left( i \int_{\tau'}^{\tau''} d\tau (p\dot{x} - NH_0) \right). \] (47)

The infinite product of Heaviside-functions indicate that in BRST theory one has to obey both the constraint eq. [13] and the further constraint $p_0(\tau) < 0$, for all $\tau$. The simplest possibility to incorporate both constraints into a single constraint function is to take the ‘square root’ of the constraint eq. [13]:
\[ \tilde{H}_0 \equiv p_0^2 + p_0 \omega_P = 0, \] (48)
where $\omega_P$ is defined by eq. [8]. It should be stressed that it is in general not allowed to choose an equivalent constraint, e.g. $p_0 + \omega_P = 0$, because the resulting BRST path integral does in general not coincide with $\Delta_+$ and furthermore it does not even have the same dimension as $\Delta_+$. This feature is not well understood and the author hopes to come back to it in a future publication.

With the constraint [15] at hand the derivation of the BRST path integral is straightforward. Therefore only the bare essentials are given here. The complete story can be found in [1].

In a first step one introduces a ghost pair $(\eta, P)$ associated with the constraint $\tilde{H}_0 = 0$. $\eta$ and $P$ are canonically conjugate anticommuting variables. $P$ is also called the ghost momentum. The Lagrange multiplier $N$
introduced to incorporate the constraint into the action (see eq. (14)) is viewed as a dynamical variable. As a further constraint the momentum Π conjugate to N is required to vanish on the constraint surface

$$\Pi = 0.$$ 

Now a second anticommuting ghost pair $$(\rho, \bar{C})$$ - called antighosts - associated with the constraint Π = 0 has to be introduced.

Finally the extended phase space is defined to be the space spanned by the original phase space coordinates $p_\mu$ and $x^\mu$ together with the ghost variables $\eta, \mathcal{P}, \rho, \bar{C}$ and with the multiplier $N$ and its conjugate momentum Π.

The Poisson-bracket is extended as follows

$$\{ \mathcal{P}, \eta \} = \{ \rho, \bar{C} \} = \{ \Pi, N \} = -1,$$

and all other Poisson-brackets involving the new variables $\eta, P, \rho, \bar{C}, N, \Pi$ are taken to vanish. The extended action appropriate to the dynamics in extended phase space is called the gauge-fixed action and is given by

$$S_E(p_\mu, x^\mu, \eta, P, \rho, \bar{C}, N, \Pi) = \int_{\tau'}^{\tau} d\tau (p_\mu \dot{x}^\mu + \dot{\eta} P + \dot{\Pi} N + \dot{\bar{C}} \rho - \{ K, \Omega \}). \quad (49)$$

Here as usual $\Omega$ denotes the BRST charge

$$\Omega = \eta \tilde{H}_0 - i\rho \Pi.$$ 

The action (49) is invariant under transformations generated by the BRST charge $\Omega$. The factor $-i$ is convention.

The BRST charge $\Omega$ is nilpotent

$$\{ \Omega, \Omega \} = 0.$$

$K$ is called the gauge-fixing fermion.

Now the BRST path integral is the formal expression

$$PI = \int Dp Dx D\eta D\bar{C} D\Pi D\rho D\bar{C} \exp(iS_E). \quad (50)$$

This path integral is thought to be defined by a time slicing procedure with the boundary conditions eq. (19) together with

$$\Pi = \bar{C} = \eta = 0$$

at the boundaries. The measure at each time slice is taken to be the Liouville measure. The famous Fradkin-Vilkovisky-Theorem \[4, 6\] states that we are free to choose $K$ as

$$K = -\mathcal{P} N,$$

without changing the value of the expression in (50). In \[4\] it is shown that this choice is compatible with eq. (13), that is, the equations of motion derived from the action (49) imply $\dot{N} = 0$.

With the choice $K = -\mathcal{P} N$ the integrations over the ghosts and antighosts decouple from the other integrations and may be performed separately

$$\int D\eta D\bar{C} D\rho D\bar{C} \exp(iS_E) \equiv \lim_{n \to \infty} \int_{h=1}^{n} d\eta_h \int_{h=1}^{n} d\bar{C}_h \int_{h=0}^{n} d\rho_h \int_{h=0}^{n} d\mathcal{P}_h \times \exp \left[ \sum_{h=0}^{n} (i(\eta_{h+1} - \eta_h)\mathcal{P}_h + i(\bar{C}_{h+1} - \bar{C}_h)\rho_h + \delta \mathcal{P}_h \rho_h) \right]. \quad (51)$$

Strictly speaking the left hand side of eq. (51) is defined through the right hand side. Expanding the exponential function and bearing in mind the definition of integration of anticommuting numbers, that is $\int d\kappa \equiv 1$ and $\int d\kappa \equiv 0$, one easily sees that only the terms containing the $\mathcal{P}_h, \rho_h$ in the form $\prod_h \mathcal{P}_h \rho_h$ survive the $\mathcal{P}, \rho$-integrations. Properly adjusting all prefactors after performing the $\mathcal{P}$ and $\rho$-integrations yields

$$(-1)^{\frac{\pi^2 a^2}{4}} \left[ \delta^{n+1} - \delta^n \sum_{h=0}^{n} (\bar{C}_{h+1} - \bar{C}_h)(\eta_{h+1} - \eta_h) \right]$$
\[ + \delta^{n-1} \sum_{h,j=0 \atop j<h}^{n} (\bar{C}_{h+1} - \bar{C}_h)(\eta_{h+1} - \eta_h)(\bar{C}_{j+1} - \bar{C}_j)(\eta_{j+1} - \eta_j) \pm \ldots + \]  
\[ + \delta(-1)^n \sum_{h=1 \atop h_1 < h_2 < \ldots < h_n}^{n} (\bar{C}_{h+1} - \bar{C}_h)(\eta_{h+1} - \eta_h) \cdots (\bar{C}_{h_n+1} - \bar{C}_{h_n})(\eta_{h_n+1} - \eta_{h_n}) \].

The factor \((-1)^{2n} \delta(n+1)(-1)^{2-n} \equiv (n+1)\delta = \tau'' - \tau'.\) (53)

(Pedestrians may check this by induction.) The II, N-integrations in eq. (54) are quite easy

\[ \int \mathcal{D}N \mathcal{D} \eta \exp \left( i \int d\tau \mathcal{N} \right) = \int \prod_{h=1}^{n} \frac{dN_h d\Pi_h}{2\pi} dN_0 \exp \left( i \sum_{h=0}^{n} (N_{h+1} - N_h) \Pi_h \right) = \int \prod_{h=0}^{n} dN_h \delta (N_{h+1} - N_h). \]

So one finally arrives at

\[ PI = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int \mathcal{D}p \mathcal{D}x \exp \left( i \int_{\tau'}^{\tau''} d\tau (p_\mu \dot{x}^\mu - N \dot{\mathcal{H}}_0) \right). \] (54)

An analysis similar to the above has already been carried out in [4] for the relativistic particle with the constraint \(\mathcal{H}_0.\) Now it is straightforward to check that \(PI = 2\Delta_+:\) Setting \(T \equiv N(\tau'' - \tau')\) one obtains

\[ PI = \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int \prod_{h=1}^{n} \frac{d^4p_h d^4x_h}{(2\pi)^4} \frac{d^4p_0}{(2\pi)^4} \exp \left\{ i \sum_{h=0}^{n} [p_h(x_{h+1} - x_h) - \delta N p_h^0 (p_h^0 + \omega_p)] \right\} \]

\[ = \int_{-\infty}^{\infty} dT \int \frac{d^4p}{(2\pi)^4} \exp \left\{ -ip(x'' - x') - iT(p^0 p^0 - p^0 \omega_p) \right\} \]

\[ = \int \frac{d^4p}{(2\pi)^4} \delta(p^0 p^0 - p^0 \omega_p) \exp(-ip(x'' - x')) \]

\[ = \int_{p^0 = +\omega_p} \frac{d^3p}{(2\pi)^3} \omega_p \exp(-ip(x'' - x')) \]

\[ = 2\Delta_+(x'' - x'). \] (55)

The final result of this subsection is therefore

\[ \Delta_+(x'' - x') = \frac{1}{2} \int_{-\infty}^{\infty} dN(\tau'' - \tau') \int \mathcal{D}p \mathcal{D}x \exp \left( i \int_{\tau'}^{\tau''} d\tau (p_\mu \dot{x}^\mu - N \dot{\mathcal{H}}_0) \right). \] (57)

\[ = \frac{1}{2} \int \mathcal{D}p \mathcal{D}x \mathcal{D}\eta \mathcal{D}p \mathcal{D}N \mathcal{D} \Pi \mathcal{D} \bar{\mathcal{C}} \exp (i S_E). \] (58)

A path integral representation for \(\Delta_-\) can be obtained similar using the constraint \(\tilde{\mathcal{H}}_{\partial_0} \equiv p_\mu^2 - p_0 \omega_p.\) Putting the path integral representations for \(\Delta_+\) and \(\Delta_-\) together yields a path-integral representation for \(\mathcal{G}_\alpha.\) This concludes the derivation of our new BRST-path-integral representation for \(\mathcal{G}_\alpha.\)
3 Composition laws for \(G_\alpha\)

The main goal in \([\text{I}]\) was to derive the relativistic composition laws for the considered Green functions from their path integral representations. For this task the path decomposition expansion (PDX) was used \([\text{I}]\). In this section the same thing will be done for the Green functions \(G_\alpha\). To this end the path decomposition expansion has to be generalized to be applicable to path-integrals of the form eq. (12).

The Path Decomposition Expansion

The path decomposition expansion allows to express the dynamics in full configuration space through the dynamics in two disjunct regions of configuration space separated by a surface \(\Sigma\). The reader not familiar with the PDX is referred to the references \([\text{1, 7}]\). In the following the path decomposition expansion for path integrals of the form

\[
K(x(\tau'), x(\tau''), N, \tau'' - \tau') \equiv \int \bar{D}x \Theta(\epsilon(N)\dot{x}_0) \exp \left( i \int_{\tau'}^{\tau''} d\tau \left( \frac{x'^2}{4N} + Nm^2 \right) \right)
\]

will be derived. The measure \(\bar{D}x\) was defined above in eq. (13) and the function \(\epsilon\) was defined in eq. (1). The following derivation of the PDX is a modification of the original derivation given in \([\text{7}]\). The PDX is derived in the Euclidian regime. The Euclidian version of (59) is obtained by rotating both \(\tau\) and \(x^0\) in the following manner

\[
\tau \rightarrow \tau_E = -\epsilon(N)i\tau; \quad x^0 \rightarrow x^0_E = -ix^0.
\]

In the following the index ‘\(E\)’ will be suppressed. Let \(\Sigma(x^0)\) be the surface in configuration space of constant \(x^0\)-coordinate \(x^0 = \tilde{x}^0\), where \(\tilde{x}^0\) lies between \(x^0(\tau'')\) and \(x^0(\tau')\).

First consider the Wick-rotated path integral

\[
\int \bar{D}x \Theta(\epsilon(N)\dot{x}_0) \exp \left( -\epsilon(N) \int_{\tau'}^{\tau''} d\tau \left( \frac{\sum_{\mu=0}(\dot{x}_\mu^2)}{4N} + Nm^2 \right) \right)
\]

\[
\equiv \lim_{n \to \infty} \frac{-i}{(2\pi)^4} \int \prod_{k=1}^{n} \frac{d^4 x_k}{(2\pi)^4} \prod_{h=0}^{n} \left( \frac{\pi}{\delta N} \right)^2 \Theta \left( \frac{x^0_{h+1} - x^0_h}{\delta} \right) \exp \left( -\frac{\sum_{\mu=0}(x^\mu_{h+1} - x^\mu_h)^2}{4\delta \epsilon(N)N} - \epsilon(N)\delta Nm^2 \right)
\]

where as above \(\delta = \tilde{x}'' - \tilde{x}'\). Now because of the factor \(\Theta(x^0)\) every path in the sum (60) crosses \(\Sigma(x^0)\) once and only once. Let \(S_m\) be the set of paths, which fulfill \(x^0_m \leq \tilde{x}^0 < x^0_{m+1}\). The contribution of \(S_m\) to the sum (60) is

\[
\frac{-i}{(2\pi)^4} \int \prod_{k=1}^{n} \frac{d^4 x_k}{(2\pi)^4} \prod_{h=0}^{n} \left( \frac{\pi}{\delta N} \right)^2 \exp \left( -\frac{(x^0_{h+1} - x^0_h)^2}{4\delta \epsilon(N)N} - \frac{\sum_{i=1}^{3}(x^i_{h+1} - x^i_h)^2}{4\delta \epsilon(N)N} - \epsilon(N)\delta Nm^2 \right)
\]

The integral in eq. (61) is in the sequel denoted by \(K^n_m(x(\tau'), x(\tau''), N, -i\epsilon(N)(\tau'' - \tau'))\). The path integral (61) is obtained by summing over all \(m\) and taking \(n \to \infty\). Therefore the expression in eq. (61) equals

\[
\lim_{n \to \infty} \sum_{m} K^n_m \]

Now the following identity is needed

\[
\left( \frac{1}{4\delta \pi \epsilon(N)N} \right)^2 \exp \left( -\frac{\sum_{\mu=0}(x^\mu_{m+1} - x^\mu_m)^2}{4\delta \epsilon(N)N} \right)
\]

\[
= \int_{0}^{\delta} d\kappa \int_{\Sigma(\tilde{x}^0)} d^3\tilde{x} \left( \frac{1}{4\kappa \pi \epsilon(N)N} \right)^2 \exp \left( -\frac{\sum_{\mu=0}(\tilde{x}^\mu - x^\mu_m)^2}{4\kappa \epsilon(N)N} \right) 2\epsilon(N)N
\]

\[
\times \left( \frac{1}{4(\delta - \kappa)\pi \epsilon(N)N} \right)^2 \frac{\partial}{\partial x^0} \exp \left( -\frac{\sum_{\mu=0}(x^\mu_{m+1} - x^\mu_m)^2}{4(\delta - \kappa)\epsilon(N)N} \right) \bigg|_{x=\tilde{x}^0}.
\]
Inserting eq. (62) into eq. (61) yields
\[ K^n_m (x', x'', N, -i\epsilon(N)(\tau'' - \tau')) = \]
\[ i \int_0^\delta \int_{\Sigma(\tilde{\varphi}')} d^3\tilde{x} K^n_{m+1} (x', \tilde{x}, N, -i\epsilon(N)(m\delta + \kappa)) 2 |N| \frac{\partial}{\partial x^0} K^n_0 (x, x'', N, -i\epsilon(N)(\tau'' - \tau' - m\delta - \kappa)) \bigg|_{x=\tilde{x}}. \] (63)

Summing over all \( m \), taking the limit \( n \to \infty \) and rotating back to real time finally yields the path decomposition expansion for \( K \)
\[ K (x', x'', N, (\tau'' - \tau')) = 2iN \int_{\tau'}^{\tau''} dt \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} K (x', x, N, t) \frac{\partial}{\partial x^0} K (x, x'', N, (\tau'' - \tau' - t)) \bigg|_{x=\tilde{x}}. \] (64)
The PDX can be written more symmetrically
\[ K (x', x'', N, (\tau'' - \tau')) = iN \int_{\tau'}^{\tau''} dt \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} \left\{ K (x', x, N, t) \delta_0 K (x, x'', N, (\tau'' - \tau' - t)) \right\} \bigg|_{x=\tilde{x}}. \] (65)

This PDX differs from the usual PDX in that no restricted Green function appears, which clearly is due to the appearance of the infinite product of Heaviside functions in the sum-over-histories eq. (69).

For path integrals of the form
\[ K' (x', x'', N, \tau'' - \tau') \equiv \int \tilde{D}x \Theta (-\epsilon(N)\tilde{x}_0) \exp \left( i \int_{\tau'}^{\tau''} d\tau \left( \frac{\tilde{x}^2}{4N} +Nm^2 \right) \right) \] (66)
the PDX has the following form
\[ K'(x', x'', N, (\tau'' - \tau')) = -iN \int_{\tau'}^{\tau''} dt \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} \left\{ K' (x', x, N, t) \delta_0 K' (x, x'', N, (\tau'' - \tau' - t)) \right\} \bigg|_{x=\tilde{x}}. \] (67)

**Composition laws**

First the composition law for \( \Delta_+ \) will be derived. The reasoning is similar to that in [1]. Setting \( T \equiv N(\tau'' - \tau') \) the path integral representation obtained above can written as
\[ \Delta_+ (x'' - x') = \int_{-\infty}^{\infty} dT \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} \Theta (-\epsilon(T)\tilde{x}_0) \exp \left( i \int_{0}^{T} dt \left( \frac{\tilde{x}^2}{4} +m^2 \right) \right), \] (68)

Insertion of the PDX eq. (67) yields
\[ \Delta_+ (x'' - x') = i \int_{-\infty}^{\infty} dT \int_{-\infty}^{T} d\tilde{t} \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} \delta_+ (x'' - \tilde{x}, T, \tilde{t}) \delta_0 \delta_+(\tilde{x} - x', \tilde{t}, 0), \] (69)
where
\[ \delta_+ (x'' - x', T_1, T_0) \equiv \int \tilde{D}x \Theta (-\epsilon(T_1 - T_0)\tilde{x}_0) \exp \left( i \int_{T_0}^{T_1} dt \left( \frac{\tilde{x}^2}{4} +m^2 \right) \right). \]

The composition law of \( \Delta_+ \) follows immediately after the substitution \( v \equiv T - \tilde{t} \) and \( u \equiv \tilde{t} \).
\[ \Delta_+ (x'' - x') = i \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} du \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} \delta_+ (x'' - \tilde{x}, v, 0) \delta_0 \delta_+(\tilde{x} - x', u, 0) \]
\[ = i \int_{\Sigma(\tilde{\varphi})} d^3\tilde{x} \Delta_+ (x'' - \tilde{x}) \delta_0 \Delta_+ (\tilde{x} - x'). \] (70)
In the same way eq. (65) yields for $\Delta_-$

$$\Delta_-(x'' - x') = -i \int_{\Sigma(\tilde{x}_0)} d^3\tilde{x} \Delta_- (x'' - \tilde{x}) \frac{\partial}{\partial \tilde{x}} \Delta_- (\tilde{x} - x'). \quad (71)$$

Using that $\Delta_+$ and $\Delta_-$ are orthogonal

$$\int_{\Sigma(\tilde{x}_0)} d^3\tilde{x} \Delta_+(x'' - \tilde{x}) \frac{\partial}{\partial \tilde{x}} \Delta_-(\tilde{x} - x') = 0$$

the 'composition law' for $G_\alpha$ follows

$$\int_{\Sigma(\tilde{x}_0)} d^3\tilde{x} G_\alpha (x'' - \tilde{x}) \frac{\partial}{\partial \tilde{x}} G_\alpha (\tilde{x} - x') = -i \Delta_+(x'' - x') + ie^{2i\alpha} \Delta_-(x'' - x'), \quad (72)$$

which can also be written as

$$G_\alpha (x'' - x') = i \int_{\Sigma(\tilde{x}_0)} d^3\tilde{x} G_{\alpha + \alpha} (x'' - \tilde{x}) \frac{\partial}{\partial \tilde{x}} G_{\alpha + \alpha} (\tilde{x} - x'). \quad (73)$$

The well known composition laws for $\Delta$ and $\Delta_1$ are obtained from eq. (72) by setting $\alpha = 0$ resp. $\alpha = \pi$.

$$\Delta(x'' - x') = - \int_{\Sigma(\tilde{x}_0)} d^3\tilde{x} \Delta (x'' - \tilde{x}) \frac{\partial}{\partial \tilde{x}} \Delta (\tilde{x} - x'), \quad (74)$$

$$\Delta(x'' - x') = + \int_{\Sigma(\tilde{x}_0)} d^3\tilde{x} \Delta_1 (x'' - \tilde{x}) \frac{\partial}{\partial \tilde{x}} \Delta_1 (\tilde{x} - x'). \quad (75)$$

The composition laws for other Green functions of free scalar field theory - e.g. the composition law for the Feynman propagator $\Delta_F$ - follow also from eq. (70) and (71).

4 Conclusion

In this paper we have obtained two different new path integral representations in phase space for the Green functions $\Delta_+$ and $\Delta_-$, respectively. The key point concerning the first path integral representation (37) is that its local measure contains an infinite product of Heaviside functions. The second new path integral representation is the BRST path integral in extended phase space associated with a special choice of the constraint. The central result of this paper is the sum-over-histories representation (42) for $\Delta_+$ and $\Delta_-$, respectively.

Putting the path integral and sum-over-histories representations for $\Delta_+$ and $\Delta_-$ together we obtained the path integral (37) and the sum-over-histories representations (44) for the $G_\alpha$'s, introduced in eqs. (35), (36).

Setting $\alpha = \pi$ we achieved the main goal of this paper, namely obtaining the sum-over-histories representation (45) for the causal Green function in compact form.

A rather complicated sum-over-histories representation for the causal Green function has already been discussed in [4]. However we do not expect this representation to be of great use in practice.

In the last section we derived a modified path decomposition expansion. Finally we derived the composition laws for $\Delta_\pm$ from their sum-over-histories representation using the modified path decomposition expansion. Furthermore en passant we improved many results already obtained in [1].

An extensive discussion of the physical motivations underlying this paper and of the relevance for quantum cosmology of the obtained results can be found in [1].

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Therefore we conclude that $\varphi^+(x_2) = i \int d^3 x \Delta_+ (x_2 - x_1) \partial_0 ^+ \varphi^+ (x_1)$.

Now let us forget for a moment this result and investigate the short-time propagator obtained from (39) by setting $n = 0$

$$\int dT \int \frac{d^4 p}{(2\pi)^4} \left( \Theta(p^0) \exp \left(-i p^0 (x'' - x') + iT(p^2 - m^2) \right) \right).$$

Imagine a wave function, whose short-time evolution is given by

$$\varphi(x', x_0 + \epsilon) = \int dT \int \frac{d^4 p}{(2\pi)^4} \int d^3 x \left( \Theta(p^0) \exp \left(-i p^0 \epsilon + ip(x' - x) + iT(p^2 - m^2) \right) P(\partial_0, \partial_k, \partial_0, \partial_k) \varphi(x, x_0) \right),$$

where $\epsilon > 0$ and $P(\partial_0, \partial_k, \partial_0, \partial_k)$ is a polynomial in the derivative operators $\partial_\mu$ and $\partial_\mu$. Expanding the left and the right hand side in a power series in $\epsilon$ and equating equal powers in $\epsilon$ one obtains

$$\frac{\partial^n \varphi}{\partial x_0^n}(x', x_0) = \int dT \int \frac{d^4 p}{(2\pi)^4} \int d^3 x \left( \Theta(p^0) e^{-ip^0 \epsilon + ip(x' - x) + iT(p^2 - m^2) + iT(p^2 - m^2)} P(\partial_0, \partial_k, \partial_0, \partial_k) \varphi(x, x_0) \right),$$

$$= \int d^3 x \frac{d^4 p}{(2\pi)^4} \frac{\delta(p^0 - \omega_p)}{2\omega_p} (-i)^n (p^0)^n \exp (ip(x' - x)) P(ip^0, -ip_k, \partial_0, \partial_k) \varphi(x, x_0),$$

$$= \int d^3 x \frac{d^4 p}{(2\pi)^4} \frac{\delta(p^0 - \omega_p)}{2\omega_p} (-i)^n (\omega_p)^n e^{ip(x' - x)} P(ip^0, -ip_k, \partial_0, \partial_k) \varphi(x, x_0),$$

$$= \int d^3 x \frac{d^4 p}{(2\pi)^4} \frac{\delta(p^0 - \omega_p)}{2\omega_p} (-i)^n \sqrt{m^2 - (\nabla')^2} e^{ip(x' - x)} P(ip^0, -ip_k, \partial_0, \partial_k) \varphi(x, x_0),$$

$$= (-i)^n \sqrt{m^2 - (\nabla')^2} \varphi(x', x_0).$$

Therefore we conclude that $\varphi$ satisfies the positive square root of the Klein-Gordon equation.
References

[1] J.J. Halliwell and M.E. Ortiz, Phys. Rev. D 48 (1993) 748.
   J.J. Halliwell and M.E. Ortiz, preprint CTP # 2235, to appear in the Proceedings of Journées Relativistes 93, World Scientific, 1993.

[2] J.J. Halliwell, Phys. Rev. D 38 (1988) 2468.

[3] C. Teitelboim, Phys. Rev. D 25 (1982) 3159.

[4] M. Henneaux and C. Teitelboim, Quantisation of Gauge Systems, (Princeton Univ. Press, 1992).
   M. Henneaux, Classical Foundations of BRST Symmetry, (Bibliopolis, Napoli, 1988).
   M. Henneaux, Phys. Rep. 126 (1985) 1.
   K. Sundermeyer, Constraint Dynamics, Lecture Notes in Physics 169, (Springer, Berlin, 1982).

[5] R.P. Feynman, Phys. Rev. 80 (1950) 440.
   V. Fock, Physik. Zeits. Sowjetunion 12 (1937) 404.

[6] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B 55 (1975) 224.
   I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B 69 (1977) 309.
   E.S. Fradkin and T.E. Fradkina, Phys. Lett. B 72 (1978) 343.

[7] A. Auerbach and S. Kivelson, Nucl. Phys. B 257 (1985) 799.

[8] A. Anderson, preprint, Imperial-TP-92-93-46.