OPTIMAL RESOURCE ALLOCATION IN THE DIFFERENCE AND DIFFERENTIAL STACKELBERG GAMES ON MARKETING NETWORKS

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Abstract. We consider difference and differential Stackelberg game theoretic models with several followers of opinion control in marketing networks. It is assumed that in the stage of analysis of the network its opinion leaders have already been found and are the only objects of control. The leading player determines the marketing budgets of the followers by resource allocation. In the basic version of the models both the leader and the followers maximize the summary opinions of the network agents. In the second version the leader has a target value of the summary opinion. In all four models we have found the Stackelberg equilibrium and the respective payoffs of the players analytically. It is shown that the hierarchical control system is ideally compatible in all cases.

1. Introduction. The most known basic model of influence in a social network which is used in this paper as well belongs to De Groot [7]. The model describes the dynamics of opinions of the agents (nodes of a network) in dependence of their interaction (characterized by weights of arcs of the network). Many modifications and refinements of the models were proposed afterwards: good reviews are presented in the monograph [10] and the paper [1]. There are also alternative approaches to the modeling and simulation of networks such as gene-environment networks, eco-finance networks, rumor propagation, Markov switching models and others. The reviews can be found in [3]-[4]. Along with the models of influence, there are also models of opinion control in social networks [6]. In this case there are one or several external control agents that exert influence on the opinions of basic network agents in their interests. In the papers which are closest to this work [15], [18] their authors built and investigated linear-quadratic game theoretic models on a network with two influence nodes. In [15] the Nash equilibrium is found for two independent nodes, and in [18] a Stackelberg equilibrium for two hierarchically
organized nodes is calculated. In the paper [2] we present briefly our approach to
the mathematical modeling of opinion control in social networks with marketing
applications. The principal idea is formulated as follows. It is proved (see, for
example, [13]) that the stable final opinions of all basic network agents (the target
audience in marketing terms) are determined exclusively by the initial opinions of
the members of strong subgroups of the network (opinion leaders). Formally, the
strong subgroups are the strong components of a network which belong to the vertex
base of its condensation [13]. Thus, the first stage of analysis of a network consists
in its segmentation on strong subgroups and satellites [6] as well as calculation
of additional quantitative characteristics of the members of strong subgroups [2].
In the second stage the problems of optimal and conflict (game theoretic) opinion
control are solved. And, the control agents exert influence only to the members
of strong subgroups that gives an essential economy while providing a sufficient
efficiency of control. In this paper we study game theoretic models of opinion control
on networks. One of the most important application domains is marketing when the
network represents a structured target audience of two or several competing firms
which produce the same goods. Certainly, other applications to political activities,
organizational and territorial control, and so on are possible as well. It is assumed
that the opinion leaders are already determined in the stage of analysis and are
the only object of control. We also assume the presence of a coordinating control
body that determines the marketing budgets of the firms by resource allocation.
Therefore, we consider Stackelberg games of the type “a leader - several followers”.
The dynamics of a controlled system (opinions of the target audience) is described by
difference or differential ordinary equations. Respectively, the payoff functionals of
the players are sums or integrals of their current payoff functions. An introduction of
the discrete or continuous time creates the complementary approaches that permit
to reflect different phenomena in real systems. Therefore, it is expedient to use
both approaches. Thus, we study difference and differential Stackelberg games
with several followers. The theory of dynamic Stackelberg games is exposed in
[5], [8]. Some examples of applications are given in [11]. An original approach
to the solution of dynamic Stackelberg games is proposed in [9]. We describe our
modification of this approach for several followers in [16], [17]. In this case a best
response of the followers to the leader’s strategy is defined as a Nash equilibrium
in the followers’ game in normal form. In this paper the difference models are
analyzed by means of induction by the number of periods of the game, and the
differential models are solved by the standard technique of Hamilton-Jacobi-Bellman
equations. The Lagrange multipliers method is used for consideration of the budget
constraints. The key problem in hierarchical systems is the coordination of interests
of the different control levels. The most widespread formulation of this problem is a
comparison of the socially optimal outcome of the game with its outcome generated
by the egoistic behavior of the players (a problem of inefficiency of equilibria) [14].
For the quantitative evaluation of the inefficiency of equilibria the notion of price
of anarchy is used [12]. The price of anarchy is a fraction which numerator is
equal to the summary payoff of all players in the worst Nash equilibrium, and the
denominator is equal to the globally maximal value of the summary payoff that
is attained in the case of complete cooperation of all players. In this paper we
use an index of system compatibility (SCI) for the characteristic of the degree of
coordination of the players’ interests. It is a fraction which numerator is equal to
the leader’s payoff in the worst for her Nash equilibrium in the game of the followers,
and the denominator is equal to the globally maximal value of the leader’s payoff. The rest of the paper is organized as follows. In Section 2 we study difference game theoretic Stackelberg models of opinion control in two versions. In the basic version both the leader and the followers maximize the summary opinions of the network agents, and the leader determines the marketing budgets of the followers by resource allocation. In the second version the leader has a target value of the summary opinion and tries to achieve it as precisely as possible. In both cases we calculate the system compatibility index which measures a degree of coordination of interests in the control system. In section 3 the differential game theoretic Stackelberg models of opinion control (in continuous time) are analyzed by the similar pattern. Thus, four dynamic Stackelberg game theoretic models of opinion control in marketing networks are investigated analytically. Section 4 concludes.

2. Difference game theoretic Stackelberg models of opinion control.

2.1. A basic difference Stackelberg game. The model has the form

\[
J_0 = \sum_{t=0}^{T} \delta^t \left[ \sum_{j=1}^{n} x_j^t - \sum_{i=1}^{m} r_i^t \right] \rightarrow \max,
\]

\[r_i^t \geq 0, \quad \sum_{i=1}^{m} r_i^t \leq R, \quad t = 0, 1, 2, \ldots, T, \quad i = 1, 2, \ldots, m,
\]

\[
J_i = \sum_{t=0}^{T} \delta^t \left[ \sum_{j=1}^{n} \left( x_j^t - s_j^i u_j^{i,t} \right) \right] \rightarrow \max,
\]

\[
\sum_{j=1}^{n} u_j^{i,t} \leq r_i^t, \quad u_j^{i,t} \geq 0, \quad t = 0, 1, 2, \ldots, T, \quad i = 1, 2, \ldots, m,
\]

\[
x_{j}^{t+1} = \sum_{i=1}^{m} b_{ij} \sqrt{u_j^{i,t}} + \sum_{l=1}^{n} a_{lj} x_{j}^{t}, \quad x_{j}^{0} = x_{j0}, \quad j = 1, 2, \ldots, n,
\]

\[
s_j^{i} = \begin{cases} 0, & \text{if } b_{j}^{i} > 0, \\ 1, & \text{if } b_{j}^{i} = 0. \end{cases}
\]

Here \(n\) – a number of basic agents (a number of target audience), \(m\) – a number of control agents (competing firms), \(T\) – a length of the game, \(J_0, J_i\) – the payoff functionals of the leader and the followers (control agents) respectively, \(r_i^t\) – a marketing budget allocated to the \(i\)-th follower by the leader in the moment (discrete period) of time \(t\), an opinion of the \(j\)-th basic agent in the moment \(t\), \(a_{ij}\) – a coefficient of influence of the \(i\)-th basic agent to the \(j\)-th basic agent, \(b_{ij}\) – a coefficient of influence of the \(j\)-th control agent to the \(j\)-th basic agent, \(\delta\) denotes a discount factor, i.e. \(\delta = e^{-\rho}\). As different firms can exert influence to different members of the strong subgroups, we simply assume that if the \(i\)-th firm (control agent) does not influence to the \(j\)-th basic agent then \(b_{ij} = 0\). Denote by \(A\) a matrix of the coefficients of influence among basic agents, i.e. \(A = \{a_{ij}\}_{i=1,2,\ldots,n,j=1,2,\ldots,n}\), \(A^T\) – a transposed matrix of influences, \(X^t\) – a column vector of the values of state variables (opinions) in the moment \(t\), \(\varepsilon\) – a row vector of the dimension \(n\) formed by units, \(I\) – the unit \(n \times n\)-matrix.
First, let us consider the problem of the \(i\)-th firm (3) – (6), and will refer to it as the problem \(FD_i\) (i-th firm discrete). Denote
\[
\left(\sum_{i=1}^{m} b_{ij} \sqrt{u_{ij}^{t}}\right)^{n}_{j=1} = \beta^{t}, \quad t = 0, 1, 2, \ldots T. \tag{7}
\]
Then we have
\[
X^1 = A^*X^0 + \beta^0, \quad X^2 = (A^*)^2 + A^* \beta^0 + \beta^1,
\]
Iterations by \(t\) give
\[
X^t = (A^*)^tX^0 + (A^*)^{t-1} \beta^0 + (A^*)^{t-2} \beta^1 + \cdots + A^* \beta^{t-2} + \beta^{t-1}. \tag{8}
\]
Each \(i\)-th firm solves the problem
\[
T \sum_{t=0}^{n} s_{ij} u_{ij}^{t} = T \sum_{t=0}^{n} \delta t \left(\varepsilon X^t - \sum_{j=1}^{n} s_{ij} u_{ij}^{t}\right) \quad \rightarrow \text{max}, \tag{9}
\]
s.t.
\[
\sum_{j=1}^{n} s_{ij} u_{ij}^{0} \leq r_{i}^{0},
\]
where \(u_{ij}^{0}, u_{ij}^{1}, \ldots, u_{ij}^{T-1} (j = 1, 2, \ldots, n)\) are the respective solutions of the \((T-1)\)-period, \((T-2)\)-period, ..., one-period problems.

Selecting in (9) only those terms that contain the variables \(u_{ij}^{0}, j = 1, 2, \ldots, n\), we receive the expression
\[
- \sum_{j=1}^{n} s_{ij} u_{ij}^{0} + \varepsilon \left[\delta I + \delta^2 A^* + \cdots + \delta^T (A^*)^{T-1}\right] \beta^0. \tag{10}
\]
Maximizing (10) with the constraint
\[
\sum_{j=1}^{n} u_{ij}^{0} \leq r_{i}^{0}, \tag{11}
\]
by the method of Lagrange multipliers, we receive the relations
\[
\frac{(1 + \delta A_{j_2} + \cdots + \delta^{T-1} A_{j_2}^{T-1}) b_{j_2}^{i}}{(1 + \delta A_{j_1} + \cdots + \delta^{T-1} A_{j_1}^{T-1}) b_{j_1}^{i}} = \frac{\sqrt{u_{ij}^{0}}}{\sqrt{u_{ij}^{0}}},
\]
for any agents \(j_1\) and \(j_2\) influenced by the \(i\)-th firm, where \(A_{j}^{t}\) denotes the sum of elements of the \(j\)-th row of the \(t\)-th power of the matrix of influences, \(t = 1, 2, \ldots, T - 1\). Denote the sum \(\sum_{j=1}^{n} u_{ij}^{0}\) by \(R_{i}^{0}\), we receive
\[
u_{ij}^{0} = \frac{\left(\sum_{t=0}^{T-1} \delta^t A_{j}^{t}\right)^{2} R_{i}^{0}}{\sum_{j=1}^{n} \left(\sum_{t=0}^{T-1} \delta^t A_{j}^{t}\right)^{2}}. \tag{12}
\]
A substitution of (12) in (10) with consideration of (7) gives
\[
- R_{i}^{0} + \delta \sum_{i=1}^{m} \sqrt{R_{i}^{0} \sum_{j=1}^{n} \left(\sum_{t=0}^{T-1} \delta^t A_{j}^{t}\right)^{2}}. \tag{13}
\]
Maximizing (13) by \( R_i^0 \) without constraints, we receive
\[
(R_i^0)_{\text{max}} = \frac{\delta^2}{4} \sum_{j=1}^{\infty} \left( b_j^T \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2.
\]

With consideration of the constraint (11) we have
\[
\begin{align*}
\sum_{j=1}^{n} \left( b_j^T \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2 &= \frac{\delta^2}{4} \sum_{j=1}^{n} \left( b_j^T \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2 \\
&\min \left\{ r_i^0 : \sum_{j=1}^{n} \left( b_j^T \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2 \right\}.
\end{align*}
\]

Let us write the strategy of the firm in the period game if the leader's strategy is \( \sum_{s=0}^{T-1} \delta^s A_j^s \). If \( T = 0 \) then independently of the leader's strategy the optimal strategy of any firm is evidently \( u_j^i, 0 = 0, j = 1, 2, \ldots, n \). Let \( T > 0 \). Then for any \( t = 0, 1, \ldots, T - 1 \) the optimal strategy of the \( i \)-th firm has the form
\[
\begin{align*}
u_i^j &= \frac{b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s}{\sum_{j=1}^{n} b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s} \left( b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s \right)^2 \\
&\min \left\{ r_i^0 : \sum_{j=1}^{n} \left( b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s \right)^2 \right\}.
\end{align*}
\]

If \( t = T \) then the optimal strategy is \( u_j^i, T = 0, j = 1, 2, \ldots, n \).

Now let us consider the leader's problem \( 32 \) – \( 2 \), \( 4 \) – \( 5 \), and will refer to it as the problem \( LD \) (leader discrete). The leader solves the problem
\[
J_0 = \sum_{t=0}^{T} \delta^t \left[ \sum_{j=1}^{n} x_j^t - \sum_{i=1}^{m} r_i^t \right] = \sum_{t=0}^{T} \delta^t \left[ \varepsilon X^t - \sum_{i=1}^{m} r_i^t \right] \rightarrow \max,
\]
s.t.
\[
\sum_{i=1}^{m} r_i^t \leq R,
\]
where \( r_i^1, r_i^2, \ldots, r_i^{T-1} (i = 1, 2, \ldots, m) \) are the respective solutions of the \( (T - 1) \)-period, \( (T - 2) \)-period, ..., one-period problems, and \( u_j^i, t \) is determined by the formula (14), \( t = 0, 1, \ldots, T - 1 \). In the period \( t \) the leader will never choose the value greater than \( \frac{\delta^2}{4} \sum_{j=1}^{n} \left( b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s \right)^2 \). Thus, we can substitute the expression (14) by
\[
\begin{align*}
u_i^j &= \frac{b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s}{\sum_{j=1}^{n} b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s} \left( b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s \right)^2 \\
&\sum_{j=1}^{n} \left( b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s \right)^2.
\end{align*}
\]

Using the denotation (7), the decomposition (8), and selecting in (15) only those terms that contain \( r_i^0 \), we receive the expression
\[
\begin{align*}
\sum_{i=1}^{m} r_i^0 + \varepsilon \left[ \delta I + \delta^2 A^2 + \cdots + \delta^T (A^T)^{T-1} \right] \beta^0 = 0,
\end{align*}
\]
where
\[
\beta^0 = \left( \sum_{i=1}^{m} b_j^T \sqrt{u_j^0} \right)^n = \sum_{i=1}^{m} \left( \frac{\sqrt{r_i^0} (b_j^T)^2 \sum_{s=0}^{T-1} \delta^s A_j^s} {\sqrt{\sum_{j=1}^{n} (b_j^T \sum_{s=0}^{T-1} \delta^s A_j^s)^2}} \right)^n.
\]
Maximizing (17) with constraint \(\sum_{i=1}^{m} r_i^0 \leq R\) by the Lagrange multipliers method, we receive the relations

\[
\frac{\sum_{j=1}^{n} \left( (1 + \delta A_j + \cdots + \delta^{t-1} A_j^{T-1}) b_j^i \right)^2}{\sum_{j=1}^{n} (1 + \delta A_j + \cdots + \delta^{T-1} A_j^{T-1}) b_j^i} = \frac{r_i^0}{r_i^0}
\]

for any control agents \(i_1\) and \(i_2\). Denote the sum \(\sum_{i=1}^{m} r_i^0\) by \(R^0\), then we receive

\[
\sum_{i=1}^{m} r_i^0 = \frac{\sum_{j=1}^{n} \sum_{i=1}^{m} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2}{\sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2} = R^0, \quad r_i^0 = \frac{\sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2}{\sum_{j=1}^{n} \sum_{i=1}^{m} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2}.
\]

Substitution (19) in (17) with consideration of (18) gives

\[
-R^0 + \delta \sqrt{R^0 \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2}.
\]

Maximizing the expression (20) by \(R^0\) without constraints, we receive

\[
(R^0)_{\text{max}} = \frac{\delta^2}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2.
\]

Therefore, with consideration of the constraint \(R^0 \leq R\) the optimal leader’s strategy is

\[
r_i^0 = \frac{\sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2}{\sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2} \min \left\{ \frac{\delta^2}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{t=0}^{T-1} \delta^t A_j^t \right)^2, R \right\}.
\]

Let us write the optimal leader’s strategy in the \(T\)-period game. If \(T = 0\) then, evidently, \(r_i^0 = 0, i = 1, 2, \ldots, m\). Let \(T > 0\). Then for any \(t = 0, 1, \ldots, T-1\) the optimal leader’s strategy is

\[
r_i^t = \frac{\sum_{j=1}^{n} \left( b_j^i \sum_{s=0}^{T-1-t} \delta^s A_j^s \right)^2}{\sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{s=0}^{T-1-t} \delta^s A_j^s \right)^2} \min \left\{ \frac{\delta^2}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{s=0}^{T-1-t} \delta^s A_j^s \right)^2, R \right\}.
\]

If \(t = T\) then, evidently, \(r_i^T = 0, i = 1, 2, \ldots, m\).

It is clear that the maximal guaranteed payoff of the leader in the \(T\)-period game is equal to

\[
\varepsilon \sum_{t=0}^{T} (\delta A^t)^t X^0 \sum_{t=1}^{T} \left[ R^{t-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{s=0}^{T-1-t} \delta^s A_j^s \right)^2 - \sum_{t=1}^{T} \delta^{t-1} R^{t-1} \right],
\]

where

\[
R^t = \sum_{i=1}^{m} r_i^t = \min \left\{ \frac{\delta^2}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b_j^i \sum_{s=0}^{T-1-t} \delta^s A_j^s \right)^2, R \right\}, \quad t = 1, 2, \ldots, T-1.
\]
Let us calculate the system compatibility index:

\[
SCI = \frac{\max_{r_i} \min_{u_{ij}} \{ \text{NE}(r_i) \} \ J_0 \left( \{ r_i \}_{i=1}^m; \{ u_{ij} \}_{i=1,j=1}^m \right) \max_{r_i} \max_{u_{ij}} \{ \text{NE}(r_i) \} \ J_0 \left( \{ r_i \}_{i=1}^m; \{ u_{ij} \}_{i=1,j=1}^m \right)}{\max_{r_i} \min_{u_{ij}} \{ \text{NE}(r_i) \} \ J_0 \left( \{ r_i \}_{i=1}^m; \{ u_{ij} \}_{i=1,j=1}^m \right) \max_{r_i} \max_{u_{ij}} \{ \text{NE}(r_i) \} \ J_0 \left( \{ r_i \}_{i=1}^m; \{ u_{ij} \}_{i=1,j=1}^m \right)}.
\]  

(24)

Given the leader’s strategy, in this case a set of equilibrium strategies of each control agent is a singleton. Therefore, the numerator of the expression (24) is equal to the right hand side of the formula (22). To calculate the denominator in (24), we assume that each control agent maximizes not its own payoff but the leader’s payoff given its strategy, i.e. it solves the problem:

\[
\tilde{J}_i = \sum_{t=0}^T \delta^t \sum_{j=1}^n x^t_j \rightarrow \max,
\]

s.t. (4)–(5).

Repeating exactly the same actions as in the solution of the problem \(FD_i\), we receive the formula that is absolutely similar to the formula (14), with the only difference that instead of the minimum of two variables there will be simply \(r_i\), i.e. we receive the formula (16). Repeating the same actions as in the game \(LD\), we receive the same optimal strategy of the leader and the same guaranteed payoff (22). In fact, during the solution of the problem \(LD\) we already used the formula (16) instead of the formula (14) because the respective leader’s logic was clear.

Thus, in the basic version of the model the system compatibility index \(SCI = 1\). In general, the leader would like that the control agents invest to the marketing actions all the allocated resources. But in fact it happens that the leader itself is not interested to allocate more resources than the control agents would use in their egoistic behavior.

### 2.2. Difference game with a target value of the summary opinion.

Now the model takes the form (problem \(HD\)):

\[
J_0 = \sum_{i=0}^T \delta^t \left( \sum_{j=1}^n z^t_j x^t_j - x^* \right) + \sum_{i=1}^m r^t_i \rightarrow \min,
\]

(25)

the other conditions are the same as in the Subsection 2.1, i.e. (2) – (6), \(x^*\) is a target value of the summary opinion that the leader tries to achieve as precise as possible.

It is assumed usually that a matrix of influences \(A\) is stochastic from the left, i.e. \(\sum_{j=1}^n a^t_{ij} = 1, \ j = 1, 2, \ldots, n\), and \(z^t_j > 0, \ j = 1, 2, \ldots, n\), are the components of a positive (right) eigenvector that corresponds to Frobenius eigenvalue 1 of the matrix \(A\). Other assumptions are the same as in the Subsection 2.1. Suppose first that the matrix \(A\) is stochastic (from the right), i.e.

\[
\sum_{j=1}^n a^t_{ij} = 1, \ i = 1, 2, \ldots, n,
\]

and, therefore, \(z^t_j = 1, \ j = 1, 2, \ldots, n\). In this case an action of the matrix \(A^\tau\) from the left on any vector does not change the sum of its components. Thus,

\[
\varepsilon (A^\tau)^T X^0 = \varepsilon (A^\tau)^{T-1} X^0 = \cdots = \varepsilon A^\tau X^0 = \varepsilon X^0,
\]

\[
A^0_j = A^2_j = \cdots = A^T_j = 1, \ j = 1, 2, \ldots, n.
\]
It is evident that the strategies of control agents in this problem is exactly the same as in the Subsection 2.1. The expression (14) takes the form
\[
u^{i,t}_{j} = \frac{(b^j_i)^2}{\sum_{j=1}^{n}(b^j_i)^2} \min \left\{ \frac{\delta^2}{4} \sum_{j=1}^{n} \left( b^j_i \sum_{s=0}^{T-t} \delta^s \right)^2, r^j_i \right\}.
\]
(26)
The leader cannot decrease the value \(\sum_{j=1}^{n} x^j_i\), and only increases it by allocating the resources to the firms. Then the optimal strategy of the leader is evident.

If \(\sum_{j=1}^{n} x^0_j \geq x^*\) then \(r^j_i = 0, i = 1, 2, \ldots, m, t = 0, 1, 2, \ldots, T - 1, T\). If \(\sum_{j=1}^{n} x^0_j < x^*\) then in fact the leader solves a problem that is equivalent to the following:
\[
\tilde{J}_0 = \sum_{t=0}^{T} \delta^t \left( \sum_{j=1}^{n} x^j - x^* - \sum_{i=1}^{m} r^i_t \right) \to \max,
\]
or, that is the same,
\[
\tilde{J}_0 = \sum_{t=0}^{T} \delta^t \left( \sum_{j=1}^{n} x^j - \sum_{i=1}^{m} r^i_t \right) \to \max,
\]
s.t. (2), (4) – (5) and \(\sum_{j=1}^{n} x^j_i \leq x^*, t = 1, 2, \ldots, T - 1, T\).

It is clear that for maximization of its payoff function the leader must increase the value \(\sum_{j=1}^{n} x^j_i\) as well as in the Subsection 2.1 but only up to the value \(x^*\), after achievement of which it should cease to allocate resources to the firms. Then the value \(\sum_{j=1}^{n} x^j_i\) will remain equal to \(x^*\) until the end of the game. For implementation of this strategy in the initial moment of time \(t = 0\) the leader must determine the period \(t = H\) when the value \(\sum_{j=1}^{n} x^j_i\) becomes greater than \(x^*\) given the respective strategy. And, in the period \(t = H - 1\) the value of allocated resources should be reduced so that from \(t = H\) the value \(\sum_{j=1}^{n} x^j_i\) will be equal to \(x^*\).

A consideration of the \(H\)-period game in the similar manner as in the problem \(LD\) shows that \(H\) is the moment when for the first time the value
\[
\Delta = \sum_{t=0}^{H-1} \sum_{i=1}^{m} \sum_{j=1}^{n} (b^j_i \sum_{s=0}^{H-1-t} \delta^s)^2 \min \left\{ \frac{\delta^2}{2} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} (b^j_i \sum_{s=0}^{H-1-t} \delta^s)^2, \sqrt{R} \right) \right\}
\]
will become greater or equal than \(x^* - \sum_{j=1}^{n} x^0_j\) (if this moment is reachable for some \(t \leq T\)). Denote in this case
\[
\tilde{R} = x^* - \sum_{j=1}^{n} x^0_j - \left( \Delta - \left( \sum_{i=1}^{m} \sum_{j=1}^{n} (b^j_i)^2 \min \left\{ \frac{\delta^2}{2} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} (b^j_i)^2, \sqrt{R} \right) \right\} \right) \right).
\]
(27)
Then the leader’s optimal strategy is
\[
r^j_i = \frac{\sum_{i=1}^{m} (b^j_i)^2}{\sum_{j=1}^{n} (b^j_i)^2} \min \left\{ \frac{\delta^2}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( b^j_i \sum_{s=0}^{H-1-t} \delta^s \right)^2, R \right\} \text{ when } 0 \leq t \leq H - 2;
\]
\[
r^{H-1}_i = \frac{\sum_{j=1}^{n} (b^j_i)^2}{\sum_{i=1}^{m} \sum_{j=1}^{n} (b^j_i)^2} \tilde{R} \text{ when } t = H - 1;
\]
\[
r^0_i = 0 \text{ when } H \leq t \leq T, \quad i = 1, 2, \ldots, m.
\]
The leader’s guaranteed payoff is equal to
\[
\left| \sum_{t=0}^{H-1} \delta^t \left( \sum_{j=1}^n x_j^0 - x^* \right) + \sum_{t=0}^{H-2} \left( \sum_{s=t+1}^{H-1} \delta^s \right) \right| R^t \sum_{i=1}^m \sum_{j=1}^n (b_j^t)^2 - \sum_{t=0}^{H-2} \delta^t R^t - \delta^{H-1} \tilde{R},
\]
(28)
when
\[
R^t = \sum_{i=1}^m r_i^t = \min \left\{ \frac{\delta^2}{4} \sum_{i=1}^m \sum_{j=1}^n (b_j^t)^2 \left( \sum_{s=0}^{H-1-t} \delta^s \right)^2, R \right\}, \quad t = 0, 1, 2, \ldots, H - 2,
\]
(29)
and the value \( \tilde{R} \) is determined by the formula (27).

Let us calculate the system compatibility index for the model with a target value:
\[
SCI = \frac{\max_{\{r_i\}_{i=1}^m} \min_{\{u_j\}_{j=1}^n \in \mathcal{N} \mathcal{E}(\{r_i\}_{i=1}^m)} \tilde{J}_0 \left( \{r_i\}_{i=1}^m, \{u_j\}_{j=1}^n, \{r_i^t\}_{t=1}^T, \{u_j^t\}_{t=1}^T \right)}{\max_{\{r_i\}_{i=1}^m} \max_{\{u_j\}_{j=1}^n} \min_{\{r_i^t\}_{t=1}^T, \{u_j^t\}_{t=1}^T} J_0 \left( \{r_i\}_{i=1}^m, \{u_j\}_{j=1}^n, \{r_i^t\}_{t=1}^T, \{u_j^t\}_{t=1}^T \right)}. \tag{30}
\]
Given the leader’s strategy, in this case a set of equilibrium strategies of each control agent is a singleton. Therefore, the numerator of the expression (30) is equal to (28). To calculate the denominator in (30), we assume that each control agent maximizes not its own payoff but the leader’s payoff given its strategy, i.e. it solves the problem:
\[
\hat{J}_i = \sum_{t=0}^T \delta^t \left( \sum_{j=1}^n x_j^t - x^* - \sum_{i=1}^m r_i^t \right) \rightarrow \max,
\]
s.t. (4) - (5) and \( \sum_{j=1}^n x_j^t \leq x^*, \ t = 1, 2, \ldots, T - 1, T. \)

If \( \sum_{j=1}^n x_j^t \geq x^* \) then the optimal strategy of each firm is evident: it should not invest at all to the marketing actions. In fact, it cannot do it because then the leader will not allocate any resource, i.e. \( u_j^t = 0. \)

Suppose that \( \sum_{j=1}^n x_j^0 < x^* \). Unlike the leader, a control agent cannot evaluate a priori the number of periods \( H \) that is necessary for the summary opinion becomes equal to \( x^* \) but in fact it is not required because optimal strategies of the firms do not depend on the length of the planning period.

The same reasoning as in the previous case gives for the optimal strategy of the \( i \)-th firm the expression that is similar to (26):
\[
u_{i,j}^t = \frac{(b_j^t)^2}{\sum_{j=1}^n (b_j^t)^2} \min \left\{ \frac{\delta^2}{4} \sum_{j=1}^n (b_j^t)^2 \left( \sum_{s=0}^{H-1-t} \delta^s \right)^2, r_i^t \right\}.
\]
Let us consider the leader’s problem in the \( H \)-period game. As the leader will never allocate to any firm more resources than
\[
\mathcal{R}_i = \frac{\delta^2}{4} \sum_{j=1}^n (b_j^t)^2 \left( \sum_{s=0}^{H-1-t} \delta^s \right)^2, \tag{31}
\]
we can accept that
\[
u_{i,j}^{t,t} = \frac{(b_j^t)^2 r_i^t}{\sum_{j=1}^n (b_j^t)^2}.
\]
Then the leader’s problem does not differ from the calculation of the numerator of the expression (30), and the value in the denominator of (30) is also expressed by
the formula (28). Therefore, in this problem the interests of all players are again ideally coordinated, and \( SCI = 1 \).

**Remark 1.** A non-trivial situation arises. The firms in the \( t \)-th period of time must choose the value of marketing expenditures equal to \( R_i \) if this value is not greater than \( r^*_t \), and choose \( r^*_t \) if \( R_i > r^*_t \). But the firms do not know the number of periods \( H \), and respectively the value of \( R_i \). However, the firms know that the leader will never allocate to them more resources than \( R_i \), therefore they always choose \( r^*_t \), \( i = 1, 2, \ldots, m \). The leader could use this circumstance and allocate to all firms in the \( t \)-th period more resources than \( \sum_{i=1}^m R_i \), and compel them to invest more resources than it is optimal for them but it is not advantageous for the leader itself. Therefore, the leader always allocate to the firms the value of resource \( \sum_{i=1}^m R_i \) if its own budget \( R \) allows for it, otherwise it allocates the resource value that is optimal to itself. It proceeds until the period \( H - 1 \), when the leader allocates a reduced value of resource that provides in the period \( H \) that the sum of opinions is equal exactly to \( x^* \). Thus, in the problem \( HD \) the interests of the leader and the followers are ideally coordinated again.

We solved the problem \( HD \), subject to a strict constraint for the matrix \( A \), namely we assumed that \( A \) is stochastic (from the right). The standard assumption is that that \( A \) is stochastic from the left and, respectively, \( A^T \) is stochastic (from the right). Let us reduce the standard assumption to the considered case. Let a matrix of influences \( A \) is stochastic from the left, and \( z \) is a positive eigenvector that corresponds to the eigenvalue 1.

If \( z^T = (z_1 \ z_2 \ \ldots \ z_n) \), \( z_i > 0, \ i = 1, 2, \ldots, n \), then, evidently, the diagonal matrix

\[
Z = \begin{pmatrix}
z_1 & 0 & \ldots & 0 \\
0 & z_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & z_n
\end{pmatrix}
\]

is a matrix of transition to the stochastic (from the right) matrix \( P: P = Z^{-1}AZ \). Then \( A = ZPZ^{-1} \), i.e. \( A^T = Z^{-1}P^T Z \), and we can rewrite the problem \( HD \) in other coordinates.

Introduce a family of matrices \( B^i, i = 1, 2, \ldots, m \), and vectors \( \sqrt{U^{i,t}} \):

\[
B^i = \begin{pmatrix}
b^i_1 & 0 & \ldots & 0 \\
0 & b^i_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & b^i_n
\end{pmatrix}, \quad \sqrt{U^{i,t}} = \begin{pmatrix}
\sqrt{u^{i,t}_1} \\
\sqrt{u^{i,t}_2} \\
\vdots \\
\sqrt{u^{i,t}_n}
\end{pmatrix}.
\]

Then the constraint (5) takes the matrix form

\[
X_{t+1} = \sum_{i=1}^m B^i \sqrt{U^{i,t}} + Z^{-1}P^TX^t,
\]

or

\[
Y_{t+1} = \sum_{i=1}^m \tilde{B}^i \sqrt{U^{i,t}} + P^TY^t,
\]
where

\[ X^t = \begin{pmatrix} x_1^t \\ x_2^t \\ \vdots \\ x_n^t \end{pmatrix}, \quad Y^t = \begin{pmatrix} y_1^t \\ y_2^t \\ \vdots \\ y_n^t \end{pmatrix} = ZX^t, \]

\[ \tilde{B}^i = ZB^i = \begin{pmatrix} z_1 b_i^1 & 0 & \ldots & 0 \\ 0 & z_2 b_i^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & z_n b_i^n \end{pmatrix} = \begin{pmatrix} \tilde{b}_i^1 & 0 & \ldots & 0 \\ 0 & \tilde{b}_i^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{b}_i^n \end{pmatrix}, \]

or in the coordinate form:

\[ y_{t+1}^j = \sum_{i=1}^{m} \tilde{b}_i^j \sqrt{u_{i,t}^j} + \sum_{l=1}^{n} p_{lj} y_l^t. \]

The expression (3) takes the form

\[ J_i = \sum_{t=0}^{T} \delta^t \left[ \sum_{j=1}^{n} (z_j^{-1} y_j^t - s_j^j u_j^i(t)) \right] \to \max, \]

and the expression (25) takes the form

\[ J_0 = \sum_{t=0}^{T} \delta^t \left[ \left| \sum_{j=1}^{n} y_j^t - x^* \right| + \sum_{i=1}^{m} r_i^t \right] \to \min. \] (32)

Other expressions in the formulation of the problem HD do not change.

3. Differential game theoretic Stackelberg models of opinion control.

3.1. A basic differential Stackelberg game. The model has the form

\[ J_0 = \int_0^T e^{-\rho t} \left[ \sum_{j=1}^{n} x_j(t) - \sum_{i=1}^{m} r_i(t) \right] dt \to \max, \] (33)

\[ r_i(t) \geq 0, \quad \sum_{i=1}^{m} r_i(t) \leq R, \quad t \in [0,T], \quad i = 1, 2, \ldots, m, \] (34)

\[ J_i = \int_0^T e^{-\rho t} \left[ \sum_{j=1}^{n} (x_j(t) - s_j^i u_j^i(t)) \right] dt \to \max, \] (35)

\[ \sum_{j=1}^{n} u_j^i(t) \leq r_i(t), \quad u_j^i(t) \geq 0, \quad t \in [0,T], \quad i = 1, 2, \ldots, m, \] (36)

\[ \dot{x}_j = \sum_{i=1}^{m} b_i^j \sqrt{u_j^i(t)} + \sum_{l=1}^{n} a_{jl} x_l(t), \quad x_j(0) = x_{j0}, \quad j = 1, 2, \ldots, n, \] (37)

\[ s_j^i = \begin{cases} 0, & \text{if } b_j^i > 0, \\ 1, & \text{if } b_j^i = 0. \end{cases} \] (38)

Here \( T \) is a length of the game, \( J_0, J_i \) are payoff functionals of the leader and the followers (control agents, firms) respectively, \( r_i(t) \) is a marketing budget allocated by the leader to the \( i \)-th agent in the moment \( t \), \( x_j(t) \) is a value of opinion of the \( j \)-the basic agent in the moment \( t \), \( u_j^i(t) \) – marketing expenditures of the \( i \)-th control
agent respective to the \(j\)-th basic agent in the moment \(t\). Other variables have the same sense as in the Subsection 2.1.

Let us consider the problem of the \(i\)-th firm \((35) - (38)\), and will refer to it as FC\(_i\) (\(i\)-th firm continuous). The Hamilton-Jacobi-Bellman equation for this problem has the form

\[
\rho V_i - \frac{\partial V_i}{\partial t} = \max_{u^i_{j,1}, \leq j \leq n} \left\{ \sum_{j=1}^{n} x_j(t) - \sum_{j=1}^{n} \rho u^i_j(x_j(t)) + \sum_{j=1}^{n} \frac{\partial V_i}{\partial x_j} \left[ \sum_{k=1}^{m} b^k_j \sqrt{w^k_j(x_j)} + \sum_{l=1}^{n} a_{ij} x_l \right] \right\}
\]

\(\text{s.t.}\)

\[
\sum_{j=1}^{n} u^i_j(x_j(t)) \leq r_i(t).
\]

Maximization by \(u^i_j, j = 1, 2, \ldots, n, b^i_j \neq 0\), gives for any \(1 \leq j_1, j_2 \leq n\):

\[
\frac{\partial V_i}{\partial x_{j_2}} b^i_{j_2} = \left( \frac{u^i_{j_2}}{u^i_{j_1}} \right)^{\frac{1}{2}}.
\]

Denote the sum

\[
\frac{u^i_{j_1}}{\left( b^i_{j_1} \frac{\partial V_i}{\partial x_{j_1}} \right)^{\frac{1}{2}}} \sum_{j=1}^{n} \left( b^i_j \frac{\partial V_i}{\partial x_j} \right)^{\frac{1}{2}}
\]

by \(R_i(t)\). Then

\[
u^i_j = \frac{R_i(t) \left( b^i_j \frac{\partial V_i}{\partial x_j} \right)^{\frac{1}{2}}}{\sum_{j=1}^{n} \left( b^i_j \frac{\partial V_i}{\partial x_j} \right)^{\frac{1}{2}}}.
\]

Choose the linear Bellman functions:

\[
V_i(x,t) = \sum_{j=1}^{n} \alpha^i_j(t)x_j + \beta^i(t),
\]

then we can right the equation \((39)\) with consideration of \((40)\) in the form

\[
\rho \sum_{j=1}^{n} \alpha^i_j(t)x_j + \rho \beta^i(t) - \sum_{j=1}^{n} \alpha^i_j(t)x_j - \beta^i(t) = \sum_{j=1}^{n} x_j(t) - R_i(t) + \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha^i_j(t)a_{ij}x_l + \sum_{k=1}^{m} \sum_{j=1}^{n} \alpha^i_j(t)b^k_j \sqrt{R_k(t)\alpha^k_j(t)b^k_j} \left( \sum_{j=1}^{n} \left( \alpha^k_j(t)b^k_j \right)^2 \right)^{-\frac{1}{2}}.
\]

Equating in the left and right hand sides of \((41)\) the coefficients of the first powers of \(x\), we receive the following differential equations for the coefficients \(\alpha\):

\[
\alpha^i_j(t) - \rho \alpha^i_j(t) + \sum_{l=1}^{n} \alpha^i_j(t)a_{jl} = -1,
\]

\(j = 1, 2, \ldots, n\). Rewrite the system of equations \((42)\) in a matrix form

\[
\alpha^i = (\rho I - A)\alpha^i - \varepsilon,
\]

where \(\alpha^i\) is a column vector of the coefficients \(\alpha^i_j\). The system of equations \((43)\) is the same for all control agents, therefore \(\alpha^1_j(t) = \alpha^2_j(t) = \cdots = \alpha^m_j(t) = \alpha_j(t)\) for any basic agent \(j = 1, 2, \ldots, n\), and from this moment the index \(i\) of the coefficients \(\alpha_j\) will be omitted.
The solution of the system of differential equations (42) with boundary conditions \(\alpha(T) = 0\) gives

\[
\alpha = \left( e^{(A - \rho I)(T - t)} - I \right) (A - \rho I)^{-1} \varepsilon. \tag{44}
\]

In particular, when \(t = 0\) we have

\[
\alpha(0) = \left( e^{(A - \rho I)T} - I \right) (A - \rho I)^{-1} \varepsilon. \tag{45}
\]

Considering that \(\alpha_j^i(t) = \alpha_k^k(t)\) for any \(k = 1, 2, \ldots, n\), we can rewrite (41) in the form:

\[
\rho \sum_{j=1}^{n} \alpha_j^i(t)x_j + \rho \beta^i(t) = \sum_{j=1}^{n} \alpha_j^i(t)x_j - \beta^i(t) = n x_i(t) - R_i(t) + \sum_{j=1}^{n} \sum_{l=1}^{m} \alpha_j^i(t) a_{ij} x_l + \sum_{k=1}^{m} \sqrt{R_k(t)} \sum_{j=1}^{n} (\alpha_j^k(t) b_j^k)^2. \tag{46}
\]

Choosing the maximal value of the right hand side of the expression (46) in dependence on \(R_i(t)\), we have

\[
(R_i(t))_{\text{max}} = \frac{1}{4} \sum_{j=1}^{n} \left( b_j^i \alpha_j^i(t) \right)^2.
\]

Thus, the value \(R_i(t)\) in (41) is equal to

\[
R_i(t) = \min \left\{ \frac{1}{4} \sum_{j=1}^{n} \left( b_j^i \alpha_j^i(t) \right)^2, r_i(t) \right\}. \tag{47}
\]

The optimal strategies of the followers are determined by the expression (40):

\[
u_j^i(t) = \frac{R_i(t) \left( b_j^i \alpha_j^i(t) \right)^2}{\sum_{j=1}^{n} \left( b_j^i \alpha_j^i(t) \right)^2}, \tag{48}
\]

where \(\alpha(t)\) and \(R_i(t)\) are determined respectively by (44) and (47).

Consider the leader’s problem (33) – (34), (36) – (37), and will refer to it as LC (leader continuous).

The Hamilton-Jacobi-Bellman equation has the form:

\[
\rho V_0 - \frac{\partial V_0}{\partial t} = \max_{r_i, 1 \leq i \leq m} \left\{ \sum_{j=1}^{n} x_j(t) - \sum_{i=1}^{m} r_i(t) + \sum_{j=1}^{n} \frac{\partial V_0}{\partial x_j} \left[ \sum_{k=1}^{m} b_j^k \sqrt{u_j^k(x_j)} + \sum_{l=1}^{n} a_{ij} x_l \right] \right\} \tag{49}
\]

s.t.

\[
\sum_{i=1}^{m} r_i(t) \leq R, \quad r_i(t) \geq 0, \quad i = 1, 2, \ldots, m,
\]

where \(u_j^i(t)\) is determined by the expression (48). The equation (49) takes the form:

\[
\rho V_0 - \frac{\partial V_0}{\partial t} = \max_{r_i, 1 \leq i \leq m} \left\{ \sum_{j=1}^{n} x_j(t) - \sum_{i=1}^{m} r_i(t) + \sum_{j=1}^{n} \frac{\partial V_0}{\partial x_j} \left[ \sum_{l=1}^{n} a_{ij} x_l + \sum_{i=1}^{m} b_j^i \sqrt{u_j^i(t)} \right] \right\}, \quad \sum_{j=1}^{n} b_j^i \sqrt{\sum_{j=1}^{n} \left( b_j^i \alpha_j^i(t) \right)^2} \min \left\{ \sqrt{\frac{1}{4} \sum_{j=1}^{n} \left( b_j^i \alpha_j^i(t) \right)^2}, \sqrt{r_i(t)} \right\}. \tag{50}
\]
Again, we choose the linear Bellman function:

\[ V_0(x, t) = \sum_{j=1}^{n} \alpha_j^0(t)x_j + \beta^0(t), \]

Similarly, we receive the differential equation for \( \alpha_j^0(t) \):

\[ \alpha_j^0(t) - \rho \alpha_j^0(t) + \sum_{l=1}^{n} \alpha_j^0(t) a_{jl} = -1, \]

which coincides with the differential equations (42) with the same boundary conditions \( \alpha_j^0(T) = 0 \), therefore \( \alpha_j^0(t) = \alpha_j^1(t) = \alpha_j^2(t) = \cdots = \alpha_j^m(t) = \alpha_j(t) \), \( j = 1, 2, \ldots, n \), \( t \in [0, T] \). Thus, we can rewrite (50) in the form:

\[
\begin{align*}
\max \left\{ \sum_{j=1}^{n} x_j(t) - \sum_{i=1}^{m} r_i(t) + \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j a_{jl} t^l : \right. \\
\left. \frac{\sum_{j=1}^{n} (b_j \alpha_j(t))^2}{1} \min \left\{ \left\lfloor \frac{1}{4} \sum_{j=1}^{n} (b_j \alpha_j(t))^2, \sqrt{r_i(t)} \right\rfloor \right\} \right\}. 
\end{align*}
\]

(51)

Notice that an unconditional maximization of the expression (51) gives the value of resource

\[ r_i(t) = \frac{1}{4} \sum_{j=1}^{n} (b_j \alpha_j(t))^2. \]

Maximizing the right hand side of the expression (51) by \( r_i(t), i = 1, 2, \ldots, m \), under the condition \( \sum_{i=1}^{m} r_i(t) \leq R \), we receive for any \( 1 \leq i_1, i_2 \leq m \):

\[ \frac{\sqrt{\sum_{j=1}^{n} (\alpha_j b_j^i)^2}}{\sqrt{\sum_{j=1}^{n} (\alpha_j b_j^{i_2})^2}} = \frac{r_{i_1}(t)}{r_{i_2}(t)}, \quad r_{i_2}(t) = \frac{\sum_{j=1}^{n} (\alpha_j b_j^{i_2})^2}{\sum_{j=1}^{n} (\alpha_j b_j^{i_1})^2} r_{i_1}(t). \]

Denote the sum \( \sum_{i=1}^{m} r_i(t) \) by \( r(t) \). Then

\[ r_{i_1}(t) \sum_{j=1}^{n} (\alpha_j b_j^i)^2 = r(t) \sum_{j=1}^{n} (\alpha_j b_j^i)^2 \quad r_i(t) = r(t) \frac{\sum_{j=1}^{n} (\alpha_j b_j^i)^2}{\sum_{k=1}^{n} \sum_{j=1}^{n} (\alpha_j b_j^k)^2}. \]

Thus, the leader’s optimal strategy is:

\[ r_i(t) = \frac{\sum_{j=1}^{n} (\alpha_j(t) b_j^i)^2}{\sum_{k=1}^{m} \sum_{j=1}^{n} (\alpha_j(t) b_j^k)^2} \min \left\{ \frac{1}{4} \sum_{k=1}^{m} \sum_{j=1}^{n} (b_j^k \alpha_j(t))^2, R \right\}, \]

(52)

\[ i = 1, 2, \ldots, m, \quad t \in [0, T], \]

where \( \alpha(t) \) is determined by the expression (44).

Given the form of the strategy (52), we can simplify (51), because

\[ \sum_{i=1}^{m} r_i(t) = \min \left\{ \frac{1}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} (b_j^i \alpha_j(t))^2, R \right\}, \]
and the last term in \((51)\) is equal to \(\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 \) if \(\sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 \leq R\), and \(\sqrt{R} \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 \) if \(\sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 > R\). Equating the free terms in the left and right hand sides of \((51)\), we receive the differential equation for \(\beta^0(t)\):

\[
\beta^0(t) - \rho \beta^0(t) = f(t),
\]

where

\[
f(t) = \begin{cases} \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2, & \text{if } \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 \leq R, \\ \sqrt{R} \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 - R, & \text{if } \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2 > R. \end{cases}
\]

The solution of the equation \((53)\) by the method of variation of parameters with the boundary condition \(\beta^0(T) = 0\) gives

\[
\beta^0(t) = \int_t^T e^{-\rho(\tau-t)} f(\tau) d\tau.
\]

When \(t = 0\) we have

\[
\beta^0(0) = \int_0^T e^{-\rho \tau} f(\tau) d\tau.
\]

Thus, the leader’s maximal guaranteed payoff is

\[
\max_{r_i, 1 \leq i \leq m} J_0 = V_0(x(0), 0) = \sum_{j=1}^{n} \alpha_j(0)x_j(0) + \int_0^T e^{-\rho \tau} f(\tau) d\tau,
\]

where \(\alpha_j(0), j = 1, 2, \ldots, n,\) are determined by \((45)\), and \(f(\tau)\) by \((54)\).

Let us calculate the system compatibility index for the basic continuous model:

\[
SCI = \frac{\max_{\{r_i\}_{i=1}^{m}} \min_{\{u_j\}_{j=1}^{n}} \mathcal{NE}(\{r_i\}_{i=1}^{m}, \{u_j\}_{j=1}^{n}) J_0 (\{r_i\}_{i=1}^{m}, \{u_j\}_{j=1}^{n})}{\max_{\{r_i\}_{i=1}^{m}} \max_{\{u_j\}_{j=1}^{n}} J_0 (\{r_i\}_{i=1}^{m}, \{u_j\}_{j=1}^{n})}.
\]

Similarly, we receive for each control agent the problem:

\[
\tilde{J}_i = \int_0^T e^{-\rho \tau} \sum_{j=1}^{n} x_j(t) dt \rightarrow \max
\]

s.t. \((36) - (37)\). The Hamilton-Jacobi-Bellman equation has the form:

\[
\rho V_i - \frac{\partial V_i}{\partial t} = \max_{u_j^i, 1 \leq j \leq n} \left\{ \sum_{j=1}^{n} x_j(t) + \sum_{j=1}^{n} \frac{\partial V_i}{\partial x_j} \left[ \sum_{k=1}^{m} b_{jk} \sqrt{u_j^k(x_j)} + \sum_{l=1}^{n} a_{jlx_l} \right] \right\}
\]

s.t.

\[
\sum_{j=1}^{n} u_j^i(x_j(t)) \leq r_i(t).
\]

Repeating exactly the same actions as in the solution of the problem \(FC_i\), we receive the optimal strategies of the control agents that differ from the respective functions \((48)\) in the problem \(FC_i\) only in that instead the value \(R_i(t)\) that is determined by \((47)\), now we have the value \(r_i(t)\):

\[
u_j^i(t) = \frac{r_i(t) (b_{ij} \alpha_j(t))^2}{\sum_{j=1}^{n} (b_{ij} \alpha_j(t))^2},
\]

where the functions \(\alpha_j(t)\) are the same as in the problem \(FC_i\).
Solving the leader’s problem similarly to the solution of the problem LC and considering that the firms’ strategies are determined by (58), we receive the same optimal controls and the same maximal guaranteed payoff of the leader as in the problem LC.

Thus, the denominator of (56) is the same as the right hand side of the expression (55), and SCI = 1. As in the discrete formulation, it seems that the leader is interested to compel the followers to invest in marketing actions all their available resources. But it happens that for the leader itself it is not advantageous to allocate to the firms more resources than they need from the point of view of their individual interests.

3.2. Differential game with a target value of the summary opinion. Now the basic model takes the form (the problem HC):

\[ J_0 = \int_0^T e^{-\rho t} \left[ \sum_{j=1}^n z_j x_j(t) - x^* \right] + \sum_{i=1}^m r_i(t) \, dt \to \min, \]  

\[ (59) \]

other conditions are the same as in the Subsection 3.1, i.e. (34) – (38). Under the transition from the discrete description to the continuous one the matrix A is substituted by the matrix \( \tilde{A} = A - I \). Usually, it is assumed that the matrix of influences A is stochastic from the left, i.e. \( \sum_{i=1}^n a_{ij} = 1 \), \( j = 1, 2, \ldots, n \), and \( z_j > 0 \), \( j = 1, 2, \ldots, n \) are the components of a positive eigenvector corresponding to the Frobenius eigenvalue 1 of the matrix A. Other assumptions are the same as in the Subsection 3.1.

For the solution of the problem HC let us first assume that the matrix A is stochastic (from the right) and, respectively, \( z_j = 1 \), \( j = 1, 2, \ldots, n \). Then solving the problem of the \( i \)-th firm we can substitute \( n \) state variables \( x_j \) by their sum and denote by \( x \) the only state variable:

\[ \sum_{j=1}^n x_j = x, \quad \sum_{j=1}^n x_{j0} = x_0, \]

then

\[ \sum_{j=1}^n \sum_{l=1}^n \tilde{a}_{lj} x_j = \sum_{j=1}^n \sum_{l=1}^n a_{lj} x_j - \sum_{j=1}^n x_j = \sum_{j=1}^n x_j - \sum_{j=1}^n x_j = x - x = 0, \]

and the conditions (35) and (37) respectively take the form

\[ J_i = \int_0^T e^{-\rho t} \left[ x - \sum_{j=1}^n s^j u^j_i(t) \right] \, dt \to \max, \]

\[ (60) \]

and

\[ \dot{x} = \sum_{j=1}^n \sum_{i=1}^m b^j_k \sqrt{u^j_i(t)}, \quad x(0) = x_0. \]

\[ (61) \]

The Hamilton-Jacobi-Bellman equation has the form

\[ \rho V_i - \frac{\partial V_i}{\partial t} = \max_{u^j_i, 1 \leq j \leq n} \left\{ u^j_i(x(t)) - \sum_{j=1}^n s^j_i u^j_i(x_j(t)) + \frac{\partial V_i}{\partial x} \sum_{j=1}^n \sum_{k=1}^m b^j_k \sqrt{u^j_k(x_j)} \right\} \]

\[ (62) \]
s.t. \[ \sum_{j=1}^{n} u_{ij}^j(x_j(t)) \leq r_i(t). \]

Maximizing by \( u_{ij} \), \( j = 1, 2, \ldots, n \), \( b^j_i \neq 0 \), we receive for any \( 1 \leq j_1, j_2 \leq n \):

\[ \frac{\partial V_i}{\partial x_j} b_{j_1}^i \frac{\partial V_i}{\partial b_{j_1}^i} = \left( u_{j_2}^i \right)^{\frac{1}{2}}. \]

Denote the sum \[ \frac{u_{j_1}^i}{(b_{j_1}^i)^2} \sum_{j=1}^{n} (b_j^i)^2 \]

by \( R_i(t) \). Then

\[ u_{ij}^i = \frac{R_i(t)(b_j^i)^2}{\sum_{j=1}^{n} (b_j^i)^2}. \] (63)

As we choose linear Bellman functions,

\[ V_i(x, t) = \alpha^i(t)x + \beta^i(t), \]

we can write the equation (62) with consideration of (63) in the form

\[ \rho \alpha^i(t)x + \rho \beta^i(t) - \alpha^i(t)x - \beta^i(t) = x - R_i(t) + \alpha^i(t) \sum_{k=1}^{m} \left( R_k(t) \sum_{j=1}^{n} (b_j^i)^2 \right). \] (64)

Equation of the coefficients of \( x \) in the left and right hand sides of (64) implies a differential equation for \( \alpha^i(t) \). Its solution gives

\[ \alpha^i(t) = \frac{1}{\rho} \left( 1 - e^{\rho(T-t)} \right). \]

Function \( \alpha^i(t) \) is the same for all firms, therefore we omit the index \( i \) in functions \( \alpha^i(t) \). Choosing the maximal value of the right hand side of the expression (64) in dependence of \( R_i(t) \), we receive

\[ - R_i(t) + \alpha^i(t) \sqrt{R_i(t)} \sum_{j=1}^{n} (b_j^i)^2 \rightarrow \text{max}, \]

therefore

\[ (R_i(t))_{max} = \frac{1}{4} \left( \alpha(t) \right)^2 \sum_{j=1}^{n} (b_j^i)^2. \]

The value is again equal to

\[ (R_i(t))_{max} = \min \left\{ \frac{1}{4} \left( \alpha(t) \right)^2 \sum_{j=1}^{n} (b_j^i)^2, r_i(t) \right\}. \] (65)

Analyze the leader’s strategy. The leader does not decrease the value \( \sum_{j=1}^{n} x_j \); it only increases the value when allocates resources to the firms. Then the leader’s optimal strategy is evident as earlier.
If \( \sum_{j=1}^{n} x_{j0} \geq x^* \) then \( r_i(t) = 0, \quad i = 1, 2, \ldots, m, \quad t \in [0, T] \). If \( \sum_{j=1}^{n} x_{j0} < x^* \) then in fact the leader solves a problem that is equivalent to the following one:

\[
\hat{J}_0 = \int_0^T e^{-\rho t} \left( \sum_{j=1}^{n} x_j(t) - x^* - \sum_{i=1}^{m} r_i(t) \right) dt \rightarrow \max,
\]

or, that is the same,

\[
\hat{J}_0 = \int_0^T e^{-\rho t} \left( \sum_{j=1}^{n} x_j(t) - \sum_{i=1}^{m} r_i(t) \right) dt \rightarrow \max,
\]

s.t. (34), (36) – (37) and \( \sum_{j=1}^{n} x_j(t) \leq x^*, \quad t \in [0, T] \).

It is clear that for maximization of its payoff function the leader must increase the value \( \sum_{j=1}^{n} x_j(t) \) as well as in the basic model but only up to the value \( x^* \), after achievement of which it should cease to allocate resources to the firms. Then the value \( \sum_{j=1}^{n} x_j(t) \) will remain equal to \( x^* \) until the end of the game. For implementation of this strategy in the initial moment of time \( t = 0 \) the leader must determine the moment \( h \) when the value \( \sum_{j=1}^{n} x_j(t) \) becomes equal to \( x^* \) given the respective strategy.

As according to (65) the leader will never allocate to any firm \( i \) in any moment of time \( t \) more resources than

\[
\frac{1}{4} (\alpha(t))^2 \sum_{j=1}^{n} (b_{ij})^2,
\]

we can rewrite the expression (63) as

\[
u^i_j = \frac{r_i(t) (b^j_i)^2}{\sum_{j=1}^{n} (b^j_i)^2}. \quad (66)
\]

Substituting to the equation (61) the expression (66) instead of the sum of state variables \( x \), we receive

\[
\dot{x} = \sum_{i=1}^{m} \sqrt{r_i(t) \sum_{j=1}^{n} (b^j_i)^2}, \quad x(0) = x_0.
\]

(67)

An integration of (67) gives

\[
x(t) = \int_0^t \sum_{i=1}^{m} \sqrt{r_i(\tau) \sum_{j=1}^{n} (b^j_i)^2} d\tau + x_0.
\]

(68)

Thus, \( h = x^{-1}(x^*) \) is the moment of time when the right hand side of (68) becomes equal to \( x^* \). Given the value of \( h \) and using an assumption about stochasticity of the matrix of influences, we can reformulate the leader’s problem as follows:

\[
\hat{J}_0 = \int_0^h e^{-\rho t} \left( x(t) - \sum_{i=1}^{m} r_i(t) \right) dt \rightarrow \max,
\]

s.t. \( r_i(t) \geq 0, \quad \sum_{i=1}^{m} r_i(t) \leq R, \quad t \in [0, h], \quad i = 1, 2, \ldots, m \) and (67).
The Hamilton-Jacobi-Bellman equation takes the form:

$$\frac{\rho V_0}{\partial t} - \frac{\partial V_0}{\partial x} = \max_{r_i, i \leq m} \left\{ x(t) - \sum_{i=1}^{m} r_i(t) + \frac{\partial V_0}{\partial x} \sum_{i=1}^{m} r_i(t) \sum_{j=1}^{n} (b_j^i)^2 \right\}$$  \hspace{1cm} (70)

s.t. $\sum_{i=1}^{m} r_i(t) \leq R$. Again, we choose the linear Bellman function:

$$V_0(x, t) = \alpha^0(t)x + \beta^0(t).$$  \hspace{1cm} (71)

Substitution of (71) into (70) gives

$$\rho \alpha^0(t)x + \rho \beta^0(t) - \alpha^0(t)x - \beta^0(t) = \max_{r_i, 1 \leq i \leq m} \left\{ x(t) - \sum_{i=1}^{m} r_i(t) + \alpha^0(t) \sum_{i=1}^{m} r_i(t) \sum_{j=1}^{n} (b_j^i)^2 \right\}. \hspace{1cm} (72)$$

Equating in the left and right hand sides of (72) the coefficients of $x$, we receive a differential equation for $\alpha^0(t)$. Its solution gives

$$\alpha^0(t) = \frac{1}{\rho} \left( 1 - e^{\rho(t-h)} \right), \quad t \in [0, h].$$  \hspace{1cm} (73)

Similarly, from (72) we receive a differential equation for $\beta^0(t)$:

$$\beta^0(t) - \rho \beta^0(t) = \sum_{i=1}^{m} r_i(t) - \alpha^0(t) \sum_{i=1}^{m} r_i(t) \sum_{j=1}^{n} (b_j^i)^2. \hspace{1cm} (74)$$

Its solution by the method of variation of parameters gives

$$\beta^0(t) = e^{\rho t} \int_t^h \left[ \alpha^0(\tau) \left( \sum_{i=1}^{m} \sqrt{r_i(\tau) \sum_{j=1}^{n} (b_j^i)^2} - \sum_{i=1}^{m} r_i(\tau) \right) - e^{-\rho \tau} dr \right].$$

In particular, when $t = 0$ we have

$$\beta^0(0) = \int_0^h \left[ \alpha^0(\tau) \left( \sum_{i=1}^{m} \sqrt{r_i(\tau) \sum_{j=1}^{n} (b_j^i)^2} - \sum_{i=1}^{m} r_i(\tau) \right) - e^{-\rho \tau} dr \right]. \hspace{1cm} (75)$$

Maximizing the right hand side of the expression (72) by $r_i(t)$, $i = 1, 2, \ldots, m$, with constraint $\sum_{i=1}^{m} r_i(t) \leq R$, we receive for any $1 \leq i_1, i_2 \leq m$:

$$\sqrt{\sum_{j=1}^{n} (b_{j_1}^{i_1})^2} \leq \sqrt{\sum_{j=1}^{n} (b_{j_2}^{i_2})^2} = \frac{\sqrt{r_{i_1}(t)}}{\sqrt{r_{i_2}(t)}}, \quad r_{i_2}(t) = \sum_{j=1}^{n} (b_{j_2}^{i_2})^2 \sum_{j=1}^{n} (b_{j_1}^{i_1})^2 r_{i_1}(t).$$

Denote the sum $\sum_{i=1}^{m} r_i(t)$ by $r(t)$. Then

$$r_{i_1}(t) \sum_{j=1}^{n} (b_{j_1}^{i_1})^2 = r(t) \sum_{j=1}^{n} (b_{j_1}^{i_1})^2, \quad r_{i_2}(t) = r(t) \sum_{j=1}^{n} (b_{j_2}^{i_2})^2 \sum_{k=1}^{m} \sum_{j=1}^{n} (b_{j_k}^{i_2})^2. \hspace{1cm} (76)$$

Substitution of (75) into the right hand side of (72) gives:

$$x(t) - r(t) + \alpha^0(t) \sqrt{r(t) \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{j}^{i})^2}. \hspace{1cm} (77)$$
An unconditional maximization of this expression by \( r(t) \) gives
\[
(r(t))_{\text{max}} = \frac{1}{4} \left( \alpha^0(t) \right)^2 \sum_{i=1}^{m} \sum_{j=1}^{n} (b^j_i)^2.
\]
Thus, the leader’s optimal strategy is:
\[
r_i(t) = \frac{\sum_{j=1}^{n} (b^j_i)^2}{\sum_{k=1}^{m} \sum_{j=1}^{n} (b^j_k)^2} \min \left\{ \frac{1}{4} \left( \alpha^0(t) \right)^2 \sum_{k=1}^{m} \sum_{j=1}^{n} (b^k_j)^2, R \right\}, \quad t \in [0, h], \quad (76)
\]
where
\[
\alpha^0(t) = \frac{1}{\rho} \left( 1 - e^{\rho(t-h)} \right), \quad i = 1, 2, \ldots, m. \quad (77)
\]
Using (73) and (74), we can write:
\[
\max_{r_i, 1 \leq i \leq m} \hat{J}_0 = V_0(x(0), 0) = \alpha^0(0)x(0) + \beta^0(0) = \frac{1}{\rho} (1 - e^{-\rho h}) x_0 + \int_0^h \left[ \alpha^0(\tau) \left( \sum_{i=1}^{m} \sqrt{r_i(\tau) \sum_{j=1}^{n} (b^j_i)^2} - \sum_{i=1}^{m} r_i(\tau) \right) e^{-\rho \tau} d\tau, \quad (78)
\]
when the value \( r_i(\tau) \) is determined by the expression (76), and \( \alpha^0(\tau) \) by the expression (77). Then according to (78) the maximal guaranteed payoff of the leader is equal to
\[
x^* \int_0^h e^{-\rho t} dt - \max_{r_i, 1 \leq i \leq m} J_0 = \frac{1}{\rho} (1 - e^{-\rho h}) x^* - \frac{1}{\rho} (1 - e^{-\rho h}) x_0 - \int_0^h \left[ \alpha^0(\tau) \left( \sum_{i=1}^{m} \sqrt{r_i(\tau) \sum_{j=1}^{n} (b^j_i)^2} - \sum_{i=1}^{m} r_i(\tau) \right) e^{-\rho \tau} d\tau = \frac{1}{\rho} (1 - e^{-\rho h}) (x^* - x_0) - \int_0^h \left[ \alpha^0(\tau) \left( \sum_{i=1}^{m} \sqrt{r_i(\tau) \sum_{j=1}^{n} (b^j_i)^2} - \sum_{i=1}^{m} r_i(\tau) \right) e^{-\rho \tau} d\tau, \quad (79)
\]
where the moment of time \( h \) is determined by the expression (68).

At last, calculate the system compatibility index in this case:
\[
SCI = \frac{\max_{\{r_i\}_{i=1}^{m} \min_{\{u_j\}_{j=1}^{n}}} \min_{\{u_j\}_{j=1}^{n} \in NE(\{r_i\}_{i=1}^{m})} \hat{J}_0 (\{r_i\}_{i=1}^{m}, \{u_j\}_{j=1}^{n})}{\max_{\{r_i\}_{i=1}^{m}} \max_{\{u_j\}_{j=1}^{n}} \hat{J}_0 (\{r_i\}_{i=1}^{m}, \{u_j\}_{j=1}^{n})}. \quad (80)
\]
Again, each control agent solves the problem
\[
\hat{J}_i = \int_0^T e^{-\rho t} \left( \sum_{j=1}^{n} x_j(t) - x^* - \sum_{i=1}^{m} r_i(t) \right) dt \to \max,
\]
s.t. (36) – (37) and \( \sum_{j=1}^{n} x_j(t) \leq x^*, \quad t \in [0, T] \).
If \( \sum_{j=1}^{n} x_j(t) \geq x^* \) then the optimal strategy of each firm is evident: it should not invest to the marketing actions at all. In fact, it cannot do it because the leader will not allocate any resource to the firms in this case, and \( u^j_i(t) = 0 \). Now suppose that \( \sum_{j=1}^{n} x_j(t) < x^* \). Unlike the leader, any control agent cannot evaluate a priori in which moment of time \( h \) the sum of the values of state variables becomes equal.
to $x^*$ but it is not required because in fact the optimal strategies of all firms do not depend on the length of the planning period.

Consider the game in the time interval $[0, h]$. As in the problem $HD$ of the Subsection 3.1, we receive for the optimal strategies of the firms the expressions similar to (63), (65):

$$u_i^j(t) = \frac{R_i(t) (b_i^j)^2}{\sum_{j=1}^{n} (b_i^j)^2}, R_i(t) = \min \left\{ \frac{1}{4} \alpha(t) \sum_{j=1}^{n} (b_i^j)^2, r_i(t) \right\}, \alpha(t) = \frac{1}{\rho} \left( 1 - e^{\rho(t-h)} \right).$$

As the leader will never allocate to any firm more resources than $\frac{1}{4} \alpha(t) \sum_{j=1}^{n} (b_i^j)^2$, we have

$$u_i^j(t) = \frac{(b_i^j)^2 r_i(t)}{\sum_{j=1}^{n} (b_i^j)^2}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad t \in [0, h].$$

Then the leader’s problem does not differ from the previous one, and the value of denominator in (80) is expressed by the formula (79). Therefore, the interests of all players are also ideally coordinated, and $SCI = 1$.

**Remark 2.** In this problem we have the same non-trivial situation which we have already discussed in details in Remark 1.

**Remark 3.** We solved the problem $HD$ subject to a strict constraint for the matrix $A$, namely we assumed that $A$ is stochastic (from the right). The standard assumption is that $A$ is stochastic from the left and, respectively, $A^\tau$ is stochastic (from the right). In the problem $HD$ the standard assumption is reduced to the considered case similarly to the problem $HC$ in the Subsection 2.2.

4. **Conclusion.** Social networks have been a popular object for the mathematical modeling and simulation since the second half of the past century. The modeling approaches are based on the various mathematical techniques that include discrete and continuous, deterministic and stochastic models of different types. These models are the graph theoretic models, the optimization, optimal control, and game theoretic models as well as the gene-environment models, eco-finance models, rumor propagation models, Markov switching models.

This paper presents formulations and solutions of the game theoretic Stackelberg models of opinion control in social groups given the network structure of interactions among agents. The problems are interpreted in marketing terms. It is assumed that in the stage of analysis the target audience is already segmented into strong subgroups and satellites. Then the members of strong subgroups become the only object of control that gives an essential economy.

We found analytical solutions for the difference and differential Stackelberg game theoretic models of opinion control in marketing networks with budget constraints. In both cases we considered both a basic model and a model with a target value of the summary opinion. We calculated the values of the system compatibility index: it happened that $SCI = 1$ that witnesses the ideal coordination of interests of all agents in the modeled systems. Thus, for the leader itself it is not advantageous to allocate to the control agents more resources than they need in their egoistic behavior.
We assume to continue the investigation of the dynamic game theoretic opinion control models for the cases of independent and cooperative behavior and hierarchically organized players for different non-linear control actions. The study of uncertainty in different aspects in such models is also an issue of big interest.

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