FRACTIONAL SERIES OPERATORS ON DISCRETE HARDY SPACES

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Abstract. For $0 \leq \gamma < 1$ and a sequence $b = \{b(i)\}_{i \in \mathbb{Z}}$ we consider the fractional operator $T_{\alpha,\beta}$ defined formally by

$$(T_{\alpha,\beta} b)(j) = \sum_{i \neq \pm j} \frac{b(i)}{|i - j|^{\alpha}|i + j|^{\beta}} \quad (j \in \mathbb{Z}),$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1 - \gamma$. The main aim of this note is to prove that the operator $T_{\alpha,\beta}$ is bounded from $H^p(\mathbb{Z})$ into $\ell^q(\mathbb{Z})$ for $0 < p < \frac{1}{\gamma}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$. For $\alpha = \beta = \frac{1-\gamma}{2}$ we show that there exists $\varepsilon \in (0, \frac{1}{3})$ such that for every $0 \leq \gamma < \varepsilon$ the operator $T_{\frac{1-\gamma}{2}, \frac{1-\gamma}{2}}$ is not bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$ for $0 < p \leq \frac{1}{1+\gamma}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$.

1. Introduction

In this note, sequences considered are complex-valued unless otherwise explicitly stated. For a sequence $b = \{b(i)\}_{i \in \mathbb{Z}}$, let

$$\|b\|_{\ell^p(\mathbb{Z})} = \begin{cases} \left( \sum_{i=-\infty}^{+\infty} |b(i)|^p \right)^{1/p}, & 0 < p < \infty, \\ \sup_{i \in \mathbb{Z}} |b(i)|, & p = \infty. \end{cases}$$

A sequence $b$ is said to belong to $\ell^p(\mathbb{Z})$, $0 < p \leq \infty$, if $\|b\|_{\ell^p(\mathbb{Z})} < +\infty$.

At a Conference held in the summer of 1907, D. Hilbert announced the inequality

$$(1) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{b(i)b(j)}{i+j} \leq 2\pi \sum_{i=1}^{\infty} |b(i)|^2$$

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for \( b(i) \geq 0 \) and \( \sum_{i=1}^{\infty} |b(i)|^2 < \infty \). Hilbert’s proof was outlined by H. Weyl in his Inaugural-Dissertation (see [22, pp. 83]). Others proofs of (1), essentially different from each other, have been published in [21], [18], and [5] (see also [8, pp. 235-236]). The inequality in (1) may be regarded as the starting point of all researches in the discrete setting.

Given a sequence \( b = \{b(i)\}_{i \in \mathbb{Z}} \) its Hilbert sequence \( \tilde{b} \) is defined by

\[
\tilde{b}(j) = \frac{1}{\pi} \sum_{i \in \mathbb{Z} \setminus \{j\}} \frac{b(i)}{i-j}.
\]

This operator is known as the discrete Hilbert transform; such an operator was introduced by D. Hilbert in 1909. There are other operators with the same name. For instance, E. C. Titchmarsh [20] studied the behavior of the sequence

\[
c(j) = \frac{1}{\pi} \sum_{i=-\infty}^{\infty} \frac{b(i)}{j+i+\frac{1}{2}}, \quad (j \in \mathbb{Z}).
\]

More precisely, he proved that if \( b = \{b(i)\}_{i \in \mathbb{Z}} \in \ell^p(\mathbb{Z}), 1 < p < \infty \), then \( c = \{c(j)\}_{j \in \mathbb{Z}} \), given by (2), belongs to \( \ell^p(\mathbb{Z}) \) with \( \|c\|_{\ell^p(\mathbb{Z})} \leq N_p \|b\|_{\ell^p(\mathbb{Z})} \), where \( N_p \) is a number depending only on \( p \). This result allowed him to obtain, by passing to the limit, that the Hilbert transform \( \mathcal{H} \) defined by

\[
\mathcal{H}f(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(t)}{t-x} \, dt \quad (x \in \mathbb{R}),
\]

is bounded on \( L^p(\mathbb{R}), 1 < p < \infty \) (see also [14]). The sequence in (2) is also known as the discrete Hilbert transform of the sequence \( b = \{b(i)\}_{i \in \mathbb{Z}} \). For the sequel, we will consider the discrete Hilbert transform \( H \) of a sequence \( b = \{b(i)\}_{i \in \mathbb{Z}} \) given by

\[
(Hb)(j) = \frac{1}{\pi} \sum_{i=-\infty}^{\infty} \frac{b(i)}{j+i+\frac{1}{2}} \quad (j \in \mathbb{Z}).
\]

E. C. Titchmarsh also proved that \( H(Hb) = b \) if \( b \in \ell^p(\mathbb{Z}), 1 < p < \infty \).

In [6] (cf. also [8, pp. 288]), G. H. Hardy, J. E. Littlewood and G. Pólya proved the inequality

\[
\left| \sum_{i \neq j} \sum_{i=j} \lambda b(i)c(j) \right| \leq C \|b\|_{\ell^p(\mathbb{Z})} \|c\|_{\ell^q(\mathbb{Z})}
\]

for

\[
p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} > 1, \quad \lambda = 2 - \frac{1}{p} - \frac{1}{q} \quad (\text{so that } 0 < \lambda < 1).
\]
With this result G. H. Hardy and J. E. Littlewood in [7] obtained the corresponding fractional integral theorem.

Given $0 < \gamma < 1$ and a sequence $b = \{b(i)\}_{i \in \mathbb{Z}}$, we define the discrete Riesz potential $I_\gamma$ by

$$ (I_\gamma b)(j) = \sum_{i \neq j} \frac{b(i)}{|i - j|^{1 - \gamma}} \quad (j \in \mathbb{Z}). $$

Taking $\lambda = 1 - \gamma$ in (3), it follows that the operator $I_\gamma$ is bounded from $\ell^p(\mathbb{Z})$ into $\ell^q(\mathbb{Z})$ for $1 < p < \gamma^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$.

The discrete Hardy space $H^p(\mathbb{Z})$, $0 < p < \infty$, consists of all sequences $b = \{b(i)\}_{i \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$ which satisfy $Hb \in \ell^p(\mathbb{Z})$. The “norm” of $b \in H^p(\mathbb{Z})$, $0 < p < \infty$, is defined as

$$ (4) \quad \|b\|_{H^p(\mathbb{Z})} = \|b\|_{\ell^p(\mathbb{Z})} + \|Hb\|_{\ell^p(\mathbb{Z})}. $$

From (4) and the boundedness on $\ell^p(\mathbb{Z})$, $1 < p < \infty$, of the discrete Hilbert transform $H$ implies that $\|b\|_{\ell^p(\mathbb{Z})} \leq \|b\|_{H^p(\mathbb{Z})} \leq C_p \|b\|_{\ell^p(\mathbb{Z})}$, with $C_p$ independent of $b$, so $H^p(\mathbb{Z}) = \ell^p(\mathbb{Z})$ when $1 < p < \infty$. For the range $0 < p \leq 1$, C. Eoff in [3] proved that $H^p(\mathbb{Z})$ is isomorphic to the Paley–Wiener space of entire functions $f$ of exponential type $\pi$ for which $\int_{\mathbb{R}} |f(x)|^p dx < +\infty$.

By definition, it is clear that for $0 < p \leq 1$ the operator $H$ is bounded from $H^p(\mathbb{Z})$ into $\ell^p(\mathbb{Z})$, with $\|Hb\|_{\ell^p(\mathbb{Z})} \leq \|b\|_{H^p(\mathbb{Z})}$ for all $b \in H^p(\mathbb{Z})$. Since $H^p(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$, for $0 < p \leq 1$, we have that $H(Hb) = b$ for every $b \in H^p(\mathbb{Z})$. Thus, $H$ is a bounded operator on $H^p(\mathbb{Z})$, $0 < p \leq 1$, with $\|Hb\|_{H^p(\mathbb{Z})} = \|b\|_{H^p(\mathbb{Z})}$ for all $b \in H^p(\mathbb{Z})$.

In [1], S. Boza and M. Carro proved equivalent definitions of the norms in $H^p(\mathbb{Z})$, $0 < p \leq 1$. One of their main goals is the discrete atomic decomposition of elements in $H^p(\mathbb{Z})$. By means of the atomic decomposition one can prove that the discrete Riesz potential $I_\gamma$ is bounded from $H^p(\mathbb{Z})$ into $\ell^q(\mathbb{Z})$ for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \gamma$ (see Remark 11 below).

Later, Y. Kanjin and M. Satake in [10] obtained the molecular decomposition for members in $H^p(\mathbb{Z})$, $0 < p \leq 1$ (see also [11]). As an application of the molecular decomposition they proved that the operator $I_\gamma$ is bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$ for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \gamma$.

Recently, Kwok-Pun Ho in [9] generalized the famous discrete Hardy inequality to $0 < p \leq 1$, by using the atomic decomposition characterization of discrete Hardy spaces. For more results about discrete Hardy spaces see [12] and references therein.
Let $0 \leq \gamma < 1$ and let $b = \{b(i)\}_{i \in \mathbb{Z}}$ be a sequence. We define the fractional series operator $T_{\alpha, \beta}$ by

$$
(T_{\alpha, \beta} b)(j) = \sum_{i \neq \pm j} \frac{b(i)}{|i - j|^\alpha |i + j|^\beta} \quad (j \in \mathbb{Z}),
$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1 - \gamma$. This operator is a discrete version of the following fractional type integral operator defined on $\mathbb{R}^n$

$$
T_\alpha f(x) = \int_{\mathbb{R}^n} |x - A_1 y|^{-\alpha_1} \cdots |x - A_m y|^{-\alpha_m} f(y) \, dy,
$$

where $0 \leq \alpha < n$, $m \in \mathbb{N} \cap (1 - \frac{n}{\gamma}, +\infty)$, the $\alpha_j$’s are positive constants such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$ and the $A_j$’s are certain $n \times n$ invertible matrices. The behavior of this kind of operators on classical and variable Hardy spaces was studied by the author and M. Urciuolo in [15], [16] and [17].

The germ of the operator in (6) appears in [13], there F. Ricci and P. Sjögren obtained the boundedness on $L^p(\mathbb{H}_1)$, $1 < p \leq +\infty$, for a family of maximal operators on the three dimensional Heisenberg group $\mathbb{H}_1$. To get this result, they studied the $L^2(\mathbb{R})$ boundedness of the operator

$$
T f(x) = \int_{\mathbb{R}} |x - y|^{|\alpha| - 1} |(\beta - 1)x - \beta y|^{-\alpha} f(y) \, dy,
$$

for $\beta \neq 0, 1$ and $0 < \alpha < 1$.

The purpose of this note is to prove the $H^p(\mathbb{Z}) - \ell^q(\mathbb{Z})$ boundedness of the operator $T_{\alpha, \beta}$, given in (5), for $0 < p < \gamma^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$. We also prove that there exists $\varepsilon \in \left(0, \frac{1}{3}\right)$ such that, for every $0 \leq \gamma < \varepsilon$, the operator $T_{\frac{1}{2}, \frac{1}{2}}$ is not bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$ for $0 < p \leq \frac{1}{1 + \gamma}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$. This is an important difference between the discrete Hilbert transform $H$ and $T_{\frac{1}{2}, \frac{1}{2}}$, and the discrete Riesz potential $I_\gamma$ and $T_{\frac{1}{2}, \frac{1}{2}}$ for $0 < \gamma < \varepsilon$.

In Section 2, we introduce the discrete maximal and we recall the atomic decomposition of discrete Hardy spaces given in [1]. We also state two corollaries, which are a consequence of the atomic decomposition. These two corollaries are necessary to make our counter-example.

In Section 3, we obtain the $H^p(\mathbb{Z}) - \ell^q(\mathbb{Z})$ boundedness of the operator $T_{\alpha, \beta}$, for $0 \leq \gamma < 1$, $0 < p < \gamma^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$.

In Section 4, we give a counter-example which proves that $T_{\alpha, \beta}$ is not bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$.

Notation. Throughout this paper, $C$ will denote a positive real constant not necessarily the same at each occurrence. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. With $\# A$ we denote the cardinality of a set $A \subset \mathbb{Z}$. Given a real number $s \geq 0$, we write $[s]$ for the integer part of $s$. 

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Given a sequence \( b = \{b(i)\}_{i \in \mathbb{Z}} \) we define the centered maximal sequence \( Mb \) by
\[
(Mb)(j) = \sup_{N \in \mathbb{N}_0} \frac{1}{2N+1} \sum_{|i-j| \leq N} |b(i)|.
\]
It is clear that \( \|Mb\|_{\ell^\infty(\mathbb{Z})} \leq \|b\|_{\ell^\infty(\mathbb{Z})} \). Since \( \ell^p(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}) \) for every \( 0 < p < \infty \), it follows that if \( b \in \ell^p(\mathbb{Z}), 0 < p \leq \infty \), then \( (Mb)(j) < +\infty \) for all \( j \in \mathbb{Z} \).

The following result is a consequence of the harmonic analysis on spaces of homogeneous type applied to the space \( (\mathbb{Z}, \mu, |\cdot|) \) where \( \mu \) is the counting measure and \( |\cdot| \) is the distance in \( \mathbb{Z} \) (see [2] or [19]). We omit its proof.

**Theorem 1.** Let \( b = \{b(i)\}_{i \in \mathbb{Z}} \) be a sequence.
(a) If \( b \in \ell^1(\mathbb{Z}) \), then for every \( \alpha > 0 \)
\[
\#\{ j : (Mb)(j) > \alpha \} \leq \frac{C}{\alpha} \|b\|_{\ell^1(\mathbb{Z})},
\]
where \( C \) is a positive constant which does not depend on \( \alpha \) and \( b \).
(b) If \( b \in \ell^p(\mathbb{Z}), 1 < p \leq \infty \), then \( Mb \in \ell^p(\mathbb{Z}) \) and
\[
\|Mb\|_{\ell^p(\mathbb{Z})} \leq C_p \|b\|_{\ell^p(\mathbb{Z})},
\]
where \( C_p \) depends only on \( p \).

We observe that if \( 0 < p \leq \infty \) and \( 0 < q \leq 1 \), then the mapping \( b \to Mb \) is not bounded from \( \ell^p(\mathbb{Z}) \) into \( \ell^q(\mathbb{Z}) \). Indeed, taking \( b = \{b(i)\} \) such that \( b(0) = 1 \) and \( b(i) = 0 \) for all \( i \neq 0 \) we have \( (Mb)(j) = \frac{1}{2|j|+1} \) for all \( j \in \mathbb{Z} \).

So \( b \in \ell^p(\mathbb{Z}) \) but \( Mb \notin \ell^q(\mathbb{Z}) \) when \( 0 < p \leq \infty \) and \( 0 < q \leq 1 \). We know that if \( 0 < p < q \leq \infty \), then \( \ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z}) \) with \( \|b\|_q \leq \|b\|_p \) for all \( b \in \ell^p(\mathbb{Z}) \). From this and Theorem 1(b), we have that if \( 1 < q \leq \infty \) and \( 0 < p \leq q \), then \( \|Mb\|_{\ell^q} \leq C_q \|b\|_{\ell^p} \). We also have that for \( 0 < q < p \leq \infty \) the inclusion \( \ell^q(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \) is strict and since \( |b(j)| \leq (Mb)(j) \) for all \( j \), it follows that the maximal operator \( M \) is not bounded from \( \ell^p(\mathbb{Z}) \) into \( \ell^q(\mathbb{Z}) \) if \( 0 < q < p \leq \infty \). In the following proposition, we summarize all this.

**Proposition 2.** Let \( 0 < p, q \leq \infty \). Then the maximal operator \( M \) is bounded from \( \ell^p(\mathbb{Z}) \) into \( \ell^q(\mathbb{Z}) \) if and only if
\[
(1/p, 1/q) \in \{ (x, y) \in [0, 1] \times [0, 1] : y \leq x \} \cup [1, \infty) \times [0, 1).
\]

Discrete atoms. Let \( 0 < p \leq 1 \leq q \leq \infty \), \( p < q \) and \( d \) be a non negative integer. A discrete \( (p,q,d) \)-atom centered at \( n_0 \in \mathbb{Z} \) is a sequence \( a = \{a(i)\}_{i \in \mathbb{Z}} \) satisfying the conditions:

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(1) $\text{supp}(a) \subseteq \{n_0 - m, \ldots, n_0, \ldots, n_0 + m\}$, $m \geq 1$,  
(2) $\|a\|_{\ell^q} \leq (2m + 1)^{1/q - 1/p}$,  
(3) $\sum_{i=-\infty}^{+\infty} i^j a(i) = 0$ for $j = 0, 1, \ldots, d$.

Here, (1) means that the support of an atom is finite, (2) is the size condition of the atom, and (3) is called the cancellation moment condition. Clearly, a $(p, \infty, d)$-atom is a $(p, q, d)$-atom, if $0 < p < q < \infty$. If $a$ is a $(p, q, d)$-atom, then $\|a\|_{\ell^p} \leq 1$.

The atomic decomposition for members in $H^p(\mathbb{Z})$, $0 < p \leq 1$, developed in [1] is as follows:

**Theorem 3** [1, Theorem 3.13]. Let $0 < p \leq 1$, $d_p = \lfloor p^{-1} - 1 \rfloor$ and $b \in H^p(\mathbb{Z})$. Then there exist a sequence of $(p, \infty, d_p)$-atoms $\{a_k\}_{k=0}^{+\infty}$, a sequence of scalars $\{\lambda_k\}_{k=0}^{+\infty}$ and a positive constant $C$, which depends only on $p$, with $\sum_{k=0}^{+\infty} |\lambda_k|^p \leq C \|b\|_{H^p(\mathbb{Z})}^p$ such that $b = \sum_{k=0}^{+\infty} \lambda_k a_k$, where the series converges in $H^p(\mathbb{Z})$.

**Corollary 4.** If $\{b(i)\}_{i=-\infty}^{+\infty} \in H^p(\mathbb{Z})$, $0 < p \leq 1$, then $\sum_{i=-\infty}^{+\infty} b(i) = 0$.

**Proof.** By Theorem 3, given $b \in H^p(\mathbb{Z})$, $0 < p \leq 1$, we can write $b = \sum_{k=0}^{+\infty} \lambda_k a_k$ where the $a_k$’s are $(p, \infty, d_p)$-atoms, and the series converges in $H^p(\mathbb{Z})$. Since $H^p(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \subset \ell^1(\mathbb{Z})$ embed continuously, the series also converges in $\ell^1(\mathbb{Z})$. Then for each $N$ fixed, by the cancellation moment condition of the atoms $a_k$, we have

\[
(7) \quad \left| \sum_{i=-\infty}^{+\infty} b(i) \right| = \left| \sum_{i=-\infty}^{+\infty} \left( b(i) - \sum_{k=0}^{N} \lambda_k a_k(i) \right) \right| \leq \sum_{i=-\infty}^{+\infty} \left| b(i) - \sum_{k=0}^{N} \lambda_k a_k(i) \right|.
\]

Finally, letting $N \to \infty$ on the right-hand side of (7), we obtain $\sum_{i=-\infty}^{+\infty} b(i) = 0$.

Given an integer $L \geq 0$, we define the set of sequences $\mathcal{D}_L$ by

\[
\mathcal{D}_L = \left\{ c = \{c(i)\}_{i \in \mathbb{Z}} : \# \text{supp}(c) < +\infty \text{ and } \sum_{i=-\infty}^{+\infty} i^j c(i) = 0 \text{ for } j = 0, 1, \ldots, L \right\}.
\]

**Corollary 5.** If $0 < p \leq 1$ and $L$ is an arbitrary integer such that $L \geq \lfloor p^{-1} - 1 \rfloor$, then the set $\mathcal{D}_L$ is dense in $H^r(\mathbb{Z})$ for each $p \leq r \leq 1$.

**Proof.** By checking the proof of Lemma 3.12 of Boza and Carro in [1], we see that in the atomic decomposition of an arbitrary element in $H^r(\mathbb{Z})$ 
one can always choose atoms with additional vanishing moments. This is, if \( L \) is any fixed integer with \( L \geq d_r \) and \( b \in H^r(\mathbb{Z}) \), then there exists an atomic decomposition for \( b \) such that all moments up to order \( L \) of the atoms are zero. For \( 0 < p \leq r \leq 1 \) we have that \( d_p \geq d_r \). If \( L \) is any fixed integer such that \( L \geq d_p \), then \( \text{span}\{(r, \infty, L)\text{-atoms}\} \subset D_L \subset H^p(\mathbb{Z}) \subset H^r(\mathbb{Z}) \) for \( p \leq r \leq 1 \), so the corollary follows. □

**Proposition 6.** Let \( 0 \leq \gamma < 1 \) and \( \alpha, \beta > 0 \) such that \( \alpha + \beta = 1 - \gamma \), and let \( K : \mathbb{R}^2 \setminus \{y = \pm x\} \to \mathbb{R} \) be the function given by

\[
K(x, y) = |x - y|^{-\alpha}|x + y|^{-\beta}.
\]

Then

\[
\left| \frac{\partial N}{\partial x^N} K(x, y) \right| + \left| \frac{\partial N}{\partial y^N} K(x, y) \right| \leq C |x - y|^{-\alpha}|x + y|^{-\beta} (|x - y|^{-1} + |x + y|^{-1})^N,
\]

for every \( N \in \mathbb{N} \), where \( C \) is a positive constant independent of \( x, y \).

**Proof.** This result is a particular case of [16, Lemma 1]. □

### 3. Main results

Let \( 0 \leq \gamma < 1, 0 < p < \gamma^{-1} \) and \( \frac{1}{q} = \frac{1}{p} - \gamma \). In this section we study the \( H^p(\mathbb{Z}) - \ell^q(\mathbb{Z}) \) boundedness of the operator \( T_{\alpha, \beta} \) defined by

\[
(T_{\alpha, \beta} b)(j) = \sum_{i \neq \pm j} \frac{b(i)}{|i - j|^\alpha |i + j|^\beta} \quad (j \in \mathbb{Z}),
\]

where \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 - \gamma \).

**Theorem 7.** Let \( 0 \leq \gamma < 1 \). If \( T_{\alpha, \beta} \) is the operator given by (8), \( 1 < p < \gamma^{-1} \) and \( \frac{1}{q} = \frac{1}{p} - \gamma \), then

\[
\|T_{\alpha, \beta} b\|_{\ell^q(\mathbb{Z})} \leq C \|b\|_{\ell^p(\mathbb{Z})},
\]

where \( C \) depends only on \( \alpha, \beta, p \) and \( q \).

**Proof.** Given a sequence \( b = \{b(i)\}_{i \in \mathbb{Z}} \) we put \( |b| = \{|b(i)|\}_{i \in \mathbb{Z}} \). We study the cases \( 0 < \gamma < 1 \) and \( \gamma = 0 \) separately. For \( 0 < \gamma < 1 \) it is easy to check that

\[
|(T_{\alpha, \beta} b)(j)| \leq (I_\gamma|b|)(j) + (I_\gamma|b|)(-j), \quad \forall j \in \mathbb{Z}.
\]

So, the \( \ell^p(\mathbb{Z}) - \ell^q(\mathbb{Z}) \) boundedness of \( T_{\alpha, \beta} \) (\( 0 < \gamma < 1 \)) follows from the boundedness of the discrete Riesz potential \( I_\gamma \).

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For the case $\gamma = 0$, we introduce the auxiliary operator $\tilde{T}_{\alpha,\beta}$ defined by $(\tilde{T}_{\alpha,\beta} b)(j) = (T_{\alpha,\beta} b)(j)$ if $j \neq 0$ and $(\tilde{T}_{\alpha,\beta} b)(0) = 0$. Since

$$||(T_{\alpha,\beta} b)(0)|| \leq 2^{1/p'} \|\{i^{-1}\}\|_{\ell^p(N)} \|b\|_{\ell^p(Z)} < +\infty, \quad \forall 1 \leq p < \infty,$$

it suffices to show that $\tilde{T}_{\alpha,\beta}$ is bounded on $\ell^p(Z)$, $1 < p < +\infty$. We will show that the operator $\tilde{T}_{\alpha,\beta}$ is bounded from $\ell^p(Z)$ into $\ell^q, \infty(Z)$ for each $1 \leq p < \infty$. Then the $\ell^p(Z)$ boundedness of $\tilde{T}_{\alpha,\beta}$ will follow from the Marcinkiewicz interpolation theorem (see [4, Theorem 1.3.2]) and, with it, that of $T_{\alpha,\beta}$.

Given $j_0 \neq 0$ fixed, we write $Z \setminus \{-j_0, j_0\} = I_1 \cup I_2 \cup I_3$ where

$$I_1 = \{ i \in Z : 0 < |i - j_0| \leq |j_0| \}, \quad I_2 = \{ i \in Z : 0 < |i + j_0| \leq |j_0| \},$$

$$I_3 = \{ i \in Z : |i| > 2|j_0| \}.$$

Then

$$|(\tilde{T}_{\alpha,\beta} b)(j_0)| = |(T_{\alpha,\beta} b)(j_0)| \leq \left( \sum_{i \in I_1} + \sum_{i \in I_2} + \sum_{i \in I_3} \right) \frac{|b(i)|}{|i - j_0|^\alpha |i + j_0|^\beta}.$$

First, we estimate the sum on $I_1$. If $i \in I_1$, then $|i + j_0| = |2j_0 + i - j_0| \geq |j_0|$. So

$$\sum_{i \in I_1} \frac{|b(i)|}{|i - j_0|^\alpha |i + j_0|^\beta} \leq \frac{1}{|j_0|^\beta} \sum_{0 < |i - j_0| \leq |j_0|} \frac{|b(i)|}{|i - j_0|^\alpha} =: S_1.$$

Now, we take $k_0 \in \mathbb{N}_0$ such that $2^{k_0} \leq |j_0| < 2^{k_0+1}$, thus

$$S_1 = \sum_{k=0}^{k_0} \frac{1}{|j_0|^\beta} \sum_{2^-(k+1) |j_0| < |i - j_0| \leq 2^{-k} |j_0|} \frac{|b(i)|}{|i - j_0|^\alpha} \leq \sum_{k=0}^{k_0} 2^{(k+1)\alpha} \sum_{|i - j_0| \leq 2^{-k} |j_0|} |b(i)|$$

$$= 2^{1+\alpha} \sum_{k=0}^{k_0} 2^{-1-\alpha k} |j_0|^\beta \sum_{|i - j_0| \leq 2^{-k} |j_0|} |b(i)|$$

$$\leq 2^{2+\alpha} \sum_{k=0}^{k_0} 2^{-1-\alpha k} \frac{1}{2 \cdot 2^{-k} |j_0|^\beta} + 1 \sum_{|i - j_0| \leq 2^{-k} |j_0|} |b(i)|,$$
this last inequality follows from that \(2^{-k}|j_0| \leq 2^{-k}|j_0|\) and that \(\frac{2^k|2^{-k}|j_0| + 1}{2^k|2^{-k}|j_0|} \leq 2\) for each \(k = 0, \ldots, k_0\). Thus

\[
\sum_{i \in I_1} \frac{|b(i)|}{|i - j_0|^\alpha|i + j_0|^\beta} \leq 2^{2+\alpha} \left(\sum_{k=0}^{+\infty} 2^{-(1-\alpha)k}\right)(Mb)(j_0).
\]

Similarly, it is seen that

\[
\sum_{i \in I_2} \frac{|b(i)|}{|i - j_0|^\alpha|i + j_0|^\beta} \leq C_\beta(Mb)(-j_0).
\]

Now, we estimate the last sum. If \(i \in I_3\), then \(|i \pm j_0| > \frac{|i|}{2}\), so

\[
\sum_{i \in I_3} \frac{|b(i)|}{|i - j_0|^\alpha|i + j_0|^\beta} \leq C \sum_{|i|>2|j_0|} |i|^{-1} |b(i)| \leq C \|b\|_{\ell^p} |j_0|^{-1/p},
\]

the last inequality follows from the H"{o}lder’s inequality. Thus (11) implies that

\[
\#\left\{ j \neq 0 : \left| \sum_{i \in I_3} |i - j|^{-\alpha}|i + j|^{-\beta} b(i) \right| > \lambda \right\} \leq \left( C \frac{\|b\|_{\ell^p}}{\lambda} \right)^p, \quad 1 \leq p < \infty.
\]

Finally, (9), (10), (12) and Theorem 1 allow us to conclude that the operator \(\tilde{T}_{\alpha,\beta}\) is bounded from \(\ell^p(Z)\) into \(\ell^{p,\infty}(Z)\), for every \(1 \leq p < \infty\). \(\square\)

**Remark 8.** Let \(0 \leq \gamma < 1\). Then the operator \(T_{\alpha,\beta}\) is not bounded from \(\ell^p(Z)\) into \(\ell^q(Z)\) for \(0 < p \leq \infty\) and \(0 < q \leq \frac{1}{1-\gamma}\). Indeed, by taking \(b = \{b(i)\}\) such that \(b(0) = 1\) and \(b(i) = 0\) for all \(i \neq 0\) we have that \((T_{\gamma} b)(0) = 0\) and \((T_{\alpha,\beta} b)(j) = |j|^\gamma - 1\) for all \(j \neq 0\). So, \(b \in \ell^p(Z)\) but \(T_{\alpha,\beta} b \notin \ell^q(Z)\) for \(0 < p \leq \infty\) and \(0 < q \leq \frac{1}{1-\gamma}\).

For \(0 \leq \gamma < 1\), one also can see that \(T_{\alpha,\beta}\) is not bounded from \(\ell^p(Z)\) into \(\ell^q(Z)\) for \(\frac{1}{\gamma} \leq p \leq \infty\) and \(0 < q \leq \infty\). For them, to consider \(b = \{b(i)\}\) with \(b(i) = 0\) for \(|i| \leq 1\), and \(b(i) = \frac{1}{|i|^{1\log(|i|)}}\) for \(|i| \geq 2\). It is easy to check that \(b \in \ell^p(Z)\) for each \(\frac{1}{\gamma} \leq p \leq \infty\), and \((T_{\alpha,\beta} b)(j) = +\infty\) for all \(j\).

Let \(0 \leq \gamma < 1\). If \(\frac{1}{1-\gamma} < q < \infty\) and \(0 < p \leq \frac{q}{1+q\gamma}\), then the operator \(T_{\alpha,\beta}\) is bounded from \(\ell^p(Z)\) into \(\ell^q(Z)\). This follows from Theorem 7 and the embedding \(\ell^{p_1}(Z) \hookrightarrow \ell^{p_2}(Z)\) valid for \(0 < p_1 < p_2 \leq \infty\).

**Theorem 9.** Let \(0 \leq \gamma < 1\). If \(T_{\alpha,\beta}\) is the operator given by (8), \(0 < p \leq 1\) and \(\frac{1}{q} = \frac{1}{p} - \gamma\), then

\[
\|T_{\alpha,\beta} b\|_{\ell^q(Z)} \leq C \|b\|_{H^p(Z)}.
\]
where $C$ depends only on $\alpha$, $\beta$, $p$ and $q$.

**Proof.** We take $p_0$ such that $1 < p_0 < \gamma^{-1}$. By Theorem 3, given $b \in H^p(\mathbb{Z})$ we can write $b = \sum_k \lambda_k a_k$ where the $a_k$’s are $(p, \infty, d_p)$ atoms, the scalars $\lambda_k$ satisfies $\sum_k |\lambda_k|^p \leq C \|b\|_{H^p(\mathbb{Z})}^p$ and the series converges in $H^p(\mathbb{Z})$ and so in $\ell^{p_0}(\mathbb{Z})$ since $H^p(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \subset \ell^{p_0}(\mathbb{Z})$ embed continuously. For $\frac{1}{q_0} = \frac{1}{p_0} - \gamma$, by Theorem 7, $T_{\alpha,\beta}$ is a bounded operator from $\ell^{p_0}(\mathbb{Z})$ into $\ell^{q_0}(\mathbb{Z})$. Since $b = \sum_k \lambda_k a_k$ in $\ell^{p_0}(\mathbb{Z})$, we have that $(T_{\alpha,\beta} b)(j) = \sum_k \lambda_k (T_{\alpha,\beta} a_k)(j)$ for all $j \in \mathbb{Z}$, and thus

\[(13) \quad \|(T_{\alpha,\beta} b)(j)\| \leq \sum_k |\lambda_k| \|(T_{\alpha,\beta} a_k)(j)\|, \quad \forall j \in \mathbb{Z}.
\]

If for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \gamma$ we obtain that $\|T_{\alpha,\beta} a_k\|_{\ell^q(\mathbb{Z})} \leq C$, with $C$ independent of the $(p, \infty, d_p)$-atom $a_k$, then the estimate (13) and the fact that $\sum_k |\lambda_k|^p \leq C \|b\|_{H^p(\mathbb{Z})}^p$ lead to

\[
\|T_{\alpha,\beta} b\|_{\ell^q(\mathbb{Z})} \leq C \left( \sum_k |\lambda_k|^{\min\{1, q\}} \right)^{\frac{1}{\min\{1, q\}}} \leq C \left( \sum_k |\lambda_k|^p \right)^{1/p} \leq C \|b\|_{H^p(\mathbb{Z})}.
\]

Since $b$ is an arbitrary element of $H^p(\mathbb{Z})$, the theorem follows.

To conclude the proof we will prove that for $0 < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \gamma$ there exists an universal constant $C > 0$, which depends on $\alpha$, $\beta$, $p$ and $q$ only, such that

\[(14) \quad \|T_{\alpha,\beta} a\|_{\ell^q(\mathbb{Z})} \leq C, \quad \text{for all } (p, \infty, d_p)\text{-atom } a = \{a(i)\}.
\]

To prove (14), let $J_{n_0} = \{n_0 - m, \ldots, n_0, \ldots, n_0 + m\}$ be the support of the atom $a = \{a(i)\}$. We put $3J_{n_0} = \{\pm n_0 - 3m, \ldots, \pm n_0, \ldots, \pm n_0 + 3m\}$. So

\[(15) \quad \sum_{j \in \mathbb{Z}} \|(T_{\alpha,\beta} a)(j)\|^q = \sum_{j \in 3J_{n_0} \cup 3J_{-n_0}} \|(T_{\alpha,\beta} a)(j)\|^q + \sum_{j \in \mathbb{Z} \setminus (3J_{n_0} \cup 3J_{-n_0})} \|(T_{\alpha,\beta} a)(j)\|^q.
\]

To estimate the first sum, by taking into account that $\frac{q_0}{q} > 1$ we apply Hölder’s inequality with that $\frac{q_0}{q}$, then from the $\ell^{p_0}(\mathbb{Z}) - \ell^{q_0}(\mathbb{Z})$ boundedness of $T_{\alpha,\beta}$, the size condition of the atom and since $\frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0} = \gamma$ we have

\[(16) \quad \sum_{j \in 3J_{n_0} \cup 3J_{-n_0}} \|(T_{\alpha,\beta} a)(j)\|^q \leq 2 \cdot 3^{(q_0 - q)/q_0} \left( \sum_{j \in \mathbb{Z}} \|(T_{\alpha,\beta} a)(j)\|^{q_0} \right)^{q/q_0} \left( \#J_{n_0} \right)^{(q_0 - q)/q_0}.
\]
where \(q_j\) is a term in the Taylor expansion there exists \(\|R\|\leq 1\) have, for every \(j\), and we put \(R = C\) with \(C\) independent of \(n_0\) and \(m\).

To estimate the second sum in (15), we denote

\[
K(x, y) = |x - y|^{-\alpha} |x + y|^{-\beta},
\]

and we put \(N - 1 = [p^{-1} - 1]\). In view of the moment condition of \(a\) we have, for \(j \in \mathbb{Z}\setminus (3J_{n_0} \cup 3J_{-n_0})\), that

\[
(T_{\alpha, \beta} a)(j) = \sum_{i \in J_{n_0}} K(i, j) a(i) = \sum_{i \in J_{n_0}} [K(i, j) - q_N(i, j)] a(i),
\]

where \(q_N(\cdot, j)\) is the degree \(N - 1\) Taylor polynomial of the function \(x \to K(x, j)\) expanded around \(n_0\). By the standard estimate of the remainder term in the Taylor expansion there exists \(\xi\) between \(i\) and \(n_0\) such that

\[
|K(i, j) - q_N(i, j)| \leq C |i - n_0|^N \left| \frac{\partial^N}{\partial x^N} K(\xi, j) \right|
\]

\[
\leq C |i - n_0|^N |j - \xi|^{-\alpha} |j + \xi|^{-\beta} (|j - \xi|^{-1} + |j + \xi|^{-1})^N,
\]

where Proposition 6 gives the last inequality. For \(j \in \mathbb{Z}\setminus (3J_{n_0} \cup 3J_{-n_0})\) we have \(|j \pm n_0| \geq 2m\), since \(\xi \in [n_0 - m, n_0 + m]\), it follows that \(|\xi - n_0| \leq m \leq \frac{1}{2}|j \pm n_0|\). So

\[
|j \pm \xi| = |j \pm n_0 \mp n_0 \pm \xi| \geq |j \pm n_0| - |n_0 - \xi| \geq \frac{|j \pm n_0|}{2}
\]

and then

(17)

\[
|K(i, j) - q_N(i, j)| \leq C |i - n_0|^N |j - n_0|^{-\alpha} |j + n_0|^{-\beta} (|j - n_0|^{-1} + |j + n_0|^{-1})^N,
\]

for every \(j \in \mathbb{Z}\setminus (3J_{n_0} \cup 3J_{-n_0})\) and \(i \in J_{n_0}\). Now we decompose the set \(R := \mathbb{Z}\setminus (3J_{n_0} \cup 3J_{-n_0})\) by \(R = R_1 \cup R_2\) where

\[
R_1 = \{ j \in R : |n_0 - j| \leq |n_0 + j| \} \quad \text{and} \quad R_2 = \{ j \in R : |n_0 + j| < |n_0 - j| \}.
\]

If \(j \in R\), then \(j \in R_k\) for some \(k = 1, 2\). From this, (17) and since \(\alpha + \beta = 1 - \gamma\), we obtain that

\[
|K(i, j) - q_N(i, j)| \leq C |i - n_0|^N |j + (-1)^k n_0|^{-\gamma - N},
\]

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for all \( i \in J_{n_0} \) and all \( j \in R_k \). This inequality gives

\[
(18) \quad \sum_{j \in \mathcal{H}} |(T_{\alpha,\beta} a)(j)|^q = \sum_{k=1}^{2} \sum_{j \in R_k} |(T_{\alpha,\beta} a)(j)|^q = \sum_{k=1}^{2} \sum_{j \in R_k} \sum_{i \in J_{n_0}} |K(i, j) a(i)|^q = \sum_{k=1}^{2} \sum_{j \in R_k} \sum_{i \in J_{n_0}} [K(i, j) - q_N(i, j)] a(i) |^q
\]

\[
\leq C \left( \sum_{i \in J_{n_0}} |i - n_0|^N |a(i)|^q \right)^2 \sum_{k=1}^{2} \sum_{j \in \mathbb{Z} \setminus 3J_{(-1)^k+1, n_0}} \left( j + (-1)^k n_0 \right)^{-(1-\gamma)q-Nq}
\]

\[
\leq C m^{qN - \frac{q}{p} + q} \int_{m}^{\infty} t^{-q((1-\gamma)+N)} \, dt \leq C
\]

with \( C \) independent of the \( p \)-atom \( a \), since \( -q((1-\gamma)+N)+1 < 0 \). Finally, the inequality in (14) follows from (16) and (18). \( \square \)

**Remark 10.** Let \( 0 \leq \gamma < 1 \). If \( 0 < q \leq \frac{1}{1-\gamma} \) and \( 0 < p \leq \frac{q}{1+q\gamma} \), then the operator \( T_{\alpha,\beta} \) is bounded from \( H^p(\mathbb{Z}) \) into \( \ell^q(\mathbb{Z}) \). This follows from Theorem 9 and the embedding \( H^{p_1}(\mathbb{Z}) \hookrightarrow H^{p_2}(\mathbb{Z}) \) valid for \( 0 < p_1 < p_2 \leq 1 \).

**Remark 11.** The argument utilized in the proof of Theorem 9 works for the discrete Riesz operator \( I_\gamma \) as well. So, if \( 0 < \gamma < 1 \), \( 0 < q \leq \frac{1}{1-\gamma} \) and \( 0 < p \leq \frac{q}{1+q\gamma} \), then the operator \( I_\gamma \) is bounded from \( H^p(\mathbb{Z}) \) into \( \ell^q(\mathbb{Z}) \).

### 4. A counter-example

For \( 0 \leq \gamma < 1 \), we consider the operator \( U_\gamma \) given by

\[
(U_\gamma b)(j) = \sum_{i \neq \pm j} \frac{b(i)}{|i - j|^{1+\gamma} |i + j|^{1-\gamma}}.
\]

It is clear that \( U_\gamma = T_{\frac{1}{2},\frac{1}{2}} \). In this section, we will prove that there exists \( \varepsilon \in (0, \frac{1}{2}) \) such that, for every \( 0 \leq \gamma < \varepsilon \), the operator \( U_\gamma \) is not bounded from \( H^p(\mathbb{Z}) \) into \( H^q(\mathbb{Z}) \) for \( 0 < p \leq (1+\gamma)^{-1} \) and \( \frac{1}{q} = \frac{1}{p} - \gamma \).

Let \( b = \{ b(i) \}_{i \in \mathbb{Z}} \) be the sequence defined by \( b(\pm 1) = 1 \), \( b(0) = -2 \) and \( b(i) = 0 \) for all \( i \neq -1, 0, 1 \). It is clear that \( b \) satisfies the first moment condition, then \( b \in H^p(\mathbb{Z}) \) for every \( \frac{1}{2} < p \leq 1 \). In particular, \( b \in H^{(1+\gamma)^{-1}}(\mathbb{Z}) \) for all \( 0 \leq \gamma < 1 \). A computation gives

\[
(U_\gamma b)(-1) = b(0) = -2, \quad (U_\gamma b)(0) = b(-1) + b(1) = 2,
\]

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\[(U_\gamma b)(1) = b(0) = -2,\]
\[(U_\gamma b)(j) = \frac{2}{|j^2 - 1|^{(1-\gamma)/2}} - \frac{2}{|j|^{1-\gamma}}, \text{ for all } j \neq 0, \pm 1.\]

So

\[\sum_{j=-\infty}^{+\infty} (U_\gamma b)(j) = -2 + 4 \sum_{j=2}^{+\infty} \left[ \frac{1}{(j^2 - 1)^{(1-\gamma)/2}} - \frac{1}{j^{1-\gamma}} \right].\]

Next, we introduce two auxiliary functions. Let \(g: [0, 1) \to (0, +\infty)\) be the function defined by \(g(\gamma) = \frac{1}{3^\gamma} - \frac{1}{2^\gamma}\), and let \(h: \left[ 0, \frac{1}{3} \right) \to (0, +\infty)\) be given by \(h(\gamma) = \frac{1}{2} - \frac{1}{8^{1-\gamma}}\). Since \(g(0) = \frac{1}{\sqrt{3}} - \frac{1}{2} < \frac{1}{2} - \frac{1}{\sqrt{8}} = h(0)\), by the continuity of \(g\) and \(h\), there exists \(\varepsilon \in \left( 0, \frac{1}{3} \right)\) such that \(g(\gamma) < h(\gamma)\) for all \(0 \leq \gamma < \varepsilon\).

On the other hand, for \(0 \leq \gamma < 1\), it is easy to check that

\[(j + 1)^{\gamma - 1} < ((j + 1)^2 - 1)^{(\gamma - 1)/2} < j^{\gamma - 1} < (j^2 - 1)^{(\gamma - 1)/2}\]

for all \(j \geq 2\) and so

\[\bigcup_{j=3}^{+\infty} \left[ \frac{1}{j^{1-\gamma}}, \frac{1}{(j^2 - 1)^{(1-\gamma)/2}} \right] \subseteq \left( 0, \frac{1}{8(1-\gamma)/2} \right),\]

being the previous union disjoint. We obtain

\[\sum_{j=2}^{+\infty} \left[ \frac{1}{(j^2 - 1)^{(1-\gamma)/2}} - \frac{1}{j^{1-\gamma}} \right] = \sum_{j=3}^{+\infty} \left[ \frac{1}{(j^2 - 1)^{(1-\gamma)/2}} - \frac{1}{j^{1-\gamma}} \right] + g(\gamma)\]

\[\leq \frac{1}{8(1-\gamma)/2} + g(\gamma) < \frac{1}{8(1-\gamma)/2} + h(\gamma) = \frac{1}{2}, \text{ for every } 0 \leq \gamma < \varepsilon,\]

this implies

\[\sum_{j=-\infty}^{+\infty} (U_\gamma b)(j) = -2 + 4 \sum_{j=2}^{+\infty} \left[ \frac{1}{(j^2 - 1)^{(1-\gamma)/2}} - \frac{1}{j^{1-\gamma}} \right] < 0,\]

for every \(0 \leq \gamma < \varepsilon\). Thus, by Corollary 4, it follows that \(U_\gamma\) is not bounded from \(H^{(1+\gamma)^{-1}}(\mathbb{Z})\) into \(H^1(\mathbb{Z})\) for every \(0 \leq \gamma < \varepsilon\).

For \(0 < p < (1 + \gamma)^{-1}\), we take \(L\) as any fixed integer with \(L \geq \lceil p^{-1} - 1 \rceil\), then by Corollary 5 the set \(D_L\) is dense in \(H^r(\mathbb{Z})\) for each \(p \leq r \leq 1\). In

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particular, there exists $c \in H^p(\mathbb{Z})$ such that
\[
\|b - c\|_{H^{(1+\gamma)^{-1}}(\mathbb{Z})} < \frac{1}{2} \|U_\gamma\|_{H^{(1+\gamma)^{-1}} \rightarrow \ell^1} \sum_{j=-\infty}^{+\infty} (U_\gamma b)(j).
\]

Then
\[
\left| \sum_{j=-\infty}^{+\infty} (U_\gamma c)(j) dx \right| \geq \left| \sum_{j=-\infty}^{+\infty} (U_\gamma b)(j) \right| - \left| \sum_{j=-\infty}^{+\infty} (U_\gamma (b - c))(j) \right| \geq \left| \sum_{j=-\infty}^{+\infty} (U_\gamma b)(j) \right| - \|U_\gamma\|_{H^{(1+\gamma)^{-1}} \rightarrow \ell^1} \|b - c\|_{H^{(1+\gamma)^{-1}}(\mathbb{Z})} > 0
\]

where the second inequality follows from Theorem 9 with $p = (1 + \gamma)^{-1}$ and $q = 1$. But then, by Corollary 4, for every $0 \leq \gamma < \varepsilon$ the operator $U_\gamma$ is not bounded from $H^p(\mathbb{Z})$ into $H^q(\mathbb{Z})$ for each $0 < p < (1 + \gamma)^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \gamma$, since $\sum_{j=-\infty}^{+\infty} (U_\gamma c)(j) \neq 0$.

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