GAUGE FIELD FLUXES AND BIANCHI IDENTITIES IN EXTENDED FIELD THEORIES

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The tensor hierarchy of exceptional field theories contains gauge fields satisfying certain Bianchi identities with sources determining the interactions with standard and exotic branes. These identities are responsible for tadpole cancellation in compactification schemes and provide consistency constraints for building cosmological models. In detail, we consider and develop an approach in which the analysis of the reduction of a (10+10)-dimensional double field theory to a (D+d+d)-dimensional split double field theory allows considering all Bianchi identities of the theory in a form analogous to the extended field theory approach.

Keywords: string theory, exotic brane, flux compactification, double field theory

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1. Introduction

A major goal of string theory as a consistent formulation of quantum gravity is to describe inflationary cosmological models in four dimensions. To comply with observational data, we consider space–time configurations where a four-dimensional space–time of de Sitter type is supplemented with a compact six-dimensional manifold whose size is sufficiently small such that it is not yet directly observable. Fortunately, the equations of motion of supergravity have solutions of precisely the required form, of which the simplest case is the six-dimensional torus. But toroidal backgrounds without fluxes cannot reproduce any realistic phenomenology, because they preserve too many supersymmetries, cannot provide a scalar potential with stable minimums, and hence generate nonchiral effective theories with too many massless scalar fields. The problem of finding a better option for the internal compact manifold that provides masses and charges for scalar fields of the effective four-dimensional theory and generates a suitable inflaton potential is usually called the problem of moduli stabilization. Finding an appropriate compactification scheme for string theory including fluxes and branes has been intensively investigated (see [1]–[3] for a review). The most famous example of a cosmological model based on compactifications with D-branes is the KKLT scenario [4] (also see [5] and the references therein for a recent discussion of the consistency of the model).

To illustrate the need for fluxes and a nontrivial geometry to provide scalar masses, we consider a toy model of a (1+5)-dimensional theory of gravity interacting with an electromagnetic field and perform...
a dimensional reduction to four dimensions as in [6]. The internal two-dimensional manifolds can all be classified by the number of handles, and we regard the volume of the internal manifold as the only scalar modulus of the resulting theory. Hence, we start with the theory

\[ S = \int d^6x \sqrt{-G} (M_6^4 R(6)[G] - M_6^2 F^2), \]  

(1.1)

where \( R(6)[G] \) is the Riemann curvature of the metric \( G \) in the full (1+5)-dimensional space–time, \( F^2 \) is the contribution from the electromagnetic field, and \( M_6 \) is the corresponding Planck mass, which is present for dimensional reasons. We consider the usual compactification ansatz for the metric

\[ ds^2 = \epsilon g_{\mu\nu} dx^\mu dx^\nu + R(x)^2 h_{mn} dy^m dy^n, \]  

(1.2)

where \( \{x^\mu\} \) and \( \{y^m\} \) are coordinates of the respective external and internal spaces, \( h_{mn} \) is the metric of unit volume on the internal space, and the field \( R(x) \) corresponds to the volume modulus in the theory. For simplicity, we assume that there are no \( G_{\mu m} \) components in the full metric. To keep the kinetic term in the canonical form, we rescale the external metric, and the final action becomes

\[ S_{\text{eff}} = \int d^4x (R(4)[h] + h^{mn} \partial_m R \partial_n R + V(R) + \ldots), \]  

(1.3)

where the potential \( V(R) \) depends on geometry of the internal manifold and on the flux of the gauge field. The ellipsis denotes contributions of higher order in the perturbation theory and interactions that have no interest for our discussion. We can write the potential as

\[ V(R) \sim (2g - 2) \frac{1}{R(x)^4} + \frac{n^2}{R(x)^6}, \quad n = \int_\Sigma F, \]  

(1.4)

where \( n \) is the flux of the field \( F_{mn} \) integrated over the internal manifold \( \Sigma \) and \( g \) is the genus of \( \Sigma \). These are topological invariants and are hence the initial parameters of the model.

The stability of the theory under small variations of the field \( R(x) \) around a potential minimum depends crucially on the chosen parameters, the genus and flux. We list essentially different cases in Table 1.

| \( g \) | \( n = 0 \) | \( n \neq 0 \) |
|---|---|---|
| 0 (sphere) | inconsistent | Freund–Rubin (stable) |
| 1 (torus) | flat | runaway |
| \( g > 1 \) | runaway | runaway |

Table 1

Potential behavior for different genera \( g \) and numbers \( n \) of fluxes.

It can be seen that for a toroidal compactification \( (g = 1) \) with no flux \( (n = 0) \), the potential is flat \( (V = 0) \) and the minimum is not represented by a single point. Hence, the effective four-dimensional theory contains a massless field for each flat direction in the scalar potential.

Compactification on a sphere \( (g = 0) \) with no flux results in a potential with a minimum and \( R(x) = 0 \). Hence, the theory dynamically tends to have no internal directions at all. This contradicts our initial assumption that the size of the internal space is small but finite.

For higher genera \( g > 1 \) and no flux, we observe runaway behavior where the potential is minimized as \( R(x) \to \infty \). This corresponds to spontaneous decompactification of the theory and a return to the (1+5)-dimensional space. As follows from Table 1, turning on fluxes and choosing a sphere as the internal
manifold allows stabilizing the field $R(x)$ at a nonzero value at a minimum of the potential. This class of solutions in supergravity is called Freund–Rubin solutions [7] and does not provide a proper background for cosmological model building because the potential is negative at the minimum. Nevertheless, this example illustrates how the presence of fluxes and the choice of geometry affects the effective lower-dimensional theory.

To provide support for a gauge field on a compact manifold, we must include branes in the compactification scheme. To do this consistently, we must ensure satisfaction of tadpole cancellation conditions, which follow from an analogue of the Gauss theorem for a compact internal manifold. Indeed, we consider a Ramond–Ramond (RR) field $C_{p+1}$ transforming as a $(p+1)$-form with the field strength $F_{p+2} = dC_{p+1}$ whose action is

$$S = -\frac{1}{4} \int d^{10}x \ast F_{p+1} \wedge F_{p+1} + \mu \sum_a \int_{\Sigma_a} d^{p+1} \xi C_{p+1},$$

(1.5)

where $\Sigma$ denotes the world-volume of the corresponding Dp-branes labeled by the index $a$. Introducing a current $J_{p+1}^a$ for each of the D-brane equations of motion, we obtain

$$d \ast dC_{p+1} = \mu \sum_a \ast J_{p+1}^a,$$

(1.6)

If the sources and the fluxes are localized on only the compact manifold, then the left-hand side of (1.6) vanishes under integration over this manifold, and the right-hand side gives the total charge. This implies that the total charge of any RR field localized on a compact manifold must be zero. This is usually called the tadpole cancellation condition, which can be satisfied by introducing orientifold planes into the model (see [2] for a review).

The magnetic dual of a Dp-brane interacts with the RR field $\tilde{C}_{7-p}$, which is Poincaré dual to the RR field $C_{p+1}$. The equation of motion for the dual field follows from the Bianchi identities for the field strength of $C_{p+1}$:

$$dF_{p+2} + \cdots = dF_{8-p} + \cdots = \mu \sum_a \ast J_{7-p}^a,$$

(1.7)

where the ellipsis denotes possible contributions from other gauge potentials. This equation implies that in the presence of branes magnetically charged with respect to a gauge potential $C_p$, the corresponding field strength is topologically nontrivial and has nontrivial Bianchi identities. For D-branes, we can consider only electrically charged objects, and all Bianchi identities can be set trivial. But in the Neveu-Schwarz–Neveu-Schwarz (NS-NS) sector, we find NS 5-branes that are magnetically charged with respect to the Kalb–Ramond 2-form field $B_2$. This field electrically couples to the fundamental string $F_1$.

In compactification schemes involving NS 5-branes, tadpole cancellation conditions cannot be satisfied by adding Op-planes to the model, but we can equivalently require that the right-hand side of the Bianchi identity vanish. Based on T-duality transformations of the Gukov–Vafa–Witten superpotential and of Bianchi identities for (constant) fluxes such cancellations were analyzed in [8]–[10]. This analysis includes not only the geometric fluxes of the NS 5-brane and the KK5 monopole but also nongeometric Q-fluxes and R-fluxes respectively sourced by $5_2^2$ and $5_2^3$-branes. Such branes interact electrically with the mixed-symmetry gauge potentials $B_{(8,2)}$ and $B_{(9,3)}$, which are magnetic duals of the bivector field $\beta^{mn}$. In [11]–[14], a world-volume Dirac–Born–Infeld action for such branes was presented, and the coupling to the mixed-symmetry potentials was analyzed.

Understood as proper gauge fields, the mixed-symmetry potentials are expected to generate field strengths that must satisfy certain Bianchi identities together with the equations of motion. Here, we show that the double field theory (DFT) formulation of supergravity allows writing such Bianchi identities in a T-duality covariant form and suggests an M-theory generalization of the result. In Sec. 2, we briefly
describe the DFT approach to nongeometric fluxes in terms of Scherk–Shwarz reduction. In Sec. 3, we review the split-form DFT as obtained from the full $O(10,10)$ theory. In Sec. 4, we describe Bianchi identities for generalized fluxes of the $O(d,d)$ theory obtained by reduction from the full DFT and interpret the identities in terms of 5-brane sources of various orientations. Finally, in Sec. 5, we discuss an extension of the described results to the case of exceptional field theory and gauge potentials of nongeometric branes of M-theory.

2. Fluxes in double field theory

The DFT provides a natural framework for addressing properties of nongeometric fluxes and the corresponding mixed-symmetry potential. This approach was mainly developed in [15]–[17], and a review of the formalism can be found in [18]–[20]. The approach is based on extending the space–time by coordinates corresponding to winding modes of the string and rewriting the field content in a T-duality-covariant form. For the fully doubled (10+10)-dimensional space–time parameterized by the coordinates $X_M = (x^\mu, \tilde{x}_\mu)$, the field content is encoded in the so-called generalized metric

$$H_{MN} = \begin{bmatrix} G_{\mu\nu} + B_{\mu\lambda} G^{\lambda\nu} & B_{\mu}{}^\sigma \\ B_{\nu}{}^\rho & G^{\rho\sigma} \end{bmatrix} \in \frac{O(10,10)}{O(1,9) \times O(1,9)}$$

and the invariant dilaton $d = \varphi - (1/4) \log \det G$.

The dynamics of the theory are given by the action first presented in [16], which has the form

$$S_{\text{HHZ}} = \int d^{20} x e^{-2d} \left( \frac{1}{8} H^{MN} \partial_M H^{KL} \partial_N H_{KL} - \frac{1}{2} H^{KL} \partial_L H^{MN} \partial_N H_{KM} - 2 \partial_M d \partial_N H^{MN} + 4 H^{MN} \partial_M d \partial_N d \right).$$

This action is invariant under global $O(10,10)$ transformations and under local transformations generated by the generalized Lie derivative defined as

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - (\partial_N \Lambda^M - \partial^M \Lambda_N) V^N.$$ 

Indices are raised and lowered by the invariant tensor of $O(10,10)$

$$\eta_{MN} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For consistency of the algebra of generalized Lie derivatives, we must impose a constraint on all fields on which it is realized [21]. This constraint is called the section condition and can be written for DFT as

$$\eta^{MN} \partial_M \bullet \partial_N \bullet = 0,$$

where bullets denote any fields or combinations of them. Effectively, this reduces to the condition that all fields can depend only on half of the total coordinates. The most natural choice is $\tilde{\partial}^m \bullet = 0$, i.e., all fields depend only on geometric coordinates $x^m$. Under this constraint, the generalized Lie derivative splits into diffeomorphisms and gauge transformations, and the action $S_{\text{HHZ}}$ reproduces the normal action of (the bosonic part of) 10-dimensional supergravity. Choosing to drop different subsets of the total coordinate set is equivalent to choosing a T-duality frame.
Based on the progress made in [22], [23], it was shown in [24] that DFT admits solutions preserving the section constraint but failing to satisfy the equations of motion of normal supergravity because they depend on dual coordinates. These correspond to backgrounds of exotic 5-branes of the NS sector: a KK vortex and Q- and R-monopoles. Backgrounds sourced by such branes can be characterized by fluxes encoded in the generalized torsion $F_{ABC}$ of DFT defined in terms of the generalized vielbein $E^A$ as

$$[E_A, E_B]_C = F_{ABC} E_C. \quad (2.6)$$

The generalized vielbein as usual is defined as $H_{MN} = E^M E^N h_{AB}$ with diagonal and constant $h_{AB}$. The NS fluxes in terms of components of the generalized torsion are

$$H_{abc} = F_{abc}, \quad f_{ab}^{\ e} = F_{ab}^{\ e}, \quad Q^a_{\ bc} = F^a_{\ bc}, \quad R^{abc} = F^{abc}. \quad (2.7)$$

From Eq. (2.6), we obtain the components of the generalized torsion in terms of the vielbein and its derivatives,

$$F_{ABC} = -3 E^M E_N \partial_M E_{C|N}, \quad (2.8)$$

and flat DFT indices are raised and lowered by the corresponding invariant tensor $\eta_{AB}$. In addition, we have the flux corresponding to the dilaton field

$$F_A = \partial_M E^M_A + 2 E^M_A \partial_M d. \quad (2.9)$$

Because these encode the same degrees of freedom as the generalized metric and the invariant dilaton, the initial action $S_{HHZ}$ can be completely rewritten in terms of $F_{ABC}$ and $F_A$. Such a flux formulation of DFT was presented in [25], and the Lagrangian has the form

$$S = \int d\mathcal{X} e^{-2d} \left( -\frac{1}{4} F_{AB}^C F_{BC}^D \mathcal{H}^{AB} - \frac{1}{12} F_{AC}^E F_{BD}^F \mathcal{H}^{AB} \mathcal{H}^{CD} \mathcal{H}_{EF} + F_A F_B \mathcal{H}^{AB} - \frac{1}{6} F^{ABC} F_{ABC} - F_A^2 \right). \quad (2.10)$$

This must preserve the generalized diffeomorphism invariance and local gauge transformations of the vielbein. These conditions imply the constraints on the fluxes

$$E^M_A \partial_M F_{BCD} - \frac{3}{4} F^{E}_{[AB} F_{|E|CD]} \equiv Z_{ABCD} = 0,$n

$$E^M_C \partial_M F_{CAB} + 2 E^M_A \partial_M F_B - F^C F_{CAB} \equiv Z_{AB} = 0, \quad (2.11)$$

$$E^M_A \partial_M F_A - \frac{1}{2} F^A F_A + \frac{1}{12} F^{ABC} F_{ABC} \equiv Z = 0.$$

Solving these Bianchi identities, we can recover the fields $F_{ABC}$ and $F_A$ in terms of the generalized vielbein as above.

The relation between the generalized vielbein and the generalized torsion has the same nature as the relation between the gauge field $B_{\mu \nu}$ and its field strength. This can be seen explicitly for the H-flux components $F_{\mu \nu \rho}$ of the torsion $F_{MNK}$ written in curved indices as

$$F_{MNK} = E^M_A E^N_B E^K_C F_{ABC}. \quad (2.12)$$
For these components, we have

\[ F_{\mu \nu \rho} = 3 \partial_{[\mu} B_{\nu] \rho} + \ldots, \]  

(2.13)

where the ellipsis denotes terms nonlinear in \( B \) that contain the metric \( G_{\mu \nu} \). The Bianchi identities for \( \mathcal{Z}_{MNKL} \) then imply that

\[ \partial_{[\mu} H_{\nu \rho \sigma]} + \cdots = 0 \]  

(2.14)

for the \( H \) flux, where the section constraint \( \tilde{\partial}^\mu = 0 \) is imposed. For the Poincaré dual of the 3-form field strength \( H(3) = * H(3) \), relation (2.14) becomes the equations of motion

\[ \nabla^\mu H_{\mu \nu_1 \ldots \nu_6} + \cdots = 0 \]  

(2.15)

and must be supplemented by a proper source in the right-hand side. In the considered case, this is the NS 5-brane, which is the magnetic dual of the fundamental string \( F1 \) and interacts with the Kalb–Ramond field \( B_{\mu \nu} \) magnetically (see, e.g., [26], [27]). Hence, it interacts with the 6-form field \( B_{\mu_1 \ldots \mu_6} \) electrically, and we can write the corresponding source contribution as

\[ \nabla^\mu H_{\mu \nu_1 \ldots \nu_6} + \cdots = j^{(0)}_{\nu_1 \ldots \nu_6}. \]  

(2.16)

Because the equations of motion for the magnetic potential \( B_{(6)} \) are a rewriting of the Bianchi identities for the electric potential \( B_{(2)} \), the latter also must be supplemented by the same source contribution:

\[ \partial_{[\mu} H_{\nu \rho \sigma]} + \cdots = (* j^{(0)})_{\mu \nu \rho \sigma}. \]  

(2.17)

The same arguments can be repeated for each component of the fluxes \( \mathcal{F}_{MNK} \) and \( \mathcal{F}_M \), and the result amounts to having a T-duality covariant source term in the right-hand side of the Bianchi identities of the DFT,

\[ \partial_{[\mu} \mathcal{F}_{MNK]} + \frac{3}{4} \mathcal{F}_{MN} P \mathcal{F}_{KLP} = T_{MNKL}. \]  

(2.18)

The Bianchi identities for exotic NS fluxes of supergravity sourced by exotic 5-branes were analyzed in [13], [28] for backgrounds of the conventional supergravity and in [24], [14] for backgrounds of the DFT.

Bianchi identities understood as conditions on T-duality covariant field strengths allow introducing the corresponding dual potentials as Lagrange multipliers in the full DFT action [29], [30],

\[ S_{Full} = S_{HHZ} + \int d^{20}x e^{-2d} \left( \mathcal{Z}_{MNKL} D^{MNKL} + \mathcal{Z}_{MN} D^{MN} + \mathcal{Z}_D \right). \]  

(2.19)

The potentials \( D^{MNKL} \) contain the Poincaré dual of the 6-form \( B_6 \), which is the magnetic partner of the Kalb–Ramond 2-form \( B_{\mu \nu} \),

\[ D^{\mu_1 \ldots \mu_4} = \epsilon^{\mu_1 \ldots \mu_{10}} B_{\mu_5 \ldots \mu_{10}}. \]  

(2.20)

The other components contain the dual graviton and potentials interacting with exotic branes, and the theory therefore cannot be written purely in terms of the dual potentials at the fully nonlinear level. A linearized version of the dual theory was presented in [30], [31], and the gauge transformations were analyzed.

If the split-form of DFT is considered, then the potentials \( D^{MNKL}, D^{MN}, \) and \( D \) pass into \( p \)-forms whose values are tensors of various rank of the remaining \( O(d, d) \) symmetry. These can be interpreted as
potentials interacting with NS 5-branes differently embedded into the partially doubled space–time. It was found in [32] that NS 5-branes in $D$ dimensions interact with the magnetic gauge potentials

\[ D_{D-4}, D_{D-3,M}, D_{D-2,MN}, D_{D-1,MNK}, D_{D,MNKL}, \]
\[ D_{D-2}, D_{D-1,M}, D_{D,MN}, \]
\[ D_{D}, \]

where the $O(d,d)$ indices are understood to be totally antisymmetric. For $D = 0$ corresponding to the full DFT, only the last column survives, which returns us to the previous case. Given the relation between potentials and branes and the fact that the split-form DFT has the same structure as exceptional field theories, it is natural to ask what the corresponding Bianchi identities are for the above potentials and what their meaning is in the exceptional field theory. We now briefly describe the split-form DFT, mainly following [17].

3. Split-form double field theory

We start with the generalized Lie derivative in $O(10,10)$ theory, which on the generalized vielbein has the form

\[ \mathcal{L}_V \mathcal{E}_M^A = V^N \partial_N \mathcal{E}_M^A + \mathcal{E}_N^A \partial_M V^N - \mathcal{E}_N^A \partial^N V_M. \]  

(3.1)

We decompose 20 coordinates $X^M$ into two sets of $2D$ and $2d$ coordinates respectively denoted by $\vec{X}$ and $\vec{X}$. The former then trivially decompose into the conventional space–time coordinates $x^\mu$ and their duals, which trivially drop from the picture to reproduce the proper section constraint for the resulting $O(d,d)$ theory.

The $O(10,10)$ invariant tensor is decomposed as

\[ \eta_{\vec{M}\vec{N}} = \begin{bmatrix} \eta_{\vec{M}\vec{N}} & 0 \\ 0 & \eta_{MN} \end{bmatrix}. \]  

(3.2)

This corresponds to the choice of the invariant tensor, less conventional but more convenient for our purposes,

\[ \eta_{MN} = \begin{bmatrix} 0 & 1_{D \times D} & 0 & 0 \\ 1_{D \times D} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{d \times d} \\ 0 & 0 & 1_{d \times d} & 0 \end{bmatrix}. \]  

(3.3)

The section condition remains the same,

\[ \eta^{MN} \partial_M \otimes \partial_N = \eta^{\vec{M}\vec{N}} \partial_{\vec{M}} \otimes \partial_{\vec{N}} + \eta^{MN} \partial_M \otimes \partial_N = \tilde{\partial}^m \otimes \partial_m + \tilde{\partial}^\mu \otimes \partial_\mu. \]  

(3.4)

In what follows, we always assume that $\tilde{\partial}^\mu = 0$, and the full $O(10,10)$ section constraint passes into the section constraint of the split $D+(d+d)$ DFT.

To decompose the fields, we impose the standard split ansatz for the metric and the 2-form field,

\[ G_{\mu \nu} = g_{\mu \nu} - A_{\mu}^m A_{\nu}^n g_{mn}, \]
\[ B_{\mu \nu} = b_{\mu \nu} - 2b_{m[\mu} A_{\nu]}^m + A_{\mu}^m A_{\nu}^n b_{mn}, \]
\[ G_{\mu m} = A_{\mu}^n g_{mn}, \]
\[ B_{\mu m} = b_{\mu m} + A_{\mu}^m b_{mn}, \]
\[ G_{mn} = g_{mn}, \]
\[ B_{mn} = b_{mn}. \]  

(3.5)
This implies that the full generalized vielbein has the block form

\[
\hat{E}_{\tilde{M}}^{\tilde{N}} = \begin{pmatrix}
\begin{array}{c|c|c|c}
\varepsilon_{\alpha} & A_{\alpha \beta}^M e_{\beta} & -b_{\mu \rho} e_{\beta} & -A_{\rho}^M e_{\beta} \\
0 & e_{\alpha} & -\frac{1}{2} A_{\mu}^M A_{\rho M} e_{\beta} & (-b_{\mu \rho} - A_{\rho}^M b_{\mu \rho}) e_{\beta} \\
0 & 0 & 0 & 0 \\
0 & 0 & -A_{\rho}^M e_{\beta} & e_{\beta} \\
\end{array}
\end{pmatrix}
\] .
\quad (3.6)

Rearranging rows and columns in the same way as for the tensor \( \eta_{\tilde{M} \tilde{N}} \), we obtain

\[
\hat{E}_{\tilde{M}}^{\tilde{N}} = \begin{pmatrix}
\begin{array}{c|c|c|c}
\varepsilon_{\alpha} & A_{\alpha \beta}^M e_{\beta} & A_{\mu}^M E_{N}^A & A_{\mu}^N E_{N}^A \\
0 & e_{\alpha} & 0 & 0 \\
0 & 0 & 0 & e_{\beta} \\
0 & 0 & -A_{\rho}^M e_{\beta} & E_{M}^A \\
\end{array}
\end{pmatrix}
\] =
\quad (3.7)

For the inverse vielbein, we have

\[
(\hat{E}^{-1})_{\tilde{N}}^{\tilde{M}} = \begin{pmatrix}
\begin{array}{c|c|c|c}
\varepsilon_{\alpha} & A_{\alpha \beta}^M e_{\beta} & -A_{\rho}^M e_{\beta} \\
0 & e_{\alpha} & 0 & 0 \\
0 & 0 & 0 & e_{\beta} \\
0 & 0 & A_{\alpha}^N E_{N}^A & E_{A}^M \\
\end{array}
\end{pmatrix}
\] .
\quad (3.8)

The NS fluxes are encoded in the generalized DFT torsion \( \mathcal{F}_{M N K} \), which is also decomposed under the split. For the generalized vielbein above, the generalized flux components

\[
\mathcal{F}_{\alpha}^{\beta \gamma}, \quad \mathcal{F}_{\alpha}^{\gamma \beta}, \quad \mathcal{F}_{\alpha}^{\alpha A}, \quad \mathcal{F}_{\alpha}^{A B}
\] (3.9)

vanish identically. For the others, we have

\[
\mathcal{F}_{\mu \nu \rho} = 3(D_{\mu} b_{\nu \rho}) + A_{\mu \nu}^M \partial_{\rho} A_{\rho M} - A_{\mu \nu}^M A_{\rho N} \partial_{N} A_{\rho M},
\]

\[
= 3(D_{\mu} b_{\nu \rho}) + A_{\mu \nu}^M \partial_{\rho} A_{\rho M} - A_{\mu}^M [A_{\nu}, A_{\rho}]_{M},
\]

\[
\mathcal{F}_{\mu \nu \rho} = 2e_{\rho}^\alpha D_{[\mu} e_{\nu]}^\alpha,
\]

\[
\mathcal{F}_{\mu \nu}^M = 2\partial_{[\mu} A_{\nu]}^M - [A_{\mu}, A_{\nu}]_{C}^M,
\]

\[
\mathcal{F}_{M \alpha}^{\beta} = 3e_{\mu}^\beta \partial_{M} e_{\mu}^\alpha = -\mathcal{F}_{M}^{\beta \alpha},
\]

\[
\mathcal{F}_{\mu N} = 6\delta_{[M} A_{N]} - 3E_{[N}^A \partial_{A} E_{A M]} + 3A_{\mu}^K E_{[N}^A \partial_{K} E_{A M]} = -3E_{[N}^A D_{\mu} E_{A M]},
\]

\[
\mathcal{F}_{ABC} = -3E_{[A}^M E_{B}^N \partial_{M} E_{C]}^N,
\]

where \([\cdot, \cdot]_C\) is the generalized Lie bracket and we define \( D_{\mu} = \partial_{\mu} - \mathcal{L}_{A_{\mu}} \). This \( D_{\mu} \) is the standard covariant derivative along the “external” coordinates \( \{x^\mu\} \) of the exceptional field theory [33], which is needed to keep the theory covariant in the split form.
The field content of the split-form DFT can be summarized as

\[ g_{\mu\nu}, \ b_{\mu\nu}, \ A_\mu^M, \ H_{MN}, \ d. \]  

(3.11)

The structure of the theory is the same as that of the exceptional field theory, and the result obtained here can therefore be expanded to the fields of 11-dimensional supergravity. In particular, we are interested in the tadpole cancellation conditions coming from exotic branes of M-theory. For that, we analyze the Bianchi identities of the split-form DFT fluxes listed above.

4. Bianchi identities and sources

The standard procedure when constructing the split-form DFT is to impose the Bianchi identities

\[ \partial [M F_{NKL}] + \frac{3}{4} F_{[MN} P F_{KL] P} = T_{M N K L}, \]

\[ D_{[\mu H_{\nu \rho \sigma]} - \frac{3}{4} F_{[\mu \nu}^M F_{\rho \sigma] M} = T_{\mu \nu \rho \sigma}, \]  

(4.1)

\[ D_{[\mu F_{\nu \rho \sigma}] M} + \partial_M H_{\mu \nu \rho} = T_{\mu \nu \rho M}, \]

where the first line represents in the full theory and the others describe the interaction of potentials (2.21) with NS 5-branes of different orientations [14]. To cover the full set of potentials, we would in addition expect to have Bianchi identities of the forms

\[ D_{[\mu \Phi_{\nu}] MN} + \partial_{[M F_{\nu N}] + \cdots = T_{\mu \nu MN}}, \]

\[ D_{\mu F_{MNK}} + \partial_{[M \Phi_{\mu NK}] + \cdots = T_{\mu MNK}}, \ etc., \]  

(4.2)

with fields \( \Phi_{\mu MN} \), which contain the components \( H_{\mu nn} \) of the full Kalb–Ramond field strength in the decomposition \( 10 = D + d \).

The most straightforward way to derive such identities is to start with the \( O(10, 10) \) theory with only Bianchi identities (2.11) and to reduce it to a \( D+(d+d) \) theory. The reduction gives all the Bianchi identities and defines the corresponding fields. We therefore start with the full covariant Bianchi identities of the \( O(10, 10) \) DFT

\[ Z_{MNKL} = \partial_{[M F_{NKL}] - \frac{3}{4} F_{[MN} P F_{KL] P}. \]  

(4.3)

For simplicity, we as before consider the DFT in the B-frame, which implies that the fluxes

\[ F_{\alpha \beta \gamma}, \ F_{\alpha \beta}^\gamma, \ F_{\alpha}^{\beta A}, \ F_{\alpha A} \]

vanish. Decomposing the full Bianchi identities, we obtain

\[ Z_{ABCD} = E_{[A}^M \partial_M F_{BCD]} - \frac{3}{4} F_{[AB} F_{CD] E}, \]

\[ Z_{\alpha ABC} = e_\alpha^\mu D_\mu F_{ABC} - 3 E_{[A}^M \partial_M F_{\alpha BC]} - 3 F_{[AB} f_{\alpha C] D} + 3 F_{[AB} F_{C}^\beta \alpha, \]

\[ Z_{\alpha \beta AB} = 3 e_\alpha^\mu D_\mu F_{\beta AB} + 3 E_A^M \partial_M F_{\alpha \beta B} - \frac{3}{2} F_{[AB} F_{\alpha \beta C} - \frac{3}{2} F_{\gamma AB} F_{\alpha \beta} - \]

\[ - 3 F_{A \alpha} F_{\beta BC} + 6 F_{\gamma A \alpha} F_{\beta \gamma B}, \]  

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\[ Z_{A\beta}^\alpha = 6E_A^M \partial_M F_{B\alpha\beta} - 3F_{C\alpha}^\beta F_{CAB} - 6F_{[B\gamma}^\beta F_{A\alpha\gamma]}, \]

\[ Z_{\alpha\beta\gamma A} = 3E_A^\mu D_\mu F_{\beta\gamma|A} - E_A^M \partial_M F_{\alpha\beta\gamma} + 3F_{B\alpha\gamma B} + 3F_{A[\alpha}^{\delta} F_{\beta]\gamma]^{\delta} - 3F_{A[\alpha}^{\delta} F_{\beta]\gamma]}^\delta, \]

\[ Z_{\alpha\beta\gamma} = -6e^{\alpha}_{\mu} D_\mu F_{\gamma A\beta} + 3E_A^M \partial_M F_{\alpha\beta\gamma} + 6F_{B\gamma}^\alpha F_{\beta]BA} + 6F_{A[\alpha}^{\delta} F_{\beta]\gamma]^{\delta} + 3F_{A\beta}^{\gamma} F_{A\beta\gamma}, \]

\[ Z_{\alpha\beta\gamma\delta} = e^{\alpha}_{\mu} D_\mu F_{\gamma\delta A} - \frac{3}{4} F_{[\alpha\beta} F_{\gamma\delta]}^\epsilon - \frac{3}{4} F_{[\alpha\beta} F_{\gamma\delta]}^\epsilon, \]

\[ Z_{\alpha\beta\gamma\delta} = 3e^{\alpha}_{\mu} D_\mu F_{\gamma\delta A} - 3F_{A\alpha}^{\beta} F_{\gamma\epsilon}^\delta + \frac{9}{4} F_{A\beta}^{\gamma} F_{\gamma\epsilon}, \]

\[ Z_{\alpha\beta\gamma\delta} = 6F_{[\gamma A}^{\alpha} F_{\beta\delta]}_{A\beta}. \]

For constant fluxes, Bianchi identities involving fluxes with only doubled indices are equivalent to quadratic constraints of the \( D \)-dimensional half-maximal gauged supergravity.

Based on the potentials in the covariant Wess–Zumino action for NS 5-branes as constructed in [32], we conclude that in the 10-dimensional space split as \( 10 = D + d \), these potentials are sourced in the corresponding \( O(d, d) \) theory by differently oriented branes [14]. For definiteness, we consider the case \( D = 6 \) and list all options for the DFT monopole:

\[
\begin{array}{cccccc|cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\times & \times & | & \times & \times & \times & D_{(2)}, & F_{(3)} \\
\times & \times & | & \times & \times & \times & D_{(3),M}, & F_{(2)}^M \\
\times & \times & \times & | & \times & \times & D_{(4)MN}, & F_{(1)}^{MN} \\
\times & \times & \times & \times & | & \times & D_{(5),MNK}, & F_{(0),MNK} \\
\times & \times & \times & \times & \times & | & D_{(6),MNLK}, & F_{(0),MNLK} \\
\end{array}
\]

where \( \times \) denotes the world-volume directions and an empty space denotes transverse directions none of which is the Taub–NUT direction of the monopole. The directions \( \{6, 7, 8, 9\} \) are doubled. The corresponding magnetic gauge potentials represented by a \( p \)-form \( D_{(p)M_1...M_p} \) with \( q \) antisymmetrized \( O(d, d) \) indices are listed in the first column on the right in (4.6). The second column contains the corresponding field strengths, of which all but the last are just the de Rahm differential of the \( p \)-form gauge potential.

The top form \( D_{(0)MNLK} \) cannot have a field strength of such a form, but it was shown in [30] that

\[ F_{(0)MNKL} = \partial^L D_{(0)LMNK} \]

at the linearized level. Indeed, the Bianchi identity for the flux \( F_{MNKL} \) can be encoded by the additional term in the action for the brane

\[ \Delta S = \int \left( \partial_M F_{NKL} - \frac{3}{4} F_{PMN} F_{QKL} \eta^{PQ} \right) D_{(6)}^{MNKL}. \]

Hence, for each Bianchi identity, we can define the corresponding magnetic potential. This procedure produces several dual potential \( p \)-forms in \( D \) external dimensions transforming in tensor representations of
the T-duality group $O(d, d)$:

\[
\begin{align*}
D_{(D-4)}, & \quad Z_{\mu\nu\rho\sigma}, \\
D_{(D-3), M}, & \quad Z_{\mu\nu\rho M}, \\
D_{(D-2), MN}, & \quad Z_{\mu\nu MN}, \quad D_{(D-2)}, \quad Z_{\mu\nu \rho}^{\rho}, \\
D_{(D-1), M NK}, & \quad Z_{\mu M NK}, \quad D_{(D-1), M}, \quad Z_{\mu\rho M}^{\rho}, \\
D_{(D), MN KL}, & \quad Z_{MN KL}, \quad D_{(D), MN}, \quad Z_{MN \rho}^{\rho}, \quad D_{(D)}, \quad Z_{\mu\nu \mu\nu}, \\
\end{align*}
\]

(4.9)

where we provide only the trace part for the flux $F_{\mu\nu\rho}$ because its traceless part corresponds to the standard dual graviton. The first column in (4.9) gives the same magnetic potentials as listed before. The second column contains additional potentials that were shown in [32] to be sourced by NS 5-branes. In the standard picture, they are nondynamical, but they seem necessary to ensure the gauge invariance of the Wess–Zumino action. They correspond to roots of zero length in the decomposition of the fundamental representation of $E_{11}$ under the subalgebra $O(d, d)$.

Finally, the last column contains the potential $D_{(D)}$, which must correspond to the Bianchi identity $Z_{\mu\nu \mu\nu}$ and which was also observed among the gauge potentials interacting with nonstandard branes. But because the corresponding Bianchi identity does not contain a derivative, it is not completely clear how to define the field strength for such a gauge potential. This corresponds to roots of negative length squared in the decomposition.

5. Discussion

Because T-duality always doubles the number of coordinates, we can develop a fully T-duality-covariant theory for 10-dimensional supergravity, which is a theory on the $(10+10)$-dimensional doubled space. The same does not seem possible for U-duality, because already for two-dimensional supergravity, the U-duality group is $E_9$, which is an infinite-dimensional affine algebra. The full 11-dimensional theory would then have $E_{11}$ as the local symmetry group, whose fundamental representation is infinite, and the number of coordinates and fields is hence infinite. As a result, we cannot write Bianchi identities for the full $E_{11}$ extended field theory as simply as (2.11). Therefore, the split-DFT construction is of interest: it has the same structure as extended field theories, and it allows drawing certain general conclusions about Bianchi identities of such theories and constructing magnetic gauge potentials interacting with nonstandard branes of M-theory and the corresponding field strengths.

The problem of defining gauge potentials for exotic branes of M-theory and their covariant field strengths was discussed in [34] for the $6^{(3,1)}$-brane. The background for such a brane was obtained as a U-dual of the KK6-monopole background in the $SL(5)$ exceptional field theory. The $6^{(3,1)}$-brane and the KK6-monopole belong to the U-duality orbit interacting with the 7-form potentials in the $5, 45,$ and $70$ of $SL(5)$. At the linearized level, their derivatives along the extended space that transform under the $10$ of $SL(5)$ give field strengths in $10, 15,$ and $40$, which presumably correspond to the nonconstant gaugings of $D = 7$ maximal supergravity:

\[
\begin{align*}
\partial_{(10)}^{\mu_1 \ldots \mu_7} \quad (5) & \rightarrow \mathcal{F}^{(10)} + \mathcal{F}^{(40)} \iff (\theta_{mn}, Z^{mn,k}), \\
\partial_{(10)}^{\mu_1 \ldots \mu_7} A_{(45)}^{\mu_1 \ldots \mu_7} & \rightarrow \mathcal{F}^{(10)} + \mathcal{F}^{(15)} \iff (\theta_{mn}, Y_{mn}), \\
\partial_{(10)}^{\mu_1 \ldots \mu_7} A_{(70)}^{\mu_1 \ldots \mu_7} & \rightarrow \mathcal{F}^{(10)} + \mathcal{F}^{(40)} \iff (Z^{mn,k}).
\end{align*}
\]

(5.1)
Bianchi identities for the $SL(5)$ extended field theory from which we could derive the desired gauge potentials are not known explicitly. Bianchi identities of the split-form DFT as described above provide a guiding principle for constructing these identities in extended field theory.

Indeed, we consider the nonzero fluxes of the split-DFT in the B-frame,

\[ \mathcal{F}_{\mu\nu\rho}, \quad \mathcal{F}^{\mu\nu}, \quad \mathcal{F}_{\mu}^{\nu M}, \quad \mathcal{F}_{ABC}, \quad \mathcal{F}_{M\alpha\beta}, \quad \mathcal{F}_{\mu}^{MN}. \]  

(5.2)

Because all these components came from the generalized flux $\mathcal{F}_{MNK}$ of the full $O(10,10)$ DFT, they can also be regarded as fluxes or field strengths in the $D+(d+d)$ DFT. The first four fluxes in (5.2) are indeed usually understood as field strengths for the corresponding fields $B_{\mu\nu}$, $e_{\mu}^{\alpha}$, $A_{\mu}^{M}$, and $E_{\mu}^{A}$. The field strength of the dual graviton is linearized in both the normal and doubled directions. The last two fluxes in (5.2) are spin-connections for the local Lorenz group $\mathcal{F}_{M\alpha\beta}$ and the local group of the generalized diffeomorphisms $\mathcal{F}_{\mu}^{MN}$. It hence follows that the full set of Bianchi identities of exceptional field theory must include connections and their derivatives understood as proper field strengths. Some progress in this direction was made in [35], where dynamical fluxes of the scalar sector of the $SL(5)$ extended field theory were analyzed using the Courant brackets of the type-II fluxes. The methods described in the present paper support extending these results to the full exceptional field theory including fluxes of the tensor sector.

The second observation is that of Bianchi identities (4.5), those that contain no derivative have the same form as the quadratic constraints of half-maximal supergravity. This allows writing the Bianchi identities directly as the quadratic constraints of the corresponding maximal supergravity supplemented by terms containing derivatives of gaugings. We thus pass to dynamical nongeometric fluxes and understand them as proper field strengths.

Conflicts of interest. The author declares no conflicts of interest.

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