New approaches for Schrödinger equations with prescribed mass: The Sobolev subcritical case and
The Sobolev critical case with mixed dispersion

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Abstract

In this paper, we prove the existence of normalized solutions for the following Schrödinger equation
\[-\Delta u - \lambda u = f(u), \quad x \in \mathbb{R}^N,\]
\[\int_{\mathbb{R}^N} u^2 dx = c\]
with $N \geq 3$, $c > 0$, $\lambda \in \mathbb{R}$ and $f \in C(\mathbb{R}, \mathbb{R})$ in the Sobolev subcritical case with weaker $L^2$-supercritical conditions and in the Sobolev critical case when $f(u) = \mu |u|^{q-2}u + |u|^{2^*-2}u$ with $\mu > 0$ and $2 < q < 2^* = \frac{2N}{N-2}$ allowing to be $L^2$-subcritical, critical or supercritical. Our approach is based on several new critical point theories on a manifold, which not only help to weaken the previous $L^2$-supercritical conditions in the Sobolev subcritical case, but present an alternative scheme to construct bounded (PS) sequences on a manifold when $f(u) = \mu |u|^{q-2}u + |u|^{2^*-2}u$ technically simpler than the Ghoussoub minimax principle [7] involving topological arguments, as well as working for all $2 < q < 2^*$. In particular, we propose new strategies to control the energy level in the Sobolev critical case which allow to treat, in a unified way, the dimensions $N = 3$ and $N \geq 4$, and fulfill what were expected by Soave [13] and by Jeanjean-Le [10]. We believe that our approaches and strategies may be adapted and modified to attack more variational problems in the constraint contexts.

Keywords: Nonlinear Schrödinger equation; Normalized solution; Sobolev critical growth; Mixed nonlinearities.

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1 Introduction

This paper is concerned with the nonlinear Schrödinger equation with an $L^2$-constraint

\[
\begin{cases}
-\Delta u - \lambda u = f(u), & x \in \mathbb{R}^N, \\
\int_{\mathbb{R}^N} u^2 \, dx = c,
\end{cases}
\]  

(1.1)

where $N \geq 3$, $f \in C(\mathbb{R}, \mathbb{R})$, $c > 0$ is a given mass, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier which depends on the solution $u \in H^1(\mathbb{R}^N)$ and is not a priori given.

The main feature of (1.1) is that the desired solutions have a priori prescribed $L^2$-norm, which are often referred to as normalized solutions in the literature, that is, for given $c > 0$, a couple $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ solves (1.1). From the physical viewpoint, these solutions often offer a good insight of the dynamical properties of the stationary solutions, such as the orbital stability or instability, see, for example, [3, 12, 13]. This type of problem has attracted much attention in the community of nonlinear PDEs in the last decades. Under mild conditions on $f$, one can introduce the $C^1$-functional $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

\[
\Phi(u) = \frac{1}{2} \| \nabla u \|_2^2 - \int_{\mathbb{R}^N} F(u) \, dx,
\]  

(1.2)

where $F(t) = \int_0^t f(s) \, ds$ for $t \in \mathbb{R}$. It is standard that for prescribed $c > 0$, a solution of (1.1) can be obtained as a critical point of the functional $\Phi$ constrained to the sphere

\[
S_c := \{ u \in H^1(\mathbb{R}^N) : \| u \|_2^2 = c \}.
\]  

(1.3)

As we know, the study of (1.1) depends on the behavior of the nonlinearity $f$ at infinity, which gives rise to a new $L^2$-critical exponent $\tilde{q} := 2 + \frac{4}{N}$, coming from the Gagliardo-Nirenberg inequality (see [5, Theorem 1.3.7]). One speaks of a $L^2$-subcritical case if $\Phi$ is bounded from below on $S_c$ for any $c > 0$, and of a $L^2$-supercritical case if $\Phi$ is unbounded from below on $S_c$ for any $c > 0$. One also refers to a $L^2$-critical case when the boundedness from below does depend on the value $c > 0$. We say that $u$ is a ground state solution to (1.1) if it is a solution having minimal energy among all the solutions which belong to $S_c$. Compared with the $L^2$-subcritical case, more efforts are always needed in the study of the $L^2$-critical and $L^2$-supercritical cases.

In this paper, we focus on not only normalized solutions of (1.1) in the $L^2$-critical and $L^2$-supercritical cases, but a more complicated situation when $f(u) = \mu |u|^{q-2} u + |u|^{2^*-2} u$ with $\mu > 0$ and $2 < q < 2^* := \frac{2N}{N-2}$, namely the following equation with Sobolev critical exponent and mixed dispersion:

\[
\begin{cases}
-\Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{2^*-2} u, & x \in \mathbb{R}^N, \\
\int_{\mathbb{R}^N} u^2 \, dx = c,
\end{cases}
\]  

(1.4)
where, particularly, $q$ is allowed to be $L^2$-subcritical $2 < q < 2 + \frac{4}{N}$, $L^2$-critical $q = 2 + \frac{4}{N}$, or $L^2$-supercritical $2 + \frac{4}{N} < q < 2^*$. Let us describe the relevant works below that motivate our researches.

1.1 $L^2$-supercritical problem (1.1)

The first contribution to the $L^2$-supercritical case was made by Jeanjean [8], where a radial solution of mountain pass type to (1.1) was found under the following conditions:

(H0) $f$ is odd;

(H1) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist $\alpha, \beta \in \mathbb{R}$ satisfying $2 + \frac{4}{N} \leq \alpha < \beta < 2^* = \frac{2N}{N-2}$ such that

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t), \quad \forall \ t \in \mathbb{R} \setminus \{0\};$$

moreover a ground state solution was obtained if $f$ also satisfies

(H2) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $C^1$ and

$$\tilde{F}'(t)t > \left(2 + \frac{4}{N}\right)F(t), \quad \forall \ t \in \mathbb{R} \setminus \{0\}.$$ 

Although the mountain pass geometry of $\Phi$ implies the existence of a Palais-Smale sequence $\{u_n\}$ (a (PS) sequence for short) at the mountain pass level, such a sequence may not be bounded. To overcome this difficulty, Jeanjean constructed a good (PS) sequence having additional property related with the Pohozaev type identity, namely $P(u_n) \to 0$, where $P : H^1(\mathbb{R}^2) \to \mathbb{R}$ is defined by

$$P(u) := \|\nabla u\|^2_2 - \frac{N}{2} \int_{\mathbb{R}^N} [f(u)u - 2F(u)]dx.$$ 

(1.6)

The inspiring part of the proofs is the application of the Ekeland principle to the fibering map $\tilde{\Phi} : H^1(\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{\Phi}(v, t) := \frac{e^{2t}}{2} \|\nabla v\|_2^2 - \frac{1}{e^{Nt}} \int_{\mathbb{R}^N} F\left(e^{Nt/2}v\right) \ dx,$$ 

(1.7)

whose mountain pass level on $\mathcal{S}_c \times \mathbb{R}$ equals to the one of $\Phi$ on $\mathcal{S}_c$. In particular, the first part of (1.5) in (H1) was used in a technical but essential way in showing not only the mountain pass geometry but the boundedness of the above (PS) sequence $\{u_n\}$ due to $\alpha \Phi(u_n) + o(1) = \alpha \Phi(u_n) - P(u_n)$. It is reminiscent of the the classical Ambrosetti-Rabinowitz condition ((AR)-condition for short) introduced in [2] for the unconstrained superlinear problem:

$$-\Delta u + u = f(u) \quad \text{in} \quad \mathbb{R}^N.$$ 

However, in contrast to unconstrained problems, the first part of (1.5), as an analogue of (AR)-condition for (1.8), is required in almost of the studies for the search of normalized
solutions. It was not until 2022 that the first part of (1.3) was relaxed by Jeanjean-Lu [11] to the condition:

\[(H3) \lim_{t \to 0} f(t)/t^{1+\frac{4}{N}} = 0 \text{ and } \lim_{t \to \infty} F(t)/t^{2+\frac{4}{N}} = +\infty, \]

in addition, provided the following monotonicity condition is satisfied:

\[(H4) [f(t)t - 2F(t)]/|t|^{1+\frac{4}{N}} t \text{ is nondecreasing on } (-\infty, 0) \text{ and } (0, +\infty). \]

Particularly, ground state solutions were found by developing new robust arguments on the Pohozaev manifold defined by

\[M(c) := \{u \in S_c : \mathcal{P}(u) = 0\}, \quad (1.9)\]

combining the techniques due to Szulkin and Weth [14, 15] with the mini-max approach in \(M(c)\) introduced by Ghoussoub [7]. To our knowledge, there seems to be little progress in this direction with the exception of [11]. Note that (H4) plays an analogous role as the Nehari-type condition for (1.8) for the search of ground state solutions. On the contrary, for the unconstrained problem (1.8), (AR)-condition can be weakened to more general superlinear conditions without the Nehari-type condition. Thus, a natural question arises:

(Q1) Can we weaken (H1) to more general \(L^2\)-supercritical conditions without imposing the monotonicity property on \(f\) like (H4)?

In the first part of this paper, we shall not only positively answer to the above question, but also provide a new approach to construct a bounded (PS) sequence of \(\Phi|_{S_c}\) at the mountain pass level. Before stating the result in this direction, let us introduce the following conditions:

(F0) \(f \in C(\mathbb{R}, \mathbb{R})\) and

\[\lim_{t \to 0} \frac{f(t)}{t} = 0, \quad \lim_{|t| \to +\infty} \frac{|f(t)|}{|t|^{2^* - 1}} < +\infty;\]

(F1) \(f \in C(\mathbb{R}, \mathbb{R})\) and

\[\lim_{t \to 0} \frac{f(t)}{|t|^{1+\frac{4}{N}}} = 0, \quad \lim_{|t| \to +\infty} \frac{|f(t)|}{|t|^{2^* - 1}} = 0;\]

(F2) \(0 < (2 + \frac{4}{N}) F(t) \leq f(t)t < \frac{2N}{N-2} F(t)\) for all \(t \in \mathbb{R} \setminus \{0\}\);

(F3) there exists \(\kappa > \frac{N}{2}\) such that

\[\limsup_{|t| \to +\infty} \frac{[f(t)t - 2F(t)]^\kappa}{t^{2\kappa}[Nf(t)t - (2N + 4)F(t)]} < +\infty;\]

(F3’) there exist \(\kappa > \frac{N}{2}\) and \(C_0 > 0\) such that

\[\left(\frac{f(t)t - 2F(t)}{t^2}\right)^\kappa \leq C_0[Nf(t)t - (2N + 4)F(t)], \quad \forall t \in \mathbb{R} \setminus \{0\}.\]
Our idea of weakening (H1) is somehow inspired by Ding [6] where the unconstrained super-linear problem (1.8) was considered. Here comes the first result of this paper.

**Theorem 1.1.** Let $c > 0$.

(i) If $f$ satisfies (F1)-(F3), then (1.1) admits a couple solution $(u_c, \lambda_c) \in H^1_{\text{rad}}(\mathbb{R}^N) \times (-\infty, 0)$.

(ii) If $f$ satisfies (F1), (F2) and (F3'), then (1.1) admits a couple solution $(\bar{u}_c, \lambda_c) \in H^1_{\text{rad}}(\mathbb{R}^N) \times (-\infty, 0)$ such that $\Phi(u_c) = \inf_{K_c} \Phi$, where

\[ K_c := \{ u \in S_c \cap H^1_{\text{rad}}(\mathbb{R}^N) : \Phi|_{S_c}(u) = 0 \}. \]

**Remark 1.2.** (i) As pointed out by Jeanjean-Lu [11, Remark 1.1 (i)], there are no existence results on (1.1) so far without imposing related conditions as (H2) or (H4). There are many functions $f(t)$ which satisfy (F1)-(F3) but do neither (H2) nor (H4). In this sense, Theorem 1.1 seems to be new, and improves and extends the related results on normalized solutions in the literature.

(ii) In the proof of Theorem 1.1 inspired by [4, 8, 18], we develop new critical point theories on a manifold (see Lemmas 2.4 and 2.13, Theorems 2.5 and 2.14, Corollaries 2.6 and 2.15), which help to generate a bounded (PS)-sequence of $\Phi|_{S_c}$ at the mountain pass level. They may be considered as counterparts of the deformation lemma and general minimax principle due to Willem [18] in the constraint context. We believe that these theories may be adapted and modified to attack more variational problems in the constraint contexts.

### 1.2 Mixed problem (1.4) with Sobolev critical exponent

The second part is devoted to the study of normalized solutions for a more complex problem (1.4) with Sobolev critical exponent and mixed dispersion, which is the heart of this paper. The study of such a problem is a very active topic nowadays, and can be as a counterpart of the Brezis-Nirenberg problem in the context of normalized solutions. It is well-known that solutions of (1.4) are critical points of the functional

\[ \Phi_\mu(u) = \frac{1}{2} \| \nabla u \|_2^2 - \frac{1}{2^*} \| u \|_{2^*}^{2^*} - \frac{\mu}{q} \| u \|_q^q, \quad \forall u \in H^1(\mathbb{R}^2) \]  (1.10)

on the constraint $S_c$, and $\Phi_\mu$ is unbounded from below on $S_c$ due to $2^* = \frac{2N}{N-2} > 2 + \frac{N}{4}$. Similarly as (1.6) and (1.9), we define the functional

\[ P_\mu(u) := \| \nabla u \|_2^2 - \| u \|_{2^*}^{2^*} - \mu \gamma_q \| u \|_q^q, \quad \forall u \in H^1(\mathbb{R}^N) \]  (1.11)
and the Pohozaev manifold

\[ \mathcal{M}_\mu(c) := \{ u \in \mathcal{S}_c : \mathcal{P}_\mu(u) = 0 \}. \]  (1.12)

Compared with the previous \( L^2 \)-supercritical problem (1.1), the study of (1.4) is more delicate since we have to not only carefully analyse how a lower order term \(|u|^{q-2}u\) affects the structure of the constrained functional \( \Phi_\mu|_{\mathcal{S}_c} \), but also solve the lack of compactness caused by Sobolev critical growth. This problem was firstly studied by Soave \cite{13}. Based on the fibration method of Pohozaev relying on the decomposition of Pohozaev manifold

\[ \mathcal{M}_\mu(c) = \left\{ u \in \mathcal{S}_c : \tilde{\phi}_u'(0) = 0 \right\} = \mathcal{M}^-_\mu(c) \cup \mathcal{M}^0_\mu(c) \cup \mathcal{M}^+_\mu(c), \]  (1.13)

where \( \tilde{\phi}_u(t) = \Phi(e^{Nt/2}u(e^t x)) \) for \( u \in H^1(\mathbb{R}^N) \) and \( t \in \mathbb{R} \),

\[ \mathcal{M}^\pm_\mu(c) := \left\{ u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) \gtrless 0 \right\} \quad \text{and} \quad \mathcal{M}^0_\mu(c) := \left\{ u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) = 0 \right\}. \]  (1.14)

Soave proved the following results:

**Theorem [S] (\cite{13, Theorems 1.1})** There exists a constant \( \alpha(N, q) > 0 \) depending on \( N, q, \gamma_q := \frac{N(q-2)}{2q} \) and the best constant for the Gagliardo-Nirenberg inequality \( C_{N,q} \) (see (1.23)) such that, if \( \mu(c) < \alpha(N, q) \), (1.4) has a ground state solution \( \tilde{u} \). Furthermore,

(i) if \( 2 < q < 2 + \frac{4}{N} \), \( \tilde{u} \) corresponds to a local minimizer satisfying \( \inf_{\mathcal{M}^+_\mu(c)} \Phi_\mu = \Phi_\mu(\tilde{u}) < 0 \);

(ii) if \( 2 + \frac{4}{N} \leq q < 2^* \), \( \tilde{u} \) is stable and characterized as a solution of mountain-pass type with \( 0 < \inf_{\mathcal{M}^-_\mu(c)} \Phi_\mu = \Phi_\mu(\tilde{u}) < \frac{1}{N} S^N \), where and in the sequel, \( S \) denotes the best constant for the Sobolev inequality (see (1.22)).

Subsequently, by introducing a set \( V(c) := \{ u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \rho_0 \} \) having the property that

\[ m_\mu(c) := \inf_{u \in V(c)} \Phi_\mu(u) < 0 < \inf_{u \in \partial V(c)} \Phi_\mu(u) \quad \text{with} \quad \partial V(c) := \{ u \in \mathcal{S}_c : \|\nabla u\|_2^2 = \rho_0 \}, \]  (1.15)

where \( \rho_0 \) and \( c_0 \) are given as follows:

\[ \rho_0 := \left[ \frac{2^* \mu \alpha_0 c_{N,q}^q S^{2^*/2}}{q \alpha_2} \right]^{\frac{\alpha_2}{\alpha_0 + \alpha_2}} \frac{\alpha_1}{c_{\alpha_0 + \alpha_2}} \]  (1.16)

and

\[ c_0 := \left[ \frac{2^* \alpha_0 S^{2^*/2}}{\alpha_0 + \alpha_2} \left( \frac{q \alpha_2}{2^* \mu \alpha_0 c_{N,q}^q S^{2^*/2}} \right) \right]^{\frac{\alpha_2}{\alpha_0 + \alpha_2}} \]  (1.17)

with

\[ \alpha_0 := 2 - \frac{N(q-2)}{2}, \quad \alpha_1 := \frac{2N - q(N-2)}{2}, \quad \alpha_2 := \frac{4}{N-2}. \]  (1.18)
Jeanjean-Jendrej-Le-Visciglia \cite{9} proved that for any $c \in (0, c_0)$, the set of ground state solutions is orbitally stable for the case $2 < q < 2 + \frac{4}{N}$. Note that such a structure of local minima for the case $2 < q < 2 + \frac{4}{N}$, suggests the possibility to search for a solution lying at a mountain pass level, which was proposed by Soave \cite{13} Remark 1.1 as a conjecture. Recently, this conjecture has been confirmed by Jeanjean-Le \cite{10} and Wei-Wu \cite{17}. Let us now recall the results obtained there. Based on the same decomposition (2.2) as that in \cite{13}, with some new energy estimates, Wei-Wu \cite{17} complemented the results of the above Theorem \cite{S} from three respects:

- in the case $2 < q < 2 + \frac{4}{N}$, a second solution $u_{\mu,c}^- \in \mathcal{M}_\mu^-(c)$ with $\Phi_\mu(u_{\mu,c}^-) = \inf_{\mathcal{M}_\mu^-(c)} \Phi_\mu$ was found if $\mu c^{\frac{(1-\gamma)q}{2}} < \alpha(N, q)$ which satisfies $0 < \Phi_\mu(u_{\mu,c}^-) < \inf_{\mathcal{M}_\mu^-(c)} \Phi_\mu + \frac{1}{N}S_N^2$;

- in the case $q = 2 + \frac{4}{N}$, it was shown that (1.4) has no ground state solutions for $\mu c^{\frac{(1-\gamma)q}{2}} \geq \alpha(N, q)$;

- in the case $2 + \frac{4}{N} < q < 2^*$, the existence range that $\mu c^{\frac{(1-\gamma)q}{2}} < \alpha(N, q)$ was extended to all $c > 0$.

Instead of the fibration method of Pohozaev used in \cite{13} \cite{17}, by introducing a new set of mountain pass level, directly connected with the decomposition

$$
\mathcal{M}_\mu(c) = \widehat{\mathcal{M}}^-\mu(c) \cup \widehat{\mathcal{M}}^+\mu(c) \quad \text{with} \quad \widehat{\mathcal{M}}^\pm\mu(c) := \{ u \in \mathcal{M}_\mu(c) : \Phi_\mu(u) \geq 0 \},
$$

and studying its relation with $V(c)$, Jeanjean-Le \cite{10} proved that for any $c \in (0, c_0)$ ($c_0$ is given in (1.17)) and $N \geq 4$, there exists a second solution $v_c \in \mathcal{S}_c$ of mountain pass type satisfying $0 < \Phi_\mu(v_c) < m_\mu(c) + \frac{1}{N}S_N^2$, which is not a ground state solution, where $V(c)$ and $m_\mu(c)$ are given by (1.15).

As turned out in the aforementioned papers, two ingredients in the search of a solution of mountain pass type for (1.4) are essential: I) obtaining a (PS) sequence $\{u_n\}$ at the mountain pass level having additional property $\mathcal{P}(u_n) \to 0$; II) proving the compactness of the obtained (PS) sequence $\{u_n\}$. Let us remark them in detail below.

To do item I), all of them adopted the Ghoussoub minimax principle \cite{7} on the manifold. This strategy is very effective for $2 < q < 2 + \frac{4}{N}$ as well as $2 + \frac{4}{N} \leq q < 2^*$, but involves technical topological arguments based on $\sigma$-homotopy stable family of compact subsets of $\mathcal{M}_\mu(c)$, moreover, the arguments are very complicated. Note that the minimax approach developed by Jeanjean \cite{8} works only for $2 + \frac{4}{N} \leq q < 2^*$, even though it is technically simpler. Thus, one may ask:
(Q2) If is it possible to establish a general minimax principle on the manifold not involving topological arguments, technically simpler than [7] and working for all $2 < q < 2^*$?

To do item II), the crucial point is to derive a strict upper bound of mountain pass level:

$$M_\mu(c) < \begin{cases} m_\mu(c) + \frac{1}{N}S^\frac{2}{q}, & \text{if } 2 < q < 2 + \frac{4}{N}, \\ \frac{1}{N}S^\frac{2}{q}, & \text{if } 2 + \frac{4}{N} \leq q < 2^* \end{cases}$$

(1.19)

through the use of testing functions, as firstly pointed out in [10] for $2 < q < 2 + \frac{4}{N}$ and in [13] for $2 + \frac{4}{N} \leq q < 2^*$. This strict inequality can help to guarantee that the obtained (PS) sequence does not carry a bubble which, by vanishing when passing to the weak limit, would prevent its strong convergence, like the classical Brezis-Nirenberg problem. In form, such a threshold of compactness, is an analogue of the unconstrained Sobolev critical problem (1.18) with concave-convex nonlinearities. Even if this idea to get (1.19) may somehow been generated, its achievement is rather involved since the choice of test functions is more delicate in the $L^2$-constrained context. Indeed, to do that, for $2 < q < 2 + \frac{4}{N}$ and $\mu c \frac{(1-q \gamma)}{2} \geq \alpha(N, q)$, Wei-Wu [17] used the radial superposition of a local minima $u^+_c$ and a suitable family of truncated extremal Sobolev functions located in a set where the local minima solution takes its greater values. The strategy in [17], recording of the one introduced by Tarantello [16], is that the interaction decreases the mountain pass value of $\Phi_\mu|_{S_c}$ with respect to the case where the two supports would be disjoint, moreover, it works for all dimensions $N \geq 3$. Instead of $\mu c \frac{(1-q \gamma)}{2} < \alpha(N, q)$ proposed by Soave [13], Jeanjean-Le [10] considered the range $c \in (0, c_0)$, and constructed non-radial test functions which could be viewed as the sum of a truncated extremal Sobolev function centered at the origin and of a local minima $u^+_c$ translated far away from the origin. Unlike [17], their construction aims at separating sufficiently the regions where the functions concentrate and to show that the remaining interaction can be assumed sufficiently. However, only when $N \geq 4$ does this strategy work, and it was pointed out in [10] Remark 1.17 that: It is not clear to us if this limitation is due to the approach we have developed or if the case $N = 3$ is fundamentally distinct from the case $N \geq 4$. We believe it would be interesting to investigate in that direction. Thus, a natural question arises:

(Q3) Can we construct alternative testing functions, working for all dimensions $N \geq 3$, to derive that $M_\mu(c) < m_\mu(c) + \frac{1}{N}S^\frac{2}{q}$ for $2 < q < 2 + \frac{4}{N}$ and $c \in (0, c_0)$?

Note that the energy estimate for the case $2 + \frac{4}{N} \leq q < 2^*$ is quite different. To do that, Soave [13] used truncated and normalized extremal Sobolev functions centered at the origin as test functions, and the dilations preserving the $L^2$-norm as test paths. Although this
choice appears to be natural and more easier in comparison to the case $2 < q < 2 + \frac{4}{N}$, to get the desired upper estimates, not only the cases $2 + \frac{4}{N} < q < 2^*$ and $q = 2 + \frac{4}{N}$, but the dimensions $N = 3$ and $N \geq 4$ in each case, all needed to be treated separately in the proofs. Later, the same strategy was employed by Wei-Wu [17]. For this case, we also refer to [1, Theorem 1.1], in which $\mu$ must be large enough and the mountain pass level was controlled small enough by the approximate scheme $M_{\mu}(c) \to 0$ as $\mu \to +\infty$. Now, pursuing the study of [13, 17], in the case $2 + \frac{4}{N} \leq q < 2^*$, it is natural to ask:

(Q4) Can we find an unified scheme to treat cases $2 + \frac{4}{N} < q < 2^*$ and $q = 2 + \frac{4}{N}$, as well as the dimensions $N = 3$ and $N \geq 4$ in each case?

In the second part of this paper, we are interested in Questions (Q2)-(Q4), and we shall solve them in turn by developing some new analytical strategies and techniques.

To present the abstract minimax principle on the manifold involving Question (Q2), inspired by [4, 8, 18], we develop new critical point theories on a manifold, see Lemmas 2.4 and 2.13 Theorems 2.5 and 2.14 Corollaries 2.6 and 2.15 which present a different approach to construct a bounded (PS) sequence on a manifold technically simpler than [7]. They may be considered as counterparts of the deformation lemma and general minimax principle due to Willem [18] in the constraint context. We believe, these results can be adapted to more $L^2$-constrained problems. In the proofs, we use the general deformation lemma on a manifold (see Lemma 2.4), instead of technical topological arguments of [7]. This shows that critical point theories introduced in the first part, mentioned in Remark 1.2 (ii), are of future development and applicability to some extend.

On Questions (Q3) and (Q4), we have the following results, respectively, in which $\Phi_{\mu}$, $M_{\mu}(c)$, $m_{\mu}(c)$ and $c_0$ are defined before by (1.10), (1.12), (1.15) and (1.17).

**Theorem 1.3.** Let $N \geq 3$, $2 < q < 2 + \frac{4}{N}$, $\mu > 0$ and $c \in (0, c_0)$. Then (1.4) has a second couple solution $(u_c, \lambda_c) \in H^1_{rad}(\mathbb{R}^N) \times (-\infty, 0)$ such that

$$0 < \Phi_{\mu}(u_c) < m_{\mu}(c) + \frac{1}{N} S^N_{2^*}.$$

**Theorem 1.4.** Let $N \geq 3$, $c > 0$ and $2 + \frac{4}{N} \leq q < 2^*$. Then, (1.4) has a couple solution $(\bar{u}_c, \lambda_c) \in H^1(\mathbb{R}^N) \times (-\infty, 0)$ such that

$$\Phi_{\mu}(\bar{u}_c) = \inf_{M_{\mu}(c)} \Phi_{\mu}$$

(i) for any $\mu > 0$ if $2 + \frac{4}{N} < q < 2^*$;

(ii) for any $0 < \mu \leq \frac{1}{2 \gamma q c_{N, \bar{q}}^2 N^2 c_{N, \bar{q}}^N}$ if $q = \bar{q} = 2 + \frac{4}{N}$.
Remark 1.5. Theorem 1.3 gives a second solution of mountain pass type by constructing a more simple geometry of the mountain pass than the one in [10], and Theorem 1.4 complements the results of [17, Theorem 1.1]. In particular, to derive the strict inequality (1.19), we introduce alternative choices of test functions in the cases $2 < q < 2 + \frac{4}{N}$ and $2 + \frac{4}{N} \leq q < 2^*$, which, together with subtle estimates and analyses, allows us to treat uniformly the dimensions $N \geq 3$, see Lemmas 4.4 and 4.14. Thus, Questions (Q3) and (Q4) are fully settled. We believe that our strategy of energy estimates is helpful to other $L^2$-constrained problems with Sobolev critical growth.

The paper is organized as follows. Section 2, is devoted to establish several new critical point theories on a manifold and finish the proof of Theorem 2.7, which will be applied to prove Theorems 1.1, 1.3 and 1.4. In Section 3, we study $L^2$-supercritical problem (1.1) and prove Theorem 1.1. In Section 4, we study Normalized solutions for mixed problem (1.4) with Sobolev critical exponent, and finish the proofs of Theorems 1.3 and 1.4.

Throughout the paper, we make use of the following notations:

- $H^1_{rad}(\mathbb{R}^2) := \{ u \in H^1(\mathbb{R}^2) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^N \}$;
- $L^s(\mathbb{R}^2)(1 \leq s < \infty)$ denotes the Lebesgue space with the norm $\| u \|_s = (\int_{\mathbb{R}^2} |u|^s \, dx)^{1/s}$;
- For any $N \geq 3$ there exists an optimal constant $S > 0$ depending only on $N$, such that
  \[ S \| u \|_2^2 \leq \| \nabla u \|_2^2, \quad \forall \ u \in D^{1,2}(\mathbb{R}^N) \quad (\text{Sobolev inequality}) \]  \hspace{1cm} (1.22)
  and $C_{N,s} > 0$ such that
  \[ \| u \|_s \leq C_{N,s} \| \nabla u \|_2^{\gamma_s} \| u \|_2^{1-\gamma_s}, \quad \forall \ u \in H^1(\mathbb{R}^N) \quad (\text{Gagliardo-Nirenberg inequality}); \]  \hspace{1cm} (1.23)
- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{ y \in \mathbb{R}^N : |y - x| < r \}$ and $B_r = B_r(0)$;
- $C_1, C_2, \cdots$ denote positive constants possibly different in different places.

2 Critical point theories on a manifold

In this section, we give several new critical point theorems on a manifold, and complete the proof of Theorem 2.7. We shall apply these theorems to obtain (PS) sequences restricted on a manifold and study problems (1.1) and (1.4).

Let $H$ be a real Hilbert space whose norm and scalar product will be denoted respectively by $\| \cdot \|_H$ and $\langle \cdot, \cdot \rangle_H$. Let $E$ be a real Banach space with norm $\| \cdot \|_E$. We assume throughout this section that

\[ E \hookrightarrow H \hookrightarrow E^* \]  \hspace{1cm} (2.1)
with continuous injections, where $E^*$ is the dual space of $E$. Thus $H$ is identified with its dual space. We will always assume in the sequel that $E$ and $H$ are infinite dimensional spaces. We consider the manifold

$$M := \{u \in E : \|u\|_H = 1\}.$$  \hfill (2.2)

$M$ is the trace of the unit sphere of $H$ in $E$ and is, in general, unbounded. Throughout the paper, $M$ will be endowed with the topology inherited from $E$. Moreover $M$ is a submanifold of $E$ of codimension 1 and its tangent space at a given point $u \in M$ can be considered as a closed subspace of $E$ of codimension 1, namely

$$T_u M := \{v \in E : (u, v)_H = 0\}.$$  \hfill (2.3)

We consider a functional $\varphi : E \to \mathbb{R}$ which is of class $C^1$ on $E$. We denote by $\varphi|_M$ the trace of $\varphi$ on $M$. Then $\varphi|_M$ is a $C^1$ functional on $M$, and for any $u \in M$,

$$\langle \varphi|_M'(u), v \rangle = \langle \varphi'(u), v \rangle, \quad \forall \ v \in T_u M.$$  \hfill (2.4)

In the sequel, for any $u \in M$, we define the norm $\|\varphi|_M'(u)\|$ by

$$\|\varphi|_M'(u)\| = \sup_{v \in T_u M, \|v\|_E = 1} |\langle \varphi'(u), v \rangle|.$$  \hfill (2.5)

Before this, we present some basic definitions and results. For any $\delta > 0$ and any set $A \subset E$, we denote $A_\delta := \{v \in E : \|u - v\| < \delta, \ \forall \ u \in A\}$. For any $a \in \mathbb{R}$ and any $\varphi : E \to \mathbb{R}$, we denote $\varphi^a := \{u \in E : \varphi(u) \leq a\}$.

**Definition 2.1.** [4, 18] A pseudo-gradient vector for $\varphi|_M$ at $u \in \tilde{M} := \{u \in M : \varphi|_M'(u) \neq 0\}$ is a vector $v \in T_u M$ such that

$$\|v\|_E \leq 2\|\varphi|_M'(u)\|, \quad \langle \varphi|_M'(u), v \rangle \geq \|\varphi|_M'(u)\|^2.$$  

Put $T(M) := \bigcup_{u \in M} (T_u M)$. A mapping (lifting) $g : \tilde{M} \to T(M)$ is called a pseudo-gradient vector field for $\varphi|_M$ on $\tilde{M}$ if $g$ is locally Lipschitz continuous, $g(u) \in T_u M$ for $u \in \tilde{M}$, and $g(u)$ is a pseudo-gradient vector for $\varphi|_M$ at any $u \in \tilde{M}$.

**Lemma 2.2.** [4, 18] If $\varphi|_M \in C^1(M, \mathbb{R})$, then there exists a pseudo-gradient vector field.

**Lemma 2.3.** [4] Let $\{u_n\} \subset M$ be a bounded sequence in $E$. Then the following are equivalent:

(i) $\|\varphi|_M'(u_n)\| \to 0$ as $n \to \infty$;

(ii) $\varphi'(u_n) - \langle \varphi'(u_n), u_n \rangle u_n \to 0$ in $E^*$ as $n \to \infty$. 

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The following two statements may be considered as counterparts of the deformation lemma and general minimax principle due to Willem [18] in the constraint context.

**Lemma 2.4.** Let $\varphi \in C^1(E, \mathbb{R})$, $S \subset M$, $a \in \mathbb{R}$, $\varepsilon, \delta > 0$ such that

$$u \in M \cap \varphi^{-1}([a - 2\varepsilon, a + 2\varepsilon]) \cap S_{2\delta} \Rightarrow \|\varphi'_M(u)\| \geq \frac{8\varepsilon}{\delta}. \quad (2.6)$$

Then, there exists $\eta \in C([0, 1] \times M, M)$ such that

(i) $\eta(t, u) = u$, if $t = 0$, or if $u \notin M \cap \varphi^{-1}([a - 2\varepsilon, a + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, \varphi^{a+\varepsilon} \cap S) \subset \varphi^{a-\varepsilon}$;

(iii) for every $t \in [0, 1]$, $\eta(t, \cdot) : M \rightarrow M$ is a homeomorphism;

(iv) $\|\eta(t, u) - u\|_E \leq \delta$, $\forall$ $u \in M$, $t \in [0, 1]$;

(v) for every $u \in M$, $\varphi(\eta(t, u))$ is non-increasing on $t \in [0, 1]$;

(vi) $\varphi(\eta(t, u)) < a$, $\forall$ $u \in M \cap \varphi^a \cap S_\delta$, $t \in [0, 1]$.

**Proof.** By Lemma 2.2 there exists a pseudo-gradient vector field $g(u)$ for $\varphi|_M$ on $\tilde{M} = \{u \in M : \varphi'_M(u) \neq 0\}$, i.e. for every $u \in \tilde{M}$, there exists $g(u) \in T_uM$ such that

$$\|g(u)|_E \leq 2\|\varphi'_M(u)\|, \quad \langle \varphi'_M(u), g(u) \rangle \geq \|\varphi'_M(u)\|^2. \quad (2.7)$$

Let

$$A := \{u \in M : a - 2\varepsilon \leq \varphi(u) \leq a + 2\varepsilon\} \cap S_{2\delta},$$

$$B := \{u \in M : a - \varepsilon \leq \varphi(u) \leq a + \varepsilon\} \cap S_{\delta}$$

and

$$\varrho(u) := \frac{\text{dist}(u, M \setminus A)}{\text{dist}(u, M \setminus A) + \text{dist}(u, B)}. \quad (2.8)$$

Define a vector field $W$ on $M$ by

$$W(u) := \begin{cases} -\varrho(u)g(u)\frac{\varrho(u)}{\|g(u)\|_E}, & \text{if } u \in A, \\ 0, & \text{if } u \in M \setminus A \end{cases} \quad (2.9)$$

Thus, $W$ is a locally Lipschitz continuous vector field on $M$ and $W(u) \in T_uM$ for $u \in \tilde{M}$.

By (2.6) and (2.7), one has

$$\|W(u)\|_E \leq \frac{\delta}{8\varepsilon}, \quad \forall$ u \in M. \quad (2.10)$$
As in [4], we let \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) be a Lipschitz continuous function such that \( h(1) = 0, h(s) = 0 \) if \( s \in [0, \frac{1}{2}] \cup [2, +\infty) \) and \( 0 \leq h(s) \leq 1 \) for \( s \in \left[ \frac{1}{2}, 2 \right] \). Define a vector field \( \tilde{W} \) on \( E \) by

\[
\tilde{W}(u) := \begin{cases} 
  h(||u||_H)W \left( \frac{u}{||u||_H} \right), & \text{if } \frac{1}{2} \leq ||u||_H \leq 2, \\
  0, & \text{otherwise.}
\end{cases}
\] (2.11)

Then \( \tilde{W}(u) \) is a locally Lipschitz continuous vector field on \( E \) and satisfies \( (u, \tilde{W}(u))_H = 0 \) for \( u \in E \).

Now consider the Cauchy problem in \( E \):

\[
\begin{cases} 
  \frac{d}{dt} \zeta(t, u) = \tilde{W}(\zeta(t, u)), \\
  \zeta(0, u) = u.
\end{cases}
\] (2.12)

By a standard arguments, for all \( u \in E \), equation (2.12) has a unique solution \( \zeta(t, u) \in E \), defined for all \( t \in \mathbb{R} \). It follows from (2.8), (2.9), (2.11) and (2.12) that

\[
\frac{d}{dt}||\zeta(t, u)||_H^2 = 2 \left( \zeta(t, u), \frac{d}{dt} \zeta(t, u) \right)_H = 2(\zeta(t, u), \tilde{W}(\zeta(t, u)))_H = 0, \quad \forall u \in E, \ t \in \mathbb{R}.
\] (2.13)

This shows that \( ||\zeta(t, u)||_H = ||u||_H = 1 \) for \( u \in M \) and \( t \in \mathbb{R} \). Thus

\[
\zeta(t, u) \in M, \quad \forall u \in M, \ t \in \mathbb{R}.
\] (2.14)

By (2.7), (2.8), (2.10), (2.12) and (2.14), we have

\[
||\zeta(t, u) - u||_E = \left| \int_0^t W(\zeta(s, u))ds \right|_E 
\leq \int_0^t ||W(\zeta(s, u))||_E ds \leq \frac{\delta t}{8\varepsilon}, \quad \forall u \in M, \ t \geq 0
\] (2.15)

and

\[
\frac{d}{dt}\varphi(\zeta(t, u)) = \langle \varphi'(\zeta(t, u)), \frac{d}{dt} \zeta(t, u) \rangle
= \langle \varphi'(\zeta(t, u)), W(\zeta(t, u)) \rangle
= -\frac{\varphi(\zeta(t, u))}{||g(\zeta(t, u))||_E^2}(\varphi'(\zeta(t, u)), g(\zeta(t, u)))
\leq -\frac{1}{4}\varphi(\zeta(t, u)), \quad \forall u \in M, \ t > 0.
\] (2.16)

Set \( \eta(t, u) := \zeta(8\varepsilon t, u) \). From the continuous dependence on \( u \) in (2.12) and (2.14), it follows that, for all \( t \in [0, 1] \), \( \eta(t, \cdot) \) is a homeomorphism: \( M \to M \), i.e. (iii) holds. By (2.8), (2.9), (2.14), (2.15) and (2.16), it is easy to verify that (i), (iv)-(vi) hold.

Finally, we prove (ii) holds also. Let \( u \in \varphi^{a+\varepsilon} \cap S \). If there is \( t_0 \in [0, 8\varepsilon] \) such that \( \varphi(\zeta(t_0, u)) < a - \varepsilon \), then it follows from (2.16) that \( \varphi(\zeta(8\varepsilon, u)) < a - \varepsilon \) and (ii) is satisfied. If

\[
a - \varepsilon \leq \varphi(\zeta(t, u)) \leq a + \varepsilon, \quad \forall t \in [0, 8\varepsilon],
\] (2.17)
then (2.14) and (2.15) implies that $\zeta(t, u) \in M \cap S_\delta$ for $t \in [0, 8\varepsilon]$, and so $\zeta(t, u) \in B$ for $t \in [0, 8\varepsilon]$, it follows from (2.16) and (2.17) that
\[
\varphi(\zeta(8\varepsilon, u)) = \varphi(u) + \int_0^{8\varepsilon} \frac{d}{dt} \varphi(\zeta(t, u)) dt 
\leq \varphi(u) - \frac{1}{4} \int_0^{8\varepsilon} \varphi(\zeta(t, u)) dt 
\leq a + \varepsilon - 2\varepsilon = a - \varepsilon,
\]
and (ii) is also satisfied.

\[\square\]

**Theorem 2.5.** Assume that $\theta \in \mathbb{R}$, $\varphi \in \mathcal{C}^1(E, \mathbb{R})$ and $\Upsilon \subset M$ is a closed set. Let
\[
\Gamma := \{ \gamma \in \mathcal{C}([0, 1], M) : \gamma(0) \in \Upsilon, \varphi(\gamma(1)) < \theta \}.
\] (2.18)

If $\varphi$ satisfies
\[
a := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \varphi(\gamma(t)) > b := \sup_{\gamma \in \Gamma} \max \{ \varphi(\gamma(0)), \varphi(\gamma(1)) \},
\] (2.19)
then, for every $\varepsilon \in (0, (a - b)/2)$, $\delta > 0$ and $\hat{\gamma} \in \Gamma$ such that
\[
\sup_{t \in [0, 1]} \varphi(\hat{\gamma}(t)) \leq a + \varepsilon,
\] (2.20)
there exists $u \in M$ such that
(i) $a - 2\varepsilon \leq \varphi(u) \leq a + 2\varepsilon$;
(ii) $\min_{t \in [0, 1]} \|u - \hat{\gamma}(t)\|_E \leq 2\delta$;
(iii) $\|\varphi'|_M(u)\| \leq \frac{8\varepsilon}{\delta}$.

**Proof.** Since $\varepsilon \in (0, (a - b)/2)$, so we have
\[
a - 2\varepsilon > b.
\] (2.21)
Suppose the conclusions are false. We apply Lemma 2.4 with $S = \hat{\gamma}([0, 1])$. Then, there exists $\eta \in \mathcal{C}([0, 1] \times M, M)$ such that (i)-(vi) in Lemma 2.4 hold. We define $\tilde{\gamma}(t) = \eta(1, \hat{\gamma}(t))$. Then it follows from (2.19), (2.21) and (i) in Lemma 2.4 that
\[
\tilde{\gamma}(0) = \eta(1, \hat{\gamma}(0)) = \hat{\gamma}(0), \quad \tilde{\gamma}(1) = \eta(1, \hat{\gamma}(1)) = \hat{\gamma}(1),
\]
so $\tilde{\gamma} \in \Gamma$. By (2.20) and (ii) in Lemma 2.4 one has
\[
\sup_{t \in [0, 1]} \varphi(\tilde{\gamma}(t)) = \sup_{t \in [0, 1]} \varphi(\eta(1, \hat{\gamma}(t))) \leq a - \varepsilon,
\]
contradicting the definition of $a$. $\square$
As a consequence of Theorem 2.5, we have:

**Corollary 2.6.** Assume that \( \theta \in \mathbb{R}, \, \Upsilon \subset M \) and \( \varphi \in C^1(E, \mathbb{R}) \) satisfy (2.18) and (2.19). Then there exists a sequence \( \{u_n\} \subset M \) satisfying

\[
\varphi(u_n) \to a, \quad \|\varphi'_{M}(u_n)\| \to 0. \tag{2.22}
\]

**Theorem 2.7.** Let \( \varphi \in C^1(E, \mathbb{R}) \) and \( A \subset E \). If there exists \( \rho > 0 \) such that

\[
a := \inf_{v \in M \cap A} \varphi(v) < b := \inf_{v \in M \cap (A_{\rho} \backslash A)} \varphi(v), \tag{2.23}
\]

then, for every \( \varepsilon \in (0, (b - a)/2) \), \( \delta \in (0, \rho/2) \) and \( w \in M \cap A \) such that

\[
\varphi(w) \leq a + \varepsilon, \tag{2.24}
\]

there exists \( u \in M \) such that

(i) \( a - 2\varepsilon \leq \varphi(u) \leq a + 2\varepsilon; \)

(ii) \( \|u - w\|_E \leq 2\delta; \)

(iii) \( \|\varphi'_{M}(u)\| \leq 8\varepsilon/\delta. \)

**Proof.** Since \( \varepsilon \in (0, (b - a)/2) \), which implies

\[
a + 2\varepsilon < b. \tag{2.25}
\]

Suppose the thesis is false. We apply Lemma 2.4 with \( S = \{w\} \), there exists \( \eta \in C([0,1] \times M, M) \) such that the conclusions (i)-(vi) in Lemma 2.4 hold. Set \( \tilde{v} = \eta(1, w) \). Then (iv) in Lemma 2.4 implies that \( \tilde{v} \in S_{\delta} \subset A_{\delta} \). If \( \tilde{v} \in A_{\delta} \backslash A \), then it follows from (i), (v) in Lemma 2.4 and (2.25) that

\[
a + \varepsilon \geq \varphi(w) = \varphi(\eta(0, w)) \geq \varphi(\eta(1, w)) = \varphi(\tilde{v}) \geq b > a + 2\varepsilon,
\]

which is absurd. If \( \tilde{v} \in A \), then both (2.24) and (ii) in Lemma 2.4 yield

\[
\varphi(\tilde{v}) = \varphi(\eta(1, w)) \leq a - \varepsilon,
\]

contradicting the definition of \( a \). \( \square \)

As direct corollaries of Theorem 2.7 we have:

**Corollary 2.8.** Let \( \varphi \in C^1(E, \mathbb{R}) \) and \( A \subset E \) and let \( \{w_n\} \subset M \cap A \) be a minimizing sequence for \( \inf_{M \cap A} \varphi \). If there exists \( \rho > 0 \) such that (2.23) holds, then there exists a sequence \( \{u_n\} \subset M \) satisfying

\[
\varphi(u_n) \to a, \quad \|u_n - w_n\|_E \to 0, \quad \|\varphi'_{M}(u_n)\| \to 0. \tag{2.26}
\]
Corollary 2.9. Let $\varphi \in C^1(E, \mathbb{R})$ and $A \subset E$. If there exist $\rho > 0$ and $\bar{u} \in M \cap A$ such that

$$\varphi(\bar{u}) = \inf_{v \in M \cap A} \varphi(v) < \inf_{v \in M \cap (A_\rho \setminus A)} \varphi(v),$$

then $\varphi'_{|M}(\bar{u}) = 0$.

Proof. Let $a := \inf_{v \in M \cap A} \varphi(v)$ and $b := \inf_{v \in M \cap (A_\rho \setminus A)} \varphi(v)$. Suppose the above conclusion is false, then there exists $\varepsilon_0 \in (0, (b - a)/2)$ such that $\|\varphi'_{|M}(\bar{u})\| \geq \varepsilon_0$. Since $\varphi \in C^1(E, \mathbb{R})$, then there exists $\delta_0 \in (0, \rho/2)$ such that

$$u \in M, \quad \|u - \bar{u}\|_E < \delta_0 \Rightarrow \|\varphi'_{|M}(u)\| \geq \frac{\varepsilon_0}{2}.$$

By Theorem 2.7 for $\varepsilon = \frac{\varepsilon_0}{n}$, $\delta = \frac{\delta_0}{4}$ and $\bar{u} \in M \cap A$ such that

$$\varphi(\bar{u}) \leq a + \frac{\varepsilon_0}{n},$$

there exists $u \in M$ such that for all $n \in \mathbb{N}$,

i) $a - \frac{2\varepsilon_0}{n} \leq \varphi(u) \leq a + \frac{2\varepsilon_0}{n}$;

ii) $\|u - \bar{u}\|_E \leq \frac{\delta_0}{2}$;

iii) $\|\varphi'_{|M}(u)\| \leq \frac{32\varepsilon_0}{n\delta_0}$.

These contradict with (2.28) for large $n \in \mathbb{N}$. 

\[ \blacksquare \]

Remark 2.10. As one will see in the next section, although a (PS) sequence $\{u_n\}$ of $\Phi|_{S_{\varepsilon}}$ can be generated by applying Corollary 2.9, it not sufficient to derive the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$. In the spirit of Jeanjean [8], to do that, we hope to find a (PS) sequence having additional property $\mathcal{P}(u_n) \to 0$. Different from that of [8], we will employ critical point theorems for the $C^1$-functional $\tilde{\Phi}(u, t) = \Phi(e^{Nt/2}u(e^t x))$ for $(u, t) \in H^1(\mathbb{R}^N) \times \mathbb{R}$, which will be given in the following.

Let $E \times \mathbb{R}$ be equipped with the scalar product

$$((u, \tau), (v, \sigma))_{E \times \mathbb{R}} := (u, v)_{H^1} + \tau \sigma, \quad \forall (u, \tau), (v, \sigma) \in E \times \mathbb{R},$$

and corresponding norm

$$\|(u, \tau)\|_{E \times \mathbb{R}} := \sqrt{\|u\|_{H^1}^2 + \tau^2}, \quad \forall (u, \tau) \in E \times \mathbb{R}.$$

Next, we consider a functional $\tilde{\varphi} : E \times \mathbb{R} \to \mathbb{R}$ which is of class $C^1$ on $E \times \mathbb{R}$. We denote by $\tilde{\varphi}'_{|M \times \mathbb{R}}$ the trace of $\tilde{\varphi}$ on $M \times \mathbb{R}$. Then $\tilde{\varphi}'_{|M \times \mathbb{R}}$ is a $C^1$ functional on $M \times \mathbb{R}$, and for any $(u, \tau) \in M \times \mathbb{R}$,

$$\langle \tilde{\varphi}'_{|M \times \mathbb{R}}(u, \tau), (v, \sigma) \rangle := \langle \tilde{\varphi}'(u, \tau), (v, \sigma) \rangle, \quad \forall (v, \sigma) \in \tilde{T}_{(u, \tau)}(M \times \mathbb{R}),$$

(2.29)
Lemma 2.13. There exists a pseudo-gradient vector field for \( \tilde{\varphi}_{\tilde{u}, \tilde{\tau}} \). By Lemma 2.12, there exists a pseudo-gradient vector field \( M \times \mathbb{R} \times \mathbb{R} \). Then, there exists a pseudo-gradient vector field \( M \times \mathbb{R} \times \mathbb{R} \). In the sequel, for any \( (u, \tau) \in M \times \mathbb{R} \), we define the norm \( \| \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau) \| \) by

\[
\| \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau) \| = \sup_{(v, \sigma) \in \tilde{T}(u, \tau)(M \times \mathbb{R}), \| (v, \sigma) \| \leq 1} |\langle \tilde{\varphi}'(u, \tau), (v, \sigma) \rangle|.
\] (2.31)

**Definition 2.11.** A pseudo-gradient vector for \( \tilde{\varphi}_{M \times \mathbb{R}} \) at \( (u, \tau) \in \tilde{M} \times \mathbb{R} \) is a vector \( (v, \sigma) \in \tilde{T}(u, \tau)(M \times \mathbb{R}) \) such that

\[
\| (v, \sigma) \|_{E \times \mathbb{R}} \leq 2 \| \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau) \| , \quad \langle \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau), (v, \sigma) \rangle \geq \| \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau) \|^2.
\]

Put

\[
\tilde{T}(M \times \mathbb{R}) := \bigcup_{(u, \tau) \in M \times \mathbb{R}} (\tilde{T}(u, \tau)(M \times \mathbb{R})).
\]

A mapping (lifting) \( g : \tilde{M} \times \mathbb{R} \rightarrow \tilde{T}(M \times \mathbb{R}) \) is called a pseudo-gradient vector field for \( \tilde{\varphi}_{M \times \mathbb{R}} \) on \( \tilde{M} \times \mathbb{R} \) if \( g \) is locally Lipschitz continuous, \( g(u, \tau) \in \tilde{T}(u, \tau)(M \times \mathbb{R}) \) for \( (u, \tau) \in \tilde{M} \times \mathbb{R} \), and \( g(u, \tau) \) is a pseudo-gradient vector for \( \tilde{\varphi}_{M \times \mathbb{R}} \) at any \( (u, \tau) \in \tilde{M} \times \mathbb{R} \).

**Lemma 2.12.** If \( \tilde{\varphi}_{M \times \mathbb{R}} \in C^1(M \times \mathbb{R}, \mathbb{R}) \), then there exists a pseudo-gradient vector field.

**Lemma 2.13.** Let \( \tilde{\varphi} \in C^1(E \times \mathbb{R}, \mathbb{R}) \), \( S \subset M \times \mathbb{R} \), \( a \in \mathbb{R} \), \( \varepsilon, \delta > 0 \) such that

\[
(u, \tau) \in (M \times \mathbb{R}) \cap \tilde{\varphi}^{-1}([a - 2\varepsilon, a + 2\varepsilon]) \cap S_{2\delta} \Rightarrow \| \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau) \| \geq \frac{8\varepsilon}{\delta}.
\] (2.32)

Then, there exists \( \eta \in C([0, 1] \times (M \times \mathbb{R}), M \times \mathbb{R}) \) such that

(i) \( \eta(t, (u, \tau)) = (u, \tau) \), if \( t = 0 \), or if \( (u, \tau) \notin (M \times \mathbb{R}) \cap \tilde{\varphi}^{-1}([a - 2\varepsilon, a + 2\varepsilon]) \cap S_{2\delta} \);

(ii) \( \eta(1, \tilde{\varphi}^{a+\varepsilon} \cap S) \subset \tilde{\varphi}^{a-\varepsilon} \);

(iii) for every \( t \in [0, 1] \), \( \eta(t, \cdot) : M \times \mathbb{R} \rightarrow M \times \mathbb{R} \) is a homeomorphism;

(iv) \( \| \eta(t, (u, \tau)) - (u, \tau) \|_{E \times \mathbb{R}} \leq \delta \), \( \forall (u, \tau) \in M \times \mathbb{R}, \ t \in [0, 1] \);

(v) \( \tilde{\varphi}(\eta(t, (u, \tau))) \) is non-increasing on \( t \in [0, 1] \);

(vi) \( \tilde{\varphi}(\eta(t, (u, \tau))) < a \), \( \forall (u, \tau) \in (M \times \mathbb{R}) \cap \tilde{\varphi}^{a} \cap S_{\delta}, \ t \in [0, 1] \).

**Proof.** By Lemma 2.12 there exists a pseudo-gradient vector field \( g(u, \tau) \in \tilde{T}(u, \tau)(M \times \mathbb{R}) \) for \( \tilde{\varphi}_{M \times \mathbb{R}} \) on \( \tilde{M} \times \mathbb{R} = \{(u, \tau) \in M \times \mathbb{R} : \tilde{\varphi}'_{M \times \mathbb{R}}(u, \tau) \neq 0\} \), i.e. for every \( (u, \tau) \in \tilde{M} \times \mathbb{R} \), there exists \( g(u, \tau) := (g_1(u, \tau), g_2(u, \tau)) \in \tilde{T}(u, \tau)(M \times \mathbb{R}) \) such that

\[
(u, g_1(u, \tau))_{H} = 0
\] (2.33)
and
\[ \|g(u, \tau)\|_{E \times \mathbb{R}} \leq 2 \|\tilde{\varphi}'|_{M \times \mathbb{R}}(u, \tau)\|, \quad \langle \tilde{\varphi}|_{M \times \mathbb{R}}(u, \tau), g(u, \tau) \rangle \geq \|\tilde{\varphi}'|_{M \times \mathbb{R}}(u, \tau)\|^2. \] (2.34)

Let
\[ A := \{(u, \tau) \in M \times \mathbb{R} : a - 2\varepsilon \leq \tilde{\varphi}(u, \tau) \leq a + 2\varepsilon \} \cap S_{2\delta}, \]
\[ B := \{(u, \tau) \in M \times \mathbb{R} : a - \varepsilon \leq \tilde{\varphi}(u, \tau) \leq a + \varepsilon \} \cap S_{\delta} \]
and
\[ \varrho(u, \tau) := \frac{\text{dist}((u, \tau), (M \times \mathbb{R}) \setminus A)}{\text{dist}((u, \tau), (M \times \mathbb{R}) \setminus A) + \text{dist}((u, \tau), B)}. \] (2.35)

Define a vector field \( W \) on \( M \times \mathbb{R} \) by
\[ W(u, \tau) := \begin{cases} -\varrho(u, \tau) \frac{\varrho(u, \tau)}{\|g(u, \tau)\|_{E \times \mathbb{R}}}, & \text{if } (u, \tau) \in A, \\ (0, 0), & \text{if } (u, \tau) \in (M \times \mathbb{R}) \setminus A. \end{cases} \] (2.36)

Thus, \( W \) is a locally Lipschitz continuous vector field on \( M \times \mathbb{R} \) and \( W(u, \tau) \in \hat{T}(u, \tau)(M \times \mathbb{R}) \) for \((u, \tau) \in \hat{M} \times \mathbb{R} \). By (2.32) and (2.34), one has
\[ \|W(u, \tau)\|_{E \times \mathbb{R}} \leq \frac{\delta}{8\varepsilon}, \quad \forall (u, \tau) \in M \times \mathbb{R}. \] (2.37)

Now let \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) be a Lipschitz continuous function such that \( h(1) = 0, h(s) = 0 \) if \( s \in [0, \frac{1}{2}] \cup [2, +\infty) \) and \( 0 \leq h(s) \leq 1 \) for \( s \in [\frac{1}{2}, 2] \). Define a vector field \( \tilde{W} \) on \( E \) by
\[ \tilde{W}(u, \tau) := \begin{cases} h(\|u\|_H)W\left(\frac{u}{\|u\|_H}, \tau\right), & \text{if } \frac{1}{2} \leq \|u\|_H \leq 2, \tau \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases} \] (2.38)

Then \( \tilde{W}(u, \tau) = (\tilde{W}_1(u, \tau), \tilde{W}_2(u, \tau)) \) is a locally Lipschitz continuous vector field on \( E \times \mathbb{R} \)
and satisfies \((u, \tilde{W}_1(u, \tau))_H = 0\) for \( u \in E \).

Consider the Cauchy problem in \( E \times \mathbb{R} \):
\[ \begin{aligned}
\frac{d}{dt}\zeta(t, (u, \tau)) &= \tilde{W}(\zeta(t, (u, \tau))), \\
\zeta(0, (u, \tau)) &= (u, \tau).
\end{aligned} \] (2.39)

By a standard arguments, for all \((u, \tau) \in E \times \mathbb{R}\), equation (2.39) has a unique solution \( \zeta(t, (u, \tau)) := (\zeta_1(t, (u, \tau)), \zeta_2(t, (u, \tau))) \in E \times \mathbb{R} \), defined for all \( t \in \mathbb{R} \). It follows from (2.32), (2.35), (2.36), (2.38) and (2.39) that
\[ \frac{d}{dt}\|\zeta_1(t, (u, \tau))\|^2_H = 2\left(\zeta_1(t, (u, \tau)) \cdot \frac{d}{dt}\zeta_1(t, (u, \tau))\right)_H = 2(\zeta_1(t, (u, \tau)), \tilde{W}_1(\zeta(t, (u, \tau))))_H = 0, \quad \forall u \in E, \tau, t \in \mathbb{R}. \] (2.40)
This shows that \( \|\zeta_1(t, (u, \tau))\|_H = \|u\|_H = 1 \) for \( u \in M \) and \( \tau, t \in \mathbb{R} \). Thus

\[
\zeta(t, (u, \tau)) \in M \times \mathbb{R}, \quad \forall \ (u, \tau) \in M \times \mathbb{R}, \quad t \in \mathbb{R}.
\] (2.41)

From (2.34), (2.35), (2.36), (2.37), (2.38), (2.39) and (2.41), we have

\[
\|\zeta(t, (u, \tau)) - (u, \tau)\|_{E \times \mathbb{R}} = \left\| \int_0^t W(\zeta(s, (u, \tau))) \right\|_{E \times \mathbb{R}} \\
\leq \int_0^t \|W(\zeta(s, (u, \tau)))\|_{E \times \mathbb{R}} ds \leq \frac{\delta t}{8\varepsilon},
\]
\(\forall \ (u, \tau) \in M \times \mathbb{R}, \ t \geq 0\) (2.42)

and

\[
\frac{d}{dt} \varphi(\zeta(t, (u, \tau))) = \left\langle \varphi'(\zeta(t, (u, \tau))), \frac{d}{dt} \zeta(t, (u, \tau)) \right\rangle \\
= \left\langle \varphi'(\zeta(t, (u, \tau))), W(\zeta(t, (u, \tau))) \right\rangle \\
= -\frac{\varphi(\zeta(t, (u, \tau)))}{\|g(\zeta(t, (u, \tau)))\|_{E \times \mathbb{R}}^2} \left\langle \varphi'(\zeta(t, (u, \tau))), g(\zeta(t, (u, \tau))) \right\rangle \\
\leq -\frac{1}{4} \varphi(\zeta(t, (u, \tau))), \ \forall \ (u, \tau) \in M \times \mathbb{R}, \ t > 0.
\] (2.43)

Set \( \eta(t, (u, \tau)) := \zeta(8\varepsilon t, (u, \tau)) \). From the continuous dependence on \( (u, \tau) \) in (2.39) and (2.41), it follows that, for all \( t \in [0, 1] \), \( \eta(t, \cdot) \) is a homeomorphism: \( M \times \mathbb{R} \rightarrow M \times \mathbb{R} \), i.e. (iii) holds. By (2.35), (2.36), (2.42) and (2.43), it is easy to verify that (i), (iv)-(vi) hold.

Finally, we prove (ii) holds also. Let \( (u, \tau) \in \varphi^{a+\varepsilon} \cap S \). If there is \( t_0 \in [0, 8\varepsilon] \) such that \( \varphi(\zeta(t_0, (u, \tau))) < a - \varepsilon \), then it follows from (2.43) that \( \varphi(\zeta(8\varepsilon, (u, \tau))) < a - \varepsilon \) and (ii) is satisfied. If

\[
a - \varepsilon \leq \varphi(\zeta(t, (u, \tau))) \leq a + \varepsilon, \ \forall \ t \in [0, 8\varepsilon],
\] (2.44)

then (2.41) and (2.42) imply that \( \zeta(t, (u, \tau)) \in (M \times \mathbb{R}) \cap S_\delta \) for \( t \in [0, 8\varepsilon] \), and so \( \zeta(t, (u, \tau)) \in B \) for \( t \in [0, 8\varepsilon] \), it follows from (2.43) and (2.44) that

\[
\varphi(\zeta(8\varepsilon, (u, \tau))) = \varphi(u, \tau) + \int_0^{8\varepsilon} \frac{d}{dt} \varphi(\zeta(t, (u, \tau))) dt \\
\leq \varphi(u, \tau) - \frac{1}{4} \int_0^{8\varepsilon} g(\zeta(t, (u, \tau))) dt \\
\leq a + \varepsilon - 2\varepsilon = c - \varepsilon,
\]

and (ii) is also satisfied.

\[\square\]

**Theorem 2.14.** Assume that \( \tilde{\theta} \in \mathbb{R}, \ \varphi \in \mathcal{C}^1(E \times \mathbb{R}, \mathbb{R}) \) and \( \tilde{Y} \subset M \times \mathbb{R} \) is a closed set. Let

\[
\tilde{\Gamma} := \left\{ \tilde{\gamma} \in \mathcal{C}([0, 1], M \times \mathbb{R}) : \tilde{\gamma}(0) \in \tilde{Y}, \ \varphi(\tilde{\gamma}(1)) < \tilde{\theta} \right\}. \] (2.45)
If \( \bar{\varphi} \) satisfies
\[
\bar{a} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \bar{\varphi}(\tilde{\gamma}(t)) > \bar{b} := \sup_{\tilde{\gamma} \in \tilde{\Gamma}} \{ \bar{\varphi}(\tilde{\gamma}(0)), \bar{\varphi}(\tilde{\gamma}(1)) \},
\] (2.46)
then, for every \( \varepsilon \in (0, (\bar{a} - \bar{b})/2) \), \( \delta > 0 \) and \( \tilde{\gamma}_* \in \tilde{\Gamma} \) such that
\[
\sup_{t \in [0,1]} \bar{\varphi}(\tilde{\gamma}_*(t)) \leq \bar{a} + \varepsilon,
\] (2.47)
there exists \((v, \tau) \in M \times \mathbb{R}\) such that
\begin{enumerate}
\item \( \bar{a} - 2\varepsilon \leq \bar{\varphi}(v, \tau) \leq \bar{a} + 2\varepsilon; \)
\item \( \min_{t \in [0,1]} \| (v, \tau) - \tilde{\gamma}_*(t) \|_{E \times \mathbb{R}} \leq 2\delta; \)
\item \( \| \bar{\varphi}'_{M \times \mathbb{R}}(v, \tau) \| \leq \frac{8\varepsilon}{\delta}. \)
\end{enumerate}
Proof. Since \( \varepsilon \in (0, (\bar{a} - \bar{b})/2) \), then one has \( \bar{a} - 2\varepsilon > \bar{b}. \) (2.48)

Suppose the conclusions of theorem are false. We apply Lemma 2.13 with \( u = v \) and \( S = \tilde{\gamma}_*(0,1) \). Then, there exists \( \eta \in C(\bar{0},1) \times (M \times \mathbb{R}), M \times \mathbb{R}) \) such that (i)-(vi) in Lemma 2.13 hold. We define \( \tilde{\gamma}_0(t) = \eta(1, \tilde{\gamma}_*(t)) \). Then it follows from (2.46), (2.48) and (i) in Lemma 2.13 that
\[
\tilde{\gamma}_0(0) = \eta(1, \tilde{\gamma}_*(0)) = \tilde{\gamma}_*(0), \quad \tilde{\gamma}_0(1) = \eta(1, \tilde{\gamma}_*(1)) = \tilde{\gamma}_*(1),
\]
so \( \tilde{\gamma}_0 \in \tilde{\Gamma}. \) By (2.47) and (ii) in Lemma 2.13 one has
\[
\sup_{t \in [0,1]} \bar{\varphi}(\tilde{\gamma}_0(t)) = \sup_{t \in [0,1]} \bar{\varphi}(\eta(1, \tilde{\gamma}_*(t))) \leq \bar{a} - \varepsilon,
\]
contradicting the definition of \( \bar{a} \).

As a consequence of Theorem 2.14 we have:

Corollary 2.15. Assume that \( \tilde{\theta} \in \mathbb{R}, \tilde{\Gamma} \subset M \times \mathbb{R} \) is closed set and \( \bar{\varphi} \in C^1(E \times \mathbb{R}, \mathbb{R}) \) satisfies (2.45) and (2.46). Let \( \{ \tilde{\gamma}_n \} \subset \tilde{\Gamma} \) be such that
\[
\sup_{t \in [0,1]} \bar{\varphi}(\tilde{\gamma}_n(t)) \leq \bar{a} + \frac{1}{n}, \quad \forall \ n \in \mathbb{N}.
\] (2.49)
Then there exists a sequence \( \{(v_n, \tau_n)\} \subset M \times \mathbb{R} \) satisfying
\begin{enumerate}
\item \( \bar{a} - \frac{2}{n} \leq \bar{\varphi}(v_n, \tau_n) \leq \bar{a} + \frac{2}{n}; \)
\item \( \min_{t \in [0,1]} \| (v_n, \tau_n) - \tilde{\gamma}_n(t) \|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}}; \)
\item \( \| \bar{\varphi}'_{M \times \mathbb{R}}(v_n, \tau_n) \| \leq \frac{8}{\sqrt{n}}. \)
\end{enumerate}
3 Normalized solutions of $L^2$-supercritical problem (1.1)

In this section, we study $L^2$-supercritical problem (1.1) and prove Theorem 1.1 by making use of Corollary 2.15.

To apply Corollary 2.15 we let $E = H^1(\mathbb{R}^N)$ (or $H^1_{\text{rad}}(\mathbb{R}^N)$) and $H = L^2(\mathbb{R}^N)$. Define the norms of $E$ and $H$ by

$$
\|u\|_E := \left[ \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right]^{1/2}, \quad \|u\|_H := \frac{1}{\sqrt{c}} \left( \int_{\mathbb{R}^N} u^2 \, dx \right)^{1/2}, \quad \forall u \in E.
$$

(3.1)

After identifying $H$ with its dual, we have $E \hookrightarrow H \hookrightarrow E^*$ with continuous injections. Set

$$
M := \left\{ u \in E : \|u\|_2^2 = \int_{\mathbb{R}^N} u^2 \, dx = c \right\}.
$$

(3.2)

Obviously, under (F0), $\Phi \in C^1(E, \mathbb{R})$, and

$$
\langle \Phi'(u), u \rangle = \|\nabla u\|^2_2 - \int_{\mathbb{R}^N} f(u)udx.
$$

(3.3)

Let us define a continuous map $\beta : H^1(\mathbb{R}^N) \times \mathbb{R} \to H^1(\mathbb{R}^N)$ by

$$
\beta(v, t)(x) := e^{Nt/2}v(e^t x) \quad \text{for} \quad v \in H^1(\mathbb{R}^N), \quad \forall \ t \in \mathbb{R}, \ x \in \mathbb{R}^N,
$$

(3.4)

and consider the following auxiliary functional:

$$
\tilde{\Phi}(v, t) := \Phi(\beta(v, t)) = \frac{e^{2t}}{2} \|\nabla v\|^2_2 - \frac{1}{e^{Nt}} \int_{\mathbb{R}^N} F(e^{Nt/2}v)dx.
$$

(3.5)

We see that $\tilde{\Phi}'$ is of class $C^1$, and for any $(w, s) \in H^1(\mathbb{R}^N) \times \mathbb{R},$

$$
\langle \tilde{\Phi}'(v, t), (w, s) \rangle = \langle \tilde{\Phi}'(v, t), (w, 0) \rangle + \langle \tilde{\Phi}'(v, t), (0, s) \rangle
$$

$$
= e^{2t} \int_{\mathbb{R}^N} \nabla v \cdot \nabla wd\mathbf{x} + e^{2t} s \|\nabla v\|^2_2 - \frac{1}{e^{Nt}} \int_{\mathbb{R}^N} f(e^{Nt/2}v)e^{Nt/2}wd\mathbf{x}
$$

$$
+ \frac{Ns}{2e^{Nt}} \int_{\mathbb{R}^N} \left[ 2F(e^{Nt/2}v) - f(e^{Nt/2}v)e^{Nt/2}v \right] \, d\mathbf{x}
$$

$$
= \langle \Phi'(\beta(v, t)), \beta(w, t) \rangle + s\mathcal{P}(\beta(v, t)).
$$

(3.6)

Let

$$
u(x) := \beta(v, t)(x) = e^{Nt/2}v(e^t x), \quad \phi(x) := \beta(w, t)(x) = e^{Nt/2}w(e^t x).
$$

(3.7)

Then

$$
(u, \phi)_H = \frac{1}{c} \int_{\mathbb{R}^N} u(x)\phi(x)dx = \frac{1}{c} \int_{\mathbb{R}^N} v(x)w(x)dx = (v, w)_H.
$$

(3.8)

This shows that

$$
\phi \in T_u(S_c) \Leftrightarrow (w, s) \in \tilde{T}_{(v, t)}(S_c \times \mathbb{R}), \quad \forall \ t, s \in \mathbb{R}.
$$

(3.9)
Proof. Let (3.14) and Lemma 3.1. Assume that (3.6), (3.7) and (3.9) hold. Then there exists a sequence \( c_\tilde{\Gamma} : \equiv \rho \) where \( \rho > 0 \)

\[ (i) \quad \text{there exists } \rho (c) > 0 \text{ small enough such that } \Phi(u) > 0 \text{ if } u \in A_{2\rho} \text{ and} \]

\[ 0 < \sup_{u \in A_{\rho}} \Phi(u) < \kappa_c : = \inf \{ \Phi(u) : u \in S_c, \| \nabla u \|_2^2 = 2\rho(c) \} \]  

where

\[ A_{\rho} = \{ u \in S_c : \| \nabla u \|_2^2 \leq \rho(c) \} \quad \text{and} \quad A_{2\rho} = \{ u \in S_c : \| \nabla u \|_2^2 \leq 2\rho(c) \} \]  

(ii) there holds:

\[ M(c) : = \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \kappa_c > \sup_{\gamma \in \Gamma_c} \max \{ \Phi(\gamma(0)), \Phi(\gamma(1)) \} \]

where \( \Gamma_c : = \{ \gamma \in C([0,1], S_c) : \| \nabla \gamma(0) \|_2^2 \leq \rho(c), \Phi(\gamma(1)) < 0 \} \).

The proof of this lemma is standard, so we omit it.

Lemma 3.2. Assume that (F0) holds, and that there exist \( \rho(c) > 0 \) and \( \kappa_c > 0 \) such that (3.14) and (3.15) hold. Then there exists a sequence \( \{ u_n \} \subset S_c \) such that

\[ \Phi(u_n) \to M(c) > 0, \quad \Phi(u_n) \to 0 \text{ and } P(u_n) \to 0. \]  

Proof. Let

\[ \tilde{\Gamma}_c : = \{ \tilde{\gamma} \in C([0,1], S_c \times \mathbb{R}) : \tilde{\gamma}(0) = (\tilde{\gamma}_1(0), 0), \| \nabla \tilde{\gamma}_1(0) \|_2^2 \leq \rho(c), \Phi(\tilde{\gamma}(1)) < 0 \} \]  

(3.17)
and

\[ \tilde{M}(c) := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)). \]  

(3.18)

For any \( \tilde{\gamma} \in \tilde{\Gamma}_c \), it is easy to see that \( \gamma = \beta \circ \tilde{\gamma} \in \Gamma_c \) defined by (3.15). Then it follows from (3.5) and (3.14) that

\[ \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) = \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \{ \Phi(\gamma(0)), \Phi(\gamma(1)) \} \]

\[ \geq \sup_{\tilde{\gamma} \in \tilde{\Gamma}_c} \{ \tilde{\Phi}(\tilde{\gamma}(0)), \tilde{\Phi}(\tilde{\gamma}(1)) \}. \] 

(3.19)

This shows that \( \tilde{M}(c) \geq M(c) \) and (2.46) holds.

On the other hand, for any \( \gamma \in \Gamma_c \), let \( \tilde{\gamma}(t) := (\gamma(t), 0) \). It is easy to verify that \( \tilde{\gamma} \in \tilde{\Gamma}_c \) and \( \Phi(\gamma(t)) = \tilde{\Phi}(\tilde{\gamma}(t)) \), and so, we trivially have \( \tilde{M}(c) \leq M(c) \). Thus \( \tilde{M}(c) = M(c) \).

For any \( n \in \mathbb{N} \), (3.14) implies that there exists \( \gamma_n \in \Gamma_c \) such that

\[ \max_{t \in [0,1]} \Phi(\gamma_n(t)) \leq M(c) + \frac{1}{n}. \] 

(3.20)

Set \( \tilde{\gamma}_n(t) := (\gamma_n(t), 0) \). Then applying Corollary 2.15 to \( \Phi \), there exists a sequence \{\( (v_n, t_n) \)\} \( \subset \mathcal{S}_c \times \mathbb{R} \) satisfying

(i) \( M(c) - \frac{2}{n} \leq \Phi(v_n, t_n) \leq M(c) + \frac{2}{n} \);

(ii) \( \min_{t \in [0,1]} \| (v_n, t_n) - (\gamma_n(t), 0) \|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}} \);

(iii) \( \| \tilde{\Phi}'_{v_n, t_n} \|_{E \times \mathbb{R}} \leq \frac{8}{\sqrt{n}} \).

Let \( u_n = \beta(v_n, t_n) \). It follows from (3.10), (3.11) and (i)-(iii) that (3.16) holds.

\[ \square \]

Similarly, we can obtain the following lemma.

**Lemma 3.3.** Assume that (F0) holds, and that there exist \( \rho(c) > 0 \) and \( \kappa_c > 0 \) such that (3.14) and (3.15) hold. Then there exists a sequence \{\( u_n \)\} \( \subset \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^N) \) such that

\[ \Phi(u_n) \to M(c) > 0, \quad \Phi|_{\mathcal{S}_c}(u_n) \to 0 \quad \text{and} \quad P(u_n) \to 0. \] 

(3.21)

**Lemma 3.4.** Assume that (F0) holds. If there exist \( u \in H^1(\mathbb{R}^N) \) and \( \lambda \in \mathbb{R} \) such that

\[ -\Delta u - \lambda u = f(u), \quad x \in \mathbb{R}^N, \] 

(3.22)

then \( P(u) = 0 \), where \( P \) is defined by (1.6).
Proof of Theorem 1.1. Let $E = H^1_{\text{rad}}(\mathbb{R}^N)$. By Lemma 3.3, there exists a sequence $\{u_n\} \subset S_c \cap H^1_{\text{rad}}(\mathbb{R}^N)$ such that

$$\|u_n\|^2_2 = c, \quad \Phi(u_n) \to M(c), \quad \Phi|_{S_c}^{'}(u_n) \to 0, \quad P(u_n) \to 0. \quad (3.23)$$

From (1.2), (1.6) and (3.23), one has

$$\int \frac{1}{2} F(u_n) dx \to 0.$$  

Then by (F1) and (F2), we have

$$M(c) + o(1) = \frac{1}{2} \|\nabla u_n\|^2_2 - \int_{\mathbb{R}^N} F(u_n) dx \quad (3.24)$$

and

$$o(1) = \|\nabla u_n\|^2_2 - \frac{N}{2} \int_{\mathbb{R}^N} [f(u_n)u_n - 2F(u_n)] dx. \quad (3.25)$$

Combining (3.24) with (3.25), we get

$$M(c) + 1 \geq \frac{N}{4} \int_{\mathbb{R}^N} [f(u_n)u_n - \left(2 + \frac{4}{N}\right) F(u_n)] dx. \quad (3.26)$$

Now, we prove that $\|\nabla u_n\|_2$ is bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \to \infty$. Let $v_n := \frac{u_n}{\|\nabla u_n\|_2}$. Then $\|\nabla v_n\|_2 = 1$, and $\|v_n\|_2 \to 0$ due to (3.23). Set $\kappa' = \frac{\kappa}{\kappa - 1}$. Then $N - (N - 2)\kappa' > 0$, it follows from (1.23) that

$$\|v_n\|^{2\kappa'}_{2\kappa'} \leq C_1 \|v_n\|^{N-(N-2)\kappa'}_2 \|\nabla v_n\|^{(N-1)\kappa'-1}_2 = o(1). \quad (3.27)$$

By (F2) and (F3), there exist $R_0 > 0$ and $C_2 > 0$ such that

$$\left| \frac{f(t)t - 2F(t)}{t^2} \right|^\kappa \leq C_2 \left( f(t)t - \left(2 + \frac{4}{N}\right) F(t) \right), \quad \forall |t| \geq R_0. \quad (3.28)$$

Set

$$\Omega_n := \{ x \in \mathbb{R}^N : |u_n(x)| \leq R_0 \}.$$  

Then by (F1) and (F2), we have

$$\int_{\Omega_n} \frac{|f(u_n)u_n - 2F(u_n)|^\kappa_{\frac{u_n^2}{2}}} {u_n^2} v_n^2 dx \leq C_3 \|v_n\|^2_2 = o(1). \quad (3.29)$$

Moreover, from (3.26), (3.27), (3.28) and the Hölder inequality, we have

$$\int_{\mathbb{R}^N \setminus \Omega_n} \frac{|f(u_n)u_n - 2F(u_n)|^\kappa_{\frac{u_n^2}{2}}} {u_n^2} v_n^2 dx \leq \left( \int_{\mathbb{R}^N \setminus \Omega_n} \left| f(u_n)u_n - 2F(u_n) \right|^\kappa dx \right)^{1/\kappa} \left( \int_{\mathbb{R}^N \setminus \Omega_n} |v_n|^{2\kappa'} dx \right)^{1/\kappa'} \leq C_2^{1/\kappa} \left( \int_{\mathbb{R}^N \setminus \Omega_n} \left( f(u_n)u_n - \left(2 + \frac{4}{N}\right) F(u_n) \right)^{1/\kappa} \right) \|v_n\|^{2\kappa'} \quad (3.30)$$

Thus, it follows from (3.25), (3.29) and (3.30) that

$$\frac{2}{N} + o(1) = \int_{\mathbb{R}^N} \frac{|f(u_n)u_n - 2F(u_n)|^\kappa_{\frac{u_n^2}{2}}} {u_n^2} v_n^2 dx$$
which is a contradiction. Hence, \(\{\|u_n\|_E\}\) is bounded. By Lemma 2.3, one has

\[
\|u_n\|_2^2 = c, \quad \Phi(u_n) \to M(c), \quad \Phi'(u_n) - \lambda_n u_n \to 0,
\]

where

\[
\lambda_n = \frac{1}{\|u_n\|_2^2} (\Phi'(u_n), u_n) = \frac{1}{c} \left[ \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} f(u_n)u_n \, dx \right].
\]

Since \(\{\|u_n\|_E\}\) is bounded, it follows from (F1), (F2) and (3.32) that \(\{\|\lambda_n\|\}\) is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that \(\lambda_n \to \lambda_c\), \(u_n \rightharpoonup \bar{u}\) in \(E\), \(u_n \to \bar{u}\) in \(L^s(\mathbb{R}^N)\) for \(s \in (2, 2^*)\) and \(u_n \to \bar{u}\) a.e. on \(\mathbb{R}^N\). By (F1), (F2) and a standard argument, we can deduce

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n)\phi \, dx = \int_{\mathbb{R}^N} f(\bar{u})\phi \, dx, \quad \forall \phi \in E,
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n)u_n \, dx = \int_{\mathbb{R}^N} f(\bar{u})\bar{u} \, dx
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} F(u_n) \, dx = \int_{\mathbb{R}^N} F(\bar{u}) \, dx.
\]

Hence, from (3.24), (3.25), (3.34) and (3.35), we deduce

\[
M(c) = \lim_{n \to \infty} \left\{ \frac{N}{4} \int_{\mathbb{R}^N} \left[ f(u_n)u_n - \left(2 + \frac{4}{N}\right) F(u_n) \right] \, dx \right\}
\]

\[
= \frac{N}{4} \int_{\mathbb{R}^N} \left[ f(\bar{u})\bar{u} - \left(2 + \frac{4}{N}\right) F(\bar{u}) \right] \, dx.
\]

This, together with (F2), shows that \(\bar{u} \neq 0\). It follows from (3.31), (3.33) and the fact that \(\lambda_n \to \lambda_c\) and \(u_n \to \bar{u}\) in \(E\) that

\[
\Phi'(\bar{u}) - \lambda_c \bar{u} = 0.
\]

Hence, (3.37) and Lemma 3.4 yield

\[
\|\nabla \bar{u}\|_2^2 - \lambda_c \|\bar{u}\|_2^2 = \int_{\mathbb{R}^N} f(\bar{u})\bar{u} \, dx = 0
\]

and

\[
\|\nabla \bar{u}\|_2^2 - \frac{N}{2} \int_{\mathbb{R}^N} [f(\bar{u})\bar{u} - 2F(\bar{u})] \, dx = 0.
\]

From (F2), (3.38) and (3.39), one has

\[
-\lambda_c \|\bar{u}\|_2^2 = \int_{\mathbb{R}^N} \left[ NF(\bar{u}) - \frac{N - 2}{2} f(\bar{u})\bar{u} \right] \, dx > 0,
\]
which implies that \( \lambda_c < 0 \). Now from (1.2), (3.31), (3.33), (3.35) and (3.37), one has

\[
0 = \lim_{n \to \infty} \left[ \langle \Phi'(u_n) - \lambda_n u_n, u_n - \bar{u} \rangle - \langle \Phi'(\bar{u}) - \lambda_c \bar{u}, u_n - \bar{u} \rangle \right]
\]

\[
\begin{align*}
&= \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^N} \left( |\nabla (u_n - \bar{u})|^2 - \lambda_c |u_n - \bar{u}|^2 \right) dx - \int_{\mathbb{R}^2} [f(u_n) - f(\bar{u})](u_n - \bar{u}) dx \right\} \\
&= \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( |\nabla (u_n - \bar{u})|^2 - \lambda_c |u_n - \bar{u}|^2 \right) dx.
\end{align*}
\]  

(3.41)

It follows that \( u_n \to \bar{u} \) in \( E \), and so \( \|\bar{u}\|_2^2 = c \), \( \Phi|_{S^c}(\bar{u}) = 0 \) and \( \Phi(\bar{u}) = M(c) \).

Next, we prove that (1.1) has a ground state solution on \( S \). To this end, let

\[
\mathcal{K}_c := \{ u \in S \cap H^1_0(\mathbb{R}^N) : \Phi'(u) = 0 \}, \quad \bar{m}(c) := \inf_{\mathcal{K}_c} \Phi.
\]  

(3.42)

Then \( \bar{u} \in \mathcal{K}_c \). It follows from (1.2), (1.6) and (F2) that

\[
\Phi(u) = \Phi(u) - \frac{1}{2} \mathcal{P}(u)
\]

\[
= \frac{N}{4} \int_{\mathbb{R}^N} \left[ f(u)u - \left( 2 + \frac{4}{N} \right) F(u) \right] dx \geq 0, \quad \forall \ u \in \mathcal{K}_c.
\]  

(3.43)

Thus \( \bar{m}(c) \geq 0 \). Let \( \{ u_n \} \subset \mathcal{K}_c \) such that

\[
\Phi(u_n) \to \bar{m}(c), \quad \Phi|_{S^c}(u_n) = 0.
\]  

(3.44)

From (1.2), (1.6), (3.41) and Lemma 3.4 one has

\[
\bar{m}(c) + o(1) = \frac{1}{2} \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} F(u_n) dx
\]  

(3.45)

and

\[
0 = \|\nabla u_n\|_2^2 - \frac{N}{2} \int_{\mathbb{R}^N} [f(u_n)u_n - 2F(u_n)] dx.
\]  

(3.46)

Combining (3.45) with (3.46), we have

\[
\bar{m}(c) + o(1) = \frac{N}{4} \int_{\mathbb{R}^N} \left[ f(u_n)u_n - \left( 2 + \frac{4}{N} \right) F(u_n) \right] dx.
\]  

(3.47)

From (F1), (F2), (1.22), (1.23) and (3.46), we get

\[
\|\nabla u_n\|_2^2 = \frac{N}{2} \int_{\mathbb{R}^N} [f(u_n)u_n - 2F(u_n)] dx
\]

\[
\leq \int_{\mathbb{R}^N} \left( \frac{1}{2} e^{-2/N} C_{1,2 + 4/N} |u_n|^{2 + \frac{4}{N}} + C_4 |u_n|^{2^*} \right) dx
\]

\[
\leq \frac{1}{2} \|\nabla u_n\|_2^2 + C_5 \|\nabla u_n\|_2^{2^*},
\]  

(3.48)

which, together with the fact that \( u_n \neq 0 \), yields that \( \|\nabla u_n\|_2 \geq C_6 \). Let \( v_n := \frac{u_n}{\|\nabla u_n\|_2} \). Then \( \|\nabla v_n\|_2 = 1 \), and \( \|v_n\|_2 \leq c/C_6 \). Let \( \kappa' = \frac{N}{\kappa - 1} \). By (1.23), one has

\[
\|v_n\|^{2\kappa'} \leq C_7 \|v_n\|_{2^{N-(N-2)\kappa'}} \|\nabla v_n\|_{2^{N\kappa' - 1}} \leq C_8 \min \left\{ 1, \|\nabla u_n\|_2^{-N + (N-2)\kappa'} \right\}.
\]  

(3.49)
Hence, it follows from (F3'), (3.40), (3.47), (3.49) and the Hölder inequality that

\[
\frac{2}{N} = \int_{\mathbb{R}^N} f(u_n) u_n - 2 F(u_n) u_n^2 dx \\
\leq \left[ \int_{\mathbb{R}^N} \left( \frac{f(u_n) u_n - 2 F(u_n)}{u_n^2} \right)^\kappa dx \right]^{1/\kappa} \|v_n\|_{2\kappa'}^2 \\
\leq C_0^{1/\kappa} \left\{ \int_{\mathbb{R}^N} \left[ N f(u_n) u_n - (2N + 4) F(u_n) \right] dx \right\}^{1/\kappa} \|v_n\|_{2\kappa'}^2 \\
\leq (4C_0)^{1/\kappa} \left[ C_8 \min \left\{ 1, \|\nabla u_n\|_2^{2-N+(N-2)\kappa'} \right\} \right]^{1/\kappa'} (\bar{m}(c) + o(1))^{1/\kappa}.
\]

This shows that $\bar{m}(c) > 0$ and $\{\|\nabla u_n\|_2\}$ is bounded. By a standard argument, we can prove that there exists $\tilde{u} \in \mathcal{K}_c$ such that $\Phi(\tilde{u}) = \bar{m}(c)$. \hfill \(\square\)

4 Normalized solutions for mixed problem (1.4) with Sobolev critical exponent

In this section, we let $f(t) = \mu |t|^{q-2} t + |t|^{2^* - 2} t, \gamma_q := N(q - 2)/2q,$ and study the Schrödinger equation (1.4) with Sobolev critical exponent and mixed dispersion.

4.1 $L^2$-subcritical perturbation

In this subsection, we always assume that $2 < q < 2 + \frac{4}{N}$ and shall prove Theorem 4.3.

Let $N \geq 3$ and let $c_0$ and $\rho_0$ are given in [9, 10], where $\rho_0$ depending only on $c_0 > 0$ but not on $c \in (0, c_0)$. As in [10], we define the sets $V(c)$ and $\partial V(c)$ as follows:

\[
V(c) := \{u \in S_c : \|\nabla u\|_2^2 < \rho_0\}, \quad \partial V(c) := \{u \in S_c : \|\nabla u\|_2^2 = \rho_0\}. \tag{4.1}
\]

For any $\mu > 0$ and any $c \in (0, c_0)$, the set $V(c) \subset S_c$ having the property:

\[
m_\mu(c) = \inf_{u \in V(c)} \Phi_\mu(u) < 0 < \inf_{u \in \partial V(c)} \Phi_\mu(u). \tag{4.2}
\]

Lemma 4.1. [10] Proposition 2.1] Let $N \geq 3$. For any $\mu > 0$ and any $c \in (0, c_0)$, $m_\mu(c)$ is reached by a positive, radially symmetric non-increasing function, denoted $u_c \in V(c) \setminus \partial V(c)$ that satisfies, for a $\lambda_c < 0$,

\[
-\Delta u_c - \mu |u_c|^{q-2} u_c - |u_c|^{2^* - 2} u_c = \lambda_c u_c, \quad x \in \mathbb{R}^N. \tag{4.3}
\]

Lemma 4.2. Let $N \geq 3$. For any $\mu > 0$ and any $c \in (0, c_0)$, there exists $\kappa_{\mu, c} > 0$ such that

\[
M_\mu(c) := \inf_{\gamma \in \Gamma_{\mu, c}} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \kappa_{\mu, c} > \sup_{\gamma \in \Gamma_{\mu, c}} \max_{t \in [0, 1]} \{ \Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1)) \}, \tag{4.4}
\]

where

\[
\Gamma_{\mu, c} = \{ \gamma \in \mathcal{C}([0, 1], S_c \cap H^1_{rad}(\mathbb{R}^N) : \gamma(0) = u_c, \Phi_\mu(\gamma(1)) < 2m_\mu(c) \}. \tag{4.5}
\]
Proof. Set \( \kappa_{\mu,c} := \inf_{u \in \partial V(c)} \Phi_{\mu}(u) \). By \((4.2)\), \( \kappa_{\mu,c} > 0 \). Let \( \gamma \in \Gamma_{\mu,c} \) be arbitrary. Since \( \gamma(0) = u_c \in V(c) \setminus (\partial V(c)) \), and \( \Phi_{\mu}(\gamma(1)) < 2m_{\mu}(c) \), necessarily in view of \((4.2)\) \( \gamma(1) \notin V(c) \). By continuity of \( \gamma(t) \) on \([0,1] \), there exists a \( t_0 \in (0,1) \) such that \( \gamma(t_0) \in \partial V(c) \), and so \( \max_{t \in [0,1]} \Phi_{\mu}(\gamma(t)) \geq \kappa_{\mu,c} \). Thus, \((4.4)\) holds.

Let \( E = H^{1}_{rad}(\mathbb{R}^{N}) \) and set

\[
\tilde{\Gamma}_{\mu,c} := \left\{ \tilde{\gamma} \in C([0,1], (\mathcal{S}_c \cap H^{1}_{rad}(\mathbb{R}^{N})) \times \mathbb{R}^{+}) : \tilde{\gamma}(0) = (u_c, 0), \tilde{\Phi}_{\mu}(\tilde{\gamma}(1)) < 2m_{\mu}(c) \right\}.
\]

Replace \( \Gamma_c \) and \( \tilde{\Gamma}_c \) in Lemmas 3.1 and 3.2 by \( \Gamma_{\mu,c} \) and \( \tilde{\Gamma}_{\mu,c} \), respectively, we can prove the following lemma by the same arguments as in the proof of Lemma 3.2.

**Lemma 4.3.** Let \( N \geq 3 \). For any \( \mu > 0 \) and any \( c \in (0,c_0) \), there exists a sequence \( \{u_n\} \subset \mathcal{S}_c \cap H^{1}_{rad}(\mathbb{R}^{N}) \) such that

\[
\Phi_{\mu}(u_n) \to M_{\mu}(c) > 0, \quad \Phi_{\mu}|_{\mathcal{S}_c}(u_n) \to 0 \quad \text{and} \quad \mathcal{P}_{\mu}(u_n) \to 0.
\]

Let \( A_N := [N(N - 2)]^{(N-2)/4} \). Now we define functions \( U_n(x) := \Theta_n(|x|) \), where

\[
\Theta_n(r) = A_N \begin{cases} 
\left( \frac{n}{1 + n^2 r} \right)^{\frac{N-2}{2}}, & 0 \leq r < 1; \\
\left( \frac{n}{1 + n^2 r} \right)^{\frac{N-2}{2}} (2 - r), & 1 \leq r < 2; \\
0, & r \geq 2.
\end{cases}
\]

Computing directly, we have

\[
\|U_n\|_{2}^{2} = \int_{\mathbb{R}^{N}} |U_n|^{2} dx = \omega_{N-1} \int_{0}^{\infty} r^{N-1} |\Theta_n(r)|^{2} dr
\]

\[
= \omega_{N-1} A_N^{2} \left[ \int_{0}^{1} \frac{n^{N-2}r^{N-1}}{(1 + n^2 r)^{N-2}} dr + \left( \frac{n}{1 + n^2} \right)^{\frac{N-2}{2}} \int_{1}^{\infty} r^{N-1} (2 - r)^{2} dr \right]
\]

\[
= \omega_{N-1} A_N^{2} \left[ \frac{1}{n^2} \int_{0}^{n} \frac{s^{N-1}}{(1 + s^2)^{N-2}} ds + \frac{2^{N+3} - (N^2 + 5N + 8)}{N(N + 1)(N + 2)} \left( \frac{n}{1 + n^2} \right)^{\frac{N-2}{2}} \right]
\]

\[
= \omega_{N-1} A_N^{2} \left[ \frac{\xi(n)}{n^2} + \frac{2^{N+3} - (N^2 + 5N + 8)}{N(N + 1)(N + 2)} \left( \frac{n}{1 + n^2} \right)^{\frac{N-2}{2}} \right]
\]

\[
= O \left( \frac{\xi(n)}{n^2} \right), \quad n \to \infty,
\]

where \( \omega_{N-1} \) is the measure of the unit sphere in \( \mathbb{R}^{N} \),

\[
\xi(n) := \int_{0}^{n} \frac{s^{N-1}}{(1 + s^2)^{N-2}} ds = \begin{cases} 
O(n), & \text{if } N = 3; \\
O(\log(1 + n^2)), & \text{if } N = 4; \\
O(1), & \text{if } N \geq 5,
\end{cases}
\]

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\[ \|\nabla U_n\|_2^2 = \int_{\mathbb{R}^N} |\nabla U_n|^2 \, dx = \omega_{N-1} \int_0^{+\infty} r^{N-1} |\Theta_n(r)|^2 \, dr \]
\[
= \omega_{N-1} A_0^2 \left[ (N-2)^2 \int_0^1 \frac{n^{N+2}r^{N+1}}{(1+n^2r^2)^N} \, dr + \left( \frac{n}{1+n^2} \right)^{N-2} \int_1^2 r^{N-1} \, dr \right] \\
= \omega_{N-1} A_0^2 \left[ (N-2)^2 \int_0^n \frac{s^{N+1}}{(1+s^2)^N} \, ds + \frac{2N-1}{N} \left( \frac{n}{1+n^2} \right)^{N-2} \right] \\
= S^{N/2} + \omega_{N-1} A_0^2 \left[ -(N-2)^2 \int_0^{+\infty} \frac{s^{N+1}}{(1+s^2)^N} \, ds + \frac{2N-1}{N} \left( \frac{n}{1+n^2} \right)^{N-2} \right] \\
= S^{N/2} + O \left( \frac{1}{n^{N-2}} \right), \quad n \to \infty, \quad (4.11) \]

\[ \|U_n\|_{2^*}^2 = \int_{\mathbb{R}^N} |U_n|^{2^*} \, dx = \omega_{N-1} \int_0^{+\infty} r^{N-1} |\Theta_n(r)|^{2^*} \, dr \]
\[
= \omega_{N-1} A_0^2 \left[ \int_0^1 \frac{n^{N-1}r^{N-1}}{(1+n^2r^2)^N} \, dr + \left( \frac{n}{1+n^2} \right)^{N} \int_1^{2^*} r^{N-1} (2-r)^{2^*} \, dr \right] \\
= \omega_{N-1} A_0^2 \left[ \int_0^n \frac{s^{N-1}}{(1+s^2)^N} \, ds + \frac{2}{N} \left( \frac{n}{1+n^2} \right)^{N-1} \right] \\
= S^{N/2} + O \left( \frac{1}{n^N} \right), \quad n \to \infty, \quad (4.12) \]

\[ \|U_n\|_q^q = \int_{\mathbb{R}^N} |U_n|^q \, dx = \omega_{N-1} \int_0^{+\infty} r^{N-1} |\Theta_n(r)|^q \, dr \]
\[
\geq \omega_{N-1} A_0^q \int_0^1 \frac{n^q(N-2)/2}{(1+n^2r^2)^{q(N-2)/2}} \, dr \\
= \frac{\omega_{N-1} A_0^q}{n^{2q(N-N-2)/2}} \int_0^n \frac{s^{N-1}}{(1+s^2)^{q(N-2)/2}} \, ds \\
= O \left( \frac{1}{n^{2q(N-N-2)/2}} \right), \quad n \to \infty. \quad (4.13) \]

Both (4.9) and (4.11) imply that \( U_n \in E \).

**Lemma 4.4.** Let \( N \geq 3, \ 2 < q < 2 + \frac{4}{N} \) and \( c \in (0, c_0) \). Then there holds:
\[ M_\mu(c) < m_\mu(c) + \frac{S^{N/2}}{N}. \quad (4.14) \]

**Proof.** Inspired by [17, Lemma 3.1], let \( u_c \in H^1_{rad}(\mathbb{R}^N) \) be given in Lemma 4.1. Then by Lemmas 3.4 and 4.1, we have
\[ \|u_c\|_2^2 = c, \quad \Phi_\mu(u_c) = m_\mu(c), \quad \lambda_c \|u_c\|_2^2 = -\mu(1-\gamma_q)\|u_c\|_q^q, \quad u_c(x) > 0, \quad \forall \ x \in \mathbb{R}^N. \quad (4.15) \]

Set \( b := \inf_{|x| \leq 1} u_c(x) \) and \( B := \sup_{|x| \leq 2} u_c(x) \). Then \( 0 < b \leq B < +\infty \). Hence, it follows from (4.8) and (4.9) that
\[ \int_{\mathbb{R}^N} u_c^{q-1} U_n \, dx \leq B^{q-1} \int_{|x| \leq 2} U_n \, dx = O \left( \frac{\sqrt{\xi(n)}}{n} \right), \quad n \to \infty, \quad (4.16) \]
\[
\int_{\mathbb{R}^N} u_c U_n \, dx \leq B \int_{|x| \leq 2} U_n \, dx = O \left( \frac{\sqrt{\xi(n)}}{n} \right), \quad n \to \infty \tag{4.17}
\]

and
\[
\int_{\mathbb{R}^N} u_c |U_n|^{2^* - 1} \, dx \geq b \omega_{N-1} \int_0^1 r^{N-1} |\Theta_n(r)|^{(N+2)/(N-2)} \, dr \\
= b \omega_{N-1} A_N^{(N+2)/(N-2)} \int_0^1 \frac{n(N+2)/2 \cdot r^{N-1}}{(1 + n^2 r^2)^{(N+2)/2}} \, dr \\
= b \omega_{N-1} A_N^{(N+2)/(N-2)} n^{-(N-2)/2} \int_0^n \frac{s^{N-1}}{(1 + s^2)^{(N+2)/2}} \, ds \\
= O \left( \frac{1}{n(N-2)/2} \right), \quad n \to \infty. \tag{4.18}
\]

By (4.3), (4.9) and (4.15), one has
\[
\int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_n \, dx = \int_{\mathbb{R}^N} \left( \mu u_c^{q-1} + u_c^{2^* - 1} + \lambda_c u_c \right) U_n \, dx \tag{4.19}
\]

and for any \( t > 0 \),
\[
\| u_c + t U_n \|_2^2 = c + t^2 \| U_n \|_2^2 + 2t \int_{\mathbb{R}^N} u_c U_n \, dx \\
= c + 2t \int_{\mathbb{R}^N} u_c U_n \, dx + t^2 \left[ O \left( \frac{\xi(n)}{n^2} \right) \right], \quad n \to \infty. \tag{4.20}
\]

Let \( \tau := \| u_c + t U_n \|_2 / \sqrt{c} \). Then
\[
\tau^2 = 1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n \, dx + t^2 \left[ O \left( \frac{\xi(n)}{n^2} \right) \right], \quad n \to \infty. \tag{4.21}
\]

Now, we define
\[
W_{n,t}(x) := \tau^{(N-2)/2} [u_c(\tau x) + t U_n(\tau x)]. \tag{4.22}
\]

Then one has
\[
\| \nabla W_{n,t} \|_2^2 = \| \nabla (u_c + t U_n) \|_2^2, \quad \| W_{n,t} \|_{2^*}^2 = \| u_c + t U_n \|_{2^*}^2 \tag{4.23}
\]

and
\[
\| W_{n,t} \|_2^2 = \tau^{-2} \| u_c + t U_n \|_2^2 = c, \quad \| W_{n,t} \|_q^2 = \tau^{q-2} \| u_c + t U_n \|_q^2. \tag{4.24}
\]

It is easy to verify the following inequalities:
\[
(1 + t)^p \geq \begin{cases} 
1 + pt + pt^{p-1} + t^p, & \text{if } p \geq 3; \\
1 + pt^{p-1} + t^p, & \text{if } p \geq 2.
\end{cases} \tag{4.25}
\]

In what follows, we distinguish two cases.
Case 1). $3 \leq N \leq 5$. In this case, we have $2^* \geq 3$. From (4.10), (4.10)-(4.12), (4.15)-(4.19) and (4.21)-(4.25), we have

$$
\Phi_\mu(W_{n,t})
$$

$$
= \frac{1}{2} \| \nabla W_{n,t} \|^2 - \frac{1}{2} \| W_{n,t} \|^2 - \frac{\mu}{q} \| W_{n,t} \|_q^q
$$

$$
= \frac{1}{2} \| \nabla (u_c + tU_n) \|^2 - \frac{1}{2} \| u_c + tU_n \|^2 - \frac{\mu \tau^{\gamma_q-q}}{q} \| u_c + tU_n \|_q^q
$$

$$
\leq \frac{1}{2} \| \nabla U_n \|^2 - \frac{\mu \tau^{\gamma_q-q}}{q} \| u_c \|_q^q + \frac{t^2}{2} \| \nabla U_n \|^2 - \frac{t^{2*}}{2} \| U_n \|_2^{2*}
$$

$$
+ t \int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_n dx - t \int_{\mathbb{R}^N} u_c^{2*} U_n dx - t^{2*} \int_{\mathbb{R}^N} u_c U_n^{2*} dx
$$

$$
- \mu \tau^{\gamma_q-q} t \int_{\mathbb{R}^N} u_c^{q-1} U_n dx
$$

$$
= \Phi(u_c) + \frac{\mu (1 - \tau^{\gamma_q-q})}{q} \| u_c \|_q^q + \frac{t^2}{2} \| \nabla U_n \|^2 - \frac{t^{2*}}{2} \| U_n \|_2^{2*}
$$

$$
+ \mu (1 - \tau^{\gamma_q-q}) t \int_{\mathbb{R}^N} u_c^{q-1} U_n dx + \lambda_c t \int_{\mathbb{R}^N} u_c U_n dx - t^{2*} \int_{\mathbb{R}^N} u_c U_n^{2*} dx
$$

$$
= m_\mu(c) + \frac{\mu \| u_c \|_q^q}{q} \left\{ 1 - \left[ 1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 \left( O \left( \frac{\xi(n)}{n^2} \right) \right) \right]^{-\frac{q-1}{q}} \right\}
$$

$$
+ \frac{t^2}{2} \| \nabla U_n \|^2 - \frac{t^{2*}}{2} \| U_n \|_2^{2*} + \lambda_c t \int_{\mathbb{R}^N} u_c U_n dx - t^{2*} \int_{\mathbb{R}^N} u_c U_n^{2*} dx
$$

$$
+ \mu \left\{ 1 - \left[ 1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 \left( O \left( \frac{\xi(n)}{n^2} \right) \right) \right]^{-\frac{q-1}{q}} \right\} \left( t \int_{\mathbb{R}^N} u_c^{q-1} U_n dx \right)
$$

$$
\leq m_\mu(c) + t^2 \left( O \left( \frac{\xi(n)}{n^2} \right) \right) + \frac{t^2}{2} \left[ S^{N/2} + O \left( \frac{1}{n^{N-2}} \right) \right] - \frac{t^{2*}}{2} \left[ S^{N/2} + O \left( \frac{1}{n^N} \right) \right]
$$

$$
- t^{2*} \left[ O \left( \frac{1}{n(N-2)^{N/2}} \right) \right] + \frac{\mu \| u_c \|_q^q}{c} \left( t \int_{\mathbb{R}^N} u_c^{q-1} U_n dx \right) \left( \int_{\mathbb{R}^N} u_c U_n dx \right)
$$

$$
\leq m_\mu(c) + S^{N/2} \left( \frac{t^2}{2} - \frac{t^{2*}}{2*} \right) + t^2 \left( O \left( \frac{\xi(n)}{n^2} \right) \right) + \frac{t^2}{2} \left[ O \left( \frac{1}{n^N} \right) \right]
$$

$$
- \frac{t^{2*}}{2*} \left[ O \left( \frac{1}{n^N} \right) \right] - t^{2*} \left[ O \left( \frac{1}{n(N-2)^{N/2}} \right) \right], \forall t > 0.
$$

(4.26)

Noting that

$$
0 < \frac{t^2}{2} - \frac{t^{2*}}{2*} \leq \frac{1}{N}, \forall t > 0.
$$

(4.27)

Since $3 \leq N \leq 5$, hence, it follows from (4.10), (4.26) and (4.27) that there exists $n \in \mathbb{N}$ such that

$$
\sup_{t > 0} \Phi_\mu(W_{n,t}) < m_\mu(c) + \frac{S^{N/2}}{N}.
$$

(4.28)

Case 2). $N \geq 6$. In this case, we have $2 < q < 2^* \leq 3$. From (4.10), (4.10)-(4.13), (4.15)-(4.17), (4.19) and (4.21)-(4.25), we have

$$
\Phi_\mu(W_{n,t})
$$

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\[ = \frac{1}{2} \| \nabla (u_c + tU_n) \|_2^2 - \frac{1}{2} \| u_c + tU_n \|_{2^*}^2 + \frac{\mu \tau^{q\gamma q - q}}{q} \| u_c + tU_n \|_q^q \]
\[ \leq \frac{1}{2} \| \nabla u_c \|_2^2 - \frac{1}{2} \| u_c \|_{2^*}^2 + \frac{\mu \tau^{q\gamma q - q}}{q} \| u_c \|_q^q + \frac{t^2}{2} \| \nabla U_n \|_2^2 - \frac{\mu \tau^{q\gamma q - q}}{q} \| U_n \|_q^q \]
\[ + t \int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_n \, dx - t \int_{\mathbb{R}^N} u_c^{q-1} U_n \, dx - \mu \tau^{q\gamma q - q} t \int_{\mathbb{R}^N} u_c^{q-1} U_n \, dx \]
\[ = \Phi(u_c) + \frac{\mu (1 - \tau^{q\gamma q - q})}{q} \| u_c \|_q^q + \frac{t^2}{2} \| \nabla U_n \|_2^2 - \frac{\mu \tau^{q\gamma q - q}}{q} \| U_n \|_q^q \]
\[ + \mu (1 - \tau^{q\gamma q - q}) t \int_{\mathbb{R}^N} u_c^{q-1} U_n \, dx + \lambda c t \int_{\mathbb{R}^N} u_c U_n \, dx \]
\[ \leq m_\mu(c) + t^2 \left[ O \left( \frac{1}{n^2} \right) \right] + \frac{t^2}{2} \left[ S^{N/2} + O \left( \frac{1}{n^{N-2}} \right) \right] - \frac{t^2}{2} \left[ S^{N/2} + O \left( \frac{1}{n^N} \right) \right] \]
\[ - t^q \left[ O \left( \frac{1}{n^{N-(N-2)q/2}} \right) \right] \]
\[ \leq m_\mu(c) + S^{N/2} \left( \frac{t^2}{2} - \frac{t^{2^*}}{2^*} \right) + t^2 \left[ O \left( \frac{1}{n^2} \right) \right] - \frac{t^2}{2} \left[ O \left( \frac{1}{n^N} \right) \right] \]
\[ - t^q \left[ O \left( \frac{1}{n^{N-(N-2)q/2}} \right) \right], \quad \forall \, t > 0. \] (4.29)

Hence, it follows from (4.27), (4.29) and the fact \( N \geq 6 \) and \( 2 < q < 2 + \frac{4}{N} \) that there exists \( \bar{n} \in \mathbb{N} \) such that (4.28) holds.

Next, we prove that (4.14) hold. Let \( \bar{n} \in \mathbb{N} \) be given in (4.28). By (4.20), (4.22), (4.23) and (4.24), we have
\[ W_{\bar{n},t}(x) := \tau^{(N-2)/2} [u_c(\tau x) + tU_\bar{n}(\tau x)], \quad \| W_{\bar{n},t} \|_2^2 = c \] (4.30)
and
\[ \| \nabla W_{\bar{n},t} \|_2^2 = \| \nabla (u_c + tU_\bar{n}) \|_2^2 = \| \nabla u_c \|_2^2 + t^2 \| \nabla U_\bar{n} \|_2^2 + 2t \int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_\bar{n} \, dx \] (4.31)
where
\[ \tau^2 = \| u_c + tU_\bar{n} \|_2^2 = \frac{1 + 2t^4}{c} \int_{\mathbb{R}^N} u_c U_n \, dx + \| U_\bar{n} \|_2^2 t^2. \] (4.32)

It follows from (4.26), (4.29), (4.30) and (4.31) that \( W_{\bar{n},t} \in S_c \) for all \( t \geq 0 \), \( W_{\bar{n},0} = u_c \) and \( \Phi_\mu(W_{\bar{n},t}) < 2m_\mu(c) \) for large \( t > 0 \). Thus, there exist \( \bar{t} > 0 \) such that
\[ \Phi_\mu(W_{\bar{n},\bar{t}}) < 2m_\mu(c). \] (4.33)

Let \( \gamma(t) := W_{\bar{n},t}^\bar{t} \). Then \( \gamma \in \Gamma_{\mu,c} \cap H^1_{\text{rad}}(\mathbb{R}^N) \) defined by (4.3). Hence, it follows from (4.14) and (4.28) that (4.14) holds. \( \square \)

**Lemma 4.5.** [10] Proposition 1.11] Let \( N \geq 3 \) and \( 2 < q < 2 + \frac{4}{N} \). For any \( c \in (0, c_0) \), if (4.14) holds, then the sequence \( \{ u_n \} \) obtained in Lemma 4.3 is, up to subsequence, strongly convergent in \( H^1_{\text{rad}}(\mathbb{R}^N) \).

**Proof of Theorem 1.3** The proof of Theorem 1.3 follows directly combining Lemmas 4.3, 4.4 and 4.5. \( \square \)
4.2 \( L^2 \)-critical and \( L^2 \)-supercritical perturbation

In this subsection, we always assume that \( 2 + \frac{4}{N} \leq q < 2^* \) and shall prove Theorem 1.4.

Let

\[
h(t) := \frac{1 - t^2}{2} - \frac{1 - t^{2*}}{2^*}.
\]  

(4.34)

It is easy to see that \( h(t) > h(1) = 0 \) for all \( t \in [0,1) \cup (1, +\infty) \).

**Lemma 4.6.** Let \( N \geq 3, c > 0, \mu > 0 \) and \( 2 + \frac{4}{N} \leq q < 2^* \). Then there hold

\[
\Phi_\mu(u) \geq \Phi_\mu \left( t^{N/2} u_t \right) + \frac{1 - t^2}{2} P_\mu(u) + h(t)\|u\|_{2^*}^{2^*}, \quad \forall \ u \in \mathcal{S}_c, \ t > 0.
\]  

(4.35)

By a simple calculation, we can prove the above lemma. From Lemma 4.6, we have the following corollary.

**Corollary 4.7.** Let \( N \geq 3, c > 0, \mu > 0 \) and \( 2 + \frac{4}{N} \leq q < 2^* \). Then for any \( u \in \mathcal{M}_\mu(c) \), there holds

\[
\Phi_\mu(u) = \max_{t > 0} \Phi_\mu \left( t^{N/2} u_t \right).
\]  

(4.36)

**Lemma 4.8.** Let \( N \geq 3, c > 0, \mu > 0 \) and \( 2 + \frac{4}{N} \leq q < 2^* \). Then for any \( u \in \mathcal{S}_c \), there exists a unique \( t_u > 0 \) such that \( t_u^{N/2} u_{t_u} \in \mathcal{M}_\mu(c) \).

The proof of Lemma 4.8 is standard, so we omit it.

From Corollary 4.7 and Lemma 4.8, we have the following lemma.

**Lemma 4.9.** Let \( N \geq 3, c > 0, \mu > 0 \) and \( 2 + \frac{4}{N} \leq q < 2^* \). Then

\[
\inf_{u \in \mathcal{M}_\mu(c)} \Phi_\mu(u) := \hat{m}_\mu(c) = \inf_{u \in \mathcal{S}_c} \max_{t > 0} \Phi_\mu \left( t^{N/2} u_t \right).
\]  

(4.37)

In what follows, we set \( \alpha(N, q) = +\infty \) if \( 2 + \frac{4}{N} < q < 2^* \) and \( \alpha(N, \tilde{q}) = \frac{1}{2 q e^{2/N c_0}} \).

**Lemma 4.10.** Let \( N \geq 3, c > 0, 0 < \mu < \alpha(N, q) \) and \( 2 + \frac{4}{N} \leq q < 2^* \). Then

(i) there exists \( \vartheta_c > 0 \) such that \( \Phi_\mu(u) > 0 \) and \( P_\mu(u) > 0 \) if \( u \in A_{2 \vartheta_c} \), and

\[
0 < \sup_{u \in A_{\vartheta_c}} \Phi_\mu(u) < \inf \left\{ \Phi_\mu(u) : u \in \mathcal{S}_c, \ \|\nabla u\|_2^2 = 2 \vartheta_c \right\},
\]  

(4.38)

where

\[
A_{\vartheta_c} = \left\{ u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq \vartheta_c \right\} \quad \text{and} \quad A_{2 \vartheta_c} = \left\{ u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq 2 \vartheta_c \right\};
\]  

(4.39)

(ii) \( \hat{\Gamma}_\mu,c = \{ \gamma \in \mathcal{C}([0, 1], \mathcal{S}_c) : \|\nabla \gamma(0)\|_2^2 \leq \vartheta_c, \Phi_\mu(\gamma(1)) < 0 \} \neq \emptyset \) and

\[
\hat{M}_\mu(c) := \inf_{\gamma \in \hat{\Gamma}_\mu,c} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \hat{\kappa}_\mu,c := \inf \left\{ \Phi_\mu(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2 \vartheta_c \right\}
\]  

> \max \max \{ \Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1)) \}.

(4.40)
Proof. i). We distinguish to two cases.

Case 1). \(2 + \frac{4}{q} < q < 2^*\). In this case, one has \(2 < q \gamma_q < 2^*\). By (1.10), (1.11), (1.22) and (1.23), one has

\[
\Phi_\mu(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{\mu}{q} \|u\|_q^q - \frac{1}{2^*} \|u\|_{2^*}^{2^*} \\
\geq \|\nabla u\|_2^2 \left[ - \frac{1}{2} - \frac{\mu}{q} C_{N,q}^q \|\nabla u\|_2^{q\gamma_q - 2} - \frac{1}{2^{2^*}} S^{-\frac{2^*}{2^*}} \|u\|_2^{-2^*} \right], \quad \forall u \in S_c
\]

and

\[
\mathcal{P}_\mu(u) = \|\nabla u\|_2^2 - \mu \gamma_q \|u\|_q^q - \|u\|_{2^*}^{2^*} \\
\geq \|\nabla u\|_2^2 \left[ 1 - \mu \gamma_q c^{(1-\gamma_q)q/2} C_{N,q}^q \|\nabla u\|_2^{q\gamma_q - 2} - S^{-\frac{2^*}{2^*}} \|u\|_2^{-2^*} \right], \quad \forall u \in S_c.
\]

Since \(q \gamma_q > 2\), the above inequalities show there exists \(\vartheta_c > 0\) such that i) holds.

Case 2). \(q = 2 + \frac{4}{N} = \bar{q} < 2^*\). In this case \(2 = q \gamma_q < 2^*\). By (1.10), (1.11), (1.22) and (1.23), one has

\[
\Phi_\mu(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{\mu}{q} \|u\|_q^q - \frac{1}{2^*} \|u\|_{2^*}^{2^*} \\
\geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{\mu}{q} C_{N,q}^q \|\nabla u\|_2^{q\gamma_q - 2} - \frac{1}{2^{2^*}} S^{-\frac{2^*}{2^*}} \|u\|_2^{-2^*} \\
\geq \|\nabla u\|_2^2 \left[ 1 - \frac{1}{4} S^{-\frac{2^*}{2^*}} \|u\|_2^{-2^*} \right], \quad \forall u \in S_c
\]

and

\[
\mathcal{P}_\mu(u) = \|\nabla u\|_2^2 - \mu \gamma_q \|u\|_q^q - \|u\|_{2^*}^{2^*} \\
\geq \|\nabla u\|_2^2 - \mu \gamma_q C_{N,q}^2 \|\nabla u\|_2^2 - S^{-\frac{2^*}{2^*}} \|u\|_2^{2^*} \\
\geq \|\nabla u\|_2^2 \left[ \frac{1}{2} - S^{-\frac{2^*}{2^*}} \|u\|_2^{-2^*} \right], \quad \forall u \in S_c.
\]

The above inequalities show there exists \(\vartheta_c > 0\) such that i) holds also.

ii). For any given \(w \in S_c\), we have \(\|t^{N/2} w_t\|_2 = \|w\|_2\), and so \(t^{N/2} w_t \in S_c\) for every \(t > 0\). Then (1.10) yields

\[
\Phi_\mu(t^{N/2} w_t) = \frac{t^2}{2} \|\nabla w\|_2^2 - \frac{\mu t^{\gamma_q}}{q} \|w\|_q^q - \frac{t^{2^*}}{2^*} \|w\|_{2^*}^{2^*} \rightarrow -\infty \quad \text{as} \quad t \rightarrow +\infty. \tag{4.41}
\]

Thus we can deduce that there exist \(t_1 > 0\) small enough and \(t_2 > 0\) large enough such that

\[
\left\| \nabla \left( t_1^{N/2} w_{t_1} \right) \right\|_2^2 = t_1^2 \|\nabla w\|_2^2 \leq \vartheta_c, \quad \text{and} \quad \Phi_\mu(t_2^{N/2} w_{t_2}) < 0. \tag{4.42}
\]

Let \(\gamma_0(t) := \left[ t_1 + (t_2 - t_1)t \right]^{N/2} w_{t_1 + (t_2 - t_1)t} \). Then \(\gamma_0 \in \hat{\Gamma}_{\mu,c}\), and so \(\hat{\Gamma}_{\mu,c} \neq \emptyset\). Now using the intermediate value theorem, for any \(\gamma \in \hat{\Gamma}_{\mu,c}\), there exists \(t_0 \in (0, 1)\), depending on \(\gamma\), such that \(\|\nabla \gamma(t_0)\|_2^2 = 2\vartheta_c\) and

\[
\max_{t \in [0,1]} \Phi_\mu(\gamma(t)) = \Phi_\mu(\gamma(t_0)) \geq \inf \{ \Phi_\mu(u) : u \in S_c, \|\nabla u\|_2^2 = 2\vartheta_c \},
\]

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which, together with the arbitrariness of \( \gamma \in \hat{\Gamma}_{\mu,c} \), implies

\[
\hat{M}_\mu(c) = \inf_{\gamma \in \hat{\Gamma}_{\mu,c}} \max_{t \in [0,1]} \Phi_\mu(\gamma(t)) \geq \inf \left\{ \Phi_\mu(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2\vartheta_c \right\} .
\] (4.43)

Hence, (4.40) follows directly from (4.38) and (4.43), and the proof is completed.

\[\square\]

**Lemma 4.11.** Let \( N \geq 3, c > 0, 0 < \mu < \alpha(N, q) \) and \( 2 + \frac{4}{N} \leq q < 2^* \). Then

\[
\hat{M}_\mu(c) = \hat{m}_\mu(c).
\] (4.44)

**Proof.** We first prove that \( \hat{M}_\mu(c) \leq \hat{m}_\mu(c) \). For any \( u \in \mathcal{M}_\mu(c) \), there exist \( t_1 > 0 \) small enough and \( t_2 > 1 \) large enough such that

\[
\|\nabla \left( t_1^{N/2} u_{t_1} \right) \|_2^2 = t_1^2 \|\nabla u\|_2^2 \leq \vartheta_c, \quad \text{and} \quad \Phi_\mu \left( t_2^{N/2} u_{t_2} \right) < 0.
\]

Letting

\[
\hat{\gamma}(t) := \left[ (1-t) t_1 + t t_2 \right]^{N/2} u_{(1-t) t_1 + t t_2}, \quad \forall \ t \in [0,1].
\]

Then \( \hat{\gamma} \in \hat{\Gamma}_{\mu,c} \). By (4.36) and the definition of \( \hat{M}_\mu(c) \), we have

\[
\hat{M}_\mu(c) \leq \max_{t \in [0,1]} \Phi_\mu(\hat{\gamma}(t)) = \Phi_\mu(u),
\]

and so \( \hat{M}_\mu(c) \leq \hat{m}_\mu(c) \).

On the other hand, by (4.35) with \( t \to 0 \), we have

\[
\mathcal{P}_\mu(u) \leq 2\Phi_\mu(u), \quad \forall \ u \in \mathcal{S}_c.
\]

which implies

\[
\mathcal{P}_\mu(\gamma(1)) \leq 2\Phi_\mu(\gamma(1)) < 0, \quad \forall \ \gamma \in \hat{\Gamma}_{\mu,c}.
\]

Since \( \|\gamma(0)\|_2^2 \leq \vartheta_c \), by (i) of Lemma 4.10 we have \( \mathcal{P}_\mu(\gamma(0)) > 0 \). Hence, any path in \( \hat{\Gamma}_{\mu,c} \) has to cross \( \mathcal{M}_\mu(c) \). This shows that

\[
\max_{t \in [0,1]} \Phi_\mu(\gamma(t)) \geq \inf_{u \in \mathcal{M}_\mu(c)} \Phi_\mu(u) = \hat{m}_\mu(c), \quad \forall \ \gamma \in \hat{\Gamma}_{\mu,c},
\]

and so \( \hat{M}_\mu(c) \geq \hat{m}_\mu(c) \) due to the arbitrariness of \( \gamma \). Therefore, \( \hat{M}_\mu(c) = \hat{m}_\mu(c) \) for any \( c > 0 \), and the proof is completed.

\[\square\]

Set

\[
\tilde{\Gamma}_{\mu,c} := \left\{ \tilde{\gamma} \in \mathcal{C}([0,1], \mathcal{S}_c \times \mathbb{R}) : \tilde{\gamma}(0) = (\gamma_1(0), 0), \|\nabla \gamma_1(0)\|_2^2 \leq \vartheta_c, \tilde{\Phi}_\mu(\tilde{\gamma}(1)) < 0 \right\} .
\]

Replace \( \Gamma_c \) and \( \tilde{\Gamma}_c \) in Lemmas 3.1 and 3.2 by \( \hat{\Gamma}_{\mu,c} \) and \( \tilde{\Gamma}_{\mu,c} \), respectively, we can prove the following lemma by the same arguments as in the proof of Lemma 3.2.

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Lemma 4.12. Let $N \geq 3$, $c > 0$, $0 < \mu < \alpha(N,q)$ and $2 + \frac{4}{N} \leq q < 2^*$. There exists a sequence $\{u_n\} \subset S_c$ such that
\[
\Phi_{\mu}(u_n) \to \hat{M}_\mu(c) > 0, \quad \Phi_{\mu}|_{S_c}(u_n) \to 0 \quad \text{and} \quad \mathcal{P}_{\mu}(u_n) \to 0.
\] (4.45)

Lemma 4.13. Let $N \geq 3$, $c > 0$, $0 < \mu < \alpha(N,\mu)$ and $2 + \frac{4}{N} \leq q < 2^*$. Then the function $c \mapsto \hat{m}_\mu(c)$ is nonincreasing on $(0, +\infty)$. In particular, if $\hat{m}_\mu(c_1)$ is achieved, then $\hat{m}_\mu(c_1) > \hat{m}_\mu(c_2)$ for any $c_2 > c_1$.

Proof. For any $c_2 > c_1 > 0$, it follows from the definition of $\hat{m}_\mu(c_1)$ that there exists $\{u_n\} \subset \mathcal{M}_\mu(c_1)$ such that
\[
\Phi_{\mu}(u_n) < \hat{m}_\mu(c_1) + \frac{1}{n}, \quad \forall \ n \in \mathbb{N}.
\] (4.46)

Let $\tau := \sqrt{c_2/c_1}$ and $v_n(x) := \tau^{(2-N)/2}u_n(x/\tau)$. Then $\|\nabla v_n\|^2 = \|\nabla u_n\|^2$, $\|v_n\|_2^2 = \|u_n\|_2^2$, and $\|v_n\|_2^2 = c_2$. By Lemma 4.8 there exists $t_n > 0$ such that $t_n^{N/2}(v_n)_{t_n} \in \mathcal{M}_\mu(c_2)$. Then it follows from (4.10), (4.46) and Corollary 4.7 that
\[
\hat{m}_\mu(c_2) \leq \Phi_{\mu}(t_n^{N/2}(v_n)_{t_n}) \leq \Phi_{\mu}(u_n) < \hat{m}_\mu(c_1) + \frac{1}{n},
\] (4.47)

which shows that $\hat{m}_\mu(c_2) \leq \hat{m}_\mu(c_1)$ by letting $n \to \infty$.

If $\hat{m}_\mu(c_1)$ is achieved, i.e., there exists $\tilde{u} \in \mathcal{M}_\mu(c_1)$ such that $\Phi_{\mu}(\tilde{u}) = \hat{m}_\mu(c_1)$. By the same argument as in (4.47), we can obtain that $\hat{m}_\mu(c_2) < \hat{m}_\mu(c_1)$.

Next, we give a precise estimation for the energy level $\hat{M}(c)$ given by (4.40), which helps us to restore the compactness in the critical case. Different from the strategy in [13], we shall introduced an alternative choice of testing functions which allows to treat, in a unified way, the case $N = 3$, the case $N \geq 4$, the $L^2$-critical case $q = \tilde{q}$ and the $L^2$-supercritical case $q > \tilde{q}$ for (1.1). To this end, for any fixed $c > 0$, we choose $R_n > n^{2/3}$ be such that
\[
c = \omega_{N-1} A_N^2 \left\{ \frac{1}{n^2} \int_0^{n^{5/3}} \frac{s^{N-1}}{(1 + s^2)^{N/2}}ds + \left( \frac{n}{1 + n^{10/3}} \right)^{N-2} \times \frac{2R_n^{N+2} - [(N + 1)(N + 2)R_n^2 + 2N(N + 2)R_n n^{2/3} - N(N + 1)n^{4/3}] n^{2N/3}}{N(N + 1)(N + 2)(R_n - n^{2/3})^2} \right\}.
\] (4.48)
Note that
\[
\int_0^{n^{5/3}} \frac{s^2}{1 + s^2} ds = n^{5/3} - \tan \left( n^{5/3} \right),
\] (4.49)
and
\[
\int_0^{n^{5/3}} \frac{s^3}{(1 + s^2)^2} ds = \frac{1}{2} \left[ \log \left( 1 + n^{10/3} \right) - \frac{n^{10/3}}{1 + n^{10/3}} \right]
\] (4.50)
From (4.48), (4.49), (4.50) and (4.51), one can deduce that
\[
\int_{\mathbb{R}^N} \nabla \cdot \left( N/\mathbf{U} \right) \times \nabla \rho \nabla \nabla \left( \frac{\mathbf{U}^2}{N} \right) = \omega A \nabla \cdot \left( \nabla \cdot \mathbf{A} \right)
\] (4.52).

Now, we define function \( \tilde{\Theta}_n(x) := \hat{\Theta}_n(|x|) \), where
\[
\tilde{\Theta}_n(r) = A_N \begin{cases} 
\left( \frac{n}{1+n^{10/3}/n} \right)^{N-2}, & 0 \leq r < n^{2/3}; \\
\left( \frac{n}{1+n^{10/3}} \right)^{N-2} \frac{R_n - r}{R_n - n^{2/3}}, & n^{2/3} \leq r < R_n; \\
0, & r \geq R_n.
\end{cases}
\] (4.53)

Computing directly, we have
\[
\begin{align*}
\| \nabla \tilde{U}_n \|_2^2 &= \int_{\mathbb{R}^N} |\nabla \tilde{U}_n|^2 dx = \omega_{N-1} \int_0^{r_n} r^{N-1} |\tilde{\Theta}_n(r)|^2 dr \\
&= \omega_{N-1} A_N^2 \left\{ \int_0^{n^{2/3}} \frac{n^{N-2} r^{N-1}}{(1 + n^{-2} r^2)^{N-2}} dr + \left( \frac{n}{1+n^{10/3}} \right)^{N-2} \int_{n^{2/3}}^{R_n} r^{N-1} \frac{(R_n - r)^2}{(R_n - n^{2/3})^2} dr \right\} \\
&= \omega_{N-1} A_N^2 \left\{ \frac{1}{n^{2/3}} \int_0^{n^{5/3}} \frac{s^{N-1}}{(1 + s^2)^{N-2}} ds + \left( \frac{n}{1+n^{10/3}} \right)^{N-2} \int_{n^{2/3}}^{R_n} r^{N-1} \frac{(R_n - r)^2}{(R_n - n^{2/3})^2} dr \right\} \\
&= \frac{2R_n^{N+2} - [(N + 1)(N + 2)R_n^2 + 2N(N + 2)R_n n^{2/3} - N(N + 1) n^{4/3} n^{2N/3}]}{N(N + 1)(N + 2)(R_n - n^{2/3})^2}
\end{align*}
\] (4.54)
\[
\|\tilde{U}_n\|_{2^*}^2 = \int_{\mathbb{R}^N} |\tilde{U}_n|^{2^*} \, dx = \omega_{N-1} \int_0^{+\infty} r^{N-1} |\tilde{\Theta}_n(r)|^{2^*} \, dr
\]
\[
= \omega_{N-1} A_N^{2^*} \left[ \int_0^{n^{2/3}} 1 \left( 1 + \frac{n^{10/3}}{1 + n^{2/3}} \right)^{N} + \left( \frac{n}{1 + n^{10/3}} \right)^{N/2} \right] \\
= \omega_{N-1} A_N^{2^*} \left[ \int_0^{n^{5/3}} 1 \left( 1 + \frac{n^{2/3}}{1 + n^{5/3}} \right)^{N} + \left( \frac{n}{1 + n^{10/3}} \right)^{N/2} \right] \\
= S^{N/2} + \omega_{N-1} A_N^{2^*} \left[ - \int_{n^{5/3}}^{+\infty} \frac{S^{N-1}}{1 + n^{5/3}} \, ds \right] \\
= S^{N/2} + O \left( \frac{1}{n^{14/3}} \right), \quad n \to \infty
\]

and
\[
\|\tilde{U}_n\|_q^q = \int_{\mathbb{R}^N} |\tilde{U}_n|^q \, dx = \omega_{N-1} \int_0^{+\infty} r^{N-1} |\tilde{\Theta}_n(r)|^q \, dr \\
\geq \omega_{N-1} A_N^q \left[ \int_0^{n^{2/3}} 1 \left( 1 + n^{2/3} \right)^{N-1} \, dr \right] \\
= \omega_{N-1} A_N^q \left[ \int_0^{n^{2/3}} 1 \left( 1 + n^{2/3} \right)^{N-1} \, dr \right] \\
= O \left( \frac{1}{n^{N-(N-2)q/2}} \right), \quad n \to \infty
\]

Both (4.54) and (4.55) imply that \( \tilde{U}_n \in S_c \).

**Lemma 4.14.** Let \( N \geq 3, \ c > 0, \ 0 < \mu < \alpha(N,q) \) and \( 2 + \frac{1}{N} \leq q < 2^* \). Then there exists \( \bar{n} \in \mathbb{N} \) such that
\[
\tilde{M}_\mu(c) \leq \sup_{t > 0} \Phi \left( t^{N/2}(\tilde{U}_n)_t \right) < \frac{S^{N/2}}{N}.
\]

**Proof.** From (1.10), (4.55), (4.56) and (4.57), we have
\[
\Phi(t^{N/2}(\tilde{U}_n)_t) = \frac{t^2}{2} \|\nabla \tilde{U}_n\|_2^2 - \frac{\mu t^{(q-2)N/2}}{q} \|\tilde{U}_n\|_q^q - \frac{t^{2^*}}{2^*} \|\tilde{U}_n\|_{2^*}^{2^*}
\]
\[
\leq \frac{t^2}{2} \left[ S^{N/2} + O \left( \frac{1}{n^{14(N-2)/3N}} \right) \right] - \frac{t^{2^*}}{2^*} \left[ S^{N/2} + O \left( \frac{1}{n^{14/3}} \right) \right] \\
- t^{(q-2)N/2} \left[ O \left( \frac{1}{n^{N-(N-2)q/2}} \right) \right] \\
\leq S^{N/2} \left( \frac{t^2}{2} - \frac{t^{2^*}}{2^*} \right) + t^{2^*} \left[ O \left( \frac{1}{n^{14(N-2)/3N}} \right) \right] - t^{(q-2)N/2} \left[ O \left( \frac{1}{n^{N-(N-2)q/2}} \right) \right], \quad \forall \ t > 0.
\]

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Hence, it follows from (4.27), (4.40), (4.59) and the fact $2 + \frac{4}{N} \leq q < 2^*$ that there exists $\bar{n} \in \mathbb{N}$ such that (1.58) holds.

\[ \boxempty \]

**Proof of Theorem 1.4.** In view of Lemmas 4.12 and 4.14, there exists \( \{ u_n \} \subset S_c \) such that
\[
\| u_n \|_2^2 = c, \quad \Phi_\mu(u_n) \rightarrow \hat{M}_\mu(c) < \frac{1}{N} S^{N/2}, \quad \Phi_\mu(u_n)|_{S_c} \rightarrow 0, \quad P_\mu(u_n) \rightarrow 0. \tag{4.60}
\]
which, together with (1.10) and (1.11) that
\[
\hat{M}_\mu(c) + o(1) = \frac{1}{2} \| \nabla u_n \|_2^2 - \frac{1}{2^*} \| u_n \|_2^{2^*} - \frac{\mu}{q} \| u_n \|_q^q \tag{4.61}
\]
and
\[
o(1) = \| \nabla u_n \|_2^2 - \| u_n \|_2^{2^*} - \mu q \| u_n \|_q^q. \tag{4.62}
\]
It follows from (4.61) and (4.62) that
\[
\hat{M}_\mu(c) + o(1) = \frac{1}{N} \| u_n \|_2^{2^*} + \mu \left( \frac{q \gamma_q - 2}{2q} \| u_n \|_q^q \right). \tag{4.63}
\]
This shows that \( \{ \| u_n \|_2^{2^*} \} \) is bounded. From (4.62), (4.63) and the H"older inequality, one has
\[
\| \nabla u_n \|_2^2 = \| u_n \|_2^{2^*} + \mu \gamma_q \| u_n \|_q^q + o(1) \leq \| u_n \|_2^{2^*} + \mu \gamma_q \| u_n \|_2^{2(2^*-q)/(2^*-2)} \| u_n \|_2^{2^*(q-2)/(2^*-2)} + o(1)
\]
and
\[
= \| u_n \|_2^{2^*} + \mu \gamma_q c^{(2^*-q)/(2^*-2)} \| u_n \|_2^{2^*(q-2)/(2^*-2)} + o(1). \tag{4.64}
\]
This shows that \( \{ u_n \} \) is bounded in \( H^1(\mathbb{R}^N) \). Let \( \delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \mathrm{d}x \). We show that \( \delta > 0 \). Otherwise, in light of Lions’ concentration compactness principle [18, Lemma 1.21], \( \| u_n \|_q \to 0 \). From (4.61) and (4.62), one can get
\[
\frac{1}{N} \| u_n \|_2^{2^*} = \hat{M}_\mu(c) + o(1), \tag{4.65}
\]
which, together with (4.62), yields
\[
\| \nabla u_n \|_2^2 = \| u_n \|_2^{2^*} + \mu \gamma_q \| u_n \|_q^q + o(1) = N \hat{M}_\mu(c) + o(1). \tag{4.66}
\]
Hence, it follows from (1.22), (4.65) and (4.66) that
\[
N \hat{M}_\mu(c) + o(1) = \| u_n \|_2^{2^*} \leq \left( \frac{\| \nabla u_n \|_2^2}{S} \right)^{\frac{N}{2^*}} = \left( \frac{N \hat{M}_\mu(c) + o(1)}{S} \right)^{\frac{N}{2^*}} = \left( \frac{N \hat{M}_\mu(c)}{S} \right)^{\frac{N}{2^*}} + o(1). \tag{4.67}
\]
Consequently, $\hat{M}_\mu(c) \geq \frac{1}{N}S^{N/2}$, which contradicts (4.58). Thus $\delta > 0$. Without loss of generality, we may assume the existence of $y_n \in \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \frac{\delta}{2}$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have

$$\|\hat{u}_n\|_2^2 = c, \quad \mathcal{P}_\mu(\hat{u}_n) \to 0, \quad \Phi_\mu(\hat{u}_n) \to \hat{M}_\mu(c), \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \frac{\delta}{2}. \quad (4.68)$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{align*}
\hat{u}_n \to \hat{u}, & \quad \text{in } H^1(\mathbb{R}^N); \\
\hat{u}_n \to \hat{u}, & \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N), \forall s \in [1,2^*]; \\
\hat{u}_n \to \hat{u}, & \quad \text{a.e. on } \mathbb{R}^N.
\end{align*} \quad (4.69)$$

By Lemma 2.3 one has

$$\Phi_\mu'(\hat{u}_n) - \lambda_n \hat{u}_n \to 0, \quad (4.70)$$

where

$$\lambda_n = \frac{1}{\|\hat{u}_n\|_2^2} \langle \Phi_\mu'(\hat{u}_n), \hat{u}_n \rangle = \frac{1}{c} \left[ \|\nabla \hat{u}_n\|_2^2 - \mu \|\hat{u}_n\|_{q^*}^2 - \|\hat{u}_n\|_{2^*}^2 \right]. \quad (4.71)$$

Since $\{\hat{u}_n\}$ is bounded in $H^1(\mathbb{R}^N)$, it follows from (1.71) that $\{\lambda_n\}$ is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that $\lambda_n \to \lambda$. By a standard argument, we can deduce

$$\Phi_\mu'(\hat{u}) - \lambda \hat{u} = 0. \quad (4.72)$$

Hence, by Lemma 3.3 one has

$$\mathcal{P}_\mu(\hat{u}) = \|\nabla \hat{u}\|_2^2 - \mu \gamma_q \|\hat{u}\|_q^2 - \|\hat{u}\|_{2^*}^2 = 0. \quad (4.73)$$

Combining (4.72) with (4.73), it is easy to deduce that $\lambda < 0$. Set $\|\hat{u}\|_2^2 := \hat{c}$. Then $0 < \hat{c} \leq c$, and (4.73) shows that $\hat{u} \in \mathcal{M}_\mu(\hat{c})$. From (1.10), (1.11), (4.35), (4.70), (4.71), Lemma 4.12 and the weak semicontinuity of norm, one has

$$\begin{align*}
\hat{M}_\mu(c) &= \lim_{n \to \infty} \left[ \Phi_\mu(\hat{u}_n) - \frac{1}{2} \mathcal{P}_\mu(\hat{u}_n) \right] \\
&= \lim_{n \to \infty} \left[ \frac{1}{N} \|\hat{u}_n\|_{2^*}^2 + \frac{\mu}{2q} (q \gamma_q - 2) \|\hat{u}_n\|_q^2 \right] \\
&\geq \frac{1}{N} \|\hat{u}\|_{2^*}^2 + \frac{\mu}{2q} (q \gamma_q - 2) \|\hat{u}\|_q^2 \\
&= \Phi_\mu(\hat{u}) - \frac{1}{2} \mathcal{P}_\mu(\hat{u}) = \Phi_\mu(\hat{u}) \\
&\geq \hat{m}_\mu(\hat{c}) \geq \hat{m}_\mu(c) = \hat{M}_\mu(c),
\end{align*}$$

which implies

$$\|\hat{u}\|_2^2 = \hat{c}, \quad \Phi_\mu(\hat{u}) = \hat{m}_\mu(\hat{c}) = \hat{m}_\mu(c), \quad \mathcal{P}_\mu(\hat{u}) = 0. \quad (4.74)$$
This shows \( m(\hat{c}) \) is achieved. In view of Lemma 4.13, \( \hat{c} = c \). Thus,

\[
\lambda_c < 0, \quad \|\tilde{u}\|^2 = c, \quad \Phi_{\mu}(\tilde{u}) = \tilde{m}_{\mu}(c), \quad P_{\mu}(\tilde{u}) = 0, \quad (4.75)
\]

Both (4.72) and (4.75) imply the conclusions of Theorem 1.4 hold.

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