A numerical algorithm to find a root of non-linear equations using Householder’s method

Srinivasarao Thota
Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Adama, Ethiopia

ABSTRACT

In this paper, we propose a three-step algorithm using Householder’s method for finding an approximate root of the given non-linear equations in one variable. Several numerical examples are presented to illustrate and validate the proposed method. Implementation of the proposed algorithm in Maple is also discussed with sample computations.

1. INTRODUCTION

Finding an approximate solution of a given non-linear equation is one of the important topics in numerical analysis. Many mathematical models of physics, economics, engineering, science and any other disciplines come up with non-linear equations of the type . In recent time, several scientists and engineers have been focused on to solve the non-linear equations both numerically and analytically. In the literature, there are several iterative methods/algorithms available using different techniques such as homotopy, interpolation, Taylor’s series, quadrature formulas, decomposition etc. with modifications and improvements, and different hybrid iterative methods, see, for example [1]-[17]. There are several methods available to find an approximate solution of non-linear systems also, see for example, [18]-[23]. In generally, the roots of non-linear or transcendental equations cannot be expressed in closed form or cannot be computed analytically. The root-finding algorithms provide us to compute approximations to the roots; these approximations are expressed either as small isolating intervals or as isolating point numbers. In this paper, we use the Householder’s technique to create a three-step iterative algorithm for solving the given non-linear equation of single variable with converging order more than or equal to four. Indeed, we focus on creating a tree-step algorithm to find a root of a given non-linear equation using the well-known techniques namely, Householder’s technique and modified Newton’s method. We also focus on its Maple implementation with sample computations. The proposed algorithm is simple to understand and efficient than some existing methods.

Since, we focus on creating an efficient and simple three-step algorithm using Householder’s method and modified Newton’s method, we recall basic concepts related to the existing methods. In this paper, we
consider the non-linear equations of the following type (1).

\[ f(x) = 0, \]  

where \( f : I \rightarrow \mathbb{R} \) is a scalar function in single variable for an open interval \( I \subseteq \mathbb{R} \) and continuously differentiable function. In numerical analysis, the order of convergence of a convergent sequence are quantities that represent how quickly the sequence approaches its limit. The order of converges of any iterative method is defined [13] as

\[ |e_{n+1}| \leq c|e_n|^p, \]

where \( p \) is the order of convergence and \( c \) is a positive finite constant. Householder’s method is one of the famous methods in producing a sequence of approximation roots of (1) with initial point \( x_0 \). The Householder’s method has cubic order of convergence [1]-[2] and is given as (2)

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(f(x_n))^2 f''(x_n)}{2(f'(x_n))^3}, \quad n = 0, 1, 2, \ldots \]  

In the literature, there are several modifications of Householder’s method have been developed and analyzed by various techniques, for example, see [1]-[3], [24]. The modified Householder’s methods give better performance than the Newton’s method.

Recall the predictor-corrector method developed by Traub [4] in 1864 as follows, for \( f'(x_n) \neq 0 \),

\[ y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \]
\[ x_n = y_n - \frac{f(y_n)}{f'(y_n)}. \]  

A modified Newton’s method developed by McDougall and Wotherspoon [25] is converged faster than the Newton’s method with a convergence order \( 1 + \sqrt{2} \), and it is given as follows: Let \( x_0 = y_0 \), for \( n = 0 \) (initially), compute

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \]

and for \( n \geq 1 \),

\[ y_n = x_n - \frac{f(x_n)}{f'(\frac{x_{n-1}+y_{n-1}}{2})}, \]
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n+y_n}{2})}. \]  

If \( x_n = y_n \), for all \( n \) in equation (4), then the modified Newton’s method developed by McDougall and Wotherspoon becomes Newton’s predictor-corrector method as given in equation (3). The rest of paper is organized as follows: Section 2, presents the methodology and steps involving in the proposed algorithm and the flow chart; Section 3, presents couple of numerical examples to illustrate and validate the proposed algorithm; and Section 4, presents the Maple implementation of the proposed algorithm with sample computations.

2. PROPOSED ITERATIVE ALGORITHM

In this section, we present a three-step iterative algorithm using the well-known Householder’s method given in (2) and a modified Newton’s method given in equation (4). The proposed iterative formula under consideration is given as: set \( x_0 = y_0 \), for \( n = 0 \) (initially), compute

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{(f(x_0))^2 f''(x_0)}{2(f'(x_0))^3}, \]
and for \( n \geq 1 \),

\[
y_n = x_n - \frac{f(x_n)}{f'(x_{n-1} + y_{n-1}/2)} = \frac{(f(x_n))^2 f''(x_n)}{2 \left(f'(x_{n-1} + y_{n-1}/2)\right)^3},
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n-1} + y_{n-1}/2)^2} = \frac{(f(x_n))^2 f''(x_n)}{2 \left(f'(x_{n-1} + y_{n-1}/2)\right)^3}.
\]

(5)

If \( x_n = y_n \), for all \( n \) and \( f''(x_n) \neq 0 \) in equation (4), then the proposed algorithm becomes Householder’s predictor-corrector method and it has fourth-order convergence [3, 24]. If \( x_n = y_n \), for all \( n \) and \( f''(x_n) = 0 \) in (4), then the proposed algorithm becomes Newton’s predictor-corrector method as given in (3).

2.1. Steps for calculating root

The following steps are involved to compute an approximate root of a given using the proposed algorithm.

1. Choose an initial approximation \( x_0 \in \mathbb{R} \) with \( x_0 = y_0 \) for \( n = 0 \).
2. Compute \( x_1 \) using formula given in (5). i.e.,

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{(f(x_0))^2 f''(x_0)}{2(f'(x_0))^3},
\]

3. For \( n \geq 1 \), compute \( y_n \) and \( x_{n+1} \) using the proposed algorithm. i.e.,

\[
y_n = x_n - \frac{f(x_n)}{f'(x_{n-1} + y_{n-1}/2)^2} = \frac{(f(x_n))^2 f''(x_n)}{2 \left(f'(x_{n-1} + y_{n-1}/2)\right)^3},
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_{n-1} + y_{n-1}/2)^2} = \frac{(f(x_n))^2 f''(x_n)}{2 \left(f'(x_{n-1} + y_{n-1}/2)\right)^3}.
\]

4. Repeat Step 3 until we get desired approximate root, for \( n = 1, 2, 3 \ldots \).

Flow chat of the proposed algorithm is presented in Figure 1.
3. NUMERICAL EXAMPLES

In this section, we present couple of numerical examples to illustrate the proposed algorithm and comparisons are made to confirm that the algorithm gives solution faster than existing methods.

Consider the following nonlinear equation to compute an approximate root using the proposed method.

\[ f(x) = xe^x - 1. \]  \hspace{1cm} (6)

The exact root of the equation is \( x = 0.5671432904 \). Suppose the initial approximation is \( x_0 = 3 \). The stopping condition is corrected to ten decimal places. Set \( x_0 = y_0 = 3 \), compute \( x_1 \) as follows

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.922456285. \]

Now, we compute \( y_n \) and \( x_{n+1} \), for \( n = 1 \), using the proposed algorithm, as follows

\[ y_1 = x_1 - \frac{f(x_1)}{f'(\frac{x_0+y_0}{2})} = 1.767472904, \]
\[ x_2 = x_1 - \frac{f(x_1)}{f'(\frac{x_1+y_1}{2})} = 0.9087794052. \]

Similarly, we can compute \( y_n \) and \( x_{n+1} \), for \( n > 1 \), using the proposed algorithm, as follows

\[ y_2 = 0.8380960197, \quad x_3 = 0.5661945014, \]
\[ y_3 = 0.5667781506, \quad x_4 = 0.5671428368, \]
\[ y_4 = 0.5671432908, \quad x_5 = 0.5671432903, \]
\[ y_5 = 0.5671432904, \quad x_6 = 0.5671432904. \]

Hence the required approximate root of the given (6) is 0.5671432904 up to ten decimal places and with the tolerance error \( 1 \times 10^{-10} \). The graphical representations (using Maple) of the iterations values at different iterations is shown in Figure 2, absolute errors at different iterations are shown in Figure 3 and absolute function values at different iterations are shown in Figure 4.

The following non-linear equations with initial approximations to compare the number of iterations taken to obtain the approximate root using existing methods and proposed method. For simplicity, we take...
We compare the proposed method with various existing methods such as Newton’s Method (NM), Kang et al. Method (KM) [26], McDougall-Wotherspoon Method (MWM) [25], Neamvdonk Method (ANM) [27]-[29] and Proposed Method (PM) to show the efficiency of the proposed method. Table 1, shows the number of iterations taken by various methods to obtain the required approximate root.

| Function | Approx. Root | NM  | KM  | MWM | ANM | PM |
|----------|--------------|-----|-----|-----|-----|----|
| $f_1(x) = x^2 - (1 - x)^5$, | 0.345954815848 | 12 | 6 | 9 | 6 | 6 |
| $f_2(x) = x^3 - e^{-x}$, | 0.772882959149 | 10 | 5 | 7 | 6 | 4 |
| $f_3(x) = -20x^5 - \frac{x}{2} + \frac{1}{2}$, | 0.427677296931 | 10 | 5 | 8 | 6 | 4 |
| $f_4(x) = \cos(x) - x^3$, | 0.865474033102 | 10 | 5 | 7 | 6 | 4 |
| $f_5(x) = (e^x + x - 20)^3$, | 0.865474033102 | 44 | 14 | 33 | 10 | 12 |
| $f_6(x) = x - 3 \ln(x)$, | 0.865474033102 | 7 | 4 | 5 | 5 | 2 |

From Table 1, one can observe that the proposed method (PM) gives the required approximate root faster than the other existing methods (i.e., in lowest iteration number).

4. MAPLE IMPLEMENTATION OF PROPOSED ALGORITHM

In this section, we present the implementation of the proposed algorithm in Maple.

Input: given $f(x)$, initial approximation $x[0]$, tolerance $\epsilon$, correct to decimal places $\delta$, maximum number of iterations $n$.
Output: approximate solution.

4.1. Pseudo-code

- Set $x[0] = y[0]$ and compute $x[1]$ using formula in (5).
- for $i$ from 1 to $n$ do
  - compute $y[i]$ and $x[i+1]$ using formulas in (5).
  - if abs($x[i+1] - x[i]$) < $\epsilon$ and abs($f(x[i+1])$) < $\delta$ then break;
- output $x[i+1]

4.2. Maple programming

We present the maple code of the proposed algorithm as follows.

```maple
> eps := TYPE; # epsilon value
> epsabs := TYPE; # delta value
> f(x) := TYPE; # given non-linear equation
> fd:=D(f);
> fdd:=D(D(f));
```
\[ x[0] := \text{TYPE}; \quad \# \text{initial approximation} \]
\[ y[0] := x[0]; \]
\[ n := \text{TYPE}; \quad \# \text{required number of iterations} \]
\[ x[1] := x[0] - (f(x[0]))/(f'(x[0])) - ((f(x[0]))^2*f''(x[0]))/(2*(f'(x[0]))^3)); \]
\[ \text{for } i \text{ from 1 to } n \text{ do} \]
\[ y[i] := x[i] - (f(x[i]))/(f'((x[i-1]+y[i-1])/2)) - ((f(x[i]))^2*f''(x[i]))/(2*(f'((x[i-1]+y[i-1])/2))^3)); \]
\[ x[i+1] := x[i] - (f(x[i]))/(f'((x[i]+y[i])/2)) - ((f(x[i]))^2*f''(x[i]))/(2*(f'((x[i]+y[i])/2))^3)); \]
\[ \text{abserror}[i] := \text{abs}(x[i+1]-x[i]); \]
\[ \text{absfun}[i] := \text{abs}(f(x[i+1])); \]
\[ \text{if abs}(x[i+1]-x[i]) < \epsilon \text{ or abs}(f(x[i+1])) < \epsilon_{\text{abs}} \text{ then break; } \]
\[ \text{end if; } \]
\[ \text{printf("Iteration No: } %g \text{ = } %10.10f, i, x[i+1]); \]
\[ \text{end do; } \]

### 4.3. Sample computations

Recall the function \( f_6(x) \) from Example 3. to compute an approximate solution using the maple implementation.

\[ \epsilon := 0.00000001; \]
\[ \epsilon_{\text{abs}} := 0.00000001; \]
\[ f(x) := x - 3 \ln(x); \]
\[ f'(x) := D(f); \]
\[ f''(x) := D(D(f)); \]
\[ x[0] := 0.5; y[0] := 0.5; \]
\[ n := 15; \]
\[ x[1] := x[0] - (f(x[0]))/(f'(x[0])) - ((f(x[0]))^2*f''(x[0]))/(2*(f'(x[0]))^3)); \]
\[ \text{for } i \text{ from 1 to } n \text{ do} \]
\[ y[i] := x[i] - (f(x[i]))/(f'((x[i-1]+y[i-1])/2)) - ((f(x[i]))^2*f''(x[i]))/(2*(f'((x[i-1]+y[i-1])/2))^3)); \]
\[ x[i+1] := x[i] - (f(x[i]))/(f'((x[i]+y[i])/2)) - ((f(x[i]))^2*f''(x[i]))/(2*(f'((x[i]+y[i])/2))^3)); \]
\[ \text{abserror}[i] := \text{abs}(x[i+1]-x[i]); \]
\[ \text{absfun}[i] := \text{abs}(f(x[i+1])); \]
\[ \text{if abs}(x[i+1]-x[i]) < \epsilon \text{ or abs}(f(x[i+1])) < \epsilon_{\text{abs}} \text{ then break; } \]
\[ \text{end if; } \]
\[ \text{printf("Iteration No: } %g \text{ = } %10.10f, i, x[i+1]); \]
\[ \text{end do; } \]
5. CONCLUSION

In this paper, we proposed a three-step iterative algorithm for finding an approximate root of a given non-linear equation in one variable with the help of Householder’s method and modified Newton’s method. Maple implementation of the proposed algorithm is also discussed with sample computations. Couple of numerical examples are presented to illustrate and validation of the proposed method. Comparisons of the proposed method with existing methods are discussed and the results showed that the proposed algorithm required less number of iterations (see, for example, Table 1). As future work, one can use this algorithm for finding the approximate solution of the non-linear equations which have multiple solutions as well as for system of non-linear equations.

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BIOGRAPHIES OF AUTHORS

Dr. Srinivasarao Thota (Corresponding Author) completed his M.Sc. in Mathematics from Indian Institute of Technology (IIT) Madras and Ph.D. in Mathematics from Motilal Nehru National Institute of Technology (NIT) Allahabad, India. Srinivasarao Thota’s areas of research interest are Computer Algebra (symbolic methods for differential equations), Numerical Analysis (root finding algorithms) and Mathematical Modelling (ecology). He has published his research work in several international journals and presented his work in different national and international conferences as oral presenter and expertise/invited presenter in different countries. Presently working at Department of Applied Mathematics, Adama Science and Technology University, Ethiopia. Further info on his homepage: https://srinivasaraothota.webs.com/