Theorems on shear-free perfect fluids with their Newtonian analogues

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Abstract

In this paper we provide fully covariant proofs of some theorems on shear-free perfect fluids. In particular, we explicitly show that any shear-free perfect fluid with the acceleration proportional to the vorticity vector (including the simpler case of vanishing acceleration) must be either non-expanding or non-rotating. We also show that these results are not necessarily true in the Newtonian case, and present an explicit comparison of shear-free dust in Newtonian and relativistic theories in order to see where and why the differences appear.

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1 Introduction

This paper deals with shear–free perfect–fluid solutions of Einstein’s field equations. The motivation for this study comes, on the one hand, from some studies on kinetic theory (see [37] and references therein), and on the other, from the relations between relativistic cosmology and Newtonian cosmology. Concerning the former, when we consider isotropic solutions of the Boltzmann equation, that is, those for which there is a timelike congruence with \( \vec{u} \) as the unit tangent vector field such that the distribution function has the form \( f(x^a, E) \) with \( E = -u^a p_a \) (where \( p_a \) denotes the particle momentum), two important results follow: i) The energy-stress tensor computed from such a distribution has the perfect-fluid form with respect to \( \vec{u} \) (see for instance [33, 35]). ii) The unit tangent vector field is shear-free and in addition its expansion \( \theta \) and rotation \( \omega \) satisfy \( \omega \theta = 0 \) (see the proof in [37]). These results led to the formulation of a conjecture whose origin seems to be the Ph. D. thesis by Treciokas [36] (see [27] for more details). This conjecture can be expressed in the following form (here \( \rho \) and \( p \) are the energy density and pressure of the perfect fluid):

**Conjecture 1**: In general relativity, if the velocity vector field of a barotropic perfect fluid \((\rho + p \neq 0 \text{ and } p = p(\rho))\) is shear-free, then either the expansion or the rotation of the fluid vanishes.

While we are still probably a long way from settling the truth or falsity of Conjecture 1, it is something short of amazing that such a conjecture might be expected at all in general relativity. Consider for example the pressure-free (dust) case for which Ellis [15] showed that \( \sigma = 0 \Rightarrow \theta \omega = 0 \). This is a purely local result to which no corresponding Newtonian result appears to hold, as counterexamples can be explicitly exhibited [22]. Ellis’s theorem holds for arbitrarily weak fields and fluids of arbitrarily low density. Why then does the Newtonian approximation fail?

Knowing whether or not this conjecture is true, or at least to what extent it is valid, might be useful in seeking and studying new perfect-fluid solutions of Einstein’s field equations with a shear-free velocity vector field. With respect to this subject, there are some interesting studies of shear-free perfect-fluid models to be found in [2, 9, 10, 11]. On the other hand, it is important to remark that there are many known cases which are shear-free and either rotation-free or expansion-free. Some examples are: the Friedmann–Lemaître–Robertson–Walker space-times (see for instance [21, 23]), the Gödel solution [19, 21], spherically symmetric shear-free perfect-fluid solutions with expansion and equation of state [23], Winicour’s stationary dust solutions [10], and other examples of perfect-fluid stationary solutions with barotropic equation of state where the velocity vector field is aligned with a timelike Killing vector field.
The conjecture has been proved in some special cases. As far as we know, these cases are the following:

i. Spatially homogeneous space-times: Schücking [31] studied the case of dust \((p = 0)\); Banerji [4] studied the case with a linear equation of state \(p = (\gamma - 1) \rho\) with \(\gamma > 1\) and \(\gamma \neq 10/9\); and finally King and Ellis [24] showed the general case with \(\rho + p > 0\).

ii. Ellis [15] proved the conjecture for dust \((p = 0)\).

iii. Treciokas and Ellis [37] showed the case for incoherent radiation \((p = 1/3 \rho)\).

iv. The case in which the acceleration and the vorticity of the perfect fluid are parallel was shown by White and Collins [39].

v. The case in which the magnetic part of the Weyl tensor with respect to the velocity vector field vanishes was proved by Collins [8].

vi. Carminati [4] showed that for the case of Petrov type N the conjecture also holds.

vii. Lang and Collins [27] proved the same for the case in which the expansion and the energy density are functionally dependent \((\theta = \theta(\rho))\).

viii. Coley [7] considered the existence of a conformal Killing vector field parallel to the velocity vector field \(\vec{u}\), proving also the conjecture for this case.

ix. Finally, there are some recent partial results on Petrov type III by Carminati [5] and by Carminati and Cyganowski [6].

It should be stressed, however, that all the above results have been proved using either a particular tetrad or coordinate system. In this sense, there is a clear need for a fully covariant proof of some of the above partial theorems. Such covariant proofs are not only desirable on aesthetical grounds, but they may also be useful for the deeper understanding of why the theorems hold. They may thereby aid in the further development of the subject, perhaps helping in eventually proving or disproving the above Conjecture 1.

In this paper, we present fully covariant proofs of the theorems in two relevant cases: when the acceleration of the fluid vanishes [including the above case (ii) of dust] and when the acceleration is parallel to the vorticity vector [case (iv)].

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1'This was emphasized by G.F.R. Ellis in his plenary lecture during the past Indian International Conference on Gravitation and Cosmology (ICGC-95), held in Pune, December 1995. Two of us (JMMS and PS) were attending this lecture which aroused our interest on the subject independently.
Analogous covariant proofs can also be given for other cases (see [32]) by using similar procedures. In Section 2 we present the main equations and Sections 3 and 4 are devoted to the proofs of the two theorems. Finally Section 5 deals with Newtonian theory, and by attempting to follow our general relativistic proof, uncovers the reason for the failure of the Newtonian limit. In addition, all shear-free Newtonian universes are revealed by this analysis.

2 General results on shear-free perfect fluids.

Let us consider a perfect fluid with unit velocity vector field \( \vec{u} \), so that the energy-stress tensor reads
\[
T_{ab} = \varrho u_a u_b + p P_{ab},
\]
where \( \varrho \) and \( p \) are the energy density and the pressure, respectively, and
\[
P_{ab} = g_{ab} + u_a u_b, \quad P_{ab} = P_{(ab)}
\]
is the projector orthogonal to \( \vec{u} \), which has the standard properties
\[
P_{ab} P_{bc} = P_{ac}, \quad P_{aa} = 3, \quad P_{ab} u^b = 0.
\]

Let us summarize briefly the main concepts for the study of the kinematics of the velocity \( \vec{u} \). The derivative along \( \vec{u} \) of any tensor quantity with components \( A_{a_1 \ldots a_p b_1 \ldots b_q} \) will be denoted by
\[
u^a \nabla_a A_{b_1 \ldots b_q} \equiv \dot{A}_{b_1 \ldots b_q}.
\]
Some different names for this derivative are used in the literature: material derivative, convective derivative, time derivative with respect to \( \vec{u} \), etc (see [38]). Along this paper we shall always use the term time propagation along \( \vec{u} \). A particular and interesting case is the time propagation along \( \vec{u} \) of \( \vec{u} \) itself, which is called the acceleration vector field of the fluid and is denoted by
\[
a^a \equiv u^b \nabla_b u^a = \dot{u}^a.
\]
From this definition and taking into account the fact that \( \vec{u} \) is a unit vector field, it follows that the acceleration is orthogonal to \( \vec{u} \) and therefore it is a spacelike vector field. Moreover, the integral curves of \( \vec{u} \) are geodesic only when its acceleration vanishes, so that the fluid is said to be geodesic if the acceleration vanishes. Physically, the acceleration vector field represents the mixed effects of gravitational as well as inertial forces (see refs.[3, 16]).

Throughout this paper we use \( a, b, c, \ldots, h = 1, 2, 3, 4 \) for spacetime indices, while we use \( i, j, k, \ldots = 1, 2, 3 \) for indices in the Newtonian theory, see section 5.
The spatial part of the covariant derivative of \( \vec{u} \) decomposes in general into its irreducible parts with respect to the rotation group (see [16, 17, 26] and references therein) as follows

\[
\nabla_b u_a + a_a u_b = \frac{1}{3} \theta P_{ab} + \sigma_{ab} + \omega_{ab}.
\]

(6)

where

\[
\theta = \nabla_a u^a,
\]

(7)

\[
\sigma_{ab} = P(a \mathcal{P}^d b) \nabla_d u_c - \frac{1}{3} \theta P_{ab}, \quad \sigma_{ab} = \sigma_{(ab)}, \quad \sigma^a = 0, \quad \sigma_{ab} u^b = 0, \quad \omega_{ab} = \frac{1}{2} \eta^{abcd} u_b \omega_{cd}.
\]

(8)

\[
\omega_{ab} = P(a \mathcal{P}^d b) \nabla_d u_c, \quad \omega_{ab} = \omega_{[ab]}, \quad \omega_{ab} u^b = 0.
\]

(9)

These objects together with the acceleration \( \ddot{\vec{u}} \) form the kinematical quantities of \( \vec{u} \). The commonly used names for them are: expansion rate for \( \theta \), shear rate for \( \sigma_{ab} \) and rotation rate for \( \omega_{ab} \).

We can also introduce the vorticity vector field by

\[
\omega^a \equiv \frac{1}{2} \eta^{abcd} u_b \omega_{cd}
\]

(10)

from where we see that the vorticity is orthogonal to \( \vec{u} \), that is, \( \omega_a u^a = 0 \). Moreover, we can invert (10) in order to obtain the rotation rate in terms of the vorticity

\[
\omega_{ab} = \eta_{abcd} \omega^c u^d
\]

(11)

where \( \eta_{abcd} \) is the canonical volume element in the spacetime (see, for instance, [14, 21, 25]). From (11) it follows directly that \( \omega_{ab} \omega^b = 0 \). We also define the rotation scalar by

\[
\omega^2 \equiv \frac{1}{2} \omega_{ab} \omega^b = \omega^a \omega_a,
\]

(12)

and, given that \( \omega_{ab} \) is a spatial tensor (that is, completely orthogonal to \( \vec{u} \)), \( \omega^2 \) is non-negative and the following equivalences hold:

\[
\omega = 0 \iff \omega^a = 0 \iff \omega_{ab} = 0.
\]

(13)

From (11) we can immediately get the following standard identities

\[
\omega_a \omega_b = \omega_a \omega_b - \omega^2 P_a^b \quad \implies \quad \omega_a \omega_c \omega_d \omega^b = -\omega^2 \omega_a^b.
\]

(14)

Finally, let us remark an important fact concerning \( \omega_{ab} \). Taking into account the following expression which comes directly from (E)

\[
\nabla_{[a} u_{b]} = a_{[a} u_{b]} + \omega_{ba},
\]

(15)

it follows that

\[
u_{[c} \nabla_a u_{b]} = 0 \iff \omega_{ab} = 0.
\]

(16)
and using Frobenius’s theorem (see [18, 25]), this means that there exist locally functions \( f \) and \( h \) such that \( u_a = h \nabla_a f \), (or equivalently, \( \vec{u} \) generates orthogonal spacelike hypersurfaces), if and only if the rotation vanishes.

The fluid (or also \( \vec{u} \)) is said to be shear-free if

\[
\sigma_{ab} = 0
\]

which we assume from now on. In this case, equation (3) becomes simply

\[
\nabla_b u_a = \frac{1}{3} \theta P_{ab} + \omega_{a b} - a_a u_b.
\]

Then, by using the Ricci identities for \( \vec{u} \)

\[
(\nabla_c \nabla_d - \nabla_d \nabla_c) u^a = R^a_{abcd} u^b
\]

we can compute the time propagation along \( \vec{u} \) of (18), which produces the following set of evolution equations [12, 13, 20] (we use units with \( 8\pi G = c = 1 \))

\[
\dot{\theta} + \frac{1}{3} \theta^2 - 2 \omega^2 - \nabla_a a^a + \frac{1}{2} (\varrho + 3p) = 0,
\]

\[
P_a^c P_b^d \omega_{cd} + \frac{2}{3} \theta \omega_{ab} - P_{[a}^c P_{b]}^d \nabla_d a_c = 0,
\]

\[
E_{ab} = -\omega_a \omega_b + \frac{1}{3} \omega^2 P_{ab} + a_a a_b + P_{[a}^c P_{b]}^d \nabla_d a_c - \frac{1}{3} P_{ab} \nabla_c a^c.
\]

Here \( E_{ab} \) is the so-called electric part of the Weyl tensor with respect to \( \vec{u} \). This and the magnetic part \( H_{ab} \) relative to \( \vec{u} \) determine completely the Weyl tensor \( C_{acbd} \), and are defined respectively by

\[
E_{ab} \equiv C_{acbd} u^c u^d \quad \Rightarrow \quad E_{ab} = E_{(ab)}, \quad E_{ab} u^b = 0, \quad E^a_a = 0,
\]

\[
H_{ab} \equiv \frac{1}{2} \eta_{ae}^{cd} C_{cdij} u^e u^f \quad \Rightarrow \quad H_{ab} = H_{(ab)}, \quad H_{ab} u^b = 0, \quad H^a_a = 0.
\]

These tensors were first introduced by Matte [29] in the context of the study of gravitational radiation.

Equation (20) is the famous Raychaudhuri equation [30], and Eq.(21) can also be given in terms of the vorticity as follows

\[
P_a^b \omega^b + \frac{2}{3} \theta \omega^a - \frac{1}{2} \eta^{abcd} u_b \nabla_d a_c = 0.
\]

Eq.(22) comes directly from the time-propagation of the absence of shear. The whole set [20,21,22] contains 9 of the 18 independent components of the Ricci identities.
The remaining 9 components are usually called constraint equations and are given by \[12, 13, 20\]

\[2P^{ab} \nabla_b \theta + 3P^a_b \nabla_d \omega^{bd} + 3\omega^a_d a^b = 0, \tag{26}\]

\[\nabla_a \omega^a = 2a_a \omega^a \iff \eta^{abcd} u_a (\nabla_b \omega_{cd} - a_b \omega_{cd}) = 0, \tag{27}\]

\[H_{ab} = 2\omega(a_b) - (a^c \omega_c) P_{ab} + P^c_{(a} P^d_{b)} \nabla_c \omega_d, \tag{28}\]

where in order to write down the magnetic part \(H_{ab}\) in the form (28) we have used the identity

\[\omega_{ab} \nabla_c \omega^{bc} = -\omega^2 a_a + (a^b \omega_b) \omega_a + P^b_a \omega^c \nabla_c \omega_b - P^b_a \omega^c \nabla_b \omega_c. \tag{29}\]

Finally, the conservation equations \(\nabla_b T^{ab} = 0\) for the energy-stress tensor (1) read

\[\dot{\rho} + (\rho + p) a^a + P^{ab} \nabla_b p = 0. \tag{31}\]

Now, we are ready to present the fully covariant proofs of the main theorems.

### 3 Case without acceleration: \(\vec{a} = \vec{0}\).

In this section we present the coordinate– and tetrad–free proof of Conjecture \[\text{I}\] for the case of vanishing acceleration

\[\vec{a} = \vec{0}, \tag{32}\]

which includes the particular case of dust because of (31). First of all, we present some lemmas which will help us in the proof of the theorem.

**Lemma 1:** If there exists a function \(f\) satisfying \(P^{ab} \nabla_b f = 0\) then either \(f = \text{const.}\) or the rotation vanishes.

**Proof:** There are two possibilities, either \(\dot{f}\) vanishes or not. If it does, then \(f = \text{const.}\) clearly. If it does not, then we can write

\[u_a = -\frac{1}{f} \nabla_a f. \tag{33}\]

But then, as we have seen in the comments following equation (16), this implies \(\omega_{ab} = 0\), proving the lemma.

**Lemma 2:** If the perfect fluid is geodesic then either the pressure \(p\) is constant or the rotation vanishes.
Proof: The proof of this lemma comes from the energy-stress conservation equation (31), which by using (32) becomes \( P^{ab} \nabla_b p = 0 \). Then Lemma 1 implies the result.

**Lemma 3:** If the perfect fluid is geodesic and shear-free, and there exist constants \( c_1 \) and \( c_2 \) with \( c_2 \neq 0 \) such that

\[
\rho = (c_1 - 1) p + c_2 \omega^2,
\]

then either the rotation or the expansion vanishes.

Proof: First of all, the geodesic condition implies that the time-propagation equations (21) and (25) for the rotation and the vorticity reduce to

\[
P^c_a P^d_b \dot{\omega}_{cd} = \dot{\omega}_{ab} = -\frac{2}{3} \theta \omega_{ab},
\]

\[
P^a_b \dot{\omega}^b = \dot{\omega}^a = -\frac{2}{3} \theta \omega^a \quad \Rightarrow \quad \dot{\omega} = -\frac{2}{3} \theta \omega.
\]

From Lemma 2 either the rotation vanishes or the pressure is constant. Thus, we need only consider the case \( p = \text{const} \). The time propagation of (34) then gives, on using (30) and (36),

\[
\theta \left( c_1 p - \frac{c_2}{3} \omega^2 \right) = 0.
\]

If \( \theta = 0 \) we are done. If the term in brackets vanishes, then \( \omega \) is constant, and (36) gives \( \theta \omega = 0 \) as required. We now pass to the proof of the theorem.

**Theorem 1:** Every shear-free and geodesic perfect fluid must be either expansion-free or rotation-free, i.e.

\[
\sigma_{ab} = 0, \quad a^b = 0 \quad \Rightarrow \quad \omega \theta = 0.
\]

This theorem was first shown by Ellis [15] for the dust case \((p = 0)\) and it was completed for the case of constant pressure by White and Collins [39]. Both theorems were proved using the so-called tetrad formalism (see, for instance, [28]).

Proof: From Lemma 2 it is clearly only necessary to consider the case \( p = \text{const} \). Moreover, from (15) restricted to the case with (32) we get immediately, for any function \( f \), the following identity

\[
\left( P^{ab} \nabla_b f \right) = P^{ab} \nabla_b \dot{f} + \omega^{ab} \nabla_b f - \frac{\theta}{3} P^{ab} \nabla_b f.
\]
Using this identity together with the relations (35-36), the time propagation of Equation (26) restricted to the case \( a^b = 0 \) results in

\[
2 P_a^b \nabla_b \theta - 13 P_a^b \omega^c \nabla_b \omega_c - 3 P_a^b \omega^c \omega_c \omega_b = 0. \tag{39}
\]

On using (26) contracted with \( \omega_{ac} \) and the identity (29), this equation can be rewritten as

\[
P_a^b \nabla_b \theta - 8 \omega P_a^b \nabla_b \omega + \omega_c \omega_b \nabla_b \theta = 0. \tag{40}
\]

This key equation provides an algebraic relation between the spatial gradients of density, rotation and expansion. By contracting it with \( \omega^{ca} \) we also get

\[
\omega_a^b \nabla_b \theta - 8 \omega_a^b \nabla_b \omega + \omega_a^c \omega_b \nabla_b \theta = 0. \tag{41}
\]

The next step is the computation of the time propagation of Equation (40). Making use of (35-36), (38), (30), (41) and the Raychaudhuri equation (20) this leads to

\[
\left( \rho + p - \frac{16}{3} \omega^2 \right) P_a^b \nabla_b \theta = \frac{1}{2} \omega_a^c \omega_b \nabla_b \theta + \frac{\theta}{3} P_a^b \nabla_b \theta. \tag{42}
\]

Time-propagating again this relation and taking into account the key equations (40) and (41) together with (35-36), (38), (20), and (42) itself we arrive at

\[
\frac{1}{2} \left( \rho + p - \frac{16}{3} \omega^2 \right) \omega_a^b \nabla_b \theta + \theta \left[ \frac{112}{9} \omega^2 - \frac{5}{3} (\rho + p) \right] P_a^b \nabla_b \theta +
+ \left( \frac{5}{9} \theta^2 - \frac{2}{3} \omega^2 + \frac{\theta + 3p}{6} \right) P_a^b \nabla_b \theta + \frac{7}{6} \theta \omega_a^c \omega_b \nabla_b \theta - \frac{1}{4} \omega_a^d \omega_c \omega_b \nabla_b \theta = 0. \tag{43}
\]

Multiplying this by \( \theta \) in order to use (42) again, and re-ordering terms it follows that

\[
\frac{\theta}{2} \left( \rho + p - \frac{16}{3} \omega^2 \right) \omega_a^b \nabla_b \theta - \frac{\theta}{4} \omega_d^c \omega_c \omega_b \nabla_b \theta + \left( \frac{\theta^2}{3} + \omega^2 - \frac{\theta + 3p}{4} \right) \omega_a^c \omega_b \nabla_b \theta +
+ \left[ \frac{32}{9} \theta^2 \omega^2 + \left( \rho + p - \frac{16}{3} \omega^2 \right) \left( \frac{\theta + 3p}{2} - 2\omega^2 \right) \right] P_a^b \nabla_b \theta = 0. \tag{44}
\]

Contracting with \( \omega^a \) we find the simple relation

\[
\left[ \frac{32}{9} \theta^2 \omega^2 + \left( \rho + p - \frac{16}{3} \omega^2 \right) \left( \frac{\theta + 3p}{2} - 2\omega^2 \right) \right] \omega^b \nabla_b \theta = 0, \tag{45}
\]

so that two different possibilities appear:

1) If \( \omega^b \nabla_b \theta \neq 0 \), then

\[
\frac{32}{9} \theta^2 \omega^2 + \left( \rho + p - \frac{16}{3} \omega^2 \right) \left( \frac{\theta + 3p}{2} - 2\omega^2 \right) = 0. \tag{46}
\]
The time propagation of this equation leads on using Eqns. (20), (30), (36) and (46) itself, either to
\[ \theta = 0 \] (in which case obviously \( \theta \omega = 0 \)) or
\[ \frac{64}{9} \omega^4 - 2\varrho \omega^2 - \frac{38}{3} p \omega^2 + p(\varrho + p) = 0. \] (47)

Once more time-propagating this relation we finally arrive at
\[ \left( \varrho + \frac{7}{3} p - \frac{40}{9} \omega^2 \right) \omega^2 = 0. \] (48)

Thus, either \( \omega = 0 \) and we are done, or the term in brackets vanishes in which case \( \omega \theta = 0 \) by Lemma 3.

2) If \( \omega^b \nabla_b \theta = 0 \), then Equation (44) together with the identities (14) lead to
\[ \theta \left( 29 \omega^2 - 6(\varrho + p) \right) \omega^b \nabla_b \theta = \left[ \frac{58}{3} \theta^2 \omega^2 + \left( 2 \omega^2 - \frac{\varrho + 3p}{2} \right) \left( 29 \omega^2 - 6(\varrho + p) \right) \right] P^b_a \nabla_b \theta. \] (49)

But the two vectors \( P^b_a \nabla_b \theta \) and \( \omega^b \nabla_b \theta \) are orthogonal to each other, as follows immediately from (11),
\[ \omega^b \nabla_b \theta = \omega_{ac} P^{cb} \nabla_b \theta = \eta_{acde} \omega^d u^e P^{cb} \nabla_b \theta. \] (50)

Hence the left-hand side of relation (49) must vanish, and there are three possibilities. Either (a) \( \theta = 0 \), in which case the theorem is proved, or (b) the term in brackets vanishes whence the theorem follows from Lemma 3, or (c) \( \omega^b \nabla_b \theta = 0 \). In this case using the condition defining case (2), namely \( \omega^b \nabla_b \theta = 0 \), it follows that either \( P^b_a \nabla_b \theta = 0 \) or \( \omega^b = 0 \). If the latter holds then \( \omega = 0 \) and the theorem is proved, while the former implies through (42) and (40) that \( P^b_a \nabla_b \omega = 0 \). Then setting \( f = \omega \) in Lemma 1 gives \( \omega \) is constant and Equation (50) implies that \( \theta \omega = 0 \). Hence the theorem holds in this case too.

This finishes the proof of Theorem 1.

It must be remarked that Theorem 1 does not need the assumptions of a barotropic equation of state and \( \varrho + p \neq 0 \). In particular, this theorem also holds for Einstein’s spaces (including vacuum) in which there is a geodesic shear-free unit timelike vector field.

**Corollary 1.1:** If \( u^a \) is a shear-free geodesic vector field in any Einstein space \( (G_{ab} = \lambda g_{ab}) \), then \( \theta \omega = 0 \).

**Proof:** Essentially this is the case
\[ \varrho = -p = -\lambda = \text{const.} \]
Equation (45) is replaced by the simpler
\[ \omega^2 \left[ \frac{2}{3} \theta^2 + \rho + 2\omega^2 \right] \omega^b \nabla_b \theta = 0. \]  (51)
If the expression in brackets vanishes we have by essentially the same discussion as given in case (1) above,
\[ \rho + 2\omega^2 = 0 \]
whence \( \theta\omega = 0 \) follows by Lemma 3. On the other hand if \( \omega^b \nabla_b \theta = 0 \) then the proof follows exactly as for case (2), with appropriate simplifications for example in Equation (49).

4 Case in which the acceleration and the vorticity vector fields are parallel: \( \vec{a} \parallel \vec{\omega} \).

In this section we study the case when the acceleration vector field \( \vec{a} \) and the vorticity vector field \( \vec{\omega} \) are parallel, that is to say, when
\[ \vec{a} = \psi \vec{\omega}, \quad \psi \neq 0. \]  (52)
Taking into account Theorem 1 we will consider \( \psi \) as an arbitrary non-vanishing function. This also means by (31) that, whenever there is a barotropic equation of state \( p = p(\rho) \), we have
\[ p' \neq 0, \quad \text{where} \quad p' \equiv dp(\rho)/d\rho. \]  (53)
As in the previous section, we collect some useful lemmas before we prove the theorem.

Lemma 4: For every shear-free perfect fluid with a barotropic equation of state and \( \rho + p \neq 0 \), we have
\[ P_{[b}^c P_{d]}^a \nabla_a \dot{a}_d = p' \theta \omega_{ab}, \]  (54)
\[ P_{[b}^a \dot{a}^b = \left[ p' - \frac{1}{3} - (\rho + p) \frac{p''}{p'} \right] \theta a^a + \omega_a^b \dot{a}^b + p' P^{ab} \nabla_b \theta. \]  (55)

Proof: If we introduce the equation of state \( p = p(\rho) \) into relation (51) and use (31) we obtain
\[ a_a = p' \theta u_a - (\rho + p)^{-1} \nabla_a p = p' \left( \theta u_a - (\rho + p)^{-1} \nabla_a \rho \right), \]  (56)
from which it is easily derived that
\[ P_{[b}^c P_{d]}^a \nabla_a \dot{a}_d = p' \theta P_{[b}^c P_{d]}^a \nabla_a u_d, \]  (57)
and now using (54) we arrive at (55). On the other hand, the time-propagation of (56), on using (31) and (31), leads directly to (53).
Lemma 5: For every shear-free perfect fluid with a barotropic equation of state and \(\varrho + p \neq 0\), the time propagation of the rotation and vorticity are

\[
P_a^c P_b^d \dot{\omega}_{cd} = \left( p' - \frac{2}{3} \right) \theta \omega_{ab},
\]

(58)

\[
P_b^a \dot{\omega}^b = \left( p' - \frac{2}{3} \right) \theta \omega^a \quad \Rightarrow \quad \dot{\omega} = \left( p' - \frac{2}{3} \right) \theta \omega.
\]

(59)

Proof: The proof is straightforward on introducing (54) into equations (21) and (25), respectively.

Theorem 2: Every shear-free perfect fluid with a barotropic equation of state and \(\varrho + p \neq 0\) in which the acceleration and vorticity vector fields are parallel is either expansion-free or rotation-free.

The proof of this theorem was given by White and Collins [39] by using the tetrad formalism [28]. Here, we present a very simple covariant proof of this result.

Proof: From the expression (11) of the rotation rate \(\omega_{ab}\) in terms of \(\vec{\omega}\), taking into account (52), and using equation (54) of lemma 4 we can write

\[
P_{ab} \nabla_d \omega^{bd} = \eta_{abcd} u^b \nabla_c (\psi^{-1} a^d) = -\psi^{-1} \left( \omega^b \nabla_b \psi + 2p' \theta \omega \right),
\]

(60)

which allows us to express Equation (26) in the form

\[
P_{ab} \nabla_b \theta = 3\psi^{-1} \left( \frac{1}{2} \omega^{ab} \nabla_b \psi + p' \theta \omega \right).
\]

(61)

Introducing this into expression (55) of Lemma 4 and using \(\omega^a \nabla_b a^b = \psi \omega^a \omega^b = 0\) we obtain

\[
P^a_{\dot{a}b} = \left[ p' - \frac{1}{3} (\varrho + p) \frac{p''}{p'} + 3 \left( \frac{p'}{\psi} \right)^2 \right] \theta a^a + \frac{3}{2} \psi^{-1} p' \omega^{ab} \nabla_b \psi.
\]

(62)

However another expression for the time propagation of the acceleration vector field \(\vec{a}\) can be obtained by putting the assumption (52) into Equation (59) of lemma 5. The result is

\[
P^a_{\dot{a}b} = \left[ \left( p' - \frac{2}{3} \right) \theta + \frac{\psi}{\psi} \right] a^a.
\]

(63)

Comparison of Equations (62) and (63) leads to

\[
\omega^{ab} \nabla_b \psi = 0,
\]

(64)

\[
\dot{\psi} = 3 \left( \frac{p'}{\psi} \right)^2 + \frac{1}{3} (\varrho + p) \frac{p''}{p'} \theta \psi,
\]

(65)
and the first of these, through Equation (61), leads in turn to the following simple formula
\[ \nabla_a \theta = -\dot{\theta} u_a + 3\psi^{-1}p' \theta \omega_a. \] (66)

Another consequence of (64) is that relation (60) simplifies to
\[ P_{ab} \nabla_d \omega^{bd} = -2 \frac{p'}{\psi} \theta \omega_a, \] (67)

and on contracting with \( \omega^a \) and using the identity (29) we get
\[ P^{ab} \omega^c \nabla_c \omega_b = 0. \] (68)

Considering this expression together with the magnetic part of the Weyl tensor (28) and the Ricci identities for \( \omega^{ab} \), the time propagation of (67) gives
\[ \dot{\theta} = 3 \left( \frac{p'}{\psi} \right)^2 \theta^2. \] (69)

Putting this into equation (66) and applying the integrability conditions \( \nabla_a \nabla_b \theta = 0 \) for \( \theta \), we obtain that either \( \theta = 0 \), which proves the theorem, or
\[ \frac{p'}{\psi^2} \omega^a \nabla_a \psi = \left[ 3 \left( \frac{p'}{\psi} \right)^2 + \frac{1}{3} - (\varrho + p) \frac{p''}{p'} \right] \omega^2. \] (70)

On the other hand, introducing (69) into the Raychaudhuri equation (20) and using the assumptions of this case we obtain another expression for \( \omega^a \nabla_a \psi \),
\[ \omega^a \nabla_a \psi = \left[ 3 \left( \frac{p'}{\psi} \right)^2 \right. + \frac{1}{3} \left. \right] \theta^2 - 2 \left( 1 + \psi^2 \right) \omega^2 + \frac{\theta + 3p}{2}. \] (71)

Comparing these two equations we arrive at the relation
\[ \frac{p'}{\psi^2} \left\{ \left[ 3 \left( \frac{p'}{\psi} \right)^2 + \frac{1}{3} \right] \theta^2 - 2 \left( 1 + \psi^2 \right) \omega^2 + \frac{\theta + 3p}{2} \right\} = \left[ 3 \left( \frac{p'}{\psi} \right)^2 + \frac{1}{3} - (\varrho + p) \frac{p''}{p'} \right] \omega^2, \] (72)

whose time propagation provides an expression containing \( p''' \). On the other hand, the \( \omega^a \nabla_a \) and \( \omega^{ab} \nabla_b \) derivatives of (72) lead to two new equations containing also \( p''' \). By eliminating \( p''' \) from these three equations two different cases appear.
• Case i):

\[2(1 + \psi^2) p' + \left[3 \left(\frac{p'}{\psi}\right)^2 + \frac{1}{3} - (\rho + p) \frac{p''}{p'}\right] \psi^2 = 0. \tag{73}\]

• Case ii):

\[
\frac{p'}{\psi} \omega^a \nabla_a \omega^2 = 2 \left(p' - \frac{2}{3}\right) \omega^4, \quad \omega^{ab} \nabla_b \omega^2 = 0. \tag{74}\]

In case i), substituting (73) into (72) we obtain

\[
\left[3 \left(\frac{p'}{\psi}\right)^2 + \frac{1}{3}\right] \theta^2 + \frac{\rho + 3p}{2} = 0, \tag{75}\]

and time-propagation of this equation leads to

\[
\theta (\rho + p) (1 + 3p') = 0. \tag{76}\]

Here, the second factor cannot vanish because of the assumptions of the theorem. If the third factor vanished, Eqns. (73) and (75) would imply \(1 + \psi^2 = 0\), which is impossible. Therefore, in case i) we must have \(\theta = 0\), proving the theorem for this case.

In case ii) the time propagation of Eqns. (74) does not give additional information. Consider instead the following Ricci identities for the vorticity

\[
2\nabla_{[a} \nabla_{b]} \omega^c = R_{c|ab} \omega^d \quad \implies \quad \nabla_c \left(\omega^b \nabla_b \omega^c\right) - \nabla^c \omega^b \nabla_b \omega_c - \omega^b \nabla_b \nabla_c \omega^c = R_{bd} \omega^b \omega^d, \tag{77}\]

Taking into account that the combination of (68) and (74) gives

\[
\omega^b \nabla_b \omega^a = \omega^2 \left[\frac{\theta}{3} u^a + \left(p' - \frac{2}{3}\right) \omega^a\right], \tag{78}\]

and considering the relations obtained until now, Equation (77) becomes

\[
(\rho + p) \omega^2 + W^{ab} W_{ab} = 0, \quad W_{ab} \equiv Q_c^{(a} Q_b^{d]} \nabla_c \omega_d - \frac{1}{2} Q_{ab} Q^{cd} \nabla_c \omega_d, \tag{79}\]

where \(Q_{ab} \equiv \omega^{-2} \omega_a^{c} \omega_{bc}\) is the projector orthogonal to both \(\vec{u}\) and \(\vec{\omega}\). Taking into account that

\[
\dot{Q}_{ab} = 0, \tag{80}\]

the time propagation of (79) leads again to the simple relation (76). If \(\theta = 0\) we are done. Finally, if \(p' = -1/3\), then the time propagation of the relation (72) leads, since \(1 + \psi^2 \neq 0\), to \(\theta \omega = 0\), which finishes the proof.
5 Comparison with the Newtonian case.

Newtonian cosmology

By a Newtonian cosmology is meant

1. A manifold \( M = E^3 \times \mathbb{R} \), where \( E^3 \) is Euclidean 3-space. For every event \((x, t)\) in \( M \) the fourth coordinate \( t \) registers its absolute time coordinate.

2. An assignment of three functions \( \varrho(x, t), \ p(x, t) \) and \( \phi(x, t) \) on \( M \), called respectively density, pressure and gravitational potential.

3. A differentiable 3-vector field \( \mathbf{v}(x, t) \) (the velocity vector field), defined on each Euclidean space \( t = \text{const} \), whose components depend differentiably on \( t \).

4. The following equations hold relating the quantities defined above (summation convention adopted even though all indices are subscripts, and units have been changed in order to restore \( G \)):

\[
\dot{\varrho} \equiv \frac{d\varrho}{dt} \equiv \frac{\partial \varrho}{\partial t} + v_i \varrho_{,i} = -\varrho v_{,i,i} \tag{81}
\]

\[
\dot{v}_i \equiv \frac{dv_i}{dt} \equiv \frac{\partial v_i}{\partial t} + v_j v_{i,j} = -\phi_{,i} - \frac{1}{\varrho} p_{,i} \tag{82}
\]

\[
\phi_{,ii} = 4\pi G \varrho. \tag{83}
\]

Homogeneous Newtonian cosmologies

A Newtonian cosmology will be called homogeneous \( ^3 \) if \( \varrho \) and \( p \) have no spatial dependence, \( \varrho = \varrho(t), \ p = p(t) \) and the velocity vector field depends linearly on the spatial coordinates,

\[
v_i = V_{ij}(t)x_j \tag{84}
\]

for some matrix of components \([V_{ij}]\) depending only on time. From \( (82) \) it follows that \( \phi_{,i} \) also is linearly dependent on \( x_i \),

\[
\phi_{,i} = -f_{ij}(t) x_j \tag{85}
\]

where

\[
f_{ij} = \dot{V}_{ij} + V_{ik}V_{kj}. \tag{86}
\]

Decomposing \( v_{i,j} \) into its standard irreducible parts

\[
v_{i,j} = \theta \frac{1}{3} \delta_{ij} + \sigma_{ij} + \omega_{ij} \tag{87}
\]
where

\[ \theta = v_{i,i} = \text{expansion} \]  
\[ \sigma_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) - \frac{1}{3}\theta \delta_{ij} = \text{shear} \]  
\[ \omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) = \text{rotation}, \]

it is found in the homogeneous case that

\[ \theta = V_{kk} = \theta(t), \quad \sigma_{ij} = V_{(ij)} - \frac{1}{3}\theta \delta_{ij} = \sigma_{ij}(t), \quad \omega_{ij} = V_{[ij]} = \omega_{ij}(t). \]  

Now suppose that one sets \( \sigma_{ij} = 0 \) in the homogeneous case, then Eqns (81-83) reduce to

\[ \dot{\rho} = -\theta \dot{\rho} \]  
\[ \dot{\omega}_{ij} = -\frac{2}{3} \theta \omega_{ij} \]  
\[ \dot{\theta} = -\frac{1}{3} \theta^2 + 2\omega^2 - 4\pi G \rho \]  
\[ f_{ij} = \omega_{ik} \omega_{kj} + \frac{2}{3}(\omega^2 - 2\pi G \rho) \delta_{ij} \]

where \( \omega^2 = \frac{1}{3} \omega_{ij} \omega_{ij} \).

Define a variable \( R = R(t) \) such that \( \theta = 3 \dot{R}/R \), then it is a straightforward matter to integrate (92) and (93)

\[ \rho = \rho(t_0) R^{-3}, \quad \omega_{ij} = \omega_{ij}(t_0) R^{-2} \]

where \( R(t_0) = 1 \) and substitution in (94) gives

\[ \ddot{R} = -\frac{4\pi G \rho(t_0)}{3} + \frac{2\omega^2(t_0)}{3R^3}. \]

This integrates to give the Heckmann-Schücking equation (92)

\[ \ddot{R}^2 = \frac{8\pi G \rho(t_0)}{3R} - \frac{2\omega^2(t_0)}{3R^2} + C \]

where \( C \) is an arbitrary constant. Solutions of this equation represent shear-free Newtonian cosmologies which are in general both expanding (\( \dot{R} \neq 0 \)) and rotating (\( \omega^2(t_0) \neq 0 \)). The gravitational potential is simply read off from (95) using (85). A point of interest is that whenever \( \omega^2(t_0) \neq 0 \) there is no singularity since \( R(t) \) always has a minimum value \( R_0 > 0 \) where \( \dot{R} = 0 \). Thus the big-bang singularity of the Newtonian equivalent to the standard FLRW models of general relativity (namely, the case \( \omega^2 = 0 \)), is easily avoided by giving the model an arbitrarily small amount of rotation. This is quite contrary to the case in general relativity where singularities are generic to all spatially homogeneous models (21).
Newtonian version of our proof

These homogeneous shear-free solutions are independent of the pressure, which may as well be set equal to zero (or constant). The question naturally arises as to why there are shear-free dust solutions in Newtonian theory having $\theta \omega \neq 0$, when Ellis’s theorem ensures that none exist in general relativity. This difference in the two theories is both surprising and interesting, since Ellis’s is a purely local result and is completely independent of the strength of the gravitational field. One would then expect it to hold in the weak field Newtonian limit, yet clearly it does not. What then is going wrong in this limit?

To investigate this question we propose to follow the proof of Ellis’s theorem given here in Section 2, but try to apply it as closely as possible to Newtonian cosmology in an attempt to pinpoint exactly where it is that the proof fails.

Consider a general pressure-free ($p = 0$) Newtonian cosmology (not necessarily homogeneous) having $\sigma_{ij} = 0$, i.e.

$$v_{i,j} = \frac{1}{3} \theta \delta_{ij} + \omega_{ij} \quad (99)$$

$$\dot{v}_i = -\phi_{,i} \quad (100)$$

where the Newtonian time evolution operator is given by

$$\cdot = \frac{d}{dt} = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x^i}$$

and

$$\phi_{,ii} = \frac{1}{2} \varrho. \quad (101)$$

In this last equation, (83) has been recast in units such that $8\pi G = 1$ which has been done in order to bring the Newtonian and general relativistic equations into closer comparison. Eqns. (99) and (100) correspond to the earlier equations (18) and (32), while Eqn. (81) reads

$$\dot{\varrho} + \varrho \theta = 0, \quad (102)$$

which is identical with the relativistic equation (31) when $p = 0$.

A useful identity is

$$\frac{d}{dt} \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} \frac{d}{dt} - \frac{1}{3} \theta \frac{\partial}{\partial x^i} + \omega_{ij} \frac{\partial}{\partial x^j} \quad (103)$$

which takes the place of (38). Now perform the time evolution of Equation (99) and use (103) to arrive at

$$\dot{\omega}_{ij} = -\frac{2}{3} \theta \omega_{ij}, \quad (104)$$
\[ \dot{\theta} = -\frac{1}{2} \ddot{\theta} - \frac{1}{3} \theta^2 + 2\omega^2, \quad (105) \]

and

\[ E_{ij} \equiv \phi_{ij} - \frac{1}{3} \phi_{kk} \delta_{ij} = \omega_{ik}\omega_{jk} - \frac{2}{3} \omega^2 \delta_{ij} = -\omega_{ij}\omega_j + \frac{1}{3} \omega^2 \delta_{ij}, \quad (106) \]

where \( \omega_i = \frac{1}{2} \epsilon_{ijk} \omega_{jk} \). These equations are clearly Newtonian versions of the relativistic equations (21), (20) and (22) when \( a^b = 0 \). From (104) there follows the equivalent of (36), i.e.

\[ \dot{\omega} = -\frac{2}{3} \theta \omega. \quad (107) \]

Applying \( v_{i,jk} = v_{i,kj} \) to (99) gives the cyclic identity

\[ \omega_{ij,k} + \omega_{jk,i} + \omega_{ki,j} = 0, \quad (108) \]

and

\[ \omega_{jk,i} = \frac{1}{3} (\theta_{k} \delta_{ij} - \theta_{j} \delta_{ik}). \quad (109) \]

Contracting Eq. (108) with \( \omega_{ij} \) gives

\[ \omega \omega_{,k} = \omega_{ij} \omega_{ik,j} \quad (110) \]

while contraction of (109) over \( ij \) gives the Newtonian equivalent of the important equation (26),

\[ \theta_{k} = \frac{3}{2} \omega_{ik,j}. \quad (111) \]

Finally contract Eq. (109) with \( \omega_{ij} \) and use (110) to give

\[ \omega_{ij} \theta_{j} = 3\omega \omega_{j}. \quad (112) \]

Now perform the time evolution of (111), apply (103) to both sides and use the identities (104), (105), (110), (111) and (112), results in the key equation

\[ \varrho_{,k} - 8\omega \omega_{,k} + \omega_{ki}\theta_{,i} = 0 \quad (113) \]

which is basically the same as Equation (10). Actually the Newtonian case is rather stronger than that in general relativity since the further identity (112), which has no relativistic equivalent, results in

\[ \varrho_{,k} = 5\omega \omega_{,k} = \frac{5}{3} \omega_{ki}\theta_{,i}. \quad (114) \]

At first sight this is a rather surprising result, since Ellis’s theorem is a theorem concerning restrictions on solutions of general relativity, so one might expect that Newtonian theory should place less stringent restrictions on its solutions.

In any case one can continue in exactly the same way as the proof given for Theorem 1 to arrive at a Newtonian version of Equation (14), and its contracted version

\[ \left[ \frac{32}{9} \theta^2 \omega^2 + \left( \varrho - \frac{16}{3} \omega^2 \right) \left( \frac{\varrho}{2} - 2\omega^2 \right) \right] \omega_i \theta_{,i} = 0. \quad (115) \]
Newtonian equivalent of Lemmas 1 and 3

In the final stages of the proof of Theorem 1 use is made of Lemmas 1 and 3. The Newtonian equivalent of Lemma 3, with $p = 0$ is

**Lemma 3':** If $\varrho = k\omega^2$ for a non-zero constant $k$, then $\theta\omega = 0$.

This lemma holds in Newtonian cosmology, for on using \((102)\) and \((107)\)

$$\dot{\varrho} = -\theta\varrho = 2k\omega\dot{\omega} = -\frac{4}{3}k\theta\omega^2.$$  

Hence

$$\frac{1}{3}k\theta\omega^2 = 0$$

which proves the result.

Lemma 1 however is a different matter. The obvious Newtonian translation would be the following:

*If a function $f(x, t)$ has zero gradient, $f_i = 0$ then $f$ is constant or $\omega_{ij} = 0$."

Of course a function whose gradient vanishes may still be dependent on time, $f = f(t)$. This can have no bearing whatsoever on the rotation of the velocity vector field. Hence as it stands, this statement cannot be true. It is worth trying to understand why the limiting process does not apply for this lemma.

Suppose we have a metric $g_{ab} = \eta_{ab} + h_{ab}$ having energy-stress tensor $T_{ab}$ given by Equation \((1)\) where (restoring $c$)

$$u^a = \left(\frac{v_i}{c} \cdot -1 + O\left(\frac{v^2}{c^2}\right)\right), \quad |h_{ab}| = O\left(\frac{v^2}{c^2}\right),$$  

\[(116)\]

where $v^2 = v_iv_i$. The Newtonian limit is the limit of all equations found on letting $c \to \infty$. For example, the rotation is given by

$$\omega_{ij} = \lim_{c \to \infty} c u_{[i,j]} = v_{[i,j]},$$  

\[(117)\]

as to be expected. However the equation $P^b_a \nabla_b f = 0$ reduces in its first three components to

$$f_i = -\frac{1}{c^2} \dot{f} v_i$$  

\[(118)\]

where $\dot{f} = \frac{\partial f}{\partial t} + f_i v_i$. The limit as $c \to \infty$ of this equation is clearly $f_i = 0$, from which no conclusions can be made concerning $\omega_{ij} = v_{[i,j]}$. In general relativity a conclusion can be reached because Equation \((33)\) couples $u_a$ and $\nabla_a f$, but in taking the Newtonian limit $v_i$ and $f_i$ become decoupled.

Physically one understands it like this. If $\omega_{ab} = 0$ then the 4-velocity field $u_a$ is hypersurface orthogonal, and in a sense there is a universal time coordinate $t$. 

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defined such that $u_a \propto \nabla_a t$ (in the lemma, $f$ could act as such a time coordinate). However in Newtonian theory such a universal time coordinate is always defined and the condition $\omega_{ij} = 0$ is equivalent to $v_i = \phi, i$ for some scalar field $\phi$ having in general nothing whatsoever to do with the universal time. For example, $\phi$ may just depend on the spatial coordinates $x, y, z$. Lemma 1 is truly a relativistic result, having no Newtonian equivalent whatsoever.

**Final stages of the proof**

Returning to the proof of Theorem 1, we attempt to follow the argument given in the final stages. Consider the two cases arising from Equation (115).

Case 1$'$: $\omega_i \theta, i \neq 0$. In place of Equation (47) the corresponding Newtonian argument arrives at

$$\left(\frac{32}{9} \omega^2 - \varrho\right) \omega^2 = 0.$$  \hfill (119)

Lemma 3' immediately applies, showing that $\theta \omega = 0$

Case 2$'$ $\omega_i \theta, i = 0$. The argument can again be continued as in Theorem 1. The Newtonian version of (119) is

$$\frac{\theta}{2} (29 \omega^2 - 6 \varrho) \omega_i \theta, j = \left[\frac{58}{3} \theta^2 \omega^2 + \left(2 \omega^2 - \frac{\varrho}{2}\right) (29 \omega^2 - 6 \varrho)\right] \theta, i.$$  \hfill (120)

Now clearly $\omega_i \theta, j$ is orthogonal to $\theta, i$ (since $\omega_i \theta, i \theta, j = 0$ and $\omega_i = -\omega_{ji}$), whence the left hand side of (120) vanishes. If $\theta = 0$ the proof is over, and by Lemma 3' this is also true if $29 \omega^2 - 6 \varrho = 0$. If $\omega_i \theta, i = 0$ then $\epsilon_{ijk} \omega_j \theta, k = 0$, and combined with $\omega_i \theta, i = 0$ it follows that $\omega_i = 0$ or $\theta, i = 0$. The former implies $\omega = 0$ and we are done, while the latter implies by (114) that

$$\varrho, i = \omega, i = 0.$$  \hfill (121)

However it is not possible to proceed any further. It is at this very last step that our attempt at a proof comes to an end since there is no equivalent to Lemma 1. The limiting process has broken down for the reasons outlined above.

**All Newtonian shear-free universes**

To conclude this Newtonian discussion, we propose to show that all pressure-free, shear-free Newtonian universes are in fact homogeneous. If $\sigma = 0$ then the above discussion has shown that $\theta \omega = 0$ or $\theta, i = 0$. In either case we see from Equation (114) that

$$\varrho, i = \theta, i = \omega, i = 0,$$  \hfill (122)

whence

$$\varrho = \varrho(t), \quad \theta = \theta(t), \quad \omega = \omega(t).$$  \hfill (123)
From (99), using (122) and \( v_{i,jk} = v_{i,kj} \) we see that \( \omega_{ij,k} = \omega_{ik,j} \), and substituting in Equation (108) gives \( \omega_{jk,i} = 0 \), i.e. \( \omega_{jk} = \omega_{jk}(t) \). Hence

\[
v_{i,j} = \frac{1}{3} \theta(t) \delta_{ij} + \omega_{ij}(t) = v_{i,j}(t)
\]

from which it follows that

\[
v_i = V_{ij} x_j \quad \text{where} \quad V_{ij} = \frac{1}{3} \theta(t) \delta_{ij} + \omega_{ij}(t) = V_{ij}(t).
\]

All conditions for a homogeneous cosmology are now satisfied, and it follows that all shear-free Newtonian cosmologies are homogeneous (and are therefore in the Heckmann–Schücking class). What this means is that although Ellis’s theorem does not hold in Newtonian cosmology, it “nearly” holds in the sense that the only solutions which violate the theorem are homogeneous.

6 Conclusions

Ellis’s theorem and its various extensions have up till now always required the use of tetrads in their proof. In this paper we have shown that it is possible to prove such results by methods which are totally covariant in character, and which do not hinge on setting up specific tetrads or coordinate systems. While in this paper we do not go further than showing already known results, our proofs of these theorems have considerable elegance and lend themselves readily for extension to more difficult cases.

Another advantage of our proofs has been the light they shed on the puzzle of the Newtonian equivalence. It seems that the basic reason these results do not hold in Newtonian cosmology is that there is already a well-defined universal time in that theory. In relativistic cosmologies, there will only be a universal time, defined for example by the energy-stress tensor, if the rotation of the 4-velocity field vanishes. Thus a rotating universe can have no non-constant scalar field whose spatial gradient with respect to the 4-velocity vanishes (the content of Lemma 1). However in the Newtonian limit this statement is quite untrue, since there exist functions of the universal time alone, even when the 3-velocity is rotating.
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