Mott phases and quantum phase transitions of extended Bose-Hubbard models in 2+1 dimensions

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We review the recent theoretical developments towards understanding the Mott phases and quantum phase transition of extended Bose-Hubbard models on lattices in two spatial dimensions. We focus on the description of these systems using the dual vortex picture and point out the crucial role played by the geometry of underlying lattices in determining the nature of both the Mott phases and the quantum phase transitions. We also briefly compare the results of dual vortex theory with quantum Monte Carlo results.

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I. INTRODUCTION

The superfluid to Mott insulators transitions of strongly correlated lattice bosons systems, described by extended Hubbard models, in two spatial dimensions have recently received a great deal of theoretical interest. One of the reasons for this renewed attention is the possibility of experimental realization of such models using cold atoms trapped in optical lattices. However, such transitions are also of interest from a purely theoretical point of view, since they provide us with a test bed for exploring the recently developed theoretical paradigm of non-Landau-Ginzburg-Wilson (LGW) phase transitions. In the particular context of lattice bosons in two spatial dimensions, a general framework for such non-LGW transitions has been developed and applied to the case of square lattice. The application of this framework to triangular, kagome lattices has also been carried out. These theoretical works has also supplemented by quantum Monte Carlo (QMC) studies which serves as a rigorous test for predicted theoretical results. Such studies, for extended Bose Hubbard models, has been carried out for square, triangular, kagome, and hexagonal lattices. In this review, we present a brief overview of the theoretical ideas developed in the above-mentioned works focusing on role of the geometry of the underlying lattices in determining the nature of both the Mott phases and the quantum phase transitions. We also briefly compare the results of dual vortex theory with QMC results.

The typical paradigm of non-LGW transitions that has been proposed in the context of lattice bosons in Refs. is the following. It is proposed that for non-integer rational fillings \( f = p/q \) of bosons per unit cell of the underlying lattice \((p \text{ and } q \text{ are integers)}\), the theory of phase transition from the superfluid to the Mott insulator state is described in terms of the vortices which are non-local topological excitations of the superfluid phase, living on the dual lattice. These vortices are not the order parameters of either superfluid or Mott insulating phases in the usual LGW sense. Thus the theory of the above mentioned phase transitions are not described in terms of the order parameters on either side of the transition which is in contrast with the usual LGW paradigm of phase transitions. Also, as first explicitly demonstrated in Refs., although these vortices are excitations of a featureless superfluid phase, they exhibit a quantum order which depends on the filling fraction \( f \). It is shown that the vortex fields describing the transition form multiplets transforming under projective symmetry group (projective representations of the space group of the underlying lattice and that this property of the vortices naturally and necessarily predicts broken translational symmetry of the Mott phase, where the vortices condense. Since this translational symmetry breaking is dependent on the symmetry group of the underlying lattice, geometry of the lattice naturally plays a key role in determining the competing ordered states of the Mott phase and in the theory of quantum phase transition between the Mott and the superfluid phases.

In what follows, we shall develop the above-mentioned ideas concentrating on square and Kagome lattices. The relevant results for the triangular lattice can be found in Ref. To this end, we consider the extended Bose-Hubbard Hamiltonian

\[
H_{\text{boson}} = -t \sum_{\langle ij \rangle} \left( b_i b_j + \text{h.c.} \right) + \frac{U}{2} \sum_i n_i (n_i - 1) + V \sum_{\langle ij \rangle} n_i n_j - \mu \sum_i n_i. \tag{1}
\]

Here \( t \) is the boson hopping amplitude between nearest neighbor sites, \( U \) is the on-site interaction, \( V \) denotes the strength of the nearest neighbor interaction between the bosons and \( \mu \) is the chemical potential. We note at this point that a simple Holstein-Primakoff transformations maps this boson model (Eq. 1.), in the limit of hardcore bosons \((U \to \infty)\), to XXZ models with ferromagnetic \( J_z \) and antiferromagnetic \( J_x \) interaction in a longitudinal magnetic field \( B_l \)

\[
H_{\text{XXZ}} = -J_z \sum_{\langle ij \rangle} \left( S_i^x S_j^x + S_i^y S_j^y \right) + J_x \sum_{\langle ij \rangle} S_i^z S_j^z - B_l \sum_i S_i^z. \tag{2}
\]
where $J_x > 0$ and $J_z > 0$ are the strengths of transverse and longitudinal nearest neighbor interactions and $B_1$ is a longitudinal magnetic field. The parameters $J_x$, $J_z$ and $B_1$ of $H_{XXZ}$ can be mapped to those of $H_{boson}$: $J_x = 2t$, $J_z = V$, and $B_1 = zV(f - \frac{1}{2})$, where $z$ denotes the coordination number of the underlying lattice and $f$ is the average boson filling.

II. COMPETING MOTT STATES AND QUANTUM PHASE TRANSITION

The basic premise underlying the theories developed in Refs. 4, 5, and 6, 19, 20 is that the quantum phase transition of the extended Bose-Hubbard model occurs due to destabilization of the superfluid phases by proliferation of vortices which are topological excitations of the superfluid phase. To analyze such a transition, one therefore needs to obtain an effective action for the vortex excitation. Such an action can be obtained by performing a duality analysis of the Bose-Hubbard model14, 16. The final form of the dual action $S_d$ can be written as

$$S_d = \frac{1}{2e^2} \sum_b (\epsilon_{\nu\lambda}\Delta_{\nu\lambda}A_{b\lambda} - f\delta_{\nu\tau})^2$$

$$- y_b \sum_b (\psi_{b+\mu}e^{2\pi i A_{b\mu}}\psi_b + \text{h.c.})$$

$$+ \sum_b \left[r|\psi_b|^2 + u|\psi_b|^4\right]$$

where $\psi_b$ are the vortex field living on the site $b$ of the dual lattice, $A_{b\mu}$ is the U(1) dual gauge field so that $\epsilon_{\tau\nu\lambda}\Delta_{\nu\lambda}A_{b\lambda} = n_i$ where $n_i$ is the physical boson density at site $i$, $\sum_\mu$ denotes sum over elementary plaquette of the dual lattice, $\Delta_{\nu\lambda}$ denotes lattice derivative along $\mu = x, y, \tau$, and $f$ is the average boson density. Note that the vortex action $S_d$ is not self-dual to the boson action obtained from $H_{boson}$. Therefore we can not, in general, obtain a mapping between the parameters of the two actions, except for identifying $\epsilon_{\tau\nu\lambda}\Delta_{\nu\lambda}A_{b\lambda}$ as the physical boson density14, 16. We shall therefore classify the phases of this action based on symmetry consideration and within the saddle point approximation as done in Refs. 4, 5, 6, 14, 16, 19, 20.

The transition from a superfluid ($\langle \psi_b \rangle = 0$) to a Mott insulating phase in $S_d$ can be obtained by tuning the parameter $f$. For $f > 0$, we are in the superfluid phase. Note that the saddle point of the gauge fields $A_{b\mu}$ in action corresponds to $\epsilon_{\tau\nu\lambda}\Delta_{\nu\lambda}A_{b\lambda} = f$, so that the magnetic field seen by the vortices is pinned to the average boson filling $f$. Now as we approach the phase transition point $f = 0$, the fluctuations about this saddle point ($\langle \psi_b \rangle = 0$, $A_{b\mu} = fx$) increase and ultimately destabilize the superfluid phase in favor of a Mott phase with $\langle \psi_b \rangle \neq 0$. Clearly, in the above scenario, the most important fluctuations of the vortex field $\psi_b$ are the ones which have the lowest energy. This prompts us to detect the minima of the vortex spectrum by analyzing the kinetic energy term of the vortices $H_{kinetic} = -y_b \sum_{b,\alpha} (\psi^{\dagger}_{b+\alpha}e^{2\pi i A_{b\mu}}\psi_b + \text{h.c.})$, where the sum over $\alpha$ is carried out over the sites of the dual lattice which are nearest neighbors to $b$. The analysis of $H_{kinetic}$ therefore amounts to solving the Hofstadter problem on the dual lattice which has been carried out in Refs. 4, 5, and 6, 19, 20 for square, triangular and Kagome lattices respectively.

A. Square lattice

The dual lattice in this case is also a square lattice with its sites being at the center of the plaquettes of the direct lattice. Thus here one needs to solve the Hofstadter problem on a square lattice with $f = p/q$ flux quanta per plaquette of the dual square lattice. It has been shown in Ref. 4 that for $f = p/q$, there are $q$ minima within the magnetic Brillouin zone at the wave-vectors $Q = (k_x a, k_y a) = (0, 2\pi l/p/q)$, where $l$ runs from 0 to $q - 1$, $k_{x,y}$ are the wavevectors of the magnetic Brillouin zone and $a$ is the lattice spacing. The eigenfunctions $\psi(k_x, k_y)$ corresponding to these minima can be easily constructed.

Once the positions of these minima are located, the next task is to identify the low lying excitations around these minima which are going to play a leading role in destabilizing the superfluid phase. These excitations, represented by the bosonic fields $\varphi_l$, has definite transformation properties under the symmetry operations of the underlying dual lattice. For the case of square lattice, the relevant symmetry operations are translations along $x$ and $y$ directions by one lattice spacing ($T_x$ and $T_y$), reflections about $x$ and $y$ axes ($I_x$ and $I_y$) and rotation by $\pi/2$ ($R_{\pi/2}$). It has been shown in Ref. 4 that the transformation properties of $\varphi_l$ under these transformations are the following: $T_x : \varphi_l \rightarrow \varphi_{l-1}$, $T_y : \varphi_l \rightarrow \varphi_{l+1}$, $R_{\pi/2} : \varphi_l \rightarrow \frac{1}{\sqrt{2}} \sum_{\omega = 0}^{\pi} \omega^{l^2} \varphi_\omega$, $I_x : \varphi_l \rightarrow \varphi_{-l}^\ast$, and $I_y : \varphi_l \rightarrow \varphi_{l+1}^\ast$, where $\omega = \exp(-2\pi i f)$.

Finally, we need to construct an effective LGW action in terms of the low-energy fields $\varphi_l$ which is invariant under all the symmetry operations. For the sake of definiteness, we shall focus on the case $q = 2$ and $q = 3$ from now on. First, let us consider the case $q = 2$ for which, the effective action reads18,

$$S_v = \int d^2 rdlt (L_2 + L_4 + L_8)$$

$$L_2 = \left(\frac{1}{4} \left( |\partial_{\mu} - iA_{\mu}|^2 \varphi_l |^2 + s|\varphi_l|^2 \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_{\nu} A_{\lambda}^2)^2 \right) \right).$$

$$L_4 = \frac{\gamma_0}{4} (|\varphi_0|^2 + |\varphi_1|^2)^2 - \frac{\gamma_1}{4} (|\varphi_0|^2 - |\varphi_1|^2)^2$$

$$L_8 = \lambda (\varphi_0 \varphi_1^*)^4 + \text{h.c.}$$

where $\varphi_0(1) = (\varphi_0 \pm \varphi_1) / \sqrt{2}$. Note that we have included a $8^{th}$ order term $L_8$ in the effective action since this is
The lowest order term, invariant under all the symmetry operations, which determines the relative phase of the vortex fields \( g_0 \) and \( g_1 \).

The Mott phases, which are stabilized for \( s < 0 \), can be obtained for \( q = 2 \) by a straightforward analysis of \( L_4 \) and \( L_8 \). To concisely present the results for these phases, it is convenient to define a generalized density \( \delta \rho(\mathbf{r}) = \sum_{m,n=-1,1} \rho_{mn} e^{i(m r_x + n r_y)} \), where \( \rho_{mn} = S (|Q_{mn}|^2 \omega_{mn}^2 / 2 \sum_{l=0}^{L} \varphi_l^* \varphi_{l+n} \omega_l^l \) denotes the most general gauge-invariant bilinear combinations of the vortex fields \( \varphi_l \) with appropriate transformation properties of square lattice space group \( \Sigma \). As shown in Ref. [1], the values of \( \delta \rho(\mathbf{r}) \) on sites of the dual lattice (integer \( r_x \) and \( r_y \)) can be considered a measure of the ring-exchange amplitude of bosons around the plaquette whereas those with \( r \) half-odd-integer co-ordinates represent sites of the direct lattice, and the values of \( \delta \rho(\mathbf{r}) \) on such sites measure the boson density on these sites. Finally, \( r \) values with \( r_x \) integer and \( r_y \) half-odd-integer correspond to vertical links of the square lattice (and vice versa for horizontal links), and the values of \( \delta \rho(\mathbf{r}) \) on the links is a measure of the mean boson kinetic energy; if the bosons represent a spin system, this is a measure of the spin exchange energy. We also note two fundamental points from the definition of \( \rho_{mn} \). First, as long as \( \langle \varphi_l \rangle \neq 0 \) for at least one non-zero \( Q_{mn} \), the Mott state is characterized by a non-trivial density wave order. Second, the relative phase of the boson fields \( g_l \) which plays a crucial role in determining the nature of the density wave order is fixed by the 8th order term \( L_8 \).

The Mott phases for \( q = 2 \) is shown in Fig. 1. We note that there are three possible phases. For \( \gamma_0 > 0 \), only one of the two vortex fields condenses and the resulting Mott state (state A in Fig. 1) has a charge density wave order. For \( \gamma_0 < 0 \), both the vortex fields condense and this leads to two possible Mott states (states B and C in Fig. 1). For these states, all sites of the direct lattice are equivalent. These therefore represent valence bond solid (VBS) states with columnar or plaquette VBS order parameters.

Next, we address the issue of quantum phase transition between the superfluid and the Mott state for \( q = 2 \). Such a transition, within the premise of dual vortex theory discussed here, is described by \( S_v \) and not, as would be expected by LGW paradigm, by an action written in terms of order parameters fields at either side of the transition. The phase transition can either be first or second order. If it turns out to be second order and if \( v < 0 \), the relative phase of the vortex fields, which is pinned in the Mott phase by the dangerously irrelevant (in the renormalization group sense) 8th order term \( L_8 \), becomes a gapless mode at the transition. Thus such a transition has an emergent gapless mode at the critical point. It provides an example of a deconfined quantum critical point and can be shown to be accompanied by boson fractionalization.

The Mott phases for \( q = 3 \) can also be similarly obtained. Here the quartic term in the effective Lagrangian which determines the Mott states are given by

\[
L_4 = \frac{\gamma_{00}}{4} (|\rho_0|^2 + |\rho_1|^2 + |\rho_2|^2)^2
+ \frac{\gamma_{01}}{2} (\rho_0 \rho_1 \rho_2^* + \rho_1^* \rho_2 \rho_0^* + \rho_2 \rho_0 \rho_1^* - \rho_0^* \rho_1 \rho_2^*) + \text{c.c.}
- 2 |\rho_0|^2 |\rho_1|^2 - 2 |\rho_2|^2 |\rho_1|^2 - 2 |\rho_2|^2 |\rho_0|^2 \cdot (5)
\]

Note that here the ordering of the Mott states are completely determined by \( L_4 \) and retaining higher order terms are not essential. It turns out that there are two sixfold degenerate VBS states with columns or diagonal VBS orders as shown in Fig. 2. The superfluid-Mott insulator quantum phase transition is again described by a dual vortex action, but does not lead to deconfined quantum criticality as pointed out in Ref. [4].

QMC studies on Bose-Hubbard models on square lattice with nearest neighbor interaction and near half-filling has been carried out in Ref. [5]. These studies indicate a checkerboard Mott state and a strong first order transition between the superfluid and checkerboard Mott state. Analogous QMC studies on triangular lattice has also been carried out in Refs. [6][7].

B. Kagome lattice

The dual lattice corresponding to Kagome is the well-known dice lattice which has three inequivalent sites denoted commonly by \( A, B \) and \( C \). To obtain the minima of the vortex spectrum, we therefore need to solve the Hofstadter problem on the dice lattice. Here we shall concentrate only for \( f = 1/2 \) and \( 2/3 \) (or equivalently \( f = 1/3 \)).

For \( f = 1/2 \), it has been shown\[19][20] that the vortex spectrum on dice lattice do not have well-defined minima and collapse to three degenerate bands. Physically,
the collapse of the vortex spectrum can be tied to localization of the vortex within the so-called Aharanov-Bohm cages as explicitly demonstrated in Ref. 19. An example of such a cage is shown in Fig. 3. A vortex whose initial wavepacket is localized at the central (white) A site can never propagate beyond the black sites which form the border of the cage. This can be understood in terms of destructive Aharanov-Bohm interference: The vortex has two paths 0 → 1 → 3 and 0 → 2 → 3 to reach the ring site 3 from the starting site 0. The amplitudes from these paths destructively interfere for f = 1/2 to cancel each other. Thus the vortex remain within the cage. Such dynamic localization of the vortex wavepackets, termed as Aharanov-Bohm caging, makes it energetically unfavorable to condense vortices. Thus superfluidity persists for arbitrarily small values of t/V. In the language of spins, this also explains the absence of S^z ordering for XXZ model in a Kagome lattice for B^z = 0 and J^z ≫ J_x. Such a persistence of superfluidity has also been confirmed by QMC studies 11,12.

In contrast to f = 1/2, the vortex spectrum has two well-defined minima for f = 2/3 within the magnetic Brillouin zone located at (k_xa, k_ya) = (0, π/3) and (2π/3, 2π/3), where k = √3k_ya/2. Thus the low energy properties of the vortex system can be characterized in terms of the two low-energy bosonic fields φ_1 and φ_2 similar to the case of square lattice at f = 1/2. However, as shown in Ref. 6, in contrast to the case of the square lattice, the space group of symmetry transformations of the dice lattice involves translations by lattice vectors u = 0 and v = 0 (T_u and T_v), rotation by π/3 (R_{π/3}), and reflections around x and y (I_x and I_y). The transformation properties of the vortex fields under these symmetry operations are the following: T_u : φ_1 → φ_1 exp(−iπ/3), φ_2 → φ_2 exp(iπ/3), T_v : φ_1 → φ_1 exp(iπ/3), φ_2 → φ_2 exp(−iπ/3), I_x : φ_1(2) → φ_1(1), I_y : φ_1(2) → φ_2(1), and R_{π/3} : φ_1(2) → φ_2(1).

The simplest Landau-Ginzburg theory for the vortex fields which respects all the symmetries is

\[ L_v = L_v^{(2)} + L_v^{(4)} + L_v^{(6)} \]  \hspace{1cm} (6)

\[ L_v^{(2)} = \sum_{\alpha=1,2} \left[ \left| (\partial_\mu - eA_\mu) \phi_\alpha \right|^2 + r |\phi_\alpha|^2 \right] \]  \hspace{1cm} (7)

\[ L_v^{(4)} = u \left( |\phi_1|^4 + |\phi_2|^4 \right) + v |\phi_1|^2 |\phi_2|^2 \]  \hspace{1cm} (8)

\[ L_v^{(6)} = w \left( |\phi_1^2\phi_2|^3 + \text{h.c.} \right) \]  \hspace{1cm} (9)

Here we find that L_v^{(6)} turns out to be the lowest-order term which breaks the U(1) symmetry associated with the relative phase of the vortex fields.

A simple power counting shows that L_v^{(6)} is marginal at tree level. Unfortunately, the relevance/irrelevance of such a term beyond the tree level, is not easily determined analytically 6. If it so turns out that L_v^{(6)} is irrelevant, the situation here will be identical to that of bosons on square lattice at f = 1/2. The relative phase of the vortices would emerge as a gapless low energy mode at the critical point. The quantum critical point would be deconfined and shall be accompanied by boson fractionalizations. On the other hand, if L_v^{(6)} is relevant, the relative phase degree of the bosons will always remain gapped and there will be no deconfinement at the quantum critical point.
Here there are two possibilities depending on the sign of \( u \) and \( v \). If 0, \( u > 0 \), and \( v > 0 \), the transition may become weakly first order whereas for \( u > 0 \), it remains second order (but without any deconfinement). QMC studies seem to indicate a weak first order transition in this case\(^{11}\). However, it is possible that a second order quantum phase transition can still be possible at special points on the phase diagram and this issue remains to be settled\(^{12,13,14}\).

The vortices condense for \( r < 0 \) signifying the onset of Mott phases. It turns out that the Mott phase for \( v > 0 \) do not lead to any density wave order\(^6\). For \( v < 0 \), there are two possible Mott phases as demonstrated in Ref.\(^{15}\). One of these phases with 9 by 9 ordering pattern is shown in Fig. 4. Here the bosons are localized in red, blue, green (closed circle) and black (open circle) sites whereas the green (open circle) and black (closed circle) sites are vacant leading to a filling of \( f = 2/3 \). The net occupation of the hexagons takes values 0, 4 and 2 as shown in Fig. 4. Note that interchanging the occupations of all the black and green sites of this state (while leaving the red and the blue sites filled) has the effect of 2 \( \leftrightarrow \) 4 for the boson occupation of the hexagons labeled 2 and 4 in Fig. 4 while leaving those for hexagons labeled 0 unchanged. The resultant state is degenerate (within mean-field theory) to the state shown in Fig. 4. This allows us to conjecture that the effect of quantum fluctuations on the state is shown in Fig. 4 might lead to stabilization of a partially resonating state with a 3 by 3 \( R - 3 - 3 \) ordering pattern\(^{22}\). Such a state is indeed found to be the true ground state in exact diagonalization\(^{22,23}\) and QMC studies\(^{11}\). Note that the mean-field analysis of vortex theory which neglects quantum fluctuations can not directly give this partially resonating state.

### III. DISCUSSION

In conclusion, we have briefly reviewed theoretical developments on duality analysis of Bose-Hubbard and equivalently XXZ models on two dimensional lattices. We have focussed mainly at predictions such an analysis for square and Kagome lattices at half and one-third filling. We have compared the predictions of such dual vortex theories with QMC results wherever possible. Finally, we would like to point out a couple of issues that still remain unsettled in this regard. It is yet unknown, whether it is possible to have a second order quantum phase transition in these models as predicted by the vortex theory. Direct QMC studies\(^{8,9,10,11,12,13}\) seem to find at best a weak first order transition. However, it has recently conjectured that at least for the Kagome lattice, a second order transition might occur at special particle-hole symmetric points in the phase diagrams\(^{23}\). Since QMC studies, at least for large enough system sizes, can only reach close to this point, this issue is not settled yet. Second, it remains to be seen whether such duality analysis for boson theories with \( U(1) \) symmetry can be extended to either models with Fermions and/or bosonic models with higher symmetry group.

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1. M. Greiner, O. Mandel, T. Esslinger, T.W. Hansch, and I. Bloch, Nature (London) 415, 39 (2002).
2. C. Orzel, A.K. Tuchman, M.L. Fenselau, M. Yasuda, and M.A. Kasevich, Science 291, 2386 (2001).
3. T. Senthil et al., Science 303, 1490 (2004); T. Senthil et al., Phys. Rev. B 70, 144407 (2004).
4. L. Balents et al., Phys. Rev. B 71, 144508 (2005); ibid 71, 144509 (2005); L. Balents et al., Prog. Theor. Phys. Suppl., bf 160, 314 (2005).
5. A. Burkov and L. Balents, Phys. Rev. B 72, 134502 (2005).
6. K. Sengupta, S. Isakov, and Y.B. Kim, Phys. Rev. B 72, 134502 (2005).
7. A. Burkov and E. Demler, Phys. Rev. Lett. 96, 180406 (2006).
8. G.G. Batrouni et al., Phys. Rev. Lett. 74, 2527 (1995); R.T. Scalettar et al., Phys. Rev. B 51, 8467 (1995); G.G. Batrouni and R.T. Scalettar, Phys. Rev. Lett. 84, 1599 (2000); F. Herbert it et al., Phys. Rev. B 65, 014513 (2002); G. Schmid et al., Phys. Rev. Lett. 88, 167208 (2002); A. Kuklov, N. Prokof’ev, and B. Svistunov, Phys. Rev. Lett. 93, 230402 (2004).
9. R. G. Melko et al., Phys. Rev. Lett. 95, 127207 (2005).
10. D. Heidarian and K. Damle, Phys. Rev. Lett. 95, 127206 (2005).
11. S. V. Isakov et al., Phys. Rev. Lett. 97, 147202 (2006).
12. K. Damle and T. Senthil, Phys. Rev. Lett. 97, 067202 (2006).
13. S. Wessel, cond-mat/0701337 (unpublished).
14. M. P. A. Fisher and D. H. Lee, Phys. Rev. B 39, 2756 (1989).
15. Z. Tesanovic, Phys. Rev. Lett. 93, 217004 (2004); A. Melikyan and Z. Tesanovic, Phys. Rev. B 71, 214511 (2005).
16. C. Dasgupta and B.I. Halperin, Phys. Rev. Lett. 47, 1556 (1981).
17. M. Wallin, E. Sorensen, A.P. Young and S.M. Girvin, Phys. Rev. B 49, 12115 (1994).
18. K. Lannert, M.P.A Fisher, and T. Senthil, Phys. Rev. B 63, 134510 (2001).
19. J. Vidal et al., Phys. Rev. B 64, 155306 (2001); M. Rizzi, V. Cataudella, R. Fazio, Phys. Rev. B 73, 144511 (2006).
20. L. Jiang and J. Ye, cond-mat/0601083 (unpublished).
21. R. Moessner, S.L. Sondhi, and P. Chandra, Phys. Rev. Lett. 84, 4457 (2000).
22. D.C. Cabra et al., Phys. Rev. B 71, 144420 (2005).
23. A. Sen, K. Damle, and T. Senthil, cond-mat/0701476.
24. A. Kuklov et al., cond-mat/0602466 (unpublished); A. Kuklov, N. Prokof’ev, and B. Svistunov, cond-mat/0501052 (unpublished).