Jordan homomorphisms and harmonic mappings

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Abstract
We show that each Jordan homomorphism \( R \to R' \) of rings gives rise to a harmonic mapping of one connected component of the projective line over \( R \) into the projective line over \( R' \). If there is more than one connected component then this mapping can be extended in various ways to a harmonic mapping which is defined on the entire projective line over \( R \).

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1 Introduction

1.1 The problem to determine all harmonic mappings (see 4.7) for projective lines over rings (see 4.1) goes back to K. von Staudt (1798–1867), who treated harmonic bijections between real projective lines; cf. [18, p. 57–58] for a survey and historical remarks. The results which have been obtained so far show that among the relevant algebraic mappings are – apart from projective transformations – not only homomorphisms but also Jordan homomorphisms of rings (see 3.1). If a ring \( R \) contains a subfield \( K \) then the point set of the associated chain geometry (see 5.1) is the projective line over \( R \). The investigation of homomorphism of chain geometries (see 5.3) has also lead to Jordan homomorphisms of rings. There is a widespread literature on the interplay of Jordan homomorphisms, harmonic mappings, and homomorphisms of chain geometries. The interested reader should consult [1], [2], [3], [4], [5], [6], [11], [13], [14], [15], [16], [20], and [21] for further references and related results.

1.2 Suppose that we are given a Jordan homomorphism \( \alpha : R \to R' \) of rings. The ring \( R \) can be embedded in the projective line \( \mathbb{P}(R) \) over \( R \) via \( t \mapsto R(t, 1) \) and there is a similar embedding \( R' \to \mathbb{P}(R') \). By virtue of these embeddings, \( \alpha \) determines a mapping

\[
R(t, 1) \mapsto R'(t^\alpha, 1') \text{ with } t \in R. \tag{1}
\]

There arises the question if (1) can be extended to a mapping \( \mathbb{P}(R) \to \mathbb{P}(R') \) in some “natural way”. It is fairly obvious that such an extension should take each point \( R(1, t) \) to \( R'(1', t'^\alpha) \). However, the projective line may also contain points of the form \( R(a, b) \), where neither \( a \) nor \( b \) are invertible, whence they cannot be written as \( R(1, t) \) or \( R(t, 1) \). For each of those points an “appropriate” definition of the image point is not immediate, since \( R(a, b) \mapsto R(a^\alpha, b'^\alpha) \) gives in general no well defined mapping.
An affirmative answer to the question above has been given by C. Bartolone [1] under the additional assumption that $R$ is a ring of stable rank 2. Among the rings with this property are, e.g., local rings, matrix rings over fields, and finite-dimensional algebras over commutative fields. In case of stable rank 2 each point of $\mathbb{P}(R)$ can be written in the form $R(t_1 t_2 - 1, t_1)$ with parameters $t_1, t_2 \in R$, and

$$R(t_1 t_2 - 1, t_1) \mapsto R'(t_1'^2 t_2^2 - 1', t_1')$$

(2)

is a well defined extension of (1).

In the present article there will be no restriction on the rings $R$ and $R'$. We shall show that the mapping (1) can be extended to a harmonic mapping $\overline{\pi} : C \to \mathbb{P}(R')$, where $C \subseteq \mathbb{P}(R)$ denotes the connected component (in the graph-theoretic sense; see 4.1) of the point $R(1, 0)$. If $C \neq \mathbb{P}(R)$ then $\overline{\pi}$ can be extended in various ways to a harmonic mapping $\mathbb{P}(R) \to \mathbb{P}(R')$.

The definition of $\overline{\pi}$ is rather involved: Each point of the connected component $C$ can be described by some finite sequence $(t_1, t_2, \ldots, t_n)$ of parameters in $R$, where $n \geq 0$ is variable. Then the Jordan homomorphism acts on these parameters, i.e., the sequence $(t_1', t_2', \ldots, t_n')$ determines the image point; cf. formula (3) below which is a generalization of (2). The number of parameters which is needed in order to describe all points of $C$ may be unbounded. Furthermore, a point of $C$ may admit many representations in terms of parameters. So the problem is to show that we have a well defined mapping. In [1] the situation is less complicated: When $R$ is a ring of stable rank 2, the projective line $\mathbb{P}(R)$ coincides with the connected component $C$ and, as has been mentioned above, each point of $\mathbb{P}(R)$ can be described with just two parameters.

1.3 The paper is organized as follows: In Section 2 we discuss the elementary subgroup $E_2(R)$ of the general linear group $GL_2(R)$ over a ring $R$. Following P.M. Cohn [12] we consider a family of matrices $E(t)$, $t \in R$, with the property that each matrix in $E_2(R)$ can be written as a product $E(t_1)E(t_2)\cdots E(t_n)$ with $t_1, t_2, \ldots, t_n \in R$ and $n \geq 0$. The entries of a matrix in $E_2(R)$ can be expressed with the help of an infinite family of polynomials in non-commuting indeterminates. Next, in Section 3, we introduce the concept of a polynomial with Jordan property and present two infinite families of such polynomials (Propositions 3.3 and 3.4). These polynomials are used in order to compare the matrices $E(t_1)E(t_2)\cdots E(t_n) \in E_2(R)$ and $E(t_1')E(t_2')\cdots E(t_n') \in E_2(R')$. For example, if the $(1, 1)$-entry of the first matrix is a unit, then so is the $(1, 1)$-entry of the second matrix (Theorem 3.5).

Unfortunately, in general there is no well defined mapping sending $E(t_1)E(t_2)\cdots E(t_n)$ to $E(t_1')E(t_2')\cdots E(t_n')$. But we can pass from $E_2(R')$ to an appropriate quotient group $E_2(\mathbb{R}')/N_{\alpha}$; here $\mathbb{R}'$ denotes the subring of $R'$ which is generated by the image of the Jordan homomorphism $\alpha$ and $N_{\alpha}$ is a normal subgroup of $E_2(\mathbb{R}')$ which depends on $\alpha$. In this way a well defined homomorphism of groups $E_2(R) \to E_2(\mathbb{R}')/N_{\alpha}$ can be obtained (Theorem 3.6). If $N_{\alpha}$ contains only the identity matrix then we have a well defined mapping $E_2(R) \to E_2(\mathbb{R}')$ (Corollary 3.7). This is the case when $\alpha$ belongs to a certain class of Jordan homomorphisms, including homomorphisms and antihomomorphisms. However, we shall see that $N_{\alpha}$ can also be non-trivial (Examples 3.8). We show that $N_{\alpha}$ is in the centre of $E_2(\mathbb{R}')$, whence the Jordan homomorphism $\alpha$ gives rise to a homomorphism

$$\alpha_{PE} : PE_2(R) \to PE_2(\mathbb{R}')$$
of *projective elementary groups* (see 4.1), which act on the connected component $C \subseteq \mathbb{P}(R)$ and a connected component of the subline $\mathbb{P}(R'' \subseteq \mathbb{P}(R')$, respectively (Theorem 4.4). The mapping $\alpha_{PE}$ is then the key to showing that a well defined mapping $\overline{\alpha}$ of the points of $C$ is given by

$$R(1,0) \cdot E(t_1)E(t_2) \cdots E(t_n) \mapsto R'(1',0') \cdot E(t'_1)E(t'_2) \cdots E(t'_n)$$

with $t_1, t_2, \ldots, t_n \in R$ and $n \geq 0$. This $\overline{\alpha}$ extends (1) and it turns $\alpha_{PE}$ into a homomorphism of transformation groups. We show some geometric properties of the mapping $\overline{\alpha}$ and present several examples to illustrate our results.

In Section 5 we examine homomorphisms of chain geometries. In particular, it will be established that the isomorphisms of affine chain geometries discussed by A. Herzer in [15, 9.1] can be extended to homomorphisms of chain geometries without any additional assumption on the underlying rings (Theorem 5.2). Thus our results yield new examples of homomorphisms of chain geometries.

## 2 The elementary subgroup $E_2(R)$

### 2.1 Throughout this paper we shall only consider associative rings with a unit element, which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules. The group of invertible elements and the centre of a ring $R$ will be denoted by $R^*$ and $Z(R)$, respectively. Also, we shall write $S(R) := R^0 \cup R^1 \cup R^2 \cup \ldots$ for the set of all finite sequences in $R$, including the empty sequence.

### 2.2 We recall that the elementary subgroup $E_2(R)$ of the general linear group $GL_2(R)$ is generated by the set of all matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \text{ with } t \in R. \quad (4)$$

Furthermore,

$$E(t)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} = E(0) \cdot E(-t) \cdot E(0), \quad (5)$$

whence each element of $E_2(R)$ can be written in the form

$$E(t_1) \cdot E(t_2) \cdots E(t_n) =: E(t_1, t_2, \ldots, t_n) =: E(T) \quad (6)$$

where $T := (t_1, t_2, \ldots, t_n) \in S(R)$ denotes a sequence of $n \geq 0$ elements; cf. [12, p. 368].

It is easily seen that a $(2 \times 2)$-matrix over $R$ commutes with all matrices $E(t)$, $t \in R$, if and only if it has the form $\text{diag}(a, a)$ with $a \in Z(R)$. Hence the centre of $E_2(R)$ is the subgroup

$$H := E_2(R) \cap \{ \text{diag}(a, a) \mid a \in Z(R)^* \}. \quad (7)$$

### 2.3 In order to describe the entries of a matrix (6) we consider an infinite sequence $X = (x_1, x_2, \ldots)$ of indeterminates over $\mathbb{Z}$ and the free $\mathbb{Z}$-algebra $\mathbb{Z} \langle X \rangle$. Its elements are polynomials in the non-commuting indeterminates $x_1, x_2, \ldots$ with coefficients in $\mathbb{Z}$. 

3
We shall frequently use the following universal property of \( \mathbb{Z}(X) \) [19, p. 6]: If \( R \) is an arbitrary ring and \( (t_1, t_2, \ldots) \) is an infinite sequence of elements in \( R \), then there is a unique homomorphism \( \mathbb{Z}(X) \to R \) such that \( x_i \to t_i \) for each \( i \in \{1, 2, \ldots\} \). The image of \( f \in \mathbb{Z}(X) \) under this homomorphism is written as \( f(t_1, t_2, \ldots) \). Also, we have a homomorphism \( E_2(\mathbb{Z}(X)) \to E_2(R) \) by the action of \( f \to f(t_1, t_2, \ldots) \) on the entries of a matrix. In addition, let \( T = (t_1, t_2, \ldots, t_n) \in \mathcal{S}(R) \) be a finite sequence which may be empty \( (n < 1) \). Then we put

\[
f(T) = f(t_1, t_2, \ldots, t_n) := f(t_1, t_2, \ldots, t_n, 0, 0, \ldots).
\] (8)

In order to avoid misinterpretations let us point out the following particular case of (8): Assume that \( f = 2x_2 + x_3 \in \mathbb{Z}(X) \), \( T = (x_2, x_3) \in \mathcal{S}(\mathbb{Z}(X)) \), and \( V = (v_1, v_2, v_3) \in \mathcal{S}(R) \). Then \( f(T) = f(x_2, x_3) \) denotes that polynomial which arises from \( f \in \mathbb{Z}(X) \) if \( X \) is substituted by \( (x_2, x_3, 0, 0, \ldots) \). As \( f(x_2, x_3) = 2x_3 \neq f \), we must not write \( f = f(x_2, x_3) \)” in order to stress that \( f \) belongs to the \( \mathbb{Z} \)-subalgebra of \( \mathbb{Z}(X) \) generated by \( \{x_2, x_3\} \). Furthermore, \( f(V) = 2v_2 + v_3 \), but \( (f(T))(V) = f(x_2, x_3))(V) = 2x_3(V) = 2v_3 = f(v_2, v_3) \).

On the other hand, for each \( g \in \mathbb{Z}(X) \) there is a sufficiently large integer \( n \) such that \( g = g(x_1, x_2, \ldots, x_n) \).

2.4 Following [12, p. 376], we define a sequence of elements in \( \mathbb{Z}(X) \) recursively by

\[
\begin{align*}
e^{(-2)} &= -1, & e^{(-1)} &= 0, & e^{(0)} &= 1, \\
e^{(n)} &= e^{(n-1)}x_n - e^{(n-2)},
\end{align*}
\] (9)

where \( n \in \{1, 2, \ldots\} \). It will turn out useful to have a short notation for polynomials that arise from the ones given in (9) by a substitution (8) as follows: Given \( i, j \in \mathbb{Z} \) with \( i \geq 1 \) and \( j \geq i - 3 \) we define

\[
e^j_i := e^{(j-i+1)}(x_i, x_{i+1}, \ldots, x_j), \quad \tilde{e}^j_i := e^{(j-i+1)}(x_j, x_{j-1}, \ldots, x_i).
\] (10)

So \( j \) is an upper index rather than an exponent. In particular, we have

\[
e^1_i = e^{(n)}(x_1, x_2, \ldots, x_n) = e^{(n)} \text{ for all } n \in \{-2, -1, \ldots\}.
\] (11)

Furthermore, each polynomial \( e^j_i \) can be written as a \( \mathbb{Z} \)-linear combination of monomials \( x_{h_1}x_{h_2}\cdots x_{h_m} \) with \( i \leq h_1 < h_2 < \cdots < h_m \leq j \) and \( m \) ranging from 0 to \( j - i + 1 \). Likewise \( \tilde{e}^j_i \) is a \( \mathbb{Z} \)-linear combination of monomials \( x_{h_1}x_{h_2}\cdots x_{h_m} \) with \( j \geq h_1 > h_2 > \cdots > h_m \geq i \) and \( m \) ranging from 0 to \( j - i + 1 \). For example, \( e^2_2 = e^{(1)}(x_2) = x_2 \) and \( e^6_5 = e^{(2)}(x_5, x_6) = x_5x_6 - 1 \).

Many of the following calculations are based upon the identities

\[
e^j_i = e^{j-1}_i x_j - e^{j-2}_i, \quad \tilde{e}^j_i = e^j_{i+1} x_i - e^j_{i+2},
\] (12)

(13)

which follow from (9) and (10) whenever \( 1 \geq i \geq j \).

Next we describe certain elements of the group \( E_2(\mathbb{Z}(X)) \):
Lemma 2.5 If \((x_1, x_2, \ldots, x_n) \in S(\mathbb{Z}(X))\) then

\[
E(x_1, x_2, \ldots, x_n) = \begin{pmatrix}
e_1^n & e_1^{n-1} \\
e_2^n & e_2^{n-1}
\end{pmatrix}, \quad (14)
\]

\[
E(x_1, x_2, \ldots, x_n)^{-1} = \begin{pmatrix}\end{pmatrix}
\]

\[
\begin{pmatrix}
e_2^{n-1} & -e_1^{n-1} \\
e_1^n & -e_2^n
\end{pmatrix}.
\] \( (15)\)

Proof: Clearly, for \(n = 0\) we have \(E() = I\), the identity in \(E_2(\mathbb{Z}(X))\). Now (14) follows easily by induction ([12, p. 376]), since for \(n \geq 1\) we infer from the induction hypothesis and (12) that

\[
E(x_1, x_2, \ldots, x_n) = \begin{pmatrix}e_1^{n-1} & e_1^{n-2} \\
e_2^{n-1} & -e_2^{n-2}
\end{pmatrix}. \begin{pmatrix}x_n & 1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}e_1^n & e_1^{n-1} \\
e_2^n & -e_2^{n-1}
\end{pmatrix}.
\]

The proof of (15) runs in a similar manner taking into account the first part of equation (5), (13), and \(E(x_1, x_2, \ldots, x_n)^{-1} = E(x_2, x_3, \ldots, x_n)^{-1} \cdot E(x_1)^{-1}\). \( \Box \)

2.6 We obtain the recursion \(e_1^n = x_1e_2^n - e_3^n\) for \(n \in \{1, 2, \ldots\}\) from the (1,1)-entry of the matrix equation \(E(x_1, x_2, \ldots, x_n) = E(x_1) \cdot E(x_2, x_3, \ldots, x_n)\) together with (14). This yields

\[
e^j_i = x_i e^j_{i+1} - e^j_{i+2}, \quad (16)
\]

\[
\text{and} \quad e^j_1 = x_j e^j_1 - e^j_2, \quad (17)
\]

for \(1 \geq i \geq j\) as counterparts of (12) and (13); cf. [12, p. 376].

2.7 We now return to an arbitrary ring \(R\). For each \(T = (t_1, t_2, \ldots, t_n) \in S(R)\) there is the homomorphism \(\mathbb{Z}(X) \rightarrow R: f \rightarrow f(T)\); see (8). So we can transfer our calculations from \(\mathbb{Z}(X)\) to \(R\) and from \(E_2(\mathbb{Z}(X))\) to \(E_2(R)\). For example, (12) yields

\[
e^n_i(T) = e^{n-1}_1(T)t_n - e^{n-2}_1(T) = e^{n-1}_1(t_1, t_2, \ldots, t_{n-1})t_n - e^{n-2}_1(t_1, t_2, \ldots, t_{n-2}).
\]

whereas (16) gives

\[
e^n_1(T) = t_1 e^n_1(T) - e^n_3(T) = t_1 e^{n-1}_1(t_2, t_3, \ldots, t_n) - e^{n-2}_1(t_3, t_4, \ldots, t_n).
\]

Observe that there are numerous ways to rewrite such identities, since we may also add irrelevant ring elements. E.g., \(e^n_1(T) = e^{n+1}_2(s, T)\) and \(e^n_1(T) = e^n_1(T, v)\) for all \(s, v \in R\).

It has been pointed out in [1, Lemma 1.5] that \(e^n_3(t_1, t_2, t_3) = t_1t_2t_3 - t_3 - t_1 \in R^*\) implies \(e^n_3(t_1, t_2, t_3) = t_3t_2t_1 - t_1 - t_3 \in R^*\) for all \((t_1, t_2, t_3) \in R^3\). The following result is a generalization of this, but we give a completely different proof using the group \(E_2(R)\):

Proposition 2.8 Let \(T \in R^n, n \geq 0\). Then \(e^n_1(T) \in R^*\) implies \(e^n_1(T) \in R^*\).

Proof: This is obvious for \(n = 0\). So let \(n \geq 1\). Since \(e^n_1(T) \in R^*\), there are elements \(s, v \in R\) such that

\[
0 = se^n_1(T) - e^n_2(T) = se^{n+1}_2(s, T) - e^{n+1}_3(s, T) = e^{n+1}_1(s, T, v),
\]

\[
0 = e^n_1(T)v - e^{n-1}_1(T) = e^n_2(T, v)v - e^{n-1}_1(T, v) = e^{n+1}_1(T, v) = e^{n+2}_2(s, T, v),
\]

where we used (16) and (12). We read off from (14) and \(e^{n+1}_2(s, T, v) = e^n_1(T)\) that

\[
E(s, T, v) = \text{diag}(e^{n+2}_1(s, T, v), -e^n_1(T)).
\]

The inverse of this matrix is diagonal, too, and by (15) its (1,1)-entry equals \(-e^{n+1}_2(s, T, v) = -e^n_1(T)\), which is therefore a unit. \( \Box \)
3 Jordan homomorphisms

3.1 Let $R$ and $R'$ be rings. We recall that a mapping $\alpha : R \to R'$ is called Jordan homomorphism if

$$(a + b)^\alpha = a^\alpha + b^\alpha, \quad 1^\alpha = 1', \quad (aba)^\alpha = a^\alpha b^\alpha a^\alpha$$

for all $a, b \in R$. (18)

See, among others, [17, p. 2] or [15, p. 832]. For such an $\alpha$ and an $a \in R^*$ the equation $1' = (aa^{-2})^\alpha = a^\alpha(a^{-2})a^\alpha$ shows that $a^\alpha \in R^*$. Also, $a^\alpha = (aa^{-1})^\alpha = a^\alpha(a^{-1})^\alpha a^\alpha$ implies $(a^{-1})^\alpha = (a^\alpha)^{-1}$ for all $a \in R^*$. We say that $\alpha$ is proper if it is neither a homomorphism nor an antihomomorphism.

In general a Jordan homomorphism is not multiplicative, but on certain expressions, like $aba$ in (18), it acts “like a ring homomorphism”. In order to generalize this we pass to the free $\mathbb{Z}$-algebra $\mathbb{Z}(X)$. We say that $f \in \mathbb{Z}(X)$ is a polynomial with Jordan property, or a $J$-polynomial for short, if $f(T)^\alpha = f(T^\alpha)$ holds for every Jordan homomorphism $\alpha : R \to R'$ between arbitrary rings $R$ and $R'$ and every (finite or infinite) sequence $T$ in $R$.

Examples 3.2 Clearly, 1, $x_i$, and $x_ix_jx_i$ are $J$-polynomials for all $i, j \in \{1, 2, \ldots\}$. The set of all $J$-polynomials forms a $\mathbb{Z}$-submodule of $\mathbb{Z}(X)$. Furthermore, suppose that $G$ is a (finite or infinite) sequence of $J$-polynomials and that $f$ is a $J$-polynomial. Then it is easily seen that also $f(G)$ has the Jordan property. In particular we obtain the $J$-polynomials

$$x_1^2 = x_1x_1,$$

$$x_1x_2 + x_2x_1 = (x_1 + x_2)^2 - x_1^2 - x_2^2,$$

$$x_1x_2x_3 + x_3x_2x_1 = (x_1 + x_3)x_2(x_1 + x_3) - x_1x_2x_1 - x_3x_2x_3.$$ 

Next we give two infinite families of $J$-polynomials.

Proposition 3.3 Let $n \in \{-1, 0, \ldots\}$. Then $e_i^n e_1^{n-1}$ is a polynomial with Jordan property.

Proof: We proceed by induction observing that $e_1^{-1}e_1^{-2} = 0$, $e_1^0 e_1^{-1} = 0$, and $e_1^1 e_1^0 = x_1$ are $J$-polynomials. Letting $n \geq 2$ we infer from the induction hypothesis and an appropriate substitution that

$$e_i^n e_i^{n-1} = e_1^{n-i+1}(x_i, x_{i+1}, \ldots, x_n)e_1^{n-i}(x_i, x_{i+1}, \ldots, x_{n-1})$$

is a $J$-polynomial for $i \in \{2, 3\}$. The keys for the following calculations are formula (16) and formula (13). So we get

$$e_1^n e_1^{n-1} = (x_1 e_2^n - e_3^n)(e_2^{n-1} x_1 - e_3^{n-1})$$

$$= x_1(e_2^n e_2^{n-1})x_1 + e_3^n e_3^{n-1} - (x_1 e_2^n e_3^{n-1} + e_3^n e_2^{n-1} x_1)$$

where, by (19) and the examples given in 3.2, $f_1$ is a $J$-polynomial. Similarly, if $n \geq 3$ then

$$x_1 e_2^n e_3^{n-1} + e_3^n e_2^{n-1} x_1 = x_1 x_2 (e_3^n e_3^{n-1} + (e_3^n e_3^{n-1}) x_2 x_1 - (x_1 e_3^n e_3^{n-1} + e_3^n e_4 e_1^{n-1} x_1)$$

$$= f_2$$
with a J-polynomial $f_2$. Also, for $n \geq 4$ we get

$$x_1 e_4^n e_3^{n-1} + e_3^n e_4^{n-1} x_1 = x_1 e_4^n e_3^{n-1} x_3 + x_3 e_4^n e_3^{n-1} x_1 - (x_1 e_4^n e_5^{n-1} + e_5^n e_4^{n-1} x_1).$$

with a J-polynomial $f_3$. Proceeding in this way we arrive either at

$$f_{n-1} - (x_1 e_{n+1}^n e_n^{n-1} + e_n^n e_{n+1}^{n-1} x_1) = f_{n-1} - (x_1 \cdot 1 \cdot 1 + x_n \cdot 0 \cdot x_1),$$

when $n$ is odd, or at

$$f_{n-1} - (x_1 e_{n+1}^n e_n^{n-1} + e_n^n e_{n+1}^{n-1} x_1) = f_{n-1} - (x_1 \cdot x_n \cdot 0 + 1 \cdot 1 \cdot x_1),$$

when $n$ is even. But $x_1$ is a J-polynomial, whence the assertion follows. \(\square\)

**Proposition 3.4** Let $n \in \{-2, -1, \ldots\}$. Then $e_1^n e_1^n$ is a polynomial with Jordan property.

**Proof:** This is clear when $n < 0$. If $n \geq 0$ then we infer from the $(1, 2)$-entry of $E(x_1, x_2, \ldots, x_n) \cdot E(x_1, x_2, \ldots, x_n)^{-1} = I$ and Lemma 2.5 that

$$-e_1^n e_1^{n-1} + e_1^{n-1} e_1^n = 0.$$

Hence $e_1^{n+1} e_1^n = e_1^n x_{n+1} e_1^n - e_1^n e_1^n = e_1^n x_{n+1} e_1^n - e_1^n e_1^{n-1}$, where we used (12) and the equation above. So

$$e_1^n x_{n+1} e_1^n = e_1^{n+1} e_1^n + e_1^n e_1^{n-1},$$

and Proposition 3.3 yields that $e_1^n x_{n+1} e_1^n$ is a J-polynomial. This property remains unaltered if we substitute $x_{n+1}$ by $1$. \(\square\)

Our next result generalizes [21, Lemma 1.2] and is based upon the previous propositions. It will be the backbone of many considerations. The theorem says that if certain entries of a matrix $E(T)$ are of a particular form, then so are the corresponding entries in $E(T^\alpha)$.

**Theorem 3.5** Let $\alpha : R \to R'$ be a Jordan homomorphism and let $T \in R^n$, $n \geq 0$. Then the following holds:

(a) $e_1^n(T) \in R^*$ implies $e_1^n(T^\alpha) \in R^*$.

(b) If $e_1^n(T) \in R^*$ and $e_1^{n-1}(T) = 0$ then $e_1^{n-1}(T^\alpha) = 0'$.

**Proof:** (a) We deduce from Proposition 2.8 that $e_1^n(T) \in R^*$. Proposition 3.4 and $R^{*\alpha} \subseteq R^*$ establish that

$$e_1^n(T^\alpha) e_1^n(T^\alpha) = (e_1^n e_1^n)(T^\alpha) = (e_1^n e_1^n)(T)^\alpha = (e_1^n(T)e_1^n(T))^\alpha \in R^*.$$

So $e_1^n(T^\alpha)$ is right invertible. Let $\tilde{T}$ be the finite sequence $T$ written in reverse order. Then

$$e_1^n(T^\alpha) e_1^n(T^\alpha) = e_1^n(\tilde{T}^\alpha) e_1^n(\tilde{T}^\alpha) = (e_1^n(\tilde{T}) e_1^n(\tilde{T}))^\alpha = (e_1^n(T) e_1^n(T))^\alpha \in R^*,$$
where we used the equation from above in the second step. Hence \( e_1^n(T^\alpha) \) is also left invertible.

(b) We read off from (12) the identity

\[
e_1^{n+1}(x_1, x_2, \ldots, x_n, 1) = e_1^n \cdot 1 - e_1^{n-1}.
\]  

(20)

So \( e_1^{n-1}(T) = 0 \) implies \( e_1^{n+1}(T, 1) = e_1^n(T) \), whereas \( e_1^n(T, 1) = e_1^n(T) \) holds trivially. Since \( e_1^{n+1} \frac{e_1}{n} \) and \( e_1^n \frac{e_1}{n} \) are J-polynomials by Proposition 3.3 and Proposition 3.4, we obtain that

\[
e_1^{n+1}(T^\alpha, 1') \frac{e_1}{n}(T^\alpha, 1') = (e_1^{n+1} \frac{e_1}{n})(T^\alpha, 1') = ((e_1^{n+1} \frac{e_1}{n})(T, 1))^{\alpha} = (e_1^n(T) \frac{e_1}{n})(T^\alpha) = e_1^n(T^\alpha).
\]

Now Proposition 2.8 and (a) allow to cancel the unit \( e_1^n(T^\alpha, 1') = e_1^n(T^\alpha) \). Hence (20) forces that \( e_1^{n-1}(T^\alpha) = 0' \).

We are now in a position to show our first main result.

**Theorem 3.6** Let \( \alpha : R \to R' \) be a Jordan homomorphism and denote by \( R'' \) the subring of \( R' \) generated by \( R'' \). Then the following statements are true:

(a) If \( T \in S(R) \) then \( E(T) \in H \) implies \( E(T^\alpha) \in H'' \), where \( H \) and \( H'' \) denote the centres of \( E_2(R) \) and \( E_2(R'') \), respectively.

(b) The set

\[
N_\alpha := \{ E(T^\alpha) \mid T \in S(R) \text{ and } E(T) = 1 \}
\]

is contained in \( H'' \).

(c) \( N_\alpha \) is a normal subgroup of \( E_2(R'') \). Furthermore, the mapping

\[
\alpha_E : E_2(R) \to E_2(R'')/N_\alpha : E(T) \mapsto N_\alpha \cdot E(T^\alpha),
\]

where \( T \in S(R) \), is a well defined homomorphism of groups.

**Proof:** (a) Let \( T \in R^n \) be a sequence such that \( E(T) \in H \). So, by (7), there is an \( a \in Z(R)^* \) with \( E(T) = \text{diag}(a, a) \). Put \( (\alpha') := E(T^\alpha) \). By virtue of (14), all entries of this matrix are in \( R'' \). The first row of \( E(T) \) is \( (e_1^n(T), e_1^n(T') = (a, 0) \). We infer from Theorem 3.5 that \( a' \) is a unit and that \( b' = 0' \). Next choose any \( s \in R \) and let \( S := (s, T, 0, -s, 0) \). By (5),

\[
E(S) = E(s)E(T)E(s)^{-1} = E(T) = \text{diag}(a, a),
\]

\[
E(S^\alpha) = E(s^\alpha)E(T^\alpha)E(s^\alpha)^{-1} = \begin{pmatrix} d' & d's^\alpha - s^\alpha a' - c' \\ 0' & a' \end{pmatrix}.
\]

(24)

As before, Theorem 3.5 shows that \( d' \) is a unit and that \( d's^\alpha - s^\alpha a' - c' \) vanishes for all \( s \in R \). Letting \( s := 0, 1 \) yields \( c' = 0' \) and \( a' = d' \). Hence \( a's^\alpha = s^\alpha a' \) for all \( s \in R \) and therefore \( a' \in Z(R'') \). We infer that \( E(T^\alpha) \in H'' \).

(b) This is immediate from (a).
(c) First, let us introduce the following notation. Given $T \in R^n$, $n \geq 0$, we define
\[
\hat{T} := (0, -t_n, 0, 0, -t_{n-1}, 0, \ldots, 0, -t_1, 0) \in R^{3n}.
\] (25)
We observe that (5) implies $E(\hat{T}) = E(T)^{-1}$. Also, when (25) is applied accordingly to sequences in $R'$, then $\hat{T}^\alpha = \hat{T}^\alpha$.

Clearly, $I' \in N_\alpha$. Next assume that $E(T) = E(V)$ for $T, V \in S(R)$. Then $E(T, \hat{V}) = I$, and $E(T^\alpha)E(V^\alpha)^{-1} = E(T^\alpha, \hat{V}^\alpha) \in N_\alpha$. Hence $E(T^\alpha) \in N_\alpha \cdot E(V^\alpha)$.

Any two matrices in $N_\alpha$ can be written in the form $E(T^\alpha), E(V^\alpha)$ with $E(T) = E(V) = I$. By the above, $E(T^\alpha)E(V^\alpha)^{-1} \in N_\alpha$. So $N_\alpha$ is a subgroup of $E_2(R'')$ which is normal by (a).

Summing up, the mapping $\alpha_E$ is well defined and obviously it is a homomorphism. \hfill \Box

The reason for introducing the subring $R''$ in the theorem above is that $N_\alpha$ need not be normal in $E_2(R')$; cf. Example 3.8 (f) below.

**Corollary 3.7** If, under the assumptions of Theorem 3.6, the group $N_\alpha$ is trivial, then
\[
\alpha_E : E_2(R) \to E_2(R'') : E(T) \mapsto E(T^\alpha),
\]
where $T \in S(R)$, is a well defined mapping.

**Examples 3.8**

(a) Let $\alpha : R \to R'$ be a homomorphism. Then there is the homomorphism $\alpha_* : \text{GL}_2(R) \to \text{GL}_2(R') : M \mapsto M^\alpha$, i.e., $\alpha$ is applied to each entry of $M$. Obviously, $E(t)^{\alpha_*} = E(t^\alpha)$ for all $t \in R$, whence $N_\alpha = \{I'\}$ and $\alpha_*$ restricts to $\alpha_E : E_2(R) \to E_2(R'')$.

(b) Let $\alpha : R \to R'$ be an antihomomorphism. The mapping $\alpha_{**} : \text{GL}_2(R) \to \text{GL}_2(R') : M \mapsto E(0')^{-1}(M^T)^{\alpha_*}^{-1}E(0')$, where $M^T$ denotes the transpose of $M$, is a homomorphism. It maps each $E(t) \in E_2(R)$ to $E(t^\alpha)$. Hence $N_\alpha = \{I'\}$ and $\alpha_{**}$ restricts to $\alpha_E$. If $R'$ is commutative then $M^{\alpha_{**}} = \det(M^\alpha)^{-1}M^\alpha$ for all $M \in \text{GL}_2(R)$.

(c) Suppose that $R = \prod_{\lambda \in \Lambda} R_\lambda$ is the direct product of rings $R_\lambda$. Then, up to isomorphism, $\text{GL}_2(R) = \prod_{\lambda \in \Lambda} \text{GL}_2(R_\lambda)$ and $E_2(R) = \prod_{\lambda \in \Lambda} E_2(R_\lambda)$. Similarly, let $R' = \prod_{\lambda \in \Lambda} R'_\lambda$ and let $\alpha_\lambda : R_\lambda \to R'_\lambda$ be a family of mappings, where each $\alpha_\lambda$ is a homomorphism or antihomomorphism. Then $\alpha := \prod_{\lambda \in \Lambda} \alpha_\lambda$ is in general a proper Jordan homomorphism $R \to R'$. Now, by (a) or (b), we can choose at least one homomorphism $\beta_\lambda : \text{GL}_2(R_\lambda) \to \text{GL}_2(R'_\lambda)$ which restricts to $\alpha_{\lambda,E}$. So $\beta := \prod_{\lambda \in \Lambda} \beta_\lambda : \text{GL}_2(R) \to \text{GL}_2(R')$ is a homomorphism with $E(t)^\beta = E(t^\alpha)$ for all $t \in R$. Finally, as above, $N_\alpha = \{I'\}$ and $\beta$ restricts to $\alpha_E$.

(d) The following class of examples is essentially due to A. Herzer [14, 4.2]. Although our assumptions are weaker, Herzer's proofs immediately carry over to our setting.

Suppose that $D$ is a commutative ring and let $B$ be a $D$-algebra with a $D$-linear homomorphism $\chi : B \to D$ of rings. Denote by $M$ a left module over $D$ which is endowed with a $D$-bilinear, alternating, and associative product $M \times M \to M$. Then $R := B \oplus M$ becomes a $D$-algebra, if a product is defined by
\[
(b_1 + m_1)(b_2 + m_2) := b_1b_2 + b_1\chi m_2 + b_2\chi m_1 + m_1m_2
\]
for all $b_1, b_2 \in B$ and $m_1, m_2 \in M$. (The commutativity of $D$ guarantees that $R$ is associative.) Assume that $B', \chi'$, and $M'$ are given as above and that $\alpha_1 : B \to B'$ is a homomorphism or antihomomorphism of $D$-algebras satisfying $\chi = \alpha_1 \chi'$. Also, let $\alpha_2 : M \to M'$ be an arbitrary $D$-linear mapping. Then

\[ \alpha : R \to R' : b + m \mapsto b^{\alpha_1} + m^{\alpha_2} \quad (b \in B, \ m \in M) \]

is a $D$-linear Jordan homomorphism.

(e) With the notation introduced in Example (d) let $B = B'$, $\chi = \chi'$, and $M = M' = D^3$. Write $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ for the canonical basis of $D^3$. A product on $D^3$ with the properties mentioned above is given by $\varepsilon_i^2 = 0$, $\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i$, $\varepsilon_1 \varepsilon_2 = \varepsilon_3$, $\varepsilon_i \varepsilon_3 = 0$ for all $i, j \in \{1, 2, 3\}$. We define a $D$-linear Jordan automorphism $\alpha$ of $R$ by $\alpha_1 = \text{id}_B$, whereas $\alpha_2$ fixes $\varepsilon_1$ and interchanges $\varepsilon_2$ with $\varepsilon_3$. Now $N_\alpha \neq \{I\}$ follows from

\[
\begin{align*}
E(\varepsilon_1)E(\varepsilon_3)E(-\varepsilon_1)E(-\varepsilon_3) & = I, \\
E(\varepsilon_1)E(\varepsilon_2)E(-\varepsilon_1)E(-\varepsilon_2) & = \text{diag}(1 - \varepsilon_3, 1 - \varepsilon_3). 
\end{align*}
\]

So $\alpha$ is a proper Jordan automorphism. Also there is no mapping $E_2(R) \to E_2(R)$ with $E(T) \mapsto \alpha E(T^\alpha)$ for all $T \in S(R)$.

(f) Let $R$ be given as in Example (e) with $B = D$ and $\chi = \text{id}_D$. (Then $R = D^4$ is isomorphic to the exterior algebra $\bigwedge D^2$.) Furthermore $\alpha : R \to R$ is defined as in (e), but we reserve the letter $R'$ for the $D$-algebra of $(4 \times 4)$-matrices over $D$. The right regular representation $\rho$ of $R$ maps each $a \in R$ to that $(4 \times 4)$-matrix which describes the linear mapping $R \to R : x \mapsto xa$ in terms of the basis $(1, \varepsilon_1, \varepsilon_2, \varepsilon_3)$. The product $\alpha \rho$ is a Jordan monomorphism, say $\beta : R \to R'$. We have $R^\alpha = R^3 = R^\beta$.

The first row of the $(4 \times 4)$-matrix $u' := (1 - \varepsilon_3)^\rho \in R'$ reads $(1, 0, 0, -1)$, since $1(1 - \varepsilon_3) = 1 - \varepsilon_3$. So it is not in the centre of $R'$ and there is a $(4 \times 4)$-matrix $r' \in R'$ that does not commute with $u'$. A straightforward calculation shows that $E(r')^{-1} \text{diag}(u', u') E(r')$ is not diagonal. On the other hand, all matrices in $N_\beta$ are diagonal, since they are in the centre $H''$ of $E_2(R'')$. Example (e) shows that $\text{diag}(u', u') \in N_\beta$, whence $N_\beta$ is not normal in $E_2(R')$.

(g) With $R$ as in Example (e) let $\alpha_1 = \text{id}_B$, whereas $\alpha_2$ is given by $\varepsilon_1 \mapsto \varepsilon_1$, $\varepsilon_2 \mapsto \varepsilon_2$, and $\varepsilon_3 \mapsto 0$. Then $R^\alpha$ is not a subring of $R$, since $\varepsilon_3 = \varepsilon_1^\alpha \varepsilon_2^\alpha$ is not in the image of $\alpha$.

4 The projective line over a ring

4.1 Consider the free left $R$-module $R^2$. The projective line over $R$ is the orbit of the free cyclic submodule $R(1,0)$ under the natural right action of $\text{GL}_2(R)$. In other words, $\mathbb{P}(R)$ is the set of all $p \leq R^2$ such that $p = R(a,b)$, where $(a, b)$ is the first row of an invertible matrix; compare [15, p. 785]. If also $(c, d)$ is the first row of an invertible matrix, then $R(a, b) = R(c, d)$ if and only if there is a unit $u \in R^*$ with $(c, d) = u(a, b)$ [8, Proposition 2.1].
Let \( \{(a, b), (c, d)\} \) be a basis of \( \mathbb{R}^2 \). Then the points \( p = R(a, b) \) and \( q = R(c, d) \) are called distant. In this case we write \( p \triangle q \). The vertices of the distant graph on \( \mathbb{P}(R) \) are the points of \( \mathbb{P}(R) \), the edges of this graph are the unordered pairs of distant points. The set \( \mathbb{P}(R) \) can be decomposed into connected components (maximal connected subsets of the distant graph); cf. [9, p. 108].

The orbit of \( R(1, 0) \in \mathbb{P}(R) \) under the group \( E_2(R) \) is exactly the connected component of \( R(1, 0) \) [9, Theorem 3.2]. It will be denoted by \( C \). If a matrix \( E(T) \in E_2(R) \) fixes \( R(1, 0) \) and all points \( R(t, 1) \) with \( t \in R \), then it is easily seen that \( E(T) = \text{diag}(a, a) \) with \( a \in Z(R)^* \). On the other hand, each such \( E(T) \) fixes all points of \( \mathbb{P}(R) \). So the kernel of the group action of \( E_2(R) \) on \( C \) (or \( \mathbb{P}(R) \)) is the centre \( H \) of \( E_2(R) \); cf. (7). As usual, we write \( PE_2(R) := E_2(R)/H \) for the projective elementary group.

**Lemma 4.2** The group \( E_2(R) \) acts \( 2\triangle \)-transitively on the connected component \( C \subseteq \mathbb{P}(R) \), i.e. transitively on the set of ordered pairs of distant points of \( C \).

**Proof:** Let \((p, q)\) be a pair of distant points in \( C \). Since all points of \( C \) are in one orbit of \( E_2(R) \), there exists a matrix in \( E_2(R) \) sending \( p \) to \( R(1, 0) \) and \( q \) to \( q_1 \triangle R(1, 0) \). So \( q_1 = R(t, 1) \) for some \( t \in R \). Now \( E(0)E(-t) \) fixes \( R(1, 0) \) and takes \( q_1 \) to \( R(0, 1) \). \( \square \)

### 4.3
Suppose that \( R'' \) is a subring of a ring \( R' \). As \( GL_2(R'') \) is a subgroup of \( GL_2(R') \), there is a mapping

\[
\mathbb{P}(R'') \to \mathbb{P}(R') : R''(a'', b'') \mapsto R'(a'', b'')
\]

which is easily seen to be injective. It will be used to identify \( \mathbb{P}(R'') \) with a subset of \( \mathbb{P}(R') \). We have to distinguish between the connected components \( C'' \) and \( C' \) of \( R'(1', 0') \) in \( \mathbb{P}(R'') \) and \( \mathbb{P}(R') \), respectively. Since \( E_2(R'') \) is a subgroup of \( E_2(R') \), we have \( C'' \subseteq C' \).

However, unless the centre \( H'' \) of \( E_2(R'') \) lies in the centre \( H' \) of \( E_2(R') \), the group \( PE_2(R'') \) cannot be considered as a subgroup of \( PE_2(R') \): In general there are two matrices in \( E_2(R'') \) that induce the same transformation on \( C'' \), but distinct transformations on \( C' \). So we cannot always consider \( PE_2(R'') \) as a group which acts on \( C' \).

The reader should keep these remarks in mind with regard to the following theorem, which is our second main result:

**Theorem 4.4** Let \( \alpha : R \to R' \) be a Jordan homomorphism and denote by \( R'' \) the subring of \( R' \) generated by \( R'' \). Then the following statements are true.

(a) The mapping

\[
\alpha_{PE} : PE_2(R) \to PE_2(R'') : H \cdot E(T) \mapsto H'' \cdot E(T''),
\]

where \( T \in S(R) \), is a well defined homomorphism of groups.

(b) Consider the connected component \( C \subseteq \mathbb{P}(R) \) and the connected component \( C'' \subseteq \mathbb{P}(R'') \subseteq \mathbb{P}(R) \). Then the mapping

\[
\overline{\alpha} : C \to C'' : R(1, 0) \cdot E(T) \mapsto R'(1', 0') \cdot E(T''),
\]

where \( T \in S(R) \), is well defined.
Proof: (a) By Theorem 3.6 (b), \(N_\alpha \subseteq H''\). So there exists the canonical epimorphism

\[ \eta : E_2(R'')/N_\alpha \to (E_2(R'')/N_\alpha)/(H''/N_\alpha) \cong E_2(R'')/H''. \]

We identify its image with \(\text{PE}_2(R'')\). So, by (22), the composition \(\alpha E\eta : E_2(R) \to \text{PE}_2(R'')\) is a homomorphism. It follows from Theorem 3.6 (a) that \(H^\alpha \subseteq H''/N_\alpha = \ker \eta\). Hence \(H \subseteq \ker \alpha E\eta\) and \(\alpha \text{PE}\) is a well defined homomorphism of groups.

(b) We regard \(\text{PE}_2(R)\) and \(\text{PE}_2(R'')\) as transformation groups on the connected components \(C\) and \(C''\), respectively.

Suppose that a matrix \(E(S)\), \(S \in \mathcal{S}(R)\), fixes \(R(1,0)\). Thus its first row has the form \((u, 0)\) with \(u \in R^*\). We infer from Theorem 3.5 that the first row of \(E(S^\alpha)\) reads \((u', 0')\) with a unit \(u' \in R''\). So \(E(S^\alpha)\) leaves \(R'(1', 0')\) invariant. This means that under \(\alpha \text{PE}\) the stabilizer of \(R(1,0)\) is mapped into the stabilizer of \(R'(1', 0')\).

For each point \(p \in C\) there is a sequence \(T \in \mathcal{S}(R)\) such that \(p = R(1,0) \cdot E(T)\). Also let \(p = R(1,0) \cdot E(V)\) with \(V \in \mathcal{S}(R)\). So the transformation \(H : E(T)E(V)^{-1} \in \text{PE}_2(R)\) fixes \(R(1,0)\), whence the transformation \(H'' \cdot E(T^\alpha)E(V^\alpha)^{-1} \in \text{PE}_2(R'')\) fixes \(R'(1', 0')\). Therefore \(\overline{\alpha}\) is well defined.

(c) By (a) and (b), the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{E(T)} & C \\
\downarrow{\alpha} & & \downarrow{\overline{\alpha}} \\
C'' & \xrightarrow{E(T^\alpha)} & C''
\end{array}
\]

commutes for each \(T \in \mathcal{S}(R)\), whence the assertion follows. \(\square\)

4.5 For each point \(p \in C\) there is a smallest integer \(n \geq 0\) such that \(p = R(1,0) \cdot E(T)\) with \(T \in R^n\). In fact, \(n\) is just the distance of \(p\) and \(R(1,0)\) in the distant graph [9, formula (10)]. The supremum of all distances between points of \(C\) is a non-negative integer or \(\infty\). It is called the \textit{diameter} of \(C\). Furthermore, we have \((1,0) \cdot E(0)^2 = (-1,0)\) and \((1,0) \cdot E(t) = (1,0) \cdot E(1,t+1)\) for all \(t \in R\). So if the diameter of \(C\) is finite, say \(d\), then it is enough to consider sequences \(T \in R^m\) with fixed length \(m := \max\{2, d\}\) in order to reach all points of \(C\). By (14), in this case \(\overline{\alpha}\) can be described by the single formula

\[ R(e_1^m(T), e_1^{m-1}(T))^{\overline{\alpha}} = R'(e_1^m(T^\alpha), e_1^{m-1}(T^\alpha)) \text{ with } T \in R^m. \]  

(29)

This generalizes [1, Theorem 2.4], where \(m = 2\) and \(R\) is a ring of stable rank 2. See also [10, Remark 5.4] for the special case of an antiisomorphism of rings. We shall see in Example 4.11 that there are rings where one needs sequences \(T \in R^n\) for infinitely many \(n \geq 0\) in order to describe \(\overline{\alpha}\).

4.6 Let us state some immediate consequences of Theorem 4.4. If \((a, b)\) and \((a', b')\) are the first rows of the matrices \(E(V) \in E_2(R)\) and \(E(V^\alpha)\), respectively, then (28) implies that

\[ (R(a, b) \cdot E(T))^{\overline{\alpha}} = R'(a', b') \cdot E(T^\alpha) \]

(30)
for all \( T \in \mathcal{S}(R) \). Letting \( E(V) = E(0) \) we get \((a, b) = (0, 1)\) and \((a', b') = (0', 1')\). Therefore also the second rows of the matrices \( E(T) \) and \( E(T^\alpha) \), with \( T \in \mathcal{S}(R) \), represent points corresponding under \( \overline{\alpha} \). We deduce from (27), (30), and (5) that
\[
R(t, 1)^{\overline{\alpha}} = (R(1, 0) \cdot E(t))^\overline{\alpha} = R'(1', 0') \cdot E(t^\alpha) = R'(t^\alpha, 1'), \tag{31}
\]
\[
R(1, t)^{\overline{\alpha}} = (R(0, 1) \cdot E(0, -t, 0))^\overline{\alpha} = R'(0', 1') \cdot E(0', -t^\alpha, 0') = R'(1', t^\alpha), \tag{32}
\]
for all \( t \in R \). In particular, \( \overline{\alpha} \) is indeed an extension of the mapping described in (1). Furthermore, \( \overline{\alpha} \) is a fundamental mapping; this means that \( R(1, 0)^{\overline{\alpha}} = R'(1', 0') \), \( R(0, 1)^{\overline{\alpha}} = R'(0', 1') \), and \( R(1, 1)^{\overline{\alpha}} = R'(1', 1') \).

If \( R^\alpha = R'' \) then each point of \( C'' \) can be written as \( R'(1', 0') \cdot E(T^\alpha) \), whence \( C^{\overline{\alpha}} = C'' \). Similarly, if \( R^\alpha = R' \) then \( C^{\overline{\alpha}} = C' = C'' \). If \( \overline{\alpha} \) injective then (31) implies the injectivity of \( \alpha \). Also, if \( \alpha \) is bijective then \( \alpha^{-1} : R' \to R \) is a Jordan isomorphism and \( \overline{\alpha}^{-1} \) is easily seen to be the inverse of \( \overline{\alpha} \). What remains open here is whether or not \( C^{\overline{\alpha}} = C'' \) implies that \( R^\alpha = R'' \) and whether or not \( \overline{\alpha} \) is injective if \( \alpha \) is injective.

4.7 A mapping \( \mathbb{P}(R) \to \mathbb{P}(R') \) is said to be harmonic if it preserves harmonic quadruples or, in other words, if it preserves cross ratio \(-1 \) \([15, 1.3.5]\). If \((p_0, p_1, p_2, p_3)\) is a harmonic quadruple in \( \mathbb{P}(R) \) then \( p_0, p_1, \) and \( p_i \) are mutually distant for \( i \in \{2, 3\} \). Furthermore, all four points are mutually distant if and only if \(-1 \neq 1 \) in \( R \). Otherwise \( p_2 = p_3 \), whence in this case harmonic mappings do not deserve our interest.

In order to state the next result we have to allow the domain and the codomain of a harmonic mapping to be a subset of a projective line.

**Proposition 4.8** Let \( \alpha : R \to R' \) and \( \overline{\alpha} : C \to C'' \) be given as in Theorem 4.4. Then the following statements are true:

(a) \( \overline{\alpha} \) takes pairs of distant points to pairs of distant points.

(b) \( \overline{\alpha} \) is a harmonic mapping.

**Proof:**

(a) Let \((p, q)\) be a pair of distant points in \( C \). By Lemma 4.2, there exists a matrix in \( E_2(R) \) sending \( p \) to \( R(1, 0) \) and \( q \) to \( R(0, 1) \). Clearly, the \( \overline{\alpha} \)-images of \( R(1, 0) \) and \( R(0, 1) \) are distant and, by (28), the points \( p^{\overline{\alpha}} \) and \( q^{\overline{\alpha}} \) are distant, too.

(b) Suppose that \( p_1, p_2 \) are points of \( \mathbb{P}(R) \). Then \((R(1, 0), R(0, 1), p_1, p_2)\) is a harmonic quadruple if and only if there is a \( u \in R^* \) with \( p_1 = R(u, 1) \) and \( p_2 = R(\overline{-u}, 1) \).

If we are given a harmonic quadruple in \( C \) then, by Lemma 4.2 and (28), we may assume without loss of generality that the first two points are \( R(1, 0) \) and \( R(0, 1) \). So the remaining two points can be described as above. We deduce from (31) and \( u^\alpha \in R^* \) that under \( \overline{\alpha} \) a harmonic quadruple is obtained. \( \square \)

The first part of Proposition 4.8 implies that
\[
\text{dist}(p^{\overline{\alpha}}, q^{\overline{\alpha}}) \leq \text{dist}(p, q) \text{ for all } p, q \in C. \tag{33}
\]
Here \( \text{dist}(p, q) \) denotes the distance of two points in the distant graph.

4.9 We turn to following question: If \( C \neq \mathbb{P}(R) \), how should one extend \( \overline{\alpha} \) to a mapping \( \gamma : \mathbb{P}(R) \to \mathbb{P}(R') \)? In view of Proposition 4.8 such an extension should at least preserve distant pairs and harmonic quadruples. Here are solutions to this problem.
**Examples 4.10**  (a) For each connected component $C_\mu$ of $\mathbb{P}(R)$ other than $C$ choose a matrix $A_\mu \in \text{GL}_2(R)$ such that the first row of $A_\mu$ represents a point of $C_\mu$. Also select a matrix $A'_\mu \in \text{GL}_2(R')$. Then $A_\mu^{-1}$ maps $C_\mu$ onto $C$, $\pi$ takes $C$ into $C'$, and $A'_\mu$ maps $C'$ into some connected component of $\mathbb{P}(R')$. In this way we obtain a solution $\gamma$ by pasting together all these mappings.

(b) In Example (a) the matrices $A'_\mu$ can be chosen at random. Suppose now that there is a homomorphism $\sigma : \text{GL}_2(R) \rightarrow \text{GL}_2(R')$ such that $\sigma$ restricts to $\alpha_E$ and such that $\sigma$ takes the stabilizer of $R(1,0)$ into the stabilizer of $R'(1',0')$. Then

$$\sigma : \mathbb{P}(R) \rightarrow \mathbb{P}(R') : R(1,0) \cdot M \mapsto R'(1',0') \cdot M^\sigma \quad \text{with} \quad M \in \text{GL}_2(R)$$

(34)

is a well defined extension of $\pi$ and $(\sigma, \pi)$ is a homomorphism of group actions. The mapping $\sigma$ fits into Example (a) by choosing $A'_\mu = A'_\mu$. The homomorphisms $\alpha_\ast$, $\alpha_{\ast\ast}$, and $\beta$ which have been introduced in 3.8 (a), (b), and (c), respectively, satisfy the conditions above: A matrix $M \in \text{GL}_2(R)$ stabilizes $R(1,0)$ if and only if there are elements $a, d \in R^*$ and $c \in R$ with $M = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. If $\alpha$ is a homomorphism of rings then the assertion is immediate. Furthermore, in this case we get the well known mapping

$$\overline{\alpha} : \mathbb{P}(R) \rightarrow \mathbb{P}(R') : R(a, b) \mapsto R'(a^\alpha, b^\alpha).$$

(35)

If $\alpha$ is an antihomomorphism then a straightforward calculation shows

$$M^{\alpha_{\ast\ast}} = \begin{pmatrix} (d^\alpha)^{-1} & 0' \\ (a^\alpha)^{-1}d^\alpha(a^\alpha)^{-1} & (a^\alpha)^{-1} \end{pmatrix},$$

whence the assertion follows also in the remaining cases, cf. also [10, Remark 5.4].

We end this section with an example where $\mathbb{P}(R)$ has more than one connected component, the connected components of $\mathbb{P}(R)$ have infinite diameter, and $\alpha$ is a proper Jordan endomorphism.

**Example 4.11** We specify and slightly modify the data of Example 3.8 (e) as follows: Let $D = K$ be a commutative field, let $B = K[x, y]$ be the algebra of polynomials in commuting indeterminates $x, y$ over $K$, and let $\chi : K[x, y] \rightarrow K : f \mapsto f(0,0)$. The module $M = K^3$ and its multiplication remain unchanged. Hence $R = K[x, y] \oplus K^3$. Again $\alpha_1 = \text{id}_{K[x,y]}$, but now $\alpha_2$ is chosen to be any $K$-linear mapping with $\varepsilon_3$ not being an eigenvector. Then $\alpha$ is a proper $K$-linear Jordan endomorphism, since $(\varepsilon_1\varepsilon_2)^\alpha = \varepsilon_3^\alpha \notin K\varepsilon_3$, whereas $\varepsilon_1^\alpha\varepsilon_2^\alpha \in K\varepsilon_3$ and $\varepsilon_2^\alpha\varepsilon_1^\alpha \in K\varepsilon_3$.

The projection $\pi : R \rightarrow K[x, y]$ is an epimorphism of $K$-algebras with kernel $\{0\} \oplus K^3$. By (35), it gives rise to a mapping $\pi^\sigma : \mathbb{P}(R) \rightarrow \mathbb{P}(K[x, y])$ which is surjective [8, Proposition 3.5 (3)]. Under $\pi^\sigma$ the connected component of $R(1,0)$ is mapped onto the connected component of $K[x, y](1,0)$, since $\pi_{\text{PE}}$ is a surjection of $\text{PE}_2(R)$ onto $\text{PE}_2(K[x, y])$. The projective line $\mathbb{P}(K[x,y])$ has more than one connected component and all its connected components have infinite diameter [9, Example 5.7 (c)]. So (33) implies that also $\mathbb{P}(R)$ has more than one connected component and that $C \subset \mathbb{P}(R)$ has infinite diameter. By [9, Theorem 3.2 (a)], then all connected components of $\mathbb{P}(R)$ have infinite diameter.
5 On homomorphisms of chain geometries

5.1 If \( K \subseteq R \) is a (not necessarily commutative) subfield, then the projective line \( \mathbb{P}(K) \) can be identified with a subset of \( \mathbb{P}(R) \); cf. 4.3. The orbit of \( \mathbb{P}(K) \) under the group \( \text{GL}_2(R) \) is the set of \( K \)-chains. It turns \( \mathbb{P}(R) \) into a chain geometry \( \Sigma(K, R) \). The following basic properties of chain geometries can be found in [7]: Any three mutually distant points are on at least one \( K \)-chain. Two distinct points are distant if and only if they are on a common \( K \)-chain. Therefore each \( K \)-chain is contained in a unique connected component. In contrast to [15] it is not assumed that \( K \) is in the centre of \( R \), whence in [7] we used the term generalized chain geometry for what is here called a chain geometry.

We now consider two chain geometries \( \Sigma(K, R), \Sigma(K', R') \) and the mapping (27). The following result is a generalization of [1, Theorem 2.4] and [15, 9.1]:

**Theorem 5.2** Let \( \alpha : R \rightarrow R' \) be a Jordan homomorphism. The mapping \( \overline{\alpha} : C \rightarrow C'' \) maps \( K \)-chains into \( K' \)-chains if and only if for each \( c \in R^* \) there is a \( u_c' \in R'^* \) such that
\[
(Kc)^\alpha \subseteq (u_c'^{-1}K'u_c)c^\alpha. \tag{36}
\]

**Proof:** For each \( c \in R^* \) the point set
\[
D_c := \{ R(kc, 1) \mid k \in K \} \cup \{ R(1, 0) \} \tag{37}
\]
is a \( K \)-chain through \( R(1, 0), R(0, 1) \), and \( R(c, 1) \). The \( K' \)-chains passing through \( R'(1', 0'), R'(0', 1') \), and \( R'(c', 1') \) are exactly the sets
\[
\{ R((u_c'^{-1}k'u')c^\alpha, 1') \mid k' \in K' \} \cup \{ R'(1', 0') \} \tag{38}
\]
where \( u' \) ranges in \( R'^* \).

Suppose that \( \overline{\alpha} \) maps \( K \)-chains into \( K' \)-chains. So for each \( c \in R^* \) the point set \( D_c^\overline{\alpha} \) is a subset of a \( K' \)-chain. Now (31) implies that this chain contains the points \( R'(1', 0'), R'(0', 1') \), and \( R'(c', 1') \), whence it can be written in the form (38) for some \( u_c' \in R'^* \) depending on \( c \). Applying (31) to each point of (37) shows that condition (36) is satisfied.

Conversely, (36) forces that each \( K \)-chain \( D_c \) given by (37) is mapped into one of the \( K' \)-chains given by (38). By Lemma 4.2, every \( K \)-chain \( D \subseteq C \) is \( \text{E}_2(R) \)-equivalent to some \( K \)-chain through \( R(1, 0) \) and \( R(0, 1) \). Such a chain has the form
\[
\{ R(ka, b) \mid k \in K \} \cup \{ R(1, 0) \} \text{ with } a, b \in R^*. \tag{39}
\]

Since \( \text{diag}(b, b^{-1}) = E(-b)E(-b^{-1})E(-b) \in \text{E}_2(R) \), the chains \( D \) and \( D_c \), where \( c := ab \), are in one orbit of \( \text{E}_2(R) \). Now (28) shows that also \( D^\overline{\alpha} \) is a subset of a \( K' \)-chain.

Condition (36) reduces to \( (Kc)^\alpha \subseteq K'c^\alpha \) provided that \( K' \) is invariant under all inner automorphisms of \( R' \). This is the case whenever \( K' \) is in the centre of \( R' \), but there are also other possibilities [7, Examples 2.5].

5.3 We close with some remarks on a mapping \( \overline{\alpha} \) where \( \alpha \) satisfies the conditions of Theorem 5.2. If \( \mathbb{P}(R) = C' \) then \( \overline{\alpha} \) is a homomorphism of chain geometries, i.e., \( K \)-chains are mapped
into $K'$-chains. If $\mathbb{P}(R) \neq C$ then $\overline{v}$ can be extended to a mapping $\gamma : \mathbb{P}(R) \to \mathbb{P}(R')$ according to Example 4.10 (a). As the general linear group preserves chains, any such $\gamma$ is a homomorphism of chain geometries. Explicit examples for this latter case arise from Example 4.11, because all Jordan endomorphisms described there are $K$-linear and thus fulfil condition (36).

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