Complexity and criticality in Laplacian growth models

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Abstract

We analyze the dynamical evolution of systems which obey simple growth laws, like diffusion limited aggregation or dielectric breakdown. We show that, if the developing patterns is sufficiently complex, a scale invariant noise spectrum is generated, in agreement with the hypothesis that the system is in a self organized critical state. The intrinsic noise generated in the evolution is shown to be independent of the (extrinsic) stochastic aspects of the growth. Instead, it is related to the complexity of the generated pattern.

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A stimulating development in the analysis of complex systems has been the hypothesis of self organized criticality [1, 2]. The main assumption is that such systems evolve towards a state where small perturbations can give rise to changes (catastrophes) of all sizes. This state describes the most unstable situation compatible with some kind of equilibrium. The distribution of catastrophes, because of its inherent scale invariance, is characterized by power laws. In previous work, we have checked that this hypothesis is well satisfied in systems which evolve into a steady state far from equilibrium [3]. The model we analyzed is Diffusion Limited Aggregation [4], for which a comprehensive amount of work is already available.

In the present work we study the mechanisms through which scale invariant noise is generated, as the complexity of the system is modified. The SOC hypothesis is only applicable to situations which cannot be described by simple, deterministic laws. When the behavior of the system is very predictable, it will show only certain kinds of catastrophes, and not a power law spectrum, as predicted by the SOC hypothesis.

To analyze in detail this question, we study a simple generalization of the DLA model: the dielectric breakdown (DB) model with a scale invariant growth law [5]. As in DLA, the growth is determined by a scalar field, $\phi$, which obeys the Laplace equation outside the aggregate. The velocity of growth, however, depends on the gradient of this field raised to some power, $v \propto |\nabla \phi|^\eta$. When $\eta = 1$, we recover DLA, and $\eta = 0$ describes the Eden model. For $\eta \gg 1$, growth takes place only in those points at the surface of the aggregate where the field, $\nabla \phi$, has a maximum. Thus, the system behaves in a simple and deterministic way. In this limit, only narrow needles can grow, with a fractal dimension close to one [6]. It is clear the the change in the growth law does not introduce any new scale in the system. Extensive calculations show that the aggregates are, indeed, self similar. On the other hand, the noise associated with the growth process can be reduced in a controlled way, while keeping its intrinsic complexity [7]. Patterns developed by this procedure are, at the same time, complex and regular, reproducing the main features of dendritic growth.

We first investigate the changes in the noise spectrum as function of $\eta$. The calculations reported below were obtained averaging results for eight aggregates, grown in a circular cell of radius 108. Preliminary results, as well as a detailed description of the methods used are given in ref. [11].

We define catastrophes following our previous work on DLA. We first an-
alyze aggregates of a given size. We calculate, once growth has take place at certain site, in how many lattice nodes outside the aggregate the diffusion field changes above a given threshold, $\epsilon$. This procedure allows us to obtain a definition of the spatial extent of the rearrangement which follows a growth event. Alternatively, we also study how many iterations are required to make the diffusion field converge to its new equilibrium value, given an overall error tolerancy in the calculations. In this way, we can define the duration of the catastrophe. We have made detailed calculations which show that the SOC regime, as measured by its critical exponents, is independent of the thresholds used. Also, modified definitions of the size and duration of the catastrophes do not alter the main results, that is, the existence of a power law distribution of catastrophes, and only change slightly the exponents. In our previous work \cite{3, 11}, we have shown that there is a close correspondence between the duration, and extension, of a given catastrophe. Moreover, these quantities are uniquely related to the strength of the field at the point where the catastrophe (growth event) occurred. This result can be intuitively understood by noting that, whenever growth takes place at a point where the field is large, a significant change in the electrostatic potential around the aggregate follows, because the new site in the aggregate has to be set at the potential of the rest of the cluster. When the field, that is, the potential drop between the aggregate and the new site, was large, the rearrangement of the potential has also to be large.

The study of the distribution of catastrophes in terms of the electrical field at the surface of the aggregate allows us to relate our work to the extensive study of the harmonic measure of growing patterns\cite{8}. In addition, it is clear that the noise associated with the growth of a given pattern can be expressed in terms of the field distribution along the perimeter of a static pattern. While the hypothesis of SOC customarily describes the noise spectrum of a dynamical process, better numerical accuracy, in our systems, can be achieved be studying static distributions.

Our results for the Dielectric Breakdown Model are summarized in figures 1 and 2. They show the distribution of the values of the electric field as the aggregates grow (curves labelled dynamical) and as function of location along the perimeter of static aggregates (curves labelled statical). Figure 1 shows the results obtained for $\eta = 0$ (Eden model) and $\eta = 1$ (DLA). All curves show well defined power laws over many decades. While in the Eden model the growth is independent of the Laplacian field outside, we use it
as representative of low $\eta$ models. In figure 2 we show results for $\eta = 5$ and $\eta = 10$. There is no possibility of defining a power law behavior in this regime. The values of the electric field found in the dynamical simulations tend to cluster at the upper range of the curves. The fact that there is no longer dispersion of the values of the fields (or catastrophe sizes) agrees with the assumption that the aggregates have lost their intrinsic complexity. The growth sites, and their associated fields are now highly predictable, despite the fact that growth still takes place via a stochastic process\cite{12}. The observed values of the fields can be used to define a scale in the system, which is not possible in the regime where scale invariant noise appears. The fields tend to have their highest possible values, so that this scale is the upper cutoff in the model, the size of the aggregate. Thus, although we find a qualitative change in the noise spectrum, it does not happen in a similar way to phase transitions in thermal equilibrium. If that were the case, we expect a transition from a scale invariant regime (characterized by one, or a line of fixed points) to a situation where a new scale, intermediate between the lattice size and the aggregate dimensions, appears. On the other hand, we cannot rule out that the system, in the high $\eta$ regime, evolves in bursts, growing first at one tip, until that tip becomes blunt and growth there is arrested, which leads to growth at another tip, and so forth. This possibility resembles the first order phase transition model proposed as an alternative to self organized criticality\cite{13}.

To complete or analysis of the role of complexity and external noise in systems growing out of equilibrium, we have analyzed the case $\eta = 1$ with reduced noise. We use counters to achieve the required degree of noise reduction\cite{10}. Each possible growth site has an attached variable. Each random choice of a given site increases the value of that variable by one. Real growth takes place when the variable exceeds a given threshold. Once growth has occurred, the variable at that point is reset to zero. Noise reduction tends to increase the role of the anisotropy of the underlying lattice, and the resulting patterns resemble crystal growth.

In figure 3, we present the results obtained, using a threshold of hundred for the noise reduction variable. The field distributions are the average of three aggregates. While the pattern is significantly different from typical DLA aggregates, the noise spectrum, or the harmonic measure, is not. The features in the pattern which are due to the randomness in the growth pro-
cess disappear, but the complexity of the system shows up in the noise and related correlation functions. The power law that we obtain is indistinguishable, within our numerical accuracy, from the results for standard DLA. The pattern shown in figure (3) is highly symmetric and regular, but contains structure on all scales. This fine structure is the cause of the scale invariant noise spectrum shown in the same figure. Thus, even when the external noise is significantly reduced, a broad band, scale invariant noise spectrum is generated. Any small noise, or inhomogeneity in the initial conditions, is amplified in the steady state. This conclusion is rather general, and should be applicable to a variety of experiments. We find confirmation to it in the fact that, in dendritic growth, the distribution of lengths of side branches seems very irregular\cite{4}. That does not exclude that the overall envelope coincides with simple, analytical, solutions.

In conclusion, we have analyzed the circumstances under which self organized critical behavior may appear. By studying models where noise and intrinsic complexity can be modified, we show that SOC is a feature directly related to the complexity of the evolving system. Models which give rise to simple patterns do not exhibit scale invariant noise, although their evolution includes noise terms. On the other hand, growth laws which give rise to complex patterns, show noise at all scales, irrespective of the stochasticity included in the simulations.

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[12] A simple case which can be solved analytically is when the aggregate can be modelled by a straight needle. Then, the density of points with field $\epsilon$ is: $\Delta n/\Delta \epsilon \propto \epsilon_{min}^2/\left(\epsilon^2 \sqrt{\epsilon^2 - \epsilon_{min}^2}\right)$, where $\epsilon_{min}$ is the value of the field at the center of the needle. This function has an intrinsic scale set by $\epsilon_{min}$, which depends on the length of the needle. See also, B. Derrida and V. Hakim, preprint.

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**Figure Captions**

**Figure 1.** Histogram of electric fields at the perimeters of clusters grown for $\eta = 0$ (upper graph) and $\eta = 1$ (lower graph) (thick curve). Histogram of the values of fields at the sites where growth actually takes place (thin curve). Inset shows the shape of one of the aggregates for each value of $\eta$.

**Figure 2.** As in figure 1, for $\eta = 5$ (upper graph) and $\eta = 10$ (lower graph).

**Figure 3.** As in figures 1 and 2, but for clusters grown with $\eta = 1$ and a noise reduction algorithm.