Covariant Hamiltonian representation of Noether’s theorem and its application to SU(N) gauge theories

Jürgen Struckmeier, Horst Stöcker, and David Vasak

Abstract We present the derivation of the Yang-Mills gauge theory based on the covariant Hamiltonian representation of Noether’s theorem. As the starting point, we re-formulate our previous presentation of the canonical Hamiltonian derivation of Noether’s theorem [1]. The formalism is then applied to derive the Yang-Mills gauge theory. The Noether currents of U(1) and SU(N) gauge theories are derived from the respective infinitesimal generating functions of the pertinent symmetry transformations which maintain the form of the Hamiltonian.

1 Introduction

Noether’s theorem establishes in the realm of the Hamilton-Lagrange description of continuum dynamics the correlation of a conserved current with a particular symmetry transformation that preserves the form of the Hamiltonian of the given system. Although usually derived in the Lagrangian formalism [2, 3], the natural context for
deriving Noether’s theorem for first-order Lagrangian systems is the Hamiltonian formalism: for all theories derived from action principles only those transformations are allowed which maintain the form of said action principle. Yet, the group of transformations which leave the action functional form-invariant coincides with the group of canonical transformations. The latter may be consistently formulated in covariant Hamiltonian field theory \[4\]. As a result, for any conserved current of a Hamiltonian system, the pertaining symmetry transformation is simply given by the canonical transformation rules. Conversely, any symmetry transformation which maintains the form of the Hamiltonian yields a conserved current if said transformation is formulated as an infinitesimal canonical transformation. Since this holds for any conserved current, we thereby obtain the covariant Hamiltonian representation of Noether’s theorem.

2 Lagrangian description of the dynamics of fields

The realm of classical continuum physics deals with the dynamics of a system of \( N \geq 1 \) fields \( \phi_I(x) \) which are functions of space \((x^1, x^2, x^3)\) and time \( t \equiv x^0/c \) as the independent variables, \( x \equiv (x^0, x^1, x^2, x^3) \) (see, e.g. Greiner, Class. Electrodyn. \([5]\)). Depending on the context of our description, an indexed quantity may denote as well the complete collection of the respective quantities. In the first-order Lagrangian description, the state of the system is completely described by the actual fields \( \phi_I(x) \) and their 4\(N\) partial derivatives \( \partial_\mu \phi_I(x), \mu = 0, \ldots, 3; I = 1, \ldots, N \). We assume the dynamical system to be described by a first-order Lagrangian density \( L \) which may explicitly depend on the independent variables, \( \mathcal{L}(\phi_I, \partial \phi_I, x) \).

Herein, \( \partial \phi_I \) denotes the complete set of partial derivatives of \( \phi_I(x) \). The Lagrangian density \( \mathcal{L} \) thus constitutes a functional as it maps \( N \) functions \( \phi_I(x) \) and their \( 4N \) partial derivatives into \( \mathbb{R} \).

The space-time evolution of a dynamical system follows from the principle of least action: the variation \( \delta S \) of the action functional, 
\[
S = \int_R \mathcal{L}(\phi_I, \partial \phi_I, x) \, d^4x,
\]
vanishes for the space-time evolution which is actually realized by nature. From the calculus of variations \([3]\), one finds that \( \delta S = 0 \) holds exactly if the fields \( \phi_I \) and their partial derivatives satisfy the Euler-Lagrange field equations
\[
\frac{\partial}{\partial x^\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi_I)} - \frac{\partial \mathcal{L}}{\partial \phi_I} = 0.
\]
3 Covariant Hamiltonian description of the dynamics of fields in the DeDonder-Weyl formalism

In order to derive the equivalent covariant Hamiltonian description of continuum dynamics, we follow the classic approach of T. De Donder and H. Weyl [6, 7] in tensor language: define for each field \( \phi_I(x) \) a conjugate momentum 4-vector field \( \pi_I^\alpha(x) \). Their components are given by

\[
\pi_I^\alpha = \frac{\partial L}{\partial \left( \frac{\partial \phi_I}{\partial \alpha} \right)} \equiv \frac{\partial L}{\partial \left( \frac{\partial \phi_I}{\partial x^\alpha} \right)}.
\] (4)

For each scalar field \( \phi_I \), the 4-vectors \( \pi_I^\alpha \) are thus induced by the Lagrangian \( L \) as the dual counterparts of the 4-covectors (1-forms) \( \frac{\partial \phi_I}{\partial \alpha} \). For the entire set of \( N \) scalar fields \( \phi_I(x) \), this establishes a set of \( N \) conjugate 4-vector fields. With this definition of the 4-vectors of canonical momenta \( \pi_I^\alpha(x) \), we now define the Hamiltonian density \( H(\phi_I, \pi_I, x) \) as the covariant Legendre transform of the Lagrangian density \( L(\phi_I, \partial \phi_I, x) \) via

\[
H(\phi_I, \pi_I, x) = \pi_I^\alpha \frac{\partial \phi_I}{\partial x^\alpha} - L(\phi_I, \partial \phi_I, x),
\] (5)

where summation over the pairs of upper and lower indices is understood. At this point suppose that \( L \) is regular, hence that for each index “I” the Hesse matrices

\[
\left( \frac{\partial^2 L}{\partial (\partial \mu \phi_I) \partial (\partial \nu \phi_I)} \right)
\]

are non-singular. This ensures that \( H \) takes over the complete information about the given dynamical system from \( L \) by means of the Legendre transformation. The definition of \( H \) by Eq. (5) is referred to in literature as the “De Donder-Weyl” Hamiltonian density [6, 7].

Obviously, the dependencies of \( H \) and \( L \) on the \( \phi_I \) and the \( x^\mu \) only differ by a sign,

\[
\frac{\partial H}{\partial \phi_I} = - \frac{\partial L}{\partial \phi_I}, \quad \frac{\partial H}{\partial x^\mu} \bigg|_{\text{expl}} = - \frac{\partial L}{\partial x^\mu} \bigg|_{\text{expl}}.
\]

In order to derive the canonical field equations, we calculate from Eq. (5) the partial derivative of \( H \) with respect to \( \pi_I^\mu \),

\[
\frac{\partial H}{\partial \pi_I^\mu} = \delta_I^\alpha \delta_j^\mu \frac{\partial \phi_j}{\partial x^\alpha} = \frac{\partial \phi_I}{\partial x^\mu}.
\]

In conjunction with the Euler-Lagrange equation (3), we obtain the set of covariant canonical field equations,
$$\frac{\partial \mathcal{H}}{\partial \pi^\alpha_I} = \frac{\partial \phi_I}{\partial x^\alpha}, \quad \frac{\partial \mathcal{H}}{\partial \phi_I} = \frac{\partial \pi^\alpha_I}{\partial x^\alpha}. \quad (6)$$

These pairs of first-order partial differential equations are equivalent to the set of second-order differential equations of Eq. (3). Provided the Lagrangian density $\mathcal{L}$ is a Lorentz scalar, the dynamics of the fields is invariant with respect to Lorentz transformations. The covariant Legendre transformation (5) passes this property to the Hamiltonian density $\mathcal{H}$. It thus ensures a priori the relativistic invariance of the fields which emerge as integrals of the canonical field equations if $\mathcal{L}$ — and hence $\mathcal{H}$ — represents a Lorentz scalar.

From the right hand side of the second canonical field equation (6), we observe that the dependence of the Hamiltonian density $\mathcal{H}$ on $\phi_I$ only determines the divergence of the conjugate vector field $\pi_I$. The canonical momentum vectors $\pi_I$ are thus determined by the Hamiltonian only up to a zero-divergence vector fields $\eta_I(x)$

$$\pi_I \mapsto \Pi_I = \pi_I + \eta_I, \quad \frac{\partial \eta_I^\alpha}{\partial x^\alpha} = 0. \quad (7)$$

This fact provides a gauge freedom for the canonical momentum fields.

### 4 Canonical transformations in the realm of field dynamics

Similar to the canonical formalism of point mechanics, we call a transformation of the fields $(\phi_I, \pi_I) \rightarrow (\Phi_I, \Pi_I)$ canonical if the form of the variational principle which is based on the action functional (3) is maintained,

$$\delta \int_R \left( \pi^\alpha_I \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\phi_I, \pi_I, x) \right) d^4x = \delta \int_R \left( \Pi^\alpha_I \frac{\partial \Phi_I}{\partial x^\alpha} - \mathcal{H}'(\Phi_I, \Pi_I, x) \right) d^4x. \quad (8)$$

For the requirement (8) to be satisfied, the integrands may differ at most by the divergence of a 4-vector field $F^\mu_I, \mu = 0, \ldots, 3$ whose variation vanishes on the boundary $\partial R$ of the integration region $R$ within space-time

$$\delta \int_R \frac{\partial F^\alpha_I}{\partial x^\alpha} d^4x = \delta \int_{\partial R} F^\alpha_I dS^\alpha = 0.$$

The obvious consequence of the form invariance of the variational principle is the form invariance of the covariant canonical field equations (6). For the integrands of Eq. (8), which are actually the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}'$, we thus obtain the condition

$$\mathcal{L} = \mathcal{L}' + \frac{\partial F^\alpha_I}{\partial x^\alpha} \quad \pi^\alpha_I \frac{\partial \phi_I}{\partial x^\alpha} - \mathcal{H}(\phi_I, \pi_I, x) = \Pi^\alpha_I \frac{\partial \Phi_I}{\partial x^\alpha} - \mathcal{H}'(\Phi_I, \Pi_I, x) + \frac{\partial F^\alpha_I}{\partial x^\alpha}. \quad (9)$$
With the definition $F_1^\mu \equiv F_1^\mu (\phi, \Phi_{I}, x)$, we restrict ourselves to a function of exactly those arguments which now enter into transformation rules for the transition from the original to the new fields. The divergence of $F_1^\mu$ reads, explicitly,

$$\frac{\partial F_1^\alpha}{\partial x^\alpha} = \frac{\partial F_1^\alpha}{\partial \phi_J} \frac{\partial \phi_J}{\partial x^\alpha} + \frac{\partial F_1^\alpha}{\partial \Phi_J} \frac{\partial \Phi_J}{\partial x^\alpha} + \frac{\partial F_1^\alpha}{\partial x^\alpha} |_{\text{expl}}.$$ \hspace{1cm} (10)

The rightmost term denotes the sum over the *explicit* dependencies of the generating function $F_1^\mu$ on the $x^\mu$. Comparing the coefficients of Eqs. (9) and (10), we find the local coordinate representation of the field transformation rules which are induced by the generating function $F_1^\mu$

$$\pi_I^\mu = \frac{\partial F_1^\mu}{\partial \phi_I}, \quad \Pi_I^\mu = -\frac{\partial F_1^\mu}{\partial \Phi_I}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial F_1^\alpha}{\partial x^\alpha} |_{\text{expl}}.$$ \hspace{1cm} (11)

In contrast to the transformation rule for the Lagrangian density $\mathcal{L}$ of Eq. (9), the rule for the Hamiltonian density is determined by the *explicit* dependence of the generating function $F_1^\mu$ on the $x^\mu$. Hence, if a generating function does not explicitly depend on the independent variables, $x^\mu$, then the value of the Hamiltonian density is not changed under the particular canonical transformation emerging thereof.

The generating function of a canonical transformation can alternatively be expressed in terms of a function of the original fields $\phi_I$ and of the new conjugate fields $\Pi_I^\mu$. To derive the pertaining transformation rules, we perform the covariant Legendre transformation

$$F_2^\mu (\phi_I, \Pi_I, x) = F_1^\mu (\phi_I, \Phi_I, x) + \Phi_J \Pi_J^\mu.$$ \hspace{1cm} (12)

We thus encounter the set of transformation rules

$$\pi_I^\mu = \frac{\partial F_2^\mu}{\partial \phi_I}, \quad \Phi_I \Phi_J = \frac{\partial F_2^\mu}{\partial \Pi_J^\mu}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial F_2^\alpha}{\partial x^\alpha} |_{\text{expl}},$$ \hspace{1cm} (13)

which is equivalent to the set of rules (11) by virtue of the Legendre transformation (12) if the Hesse matrices $(\partial^2 F_1^\mu / \partial \phi_I \partial \Phi_J)$ are non-singular for all indices $\mu$.

5 Noether’s theorem in the Hamiltonian description of field dynamics

Canonical transformations are defined as the particular subset of general transformations of the fields $\phi_I$ and their conjugate momentum vector fields $\pi_I$ which preserve the form of the action functional (9). Such a transformation depicts a symmetry transformation which is associated with a conserved four-current vector, hence with a vector with vanishing space-time divergence. In the following, we work out the correlation of this conserved current by means of an *infinitesimal* canonical transfor-
mation of the field variables. The generating function $F^\mu_2$ of an infinitesimal transformation differs from that of an identical transformation by an infinitesimal parameter $\varepsilon \neq 0$ times an—as yet arbitrary—function $j^\mu(\phi_I, \pi_I, x)$:

$$F^\mu_2(\phi_I, \Pi_I, x) = \phi_I \Pi^\mu_I + \varepsilon j^\mu(\phi_I, \pi_I, x). \quad (14)$$

The subsequent transformation rules follow to first order in $\varepsilon$ from the general rules (13) as

$$\Pi^\mu_I = \pi^\mu_I - \varepsilon \frac{\partial j^\mu}{\partial \phi_I}, \quad \Phi_I \delta^\mu = \phi_I \delta^\mu + \varepsilon \frac{\partial j^\mu}{\partial \pi^\mu_I}, \quad \mathcal{H}' = \mathcal{H} + \varepsilon \frac{\partial j^\alpha}{\partial x^\alpha}|_{\text{expl}},$$

hence

$$\delta \pi^\mu_I = -\varepsilon \frac{\partial j^\mu}{\partial \phi_I}, \quad \delta \phi_I \delta^\mu = \varepsilon \frac{\partial j^\mu}{\partial \pi^\mu_I}, \quad \delta \mathcal{H}|_{\text{CT}} = \varepsilon \frac{\partial j^\alpha}{\partial x^\alpha}|_{\text{expl}}. \quad (15)$$

As the transformation does not change the independent variables, $x^\mu$, both the original as well as the transformed fields refer to the same space-time event $x^\mu$, hence $\delta x^\mu = 0$. With the transformation rules (15), the divergence of the four-vector of characteristic functions $j^\mu$ is given by

$$\varepsilon \frac{\partial j^\alpha}{\partial x^\alpha} = \varepsilon \frac{\partial j^\alpha}{\partial \phi_I} \delta \phi_I - \frac{\partial \mathcal{H}}{\partial \pi^\alpha_I} \delta \pi^\alpha_I - \frac{\partial \mathcal{H}}{\partial \phi_I} \delta \phi_I + \delta \mathcal{H}|_{\text{CT}}.$$  

The canonical field equations (6) apply along the system’s space-time evolution. The derivatives of the fields with respect to the independent variables may be then replaced accordingly to yield

$$\varepsilon \frac{\partial j^\alpha}{\partial x^\alpha} = -\frac{\partial \mathcal{H}}{\partial \pi^\alpha_I} \delta \pi^\alpha_I - \frac{\partial \mathcal{H}}{\partial \phi_I} \delta \phi_I + \delta \mathcal{H}|_{\text{CT}}.$$  

On the other hand, the variation $\delta \mathcal{H}$ of the Hamiltonian due to the variations $\delta \phi_I$ and $\delta \pi_I$ of the canonical fields is given by

$$\delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial \phi_I} \delta \phi_I + \frac{\partial \mathcal{H}}{\partial \pi_I} \delta \pi_I.$$  

If and only if the infinitesimal transformation rule $\delta \mathcal{H}|_{\text{CT}}$ for the Hamiltonian from Eqs. (15) coincides with the variation $\delta \mathcal{H}$ from Eq. (16), then the set of infinitesimal transformation rules is consistent and actually does define a canonical transformation. We thus have

$$\delta \mathcal{H}|_{\text{CT}} \equiv \delta \mathcal{H} \iff \frac{\partial j^\alpha}{\partial x^\alpha} = 0. \quad (17)$$
Thus, the divergence of \( j^\mu (x) \) must vanish in order for the transformation (15) to be canonical, and hence to preserve the Hamiltonian according to Eq. (17). The \( j^\mu (x) \) then define a conserved four-current vector, commonly referred to as Noether current. The canonical transformation rules (15) then furnish the corresponding infinitesimal symmetry transformation. Noether’s theorem and its inverse can now be formulated in the realm of covariant Hamiltonian field theory as:

**Theorem 1 (Hamiltonian Noether)** The characteristic vector function \( j^\mu (\phi, \pi, x) \) in the generating function \( F_2^\mu \) from Eq. (14) must have zero divergence in order to define a valid canonical transformation. The subsequent transformation rules (15) then comprise an infinitesimal symmetry transformation which preserves the action functional.

Conversely, if a symmetry transformation is known to preserve the action functional, then the transformation is canonical and hence can be derived from a generating function. The characteristic 4-vector function \( j^\mu (\phi, \pi, x) \) in the corresponding infinitesimal generating function (14) then represents a conserved current, hence \( \partial j^\alpha / \partial x^\alpha = 0 \).

### 6 Example 1: \( U(1) \) gauge theory

#### 6.1 Finite symmetry transformation

As an example, we consider the covariant Hamiltonian density \( \mathcal{H}_{\text{KGM}} \) of a complex Klein-Gordon \( \phi \) field that couples to an electromagnetic 4-vector potential \( a_\mu \)

\[
\mathcal{H}_{\text{KGM}} = \bar{\pi}_\alpha \pi^\alpha + i q (\bar{a}_\alpha a_\alpha \phi - \phi a_\alpha \pi^\alpha) + m^2 \phi \phi - \frac{1}{4} p^{a\beta} p_{a\beta}, \quad p^{a\beta} = -p^{\beta a}.
\]  

(18)

Herein, the \((2,0)\)-tensor field \( p^{a\beta} \) denotes the conjugate momentum field of \( a_\alpha \). We now define for this Hamiltonian density a local symmetry transformation by means of the generating function

\[
F_2^\mu = \Pi^\mu \phi e^{\Lambda(x)} + \bar{\Phi} \Pi^\mu e^{-\Lambda(x)} + p^{a\mu} \left( a_\alpha + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^\alpha} \right),
\]

(19)

In this context, the notation “local” refers to the fact that the generating function (19) depends explicitly on \( x \) via \( \Lambda = \Lambda(x) \). The general transformation rules (15) applied to the actual generating function yield for the fields

\[
\begin{align*}
p^{\mu\nu} &= p^{\mu\nu}, & A_\mu &= a_\mu + \frac{1}{q} \frac{\partial \Lambda}{\partial x^\mu} \\
\Pi^\mu &= \pi^\mu e^{\Lambda(x)}, & \Phi &= \phi e^{\Lambda(x)} \\
\bar{\Pi}^\mu &= \bar{\pi}^\mu e^{-\Lambda(x)}, & \bar{\phi} &= \bar{\phi} e^{-\Lambda(x)}
\end{align*}
\]

(20)
and for the Hamiltonian from the explicit $x^\mu$-dependency of $F_2^\mu$

\[ \mathcal{H}'_{\text{KGM}} - \mathcal{H}_{\text{KGM}} = \left. \frac{\partial F_2^\mu}{\partial x^\alpha} \right|_{\text{expl}} \]

\[ = i \left( \tilde{\pi}^\mu \phi - \tilde{\phi} \pi^\mu \right) \frac{\partial \Lambda(x)}{\partial x^\alpha}(A_a - a_a) \]

\[ = i q \left( \Pi^\alpha A_\alpha \Phi - \Phi A_\alpha \Pi^\alpha \right) - i q \left( \pi^\alpha a_\alpha \phi - \phi a_\alpha \pi^\alpha \right). \]

In the transformation rule for the Hamiltonian density, the term $P_{\alpha \beta} \partial^2 \Lambda / \partial x^\alpha \partial x^\beta$ vanishes as the momentum tensor $P_{\alpha \beta}$ is skew-symmetric. The transformed Hamiltonian density $\mathcal{H}'_{\text{KGM}}$ is now obtained by inserting the transformation rules into the Hamiltonian density $\mathcal{H}_{\text{KGM}}$

\[ \mathcal{H}'_{\text{KGM}} = \Pi_\alpha \Phi + i q \left( \Pi^\alpha A_\alpha \Phi - \Phi A_\alpha \Pi^\alpha \right) + m^2 \Phi \Phi \sqrt{1 - P_{a \mu} P^a_\mu}. \]

We observe that the Hamiltonian density (18) is form-invariant under the local canonical transformation generated by $F_2^\mu$ from Eq. (19) — which thus defines a symmetry transformation of the given dynamical system.

### 6.2 Field equations from Noether’s theorem

In order to derive the conserved Noether current which is associated with the symmetry transformation (20), we first set up the generating function of the infinitesimal canonical transformation corresponding to (19) by letting $\Lambda \rightarrow \varepsilon \Lambda$ and expanding the exponential function up to the linear term in $\varepsilon$

\[ F_2^\mu = \Pi^\mu \phi + i \varepsilon \Lambda \phi \left( 1 + i \right) + \tilde{\phi} \Pi^\mu \left( 1 - \varepsilon \right) A_\alpha - a_\alpha + \frac{\varepsilon}{q} \frac{\partial \Lambda}{\partial x^\alpha} \]

\[ = \Pi^\mu \phi + \tilde{\phi} \Pi^\mu + P_{a \mu} a_\alpha + \frac{\varepsilon}{q} \left[ i q \left( \pi^\mu \phi - \tilde{\phi} \pi^\mu \right) \Lambda + \rho_{a \mu} \frac{\partial \Lambda}{\partial x^\alpha} \right]. \]

According to Noether’s theorem (17), the expression in brackets represents the conserved Noether current $J^\mu(x)$

\[ J^\mu(x) = i q \left( \pi^\mu \phi - \tilde{\phi} \pi^\mu \right) \Lambda + \rho_{a \mu} \frac{\partial \Lambda}{\partial x^\alpha}. \]

As the system’s symmetry transformation (20) holds for arbitrary differentiable functions $\Lambda = \Lambda(x)$, the Noether current (22) must be conserved for all $\Lambda(x)$. The divergence of $J^\mu(x)$ is given by:
\[ \frac{\partial j_\alpha}{\partial x^\alpha} = \Lambda \left[ \frac{\partial}{\partial x^\alpha} iq \left( \bar{\pi}^\alpha \phi - \bar{\phi} \pi^\alpha \right) \right] \]

\[ + \frac{\partial \Lambda}{\partial x^\beta} \left[ iq \left( \bar{\pi}^\beta \phi - \bar{\phi} \pi^\beta \right) + \frac{\partial p^\beta}{\partial x^\alpha} \right] + \frac{\partial^2 \Lambda}{\partial x^\beta \partial x^\alpha} p^{\beta \alpha}. \] (23)

With \( \Lambda(x) \) an arbitrary function of space-time, the divergence of \( j^\mu(x) \) vanishes if and only if the three terms associated with \( \Lambda(x) \) and its derivatives in Eq. (23) separately vanish. This means in particular that the term \( j_1^\alpha \) proportional to \( \Lambda \) of the divergence (23) of the Noether current is separately conserved

\[ j_1^\alpha = iq \left( \bar{\pi}^\alpha \phi - \bar{\phi} \pi^\alpha \right), \quad \frac{\partial j_1^\alpha}{\partial x^\alpha} = 0. \] (24)

The second term depicts the inhomogeneous Maxwell equation:

\[ \frac{\partial p^{\beta \alpha}}{\partial x^\alpha} = -j_1^\beta. \] (25)

The third term demands the canonical momentum tensor to be skew-symmetric:

\[ p^{\alpha \beta} = -p^{\beta \alpha}, \]

which entails Eq. (25) to satisfy the consistency requirement:

\[ \frac{\partial^2 p^{\alpha \beta}}{\partial x^\alpha \partial x^\beta} = \frac{\partial j_1^\alpha}{\partial x^\alpha} = 0. \]

The explicit proof of a vanishing divergence of the Noether current \( j_1^\mu \) from Eq. (24) is obtained here only if we insert the canonical field equations (6) emerging from the Hamiltonian (18)

\[ \frac{1}{iq} \frac{\partial j_1^\alpha}{\partial x^\alpha} = \bar{\pi}^\alpha \frac{\partial \phi}{\partial x^\alpha} - \bar{\phi} \frac{\partial \bar{\pi}^\alpha}{\partial x^\alpha} + \bar{\phi} \frac{\partial \pi^\alpha}{\partial x^\alpha} \]

\[ = \bar{\pi}^\alpha \frac{\partial \mathcal{H}_{KGM}}{\partial \pi^\alpha} - \frac{\partial \mathcal{H}_{KGM}}{\partial \phi} \phi + \bar{\phi} \frac{\partial \mathcal{H}_{KGM}}{\partial \bar{\phi}} \]

\[ = \bar{\pi}^\alpha \left( \pi_a + iq a_a \phi \right) - \left( \bar{\pi}_a - iq a_a \bar{\phi} \right) \pi^\alpha \]

\[ - (iq \bar{\pi}^a a_a + m^2 \bar{\phi}) \phi + \bar{\phi} \left( m^2 \phi - iq a_a \pi^a \right) \]

\[ = 0. \]

Hence, \( j_1^\mu(x) \) from Eq. (24) is indeed a conserved current along the system’s spacetime evolution, as described by the canonical field equations for the Hamiltonian (18).

In the actual case, the Noether current \( j^\mu \) from Eq. (22) does not depend on the gauge field \( a_\mu \). As a consequence the correlation of \( a_\mu \) to its momentum field \( p^{\mu \nu} \) does not follow from Noether’s theorem. This does not apply for the SU(N) gauge
theory, to be sketched in the following. The canonical fields equations then follow without any reference to the Yang-Mills Hamiltonian $H_{YM}$. Moreover, the subsequent restriction to the particular case of a $U(1)$ gauge theory then does provide the missing correlation of $a_\mu$ to $p^{\mu\nu}$ and, subsequently, the homogeneous Maxwell equation.

7 Example 2: SU($N$) gauge theory

7.1 Finite symmetry transformation

Similarly to the $U(1)$ case of Eq. (18), the Yang-Mills Hamiltonian $H_{YM}$ with $p^{\mu\nu} = -p^{\nu\mu}$,

$$H_{YM} = \bar{\pi}^\alpha_{J} \pi^\alpha_{J} + m^2 \bar{\phi} \phi - 1/4 p^{\mu\alpha}_{J} p^{\mu\beta}_{K} p_{KJ}^{\alpha\beta} + i q \left( \bar{\pi}^\alpha_{K} a_{KJ}^{\alpha} \phi_{J} - p^{\mu\alpha}_{J} a_{KJ}^{\mu} \phi_{J} - p^{\mu\alpha}_{J} a_{KJ}^{\mu} a_{IJ}^{\alpha} \right)$$

can be shown to be form-invariant under the local transformation of a set of $I = 1, \ldots, N$ complex fields $\phi_{I}$, provided that $H'(\phi_{I}, \bar{\phi}_{I}, \pi^\mu_{I}, \bar{\pi}^\mu_{I}) = \bar{\pi}^\alpha_{J} \pi^\alpha_{J} + m^2 \bar{\phi} \phi_{J}$ is form-invariant under the corresponding global transformation

$$\Phi_{I} = u_{IJ} \phi_{J}, \quad \bar{\Phi}_{I} = \bar{\phi}_{J} u_{JI}^*.$$

The $u_{IJ}$ are supposed to represent the coefficients of a unitary matrix and hence satisfy

$$u_{IJ} u_{IK} = \delta_{JK} = u_{IJK} u_{IK}^*.$$

At this point, the unitary matrix $U = (u_{IJ})$ is usually expressed in textbooks in terms of its representation

$$U = \exp \left( \frac{i}{2} \sum \tau \cdot \alpha \right), \quad (26)$$

where $\alpha$ denotes an $N$-vector of phase angles—which corresponds to the phase factor $\Lambda$ of $U(1)$ gauge theory. The $N \times N$-matrices $\tau$ stand for the generators of the given symmetry group (i.e. for the Pauli matrices, Gell-Mann matrices, ...). Yet, for the sake of simplicity of the derivation, we do not pursue this formulation here, but continue to work with the coefficients $u_{IJ}$. Their particular representation (26) can be inserted at any point later in the derivation. On the other hand, it is the spirit of all gauge theories to finally replace all dependencies on the arbitrary coefficients of a particular symmetry transformation by gauge fields, which finally yields a Lagrangian/Hamiltonian completely independent of those coefficients. For this reason, there is no need to specify an explicit representation of the unitary matrix $U = (u_{IJ})$ in the actual context.

The generating function of the local symmetry transformation is given by
Hamiltonian representation of Noether’s theorem

\[ F_2^\mu = \tilde{\Pi}_K^\mu u_{KJ} \phi_J + \tilde{\phi}_K u_{KJ}^* \Pi_J^\mu + P_{JK}^{a\mu} \left( u_{KL} a_{LJa} u_{IJ}^* + \frac{1}{i\hbar} \frac{\partial u_{KL}}{\partial \chi^\alpha} u_{IJ}^* \right). \]  

(27)

It entails the canonical transformation rules for the complex fields and their conjugates

\[
\begin{align*}
\pi_i^\mu &= \tilde{\Pi}_K^\mu u_{KL}, & \bar{\pi}_i^\mu &= \tilde{\phi}_K u_{KL} \\
\pi_i^\mu &= u_{iJ}^* \Pi_J^\mu, & \bar{\phi}_i &= u_{iJ} \phi_J 
\end{align*}
\]

(28)

and the following rules for the Hermitian \( N \times N \) matrix of 4-vector gauge fields \( a_{LJa} \) and their conjugates

\[
\begin{align*}
A_{KJa} &= u_{KL} a_{LJa} u_{IJ}^*, & \frac{1}{i\hbar} \frac{\partial u_{KL}}{\partial \chi^\alpha} u_{IJ}^* \\
p_{KL}^{a\mu} &= u_{ij}^* p_{ij}^{a\mu} u_{KL}. 
\end{align*}
\]

(29)

The transformation rule for the Hamiltonian is obtained from the explicit \( x^\mu \)-dependency of the generating function \( F_{\alpha}^\mu \)

\[
\mathcal{H}'_{YM} - \mathcal{H}_{YM} = \left. \frac{\partial F_{\alpha}^\mu}{\partial \chi^\alpha} \right|_{\text{expl}}.
\]

Expressing all \( u_{ij} \)-dependent terms in this equation in terms of the fields and their conjugates according to the above canonical transformation rules \( 28 \) and \( 29 \) finally yields

\[
\mathcal{H}'_{YM} - \mathcal{H}_{YM} = i\hbar \left[ \Pi_K^{a\alpha} A_{KJa} \phi_J - \tilde{\phi}_K A_{KJa} \Pi_J^{a\alpha} - p_{JK}^{a\alpha} A_{KJa} A_{LJa} - \left( \bar{\pi}_K^{a\alpha} a_{KJa} \phi_J - \tilde{\phi}_K a_{KJa} \pi_J^{a\alpha} - p_{JK}^{a\alpha} a_{KJa} a_{LJa} \right) \right].
\]

Again, we made use of the fact that the momentum fields \( p_{JK}^{a\alpha} \) are skew-symmetric in \( \alpha, \beta \). The transformed Hamiltonian now follows with \( p_{JK}^{\alpha\beta} = -p_{JK}^{\beta\alpha} \) as

\[
\mathcal{H}'_{YM} = \tilde{\Pi}_{Ja} \Pi_J^{a\alpha} + m^2 \bar{\phi}_J \phi_J - \frac{1}{4} p_{JK}^{a\alpha} p_{JK}^{a\beta} \\
+ i\hbar \left( \tilde{\Pi}_K^{a\alpha} A_{KJa} \phi_J - \tilde{\phi}_K A_{KJa} \Pi_J^{a\alpha} - p_{JK}^{a\alpha} A_{KJa} A_{LJa} \right),
\]

which has the same form as the original one, \( \mathcal{H}_{YM} \). Thus, the generating function \( F_{\alpha}^\mu \) defines a local symmetry transformation of the Yang-Mills Hamiltonian.

### 7.2 Field equations from Noether’s theorem

In order to derive the conserved Noether current which is associated with the symmetry transformation given by Eqs. \( 28 \) and \( 29 \), we again set up the generating
function of the corresponding *infinitesimal* transformation by letting
\[
  u_{IJ} \to \delta_{IJ} + i \varepsilon u_{IJ}, \quad u^*_{JI} \to \delta_{JI} - i \varepsilon u_{JI},
\]

hence
\[
  \Phi_I = (\delta_{IJ} + i \varepsilon u_{IJ}) \phi_J, \quad \bar{\Phi}_I = \bar{\phi}_J (\delta_{JI} - i \varepsilon u_{JI}).
\]

For the *local* transformation, \(u_{IJ}\) denotes an \(N \times N\) matrix of arbitrary space-time dependent and now real coefficients with \(\text{det}(u_{IJ}) = 1\). The generating function (27) is then transposed into the generating function of the corresponding *infinitesimal* canonical transformation
\[
  F_2^\mu = \Pi_k^\mu (\delta_{KJ} + i \varepsilon u_{KJ}) \phi_J + \bar{\phi}_K (\delta_{KJ} - i \varepsilon u_{KJ}) \bar{\Pi}_J^\mu
  \quad + p_{JK}^{\alpha \mu} (\delta_{KJ} + i \varepsilon u_{KJ}) a_{KJ}^\alpha (\delta_{IJ} - i \varepsilon u_{IJ}) + \frac{\varepsilon}{q} \frac{\partial u_{KJ}}{\partial x^\alpha} (\delta_{IJ} - i \varepsilon u_{IJ}).
\]

Omitting the quadratic terms in \(\varepsilon\), the generating function of the sought-for infinitesimal canonical transformation is obtained as
\[
  F_2^\mu = \Pi_k^\mu \phi_J + \bar{\phi}_K \Pi^\mu_J + p_{JK}^{\alpha \mu} a_{KJ}^\alpha + \frac{\varepsilon}{q} j^\mu, \quad (30)
\]

with the Noether current of the SU\((N)\) gauge theory
\[
  j^\mu = i q \left[ \bar{\Phi}_K u_{KJ} \phi_J - \bar{\phi}_K u_{KJ} \bar{\Phi}_J + p_{JK}^{\alpha \mu} \left( u_{KJ} a_{KJ}^\alpha - a_{KJ} \bar{u}_{KJ} + \frac{1}{i q} \frac{\partial u_{KJ}}{\partial x^\alpha} \right) \right]. \quad (31)
\]

As this defines the corresponding *infinitesimal* symmetry transformation of the Hamiltonian, \(j^\mu\) from Eq. (31) must represent a conserved current according to Noether’s theorem, hence \(\partial j^\mu / \partial x^\beta = 0\) for all differentiable functions \(u_{KJ} = u_{KJ}(x)\). Calculating its divergence and ordering the terms according to zeroth, first and second derivatives of the \(u_{KJ}(x)\) yields
\[
  \frac{1}{i q} \frac{\partial}{\partial x^\beta} j^\beta_{JK} = u_{KJ} \frac{\partial}{\partial x^\beta} \left( \bar{\Pi}_K^{\beta \mu} \phi_J - \bar{\phi}_K \Pi^\mu_J + a_{KJ} p_{JK}^{\beta \alpha} a^\alpha_{KJ} \bar{a}_{KJ}^\alpha \right)
  + \frac{\partial u_{KJ}}{\partial x^\beta} \left( \bar{\Pi}_K^{\beta \mu} \phi_J - \bar{\phi}_K \Pi^\mu_J + a_{KJ} p_{JK}^{\beta \alpha} a^\alpha_{KJ} \bar{a}_{KJ}^\alpha \right)
  + \frac{1}{i q} \frac{\partial^2 u_{KJ}}{\partial x^\alpha \partial x^\beta} p_{JK}^{\beta \alpha} a^\alpha_{KJ} \bar{a}_{KJ}^\alpha. \quad (32)
\]

With \(u_{KJ}(x)\) *arbitrary* functions of space-time, the divergence of \(j^\mu(x)\) vanishes if and only if the three terms associated with the \(u_{KJ}(x)\) and their derivatives vanish separately. This means in particular that the term \(j^\beta_{JK}\) proportional to \(u_{KJ}\) of the divergence of the Noether current (32) is separately conserved.
where the divergence of the momenta \( p^\alpha_{jk} \) were replaced by the SU(N) gauge currents \( j^\mu_{jk} \) according to Eq. (34). Inserting finally the explicit representation (33) of the SU(N) gauge currents yields

\[
0 = \frac{1}{iq} \frac{\partial j^\mu_{jk}}{\partial x^\beta} = \pi^\alpha_{jk} \left( \frac{\partial \phi_I}{\partial x^\alpha} - i q a_{jIa} \phi_I \right) - \left( \frac{\partial \pi^\beta_{jk}}{\partial x^\alpha} + i q a_{jIk} \right) \pi^\alpha_{jk} + \frac{1}{i} \left[ \frac{\partial a_{j\alpha}}{\partial x^\beta} - \frac{\partial a_{j\beta}}{\partial x^\alpha} + i q \left( a_{j\alpha} a_{NI\beta} - a_{jN\beta} a_{N\alpha} \right) \right] \left( \frac{\partial p^\beta_{jk}}{\partial x^\alpha} - i q a_{j\alpha} \pi^\alpha_{jk} \right) + \frac{1}{i} \left[ \frac{\partial a_{j\alpha}}{\partial x^\beta} - \frac{\partial a_{j\beta}}{\partial x^\alpha} + i q \left( a_{j\alpha} a_{NI\beta} - a_{jN\beta} a_{N\alpha} \right) \right] \left( \frac{\partial p^\beta_{jk}}{\partial x^\alpha} - i q a_{j\alpha} \pi^\alpha_{jk} \right) \tag{35} \]

For a vanishing coupling constant \( q \), Eq. (35) must provide the field equations of the original, globally form-invariant Klein-Gordon system

\[
\mathcal{H} = \mathcal{H}_{\text{tor}} + m^2 \phi_I \partial^2 \phi_I, \]
hence

\[
\frac{\partial \phi_I}{\partial x^\alpha} = \frac{\partial \mathcal{H}}{\partial \pi_J^\alpha} = \pi_J^\alpha, \quad \frac{\partial \pi_J^\alpha}{\partial x^\alpha} = -\frac{\partial \mathcal{H}}{\partial \phi_I} = -m^2 \phi_I.
\]

Equation (35) thus vanishes exactly if the amended canonical equations of the locally form-invariant system

\[
\pi_J^\alpha = \frac{\partial \phi_I}{\partial x^\alpha} - i q a^I_{J \alpha} \phi_I
\]

\[
\bar{\pi}_K^\alpha = \frac{\partial \phi_K}{\partial x^\alpha} + i q \bar{\phi}_I a^I_{K \alpha}
\]

\[
\frac{\partial \pi_J^\alpha}{\partial x^\alpha} = -m^2 \phi_I + i q a^I_{J \alpha} \pi_J^\alpha
\]

\[
\frac{\partial \bar{\pi}_K^\alpha}{\partial x^\alpha} = -m^2 \bar{\phi}_I - i q \bar{\pi}_\alpha^I a^I_{K \alpha}
\]

(36)

and

\[
p_{J \beta \alpha} = \frac{\partial a_{J \alpha}}{\partial x^\beta} - \frac{\partial a_{J \beta}}{\partial x^\alpha} + i q \left( a_{J N \alpha} a^N_{J \beta} - a_{J N \beta} a^N_{J \alpha} \right)
\]

(37)

hold. The canonical momenta \(\pi_J^\alpha\) and \(\bar{\pi}_K^\alpha\) turn out to represent the gauge-covariant derivatives of the pertaining fields \(\phi_I\) and \(\bar{\phi}_K\), respectively. In conjunction with Eqs. (33) and (34), the dynamics of the system is thus completely determined by Noether’s theorem on the basis of the local symmetry transformation defined by Eqs. (30) and (31).

Remarkably, the missing correlation of the derivatives of \(a_\mu\) to their duals \(p^{\mu \nu}\) encountered in the previously presented U(1) gauge formalism based on Noether’s theorem is now provided by Eq. (37). Restricting the range of the field indices to \(I = J = N = 1\)—hence to one (real) gauge field \(a_\mu \equiv a_{11\mu}\) and thus one canonical momentum tensor \(p^{\mu \nu} \equiv p_{11}^{\mu \nu}\)—corresponds to the transition \(SU(N) \rightarrow U(1)\). As only the self-coupling terms cancel for this case, we get

\[
p_{\beta \alpha} = \frac{\partial a_\alpha}{\partial x^\beta} - \frac{\partial a_\beta}{\partial x^\alpha},
\]

(38)

which did not follow from Eq. (22). As a consequence of Eq. (38), one encounters the homogeneous Maxwell equation:

\[
\frac{\partial p_{\nu \mu}}{\partial x^\alpha} + \frac{\partial p_{\mu \alpha}}{\partial x^\nu} + \frac{\partial p_{\alpha \nu}}{\partial x^\mu} = 0,
\]

which now completes the set of field equations derived in Sect. 6.2.
8 Conclusions and outlook

Our presentation shows that the field equations usually obtained by setting up the canonical field equations of the locally form-invariant Hamiltonian can be obtained directly from Noether’s theorem on the basis of the system’s local symmetry transformation. Given a theory’s field equations, the pertaining Hamiltonian is not uniquely fixed. In a recent paper, Koenigstein et al. [8] have worked out an alternative approach to the U(1) gauge theory, yielding an equivalent form-invariant Hamiltonian and the pertaining symmetry transformation.

The actual representation of the Hamiltonian Noether theorem has also found a theoretically fruitful generalization. Treating the space-time geometry as an additional dynamical quantity, the Noether approach yields a fully consistent formalism based on the requirement a form-invariance of the given system under local space-time transformations. Noether’s theorem then yields the pertaining field equations which describe in addition the dynamics of the space-time geometry [9]. In order to include the coupling of spin and a torsion of space-time, the formalism can be further generalized in the tetrad formalism [10].

Acknowledgements This paper is prepared for the Symposium on Exciting Physics, which was held in November 2015 at Makutsi Safari Farm, South Africa, to honor our teacher, mentor, and friend Prof. Dr. Dr. h.c. mult. Walter Greiner on the occasion of his 80th birthday. We thank Walter for stimulating generations of young scientists for more than 100 semesters, both at the Goethe Universität Frankfurt am Main and internationally. We wish him good health to further take part in the progress of physics in the years to come.

We furthermore thank the present members of our FIAS working group on the Extended canonical formalism of field theory, namely Michail Chabanov, Matthias Hanauske, Johannes Kirsch, Adrian Koenigstein, and Johannes Muench for many fruitful discussions.

References

1. J. Struckmeier, H. Reichau, General U(N) gauge transformations in the realm of covariant Hamiltonian field theory, in: Exciting Interdisciplinary Physics. FIAS Interdisciplinary Science Series (Springer, New York, 2013). URL http://arxiv.org/abs/1205.5754, p. 367
2. E. Noether, Nachr. Königl. Ges. Wiss. Göttingen, Math.-Phys. Kl. 57, 235 (1918)
3. J.V. José, E.J. Saletan, Classical Dynamics (Cambridge University Press, Cambridge, 1998)
4. J. Struckmeier, A. Redelbach, Covariant Hamiltonian Field Theory, Int. J. Mod. Phys. E 17, 435 (2008). URL http://arxiv.org/abs/0811.0508
5. W. Greiner, Classical Electrodynamics (Springer, 1998)
6. T. De Donder, Théorie Invariantive Du Calcul des Variations (Gauthier-Villars & Cie., Paris, 1930)
7. H. Weyl, Annals of Mathematics 36, 607 (1935)
8. A. Koenigstein, J. Kirsch, H. Stoecker, J. Struckmeier, D. Vasak, M. Hanauske, Int. J. Mod. Phys. E 25, 1642005 (2016). DOI 10.1142/S0218301316420052
9. J. Struckmeier, D. Vasak, H. Stoecker, A. Koenigstein, J. Kirsch, M. Hanauske, J.a. Muench, in preparation (2016)
10. D. Vasak, J. Struckmeier, H. Stoecker, A. Koenigstein, J. Kirsch, M. Hanauske, in preparation (2016)