Collisions of Einstein-Conformal Scalar Waves

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Abstract

A large class of solutions of the Einstein-conformal scalar equations in D=2+1 and D=3+1 is identified. They describe the collisions of asymptotic conformal scalar waves and are generated from Einstein-minimally coupled scalar spacetimes via a (generalized) Bekenstein transformation. Particular emphasis is given to the study of the global properties and the singularity structure of the obtained solutions. It is shown, that in the case of the absence of pure gravitational radiation in the initial data, the formation of the final singularity is not only generic, but is even inevitable.

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1. Introduction

Colliding gravitational waves have attracted a lot of interest in the last two decades \([1]-[10]\). Apart from the character of the nonlinearities of the gravitational interaction, the interest was probably caused by characteristic curvature singularities occurring as the result of the collision of two waves. Much work has been done on the structure of the singularities \([6]-[9]\) in the case of collision of either sourceless or various source waves, with the result that the final singularity formation is, in fact, generic.

A relevant contribution concerning the singularity formation was made by Hayward \([11]\), who formulated the criterion of “incoming” regularity. In other words, he proposed to make a clear distinction whether the singularity formation occurs for the collision of waves which are initially regular or singular. Then the problem reads: Under what conditions may the initially regular waves avoid the singularity formation after the collision? For the case of the purely gravitational (sourceless) waves Hayward himself found that the regular waves generically produce the curvature singularities. However, there were also exceptional cases where the singularities were avoided.

In \(D=2+1\) the present authors found that after the collisions of regular asymptotic scalar waves the singularity is always formed \([12]\). We considered the minimally coupled scalar field. A similar conclusion was then obtained in \(D=3+1\) by Hayward \([13]\), who has shown that if the pure gravitational radiation is absent in the initial data then the collisions of the minimally coupled scalar waves always end up in a singularity.

In this paper, we wish to study a source field of a different type and find whether similar conclusions about the inevitability of the singularity formation can be reached. We shall work with the conformal scalar field with the field equation (in the \(D\)-dimensional spacetime) \([14], [15]\)

\[
\nabla_\alpha \nabla^\alpha \phi - \frac{D-2}{4(D-1)} R \phi = 0,
\]

following from the action

\[
S = \int \left[ \left( \frac{1}{2\kappa} - \frac{D-2}{8(D-1)} \phi^2 \right) R - \frac{1}{2} \left( \nabla \phi \right)^2 \right] \sqrt{-g} d^D x.
\]

Unlike the other massless field equations (i.e. Maxwell, Dirac or Weyl), the minimally coupled massless scalar equation is not conformally invariant. The
coupling according to (1.1) cures this “deficiency” and, in any case, it is a reasonable alternative for gravitational coupling of the scalar field. The stress tensor for the conformal scalar field is quite different from the ordinary one, and we may therefore test the singularity formation problem in a different setting to previously.

From a technical point of view, it is not difficult to generate solutions of the Einstein-conformal scalar equations from the minimally coupled Einstein-scalar solutions via the generalized Bekenstein transformation [14], [15]. However, the structure of singularities requires independent analysis, since the Bekenstein transformation multiplies the original metric by a nontrivial conformal factor. This operation, in general, may change the asymptotic behaviour of the Riemann tensor components. Indeed, consider for instance the dilaton gravity in D=2+1 with the action

\[ S = \int d^3x \sqrt{-g} e^\phi R. \]

There is the following (black hole) solution of the corresponding field equations [16]

\[ ds^2 = -\frac{r-2m}{r} dt^2 + \frac{r}{r-2m} dr^2 + r^2 d\vartheta^2, \quad \phi = \log |r \sin \vartheta|. \]  

(1.2)

On the other hand under the transformation

\[ \tilde{g}_{\alpha\beta} = e^{2\phi} g_{\alpha\beta}, \quad \tilde{\phi} = \phi, \]

the action \( S \) changes into the action of the minimally coupled scalar field

\[ S = \int d^3x \sqrt{-\tilde{g}} (\tilde{R} - 2\tilde{\phi}, \tilde{\phi}). \]

The metric corresponding to \( (1.2) \) becomes

\[ ds^2 = r^2 \sin^2 \vartheta \left( -\frac{r-2m}{r} dt^2 + \frac{r}{r-2m} dr^2 + r^2 d\vartheta^2 \right). \]

It is not difficult to demonstrate that the singularity structures of both metrics differ considerably from each other.

In sections 2 and 3 we study the singularity structure of the Einstein-conformal scalar spacetimes in 2+1 and 3+1 dimensions, respectively. We arrive again to the same conclusions as for the minimally coupled scalar field, namely, for initially regular waves (without sourceless part) the singularity formation is inevitable.
2. Conformal scalar waves in $D=2+1$

We start with a reminder of some properties of the spacetimes describing the collision of line-fronted scalar waves. (It is known that the formation of singularities occurs as the result of these processes [12].) In such spacetimes, global coordinates can be introduced such that the metric has the form

$$d^2 = -e^{-\tilde{K}(u,v)} du dv + e^{-\tilde{N}(u,v)} dx^2$$

(2.1)

with one space-like coordinate $x$ and a pair of null coordinates $u$, $v$. The form of the metric functions $\tilde{N}$, $\tilde{K}$ follows from the Einstein equations where on the r.h.s. the stress tensor for the scalar field $\tilde{\phi}$ stands:

$$\tilde{T}_{\mu\nu} = \tilde{\phi}_{\mu} \tilde{\phi}_{\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \tilde{\phi}_\rho \tilde{\phi}_\sigma.$$  

(2.2)

Restricting ourselves to the case of the so-called asymptotic waves spacetimes, for which all functions $\tilde{N}$, $\tilde{K}$, $\tilde{\phi}$ are smooth and fulfill asymptotic conditions

$$(\tilde{K}, \tilde{N}, \tilde{\phi})(u \to -\infty, v \to -\infty) = 0,$$

$$(\tilde{K}, \tilde{N}, \tilde{\phi})(u, v \to -\infty) = (\tilde{K}, \tilde{N}, \tilde{\phi})(u),$$

$$(\tilde{K}, \tilde{N}, \tilde{\phi})(u \to -\infty, v) = (\tilde{K}, \tilde{N}, \tilde{\phi})(v),$$

(2.3)

we can write down the general solution

$$\tilde{N} = -2 \ln (1 - f(u) - g(v)), $$

$$\tilde{\phi} = k \ln(1 - f - g) + p \cosh^{-1} \left[ \frac{1 + f - g}{1 - f - g} \right] + q \cosh^{-1} \left[ \frac{1 - f + g}{1 - f - g} \right]$$

$$+ \int_0^\infty [A(\omega) J_0 (\omega(1-f-g)) + B(\omega) N_0 (\omega(1-f-g))] \sin (\omega(f-g)) d\omega$$

$$+ \int_0^\infty [C(\omega) J_0 (\omega(1-f-g)) + D(\omega) N_0 (\omega(1-f-g))] \cos (\omega(f-g)) d\omega,$$

(2.4)

subject to

$$\int_0^\infty [C(\omega) J_0 (\omega) + D(\omega) N_0 (\omega)] d\omega = 0.$$
where \( f(u) \) and \( g(v) \) are functions, \( k, p \) and \( q \) are real numbers, \( A(\omega), B(\omega), C(\omega) \) and \( D(\omega) \) may be integrable functions or distributions and \( J_0 \) and \( N_0 \) are zero-order Bessel and Neumann functions.

The function \( \tilde{K} \) can be obtained by integration from the relevant Einstein equations, viz.

\[
\tilde{N}_{uu} - \frac{1}{2} \tilde{N}_u^2 + \tilde{N}_u \tilde{K}_u = 2 \kappa \tilde{\phi}_u^2, \\
\tilde{N}_{vv} - \frac{1}{2} \tilde{N}_v^2 + \tilde{N}_v \tilde{K}_v = 2 \kappa \tilde{\phi}_v^2, \\
\tilde{K}_{uv} = \kappa \tilde{\phi}_u \tilde{\phi}_v. 
\] (2.5)

The asymptotic conditions [2.3], requiring certain asymptotic behaviour of the functions \( f(u) \) and \( g(v) \), are met by the choice

\[
f(u) = [-a(u - u_s)]^{1/(1 - \kappa p^2)} \quad \text{for} \quad \kappa p^2 > 1, \\
f(u) = \exp[a(u - u_s)] \quad \text{for} \quad \kappa p^2 = 1,
\]

and

\[
g(v) = [-b(v - v_s)]^{1/(1 - \kappa q^2)} \quad \text{for} \quad \kappa q^2 > 1, \\
g(v) = \exp[b(v - v_s)] \quad \text{for} \quad \kappa q^2 = 1.
\]

Now, the plan of the investigation of the properties of these spacetimes was the following [12]: firstly we had to exclude the cases when the asymptotic waves were singular themselves, for the “good physical situation” should avoid singularities in the initial data. Then the formation of final singularities was questioned. It turned out that in \( D=2+1 \), as the result of the collision of regular asymptotic scalar waves, the final singularity always appeared [12].

Now, what is the situation when the source of colliding waves is not the scalar field, but the conformal scalar field? The solutions for the selfgravitating conformal scalar field can be easily obtained from the selfgravitating minimal scalar spacetimes via a generalized Bekenstein transformation [4], [14], linking the D-dimensional scalar field \( \phi \) and metric \( g_{\alpha\beta} \) to the D-dimensional conformal scalar field \( \tilde{\phi} \) and metric \( \tilde{g}_{\alpha\beta} \) as follows

\[
\tilde{\phi} = \left( \kappa \frac{D - 2}{4(D - 1)} \right)^{-1/2} \tanh \left[ \left( \kappa \frac{D - 2}{4(D - 1)} \right)^{-1/2} \tilde{\phi} \right], \\
g_{\alpha\beta} = \left( \cosh \left[ \left( \kappa \frac{D - 2}{4(D - 1)} \right)^{-1/2} \tilde{\phi} \right] \right)^{4/(D - 2)} \tilde{g}_{\alpha\beta}. 
\] (2.6)
In terms of metric functions $\tilde{N}$, $\tilde{K}$ and analogously defined $N$ and $K$ we have (for D=2+1) the following transformation rules

$$
\phi = \sqrt{\frac{8}{\kappa}} \tanh \left[ \sqrt{\frac{K}{8\phi}} \right],
$$
$$
N = \tilde{N} - 4 \ln \cosh \left[ \sqrt{\frac{K}{8\phi}} \right],
$$
$$
K = \tilde{K} - 4 \ln \cosh \left[ \sqrt{\frac{K}{8\phi}} \right].
$$

(2.7)

We see that (2.3) imply fulfillment of the same asymptotic conditions for the new metric functions and the new (conformal scalar) field. It means that the metric $g_{\alpha\beta}$ also describes the colliding asymptotic waves spacetimes and we can study the conditions for the regularity of initial data as well as the consequent creation of singularities after collision.\footnote{Actually, there is one more branch of the generalized Bekenstein transformation (see \cite{15}) which, however, does not preserve the asymptotic conditions, so we shall not deal with it anymore.} What is the criterion for the regularity of the initial data?

Consider the (asymptotic) regions where only one of the initial waves is present while the influence of the other wave vanishes, i.e. the regions $(u \to -\infty, v \neq -\infty)$ or $(v \to -\infty, u \neq -\infty)$. Now the metric is that of the single wave there (i.e. $u$- or $v$-independent) and we can introduce the amplitude $u^h(u)$

$$
^uh(u) = \frac{1}{2} e^{2K} \left( \frac{1}{2} N_u^2 - N_{uu} - N_u K_u \right),
$$

resp. $v^h(v)$

$$
^vh(v) = \frac{1}{2} e^{2K} \left( \frac{1}{2} N_v^2 - N_{vv} - N_v K_v \right).
$$

These are the amplitudes of the single line-wave in the so-called Brinkmann coordinates. It was shown in \cite{12} that the boundedness of these amplitudes is the necessary and sufficient condition for the absence of the curvature singularities in the asymptotic (incoming) region.

Using the transformation rules (2.7) and the Einstein equations for the ordinary scalar field (2.5) we can write

$$
^uh(u) = \frac{1}{2} CH^{-8} \sqrt{2} \tilde{\phi} \left( \frac{1}{2} \phi^2 u (-2 - TH^2 + \frac{1}{2CH^2}) + \sqrt{2\kappa} TH (\tilde{\phi} uu + \tilde{\phi}_u \tilde{K}_u) \right),
$$
where \( CH = \cosh(\bar{\phi}\sqrt{\kappa/8}) \), \( TH = \tanh(\bar{\phi}\sqrt{\kappa/8}) \) and the other amplitude \( \nu h(v) \) has the analogous form. Now, we need the asymptotic behaviour of the Bessel and Neumann functions near zero

\[
J_0(w) \sim 1 - \frac{w^2}{4} + \cdots, \\
J'_0(w) \sim -\frac{w}{2} + \cdots, \\
N_0(w) \sim (1 - \frac{w^2}{4}) \ln w + \cdots, \\
N'_0(w) \sim \frac{1}{w} - \frac{w}{2} \ln w + \cdots
\]

(2.8)

From the expressions (2.4) taken in \( g = 0, f \sim 1 \) it follows that

\[
\bar{\theta}_u \sim f_u \left[ \frac{c}{1-f} + d \ln(1-f) + e + h(1-f) \ln(1-f) + \cdots \right],
\]

and then from (2.5) we can obtain

\[
\bar{K}_u = f_u \left[ \frac{\kappa c^2}{1-f} + 2\kappa cd \ln(1-f) + \cdots \right],
\]

and hence

\[
\bar{K} = -\kappa c^2 \ln(1-f) + \text{bounded}.
\]

Here

\[
c = p - k - \int_0^\infty [B(\omega) \sin(\omega) + D(\omega) \cos(\omega)]d\omega,
\]

\[
d = \int_0^\infty [\omega B(\omega) \cos(\omega) - \omega D(\omega) \sin(\omega)]d\omega,
\]

\[
e = \frac{p}{2} + \int_0^\infty \omega [A(\omega) + B(\omega) \ln \omega] \cos(\omega)d\omega
\]

\[
- \int_0^\infty \omega [C(\omega) + D(\omega) \ln \omega] \sin(\omega)d\omega,
\]

\[
h = \frac{1}{2} \int_0^\infty \omega^2 [B(\omega) \sin(\omega) + D(\omega) \cos(\omega)]d\omega.
\]

(2.9)

Thus if \( c \neq 0 \) the leading term in \( \nu h(u) \) is

\[
2^7 f_u^2 (1-f)^8 |c| \sqrt{\kappa/8 - 2\kappa c^2 - 2} |c| \left[ -3\kappa |c| + \sqrt{2\kappa(1+\kappa c^2)} \right].
\]

7
It vanishes for $|c| = 1/\sqrt{2\kappa}$ or $|c| = 2/\sqrt{2\kappa}$, but then we cannot suppress the subleading term proportional to $(1 - f)^{-1/2}$ or $(1 - f)^{-1}$ respectively. Hence, we arrived at the first necessary condition for the incoming regularity, viz. $c = 0$. In this case the asymptotic behaviour of $u h(u)$ is

$$u h(u) \sim \text{bounded} \times \left[ -\frac{1}{2} \kappa (3 d^2 \ln(1 - f) + (6 d e - d^2) \ln(1 - f)) + \cdots \right],$$

from which the second necessary condition for the incoming regularity can be extracted: $d = 0$. It is easy to see that the conditions $c = d = 0$ are also sufficient.

After a similar analysis for $v h(v)$ we conclude that for the incoming regularity it is necessary and sufficient to fulfil

$$c = d = c' = d' = 0,$$

where $c$ and $d$ are as above and

$$c' = q - k + \int_0^\infty [B(\omega) \sin(\omega) - D(\omega) \cos(\omega)] d\omega,$$

$$d' = -\int_0^\infty [\omega B(\omega) \cos(\omega) + \omega D(\omega) \sin(\omega)] d\omega.$$

Now, we wish to study the region of interaction, i.e. the region where the metric is both $u$- and $v$-dependent. There are special points (forming the so-called caustic) in which the metric functions are singular, namely the points where

$$1 - f - g = 0.$$

We have to elucidate what kind of singularity this is, or – in other words – how the curvature behaves in its vicinity. In particular, we find the conditions under which the scalar curvature $R$ is unbounded while approaching the caustic\(^2\). In terms of metric functions the scalar curvature $R$ reads

$$R = -4 e^\kappa K_{uv},$$

which we can rewrite using (2.7) and (2.3)

$$R = -4 e^\kappa CH^{-4} [\kappa \tilde{\phi}_u \tilde{\phi}_v - \sqrt{2\kappa} \tilde{\phi}_{uv} TH - \frac{1}{2} \kappa \tilde{\phi}_u \tilde{\phi}_v CH^{-2}].$$

\(^2\)This case is often referred to as the $C^0$ scalar curvature singularity (see \cite{17}).
In the neighbourhood of the caustic it is convenient to introduce other functions $t$ and $z$ instead of $f$ and $g$, given by

$$ t = 1 - f - g, \quad z = f - g. \quad (2.11) $$

Then the caustic is formed by the points $t = 0$, $|z| \neq 1$, the second condition guaranteeing that we do not deal with the asymptotic caustic studied previously. Near the caustic the following behaviour of $\bar{K}$ and $\bar{\phi}$ can be derived from (2.4) and (2.8)

$$ \bar{\phi} \sim E(z) \ln t + \cdots, $$
$$ \bar{K} \sim -\kappa E^2(z) \ln t + \cdots, $$

where

$$ E(z) = k - p - q + \int_0^\infty [B(\omega) \sin(\omega z) + D(\omega) \cos(\omega z)]d\omega. \quad (2.12) $$

So unless $E(z) \equiv 0$ the leading term in $R$ is

$$ f_u g_v \text{const} t^{-\kappa E^2(z)+4|E(z)|\sqrt{\kappa/8-2}[\kappa E^2(z) - \sqrt{2} \kappa (\text{sign } E(z)) E(z)]}, $$

which vanishes if $|E(z)| = \sqrt{2}/\kappa$. But then the subleading term

$$ -4 f_u g_v \text{const} t^{-1} $$

is present in $R$, so $R$ is unbounded. Hence the only case when $R$ may be bounded near the caustic is if $E(z) \equiv 0$. (This, in turn, can only be true for $B(\omega) = D(\omega) = 0$ because of the nontrivial dependence of $E(z)$ on $z$.)

However, the condition $E = 0$ is incompatible with the conditions (2.10) for the asymptotic (incoming) regularity. We can conclude that after collision of regular asymptotic conformal scalar waves the scalar curvature singularity at the caustic is always produced. Hence, the conclusion is the same as in the case of minimal scalar waves.

### 3. Conformal scalar waves in D=3+1

In this section, we shall proceed to the collisions of asymptotic conformal scalar waves in D=3+1. In order to obtain a solution which describes such a process, we again apply the Bekenstein transformation (2.6). The metric with the minimally coupled scalar field $\bar{\phi}$ as the source has the form (see

$^3$Unfortunately, the exact solutions with scalar sources are known only for the case of so-called collinear waves, it means $g_{xy} = 0$ in terms of the metric given below.
\[ ds^2 = -2e^{-\tilde{M}}\,du\,dv + e^{-\tilde{P}} + \tilde{Q}\,dx^2 + e^{-\tilde{P}} - \tilde{Q}\,dy^2. \]

The metric functions and \( \tilde{\phi} \), fulfilling the asymptotic conditions analogous to (2.3), can be expressed in the form

\[
\tilde{P} = -\ln (1 - f(u) - g(v)),
\]

\[
\tilde{Q} = k \ln(1 - f - g) + p \cosh^{-1} \left[ \frac{1 + f - g}{1 - f - g} \right] + q \cosh^{-1} \left[ \frac{1 - f + g}{1 - f - g} \right] + \int_0^\infty \left[ A(\omega)J_0(\omega(1 - f - g)) + B(\omega)N_0(\omega(1 - f - g)) \right] \sin(\omega(f - g)) \, d\omega,
\]

\[
\tilde{\phi} = \lambda \ln(1 - f - g) + \pi \cosh^{-1} \left[ \frac{1 + f - g}{1 - f - g} \right] + \chi \cosh^{-1} \left[ \frac{1 - f + g}{1 - f - g} \right] + \int_0^\infty \left[ A(\omega)J_0(\omega(1 - f - g)) + B(\omega)N_0(\omega(1 - f - g)) \right] \cos(\omega(f - g)) \, d\omega,
\]

subject to the constraints

\[
\int_0^\infty [C(\omega)J_0(\omega) + D(\omega)N_0(\omega)] \, d\omega = 0,
\]

\[
\int_0^\infty [C(\omega)J_0(\omega) + D(\omega)N_0(\omega)] \, d\omega = 0.
\]

All symbols have analogous meaning as in the D=2+1 case. The last – unexpressed – metric function \( \tilde{M} \) is given by direct integration of the relevant Einstein equations

\[
2\tilde{P}_{\bar{u}\bar{u}} - \tilde{P}_u^2 + 2\tilde{P}_u\tilde{M}_u = \tilde{Q}_u^2 + 2\kappa\tilde{\phi}_u^2,
\]

\[
2\tilde{P}_{\bar{v}\bar{v}} - \tilde{P}_v^2 + 2\tilde{P}_v\tilde{M}_v = \tilde{Q}_v^2 + 2\kappa\tilde{\phi}_v^2,
\]

\[
2\tilde{M}_{\bar{u}\bar{v}} = \tilde{Q}_u\tilde{Q}_v - \tilde{P}_u\tilde{P}_v + 2\kappa\tilde{\phi}_u\tilde{\phi}_v,
\]  

(3.1)
(in our conventions the stress tensor is double of that considered by Hayward \[13\]). In what follows we shall restrict ourselves to the special type of functions \(f(u)\) (and \(g(v)\)) \[13\]

\[
\begin{align*}
    f(u) &= \left[-a(u - u_s)\right]^{2/(2-p^2-2\kappa\pi^2)} \quad \text{for } p^2 + 2\kappa\pi^2 > 2, \\
    f(u) &= \exp[a(u - u_s)] \quad \text{for } p^2 + 2\kappa\pi^2 = 2,
\end{align*}
\]

ensuring the proper asymptotic behaviour of the metric functions and of the field in past timelike infinity. Then the Bekenstein transformation (2.6) yields the new metric functions \(M, P\) and \(Q\) and new (conformal scalar) field \(\phi\):

\[
\begin{align*}
    \phi &= \sqrt{\frac{6}{\kappa}} \tanh \left[ \sqrt{\frac{\kappa}{6}} \tilde{\phi} \right], \\
    P &= \tilde{P} - 2 \ln \cosh \left[ \sqrt{\frac{\kappa}{6}} \tilde{\phi} \right], \\
    M &= \tilde{M} - 2 \ln \cosh \left[ \sqrt{\frac{\kappa}{6}} \tilde{\phi} \right], \\
    Q &= \tilde{Q}.
\end{align*}
\]

We turn to the study of the singularity structure of the new spacetimes (3.2).

Near the asymptotic caustic, e.g. \((v = -\infty, u = 0)\), where the metric is \(v\)-independent, it is convenient to take a new null coordinate \(u'\) instead of \(u\) such that

\[
du' = e^{-M} du.
\]

Then the following vierbein is orthonormal and parallelly propagated along the incomplete geodesics respecting \(x\)- and \(y\)-symmetry and hitting the asymptotic caustic:

\[
\begin{align*}
    e^\alpha_{(0)} &= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right), \\
    e^\alpha_{(1)} &= \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0 \right), \\
    e^\alpha_{(2)} &= \left( 0, 0, e^{\frac{1}{2}(P-Q)(u')}, 0 \right), \\
    e^\alpha_{(3)} &= \left( 0, 0, 0, e^{\frac{1}{2}(P+Q)(u')} \right).
\end{align*}
\]
It is straightforward to compute the only two (mutually independent) nonzero vierbein components of the Riemann curvature tensor as

\[ R_{2020} = -\frac{1}{8}[2Q_u' - 2P_u' - (Q_u - P_u)^2], \]

\[ R_{3030} = \frac{1}{8}[2Q_u' + 2P_u' - (Q_u + P_u)^2]. \]

Going back to the original coordinates we have

\[ R_{2020} = -\frac{1}{8}e^{2M}[2Q_u + 2Q_M u - 2P_u - 2P_M + (Q_u - P_u)^2], \]

\[ R_{3030} = \frac{1}{8}e^{2M}[2Q_u + 2Q_M u + 2P_u + 2P_M - (Q_u + P_u)^2]. \]

The incoming regularity now requires the boundedness of both components. Using (3.2), we can write

\[ uR^- \equiv R_{2020} - R_{3030} = -\frac{1}{8}Ch^{-4}e^{2M}[\tilde{Q}_u + \tilde{Q}_M u - \tilde{Q}_u \tilde{P}_u], \]

\[ uR^+ \equiv R_{2020} + R_{3030} = \frac{1}{4}Ch^{-4}e^{2M}[2\tilde{P}_u - 4\sqrt{\kappa/6} \tilde{\phi}_{uu} Th - \frac{2}{3}\kappa \tilde{\phi}_u^2 Ch \]

\[ + (\tilde{P}_u - 2\sqrt{\kappa/6} \tilde{\phi}_u Th)(2\tilde{M}_u - \tilde{P}_u - 2\sqrt{\kappa/6} \tilde{\phi}_u Th) - \tilde{Q}_u^2], \]

with Ch and Th standing instead of \( \cosh(\tilde{\phi}\sqrt{\kappa/6}) \) and \( \tanh(\tilde{\phi}\sqrt{\kappa/6}) \), respectively.

We have to identify the behaviour of \( uR^\pm \) for \( f \sim 1 \). Taking again into account the asymptotic behaviour of the Bessel and Neumann functions (2.8), we obtain (for \( f \sim 1 \)):

\[ \tilde{Q}_u \sim f_u \left[ \frac{c}{1-f} + d \ln(1-f) + e + h(1-f) \ln(1-f) + \cdots \right], \]

\[ \tilde{\phi}_u \sim f_u \left[ \frac{c^*}{1-f} + d^* \ln(1-f) + e^* + h^*(1-f) \ln(1-f) + \cdots \right], \]

where \( c, d, e, h \) have the same form as in (2.9) and \( c^*, d^*, e^*, h^* \) are given by

\[ c^* = \pi - \lambda - \int_0^\infty [B(\omega) \sin(\omega) + D(\omega) \cos(\omega)] d\omega, \]

\[ 12 \]
\[
    d^* = \int_0^\infty [\omega B(\omega) \cos(\omega) - \omega D(\omega) \sin(\omega)] d\omega,
\]

\[
e^* = \frac{\pi}{2} + \int_0^\infty \omega [A(\omega) + B(\omega) \ln \omega] \cos(\omega) d\omega
    - \int_0^\infty \omega [C(\omega) + D(\omega) \ln \omega] \sin(\omega) d\omega,
\]

\[
h^* = \frac{1}{2} \int_0^\infty \omega^2 [B(\omega) \sin(\omega) + D(\omega) \cos(\omega)] d\omega.
\]  

(3.3)

From (3.1) we have

\[
\tilde{M}_u = f_u \left[ \frac{c^2 + 2\kappa c^*}{2} - 1 \right] \frac{1}{1 - f} + (cd + 2\kappa c^* d^*) \ln(1 - f) + \cdots,$

hence

\[
\tilde{M} = -\frac{c^2 + 2\kappa c^*}{2} \ln(1 - f) + \text{bounded}.
\]

Therefore, the leading singular terms in \( u^- \) and \( u^+ \) read

\[
-8f_u^2 \frac{c^2 + 2\kappa c^*}{2} - 1 \frac{1}{1 - f} \left[ \left( \frac{8\kappa}{3} |c^*| - 2 \sqrt{\frac{R}{6}} (c^2 + 2\kappa c^* + 1) \right) (1 - f)^{4\sqrt{\kappa/6} |c^*| - (c^2 + 2\kappa c^* - 1) - 2},
\]

Both coefficients of proportionality vanish if

\[
c = 0, \quad |c^*| = (2 \pm 1)/\sqrt{6\kappa},
\]

\[
|c| = \frac{1}{2}, \quad |c^*| = \sqrt{\frac{3}{8\kappa}},
\]

\[
c = c^* = 0,
\]

\[
c^* = 0, \quad |c| = 1.
\]

In the first two cases there are subleading singular terms, which cannot be eliminated, but in the remaining cases we can exclude them by fitting some other coefficients in the expansions of \( \tilde{\phi} \) and \( \tilde{Q} \). Hence, the initial data are free of curvature singularities if \( c = c^* = d = 0 \) or if \( c^2 = 1, \ c^* = d = d^* = h = 0 \). The analogous conditions have to be satisfied at the other asymptotic caustic.

Now, we have to study the components of the curvature tensor at the caustic \( 1 - f - g = 0 \). It is sufficient \( \ref{3.1} \) to consider the scalar curvature \( R \) given by

\[
R = -e^M (P_u P_v + 2M_{uv} - Q_u Q_v),
\]

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and the component $\Psi_2$ of the Weyl spinor in the null spin-frame [13]

$$\Psi_2 = \frac{1}{3} e^M (Q_u Q_v - P_u P_v + M_{uv}).$$

If one of them is unbounded, then there is a (final) curvature singularity at the caustic. We take the suitable combinations of $\Psi_2$ and $R$:

$$V_1 = e^M M_{uv},$$
$$V_2 = e^M (Q_u Q_v - P_u P_v).$$

Using (3.2) we have

$$V_1 = \text{Ch}^{-2} e^{\tilde{M}} \left[ \tilde{M}_{uv} - 2 \sqrt{\frac{\kappa}{6}} \tilde{\phi}_{uv} \text{Th} - \frac{1}{3} \kappa \tilde{\phi}_u \tilde{\phi}_v \text{Ch}^{-2} \right],$$
$$V_2 = \text{Ch}^{-2} e^{\tilde{M}} \left[ \tilde{Q}_u \tilde{Q}_v - (\tilde{P}_u - 2 \sqrt{\frac{\kappa}{6}} \tilde{\phi}_u \text{Th})(\tilde{P}_v - 2 \sqrt{\frac{\kappa}{6}} \tilde{\phi}_v \text{Th}) \right].$$

We again introduce the functions $t$ and $z$ as in (2.11). The asymptotic behaviour of $\tilde{\phi}$ and $\tilde{Q}$ near $t = 0$ is

$$\tilde{\phi} \sim \mathcal{E}(z) \ln t + F(z) t^2,$$
$$\tilde{Q} \sim \mathcal{E}(z) \ln t + F(z) t^2,$$

where

$$\mathcal{E}(z) = \lambda - \pi - \chi + \int_0^\infty [\mathcal{B}(\omega) \sin(\omega z) + \mathcal{D}(\omega) \cos(\omega z)] d\omega,$$

$E(z)$ is given as (2.12) and the forms of $F(z)$ and $F(z)$ are not important. Then the leading singular terms of $V_1, V_2$ are proportional to

$$\left[ E_2(z) + 2 \kappa \mathcal{E}^2(z) - 1 - 4 \sqrt{\frac{\kappa}{6}} |\mathcal{E}(z)| \right] t^2 \frac{1}{\sqrt{\kappa/6}} |\mathcal{E}(z)| + \frac{1}{2} (1 - E_2(z) - 2 \kappa \mathcal{E}^2(z) - 2),$$
$$\left[ E_2(z) - \frac{2}{3} \kappa \mathcal{E}^2(z) - 1 + 4 \sqrt{\frac{\kappa}{6}} |\mathcal{E}(z)| \right] t^2 \frac{1}{\sqrt{\kappa/6}} |\mathcal{E}(z)| + \frac{1}{2} (1 - E_2(z) - 2 \kappa \mathcal{E}^2(z) - 2),$$

respectively. Both coefficients of proportionality vanish if

$$\mathcal{E} = 0, \ |E| = 1,$$
$$E = 0, \ |\mathcal{E}| = \sqrt{\frac{3}{2\kappa}}.$$
In the second case, $V_1$ is unbounded due to the subleading term. Hence the only way to keep both $V_1$ and $V_2$ bounded is to set $\mathcal{E}(z) \equiv 0$ and $|E(z)| \equiv 1$.

Therefore both the criterion for the incoming regularity and the necessary condition for the avoiding of the final singularity are in the case of conformal scalar waves the same as in the case of minimal scalar waves [13]. Hence the conclusions have to be the same, too. In particular, the formation of final singularities in the collision of regular asymptotical conformal scalar waves is generic. Moreover, if the pure gravitational radiation is absent in the initial data, i.e.

\[
(-Q_{uu} + P_u Q_u - M_u Q_u) = 0, \ v \to -\infty,
\]
\[
(-Q_{vv} + P_v Q_v - M_v Q_v) = 0, \ u \to -\infty,
\]

the final singularities are even inevitable.

4. Concluding remarks

In 2+1 dimensions, where there is no pure gravitational radiation, the only gravitational waves are those accompanying light-like matter sources. For the massless minimally coupled scalar field [12] previously and for the conformal scalar field now, we have shown that collisions of regular waves necessarily form the final curvature singularities. Since the electrovacua in $D=2+1$ are (up to some global obstructions of the cohomological origin) equivalent to massless minimally coupled scalar vacua [18], it is reasonable to conjecture that collisions of regular general light-like matter waves always end up in a curvature singularity. Of course, systems with more exotic stress tensors have to be studied in order to prove this conjecture. From the point of view of quantum theory it may be worth remarking that the classical phase space, which is to be quantized, does not possess too complicated a structure from the geometrical point of view. Indeed, the spacetimes for all regular initial data have the same global structure.

In 3+1 dimensions the situation is slightly more complicated due to the presence of the pure (sourceless) gravitational radiation. Indeed, within the framework of pure gravity there are regular initial data the evolution of which does not lead to a curvature singularity at the caustic. However, if we exclude the sourceless waves in the initial data, we have shown that the collision of the regular pure conformal scalar waves inevitably leads to the formation of
a curvature singularity. A similar conclusion was made for the case of the massless minimally coupled scalar field \([3]\).

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