Some Comments on Entanglement and Local Thermofield Theory

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We combine recent results of Clifton and Halvorson [1] with structural results of the author [2–5] concerning the local observables in thermofield theory. An number of interesting consequences are discussed.

1. INTRODUCTION

Entangled states are the starting point for quantum information theory, quantum cryptography and quantum teleportation. As long as there are only a few observables experimentally accessible it can be rather difficult to entangle a given system with respect to these observables†. If however, the energy–momentum density distributions for bounded space–time regions are considered to be observable (as one usually does in quantum field theory), then entanglement is a generic feature (see e.g., [6][7]). As we will discuss in this short letter, a thermal background sustains these effects:

- with respect to measurements in spacelike separated space–time regions the set of entangled states is norm dense (see Corollary 5.2 below) in the set of all states which only locally differ from the thermal background (these states are described by density matrices in the Hilbert space \( \mathcal{H}_\beta \) distinguished by the thermal state);
- any state, which is described by a density matrix acting on the Hilbert space \( \mathcal{H}_\beta \), can approximately (in the norm topology) be prepared (see Theorem 3.1 below) by a strictly local operation (i.e., an operation performed in an arbitrary bounded open space–time region \( O \subset \mathbb{R}^4 \)) acting on the thermal equilibrium state;
- if \( P \) and \( Q \) are two nontrivial projections (representing YES-NO experiments) which are localized in spacelike separated space–time regions, then the product \( PQ \) can not vanish identically (see Theorem 4.1 below);

† Quantum information theory is usually set up in a finite dimensional Hilbert space. It is left up to the experimental physicists to realize an apparatus which can approximately be described by such a theory (by neglecting the other degrees of freedom of the system).
Let $E$ represent a single YES-No experiment which can be performed in a bounded space–time region $O$. Given an arbitrary density matrix $\rho \in B(H_\beta)$ one can perform (see Theorem 4.3 below) an operation $V$ (more precisely, one can find an isometry) in a slightly larger space–time region $\hat{O}$ such that the density matrix $\rho_V := V\rho V^*$ satisfies

$$\text{Tr} \rho_V E = 1 \quad \text{and} \quad \text{Tr} \rho_V B = \text{Tr} \rho B \quad \forall B \text{ observable in } \hat{O}. \quad (1)$$

The state given by the density matrix $\rho$ remains completely unchanged in the spacelike complement $\hat{O}'$ of $\hat{O}$.

For any pair $E, F$ of nontrivial projections (i.e., YES-NO experiments) which are localized in spacelike separated space–time regions and for all $\lambda, \mu \in [0, 1]$ there exists (see (25) below) a density matrix $\rho \in B(H_\beta)$ such that

$$\text{Tr} \rho E = \lambda \quad \text{and} \quad \text{Tr} \rho F = \mu. \quad (2)$$

Moreover, for any pair of density matrices $\rho_1, \rho_2 \in B(H_\beta)$ and any pair of spacelike separated space–time regions $O_1, O_2$ there exists a density matrix $\rho$ such that

$$\text{Tr} \rho_1 A = \text{Tr} \rho A \quad \forall A \text{ observable in } O_1 \quad (3)$$

and

$$\text{Tr} \rho_2 B = \text{Tr} \rho B \quad \forall B \text{ observable in } O_2. \quad (4)$$

Product states for the observables associated with two bounded spacelike separated space–time regions $O_1, O_2$ exist if (see Theorem 6.1 below) and only if (see Theorem 6.2 below) the model has decent phase space properties.

If a single product state exists, then very specific product states exist. In fact, for any pair of partial states on the sub algebras there exists a product state which can not be distinguished from the partial states by measurements in the sub algebras alone (see [4] for a proof of these statements).

We conclude this introduction with some comments concerning thermal equilibrium states, the Hilbert spaces associated with them and the density matrices describing deviations from the thermal equilibrium state (see e.g. [8]):

Just like in standard quantum mechanics, the observable quantities of a thermal theory are modelled by linear operators which are embedded into an algebra $R_\beta \subset B(H_\beta)$ of bounded operators acting on a separable Hilbert space $H_\beta$. One may in fact start from an abstract algebra of observables $A$ (which might e.g. be constructed from the $C^*$-algebra associated with the Poincaré group (see [9])). The algebra $R_\beta$ and the Hilbert state $H_\beta$ may then be viewed as secondary objects which arise via the GNS construction, once a thermal state $\omega_\beta$ on $A$ has been distinguished. More precisely, the algebra $R_\beta$ can be
viewed as the weak closure $\pi(\mathcal{A})''$ of $\pi(\mathcal{A})$ in $\mathcal{B}(\mathcal{H}_\beta)$. The vector $\omega_\beta$ can be identified as the GNS vector associated with the pair $(\mathcal{A}, \omega_\beta)$, which satisfies

$$\omega_\beta(a) = (\Omega_\beta, \pi_\beta(a)\Omega_\beta) \quad \forall a \in \mathcal{A}. \quad (5)$$

Thermal states are always mixed states, i.e., they can be decomposed into a convex combination of states (which are not necessarily equilibrium states). However, if we restrict the decomposition to thermal states, then it is unique. An equilibrium state is called extremal, if it cannot be decomposed into other equilibrium states. The decomposition into extremal equilibrium states corresponds to the physical separation of an equilibrium state into pure thermodynamic phases. The symmetry, or lack of symmetry, of these phases is thereby automatically determined. In this short letter we will concentrate on pure phases. Therefore we will always assume that the given equilibrium state is an extremal equilibrium state.

Extremal equilibrium states are distinguished within the set of all (physical) states of $\mathcal{A}$ by first principles: they are precisely those states which are distinguished among (possible other) stationary states by the fact that they turn continuously into the unperturbed states as a certain family of perturbations tends to zero [10]. The same condition may also be interpreted as adiabatic invariance [11]: Extremal equilibrium states return to their original form at the end of a procedure in which the dynamical law is changed by a local perturbation which is slowly switched on and, as $t \to \infty$, slowly switched off again. A second important characteristic of equilibrium states is their passivity, which is the requirement that the energy of the system at time $t$ can only have increased if the Hamiltonian $H_\beta$ (which generates the time evolution in the thermal representation) depends on the time and has returned to its initial form at time $t$ [12]. This condition is just the second law of thermodynamics; it fixes the sign of $\beta$ and means that no energy can be removed from an equilibrium state having $\beta > 0$, just as a periodic process can extract no energy from the ground state.

As a sub algebra of $\mathcal{B}(\mathcal{H}_\beta)$ the algebra $\mathcal{R}_\beta$ generated by the observables of a thermal theory acts irreducibly, i.e., its commutant

$$\mathcal{R}'_{\beta} := \{ A \in \mathcal{B}(\mathcal{H}_\beta) : [A, B] = 0 \quad \forall B \in \mathcal{R}_\beta \} \quad (6)$$

is unequal to $\mathbb{C} \cdot \mathbb{1}$. (The underlying reason is that thermal states are mixed states.) In fact, $\mathcal{R}'_{\beta}$ is as large as (more precisely, isomorphic to) $\mathcal{R}_\beta$. Consequently the structural properties of a thermofield theory are somehow complementary to the ones known from zero temperature quantum field theory. $\Omega_\beta \in \mathcal{H}_\beta$ induces an extremal equilibrium state if, and only if,

$$\mathcal{R}_\beta \cap \mathcal{R}'_{\beta} = \mathbb{C} \cdot \mathbb{1}; \quad (7)$$

in this case $\Omega_\beta$ is said to induce a factor state and $\mathcal{R}_\beta$ is called a factor. Two extremal equilibrium states are either equal or disjoint, which means that they represent different global circumstances. If at a certain temperature there is one and only one equilibrium state, this state is automatically a factor state. Factor states have characteristic cluster properties which reflect the absence of long-range correlations, or the absence of large fluctuations for the values of space-averaged observables. In fact, it has been shown by
the author [2] that there is a tight relation between the infrared properties and the decay of spatial correlations in any extremal equilibrium state, in complete analogy to the well understood case of the vacuum state. To be more precise, since the energy spectrum does not have a mass gap, the correlations between two spacelike separated measurements are bounded by some inverse power of their spatial distance. (The correlations of free massless bosons in two dimensions saturate these bounds.) These various points all indicate that pure phases should correspond to factor states and more precisely to extremal equilibrium states.

2. LOCAL THERMOFIELD THEORIES

In the framework of local quantum physics (see e.g. [13][14][15]) a thermofield theory is specified by a map (usually called a net)

\[
\mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^4,
\]

which associates bounded space–time regions with algebras of bounded operators (von Neumann algebras, to be precise) acting on a Hilbert space \(\mathcal{H}_\beta\). The Hermitian elements of \(\mathcal{R}_\beta(\mathcal{O})\) are interpreted as the observables which can be measured at times and locations in \(\mathcal{O}\). If \(\mathcal{O}_1 \subset \mathcal{O}_2\), then there exists an embedding

\[
\mathcal{R}_\beta(\mathcal{O}_1) \hookrightarrow \mathcal{R}_\beta(\mathcal{O}_2).
\]

We assume that the net \(\mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O})\) is additive, i.e., if

\[
\bigcup_i \mathcal{O}_i = \mathcal{O} \Rightarrow \bigvee_i \mathcal{R}_\beta(\mathcal{O}_i) = \mathcal{R}_\beta(\mathcal{O}).
\]

Here \(\mathcal{R}_1 \vee \mathcal{R}_2\) denotes the von Neumann algebra generated by the algebras \(\mathcal{R}_1\) and \(\mathcal{R}_2\) (i.e., the smallest von Neumann algebra containing \(\mathcal{R}_1\) and \(\mathcal{R}_2\)). The algebra

\[
\mathcal{R}_\beta := \vee_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{R}_\beta(\mathcal{O})
\]

is defined as the inductive limit of the local algebras (see [16].). Observables localized in spacelike separated space–time regions commute:

\[
\mathcal{R}_\beta(\mathcal{O}_1) \subset \mathcal{R}_\beta(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'.
\]

Here \(\mathcal{O}'\) denotes the spacelike complement of \(\mathcal{O}\) and \(\mathcal{R}_\beta(\mathcal{O})'\) denotes the set of operators in \(\mathcal{B}(\mathcal{H}_\beta)\) which commute with all operators in \(\mathcal{R}_\beta(\mathcal{O})\). This property is traditionally called locality, since it reflects the local character of the interaction. However, in order to avoid confusion with the usage of this term in the current literature on quantum information theory, we will prefer the german term “Nahwirkungsprinzip”.

As mentioned in the introduction, the decay of correlations or the absence of large fluctuations is typical of pure thermodynamic phases. But decent cluster properties can
only be expected if the KMS state can not be decomposed into time invariant states. An appropriate criterion is available: An extremal KMS state is an extremal time invariant state, if and only if, it is weakly asymptotically abelian, i.e.,

$$\lim_{s \to r \to \infty} \int_{s}^{r} \mathrm{d}t \left( \Omega_{\beta} , A[e^{-itH_{\beta}}Be^{itH_{\beta}}, C]D\Omega_{\beta} \right) = 0 \quad (13)$$

for all $A, B, C, D \in \mathcal{R}_{\beta}$ [8, 2.5.31]. (We should note that the Hamiltonian $H_{\beta}$ which generates the (weakly continuous) one-parameter unitary group of time evolutions in the thermal representation is not bounded from below for $T > 0$.) The property (13) excludes finite quantum systems; a purely discrete energy spectrum leads to a quasi-periodic motion violating the condition of weak asymptotic abelianess. For infinite quantum systems, however, weak asymptotic abelianess can be considered as a generic feature, and if it holds, then the algebra of observables $\mathcal{R}_{\beta}$ is of type III in the classification of Murray and von Neumann [8, 5.3.36]. Consequently, it does not contain any finite dimensional projection. (A projection $P = P^{2} = P^{*} \in \mathcal{B}(\mathcal{H}_{\beta})$ is called finite dimensional if $PH_{\beta}$ is a finite dimensional subspace of $\mathcal{H}_{\beta}$.)

If the time evolution is norm asymptotically abelian, i.e.,

$$\lim_{t \to \infty} \| [A, e^{-itH_{\beta}}Be^{itH_{\beta}}] \| = 0 \quad \forall A, B \in \mathcal{R}_{\beta}, \quad (14)$$

then $\Omega_{\beta}$ is the unique—up to a phase—time invariant vector in $\mathcal{H}_{\beta}$. We will assume the latter property in the sequel.

3. THE REEH–SCHLIEDER PROPERTY

A relativistic theory requires a drastic departure from ‘classical’ quantum mechanics and its interpretation. The famous Reeh–Schlieder theorem [17] states that if there were no restrictions on the available energy–momentum transfer, then one could prepare any vector state with arbitrary accuracy using only strictly local operations; i.e., operations performed in an arbitrary bounded space–time region. However, we emphasize that the cluster theorems put severe limits on the size of affordable effects, if the available energy–momentum transfer is restricted.

In thermofield theory the thermal equilibrium state itself selects (via the GNS construction) an appropriate Hilbert space $\mathcal{H}_{\beta}$. All states which represent the same global circumstances (mean particle density, mean energy density, temperature, etc.) can be described by density matrices acting on $\mathcal{H}_{\beta}$. The thermal equilibrium state itself is induced by a vector $\Omega_{\beta} \in \mathcal{H}_{\beta}$ which satisfies [18] (and is completely characterized by) the Kubo–Martin–Schwinger (KMS) boundary condition [19][20]

$$\left( \Omega_{\beta} , Ae^{-\beta H_{\beta}}B\Omega_{\beta} \right) = \left( \Omega_{\beta} , BA\Omega_{\beta} \right) \quad \forall A, B \in \mathcal{R}_{\beta}. \quad (15)$$

The KMS vector $\Omega_{\beta}$ (sometimes called the thermal vacuum vector) is cyclic, i.e.,

$$\mathcal{R}_{\beta}\Omega_{\beta} = \mathcal{H}_{\beta} \quad (16)$$
and separating for $\mathcal{R}_\beta$, i.e.,

$$A\Omega_\beta = 0 \Rightarrow A = 0 \quad \forall A \in \mathcal{R}_\beta. \quad (17)$$

Thus any state, which is described by a density matrix $\rho \in \mathcal{B}(\mathcal{H}_\beta)$ (i.e., any state describing the same global circumstances as the thermal state w.r.t. mean particle density, mean energy density, temperature, etc.) is a vector state (see e.g. [8][16]). As we will discuss next, $\Omega_\beta$ is even cyclic for $\mathcal{R}_\beta(\mathcal{O})$, if $\mathcal{O}$ contains an open space–time region $\mathcal{O}_0$. In order to derive this result, we have to take a closer look at the characteristic analyticity properties of a relativistic KMS states.

Lorentz invariance is always broken by a KMS state [21][22]. A KMS state might also break spatial translation or rotation invariance, but the maximal propagation velocity of signals, which is characteristic for a relativistic theory, is not affected by such a lack of symmetry. It was first recognized by Bros and Buchholz that a finite maximal propagation velocity of signals implies that the KMS states of a relativistic theory have stronger analyticity properties in configuration space than those imposed by the traditional KMS condition [23].

Definition. A vector $\Omega_\beta$ satisfies the relativistic KMS condition at inverse temperature $\beta > 0$ if and only if there exists some positive timelike vector $e \in V_+$, $e^2 = 1$, such that for every pair of elements $A, B$ of $\mathcal{R}_\beta$ there exists a function $F_{A,B}$ which is analytic in the domain

$$-\mathcal{T}_{\beta e/2} \times \mathcal{T}_{\beta e/2}, \quad (18)$$

where $\mathcal{T}_{\beta e/2} = \{ z \in \mathbb{C} : \exists z \in V_+ \cap (\beta e/2 + V_-) \}$ is a tube, and continuous at the boundary sets $\mathbb{R}^4 \times \mathbb{R}_4$ and $(\mathbb{R}^4 - \frac{i}{2}\beta e) \times (\mathbb{R}^4 + \frac{i}{2}\beta e)$ with boundary values given by

$$F_{A,B}(x_1, x_2) = (\Omega_\beta, \alpha_{x_1}(A)\alpha_{x_2}(B)\Omega_\beta)$$

$$F_{A,B}(x_1 - \frac{i}{2}\beta e, x_2 + \frac{i}{2}\beta e) = (\Omega_\beta, \alpha_{x_2}(B)\alpha_{x_1}(A)\Omega_\beta) \quad \forall x_1, x_2 \in \mathbb{R}^4. \quad (19)$$

Here $\alpha_x(A)$ means that the element $A \in \mathcal{R}_\beta$ has been shifted in space–time by $x \in \mathbb{R}^4$.

Remark. The relativistic KMS condition can be understood as a remnant of the relativistic spectrum condition in the vacuum sector. It has been rigorously established (see [23]) for the KMS states constructed by Buchholz and Junglas [24]. We would like to emphasize that the relativistic KMS condition does not exclude the possibility that the thermal state breaks translation or rotation symmetry. (Consequently, a unitary implementation of the spatial translations may not exist. This is why we had introduce the automorphisms $\alpha_x : \mathcal{R}_\beta(\mathcal{O}) \to \mathcal{R}_\beta(\mathcal{O} + x), \mathcal{O} \subset \mathbb{R}^4$, in the previous definition.)

As has been demonstrated by the author [5, Th. 3.9], the relativistic KMS condition implies the Reeh–Schlieder property: $\Omega_\beta$ is cyclic for $\mathcal{R}_\beta(\mathcal{O})$, i.e.,

$$\mathcal{R}_\beta(\mathcal{O})\Omega_\beta = \mathcal{H}_\beta, \quad (20)$$
where $\mathcal{O}$ is any open subset of $\mathbb{R}^4$.

We can reformulate this result:

**Theorem 3.1.** Given a density matrix $\rho \in \mathcal{B}(\mathcal{H}_\beta)$ and any $\epsilon > 0$ we can find an element $A_{\rho,\epsilon} \in \mathcal{R}_\beta(\mathcal{O})$ (representing a strictly local operation in $\mathcal{O}$) such that

$$\sup_{\|B\|=1, B \in \mathcal{R}_\beta} \left| \text{Tr} \rho B - (A_{\rho,\epsilon} \Omega_\beta, B A_{\rho,\epsilon} \Omega_\beta) \right| < \epsilon. \quad (21)$$

**Remark.** As mentioned before, any state which is described by a density matrix is a vector state. The theorem given is an immediate consequence of (20) and this fact. It has been shown in [5, Th. 3.10] that $\Omega_\beta$ shares the “Reeh-Schlieder property” (20) with a dense set of vectors in $\mathcal{H}_\beta$.

As has been shown by Kadison [25], the Reeh–Schlieder property implies that the local algebras cannot be finite dimensional matrix algebras:

**Proposition 3.2.** (Kadison). If $\mathcal{R}_\circ$ is a proper sub von Neumann algebra of the von Neumann algebra $\mathcal{R}$, and $\Omega$ is a separating and cyclic vector for both $\mathcal{R}$ and $\mathcal{R}_\circ$, then $\mathcal{R}$ (and $\mathcal{R}_\circ$) are of infinite type.

**4. THE SCHLIEDER AND THE BORCHERS PROPERTY**

Without any further assumptions on e.g., the energy spectrum of excitations of a given extremal KMS state, the following statement is valid:

**Theorem 4.1.** (Schlieder property). Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ denote two open (not necessarily bounded) space–time regions such that

$$\mathcal{O} + te \subset \hat{\mathcal{O}} \quad \forall |t| < \delta, \quad \delta > 0. \quad (22)$$

It follows that $0 \neq A \in \mathcal{R}_\beta(\mathcal{O})$ and $0 \neq B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$ implies $AB \neq 0$.

**Remark.** A proof of the thermal case given here can be found in [3, Th. II.4]; the original result for the vacuum sector can be found in [26].
Thus, if $P \in \mathcal{R}_\beta(\mathcal{O})$ and $Q \in \mathcal{R}_\beta(\hat{\mathcal{O}}') \subset \mathcal{R}_\beta(\hat{\mathcal{O}})'$ are two nontrivial projections, then the product $PQ$ can not vanish identically. The Schlieder property implies that $\mathcal{R}_\beta(\mathcal{O})$ is almost a factor, namely
\[
\mathcal{R}_\beta(\mathcal{O}) \cap \mathcal{R}_\beta(\hat{\mathcal{O}})' = \mathbb{C} \cdot \mathbb{1}.
\] It is a first step towards the “statistical independence” of $\mathcal{R}_\beta(\mathcal{O})$ and $\mathcal{R}_\beta(\hat{\mathcal{O}})'$:

**Corollary 4.2.** (Florig and Summers [27]): Let $\mathcal{O}, \hat{\mathcal{O}}$ denote a pair of space–time regions such that the closure of the open region $\mathcal{O}$ is contained in the interior of $\hat{\mathcal{O}}$. It follows that

(i) For any nonzero vectors $\Phi, \Psi \in \mathcal{H}_\beta$ there exist $A' \in \mathcal{R}_\beta(\mathcal{O})'$ and $B' \in \mathcal{R}_\beta(\hat{\mathcal{O}})$ such that
\[
A' \Phi = B' \Psi \neq 0.
\] (24)

(ii) $\|AB\| = \|A\| \|B\|$ for all $A \in \mathcal{R}_\beta(\mathcal{O})$ and all $B \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$.

(iii) The von Neumann algebras $\mathcal{R}_\beta(\mathcal{O})$ and $\mathcal{R}_\beta(\hat{\mathcal{O}})'$ are algebraically independent; i.e., given two arbitrary sets $\{A_i : i = 1, \ldots, m\}$ and $\{B_j : j = 1, \ldots, n\}$ of linear independent elements of $\mathcal{R}_\beta(\mathcal{O})$ and $\mathcal{R}_\beta(\hat{\mathcal{O}})'$, respectively, the collection $\{A_i B_j : i = 1, \ldots, m; j = 1, \ldots, n\}$ is linearly independent in the algebraic tensor product $\mathcal{R}_\beta(\mathcal{O}) \odot \mathcal{R}_\beta(\hat{\mathcal{O}})'$ of $\mathcal{R}_\beta(\mathcal{O})$ and $\mathcal{R}_\beta(\hat{\mathcal{O}})'$.

The Schlieder property has interesting consequences for the existence of certain states. E.g., for any pair of nontrivial projections (i.e., YES-NO experiment) $E \in \mathcal{R}_\beta(\mathcal{O})$, $F \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$ and $\lambda, \mu \in [0, 1]$ there exists a density matrix $\rho \in \mathcal{B}(\mathcal{H}_\beta)$, such that
\[
\text{Tr} \rho E = \lambda \quad \text{and} \quad \text{Tr} \rho F = \mu.
\] (25)

Moreover, for any pair of density matrices $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}_\beta)$ there exists a density matrix $\rho \in \mathcal{B}(\mathcal{H}_\beta)$ such that
\[
\text{Tr} \rho_1 A = \text{Tr} \rho A \quad \forall A \in \mathcal{R}_\beta(\mathcal{O})
\] (26)

and
\[
\text{Tr} \rho_2 B = \text{Tr} \rho B \quad \forall B \in \mathcal{R}_\beta(\hat{\mathcal{O}}').
\] (27)

A proof of these statements can be found in [27].

Another consequence of the “Nahwirkungsprinzip”, the Reeh–Schlieder property and the Schlieder property is the so-called Borchers property:

**Theorem 4.3.** (Borchers property). Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ denote two open and bounded space–time regions such that
\[
\mathcal{O} + te \subset \hat{\mathcal{O}} \quad \forall |t| < \delta, \quad \delta > 0.
\] (28)

Given a nonzero projection $E \in \mathcal{R}_\beta(\mathcal{O})$, there exists a partial isometry $V \in \mathcal{R}_\beta(\hat{\mathcal{O}})$ such that $V^* V = \mathbb{1}$ and $V V^* = E$. 
Remark. One writes
\[ E \sim 1 \mod \mathcal{R}_{\beta} (\hat{O}) \cdot \] (29)
Recall that a factor \( \mathcal{M} \) is called type III, if \( E \sim 1 \mod R \) for all self-adjoint projections \( E \in \mathcal{M} \). Thus \( \mathcal{R}_{\beta}(O) \) is of infinite type, and “almost” a factor of type III. A proof of the thermal case given here can be found in [3, Th. II.6]; the original result for the vacuum sector can be found in [28].

The Borchers property has interesting consequences for the actual preparation of states: Given a density matrix \( \rho \), we set \( \rho_V := V \rho V^* \). Then
\[ \text{Tr} \rho_V E = 1 \quad \text{and} \quad \text{Tr} \rho_V B = \text{Tr} \rho B \quad \forall B \in R(\hat{O}'). \] (30)
This demonstrates that the Borchers property allows us to prepare a state \( \omega_V \) which satisfies the properties (28) by a strictly local operation. The state given remains completely unchanged in the spatial complement of \( \hat{O} \). This is a remarkable difference to the “collapse of the wave-function” type of preparation.

Remark. While finite quantum systems may very well allow a tensor product decomposition of their observables into sub algebras, which are statistically independent (so that they satisfy the Schlieder and even the split property (see Sec. 6)), the Borchers property is reserved for infinite quantum systems (as can be seen from the previous remark). Thinking of photon polarization states, which are entangled into a Bell state \( \rho \) on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), it will not always be possible to find an isometry \( V \in B(\mathbb{C}^2 \otimes \mathbb{C}^2) \) (isometries are unitary operators in the finite dimensional case) such that for some given projection \( E \otimes 1 \) the resulting state \( \rho_V \) satisfies the properties
\[ \text{Tr} \rho_V E = 1 \quad \text{and} \quad \text{Tr} \rho_V B = \text{Tr} \rho B \quad \forall B \in 1 \otimes B(\mathbb{C}^2). \] (31)
The equation on the r.h.s. implies that \( V \in B(\mathbb{C}^2) \otimes 1 \). If, for example the restriction of \( \rho \) to \( B(\mathbb{C}^2) \otimes 1 \) coincides with \( \frac{1}{2} E \otimes 1 + \frac{1}{2} (1 - E) \otimes 1 \) then one can not find a unitary operator in \( B(\mathbb{C}^2) \otimes 1 \) such that the equation on the l.h.s. in (31) is fulfilled. The obvious difference is that in the case discussed in Theorem 4.3 an intermediate region \( \hat{O} \ \backslash \ O \) is at our free disposal, which may be used to decouple the inside from the outside.

5. ENTANGLED STATES

Definition. Consider two sub algebras \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) of a von Neumann algebra \( \mathcal{R} \subset B(H) \). A state induced by a density matrix \( \rho \in B(H) \) on \( \mathcal{R} \) is called a product state for the pair \((\mathcal{R}_1, \mathcal{R}_2)\) if there exist two density matrices \( \rho_1, \rho_2 \in B(H) \) such that
\[ \text{Tr} \rho AB = \text{Tr} \rho_1 A \cdot \text{Tr} \rho_2 B \] (32)
for all $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$. A state induced by a density matrix $\rho \in \mathcal{B}(\mathcal{H})$ is called \textit{separable} for the pair $(\mathcal{R}_1, \mathcal{R}_2)$, if, and only if, it is in the weak*-closed convex hull of product states. I.e., there exists a weak*-limit of convex combinations of product states $\{\rho_\gamma\}_{\gamma \in I}$ for the pair $(\mathcal{R}_1, \mathcal{R}_2)$ such that

$$\text{Tr} \rho AB = \sum_{\gamma \in I} \lambda_\gamma (\text{Tr} \rho_\gamma AB), \quad \sum_{\gamma \in I} \lambda_\gamma = 1, \quad 0 < \lambda_\gamma < 1,$$

for all $A \in \mathcal{R}_1$ and $B \in \mathcal{R}_2$. A density matrix $\rho \in \mathcal{B}(\mathcal{H})$ is called \textit{entangled} w.r.t. the pair $(\mathcal{R}_1, \mathcal{R}_2)$, if it is not separable.

In a recent paper Clifton and Halvorson [1] derived the following result:

**Proposition 5.1.** (Clifton and Halvorson). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be nonabelian von Neumann algebras acting on a Hilbert space $\mathcal{H}$ such that $\mathcal{R}_1 \subset \mathcal{R}_2'$ and $\mathcal{R}_1, \mathcal{R}_2$ satisfy the Schlieder property. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are of infinity type, then there is an open dense subset of vectors in $\mathcal{H}$, which induce entangled states w.r.t. the pair $(\mathcal{R}_1, \mathcal{R}_2)$. If a vector $\Phi \in \mathcal{H}$ is cyclic for $\mathcal{R}_1$, then $(\Phi, . \Phi)$ is one of these entangled states w.r.t. the pair $(\mathcal{R}_1, \mathcal{R}_2)$.

As mentioned before, any state which is described by a density matrix acting on $\mathcal{H}_\beta$ is a vector state. Moreover, $\Omega_\beta$ shares the Reeh–Schlieder property with a dense set of vectors. Thus the following corollary is an immediate consequence of Proposition 5.1.

**Corollary 5.2.** Let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote two open (not necessarily bounded) space–time regions such that

$$\mathcal{O}_1 + te \subset \mathcal{O}_2' \quad \forall |t| < \delta, \quad \delta > 0. \quad (34)$$

Then there exists a norm dense set of entangled states w.r.t. the pair $(\mathcal{R}_\beta(\mathcal{O}_1), \mathcal{R}_\beta(\mathcal{O}_2))$.

**Proof.** As mentioned before, Kadison’s result (see Prop. 3.2) and the Reeh–Schlieder theorem together imply that $\mathcal{R}_\beta(\mathcal{O}_1)$ and $\mathcal{R}_\beta(\mathcal{O}_2)$ are of infinite type. There are now two distinct lines of arguments which may be used to prove the Corollary given:

i.) According to Prop. 2 of [1], if $\Phi$ is a cyclic vector for $\mathcal{R}_1$, then $\Phi$ is entangled w.r.t. the pair $(\mathcal{R}_1, \mathcal{R}_2)$ (as long as $\mathcal{R}_1$ and $\mathcal{R}_2$ are nonabelian.) Consequently, the state induced by $\Omega_\beta$ is entangled. Moreover, there is a dense set of vectors in $\mathcal{H}_\beta$ which share the Reeh–Schlieder property with $\Omega_\beta$. These vectors induce entangled states w.r.t. the pair $(\mathcal{R}_\beta(\mathcal{O}_1), \mathcal{R}_\beta(\mathcal{O}_2))$.

ii.) According to Prop. 1 of [1], if $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy the Schlieder property and are both of infinite type, then there is a dense set of Bell correlated (hence entangled) states w.r.t. the pair $(\mathcal{R}_1, \mathcal{R}_2)$. This result is in one sense stronger then the first, since a mixed entangled state need not display Bell correlations. But this second line of arguments fails to establish that the vector state induced by $\Omega_\beta$ (or any other experimentally feasible vector state is entangled. \qed
This result may be compared with the finite dimensional situation: If \( R_1 := B(\mathcal{C}^2) \otimes I_2 \), \( R_2 := I_2 \otimes B(\mathcal{C}^2) \), then any entangled state vector is cyclic for \( R_1 \). But the set of entangled states of \( B(\mathcal{C}^2) \otimes B(\mathcal{C}^2) \) is not norm dense in the set of all density matrices on \( B(\mathcal{C}^2) \otimes B(\mathcal{C}^2) \) [29].

6. THE SPLIT PROPERTY

While the existence of a norm dense set of entangled states is a direct consequence of the generic properties of any thermofield theory, the existence of (normal) product states is a much more delicate property. If one carefully studies (Sect. 3 in [4] is entirely devoted to this task) the distribution of the energy eigenvalues of a system consisting of relativistic particles which are confined in a finite volume as the size of the “box” goes to infinity, then one is led to the assumption that a thermal state should have the following characteristic phasespace properties: for any bounded space–time region \( O \) the maps \( \Theta_{\alpha,O}^+: R_\beta(O) \to \mathcal{H}_\beta \) given by

\[
\Theta_{\alpha,O}^+(A) = e^{-\alpha H_\beta} P^+ A \Omega_\beta, \quad \alpha > 0,
\]

are of type \( s \). Here \( P^+ \) denotes the projection onto the strictly positive spectrum of the Hamiltonian \( H_\beta \). We recall (see e.g. [30]) that \( \Theta_{\alpha,O}^+ \) is said to be of type \( l^p \), \( p > 0 \), if there exists a sequence of linear mappings \( \Theta_k \) of rank \( k \) such that

\[
\sum_{k=0}^{\infty} \| \Theta_{\alpha,O}^+ - \Theta_k \|^p < \infty.
\]

(36)

\( \Theta_{\alpha,O}^+ \) is said to be nuclear, if \( \Theta_{\alpha,O}^+ \) is of type \( l^p \) for \( p = 1 \). It is said to be of type \( s \), if it is of type \( l^p \) for all \( p > 0 \). Quantitative information can be extracted from the nuclear norm of \( \Theta_{\alpha,O}^+ \), but we will not need this kind of information here.

Theorem 6.1. (Split Property). Assume that for any bounded space–time region \( O \) the maps \( \Theta_{\alpha,O}^+ \) are of type \( s \) for all \( \alpha > 0 \). It follows that for any pair \( \Lambda = (O, \hat{O}) \) of open bounded space–time regions which satisfies

\[
O + t e \subset \hat{O} \quad \forall |t| < \delta, \quad \delta > 0,
\]

(i) there exists a vector \( \eta_\Lambda \in \mathcal{H}_\beta \) such that

a.) \( (\eta_\Lambda, AB \eta_\Lambda) = (\Omega_\beta, A \Omega_\beta)(\Omega_\beta, B \Omega_\beta) \) for all \( A \in R_\beta(O) \) and \( B \in R_\beta(\hat{O})' \).

b.) \( \eta_\Lambda \) is cyclic and separating for \( R_\beta(O) \lor R_\beta(\hat{O})' \).

(ii) the von Neumann algebra generated by \( R_\beta(O) \) and \( R_\beta(\hat{O})' \) is isomorphic to the \( W^* \)-tensor product of the two algebras. I.e., there exists a unitary operator \( W_\Lambda: \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta \) such that

\[
W_\Lambda AB W_\Lambda^* = A \otimes B
\]

(38)

for all \( A \in R_\beta(O) \) and \( B \in R_\beta(\hat{O})' \).
Remark. This result has been proven by the author in [4, Sect. 4]; the proof is based on previous work by different authors, see for instance [31]. We emphasize, that the requirement that $O$ is bounded can not be removed (see e.g. [32] for counterexamples, which may easily be adapted to the thermal case).

The split property (see [33] for a general account) has far reaching consequences for the preparation of states: It implies that for any pair of density matrices $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}_\beta)$ there exists a density matrix $\rho \in \mathcal{B}(\mathcal{H}_\beta)$ such that

$$\text{Tr} \rho AB = \text{Tr} \rho_1 A \cdot \text{Tr} \rho_2 B$$  \hspace{1cm} (39)

for all $A \in \mathcal{R}_\beta(O)$ and $B \in \mathcal{R}_\beta(\hat{O})'$. Moreover, it allows us to select a set of states which represents local excitations of the thermal states: the set

$$L_\beta(O, \hat{O}) := \mathcal{R}_\beta(O)\eta_A$$  \hspace{1cm} (40)

has the following properties:

(i) $L_\beta(O, \hat{O})$ is a closed subspace of $\mathcal{H}_\beta$;

(ii) $L_\beta(O, \hat{O})$ is invariant under the action of $\mathcal{R}_\beta(O)$;

(iii) If $\Psi \in L_\beta(O, \hat{O})$, then

$$(\Psi, AB\Psi) = (\Psi, A\Psi)(\Omega_\beta, B\Omega_\beta)$$  \hspace{1cm} (41)

for all $A \in \mathcal{R}_\beta(O)$ and $B \in \mathcal{R}_\beta(\hat{O})'$.

(iv) $L_\beta(O, \hat{O})$ is complete in the following sense: to every density matrix $\rho \in \mathcal{B}(\mathcal{H}_\beta)$ there exists a vector $\Phi \in L_\beta(O, \hat{O})$ such that

$$(\Phi, A\Phi) = \text{Tr} \rho A$$  \hspace{1cm} (42)

for all $A \in \mathcal{R}_\beta(O)$.

Remark. Similar properties have been derived for a subspace of the vacuum Hilbert space representing local excitations of the vacuum (see [24]). The generalization to the present thermal case is straightforward.

It was noticed by Buchholz, D’Antoni and Longo that the split property imposes certain restrictions on the energy level density of excitations of the KMS state [31, Proposition 4.2 and Lemma 3.1.i):]

**Theorem 6.2.** (Buchholz, D’Antoni and Longo). Consider a TFT, specified by a von Neumann algebra $\mathcal{R}_\beta$ with a cyclic and separating vector $\Omega_\beta$ and a net of sub algebras $O \rightarrow \mathcal{R}_\beta(O)$, subject to the conditions specified in Sect. 2. Assume the inclusion $\mathcal{R}_\beta(O) \subset \mathcal{R}_\beta(\hat{O})$ is split. Then the maps

$$\Theta^{+}_{\alpha, O}(A) = e^{-\alpha H_\beta} P^+ A\Omega_\beta, \quad \alpha > 0,$$  \hspace{1cm} (43)

are compact for $\alpha > 0$.  
Remark. Let us summarize: If the map $\Theta^{+}_{\alpha,\mathcal{O}}$ is of type $s$, then the split property holds. The latter implies that $\Theta^{+}_{\alpha,\mathcal{O}}$ is a least compact. Whether or not the first assumption can be relaxed and whether or not the second conclusion can be sharpened is unknown to the author.

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