DUALITY AND MASSIVE GAUGE INVARIANT THEORIES

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ABSTRACT

Two different massive gauge invariant spin-one theories in 3 + 1 dimensions, one Stuckelberg formulation and the other ‘$B^\wedge F$’ theory, with Kalb-Ramond field are shown to be related by duality. This is demonstrated by gauging the global symmetry in the model and constraining the corresponding dual field strength to be zero by a Lagrange multiplier, which becomes a field in the dual theory. Implication of this equivalence to the 5 dimensional theories from which these theories can be obtained is discussed. The self-dual Deser-Jackiw model in 2 + 1 dimensions, is also shown to result by applying this procedure to Maxwell-Chern-Simon theory.

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Massive gauge invariant spin-one theories has been studied for a long time with two principle procedure for it: Schwinger mechanism in $1 + 1$ dimension [1] of 2d quantum electrodynamics of massless fermion, yielding massive gauge field through axial anomaly and Higgs mechanism. In $3 + 1$ dimensions massive spin-one theories are generally considered following one of the two procedures: one by Stuckelberg formulation [2] and the other by using Kalb-Ramond field (rank two antisymmetric tensor gauge field) [3] in a Chern-Simons like formulation known as $'B \wedge F'$ theory [4]. The latter is well studied in different contexts, including as realizations of certain condensed matter system [5] by Sodano et al., as an alternate to Higgs mechanism [6] and as realization of Bosonised Schwinger model in $3 + 1$ dimensions by Aurilia and Takahashi [7]. On the other hand, the Stuckelberg formulation of spin-one, (and also for higher spin fields) [8] have been studied in various contexts, like for consistency problems in higher spin fields [9] and in string field theory as description of massive modes [10]. Though they appear as different construction for maintaining gauge invariance in the presence of mass terms, in this paper we show that the two theories are related by duality transformation.

In recent times duality is being studied in detail, due to developments in string theory, where different inequivalent vacua are shown to be related by duality [10]. In the context of sigma models, a procedure for constructing dual theory was given by Busher [12] and generalized by Rocek and Verlinde[13]. Basically, the procedure consists in gauging the global symmetry with gauge fields, whose field strength is constrained to be zero by means of a Lagrange multiplier. Integrating the multiplier field and then the gauge field, original action was recovered. Instead, if one integrates the original and gauge fields, keeping the multiplier field, dual theory was obtained. This procedure of obtaining dual theory was applied to $1 + 1$ dimensional Dirac theory and Bosanisation rules were obtained from this duality procedure by gauging the global phase symmetry by Burgess and Quvedo [14].
This procedure has also been applied to gauge theories and shown to lead to S-duality, which relates strong and weak coupling [15]. This procedure has also been shown to be related to canonical transformations [16].

In this paper, we apply this procedure of dualization to topologically massive gauge theories. We first apply this method to $B^\wedge F$ theory in $3 + 1$ dimensions where the global symmetry is shift of the field and show that Stuckelberg type massive theory is obtained. First we note that, in the former theory, current due to local gauge symmetry is conserved as an algebraic identity, like that for topological currents and in the latter case it is conserved due to the equation of motion for Stuckelberg field. Since this interchange between topological and Noether current, generally takes place under duality transformation, it is plausible that the two theories are related by duality transformation. This is demonstrated in this paper by the procedure outlined above. Next topologically massive $2 + 1$ dimensional Maxwell-Chern-Simons theory is considered and this procedure is shown to lead to Deser-Jackiw model [17] of self dual massive theory. This paper is organized as follows:

Section 1 deals with equivalence between ‘$B^\wedge F$’ theory and Stuckelberg theory. Section 2 applies it to Maxwell-Chern-Simons theory. Finally we end with discussion. Both these models, i.e., ‘$B^\wedge F$’ model and Stuckelberg model has earlier [18,19] been shown to be obtained from different $5d$-theories. This result, in the context of $5d$-theories is also discussed, in the final section.

We use the metric $g_{\mu\nu} = diag(1, -1, -1, -1)$ and $\epsilon_{0123} = 1$
Section I:

Consider the Lagrangian for topologically massive spin-one theory involving Kalb-Romand field $B_{\mu\nu}$ and a vector field $A_\mu$ known as $B^\wedge F$ theory.

$$L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2(3!)} H_{\mu\nu\lambda}^2 - \frac{m}{3!} H_{\mu\nu\lambda} \epsilon^{\mu\nu\lambda\rho} A_\rho .$$

(1)

This Lagrangian has local invariance under

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \quad (2)$$

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu) \quad (3)$$

where $H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\nu\mu} \quad (6)$

The field equations following from this are

$$\partial_\mu F^{\mu\nu} = J^\nu \quad (4)$$

$$\partial_\mu H^{\mu\nu\lambda} = J^{\nu\lambda} \quad (5)$$

where $J^\mu = \frac{m}{3!} \epsilon^{\mu\nu\lambda\rho} H_{\nu\lambda\rho}$ and $j^{\mu\nu} = \frac{m}{3!} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$ are the currents associated with local gauge symmetry (2), (3).

Note that both the currents are conserved as an algebraic identity, like that of topological current. The fact that this describes massive spin-one theory can be shown easily by solving the coupled differential equations [7].

Next we consider the Stuckelberg formulation of massive vector theory whose Lagrangian is

$$L = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2}(\partial_\mu \Phi - mA_\mu)^2$$

(6)

This has invariance under

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \quad (7)$$

$$\Phi \rightarrow \Phi + mA \quad (8)$$
The equation of motion following from this Lagrangian (6) for $A_\mu$ and $\Phi$ are

\[
\partial_\mu F^{\mu\nu} + K^\nu = 0 \quad \text{(9)}
\]
\[
\partial^\mu (\partial_\mu \Phi - m A_\mu) = 0. \quad \text{(10)}
\]

where $K^\nu \equiv m(\partial^\nu \Phi - m A^\nu)$. Now note that the current associated with $A_\mu$ field is conserved due to the equation of motion of $\Phi$ field, like that of Noether current. The fact that this describes massive spin-one theory can be seen by using the gauge invariance (8) and fixing the field $\Phi$ to zero.

Thus we have two (apparently) different formulation of spin-one theory. But the nature of currents in the two theories and physical equivalence of the system they describe, viz., massive spin-one particle forces one to enquire if both these formulations are related by duality transformation. We next show, indeed that is the case.

The dual theory is obtained by the procedure of gauging the global symmetry in the model by a gauge field and constraining its dual field strength to be zero by means of a Lagrange multiplier and by integrating the original and the gauge field and expressing the theory in terms of the multiplier field, the dual theory is obtained. The global symmetry, in question in this model (1) is $\delta B_{\mu\nu} = \epsilon_{\mu\nu} \; , \; \delta A_\mu = 0$ (Note by dropping a surface term, the global symmetry is on the vector field. This is discussed later). This symmetry is gauged by introducing a three form gauge potential, $G_{\mu\nu\lambda}$ in the Lagrangian (1), whose dual field strength is $\epsilon^{\mu\nu\lambda\rho} \partial_\mu G_{\nu\lambda\rho}$, which is gauge invariant under $\delta G_{\mu\nu\lambda} = \partial_\mu \eta_{\nu\lambda}$. By adding a scalar field as a Lagrange multiplier, its dual field strength is constrained to be flat. The Lagrangian is

\[
L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2(3!)} (H_{\mu\nu\lambda} - G_{\mu\nu\lambda})^2 - \frac{m}{3!} (H_{\mu\nu\lambda} - G_{\mu\nu\lambda}) \epsilon^{\mu\nu\lambda\rho} A_\rho + \frac{1}{3!} \Phi \epsilon^{\mu\nu\lambda\rho} \partial_\mu G_{\nu\lambda\rho}. \quad \text{(11)}
\]

Note that the original gauge invariance of the vector theory, $\delta A_\mu = \partial_\mu \Lambda$, is recovered
only when, under $A_\mu$ gauge transformation, scalar field also transforms

$$\Phi \rightarrow \Phi + mA.$$  \hfill (12)

This transformation is the same as that of Stuckelberg formulation of the theory. Indeed, by integrating over $B_{\mu\nu}$ and $A_\mu$ fields, which appear as Gaussian, Stuckelberg theory (6) results.

Instead of considering the global symmetry in 2-form $B$ field, one could start, after omitting a surface term in the Lagrangian (1), which has a global symmetry in $A_\mu$ field of the form $\delta A_\mu = \epsilon_\mu$ and $\delta B_{\mu\nu} = 0$. Gauging this symmetry, one gets

$$L = -\frac{1}{4}(F_{\mu\nu} - G_{\mu\nu})^2 + \frac{1}{2(3!)}H^2_{\mu\nu\lambda} + \frac{1}{2}mB_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}(F_{\chi\rho} - G_{\chi\rho}) + \Phi_\mu\epsilon^{\nu\lambda\rho}\partial_\nu G_{\chi\rho}$$  \hfill (13)

Here $G_{\mu\nu}$ is a two form gauge field, with transformation $\delta G_{\mu\nu} = \partial_\mu \epsilon_{\nu} - \partial_\nu \epsilon_{\mu}$. Repeating the procedure of integrating over $G_{\mu\nu}$ and $A_\mu$ which are again Gaussian, the following Lagrangian is obtained.

$$\frac{1}{2(3!)}H^2_{\mu\nu\lambda} + (mB_{\mu\nu} - \Phi_{\mu\nu})^2$$  \hfill (14)

where $\Phi_{\mu\nu} = (\partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu)$. This again has invariance under $\delta B_{\mu\nu} = (\partial_\mu \epsilon_{\nu} - \partial_\nu \epsilon_{\mu})$ and $\delta \Phi_\mu = m\epsilon_\mu + \partial_\mu \chi$.

This Stuckelberg type action for 2-form field (14), was constructed and studied earlier by Aurilia and Takahashi. This also describes a massive spin-one field, as can be seen in the gauge where $\Phi_\mu$ is zero when it becomes Takahashi-Palmer equation for spin-1 [21].

The interesting aspect about the actions (1) and (6) is that both are obtainable from 5d theories. The 5d theory which gives rise to the action (1) is the topologically massive theory formed out of Kalb-Ramond field in 5d, wherein, only zero-mode is kept upon dimensional reduction in 4d [18]. Whereas, 5d Maxwell theory, upon dimensional reduction, keeping non-zero modes give the Stuckelberg action [19]. Thus different 5d
gauge invariant theories give 4d theories which are dually related. Interestingly zero-mode of topologically massive 5d Chern-Simons theory is seen to be dual of non-zero mode of Maxwell theory in 5d. The corresponding dual relation between the two 5d theories, if any, is not clear.

Section 2:

Next we consider the 2 + 1 dimensional Maxwell Chern-Simons theory, whose Lagrangian is

\[ L = -\frac{1}{4g^2} F_{\mu\nu}^2 + \theta A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \]  
(15)

Apart from the usual gauge invariance, \( \delta A_\mu = \partial_\mu \Lambda \), there is also a global symmetry,

\[ A_\mu \rightarrow A_\mu + \epsilon_\mu \]  
(16)

upto surface terms. Instead of gauging this global symmetry directly, it is needed to first linearize the Chern-Simons term as

\[ A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = P_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda - \frac{1}{4} P_\mu \epsilon^{\mu\nu\lambda} \partial_\nu P_\lambda \]  
(17)

where \( P_\mu \) is an auxiliary vector field. This action(15) with (17) used for Chern-Simons term has still the global symmetry in (16). Also the auxiliary vector field \( P_\mu \) has a local gauge invariance \( \delta P_\mu = \partial_\mu \chi \).

Now gauging the global symmetry, using a 2-form gauge field, the action is given by

\[ L = -\frac{1}{4g^2} (F_{\mu\nu} - B_{\mu\nu})^2 + \frac{1}{2} \theta P_\mu \epsilon^{\mu\nu\lambda} (F_{\nu\lambda} - B_{\nu\lambda}) - \frac{1}{4} \theta P_\mu \epsilon^{\mu\nu\lambda} \partial_\nu P_\lambda \]  
(18)

Adding the constraint, which makes the field strength of \( B_{\mu\nu} \) flat, the final action is

\[ L = -\frac{1}{4g^2} (F_{\mu\nu} - B_{\mu\nu})^2 + \frac{\theta}{2} P_\mu \epsilon^{\mu\nu\lambda} (F_{\nu\lambda} - B_{\nu\lambda}) - \frac{1}{4} \theta P_\mu \epsilon^{\mu\nu\lambda} \partial_\nu P_\lambda + \Phi \epsilon^{\mu\nu\lambda} \partial_\mu B_{\nu\lambda} \]  
(19)

Where \( \Phi \) is the Lagrange multiplier field. Note that to maintain gauge invariance of \( P_\mu \), the multiplier field also undergoes corresponding transformation \( \delta \Phi = -\frac{1}{2} \theta \chi \). Integrating
the $B_{\mu\nu}$ and $A_{\mu}$ field we get the Lagrangian
\begin{equation}
L = 2g^2(\partial_{\mu}\Phi + \frac{1}{2}\theta P_{\mu})^2 - \frac{1}{4}\theta P_{\mu}\epsilon^{\mu\nu\lambda}\partial_{\nu}P_{\lambda}.
\end{equation}
(20)

By redefining $P_{\mu}' = (\partial_{\mu}\Phi + \frac{1}{2}\theta P_{\mu})$ we get
\begin{equation}
L = 2g^2P_{\mu}'^2 - \frac{1}{\theta}\epsilon_{\mu\nu\lambda}P_{\nu}'\partial_{\nu}P_{\lambda}'.
\end{equation}
(21)

This is the self-dual Lagrangian to Maxwell-Chern-Simons theory due to Deser-Jackiw, who obtained it using Legendre transformation. Note that $g^2$ and $\theta$ have appeared as a reciprocal as in that of (15). This Lagrangian is, thus shown to be obtained by the usual duality procedure.

Conclusion:

In this paper we have shown that two massive gauge invariant theories, Stuckelberg formulation and topologically massive $'B\wedge F'$ theory are dually equivalent. Maxwell-Chern-Simon theory in 2 + 1 dimensions, which has a global shift symmetry of the gauge field, is used to obtain Deser-Jackiw model as its dual theory. It has been argued by Aurilia and Takahashi [7], that the $'B\wedge F'$ theory (called as gauge mixing mechanism by these authors) is the four dimensional analog of Bosonised Schwinger model known in 1 + 1 dimensions. Hence it should be interesting to see if the Schwinger mechanism has a duality relation to Higgs mechanism. Both these theories defined by the lagrangian (1) and (6) have as massless limit, uncoupled massless spin-1 field and massless spin-0 field. The latter is described in (1) by Kalb-Ramond gauge field and in (6) by scalar field. Thus the distinction between the massive gauge invariant spin-1 theories (1) and (6), seems to be that in the former, the massless spin-0 field ‘eaten’ by the massless vector field is described by Kalb-Ramond gauge field and in the latter by scalar field. It is well-known that massless Kalb-Ramond description of spin-0 particle is dual to that of scalar field description. Hence this difference in description appears as duality equivalence.
Since, both these theories, are obtained as dimensional reduction of different 5d theories, i.e., one topologically massive Kalb-Ramond theory [18] and other Maxwell theory [19], with the former having zero-modes only and the other non-zero modes only, the equivalence shown may have some implication for the 5d theory. It should be interesting to investigate the relation, if any, between the two 5d theories. Since the 2 + 1 dimensional version of (14), with two vector field, instead of a vector and anti-symmetric tensor in 3 + 1 dimension, has been shown recently to be a realization of Josephson junction arrays [5], the equivalence shown may have an implication there also. Also there has been generalization to higher spin theories of the Stuckelberg formulation, it should be interesting to obtain generalization of such topologically massive theories to higher spin fields also. It should also be interesting to extend this method of obtaining dual theory, also to non-abelian ‘$B^F$’ theory [20]. Work along these lines is in progress.

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