Higher Regularity of Weak Limits of Willmore Immersions II

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Abstract

We obtain in arbitrary codimension a removability result on the order of singularity of Willmore surfaces realising the width of Willmore min-max problems on spheres. As a consequence, out of the twelve families of non-planar minimal surfaces in $\mathbb{R}^3$ of total curvature greater than $-12\pi$, only three of them may occur as conformal images of bubbles in Willmore min-max problems.

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1 Introduction

The main result of the paper is a removability result on the second residue (defined in [1], see also the introduction of [18]) of branched Willmore spheres solving min-max problems (see [20]).

**Theorem A.** Let $n \geq 3$ and $\mathcal{A}$ be an admissible family of $W^{2,4}$ immersions of the sphere $S^2$ into $\mathbb{R}^n$. Assume that

$$
\beta_0 = \inf_{A \in \mathcal{A}} \sup W(A) > 0.
$$

Then there exists finitely many true branched compact Willmore spheres $\vec{\Phi}_1, \ldots, \vec{\Phi}_p : S^2 \to \mathbb{R}^n$, and true branched compact Willmore spheres $\vec{\Psi}_1, \ldots, \vec{\Psi}_q : S^2 \to \mathbb{R}^n$ such that

$$
\beta_0 = \sum_{i=1}^{p} W(\vec{\Phi}_i) + \sum_{j=1}^{q} \left( W(\vec{\Psi}_j) - 4\pi \theta_j \right) \in 4\pi \mathbb{N},
$$

where $\theta_j(p_j) \geq 1$ is the multiplicity of $\vec{\Psi}_j$ at some point $p_j \in \vec{\Psi}_j(S^2) \subset \mathbb{R}^n$. Then at every branch point $p$ of $\vec{\Phi}_i, \vec{\Psi}_j$ of multiplicity $\theta_0 = \theta_0(p) \geq 2$, the second residue $r(p)$ satisfies the inequality $r(p) \leq \theta_0 - 2$.

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As a corollary, we deduce as in [18] that most of possible minimal spheres of small absolute total curvature cannot occur. For the convenience of the reader, we recall the table below.

| Minimal surface | Total curvature | Non-zero flux? | Number of ends | Multiplicities of ends | Second residues | Possible min-max Willmore bubble? |
|-----------------|----------------|---------------|---------------|------------------------|----------------|----------------------------------|
| Catenoid        | $-4\pi$        | Yes           | $d = 2$       | $m_1 = 1$              | $r_1 = 0$      | No                               |
|                 |                |               | $m_2 = 1$     | $r_2 = 0$              |                |                                  |
| Enneper surface | $-4\pi$        | No            | $d = 1$       | $m_1 = 3$              | $r_1 = 2$      | No                               |
| Trinoid         | $-8\pi$        | Yes           | $d = 3$       | $m_j = 1$              | $r_j = 0$      | No                               |
|                 |                |               | $1 \leq j \leq 3$ | $1 \leq j \leq 3$ |                |                                  |
| López surface I | $-8\pi$        | No            | $d = 1$       | $m_1 = 5$              | $r_1 = 4$      | No                               |
| López surface II| $-8\pi$        | No            | $d = 1$       | $m_1 = 5$              | $r_1 = 3$      | Yes                              |
| López surface III| $-8\pi$      | Yes           | $d = 2$       | $m_1 = 2$              | $r_1 = 1$      | No                               |
| López surface IV| $-8\pi$        | Yes           | $d = 2$       | $m_1 = 2$              | $r_1 = 1$      | No                               |
| López surface V | $-8\pi$        | Yes           | $d = 2$       | $m_1 = 2$              | $r_1 = 0$      | No                               |
| López surface VI| $-8\pi$        | Yes           | $d = 2$       | $m_1 = 1$              | $r_1 = 0$      | No                               |
| López surface VII| $-8\pi$       | Yes           | $d = 2$       | $m_1 = 2$              | $r_1 = 0$      | No                               |
| López surface VIII| $-8\pi$      | No            | $d = 2$       | $m_1 = 1$              | $r_1 = 0$      | Yes                              |
| López surface IX| $-8\pi$        | No            | $d = 2$       | $m_1 = 1$              | $r_1 = 0$      | Yes                              |
|                 |                |               | $m_2 = 3$     | $r_2 = 1$              |                |                                  |

Figure 1: Geometric properties of complete minimal surface with total curvature greater than $-12\pi$

The main application of this Theorem is to restrict the possibilities of blow-up for Willmore surfaces realising the cost of the sphere eversion ([10]).

**Theorem B.** Let $\iota_+ : S^2 \to \mathbb{R}^3$ be the standard embedding of the round sphere into Euclidean 3-space, let $\iota_- : S^2 \to \mathbb{R}^3$ be the antipodal embedding and let $\text{Imm}(S^2, \mathbb{R}^3)$ be the space of smooth immersions from $S^2$ to $\mathbb{R}^3$. Furthermore, we denote by $\Omega$ the set of paths between the two immersions, defined by

$$\Omega = C^0([0,1], \text{Imm}(S^2, \mathbb{R}^3)) \cap \{ \Phi = \{ \Phi_t \}_{t \in [0,1]}, \bar{\Phi}_0 = \iota_+, \bar{\Phi}_1 = \iota_- \},$$

and we define the cost of the min-max sphere eversion by

$$\beta_0 = \min_{\Phi \in \Omega} \max_{t \in [0,1]} W(\Phi_t) \geq 16\pi.$$

Let $\Phi_1, \ldots, \Phi_p, \bar{\Psi}_1, \ldots, \bar{\Psi}_q : S^2 \to \mathbb{R}^3$ be branched Willmore spheres such that

$$\beta_0 = \sum_{i=1}^{p} W(\Phi_i) + \sum_{j=1}^{q} (W(\bar{\Psi}_j) - 4\pi \theta_j) \geq 16\pi. \quad (1.2)$$

If $\beta_0 = 16\pi$, then we have either:
(1) $p = 1$, $q = 0$ and $\tilde{\Phi}_1$ is the inversion of a Bryant minimal sphere with four embedded planar ends.

(2) $1 \leq p \leq 4$, $q = 1$, $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_p$ are round spheres and $\tilde{\Phi}_1$ is the inversion of a Bryant minimal sphere with four embedded planar ends and $\theta_1 = p$.

Remarks 1. (1) In the second case, the non-compact Willmore surface $\tilde{\chi}_1 : S^2 \to \mathbb{R}^3$ arising in the bubble tree such that $W(\tilde{\chi}_1) = W(\tilde{\Psi}_1) - 4\pi \theta_1$ is obtained by inverting $\tilde{\Psi}_1$ at a point of multiplicity $\theta_1 = p \in \{1, \ldots, 4\}$. It corresponds to a bubbling where a sphere is glued to each non-compact end of $\tilde{\chi}_1$ (there are exactly $p$ of them).

(2) The inequality $\beta_0 \geq 16\pi$ is a direct consequence of Li-Yau inequality (\cite{13}) and of a celebrated result of Max and Banchoff \cite{15} (see also \cite{7}).

Proof. By \cite{16}, \cite{17}, $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_p, \tilde{\Psi}_1, \ldots, \tilde{\Psi}_q : S^2 \to \mathbb{R}^3$ are conformally minimal.

First assume that not all $\tilde{\Phi}_1, \ldots, \tilde{\Phi}_p$ are round spheres.

Let $\bar{n} : S^2 \setminus \{p_1, \ldots, p_d\} \to S^2$ be the associated Gauss map of the dual minimal surface of (say) $\tilde{\Phi}_1$, and $m_1, \ldots, m_d \geq 1$ be the respective multiplicities of the ends $p_1, \ldots, p_d$.

Thanks to the analysis of Theorem B of \cite{18}, we first assume (using the Jorge-Meeks formula \cite{9})

$$0 - 1 + \frac{1}{2} \sum_{j=1}^{d} (m_j + 1) = \deg(\bar{n}) = \frac{1}{4\pi} \int_{S^2} -K_g dvol_g \geq 3. \quad (1.3)$$

Therefore, we have

$$2 \sum_{j=1}^{d} m_j \geq \sum_{j=1}^{d} (m_j + 1) \geq 8.$$

The conformal invariance of the Willmore energy coupled with the Li-Yau (\cite{13}) inequality imply that

$$W(\tilde{\Phi}_1) = 4\pi \sum_{j=1}^{d} m_j \geq 16\pi. \quad (1.4)$$

Now, if

$$\int_{S^2} K_g dvol_g = -8\pi,$$

then (by López’s classification \cite{14} and Figure 1) $\tilde{\Phi}_1$ is the inversion of a minimal sphere with one end of multiplicity 5, which has Willmore energy $W(\tilde{\Phi}_1) = 4\pi \times 5 = 20\pi$, or of a minimal sphere with two ends, one of multiplicity 3 and the other planar (multiplicity 1 with 0 logarithmic growth), with Willmore energy $W(\tilde{\Phi}_1) = 4\pi \times (3 + 1) = 16\pi$. Therefore, in all cases, we have

$$W(\tilde{\Phi}_1) \geq 16\pi.$$

In particular, $\beta_0 = 16\pi$ (or $\beta_0 < 32\pi$ with a minimal bubbling) always implies that $p = 1$ in (1.2), and that the bubbling is minimal, i.e. $W(\tilde{\Psi}_j) = 4\pi \theta_j$ for all $1 \leq j \leq q$.

Now let $\tilde{\chi}_1, \ldots, \tilde{\chi}_q : S^2 \to \mathbb{R}^3 \cup \{\infty\}$ be the dual minimal surfaces of $\tilde{\Psi}_1, \ldots, \tilde{\Psi}_q$, and let also $\tilde{\phi} : S^2 \setminus \{p_1, \ldots, p_d\} \to \mathbb{R}^3$ be the dual minimal surface of $\tilde{\phi}_1$. If $m_1, \ldots, m_d \geq 1$ are the multiplicities of the ends $p_1, \ldots, p_d$ of $\tilde{\phi}_1$, then by the Li-Yau inequality, we have (\cite{13})

$$W(\tilde{\phi}_1) = 4\pi \sum_{j=1}^{d} m_j = 16\pi, \quad \int_{S^2} K_g dvol_g = 4\pi + 2\pi \sum_{j=1}^{d} (m_j - 1).$$

In particular, $d \leq 4$, and by the Jorge-Meeks formula (\cite{9}), $d$ must be even.
**Case 1.** $d = 4$. Then by [3], $\Phi_1$ is the inversion of Bryant’s minimal surface with four embedded planar ends, and $q = 0$.

**Case 2.** $d = 2$. Then by [18], $\Phi_1$ is the inversion of López minimal surface with one planar end and one end of multiplicity 3. Therefore, we deduce that

$$\int_{S^2} K_{g_{\Phi_1}} \, d\text{vol}_{g_{\Phi_1}} = 4\pi + 2\pi(1 - 1) + 2\pi(3 - 1) = 8\pi,$$

and the quantization of Gauss curvature shows that

$$\sum_{j=1}^q \int_{S^2} K_{g_{\Phi_j}} \, d\text{vol}_{g_{\Phi_j}} = -4\pi,$$

but this is excluded by [18]. Indeed, this would correspond to the Enneper surface, which has one end of multiplicity 3 and second residue $r = 2 = 3 - 1$ forbidden by Theorem A.

**Now assume that $\Phi_1, \ldots, \Phi_p$ are all round spheres.**

Then $1 \leq p \leq 4$, and

$$\sum_{i=1}^p \int_{S^2} K_{g_{\Phi_i}} \, d\text{vol}_{g_{\Phi_i}} = 4\pi p.$$

**Case 1.** $p = 4$. Then the bubbling is minimal, and by the quantization of the Gauss curvature ([2])

$$\sum_{j=1}^q \int_{S^2} K_{g_{\Phi_j}} \, d\text{vol}_{g_{\Phi_j}} = -12\pi.$$

Therefore, we have $1 \leq q \leq 3$ (counting only non-planar bubbles), and as by [18] (see Figure 1)

$$\sum_{j=1}^q \int_{S^2} K_{g_{\Phi_j}} \, d\text{vol}_{g_{\Phi_j}} \geq -8\pi,$$

we deduce that $q = 1$ and

$$\int_{S^2} K_{g_{\Phi_1}} \, d\text{vol}_{g_{\Phi_1}} = -12\pi.$$

Now, by the Jorge-Meeks formula, if $\chi_1$ has $d$ ends of multiplicities $m_1, \ldots, m_d \geq 1$, we deduce that

$$\int_{S^2} K_{g_{\chi_1}} \, d\text{vol}_{g_{\chi_1}} = 4\pi \left( 0 - 1 + \frac{1}{2} \sum_{j=1}^d (m_j + 1) \right).$$

In particular, we deduce that

$$8 = \sum_{j=1}^d (m_j + 1) \geq 2d,$$

or $d \leq 4$. Furthermore, as there are four bubbles of multiplicity 1, we deduce that $\chi_1$ must have at least four ends, so all of them are planar, with exactly says that $\chi_1$ is a minimal sphere of Bryant with four planar ends.

**Case 2.** $p = 3$. Then we have

$$\sum_{j=1}^q W(\chi_j) = 4\pi, \quad \sum_{j=1}^q \int_{S^2} K_{g_{\chi_j}} \, d\text{vol}_{g_{\chi_j}} = -8\pi.$$

(1.5)
Here the $\vec{\chi}_j$ are not necessarily minimal and are the non-compact bubbles occurring in the min-max process. Here, without loss of generality, we can assume that $W(\vec{\chi}_1) = 4\pi$. As $\vec{\chi}_1$ is not compact, we deduce that its dual minimal surface is non-planar. Furthermore, we deduce that $\vec{\chi}_2, \cdots, \vec{\chi}_q$ are minimal. In particular, for all $2 \leq j \leq q$, either $\vec{\chi}_j$ is a plane, or

$$\int_{S^2} K_{\vec{\chi}_j} d\text{vol}_{\vec{\chi}_j} \leq -8\pi. \quad (1.6)$$

If $\vec{\xi}_1$ is the dual minimal surface of $\vec{\chi}_1$, we have by the conformal invariance of the Willmore energy

$$\int_{S^2} K_{\vec{g}_{\vec{\xi}_1}} d\text{vol}_{\vec{g}_{\vec{\xi}_1}} = \int_{S^2} K_{\vec{g}_{\vec{\xi}_1}} d\text{vol}_{\vec{g}_{\vec{\xi}_1}} + W(\vec{\chi}_1) \leq -8\pi + 4\pi = -4\pi. \quad (1.7)$$

Therefore, comparing (1.5), (1.6) and (1.7), we deduce that $\vec{\chi}_2, \cdots, \vec{\chi}_q$ are all planes, and

$$\int_{S^2} K_{\vec{g}_{\vec{\xi}_1}} d\text{vol}_{\vec{g}_{\vec{\xi}_1}} = -12\pi.$$ 

Furthermore, as $\vec{\Phi}_1$, $\vec{\Phi}_2$ and $\vec{\Phi}_3$ are round spheres, we deduce that all ends of $\vec{\xi}_1$ must be planar. Therefore, we deduce that $\vec{\chi}_1$ is the inversion of the compactification of Bryant surface with four planar ends at a point of multiplicity 3 (if any).

**Case 3.** $p = 2$. Then we have

$$\sum_{j=1}^q W(\vec{\chi}_j) = 8\pi$$

$$\sum_{j=1}^q \int_{S^2} K_{g_{\vec{\xi}_j}} d\text{vol}_{g_{\vec{\xi}_j}} = -4\pi. \quad (1.8)$$

We have either $W(\vec{\chi}_1) = 8\pi$ and $\vec{\chi}_2, \cdots, \vec{\chi}_q$ are minimal, or $W(\vec{\chi}_1) = W(\vec{\chi}_2) = 4\pi$. Notice that the first possibility shows as the dual minimal surface of $\vec{\chi}_1$ is not planar that (denote by $\vec{\xi}_1$ its dual minimal surface)

$$\int_{S^2} K_{g_{\vec{\xi}_1}} d\text{vol}_{g_{\vec{\xi}_1}} = \int_{S^2} K_{g_{\vec{\xi}_1}} d\text{vol}_{g_{\vec{\xi}_1}} + W(\vec{\chi}_1) \leq -8\pi + 8\pi \leq 0$$

while in the second case,

$$\int_{S^2} K_{g_{\vec{\xi}_1}} d\text{vol}_{g_{\vec{\xi}_1}} \leq -4\pi, \quad \int_{S^2} K_{g_{\vec{\xi}_2}} d\text{vol}_{g_{\vec{\xi}_2}} \leq -4\pi$$

so the second case is excluded by (1.6) and (1.8). Therefore, we deduce as previously that $\vec{\xi}_1$ is the inversion of the compactification of Bryant surface with four planar ends at a point of multiplicity 2 (if any).

**Case 3.** $p = 1$. Then we have

$$\sum_{j=1}^q W(\vec{\chi}_j) = 12\pi$$

$$\sum_{j=1}^q \int_{S^2} K_{g_{\vec{\xi}_j}} d\text{vol}_{g_{\vec{\xi}_j}} = 0. \quad (1.9)$$

If $\vec{\chi}_j$ is not minimal and $W(\vec{\chi}_j) = 4\pi m$, where $1 \leq m \leq 3$, then we have with the previous notations

$$\int_{S^2} K_{g_{\vec{\xi}_j}} d\text{vol}_{g_{\vec{\xi}_j}} = \int_{S^2} K_{g_{\vec{\xi}_j}} d\text{vol}_{g_{\vec{\xi}_j}} + W(\vec{\chi}_j) \leq -8\pi + 4\pi m = 4\pi(m - 2) \leq 4\pi,$$

so (1.6) implies that the minimal $\vec{\chi}_j$ must be planes.
Sub-case 1. $W(\vec{\chi}_1) = 12\pi$. Then we get as previously
\[ \int_{S^2} K_{\vec{\chi}_1} d\text{vol}_{\vec{\chi}_1} = -12\pi, \]
and $\vec{\chi}_1$ is the inversion of the compactification of Bryant surface with four planar ends at a point of multiplicity 1.

Sub-case 2. $W(\vec{\chi}_1) = 8\pi$, $W(\vec{\chi}_2) = 4\pi$, then we get
\[ \int_{S^2} K_{\vec{\chi}_1} d\text{vol}_{\vec{\chi}_1} \leq 0, \quad \int_{S^2} K_{\vec{\chi}_2} d\text{vol}_{\vec{\chi}_2} \leq -4\pi, \]
contradicting (1.9).

Sub-case 3. $W(\vec{\chi}_1) = W(\vec{\chi}_2) = W(\vec{\chi}_3) = 4\pi$, so for all $1 \leq j \leq 3$, we have
\[ \int_{S^2} K_{\vec{\chi}_j} d\text{vol}_{\vec{\chi}_j} \leq -4\pi, \]
contradicting once more (1.9).

This analysis concludes the proof of the Theorem.

Remarks 2. (1) Without Theorem A, one could have had $p = q = 1$ in (1.2), where $\vec{\Phi}_1$ is the inversion of a minimal surface of López with one planar end and one end of multiplicity 3 (either the López surface VIII or IX in Figure 1), and $\vec{\Psi}_1$ is the inversion of Enneper surface, denoted by $\vec{\chi}_1$. Indeed, as
\[ W(\vec{\Phi}_1) = 4\pi(1 + 3) = 16\pi, \quad \int_{S^2} K_{\vec{\Phi}_1} d\text{vol}_{\vec{\Phi}_1} = 4\pi + 2\pi(1 - 1) + 2\pi(3 - 1) = 8\pi, \]
\[ \int_{S^2} K_{\vec{\chi}_1} d\text{vol}_{\vec{\chi}_1} = -4\pi \]
and $\vec{\Psi}_1$ is minimal, we indeed have
\[ W(\vec{\Phi}_1) + (W(\vec{\Psi}_1) - 12\pi) = W(\vec{\Phi}_1) = W(\vec{\Phi}_1) = 16\pi \]
\[ \int_{S^2} K_{\vec{\Phi}_1} d\text{vol}_{\vec{\Phi}_1} + \int_{S^2} K_{\vec{\chi}_1} d\text{vol}_{\vec{\chi}_1} = 4\pi, \]
so with respect to the quantization of energy this would have been a legitimate candidate.

(2) Recall by the non-existence of minimal surfaces with 5 planar ends ([4]) that $\beta_0 = 20\pi$ implies that a non-trivial bubbling occurs. A possible minimal bubbling is given by the López minimal surface with one end of multiplicity 5 and its inversion (see the López surface II in Figure 1).

(3) We do not expect by the $\varepsilon$-regularity depending only on the trace-free second fundamental form of Kuwert-Schätzle ([11]) to have a limiting macroscopic surface $\vec{\Phi}_1$ to be a round sphere in case of bubbling. This would show that the only candidate with energy $\beta_0 = 16\pi$ is the compactification of Bryant’s sphere with four planar ends. However, we cannot exclude for now this possibility.

2 The viscosity method for the Willmore energy

We first introduce for all weak immersion $\vec{\Phi} : S^2 \rightarrow \mathbb{R}^n$ of finite total curvature the associated metric $g = \vec{\Phi}^* g_{\mathbb{R}^n}$ on $S^2$. By the uniformisation theorem, there exists a function $\omega : S^2 \rightarrow \mathbb{R}$ such that
\[ g = e^{2\omega} g_0, \]
where \( g_0 \) is a metric of constant Gauss curvature \( 4\pi \) and unit volume on \( S^2 \). Furthermore, in all fixed chart \( \varphi : B(0,1) \to S^2 \), we define \( \mu : B(0,1) \to \mathbb{R} \) such that
\[
\lambda = \alpha + \mu,
\]
where in the given chart
\[
g = e^{2\lambda} |dz|^2.
\]

For technical reasons, we will have to make a peculiar choice of \( \omega \) (see [20], Definition III.2).

**Definition 2.1.** Under the preceding notations, we say that a choice \((\omega, \varphi)\) of a map \( \omega : S^2 \to \mathbb{R} \) and of a diffeomorphism \( \varphi : S^2 \to S^2 \) is an Aubin gauge if
\[
\varphi^* g_0 = \frac{1}{4\pi} g_{S^2} \quad \text{and} \quad \int_{S^2} x_j e^{2\omega \varphi(x)} d\text{vol}_{g_{S^2}}(x) = 0 \quad \text{for all} \quad j = 1, 2, 3,
\]
where \( g_{S^2} \) is the standard metric on \( S^2 \).

We also recall that the limiting maps arise from a sequence of critical point of the following regularisation of the Willmore energy (see [20] for more details):
\[
W_\sigma(\vec{\Phi}) = W(\vec{\Phi}) + \sigma^2 \int_{S^2} \left( 1 + |\vec{H}|^2 \right) d\text{vol}_g
\]
\[
+ \frac{1}{\log \left( \frac{1}{2} \right)} \left( \frac{1}{2} \int_{S^2} |d\omega|^2 d\text{vol}_g + 4\pi \int_{S^2} \omega e^{-2\omega} d\text{vol}_g - 2\pi \log \int_{S^2} d\text{vol}_g \right)
\]
where \( \omega : S^2 \to \mathbb{R} \) is as above.

We need a refinement of a standard estimate (see [8], 3.3.6).

**Lemma 2.2.** Let \( \Omega \) be a open subset of \( \mathbb{R}^2 \) whose boundary is a finite union of \( C^1 \) Jordan curves. Let \( f \in L^1(\Omega) \) and let \( u \) be the solution of
\[
\begin{align*}
\Delta u &= f \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
(2.1)

Then \( \nabla u \in L^{2,\infty}(\Omega) \), and
\[
\| \nabla u \|_{L^{2,\infty}(\Omega)} \leq 3 \sqrt{\frac{2}{\pi}} \| f \|_{L^1(\Omega)}.
\]

**Remark 2.3.** We need an estimate independent of the domain for a sequence of annuli of conformal class diverging to \( \infty \), but the argument applies to a general domain (although some regularity conditions seem to be necessary).

**Proof.** First assume that \( f \in C^{0,\alpha}(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \). Then by Schauder theory, \( u \in C^{2,\alpha}(\overline{\Omega}) \), and by Stokes theorem ([6] 1.2.1), we find as \( u = 0 \) on \( \partial \Omega \) that for all \( z \in \Omega \)
\[
\partial_z u(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{\partial_{\zeta} (\partial_{\zeta} u(\zeta))}{\zeta - z} d\zeta \wedge d\overline{\zeta}.
\]
(2.2)

As \( \Delta u = 4 \partial_{\zeta}^2 u \) and \( |d\zeta|^2 = \frac{d\zeta \wedge d\overline{\zeta}}{2i} \), the pointwise estimate (2.2) implies that
\[
\partial_z u(z) = -\frac{1}{4\pi} \int_{\Omega} \frac{\Delta u(\zeta)}{\zeta - z} |d\zeta|^2 = -\frac{1}{4\pi} \int_{\Omega} \frac{f(\zeta)}{\zeta - z} |d\zeta|^2.
\]
(2.3)

Now, define \( \overline{f} \in L^1(\mathbb{R}^2) \) by
\[
\overline{f}(z) = \begin{cases} f(z) & \text{for all } z \in \Omega \\ 0 & \text{for all } z \in \mathbb{R}^2 \setminus \Omega. \end{cases}
\]
and $U : \mathbb{R}^2 \to \mathbb{C}$ by

$$U(z) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \overline{T(\zeta)} \frac{d\zeta}{z-\zeta} = -\frac{1}{4\pi} \left( \left( \zeta \mapsto \frac{1}{z} \right) \ast \overline{T} \right)(z),$$

(2.4)

where $\ast$ indicates the convolution on $\mathbb{R}^2$. Now, recall that for all $1 \leq p < \infty$ and $g \in L^p(\mathbb{R}^2, \mathbb{C})$, we have

$$\|T \ast g\|_{L^p(\mathbb{R}^2)} \leq \|T\|_{L^1(\mathbb{R}^2)} \|g\|_{L^p(\mathbb{R}^2)}.$$ Interpolating between $L^1$ and $L^p$ for all $p > 2$ shows by the Stein-Weiss interpolation theorem ([8] 3.3.3) that for all $g \in L^{2,\infty}(\mathbb{R}^2)$,

$$\|T \ast g\|_{L^{2,\infty}(\mathbb{R}^2)} \leq \sqrt{2} \left( \frac{2 \times 1}{2 - 1} + \frac{p \cdot 1}{p - 2} \right) \|T\|_{L^1(\mathbb{R}^2)} \|g\|_{L^{2,\infty}(\mathbb{R}^2)} = \sqrt{2} \left( 2 + \frac{p}{p - 2} \right) \|T\|_{L^1(\mathbb{R}^2)} \|g\|_{L^{2,\infty}(\mathbb{R}^2)}.$$

Taking the infimum in $p > 2$ (that is, $p \to \infty$) shows that for all $g \in L^{2,\infty}(\mathbb{R}^2)$,

$$\|T \ast g\|_{L^{2,\infty}(\mathbb{R}^2)} \leq 3\sqrt{2} \|T\|_{L^1(\mathbb{R}^2)} \|g\|_{L^{2,\infty}(\mathbb{R}^2)}.$$ (2.5)

Therefore, we deduce from (2.3) and (2.5) that

$$\|U\|_{L^{2,\infty}(\mathbb{R}^2)} \leq \frac{3\sqrt{2}}{4\pi} \|T\|_{L^1(\mathbb{R}^2)} \left\| \frac{1}{\cdot} \right\|_{L^{2,\infty}(\mathbb{R}^2)} = \frac{3}{\sqrt{2\pi}} \|f\|_{L^1(\Omega)}.$$

Now, as $U = \partial_2 u$ on $\Omega$ and $2|\partial_2 u| = |\nabla u|$, we finally deduce that

$$\|\nabla u\|_{L^{2,\infty}(\Omega)} \leq \frac{3\sqrt{2}}{\pi} \|f\|_{L^1(\Omega)}.$$ (2.6)

In the general case $f \in L^1(\Omega)$, by density of $C_c^\infty(\Omega)$ in $L^1(\Omega)$, let $\{f_k\}_{k \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that

$$\|f_k - f\|_{L^1(\Omega)} \to 0.$$ (2.7)

Then $u_k \in C_c^\infty(\Omega)$ (defined to be the solution of the system (2.1) with $f$ replaced by $f_k$ and the same boundary conditions) so for all $k \in \mathbb{N}$, $\nabla u_k \in L^{2,\infty}(\Omega)$ and

$$\|\nabla u_k\|_{L^{2,\infty}(\Omega)} \leq \frac{3\sqrt{2}}{\pi} \|f_k\|_{L^1(\Omega)}.$$ (2.8)

As $\left\{\|f_k\|_{L^1(\Omega)}\right\}_{k \in \mathbb{N}}$ is bounded, up to a subsequence $u_k \rightharpoonup u_\infty$ in the weak topology of $W^{1,2,\infty}(\Omega)$. Therefore, (2.7) and (2.8) yield

$$\|\nabla u_\infty\|_{L^{2,\infty}(\Omega)} \leq \liminf_{k \to \infty} \|\nabla u_k\|_{L^{2,\infty}(\Omega)} \leq \frac{3\sqrt{2}}{\pi} \|f\|_{L^1(\Omega)}.$$ Furthermore, as $f_k \to f$ in $L^1(\Omega)$, we have $\Delta u_\infty = f$ in $\mathcal{D}'(\Omega)$, so we deduce that $u_\infty = u$ and this concludes the proof of the lemma. \hfill \Box

Finally, recall the following Lemma from [2] (see also [5]).

**Lemma 2.4.** Let $\Omega$ be a Lipschitz bounded open subset of $\mathbb{R}^2$, $1 < p < \infty$ and $1 \leq q \leq \infty$, and $(a, b) \in W^{1,(p,q)}(B(0,1)) \times W^{1,(2,\infty)}(B(0,1))$. Let $u : B(0,1) \to \mathbb{R}$ be the solution of

$$\begin{cases}
\Delta u = \nabla a \cdot \nabla b & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

Then there exists a constant $C_{p,q}(\Omega) > 0$ such that

$$\|\nabla u\|_{L^{p,q}(\Omega)} \leq C_{p,q}(\Omega) \|\nabla a\|_{L^{p,q}(\Omega)} \|\nabla b\|_{L^{2,\infty}(\Omega)}.$$**

**Remark 2.5.** Notice that by scaling invariance, we have for all $R > 0$ if $\Omega_R = B(0,R)$

$$\|\nabla u\|_{L^{2,1}(B(0,R))} \leq C_{2,1}(B(0,1)) \|\nabla a\|_{L^{2,1}(B(0,R))} \|\nabla b\|_{L^{2,\infty}(B(0,R))}.$$
3 Improved energy quantization in the viscosity method

**Theorem 3.1.** Under the hypothesis of Theorem A, and by [20] let \( \{\sigma_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) be such that \( \sigma_k \to 0 \) and let \( \{\Phi_k\}_{k \in \mathbb{N}} : S^2 \to \mathbb{R}^n \) be a sequence of critical points associated to \( W_{\sigma_k} \) such that

\[
\begin{align*}
W_{\sigma_k}(\Phi_k) &= \beta(\sigma_k) \to \beta_0 \\
W_{\sigma_k}(\Phi_k) - W(\Phi_k) &= o \left( \frac{1}{\log \left( \frac{1}{\sigma_k} \right) \log \left( \frac{1}{\sigma_k^e} \right)} \right).
\end{align*}
\]

(3.1)

Let \( \{R_k\}_{k \in \mathbb{N}}, \{r_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) be such that

\[
\lim_{k \to \infty} \frac{R_k}{r_k} = 0, \quad \limsup_{k \to \infty} R_k < \infty,
\]

and for all \( 0 < \alpha < 1 \) and \( k \in \mathbb{N} \), let \( \Omega_k(\alpha) = B_{\alpha R_k} \setminus B_{\alpha^{-1} r_k}(0) \) be a neck region, i.e. such that

\[
\lim_{\alpha \to 0} \lim_{k \to \infty} \sup_{2 \alpha^{-1} r_k < s < \alpha R_k/2} \int_{B_{2s} \setminus B_{s}(0)} |\nabla \tilde{n}_k|^2 \, dx = 0.
\]

Then we have

\[
\lim_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \tilde{n}_k\|_{L^2,1(\Omega_k(\alpha))} = 0.
\]

**Proof.** As in [20], we give the proof in the special case \( n = 3 \). By Theorem 3.1 of [18], this is not restrictive.

\[
\Lambda = \sup_{k \in \mathbb{N}} \left( \|\nabla \lambda_k\|_{L^\infty(B(0,1))} + \int_{B(0,1)} |\nabla \tilde{n}_k|^2 \, dx \right) < \infty
\]

and

\[
\ell(\sigma_k) = \frac{1}{\log \left( \frac{1}{\sigma_k} \right)}, \quad \tilde{\ell}(\sigma_k) = \frac{1}{\log \log \left( \frac{1}{\sigma_k} \right)}.
\]

Furthermore, the entropy condition (3.1) and the improved Onofri inequality show (see [2] III.2)

\[
\begin{align*}
\frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \|\omega_k\|_{L^\infty(B(0,1))} &= o \left( \frac{1}{\log \log \left( \frac{1}{\sigma_k} \right)} \right) \\
\frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \int_{S^2} |\omega_k|_{g_k}^2 \, d\text{vol}_{g_k} &= o \left( \frac{1}{\log \log \left( \frac{1}{\sigma_k} \right)} \right) \\
\frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} \int_{S^2} |\omega_k|_{g_k}^2 \, d\text{vol}_{g_k} + 4\pi \int_{S^2} \omega_k e^{-2\omega_k} \, d\text{vol}_{g_k} - 2\pi \log \int_{S^2} d\text{vol}_{g_k} \right) &= o \left( \frac{1}{\log \log \left( \frac{1}{\sigma_k} \right)} \right).
\end{align*}
\]

(3.2)

Thanks to [20], we already have

\[
\lim_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \tilde{n}_k\|_{L^2(\Omega_k(\alpha))} = 0.
\]

Therefore, as in Lemma IV.1 in [2] (and using the same argument as in Lemma 3.3 of [18]), there exists a controlled extension \( \tilde{n}_k : B(0, \alpha R_k) \to \mathcal{G}_{n-2}(\mathbb{R}^n) \) such that \( \tilde{n}_k = \tilde{n}_k \) on \( \Omega_k(\alpha) = B(0, \alpha R_k) \setminus \mathcal{B}(0, \alpha^{-1} r_k) \) and

\[
\|\nabla \tilde{n}_k\|_{L^2(B(0, \alpha R_k))} \leq C_0(n) \|\nabla \tilde{n}_k\|_{L^2(\Omega_k(\alpha))}.
\]
By scaling invariance and the inequality of Lemma 2.4, we deduce by (3.7) that for some universal 
Then following [20], we have 
Furthermore, we have by Lemma 2.4 and scaling invariance 

Now, by [20], let \( \bar{L}_k : B(0, 1) \to \mathbb{R}^3 \) be such that 
\[
d\bar{L}_k = d\left( \tilde{H}_k + 2\sigma_k^2 (1 + |\tilde{H}_k|^2) \tilde{H}_k \right) - 2\left( 1 + 2\sigma_k^2 (1 + |\tilde{H}_k|^2) \right) H_k \ast d\tilde{n}_k \]
\[
+ \left( - |\tilde{H}_k|^2 + \sigma_k^2 (1 + |\tilde{H}_k|^2) \right) \frac{1}{\log \left( \frac{1}{|\tilde{n}_k|} \right)} \left( \frac{1}{2} |d\omega_k|^2 - 2\pi \omega_k e^{-2\omega_k} + \frac{2\pi}{\text{Area}(\tilde{\Phi}_k(S^2))} \right) \ast d\tilde{\Phi}_k \]
\[
- \frac{1}{\log \left( \frac{1}{|\tilde{n}_k|} \right)} \langle d\tilde{\Phi}_k, d\omega_k \rangle_{\mathbb{R}^3} \ast d\omega_k + \frac{1}{\log \left( \frac{1}{|\tilde{n}_k|} \right)} \tilde{\Phi}_k \ast d\omega_k. \quad (3.4)
\]

Then following [20], we have 
\[
e^{\lambda_k(z)|\bar{L}_k(z)|} \leq \left( C_1(n) (1 + \lambda) e^{C_1(n)\lambda} \| \nabla \tilde{n}_k \|_{L^2(\Omega_k(\alpha))} + \tilde{I}(\sigma_k) \right) \frac{1}{|z|} \quad \text{for all } z \in \Omega_k(\alpha/2),
\]
so that 
\[
\left\| e^{\lambda_k |\bar{L}_k|} \right\|_{L^2(\Omega_k(\alpha/2))} \leq 2\sqrt{\pi} \left( C_1(n) (1 + \lambda) e^{C_1(n)\lambda} \| \nabla \tilde{n}_k \|_{L^2(\Omega_k(\alpha))} + \tilde{I}(\sigma_k) \right). \]

Now let \( Y_k : B(0, \alpha R_k) \to \mathbb{R} \) (see [20], VI.21) be the solution of 
\[
\begin{cases}
\Delta Y_k = -4e^{2\lambda_k} \sigma_k^2 (1 - H_k^2) - 2l(\sigma_k) K_{90} \omega_k e^{2\mu_k} + 8\pi l(\sigma_k) e^{2\lambda_k} \text{Area}(\tilde{\Phi}(S^2))^{-1} & \text{in } B(0, \alpha R_k) \\
Y_k = 0 & \text{on } \partial B(0, \alpha R_k).
\end{cases} \quad (3.5)
\]

Then we have (recall that \( K_{90} = 4\pi \) by the chosen normalisation in Definition 2.1) 
\[
\| \Delta Y_k \|_{L^1(B(0, \alpha R_k))} \leq 4\sigma_k^2 \int_{B(0, \alpha R_k)} (1 + H_k^2) \, d\mathcal{L} + 8\pi l(\sigma_k) \| \omega_k \|_{L^\infty(B(0, \alpha R_k))} \int_{B(0, \alpha R_k)} e^{2\mu_k} \, dx \\
+ 8\pi l(\sigma_k) \frac{\text{Area}(\tilde{\Phi}_k(B(0, \alpha R_k)))}{\text{Area}(\tilde{\Phi}_k(S^2))} = o(\tilde{I}(\sigma_k)). \quad (3.6)
\]

Therefore, Lemma 2.2 implies by (3.6) that 
\[
\| \nabla Y_k \|_{L^\infty(B(0, \alpha R_k))} \leq 3 \left\| \Delta Y_k \right\|_{L^\infty(B(0, \alpha R_k))} = o(\tilde{I}(\sigma_k)) \leq \tilde{I}(\sigma_k). \quad (3.7)
\]
for \( k \) large enough. Now, let \( \tilde{v}_k : B(0, \alpha R_k) \to \mathbb{R}^3 \) be the solution of 
\[
\begin{cases}
\Delta \tilde{v}_k = \nabla \tilde{n}_k \cdot \nabla^2 Y_k & \text{in } B(0, \alpha R_k) \\
\tilde{v}_k = 0 & \text{on } \partial B(0, \alpha R_k).
\end{cases} \quad (3.8)
\]
By scaling invariance and the inequality of Lemma 2.4, we deduce by (3.7) that for some universal constant \( C_2 > 0 \) 
\[
\left\| \nabla \tilde{v}_k \right\|_{L^2(\Omega_k(\alpha))} \leq C_2 \left\| \nabla \tilde{n}_k \right\|_{L^2(B(0, \alpha R_k))} \left\| \nabla Y_k \right\|_{L^\infty(B(0, \alpha R_k))} \leq C_2 \left( C_0(n) \left\| \nabla \tilde{n}_k \right\|_{L^2(B(0, \alpha R_k))} \left\| \nabla Y_k \right\|_{L^\infty(B(0, \alpha R_k))} \leq \tilde{I}(\sigma_k) \right\| \nabla \tilde{n}_k \|_{L^2(\Omega_k(\alpha))}. \quad (3.9)
\]
Furthermore, we have by Lemma 2.4 and scaling invariance 
\[
\left\| \nabla \tilde{v}_k \right\|_{L^2(B(0, \alpha R_k))} \leq C_3 \left\| \nabla \tilde{n}_k \right\|_{L^2(B(0, \alpha R_k))} \left\| \nabla Y_k \right\|_{L^\infty(B(0, \alpha R_k))} \leq C_3 C_0(n) \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} \tilde{I}(\sigma_k) \right) \}
\]
As previously, the improved Wente implies that
\[
\tilde{l}(\sigma_k) \|\nabla\tilde{n}_k\|_{L^2(\Omega_k(\alpha))} \leq (3.10)
\]
Now, recall that the Codazzi identity ([20], III.58) implies that
\[
\text{div} \left( e^{-2\lambda_k} \sum_{j=1}^{2} L_{2,j} \partial_{x_j} \Phi_k, -e^{-2\lambda_k} \sum_{j=1}^{2} \Pi_{1,j} \partial_{x_j} \Phi_k \right) = 0 \quad \text{in } B(0, \alpha R_k)
\]
(3.11)
Therefore, by the Poincaré Lemma, there exists \( \tilde{D}_k : B(0, \alpha R_k) \to \mathbb{R}^3 \) such that
\[
\nabla \tilde{D}_k = \left( e^{-2\lambda_k} \sum_{j=1}^{2} \Pi_{1,j} \partial_{x_j} \Phi_k, e^{-2\lambda_k} \sum_{j=1}^{2} \Pi_{2,j} \partial_{x_j} \Phi_k \right).
\]
Notice that we have the trivial estimate
\[
\|\nabla \tilde{D}_k\|_{L^2(B(0, \alpha R_k))} \leq 2 \|\nabla\tilde{n}_k\|_{L^2(B(0, \alpha R_k))} \leq 2\sqrt{\Lambda}.
\]
(3.12)
Furthermore,
\[
l(\sigma_k) \|\nabla \tilde{D}_k\|_{L^{2,1}(\Omega_k(\alpha))} \leq 2l(\sigma_k) \|\nabla\tilde{n}_k\|_{L^{2,1}(\Omega_k(\alpha))}.
\]
(3.13)
Now, let \( \tilde{E}_k : B(0, \alpha R_k) \to \mathbb{R}^3 \) be the solution of
\[
\begin{cases}
\Delta \tilde{E}_k = 2 \nabla l(\sigma_k) \omega_k \cdot \nabla \tilde{D}_k & \text{in } B(0, \alpha R_k), \\
\tilde{E}_k = 0 & \text{on } \partial B(0, \alpha R_k).
\end{cases}
\]
(3.14)
The improved Wente estimate, the scaling invariance and the estimates (3.1) and (3.12) imply that
\[
\|\nabla \tilde{E}_k\|_{L^{2,1}(B(0, \alpha R_k))} \leq C_0 l(\sigma_k) \|\nabla \omega_k\|_{L^2(B(0, \alpha R_k))} \|\nabla \tilde{D}_k\|_{L^{2,1}(B(0, \alpha R_k))} \leq 4C_0 \sqrt{\Lambda} \sigma_k \|\nabla \tilde{n}_k\|_{L^2(B(0, \alpha R_k))} \leq \sqrt{I(\sigma_k)}.
\]
(3.15)
Now, let \( \tilde{F}_k : B(0, \alpha R_k) \to \mathbb{R}^3 \) be such that
\[
2\omega_k l(\sigma_k) \nabla \tilde{D}_k = \nabla \tilde{F}_k + \nabla \tilde{E}_k.
\]
Combining (3.13), (3.15), and recalling that \( l(\sigma_k) \|\omega_k\|_{L^\infty(B(0, \alpha R_k))} = o(\tilde{l}(\sigma_k)) \) (by (3.1)), we deduce that
\[
\|\tilde{F}_k\|_{L^{2,1}(\Omega_k(\alpha))} \leq 2l(\sigma_k) \|\omega_k\|_{L^\infty(\Omega_k(\alpha))} \|\nabla \tilde{D}_k\|_{L^{2,1}(\Omega_k(\alpha))} + \|\nabla \tilde{E}_k\|_{L^{2,1}(B(0, \alpha R_k))} \leq \tilde{l}(\sigma_k) \|\nabla\tilde{n}_k\|_{L^{2,1}(\Omega_k(\alpha))} + \sqrt{I(\sigma_k)}.
\]
(3.16)
Finally, let \( \tilde{w}_k : B(0, \alpha R_k) \to \mathbb{R}^3 \) be the solution of
\[
\begin{cases}
\Delta \tilde{w}_k = \nabla \tilde{n}_k \cdot \nabla \left( \tilde{v}_k - \tilde{E}_k \right) & \text{in } B(0, \alpha R_k), \\
\tilde{w}_k = 0 & \text{on } \partial B(0, \alpha R_k).
\end{cases}
\]
As previously, the improved Wente implies that
\[
\|\nabla \tilde{w}_k\|_{L^{2,1}(B(0, \alpha R_k))} \leq C_0 \|\nabla \tilde{n}_k\|_{L^2(B(0, \alpha R_k))} \|\nabla (\tilde{v}_k - \tilde{E}_k)\|_{L^2(B(0, \alpha R_k))} \leq C_0 \|\nabla \tilde{n}_k\|_{L^2(\Omega_k(\alpha))} \left( \|\nabla \tilde{v}_k\|_{L^2(B(0, \alpha R_k))} + \|\nabla \tilde{E}_k\|_{L^2(B(0, \alpha R_k))} \right)
\]
for \( k \) large enough. Finally, if \( \tilde{Z}_k : \Omega_k(\alpha) \to \mathbb{R}^3 \) satisfies

\[
\nabla \cdot \tilde{Z}_k = \tilde{n}_k \times \nabla \Big( \tilde{\sigma}_k - \tilde{E}_k \Big) - \nabla \tilde{w}_k,
\]

the estimates (3.9), (3.15), (3.17) show that (as \( \tilde{n}_k = \tilde{n}_k \) on \( \Omega_k(\alpha) \))

\[
\left\| \nabla \tilde{Z}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} \leq \tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} + \sqrt{\tilde{l}(\sigma_k)} + \tilde{l}(\sigma_k). \tag{3.18}
\]

Finally, following constants and using the controlled extension \( \tilde{n}_k \) of \( \tilde{n}_k \), we deduce as in [20] (see (VI.75)) that

\[
\left\| 2 \left( 1 + 2\sigma_k^2 (1 + H_k^2) - l(\sigma_k)\omega_k \right) e^{\lambda_k} \tilde{H}_k + \left( \nabla \tilde{n}_k + \nabla \left( \tilde{F}_k + \tilde{Z}_k \right) \right) \times \nabla \tilde{\Phi}_k e^{-\lambda_k} + l(\sigma_k) \nabla \tilde{\Phi}_k \cdot \nabla \tilde{\Phi}_k e^{-\lambda_k} \right\|_{L^{2,1}(\Omega_k(\alpha))} \\
\leq C_4(n) e^{C_4(n)\Lambda} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} + \tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} \\
+ \tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)} + \tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)} + 3\tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} + 3\tilde{l}(\sigma_k). \tag{3.19}
\]

Furthermore, as \( l(\sigma_k) \left\| \omega_k \right\|_{L^\infty(\Omega_k(\alpha))} = o(\tilde{l}(\sigma_k)) \), we have \( 2(1 + 2\sigma_k^2 (1 + H_k^2) - l(\sigma_k)\omega_k) \geq 1 \) for \( k \) large enough and by the estimates (3.9), (3.13), (3.16), (3.18), (3.19), we deduce that

\[
\left\| e^{\lambda_k} \tilde{H}_k \right\|_{L^{2,1}(\Omega_k(\alpha/2))} \leq C_5(n) e^{C_5(n)\Lambda} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha/2))} + 5\tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha/2))} + 3\tilde{l}(\sigma_k). \tag{3.20}
\]

Thanks to the proof of Theorem 3.1 and (3.20), we have

\[
\left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha/2))} \leq C_6(n) e^{C_6(n)\Lambda} \left( \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} + \left\| e^{\lambda_k} \tilde{H}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} \right) \\
\leq C_6(n) e^{C_6(n)\Lambda} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha/2))} + 5\tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha/2))} + 3\tilde{l}(\sigma_k). \tag{3.21}
\]

Furthermore, thanks to the \( \varepsilon \)-regularity (19)], we obtain

\[
\left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(B(0,0R_k) \setminus B(0,0R_k/4))} \leq C_7(n) \left\| \nabla \tilde{n}_k \right\|_{L^2(2R_k/3) \setminus B(0,0R_k/4)} \\
\left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(B(0,0R_k/2) \setminus B(0,0R_k/4))} \leq C_7(n) \left\| \nabla \tilde{n}_k \right\|_{L^2(B(0,0R_k/2) \setminus B(0,0R_k/4))} \tag{3.22}.
\]

Finally, by (3.21) and (3.22), we have

\[
\left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} \leq C_8(n) e^{C_8(n)\Lambda} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha/2))} + 5\tilde{l}(\sigma_k) \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha/2))} + 3\tilde{l}(\sigma_k),
\]

which directly implies as \( \tilde{l}(\sigma_k) \to 0 \) that for \( k \) large enough

\[
\left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} \leq 2C_8(n) e^{C_8(n)\Lambda} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha/2))}
\]

and the improved no-neck energy

\[
\lim_{\alpha \to 0} \limsup_{k \to \infty} \left\| \nabla \tilde{n}_k \right\|_{L^{2,1}(\Omega_k(\alpha))} = 0.
\]

This concludes the proof of the Theorem.
4 Removability of the second residue

First, recall the following two Schwarz-type Lemmas from [18].

**Lemma 4.1.** Let $0 < 4r < R < \infty$, let $\bar{u} : \Omega = B_R \setminus B_r(0) \to \mathbb{C}^m$ be a vector-valued holomorphic function and let $\delta \geq 0$ be such that

$$\|\bar{u}\|_{L^\infty(\partial B_r)} \leq \delta.$$

Then for all $1 \leq j \leq m$, we have

$$|u_j(z)| \leq \frac{5}{R} \left( \frac{1}{m} \|u\|_{L^\infty(\partial B_R(0))} + \delta \right) |z| + 2\delta.$$

**Lemma 4.2.** Let $0 < r_1, \ldots, r_m < R < \infty$ be fixed radii, $a_1, \ldots, a_m \in B(0, R)$ be such that $\overline{B}(a_j, r_j) \subset B(0, R)$, $\overline{B}(a_j, r_j) \cap B(a_k, r_k) = \emptyset$ for all $1 \leq j \neq k \leq m$ and

$$4r_j < R - |a_j| \quad \text{for all } 1 \leq j \leq m. \quad (4.1)$$

Furthermore, define

$$\Omega = B(0, R) \setminus \bigcup_{j=1}^m B(a_j, r_j) \quad \Omega' = \bigcap_{j=1}^m B(a_j, R - |a_j|) \setminus B(a_j, r_j).$$

Let $u : \Omega \to \mathbb{C}$ be a holomorphic function and for all $1 \leq j \leq m$ let $\delta_j \geq 0$ such that

$$\|u\|_{L^\infty(\partial B_r(a_j))} \leq \delta_j.$$ 

Then we have for all $z \in \Omega'$

$$|u(z)| \leq \sum_{j=1}^m \frac{5}{R - |a_j|} \left( \frac{1}{m} \|u\|_{L^\infty(\partial B_R(0))} + 2\delta_j \right) |z - a_j| + 4 \sum_{j=1}^m \delta_j + \sum_{j=1}^m \frac{5}{R - |a_j|} \left( \frac{1}{m} \|u\|_{L^\infty(\partial B_R(0))} + 2\delta_j \right) \max_{1 \leq j \neq k \leq m} \text{dist}_{\mathcal{S}}(a_k, \partial B_r(a_j)), \quad (4.2)$$

where $\text{dist}_{\mathcal{S}}$ is the Hausdorff distance.

**Theorem 4.3.** Let $n \geq 3$ and $\mathcal{A}$ be an admissible family of $W^{2,1}$ immersions of the sphere $S^2$ into $\mathbb{R}^n$. Assume that

$$\beta_0 = \inf_{A \in \mathcal{A}} \sup W(A) > 0.$$

Then there exists finitely many true branched compact Willmore spheres $\bar{\Phi}_1, \ldots, \bar{\Phi}_p : S^2 \to \mathbb{R}^n$, and true branched compact Willmore spheres $\bar{\tilde{\Phi}}_1, \ldots, \bar{\tilde{\Phi}}_q : S^2 \to \mathbb{R}^n$ such that

$$\beta_0 = \sum_{i=1}^p W(\bar{\Phi}_i) + \sum_{j=1}^q (W(\bar{\tilde{\Phi}}_j) - 4\pi \theta_j) \in 4\pi \mathbb{N}, \quad (4.3)$$

where $\theta_0(\bar{\tilde{\Phi}}_j, p_j) \geq 1$ is the multiplicity of $\bar{\tilde{\Phi}}_j$ at some point $p_j \in \bar{\tilde{\Phi}}_j(S^2) \subset \mathbb{R}^n$. Then at every branch point $p$ of $\bar{\Phi}_1, \bar{\tilde{\Phi}}_j$ of multiplicity $\theta_0 = \theta_0(p) \geq 2$, the second residue $r(p)$ satisfies the inequality $r(p) \leq \theta_0 - 2$.

**Proof.** Using the main result of [20], we see that there exists a sequence $\{\sigma_k\}_{k \in \mathbb{N}} \subset (0, \infty)$ such that $\sigma_k \to 0$ and a sequence of smooth immersions $\{\bar{\Phi}_k\}_{k \in \mathbb{N}} \subset \text{Imm}(S^2, \mathbb{R}^n)$ such that $\bar{\Phi}_k$ be a critical point of $W_{\sigma_k}$ such that

$$W_{\sigma_k}(\bar{\Phi}_k) = \beta(\sigma_k),$$

where $\beta(\sigma_k)$ is...
where

$$\beta(\sigma_k) = \inf_{A \in \mathcal{A}} \sup_{k \to \infty} \frac{W_{\sigma_k}(A)}{k} \to \beta_0.$$  

Now, we can consider that $\tilde{\Phi}_k : B(0, 1) \to \mathbb{R}^3$ to be a critical point of $W_{\sigma_k}$ such that $\tilde{\Phi}_k \to \tilde{\Phi}_\infty$ in $C^1_{\text{loc}}(B(0, 1) \setminus \{0\})$ for all $l \in \mathbb{N}$. By [1], there exists $\tilde{C}_0 \in \mathbb{C}^n$ such that

$$\tilde{H}_{\tilde{\Phi}_\infty} = \text{Re} \left( \frac{\tilde{C}_0}{z^{\theta_0-1}} \right) + O(|z|^{2-\theta_0} \log^2 |z|).$$

Then $r(0) \leq \theta_0 - 2$ if and only if $\tilde{C}_0 = 0$, and this is what we will show in the rest of the proof. Now, define $\Omega_k(\alpha)$ to be the neck-region

$$\Omega_k(\alpha) = B_\alpha(0) \setminus \bigcup_{j=1}^m B_{\rho_k^{-1}, \rho_k}(x_j^k).$$

Then notice that Theorem D of [18] still applies. In particular, there exists $\theta_1^0, \ldots, \theta_m^0 \in \mathbb{Z} \setminus \{0\}$ and a universal constant $C = C(n, \Lambda)$ independent of $k$ and $0 < \alpha < 1$ such that for all $k \in \mathbb{N}$ large enough and for all $z \in \Omega_k(\alpha)$

$$\frac{1}{C} \leq \frac{\varphi_{\lambda_k}(z)}{\prod_{j=1}^m |z - x_j^k|^{\theta_j^0}} \leq C. \quad (4.4)$$

Now, let $\varphi_k(z) : \Omega_k(\alpha) \to \mathbb{C}$ be the holomorphic function such that

$$\varphi_k(z) = \prod_{j=1}^m (z - x_j^k)^{\theta_j^k} \quad \text{for all } z \in \Omega_k(\alpha).$$

To simplify notations, we state the proof in codimension 1 ($n = 3$). Furthermore, to simplify the proof, we will use Lemma 4.1 instead of 4.2 and assume that we have only one bubble. We assume then in the following that

$$\Omega_k(\alpha) = B_\alpha(0) \setminus \overline{B}_{\rho_k}(0).$$

First recall that the invariance by translation ([20], Lemma III.8) shows that there exists $\tilde{L}_k : B(0, 1) \to \mathbb{R}^3$ such that

$$
\begin{align*}
\text{d} \tilde{L}_k &= * \text{d} \left( \tilde{H}_k + 2\sigma_k^2(1 + |\tilde{H}_k|^2) \tilde{H}_k \right) - 2 \left( 1 + 2\sigma_k^2(1 + |\tilde{H}_k|^2) \right) \tilde{H}_k \ast d\tilde{n}_k \\
&\quad + \left( - \left( \langle \tilde{H}_k, \tilde{H}_k \rangle^2 \tilde{H}_k \right) + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} |d\omega(\sigma_k^2 \tilde{H}_k, \tilde{H}_k, \tilde{H}_k) - 2\pi \omega_0 e^{-\pi \omega_0} + \frac{2\pi}{\text{Area}(\Sigma_0)} \right) \right) \ast d\tilde{\Phi}_k \\
&\quad - \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \text{d} \tilde{\Phi}_k, d\omega_k \right)_{\rho_k} \ast d\omega_k + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \tilde{L}_k(\ast d\omega_k). \quad (4.5)
\end{align*}
$$

Now, recall that $d = \partial + \bar{\partial}$, and $\ast \partial = -i \partial$, we deduce that

$$
\begin{align*}
i \partial \tilde{L}_k &= \partial \left( 1 + 2\sigma_k^2(1 + |\tilde{H}_k|^2) \tilde{H}_k \right) - 2 \left( 1 + 2\sigma_k^2(1 + |\tilde{H}_k|^2) \right) \tilde{H}_k \partial \tilde{n}_k \\
&\quad + \left( - \left( \langle \tilde{H}_k, \tilde{H}_k \rangle^2 \tilde{H}_k \right) + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} |d\omega_k(\sigma_k^2 \tilde{H}_k, \tilde{H}_k, \tilde{H}_k) - 2\pi \omega_0 e^{-\pi \omega_0} + \frac{2\pi}{\text{Area}(\Sigma_0)} \right) \right) \partial \tilde{\Phi}_k
\end{align*}
$$
\[- \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \langle d\bar{\Phi}_k, d\omega_k \rangle_{g_k} \omega_{\partial} + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \bar{n}_k \omega_{\partial}.\]

Finally, we can recast this equation as
\[
\Phi \left( \left( 1 + 2\sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) \bar{\Phi}_k + i\tilde{L}_k \right) = - \left( 1 + 2\sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) H_k \bar{\Phi}_k
\]
\[
+ \left( - \left( \left| \bar{\Phi}_k \right|^2 + \sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} \langle d\omega_k \rangle_{g_k}^2 - 2\pi \omega_k e^{-2\omega_k} + \frac{2\pi}{\text{Area}(\Phi_k(S^2))} \right) \right) \bar{n}_k
\]
\[
= \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \langle d\Phi_k, d\omega_k \rangle_{g_k} \bar{n}_k + \frac{\bar{n}_k - \bar{\Phi}_k \bar{n}_k}{\log \left( \frac{1}{\sigma_k} \right)} \omega_{\partial}. \tag{4.6}
\]

Now, let \( \bar{\Psi}_k : \Omega_k(\alpha) \to \mathbb{C} \) be defined as
\[
\bar{\Psi}_k(z) = \varphi_k(z) \left\{ \left( 1 + 2\sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) \bar{\Phi}_k + i\tilde{L}_k \right\}. \tag{4.7}
\]

Then by (4.6), as \( \varphi_k \) is holomorphic, \( \bar{\Psi}_k \) solves
\[
\Phi \left( \left( 1 + 2\sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) \bar{\Phi}_k + i\tilde{L}_k \right) = - \left( 1 + 2\sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) H_k \bar{\Phi}_k
\]
\[
+ \varphi_k \left( - \left( \left| \bar{\Phi}_k \right|^2 + \sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} \langle d\omega_k \rangle_{g_k}^2 - 2\pi \omega_k e^{-2\omega_k} + \frac{2\pi}{\text{Area}(\Phi_k(S^2))} \right) \right) \bar{n}_k
\]
\[
= - \varphi_k H_k \bar{\Phi}_k \left( 1 + 2\sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) \bar{\Phi}_k
\]
\[
+ \varphi_k \left( - \left( \left| \bar{\Phi}_k \right|^2 + \sigma_k^2 \left( 1 + \left| \bar{\Phi}_k \right|^2 \right) \right) + \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} \langle d\omega_k \rangle_{g_k}^2 - 2\pi \omega_k e^{-2\omega_k} + \frac{2\pi}{\text{Area}(\Phi_k(S^2))} \right) \right) \bar{n}_k
\]
\[
\left\{ \begin{array}{l}
\bar{\Phi}_k = - \varphi_k H_k \bar{\Phi}_k - \varphi_k |\bar{\Phi}_k|^2 \bar{\Phi}_k \quad \text{in } \Omega_k(\alpha)
\\
\bar{\Phi}_k = 0 \quad \text{on } \partial \Omega_k(\alpha)
\end{array} \right. \tag{4.8}
\]

Now, write \( \bar{\Psi}_k = \bar{u}_k + \bar{v}_k + \bar{w}_k \), where
\[
\left\{ \begin{array}{l}
\bar{\Phi}_k = - \varphi_k H_k \bar{\Phi}_k - \varphi_k |\bar{\Phi}_k|^2 \bar{\Phi}_k \quad \text{in } \Omega_k(\alpha)
\\
\bar{\Phi}_k = 0 \quad \text{on } \partial \Omega_k(\alpha)
\end{array} \right.
\]

and
\[
\left\{ \begin{array}{l}
\bar{\Phi}_k = - \varphi_k H_k \bar{\Phi}_k - \varphi_k |\bar{\Phi}_k|^2 \bar{\Phi}_k \quad \text{in } \Omega_k(\alpha)
\\
\bar{\Phi}_k = 0 \quad \text{on } \partial \Omega_k(\alpha)
\end{array} \right. \tag{4.9}
\]

Finally, \( \bar{v}_k : \Omega_k(\alpha) \to \mathbb{C} \) is the holomorphic function
\[
\left\{ \begin{array}{l}
\bar{\Phi}_k = 0 \quad \text{in } \Omega_k(\alpha)
\\
\bar{\Phi}_k = \bar{\Psi}_k \quad \text{on } \partial \Omega_k(\alpha).
\end{array} \right.
\]

As \( \bar{\Phi}_k \in L^2(\Omega_k(\alpha)) \), the Sobolev embedding only shows that \( \bar{w}_k \in L^{2,\infty}(\Omega_k(\alpha)) \), so we have to obtain an estimate using this norm. The duality \( L^{2,1}/L^{2,\infty} \) shows that
\[
\| \bar{\Phi}_k \|_{L^2(\Omega_k(\alpha))} \leq \| \varphi_k H_k \|_{L^{2,\infty}(\Omega_k(\alpha))} \| \bar{\Phi}_k \|_{L^2(\Omega_k(\alpha))} + \| \varphi_k H_k \|_{L^{2,\infty}(\Omega_k(\alpha))} \| \Phi_k \|_{L^{2,\infty}(\Omega_k(\alpha))}
\]
\[
\leq 2 \| \nabla \bar{w}_k \|_{L^{2,1}(\Omega_k(\alpha))} \| \bar{\Phi}_k \|_{L^2(\Omega_k(\alpha))} \| \bar{\Phi}_k \|_{L^2(\Omega_k(\alpha))} \| \Phi_k \|_{L^{2,\infty}(\Omega_k(\alpha))} \| \bar{\Phi}_k \|_{L^{2,\infty}(\Omega_k(\alpha))} \tag{4.10}
\]
This implies by Lemma 2.2 that
\[ \|\vec{u}_k\|_{L^\infty(\Omega_k(\alpha))} \leq 6 \sqrt{\frac{\pi}{\alpha}} \|\nabla \vec{u}_k\|_{L^2(\Omega_k(\alpha))} \|\bar{\Psi}_k\|_{L^2(\Omega_k(\alpha))} \] (4.11)

Finally, thanks to the maximum principle
\[ \|\vec{u}_k\|_{L^\infty(\Omega_k(\alpha))} \leq \|\bar{\Psi}_k\|_{L^\infty(\partial\Omega_k(\alpha))}. \] (4.12)

Now, we will recall the estimates obtained in [20] imply that
\[ \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \|\omega_k\|_{L^\infty(B(0,1))} = o \left( \tilde{l}(\sigma_k) \right) \]
\[ \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \int_{S^2} |d\omega_k|_{g_k}^2 dv_{g_k} = o \left( \tilde{l}(\sigma_k) \right) \] (4.13)
\[ \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left( \frac{1}{2} \int_{S^2} |d\omega_k|_{g_k}^2 dv_{g_k} + 4\pi \int_{S^2} \omega_k e^{-2\omega_k} dv_{g_k} - 2\pi \log \int_{S^2} dv_{g_k} \right) = o \left( \tilde{l}(\sigma_k) \right). \]

Furthermore, as Area(\bar{\Phi}_k(S^2)) = 1 and by (4.4), we deduce that
\[ \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left\| \frac{1}{2} \left[ |dx|_{g_k}^2 - 2\pi \omega_k e^{-2\omega_k} + \frac{2\pi}{\text{Area}(\bar{\Phi}_k(S^2))} \right] \varphi_k \bar{\Phi}_k \right\|_{L^1(\Omega_k(\alpha))} = o \left( \tilde{l}(\sigma_k) \right), \] (4.14)

while
\[ \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left\| (\bar{\Phi}_k, d\omega_k) \varphi_k \bar{\Phi}_k \right\|_{L^1(\Omega_k(\alpha))} \leq \frac{C}{\log \left( \frac{1}{\sigma_k} \right)} \int_{\Omega_k(\alpha)} |d\omega_k|_{g_k}^2 dv_{g_k} = o \left( \tilde{l}(\sigma_k) \right). \] (4.15)

Finally, we have by the Cauchy-Schwarz inequality and (4.13)
\[ \frac{1}{\log \left( \frac{1}{\sigma_k} \right)} \left\| \varphi_k \left( \bar{\Phi}_k \nabla \omega_k \right) \right\|_{L^1(\Omega_k(\alpha))} \leq \frac{C}{\log \left( \frac{1}{\sigma_k} \right)} \|\nabla \vec{u}_k\|_{L^2(\Omega_k(\alpha))} \|\nabla \omega_k\|_{L^2(\Omega_k(\alpha))} = o \left( \sqrt{l(\sigma_k)} \right). \] (4.16)

Therefore, by (4.9), (4.14), (4.15) and (4.16)
\[ \|\bar{\nabla} \vec{u}_k\|_{L^1(\Omega_k(\alpha))} = o \left( \tilde{l}(\sigma_k) \right) \xrightarrow{k \to \infty} 0. \]

Now, thanks to Lemma 2.2, we deduce that
\[ \|\vec{u}_k\|_{L^2(\Omega_k(\alpha))} = o \left( \tilde{l}(\sigma_k) \right) \xrightarrow{k \to \infty} 0. \] (4.17)

Now, notice that
\[ \sigma_k e^{-\lambda_k} = \sigma_k \exp \left( o \left( \log \left( \frac{1}{\sigma_k} \right) \right) \right) = o_k(1) \]
so that
\[ \sigma_k |\bar{H}_k| = \sigma_k e^{-\lambda_k} e^{\lambda_k} |\bar{H}_k| = o_k(1) e^{\lambda_k} |\bar{H}_k|. \]

Therefore, by the conformal invariant of $e^{\lambda_k} |\bar{H}_k|$, the argument of **Step 3** of the proof of Theorem 4.1 (of [18]) still applies and shows that
\[ |\bar{\Psi}_k| = O(\alpha^2) + o_k(1) \quad \text{on } \partial B_{\alpha^{-1} \rho_k}(0). \]
Therefore, Lemma 4.1 shows that there exists a universal constant $C_1 = C_1(n, \Lambda)$ (independent of $k \in \mathbb{N}$ and $0 < \alpha < 1$) such that for all $z \in \Omega_k(\alpha) = B_\alpha \setminus B_{\alpha^{-1}p_k}(0)$

$$|\tilde{v}_k(z)| \leq \frac{C_1}{\alpha} |z| + C_1(\alpha^2 + o_k(1)) \quad (4.18)$$

Now observe that

$$\int_{B_\alpha \setminus B_{\alpha^{-1}p_k}} |z|^2 |dz|^2 = 2\pi \int_{\alpha^{-1}p_k} r^3 dr \leq \frac{\pi}{2} \alpha^4.$$  

In particular, we deduce that

$$\|\tilde{v}_k\|_{L^2(\Omega_k(\alpha))} \leq 2 \|\tilde{v}_k\|_{L^2(\Omega_k(\alpha))} \leq \sqrt{2\pi} C_1 \alpha + \sqrt{\pi} C_1 \alpha^2 (\alpha^2 + o_k(1)) \quad (4.19)$$

Now, as $\tilde{\Psi}_k = \tilde{u}_k + \tilde{\nu}_k + \tilde{\omega}_k$, we have for fixed $0 < \alpha < 1$ and all $k \in \mathbb{N}$ large enough

$$\|\tilde{\Psi}_k\|_{L^2(\Omega_k(\alpha))} \leq \frac{6}{\sqrt{2\pi}} \|\nabla \tilde{\nu}_k\|_{L^2(\Omega_k(\alpha))} \|\tilde{\Psi}_k\|_{L^2(\Omega_k(\alpha))} + \sqrt{2\pi} C_1 \alpha + \sqrt{\pi} C_1 \alpha^2 (\alpha^2 + o_k(1)) + \frac{1}{\log \log \left( \frac{1}{\alpha} \right)}.$$  

Thanks to the improved no-neck energy of Theorem 3.1

$$\lim_{\alpha \to 0} \limsup_{k \to \infty} \|\nabla \tilde{\nu}_k\|_{L^2(\Omega_k(\alpha))} = 0,$$  

we deduce that (for some $C_2 = C_2(n, \Lambda)$)

$$\|\tilde{\Psi}_k\|_{L^2(\Omega_k(\alpha))} \leq C_2 \alpha + C_2 \alpha^2 (\alpha^2 + o_k(1)) + \frac{2}{\log \log \left( \frac{1}{\alpha} \right)}.$$  

Therefore, coming back to the estimate (4.11), we deduce that for some universal constant $C_2 = C_2(n, \Lambda)$

$$\|\tilde{u}_k\|_{L^2(\Omega_k(\alpha))} \leq \left( C_2 \alpha + C_2 \alpha^2 (\alpha^2 + o_k(1)) + \frac{2}{\log \log \left( \frac{1}{\alpha} \right)} \right) \|\nabla \tilde{\nu}_k\|_{L^2(\Omega_k(\alpha))}. \quad (4.21)$$

Now, let $p = p(\alpha) > 1$ be a fixed positive number independent of $k \in \mathbb{N}$, to be determined later. The estimate

$$\|z\|_{L^2(B_{\alpha^p} \setminus B_{\alpha^{-1}p_k}(0))}^2 = \int_{B_{\alpha^p} \setminus B_{\alpha^{-1}p_k}(0)} |z|^2 |dz|^2 = 2\pi \int_{\alpha^{-1}p_k} r^3 dr \leq \frac{\pi}{2} \alpha^{4p} \quad (4.22)$$

implies by (4.18) that

$$\|\tilde{u}_k\|_{L^2(\Omega_k(\alpha))} \leq \frac{2C_1}{\alpha} \times \sqrt{\frac{\pi}{2}} \alpha^{2p} + 2\sqrt{\pi} C_1 \alpha^p (\alpha^2 + o_k(1)) \leq 2\sqrt{\pi} C_1 \alpha^{2p-1} + 2\sqrt{\pi} C_1 \alpha^p (\alpha^2 + o_k(1)). \quad (4.23)$$

Furthermore, the uniform $\varepsilon$-regularity implies that for all $z \in B(0, 1) \setminus \{0\}$, we have

$$\lim_{k \to \infty} \tilde{\Psi}_k(z) = C_0 + O(|z|).$$

Therefore, for all $\beta < \alpha^p$, we deduce thanks to the triangle inequality and (4.22) that for all $k \in \mathbb{N}$ large enough

$$\|\tilde{\Psi}_k\|_{L^2(\Omega_k(\alpha))} \geq \|\tilde{\Psi}_k\|_{L^2(B_{\alpha^p} \setminus B_{\alpha^{-1}p_k}(0))} \quad \overset{k \to \infty}{\longrightarrow} \quad \|\tilde{C}_0 + O(|z|)\|_{L^2(\Omega_k(\alpha))} \geq \sqrt{\pi} \alpha^p \sqrt{\frac{\beta}{\alpha^p}} |C_0| - C_3 \alpha^{2p} \quad (4.24)$$
for some universal constant $C_3 = C_3(n, \Lambda)$ independent of $0 < \alpha < 1$ and $p \geq 1$. Taking $\beta \to 0$ in (4.24) yields

$$\liminf_{k \to \infty} \left\| \tilde{\Psi}^k \right\|_{L^2(\mathbb{R}^n \setminus \overline{T_{n-1} T^k}(0))} \geq \sqrt{\pi} \alpha^p |C^0| - C' \alpha^{2p}. \quad (4.25)$$

Furthermore, as $\tilde{\Psi}_k = \tilde{u}_k + \tilde{v}_k + \tilde{w}_k$, and as $\lim_{k \to \infty} \left\| \tilde{u}_k \right\|_{L^2(\Omega_k(\alpha))} = 0$ by (4.17)

$$\limsup_{k \to \infty} \left\| \tilde{\Psi}_k \right\|_{L^2(\mathbb{R}^n \setminus \overline{T_{n-1} T^k}(0))} \leq \limsup_{k \to \infty} \left\| \tilde{u}_k \right\|_{L^2(\mathbb{R}^n \setminus \overline{T_{n-1} T^k}(0))} + 2\sqrt{\pi} C_1 \alpha^{2p-1} + 2\sqrt{\pi} C_1 \alpha^{p+2}. \quad (4.26)$$

Therefore, we deduce by (4.25) and (4.26)

$$\sqrt{\pi} \alpha^p |C_0| - C_3 \alpha^{2p} \leq \limsup_{k \to \infty} \left\| \tilde{u}_k \right\|_{L^2(\mathbb{R}^n \setminus \overline{T_{n-1} T^k}(0))} + 2\sqrt{\pi} C_1 \alpha^{2p-1} + 2\sqrt{\pi} C_1 \alpha^{p+2}. \quad (4.27)$$

By (4.21), we deduce that

$$\sqrt{\pi} \alpha^p |C_0| - C_3 \alpha^{2p} \leq 2C_2 \alpha \limsup_{k \to \infty} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} + 2\sqrt{\pi} C_1 \alpha^{2p-1} + 2\sqrt{\pi} C_1 \alpha^{p+2}. \quad (4.28)$$

Therefore, we find by dividing both inequalities by $\sqrt{\pi} \alpha^p$ (using $p \geq 1$ for the second inequality) for some universal constant $C_4 = C_4(n, \Lambda)$

$$|C_0| \leq C_4 \left( \frac{1}{\alpha^{p-1}} \liminf_{k \to \infty} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} + \alpha^{p-1} + \alpha^p + \alpha^2 \right)$$

$$\leq C_4 \left( \frac{1}{\alpha^{p-1}} \liminf_{k \to \infty} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} + \alpha^{p-1} + \alpha^p + \alpha^2 \right). \quad (4.29)$$

We will now have to distinguish two cases

**Case 1.** Now, if for all $0 < \alpha < 1$ small enough

$$\limsup_{k \to \infty} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} = 0,$$

then (4.29) implies by taking $p = 2$ that

$$|C_0| \leq C_4 (2\alpha + \alpha^2) \not\to 0,$$

which concludes the proof of the Theorem.

**Case 2.** Otherwise, assume that

$$f(\alpha) = \limsup_{k \to \infty} \left\| \nabla \tilde{n}_k \right\|_{L^2(\Omega_k(\alpha))} > 0 \quad \text{for all } 0 < \alpha < 1,$$

and choose

$$p(\alpha) = 1 + \frac{\log \left( \frac{1}{f(\alpha)} \right)}{2 \log \left( \frac{1}{\alpha} \right)} > 1,$$

so that

$$\alpha^{p-1} = \sqrt{f(\alpha)}.$$

Then (4.29) implies that

$$|C_0| \leq C_4 \left( \frac{1}{\sqrt{f(\alpha)}} \times f(\alpha) + \sqrt{f(\alpha)} + \alpha + \alpha^2 \right) = C_4 \left( 2\sqrt{f(\alpha)} + \alpha + \alpha^2 \right) \not\to 0$$

and this concludes the proof of the Theorem by the improved no-neck energy (4.20).
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