ON A CLASS OF DYNAMICAL SYSTEMS BOTH QUASI-BI-HAMILTONIAN AND BI-HAMILTONIAN

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ABSTRACT. It is shown that a class of dynamical systems (encompassing the one recently considered by F. Calogero in [1]) is both quasi-bi-Hamiltonian and bi-Hamiltonian. The first formulation entails the separability of these systems; the second one is obtained through a non canonical map whose form is directly suggested by the associated Nijenhuis tensor.
1. INTRODUCTION AND PRELIMINARIES

The aim of this Letter is to point out some properties of a class of dynamical systems which admit both a quasi-bi-Hamiltonian (QBH) formulation and a bi-Hamiltonian (BH) formulation.

Let $M$ be an even dimensional differentiable manifold ($\dim M = 2n$), $TM$ and $T^*M$ its tangent and cotangent bundle: a bi-Hamiltonian structure on $M$ is a pair $(P_0, P_1)$ ($P_0, P_1 : T^*M \rightarrow TM$) of two compatible Poisson tensors [3]. If $P_0$ is invertible and the Nijenhuis tensor $N := P_1 P_0^{-1}$ has $n$ functionally independent eigenvalues $(\lambda_1, \ldots, \lambda_n)$ one can introduce a set of canonical coordinates $(\lambda; \mu)$ ($\lambda := (\lambda_1, \ldots, \lambda_n); \mu := (\mu_1, \ldots, \mu_n)$) referred to as Darboux-Nijenhuis coordinates [4] such that $P_0, P_1$ and $N$ take the matrix form

$$
P_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}, \tag{1.1}$$

where $I$ denotes the $n \times n$ identity matrix and $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n)$. The above form of the BH structure will be referred to as a canonical BH structure.

We recall that the local model of a BH structure was studied in [5], where it was shown that a compatible pair $(P_0, P_1)$ of Poisson tensors (being $P_0$ invertible) admits a representation with $P_0$ constant and $P_1$ depending linearly on the coordinates; the Darboux–Nijenhuis coordinates are just a particular realization of this representation: the eigenvalues of $N$ are the first $n$ coordinates and the remaining ones are constructed by quadratures [4].

A vector field $X$ is said to be bi-Hamiltonian w.r.t. $(P_0, P_1)$ if there exist two smooth functions $h_0$ and $h_1$ such that

$$
X = P_0 dh_1 = P_1 dh_0, \tag{1.2}
$$

d denoting the exterior derivative.

Under the above assumption on the eigenvalues of $N$, a BH vector field is completely integrable, a set of independent involutive integrals being just the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ [6]. However, it has been proved in [7, 8] that a very strong condition has to be satisfied by a completely integrable system (Hamiltonian w.r.t. $P_0$) in order to have a BH formulation in a neighborhood of an invariant Liouville torus, at least if one searches for a second Poisson tensor compatible with $P_0$. Nevertheless, the property of Liouville integrability, can be related with geometrical structures which are actually different from the canonical BH structure. This can be done in (at least) three distinct ways.

i) Searching for a BH formulation without the canonical Poisson structure $P_0$ [9] (hereafter referred to as a non canonical BH formulation). The BH structure
constructed in Sect. 3 and the one considered in Sect. 4 are just applications of such a construction.

ii) Admitting a degenerate BH formulation [10]; for instance, this is the case of the rigid body with a fixed point [11, 12, 13] and of the stationary flows of the KdV hierarchy [14, 15].

iii) Searching for a quasi–bi–Hamiltonian formulation of $X$ [16, 17].

In connection with the third approach, we recall that the vector field $X$ is said to be quasi-bi-Hamiltonian w.r.t. $(P_0, P_1)$ [16] if there exist three smooth functions $H$, $K$ and $\rho$ such that

$$X = P_0 dH = \frac{1}{\rho} P_1 dK;$$

(1.3)
in particular, if $\rho$ is the product of the eigenvalues of the Nijenhuis tensor $N = P_1 P_0^{-1}$, i.e.,

$$\rho = \prod_{i=1}^{n} \lambda_i,$$

(1.4)
the QBH vector field $X$ is said to be Pfaffian.

It has been proved in [16] that any completely integrable system with two degrees of freedom has a QBH formulation in a neighborhood of a Liouville torus. Furthermore, for a Pfaffian QBH vector field with $n$ degrees of freedom we proved [17] that the general solution of Eq.(1.3), written in the Darboux-Nijenhuis coordinates, is

$$H = \sum_{i=1}^{n} \frac{f_i(\lambda_i; \mu_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} , \quad K = \sum_{i=1}^{n} \frac{\rho_i f_i(\lambda_i; \mu_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} (\rho_i := \prod_{j \neq i} \lambda_j)$$

(1.5)
where each function $f_i$ is an arbitrary smooth function, at most depending on one pair $(\lambda_i; \mu_i)$. A remarkable feature of $H$ and $K$ is that they are separable (in the sense of Hamilton-Jacobi) as they verify the Levi–Civita condition [18], therefore the corresponding Hamilton equations are integrable by quadratures. We stress the fact that, owing to the arbitrariness of $f_i$, the functions $H$ and $K$ (1.5) provide a class of separable functions different from the known Stäckel class (e.g., see [19, p. 101]).

The previous results have been completed in [20], where it has been shown that a QBH vector field $X$ admits $n$ integrals of motion in involution $F_k (k = 1, \ldots, n)$

$$F_k = \sum_{i=1}^{n} \frac{\partial c_k}{\partial \lambda_i} \frac{f_i(\lambda_i; \mu_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} ,$$

(1.6)
where \( c_1, \ldots, c_n \) are the coefficients of the minimal polynomial of the Nijenhuis tensor \( N \).

\[
\lambda^n + \sum_{i=1}^{n} c_i \lambda^{n-i} = \prod_{i=1}^{n} (\lambda - \lambda_i) ;
\]

(1.7)
in particular, \( F_1 = -H, F_n = (-1)^n K \). Furthermore, each function \( F_k \) turns out to be separable.

A natural question arises about the mutual relations between the different formulations of the above items. In this regard, we observe that a few examples of QBH systems, such those considered in \([17, 20]\) can be obtained as highly non trivial reductions of degenerate BH systems \([15]\). However the relation between the BH and the QBH formulation for a given vector field and the very existence of one or both structures is not yet completely clarified. An open problem, which we are not going to examine here, is to give conditions assuring that a given integrable vector field with \( n \)-degrees of freedom admits globally a QBH formulation. Since a theoretical result for \( n > 2 \) is still lacking, it seems to us of some interest to collect and classify examples of such systems. In this Letter our aim is just to discuss in some details an explicit example of a system admitting both formulations. Its phase space is an open dense submanifold of \( R^{2n} \), so that its QBH and BH formulations are globally defined.

2. THE DYNAMICAL SYSTEM AND ITS CANONICAL QBH FORMULATION

Let \( M = R^{2n}, M \ni u = (\lambda; \mu) \). Let us consider the Hamiltonian dynamical system \( \dot{u} = X(u) \), with \( X = P_0 dH; P_0 \) is the canonical Poisson tensor and \( H \) is given by

\[
H = \sum_{i=1}^{n} \frac{g_i(\lambda_i)}{\prod_{j \neq i} (\lambda_i - \lambda_j)} e^{a\mu_i} ,
\]

(2.1)
where \( g_i \) are arbitrary smooth functions, each one depending only on the corresponding coordinate \( \lambda_i \), and \( a \) is an arbitrary constant.

The related Newton equations of motion take the form

\[
\ddot{\lambda}_k = 2 \sum_{i \neq k} \frac{\dot{\lambda}_i \dot{\lambda}_k}{\lambda_k - \lambda_i} \quad (k = 1, \ldots, n) .
\]

(2.2)
They were found to be solvable by F. Calogero \([2]\) and recently they have been shown \([1]\) to describe a solvable \( n \)-body systems in the plane. Indeed, as remarked in \([1]\), the previous equations describe also a special case of the integrable relativistic \( n \)-body problems introduced by S.N. Ruijsenaars and H. Schneider \([21]\).
Now, comparing (2.1) with (1.5) one immediately concludes that \( \{\lambda; \mu\} \) is a Darboux–Nijenhuis chart for the Hamiltonian \( H \), which is consequently separable (a property unnoticed in [1]):

**Proposition 2.1.** The vector field \( X = P_0 \, dH \) is a Pfaffian QBH vector field; \( P_0, P_1 \) and \( N \) are given by (1.1), \( H \) is the Hamiltonian (2.1) and \( K \) is given by

\[
K = \sum_{i=1}^{n} \frac{\rho_i \, g_i(\lambda_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} e^{\mu_i}.
\]  

(2.3)

Furthermore, the corresponding Hamilton–Jacobi equation is separable; a complete integral is \( S(\lambda; b_1, \ldots, b_n) = -b_1 t + W(\lambda; b_1, \ldots, b_n) \) with

\[
W = \frac{1}{a} \sum_{i=1}^{n} \int^{\lambda_i} \log \left( \frac{1}{g_i(\xi)} \sum_{j=1}^{n} b_j \xi^{n-j} \right) \, d\xi,
\]  

(2.4)

and the Hamilton equations of motion can be solved by quadratures. \( \square \)

As it was shown in [2], the Newton equations (2.2) can be linearized by introducing a suitable set of coordinates. As a matter of fact, these coordinates are strictly related with the Nijenhuis tensor (1.1). Indeed, let us consider the minimal polynomial of \( N \) given by (1.7): expressing its coefficients \( c := (c_1, \ldots, c_n) \) in terms of its roots \( \lambda := (\lambda_1, \ldots, \lambda_n) \) by means of the Viète’s formulae

\[
c_1 = -\sum_{i=1}^{n} \lambda_i
\]
\[
c_2 = \sum_{i<j}^{n} \lambda_i \lambda_j
\]
\[
c_3 = -\sum_{i<j<k}^{n} \lambda_i \lambda_j \lambda_k
\]
\[\vdots\]
\[
c_n = (-1)^n \prod_{i}^{n} \lambda_i,
\]  

(2.5)

we can introduce the map

\[
\Phi : \lambda \mapsto c : c_k = \Phi_k(\lambda_1, \ldots, \lambda_n) \quad (k = 1, \ldots, n)
\]  

(2.6)

Now, we can easily obtain the following result:
Proposition 2.2. The evolution of $c$ along the flow of $X$ is given by $\dot{c}_k = aF_k$, where $F_k$ are given by (1.6), with $f_i(\lambda_i; \mu_i) = g_i(\lambda_i) e^{a\mu_i}$. So, one has $\ddot{c}_k = 0$.

Proof. On account of Eq. (2.1), one gets

$$\dot{c}_k = \sum_{i=1}^n \frac{\partial c_i}{\partial \lambda_i} \dot{\lambda}_i = a \sum_{i=1}^n \frac{\partial c_k}{\partial \lambda_i} \frac{g_i(\lambda_i)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} e^{a\mu_i} = aF_k .$$

(2.7)

This shows that the dynamics associated with $X$ is trivial when expressed in terms of $c$, as it was already remarked in [1, 2]. It seems to us of some interest to point out the algebraic meaning of the map $\Phi$ (2.6), i.e., its relation with the Nijenhuis tensor (1.1).

3. THE DYNAMICAL SYSTEM AND ITS NON CANONICAL BH FORMULATION

Let $\Psi : R^{2n} \rightarrow R^{2n}$ be the non canonical map

$$\Psi : u = (\lambda; \mu) \mapsto v = (c; \gamma) \quad c_k = \Phi_k(\lambda) ; \gamma_k = aF_k(\lambda; \mu) .$$

(3.1)

On account of Prop. 2.2 and of the fact that $F_k$ are integrals of motion for $X$, it is $\dot{c}_k = \gamma_k$ and $\dot{\gamma}_k = a\dot{F}_k = 0$; so, the vector field $X$ is mapped by $\Psi$ into the vector field $Y = (\gamma, 0)^T$ (of course, the whole QBH structure $(P_0, P_1)$ could as well be transformed). Easily enough, for any dynamical system $\dot{v} = Y(v)$ of this form one has the following result:

Proposition 3.1. The system $\dot{v} = Y(v)$, with $Y = (\gamma, 0)^T$, is BH w.r.t. $Q_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ and $Q_1 = \begin{bmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{bmatrix}$, where $\Gamma := diag(\gamma_1, \ldots, \gamma_n)$. The BH chain is

$$Q_0 dh_{j+1} = Q_1 dh_j \quad (j = 0, 1, \ldots) ,$$

(3.2)

with $h_0 = \log(\det \Gamma)$ and $h_j = \frac{1}{2j} Tr(Q_1 Q_0^{-1})^j = \frac{1}{j} \sum_{i=1}^n \gamma_i^j \quad (j = 1, 2, 3, \ldots) . \quad \Box$

Remark 3.1. For the sake of completeness, we recall that $X$ admits also a Virasoro algebra of graded conformal symmetries $\tau_j (j = -1, 0, 1, \ldots)$ with $\tau_j = (0, \gamma^{j+1})^T (\gamma^j := (\gamma_1^j, \ldots, \gamma_n^j)^T)$. The relation between this algebraic structure and the BH structure is well known [22]. \quad \Box

Since the BH structure can be transformed from the chart $\{c; \gamma\}$ to the chart $\{\lambda; \mu\}$, we conclude that the system $\dot{u} = X(u)$, which is QBH w.r.t. $(P_0, P_1)$, is also BH w.r.t. $(Q_0, Q_1)$. The Poisson tensors of the BH formulation can be obtained by transformation (under $\Psi$) of the Poisson tensors $Q_0, Q_1$ of Prop. 3.1. We observe that the map $\Psi$ is non canonical w.r.t. $Q_0$, so in the chart $\{\lambda; \mu\}$ it is $Q_0 \neq P_0$; as a
matter of fact, the Poisson pair \((Q_0, Q_1)\) take a quite complicated form in the chart \(\{\lambda; \mu\}\), being simpler in the chart \(\{c; \gamma\}\) (the opposite situation occurs for the pair \((P_0, P_1)\)).

For the sake of clarity, let us consider explicitly the case \(n = 2\). The dynamical system \(\dot{u} = X(u)\) is of the form (1.3), with \(H\) and \(K\) given respectively by (2.1) and (2.3):

\[
H = \frac{g_1}{\lambda_{12}} e^{a\mu_1} - \frac{g_2}{\lambda_{12}} e^{a\mu_2}, \quad K = \frac{\lambda_2 g_1}{\lambda_{12}} e^{a\mu_1} - \frac{\lambda_1 g_2}{\lambda_{12}} e^{a\mu_2} \quad (\lambda_{12} := \lambda_1 - \lambda_2). \quad (3.3)
\]

So, we have

\[
\begin{align*}
\dot{\lambda}_1 &= \frac{ag_1}{\lambda_{12}} e^{a\mu_1}, \\
\dot{\lambda}_2 &= -\frac{ag_2}{\lambda_{12}} e^{a\mu_2}, \\
\dot{\mu}_1 &= -\frac{\partial}{\partial \lambda_1} \left( \frac{g_1}{\lambda_{12}} \right) e^{a\mu_1} - \frac{g_2}{\lambda_{12}} e^{a\mu_2}, \\
\dot{\mu}_2 &= -\frac{g_1}{\lambda_{12}^2} e^{a\mu_1} - \frac{\partial}{\partial \lambda_2} \left( \frac{g_2}{\lambda_{12}} \right) e^{a\mu_2}.
\end{align*}
\]

(3.4)

This system is separable; a complete integral of the corresponding Hamilton-Jacobi equation is \(S(\lambda_1, \lambda_2; b_1, b_2) = -b_1 t + W\). According to (2.4), \(W\) is given by

\[
W = \frac{1}{a} \int_{\lambda_1}^{\lambda_2} d\xi \log \frac{b_2 + b_1 \xi}{g_1(\xi)} + \frac{1}{a} \int_{\lambda_1}^{\lambda_2} d\xi \log \frac{b_2 + b_1 \xi}{g_2(\xi)}, \quad (3.5)
\]

therefore the general solution of Hamilton equations is

\[
\begin{align*}
t - t_0 &= \frac{1}{a} \int_{\lambda_1}^{\lambda_2} d\xi \frac{\xi}{b_2 + b_1 \xi} + \frac{1}{a} \int_{\lambda_1}^{\lambda_2} d\xi \frac{\xi}{b_2 + b_1 \xi}, \\
\beta &= \frac{1}{a} \int_{\lambda_1}^{\lambda_2} d\xi \frac{1}{b_2 + b_1 \xi} + \frac{1}{a} \int_{\lambda_1}^{\lambda_2} d\xi \frac{1}{b_2 + b_1 \xi}, \\
\mu_1 &= \frac{1}{a} \log \frac{b_2 + b_1 \lambda_1}{g_1(\lambda_1)}, \\
\mu_2 &= \frac{1}{a} \log \frac{b_2 + b_1 \lambda_2}{g_2(\lambda_2)}.
\end{align*}
\]

(3.6)

As for the map \(\Psi\), it is

\[
c_1 = -(\lambda_1 + \lambda_2), \quad c_2 = \lambda_1 \lambda_2, \quad \gamma_1 = -aH, \quad \gamma_2 = aK; \quad (3.7)
\]
the BH dynamical system \( \dot{v} = Y(v) \) is given by

\[
\dot{c}_1 = \gamma_1 , \quad \dot{c}_2 = \gamma_2 , \quad \dot{\gamma}_1 = 0 , \quad \dot{\gamma}_2 = 0 ,
\]

with Hamiltonians

\[
h_1 = \gamma_1 + \gamma_2 , \quad h_2 = \frac{1}{2}(\gamma_1^2 + \gamma_2^2).
\]

In order to transform this BH structure, it suffices to consider the inverse map \( \Psi^{-1} \) given by

\[
\lambda_{1,2} = -\frac{1}{2} c_1 \pm \frac{1}{2} (c_1^2 - 4c_2)^{1/2} ,
\]

\[
\mu_{1,2} = \frac{1}{a} \log(c_1 \gamma_1 - 2 \gamma_2 \mp \gamma_1 (c_1^2 - 4c_2)^{1/2}) - \frac{1}{a} \log(2a) .
\]

We obtain \( h_1 = a(K - H) , h_2 = \frac{1}{2} a^2(K^2 + H^2) \) and

\[
Q_0 = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix} , \quad Q_1 = \begin{bmatrix} 0 & C \\ -C^T & D \end{bmatrix} ;
\]

\( A \) and \( C \) are \((2 \times 2)\) matrices with entries

\[
A_{ij} = \frac{(-1)^{i+1}}{a^2 \lambda_{12}} \frac{1 + \lambda_i \lambda_j e^{-a \mu_j}}{g_j} \quad (i, j = 1, 2)
\]

\[
C_{ij} = (-1)^{i+j} \left( \lambda_{j+1} - \lambda_i \lambda_j + \lambda_j (\lambda_i - 1) \frac{g_{j+1}}{g_j} e^{a(\mu_{j+1} - \mu_j)} \right) \quad (i, j = 1, 2)
\]

where \( \lambda_3 := \lambda_1 , g_3 := g_1 , \mu_3 := \mu_1 \); \( B \) and \( D \) are \((2 \times 2)\) skew-symmetric matrices with

\[
B_{21} = \frac{1 + \lambda_1 \lambda_2}{a^3 g_1 g_2} \left( \frac{\partial}{\partial \lambda_1} \left( \frac{g_1}{\lambda_{12}} \right) e^{-a \mu_2} + \frac{\partial}{\partial \lambda_2} \left( \frac{g_2}{\lambda_{12}} \right) e^{-a \mu_1} \right)
\]

\[
D_{21} = \frac{1}{a^2 \lambda_{12}^3 g_1 g_2} \left( (2g_2 + g_1 \lambda_{12}) \left( g_1 \lambda_3 (1 - \lambda_1) + g_2 \lambda_1 (\lambda_2 - 1) e^{a(\mu_2 - \mu_1)} \right) \right) \quad (3.13)
\]

\[- \frac{1}{a^2 \lambda_{12}^3 g_1 g_2} \left( (2g_1 - g_1 \lambda_{12}) \left( g_2 \lambda_1 (\lambda_2 - 1) + g_1 \lambda_3 (1 - \lambda_1) e^{a(\mu_3 - \mu_2)} \right) \right) .
\]

Clearly, this BH structure is much more involved than the QBH structure \((P_0, P_1)\).
4. Concluding remarks

The dynamical system considered by F. Calogero in [1], as well as its generalization analyzed in this Letter, is naturally a Pfaffian QBH system; consequently, it is separable and therefore integrable by quadratures, the physical coordinates being just the Darboux-Nijenhuis coordinates w.r.t. \((P_0, P_1)\) (as for other examples of systems which admit this type of formulation, see [23, 24]).

A property of this system, which seems to be rather peculiar, is that the minimal polynomial of the Nijenhuis tensor \(N\) directly suggests the introduction of a non canonical map which provides the non canonical BH structure \((Q_0, Q_1)\); this procedure is an example of the strategy mentioned in item i) of the Introduction. In this regard, it is known that any Liouville integrable system can be given infinitely many non canonical BH formulations [9] in a neighborhood of an invariant Liouville torus; to this end, a non canonical map has to be constructed starting from a set of action-angles variables [25]. This procedure is quite similar to the one we have used in this Letter; the main difference is given by the fact that we can construct the non canonical map between the Darboux–Nijenhuis coordinates \(\{\lambda; \mu\}\) and the coordinates \(\{c; \gamma\}\) directly, without passing through the action–angle variables.

As a final remark, let us observe that a quite similar situation occurs if one considers the dynamical system

\[
\dot{u} = P_0 dH, \quad u^T = (s_1, s_2, \theta_1, \theta_2), \quad H = s_1(1 + s_2^2), \tag{4.1}
\]

where \(P_0\) is the canonical Poisson tensor, \((s_1, s_2), (\theta_1, \theta_2)\) are action-angles variables respectively. This system was introduced by Brouzet [7] as a counterexample to the existence of a canonical BH structure. However, since \(n = 2\), it can be endowed with a canonical (not Pfaffian) QBH structure, which is explicitly given by

\[
\dot{u} = \frac{1}{H} P_1 dK, \quad K = -\frac{1}{s_1} + 2 \arctan s_2, \quad P_1 = \begin{bmatrix} 0 & S \\ -S & 0 \end{bmatrix}, \tag{4.2}
\]

with \(S = \text{diag}(s_1, s_2)\). Thus, the Brouzet’s counterexample has both a non canonical BH structure and a canonical QBH one, as well as the system we have considered in this Letter.

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