Dissipativity Analysis for Neural Networks With Time-Varying Delays Based on Augmented Second-Order Delay-Product-Type Functionals

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ABSTRACT This paper investigates the problem of dissipativity of a class of neural networks with time-varying delays. First, a suitable augmented Lyapunov functional containing the second-order delay-product term is constructed. Based on the integral inequality method and the recently developed relaxed quadratic function negative-determination lemma, a strictly $(Q,S,R)$-$\gamma$-dissipative criterion is derived in forms of linear matrix inequality (LMI). Finally, a numerical example is used to verify the advantages of the proposed method.

INDEX TERMS Neural networks, delay-product-type Lyapunov-Krasovskii functional, dissipativity, time-varying delays.

I. INTRODUCTION

In the past few decades, neural networks have been widely used in various fields, such as pattern recognition, image processing, intelligent control, and optimization problems [1]–[8]. In the implementation of artificial neural networks, the finite switching speed of electronic amplifier and the inherent transmission time of information inevitably cause time delay. Time delay usually causes system instability or poor system performance. Thus, many delay-dependent criteria have been proposed based on the dynamic performance analysis of delayed neural networks [9]–[19]. In order to reduce the conservativeness of these criteria, many useful techniques can be applied to the performance analysis of time-delay neural networks, such as augmented Lyapunov-Krasovskii functional (LKF) [20], [21], free-matrix-based inequalities [22]–[24] and reciprocal convex combination technique [25]–[27].

Dissipative topics, as an essential property in physical systems, can intuitively reflect the loss or dissipation of energy. In the meantime, the dissipative theory provides a framework for performance analysis and synthesis design based on the input-output mathematical description related to energy [28], [29]. Therefore, since it was first proposed in [30], the topic of dissipativity has attracted extensive attention [31]–[38]. Ren et al. investigates the finite-time $L_2 - L_\infty$ gain asynchronous control problem of continuous-time positive hidden Markov jump systems by using the Takagi-Sugeno fuzzy model method [39]. Cheng et al. used $H_\infty$ performance indicators to develop a multi-targets asynchronous fault detection strategy for conic-type nonlinear jumping systems [40]. In [41], an extended dissipation concept contains $L_2 - L_\infty$ performance is proposed, which realizes the unification of passivity, dissipation, $H_\infty$ performance, and $L_2 - L_\infty$ performance. By introducing a triple-summable term in the Lyapunov functional and using a reciprocal convex combination method, the extended dissipativity problem of discrete-time neural network with time-varying delay is studied in [42]. In [43], a delay-product-type functional method is used to conduct an extended dissipativity analysis of Markovian jump neural networks with time-varying delay. In [44] and [45], a class of time-varying delay neural network $(Q,S,R)$-$\gamma$-dissipative issue is studied, using the free-matrix-based inequality method and the delay-product-type functional method, respectively, where $\gamma$ denotes the dissipativity level. It can be seen that the delay-product-type functionals play a crucial role in the dissipative analysis. But it is wondered whether the conservativeness of the
derived results can be further reduced by employing a second-order delay-product-type functional. This inspired the current research motivation.

This paper mainly constructs a second-order delay-product-type functional for dissipative analysis of delayed neural networks. Different from traditional delay-product-type functionals, this paper introduces the second-order delay product term \( d^2(t) \). The introduction of the second-order delay-product-type functional not only makes the LKF functional contain the time-delay \( d(t) \) information but also includes the time-delay squared \( d^2(t) \) information. At the same time, \( d^2(t) \) is coupled with the augmented vector, which will produce more useful information and thus obtain a less conservative result. Then, the integral term in the functional derivative is estimated by employing the integral inequality lemma is used to convert the second-order delay-dependent dissipative criterion into a dissipative criterion in the form of LMI. Finally, a numerical example is used to prove the advantages of the constructed second-order delay-product-type functional. The main contributions of this paper are listed as follows.

1) Construct an LKF that contains more useful information, such as second-order delay-product terms and augmented terms.

2) Introduce \( \nu_2(t) \) and \( \nu_3(t) \) to \( \xi(t) \) so that \( \xi(t) \) contains higher-dimensional information. Meanwhile, add two zero equations connecting the existing vectors \( \frac{1}{\alpha(t) - \beta(t)} \nu_2(t), \frac{1}{\alpha(t) - \beta(t)} \nu_3(t) \) and the augmented vectors \( \nu_2(t), \nu_3(t) \).

3) Based on the constructed LKF and integral inequalities, some improved dissipative conditions for time-delay neural networks are proposed.

Throughout the paper, \( \mathbb{R}^n \) denotes the n-dimensional Euclidean space and \( \mathbb{R}^{n \times m}, \mathbb{S}^n, \mathbb{S}^n_+ \) and \( \mathbb{D}^n_+ \) are the set of \( n \times m \) real matrices, the set of \( n \times n \) real symmetric matrices, the set of \( n \times n \) symmetric positive definite matrices and the set of \( n \times n \) diagonal positive-definite matrices, respectively; \( \mathbb{N} \) means positive integer; the superscripts \( 'T' \) stand for transpose of a matrix, respectively; \( \text{diag} \{ \cdots \} \) refers to a block-diagonal matrix; the notation \( * \) represents the symmetric term in a symmetric matrix; and the notation \( \text{Sym}[X] = X + X^T \).

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following neural networks with time-varying delay:

\[
\begin{align*}
\dot{x}(t) &= -Ax(t) + W_0 g(Wx(t)) \\
&\quad + W_1 g(Wx(t - d(t))) + u(t) \\
z(t) &= g(Wx(t))
\end{align*}
\]

where \( x(\cdot) = [x_1(\cdot) \ x_2(\cdot) \ \cdots \ x_n(\cdot)]^T \in \mathbb{R}^n \) is the neuron state vector, \( g(\cdot) = [g_1(\cdot) \ g_2(\cdot) \ \cdots \ g_m(\cdot)]^T \in \mathbb{R}^m \) denotes the neuron activation function. \( A = \text{diag}\{a_1, a_2, \cdots, a_n\} \) is a diagonal matrix with \( a_i > 0, i = 1, 2, \cdots, n \), and \( W_0, W_1 \) and \( W \) are the connection weight matrices between neurons.

The time delay, \( d(t) \), is a time-varying differentiable function that satisfies

\[
\begin{align*}
h_1(\alpha) &\leq d(t) \leq h_2(\alpha) \quad (2) \\
\mu_1 &\leq \dot{d}(t) \leq \mu_2 \quad (3)
\end{align*}
\]

where \( h_1, h_2 \) and \( \mu \) are constants. In addition, it is assumed that each neuron activation function \( g_i(\cdot) \) in (1) is satisfied the following assumption.

Assumption 1: The function \( g_i(\cdot) \) in (1) are continuous and bounded, and satisfy

\[
k_i^1 \leq \frac{g_i(\alpha_1) - g_i(\alpha_2)}{\alpha_1 - \alpha_2} \leq k_i^2, \quad i = 1, 2, \cdots, n \quad (4)
\]

where \( g_i(0) = 0, \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 \neq \alpha_2 \), and \( k_i^1 \) and \( k_i^2 \) are known real scalars. For convenience, we define \( K_1 = \text{diag}(k_1^1, k_2^1, \cdots, k_n^1) \) and \( K_2 = \text{diag}(k_1^2, k_2^2, \cdots, k_n^2) \). For any \( H, U \in \mathbb{D}^n_+ \) and \( s, s_1, s_2 \in \mathbb{R} \), the following inequalities can be obtained by (4):

\[
\rho_1(s, H) \geq 0, \quad \rho_2(s_1, s_2, U) \geq 0 \quad (5)
\]

where

\[
\begin{align*}
\rho_1(s, H) &= 2[K_1 Wx(s) - g(Wx(s))]^T H \times [g(Wx(s)) - K_2 Wx(s)] \\
\rho_2(s_1, s_2, U) &= 2[K_1 W(x(s_1) - x(s_2)) - g(Wx(s_1)) \\
&\quad + g(Wx(s_2))]^T U [g(Wx(s_1)) \\
&\quad - g(Wx(s_2)) - K_2 W(x(s_1) - x(s_2))].
\end{align*}
\]

To introduce the property of dissipativity, let us define an energy supply function as follow:

Definition 1: The system \( (1) \) is said to be strictly \((Q, S, R)\)-\(\gamma\)-dissipative if, for any \( \gamma > 0 \), the following inequality holds under zero initial condition:

\[
\int_0^\tau r(u(t), z(t))dt \geq \gamma \int_0^{\tau f} u^T(t)u(t)dt, \quad \forall \tau f > 0 \quad (6)
\]

where \( r(u(t), z(t)) \) with \( r(0, 0) = 0 \) is a defined energy supply rate function for the system \( (1) \) and satisfies

\[
r(u(t), z(t)) = z^T(t)Qz(t) + 2z^T(t)Su(t) + u^T(t)Ru(t) \quad (7)
\]

where \( Q, S, R \) are real matrices with \( Q^T = Q \) and \( R^T = R \), without loss of generality, it is assumed that \( Q \preceq 0 \).

Before presenting the main result, the following lemmas are given first.

Lemma 1: ( [46], [47]) Let \( N \in \mathbb{N}, \xi \in \mathbb{R}^m, \) and \( x \) be a continuous differentiable function: \([a, b] \rightarrow \mathbb{R}^m \). For any matrices \( Z \in \mathbb{R}^{n \times n} > 0, N \in \mathbb{R}^{3n \times m} \), the following inequalities hold:

\[
\begin{align*}
- \int_0^\beta \dot{z}^T(s)Z\dot{z}(s)ds &\leq -\frac{1}{\beta - a} \dot{z} \dot{z}^T \dot{z} \dot{z}^T \xi \\
- \int_0^\beta \dot{z}^T(s)Z\dot{z}(s)ds &\leq (\beta - \alpha)\xi^T N \xi + 2\dot{z}^T \dot{z} \dot{z}^T N \xi
\end{align*}
\]
where

\[
\begin{align*}
\tilde{\xi} &= \text{col}(x(\beta), x(\alpha), \omega_1(s), \omega_2(s)) \\
\omega_1(s) &= \frac{1}{\beta - \alpha} \int_{\alpha}^{s} x(s)ds \\
\omega_2(s) &= \frac{1}{(\beta - \alpha)^2} \int_{\alpha}^{s} \int_{\alpha}^{\theta} x(s)dsd\theta \\
\Gamma &= [\tilde{e}_1^T - \tilde{e}_2^T, \tilde{e}_1^T + \tilde{e}_2^T - 2\tilde{e}_1^T, \tilde{e}_1^T - \tilde{e}_2^T + 6\tilde{e}_1^T - 12\tilde{e}_1^T]^T \\
\bar{Z} &= \text{diag}(Z, 3Z, SZ) \\
\tilde{e}_j &= [0_{n \times (j-1)n}, I_n, 0_{n \times (4-j)n}], \ j = 1, 2, \ldots, 4
\end{align*}
\]

Lemma 2: [48] Let \( f(s) = a_2s^2 + a_1s + a_0 \), where \( s \in [h_1, h_2] \) and \( a_2, a_1, a_0 \in \mathbb{R} \). For a given positive integer \( N \), suppose that the following conditions are satisfied for \( i = 1, 2, \ldots, 2^N \),

\[
\begin{align*}
&\text{(i) } f(h_1) < 0, \\
&\text{(ii) } f(h_2) < 0, \\
&\text{(iii) } h_{12} < 2^{n+1} h_1 + h_2 + f(-\frac{1}{2^N} h_1 + h_2) < 0.
\end{align*}
\]

Then \( f(s) < 0 \).

III. MAIN RESULTS

In this section, by using second-order delay-product-type functionals, some new delay neural network dissipative conditions are established. For simplifying the presentation, we define the following notations:

\[
\begin{align*}
v_1(t) &= \int_{t-h_1}^{t} x(s)ds, \quad v_2(t) = \int_{t-h_2}^{\min\{t-d(t), h_2\}} x(s)ds \\
v_3(t) &= \int_{t-h_1}^{\min\{t-d(t), h_2\}} x(s)ds, \quad v_4(t) = \frac{1}{h_1} \int_{t-h_1}^{h_1} x(s)ds d\theta \\
v_5(t) &= \frac{1}{d(t) - h_1} \int_{t-d(t)}^{t-h_1} x(s)ds d\theta \\
v_6(t) &= \frac{1}{h_2 - d(t)} \int_{h_2}^{\min\{t-d(t), h_2\}} x(s)ds d\theta \\
\eta(t) &= [x^T(t) \gamma^T(t-h_1) \gamma^T(t-d(t)) \gamma^T(t-h_2)]^T \\
\delta(s) &= [x^T(s) \gamma^T(Wx(s))]^T \\
\eta_0(t) &= [\eta_0^T(t) \nu_1^T(t) \nu_2^T(t) \nu_3^T(t) \nu_4^T(t)]^T \\
\eta_1(s) &= [\tilde{x}^T(s) \gamma^T(s)]^T, \quad \eta_2(s) = [\eta_0^T(s) \nu_1^T(s) \nu_2^T(s)]^T \\
\eta_3(s) &= [\eta_0^T(s) \nu_2^T(s) \nu_1^T(s)]^T \\
\eta_4(t) &= [x^T(t) \nu_2^T(t)]^T, \quad \eta_5(t) = [x^T(t) \nu_3^T(t)]^T \\
\xi &= [\delta^T(t) \delta^T(t-h_1) \delta^T(t-d(t)) \delta^T(t-h_2)]^T \\
&= \frac{1}{h_1} \nu_1^T(t) \frac{1}{d(t) - h_1} \nu_1^T(t) \frac{1}{h_2 - d(t)} \nu_1^T(t) \\
&\frac{1}{h_1} \nu_1^T(t) \frac{1}{h_1} \nu_2^T(t) \frac{1}{h_2 - d(t)} \nu_2^T(t) \\
&\nu_3^T(t) \hat{\nu}^T(t-h_1) \hat{\nu}^T(t-d(t)) \hat{\nu}^T(t-h_2) \nu^T(t)]^T \\
e_i &= [0_{n \times (j-1)n}, I_n, 0_{n \times (20-j)n}], \ i = 1, 2, \ldots, 20
\end{align*}
\]

Theorem 1: For given \( h_1, h_2, \mu_1 \) and \( \mu_2, \) neural network (1) with the time delay satisfying (2) and (3) and activation function satisfying (4) is strictly \((Q, S, R)\)-dissipative, if there exist \( P \in S_+^{2n}, Q_1, Q_2, Q_3 \in S_+^{n}, R_1, R_2 \in S_+^{n}, M_1, M_2 \in S_+^{n}, M_3, M_4 \in S_+^{2n}, Z_1, Z_2 \in \mathbb{R}^{20n \times n}, N_1, N_2 \in \mathbb{R}^{3n \times 20n} \) and \( \Lambda_i, H_i, U_k \in \mathbb{R}^{n}, i \in \{1, 2, \ldots, 4\}, k \in \{1, 2, \ldots, 6\} \) such that inequality (8) is satisfied for \( d(t) \in [h_1, h_2] \) and \( d(t) \in [\mu_1, \mu_2] \),

\[
d^2(t)\Xi_2(\dot{d}(t)) + d(t)\Xi_1(\dot{d}(t)) + \Xi_0(\dot{d}(t)) < 0
\]

where

\[
\begin{align*}
\Xi_0(\dot{d}(t)) &= \gamma_0(\dot{d}(t)) + \Psi_0, \quad \Xi_1(\dot{d}(t)) = \gamma_1(\dot{d}(t)) + \Psi_1 \\
\Xi_2(\dot{d}(t)) &= \text{Sym}\{\Pi_0^T M_1 \lambda_3 + \Pi_1^T M_2 \lambda_3\} \\
\gamma_0(\dot{d}(t)) &= \text{Sym}\{\Pi_{10}^T P_{12} + h_{12}(E_1^T N_1 + E_2^T N_2) \} \\
&+ ((K_i W_1 - e_1^T T_1 A_1 + (e_2 - K_2 W_1)^T A_2) \\
&\times W_0 + h_i^T P_{13} M_2 + h_i^T P_{12} M_3 - e_i^T S_0) \\
&- Z_1(h_1 e_{10} + e_{15}) + Z_2(h_2 e_{11} - e_{15}) + \Phi \\
&- h_1^T P_{13} M_3 + h_2^T P_{12} M_3 + h_2^T P_{13} M_2 \} \\
&+ \lambda_1 \left( Q_2 - Q_1 \right) P_1 - \lambda_2 \left( Q_3 - Q_1 \right) P_2 \\
&+ \lambda_3 \left( P_{12} - P_{13} \right) Q_3 P_5 + h_1^T e_{10} R_1 e_0 + h_2^T e_{10} R_2 e_0 + (1 - d(t)) Q_4 P_4 \\
&- E_1^T R_3 e_{20} (R - y) e_{20} - e_1^T Q_5 \\
&- 2\dot{d}(t)(h_1 P_0 M_1 P_1 + h_2 P_0 M_2 P_2) \\
&+ d(t)(P_0^T S_0 - d(t)) P_0 \\
&+ \gamma_1(\dot{d}(t)) = \text{Sym}\{\Pi_{11}^T P_{12} \lambda_4 + \Pi_{13}^T P_{14} \lambda_4 - \lambda_3 P_{14} M_4 \} \\
&+ Z_1 e_{10} - 2h_1 P_{12}^T M_1 \lambda_2 - 2h_2 P_{13}^T M_2 \lambda_3 - Z_2 e_{11} \\
&+ 2d(t)(P_0^T M_1 P_0 + 2d(t)P_0^T M_2 P_2) \\
&+ \Psi_0 = h_2 h_1 N_1 R_1^{-1} N_1 - h_2 h_1 N_2 R_2^{-1} N_2 \\
&+ \Psi_1 = h_1 h_2 N_2 R_2^{-1} N_2 - h_1 h_2 N_1 R_2^{-1} N_1
\end{align*}
\]
where

\[ V(t) = \sum_{i=1}^{4} V_i(t) + V_S(t) \]

where

\[
V_i(t) = \eta_i(t)P\eta_i(t)
\]

\[
V_2(t) = \int_{t-h_1}^{t} \eta_1^T(s)Q_1\eta_1(s)ds
\]

\[
+ \int_{t-d(t)}^{t-h_1} \eta_1^T(s)Q_2\eta_1(s)ds
\]

\[
+ \int_{t-h_2}^{t-d(t)} \eta_1^T(s)Q_3\eta_1(s)ds
\]

\[
V_3(t) = h_1^2 \int_{-h_1}^{t} \dot{x}(s)R_1\dot{x}(s)ds d\theta
\]

\[
+ h_1 \int_{t-h_1}^{t} \dot{x}(s)R_1\dot{x}(s)ds d\theta
\]

\[
V_4(t) = 2 \sum_{i=0}^{n} \int_{0}^{w_{i}(t)} [\lambda_{1i}f_i^+(t) + \lambda_{2i}f_i^-(t)] ds
\]

\[
V_S(t) = (d(t) - h_1)^2 \eta_2^2(t)M_1\eta_2(t)
\]

\[
+ (h_2 - d(t))^2 \eta_2^2(t)M_2\eta_2(t)
\]

\[
+ (d(t) - h_1)\eta_2^2(t)M_3\eta_4(t)
\]

\[
+ (h_2 - d(t))^2 \eta_2^2(t)M_4\eta_5(t)
\]

with \( f_i^{+} = k_i^+ s - g_i(s), f_i^{-} = g_i(s) - k_i^- s, \Lambda_k = \text{diag}(\lambda_{k_1}, \lambda_{k_2}, \ldots, \lambda_{k_n}), k = \{1, 2\} \)

The derivative of the \( V(t) \) can be obtained as

\[
\dot{V}_1(t) = 2\eta_0^T(t)P\dot{\eta}_0(t)
\]

\[
\dot{V}_2(t) = \eta_1^T(t)Q_1\eta_1(t) + \eta_1^T(t - h_1)(Q_2 - Q_1)\eta_1(t - h_1)
\]

\[
+ (1 - d(t))\eta_1^T(t - d(t))(Q_1 - Q_2)\eta_1(t - d(t))
\]

\[
- \eta_1^T(t - h_2)Q_3\eta_1(t - h_2)
\]

\[
\dot{V}_3(t) = h_1^2 \dot{x}(t)R_1\dot{x}(t) - \theta_1
\]

\[
+ h_1^2 \dot{x}(t)R_1\dot{x}(t) - \theta_2
\]

\[
\dot{V}_4(t) = 2 \sum_{i=0}^{n} [\lambda_{1i}(k_i^+ W_i x(t) - g_i(W_i x(t)))
\]

\[
+ \lambda_{2i}(g_i(W_i x(t)) - k_i^- W_i x(t))]W_i \dot{x}(t)
\]

\[
\dot{V}_S(t) = 2 d(t)(d(t) - h_1)^2 \eta_2^2(t)M_1\eta_2(t)
\]

\[
+ 2(d(t) - h_1)^2 \eta_2^2(t)M_1\eta_2(t)
\]

\[
- 2d(t)(h_2 - d(t))^2 \eta_2^2(t)M_2\eta_3(t)
\]

\[
+ 2(h_2 - d(t))^2 \eta_2^2(t)M_2\eta_3(t)
\]

\[
+ \dot{d}(t)\eta_2^2(t)M_3\eta_4(t) - \dot{d}(t)\eta_2^2(t)M_4\eta_5(t)
\]

\[
+ 2(d(t) - h_1)\eta_2^2(t)M_3\eta_4(t)
\]

\[
+ 2(h_2 - d(t))^2 \eta_2^2(t)M_4\eta_5(t)
\]

\[
\dot{\theta}_1 = h_1 \int_{-h_1}^{t} \dot{x}(s)R_1\dot{x}(s)ds
\]

\[
\dot{\theta}_2 = h_1 \int_{-h_1}^{t} \dot{x}(s)R_2\dot{x}(s)ds
\]

Using Lemma 1 to estimate (14) and (15) yield

\[
\dot{\theta}_1 \geq \xi^T(t) E_1^T \tilde{R}_1 E_3 \xi(t)
\]

\[
\dot{\theta}_2 \geq \xi^T(t) E_2^T \tilde{R}_2 E_5 \xi(t)
\]

where

\[
\tilde{R}_1 = \text{diag}(R_1, 3R_1, 5R_1)
\]

\[
\tilde{R}_2 = \text{diag}(R_2, 3R_2, 5R_2)
\]

According to (5), the following inequalities hold:

\[
\rho_1(t, H_1) \geq 0, \rho_1(t - h_1, H_2) \geq 0
\]

\[
\rho_1(t - d(t), H_3) \geq 0, \rho_1(t - h_2, H_4) \geq 0
\]

\[
\rho_2(t - h_1, U_1) \geq 0, \rho_2(t - h_2, U_2) \geq 0
\]

\[
\rho_2(t - d(t), U_3) \geq 0, \rho_2(t - d(t), U_4) \geq 0
\]

\[
\rho_2(t - h_1, U_5) \geq 0, \rho_2(t - h_2, U_6) \geq 0
\]

which means

\[
\xi^T(t)(\tilde{\Phi} + \tilde{\Phi}^T)\xi(t) \geq 0
\]

where \( \tilde{\Phi} \) is defined in Theorem 1.
Adding the left-hand side of (18) - (20) to ̇V(t) and applying (16)-(17), we have

\[ ̇V(t) - r(u(t), z(t)) + γ u^T(t)u(t) \leq ξ^T(t)(d^2(t)ξ_d(t)) + d(t)ξ_1( ̇d(t)) + Ξ_0( ̇d(t)))ξ(t) \]

with Ξ2( ̇d(t)), Ξ1( ̇d(t)), Ξ0( ̇d(t)) are defined in Theorem 1. It follows from (8) that

\[ ̇V(t) - r(u(t), z(t)) + γ u^T(t)u(t) \leq 0 \quad (21) \]

Integrating (21) with an integration interval of [0, t_f] yields

\[ γ \int_0^{t_f} u^T(t)u(t)dt - \int_0^{t_f} r(u(t), z(t))dt \leq -V(t_f) + V(0) \]

Under the zero initial condition V(0) = 0 and ∀t_f > 0, it can be clearly obtained:

\[ γ \int_0^{t_f} u^T(t)u(t)dt - \int_0^{t_f} r(u(t), z(t))dt \leq -V(t_f) \leq 0, \]

which implies that (6) holds. Thus, neural network (1) is strictly (Q, S, R)-γ-dissipative. The proof is completed.

**Remark 1:** By introducing u2(t) and u3(t) in ̇ξ(t), the constructed LKF can process the time-delay information of a higher dimension in the same negative-determination method. For example, the functional term ( ̇d(t) - h1)υ2(t)Mυ2(t), if there is no u2(t) term in ̇ξ(t), then the functional derivative will be a formula containing d^2(t). However, with the addition of u2(t), the LKF derivative is just an expression containing ̇d(t), which means that it is possible to handle a functional containing ( ̇d(t) - h1)υ2(t)Mυ2(t) without using the negative-determination method like Lemma 2. Therefore, adding u2(t) and u3(t) to ̇ξ(t) in this paper will make LKF contain more information.

**Remark 2:** The introduced zero equation (19) builds a bridge for the connection between \( \frac{1}{d(t) - h1}υ2(t) \), \( \frac{1}{h2_d(t)}υ3(t) \), and u2(t), u3(t), so that the dissipative criterion can be solved. At the same time, the free variables Z1 and Z2 are introduced to consider the relationships between these vectors.

**Remark 3:** Compared with the existing delay-product-type functional \( d(t)υ2(t)^T(t)Mυ2(t) \), the second-order delay-product-type functional V(t) in this paper also includes d^2(t)υ2(t)^T(t)Mυ2(t). This makes LKF consider not only the information of time delay d(t) but also the information of second-order time-delay d^2(t). In the meantime, the augmented items u2(t) and u3(t) are included in ̇ξ(t). Therefore, the second-order delay-product-type functional V(t) considers more comprehensive time-delay information. It helps reduce the conservativeness of dissipative conditions.

**Remark 4:** Since the LMI toolbox in Matlab cannot solve the quadratic function inequality (8), Lemma 2 is adopted to transform this inequality into LMIs. By using Lemma 2 with N = 1 and Schur complement, the following condition is obtained.

**Corollary 1:** For given h1, h2, μ1 and μ2, neural network (1) with the time delay satisfying (2) and (3) and activation function satisfying (4) is strictly (Q, S, R)-γ-dissipative, if there exist P ∈ S^m, Q1 ∈ S^m, Q2 ∈ S^m, Q3 ∈ S^m, R1 ∈ S^m, R2 ∈ S^m, M1, M2 ∈ S^m, Z1, Z2 ∈ R^{2m×n}, N1, N2 ∈ R^{3n×2n} and Λ, H1, H2 ∈ S^m, i ∈ [1, 2], j ∈ [1, 2, 3, 0], k ∈ [1, 2, 3, 0], such that inequalities (22) - (25) are satisfied for ̇d(t) ∈ [μ1, μ2]

\[ \begin{bmatrix} Ω1( ̇d(t)) & h12N1^T_1 \\ * & -R2 \end{bmatrix} < 0 \quad (22) \]

\[ \begin{bmatrix} Ω2( ̇d(t)) & h12N2^T_1 \\ * & -R2 \end{bmatrix} < 0 \quad (23) \]

\[ \begin{bmatrix} Ω3( ̇d(t)) & \frac{\sqrt{3}}{2}h12N1^T_1 \\ * & -R2 \\ * & -R2 \end{bmatrix} < 0 \quad (24) \]

\[ \begin{bmatrix} Ω4( ̇d(t)) & \frac{1}{2}h12N1^T_1 \\ * & -R2 \\ * & -R2 \end{bmatrix} < 0 \quad (25) \]

where

\[ Ω1( ̇d(t)) = Ξ2( ̇d(t))h12 + Υ1( ̇d(t))h1 + Υ0( ̇d(t)) \]

\[ Ω2( ̇d(t)) = Ξ2( ̇d(t))h12 + Υ1( ̇d(t))h2 + Υ0( ̇d(t)) \]

\[ Ω3( ̇d(t)) = Ξ2( ̇d(t))(h12^2 + \frac{1}{2}h12h12) \]

\[ + Υ1( ̇d(t))(h1 + \frac{1}{4}h12) + Υ0( ̇d(t)) \]

\[ Ω4( ̇d(t)) = Ξ2( ̇d(t))\left(\frac{1}{2}h12 + \frac{3}{4}h12 + h12^2\right) \]

\[ + Υ1( ̇d(t))(\frac{3}{4}h12 + h1) + Υ0( ̇d(t)) \]

To better reflect the advantages of the second-order delay-product-type functional constructed in this paper, the following Corollary can be obtained by setting M1 = M2 = 0 in Corollary 1.

**Corollary 2:** For given h1, h2, μ1 and μ2, neural network (1) with the time delay satisfying (2) and (3) and activation function satisfying (4) is strictly (Q, S, R)-γ-dissipative, if there exist P ∈ S^m, Q1, Q2, Q3 ∈ S^m, R1, R2 ∈ S^m, M1, M2 ∈ S^m, Z1, Z2 ∈ R^{2m×n}, N1, N2 ∈ R^{3n×2n} and Λ, H1, H2 ∈ S^m, i ∈ [1, 2], j ∈ [1, 2, 3, 0], k ∈ [1, 2, 3, 0], such that inequalities (26) - (27) are satisfied for ̇d(t) ∈ [μ1, μ2]

\[ \begin{bmatrix} Ω1( ̇d(t)) & h12N1^T_1 \\ * & -R2 \end{bmatrix} < 0 \quad (26) \]

\[ \begin{bmatrix} Ω2( ̇d(t)) & h12N2^T_1 \\ * & -R2 \end{bmatrix} < 0 \quad (27) \]

where

\[ Ω1( ̇d(t)) = Υ1( ̇d(t))h1 + Υ0( ̇d(t)) \]

\[ Ω2( ̇d(t)) = Υ1( ̇d(t))h2 + Υ0( ̇d(t)) \]

**Remark 5:** In terms of LKF, the difference between Corollary 2 and [45] is that Corollary 2 adds augmentation terms u2(t) and u3(t) to the delay-product-type functionals. Therefore, the advantages of the proposed augmented terms
can be verified by comparing the results obtained by Corollary 2 with those in [45].

IV. NUMERICAL EXAMPLES

This section provides a numerical example to show the effectiveness and advantages of the presented method.

Example 1: Consider neural network (1) with \( W_0 = 0, W_1 = I \), and the other parameters are given as follows:

\[
A = \text{diag}(7.0214, 7.4367), \\
W = \begin{bmatrix}
-6.4493 & -12.0275 \\
-0.6867 & 5.6614
\end{bmatrix}, \\
K_1 = \text{diag}(0.4, 0.4), \ K_2 = \text{diag}(0, 0), \\
Q = \text{diag}(-1, -1), \ R = \text{diag}(3, 3), \\
S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.
\]

This example was recently employed in [45] and [49] to compare the superiority of dissipative conditions. Given the two cases of parameters, \( h_1 = 0.8, h_2 = 1.0 \), and \( h_1 = 0.1, h_2 = 0.3 \), the optimal dissipation performance \( \gamma^* \) is listed in Table 1 under different methods. At the same time, Figure 1 provides an intuitive diagram of the optimal dissipation performance index \( \gamma^* \) obtained by different methods under \( h_1 = 0.8, h_2 = 1.0 \). By comparing the results in Table 1 and Figure 1, the following conclusions can be obtained.

- The results obtained in Corollary 1 and Corollary 2 outperform others in the existing literature, indicating that the augmented second-order delay-product-type LKF proposed in this paper is helpful to improve the dissipative index \( \gamma^* \).
- By comparing the index \( \gamma^* \) obtained by Corollary 1 and Corollary 2, it illustrates the advantages of the second-order delay-product-type functional \( V_2(t) \).
- The advantages of the augmented terms in this paper can be proved by comparing the results obtained in Corollary 2 and [45].
- In particular, when estimating the integral term of the functional derivative, the third-order free-matrix-based inequality used in [45], and this paper uses the second-order ones. Obviously, the \( \gamma^* \) obtained by Corollary 1 is still better than [45]. This shows that the augmented second-order delay-product-type LKF constructed in this paper are superior to the one employed in [45].

V. CONCLUSION

This paper studies the \((\mathcal{Q},\mathcal{S},\mathcal{R})\)-\(\gamma\)-dissipative problem of a class of delayed neural networks. By introducing the time-delay square term, a second-order delay-product-type functional is constructed. The Bessel-Legendre inequality and generalized free-matrix-based inequality method are used to estimate the integral term in the derivative of LKF functional. Then, some dissipative criteria in the form of LMI are proposed by using a relaxed quadratic function negative-determination lemma. Finally, a numerical example proves the advantages and effectiveness of the proposed method.

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