The limit point in the Jante’s law process has a continuous distribution

Edward Crane* and Stanislav Volkov†

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Abstract

We study a stochastic model of consensus formation, introduced in 2015 by Grinfeld, Volkov and Wade, who called it a multidimensional randomized Keynesian beauty contest. The model was generalized by Kennerberg and Volkov, who called their generalization the Jante’s law process. We consider a version of the model where the space of possible opinions is a convex body $B$ in $\mathbb{R}^d$. $N$ individuals in a population each hold a (multidimensional) opinion in $B$. Repeatedly, the individual whose opinion is furthest from the center of mass of the $N$ current opinions chooses a new opinion, sampled uniformly at random from $B$. Kennerberg and Volkov showed that the set of opinions that are not furthest from the center of mass converges to a random limit point. We show that the distribution of the limit opinion is continuous, thus proving the conjecture made after Proposition 3.2 in Grinfeld et al.

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1 Introduction

1.1 The multidimensional randomized Keynesian beauty contest

Let $N \geq 3$ and $d \geq 1$ be integers. Let $\lambda$ denote Lebesgue measure on $\mathbb{R}^d$ and let $B$ be a convex body, i.e. a closed convex subset of $\mathbb{R}^d$ with non-empty interior. In particular, $\lambda(B) > 0$. For

*School of Mathematics, University of Bristol, BS8 1TH, UK
†Centre for Mathematical Sciences, Lund University, Box 118 SE-22100, Lund, Sweden
the moment, while describing the model studied in [1], we will also assume that \( B \) is bounded, so that \( \lambda(B) < \infty \), but later we will drop this assumption. We say that a random variable is uniform on \( B \), i.e. it is a \( U(B) \) random variable, when its distribution is proportional to \( \lambda|_B \). (This notion requires \( \lambda(B) < \infty \), of course.)

A discrete time Markov process \( X(t)_{t=0}^\infty \), taking values in \( ([0,1]^d)^N \), called the multidimensional randomized Keynesian beauty contest was studied in [1]. In the sequel [2] a generalization of the process was studied which the authors called the Jante’s law process \(^1\) (see §1.4). In the present paper the process \( X(t) \) will take values in \( B^N \), so our setting is only a little more general than that of [1]; however we prefer to retain the shorter name for the process.

If \( X \) is either a finite sequence \( (x_1,\ldots,x_m) \) of \( m \geq 1 \) points of \( \mathbb{R}^d \), or a set \( \{x_1,\ldots,x_m\} \) of \( m \geq 1 \) distinct points of \( \mathbb{R}^d \), then we denote by \( \mu(X) \) its center of mass, i.e.

\[
\mu(X) = \frac{1}{m}(x_1 + \cdots + x_m).
\]

We denote by \( \|x-y\| = \sqrt{\sum_{i=1}^d (x_i - y_i)^2} \) the Euclidean distance between \( x \) and \( y \) in \( \mathbb{R}^d \), and by

\[
B(x,\rho) = \{y \in \mathbb{R}^d : \|x-y\| < \rho\}
\]

the open ball of radius \( \rho \) centered at \( x \).

Let \( X(0) = (X_1(0),\ldots,X_N(0)) \in B^N \) be a possibly random initial state. We will always assume that a.s. the \( N \) points of \( X(0) \) are distinct. An informal description of each step of the Markov chain \( X(t) \) is that a new point arrives, distributed uniformly in \( B \), so that now there are \( N+1 \) points, and then the point among these \( N+1 \) that is furthest from the new center of mass is thrown out, so that \( N \) points remain.

Formally, let \( (\zeta_t)_{t=1}^\infty \) be an i.i.d. sequence of random variables uniformly distributed on \( B \), denoted by \( U(B) \), independent of the initial state \( X(0) \). For each time \( t = 0,1,2,\ldots \) in turn, let \( j(t) \) be the index of the furthest point of \( X(t) \) from \( \mu(X(t)) \), that is, \( j(t) \) is defined by

\[
\|X_{j(t)}(t) - \mu(X(t))\| = \max \{\|X_i(t) - \mu(X(t))\| : 1 \leq i \leq N\} ;
\]

in case of a tie, choose \( j(t) \) uniformly at random among the tied indices\(^2\). Define \( X(t+1) \) by

\[
X_i(t+1) = \begin{cases} X_i(t), & \text{if } i \in \{1,\ldots,N\} \setminus \{j_t\}, \\ \zeta_t, & \text{if } i = j_t, \end{cases}
\]

\(^1\)The law of Jante is literary caricature of the virtue in Scandinavian culture of not standing out from the crowd. It appeared in Aksel Sandemose’s satirical novel En flyktning krysser sitt spor, (A fugitive crosses his tracks), published in 1933.

\(^2\)Any other way of breaking ties would also be acceptable, since a.s. ties do not occur.
i.e., the point of \( X(t) \) that is furthest from the center of mass of \( (X_1(t), \ldots, X_N(t)) \) is removed and its place is taken by the new point \( \zeta_t \). Note that although \( X(t + 1) \) and \( \mu(X(t + 1)) \) depend continuously on the new point \( \zeta_t \), they depend discontinuously on \( \zeta_{t-1} \).

Let \( X'(t) \) denote the core of the configuration \( X(t) \), that is the set \( \{X_i(t) : i \neq j_t\} \). It has \( M := N - 1 \) points.

**Definition 1.** We say that \( X'(t) \) converges to some point \( \xi \in \mathbb{R}^d \) if

\[
\text{for all } \varepsilon > 0 \text{ there is } t_\varepsilon < \infty \text{ such that } X'(t) \subset B(\xi, \varepsilon) \text{ for all } t > t_\varepsilon .
\]

It was shown in [1] that \( X'(t) \) converges a.s. to a random limit point \( \xi \) in the above sense; additionally, in the case \( d = 1 \), the distribution of \( \xi \) has support \([0, 1]\). It was conjectured\(^3\) that the distribution of \( \xi \) is, in fact, continuous. **The purpose of this paper is to prove this continuity conjecture in the \( d \)-dimensional setting described above.**

A.s. for every \( t \geq 0 \), \( X(t) \) is a sequence of \( N \) distinct points, and it will be convenient to discard from the underlying probability space the null set where this fails. Hence, from now on, we work with a version of the process where for each \( t \) the points of \( X(t) \) are surely distinct. In particular, the core process \( X'(t) \) a.s. takes its values in the set of subsets of \( \mathcal{B} \) of cardinality \( M \). Assuming \( X'(t) = S \), we can ask what is the set of possible values of \( \zeta_t \) which would result in \( \zeta_t \in X'(t + 1) \), that is, that we keep the newly sampled point in the core while throwing out one of the original points. The answer depends only on \( S \), and not on the point \( X_{j_t}(t) \). The set of possible locations which could enter the core at time \( t + 1 \) is denoted by

\[
\text{Keep}(S; \mathcal{B}) := \{ x \in \mathcal{B} : \| x - \mu(\{x\} \cup S) \| < \max(\{\| s - \mu(\{x\} \cup S) \| : s \in S\}) \}. \quad (1)
\]

For later use, we extend the definition \([1]\) to allow \( \mathcal{B} \) to be unbounded.

Observe that in the one-dimensional case \( \mathcal{B} \) is necessarily an interval \( [a, b] \subset \mathbb{R} \); and then \( \text{Keep}(S; \mathcal{B}) \) is also a non-empty subinterval of \([a, b]\). If \( d \geq 2 \) the geometry of this set is a little more complicated, see e.g. Figure\([1]\). Later, in Lemma\([2]\) we will show that \( \text{Keep}(S; \mathcal{B}) \) is in fact always the intersection of \( \mathcal{B} \) with a non-empty union of at most \( M \) bounded open balls which only depend on \( S \). Moreover, \( \text{Keep}(S, \mathcal{B}) \) always contains a nonempty open ball whose center is \( \mu(S) \). As a result, \( \text{Keep}(S; \mathcal{B}) \) always has finite and positive Lebesgue measure, so that it makes sense to sample a point uniformly from \( \text{Keep}(S; \mathcal{B}) \). In fact, conditional on \( X'(t) = S \) and \( X'(t + 1) \neq X'(t) \), the new point \( \zeta_t \) is uniformly distributed on \( \text{Keep}(S; \mathcal{B}) \).

\(^3\)The conjecture is stated immediately after [1, Proposition 3.2].
Figure 1: $d = 2$ and $M = 3$. In this example, the core consists of points $(0, 0)$, $(10, 0)$, and $(4, 6)$. A newly sampled point will join the core if and only if it lies in one of the three shaded areas; the colors indicate which point will be removed.

1.2 Time-changed models

For convenience, we will work with a time-change of the process $X'(t)$, namely the sequence $Y(0), Y(1), Y(2), \ldots$ defined by $Y(0) = X'(0)$ and for $n \geq 1$, $Y(n) = X'(t_n)$, where $t_1 < t_2 < t_3 < \ldots$ is the increasing sequence of all times $t \geq 1$ for which $X'(t) \neq X'(t - 1)$. The sequence $Y(n)_{n \geq 0}$ is itself a Markov chain. It is an instance of a $B$-valued Jante's law process defined below. When studying $Y(\cdot)$, there is no longer any reason to insist that $B$ be bounded, since we can describe the law of $Y(\cdot)$ without making any reference to $U(B)$ random variables. Instead, the definition involves sampling uniformly from sets of the form $\text{Keep}(S; B)$, which always have finite positive Lebesgue measure, regardless of whether $B$ is bounded. So in the following definition and for the rest of the paper we allow $B$ to be unbounded.

Definition 2 ($B$-valued Jante's law process $Y(n)_{n \in \mathbb{N}_0}$). Let $B$ be any closed convex subset of $\mathbb{R}^d$ with non-empty interior. Let $Y(0)$ be any (possibly random) set of $M \geq 2$ distinct points in $B$. For each $n \geq 1$, sample a point $y_n$ uniformly from $\text{Keep}(Y(n-1); B)$, conditionally independently of $(Y(i) : 0 \leq i < n - 1)$ given $Y(n-1)$, and set

$$Y(n) := \{y_n\} \cup Y(n-1) \setminus \{r_n\}, \quad (2)$$

where $r_n$ is the (a.s. unique) point $r \in Y(n-1)$ which maximizes $\|r - \mu(Y(n-1) \cup \{y\})\|$. 

By definition, a $B$-valued Jante's law process is a Markov chain. When working with $B$-valued Jante's law processes, it is convenient to discard from our probability space the null set
where for any $n$ either there is a tie in choosing $r_n$, or $y_n \in \bigcup_{i=0}^{n-1} Y(i)$. From the definition of \( \text{Keep}(S; \mathcal{B}) \), it is surely the case that $r_n \neq y_n$, so $Y(n) \neq Y(n-1)$.

By a slight modification of the proof in [1] we will show that even when $\mathcal{B}$ is unbounded, the $\mathcal{B}$-valued Jante’s law process $Y(\cdot)$ a.s. converges to a random limit point $\xi$, in the sense that

\[(\forall \varepsilon > 0)(\exists n_\varepsilon < \infty)(n > n_\varepsilon \implies Y(n) \subset B(\xi, \varepsilon)).\]

Our strategy for proving that the limit point $\xi$ is a continuous random variable is to compare $Y(\cdot)$ with an $\mathbb{R}^d$-valued Jante’s law process denoted by $Z(n)_{n \in \mathbb{N}_0}$; we call the Markov chain $Z(\cdot)$ the \textit{scale-free core process}. It eliminates the complicated effect of the boundary of $\mathcal{B}$ and has the great advantage of being invariant in law with respect to scaling and translation.\(^4\) It will be useful to have notation for the points added and removed at each step of the scale-free core process $Z(n)$:

\[Z(n) = \{z_n\} \cup (Z(n-1) \setminus \{s_n\}).\]

Also let $z_\infty$ denote the a.s. limit of the process $Z(n)$, in the sense of Definition [1].

### 1.3 Outline and results

In §2 we collect a number of basic results, including the fact that every $\mathcal{B}$-valued Jante process almost surely converges to a random limit (Proposition [1]). We give a standalone proof because for the case where $\mathcal{B}$ is unbounded this is not a special case of the results of [1, 2], and because the required lemmas are needed again later in the paper.

In §3 we discuss a property of convex bodies called \textit{uniform geometry}. This property is used later in the paper to avoid some geometric complications. Bounded convex sets have uniform geometry. We reduce the continuity of the distribution of the limit point of a $\mathcal{B}$-valued Jante process to the case where $\mathcal{B}$ is bounded.

Our first continuity result is for the scale-free core process:

**Theorem 1.** For an arbitrary deterministic initial condition $Z(0) = S$, where $S$ is any set of $M$ distinct points in $\mathbb{R}^d$, the distribution $\pi_S$ of $z_\infty$ is continuous.

Our proof of Theorem [1] is less straightforward than one might expect. The main difficulty is to show that a.s. all of the original points are eventually removed; we isolate this statement as Proposition [2] and prove it in §4 using a supermartingale argument, a trick using translational symmetry, and a compactness argument. Then in §5 we complete the proof of Theorem [1].

\(^4\) An analogous modification of the original process $X(t)$ was already used as a tool in [1, §3.4].
by showing that the distribution of the limit is a mixture of continuous distributions. Each
distribution in the mixture is seen to be continuous because it is a convolution with a uniform
distribution over a set of positive Lebesgue measure.

In §6 we use another supermartingale argument and a coupling argument to deduce the
continuity of the distribution of the limit point for any \( B \)-valued Jante’s law process.

**Theorem 2.** Let \( B \subset \mathbb{R}^d \) be any convex body, and let \( S \) be any set of \( M \) distinct points of \( B \). Let \( \xi \) be the limit point of the \( B \)-valued Jante’s law process \( Y(\cdot) \) started at \( Y(0) = S \). Then the
distribution of \( \xi \) is continuous.

Of course this also shows that the distribution of the random limit \( \xi \) of the original core
process \( X'(t) \) started at \( X'(0) = S \) has a continuous distribution. Theorem 2 covers the original
setting of [1] where \( B = [0,1]^d \), so it resolves the continuous distribution conjecture made in
that paper. We remark that if \( X'(0) \) is random, then the distribution of the limit point \( \xi \) is
still continuous, since it is a mixture of such distributions. We also remark that \( P(\xi \in \partial B) = 0 \)
because \( \lambda(\partial B) = 0 \) for any convex body \( B \subseteq \mathbb{R}^d \).

1.4 Related models

In the paper [2], the authors generalized the original model of [1] by allowing the common dis-
btribution of the new points \( \zeta_t \) to be an arbitrary continuous distribution on \( \mathbb{R}^d \). They also allowed
for more than one point to be replaced simultaneously. It was shown that if the distribution
of \( \zeta_t \) has bounded support then a.s. a limit point \( \xi \) exists. However, although the support of \( \xi \) is
certainly a subset of the support of \( \zeta \), it may not be the whole of the support of \( \zeta \). For example,
if \( \zeta_t \) is uniform on \([0,1] \cup [2,3] \) and a.s. all coordinates of \( X(0) \) lie in \([0,1] \), then \( X'(t) \subset [0,1] \)
for all \( t \), and after passing to the subsequence of times at which \( \zeta_t \in [0,1] \), we recover the original
process with \( \zeta \sim U([0,1]) \). In particular, in this case the support of \( \xi \) is \([0,1] \). In case \( \zeta \)
has unbounded support, [2] shows that a.s. either \( X'(t) \) converges to a random limit point \( \xi \) or
\( X'(t) \to \infty \), i.e. \( \inf\{|x| : x \in X'(t)\} \to \infty \). However, no examples of distributions of \( \zeta_t \) are
known to satisfy \( X'(t) \to \infty \) with positive probability. For an arbitrary common distribution of
the random variables \( \zeta_t \), and an arbitrary distribution of \( X(0) \) on \( \mathbb{R}^d \), such that there is a positive
probability that \( X(t) \) converges, one can ask whether the distribution of the limit \( \xi \) conditional
on its existence is absolutely continuous with respect to the distribution of the \( \zeta_t \). We do not
address this more general continuity problem in this paper, although it might be possible to
extend our techniques to other cases where \( \zeta_t \) has a continuous distribution with a sufficiently
well-behaved density.
In another paper \cite{3}, the authors studied asymptotic properties of a modified version of Jante’s law process. In this model, called the \textit{p-contest}, at each moment of time the point farthest from \(p\mu\), where \(\mu\) is the current centre of mass, is removed and replaced by an independent \(\zeta\)-distributed point; note that the case \(p = 1\) would correspond to the original Jante’s law process. Finally, in \cite{4}, a local version of the process was considered. In this version, \(N\) vertices are placed on a circle, so that each vertex has exactly two neighbours. To each vertex assign a real number, called \textit{fitness}. At each unit of time the vertex whose fitness deviates most from the average of the fitnesses of its two immediate neighbours has its fitness replaced by a random value drawn independently according to some distribution \(\zeta\). The authors showed that if \(\zeta\) has a uniform or a discrete uniform distribution, all fitnesses but at most one converge to the same limit. We would also like to mention that there is a related continuous-time model, called “Brownian bees”, which involves a fixed number \(N\) of particles which perform independent Brownian motions, and branch from time to time simultaneously with removal of the particle most distant from the origin, thus keeping the total number of particles constant; see \cite{6}.

2 Preliminaries

For any set \(X \subset \mathbb{R}^d\) we denote by \(\text{Conv}(X)\) the convex hull of \(X\), i.e. the smallest convex set containing \(X\). We denote by \(\partial X\) the boundary of \(X\) and by \(X^o\) the interior of \(X\). For any \(\delta > 0\) we denote by \(N_\delta(X)\) the \(\delta\)-neighbourhood of \(X\), i.e. the set of points whose Euclidean distance to \(X\) is less than \(\delta\).

Some basic facts about continuity of \(\mathbb{R}^d\)-valued random variables and continuity of Borel probability measures on \(\mathbb{R}^d\) are collected in the appendix. The main two facts that we will use are that any mixture of continuous random variables is continuous, and that if a continuous random variable is conditioned on an event of positive probability then the conditioned random variable is continuous. We will see a simple argument that uses these two facts in the proof of Lemma \ref{lemma10} at the end of section \ref{section3}.

For any set \(X = \{x_1, \ldots, x_M\} \subset \mathbb{R}^d\) of cardinality \(M = N - 1\) we let

\[
\Sigma(X) = M \mu(X) = \sum_{i=1}^{M} x_i
\]

and we define the following geometric functionals of \(X\):

\[
\begin{align*}
F(X) &= \sum_{1 \leq i < j \leq M} \|x_i - x_j\|^2 = M \sum_{i=1}^{M} \|x_i - \mu(X)\|^2, \\
A(X) &= \max_{w \in X} \|w - \mu(X)\|, \\
D(X) &= \max_{i \neq j} \|x_i - x_j\|.
\end{align*}
\]
We call $F(X)$ the moment of inertia of $X$. The functional $F$ serves as a Lyapunov function for the Jante’s law processes $Y(\cdot)$ and $Z(\cdot)$. One can easily verify the following inequalities among the geometric functionals:

\[
\frac{M}{M-1} A(X) \leq D(X) \leq 2A(X), \\
\frac{M(M-1)}{2} d(X)^2 \leq F(X) \leq \frac{M(M-1)}{2} D(X)^2, \\
\frac{M^2}{(M-1)^2} A(X)^2 \leq F(X) \leq M^2 A(X)^2.
\]

(4)

**Lemma 1.** Let $X = \{x_1, \ldots, x_M\}$ be any set of $M \geq 2$ distinct points in $\mathbb{R}^d$, and suppose $z \in \mathbb{R}^d \setminus X$. Then

\[
F(X) - F(\{z\} \cup X \setminus \{x_j\}) = (M+1) \left( \|x_j - \frac{z + \Sigma(X)}{M+1}\|^2 - \|z - \frac{z + \Sigma(X)}{M+1}\|^2 \right).
\]

(5)

\[
= (M-1) \left( \|x_j - \frac{\Sigma(X) - x_j}{M-1}\|^2 - \|z - \frac{\Sigma(X) - x_j}{M-1}\|^2 \right). \quad (6)
\]

From (5) it follows that a point $z \in \mathcal{B} \setminus X$ belongs to $\text{Keep}(X; \mathcal{B})$ if and only if there exists $x \in X$ such that $F(\{z\} \cup (X \setminus \{x\})) < F(X)$. Moreover, if $z \in \text{Keep}(X; \mathcal{B})$ then any choice of $j$ which maximizes $\|x_j - \frac{\Sigma(X) + x}{M+1}\|$ is also a choice which minimizes $F(\{z\} \cup (X \setminus \{x_j\}))$.

**Proof.** We have

\[
F(X) - F(\{z\} \cup (X \setminus \{x_j\})) = \sum_{i: i \neq j} \|x_j - x_i\|^2 - \sum_{i: i \neq j} \|z - x_i\|^2
\]

\[
= \sum_{i: i \neq j} (\|x_j\|^2 - \|z\|^2 - 2(x_j - z) \cdot x_i)
\]

\[
= (M-1)(x_j - z) \cdot (x_j + z) - 2(x_j - z) \cdot (\Sigma(X) - x_j)
\]

\[
= (x_j - z) \cdot ((M+1)(x_j + z) - 2z - 2\Sigma(X)).
\]

On the other hand, expanding the squared distances on the RHS of (5) as dot products, the terms $\pm (M+1) \frac{\Sigma(X)}{M+1} \cdot z$ cancel, and we are left with the same quantity:

\[
(M+1)(\|x_j\|^2 - \|z\|^2 - 2(x_j - z) \cdot (z + \Sigma(X))) = (x_j - z) \cdot ((M+1)(x_j + z) - 2z - 2\Sigma(X)).
\]

The proof of (6) is a similar calculation. The remaining statements follow immediately from (5).

**Corollary 1** (Essentially Lemma 2.1 in [1]). Let $Y(\cdot)$ be a $\mathcal{B}$-valued Jante’s law process with $M \geq 2$ (where $\mathcal{B}$ may be $\mathbb{R}^d$). Then $F(Y(n)) < F(Y(n-1))$ for each $n \geq 1$. 

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Lemma 2. Let $X = \{x_1, \ldots, x_M\} \subset \mathcal{B}$, where $M \geq 2$. Then

$$\text{Keep}(X; \mathcal{B}) = \mathcal{B} \cap \bigcup_{j=1}^{M} B \left( \frac{\Sigma(X) - x_j}{M - 1}, \frac{M}{M - 1}\|x_j - \mu(X)\| \right),$$

(7)

Proof. Let $X = \{x_1, \ldots, x_M\}$. For any point $z \in \mathbb{R}^d$, $z \in \text{Keep}(X; \mathbb{R}^d)$ if and only if there exists $j \in \{1, \ldots, M\}$ such that

$$\left\| z - \frac{z + \Sigma(X)}{M + 1} \right\| < \left\| x_j - \frac{z + \Sigma(X)}{M + 1} \right\|.$$  

(8)

Straightforward algebraic manipulation shows that each of the following three inequalities is equivalent to (8).

$$\| Mz - \Sigma(X) \| < \|(M + 1)x_j - z - \Sigma(X)\|$$

$$\sum_{i \neq j} \|z - x_i\|^2 < \sum_{i \neq j} \|x_j - x_i\|^2$$

$$\left\| z - \frac{\Sigma(X) - x_j}{M - 1} \right\| < \left\| x_j - \frac{\Sigma(X) - x_j}{M - 1} \right\| = \frac{M}{M - 1}\|x_j - \mu(X)\|. $$

(9)

The expression (7) follows from the last of these inequalities. \qed

Lemma 3. Let $X = \{x_1, \ldots, x_M\} \subset \mathcal{B}$, where $M \geq 2$. Then

$$\text{Keep}(X; \mathcal{B}) \subseteq \mathcal{B} \cap B \left( \mu(X), \frac{M + 1}{M - 1}A(X) \right).$$

(10)

Proof. For every $j \in \{1, \ldots, M\}$ we have

$$\left\| \mu(X) - \frac{\Sigma(X) - x_j}{M - 1} \right\| = \frac{1}{M - 1}\|x_j - \mu(X)\| \leq \frac{1}{M - 1}A(X).$$

(11)

Suppose $z \in \text{Keep}(X; \mathbb{R}^d)$ and let $j$ be such that the equivalent inequalities (8)-(9) hold. Then by (9) we have

$$\left\| z - \frac{\Sigma(X) - x_j}{M - 1} \right\| < \frac{M}{M - 1}\|x_j - \mu(X)\| \leq \frac{M}{M - 1}A(X).$$

(12)

Summing the inequalities (11) and (12), and using the triangle inequality, we obtain

$$\|z - \mu(X)\| \leq \left\| z - \frac{\Sigma(X) - x_j}{M - 1} \right\| + \left\| \mu(X) - \frac{\Sigma(X) - x_j}{M - 1} \right\| < \frac{M + 1}{M - 1} \cdot A(X),$$

i.e. $z \in B \left( \mu(X), \frac{M + 1}{M - 1}A(X) \right).$ \qed
Lemma 4. Let \( X = \{x_1, \ldots, x_M \} \subset \mathcal{B} \), where \( M \geq 2 \) and the \( x_i \) are distinct. Then
\[
\mathcal{B} \cap B(\mu(X), A(X)) \subseteq \text{Keep}(X; \mathcal{B}).
\] (13)

Proof. Let \( x_j \) be a point of \( X \) furthest from \( \mu(X) \). Then \( \mu(X) \) lies on the line segment joining \( \frac{\Sigma(X) - x_j}{M-1} \) and \( x_j \), since \( \mu(X) = (1 - \frac{1}{M}) \left( \frac{\Sigma(X) - x_j}{M-1} \right) + \frac{1}{M} x_j \). We have \( \|\mu(X) - x_j\| = A(X) \) and \( \left\| \frac{\Sigma(X) - x_j}{M-1} - x_j \right\| = \left\| \frac{M\mu(X) - M^2 x_j}{M^2 - 1} \right\| = \frac{M}{M^2 - 1} A(X) \). Therefore
\[
B(\mu(X), A(X)) \subseteq B\left( \frac{\Sigma(X) - x_j}{M - 1}, \frac{M}{M - 1}\right) \cdot x_j - \mu(X)\right\}.
\]
Indeed, the boundary spheres of these two balls are tangent at the point \( x_j \). The result then follows from Lemma 2.

Lemma 5. Let \( X = \{x_1, \ldots, x_M \} \subset \mathcal{B} \), where \( M \geq 2 \) and the \( x_i \) are distinct. For each \( x_i \in X \) we have
\[
\mathcal{B} \cap B\left( \mu(X \setminus \{x_i\}), \frac{M}{M + 1} A(X) \right) \subseteq \text{Keep}(X; \mathcal{B}).
\]
Proof. \( \mu(X \setminus \{x_i\}) = \frac{\Sigma(X) - x_i}{M-1} \) so \( \|\mu(X) - \mu(X \setminus \{x_i\})\| = \frac{1}{M-1} \|\mu(X) - x_i\| \). Therefore by Lemma 4 we have
\[
\mathcal{B} \cap B\left( \mu(X \setminus \{x_i\}), A(X) - \frac{1}{M - 1} \|\mu(X) - x_i\| \right) \subseteq \text{Keep}(X; \mathcal{B}).
\]
On the other hand from Lemma 2 we have
\[
\mathcal{B} \cap B\left( \mu(X \setminus \{x_i\}), \frac{M}{M - 1} \|\mu(X) - x_i\| \right) \subseteq \text{Keep}(X; \mathcal{B}).
\]
The maximum of the two radii defining these sets is at least \( \frac{M}{M + 1} A(X) \).

Lemma 6. Let \( X = \{x_1, \ldots, x_M \} \subset \mathcal{B} \), where \( M \geq 2 \). Suppose \( z \in B(\mu(X), \alpha A(X)) \) for some \( 0 < \alpha < 1 \), and \( z \notin X \). Let \( x_k \) be a point of \( X \) farthest from \( (z + \Sigma(X))/(M + 1) \). Then
\[
F(X) - F(\{z\} \cup (X \setminus \{x_k\})) \geq (1 - \alpha)((M - 1)\alpha + M + 1)A(X)^2 > \frac{1 - \alpha}{M} F(X). \quad (14)
\]
Proof. Let \( x_j \) be a point of \( X \) farthest from \( \mu(X) \), so that \( \|x_j - \mu(X)\| = A(X) \) and
\[
\left\| x_j - \frac{\Sigma(X) - x_j}{M - 1} \right\| = \frac{M}{M - 1} A(X).
\]
By the triangle inequality,
\[
\left\| z - \frac{\Sigma(X) - x_j}{M - 1} \right\| \leq \|z - \mu(X)\| + \left\| \mu(X) - \frac{\Sigma(X) - x_j}{M - 1} \right\| \leq \alpha A(X) + \frac{1}{M - 1} A(X).
\]
By Lemma 1 and equation (6) we have
\[ F({z} \cup (X \setminus \{x_k\})) \leq F({z} \cup (X \setminus \{x_j\})) \]
\[ = F(X) - (M-1) \left( \left\| x_j - \frac{\Sigma(X) - x_j}{M-1} \right\|^2 - \left\| z - \frac{\Sigma(X) - x_j}{M-1} \right\|^2 \right) \]
\[ \leq F(X) - (M-1) \left( \left( \frac{M}{M-1} \right)^2 - \left( \alpha + \frac{1}{M-1} \right)^2 \right) A(X)^2 \]
\[ = F(X) - (1-\alpha)((M-1)(M+1))A(X)^2. \]

The final strict inequality in (14) follows from the final inequality in (4).

The following estimate is well-known (see e.g. [5, Lemma 2.6]).

**Lemma 7.** Let \( \mathcal{B} \subset \mathbb{R}^d \) be a convex body. Let \( x \in \mathcal{B} \) and \( 0 < r_1 < r_2 \). Then
\[ \frac{\lambda(\mathcal{B} \cap B(x, r_1))}{\lambda(\mathcal{B} \cap B(x, r_2))} \geq \left( \frac{r_1}{r_2} \right)^d. \]

**Proof.** Since \( d \)-dimensional Lebesgue measure \( \lambda \) is translation-invariant, we may assume w.l.o.g. that \( x = 0 \). Since \( \mathcal{B} \) is convex and contains \( 0 \), the dilation \( (r_2/r_1)\mathcal{B} \) contains \( \mathcal{B} \), so
\[ \mathcal{B} \cap B(0, r_2) \subseteq \left( \frac{r_2}{r_1} \right) (\mathcal{B} \cap B(0, r_1)). \]
Comparing volumes yields the desired inequality.

**Corollary 2** (Adapted from Lemma 2.3 and Lemma 2.5 in [1]). Let \( Y(0), Y(1), Y(2), \ldots \) be a \( \mathcal{B} \)-valued Jante’s law process, where \( \mathcal{B} \) is any convex subset of \( \mathbb{R} \) with non-empty interior. Let \( \mathcal{F}_n \) be the \( \sigma \)-algebra generated by \( Y(i), i \leq n \). Then for every \( n \geq 1 \),
\[
\mathbb{P} \left( F(Y(n+1)) - F(Y(n)) < -\frac{1}{4M} F(Y(n)) \mid \mathcal{F}_n \right) \geq 4^{-d}. \tag{15}
\]
Consequently,
\[
\mathbb{E} \log F(Y(n+1)) - \log F(Y(n)) \mid \mathcal{F}_n \leq -\frac{1}{4M} 4^{-d}. \tag{16}
\]

**Proof.** Suppose \( Y(n) = X \). Since \( \mathcal{B} \) is convex, \( \mu(X) \in \mathcal{B} \). The new point \( y_{n+1} \) in \( Y(n+1) \) is distributed uniformly in \( \text{Keep}(X; \mathcal{B}) \), and by Lemmas 3 and 4 we have
\[ \mathcal{B} \cap B(\mu(X), A(X)) \subseteq \text{Keep}(X; \mathcal{B}) \subseteq \mathcal{B} \cap B(\mu(X), \frac{M+1}{M-1} A(X)). \]

\( ^5 \)In fact one could obtain \( F(X) - F({z} \cup (X \setminus \{x_k\})) \geq \frac{(1-\alpha)(M+1)}{M^2} F(X) \), but the simpler lower bound given in (14) will suffice for the applications.
Since $M \geq 2$, we have $\frac{M+1}{4(M-1)} A(X) \leq A(X)$, so

$$
\mathbb{P} \left( y_{n+1} \in \mathcal{B} \cap B \left( \mu(X), \frac{M+1}{4(M-1)} A(X) \right) \right) \geq \frac{\lambda \left( \mathcal{B} \cap B \left( \mu(X), \frac{M+1}{4(M-1)} A(X) \right) \right)}{\lambda \left( \mathcal{B} \cap B \left( \mu(X), \frac{M+1}{4(M-1)} A(X) \right) \right)} \geq 4^{-d},
$$

where the second inequality follows from Lemma 7. When $y_{n+1} \in B \left( \mu(X), \frac{M+1}{4(M-1)} A(X) \right)$, we may take $X = Y(n)$ and $\alpha = \frac{M+1}{4(M-1)}$ in Lemma 6, so that inequality (14) becomes

$$
F(Y(n)) - F(Y(n+1)) > \frac{3M - 5}{4M(M-1)} F(Y(n)) \geq \frac{1}{4M} F(Y(n)). \quad (17)
$$

The final consequence follows from the fact that $F(Y(n+1))/F(Y(n)) \leq 1$ (by Corollary 1) and the inequality $\log x \leq x - 1$ for $0 < x \leq 1$.

Inequality (15) implies that a.s. $Y(n)$ converges exponentially fast.

**Lemma 8** (essentially Lemma 2.5 in [1]). Let $N \geq 3$. For any fixed choice of $Y(0)$, there exists $\alpha > 0$ such that, a.s., $D(Y(n)) \leq e^{-\alpha n}$ for all sufficiently large $n$.

**Proof.** From Corollary 1 and (15) it follows that $\mathbb{E}[F(Y(n))|F(Y(0))] \leq \gamma^n F(Y(0))$, where $\gamma = 1 - \frac{4^{1-d}}{M} < 1$. Hence $\mathbb{P}[F(Y(n)) \geq \gamma^{n/2} F(Y(0))] \leq \gamma^{n/2}$, and therefore a.s. for all sufficiently large $n$ we have $F(Y(n)) < \gamma^{n/2} F(Y(0))$. The lemma follows using the inequalities (4) to bound $D(Y(n))$ above in terms of $\sqrt{F(Y(n))}$; we may take any $\alpha > 0$ such that $e^{-\alpha} < \gamma^{1/4}$.

From the preliminary results established above it is now easy to recover the almost sure convergence result [1, Theorem 1.1] (for $\mathcal{B} = [0,1]^d$) extended to the general setting of a $\mathcal{B}$-valued Jante’s law process $Y(n)$; we state it below as Proposition 1. For the case of a bounded convex body $\mathcal{B}$, this is in fact a special case of [2, Theorem 2]: the regularity assumption required by that theorem is satisfied by the uniform distribution on $\mathcal{B}$, as an immediate consequence of Lemma 7. In the case of unbounded $\mathcal{B}$ the exponential decay of $D(Y(n))$ ensures that the process will not escape to infinity, a fact which was also implicitly used in [1, §3.4].

**Proposition 1** (essentially Theorem 1.1 in [1]). Let $d \geq 1$ and $M = N - 1 \geq 2$. Let $Y(0)$ consist of $M$ distinct points in $\mathcal{B}$. Then there exists a random $\xi \in \mathcal{B}$ such that $Y(n)$ converges to $\xi$ a.s. in the sense of Definition 7.

It will be useful to know that if $F(Y(n))$ is small, then with high probability the $\mathcal{B}$-valued Jante’s law process never travels too far from $\mu(Y(n))$. 

---

1. [1]
2. [2]
**Lemma 9.** Fix any $\epsilon > 0$ and $n \in \mathbb{N}$. Let $\gamma = 1 - 4^{1-d}/M$ and $n_0(\epsilon) = \lceil 2 \log_4(\epsilon(1 - \sqrt{\gamma})) \rceil$.

Denote

$$E_j = \left\{ \|\mu(Y(n+j)) - \mu(Y(n))\| \leq \frac{2}{M\sqrt{M-1}} \left( n_0(\epsilon) + \frac{1}{1 - \gamma^{1/4}} \right) \sqrt{F(Y(n))} \right\}$$

Then

$$\mathbb{P}\left( \bigcap_{j=1}^{\infty} E_j \mid Y(n) \right) \geq 1 - \epsilon.$$ 

As a result,

$$\mathbb{P}\left( \|\xi - \mu(Y(n))\| \leq \frac{2}{M\sqrt{M-1}} \left( n_0(\epsilon) + \frac{1}{1 - \gamma^{1/4}} \right) \sqrt{F(Y(n))}\mid Y(n) \right) \geq 1 - \epsilon$$
as well.

**Proof.** From Lemma 3 and 4, we have for every $t \geq 1$

$$\|\mu(Y(t)) - \mu(Y(t-1))\| = \left\| \frac{y_t - r_t}{M} \right\| \leq \left( 1 + \frac{M+1}{M-1} \right) \frac{A(Y(t-1))}{M} = \frac{2}{M-1} A(Y(t-1)) \leq \frac{2}{M\sqrt{M-1}} \sqrt{F(Y(t-1))}.$$ 

Now note that $n_0(\epsilon)$ is the least natural number $k$ such that $\sum_{i=k}^{\infty} \gamma^{i/2} \leq \epsilon$. As we showed in the proof of Lemma 8, $\mathbb{P}(F(Y(n+i)) > \gamma^{i/2}F(Y(n))) \leq \gamma^{i/2}$, so with probability at least $1 - \epsilon$ it holds for all $i \geq n_0(\epsilon)$ that $F(Y(n+i)) \leq \gamma^{i/2}F(Y(n))$. Suppose that this event occurs. Then since $F(Y(n+i)) \leq F(Y(n))$ for all $i \geq 0$, we have

$$\|\mu(Y(n+n_0(\epsilon))) - \mu(Y(n))\| \leq n_0(\epsilon) \frac{2}{M\sqrt{M-1}} \sqrt{F(Y(n))}.$$ 

We bound the remaining increments by a geometric series. For every $j > n_0(\epsilon)$ we have

$$\|\mu(Y(n+j)) - \mu(Y(n+n_0(\epsilon)))\| \leq \left( \sum_{i=n_0(\epsilon)}^{j} \gamma^{i/4} \right) \frac{2}{M\sqrt{M-1}} \sqrt{F(Y(n))}.$$ 

For all $j > n_0(\epsilon)$ the geometric series is bounded by $1/(1 - \gamma^{1/4})$. Applying the triangle inequality and using $\xi = \lim_{j\to\infty} Y(n+j)$ gives the claimed bounds. 

\[\square\]
3 Reduction to the case of uniform geometry

Following the terminology of [5, Ch. 3], a convex body \( B \) is said to have uniform geometry when there exists some \( r_0 > 0 \) such that

\[
b(r_0) := \inf_{x \in B} \lambda(B(x, r_0) \cap B) > 0
\]  

(18)

(see [5, Ch. 3].) Since the volume \( \lambda(B(x, r_0) \cap B) \) depends continuously on \( x \), when \( B \) is compact the infimum is achieved and is positive because \( B \) is the closure of its interior. That is, every bounded convex body has uniform geometry. Examples of unbounded convex bodies with bounded geometry are \( \mathbb{R}^d \) itself, and any convex body which is the intersection of finitely many closed half-spaces. However, not every unbounded convex body has uniform geometry, see for example [5, Example 3.12].

Suppose \( B \) is a convex body of uniform geometry, with \( r_0 \) and \( b(r_0) > 0 \) as in (18). Let \( V(d) \) denote the volume of the unit ball in \( \mathbb{R}^d \), and define

\[
c = c(B, r_0) := \frac{b(r_0)}{V(d)r_0^d}.
\]  

(19)

Note that in the case \( B = \mathbb{R}^d \) we have \( c = 1 \), and in all other cases we have \( c \leq 1/2 \). For the case \( B = [0, 1]^d \) which was studied in [1], we have \( c = 2^{-d} \). By Lemma 7, for all \( r \in (0, r_0) \) and all \( x \in B \) we have

\[
\frac{\lambda(B \cap B(x, r))}{\lambda(B(x, r))} \geq c > 0.
\]  

(20)

Recalling Lemma 3 and Lemma 4 for \( X = \{x_1, \ldots, x_M\} \subset B \) with \( M \geq 2 \) we have

\[
B \cap B(\mu(X), A(X)) \subseteq \text{Keep}(X; B) \subseteq B(\mu(X), \frac{M+1}{M-1} A(X)).
\]

It follows that when \( A(X) \leq r_0 \),

\[
\frac{\lambda(\text{Keep}(X; B))}{\lambda(B(\mu(X), \frac{M+1}{M-1} A(X)))} \geq \left( \frac{M-1}{M+1} \right)^d c > 0.
\]

This means that conditional on \( Y(n) = X \), we may sample from the distribution of \( y_{n+1} \) by rejection sampling with success probability bounded away from 0. That is, we repeatedly sample a point uniformly from \( B(\mu(X), \frac{M+1}{M-1} A(X)) \) until we obtain a sample that lies in \( \text{Keep}(X; B) \), and for each trial the probability of success is at least \( \left( \frac{M-1}{M+1} \right)^d c \). This property will be used several times. If \( B \) did not have uniform geometry then the success probability for this rejection sampling procedure would not be bounded away from 0 as \( X \) ranges over all sets with sufficiently small \( A(X) \). Fortunately, we may reduce Theorem 2 to the case where \( B \) is bounded and therefore has uniform geometry, as follows.
Lemma 10. Suppose the conclusion of Theorem 2 holds subject to the extra hypothesis that $\mathcal{B}$ is bounded. Then it also holds without this hypothesis.

Proof. Let $\mathcal{B}$ be an unbounded convex body and consider the $\mathcal{B}$-valued Jante’s law process $Y(n)_{n\in\mathbb{N}_0}$ started at $Y(0) = S$ where $S$ is a set of $M$ distinct points of $\mathcal{B}$. For any $R > 0$, let $\mathcal{B}_R = \mathcal{B} \cap B(0, R)$. Define the random variable

$$K := \inf\{ R \in \mathbb{N} \mid (\forall n)(Y(n) \subset \mathcal{B}_R) \}.$$  

Then $K < \infty$ (and the inf is actually a min) a.s., by Lemma 8 and Proposition 1. Let $C = \left\lceil \frac{M + 1}{M - 1} \sqrt{\frac{(M - 1)F(S)}{M^2}} \right\rceil$. Then by Corollary 1 and (4), we have $\frac{M - 1}{M + 1} A(Y(n)) \leq C$ for all $n \geq 0$. Suppose $R \in \mathbb{N}$ satisfies $\mathbb{P}(K = R) > 0$. Let $\tilde{Y}(\cdot)$ be the $\mathcal{B}_{R+C}$-valued Jante’s law process started at $\tilde{Y}(0) = S$, and define $\tilde{K} = \inf\{ R \in \mathbb{N} \mid (\forall n)(\tilde{Y}(n) \subset \mathcal{B}_R) \}$. We claim that $\mathbb{P}(K = R) = \mathbb{P}(\tilde{K} = R)$ and that the distribution of $Y(\cdot)$ conditioned on $K = R$ is the same as the distribution of $\tilde{Y}(\cdot)$ conditioned on $\tilde{K} = R$. Indeed, $Y(\cdot)$ and $\tilde{Y}(\cdot)$ may be coupled by a maximal coupling, meaning that for each $n \geq 0$, on the event $Y(n) = \tilde{Y}(n)$, conditional on $Y(n)$ we have $Y(n + 1) = \tilde{Y}(n + 1)$ with the largest possible probability. When $Y(n) = \tilde{Y}(n)$ and $\text{Keep}(Y(n); \mathcal{B}) = \text{Keep}(\tilde{Y}(n); \mathcal{B}_{R+C})$, this maximal coupling probability is 1. On the event that $K = R$, this occurs for every $n \geq 0$. For then $\mu(Y(n)) \in \mathcal{B}_R$ for every $n \geq 0$ (since $\mathcal{B}_R$ is convex), and therefore $B(\mu(Y(n), C) \subseteq B(0, R + C)$ and so

$$\text{Keep}(Y(n); \mathcal{B}) = \mathcal{B} \cap \text{Keep}(Y(n); \mathbb{R}^d) \subseteq \mathcal{B} \cap B\left(\mu(Y(n)), \frac{M + 1}{M - 1} A(Y(n))\right) \subseteq \mathcal{B} \cap B(\mu(Y(n), C) \subseteq \mathcal{B}_{R+C}.$$  

It follows that for every $n \geq 0$

$$\text{Keep}(\tilde{Y}(n); \mathcal{B}_{R+C}) = \mathcal{B}_{R+C} \cap \text{Keep}(\tilde{Y}(n); \mathcal{B}) = \text{Keep}(\tilde{Y}(n); \mathcal{B}) = \text{Keep}(Y(n); \mathcal{B})$$  

as required.

Let $\xi$ be the limit point of $\tilde{Y}(\cdot)$. By the hypothesis that Theorem 2 holds for bounded convex bodies, the distribution of $\xi$ conditioned on $\tilde{K} = R$ is continuous: it is a continuous distribution conditioned on an event of positive probability. Now the distribution of $\xi$ is a mixture of these conditioned distributions over the set of $R \in \mathbb{N}$ for which $\mathbb{P}(K = R) > 0$. Therefore the distribution of $\xi$ is continuous, as required. \qed
4 All original points are eventually removed, a. s.

In this section, we will study the \( \mathbb{R}^d \)-valued Jante’s law process \( Z(n)_{n \geq 0} \). In particular this is an instance of a \( \mathcal{B} \)-valued Jante’s law process where \( \mathcal{B} \) has uniform geometry. For some of the lemmas in this section it takes very little extra work to make the proofs apply to \( Y(n)_{n \geq 0} \) subject to the uniform geometry assumption, and we will state them in that generality even though we will only apply them to the process \( Z(\cdot) \). So for the whole of this section we will assume that \( Y(\cdot) \) is a \( \mathcal{B} \)-valued Jante’s law process where \( \mathcal{B} \) is a convex body with uniform geometry and \( c = c(\mathcal{B}, r_0) > 0 \).

Let \( \tau \) be the first time when all the original points of the configuration are removed, i.e.

\[
\tau = \inf\{n > 0 : Z(n) \cap Z(0) = \emptyset\},
\]

with the usual convention that \( \tau = \infty \) if there is no \( n \) for which \( Z(n) \cap Z(0) = \emptyset \). We know from Proposition 1 that for every \( \epsilon > 0 \) there exists a random \( n(\epsilon) \) such that for all \( n \geq n(\epsilon) \) we have \( Z(n) \subset B(\xi, \epsilon) \), so if for some deterministic choice of \( Z(0) \) it happened that \( \Pr(\tau = \infty) > 0 \) then we would have \( \Pr(\xi \in Z(0)) > 0 \), and the distribution of \( \xi \) would not be continuous. Therefore Theorem 1 can only hold if it is true that that \( \tau < \infty \) a.s.. It will be useful to prove this fact separately first.

**Proposition 2.** \( \tau < \infty \) a.s.

The rest of this section is devoted to the proof of Proposition 2.

In the case \( N = 3 \), i.e. \( M = 2 \), a.s. for each \( n \geq 0 \) the set \( Z(n) \) consists of two distinct points, which are equally likely to be removed at the next step. Thus in this simple case \( \tau - 1 \) is a geometric random variable, in particular a.s. finite.

The case \( M = 2 \) is dealt with, so from now on we will assume that \( M \geq 3 \). Let \( V(d) \) be the volume of the unit ball in \( \mathbb{R}^d \).

Let \( \mathcal{F}_n \) denote the \( \sigma \)-algebra generated by the process \( Y(i)_{i \leq n} \). Recall from Corollary 2 that

\[
\mathbb{E}(\log F(Y(n + 1)) - \log F(Y(n)) \mid \mathcal{F}_n) \leq -\frac{4^{-1-d}}{M} \cdot \tag{21}
\]

To complement this we need to bound the downward drift of the smallest inter-point distance \( d(Y(n)) \).

**Lemma 11.** Suppose that \( X \) is a set of \( M \) distinct points of \( \mathcal{B} \) such that \( A(X) \leq r_0 \). Then

\[
\mathbb{E}(\log d(Y(n)) - \log d(Y(n + 1)) \mid Y(n) = X) \leq \frac{M}{cd} \left( \frac{d(X)}{A(X)} \right)^d .
\]
Proof. Suppose $Y(n) = X$, where $A(X) \leq r_0$. The only way the smallest inter-point distance can decrease is for the newly sampled point $y_{n+1}$ to lie closer than $d(Y(n))$ to one of the existing $M$ points. By Lemma 4, together with (20),

$$\lambda(\text{Keep}(X; \mathcal{B})) \geq \lambda(\mathcal{B} \cap B(\mu(X), A(X))) \geq \frac{b(r_0)A(X)^d}{r_0^d} = C d(X) A(X)^d.$$ 

Therefore

$$\mathbb{E} \left( \log \frac{d(Y(n))}{d(Y(n+1))} \middle| Y(n) = X \right) \leq \frac{M}{\lambda(\text{Keep}(X; \mathcal{B}))} \int_0^{d(X)} V(d) \, d u^{d-1} \log \frac{d(X)}{u} \, du \leq \frac{M}{cd} \left( \frac{d(X)}{A(X)} \right)^d.$$ 

Next, for any set $X$ of $M$ distinct points in $\mathbb{R}^d$, define

$$h(X) := \log \left( \frac{\sqrt{F(X)}}{d(X)} \right) = - \log d(X) + \frac{\log F(X)}{2}.$$ 

Note that by (4), we have

$$h(X) \geq \frac{1}{2} \log \left( \frac{M(M-1)}{2} \right) \geq \frac{1}{2} \log 3 > 0.$$ 

Let $S(d, M)$ be the space of subsets of $\mathbb{R}^d$ of cardinality $M$, with the obvious topology. For any $X \in S(d, M)$, denote by $\hat{X}$ the recentered and rescaled version of $X$, defined by

$$\hat{X} = \left\{ \frac{x - \mu(X)}{\sqrt{F(X)}} : x \in X \right\}.$$ 

We have $\mu(\hat{X}) = 0$, $F(\hat{X}) = 1$, and at the same time $h(\hat{X}) = h(X)$. For any $\rho \in [0, \infty)$, define

$$S(d, M, \rho) := \{ X \in S(d, M) : h(X) \leq \rho \},$$

$$\hat{S}(d, M, \rho) := \{ X \in S(d, M) : h(X) \leq \rho, \mu(X) = 0, F(X) = 1 \}.$$ 

Then $S(d, M, \rho)$ is invariant under translation and scaling, while $\hat{S}(d, M, \rho)$ is a compact subset of $S(d, M)$.

Our next goal is to show that for the $\mathcal{B}$-valued Jante’s law process $Y(\cdot)$, there is a constant $\rho_2 < \infty$ such that a.s. the rescaled and recentered set $\hat{Y}(n)$ visits the compact set $\hat{S}(d, M, \rho_2)$ infinitely often. The proof uses $h$ as a Lyapunov function.

---

6i.e. the topology which it inherits as a quotient of $(\mathbb{R}^d)^M \setminus \Delta$ by the permutation action of the symmetric group $S_M$, where $\Delta \subset (\mathbb{R}^d)^M$ is the subset consisting of sequences in which two or more terms coincide. Note that $S(d, M)$ is not complete with respect to the Hausdorff metric.

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Proposition 3. There exist \( \rho_1 = \rho_1(d, M) < \infty \) and \( \gamma_i = \gamma_i(d, M) > 0 \), such that on the event that \( h(Y_n) \geq \rho_1 \) and \( A(Y(n)) \leq r_0 \), we have

(a) \( \mathbb{E}(h(Y(n+1)) - h(Y(n)) \mid \mathcal{F}_n) \leq 0 \), and

(b) \( \mathbb{P}(h(Y(n+1)) - h(Y(n)) < -\gamma_1 \mid \mathcal{F}_n) \geq \gamma_2 \).

Proof. For part (a) we use inequality (21) and Lemma 11, along with (4) to compare \( A(Y(n)) \) and \( \sqrt{F(Y(n))} \). This requires us to take \( \rho_1 \geq \frac{1}{d} \left( \log \left( 2^{2d+3}M^{d+2} - \log(cd) \right) \right) \).

For part (b), consider the ball “with holes” (see Figure 2)

\[
B^* := \mathcal{B} \cap \left( B(\mu(Y(n)), r) \setminus \bigcup_{w \in Y(n)} B(w, \left( \frac{c}{2M} \right)^{1/d} r) \right),
\]

where

\[
r := \frac{A(Y(n))}{2}
\]

and \( c \) is the constant defined in (19), which depends on the geometry of \( \mathcal{B} \).

Recall that \( Y(n+1) \setminus Y(n) = \{ y_{n+1} \} \). By Lemma 6 (with \( \alpha = 1/2 \), if \( y_{n+1} \in B^* \), then

\[
F(Y(n+1)) \leq \left( 1 - \frac{1}{2M} \right) F(Y(n)).
\]

On the other hand, using Lemmas 3 and 4 with Lemma 7

\[
\mathbb{P}(y_{n+1} \in B(\mu(Y(n)), r)) \geq \frac{\lambda(\mathcal{B} \cap B(\mu(Y(n)), r))}{\lambda(\mathcal{B} \cap B(\mu(Y(n)), \frac{M+1}{M-1} A(Y(n))))} \geq \frac{(M-1)^d}{2d(M+1)^d}.
\]

At the same time, the relative volume of the “holes” in \( B^* \) is bounded. We have \( \mu(Y(n)) \in \mathcal{B} \) and \( r < r_0 \), so we may use the hypothesis of uniform geometry to compare \( \lambda(\mathcal{B} \cap B(\mu(Y(n)), r)) \) with \( \lambda(B(\mu(Y(n)), r)) \). We find

\[
\frac{\text{Volume}(B^*)}{\text{Volume}(\mathcal{B} \cap B(\mu(Y(n)), r))} \geq 1 - \frac{M}{c} \left( \frac{c}{2M} \right) = \frac{1}{2}.
\]
Hence, \( \mathbb{P}(z \in B^*) \geq \gamma_2 := \frac{(M-1)^d}{2^{d+1}(M+1)^d} > 0 \). Next, on the event that \( y_{n+1} \in B^* \),
\[
\min_{w \in Y(n)} \|y_{n+1} - w\| \geq \left( \frac{c}{2M} \right)^{1/d} r \geq \left( \frac{c}{2M} \right)^{1/d} \frac{1}{2M} \sqrt{F(Y(n))}.
\]
by (4). Let \( C := \left( \frac{c}{2M} \right)^{1/d} \frac{1}{2M} \). Then
\[
d(Y(n + 1)) \geq \min \{ d(Y(n)), C \sqrt{F(Y(n))} \}
\]
and thus
\[
h(Y(n + 1)) - h(Y(n)) = \frac{1}{2} \log \frac{F(Y(n + 1))}{F(Y(n))} - \log d(Y(n + 1)) + \log d(Y(n))
\leq - \frac{1}{4M} - \min \left\{ 0, \log \frac{C \sqrt{F(Y(n))}}{d(Y(n))} \right\}
= - \frac{1}{4M} + \max \left\{ 0, + \log \frac{1/C}{h(Y(n))} \right\}
\]
which equals \( -\gamma_1 := -\frac{1}{4M} < 0 \) provided \( h(Y(n)) \geq 1/C \). We see that both parts of the lemma hold when we take
\[
\rho_1 = \max \left( \frac{1}{C}, \frac{1}{d} \log \left( \frac{M^{d+2}d+1}{cd} \right) \right).
\]

Let \( \rho_2 := \frac{M - \rho_1}{2(M - 1)} \) where \( \rho_1 \) is the constant from Proposition \ref{prop:3}. Then define
\[
\tilde{S}(d, M, \rho_2) := \{ X \in S(d, M) : d(X) \geq \rho_2 D(X) \}.
\]

**Proposition 4.** The event \( \{ Y(n) \in \tilde{S}(d, M, \rho_2) \} \) occurs infinitely often a.s.

**Proof.** First of all, by (4),
\[
X \in S(d, M, \rho_1) \implies X \in \tilde{S}(d, M, \rho_2).
\]
For each \( m \geq 1 \) let \( \tau_m = \inf \{ n \geq m : Y(n) \in S(d, M, \rho_1) \} \). By Proposition \ref{prop:3} we have that \( \eta_{nm} = h(Y(n \wedge \tau_m)) \) is a non-negative supermartingale for \( n \geq m \), for any fixed \( m \). From the supermartingale convergence theorem it follows from that a.s. \( \eta_{nm} \) converges as \( n \to \infty \). Since on \( \{ \tau_m > n \} \) we have \( \mathbb{P}(h(Y(n + 1)) \leq h(Y(n)) - \gamma_1) \geq \gamma_2 \), where \( \gamma_i > 0, i = 1, 2 \), are from Proposition \ref{prop:3} we conclude that a.s. \( \tau_m < \infty \). As a result, \( Y(n) \) returns to \( S(d, M, \rho_1) \) and hence also to \( \tilde{S}(d, M, \rho_2) \) infinitely often, a.s.

For the remainder of this section we focus on the process \( Z(n)_{n \geq 0} \).
Lemma 12. For every set $X$ of $M$ distinct points in $\mathbb{R}^d$, there exists a finite $n_0$ and a possible finite trajectory of the process $Z(\cdot)$, say $Z^*(0), \ldots, Z^*(n)$, such that $Z^*(0) = X$, $Z^*(n_0) \cap X = \emptyset$, along the trajectory there are no ties, i.e. at each step there is a unique legal choice of point $s_n^*$ to remove, $s_n^* \in Z^*(n-1)$, and the added points $z^*_1, \ldots, z^*_n$ are distinct and are not in $X$.

Proof. We will identify a trajectory meeting the conditions of the lemma through a combination of probabilistic and constructive reasoning. Note that the process $Z(\cdot)$, started at $Z(0) = X$, a.s. obeys $F(Z_n) \to 0$, implying that $D(Z_n) \to 0$. Let $\tau_A$ be the least $n$ such that $D(Z_n) < d(X)$. Then $\tau_A < \infty$ a.s., and a.s. the points $z_1, \ldots, z_{\tau_A}$ are distinct and not in $X$. Therefore there exists some $m \in \mathbb{N}$ and a trajectory $Z_0, \ldots, Z_m$, with $Z_i \setminus Z_{i-1} = \{z_i^*\}$ and $Z_{i-1} \setminus Z_i = \{s_i^*\}$ for $i = 1, \ldots, m$, such that $D(Z_m^*) < d(X)$, hence $|Z_m^* \cap X| \leq 1$, and the points $z_1^*, \ldots, z_m^*$ are distinct and not in $X$. If $Z_m^* \cap X = \emptyset$ then we are done. So from now on assume that $Z_m^* \cap X = \{x^*\}$.

The end of the construction falls into two cases, depending on whether $x^*$ is a furthest point in $Z_m^*$ from the point

$$
\mu_0 := \frac{M \mu(Z_m^*) + x^*}{M + 1}.
$$

First, suppose we are in case 1, where $x^*$ is indeed a furthest point in $Z_m^*$ from the point $\mu_0$. If we had $x^* = \mu(Z_m^*)$ then we would also have $x^* = \mu_0$ and hence all points of $Z_m^*$ would have to coincide, which was ruled out in the construction of $Z_m^*$. Therefore $x^* \neq \mu(Z_m^*)$.

Let $z_{m+1}^* = \epsilon \mu(Z_m^*) + (1 - \epsilon)x^*$, for some very small $\epsilon \in (0, 1)$, chosen so that $z_{m+1}^*$ is not in $\bigcup_{i=0}^m Z_i^*$. (This can fail only for finitely many values of $\epsilon$.) Then $x^*$ is the unique furthest point of $Z_m \cup \{z_{m+1}^*\}$ from its center of mass, which is

$$
\mu_\epsilon := \frac{(M + \epsilon) \mu(Z_m^*) + (1 - \epsilon)x^*}{M + 1}.
$$

To see this, note first that

$$
\|z_{m+1}^* - \mu_\epsilon\| = \frac{M}{M + 1} (1 - \epsilon) \|x^* - \mu(Z_m^*)\| > 0,
$$

while

$$
\|x^* - \mu_\epsilon\| = \frac{M + \epsilon}{M + 1} \|x^* - \mu(Z_m^*)\|.
$$

Thus $\|x^* - \mu_\epsilon\| > \|z_{m+1}^* - \mu_\epsilon\|$ if $\epsilon > 0$. Secondly, note that the ball $B(\mu_\epsilon, \|x^* - \mu_\epsilon\|)$ contains the ball $B(\mu_0, \|x^* - \mu_0\|)$, and their boundary spheres meet only at the point $x^*$. Hence $\|x^* - \mu_\epsilon\| > \|z - \mu_\epsilon\|$ for every $z \in Z_m^* \setminus \{x^*\}$. Hence the point $x^*$ is the unique point which may be removed, and we obtain $Z_{m+1}^* = (Z_m \cup \{z_{m+1}^*\}) \setminus \{x^*\}$. The trajectory $Z_0^*, \ldots, Z_{m+1}^*$ is a permissible trajectory with no tiebreaks and no repeated points, and $Z_m^* \cap X = \emptyset$, as required.

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The arriving points are all perturbed by the vector $0 \in \mathbb{R}^d$. In the proof we assume that we are in this case. We claim that if $v$ is permissible means that $s_i$ is sufficiently close to 0 then the corresponding points are removed at each step, by which we see this, note that each point $z$ respectively, and which are related by $s_i$. It follows that when $z = (Z_{i-1} \cup \{z_i\}) \setminus \{s_i\}$, where $z_i = z^*_i + v$ and $s_i$ is a point of $Z_{i-1} \cup \{z_i\}$ furthest from $\mu(Z_{i-1} \cup \{z_i\})$. To say that the trajectory $Z_0, \ldots, Z_m$ is permissible means that $z^*_i \in \text{Keep}(Z_{i-1}; \mathbb{R}^d)$ for each $i = 1, \ldots, m$, so that $z^*_i \neq s_i$. In fact, if $v$ is sufficiently close to 0 then the corresponding points are removed at each step, by which we mean that $Z_i = Z_{i-1} \cup \{z_i\} \setminus \{s_i\}$, where either $s_i = z^*_i$ (if $s_i \in X$), or $s_i = z^*_i + v$ (if $s_i \notin X$). To see this, note that each point $z^*_i$ or $s_i$ depends continuously on $v$, as does $\mu(Z_i)$. The (assumed) statement that $z^*_i$ is the unique point of $Z_{i-1} \cup \{z^*_i\}$ furthest from $\mu(Z_{i-1} \cup \{z^*_i\})$ is equivalent to a finite collection of strict inequalities: for each $z \in (Z_i \cup \{z^*_i\}) \setminus \{s^*_i\}$,

$$\|s^*_i - \mu(Z_i \cup \{z^*_i\})\| > \|z - \mu(Z_i \cup \{z^*_i\})\|.$$ 

It follows that when $v \in \mathbb{R}^d \setminus \{0\}$ is sufficiently close to 0 then for each $z \in (Z_{i-1} \cup \{z_i\}) \setminus \{s_i\}$,

$$\|s_i - \mu(Z_i \cup \{z_i\})\| > \|z - \mu(Z_i \cup \{z_i\})\|,$$

i.e. $s_i$ is the unique point of $Z_{i-1} \cup \{z_i\}$ furthest from $\mu(Z_{i-1} \cup \{z_i\})$ for each $i = 1, \ldots, m$. In particular the trajectory $Z_0, \ldots, Z_m$ involves no tiebreaks when $v$ is sufficiently close to 0. By similar reasoning, since the trajectory $Z_0, \ldots, Z_m$ involves no repeated points, the same is true of the trajectory $Z_0, \ldots, Z_m$ when $v$ is sufficiently close to 0.

Having chosen a vector $v$ suitably close to 0, we now extend our two trajectories one further step, so that $Z_{m+1} = Z_m + v$, i.e. $Z_m$ is the translate of $Z_m$ by the vector $v$. To do this, we choose $z^*_{m+1} = x_m - v$ and $z^*_{m+1} = x_m + v$. Then

$$Z_m \cup \{z^*_{m+1}\} = (Z_m \cup \{z^*_{m+1}\}) + v.$$ 

(22)

Notice that in the translation which relates the two configurations, the roles of $x_m$ and the new point ($z^*_{m+1}$ and $z^*_{m+1}$ respectively) are swapped. We may choose $v$ so that there is no tiebreak in either configuration when selecting the point furthest from the center of mass. This is because the set of vectors $v$ which would cause such a tie is contained in a finite union of hypersurfaces of dimension less than $d$, so we may choose $v$ as close as we like to 0 but not in any of these hypersurfaces. From (22), we see that there are points $s^*_{m+1} \in Z_m \cup \{z^*_{m+1}\}$ and $s^*_{m+1} \in Z_m \cup \{z^*_{m+1}\}$ which are furthest from the centers of mass $\mu(Z_m \cup \{z^*_{m+1}\})$ and $\mu(Z_m \cup \{z^*_{m+1}\})$ respectively, and which are related by $s^*_{m+1} = s^*_{m+1} + v$. Hence $Z_{m+1} = Z_{m+1} + v$. Before
Definition 3. For two sets proceeding, we must check that the trajectories $Z^*_{m,\ldots,m+1}$ and $Z^\dagger_0,\ldots,Z^\dagger_{m+1}$ are permissible. This is where we use the assumption that we are in case 2, together with the continuity of all defined points as functions of $v$. Let $u^*$ be a point of $Z^*_m$ which is strictly further from $\mu_0$ than $x_\bullet$ is from $\mu_0$, and let $u^\dagger = u^* + v$. As $v \to 0$, we have $\mu(Z^*_m \cup \{z^*_i\}) \to \mu_0$ and $\mu(Z^\dagger_m \cup \{z^\dagger_{m+1}\}) \to \mu_0$, while $z^*_m \to x_\bullet$, $z^\dagger_{m+1} \to x_\bullet$ and $u^\dagger \to u^*$. Therefore for $v$ close enough to 0 we have

$$\|u^* - \mu(Z^*_m \cup \{z^*_i\})\| > \|z^*_m - \mu(Z^*_m \cup \{z^*_i\})\|$$

and

$$\|u^\dagger - \mu(Z^\dagger_m \cup \{z^\dagger_{m+1}\})\| > \|z^\dagger_{m+1} - \mu(Z^\dagger_m \cup \{z^\dagger_{m+1}\})\|,$$

showing that both trajectories are indeed permissible.

Finally, by the same argument as in the first paragraph of this proof, there exists a possible trajectory $Z^*_m \cup \{z^*_i\}$ such that $|Z^*_m \cup Z^*_{m+m'}| \leq 1$, and having no tiebreaks, no repeated points, and no arriving points which were already in $\bigcup_{i=0}^m Z^*_i$. There are now two possibilities: either $x_\bullet \notin Z^*_{m+m'}$, in which case we are done, or $x_\bullet \notin Z^*_{m+m'}$. In the latter case, consider the trajectory $Z^\dagger_m \cup \{z^\dagger_{m+1}\}$ defined by $Z^\dagger_i = Z^\dagger_i + v$ for $i = m+1,\ldots,m+m'$. It is also a permissible trajectory, being a translate of a permissible trajectory. Moreover, $Z^\dagger_m \cup Z^\dagger_{m+m'} = \{x_\bullet + v\}$, and in particular $x_\bullet \notin Z^\dagger_{m+m'}$, so that $Z^\dagger_{m+m'} \cap X = \emptyset$. Therefore $Z^\dagger_0,\ldots,Z^\dagger_{m+m'}$ is a trajectory satisfying the conclusion of the lemma, and we are done.

$\square$

Recall the following definition.

**Definition 3.** For two sets $X$ and $X'$ the Hausdorff distance is defined as

$$d_H(X,X') = \max \left\{ \sup_{x \in X} \left[ \inf_{x' \in X'} ||x-x'|| \right], \sup_{x' \in X'} \left[ \inf_{a \in X} ||x'-a|| \right] \right\}$$

$$= \inf \left\{ \epsilon > 0 : X \subseteq \bigcup_{a' \in X'} B(a',\epsilon) \text{ and } X' \subseteq \bigcup_{a \in X} B(a,\epsilon) \right\}.$$ 

Note that when $X$ and $X'$ are both finite, then $d_H(X,X') = 0$ if and only if $X = X'$.

**Lemma 13.** For every set $X \in S(d,M)$, there is a finite $n_0 = n_0(X)$, an $\epsilon_0 = \epsilon_0(X) \in (0,d(X)/2)$, and a $\delta_0 = \delta_0(X) > 0$ such that if $X' \in S(d,M)$ has Hausdorff distance less than $\epsilon_0$ from $X$ then

$$\mathbb{P}(Z(n_0) \cap Z(0) = \emptyset \mid Z(0) = X') > \delta_0.$$ 

**Proof.** Let $z^*_{-M},\ldots,z^*_0$ be any $M$ distinct points in $\mathbb{R}^d$, and let $X = \{z^*_{-M},\ldots,z^*_0\}$. Take a trajectory as provided by Lemma [12]. Since at each step $1 \leq n \leq n_0$ there is a unique choice of
point $s^*_n \in Z^*(n - 1) \cup \{z^*_n\}$ such that $s^*_n$ maximizes the distance from $\mu(Z^*(n - 1) \cup \{z^*_n\})$, and there are only finitely many steps, there is an $\epsilon > 0$ such that for all $1 \leq n \leq n_0$ we have for any $z \in Z^*(n)$ that

$$\|s^*_n - \mu(Z^*(n - 1) \cup \{z^*_n\})\| \geq \|z - \mu(Z^*(n - 1) \cup \{z^*_n\})\| + 4\epsilon.$$ 

By making $\epsilon$ smaller if necessary, we may also assume that

$$\min(\|z - x\| : z \in Z^*(n_0), x \in X) > 2\epsilon.$$

Consider any perturbed trajectory $Z'(0), \ldots, Z'(n_0)$ where $Z'_0 = \{z'_{1-M}, \ldots, z'_0\}$ and for $1 \leq n \leq n_0$ we have $Z'(n) = (\{z'_n\} \cup Z'(n - 1)) \setminus \{s'_n\}$, and for each $i = 1 - M, \ldots, n_0$ we have $\|z'_i - z'_i\| < \epsilon$. Then one may prove by induction over $n$ that in the perturbed trajectory, the thrown out points $s'_n$ correspond to the original points $s_n$ in the sense that for each $1 \leq n \leq n_0$, the index $j$ such that $s^*_n = z^*_j$ also satisfies $s'_n = z'_j$. This is because the perturbation moves each point by a distance less than $\epsilon$ and also moves each center of mass by a distance at most $\epsilon$. It therefore alters corresponding distances by less than $2\epsilon$, and differences of distances by less than $4\epsilon$.

In particular, the points of $Z'(n_0)$ are in one-to-one correspondence with points of $Z^*(n_0)$ so that corresponding pairs of points are at distance less than $\epsilon$. It follows that $Z'(n_0) \cap X' = \emptyset$.

Finally, consider the random process $Z(\cdot)$ started at $Z(0) = X'$. The probability that each new point $z_i$, $i = 1, \ldots, n_0$, lies in the ball $B(z_i^*, \epsilon)$ is bounded away from 0, uniformly over all choices of $X'$ that are $\epsilon$-perturbations of $X$. This is because the volumes of the sets $\text{Keep}(Z(n); \mathbb{R}^d)$ are bounded above a priori in terms of $X$, using the fact that

$$\frac{M^2}{M - 1} A(Z_n)^2 \leq F(Z_n) \leq F(X') \leq \frac{M(M - 1)}{2} D(X')^2 \leq \frac{M(M - 1)}{2} (D(X) + 2\epsilon)^2$$

together with Lemma 3. When this occurs, we have $Z(n_0) \cap Z(0) = \emptyset$. □

**Lemma 14.** There exist $n_1 = n_1(d, M) < \infty$ and $\delta_1 = \delta_1(d, M) > 0$ such that for every $X \in \mathcal{S}(d, M, \rho_2)$,

$$\mathbb{P}(Z(n + n_1) \cap Z(n) = \emptyset \mid Z(n) = X) \geq \delta_1.$$  

**Proof.** First, observe that since the process $Z(\cdot)$ is time-homogeneous and invariant under translation and scaling, we have

$$\mathbb{P}(Z(n + n_1) \cap Z(n) = \emptyset \mid Z(n) = X) = \mathbb{P}(Z(n_1) \cap Z(0) = \emptyset \mid Z(0) = \tilde{X}).$$
So it suffices to prove that there exist $n_1 < \infty$ and $\delta_1 > 0$ such that for all $X \in \hat{S}(d, M, \rho_2)$ we have
\[
P(Z(n_1) \cap Z(0) = \emptyset \mid Z(0) = X) \geq \delta_1.
\]
This follows from Lemma~13 since for each point $X$ in the compact set $\hat{S}(d, M, \rho_2)$, that corollary specifies a neighbourhood $U_X$ of $X$ and constants $n_0(X) < \infty$, $\delta_0(X) > 0$ such that for all $X' \in U_X$,
\[
P(Z(n_0) \cap Z(0) = \emptyset \mid Z(0) = X') \geq \delta_0(X).
\]
Since $\hat{S}(d, M, \rho_2)$ is compact, there exists a covering of $\hat{S}(d, M, \rho_2)$ by some finite sequence of such neighbourhoods $U_{X_1}, \ldots, U_{X_k}$, and we may take
\[
n_1 = \max(n_0(X_1), \ldots, n_0(X_k)) < \infty
\]
and
\[
\delta_1 = \min(\delta_0(X_1), \ldots, \delta_0(X_k)) > 0.
\]

We can now prove Proposition~2 which claims that $\tau < \infty$ a. s.

**Proof of Proposition 2** Starting from any configuration $Z(0) = X$ of $M$ distinct points in $\mathbb{R}^d$, the process $Z(\cdot)$ a.s. eventually enters $\hat{S}(d, M, \rho_2)$, by Proposition 4, say at stopping time $\tau_1$. Let $n_1 = n_1(d, M)$ be the constant provided by Lemma 14. Inductively define $\tau_k$ for $k = 2, 3, \ldots$, by
\[
\tau_k = \min\{t : t \geq \tau_{k-1}, Z(\tau_k) \in \hat{S}(d, M, \rho_2)\}.
\]
For each $k \geq 1$, conditional on $\tau_k$ and $Z(\tau_k)$, in time steps $\tau_k + 1, \ldots, \tau_k + n_1$ all the points of $Z(\tau_k)$ are removed, with probability at least $\delta_1$, regardless of the configuration $Z(\tau_k)$. Therefore with probability 1 there exists a finite $k$ for which this event occurs. We then have $\tau < \tau_k + n_1 < \infty$, as required. \hfill $\Box$

5 Proof of Theorem 1

Without loss of generality, we assume that the initial state $Z(0)$ is deterministic. This is harmless since if for each deterministic choice of $Z(0)$ the limit $z_\infty$ exists a.s. and has a continuous distribution, then if instead $Z(0)$ is random, $z_\infty$ still exists a.s. and its distribution is a mixture of continuous distributions, which is necessarily continuous.
Figure 3: Area where \((z_1, z_2)\) can be sampled in the example of Remark 1

List the elements of \(Z(0)\) in an arbitrary order as \(z_{-(N-2)}, z_{-(N-3)}, \ldots, z_0\). Recalling that for \(n \geq 1\), \(z_n\) is the unique point of \(Z(n) \setminus Z(n-1)\), we observe that for each \(m \geq 0\) we have

\[
\{z_i : i = -(N-2), \ldots, m\} = \bigcup_{n=0}^{m} Z(n).
\]

We discarded a null set to ensure that the points of the sequence \((z_i)_{i=-\infty}^{-(N-2)}\) are pairwise distinct. Thus for each \(n \geq 1\), the unique point \(s_n\) of \(Z(n-1) \setminus Z(n)\) is equal to \(z_{\alpha(n)}\) for a unique index \(\alpha(n)\) in the range \(-(N-2) \leq \alpha(n) \leq n - 1\). Moreover, the indices \(\alpha(1), \alpha(2), \alpha(3), \ldots\) are pairwise distinct random variables since each point is removed at most once.

**Lemma 15.** Fix any \(K \geq N - 1\) and any deterministic sequence of distinct indices \(i = (i(1), \ldots, i(K))\) such that for each \(n = 1, \ldots, K\) we have \(-(N-2) \leq i(n) \leq n - 1\) and \(K\) is the least positive integer such that \(\{-(N-2), \ldots, 0\} \subseteq \{i(1), \ldots, i(K)\}\). Let \(\bar{\alpha} = (\alpha(1), \ldots, \alpha(K))\). Then the event that \(\tau = K\) and \(\bar{\alpha} = i\) is equivalent to the event that a certain finite collection of linear or quadratic inequalities are satisfied, which involve the coordinates of the random points \(z_1, \ldots, z_K\), as well as the coordinates of the deterministic points \(z_{-(N-2)}, \ldots, z_0\).

**Remark 1.** For example, take \(d = 1\), \(N = 3\), and w.l.o.g. \(z_{-1} = -1\), \(z_0 = 1\). Then on \(\tau = 2\), \(\alpha(1) = -1\), \(\alpha(2) = 0\), we have \(z_1 \in (0, 1)\) and \(z_2 \in \left(2z_1 - 1, \frac{3z_1 + 1}{2}\right)\) or \(z_1 \in (1, 3)\) and \(z_2 \in \left(\frac{z_1 + 1}{2}, 2z_1 - 1\right)\). See Figure 3.
These inequalities define a bounded semi-algebraic set $\mathcal{P}_{K,1} \subset (\mathbb{R}^d)^K$. (When $d = 1$, $\mathcal{P}$ is a polytope, since the defining inequalities may be reduced to a collection of affine linear inequalities.) For some sequences $i$, $\mathcal{P}_{K,1}$ may be empty or have empty interior. Note that the boundary of $\mathcal{P}_{K,1}$ is contained in a finite union of algebraic hypersurfaces where at least one of the defining inequalities becomes an equality. These hypersurfaces have trivial $(Kd)$-dimensional Lebesgue measure. Thus $\mathcal{P}_{K,1}$ has positive $(Kd)$-dimensional Lebesgue measure if and only if it has non-empty interior.

Let $E_{K,1}$ be the event that $\tau = K$ and $\bar{\alpha} = i$. If $\mathbb{P}(E_{K,1}) > 0$ then conditional on $E_{K,1}$ the sequence $\bar{\alpha}$ is distributed according to $U(\mathcal{P}_{K,1})$; in particular it lies a.s. in the interior $\mathcal{P}_{K,1}^\circ$.

Define a $d$-dimensional linear subspace $L_K$ of $(\mathbb{R}^d)^K$ by

$$L_K := \{(v, \ldots, v) \in (\mathbb{R}^d)^K : v \in \mathbb{R}^d\}.$$

Consider the orthogonal projection $\pi_K$ of $(\mathbb{R}^d)^K$ onto $(L_K)\perp$. We have

$$\lambda_{Kd}(\mathcal{P}_{K,1}) = \int_{\pi_K(\mathcal{P}_{K,1})} \lambda_d(\mathcal{P}_{K,1} \cap \pi_K^{-1}(x)) d\lambda_{(K-1)d}(x).$$

We conclude that if $\lambda_{Kd}(\mathcal{P}_{K,1}) > 0$ then for $\lambda_{Kd}$-a.e. $p = (p_1, \ldots, p_K) \in \mathcal{P}_{K,1}$, the set

$$T(p, K, i) := \{v \in \mathbb{R}^d : (p_1 + v, \ldots, p_K + v) \in \mathcal{P}_{K,1}\}$$

has $\lambda_d(T(p, K, i)) > 0$. Therefore we can sample a $U(\mathcal{P}_{K,1})$ random variable $q$ by sampling first a $U(\mathcal{P}_{K,1})$ random variable $p$, then (conditional on $p$) sampling a $U(T(p, K, i))$ random variable $\delta$ and setting $q := p + (\delta, \delta, \ldots, \delta)$.

Let $\mathcal{F}_\tau$ be the $\sigma$-algebra generated by the stopped process $Z(\cdot \land \tau)$. Let $V$ be a random variable whose distribution conditional on $\mathcal{F}_\tau$ is $U(T((z_1, \ldots, z_\tau), \tau, (\alpha(1), \ldots, \alpha(\tau))))$, such that $V$ is conditionally independent of $(z_{\tau+1}, z_{\tau+2}, \ldots)$ given $\mathcal{F}_\tau$. That is to say, on the event $E_{K,1}$, given $z_{-(N-2)}, \ldots, z_K$, we let $V \sim U(T((z_1, \ldots, z_K), K, i))$, independent of $(z_{K+1}, z_{K+2}, \ldots)$. Proposition 2 and the fact that ties a.s. do not occur, imply that a.s. $E_{K,1}$ holds for exactly one $(K, i)$, so $V$ is well-defined. The crucial property of $V$ is that its conditional distribution of $V$ on $\mathcal{F}_\tau$ is a.s. continuous, being the uniform distribution on a subset of $\mathbb{R}^d$ of positive and finite $d$-dimensional Lebesgue measure.

Now define a new random sequence $(z'_n)_{n=-(N-2)}^\infty$ as follows.

$$z'_n = \begin{cases} z_n, & \text{if } n \leq 0, \\ z_n + V, & \text{if } n \geq 1, \end{cases}$$

\(\text{i.e., a set defined by a number of polynomial inequalities and equalities; in our case, a.s. these will be just inequalities.}\)
Conditional on \( z_{-(N-2)}, \ldots, z_0 \) and on \( E_{K,i} \), the sequences \( (z_n)_{n=1}^\tau \) and \( (z'_n)_{n=1}^\tau \) are identically distributed: both are \( U(\mathcal{P}_{K,i}) \) random variables.

Let \( Z'(0) = Z(0) \), then for \( n \geq 1 \) let
\[
Z'(n) = \{ z'_n \} \cup \left( Z'(n-1) \setminus \{ z_{a(n)} \} \right).
\]
For all \( n \geq \tau \) we have
\[
Z'(n) = \{ z + V : z \in Z(n) \}
\]
Moreover \( Z'(\cdot) \) is a Markov chain with the same distribution as \( Z(\cdot) \): by construction they have the same distribution up to the random time \( \tau \), and from time \( \tau \) onwards, the transitions have the correct distribution by equation (23) together with the translation-invariance of the transition law of \( Z(\cdot) \).

Therefore the sequence \( (z'_n) \) has the same distribution as the original sequence \( (z_n) \). Since \( (z_n) \) a.s. converges to a limit \( z_{\infty} \), it also holds that \( (z'_n) \) a.s. converges; let \( z'_\infty \) be its limit. The distribution of \( z'_\infty \) coincides with the distribution of \( z_{\infty} \).

On the other hand, the distribution of \( z'_\infty \) is also the mixture of its conditional distributions on \( \mathcal{F}_\tau \). But we have
\[
z'_\infty = z_{\infty} + V
\]
and \( z_{\infty} \) and \( V \) are conditionally independent given \( \mathcal{F}_\tau \). Thus the distribution of \( z'_\infty \) is a mixture of continuous distributions, and it is therefore continuous. This completes the proof of Theorem 1.

6 Coupling \( Y(\cdot) \) and \( Z(\cdot) \)

In this section we complete the proof of Theorem 2. By Lemma 10 it suffices to prove the continuity of the limit point \( \xi \) for a \( \mathcal{B} \)-valued Jante’s law process where \( \mathcal{B} \) is bounded. So we shall assume throughout this section that \( \mathcal{B} \) is bounded, and therefore has uniform geometry.

We will need an isoperimetric inequality for inner shells of convex bodies, for which we have been unable to find a reference. It concerns the following problem. Suppose you have a (possibly) hollow chocolate egg whose outer boundary is the boundary of a convex body. If all the chocolate is within distance \( r \) of the outer boundary of the egg, what is the maximum quantity of chocolate that can possibly be contained within a ball of radius \( R \)? See Figure 4.

**Lemma 16.** Let \( R > r > 0 \) and let \( \mathcal{B} \) be a convex body in \( \mathbb{R}^d \). For every \( y \in \mathbb{R}^d \), we have
\[
\lambda(B(y,R) \cap \mathcal{B} \cap N_r(\mathcal{B}^c)) \leq 2rd\sqrt{d}V(d-1)R^{d-1},
\]
where \( V(d-1) \) is the volume of the unit ball in \( \mathbb{R}^{d-1} \), with the convention that \( V(0) = 1 \).
Figure 4: Lemma 16 bounds the quantity of chocolate in the ball of radius $R$.

Proof. For $d = 1$ the inequality holds with $V(0) = 1$, since the set $N_r(\mathcal{B}^c)$ is a union of at most two intervals of length at most $r$ and therefore has length at most $2r$. Now assume that $d \geq 2$. Let $A = \mathcal{B} \cap B(y, R) \cap N_r(\mathcal{B}^c)$. Consider any point $x \in A$. Then $\|x - z\| \leq r$ for some $z \in \partial \mathcal{B}$. Let $H$ be a supporting hyperplane through $z$, meaning that $H \cap \mathcal{B}^o = \emptyset$. Then (as for any hyperplane through $z$) $H$ cuts at least one of the $d$ axis-parallel lines through $x$ at a point whose distance from $x$ is at most $r\sqrt{d}$. It follows that there is a point of $\partial \mathcal{B}$ that lies between $x$ and $H$ on one of the axis-parallel lines through $x$. The situation is illustrated in Fig. 5 below.

Figure 5: The supporting hyperplane at $z$ to the green convex body $\mathcal{B}$ cuts at least one axis-parallel line through $x$ within distance $\sqrt{d}\|z - x\|$ of $x$.

Thus $A$ is covered by the $d$ sets $A_1, \ldots, A_d$, where

$$A_i = \left\{ x \in A : \left( \exists t \in [-r\sqrt{d}, r\sqrt{d}] \right) (x + te_i \in \partial \mathcal{B}) \right\},$$

where $e_1, \ldots, e_d$ are the standard basis vectors of $\mathbb{R}^d$. Since $A_i \subset B(y, R)$, the orthogonal projection of $A_i$ onto a hyperplane orthogonal to $e_i$ has $d - 1$-dimensional Lebesgue measure at most $V(d - 1)R^{d-1}$. Each line parallel to $e_i$ that meets $B(y, R)$ meets $A_i$ in at most two intervals, whose total length is at most $2r\sqrt{d}$. Hence $\lambda(A_i) \leq 2r\sqrt{d}V(d - 1)R^{d-1}$ for each $i = 1, \ldots, d$. 

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Summing over $i$ we get
\[
\lambda(A) \leq \lambda \left( \bigcup_{i=1}^{d} A_i \right) \leq \sum_{i=1}^{d} \lambda(A_i) \leq 2rd\sqrt{d}V(d-1)R^{d-1}.
\]

Lemma 16 will suffice for our purposes, but we remark that the sharp version of the inequality is as follows. (A proof is given in the appendix.)

**Lemma 17.** Let $R > r > 0$ and let $\mathcal{B}$ be a convex body in $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$, we have
\[
\lambda(B(x,R) \cap \mathcal{B} \cap N_r(\mathcal{B}^c)) \leq V(d)(R^d - (R-r)^d),
\]
where $V(d)$ is the volume of the unit ball in $\mathbb{R}^d$.

Let $\mathcal{B} \subset \mathbb{R}^d$ be a bounded convex body. For any set $X$ of $M$ distinct points of $\mathcal{B}$ define
\[
d_\mathcal{B}(X) = \text{dist}(X, \mathcal{B}^c),
\]
and for any set $X$ of $M$ distinct points of $\mathcal{B}$ that is not entirely contained in $\partial \mathcal{B}$, define
\[
d_\mathcal{B}^o(X) = \text{dist}(X \cap \mathcal{B}^o, \mathcal{B}^c),
\]
For $Y(n)_{n \geq 0}$ a $\mathcal{B}$-valued Jante’s law process we define
\[
g_n := \begin{cases} 
\frac{1}{2} \log(F(Y(n))) - \log \left( d_\mathcal{B}^o(Y(n)) \right), & \text{if } Y(n) \not\subseteq \partial \mathcal{B}; \\
+\infty, & \text{if } Y(n) \subseteq \partial \mathcal{B}.
\end{cases}
\]

**Lemma 18.** There exists $\delta > 0$ (depending on $M$, $d$, and $\mathcal{B}$) such that for any set $X_0$ of $M$ distinct points of $\mathcal{B}$ and any $n_0$,
\[
\mathbb{P}(\exists n \geq n_0 : g_n \leq \log(\delta^{-1}) \mid Y(n_0) = X_0) = 1,
\]
and hence almost surely for infinitely many $n$ we have
\[
d_\mathcal{B}^o(Y(n)) \geq \delta \sqrt{F(Y(n))}.
\]

**Proof.** We may assume without loss of generality that $X_0$ has at most one point in $\partial \mathcal{B}$. Indeed, a.s. all new points are not in $\partial \mathcal{B}$, and a.s. $D(Y(n)) \to 0$ as $n \to \infty$, so there exists an $n \geq n_0$ such that $D(Y(n)) < d(Y(n_0))$ and therefore $Y(n) \cap Y(n_0)$ contains at most one point; now apply the lemma with $X_0 = Y(n)$. It follows from these assumptions that a.s. $g_n \in \mathbb{R}$ for every
\( n \geq 0 \). Likewise we may assume that \( \sqrt{\frac{M-1}{M}} \sqrt{F(X_0)} < r_0 \), where \( r_0 \) and \( c \) are the constants used to specify the uniform geometry of \( \mathcal{B} \), so that by Corollary \ref{cor:uniform} and inequality \ref{ineq:uniform} we have \( A(Y(n)) < r_0 \) for all \( n \geq 0 \).

We will use Lemma \ref{lemma:upper_bound} to get an upper bound for the expectation of the positive part of the increment of \(-\log d_{\phi}^\circ(Y(\cdot))\).

\[
E \left( \log^+ \frac{d_{\phi}^\circ(Y(n))}{d_{\phi}(Y(n+1))} \bigg| Y(n) = X \right) = \int 1(y \in \text{Keep}(X; \mathcal{B})) \log^+ \frac{d_{\phi}^\circ(X)}{\lambda(\text{Keep}(X; \mathcal{B}))} \frac{d\mu_n(y)}{\lambda(\text{dist}(y, \mathcal{B}^c))} \, d\lambda(y) \tag{26}
\]

Let

\[
R := \frac{M}{M-1} A(X) \text{.}
\]

By \ref{ineq:uniform} we have \( R \leq \frac{M+1}{M^{1/2}} \sqrt{F(X)} \) and by Lemma \ref{lemma:keep} we have \( \text{Keep}(X; \mathcal{B}) \subseteq B(\mu(X), R) \cap \mathcal{B} \).

The numerator of the RHS of (26) may be rewritten as

\[
\int \int 1(y \in \text{Keep}(X; \mathcal{B})) \cdot 1(\text{dist}(y, \mathcal{B}^c) \leq r \leq d_{\phi}^\circ(X)) \frac{dr}{r} \, d\lambda(y)
\]

\[
= \int_0^{r_1} \frac{1}{r} \lambda(\text{Keep}(X; \mathcal{B}) \cap N_\epsilon(\mathcal{B}^c)) \, dr
\]

\[
\leq \int_0^{r_1} \frac{1}{r} \lambda(B(\mu(X), R) \cap \mathcal{B} \cap N_\epsilon(\mathcal{B}^c)) \, dr
\]

\[
\leq \int_0^{r_1} 2r dV(d-1) R^{d-1} \, dr = 2r_1 dV(d-1)R^{d-1},
\]

where the first equality above uses Tonelli’s theorem, the next inequality uses Lemma \ref{lemma:keep}, and the final inequality uses Lemma \ref{lemma:uniform}.

On the other hand, by our assumption that \( A(X) < r_0 \), we have the uniform geometry bound

\[
\lambda(\text{Keep}(X; \mathcal{B})) \geq c V(d) A(X)^d,
\]

where \( c \) is the constant from \ref{cor:uniform}. Therefore the RHS of (26) is bounded above by

\[
\frac{2 d_{\phi}^\circ(X) d^{3/2} V(d-1) \left( \frac{M+1}{M} A(X) \right)^{d-1}}{c V(d) A(X)^d} = c_1 \frac{d_{\phi}^\circ(X)}{A(X)} \leq c_1 M \frac{d_{\phi}^\circ(X)}{\sqrt{F(X)}}
\]

where \( c_1 \) is a constant depending on \( d \), \( M \) and \( c \). Thus we have shown that

\[
E \left( \log^+ \frac{d_{\phi}^\circ(Y(n))}{d_{\phi}(Y(n+1))} \bigg| Y(n) = X \right) \leq c_1 M \frac{d_{\phi}^\circ(X)}{\sqrt{F(X)}}. \tag{27}
\]

By almost the same argument,

\[
\mathbb{P}(d_{\phi}^\circ(Y(n+1)) < d_{\phi}^\circ(Y(n)) \mid Y(n) = X) \leq \frac{\lambda(B(\mu(X), R) \cap \mathcal{B} \cap N_\epsilon(\mathcal{B}^c))}{\lambda(\text{Keep}(X; \mathcal{B}))} \leq c_1 M \frac{d_{\phi}^\circ(X)}{\sqrt{F(X)}}.
\]
Define
\[ \delta = \frac{1}{8c_1M^24^d}, \quad L = \log(\delta^{-1}) + \frac{1}{2}\log\left(1 - \frac{1}{4M}\right), \]
(note that \( L < \log(\delta^{-1}) \)) and the stopping time
\[ \tau_g := \inf\{n \geq n_0 : g_n \leq \log(\delta^{-1})\}. \]

We aim to prove that \( L \vee g_{n\wedge \tau_g} \) is a supermartingale. Noting that \( \log(1 - 1/(4M)) < 0 \) and \( \log F(Y(n + 1)) - \log F(Y(n)) \leq 0 \), inequality (15) in Corollary 2 implies
\[
\mathbb{E}(\max(\log(F(Y(n + 1))) - \log F(Y(n)), 2(L - \log(\delta^{-1}))) \mid \mathcal{F}_n)
= \mathbb{E}\left(\max\left(\log(F(Y(n + 1))) - \log F(Y(n)), \log\left(1 - \frac{1}{4M}\right)\right) \mid \mathcal{F}_n\right)
\leq \mathbb{E}\left(1 \left(\log(F(Y(n + 1))) - \log F(Y(n)) \leq \log\left(1 - \frac{1}{4M}\right)\right) \cdot \log\left(1 - \frac{1}{4M}\right) \mid \mathcal{F}_n\right)
= \log\left(1 - \frac{1}{4M}\right) \mathbb{P}\left(F(Y(n + 1)) - F(Y(n)) \leq -\frac{1}{4M}F(Y(n)) \mid \mathcal{F}_n\right) \leq -\frac{1}{4M}4^{-d}.
\]

Combining this with (27), we find that on the event \( g_n \geq \log(\delta^{-1}) \), we have
\[
\mathbb{E}((L \vee g_{n+1}) - (L \vee g_n) \mid \mathcal{F}_n) = \mathbb{E}((L \vee g_{n+1}) - g_n \mid \mathcal{F}_n)
= \mathbb{E}((L - g_n) \vee (g_{n+1} - g_n) \mid \mathcal{F}_n)
\leq \mathbb{E}((L + \log \delta) \vee (g_{n+1} - g_n) \mid \mathcal{F}_n)
= \mathbb{E}\left(\frac{1}{2}\log\left(1 - \frac{1}{4M}\right) \vee (g_{n+1} - g_n) \mid \mathcal{F}_n\right)
\leq \mathbb{E}\left(\frac{1}{2}\log\left(1 - \frac{1}{4M}\right) \vee \frac{1}{2}\log\left(F(Y(n + 1)) \mid \mathcal{F}_n\right) + \log\left(1 - \frac{1}{4M}\right) + \log\left(1 - \frac{1}{4M}\right) \mid \mathcal{F}_n\right)
\leq \mathbb{E}\left(\frac{1}{2}\log\left(1 - \frac{1}{4M}\right) \vee \frac{1}{2}\log\left(F(Y(n + 1)) \mid \mathcal{F}_n\right) + \log\left(1 - \frac{1}{4M}\right) \mid \mathcal{F}_n\right)
\leq \frac{1}{8M}4^{-d} + c_1M \frac{d_{\mathcal{F}}(Y(n))}{\sqrt{F(Y(n))}} = \frac{4^{-d}}{8M} + c_1Me^{-g_n} \leq -\frac{4^{-d}}{8M} + c_1M\delta = 0.
\]
(Note that we used the fact that \( \max(a + b, c) \leq \max(a, b) + c \) whenever \( c \geq 0 \) when we went from the fifth to the sixth line.) We therefore have
\[
\mathbb{E}((L \vee g_{(n+1)\wedge \tau_g}) - (L \vee g_{n\wedge \tau_g}) \mid \mathcal{F}_n) \leq 0.
\]

Using the first part of Corollary 2, we find
\[
\mathbb{P}(g_{n+1} - g_n \leq \log(1 - 1/(4M))) \geq 4^{-d} - c_1Me^{-g_n},
\]

so under the same condition that \( g_n \geq \log(\delta^{-1}) \) we also have

\[
\mathbb{P} \left( (L \lor g_{n+1}) - (L \lor g_n) \leq \log \left( 1 - 1/(4M) \right) \right) \geq 4^{-d} \left( 1 - \frac{1}{8M} \right) > 0. \tag{29}
\]

Inequality (28) shows that the random sequence \( L \lor g_n \) is a supermartingale that is bounded below by \( L \), so it almost surely converges, but inequality (29) shows that it can only converge to a value less than or equal to \( \log(\delta^{-1}) \). Therefore \( \tau_g < \infty \text{ a.s.} \), and indeed a.s. \( g_n \leq \log(\delta^{-1}) \) for infinitely many \( n \).

Recall the definition of \( d_{\mathcal{G}}(\cdot) \) from before Lemma 18.

**Lemma 19.** Let \( \delta > 0 \) and \( \Delta > 0 \). Then there exist \( n_1 \in \mathbb{N} \) and \( \epsilon > 0 \) (both depending on \( d, M, \delta \) and \( \Delta \)) such that for any \( n \in \mathbb{N} \),

\[
\mathbb{P} \left( \frac{d_{\mathcal{G}}(Y(n + n_1))}{\sqrt{F(Y(n + n_1))}} \geq \Delta \left| \frac{d_{\mathcal{G}}(Y(n))}{\sqrt{F(Y(n))}} > \delta \right. \right) > \epsilon.
\]

**Proof.** Before we proceed with the formal proof, let us explain the idea behind it. The main ingredient is that there is a positive probability \( \epsilon \) that for some number \( n_1 \) of consecutive steps the new point arrives very close to the center of mass of the existing configuration, relative to its diameter. When this occurs, the moment of inertia decreases by a definite multiplicative factor at each step. As a result, after \( n_1 \) steps the square-root of the moment of inertia will have decreased by at least a factor \( \delta/(2\Delta) \). At the same time, the convex hull of the configuration stays inside a bounding region that grows from step to step, by a geometrically decreasing sequence of increments. It can be arranged that the bounding region never gets closer to \( \mathcal{B}^c \) than \( \frac{1}{2}d_{\mathcal{G}}(Y(n)) \), and hence \( d_{\mathcal{G}}(Y(n + n_1)) \geq \frac{1}{2}d_{\mathcal{G}}(Y(n)) \).

We will combine ideas from the proofs of Corollary 2 and Lemma 9. Define

\[
\alpha := \min \left( \frac{M - 1}{2M(M + 1)}, \frac{\delta}{48(M + 1)} \right), \quad \gamma := \sqrt{1 - \frac{1}{12(M + 1)}} \in (0, 1)
\]

and

\[
n_1 := \left\lceil \frac{\log \left( \frac{\delta}{2\Delta} \right)}{\log \gamma} \right\rceil.
\]

For each \( i \in \{1, \ldots, n_1\} \), define the event \( E_i \) by

\[
E_i := \left\{ y_{n+i} \in B \left( \mu(Y(n+i-1)), \alpha\sqrt{F(Y(n+i-1))} \right) \right\}.
\]

*Think of \( \delta \) small and \( \Delta \) large.*
We have
\[ \alpha \sqrt{F(Y(n+i-1))} \geq \alpha \frac{M}{\sqrt{M-1}} A(Y(n+i-1)), \]
so
\[ \mathbb{P}(E_i \mid Y(n+i-1)) \geq \mathbb{P}\left( y_{n+i} \in B\left( \mu(Y(n+i-1)), \alpha \frac{MA(Y(n+i-1))}{\sqrt{M-1}} \right) \mid Y(n+i-1) \right). \]

We have \( \frac{\alpha M}{\sqrt{M-1}} < 1 \) because \( \alpha \leq \frac{M-1}{2M(M+1)} \). Hence, using Lemmas 3, 4 and 7 (as in the proof of Corollary 2), we have
\[ \mathbb{P}(E_i \mid Y(n+i-1)) \geq \left( \frac{\alpha M \sqrt{M-1}}{M+1} \right)^d. \]

Therefore
\[ \mathbb{P}\left( \bigcap_{i=1}^{n_1} E_i \mid Y(n) \right) \geq \left( \frac{\alpha M \sqrt{M-1}}{M+1} \right)^{dn_1} =: \epsilon. \]

Now suppose that \( d_{\theta}(Y(n)) \geq \delta \sqrt{F(Y(n))} \) and suppose that the event \( \bigcap_{i=1}^{n_1} E_i \) occurs. Our goal is to show that \( d_{\theta}(Y(n+n_1)) \geq \Delta \sqrt{F(Y(n+n_1))} \).

Using the condition \( \alpha \leq \frac{M-1}{2M(M+1)} \) together with \( \sqrt{F(Y(n+i-1))} \leq MA(Y(n+i-1)) \), see [4], we obtain for each \( i = 1, \ldots, n_1 \) that
\[ \alpha \sqrt{F(Y(n+i-1))} < \frac{M-1}{2(M+1)} A(Y(n+i-1)). \]

Then by inequality [17] from the proof of Corollary 2 we have
\[ F(Y(n+i)) < \gamma^2 F(Y(n+i-1)). \]

By induction we find that for \( i = 1, \ldots, n_1 \),
\[ F(Y(n+i)) < \gamma^{2i} F(Y(n)) \]
and in particular
\[ \sqrt{F(Y(n+n_1))} \leq \frac{\delta}{2\Delta} \sqrt{F(Y(n))}. \] (30)

For every \( i = 1, \ldots, n_1 \), since \( \mu(Y(n+i-1)) \in \text{Conv}(Y(n+i-1)) \),
\[ y_{n+i} \in N_{\alpha \sqrt{F(Y(n+i-1))}}(\text{Conv}(Y(n+i-1))), \]
so
\[ \text{Conv}(Y(n+i)) \subseteq N_{\alpha \gamma^{i-1} \sqrt{F(Y(n))}}(\text{Conv}(Y(n+i-1))). \]

Since
\[ \sum_{i=1}^{n_1} \alpha \gamma^{i-1} \sqrt{F(Y(n))} < \beta \sqrt{F(Y(n))}, \]

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where
\[ \beta := \frac{\alpha}{1 - \gamma} \]
it follows that
\[ \text{Conv}(Y(n + n_1)) \subseteq N_{\beta \sqrt{F(Y(n))}}(\text{Conv}(Y(n))). \]
Moreover,
\[ \beta < 24(M + 1) \alpha < \frac{\delta}{2}, \]
so
\[ d_{\mathcal{B}}(Y(n + n_1)) \geq d_{\mathcal{B}}(\text{Conv}(Y(n))) - \beta \sqrt{F(Y(n))} = d_{\mathcal{B}}(Y(n)) - \beta \sqrt{F(Y(n))} > (\delta - \delta/2)\sqrt{F(Y(n))}. \tag{31} \]
Taking square roots of both sides of (30) and dividing by (31) we obtain
\[ \frac{d_{\mathcal{B}}(Y(n + n_1))}{\sqrt{F(Y(n + n_1))}} > \Delta, \]
as required. \hfill \square

**Lemma 20.** Let \( \delta > 0 \). Then there exist \( n_1' \in \mathbb{N} \) and \( \epsilon' > 0 \) (both depending on \( d, M, \) and \( \delta \)) such that for any set \( X \) of \( M \) points in \( \mathcal{B} \) such that \( |X \cap \partial \mathcal{B}| = 1 \) and \( d_{\mathcal{B}}(X) > \delta \sqrt{F(X)} \), and any \( n \in \mathbb{N} \), we have
\[ \mathbb{P}(Y(n + n_1') \subset \mathcal{B}^c | Y(n) = X) > \epsilon'. \]

**Proof.** The proof is very similar to the proof of Lemma 19. We take \( \Delta = \frac{1}{\sqrt{M - 1}} \) and then make the same choices of constants in terms of \( M, d, \delta \) and \( \Delta \) as we did there:
\[ \alpha := \min \left( \frac{M - 1}{2M(M + 1)}, \frac{\delta}{48(M + 1)} \right), \quad \gamma := \sqrt{1 - \frac{1}{12(M + 1)}}, \]
\[ n_1' := \left\lceil \log \left( \frac{\delta}{2\Delta} \right) \right\rceil \frac{\log \gamma}{\log \gamma}, \quad \epsilon' = \left( \frac{\alpha M \sqrt{M - 1}}{M + 1} \right)^{dn_1'}. \]
Let \( y_b \) be the unique point in \( Y(n) \cap \partial \mathcal{B} \). This time, the good sequence of events whose probability is at least \( \epsilon \) is
\[ E_i' := \left\{ y_{n+i} \in B \left( \mu(Y(n + 1 - i) \setminus \{y_b\}), \alpha \sqrt{F(Y(n + i - 1))} \right) \right\}. \]
Instead of applying Lemma 4, we apply Lemma 5 to see that
\[ \mathcal{B} \cap B \left( \mu(Y(n + i - 1) \setminus \{y_b\}), \frac{\alpha MA(Y(n + i - 1))}{\sqrt{M - 1}} \right) \subseteq \text{Keep}(Y(n + i - 1); \mathcal{B}). \]
This requires us to check that \( \frac{\alpha M}{\sqrt{M+1}} < \frac{M}{M+1} \), which follows from \( \alpha \leq \frac{M-1}{2M(M+1)} \). It follows as before that on the event \( Y(n) \cap \partial \mathscr{B} = \{ y_b \} \) we have

\[
P \left( \bigcap_{i=1}^{n_1'} E_i' \ \middle| \ Y(n) \right) \geq \epsilon'.
\]

It also follows as before that on the events \( d_{\mathscr{B}}(Y(n) \setminus \{ y_b \}) \geq \delta \) and \( \bigcap_{i=1}^{n-1} E_i' \) we have

\[
\sqrt{F(Y(n+n_1'))} \leq \frac{\delta}{2\Delta} \sqrt{F(Y(n))}.
\]

Throughout the rest of the proof, we replace each set of the form \( \text{Conv}(Y(k)) \) by \( \text{Conv}(Y(k) \setminus \{ y_b \}) \). Note that \( \mu(Y(k) \setminus \{ y_b \}) \in \text{Conv}(Y(k) \setminus \{ y_b \}) \) so that on the events \( E_i' \) we obtain

\[
\text{Conv}(Y(n+n_1') \setminus \{ y_b \}) \subseteq N_{\beta \sqrt{F(Y(n))}} \text{Conv}(Y(n) \setminus \{ y_b \})
\]

and it follows that

\[
d_{\mathscr{B}}(Y(n+n_1') \setminus \{ y_b \}) \geq \frac{\delta}{2} \sqrt{F(Y(n))},
\]

as before, and hence

\[
\frac{d_{\mathscr{B}}(Y(n+n_1'))}{\sqrt{F(Y(n+n_1'))}} > \Delta = \frac{1}{\sqrt{M-1}}. \tag{32}
\]

On the other hand, if \( y_b \in Y(n+n_1') \) then

\[
F(Y(n+n_1')) \geq \sum_{x \in Y(n+n_1') \setminus \{ y_b \}} \| y_b - x \|^2
\]

\[
= (M-1) \| y_b - \mu(Y(n+n_1') \setminus \{ y_b \}) \|^2 + \sum_{x \in Y(n+n_1') \setminus \{ y_b \}} \| x - \mu(Y(n+n_1') \setminus \{ y_b \}) \|^2
\]

\[
\geq (M-1) \| y_b - \mu(Y(n+n_1') \setminus \{ y_b \}) \|^2
\]

\[
\geq (M-1) \left[ \text{dist}(\mu(Y(n+n_1') \setminus \{ y_b \}), \partial \mathscr{B}) \right]^2
\]

\[
\geq (M-1) \left[ d_{\mathscr{B}}(\text{Conv}(Y(n+n_1') \setminus \{ y_b \})) \right]^2
\]

\[
= (M-1) \left[ d_{\mathscr{B}}(Y(n+n_1')) \right]^2,
\]

contrary to (32). We have shown that on the event \( d_{\mathscr{B}}(Y(n) \setminus \{ y_b \}) \geq \delta \), the event \( \bigcap_{i=1}^{n-1} E_i' \) occurs with probability at least \( \epsilon' \), and when it does, we have \( y_b \not\in Y(n+n_1') \), as required.

**Corollary 3.** Suppose \( \mathscr{B} \) is a convex body in \( \mathbb{R}^d \) with uniform geometry and let \( Y(n)_{n \geq 0} \) be a \( \mathscr{B} \)-valued Jante’s law process. There exists \( \delta > 0 \) (depending on \( d \) and \( \mathscr{B} \)) such that for any set \( X \) of \( M \) distinct points of \( \mathscr{B} \), conditional on \( Y(n_0) = X \), a.s. there exists \( n \geq n_0 \) such that

\[
d_{\mathscr{B}}(Y(n)) > \delta \sqrt{F(Y(n))}.
\]
Proof. Consider a \( \mathcal{B} \)-valued Jante’s law process started at \( Y(n_0) = X \). Define the stopping time

\[
\tau_b = \inf\{n \geq n_0 : Y(n) \cap \partial \mathcal{B} = \emptyset\}.
\]

Note that if \( n \geq \tau_b \) then \( g_n = \frac{1}{2} \log F(Y(n)) - \log d_B(Y(n)) \). Let \( \delta \) be as in Lemma 18 so that a.s. infinitely often \( g_n < \log(\delta^{-1}) \). Note that if \( n \geq \tau_b \) and \( g_n < \log(\delta^{-1}) \) then \( d_B(Y(n)) > \delta \sqrt{F(Y(n))} \). So it suffices to show that a.s. \( \tau_b < \infty \).

We define a sequence of stopping times \((\sigma_i)_{i \geq 0}\). Let

\[
\sigma_0 = \inf\{n \geq n_0 : |Y(n) \cap \partial \mathcal{B}| \leq 1\}.
\]

Since \( D(Y(n)) \to 0 \) a.s., and a.s. all new points are in \( \mathcal{B}^o \), we have a.s. \( \sigma_0 < \infty \). Let \( n_1 \) and \( \epsilon \) be as provided by Lemma 20. Now inductively define for \( i = 1, 2, 3, \ldots \)

\[
\sigma_i = \begin{cases} 
\inf \{n > \sigma_{i-1} : d_B^o(Y(n)) > \delta \sqrt{F(Y(n))}\} & \text{if } i \text{ is odd}, \\
\sigma_{i-1} + n_1 & \text{if } i \text{ is even}.
\end{cases}
\]

By Lemma 18 a.s. \( \sigma_i < \infty \) for all \( i \geq 1 \). Then by Lemma 20, for each \( m \geq 1 \), on the event \( |Y(\sigma_{2m-1}) \cap \partial \mathcal{B}| = 1 \) we have

\[
\mathbb{P}(Y(\sigma_{2m}) \subset \mathcal{B}^o \mid \mathcal{F}_{\sigma_{2m-1}}) \geq \epsilon.
\]

Hence

\[
\mathbb{P}(Y(\sigma_{2m}) \not\subset \mathcal{B}^o) < (1 - \epsilon)^m,
\]

so \( \tau_b < \infty \) a.s., as required.

\[ \square \]

**Corollary 4.** Suppose \( \mathcal{B} \) is a convex body in \( \mathbb{R}^d \) with uniform geometry and let \( Y(n)_{n \geq 0} \) be a \( \mathcal{B} \)-valued Jante’s law process. Let \( \Delta > 0 \) be given. Then for any set \( X \) of \( M \) distinct points of \( \mathcal{B} \), conditional on \( Y(n_0) = X \), a.s. there exists \( n \geq n_0 \) such that \( d_{\mathcal{B}}(Y(n)) > \Delta \sqrt{F(Y(n))} \).

**Proof.** We combine Lemma 19 with Corollary 3. Consider a \( \mathcal{B} \)-valued Jante’s law process started at \( Y(n_0) = X \). Now let \( n_1 \) and \( \epsilon \) be as given by Lemma 19 applied to the \( \Delta \) specified in the statement of Corollary 4. Define a sequence of stopping times \((\kappa_i)_{i \geq 0}\) as follows. \( \kappa_0 = n_0 \), and then inductively for \( i = 1, 2, 3, \ldots \)

\[
\kappa_i = \begin{cases} 
\inf \{n > \kappa_{i-1} : d_{\mathcal{B}}(Y(n)) > \delta \sqrt{F(Y(n))}\} & \text{if } i \text{ is odd}, \\
\kappa_{i-1} + n_1 & \text{if } i \text{ is even}.
\end{cases}
\]

By Corollary 3 a.s. \( \kappa_i < \infty \) for all \( i \). By Lemma 19, for each \( m \geq 1 \),

\[
\mathbb{P} \left( d_{\mathcal{B}}(Y(\kappa_{2m})) > \Delta \sqrt{F(Y(\kappa_{2m}))} \mid \mathcal{F}_{\kappa_{2m-2}} \right) > \epsilon,
\]

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so
\[ \mathbb{P}\left( d_{\mathcal{B}}(Y(\kappa_{2m})) \leq \Delta \sqrt{\frac{F(Y(\kappa_{2m}))}{m}} \right) \leq (1 - \epsilon)^m, \]
giving the conclusion by letting \( m \to \infty. \)

We can now give the proof of Theorem 2.

\textbf{Proof.} In Lemma 9, we take \( \epsilon = \frac{1}{2}. \) Define
\[ \Delta = \frac{2}{M\sqrt{M-1}} \left( n_0(1/2) + \frac{1}{1 - \gamma^{1/4}} \right) + \frac{M + 1}{M\sqrt{M-1}}. \]
On the event that \( d_{\mathcal{B}}(Y(n)) > \Delta \sqrt{F(Y(n))}, \) we have
\[ \mathbb{P}\left( \| \mu(Y(n+j)) - \mu(Y(n)) \| + \frac{M + 1}{M\sqrt{M-1}} \sqrt{F(Y(n))} \leq \Delta \sqrt{F(Y(n))} \right) \text{ for all } j \geq 0 \bigg| \mathcal{F}_n \geq \frac{1}{2} \]
and hence, using (4) to compare \( A(Y(n+j)) \) with \( \sqrt{F(Y(n+j))}, \)
\[ \mathbb{P}\left( \| \mu(Y(n+j)) - \mu(Y(n)) \| + \frac{M + 1}{M\sqrt{M-1}} A(Y(n)) \leq \Delta \sqrt{F(Y(n))} \right) \text{ for all } j \geq 0 \bigg| \mathcal{F}_n \geq \frac{1}{2}. \]
Now by Lemma 3,
\[ \mathbb{P}\left( \text{Keep } (Y(n+j); \mathbb{R}^d) = \text{Keep}(Y(n+j); \mathcal{B}) \right) \text{ for all } j \geq 0 \bigg| \mathcal{F}_n \geq \frac{1}{2}. \]
Define an increasing sequence of stopping times \((\theta_i)_{i \geq 0}\) as follows. \( \theta_0 = 0 \) and then inductively for \( i = 1, 2, 3, \ldots \)
\[ \theta_i = \begin{cases} \inf \left\{ n > \theta_{i-1} : d_{\mathcal{B}}(Y(n)) > \Delta \sqrt{F(Y(n))} \right\} & \text{if } i \text{ is odd}, \\ \inf \left\{ n > \theta_{i-1} : \text{Keep } (Y(n); \mathbb{R}^d) \neq \text{Keep } (Y(n); \mathcal{B}) \right\} & \text{if } i \text{ is even}. \end{cases} \]
(If for some \( i \) we have \( \theta_i = \infty \) then for all \( j > i \) we also have \( \theta_j = \infty. \)) For each \( m \geq 1, \) on the event that \( \theta_{2m-2} < \infty, \) firstly \( \theta_{2m-1} < \infty \) a.s., by Corollary 4, and then \( \mathbb{P}\left( \theta_{2m} = \infty \bigg| \mathcal{F}_{\theta_{2m-1}} \right) \geq \frac{1}{2}. \)
Hence a.s. there is a finite \( m \) such that \( \theta_{2m-1} < \infty \) but \( \theta_{2m} = \infty. \)

We have shown that the distribution of \( \xi \) is a mixture of distributions, each of which is (for some \( m, n, \) and \( X \)) the conditional distribution of \( \xi \) given \( \theta_{2m-1} = n, \ Y(n) = X \) and \( \theta_{2m} = \infty. \)
Each such distribution is equal to the conditional distribution of \( z_{\infty} \) for the \( \mathbb{R}^d \)-valued Jante’s law process \( Z(\cdot) \) started at \( Z(0) = X, \) conditioned on the positive probability event that for all \( j \geq 0, \) \( \text{Keep } (Z(j); \mathbb{R}^d) = \text{Keep } (Z(j); \mathcal{B}). \) By Theorem 1, this distribution is continuous. Thus the distribution of \( \xi \) is a mixture of continuous distributions, and is therefore continuous. \( \square \)
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Appendix 1: Continuity of probability distributions

Here are the basic facts which we use about continuous \( \mathbb{R}^d \)-valued random variables.

1. **Definitions.** Let \( \mathcal{B}(\mathbb{R}^d) \) be the Borel \( \sigma \)-algebra of Euclidean space \( \mathbb{R}^d \). Let \( W : M \to \mathbb{R}^d \) be a Borel-measurable random variable on a probability space \((M, \mathcal{S}, \mathbb{P})\), meaning that for every \( A \in \mathcal{B}(\mathbb{R}^d) \), \( W^{-1}(A) \in \mathcal{S} \). The distribution of \( W \) is the unique Borel probability measure \( \nu \) on \( \mathcal{B}(\mathbb{R}^d) \) such that for every \( A \in \mathcal{B}(\mathbb{R}^d) \) we have \( \nu(A) = \mathbb{P}(W \in A) \). (It suffices to check that this holds for every open set \( A \subset \mathbb{R}^d \), or for every closed set.) \( W \) and \( \nu \) are
both said to be *continuous* when \( \nu \) is absolutely continuous with respect to the Lebesgue measure \( \lambda \). This means that for every Borel set \( A \subset \mathbb{R}^d \),

\[
(\lambda(A) = 0) \implies (\nu(A) = \mathbb{P}(W \in A) = 0).
\]

Equivalently, there exists an element \( f \in L^1(\mathbb{R}^d, \lambda) \) of norm 1 such that for every Borel set \( A \subseteq \mathbb{R}^d \) we have

\[
\mathbb{P}(W \in A) = \nu(A) = \int_A f(x) \, d\lambda(x).
\]

This element \( f \) is called the density of \( \nu \), or the density of \( W \). Every element of \( L^1(\mathbb{R}^d, \lambda) \) has a Borel-measurable representative, so we may assume that \( f \) is Borel-measurable. The density of a continuous random variable is not necessarily representable by a continuous function, and if \( M \) happens to be equipped with a topology which generates \( \mathcal{G} \), the continuity of \( W \) as a random variable is unrelated to the continuity of \( W \) as a function.

2. **Mixtures of continuous random variables are continuous.** Suppose that \( \pi \) is a Borel probability measure on \( L^1(\mathbb{R}^d) \) and \( \nu \) is a random continuous probability measure defined on \( (M, \mathcal{G}, \mathbb{P}) \) whose density \( d\nu/d\lambda \) with respect to Lebesgue measure \( \lambda \) is distributed according to \( \pi \). Then for any Lebesgue measurable set \( A \subseteq \mathbb{R}^d \) we have (by Tonelli’s theorem)

\[
\mathbb{E}^{\pi}(\nu)(A) := \int \nu(A) \, d\pi(\nu) = \int \int \frac{d\nu}{d\lambda}(x) 1_A(x) \, d\lambda(x) \, d\pi(\nu) = \int \int \frac{d\nu}{d\lambda}(x) \, d\pi(\nu) \, 1_A(x) \, d\lambda(x),
\]

so the mixture \( \mathbb{E}^{\pi}(\nu) \) is a continuous probability measure with density \( \int \frac{d\nu}{d\lambda} \, d\pi(\nu) \). To see that the integral of this density with respect to \( \lambda \) is 1, take \( A = \mathbb{R}^d \) above. Even if \( \nu \) a.s. has a \( C^\infty \) density, it need not be true that \( \mathbb{E}^{\pi}(\nu) \) has a continuous density.

3. **Conditioning on positive probability events.** Let \( \nu \) be a Borel probability measure on \( \mathbb{R}^d \) and \( W \) an \( \mathbb{R}^d \)-valued random variable on \( (M, \mathcal{G}, \mathbb{P}) \) with distribution \( \nu \). If \( E \in \mathcal{G} \) is an event such that \( \mathbb{P}(E) > 0 \), then the conditional distribution of \( W \) given \( E \) is also continuous, with density \( \frac{\mathbb{P}(E|W=x) \, dx}{\mathbb{P}(E)} \), since for any Lebesgue-measurable set \( A \subseteq \mathbb{R}^d \) we have

\[
\mathbb{P}(W \in A \mid E) = \frac{\mathbb{P}(E \cap (W \in A))}{\mathbb{P}(E)} = \int 1_A(x) \frac{\mathbb{P}(E|W=x) \, dx}{\mathbb{P}(E)} \, d\lambda(x).
\]

Again, taking \( A = \mathbb{R}^d \) shows that the integral of this density is 1.
Appendix 2: Proof of Lemma \[17\]

**Proof.** By replacing \(B\) with \(B \cap B(x, R + r + 1)\), which does not alter the set whose volume is to be estimated, we may assume that \(B\) is bounded.

Next, we reduce to the case where \(B\) is a polyhedron. Let \(\epsilon > 0\). We claim that there is a convex polyhedron \(P\) (with finitely many facets) such that \(B \subseteq P \subset N_\epsilon(B)\). To see this, let \(y\) be any point in the interior of \(B\) and let \(\pi_y\) denote the radial projection about \(y\) from \(\mathbb{R}^d \setminus \{y\}\) onto the unit sphere \(S_y\) that is centered on \(y\). For each point \(z \in \partial B\) and each supporting hyperplane \(H\) of \(B\) at \(z\), the radial projection \(\pi_y(H \cap N_\epsilon(B))\) is a neighbourhood of \(\pi_y(z)\) in the sphere \(S_y\). Since \(S_y\) is compact, some finite collection of such projections covers it. The corresponding finite collection of support hyperplanes defines a suitable polyhedron \(P\) (by taking the intersection of the half-spaces that they bound which contain \(B\)).

For each \(z \in B\), there exists a point \(b \in \partial B\) such that \(\|z - b\| = \text{dist}(z, B^c)\), and a supporting hyperplane \(H\) such that \(H\) passes through \(b\) and if \(z \neq B\) then \(H\) is orthogonal to \(b - z\). Then the translate of \(H\) by distance \(\epsilon\) in the direction \(b - z\) is a supporting hyperplane of \(N_\epsilon(B)\) and it follows that

\[
\text{dist} (z, N_\epsilon (B^c)) = \text{dist} (z, B^c) + \epsilon.
\]

Since \(P \subseteq N_\epsilon(B)\), we have

\[
\text{dist} (z, P^c) \leq \text{dist} (z, B^c).
\]

It follows that

\[
B \cap N_r (B^c) \subset P \cap N_{r+\epsilon} (P^c).
\]

Hence if inequality \[25\] always holds for a bounded convex polyhedron \(P\) with finitely many faces in place of \(B\), (and \(r + \epsilon\) in place of \(r\)), then by taking the limit as \(\epsilon \downarrow 0\), we obtain \(25\) for a general bounded convex body \(B\) and hence also for any convex body \(B\).

Finally, we prove \[25\] in the case where \(B\) is a bounded convex polyhedron \(P\) which has finitely many faces. Suppose the facets of \(P\) are \(F_1, \ldots, F_n\). (A facet is a face of co-dimension one.) For each \(i = 1, \ldots, n\), let \(u_i\) be the outward-pointing normal vector of \(F_i\), and let the facet \(F_i\) be contained in the hyperplane \(H_i = \{z \in \mathbb{R}^d : z.u_i = h_i\}\), so that

\[
P = \{z \in \mathbb{R}^d : z.u_i < h_i \text{ for } i = 1, \ldots, n\}.
\]

Let \(\text{Cut}(P)\) be the cut-set of \(P\), which is the set of points in \(P\) that are equidistant from at least two points in \(\partial P\). Note that \(\lambda(\text{Cut}(P)) = 0\) since \(\text{Cut}(P)\) is contained in the union of the finite set of hyperplanes which bisect the dihedral angles of pairs of hyperplanes \(H_i, H_j\).
Also, $\lambda(\partial P) = 0$. So to prove \cite{25}, if suffices to define a volume-preserving injection $\varphi$ from $B(x, R) \cap (P^o \setminus \text{Cut}(P)) \cap N_r(P^c)$ to $B(x, R) \setminus B(x, R - r)$.

Partition the open set $(P^o \setminus \text{Cut}(P)) \cap N_r(P^c)$ into finitely many pieces $Q_1, \ldots, Q_n$, where each piece $Q_i$ consists of the points whose closest facet of $P$ is a given facet $F_i$. $Q_i$ is an open convex polyhedron contained in the $r$-neighbourhood of the facet $F_i$.

The map $\varphi$ will translate each point of $Q_i \cap B(x, R)$ by a non-negative multiple of $u_i$, where the multiple may vary from point to point. Specifically, for any $z \in Q_i \cap B(x, R)$, define

$$a(z) = \sup(\{t \in \mathbb{R} : z + tu_i \in Q_i\}) = h_i - z.u_i = \text{dist}(z, P^c),$$

and

$$b(z) = \sup(\{t \in \mathbb{R} ; z + tu_i \in B(x, R)\}).$$

Then define

$$\varphi(z) := z + \max(0, b(z) - a(z)).$$

Note that if $z' = z + tu_i$ is also in $Q_i \cap B(x, R)$ then $b(z') = b(z) - t$ and $a(z') = a(z) - t$, since both $B(x, R)$ and $Q_i$ are convex. Hence $\varphi(z') = \varphi(z) + tu_i$. Informally, the map $\varphi$ slides each line segment of the form $\{z + tu_i : t \in \mathbb{R}, z + tu_i \in B(x, R) \cap Q_i\}$ as far as possible in the direction $u_i$ (parallel to the line segment) such that it remains a subset of $B(x, R)$. Each of these line segments has length at most $r$. It follows that every point in the image of $\varphi$ is within distance $r$ of the boundary of $B(x, R)$. The restriction of $\varphi$ to $Q_i \cap B(x, R)$ is a smooth volume-preserving map, since both $a(z)$ and $b(z)$ depend smoothly on $z$, and in an appropriate coordinate system its Jacobian matrix is upper unitriangular.

Finally, we must check that $\varphi$ is an injection. From the description in terms of sliding line segments, we see that the restriction of $\varphi$ to each piece $Q_i \cap B(x, R)$ is an injection. Secondly, if $z \in Q_i \cap B(x, R)$, then the unique closest facet of $P$ to $\varphi(z)$ is $F_i$. Therefore the images of distinct pieces under $\varphi$ are disjoint and we are done.