Colored Tobit Kalman filter

Kostas Loumponias

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki, Greece

ABSTRACT
This article deals with the Tobit Kalman filtering (TKF) process when the one-dimensional measurements are censored and the noises of the state-space model are colored. Two improvements of the standard TKF process are proposed. Firstly, the exact moments of the censored measurements are calculated via the moment generating function of the censored measurements. Secondly, colored noises are considered in the proposed method in order to tackle real-life problems, where the white noises are not common. The designed process is evaluated using two experiments-simulations. The results show that the proposed method outperforms other methods in minimizing the Root Mean Square Error (RMSE) in both experiments.

ARTICLE HISTORY
Received 29 July 2020
Accepted 5 May 2021

KEYWORDS
Censored data; moment generating function; Kalman filtering; colored noise

1. Introduction
The well-known Kalman filter (KF) (Brow and Hwang 1992) process is a recursive linear filter, which provides estimates of the hidden state vectors by using a series of measurements observed over time. The KF process provides an optimal performance (i.e., unbiased estimations with minimum variance errors) when i) the state and measurement equations are linear and ii) the measurement and process noises are white (usually normally distributed). However in many real-life problems the state-space model is not linear, therefore, KF is not suitable for providing optimal estimates. The most known filters, that have been proposed in order to overcome the problem of non-linearity, are the Extended Kalman filter (EKF) (Ljung 1979), the Unscented Kalman Filter (UKF) (Wan and Van Der Merwe 2000) and the Particle filter (PF) (Djuric et al. 2003).

In the case where non-linearity is due to censoring in the measurements (Dong, Gould, and Kaiser 2004), the UKF and EKF result in calculating biased estimates for the hidden state vectors, as it has been shown in Allik (2014) and Allik et al. (2014). PF is able to cope with censored measurements, however, it imposes a heavy computational burden. To that end, Tobit Kalman filter (TKF) (Allik et al. 2016) has been proposed to cope with censored measurements with a low computational cost. More specifically, the TKF process provides unbiased estimates when dealing with censored measurements using the Tobit model of type I with two censoring limits (Loumponias et al. 2016a, 2016b). In Loumponias et al. (2018), a multi-object tracking algorithm based on the TKF process was proposed, where the exact variance of the censored measurement is
calculated, while in Allik et al. (2016) and Loumponias et al. (2016a, 2016b), an approximated censored variance is used. In Loumponias et al. (2018), a brief proof for the calculation of the exact censored variance is provided.

Next, methods dealing with variants of TKF are presented. However, in all the methods mentioned, the exact variance of the censored measurement is not calculated. In Geng et al. (2017b), a modified TKF process, which deals with intermittent failures in data transmissions is presented. To that end, the TKF process is designed, based on a modified Tobit regression model, which takes into account the intermittent failures as well as the time correlated multiplicative measurement noises under the redundant channel transmission protocol. In Geng et al. (2017a), the TKF process with censored and fading measurements is proposed. The censored measurements are described by the Tobit model of type I, while the fading measurements are described by the Lth-order Rice fading channel model. To that end, TKF is designed in the presence of fading measurements. Furthermore, several state augmentation induced terms are introduced, which can be calculated recursively or off-line. In Li, Jia, and Du (2017), the TKF process considers time-correlated multiplicative measurement noise, while the measurements are censored. Several new terms are introduced including the estimates for the product of multiplicative measurement noise and the system state as well as their error covariance matrices. Recursive update computations for these terms are also presented.

In all the aforementioned methods, the measurement and process noises are assumed to be white. Nevertheless, in many real problems (Kuhlmann 2003), this assumption fails and the noises can be adequately described by AR(p) models (Akaike 1998); such noises are called “coloured” noises. In the case where the noises are colored, the standard KF, EKF, UKF etc. provide biased estimations, since, they cope with white noises. In order to overcome this drawback, the state-space model with colored noises is written in the form of a system driven by white noises. To that end, two non-numerical methods are proposed, the augmented (Bryson and Henrikson 1968; Popescu and Zeljkovic 1998) and the measurement differencing (Anderson and Moore 2012; Chen and Chui 1986) approach.

In the augmented approach, the colored noise of the measurement is included into the state vector. By doing so, the measurements of the augmented system are perfect, i.e., they do not longer contain noise. Hence, the covariance matrix of the measurement at time t given the a priori estimations of the state vector up to time t – 1 may become ill-conditioned, i.e., a singular matrix. In the measurement differencing approach, the derived measurements are expressed in terms of the state vector by considering white noise. In effect, a linear combination of two measurements in sequence is determined in order to eliminate the colored noise. However, the differencing approach leads to a risk of unstable solution when inaccurate observations occur (Gazit 1997).

The main contribution of this article is the establishment of the colored TKF (ColTKF) dealing with censored measurements when the process and measurement noises are colored. In accordance with other studies dealing with censored measurements (Allik et al. 2014, 2016; Li, Jia, and Du 2017; Loumponias et al. 2016a), the exact censored moments (order one to three) are calculated and also their properties are described. Furthermore, this article deals with a) multi-dimensional hidden state vector, b) one-dimensional censored measurement (Tobit Type I) (Wooldridge 2002) and c)
colored noises described by the AR(1) model. To that end, the moment generating function (mgf) of the multivariate Gaussian distribution is calculated when a marginal variable is censored with two censoring limits. Then, the marginal mgfs of the censored and uncensored variables are derived. Therefore, censored moments -of order one to three- can be calculated and it is proved that the rest variables (the uncensored) are still normally distributed. Next, ColTKF is derived by using the augmented approach and the moments of the censored measurements. Finally, the likelihood function of the censored measurements is utilized in order to estimate the unknown parameters of AR(1) models in colored noises.

In this article, the augmented approach is preferred, since, the censored measurements are one-dimensional, therefore, the computational burden is only slightly increased. Furthermore, in the measurement differencing approach, two latent measurements (not censored) in sequence are utilized in order the derived measurements to be expressed in terms of white noises; nevertheless, in the case of censoring, the latent measurements are not provided. The results in the simulations show that ColTKF has a better performance than TKF and the augmented KF, as it was expected.

The rest of the article is organized as follows: In Section 2, the moments of a censored one-dimensional variable are calculated by means of the mgf. In Section 3, the proposed method (ColTKF) is provided. In Section 4, experimental results are illustrated using artificial data to show up the effectiveness of the proposed process. Finally, in Section 5, conclusions are provided.

2. Moment generating function: censored case

Let \( x \sim N(m, S) \) and \( x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)^T \), where \( x \in \mathbb{R}^n \), \( m = (m_i)_{i=1}^n \) and \( S = \{S_{i,j}\}_{i,j=1}^n \). We consider the case where the \( k^{th} \) random variable (rv) of \( x \) is censored, symbolized by \( x^c_k \). More specifically, let

\[
    x^c_k = \begin{cases} 
    x_k, & a < x_k < b \\
    a, & x_k \leq a \\
    b, & x_k \geq b 
    \end{cases}
\]

(1)

where \( a \) and \( b \) are the censoring limits.

Then, the probability function of the random vector \( x^c = (x_1, \ldots, x_{k-1}, x^c_k, x_{k+1}, \ldots, x_n)^T \) is given by

\[
    f_{x^c}(x^c) = f_k(x^c)u_{(a,b)}(x^c_k) \\
    + \int_{-\infty}^{a} f_k(x)dx_k \cdot \delta(x^c_k - a) \\
    + \int_{b}^{\infty} f_k(x)dx_k \cdot \delta(x^c_k - b), 
\]

(2)

where \( f_k(x) \) is the probability density function (pdf) of \( x \), \( \delta \) stands for the Kronecker delta function and the function \( u_{(a,b)}(x^c_k) \) is equal with 1 when \( x^c_k \in (a, b) \) and 0 otherwise. Then, the moment generating function (mgf) of the censored random vector \( x^c \) can be derived by using Equation (2):
Proposition 1. For a normally distributed random vector \( x \sim N(\mathbf{m}, \mathbf{S}) \), the mgf of \( x^\prime \) is given by
\[
M_{x^\prime}(t) = \exp \left( t^\prime \mathbf{m} + \frac{1}{2} t^\prime \mathbf{S} t \right) \left( F_{e_k} \left( b - m_k - \sum_{i=1}^{n} S_{k,i} t_i \right) - F_{e_k} \left( a - m_k - \sum_{i=1}^{n} S_{k,i} t_i \right) \right) \\
+ \exp \left( t_k a + t_0^\prime \mathbf{m} + t_0^\prime \mathbf{S} t_0 / 2 \right) F_{e_k} \left( a - m_k - \sum_{i=1}^{n} S_{k,i} t_0,i \right) \\
+ \exp \left( t_k b + t_0^\prime \mathbf{m} + t_0^\prime \mathbf{S} t_0 / 2 \right) \left( 1 - F_{e_k} \left( b - m_k - \sum_{i=1}^{n} S_{k,i} t_0,i \right) \right),
\]
where \( t_0 = (t_1, ..., t_{k-1}, 0, t_{k+1}, ..., t_n)^T \), \( e_k \sim N(0, S_{k,k}) \) and \( F_{e_k} \) stands for the cumulative distribution function of \( e_k \).

Proof. We have that
\[
M_{x^\prime}(t) = \mathbb{E} e^{t^\prime x^\prime} = \int_{\mathbb{R}^n} \int_{a}^{b} e^{t^\prime x} f_x(x) dx \\
+ \int_{\mathbb{R}^n} e^{t^\prime x} f_{x-k}(x) \left( \int_{-\infty}^{a} f_x(x) dx \right) dx_{-k} \\
+ \int_{\mathbb{R}^n} e^{t^\prime x} f_{x-k}(x) \left( \int_{b}^{\infty} f_x(x) dx \right) dx_{-k},
\]
where \( t_{-k} = (t_1, ..., t_{k-1}, t_{k+1}, ..., t_n)^T \). The first term on the right hand side of Equation (3) reads
\[
A_1(t) = \int_{\mathbb{R}^n} \int_{a}^{b} e^{t^\prime x} \frac{1}{(2\pi)^{n/2} |\mathbf{S}|^{1/2}} \exp \left( -\frac{1}{2} (x - \mathbf{m})^T \mathbf{S}^{-1} (x - \mathbf{m}) \right) dx,
\]
which for \( x^\prime = x - \mathbf{m} \) yields
\[
A_1(t) = \int_{\mathbb{R}^n} \int_{a-m_k}^{b-m_k} e^{t^\prime (x^\prime + \mathbf{m})} \frac{1}{(2\pi)^{n/2} |\mathbf{S}|^{1/2}} \exp \left( -\frac{1}{2} x^\prime^T \mathbf{S}^{-1} x^\prime \right) dx^\prime \\
= \frac{e^{t^\prime \mathbf{m}}}{(2\pi)^{n/2} |\mathbf{S}|^{1/2}} \int_{\mathbb{R}^n} \int_{a-m_k}^{b-m_k} \exp \left( -\frac{1}{2} x^\prime^T \mathbf{S}^{-1} x^\prime + t^\prime x^\prime \right) dx^\prime.
\]
Then, for \( j = \mathbf{S} t \) we get
\[
A_1(t) = \frac{e^{t^\prime \mathbf{m}}}{(2\pi)^{n/2} |\mathbf{S}|^{1/2}} \int_{\mathbb{R}^n} \int_{a-m_k}^{b-m_k} \exp \left( -\frac{1}{2} (x^\prime - j)^T \mathbf{S}^{-1} (x^\prime - j) + \frac{1}{2} t^T \mathbf{S} t \right) dx^\prime \\
= \frac{\exp \left( t^\prime \mathbf{m} + \frac{1}{2} t^T \mathbf{S} t \right)}{(2\pi)^{n/2} |\mathbf{S}|^{1/2}} \int_{\mathbb{R}^n} \int_{a-m_k}^{b-m_k} \exp \left( -\frac{1}{2} (x^\prime - j)^T \mathbf{S}^{-1} (x^\prime - j) \right) dx^\prime.
\]
Finally, we get the analytic form of the mgf by substituting Equations (5), (7) and (8) into (3).

Now, for \( \mathbf{e} = \mathbf{x}^* - \mathbf{j} \), Equation (4) leads to

\[
A_1(t) = \frac{\exp \left( \mathbf{t}^T \mathbf{m} + \frac{1}{2} \mathbf{t}^T \Sigma \mathbf{t} \right)}{\left(2\pi\right)^{n/2}|\Sigma|^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \mathbf{e}^T \Sigma^{-1} \mathbf{e} \right) d\mathbf{e}
\]

where \( F_{\mathbf{x}_k}(\mathbf{e}) \) is the marginal cumulative function of the random variable \( \mathbf{e}_k \sim N(0, S_{k,k}) \), and \( j_k = \sum_{i=1}^{n} S_{k,i} t_i \).

Next, the second term of Equation (3) is computed as follows:

\[
A_2(t) = \frac{\exp \left( t_k a + \mathbf{t}_0^T \mathbf{m} + \frac{1}{2} \mathbf{t}_0^T \Sigma \mathbf{t}_0 / 2 \right)}{\left(2\pi\right)^{n/2}|\Sigma|^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \left( \mathbf{x}_k - \mathbf{j}_k \right)^T \Sigma^{-1} \left( \mathbf{x}_k - \mathbf{j}_k \right) \right) d\mathbf{x}_k
\]

where \( \mathbf{t}_0 = (t_1, \ldots, t_{k-1}, 0, t_{k+1}, \ldots, t_n)^T \). Then, for \( \mathbf{j}_0 = \Sigma t_0 \), Equation (6) becomes

\[
A_2(t) = \frac{\exp \left( t_k a + \mathbf{t}_0^T \mathbf{m} + \frac{1}{2} \mathbf{t}_0^T \Sigma \mathbf{t}_0 / 2 \right)}{\left(2\pi\right)^{n/2}|\Sigma|^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \left( \mathbf{x}_k - \mathbf{j}_0 \right)^T \Sigma^{-1} \left( \mathbf{x}_k - \mathbf{j}_0 \right) \right) d\mathbf{x}_k
\]

where \( j_{0,k} = \sum_{i=1}^{n} S_{k,i} t_{0,i} \).

In the same way as for the second term, the third term of Equation (3) is given by

\[
A_3(t) = \exp \left( \mathbf{t}_k b + \mathbf{t}_0^T \mathbf{m} + \frac{1}{2} \mathbf{t}_0^T \Sigma \mathbf{t}_0 / 2 \right) \left( 1 - F_{\mathbf{x}_k}(b - m_k - j_{0,k}) \right).
\]

Finally, we get the analytic form of the mgf by substituting Equations (5), (7) and (8) into (3).

It is derived from Proposition 1, that the mgf of the marginal \( \mathbf{x}_{-k} \) is equal with

\[
M_{\mathbf{x}_{-k}}(t) = M_{\mathbf{x}}(t_0) = \exp \left( \mathbf{t}_0^T \mathbf{m} + \frac{1}{2} \mathbf{t}_0^T \Sigma \mathbf{t}_0 \right)
\]

thus, \( \mathbf{x}_{-k} \sim N(\mathbf{m}_{-k}, \Sigma_{-k,-k}) \), where \( \mathbf{m}_{-k} = \{m_i\}_{i=1}^n \), \( i \neq k^n \) and \( \Sigma_{-k,-k} = \{S_{ij}\}_{i=1}^n \), \( j = 1, i, j \neq k^n \). We notice that this result does not hold in the case where the random variable \( x_k \) is truncated (Arismendi 2013).
In the same way, the mgf of the censored variable \( x'_k \) is given by

\[
M_{x'_k}(t_k) = \exp \left( t_k m_k + \frac{1}{2} t_k^2 S_{k,k} \right) (F_{c_k} (b - m_k - S_{k,k} t_k) - F_{c_k} (a - m_k - S_{k,k} t_k)) + \exp(t_k a) F_{c_k} (a - m_k) + \exp(t_k b) (1 - F_{c_k} (b - m_k)),
\]

which has the same form as in Loumponias et al. (2018).

Next, the censored mean, variance and skewness of \( x'_k \), can be calculated by Equation (10):

\[
\mathbb{E}(x'_k) = \left. \frac{dM_{x'_k}(t)}{dt} \right|_{t=0} = a F_{c_k} (a - m_k) + b (1 - F_{c_k} (b - m_k)) + (F_{c_k} (b - m_k) - F_{c_k} (a - m_k)) m_k + S_{k,k} (F_{c_k} (a - m_k) - F_{c_k} (b - m_k)),
\]

\[
\text{Var}(x'_k) = \left. \frac{d^2 M_{x'_k}(t)}{dt^2} \right|_{t=0} - (\mathbb{E}(x'_k))^2 = a^2 F_{c_k} (a - m_k) (1 - F_{c_k} (a - m_k)) + b^2 F_{c_k} (b - m_k) (1 - F_{c_k} (b - m_k)) + m_k P (1 - P) + S_{k,k}^2 P + 2 m_k S_{k,k} (F_{c_k} (a - m_k) - F_{c_k} (b - m_k)) + S_{k,k} ((a - m_k) f_{a_k} (a - m_k) - (b - m_k) f_{a_k} (b - m_k)) - 2 ab f_{a_k} (a - m_k) (1 - F_{c_k} (b - m_k)) - S_{k,k}^2 (F_{c_k} (a - m_k) - F_{c_k} (b - m_k))^2 - 2 [P m_k + S_{k,k} (F_{c_k} (a - m_k) - F_{c_k} (b - m_k))]: [a F_{c_k} (a - m_k) + b (1 - F_{c_k} (b - m_k))],
\]

where \( P = F_{c_k} (b - m_k) - F_{c_k} (a - m_k) \).

The third moment of \( x'_k \) is given by

\[
\mathbb{E}(x'^3_k) = \left. \frac{d^3 M_{x'_k}(t)}{dt^3} \right|_{t=0} = (m^3_k + 3 m_k^2 s) P + a^3 F_{c_k} (a - m_k) + b^3 (1 - F_{c_k} (b - m_k)) + (2 s^3 + 3 m_k^2 s) (\phi (a^*) - \phi (b^*)) + 3 m_k s^2 (a^* \phi (a^*) - b^* \phi (b^*)) + s^3 (a^*^2 \phi (a^*) - b^*^2 \phi (b^*))),
\]

where \( s = \sqrt{S_{k,k}} \), \( \phi (x) \) stands for the probability density function of the standard normal distribution, \( a^* = (a - m_k)/s \) and \( b^* = (b - m_k)/s \). Then, the coefficient of the censored skewness, \( \gamma'_k \), (Azzalini 2013) is calculated by substituting Equations (11)–(13) into

\[
\gamma'_k = \frac{\mathbb{E}(x'^3_k) - 3 \mathbb{E}(x'_k) \text{Var}(x'_k) - (\mathbb{E}(x'_k))^3}{\text{Var}(x'_k)^{3/2}}.
\]

Furthermore, the covariance of the variables \( x_i \) and \( x'_k \), for \( i \neq k \), can be calculated by means of Proposition 1 and Equation (11):

\[
\text{Cov}(x_i, x'_k) = \mathbb{E}(x_i x'_k) - \mathbb{E}(x_i) \mathbb{E}(x'_k) = \left. \frac{dM_{x_i x'_k}(t, t_k)}{dt dt_k} \right|_{(t, t_k) = (0, 0)} - m_i \mathbb{E}(x'_k) = P \cdot S_{i,k},
\]
where \( S_{t,k} = \text{Cov}(x_t, x_k) \). Furthermore, it is derived by Equations (11) and (14) that

\[
m_c^k = m_k
\]

and

\[
\gamma_c^k = 0,
\]

for the censoring limits \( a \) and \( b \), which fulfill the conditions, \( a < m_k \) and \( m_k = (a + b)/2 \). **An illustrative example:** In order to verify the aforementioned results, let us consider the censored mean vector \( m_c^i \), covariance matrix \( S^c \), and the coefficients of skewness \( g^c = \{g_i^c\}_{i=1}^3 \) of the random variable \( X \sim N(m, S) \), where \( m = (1,1,1)^T \) and \( S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \), when the censoring limits for the r.v. \( X_3 \) are \( a = 0.5 \) and \( b = 2 \). Then, we proceed as follows: 1) we produce \( 10^6 \) random measurements from \( N(m, S) \) 100 times. 2) Each time, we calculate the sampling mean vector, the covariance matrix and the coefficients of skewness, derived from the censored measurements. 3) We calculate the average, \( m_{sam}, S_{sam} \) and \( g_{sam} \) of the 100 samples mean vectors, covariance matrices and coefficients of skewness, respectively. 4) The mean vector, \( m^c \), the covariance matrix, \( S^c \) and the coefficients of skewness, \( g^c \), are calculated by Equations (9) and (11)–(15). As it can be seen in Equations (16)–(21), the proposed \( m^c, S^c \) and \( g^c \), are almost identical with the corresponding results from the samples,

\[
m_{sam} = (0.9999, 1.0000, 1.1495)^T
\]

\[
m^c = (1.0000, 1.0000, 1.1494)^T
\]

\[
S_{sam} = \begin{bmatrix} 2.0003 & 1.002 & 0.3985 \\ 1.0002 & 1.9998 & 0.7968 \\ 0.3985 & 0.7968 & 0.4003 \end{bmatrix}
\]

\[
S^c = \begin{bmatrix} 2.0000 & 1.0000 & 0.3984 \\ 1.0000 & 2.0000 & 0.7968 \\ 0.3984 & 0.7968 & 0.4003 \end{bmatrix}
\]

\[
g_{sam} = (0.0001, -0.0001, 0.2654)^T
\]

\[
g^c = (0.0000, 0.0000, 0.2657)^T.
\]

**3. Tobit Kalman filter with colored noise**

The state-space model with censored measurements and colored noises is defined as

\[
x_{t+1} = Ax_t + u_t,
\]

\[
u_t = Cu_{t-1} + w_{t,i},
\]

\[
y_t = \begin{cases} a, & y_t^c \leq a \\ y_t^c = Hx_t + v_t, & a < y_t^c < b \\ b, & y_t^c \geq b \end{cases}
\]
\[ v_t = g v_{t-1} + w_{2,t}, \tag{25} \]

where \( A \) and \( H \) are the transition and observation matrix, respectively, \( y_t, y_t^*, y_t, \) and \( x_t \in \mathbb{R}^n \) are the latent measurement, the censored measurement and the unknown state vector at time frame \( t \), respectively, while \( a \) and \( b \) are the censoring limits. \( w_{1,t}, w_{2,t} \) are white noises (hence, of zero mean) with covariance matrix \( Q \) and variance \( r^2 \), respectively, while \( u_t, v_t \) are colored noises generated by the associated AR(1) models driven by matrix \( C \) and scalar \( g \), respectively. To overcome the problem of the colored noises, the system given by Equations (22)–(25) is expressed as a system with white noise using the augmented approach. For that purpose let

\[
A_{\text{aug}} = \begin{bmatrix} A & I_n & 0 \\ 0_n & C & 0 \\ 0^T & 0^T & g \end{bmatrix}, \quad z_t = \begin{bmatrix} x_t \\ u_t \\ v_t \end{bmatrix},
\]

\[
H_{\text{aug}} = [H \ 0^T \ 1], \quad w_{\text{aug},t} = \begin{bmatrix} 0 \\ w_{1,t} \\ w_{2,t} \end{bmatrix} \sim N(0, Q_{\text{aug}}), \quad Q_{\text{aug}} = \begin{bmatrix} 0_n & 0 \\ 0_n & Q \\ 0^T & 0^T & r^2 \end{bmatrix},
\]

where \( 0_n \) and \( 0 \) denote the \( n \times n \) zero matrix and the \( n \times 1 \) zero vector, respectively, while \( I_n \) stands for the identity matrix \( n \times n \). Then, the state-space model (22)–(25) can be written in the form:

\[
z_{t+1} = A_{\text{aug}} z_t + w_{\text{aug},t+1}, \tag{26}\]

\[
y_t = \begin{cases} a, & y_t^* \leq a \\ y_t^* = H_{\text{aug}} z_t, & a < y_t^* < b \\ b, & y_t^* \geq b \end{cases} \tag{27}
\]

As it can be seen in Equation (27) the latent measurement \( y_t^* \) has not added noise as in case of Equation (24). The linear optimal estimates for the state-space model (26)–(27) have the same form (with the corresponding new matrices in the augmented model) as in the case of the state-space model with white noises, except that the variance of measurement noise in the augmented model equals 0.

In this article, as in Allik et al. (2016), Geng et al. (2017a) and Loumponias et al. (2018), the a posteriori estimation of the augmented state vector, \( \hat{z}_t \) (hence the estimation of state vector \( \hat{x}_t \)), is calculated as a linear combination of the a priori estimation of the augmented state vector, \( \hat{z}_{-t}^- \), and the censored measurement \( y_t \). Although these estimations are not optimal, it is proved that they minimize the trace of the state error covariance matrix (Anderson and Moore 2012). More specifically, the proposed method -as in standard KF- evolves in two stages, the predict and the update stage, respectively:

**Predict Stage**

\[
\hat{z}_{-t}^- = \mathbb{E}(z_t|y_{1:t-1}), \quad P_{-t}^- = \text{Cov}(z_t - \hat{z}_{-t}^-|y_{1:t-1})
\]

**Update Stage**

\[
K_t = \text{Cov}(z_t, y_t|y_{t-1})\text{Var}(y_t|y_{t-1})^{-1}, \quad \hat{z}_t = \hat{z}_{-t}^- + K_t(y_t - \mathbb{E}(y_t|y_{t-1})),
\]

\[
P_t = P_{-t}^- - K_t \text{Cov}(x_t, y_t|y_{k-1})^T,
\]

where \( P_{-t}^- \) and \( P_t \) are the covariance matrices of the a priori and a posteriori error
estimations, respectively. $\mathbb{E}(y_t | y_{t-1})$ and $\text{Var}(y_t | y_{t-1})$ are the mean and variance of the censored measurement $y_t$ given the censored measurements up to time $t-1$, while $\text{Cov}(z_t, y_t | y_{t-1})$ is the cross-covariance matrix of the augmented state and the censored measurement at time $t$.

The predict stage is the same as in the case of the standard KF, since the censored measurements are not used in this stage. Therefore the a priori estimations are given by

$$\hat{z}_t^- = A_{\text{aug}} \hat{z}_{t-1},$$

$$P_t^- = A_{\text{aug}} P_{t-1} A_{\text{aug}}^T + Q_{\text{aug}}.$$  \hspace{1cm} (30)

(31)

It is clear by Equations (26) and (27) that the joint distribution of $z_t$ and $y_t^c$ is Gaussian and more specifically $(z_t, y_t^c | t-1) \sim N(\mathbf{m}, \mathbf{S})$, where

$$\mathbf{S} = \begin{bmatrix} P_t^- & \text{Cov}(z_t, y_t^c | t-1) \\ \text{Cov}(z_t, y_t^c | t-1)^T & \text{Var}(y_t^c | t-1) \end{bmatrix}$$

and $\mathbf{m} = \begin{bmatrix} \hat{z}_t^- \\ H_{\text{aug}} \hat{z}_t^- \end{bmatrix}$. Furthermore, it can be proven that $\text{Cov}(z_t, y_t^c | t-1) = P_t^- H_{\text{aug}}^T$ and $\text{Var}(y_t^c | t-1) = H_{\text{aug}} P_t^T H_{\text{aug}}^T$. Next, the results of Proposition 1 are utilized to cope with the censored moments in the Update Stage. The censored moments $\mathbb{E}(y_t | t-1), \text{Var}(y_t | t-1)$ and $\text{Cov}(z_t, y_t | t-1)$ are calculated by Equations (11), (12) and (15), respectively. Summarizing, the proposed Update Stage for censored measurements is calculated as follows:

1. $\mathbb{E}(y_t | t-1)$: is calculated by Equation (11), by substituting $m_k = H_{\text{aug}} \hat{z}_t^-$ and $S_{k,k} = H_{\text{aug}} P_t^- H_{\text{aug}}^T$.
2. $\text{Var}(y_t | t-1)$: is calculated by Equation (12).
3. $\text{Cov}(z_t, y_t | t-1)$: is calculated by Equation (15) and is equal with,

$$\text{Cov}(z_t, y_t | t-1) = P_t^- H_{\text{aug}}^T \cdot P,$$  \hspace{1cm} (32)

where $P$ is the probability of $y_t^c$ to belong into the uncensored interval $(a, b)$ and is given by,

$$P = F_{c_1}(b - m_k) - F_{c_1}(a - m_k),$$

where $c_k \sim N(0, S_{k,k})$.

Then, the estimation of the augmented state vector $z_t$ and the corresponding covariance matrix of the estimation error are calculated by Equations (28) and (29), respectively. Hereinafter, we denote the filtering process that takes into account the corrected censored moments and does not consider colored noises, by TKF$^c$.

The proposed process described by Equations (28)–(31) can only be applied when the AR(1) models of colored noises (C and g) are assumed to be known. However, in real-life problems these parameters are unknown. In order to overcome this problem, the Likelihood Function (LF) of the censored measurements is utilized to estimate the parameters of the colored noises. The LF for the censored measurements $\{y_t\}_{t=1}$ given in Equation (27) with censoring limits $a$ and $b$ has the form (Loumponias et al. 2016a, 2016b),
\[
L(y) = \prod_{a < y < b} \frac{1}{(H_{aug} P_t^T H_{aug}^T)^{1/2}} \Phi\left( \frac{y_t - H_{aug} \hat{z}_{t}^-}{(H_{aug} P_t^T H_{aug}^T)^{1/2}} \right) \\
\times \prod_{y_t = a} \Phi\left( \frac{a - H_{aug} \hat{z}_{t}^-}{(H_{aug} P_t^T H_{aug}^T)^{1/2}} \right) \\
\times \prod_{y_t = b} \left( 1 - \Phi\left( \frac{b - H_{aug} \hat{z}_{t}^-}{(H_{aug} P_t^T H_{aug}^T)^{1/2}} \right) \right)
\]

(33)

The steps to estimate the parameters \( C \) and \( g \) are:

1. The proposed method, described by Equations (28)–(31), is initially used with arbitrary initial values for the (unknown) parameters \( C \) and \( g \).
2. The negative logarithm of the likelihood function, \(-\log(L)\) (33), is calculated as a function of parameters \( C \) and \( g \).
3. Finally, the local minimum of function \(-\log(L)\) with respect to parameters \( C \) and \( g \) is calculated. To that end, a numerical method for finding local minimum is utilized; in this article, the Maltab function \texttt{fminsearch} is used.

4. Experiments

In this section, two experiments-simulations are conducted to evaluate ColTKF in comparison to the standard augmented KF (AKF) and TKF\(^c\). More specifically, two oscillators (without damping) are considered, which have been utilized frequently in literature (Allik et al. 2016; Geng, Wang, and Cheng 2018 Geng et al. 2017a; Han et al. 2018). In the first experiment, the noises of the state-space model are colored, while, in the second, they are white.

Let the state space equations have the form of Equations (22)–(25) with \( H = [1 \ 0.5] \), \( A = \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix} \), where \( \omega = 0.005 \cdot 2\pi \). The disturbances \( w_{1,t} \) and \( w_{2,t} \) are assumed to be normally distributed, i.e., \( w_{1,t} \sim N(0, Q) \) and \( w_{2,t} \sim N(0, r^2) \), where \( Q = \begin{bmatrix} 0.01^2 & 0 \\ 0 & 0.01^2 \end{bmatrix} \) and \( r^2 = 1 \). In the first experiment, the colored noise parameters are set as \( C_1 = \begin{bmatrix} 0.9 \\ 0 \\ 0.9 \end{bmatrix} \) and \( g_1 = 0.99 \), while in the second experiment, they are equal with \( C_2 = 0_2 \) and \( g_2 = 0 \), i.e., colored noises are not considered. Moreover, in the first experiment, the censoring limits are equal with \( a_1 = -5 \) and \( b_1 = 5 \), while in the second experiment, they are equal with \( a_2 = -1 \) and \( b_2 = 1 \).

Let the initial augmented state vector be \( z_0 = (5, 0, 0, 0, 0)^T \) with covariance matrix \( P_0 = 10^{-3} \cdot I_5 \). Then, by the above parameters, censored (saturated) measurements, \( y_t \), are produced for \( t = 1, 2, ..., 500 \). In order the results of the experiments to be more valuable, the above process is repeated 100 times (Monte Carlo simulations) and in each Monte Carlo simulation the root mean square errors (RMSEs) of the three methods (i.e., AKF, TKF\(^c\) and ColTKF) are calculated. Moreover, only in the case of ColTKF, the parameters of AR(1) models, i.e., \( \{C_1, g_1\} \) and \( \{C_2, g_2\} \) are assumed to
The means of the filters’ RMSEs for the first experiment (for the 100 simulations) are presented in Table 1, where the means of RMSEs for both coordinates of the state vector $x_t$ are provided. It is clear by Table 1 that the AKF process has a poor performance, since it is not able to deal with censored measurements (see Figure 1). TKF$^c$ has a better performance than AKF, since it considers the censoring measurements, but, it can not cope with the colored noises. Finally, the proposed method (ColTKF) has the best performance overall, since it takes into account: a) the censoring limits in the measurements by calculating the accurate censored moments (Section 2) and b) the heteroskedasticity by estimating the parameters $\{c_1, g_1\}$ via LF (33). In Figure 2 the methods’ estimations for the hidden states vector $x_t$ (yellow plot) are illustrated. It is clear, that AKF and TKF$^c$ provide biased estimations due to the censoring and the colored noises, while ColTKF tackles both problems.

In the same way as in the first experiment, the means of the filters’ RMSEs for the second experiment are presented in Table 2. In the second experiment, the AKF process coincides with the corresponding one of the standard KF, since the system noises are not colored. For the same reason, TKF$^c$ process coincides with the proposed ColTKF. It is clear by Table 2 that the AKF process has a poor performance, since it is not able to deal with censored measurements (see Figure 3). TKF$^c$ has almost the same

| Filter   | Mean RMSE of $\hat{x}_1$ | Mean RMSE of $\hat{x}_2$ |
|----------|--------------------------|--------------------------|
| AKF      | 10.1292                  | 10.4497                  |
| TKF$^c$  | 8.7346                   | 9.0072                   |
| ColTKF   | 6.2879                   | 6.9183                   |

Figure 1. Censored measurements of the first experiment.
performance (not exactly the same) as CoITKF, since, in the proposed method the parameters \( \{ C_2, g_2 \} \) are estimated by LF (33) and they are not assumed to be known. In Figure 4 the methods’ estimations for the hidden states vector \( x_t \) (yellow plot) are
illustrated. As it can be seen, TKF^c and ColTKF have the same performance, while AKF provides biased estimates when the measurements belong into the censored region.

5. Conclusion

The aim of this article is to improve the TKF process by 1) calculating the exact censored moments and 2) by considering colored noises for the state-space model. To that end, the mgf of a censored normal distribution with two censoring limits was calculated. Then, the exact censored moments were calculated by utilizing the associated mgf. Next, in the proposed method, the augmented approach was used in order to deal with colored noises which are described by AR(1) models. Furthermore, LF of the censored measurements was provided in order to estimate the unknown parameters of the AR(1) models.

The proposed method, ColTKF, was evaluated against TKF^c and AKF in two different simulations-experiments. In the first experiment, the state-space model describes the motion of an oscillator, where system’s noises are assumed to be colored, while in the second experiments the noises are assumed to be white. In the proposed method, the AR(1) parameters were set to be unknown, thus, LF was utilized in advance to estimate them. It is worth to mention that in each Monte Carlo simulation, only the estimated parameters of \( \{C_1, g_1\} \) and \( \{C_2, g_2\} \) were utilized in ColTKF. In the first experiment, the results showed that ColTKF outperforms (minimum RMSE) both TKF^c and AKF. This result was expected, since AKF cannot handle the censored measurements and TKF^c cannot handle colored noises. In the second experiment, ColTKF and TKF^c appear to have almost the same performance, since the noises are white, while AKF provides biased estimations, when the measurements are censored. Therefore, the proposed method, ColTKF, in both experiments is able to detect whether colored noises are present in the system (first experiment) or not (second experiment) and then to
estimate the hidden state vectors \( \{x_t\}_{t=1}^T \). Moreover, as a step further it would be interesting to extend the proposed method in multidimensional censored measurements with correlated coordinates, to describe efficiently real-life problems with censored measurements, when the noises in the state-space model are colored.

**Acknowledgement**

I would like to thank Prof. George Tsaklidis, (Department of Mathematics, Aristotle University of Thessaloniki, Greece) for his useful remarks and indications in writing this article.

**References**

Akaike, H. 1998. Autoregressive model fitting for control. In *Selected papers of Hirotugu Akaike*, 153–70. New York: Springer.

Allik, B. 2014. The Tobit Kalman filter: An estimator for censored data. PhD. thesis, University of Delaware.

Allik, B., C. Miller, M. J. Piovoso, and R. Zurakowski. 2014. Estimation of saturated data using the Tobit Kalman filter. In *2014 American control conference*, 4151–4156. Portland, OR: IEEE.

Allik, B., C. Miller, M. J. Piovoso, and R. Zurakowski. 2016. The Tobit Kalman filter: An estimator for censored measurements. *IEEE Transactions on Control Systems Technology* 24 (1): 365–71. doi:10.1109/TCST.2015.2432155.

Anderson, B. D., and J. B. Moore. 2012. *Optimal filtering*. Englewood Cliffs, NJ: Courier Corporation.

Arismendi, J. C. 2013. Multivariate truncated moments. *Journal of Multivariate Analysis* 117: 41–75. doi:10.1016/j.jmva.2013.01.007.

Azzalini, A. 2013. *The skew-normal and related families*, vol. 3. Cambridge, UK: Cambridge University Press.

Brow, R. G., and P. Y. Hwang. 1992. *Introduction to random signals and applied Kalman filtering*, vol. 3. New York: Wiley.

Bryson, A., Jr, and L. Henrikson. 1968. Estimation using sampled data containing sequentially correlated noise. *Journal of Spacecraft and Rockets* 5 (6):662–5. doi:10.2514/3.29327.

Chen, G., and C. K. Chui. 1986. Design of near-optimal linear digital tracking filters with colored input. *Journal of Computational and Applied Mathematics* 15 (3):353–70. doi:10.1016/0377-0427(86)90226-8.

Djuric, P. M., J. H. Kotecha, J. Zhang, Y. Huang, T. Ghirmai, M. F. Bugallo, and J. Miguez. 2003. Particle filtering. *IEEE Signal Processing Magazine* 20 (5):19–38. doi:10.1109/MSP.2003.1236770.

Dong, D., B. W. Gould, and H. M. Kaiser. 2004. Food demand in Mexico: An application of the amemiya-tobin approach to the estimation of a censored food system. *American Journal of Agricultural Economics* 86 (4):1094–107. doi:10.1111/j.0002-9092.2004.00655.x.

Gazit, R. 1997. Digital tracking filters with high order correlated measurement noise. *IEEE Transactions on Aerospace and Electronic Systems* 33 (1):171–7. doi:10.1109/7.570736.

Geng, H., Z. Wang, and Y. Cheng. 2018. Distributed federated Tobit Kalman filter fusion over a packet-delaying network: A probabilistic perspective. *IEEE Transactions on Signal Processing* 66 (17):4477–89. doi:10.1109/TSP.2018.2853098.

Geng, H., Z. Wang, Y. Liang, Y. Cheng, and F. E. Alsaadi. 2017a. Tobit Kalman filter with fading measurements. *Signal Processing* 140:60–8. doi:10.1016/j.sigpro.2017.04.016.

Geng, H., Z. Wang, Y. Liang, Y. Cheng, and F. E. Alsaadi. 2017b. Tobit Kalman filter with time-correlated multiplicative sensor noises under redundant channel transmission. *IEEE Sensors Journal* 17 (24):8367–77. doi:10.1109/JSEN.2017.2766077.
Han, F., H. Dong, Z. Wang, G. Li, and F. E. Alsaadi. 2018. Improved Tobit Kalman filtering for systems with random parameters via conditional expectation. *Signal Processing* 147:35–45. doi: 10.1016/j.sigpro.2018.01.015.

Kuhlmann, H. 2003. Kalman-filtering with coloured measurement noise for deformation analysis. Paper presented at the Proceedings, 11th FIG Symposium on Deformation Measurements, Santorini, Greece.

Li, W., Y. Jia, and J. Du. 2017. Tobit Kalman filter with time-correlated multiplicative measurement noise. *IET Control Theory & Applications* 11 (1):122–28. doi:10.1049/iet-cta.2016.0624.

Ljung, L. 1979. Asymptotic behavior of the extended Kalman filter as a parameter estimator for linear systems. *IEEE Transactions on Automatic Control* 24 (1):36–50. doi:10.1109/TAC.1979.1101943.

Loumponias, K., A. Dimou, N. Vretos, and P. Daras. 2018. Adaptive Tobit Kalman-based tracking. Paper presented at the 2018 14th International Conference on Signal-Image Technology & Internet-Based Systems (SITIS), 70–76. IEEE. doi:10.1109/SITIS.2018.00021.

Loumponias, K., N. Vretos, P. Daras, and G. Tsaklidis. 2016a. Using Kalman filter and Tobit Kalman filter in order to improve the motion recorded by kinect sensor II. Paper presented at the Proceedings of the 29th Panhellenic Statistics Conference, vol. 1, 2.

Loumponias, K., N. Vretos, P. Daras, and G. Tsaklidis. 2016b. Using Tobit Kalman filtering in order to improve the motion recorded by microsoft kinect. Paper presented at the Proceedings of the International Workshop on Applied Probability IWAP 2016.

Popescu, D. C., and I. Zeljkovic. 1998. Kalman filtering of colored noise for speech enhancement. Paper presented at the Proceedings of the 1998 IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP’98 (Cat. No. 98CH36181), vol. 2, 997–1000. IEEE.

Wan, E. A., and R. Van Der Merwe. 2000. The unscented kalman filter for nonlinear estimation. Paper presented at the Proceedings of the IEEE 2000 Adaptive Systems for Signal Processing, Communications, and Control Symposium (Cat. No. 00EX373), 153–8. IEEE.

Wooldridge, J. M. 2002. *Econometric analysis of cross section and panel data*, 108. Cambridge, MA: MIT Press.