Convex effective potential of $O(N)$-symmetric $\phi^4$ theory for large $N$

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Abstract

We obtain effective potential of $O(N)$-symmetric $\phi^4$ theory for large $N$ starting with a finite lattice system and taking the thermodynamic limit with great care. In the thermodynamic limit, it is globally real-valued and convex in both the symmetric and the broken phases. In particular, it has a flat bottom in the broken phase. Taking the continuum limit, we discuss renormalization effects to the flat bottom and exhibit the effective potential of the continuum theory in three and four dimensions. On the other hand the effective potential is nonconvex in a finite lattice system. Our numerical study shows that the barrier height of the effective potential flattens as a linear size of the system becomes large. It decreases obeying power law and the exponent is about $-2$. The result is clearly understood from dominance of configurations with slowly-rotating field in one direction.

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1 Introduction

Effective potential is widely employed in various contexts of physics for studying phase transition. In the language of statistical physics, it is a free-energy density with some dynamical variable fixed. The fixed dynamical variable is no longer “dynamical”, and is called an order parameter. A value of the order parameter in the thermodynamic limit specifies a phase of a model. As shown in later section, the value of the order parameter is dynamically determined in a thermal equilibrium such that it gives a global minimum of the effective potential. Therefore we can recognize which phase is realized by looking at a form of the effective potential.

For example, let us consider a classical ferromagnetic spin system. The order parameter in this case is a component of the spin variable averaged over all lattice sites. According to the Landau-Ginzburg theory of phase transition, the effective potential is phenomenologically given as a polynomial of the order parameter: it has the unique minimum at the origin in the high-temperature (symmetric) phase. That is, a value of the order parameter is zero in this phase. As temperature is lowered, the effective potential continuously changes and it eventually comes to have form of a double-well. Namely the order parameter does not vanish any more and we recognize that the system is in the low-temperature (broken) phase.

Although the Landau-Ginzburg theory gives a good qualitative picture of phase transition, the form of the phenomenological effective potential is not globally correct because the effective potential of the spin system must be convex even in the low-temperature phase. More precisely, as proved by O’Raifeartaigh et al., the effective potential of a lattice scalar model with a nearest neighbor interaction plus on-site potential must be always convex even if the classical potential is not. It means that a form like a double-well is ruled out.

How does the effective potential look like in the low-temperature phase? Numerical simulations for $\phi^4$ theory on a lattice suggest that it has a flat bottom. Similar effective potential is analytically obtained by Langer in the spherical model with a help of the so-called sticking argument. The flat region reflects dominance of configuration with domain structures, or dominance of spatially slowly varying configuration.

A usual way of computing the effective potential of $\phi^4$ theory is a loop expansion from a uniform background, which is considered to be valid in a weak coupling region. It gives the classical potential in the leading order. However, when the classical potential is not convex, the loop expansion develops an imaginary part due to unstable modes of fluctuation. Systematic formulations that circumvent the unstable fluctuation have been proposed and one can obtain the (almost) convex effective potential by these methods. However, these results do not give us quantitative understanding because one cannot explicitly find any controllable expansion parameters. Necessity of non-perturbative and quantitative approach naturally leads us to investigating the case where the
number of fields, \(N\), is sufficiently large and to carrying out the \(1/N\) expansion.

Investigation of phase structure of the large-\(N\) \(\phi^4\) theory using the effective potential has been extensively carried out for more than two decades\[13, 14, 15, 9, 16, 17\]. In order to determine a value of the order parameter by the effective potential, we need to know where a global minimum is. However, most of the authors encountered trouble, i.e., an imaginary part appears in the broken phase and they fails to obtain the effective potential globally. In our knowledge, only exception is ref.\[15\], where a non-uniform external field is used as a “regularization”. Although the effective potential seems to be globally obtained in that paper, the ansatz used to solve a saddle-point equation is incorrect. This point will be discussed more detail in sect. 5.

Thus, we attempt to find a global form of the effective potential of the \(\phi^4\) model in the large-\(N\) limit, which is the main subject of this paper.

We solve the saddle-point equation with giving a special attention to asymptotic property of the solution when the volume of the system tends to infinity. If one solves it after taking the thermodynamic limit, there are no real solutions for some range of order parameters in the broken phase. This is the reason why the appearance of the imaginary part explained above. On the other hand, if one solves it in a finite system and then takes the thermodynamic limit, one can obtain a globally real-valued effective potential. The idea of going back to a finite system when one solves a saddle-point equation was used to reduce mathematical manipulation of the sticking argument\[18, 19\].

The rest of the paper is organized as follows. In sect. 2, we define \(O(N)\)-symmetric \(\phi^4\) model on a lattice and give a formulation of the \(1/N\) expansion of the effective potential in a finite lattice system. In sect. 3, we study asymptotic property of a solution of the saddle point equation derived in sect. 2 when the number of lattice sites of the system tends to infinity. Then we exhibit forms of the effective potential in the thermodynamic limit for various dimensions. Also, we numerically solve the saddle-point equation in a finite system and quantitatively study how effective potential in a finite system approaches to that in the thermodynamic limit. In sect. 4, we consider the continuum limit of the model and discuss the renormalization effects to the effective potential in three and four dimensions. Section 5 is devoted to summary and discussion.

Here we would like to comment on definition of effective potential. The conventional way to define the effective potential is to introduce an external source coupled to some dynamical variable. The expectation value of the dynamical variable, which is the order parameter in this case, and also the free-energy density, are a function of the external source. We can regard the free-energy density as a function of the order parameter by the Legendre transformation\[1\]. The free-energy density constructed in this way is called the effective potential.

Alternatively we can fix a dynamical variable by imposing a constraint on a measure of the partition function so that the dynamical variable takes an given value(see eq.(4)). The free-energy density derived from the partition function using this measure is also called the effective potential\[22\].
(or called the constraint effective potential[1]).

The relationship between the two is clarified in refs.[3, 4]: the former is the convex hull of the latter. O’Raifeartaigh et al. also show that the effective potential of the lattice \( \phi^4 \) theory derived from the second definition is convex. Therefore the two definitions give the same effective potential in the case of the lattice \( \phi^4 \) theory. In this paper we adopt the second definition.

2 The \( 1/N \) expansion

We study the \( O(N) \)-symmetric \( \phi^4 \) model on a \( D \)-dimensional hypercubic lattice. The lattice spacing is set to be unity. There are \( N \)-scalar fields \( \phi^a_x, a = 1, \ldots, N \) defined on each lattice site \( x \). We impose periodic boundary conditions: 
\[
\phi^a_x = \phi^a_{x + L e^\mu}, \quad \mu = 1, \ldots, D,
\]
where \( e^\mu \) is the unit vector pointing to the \( \mu \)-direction and \( L \) is an integer standing for the length of the side of the hypercubic lattice. The model is defined by the action
\[
S_0 = \sum_x \left\{ \frac{1}{2} \phi^a_x (-\Delta + m_0^2) \phi^a_x + \frac{g_0}{8N} (\phi^a_x \phi^a_x)^2 \right\}, \tag{1}
\]
where the summation over \( a \) is implicitly taken. The lattice Laplace operator \( \Delta \) acts on fields as
\[
\Delta \phi^a_x = \sum_{\mu = 1}^D \left( \phi^a_{x + e^\mu} + \phi^a_{x - e^\mu} - 2 \phi^a_x \right). \tag{2}
\]
Since we are interested in the broken phase, \( m_0^2 \) is set to be negative, while \( g_0 \) is taken to be positive because of the requirement that \( S_0 \) should be bounded from below.

We can replace \( S_0 \) with the following action by introducing an auxiliary field \( \chi_x \) as follows[13]:
\[
S = \sum_x \left\{ \frac{1}{2} \phi^a_x (-\Delta + \chi_x) \phi^a_x - \frac{N}{2 g_0} \chi_x^2 + \frac{N m_0^2}{g_0} \chi_x \right\}. \tag{3}
\]
In fact, the integration over \( \chi_x \) in the partition function is easily performed and the resulting action equals \( S_0 \) up to a constant. Note that the sign of the \( \chi_x^2 \) term is negative: \( \chi_x \) is integrated along the imaginary axis.

The effective potential \( U \) of the finite system is defined by[22, 13, 1]
\[
\exp \{-\Omega U(\Omega, \phi^2)\} \equiv \int \prod_{x,a} d\phi^a_x \delta(\phi^a_c - \sqrt{N} \phi^a) \prod_x d\chi_x \exp \{-S\}, \tag{4}
\]
where \( \phi^2 \equiv \phi^a \phi^a \) and \( \Omega \) is the total number of the lattice sites. We choose
\[
\phi^a_c \equiv \frac{1}{\Omega} \sum_x \phi^a_x, \quad a = 1, \ldots, N \tag{5}
\]
as order parameters. It is obvious that $U$ is a function of $\varphi^2$ due to the $O(N)$ symmetry.

The partition function $Z$ of the system is written in terms of $U(\Omega, \varphi^2)$ as

$$Z = N^{N/2} \int \prod_a d\varphi^a \exp\{-\Omega U(\Omega, \varphi^2)\}. \quad (6)$$

In the limit of $\Omega \to \infty$, the integration on the right-hand-side of eq.(6) is evaluated by a global minimum of $U$, i.e., the global minimum of $U(\infty, \varphi^2)$ determines a value of the order parameter realized in the thermodynamic limit. If there are multiple global minima, we need to choose one of them for looking at spontaneously symmetry breaking. A practical way of doing this is presented in sect. 3 (see also [1]). A physical meaning of this procedure is explained in the literature [24, 25, 26], for example.

The classical potential

$$\frac{1}{2} m_0^2 \varphi_0^a \varphi_0^a + \frac{g_0}{8N} (\varphi_0^a \varphi_0^a)^2 \quad (7)$$

takes the minimum on the $(N-1)$-dimensional sphere with the radius $\sqrt{-2N m_0^2 / g_0}$ in $\varphi$-space (the target space). We expect that the values of $N \varphi^2$ minimizing $U$ lies in a range $N \varphi^2 \leq O(N)$ as well as those of the classical potential, so that we shall search a minimum of $U$ in that range: $\varphi^2 \leq O(N^0)$.

The main purpose of this section is to derive a formulation of the $1/N$ expansion of $U(\Omega, \varphi^2)$ following the spirit of refs. [7, 23]. That is, we perform integration over $\varphi$ and obtain a theory described by $\chi$. The integration over $\chi$ is approximated by the saddle point method, which is applicable for large $N$.

In order to carry out this program, we use an integral representation of the $\delta$-function in eq.(4)

$$\prod_a \delta(\varphi^a - \sqrt{N} \varphi^a) = \int_{-i\infty}^{i\infty} \frac{d\eta}{2\pi i \Omega} \exp\{\eta^a (\sum_x \varphi^a_x - \Omega \sqrt{N} \varphi^a)\}. \quad (8)$$

We can readily integrate over $\varphi^a$ and $\eta^a$. The result is, up to a constant,

$$\exp\{-\Omega U(\Omega, \varphi^2)\} = \int \prod_x d\chi_x \exp\{-NS_1\},$$

$$S_1[\chi] = \frac{1}{2} \text{Tr} \ln(-\Delta + \tilde{\chi}) + \frac{1}{2} \ln f + \sum_x \left(-\frac{1}{2g_0} \chi^2_x + \frac{m_0^2}{g_0} \chi_x \right) + \frac{1}{2} \Omega^2 f^{-1} \varphi^2. \quad (9)$$

Here we defined the operator $\tilde{\chi}$

$$\tilde{\chi}|x\rangle = \chi_x |x\rangle, \quad (10)$$

and the functional $f$

$$f \equiv \sum_{x,y} \langle x| \left(\frac{1}{-\Delta + \tilde{\chi}}\right) |y\rangle. \quad (11)$$

The next step is to find the solution of the saddle-point equation

$$\frac{\delta S_1[\chi]}{\delta \chi_x} = 0. \quad (12)$$
We assume translationally invariant solution $\chi^*$. In this case, the above equation (12) is reduced to

$$
\varphi^2 + \frac{2m_0^2}{g_0} = -\frac{1}{\Omega} \sum_{p \neq 0} \frac{1}{-\hat{\Delta}(p) + \chi^*} + \frac{2}{g_0} \chi^*.
$$

(13)

The solution $\chi^*$ is regarded as a function of $\varphi^2$ and $\Omega$. We get the leading-order effective potential, $U_0(\Omega, \varphi^2)$, by inserting $\chi^*$ into $S_1[\chi]$:

$$
U_0(\Omega, \varphi^2) = \frac{N}{\Omega} S_1[\chi^*] = \frac{N}{2\Omega} \sum_{p \neq 0} \ln(-\hat{\Delta}(p) + \chi^*) - \frac{N}{2g_0} \chi^{*2} + \frac{Nm_0^2}{g_0} \chi^* + \frac{N}{2} \varphi^2 \chi^*.
$$

(14)

Further investigation to the leading order will be extensively carried out in later section.

In the following we formulate calculation of higher-order corrections. The $1/N$ expansion is the expansion around the saddle point. Writing

$$
\chi_x = \chi^* + \frac{1}{\sqrt{N}} \xi_x,
$$

(15)

then we expand $S_1$ in terms of $\xi_x$:

$$
NS_1[\chi] = NS_1[\chi^*] + \frac{N}{2} \sum_{x,y} \frac{\delta^2 S_1}{\delta \xi_x \delta \xi_y} [\chi^*] \xi_x \xi_y + \sum_{n \geq 3} \frac{N}{n!} \frac{\delta^n S_1}{\delta \xi_{x_1} \cdots \delta \xi_{x_n}} [\chi^*] \xi_{x_1} \cdots \xi_{x_n}.
$$

(16)

Note that a linear term in $\xi_x$ vanishes due to the saddle-point equation.

The explicit form of the quadratic term is computed using the following formulae:

$$
\text{Tr} \ln(-\Delta + \tilde{\chi}) = \sum_p \ln(-\hat{\Delta}(p) + \chi^*) + \frac{i}{\sqrt{N\Omega}} \sum_p \hat{G}_0(p) \sum_x \xi_x + \frac{1}{2N} \sum_{x,y} \xi_x \xi_y \Pi(x - y) + \cdots,
$$

$$
f = \frac{\Omega}{\chi^*} \{ 1 - \frac{i}{\sqrt{N\Omega} \chi^*} \sum_x \xi_x - \frac{1}{N\Omega \chi^*} \sum_{x,y} \xi_x \xi_y G_0(x - y) \} + \cdots.
$$

(17)

Here $\hat{\Delta}(p)$ is a Fourier component of the Laplace operator given by

$$
\hat{\Delta}(p) = 2(\sum_{\mu=1}^{D} \cos p_\mu - D),
$$

(18)

and

$$
G_0(x) \equiv \frac{1}{\Omega} \sum_p e^{ipx} \hat{G}_0(p), \quad \hat{G}_0(p) \equiv \frac{1}{-\hat{\Delta}(p) + \chi^*},
$$

$$
\Pi(x) \equiv \frac{1}{\Omega} \sum_p e^{ipx} \hat{\Pi}(p), \quad \hat{\Pi}(p) \equiv \frac{1}{\Omega} \sum_q \hat{G}_0(p - q) \hat{G}_0(q).
$$

(19)
The result is
\[
\frac{\delta^2 S_1}{\delta \xi_x \delta \xi_y} [\chi^*] = \frac{1}{\Omega} \sum_p \hat{D}^{-1}(p) e^{ip(x-y)},
\]
\[
\hat{D}^{-1}(p) = \frac{1}{2}(\hat{\Pi}(p) - \frac{1}{\Omega \chi^2} \delta_{p,0}) + \frac{1}{g_0} + (\varphi^2 - \frac{1}{\Omega \chi^*})(\hat{G}_0(p) - \frac{1}{\chi^*} \delta_{p,0}).
\]  
(20)

We can read the propagator of \(\xi_x\) from the quadratic term:
\[
\langle \xi_x \xi_y \rangle = \frac{1}{\Omega} \sum_p e^{ip(x-y)} D(p).
\]  
(21)

Integrating over \(\xi_x\) with ignoring the cubic and higher-order vertices in (16) gives the next-to-leading correction, which is of order \(N^0\). Higher-order corrections are calculated in a diagrammatic way. We see that the \(p\)-point vertex \((p \geq 3)\) is suppressed by the factor of \(N^{-p/2+1}\). Therefore a loop diagram that consists in \(p_1, \cdots, p_k\) point vertices contributes to the order of \(N^{-(p_1+\cdots+p_k)/2+k}\). The first nontrivial two-loop diagrams are generated by one 4-point vertex, or two 3-point vertices, which gives correction of order of \(N^{-1}\).

Finally we go back to the integration over \(\phi^a_x\) that leads to eq.(14) and comment on an allowed range of \(\chi^*\). The first term of \(U_0(\Omega, \varphi^2)\) in eq.(14) is derived from the integration over the oscillation modes of \(\phi^a_x\). The condition that the integration should converge is that
\[
\delta + \text{Re} \chi^* > 0,
\]  
(22)
where \(\delta\) is the smallest eigenvalue of \(-\Delta\) in the oscillating modes:
\[
\delta \equiv 2(1 - \cos\left(\frac{2\pi}{L}\right)).
\]  
(23)

3  Solution of the saddle point equation and a form of the effective potential

3.1  Effective potential in the thermodynamic limit

In this subsection we shall study the leading-order effective potential of the lattice \(\phi^4\) theory in the thermodynamic limit \(\Omega \to \infty\). In particular, we would like to clarify the form of the effective potential in the broken phase. We denote
\[
u_0(\Omega, \varphi^2) \equiv \frac{1}{N} U_0(\Omega, \varphi^2),
\]  
(24)
for convenience. Since \( u_0(\Omega, \varphi^2) \) is an \( O(N) \)-symmetric function, we can put \( \varphi^1 = t, \varphi^k = 0 \) \((2 \leq k \leq N)\) without loss of generality. From eqs. (13) and (14), we have

\[
\begin{align*}
\frac{d}{dt} u_0(\Omega, t^2) &= \chi^* t, \\
\frac{d^2}{dt^2} u_0(\Omega, t^2) &= \chi^* + \chi^{*'} t,
\end{align*}
\]

(25)
(26)

where \( ' \) means differentiation with respect to \( t \). We wish to know asymptotic behavior of \( \chi^* \) when \( \Omega \to \infty \). Let

\[
\begin{align*}
h(x) &\equiv -\frac{1}{\Omega} \sum_{p \neq 0} \frac{1}{-\hat{\Delta}(p) + x} + \frac{2}{g_0} x.
\end{align*}
\]

(27)

Namely, \( h(\chi^*) \) is the right-hand-side of the saddle-point equation (13). The solution \( \chi^* \) is read from the crossing point having the graph of \( y = h(x) \) and a horizontal line representing \( y = t^2 + 2m^2_0/g_0 \), as is shown in Fig. 1.

Since \( h(x) \) is strictly increasing and takes any real value for \( x \in (-\delta, \infty) \), there is a unique real solution satisfying (22) for arbitrary \( t \) and \( \Omega \). The sign of \( \chi^* \) depends on whether the horizontal line lies above or below \( h(0) \). If \( \chi^* \) is positive, the summation over \( p \) in \( h(\chi^*) \) becomes the integration in the thermodynamic limit:

\[
\begin{align*}
\lim_{\Omega \to \infty} h(\chi^*) &= - \int \frac{d^D p}{(2\pi)^D} \frac{1}{-\hat{\Delta}(p) + \chi^*} + \frac{2}{g_0} \chi^*,
\end{align*}
\]

(28)

where the momentum \( p_\mu \) \((1 \leq \mu \leq D)\) runs from \(-\pi\) to \(\pi\). The saddle-point equation (13) in this case was essentially studied in ref. [13]. On the other hand, if \( \chi^* \) is negative, we cannot employ eq. (28) any more and have to take the thermodynamic limit carefully. For this reason, we need to know asymptotic behavior of the graph of \( h(x) \) in the neighborhood of \( x = 0 \), which depends on the dimension of the system.
For $D \leq 2$, $h(0)$ tends to $-\infty$ as $\Omega$ becomes large. Hence, for arbitrary $\varphi^2$, $\chi^*$ is positive if $\Omega$ is sufficiently large. Further, it is shown that

$$\chi^{**} t \geq 0$$

(29)

because $h(x)$ is strictly increasing. Combining eqs. (25), (26) and (29) we conclude that $u_0(\infty, \varphi^2)$ is strictly convex and has the unique minimum at the origin. Therefore the order parameter vanishes in the thermodynamic limit. It is a typical feature of the effective potential in the symmetric phase. The result corresponds to the fact that a continuous symmetry is not broken when $D \leq 2$ in a sense that a system has a non-vanishing order parameter.

For $D \geq 3$, $h(0)$ remains finite. If $x > 0$ and sufficiently small, the integral in (28) is evaluated as follows [23]:

$$- \int \frac{d^D p}{(2\pi)^D} \frac{1}{-\Delta(p) + x} \sim \begin{cases} -c_3 + \frac{1}{4\pi^2} \sqrt{x} & (D = 3) \\ -c_4 - \frac{1}{16\pi^2} x \ln x & (D = 4) \\ -c_D + K_D x & (D > 4) \end{cases}$$

(30)

where

$$c_D \equiv \int \frac{d^D p}{(2\pi)^D} \frac{1}{-\Delta(p)}, \quad (D \geq 3)$$

(31)

and $K_D$ is a non-universal positive constant.

On the other hand, if $-\delta < x < 0$, the graph of $h(x)$ drags a tail as seen in Fig. 1. Since $\delta \to 0$ as $\Omega \to \infty$, the tail sticks along the $y$-axis, from $(0, h(0))$ towards $(0, -\infty)$ in the thermodynamic limit. This is the heart of the sticking argument [20, 18] used in the spherical model [21]. We emphasize that we could not have found the tail and have lost a real solution for $t^2 + 2m_0^2/g_0 < h(0)$ if we took the thermodynamic limit before solving the saddle-point equation [13, 14, 15, 16].

From the above consideration for the asymptotic behavior of $h(x)$, we conclude

$$\lim_{\Omega \to \infty} \chi^* \begin{cases} > 0 & (t^2 > \varphi_0^2) \\ = 0 & (t^2 \leq \varphi_0^2) \end{cases}$$

(32)

for $D \geq 3$. Here we defined the constant $\varphi_0^2$ as

$$\varphi_0^2 \equiv -\frac{2m_0^2}{g_0} - c_D.$$  

(33)

Note that $\varphi_0^2$ can be both positive and negative. The first term of the right-hand-side in eq. (33) is related to the radius of the sphere in the target space where the classical potential takes the minimum. The second term originates from statistical fluctuations. If $\varphi_0^2 < 0$, namely, if the statistical fluctuations dominate over the classical radius, the sphere “disappears” and the $O(N)$ symmetry will be restored.
In fact, if $\varphi_0^2 < 0$, then $\chi > 0$ for all $t$. Namely the effective potential in this case has the unique minimum at the origin and the symmetry is not spontaneously broken as in the lower dimensional case.

In contrast, if $\varphi_0^2 \geq 0$, $\chi$ vanishes when $t^2 \leq \varphi_0^2$. This indicates that $u_0(\infty, \varphi^2)$ is flat inside the $(N - 1)$-dimensional sphere with the radius $\varphi_0$ in $\varphi$-space according to eq.(25). This is an expected result in accord with numerical simulations\[2, 1] or the Maxwell construction\[4] when the symmetry is spontaneously broken. The reason why the flat region appears is that configurations with a domain structure contribute.\[5, 12].

We can quantitatively study the form of $u_0(\infty, \varphi^2)$ outside of the flat region near the boundary, where we can assume $(\varphi^2 - \varphi_0^2)$ is positive and sufficiently small. Since $0 < \chi^* < 1$ in this case and the formulae (30) are available. For $D = 3$, we have

$$ \chi^* = 16\pi^2 (\varphi^2 - \varphi_0^2)^2 + \cdots, $$

where $\cdots$ stands for terms higher than $(\varphi^2 - \varphi_0^2)^2$. This provides, ignoring a constant term,

$$ u_0(\infty, \varphi^2) = \frac{1}{2} \chi^* (\varphi^2 - \varphi_0^2) - \frac{1}{12\pi} \chi^{3/2} + \cdots, $$

$$ = \frac{8\pi^2}{3} (\varphi^2 - \varphi_0^2)^3 + \cdots. $$

Similarly, we have

$$ u_0(\infty, \varphi^2) = \begin{cases} 
4\pi^2 (\varphi^2 - \varphi_0^2)^2 (-\ln \{16\pi^2 (\varphi^2 - \varphi_0^2)\})^{-1} + \cdots, & D = 4 \\
\frac{1}{4} (K_D + \frac{2}{9})^{-1} (\varphi^2 - \varphi_0^2)^2 + \cdots, & D > 4. 
\end{cases} $$

Using $u_0(\infty, \varphi^2)$, we can compute the one-point function in a usual manner\[3, 12, 23]. First we introduce a uniform external field $\sqrt{N}J$ while $\Omega$ is kept finite. That gives the following change in the partition function\[3] :

$$ U_0(\Omega, \varphi^2) \to U_0(\Omega, \varphi^2) - NJ^a \varphi^a. $$

The one-point function is computed as

$$ \langle \frac{1}{\Omega} \sum_x \phi \rangle = \lim_{\Omega \to 0, \Omega \to \infty} \sqrt{N} \int Z[J] \prod_a d\varphi^a \exp \{-\Omega N (u_0(\Omega, \varphi^2) - J^a \varphi^a)\}, $$

where $Z[J]$ is the partition function in the presence of the external field $J$. Note that the limiting procedure does not commute. Since the external field explicitly breaks the $O(N)$ symmetry, $u_0(\infty, \varphi^2) - J^a \varphi^a$ takes the minimum at a unique point in $\varphi$-space, say, $\varphi^*$. The integration over $\varphi^a$ in (38) can be exactly evaluated at $\varphi^*$ in the thermodynamic limit. As $J \to 0$, $\varphi^*$ approaches to $\varphi_0 e$, where $e$ is the unit vector pointing to the same direction as the external field. Thus we get

$$ \langle \frac{1}{\Omega} \sum_x \phi \rangle = \sqrt{N} \varphi_0 e. $$
It shows that the symmetry is spontaneously broken.

To summarize, in the case of $D \geq 3$, the effective potential has a flat region and the order parameter takes the value of $\varphi_0 \mathbf{e}$ if and only if $\varphi_0^2 > 0$. Namely, the symmetry is spontaneously broken if and only if $\varphi_0^2 > 0$. The direction $\mathbf{e}$ is determined by a uniform external field.

![Figure 2](image_url)

Fig. 2. The effective potential of a finite system in $D = 4$. The classical potential (long-dashed line), $\Omega = 2^4$ (dashed line) and $\Omega = 8^4$ are presented. The coupling constants are chosen as $g_0 = 80$, $m_0^2 = -15$, which are identical with those of the Monte Carlo simulations in ref.[1] if we put $N = 1$.

### 3.2 Effective potential in a finite-size system

In this subsection, we show numerical results for finite systems in order to see how the effective potential approaches to the form clarified in the previous subsection, as $\Omega \to \infty$. We can numerically solve the saddle-point equation(13) for finite systems and obtain Fig. 2. A similar result is obtained in the context of renormalization-group analysis[27]. We wish to understand, from a quantitative point of view, how the barrier of the effective potential flattens as the system size becomes large. To this end, we define the barrier height $H$ as follows:

$$H \equiv u_0(0, \Omega) - \min u_0,$$

where we denote $\min u_0$ as a minimum value of the effective potential. It is noted that an extreme value of the effective potential satisfies the condition $\varphi = 0$ or $\chi^* = 0$ from eq. (25). When the system is in the broken phase, $\chi^*$ is negative at $\varphi = 0$ for all $\Omega$, so that $u_0$ has a local maximum at $\varphi = 0$. On the other hand, since $h(x)$ of eq. (27) is strictly increasing as discussed above, there exists a solution $\chi^* = 0$ for some $|\varphi|^2 > 0$, where $u_0$ has a local minimum. Thus, from eqs.(14) and
the barrier height $H$ is explicitly written as

$$H = \frac{1}{2\Omega} \sum_{p \neq 0} \{ \ln(-\Delta(p) + \chi^*(0)) - \ln(-\hat{\Delta}(p)) \} - \frac{1}{2g_0} \chi'^2(0) + \frac{m_0^2}{g_0} \chi^*(0), \quad (41)$$

where

$$\chi^*(0) \equiv \chi^*|_{\varphi=0}. \quad (42)$$

Let us first consider how $H$ depends on the coupling constants in a finite-size system. Since $-\delta < \chi^*(0) < 0$ in the broken phase, $\chi^*(0)$ goes to zero as $\Omega \to \infty$. Therefore we can neglect the $\chi'^2(0)$-term in eq. (41) and also, due to the same reason, the $\chi^*$-term in eq. (13). That means that $H$ depends only on the combination $m_0^2/g_0$. Neglecting these terms can be effectively carried out by taking the limit $g_0 \to \infty$ with $\beta \equiv -m_0^2/g_0$ kept finite. In this limit the $O(N)$ $\phi^4$ model becomes the $O(N)$ nonlinear $\sigma$-model, where a length of a field variable is fixed to $\sqrt{2N\beta}$.

We numerically compute the barrier height varying coupling constants with $\beta$ kept fixed and plot the ratio of these barrier heights to that of the $O(N)$ nonlinear $\sigma$-model as a function of $L$ in $D = 3$(Fig. 3) and $D = 4$(Fig. 4).

![Fig. 3. The ratio of the barrier height $H/H_0$ as a function of $L$ from $L = 10$ to 100 in $D = 3$. The coupling constants are $(g_0, m_0^2) = (40, -7.5)$(boxes), $(g_0, m_0^2) = (80, -15)$(crosses), and $(160, -30)$(triangles), where $\beta$ is kept at 0.1875. $H_0$ is the barrier height of the $O(N)$ nonlinear $\sigma$-model.](image-url)
The result shows that they have the same asymptotic behavior as that of the $O(N)$ nonlinear $\sigma$-model for sufficiently large $L$, as we expected. Hence we conclude that, when the system is in the broken phase, the length of the field at each lattice is $\sqrt{2N/\beta}$ in dominant configurations for sufficiently large $L$.

In order to clarify dominant configuration at $\varphi = 0$ more definitely, we treat $H$ as a function of $L$ and study its asymptotic behavior. When the parameters $\beta$ and $L$ are sufficiently large, contribution from energy to the effective potential will dominate over that from entropy in the broken phase. If the energy density (i.e., the value of the action per unit volume) of most dominant configurations at $\varphi = 0$ behaves as

$$\min u_0 + \text{const.} \times L^{-\gamma},$$

then $H$ will indicate asymptotic behavior as $\text{const.} \times L^{-\gamma}$.

The barrier heights as a function of $L$ in $D = 3$ and $D = 4$ are shown in Fig. 5. We find that for $L \geq 10$ each $H$ is fitted very well by the power function eq. (43). The fitting exponent is summarized in Table 1. The result is that the all exponents are very close to 2.
Fig. 5. The barrier height $H = u_0(0, \Omega) - \min u_0$ as a function of $L$ from $L = 2$ to 100 with $(g_0, m^2_0) = (80, -15)$ in $D = 3$(daggers), $(g_0, m^2_0) = (80, -150)$ in $D = 3$(boxes), $(g_0, m^2_0) = (80, -15)$ in $D = 4$(crosses), $(g_0, m^2_0) = (80, -150)$ in $D = 4$(triangles). Each line is a fitting power function.

| $D$ | $(g_0, m^2_0)$ | $\gamma$ |
|-----|----------------|---------|
| 3   | $80, -15$      | 1.888   |
| 3   | $80, -150$     | 1.980   |
| 4   | $80, -15$      | 1.953   |
| 4   | $80, -150$     | 1.986   |

Table 1: Fitting exponent $\gamma$ in the case of $(g_0, m^2_0) = (80, -15), (80, -150)$ in $D = 3$ and 4.
In order to understand the above asymptotic property, let us find a configuration satisfying the following conditions:

1. $\varphi = 0$,
2. the length of the field at each lattice is $\sqrt{2N\beta}$,
3. its energy density behaves as $\text{const.} \times L^{-2}$.

Let us observe a configuration where $\phi^a_x$ with length $\sqrt{2N\beta}$ slowly rotates for one direction and is uniform for all the other directions in the lattice. For example,

$$
\begin{align*}
\phi^1_x &= \sqrt{2N\beta} \cos \frac{2\pi x^1}{L}, \\
\phi^2_x &= \sqrt{2N\beta} \sin \frac{2\pi x^1}{L}, \\
\phi^i_x &= 0 \quad (3 \leq i \leq N).
\end{align*}
$$

(44)

We find from eq. (1) that the energy density of this configuration measured from that of the uniform configuration $\phi^1_x = \sqrt{2N\beta}$ is $N\beta\delta$, which asymptotically behaves as $L^{-2}$. Hence such configurations satisfy the above conditions.

It is noted that energy density of a configuration composed of some macroscopic uniform domains can be roughly estimated to be proportional to $L^{-1}$.

Therefore our numerical study strongly suggests that the barrier height of the effective potential in a finite-size system flattens as the system size becomes large because the slowly rotating configuration described above becomes dominant as the size becomes large.

4 The continuum limit and renormalization effects

So far, we have treated the effective potential of the lattice model. In this section, we take the continuum limit and study the renormalization effects. The global form of the effective potential of the continuum theory in dimension lower than three is obtained in refs. [7,15]. Here we restrict the dimension of the system to three or four. Our main interest is to see a fate of the flat region.

In order to take the continuum limit, we introduce the lattice spacing parameter $a$, which measures dimensions of physical quantities. We define

$$
\begin{align*}
V &\equiv a^D \Omega, \\
\varphi^b &\equiv a^{-\frac{D-2}{2}} \varphi^b, \\
\phi^b_R &\equiv a^{-\frac{D-2}{2}} \phi^b, \\
\chi^*_R &\equiv a^{-2} \chi^*,
\end{align*}
$$

(45)

$^3$Similar configuration was considered by Ringwald and Wetterich [11].
where we assign the canonical dimension to the scalar fields. The effective potential of the lattice theory with the finite volume $V$ and with the lattice parameter $a$, $u_R(a, V, \varphi_R^2)$, is defined by

$$u_R(a, V, \varphi_R^2) \equiv a^{-D}u_0(\Omega, \varphi^2).$$

(46)

It is explicitly written in terms of dimensionful quantities defined in eq.(45). Using eqs.(14) and (24), we get

$$u_R(a, V, \varphi_R^2) = \frac{1}{2} \varphi_R^2 \chi_R^* - \frac{1}{2g_0(a)a^{D-4}} \chi_R^2 + \frac{m_0^2(a)a^{-2}}{g_0(a)a^{D-4}} \chi_R^* + \frac{1}{2V} \sum_{\bar{p}\neq 0} \ln(-\hat{\Delta}(\bar{p}a)a^{-2} + \chi_R^*).$$

(47)

Here $\chi_R^*$ is the solution of the saddle-point equation of the theory with the lattice spacing $a$, which is derived from eq.(13):

$$\varphi_R^2 + \frac{2m_0^2(a)a^{-2}}{g_0(a)a^{D-4}} = -\frac{1}{V} \sum_{\bar{p}\neq 0} \frac{1}{-\hat{\Delta}(\bar{p}a)a^{-2} + \chi_R^*} + \frac{2}{g_0(a)a^{D-4}} \chi_R^*.$$  

(48)

where the momentum $\bar{p}_\mu$ is quantized by $2\pi V^{-1/D}$ and runs from $-\pi/a$ to $\pi/a$. We adopt a renormalization prescription to keep $u_R(a, V, \varphi_R^2)$ finite in the continuum limit $a \to 0$. This determines lattice-spacing dependence of the bare parameters, $g_0(a)$ and $m_0^2(a)$.

We shall first consider the case of $D = 3$. The bare parameters are chosen in such a way that the following equations hold:\[13:\]

$$g_R = \frac{g_0(a)a^{-1}},$$
$$\frac{2m_R^2}{g_R} = \frac{2m_0^2(a)a^{-2}}{g_0(a)a^{-1}} + \frac{1}{V} \sum_{\bar{p}\neq 0} \frac{1}{-\hat{\Delta}(\bar{p}a)a^{-2} + \chi_R^*},$$

(49)

where we introduced the renormalized mass and the coupling constant, $m_R^2$ and $g_R$. They are directly related to physical observables of the continuum theory. The continuum limit $a \to 0$ is taken keeping $\varphi_R^b, m_R^2, g_R$, and $V$ finite. In this limit, we can make the following substitution in (47) and (48)

$$-\hat{\Delta}(\bar{p}a)a^{-2} + \chi_R^* \to \bar{p}^2 + \chi_R^*,$$

(50)

where $\bar{p}^2 \equiv \sum_\mu \bar{p}_\mu \bar{p}_\mu$. The continuum effective potential is written in terms of the renormalized coupling constants

$$u_R(0, V, \varphi_R^2) = \frac{1}{2} \varphi_R^2 \chi_R^* - \frac{1}{2g_R} \chi_R^2 + \frac{m_R^2}{g_R} \chi_R^* + \frac{1}{2V} \sum_{\bar{p}\neq 0} \ln(\bar{p}^2 + \chi_R^*) - \frac{1}{\bar{p}^2} \chi_R^*,$$

(51)

where $\chi_R^*$ satisfies

$$\varphi_R^2 + \frac{2m_R^2}{g_R} = -\frac{1}{V} \sum_{\bar{p}\neq 0} \left(\frac{1}{\bar{p}^2 + \chi_R^*} - \frac{1}{\bar{p}^2}\right) + \frac{2}{g_R} \chi_R^*.$$  

(52)
Our aim is to clarify the global form of the effective potential in the infinite-volume limit. Behavior of the solution in the limit $V \to \infty$ is the same as that in the lattice model. Namely, if $m_R^2 > 0$,
\[
\lim_{V \to \infty} \chi_R^* > 0,
\]
and if $m_R^2 < 0$,
\[
\lim_{V \to \infty} \chi_R^* = \begin{cases} 
> 0 & (\varphi_R^2 > -2m_R^2/g_R) \\
= 0 & (\varphi_R^2 < -2m_R^2/g_R)
\end{cases}.
\]
We conclude from this behavior that $u_R(0, \infty, \varphi_R^2)$ is flat inside the sphere with the radius $\sqrt{-2m_R^2/g_R}$ when the symmetry is broken ($m_R^2 < 0$), as is the case of the lattice model. When $\chi_R^* > 0$, the summation over $\bar{p}$ in (52) is replaced by the integration and the result is
\[
\varphi_R^2 + \frac{2m_R^2}{g_R} = \frac{1}{4\pi} \sqrt{\chi_R^*} + \frac{2}{g_R} \chi_R^*.
\]
We can explicitly solve the equation. Inserting the solution into (51), we obtain $u_R(0, \infty, \varphi_R^2)$ outside the flat region. We depict the result in Fig. 6.

It may possible to take other limiting procedures for obtaining $u_R(0, \infty, \varphi_R^2)$. For example, one may first take the limit $\Omega \to \infty$ with fixed $a > 0$ and next take the limit $a \to 0$. It corresponds to taking the infinite-volume limit before taking the continuum limit. We can obtain the same result in this limiting procedure if we solve the saddle-point equation before taking the limit $\Omega \to \infty$.

Let us turn to the case of $D = 4$. We introduce the renormalized coupling constants by the following equations[13]
\[
\frac{2}{g_R} = \frac{2}{g_0(a)} + \frac{1}{V} \sum_{\bar{p} \neq 0} \frac{1}{(-\Delta(\bar{p}a)a^{-2} + M^2)(-\Delta(\bar{p}a)a^{-2})},
\]
where $M^2$ is an arbitrary positive constant. The saddle-point equation can be written in the finite form as

$$
\frac{2m_R^2}{g_R} = \frac{2m_R^2(a)}{g_0(a)} + \frac{1}{V} \sum_{\bar{p} \neq 0} \frac{1}{-\Delta(\bar{p}a)a^{-2}},
$$

(56)

Since the summation over $\bar{p}$ in the first equation in eq.(56) diverges as $a \to 0$, $g_R$ must be chosen to zero as long as we start from a positive $g_0(a)$\cite{13,16}. This result corresponds to the conjecture that the $\phi^4$ theory becomes trivial in the continuum limit. The saddle-point equation becomes

$$
m_R^2 = \chi^*_R.
$$

(57)

Namely, the theory reduces to the Gaussian model. The effective potential in the continuum limit is

$$
u_R(0,V,\phi_R^2) = \frac{1}{2}m_R^2\phi_R^2,
$$

(58)

up to an irrelevant additive constant. We must choose $m_R^2$ to be non-negative for well-defined continuum field theory.

### 5 Summary and discussion

In this paper, we obtained the effective potential of the lattice $\phi^4$ theory in the large-$N$ limit. In the thermodynamic limit, we found out that it is globally real-valued and convex in both the symmetric and the broken phases even if classical potential is not convex. In particular, the effective potential has a flat region in the broken phase, which is consistent with numerical simulations and with the Maxwell construction.

The effective potential in a finite system has a potential barrier at the origin although it is much lower than that of the classical potential. The height of the barrier is related to energy of defects of non-uniform configurations, and asymptotically flattens as $L^{-2}$, where $L$ is the linear size of the system. The result concerning with a finite lattice system can be applied to checking the validity of an $1/N$ expansion\cite{29}. The effective potential with sufficiently flat maximum as shown in Fig. 2 may shed light on global structure of inflationary universe\cite{30}.

We also studied the continuum limit of this model and discussed effects of the renormalization to the flat region in the case of $D = 3$ and $D = 4$. In the case of $D = 3$, the effective potential in the continuum limit has the flat region as in the case of the lattice model. On the other hand, in the case of $D = 4$, the flat region disappears because the $\phi^4$ coupling is renormalized to zero.
Here we discuss a role of the effective potential in continuum quantum field theory. After taking the continuum limit, it is believed that a minimum of the effective potential corresponds to the ground state (vacuum) in quantum field theory. If there are multiple points giving the minimum, we need to choose one point. A standard way of doing this is to turn on a uniform external field as we demonstrated in sect. 3. The vacuum selected in this way is homogeneous and corresponds to a point on the boundary of the flat region. Since it is the vacuum that is used in the previous literature, we can rederive the one-point function obtained in those works. However, physical observables concerned with higher-point correlation functions such as a mass matrix (or a susceptibility tensor) may be changed if we take the flat region into account. We hope to report on this point more detail in a subsequent publication.

In addition to reproducing the previous results, the effective potential obtained here suggests possibility of a non-uniform ground state such as a rotating configuration with a long-wave length in eq.(44) or as kink ground states in the quantum ferromagnetic XXZ chain. In order to study such a non-homogeneous vacuum, it is expected that a non-uniform external field will be needed, which will pick out a point inside the flat region.

Related with this topic, we comment on the $1/N$ expansion in the presence of a non-uniform external field $J_x$. In this case, the action of the system changes to $S - \sqrt{N} \sum \phi_x^a$. The strategy adopted in sect. 2 leads us to the following saddle-point equation

$$\frac{1}{2} \sum_{x,y} \left( \{ J^a_y + A^{-1} (\Omega \varphi^a - B^a) \} \{ J^a_x + A^{-1} (\Omega \varphi^a - B^a) \} - A^{-1} \right) G_{xz} G_{zy} + \frac{1}{2} G_{zz} - \frac{1}{g_0} \chi + \frac{m_0^2}{g_0} = 0,$$

(60)

where

$$G_{xy} \equiv \langle x | \frac{1}{-\Delta + \chi} | y \rangle, \quad A \equiv \sum_{x,y} G_{xy}, \quad B^a \equiv \sum_{x,y} G_{xy} J^a_y.$$

(61)

Although we need to solve this equation for carrying out the $1/N$ expansion, it is difficult because we cannot employ the ansatz of a uniform solution due to the first term. In fact, using the ansatz, we find that the constant solution $\chi^*$ must satisfy

$$\frac{1}{\Omega} \sum_{p \neq 0} \hat{G}_0(p) - \frac{2}{g} \chi^* + \frac{2m_0^2}{g_0} + \left( \frac{1}{\Omega} \sum_p \hat{J}^a(p) \hat{G}_0(p) e^{ipz} \right)^2 = 0,$$

(62)

for an arbitrary site $z$. Here $\hat{J}^a(p), p \neq 0$ are identical with Fourier components of $J_x$ and $\hat{J}^a(0) \equiv \Omega \chi^* \varphi^a$.

(63)

However it is obviously impossible if an oscillating mode of $J_x$ is present.

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Footnote 4: In ref., where a non-uniform external field is employed as a “regularization”, the first term is oversimplified and it misleads to a uniform solution.
Moreover, a perturbative expansion in the external source fails even if the magnitude of it is infinitesimal because the expansion induces the infrared divergence in the broken phase.

We feel that a constant mode of $\phi_\infty$ is not appropriate for studying a non-uniform ground state. The order parameters cannot resolve whether an inner point of the flat region corresponds to an inhomogeneous states or a superposition of pure states selected by a uniform external source. Some oscillating modes will be chosen as order parameters.

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