Finite groups with generalized Ore supplement conditions for primary subgroups

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Abstract
We consider some applications of the theory of generalized Ore supplement conditions in the study of finite groups.

1 Introduction
Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $p$ is always supposed to be a prime. A subgroup $H$ of $G$ is said to be $S$-quasinormal in $G$ if $H$ permutes with each Sylow subgroup $P$ of $G$, that is, $HP = PH$. We use $\mathfrak{U}$ to denote the class of all supersoluble groups.

Let $\mathfrak{F}$ be a class of groups. A chief factor $H/K$ of $G$ is said to be $\mathfrak{F}$-central in $G$ if $(H/K) \times (G/C_G(H/K)) \in \mathfrak{F}$. The product of all normal subgroups $N$ of $G$ such that every chief factor of $G$ below $N$ is $\mathfrak{F}$-central in $G$ denoted by $Z_{\mathfrak{F}}(G)$ and called the $\mathfrak{F}$-hypercentre of $G$. By the Barnes-Kegel theorem [1, IV, 1.5], for any group $G \in \mathfrak{F}$ we have $Z_{\mathfrak{F}}(G) = G$ provided $\mathfrak{F}$ is a formation.

If $G = HB$, then $B$ is said to be a supplement of $H$ in $G$. Since $HG = G$, it makes sense to consider only the supplements $B$ with some restrictions on $B$. For example, we often deal with

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the situation when for a supplement $B$ of $H$ we have $H \cap B = 1$. In this case, $B$ is said to be a complement of $H$ in $G$ and $H$ is said to be complemented in $G$; if, in addition, $B$ is also normal in $G$, then $B$ is said to be a normal complement of $H$ in $G$. Note that if $H$ either is normal in $G$ or has a normal complement in $G$, then, clearly, $H$ satisfies the following: there are normal subgroups $T$ and $S$ of $G$ such that $G = HT$, $S \leq H$ and $H \cap T \leq S$ (see Ore [2]); in this case we say $H$ satisfies the Ore supplement condition in $G$. It is clear that $H$ satisfies the Ore supplement condition if and only if $H/H_G$ has a normal complement in $G/H_G$. In the paper [3], the subgroups which satisfy the Ore supplement condition were called $c$-normal.

It was discovered that many important for applications classes of groups (for example, the classes of all soluble, supersoluble, nilpotent, $p$-nilpotent, metanilpotent, dispersive in the sense of Ore [2] groups (i.e., groups having a Sylow tower)) may be characterized in the terms of the Ore supplement condition or in the terms of some generalized Ore supplement conditions. It was the main motivation for introducing, studying and applying the generalized Ore supplement conditions of various type. But, in fact, all recent results in this line of researches are based on the ideas of the papers [4–6], in which the authors analyze three fundamentally different generalizations of the Ore supplement condition. A subgroup $H$ of $G$ is called: $c$-supplemented in $G$ [4] provided $H/H_G$ is complemented in $G/H_G$; $\mathfrak{F}$-supplemented [5] in $G$ provided there is a supplement $T/H_G$ of $H/H_G$ in $G/H_G$ such that $(H/H_G) \cap (T/H_G) \leq Z_{\mathfrak{F}}(G/H_G)$; weakly $S$-permutable in $G$ [6] provided there is a subnormal subgroup $T$ of $G$ such that $HT = G$ and $H \cap T \leq S \leq H$ for some $S$-quasinormal subgroup $S$ of $G$. Finally, also we often meet the situation when a subgroup $H$ has a supplement $T$ in $G$ such that $T \in \mathfrak{F}$. In spite of the four supplement conditions are quite different, there are a lot of similar results, in which we meet one of these ones.

It is known for example that a soluble group $G$ is supersoluble provided $G$ has a normal subgroup $E$ with supersoluble quotient $G/E$ such that for every maximal subgroup $H$ of every Sylow subgroup of $F(E)$ at least one of the following holds:

(I) $H$ is a $\text{CAP}$-subgroup of $G$ [7], that is, $H$ either covers or avoids each chief factor of $G$ (see [11, p. 37]);

(II) $H$ is complemented [8] or, at least, $c$-complemented in $G$ [4];

(III) $H$ has a supersoluble supplement in $G$ [9];

(IV) $H$ is $\text{UL}$-supplemented in $G$ [5];

(V) $H$ is weakly $S$-permutable in $G$ [6];

(VI) $H$ is a modular element (in the sense of Kurosh [10, p. 43]) of the lattice of all subgroups of $G$ [11].

The similarity of these results, as well as the similarity of many other results of this kind, makes natural to ask:

**Question A.** Is there a condition which generalizes all these conditions on the maximal subgroups
of the Sylow subgroups, and under which $G$ is still supersoluble?

In fact, the solution of this problem is based on the concept of the $\mathfrak{S}$-hypercentre and on the idea of the subgroup functor.

**Definition 1.1.** Let $K \leq H$ be subgroups of $G$. Then we say that the pair $(K, H)$ satisfies the $\mathfrak{S}$-supplement condition in $G$ if $G$ has a subgroup $T$ such that $HT = G$ and $H \cap T \leq KZ_\mathfrak{S}(T)$.

Recall that a subgroup functor is a function $\tau$ which assigns to each group $G$ a set $\tau(G)$ subgroups (perhaps consisting of a single element) of $G$ satisfying $1 \in \tau(G)$ and $\theta(\tau(G)) = \tau(\theta(G))$ for any isomorphism $\theta : G \to G^*$. If $H \in \tau(G)$, then we say that $H$ is a $\tau$-subgroup of the group $G$.

For our goal, we need the following realization of Definition 1.1.

**Definition 1.2.** Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$, where $H$ is a subgroup of $G$. Let $\tau$ be a subgroup functor. Then we say that $H$ is $\mathfrak{S}_{\tau}$-supplemented in $G$ if for some $\tau$-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$ the pair $(\bar{S}, \bar{H})$ satisfies the $\mathfrak{S}_{\tau}$-supplement condition in $\bar{G}$.

We show that this concept gives the positive answer to the first part of Question A. First note that if $\tau(G)$ is the set of $\text{CAP}$-subgroups of $G$ and $H \in \tau(G)$, then $H/H_G$ is a $\text{CAP}$-subgroup of $G/H_G$ (see Example 1.4 below), so the triple $(\bar{H}, \bar{S}, \bar{T})$, where $\bar{S} = \bar{H}$ and $\bar{T} = \bar{G}$, satisfies Definition 1.2. Clearly, a subgroup $H$ is $c$-supplemented in $G$ if and only if it is $\mathfrak{S}_{\tau}$-supplemented in $G$, where $\mathfrak{S} = (1)$ is the class of all identity groups and $\tau(G)$ is the set of all normal subgroups of $G$. If $H$ has a supersoluble supplement $T$ in $G$, then $(H/H_G) \cap (H_GT/H_G) = (H \cap T)H_G/H_G \leq TH_G/H_G \simeq T/H_G \cap T$, so $H$ is $\mathfrak{S}_{\tau}$-supplemented in $G$ for any subgroup functor $\tau$. If $H$ is $\mathfrak{S}_{\tau}$-supplemented in $G$, then similarly we get that $H$ is $\mathfrak{S}_{\tau}$-supplemented in $G$ for any subgroup functor $\tau$. Finally, if $H$ is weakly $S$-permutable in $G$, and $T$ is a subnormal subgroup of $G$ such that $HT = G$ and $H \cap T \leq S \leq H$ for some $S$-quasinormal subgroup $S$ of $G$, then $(H/H_G) \cap (TH_G/H_G) = (H \cap T)H_G/H_G \leq SH_G/H_G$, where $SH_G/H_G$ is a $S$-quasinormal subgroup of $G/H_G$ (see Example 1.6 below). Hence $H$ is $\mathfrak{S}_{\tau}$-supplemented in $G$ for any formation $\mathfrak{S}$ and for the subgroup functor $\tau$, which assigns to each group $G$ the set $\tau(G)$ of all $S$-quasinormal subgroup of $G$.

Our next goal is to give the positive answer to the second part of Question A.

But first, we define some subgroup functors which will be used in applications of the results.

**Definition 1.3.** Let $\tau$ be a subgroup functor. Then we say that $\tau$ is:

1. **Inductive** provided $HN/N \in \tau(G/N)$ whenever $H \in \tau(G)$ and $N \leq G$.
2. **Hereditary** provided $\tau$ is inductive and $H \in \tau(E)$ whenever $H \leq E \leq G$ and $H \in \tau(G)$.
3. **$\Phi$-regular** (respectively **$\Phi$-quasiregular**) provided for any primitive group $G$, whenever $H \in \tau(G)$ is a $p$-group and $N$ is a minimal normal (minimal normal abelian, respectively) subgroup of $G$, then $|G : N_G(H \cap N)|$ is a power of $p$.
4. **Regular** or a **Li-subgroup functor** $^{[12]}$ provided for any group $G$, whenever $H \in \tau(G)$ is a $p$-group and $N$ is a minimal normal subgroup of $G$, then $|G : N_G(H \cap N)|$ is a power of $p$. 

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(5) Quasiregular provided for any group $G$, whenever $H \in \tau(G)$ is a $p$-group and $N$ is an abelian minimal normal subgroup of $G$, then $|G : N_G(H \cap N)|$ is a power of $p$.

Example 1.4. For any group $G$, let $\tau(G)$ be the set of all CAP-subgroups of $G$. Then $\tau$ is regular inductive by Lemma 1 in [7].

Example 1.5. A subgroup $H$ of $G$ is called completely $c$-permutabe [13] provided for any two subgroups $A \leq E$ of $G$, where $H \leq E$, there is an element $x \in E$ such that $HA^x = A^xH$. Let $\tau(G)$ be the set of all completely $c$-permutabe subgroups of $G$. Then in view of [13] Lemma 2.1(3) and Corollary 2.2(1)], the functor $\tau$ is hereditary inductive. Now let $H$ be a completely $c$-permutabe $p$-subgroup of $G$ and $N$ any abelian minimal normal subgroup of $G$. For any prime $q \neq p$ dividing $|G|$, there is a Sylow $q$-subgroup $Q$ of $G$ such that $HQ = QH$. Then $H \cap N \neq 1$, $H \cap N = HQ \cap N$ and so $q$ does not divide $|G : N_G(H \cap N)|$. Hence $\tau$ is quasiregular.

Example 1.6. Let $\tau(G)$ be the set of all normal or of all $S$-quasinormal subgroups of $G$, for any group $G$. Then by [14] 1.2.7 and 1.2.8), the functor $\tau$ is hereditary inductive and regular (see [14] 1.2.14 and Example 1.4).

Example 1.7. We say, by analogy with CAP-subgroups, that a subgroup $H$ of $G$ is a CAP-group of $G$ if for every subgroups $K \leq L \leq G$, where $K$ is a maximal subgroup of $L$, $H$ either covers the pair $(K, L)$ (that is, $HK = HL$) or avoids this one (that is, $H \cap K = H \cap L$). Let $\tau(G)$ be the set of all CAP-subgroups of $G$ for any group $G$. Then $\tau$ is hereditary inductive by [15] Lemma 2.3]. Now let $H$ be a $p$-subgroup of a primitive group $G$ which is a CAP-subgroup of $G$. Then $H$ is subnormal in $G$ by [15] (Lemma 2.5)]. Let $N$ be a minimal normal subgroup of $G$. Suppose that $L = H \cap N \neq 1$. Then $N$ is a $p$-group, so $NH$ is a subnormal $p$-subgroup of $G$. Now let $M$ be a maximal subgroup of $G$ such that $M_G = 1$. Then $G = N \rtimes M$ and $H$ either covers or avoids the pair $(M, G)$. But since $L = H \cap N \neq 1$, $H \notin M$ and so $G = HM$. On the other hand, $NH \cap M = 1$ by [16] Lemma 7.3.16] since $M_G = 1$. Therefore $NH = N = H$. This shows that $\tau$ is $\Phi$-regular. It is not difficult to find a example which show that $\tau$ is not quasiregular.

Example 1.8. Recall that $H$ is said to be $S$-quasinormally (respectively subnormally) embedded [17] in $G$ if every Sylow subgroup of $H$ is also a Sylow subgroup of some $S$-quasinormal (respectively subnormal) subgroup of $G$. Note that in view of Kegel’s result [18], every $S$-quasinormal subgroup is subnormal, so every $S$-quasinormally embedded subgroup is also subnormally embedded. If $\tau(G)$ is the set of all $S$-quasinormally embedded subgroups of $G$ for any group $G$, then $\tau$ is a hereditary inductive by [17] Lemma 1] and $\tau$ is quasiregular (see Example 1.5 and [14] 1.2.19]). It is clear that this functor is not regular since every Sylow subgroup is $S$-quasinormally embebedded.

Example 1.9. Let $\tau(G)$ be the set of all modular subgroups of $G$ for any group $G$. Then $\tau$ is hereditary by [10] p. 201]. From Theorem 5.2.5 in [10] it easily follows that $\tau$ is regular.

Example 1.10. A subgroup $H$ of $G$ is called SS-quasinormal in $G$ [19] provided there is a subgroup $B$ of $G$ such that $HB = G$ and $H$ permutes with all Sylow subgroups of $B$. If $\tau(G)$ is the set of all $S$-quasinormal subgroups of $G$ for any group $G$, then the functor $\tau$ is hereditary inductive
by [19, Lemma 2.1] and it is also regular by [20, Lemma 7.1(6)].

In what follows, $\tau$ is always supposed to be an inductive subgroup functor.

Now we can state our first result.

**Theorem 1.11.** A soluble group $G$ is supersoluble if and only if $G$ has a normal subgroup $E$ with supersoluble quotient $G/E$ such that every maximal subgroup of every Sylow subgroup of $F(E)$ is $\mathfrak{U}_\tau$-supplemented in $G$ for some $\Phi$-regular subgroup functor $\tau$.

In view of the remarks after Definition 1.2, Theorem 1.11 gives the positive answer to the second part of Question A.

If $1 \in \mathfrak{F}$, then we write $G^\mathfrak{F}$ to denote the intersection of all normal subgroups $N$ of $G$ with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a formation if either $\mathfrak{F} = \emptyset$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^\mathfrak{F}$ belongs to $\mathfrak{F}$ for any group $G$. The formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$.

In fact, Theorem 1.11 is a special case of the next our result.

**Theorem A.** Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $X \leq E$ normal subgroups of $G$ such that $G/E \in \mathfrak{F}$. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of $X$ is $\mathfrak{U}_\tau$-supplemented in $G$ for some $\Phi$-regular hereditary or regular subgroup functor $\tau$ such that every $\tau$-subgroup of $G$ contained in $X$ is subnormally embedded in $G$. If $X = E$ or $X = F^*(E)$, then $G \in \mathfrak{F}$. Moreover, in the case when $\tau$ is regular, then $E \leq Z_\mathfrak{U}(G)$.

The following theorem is an analogue of the previous one. But the methods of their proofs are quite different (see Sections 3 and 4).

**Theorem B.** Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $X \leq E$ a normal subgroup of $G$ such that $G/E \in \mathfrak{F}$. Suppose that for every non-cyclic Sylow subgroup $P$ of $X$ every cyclic subgroup of $P$ of prime order or order 4 (if $P$ is a non-abelian group) is $\mathfrak{U}_\tau$-supplemented in $G$. Suppose that at least one of the following holds:

(i) $\tau$ is hereditary $\Phi$-quasiregular and $X = E$;

(ii) $\tau$ is hereditary quasiregular, $E$ is soluble and $X = F(E)$;

(iii) $\tau$ is regular, and $X = F^*$ or $X = E$.

Then $G \in \mathfrak{F}$.

By analogy with Theorem 1.11, Theorems A and B cover and unify the results in many papers. Some of them we discuss in Section 5.

Finally, note that Theorem A and B remain to be new for each concrete subgroup functor $\tau$, for example, if we supposed that $\tau$ is one of the functors in Examples 1.4–1.10.

All unexplained notation and terminology are standard. The reader is referred to [21], [1], [22], or [23], if necessary.
2 Base lemmas

Lemma 2.1. Let $\mathcal{F}$ be a saturated formation, $K \leq H \leq G$ and $N \leq G$. Suppose that the pair $(K, H)$ satisfies the $\mathcal{F}$-supplement condition in $G$.

(1) If either $N \leq H$ or $(|H|, |N|) = 1$, then the pair $(KN/N, HN/N)$ satisfies the $\mathcal{F}$-supplement condition in $G/N$.

(2) If $H \leq E \leq G$ and $\mathcal{F}$ is hereditary, then the pair $(K, H)$ satisfies the $\mathcal{F}$-supplement condition in $E$.

(3) If $K \leq V \leq H$, then the pair $(V, H)$ satisfies the $\mathcal{F}$-supplement condition in $G$.

Proof. Let $T$ be a subgroup of $G$ such that $HT = G$ and $H \cap T \subseteq KZ\mathcal{F}(T)$.

(1) Clearly, $(HN/N)(TN/N) = G/N$. Moreover, $HN \cap HT = (H \cap T)N$. Indeed, if either $N \leq H$ or $N \leq T$, it is clear. But if $N \not\leq H$, we have $N \leq T$ since in this case $(|H|, |N|) = 1$ by hypothesis. Hence

$$(HN/N) \cap (NT/N) = (NH \cap NT)/N = N(H \cap T)/N \subseteq N(KZ\mathcal{F}(T))/N = (NK/N)(Z\mathcal{F}(T)N/N) \leq (KN/N)(Z\mathcal{F}(TN/N))$$

since $Z\mathcal{F}(T)N/N \leq Z\mathcal{F}(TN/N)$ by [24] Lemma 2.2(4)]. Hence the pair $(KN/N, HN/N)$ satisfies the $\mathcal{F}$-supplement condition in $G/N$.

(2) Let $T_0 = T \cap E$. Then $E = E \cap HT = H(T \cap E) = HT_0$. Moreover,

$$H \cap T_0 = H \cap T \subseteq KZ\mathcal{F}(T) \leq K(Z\mathcal{F}(T) \cap E) \leq KZ\mathcal{F}(T \cap E) = KZ\mathcal{F}(T_0)$$

by [24] Lemma 2.2(5)]. This shows that the pair $(K, H)$ satisfies the $\mathcal{F}$-supplement condition in $E$.

(3) This is clear.

Lemma 2.2. Let $\mathcal{F}$ be a saturated formation. Let $H \leq G$ and $N$ be a normal subgroup of $G$.

(1) If $N \leq H$, then $H/N$ is $\mathcal{F}_r$-supplemented in $G/N$ if and only if $H$ is $\mathcal{F}_r$-supplemented in $G$.

(2) Suppose that $S$ is a $\tau$-subgroup of $G$ such that $H_G \leq S \leq H$ and the pair $(S, H)$ satisfies the $\mathcal{F}$-supplement condition in $G$. Then $H$ is $\mathcal{F}_\tau$-supplemented in $G$.

(3) If $H$ is $\mathcal{F}_\tau$-supplemented in $G$, and either $N \leq H$ or $(|H|, |N|) = 1$, then $HN/N$ is $\mathcal{F}_\tau$-supplemented in $G/N$.

(4) If $H$ is $\mathcal{F}_\tau$-supplemented in $G$, $\mathcal{F}$ is hereditary, $H \leq E \leq G$ and $\tau$ is hereditary, then $H$ is $\mathcal{F}_\tau$-supplemented in $E$.

Proof. For any subgroup $V$ of $G$ we put $\tilde{V} = VN/N$ and $\tilde{V} = VH_G/H_G$. Let $f$ be the canonical isomorphism from $(G/N)/(H_G/N)$ onto $G/H_G$. 6
Let $\hat{T}/\hat{H}_{G}$ be a subgroup of $G/\hat{H}_{G}$ such that $G/\hat{H}_{G} = (\hat{H}/\hat{H}_{G})(\hat{T}/\hat{H}_{G})$ and 

$$(\hat{H}/\hat{H}_{G}) \cap (\hat{T}/\hat{H}_{G})) \subseteq (\hat{S}/\hat{H}_{G})Z_{\tau}(\hat{T}/\hat{H}_{G})$$

for some $\tau$-subgroup $\hat{S}/\hat{H}_{G}$ of $G/\hat{H}_{G}$ contained in $\hat{H}/\hat{H}_{G}$.

Then

$$f((\hat{H}/\hat{H}_{G}) \cap (\hat{T}/\hat{H}_{G})) \leq f(\hat{S}/\hat{H}_{G})f(Z_{\tau}(\hat{T}/\hat{H}_{G})).$$

Note that

$$f((\hat{H}/\hat{H}_{G}) \cap (\hat{T}/\hat{H}_{G}))) = \hat{H} \cap \hat{T}$$

and $\hat{H}_{G} = (H/N)_{G/N} = H_{G}/N$.

On the other hand,

$$f(\hat{S}/\hat{H}_{G}) = S/H_{G} = \hat{S},$$

where $\hat{S}$ is a $\tau$-subgroup of $\hat{G}$ since $\hat{S}/\hat{H}_{G}$ is a $\tau$-subgroup of $G/\hat{H}_{G}$.

Similarly, $f(Z_{\delta}(\hat{T}/\hat{H}_{G})) = Z_{\delta}(\hat{T})$. Therefore $H$ is $\mathfrak{F}_{\tau}$-supplemented in $G$.

Now suppose that $H$ is $\mathfrak{F}_{\tau}$-supplemented in $G$. Then by considering the canonical isomorphism $f^{-1}$ from $G/H_{G}$ onto $(G/N)(H_{G}/N)$, one can prove analogously that $H/N$ is $\mathfrak{F}_{\tau}$-supplemented in $G/N$. The second assertion of (1) can be proved similarly.

(2) By the hyperthesis, the pair $(S,H)$ satisfies the $\mathfrak{F}$-supplement condition in $G$. Hence by Lemma 2.1(1), $(S/H_{G},H/H_{G})$ satisfies the $\mathfrak{F}$-supplement condition in $G/H_{G}$. Thus $H$ is $\mathfrak{F}_{\tau}$-supplemented in $G$.

(3) Let $\hat{S}$ be a $\tau$-subgroup of $\hat{G}$ contained in $\hat{H}$ such that the pair $(\hat{S},\hat{H})$ satisfies the $\mathfrak{F}$-supplement condition in $\hat{G}$.

Then the pair $(\hat{S}\hat{N}/\hat{N},\hat{H}\hat{N}/\hat{N})$ satisfies the $\mathfrak{F}$-supplement condition in $\hat{G}/\hat{N}$ by Lemma 2.1(1). Let $h$ be the canonical isomorphism from $(G/H_{G})/(H_{G}N/H_{G})$ onto $G/NH_{G}$. Then $h(\hat{S}\hat{N}/\hat{N}) = SN/NH_{G}$ and $h(\hat{H}\hat{N}/\hat{N}) = HN/NH_{G}$. Hence the pair $(SN/NH_{G},HN/NH_{G})$ satisfies the $\mathfrak{F}$-supplement condition in $G/NH_{G}$. Note also that $SN/NH_{G}$ is a $\tau$-subgroup of $G/NH_{G}$ since $\tau$ is inductive and $\hat{S}$ is a $\tau$-subgroup of $\hat{G}$. Hence, $HN/NH_{G}$ is $\mathfrak{F}_{\tau}$-supplemented in $G/NH_{G}$ and so $(HN/N)/(H_{G}N)/N$ is $\mathfrak{F}_{\tau}$-supplemented in $(G/N)/(N/H_{G}/N)$, which implies that $HN/N$ is $\mathfrak{F}_{\tau}$-supplemented in $G/N$ by (1).

(4) By hypothesis, for some $\hat{S} \leq \hat{H}$, where $\hat{S}$ is $\tau$-subgroup of $\hat{G}$, the pair $(\hat{S},\hat{H})$ satisfies the $\mathfrak{F}$-supplement condition in $\hat{G}$. Then, by Lemma 2.1(2), the pair $(\hat{S},\hat{H})$ satisfies the $\mathfrak{F}$-supplement condition in $\hat{E}$. Hence, by Lemma 2.1(3), the pair $(\hat{SH}_{E}/\hat{H}_{E},\hat{H}/\hat{H}_{E})$ satisfies the $\mathfrak{F}$-supplement condition in $\hat{E}/\hat{H}_{E}$, where $(\hat{SH}_{E}/\hat{H}_{E})$ is $\tau$-subgroup of $\hat{E}/\hat{H}_{E}$. Hence $SH_{E}/H_{E}$ is a $\tau$-subgroup of $E/H_{E}$ and the pair $(SH_{E}/H_{E},H/H_{E})$ satisfies the $\mathfrak{F}$-supplement condition in $E/H_{E}$. This shows that $H$ is $\mathfrak{F}_{\tau}$-supplemented in $E$. 

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3 Proof of Theorem A

The following lemma is a corollary of [1], IV, (6.7).

Lemma 3.1 Let $\mathfrak{F}$ be a saturated formation and $F$ the canonical local satellite of $\mathfrak{F}$ (See [25] Lemma 2.7 or [1, p. 361]). Let $P$ be a normal $p$-subgroup of $G$. If $P/\Phi(P) \leq Z_3(G/\Phi(P))$, then $P \leq Z_3(G)$.

Lemma 3.2 (See Lemma 2.14 in [25]). Let $\mathcal{F}$ be a saturated formation and $F$ the canonical local satellite of $\mathcal{F}$. Let $E$ be a normal $p$-subgroup of $G$. Then $E \leq Z_3(G)$ if and only if $G/C_G(E) \in F(p)$.

Lemma 3.3. Let $\mathfrak{F}$ be a saturated formation and $G = PT$, where $P$ is a normal $p$-subgroup of $G$. If $P \cap Z_3(T)$ is normal in $P$, then $P \cap Z_3(T) \leq Z_3(G)$.

Proof. First note that since $G = PT$ and $P \cap Z_3(T)$ is normal in $P$, $P \cap Z_3(T)$ is normal in $G$. Let $F$ be the canonical local satellite of $\mathfrak{F}$ and $H/K$ a chief factor of $G$ below $P \cap Z_3(T)$. Then $T/C_T(H/K) \in F(p)$ by Lemma 3.2. Hence $G/C_G(H/K) = PT/C_G(H/K) = PT/P(C_G(H/K) \cap T) \simeq T/P(C_G(H/K) \cap T) \cap T = T/C_T(H/K) \in F(p)$ by [26, Lemma 2.11] or [1, A, (10.6)(b)]. Hence $P \cap Z_3(T) \leq Z_3(G)$.

Lemma 3.4. Let $L \leq V \leq P$, where $P$ is a Sylow $p$-subgroup of a group $G$, and $N$ and $M$ are different normal subgroups of $G$. Suppose that $|G/M : N_{G/M}((LM/M) \cap (NM/M))|$ is a power of $p$. Then:

1. $((LM/M) \cap (NM/M))^{G/M} \leq VM/M$.
2. If $N$ is a non-abelian minimal normal subgroup of $G$, then $L \cap N = 1$.
3. If $NL \cap M = 1$, then $(L \cap N)^G \leq VM$.

Proof. (1) It is clear that $LM/M \leq VM/M \leq PM/M$ where $PM/M$ is a Sylow $p$-subgroup of $G/M$. On the other hand, since $|G/M : N_{G/M}((LM/M) \cap (NM/M))|$ is a power of $p$, we have $((LM/M) \cap (NM/M))^{G/M} = ((LM/M) \cap (NM/M))^{N_{G/M}((LM/M) \cap (NM/M))}^{PM/M} = ((LM/M) \cap (NM/M))^{PM/M}$. Hence we have (1).

(2) Suppose that $B = L \cap N \neq 1$. Then $(LM/M) \cap (NM/M) \neq 1$ and so $N \simeq NM/M \leq ((LM/M) \cap (NM/M))^{G/M} \leq PM/M$, which implies that $N$ is a $p$-group.

(3) Since $NL \cap M = 1$, $(LM/M) \cap (NM/M) = (L \cap N)M/M$. On the other hand, $((L \cap N)M/M)^{G/M} = (L \cap N)^G M/M$. Hence (3) is a corollary of (1).

A normal subgroup $N$ of $G$ is said to be $\mathfrak{F}$-$\Phi$-hypertcentral in $G$ [27] if either $N = 1$ or $N \neq 1$ and every non-Frattini chief factor of $G$ below $N$ is $\mathcal{F}$-central in $G$. The product of all normal $\mathfrak{F}$-$\Phi$-hypertcentral subgroups is denoted by $Z_{\mathfrak{F}}(G)$ [27].

Proposition 3.5. Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, $\tau$ be $\Phi$-quasiregular (quasiregular, respectively) and $P$ a non-identity normal $p$-subgroup of $G$. Suppose that every maximal subgroup of $P$ is $\mathfrak{F}_\tau$-supplemented in $G$. Then $P \leq Z_{\mathfrak{F}}(G)$ ($E \leq Z_{\mathfrak{F}}(G)$,
Proof. Suppose that the proposition is false and let \((G, P)\) be a counterexample with \(|G| + |P|\) minimal. Let \(Z = Z_{\Phi}(G)\) (\(Z = Z_{\Phi}(G)\), respectively). Let \(G_p\) be a Sylow \(p\)-subgroup of \(G\).

(1) \(P \not\leq Z_{\Phi}(G)\) (\(P \not\leq Z_{\Phi}(G)\), respectively). (This follows from the hypothesis that \(\mathfrak{F}\) contains all supersoluble groups and the choice of \(G\).

(2) \(P\) is not a minimal normal subgroup of \(G\).

Suppose that \(P\) is a minimal normal subgroup of \(G\). Then \(P \cap Z = 1\). Let \(H\) be a maximal subgroup of \(P\) such that \(H\) is normal in \(G_p\). Then \(H_G = 1\). Let \(S \in \tau(G)\) and \(T\) be subgroups of \(G\) such that \(S \leq H\), \(HT = G\) and \(H \cap T \subseteq SZ_3(T)\). Suppose that \(T \neq G\). Then \(1 < P \cap T < P\), where \(P \cap T\) is normal in \(G\) since \(P\) is abelian, which contradicts the minimality of \(P\). Hence \(T = G\), so \(H = H \cap T \leq SZ\), which implies that \(H = S(H \cap Z) = S\) is a \(\tau\)-subgroup of \(G\). It is clear that \(P \not\leq \Phi(G)\). Hence for some maximal subgroup \(M\) of \(G\) we have \(G = P \times M\). Since \(\tau\) is \(\Phi\)-quasiregular and \(HM_G/M_G\) is normal in \(G_pM_G/M_G\), \(HM_G/M_G\) is normal in \(G/M_G\) by Lemma 3.4. Hence \(HM_G/M_G = PM_G/M_G\), which implies that \(H = P\), a contradiction. Hence we have (3).

(3) If \(N\) is a minimal normal subgroup of \(G\) contained in \(P\), then \(P/N \leq Z_{\Phi}(G/N)\) (\(P/N \leq Z_{\Phi}(G/N)\), respectively) and \(Z \cap P = 1\).

Indeed, by Lemma 2.2 the hypothesis holds for \((G/N, P/N)\). Hence \(P/N \leq Z_{\Phi}(G/N)\) (\(P/N \leq Z_{\Phi}(G/N)\), respectively) by the choice of \((G, E)\). Hence \(N \not\leq Z\) by [27] Lemma 2.2 and the choice of \((G, P)\).

(4) \(P \leq Z_{\Phi}(G)\).

Suppose that \(P \not\leq Z_{\Phi}(G)\). Then, in view of (3) and [27] Lemma 2.2], \(\Phi(G) \cap P = 1\). Hence \(P = N \times D\) for some normal subgroup \(D\) of \(G\) by [25] Lemma 2.15, where \(D \neq 1\) by (2). Let \(R\) be a minimal normal subgroup of \(G\) contained in \(D\). Then, by [1] A, 9.11, \(RN/N \not\leq \Phi(G/N)\). Hence in view of (3) and the \(G\)-isomorphism \(R \simeq RN/N\), \(R\) is \(\mathfrak{F}\)-central in \(G\), and so \(P \leq Z_{\Phi}(G)\) by (3) and [27] Lemma 2.2]. Hence we have (4).

(5) \(\tau\) is quasiregular (This follows from (4) and the choice of \((G, P)\)).

(6) If \(N\) is a minimal normal subgroup of \(G\) contained in \(P\), then \(N\) is the unique minimal normal subgroup of \(G\) contained in \(P\) (see the proof of (3)).

(7) \(\Phi(P) \neq 1\).

Suppose that \(\Phi(P) = 1\). Then \(P\) is an elementary abelian \(p\)-group. Let \(W\) be a maximal subgroup of \(N\) such that \(W\) is normal in \(G_p\). We show that \(W\) is normal in \(G\). Let \(B\) be a complement of \(N\) in \(P\) and \(V = WB\). Then \(V\) is a maximal subgroup of \(P\) and, evidently, \(V_G = 1\) by (6).

Let \(S \in \tau(G)\) and \(T\) be subgroups of \(G\) such that \(S \leq V\), \(VT = G\) and \(V \cap T \subseteq SZ_3(T)\). Assume that \(T = G\). Then \(V = V \cap T \leq SZ\) and so \(V = S(V \cap Z)\). But in view of (3), \(Z \cap P = 1\). Hence \(V = S\) and thereby \(W = WB \cap N = V \cap N = S \cap N\). Since \(\tau\) is quasiregular by (5) and \(W\) is
normal in $G_p$, we have that that $W$ is normal in $G$ by Lemma 3.4. Let $T \neq G$. Then $1 \neq T \cap P < P$. Since $G = VT = PT$ and $P$ is abelian, $T \cap P$ is normal in $G$. Hence $N \leq T$, which implies that $W \leq T \cap V \leq S\bar{Z}(T) \cap P = S(Z\bar{3}(T) \cap P)$. Since $P$ is abelian, $Z\bar{3}(T) \cap P$ is normal in $P$, so $Z\bar{3}(T) \cap P \leq Z \cap P = 1$ by Lemma 3.3. This implies that $W = S \cap N$ and so $W$ is normal in $G$ by Lemma 3.4.

Finally, as above, we get the same conclusion in the case when $N \leq T$. Therefore $W$ is normal in $G$ and so $W = 1$, which implies that $|N| = p$. This contradiction shows that we have $\Phi(P) \neq 1$.

The final contradiction.

By (7), $\Phi(P) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $\Phi(P)$. Then $P/N \leq Z\bar{3}(G/N)$ by (3). It follows that $P/\Phi(P) \leq Z\bar{3}(G/\Phi(P))$. Thus $P \leq Z$ by Lemma 3.1. This contradiction completes the proof.

**Theorem 3.6.** Let $\mathfrak{F}$ be the class of all $p$-supersoluble groups. Let $E$ be a normal subgroup of $G$ and $P$ a Sylow $p$-subgroup of $E$ of order $|P| = p^n$, where $n > 1$ and $(|E|, p - 1) = 1$. Suppose that $\tau$ is $\Phi$-regular and every $\tau$-subgroup of $G$ contained in $P$ is subnormally embedded in $G$. If every maximal subgroup of $P$ in $\mathfrak{F}$ is $\tau$-supplemented in $G$, then $E$ is $p$-nilpotent.

**Proof.** Suppose that this theorem is false and let $(G, E)$ be a counterexample with $|G| + |E|$ minimal. We proceed via the following steps.

1. $O_p'(E) = 1$.

Suppose that $O_p'(E) \neq 1$. Since $O_p'(E)$ is characteristic in $E$, it is normal in $G$ and the hypothesis holds for $(G/O_p'(E), E/O_p'(E))$ by Lemma 2.2. The choice of $G$ implies that $E/O_p'(E)$ is $p$-nilpotent. It follows that $E$ is $p$-nilpotent, a contradiction.

2. If $O_p(E) \neq 1$, then $E$ is $p$-soluble.

Indeed, by Lemma 2.2, the hypothesis holds for $(G/O_p(E), E/O_p(E))$. Hence in the case when $O_p(E) \neq 1$, $E/O_p(E)$ is $p$-nilpotent by the choice of $(G, E)$, which implies the $p$-solubility of $E$.

3. $O_p(E) \neq 1$.

Suppose that this is false. Then, in view of [25] Lemma 3.4(3)], for any subnormal subgroup $L$ of $G$ contained in $E$ we have neither $L$ is a $p$-group nor a $p'$-group. Let $N$ be a minimal normal subgroup of $G$ contained in $E$. Then $N$ is non-abelian group. Then since $(|E|, p - 1) = 1$, we have that $p = 2$ by Feit-Thompson theorem. It is clear that $|N_2| > 2$.

We claim that for any minimal normal subgroup $L$ of $G$ contained in $E$ and for any $\tau$-subgroup $S$ of $G$ we have $S \cap L = 1$. Indeed, assume that $S \cap L \neq 1$. Let $M$ be a maximal subgroup of $G$ such that $LM = G$. Since $\tau$ is $\Phi$-regular, $|G/M_G : N_{G/M_G}((SM_G/M_G) \cap (LM_G/M_G))|$ is a power of 2. Then $L$ is abelian by Lemma 3.4(2) and so $L \leq O_2(E) = 1$, a contradiction.

Let $H$ be an arbitrary maximal subgroup of $P$. It is clear that $H_G = 1$. Hence there exists a subgroup $T$ such that $G = HT$ and $H \cap T \leq S\bar{Z}(T)$ for some $\tau$-subgroup $S$ of $G$ contained in $H$. 

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Suppose that $S \neq 1$. Let $W$ be a subnormal subgroup of $G$ such that $S$ is a Sylow 2-subgroup of $W$. In view of [25] Lemma 3.4(2)], we may assume, without loss of generality, that $W \leq E$. Let $L$ be a minimal subnormal subgroup of $G$ contained in $W$. Then $L$ neither is a 2-group nor a $2'$-group. Therefore $L$ is a non-abelian simple group and $L_2 = S \cap L$ is a Sylow 2-subgroup of $L$ since $S$ is a Sylow 2-subgroup of $W$. It is clear that $R = L^G$ is a minimal normal subgroup of $G$ and $S \cap R \neq 1$, contrary to (a). Therefore $S = 1$. Hence every maximal subgroup $H$ of $P$ has a supplement $T$ in $G$ such that $H \cap T \leq Z_3(T)$.

We now show that $V = T \cap E$ is 2-nilpotent. Let $V_2$ be a Sylow 2-subgroup of $V$ containing $M \cap V$. Then $|V_2 : V \cap M| \leq |P : M| = 2$. Therefore for a Sylow 2-subgroup $Q$ of $VZ_3(T)/Z_3(T)$ we have $|Q|$ divides 2. This induces that $VZ_3(T)/Z_3(T) \simeq V/V \cap Z_3(T)$ is 2-nilpotent. It is well-known that the class of all 2-nilpotent groups is a hereditary saturated formation. Hence in view of [24, Lemma 2.2(5)], $V \cap Z_3(T) \leq Z_3(V)$. Thus $V = T \cap E$ is 2-nilpotent. But $E = E \cap TM = M(T \cap E)$, so every maximal subgroup of $P$ has a 2-nilpotent supplement $T$ in $E$. It is clear that a Hall $2'$-subgroup of $T \cap E$ is also a Hall $2'$-subgroup of $E$. Therefore $E$ is 2-nilpotent by [25] Lemmas 3.7 and 3.8]. It follows that $N$ is a 2-group, a contradiction. Hence we have (3)

(4) There is a maximal subgroup $D$ of $G$ such that $ND = G$, $D \cap E = 1$ and $E = N \times M$, where $M = D \cap E$ and $N = O_p(E) = C_E(N)$ is a minimal normal subgroup of $G$ and $M$ is $p$-nilpotent. In particular, $E$ is $p$-soluble.

In view of (3), $O_p(E) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(E)$. Then the hypothesis holds for $(G/N, E/N)$ by Lemma 2.2. Therefore $E/N$ is $p$-nilpotent by the choice of $(G.E)$, and so $E$ is $p$-soluble. It follows that $N$ is the unique minimal normal subgroup of $G$ contained in $E$. If $N \leq \Phi(G)$, then $E$ is $p$-nilpotent by [25] Corollary 1.6. Hence $N \not\leq \Phi(G)$ and so $G = N \rtimes D$ for some maximal subgroup $D$ of $G$. Since $O_p(G) \leq C_G(N)$ by [26] Lemma 2.11 or [1] A, 10.6(b)], we have that $O_p(G) \cap D$ is normal in $G$. Hence $O_p(G) \cap D \cap E$ is normal in $G$. Note that $E = N \times (D \cap E)$, so $D \cap M = 1$ and

\[ O_p(E) = O_p(G) \cap E = N(O_p(E) \cap D \cap E), \]

where $O_p(E) \cap D \cap E = O_p(G) \cap D \cap E$ is normal in $G$. Hence $O_p(E) \cap D \cap E = 1$, and so $N = O_p(E)$. Finally, since $E$ is $p$-soluble and $O_{p'}(E) = 1$ by (1), we have $C_E(N) = N$ by [28] Chapter 6, 3.2.

(5) If $H/K$ is a chief factor of $E$ below $N$, then $|H/K| > p$

By Proposition 4.13(c) in [1] A], $N = N_1 \times \cdots \times N_t$, where $N_1, \ldots, N_t$ are minimal normal subgroups of $E$, and from the proof of this proposition we see that $|N_i| = |N_j|$ for all $i, j \in \{1, \ldots, t\}$. Hence for any chief factor $H/K$ of $E$ below $N$ we have $|H/K| = |N_1|$ by [27] Lemma 2.2]. Suppose that $|H/K| = p$. Since $(p-1, |E|) = p$, $C_E(H/K) = E$. Hence $N \leq Z_\infty(E)$, which implies the $p$-nilpotency of $E$ by (4). This contradiction shows that (5) holds.

(6) If a non-identity subgroup $S$ of $P$ is subnormally embedded in $G$, then $S \cap N \neq 1$.

Indeed, let $W$ be a subnormal subgroup of $G$ such that $S$ is a Sylow $p$-subgroup of $W$. If $S \cap N = 1$,
then $W \cap N = 1$. Hence by (4) and [25 Lemma 3.4(4)], $W \leq C_E(N) = N$, a contradiction. Thus (6) holds.

(7) $M = N_E(M_{p'})$, where $M_{p'}$ is the Hall $p'$-subgroup of $M$.

Let $J = N_E(M_{p'})$. Suppose that $M < J$. Then $J = J \cap NM = M(J \cap N)$ and therefore $J \cap N \neq 1$. Since $E = NJ$, $J \cap N$ is normal in $E$ and $E/C_E(J \cap N)$ is a $p$-group. If $F$ is the canonical local satellite of the saturated formation of all nilpotent, the $F(p)$ is the class of all $p$-groups by (see [1 IV]). Hence in view of Lemma 3.2, $J \cap N \leq Z_\infty(E)$. Hence for some minimal normal subgroup $C$ of $E$ contained in $N$ we have $|C| = p$, which contradicts (5).

**Final contradiction.**

Let $M_p \leq D_p$, where $M_p$ is a Sylow $p$-subgroup of $M$ and $D_p$ is a Sylow $p$-subgroup of $D$. Without loss of generality, we may suppose that $M_p \leq P$. Then $NM_p = P$ and $ND_p$ is a Sylow $p$-subgroup of $G$. Let $N_0 \leq N$ be a normal subgroup of $ND_p$ such that $|N : N_0| = p$. Let $W = N_0D_p$ and $V = N_0M_p$. Then $W$ is maximal in $ND_p$ and $V$ is maximal in $P$ such that $V_G = 1$.

(i) For any $\tau$-subgroup $S$ of $G$ contained in $V$, we have $S \cap N = 1$.

Assume that this is false. It is clear that $SN \cap D_G = 1$. Since $\tau$ is $\Phi$-regular, $N \leq (S \cap N)^GD_G \leq WD_G$ by Lemma 3.4. Hence $N = N \cap N_0D_pD_G = N_0(N \cap D_pD_G) = N_0$. This contradiction shows that we have (i).

(ii) $V$ has no a $p$-nilpotent supplement in $E$.

Assume that $V$ has a $p$-nilpotent supplement $T_0$ in $E$. Then a Hall $p'$-subgroup $T_{p'}$ of $T_0$ is a Hall $p'$-subgroup of $E$. By (4), $E$ is $p$-soluble and so any two Hall $p'$-subgroup of $E$ are conjugate in $E$. Without loss of generality, we may assume that $T_{p'} \leq M$, so $T_0 \leq M$ by (7). It follows that $E = VT_0 = VM$. But since $M_p \leq V$ and $V$ is maximal in $P$, $VM \neq E$. This contradiction shows that we have (ii).

By hypothesis, $V$ is $\mathfrak{Z}_\tau$-supplemented in $G$, so there exists a subgroup $T$ such that $G = VT$ and $V \cap T \leq SZ_\tau(T)$ for some $\tau$-subgroup $S$ of $G$ contained in $V$.

Assume that $S \neq 1$. Then $S \cap N \neq 1$ by (6), contrary to (i). Hence $S = 1$, so

$$V \cap T_0 = V \cap T \leq Z_\tau(T) \cap T_0 \leq Z_\tau(T_0)$$

by [24 Lemma 2.2(5)]. Hence, as in the proof of (3), one can show that $T_0 = T \cap E$ is $p$-nilpotent. But this contradicts (ii).

The theorem is proved.

**Corollary 3.7.** Let $E$ be a non-identity normal subgroup $G$. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of $E$ is $\Phi_\tau$-supplemented in $G$ for some regular subgroup functor $\tau$ such that every $\tau$-subgroup of $G$ contained in $E$ is subnormally embedded in $G$. Then $E \leq Z_\delta(G)$.

**Proof.** Suppose that this corollary is false and let $(G, E)$ be a counterexample with $|G| + |E|$
implies that \( E/N \) holds for (1). Let \( H \) to prove \( E \) is supersoluble by (25, Theorem C). On the other hand, the hypothesis holds for \((G/V, E/V)\) by Lemma 2.2, so \( E/V \leq Z_\Phi(G/V) \). Therefore \( E \leq Z_\Phi(G) \) by (27, Lemma 2.2). This contradiction completes the proof.

**Lemma 3.8.** Let \( P \) be a normal non-identity \( p \)-subgroup of \( G \) with \( |P| > p \) and \( P \cap \Phi(G) = 1 \). Suppose that \( \tau \) is \( \Phi \)-quasiregular and every maximal subgroup of \( P \) is \( \Omega_\tau \)-supplemented in \( G \). Then some maximal subgroup of \( P \) is normal in \( G \).

**Proof.** Let \( G_p \) be a Sylow \( p \)-subgroup of \( G \) and \( Z = Z_\Phi(G) \). Since \( P \cap \Phi(G) = 1, P = N_1 \times \cdots \times N_t \), where \( N_i \) is a minimal normal subgroup of \( G \), for all \( i = 1, \ldots, t \) by (25, Lemma 2.15). Hence \( |N_i| \neq p \) for all \( i = 1, \ldots, t \). Then \( P \cap Z = 1 \) and \( t > 1 \) (see (2) in the proof of Proposition 3.5). Moreover, the hypothesis holds for \((G/N_1, P/N_1)\) by Lemma 2.2 and (27, Lemma 2.2), so by induction some maximal subgroup \( M/N_1 \) of \( P/N_1 \) is normal in \( G/N_1 \). Then a maximal subgroup \( M \) of \( P \) is normal in \( G \).

**Lemma 3.9** Let \( E \) be a normal subgroup of \( G \) and \( \tau \) a \( \Phi \)-regular inductive subgroup functor such that every primary \( \tau \)-subgroup of \( G \) is subnormally embedded in \( G \). If every maximal subgroup of every non-cyclic Sylow subgroup of \( E \) is \( \Omega_\tau \)-supplemented in \( G \), then \( E \) is supersoluble.

**Proof.** Suppose that this lemma is false and let \((G, E)\) be a counterexample with \(|G| + |E| \) minimal. Let \( P \) be a Sylow \( p \)-subgroup of \( E \) where \( p \) is the smallest prime dividing \(|E|\). By Theorem 3.6, \( E \) is \( p \)-nilpotent. Let \( V \) be the Hall \( p' \)-subgroup of \( E \). Then \( V \) is normal in \( G \) and the hypothesis holds for \((G, V)\). Hence \( V \) is supersoluble by the choice of \((G, E)\). Then a Sylow \( q \)-subgroup \( Q \) of \( V \), where \( q \) is the largest prime dividing \(|V|\), is normal and so it is characteristic in \( V \). Hence \( Q \) is normal in \( G \) and the hypothesis holds for \((G/Q, E/Q)\) by Lemma 2.2. The choice of \((G, E)\) implies that \( E/N \) is supersoluble. On the other hand, by Proposition 3.5, \( Q \leq Z_\Phi(G) \). Thus \( E \) is supersoluble by (25, Theorem C).

**Proof of Theorem A.** Firstly, suppose that \( \tau \) is regular. Then \( X \leq Z_\Phi(G) \) by Corollary 3.7. Hence \( E \leq Z_\Phi(G) \) by (26, Theorem B). Since \( G/E \in \mathfrak{F} \), we obtain \( G \in \mathfrak{F} \). Therefore, we only need to prove \( G \in \mathfrak{F} \) in the case when \( \tau \) is \( \Phi \)-regular hereditary.

Assume that this is false and let \((G, E)\) be a counterexample with \(|G| + |E| \) minimal. Let \( F = F(E) \) and \( F^* = F^*(E) \). Let \( p \) be prime divisor of \(|F^*|\) and \( P \) the Hall \( p \)-subgroup of \( F^* \).

(1) \( X \) is supersoluble (This follows from Lemma 3.9).

(2) \( X = F^* \neq E \).

Indeed, suppose that \( X = E \). Then \( E \) is \( q \)-nilpotent, where \( q \) is smallest prime divisor of \(|E|\), by (1). Let \( V \) be the Hall \( q' \)-subgroup of \( X \). If \( V = 1 \), then \( E \leq Z_\Phi(G) \) by Lemma 3.5, so \( G \in \mathfrak{F} \).
But this contradicts the choice of \((G, E)\). Hence \(V \neq 1\). Since \(V\) is characteristic in \(X\), it is normal in \(G\). Moreover, the hypothesis holds for \((G/V, X/V)\) by Lemma 2.2. Hence \(G/V \in \mathcal{F}\) by the choice of \((G, E)\). Now we see that the hypothesis holds also for \((G, V)\) and so \(G \in \mathcal{F}\) again by the choice of \((G, E)\). This contradiction shows that we have (2).

\((3)\) \(F^* = F\) and \(C_G(F) = C_G(F^*) \leq F\).

Since by (1) and (2), \(X = F^*\) is soluble, \(F^* = F\) by \([29, X, 13.6]\). We have also \(C_G(F) = C_G(F^*) \leq F\) by \([29, X, 13.12]\).

\((4)\) Every proper normal subgroup \(W\) of \(G\) with \(F \leq W \leq E\) is supersoluble.

By \([29, X, 13.11]\), \(F^*(E) = F^*(F^*(E)) \leq F^*(W) \leq F^*(E)\). It follows that \(F^*(W) = F^*(E) = F^*\). Thus the hypothesis is still true for \((W, W)\) by Lemma 2.2(4). The minimal choice of \(G\) implies that \(W\) is supersoluble.

\((5)\) If \(E \neq G\), then \(E\) is supersoluble (It follows directly from (4)).

\((6)\) If \(L\) is a minimal normal subgroup of \(G\) and \(L \leq P\), then \(|L| > p\).

Assume that \(|L| = p\). Let \(C_0 = C_E(L)\). Then the hypothesis is true for \((G/L, C_0/L)\). Indeed, clearly, \(G/C_0 = G/(E \cap C_G(L))\) is supersoluble. Besides, since \(L \leq Z(C_0)\) and evidently \(F = F^* \leq C_0\) and \(L \leq Z(F)\), we have \(F^*(C_0/L) = F^*(C_0)/L = F^*/L\). Hence the hypothesis is still true for \((G/L, C_0/L)\). This implies that \(G/L \in \mathcal{F}\) and thereby \(G \in \mathcal{F}\) since \(|L| = p\) and \(U \subseteq \mathfrak{F}\), a contradiction.

\((7)\) \(\Phi(G) \cap P \neq 1\) and \(F^*(E/L) \neq F^*/L\) for every minimal normal subgroup \(L\) of \(G\) contained in \(\Phi(G) \cap P\).

Suppose that \(\Phi(G) \cap P = 1\). Then \(P\) is the direct product of some minimal normal subgroups of \(G\) by \([25, Lemma 2.15]\). Hence by Lemma 3.8, \(P\) has a maximal subgroup \(M\) which is normal in \(G\). Now by \([27, Lemma 2.2]\), \(G\) has a minimal normal subgroup \(L\) with order \(p\) contained in \(P\), which contradicts (6). Thus \(\Phi(G) \cap P \neq 1\). Let \(L \leq \Phi(G) \cap P\) and \(L\) be a minimal normal subgroup of \(G\). Assume that \(F^*(E/L) = F^*/L\). Then the hypothesis is true for \(G/L\) and so \(G/L \in \mathcal{F}\) by the choice of \(G\). But then \(G \in \mathcal{F}\) since \(L \leq \Phi(G)\). This contradiction shows that \(F^*(E/L) \neq F^*/L\).

\((8)\) \(E\) is not soluble and \(E = G\).

Assume that \(E\) is soluble. Let \(L\) be a minimal normal subgroup of \(G\) contained in \(\Phi(G) \cap P\). By \([1, A, 9.3(c)]\), \(F/L = F(E/L)\). On the other hand, \(F^*(E/L) = F(E/L)\) by \([29, X, 13.6]\). Hence \(F^*(E/L) = F(E/L) = F^*/L\) by (3), which contradicts (7). Therefore \(E\) is not soluble and so \(E = G\) by (5).

\((9)\) \(G\) has a unique maximal normal subgroup \(M\) containing \(F\), \(M\) is supersoluble and \(G/M\) is a non-abelian simple group (This directly follows from (4) and (8)).

\((10)\) \(G/F\) is a non-abelian simple group and \(G/L\) is a quasinilpotent group if \(L\) is a minimal normal subgroup of \(G\) contained in \(\Phi(G) \cap P\).

Let \(L\) be a minimal normal subgroup of \(G\) contained in \(\Phi(G) \cap P\). Then by (7), \(F^*(E/L) \neq F^*/L\).
Thus $F/L = F^*/L$ is a proper subgroup of $F^*(G/L)$. By [29] X, 13.6, $F^*(G/L) = F(G/L)E(G/L)$, where $E(G/L)$ is the layer of $G/L$. By (9), every chief series of $G$ has only one non-abelian factor. But since $E(G/L)/Z(E(G/L))$ is a direct product of simple non-abelian groups, we see that $F^*(G/L) = G/L$ is a quasinilpotent group. Since $F(G/L) \cap E(G/L) = Z(E(G/L))$ by [29] X, (13.18), $G/F \simeq (G/L)/(F/L)$ is a simple non-abelian group.

(11) $F^* = P$.

Assume that $P \neq F$ and let $Q$ be a Sylow $q$-subgroup of $F$, where $q \neq p$. By (10), $Q \leq Z_\infty(G)$. Hence by [29] X, 13.6, $F^*(G/Q) = F^*/Q$ and so the hypothesis is still true for $(G/Q, G/Q)$. Hence $G/Q$ is supersoluble by the choice of $Q$. It follows that $G$ is soluble, which contradicts (8).

(12) $p$ is the largest prime dividing $|G|$ and every Sylow $q$-subgroup $Q$ of $G$ where $q \neq p$ is abelian.

Let $D = PQ$. Then $D < G$ by (8). By Lemma 3.9, $D = PQ$ is supersoluble. Since $O_q(D) \leq C_G(P)$, we have $C_G(P) \leq P$ by (3) and (11). Hence $O_q(D) = 1$. Consequently, $p > q$ and $F(D) = P$. Hence $p$ is the largest prime dividing $|G|$ and $D/P \simeq Q$ is abelian.

Final contradiction.

By (8) and the Feit-Thompson theorem, 2 divides $||G|$. By (12), a Sylow 2-subgroup of $G/P$ is abelian. Hence by [29] XI, 13.7, $G/P$ is isomorphic to one of the following: a) $PSL(2, 2^f)$; b) $PSL(2, q)$, where 8 divides $q - 3$ or $q - 5$; c) The Janko group $J_1$; d) A Ree group. It is not difficult to show that in any case $G/P$ has a non-abelian supersoluble subgroup $V/P$ such that $p$ does not divide $|V/P|$. Hence in view of (3) and (11), we have $C_V(P) \leq P$ and so $P = F(V)$. On the other hand, $V$ is supersoluble by Lemma 3.9. Thus $V/P$ is abelian, a contradiction. Hence $G \in \mathfrak{F}$. The theorem is thus proved.

4 Proof of Theorems B

Lemma 4.1. Let $\mathfrak{F}$ be a saturated formation, $P$ be a normal $p$-subgroup of a group $G$, where $p$ is a prime. Let $D$ be a characteristic subgroup of $P$ such that every non-trivial $p'$-automorphism of $P$ induces a non-trivial automorphism of $D$. Suppose that $D \leq Z_\tau(G)$. Then $P \leq Z_\mathfrak{F}(G)$.

Proof. Let $F$ be the canonical local satellite of $\mathfrak{F}$. Let $C = C_G(P)$. Since $D \leq Z_\mathfrak{F}(G)$, then $G/C_G(D) \in F(p)$ by Lemma 3.2. On the other hand, since every non-trivial $p'$-automorphism of $P$ induces a non-trivial automorphism of $D$, $C_G(D)/C_G(P)$ is a $p$-group. Hence from the definition of $F$ we have $G/C_G(P) \in F(p)$. It follows that $P \leq Z_\mathfrak{F}(G)$.

Let $P$ be a non-identity $p$-group. If $P$ is not a non-abelian 2-group, we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 4.2 (see [30] Theorem 2.4]. Let $P$ be a group, $\alpha$ a $p'$-automorphism of $P$. If $[\alpha, \Omega(P)] = 1$, then $\alpha = 1$.

Lemma 4.3. Let $C$ be a Thompson critical subgroup of a $p$-group $P$ (see [28] p. 185]). Then
the group \( D := \Omega(C) \) is of exponent \( p \) if \( p \) is an odd prime, or exponent 4 if \( P \) is non-abelian 2-group. Moreover, every non-trivial \( p' \)-automorphism of \( P \) induces a non-trivial automorphism of \( D \).

**Proof.** See the proof of [28 Chapter 5, 3.13] and use the fact that if \( C \) is a non-abelian 2-group, then \( \Omega(C) \) is of exponent 4 (see [31 p. 3]).

**Lemma 4.4.** Let \( \mathfrak{F} \) be a saturated formation, Let \( P \) be a normal \( p \)-subgroup of a group \( G \) and \( D = \Omega(C) \), where \( C \) is a Thompson critical subgroup of \( P \). If \( C < Z_{\mathfrak{F}}(G) \) or \( D < Z_{\mathfrak{F}}(G) \), then \( P < Z_{\mathfrak{F}}(G) \).

**Proof.** Let \( Z = Z_{\mathfrak{F}}(G) \). Suppose that \( C \leq Z \). Then \( G/C_G(C) \in F(p) \), where \( F \) is the canonical local satellite of \( \mathfrak{F} \) by Lemma 3.2. On the other hand, \( C_G(C)/C_G(P) \) is a \( p \)-group by [28 Chapter 5, 3.11]. Hence \( G/C_G(P) \in F(p) \). Consequently, \( P \leq Z \). On the other hand, by Lemmas 4.2 and 4.3, \( C_G(C)/C_G(D) \) is also a \( p \)-group, and so in the case when \( D < Z \) we similarly get that \( C \leq Z \).

**Lemma 4.5.** Let \( P/R \) be a chief factor of a group \( G \) with \( |P/R| = p^n \), where \( p \) is a prime and \( n > 1 \). Suppose that for every normal subgroup \( V \) of \( G \) with \( V < P \) we have \( V \leq R \). Let \( H \) be a subgroup of \( P \) such that \( R < RH < P \). If \( H \) is a cyclic group of prime order or order 4, and \( T \) is a supplement of \( H \) in \( G \), then \( T = G \).

**Proof.** Assume that \( T \neq G \). Then \( P = H(P \cap T) \), where \( P \cap T \neq P \) and \( |P : P \cap T| = |G : T| \). Let \( N = N_G(P \cap T) \). Since \( T \leq N \) and \( P \cap T < N_P(P \cap T) \), we have that either \( P \cap T \) is a normal subgroup of \( G \) or \( |G : N| = 2 \). The first case implies that \( P \cap T \leq R \) and hence \( P = RH \), a contradiction. In the second case, \( N \) is normal in \( G \) and so \( N \cap P \) is a normal subgroup of \( G \) with \( |P : P \cap N| = 2 \). Therefore \( P \cap N \leq R \) and thereby \( |P/R| = 2 \), a contradiction. Hence \( T = G \).

**Proposition 4.6.** Let \( \mathfrak{F} \) be a saturated formation containing all supersoluble groups and \( P \) a non-identity normal \( p \)-subgroup of \( G \). Suppose that every cyclic subgroup of \( P \) of prime order or order 4 (if \( P \) is a non-abelian group) is \( \mathfrak{F} \)-supplemented in \( G \).

(a) If \( \tau \) is \( \Phi \)-quasiregular and \( P \) is of exponent \( p \) or exponent 4, then \( P \leq Z_{\mathfrak{F}}(G) \).

(b) If \( \tau \) is quasiregular, then \( P \leq Z_{\mathfrak{F}}(G) \).

**Proof.** Suppose that in this theorem is false and let \((G, P)\) be a counterexample with \(|G| + |P|\) minimal. We write \( Z = Z_{\mathfrak{F}}(G) \) if \( \tau \) is \( \Phi \)-quasiregular and \( P \) is of exponent \( p \) or exponent 4, and \( Z = Z_{\mathfrak{F}}(G) \) if \( \tau \) is quasiregular. Let \( G_p \) a Sylow \( p \)-subgroup of \( G \).

(1) \( G \) has a normal subgroup \( R \leq P \) such that \( P/R \) is an \( \mathfrak{F} \)-eccentric chief factor of \( G \), \( R \leq Z \) and \( V \leq R \) for any normal subgroup \( V \neq P \) of \( G \) contained in \( P \). In particular, \(|P/R| > p| \).

Let \( P/R \) be a chief factor of \( G \). Then the hypothesis holds for \((G, R)\). Therefore \( R \leq Z \) and so \( P/R \) is \( \mathfrak{F} \)-eccentric in \( G \) by the choice of \((G, P)\) and [27 Lemma 2.2]. It follows that \(|P/R| > p| \). Now let \( V \neq P \) be any normal subgroup of \( G \) contained in \( P \). Then \( V \leq Z \). If \( V \nleq R \), then \( VR = P \leq Z \) by [27 Lemma 2.2]. This contradiction shows that \( V \leq R \).

(2) \( P \) is of exponent \( p \) or exponent 4.
Assume that this is false and let $\tau$ be quasiregular. Let $L$ be a Thompson critical subgroup of $P$ and $\Omega = \Omega(L)$. Then $\Omega < P$, and so $\Omega \leq Z$ by (1), where $Z = Z_\Phi(G)$. Hence $P \leq Z$ by Lemma 4.4, which contradicts the choice of $(G, P)$. Hence $\Omega = P$, so we have (2) by Lemma 4.3.

Now let $L/R$ be any minimal subgroup of $(P/R) \cap Z(G_p/R)$. Let $x \in L \setminus R$ and $H = \langle x \rangle$. Then $L/R = HR/R$, so $H$ is not normal in $G$ and $H_G \leq R$ by (1). Moreover, $|H|$ is either prime or 4 by (2). Hence $H$ is $\Phi$-supplemented in $G$, so there is a subgroup $T/H_G$ of $G/H_G$ such that $(T/H_G)(H/H_G) = G/H_G$ and $(T/H_G) \cap (H/H_G) \subseteq (S/H_G)Z_\Phi(T/H_G)$ for some $\tau$-subgroup $S/H_G$ of $G/H_G$ contained in $H/H_G$.

(3) $H/H_G = S/H_G$ and $L/R \in \tau(G/R)$.

By Lemma 4.5, we have $T = G$. Hence $H/H_G = (S/H_G)(H/H_G \cap Z_\Phi(G/H_G))$. On the other hand, since $H$ is cyclic, we have either $H/H_G \leq Z_\Phi(G/H_G)$ or $H/H_G = S/H_G$. Note that $(R/H_G)/(R/H_G) = (RH/G)/(R/H_G) = (L/H_G)/(R/H_G)$, so if we have the former case, then $(L/H_G)/(R/H_G) \leq (P/H_G)/(R/H_G) \cap Z_\Phi(G/H_G)/(R/H_G)$ by [24, Lemma 2.2(4)]. But then $L/R \leq (P/R) \cap Z_\Phi(G/R)$. This implies that $P/R$ is $\Phi$-central in $G$, contrary to (1). Therefore we have $H/H_G = S/H_G$, so $(L/H_G)/(R/H_G) = (H/H_G)(R/H_G)/(R/H_G)$ is a $\tau$-subgroup of $(G/H_G)/(R/H_G)$ and hence $L/R$ is a $\tau$-subgroup of $G/R$. Hence we have (3).

(4) $\tau$ is not quasiregular.

Assume that this is false. In view of (3), $L/R$ is a $\tau$-subgroup of $G/R$. But since $L/R$ is normal in $G$ and $L/R$ is normal in $G/R$ by Lemma 3.4. Hence $L/R = P/R$, which contradicts (1). Hence we have (4).

Now, in view of (4), we have only to prove that $P \leq Z_\Phi(G)$.

(5) There is a maximal subgroup $M$ of $G$ such that $R = P \cap M$ and $MP = G$.

Indeed, if $P/R \leq \Phi(G/R)$, then in view of (1) and [24 Lemma 2.2], $P \leq Z_\Phi(G)$, which contradicts the choice of $(G, P)$. Therefore for some maximal subgroup $M/R$ of $G/R$ we have $(M/R)(P/R) = G/R$. Then $PM = G$ and $M \cap P = R$ since $P/R$ abelian.

Final contradiction.

Let $D = M_G$. Clearly, $R \leq D$. By (3), $LD/D$ is a $\tau$-subgroup of $G/D$. On the other hand, since $L \not\leq R$, $L \not\leq D$ and so $|LD/D| = |L/L \cap D| = |L/R| = p$. Since $\tau$ is $\Phi$-quasiregular, $|G/D : N_{G/D}(LD/D)|$ is a power of $p$. On the other hand, since $L/R$ is normal in $G_p/R$, $LD/D$ is normal in $G_p/D$. Hence in view of Lemma 3.4, $L^G/D = LD$. This induces that $PD = LD$ and $P = L(P \cap D) = LR = L$, which contradicts (1). The proposition is thus proved.

Lemma 4.7 If $G = NT$, where $T$ is a proper subgroup of $G$, $N \leq Z_\infty(G)$ and $N$ is a $p$-group, then $O_p(G) \neq G$.

Proof. Let $T \leq M$ where $M$ is a maximal subgroup of $G$. Then $G/M_G = (NM_G/M_G)(M/M_G)$ is a primitive group and $NM_G/M_G \leq Z_\infty(G/M_G)$. Hence $NM_G/M_G \leq Z(G/M_G)$ and so $M/M_G$ is
normal in $G/M_G$. Therefore $O^p(G) \leq M$.

**Lemma 4.8.** Let $G = NT$, where $N$ is a minimal normal subgroup of $G$ and $T$ is a maximal subgroup of $G$.

1. If $|G : T|$ divides 4, then $N$ is abelian.
2. If $N \leq Z_U(G)$, then $|G : T|$ is a prime.

**Proof.** (1) If $|G : T| = 4$, then $G/T_G$ is isomorphic with a subgroup of the symmetric group $S_4$ of degree 4, which implies that $N \simeq NT_G/T_G$ is abelian. If $|G : T| = 2$, then clearly $N \leq Z(G)$.

2. Assume that $N \leq Z_U(G)$. Since $N$ is a minimal normal of $G$, the order of $N$ is a prime. Thus $|G : T|$ is a prime.

**Theorem 4.9.** Let $\mathfrak{F}$ be the class of all $p$-supersoluble groups. Let $E$ be a normal subgroup $G$ and $P$ a Sylow $p$-subgroup of $E$ of order $|P| = p^n$, where $n > 1$ and $(|E|, p - 1) = 1$. Suppose that every cyclic subgroup of $P$ of prime order or 4 (if $P$ is a non-abelian 2-group) is $\mathfrak{F}_\tau$-supplemented in $G$. If $\tau$ is either hereditary or regular, then $E$ is $p$-nilpotent.

**Proof.** Suppose that this theorem is false and let $(G, E)$ be a counterexample with $|G| + |E|$ minimal. Let $Z = Z_U(G)$.

1. $O_p(E) = 1$ (see (1) in the proof of Theorem 3.6).
2. $\tau$ is regular.

Assume that this is false. Then, by hypothesis, $\tau$ is hereditary. Therefore the hypothesis holds for every subgroup $B$ of $G$ containing $P$. Hence $E = G$ and every maximal subgroup of $G$ is $p$-nilpotent by the choice of $G$. Hence by [32, IV, 5.4], $G = P \times Q$ is a $p$-closed Schmidt group, where $Q$ is a Sylow $q$-subgroup of $G$ ($q \neq p$) and $P$ is of exponent $p$ or exponent 4 (if $P$ is a non-abelian 2-group). But then, by Proposition 4.6, $P \leq Z_U(G) = Z_\infty(G)$ since $(|G|, p - 1) = 1$. Hence $G$ is nilpotent. This contradiction shows that we have (2).

3. $O_p(E) \leq Z_\infty(E)$.

Assume $O_p(E) \neq 1$. Let $H$ be a subgroup of $O_p(E)$ which is $\mathfrak{F}_\tau$-supplemented in $G$. Then there is a subgroup $T/H_G$ of $G/H_G$ such that $(H/H_G)(T/H_G) = G/H_G$ and $(H/H_G) \cap (T/H_G) \subseteq (S/H_G)Z_\mathfrak{F}(T/H_G)$ for some $\tau$-subgroup $S/H_G$ of $G/H_G$ contained in $H/H_G$. Then $(H/H_G) \cap (T/H_G) \subseteq (S/H_G)(Z_\mathfrak{F}(T/H_G) \cap (O_p(E)/H_G)) \leq (S/H_G)Z_U(T/H_G)$. Hence $H$ is $\mathfrak{F}_\tau$-supplemented in $G$. Now since $\tau$ is regular, $O_p(E) \leq Z_U(G)$ by Lemma 4.6. But since $(|E|, p - 1) = 1$, $Z_U(G) = Z_\infty(G)$. Thus $O_p(E) \leq Z_\infty(G) \cap E \leq Z_\infty(E)$.

4. If $V/D$ is a chief factor of $G$ where $D \leq O_p(E)$, then $V \leq O_p(E)$.

Assume that this is false and let $(V, D)$ be the pair with $|V| + |D|$ minimal such that $V/D$ is a chief factor of $G$, $D \leq O_p(E)$ and $V \not\leq O_p(E)$. Then:

(a) $p$ divides $|V/D|$. 

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Assume that $V/D$ is a $p'$-group. Then $V$ is $p$-soluble. By (3), $D \leq Z_\infty(E) \cap V \leq Z_\infty(V)$. Hence $V/C_V(D)$ is a $p$-group by Lemma 3.2. Hence for a Hall $p'$-subgroup $W$ of $V$ we have $V = D \times W$. Then $W$ is characteristic in $V$. Hence $W \leq O_p(E)$, contrary to (1). Therefore we have (a).

(b) $p = 2$, $V/D$ is non-abelian and $O_2(E) \leq Z_\infty(G)$.

Since $V \nleq O_p(E)$, the choice of $(V, D)$ and Claim (a) imply that $V/D$ is non-abelian. But since $(|E|, p - 1) = 1 = (|V|, p - 1)$, by the Feit-Thompson’s theorem we have $p = 2$. By Lemma 4.6, $O_2(E) \leq Z_\infty(G)$.

(c) $V$ has a 2-closed Schmidt subgroup $A$ such that for some cyclic subgroup $H$ of $A$ of order 2 or order 4 we have $H \nleq D$.

In view of (b) and [22] Theorem 3.4.2, $V$ has a 2-closed Schmidt subgroup $A$ and for the Sylow 2-subgroup $A_2$ of $A$ the following hold: (i) $A_2 = A_2^3$; (ii) $A_2$ is of exponent 2 or exponent 4 (if $A_2$ is non-abelian); (iii) $\Phi(A) = Z_\infty(A)$; $A_2/\Phi(A_2)$ is a non-cyclic chief factor of $A$. Therefore for some cyclic subgroup $H$ of $A$ of order 2 or order 4 we have $H \nleq \Phi(A_2)$. But $\Phi(A_2) = Z_\infty(A) \cap A_2$, so $H \nleq Z_\infty(V) \cap A \leq Z_\infty(A)$, which implies that $H \nleq D$ by (3). It is also clear that $H_G \leq D$.

Without loss of generality, we may assume that $H \leq P$, so $H$ is $\mathfrak{F}_\tau$-supplemented in $G$.

For any subgroup $L$ of $G$, we put $\hat{L} = LD/D$.

(d) For any non-identity subgroup $S/H_G$ of $G/H_G$ not contained in $D/H_G$ we have $S/H_G \nleq \tau(G/H_G)$.

Suppose that this is false. Since $\tau$ is regular, $|\hat{G} : N_{\hat{G}}(\hat{H})|$ is a power of 2, and so $\hat{H} \leq O_2(\hat{G})$ by Lemma 3.4. Hence $\hat{V} \leq O_2(\hat{G})$, a contradiction. Thus (d) holds.

(e) For any $\tau$-subgroup $S/H_G$ of $G/H_G$ contained in $H/H_G$ we have $H/H_G \nleq (S/H_G)Z_{\hat{\tau}}(G/H_G)$.

Suppose that $H/H_G \leq (S/H_G)Z_{\hat{\tau}}(G/H_G)$. Then $H/H_G = (S/H_G)((H/H_G)\cap Z_{\hat{\tau}}(G/H_G))$. Since $H/H_G$ is cyclic, either $H/H_G = S/H_G$ or $H/H_G \leq Z_{\hat{\tau}}(G/H_G)$. But the former case is impossible by (d). Hence $H/H_G \leq Z_{\hat{\tau}}(G/H_G)$. But then $(H/H_G)(D/H_G)(D/H_G) \leq Z_{\hat{\tau}}((G/H_G)/(D/H_G))$ by [23] Lemma 2.2(4)]. Hence $V/D \simeq (V/H_G)/(D/H_G)$ is a 2-group, a contradiction. Therefore we have (e).

(f) $O^2(V) = V$.

Assume that $O^2(V) \neq V$. Since $O^2(V)$ is characteristic in $V$, it is normal in $G$. Hence $DO^2(V) = V$. Then in view of the $G$-isomorphism $V/D \simeq O^2(V)/D \cap O^2(V)$, $O^2(V)/D \cap O^2(V)$ is a non-abelian chief factor of $G$ such that $O^2(V) \nleq O_2(V)$ and $D \cap O^2(V) \leq O_2(E)$, which contradicts the choice of $(V, D)$. Hence we have (f).

final contradiction.

Since $H$ is $\mathfrak{F}_\tau$-supplemented in $G$, there is a subgroup $T/H_G$ of $G/H_G$ such that $(H/H_G)(T/H_G) = G/H_G$ and $(H/H_G) \cap (T/H_G) \subseteq (S/H_G)Z_{\hat{\tau}}(T/H_G)$ for some $\tau$-subgroup $S/H_G$ of $G/H_G$ contained in $H/H_G$. 

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Suppose that $T = G$. Then $(H/H_G) \leq (S/H_G)Z_\mathfrak{F}(G/H_G)$, which contradicts (e). Therefore $T \neq G$.

Since $HT = G$, $|G : T|$ divides 4. Let $T \leq M$, where $M$ is a maximal subgroup of $G$. Then $V \nleq M$. Suppose that $|G : M| = 2$. Then from the isomorphism $G/M \simeq V/V \cap M$ we get $O^2(V) \neq V$, which contradicts (f). Hence $M = T$ is a maximal subgroup of $G$ and $|G : T| = 4$.

Note that if $TD = G$, then $V = D(T \cap V)$. Hence $O^2(V) \neq V$ by Lemma 4.7, contrary to (f). Therefore $TD \neq G$, which in view of maximality of $T$ implies that $D \leq T$. Therefore $\tilde{G} = \tilde{H}\tilde{T} = \tilde{V}\tilde{T}$. Then by Lemma 4.8, $\tilde{V}$ is abelian, which contradicts (b). This final contradiction completes the proof.

**Corollary 4.10.** Let $E$ be a non-identity normal subgroup $G$. Suppose that for every non-cyclic Sylow subgroup $P$ of $E$, every cyclic subgroup of $P$ of prime order or 4 (if $P$ is a non-abelian 2-group) is $\Delta_\tau$-supplemented in $G$ for some hereditary quasiregular subgroup functor $\tau$. Then $E \leq Z_\mathfrak{F}(G)$.

**Proof.** See the proof of Corollary 3.7 and use Proposition 4.6 and Theorem 4.9 instead of Proposition 3.5 and Theorem 3.6, respectively.

**Proof of Theorem B.** Suppose that this theorem is false and let $(G, E)$ be a counterexample with $|G| + |E|$ minimal. Then $E = G^{\mathfrak{F}}$. Let $p$ be the smallest prime dividing $|X|$. Then $X$ is $p$-nilpotent by Theorem 4.9 and [28, Chapter 7, 6.1]. Hence $X$ is soluble by the Feit-Thompson’s theorem. Let $V$ be the Hall $p'$-subgroup of $X$. Then $V$ is characteristic in $X$ and so it is normal in $G$.

Suppose that Assertion (i) is true. Then the choice of $G$ implies that $E \nleq Z_\mathfrak{F}(G)$ and so $V \neq 1$ by Proposition 4.6. It is clear that the hypothesis holds for $(G/V, E/V)$ by Lemma 2.2. Hence $G/V \in \mathfrak{F}$ by the choice of $(G, E)$. Then the hypothesis holds for $(G, V)$. Hence the choice of $(G, E)$ implies that $G \in \mathfrak{F}$, a contradiction. Hence at least one of the Assertions (ii) or (iii) is true. In this case $\tau$ is either hereditary quasiregular or regular. By Corollary 4.10, $X \leq Z_{\Delta}(G)$. It follows from [26, Proposition C] that $E \leq Z_{\Delta}(G)$. Then $G \in \mathfrak{F}$. This contradiction completes the proof.

### 5 Final remarks

I. A large number of known results are corollaries of Theorems A and B. In particular, in view of Example 1.4, Theorem A covers Theorem D in [7] and Theorem 4.7 in [33]; in view of Example 1.5, Theorems B covers Theorem 1.3 in [34]; in view of the remarks after Definition 1.2, Theorems A and B cover Theorems 1.2 and 1.4 in [35], Theorem 3.1 and 3.2 in [5] and Theorem 3.3 in [36]; in view of Example 1.6 and the remarks after Definition 1.2, Theorems A and B cover Theorem 4.1 in [37], Theorems 1.1 and 1.2 in [38], Theorems 3.1, 3.4 and 3.6 in [39], Theorem A in [26] and Theorems A and B in [40]; in view of Example 1.7, Theorems A and B cover Theorems 5.1 and 5.2 in [15]; in view of Example 1.8, Theorem B covers Theorems 3.1 and 3.2 in [11]; in view of Example 1.9, Theorem A and B cover Theorem 1.3 in [11] and Theorem 1.2 in [42]; in view of Example 1.10, Theorems A and B cover Theorems 3.1, 3.4-3.7 in [13] and so on.
II. We do not know now the answer to the following

**Question B.** Is the subgroup functor in (1.5) regular? (see Question 17.112 in [44]).

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