Statistical Mechanics and Black Hole Thermodynamics
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Black holes are thermodynamic objects, but despite recent progress, the ultimate statistical mechanical origin of black hole temperature and entropy remains mysterious. Here I summarize an approach in which the entropy is viewed as arising from “would-be pure gauge” degrees of freedom that become dynamical at the horizon. For the (2+1)-dimensional black hole, these degrees of freedom can be counted, and yield the correct Bekenstein-Hawking entropy; the corresponding problem in 3+1 dimensions remains open.

It has been nearly 25 years since Bekenstein and Hawking first demonstrated that black holes are thermodynamic objects, characterized by a temperature and an entropy \[1,2\]. Despite considerable effort, however, the underlying statistical mechanical source of these thermal properties is not yet understood. Recent progress in string theory notwithstanding \[3\], the roots of black hole entropy remain mysterious.

Here I would like to describe a new approach to this problem, developed over the past few years by several groups \[4–7\]. This approach attributes black hole statistical mechanics to a collection of previously unappreciated quantum gravitational degrees of freedom, “would-be pure gauge” excitations that would normally be discarded as unphysical, but that become dynamical at the black hole horizon. The analysis has been developed most fully for the (2+1)-dimensional black hole of Bañados, Teitelboim, and Zanelli \[8\], but there has been a bit of progress in extending the results to other dimensions.

It is appropriate that I should present this work at a conference that honors Tullio Regge. Regge’s work touches on this subject in two important ways. First, he and Claudio Teitelboim were the first to recognize the importance of (2+1)-dimensional general relativity as a model for realistic quantum gravity, and to analyze its degrees of freedom in the Chern-Simons formalism \[10,11\]. The approach I present here can be viewed as a combination of these two pieces of work, albeit in a slightly novel context.

1. EDGE DEGREES OF FREEDOM

We can gain some insight into the problem of black hole entropy by considering the simpler problem of black hole mass. There is no doubt that black holes have mass. But until 1974, it was widely accepted that the Hamiltonian of general relativity was simply a sum of constraints, and therefore vanished on physical states. Where, then, could black hole mass come from?

In a seminal paper, Regge and Teitelboim resolved this problem by showing that the Hamiltonian of general relativity must include boundary terms at spatial infinity \[9\]. Consider, for instance, the role of spatial diffeomorphisms in canonical gravity. Let \(\Sigma\) denote a constant time hypersurface, perhaps with boundary. An infinitesimal diffeomorphism of \(\Sigma\) acts on the spatial metric \(g_{ij}\) as

\[
\delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i
\]

\[
= \left\{ 2 \int_\Sigma \nabla_i \xi_k \pi^{kl}, g_{ij} \right\} = \left\{ \int_\Sigma \xi_k \mathcal{H}^k, g_{ij} \right\}
\]

where \(\mathcal{H}^k\) is the momentum constraint of canonical gravity,

\[
\mathcal{H}^k = -2\nabla_i \pi^{kl}.
\]
But the momentum constraint vanishes on physical states, so the last term in (3) is zero. Canonical gravity thus predicts that physical states are diffeomorphism-invariant: the metrics $g_{ij}$ and $g_{ij} + \delta g_{ij}$ are indistinguishable.

Note, however, that the last equality of equation (3) involves a partial integration, which can potentially introduce a boundary term. Indeed, the final Poisson bracket in (3) may not be well-defined: we should really write

$$\delta g_{ij} = \left\{ \left( \int_{\Sigma} \xi_k \mathcal{H}^k + 2 \int_{\partial \Sigma} \xi_k \pi^{k\perp} \right), g_{ij} \right\}. \quad (3)$$

The momentum constraint has thus acquired a boundary term, which need not vanish on physical states. The Hamiltonian constraint is slightly more difficult to analyze, but it, too, picks up a boundary term, which need not vanish on physical states. New boundary degrees of freedom have appeared, of the form

$$\delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i, \quad \xi^i \big|_{\partial \Sigma} \neq 0, \quad (4)$$

which can no longer be discarded as "pure gauge." The existence of such new degrees of freedom was already recognized by Regge and Teitelboim, who wrote of "a new set of canonical pairs which describe the asymptotic location of the spacelike surface on which the state is defined."

While this argument is clearest in the Hamiltonian formalism, a Lagrangian version also exists. A fluctuation of the spacetime metric $g_{\mu\nu}$ may be decomposed as

$$\delta g_{\mu\nu} = (K \xi)_{\mu\nu} + h_{\mu\nu}, \quad (K^\dagger h)_{\mu} = 0 \quad (5)$$

with

$$(K \xi)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad (6)$$

where $\nabla_\mu$ is now the full spacetime covariant derivative. For a closed manifold, this splitting is unique, and provides the standard division into "physical" and "gauge" degrees of freedom [12][13]. If $M$ has a boundary, however, a unique decomposition requires boundary conditions that make $K^\dagger K$ self-adjoint. The simplest choice is

$$\xi^\mu \big|_{\partial \Sigma} = 0. \quad (7)$$

Just as in the Hamiltonian formalism, the "would-be gauge" degrees of freedom

$$\delta g_{\mu\nu} = (K \xi)_{\mu\nu}, \quad \xi^\mu \big|_{\partial \Sigma} \neq 0, \quad (8)$$

become dynamical at the boundary.

2. HORIZONS AS BOUNDARIES

The approach to black hole thermodynamics I am advocating is based on these same degrees of freedom, now pushed inward to the black hole horizon. The obvious objection is that a horizon is not a boundary. This is certainly true. Nevertheless, an event horizon in quantum gravity is a location at which one imposes "boundary conditions," and these are sufficient to require the introduction of boundary terms.

Consider, for example, a question about black hole radiation. In semiclassical gravity, one can ask, "Here is a metric. What is the probability of observing Hawking radiation with a given spectrum?" In a full quantum theory, however, such a question makes no sense—the metric is a quantum variable, and cannot be fixed in advance. Moreover, if one is only interested in the region near the horizon, the metric far from the black hole should be irrelevant. The appropriate question is thus, "Suppose the metric satisfies geometric conditions that represent the existence of a horizon with given characteristics. Then what is the probability of observing Hawking radiation with a given spectrum?" This is a question about conditional probability, and the condition—the existence of a horizon with certain geometric properties—is a boundary condition.

This condition can perhaps be best understood in a path integral formalism. The simplest way to impose such a requirement is to split the spacetime $M$ into two pieces, $M_1$ and $M_2$, along a hypersurface $\Sigma$, the putative event horizon. If $h$
denotes the metric on $\Sigma$, the total partition function is, schematically,

$$Z_M = \int [dh] Z_{M_1}[h] Z_{M_2}[h],$$

(9)

where $Z_{M_1}[h]$ and $Z_{M_2}[h]$ are the partition functions for $M_1$ and $M_2$ with the specified induced metric $h$ on $\Sigma$, and the integral (9) is restricted to boundary metrics that satisfy the required conditions for $\Sigma$ to be a horizon.

The question is now whether the actions used to compute $Z_{M_1}[h]$ and $Z_{M_2}[h]$ should include boundary terms. This can be answered by considering the requirement of “sewing”; if the range of integration in (9) is extended to include all intermediate metrics on $\Sigma$, the result should be equivalent to the ordinary path integral over $M$, independent of $\Sigma$. This sewing condition has been examined for a number of exactly soluble systems, including free fields [14] and Chern-Simons theories [15], and in all cases it has been shown that the action must include boundary terms, guaranteeing the appearance of “would-be pure gauge” degrees of freedom at the horizon.

3. CHERN-SIMONS THEORY

To make this discussion less abstract, let us look at the best-understood example, Chern-Simons gauge theory. Let $A_\mu = A_\mu^a T_a$ be a gauge field for a nonabelian group $G$, defined on a three-manifold $M$ with boundary. Fix a complex structure on the surface $\partial M$. The Chern-Simons action is then

$$I_{CS} = \frac{k}{4\pi} \int_M Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$+ \frac{k}{4\pi} \int_{\partial M} Tr A_z A_{\bar{z}},$$

(10)

where the boundary term is the one appropriate for fixing the component $A_z$ at $\partial M$.

The equations of motion arising from this action are

$$F_{\mu\nu} = 0,$$

(11)

where $F$ is the field strength. On a closed topologically trivial manifold, Chern-Simons theory thus has no degrees of freedom. If $M$ has a non-trivial fundamental group, on the other hand, the model possesses global degrees of freedom, corresponding to Wilson loops or Aharonov-Bohm phases around noncontractible loops.

The action (10) depends explicitly on the potential, and is not manifestly gauge invariant. However, a simple computation shows that under a transformation

$$A = g^{-1} dg + g^{-1} A g,$$

(12)

the action becomes

$$I_{CS}[A] = I_{CS}[\tilde{A}] + k I_{WZW}^{+}[g, \tilde{A}],$$

(13)

where $I_{WZW}^{+}[g, \tilde{A}]$ is the action of a chiral Wess-Zumino-Witten model on $\partial M$,

$$I_{WZW}^{+}[g, \tilde{A}] = \frac{1}{4\pi} \int_{\partial M} Tr \left( g^{-1} \partial_z g^{-1} \partial_{\bar{z}} g - 2g^{-1} \partial_{\bar{z}} g \tilde{A} \right)$$

$$+ \frac{1}{12\pi} \int_M Tr (g^{-1} dg)^3.$$

(14)

If $M$ is closed, the first term in (14) disappears, and the second is a topological invariant, the winding number of the gauge transformation $g : M \to G$. For a suitably choice of $k$, this term always contributes an integral multiple of $2\pi$, so $\exp\{i I_{CS}[A]\}$ is indeed gauge invariant.

In the presence of a boundary, however, this invariance is lost, and the “would-be pure gauge” degrees of freedom become dynamical on the boundary, with an action given by the WZW action (14). These new degrees of freedom are closely related to those described in the first section. Indeed, recall that the Lie derivative of the one-form $A_\mu dx^\mu$ satisfies the identity

$$\mathcal{L}_{\xi} A = d(\iota_\xi A) + \iota_\xi dA,$$

(15)

where $\iota_\xi$ denotes the interior product. It is then easy to show that

$$\mathcal{L}_{\xi} A = D(\iota_\xi A) + \iota_\xi F,$$

(16)

where $D$ is the gauge-covariant derivative. On shell, the field strength $F$ vanishes, and a diffeomorphism, the left-hand side of (16), is equivalent to a gauge transformation, the right-hand side. The boundary diffeomorphisms of section 1 are thus equivalent, at least on shell, to the dynamical gauge transformations of this section.


4. (2+1)-DIMENSIONAL GRAVITY

Chern-Simons theory is a fascinating model, but we are really interested in gravity. In three spacetime dimensions, however, we need look no further: as Achúcarro and Townsend observed in 1986 [16] and Witten spectacularly rediscovered a few years later [17], (2+1)-dimensional general relativity is a Chern-Simons theory. In particular, for Lorentzian gravity with a negative cosmological constant \( \Lambda = -1/\ell^2 \), we can define an \( \text{SU}(1, 1) \times \text{SU}(1, 1) \) gauge field

\[
A^\pm = \left( \omega^a \pm \frac{1}{\ell} e^a \right) T_a, \tag{17}
\]

where \( \omega^a = \frac{1}{k} \epsilon^{abc} \omega_{\mu bc} dx^\mu \) is the spin connection and \( e^a = e^\mu_a dx^\mu \) is the triad. The standard first-order form of the Einstein action can then be written as

\[
I_{\text{grav}} = I_{\text{CS}}[A^+] - I_{\text{CS}}[A^-], \tag{18}
\]

where \( I_{\text{CS}}[A] \) is the Chern-Simons action [10] with a coupling constant

\[
k = -\frac{\ell}{4 G}. \tag{19}
\]

As in a general Chern-Simons theory, the physical degrees of freedom of this model are Wilson loops

\[
R^\pm_\gamma = \text{Tr} \exp \int_\gamma A^\pm \tag{20}
\]

around closed noncontractible paths \( \gamma \). Nelson and Regge have studied the algebra of these observables extensively [11,12], and it is clear that they do not provide enough degrees of freedom to account for the entropy of a (2+1)-dimensional black hole. But by the discussion of the preceding section, we also expect an \( \text{SU}(1, 1) \times \text{SU}(1, 1) \) WZW action to be induced at the horizon of a black hole. The degrees of freedom provided by this action are our candidates for explaining black hole statistical mechanics.

At first sight, we have been too successful: a WZW model has an infinite number of degrees of freedom, not the finite number needed to account for black hole entropy. We must be careful, however, about which states we count as physical. Recall that in the metric formalism, the new physical excitations are given by equation (3). Not all boundary diffeomorphisms appear in this equation: if \( \chi \) satisfies the Killing equation \( K \chi = 0 \) at \( \partial M \), the right-hand side of (3) vanishes, and the corresponding constraint \( \int \chi^a H_a \) remains a genuine constraint even at \( \partial M \). In other words, a remnant of the Wheeler-DeWitt equation survives at the boundary: states must be invariant under those diffeomorphisms that reduce to isometries at the horizon.

We can now proceed to count states. I will only sketch the argument here; the reader is referred to references [3] and [4] for details. Note first that a WZW model is a conformal field theory, and that diffeomorphisms of \( \partial M \) are therefore described by Virasoro operators \( L_n \) and \( \bar{L}_n \), whose properties are well understood. In particular, the isometries of the horizon are rigid rotations and time translations, which are generated by \( L_0 \) and \( \bar{L}_0 \), so the physical state condition is

\[
L_0|\text{phys}\rangle = \bar{L}_0|\text{phys}\rangle = 0. \tag{21}
\]

For convenience, let us analytically continue from our SU(1, 1) \( \times \) SU(1,1) WZW model with \( k < 0 \) to the better understood SL(2, C) model with \( k > 0 \). Let \( \bar{A} \) denote the boundary values of the gauge field \( A \) at the horizon, which may be determined from the Chern-Simons form of the classical Euclidean black hole solution [18]. The partition function for this model,

\[
Z_{\text{SL}(2, C)}(\tau) |\bar{A}, \bar{\bar{A}}\rangle = \text{Tr} \left\{ e^{2\pi i \tau L_0} e^{-2\pi i \bar{\bar{L}}_0} \right\}, \tag{22}
\]

is known from conformal field theory, and can be expressed in terms of Weyl-Kac characters for affine SU(2) [14,21]. Moreover, standard results from WZW theory tell us that

\[
Z_{\text{SL}(2, C)}(\tau) |\bar{A}, \bar{\bar{A}}\rangle = \sum \rho(N, \bar{N}) q_1^{N-\bar{N}} q_2^{N+\bar{N}}, \tag{23}
\]

where \( q_1 = e^{2\pi i \tau_1} \), \( q_2 = e^{-2\pi i \tau_2} \), and \( \rho(N, \bar{N}) \) is the number of states for which the Virasoro generators \( L_0 \) and \( \bar{L}_0 \) have eigenvalues \( N \) and \( \bar{N} \). The number of states satisfying the physical state condition (21) is thus \( \rho(0, 0) \), which can be extracted from (23) by contour integration.
This computation is carried out in reference [3]. The outcome is that
\[
\ln \rho(0,0) = \frac{2\pi r_+}{4G} + \frac{\pi r_+}{\ell} + \ldots, \tag{24}
\]
where \(r_+\) is the radius of the event horizon. The first term in this expression is precisely the right Bekenstein-Hawking entropy, while the second is a one-loop correction. Our counting argument has thus succeeded. A similar computation can be performed directly in Lorentzian signature, again yielding the correct entropy [5].

5. THE REAL WORLD

The evidence from 2+1 dimensions is certainly suggestive, but it is not conclusive. The obvious question is whether these results can be generalized to 3+1 dimensions. In this simple form, they certainly cannot. The Chern-Simons formulation of (2+1)-dimensional gravity allowed us to trade the complicated diffeomorphism group for a much simpler gauge group, via equation (16). No such procedure is known in 3+1 dimensions, and we have no simple splitting of the action into “bulk” and “boundary” terms comparable to that of equation (13). On the other hand, the arguments of section 1 hold in any number of dimensions. There are certainly “would-be pure gauge” degrees of freedom in 3+1 dimensions; the problem is that we do not know how to count them.

One interesting place to test these ideas is (1+1)-dimensional dilaton gravity. It is not too hard to show that a suitable choice of boundary conditions induces a dynamical theory on the horizon of a (1+1)-dimensional black hole, but no analog of the physical state conditions [14] is yet known. It is also possible that state-counting arguments in the (3+1)-dimensional loop representation are looking at these “would-be pure gauge” degrees of freedom [23], but the connection is still somewhat speculative.

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