TENSOR PRODUCT OF CYCLIC $A_{\infty}$-ALGEBRAS AND THEIR KONTSEVICH CLASSES

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ABSTRACT. Given two cyclic $A_{\infty}$-algebras $A$ and $B$, we prove that there exists a cyclic $A_{\infty}$-algebra structure on their tensor product $A \otimes B$ which is unique up to a cyclic $A_{\infty}$-quasi-isomorphism. Furthermore, the Kontsevich class of $A \otimes B$ is equal to the cup product of the Kontsevich classes of $A$ and $B$ on the moduli space of curves.

1. Introduction

1.1. Tensor products. Let $A$ and $B$ be two associative algebras with product maps $m^A_2$ and $m^B_2$. Then there is a natural associative algebra structure on $A \otimes B$ defined by

$$m^\otimes_2(a_1 \otimes b_1, a_2 \otimes b_2) := m^A_2(a_1, a_2) \otimes m^B_2(b_1, b_2).$$

This tensor product construction is much less obvious if one replaces associative algebras by their homotopy version: $A_{\infty}$-algebras, introduced by Stasheff [18]. Explicitly, this structure consists of a graded vector space $A$ and a family of multi-linear maps of degree $k-2$

$$m_k : A \otimes^k \to A$$

for each $k \geq 1$, which satisfy a certain homotopy version of associativity. In this case, the naive definition of a tensor product structure

$$m^\otimes_k := m^A_k \otimes m^B_k$$

is clearly false, simply by degree considerations.

In [16], Saneblidze and Umble constructed a tensor product structure on $A \otimes B$ whose structure maps $m^\otimes_k$ are given by a beautiful formula involving various compositions of $m_j$ ($j \leq k$) in each tensor component. Later in [12], Markl and Shnider interpreted the Saneblidze-Umble’s formula as giving a diagonal map on the associahedra (the cellular complex which governs $A_{\infty}$ algebras). Their construction used cubical decompositions of the associahedra. A similar approach, using simplicial decompositions of the associahedra instead, was carried out by Loday [11].

In this paper, we are interested in the tensor product of $A_{\infty}$ algebras with an additional structure: a cyclic inner product. Namely, a cyclic $A_{\infty}$ algebra is an $A_{\infty}$ algebra $A$, together with a non-degenerate inner product $\langle , \rangle$ such that the expression

$$\langle m_k(a_0, \ldots, a_{k-1}, a_k) \rangle$$

is invariant under cyclic permutations of the indices $0, 1, \ldots, k$ (up to a sign).

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In the case when the only nonzero structure map is $m_2$, this is equivalent to the notion of a Frobenius algebra. Given two Frobenius algebras $A$ and $B$, the natural tensor product algebra structure $m_2^\otimes$ on $A \otimes B$ is again Frobenius with respect to the natural inner product defined by

\[
\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle_{A \otimes B} := (-1)^{|b_1||a_2|} \langle a_1, a_2 \rangle_A \langle b_1, b_2 \rangle_B.
\]

Thus a natural question is whether Saneblidze-Umble’s formula $m_k^\otimes$ on the tensor product is cyclic with respect to the pairing above. Unfortunately, this is not the case. As was observed in [19], the third product $m_3^\otimes$ of Saneblidze-Umble is already not cyclic.

In [3], Cho introduced the notion of strong homotopy inner product, which is a cyclic $A_\infty$-algebra up to homotopy. It consists of an infinity inner product as defined by Tradler (see [19], for a definition) satisfying some additional properties, in particular being closed. He also shows that an $A_\infty$-algebra with a strong homotopy inner product, is quasi-isomorphic to a cyclic $A_\infty$-algebra. In [19], the authors show that the tensor product of cyclic $A_\infty$-algebras admits an infinity inner product, but did not check if it is closed. Therefore we cannot use Cho’s result to conclude the tensor product is cyclic up to quasi-isomorphism.

In this paper, we take a different approach to this problem. Namely, we use a different definition of tensor product and we prove the following

**Theorem 1.1.** Let $A$ and $B$ be two cyclic $A_\infty$ algebras. Then there exists a cyclic $A_\infty$ algebra structure on $A \otimes B$ with respect to the inner product defined by Formula 1. Furthermore, this structure is uniquely defined up to cyclic homotopy.

The theorem is proved by showing that the cyclic differential graded operad $A_\infty$ admits a cyclic diagonal morphism

\[ \Delta : A_\infty \to A_\infty \otimes A_\infty \]

which is unique up to homotopy of morphisms between cyclic operads. Our proof is to inductively construct $\Delta$ by using the simple fact that the associahedra are contractible. Unlike the case of tensor products without cyclic structure, we do not obtain a general formula for $m_k^\otimes$. See Section 3.3 for a description of $m_k^\otimes$ for $k \leq 4$.

**1.2. Kontsevich classes.** It is well-known that the associative graph homology complex $G_*$ with $\mathbb{Q}$ coefficients computes the rational homology of $\bigsqcup_{g,n} \mathcal{M}_{g,n}$, the union of the moduli spaces of smooth Riemann surfaces of genus $g$ and $n$ unlabelled boundary components. In [9], Kontsevich associated a cocycle $c_A$ on the complex $G_*$, to a given cyclic $A_\infty$ algebra whose inner product is even and symmetric. Thus the cocycle $c_A$ also defines a cohomology class $[c_A] \in H^*(\bigsqcup \mathcal{M}_{g,n}, K)$ if the algebra $A$ is defined over $K$, a field of characteristic zero. In the case when the inner product is odd, but still symmetric, one can define a class $[c_A] \in H^*(\bigsqcup \mathcal{M}_{g,n}, \text{det})$ where $\text{det}$ is a natural local system on $\bigsqcup \mathcal{M}_{g,n}$ whose fiber over a Riemann surface $\Sigma$ of type $(g,n)$ is

\[ \text{det} H^1(\Sigma, K), \]

the determinant of the first cohomology group of $\Sigma$. From the definition, it is clear that $\text{det}^{\otimes 2}$ is trivial.

In summary, for a cyclic $A_\infty$ algebra $A$ whose inner product is of parity $\epsilon$, there is an associated class $c_A \in H^*(\bigsqcup \mathcal{M}_{g,n}, \text{det}^{\otimes \epsilon})$. We can now state our second main
Theorem 1.2. Let $A$ and $B$ be two cyclic $A_\infty$ algebras over a field $K$ of characteristic zero. Assume that the inner products on $A$ and $B$ are both symmetric and of parity $\epsilon_1$ and $\epsilon_2$ respectively. Then we have

$$[c_{A \otimes B}] = [c_A] \cup [c_B]$$

where $\cup : H^*(\prod M_{g,n}, \det^{\otimes \epsilon_1}) \otimes H^*(\prod M_{g,n}, \det^{\otimes \epsilon_2}) \to H^*(\prod M_{g,n}, \det^{\otimes \epsilon_1+\epsilon_2})$ is the cup product map on cohomology with local coefficients.

The major part of the proof of this result is to construct a cellular “diagonal”

$$\delta : G_* (\det^{\otimes \epsilon_1+\epsilon_2}) \to G_* (\det^{\otimes \epsilon_1}) \otimes G_* (\det^{\otimes \epsilon_2})$$
on the graph homology complex (twisted by appropriate local systems). For this, we need to fix a cyclic diagonal $\Delta$ for $A_\infty$. Then we define $\delta$ applied to an oriented graph $(\Gamma, \sigma)$ to be a kind of direct product of $\Delta$ applied to every vertex of $\Gamma$. Then we show that, if one uses $\Delta$ to define the cyclic $A_\infty$ structure on $A \otimes B$, there is a chain level equality $c_{A \otimes B} = (c_A \otimes c_B) \circ \delta$, which implies the theorem.

1.3. Contents. The paper is organized as follows. In Section 2, we review basic definitions about $A_\infty$ algebras and the operad $A_\infty$. In Section 3, we show the existence and uniqueness of a cyclic diagonal of $A_\infty$, and use it to prove Theorem 1.1. Section 4 contains the construction of a diagonal $\delta : G_* \to G_* \otimes G_*$, which is then used to prove Theorem 1.2.

1.4. Conventions. Throughout the paper we work over the field $\mathbb{Q}$. This is just for simplicity, all the results work over any field of characteristic 0. For multi-linear operations with graded vector spaces, Koszul sign convention is assumed. We work with homological complexes where the differentials are of degree $-1$. We would also like to point out that everything in this paper applies to the $\mathbb{Z}/2$-graded case. Finally, in this paper, an inner product on a graded finite dimensional vector space is always assumed to be symmetric and either even or odd. More precisely, we require that, given homogeneous elements $a$ and $b$

$$\langle a, b \rangle = (-1)^{|a||b|} \langle b, a \rangle,$$

$$\langle a, b \rangle \neq 0 \Rightarrow |a| \equiv |b| + \epsilon \pmod{2}$$

where $\epsilon = 0$ in the even case and $\epsilon = 1$ in the odd case.

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2. The $A_\infty$ operad

In this section, we recall basic definitions and facts about the operad $A_\infty$. The book [13] is a good reference for this material.

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1This is, for example, necessary for $A_\infty$ algebras appearing in Lagrangian Floer Theory.
2.1. **The \( A_\infty \) operad.** Stasheff introduced in [18], for each \( n \geq 2 \), a cellular complex \( K(n) \) of dimension \( n - 2 \). They are known as the associahedra or the Stasheff polytopes. The \( A_\infty \)-operad is a non-symmetric, non-unital operad on the category of differential graded vector spaces, defined by setting

\[
A_\infty(n) := C_*(K(n), \mathbb{Q}), \quad n \geq 2,
\]

where \( C_*(K(n), \mathbb{Q}) \) is the cellular chain complex of the \( n \)-th Stasheff polytope \( K(n) \). Cells in \( K(n) \) of dimension \( k \) are in one-to-one correspondence with planar, rooted trees with \( n \) leaves and \( n - k - 2 \) internal edges.

To describe the differential on \( A_\infty(n) \) in terms of planar trees, one needs to define appropriate orientations on them. Following [12], we define an orientation of a tree \( T \) with internal edges \( e_1, \ldots, e_k \), to be an orientation of the vector space spanned by \( \{e_1, \ldots, e_k\} \). With this notation we define \( A_\infty(n) \) to be the vector space generated by pairs \( (T, \omega) \) where \( T \) is a planar rooted tree with \( n \) leaves and \( \omega \) is an orientation of \( T \), together with the relation \( (T, -\omega) = - (T, \omega) \). The operadic composition

\[
\circ_i : C_k(K(n_1)) \otimes C_l(K(n_2)) \to C_{k+l}(K(n_1 + n_2 - 1))
\]

is defined by

\[
(U, \sigma) \circ_i (V, \tau) = (-1)^{i(n_2+1)+n_1} (U \circ_i V, \sigma \wedge \tau \wedge e)
\]

where \( U \circ_i V \) is the tree obtained by grafting the root of \( V \) to the \( i \)-th leaf of \( U \) and \( e \) is the new internal edge created by grafting.

The differential

\[
\partial : C_k(K(n)) \to C_{k-1}(K(n))
\]

is defined by

\[
\partial(U, \sigma) = \sum_{\{V | V/e=U\}} (V, e \wedge \sigma)
\]

with the sum taken over all trees \( V \) with an edge \( e \) such that when we collapse \( e \) on \( V \) we obtain \( U \).

It is a well-known fact that, as an operad of graded vector spaces, \( A_\infty \) is freely generated by the corollas \( c_k \in A_\infty(k) \), the trees with only one internal vertex.

2.2. **Cyclic structures.** The notion of a cyclic structure on an operad \( O \) was introduced in [5]. In the context of non-symmetric operads, this structure reduces to an action of the cyclic group \( \mathbb{Z}/(n+1)\mathbb{Z} \) on \( O(n) \) satisfying certain compatibility conditions. We refer to [13] or [5] for a precise definition. It was proved in [5] that \( A_\infty \) is cyclic. Denote by \( r \) a generator of \( \mathbb{Z}/(n+1)\mathbb{Z} \). The action of \( r \) on \( A_\infty(n) \) determined by setting

\[
r(c_n) = (-1)^n c_n
\]

and using the rule

\[
 r((U, \sigma) \circ_i (V, \tau)) = \begin{cases} r((U, \sigma)) \circ_{i+1} (V, \tau), & i < k, \\ (-1)^{U||V} r((V, \tau)) \circ_1 r((U, \sigma)), & i = k, \end{cases}
\]

where \( U \in A_\infty(k) \).
Geometrically, the operator $r$ acts on a tree $T$ by rotating it in the counterclockwise direction. That is, we relabel the leaves and root of $T$ so that the rightmost leaf of $T$ becomes the root of $r(T)$. See the following picture for an example:

By induction on the number of vertices, one can deduce the following formula.

**Lemma 2.1.** Assume that $(T, \sigma) \in \mathcal{A}_\infty(n)$. Then we have

$$r((T, \sigma)) = (-1)^n (r(T), \sigma)$$

where $r(T)$, as in the previous example, is the ribbon tree obtained from $T$ by a counterclockwise rotation.

2.3. $\mathcal{A}_\infty$-algebras. Given a differential graded vector space $A$ we can define the operad $\text{End}_A$ by taking

$$\text{End}_A(n) = \text{Hom}(A^\otimes n, A)$$

and defining the operadic composition and differential as follows:

$$f \circ_i g(a_1, \ldots, a_n) = (-1)^{|g|\sum_{i=1}^{n-1}|a_i|} f(a_1, \ldots, g(a_i, \ldots, a_{i+j-1}), \ldots, a_n),$$

$$\partial f(a_1, \ldots, a_n) = df(a_1, \ldots, a_n) - \sum_{i=1}^n (-1)^{|f|+\sum_{l=1}^{i-1}|a_l|} f(a_1, \ldots, da_i, \ldots, a_n),$$

where $d$ is the differential on $A$.

**Definition 2.2.** An $A_\infty$-algebra structure $(A, \rho)$ on $A$ is an operad homomorphism

$$\rho : \mathcal{A}_\infty \to \text{End}_A.$$

Recall that $\mathcal{A}_\infty$ is generated as an operad by $c_j$, $j \geq 2$. Thus $\rho$ is equivalent to the choice of maps $m_k : A^\otimes k \to A$ of degree $k-2$, with $m_1 = d$, satisfying, for each $n$, $n \geq 2$,

$$\sum_{i,j} (-1)^{i(j+1)+jn+\sum_{l=1}^{i-1}|a_l|} m_{n-j+1}(a_1, \ldots, m_j(a_i, \ldots, a_{i+j-1}), \ldots, a_n) = 0.$$

Here $m_k = \rho(c_k)$, for $k \geq 2$ and the above equation is a consequence of the relation

$$\partial(c_n) = \sum_{i,j} (-1)^{i(j+1)+jn} c_{n-j+1} \circ_i c_j.$$

2.4. Cyclic $A_\infty$ algebras. Let $A$ be a finite dimensional differential graded vector space with a non-degenerate inner product $\langle \cdot, \cdot \rangle$. Furthermore, we assume that the differential $d$ on $A$ is (graded) self-adjoint:

$$\langle da_0, a_1 \rangle = (-1)^{|a_0|+1} \langle a_0, da_1 \rangle.$$
This is equivalent to requiring that \( \langle, \rangle : A \otimes A \rightarrow Q \) is a chain map. Then the endomorphism operad \( \text{End}_A \) is a cyclic operad (see [5]). Indeed, the inner product induces a natural identification
\[
\text{Hom}(A^{\otimes n}, A) \cong (A^V)^{\otimes n+1},
\]
where \( A^V = \text{Hom}(A, Q) \) is the dual vector space of \( A \). Since the group \( \mathbb{Z}/(n+1)\mathbb{Z} \) acts on the right hand side by permuting factors, we also get an action on \( \text{Hom}(A^{\otimes n}, A) \) via the above identification.

A cyclic \( A_\infty \)-algebra structure on \( A \) is then given by an operad homorphism
\[
\rho : A_\infty \rightarrow \text{End}_A
\]
which is compatible with the cyclic structure. More explicitly, the following equation must hold for any \((T, \sigma) \in A_\infty(n)\)
\[
\langle \rho((T, \sigma))(a_0, \ldots, a_{n-1}), a_n \rangle = (-1)^{|a_0| \sum_{i=1}^{n} |a_i|} \langle \rho((T, \sigma))(a_1, \ldots, a_n), a_0 \rangle.
\]
Again, since \( A_\infty(n) \) is generated by corollas, this is equivalent to
\[
\langle m_n(a_0, \ldots, a_{n-1}), a_n \rangle = (-1)^{n + |a_0| \sum_{i=1}^{n} |a_i|} \langle m_n(a_1, \ldots, a_n), a_0 \rangle.
\]

3. Tensor product of cyclic \( A_\infty \)-algebras

In this section, we prove our first main result, Theorem 1.1. We will do this by showing the existence and uniqueness (up to homotopy) of a cyclic diagonal of the operad \( A_\infty \).

3.1. Diagonals of the \( A_\infty \) operad. Given two cyclic \( A_\infty \)-algebras \((A, \rho_A, \langle, \rangle_A)\) and \((B, \rho_B, \langle, \rangle_B)\) we want to construct a cyclic \( A_\infty \)-algebra on the differential vector space \((A \otimes B, d = d_A \otimes \text{id} + \text{id} \otimes d_B)\) equipped with the inner product
\[
\langle a_1 \otimes a_2, b_1 \otimes b_2 \rangle_{A \otimes B} := (-1)^{|b_1||a_2|} \langle a_1, b_1 \rangle_A \langle a_2, b_2 \rangle_B.
\]

Let us recast this problem in operadic terms. First observe that given operads \( O_1 \) and \( O_2 \) we can define their tensor product as
\[
(O_1 \otimes O_2)(n) = O_1(n) \otimes O_2(n)
\]
with the operadic composition and differential
\[
(U_1 \otimes U_2) \circ_i (V_1 \otimes V_2) = (-1)^{|U_2||V_1|} U_1 \circ_i V_1 \otimes U_2 \circ_i V_2,
\]
\[
\partial(U_1 \otimes U_2) = \partial U_1 \otimes U_2 + (-1)^{|U_1|} U_1 \otimes \partial U_2.
\]
Also note that there is a map of operads
\[
i : \text{End}_A \otimes \text{End}_B \rightarrow \text{End}_{A \otimes B}
\]
given by
\[
i(f \otimes g)(a_1 \otimes b_1, \ldots, a_n \otimes b_n) = (-1)^{\sum_{i<j} |b_i||a_j| + \sum_{j=1}^{n} |a_j||g(f(a_1, \ldots, a_n)) \otimes g(b_1, \ldots, b_n)}.
\]

With these observations we see that if there exists a canonical (in an appropriate sense) map of operads
\[
\Delta : A_\infty \rightarrow A_\infty \otimes A_\infty,
\]
then we can simply use the composition
\[
\rho = i \circ \rho_A \otimes \rho_B \circ \Delta : A_\infty \rightarrow A_\infty \otimes A_\infty \rightarrow \text{End}_A \otimes \text{End}_B \rightarrow \text{End}_{A \otimes B}
\]
to define the desired $A_\infty$-algebra structure on $A \otimes B$.

Furthermore, if one can construct $\Delta$ which is also compatible with the cyclic group actions, then the resulting $A_\infty$-algebra structure on $A \otimes B$ will also be cyclic. This follows from the observation that in order for the composition $i \circ \rho_A \otimes \rho_B \circ \Delta$ to be compatible with the cyclic structures, it is enough to require that $\Delta$ is so. This is because the morphisms $i$ and $\rho_A \otimes \rho_B$ are already compatible.

3.2. Existence of a cyclic diagonal. To characterize the morphism $\Delta$, we first recall some basic facts from [13]. The operad $A_\infty$ is a minimal resolution of the associative operad $A_s$, i.e. there is a morphism of dg operads (with $A_s$ endowed with the trivial differential)

$$A_\infty \longrightarrow A_s,$$

inducing an isomorphism on the homology operads $H_*(A_\infty) \cong A_s$. Furthermore, the operad $A_s$ admits a canonical diagonal morphism

$$A_s \longrightarrow A_s \otimes A_s$$

which is coassociative. (In fact $A_s$ is a Hopf operad.)

**Definition 3.1.** A morphism of operads $\Delta : A_\infty \longrightarrow A_\infty \otimes A_\infty$ is called a diagonal of $A_\infty$ if the induced map on the corresponding homology operads

$$H_*(A_\infty) \longrightarrow H_*(A_\infty \otimes A_\infty) \cong H_*(A_\infty) \otimes H_*(A_\infty)$$

is the canonical diagonal of the associative operad $A_s$ under the canonical isomorphism $H_*(A_\infty) \cong A_s$. If, in addition, the map $\Delta$ commutes with the cyclic action of $\mathbb{Z}/(n+1)\mathbb{Z}$, i.e.

$$\Delta \circ r = r \otimes r \circ \Delta,$$

then we say $\Delta$ is a cyclic diagonal of $A_\infty$.

The existence of a diagonal (without the requirement of being cyclic) and its uniqueness up to homotopy follow from general results on Hopf operads. See [13] for a proof of this. Here we give a different proof of the existence of a diagonal which will also be cyclic. The proof is based on the following fact:

**Proposition 3.2.** The spaces $K(n)$ are contractible. Therefore

$$H_*(A_\infty(n)) = H_*(C_*(K(n))) = \begin{cases} \mathbb{Q}, & * = 0, \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** For example, see [17] for a proof of the fact that $K(n)$ can be realized as convex polytopes in $\mathbb{R}^N$, which implies the result. \hfill \Box

We will also need the following

**Lemma 3.3.** Let $(C_*, \partial)$ be a chain complex with a $\mathbb{Z}/(n+1)$ action (generated by $r$) and let $\alpha, \beta \in C_*$. The equation $\partial \alpha = \beta$ has a solution $\alpha$ satisfying $r(\alpha) = (-1)^n \alpha$ if and only if there exists $\gamma$ such that $\partial \gamma = \beta$ and $r(\beta) = (-1)^n \beta$. 

Proof. For the only if part, observe that
\[ r(\beta) = r(\partial \alpha) = \partial r(\alpha) = \partial((-1)^n \alpha) = (-1)^n \beta. \]
Conversely, we define
\[ \alpha = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i r^i(\gamma) \]
and compute
\[
\partial \alpha = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i \partial r^i(\gamma) = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i r^i(\partial \gamma)
\]
\[ = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i r^i(\beta) = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i (-1)^i \beta
\]
\[ = \beta. \]
Moreover
\[
\partial \alpha = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i r^i(\gamma)
\]
\[ = \frac{(-1)^n}{n+1} \sum_{i=0}^{n} (-1)^{(i+1)n} r^{i+1}(\gamma)
\]
\[ = \frac{(-1)^n}{n+1} \left( \sum_{j=1}^{n} (-1)^n r^j(\gamma) + (-1)^{(n+1)n} r^{n+1}(\gamma) \right)
\]
\[ = \frac{(-1)^n}{n+1} \sum_{j=0}^{n} (-1)^n r^j(\gamma) = (-1)^n \alpha,
\]
since \( r^{n+1} = id \).

\[ \Box \]

**Theorem 3.4.** There exists a cyclic diagonal for \( \mathcal{A}_\infty \).

Proof. The operad \( \mathcal{A}_\infty \) is generated by the corollas \( \{c_k\}_{k \geq 2} \). Therefore it is enough to define \( \Delta(c_k) \) for \( k \geq 2 \). To ensure that \( \Delta \) is a map of operads we just need to ensure that \( \Delta \) is a chain map, that is, \( \Delta(\partial c_k) = \partial \Delta(c_k) \). Also, to ensure that \( \Delta \) respects the cyclic action, we only need to check it on corollas:
\[ \Delta(r(c_k)) = (r \otimes r)(\Delta(c_k)), \ k \geq 2. \]
We will proceed by induction on \( k \). For \( k = 2 \) we define
\[ \Delta(c_2) = c_2 \otimes c_2 \]
and observe that it is a chain map since the differential is trivial on both sides. Moreover \( r(c_2) = c_2 \), which implies \( \Delta \) commutes with \( r \).
Now assume we have defined \( \Delta(c_k) \) for all \( k \leq n-1 \) in such a way that it commutes with both \( r \) and \( \partial \). We want to define \( \Delta(c_n) \) so that
\[ \partial \Delta(c_n) = \Delta(\partial c_n) \text{ and } r \otimes r(\Delta(c_n)) = \Delta(r(c_n)). \]

By Proposition 3.2, \( A_\infty(n) \) is contractible, then so is \( A_\infty(n) \otimes A_\infty(n) \) by the Künneth theorem. Therefore to prove that the first equation has a solution we only need to check that \( \Delta(\partial c_n) \) is closed. By definition

\[
\Delta(\partial c_n) = \Delta \left( \sum (-1)^{i(j+1)+jn} c_{n-j+1} \circ_i c_j \right)
\]

= \[
\sum (-1)^{i(j+1)+jn} \Delta(c_{n-j+1}) \circ_i \Delta(c_j).
\]

By induction hypothesis the right hand side involves only \( c_j \) with \( j < n \). For these, \( \Delta \) was already defined and commutes with \( r \) and \( \partial \). Therefore

\[
\partial \Delta(\partial c_n) = \Delta(\partial^2 c_n) = 0.
\]

Hence we conclude that we can find \( \gamma \in A_\infty(n) \otimes A_\infty(n) \) such that

\[
\partial \gamma = \Delta(\partial c_n).
\]

Now, in order to use Lemma 3.3, we need to compute

\[
r(\Delta(\partial c_n)) = r \left( \Delta \left( \sum (-1)^{i(j+1)+jn} c_{n-j+1} \circ_i c_j \right) \right)
\]

= \[
\sum (-1)^{i(j+1)+jn} r(\Delta(c_{n-j+1}) \circ_i \Delta(c_j))
\]

= \[
\sum_j \left( \sum_{1 \leq i < n-j+1} (-1)^{i(j+1)+jn} r(\Delta(c_{n-j+1}) \circ_i \Delta(c_j)) + (-1)^{(n-j+1)(j+1)+jn+j(n-j+1)} r(\Delta(c_j)) \circ_1 r(\Delta(c_{n-j+1})) \right)
\]

= \[
\sum_{2 \leq j \leq n-j+1} (-1)^{(j+1)+jn+n+1} \Delta(c_{n-j+1}) \circ_i \Delta(c_j) + \sum_j (-1)^{(n+j+1)(j+1)+jn+j(n+j+1)+j+n+j+1} \Delta(c_j) \circ_1 \Delta(c_{n-j+1})
\]

= \[
(-1)^n \sum_{2 \leq i \leq n-j+1} (-1)^{i(j+1)+jn} \Delta(c_{n-j+1}) \circ_i \Delta(c_j) + \sum_k (-1)^{n+k+1+kn} \Delta(c_{n-k+1}) \circ_1 \Delta(c_k)
\]

= \[
(-1)^n \sum (-1)^{i(j+1)+jn} \Delta(c_{n-j+1}) \circ_i \Delta(c_j)
\]

= \[
(-1)^n \Delta(\partial c_n).
\]

Lemma 3.3 now implies that there is \( \alpha \in A_\infty(n) \otimes A_\infty(n) \) satisfying

\[
\partial \alpha = \Delta(\partial c_n) \quad \text{and} \quad r(\alpha) = (-1)^n \alpha.
\]

Finally we define \( \Delta(c_n) = \alpha \), which concludes the induction step. \( \square \)
Remark 3.5. This proof can be immediately adapted to prove that we can choose a commutative diagonal. This means we can choose \( \Delta \) so that

\[ \tau \circ \Delta = \Delta \]

where \( \tau : A_\infty(n) \otimes A_\infty(n) \to A_\infty(n) \otimes A_\infty(n) \) is the map that interchanges both factors of the tensor product.

Observe that Lemma 3.3 works for the action of any finite group on a chain complex. Since the \( \mathbb{Z}_2 \)-action generated by \( \tau \) and the action generated by \( r \) commute, we can in fact prove the existence of a diagonal which is simultaneously cyclic and commutative. With such diagonal, the tensor product of \( A_\infty \)-algebras is naively commutative. In other words, given \( A_\infty \)-algebras \( A \) and \( B \) the map \( t = A \otimes B \to B \otimes A \), \( t(a \otimes b) = (-1)^{|a||b|} b \otimes a \) is an isomorphism of \( A_\infty \)-algebras. This is in contrast with the fact that we cannot construct a coassociative diagonal, this meaning

\[ (\Delta \otimes id)\Delta \neq (id \otimes \Delta)\Delta, \]

for any choice of diagonal \( \Delta \). See [12] for a proof of this fact.

3.3. Explicit formula for the diagonal. As we mentioned in the Introduction there are three constructions of a diagonal on \( A_\infty \). The first is due to Saneblidze and Umble [16], the second to Markl and Schider [12] and the third to Loday [11]. It seems these three constructions give the same diagonal, but this has not been checked for \( \Delta(c_k) \), with \( k \) larger than 5.

These diagonals are not cyclic. Here we will give explicit formulas for \( \Delta(c_k) \), \( k \leq 4 \), which are cyclic. Unfortunately we do not know how to construct \( \Delta(c_k) \) for \( k \geq 6 \) (we have constructed the case \( k = 5 \) but will not present it in the present article due to its length). This appears to be a very complicated combinatorial problem.

**Diagonal for \( k = 2 \).** We take \( \Delta(c_2) = c_2 \otimes c_2 \).

**Diagonal for \( k = 3 \).** If we denote by \( B_1 \) and \( B_2 \) the two trees with tree leaves and all internal vertices trivalent we have

\[ \Delta(c_3) = \frac{1}{2} (B_1 \otimes c_3 + B_2 \otimes c_3 + c_3 \otimes B_1 + c_3 \otimes B_2). \]

Note that this is also cocommutative.

**Diagonal for \( k = 4 \).** Denote by \( B_1 \) trough \( B_5 \) the five trivalent trees with five leaves with the orientation \( e_1 \wedge e_2 \), where \( e_1 \) and \( e_2 \) are the internal edges indicated in the diagram below:

Additionally consider the denominations of the following trees:
Let $x \in \mathbb{Q}$. The most general form of a cyclic $\Delta(c_4)$ is

$$\Delta(c_4) = \frac{1}{5} c_4 \otimes \left( \sum_{i=1}^{5} B_i \right) + \frac{1}{5} \left( \sum_{i=1}^{5} B_i \right) \otimes c_4 + \sum_{i,j} \gamma_{ij} E_i \otimes E_j$$

with

$$\left\{ \begin{align*}
\gamma_{1,1} &= -1/10 - x, \\
\gamma_{1,2} &= x, \\
\gamma_{1,3} &= -2/5 - x, \\
\gamma_{1,4} &= -1/5 + x, \\
\gamma_{1,5} &= 1/5 + x,
\end{align*} \right.$$ 

together with the relations

$$\left\{ \begin{align*}
\gamma_{1,1} &= \gamma_{2,2} = \gamma_{3,3} = \gamma_{4,4} = \gamma_{5,5}, \\
\gamma_{1,2} &= \gamma_{2,3} = \gamma_{3,4} = -\gamma_{4,5} = \gamma_{5,1}, \\
\gamma_{1,3} &= \gamma_{2,4} = -\gamma_{3,5} = -\gamma_{4,1} = \gamma_{5,2}, \\
\gamma_{1,4} &= -\gamma_{2,5} = -\gamma_{3,1} = -\gamma_{4,2} = \gamma_{5,3}, \\
\gamma_{1,5} &= \gamma_{2,1} = \gamma_{3,2} = \gamma_{4,3} = -\gamma_{5,4}.
\end{align*} \right.$$ 

Thus we see that the cyclicity condition uniquely determines $\Delta(c_3)$, but not $\Delta(c_4)$. However if we require that $\Delta$ must also be cocommutative, it forces $x = -1/10$ and so it fixes $\Delta(c_4)$.

**Diagonal for $k = 5$.** The most general form of $\Delta(c_5)$ has five degrees of freedom. Cocommutativity cuts down the ambiguity to four dimensions.

### 3.4. Uniqueness up to cyclic homotopy

The goal of this subsection is to show that a cyclic diagonal of $\mathcal{A}_\infty$ is unique up to cyclic homotopy. On the level of algebras, this implies that the cyclic tensor product structure defined by any cyclic diagonal is unique up to a cyclic quasi-isomorphism.

We first recall the notion of a homotopy between morphisms of dg operads from [13]. Denote by $\Omega^*_{[0,1]}$ the commutative dg algebra of algebraic differential forms on the unit interval $[0, 1]$, with coefficients in $\mathbb{Q}$, equipped with the standard de Rham differential. Note that since we are using the homological degree, we require the the space of one forms $\Omega^1_{[0,1]}$ to have degree $-1$. An elementary homotopy of between two morphisms $f, g : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ of dg operads is another homomorphism

$$h : \mathcal{O}_1 \longrightarrow \mathcal{O}_2 \otimes_{\mathbb{Q}} \Omega^*_{[0,1]}$$

such that $h(0) = f$ and $h(1) = g$.\footnote{A homotopy between $f$ and $g$ is a sequence of elementary homotopies $(h_1, \ldots, h_k)$ with $h_i(1) = h_{i+1}(0)$ where $1 \leq i < k-1$, and $h_1(0) = f$ and $h_k(1) = g$. For purposes of this paper, the notion of elementary homotopy is enough.}
For cyclic operads $\mathcal{O}_1$, $\mathcal{O}_2$, and cyclic morphisms $f$, $g$, we call an elementary homotopy $h$ cyclic if it intertwines with the cyclic action, where we put the trivial action on the $\Omega_{[0,1]}$ component.

**Theorem 3.6.** Let $\Delta_1, \Delta_2 : A_\infty \to A_\infty \otimes A_\infty$ be two cyclic diagonals, then there exists a cyclic homotopy $h : A_\infty \to A_\infty \otimes A_\infty \otimes \Omega_{[0,1]}^*$ between them.

**Proof.** The proof is very similar to that of Theorem 3.4. We shall only sketch it here. Indeed, the idea is to use the acyclicity of $A_\infty \otimes A_\infty \otimes \Omega_{[0,1]}^*$ (which follows from acyclicity of $A_\infty$ and $\Omega_{[0,1]}^*$) to inductively construct $h$. The only difference is that when constructing $h(c_n)$, we also need to impose the boundary condition

$$h(c_n)(0) = \Delta_1(c_n), \quad h(c_n)(1) = \Delta_2(c_n).$$

For this we argue as follows. For $n = 2$ we simply take $h(c_2) = \Delta_1(c_2) = \Delta_2(c_2)$. As in the proof of Theorem 3.4 we can find $h'(c_n)$, some extension of $h(c_2), \ldots, h(c_{n-1})$ as a morphism of dg operads (ignoring cyclicity and the boundary condition). Since $h(c_j)$ ($2 \leq j \leq n-1$) has the required boundary condition, we conclude that

$$\partial(h'(c_n)(0) - \Delta_1(c_n)) = 0,$$

$$\partial(h'(c_n)(1) - \Delta_2(c_n)) = 0.$$

Thus by acyclicity, there exist elements $a, b \in (A_\infty \otimes A_\infty)(n)$ of degree $n - 1$ such that

$$\partial a = h'(c_n)(0) - \Delta_1(c_n),$$

$$\partial b = h'(c_n)(1) - \Delta_2(c_n).$$

Next we set

$$h''(c_n) = h'(c_n) - [(1 - t)\partial a + t\partial b] - (-1)^n(a - b)dt,$$

and, finally, take $h(c_n)$ to be the average of $h''(c_n)$ by the cyclic group $\mathbb{Z}/(n + 1)\mathbb{Z}$. Note that $h(c_n)$ gives a cyclic dg-operad map with the correct boundary condition.

On the level of algebras, we can see that the cyclic $A_\infty$ algebra structure on $A \otimes B$ defined using a cyclic diagonal of $A_\infty$ is unique up to cyclic quasi-isomorphisms. Indeed, a cyclic homotopy $h$ induces a morphism of dg operads defined by the composition

$$(i \otimes \text{id})(\rho_1 \otimes \rho_2 \otimes \text{id}) \circ h : A_\infty \to A_\infty \otimes A_\infty \otimes \Omega_{[0,1]}^* \to \text{End}_{A \otimes \text{End}_B \otimes \Omega_{[0,1]}^*} \to \text{End}_{A \otimes B \otimes \Omega_{[0,1]}^*},$$

which we denote by $\theta$. Using the isomorphism $A_\infty \cong \Omega A s^i$ where $A s^i$ is the Koszul dual cooperad of $A s$ and $\Omega$ is the coBar construction, we conclude that

$$\text{Hom}_{\text{dg. op.}}(A_\infty, \text{End}_{A \otimes B \otimes \Omega_{[0,1]}^*}) \cong \text{Tw}(A s^i, \text{End}_{A \otimes B \otimes \Omega_{[0,1]}^*})$$

$$\cong \text{Hom}_{\text{dg. coaug. coop.}}(A s^i, \mathbb{B} \text{End}_{A \otimes B \otimes \Omega_{[0,1]}^*})$$

$$\cong \text{Codiff}(A s^i(\text{End}_{A \otimes B} \otimes \Omega_{[0,1]}^*).$$

Here the functor $\mathbb{B}$ is the Bar construction, which is the right adjoint of the coBar functor $\Omega$. Via the above isomorphism, the morphism $\theta$ corresponds to an element of the set $\text{Codiff}(A s^i(\text{End}_{A \otimes B} \otimes \Omega_{[0,1]}^*) \otimes \Omega_{[0,1]}^*)$ which is compatible with the inner product on $A \otimes B$. In [4], such an element was called a cyclic pseudo-isotopy between the two cyclic $A_\infty$-structures on $A \otimes B$ defined using $\Delta_1$ and $\Delta_2$. It was proved in Proposition 9.2 of [4] that cyclic
pseudo-isotopies give rise to cyclic quasi-isomorphisms between the boundary cyclic $A_\infty$ structures defined by $\theta(0)$ and $\theta(1)$.

4. Tensor products and cup products

In this section, we prove our second main result, Theorem 1.2. The major part of the work is to use a cyclic diagonal of $A_\infty$ to construct a diagonal of the graph complex $G_\ast$.

4.1. The graph homology complex. We recall the definition of the graph homology complex $G_\ast$, following Igusa’s careful treatment of signs and orientations [8].

**Definition 4.1.** A ribbon graph is a (finite) connected graph whose vertices have valency $\geq 3$, together with a fixed cyclic ordering on the half-edges incident to each vertex.

An orientation of a ribbon graph $\Gamma$ is an orientation on the vector space spanned by the vertices and half-edges of $\Gamma$.

We define $G_\ast$ to be the vector space generated by isomorphism classes of pairs $(\Gamma, \sigma)$ where $\Gamma$ is a ribbon graph and $\sigma$ is an orientation of $\Gamma$, with the relation $(\Gamma, -\sigma) = -(\Gamma, \sigma)$. Here an isomorphism between $(\Gamma, \sigma)$ and $(\Gamma', \sigma')$ is an isomorphism $\varphi : \Gamma \to \Gamma'$ of ribbon graphs which also preserves orientation.

The grading on $G_\ast$ is defined as follows. Given a pair $(\Gamma, \sigma)$, we denote by $v_1, \ldots, v_k$ the vertices of $\Gamma$, and define

$$| (\Gamma, \sigma) | = \sum_{i=1}^{k} (\text{val}(v_i) - 3)$$

where $\text{val}(v_i)$ is the valency of the vertex $v_i$.

To define the boundary map on $G_\ast$, we need to introduce the contraction and expansion operations. Given an edge $e$ of a ribbon graph $\Gamma'$ which is not a loop, we define another ribbon graph $\Gamma := \Gamma'/e$ to be the ribbon graph obtained from $\Gamma'$ by contracting the edge $e$. Let us denote by $v^-$ and $v^+$ the boundary vertices of $e$, and $e^-, e^+$ the two half edges of $e$ incident to $v^-$ and $v^+$ respectively. Denote by $v$ the vertex of $\Gamma$ obtained from identifying $v^-$ and $v^+$. Assume that the cyclic orderings of the half-edges at $v^-$ and $v^+$ were:

$$e_1 \cdots e_k e^-, \text{ and } e^+ f_1 \cdots f_n.$$

We define the cyclic ordering at the vertex $v$ of $\Gamma$ by

$$e_1 \cdots e_k f_1 \cdots f_n.$$

Also, given an orientation $\sigma'$ of $\Gamma'$ of the form $\epsilon \langle v^- v^+ e^- e^+ \ldots \rangle$ where $\epsilon = \pm 1$ $^3$, we define a natural orientation on $\Gamma = \Gamma'/e$ by setting $\sigma'/e := \epsilon \langle v \ldots \rangle$.

In the reverse direction, given a vertex $v$ of a ribbon graph $\Gamma$, one can expand $v$ into two vertices $v^-$, $v^+$ together with an edge $e$ joining them, in $\frac{\text{val}(v)^2 - 3\text{val}(v)}{2}$ different ways. This way we obtain a new oriented ribbon graph $(\Gamma', \sigma')$ with a distinguished edge $e$, satisfying $(\Gamma'/e, \sigma'/e) = (\Gamma, \sigma)$.

Finally we define

$$\partial(\Gamma, \sigma) = \sum (\Gamma', \sigma')$$

---

$^3$This can always be achieved by a permutation action on $\sigma'$, then put $\epsilon$ to be the sign of this permutation.
where the sum is taken over pairs consisting a vertex $v$ of $\Gamma$ and a choice of expansion at $v$.

**Definition 4.2.** We define the ribbon graph complex to be the pair $(\mathcal{G}_*, \partial)$. Given a ribbon graph $\Gamma$, we define $\mathcal{G}_*(\Gamma) \subseteq \mathcal{G}_*$ to be the subcomplex generated by all ribbon graphs, which after collapsing a subtree (possibly disconnected) equal $\Gamma$.

We will write $(\Gamma' \to \Gamma)$ if $\Gamma = \Gamma'/e$ for some non-loop edge $e$ of $\Gamma'$. Observe that for such graph $\Gamma'$, there is an inclusion of complexes $\mathcal{G}_*(\Gamma') \subset \mathcal{G}_*(\Gamma)$.

With this notation, we have

$$\mathcal{G}_* = \lim_{\Gamma' \to \Gamma} \mathcal{G}_*(\Gamma) \quad (3)$$

4.2. **Diagonals of the graph complex.** In [2], Costello constructed compact orbi-spaces $D_{g,h,r,s}$ with cellular decompositions. Moreover, if we denote by $\Gamma_{g,n}$ the space $D_{g,n,0,0}$ and consider its associated cellular chain complex $C_*(\Gamma_{g,n})$ we have that

$$(\mathcal{G}_*, \partial) = \bigoplus_{g,n} C_*(\Gamma_{g,n}).$$

Our purpose is to define a diagonal on $\mathcal{G}_*$, that is, a degree zero chain map

$$\delta : \mathcal{G}_* \to \mathcal{G}_* \otimes \mathcal{G}_*$$

which is a cellular approximation to the geometric diagonal map

$$\prod \Gamma_{g,n} \to \prod \Gamma_{g,n} \times \prod \Gamma_{g,n}.$$

The definition of $\delta$ will depend on the choice of a cyclic diagonal of $A_\infty$ constructed in the previous section. In the following we shall fix, once and for all, such a diagonal $\Delta$.

We introduce the following

**Construction 4.3.** Let $(\Gamma, \sigma) \in \mathcal{G}_*$, pick an ordering of the vertices of $\Gamma$, $v_1, \ldots, v_k$ and pick a half-edge incident at each vertex $v_i$ and label it by $e_{i,0}$. Then label the other half-edges following the cyclic ordering $e_{i,0}, e_{i,1}, \ldots, e_{i,n_i}$ (if $\text{val}(v_i) = n_i + 1$). We define $A_{\Gamma} \in \{\pm 1\}$ by the following equality

$$\sigma = A_{\Gamma} (v_1 e_{1,0} e_{1,1} \ldots e_{1,n_1} v_2 e_{2,0} e_{2,1} \ldots e_{2,n_2} \ldots v_k e_{k,0} e_{k,1} \ldots e_{k,n_k}).$$

Now, given oriented trees $(T_i, \sigma_i = f_{i,1} \wedge \ldots \wedge f_{i,t_i}) \in C_{n_i - t_i - 2}(K(n_i))$ with $n_i$ leaves for $i = 1, \ldots, k$ we define the ribbon graph $\Gamma(T_1, \ldots, T_k)$ by replacing each vertex $v_i$ by the tree $T_i$, gluing the root of $T_i$ to $e_{i,0}$ and the leaves of $T_i$ to $e_{i,1}, \ldots, e_{i,n_i}$, following the cyclic ordering. We define the orientation $\sigma_{\Gamma(T_1, \ldots, T_k)}$ on $\Gamma(T_1, \ldots, T_k)$ so that

$$\sigma_{\Gamma(T_1, \ldots, T_k)}/f_{1,1}/ \ldots/f_{1,t_1}/f_{2,t_1}/f_{2,t_2}/ \ldots/f_{k,1}/f_{k,t_k} = \sigma.$$

**Lemma 4.4.** Let $(\Gamma, \sigma)$ and $T_i (i = 1, \ldots, k)$ be as in Construction 4.3. Then we have

$$\partial (\Gamma(T_1, \ldots, T_k)) = \sum_{i=1}^{k} (-1)^{\sum_{j < m} t_j} \Gamma(T_1, \ldots, \partial T_m, \ldots, T_k).$$

Here $t_j$ is the number of internal edges of $T_j$. 

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Proof. Assume that
\[ \partial T_m = \sum (U, \sigma_k = e \wedge \sigma_{T_m}) \]
where \( U/e \simeq T_m \). Then the orientation \( \sigma_{T_1, \ldots, U, \ldots, T_k} \) on \( \Gamma(T_1, \ldots, T_{m-1}, U, T_{m+1}, \ldots, T_k) \) is determined by
\[ \sigma_{T_1, \ldots, U, \ldots, T_k} / \sigma_1 / \ldots / e / \sigma_{T_m} / \ldots / \sigma_k = \sigma. \]
Then we have
\[ (-1)^{\sum_{j<m} t_j} \sigma_{T_1, \ldots, U, \ldots, T_k} / e / \sigma_1 / \ldots / \sigma_k = \sigma, \]
which implies the lemma.

\[ \square \]

Corollary 4.5. There is an isomorphism of chain complexes
\[ \phi_\Gamma : C_*(K(n_1)) \otimes \cdots \otimes C_*(K(n_k)) \to G_*(\Gamma) \]
defined by
\[ \phi_\Gamma(T_1, \ldots, T_k) := (-1)^{|\Gamma|} A_\Gamma (-1)^{\sum_{i<j} n_i t_j} \Gamma(T_1, \ldots, T_k) \]
In particular \( G_*(\Gamma) \) is acyclic.

Proof. This follows by straightforward calculation using Lemma 4.4.

Definition 4.6. We define the morphism \( \delta_\Gamma : G_*(\Gamma) \to G_*(\Gamma) \otimes G_*(\Gamma) \) as the composition
\[ G_*(\Gamma) \xrightarrow{\phi^{-1}_\Gamma} C_*(K(n_1)) \otimes \cdots \otimes C_*(K(n_k)) \xrightarrow{\Delta \otimes \cdots \otimes \Delta} (C_*(K(n_1)) \otimes C_*(K(n_1))) \otimes \cdots \otimes (C_*(K(n_k)) \otimes C_*(K(n_k))) \xrightarrow{\tau} G_*(\Gamma) \otimes G_*(\Gamma) \]
Here \( \tau \) is the permutation of tensor components as indicated in the diagram. It follows from Corollary 4.5 that \( \delta_\Gamma \) is a chain map.

Lemma 4.7. The map \( \delta_\Gamma \) does not depend on the choice of the ordering of \( v_l \)’s, nor on the choices of roots \( e_i, 0 \)’s.

Proof. Suppose we interchange two consecutive vertices \( v_1 \) and \( v_2 \). Then \( A_\Gamma \) changes by a factor of \((-1)^{n_1 n_2}\). The sign \((-1)^{\sum_{i<j} n_i t_j}\) changes by a factor of \((-1)^{n_1 t_2 + n_2 t_1}\). The two graphs’ orientations differ by \((-1)^{t_1 t_2}\) due to Construction 4.3. Putting these signs together we get
\[ (-1)^{n_1 n_2 + n_1 t_2 + n_2 t_1 + t_1 t_2} = (-1)^{(n_1 + t_1)(n_2 + t_2)}. \]
Observe that \( \text{deg}(T_i) = n_i + t_i \) (mod 2), hence the above sign is precisely the Koszul sign that we get from switch the first two tensor product. So we conclude that \( \delta \) does not depend on the choice of ordering of the vertices.

Next suppose we cyclically permute the half-edges around \( v_1 \),
\[ e_{1, 0} e_{1, 1} \cdots e_{1, n_1} \to e_{1, n_1} e_{1, 0} e_{1, 1} \cdots e_{1, n_1 - 1}. \]
Then $A_\Gamma$ changes by $(-1)^{n_1}$. By Lemma 2.1 this extra sign also appears from the rotation action on $T_1$. The rest follows from the cyclicity of $\Delta$.

Recall (from 3) that $\mathcal{G}_* = \lim_{\Gamma' \to \Gamma} \mathcal{G}_*(\Gamma)$, thus to define a diagonal morphism for $\mathcal{G}_*$, it suffices to prove that the morphisms $\delta_\Gamma$ and $\delta_{\Gamma'}$ are compatible with each other.

More precisely, let $(\Gamma, \sigma)$ be as in Construction 4.3 and consider a term $(\Gamma', \sigma')$ in the sum

$$\partial(\Gamma, \sigma) = \sum (\Gamma', \sigma').$$

By definition there exists an unique $e \in E(\Gamma')$ such that $(\Gamma', e, \sigma') = (\Gamma, \sigma)$. Assume $\Gamma'$ is an expansion of $\Gamma$ at the vertex $v_m$. We order the vertices of $\Gamma'$ as $v_1 \ldots v_{m-1} v_m^- v_{m+1} \ldots v_k$.

where $v_m^+$ is the unique vertex that contains the half-edge $e_m, 0$ after expansion. We then order half-edges incident at $v_1$ and $v_k$ as $e_{m,0} e_{m,1} \ldots e_{m,i-1} e_{m,i}^+ e_{m,i+1} \ldots e_{m,n}$ and $e^-_{m,i} e_{m,i+1} \ldots e_{m,n}$ respectively. Here $e^-$ and $e^+$ are the new half-edges incident to $v_m^-$ and $v_m^+$.

Using these choices of orderings we prove the following

**Proposition 4.8.** Let $\Gamma$ and $\Gamma'$ be as above. Then the following diagram is commutative.

$$
\begin{array}{ccc}
\mathcal{G}_s(\Gamma') & \longrightarrow & \mathcal{G}_s(\Gamma) \\
\delta_{\Gamma'} & \downarrow & \delta_{\Gamma} \\
\mathcal{G}_s(\Gamma') \otimes \mathcal{G}_s(\Gamma') & \longrightarrow & \mathcal{G}_s(\Gamma) \otimes \mathcal{G}_s(\Gamma)
\end{array}
$$

Where the horizontal arrows are simply inclusions.

The proposition is clear on the level of ribbon graphs. So its content is to match the orientations. For this we need to prove two lemmas.

**Lemma 4.9.** Given $U \in C_*(K(n_m - j + 1))$ and $V \in C_{j-t_V-2}(K(j))$ (with $t_V$ the number of internal edges of $V$) we have

$$
\Gamma'(T_1, \ldots, T_m-1, U, V, T_{m+1}, \ldots, T_k) = (-1)^{*}(T_1, \ldots, T_m-1, U \circ_i V, T_{m+1}, \ldots, T_k)
$$

with $* = i(j+1) + (j + t_V)(n_m - j + 1) + \sum_{l>m} t_l$. Additionally,

$$
A_{\Gamma'} = (-1)^{\sum_{l<m} (n+1) + \sum_{l>m} (j+1)} A.
$$

**Proof.** The equality as ribbon graphs is clear, we just need to compare orientations. Denote by $\sigma_U$ and $\sigma_V$ the orientations of $U$ and $V$. By definition of composition we have

$$
\sigma_{U \circ V} = (-1)^{i(j+1) + (j+t_V)(n_m - j+1)} \sigma_U \land \sigma_V \land e.
$$

Also by definition,

$$
\sigma'_{T_1, \ldots, U, V, \ldots, T_k} / \sigma_1 / \ldots / \sigma_U / \sigma_V / \ldots / \sigma_k = \sigma',
$$

$$\sigma' / e = \sigma,$$

and

$$
\sigma_{T_1, \ldots, U \circ V, \ldots, T_k} / \sigma_1 / \ldots / \sigma_{U \circ V} / \ldots / \sigma_k = \sigma.
$$
Then

\[-1\]^{(j+1)+(j+n)}(n_m-j+1)\sigma T_{1\ldots U_{0\ldots V_{\ldots T_k}}}/\sigma_1/\ldots/\sigma_U/\sigma_V/e/\ldots/\sigma_k = \sigma

\Rightarrow -1\]^{(j+1)+(j+n)}(n_m-j+1)+\sum l>m u\sigma T_{1\ldots U_{0\ldots V_{\ldots T_k}}}/\sigma_1/\ldots/\sigma_U/\sigma_V/\ldots/\sigma_k = e,

which implies

\[\sigma' = -1\]^{(j+1)+(j+n)}(n_m-j+1)+\sum l>m u\sigma T_{1\ldots U_{0\ldots V_{\ldots T_k}}}

For the second statement note that, by definition of \((\Gamma', \sigma')\) we have \(\sigma' = v_m^+ \wedge e^+ \wedge e^- \wedge \sigma\), if we identify the vertex \(v_m^-\) in \(\Gamma'\) with \(v_m^-\) in \(\Gamma\). Also by definition of \(A\), we have:

\[\sigma' = A_{\Gamma'}(v_1e_1 \ldots e_k \ldots v_m e_{m,0} \ldots e_{m,i-1} e^+ e_{m,i+j} \ldots e_{m,n_m} v_m^- e^- e_{m,i} \ldots e_{m,i+j-1} \ldots)
\]

\[= (-1)^{(j+1)+j+n_m+1} A_{\Gamma'}(v_1 \ldots v_m^- v_m^+ e^+ e^- e_{m,0} \ldots e_{m,n_m} \ldots)
\]

\[= (-1)^{(j+1)+j+n_m+1} A_{\Gamma'}(-1)\sum_{l<m} n_l (v_m^+ e^+ e^- v_1 \ldots v_m^- e^- e_{m,0} \ldots e_{m,n_m} \ldots)
\]

\[= (-1)^{(j+1)+j+n_m+1} \sum_{l<m} A_{\Gamma'} v_m^+ e^+ e^- \wedge (A_{\Gamma} \sigma).
\]

Hence

\[A_{\Gamma'} = A_{\Gamma}(-1)^{(j+1)+j+n_m+1} \sum_{l<m} n_l.
\]

and the lemma is proved. 

\[\square\]

**Lemma 4.10.** Let \(\Gamma\) and \(\Gamma'\) be as in Proposition 4.8. Then the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{G}_* (\Gamma') & \xrightarrow{\phi_{\Gamma'}^{-1}} & C_* (K(n_1)) \otimes \ldots \otimes C_* (K(n_m-j+1)) \otimes \ldots \otimes C_* (K(n_k)) \\
\downarrow & & \downarrow \text{id} \otimes \otimes \text{id} \\
\mathcal{G}_* (\Gamma) & \xrightarrow{\phi_{\Gamma}^{-1}} & C_* (K(n_1)) \otimes \ldots \otimes C_* (K(n_m)) \otimes \ldots \otimes C_* (K(n_k))
\end{array}
\]

Here the left vertical arrow is the inclusion morphism.

**Proof.** Let \(T_l \in C_* (K(n_l))\) if \(1 \leq l \leq k, l \neq m\), \(U \in C_* (K(n_m-j+1))\), and \(V \in C_* (K(n_j))\). By definition of \(\phi\) in Corollary 4.5, we have

\[\phi_{\Gamma'}^{-1} (\Gamma' (T_1, \ldots, U, V, \ldots, T_k)) = (-1)^{|\Gamma|} \cdot A_{\Gamma'} \cdot (-1)^{\bullet} T_1 \otimes \ldots \otimes U \otimes V \otimes \ldots \otimes T_k,
\]

where the sign is given by

\[\bullet = \sum_{1 \leq p < q < m} (n_p t_q + (n_1 + \ldots + n_{m-1}) t_U + (n_1 + \ldots + n_m - j + 1) t_V + \sum_{m < l} (n_1 + \ldots + n_m + 1) t_l.
\]

Here, as before, \(t_l\) is the number of internal vertices of \(T_l\).

Apply the morphism \((\phi_{\Gamma}) \circ (\text{id} \otimes \otimes \text{id})\). We obtain

\[
[(\phi_{\Gamma}) \circ (\text{id} \otimes \otimes \text{id})] \phi_{\Gamma'}^{-1} (\Gamma' (T_1, \ldots, U, V, \ldots, T_k)) = \phi_\Gamma (-1)^{|\Gamma|} A_{\Gamma'} (-1)^{\bullet} T_1 \otimes \ldots \otimes U \otimes V \otimes \ldots \otimes T_k
\]

\[= (-1)^{|\Gamma|} A_{\Gamma'} (-1)^{\bullet} (-1)^{|\Gamma|} A_{\Gamma} (-1) \sum_{1 \leq p < q < k} n_p t_q \Gamma (T_1, \ldots, U \otimes V, \ldots, T_k)
\]

\[= (-1)^{|\Gamma|} A_{\Gamma'} (-1)^{\bullet} (-1)^{|\Gamma|} A_{\Gamma} (-1) \sum_{1 \leq p < q < k} n_p t_q (-1)^* \Gamma' (T_1, \ldots, U, V, \ldots, T_k)
\]

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where * is as in Lemma 4.9, and \( t_m := t_V + t_U + 1 \). We need to show that the sign in the last equation all cancel. For this we observe that

\[
\bigoplus_{1 \leq p < q \leq k} n_p t_q = \sum_{l < m} n_l + (n_m - j + 1)t_V + \sum_{l > m} t_l \pmod{2},
\]

and so

\[
\bigoplus_{1 \leq p < q \leq k} n_p t_q + * = \sum_{l < m} n_l + i(j + 1) + j(n_m - j + 1) \pmod{2}.
\]

Finally, by the second part of Lemma 4.9 we have

\[
A_\Gamma A_{\Gamma'} = (-1)^{\sum_{l < m} n_l + i(j + 1) + jn_m + 1}.
\]

Hence we arrived at the desired conclusion that

\[
(-1)^{|\Gamma'|} A_{\Gamma'} (-1)^* = 1,
\]

since \( j(-j + 1) = 0 \pmod{2} \) and \( |\Gamma'| = |\Gamma| + 1 \).

\[\square\]

**Proof of Proposition 4.8.** To simplify the notations, in this proof, we set

\[
C_{n_1, \ldots, n_k} := C_*(K(n_1)) \otimes \cdots \otimes C_*(K(n_k))
\]

for a sequence of (positive) integers \( n_1, \ldots, n_k \). Then the proposition follows from the commutativity of the following diagram.

\[
\begin{array}{ccc}
G_*(\Gamma') & \longrightarrow & G_*(\Gamma) \\
\phi_{\Gamma'}^{-1} \downarrow & & \phi_{\Gamma}^{-1} \downarrow \\
C_{n_1, \ldots, n_m-j+1, j, \ldots, n_k} & \longrightarrow & C_{n_1, \ldots, n_m, \ldots, n_k} \\
\Delta \otimes \cdots \otimes \Delta \downarrow & & \Delta \otimes \cdots \otimes \Delta \\
C_{n_1, n_1, \ldots, n_m-j+1, n_m-j+1, j, \ldots, n_k, n_k} & \longrightarrow & C_{n_1, n_1, \ldots, n_m, n_m, \ldots, n_k, n_k} \\
\tau \downarrow & & \tau \\
(C_{n_1, n_1, \ldots, n_m-j+1, j, \ldots, n_k})^\otimes 2 & \longrightarrow & (C_{n_1, n_1, \ldots, n_m, \ldots, n_k})^\otimes 2 \\
(\phi_{\Gamma'})^\otimes 2 \downarrow & & (\phi_{\Gamma})^\otimes 2 \\
G_*(\Gamma')^\otimes 2 & \longrightarrow & G_*(\Gamma)^\otimes 2
\end{array}
\]

Here the top and bottom horizontal arrows are inclusions, the left vertical composition is by definition \( \delta_{\Gamma'} \), while the right vertical composition is \( \delta_{\Gamma} \). The commutativity of the top and bottom squares follows from Lemma 4.10. The commutativity of the middle square follows from the Koszul sign convention together with the fact that \( \Delta \) is an operad map (that is, it commutes with operadic composition \( \circ_i \)) of degree zero.

\[\square\]

**Definition 4.11.** We define the morphism

\[
\delta : G_* \rightarrow G_* \otimes G_*,
\]

for a sequence of (positive) integers \( n_1, \ldots, n_k \). Then the proposition follows from the commutativity of the following diagram.
by taking the limit of morphisms
\[ \delta_\Gamma : \mathcal{G}_*(\Gamma) \to \mathcal{G}_*(\Gamma) \otimes \mathcal{G}_*(\Gamma) \]
over \((\Gamma' \to \Gamma)\). Note that this is possible by Proposition 4.8. Also since \(\delta_\Gamma\)'s are chain maps, so is \(\delta\).

**Remark 4.12.** For a cyclic diagonal \(\Delta\) of \(\mathcal{A}_\infty\), using the Sweedler notation, we write
\[ \Delta(c_n) := c_n^{(1)} \otimes c_n^{(2)} \]
to denote the image of \(\Delta\) applied to the \(n\)-th corolla, and we denote by \(t_n^{(i)}\) \((i = 1, 2)\) the number of internal edges of \(c_n^{(i)}\). Then we can write down an explicit formula:
\[ \delta(\Gamma) = (-1)^{||\Gamma||} A_{\Gamma} (-1)^{\sum_{i<j} n_{i}^{(1)} + n_{j}^{(2)}} \Gamma(c_{n_{1}}^{(1)}, \ldots, c_{n_{k}}^{(1)}) \otimes \Gamma(c_{n_{1}}^{(2)}, \ldots, c_{n_{k}}^{(2)}). \]

As noted before, the complex \(\mathcal{G}_*\) is the cellular complex associated to the cell complex \(\coprod \Gamma_{g,n}\). By the cellular approximation theorem, the geometric diagonal map
\[ \coprod \Gamma_{g,n} \to \coprod \Gamma_{g,n} \times \coprod \Gamma_{g,n}, \]
being continuous, admits a cellular approximation. Any such cellular approximation extends to a morphism of complexes
\[ \delta^\dagger : \mathcal{G}_* \to \mathcal{G}_* \otimes \mathcal{G}_*. \]

**Proposition 4.13.** Let \(\delta^\dagger\) be any cellular approximation of the geometric diagonal map. Then the two morphisms \(\delta^\dagger\) and \(\delta\) are homotopic.

**Proof.** The theorem follows from the acyclic carrier theorem in [14]. Indeed, the assignment
\[ \Psi : \Gamma \mapsto \mathcal{G}_*(\Gamma) \otimes \mathcal{G}_*(\Gamma) \]
forms an acyclic carrier from \(\mathcal{G}_*\) to \(\mathcal{G}_* \otimes \mathcal{G}_*\), since we have that
1. if \((\Gamma' \to \Gamma)\), then \(\mathcal{G}_*(\Gamma') \otimes \mathcal{G}_*(\Gamma') \subset \mathcal{G}_*(\Gamma) \otimes \mathcal{G}_*(\Gamma)\);
2. acyclicity follows from Corollary 4.5.

Recall that a morphism of complexes \(f : \mathcal{G}_* \to \mathcal{G}_* \otimes \mathcal{G}_*\) is called carried by \(\Psi\) if we have
\[ f(\Gamma) \in \Psi(\Gamma) = \mathcal{G}_*(\Gamma) \otimes \mathcal{G}_*(\Gamma). \]
Then the acyclic carrier theorem asserts that there exists a unique such morphism up to a homotopy (which is also carried by \(\Psi\)). Since both \(\delta\) and \(\delta^\dagger\) are carried by \(\Psi\) by definition, it follows that they are homotopic.

It was proved in [2], or [8] that the space \(\Gamma_{g,n}\) is weakly homotopy equivalent to \(\mathcal{M}_{g,n}\), which implies that their rational cohomology rings are isomorphic. Proposition 4.13 enables us to describe the cup products on these geometric spaces in terms of the graph complex.
Corollary 4.14. Given maps \( c_1, c_2 : G_* \to \mathbb{Q} \) representing cohomology classes

\[
[c_1], [c_2] \in H^*(G_*, \mathbb{Q}) \cong H^*(\coprod_{g,n} \Gamma_{g,n}, \mathbb{Q}) \cong H^*(\coprod_{g,n} M_{g,n}, \mathbb{Q}),
\]

then we have

\[
[c_1] \cup [c_2] = [(c_1 \otimes c_2) \circ \delta].
\]

4.3. Kontsevich classes. Let \((A, \rho, \langle \cdot, \cdot \rangle)\) be a finite dimensional cyclic \(A_\infty\) algebra over \(\mathbb{Q}\). In this subsection, we shall assume that the inner product \(\langle -, - \rangle\) on \(A\) is even (and symmetric by our convention). The odd case will be dealt with in the next subsection.

In [9], Kontsevich constructed a cohomology class \([c_A] \in H^*(G_*, \mathbb{Q}) \cong H^*(\coprod_{g,n} M_{g,n}, \mathbb{Q})\).

It is known this is a homotopy invariant of \(A\), see [6].

We first recall the definition of \(c_A : G_* \to \mathbb{Q}\), following [8, Section 2]. In [8], Igusa wrote down the formula for \(c_A\) explicitly by choosing a basis of \(A\), and dealing carefully with signs involved. We shall use a more diagramatic approach instead. The equivalence between the approaches will be clear from our construction. A notable feature of our definition is that we do not need to assume that \(m_1 = 0\).

Definition 4.15. Let \((\Gamma, \sigma) \in G_*\) be a ribbon graph as in Construction 4.3. We define the chain map \(c_{A, \Gamma} : G_*(\Gamma) \to \mathbb{Q}\) as the composition

\[
G_*(\Gamma) \xrightarrow{\phi_{\Gamma}^{-1}} C_*(K(n_1)) \otimes \cdots \otimes C_*(K(n_k)) \xrightarrow{\rho \otimes \cdots \otimes \rho} \operatorname{Hom}(A^{\otimes n_1}, A) \otimes \cdots \otimes \operatorname{Hom}(A^{\otimes n_k}, A)
\]

\[
\xrightarrow{\cong} (A^\vee)^{\otimes n_1 + 1} \otimes \cdots \otimes (A^\vee)^{\otimes n_k + 1} \xrightarrow{\eta} (A^\vee)^{\otimes 2} \otimes \cdots \otimes (A^\vee)^{\otimes 2} \xrightarrow{\langle -, - \rangle^{-1} \otimes \cdots \otimes \langle -, - \rangle^{-1}} \mathbb{Q} \otimes \cdots \otimes \mathbb{Q} \cong \mathbb{Q}.
\]

Here we fix an ordering of edges of \(\Gamma\), as well as an orientation for each edge and take the permutation \(\eta\) associated to the permutation of half edges

\[
(e_{1,1}, \ldots, e_{1,n_1}, e_{1,0}, e_{2,1}, \ldots, e_{k,1}, \ldots, e_{k,n_k}, e_{k,0}) \to (h_1^+, h_1^-, \ldots, h_E^+, h_E^-)
\]

where \(E\) is the number of edges of \(\Gamma\) and \((-)^+\) and \((-)^-\) are the two half-edges of an edge.

\[\text{This is not a big generalization since we assume finite dimensionality. In this case, cyclic } A_\infty\text{ algebras with non-vanishing } m_1 \text{ always admits a self-adjoint homotopy. Using the tree formula to transfer the } A_\infty\text{ structure, we obtain a homotopy equivalent cyclic } A_\infty\text{ algebra whose } m_1 = 0.\]
Lemma 4.16. The map $c_{A,\Gamma}$ is independent of all the choices made, namely, orderings of vertices and edges of $\Gamma$, orientations of each edge and choices of roots $e_{i,0}$.

Moreover, $c_{A,\Gamma}$ is a chain map, that is $c_{A,\Gamma}(\partial G) = 0$.

PROOF. The independence of the choices of ordering of vertices and roots $e_{i,0}$ is proved in the same way as Lemma 4.7, this time using the cyclicity of $\rho$.

Next note that the morphism $\langle -,- \rangle^{-1} \otimes \ldots \otimes \langle -,- \rangle^{-1}$ is invariant under permutations since $\langle -,- \rangle$ is even, therefore the composition $c_{A,\Gamma}$ is independent of ordering of edges. Also, since $\langle -,- \rangle$ is symmetric, $c_{A,\Gamma}$ does not depend on the choices of orientation of each edge. Thus $c_{A,\Gamma}$ is well-defined.

Finally, since all morphisms in the definition of $c_{A,\Gamma}$ are chain maps, it is also a chain map.

Proposition 4.17. Let $\Gamma$ and $\Gamma'$ be as in Proposition 4.8. Then the following diagram is commutative.

$$
\begin{array}{ccc}
G_*(\Gamma') & \longrightarrow & G_*(\Gamma) \\
c_{A,\Gamma'} & \downarrow & c_{A,\Gamma} \\
Q & \longrightarrow & Q
\end{array}
$$

PROOF. The proof is entirely analogous to the proof of Proposition 4.8. It follows from Lemma 4.10 and the fact that $\rho$ is an operad map.

This Proposition implies that we can take the limit over $(\Gamma' \to \Gamma)$ to define a chain map

$$c_A : G_* \to Q,$$

which is the Kontsevich class of $A$.

Theorem 4.18. Let $A$ and $B$ be two finite dimensional cyclic $A_\infty$ algebras. Assume that the inner products on both $A$ and $B$ are even. Fix a diagonal $\Delta$ of $A_\infty$ to define the cyclic tensor product $A \otimes B$ and the morphism $\delta : G_* \to G_* \otimes G_*$. Then we have $c_{A \otimes B} = (c_A \otimes c_B) \circ \delta$.

In view of Corollary 4.14, this implies $[c_{A \otimes B}] = [c_A] \cup [c_B]$.

PROOF. To simplify the notations, we denote

$$C_{n_1,\ldots,n_k} := C_*(K(n_1)) \otimes \ldots \otimes C_*(K(n_k)),$$

and $\text{End}^0_v := \text{Hom}(V^\otimes n, V)$ for a vector space $V$. Moreover, it is clear that we only need to prove the equality

$$c_{A \otimes B,\Gamma} = (c_{A,\Gamma} \otimes c_{B,\Gamma}) \circ (\delta_\Gamma)$$

for a fixed $\Gamma \in G_*$, since both $\delta_\Gamma$ and $c_{A,\Gamma}$ are compatible with inclusions. The proof then follows from commutativity of the following diagram.
4.4. **Twisted cases.** In the case when $A$ is a cyclic $A_\infty$ algebra with an *odd* inner product, one can still define a Kontsevich class $[c_A]$ in the cohomology of $\coprod_{g,n} M_{g,n}$ with coefficients in a local system $\det$. The fiber of $\det$, over a Riemann surface $\Sigma$, is

$$\det(\Sigma) = \det H^1(\Sigma),$$

the determinant (the top exterior power) of the first cohomology group of $\Sigma$. By definition, it is clear that $\det \otimes^2 = 0$.

In terms of the graph complex, twisting by $\det$ amounts to a changing of the definition of orientation for a graph. A good reference for this is [10].
We define a twisted graph complex $G_\ast(\det)$ as follows. Let $\Gamma$ be a ribbon graph, a twisted orientation $\mu$ on $\Gamma$ is an element of $\det E(\Gamma)$, i.e. an orientation on the vector space spanned by the set of edges of $\Gamma$. We note that, since $\Gamma$ is a connected graph, there is a canonical identification
\[
\det E(\Gamma) \cong \det(V(\Gamma)) \otimes \det(H(\Gamma)) \otimes \det H_1(\Gamma)
\]
where $V(\Gamma)$ is the set of vertices, $H(\Gamma)$ the set of half edges, and $H_1(\Gamma)$ the first homology group of $\Gamma$ (seen as a one dimensional cell complex). We refer the reader to [1] for a proof of this fact. In the following we shall freely use this identification.

The vector space $G_\ast(\det)$ is the span of isomorphism classes of pairs $(\Gamma, \mu)$ modulo the relation $- (\Gamma, \mu) = (\Gamma, -\mu)$. The degree of $(\Gamma, \mu)$ is the same as before: $|\Gamma| = \sum_{v \in V(\Gamma)} \text{val}(v) - 3$. The differential is again given by summing over expansion of vertices:
\[
\partial(\Gamma, \mu) := \sum (\Gamma', \mu')
\]

The two orientations are related as $\mu' = e \wedge \mu$, where $e$ is the unique new edge of $\Gamma'$. As in the untwisted case, given a twisted graph $\Gamma$, we define $G_\ast(\Gamma, \det)$ as the subcomplex of $G_\ast(\det)$ spanned by those $\Gamma'$ which after contracting a subtree equal $\Gamma$.

The construction of the Kontsevich class and the diagonal run in parallel to the untwisted case we discussed in the previous sections. We start by introducing the twisted version of the Construction 4.3 and $\phi_\Gamma$.

**Construction 4.19.** Let $(\Gamma, \mu)$ be a twisted graph in $G_\ast(\det)$. We choose an ordering of the vertices $v_1, \ldots, v_k$. And for each $v_i$, choose a starting half edge $e_{i,0}$. We also choose an ordering of the cycles $s_1, \ldots, s_n$ that span $H_1(\Gamma)$. We define a $B_\Gamma \in \pm 1$ by
\[
\mu = B_\Gamma \langle v_1 e_{1,0} \ldots e_{1,n_1} \ldots v_k e_{k,0} \ldots e_{k,n_k} s_1 \ldots s_n \rangle.
\]

For oriented trees $(T_i, \sigma_i = f_{i,1} \wedge \ldots \wedge f_{i,t_i}) \in C_n_{n_i-t_i-2}(K(n_i))$ ($i = 1, \ldots, k$), we define a twisted graph $\Gamma(T_1, \ldots, T_k)$ by gluing the root of $T_i$ to $e_{i,0}$ and leaves of $T_i$ to $e_{i,1}, \ldots, e_{i,n_i}$ following the cyclic ordering. We define the orientation on $\Gamma(T_1, \ldots, T_k)$ by
\[
\mu_{\Gamma(T_1, \ldots, T_k)} := f_{1,1} \ldots f_{1,t_1} \ldots f_{k,1} \ldots f_{k,t_k} \wedge \mu.
\]

One can easily check that Lemma 4.4 also holds in the twisted complex. Therefore, as in Corollary 4.5, we can define the chain isomorphism
\[
\psi_\Gamma : C_\ast(K(n_1)) \otimes \ldots \otimes C_\ast(K(n_k)) \to G_\ast(\Gamma, \det)
\]
by the formula
\[
\psi_\Gamma(T_1, \ldots, T_k) := (-1)^{|\Gamma|} B_\Gamma (-1)^{\sum_{i<j} n_i t_j} \Gamma(T_1, \ldots, T_k).
\]
Definition 4.20. Assume that $A$ is a cyclic $A_{\infty}$ algebra endowed with an odd inner product. Define $c_{A,\Gamma} : G_*(\Gamma) \to \mathbb{Q}$ as the composition

$$
\begin{align*}
G_*(\Gamma, \det) & \xrightarrow{\psi^{-1}_\Gamma} C_*(K(n_1)) \otimes \ldots \otimes C_*(K(n_k)) \\
& \xrightarrow{\rho \otimes \ldots \otimes \rho} \text{Hom}(A^\otimes n_1, A) \otimes \ldots \otimes \text{Hom}(A^\otimes n_k, A) \\
& \xrightarrow{\cong} (A^\vee)^{\otimes n_1+1} \otimes \ldots \otimes (A^\vee)^{\otimes n_k+1} \\
& \xrightarrow{\eta} (A^\vee)^{\otimes 2} \otimes \ldots \otimes (A^\vee)^{\otimes 2} \\
& \xrightarrow{(-,-)^{-1} \otimes \ldots \otimes (-,-)^{-1}} \mathbb{Q} \otimes \ldots \otimes \mathbb{Q} \cong \mathbb{Q}.
\end{align*}
$$

Here the permutation $\eta$ is the one associated with the permutation of half edges

$$(e_{1,1}, \ldots, e_{1,n_1}, e_{1,0}, e_{2,1}, \ldots, e_{k,1}, \ldots, e_{k,n_k}, e_{k,0}) \to (h_1^+, h_1^-, \ldots, h_E^+, h_E^-),$$

where we choose an ordering of the edges of $\Gamma$ that is compatible with the orientation given by

$$\langle v_{1}e_{1,0} \ldots e_{1,n_1} \ldots v_{k}e_{k,0} \ldots e_{k,n_k}s_1 \ldots s_n \rangle.$$

The definition of $c_{A,\Gamma}$ is independent of all choices made. Indeed, independence on the ordering of the vertices can be proved as in the untwisted case and cyclicity of $\rho$ implies the independence on choices of starting edges $v_i,0$ (see Lemma 4.16). Finally we note that when flipping two cycles $s_i$ and $s_{i+1}$, the sign $B_{\Gamma}$ changes by $(-1)$, which cancels the sign from interchanging two consecutive $\langle -,- \rangle^{-1}$ due to oddness of the inner product.

To take limit over $(\Gamma' \to \Gamma)$, we need to show that $c_{A,\Gamma'}$’s are compatible with inclusions. For this, it is enough to note that Lemma 4.10 (with $A_\Gamma$ replaced by $B_\Gamma$) remains valid. This is because when expanding a graph $\Gamma$ to $\Gamma'$, the set of cycles in the two graphs are canonically identified. Therefore Proposition 4.8 also holds in $G_*(\text{det})$. This produces a chain map

$$c_A : G_*(\text{det}) \to \mathbb{Q},$$

the Kontsevich class of $A$.

As we have seen, in the untwisted case, the proof of the tensor product formula for Kontsevich classes follows easily after the construction of a diagonal map $\delta$ on the graph complex. In the twisted situation, there are two cases to consider:

1. both $\langle -,- \rangle_A$ and $\langle -,- \rangle_B$ are odd;
2. the inner product $\langle -,- \rangle_A$ is even, and $\langle -,- \rangle_B$ is odd (the case when $\langle -,- \rangle_A$ is odd and $\langle -,- \rangle_B$ is even is symmetric).

In the following, for a fixed graph $\Gamma$, as in Definition 4.6, we first define $\delta_\Gamma$ in each case separately.
Case (1): We define \( \delta : \mathcal{G}_s \to \mathcal{G}_s(\det) \otimes \mathcal{G}_s(\det) \) as the composition

\[
\begin{array}{ccc}
\mathcal{G}_s(\Gamma) & \rightarrow^\phi^{-1} & C_{n_1,\ldots,n_k} \\
\downarrow & & \downarrow \Delta \otimes \cdots \otimes \Delta \\
C_{n_1,n_1,\ldots,n_k,n_k} & \rightarrow^\tau & C_{n_1,\ldots,n_k} \otimes C_{n_1,\ldots,n_k}
\end{array}
\]

\[
\mathcal{G}_s(\Gamma, \det) \otimes \mathcal{G}_s(\Gamma, \det) \rightarrow C_{n_1,\ldots,n_k} \otimes C_{n_1,\ldots,n_k}
\]

Case (2): We define \( \delta_{\Gamma} : \mathcal{G}_s(\Gamma, \det) \to \mathcal{G}_s(\Gamma) \otimes \mathcal{G}_s(\Gamma, \det) \) as the composition

\[
\begin{array}{ccc}
\mathcal{G}_s(\Gamma, \det) & \rightarrow^\psi^{-1} & C_{n_1,\ldots,n_k} \\
\downarrow & & \downarrow \Delta \otimes \cdots \otimes \Delta \\
C_{n_1,n_1,\ldots,n_k,n_k} & \rightarrow^\tau & C_{n_1,\ldots,n_k} \otimes C_{n_1,\ldots,n_k}
\end{array}
\]

\[
\mathcal{G}_s(\Gamma) \otimes \mathcal{G}_s(\Gamma, \det) \rightarrow C_{n_1,\ldots,n_k} \otimes C_{n_1,\ldots,n_k}
\]

In both cases the definition of \( \delta_{\Gamma} \) does not depend on the choices involved. For ordering of vertices, this is as before. For the choices of \( e_{i,0} \), independence follows from cyclicity of \( \Delta \). The map \( \psi_{\Gamma} \) also depends on an ordering of the cycles, but note that the map \( \psi_{\Gamma} \) (or its inverse) appears twice in each composition, so this ambiguity is canceled.

Again by the twisted version of Lemma 4.10, the maps \( \delta_{\Gamma} \) are compatible with the inclusions \( \mathcal{G}_s(\Gamma', \det) \subseteq \mathcal{G}_s(\Gamma, \det) \). Hence we can take the limit over \( (\Gamma' \to \Gamma) \) to obtain morphisms

\[
\delta : \mathcal{G}_s(\det) \to \mathcal{G}_s \otimes \mathcal{G}_s(\det).
\]

\[
\delta : \mathcal{G}_s \to \mathcal{G}_s(\det) \otimes \mathcal{G}_s(\det).
\]

With this construction, the proof of the tensor product formula is again the same as that of Theorem 4.18. Thus we have the following

**Theorem 4.21.** Let \( A \) and \( B \) be two finite dimensional cyclic \( A_\infty \) algebras and fix a diagonal \( \Delta \) of \( A_\infty \). Then we have

\[
c_{A \otimes B} = (c_A \otimes c_B) \circ \delta.
\]

This completes the proof of Theorem 1.2.

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