A UNITARY TRANSFORM BASED GENERALIZED APPROXIMATE MESSAGE PASSING

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ABSTRACT

We consider the problem of recovering an unknown signal from general nonlinear measurements obtained through a generalized linear model (GLM). Based on the unitary transform approximate message passing (UAMP) and expectation propagation, a unitary transform based generalized AMP (GUAMP) algorithm is proposed for general measurement matrices, in particular highly correlated matrices. Experimental results on quantized compressed sensing demonstrate that the proposed GUAMP significantly outperforms state-of-the-art Generalized AMP (AMP) and generalized vector AMP (GVAMP) under correlated matrices.

Index Terms— GLM, AMP, GAMP, message passing, quantized compressed sensing

1. INTRODUCTION

We consider the general problem of inference on generalized linear models (GLM) [1], i.e., recovering an unknown signal \( x \in \mathbb{R}^n \) from a noisy linear transform followed by componentwise nonlinear measurements

\[
y = f(Ax + w),
\]

where \( A \in \mathbb{R}^{m \times n} \) is a known linear mixing matrix, \( w \sim \mathcal{N}(w; 0, \Sigma^2 I_m) \) is an i.i.d. Gaussian noise with known variance, and \( f(\cdot) \) is an componentwise nonlinear link function. Denote \( z = Ax \in \mathbb{R}^m \) as the hidden linear transform outputs, the componentwise nonlinear function \( f(\cdot) \) can be equivalently described in a probabilistic way using a fully factorized likelihood function \( p(y | z) = \prod_{i=1}^{n} p(y_i | z_i) \), where \( p(y_i | z_i) \) is determined by the specific nonlinear function \( f(\cdot) \). The prior distribution \( p(x) \) of \( x \) is also assumed to be fully factorized

\[
p(x) = \prod_{j=1}^{n} p(x_j)
\]

for simplicity. GLM inference has wide applications in science and engineering such as wireless communications, signal processing, and machine learning. In the special case of identity function \( f(\cdot) \), the nonlinear GLM in (1) will reduce to the popular standard linear model (SLM) as follows

\[
y = Ax + w.
\]

Thus, GLM is actually an extension of SLM from linear measurements to nonlinear measurements, which are prevalent in some real-world applications such as quantized compressed sensing, pattern classification, phase retrieval, etc.

A variety of algorithms have been proposed for inference over GLMs and SLMs. Among them, the past decade has witnessed an advent of one distinguished family of probabilistic algorithms called message passing algorithm. Among them, the most famous ones are the approximate message passing (AMP) algorithm [2, 3] for SLM and generalized approximate message passing (GAMP) [4] for GLM, which have been proved to be optimal under i.i.d. Gaussian matrices \( A \). However, both AMP and GAMP diverge for general \( A \). In [5–7], the GAMP is incorporated into the sparse Bayesian learning (SBL) to improve the robustness of GAMP and reduce the computation complexity of SBL. Vector AMP (VAMP) [8] (or Orthogonal AMP (OAMP) [9], one similar algorithm to VAMP) and generalized VAMP (GVAMP) [10] have been proposed to improve the performance with general \( A \) for SLM and GLM, respectively. The VAMP is derived from expectation propagation (EP) [11], another powerful message passing algorithm. Remarkably, it has been demonstrated in [12, 13] that all the AMP, GAMP, and GVAMP, can be derived concisely as special instances of EP under different assumptions and thus be unified within a single EP framework. Despite significant improvement in robustness, in particular for right-orthogonally invariant matrices, VAMP and GVAMP still suffer from poor convergence in some more challenging scenarios, e.g., the measurement matrix is highly correlated [14, 15].

In this work, we focus on the AMP variant: unitary approximate message passing (UAMP) [16], which was formerly called UTAMP [16–18]. The motivation behind UAMP is this: since AMP has proven to work well when the elements of \( A \) in (2) are uncorrelated, one can artificially construct such a “good” and equivalent measurement model simply by first performing a singular value decomposition (SVD) on \( A = U \Sigma V^\top \), where \( U \in \mathbb{R}^{m \times r}, \Sigma \in \mathbb{R}^{r \times r}, V \in \mathbb{R}^{r \times r} \) and \( r = \text{rank}(A) \), and then by left multiplying

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$\mathbf{U}^T$ on the original SLM (2), thus leading to an equivalent SLM as [16]

$$\mathbf{b} = \mathbf{Q} \mathbf{x} + \tilde{\mathbf{w}}, \quad (3)$$

where $\tilde{\mathbf{b}} = \mathbf{U}^T \mathbf{y} \in \mathbb{R}^r$, $\mathbf{Q} = \mathbf{\Sigma V}^T \in \mathbb{R}^{r \times n}$ and $\tilde{\mathbf{w}} = \mathbf{U}^T \mathbf{\tilde{w}} \in \mathbb{R}^r$ correspond to unitary-transformed pseudo linear measurements, linear mixing matrix, and additive noise, respectively. Subsequently, one can readily run standard AMP on the equivalent model (3), which leads to one version of UAMP. Further, two averaging operations can be performed to obtain the other version of UAMP, where the number of matrix-vector products per iteration is reduced from 4 to 2 [16–18]. Despite its simplicity, UAMP has been shown to be more robust than AMP and VAMP under some types of “tough” measurement matrices, e.g., highly correlated matrices [17, 18]. Nevertheless, the extension of UAMP to the case of GLM has remained lacking.

In this paper, we extend UAMP from SLM to GLM and propose the generalized UAMP (GUAMP). The key idea is to utilize the unified Bayesian inference framework in [13, 19] and EP [11] to iteratively decompose the original nonlinear measurement model (1) into a series of pseudo SLMs, whereby UAMP could be conducted. Conceptually, GUAMP consists of two modules, namely AMP module and GAMP module, as shown in Fig. 1. Extrinsic information [20] is exchanged per iteration between the two modules before GUAMP finally converges. Experimental results demonstrate that GUAMP significantly outperforms both GAMP and GVAMP under highly correlated matrices $\mathbf{A}$.

2. THE GUAMP ALGORITHM

The key idea is that, inspired by UAMP [16], we introduce an additional hidden variable $\mathbf{b} \in \mathbb{R}^r$ and thus an equivalent representation of $\mathbf{z} \triangleq \mathbf{A} \mathbf{x}$ as follows

$$\mathbf{b} = \mathbf{\Sigma V}^T \mathbf{x} \triangleq \mathbf{Q} \mathbf{x}, \quad (4a)$$

$$\mathbf{z} = \mathbf{U} \mathbf{b}, \quad (4b)$$

where, as in the original UAMP [16], $\mathbf{U} \in \mathbb{R}^{m \times r}$, $\mathbf{\Sigma} \in \mathbb{R}^{r \times r}$, $\mathbf{V} \in \mathbb{R}^{n \times r}$ are obtained from SVD $\mathbf{A} = \mathbf{U} \mathbf{\Sigma V}^T$ when rank($\mathbf{A}$) = $r$. Note that $m < n$ and $m \geq n$ cases are treated in a unified way.

The corresponding factor graph of the original GLM (1) can be equivalently shown in Fig. 1 (a). Subsequently, using the unified modular framework in [13], the inference on the factor graph in Fig. 1 (a) can be decomposed into two modules, namely one AMP module and one GAMP module, as shown in Fig. 1 (b). At a high level, in each iteration, module A performs standard AMP w.r.t. $\mathbf{x}$ on an equivalent SLM with a pseudo measurement matrix $\mathbf{Q}$ while module B performs GAMP w.r.t. $\mathbf{b}$ on an equivalent GLM with a pseudo measurement matrix $\mathbf{U}$, rather than the original measurement matrix $\mathbf{A}$. Module A and Module B exchange extrinsic information with each other in the same way as [13] and this process proceeds before convergence. Please refer to [13] for further details of the modular representation perspective via EP [11].

In the following, we describe the implementation details of GUAMP. First of all, we initialize $\mathbf{\tau}_B^0(0) \in \mathbb{R}^r$, $\mathbf{\tilde{s}}_B(-1) \in \mathbb{R}^m$, $\mathbf{b}_A^{ext}(1) \in \mathbb{R}^r$ and variances $\nu_{A,ext}^{b}(1) \in \mathbb{R}^r$ in module B, and initialize $\hat{\mathbf{p}}_A(0) \in \mathbb{R}^r$, $\hat{\mathbf{\tau}}_B^{ext}(0) \in \mathbb{R}^r$, $\hat{\mathbf{x}}(0) \in \mathbb{R}^n$. The number of outer iterations between module A and module B is set as $T_{max}$, while the number of inner iterations in module A and B during an outer iteration are set as $T_A$ and $T_B$, respectively. Next, we present the details of the operation in the two modules.

2.1. The GAMP Module

The extrinsic means $\mathbf{b}_A^{ext}(t) \in \mathbb{R}^r$ and variances $\nu_{A,ext}^{b}(t) \in \mathbb{R}^r$ transmitting from module A to module B can be viewed as the prior means and variances of $\mathbf{b}$, i.e.,

$$p(\mathbf{b}) = \mathcal{N}(\mathbf{b}; \mathbf{b}_A^{ext}(t), \text{diag}(\nu_{A,ext}^{b}(t))). \quad (5)$$

In addition, $\mathbf{z} = \mathbf{U} \mathbf{b}$ and $\mathbf{y} | \mathbf{z} \sim p(y | z)$. Thus we could run the GAMP algorithm [4] treating $\mathbf{b}$ as the unknown signal and $\mathbf{U}$ as the measurement matrix as follows.

- [Step 1 (B)] Perform output linear step to obtain $\hat{\mathbf{p}}_B(t) \in \mathbb{R}^m$ and $\mathbf{\tau}_B(t) \in \mathbb{R}^m$ as

$$\mathbf{\tau}_B^{t}(t) = |U|^2 \mathbf{\tau}_B(t), \quad (6a)$$

$$\hat{\mathbf{p}}_B(t) = \mathbf{U} \hat{\mathbf{b}}(t) - \mathbf{\tau}_B^{t}(t) \hat{\mathbf{s}}_B(t - 1). \quad (6b)$$

- [Step 2 (B)] Perform output linear step to obtain $\hat{\mathbf{s}}_{B,i}(t)$ and $\mathbf{\tau}_{B,i}(t)$ as

$$\hat{\mathbf{s}}_{B,i}(t) = g_{out}(\hat{\mathbf{p}}_{B,i}(t), y_i, \mathbf{\tau}_{B,i}(t)), \quad (7a)$$

$$\mathbf{\tau}_{B,i}(t) = - \frac{\partial g_{out}(\hat{\mathbf{p}}_{B,i}(t), y_i, \mathbf{\tau}_{B,i}(t))}{\partial \hat{\mathbf{p}}}, \quad (7b)$$

for $i = 1, 2, \cdots, m$, where

$$g_{out}(\hat{\mathbf{p}}, y, \mathbf{\tau}) = \frac{\mathbf{\tilde{z}}_0 - \hat{\mathbf{p}}}{\mathbf{\tau}}, \quad (8)$$

$$- \frac{\partial g_{out} (\hat{\mathbf{p}}, y, \mathbf{\tau})}{\partial \hat{\mathbf{p}}} = \frac{\mathbf{\tau}_p - \text{Var}(z | \hat{\mathbf{p}}, y)}{(\mathbf{\tau}_p)^2}, \quad (9)$$

Fig. 1: (a). Equivalent factor graph representation of GLM. (b) Modular representation of the GUAMP algorithm.
and the posterior means $\hat{z}^0$ and variances $\text{Var}(z|\hat{p}, y)$ are computed w.r.t. the posterior $\propto \mathcal{N}(z; \hat{p}, \tau_p)p(y|z)$. See [4] for further details.

- [Step 3 (B)] Perform input linear step as
  \[
  \tau_B(t) = (U^T \mathbf{r}_B(t))^{-1},
  \]
  \[
  \hat{r}_B(t) = \mathbf{b}_B(t) + \tau_B(t) \odot (U^T \hat{s}_B(t-1)).
  \]

- [Step 4 (B)] Perform input nonlinear step in Module B to obtain the posterior means and variances of variable $b$ as: For $j = 1, 2, \ldots, r$,
  \[
  \hat{b}_j(t) = \left[ \frac{\hat{r}_{b,j}(t)v_{A,j}^{\text{ext},b}(t) + b_{A,j}(t)\tau^{\text{ext},b}_{B,j}(t)}{\tau^{\text{ext},b}_{B,j}(t) + v_{A,j}^{\text{ext},b}(t)} \right],
  \]
  \[
  \tau^{\text{b}}_j(t) = \left[ \frac{v_{A,j}^{\text{ext},b}(t)\tau^{\text{ext},b}_{B,j}(t)}{\tau^{\text{ext},b}_{B,j}(t) + v_{A,j}^{\text{ext},b}(t)} \right].
  \]

After running $T_B$ iterations, the extrinsic means $b^{\text{ext}}_B(t)$ and variances $v^{\text{ext},b}_B(t)$ of $b$ from module $B$ to module $A$ can be calculated as follows [13]
\[
b^{\text{ext}}_B(t) = \hat{r}_B(t), v^{\text{ext},b}_B(t) = \tau^{\text{b}}_B(t).
\]

**Remark:** It is worth noting that if the GLM (1) degenerates to the SLM (2), it can be verified that the extrinsic information $b^{\text{ext}}_B$ and $v^{\text{ext},b}_B(t)$ from module $B$ to module $A$ are always $b^{\text{ext}}_B = U^T \mathbf{y}$ and $v^{\text{ext},b}_B = \sigma^2 \mathbf{1}_r$. Consequently, the GUAMP reduces to the UAMP in the special case of SLM.

### 2.2. The AMP Module

As shown in [13], the extrinsic means $b^{\text{ext}}_B(t)$ and variances $v^{\text{ext},b}_B(t)$ can be regarded as the pseudo observations and variances of $b$ in module $A$, i.e.,
\[
\tilde{b}(t) = Qx + \tilde{c}(t),
\]
where $\tilde{b}(t) \triangleq b^{\text{ext}}_B(t)$, $\tilde{c}(t) \sim \mathcal{N}(0, \text{diag}(v^{\text{ext},b}_B(t)))$. Consequently, we could run standard AMP with $T_A$ iterations on this pseudo-linear model (13), as shown below.

- [Step 1 (A)] For $i = 1, 2, \ldots, r$, $\hat{s}_{A,i}(t)$ and $\tau^{\text{a}}_{A,i}(t)$ can be calculated as
  \[
  \hat{s}_{A,i}(t) = \left[ \frac{b^{\text{ext}}_{B,i}(t) - \hat{p}_{A,i}(t-1)}{\tau^{\text{ext},b}_{B,i}(t) + \tau^{\text{a}}_{A,i}(t-1)} \right],
  \]
  \[
  \tau^{\text{a}}_{A,i}(t) = \left[ \frac{1}{\tau^{\text{ext},b}_{B,i}(t) + \tau^{\text{a}}_{A,i}(t-1)} \right].
  \]

- [Step 2 (A)] Perform input linear step to obtain
  \[
  \hat{r}_A(t) \in \mathbb{R}^n \text{ and } \tau^{\text{a}}_A(t) \in \mathbb{R}^n \text{ as}
  \]
  \[
  \tau^{\text{a}}_A(t) = (Q^T \mathbf{r}_A(t))^{-1},
  \]
  \[
  \hat{r}_A(t) = \hat{x}(t-1) + \tau^{\text{a}}_A(t) \odot (Q^T \hat{s}_A(t-1)).
  \]

which can be characterized by a pseudo model
\[
\hat{r}_A(t) = x + \tilde{w}(t),
\]
where $\tilde{w}(t) \sim \mathcal{N}(0, \text{diag}(\tau^{\text{a}}_A(t)))$.

- [Step 3 (A)] Perform input nonlinear step in Module A to obtain the posterior means and variances of $x$.
  \[
  \hat{x}_j(t) = E[x_j|\hat{r}_{A,j}(t), \tau^{\text{a}}_{A,j}(t)],
  \]
  \[
  \tau^{\text{a}}_j(t) = \text{Var}[x_j|\hat{r}_{A,j}(t), \tau^{\text{a}}_{A,j}(t)].
  \]

for $j = 1, 2, \ldots, n$, where $E[\cdot]$ and $\text{Var}[\cdot]$ denotes the posterior means and variances with respect to the likelihood calculated in (16) and the prior $p(x)$. Even if the nuisance parameters in $p(x)$ are unknown, EM algorithm [21] can be easily incorporated into the GUAMP to learn them similarly as [22].

- [Step 4 (A)] Perform output linear step to obtain

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**Algorithm 1 GUAMP Algorithm**

1. Initialize $\tau^{\text{a}}(0) \in \mathbb{R}^r$, $\hat{s}_B \in \mathbb{R}^m$, $b^{\text{ext}}(1) \in \mathbb{R}^r$ and variances $v^{\text{ext},b}(1) \in \mathbb{R}^r$ in module $B$. Initialize $\hat{p}_A(0) \in \mathbb{R}^r$, $\hat{x}(0) \in \mathbb{R}^n$. Set $T_{\text{max}}, T_A$ and $T_B$.
2. **for** $t = 1, \ldots, T_{\text{max}}$ **do**
   1. // [Step 1 (B)] Perform output linear step
      **Calculate** $\hat{p}_B(t)$ (6b) and $\tau^{\text{b}}_B(t)$ (6a).
   2. // [Step 2 (B)] Perform output linear step
      **Calculate** $\hat{s}_{B,i}(t)$ (7a) and $\tau_{B,i}(t)$ (7b).
   3. // [Step 3 (B)] Perform input linear step
      **Calculate** the posterior means $\hat{r}_B(t)$ (10b) and variances $\tau^{\text{b}}_B(t)$ (10a) of $b$.
   4. // [Step 4 (B)] Perform input nonlinear step
      **Calculate** the posterior means (11a) and variances (11b) of $b$.
3. **end for**
4. Set $b^{\text{ext}}_B(t) = \hat{r}_B(t)$ and $v^{\text{ext},b}_B(t) = \tau^{\text{b}}_B(t)$.
5. **for** $t_A = 1, \ldots, T_A$ **do**
   1. // [Step 1 (A)] Perform output nonlinear step.
      **Calculate** $\hat{s}_A(t) \in \mathbb{R}^m$ (14a) and $\tau^{\text{a}}_A(t) \in \mathbb{R}^m$ (14b).
   2. // [Step 2 (A)] Perform input linear step
      **Calculate** $\hat{r}_A(t) \in \mathbb{R}^m$, (15b) and $\tau^{\text{a}}_A(t) \in \mathbb{R}^m$ (15a).
   3. // [Step 3 (A)] Perform input nonlinear step
      **Calculate** the posterior means (17a) and variances (17b) of $x$.
   4. // [Step 4 (A)] Perform output linear step
      **Calculate** $\tau^{\text{a}}_A(t)$ (17a) and variances $p_A(t)$ (18b).
6. **end for**
7. Return $\hat{x}$.
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