Maximally correlated matter from many-body quantum coherence

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We construct quantum coherence resource theories in symmetrized Fock space (QCRTF) that unify previous frameworks for the analysis of coherence in discrete-variable (DV) and continuous variable (CV) quantum systems. Unlike traditional finite dimensional or CV quantum coherence resource theories, QCRTF can be made independent of the single-particle basis and allows to quantify coherence within and between particle number sectors. In a basis-independent QCRTF, we construct free quantum channels that are associated with Stinespring isometries that preserve the set of free states and show that an energy density constraint can be imposed on the free channels that still allows a wide range of protocols to be implemented within the resource theory. The QCRTF framework is utilized to calculate the optimal asymptotic distillation rate of maximally correlated states both for particle number conserving resource states and resource states of indefinite particle number. In particular, we show that energy density preserving manipulations of bosonic insulating states allow the extraction of a uniform superposition of maximally correlated states from a state of maximal bosonic coherence with asymptotically unit efficiency.

I. INTRODUCTION

Protocols for transferring quantum coherence from discrete variable (DV) quantum systems to continuous variable (CV) quantum systems, and vice versa, provide vital links between quantum information processing platforms [1]. Although most proposed protocols involve explicit models for light-matter dynamics [2, 3], for analyses of fundamental limits on processing quantum coherence it is useful to restrict the availability of quantum states and channels according to a resource-theoretic framework [4–6]. For example, utilizing local Gaussian operations (i.e., quantum channels that map Gaussian states to Gaussian states), it is possible to produce Gaussian entanglement from entangled DV quantum systems [7]. The recent development of quantum coherence resource theories (QCRT) for DV and CV quantum systems allow to place such protocols within the more general context of coherence manipulations [6, 8–10].

In previously studied QCRTs, the free states of composite quantum systems are constructed from the tensor product of free states of a single subsystem [11–14]. When the free states of a single subsystem are separable, this necessarily implies that free states of the composite system are separable. However, in the case of bosonic matter, all states that are not statistical mixtures of Bose-Einstein condensates (BEC) exhibit entanglement. This fact motivates the analysis of QCRTs for identical BEC exhibit entanglement. Matter, all states that are not statistical mixtures of free states, showing that in certain instances of QCRTF, dynamical resources are not necessary to implement free quantum operations. In Section II A we discuss some features of QCRTF that are relevant to practical implementations, including the possibility of constraining the set of free quantum channels to preserve the energy density according to a partition of the single-particle modes. Among the quantum coherence manipulation protocols that can be formulated within the QCRTF, we focus in Section III on the distillation of maximal correlations between pairs of modes (i.e., determination of the largest \( R > 0 \) such that \( nR \) copies of a maximally correlated (MC) state with energy \( E \) are produced from \( n \) copies of a given quantum state with energy \( E \) by using only free quantum channels in the QCRTF). This approach yields the asymptotic MC distillation rate for Bose-Einstein condensates, pair correlated states [15], and maximally coherent states of indefinite particle number. In the latter case, we show that preparation of maximal coherence with respect to QCRTF allows extraction of intermediate correlations with asymptotically unit efficiency.

The setting of our analysis is the symmetrized Fock space \( \mathcal{F}_M = \mathbb{C} \text{VAC} \mathcal{P}_B \bigoplus_{N=1}^{\infty} \mathcal{H}_N^\otimes N \) associated to an \( M \)-dimensional single-particle Hilbert space \( \mathcal{H}_B \), with \( \mathcal{P}_B \) the Bose projector and \( N \)-particle sectors labeled \( \mathcal{F}_M^{(N)} := Q_N \mathcal{F}_M \) where \( Q_N \) is the \( N \)-particle sector projection. For a physical picture, one can consider, e.g., optically trapped ultracold bosons in the grand canonical ensemble. Since the quantum coherence resource theories that we consider interpolate between descriptions of DV quantum coherence and CV quantum coherence, we use the generic notation \( S(\mathcal{K}) \) for quantum states on a given Hilbert space \( \mathcal{K} \). We use an asterisk to denote the adjoint of a linear operator on \( \mathcal{K} \).

II. QUANTUM COHERENCE IN BOSONIC FOCK SPACE

We motivate and define three versions of QCRTF which have free state sets that are related by inclusion.
However, the corresponding sets of free operations are not so simply related, and thus allow to analyze a wide variety of information processing protocols.

As a first step, consider the restricted quantum coherence resource theory QCRT\(^{(N)}\), which is associated to a particle number sector \(\mathcal{F}_M^{(N)}\), \(N \in \mathbb{Z}_{\geq 0}\). The free states in QCRT\(^{(N)}\) are discrete probability measures on the Fock states \(|\vec{\mu}\rangle/\langle \vec{\mu}|\) where \(\vec{\mu} = (m_1, \ldots, m_M) \in \mathbb{Z}_{\geq 0}^M\) and \(|\vec{\mu}| := \sum_{j=1}^M m_j = N\). We refer to this set of free states as \(\Delta^{(N)}_M\), its extreme points (pure free states) as \(\Phi^{(N)}_M\) and take \(\Delta^{(0)}_M := \Phi^{(0)}_M = |\text{VAC}\rangle/\langle \text{VAC}|\), the empty vacuum, for all \(M\). The set of free operations \(\mathcal{E}^{(N)}_M\) is taken to be those completely positive, trace perserving (CPTP) quantum channels \(\Phi\) for which the indexed set of bounded operators \(\{K_\beta^\star\}_\beta\) acting on \(\mathcal{F}_M^{(N)}\) provide a Kraus decomposition for \(\Phi\) and further satisfies

\[
\sum_\beta K_\beta^\star K_\beta = \mathbb{I}^{(N)}_M
\]

for all indices \(\beta\) and \(\rho \in \Delta^{(N)}_M\). \(\mathcal{E}^{(N)}_M\) is a proper subset of the set of CPTP \(\Phi\) such that \(\Phi(\Delta^{(N)}_M) \subset \Delta^{(N)}_M\), except in the cases of QCRT\(^{(1)}\) and QCRT\(^{(1)}\), in which the respective identity channels are the only free channels. Note that due to the bosonic symmetry, QCRT\(^{(N)}\) is not equivalent to a tensor product of quantum coherence resource theories of a finite dimensional Hilbert space with orthonormal basis \([1], \ldots, [M]\) defining the free states. For example, in such a tensor product, the two-particle, non-bosonic state \([1]_A \otimes [2]_B\) is free. With these definitions in hand, the first version of QCRT\(_M\), called QCRT\(_M^{(1)}\), is straightforwardly defined by the set of free states \(\Delta^{A}_{M}:=\oplus_{N\geq 0} p_N \Delta^{(N)}_M\), where \(p_N\) is a discrete probability density on \(\mathbb{Z}_{\geq 0}\), and the set of free operations \(\mathcal{E}^{A}_{M}:=\oplus_{N\geq 0} \mathcal{E}^{(N)}_M\) are restricted to states of free, non-relativistic bosons on the lattice \([1, \ldots, L]^d\) at temperature \(T \geq 0\) and chemical potential \(\mu \in \mathbb{R}\). Free quantum channels from \(\Delta^{A}_{M}\) to \(\Delta^{A}_{M}\) are constructed via \(\Phi_2 \circ \mathcal{Q} \circ \Phi_1\) where \(\Phi_1(2) \in \mathcal{E}^{A}_{M(M')}\), and \(\mathcal{Q}\) is the partial trace over modes or bosonic mode appending channel. Explicitly, incoherent quantum systems are appended via tensor product followed by Boson symmetrization, e.g., \([n_1, \ldots, n_M]\) \(\in \Delta^{A}_{M}\) with \(|\vec{n}| = N\) defines the appending channel \(\Delta^{A}_{M} \rightarrow \oplus_{N \geq N'} \Delta^{A}_{M+M'} \subset \Delta^{A}_{M+M'}\) given by \(|\vec{\mu}\rangle \rightarrow |\vec{n}\rangle \langle \vec{n}|\).

Not every element of the set of free operations in QCRT\(_M^{(1)}\) given in Eq.\([1]\) can be written as a concatenation of the following three quantum channels: appending a free environment state, evolving by a free system-plus-environment interaction, and tracing over the environment degrees of freedom (such a concatenation defines the existence of a free Stinespring dilation). However, because the \(\Delta^{(N)}_M\) has a natural action by the permutation group on \(\binom{N+M-1}{N}\) letters, the subset of \(\mathcal{E}^{(N)}_M\) that can be expressed in such a free Stinespring form can be constructed following Proposition 1 of Ref.\([20]\).

In the examples that we consider in Sec.\([III]\) it will be useful to define a version of QCRT\(_M^{(B)}\) that maintains the particle number superselection sectors as in QCRT\(_M^{(A)}\), but is independent of the single-particle basis. A quantum coherence resource theory QCRT\(_M^{(B)}\) that allows to do this is defined by a set of free states \(\Delta^{B}_{M}\) such that \(\rho \in \Delta^{B}_{M}\) if and only if for any \(0 < \epsilon < 2\), there is an \(N \in \mathbb{Z}_{\geq 0}\), a joint probability density \(P: U(M) \times (\Phi^{(0)}_M \oplus \cdots \oplus \Phi^{(N)}_M) \rightarrow \mathbb{R}\), and \(\rho' \in \mathcal{S}(\mathcal{F}_M)\) having the form

\[
\rho' = \sum_\sigma \int \mu(d(\rho, \sigma)) U_g \sigma U_g^\star
\]

such that \(\|\rho - \rho'\|_1 < \epsilon\). In Eq.\([7]\), \(\mu(d(\rho, \sigma))\) is the normalized Haar measure on \(U(M)\) so that \(\Delta^{B}_{M}\) consists of statistical mixtures of linear optical quantum computations on \(\Delta^{A}_{M}\). Note that applying the quantum channel \(\rho \mapsto \lim_{\epsilon \rightarrow 0} \int \sigma \mu(d(\rho, \sigma)) U_g \sigma U_g^\star \) gives \(\Delta^{A}_{M}\). The free set \(\Delta^{B}_{M}\) has striking features which make it appropriate for the consideration of many-body quantum coherence, as opposed to single-particle quantum coherence. For example, unlike QCRT\(_M^{(1)}\), all single-particle quantum states are contained in \(\Delta^{B}_{M}\). By the same reasoning, all Bose-Einstein condensates are contained in \(\Delta^{B}_{M}\). The general consequence of these facts is that the definition Eq.\([2]\) is independent of how the single-particle basis of \(\mathcal{H}\) is chosen. In general, any pure element of \(\Delta^{B}_{M}\) is given by the ground state of a twisted Bose-Hubbard interaction of the form

\[
H_g := \sum_{j=1}^M U_g (a_j^\star a_j - m_j) U_g^\star.
\]

Such an interaction can be compared to the locally varying potentials that arise in the physics of ultracold atoms in random optical traps.\([22]\).

The set of free operations in QCRT\(_M^{(B)}\), \(\mathcal{E}^{B}_{M}\), is taken to consist of quantum channels \(\Phi\) having the composite form \(\Phi = (\oplus_{N=0}^\infty \Phi_N)\) where, for each \(N\), \(\Phi_N(\rho) = \sum_\beta K_\beta^{(N)} \rho K_\beta^{(N)^\star}\), where \(K_\beta^{(N)}: \mathcal{F}_M^{(N)} \rightarrow \mathcal{F}_M^{(N)}\) are bounded and

\[
\sum_\beta K_\beta^{(N)^\star} K_\beta^{(N)} = \mathbb{I}^{(N)}_M
\]

for all \(\rho \in \Delta^{B}_{M}\) and for all \(\beta\). Despite the proper subset relation of free states given by \(\Delta^{A}_{M} \subset \Delta^{B}_{M}\), the set \(\mathcal{E}^{A}_{M}\) is not contained in \(\mathcal{E}^{B}_{M}\). For example, the projection-valued
**measurement channel**

\[
\rho \mapsto (I_{F_2} - P) \rho (I_{F_2} - P) + P \rho P
\]

where \( P := |0,N\rangle \langle 0,N| + |N,0\rangle \langle N,0| \), is an element of \( \mathcal{E}^A \) but not an element of \( \mathcal{E}^B \) since, e.g., it contains a Kraus operator that maps the Bose-Einstein condensate \( |a^+_N a^+_N\rangle \langle 0| \) to the GHZ state (NOON state) \( |a^+_N a^+_N\rangle \langle 0| \) belonging to \( \mathcal{E}^B \), which means that the probability distribution over displacements of \( M \)-mode bosonic vacua. Explicitly, \( \rho \in \Delta_M \) if and only if for any \( 0 < \epsilon < 2 \), there is an \( N \in \mathbb{Z}_{\geq 0} \) and a joint probability density \( P \in \mathbb{R}^M \) such that \( \rho' \) having the form

\[
\rho' = \frac{1}{\pi^M} \int d^2\mathbf{a} \rho(\mathbf{a}, \sigma) D(\mathbf{a}) U_{\sigma} \mathbf{a} D(\mathbf{a})^* \quad (7)
\]

such that \( \| \rho - \rho' \|_1 < \epsilon \). In Eq. (7), \( D(\mathbf{a}) := \exp i R\mathbf{Z}_{\mathbf{a}} \) and \( \mathbf{Z}_{\mathbf{a}} \in \mathbb{R}^M \) is the solution of the linear equation \( \alpha_j = (-z_{2j} + i z_{2j-1})/\sqrt{2} \).

**Proposition 1.** \( \mathcal{W}^B \subset \mathcal{E}^B \) and \( \mathcal{W}^B(\Delta^B_M) \subset \Delta^B_M \). If \( \Phi \in \mathcal{E}^B_M \) and for each \( N \in \mathbb{N}_M \), there is a \( U_N \in \mathcal{U}M \) and sequence \( \{ k_j(N) \}_{j=1}^M \in \mathbb{Z}_{\geq 0}^M \) such that the Kraus operators \( \{ K_j(N) \}_{j=1}^M \) of the restriction of \( \Phi \) to the particle sector satisfy \( \sum_{j=1}^M a_j a_j^0 \) and hence, its elements have \( P \)-function given by a probability density on \( \mathbb{C}^M \). However, the usual mathematical setting of quantum optics is a tensor product of copies of \( \mathcal{L}(\mathbb{C}) \), not the image of that tensor product under \( P \).

The connection between QCRTM and quantum optics is made by going from the Fock representation \( \mathcal{F}_M \) to the Schrödinger representation \( \mathcal{L}^M \) of the canonical commutation relations (CCR), i.e., by mapping the bosonic state \( |\tilde{m}\rangle \in \Delta_M^N \) to \( |m_1 \rangle \otimes \cdots \otimes |m_M \rangle \) of an \( M \)-dimensional quantum harmonic oscillator. Finally, from the set of factorizable \( P(\tilde{a}, \sigma) \) with the form \( \sum_{j=1}^M a_j a_j^0 \) in Eq. (7), it is clear that \( \Delta_M^N \) is a proper subset of \( \Delta_M^N \) (e.g., a coherent state \( D(\tilde{a}) |\tilde{m}\rangle \in \Delta_M^N \) is not in \( \Delta_M^N \)).

A simple set of operations \( \mathcal{E}^B_M \) in QCRTM is constructed by a method analogous to the previous examples. The set \( \mathcal{E}^B_M \) is defined to be those CPTP \( \Phi \), for
which there exists an indexed set of bounded operators \( \{K_β\}_β \) acting on \( F_M \) and a \( V \in \mathcal{U}_M \) such that

\[
\sum_β K_β^* K_β = I_{FM}
\]

\[
K_β K_β^* = \frac{\text{tr} K_β \rho K_β^*}{\Delta^C_M} \quad (8)
\]

for all \( ρ \in Δ^A_M \), where \( I_{FM} := \oplus_{N=0}^\infty I^{(N)} \) is the Fock space identity. Note that there are no noiseless elements of \( E_M \) that map one particle number sector into another (the image of a displacement map has infinite dimensional support). The idea in the second part of the proof of Proposition 1 leads to an analogous partial result for QCRF^C_M. We define \( WC \) by replacing “B” by “C” in definition[8].

**Proposition 2.** \( WC \subset E^C_M \) and \( WC(Δ^C_M) \subset Δ^C_M \). If \( Φ \in E^C_M \) has Kraus operators \( \{K_β\}_β \) where \( J \) is an index set, and if \( Φ \in U_M \), sequences \( \{(k_j)_{j=1}^\infty \in \mathbb{Z}_{≥0}^M \) and \( \{(α_j)_{j=1}^\infty \in C^M \) such that \( \sum_β K_β = U \sum_β (a_β^* + α_β)^{β_0} \), then \( Φ \in WC \).

Quantum states in \( S(F_M) \) exhibit coherence with respect to QCRF^A_M in two ways, viz., a state can have coherence between particle number sectors or within a particle number sector. For example, in QCRF^A_M, average coherence within the particle number sectors can be defined via the weighted entropy of coherence \( C(ρ) = \sum_{N≥0} (\text{tr} Q_{Nρ}) C_N(ρ) \) where \( C_N(ρ) = H(\sum_{σ∈\Psi_M^{(N)}N} |\frac{Q_{Nσ}^N}{Q_{Nρ}^N}|^2) - H(\frac{Q_{Nρ}^N}{Q_{Nρ}^N}) \), where \( H \) is the von Neumann entropy function on quantum states [12]. On the other hand, it is not possible to quantize quantum coherence with respect to QCRF^B(C) if they exhibit superposition in the basis of coherent states with various bosonic vacua. For QCRF^B(C), coherence measures that satisfy widely adopted axioms for the quantification of quantum coherence can be constructed by generalizing the algorithm in Ref.[8]. Since our examples in Sec III make use of the aforementioned entropy of coherence \( C \) and \( C^A \), we do not flesh out the details of extending the algorithm here.

### A. Physical considerations

We conclude this section with a discussion of some features of the QCRF^A,B,C_M that are relevant to the physical aspects of information processing within these theories. Firstly, unlike for finite-dimensional quantum coherence resource theories on spin-1/2 chains, the CNOT operation acting on two particles is not free in any of the QCRF_M because it is not a linear transformation of the bosonic Fock space (e.g., it maps the bosonic state \( a_A^* a_B^* |0⟩ \) to \( \frac{1}{\sqrt{2}} (|0⟩ \otimes |1⟩_B + |1⟩_A \otimes |0⟩_B) \). The incoherent nature of the CNOT operation is central to analyses of the relation between coherence and entanglement in DV systems[28,29]. However, \( E^A,B,C_M \) contain bosonic analogues of controlled operations. For example, in a bosonic system of \( M' > M \) modes, the sequence \( \{L_N := |m⟩_N ⟨m_N|^N = 0 \} \) together with the sequence \( \{U_N\} \in \mathcal{U} \) of linear optical unitaries acting on modes \( M+1, \ldots, M' \) can be used to define the unitary operator \( \sum_{N=0}^\infty L_N \otimes \{U_N\} U_N \in E^C_M \).

Finally, a principal criticism regarding the definition of free operations as given in Eq. (11) is that there is no a priori constraint on the energy density of the set of free states. For example, in QCRF^A_M there are free quantum channels that transform the homogeneous state \( |1,1,\ldots,1⟩ \) to \( |M,0,\ldots,0⟩ \), whereas such an operation is not always feasible in a given optical trap setup. In the protocols discussed in Section III for distillation of maximally correlated states from initial states in \( S(F_2) \), we only consider free quantum channels that preserve the energy density, i.e., preserve the expected local number of particles with respect to a specified partition of the single particle modes (even if the total number of modes varies).

### III. EXAMPLES AND MAXIMALLY CORRELATED STATE DISTILLATION

The distillation of Gaussian entanglement from non-Gaussian states reported in Ref.[2] involves only linear optical unitary operations and local vacuum projections, both of which are examples of quantum channels in \( E^D_M \). More generally, since particle detection measurements are defined by projections in \( \Psi_M^{(0)} + \cdots + \Psi_M^{(N)} \), which are in \( E^B_M \), it follows that protocols that implement linear optical dynamics supplemented by particle detection measurements or their coarse-grainings are also in \( E^B_M \) (e.g., boson sampling[30], and cat state amplification[31]). We now proceed to consider protocols for distillation of maximally correlated states within the QCRF framework. Our analysis will treat both cases of quantum states with fixed particle number and indefinite particle number, and will make use of free quantum channels that preserve the local particle number in expectation, in accordance with the discussion in Section IIIA.

#### A. Number conserving correlation distillation

We now consider the task of producing maximal correlations from DV and CV mode-symmetric states within the QCRF framework. For the analysis of the present subsection, which concerns states of fixed particle number \( N \), we focus on a target state that consists of many copies of a maximally correlated state in \( F_2^N \), e.g., \( |MC_N⟩ \propto \sum_{m=0}^N |N - m, m⟩ \). Note that \( MC_1 \in Δ^B,C_2 \) since it is simply a maximally coherent qubit state, but for \( N > 1 \), \( MC_N \notin Δ^B,C_2 \). By imposing the constraint
that initial states and target state have the same particle number, the distillation rate provides an asymptotically energy-independent quantity that characterizes the usefulness of the initial state as a resource for producing quantum correlations.

To begin, consider $n$ copies of a Bose-Einstein condensed state describing $N$ particles condensed into an equal-amplitude superposition of $2$ orthogonal single-particle states (i.e., $n$ copies of an $SU(2)$ coherent state)

$$\text{BEC}_{N,n} \propto \prod_{j=1}^{n} (a_{2j-1}^* + a_{2j})^{N}|\text{VAC}\rangle \in \Delta_{2n}^B.$$  \hspace{1cm} (9)

This is a bosonic version of $n$ independent, identically distributed Bose-Einstein condensates $\times$ BEC$_N^c$ that would be considered in a traditional quantum communication task. In order to derive the optimal asymptotic rate $R$ for extraction of $nR$ copies of a MC$_N$ state from BEC$_{N,n}$, we proceed by analogy with the optimal protocol for pure state distillation in the quantum coherence resource theory of spin chains [12 32]. We denote $\vec{m}^n$ a list $(\vec{m}_1, \ldots, \vec{m}_n)$ such that $\vec{m}_n = (m_{1n}, m_{2n}) \in \mathbb{Z}^2$, and $|\vec{m}_n| = N$. It is clear that every projection onto a type class $T_1 := \{\vec{m}^n : t_{\vec{m}^n} = 1\}$ is in $\Delta_n^A$, and therefore the type class measurement is in $\mathcal{E}_n^A$. Because the amplitudes of the state $\text{BEC}_{N,1}$ in the Fock basis are the square root of the binomial distribution $B(N, 1/2)$, application of the type class measurement to the state $\text{BEC}_{N,n}$ gives the state

$$\frac{1}{\sqrt{2^{nH(B(N,1/2))}}} \sum_{|\vec{m}_n| = N} |\vec{m}^n\rangle$$  \hspace{1cm} (10)

for large $n$, where for a discrete probability density $p$, $H(p)$ is the Shannon entropy. To convert this state to $nR$ copies of a MC$_N$ state, one defines an isometry that takes $|\vec{m}^n\rangle \in T_B(N,1/2) \subset \Delta_n^A$ to $|\vec{m}^{n+1}\rangle \in \Delta_n^A$ where $\vec{m}^{n+1}_j = (\ell_{j,1}, \ell_{j,2}) \in \{(N - r, r)\}_{r=0}^N$. Such an isometry maps the state in (10) to $\prod_{j=1}^{n+1} T_B(N,1/2) |\text{MC}_N^{2j-1,2j}N\rangle$. The isometry also preserves the expected particle number $N$ in each mode, therefore keeping the energy density constant (see Section 1A). Therefore, the optimal rate is given by $R = \frac{1}{n} \log_{N+1} |T_B(N,1/2)| = s(N)^{-1} H(B(N, 1/2))$ ($s(N) := \log_2(N+1)$ which goes to $1/2$ as $N \to \infty$). Note that if the two orthogonal single particle modes that define MC$_N$ are are taken to be different from the initial modes of the $n$ copies of BEC$_{N,n}$, the required unitary rotation can be freely implemented in $\mathcal{E}_n^B$. Finally, it follows from the asymptotics of the Shannon entropy of the multinomial distribution [33] that the two mode, $N$ particle BEC in Eq. (9) by a $M$ mode, $N$ particle BEC gives an optimal rate of $\text{MC}_N$ distillation that scales linearly with $M$.

This example leads to natural questions concerning which states allow an optimal asymptotic rate of maximal correlation production. For instance, one may seek elements of $\Delta_n^B$ that maximize the MC$_N$ distillation rate. With this motivation, we consider the optimal asymptotic rate over all $Q_N \Delta_n^B Q_N$. As seen from the previous examples, the distillation rates for states of fixed particle number are equal to the entropy of coherence $C$ with respect to QCRTF$^{(A)}$ (and scaled by an appropriate factor $12$). The quantity $C$ is convex on the quantum state space and invariant under argument shifts of complex Fock state amplitudes. Therefore, the search can be restricted to pure states in $Q_N \Delta_n^B Q_N$ with real amplitudes, without loss of generality. These states take the form $\Psi(\theta, m)_N \propto (a_1^* + \tan \theta a_2^*)^m (a_1^* - \tan \theta a_2^*)^{N-m} |\text{VAC}\rangle$, and we search over $\theta \in [0, \pi/4]$ and $m \in \{0, 1, \ldots, N/2\}$ due to the symmetry of the factors. From numerical computation, we find that the greatest entropy of coherence over $Q_N \Delta_n^B Q_N$ occurs for $\theta = \pi/4$, but neither for $m = 0$ (viz., BEC$_{N,1}$), nor $m = N/2$ (viz., the pair correlated state $(a_1^2 + a_2^2 \propto |\text{VAC}\rangle$ [13]). In Fig.1, we show the entropy of coherence data for $N = 4000$ for $\theta = \pi/4$ and selected values of $m$. Although $m = N/2$, $\theta = \pi/4$ does not give the greatest entropy of coherence over $Q_N \Delta_n^B Q_N$ (and, therefore, does not allow the greatest MC$_N$ distillation rate in $Q_N \Delta_n^B Q_N$), it is an interesting case for two reasons: 1. the $C(\Psi(\pi/4, N/2) \sim \log_2 N^2$ scaling with $x > 1/2$ can be analytically verified (see Appendix B), and 2. this state can be obtained from a two-mode CV quadrature squeezed state by applying a beamsplitter followed by the particle number sector projection $Q_N$, which indicates that maximal correlations can be distilled at a greater rate from quadrature squeezed states than from coherent matter waves using operations from $\mathcal{E}_n^A$.

\textbf{B. Non-number-conserving correlation distillation}

We now consider the asymptotically optimal distillation rate of the following correlated state of indefinite
particle number
\[ |\tilde{MC}_N\rangle := \frac{1}{\sqrt{2N+1}} \sum_{k=0}^{2N} |MC_k\rangle \in \mathcal{F}_2. \]  

Although it is possible to calculate the distillation rate of \(|MC_N\rangle\) from a CV coherent state (in \(\Delta^A_M\)) or CV quadrature squeezed state (not in \(\Delta^A_M\)), which are analogous to the cases of BEC and \(\Psi(\frac{\pi}{4}, \frac{\pi}{2})\) considered in Subsection III.A, we focus in this section on a resource state which exhibits maximal coherence with respect to QCRTF \((A)\).

To do this, it is necessary to identify a maximally coherent element of \(S(\mathcal{F}_2)\) that has expected particle number \(N\). Since this is a simple exercise in using the Karush-Kuhn-Tucker conditions \((B)\), we just state the result.

**Lemma 1.** An optimizer for max \(C^A(\rho)\) subject to \(\rho \in S(\mathcal{F}_2)\) and \(N_2(\rho) = N\) is given by
\[ |\Phi_N\rangle := \sum_{k=0}^{\infty} \frac{2}{N+2} \left( \frac{N}{N+2} \right)^{\frac{k}{2}} \sum_{r=0}^{k} |k-r, r\rangle. \]  

Note that the amplitudes of \(\Phi_N\) are not uniform across particle number sectors. Further, it is clear that \(Q_N \Phi_N \sim MC_N\). In Appendix C we derive the asymptotic scaling \(C^A(\Phi_N) \sim \log_2 N^2\), which shows that superposition of particle number sectors allows a greater maximal coherence compared to the case of fixed particle number analyzed in the previous section. There, we also derive \(C^A(MC_N)\) for completeness.

A general state \(|G_N\rangle \in \mathcal{F}_2\) such that \(N_2(G_N) = N\), defines a joint probability distribution \(p_{X,Y}(x, y)\) on the subset of \(\mathbb{R}^2\) defined by \(\{(x, y) : x \geq 0, 0 \leq y \leq x\}\). This can be seen by writing \(|G_N\rangle = \sum_{y=0}^{\infty} \sum_{x=0}^{\infty} \sqrt{p_{X,Y}(x, y)} |x-y, y\rangle\). Note that \(X\) and \(Y\) are not necessarily independent random variables. Starting from \(n\) bosonic copies of \(|G_N\rangle\) (formed in the same way as in Eq.\((12)\)) one applies the \(\delta\)-typical measurement defined by the jointly typical subspace \(T^T\) corresponding to \(p_{X,Y}(x, y)\), which produces the following state with high probability for large \(n\):
\[ \frac{1}{\sqrt{|T^T|}} \sum_{\tilde{n} \in T^T} |\tilde{n}\rangle \]  

where \(\tilde{n} \in \{(N'-r, r) : N' \in \mathbb{Z}_{\geq 0}, r \in \{0,\ldots,N'\}\}\).

Note that although \(|G_N\rangle\) may have infinite dimensional support, the state in \((13)\) has finite dimensional support because of the exponentially smaller cardinality property of the jointly typical set \((13)\). The final step is to apply an isometry that maps \(\tilde{n}\) to \(\tilde{\ell}\) in \(T^T\) to
\[ |\tilde{\ell}\rangle := \left( \tilde{\ell}_1, \ldots, \tilde{\ell}_n \right) \text{ such that } \tilde{\ell}_j \in \mathbb{Z}_{\geq 0}, \]  

and \(|\tilde{\ell}\rangle \in \{0,\ldots,2N\}\). Note that the image of this isometry has the same dimension as \(T^T\) and that the expected number of particles in each mode is \(N\), which shows that the isometry conserves the energy density. The output of the isometry is \(\log_{2(2N+1)}(N+1) |T^T|\) bosonic copies of the state \(|MC_N\rangle\). It follows that the \(n \to \infty\) distillation rate is
\[ R = (\log_2(2N+1)(N+1))^{-1} H(p_{X,Y}). \]  

For \(|G_N\rangle\) given by \(|\Phi_N\rangle\) in Eq.\((12)\), this rate is shown in the inset of Fig.\(\|\). The result shows that the maximally coherent state with respect to QCRTF \((A)\) can be used to extract a uniform superposition of maximally correlated states with almost unit efficiency.

**IV. CONCLUSION**

In this work, we have introduced and analyzed three versions of QCRTF that allow a wide variety of DV and CV quantum information processing protocols to be considered within a unified resource theoretic framework. QCRTF\((B)\) is of particular interest due to the fact that the free quantum channels can be implemented by free system-environment interactions (Proposition 1). This result shows that linear quantum computations on bosonic insulating states are sufficient to implement any quantum dynamics that preserves \(\Delta^B_M\).

Our results on optimal distillation of maximally correlated states indicate that quantum coherence with respect to QCRTF \((A)\) is a resource for producing such correlations. Because the sets of free quantum states and free operations in the QCRTF framework are not limited to classical states (e.g., states with positive \(P\) function) and classical dynamics (e.g., convex combinations of passive canonical transformations), respectively, they allow to circumvent certain no-go theorems of other many-body resource theories that utilize such a classical or quasiclassical framework, such as the impossibility of Gaussian entanglement distillation by Gaussian local operations supplemented by classical communication \((C)\). We note that it is also possible to consider asymptotic distillation rates of other entangled quantum states, e.g., GHZ states of \(N\) particles in two orthogonal modes, within the QCRTF framework and that, in such cases, the distillation rate exhibits asymptotically non-constant scaling with \(N\). Potential directions of future research include analyses of fully bosonic one-shot distillation and formation \((D)\), and localization of the QCRTF \((E)\). The results reported here are expected to stimulate future work on optimal distillation of many-body DV and CV quantum states that are useful for quantum communication, quantum error correction, or quantum metrology \((E)\), within the framework of experimentally-motivated QCRTFs.
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Appendix A: Proofs of Propositions

Proof of Proposition 1. We provide a constructive proof that $\mathcal{E}^B_M \supset W^B$ given $\eta \in W^B$ associated with an isometry $V$ as in Eq. (3) and an integer $M' > M$, we define an orthonormal basis $|M + 1\rangle, \ldots, |M' - 1\rangle, |M'\rangle$ for the ancilla modes. The basic idea of the proof is, for each positive integer $a$ to construct a set of Kraus operators $\{K^{(a)}_{\gamma}\}_\gamma$ such that $\eta(\rho) = \sum_\gamma K^{(a)}_{\gamma} \rho K^{(a)\dagger}_{\gamma}$ for all $\rho \in Q_a \Delta^B_M Q_a$. We take $a = N$ for generality and remove the superscript on $K^{(a)}_{\gamma}$, while remembering that the construct must be carried out for all particle number sectors. Note that for $\bar{m}$ with $|\bar{m}| = N$, we can write

$$K_{\gamma} U |\bar{m}\rangle = \langle \bar{n}_\gamma | VU |\bar{m}\rangle \quad (A1)$$

where $U \in \hat{U}_M$ and $|\bar{n}_\gamma\rangle$ is a Fock state on the ancilla modes. Due to definition (4), there is an $N' \geq N$ and $U' \in \hat{U}_M$ such that $VU|\bar{m}\rangle = U'|n'\rangle$ with $|n'| = N'$. Therefore, we take $|\bar{n}_\gamma\rangle = |N'\rangle - |N\rangle$, so that $K_{\gamma}(|\bar{m}\rangle \langle \bar{m}| K_{\gamma}/trK_{\gamma}|\bar{m}\rangle \langle \bar{m}| K_{\gamma} \in Q_N \Delta^B_M Q_N$.

Note also that one can write $U' = RV_{M+1} \ldots V_{M'-1} V_{M'}$, where $R \in \hat{U}_M$ and $V_\ell := \prod_{j=1}^{\ell-1} V^{(\ell,j)}$, where $V^{(\ell,j)}$ is a particle number-conserving linear optical unitary on modes $|\ell\rangle$ and $|j\rangle$. This decomposition provides the basis for the following iterative construction.

1. There exists $z_1 \in \mathbb{C}$, $V' \in \hat{U}_{M-1}$, and $|\psi^{(1)}\rangle \in \text{span}_{\mathbb{C}}\{|1\rangle, |2\rangle, \ldots, |M'-1\rangle\}$ such that the unitary $V_{M'} = V^{(1)} Z^{(1)}$ where

$$Z^{(1)} := e^{zi_{\psi^{(1)}} a_{M'}^\dagger - \overline{z}_{\psi^{(1)}} a_M} \quad (A2)$$

where $a_{\psi^{(1)}}$ is the creation operator of the single-particle mode $|\psi^{(1)}\rangle$.

Note that $Z^{(1)}$ is a particle number-conserving linear optical unitary on modes $|\psi^{(1)}\rangle$ and $|M'\rangle$.

2. Construct a unitary Gram-Schmidt process $G^{(1)}$ in span$_{\mathbb{C}}\{Z^{(1)} |1\rangle, \ldots, Z^{(1)} |M'-1\rangle\}$ such that $G^{(1)} Z^{(1)} |M' - 1\rangle = |\psi^{(1)}\rangle$. For any particle number-conserving linear optical unitary $S$ on two modes, and Fock state $|\ell_1, \ell_2\rangle$ on the two modes, $|n|\ell_1, \ell_2 - \bar{n}| \not\in \mathbb{N}$. Therefore,

$$\langle \bar{n}_\gamma | U' |\bar{m}'\rangle = \langle \bar{n}_\gamma | RV_{M+1} V_{M+2} \ldots V_{M'-1} V^{(1)} G^{(1)*} |\bar{m}'\rangle \quad (A3)$$

where $\bar{n}_\gamma^{(1)} = (|\bar{n}_\gamma\rangle_{M+1} \ldots, |\bar{n}_\gamma\rangle_{M'-1})$ and $|\bar{m}'^{(1)}\rangle$ is a Fock state on modes $G^{(1)} Z^{(1)} |1\rangle, \ldots, G^{(1)} Z^{(1)} |M'-2\rangle, |\psi^{(1)}\rangle$.

3. Rewrite the right hand side of Eq. (A3) as

$$\langle \bar{n}_\gamma^{(1)} | Z^{(1)*} G^{(1)*} G^{(1)} Z^{(1)} RV_{M+1} V_{M+2} \ldots V_{M'-1} V^{(1)} G^{(1)*} |\bar{m}'^{(1)}\rangle \quad (A4)$$

and note that this expression has the form (again from Ref. (39)):

$$\langle \bar{n}_\gamma^{(2)} | R^{(1)} V_{M+1} V_{M+2} \ldots V_{M'-1} |\bar{m}'^{(1)}\rangle \quad (A5)$$

where $|\bar{n}_\gamma^{(2)}\rangle$ is a Fock state on $G^{(1)} Z^{(1)} |1\rangle, \ldots, G^{(1)} Z^{(1)} |M'-2\rangle, |\psi^{(1)}\rangle$ and $V_j^{(1)}$ is a product of two mode unitaries on mode pairs $(G^{(1)} Z^{(1)} |1\rangle, G^{(1)} Z^{(1)} |j\rangle), \ldots, (G^{(1)} Z^{(1)} |j-1\rangle, G^{(1)} Z^{(1)} |j\rangle)$ (with $G^{(1)} Z^{(1)} |M' - 1\rangle = |\psi^{(1)}\rangle$).

The result of Step 3 allows to apply Step 1 to $V_{M'-1}$, et cetera. The end result after iterating this procedure $M' - M$ times is a state $R^{(M' - M)} |\bar{m}'^{(M' - M)}\rangle$ which is an element of $Q_N \Delta^B_M Q_N$.

For the second statement in Proposition 1, we let $\Phi \in \mathcal{E}^B_M$, $\Phi(\rho) = \oplus_{N \geq 0} \sum_{\beta \in J_N} K^{(N)}_{\beta} \rho K^{(N)*}_{\beta}$ where $J_N$ is a sequence of index sets and show that if for all $N$, there exists $U^{(N)} \in \hat{U}_M$ and sequence $(k^{(N)}_{j})_{j=1}^{N} \in \mathbb{Z}^M_{\geq 0}$ such that $\sum_{\beta \in J} K^{(N)}_{\beta} = U^{(N)} \prod_{j=1}^{M} a_{\beta}^{(N)}$, then $\Phi \in W^B$. Define the action of an isometry $W$ by $W|\bar{m}\rangle = \sum_{\beta \in J_{|\bar{m}|}} K^{(N)}_{\beta} |\bar{m}\rangle |\bar{n}(\beta)\rangle$ where $\bar{n}(\beta) = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 at position $\beta$ and $|\bar{n}(\beta)\rangle = |J_{|\bar{m}|}\rangle$. Define $V$ corresponding to an element of $SO(2, |J_{|\bar{m}|}|, \mathbb{R})$ such that $V a_{M+1} V^* = \sum_{\beta \in J_{|\bar{m}|}} \frac{a_{\beta}^{(N)} + \ldots + a_{\beta}^{(N)}}{\sqrt{|J_{|\bar{m}|}|}}$. Then $W|\bar{m}\rangle \propto \sum_{\beta \in J_{|\bar{m}|}} K^{(N)}_{\beta} |\bar{m}\rangle |\bar{n}(\beta)\rangle$ with $|\bar{m}| = N$, we can write
Define $V_{\pi/N}$ the asymptotics for the central binomial coefficient. Note that the asymptotic value of $\min_{\gamma} \langle \hat{n}|K_{\gamma}|\hat{m}\rangle$ relative to $\Delta_{\sum_{A}}^{\pi/N}$ is given by $\Psi(\hat{m})$. Let $W_{\sum_{A}}^{\pi/N}$ be a projector $U_{\sum_{A}}^{\pi/N}$, element $U_{\sum_{A}}^{\pi/N}$ of $U$ such that $K_{\gamma}D(\hat{a})U_{\sum_{A}}^{\pi/N}|\hat{m}\rangle = |0_{M+1}|D(\gamma)^{\ast}U_{\sum_{A}}^{\pi/N}|\hat{m}\rangle$, where $U := D(\gamma)^{\ast}U_{\sum_{A}}^{\pi/N}$. The operator $0_{M+1}|U^\prime\rangle$ is proportional to a unitary operator in $U$. To see this, just note that the projector $U^\prime|0_{M+1}\rangle|0_{M+1}\rangle$ maps an $M^\prime$-mode coherent state $D(\hat{m})|\pi/N\rangle$, $\hat{m} \in \mathbb{C}^{M^\prime}$, to another $M^\prime$-mode coherent state of the same form (i.e., it occurs as a selected outcome of a heterodyne measurement on $M^\prime$ modes). This proves the first statement in Proposition 2.

To prove the second statement, let $\Phi \in \mathcal{E}_{\pi/N}$ satisfy the assumptions in the statement of the proposition and let $W_{\sum_{A}}^{\pi/N} = \sum_{\beta \in J} K_{\beta}|\hat{m}(\beta)\rangle$ where $|\hat{m}(\beta)\rangle = (0, 0, 1, 0, \ldots, 0)$ with the 1 at position $\beta$ and $|\hat{m}(\beta)| = |J|$. Define $V$ corresponding to an element of $SO(2|J|, \mathbb{R})$ such that $V_{\sum_{A}^{\ast}}^{\pi/N}V_{\sum_{A}}^{\pi/N} = \frac{\hat{a}_{M+1}^{\ast} \cdots \hat{a}_{M+J}^{\ast}}{\sqrt{|J|}}$. Then $W_{\sum_{A}}^{\pi/N} \propto \sum_{\beta \in J} K_{\beta}V_{\sum_{A}}^{\pi/N}|\hat{m}, 1, 0, \ldots, 0\rangle = U \prod_{j=1}^{M} (a_{j} + \alpha_{j})^{\ast k_{j}}V_{\sum_{A}}^{\pi/N}|\hat{m}, 1, 0, \ldots, 0\rangle = U^\prime \prod_{j=1}^{M} a_{j} \alpha_{j}^{\ast}D(\hat{a})V_{\sum_{A}}^{\pi/N}|\hat{m}, 1, 0, \ldots, 0\rangle \in \Delta_{\sum_{A}^{\ast}}^{\pi/N}$, where in the last equality $D(\hat{a})$ is a displacement of modes $|1, \ldots, |M\rangle$ and $U^\prime = UD(\hat{a})$. Therefore, $\Phi \in \mathcal{W}_{\pi/N}$.

Appendix B: Scaling of entropy of coherence for $\Psi(\theta = \pi/4, m = N/2)_{N}$

To derive an analytical lower bound for the asymptotics of the entropy of coherence for $\Psi(\pi/4, N/2)_{N}$, one can use the asymptotics for the central binomial coefficient. Note that the asymptotic value of $\min_{k} p(k) := \min_{k} \{|k, N-k|\Psi(\pi/4, N/2)_{N}\rangle\rangle$ is $\frac{1}{N} \log_{2} N + \log_{2} \frac{\pi}{4}$ for $k = N/2$. Since $-x \log_{2} x$ is an increasing function on $(0, e^{-1})$ and the Fock state amplitudes of $\Psi(\pi/4, N/2)_{N}$ are increasing in modulus away from $(\frac{\pi}{4}, \frac{\pi}{2})$, we have

$$\liminf_{N \to \infty} \sum_{k=0}^{N} -p(k) \log_{2} p(k) \geq \left(\frac{N}{2} + 1\right) \min_{k} p(k) = \frac{4 (\frac{N}{2} + 1)}{\pi N} \left(\log_{2} N + \log_{2} \frac{\pi}{4}\right)$$

(B1)

because $\Psi(\pi/4, N/2)_{N}$ has only $\frac{N}{2} + 1$ nonvanishing amplitudes. Therefore, the entropy of coherence of $\Psi(\pi/4, N/2)_{N}$ relative to $\Delta_{A}^{N}$ is at least $O \left(\log_{2} N \frac{\pi}{4}\right)$.

Appendix C: Entropy of coherence for $\Phi_{N}$

For $\Phi_{N}$, the joint probability density $p_{X,Y}(x, y)$ on the subset $K \subset \mathbb{Z}^{2}$ defined by $K = \{(x, y) : x \geq 0, 0 \leq y \leq x\}$ is given by $p_{X,Y}(x, y) = \left(\frac{2}{N+2}\right)^{2} \left(\frac{N}{N+2}\right)^{x}$, independent of $y$. It follows that

$$C^{A}(\Phi_{N}) = - \sum_{(x,y) \in K} p_{X,Y}(x,y) \log_{2} p_{X,Y}(x,y)$$

$$= \sum_{(x,y) \in K} \left(\frac{2}{N+2}\right)^{2} \left(\frac{N}{N+2}\right)^{x} \log_{2} \left(\frac{2}{N+2}\right)^{2} \left(\frac{N}{N+2}\right)^{x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{2}{N+2}\right)^{2} \left(\frac{N}{N+2}\right)^{x} (x+1) \log_{2} \left(\frac{2}{N+2}\right)^{2} \left(\frac{N}{N+2}\right)^{x}$$

$$= \frac{2N}{N+2} \log_{2} \frac{N+2}{2} + \frac{N}{N+2} \log_{2} \frac{N+2}{2} + \frac{4}{N+2} \log_{2} \frac{N+2}{2} + (N+1) \log_{2} \frac{N+2}{N}$$

$$\sim \log_{2} N^{2}.$$  

(C1)
Since \( \log_2(2N + 1)(N + 1) \in \log_2 N^2 + O(1) \), Eq. [15] shows that the asymptotically optimal distillation rate of \( \tilde{MC}_N \) from \( \Phi_N \) is 1.

The state \( \tilde{MC}_N \) also exhibits large coherence in QCRTF\(^{(A)}\), although it is not a state of maximal coherence in \( S(\mathcal{F}_2) \).

\[
C^A(\tilde{MC}_N) = \sum_{x=0}^{2N} \sum_{y=0}^{x} \frac{1}{(2N + 1)(x + 1)} \log_2(2N + 1)(x + 1)
= \frac{1}{2N + 1} \log_2 2N + 1 + \log_2 2N + 1 + \frac{1}{2N + 1} \log_2 2N + 1
\] (C2)