Obstacle problem for semilinear parabolic equations with measure data

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Abstract
We consider the obstacle problem with two irregular reflecting barriers for the Cauchy-Dirichlet problem for semilinear parabolic equations with measure data. We prove the existence and uniqueness of renormalized solutions of the problem and also results on approximation of the solutions by the penalization method. In the proofs we use probabilistic methods of the theory of Markov processes and the theory of backward stochastic differential equations.

1 Introduction
Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded set and let $\mu$ be a bounded soft measure on $D_T \equiv [0,T] \times D$ (we call a Radon measure $\mu$ soft if it does not charge sets of zero parabolic capacity). Suppose we are also given $f : D_T \times \mathbb{R} \to \mathbb{R}$, $\varphi : D \to \mathbb{R}$ and two functions $h_1, h_2 : D_T \to \mathbb{R}$ such that $h_1 \leq h_2$. In the present paper we investigate the obstacle problem

\begin{align}
\begin{cases}
\frac{\partial u}{\partial t} + A_t u &\leq - f_u - \mu \quad \text{on } \{u \geq h_1\}, \\
\frac{\partial u}{\partial t} + A_t u &\geq - f_u - \mu \quad \text{on } \{u \leq h_2\}, \\
h_1 &\leq u \leq h_2 \quad \text{on } D_T, \\
 u(T, \cdot) = \varphi, & u(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T).
\end{cases}
\end{align}

(1.1)

Here $f_u(t, x) = f(t, x, u(t, x))$, $(t, x) \in D_T$, and

\begin{align}
A_t = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} (a_{ij}(t, x) \frac{\partial}{\partial x_i}),
\end{align}

(1.2)

where $a : D_T \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function such that for some $\Lambda \geq 1$,

\begin{align}
\Lambda^{-1} \leq |y|^2 \sum_{ij=1}^d a_{ij}(t, x) y_i y_j \leq \Lambda |y|^2, \quad y \in \mathbb{R}^d
\end{align}

(1.3)

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for a.e. \((t,x) \in D_T\).

Problem (1.1) with regular barriers and \(L^2\) data is quite well investigated (see, e.g., the classical monograph [2]). There are only few papers devoted to problem (1.1) with one irregular time-dependent barrier and to our knowledge there is no paper on problem (1.1) with two irregular barriers and general soft measure on the right-hand side. In [18, 22] the linear problem with one barrier and \(L^2\) data is considered. Semilinear problem (1.1) with \(L^2\) data and \(f\) satisfying the Lipschitz and the linear growth condition is considered in [9] in the case of one barrier and in [12] in the case of two barriers. Note, however, that unlike the present paper, in [9, 12] the function \(f\) may depend on the solution of (1.1) as well as its gradient.

The main goal of the paper is to prove the existence and uniqueness of solutions of (1.1) in the case where the barriers \(h_1, h_2\) are merely measurable and satisfy some kind of separation condition, \(\varphi \in L^1(D)\), \(f\) satisfies the monotonicity condition in \(u\) and some mild growth condition considered earlier in the theory of PDEs with measure date (see, e.g., [3]). We are also interested in approximation of solutions of (1.1) by the penalization method. As in [9, 12] to study these problems we adopt a stochastic approach based on the theory of backward stochastic differential equations.

The problem of existence and uniqueness of solutions of (1.1) with measure data is delicate even for regular barriers. If the barriers are irregular then the additional serious problem is to define solutions of (1.1) in such a way to guarantee their uniqueness. The classical approach via variational inequalities is not suitable even in the case of one barrier and \(L^2\) data, because even in that case when the barrier is irregular then in general a solution of the variational inequality associated with (1.1) is not unique.

To overcome the difficulty with uniqueness of solutions one could try to adapt to the parabolic case the method of minimal solutions, which was applied successfully in [15] to one-sided elliptic obstacle problem with general Radon measure on the right-hand side. Unfortunately, the concept of minimal solutions is not directly applicable to the two-sided obstacle problem.

To address the nonuniqueness problem one can also try to adopt the approach from the paper [22] devoted to linear parabolic one-sided obstacle problem with \(L^2\) data and define a solution of (1.1) as a pair \((u, \nu)\) consisting of a measurable function \(u : D_T \to \mathbb{R}\) and a soft measure \(\nu\) on \(D_T\) such that

\[
\begin{cases}
\frac{\partial u}{\partial t} + A_t u = -f_u - \mu - \nu & \text{in } D, \\
u(T, \cdot) = \varphi, & u(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T)
\end{cases}
\]  

(1.4)

is satisfied in some weak sense, \(h_1 \leq u \leq h_2\) a.e. and \(\nu\) satisfies some minimality condition. In case \(h_1, h_2\) and \(u\) are regular, the natural minimality condition says that

\[
\int_{D_T} (u - h_1) \, d\nu^+ = \int_{D_T} (h_2 - u) \, d\nu^- = 0,
\]

(1.5)

where \(\nu^+, \nu^-\) denote the positive and the negative part of the Jordan decomposition of \(\nu\), i.e. the reaction measure acts only when \(u\) touches the barriers. If \(h_1, h_2\) are merely measurable, the problem is to make sense of (1.5), because in general \(\nu\) is not absolutely continuous with respect to the Lebesgue measure. A further complication arises from the fact that in general \(u\) is not quasi-continuous. It is, however, quasi-l.s.c., so determined q.e. Therefore one can hope that when the barriers are quasi-l.s.c.
or quasi-u.s.c., and hence determined q.e., then then minimality condition holds (the
integrals in (1.5) are then well defined since \(\nu\) is soft). Unfortunately, even in that case
the integrals in (1.5) may be strictly positive (a simple example is to be found in [9]).
Our definition of a solution of (1.1) is based on a nonlinear Feynman-Kac formula.
It may be viewed as a probabilistic extension of the definition described above, because
in the linear case with \(L^2\) data our probabilistic solution \((u,\nu)\) coincides with the
solution considered in [22]. Let us also note that in the case of one-sided problem the
first component \(u\) of the probabilistic solution is a minimal solution in the sense that

\[
u = \text{quasi-essinf}\{v \geq h_1, \text{m}_1\text{-a.e.}: v \text{ is a supersolution of problem (1.7)}\}, \tag{1.6}\]

where \(m_1\) is the Lebesgue measure on \(\mathbb{R}_+ \times \mathbb{R}^d\) and

\[
\begin{aligned}
\frac{\partial u}{\partial t} + A_t u &= -f u - \mu \text{ in } D, \\
u(T,\cdot) &= \varphi, \quad u(t,\cdot)|_{\partial D} = 0, \quad t \in (0, T).
\end{aligned} \tag{1.7}
\]

Thus, our probabilistic approach leads to a parabolic analogue of a solution considered
in [15]. In the case of two-sided problem it leads to some variant of (1.6).

Let \(X = (X, P_{s,x})\) be a Markov family associated with the operator \(A_t\) and for a
soft measure \(\gamma\) on \(D_T\) let \(A^\gamma\) denote the additive functional of \(X\) associated with \(\gamma\) in
the Revuz sense (see Section 2). Set

\[
\xi^s = \inf\{t \geq s : X_t \notin D\}. \tag{1.8}
\]

By a solution of (1.1) we mean a pair \((u,\nu)\) consisting of a function \(u\) on \(D_T\) and a
bounded soft measure \(\nu\) on \(D_T\) such that

\[
u(s, x) = E_{s,x}\left(1_{[\xi^s > T]}\varphi(X_T) + \int_{\xi^s \wedge T}^{\xi^s \wedge T} f_u(t, X_t) \, dt + \int_{\xi^s \wedge T}^{\xi^s \wedge T} d(A_{s,t}^\mu + A_{s,t}^\nu)\right) \tag{1.9}
\]

for q.e. \((s, x) \in D_T\) and \(\nu\) satisfies a minimality condition introduced in [9]. The
minimality condition says that for any measurable functions \(h_1^*, h_2^*\) on \(D_T\) such that
\(h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2\) a.e. and the processes \([s, T] \mapsto h_i^*(t, X_t), i = 1, 2,\) are càdlàg
under \(P_{s,x}\) for q.e. \((s, x) \in D_T\) we have

\[
\int_{s}^{\xi^s \wedge T} (u_-(t, X_t) - h_1^*(t, X_t)) \, dA_{s,t}^\nu = \int_{s}^{\xi^s \wedge T} (h_2^*(t, X_t) - u_-(t, X_t)) \, dA_{s,t}^\nu = 0 \tag{1.10}
\]

\(P_{s,x}\)-a.s. for q.e. \((s, x) \in D_T\), where \(g_-(t, X_t) = \lim_{s \downarrow t, s \to t} g(s, X_s)\) for \(g \equiv u, h_1^*, h_2^*\).

Our main result says that under natural mild assumptions on the data there exists
a unique solution \((u,\nu)\) of (1.1). Moreover, from the nonlinear Feynman-Kac formula
(1.9) we deduce that \(u\) is a renormalized (and an entropy as well) solution of problem
(1.4). If \(h_1, h_2\) are quasi-continuous, condition (1.10) has purely analytic interpretation.
We show that under this additional assumption \(u\) is quasi-continuous and (1.10) reduces
to condition (1.5). In the case of irregular barriers an analytical formulation of the
minimality condition is possible in the linear case with \(L^2\) data. In that case (1.10)
may be expressed by using the notion of precise version of function introduced in [23].

The reason for adopting here probabilistic definition of a solution of (1.1) not only
pertains to the difficulties with the analytic formulation of the minimality condition
for the reaction measure $\nu$. One major advantage of the probabilistic definition is that it fits well to a general scheme of proving existence of solutions of equations with measure data which was successfully adopted in the paper [13] devoted to semilinear elliptic equations with operators associated with general symmetric regular Dirichlet forms. The scheme comprises two essentially different parts. In the present context the first part consists in using stochastic methods to show existence of a pair $(u, \nu)$ satisfying (1.9), (1.10) and such that the process $t \mapsto u(t, X_t)$ has some integrability properties. As a matter of fact this part follows rather easily from results on doubly reflected BSDEs proved recently in [10]. The second part consists in using the nonlinear Feynman-Kac formula (1.9) to prove additional regularity properties of $u$ and to show that $u$ is a renormalized solution of (1.4).

Results from [10] are also used to show that the solution of (1.1) can be approximated by the penalization method. For instance, we show that under the same assumptions under which there exists a unique solution $(u, \nu)$ of (1.1), if $u_n$ is a renormalized solution of the problem

$$
\begin{aligned}
\frac{\partial u_n}{\partial t} + A_t u_n &= -f u_n - \mu - n(u_n - h_1) + n(u_n - h_2), \\
 u_n(T, \cdot) &= \varphi, \quad u_n(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T),
\end{aligned}
$$

then $u_n \to u$ q.e. on $D_T$ and $\nabla u_n \to \nabla u$ in the Lebesgue measure. From [10] and the main result of the present paper it also follows that q.e. on $D_T$,

$$
u = \text{quasi-essinf}\{v \geq h_1, m_1\text{-a.e.: } v \text{ is a supersolution of (1.7) with } \mu \text{ replaced by } \mu - \nu^-\}. \tag{1.11}
$$

If $h_2 \equiv +\infty$ then $\nu^- = 0$, and so (1.11) reduces to (1.6). Finally, let us mention that from [10] and our main result it follows that $u$ can be also characterized as a solution of the following stopping time problem (sometimes called Dynkin game): for any $h_1^*, h_2^*$ as in condition (1.10),

$$
u(s, x) = \text{ess sup}_{s \in \mathcal{T}^s} \text{ess inf}_{\delta \in \mathcal{T}^\delta} E_{s,x} \left( \int_s^{\sigma \land \xi^s \land T} f_u(t, X_t) \, dt + \int_s^{\sigma \land \xi^s \land T} dA_{s,t}^t \right.
$$

$$
+ h_1^*(\delta, X_\delta) \mathbf{1}_{\{\delta < \sigma \land T\}} + h_2^*(\sigma, X_\sigma) \mathbf{1}_{\{\sigma < \xi^s \land T\}} + \varphi(X_T) \mathbf{1}_{\{\sigma = \delta = T\}} \left. \right)
$$

for q.e. $(s, x) \in D_T$, where $\mathcal{T}^s$ denotes the set of all stopping times with values in $[s, T]$ with respect to the completion of the filtration generated by $X$.

In the present paper we are mostly interested in investigation of renormalized (or entropy) solutions of (1.1). But it is worth mentioning that as a byproduct of our proofs we obtain new results on stochastic representation of solutions of (1.1) and the Cauchy-Dirichlet problem (1.7), which can be regarded as problem (1.1) with $h_1 \equiv -\infty$, $h_2 \equiv +\infty$. Some of these results seem to be new even in the case of problem (1.7) with $L^2$ data. To our knowledge in all the existing results on stochastic representation of solutions to that problem some regularity of the boundary of the domain is assumed. In the present paper we do not require any regularity of $D$. Let us also note that in the recent paper [21] the existence and uniqueness of renormalized solutions of the
In this section we have compiled some basic facts on diffusions associated with the operator $A$ defined by (1.2) and their additive functionals associated with soft measures on $\mathbb{R}_+ \times \mathbb{R}^d$. Here and in the next sections we assume that $a$ satisfies (1.3). By putting $a_{ij} = \delta_j^i$ outside $D_T$ we can and will assume that $a$ is defined and satisfy (1.3) in all $\mathbb{R}_+ \times \mathbb{R}^d$.

2 Preliminaries

In this section we have compiled some basic facts on diffusions associated with the operator $A$ defined by (1.2) and their additive functionals associated with soft measures on $\mathbb{R}_+ \times \mathbb{R}^d$. Here and in the next sections we assume that $a$ satisfies (1.3). By putting $a_{ij} = \delta_j^i$ outside $D_T$ we can and will assume that $a$ is defined and satisfy (1.3) in all $\mathbb{R}_+ \times \mathbb{R}^d$.

2.1 Time-inhomogeneous diffusions and additive functionals

Let $\Omega = C(\mathbb{R}_+; \mathbb{R}^d)$ be the space of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+ = [0, +\infty)$, $X$ be the canonical process on $\Omega$, $\mathcal{F}_t^s = \sigma(X_u, u \in [s, t])$ and for given $T > 0$ let $\mathcal{F}_t^T = \sigma(X_u, u \in [T + s - t, T])$. We define $\mathcal{G}$ as the completion of $\mathcal{F}_t^s$ with respect to the family $\mathcal{P} = \{P_{s, \mu} : \mu$ is a probability measure on $\mathcal{B}(\mathbb{R}^d)\}$, where $\mathcal{P}_{s, \mu} = \int_{\mathbb{R}^d} P_{s, x}(\cdot) \mu(dx)$, and then we define $\mathcal{G}_t^s$ ($\mathcal{G}_t^T$) as the completion of $\mathcal{F}_t^s$ ($\mathcal{F}_t^T$) in $\mathcal{G}$ with respect to $\mathcal{P}$.

Let $p$ denote the fundamental solution for the operator $A$ and let $X = \{(X, P_{s, x}) : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ be a time-inhomogeneous Markov process for which $p$ is the transition density function, i.e.

$$P_{s, x}(X_t = x; 0 \leq t \leq s) = 1, \quad P_{s, x}(X_t \in \Gamma) = \int_{\Gamma} p(s, x, t, y) dy, \quad t > s$$

for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. The process $X$ admits the so-called strict Fukushima decomposition, i.e. for any $T > 0$,

$$X_t = X_s + A_{s, t} + M_{s, t}, \quad t \in [s, T], \quad P_{s, x}\text{-a.s.} \quad (2.1)$$

for every $(s, x) \in [0, T] \times \mathbb{R}^d$, where $\{A_{s, t} : 0 \leq s \leq t \leq T\}$ is a two parameter continuous additive functional (CAF) of $X$ (with respect to the filtration $\{\mathcal{G}_t^s\}$) of zero energy and $\{M_{s, t} : 0 \leq s \leq t \leq T\}$ is a two parameter martingale additive functional (MAF) of $X$ (with respect to $\{\mathcal{G}_t^s\}$) of finite energy (see [25]). In particular, $M_{s, t}$ is a $\{(\mathcal{G}_t^s), P_{s, x}\}$-martingale. Moreover, for every $(s, x) \in [0, T] \times \mathbb{R}^d$,

$$\langle M_{s, t}^i, M_{s, t}^j \rangle_t = \int_s^t a_{ij}(\theta, X_\theta) d\theta, \quad t \in [s, T], \quad P_{s, x}\text{-a.s.} \quad (2.2)$$

(see [25]). It follows that under $P_{s, x}$ the process $B_{s, t}$ defined as

$$B_{s, t} = \int_s^t \sigma^{-1}(\theta, X_\theta) dM_{s, \theta}, \quad t \in [s, T],$$

where $\sigma \cdot \sigma^* = a$, is a Brownian motion on $[s, T]$. It is also known (see [11]) that $X$ admits the so-called Lyons-Zheng decomposition, i.e. for any $T > 0$,

$$X_t - X_u = \frac{1}{2} M_{u, t} + \frac{1}{2} (N^{s, x}_{t, T + s - t} - N^{s, x}_{s, T + s - u}) - a_{u, t}^{s, x}, \quad s \leq u \leq t \leq T$$

Cauchy-Dirichlet problem with more general than (1.2) divergence form operator (for instance $p$-Laplace operator) and $f$ not depending on $x$ and satisfying the so-called sign condition is proved. In our paper we consider equations with $A$ given by (1.2) but we allow $f$ to depend on $x$. 
for every \((s, x) \in [0, T] \times \mathbb{R}^d\), where \(N_{s,x}^{s,x}\) is an \((\{\bar{G}_t^s\}, P_{s,x})\)-martingale and

\[
\alpha_{u,t}^{s,x} = \sum_{j=1}^d \int_u^t \frac{1}{2} a_{ij}(\theta, X_\theta) p^{-1} \frac{\partial p}{\partial y_j}(s, x, \theta, X_\theta) \, d\theta.
\]

For a measurable \(\mathbb{R}^d\)-valued function \(f\) and \(s \leq r \leq t \leq T\) we put

\[
\int_r^t f(\theta, X_\theta) \, d^s X_\theta \equiv -\int_r^t f(\theta, X_\theta) (dM_{s,\theta} + d\alpha_{s,\theta}^{s,x}) - \int_{T+s-r}^{T+s-t} f(T+s-\theta, X_{T+s-\theta}) \, dN_{s,\theta}^{s,x}.
\]

The usefulness of the backward integral defined above comes from the fact that for regular \(f\),

\[
\int_u^t \text{div} f(\theta, X_\theta) \, d\theta = \int_u^t a^{-1} f(\theta, X_\theta) \, d^s X_\theta, \quad s \leq u \leq t, \quad P_{s,x}\text{-a.s.}
\]

(see, e.g., [26]).

Let us recall that for every \(\Phi \in L^2(0, T; H^{-1}(\mathbb{R}^d))\) there exist \(f \in L^2([0, T] \times \mathbb{R}^d)\) and \(G \in L^2([0, T] \times \mathbb{R}^d)\) such that

\[
\Phi = F + \text{div}(G),
\]

i.e. for every \(\eta \in L^2(0, T; H^1(\mathbb{R}^d))\),

\[
\Phi(\eta) = \int_{S_T} \eta f \, dm_1 - \int_{S_T} G \cdot \nabla \eta \, dm_1.
\]

The following lemma will be needed in the next section to prove regularity of parabolic potentials.

**Lemma 2.1.** Assume that \(\Phi_1, \Phi_2 \in L^2(0, T; H^{-1}(\mathbb{R}^d))\), \(D \subset \mathbb{R}^d\) is an open bounded set and \(\Phi_1 = \Phi_2\) in \(L^2(0, T; H^{-1}(D))\). Then for q.e. \((s, x) \in D_T\),

\[
\int_s^{t+\xi} f_1(\theta, X_\theta) \, d\theta + \int_s^{t+\xi} a^{-1} G_1(\theta, X_\theta) \, d^s X_\theta = \int_s^{t+\xi} f_2(\theta, X_\theta) \, d\theta + \int_s^{t+\xi} a^{-1} G_2(\theta, X_\theta) \, d^s X_\theta, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.,}
\]

where \(\Phi_1 = f_1 + \text{div}(G_1)\), \(\Phi_2 = f_2 + \text{div}(G_2)\).

**Proof.** Let \(D_\varepsilon = \{x \in D : \text{dist}(x, D^c) > \varepsilon\}\) and let \(\Phi_i^\varepsilon = f_i^\varepsilon + \text{div}(G_i^\varepsilon), i = 1, 2\), where \(f_i^\varepsilon, G_i^\varepsilon\) are standard regularizations of \(f_i, G_i\), respectively. It is an elementary check that

\[
\Phi_1^\varepsilon = \Phi_2^\varepsilon \text{ on } L^2(0, T; H^{-1}(D_\varepsilon)).
\]

Since \(\Phi_1^\varepsilon, \Phi_2^\varepsilon \in L^2([0, T] \times \mathbb{R}^d)\), it follows from the above equality that

\[
\Phi_1^\varepsilon = \Phi_2^\varepsilon, \quad m_1\text{-a.e. on } [0, T] \times D_\varepsilon.
\]
In [11] it is proved that
\[
\int_s^\infty f_1^e(\theta, X_\theta) + \int_s^\infty a^{-1} G_1^e(\theta, X_\theta) \, d^* X_\theta \rightarrow \int_s^\infty f_1(\theta, X_\theta) \, d\theta + \int_s^\infty a^{-1} G_1(\theta, X_\theta) \, d^* X_\theta
\]
uniformly on \([s, T]\) in probability \(P_{s,x}\) for q.e. \((s, x) \in [s, T] \times \mathbb{R}^d\). Let \(\xi^{s,e} = \inf\{t \geq s, X_t \notin D_e\}\). Then
\[
\int_0^{t \wedge \xi^{s,e}} f_1^e(\theta, X_\theta) \, d\theta + \int_0^{t \wedge \xi^{s,e}} a^{-1} G_1^e(\theta, X_\theta) \, d^* X_\theta = \int_0^{t \wedge \xi^e} \Phi_1^e(\theta, X_\theta) \, d\theta
\]
from which the desired result follows, because \(\xi^{s,e} \wedge \xi^e, P_{s,x}\)-a.s. as \(\varepsilon \searrow 0\).

### 2.2 Time-homogeneous diffusions

In the next sections it will be advantageous to consider a time-homogeneous diffusion determined by \(\bar{X}\). The standard construction of it is as follows. We set
\[
\Omega' = \mathbb{R}_+ \times \Omega, \quad P'_{s,x}(B) = P_{s,x}(\omega \in \Omega : (s, \omega) \in B) \quad (2.3)
\]
and consider the process \(\bar{X}\) on \(\Omega'\) defined as
\[
\bar{X}_t(s, \omega) = (s + t, X_{s+t}(\omega)), \quad t \geq 0. \quad (2.4)
\]
Let \(\mathcal{F}'_t = \sigma(\bar{X}_u, u \leq t), \mathcal{F}'_\infty = \sigma(\bar{X}_u, u < +\infty)\) and let \(\mathcal{G}'_\infty\) denote the completion of \(\mathcal{F}'_\infty\) with respect to the family \(\mathcal{P}' = \{P'_\mu : \mu \text{ is a probability measure on } \mathbb{R}_+ \times \mathbb{R}^d\}\) and \(\mathcal{G}'_t\) denote the completion of \(\mathcal{F}'_t\) in \(\mathcal{G}'_\infty\) with respect to \(\mathcal{P}'\). Then \(\bar{X}' = \{(\bar{X}_t, P'_{s,x}) : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}\) is a time-homogeneous Markov process with respect to the filtration \(\{\mathcal{G}'_t\}\) with the transition density
\[
P'(t, (s, x), \Gamma) = P(s, x, s + t, \Gamma_{s+t}), \quad (2.5)
\]
where \(\Gamma_{s+t} = \{x \in \mathbb{R}^d : (s + t, x) \in \Gamma\}\).

For \(h, t \geq 0\) we define \(\theta_h : \Omega' \rightarrow \Omega'\) and \(\tau(t) : \Omega' \rightarrow \mathbb{R}_+\) by putting
\[
\theta_h \omega' = (s + h, \omega), \quad \tau(t)(\omega') = s + t = \tau(0)(\omega') + t
\]
for \(\omega' = (s, \omega)\). Then
\[
\bar{X}_t(\theta_h \omega') = \bar{X}_{t+h}(\omega'), \quad \tau(t) \circ \theta_h = \tau(t) + h, \quad h, t \geq 0.
\]
Since the decomposition (2.1) is unique, we may assume that the two parameter AFs \(A, M\) of (2.1) are defined for all \(0 \leq s \leq t\) and that for every \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) the decomposition (2.1) holds true for all \(t \geq s\). Set
\[
A_t(\omega') = A_{s,s+t}(\omega), \quad M_t(\omega') = M_{s,s+t}(\omega), \quad t \geq 0 \quad (2.6)
\]
for \(\omega' = (s, \omega)\). Since \(P_{s,x}(A_{s,t} = A_{s,u} + A_{u,t}, 0 \leq s \leq u \leq t) = 1\) for every \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\) and \(M\) has the same property, it follows from (2.6) and the fact that the
The parabolic capacity of a Borel set $B$ is a $(\mathcal{M}, \mathcal{P})$-martingale for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Moreover, from (2.1), (2.4) it follows that for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$X_{\tau(t)}(s, x) = X_{\tau(0)} + A_t + M_t, \quad t \geq 0 \quad P'_{s,x} \text{-a.s.}$$

(By convention, if $\xi$ is a random variable defined on $\Omega$ we set $\xi(\omega') = \xi(\omega)$ for any $\omega' = (s, \omega) \in \Omega'$. Thus, in particular, $X_{\tau(t)}(\omega') = X_{\tau(t)(\omega')}(\omega)$).

Set

$$B_t = \int_0^t \sigma^{-1}(X_\theta) \, d\theta, \quad t \geq 0. \quad (2.7)$$

By (2.2) and (2.6),

$$\langle M^i, M^j \rangle_t = \langle M^i_s, M^j_s \rangle_{s+t} = \int_s^{s+t} a_{ij}(\theta, X_\theta) \, d\theta, \quad t \geq 0, \quad P'_{s,x} \text{-a.s.}$$

for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Therefore $B$ is a Brownian motion under $P'_{s,x}$ for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. In fact, by [14, Theorem 12], it is an $\{\mathcal{G}_t\}$-Brownian motion.

### 2.3 Capacity and soft measures

Let $\mathcal{W}$ be the space of all $u \in L^2(\mathbb{R}_+; H^1_0(\mathbb{R}^d))$ such that $\frac{\partial u}{\partial t} \in L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}^d))$, endowed with the usual norm $\|u\|_{\mathcal{W}} = \|u\|_{L^2(\mathbb{R}_+; H^1_0(\mathbb{R}^d))} + \|\frac{\partial u}{\partial t}\|_{L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}^d))}$.

We define the parabolic capacity of an open set $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ as

$$\text{cap}(U) = \inf\{\|u\|_{\mathcal{W}} : u \in \mathcal{W}, u \geq 1_U \text{ a.e. in } \mathbb{R}_+ \times \mathbb{R}^d\}.$$ 

The parabolic capacity of a Borel set $B \subset \mathbb{R}_+ \times \mathbb{R}^d$ is defined as

$$\text{cap}(B) = \inf\{\text{cap}(U) : U \text{ is an open subset of } \mathbb{R}_+ \times \mathbb{R}^d, B \subset U\}.$$ 

It is known that for a Borel $B \subset \mathbb{R}_+ \times \mathbb{R}^d$, $\text{cap}(B) = 0$ iff

$$\int_{\mathbb{R}_+} P_{s,m}(\exists t > s : (t, X_t) \in B) \, ds = 0,$$

where $P_{s,m}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \, dx$ and $m$ denotes the Lebesgue measure on $\mathbb{R}^d$.

From now on we say that some property is satisfied quasi-everywhere (q.e. for short) if it is satisfied except for some Borel subset of $\mathbb{R}_+ \times \mathbb{R}^d$ of capacity zero.

Let $\mu$ be a Borel measure on $\mathbb{R}_+ \times \mathbb{R}^d$. We say that $\mu$ is smooth if $\mu(B) = 0$ for every Borel set $B \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that $\text{cap}(B) = 0$ and there exists an ascending sequence $\{F_n\}$ of closed subsets of $\mathbb{R}_+ \times \mathbb{R}^d$ such that for every compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$, $\lim_{n \to \infty} \text{cap}(K \setminus F_n) = 0$, $|\mu|(F_n) < \infty$, $n \geq 1$, and $\mu(\mathbb{R}_+ \times \mathbb{R}^d \setminus \bigcup F_n) = 0$. We say that a smooth measure $\mu$ is soft if $\mu$ is a Radon measure. We say that $\mu$ is a soft measure on $V \subset \mathbb{R}_+ \times \mathbb{R}^d$ if it is soft and $\mu(\mathbb{R}_+ \times \mathbb{R}^d \setminus V) = 0$.

In what follows by $\mathcal{M}_0$ we denote the set of all soft measures on $(0, T] \times D$, by $\mathcal{M}_{0,b}$ the set of all bounded soft measures on $(0, T] \times D$ and by $\mathcal{M}_{0,b}^*\text{ the subset of } \mathcal{M}_{0,b}$ consisting of positive measures.

Let us note that in the literature soft measures are usually defined on $(0, T] \times D$. In the present paper we define soft measures on $(0, T] \times D$ because in general the obstacle reaction measure is concentrated on that set.
2.4 Additive functionals and soft measures

Let $E_{s,x}$ (resp. $E'_{s,x}$) denote the expectation with respect to $P_{s,x}$ (resp. $P'_{s,x}$) and let $m_1$ denote the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Let us recall that a positive CAF $A$ of $\mathbb{X}'$ and a positive soft measure $\mu$ on $\mathbb{R}_+ \times \mathbb{R}^d$ are in the Revuz correspondence if

$$\langle \mu, f \rangle = \lim_{\alpha \to \infty} \alpha \int_{\mathbb{R}_+ \times \mathbb{R}^d} (E'_{s,x}(t) \int_0^{\tau^-} e^{-\alpha t} f(X_t) dA_t) dm_1(s, x)$$

(2.8)

for every $f \in \mathcal{B}^+(\mathbb{R}_+ \times \mathbb{R}^d)$. If $\langle \mu, 1 \rangle < +\infty$ then $A$ is called integrable. It is known (see [17, 24]) that under (2.8) the family of all integrable positive CAFs of $\mathbb{X}'$ and the family of all bounded positive soft measures on $\mathbb{R}_+ \times \mathbb{R}^d$ are in one-to-one correspondence. In what follows the additive functional corresponding to a positive bounded soft measure $\mu$ will be denoted by $A^\mu$.

Let

$$\zeta = \inf \{ t \geq 0, X_t \notin \mathbb{R}_+ \times D \}, \quad \zeta_r = \zeta \wedge (T - \tau(0))$$

and let $p'_D$ denote the transition density of the process $\mathbb{X}'$ killed at first exit from $\mathbb{R}_+ \times D$, i.e.

$$p'_D(t, (s, x), z) = p'(t, (s, x), z) - E'_{s,x}(\zeta < t, p'(t - \zeta, X_{\zeta}, z)),$$

where $p'$ is the transition density of $\mathbb{X}'$. It is known that $A^\mu$ corresponds to $\mu$ iff for quasi-every $(s, x) \in D_T$,

$$E_{s,x} \int_0^{\zeta_r} f(X_t) dA^\mu_t = \int \int_{[0, T - s] \times D} f(z) p'_D(t, (s, x), z) d\mu(z) dt$$

(2.9)

for every $f \in \mathcal{B}^+(D_T)$ (see [17]).

In [9] a correspondence between two parameter additive functionals of $\mathbb{X}$ and soft measures on $D_T$ is considered. An AF $\{A^\mu_{s,t}, 0 \leq s \leq t \leq T\}$ of $\mathbb{X}$ corresponds to a bounded soft measure $\mu$ on $D_T$ in the sense of [9] if for q.e. $(s, x) \in D_T$,

$$E_{s,x} \int_s^{s \wedge T} f(t, X_t) dA^\mu_{s,t} = \int \int_{[s, T] \times D} f(t, y) p_D(s, x, t, y) d\mu(t, y)$$

(2.10)

for every $f \in \mathcal{B}^+(D_T)$, where $\xi^s$ is defined by (1.8) and $p_D$ is the transition density of the process $\mathbb{X}$ killed at the first exit from $D$, i.e.

$$p_D(s, x, t, y) = p(s, x, t, y) - E_{s,x}[\xi^s < t, p(t - \xi^s, X_{\xi^s}, y)].$$

Let us define $A = \{A_t, t \geq 0\}$ as

$$A_t(\omega') = A^\mu_{s', s+t}(\omega'), \quad \omega' = (s, \omega), \ t \geq 0.$$  

(2.11)

Using (2.3)–(2.5) and the fact that $\zeta_r(\omega') = (\xi^s(\omega) - s) \wedge (T - s)$ one can show that (2.10) is satisfied iff (2.9) is satisfied with $A^\mu$ replaced by $A$. Therefore a two parameter AF $\{A^\mu_{s,t}\}$ of $\mathbb{X}$ corresponds to $\mu$ in the sense of [9] iff the one-parameter AF $A$ of $\mathbb{X}'$ defined by (2.11) corresponds to $\mu$ in the sense of (2.8).
Lemma 2.2. Assume that $\mu \in \mathcal{M}^+_0(D_T)$. Then for q.e. $(s, x) \in D_T$,

$$E'_{s, x} \int_0^{\zeta_r} dA^\mu_t < +\infty.$$ 

Proof. Follows from [9].

We say that a measurable function $f : D_T \to \mathbb{R}$ is quasi-integrable if the function $\mathbb{R}_+ \ni t \mapsto f(X_t)$ belongs to $L^1_{loc}(\mathbb{R}_+)$ $P_{s, x}$-a.s. for q.e. $(s, x) \in D_T$, where we put $f(s, x) = 0$ for $(s, x) \in D_T$. The set of all quasi-integrable functions on $D_T$ will be denoted by $qL^1(D_T)$.

Corollary 2.3. If $f \in L^1(D_T)$ then $f \in qL^1(D_T)$.

Remark 2.4. Analysis similar to that in [13, Remark 4.4] shows that if for every $\varepsilon > 0$ there exists an open set $G_\varepsilon \subset D_T$ such that $\text{cap}(G_\varepsilon) < \varepsilon$ and $f|_{D_T \setminus G_\varepsilon} \in L^1(D_T \setminus G_\varepsilon)$ then $f \in qL^1(D_T)$. Since $D_T$ is bounded the reverse implication is also true. Indeed, assume that $f \in qL^1(D_T)$. Then extending $f$ on $\mathbb{R}_+ \times \mathbb{R}^d$ by putting zero outside $D_T$ we get that $A_t = \int_0^t f(X_r) dr$ is finite for every $t \geq 0$, which implies that it is a PCAF of $\mathcal{X}$. Directly from the definition of the Revuz duality it follows that $f \cdot m_1$ corresponds to the PCAF $A$. This implies in particular that $f \cdot m_1$ is a smooth measure. Therefore for every compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$ and for every $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that $\text{cap}(U_\varepsilon) \leq \varepsilon$ and $f \in L^1(K \setminus U_\varepsilon)$. Taking $K = D_T$ we get the desired result.

Lemma 2.5. Let $\mu \in \mathcal{M}^+_0(D_T)$, $\nu \in \mathcal{M}^+_0(D_T)$. If

$$E'_{s, x} \int_0^{\zeta_r} dA^\mu_t \leq E'_{s, x} \int_0^{\zeta_r} dA^\nu_t$$

for q.e. $(s, x) \in D_T$ then $\|\mu\|_{TV} \leq \|\nu\|_{TV}$.

Proof. By the assumptions, for q.e. $(s, x) \in D_T$,

$$\int_0^T \int_D p_D(s, x, r, y) d\mu(r, y) \leq \int_0^T \int_D p_D(s, x, r, y) d\nu(r, y). \quad (2.12)$$

Let $D^k = \{x \in D; \text{dist}(x, \partial D) > 1/k\}$, $D^k_T = [1/k, T] \times D^k$ and $h_k = 1_{\{D^k_T\}}$. By [18], for every $k$ there exists a minimal solution $e_k \in L^2(0, T; H^1(D))$, in the variational sense, of the obstacle problem

$$\begin{cases}
\frac{\partial e_k}{\partial t} - A_t e_k \geq 0, & \frac{\partial e_k}{\partial t} - A_t e_k = 0 \text{ on } \{e_k > h_k\}, \\
e_k \geq h_k, \\
e_k(0, \cdot) = 0, & e_k(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T).
\end{cases}$$

In particular $\beta_k = \frac{\partial e_k}{\partial t} - A_t e_k$ belongs to $\mathcal{M}^+_0(D_T) \cap \mathcal{W}'(D_T)$, where $\mathcal{W}'(D_T)$ is the space dual to $\mathcal{W}(D_T)$. In fact, $\beta_k \in \mathcal{M}^+_0(D_T) \cap \mathcal{W}'(D_T)$ because it is known that $\mathcal{M}^+_0(D_T) \cap \mathcal{W}'(D_T) = \mathcal{M}^+_0(D_T) \cap \mathcal{W}'(D_T)$ (see, e.g., [7]). By [1], for q.e. $(s, x) \in D_T$,

$$e_k(s, x) = \int_0^T \int_D p_D(r, y, s, x) d\beta_k(r, y). \quad (2.13)$$
From (2.12), (2.13) and the fact that $0 \leq e_k \leq 1$ q.e. on $D_T$ it follows that

$$\int_{D_T} e_k(s, x) d\mu(s, x) \leq \int_{D_T} e_k(s, x) d\nu(s, x) \leq \|\nu\|_{TV}. \quad (2.14)$$

The fact that $0 \leq e_k \leq 1$ q.e. follows from the fact that $e_k$ is the minimal potential majorizing $h_k$ (see [2]). Indeed, it is an elementary check that $e_k \wedge 1$ is a potential majorizing $h_k$. Therefore $e_k \leq e_k \wedge 1$, which implies that $e_k \leq 1$. Of course $e_k \geq 0$ since $e_k \geq h_k$. From this we get in particular that $e_k(s, x) = 1$ for $(s, x) \in D_T^f$. Therefore $e_k(s, x) \not\leq 1$ q.e. on $D_T$. Consequently, letting $k \to \infty$ in (2.14) and using Fatou’s lemma we get the desired result. \hfill \Box

3 Backward stochastic differential equations

Results of this section together with some results on BSDEs proved in [10] form the basis for our main theorems on the obstacle problem (1.1).

3.1 General BSDEs

Suppose we are given a filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq 0}, P)$. We will need the following spaces of processes.

$\mathcal{V}$ (resp. $\mathcal{V}^+$) is the space of all progressively measurable càdlàg processes $V$ of finite variation (resp. increasing processes) such that $V_0 = 0$. $\mathcal{V}$ is the space of all processes $V \in \mathcal{V}$ such that $E|V|_{T} < +\infty$, where $|V|_{T}$ denotes the total variation of $V$ on $[0, T]$.

$\mathcal{D}^q$, $q > 0$, is the space of all progressively measurable càdlàg processes $\eta$ such that $E\sup_{0 \leq t \leq T} |\eta_t|^q < +\infty$ for every $T > 0$.

$\mathcal{M}^q$, $q > 0$, is the space of all progressively measurable processes $\eta$ such that $E(\int_0^T |\eta_t|^2 dt)^{q/2} < +\infty$ for every $T > 0$.

Let us also recall that a càdlàg adapted process $\eta$ is said to be of Doob’s class (D) if the collection $\{\eta_\tau : \tau \text{ is a finite stopping time}\}$ is uniformly integrable.

Let $B$ be a $\{\mathcal{G}_t\}$-Brownian motion, $\sigma$ be a bounded stopping time, $\xi$ be a $\mathcal{G}_\sigma$-measurable random variable, $A \in \mathcal{V}$ and let $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a function such that $f(\cdot, y, z)$ is progressively measurable for $y \in \mathbb{R}, z \in \mathbb{R}^d$. Let us recall that a pair $(Y, Z)$ of $\mathbb{R} \times \mathbb{R}^d$-valued processes is called a solution of BSDE$(\xi, \sigma, f + dA)$ if

(a) $Y, Z$ are $\{\mathcal{G}_t\}$-progressively measurable, $Y$ is càdlàg, $P(\int_0^\sigma |Z_t|^2 dt < +\infty) = 1$,

(b) $t \mapsto f(t, Y_t, Z_t) \in L^1(0, \sigma), P$-a.s.,

(c) $Y_t = \xi + \int_0^t f(s, Y_s, Z_s) ds + \int_t^\sigma dA_s - \int_0^\sigma Z_s dB_s, 0 \leq t \leq \sigma, P$-a.s.

Remark 3.1. The processes $Y, Z$ in the above definition may be considered as processes on $\mathbb{R}_+$. Indeed, if we set $Z_t = 0$ for $t \geq \sigma$ and $Y_t = Y_\sigma$ for $t \geq \sigma$ then the equation in condition (c) is satisfied for every $t \geq 0$ if we admit the convention that $\int_0^b 0$ is 0 for $a \geq b$.

Let us consider the following assumptions.

(A1) $f(\cdot, y, z)$ is $\{\mathcal{G}_t\}$-progressively measurable for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $f(t, \cdot, z)$ is continuous for every $(t, z) \in \mathbb{R} \times \mathbb{R}^d, P$-a.s.,
(A2) There is $L > 0$ such that \(|f(t, y, z) - f(t, y, z')| \leq L|z - z'|\) for a.e. $t \geq 0$ and every $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $P$-a.s.,

(A3) There is $\kappa \in \mathbb{R}$ such that \((f(t, y, z) - f(t, y', z))(y - y') \leq \kappa|y - y'|^2\) for a.e. $t \geq 0$ and every $y, y' \in \mathbb{R}$, $z \in \mathbb{R}^d$, $P$-a.s.,

(A4) $E[|\xi|] + E\int_0^T |f(t, 0, 0)| dt + E\int_0^T d|A_t| < +\infty$,

(A5) For every $r > 0$, $t \mapsto \sup_{|y| \leq r} |f(t, y, 0)| \in L^1(0, \sigma)$, $P$-a.s.,

(AZ) There exist $\alpha \in (0, 1)$, $\gamma \geq 0$ and a nonnegative progressively measurable process $g$ such that $E\int_0^T |g_t| dt < +\infty$ and $P$-a.s.,

\[ |f(t, y, z) - f(t, y, 0)| \leq \gamma(g_t + |y| + |z|)^\alpha \]

for a.e. $t \geq 0$ and every $y \in \mathbb{R}$, $z \in \mathbb{R}^d$.

It is known (see [10, Theorem 3.11]) that under (A1)–(A5), (AZ) there exists a unique solution $(Y, Z)$ of BSDE$(\xi, \sigma, f + dA)$ such that $Y$ is of class (D), $Y \in \mathcal{D}^q$ for $q \in (0, 1)$ and $Z \in M^q$ for $q \in (0, 1)$.

### 3.2 Markov type BSDEs

Let $T > 0$, $f : D_T \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function, $\varphi \in L^1(D)$. By putting $\varphi(t, x) = \varphi(x)$ for $(t, x) \in D_T$ we will regard $\varphi$ as defined on $D_T$. Let $\mu$ be a soft measure on $D_T$ and $B$ be the Brownian motion defined by (2.7).

Let $(s, x) \in D_T$. We say that a pair $(Y^{s, x}, Z^{s, x})$ consisting of an $\mathbb{R}$-valued process $Y^{s, x}$ and an $\mathbb{R}^d$-valued process $Z^{s, x}$ is a solution of BSDE$_{s, x} (\varphi, D, f + d\mu)$ if $Y^{s, x}, Z^{s, x}$ are $\{G_t\}$-progressively measurable, $Y^{s, x}$ is càdlàg, $t \mapsto f(X_t, Y^{s, x}_t, Z^{s, x}_t) \in L^1(0, \zeta_t)$, $P'_{s, x}$-a.s., $P'_{s, x}(\int_0^{\zeta_t} |Z^{s, x}_r|^2 dr < +\infty) = 1$ and

\[ Y^{s, x}_t = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta_t} f(X_r, Y^{s, x}_r, Z^{s, x}_r) dr \]

\[ + \int_t^{\zeta_t} dA_r^\mu - \int_t^{\zeta_t} Z^{s, x}_r dB_r, \quad t \in [0, \zeta_t] \]  

(3.1)

holds true $P'_{s, x}$-a.s. In other words, $(Y^{s, x}, Z^{s, x})$ is a solution of (3.1) if it is a solution of BSDE$(1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}), \zeta_t, f(X, \cdot, \cdot) + dA^\mu)$ on the filtered probability space $(\Omega', G_\infty', \{G_t^\prime\}, P'_{s, x})$ with the Brownian motion $B$.

**Proposition 3.2.** Let $(s, x) \in D_T$ and let $(Y^{s, x}, Z^{s, x}) = (Y, Z)$ be a solution of (3.1). Assume that (A1), (A2), (AZ) are satisfied, $Z \in M^q$ for some $q > \alpha$, there exist $0 < h \leq T - s$ and $\Lambda \in G_\infty'$ such that $P'_{s, x}(\Lambda) = 1$, $P'_{s, x}(\theta_h^{-1}(\Lambda)) = 1$ and (3.1) is satisfied for every $\omega \in \Lambda$. Then

\[ Y_{t-h} \circ \theta_h = Y_t, \quad t \geq h, \quad P'_{s, x}$-a.s. \]

and

\[ Z_{-h} \circ \theta_h = Z, \quad dt \otimes P'_{s, x}$-a.s. on $[h, \zeta_t] \times \Omega'. \]
Proof. Put $\kappa_h = (\zeta \circ \theta_h) \wedge (T - \tau(0) \circ \theta_h)$. Then

$$Y_t \circ \theta_h = 1_{\{\zeta \circ \theta_h > T - \tau(0) \circ \theta_h\}} \varphi(X_{T - \tau(0) \circ \theta_h}) + \int_t^{\kappa_h} f(X_r \circ \theta_h, Y_r \circ \theta_h, Z_r \circ \theta_h) \, dr$$

$$+ \int_t^{\kappa_h} d(A_r^\mu \circ \theta_h) - \left( \int_t^{\zeta} Z_r \, dB_r \right) \circ \theta_h, \quad t \geq 0, \quad P'_{s,x}-a.s.$$

Fix $t \geq h$. Since $\zeta \circ \theta_h = (\zeta - h)^+$ and $\tau(0) \circ \theta_h = (\zeta - h) + h$, we have $\kappa_h = (\zeta - h) \wedge (T - \tau(0) - h)$ on the set $\{\zeta \geq h\}$. Consequently, $h + \kappa_h = \zeta$ on the set $\{\zeta \geq h\}$. Therefore, since $X_r \circ \theta_h = X_{r+h}$ and $A^\mu$ is additive, we have

$$\int_t^{\kappa_h} f(X_r \circ \theta_h, Y_r \circ \theta_h, Z_r \circ \theta_h) \, dr = \int_t^{\zeta} f(X_r, Y_{r-h} \circ \theta_h, Z_{r-h} \circ \theta_h) \, dr$$

and

$$\int_t^{\kappa_h} d(A_r^\mu \circ \theta_h) = \int_t^{\zeta} dA_r^\mu$$

$P'_{s,x}$-a.s. on $\{\zeta \geq h\}$. Let $t = t_0 < t_1 < \ldots < t_n = \zeta$ and let $t_i' = t_i + h$, $i = 0, \ldots, n$. Since $B$ is additive, $(Z_{t_i}(B_{t_{i+1}} - B_{t_i})) \circ \theta_h = Z_{t_i-h} \circ \theta_h(B_{t_{i+1}} - B_{t_i})$, from which it follows that

$$(\int_t^{\zeta} Z_r \, dB_r) \circ \theta_h = \int_{t+h}^{\zeta} Z_{r-h} \circ \theta_h \, dB_r$$

$P'_{s,x}$-a.s. on $\{\zeta \geq h\}$. Thus,

$$Y_t \circ \theta_h = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta} f(X_r, Y_{r-h} \circ \theta_h, Z_{r-h} \circ \theta_h) \, dr$$

$$+ \int_{t+h}^{\zeta} dA_r^\mu - \int_t^{\zeta} Z_{r-h} \circ \theta_h \, dB_r,$$

and hence

$$Y_{t-h} \circ \theta_h = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta} f(X_r, Y_{r-h} \circ \theta_h, Z_{r-h} \circ \theta_h) \, dr$$

$$+ \int_t^{\zeta} dA_r^\mu - \int_t^{\zeta} Z_{r-h} \circ \theta_h \, dB_r$$

$P'_{s,x}$-a.s. on $\{\zeta \geq h\}$. On the set $\{\zeta < h\}$ we have $Y_{t-h} \circ \theta_h = 0 = Y_t$ for $t \geq h$, because by the assumption, $h \leq T - s$. By what has been proved the pair $(Y_{t-h} \circ \theta_h, Z_{t-h} \circ \theta_h)$ solves BSDE (3.1) on $[h, \zeta]$, so the desired result follows from uniqueness of solutions of (3.1). □

In the sequel we say that a Borel set $N \subset D_T$ is properly exceptional if $m_1(N) = 0$ and $P'_{s,x}(\exists t \geq 0 : X_t \in N) = 0$ for every $(s, x) \in N^c$.

**Lemma 3.3.** Let $\Lambda \in \mathcal{G}'_{\infty}$. If $P'_{s,x}(\Lambda) = 1$ for q.e. $(s, x) \in D_T$ then there exists a properly exceptional set $N \subset D_T$ such that $P'_{s,x}(\theta_h^{-1}(\Lambda)) = 1$ for every $h > 0$ and $(s, x) \in N^c$. 

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Proof. By the assumption, \( P_{s,x}'(\Lambda^c) = 0 \) for q.e. \((s,x) \in D_T\). It is known (see [8, Theorem 4.1.1]) that there exists a properly exceptional set \( N \subset D_T \) such that \( P_{s,x}'(\Lambda^c) = 0 \) for every \((s,x) \in N^c\). Let \((s,x) \in N^c\). Then by the Markov property, \( P_{s,x}'(\theta_h^{-1} \Lambda^c) = E_{s,x}'(P_{X_h}(\Lambda^c)) \). By the assumption and the definition of \( N \), \( E_{s,x}' P_{X_h}(\Lambda^c) = 0 \) for every \((s,x) \in N^c\), which completes the proof. \( \square \)

In the case of Markov type equations the following hypotheses are analogues of (A1)--(A5) and (AZ).

(H1) \( f(\cdot,\cdot,y,z) \) is measurable for every \( y \in \mathbb{R}, z \in \mathbb{R}^d \) and \( f(t,x,\cdot,\cdot) \) is continuous for a.e. \((t,x) \in D_T\).

(H2) There is \( L > 0 \) such that \( |f(t,x,y,z) - f(t,x,y',z')| \leq L|z-z'| \) for every \( y \in \mathbb{R}, z, z' \in \mathbb{R}^d \) and a.e. \((t,x) \in D_T\).

(H3) There is \( \kappa \in \mathbb{R} \) such that \( (f(t,x,y,z) - f(t,x,y',z))(y-y') \leq \kappa|y-y'|^2 \) for every \( y, y' \in \mathbb{R}, z \in \mathbb{R}^d \) and a.e. \((t,x) \in D_T\).

(H4) \( f(\cdot,0,0) \in L^1(D_T), \mu \in M_{0,b}(D_T), \varphi \in L^1(D)\).

(H5) \( \forall \tau > 0 \) \( (t,x) \mapsto \sup_{|y| \leq \tau} |f(t,x,y,z)| \in qL^1(D_T)\).

(HZ) There exists \( \alpha \in (0,1), \gamma \geq 0 \) and \( \varrho \in L^1(D_T), \varrho \geq 0 \) such that

\[
|f(t,x,y,z) - f(t,x,y,0)| \leq \gamma(g(t,x) + |y| + |z|)^\alpha
\]

for every \( y \in \mathbb{R}, z \in \mathbb{R}^d \) and a.e. \((t,x) \in D_T\).

Lemma 3.4. Assume that (H1)--(H5) are satisfied. Then for q.e. \((s,x) \in D_T\) the data \( \varphi(X_{T-\tau(0)})1_{\{\zeta > T-\tau(0)\}}, \zeta, f(X,\cdot,\cdot)\), \( A^t\) satisfy (A1)--(A5) under the measure \( P_{s,x}'\).

Proof. Follows from Lemma 2.2. \( \square \)

Proposition 3.5. Assume (H1)--(H5) and (HZ). Then for q.e. \((s,x) \in D_T\) there exists a solution \((Y^{s,x},Z^{s,x})\) of (3.1) and there exists a pair of processes \((Y,Z)\) such that for q.e. \((s,x) \in D_T\),

\[
Y_t^{s,x} = Y_t, \quad t \geq 0, \quad P_{s,x}'-a.s., \quad Z^{s,x} = Z, \quad dt \otimes P_{s,x}'-a.e. \text{ on } [0,\zeta] \times \Omega'.
\]

Moreover, for q.e. \((s,x) \in D_T\) and every \( t \in [0,T-\tau(0)]\),

\[
Y_t = u(X_t), \quad P_{s,x}'-a.s.,
\]

where

\[
u(s,x) = E_{s,x}' \left( 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) + \int_0^{\zeta} f(X_r,u(X_r),\psi(X_r)) \, dr + \int_0^{\zeta} dA_r^t \right)
\]

and \( \psi : D_T \rightarrow \mathbb{R} \) is a Borel measurable function such that

\[
\psi(X) = Z, \quad dt \otimes P_{s,x}'-a.e. \text{ on } [0,\zeta] \times \Omega'.
\]
If $f$ does not depend on $z$ then there is $C$ depending only on $\kappa, T$ such that

$$
E'_{s,x} \int_0^{\xi} |f(X_r, u(X_r))| \, dr \leq CE'_{s,x} \left(1 \{ \xi > T-\tau(0) \} \right) \varphi(X_{T-\tau(0)})
$$

$$
+ \int_0^{\xi} |f(X_r, 0)| \, dr + \int_0^{\xi} d|A'_{\xi}|(r)
$$

(3.3)

for q.e. $(s, x) \in D_T$.

**Proof.** By Lemma 2.2, $E'_{s,x}|A'|_{\xi} < +\infty$ for q.e. $(s, x) \in D_T$. Moreover, for $(s, x) \in D_T$,

$$
E'_{s,x} 1_{(\xi > T-\tau(0))} |\varphi(X_{T-\tau(0)})| \leq E'_{s,x} |\varphi(X_{T-\tau(0)})| \leq \int_D p(s, x, T, y)|\varphi(y)| \, dy \leq c \int_D |\varphi(y)| \, dy.
$$

Therefore by [10, Proposition 3.10], for q.e. $(s, x) \in D_T$ there exists a solution $(\overline{Y}_{s,x}, \overline{Z}_{s,x})$ of the equation

$$
\overline{Y}_{t}^{s,x} = 1_{(\xi > T-\tau(0))} \varphi(X_{T-\tau(0)}) + \int_t^{\xi} f(X_r, 0, 0) \, dr + \int_t^{\xi} dA'_{r} - \int_t^{\xi} Z_r \, dB_r, \quad t \geq 0
$$

such that $\overline{Y}_{s,x}$ is of class (D) and $(\overline{Y}_{s,x}, \overline{Z}_{s,x}) \in D^q \otimes M^q$ for $q \in (0, 1)$. Hence

$$
\overline{Y}_{t}^{s,x} = E'_{s,x} \left(1_{(\xi > T-\tau(0))} \varphi(X_{T-\tau(0)}) + \int_t^{\xi} f(X_r, 0, 0) \, dr + \int_t^{\xi} dA'_{r}|G'_t)\right),
$$

so from [8, Lemma A.3.5] it follows that there exists a process $\overline{Y}$ such that $\overline{Y}_{t}^{s,x} = \overline{Y}_t$, $t \geq 0$, $P'_{s,x}$-a.s. for q.e. $(s, x) \in D_T$. Since

$$
\overline{Z}_{t}^{s,x} = E'_{s,x} \left(1_{(\xi > T-\tau(0))} \varphi(X_{T-\tau(0)}) + \int_0^{\xi} f(X_r, 0, 0) \, dr + \int_0^{\xi} dA'_{r}|G'_t) - \overline{Y}_0,
$$

using once again [8, Lemma A.3.5] we conclude that there exists a process $\overline{Z}$ such that $\overline{Z}_{t}^{s,x} = \overline{Z}$, $dt \otimes P'_{s,x}$-a.e. on $[0, \xi] \times \Omega'$ for q.e. $(s, x) \in D_T$. Thus, for q.e. $(s, x) \in D_T$,

$$
\overline{Y}_t = 1_{(\xi > T-\tau(0))} \varphi(X_{T-\tau(0)}) + \int_t^{\xi} f(X_r, 0, 0) \, dr + \int_t^{\xi} dA'_{r} - \int_t^{\xi} Z_r \, dB_r, \quad t \geq 0
$$

$P'_{s,x}$-a.s. From this, the method of construction of the solution of BSDE$_{s,x}(\varphi, D, f + d\mu)$ (see the proof of [10, Proposition 3.10]) and repeated application of [8, Lemma A.3.3] we deduce existence of a pair $(Y, Z)$ such that (3.2) is satisfied. Let $\Lambda \subset \Omega'$ be a set of those $\omega \in \Omega'$ for which $Y$ is càdlàg and

$$
Y_t = 1_{(\xi > T-\tau(0))} \varphi(X_{T-\tau(0)}) + \int_t^{\xi} f(X_r, Y_r, Z_r) \, dr + \int_t^{\xi} dA'_{r} - \int_t^{\xi} Z_r \, dB_r, \quad t \geq 0.
$$

Then $\Lambda \in G_{\infty}^1$ and of course $P'_{s,x}(\Lambda) = 1$ for q.e. $(s, x) \in D_T$. Therefore by Proposition 3.2 and Lemma 3.3, for every $t \in [0, T - s],

$$
u(X_t) = E'_{X_t} Y_0 = E'_{s,x}(Y_0 \circ \theta_t|G'_t) = E'_{s,x} (Y_t|G'_t) = Y_t, \quad P'_{s,x}$-a.s.
for q.e. \((s, x) \in D_T\). Put \(C_t = \int_0^t Z_r \, dr\), \(t \geq 0\). Since \(Z1_{[0, \zeta_t]} = Z\), using the property of \(Z\) proved in Proposition 3.2 one can check that for every \(h \geq 0\),

\[
C_{(t+h)\wedge (T-\tau(0))} = C_{t\wedge (T-\tau(0))} + C_{h\wedge (T-\tau(0))} \circ \theta_{t\wedge (T-\tau(0))}.
\]

Accordingly, \(C\) is a CAF of the process \(X\) killed at first exit from \(D_T\). Therefore it follows from [27, Theorem 66.2] that there exists a Borel measurable function \(\psi\) on \(D_T\) such that \(Z = \psi(X)\), \(dt \otimes P'_{s,x}\)-a.s. on \([0, \zeta]\) \(\times \Omega'\) for q.e. \((s, x) \in D_T\). Finally, by the Itô-Tanaka formula,

\[
-\int_0^t \dot{Y}_s f(r, X_r, Y_r) \, dr \leq |Y_t| - |Y_0| + \int_0^t \dot{Y}_r \, dA^r_\theta - \int_0^t \dot{Y}_r \, dB_r, \quad 0 \leq t \leq \zeta_r. \tag{3.4}
\]

If \(\kappa \leq 0\) then by (H3), \(-\dot{Y}_r(f(r, X_r, Y_r)) \geq |f(r, X_r, Y_r)| - |f(r, X_r, 0)|\). On substituting this into (3.4) and then taking expectation on both sides of the inequality we get (3.3) with \(C = 1\). The general case can be reduced to the case \(\kappa \leq 0\) by the standard change of variables. \(\square\)

In what follows we denote by \(T_k\), \(k \geq 0\), the truncature operator, i.e.

\[
T_k(y) = \max\{-k, \min\{y, k\}\}, \quad y \in \mathbb{R}.
\]

**Lemma 3.6.** Assume that \(\psi_n, \psi\) are Borel measurable functions on \(D_T\) such that \(\psi_n(X) \to \psi(X)\), \(dt \otimes P'_{s,x}\)-a.e. on \([0, \zeta]\) \(\times \Omega'\) for q.e. \((s, x) \in D_T\). Then for some subsequence (still denoted by \(n\)), \(\psi_n \to \psi\), \(m_1\)-a.e. on \(D_T\).

**Proof.** Let \((s, x) \in D_T\) be such that \(\psi_n(X) \to \psi(X)\), \(dt \otimes P'_{s,x}\)-a.s. on \([0, \zeta]\) \(\times \Omega'\). Then

\[
E'_{s,x} \int_0^{\zeta_r} T_k(|\psi_n - \psi|)(X_r) \, dr \to 0
\]
as \(n \to \infty\). Since

\[
E'_{s,x} \int_0^{\zeta_r} T_k(|\psi_n - \psi|)(X_r) \, dr = \int_{D_T} p_D(s, x, \theta, y) T_k(|\psi_n - \psi|)(\theta, y) \, d\theta \, dy,
\]

there exists a subsequence (still denoted by \(n\)) such that \(p_D(s, x, \theta, y) T_k(|\psi_n - \psi|) \to 0\), \(m_1\)-a.e. on \([s, T] \times D\), which by the positivity of \(p_D(s, x, \cdot, \cdot)\) and the definition of \(T_k\) implies that \(\psi_n \to \psi\), \(m_1\)-a.e. on \([s, T] \times D\). Since \((s, x)\) can be chosen so that \(s\) is arbitrary close to zero, one can choose a further subsequence (still denoted by \(n\)) such that \(\psi_n \to \psi\), \(m_1\)-a.e. on \(D_T\). \(\square\)

We say that a measurable function \(u : D_T \to \mathbb{R}\) is of class (FD) if for q.e. \((s, x) \in D_T\) the process \([s, T] \ni t \mapsto u(t, X_t)\) is of Doob’s class (D) under the measure \(P_{s,x}\).

We say that a measurable function \(u : D_T \to \mathbb{R}\) is quasi-càdlàg if for q.e. \((s, x) \in D_T\) the process \([s, T] \ni t \mapsto u(t, X_t)\) is càdlàg under \(P_{s,x}\).

\(\mathcal{FD}\) is the set of all quasi-càdlàg functions \(u : D_T \to \mathbb{R}\). \(\mathcal{FS}'\) (resp. \(\mathcal{FD}'\)), \(q > 0\), is the set of all quasi-continuous (resp. quasi-càdlàg) functions \(u : D_T \to \mathbb{R}\) such that for q.e. \((s, x) \in D_T\),

\[
E_{s,x} \sup_{s \leq t \leq T} |u(t, X_t)|^q < +\infty.
\]
$FM^q$, $q > 0$, is the set of all measurable functions $u : D_T \to \mathbb{R}$ such that for q.e. $(s, x) \in D_T$,

$$E_{s,x}(\int_s^T |f(t, X_t)|^2 dt)^{q/2} < +\infty.$$  

$T_{q,1}^0$ is the set of all measurable functions $u$ on $D_T$ such that for every $k \geq 0$, $T_k(u) \in L^2(0, T; H^q_0(D_T))$. From [3] it follows that for every $u \in T_{q,1}^0$ there exists a unique measurable function $v$ on $D_T$ such that $\nabla T_k(u) = 1_{[|u| < k]}v$, $m_1$-a.e. We shall set $\nabla u = v$.

**Proposition 3.7.** Assume that $\mu \in \mathcal{M}^+_0(D_T)$, $\varphi \in L^1(D)$, $\varphi \geq 0$. Then $v : D_T \to \mathbb{R}$ defined as

$$v(s, x) = E'_{s,x}\left(1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta} dA^\mu_r\right), \quad (s, x) \in D_T$$  

(3.5)

is of class (FD) and belongs to $\mathcal{F}D^q$, $q \in (0, 1)$. Moreover, $v \in T_{q,1}^0$, $v \in L^q(0, T; W_{0,q}^1)$ for $q \in (1, \frac{d+2}{d+1})$, $\nabla v \in FM$ for $q \in (0, 1)$ and for every $k \geq 0$,

$$E'_{s,x}\int_0^{\zeta} |\sigma \nabla T_k(v)|^2(r, X_r) \, dr \leq 4kE'_{s,x}\left(1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta} dA^\mu_r\right)$$  

(3.6)

for q.e. $(s, x) \in D_T$. Finally, for q.e. $(s, x) \in D_T$ the pair $(v(X), \sigma \nabla v(X))$ is a solution of BSDE$_{s,x}(\varphi, D, d\mu)$.

**Proof.** By Proposition 3.5 there exists a pair of processes $(Y, Z)$ such that $Y$ is of class (D), $(Y, Z) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and for q.e. $(s, x) \in D_T$ the pair $(Y, Z)$ is a solution of BSDE$_{s,x}(\varphi, D, d\mu)$, i.e.

$$Y_t = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta} dA^\mu_r - \int_t^{\zeta} Z_r \, dB_r, \quad t \in [0, \zeta].$$

By Proposition 3.5 again, for $n \in \mathbb{N}$ there is a pair of processes $(Y^n, Z^n)$ such that $Y^n$ is of class (D), $(Y^n, Z^n) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and $(Y^n, Z^n)$ is a solution of the following BSDE$_{s,x}$

$$Y^n_t = 1_{\{\zeta > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_t^{\zeta} n(Y^n_r - Y_r) - \int_t^{\zeta} Z^n_r \, dB_r, \quad t \in [0, \zeta]$$  

(3.7)

for q.e. $(s, x) \in D_T$. By [10, Theorem 5.7], for q.e. $(s, x) \in D_T$,

$$Y^n_t \nearrow Y_t, \quad t \geq 0, \, P'_{s,x}$$.a.s.,  

$$Z^n \to Z, \quad dt \otimes P'_{s,x}$$.a.e. on $[0, \zeta] \times \Omega'$.  

(3.9)

Let $A^n_t = \int_0^t n(Y^n_r - Y_r) - \int_t^r Z^n_r \, dB_r$. By Proposition 3.5, $Y_t = v(X_t)$, $P'_{s,x}$-a.s. for $t \in [0, T - \tau(0)]$ and $Z = \psi(X)$, $dt \otimes P'_{s,x}$-a.e. on $[0, \zeta] \times \Omega'$ for some measurable function $\psi$ on $D_T$. Therefore

$$A^n_t = \int_0^t n(Y^n_r - v(X_r)) - \int_t^r Z^n_r \, dB_r, \quad t \in [0, \zeta], \quad P'_{s,x}$$.a.s.
From this and (3.7), once again by Proposition 3.5,
\[
Y_t^n = v_n(X_t), \quad P_{s,x}^t\text{-a.s. for } t \in [0, \zeta_t] \tag{3.10}
\]
and
\[
Z^n = \psi_n(X), \quad dt \otimes P_{s,x}^t\text{-a.e. on } [0, \zeta_t] \times \Omega', \tag{3.11}
\]
where
\[
v_n(s, x) = E_{s,x}'\left(1_{\{\zeta > T-\tau(0)\}}\varphi(X_{T-\tau(0)}) + \int_0^{\zeta_t} \phi_n(X_r) \, dr\right), \quad \phi_n = n(v_n - v)
\]
and \(\psi_n\) is a measurable function on \(D_T\). Now fix \(n \in \mathbb{N}\) and put \((\bar{Y}, \bar{Z}) = (Y^n, Z^n)\), \(\bar{\varphi} = \phi_n\). Then for q.e. \((s, x) \in D_T\) the pair \((\bar{Y}, \bar{Z})\) is a solution of BSDE\(_{s,x}(\varphi, D, \bar{\varphi})\).

Let \(\varphi_m = \varphi \wedge m, \bar{\varphi}_m = \bar{\varphi} \wedge m\). By Proposition 3.5, for q.e. \((s, x) \in D_T\) there exists a solution \((\bar{Y}_m^n, \bar{Z}_m^n)\) of BSDE\(_{s,x}(\varphi_m, D, \bar{\varphi}_m)\) such that \(\bar{Y}_m^n\) is of class (D) and \((\bar{Y}_m^n, \bar{Z}_m^n) \in \mathcal{D}^2 \otimes M^q\) for \(q \in (0, 1)\) (it is known that in fact \(\bar{Y}_m^n\) is continuous). By [10, Theorem 5.7],
\[
\bar{Y}_m^n = Y_t, \quad t \geq 0, \quad P_{s,x}^t\text{-a.s.}, \tag{3.12}
\]
\[
\bar{Z}_m^n = Z_t, \quad dt \otimes P_{s,x}^t\text{-a.e. on } [0, \zeta_t] \times \Omega'. \tag{3.13}
\]

The proof that \(v\) has the desired regularity properties we divide into 3 steps.

Step 1. We assume that \(\mu = g \cdot m_1\) for some \(g \in L^2(D_T)\) and \(\varphi \in L^2(D)\). Let \(w \in \mathcal{W}(D_T)\) be a weak solution of the problem
\[
\begin{cases}
\frac{\partial w}{\partial t} + A_tw = -g \text{ in } D_T, \\
w(T, \cdot) = \varphi, \quad w(t, \cdot)_{\partial D} = 0, \quad t \in (0, T).
\end{cases}
\]
Let \(\bar{w} : S_T \equiv [0, T] \times \mathbb{R}^d \to \mathbb{R}\) be an extension of \(w\) on \(S_T\) such that
\[
\bar{w}(t, x) = \begin{cases}
w(t, x), & (t, x) \in D_T, \\
w(t, x) = 0, & (t, x) \in S_T \setminus D_T.
\end{cases}
\]
Then \(\bar{w} \in \mathcal{W}(S_T)\) and hence \(\frac{\partial \bar{w}}{\partial t} + A_t \bar{w} \in L^2(0, T; H^{-1}(\mathbb{R}^d))\). Let \(g_0, G \in L^2(S_T)\) be such that \(\frac{\partial \bar{w}}{\partial t} + L_t \bar{w} = -g_0 - \text{div}(G)\). By [11, Theorem 4.3], \(\bar{w}\) has a quasi-continuous \(m_1\) version (still denoted by \(\bar{w}\)) such that \(\bar{w} \in \mathcal{F}S^2, \nabla \bar{w} \in M^2\) and for q.e. \((s, x) \in S_T,\)
\[
\bar{w}(t, X_t) = \bar{w}(s, x) - \int_s^t g_0(r, X_r) \, dr
- \int_s^t G(r, X_r) \, d^n X_r + \int_s^t \sigma \nabla \bar{w}(r, X_r) \, dB_{s,r}, \quad s \leq t \leq T, \quad P_{s,x}^t\text{-a.s.}
\]
Hence, by Lemma 2.1,
\[
\bar{w}(X_t) = 1_{\{\zeta > T-\tau(0)\}} \bar{w}(X_{T-\tau(0)}) + \int_t^{\zeta_t} g(X_r) \, dr
- \int_t^{\zeta_t} \sigma \nabla \bar{w}(X_r) \, dB_r, \quad t \in [0, \zeta_t], \quad P_{s,x}^t\text{-a.s.}
\]
Observe also that $E'_{s,x}|(\bar{w} - \varphi)(X_{T-\tau(0)})| = \int_D p_D(s,x,T,y) |\bar{w} - \varphi|(T,y) \, dy = 0$. Thus,

$$\bar{w}(s,x) = E'_{s,x} \left( 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) + \int_0^{\zeta_T} g(X_r) \, dr \right). \quad (3.14)$$

Therefore $v = w$ satisfies all assertions of the proposition except from (3.6). To prove (3.6) let us fix $k > 0$ and $z \in \mathbb{R}$. By the Itô-Tanaka formula,

$$|\bar{w}(s,x) - z| + E'_{s,x} L_{\zeta_T}^z (\bar{w}(X)) = E'_{s,x} 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) - z$$

$$+ E'_{s,x} \int_0^{\zeta_T} \text{sgn}(\bar{w}(X_r) - z) g(X_r) \, dr,$$

where $\{L_t^z(\bar{w}(X)), t \geq 0\}$ denote the local time of $\bar{w}(X)$ at $z$ and $\text{sgn}(x) = 1_{x \neq 0} \frac{x}{|x|}$. Hence

$$E'_{s,x} L_{\zeta_T}^z (\bar{w}(X)) \leq E'_{s,x} 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) - \bar{w}(s,x) + E'_{s,x} \int_0^{\zeta_T} g(X_r) \, dr.$$

Multiplying the above inequality by the function $i(z) = 1_{[-k,k]}(z)$, integrating with respect to $z$ and applying the occupation time formula we get

$$E'_{s,x} \int_0^{\zeta_T} |\sigma \nabla T_k(\bar{w})|^2(X_r) \, dr \leq 2k E'_{s,x} \left( 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) + \bar{w}(s,x) + \int_0^{\zeta_T} g(X_r) \, dr \right),$$

which when combined with (3.14) yields (3.6).

Step 2. We are going to show that $v_n$ satisfies all the assertions of the proposition. To shorten notation, we write $\overline{\nu}$ (resp. $\bar{\nu}$) instead of $v_n$ (resp. $\psi_n$). Since $\varphi_m \in L^2(D)$, and $\overline{\phi}_m \in L^2(D_T)$, it follows from Step 1 that there exists $\overline{\nu}_m \in \mathcal{FS}^2$ such that $\overline{\nu}_m \in \mathcal{W}(D_T)$ and for q.e. $(s,x) \in D_T$,

$$\overline{\nu}_m(X_t) = \overline{\nu}_t^m, \quad t \in [0,\zeta_T], \quad P'_{s,x}$-a.s.,$$

$$\sigma \nabla \overline{\nu}_m(X) = \overline{Z}_r^m, \quad \text{dt} \otimes P'_{s,x}$-a.s. \text{ on } [0,\zeta_T] \times \Omega'_{s,x}.$$

Put $\overline{\nu}(s,x) = \limsup_{m \to \infty} \overline{\nu}_m(s,x), (s,x) \in D_T$. Then by (3.12), $\overline{\nu}(X_t) = \overline{\nu}_t, t \in [0,\zeta_T], P'_{s,x}$-a.s. for q.e. $(s,x) \in D_T$. This implies that $\overline{\nu} \in \mathcal{FD}^q$, $q \in (0,1)$, $\overline{\nu}'$ is of class (FD) and $\overline{\nu}' = \overline{\nu}$ q.e. The last statement implies that $\overline{\nu}$ belongs to the same spaces of functions as $\overline{\nu}'$. We have proved that (3.6) is true if $\mu = g \cdot m_1$ for some $g \in L^2(D_T)$ and $\varphi \in L^2(D)$. Therefore for q.e. $(s,x) \in D_T$,

$$E'_{s,x} \int_0^{\zeta_T} |\sigma \nabla T_k(\overline{\nu}_m)||X_r|| \, dr$$

$$\leq 4k E'_{s,x} \left( 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) + \int_0^{\zeta_T} \bar{\phi}_m(X_r) \, dr \right)$$

$$\leq 4k E'_{s,x} \left( 1_{\{\zeta > T-\tau(0)\}} \varphi(X_{T-\tau(0)}) + \int_0^{\zeta_T} \bar{\phi}(X_r) \, dr \right). \quad (3.15)$$

From this and Lemma 2.5,

$$\int_{D_T} |\sigma \nabla (T_k(\overline{\nu}_m))|^2 \, dm_1 \leq 4k(\|\varphi\|_{L^1} + \|\bar{\phi}_m\|_{L^1}) \leq 4k(\|\varphi\|_{L^1} + \|\bar{\phi}\|_{L^1}). \quad (3.16)$$

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Due to [20], from (3.16) it follows that \( \overline{v} \in T_{2}^{0,1} \cap L^{q}(0, T; W_{0}^{1,q}(D)) \) and for some subsequence (still denoted by \( n \)), \( \sigma \nabla \tau_{m} \to \sigma \nabla \tau \) weakly in \( L^{q}(D; T) \) for \( q \in \left[ 1, \frac{d+2}{d+1} \right] \). On the other hand, by (3.13) and Lemma 3.6, \( \sigma \nabla \tau_{m} \to \overline{\psi}, m_{1}\text{-a.e. on } D_{T} \). Hence \( \overline{\psi} = \sigma \nabla \tau, m_{1}\text{-a.e. on } D_{T} \). Summarizing, \( \overline{v} \) satisfies all the assertions of the proposition.

Step 3. Using (3.8)–(3.11) one can show in the same manner as in Step 2 that \( v \) satisfies all the assertions of the proposition. The only difference from Step 2 is that in estimates of \( \sigma \nabla T_{k}^{n}(v_{n}) \) of the form (3.15), (3.16) the function \( \phi_{m} \) is replaced by \( \phi_{n} \), so to obtain the estimates for \( \sigma \nabla T_{k}(v_{n}) \) which do not depend on \( n \) we have to show that

\[
E_{s,x}^{{x_{*}}} \int_{0}^{{x_{*}}} \phi_{n}(X_{r}) \, dr \leq E_{s,x}^{{x_{*}}} \int_{0}^{{x_{*}}} dA_{r}^{n},
\]

for q.e. \((s, x) \in D_{T}\) and that \( \|\phi_{n}\|_{TV} \leq \|\mu\|_{TV} \). But (3.17) follows from the fact that \( v_{n}(s, x) \leq v(s, x) \) for q.e. \((s, x) \in D_{T}\) and the estimate for \( \|\phi_{n}\|_{TV} \) follows from (3.17) and Lemma 2.5.

4 Cauchy-Dirichlet problem

It is convenient to begin the study of the obstacle problem (1.1) with the study of the Cauchy-Dirichlet problem

\[
\begin{cases}
\frac{\partial u}{\partial t} + A_{t}u = -f_{u} - \mu, \\
u(T, \cdot) = \varphi, \quad u(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T),
\end{cases}
\]

which can be regarded as problem (1.1) with \( h_{1} \equiv -\infty, h_{2} \equiv +\infty \).

Let us recall that every functional \( \Phi \in \mathcal{W}'(D_{T}) \) admits decomposition of the form

\[
\Phi = (g)_{t} + \text{div}(G) + f,
\]

where \( g \in L^{2}(0, T; H_{0}^{1}(D)), G, f \in L^{2}(D_{T}), \) i.e. for every \( \eta \in \mathcal{W}(D_{T}) \),

\[
\Phi(\eta) = -\langle g, \frac{\partial \eta}{\partial t} \rangle - \langle G, \nabla \eta \rangle_{L^{2}} + \langle f, \eta \rangle_{L^{2}},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( L^{2}(0, T; H_{0}^{1}(D)) \) and \( L^{2}(0, T; H^{-1}(D)) \). It is also known that every measure \( \mu \in \mathcal{M}_{0,b}(D_{T}) \) admits decomposition of the form

\[
\mu = \Phi + f,
\]

where \( \Phi \in \mathcal{W}'(D_{T}), f \in L^{1}(D_{T}), \) i.e. for every \( \eta \in \mathcal{C}_{0}^{\infty}(D_{T}) \),

\[
\int_{D_{T}} \eta \, d\mu = \Phi(\eta) + \int_{D_{T}} f \eta \, dm_{1}.
\]

Accordingly, \( \mu \in \mathcal{M}_{0,b}(D_{T}) \) can be written in the form

\[
\mu = -(g)_{t} + \text{div}(G) + f
\]

for some \( g \in L^{2}(0, T; H_{0}^{1}(D)), G \in L^{2}(D_{T}) \) and \( f \in L^{1}(D_{T}) \). Let us stress that in general, \( \Phi, f \) of the decomposition (4.3) cannot be taken nonnegative even if \( \mu \) is nonnegative.

We say that a triple \((g, G, f)\) is the decomposition of \( \mu \in \mathcal{M}_{0,b}(D_{T}) \) if (4.4) is satisfied.

Let \( \mu \in \mathcal{M}_{0,b}(D_{T}), \varphi \in L^{1}(D) \) and let \( f : D_{T} \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function.
Definition. We say that a measurable function \( u : D_T \to \mathbb{R} \) is a renormalized solution of the problem (4.1) if

(a) \( f_u \in L^1(D_T) \),

(b) For some decomposition \((g,G,f)\) of \( \mu \), \( u - g \in L^\infty(0,T;L^2(D)) \), \( T_k(u - g) \in L^2(0,T;H^1_0(D)) \) for \( k \geq 0 \) and

\[
\lim_{k \to +\infty} \int_{\{k \leq |u-g| \leq k+1\}} |\nabla u| \, dm_1 = 0,
\]

(c) For any \( S \in W^{2,\infty}(\mathbb{R}) \) with compact support,

\[
\frac{\partial}{\partial t}(S(u-g)) + \text{div}(a\nabla u S'(u-g)) - S''(u-g) a\nabla u \cdot \nabla (u-g) = -S'(u-g)f - \text{div}(GS'(u-g)) + GS''(u-g) \cdot \nabla (u-g)
\]

in the sense of distributions,

(d) \( T_k(u - g)(T) = T_k(\varphi) \) in \( L^2(D) \) for all \( k \geq 0 \).

Note that different but equivalent definition of a renormalized solution of (4.1) is given in [21].

Set

\[
\Theta_k(s) = \int_0^s T_k(y) \, dy, \quad s \in \mathbb{R}
\]

and

\[
E = \{ \eta \in L^2(0,T;H^1_0(D)) \cap L^\infty(D_T) : \frac{\partial \eta}{\partial t} \in L^2(0,T;H^{-1}(D)) + L^1(D_T) \}.
\]

Definition. We say that a measurable function \( u : D_T \to \mathbb{R} \) is an entropy solution of (4.1) if \( f_u \in L^1(D_T) \), for some decomposition \((g,G,f)\) of \( \mu \), \( T_k(u-g) \in L^2(0,T;H^1_0(D)) \) for \( k \geq 0 \), \( [0,T] \ni t \mapsto \int_D \Theta_k(u-g-\eta)(t,\cdot) \, dm \) is continuous for any \( \eta \in E \), \( k \geq 0 \), and

\[
\int_D \Theta_k(u-g-\eta)(0,\cdot) \, dm - \int_D \Theta_k(\varphi-\eta(T,\cdot)) \, dm - \langle \eta, T_k(u-g-\eta) \rangle + \int_{D_T} a\nabla u \cdot \nabla T_k(u-g-\eta) \, dm_1 \leq \int_{D_T} fT_k(u-g-\eta) \, dm_1
\]

\[
- \int_{D_T} G \cdot \nabla T_k(u-g-\eta) \, dm_1 + \int_{D_T} f_u T_k(u-g-\eta) \, dm_1. \tag{4.5}
\]

Remark 4.1. (i) From the general results proved in [6] it follows that \( u \) is a renormalized solution of (4.1) if it is an entropy solution of (4.1).

(ii) If \( u \) is a renormalized solution of (4.1) then it is a distributional solution of (4.1) in the sense that \( u, \nabla u \in L^1(D_T) \) and for any \( \eta \in C_0^\infty((0,T] \times D) \),

\[
\int_{D_T} u \frac{\partial \eta}{\partial t} \, dm_1 + \int_{D_T} a\nabla u \cdot \nabla \eta \, dm_1 = \int_D \varphi \eta(T,\cdot) \, dm + \int_{D_T} f_u \eta \, dm_1 + \int_{D_T} \eta \, d\mu \tag{4.6}
\]

(see Proposition 4.5 and Theorem 4.11 in [21]).
**Lemma 4.2.** Assume that $\mu_n, \mu \in \mathcal{M}_{0,b}(D_T)$ and $\|\mu_n - \mu\|_{TV} \to 0$. Then there exist $g_n, g \in L^2(0,T;H^1_0(D))$, $G_n, G \in L^2(D_T)$, $f_n, f \in L^1(D_T)$ such that

$$\mu_n = (g_n)_t + \text{div}(G_n) + f_n, \quad \mu = (g)_t + \text{div}(G) + f$$

and

$$G_n \to G \text{ in } L^2(D_T), \quad f_n \to f \text{ in } L^1(D_T), \quad g_n \to g \text{ in } L^2(0,T;H^1_0(D)).$$

**Proof.** From the proof of [7, Theorem 2.7] it follows that each $\mu \in \mathcal{M}_{0,b}(D_T)$ admits decomposition of the form (4.3) with $\Phi, f$ such that $\|\Phi\|_{W'(D_T)} \leq 1$, $\|f\|_{L^1} \leq \|\mu\|_{TV}$. Moreover, by [7, Lemma 2.24], $\Phi$ admits decomposition (4.2) with $g, G, h$ such that

$$\|g\|_{L^2(0,T;H^1_0(D))} + \|G\|_{L^2} + \|h\|_{L^2} \leq \|\Phi\|_{W'(D_T)}.$$

Therefore repeating arguments from the proof of [15, Corollary 3.2] we get the desired result. \hfill \Box

**Lemma 4.3.** Let $\{\mu_n\} \subset \mathcal{M}_{0,b}(D_T)$, $\mu \in \mathcal{M}_{0,b}(D_T)$, $\{\varphi_n\} \subset L^1(D), \varphi \in L^1(D)$ and let $u_n$ (resp. $u$) be a renormalized solution of (4.1) with terminal condition $\varphi_n$ (resp. $\varphi$), $f \equiv 0$ and with $-\mu_n$ (resp. $-\mu$) on the right-hand side. If $\|\mu_n - \mu\|_{TV} \to 0$ and $\|\varphi_n - \varphi\|_{L^1} \to 0$ then $u_n \to u$, $m_1$-a.e.

**Proof.** It is enough to use Lemma 4.2, observe that $u_n$ satisfies all the properties of Proposition 4 in [20], apply equivalence between renormalized and entropy solutions for parabolic equations proved in [6] and uniqueness of them (see [20]). \hfill \Box

**Proposition 4.4.** Let $\varphi \in L^1(D), \mu \in \mathcal{M}_{0,b}(D_T)$ and let $v$ be defined by (3.5). Then $v \in T^{0,1}_2$, $v \in L^q(0,T; W^{1,q}_0(D))$ for $q \in [1, \tfrac{d+2}{d+1})$ and $v$ is a renormalized solution of the problem

$$\begin{aligned}
\frac{\partial v}{\partial t} + A_tv &= -\mu, \\
v(T, \cdot) &= \varphi, \quad v(t, \cdot)|_{\partial D} = 0, \quad t \in (0,T).
\end{aligned} \quad (4.7)$$

**Proof.** Without loss of generality we may assume that $\varphi \geq 0$ and $\mu \geq 0$. Assume for a moment that $\varphi \in L^2(D_T)$ and $\mu \in \mathcal{M}_{0,b}^+(D_T)$. From the proof of Proposition 3.7 it follows that $v$ is the q.e. limit of $v_n$, where $v_n$ is a weak solution of

$$\begin{aligned}
\frac{\partial v_n}{\partial t} + A_tv_n &= -n(v_n - v)^-, \\
v_n(T, \cdot) &= \varphi, \quad v_n(t, \cdot)|_{\partial D} = 0, \quad t \in (0,T).
\end{aligned}$$

But it is known (see [18, 23]) that $\{v_n\}$ converges in $L^2(D_T)$ to a unique weak solution of (4.7). Therefore $v$ is a weak solution of (4.7). Since a weak solution of (4.7) is a renormalized solution, this proves the proposition under the additional assumptions on $\varphi, \mu$. Assume now that $\varphi \in L^1(D)$ is nonnegative and $\mu \in \mathcal{M}_{0,b}^+(D_T)$. By [19] there exists a generalized nest, i.e. an ascending sequence $\{F_n\}$ of compact subsets of $D_T$ such that $\text{cap}(K \setminus F_n) \to 0$ for every compact $K \subset D_T$, with the property that $1_{F_n}, \mu \in \mathcal{M}_{0,b}^+(D_T) \cap W'(D_T)$ and $\mu(D_T \setminus \bigcup_n F_n)$. Let $\varphi_n = \varphi \cap n$. By what has already been proved, $v_n$ defined as

$$v_n(s,x) = E_{s,x}^{\mu} \left( 1_{\{\zeta > T - \tau(0)\}} \varphi_n(X_{T - \tau(0)}) + \int_0^{\tau(0)} 1_{F_n}(X_r) dA^\mu_r \right), \quad (s,x) \in D_T \quad (4.8)$$



\hfill 22
is a renormalized solution of (4.7) with \( \varphi, \mu \) replaced by \( \varphi_n \) and \( \mu_n \), respectively. Since 
\[ \|\varphi_n - \varphi\|_{L^1} \to 0 \] and \( \{F_n\} \) is a generalized nest we conclude from (4.8) that \( v_n(s, x) \to v(s, x) \) for q.e. \( (s, x) \in D_T \). This completes the proof because by Lemma 4.3, \( \{v_n\} \) converges to the renormalized solution of (4.7).

\textbf{Theorem 4.5.} Assume (H1)–(H5). Then there exists a unique renormalized solution \( u \) of (4.1). Moreover, \( u \in FD; u \in L^q(0, T; W_0^{1,q}(D)) \) for \( q \in \left[1, \frac{d+2}{d+1}\right] \) and

\[ u(s, x) = E_{s,x}^* \left( 1_{\{\xi > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta_r} f_u(X_r) \, dr + \int_0^{\zeta_T} dA_r^u \right) \tag{4.9} \]

for q.e. \( (s, x) \in D_T \). Finally, there exists \( C > 0 \) depending only on \( \kappa, T \) such that

\[ \|f_u\|_{L^1(D_T)} \leq C(\|\varphi\|_{L^1(D)} + \|f(\cdot, \cdot, 0)\|_{L^1(D_T)} + \|\mu\|_{TV}). \tag{4.10} \]

\textbf{Proof.} By Lemma 3.4 and [10, Proposition 3.10], for q.e. \( (s, x) \in D_T \) there is a unique solution \( (Y^{s,x}, Z^{s,x}) \) of BSDE \( s, x \) \( \varphi, D, f + d\mu \). By Proposition 3.5, \( Y_t^{s,x} = u(X_t) \), \( P^s_0 \text{-a.s.} \) for every \( t \in (0, T - \tau(0)) \) and q.e. \( (s, x) \in D_T \), where \( u : D_T \to \mathbb{R} \) satisfies (4.9). Furthermore, by Lemma 2.5 and (3.3), \( f_u \) satisfies (4.10). Hence \( f_u \cdot m_1 + \mu \in M_{0,0}(D_T) \) and from Proposition 4.4 it follows that \( u \) is a renormalized solution of (4.1) and has the other regularity properties. The uniqueness part of the theorem follows from the uniqueness of solutions of BSDEs, \( s, x, (\varphi, D, f + d\mu) \).

\textbf{Remark 4.6.} Under the assumptions of Theorem 4.5 for every \( q \in \left[1, \frac{d+2}{d+1}\right] \) there exist \( C, \gamma > 0 \) such that

\[ \|u\|_{L^q(0, T; W_0^{1,q}(D))} \leq C(\|\varphi\|_{L^1(D)} + \|f(\cdot, \cdot, 0)\|_{L^1(D_T)} + \|\mu\|_{TV})^\gamma, \]

because from Remark 4.1(ii) and results proved in [4, Section 3] (see also [5]) it follows that if \( u \) is a solution of (4.1) then \( \|u\|_{L^q(0, T; W_0^{1,q}(D))} \leq c\|f_u \cdot m_1 + \mu\|_{TV}^\gamma \) for some \( c, \gamma > 0 \).

5 \hspace{1cm} \textbf{Obstacle problem}

We begin with a probabilistic definition of a solution of the obstacle problem.

\textbf{Definition.} Assume (H1), (H4) and let \( h_1, h_2 \) be measurable functions on \( D_T \) such that \( h_1 \leq h_2 \), \( m_1 \text{-a.e.} \). We say that a pair \( (u, \nu) \) consisting of a measurable function \( u : D_T \to \mathbb{R} \) and a measure \( \nu \) on \( D_T \) is a solution of the obstacle problem with terminal condition \( \varphi \), the right-hand side \( f + d\mu \) and obstacles \( h_1, h_2 \) (OP(\( \varphi, f + d\mu, h_1, h_2 \)) for short) if

\begin{enumerate}[(a)]  
\item \( f_u \in L^1(D_T), \nu \in M_{0,0}(D_T), h_1 \leq u \leq h_2, m_1 \text{-a.e.}, \)
\item \( \text{For q.e. } (s, x) \in D_T, \]
\end{enumerate}

\[ u(s, x) = E_{s,x}^* \left( 1_{\{\xi > T - \tau(0)\}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta_r} f_u(X_r) \, dr + \int_0^{\zeta_T} dA_r^u \right), \]
(c) For every \( h_1^*, h_2^* \in \mathcal{FD} \) such that \( h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2, \) \( m_1 \text{-a.e.} \) we have
\[
\int_0^{\zeta_r} (u_-(X_r) - h_1^-(X_r)) \, dA_r^\nu = \int_0^{\zeta_r} (h_2^-(X_r) - u_-(X_r)) \, dA_r^\nu = 0, \quad P'_{s,x} \text{-a.s.}
\]
for q.e. \((s,x) \in D_T\), where \( g_-(X_r) = \lim_{t \downarrow r, t \to r} g(X_t) \) for \( g \equiv u, h_1^*, h_2^* \).

We say that \((u, \nu)\) is a solution of the obstacle problem with one lower (resp. upper) barrier \( h \) (\( \Omega \cup \varphi, f + d\mu, h \)) (resp. \( \Omega \cup \varphi, f + d\mu, h \)) for short) if \((u, \nu)\) satisfies the conditions of the above definition with \( h_1 = h, h_2 = +\infty \) (resp. \( h_1 = -\infty, h_2 = h \)) and \( \nu \in \mathcal{M}_{0,b}^+(D_T) \) (resp. \( -\nu \in \mathcal{M}_{0,b}^+(D_T) \)).

**Remark 5.1.** Let us note that in view of Proposition 4.4, condition (b) in the above definition says that \( u \) is a renormalized solution of (4.1) with \( \mu \) replaced by \( \mu + \nu \). Condition (c) provides a probabilistic formulation of minimality of \( \nu \). In [22] minimality of \( \nu \) was expressed by using the notion of precise versions of functions introduced in [23]. In fact, in the linear case with \( L^2 \) data, condition (c) coincides with that introduced in [22], because for every parabolic potential \( h, \hat{h}(X_t) = h_-(X_t), \) \( t \in [0, \zeta_r) \), where \( \hat{h} \) is the precise version of \( h \) (see [9] for details).

As in the case of the Cauchy-Dirichlet problem considered in the previous section, the proof of the existence of a solution of the obstacle problem relies heavily on the results on BSDEs proved in [10].

Suppose we are given a filtered probability space and \( A, B, \sigma, \xi \) as in Subsection 3.1, and moreover, we are given two progressively measurable processes \( U, L \) such that \( U \leq L \) and \( f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) such that \( f(\cdot, y) \) is progressively measurable (\( f \) does not depend on \( z \)).

**Definition.** A triple \((Y, Z, R)\) of progressively measurable processes is a solution of the reflected backward stochastic differential equation with terminal condition \( \xi \), right-hand side \( f + dA \) and two reflecting barriers \( L, U \) (\( \text{RBSDE}(\xi, \sigma, f + dA, L, U) \) for short) if

(a) \( Z \in \mathcal{M}, t \mapsto f(t, Y_t, Z_t) \in L^1(0, \sigma), P\text{-a.s.}, \)

(b) \( L_t \leq Y_t \leq U_t \) for a.e. \( t \in [0, \sigma], P\text{-a.s.}, \)

(c) \( Y_t = \xi + \int_t^\sigma f(s, Y_s, Z_s) \, ds + \int_t^\sigma dA_s - \int_t^\sigma dR_s - \int_t^\sigma Z_s \, dB_s, 0 \leq t \leq \sigma, P\text{-a.s.,} \)

(d) \( R \in \mathcal{V} \) for any \( \tilde{L}, \tilde{U} \in \mathcal{D} \) such that \( L_t \leq \tilde{L}_t \leq Y_t \leq \tilde{U}_t \leq U_t \) for a.e. \( t \in [0, \sigma], \)
\[
\int_0^\sigma (Y_t - \tilde{Y}_t) \, dR_t^+ + \int_0^\sigma (\tilde{L}_t - Y_t) \, dR_t^- = 0, P\text{-a.s.}
\]

We say that \((Y, Z, R)\) is a solution of RBSDE with one lower (resp. upper) barrier \( L \) (resp. \( U \)), terminal condition \( \xi \), the right-hand side \( f + dA \) (\( \text{RBSDE}(\xi, \sigma, f + dA, L) \)) (resp. \( \text{RBSDE}(\xi, \sigma, f + dA, U) \)) for short) if \((Y, Z, R)\) satisfies the conditions of the above definition with \( U \equiv +\infty \) (resp. \( L \equiv -\infty \)) and \( R \in \mathcal{V}^+ \) (resp. \( -R \in \mathcal{V}^+ \)).

To shorten notation, in what follows for given \( \mu, \varphi, f \) and \( h_1, h_2 \) we denote by \( \text{RBSDE}_{s,x}(\varphi, D, f + d\mu, h_1, h_2) \) the reflected BSDE with data \( 1_{\{\tau_\tau - \tau(0)\}} \varphi(X_{\tau_\tau - \tau(0)}), \)
\( \zeta_r, f(X, \cdot) + d\mu, h_1(X), h_2(X) \) considered on the space \((\Omega', G_\infty', \{G_t\}_{t \in [0,T]}), P_{s,x}' \).

Let \( \mathcal{M}_q, q > 0, \) denote the space of continuous martingales such that \( E((M_T)^q) < +\infty \) for every \( T > 0 \).

We will need the following general growth conditions for \( f \).
(A6) There exists a semimartingale Γ such that Γ is of class (D), Γ ∈ \( M^q \oplus \mathcal{V}^1 \) for \( q \in (0, 1) \), \( L_t \leq \Gamma_t \) for a.e. \( t \in [0, \sigma] \) and \( E \int_0^\sigma f^-(t, \Gamma_t) \, dt < +\infty \).

(A6') There exists a semimartingale Γ such that Γ is of class (D), Γ ∈ \( M^q \oplus \mathcal{V}^1 \) for \( q \in (0, 1) \), \( L_t \leq \Gamma_t \leq U_t \) for a.e. \( t \in [0, \sigma] \) and \( E \int_0^\sigma |f(t, \Gamma_t)| \, dt < +\infty \).

It is known that under (A1)–(A6) (resp. (A1)–(A5), (A6')) there exists a unique solution of BSDE\( (\xi, \sigma, f + dV, L) \) (resp. RBSDE \( (\xi, \sigma, f + dA, L, U) \)) (see Theorems 5.7, 6.6 in [10]). By Lemma 3.4, assumptions (H1)–(H5) imply that (A1)–(A5) hold under the measure \( P_{s,x}' \) for q.e. \((s, x) \in D_T\).

Analytic analogues of conditions (A6), (A6') are as follows.

(H6) There exists a measurable function \( v : D_T \to \mathbb{R} \), a measure \( \lambda \in \mathcal{M}_{0,b}(D_T) \) and \( \phi \in \mathbf{L}^1(D) \), \( \phi \geq \varphi \), such that \( v \) is a renormalized solution of the problem

\[
\begin{align*}
\frac{\partial v}{\partial t} + A_t v &= -\lambda, \\
v(T, \cdot) &= \phi, \quad v(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T)
\end{align*}
\]

and \( f^-v \in \mathbf{L}^1(D_T) \), \( v \geq h_1, m_1 \)-a.e. on \( D_T \).

(H6') There exists a measurable function \( v : D_T \to \mathbb{R} \), a measure \( \lambda \in \mathcal{M}_{0,b}(D_T) \) and \( \phi \in \mathbf{L}^1(D) \), \( \phi \geq \varphi \), such that \( v \) is a renormalized solution of (5.1) and \( f^v \in \mathbf{L}^1(D_T) \), \( h_1 \leq v \leq h_2, m_1 \)-a.e. on \( D_T \).

**Lemma 5.2.** Let \( L = h_1(X), U = h_2(X) \). If \( v \) satisfies (H6) (resp. (H6')) then \( \Gamma = v(X) \) satisfies (A6) (resp. (A6')) under the measure \( P'_{s,x} \) for q.e. \((s, x) \in D_T\).

**Proof.** Follows immediately from Propositions 3.7 and 4.4.

We first prove the comparison and uniqueness results for solutions of the obstacle problem.

**Proposition 5.3.** Let \((u_i, \nu_i)\) be a solution of \( \text{OP}(\varphi_i, f_i + d\mu_i, h_1^i, h_2^i), i = 1, 2 \). If \( \varphi_1 \leq \varphi_2, m\text{-a.e.}, d\mu_1 \leq d\mu_2, h_1^i \leq h_2^i, h_1^i \leq h_2^i, m_1\text{-a.e.} \) and either

\[
\text{if } f_2 \text{ satisfies (H3) and } 1_{\{u_1 > u_2\}}(f_{u_1}^1 - f_{u_2}^2) \leq 0, \quad m_1\text{-a.e.}
\]

or

\[
\text{if } f_1 \text{ satisfies (H3) and } 1_{\{u_1 > u_2\}}(f_{u_1}^1 - f_{u_2}^2) \leq 0, \quad m_1\text{-a.e.}
\]

then \( u_1 \leq u_2 \) q.e.

**Proof.** The desired result follows from [10, Corollary 6.2], because by the definition of a solution of the obstacle problem and Proposition 3.7, for q.e. \((s, x) \in D_T\) the triple \((u_i(X), \sigma\nabla u_i(X), A_{u_i})\) is a solution of RBSDE\(\mathcal{S}_{s,x}(\varphi_i, D, f^i + d\mu_i, h_1^i, h_2^i)\).

**Corollary 5.4.** Let assumption (H3) hold. Then there exists at most one solution of \( \text{OP}(\varphi, f + d\mu, h_1, h_2) \).
proof. Follows immediately from Proposition 5.3.

Before proving our main result on existence and approximation by the penalization method of solutions of the obstacle problem with one barrier let us recall that a function \( v \) on \( D_T \) is a supersolution of \( \text{PDE}(\varphi, f + d\mu) \) if there exists a measure \( \lambda \in \mathcal{M}_{0,b}^+(D_T) \) such that \( v \) is a renormalized solution of the problem

\[
\begin{aligned}
\frac{\partial v}{\partial t} + A_t v &= -f_v - \mu - \lambda, \\
v(T, \cdot) &= \varphi, \quad v(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T).
\end{aligned}
\]

Theorem 5.5. Assume (H1)–(H6).

(i) There exists a solution \((u, \nu)\) of \( \text{OP}(\varphi, f + d\mu, h_1) \) such that \( u \) is of class (FD), \( u \in \mathcal{FD}_q \) for \( q \in (0, 1) \), \( \nabla u \in FM^q \) for \( q \in (0, 1) \), \( u \in \mathcal{T}^2_{0,1} \), \( u \in L^q(0, T; W^{1,q}_0(D)) \) for \( q \in [1, \frac{d+2}{d-1}] \).

(ii) For \( n \in \mathbb{N} \) let \( u_n \) be a renormalized solution of the problem

\[
\begin{aligned}
\frac{\partial u_n}{\partial t} + A_t u_n &= -f_n - n(u_n - h_1)^- - \mu, \\
\nu_n(T, \cdot) &= \varphi, \quad \nu_n(t, \cdot)_{|\partial D} = 0, \quad t \in (0, T).
\end{aligned}
\]

Then \( u_n \not\to u \) q.e. on \( D_T \), \( u_n \to u \) in \( L^q(0, T; W^{1,q}_0(D)) \), \( q \in [1, \frac{d+2}{d-1}] \) and \( \nu_n \to \nu \) weakly, where \( d\nu_n = n(u_n - h_1)^- dm_1 \).

Proof. (i) By Lemmas 3.4, 5.2 and [10, Theorem 5.7], for q.e. \((s, x) \in D_T\) there exists a unique solution \((Y^{s,x}, Z^{s,x}, R^{s,x})\) of \( \text{RBSDE}_{s,x}(\varphi, D, f + d\mu, h_1) \) such that \( Y^{s,x} \) is of class (D), \( Y^{s,x}, Z^{s,x} \in \mathcal{D}^q \otimes M^q \) for \( q \in (0, 1) \) and \( R^{s,x} \in V^{+,1} \). By [10, Theorem 5.7], for q.e. \((s, x) \in D_T\),

\[
Y_t^{n,s,x} \not\to Y_t^{s,x}, \quad t \in [0, \zeta_T], \quad P_{s,x}^{s,x}\text{-a.s.,} \quad (5.3)
\]

\[
Z_t^{n,s,x} \to Z_t^{s,x}, \quad dt \otimes P_{s,x}^{s,x}\text{-a.e. on } [0, \zeta_T] \times \Omega', \quad (5.4)
\]

where \((Y^{n,s,x}, Z^{n,s,x})\) is a solution of the equation

\[
Y_0^{n,s,x} = \mathbb{1}_{\{\zeta_T = 0\}} \varphi(X_{T-}) + \int_0^{\zeta_T} (f(r, X_r, Y_r^{n,s,x}) + n(Y_r^{n,s,x} - h_1(X_r))^-) \, dr + \int_0^{\zeta_T} dA_r - \int_0^{\zeta_T} Z_r^{n,s,x} dB_r, \quad t \in [0, \zeta_T], \quad P_{s,x}^{s,x}\text{-a.s.}
\]

By Proposition 3.7, \( u_n(X_t) = Y_t^{n,s,x}, \quad t \in [0, \zeta_T], \) \( Z_t^{n,s,x} = \sigma \nabla u_n(X), \) \( dt \otimes P_{s,x}^{s,x}\text{-a.e. on } [0, \zeta_T] \times \Omega' \) for q.e. \((s, x) \in D_T\). For \((s, x) \in D_T\) set \( u(s, x) = \limsup_{n \to +\infty} u_n(s, x) \). Then by (5.3), for q.e. \((s, x) \in D_T\),

\[
u(X_t) = Y_t^{s,x}, \quad u(X_t) = Y_t^{s,x}, \quad t \in [0, \zeta_T], \quad P_{s,x}^{s,x}\text{-a.s.} \quad (5.5)
\]

Moreover, by (5.4) and Lemma 3.6, there exists a Borel measurable function \( \psi \) on \( D_T \) such that for q.e. \((s, x) \in D_T\),

\[
\psi(X) = Z^{s,x}, \quad dt \otimes P_{s,x}^{s,x}\text{-a.e. on } [0, \zeta_T] \times \Omega'. \quad (5.6)
\]
Hence, it follows from (5.9) and Lemma 3.6 that $D$ for q.e. $(s,x)$ it follows that $u$ that $\mu$ by (4.4) it follows now that $u$ is a positive functional. Therefore there exists a positive smooth measure $\nu$ on $D_T$ such that $A = A'$. From this and (5.5) it follows that

$$u(s,x) = E'_{s,x} \left( 1_{\{ \zeta > T - \tau(0) \}} \varphi(X_{T - \tau(0)}) + \int_0^{\zeta_T} f_u(X_r) \, dr + \int_0^{\zeta_T} d(A^\mu_r + A^\nu_r) \right)$$  

(5.8)

for q.e. $(s,x) \in D_T$. Let $v$ denote the function from condition (H6). By Lemma 5.2, $\nu(X)$ satisfies (A6) (with $L = h_1(X)$) and $P_{s,x}$ for q.e. $(s,x) \in D_T$. Therefore arguing as at the beginning of the proof of [10, Theorem 5.7] and using Theorem 4.1 one can show that there exists a supersolution $\tilde{v}$ of PDE$(\varphi \vee \phi, f + d\mu)$ such that $u_n \leq \tilde{v}$ q.e. on $D_T$ for $n \in \mathbb{N}$. From this, the fact that $u_1 \leq u$ q.e. on $D_T$ and (H3) it follows that $f_0 - \kappa \tilde{v} \leq f_0 - \kappa u \leq f_{u_1} - u_1$, $m_1$-a.e. Hence $u_f \in L^1(D_T)$, because $u_1, \tilde{v}, f_{u_1}, f_0 \in L^1(D_T)$. Since $u \leq \tilde{v}$ q.e. on $D_T$, it follows from (5.8) that

$$E'_{s,x} \int_0^{\zeta_T} dA^\nu_r \leq 2E'_{s,x} |\varphi(X_{T - \tau(0)})| + E'_{s,x} |\phi(X_{T - \tau(0)})|$$

$$+ E'_{s,x} \int_0^{\zeta_T} |f_u(X_r)| + |f_0(X_r)| \, dr + 2E'_{s,x} \int_0^{\zeta_T} d(|A^\mu_r + A^\lambda_r|)$$

for q.e. $(s,x) \in D_T$. Hence, by Lemma 2.5, $\nu \in M^+_0(D_T)$. From (5.8) and Proposition 4.4 it follows now that $u$ is a renormalized solution of the problem (4.1) with $\mu$ replaced by $\mu + \nu$, $\nu \in T_{2,1}^{0,1}$ and $u \in L^q(0,T;W^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1}]$. Actually, from the definition of a solution of reflected BSDE and (5.5), (5.7) it follows that the pair $(u, \nu)$ is a solution of OP$(\varphi, f + d\mu, h_1)$. Moreover, from (5.5) and the fact that for q.e. $(s,x) \in D_T$ the process $Y^{s,x}$ is of class (D) and $Y^{s,x} \in D^q$ for $q \in (0,1)$ we conclude that $u$ is of class (FD) and $u \in F^D$ for $q \in (0,1)$. Furthermore, since $(u(X), \psi(X))$ is a solution of BSDE$(s,x)(\varphi, D, f_u + d(\mu + \nu))$ it follows from Proposition 3.7 that for q.e. $(s,x) \in D_T$, $\sigma \nabla u(X) = \psi(X), \ dt \otimes P_{s,x}$-a.e. on $[0, \zeta_T] \times \Omega'$.  

(5.9)

Hence $\nabla u \in FM^q$ for $q \in (0,1)$, because $Z^{s,x} \in M^q$, $q \in (0,1)$, for q.e. $(s,x) \in D_T$.

(ii) From (5.3) and (5.5) it follows that $u_n \not\to u$ q.e. on $D_T$, whereas from (5.4), (5.6), (5.9) and Lemma 3.6 it follows that $\nabla u_n \to \nabla u$ in measure $m_1$. Since $u_n \leq u$ q.e. on $D_T$, we have

$$E'_{s,x} \int_0^{\zeta_T} dA^\nu_r \leq C(\kappa, T) \left( E'_{s,x} |\varphi(X_{T - \tau(0)})| \right)$$

$$+ E'_{s,x} \int_0^{\zeta_T} |f_u(X_r)| \, dr + E'_{s,x} \int_0^{\zeta_T} d(|A^\mu_r + A^\lambda_r|).$$
Hence \( \sup_{n \geq 1} \| \nu_n \|_{TV} < +\infty \) by Lemma 2.5, and consequently, by Remark 4.6 and Vitali's theorem, \( u_n \to u \) in \( L^q(0,T;W_0^{1,q}(D)) \) for every \( q \in [1, \frac{d+2}{d+1}) \). Suppose that for some subsequence, still denoted by \( n \), \( \{\nu_n\} \) converges weakly on \( DT \) to some measure \( \nu' \). By Remark 4.1, \( u_n \) is a distributional solution of (5.2) and \( u \) is a solution of (4.1) with \( \mu \) replaced by \( \mu + \nu \), i.e. for any \( \eta \in C_c^\infty((0,T) \times D) \),

\[
\int_{DT} (u_n \frac{\partial \eta}{\partial t} + a \nabla u_n \cdot \nabla \eta) \, dm_1 = \int_D \varphi \eta(T,\cdot) \, dm + \int_{DT} f_{u_n} \eta \, dm_1 + \int_{DT} \eta \, d(\mu + \nu_n)
\]

and (4.6) is satisfied with \( \mu \) replaced by \( \mu + \nu \). Since \( f_{u_n} \to f_{u} \), \( m_1 \)-a.e. and by (H3), \( f + \kappa(u_1 - v) \leq f_{u_n} \leq f_{u_1} + \kappa(u - u_1) \), applying the Lebesgue dominated convergence theorem we conclude that \( f_{u_n} \to f_{u} \) in \( L^1(D_T) \). Since we also know that \( u_n \to u \) and \( \nabla u_n \to \nabla u \) in \( L^1(D_T) \), it follows that \( \int_{D_T} \eta \, dv = \int_{D_T} \eta \, dv' \), and hence that \( \nu = \nu' \). Thus, \( \nu_n \to \nu \) weakly on \( D_T \), and the proof is complete.

\(\square\)

**Corollary 5.6.** Assume (H1)–(H4). Let \((u, \mu)\) be a solution of \( \text{OP}(\varphi, f + d\mu, h_1) \). Then

\[ u = \text{quasi-essinf}\{v \geq h_1, m_1 \text{-a.e. : } v \text{ is a supersolution of } \text{PDE}(\varphi, f + d\mu)\} \]

q.e. on \( D_T \).

**Proof.** Follows from Theorem 5.5, Proposition 3.7, Theorem 4.5 and [10, Lemma 4.9]. \(\square\)

**Corollary 5.7.** Let \((u, \nu)\) be a solution of \( \text{OP}(\varphi, f + d\mu, h_1) \) and let \( h^* \in \mathcal{FD} \) be such that \( h \leq h^* \leq u \), \( m_1 \)-a.e. Then for q.e. \( (s, x) \in D_T \),

\[
u(s, x) = \sup_{\sigma \in T'} E_{s,x} \left( \int_0^{\sigma \wedge \tau} f_u(X_r) \, dr + \int_0^{\sigma \wedge \tau} dA^\mu_r \right. \\
+ h^*(X_{\sigma}) 1_{\{\sigma < \tau, s \leq T - \tau(0)\}} + \varphi(X_{T - \tau(0)}) 1_{\{\sigma = T - \tau(0)\}} \right),
\]

where \( T' \) denotes the set of all \( \{\mathcal{G}_t\} \)-stopping times.

**Proof.** Follows from Theorem 5.5 and [10, Lemma 4.9]. \(\square\)

We now turn to the obstacle problem with two barriers.

**Theorem 5.8.** Assume (H1)–(H5), (H6').

(i) There exists a solution \((u, \nu)\) of \( \text{OP}(\varphi, f + d\mu, h_1, h_2) \) such that \( u \) is of class (FD), \( u \in \mathcal{FD}^q \) for \( q \in (0, 1) \), \( u \in T^{1,1}_{d+1}, L^q(0,T;W_0^{1,q}(D)) \) for \( q \in [1, \frac{d+2}{d+1}) \) and \( \nabla u \in FM^q \) for \( q \in (0, 1) \).

(ii) If \( h_1, h_2 \) are quasi-continuous and \( h_1(T, \cdot) \leq \varphi \leq h_2(T, \cdot) \), m.a.e. then \( u \) is quasi-continuous.

(iii) Let \( u_n \) be a renormalized solution of the problem

\[
\begin{cases}
\frac{\partial u_n}{\partial t} + A_t u_n = -f_{u_n} - \mu - n(u_n - h_1)^- + n(u_n - h_2)^- , \\
\ \ \ \ u_n(T, \cdot) = \varphi , \quad u_n(t, \cdot)_{|\partial D} = 0 , \quad t \in (0,T).
\end{cases}
\]

Then \( u_n \to u \) q.e. on \( D_T \) and \( u_n \to u \) in \( L^q(0,T;W_0^{1,q}(D)) \), \( q \in [1, \frac{d+2}{d+1}) \).
(iv) Let \( (\gamma_n, \beta_n) \) be a solution of \( \mathbf{OP}(\varphi, f_n + d\mu, h_1) \) with
\[
\tilde{f}_n(t, x, y) = f(t, x, y) - n(y - h_2(t, x))^+.
\]

Then \( w_n \searrow u \) q.e. on \( D_T \), \( u \) in \( L^q(0, T; W_0^{1,q}(D)) \), \( q \in [1, \frac{d+2}{d+1}] \), \( \beta_n \geq \beta_{n+1} \), \( \beta_n \searrow \nu^+ \) setwise, \( \gamma_n \to \nu^- \) weakly on \( D_T \), where \( \gamma_n = n(u_n - h_2)^+ \cdot m_1 \).

**Proof.** By Lemma 3.4, Lemma 5.2 and [10, Theorem 6.6], for q.e. \( (s, x) \in D_T \) there exists a solution \( (Y^{s,x}, Z^{s,x}, R^{s,x}) \) of RBSDE_{s,x}(\varphi, D, f + d\mu, h_1, h_2) such that \( Y^{s,x} \) is of class \( D \), \( (Y^{s,x}, Z^{s,x}) \in D^q \otimes M^q \) for \( q \in (0, 1) \) and \( R^{s,x} \in \mathcal{V}^1 \). Moreover, by [10, Theorem 6.6] again, for q.e. \( (s, x) \in D_T \),
\[
\begin{align*}
Y_t \in [0, \zeta_t], & \quad P_{s,x}^t - \text{a.s.,} \quad (5.10) \\
Z_t \to Z^{s,x}_t, & \quad dt \otimes P_{s,x}^t - \text{a.e. on } [0, \zeta_t] \times \Omega', \quad (5.11) \\
K_t \to R^{s,x}_t, & \quad t \in [0, \zeta_t] \times \Omega', \quad (5.12)
\end{align*}
\]
where \( (Y^{n,s,x}_t, Z^{n,s,x}_t, K^{n,s,x}_t) \) is a solution of RBSDE_{s,x}(\varphi, D, f_n + d\mu, h_1). By Theorem 5.5, \( u_n(X_t) = Y^{n,s,x}_t, t \in [0, \zeta_t] \), \( P_{s,x}^t - \text{a.s.} \), \( \sigma \nabla u_n(X) = Z^{n,s,x}_t, dt \otimes P_{s,x}^t - \text{a.e. on } [0, \zeta_t] \times \Omega' \) and \( K^{n,s,x}_t = A^\beta_n, t \in [0, \zeta_t] \), \( P_{s,x}^t - \text{a.s.} \). Put \( u(s, x) = \limsup_{n \to +\infty} u_n(s, x), (s, x) \in D_T \). Then by (5.10),
\[
\begin{align*}
u(s, x) = Y^{s,x}_t, & \quad t \in [0, \zeta_t] \times \Omega'. \quad (5.13)
\end{align*}
\]
for q.e. \( (s, x) \in D_T \). By Lemma 3.6 and (5.11) there exists a Borel measurable function \( \psi \) on \( D_T \) such that
\[
\begin{align*}
\psi(X) = Z^{s,x}_t, & \quad dt \otimes P_{s,x}^t - \text{a.e. on } [0, \zeta_t] \times \Omega'. \quad (5.14)
\end{align*}
\]
for q.e. \( (s, x) \in D_T \). Since the Revuz correspondence is one-to-one, (5.12) implies that \( \{\beta_n\} \) is a nonincreasing sequence and \( R^{s,x}_{t,\zeta_t} = A^\nu_1, t \in [0, \zeta_t] \), \( P_{s,x}^t - \text{a.s.} \), where \( \nu_1 \) is a setwise limit of \( \{\beta_n\} \). Put
\[
A_t = u(X_0) - u(X_t) - \int_0^t f_u(X_r) \, dr + A^\nu_t + \int_0^t \psi(X_r) \, dB_r, \quad t \geq 0.
\]
As in the proof of Theorem 5.5 one can show that there exists a nonnegative smooth measure \( \nu_2 \) such that \( A = A^{\nu_2} \). Let \( \nu = \nu^1 - \nu^2 \). From the construction of \( \nu^1, \nu^2 \) it follows that \( \nu^1 = \nu^+, \nu^2 = \nu^- \) and
\[
A'_t = R^{s,x}_t, \quad t \in [0, \zeta_t] \times \Omega'. \quad (5.15)
\]
for q.e. \( (s, x) \in D_T \). By (5.13) and (5.15),
\[
\begin{align*}
\begin{align*}
\nu(s, x) = E'_{s,x}(1_{\{\zeta_T < \tau(0)\}} \varphi(X_{T-\tau(0)}) + \int_0^{\zeta_T} f_u(X_r) \, dr + \int_0^{\zeta_T} d(A^\nu_t + A^\nu_t)) \quad (5.16)
\end{align*}
\end{align*}
\]
for q.e. \( (s, x) \in D_T \). By (6.8) in [10] (see the beginning of the proof of [10, Theorem 6.6]) and Theorems 4.5 and 5.5 and Lemma 2.5,
\[
\|\beta_n\|_{TV} \leq \|\beta_1\|_{TV}, \quad \|\gamma_n\|_{TV} \leq \|\delta_2^0\|_{TV}, \quad (5.17)
\]
where $\delta_n = n(v_n^2 - h_2^2) \cdot m_1$, $v_n^2$ is a renormalized solution of PDE($\phi \vee \varphi, f_n + d\lambda_2 + d\mu$) with $\lambda_2 = f_n \cdot m_1 + \lambda^+ + \mu^-$ ($\lambda$ is the measure from condition (H6')). From the first inequality in (5.17) it follows that $\|v\|_T V < +\infty$. Furthermore, from Theorem 5.5 we know that sup$_{n \geq 1} \|\delta_n\|_T V < +\infty$. Therefore $\{\gamma_n\}$ is tight, which in fact implies that $\gamma_n \to \nu^-$ weakly on $D_T$ (see the reasoning at the end of the proof of Theorem 5.5). Hence $\|\nu\|_T V < +\infty$, and consequently $\nu \in M_{0,b}$. By (6.32) in [10] and Theorem 5.5, $u_n \geq \bar{v}$, where $\bar{v}$ is the first component of the solution of OP($\phi, f + d\mu, h_1, h_2$). Using (5.13)–(5.16) and Proposition 4.4 one can deduce now in much the same way as in the proof of Theorem 5.5 that $(u, \nu)$ is a solution of OP($\varphi, f + d\mu, h_1, h_2$), $u$ has the regularity properties stated in (i) and

$$\sigma \nabla u(X) = \psi(X), \quad dt \otimes P_{s,x}' - \text{a.e. on } [0, \zeta_T) \times \Omega'$$

for q.e. $(s, x) \in D_T$. From what has already been proved, (5.11), (5.14), (5.18) and Lemma 3.6 we get (iv). Finally, assertions (ii) and (iii) follow from [10, Theorem 6.6(ii),(v)], Lemma 3.6, Remark 4.6 and the stochastic representation (5.13)–(5.15), (5.18) of the solution $(u, \nu)$. 

Corollary 5.9. Assume that barriers $h_1, h_2$ are quasi-continuous and $h_1(T, \cdot) \leq \varphi \leq h_2(T, \cdot)$, m.a.e., then the minimality condition (c) in the definition of a solution of the obstacle problem is equivalent to (1.5).

Proof. Since $u, h_1, h_2$ are quasi-continuous, from condition (c) it follows that

$$E'_{s,x} \int_0^{\zeta_T} (u - h_1)(X_r) dA_r^{\nu^+} = E'_{s,x} \int_0^{\zeta_T} (h_2 - u)(X_r) dA_r^{\nu^-} = 0.$$  

Hence

$$\int_s^T \int_D (u - h_1)(t, y)p_D(s, x, t, y) d\nu^+(t, y)$$

$$= \int_s^T \int_D (h_2 - u)(t, y)p_D(s, x, t, y) d\nu^-(t, y) = 0,$$

which implies (1.5), because $p_D(s, x, \cdot, \cdot)$ is positive on $(s, T) \times D$. Conversely, assume that (1.5) is satisfied. Then (5.20), and consequently (5.19) is satisfied. Of course (5.19) implies (c).

Let us note that in general, even in the case of one quasi-l.s.c. or quasi-u.s.c. reflecting barrier, the integrals in (1.5) may be strictly positive (see [9, Example 5.5]).

Corollary 5.10. Assume (H1)–(H4). Let $(u, \nu)$ be a solution of OP($\varphi, f + d\mu, h_1, h_2$). Then

$$u = \text{quasi-essinf} \{v \geq h_1, m_1\text{-a.e. : } v \text{ is a supersolution of } \text{PDE}($$}

$$\varphi, f + d\mu - d\nu^-)\}$$

q.e. on $D_T$.

Proof. Follows from Theorem 5.8 and [10, Lemma 4.9].
Corollary 5.11. Let \((u, \nu)\) be a solution of \(\text{OP}(\varphi, f + d\mu, h_1, h_2)\) and let \(h_1^*, h_2^* \in F\) be such that \(h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2, m_1\)-a.e. Then

\[
 u(s, x) = \text{ess sup} \text{ess inf}_{\sigma \in T'} E^\mu_{s,x} \left( \int_0^{\sigma \wedge \delta \wedge \zeta} f_u(X_r) \, dr + \int_0^{\sigma \wedge \delta \wedge \zeta} dA^\mu_r 
+ h_1^*(X_\delta) 1_{\{\delta < \sigma < T - \tau(0)\}} 1_{\{\delta < \zeta\}} 
+ h_2^*(X_\sigma) 1_{\{\sigma < \delta\}} 1_{\{\sigma < \zeta\}} 
+ \varphi(X_{T - \tau(0)}) 1_{\{\sigma = \delta = T - \tau(0)\}} \right) 
q.e. \text{ on } D_T, \text{ where } T' \text{ is the set of all } \{G_t\}-\text{stopping times.}
\]

Proof. Follows from Theorem 5.8 and [16].

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