Gödel -type space-time metrics

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Abstract

A simple group theoretic derivation is given of the family of space-time metrics with isometry group $SO(2, 1) \times SO(2) \times \mathbb{R}$ first described by Gödel, of which the Gödel stationary cosmological solution is the member with a perfect-fluid stress-energy tensor. Other members of the family are shown to be interpretable as cosmological solutions with a electrically charged perfect fluid and a magnetic field.

1 Introduction

The Gödel metric \cite{2} was the first known cosmological solution of the Einstein field equations with rotating matter and closed timelike curves. By ‘Gödel -type’ metrics we mean a one-parameter family of space-time metrics of the form

$$ds^2 = ds^2 - dz^2$$

(1)

where $ds^2$ is the metric of a signature $(+--)$ space $\mathcal{G}S$ with isometry group $SO(2, 1) \times SO(2)$. Since $ds^2$ is independent of $z$ the isometry group
of $ds^2$ is then $SO(2,1) \times SO(2) \times \mathbb{R}$ where the factor $\mathbb{R}$ is the one-dimensional translation group $z \rightarrow z + \text{constant}$. (The simply transitive subgroup $SO(2,1) \times \mathbb{R}$ of $SO(2,1) \times SO(2) \times \mathbb{R}$ shows that G"odel-type solutions are four-dimensionally uniform, i.e., spatially uniform and stationary.)

G"odel [1] (footnote 12) derived $\mathcal{G}S$ by starting with three-dimensional anti-deSitter (adS) space, a space of constant positive curvature and signature (+ --), realizable as a pseudosphere in a flat $2+2$ space, and ‘stretched’ it by a factor $\mu$ in the direction of a system of time-like Clifford parallels (= Hopf lines). [ $\mu$ is the ‘one parameter’ of the family of G"odel-type metrics; throughout this paper we ignore any trivial overall (scale) factor of $ds^2$.] The ‘stretch’ reduces the the adS-space isometry $SO(2,1) \times SO(2,1)$ to $SO(2,1) \times SO(2)$. In section 2 we exploit the fact that the three-dimensional adS space is a group space, namely of $SO(2,1)$, to give an alternative derivation of $ds^2$ starting with a group-space metric.

The G"odel cosmological solution is the one member (namely with stretch $\mu = \sqrt{2}$) of the G"odel-type metrics, Eq.(1), for which the stress tensor $T_{\mu\nu}$ is that of a perfect fluid. In section 3 we show that other amounts of stretch yield cosmological solutions which contain also a uniform magnetic field and a uniform electrical charge density.

### 2 Construction of the metric tensor

Given a matrix representation $M$ of a continuous group $G$, a metric $dl^2$ which is left and right invariant under the action of $G$ is given by

$$dl^2 = \text{tr} (dM M^{-1} dM M^{-1}).$$

[ Of course this is the Cartan-Killing metric (up to a constant factor), the unique invariant quadratic form of a Lie-group space. ] Insertion of a matrix $H$ between $dM$ and $M^{-1}$ gives a modified metric

$$dL^2 = \text{tr} (dM H M^{-1} dM H M^{-1})$$

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which is still left invariant, but generally no longer right invariant. In fact under right multiplication $M \rightarrow M g$, $g \in G$, $dL^2$ transforms to

$$\text{tr} \left( dM g H g^{-1} M^{-1} dM g H g^{-1} M^{-1} \right),$$

so the only remaining right invariance of the modified metric is under the subgroup of $G$ which commutes with $H$.

If we take $G$ to be $SO(2,1)$ (this is Gödel’s ‘hyperbolic quaternion’ group), then $dl^2$, Eq. (2), is the metric of a $2+1$ adS space. To obtain a metric with the isometry group $SO(2,1) \times SO(2)$ it suffices to choose an $H$ which commutes only with an $SO(2)$ subgroup of $G$.

Because $SO(2,1)$ is isomorphic to $SL(2, \mathbb{R})$, we can take

$$M = e^{\varepsilon \phi/2} e^{\sigma_3 r/2} e^{\varepsilon \psi/2}, \quad \varepsilon \equiv i \sigma_2$$

(4)

where $\sigma_j, j = 1, 2, 3$ are the Pauli matrices and $\phi, \psi, r$ are real parameters.

The general form of a matrix $H$ which commutes with the $SO(2)$ subgroup generated by $\varepsilon$ is

$$H = a + b \varepsilon = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad (5)$$

with $a, b$ complex numbers. Writing for short

$$\gamma \equiv \cosh r, \quad \sigma \equiv \sinh r, \quad c \equiv \cos \phi, \quad s \equiv \sin \phi \quad (6)$$

(note $\gamma^2 - \sigma^2 = 1$, $c^2 + s^2 = 1$)

we have

$$e^{\sigma_3 r/2} \varepsilon e^{-\sigma_3 r/2} = \gamma \varepsilon + \sigma \sigma_1$$
$$e^{\varepsilon \phi/2} \sigma_3 e^{-\varepsilon \phi/2} = c \sigma_3 - s \sigma_1$$
$$e^{\varepsilon \phi/2} \sigma_1 e^{-\varepsilon \phi/2} = c \sigma_1 + s \sigma_3$$

(7)

and hence an easy calculation gives

$$2 \frac{dM}{dr} HM^{-1} = a (c \sigma_3 - s \sigma_1) + b [\gamma (c \sigma_1 + s \sigma_3) + \sigma \varepsilon]$$
$$2 \frac{dM}{d\phi} HM^{-1} = a \varepsilon - b [\gamma - \sigma , (c \sigma_3 - s \sigma_1)]$$
$$2 \frac{dM}{d\psi} HM^{-1} = a [\gamma \varepsilon + \sigma (c \sigma_1 + s \sigma_3)] - b.$$
Substituting into Eq. (3) and using \( \text{tr}(\sigma_i \sigma_j) = 2 \delta_{ij} \) we obtain for \(-4 \, dL^2\)

\[
(a^2 - b^2) \, d\psi^2 + 2(a^2 - b^2) \gamma \, d\psi \, d\phi + (a^2 + b^2 - 2b^2 \gamma^2) \, d\phi^2 - (a^2 + b^2) \, dr^2
\]

\[
= (a^2 - b^2) \,(d\psi + \gamma \, d\phi)^2 - (a^2 + b^2) \,(\gamma^2 - 1) \, d\phi^2 - (a^2 + b^2) \, dr^2
\]

\[
\simeq \mu^2 \,(d\psi + \gamma \, d\phi)^2 - \sigma^2 \, d\phi^2 - dr^2 \quad \text{where} \quad \mu^2 \equiv \frac{a^2 - b^2}{a^2 + b^2}
\]

\[
= \mu^2 \,[d\psi + d\phi + (\gamma - 1) \, d\phi]^2 - \sigma^2 \, d\phi^2 - dr^2
\]

\[
= [dt + \mu \,(\gamma - 1) \, d\phi]^2 - \sigma^2 \, d\phi^2 - dr^2 \quad \text{where} \quad dt \equiv \mu (d\psi + d\phi). \quad (9)
\]

This metric and its curvature are nonsingular at \( r = 0 \) if \( r \) and \( \phi \) are regarded as polar coordinates, i.e., \( \phi \) and \( \phi + 2\pi \) specify the same point and \( r \) is nonnegative. It is the metric of the space \( G\mathcal{S} \) which Gödel [1] (footnote 12) obtained by stretching by the factor \( \mu \) a three-dimensional anti-deSitter space in the direction of a system of time-like Clifford parallels. [For \( \mu = 1 \), hence \( b = 0 \) in Eq.(3), Eq.(9) is the metric of the adS space; upon multiplying the first term by \( \mu^2 \) and then redefining \( dt \rightarrow dt/\mu \) the metric Eq.(9) for general \( \mu \) is obtained.] The metric Eq.(9) was also obtained in [2] (in their Eq.(38) replace \( \tau \) by \( t/2 \) and \( r \) by \( r/2 \)) by what they call a ‘squash’, rather than a stretch. And finally, it is a special case of the metric (III.14) of [3], for \( e = -1 \), \( F \) and \( G \) = constant.

Using the metric Eq.(9) for the \( ds^2 \) of Eq.(11),

\[
ds^2 = [dt + \mu \,(\gamma - 1) \, d\phi]^2 - \sigma^2 \, d\phi^2 - dr^2 - dz^2
\]

is a one-parameter family of metrics with the isometry \( SO(2,1) \times SO(2) \times \mathbb{R} \).

The corresponding metric tensor is

\[
g_{\mu\nu} = \begin{pmatrix}
1 & \mu(\gamma - 1) & 0 & 0 \\
\mu(\gamma - 1) & \mu^2(\gamma - 1)^2 - (\gamma^2 - 1) & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\quad | 0 \quad t \\
1 \quad \phi \\
2 \quad r \\
3 \quad z
\]

(11)

The Ricci tensor of this metric is found to be

\[
R_{\mu\nu} = \frac{1}{2} \begin{pmatrix}
\mu^2 & \mu^3(\gamma - 1) & 0 & 0 \\
\mu^3(\gamma - 1) & (\mu^2 - 2)(\gamma^2 - 1) + \mu^4(\gamma - 1)^2 & 0 & 0 \\
0 & 0 & \mu^2 - 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(12)
This can be written as
\[ R_{mn} = (\mu^2 - 1)u_m u_n - \frac{1}{2}(\mu^2 - 2)g_{mn} \quad ; \quad R_{3\nu} = 0 \quad (13) \]
where \( R_{mn} \) is \( R_{\mu\nu} \) with \( \mu, \nu \) restricted to the range 0, 1, 2 (i.e., the Roman indices \( m, n \) take on the values \( 0, 1, 2 \)) and \( u^\mu \) is the unit tangent of fixed-\((\phi, r, z)\) lines,
\[ u^\mu = (1 \ 0 \ 0 \ 0) \quad , \quad u_\mu = (1 \ \mu(\gamma - 1) \ 0 \ 0) \quad (14) \]
Note that the curl of \( u_\mu \) has the nonvanishing component
\[ u_{[1,2]} = u_{1,2} = \mu(\gamma - 1), r = \mu \sigma, \quad \approx \mu r \quad \text{for small} \ r, \quad (15) \]
which means that points at fixed \((\phi, r, z)\) are rotating at the angular velocity
\[ \omega = -\frac{1}{2}\mu, \quad \text{where positive} \ \omega \quad \text{means rotation in the} \ +\phi \ \text{direction.} \]

It also may be mentioned that when \( \mu^2 > 1 \), the metric allows closed time-like curves, for example \( \phi\)-curves (fixed \( t, r, z \)), with \( r > \ln \sqrt{\frac{\mu + 1}{\mu - 1}} \).

From Eq. (13) it follows that the Ricci scalar is
\[ R = R^\mu_\nu = (\mu^2 - 1) - \frac{3}{2}(\mu^2 - 2) = -\frac{1}{2}(\mu^2 - 4) \quad (16) \]
and the Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \) is
\[ G_{mn} = (\mu^2 - 1)u_m u_n - \frac{1}{4}\mu^2 g_{mn} \quad ; \quad G_{3n} = 0 \quad ; \quad G_{33} = \frac{1}{4}(\mu^2 - 4)g_{33} \quad (17) \]
If \( \mu^2 = 2 \), \( R_{\mu\nu} \) and \( G_{\mu\nu} \) are of the form \( C_1 u_\mu u_\nu + C_2 g_{\mu\nu} \) and so (as observed by Gödel) the metric is a solution of the Einstein equation \( G_{\mu\nu} = T_{\mu\nu} \) where \( T_{\mu\nu} \) is the stress tensor \( T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \) of a perfect fluid where its four-velocity \( u^\mu \) is given in Eq. (14) and its (proper) energy density \( \rho \) and pressure \( p \) are
\[ \rho = p = \frac{1}{2} \quad (18) \]
If the perfect fluid is vacuum (‘cosmological constant’) plus matter then
\[ \rho = \rho_{\text{vac}} + \rho_{\text{matter}} \quad \text{and} \quad p = -\rho_{\text{vac}} + p_{\text{matter}} \quad \text{so} \]
\[ \rho_{\text{matter}} = \frac{1}{1 + f} \quad ; \quad \rho_{\text{vac}} = -\frac{1}{2} \frac{1 - f}{1 + f} \quad \text{where} \quad f = \frac{p_{\text{matter}}}{\rho_{\text{matter}}} \quad (19) \]
If the matter is dust, \( f = 0 \), then \( \rho_{\text{matter}} = 1 \); \( \rho_{\text{vac}} = -\frac{1}{2} \).
3 Solutions with $\mu \neq 0$

We now show that when $\mu^2 > 2$, the $T^{\mu\nu}$ implied by the metric Eq. (11) and its Einstein tensor Eq. (17) can be realized by adding to a perfect fluid a uniform magnetic field and a uniform electric charge density.

As will be evident later,

$$F_{\mu\nu} = \left( \delta_\mu^1 \delta_\nu^2 - \delta_\mu^2 \delta_\nu^1 \right) \sigma B \quad ; \quad B = \text{constant} \quad (20)$$

describes a uniform magnetic field in the 3 (‘z’) direction. The resulting electromagnetic $T^{\mu\nu}$ is

$$T^{\mu\nu}_{\text{EM}} = F_{\mu\rho} g^{\rho\sigma} F_{\sigma\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\alpha} g^{\alpha\beta} F_{3\alpha} g^{\beta\sigma},$$

$$= \left( \sigma^2 \delta_\mu^1 \delta_\nu^1 + \delta_\mu^2 \delta_\nu^2 + \frac{1}{2} g_{\mu\nu} \right) B^2. \quad (21)$$

By using

$$g_{mn} = u_m u_n - \left( \sigma^2 \delta_m^1 \delta_n^1 + \delta_m^2 \delta_n^2 \right) \quad ; \quad g_{3n} = 0, \quad (22)$$

which can be seen from Eq. (11) and Eq. (14), we can write $T^{\mu\nu}_{\text{EM}}$ as

$$T^{\mu\nu}_{\text{EM}} = \left( u_m u_n - \frac{1}{2} g_{mn} \right) B^2 \quad ; \quad T^{\mu\nu}_{3n} = 0 \quad ; \quad T^{\mu\nu}_{33} = \frac{1}{2} g_{33} B^2. \quad (23)$$

The total $T^{\mu\nu}$ of a perfect fluid plus the magnetic field is then

$$T_{mn} = \left( \rho + p + B^2 \right) u_m u_n - \left( p + \frac{1}{2} B^2 \right) g_{mn}$$

$$T_{3n} = 0 \quad ; \quad T_{33} = -\left( p - \frac{1}{2} B^2 \right) g_{33} \quad (24)$$

and Einstein’s equation for $G_{\mu\nu}$, Eq. (17), is satisfied by

$$\rho = \frac{1}{2}(\mu^2 - 1) \quad ; \quad p = \frac{1}{2} \quad ; \quad B^2 = \frac{1}{2}(\mu^2 - 2). \quad (25)$$

Note that $B$ is real only if $\mu^2 \geq 2$. Also note that if $\mu^2 \geq 4$ then $\rho \geq 3p$ and a vacuum contribution to $T^{\mu\nu}$ is not needed.

We now consider Maxwell’s equations, which in terms of ordinary gradients are

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho,\mu,\nu} = 0 \quad ; \quad \mathcal{F}^{\mu\nu} = -\mathcal{J}^\mu \quad (26)$$
where \( \mathcal{F}^{\mu \nu} = \sqrt{-g} F^{\mu \nu} \) and \( \mathcal{J}^{\mu} = \sqrt{-g} J^{\mu} \), where \( g \equiv \det g_{\alpha \beta} = -\sigma^2 \) for the metric Eq.(11). The first (‘curl’) equation, which states the vanishing of magnetic four-current, is obviously satisfied by the field Eq.(20), since the only nonvanishing gradient is \( \partial_2 (= \partial_r) \) and all nonvanishing \( F_{\mu \nu} \) have an index ‘2’. As for the ‘div’ equation, since
\[
\mathcal{F}^{21} = \sqrt{-g} g^{22} g^{11} F_{21} = -B \\
\mathcal{F}^{20} = \sqrt{-g} g^{22} g^{01} F_{21} = \mu(\gamma - 1)B ,
\]
the only nonvanishing component of \( \mathcal{J}^{\mu} \) is
\[
\mathcal{J}^0 = \mathcal{F}^{20,2} = \mu \sigma B , \quad \text{i.e.,} \quad J^0 = \mu B ;
\]
a constant electric charge density. The physical reason for this charge density is that although in the rest frame of the matter the field is pure magnetic, in an inertial (nonrotating) frame with spatial origin fixed at, say, \( r = 0 \), there is an electric field in the radial (\( dr \)) direction, with magnitude \( \sim r \) at small \( r \), hence with \( \nabla \cdot \mathbf{E} \) constant. [It might be remarked that in a Friedman (homogeneous, isotropic) cosmological solution a nonvanishing charge density is impossible because it would imply an electric field with nonuniform magnitude and hence a nonuniform (inhomogeneous) electromagnetic stress tensor.]

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References

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