An axiomatic theory for anonymized risk sharing

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Abstract

We study an axiomatic framework for anonymized risk sharing. In contrast to traditional risk sharing settings, our framework requires no information on preferences, identities, private operations and realized losses from the individual agents, and thereby it is useful for modeling risk sharing in decentralized systems. Four axioms natural in such a framework – actuarial fairness, risk fairness, risk anonymity, and operational anonymity – are put forward and discussed. We establish the remarkable fact that the four axioms characterizes the conditional mean risk sharing rule, revealing the unique and prominent role of this popular risk sharing rule among all others in relevant applications of anonymized risk sharing. Several other properties and their relations to the four axioms are studied, as well as their implications in rationalizing the design of Bitcoin mining pools.

Keywords: Conditional expectation, anonymity, fairness, P2P insurance, Bitcoin mining pools

1 Introduction

Risk sharing, as one of the most popular risk management mechanisms, refers to pooling risks from several participants in a group and reallocating the total risk based in a specific way. A risk sharing scheme arises in different forms, such as insurance, tontines, taxation, founders stock, investment profit sharing, and Bitcoin mining pools, to name a few. In these contexts, either wealth or losses, or both of them, may be shared among participants.

The participants of a risk sharing scheme, such as individual investors, co-workers, financial institutions, policyholders and an insurer, peer-to-peer (P2P) insureds, and miners in a Bitcoin

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mining pool, are generally referred to as agents. Each agent has an initial risk contribution, and it will be exchanged to a new position after risk sharing, which we call an allocation to the agent.

A sensible, or even optimal in some sense, risk sharing arrangement can be obtained in several ways. Two common approaches studied in the literature are either through a centralized planner or through a trading mechanism such as an exchange market to arrive at some forms of equilibria. These equilibria are often Pareto or competitive equilibria; see e.g., Starr (2011) for a general treatment. In either form of equilibria, information on the preferences of the agents is required to define and compute an equilibrium. Commonly used preferences include expected utility, rank-dependent utility, cumulative prospect theory, risk measures, and many more advanced models; see Wakker (2010) for decision models and Föllmer and Schied (2016) for risk measures. Equilibrium risk sharing is studied in the classic work of Arrow and Debreu (1954) and Borch (1962) among a very rich literature.¹ In practical situations, however, one rarely has precise information on the preferences, since elicitation of preferences can be challenging and costly (e.g., Leonard (1983)), and preferences may be incomplete, ambiguous, or falsely supplied (e.g., Delage and Li (2018)).

In this paper, we consider a framework of anonymized risk sharing, where no information on preferences is required or used. The key feature of this framework is that agents do not need to disclose their preferences, identity, or wealth level.² More precisely, the allocation to an agent is determined by the initial risk contributions of all agents, but not the specification of these agents. For this reason, anonymized risk sharing schemes are desirable in several application such as P2P insurance (e.g., Denuit (2019), Abdikerimova and Feng (2022) and ), Bitcoin mining pools (e.g., Eyal and Sirer (2018) and Leshno and Strack (2020)), and tontines (e.g., Chen et al. (2019) and Hieber and Lucas (2022)). Several examples of risk sharing rules within our framework are presented in Section 2.

To better understand a suitable anonymized risk sharing rule, we put forward four natural axioms, namely, actuarial fairness, risk fairness, risk anonymity, and operational anonymity. The interpretation and desirability of these axioms will be discussed in detail in Section 3. Quite remarkably, we show in Section 4 that these four axioms uniquely identify one risk sharing rule (Theorem 1), the conditional mean risk sharing (CMRS). As far as we know, this paper provides the first axiomatic result for any risk sharing rules.

¹See the later work on risk sharing by Barrieu and El Karoui (2005) for convex risk measures, Carlier et al. (2012) for multivariate stochastic dominance, Xia and Zhou (2016) for rank-dependent utilities, Cai et al. (2017) for reinsurance arrangements, and Embrechts et al. (2018) for quantile-based risk measures.

²We chose the term “anonymized risk sharing” over “anonymous risk sharing”, as the former emphasizes that individual information is deliberately masked (but it could be available), and the latter stresses that such information is not known or supplied.
As an important risk sharing rule in economic theory with many attractive properties, CMRS was used by Landsberger and Meilijson (1994) to study Pareto optimality of comonotonic risk allocations, and its properties were studied in detail by Denuit and Dhaene (2012); see Denuit et al. (2022a) for a summary of these properties. Our characterization hence provides a first axiomatic foundation for CMRS and its applications in economic theory and decentralized finance and insurance.  

On the technical side, the proof of Theorem 1 relies on a new characterization of the conditional expectation which we present in Theorem 2. We further show that the four axioms are independent (Proposition 2). Several other properties related to our axioms are studied in Section 5, including backtracking, universal improvement, comonotonicity, and symmetry. In particular, we show that CMRS is the unique risk sharing rule satisfying universal improvement, risk anonymity, and operational anonymity, complementing the characterization in Theorem 1. In Section 7, we discuss some applications in cryptocurrency mining and revenue sharing, highlighting the suitability of the four axioms and their implications on the unique choice of the reward sharing mechanism. Proofs of all results are relegated to the appendices.

Research on axiomatic approaches for decision models and risk measures has a long history. For a specimen, see the monographs by Gilboa (2009), Wakker (2010) and the extensive lists of references therein. Axiomatic studies on risk functionals have been prolific in decision theory (e.g., Yaari (1987), Schmeidler (1989), Maccheroni et al. (2006) and Gilboa et al. (2010)) and risk measures (e.g., Artzner et al. (1999), Föllmer and Schied (2002) and Wang and Zitikis (2021)). Gilboa et al. (2019) had a recent discussion on the usefulness of axiomatic approaches in modern economic theory. Despite the huge success of axiomatic theories for risk functionals, an axiomatic study for risk sharing rules is missing from the literature; our work fills in this gap. Our new framework imposes substantial technical challenges compared to the above literature, as the risk sharing rules are multi-dimensional and random-vector-valued, as opposed to preference functionals or risk measures, which are typically real- or vector-valued.

2 Risk sharing rules: Definition and examples

We describe in this section the main object of the paper, the risk sharing rules. For this, we first need to fix some notation. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\mathcal{X}\) be a set of random variables on this space, representing the set of possible random losses of interest. We assume that

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3Examples of decentralized insurance include P2P insurance, mutual aid, and catastrophic risk pooling; see Feng et al. (2022) for a summary of models for decentralized insurance.
\(\mathcal{X}\) is closed under addition and \(0 \in \mathcal{X}\). Positive values of random variables represent losses and negative values represent gains; flipping this convention makes no difference in all mathematical results. We always treat almost surely (a.s.) equal random variables as identical, and we use \(\sup X\) for the essential supremum of \(X\), that is, \(\sup X = \inf \{ x \in \mathbb{R} : P(X > x) = 0 \}\).

In our framework, \(n\) economic agents share a total risk, where \(n \geq 3\) is an integer, and we write \([n] = \{1, \ldots, n\}\). Each agent \(i \in [n]\) faces an initial risk \(X_i\), which is the risk contribution of agent \(i\) to the risk sharing pool. We use the term “risk” to reflect that the random variable \(X_i\) may be positive or negative, and sometimes we use the term “loss” to emphasize its positive side.

For any random variable \(S\), the set of all allocations of \(S\) is denoted by

\[A_n(S) = \left\{ (Y_1, \ldots, Y_n) \in \mathcal{X}^n : \sum_{i=1}^{n} Y_i = S \right\}.

Throughout, we write \(X = (X_1, \ldots, X_n)\) for the initial risk (contribution) vector, and \(S^X = \sum_{i=1}^{n} X_i\) for the total risk.

A risk sharing rule is a mapping \(A : \mathcal{X}^n \to \mathcal{X}^n\) satisfying \(A^X = (A_1^X, \ldots, A_n^X) \in A_n(S^X)\) for each \(X \in \mathcal{X}^n\). The requirement \(A^X \in A_n(S^X)\) means that \(A^X\) sums up to the total risk. In other words, there is no external fund coming in or out of the risk sharing pool except for the initial contributions of the agents, a most natural requirement for defining an allocation rule. Each component of \(A^X\) represents the (random) allocation of risk to an agent. Through the rule \(A\), the initial risk vector \(X\) enters the sharing pool as an input, and the allocation vector \(A^X\) comes out as the output. Given each scenario \(\omega \in \Omega\), the actual payment is settled as the vector \(A^X(\omega) \in \mathbb{R}^n\). A positive payment \(A_i^X(\omega) = x > 0\) means that agent \(i\) needs to pay the amount of \(x\), because positive values represent losses. This simple procedure is illustrated in Figure 1.

![Figure 1: Risk sharing](image)

As a key feature of this framework, different from the large body of risk sharing problems studied in the literature, a risk sharing rule \(A\) does not require any information on the preferences of the agents, a risk exchange market, or subjective decisions of a central planner. The risk allocation will be determined completely through the mechanism design and the input risk vector.

\(\textsuperscript{4}\)We assume \(n \geq 3\) since the case \(n = 2\) is technically different; see Example 3.
We next provide several simple examples of risk sharing rules; see Denuit et al. (2022a) for a collection of risk sharing rules and their properties. Throughout, for $q \in [0, \infty)$, denote by $L^q = L^q(\Omega, \mathcal{F}, \mathbb{P})$ the set of all random variables with a finite $q$-th moment, and $L^q_+$ be the set of non-negative elements of $L^q$. We use the shorthand $L^q$, and we will write the full $L^q(\Omega, \mathcal{G}, Q)$ when we encounter another probability space $(\Omega, \mathcal{G}, Q)$. Some of the risk sharing rules below require $\mathcal{X}$ to be a subset of some specific spaces. We always use the convention $0/0 = 0$ which may appear in degenerate cases of (vi) and (vii).

(i) The identity risk sharing rule

$$A_{\text{id}}^X = X \quad \text{for } X \in \mathcal{X}^n.$$ 

(ii) The all-in-one risk sharing rule

$$A_{\text{all}}^X = (S^X, 0, \ldots, 0) \quad \text{for } X \in \mathcal{X}^n.$$ 

(iii) The mean-adjusted all-in-one risk sharing rule

$$A_{\text{ma}}^X = (S^X - \mathbb{E}[S^X], 0, \ldots, 0) + \mathbb{E}[X] \quad \text{for } X \in \mathcal{X}^n \subseteq (L^1)^n.$$ 

(iv) The uniform risk sharing rule

$$A_{\text{unif}}^X = S^X \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \quad \text{for } X \in \mathcal{X}^n.$$ 

(v) The conditional mean risk sharing rule (CMRS)

$$A_{\text{cm}}^X = \mathbb{E} [X|S^X] \quad \text{for } X \in \mathcal{X}^n \subseteq (L^1)^n.$$ 

(vi) The mean proportional risk sharing rule

$$A_{\text{prop}}^X = \frac{S^X}{\mathbb{E}[S^X]} \mathbb{E}[X] \quad \text{for } X \in \mathcal{X}^n \subseteq (L^1_+)^n.$$ 

(vii) The covariance risk sharing rule

$$A_{\text{cov}}^X = \frac{S^X - \mathbb{E}[S^X]}{\text{Var}(S^X)} \text{Cov}(X, S^X) + \mathbb{E}[X] \quad \text{for } X \in \mathcal{X}^n \subseteq (L^2)^n.$$ 

These examples will be revisited repeatedly in the paper. Among them, CMRS in (v) is the
most important for our theory of anonymized risk sharing.

3 Four axioms for anonymized risk sharing

We next discuss desirable criteria for risk sharing rules by addressing the considerations of both fairness and anonymity. In a few senses to be made precise below, fairness refers to the feature that each agent does not receive an absurd or unjustified allocation, and anonymity refers to the feature that agents do not need to disclose information on their identity, wealth, preferences, private operations, and final realized losses. Given a risk sharing rule $A$, the only information required to determine the risk allocation is the initial risk vector $X$. Anonymity also guarantees that each agent will not be treated differently and reduces discrimination. As such, anonymity is closely related to fairness, although the two concepts have different motivations. To reflect these key features, we propose four natural axioms on a candidate risk sharing rule $A$. Two of these axioms may be categorized as fairness axioms, and two may be categorized as anonymity axioms.

**Axiom AF (Actuarial fairness).** The expected value of each agent’s allocation coincides with the expected value of the initial risk. That is, $E[A^X] = E[X]$ for $X \in \mathcal{X}^n$.

Axiom AF is one of the most ancient and formidable idea in risk management, which dates back to at least the 16th century; see Heras et al. (2020) for a history. AF serves as the basis for premium pricing in insurance, and this served as one of the earliest sources for studying probability and statistics. Certainly, not all risk exchanges in practice are actuarially fair. In our framework, because of no information on the preferences or identities of the agents, it should not happen that one agent would receive an allocation with a higher expected value than her contribution, and some others receive allocations with lower expected values. Recall that the sum of these expected values is equal to the sum of the total risk, and hence agents on average receive the same expected value before and after risk sharing. Based on the above reasons, AF is a most natural requirement for anonymized risk sharing, and here we observe a joint effect of fairness and anonymity. The recent book Friedman (2020) has a comprehensive non-technical treatment on the historical importance of actuarial fairness and probability theory in insurance and social welfare.

Axiom AF can be alternatively formulated via incentive compatibility of certain agents in the risk sharing pool. Recall that a risk-neutral agent would not join the risk sharing pool if the expected value of their loss increases after the risk exchange. If AF fails, then some agents will have

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5 As we know, another important early source, roughly around the same time, is gambling, which motivated some work of Blaise Pascal, Pierre de Fermat, and Jacob Bernoulli.
a higher expected loss. Since preferences are not revealed or used, any agents could potentially be risk-neutral, and it would be suboptimal for them to stay in the pool. A risk sharing rule should not exclude by design risk-neutral agents;\(^6\) recall that a fundamental model of insurance (Arrow (1963)) involves a risk-neutral insurer to help share losses from risk-averse insureds.

**Axiom RF (Risk fairness).** The allocation to each agent should not exceed their maximum possible loss. That is, for \(X \in \mathcal{X}^n\) and \(i \in [n]\), it holds that \(A^X_i \leq \sup X_i\).

Axiom RF reflects the idea that agents join the pool to share their risk, and they should not have to suffer more than their worst-case loss.\(^7\) For each realization of the actual losses, the allocation satisfies the no rip-off condition in the insurance pricing literature (Deprez and Gerber (1985)), which says that an insured will never pay more premium than their maximum possible loss. If \(A^X_i \leq \sup X_i\) does not hold, then an agent with no risk of default may introduce positive probability of default after risk sharing, a clearly undesirable situation. For instance, using a power or logarithmic utility function, an agent’s potential loss should never exceed her total wealth level (this may be called bankruptcy aversion), and Axiom RF says that there is no bankruptcy after the risk exchange if the initial risk is safe in this regard. Hence, formulated via incentive compatibility, this axiom means that not all bankruptcy-averse agents are excluded, which is arguably a weak requirement.

Two special implications of RF may be useful. First, if the agent brings a pure surplus to the pool, i.e., \(X_i \leq 0\), its allocation should also be a pure surplus; this is certainly true if \(\mathcal{X}\) is contained in a half space such as the space of negative random variables. Second, in conjunction with Axiom AF, RF yields

\[
\text{for } X \in \mathcal{X}^n \text{ and } i \in [n], \text{ if } X_i = x \text{ is a constant, then } A^X_i = x; \quad (1)
\]

this follows from \(A^X_i \leq x\) and \(E[A^X_i] = x\). That is, if the initial risk of an agent is a constant, then there is no risk exchange for this agent; this is quite intuitive since any risk-averse agent (in the sense of Rothschild and Stiglitz (1970)) would not trade a constant risk with a non-constant risk with the same mean. As a particular example, for a risk vector \((X, 0, \ldots, 0)\), i.e., only the first

\(^6\)Certainly, the design of any risk sharing mechanism excludes some types of agents; however, it seems that excluding risk-neutral agents would be quite undesirable.

\(^7\)We can alternatively formulate Axiom RF using \(A^X_i \geq \inf X_i\), where \(\inf\) is the essential infimum. This alternative formulation has a different interpretation, and with this formulation, mathematical results in the paper remain the same due to symmetry.
agent having a non-zero initial risk, this property implies

\[ A_1^{(X,0,\ldots,0)} = X \] and \[ A_j^{(X,0,\ldots,0)} = 0 \text{ for } j \neq 1, \] (2)

which is arguably the only reasonable allocation in this particular case. On a point related to (1) and (2), our framework does not include the mechanism of side-payments, as in e.g., selling insurance, because deciding side-payments usually requires the knowledge of specific identities or preferences (e.g., which agent is institutional, more risk averse, or with more bargaining power).

**Axiom RA (Risk anonymity).** The realized value of the allocation to each agent is determined by that of the total risk. That is, for \( X \in \mathcal{X}^n \), \( A^X \) is \( \sigma(S^X) \)-measurable, where \( \sigma(S) \) is the \( \sigma \)-field generated by \( S \in \mathcal{X} \).

Axiom RA is central to the idea of designing a risk sharing mechanism. It means that the total realized allocation is determined only by the total loss suffered by the risk sharing pool, and not by specific losses from the individual participants. This resembles the earliest idea in insurance and risk sharing: Individuals get together to share their total future losses (in early years, these losses are typically caused by unexpected deaths, diseases or injuries), regardless of which one of them is the realized cause of the future loss. In other words, once an agent enters the pool, her own realized loss no longer matters, and only the realized loss of the pool matters. This reflects anonymity, as each agent does not need to disclose what is the realized loss; all individual losses are masked and only the total loss is revealed. The knowledge of the initial risk vector is only used for the design of the risk sharing mechanism, but not for the settlement of actual losses (see Figure 1).

Technically, RA holds for \( X \in \mathcal{X}^n \) satisfying that \( A^X \) is comonotonic. As studied by Borch (1962) and Landsberger and Meilijson (1994), comonotonicity is closely related to Pareto optimality for risk-averse agents; see Section 5.3 for details.

**Axiom OA (Operational anonymity).** The allocation to one agent is not affected if risks of two other agents merge. That is, \( A^Y_k = A^X_k \) for \( k \neq i, j \) for \( X \in \mathcal{X}^n, i, j \in [n] \) and \( Y = X + X_j e_i - X_i e_j \), where \( e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \) is the unit vector along the \( k \)-th axis (the \( k \)-th component is 1).

In the definition of Axiom OA, the risk vector \( Y \) can be written by \( Y_i = X_i + X_j, Y_j = 0 \) and \( Y_k = X_k \) for \( k \neq i, j \). Axiom OA means that merging the risks of two agents will not affect the allocation components of uninvolved agents. This also implies that a redistribution of risks between agent \( i \) and \( j \) does not affect agent \( k \) for \( k \neq i, j \). In an anonymized risk sharing framework, two agents may be two different accounts of the same family, same organization, or even the same person. Their internal (private) operations do not need to be disclosed and should not affect the allocation.
to other agents. OA is closely related to the fair-merging property of Denuit et al. (2022a), which clearly has a connection to fairness, although our motivation is different from the latter paper. In the context of Bitcoin mining, Leshno and Strack (2020) formulated two axioms, called robustness to Sybil attacks and robustness to merging, which together reflect the same consideration as OA. This property is further explained in the following simple example.

**Example 1.** Assume $X = (X_1, X_2, X_3)$ and $Y = (X_1, X_2 + X_3, 0)$. In this setting, we have $A^Y_1 = A^X_1$ if Axiom OA holds. Further, we have $A^Y_3 = 0$ from (1) implied by AF and RF, leading to $A^Y_2 = A^X_2 + A^X_3$. Therefore, by merging risks from agents 2 and 3, agent 2 now takes up the total allocation to the two agents, and the allocation to agent 1 is unaffected by this operation.

Axiom OA can be alternatively formulated by another intuitive property that $A^X_i$ is determined by $(X_i, S^X)$ for each $i$ and $X$. This latter property implies OA by definition. To see that OA implies this property, it suffices to observe, by repeatedly merging all agents except for agent 1, that

$$A^X_i = A_1^{(X_i, S^X - X_1, 0, \ldots, 0)}$$

(3)

holds. We summarize the above observation in the following proposition, which is convenient to use for our later discussions.

**Proposition 1.** A risk sharing rule $A$ satisfies Axiom OA if and only if for all $X \in \mathcal{X}^n$ and $i \in [n]$, $A^X_i$ is determined by $(X_i, S^X)$.

As discussed in this section, the four axioms are mathematically very simple and arguably natural in the framework of anonymized risk sharing. Quite remarkably, these four axioms uniquely pin down one risk sharing rule, which will be studied in the next section.

### 4 An axiomatic characterization

In this section, we show that Axioms AF, RF, RA and OA uniquely identify CMRS among all risk sharing rules. Recall that CMRS is defined as

$$A^X_{\text{cm}} = \mathbb{E}[X|S^X] \quad \text{for } X \in \mathcal{X}^n \subseteq (L^1)^n.$$  

For the ease of presentation, we take $\mathcal{X} = L^1$ or $L^1_+$ in all our results; these results hold true for $\mathcal{X} = L^q$ and $\mathcal{X} = L^q_+$ with $q \in [1, \infty]$ following the same proof; see Remark 2.

We first briefly check that CMRS satisfies the four axioms by using properties of the conditional expectation $\mathbb{E}[X|S]$ for any $(X, S) \in \mathcal{X}^2$ which will be chosen as $(X_i, S^X)$ for $i \in [n]$. First, AF
holds by the tower property $\mathbb{E}[\mathbb{E}[X|S]] = \mathbb{E}[X]$. Second, RF holds since $\mathbb{E}[X|S] \leq \sup X$. Third, RA holds by definition since $\mathbb{E}[X|S]$ is a function of $S$. Fourth, OA holds since $\mathbb{E}[X|S]$ is determined solely by $(X, S)$. See also Denuit et al. (2022a) for these and other properties of CMRS.

**Theorem 1.** Assume $\mathcal{X} = L^1$ or $\mathcal{X} = L_{+1}$. A risk sharing rule satisfies Axioms AF, RF, RA and OA if and only if it is CMRS.

Theorem 1 is the main result of the paper, showing that the four fairness and anonymity axioms allow for only one risk sharing rule. As far as we know, Theorem 1 is the first axiomatic characterization of risk sharing rules in the literature.

The “if” statement in Theorem 1, that CMRS satisfies the four axioms, has been checked above. The “only if” statement, which is the most important part of Theorem 1, requires a much more involved proof based on some advanced results from functional analysis. Below we provide an intuitive sketch of the proof in the case that $(\Omega, \mathcal{F}, \mathbb{P})$ is discrete and $\mathcal{X}$ is the set of all random variables on this probability space. A full proof is presented in Appendix A.

For a discrete $\Omega$, the main idea is to analyze each possible realized value $s \in \mathbb{R}$ of $S^X$ one by one. There are at most countably many such $s$. Let $A$ be a risk sharing rule satisfying the four axioms. We focus on the allocation to agent 1, and aim to show $A_1^X = \mathbb{E}[X_1|S^X]$ for all $X \in \mathcal{X}^n$; the allocations to the other agents are similar. Fix $S \in \mathcal{X}$, and we will first consider the risk vector $(X, S - X, 0, \ldots, 0)$ by allowing $X$ to vary within $\mathcal{X}$. Let $V$ be the set of possible values taken by $S$. By RA, the value of the allocation $A_1^{(X, S - X, 0, \ldots, 0)}$ to agent 1 is determined by the realized value $s \in V$ of $S$. Denote this value by $h^{S,s}(X)$, that is, for fixed $S \in \mathcal{X}$ and $s \in V$,

$$h^{S,s}(X) = A_1^{(X, S - X, 0, \ldots, 0)} \text{ given } S = s. \quad (4)$$

We can carefully check that $h^{S,s} : \mathcal{X} \rightarrow \mathbb{R}$ satisfies the following properties (in the last property we allow $s$ to vary in $V$):

(a) normalization: $h^{S,s}(t) = t$ for all $t \in \mathbb{R}$; \hspace{1cm} (by (1))

(b) additivity: $h^{S,s}(X + Y) = h^{S,s}(X) + h^{S,s}(Y)$ for $X, Y \in \mathcal{X}$; \hspace{1cm} (by OA and (a))

(c) monotonicity: $h^{S,s}(Y) \geq h^{S,s}(X)$ if $Y \geq X$; \hspace{1cm} (by RF and (b))

(d) $h^{S,s}(S) = s$; \hspace{1cm} (by RF and (2))

(e) $\sum_{t \in V} h^{S,t}(X)\mathbb{P}(S = t) = \mathbb{E}[X]$ for $X \in \mathcal{X}$. \hspace{1cm} (by AF)
The properties (a), (b) and (c) together guarantee that \( h^{S,s} \) is linear, monotone, and normalized. Using a standard representation theorem (such as that of Riesz), there exists a probability measure \( P_{S,s} \) such that
\[
h^{S,s}(X) = \int X dP_{S,s} \quad \text{for all } X \in \mathcal{X}. \tag{5}
\]
The next task is to show that \( P_{S,s} \) is precisely the conditional probability \( \mathbb{P}(\cdot | S = s) \). Let \( x \wedge y \) represent the minimum of \( x, y \in \mathbb{R} \). Using (d) and taking \( X = S \wedge x \) in (5), we arrive at
\[
h^{S,s}(S \wedge x) \leq \int S dP_{S,s} \wedge \int x dP_{S,s} = s \wedge x.
\]
This inequality further implies
\[
\mathbb{E}[S \wedge x] = \sum_{t \in V} (t \wedge x) \mathbb{P}(S = t) \geq \sum_{t \in V} h^{S,t}(S \wedge x) \mathbb{P}(S = t) = \mathbb{E}[S \wedge x],
\]
where the last equality is due to (e). Hence, we get \( h^{S,s}(S \wedge x) = s \wedge x \) for each \( s \in V \), and this gives, in particular, \( \int (S \wedge s) dP_{S,s} = s \). Therefore, \( P_{S,s}(S > s) = 1 \). Using symmetric arguments, we can show \( P_{S,s}(S < s) = 1 \). As a result, \( P_{S,s}(S = s) = 1 \). Using this equality and (e), for any \( B \subseteq \{S = s\} \), we have
\[
\mathbb{P}(B|S = s) = \frac{\mathbb{E}[1_B]}{\mathbb{P}(S = s)} = \frac{\sum_{t \in V} h^{S,t}(1_B) \mathbb{P}(S = t)}{\mathbb{P}(S = s)} = \frac{\sum_{t \in V} P_{S,t}(B) \mathbb{P}(S = t)}{\mathbb{P}(S = s)} = P_{S,s}(B).
\]
Therefore, \( P_{S,s}(\cdot) = \mathbb{P}(\cdot|S = s) \) and \( h^{S,s}(X) = \mathbb{E}[X|S = s] \) for \( X \in \mathcal{X} \). Based on this result, we can finally get \( A_1^X = \mathbb{E}[X_1|S^X] \) for a general \( X \) using (3) guaranteed by OA. This concludes the proof of Theorem 1 in case \( \Omega \) is discrete.

It is clear that the above proof sketch heavily relies on the assumption that \( \mathbb{P}(S = s) > 0 \) for \( s \in V \), and it cannot be directly generalized to non-discrete spaces. For a proof of Theorem 1 on general probability spaces, we need a more refined representation result in functional analysis. We obtain such a result in Theorem 2 below, which may be of independent interest.

**Theorem 2.** For a random variable \( S \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( \mathcal{G} = \sigma(S) \), let \( \phi : L^1(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow L^1(\Omega, \mathcal{G}, \mathbb{P}) \). The equality \( \phi(X) = \mathbb{E}[X|S] \) holds for all \( X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) if and only if \( \phi \) satisfies the following properties: (a) \( \phi(t) = t \) for all \( t \in \mathbb{R} \); (b) \( \phi(X + Y) = \phi(X) + \phi(Y) \) for all \( X, Y \); (c) \( \phi(Y) \geq \phi(X) \) if \( Y \geq X \); (d) \( \phi(S) = S \), and (e) \( \mathbb{E}[\phi(X)] = \mathbb{E}[X] \) for all \( X \).

The \( \mathcal{G} \)-conditional expectation as a mapping from \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \) to \( L^1(\Omega, \mathcal{G}, \mathbb{P}) \) admits a few different sets of characterizations; Pfanzagl (1967) has a collection of several early results. For a more
recent account, see Eisner et al. (2015, Chapter 13) in the context of Markov operators. Theorem 2 extends the above literature by offering a new characterization of the conditional expectation.

Theorem 1 treats both the spaces $\mathcal{X} = L^1$ and $\mathcal{X} = L^1_+$ (the space of non-positive random variables is similar). These two cases represent different contexts. In some applications, risk allocation and risk contributions are restricted to being all positive or all negative, depending on the context. For instance, if agents are sharing P2P insurance losses, then it may be sensible to assume $X$ and $A^X$ both take non-negative vector values; if agents are sharing profits from an investment, then it is the opposite; recall that positive values represent losses and negative values represent gains. By working with the positive half space, Axiom RF can be replaced by the simpler Property CP stated in (1). This is because CP and OA imply RF in case $\mathcal{X} = L^1_+$.

Property CP (Constant preserving). A constant initial risk results in constant allocation. That is, for $X \in \mathcal{X}^n$ and $i \in [n]$, $X_i = x \in \mathbb{R}$ implies $A_i^X = x$.

In the next proposition, we verify that the four axioms are independent, and thus none of them can be removed from Theorem 1.

Proposition 2. Axioms AF, RF, RA and OA are independent. That is, any combination of three of Axioms AF, RF, RA and OA does not imply the remaining fourth axiom.

For each axiom, we will provide in Example 2 a risk sharing rule that only satisfies three of them but not the fourth one; some of these examples have been listed in Section 2. The technical details of these claims are in Appendix A.

Example 2. (i) The $Q$-CMRS $A^X_{Q-cm} = \mathbb{E}^Q[X|S^X]$ with $\mathcal{X} \subseteq L^1(\Omega, \mathcal{F}, Q)$ for a probability measure $Q \neq \mathbb{P}$ satisfies RF, RA and OA, but not AF.

(ii) The mean-adjusted all-in-one risk sharing rule

$$A^X_{ma} = (S^X - \mathbb{E}[S^X], 0, \ldots, 0) + \mathbb{E}[X]$$

with $\mathcal{X} \subseteq L^1$ satisfies AF, RA and OA, but not RF. As further examples, both the covariance risk sharing rule and the mean proportional risk sharing rule in Section 2 satisfy AF, RA and OA but not RF.

(iii) The identity risk sharing rule $A^X_{id} = X$ satisfies AF, RF and OA, but not RA.

(iv) A combination of $A_{all}$ and $A_{cm}$, defined by $A^X = A^X_{all} = (S^X, 0, \ldots, 0)$ if $X$ is standard Gaussian, and $A^X = A^X_{cm} = \mathbb{E}[X|S^X]$ otherwise, satisfies AF, RF and RA, but not OA.
5 Other properties and their connection to the four axioms

In this section, we discuss several further properties that CMRS satisfies or does not satisfy. These properties are known and straightforward to check. The purpose of this section is to clarify their relationship with the four axioms in Section 3.

5.1 Universal improvement

For two random variables $X$ and $Y$, we say $X$ is improved compared to $Y$ in convex order if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for any convex function $u : \mathbb{R} \to \mathbb{R}$; this is denoted by $X \preceq_{cx} Y$. The most appealing feature of CMRS, as argued by Landsberger and Meilijson (1994) and Denuit and Dhaene (2012), is that it universally improve the risk for a larger class of decision makers, via the following property.

Property UI (Universal improvement). The allocation improves the initial risk in convex order. That is, for any $X \in \mathcal{X}^n$ and $i \in [n]$, it holds that $A^X_i \preceq_{cx} X_i$.

CMRS satisfies UI as a direct result of conditional Jensen’s inequality. Intuitively, UI means that the initial risk for each agent has larger variability than the allocation to that agent. As a consequence, risk-averse agents in the classic sense of Rothschild and Stiglitz (1970), i.e., those who prefer both an improvement of convex order and a sure gain, will prefer their UI allocations over their initial risks. In a similar spirit, Denuit and Robert (2020, Proposition 4.2) showed that if risks in the pool are independent then the CMRS allocation improves in convex order for each existing agent when the pool is enlarged.

To illustrate the important role of UI for CMRS, we note that UI implies both AF and RF, since $X \preceq_{cx} Y$ implies $X \preceq \sup X \preceq \sup Y$ and $\mathbb{E}[X] = \mathbb{E}[Y]$ for any random variables $X, Y \in L^1$. We summarize this observation in the following proposition.

Proposition 3. Property UI implies Axioms AF and RF and Property CP.

Combining Proposition 3 with Theorem 1, we immediately obtain in Corollary 1 another characterization of CMRS with AF and RF replaced by UI. Proposition 3 and Corollary 1 also illustrate that UI is a very strong property. Recall that our characterization in Theorem 1 relies on the weaker axioms of AF and RF, and thus the more important “only if” statement is stronger than that of Corollary 1.

Corollary 1. Assume $\mathcal{X} = L^1$ or $L^1_+$. A risk sharing rule satisfies Axioms RA and OA and Property UI if and only if it is CMRS.
Finally, we provide a subtle example, showing that the condition $n \geq 3$ which we assumed from the beginning is indispensable, and this remains true even if we further assume the stronger property of UI. The intuition is that in case $n = 2$, Axiom OA is empty since no merging operation is possible when one agent’s risk is fixed. As a result, we cannot obtain the additivity of $h^{S,s}$ in (4) which requires some “wiggle room” provided by the third dimension.

**Example 3.** Let $n = 2$. We design a risk sharing rule $A$ which satisfies all of AF, RF, RA, OA and UI, but it is not CMRS. Let $A = A_{cm}$ for all $X \in \mathcal{X}^n$ except for a specific $Y = (Y_1, Y_2)$, which is given by

$$A^Y_1 = E[Y_1|S^Y] + h(S^Y) \text{ and } A^Y_2 = E[Y_2|S^Y] - h(S^Y),$$

where $h$ satisfies $E[h(S^Y)] = 0$. The intuition is that, if $h$ is sufficiently small and $E[Y_i|S^Y]$ is sufficiently different from $Y_i$, then $A^Y_i \leq_{cx} Y_i$ still holds, thus satisfying RA, OA and UI. To make the example explicit, let us take $Y_1 \sim N(0,1)$ and $Y_2 \sim N(0,2)$, and $Y_1, Y_2$ are independent. Let $h(s) = s/6$. Note that $S^Y \sim N(0,3)$. We can compute

$$A^Y_1 = \frac{1}{3}S^Y + \frac{1}{6}S^Y = \frac{1}{2}S^Y \sim N(0,0.75) \text{ and } A^Y_2 = \frac{2}{3}S^Y - \frac{1}{6}S^Y = \frac{1}{2}S^Y \sim N(0,0.75).$$

Hence, for this particular $Y$, everything in Axiom RA and Property UI (hence Axioms AF and RF) is satisfied. Axiom OA holds trivially as its statement is empty. Therefore, $A$ is not CMRS but it satisfies the four axioms and Property UI.

### 5.2 Backtracking

The second property we discuss is the backtracking property, which means that, for any $i \in [n]$ if $S^X$ is able to determine $X$, then $A^X = X$, and thus there is no risk exchange. It is straightforward to verify that CMRS satisfies this property.

**Property BT (Backtracking).** For each $X \in \mathcal{X}^n$, if $X$ is $\sigma(S^X)$-measurable, then $A^X = X$.

Property BT is sometimes argued as an undesirable property; see Denuit et al. (2022b). We give a simple example below for the purpose of discussion.

**Example 4.** Suppose that $X_1 = \sigma_1 Y_1$, $X_2 = \sigma_2 Y_2$, and $X_3 = \sigma_3 Y_3$, where $Y_1, Y_2, Y_3$ are iid taking values in $\{0,\ldots,9\}$, and $\sigma_1 = 1001$, $\sigma_2 = 1010$ and $\sigma_3 = 1100$. Note that $S^X$ uniquely determines the value of $(X_1, X_2, X_3)$ since the last three digits of $S^X$ are precisely $Y_3, Y_2, Y_1$. In this example, $X_1, X_2, X_3$ have similar distributions, and they are independent. Intuitively, some risk sharing effect is possible for such $X$, but $A^X_{cm} = X$ due to the backtracking property.
Property BT intuitively means that there is no risk sharing effect if $S^X$ is too informative compared to the individual contributions. As a consequence, there are some situations, although perhaps rare, in which CMRS discourages some participants to enter the risk sharing pool, even if they bring in risks independent of the other participants. Theorem 1 provides the additional insight that BT is unavoidable, given the four natural axioms of fairness and anonymity. If some applications demand BT to be avoided, then one has to relax some axioms. For this, one naturally wonders which of the four axioms are responsible for Property BT.

The axiom which involves $\sigma(S^X)$ is RA, and a first guess may be that RA is connected to BT. Somewhat surprisingly, this is not true. In the next result we establish that AF, RF and OA are sufficient for BT if $A^X$ is further assumed $\sigma(X)$-measurable; the last assumption is quite weak as the risk settlement usually do not involve extra randomness outside $\sigma(X)$.

**Proposition 4.** Assume $\mathcal{X} = L^1$ or $L^1_+$. If a risk sharing rule $A$ satisfies Axioms AF, RF and OA, and $A^X$ is $\sigma(X)$-measurable for all $X \in \mathcal{X}^n$, then it satisfies Property BT.

An example of a risk sharing rule satisfying AF, RF and OA but not RA is the mixture $A = \lambda A_{id} + (1 - \lambda) A_{cm}$ for some $\lambda \in (0, 1]$; such a rule satisfies BT. On the other hand, the mean-adjusted all-in-one and covariance risk sharing rules in Example 2 satisfy AF, RA and OA, and it does not satisfy BT or RF.

### 5.3 Comonotonicity

Next, we discuss comonotonicity, an important concept in risk sharing. A random vector $(X_1, \ldots, X_n)$ is **comonotonic** if there exists increasing (in the non-strict sense) functions $g_1, \ldots, g_n$ and a random variable $Z$ such that $X_i = g_i(Z)$ a.s. for $i \in [n]$.

**Property CM (Comonotonicity).** For each $X \in \mathcal{X}^n$, $A^X$ is comonotonic.

Property CM implies Axiom RA since each component of a comonotonic random vector can be written as an increasing function of the sum; see Denneberg (1994). Therefore, Property CM can be equivalently formulated as that for $X \in \mathcal{X}^n$ and $i \in [n]$, there exists an increasing function $g_i^X : \mathbb{R} \to \mathbb{R}$ such that $A_i^X = g_i^X(s^X)$.

If Property CM holds, then the allocation to each agent increases as the total realized risk increases. Under belief homogeneity and mild assumptions, Property CM is also a necessary condition for a risk sharing rule to be Pareto optimal for risk-averse agents or for an exchange market with linear prices; see e.g., Borch (1962), Landsberger and Meilijson (1994) and Boonen et al.
As notable exceptions, quantile-based risk sharing and belief heterogeneity both result in non-comonotonic Pareto-optimal allocations; see Embrechts et al. (2018, 2020).

CMRS does not generally satisfy Property CM, which may be seen as a drawback of CMRS when the preferences of the agents are specified and risk averse, as the allocation is suboptimal. In our context of anonymized risk sharing, optimality cannot be discussed this way, since agents’ specific preferences are not relevant. Nevertheless, if \( s \mapsto \mathbb{E}[X_i|S = s] \) is increasing for each \( i \in [n] \), then CMRS is comonotonic. There are many specific models of \( \mathbf{X} \) for which comonotonicity of CMRS holds; see Denuit et al. (2022a) and the references therein. In several contexts, such as those with risk-averse agents or moral hazard, comonotonicity is desirable. On this point, our Theorem 1 implies the negative result that the four axioms and Property CM conflict each other. We further strengthen this result by showing that OA, CM, and a weak version of CP cannot be satisfied by the same risk sharing rule. This weak version of CP is the following property.

**Property ZP (Zero preserving).** For \( \mathbf{X} \in \mathcal{X}^n \) and \( i \in [n] \), if \( X_i = 0 \), then \( A_i^X = 0 \).

It might be useful to recall some logical relationship among some properties and axioms mentioned above, that is,

\[
\text{UI} \implies \text{AF} + \text{RF} \implies \text{CP} \implies \text{ZP} ; \quad \text{CM} \implies \text{RA}.
\]

**Proposition 5.** Assume \( \mathcal{X} = L^1 \). There is no risk sharing rule satisfying Axiom OA and Properties CM and ZP.

If Property CM is needed in a specific application, one may need to relax some of the axioms. In the following example, we provide two relaxations to show that it is possible to have both CM and OA or both CM and UI.

**Example 5.**

(i) The mean-adjusted all-in-one risk sharing rule satisfies CM (implying RA), AF, and OA, but not RF or ZP, as we see from Example 2.

(ii) For each \( \mathbf{X} \in \mathcal{X}^n \subseteq (L^1)^n \), the comonotonic improvement of Landsberger and Meilijson (1994) gives a comonotonic vector \( \mathbf{X}' \) such that each component of \( \mathbf{X}' \) is dominated by the corresponding component of \( \mathbf{X} \) and \( S^X = S^{X'} \); see also Rüschendorf (2013). The risk sharing rule given by \( A^X = X' \) satisfies CM (implying RA) and UI (implying AF and RF), but not OA.

**Remark 1.** Property CM is closely related to Pareto optimality for risk-averse decision makers with specified preferences. For instance, any comonotonic allocation is Pareto optimal for agents with
an identical dual utility of Yaari (1987). Recall that a risk-averse dual utility $U$ is comonotonic-
additive and superadditive; see Yaari (1987) and Wang et al. (2020, Theorems 1 and 3). For any
comonotonic risk vector $X$ and any $Y = (Y_1, \ldots, Y_n) \in A_n(S^X)$,

$$\sum_{i=1}^{n} U(-X_i) = U\left(-\sum_{i=1}^{n} X_i\right) = U\left(-\sum_{i=1}^{n} Y_i\right) \geq \sum_{i=1}^{n} U(-Y_i),$$

where the negative sign reflects that our risks are losses and $U$ is applied to wealth. Therefore, $Y$
do not dominate $X$, showing that $X$ is Pareto optimal.

5.4 Symmetry

Symmetry is another important property reflecting the spirit both fairness and anonymity. Let
$\Pi_n$ be the set of $n$-permutations, and we write $X_{\pi} = (X_{\pi(1)}, \ldots, X_{\pi(n)})$ for $\pi \in \Pi_n$ and $X \in X^n$.

Property SM (Symmetry). For each $X \in X^n$ and $\pi \in \Pi_n$, $(A^X)_{\pi} = A^{X_{\pi}}$.

Property SM reflects that consideration that if agents $i$ and $j$ exchange their initial risk con-
tributions, then they also exchange their allocations. Hence, their identities or positions in the risk
sharing pool does not matter; this clearly relates to both fairness and anonymity. Property SM is
called the reshuffling property by Denuit et al. (2022a), and a similar property is called anonymity
by Leshno and Strack (2020) in the setting of Bitcoin reward sharing.

Property SM is not directly assumed among our axioms, and CMRS satisfies Property SM
by definition. Therefore, SM must follow from some of the axioms we impose. Since SM is very
intuitive for an anonymized risk sharing rule, we wonder which axioms yield SM. It turns out that
OA and ZP are sufficient for SM.

Proposition 6. Axiom OA and Property ZP imply Property SM.

We can briefly verify that SM does not come from any one of the four axioms alone. The
mean-adjusted all-in-one risk sharing rule in Example 2 satisfies AF, RA and OA, but not SM or
ZP. The combination of $A_{all}$ and $A_{cm}$ in Example 2 satisfies AF, RF (hence ZP) and RA, but not
SM or OA.

6 Generalized risk sharing rules with target information

In some applications, more information than simply the realized value of the total risk is
observable, and one may wish to allocate risks according to such information. This leads to a
generalization of risk sharing rules. Denote by $\Sigma$ the set of sub-$\sigma$-fields of $\mathcal{F}$. A generalized risk sharing rule is a mapping $\hat{A} : (\mathcal{X}^n \times \Sigma) \to \mathcal{X}^n$ satisfying $\hat{A}^{X|G} = (\hat{A}_1^{X|G}, \ldots, \hat{A}_n^{X|G}) \in A_n(S^X)$ for each $X \in \mathcal{X}^n$ and $G \in \Sigma$.

The input $\sigma$-field $G$ represents the information used to determine the realized values of the allocation, called target information. Note that $\sum_{i=1}^n \hat{A}_i^{X|G} = S^X$ implies that the $\sigma$-field of $\hat{A}^{X|G}$ must contain $\sigma(S^X)$ regardless of the choice of $G$. To address this issue, we merge the information in $\sigma(S^X)$ into $G$, and denote by $G^X = \sigma(S^X, G)$ the $\sigma$-field generated by $S^X$ and $G$. Below, we present two properties describing how the information modelled by $G$ and $G^X$ is used for the generalized risk sharing rule $\hat{A}$.

Property IA (Information anonymity). For $X \in \mathcal{X}^n$ and $G \in \Sigma$, $\hat{A}^{X|G}$ is $G^X$-measurable.

Property IA gives $G^X$-measurability instead of $G$-measurability. As discussed above, there does not exist $\hat{A}$ such that $\hat{A}^{X|G}$ is $G$-measurable for every $G \in \Sigma$ and every $X \in \mathcal{X}^n$. Property IA reflects on the idea that the risk allocations may not be determined by the realized value of $S^X$ but it may also depend on other information represented by the set $G$. Property IA is a generalization of Axiom RA. For a given risk sharing rule $A$, we can define $\hat{A}$ by $\hat{A}^{X|G} = A^X$ for each $X \in \mathcal{X}^n$ and $G \in \Sigma$; that is, the information $G$ is ignored. In this case, $A$ satisfies RA if and only if $\hat{A}$ satisfies IA. For instance, the generalized risk sharing rule $\hat{A}$ defined by $\hat{A}^{X|G} = A^X_{cm}$ for each $(X, G)$ satisfies Property IA.

Property IB (Information backtracking). For each $X \in \mathcal{X}^n$ and $G \in \Sigma$, $\hat{A}^{X|G}$ is $G^X$-measurable.

Property IB is a generalization of Property BT. It reflects on the consideration that getting an allocation determined by $G^X$ is our target, and no risk exchange happens if the initial risk is already determined by $G^X$. This property excludes the example above given by $\hat{A}^{X|G} = A^X_{cm}$ for each $(X, G)$, since $\hat{A}^{X|\sigma(X)} = X$ needs to hold by Property IB.

For a generalized risk sharing rule $\hat{A}$, we say that it satisfies an axiom or property introduced for risk sharing rules, if the mapping $X \mapsto \hat{A}^{X|G}$ satisfies the corresponding axiom or property for each $G \in \Sigma$. The next result characterizes the generalized CMRS, defined by

$$\hat{A}^{X|G}_{cm} = \mathbb{E}[X|G^X] \quad \text{for } X \in \mathcal{X}^n \text{ and } G \in \Sigma,$$

among all generalized risk sharing rules.

**Theorem 3.** Assume $\mathcal{X} = L^1$ or $L^1_+$. A generalized risk sharing rule satisfies Axioms AF, RF and OA and Properties IA and IB if and only if it is the generalized CMRS.
Theorem 3 indicates that, assuming AF, RF, OA, IA and IB, the only generalized risk sharing rule with a given target information $G^X$ needs to be calculated based on the conditional expectation with respect to $G^X$. This interpretation is similar to the result of Theorem 1. The generalized CMRS characterized in Theorem 3 will be useful as many practical applications involve allocations that are not solely determined by the total risk. We discuss some of them in Section 7.

7 Applications

In this section, we discuss a few examples of risk or reward sharing, as illustrative examples of the features of our axioms for anonymized risk sharing. We begin with the most simple application of a Bitcoin mining pool which is best described by CMRS, and proceed to three other contexts where the generalized CMRS in Section 6 appears to be suitable.

7.1 A single Bitcoin mining pool

By the design of the Bitcoin protocol (Nakamoto (2008)), when a computational puzzle is solved by a decentralized network of anonymous computers, which are commonly called miners, a block in the Bitcoin blockchain is issued by a randomly selected miner, to which a block of bitcoins is rewarded.\(^8\) Since the bitcoin price has increased drastically over the past few years (with a peak at more than 60,000 USD per coin in 2021), mining activities become very risky, with a large monetary value of the reward and a small probability of success, for individual miners. For this reason, mining pools are formed by groups of miners to share the risk. Risk-averse miners always have incentives to join mining pools to improve their utility; for this statement and more background on Bitcoin, including criticisms on its environmental and economic impact and the conflict between mining pools and decentralization, we refer to Chiu and Koeppel (2017), Eyal and Sirer (2018) and Leshno and Strack (2020). It is not our intention to say whether mining pools are good or bad; taking their existence as given, our focus is the design of reward sharing mechanisms within mining pools.

Suppose that $n$ miners form a mining pool to share the possible reward from mining the next block. Let the random variable $P > 0$ represent the monetary value of the next block at the time of solving the block. The miners’ initial contribution vector is $X = (1_{D_1}, \ldots, 1_{D_n})$, where $D_i \subseteq \Omega$ is the event that miner $i$ successfully issues the next block,\(^9\) and the probability $P(D_i)$ represents the

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\(^8\)To be more accurate, each computation of a hash is equally likely to lead to a value that allows the miner to write the next block and receive the reward.

\(^9\)All sets $D \subseteq \Omega$ that appear in this section are assumed measurable, i.e., $D \in \mathcal{F}$.
computational contribution of the miner $i$, measured by the number of hashes tried, divided by that of all miners in the world mining the block. Since the reward has a non-negative monetary value, we interpret positive values of allocations as rewards in this section, a different sign convention from the rest of the paper; this causes no technical problem as all our results are invariant with respect to their signs. The events $D_1, \ldots, D_n$ are mutually exclusive because at most one miner can issue the next block. We assume that these events are independent of the bitcoin price $P$ because $P$ is determined by market activities and $D_1, \ldots, D_n$ are determined by randomly trying solutions.

Denote by $D = \bigcup_{i=1}^n D_i$ the event that any miner from this pool issues the block.

Our setting is similar to the one used by Leshno and Strack (2020) which focuses on the reward mechanism of home miners, i.e., those outside mining pools. In the setting of the latter paper, rewards to individual miners are characterized by the probability of receiving 1 block, whereas in our framework of a mining pool, miners can receive any amount between 0 and $P$. This distinction is useful in our later analysis of application of mining multiple cryptocurrencies in Section 7.3. Our reward mechanism within a mining pool complements the one for home miners studied by Leshno and Strack (2020).

To make our analysis of reward sharing rigorous, let $P \in \mathcal{X}$ be a fixed positive random variable, and denote by $\mathcal{B}_n$ the set of all possible contribution vectors from the miners, that is,

$$\mathcal{B}_n = \{(\mathbf{1}_{D_1}, \ldots, \mathbf{1}_{D_n}) : D_1, \ldots, D_n \subseteq \Omega \text{ are disjoint and independent of } P\}.$$

We assume that the probability space is rich enough so that a continuously distributed random variable independent of $P$ exists. A reward sharing rule is a mapping $A : \mathcal{B}_n \to \mathcal{X}^n$ satisfying $A^X = A_n(S^X)$ for each $X \in \mathcal{B}_n$ and $A^X_i = A^X_j$ for $i, j \in [n]$ with $\mathbb{P}(D_i) = \mathbb{P}(D_j)$ where $X = P(\mathbf{1}_{D_1}, \ldots, \mathbf{1}_{D_n})$. The last requirement reflects that only the amount of computational contribution of each miner is supplied (instead of the specification of the events $D_i$ and $D_j$), as in the settings of Eyal and Sirer (2018) and Leshno and Strack (2020). The four axioms of fairness and anonymity have natural interpretations and desirability in this setting.

(i) Axiom AF means that no agent gets less (or gets more) than their initial contribution in expectation, a simple form of fairness among anonymous participants.

(ii) Axiom RF (with a sign flip, i.e., $A^X_i \geq \inf X_i$ for $i \in [n]$) means that any miner has a non-negative reward, since $\inf(P\mathbf{1}_{D_i}) = 0$ if $\mathbb{P}(D_i) \in [0, 1)$. Note that the case $\mathbb{P}(D_i) = 1$ is trivial since all other agents have 0 contribution and 0 reward (using AF), and agent $i$ receives the whole reward $P$. 

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(iii) Axiom RA means that the reward does not depend on which miner issues the block. If the block is issued by the pool, the rewards to miners depend on their computational contributions and the bitcoin price, but not the actual issuing miner. This feature is central to the idea of creating a mining pool and joining computational resources.

(iv) Axiom OA means that the mechanism is safe against merging and Sybil attacks, i.e., creating multiple accounts of the same participant. Recall that miners are represented by computers and online accounts, and merging, splitting, or creating them is not disclosed to other miners. Hence, such operations between some miners should not affect the reward to an uninvolved miner.

Anticipated from Theorem 1, the only reward sharing rule satisfying Axioms RA, RF, AF and OA should be CMRS. This is indeed true, although a separate proof is needed, as the set \( \mathcal{B}_n \) is much smaller than \( \mathcal{X}^n \), preventing us from directly applying Theorem 1.

**Proposition 7.** Assume \( P \in \mathcal{X} = L^1 \) and \( P > 0 \). A reward sharing rule \( \mathbf{A} : \mathcal{B}_n \to \mathcal{X}^n \) satisfies Axioms RA, RF, AF and OA if and only if it is specified by

\[
A_i^X = \frac{\mathbb{P}(D_i)}{\mathbb{P}(D)} P \mathbb{1}_D, \quad i \in [n], \quad X = P(\mathbb{1}_{D_1}, \ldots, \mathbb{1}_{D_n}) \in \mathcal{B}_n, \tag{7}
\]

which is CMRS.

The allocation (7) is precisely the common practice in mining pools; see Eyal and Sirer (2018) and Leshno and Strack (2020) in case \( P = 1 \). The total value \( P \) is shared proportionally to the computational contribution of each miner if this mining pool successfully issues the block (i.e., \( \mathbb{1}_D = 1 \)), and the rewards are 0 otherwise. This reward sharing rule is CMRS, since \( \mathbb{E}[P \mathbb{1}_{D_i} | P \mathbb{1}_D] = P \mathbb{1}_D \mathbb{E}[\mathbb{1}_{D_i} | \mathbb{1}_D] = P \mathbb{1}_D \mathbb{P}(D_i)/\mathbb{P}(D) \). An example is shown in Figure 2 to illustrate how (7) works for three miners. Before joining a mining pool, miner 1 gets the reward \( P \) if she issues the block (purple area in Figure 2) and otherwise she receives nothing. After miner 1 joins the group, the reward for her will be \( P \mathbb{P}(D_1)/\mathbb{P}(D) \) if any of the three miners issues the block (brown area in Figure 2). The new insight offered by Proposition 7 is that the reward sharing rule (7) is the unique possible mechanism if our four axioms are considered as desirable, and thus they fully rationalize the choice of this mechanism in practice.

\[\text{In this simple setting, CMRS also coincides with the mean proportional risk sharing rule.}\]
7.2 Multiple mining pools

Next, suppose that miners can choose to participate in multiple pools by allocating their computational resources among these pools. We would like to pin down a suitable allocation rule in this setting with the help of the generalized CMRS and a specific choice $G$ of target information.

There are $m$ mining pools. Let $E_1, \ldots, E_m$ be mutually exclusive events where $E_j$ represents the event that pool $j$ successfully issues the block. Miners can choose to join one or several mining pools, and their initial risk vector is given by $X = P(1_{D_1}, \ldots, 1_{D_n})$ as in Section 7.1. Since the $n$ miners form the $m$ mining pools, we have the equality $\bigcup_{i=1}^{n} D_i = \bigcup_{j=1}^{m} E_j$ and the decomposition $D_i = \sum_{j=1}^{m} 1_{D_i \cap E_j}$ where $D_i \cap E_j$ is the contribution of miner $i$ to pool $j$.

Due to the separation of mining pools, we consider a generalized reward sharing rule with target information $G = \sigma(P, 1_{E_1}, \ldots, 1_{E_m})$, that is, the information of the Bitcoin price $P$ and the winning pool which successfully mines the block. For this choice of $G$, the generalized CMRS $\hat{A}$ in (6) is given by

$$\hat{A}^{X|G} = \sum_{j=1}^{m} \frac{P(D_i \cap E_j)}{P(E_j)} P1_{E_j}. \quad (8)$$

Note that this rule can be easily implemented in practice as $P(D_i \cap E_j)/P(E_j)$ is the relative share of computational contribution of worker $i$ to pool $j$. This rule can be equivalently explained by the mechanism in which all $m$ pools are allocated separately and each of them uses CMRS.

Similarly to Proposition 7, we can show that (8) is the only generalized reward sharing rule with target information $G$ satisfying conditions similar to those in Section 7.1, and this rationalizes the practice of allocating rewards across multiple mining pools.
### 7.3 Multiple cryptocurrencies

We proceed to consider a pool of \(n\) miners with a collection of \(m\) cryptocurrencies (which we call coins) with random prices \(P_1, \ldots, P_m\) in a pre-specified period of time. The computational contribution of miner \(i\) to coin \(j\) is fixed during this period of time. For simplicity, we assume that for each of these coins at most one block may be issued during this period of time. Denote by \(D_{ij}\) the event that miner \(i\) issues the block for coin \(j\), and by \(D_j = \bigcup_{i=1}^n D_{ij}\) is the event that coin \(j\) is successfully mined by the pool. The events \(D_{ij}\) are mutually exclusive across \(i \in [n]\) for the same \(j\). We further assume that \(D_{ij}\) is independent of \(\{D_{k\ell} : k \in [n], \ell \in [m] \setminus \{j\}\}\), because issuing the block of one coin should not affect issuing the block of another one. Similarly to Section 7.1, we assume that the prices \(P_1, \ldots, P_m\) are independent to the issuance events. The initial risk vector is given by \(X = \sum_{j=1}^m P_j (1_{D_{1j}}, \ldots, 1_{D_{nj}})\). Finally, we consider the target information \(G = \sigma(P_1, \ldots, P_m, 1_{D_1}, \ldots, 1_{D_m})\), which is the information of the coin prices and the events of whether each of them is successfully mined.

For this choice of \(G\), the generalized CMRS \(\hat{A}\) in (6) is given by

\[
\hat{A}^X|G = \sum_{j=1}^m \frac{\mathbb{P}(D_{ij})}{\mathbb{P}(D_j)} P_j 1_{D_j},
\]

because \(\mathbb{E}[P_j 1_{D_{ij}}|G] = \mathbb{E}[P_j 1_{D_{ij}}|P_j 1_{D_j}] = P_j \mathbb{P}(D_{ij})/\mathbb{P}(D_j)\). In other words, each miner gets a proportion \(\mathbb{P}(D_{ij})/\mathbb{P}(D_j)\) of each successfully mined coin, where the proportion is determined by its relative contribution to the pool for that particular coin.

### 7.4 Revenue sharing

Our final application concerns revenue sharing in subscription-based online platforms, and our primary examples are music platforms such as Spotify, Deezer, or Apple Music; see Meyn et al. (2023) for a description of revenue sharing in subscription-based music platforms. Suppose that there are \(n\) artists and \(m\) potential users in a specific month (many platforms collect subscription fees monthly). In this context, \(m\) is usually much larger than \(n\).

We assume that each user can subscribe to the platform because of one artist \(i\), which is unobservable from the platform or the artist. Let \(D_{ij}\) be the event that user \(j\) subscribes because of artist \(i\), and \(D_{ij}\) are mutually singular across \(i \in [n]\). Assume that the subscription events across different users are independent; i.e., \(D_{ij}\) is independent of \(\{D_{k\ell} : k \in [n], \ell \in [m] \setminus \{j\}\}\) for each \(i \in [n]\) and \(j \in [m]\). Let \(D^j = \bigcup_{i=1}^n D_{ij}\) be the event that user \(j\) subscribes to the platform, which is observable, and it generates a non-random revenue \(q_j > 0\) (i.e., subscription fee, which may vary

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across users). If $D^j$ does not occur, then user $j$ does not subscribe to the platform during the considered month. Suppose that for $j \in [m]$, a proportion $\delta_j$ of $q_j$ will be shared by the artists (the other proportion is kept by the platform or used to cover costs), and we denote by $p_j = \theta_j q_j$.

In this model, the initial risk vector is given by $X = \sum_{j=1}^m p_j (\mathbb{1}_{D^j_1}, \ldots, \mathbb{1}_{D^j_{n_j}})$, which is not observable to the platform. The target information is modelled by $G = \sigma(\mathbb{1}_{D^1}, \ldots, \mathbb{1}_{D^m})$, that is, the information based on the events of subscription. For this choice of $G$, the generalized CMRS $\hat{A}_i$ in (6) is given by

$$\hat{A}_i^X|G = \sum_{j=1}^m \frac{\mathbb{P}(D^j)}{\mathbb{P}(D^i)} p_j \mathbb{1}_{D^i},$$

similarly to the model in Section 7.3. Although $\mathbb{P}(D^i)$ and $\mathbb{P}(D^j)$ are not directly observable, their ratio $\mathbb{P}(D^i)/\mathbb{P}(D^j)$ can be estimated the ratio $s_{ij}$ of the number of streams of user $j$ using (e.g., listening to) the work of artist $i$ to that of all streams of user $j$, and such data are available to the platform. Intuitively, the more user $j$ uses the work of artist $i$, the more likely that user $j$ subscribed because of artist $i$. With the ratio $\mathbb{P}(D^i)/\mathbb{P}(D^j)$ estimated by $s_{ij}$, the revenue sharing mechanism (9) is the user-centric remuneration model promoted by some platforms based on an argument of fairness.\footnote{For instance, the platform Deezer is promoting the user-centric payment system; see https://www.deezer-blog.com/how-much-does-deezer-pay-artists/ (accessed April 2023).} We refer to Meyn et al. (2023) for a comparison of this revenue sharing rule and others, and our framework provides a theoretical reasoning for the user-centric system.

## 8 Concluding remarks

Four axioms of fairness and anonymity, Axioms AF, RF, RA and OA, are proposed in the paper. As the main result of the paper (Theorem 1), the four axioms uniquely identify CMRS, making CMRS a unique desirable rule to use in many applications of anonymized risk sharing. Among the four axioms, AF and RF reflect fairness in the most natural sense. OA reflects irrelevance of certain unseen operations, and it is desirable if agents are simply online accounts without disclosing their identity. RA, requiring the realized risk allocation to be determined by the sum, is the least straightforward requirement among the four, although it is quite commonly seen in many applications. The application of reward sharing in a Bitcoin mining pool illustrates the desirability of all four axioms. We do not see any general reasons to dispute any of the four axioms in the framework of anonymized risk sharing, although in certain specific applications CMRS may be suboptimal for some agents.
Decentralization in finance and insurance is getting increasing attention from both academia and the financial industry. As one of the most important features of decentralization, anonymity guarantees that agents are not distinguished by their preference, identity, private operations, and realized losses. As such, our paper serves as a theoretical support to the wide applications of CMRS as a standard tool in many relevant applications in decentralized risk sharing.

As a potential limitation, CMRS requires a full specification of the joint distribution of the risk contributions from the agents to compute. This is not a problem for the applications discussed in Section 7 due to their specific settings of available information. For some other applications, computational issues can be cumbersome for a large set of heterogeneous agents; for computing CMRS in some specific models, see Denuit (2019) and Denuit et al. (2022a) and the references therein.

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References

Abdikerimova, S. and Feng, R. (2022). Peer-to-Peer multi-risk insurance and mutual aid. European Journal of Operational Research, 299(2), 735–749.
Aliprantis, C. D. and Burkinshaw, O. (2006). Positive Operators. Springer.
Arrow, K. J. (1963). Uncertainty and the welfare economics of medical care. American Economic Review, 53(5), 941–973.
Arrow, K. J. and Debreu, G. (1954). Existence of an equilibrium for a competitive economy. Econometrica, 22(3), 265–290.
Artzner, P., Delbaen, F., Eber, J. M. and Heath, D. (1999). Coherent measures of risk. Mathematical Finance, 9(3), 203–228.
Barrieu, P. and El Karoui, N. (2005). Inf-convolution of risk measures and optimal risk transfer. Finance and Stochastics, 9, 269–298.
Boonen, T. J., Liu, F. and Wang, R. (2021). Competitive equilibria in a comonotone market. Economic Theory, 72(4), 1217–1255.
Borch, K. (1962). Equilibrium in a reinsurance market. Econometrica, 30(3), 424–444.
Carlier, G., Dana, R.-A. and Galichon, A. (2012). Pareto efficiency for the concave order and multivariate comonotonicity. Journal of Economic Theory, 147, 207–229.
Cai, J., Liu, H. and Wang, R. (2017). Pareto-optimal reinsurance arrangements under general model settings. *Insurance: Mathematics and Economics, 77*, 24–37.

Chen, A., Hieber, P. and Klein, J. K. (2019). Tonuity: A novel individual-oriented retirement plan. *ASTIN Bulletin, 49*(1), 5–30.

Chiu, J. and Koeppel, T. V. (2017). The economics of cryptocurrencies – Bitcoin and beyond. *SSRN*: 3048124.

Delage, E. and Li, Y. (2018). Minimizing risk exposure when the choice of a risk measure is ambiguous. *Management Science, 64*(1), 327–344.

Denneberg, D. (1994). *Non-additive Measure and Integral*. Springer Science & Business Media.

Denuit, M. (2019). Size-biased transform and conditional mean risk sharing, with application to P2P insurance and tontines. *ASTIN Bulletin, 49*(3), 591–617.

Denuit, M. and Dhaene, J. (2012). Convex order and comonotonic conditional mean risk sharing. *Insurance: Mathematics and Economics, 51*(2), 265–270.

Denuit, M., Dhaene, J. and Robert, C. Y. (2022a). Risk-sharing rules and their properties, with applications to peer-to-peer insurance. *Journal of Risk and Insurance, 89*(3), 615–667.

Denuit, M., Hieber, P. and Robert, C. Y. (2022b). Mortality credits within large survivor funds. *ASTIN Bulletin, 52*(3), 813–834.

Denuit, M. and Robert, C. Y. (2020). Risk reduction by conditional mean risk sharing with application to collaborative insurance. Available at [https://ideas.repec.org/p/aiz/louvad/2020024.html](https://ideas.repec.org/p/aiz/louvad/2020024.html).

Deprez, O. and Gerber, H. U. (1985). On convex principles of premium calculation. *Insurance: Mathematics and Economics, 4*(3), 179–189.

Eisner, T., Farkas, B., Haase, M. and Nagel, R. (2015). *Operator Theoretic Aspects of Ergodic Theory*. Springer, Heidelberg.

Embrechts, P., Liu, H. and Wang, R. (2018). Quantile-based risk sharing. *Operations Research, 66*(4), 936–949.

Embrechts, P., Liu, H., Mao, T. and Wang, R. (2020). Quantile-based risk sharing with heterogeneous beliefs. *Mathematical Programming Series B, 181*(2), 319–347.

Eyal, I. and Sirer, E. G. (2018). Majority is not enough: Bitcoin mining is vulnerable. *Communications of the ACM, 61*(7), 95–102.

Feng, R., Liu, M. and Zhang, N. (2022). A unified theory of decentralized insurance. *SSRN*: 4013729.

Filipović, D., Kupper, M. and Vogelpoth, N. (2012). Approaches to conditional risk. *SIAM Journal on Financial Mathematics, 3*(1), 402–432.

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics, 6*(4), 429–447.

Föllmer, H. and Schied, A. (2016). *Stochastic Finance. An Introduction in Discrete Time*. Fourth Edition. Walter de Gruyter, Berlin.

Friedman, R. Z. (2020). *Probable Justice: Risk, Insurance, and the Welfare State*. University of Chicago Press, Chicago.

Gilboa, I. (2009). *Theory of Decision under Uncertainty*. Cambridge University Press, Cambridge.
Gilboa, I., Maccheroni, F., Marinacci, M. and Schmeidler, D. (2010). Objective and subjective rationality in a multiple prior model. *Econometrica, 78*(2), 755–770.

Gilboa, I., Postlewaite, A., Samuelson, L., and Schmeidler, D. (2019). What are axiomatizations good for? *Theory and Decision, 86*(3), 339–359.

Heras, A. J., Pradier, P. C. and Teira, D. (2020). What was fair in actuarial fairness? *History of the Human Sciences, 33*(2), 91–114.

Hieber, P. and Lucas, N. (2022). Modern life-care tontines. *ASTIN Bulletin, 52*(2), 563–589.

Landsberger, M. and Meilijson, I. (1994). Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion. *Annals of Operations Research, 52*(2), 97–106.

Leonard, H. B. (1983). Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy, 91*(3), 461–479.

Leshno, J. D. and Strack, P. (2020). Bitcoin: An axiomatic approach and an impossibility theorem. *American Economic Review: Insights, 2*(3), 269–286.

Maccheroni, F., Marinacci, M. and Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences *Econometrica, 74*(6), 1447–1498.

Meyn, J., Kandziora, M., Albers, S. and Clement, M. (2023). Consequences of platforms’ remuneration models for digital content: initial evidence and a research agenda for streaming services. *Journal of the Academy of Marketing Science, 51*(1), 114–131.

Nakamoto, S. (2008). Bitcoin: A peer-to-peer electronic cash system. *Decentralized Business Review, 21260*.

Pfanzagl, J. (1967). Characterizations of conditional expectations. *Annals of Mathematical Statistics, 38*(2), 415–421.

Rothschild, M. and Stiglitz, J. E. (1970). Increasing risk: I. A definition. *Journal of Economic Theory, 2*(3), 225–243.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios.* Springer, Heidelberg.

Starr, R. M. (2011). *General Equilibrium Theory: An Introduction.* Second Edition, Cambridge University Press.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica, 57*(3), 571–587.

Wakker, P. P. (2010). *Prospect Theory: For Risk and Ambiguity.* Cambridge University Press.

Wang, R., Wei, Y. and Willmot, G. E. (2020). Characterization, robustness and aggregation of signed Choquet integrals. *Mathematics of Operations Research, 45*(3), 993–1015.

Wang, R. and Zitikis, R. (2021). An axiomatic foundation for the Expected Shortfall. *Management Science, 67*, 1413–1429.

Xia, J. and Zhou, X. Y. (2016). Arrow–Debreu equilibria for rank-dependent utilities. *Mathematical Finance, 26*(3), 558–588.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica, 55*(1), 95–115.
A Proofs in Section 4

We first prove Theorem 2, as it will be used in the proof of Theorem 1.

Proof of Theorem 2. First, we prove that \( \phi \) is continuous. Suppose that \( X_n \to X \) in \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \).

By using (b) and (c), we have

\[
\phi(X_n) - \phi(X) = \phi(X_n - X) \leq \phi(|X_n - X|).
\]

Similarly,

\[
\phi(X_n) - \phi(X) = \phi(X_n - X) \geq \phi(-|X_n - X|) = -\phi(|X_n - X|).
\]

Hence,

\[
|\phi(X_n) - \phi(X)| \leq |\phi(|X_n - X|)| = \phi(|X_n - X|),
\]

where the last equality is due to \( \phi(|X_n - X|) \geq \phi(0) = 0 \) by (a). Using (e), we have \( \mathbb{E}[\phi(|X_n - X|)] = \mathbb{E}[|X_n - X|] \to 0 \). Therefore, \( \mathbb{E}[|\phi(X_n) - \phi(X)|] \leq \mathbb{E}[\phi(|X_n - X|)] \to 0 \). This means \( \phi(X_n) \to \phi(X) \) in \( L^1(\Omega, \mathcal{G}, \mathbb{P}) \), thus showing the continuity of \( \phi : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to L^1(\Omega, \mathcal{G}, \mathbb{P}) \).

Next, we prove that \( \phi \) is linear. Based on the fact that \( \phi \) is (b) additive and (c) monotone, we have

\[
\phi(X) \geq \phi(0) = 0 \quad \text{for any } X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \text{ and } X \geq 0,
\]

which implies that \( \phi \) is linear (see, e.g., Theorem 1.10 of Aliprantis and Burkinshaw (2006)):

\[
\phi(\alpha X + \beta Y) = \alpha \phi(X) + \beta \phi(Y) \quad \text{for any } \alpha, \beta \in \mathbb{R} \text{ and } X, Y \in L^1(\Omega, \mathcal{F}, \mathbb{P}).
\]

It further follows that \( \phi \) is a positive operator on \( \mathcal{X} \). Recall that a linear operator between two ordered vector spaces is a positive operator if it maps positive elements to positive elements.

Finally, we show that \( \phi \) satisfies the following property

\[
\phi(X) = X \text{ for all } \sigma(S)\text{-measurable } X. \quad (10)
\]

For \( t \in \mathbb{R} \), by (c) we have \( \phi(S \vee t) \geq \phi(S) \vee \phi(t) \), and by (a) and (d) we get \( \phi(S) \vee \phi(t) = S \vee t \). Since \( \phi(S \vee t) \geq S \vee t \) and they have the same mean by (e), we know

\[
\phi(S \vee t) = S \vee t \quad \text{for all } t \in \mathbb{R}. \quad (11)
\]
Write $T_{s,t} = \frac{1}{t-s}(S \vee t - S \vee s)$ for $t > s$. Note that for all $t > s$,

$$\mathbb{1}_{\{S \leq s\}} \leq T_{s,t} \leq \mathbb{1}_{\{S \leq t\}}.$$  \hfill (12)

The linearity of $\phi$ and (11) imply that $\phi(T_{s,t}) = T_{s,t}$. Using the above equality, (c) and (12), we have, for all $t > s$,

$$\phi\left(\mathbb{1}_{\{S \leq s\}}\right) \leq \phi(T_{s,t}) = T_{s,t} \leq \mathbb{1}_{\{S \leq t\}},$$

and

$$\phi\left(\mathbb{1}_{\{S \leq t\}}\right) \geq \phi(T_{s,t}) = T_{s,t} \geq \mathbb{1}_{\{S \leq s\}}.$$  

It follows that for all $\varepsilon > 0$,

$$\mathbb{1}_{\{S \leq t-\varepsilon\}} \leq \phi\left(\mathbb{1}_{\{S \leq t\}}\right) \leq \mathbb{1}_{\{S \leq t+\varepsilon\}}.$$  

Hence, $\mathbb{1}_{\{S < t\}} \leq \phi\left(\mathbb{1}_{\{S \leq t\}}\right) \leq \mathbb{1}_{\{S < t\}}$ for all $t \in \mathbb{R}$. Using (e), we know $\phi\left(\mathbb{1}_{\{S \leq t\}}\right) = \mathbb{1}_{\{S \leq t\}}$. From this equality, using (a) and linearity of $\phi$, it follows that $\phi\left(\mathbb{1}_{\{S > t\}}\right) = \mathbb{1}_{\{S > t\}}$ for all $t \in \mathbb{R}$.

Define the class

$$\mathcal{C} = \{C \in \mathcal{F} : \phi(\mathbb{1}_C) = \mathbb{1}_C\}.$$  

Hence, $\{S \leq t\} \in \mathcal{C}$ for any $t \in \mathbb{R}$. We have $\Omega \in \mathcal{C}$. Using linearity of $\phi$, we have that if $C \in \mathcal{C}$, then

$$\phi(\mathbb{1}_{C^c}) = \phi(1 - \mathbb{1}_C) = \phi(1) - \phi(\mathbb{1}_C) = 1 - \mathbb{1}_C = \mathbb{1}_{C^c},$$

which implies that the complement set $C^c \in \mathcal{C}$. Suppose that $\{C_i\}_{i \geq 1} \subseteq \mathcal{C}$ are disjoint. We proceed to show that $\bigcup_{i=1}^{\infty} C_i \in \mathcal{C}$. Indeed, using monotonicity and additivity of $\phi$, we have

$$\phi\left(\mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}\right) \geq \phi\left(\mathbb{1}_{\bigcup_{i=1}^{m} C_i}\right) = \phi\left(\sum_{i=1}^{m} \mathbb{1}_{C_i}\right) = \sum_{i=1}^{m} \mathbb{1}_{C_i}, \quad \text{for all } m \geq 1.$$  

Letting $m \to \infty$, we have

$$\phi\left(\mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}\right) \geq \sum_{i=1}^{\infty} \mathbb{1}_{C_i} = \mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}.$$  

Based on (e), we have

$$\mathbb{E}\left[\phi\left(\mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}\right)\right] = \mathbb{E}\left[\mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}\right],$$

which implies $\phi\left(\mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}\right) = \mathbb{1}_{\bigcup_{i=1}^{\infty} C_i}$ and $\bigcup_{i=1}^{\infty} C_i \in \mathcal{C}$. Hence, the class $\mathcal{C}$ is a $\sigma$-field and $\sigma(S) \subseteq \mathcal{C}$ based on the monotone class theorem. It follows that $\phi(\mathbb{1}_B) = \mathbb{1}_B$ for all $B \in \mathcal{G}$. Since any $\mathcal{G}$-measurable $X$ can be upper and lower approximated by the summation of simple functions,
using linearity and monotonicity we conclude that \( \phi(X) = X \) for all \( G \)-measurable \( X \).

The conditions that \( \phi \) is continuous, linear and monotone and satisfies (10) guarantee the representation of \( \phi \) (see Proposition 2.6 of Filipović et al. (2012) or Theorem 1 of Pfanzagl (1967)), as

\[
\phi(X) = \mathbb{E}[ZX|S] \quad \text{for all } X \in L^1(\Omega, \mathcal{F}, \mathbb{P}),
\]

for some \( Z \geq 0 \) satisfying \( \mathbb{E}[Z|S] = 1 \). Using (e), we get \( 1 = \mathbb{E}[Z] = \mathbb{E}[\phi(Z)] = \mathbb{E}[\mathbb{E}[Z^2|S]] = \mathbb{E}[Z^2] \). Since \( \mathbb{E}[Z^2] = \mathbb{E}[Z] = 1 \), we know \( Z = 1 \). Hence, we have \( \phi(X) = \mathbb{E}[X|S] \) for \( X \in \mathcal{X} \) and this completes the proof. \( \square \)

**Remark 2.** The key step in the proof of Theorem 2 is to obtain the property (10); \( \phi \) with such a property is sometimes called a projection. Several characterizations of the conditional expectation directly rely on this property; see Pfanzagl (1967) and Eisner et al. (2015). In particular, Theorem 1 of Pfanzagl (1967) holds for subspaces of \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \), and hence our result in Theorem 1 holds for general \( \mathcal{X} = L^q \) where \( q \in [1, \infty] \).

**Proof of Theorem 1.** The “if” statement is checked in Section 4. To show the the “only if” statement, we separate the two cases of \( \mathcal{X} = L^1 \) and \( \mathcal{X} = L^1_+ \).

(i) **The case \( \mathcal{X} = L^1 \).**

Let \( A \) be a risk sharing rule satisfying Axioms AF, RF, RA and OA. Fix any \( S \in \mathcal{X} \). Define the mapping

\[
h^S : \mathcal{X} \to L^1(\Omega, \sigma(S), \mathbb{P}), \quad X \mapsto A^{(X,S-X,0,...,0)}_1.
\]

Note that RA guarantees that \( h^S \) takes values in \( L^1(\Omega, \sigma(S), \mathbb{P}) \). We will verify that \( h^S \) satisfies the following properties on \( \mathcal{X} \):

(a) constant preserving: \( h^S(t) = t \) for all \( t \in \mathbb{R} \);

(b) additivity: \( h^S(X + Y) = h^S(X) + h^S(Y) \) for \( X, Y \in \mathcal{X} \);

(c) monotonicity: \( h^S(Y) \geq h^S(X) \) if \( Y \geq X \);

(d) \( h^S(S) = S \);

(e) \( \mathbb{E}[h^S(X)] = \mathbb{E}[X] \) for \( X \in \mathcal{X} \);

First, (a) follows directly from (1) and the definition of \( h^S \). Next, we proceed to prove (b). By
using Axiom OA, we have, for any $X, Y \in \mathcal{X}$ (note that here we use the fact that $n \geq 3$),

$$h^S(X + Y) = A_1^{(X + Y), S - X - Y, 0, \ldots, 0}$$
$$= A_1^{(X, S - X - Y, Y, 0, \ldots, 0)} + A_3^{(X, S - X - Y, Y, 0, \ldots, 0)}$$
$$= A_1^{(X, S - X, 0, \ldots, 0)} + A_3^{(0, S - Y, Y, 0, \ldots, 0)} = h^S(X) + A_3^{(0, S - Y, Y, 0, \ldots, 0)}$$

(14)

where Axiom OA is used in the second and third equalities. In particular, by choosing $X = 0$ and using the fact that $h^S(0) = 0$ in (a), (14) implies $h^S(Y) = A_3^{(0, S - Y, Y, 0, \ldots, 0)}$. Using this relationship and (14), we further have

$$h^S(X + Y) = h^S(X) + h^S(Y),$$

and hence (b) holds. Next, we show (c). Using Axiom RF, we have $h^S(X - Y) \leq 0$ if $X - Y \leq 0$. Hence, by (b), we obtain (c). Moreover, (d) is implied by the equality $A_1^{(S, 0, \ldots, 0)} = S$ from (2). Finally, (e) follows from Axiom AF.

Using Theorem 2, (a)-(e) imply that $h^S$ admits the representation

$$h^S(X) = \mathbb{E}[X | S] \text{ for all } X \in \mathcal{X}.$$ 

For any $X \in \mathcal{X}^n$, let $S = S^X = \sum_{i=1}^n X_i$. Using Axiom OA and the representation of $h^S$, we have

$$A_1^X = A_1^{(X_1, S - X_1, 0, \ldots, 0)} = h^S(X_1) = \mathbb{E}[X_1 | S] = \mathbb{E}[X_1 | S^X].$$

Similarly, we have $A_j^X = \mathbb{E}[X_j | S^X]$ for any $j = 2, \ldots, n$, which gives that $A$ is CMRS.

(ii) The case $\mathcal{X} = L^1_+$. The gap between the proof in case $\mathcal{X} = L^1$ and this case is that we need the following extension argument.

Let $A$ be a risk sharing rule satisfying Axioms AF, RF, RA and OA. Fix any $S \in \mathcal{X} = L^1_+$. Denote by $B_S = \{X \in L^1_+: X \leq S\}$, which is the set of random variables between 0 and $S$. Define the mapping as in the proof of case (i),

$$h^S : B_S \to L^1(\Omega, \sigma(S), \mathbb{P}), \ X \mapsto A_1^{(X, S - X, 0, \ldots, 0)}.$$ 

It is clear that $h^S$ is well-defined on $B_S$ and satisfies additivity

$$h^S(X + Y) = h^S(X) + h^S(Y) \text{ for } X, Y, X + Y \in B_S,$$
which can be checked by the same argument as in the proof of Theorem 1. Define \( C_S = \{\lambda X : \lambda \in \mathbb{R}_+, \ X \in B_S\} \) which is the cone generated by \( B_S \), and \( L_S = \{\lambda X : \lambda \in \mathbb{R}, \ X \in B_S\} \) which is the linear space generated by \( B_S \). According to Lemma 1 below, \( h^S \) can be uniquely extended on \( C_S \) and \( L_S \) and it is linear on \( L_S \). This allows us to use the same arguments in (i) to get

\[
h^S(X) = E[X|S], \quad X \in L_S,
\]

and following the rest of the steps for the proof of Theorem 1 yields that \( A \) is CMRS.

**Lemma 1.** Fix \( S \in L^1_+ \). Any additive function \( \phi : B_S \to L^1_+ \) has a unique additive extension on \( C_S \) and a unique linear extension on \( L_S \).

**Proof.** For \( X \in C_S \), denote by \( \gamma_X = \sup\{\gamma \in [0,1] : \gamma X \in B_S\} \). Note that there exists \( \lambda_X \in \mathbb{R}_+ \) and \( Y \in B_S \) such that \( \lambda_X Y = X \), and hence \( \gamma_X \geq 1/\lambda_X > 0 \). Moreover, we have \( \gamma_X X \in B_S \) since \( B_S \) is closed. Define \( \hat{\phi}(X) = \phi(\gamma_X X)/\gamma_X \) for \( X \in C_S \). It is clear that \( \hat{\phi} = \phi \) on \( B_S \) because \( \lambda_X = 1 \) for all \( X \in B_S \). We next verify that \( \hat{\phi} \) is additive.

Take \( m, k \in \mathbb{N} \) such that \( m \leq k \). By additivity of \( \phi \) on \( B_S \), we have \( \phi(mX/k) = m\phi(X/k) \) for \( X \in B_S \). By taking \( m = 1 \), we get \( \phi(X/k) = \phi(X)/k \), which in turn gives \( \phi(mX/k) = m\phi(X)/k \). Since \( X \) is non-negative, positivity (monotonicity) of \( \phi \) further gives \( \phi(\lambda X) = \lambda \phi(X) \) for any real number \( \lambda \in [0,1] \).

For any \( X, Z \in C_S \) such that \( Z \geq X \), since \( \gamma_Z \leq \gamma_X \), we obtain, by choosing \( \lambda = \gamma_Z/\gamma_X \),

\[
\phi(\gamma_Z X) = \phi(\lambda \gamma_X X) = \lambda \phi(\gamma_X X) = \frac{\gamma_Z}{\gamma_X} \phi(\gamma_X X).
\]

Take any \( X, Y \in C_S \) and write \( Z = X + Y \). Using (15) and additivity of \( \phi \) on \( B_S \),

\[
\hat{\phi}(X + Y) = \frac{1}{\gamma_Z} \phi(\gamma_Z(X + Y)) = \frac{1}{\gamma_Z} \phi(\gamma_Z X) + \frac{1}{\gamma_Z} \phi(\gamma_Z Y) = \frac{1}{\gamma_X} \phi(\gamma_X X) + \frac{1}{\gamma_Y} \phi(\gamma_Y Y) = \hat{\phi}(X) + \hat{\phi}(Y).
\]

Therefore, \( \hat{\phi} \) is additive on \( C_S \). The extension is unique because any two additive and monotone functions agreeing on \( B_S \) must agree on \( C_S \). The unique linear extension to \( L_S \) follows from Theorem 1.10 of Aliprantis and Burkinshaw (2006).

**Proof of Proposition 2.** (i) The Q-CMRS rule \( A^{X}_{Q-cm} = E^Q[X|S^X] \) satisfies Axioms RA, RF and OA with the same reasoning as the CMRS. Since \( E[A^{X}_{Q-cm}] = E^Q[X] \), AF does not hold as long as \( Q \neq \mathbb{P} \).
(ii) For the mean-adjusted all-in-one risk sharing rule

\[ A_{ma}^X = (S^X - \mathbb{E}[S^X], 0, \ldots, 0) + \mathbb{E}[X], \]

it is clear that Axioms RA and AF hold by definition. Axiom OA holds because the allocation to agent \( i \in [n] \) is determined only by \((X_i, S)\). Axiom RF does not hold because the allocation to agent 1 is not a constant if \( S^X \) is not a constant, regardless of whether \( X_1 \) is a constant, violating (1).

(iii) For the identity risk sharing rule \( A_{id}^X = X \), it is clear that Axioms AF, RF and OA hold. Axiom RA does not hold because \( A_{id}^X \) is not necessarily a function of \( S^X \).

(iv) Consider a combination of \( A_{all} \) and \( A_{cm} \), defined by \( A_X = A_{all}^X = (S^X, 0, \ldots, 0) \) if \( X \) is standard Gaussian, and \( A_X = A_{cm}^X = \mathbb{E}[X|S^X] \) otherwise. Axioms AF, RF and RA and be checked separately for \( A_{all} \) and \( A_{cm} \), by noting that RF only needs to be checked for \( A_{cm} \) since the standard Gaussian \( X \) is not included in the statement of RF.

To verify that OA does not hold, it suffices to consider \( n = 3 \). Let \( X = (X_1, X_2, X_3) \) follow a standard Gaussian distribution. By definition, \( A_1^X = S^X \). However, for \( Y = (X_1, X_2 + X_3, 0) \), we have \( A_Y^1 = \mathbb{E}[X_1|S^X] = S^X/3 \neq A_1^X \), thus violating OA.

\[ \Box \]

B Proofs in Section 5

Proof of Proposition 3. Since \( X \leq_{cx} Y \) implies \( X \leq \sup X \leq \sup Y \) and \( \mathbb{E}[X] = \mathbb{E}[Y] \), UI implies both RF and AF. Property CP follows from AF and RF as discussed in (1).

Proof of Proposition 4. We only show the case that \( \mathcal{X} = L^1 \), as the case \( \mathcal{X} = L^1_+ \) is analogous. Let \( A : \mathcal{X}^n \to \mathcal{X}^n \) be a risk sharing rule satisfying Axioms AF, RF and OA. Fix \( S \in \mathcal{X} \). For any \( X \in L^1(\Omega, \sigma(S), \mathbb{P}) \), \( A_1^{(X,S-X,0,\ldots,0)} \) is \( \sigma(S) \)-measurable, because it is \( \sigma(X,S-X) \)-measurable by assumption and \( \sigma(S) = \sigma(X,S-X) \). Define the mapping

\[ h^S : L^1(\Omega, \sigma(S), \mathbb{P}) \to L^1(\Omega, \sigma(S), \mathbb{P}), \quad X \mapsto A_1^{(X,S-X,0,\ldots,0)}, \]

where we use the fact that \( h^S(X) \) is \( \sigma(S) \)-measurable for \( X \in L^1(\Omega, \sigma(S), \mathbb{P}) \). The arguments in the proof of Theorem 1 yield that \( h^S \) satisfies the conditions in Theorem 2. By Theorem 2, \( h^S \) is the identity on \( L^1(\Omega, \sigma(S), \mathbb{P}) \). Using this and OA, for any \( X \in \mathcal{A}_n(S) \) and \( X_1 \in L^1(\Omega, \sigma(S), \mathbb{P}) \), we
have
\[ A_1^X = A_1^{(X_1,S-X_1,0,\ldots,0)} = h^S(X_1) = X_1. \]
The other case of \( A_j^X \) for \( j \in [n] \) are similar. \( \square \)

**Proof of Corollary 1.** The proof follows directly from Theorem 1, Proposition 3, and the fact that CMRS satisfies Property UI. \( \square \)

**Proof of Proposition 5.** Fix a non-constant \( S \in \mathcal{X} \), and write \( h^S : \mathcal{X} \to \mathcal{X}, \ X \mapsto A_1^{(X,S-X,0,\ldots,0)} \).
By ZP, we have \( h^S(S) = S \). Using additivity (14) guaranteed by OA in the proof of Theorem 1, we have \( h^S(2S) = 2h^S(S) = 2S \). This and additivity give \( h^S(-S) = -S \), and therefore, \( A^{(-S,2S,0,\ldots,0)} = (-S,2S,0,\ldots,0) \) is not comonotonic. \( \square \)

**Proof of Proposition 6.** By Proposition 1, OA implies that \( A_i^X \) is determined by \((X_i,S^X)\) and \( i \in [n] \). It suffices to show that \( i \in [n] \) is also not relevant. Using ZP and OA, we have, for the pair \((1,3)\) and any \( X,S \in \mathcal{X} \),
\[
A_1^{(X,S-X,0,\ldots,0)} = S - A_2^{(X,S-X,0,\ldots,0)} = S - A_2^{(0,S-X,X,0,\ldots,0)} = A_3^{(0,S-X,X,0,\ldots,0)}.
\]
The other pairs \((i,j)\) are similar. Therefore, \( A_i^X \) is determined by \((X_i,S^X)\) regardless of \( i \in [n] \), showing that SM holds. \( \square \)

**C Proofs in Section 6**

**Proof of Theorem 3.** In Section 4, we used properties of the conditional expectation to check Axioms AF, RF and OA for CMRS, and the same properties hold for the generalized CMRS. Properties IA and IB are straightforward from basic properties of \( \mathbb{E}[X|\mathcal{G}^X] \). Therefore, the “if” statement holds.

We proceed to prove the “only if” statement. We will only show the case \( \mathcal{X} = L^1 \). The case \( \mathcal{X} = L^1_+ \) is similar as we see in the proof of Theorem 1. Let \( \hat{A} \) be a generalized risk sharing rule satisfying Axioms AF, RF and OA and Properties IA and IB. Fix any \( S \in \mathcal{X} \). Define the mapping
\[
h^S|\mathcal{G} : \mathcal{X} \to L^1(\Omega, \mathcal{G}^S, \mathbb{P}), \ X \mapsto \hat{A}_1^{(X,S-X,0,\ldots,0)|\mathcal{G}},
\]
where \( \mathcal{G}^S = \sigma(S, \mathcal{G}) \). Note that IA guarantees that \( h^S|\mathcal{G} \) takes values in \( L^1(\Omega, \mathcal{G}^S, \mathbb{P}) \). We will verify that \( h^S|\mathcal{G} \) satisfies the following properties on \( \mathcal{X} \):

(a) constant preserving: \( h^S|\mathcal{G}(t) = t \) for all \( t \in \mathbb{R} \);
(b) additivity: \( h^{S|G}(X + Y) = h^{S|G}(X) + h^{S|G}(Y) \) for \( X, Y \in \mathcal{X} \);

(c) monotonicity: \( h^{S|G}(Y) \geq h^{S|G}(X) \) if \( Y \geq X \);

(d) \( h^{S|G}(X) = X \) for any \( X \) that is \( G^S \)-measurable;

(e) \( \mathbb{E}[h^{S|G}(X)] = \mathbb{E}[X] \) for \( X \in \mathcal{X} \).

The properties (a)-(c) and (e) can be shown analogously to the proof of Theorem 1. The property (d) is implied by Property IB. Note that (d) corresponds to (10) in the proof of Theorem 2. Using the same argument there, we obtain that (a)-(e) imply that \( h^{S|G} \) admits the representation

\[
 h^{S|G}(X) = \mathbb{E} \left[ X | G^S \right] \text{ for all } X \in \mathcal{X}.
\]

For any \( X \in \mathcal{X}^n \), let \( S = S^X = \sum_{i=1}^n X_i \). Using Axiom OA and the representation of \( h^{S|G} \), we have

\[
 \hat{A}_1^X = \hat{A}_1^{[X_1,S-X_1,0,\ldots,0]} = h^{S|G}(X_1) = \mathbb{E} \left[ X_1 | G^X \right].
\]

Similarly, we have \( A_j^X = \mathbb{E}[X_j|G^X] \) for any \( j = 2, \ldots, n \), which completes the proof. \( \square \)

D Proofs in Section 7

Proof of Proposition 7. We have seen that CMRS satisfies the four axioms for mappings on general spaces, and hence also on \( B_n \), and it clearly satisfies the definition of a reward sharing rule. Below we show that the four axioms are sufficient for CMRS. Fix \( D \subseteq \Omega \) independent of \( P \) and denote by

\[
 I_D = \{ C \subseteq D : C \text{ is independent of } P \} \text{ and } M_D = \{ P 1_C : C \in I_D \}.
\]

Define the mapping as in the proof of Theorem 1,

\[
 h^D : M_D \to L^1(\Omega, \sigma(P 1_D), P), \quad P 1_C \mapsto A_1^{[P 1_C, P 1_D - 1_C, 0, \ldots, 0]}.
\]

We can check that \( h^D \) satisfies additivity

\[
 h^D(X + Y) = h^D(X) + h^D(Y) \text{ for } X, Y \in M_D \text{ with } X + Y \in M_D,
\]

and monotonicity

\[
 h^D(X) \leq h^D(Y) \text{ for } X, Y \in M_D \text{ with } X \leq Y;
\]
these statements can be shown by arguments using AF, RF and OA as in the proof of Theorem 1. For \( m \in \mathbb{N} \), take \( C_1, \ldots, C_m \in I_D \) such that \( \mathbb{P}(C_1) = \cdots = \mathbb{P}(C_m) \) and \( \bigcup_{j=1}^{m} C_j = D \). Using \( h^D(P\mathbb{1}_D) = P\mathbb{1}_D \) (guaranteed by AF and RF) and \( h^D(P\mathbb{1}_{C_1}) = h^D(P\mathbb{1}_{C_j}) \) for \( j \in [m] \) (by the definition of a reward sharing rule), we get from additivity of \( h^D \) that

\[
h^D(P\mathbb{1}_{C_1}) = \frac{P\mathbb{1}_D}{m} = P\mathbb{1}_D \frac{\mathbb{P}(C_1)}{\mathbb{P}(D)}.
\]

Since \( C_1 \) is arbitrary, we get that, for any \( C \in I_D \) with \( \mathbb{P}(C) = \mathbb{P}(D)/m \),

\[
h^D(P\mathbb{1}_C) = P\mathbb{1}_D \frac{\mathbb{P}(C)}{\mathbb{P}(D)} = P\mathbb{1}_D \mathbb{E}[\mathbb{1}_C|P\mathbb{1}_D] = \mathbb{E}[P\mathbb{1}_C|P\mathbb{1}_D]. \quad (16)
\]

Using additivity again, we know that (16) holds for any \( j \in [m] \) and \( C \in I_D \) with \( \mathbb{P}(C) = j\mathbb{P}(D)/m \), and finally, by monotonicity of \( h^D \), we get (16) for all \( C \in I_D \). Following the rest of the steps for the proof of Theorem 1 yields that \( A \) is CMRS on \( \mathcal{B}_n \). \qed