Field redefinitions and massive BF models in arbitrary space-time dimensions

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Abstract

We show that the topological massive BF theories can be written as a pure BF term through field redefinitions. The fields are rewritten as power expansion series in the inverse of the mass parameter $m$. We also give a cohomological justification of this expansion through BRST framework. In this approach the BF term can be seen as a topological generator for massive BF theories.

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1 Introduction

After the work of Deser, Jackiw and Templeton [1], topological massive theories have been object of continuous source of investigation from both mathematical and physical point of views. Such theories represent an alternative

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to give mass to gauge fields without the Higgs mechanism. In three dimensions they can be formulated with the Chern-Simons term in an abelian and non-abelian version [1]. Its generalization to any dimension is possible via the topological BF term, where B is a \((D - 2)\)-form gauge field and F is the field strength of the usual vector gauge field [2, 3, 4]. It is also worthwhile to mention that other topological terms have been used to construct topological massive theories [5, 6].

A lot of work has been done in order to have a better understanding of topological massive theories, leading to interesting and promising results. An important result was obtained a few years ago by Giavarini and collaborators about the finiteness of topological massive Yang-Mills theory in three dimensions, with a careful analysis of higher loops of the Feynman integrals [7]. More recently, Lemes et al [8, 9], showed that the topological three dimensional massive Yang-Mills theory can be cast in the form of a pure Chern-Simons action through a non-linear but local field redefinition,

\[
S_{YM}(A) + S_{CS}(A) = S_{CS}(\hat{A}),
\]

where

\[
S_{YM} = \frac{1}{4m} \text{Tr} \int d^3x \, F_{\mu\nu}F^{\mu\nu},
\]

\[
S_{CS}(A) = \frac{1}{2} \int d^3x \, \varepsilon^{\mu\nu\rho} \left( A_{\mu} \partial_\nu A_\rho + \frac{2}{3} g A_\mu A_\nu A_\rho \right),
\]

and

\[
\hat{A}_\mu = A_\mu + \sum_{n=1}^{\infty} \frac{1}{m^n} \vartheta^n_\mu(D, F).
\]

As shown by the authors [9], the coefficients \(\vartheta^n_\mu(D, F)\) are local and covariant and depend only on the field strength \(F_{\mu\nu}\) and the covariant derivative \(D_\mu = \partial_\mu + g[A_\mu, \cdot]\). This property, also valid for the abelian Chern-Simons theory, was used in bosonization in three dimensions [10].

In four dimensions, the contribution from one loop diagrams to the effective action in theories with four fermions reproduces the topological massive BF theory [11]. The topological massive BF theory also appears in D-dimensional bosonization of massive Thirring model [12].

The BF term is not important only in the construction of topological massive theories in any dimension, but also in other branches of physics. They are related to the Ray-Singer torsion [13] and the intersection number.
of manifolds in any dimension [14]. Also they provide an example of ultra-violet finite field theories [15]. A few years ago a generalization of anyons to (3+1) dimensions, making use of the BF term, was proposed in [16]. More recently, Smailagic and Spallucci, have studied the dualization of abelian [17] and nonabelian [18] BF models of arbitrary $p-$forms to a Stueckelberg-like massive gauge invariant theories.

The aim of this work is to show that the topological massive BF theories in $D$-dimensions can also be written as a pure BF term through field redefinitions. This feature is not exclusive to the topological three dimensional massive Yang-Mills theory, but seems to be a property of all topological massive theories in any dimension.

The paper is organized as follows. In section 2 we analyse the field redefinitions in the abelian topological BF model, where a cohomological justification through the BRST framework is established. The section 3 is devoted to a brief discussion about the field redefinitions of the non-abelian topological massive BF model. Further possible applications will be discussed.

## 2 The abelian case

Let us consider the abelian massive BF model in $D$-dimensions, described by a real-valued $(D-2)$–form field $B$ and a real-valued 1–form field $A$, both with canonical dimension $(D-2)/2$, defined in a $D$-dimensional space-time manifold $\mathcal{M}_D$ with metric $g_{\mu\nu} = \text{diag}(-++\cdots++)$, with the action given by

$$S_0 = \int_{\mathcal{M}_D} \left( \frac{1}{2} H \wedge H + mB \wedge F - \frac{1}{2} F \wedge^* F \right), \quad (2.1)$$

where $F = dA$ and $H = dB$ are the field strengths of $A$ and $B$ respectively, $m$ is a mass parameter, $d = dx^\mu(\partial/\partial x^\mu)$ is the exterior derivative and $*$ is the Hodge star operator. The adjoint operator acting in a $p-$form is defined as $d^\dagger = (-1)^{Dp+D} * ds$ [13].

The action (2.1) is invariant under the gauge transformations

$$\delta A = d\theta, \quad \delta B = d\Omega, \quad (2.2)$$

-We shall use the form representation of the fields in order to simplify our analysis.
where \( \theta \) is a 0-form and \( \Omega \) is a \((D - 3)\)-form.

We will show that the action (2.1) is indeed a pure BF term through a local and linear field redefinition of the fields \( A \) and \( B \), namely

\[
S_0 = \int_{M_D} \left( \frac{1}{2} H \wedge *H + mB \wedge F - \frac{1}{2} F \wedge *F \right) = \int_{M_D} m \hat{B} \wedge \hat{F}, \tag{2.3}
\]

where \( \hat{F} = d\hat{A} \), and

\[
\hat{B} = B + \sum_{i=1}^{\infty} \frac{B_i}{m^i}, \tag{2.4}
\]
\[
\hat{A} = A + \sum_{i=1}^{\infty} \frac{A_i}{m^i}. \tag{2.5}
\]

The terms \( B_i \) and \( A_i \) are \((D - 2)\)-form and 1-form respectively, constructed only with the field strength \( H \) and \( F \), the Hodge operator and the exterior derivative. The power in the inverse of mass parameter \( m \) in (2.4) and (2.5) gives the correct mass dimension for \( \hat{B} \) and \( \hat{A} \).

The properties given by equations (2.3), (2.4) and (2.5) seem to be fundamental not only for the Chern-Simons theory but for all topological massive theories.

In order to provide an explicit form for the terms \( B_i \) and \( A_i \), we insert the redefined field given by eqs. (2.4) and (2.5) into the eq.(2.3) and identify the terms with the same power in \( 1/m \). We find the following expressions,

\[
B_1 = -\frac{1}{2} *F, \quad A_1 = \frac{1}{2} (\cdot)^{-1} D^{-1} *H, \tag{2.6}
\]

and the integral consistency condition

\[
\int_{M_D} \left( B_{k+1} \wedge dA + B \wedge dA_{k+1} + \sum_{i=1}^{k} B_i dA_{k+i-1} \right) = 0, \quad \text{for } k \geq 1. \tag{2.7}
\]

Contrary to the Chern-Simons case in three dimensions [8, 9], we have many solutions to eq. (2.7). This is due to the fact that we start with two fields, \( A \) and \( B \), in the original theory. We give one solution of (2.7) explicitly

\[
A_{2j} = 0, \tag{2.8}
\]
The term $A_1$ in $D = 4$, which corresponds to $j = 0$ in eq.(2.9), was used in [20] to couple fermions to Kalb-Ramond field. As shown by the authors, this coupling leads an anomalous axial current.

Let us underline that the equation (2.3) has to be understood here in pure classical terms. The presence of the expansion parameter $1/m$ in Eqs.(2.4) and (2.5), will introduce an infinite number of power counting nonrenormalizable interactions and a more careful quantum analysis should be done. In the present work we are interested only classical aspects of the field redefinitions.

As in the Chern-Simons case [8, 9], there is an analog cohomological justification to eq. (2.3). This justification is based on [8, 9] and is quite formal. Let $\tilde{B}$ and $\tilde{A}$ be the anti-fields of $B$ and $A$. Obviously we need a pyramid of ghosts to take into account of the reducibility of the gauge transformation of $B$, but not necessary for our purposes. Since we are interested only in the classical aspects of (2.3), there is no necessity of a gauge fixing term. The dependence of $\tilde{B}$ and $\tilde{A}$ is given by [21, 22]

$$ S(\tilde{A}, \tilde{B}) = \int_{M_D} \tilde{A} \wedge *dc + \tilde{B} \wedge *d\eta, \quad (2.12) $$

where $c$ and $\eta$ are the ghosts for the gauge transformations (2.2), namely

$$ sA = dc, \quad (2.13) $$
$$ sB = d\eta. \quad (2.14) $$

Following the standard procedure [21, 22], the BRST transformation of the anti-fields $\tilde{B}$ and $\tilde{A}$ are

$$ s\tilde{B} = d^\dagger H - m *F, \quad (2.15) $$
$$ s\tilde{A} = -d^\dagger F + (-1)^D m *H. \quad (2.16) $$

The equations (2.15) and (2.16) can be writing in a suitable form

$$ H = \frac{1}{m^2} (-1)^D dd^\dagger H + \frac{1}{m} \left( \frac{1}{m} (-1)^D d\tilde{B} + *\tilde{A} \right), \quad (2.17) $$
$$ F = \frac{1}{m^2} dd^\dagger F - \frac{1}{m} \left( \frac{1}{m} d\tilde{A} + (-1)^D *\tilde{B} \right). \quad (2.18) $$
The equations (2.17) and (2.18) are recursive formulas for $H$ and $F$ respectively. This allows us to write $H$ and $F$ as a BRST trivial, namely

$$H = s \left( \sum_{i=1}^{\infty} \frac{H_i}{m^i} \right),$$

(2.19)

$$F = s \left( \sum_{i=1}^{\infty} \frac{F_i}{m^i} \right).$$

(2.20)

This property enables us to express the kinetic terms of (2.1) as a pure BRST variation. It is well known, from the BRST algebraic framework, that terms of the action which are BRST trivial correspond to field redefinitions. Therefore, the expressions (2.3), (2.4) and (2.5) are consequences of the BRST triviality of $F$ and $H$. Let us underline that the triviality of $F$ and $H$ is due to the presence of the BF term in the action (2.1). As we can see from (2.13) and (2.14), the absence of the BF term spoils the possibility of writing $F$ and $H$ as a recursive formula.

3 The non-abelian case

In this section we consider the topologically massive non-abelian BF model in D-dimensions. We will use the same convention of [4]. The gauge fields are written as $A = A^a T^a$, $B = B^a T^a$, where $T^a$ are generators of a Lie algebra $G$ of a semi-simple Lie group $G$. The action reads

$$S = \text{Tr} \int_{\mathcal{M}_D} \left( \frac{1}{2} H \wedge^* H + m B \wedge F - \frac{1}{2} F \wedge^* F \right),$$

(3.1)

where $F = dA + A \wedge A$ and $H = DB + [F, V]$. $D = d + [A, \cdot]$ is the covariant derivative and $V = V^a T^a$ is an auxiliary $(D - 3)$-form Lie algebra valued, with canonical dimension $(D - 4)/2$. This auxiliary field is necessary in order to implement the gauge invariance of the model and to eliminate a constraint that appears in the equations of motion [4]. The action (3.1) is invariant under the gauge transformations\(^2\)

$$\delta B = D \Omega + [B, \theta],$$

(3.2)

$$\delta A = D \theta,$$

(3.3)

$$\delta V = - \Omega + [V, \theta].$$

(3.4)

\(^2\)The commutator between two Lie algebra valued forms $P$ and $Q$ is defined as $[P, Q] = g (P \wedge Q - (-1)^{d(P)d(Q)} Q \wedge P)$, where $d(X)$ is the form degree of $X$ and $g$ is a parameter with mass dimension $(4 - D)/2$. 

6
We shall show that the action \((3.1)\) can be written as a pure BF term, namely

\[
S = \operatorname{Tr} \int_{M_D} \left( \frac{1}{2} H \wedge *H + mB \wedge F - \frac{1}{2} F \wedge *F \right) = \operatorname{Tr} \int_{M_D} m\hat{B} \wedge \hat{F},
\]

(3.5)

where \(\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A}\), and

\[
\hat{B} = B + \sum_{i=1}^{\infty} \frac{1}{m^i} (B_i - [A_i, V]),
\]

(3.6)

\[
\hat{A} = A + \sum_{i=1}^{\infty} \frac{A_i}{m^i}.
\]

(3.7)

The terms \(A_i\) and \(B_i\) are 1-form and \((D - 2)\)-form respectively, constructed with \(H, F\), the covariant derivative \(D\) and the hodge operator. They are non-linear due to the non-abelian character of the model. Proceeding as in the abelian case, we show bellow some terms of the expansions \((3.6)\) and \((3.7)\),

\[
A_1 = \frac{1}{2} (-1)^{D-1} *H,
\]

\[
A_2 = \frac{1}{8} (-1)^{D-2} *[B + DV, *H],
\]

\[
A_3 = -\frac{1}{2} *[B + DV, A_2] - \frac{1}{8} *D*D'H,
\]

(3.8)

\[
B_1 = -\frac{1}{2} *F,
\]

\[
B_2 = \frac{1}{2} *DA_1,
\]

\[
B_3 = \frac{1}{2} *DA_2 + \frac{1}{4} *[A_1, A_1].
\]

Let us underline that the terms \(A_i\) and \(B_i\) transforms covariantly, i.e,

\[
\delta A_i = [A_i, \theta],
\]

(3.9)

\[
\delta B_i = [B_i, \theta].
\]

(3.10)

This property can be easily obtained by the gauge invariance of both sides of \((3.9)\). Note that the field \(V\) do not need to be redefined. This is due to
the fact that $V$ is an auxiliary field. The covariant transformation of $A_i$ and $B_i$ leads to the following gauge transformation for $\hat{A}$ and $\hat{B}$,

$$\delta \hat{B} = \hat{D}\Omega + [\hat{B}, \theta], \quad (3.11)$$

$$\delta \hat{A} = \hat{D}\theta, \quad (3.12)$$

where $\hat{D}$ is the redefined covariant derivative,

$$\hat{D} = d + [\hat{A}, \cdot]. \quad (3.13)$$

Just as in the abelian case, the field redefinition is a consequence of the BRST triviality of the fields strengths $H$ and $F$, due to the presence of the BF term. In order to justify this, we follow the same arguments used for the Chern-Simons case [9]. The full BRST differential $s$, in this case, has non-linear dependence on the fields and anti-fields. We can filter the linear part, say $s_0$, from the complete BRST operator $s$. There is a theorem of the BRST cohomology [21, 22], which states that the cohomology of the full BRST operator $s$ is isomorphic to the cohomology of the corresponding linear operator $s_0$. The linear operator $s_0$ is just the BRST operator of the abelian case discussed in section 2. As proved in section 2, the abelian fields strengths $H$ and $F$ are BRST trivial in a formal power series expansion in $1/m$. Hence, making the use of the above theorem, the non-abelian $H$ and $F$ can be written as a pure $s$ variation in a formal power series expansion in the parameter $1/m$.

4 Conclusion

In this paper we have shown that both abelian and non-abelian topological massive BF theories can be written as a pure BF term through field redefinitions. This result, valid for any dimension, represents a generalization of the results obtained in [3, 9] for topological three dimensional massive Yang-Mills theory. Let us underline that this property seems to be valid for all topological massive theories in any dimension. In this framework the BF term can be seen as a topological generator for massive BF theories.

Let us comment about a possible application of our result. As showed in [10] the three dimensional abelian fermionic determinant is a pure Chern-Simons term through local and non-local field redefinitions. This suggest that abelian fermionic determinant in $D$ dimension could be written in a
suitable way as a pure BF term through field redefinitions. Also our result can be applied to the study of the finiteness of the topologically massive non-abelian BF models.

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