Infrared Effects and the Soft Photon Theorem in Massive QED

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Abstract

Stuckelberg QED with massive photon is known to be renormalizable. But the limit of the mass going to zero is interesting because it brings the resolution to infrared questions through the role of Stuckelberg field at null infinity in addition to providing new asymptotic symmetries. Such symmetries facilitate the soft photon theorems also.

1 Introduction

Asymptotic symmetries depend on the exact masslessness of the photon. But the current bound on the Compton wavelength of photon is that of solar system [1, 2]. At no stage we may be able to conclude operationally exact masslessness of the photon. Given this scenario it is more appropriate to study the massive photon QED theory. But gauge invariance is crucial for studying interactions. Such a massive QED with gauge invariance is Stuckelberg theory. This can be considered as a limit of Higgs model where phase of the Higgs alone survives. In this model the infrared questions can also be regulated. The extra degree of freedom play an important role providing sky photons. We study the soft photon theorem which is another manifestation of the asymptotic symmetry and its consequences.
The soft photon and soft graviton theorems, described by Weinberg [3, 4] are rules for the modification of transition rates when an investigation is carried out to deal with the infrared divergences which arise due to the exact masslessness of these particles. Referring to the soft photon theorem in particular, infrared divergences manifest themselves in the virtual transmission of very soft photons between external particle legs. By taking into account all such communications and then proceeding to include the effects of all possible real soft photon emissions, while maintaining a cut off that corresponds to some energy resolution, a cancellation of the infrared cutoff is achieved, thereby supplying the soft photon theorem. The actual statement of the theorem is as follows. If one considers a reaction $\alpha \rightarrow \beta$ involving photons, the reaction rate has to be modified to account for the possible emissions of soft photons having total energy less than or equal to a minimum energy resolution $\Delta E$. The modification takes the form:

$$\Gamma_{\beta\alpha} = \left(\frac{\Delta E}{\Lambda}\right)^A b(A)\Gamma^0_{\beta\alpha}$$ (1.1)

Here, $\Gamma^0$ refers to the bare reaction rate which does not refer to any soft photons. $\Lambda$ is a renormalization scale. The functions $A$ and $b(x)$ are defined in the book of Weinberg [3] Eqs.2.16 and 2.50 respectively.

While attempting the calculation of a Feynman diagram that is characterised by such divergences (for example, the electron self energy contribution or the vertex), the problem can be temporarily circumvented by appealing to a small mass parameter $m_\gamma$. This parameter serves as a convenient regulator of the integrals, which despite its appeal, violates gauge invariance. Hence, an approach which offers both the technical ease of a photon mass which nonetheless respects gauge invariance would be extremely desirable.

These qualitative points being made, it may be noted that originally, the infrared divergences were regarded as essentially unphysical, which ultimately had no bearing on the final answers, so long as one took into account the possibility of soft photon emissions, keeping in mind the right physical picture. In recent years however, there has been a resurgence of interest in soft photon theorems and their relationship to asymptotic symmetries and the Ward identities, described in Strominger[5, 6, 7, 8]. Following there are several interesting questions that have been posed in understanding BMS symmetries in gravity[9], effective action in QFT’s and gauge theories including non abelian gauge theories. Balachandran et al[10] have pointed out
the exact massless photon QED may violate Lorentz symmetry. Similar analysis also leads to color symmetry being broken [11]. To note one point of interest, the classical Lienard-Wiechert potential displays a discontinuity at infinity, which when investigated, reveals a family of conserved quantities in QED.

Having noted these aspects of the infrared regime of QED, it may be interesting to consider what we obtain when we introduce a very tiny photon mass in a gauge invariant fashion and proceed to probe the infrared regime of the theory. To do this, recourse may be had to the Stuckelberg mechanism.

The Stuckelberg mechanism, simply put, is an abelian Higgs mechanism. It involves the introduction of an auxiliary scalar field, supplying the photon a small mass $m_\gamma$, which arises as a coupling constant between the photon and the Stuckelberg particle. Once we introduce this mechanism, we will study the analogue of the soft photon theorems in this theory. It makes sense then, to speak of the infrared regime since the photon mass, of $\lesssim 10^{-18}$eV constrained by experiments [1, 12, 13], may be treated as extremely small. The actual experimental inaccuracy may in fact be much higher than this limit.

The goal of this paper is twofold. Firstly, we will investigate the soft photon theorem for a massive photon. In doing so, naturally, the infrared cutoff will be taken as the photon mass, now suitably regulated in a gauge invariant fashion. When real and virtual transitions are taken into account and the limit $m_\gamma \to 0$ is attempted, a divergence is encountered, the vanishing of which will be tied up with charge conservation. We unambiguously obtain the original result then, this time with no reference to an arbitrary minimum wavelength, the theory now being effectively regulated by the photon mass parameter. In principle $O(m^2)$ corrections would be there which will be negligible.

Secondly, the classical Lienard-Wiechert potentials for the massive photon field is investigated, with the aim of recovering the correct antipodal matching points in a suitable limit. In doing so, we obtain another regulating parameter, this time, encoding the relationship between the bound of the photon mass and the limit of the radius going to infinity. Conditions on the photon mass are obtained once the antipodal matching condition is identified and the implications of this on the regulating parameter are investigated.

We briefly introduce the Stuckelberg model in Sec 2. In Sec 3, we explore infrared divergences and soft theorems, In Sec 4, we consider modified Lienard Wiechert potential relevant for our purpose and consider asymptotic
limits. In Sec.5 we present our conclusions taking into account several developments in the recent revisit of this question and some details of future works and publications.

2 Review of Stuckelberg theory

Stuckelberg theory is defined by a modified Maxwell/Proca lagrangian given by:

$$\mathcal{L}_S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m_\gamma^2}{2} \left(A_\mu + \frac{1}{m_\gamma} \partial_\mu S\right) \left(A^\mu + \frac{1}{m_\gamma} \partial^\mu S\right). \quad (2.1)$$

The lagrangian is invariant under the gauge transformations:

$$A_\mu \rightarrow A_\mu - \partial_\mu \lambda, \quad S \rightarrow S - m_\gamma \lambda \quad (2.2)$$

where $\lambda$ is the gauge parameter. In this theory, the Lorenz gauge is generalised to the mass shell condition on $S$.

Stuckelberg theory is known to be ultraviolet renormalizable for all values of the mass parameter $m_\gamma$ as explained in the review paper by Ruegge and Altaba[14]. The infrared divergence when $m_\gamma \rightarrow 0$ would be of interest.

There are two avenues which may be pursued if one wishes to study the infrared effects that arise in Stuckelberg theory. One may study the photon-Stuckelberg vertex and include its effects in radiative corrections. This procedure however is cumbersome and not very illuminating. Instead, a suitable gauge transformation will make the analysis simpler. By effecting the gauge transformation

$$A_\mu \rightarrow A_\mu + \frac{1}{m_\gamma} \partial_\mu S = \widetilde{A}_\mu, \quad (2.3)$$

the covariant derivative is modified to,

$$\partial_\mu \rightarrow ie\widetilde{\alpha}_\mu + \frac{ie}{m_\gamma} \partial_\mu S. \quad (2.4)$$

This is just a convenient writing of the covariant derivative. It shows that the gauge part of the derivative coming from transformations of the matter field are eliminated by the corresponding transformations of the Stuckelberg field.
We study scalar QED to avoid technical complications of spinorial polarisations. The additional term in the Lagrangian is

$$\mathcal{L}_\phi = (D_\mu \phi)^* (D^\mu \phi) + m^2 \phi^* \phi$$

(2.5)

The relevant interaction term for scalar QED may then be written as,

$$\mathcal{H}_I = ieA_\mu (\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi) - \frac{i e}{m_\gamma} S (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) .$$

(2.6)

Now, the photon scalar and the Stuckelberg scalar vertices read as follows.

![Feynman Rules for Stuckelberg Theory.](image)

The propagators are,

$$-i \left( \frac{g_{\mu \nu} + \frac{q_\mu q_\nu}{m_\gamma^2}}{(2\pi)^4 q^2 + m^2 - i\epsilon} \right)$$

(2.7)

for the photon and,

$$-i \frac{1}{(2\pi)^4 q^2 + m^2 - i\epsilon}$$

(2.8)

for the Stuckelberg field.

As pointed earlier, renormalizability of the Stuckelberg scalar QED can be demonstrated by constructing nilpotent BRST operator [14]. But we will focus on questions of infrared behaviour. Since there is mass for the photon there is no infrared divergence however small the mass is. We would like to analyse $m_\gamma \to 0$ limit where potentially it can appear.

### 3 Infrared limit and soft theorems

With the Feynman rules in hand, the calculation of the soft factors proceeds in much the same way as that of Weinberg. Consider an arbitrary Feynman
diagrams with legs labelled as either incoming or outgoing. Defining the symbol $\eta$, which takes values $+1$ for outgoing and $-1$ for incoming legs, we may proceed to compute the soft factors.

First, we should qualify the meaning of the term soft factor. A soft factor is a correction introduced into a matrix element due to the addition of an outgoing photon leg onto any of the external charged particle legs of the amplitude in question. This addition of this so called soft photon leg is quantified by the multiplication by the charged particle-photon vertex and the inclusion of a propagator of the form $((p + \eta q) + m^2 - i\epsilon)^{-1}$, where $p$ is the momentum of the particle leg and $q$ is the photon (or Stuckelberg) momentum, which is very close to zero. Due to the smallness of $q$, $p$ is essentially on the mass shell. $q^2$ may be neglected. These leaves behind $(2p \cdot q\eta - i\epsilon)^{-1}$. This has to now be multiplied by a suitable vertex factor.

The photon soft factor remains unchanged,

$$S^1_{\gamma}(q) = e^{\eta p \cdot q / p \cdot q - i\eta \epsilon}. \quad (3.1)$$

The Stuckelberg soft factor takes the form,

$$S^1_{S}(q) = -ie^{\eta p \cdot q / p \cdot q - i\eta \epsilon} = -i\eta e^{m^2 / m_{\gamma}^2}. \quad (3.2)$$

It is clear from the form of these soft factors that the factorization necessary to sum over all possible virtual or real emissions of soft particles is ensured and this holds true in the new theory as well. The factors that correct the $S$-matrix that correspond to a delivery of one soft particle from one leg to another is expressed by the following for a photon.

$$A_{\gamma}(q) = -ie^{g_{\mu\nu} q_{\mu} q_{\nu} / m_{S}^2} \sum_{ij} e_{i j} \eta_{i} \eta_{j} p_{i}^{\mu} p_{j}^{\nu} \frac{q_{\mu} q_{\nu}}{q_{\mu} q_{\nu} - i\eta_{i} \epsilon} (p_{j}^{\nu} q_{i} - i\eta_{j} \epsilon) (p_{j}^{\mu} q_{i} - i\eta_{j} \epsilon) \quad (3.3)$$

and by the following sum for a Stuckelberg particle,

$$A_{S}(q) = -i \frac{1}{(2\pi)^4} \frac{1}{q^2 + m^2 - i\epsilon m_{S}^2} \sum_{ij} e_{i j} \eta_{i} \eta_{j} (p_{i} \cdot q) (p_{j} \cdot q) \frac{q^2}{q^2 - i\eta_{i} \epsilon} (p_{j} \cdot q - i\eta_{j} \epsilon) (p_{j} \cdot q - i\eta_{j} \epsilon) \quad (3.4)$$

where $i, j$ run over all the external legs.

Consider the virtual communication of $N$ particles, $m$ of which are soft photons and $n$ of which are soft Stuckelberg particles. We sum over all
permutations and dividing by a factor $m!n!$ to account for spurious sums over identical soft photon and Stuckelberg particles and then sum over all $m$ and $n$ that add up to $N$. This gives the following contribution to the $S$-matrix,

$$\frac{1}{N!} \sum_{m+n=N} \frac{N!}{m!n!} A_\gamma(q)^m A_S(q)^n = \frac{1}{N!} (A_\gamma(q) + A_S(q))^N \text{.}$$  

(3.5)

We pause here to note that in the photon soft factor in Eq.(3.3), the term arising from the mass correction in the propagator has the same form as the soft factor in Eq.(3.4). This term, proportional to $m^{-2}$, diverges in the massless limit. It can be seen immediately that the condition for finiteness takes the form,

$$\sum_{ij} \eta_i\eta_j e_i e_j = \left( \sum_i \eta_i e_i \right)^2 = 0 \text{.}$$  

(3.6)

which is simply charge conservation arising from global gauge invariance. On account of this, we drop this term henceforth, noting charge conservation as arising from global $U(1)$ invariance in the Stuckelberg QED. Now we may proceed to study the modifications experienced by the soft theorems due to the inclusion of mass coming from the denominators.

Summing $N$ from 0 to $\infty$, we obtain a correction to the $S$-matrix in the form of an exponential. Only the real part of the term in the exponential contributes to lifetimes with the mass-shell condition imposed by the propagator. It is given by,

$$- \int_{E=m_0}^{E=\Lambda} \frac{1}{(2\pi)^3 E} \sum_{ij} \frac{e_i e_j \eta_i \eta_j p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} d^3q \text{.}$$  

(3.7)

Now, employing the substitution $|p| = m_\gamma \sinh(x)$, the integral may be rewritten as,

$$- \int_{x=0}^{x=\xi_\Lambda} \frac{1}{2(2\pi)^3} \sum_{ij} \frac{e_i e_j \eta_i \eta_j p_i \cdot p_j}{(E_i \coth(x) - p_i \cdot \hat{q})(E_j \coth(x) - p_j \cdot \hat{q})} dx d\Omega \text{.}$$  

(3.8)

where $\xi_\Lambda = \cosh^{-1}(\frac{\Lambda}{m_\gamma})$ and the solid angle integral is carried out over all directions of $\hat{q}$. 

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For future reference, we define the following function,

$$N(x, \hat{q}) = \frac{1}{2(2\pi)^3} \sum_{ij} \left( \eta_i \eta_j p_i \cdot p_j \right) \cdot \left( E_i \coth(x) - p_i \cdot \hat{q} \right) \left( E_j \coth(x) - p_j \cdot \hat{q} \right).$$  \hspace{1cm} (3.9)$$

To compute the effects of the emissions of soft photons and Stuckelberg particles, the calculation involved is entirely analogous to that carried out by Weinberg[3, 15]. The factorization employed is identical to the one shown in the virtual case. We will not repeat the calculation here, as the procedure carried out in the same way. We may note however that the polarization sum in the massive case supplies a corrected propagator,

$$\sum_\lambda \epsilon(q, \lambda) \epsilon^\ast(q, \lambda) = g_{\mu\nu} + \frac{q_\mu q_\nu}{m_\gamma^2}. \hspace{1cm} (3.10)$$

As mentioned earlier, the terms dependent on $m_\gamma^{-2}$ drop out of the correction terms, due to the global conservation of charge in the theory. This means that the terms that diverge as the mass of the photon vanishes, drop out of the calculation, the second term coming from the photon propagator and the Stuckelberg field.

It is worthwhile to note here that the effects of the Stuckelberg field are not at all observed in cross sections and decay rates insofar as an explicit interaction is concerned. Its existence is obtained from the introduction of the photon mass while preserving charge conservation but it is otherwise undetectable as a soft particle. The production of longitudinal/Stuckelberg particles are completely suppressed.

Having made these points, we may now note the correction to the transition rate that is effected on account of real emissions,

$$\Gamma_{\alpha\beta} = F(\Delta E) \Gamma^0_{\alpha\beta} \hspace{1cm} (3.11)$$

where $\Delta E$ is the energy resolution of the theory and the function $F$ is defined by the transcendental expression,

$$F(\Delta E) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\sigma \Delta E)}{\sigma} \exp \left( \int_{x=\xi \Delta E}^x N(x, \hat{q}) e^{i m_\gamma \cosh(x)} dx d\Omega \right) d\sigma. \hspace{1cm} (3.12)$$

Here, $\xi_{\Delta E} = \cosh^{-1}\left(\frac{\Delta E}{m_\gamma}\right)$. 
We may recall that the inclusion of virtual processes entailed multiplication by a factor,

\[ \exp \left( -\int_{x=0}^{x=\Lambda} N(x, \hat{q})dx d\Omega \right). \quad (3.13) \]

Finally, then, we may include both virtual and real processes. This means multiplying the transition rate by the following function,

\[ \exp \left( \int_{x=\Lambda}^{x=\Delta E} N(x, \hat{q})dx d\Omega \right) G(\Delta E) \quad (3.14) \]

where,

\[ G(\Delta E) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \sigma}{\sigma} \exp \left( \int_{x=0}^{x=\Delta E} N(x, \hat{q}) \left( e^{i\sigma \frac{m_\gamma}{2x} \cosh(x)} - 1 \right) dx d\Omega \right) d\sigma. \quad (3.15) \]

At this point, it must be noted that a naive limit of \( m_\gamma \to 0 \) cannot be taken, since the integral has been regularized effectively, taking such a simple limit supplies the incorrect answer. One must instead replace \( x \) with \( E \) and then appeal to the smooth limit of vanishing photon mass. When one does this, the large values of \( \Lambda/m_\gamma \) and \( \Delta E/m_\gamma \) effectively impose \( \coth(x) \sim 1 \). Then, the first exponential factor reduces to Weinberg’s energy dependent factor in eq. 2.51 of Weinberg’s 1965 calculation [3] and the second factor \( G \) reduces to the soft function.

### 4 Modified Lienard Wiechert potentials and antipodal matching

In the presence of a photon mass, modifications to the classical formulae of Lienard and Wiechert can be obtained. As was noted in [5], there are discontinuities in the Lienard-Wiechert potentials at infinity, and it would be interesting to see if they indeed arise in our theory as well if the massless limit is taken.

As we shall see, the situation is not quite as simple as taking a naive massless limit. A suitable regulating procedure must be followed, and only under suitable relations being obeyed between the mass bound and the radius of the sphere taken formally at infinity do the discontinuities arise.
Having said this, we may actually calculate the potentials for a massive photon. For a charged particle at rest, the potential takes the form,

\[ A_t = -e^2 Q e^{-m\gamma} \frac{r}{4\pi r}. \]  

(4.1)

The field strength then takes the form,

\[ F_{rt} = \frac{e^2 Q (m_r r + 1)}{4\pi r^2} e^{-m\gamma} r. \]  

(4.2)

Now, in order to obtain the electric field due to the electron moving at constant velocity, we have to effect a Lorentz boost. Analogous to the field in eq 2.1.8 in [5], we obtain the following formula,

\[ F_{rt} = \frac{e^2 Q}{4\pi} \frac{m_r R(\vec{\beta}) + 1}{R(\vec{\beta})^3} e^{-m\gamma} R(\vec{\beta}) \gamma (r - t\hat{x} \cdot \vec{\beta}) \]  

(4.3)

where \( \gamma \) is the Lorentz factor, \( \vec{\beta} \) is the rapidity and,

\[ R(\vec{\beta}) = |\gamma^2 (t - r\hat{x} \cdot \vec{\beta})^2 - t^2 + r^2|^{1/2}. \]  

(4.4)

The derivation is almost similar to the standard one. We may now look at the formula in retarded coordinates \( u = t - r \) (the advanced coordinate treatment is analogous) and take the limit \( r \rightarrow \infty \). This supplies the following equation,

\[ \lim_{r,t \rightarrow \infty, u = \text{const}} F_{rt} = \frac{e^2 Q}{4\pi r^2} \frac{\alpha + 1}{\gamma^2 (1 - \hat{x} \cdot \vec{\beta})^2} e^{-\alpha} \]  

(4.5)

where,

\[ \alpha = \gamma (1 - \hat{x} \cdot \vec{\beta}) m_r r. \]  

(4.6)

It is here that the precise relationship between the photon mass and the regulating procedure becomes clear. If \( \alpha \) is taken to be unbounded, the expression is rapidly decreasing and vanishes at infinity. If however \( \alpha \) is some positive number, the discontinuity manifests again.

The source of the discontinuity is on account of the fact that the order of the limits have bearing on the final answer. If the limit of large \( r \) is taken first, the expression vanishes and no discontinuity is observed. If however,
the limit of vanishing mass is first assumed, the answer reduces to the class-
cical expression, and the discontinuity is recovered along with the antipodal
matching condition.

These two limits correspond to leaving $\alpha$ unbounded and fixing $\alpha$ to be
finite respectively. Note that we are ultimately interested in the limit,

$$\alpha \ll 1$$ (4.7)

which implies,

$$m_\gamma \ll \frac{1}{r}.$$ (4.8)

The current bounds on the mass of the photon will be consistent with this
expectation.

5 Conclusions

In this note, we revisited the question of infrared divergence in QED, soft
photons and asymptotic symmetries. This we approach in a gauge invariant
way in massive photon theory. This removes the discontinuity in the degrees
of freedom as we take the limit of photon mass to zero. This was done
using Stuckelberg theory which is the phase of the gauge field. We have a
current limit of photon mass being less than $10^{-18}$ eV. Such a mass even
though extremely small introduces extra degrees of freedom. This extra
degree of freedom is coupled to the matter with very little strength and does
not affect any of the known results of QED. Such a question was posed by
Schroedinger [2] in relation to blackbody radiation. But the field is essential
for the asymptotic region as well as providing new Hilbert space in which
QED will be free of infrared divergence along the lines of Faddeev and Kulish
[16] coherent states. Using this we can define gauge in variant matter field
operator as $\tilde{\varphi} = e^{-iS}\varphi$. This will define the asymptotic states of the
theory. In the limit of mass to zero carefully, the limit will still provide
$|in>$ and $|out>$ states. This can also be compared with the formulation of
Bagan et al[17]. As pointed out in the Introduction this analysis with exact
massless QED is beset with problems like Lorentz symmetry violation[10, 18].
The analysis of limit of massive QED to massless is better understood when
we use lightcone quantisation. Such an analysis has been done and will be
presented elsewhere[19]. We can also consider 2+1 dimensional QED where
there are novel mechanisms of generations mass are available in addition to
Stuckelberg theory like Chern Simons term. The role of edge states in such a program playing the role of a Stuckelberg field is interesting which again will be presented soon [20]. The lesson we learn from these studies is that the gauge non-invariant longitudinal component or Stueckelberg field can be made to be absent in the bulk inside a sphere of radius \( \propto \frac{1}{m} \) the compton wavelength, but the generalised Robin conditions will provide localised states at the boundary. In the limit of \( m \to 0 \) they survive as zero energy modes and provide the required changes in the Hilbert space as well as contribute to asymptotic symmetries. This requires subtle conditions on the asymptotic behaviour along with the radius reaching infinity.

One can ask whether the extra degree of freedom or Stuckelberg field has any role to play. Since the electromagnetic interaction is completely suppressed it can have only gravitational interaction. It can play the part of dark matter component. Whether it is a substantial component or gets decoupled during evolution of the universe will be determined by the relic density. Very low mass of the photon can naively make it impossible to survive with appropriate densities. But recent analysis provide novel BEC condensate being formed to facilitate the role as fuzzy dark matter candidates. This will be presented elsewhere. In this connection it is worth pointing out the possibility of ultralight dark matter candidates have been explored [21, 22].

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