Upper and lower estimates for numerical integration errors on spheres of arbitrary dimension

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\textbf{Abstract}
In this paper we study the worst-case error of numerical integration on the unit sphere $S^d \subset \mathbb{R}^{d+1}$, $d \geq 2$, for certain spaces of continuous functions on $S^d$. For the classical Sobolev spaces $H^s(S^d) \ (s > \frac{d}{2})$ upper and lower bounds for the worst case integration error have been obtained in \cite{2,10,12}. We investigate the behaviour for $s \to \frac{d}{2}$ by introducing spaces $H^{n-\gamma}(S^d)$ with an extra logarithmic weight. For these spaces we obtain similar upper and lower bounds for the worst case integration error.

\textit{Keywords:} Worst-case error, numerical integration, cubature rules, reproducing kernel, $t$-design, QMC design, sphere.

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\section{1. Introduction}
Let $S^d \subset \mathbb{R}^{d+1}$, where $d \geq 2$ denote the unit sphere in the Euclidean space $\mathbb{R}^{d+1}$. The integral of a continuous function $f : S^d \to \mathbb{R}$, denoted by

$$I(f) := \int_{S^d} f(x) d\sigma_d(x),$$

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where $d\sigma_d(x)$ is the normalised surface (Lebesgue) measure on $S^d$ (i.e., $\sigma_d(S^d) = 1$), is approximated by an $N$-point numerical integration rule $Q[X_N, \omega](f)$

$$Q[X_N, \omega](f) = Q[X_N, (\omega_j)_{j=1}^N](f) := \sum_{i=1}^N \omega_i f(x_i)$$

with nodes $x_1, \ldots, x_N \in S^d$ and associated weights $\omega_1, \ldots, \omega_N \in \mathbb{R}$. We will always assume that the weights satisfy the relation

$$\sum_{i=1}^N \omega_i = 1.$$

By $Q[X_N](f)$ we will denote the equal weight numerical integration rule:

$$Q[X_N](f) := \frac{1}{N} \sum_{i=1}^N f(x_i).$$

The worst-case (cubature) error of the cubature rule $Q[X_N, \omega]$ in a Banach space $B$ of continuous functions on $S^d$ with norm $\| \cdot \|_B$ is defined by

$$\text{wce}(Q[X_N, \omega]; B) := \sup_{f \in B, \| f \|_B \leq 1} |Q[X_N](f) - I(f)|. \quad (1)$$

In this work we consider reproducing kernel Hilbert spaces $H_2^d(\mathbb{R}^d)$, which interpolate the classical spaces $H^s(S^d)$ for $s \rightarrow \frac{d}{2}$, (see Section 2 for a precise definition).

The paper is organised as follows.

Section 2 provides necessary background for Jacobi polynomials, the spaces $H^s(S^d)$ and $H^d_2(\mathbb{R}^d)$, their associated reproducing kernels and an expression for the worst-case error.

In Section 3 we find upper and lower bounds of equal weight numerical integration over the unit sphere $S^d \subset \mathbb{R}^{d+1}$ for functions in the space $H^d_2(S^d)$, $\gamma > \frac{1}{2}$. In this section we consider sequences $X_N(t)$ of well-separated $t$-designs. Here we also assume that $t \asymp N^{\frac{d}{2}}$. Such $t$-designs exist by [3]. We write $a_n \asymp b_n$ to mean that there exist positive constants $C_1$ and $C_2$ independent of $n$ such that $C_1 a_n \leq b_n \leq C_2 a_n$ for all $n$.

We show that

$$C_{d, \gamma} N^{-\frac{d}{2}} (\ln N)^{-\gamma} \leq \text{wce}(Q[X_N, \omega]; H^d_2(S^d)) \quad (2)$$
for all quadrature rules $Q[X_N, \omega]$ and provide examples of quadrature rules which satisfy

$$C^{(1)}_{d, \gamma} N^{-\frac{1}{2}} (\ln N)^{-\gamma + \frac{1}{2}} \leq \text{wce}(Q[X_N]; \mathbb{H}^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d)) \leq C^{(2)}_{d, \gamma} N^{-\frac{1}{2}} (\ln N)^{-\gamma + \frac{1}{2}} ,$$

where the positive constants $C^{(1)}_{d, \gamma}$ and $C^{(2)}_{d, \gamma}$ depend only on $d$ and $\gamma$, but are independent of the rule $Q[X_N]$ and the number of nodes $N$ of the rule.

Here and further by $C_{\gamma,d}$, $C^{(1)}_{\gamma,d}$ and $C^{(2)}_{\gamma,d}$ we denote some positive constants, which depend only on $d$ and $\gamma$ and can be different in different relations.

The upper estimate of this result is an extension of results in [7, 12], where the upper bound for the worst-case error in the Sobolev space $\mathbb{H}^s(\mathbb{S}^d)$, $s > \frac{d}{2}$, (see Section 2 for a precise definition) of a sequence of cubature rules $Q[X_N]$ was found. In these papers the sequence $Q[X_N]$ integrates all spherical polynomials of degree $\leq t$ exactly and satisfies a certain local regularity property.

In Section 4 we show that the worst-case error for functions in the space $\mathbb{H}^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d), \gamma > \frac{1}{2}$, for an arbitrary $N$-point cubature rule $Q[X_N, \omega]$ has the lower bound

$$\text{wce}(Q[X_N, \omega]; \mathbb{H}^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d)) \geq C_{d,\gamma} N^{-\frac{1}{2}} (\ln N)^{-\gamma},$$

where the positive constant $C_{d,\gamma}$ depends only on $d$ and $\gamma$, but is independent of the rule $Q[X_N]$ and the number of nodes $N$ of the rule. On the basis of the estimate (3), we can make a conjecture that the order of convergence $O(N^{-\frac{1}{2}} (\ln N)^{-\gamma + \frac{1}{2}})$ is optimal for classes $\mathbb{H}^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d)$.

In Section 5 we analyse QMC designs for $\mathbb{H}^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d)$ and compare them with QMC designs for Sobolev spaces $\mathbb{H}^s(\mathbb{S}^d)$. We prove that if $X_N$ is a sequence of QMC designs for Sobolev spaces $\mathbb{H}^s(\mathbb{S}^d)$, $s > \frac{d}{2}$, it is also a sequence of QMC designs for $\mathbb{H}^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d)$ for all $\gamma > \frac{1}{2}$.

We remark here that J. Beck [1, 2] could show a lower bound for the spherical cap discrepancy of order $N^{-1/2-1/2d}$; he proved by probabilistic means that for every $N$ there exists a point set $X_N$ with discrepancy of order $N^{-1/2-1/2d} \sqrt{\log N}$. Beck’s lower bound can be reproved by using the techniques found by D. Bilyk and F. Dai [3], which we will refer to in more detail in Section 4. The $\sqrt{\log N}$-factor between the lower and the upper bound in (2) and (3) resembles the difference between Beck’s general lower bound and the upper bound achieved by a probabilistic construction.
2. Preliminaries

2.1. Background and basic notations

We denote the Euclidean inner product of $x$ and $y$ in $\mathbb{R}^{d+1}$ by $\langle x, y \rangle$.

We use the Pochhammer symbol $(a)_n$, where $n \in \mathbb{N}_0$ and $a \in \mathbb{R}$, defined by

$$(a)_0 := 1, \quad (a)_n := a(a+1)\ldots(a+n-1) \quad \text{for} \quad n \in \mathbb{N},$$

which can be written in the terms of the gamma function $\Gamma(z)$ by means of

$$(a)_\ell = \frac{\Gamma(\ell + a)}{\Gamma(a)}. \quad (4)$$

The following asymptotic relation holds

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \quad \text{as} \quad z \to \infty \quad \text{in the sector} \quad |\arg z| \leq \pi - \delta \quad (5)$$

for $\delta > 0$. Here, $f(x) \sim g(x), x \to \infty$, means that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

We denote, as usual, by $\{Y^{(d)}_{\ell,k} : k = 1, \ldots, Z(d, \ell)\}$ a collection of $L_2$-orthonormal real spherical harmonics (homogeneous harmonic polynomials in $d + 1$ variables restricted to $S^d$) of degree $\ell$ (see, e.g., [14]). The space of spherical harmonics of degree $\ell \in \mathbb{N}_0$ on $S^d$ has the dimension

$$Z(d, 0) = 1, \quad Z(d, \ell) = (2\ell + d - 1) \frac{\Gamma(\ell + d - 1)}{\Gamma(d)\Gamma(\ell + 1)} \sim \frac{2}{\Gamma(d)} \ell^{d-1}, \quad \ell \to \infty. \quad (6)$$

Each spherical harmonic $Y^{(d)}_{\ell,k}$ of exact degree $\ell$ is an eigenfunction of the negative Laplace-Beltrami operator $-\Delta^*_d$ with eigenvalue

$$\lambda_\ell := \ell(\ell + d - 1). \quad (7)$$

The spherical harmonics of degree $\ell$ satisfy the addition theorem:

$$\sum_{k=1}^{Z(d,\ell)} Y^{(d)}_{\ell,k}(x) Y^{(d)}_{\ell,k}(y) = Z(d,\ell) P^{(d)}_{\ell}(\langle x, y \rangle), \quad (8)$$
where $P^{(d)}_{\ell}$ is the $\ell$-th generalised Legendre polynomial, normalised by $P^{(d)}_{\ell}(1) = 1$ and orthogonal on the interval $[-1,1]$ with respect to the weight function $(1-t^2)^{d/2-1}$. These functions are zonal spherical harmonics on $S^d$. Notice that

$$Z(d, n) P^{(d)}_{\ell}(x) = \frac{n + \lambda}{\lambda} C_n^{\lambda}(x), \quad P^{(d)}_{\ell}(x) = \frac{n!}{(d/2)_n} P^{\lfloor d/2 \rfloor - 1/2}_{n}(x), \quad (9)$$

where $C_n^{\lambda}(x)$ is the $n$-th Gegenbauer polynomial with index $\lambda = \frac{d-1}{2}$ and $P^{\lfloor d/2 \rfloor - 1/2}_{n}(x)$ are the Jacobi polynomials with the indices $\alpha = \beta = \frac{d}{2} - 1$.

### 2.2. Jacobi polynomials

The Jacobi polynomials $P^{(\alpha, \beta)}_{\ell}(x)$ are the polynomials orthogonal over the interval $[-1,1]$ with respect to the weight function $w_{\alpha, \beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ and normalised by the relation

$$P^{(\alpha, \beta)}_{\ell}(1) = \left(\frac{\ell + \alpha}{\ell}\right) = \frac{(1 + \alpha)_{\ell}}{\ell!} \sim \frac{1}{\Gamma(1 + \alpha)} \ell^{\alpha}, \quad \alpha, \beta > -1. \quad (10)$$

(see, e.g., [13, (5.2.1)]).

Also the following equality holds

$$P^{(\alpha, \beta)}_{\ell}(-x) = (-1)^{\ell} P^{(\beta, \alpha)}_{\ell}(x). \quad (11)$$

For fixed $\alpha, \beta > -1$ and $0 < \theta < \pi$, the following relation gives an asymptotic approximation for $\ell \to \infty$ (see, e.g., [17, Theorem 8.21.13])

$$P^{(\alpha, \beta)}_{\ell}(\cos \theta) \approx \frac{1}{\sqrt{\pi}} \ell^{-1/2} \left(\sin \frac{\theta}{2}\right)^{-\alpha-1/2} \left(\cos \frac{\theta}{2}\right)^{-\beta-1/2} \times \left\{ \cos \left(\ell + \frac{\alpha + \beta + 1}{2}\right) \theta - \frac{2\alpha + 1}{4 \pi} \right\} + O(\ell \sin \theta)^{-1} \right\}. \quad (12)$$

Thus, for $c_{\alpha, \beta} \ell^{-1} \leq \theta \leq \pi - c_{\alpha, \beta} \ell^{-1}$ the last asymptotic equality yields

$$|P^{(\alpha, \beta)}_{\ell}(\cos \theta)| \leq \tilde{c}_{\alpha, \beta} \ell^{-1/2}(\sin \theta)^{-\alpha-1/2} + \tilde{c}_{\alpha, \beta} \ell^{-3/2}(\sin \theta)^{-\alpha-3/2}, \quad \alpha \geq \beta. \quad (12)$$

If $\alpha, \beta$ are real and $c$ is fixed positive constant, then as $\ell \to \infty$ (see, e.g., [13, (5.2.3)])

$$|P^{(\alpha, \beta)}_{\ell}(\cos \theta)| = \begin{cases} O\left(\theta^{-\frac{\alpha}{2}} \ell^{-\frac{1}{2}}\right) & \text{if } \frac{\pi}{2} \ell \leq \theta \leq \frac{\pi}{2}, \\ O\left(\ell^{\alpha}\right) & \text{if } 0 \leq \theta \leq \frac{\pi}{2}. \end{cases} \quad (13)$$
We will also use the formula (see, e.g., [17, (4.5.3)])

\[ \sum_{\ell=0}^{n} \frac{2\ell + \alpha + \beta + 1}{\alpha + \beta + 1} \frac{(\alpha + \beta + 1)_{\ell}}{(\beta + 1)_{\ell}} P_\ell^{(\alpha,\beta)}(t) = \frac{(\alpha + \beta + 2)_n}{(\beta + 1)_n} P_n^{(\alpha+1,\beta)}(t). \quad (14) \]

Choosing \( \alpha = \beta = \frac{d}{2} - 1 \) and taking into account the relations (6) and (9), formula (14) also reads

\[ \sum_{r=0}^{\ell} 2r + d - 1 \frac{(d-1)_r}{(d/2)_r} P_r^{(\frac{d}{2}-1,\frac{d}{2}-1)}(t) = \frac{(d)_{\ell}}{(d/2)_{\ell}} P_\ell^{(\frac{d}{2},\frac{d}{2})}(t). \quad (15) \]

Substituting \( \alpha = \frac{d}{2} - 1 + k \) and \( \beta = \frac{d}{2} - 1 \), formula (14) gives

\[ \sum_{r=0}^{\ell} 2r + d - 1 + k \frac{(d-1+k)_r}{(d/2)_r} P_r^{(\frac{d}{2}+k,\frac{d}{2})}(t) = \frac{(d+k)_{\ell}}{(d/2)_{\ell}} P_\ell^{(\frac{d}{2}+k,\frac{d}{2})}(t). \quad (16) \]

For any integrable function \( f : [-1, 1] \to \mathbb{R} \) (see, e.g., [14])

\[ \int_{\mathbb{S}^d} f(\langle x, y \rangle) d\sigma_d(x) = \frac{\Gamma(d+1)}{\sqrt{\pi} \Gamma(d/2)} \int_{-1}^{1} f(t)(1 - t^2)^{d/2 - 1} dt \quad \forall y \in \mathbb{S}^d. \quad (17) \]

For \( \alpha > 1 \) and \( L \in \mathbb{N}_0 \), we have (see, e.g., formula (2.18) in [7])

\[ \int_{-1}^{1} P_\ell^{(\alpha+L,\alpha)}(t)(1 - t^2)^{\alpha} dt = 2^{2\alpha+1} \frac{(L)_{\ell}}{\ell!} \frac{\Gamma(\alpha + 1) \Gamma(\alpha + \ell + 1)}{\Gamma(2\alpha + \ell + 2)}. \quad (18) \]

This formula also can be easily derived with the help of Rodrigues’ formula (see, e.g., [17, (4.3.1)]).

In particular (17), (18) and (5) imply

\[ \int_{\mathbb{S}^d} P_\ell^{(\frac{d}{2}+L,\frac{d}{2}-1)}(\langle x, y \rangle) d\sigma_d(x) = 2^{d-1} \frac{\Gamma(d+1)}{\sqrt{\pi}} \frac{(L+1)_{\ell}}{\ell!} \frac{\Gamma(\frac{d}{2} + L) \Gamma(\frac{d}{2} + \ell)}{\Gamma(d + \ell)} \times \frac{\Gamma(L + \ell + 1) \Gamma(\frac{d}{2} + \ell)}{\Gamma(\ell + 1) \Gamma(d + \ell)} \approx \ell^{L-\frac{d}{2}}. \quad (19) \]
2.3. The space of continuous functions on $\mathbb{S}^d$ and representation of worst-case error

The Sobolev space $H^s(\mathbb{S}^d)$ for $s \geq 0$ consists of all functions $f \in L_2(\mathbb{S}^d)$ with finite norm

$$
\|f\|_{H^s} = \left( \sum_{\ell=0}^{\infty} \sum_{k=1}^{Z(d,\ell)} (1 + \lambda_\ell)^s |\hat{f}_{\ell,k}|^2 \right)^{\frac{1}{2}},
$$

(20)

where the Laplace-Fourier coefficients are given by the formula

$$
\hat{f}_{\ell,k} := (f, Y^{(d)}_{\ell,k})_{\mathbb{S}^d} = \int_{\mathbb{S}^d} f(x) Y^{(d)}_{\ell,k}(x) d\sigma_d(x).
$$

For $s > \frac{d}{2}$ the space $H^s(\mathbb{S}^d)$ is embedded into the space of continuous functions $C(\mathbb{S}^d)$. This fact also implies that point evaluation in $H^s(\mathbb{S}^d)$, $s > \frac{d}{2}$, is bounded and $H^s(\mathbb{S}^d)$, $s > \frac{d}{2}$, is a reproducing kernel Hilbert space.

In the row of papers [7, 8, 10–12], the worst-case error for Sobolev spaces $H^s(\mathbb{S}^d)$ in the case $s > \frac{d}{2}$ was studied. Our aim is to consider the class of functions, which are less smooth than functions from $H^s(\mathbb{S}^d)$, $s > \frac{d}{2}$.

We define the space $H^{(d,\gamma)}(\mathbb{S}^d)$ for $\gamma > \frac{1}{2}$ as the set of all functions $f \in L_2(\mathbb{S}^d)$ whose Laplace-Fourier coefficients satisfy

$$
\|f\|_{H^{(d,\gamma)}}^2 := \sum_{\ell=0}^{\infty} w_\ell(d, \gamma) \sum_{k=1}^{Z(d,\ell)} |\hat{f}_{\ell,k}|^2 < \infty,
$$

(21)

where

$$
w_\ell(d, \gamma) := (1 + \lambda_\ell)^{\frac{d}{2}} (\ln (3 + \lambda_\ell))^{2\gamma}.
$$

The space $H^{(d,\gamma)}(\mathbb{S}^d)$ is a Hilbert space with a corresponding inner product denoted by $(f, g)_{H^{(d,\gamma)}}$. For $\gamma > \frac{1}{2}$ the space $H^{(d,\gamma)}(\mathbb{S}^d)$ is embedded into the space of continuous functions $C(\mathbb{S}^d)$. Indeed, using the Cauchy-Schwarz inequality we can show in the same way as in [10], that for $f \in H^{(d,\gamma)}(\mathbb{S}^d)$

$$
\sup_{x \in \mathbb{S}^d} |f(x)| \leq C_{d,\gamma} \|f\|_{H^{(d,\gamma)}}.
$$

Embedding into $C(\mathbb{S}^d)$ implies that point evaluation in $H^{(d,\gamma)}(\mathbb{S}^d)$ with $\gamma > \frac{1}{2}$ is bounded, and consequently $H^{(d,\gamma)}(\mathbb{S}^d)$ is a reproducing kernel.
Hilbert space. That is to say there exists a kernel $K_{d,\gamma} : S^d \times S^d \to \mathbb{R}$, with the following properties: (i) $K_{d,\gamma}(x, y) = K_{d,\gamma}(y, x)$ for all $x, y \in S^d$; (ii) $K_{d,\gamma}(\cdot, x) \in \mathbb{H}^{(d/2)}(S^d)$ for all fixed $x \in \mathbb{H}^{(d/2)}(S^d)$; and (iii) the reproducing property

$$(f, K_{d,\gamma}(\cdot, x))_{\mathbb{H}^{(d/2)}(S^d)} = f(x) \quad \forall f \in \mathbb{H}^{(d/2)}(S^d) \quad \forall x \in S^d.$$ 

The reproducing kernel $K_{d,\gamma}$ in $\mathbb{H}^{(d/2)}(S^d)$ is given by

$$K_{d,\gamma}(x, y) = \sum_{\ell=0}^{\infty} w_{\ell}(d, \gamma)^{-1} Z(d, \ell) P_{\ell}^d(\langle x, y \rangle).$$

(22)

It is easily verified, that $K_{\gamma,d}$, defined by (22) has the reproducing kernel properties.

This kernel is a zonal function: $K_{\gamma,d}(x, y)$ depends only on the inner product $\langle x, y \rangle$.

Using arguments, as in (8) or (12), it is possible to write down an expression for the worst-case error. Indeed

$$\text{wce}(Q[X_N, \omega]; \mathbb{H}^{(d/2)}(S^d))^2 = \sum_{i,j=1}^{N} \omega_i \omega_j K_{d,\gamma}(x_i, x_j) - \int_{S^d} K_{d,\gamma}(x, y) d\sigma_d(y),$$

where we have used the reproducing property of $K_{d,\gamma}$.

Therefore,

$$\text{wce}(Q[X_N, \omega]; \mathbb{H}^{(d/2)}(S^d))^2 = \sum_{i,j=1}^{N} \omega_i \omega_j \tilde{K}_{d,\gamma}(x_i, x_j),$$

(23)

where

$$\tilde{K}_{d,\gamma}(x, y) = \sum_{\ell=1}^{\infty} w_{\ell}(d, \gamma)^{-1} Z(d, \ell) P_{\ell}^d(\langle x, y \rangle).$$

(24)

3. Upper and lower bounds for the worst-case error for well-separated $t$-designs

**Definition 1.** A spherical $t$-design is a finite subset $X_N \subset S^d$ with the characterising property that an equal weight integration rule with nodes from $X_N$ integrates all polynomials $p$ with degree $\leq t$ exactly; that is,$$\frac{1}{N} \sum_{x \in X_N} p(x) = \int_{S^d} p(x) d\sigma_d(x), \quad \deg(p) \leq t.$$
Here $N$ is the number of points of the spherical design.

A concept of $t$-design was introduced in the paper [9] by Delsarte, Goethals and Seidel. There it was proved that the number of points for a $t$-design has to satisfy $N \geq C_d t^d$ for a positive constant $C_d$.

Bondarenko, Radchenko and Viazovska [4] proved that there always exist spherical $t$-designs with $N \approx t^d$ points. That is why in this section we always assume that

$$N = N(t) \approx t^d.$$  \hfill (25)

Then

$$\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} P_{\ell}^{(d)}(\langle x_i, x_j \rangle) = 0, \quad \text{for } \ell = 1, \ldots, t.$$  

Thus for such sequences $Q[X_{N(t)}]$ [23] simplifies to

$$\text{wce}(Q[X_{N(t)}]; \mathbb{H}^{(2)}(S^d))^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=t+1}^{\infty} w_{\ell}(d, \gamma)^{-1} Z(d, \ell) P_{\ell}^{(d)}(\langle x_i, x_j \rangle).$$  \hfill (26)

By a spherical cap $S(x; \varphi)$ of centre $x$ and angular radius $\varphi$ we mean

$$S(x; \varphi) := \{ y \in S^d | \langle x, y \rangle \geq \cos \varphi \}.$$  

The normalised surface area of a spherical cap is given by

$$|S(x; \varphi)| = \frac{\Gamma((d+1)/2)}{\sqrt{\pi} \Gamma(d/2)} \int_{\cos \varphi}^{1} (1 - t^2)^{(d-1)/2} \, dt \asymp (1 - \cos \varphi)^{d/2} \quad \text{as } \varphi \to 0. \quad (27)$$  

Here and in the sequel we use $|S|$ as a shorthand for $\sigma_d(S)$ for $S \subset S^d$.

**Definition 2 (Property (R)).** A sequence $(Q[X_{N(t)}, \omega])_{t \in \mathbb{N}}$ of numerical integration rules $Q[X_{N(t)}, \omega]$, which integrates all spherical polynomials of degree $\leq t$ exactly, that is

$$\sum_{j=1}^{N(t)} \omega_j p(x_j) = \int_{S^d} p(x) d\sigma_d(x), \quad \deg(p) \leq t.$$  

is said to have property (R) (or to be “quadrature regular”), if there exist positive numbers $c_1$ and $c_2$ independent of $t$ with $c_1 \leq \pi/2$, such that for all
\[ t \geq 1 \] the weights \( \omega_j \) associated with the nodes \( x_j, j = 1, \ldots, N(t) \) of \( Q[X_N(t)] \) satisfy
\[
\sum_{j=1}^{N(t)} \left| \omega_j \right| \leq c_2 \left| S(x; c_1 t) \right| \quad \forall x \in \mathbb{S}^d.
\tag{28}
\]

Reimer [15] has shown that every sequence of positive weight cubature rules \( Q[X_{N(t)}, \omega]\), with \( Q[X_{N(t)}, \omega](p) = I(p) \) for all polynomials \( p \) with \( \deg p \leq t \) satisfies property (R) automatically with positive constants \( c_1 \) and \( c_2 \) depending only on \( d \).

**Definition 3.** A sequence of \( N \)-point sets \( X_N = \{x_1, \ldots, x_N\} \), is called well-separated if there exists a positive constant \( c_3 \) such that
\[
\min_{i \neq j} |x_i - x_j| > \frac{c_3}{N^{\frac{\gamma}{2}}}. \tag{29}
\]

It should be noticed, that a well-separated sequence \( X_N \) of numerical integration rules with equal weights \( \omega_i = \frac{1}{N} \) satisfies property (R), but not conversely. Indeed, from the inequality (29) it follows, that for all \( x_i, x_j \in X_{N(t)}, i \neq j \),
\[
\langle x_i, x_j \rangle < 1 - \frac{c_3^2}{2N^{\frac{\gamma}{2}}}
\]
Thus the spherical cap \( S\left(x_i; \arccos \left(1 - \frac{c_3^2}{2N^{\frac{\gamma}{2}}} \right)\right) \) contains no points of \( X_N \) in its interior, except of the point \( x_i \).

Using (27) we deduce the following estimate
\[
\frac{1}{N} \# \left\{ x_j \in X_{N(t)} \cap S(x; \frac{c_1}{t}) \right\} \leq \frac{1}{N} \left| S(x; \arccos \left(1 - \frac{c_3^2}{2N^{\frac{\gamma}{2}}} \right)\right) \ll \left| S(x; \frac{c_1}{t})\right|.
\]

Here we write \( a_n \ll b_n \) (\( a_n \gg b_n \)) to mean that there exists positive constant \( K \) independent of \( n \) such that \( a_n \leq K b_n \) (\( a_n \geq K b_n \)) for all \( n \).

**Theorem 1.** Let \( d \geq 2, \gamma > \frac{1}{2} \) be fixed, and \( (X_{N(t)})_t \) be a sequence be a well-separated spherical \( t \)-designs, \( t \) and \( N(t) \) satisfy relation (25). Then there exist positive constants \( C_{d,\gamma}^{(1)} \) and \( C_{d,\gamma}^{(2)} \), such that
\[
C_{d,\gamma}^{(1)} N^{-\frac{1}{2}} (\ln N)^{-\gamma + \frac{1}{2}} \leq \wce(Q[X_N]; H^{(\frac{d}{2}-\gamma)}(\mathbb{S}^d)) \leq C_{d,\gamma}^{(2)} N^{-\frac{1}{2}} (\ln N)^{-\gamma + \frac{1}{2}}. \tag{30}
\]
The constants $C_{d,\gamma}^{(1)}$ and $C_{d,\gamma}^{(2)}$ depend only on $d$, $\gamma$ and on the constants $c_i$, $i = 1, \ldots, 3$, from the relations (28) and (29).

In (30) and further in this section for brevity we write $N$ instead $N(t)$ for the number of nodes in $X_{N(t)}$.

Theorem 1 is a consequence of the following lemmas:

**Lemma 1.** Let $d \geq 2$ and $\gamma > \frac{1}{2}$ be fixed. Then for any sequence $X_N$, $K \in \mathbb{N}_0$ and for any $n \in \mathbb{N}$ the following relation holds

$$
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} w_{\ell}(d, \gamma)^{-1} Z(d, \ell) P_{\ell}^{(d)}((x_i, x_j))
\leq \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} \ell^{-\frac{d}{2}-K} (\ln \ell)^{-2\gamma} \ell^{(\frac{d}{2}+K-1, \frac{d}{2}-1)} ((x_i, x_j)).
$$

**Lemma 2.** Let $d \geq 2$ and $\gamma > \frac{1}{2}$ be fixed, let $(X_{N(t)})_t$ be a sequence of spherical $t$-designs, $t$ and $N(t)$ satisfy relation (25). Then for any $K \in \mathbb{N}_0$ there exists a positive constant $C_{d,\gamma}$, such that

$$
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K} (\ln \ell)^{-2\gamma} \ell^{(\frac{d}{2}+K-1, \frac{d}{2}-1)} ((x_i, x_j)) - C_{d,\gamma}t^{-d} (\ln t)^{-2\gamma}
\leq \text{wce}(Q[X_N]; H(\frac{d}{2},\gamma))^2
\leq \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K} (\ln \ell)^{-2\gamma} \ell^{(\frac{d}{2}+K-1, \frac{d}{2}-1)} ((x_i, x_j)).
$$

The constant $C_{d,\gamma}$ depends only on $d$ and $\gamma$.

**Lemma 3.** Let $d \geq 2$ and $\gamma > \frac{1}{2}$ be fixed, $(X_{N(t)})_t$ be a well-separated sequence, $t$ and $N(t)$ satisfy relation (25). Then for any $K > \frac{d}{2}$, $K \in \mathbb{N}$, there exist positive constants $C_{d,\gamma}^{(1)}$ and $C_{d,\gamma}^{(2)}$, such that

$$
C_{d,\gamma}^{(1)} N^{-1} (\ln N)^{-2\gamma+1}
\leq \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K} (\ln \ell)^{-2\gamma} \ell^{(\frac{d}{2}+K-1, \frac{d}{2}-1)} ((x_i, x_j)) \leq C_{d,\gamma}^{(2)} N^{-1} (\ln N)^{-2\gamma+1}.
$$

The constants $C_{d,\gamma}^{(1)}$ and $C_{d,\gamma}^{(2)}$ depend only on $d$ and $\gamma$. 

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Remark 1. Let $d \geq 2$, $\gamma > \frac{1}{2}$ be fixed and let the sequence $(X_N)_N$ have property (R). Then

\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell = \lfloor N^{\frac{1}{d}} \rfloor + 1}^{\infty} \ell^{-\frac{d}{2} - K} (\ln \ell)^{-2\gamma} P_{\ell}^{(\frac{d}{2} + K - 1, \frac{d}{2} - 1)}((x_i, x_j)) \ll N^{-1} (\ln N)^{-2\gamma + 1}.
\]

(34)

Lemma 1 and Remark 1 allow us to write down the following estimate.

**Theorem 2.** Let $d \geq 2$, $\gamma > \frac{1}{2}$ be fixed and let the sequence $(X_N)_N$ have property (R). Then

\[
\text{wce}(Q[X_N]; \mathbb{H}^{(\frac{d}{2}, \gamma)}(S^d))^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell = 1}^{\lfloor N^{\frac{1}{d}} \rfloor} w_{\ell}(d, \gamma)^{-1} Z(d, \ell) P_{\ell}^{(d)}((x_i, x_j)) + O\left(\frac{1}{N (\ln N)^{2\gamma - 1}}\right).
\]

From the proofs of Lemmas 1–3 one can easily get an estimate.

**Theorem 3.** Let $d \geq 2$, $\gamma > \frac{1}{2}$ be fixed and let $(X_{N(t)})_t$ be a sequence of spherical $t$-designs. Then there exists a positive constant $C_{d, \gamma}$ such that

\[
\text{wce}(Q[X_N]; \mathbb{H}^{(\frac{d}{2}, \gamma)}(S^d)) \leq C_{d, \gamma} t^{-\frac{d}{2}} (\ln t)^{-\gamma + \frac{1}{2}}.
\]

(35)

The constant $C_{d, \gamma}$ depends only on $d$ and $\gamma$.

**Proof of Lemma 1.** We write

\[
\Delta a_{\ell} := a_{\ell} - a_{\ell+1}.
\]

For all $K \in \mathbb{N}_0$ denote by $a_{\ell}^{(K)}$ the following quantity

\[
a_{\ell}^{(K)} = a_{\ell}^{(K)}(\gamma, d) := (1 + \lambda_{\ell})^{-\frac{d}{2} - K} (\ln (3 + \lambda_{\ell}))^{-2\gamma}.
\]

(36)
An application of Abel summation yields

\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} a_{\ell}^{(0)} Z(d, \ell) P_{\ell}^{(d)}(\langle x_i, x_j \rangle) \]

\[
- \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} \Delta a_{\ell}^{(0)} \sum_{k=0}^{\ell} Z(d, k) P_{k}^{(d)}(\langle x_i, x_j \rangle) \]

\[
- a_{n+1}^{(0)} \sum_{k=0}^{n} Z(d, k) \frac{1}{N^2} \sum_{i,j=1}^{N} P_{k}^{(d)}(\langle x_i, x_j \rangle). \quad (37)
\]

Here and further we use that for all \(k, \ell \in \mathbb{N}_0\)

\[
\sum_{i,j=1}^{N} P_{\ell}^{(d/2 - 1 + k; d/2 - 1)}(\langle x_i, x_j \rangle) \geq 0, \quad (38)
\]

which follows from the fact that all coefficients in (15) and (16) are positive and the fact that \(P_{\ell}^{(d)}\) is a positive definite function in the sense of Schoenberg [16].

From (37) we obtain the following upper estimate

\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} a_{\ell}^{(0)} Z(d, \ell) P_{\ell}^{(d)}(\langle x_i, x_j \rangle) \]

\[
\leq \frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} \Delta a_{\ell}^{(0)} \sum_{k=0}^{\ell} Z(d, k) P_{k}^{(d)}(\langle x_i, x_j \rangle). \quad (39)
\]

Taking into account (15), applying Abel transform and formulas (16) and (36) \(K - 1\) times and using positive definiteness in every step we arrive at

\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=n+1}^{\infty} a_{\ell}^{(K)} \frac{2\ell + d - 1 + K(d + K - 1)\ell}{d - 1 + K} P_{\ell}^{(d+K-1; d-1)}(\langle x_i, x_j \rangle). \quad (40)
\]
From formulas (4) and (5) we get
\[
\frac{(d + K - 1) \ell}{(d/2)\ell} = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d + K - 1)} \frac{\Gamma(d + K - 1 + \ell)}{\Gamma\left(\frac{d}{2} + \ell\right)} \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d + K - 1)} \ell^{\frac{d}{2} + K - 1}. \tag{41}
\]

Relations (36), (39)-(41) yield (31) and Lemma 1 is proved.

Proof of Lemma 2. The upper estimate in (32) follows from (31). Let us show that the lower estimate is true.

Rewriting the squared worst-case error as above using \( K \) times iterated Abel transform and formulas (15), (16), (36) and (38), we obtain

\[
\text{wce}(Q[X_N]; H(d^2, \gamma)(S^d))^2 \gg \frac{1}{N^2} \sum_{i,j=1}^N \sum_{\ell=t+1}^{\infty} a_{t+1}^{(K)} \frac{2\ell - d + 1 + K \ell (d + K - 1)}{d - 1 + K} P_{\ell}^{(\frac{d}{2} + K - 1)}(\langle x_i, x_j \rangle)
\]
\[
- \sum_{m=0}^{K-1} a_{t+1}^{(m)} \frac{1}{N^2} \sum_{i,j=1}^N \frac{(d + m)\ell}{(d/2)\ell} P_{\ell}^{(\frac{d}{2} + m, \frac{d}{2} - 1)}(\langle x_i, x_j \rangle). \tag{42}
\]

Because of the exactness of the numerical integration rule for polynomials of degree \( \leq t \) and of (19), we have

\[
\frac{1}{N} \sum_{i=1}^N P_{\ell}^{(\frac{d}{2} + m, \frac{d}{2} - 1)}(\langle x_i, x_j \rangle) = \int_{S^d} P_{\ell}^{(\frac{d}{2} + m, \frac{d}{2} - 1)}(\langle x_i, x_j \rangle) d\sigma_d(x)
\]
\[
= 2^{d-1} \frac{\Gamma\left(\frac{d+1}{2}\right) (m + 1)\ell \Gamma\left(\frac{d}{2} + t\right)}{t! \Gamma(d + t)} \approx \ell^{m-\frac{d}{2}}. \tag{43}
\]

From (11) and (36) we obtain the order estimate

\[
a_{t+1}^{(m)} (d + m)\ell \ell^{m-\frac{d}{2}} \approx t^{-d-2m} (\ln t)^{-2\gamma} t^{m+\frac{d}{2}} \ell^{m-\frac{d}{2}} = t^{-d} (\ln t)^{-2\gamma}. \tag{44}
\]

Formulas (11), (42) - (44) imply that

\[
\text{wce}(Q[X_N]; H(d^2, \gamma)(S^d))^2 \gg \frac{1}{N^2} \sum_{i,j=1}^N \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K (\ln \ell)^{-2\gamma}} P_{\ell}^{(\frac{d}{2} + K - 1, \frac{d}{2} - 1)}(\langle x_i, x_j \rangle) - C_{d,\gamma} t^{-d} (\ln t)^{-2\gamma}. \tag{45}
\]
Thus, Lemma 2 is proved.

\textbf{Proof of Lemma 3.} For each \(i \in \{1, \ldots, N\}\) we divide the sphere \(S^d\) into an upper hemisphere \(H_i^+\) with 'north pole' \(x_i\) and a lower hemisphere \(H_i^-\):
\[
H_i^+ := \left\{ x \in S^d \mid \langle x_i, x \rangle \geq 0 \right\},
\]
\[
H_i^- := S^d \setminus H_i^+.
\]

Because the spherical cap \(S(x_i; \alpha_N)\), where \(\alpha_N := \arccos \left(1 - \frac{c^2}{8N^2}\right)\), contains no points of \(X_N\) in its interior, except for the point \(x_i\), we obtain
\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K} (\ln \ell)^{-2\gamma} P_{\ell}^{\left(\frac{d}{2} + K - 1, \frac{d}{2} - 1\right)}(\langle x_i, x_j \rangle)
\]
\[
= \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in H_j^+ \setminus \{x_j; \alpha_N\}}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K} (\ln \ell)^{-2\gamma} P_{\ell}^{\left(\frac{d}{2} + K - 1, \frac{d}{2} - 1\right)}(\langle x_i, x_j \rangle)
\]
\[
\quad + \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in S(-x_j; \alpha_N)}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K} (\ln \ell)^{-2\gamma} P_{\ell}^{\left(\frac{d}{2} + K - 1, \frac{d}{2} - 1\right)}(\langle x_i, x_j \rangle)
\]
\[
\quad + \frac{1}{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K} (\ln \ell)^{-2\gamma} P_{\ell}^{\left(\frac{d}{2} + K - 1, \frac{d}{2} - 1\right)}(1). \quad \text{(46)}
\]

From \((10)\) and the relation
\[
\sum_{j=n+1}^{\infty} \xi(j) = \int_{n}^{\infty} \xi(u) du + O(\xi(n)),
\]
which holds for any positive and decreasing function \(\xi(u), u \geq 1\), such that \(\int_{n}^{\infty} \xi(u) du < \infty\), we have
\[
\frac{1}{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K} (\ln \ell)^{-2\gamma} P_{\ell}^{\left(\frac{d}{2} + K - 1, \frac{d}{2} - 1\right)}(1) \sim \frac{1}{\Gamma\left(\frac{d}{2} + K\right)} \frac{1}{N} \sum_{\ell=t+1}^{\infty} \ell^{-1} (\ln \ell)^{-2\gamma}
\]
\[
= C_{d, \gamma} \left(\frac{1}{N} (\ln t)^{1-2\gamma} + O\left(\frac{1}{N} (\ln t)^{-2\gamma}\right)\right). \quad \text{(47)}
\]
Now we estimate the second term from the equality (46). An application of equality (11) yields

\[
\frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K}(\ln \ell)^{-2\gamma} P_{\ell}^{\frac{d}{2}+K-1,\frac{d}{2}-1}\left(\langle x_i, x_j \rangle\right)
\]

\[
= \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in S(-x_j; \alpha_N)}^{N} \sum_{\ell=t+1}^{\infty} (-1)^\ell \ell^{-\frac{d}{2}-K}(\ln \ell)^{-2\gamma} P_{\ell}^{\frac{d}{2}-1,\frac{d}{2}+K-1}\left(-\langle x_i, x_j \rangle\right).
\]

(48)

If \(x_i \in S(-x_j; \alpha_N)\), then

\[- \langle x_i, x_j \rangle \geq \cos \alpha_N.\]

(49)

From the elementary estimates

\[\sin \theta \leq \theta \leq \frac{\pi}{2} \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2},\]

we obtain

\[
\left(1 - \frac{c_3^2}{16N^2\pi}\right)^{\frac{1}{2}} \frac{c_3}{2N^{\frac{3}{2}}} \leq \alpha_N \leq \frac{\pi}{4} \left(1 - \frac{c_3^2}{16N^2\pi}\right)^{\frac{1}{2}} \frac{c_3}{N^{\frac{3}{2}}}.\]

(50)

As for the sequence \(X_N\), condition (23) holds, it means that the spherical cap \(S(-x_j; \alpha_N), j = 1, \ldots, N\), contains at most one point of the sequence \(X_N\). This fact and formulas (48)–(50) imply

\[
\left| \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in S(-x_j; \alpha_N)}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K}(\ln \ell)^{-2\gamma} P_{\ell}^{\frac{d}{2}+K-1,\frac{d}{2}-1}\left(\langle x_i, x_j \rangle\right) \right|
\]

\[
\leq \frac{1}{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K}(\ln \ell)^{-2\gamma} \left| P_{\ell}^{\frac{d}{2}-1,\frac{d}{2}+K-1}\left(\cos \theta_N\right) \right|,
\]

(51)

for some \(\theta_N\) satisfying

\[0 \leq \theta_N \leq \frac{\pi}{4} \left(1 - \frac{c_3^2}{16N^2\pi}\right)^{\frac{1}{2}} \frac{c_3}{N^{\frac{3}{2}}}.\]

(52)
Let \( \theta_N > 0 \) and \( \ell^* \in \mathbb{N} \) is such that
\[
\frac{1}{\ell^* + 1} \leq \theta_N \leq \frac{1}{\ell^*},
\]
and \( \ell^* = \infty \), if \( \theta_N = 0 \).

Then, applying the estimates (13), (25) and (52), we get
\[
\frac{1}{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d-K}{2}} \ln \ell^{-2\gamma} P_{\ell}^{\left(\frac{d}{2}-1,\frac{d}{2}+K-1\right)}(\cos \theta_N) \ll \frac{1}{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d-K}{2}} \ln \ell^{-2\gamma} \ell^\frac{d}{2} - 1
\]
\[+ \frac{1}{N} \theta_N^{-\frac{1}{2}-\frac{d}{2}+1} \sum_{\ell=\ell^*+1}^{\infty} \ell^{-\frac{d-K}{2}} \ln \ell^{-2\gamma} \ell^{-\frac{1}{2}} \ll N^{-\frac{K}{2}-1} (\ln N)^{-2\gamma}. \tag{53} \]

Now let us show that
\[
\frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d-K}{2}} \ln \ell^{-2\gamma} P_{\ell}^{\left(\frac{d}{2}+K-1,\frac{d}{2}-1\right)}(\langle x_i, \pm x_j \rangle) \ll \frac{1}{N} \ln (t)^{-2\gamma}. \tag{54} \]

Using formula (12), we have that for \( 0 < \theta < \pi \)
\[
|P_{\ell}^{\left(\frac{d}{2}+K-1,\frac{d}{2}-1\right)}(\cos \theta)| \ll \ell^{-\frac{d}{4}} (\sin \theta)^{-\frac{d}{4} - K + \frac{1}{2}} + \ell^{-\frac{d}{4}} (\sin \theta)^{-\frac{d}{4} - K - \frac{1}{2}}. \tag{55} \]

Then
\[
\left| \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d-K}{2}} \ln \ell^{-2\gamma} P_{\ell}^{\left(\frac{d}{2}+K-1,\frac{d}{2}-1\right)}(\cos \theta) \right| \ll (\sin \theta)^{-\frac{d}{4} - K + \frac{1}{2}} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d-K}{2}} \ln \ell^{-2\gamma}
\]
\[+ (\sin \theta)^{-\frac{d}{2} - K - \frac{1}{2}} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2} - K - \frac{3}{2}} \ln \ell^{-2\gamma}
\]
\[\ll (\sin \theta)^{-\frac{d}{4} - K + \frac{1}{2}} t^{-\frac{d}{2} - K + \frac{1}{2}} (\ln t)^{-2\gamma} + (\sin \theta)^{-\frac{d}{4} - K - \frac{1}{2}} t^{-\frac{d}{2} - K - \frac{1}{2}} (\ln t)^{-2\gamma}. \]

We define \( \theta_{ij}^\pm \in [0, \pi] \) by \( \cos \theta_{ij}^\pm := \langle x_i, \pm x_j \rangle \). Then \( \sin \theta_{ij}^+ = \sin \theta_{ij}^- \).
So,
\[
\left| \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{\ell=t+1}^{\infty} \ell^{-\frac{d}{2}-K} (\ln \ell)^{-2\gamma} P_{\ell}^{\left(\frac{d}{2}+K-1,\frac{d}{2}-1\right)} \left( (x_i, x_j) \right) \right| \\
\ll t^{-\frac{d}{2}-K+\frac{1}{2}} (\ln t)^{-2\gamma} \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in H^0 \setminus S(\pm x_j, \alpha_N)} (\sin \theta_{ij}^\pm)^{-\frac{d}{2}-K+\frac{1}{2}} \\
+ t^{-\frac{d}{2}-K-\frac{1}{2}} (\ln t)^{-2\gamma} \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in H^0 \setminus S(\pm x_j, \alpha_N)} (\sin \theta_{ij}^\pm)^{-\frac{d}{2}-K-\frac{1}{2}}. 
\tag{56}
\]

From [7, (3.30) and (3.33)], it follows that
\[
\frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in H^0 \setminus S(\pm x_j, \alpha_N)} (\sin \theta_{ij}^\pm)^{-\frac{d}{2}+\frac{1}{2}+k-L} \\
\ll 1 + n^{L+k-(d+1)/2}, \quad k = 0, 1, \ldots \quad \text{for } L > \frac{d+1}{2}. 
\tag{57}
\]

Choosing \(K > \frac{d+1}{2}\), applying (25) and (57), we obtain
\[
\frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1, x_i \in H^0 \setminus S(\pm x_j, \alpha_N)} (\sin \theta_{ij}^\pm)^{-\frac{d}{2}-K+\frac{1}{2}} \ll 1 + (N^{\frac{1}{2}})^{K-\frac{d}{2}+\frac{1}{2}} \ll (N^{\frac{1}{2}})^{K-\frac{d}{2}+\frac{1}{2}}. 
\tag{58}
\]

Formulas (25), (56) and (58) now imply that estimate (54) holds.

From (25), (47), (53) and (54) we obtain (33) and Lemma 3 is proved. \(\square\)

4. Lower bounds for the worst-case error

The main result of this section is the following theorem.

**Theorem 4.** Let \(d \geq 2\), \(\gamma > \frac{1}{2}\), \(Q[X_N, \omega]\) is an arbitrary \(N\)-point cubature rule. Then, there exists a positive constant \(C_{d,\gamma}\) such that
\[
\text{wce}(Q[X_N, \omega], \mathcal{H}^{\alpha}(S^d)) \geq C_{d,\gamma} N^{\frac{1}{2}} (\ln N)^{-\gamma}. 
\tag{59}
\]

The constant \(C_{d,\gamma}\) depends only on \(d\) and \(\gamma\), but is independent of the rule \(Q[X_N, \omega]\) and the number of nodes \(N\) of the rule.
In [12] for case \( d = 2 \) and in [10] for all \( d \geq 2 \) the lower bound
\[
\text{wce}(Q[X_N, \omega]; \mathbb{H}^s(\mathbb{S}^d)) \gg N^{-\frac{d}{4}}
\]
was found.

Before we actually give the proof of Theorem 4, we formulate a packing result [10, Lemma 1].

**Statement 1.** Let \( d \geq 2 \). Then there exist constants \( \tilde{c}_1 > 0 \) and \( \tilde{c}_2 \geq 1 \) depending only on \( d \), such that for any \( N \in \mathbb{N} \), there exist \( N_0 \) points \( y_1, \ldots, y_{N_0} \) on \( \mathbb{S}^d \) and an angle \( \beta_N \), with
\[
\beta_N = \tilde{c}_1(2N)^{-\frac{1}{4}},
\]
\[
2N \leq N_0 \leq \tilde{c}_22N,
\]
such that the caps \( S(y_i; \beta_N) \), \( i = 1, \ldots, N_0 \) form a packing of \( \mathbb{S}^d \) (that is \( S(y_i; \beta_N) \) and \( S(y_j; \beta_N) \) with \( i \neq j \) touch at most at their boundaries).

As we consider a packing with \( 2N \geq 2 \) caps in Statement 1, the angle \( \beta_N \) can be at most \( \frac{\pi}{2} \) (which is achieved for a packing with 2 caps with opposite centres).

**Proof of Theorem 4.** To prove the lower bound we will use the same 'fooling' function as in [10], that is a function which vanishes in all nodes of the cubature rule \( Q[X_N, \omega] \) but has large integral and small norm.

At the beginning we construct the function \( \Phi \in \mathbb{C}^\infty(\mathbb{R}) \) with the following properties: (i) \( \Phi(t) \geq 0 \) for all \( t \in \mathbb{R} \); (ii) \( \max_{t \in \mathbb{R}} \Phi(t) = \Phi(0) = 1 \); (iii) \( \Phi \) has the compact support \( \text{supp}(\Phi) = [-1, 1] \).

Statement 1 guarantees that there exist at least \( 2N \) spherical caps \( S(y_i; \beta_N) \), which touch at most at their boundaries. Consequently, at least \( N \) of these spherical caps do not contain any node of the cubature rule in their interior.

We shift the argument of the function \( \Phi \) in such a way, that the support of the function will be \( [\cos \beta_N, \cos \frac{\beta_N}{2}] \).

The scaled version of \( \Phi \) is given by
\[
\Phi_N(t) := \Phi \left( \frac{2t - \left( \cos \frac{\beta_N}{2} + \cos \beta_N \right)}{2 \sin \frac{3\beta_N}{4} \sin \frac{\beta_N}{4}} \right), \quad t \in \mathbb{R}.
\]

We define our 'fooling' function \( f_N \in \mathbb{C}^\infty(\mathbb{S}^d) \) by
\[
f_N(x) := \sum_{i=1}^{N} \Phi_N(\langle x, y_i \rangle), \quad x \in \mathbb{S}^d.
\]
In [10] it was proved that for all \( s \geq 0 \)
\[
\|f_N\|_{H^s} \leq C_{s,d} N^{\frac{s}{2}}.
\] (61)

The function \( f_N \) vanishes in all nodes of the cubature rule, that is, \( Q[X_N, \omega](f_N) = 0 \). And (see formula (33) of [10])
\[
I(f_N) \geq c_d.
\]

Hence,
\[
\text{wce}(Q[X_N, \omega]; \mathbb{H}^{(\frac{d}{2},\gamma)}(S^d)) \geq \left| Q[X_N, \omega] \left( \frac{f_N}{\|f_N\|_{H^{(\frac{d}{2},\gamma)}}} \right) - I \left( \frac{f_N}{\|f_N\|_{H^{(\frac{d}{2},\gamma)}}} \right) \right| = \frac{I(f_N)}{\|f_N\|_{H^{(\frac{d}{2},\gamma)}}} \gg \frac{1}{\|f_N\|_{H^{(\frac{d}{2},\gamma)}}}. \quad (62)
\]

The function \( \Phi_N \) can be expanded on \([-1, 1]\) into an \( L^2([-1, 1]) \) convergent Laplace series
\[
\Phi_N = \sum_{\ell=0}^{\infty} Z(d, \ell) \left( \int_{-1}^{1} \Phi_N(t) P^{(d)}_\ell(t) dt \right) P^{(d)}_\ell.
\]

Hence,
\[
f_N(x) = \sum_{i=1}^{N} \sum_{\ell=0}^{\infty} Z(d, \ell) \left( \int_{-1}^{1} \Phi_N(t) P^{(d)}_\ell(t) dt \right) P^{(d)}_\ell(\langle x, y_i \rangle).
\] (63)

Due to the definition (21), representation (63), the addition theorem (8) and inequality (61), we have the following estimate
\[
\|f_N\|_{H^s}^2 = \sum_{\ell=0}^{\infty} \left( \int_{-1}^{1} \Phi_N(t) P^{(d)}_\ell(t) dt \right)^2 (1 + \lambda \ell)^s Z(d, \ell) \sum_{i,j=1}^{N} P^{(d)}_\ell(\langle y_i, y_j \rangle) \ll N^{\frac{s}{2}},
\] (64)
which holds for \( s > 0 \) by [7].
The corresponding norm of the function $f_N$ in $\mathbb{H}^{(d, \gamma)}$ has the form

$$
\|f_N\|^2_{\mathbb{H}^{(d, \gamma)}} = \sum_{\ell=0}^{\lfloor N^{\frac{d}{2}} \rfloor} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 w_{\ell}(d, \gamma) Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle)
$$

$$
+ \sum_{\ell=\lfloor N^{\frac{d}{2}} \rfloor + 1}^{\infty} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 w_{\ell}(d, \gamma) Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle). 
$$

Taking into account (38) and setting $s = 1$ in (64), we obtain

$$
\sum_{\ell=0}^{\lfloor N^{\frac{d}{2}} \rfloor} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 (1 + \lambda_\ell) Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle) \ll N^{\frac{d}{2}}. 
$$

Thus, (66) yields

$$
\sum_{\ell=0}^{\lfloor N^{\frac{d}{2}} \rfloor} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 w_{\ell}(d, \gamma) Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle)
$$

$$
\ll (N^{\frac{d}{2}})^{d-2} (\ln N)^{2\gamma} \sum_{\ell=0}^{\lfloor N^{\frac{d}{2}} \rfloor} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 (1 + \lambda_\ell) Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle)
$$

$$
\ll N (\ln N)^{2\gamma}. 
$$

(67)

Setting $s = \frac{d+1}{2}$ in (64), we get

$$
\sum_{\ell=\lceil N^{\frac{d}{2}} \rceil + 1}^{\infty} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 (1 + \lambda_\ell)^{\frac{d+1}{2}} Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle) \ll N^{1+\frac{d}{2}}. 
$$

(68)
Thus, (68) yields

\[ \sum_{\ell = \lceil \frac{N}{2} \rceil + 1}^{\infty} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 w_{\ell}(d, \gamma) Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle) \]

\[ \ll N^{-\frac{1}{2}} (\ln N)^{2\gamma} \sum_{\ell = \lceil \frac{N}{2} \rceil + 1}^{\infty} \left( \int_{-1}^{1} \Phi_N(t) P_{\ell}^{(d)}(t) dt \right)^2 (1 + \lambda_{\ell}) \frac{d+1}{d} Z(d, \ell) \sum_{i,j=1}^{N} P_{\ell}^{(d)}(\langle y_i, y_j \rangle) \]

\[ \ll N (\ln N)^{2\gamma}. \] (69)

Estimates (65), (67) and (69) imply

\[ \| f_N \|_{\tilde{L}^{2\gamma}} \ll N^{\frac{1}{2}} (\ln N)^{\gamma}. \] (70)

From (62) and (70) we obtain (59) and Theorem 4 is proved. \quad \Box

We should remark, that we can obtain Theorem 4 in the case of equal weights by simply applying [3, Theorem 4.2].

Let the zonal function \( F: F(x, y) = F(\langle x, y \rangle) \), \( x, y \in S^d \) be continuous on the segment \([-1, 1]\) and have the form

\[ F(x, y) = \sum_{\ell=0}^{\infty} \hat{F}(d, \ell) Z(d, \ell) P_{\ell}^{(d)}(\langle x, y \rangle), \] (71)

where \( \hat{F}(d, \ell) \geq 0 \).

The following Statement 2 is [3, Theorem 4.2].

**Statement 2.** Let \( \lambda = \frac{d-1}{2} \). Assume that \( F \) satisfies relation (71). Then there exists positive constants \( c_d \) and \( C_d \) depending only on \( d \) and \( F \), such that for any \( N \in \mathbb{N} \) and a given set of \( N \) points \( X_N = \{x_1, ..., x_N\} \subset S^d \) the inequality

\[ C_d \min_{1 \leq \ell \leq c_d N^{1/d}} \hat{F}(d, \ell) \leq \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} F(\langle x_i, x_j \rangle) - \hat{F}(d, 0) \] (72)

holds.
Applying this statement to \( F = \tilde{K}_{d,\gamma} \) gives

\[
\text{wce}(Q[X_N]; \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d))^2 = \sum_{i,j=1}^{N} \sum_{\ell=1}^{\infty} w_\ell^{-1}(d,\gamma) Z(d,\ell) P^{(d)}((x_i, x_j)) \geq C_d \min_{1 \leq \ell \leq c_d N^{1/d}} w_\ell^{-2}(d,\gamma) \gg C_d N^{-2\gamma}. \tag{73}
\]

And, therefore,

\[
\text{wce}(Q[X_N]; \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d)) \geq C_{d,\gamma} N^{-1/2} (\ln N)^{-\gamma}. \tag{74}
\]

In the same way, by applying (72), one can easily obtain estimate (60) in the case of equal weights.

5. QMC designs for \( \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d) \) and their properties

5.1. QMC designs for \( \mathbb{H}^{s}(\mathbb{S}^d) \) and \( \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d) \)

Let us formulate at the beginning the definition of QMC-designs for Sobolev spaces \( \mathbb{H}^{s}(\mathbb{S}^d) \) (see, e.g. [8]).

**Definition 4.** Given \( s > \frac{d}{2} \), a sequence \( X_N \) of \( N \)-point configurations on \( \mathbb{S}^d \) with \( N \to \infty \) is said to be a sequence of QMC designs for \( \mathbb{H}^{s}(\mathbb{S}^d) \) if there exists \( c(s, d) > 0 \), independent of \( N \), such that

\[
\sup_{f \in \mathbb{H}^{s}, \|f\|_{\mathbb{H}^{s}} \leq 1} \left| \frac{1}{N} \sum_{x \in X_N} f(x) - \int_{\mathbb{S}^d} f(x) d\sigma_d(x) \right| \leq \frac{c(s, d)}{N^{s}}. \tag{75}
\]

We define the notion of a sequence of QMC designs for \( \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d), \gamma > \frac{1}{2} \), as it was defined for Sobolev classes \( \mathbb{H}^{s}(\mathbb{S}^d), s > \frac{d}{2} \).

**Definition 5.** Given \( \gamma > \frac{1}{2} \), a sequence \( (X_N)_N \) of \( N \)-point configurations on \( \mathbb{S}^d \) with \( N \to \infty \) is said to be a sequence of QMC designs for \( \mathbb{H}^{(\frac{d}{2},\gamma)}(\mathbb{S}^d) \) if there exists \( c(\gamma, d) > 0 \), independent of \( N \), such that

\[
\sup_{f \in \mathbb{H}^{(\frac{d}{2},\gamma)}, \|f\|_{\mathbb{H}^{(\frac{d}{2},\gamma)}} \leq 1} \left| \frac{1}{N} \sum_{x \in X_N} f(x) - \int_{\mathbb{S}^d} f(x) d\sigma_d(x) \right| \leq \frac{c(\gamma, d)}{N^{\frac{1}{2}} (\ln N)^{\gamma - \frac{1}{2}}} \tag{76}
\]
Theorem 5. Given $s > \frac{d}{2}$, let $(X_N)_N$ be a sequence of QMC designs for $H^s(S^d)$. Then $(X_N)_N$ is a sequence of QMC designs for $H^{(\frac{d}{2}, \gamma)}(S^d)$, for all $\gamma > \frac{1}{2}$.

Theorem 6. Given $\gamma > \frac{1}{2}$, let $(X_N)_N$ be a sequence of QMC designs for $H^{(\frac{d}{2}, \gamma)}$. Then $(X_N)_N$ is a sequence of QMC designs for $H^{(\frac{d}{2}, \gamma')} (S^d)$, for all $\frac{1}{2} < \gamma' \leq \gamma$.

We will prove here only Theorem 5. The proof of Theorem 6 follows the same lines as that of Theorem 5 with some additional estimations.

Proof of Theorem 5 is based on the following lemma.

Lemma 4. Assume that there exists a $\delta > 0$, such that

$$\text{wce}(Q[X_N]; H^s(S^d)) \ll N^{-\delta},$$

holds for some $s > \frac{d}{2}$. Then for $\gamma > \frac{1}{2}$ there exists a constant $C(d, s, \delta, \gamma)$ such that for all $N$

$$\text{wce}(Q[X_N]; H^{(\frac{d}{2}, \gamma)}(S^d)) < C(d, s, \delta, \gamma)[\text{wce}(Q[X_N]; H^s(S^d))]^{\frac{d}{2}}(\ln N)^{-\gamma + \frac{1}{2}}$$

holds.

Proof of Lemma 4. The proof of (78) goes along the lines as that of Lemma 26 in [8] and Theorem 3.1 in [6].

We write

$$\frac{1}{(1 + \lambda t)^2(\ln(3 + \lambda t))^{2\gamma}} = \int_0^\infty e^{-(1 + \lambda t)t} g(t)dt,$$

in terms of the Laplace transform of the function $g$ given by the inverse Laplace transform

$$g(t) = g(d, \gamma, t) := \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} z^{-\frac{d}{2}}(\ln(z + 2))^{-2\gamma} e^{tz}dz.$$
First of all, let us show, that the function \( g \) satisfies
\[
|g(t)| \ll \begin{cases} 
 t^{d-1}, & \text{if } t \geq 1, \\
 t^{d-1}(\ln \frac{1}{t})^{-2\gamma}, & \text{if } 0 < t < 1.
\end{cases}
\] (81)

Indeed, substituting \( tz = 1 + ix \) and integrating by parts, we obtain
\[
\frac{1}{2\pi i} \int_{\frac{1}{t} + i\infty}^{\frac{1}{t} - i\infty} z^{-\frac{d}{2}}(\ln(z + 2))^{-2\gamma} e^{tx} dz
\]
\[
= \frac{e}{2\pi} t^{d-1} \int_{-\infty}^{\infty} (1 + ix)^{-\frac{d}{2}} \left( \ln \left( 2 + \frac{1 + ix}{t} \right) \right)^{-2\gamma} e^{ix} dx
\]
\[
= \frac{e}{2\pi} t^{d-1} \int_{-\infty}^{\infty} e^{ix} \left[ \frac{d}{2}(1 + ix)^{-\frac{d}{2}-1} \left( \ln \left( 2 + \frac{1 + ix}{t} \right) \right)^{-2\gamma} + 2\gamma(1 + ix)^{-\frac{d}{2}}(2t + 1 + ix)^{-1} \left( \ln \left( 2 + \frac{1 + ix}{t} \right) \right)^{-2\gamma-1} \right] dx. \tag{82}
\]

For large values of \( t : t \geq 1 \) from (82) one can easily get \( |g(t)| \ll t^{d-1} \).
In turn, for small values of \( t : 0 < t < 1 \), the inequalities
\[
\left| \ln \left( 2 + \frac{1 + ix}{t} \right) \right| > \ln \frac{1}{t}, \quad \left( \ln \frac{1}{t} \right)^{-2\gamma-1} < \left( \ln \frac{1}{t} \right)^{-2\gamma}, \quad 0 < t < 1,
\]
and relation (82) imply that \( |g(t)| \ll t^{d-1}(\ln \frac{1}{t})^{-2\gamma} \).

The representation of the worst-case error (23) allows to write
\[
\text{wce}(Q[X_N]; \mathbb{H}(\frac{d}{2},\gamma)(\mathbb{S}^d))^2 = \int_0^\infty e^{-t} g(t) h(t) dt, \tag{83}
\]
where
\[
h(t) = h(t; x_1, ..., x_N) := \frac{1}{N^2} \sum_{i,j=1}^N \tilde{H}(t, (x_i, x_j)), \tag{84}
\]
and \( \tilde{H} \) denotes the heat kernel with the constant term removed:
\[
1 + \tilde{H}(t, x, y) := \sum_{\ell=0}^\infty e^{-\lambda^t} Z(d, \ell) P^{(d)}_\ell((x, y)), \quad x, y \in \mathbb{S}^d. \tag{85}
\]
which is fundamental solution to the heat equation $\frac{\partial u}{\partial t} + \Delta u = 0$ on $\mathbb{R}_+ \times \mathbb{S}^d$.

The worst-case error for Sobolev spaces in terms of Laplace transform can be written in the form (see formula (46) in [8])

$$\text{wce}(Q[X_N]; \mathbb{H}^s(\mathbb{S}^d))^2 = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1} h(t) dt.$$  \hspace{1cm} (86)

Let $\varepsilon := [\text{wce}(Q[X_N]; \mathbb{H}^s(\mathbb{S}^d))]^\frac{1}{2}$, and $\varepsilon \ll N^{-\delta} < 1$ by assumption. The first inequality from (81) and (86) yield

$$\left| \int_1^\infty e^{-t} g(t) h(t) dt \right| \ll \int_1^\infty e^{-t} t^{\frac{d}{2}-1} h(t) dt \ll \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1} h(t) dt = \varepsilon^s, \ s > \frac{d}{2}.$$  \hspace{1cm} (87)

Taking into account the second inequality from (81), (77) and (86), we get

$$\left| \int_{\frac{d}{2}}^1 e^{-t} g(t) h(t) dt \right| \ll \int_{\frac{d}{2}}^1 e^{-t} t^{\frac{d}{2}-1} \left( \ln \frac{1}{t} \right)^{-2\gamma} h(t) dt$$

$$\leq \left( \frac{\varepsilon}{2} \right)^{\frac{d}{2}-s} \left( \ln \left( \frac{2}{\varepsilon} \right) \right)^{-2\gamma} \int_{\frac{d}{2}}^1 e^{-t} t^{s-1} h(t) dt$$

$$\ll \varepsilon^{\frac{d}{2}-s} (\ln N)^{-2\gamma} \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{s-1} h(t) dt = \varepsilon^{\frac{d}{2}} (\ln N)^{-2\gamma}. \hspace{1cm} (88)$$

In [8] it was proved, that $h(t)$ is uniformly bounded on $[0, 1)$, and for $0 < t < \frac{d}{2}$ the following estimate holds

$$t^{\frac{d}{2}} h(t) \ll \varepsilon^{\frac{d}{2}}. \hspace{1cm} (89)$$
Applying \((81)\), relations \((77)\) and \((89)\), we arrive at the estimate
\[
\left| \int_0^\varepsilon e^{-t} g(t) dt \right| \ll \int_0^\varepsilon e^{-t} \frac{1}{t} \left( \ln \frac{1}{t} \right)^{-2\gamma} h(t) dt
\]
\[
\ll \varepsilon^\frac{d}{2} \int_0^\frac{\varepsilon}{2} e^{-t} \frac{1}{t} dt < \varepsilon^\frac{d}{2} \int_0^1 \left( \ln \frac{1}{t} \right)^{-2\gamma} h(t) dt
\]
\[
= \frac{1}{2\gamma - 1} \varepsilon^\frac{d}{2} \left( \ln \frac{2}{\varepsilon} \right)^{-2\gamma + 1} \ll \varepsilon^\frac{d}{2} \left( \ln N \right)^{-2\gamma + 1}.
\]
\[(90)\]

Formulas \((83)\), \((87)\), \((88)\) and \((90)\) imply
\[
\text{wce}(Q[X_N]; \mathbb{H}^{(d, \gamma)}(S^d))^2 < C(d, s, \gamma) \text{wce}(Q[X_N]; \mathbb{H}^{s}(S^d))^\frac{d}{2} \left( \ln N \right)^{-\gamma + \frac{1}{2}}
\]
\[
\ll (N^{-\frac{d}{2}}) \left( \ln N \right)^{-\gamma + \frac{1}{2}} = N^{-\frac{1}{2}} \left( \ln N \right)^{-\gamma + \frac{1}{2}}
\]
and Theorem 5 is proved.

\[\square\]

5.2. Examples of QMC designs for classes \(\mathbb{H}^{(d, \gamma)}(S^d)\)

In \[8\] it was shown, that the maximisers of the generalised sum of distances
\[
\sum_{i,j=1}^N |x_i - x_j|^{2s-d}, \quad N = 2, 3, 4, \ldots
\]
form a sequence of QMC designs for \(\mathbb{H}^{s}(S^d)\) for \(s\) in the interval \((\frac{d}{2}, \frac{d}{2} + 1)\).

Consequently, from this fact and from Theorem 5 we obtain the statement.
**Theorem 7.** Let $\gamma > \frac{1}{2}$ and $0 < \alpha < 2$. Then, the maximisers of generalised sum of distances

$$\sum_{i,j=1}^{N} |x_i - x_j|^\alpha, \quad N = 2, 3, 4, \ldots$$

form a sequence of QMC designs for $\mathbb{H}(\frac{d}{2}\gamma)(\mathbb{S}^d)$.

**Theorem 8.** If $X_N^*$, $N = 2, 3, \ldots$, minimise the energy functional

$$\sum_{i,j=1}^{N} \tilde{K}_{\gamma,d}(x_i, x_j),$$

where $\tilde{K}_{\gamma,d}(x, y)$ is defined by (22), then there exists $C_{d,\gamma} > 0$, such that for all $N \geq 2$

$$\text{wce}(Q[X_N^*]; \mathbb{H}(\frac{d}{2}\gamma)(\mathbb{S}^d)) \leq \frac{C_{d,\gamma}}{N^{\frac{1}{\gamma}}(\ln N)^{\gamma - \frac{1}{2}}}.$$

Consequently, $X_N^*$ is a sequence of QMC designs for $\mathbb{H}(\frac{d}{2}\gamma)(\mathbb{S}^d)$.

**References**

[1] J. Beck, *Sums of distances between points on a sphere – an application of the theory of irregularities of distribution to discrete geometry*, Mathematika 31 (1984), 33–41.

[2] J. Beck and W. Chen, *Irregularities of distribution*, Tracts in Mathematics, vol. 89, Cambridge University Press, 1987.

[3] D. Bilyk and F. Dai, *Geodesic distance Riesz energy on the sphere*, arXiv:1612.08442v1, 2016.

[4] A. Bondarenko, D. Radchenko, and M. Viazovska, *Optimal asymptotic bounds for spherical designs*, Ann. of Math. (2) 178 (2013), no. 2, 443–452.

[5] ______, *Well-separated spherical designs*, Constr. Approx. 41 (2015), no. 1, 93–112.
[6] L. Brandolini, Ch. Choirat, L. Colzani, G. Gigante, R. Seri, and Travaglini G., *Quadrature rules and distribution of points on manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 13 (2014), no. 4, 889–923.

[7] J. S. Brauchart and K. Hesse, *Numerical integration over spheres of arbitrary dimension*, Constr. Approx. 25 (2007), no. 1, 41–71.

[8] J. S. Brauchart, E. B. Saff, I. H. Sloan, and R. S. Womersley, *QMC designs: optimal order quasi Monte Carlo integration schemes on the sphere*, Math. Comp. 83 (2014), no. 290, 2821–2851.

[9] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geometriae Dedicata 6 (1977), no. 3, 363–388.

[10] K. Hesse, *A lower bound for the worst-case cubature error on spheres of arbitrary dimension*, Numer. Math. 103 (2006), no. 3, 413–433.

[11] K. Hesse and I. H. Sloan, *Optimal lower bounds for cubature error on the sphere $S^2$*, J. Complexity 21 (2005), no. 6, 790–803.

[12] _____, *Cubature over the sphere $S^2$ in Sobolev spaces of arbitrary order*, J. Approx. Theory 141 (2006), no. 2, 118–133.

[13] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, 1966.

[14] Claus Müller, *Spherical harmonics*, Lecture Notes in Mathematics, vol. 17, Springer-Verlag, Berlin-New York, 1966.

[15] M. Reimer, *Hyperinterpolation on the sphere at the minimal projection order*, J. Approx. Theory 104 (2000), no. 2, 272–286.

[16] I. J. Schoenberg, *Positive definite functions on spheres*, Duke Math. J. 9 (1942), 96108.

[17] G. Szegő, *Orthogonal polynomials*, fourth ed., American Mathematical Society, Providence, R.I., 1975, American Mathematical Society, Colloquium Publications, Vol. XXIII.