T-Duality in Lattice Regularized $\sigma$-Models

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Abstract

It is shown that when the underlying sigma model of bosonic string theory is written in terms of single-valued fields, which live in the covering space of the target space, Abelian $T$-duality survives lattice regularization of the world-sheet. The projection onto the target-space is implemented through a sum over cohomology, which bears resemblance to summing over topological sectors in Yang-Mills theories. In particular, the case of string theory on a circle is shown to be explicitly self-dual in the lattice regulated model and automatically forbids vortex excitations which would otherwise destroy the duality. For other target spaces a generalized notion of $T$-duality is observed in which the target space and the cohomology coefficient group are interchanged under duality. Specific examples show that the fundamental group of the target space may not be preserved in the $T$-dual theory. Generalized models which exhibit $T$-duality behaviour, with dynamical variables that live on the $k$-dimensional cells of $(p+1)$-dimensional world-volumes, are also constructed. These models correspond to gauge theories, and higher-dimensional analogues, in which one sums over various topological sectors of the theory.

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Target space duality [4] is a symmetry of string theory which maps models defined on classically distinct target manifolds into one another. This is a rather surprising result when observed from the point of view of the target space. However, it has been known for some time now that the underlying principle of T-duality is intimately connected with the Hodge duality of forms on the world-sheet and is manifest in the sigma model that defines the theory [2, 3]. An interesting and useful question to ask is whether this duality survives lattice regularization of the world-sheet. The authors of [4] determined the potential, for a $D = 0$ matrix model, which preserved T-duality at the level of Feynman graphs. The question of whether this duality survives when spins are included on the sites of the graph was first studied by [5] where a string partition function for a discretized world-sheet with target space $S^1$ was formulated. They noticed that when the Hodge duality was applied on the lattice, which amounts to performing a Kramers-Wannier $S$-duality transformation [7], the T-duality of the continuum model was lost due to the existence of vortices, and the model undergoes a Kosterlitz-Thouless phase transition at a critical radius of the target space. This loss of T-duality was seen as a lattice artifact and was solved by altering the string partition function to forbid all vortex configurations. An ansatz which implemented this restriction was inserted by hand and the model regained its self-dual nature. In this letter we show that if one defines the partition function in terms of single-valued fields that are elements of the cover of the target space, while a sum over harmonic forms induces the projection down to the target space, then a straightforward lattice regularization of the model immediately implements such a constraint. The sum over harmonic forms can be thought of as a sum over large gauge transformations and is analogous to the schemes implemented in refs. [7, 8, 9] in which sums over the different theta sectors of (Super)-Yang-Mills theory was introduced in order to produce the correct $2\pi$ periodicity of the various correlators.

The idea of lifting from the target space, $S^1$, to the covering space, $\mathbb{R}$, is quite similar to what one does when quantizing on the circle [10]. This general strategy will be applied to non-linear sigma-models with target space $G/H$ (throughout this letter $G$ and $H$ are Abelian groups) where the world-sheet has been regulated by a lattice. The spins will be taken to be elements of the natural cover of $G/H$, which is $\tilde{G}$, and the projection onto the target space will be implemented through a sum over $H$-valued cohomology. On the lattice a generalized idea of duality is observed in which the target space and the coefficient group of the cohomology are interchanged. This is the analogue of the momentum and winding modes being interchanged under duality in the continuum theory. We determine choices of $G$ and $H$ which renders the model explicitly self-dual. In addition to the usual self-dual $S^1$ target space, we identity the target space $\mathbb{Z}_N$ with cover $\mathbb{Z}_{N^2}$ as new self-dual models. These models are the discrete versions
of the circular target space. We also construct generalized models in which a \( \mathcal{G}/\mathcal{H} \)-valued “spin” lives on the \((k - 1)\)-dimensional cells of a \((p + 1)\)-dimensional triangulated world-volume. These models have the interpretation of a \( p \)-brane on which a \( \mathcal{G}/\mathcal{H} \)-valued \((k - 1)\)-form field is defined. If, however, the world-volume is viewed as space-time, then the models correspond to modified theories of spins, gauge fields, antisymmetric tensors fields, etc..., where one sums over various topological sectors of the theory.

We begin with a discussion of \( T \)-duality in the continuum (for a review see for instance \([11, 12]\)), where the anti-symmetric tensor field is absent. The sigma model action is given by,

\[
S = \frac{1}{\sqrt{\alpha'}} \int_{\Sigma} d^2 \sigma \ G_{\mu\nu} \partial X^\mu \partial X^\nu
\]

(1)

here \( \Sigma \) is a fixed, but arbitrary, orientable two-dimensional world-sheet of genus \( g \), \( G_{\mu\nu} \) is the target space metric and we have trivialized the world-sheet metric. Consider the case in which the target space is \( S^1 \) and write \( X^0 = \theta \). The mode expansion of the \( \theta \) co-ordinate contains, in addition to the vibrational modes, winding modes corresponding to the string wrapping around the target space. Hence, \( \theta \) can be multi-valued, consequently as one moves along a non-contractable loop of the world-sheet \( \theta \) can pick up an extra factor of \( 2\pi \times \text{integer} \), i.e.,

\[
\oint_{\gamma_a} d\theta \in 2\pi \mathbb{Z}
\]

(2)

where \( \{\gamma_a : a = 1, \ldots, 2g\} \) are the canonical set of cycles which generate the first singular homology group of \( \Sigma \). Defining the partition function in terms of multi-valued fields is undesirable both from a pedagogical point of view, and since multi-valued fields do not exist in lattice regularization. Lifting \( \theta \) to the covering space of the circle, i.e. the real line, allows a natural decomposition into a smooth single-valued function and an element of the integer cohomology of the world-sheet: \( d\theta \rightarrow d\theta + 2\pi h \). The cohomology elements are the analogues of the winding modes in the mode expansion. It is important to realize that this ansatz for introducing single-valued fields is quite general. If the \( \theta \) co-ordinate takes values in \( \mathcal{G} \) and the cohomology in \( \mathcal{H} \), then the target space is the quotient space \( \mathcal{G}/\mathcal{H} \), which has \( \mathcal{G} \) as a natural cover. To make contact with the work of \([5]\) we first develop the case when \( \mathcal{G} = \mathbb{R} \) and \( \mathcal{H} = \mathbb{Z} \), in the latter part of this letter we will introduce the general models on the lattice. The new partition function (for a fixed surface \( \Sigma \)) is then written as,

\[
Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \int \mathcal{D}\theta \exp \left\{ -\frac{1}{\sqrt{\alpha'}} \int_{\Sigma} G^{00}(d\theta + 2\pi h) \wedge *(d\theta + 2\pi h) \right\}
\]

(3)

This will be the defining continuum theory, and all models introduced in this letter are straightforward lattice regularizations of this partition function and simple modifications of the coefficient groups. Notice that here there is an explicit sum over several partition functions each
defined on the cover of the target space and from the outset multi-valued fields are absent. Although it is a simple re-writing of the model, we will see that its lattice regularization leads to an explicitly self-dual theory without the insertion of any extra constraints.

We now demonstrate that (3) is explicitly self-dual. The strategy is a familiar one, first introduce a one-form $V$ which satisfies a Bianchi constraint plus holonomy constraints,

$$ Z = \int D V \, \delta (*dV) \left( \prod_{a=1}^{2g} \delta_{2\pi} \left( \oint_{\gamma_a} V \right) \right) \exp \left\{ -\frac{1}{\sqrt{\alpha'}} \int_{\Sigma} G^{00}(V \wedge *V) \right\} $$

(4)

Notice that there are two distinct types of delta-functions here, the first constraint implies that $V$ is the sum of an exact form plus cohomology elements with real coefficients, while the second periodic constraint forces the coefficients of the cohomology to be elements of $2\pi \mathbb{Z}$. Solving the constraints on $V$ leads back to the original model (3). Alternatively one can introduce Lagrange multipliers to implement the constraints,

$$ Z = \int D V \, D \tilde{\theta} \sum_{\tilde{h} \in H^1(\Sigma, \mathbb{Z})} \exp \left\{ -\frac{1}{\sqrt{\alpha'}} \int_{\Sigma} G^{00}(V \wedge *V) + i \, dV \wedge \tilde{\theta} + i \, V \wedge \tilde{h} \right\} $$

To represent the holonomy constraints we have used: $\oint_{\gamma_a} V = \oint_{\Sigma} V \wedge h^a$ where $\{h^a\}$ are the canonical set of cohomology elements dual to the cycles $\{\gamma_a\}$: $\oint_{\gamma_a} h^b = \delta^b_a$. The one-form $V$ now appears only quadratically in the action, and it can be eliminated via its equations of motion to give the dual partition function\[1\],

$$ Z = \sum_{\tilde{h} \in H^1(\Sigma, \mathbb{Z})} \int D \tilde{\theta} \exp \left\{ -\sqrt{\alpha'} \int_{\Sigma} \frac{1}{4G^{00}} \left( d\tilde{\theta} + \tilde{h} \right) \wedge * (d\tilde{\theta} + \tilde{h}) \right\} $$

Here the replacement of the old fields with the Lagrange multiplier fields is the analogue of interchanging the winding and momentum modes in the mode expansion of the string coordinates. Of course, in addition, the target space metric transformed as $G^{00} \leftrightarrow \alpha'/4G^{00}$.

It is instructive to demonstrate that a lattice regularization of (3) eliminates vortex configurations and leads directly to the constrained model in [5]. In order to express the results in a manner which is easily generalized, we introduce some notations of simplicial homology (see for example [13]). Let $\Sigma$ be a triangulation of a smooth manifold and $\{c_k^{(i)}\}$ be the generators of the chain complex $(C_*(\Sigma, \mathcal{G}), \partial)$ with Abelian coefficient group $\mathcal{G}$ (group multiplication is written additively) and boundary operator $\partial$ defined by the incidence numbers $[,]$,

$$ \partial_k c_k^{(i)} = \sum_{j=1}^{N_k} [c_k^{(i)} : c_{k-1}^{(j)}] c_{k-1}^{(j)} $$

\footnote{The determinant factor only serves to shift the dilaton which we ignore here.}
Here $\mathcal{N}_k$ denotes the number of $k$-cells in the lattice $\Sigma$ and $[c_k^{(i)}, c_{k-1}^{(j)}]$ is $\pm 1$ if the cell $(j)$ is contained in the cell $(i)$ and zero otherwise. There exists a natural inner product between the generators,

$$\langle c_k^{(i)}, c_l^{(j)} \rangle = \delta_{k,l}\delta^{i,j}$$

which acts linearly on elements of the chain complex. This inner product induces the operation of the co-boundary operator, $\delta$,

$$\langle \partial g, h \rangle = \langle g, \delta h \rangle$$

where $g$ and $h$ are arbitrary chains (elements of $C_*(\Sigma, G)$). The simplicial homology and cohomology groups are then given by,

$$H_k(\Sigma, G) = \ker \partial_k / \text{Im} \partial_{k+1}, \quad H^k(\Sigma, G) = \ker \delta_k / \text{Im} \delta_{k-1}$$

A straightforward lattice regularization of (3) can then be written as,

$$Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\sigma \in C_0(\Sigma, \mathbb{R})} \mathcal{N}_1 \prod_{l=1}^{\mathcal{N}_1} B \left( \langle \delta \sigma + 2\pi h, c_1^{(l)} \rangle \right)$$

Here the spins $\sigma$ are the analogue of $\theta$ in the continuum and the Boltzmann weight is defined by $B(g) \equiv \exp\{-\alpha'/2 G^{00} g\}$. It is possible to introduce a real-valued one-chain in place of the spins and cohomology, much like introducing the real-valued one-form $V$ in the continuum. This leads to the following representation,

$$Z = \sum_{v \in C_1(\Sigma, \mathbb{R})} \left( \prod_{l=1}^{\mathcal{N}_1} \delta_R \left( \langle \delta v, c_1^{(l)} \rangle \right) \right) \left( 2g \prod_{a=1}^{2g} \delta_{U(1)} \left( \langle v, h_a \rangle \right) \right) \prod_{l=1}^{\mathcal{N}_1} B \left( \langle v, c_1^{(l)} \rangle \right)$$

where $h_a \equiv \sum_{l \in \gamma_a} c_1^{(l)}$ are the generators of the first homology group. In the above $\delta_G$ represents a $G$ invariant delta function. This form of the partition function is the lattice analogue of (3), the first constraints are the Bianchi constraints forcing $v$ to be the sum of a co-exact chain and a cohomology element both with real coefficients; while the second constraints forces the coefficients of the cohomology to be elements of $2\pi \mathbb{Z}$. Consequently, solving the constraints on $v$ reproduces the original model much like in the continuum. There exists a slightly different representation of the model which makes direct contact with the work of [3]. This involves decomposing $\sigma$ into an integer valued chain, $\tilde{\sigma}$, (integer valued fields pose no problem on the lattice) and a $U(1)$ valued chain, $\theta$. Decomposing $\sigma$ in this manner allows one to introduce an integer-valued one-chain in place of $\tilde{\sigma}$ and $h$ while leaving $\theta$ untransformed. In this representation there are dynamical variables on the sites and links of the lattice. The partition function
in this decomposition is given by the following,

\[
Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\theta \in C_0(\Sigma, U(1))} \sum_{\tilde{\sigma} \in C_0(\Sigma, \mathbb{Z})} N_1 \prod_{l=1}^{N_1} B \left( \left\langle (\delta \theta + 2\pi(\delta \tilde{\sigma} + h)), c_{1}^{(l)} \right\rangle \right)
\]

\[
= \sum_{\theta \in C_0(\Sigma, U(1))} \sum_{v \in C_1(\Sigma, \mathbb{Z})} \left( \prod_{l=1}^{N_1} \delta_Z \left( \left\langle \delta v, c_{1}^{(l)} \right\rangle \right) \right) \prod_{l=1}^{N_1} B \left( \left\langle (\delta \theta + 2\pi v), c_{1}^{(l)} \right\rangle \right)
\]  

(7)

Notice that here there are no lattice analogues of the holonomy constraints as in (6). The advantage in writing the model in its present form is that the spin variables take values in the target space itself rather than its covering space. This is a desirable prescription, however, the decomposition which leads to this model is not well-defined in the continuum as integer valued fields are problematic, and is thus only valid on the lattice. In this representation the winding modes are implemented through the action of the link-valued objects \( v \). As one moves around elementary plaquettes the winding number, given by \( \sum_{l \in p} v_l \), must vanish due to the Bianchi constraint, while along the canonical cycles there is no restriction on the winding number. This is precisely the constraint that the authors of [5] inserted into the discrete version of (3), which they wrote as an \( X - Y \) model on \( \Sigma \), in order to suppress vortex configurations. It is, however, clear that these constraints follow directly from a lattice regularization of (3) and there is no need to insert it by hand. This feature is a direct consequence of writing the continuum variables as single valued fields in the covering space of the circle and then projecting onto the target space through a sum over cohomology elements.

Let us now perform the duality transformation on this model. On the lattice it is easiest to perform the transformations directly on (5) rather than on (6) or (7). Inserting a character expansion (in this case a character expansion amounts to a Fourier transformation) of the Boltzmann weights in the partition function introduces a representation on every link, encoding this information into a one-chain, denoted by \( r \), one finds,

\[
Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\sigma \in C_0(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} \sum_{r \in \mathbb{R}} b(r_l) \chi_{r_l} \left( \left\langle (\delta \sigma + 2\pi h), c_{1}^{(l)} \right\rangle \right)
\]

\[
= \sum_{r \in C_1(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} b \left( \left\langle r, c_{1}^{(l)} \right\rangle \right) \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\sigma \in C_0(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} \chi_{r, c_{1}^{(l)}} \left( \left\langle (\delta \sigma + 2\pi h), c_{1}^{(l)} \right\rangle \right)
\]  

(8)

here the character coefficients of the Boltzmann weights are given by \( b(r) \equiv \sum_{g \in \mathbb{R}} \chi_r(g) B(g) \) (throughout we ignore overall constants which can easily be restored). The characters satisfy simple factorization properties,

\[
\chi_{r_1}(g) \chi_{r_2}(g) = \chi_{r_1+r_2}(g), \quad \chi_r(\arg) = \chi_{\arg}(g)
\]
which can be used to re-arrange the sums over $h$ and $\sigma$ in (8) into the form,

$$
\sum_{h, \sigma} \cdots = \left( \prod_{i=1}^{N_0} \sum_{\sigma_i \in \mathbb{R}} \chi_{(\partial r, c^{(i)}_a)} (\sigma_i) \right) \left( \prod_{a=1}^{2g} \sum_{m_a \in \mathbb{Z}} \chi_{(r, h^a)} (2\pi m_a) \right) = \delta_{\mathbb{R}} (\partial r) \prod_{a=1}^{2g} \delta_{U(1)} (2\pi \langle r, h^a \rangle)
$$

The orthogonality of the characters was used to obtain the last equality and $h^a$ is the generator of the cohomology dual to homology generator $h_a$: $\langle h_a, h^b \rangle = \delta^b_a$. The first constraint forces $r$ to be closed and is therefore a sum of an exact chain and an element of the homology group both with real coefficients: $r = \partial \tilde{\sigma} + \tilde{h}$, while the second constraint forces the coefficients of the homology to be integers. Inserting this solution of the constraints into (8) yields,

$$
Z = \sum_{\tilde{h} \in H_1(\Sigma, \mathbb{Z})} \sum_{\tilde{\sigma} \in C_0(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} b \left( \langle \partial \tilde{\sigma} + \tilde{h}, c^{(l)}_1 \rangle \right)
$$

Interpreting the generators, $\{c^{(i)}_k\}$, of the chain complex on the dual lattice transforms the boundary operator to a co-boundary operator and homology to cohomology, the dual partition function then reads,

$$
Z = \sum_{\tilde{h} \in H^1(\Sigma^*, \mathbb{Z})} \sum_{\tilde{\sigma} \in C_0(\Sigma^*, \mathbb{R})} \prod_{l=1}^{N_1} b \left( \langle \partial \tilde{\sigma} + \tilde{h}, c^{*(l)}_1 \rangle \right)
$$

where the starred objects are on the dual lattice. This is clearly equivalent to the original model (8) with the Boltzmann weights being replaced by their character coefficients,

$$
B(g) = \exp \left\{ - \frac{R^2}{\sqrt{\alpha'}} g^2 \right\}, \quad b(\tilde{g}) = \sqrt{\frac{\pi \alpha'}{4R^2}} \exp \left\{ - \frac{\sqrt{\alpha'}}{2R} \tilde{g}^2 \right\}
$$

Thus we recover the $T$-duality transformation of the continuum model, with the lattice being replaced by the dual lattice and $R \leftrightarrow \sqrt{\alpha'}/2R$. We have demonstrated that writing the continuum model in terms of the covering space of the circle and performing a straightforward lattice regularization, leads to an automatic suppression of vortex configurations and to an explicitly self-dual model.

We would like to generalize the model to include target spaces which are the quotient of two arbitrary Abelian groups $\mathcal{G}/\mathcal{H}$ in which $\mathcal{H}$ acts freely on $\mathcal{G}$. With the world-sheet regulated by a lattice the model is a trivial extension of (8) and is written as,

$$
Z = \sum_{h \in H_1(\Sigma, \mathcal{H})} \sum_{\sigma \in C_0(\Sigma, \mathcal{G})} \prod_{l=1}^{N_1} B \left( \langle (\delta \sigma + h), c^{(l)}_1 \rangle \right)
$$

Here elements of $\mathcal{H}$ are written in such a way that addition in the Boltzmann weight is well-defined. For example, if $\mathcal{G} = U(1)$ and $\mathcal{H} = \mathbb{Z}_N$, then the argument of the Boltzmann
weight should be: $\delta \sigma + (2\pi/N)h$. These models are very similar to the ones considered in where the authors consider a $\mathcal{G}$-valued spin model and introduced a sum over a subset of the generators of the cohomology in order to generate self-dual models. The difference here is that the sum extends over the entire cohomology and its coefficient group differs from the spins coefficient group, while in they were taken to be identical.

Performing the dual transformations on is a simple generalization of the previous calculation and in lieu of repeating the steps we mention the relevant points. A character expansion of the Boltzmann weights is carried out and the one-chain $r$ carries an element of $\mathcal{G}^*$ (the group of irreducible representations of $\mathcal{G}$, for Abelian groups $\mathcal{G}^*$ inherits the groups Abelian structure) on every link. The factorization properties of the characters allows the sum over $h$ and $\sigma$ to be performed and constrains $\partial r$ to vanish in $\mathcal{G}^*$ and $\langle r, h^a \rangle$ to vanish in $\mathcal{H}^*$. The first constraint forces $r$ to be a sum of an exact chain and an element of the homology group with $\mathcal{G}^*$ coefficients, while the second set of constraints forces the coefficient group of the homology to be $\mathcal{G}^*/\mathcal{H}^*$. Interpreting the objects on the dual lattice leads to the dual model,

$$Z = \sum_{\tilde{h} \in H^1(\Sigma, \mathcal{G}^*/\mathcal{H}^*)} \sum_{\tilde{\sigma} \in C_0(\Sigma, \mathcal{G}^*)} \prod_{l=1}^{N_b} b \left( \langle \delta \tilde{\sigma} + \tilde{h}, c_1^{(l)} \rangle \right)$$  \hspace{1cm} (10)

The effects of the duality transformation on the various coefficient groups and the target space are shown in Table I. This table illustrates an interesting generalized version of T-duality. Unfortunately, if either the original or dual spin variables take values in a discrete group, then these models can only be defined on the lattice. This is simply because a continuum theory cannot have discrete valued fields. Nevertheless, the lattice models are perfectly well-defined, and we should investigate what the duality implies. It is interesting to identify the groups which lead to explicitly self-dual models. Certainly a necessary condition is that $\mathcal{G} \cong \mathcal{G}^*$ (spin models on 2-d infinite lattices also have this self-dual restriction). In that case, under duality the coefficient group of the cohomology and the target space are interchanged and then replaced by their dual group. This is the analogue of the interchanging of the momentum and winding modes and $R \leftrightarrow \sqrt{\alpha'/R}$ in the mode expansion of the string co-ordinates. It is also the analogue of the cohomology group being replaced by the Lagrange multipliers which implement the holonomy constraints in the path integral.

Consider the model defined in eq. (5). We will obtain its dual using Table I. In that case $\mathcal{G} = \mathbb{R}$ and $\mathcal{H} = 2\pi R\mathbb{Z}$ so that the target space is $\mathbb{R}/2\pi R\mathbb{Z} \cong S^1_R$ where the subscript identifies the radius of the circle. The dual model has $\mathcal{G}' = \mathbb{R}^* \cong \mathbb{R}$, $\mathcal{H}' = \mathcal{G}'/2\pi R\mathbb{Z}^* \cong R^{-1}\mathbb{Z}$ and target space $\mathcal{G}'/\mathcal{H}' \cong S^1_{(2\pi R)^{-1}}$. We have thus recovered the earlier result that the duality transformation only serves to invert the radius of the target space. Notice that it is straightforward to write down the result for a toroidal target space $T^n = S^1 \times \ldots \times S^1$. In
Table 1: Transformations of the various groups under duality.

| Original Model | Spin Variable | Cohomology Coefficient | Target Space |
|----------------|---------------|-------------------------|--------------|
|                | $G$           | $H$                     | $G/H$        |
| Dual Model     | $G^*$         | $G^*/H^*$               | $H^*$        |

that case one chooses $G = \mathbb{R} \oplus \ldots \oplus \mathbb{R}$ and $H = 2\pi R_1 \mathbb{Z} \oplus \ldots \oplus 2\pi R_n \mathbb{Z}$ the target space is obviously the n-tori with compactification radii $R_i$ in the i-th direction. The dual model leaves $G$ invariant as $G' = G^* \cong G$ while $H' = G^*/H^* = R_1^{-1} \mathbb{Z} \oplus \ldots \oplus R_n^{-1} \mathbb{Z}$ and the dual target space is an n-tori with radii $(2\pi R_i)^{-1}$ in the i-th direction. This case corresponds to taking the target space metric to be diagonal. Of course it is possible to consider metrics with off diagonal elements. To incorporate this into our formalism the Boltzmann weights should be defined as follows,

$$B((g_1, \ldots, g_n)) = \exp \left\{ -G^{ij} g_i g_j \right\}, \quad b((\tilde{g}_1, \ldots, \tilde{g}_n)) = \frac{1}{4\pi \sqrt{G}} \exp \left\{ -\frac{1}{4}(G^{-1})^{ij} \tilde{g}_i \tilde{g}_j \right\}$$

here the n-tuple $(g_1, \ldots, g_n)$ and $(\tilde{g}_1, \ldots, \tilde{g}_n)$ represent elements of $G = \mathbb{R} \oplus \ldots \oplus \mathbb{R}$ and $G' \cong G$ respectively. Also, choose $H = 2\pi \mathbb{Z} \oplus \ldots \oplus 2\pi \mathbb{Z}$ so that the metric information is contained solely in the Boltzmann weight. This demonstrates that under duality the target space metric is replaced by its inverse and reduces to one of the Buscher formulæ\[15\] in the case of vanishing torsion.

The toroidal compactifications are the simplest example of a self-dual model. There are other groups which satisfy the necessary condition $G \cong G^*$ namely the cyclic groups $\mathbb{Z}_P$. For this choice of $G$ the coefficient group of the cohomology is forced to be cyclic as well, $H = \mathbb{Z}_N$ where $N$ is a factor of $P$ (let $P = NM$). This is necessary so that the action of $H$ on $G$ (identifying elements of $G$ which differ by angle of $2\pi/N$) is well-defined. These choices lead to the discrete target space $\mathbb{Z}_M$, i.e. $M$ points on a circle. The coefficient group of the dual model is $\mathbb{Z}_M$ while the dual target space is $\mathbb{Z}_N$. Thus under duality the number of points in the target space is interchanged with the number of points in the coefficient group of the cohomology. This is a novel feature of these models. In addition to this interchanging, the radius of the target space also undergoes a transformation. Rather than including the radius in the defining group, it appears in the Boltzmann weights. We illustrate the relation of the Boltzmann weights and its character coefficients for the case of a direct product of discrete
groups: \( \mathcal{G} = \mathbb{Z}_{P_1} \oplus \ldots \oplus \mathbb{Z}_{P_n} \) and \( \mathcal{H} = \mathbb{Z}_{M_1} \oplus \ldots \oplus \mathbb{Z}_{M_n} \),

\[
B((g_1, \ldots, g_n)) = \left( \prod_{a=1}^{n} \sum_{m_a \in \mathbb{Z}} \right) \exp \left\{ -G^{ij} \left( g_i^{P_i} + m_i \right) \left( g_j^{P_j} + m_j \right) \right\}
\]

\[
b((\tilde{g}_1, \ldots, \tilde{g}_n)) = \sqrt{\frac{\pi}{G}} \left( \prod_{a=1}^{n} \sum_{\tilde{m}_a \in \mathbb{Z}} \right) \exp \left\{ -\frac{1}{4} (\tilde{G}^{-1})^{ij} \left( \tilde{g}_i^{P_i} + \tilde{m}_i \right) \left( \tilde{g}_j^{P_j} + \tilde{m}_j \right) \right\}
\]

where \( \tilde{G}^{ij} = G^{ij}/P_iP_j \) is the normalized “metric” on the original target space. This demonstrates that even in these models an inversion of the “metric” occurs, much like in the toroidal compactifications, as one would expect. For the model to be self-dual the number of points in both the original and dual theory should be identical. This is achieved if \( N_i = M_i \) so that \( P_i \) is a perfect square. Notice that in the limit of very large \( N_i \) there are a large number of points on the target space while the number of elements in the spin group is of order \( N_i^2 \). One can then roughly view the limit of infinite \( N_i \) as the case where the target space becomes a continuous circle while the spin variable reduces to the reals, thus recovering the \( S^1 \) case.

Of course it is possible to choose \( \mathcal{G} \) to be products of \( \mathbb{Z}_{N_2} \) and \( \mathbb{R} \), and \( \mathcal{H} \) to be products of \( \mathbb{Z} \) and \( \mathbb{Z}_{N_i} \) to obtain mixed target spaces which are explicitly self-dual. Some other choices for \( \mathcal{G} \) and \( \mathcal{H} \) which are interesting on there own, but are not self-dual, are \( \mathcal{G} = U(1) \) and \( \mathcal{H} = \mathbb{Z}_{N} \). With such target spaces the string is allowed to wrap around the space only a finite number of times before it is homotopically equivalent to zero windings. In this case, the dual target space is a discrete space, \( \mathbb{Z}_{N} \), even though the original target space was continuous. This nicely demonstrates that even for Abelian isometries the dual need not have the same fundamental group as the original target space (for the non-Abelian case see \cite{10}).

It is possible to generalize these results to the case where the lattice \( \Sigma \) is a triangulation of an arbitrary \((p+1)\)-dimensional orientable manifold. There is one technical constraint on \( \Sigma \): \( H^{k}(\Sigma, \mathbb{Z}) \) must be free Abelian, this is automatic for the case of two-dimensional orientable manifolds but not for higher dimensional spaces. The models in this case are written as,

\[
Z = \sum_{h \in H^{k}(\Sigma, \mathcal{H})} \sum_{\sigma \in C_{k-1}(\Sigma, \mathcal{G})} \prod_{l=1}^{N_k} B \left( \langle \{\delta \sigma + h\}, c^{(l)}_{k} \rangle \right)
\]

These are models in which a \( \mathcal{G}/\mathcal{H} \)-valued “spin” lives on the \((k-1)\)-dimensional cells of a \((p+1)\)-dimensional lattice. For example, if \( k = 2 \) this describes a gauge theory on a \((p+1)\)-dimensional world-volume. The duality transformations can be applied to this model with very little effort. Simply replace the 0-dimensional objects with \((k-1)\)-dimensional ones and 1-dimensional objects with \(k\)-dimensional ones. The dual model is a trivial extension of the
previous dual model,

\[
Z = \sum_{\tilde{h} \in H^{p+1-k}(\Sigma, G^*/H^*)} \sum_{\tilde{\sigma} \in C_{p-k}(\Sigma, G^*)} \prod_{l=1}^{N_{p+1-k}^*} b \left( \langle (\delta \tilde{\sigma} + \tilde{h}), c_{p+1-k}^* \rangle \right)
\]

Self-dual models exist only when the previous relations among the groups are satisfied and \( p + 1 = 2k \). Clearly \( p = 1, k = 1 \) is among those and reproduces the string-theory case. The next case is \( k = 2 \) and \( p = 3 \). This is a gauge theory, with gauge group \( G \) defined on the 4-dimensional world-volume \( \Sigma \) and the sum over \( H \)-valued cohomology is akin to summing over the topological sectors of the theory. A continuum example is given by,

\[
Z = \sum_{\tilde{h} \in H^2(\Sigma, \mathbb{Z})} \int \mathcal{D}X \exp \left\{ -g^2 \int (dX + 2\pi h) \wedge \ast (dX + 2\pi h) \right\}
\]

where the field \( X \) is a real-valued one-form on \( \Sigma \). There are of course many higher-dimensional analogues of such self-dual models.

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