On the origin of duality in the quantum Hall system

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(Dated: September 15, 2010)

We discuss the possible origin of the duality observed in the quantum Hall current-voltage characteristics. We clarify the difference between “particle-vortex” (complex modular) duality, which acts on the full transport tensor, and “charge-flux” (“real”) duality, which acts directly on the filling factor. Comparison with experiment strongly favors the form of duality which descends from the modular symmetry group acting holomorphically on the complexified conductivity.

PACS numbers: 73.20.-r

In a remarkable experiment1 Shahar et al. [1] found clear evidence for a duality between current-voltage (IV) characteristics obtained on opposite sides of the quantum Hall (QH) liquid to insulator transition. They identified six pairs \((B, B_d)\) of “dual” magnetic field values, with \(B\) and \(B_d\) on opposite sides of the quantum phase transition at \(B = B_c \approx 9.1\ T\), separating the \(\nu = 1/3\) fractional QH liquid from the QH insulator phase \((\nu = 0)\). With \(B\)-values in this range the transverse IV-characteristics \(V_y(B, I) = R_{yx}(B, I)I\) was found to be independent of \(B\) and linear in \(I\), with slope \(R_H = R_{yx} \approx 3 \ [h/e^2]\) (Fig. 1, bottom inset). The critical value of the Hall resistivity is therefore \(\rho_H^c = \rho_{yx}(B_c) \approx 3 \ [h/e^2]\). The dissipative IV-characteristics \(V_x(B, I) = R_{xx}(B, I)I\), on the other hand, are extremely non-linear in both phases, degenerating to a linear (Ohmic) relation only when \(B \to B_c\) (Fig. 1, top inset).

Shahar et al. [1] discovered that to each \(B\) there exists a dual field value \(B_d\), such that the dissipative IV-curve \(V_x(B, I)\) (suppressing the now superfluous subscript on \(V_x\)) after reflection in the diagonal \(V = I\) is virtually identical to the dual IV-curve \(V_d(B_d, I_d)\) in the opposite phase. This is implied that

\[
\rho_D^d(B_d, I_d) = 1/\rho_D(B, I) ,
\]

a remarkable relation providing unambiguous evidence of a duality symmetry in the QH system, for a particularly simple quantum phase transition. It reveals a profound connection between the transport mechanisms deep inside the quantum liquid and insulator phases, a symmetry which holds to great accuracy even far from the quantum critical point found at:

\[
(\rho_H^c, \rho_D^c) \approx (3, 1) \ [h/e^2] ,
\]

consistent with the self-dual value of eq. [1].

We are only aware of two (related) theoretical frameworks that aspire to account for these data (and which motivated the experimental investigation of duality). The first [1] is based on an effective description of the macroscopic theory that possesses a holomorphic modular symmetry relating the complexified response functions

\[
\sigma = \sigma_{xy} + i\sigma_{xx} = \sigma_H + i\sigma_D
\]

\[
\rho = \rho_{xy} + i\rho_{xx} = -\rho_H + i\rho_D
\]

in different QH phases. It successfully predicts the full phase diagram of the QH system, both integer and fractional, including the position of the quantum critical points governing the scaling behaviour of transitions between QH levels [4]. A “microscopic” [3] interpretation of the symmetry as “particle-vortex duality” was given in ref. [5].

The second framework is provided by a “microscopic” model [6] in which a so-called “flux attachment transformation” maps the two-dimensional electron system in a magnetic field onto a bosonic system in a different “effective” field. This “charge-flux duality” resulted in a set of rules, known collectively as “the law of corresponding states”, which relates QH states carrying different filling factors \(\nu\). It also determines the topology of the phase diagram, but neither the location of quantum critical points

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1 The Hall bar is a high mobility (\(\mu = 5.5 \times 10^7\ cm^2/sV\)) low density \((n = 6.5 \times 10^{10}/cm^2)\) GaAs/AlGaAs hetero-structure of size \(L_x \times L_y\), aligned with a current \(I\) in the \(x\)-direction so that \(R_{xx} = (L_y/L_x)\rho_{xx}\). The Hall resistance \(R_H = \rho_H = \rho_{yx}\) is quantized in the fundamental unit of resistance, \(h/e^2 \approx 25.8\ k\Omega\), while the dissipative resistance \(R_D = \rho_{xx}/\Omega\) is rescaled by the aspect ratio \(\Omega = L_y/L_x\). In this experiment \(R_D^c \approx 23\ k\Omega\) and \(\rho_{xx}^c \approx 1 \ [h/e^2]\). [2]

2 Since no errors are given in ref. [1], our least biased estimate is to take the largest error consistent with the published value: \(B_c = 9.1 \pm 0.05\ [T]\), and similarly for the other values of \(B\).

3 Our timid use of the term “microscopic” signals that a rigorous derivation of these models from the quantum electrodynamics of disordered media is far beyond current theory. For each model assumptions about the effective degrees of freedom at some intermediate scale are made.
nor the geometry of renormalization group (RG) flows in the complex conductivity plane are obtained by this method.

In refs. [1, 3] the law of corresponding states was used to determine the pairs of filling factors \((\nu, \nu_d)\) that correspond to states related by charge-flux (particle-vortex) duality. For the \(\nu = 0\) to \(\nu = 1\) transition the corresponding states are related by particle-hole duality giving \(\nu_d = 1 - \nu\). The flux attachment transformation \(1/\nu' = 1/\nu + 2m\) \((m\) integer) maps this transition to the \(\nu = 0\) to \(\nu = 1/k\) transition, with \(k = 2m + 1\), giving the duality relation: \(1/\nu_d - k = (1/\nu - k)^{-1}\). For the \(\nu = 0\) to \(\nu = 1/3\) transition this gives

\[
\nu_d(\nu) = \frac{1 - 3\nu}{3 - c_\nu} \quad (c_\nu = 8), \tag{3}
\]

which relates the filling \(\nu = \nu(B)\) to a dual filling \(\nu_d = \nu(B_\nu)\).

The self-dual point of this transformation is \(\nu_s = 0.25\), distinct from the critical value \(\nu_c \approx 0.28\) found experimentally [2]. In fact, this argument fails to reproduce any of the main features observed experimentally. It has nothing to say about the Hall response of the system, including the fact that it remains constant at its critical value \(\rho_H^c \approx 3 [h/e^2]\) across the transition. It is also not clear that eq. (1) relating dual values of \(\rho_D\) applies for the dual pairs of filling factors given by eq. (3), because the mapping used to derive the latter strictly applies only on the plateaux where \(\rho_D\) vanishes. Also, as discussed below, it does not reproduce the measured values of the dual filling factors.

The situation is quite different for the modular symmetry [4]. The transformations relating corresponding states are fractional linear (Möbius) maps that move points on the upper half of the complex \(\rho\) (or \(\sigma\)) plane \((\rho_D\) and \(\sigma_D\) are positive). It swaps the entire quantum liquid phase, a region of the \(\rho\) (or \(\sigma\)) plane “attached” only to \(\rho_\sigma = \rho_\mu = -3\) \((\sigma_\sigma = \sigma_\mu = 1/3)\) on the real axis,\(^4\) with the entire QH insulator phase, a region of the \(\rho\) (or \(\sigma\)) plane attached only to \(\rho_D = \rho_\mu = \infty\) \((\sigma_\sigma = 0)\). Since these transformations act on complex coordinates rather than on the real filling factor \(\nu\), the prediction for the dual pairs is different.

The particular modular transformation that maps the (infrared) plateaux fixed points \(\rho_\sigma = -3\) and \(\rho_D = \infty\) into each other has the form:

\[
\rho_D^d(\rho) = -\frac{c + 3\rho}{3 + \rho}. \tag{4}
\]

This transformation is modular iff \(\text{det } \rho_D^d = 1\), which fixes

\(^4\) In other words, the domain of attraction (universality class) of the RG flow towards the real infra-red fixed point \(\rho_D\).
should exist dual pairs of $B$-fields for which the experimental result recorded in eq. (1) should hold with great precision, i.e., at the same level of accuracy that the emergent symmetry holds. In a transition to the insulator phase there must of course be pairs of points with inverse values of the resistivities, but duality implies that the physics of the system at dual parameter values is the same, explaining why the full nonlinear structure of the IV-curves evident in Fig. 1 coincide so precisely.

To gain further insight into the origin of the differences between the two approaches it is instructive to consider the structure in the conductivity plane. The modular symmetry transformation swapping the plateaux fixed points $\sigma_d = 0$ and $\sigma_d = 1/3$ follows immediately from eq. (4):

$$
\sigma^d(\sigma) = \frac{1 - 3\sigma}{3 - \sigma}.
$$

(7)

This is superficially similar to eq. (3), but these two transformations are in fact very different. Not only does modularity (det $\sigma^d = 1$) fix $c = 10$, but more importantly eq. (7) is a much stronger statement about emergent global symmetries in the QH system than eq. (3).

Decompressing the complex transformation in eq. (7) into components gives two non-linear functions $\sigma_H^d(\sigma_H, \sigma_D)$ and $\sigma_D^d(\sigma_H, \sigma_D)$ that in general are inextricably entangled. The constraint $\rho_H = 3$ restricts the flow to the semi-circle $|\sigma|^2 = \sigma_H/3$, giving the constrained modular duality transformation of interest here:

$$
\sigma_H^d = \frac{1 - 3\sigma_H}{3 - \sigma_H} (c_m = 80/9).
$$

(8)

Since $\sigma_H^d$ now depends only on $\sigma_H$, this is as close to a formal similarity to the $\nu$-duality in eq. (3), following from the law of corresponding states, as we can get. However, since $\sigma_H$ is only given by the filling factor on the plateaux, and $c_m$ is quite different from $c_\nu$, there is no way to reconcile these transformations.

In order to exhibit a more quantitative comparison with the data we show in Fig. 2(a) the observed values for $\Delta \nu = \nu - \nu_c$ reported in ref. [1]. The boxed values are the images of the low-field data (the black bullets, obtained in the $\nu = 1/3$ phase) under the transformation following immediately from eq. (7):

$$
\Delta \nu_d(\Delta \nu) = \frac{-(1 - 6\nu_c + 8\nu_c^2) + (8\nu_c - 3)\Delta \nu}{(8\nu_c - 3) + 8\Delta \nu}.
$$

(9)

Clearly the image points $\Delta \nu_d = \Delta \nu_d(\Delta \nu_\bullet)$ do not coincide with the insulator fillings $\Delta \nu_c$ reported in ref. [1].

By comparison, the full non-linear content of modular duality is contained in eq. (4), which is in perfect agreement with the experiment (within experimental accuracy), as already discussed. The prediction following from the emergent symmetry is that there should exist dual parameter values so that the full non-linear resistivities collapse as shown in Fig. 1. In short, the experimentally observed duality is in clear disagreement with the form obtained using the law of corresponding states, and in excellent agreement with the structure predicted by the modular symmetry $\Gamma_H$.

The modular symmetry by itself does not predict the values of the dual magnetic fields. However, we expect that the relation between $B_d$ and $B$ should, to a good approximation, be fractional linear (a Padé approximant of order (1,1)) of the form

$$
B_d(B) = \frac{aB - b}{B - c},
$$

(10)

with $a$, $b$ and $c$ are constants. This form has a simple pole at $B = c$ corresponding to the need to have dual field val-
ues deep in the insulator phase. It also has the property that its form is maintained when expressing $B$ as a function of $B_\odot$ as is required from the duality property of the theory. We can fix the three constants by requiring that the self dual point should be at $B = B_\odot$, and by demanding that the duality relation swap the plateau value, $B = B_\odot$ ($\nu = 1/m$), with the insulator value, $B = \infty$ ($\nu = 0$). The resulting duality relation has the form

$$B_d(B) = \frac{BB_\odot + (B_\odot - 2B_\odot)B_\odot}{B - B_\odot}.$$  \hfill (11)

Both the parameters $B_\odot$ and $B_\oplus$ are of course non-universal, but can be extracted from each experiment as follows. $B_\odot \approx B_\ast$ is the critical field value, identified in the experiment from the temperature independent crossing point of all the resistivity traces. The value of $B_\oplus$ can be directly obtained from the measured $B$ value at the centre of the plateau.

For the case of the transition from the $\nu = 1/3$ plateau $m = 3$. In the experiment discussed above $B_\odot = 9.1 T$, and we estimate that $B_\odot \approx 7.6 T$, corresponding to the centre of the $\nu = 1/3$ plateau. (For $\nu \propto B^{-1}$ this value agrees very well with the positions of the other plateau values.)

Fig.2(b) shows the dual pairs of field values found experimentally as black and white bullets. The boxed values are the images of the low-field data (the black bullets, obtained in the $\nu = 1/3$ phase) under the transformation in eq. (11). The image points $B_\ominus = B_d(B_\ast)$ fall on top of the insulator values $B_\odot$ reported in ref. [1], even in this linearized approximation.

One way to see why this linear approximation gives such a good result is to observe that a simple power law gives an accurate model of the data. The function:

$$\rho_\odot^{b_1}(B) = (\Delta B/\Delta B_\odot)^a,$$  \hfill (12)

with $\Delta B = B - B_\odot$, $\Delta B_\odot = B_\odot - B_\oplus$ and the fitted value $a = 3.8$, is shown as the dashed curve in Fig.4. We see that it gives a good fit to the real resistivity trace (solid red curve, adapted from the lowest temperature trace in ref. [1]) for a reasonable range of $B$-fields, and only deviates when $B$ goes deeply into the insulator phase ($B \to \infty$). If we set $\rho_\odot^{b_1}(B_d) = 1/\rho_\odot^{b_1}(B)$, the fitting parameter drops out and eq. (11) follows.

Since duality relates all the plateau-insulator transitions, we expect the fractional linear approximation in eq. (11) to work equally well for any transition of this type. It should therefore provide an accurate prediction for dual pairs of $B$-fields across any insulator-plateau ($\nu = 0 \leftrightarrow \nu = 1/m$) quantum phase transition.

In summary, there is a significant difference between the duality predictions derived from the law of corresponding states, and the predictions derived from the modular symmetry acting holomorphically on the complexified conductivity or resistivity. While the former is strongly disfavoured by experiment, the latter gives a detailed explanation for the striking features of duality observed in the experiment, including the exact agreement (within experimental accuracy) of the modular predictions in eqs. (5) and (6), with the data recorded in eqs. (2) and (1). This provides further strong evidence for an emergent modular symmetry in the QH system.

Since the two proposed emergent symmetries are distinct, effective field theories possessing these symmetries must also be different. The nature of the effective degrees of freedom, i.e., the composite quasiparticles (possibly anyons), at large in the QH system is still an open problem, three decades after the discovery of the QH effect. We have explained how duality experiments are well suited to inform this issue, by efficiently discriminating between candidate models dependent on specific degrees of freedom.

To date we are not aware of any experimental study of another plateau-insulator transition that probes the duality relations in the quantum regime where the temperatures are low enough for the fixed point structure to be reached [9]. Given that the duality predictions following from modular symmetry are rigid and precise, such measurements are capable of providing a definitive test of the emergent modular symmetry in the quantum Hall system.

**Epilog:** We can write the $B$-duality relation in a universal form that can be easily tested when duality is investigated for other plateau-insulator transitions. Start by introducing the “reduced” $B$-field $b = (B - B_\odot)/B_\odot$, which is analogous to the reduced temperature $t =
\[(T - T_c)/T_c\] used in the study of classical phase transitions. We also choose to measure all fields in the unit \(\alpha = -b = (B - B)/B > 0\), whence \(b = \beta \alpha \approx \beta\) and \(b_d = \beta_d \alpha \approx \beta_d\). With this notation all non-universal (system- and transition-specific) data have been absorbed in the units, and the duality transformation takes the universal form:

\[\beta_d(\beta) = -\frac{\beta}{1 + \beta} . \] (13)

The prediction is now that the dual \(B\)-field data for any plateau-insulator transition will collapse onto this curve, shown in Fig. 3 where we have also added the only available duality data on \(B\)-fields [1].

[1] D. Shahar, D.C. Tsui, M. Shayegan, E. Shimshoni, S.L. Sondhi, Science 274, 589 (1996)