Cluster Superalgebras and Stringy Integrals

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ABSTRACT: We take some initial steps to explore physical applications of the cluster superalgebras recently defined by Ovsienko and Shapiro. Our primary example is a fermionic extension of the $A_2$ cluster algebra, having fifteen cluster supervariables instead of the usual five. We also explore an alternate definition of cluster superalgebras based on the promotion of cluster variables to superfields.
1 Introduction

In recent years several connections have been found between various aspects of scattering amplitudes in quantum field theory and cluster algebras, which were discovered by Fomin and Zelevinsky [1] in 2002 and have since been under intense investigation by mathematicians (see for example [2] for a comprehensive modern reference).

In this paper we take a few tentative steps towards asking whether there might be any interesting connections between superamplitudes and cluster superalgebras, which have very recently begun to be explored by mathematicians [3–6].

So far, the known appearances of cluster algebras in scattering amplitudes can be organized into four broad themes: (1) it has been observed [7] that the singularities of (certain) amplitudes are dictated by cluster variables of the Gr(4, n) cluster algebra; (2) cluster structures appear naturally in the positive Grassmannian description
of integrands [8] (and amplituhedra); (3) finite-type cluster algebras provide natural examples of “stringy” integrals that generalize [9] the Koba-Nielsen amplitude [10]; and (4) certain cluster polytopes are amplituhedra for the amplitudes of bi-adjoint $\phi^3$ theory [11].

The first two of these four connections are currently confined (see however [12]) to the realm of planar maximally supersymmetric Yang-Mills theory and are tied, in particular, to the rich mathematical structure of the Grassmannian $\text{Gr}(k, n)$. It is natural to wonder whether, super versions of these connections could be described in terms of cluster superalgebras associated to the super Grassmanian $\text{Gr}(k|l, m|n)$ (the space of $k|l$ planes in $\mathbb{C}^{m|n}$). We postpone this interesting but ambitious question to future work, in part because the mathematics of cluster superalgebras is not yet sufficiently well developed, though we note that the approaches of [4, 6] might provide first steps in that direction.

Instead we largely focus on the broader and more general third connection, between cluster algebras and stringy integrals, since it has interesting things to say even for the simplest nontrivial cluster algebra $A_2$. We also defer consideration of the fourth connection to future work since $\phi^3$ theory is not amenable to supersymmetrization, although we note that it might be interesting to look at a suitable supersymmetrization of $\phi^4$ theory, whose tree-level amplitudes are geometrically encoded in the structure of Stokes polytopes [13]. Finally we note that there has also been recent interest in the connection between string amplitudes and cluster algebras associated to surfaces following [14, 15] and work in progress by Arkani-Hamed et al. It would be interesting to explore whether there is a natural way to attach cluster superalgebras to super Riemann surfaces (see for example [16, 17]), and to connect those to superstring amplitudes.

The structure of this paper is as follows. In Sec. 2 we review the definition of cluster superalgebras given by Ovsienko and Shapiro in [3] and study in detail the 15 cluster supervariables associated to the simplest nontrivial cluster superalgebra that extends the ordinary $A_2$ algebra. In Sec. 3 we propose an alternate, but perhaps more physically motivated, definition of cluster superalgebras that is based on promoting ordinary cluster variables to superfields. In Sec. 4 we note that the famous five-term dilogarithm identity remains valid when it is extended to the $A_2$ superalgebras discussed in Secs. 2 and 3. Finally in Sec. 5 we explore a few different cluster superalgebra generalizations of the stringy integrals introduced in [9].

2 Cluster superalgebras: Ovsienko-Shapiro definition

We assume the reader has basic familiarity with the definition of cluster algebras in terms of quivers and mutations [1, 2]. In [3] Ovsienko proposed a definition of cluster
superalgebras via extended quivers and their mutations and proved the super analog of the Laurent phenomenon. In [5] Ovsienko and Shapiro refined the definition of [3] and relaxed some of its constraints. We begin this section by briefly reviewing the construction of [5].

First, we define an extended quiver $\tilde{Q}$ associated to an ordinary quiver $Q$, by adding:

1. an even number of frozen nodes with Grassmann variables $\xi_i$, and
2. one or more 2-paths which take the form $(\xi_i \to x_k \to \xi_j)$, where $x_k$ are (bosonic) variables associated to nodes of $Q$.

Note that 2-paths of opposite orientations $(\xi_i \to x_k \to \xi_j)$ and $(\xi_j \to x_k \to \xi_i)$ are not allowed.

An example of an extended quiver is

Here $x_1$ is a bosonic variable and $\xi_1, \xi_2$ are frozen Grassmann variables. Following the standard convention, boxes in the quiver diagram denote frozen nodes. In this example there is only a single 2-path, so no ambiguity can arise, but in general it is not enough to draw only nodes and arrows; one must specifically indicate all 2-paths.

Next we define how to mutate an extended quiver $\tilde{Q}$ on node $x_k$ (this operation will be denoted $\tilde{\mu}_k$):

0. The ordinary quiver $Q \subset \tilde{Q}$ mutates according to the classical rules.

1. For each 2-path $(\xi_i \to x_k \to \xi_j)$: for all $(x_k \to x_l)$, add the 2-path $(\xi_i \to x_l \to \xi_j)$.

2. Reverse all 2-paths through $x_k$, i.e. change $(\xi_i \to x_k \to \xi_j)$ to $(\xi_j \to x_k \to \xi_i)$.

3. Remove any pair of 2-paths with opposite orientations, i.e. $(\xi_i \to x_k \to \xi_j)$ and $(\xi_j \to x_k \to \xi_i)$ cancel each other.

An example of quiver mutation is
The mutation $\tilde{\mu}_k$ replaces $x_k$ by $x'_k$ according to the exchange relation

$$x_kx'_k = \prod_{x_k \rightarrow x_l} x_l + \sum_{\xi_i \rightarrow x_k \rightarrow \xi_j} (1 + \xi_i\xi_j) \prod_{x_l \rightarrow x_k} x_l$$

(2.1)

leaving all other variables unchanged.

For classical cluster algebras mutation is an involution, meaning that if you mutate twice on the same node, you come back to the cluster you started with. It is evident from the above definition that this is not the case for cluster superalgebras; indeed mutating over and over on the same node would generate an infinite number of quivers and cluster variables in general. In order to avoid this problem we adopt a rule of thumb whereby we never mutate twice in a row on the same node, but instead “walk through” the algebra following some definite mutation sequence adapted from the classical case. For example, for the $A_2$ algebra we will use a mutation sequence that alternates between the two nodes, while for the Somos-$n$ example discussed in [2, 5] it was natural for them to use a cyclic mutation sequence. It is not immediately clear how to construct finite cluster superalgebras based on more complicated classical algebras, such as $A_3$, where there is no suitable choice of mutation sequence for walking through the algebra and consistently assigning cluster supervariables to the nodes of each quiver.

In order to demonstrate these ideas we now present two cluster superalgebras $A_{2}^{2 OS_1}$ and $A_{2}^{2 OS_2}$ based on $A_2$; remarkably, we will see that each contains exactly the same 15 unique cluster supervariables.

### 2.1 Example: $A_{2}^{OS_1}$

Consider the following initial quiver with two bosonic mutable variables $x_1$, $x_2$, and two Grassmann variables $\xi_1$, $\xi_2$.

![Diagram of a quiver with nodes $\xi_1$, $\xi_2$, and edges $x_1 \rightarrow x_2$](image)

- 4 -
Alternately performing the mutations $\tilde{\mu}_1$ and $\tilde{\mu}_2$ gives the sequence:

\[
\begin{align*}
\xi_1 & \xrightarrow{\tilde{\mu}_1} \xi_2 \xrightarrow{\tilde{\mu}_2} \xi_1 \xrightarrow{\tilde{\mu}_1} \xi_2 \\
\quad & \cdots \quad \xrightarrow{\tilde{\mu}_1} \xi_1 \xrightarrow{\tilde{\mu}_2} \xi_2
\end{align*}
\]

which, remarkably, returns to the original quiver after precisely 6 mutations. Now let us look at the variables encountered along the way. Using the exchange relation (2.1), starting from $\tilde{x}_1 \equiv x_1$ and $\tilde{x}_2 \equiv x_2$, we have

\[
\begin{align*}
\tilde{x}_3 &= x'_1 = \frac{1 + x_2}{x_1} + \frac{1}{x_1} \xi_1 \xi_2 \\
\tilde{x}_4 &= x'_2 = \frac{1 + x_1 + x_2}{x_1 x_2} + \frac{1 + x_1}{x_1 x_2} \xi_1 \xi_2 \\
\tilde{x}_5 &= x''_1 = \frac{1 + x_1}{x_2} \\
\tilde{x}_6 &= x''_2 = x_1 (1 - \xi_1 \xi_2) \\
\tilde{x}_7 &= x''_1 = x_2 (1 - \xi_1 \xi_2) \\
\tilde{x}_8 &= x''_2 = \frac{1 + x_2}{x_1} + \frac{1}{x_1} \xi_1 \xi_2 \quad \text{etc.}
\end{align*}
\]

We present the resulting cluster supervariables $\tilde{x}_n$ in Tab. 1. There we highlight the fact (manifest from (2.1)) that if we set the odd variables to zero ($\xi_i \to 0$), they reduce to the ordinary $A_2$ cluster variables

\[
\begin{align*}
x_1, \quad x_2, \quad x_3 &= \frac{1 + x_2}{x_1}, \quad x_4 = \frac{1 + x_1 + x_2}{x_1 x_2}, \quad x_5 = \frac{1 + x_1}{x_2}.
\end{align*}
\]

In the table one sees some interesting patterns. For example in each column there are only three distinct "fermionic corrections", each repeated twice. However, the rows in which they repeat are different for different columns, but amazingly it turns out that

\[
\tilde{x}_{31} = x_1, \quad \tilde{x}_{32} = x_2
\]
Again we consider the alternating mutation sequence starting from $\tilde{\mu}_1$:

$\begin{align*}
\xi_1 &\quad \xi_2 \\
1 &\rightarrow x_1 \quad x_2 \\
2 &\rightarrow x_1' \quad x_2 \\
3 &\rightarrow x_1'' \quad x_2 \\
4 &\rightarrow x_1''' \quad x_2'' \\
5 &\rightarrow x_1'''' \quad x_2'''
\end{align*}$

and applying $\tilde{\mu}_2$ to the fourth quiver brings us back to the first. The quiver period is therefore 4, while the bosonic $A_2$ period is still 5; therefore the overall period for this
The 20 cluster supervariables of $A^G_{2S2}$; the 15 unique entries in this table are precisely the same as the 15 unique entries in Tab. 1.

Cluster superalgebra is $4 \times 5 = 20$. The cluster supervariables are listed in Tab. 2, and it is clear that taking the $\xi \to 0$ limit reduces to 4 copies of the classical $A_2$ algebra. There is one repeated variable in each column, so the total number of distinct variables is 15. In fact these are the precisely the same as the 15 variables of $A^G_{2S1}$!

3 Cluster superalgebras: superfield definition

In this section, we will propose a different construction of cluster superalgebras by promoting cluster variables to superfields. We consider $N = 1$ superfunctions

$$X_k = a_k(1 + \eta \theta_k)$$

(3.1)

where $\eta$ is a fixed Grassmann parameter and $\theta_k$ is the Grassmann partner of $a_k$; the fact that the overall factor of $a_k$ is pulled out is a convenient choice of normalization (we adopted the same convention already in Tables 1 and 2).

We propose to apply the ordinary cluster algebra exchange relation to superfields:

$$X_k X'_k = \prod_{i \rightarrow k} X_i + \prod_{k \rightarrow i} X_i .$$

(3.2)

This can be expanded into individual exchange relations

$$a_k a'_k = \prod_{i \rightarrow k} a_i + \prod_{k \rightarrow i} a_i$$

(3.3)

$$a_k a'_k (\theta_k + \theta'_k) = \left( \prod_{i \rightarrow k} a_i \right) \left( \sum_{i \rightarrow k} \theta_i \right) + \left( \prod_{k \rightarrow i} a_i \right) \left( \sum_{k \rightarrow i} \theta_i \right)$$

(3.4)

which define $a'_k$ and $\theta'_k$ in terms of the unprimed variables. The exchange relation for the bosonic components $a_k$ is identical to that of classical cluster algebras.

Table 2. The 20 cluster supervariables of $A^G_{2S2}$; the 15 unique entries in this table are precisely the same as the 15 unique entries in Tab. 1.
We could also consider $\mathcal{N} = 2$ superfunctions of the form

$$X_k = a_k (1 + \eta_1 \theta_k + \eta_2 \tilde{\theta}_k + \eta_1 \eta_2 z_k). \quad (3.5)$$

In this case the exchange relation (3.2) can be expanded into four component exchange relations. The relations for $a_k$, $\theta_k$ and $\tilde{\theta}_k$ would be the same as in the $\mathcal{N} = 1$ case, while the mutation of $z_k$ would be governed by

$$a_k a'_k (z_k + z'_k - \theta_k \theta'_k - \theta'_k \tilde{\theta}_k) = \left( \prod_{i \rightarrow k} a_i \right) \sum_{i \rightarrow k} \left( z_i - \sum_{j \rightarrow k} \theta_i \theta_j \right)$$

$$+ \left( \prod_{k \rightarrow i} a_i \right) \sum_{k \rightarrow i} \left( z_i - \sum_{(k \rightarrow j) \neq i} \theta_i \tilde{\theta}_j \right). \quad (3.6)$$

Of course one could just as easily also consider higher $\mathcal{N}$. Unlike the OS definition of cluster superalgebras reviewed in Sec. 2, it is manifest that applying the classical exchange relation to superfields makes mutation an involution. The superfield construction therefore provides a family (indexed by $\mathcal{N}$) of manifestly finite cluster superalgebras to any finite classical cluster algebra. We now work out two examples; some others are given in the appendix.

### 3.1 Example: $A_1^{SF}$

The $A_1$ cluster variables are

$$X_1 \quad X_2 = \frac{1}{X_1}, \quad (3.7)$$

in the classical case. Promoting these to $\mathcal{N} = 1$ superfields gives the cluster supervariables

$$X_1 = a_1 (1 + \eta \theta_1)$$

$$X_2 = \frac{1}{a_1} (1 - \eta \theta_1), \quad (3.8)$$

while promoting them to $\mathcal{N} = 2$ superfields gives

$$X_1 = a_1 (1 + \eta_1 \theta_1 + \eta_2 \tilde{\theta}_1 + \eta_1 \eta_2 z_1)$$

$$X_2 = \frac{1}{a_1} \left( 1 - \eta_1 \theta_1 - \eta_2 \tilde{\theta}_1 - \eta_1 \eta_2 (z_1 + 2 \theta_1 \tilde{\theta}_1) \right). \quad (3.9)$$
3.2 Example: $A_2^{SF}$

The classical $A_2$ cluster variables are

$$X_1, \ X_2, \ X_3 = \frac{1 + X_2}{X_1}, \ X_4 = \frac{1 + X_1 + X_2}{X_1X_2}, \ X_5 = \frac{1 + X_1}{X_2}. \quad (3.10)$$

If we write cluster variables in the form $X_i = a_i(1 + \eta \theta_i)$, the bosonic components $a_i$ would be the $A_2$ cluster variables, and the fermionic components are

$$\theta_3 = - \theta_1 + \frac{a_2}{1 + a_2} \theta_2,$$

$$\theta_4 = - \frac{1 + a_2}{1 + a_1 + a_2} \theta_1 - \frac{1 + a_1}{1 + a_1 + a_2} \theta_2,$$

$$\theta_5 = \frac{a_1}{1 + a_1} \theta_1 - \theta_2. \quad (3.11)$$

For the $\mathcal{N} = 2$ case, if we let $X_i = a_i(1 + \eta_1 \theta_i + \eta_2 \tilde{\theta}_i + \eta_1 \eta_2 z_i)$, we find the same $a_k$’s as in the classical case, two copies of (3.11) (one copy for $\theta$ and one for $\tilde{\theta}$), and finally

$$z_3 = - z_1 + \frac{a_2}{1 + a_2} z_2 - 2 \theta_1 \tilde{\theta}_1 + \frac{a_2}{1 + a_2} (\theta_1 \tilde{\theta}_2 + \theta_2 \tilde{\theta}_1),$$

$$z_4 = - \frac{1 + a_2}{1 + a_1 + a_2} z_1 - \frac{1 + a_1}{1 + a_1 + a_2} z_2 - \frac{2(1 + a_2)}{1 + a_1 + a_2} \theta_1 \tilde{\theta}_1 - \frac{2(1 + a_1)}{1 + a_1 + a_2} \theta_2 \tilde{\theta}_2 - \frac{1}{1 + a_1 + a_2} (\theta_1 \tilde{\theta}_2 + \theta_2 \tilde{\theta}_1),$$

$$z_5 = \frac{a_1}{1 + a_1} z_1 - z_2 - 2 \theta_2 \tilde{\theta}_2 + \frac{a_1}{1 + a_1} (\theta_1 \tilde{\theta}_2 + \theta_2 \tilde{\theta}_1). \quad (3.12)$$

4 A super cluster polylogarithm identity

Here we pause to consider one aspect of the connection between cluster algebras and amplitudes that emerged from the study of planar $\mathcal{N} = 4$ super-Yang-Mills (pSYM) theory. Namely, cluster algebras have provided an important tool for identifying and elucidating the many nontrivial functional relations satisfied by multiple polylogarithm functions. The simplest of these is the remarkable relation

$$\sum_{i=1}^{5} \left( \text{Li}_2(-x_i) + \ln x_i \ln x_{i+1} + \frac{\pi^2}{10} \right) = 0 \quad (4.1)$$
for the dilogarithm function

$$\text{Li}_2(z) = -\int_0^z \frac{dt}{t} \ln(1-t). \quad (4.2)$$

In (4.1) the sum is taken over the five cluster variables $x_i$ of the $A_2$ cluster algebra defined by the exchange relation

$$1 + x_i = x_{i-1}x_{i+1} \quad (4.3)$$

(which implies that $x_{i+5} = x_i$). The identity (4.1) is equivalent to a form attributed to Abel [18], though the geometric properties of the pentagram of arguments were studied already by Gauss [19]. In fact (4.1) is the only non-trivial identity for the dilogarithm, in the sense that every other identity one can write is a functional consequence of it and the "trivial" identities that relate $\text{Li}_2(1-x)$ or $\text{Li}_2(-1/x)$ to $\text{Li}_2(x)$.

Polylogarithm identities involving cluster variables emerge naturally from the study of perturbative scattering amplitudes in pSYM theory. For example, by expressing its 2-loop 7-particle MHV amplitude in two different ways, guaranteed to be equal to each other as a simple physical consequence of parity symmetry, the authors of [7] discovered a mathematically nontrivial 40-term functional equation for the trilogarithm function $\text{Li}_3(z) = \int_0^z \frac{dt}{t} \text{Li}_2(t)$ whose arguments are cluster Poisson coordinates on the moduli space of 6 cyclically ordered points in $\mathbb{P}^2$ (the $D_4$ cluster algebra). More generally, numerous identities at various weights have emerged from the study of Feynman integrals in quantum field theory; for recent developments see for example [20, 21] and references therein. In a recent mathematical breakthrough, Goncharov and Rudenko have used the link between cluster varieties and polylogarithms to prove Zagier’s polylogarithm conjecture in weight 4 [22].

It is therefore natural to ask whether the cluster superalgebras we have explored in the previous sections have any interesting implications for polylogarithm identities. Interestingly, it is easy to check that the Abel identity (4.1) remains valid if we take the sum over 30 terms, with the $x_i$ consisting of the variables from Sec. 2.1; or over 20 terms using the $x_i$ from Sec. 2.2; or over the 15 unique cluster supervariables from either of those two lists.

A moment’s reflection reveals that in fact (4.1) remains satisfied under any first-order infinitesimal deformation of the cluster variables. If we take $x_i \to x_i + \epsilon$ (for any single individual $x_i$), then (4.1) changes by the amount

$$-\frac{\epsilon}{x_i} \ln(1+x_i) + \frac{\epsilon}{x_i} \ln x_{i+1} + \frac{\epsilon}{x_i} \ln x_{i-1} + \mathcal{O}(\epsilon^2) = 0 + \mathcal{O}(\epsilon^2) \quad (4.4)$$
by virtue of (4.3). This simple calculation suggests that the Abel identity remains satisfied by any fermionic extension of the \( A_2 \) cluster algebra, for which the higher terms in (4.4) would automatically vanish by Grassmann statistics.

However, in general the second (and higher) order terms in (4.4) would not vanish unless the \( x_i \) are deformed in a way that conspires to produce miraculous cancellation between various terms in the sum. One way to guarantee this nontrivial cancellation is to take the \( x_i \) to be superfields (with arbitrary \( \mathcal{N} \)) satisfying (4.3), as suggested in Sec. 3. It would be interesting to investigate whether there are any other “\( \mathcal{N} > 1 \)” fermionic extensions of \( A_2 \) that preserve the Abel identity. Finally, of course it would be very interesting to study fermionic extensions of higher cluster polylogarithm identities such as the 40-term \( \text{Li}_3 \) identity from [7].

\section{Super-stringy integrals}

In this section we turn our attention to the stringy integrals defined in [9], specifically those associated to finite cluster algebras. For an algebra \( \mathcal{A} \) of rank \( d \), the associated stringy integral is defined by

\[
I_{\mathcal{A}} = (\alpha')^d \int_0^\infty \prod_{i=1}^d \frac{dy_i}{y_i} y_i^{\alpha' X_i} \prod_j (F_j(y))^{-\alpha' c_j}
\]  

where \( y = (y_1, \ldots, y_d) \), \( \alpha' \) and the \( c_j \)'s are positive real parameters and the \( F_j \) are the \( F \)-polynomials [24] of the algebra (to be reviewed shortly).

The structure of the integral \( I_{\mathcal{A}} \) is naturally encoded in a polytope \( \mathcal{P}_{\mathcal{A}} \) defined as follows. If we let \( \mathcal{N}[F] \) denote the Newton polytope in \( \mathbb{R}^d \) associated to a polynomial \( F(y) \), then \( \mathcal{P}_{\mathcal{A}} \) is the Minkowski sum (over \( j \)) of \( c_j \mathcal{N}[F_j] \). The main results of [9] are

1. The integral \( I_{\mathcal{A}} \) converges if and only if \( (X_1, \ldots, X_d) \) lies inside \( \mathcal{P}_{\mathcal{A}} \), and

2. \( \lim_{\alpha' \to 0} I_{\mathcal{A}} \) is the canonical function associated to \( \mathcal{P}_{\mathcal{A}} \).

The convergence criterion requires \( \mathcal{P}_{\mathcal{A}} \) to be full-dimensional, and the canonical function is the coefficient of the \( \mathbb{R}^d \) top-form in the canonical form [25] associated to \( \mathcal{P}_{\mathcal{A}} \).

Integrals of the type (5.1) are of interest to physicists because they generalize the classic tree-level Koba-Nielsen string scattering formula [10] in a way that manifests factorization at arbitrary \( \alpha' \).

Now let us review the definition of \( F \)-polynomials, since this will enable a generalization of (5.1) to cluster superalgebras. Given an initial quiver for the cluster algebra \( \mathcal{A} \), we add, for every mutable node \( x_i \), a frozen node labeled by a coefficient \( y_i \) and an
The F-polynomials are then the cluster variables with all $x_i$ set to 1. (We don’t include the trivial F-polynomials associated to the cluster variables in the initial quiver, since these are just 1.)

F-polynomials for cluster superalgebras can be defined in the same way and will, for the cases we consider (with only two fermionic variables $\xi_1, \xi_2$), always take the form

$$F_j(y) \rightarrow F_j(y, \xi_1, \xi_2) \equiv F_j(y) \left[1 + G_j(y) \xi_1 \xi_2\right]$$  \hspace{1cm} (5.2)

where the right-hand side defines the quantities $G_j(y)$. For a cluster superalgebra with more than two fermions, there would in general be several additional possible structures inside the brackets.

Compared to the standard (bosonic) stringy integral, when we pass to the superalgebra the term in the stringy integrand involving $F_j(y)$ therefore picks up the factor

$$\left[1 + G_j(y) \xi_1 \xi_2\right]^{-\alpha c} = 1 - \alpha' c_j G_j(y) \xi_1 \xi_2.$$  \hspace{1cm} (5.3)

Since the terms involving fermions are all manifestly proportional to $\alpha'$, $\lim_{\alpha' \to 0} \mathcal{I}_A$ would be completely unchanged, compared to the bosonic case, if we made no other modifications.

Instead we propose to study a slight modification of (5.1) that probes the $O(\alpha')$ effects of the fermionic contributions. Our working definition of the super-stringy integral for an OS-type cluster superalgebra $A$ with two fermions $\xi_1, \xi_2$ is

$$\mathcal{I}_A^f = (\alpha')^{d-1} \int d\xi_1 d\xi_2 \int_0^\infty \prod_{i=1}^d \frac{dy_i}{y_i} y_i^{\alpha' x_i} \prod_j (F_j(y, \xi_1, \xi_2))^{-\alpha' c_j}.$$  \hspace{1cm} (5.4)

For the superfield-type cluster superalgebras we will see in Sec. 5.3 that instead of a 2-fold integral over both $\xi_1$ and $\xi_2$, it is more natural to look at 1-fold integrals over each of the $\xi_i$ separately.

5.1 $A_2$

For the purpose of review let us consider first the classic $A_2$ example before moving to cluster superalgebras. The $F$-polynomials are

$$F_3 = 1 + y_1 \ , \quad F_4 = 1 + y_1 + y_1 y_2 \ , \quad F_5 = 1 + y_2$$  \hspace{1cm} (5.5)

and the stringy integral is therefore

$$\mathcal{I}_{A_2} = (\alpha')^2 \int_0^\infty \frac{dy_1}{y_1} \frac{dy_2}{y_2} y_1^{\alpha' x_1} y_2^{\alpha' x_2} (1 + y_1)^{-\alpha' c_3} (1 + y_1 + y_1 y_2)^{-\alpha' c_4} (1 + y_2)^{-\alpha' c_5}.$$  \hspace{1cm} (5.6)

The Newton polytopes associated to $c_3 F_3$, $c_4 F_4$ and $c_5 F_5$ are respectively
and their Minkowski sum is the pentagon

The integral converges for \((X_1, X_2)\) taking values in the interior of this pentagon, and the \(\alpha' \to 0\) limit gives the canonical function associated to this pentagon:

\[
\lim_{\alpha' \to 0} I_{A_2} = 1 + \frac{1}{X_1 X_2} + \frac{1}{(c_3 + c_4 - X_1)X_2} + \frac{1}{(c_3 + c_4 - X_1)(c_4 + c_5 - X_2)} + \frac{1}{(c_3 - X_1)(c_4 + c_5 - X_2)} + \frac{1}{X_1(c_5 - X_2)}. \quad (5.7)
\]

5.2 \(A_2^{\text{OS}}\)

In Sec. 2 we found that \(A_2^{\text{OS1}}\) and \(A_2^{\text{OS2}}\) have the same 15 distinct cluster variables. It is straightforward to check that they also have the same \(F\)-polynomials. Therefore, we henceforth do not distinguish between these two cases. The 15 \(F\)-polynomials of \(A_2^{\text{OS}}\) are summarized in Tab. 3. As always, we exclude the trivial “1”s that are connected to the cluster variables in the initial cluster. A further four \(F\)-polynomials are independent of \(y\) and hence are uninteresting (they give factors that pull out of the stringy integral), so we set their corresponding \(c_j = 0\). The super-stringy integral (5.4) is then
In such a way that some edges of the pentagon move off to infinity as \( \alpha' \to 0 \). The region on which all three terms converge for arbitrary \( 1/\alpha' \) is a rectangle: specifically, the “bottom half” of the pentagon drawn in Sec. 5.1.
5.3 $A_2^{SF}$

Next let us take a look at the superfield version $A_2^{SF}$ of the $A_2$ cluster superalgebra. We will do the $\mathcal{N} = 1$ case. For the superfield definition, there are a few different possible ways to define the $F$-polynomials. We can take the coefficients in the frozen nodes to be (A) ordinary (bosonic) variables $y_i$, or (B) superfields $Y_i = y_i(1 + \eta \theta_i)$, as shown here:

![Diagram](Type A and B)

Moreover, in the last step of calculating the $F$-polynomials, one can choose to set (I) $X_i \rightarrow 1$ or (II) $a_i \rightarrow 1$ (recall that $X_i = a_i(1 + \eta \xi_i)$). Let us examine a few of these possibilities case by case.

Case (B)(I) would give the ordinary $A_2$ $F$-polynomials but with $y \rightarrow Y$, i.e. the bosonic $y$'s are promoted to superfields $Y$. That means the super-stringy integral would be

$$I_{A_2^{SF}(B)(I)} = (\alpha')^2 \int_0^\infty dy_1 dy_2 \ Y_1^{\alpha' X_1 - 1} Y_2^{\alpha' X_2 - 1} Y_3^{\alpha' c_3} Y_4^{\alpha' c_4} Y_5^{\alpha' c_5}.$$  \hfill (5.12)

Substituting the $Y$-superfield expansion would give

$$I_{A_2^{SF}(B)(I)} = (\alpha')^2 \int_0^\infty dy_1 dy_2 \ y_1^{\alpha' X_1 - 1} y_2^{\alpha' X_2 - 1} y_3^{\alpha' c_3} y_4^{\alpha' c_4} y_5^{\alpha' c_5}$$

$$\times \left\{ 1 + \eta \left[ (\alpha' X_1 - 1 - \alpha' c_3 - \alpha' c_4) \theta_1 + (\alpha' X_2 - 1 - \alpha' c_4 - \alpha' c_5) \theta_2 \right] \right.$$  \hfill (5.13)

$$+ \alpha' c_3 \theta_1 \frac{1}{y_3} + \alpha' c_4 (\theta_1 + \theta_2) \frac{1}{y_4} + \alpha' (c_4 + c_5) \theta_2 \frac{1}{y_5} - \alpha' c_4 \theta_2 \frac{1}{y_4 y_5} \right\}.$$

Note that

$$\lim_{\alpha' \rightarrow 0} I_{A_2^{SF}(B)(I)} = I_{A_2}(X_1, X_2, c_3, c_4, c_5) \left[ 1 - \eta (\theta_1 + \theta_2) \right]$$  \hfill (5.14)

This is the only case in which a fermionic part survives (and moreover, is finite) in the $\alpha' \rightarrow 0$ limit. Since in this case we have $X_i \rightarrow 1$, there are no fermionic components of the $X$'s to integrate over, and we do not study an analogue of the modification (5.4). This integral has the same pentagonal convergence range as the ordinary $A_2$ stringy integral.
Another interesting case appears to be (A)(II), which would give the five $F$-polynomials

\[
1 + \eta \xi_1 , \quad 1 + \eta \xi_2 , \quad y_3 \left[ 1 + \eta \left( -\xi_1 + \frac{1}{y_3} \xi_2 \right) \right] , \\
y_4 \left[ 1 + \eta \left( -\xi_1 - \xi_2 + \frac{1}{y_4} (\xi_1 + \xi_2) \right) \right] , \quad y_5 \left[ 1 + \eta \left( -\xi_2 + (1 - \frac{1}{y_5}) \xi_1 \right) \right]
\] (5.15)

Note that since we are only setting $a_i \to 1$ instead of $X_i \to 1$, the first two $F$-polynomials are not just 1; however, they are independent of $y$, so we do not need to include them in the super-stringy integral (they pull out as overall, uninteresting factors). Therefore we consider

\[
\mathcal{I}_{A^2F(A)(II)} = (\alpha')^2 \int_0^\infty \frac{dy_1}{y_1} \frac{dy_2}{y_2} y_1^\alpha x_1 y_2^\alpha x_2 y_3^{\alpha' c_3} y_4^{\alpha' c_4} y_5^{\alpha' c_5} \\
\times \left\{ 1 - \alpha' \eta \left[ (-c_3 - c_4 + c_5) \xi_1 + (-c_4 - c_5) \xi_2 \right.ight.
\left. + c_3 \frac{1}{y_3} + c_4 (\xi_1 + \xi_2) \frac{1}{y_4} - c_5 \frac{1}{y_5} \right\}.
\] (5.16)

Since all the fermionic terms vanish as $\alpha' \to 0$ in this case, let us study a modification already advertised just below the integral (5.4). Replacing $(\alpha')^2$ in (5.16) by $\alpha' \int d\xi_1$ and evaluating the $y$-integrals using the definition (5.6) gives

\[
\mathcal{I}_{A^2F(A)(II)}^{f(1)} = \eta \left[ (c_3 + c_4 - c_5) \mathcal{I}_{A_2}(X_1, X_2, c_3, c_4, c_5) \right. \\
- c_4 \mathcal{I}_{A_2}(X_1, X_2, c_3, c_4 + \frac{1}{\alpha'}, c_5) + c_5 \mathcal{I}_{A_2}(X_1, X_2, c_3, c_4, c_5 + \frac{1}{\alpha'}) \right].
\] (5.17)

On the other hand, replacing $(\alpha')^2$ by $\alpha' \int d\xi_2$ leads to

\[
\mathcal{I}_{A^2F(A)(II)}^{f(2)} = \eta \left[ (c_4 + c_5) \mathcal{I}_{A_2}(X_1, X_2, c_3, c_4, c_5) \right. \\
- c_3 \mathcal{I}_{A_2}(X_1, X_2, c_3 + \frac{1}{\alpha'}, c_4, c_5) - c_4 \mathcal{I}_{A_2}(X_1, X_2, c_3, c_4 + \frac{1}{\alpha'}, c_5) \right].
\] (5.18)

These results are qualitatively similar to those in the previous subsection: in each case we get a sum of three terms, in two of which the convergence region is changed by shifting some edge(s) of the pentagon outward by $+1/\alpha'$, and each of the results is finite in the limit $\alpha' \to 0$.

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A More examples: superfield definition

A.1 $B_2^{SF}$ and $C_2^{SF}$

The $B_2$ cluster variables are

\[
X_1, \ X_2, \ X_3 = \frac{1+X_2}{X_1}, \ X_4 = \frac{X_1^2 + (1+X_2)^2}{X_1^2 X_2}, \ X_5 = \frac{1+X_2^2 + X_2}{X_1 X_2}, \ X_6 = \frac{1+X_1^2}{X_2}.
\]

(A.1)

If we start with $N = 1$ initial cluster variables, and define $X_i = a_i(1 + \eta \theta_i)$, the fermionic components will be

\[
\theta_3 = - \theta_1 + \frac{a_2}{1 + a_2} \theta_2
\]
\[
\theta_4 = - \frac{2(1 + a_2)^2}{a_1^2 + (1 + a_2)^2} \theta_1 - \frac{1 + a_1^2 - a_2^2}{a_1^2 + (1 + a_2)^2} \theta_2
\]
\[
\theta_5 = - \frac{1 - a_1^2 + a_2}{1 + a_1^2 + a_2} \theta_1 - \frac{1 + a_1^2}{1 + a_1^2 + a_2} \theta_2
\]
\[
\theta_6 = \frac{2a_1^2}{1 + a_1^2} \theta_1 - \theta_2.
\]

(A.2)

Note that $C_2$ cluster variables can be obtained from those of $B_2$ by exchanging $X_1 \leftrightarrow X_2$ (and reversing the order of the 6 variables).

A.2 $A_3^{SF}$

The $A_3$ cluster variables are

\[
X_1, \ X_2, \ X_3, \ X_4 = \frac{1 + X_2}{X_1}, \ X_5 = \frac{X_1 + X_3}{X_2}, \ X_6 = \frac{1 + X_2}{X_3}, \ X_7 = \frac{X_1 + (1 + X_2)X_3}{X_1 X_2}, \ X_8 = \frac{X_3 + (1 + X_2)X_1}{X_2 X_3}, \ X_9 = \frac{(1 + X_2)(X_1 + X_3)}{X_1 X_2 X_3}
\]

(A.3)
If we start with $N = 1$ initial cluster variables and define $X_i = a_i(1 + \eta \theta_i)$, the fermionic components will be

\[
\begin{align*}
\theta_4 &= -\theta_1 + \frac{a_2}{1 + a_2} \theta_2 \\
\theta_5 &= \frac{a_1}{a_1 + a_3} \theta_1 - \frac{a_3}{1 + a_2} \theta_3 \\
\theta_6 &= \frac{a_2}{1 + a_2} \theta_2 - \theta_3 \\
\theta_7 &= -\frac{(1 + a_2)a_3}{a_1 + (1 + a_2)a_3} \theta_1 - \frac{a_1 + a_3}{a_1 + (1 + a_2)a_3} \theta_2 + \frac{(1 + a_2) a_3}{a_1 + (1 + a_2) a_3} \theta_3
\end{align*}
\]

\[\theta_8 = \frac{a_1 (1 + a_2)}{a_1 (1 + a_2) + a_3} \theta_1 - \frac{a_1 + a_3}{a_1 (1 + a_2) + a_3} \theta_2 - \frac{a_1 (1 + a_2)}{a_1 (1 + a_2) + a_3} \theta_3 \]

\[\theta_9 = -\frac{a_3}{a_1 + a_3} \theta_1 - \frac{1}{1 + a_2} \theta_2 - \frac{a_1}{a_1 + a_3} \theta_3. \]

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