Multisetting Bell-type inequalities for detecting genuine tripartite entanglement

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In a recent paper, Bancal et al. put forward the concept of device-independent witnesses of genuine multipartite entanglement. These witnesses are capable of verifying genuine multipartite entanglement produced in a lab without resorting to any knowledge of the dimension of the state space or of the specific form of the measurement operators. As a by-product they found a three-party three-setting Bell inequality which enables to detect genuine tripartite entanglement in a noisy 3-qubit Greenberger-Horne-Zeilinger (GHZ) state for visibilities as low as 2/3 in a device-indepen-dent way. In this paper, we generalize this inequality to an arbitrary number of settings, demonstrating a threshold visibility of $2/\pi \sim 0.6366$ for number of settings going to infinity. We also present a pseudo-telepathy Bell inequality achieving the same threshold value. We argue that our device-independent witnesses are optimal in the sense that the above value cannot be beaten with three-party-correlation Bell inequalities.

I. INTRODUCTION

Quantum theory allows correlations between remote systems, which are fundamentally different from classical correlations \[1\]. Quantum entanglement is in the heart of this phenomenon \[2\]. Already two entangled particles give rise to correlations not reproducible within any local realistic theory \[3\]. However, moving to more particles a much richer structure and various types of entanglement arise \[4\] suggesting novel applications such as quantum computation using cluster states \[5\], sub-shotnoise metrology \[6\], or multiparty quantum networking \[7\]. In these tasks, genuinely entangled particles offer enhanced performance. Hence, it is a central problem to decide whether in an actual experiment genuinely multipartite entanglement has been produced, or alternatively, the entangled state prepared in the laboratory could be explained without requiring the interaction of all particles. In the latter case, we say that the state created is biseparable. Focussing on the tripartite case, a biseparable state $\rho_{bs}$ can be written as

$$\rho_{bs} = \sum_i p_i |\phi_i\rangle\langle\phi_i|,$$

where the pure states $\phi_i$ are separable with respect to one of the three bipartitions $1|23$, $12|3$, $13|2$, and the weights $p_i > 0$ add up to 1. For more than three parties, the generalization is straightforward.

Several experiments have been conducted so far generating multipartite entangled photonic states up to six photons (for instance, Ref. \[8\] generated a Dicke state of six photons). One of the traditional approaches to decide on the existence of genuine multipartite entanglement consists in performing a complete state tomography, and then deducing the kind of entanglement directly from the density matrix using witness operators. Alternatively, the experimentalist may measure cleverly chosen witness operators, thereby reducing the number of correlation terms to be measured in the actual experiment \[9\]. However, a common drawback is that in both cases the experimentalist needs to have a precise control over the system on which the measurements are performed.

Remarkably, there is another route, avoiding the above problem, building on the seminal work of John Bell \[1\]: Bell expressions are linear functions of joint correlations enabling one to say important things in a black box scenario about the dimension of the systems, the states involved, or the kind of measurements performed. In particular, it is possible to decide on the presence of genuine multipartite entanglement based on merely statistical data (that is, without relying on any knowledge of the implementation of the devices involved in the measurement process) \[10\]; if a Bell value, coming from the statistics of a Bell experiment, is bigger than a certain value achievable with measurements acting on biseparable quantum states, then we can be sure that the state in question is genuinely multipartite entangled. This approach has been formalized more recently by Bancal et al. \[11\], coining the term device-independent witnesses of genuine multipartite entanglement witnesses for such Bell expressions (for more details we refer the reader to that paper).

As a simplest illustration of a device-independent witness of genuine tripartite entanglement, let us represent the Mermin polynomial \[12\] in terms of three-party correlators,

$$I_2 = \langle A_0 \otimes \hat{B}_0 \otimes \hat{C}_0 \rangle - \langle \hat{A}_0 \otimes \hat{B}_1 \otimes \hat{C}_1 \rangle - \langle \hat{A}_1 \otimes \hat{B}_0 \otimes \hat{C}_1 \rangle - \langle \hat{A}_1 \otimes \hat{B}_1 \otimes \hat{C}_0 \rangle,$$

where $\langle A_n \otimes \hat{B}_\beta \otimes \hat{C}_\gamma \rangle$ designate the expected value of the product of three ±1-observables, $\hat{A}_n$, $\hat{B}_\beta$, $\hat{C}_\gamma$. It has been shown in Ref. \[10\], that $I_2 \leq 2\sqrt{2}$ for biseparable quantum states ($B_2 = 2\sqrt{2}$), whereas the maximum quantum

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value saturates the algebraic limit of 4 ($Q_2 = 4$), hence the violation of the bound $B_2$ implies genuine tripartite entanglement. Note that this reasoning holds true independently on the size of the Hilbert space dimension or on the type of measurements carried out. Hence, Mermin inequality serves as a device-independent witness of genuine tripartite entanglement [11]. Let us now take the noisy 3-qubit GHZ state,
\[
\rho(V) = V |GHZ\rangle \langle GHZ| + (1-V) \frac{I}{S},
\]
where \(|GHZ\rangle = (|000\rangle + |111\rangle) / \sqrt{2}\) is the 3-qubit GHZ state [13], and $V$ is the visibility parameter. The measurements achieving the bounds $Q_2$ and $B_2$ correspond to traceless observables, entailing the threshold visibility $V = B_2/Q_2 = 1/\sqrt{2}$. Hence, genuine tripartite entanglement in the noisy GHZ state (for $V > 1/\sqrt{2}$) can be detected in a device-independent way.

More recently, however, Bancal et al. [11] managed to lower the threshold visibility of the noisy 3-party GHZ state to $V = 2/3$ by considering a three-party three-setting Bell inequality, which can be considered as a three-setting generalization of the two-setting Mermin inequality. Note that similarly to the Mermin inequality, the Bancal et al. inequality extends to more than three parties as well [11].

In the present paper, we generalize the three-setting three-party Bancal et al. inequality to an arbitrary number of settings $m$, exhibiting the threshold visibility $V = 1/(m \sin(\pi/2m))$, which approaches $V = 2/\pi$ for large number of settings. This generalization is discussed in section III whereas another family of Bell inequalities, based on the extended parity game [14], is discussed in section III. Notably, this game exhibits pseudo-telepathy [15], and the corresponding Bell inequality has the same performance (for $m$ a power of 2) as our Bell inequality of section III.

II. MULTISETTING TRIPARTITE BELL-TYPE INEQUALITIES

Let us introduce the $m$-setting tripartite Bell expression,
\[
I_m = \sum_{\alpha, \beta, \gamma=0}^{m-1} M_{\alpha \beta \gamma} \langle \hat{A}_\alpha \otimes \hat{B}_\beta \otimes \hat{C}_\gamma \rangle,
\]
where the matrix of Bell coefficients is defined by
\[
M_{\alpha \beta \gamma} = \cos \left( \frac{\pi}{m} (\alpha + \beta + \gamma - \Delta) \right),
\]
where indices $\alpha$, $\beta$ and $\gamma$ may take the values of $0, 1, \ldots, m-1$, and $\Delta$ may be any real number. By choosing $m = 2$ and $\Delta = 0$, the Mermin polynomial [2] is recovered. On the other hand, for $m = 3$ and $\Delta = -1/2$, we obtain the polynomial of Bancal et al. [11] (apart from an irrelevant multiplicative factor).

We next exhibit a lower bound on the quantum maximum, $Q_m^l = m^3/2$, as a function of number of settings $m$. Then an upper bound is given on the biseparable quantum maximum, which is shown to be attained by von Neumann-type projective measurements, $B_m = m^2/(2 \sin(\pi/2m))$. This implies the threshold visibility $V = B_m/Q_m^l \leq B_m/Q_m^l = 1/(m \sin(\pi/2m))$, tending to $2/\pi$ in the limit of large number of measurement settings.

We wish to note that Bancal et al. (Appendix C in [11]) presented a biseparable model simulating all the single-party expectations $\langle \hat{A}\rangle$, $\langle \hat{B}\rangle$, $\langle \hat{C}\rangle$, two-party correlators $\langle \hat{A} \otimes \hat{B}\rangle$, $\langle \hat{A} \otimes \hat{C}\rangle$, $\langle \hat{B} \otimes \hat{C}\rangle$ and three-party correlators $\langle \hat{A} \otimes \hat{B} \otimes \hat{C}\rangle$, achievable with von Neumann measurements on the noisy 3-qubit GHZ state [8] of visibility $V \leq 1/2$. Within this biseparable model all of the three parties may share local random variables, but at most two parties can share a quantum state at a given time. It can be shown that if we are content with simulating only the three-party correlators, then the threshold visibility becomes a higher value, $V = 2/\pi$. This implies that it is not possible to detect genuine tripartite entanglement in the 3-qubit GHZ state in the range $V \leq 2/\pi$ by applying von Neumann measurements and considering Bell expressions which are sums of three-party correlators. In this sense, our family of Bell inequalities is optimal, giving $V \rightarrow 2/\pi$ when $m$ goes to infinity.

Lower bound on the quantum maximum, $Q_m^l$. If each of the participants performs a von Neumann projective measurement on one component of a shared 3-qubit GHZ state, the tripartite correlation of their measurement values can be written as (see for instance Appendix C in Ref. [11]):
\[
\langle \hat{A}_\alpha \otimes \hat{B}_\beta \otimes \hat{C}_\gamma \rangle = \sin \theta_\alpha \sin \theta_\beta \sin \theta_\gamma \cos (\varphi_\alpha + \varphi_\beta + \varphi_\gamma),
\]
where the $\alpha$th, $\beta$th and $\gamma$th measurement operator $\hat{A}_\alpha$, $\hat{B}_\beta$ and $\hat{C}_\gamma$ of Alice, Bob and Cecil, respectively are given as:
\[
\hat{A}_\alpha = \cos \varphi_\alpha \sin \theta_\alpha \hat{x} + \sin \varphi_\alpha \sin \theta_\alpha \hat{y} + \cos \theta_\alpha \hat{z},
\hat{B}_\beta = \cos \varphi_\beta \sin \theta_\beta \hat{x} + \sin \varphi_\beta \sin \theta_\beta \hat{y} + \cos \theta_\beta \hat{z},
\hat{C}_\gamma = \cos \varphi_\gamma \sin \theta_\gamma \hat{x} + \sin \varphi_\gamma \sin \theta_\gamma \hat{y} + \cos \theta_\gamma \hat{z},
\]
where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the Pauli operators.

With the choice of $\theta_\alpha = \theta_\beta = \theta_\gamma = 0$ and $\varphi_\alpha = \varphi_\beta = \varphi_\gamma = \pi (\mu - \Delta/3)/m$ each tripartite correlation will take the same value as the Bell coefficient to be multiplied with, and the quantum value of the Bell expression will
be easy to calculate:

\[
Q_m' = \sum_{\gamma=0}^{m-1} \sum_{\beta=0}^{m-1} A_{\alpha \beta \gamma} \langle \hat{A}_\alpha \hat{B}_\beta \hat{C}_\gamma \rangle
\]

\[
= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} \cos^2 \left( \frac{\pi}{m} (\alpha + \beta + \gamma - \Delta) \right)
\]

\[
= \frac{1}{2} \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} \left( 1 - \cos \left( \frac{2\pi m}{m} (\alpha + \beta + \gamma - \Delta) \right) \right)
\]

\[
= m^3.
\] (8)

This value \( Q_{m}' \) is a lower bound for the maximum quantum value \( Q_m \).

For the maximum of the biseparable value first we will give an upper bound \( (B_m') \), then we will prove that this bound can be saturated, that is, \( B_m' = B_m = B_m'' \).

**Upper bound on the biseparable quantum value, \( B_m'' \).** The value to be calculated is,

\[
B_m = \max \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} M_{\alpha \beta \gamma} \langle \hat{B}_\beta \hat{C}_\gamma \rangle
\]

\[
= \max \sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} M_{\beta \gamma}^{(A)} \langle \hat{B}_\beta \hat{C}_\gamma \rangle,
\] (9)

where we used the fact that Bell inequality (4) is linear in the correlators, and that a biseparable density matrix is a convex combination of pure states, hence it is enough to take the 3-party correlators in the form \( \langle \hat{A}_\alpha \rangle \langle \hat{B}_\beta \hat{C}_\gamma \rangle \). In Eq. (9) each of \( A_\alpha \) may take the value of either +1 or -1, Bob and Cecil may share any quantum state and perform measurements on them, the operators of their measurement settings are \( \hat{B}_\beta \) and \( \hat{C}_\gamma \), respectively, and the coefficients of the two-partite Bell inequality, which depends on the actual choice of \( A_\alpha \) are:

\[
M_{\beta \gamma}^{(A)} = \sum_{\alpha=0}^{m-1} A_\alpha \cos \left( \frac{\pi}{m} (\alpha + \beta + \gamma - \Delta) \right)
\] (10)

We note that due to the symmetry of the Bell expression under party exchange, it is enough to consider the case when it is Alice who may not share an entangled quantum object with the others.

From the work of Ref. [16] it easily follows that for bipartite correlation type Bell inequalities with an equal number of measurement settings per party, an upper bound for the maximum quantum value is the largest of the singular values of the matrix defined by the Bell coefficients multiplied by the number of measurement settings. In the present case the matrix depends on the sum of its indices. Therefore, \( M_{\beta \gamma}^{(A)} = M_{\beta (\gamma+1)}^{(A)} \), that is each row contains the elements of the preceding row, shifted to the left. From Eq. (10) it is also clear, that

\[
M_{\beta (\gamma+1)}^{(A)} = -M_{\beta \gamma}^{(A)},
\]

that is the last element of each row is the same as minus one times the first element of the preceding row. These properties are very similar to the properties defining circulant matrices [13], whose eigenvectors are independent of the actual values of its elements, and therefore whose eigenvalues are very easy to derive. There are just two differences. In the case of the circulant matrices the elements are shifted not to the left, but to the right. Furthermore, they do it cyclically, that is there is no change of sign when the last element takes the first place in the next row. The first difference is easily corrected if we rearrange Cecil’s measurement settings into the opposite order. Fortunately, the change of sign of the matrix element poses no serious problem either, because it can be shown that the eigenvectors of these modified circulant matrices are also independent of the actual values of the elements of the matrix, they are given as:

\[
v_j = \left( \omega_m, \omega_{m^2}, \ldots, \omega_m^{m-1} \right)^T,
\]

\[
\omega_j = e^{\frac{2\pi i (j+1/2)}{m}},
\] (11)

where \( j = 0, \ldots, m-1 \). The difference from the circulant case [17] is the 1/2 term in the exponent. To calculate the eigenvalues we only need the first row of the matrix. Therefore, we get the upper bound for the biseparable value as:

\[
B_m'' = \max m \sum_{\alpha=0}^{m-1} \sum_{\gamma=0}^{m-1} A_\alpha \cos \left( \frac{\pi (\alpha - \Delta' - \gamma)}{m} \right) \omega_j^\gamma
\] (12)

where we introduced the notation \( \Delta' = \Delta - m + 1 \). By substituting the \( \omega_j \) from Eq. (11) and using identity

\[
\cos \frac{\pi (\alpha - \Delta' - \gamma)}{m} = \cos \frac{\pi (\alpha - \Delta')}{m} \cos \frac{\pi \gamma}{m} - \sin \frac{\pi (\alpha - \Delta')}{m} \sin \frac{\pi \gamma}{m}
\] (13)

we arrive at:

\[
B_m'' = \max m \sum_{\alpha=0}^{m-1} A_\alpha \left[ \cos \frac{\pi (\alpha - \Delta')}{m} (D_1^1 + iD_1^2) + \sin \frac{\pi (\alpha - \Delta')}{m} (D_3^1 + iD_3^2) \right],
\] (14)
if we have taken $\sum_{j=0}^{m-1} \cos \frac{\pi \gamma}{m} \cos \frac{2\pi(j + 1/2)\gamma}{m}$ we get:

$$D^{j}_1 = \sum_{\gamma=0}^{m-1} \cos \frac{\pi \gamma}{m} \cos \frac{2\pi(j + 1/2)\gamma}{m}$$

$$D^{j}_2 = \sum_{\gamma=0}^{m-1} \cos \frac{\pi \gamma}{m} \sin \frac{2\pi(j + 1/2)\gamma}{m}$$

$$D^{j}_3 = \sum_{\gamma=0}^{m-1} \sin \frac{\pi \gamma}{m} \cos \frac{2\pi(j + 1/2)\gamma}{m}$$

$$D^{j}_4 = \sum_{\gamma=0}^{m-1} \sin \frac{\pi \gamma}{m} \sin \frac{2\pi(j + 1/2)\gamma}{m}.$$  \hfill (15)

However,

$$D^{j}_2 + D^{j}_4 = \sum_{\gamma=0}^{m-1} \sin \frac{2\pi(j + 1/2 \pm 1/2)\gamma}{m} = 0,$$  \hfill (16)

therefore $D^{j}_2 = D^{j}_4 = 0$, and

$$D^{j}_1 + D^{j}_3 = \sum_{\gamma=0}^{m-1} \cos \frac{2\pi(j + 1/2 \mp 1/2)\gamma}{m}.$$  \hfill (17)

from which it follows that $D^{j}_1 = D^{j}_3 = 0$ for $1 \leq j \leq m-2$, $D^{0}_1 = D^{0}_3 = m/2$, and $D^{m-1}_1 = -D^{m-1}_3 = m/2$. To get the maximum we must take either $j = 0$ or $j = m - 1$. For $j = 0$ we get:

$$B^u_m = \max \frac{m^2}{2} \sum_{\alpha=0}^{m-1} A^\beta \bar{\gamma} \bar{\gamma} e^{i\pi(a - \Delta')/m}.$$  \hfill (18)

If we have taken $j = m - 1$ instead of $j = 0$, we would have got the complex conjugate of the numbers whose absolute value has to be taken, which would have given the same result. In Eq. (18) we have to add $m$ vectors on the complex plane, each pointing towards corners of a regular polygon of $2m$ sides, and then we have to take the length of this vector. Each vector lies on a different diagonal of the polygon, but may point towards either direction depending on the value of $A_\alpha$. It can be shown that we get the largest value if the vectors taken in some order point towards consecutive corners. All such arrangements give obviously the same result. We get one of those arrangements if we take $A_\alpha = 1$. The result does not depend on $\Delta'$, as changing $\Delta'$ means only an overall rotation of the arrangement. Let us take $\Delta' = -1/2$. Then the set of numbers will be symmetric with respect to the imaginary axis, therefore the real part of the sum will be zero, while the imaginary part will be positive.

Then we get

$$B^u_m = \frac{m^2}{2} \sum_{\alpha=0}^{m-1} \sin \frac{\pi(a + 1/2)}{m}$$

$$= \frac{m^2}{2} \sum_{\alpha=0}^{m-1} \left( \sin \frac{\pi(a + 1/2)}{m} \sin \frac{\pi}{2m} \right. + \cos \left. \frac{\pi(a + 1/2)}{m} \cos \frac{\pi}{2m} \right)$$

$$= \frac{m^2}{2} \sum_{\alpha=0}^{m-1} \cos \frac{\pi\alpha}{m}$$

$$= \frac{m^2}{2} \sin \frac{\pi}{2m}. \hfill (19)$$

Here we have used that $\sum_{\alpha=0}^{m-1} \cos[\pi(a + 1/2)/m] = 0$, and that $\cos[\pi\alpha/m] = -\cos[\pi(m - \alpha)/m]$.

Now we will show that this upper bound can be saturated.

**Lower bound on the biseparable quantum value, $B^v_m$.** If $\sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} M_{\beta\gamma} C_{\beta} \cdot \bar{C}_{\gamma}$ is a certain number, where $C_{\beta}$ and $\bar{C}_{\gamma}$ are Euclidean unit vectors, then there exist measurement operators giving the same number as the quantum value of the bipartite correlation type Bell inequality of coefficients $\bar{M}_{\beta\gamma}$, applied on the maximally entangled state [18]. In case of two dimensional vectors pairs of real qubits are sufficient. Let $\bar{M}_{\beta\gamma} \equiv \bar{M}_{\beta\gamma}^{(A)}$ with all $A_\alpha = +1$ (see Eq. (10)), let Cecil’s vectors be $\bar{C}_{\gamma} = \left( \cos \frac{\pi \alpha}{m}, \sin \frac{\pi \alpha}{m} \right)$, \hfill (20)

and let us choose $\bar{B}_\beta$ optimally, that is $\bar{B}_\beta = \sum_{\gamma=0}^{m-1} \bar{M}_{\beta\gamma} \bar{C}_{\gamma}$. Then the corresponding quantum value is:

$$B^v_m = \max \frac{1}{2} \sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} \bar{M}_{\beta\gamma} \bar{C}_{\gamma}$$

$$= \sum_{\beta=0}^{m-1} \sum_{\alpha=0}^{m-1} \sum_{\gamma=0}^{m-1} \cos \frac{\pi(a + \beta + \gamma - \Delta)}{m} \left( \cos \frac{\pi\alpha}{m} \right). \hfill (21)$$

Now if we follow analogous steps to the ones we have taken calculating the value of $B^u$, we will arrive at the same result. We can also easily see this if we compare Eq. (21) to Eq. (12). The maximum value of the latter expression has been attained with $A_\alpha = 1$ and $j = 0$. By substituting these values, and also $\omega_0$ from Eq. (11), we get almost the same formula as Eq. (21). Indeed, the $\omega_0$ complex numbers correspond to the same vectors on the complex plane as the two dimensional vectors appearing in Eq. (21). From the calculation of $B^u$ it turns out that the value does not depend on $\Delta'$, so the absolute value in Eq. (21) does not depend on $\Delta - \beta$ either, therefore,
we may replace the summation in terms of $\beta$ for a multiplicative factor of $m$. The only remaining difference is the opposite sign of $\gamma$ in the cosine, but that will not affect the result either.

As the upper and lower bound for the biseparable case are equal, the biseparable value itself is given by Eq. (19). For the quantum value we have only proven a lower bound (see Eq. (5)). Therefore, for this family of Bell inequalities the ratio of the quantum and the biseparable values (which equals the visibility threshold) satisfies:

$$V = \frac{\mathcal{B}_m}{\mathcal{Q}_m} \leq \frac{1}{m \sin \frac{\pi}{2m}}. \quad (22)$$

We believe that the lower bound $\mathcal{Q}_m^l$ we have given in [5] is actually the quantum maximum itself, and the above expression is valid as an equality.

III. BELL-TYPE INEQUALITIES BASED ON THE EXTENDED PARITY GAME

Now we define another family of multisetting tripartite inequalities giving the same ratio of $\mathcal{B}_m/\mathcal{Q}_m$ as the right hand side of Eq. (22), at least when $m$ is a power of 2.

The Bell coefficients may only take the values of zero, one and minus one, namely:

$$M_{\alpha\beta\gamma} = \begin{cases} 
0 & \text{if } (\alpha + \beta + \gamma) \mod m \neq 0, \\
1 & \text{if } (\alpha + \beta + \gamma)/m \text{ is even,} \\
-1 & \text{if } (\alpha + \beta + \gamma)/m \text{ is odd,}
\end{cases} \quad (23)$$

and $\alpha, \beta, \gamma = 1, \ldots, m - 1$. These Bell coefficients correspond to the so-called extended parity game considered in Ref. [14]. An equivalent definition, more similar to the definition of the Bell inequality [5] treated in section II is that $M_{\alpha\beta\gamma} = \cos(\pi(\alpha + \beta + \gamma)/m)$, whenever the absolute value of this expression is one, and $M_{\alpha\beta\gamma} = 0$ is otherwise.

Maximum quantum value, $\mathcal{Q}_m$. We get a lower bound for the quantum value with measurement operators given in Eq. (7) applied to components of a 3-qubit GHZ state, with $\theta^A = \theta^B = \theta^C = 0$ and $\varphi^A = \varphi^B = \varphi^C = \pi/2$. Using Eq. (10) it is clear that each nonzero Bell coefficient will be multiplied by the same value as itself, therefore, the quantum value will be equal to the sum of the absolute values of the Bell coefficients, that is with the no signalling limit, which is an upper bound for the quantum value. Hence it has the property of pseudo-telepathy [12]. From this it follows, that the quantum value will be nothing else than the number of nonzero Bell coefficients, which is actually $m^2$. To see this, it is enough to note that however we slice up the $m \times m \times m$ arrangement, each resulting $m \times m$ matrices will have exactly one nonzero number (plus or minus one) in each of its rows and columns. We can get such a row or column by fixing two of the indices of $M_{\alpha\beta\gamma}$. The sum of the indices we get this way are $m$ consecutive numbers, exactly one of them will be divisible by $m$. Such a matrix will have $m$ nonzero elements, the $m$ slices together will contain $m^2$ such elements, therefore the quantum value and the no signalling limit will be $\mathcal{Q}_m = m^2$.

We now place an upper bound on the maximum of the biseparable value ($\mathcal{B}_m^l$), and then we prove that this bound can be saturated, that is, ($\mathcal{B}_m^l = \mathcal{B}_m$).

Upper bound on the biseparable quantum value, $\mathcal{B}_m^l$. A further property of the slices of the present $m \times m \times m$ arrangement is that they are modified circulant matrices like in the case of the previous family, which can be shown exactly the same way as we have shown there. To get the matrices relevant to the biseparable value, we have to add up the slices with different signs. If it is Alice who is not allowed to share entangled state with the others, this sum is $M_{\beta\gamma}^{(A)} = \sum_{\alpha=0}^{m-1} A_\alpha M_{\alpha\beta\gamma}$. Due to the property of the arrangement, for each matrix element, all terms of the sum but one will be zero. Therefore, each entry of $M_{\beta\gamma}^{(A)}$ will either be one or minus one. Moreover, this matrix will also be a modified circulant one, and its first line, which determines all the others, may contain any combination of plus and minus one values, depending on $A_\alpha$. Let $a_{m-1} - \gamma = \sum_{\alpha=0}^{m-1} A_\alpha M_{\alpha\beta\gamma}$, that is the first line of $M_{\beta\gamma}^{(A)}$, written in opposite order. Then an upper bound for the biseparable value may be written as:

$$\mathcal{B}_m^l = \max m \sum_{\gamma=0}^{m-1} a_\gamma \exp \left(\frac{i\pi(\gamma+1)}{m}\right), \quad (24)$$

Let us consider the case of $j = 0$. Then what we get is the same as Eq. (18) but with $\Delta' = 0$ (which is irrelevant), and a prefactor of $m$ instead of $m^2/2$. Then, according to Eq. (19), the result is $m/\sin(\pi/2m)$. We will show that this actually is the upper bound, whenever $m$ is a power of two. As we have discussed earlier, $\exp(i\pi \gamma/m)$, which corresponds to $j = 0$, will point towards consecutive corners of a regular polygon of $2m$ edges on the complex plane while $\gamma$ takes all values between zero and $m - 1$. If $m$ is a power of two, then for any $j$, $\exp(i\pi(2j+1)/m)$ will point towards different corners for the different $\gamma$ values, moreover if one of them will point towards one corner, there will be none pointing towards the opposite corner. The reason is that $(2j+1)\gamma$ is never divisible with $m$ in this case. Choosing $a_\gamma$ appropriately one can achieve that the terms to be added point towards consecutive corners, if taken in some order, which maximizes the absolute value of the sum. This is not true if $m$ is divisible with an odd number. When $2j + 1$ is equal to this number, for $\gamma = m/(2j + 1)$ the value of $\exp(i\pi(2j+1)/m) = -1$, which lies opposite to $+1$, the value for $\gamma = 0$. In this case not all corners can be reached with appropriate choices of $a_\gamma$, and the other corners can be reached more than once, and $\mathcal{B}_m^l$ may be larger than what we have calculated. If $m$ is odd, for $2j + 1 = m$ with $a_\gamma = -1$ we even reach the no signalling limit.

Now we show that we can actually reach the value of $\mathcal{B}_m^l = m/\sin(\pi/2m)$. 


Lower bound on the biseparable quantum value, $\mathcal{B}_m^\dagger$. The coefficients of the reduced Bell inequality are the elements of the modified circulant matrix $\bar{M}_{j\gamma}$, whose entries in the first line are all +1. The appropriate Euclidean vectors $\tilde{C}_s$ are the same as the ones already defined in Eq. (23), and analogously to Eq. (21) we may write

$$\tilde{B}_m^\dagger = \sum_{\beta=0}^{m-1} \sum_{\gamma=0}^{m-1} \bar{M}_{j\gamma} \left( \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{m}} \right).$$

(25)

For $\beta = 0$, $\bar{M}_{0\gamma} = 1$, and we have to sum $m$ unit vectors pointing towards consecutive corners of a polygon of 2$m$ sides, the usual formation, the length of the resulting vector is $1/\sin(\pi/2m)$. For $\beta = 1$ only the last element of the row will be −1. But if we change the sign of just the last vector of the formation, we get the same formation rotated by an angle of $\pi/2m$. This formation will give the same result. The next line will give a formation rotated further by $\pi/2m$, and so on, therefore the result is $m/\sin(\pi/2m)$. This is a lower bound for the biseparable value, which is equal to the upper bound if $m$ is a power of 2. In this case the ratio of the quantum and the biseparable limits is $\tilde{Q}_m/\tilde{B}_m = m\sin(\pi/2m)$, resulting in the threshold visibility $V = \tilde{B}_m/\tilde{Q}_m = 1/(m\sin(\pi/2m))$. This is the same visibility obtained under Eq. (22) by means of the Bell inequality (5) of section II.

However, we would like to mention that this family of inequalities is more economical than our previous one. The number of joint measurements involved in Bell inequality (5) scales as $m^3$, whereas the present Bell inequality defined by (23) consists of only $m^2$ joint measurements. Even for smaller number of measurements, the case which is more relevant to experiments, the difference is not negligible: Inequality (5) (or equivalently the Bancal et al. inequality [11]) gives the threshold visibility $V = 0.666$, requiring 18 joint correlation terms. On the other hand, the inequality defined by (23) yields the lower threshold $V = 0.653$, using only 16 joint terms.

### IV. CONCLUSION

In this paper we extended the three-party three-setting inequality of Bancal et al. [11], which serves as a device-independent genuine tripartite entanglement witness, to an arbitrary number of settings. Our Bell inequalities (see Eq. (5) and Eq. (23) for their definitions) can detect genuine tripartite entanglement in the noisy 3-qubit GHZ state with a visibility threshold of $V = 1/(m\sin(\pi/2m))$, where $m$ denotes the number of settings per party. For $m = 2, 3$ our result recovers the threshold values corresponding to the Mermin inequality [12] and the Bancal et al. inequality [11], respectively. Numerical optimization suggests that these threshold values are optimal for $m = 2$ and $m = 3$. However, it is still an open question whether our family of inequalities (defined by Eq. (5)) is optimal for any value of $m$. The optimality of these inequalities for $m > 3$ is supported by the fact that for $m$ going to infinity the visibility $V$ approaches the value of $2/\pi$, achievable by a biseparable model simulating three-party correlators. Also, it would be desirable to generalize our families either to more parties (here the method of Appendix C in Ref. [11] might be instructive) or to more outcomes. One may also wonder whether the inequalities presented in this work are optimal for important states different from the 3-qubit GHZ state. Furthermore, it would be of interest to find a Bell inequality, which is not the sum of three-party-correlators, giving a threshold visibility lower than $2/\pi$ for the noisy 3-qubit GHZ state.

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land, 1989), pp. 69-72.

[14] H. Buhrman, P. Høyer, S. Massar, and H. Röhrig, Phys. Rev. Lett. 91, 047903 (2003).

[15] G. Brassard, A. Broadbent, A. Tapp, Foundations of Physics, 35, 1877 (2005).

[16] S. Wehner, Phys. Rev. A 73, 022110 (2006).

[17] R.M. Gray, Toeplitz and Circulant Matrices: A review, http://www-ee.stanford.edu/~gray/toeplitz.html

[18] B.S. Tsirelson, J. Soviet. Math., 36, 557, (1987).