Bivectorial Mesoscopic Nonequilibrium Thermodynamics: Landauer-Bennett-Hill Principle, Cycle Affinity and Vorticity Potential

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In mesoscopic nonequilibrium thermodynamics (NET), Landauer-Bennett-Hill principle emphasizes the importance of kinetic cycles. For continuous stochastic systems, a NET in phase space is formulated in terms of cycle affinity $\nabla \wedge (D^{-1} b)$ and vorticity $A(x)$ representing the stationary flux $J = \nabla \times A$. Each bivectorial cycle couples two transport processes represented by vectors and gives rise to Onsager’s reciprocity; the scalar product of the two vectors $A \cdot \nabla \wedge (D^{-1} b)$ is the rate of local entropy production in the nonequilibrium steady state. An Onsager operator that maps vorticity to cycle affinity is introduced.

Introduction. Nonequilibrium thermodynamics (NET) pioneered by L. Onsager [1] is concerned with a diverse array of macroscopic physical and chemical processes: mass transport, heat conduction, chemical reactions, etc. A unified treatment in continuous systems was developed since the 1960s [2]. In recent years, introducing a NET of mesoscopic stochastic dynamics in its phase space has provided a more fundamental formulation in which the different physical and chemical fluxes are all represented by a single probability flux. Positivity of mean entropy production can be mathematically demonstrated and large deviation fluctuation theorems were discovered [3]. The local equilibrium assumption required in [2] does not enter the stochastic theory per se until its application when constitutive models for real world processes are required. In discrete-state systems, cycle flux and cycle affinity play fundamental roles in its NET; the initial idea goes back to [4]. See [5] for a recent synthesis.

Landauer-Bennett-Hill Principle. Consider a stochastic trajectory of a discrete-state Markov process one step short of the completion of a full cycle: $i_0, i_1, \ldots, i_k$ which are all distinct. Before the transition from $i_k \rightarrow i_0$, the trajectory can always be reversible. Actually, denoting the ratio of conditional probabilities of the forward and its reversed paths, 

$$\sigma = \frac{q_{i_{k+1}i_{k+2}} \cdots q_{i_1i_0}}{q_{i_1i_0}q_{i_2i_1} \cdots q_{i_{k+1}i_k}},$$

there is always the possibility that the last step balanced out the probability difference, $q_{i_{k+1}i_{k+2}/(q_{i_0i_{k}}) = \sigma^{-1}}$. In that case, the trajectory would have completed a reversible cycle with no dissipation. In other words, before the completion of a cycle, one cannot know whether a system is detailed balanced or not. We shall call this observation Landauer-Bennett-Hill (LBH) principle: In the theory of computation, Landauer applied the second law of thermodynamics to point out the necessary accompanied heat dissipation of “erasing one bit” [6]; Bennett then used Landauer’s principle to argue that it is the last step of “erasing bits” in a cyclic Maxwell demon that “saves” the second law [7]; Independently in the theory of cycle kinetics driven by chemostat chemical potential, T. L. Hill introduced the concept of cycle completion [8] and argued that cycles are more fundamental object in mesoscopic NET than transitions [4]. The notion of “erasing one bit” of Landauer’s and Bennett’s matches exactly the idea of “completing one cycle”! 

Parallel to the cycle representation of discrete-state Markov processes which has been extensively studied [9], here we present a cycle representation for the NET of continuous Markovian stochastic dynamics in its phase space $\mathbb{R}^n$ and discuss how the LBH principle comes in. It turns out that both the cycle flux and cycle affinity in the continuous system are bivectors (see Appendix); they can be represented by skew-symmetric $n \times n$ matrices. More importantly, while nonequilibrium steady state (NESS) cycle flux as a kinematic concept is nonlocal and requires highly nontrivial computation, the cycle affinity that quantifies NET thermodynamic force is locally determined and completely independent of the kinematics. The bivectorial nature of a cycle reflects the coupling between two different transport processes in real world. This further implies the fundamental importance of cycles: A nonequilibrium device converts the force in one dimension to the transport in other, and a symmetry naturally follows. This yields a clear physical picture and the mathematical representation of reciprocity in NET envisioned by Onsager.

Diffusion Processes. Consider a mesoscopic system represented by a continuous stochastic process with diffusion matrix $D(x)$ and drift $b(x)$, $x \in \mathbb{R}^n$. The stochastic dynamics is described by a time-dependent probability density function $p(x,t)$ that follows the Fokker-Planck equation (FPE)

$$\frac{\partial}{\partial t} p(x,t) = -\nabla \cdot [b(x)p(x,t) - D(x)\nabla p(x,t)].$$

With Ito’s calculus, this has a corresponding trajectory-based stochastic differential equation,

$$dX_t = [b(X_t) + \nabla \cdot D(X_t)] dt + \Gamma(X_t) dB_t$$

where $D = \Gamma \Gamma^T / 2$, $((\nabla \cdot D)_{ij} = \sum_{j=1}^{n} \partial_j D_{ji}$, and $B_t$ is the $n$-D Brownian motion. We’ve denoted $\partial_j$ as the partial derivative with respect to $x_j$.

With Eq. (2), the probability flux at $t$ is defined as

$$J(x,t) = b(x)p(x,t) - D(x)\nabla p(x,t),$$

and the notion of “probability velocity” can be introduced as $\gamma(x,t) = J(x,t)/p(x,t)$. In the stationary state, we
have an invariant probability density $\pi(x)$, a divergence-free stationary flux
\[ J^*(x) = b(x)\pi(x) - D(x)\nabla\pi(x), \tag{5} \]
and a “stationary probability velocity” $\gamma^*(x) = J^*(x) / \pi(x)$. An equilibrium corresponds to detailed balanced condition: $J^*(x) = 0 = \gamma^*(x)$. 

Infinitesimal change and cyclic change of thermodynamic quantities. Mesoscopic thermodynamics concerns the rate of change, production and dissipation of mainly three thermodynamic quantities: the (stochastic) Shannon entropy $S(x, t) := -\ln p(x, t)$, the nonequilibrium potential energy $\Phi(x) := -\ln \pi(x)$, and the free energy $F(x, t) := \Phi(x) - S(x, t)$ [10, 11]. Their infinitesimal change along $X_t$ from $t$ to $t + dt$ can be expressed as
\[ d\Phi(X_t) = \nabla\Phi(X_t) \circ dX_t \tag{6a} \]
\[ dS(X_t, t) = \partial_t S(X_t, t) dt + \nabla S(X_t, t) \circ dX_t \tag{6b} \]
\[ dF(X_t, t) = -\partial_t S(X_t, t) dt + \nabla F(X_t, t) \circ dX_t. \tag{6c} \]
Here $\circ$ denotes the Stratonovich midpoint integration $u(X_t, t) \circ dX_t = u(X_t + \frac{1}{2}dX_t, t) \cdot dX_t$, which takes care of the extra term in Ito’s calculus due to the $\sqrt{dt}$ scaling of $dB_t$.

The instantaneous production of entropy $dS$ has a decomposition $dS = dS_{\text{tot}} - dQ$ in terms of two new quantities,
\[ dQ = D^{-1}b \circ dX_t, \quad dS_{\text{tot}} = \partial_t S dt + D^{-1}\gamma \circ dX_t. \tag{7} \]
They are the total amount of heat dissipated from the system to the environment [3] and the total entropy production of the system and the environment. Note the important distinction: Infinitesimal change of a function $A(X_t, t)$ is $dA \equiv A(X_{t+dt}, t + dt) - A(X_t, t)$; but there is no such a function for $dB$ in general. The latter represents work against a non-conservative force, or a “source” term.

When the system reaches its NESS, the total entropy production at the steady state is the difference between the total heat dissipation $dQ$ and the excess heat dissipation associated with the change in the nonequilibrium potential $dQ_{\text{ex}} = -\nabla \Phi \circ dX_t$. It is the amount of energy needed to sustain the steady state, called housekeeping heat,
\[ dQ_{\text{hk}} \equiv dQ - dQ_{\text{ex}} = D^{-1}\gamma^*(X_t) \circ dX_t. \tag{8} \]
From Eqs (6c), (7) and (8), one gets the entropy production decomposition $dS_{\text{tot}} = dQ_{\text{hk}} - dF$.

We thus consider the infinitesimal change of a “work”-like quantity $W$ associated with a “force” field $f(X_t, t)$,
\[ dW = f(X_t, t) \circ dX_t. \tag{9} \]
For a smooth cyclic path $\Gamma : x(t)$, $0 \leq t \leq T$ where $x(0) = x(T) = \xi$ in $\mathbb{R}^n$, the cyclic “work” can be rewritten by Stoke’s theorem [12],
\[ W(\Gamma) = \int_{\Gamma} f \cdot dx = \int_{\Sigma} \nabla \wedge f \cdot d\sigma. \tag{10} \]
where $\Gamma$ is the boundary of a surface $\Sigma$, $(d\sigma)_{ij} = dx_i \wedge dx_j$, and $(\nabla \wedge f)_{ij} = \partial_i f_j - \partial_j f_i$ is the “curl” of $f$. Note that $\nabla \wedge f$ is not a vector in $\mathbb{R}^n$ in general; rather it is a bivector, a skew-symmetric matrix.

The cyclic changes of the thermodynamic quantities are then given by,
\[ \Delta \Phi(\Gamma) = -Q_{\text{ex}}(\Gamma) = 0 \tag{11a} \]
\[ \Delta S(\Gamma) = -\Delta F(\Gamma) = S(\xi, T) - S(\xi, 0) \tag{11b} \]
\[ Q(\Gamma) = Q_{hk}(\Gamma) = \int_{\Sigma} \nabla \wedge (D^{-1}b) \cdot d\sigma \tag{11c} \]
\[ S_{\text{tot}}(\Gamma) = \Delta S(\Gamma) + Q(\Gamma). \tag{11d} \]
If path probability of the Markovian $\Gamma$ starts with the invariant probability as the initial distribution, $\Delta S(\Gamma) = 0$ for all cycles $\Gamma$, over which the total entropy production equals to the heat dissipation:
\[ S_{\text{tot}}^*(\Gamma) = Q(\Gamma) = \int_{\Sigma} \nabla \wedge (D^{-1}b) \cdot d\sigma. \tag{12} \]
$S$ and $\Phi$ are state functions, but $Q$ and $S_{\text{tot}}^*$ are not. These are direct consequences of (6) and (7) in the cycle representation.

The mean rate of $W(t)$ in (10) can be computed following Ito’s calculus [13],
\[ w \equiv \frac{1}{dt} \mathbb{E}[dW] = \int_{\mathbb{R}^n} J(x, t) \cdot f(x, t) dx \tag{13} \]
where $dx = \prod_{i=1}^n dx_i$ and $\mathbb{E}[\cdot]$ denotes expectation. The mean rates of $\Phi$, $S$, $F$, $S_{\text{tot}}$, $Q_{hk}$ can then be obtained by plugging in the corresponding forces $f$. We note that since $\mathbb{E}[\partial_t S] = 0$, the first terms in Eqs. (6b) and (6c) do not contribute to the mean rate.

Cycle representation of kinematic NESS flux. The divergence-free stationary flux $J^*$ can be expressed in terms of a bivector potential $A(x)$, $\nabla \times A = J^*$. Note that $A$ is also not a vector in $\mathbb{R}^n$ in general; rather it is a bivector whose components satisfy
\[ J^*_i(x) = (\nabla \times A)_i = \sum_{j=1}^n \partial_j A_{ij}(x). \tag{14} \]
It is straightforward to verify that $\nabla \cdot (\nabla \times A) = 0$. See Appendix for more discussion on the generalized vector potential and curl in $\mathbb{R}^n$. We fix the notion $\nabla \times A$ to denote a vector from a bivector $A$, $(\nabla \times A)_i := \sum_{j=1}^n \partial_j A_{ij}$ and $\nabla \times v$ to map a vector $v$ to a bivector: $(\nabla \times v)_{ij} := \partial_i v_j - \partial_j v_i$.

The physical meaning of Eq. (14) is rather clear. For every infinitesimal vector $dx_i$ in the $x_i$ direction at the point $x \equiv (x_1, \cdots, x_n) \in \mathbb{R}^n$, there are $(n-1)$ orthogonal vectors $dx_j, j \neq i$, and $dx_i \wedge dx_j$ forms a bivector, an infinitesimal planar element, as shown in Fig. 1(a). $A_{ij}(x)$ then denotes the stationary cycle flux around the oriented infinitesimal planar element $dx_i \wedge dx_j$ at $x$. The $i$th component of $J^*$, $J^*_i$, is then determined from all the neighboring infinitesimal planes $dx_i \wedge dx_j, j \neq i$. And, $\partial_j A_{ij}$ is thus the net edge flux.
along $x_i$ due to the pair of $A_{ij}$ at $(x_1, \ldots, x_j, \ldots, x_n)$, and at $(x_1, \ldots, x_j - dx_j, \ldots, x_n)$ as shown in Fig. 1(b). An increasing $A_{ij}$ in the $j$th direction leads to a positive net flow $\partial_j A_{ij}$ in the $x_i$ direction. Eq. 14 gives a cycle representation of the steady state fluxes $J^*_{ij}$ along the edges in terms of the cycle fluxes around the planar elements. An earlier discussion for 3-D cases can be found in [14]. $A(x)$ is a potential of $J^*$ in terms of vorticity components at $x$.

**Landauer-Bennett-Hill principle for diffusion.** In NESS, the $J(x, t)$ in Eq. (13) is replaced by the divergence-free stationary flux $J^*(x)$. With our bivector potential, the mean rate of $W$ in Eq. (9) with corresponding force $f$ can be rewritten as

$$w^* = \int_{\mathbb{R}^n} J^* \cdot f \, dx = \int_{\mathbb{R}^n} ( \nabla \times A ) \cdot f \, dx \quad (15a)$$
$$= \int_{\mathbb{R}^n} A \cdot ( \nabla \times f ) \, dx. \quad (15b)$$

The scalar product in (15a) is between two vectors in $\mathbb{R}^n$, but is between two bivectors in (15b). It is the half of the Frobenius product between two matrices. See Appendix for detailed calculation.

This immediately implies that thermodynamics quantities with a gradient force would have zero mean rate in NESS. That includes all the functions $\Phi$, $S$, and $F$, implying that the mean rates of $s_{\text{tot}}$, $Q$, and $Q_{hk}$ are all identical at NESS, $s_{\text{tot}}^* = q^* = q_{hk}^*$. This has been termed as a “gauge freedom” in [12, 15].

Thus, the average total entropy production rate at NESS can be written as

$$s_{\text{tot}}^* = \int_{\mathbb{R}^n} A \cdot ( \nabla \times (D^{-1}b) ) \, dx. \quad (16)$$

The stationary cycle flux $A$ is a purely kinematic concept that devioes any thermodynamic content. A closed loop $\Gamma$ in $\mathbb{R}^n$ contains a surface $\Sigma$ which can be tiled by an array of tiny oriented infinitesimal planar elements at $x$, for all $x \in \Sigma$. $A(x)$ then decomposes $J^*(x)$, following Kirchhoff’s law, in terms of the occurrence rate of these tiny oriented elements along the infinitely long ergodic path $X_t$. As a vorticity description of the NESS, $A$ is nonlocally determined.

On the other hand, the cycle affinity [5], as the Onsager’s thermodynamic force corresponding to the cycle flux, is locally determined through $\nabla \cdot (D^{-1}b)$. $D^{-1}b$ should be identified as the vector potential of the cycle affinity. This is in sharp contrast to the standard expression $s_{\text{tot}} = \int_{\mathbb{R}^n} J^* \cdot D^{-1} \gamma^* \, dx$ where the thermodynamic force corresponding to $J^*$ is nonlocally defined by $D^{-1} \gamma^*$. Note that the cycle affinity is a bivector with components

$$[\nabla \times (D^{-1}b)]_{ij} = \partial_i (D^{-1}b)_j - \partial_j (D^{-1}b)_i, \quad (17)$$

representing how the two dimensions $x_i$ and $x_j$ are coupled.

This constitutes the the LBH principle for diffusion processes: Entropy production in NESS is characterized by the locally-defined cycle affinity; entropy production of a bigger loop is the integral of the cycle affinity of infinitesimal cycles; and the average entropy production rate is the average cycle affinity, weighted by the cycle flux of infinitesimal cycles. The fundamental unit of NESS is the non-detailed-balanced kinetic cycle [4], in terms of bivectors.

**Mean rate decomposition outside of NESS.** For the mean rate of thermodynamics quantities outside of NESS, we rewrite Eq. (13) as $w = \mathbb{E}[\gamma(X(t), t) \cdot f(X(t), t)]$. By $\gamma(x, t) = \gamma^*(x) - D \nabla F(x, t)$, the $w$ has a decomposition,

$$w = \mathbb{E}[f \cdot \gamma^*] = \mathbb{E}[f \cdot D \nabla F]. \quad (18)$$

Outside of NESS, the two terms within can be rewritten as

$$\mathbb{E}[f \cdot \gamma^*] = \int_{\mathbb{R}^n} A \cdot (\nabla \times (\frac{\pi}{\pi} f)) \, dx \quad (19a)$$
$$\mathbb{E}[f \cdot D \nabla F] = \int_{\mathbb{R}^n} \frac{\pi}{\pi} \nabla \cdot (\pi D f) \, dx. \quad (19b)$$

This implies an average perpendicularity between $-\nabla F$ and $\gamma^*$, $\mathbb{E}[-\nabla F \cdot \gamma^*] = 0$.

With inner product defined as $\langle u, v \rangle = \mathbb{E}[u \cdot D^{-1}v]$, this average perpendicularity becomes $\langle -D \nabla F, \gamma^* \rangle = 0$.

The decomposition in Eq. (18) then have a rather nice geometric interpretation under the provided inner product. The mean rate of a “work”-like quantity $W$ of Eq. (9) with the force $f$ is determined by the inner product of $D f$ with two perpendicular vectors, $\gamma^*$ and $-D \nabla F$,

$$w = \langle D f, \gamma^* \rangle + \langle D f, -D \nabla F \rangle. \quad (20)$$

This gives the Pythagorean-like relation between $\gamma^*$, $\gamma^*$, and $-D \nabla F$ [16] hidden behind the famous entropy production rate decomposition [17].

$$\langle \gamma, \gamma \rangle = \langle \gamma^*, \gamma^* \rangle + \langle -D \nabla F, -D \nabla F \rangle. \quad (21)$$
Results above indicate that $\gamma^*$ and $-D\nabla F$ originate from two rather disjoint irreversibilities, and that the geometry defined through the Riemannian metric $D^{-1}(x)$ may be the most natural one in thermodynamics.

Onsager’s reciprocality and the Onsager operator. Diffusion process in $\mathbb{R}^n$ always has its NESS thermodynamic force $D^{-1}\gamma^*$ linearly related to transport flux $J^*$: $(D^{-1}\gamma^*)_i = e^\Phi D^{-1}j^*_i$. Many previous studies have explore this unique feature [13, 18]. In the bivectorial representations there is a further linear affinity-vorticity relationship

$$\nabla \wedge (D^{-1}b)(x) = OA(x). \quad (22)$$

where $O = \nabla \wedge (e^\Phi D^{-1}\nabla \times)$. We shall call the operator $O$ the Onsager operator. It linearly maps the vorticity bivector to the cycle affinity bivector.

For discrete-state systems, it was understood in [5, 19] that such a relation at the cycle level is the fundamental origin of the Onsager’s reciprocal relation. We note that the mean NESS entropy production rate has a simple bilinear form:

$$s_{tot}^* = \int_{\mathbb{R}^n} A \cdot OA \, dx. \quad (23)$$

Incidentally, Onsager has also considered tiny vortices “who wanted to play” as the fundamental objects in hydrodynamic turbulent flow [20].

The probabilistic gauge of the bivector potential $A$. The bivector potential $A$ obtained for the divergence-free $J^*(x) = \nabla \times A(x)$ is not unique: It has a gauge freedom with an arbitrary curl-free bivector. The situation has an analogue to that of discrete-state Markov process [21, 22], and the vector potential in classical electrodynamics. Interestingly, for discrete-state Markov process, Qian and Qian have proven the existence and uniqueness of a gauge with a probabilistic meaning: Cycles are not just represented in terms of Kirchhoff decomposition via linearly independent bases; rather the space of all possible cycles are considered, on which the unique probabilistic gauge, as NESS cycle flux, is the occurrence rate of a given cycle along the infinitely long, ergodic path [22]. Whether such a unique probabilistic gauge also exists for bivector potential $A$ on $\mathbb{R}^n$, or a more extended space of loops [23], remains to be further investigated.

Conclusions and discussion. This study clearly points to the importance of cycle representation for mesoscopic nonequilibrium thermodynamics (NET) in term of cycle flux $A$ and cycle affinity $\nabla \wedge (D^{-1}b)$. The former is a pure kinematic concept and the latter contains all the fundamental information on NET. We discover that the cycle flux and cycle affinity are not simple vectors in $\mathbb{R}^n$; rather they are bivectors, e.g., skew-symmetric matrices. The cycle flux is the bivector potential of the conventional NESS flux; and the cycle affinity has a vector potential $D^{-1}(x)b(x)$ which is obtained locally.

Some of the mathematics in the present work is contained in the diffusion process on a manifold [24] and the gauge field formulations of NET [12, 15]. The present work provides a clearer physics of NET in phase space as a formulation of Onsager’s general principle for entropy production. We identify the bivector nature of the cycle representation in terms of a local cycle affinity and a nonlocal kinematic cycle flux; and reveal a unified Landauer-Bennett-Hill thermodynamic principle for stationary nonequilibrium systems.

Finally, we noted a parallel between quantum mechanical phase giving a reality to the “indeterminate” vector potential in electromagnetism [25] and our stochastic formulation giving a vorticity interpretation to the bivector $A$ in stochastic thermodynamics: Steady state flux $J^*$ turns out to be a derivative.

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APPENDIX: CURL AND BIVECTOR POTENTIAL IN n-DIMENSION

To summarize the mathematics to be used in the present work: According to the Poincaré lemma, on a contractible domain, every closed form is exact, implying the existence of a “vector” potential of a divergence-free vector field. The “vector” potential of a $n$-dimensional divergence-free vector field $\mathbf{F}(\mathbf{x})$ is actually a bivector, a skew-symmetric matrix $\mathbf{A}(\mathbf{x})$: $F_i(\mathbf{x}) = \sum_{j=1}^{\infty} \partial_j A_{ij}(\mathbf{x})$ where $\partial_j$ denote the partial derivative w.r.t. $x_j$. The curl of a $n$-dimensional vector $\mathbf{v}(\mathbf{x})$ is also a bivector $[\nabla \wedge \mathbf{v}(\mathbf{x})]_{ij} = \partial_i v_j(\mathbf{x}) - \partial_j v_i(\mathbf{x})$. The scalar product of the vector potential of $\mathbf{F}$ and the curl of $\mathbf{v}$ then is half of the Frobenius product between matrices,

$$\mathbf{A} \cdot (\nabla \wedge \mathbf{v}) = \sum_{i<j} A_{ij}(\mathbf{x}) \left[ \partial_i v_j(\mathbf{x}) - \partial_j v_i(\mathbf{x}) \right]$$

(24)

The concept of exterior calculus is needed to generalize the curl and cross product to dimensions higher than 3 [26]. One takes the dual of a vector space, the space of linear functions, in which elements are called covectors or 1-forms. Therefore corresponding to the vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ we have the 1-form

$$\sum_{k=1}^{n} v_k \, dx_k.$$  

(25)

The exterior derivative of a differential form can be interpreted, geometrically, as the integral over the boundary of an infinitesimal parallelepiped. The exterior derivative of a $k$-form is a $(k + 1)$-form. The Stokes-Cartan theorem states that the integral of a differential form $\omega$ over the boundary of some orientable manifold $\Omega$ is equal to the integral of its exterior derivative $d\omega$ over the whole of $\Omega$:

$$\oint_{\partial \Omega} \omega = \int_{\Omega} d\omega.$$  

(26)

One of the most important consequences of the Stokes-Cartan theorem in Eq. (26) is that if $\Omega$ is a closed manifold without boundary, then $\partial \Omega = \emptyset$ and

$$\int_{\Omega} d\omega = 0.$$  

(27)

For example, the integral over every closed curve $\Gamma$

$$\oint_{\Gamma} \mathbf{F}(\mathbf{x}) \cdot d\ell = \oint_{\Gamma} d\varphi(\mathbf{x}) = \oint_{\Gamma} \sum_{k=1}^{n} (\partial_k \varphi) \, dx_k = 0$$

(28)

if and only if the vector field $\mathbf{F}(\mathbf{x})$ is the exterior derivative of a 0-form, a scalar $\varphi(\mathbf{x})$.

Using exterior derivatives and differential forms, the integral of an $n$-dimensional vector field $\mathbf{F}(\mathbf{x})$ over an $(n - 1)$-dimensional closed surface $\Sigma$

$$\oint_{\Sigma} \sum_{k=1}^{n} F_k(\mathbf{x}) \, d\sigma_k = \oint_{V} \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} (\partial_j F_k) \, dx_j \right\} \, d\sigma_k$$

$$= \oint_{V} \sum_{k,j=1}^{n} (\partial_j F_k) \, dx_j \wedge d\sigma_k$$

$$= \oint_{V} \sum_{j=1}^{n} (\partial_j F_j) \, dx_j,$$  

(29)

where $V$ is the $n$-volume contained by the closed $(n - 1)$-surface $\Sigma$, and

$$d\sigma_k = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n$$

(30)

with $dx_k$ missing. Thus, $dx_j \wedge d\sigma_k = (-1)^{j-k} \delta_{jk} \, dx$. Note that according to the rule of geometric product of multivectors, $(dx_i)(dx_j) = dx_i \cdot dx_j + dx_i \wedge dx_j$ in which the first term is a lower grade quantity which is negligible in a higher grade integration. Now if a vector field $\mathbf{F}(\mathbf{x})$ is divergence free, then the integral on the left-hand-side of (29) is zero for every closed surface $\Sigma$. Applying the Stokes-Cartan theorem again, this implies that

$$\sum_{k=1}^{n} F_k(\mathbf{x}) \cdot d\sigma_k = d\omega,$$  

(31)

where $\omega$ is expected to be a $(n - 2)$-form with the general expression

$$\omega = \sum_{i,j=1}^{n} u_{ij}(\mathbf{x}) d\eta_{ij},$$  

(32)

in which

$$d\eta_{ij} = dx_1 \cdots dx_{i-1} \wedge dx_{i+1} \cdots dx_j \wedge dx_{j+1} \cdots dx_n$$

(33)

with $dx_i$ and $dx_j$ missing. We shall assume that $i < j$. Then,
\[
d\omega = \sum_{i,j=1: i<j}^{n} \sum_{k=1}^{n} (\partial_k u_{ij}(x)) \, dx_k \wedge d\eta_{ij} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \{ (\partial_i u_{ij}(x)) \, dx_i \wedge d\eta_{ij} + (\partial_j u_{ij}(x)) \, dx_j \wedge d\eta_{ij} \}
\]

\[
= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{i-1} (-1)^{j-1} (\partial_j u_{ji}(x)) + \sum_{j=i+1}^{n} (-1)^{j} (\partial_j u_{ij}(x)) \right\} \, d\sigma_i = \sum_{i=1}^{n} \sum_{j=1}^{n} (\partial_j A_{ij}(x)) \, d\sigma_i. \tag{34}
\]

In the last step we have introduced \( A_{ij}(x) = (-1)^{j} u_{ij}(x) \) for \( i < j \), \( A_{ij}(x) = -A_{ji}(x) \) for \( i > j \) and \( A_{ii}(x) = 0 \). It is easy to verify that

\[
F_i(x) = \sum_{j=1}^{n} (\partial_j A_{ij}(x)) \tag{35}
\]

is a divergence free vector field:

\[
\nabla \cdot F(x) = \sum_{i=1}^{n} \partial_i F_i(x) = \sum_{i,j=1}^{n} (\partial_i \partial_j A_{ij}(x)) = 0. \tag{36}
\]

The vector potential \( A(x) \) of a divergence-free field is a bivector, an anti-symmetric matrix.

In terms of the bivector potential, the curl is best understood via the integration by parts:

\[
\int_{\mathbb{R}^n} F(x) \cdot v(x) \, dx = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \partial_j A_{ij}(x) v_i(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} A_{ij}(x) (\partial_i v_j(x) - \partial_j v_i(x)) \, dx. \tag{37}
\]

We have assumed that the integrand vanishes at infinity. The term inside \((\cdots)\) in (37) can be formally thought as \(\nabla \wedge v\).

Eq. (37) is the \(n\)-dimensional generalization of

\[
\int_{\mathbb{R}^3} F(x) \cdot v(x) \, dx = \int_{\mathbb{R}^3} (\nabla \times A) \cdot v(x) \, dx
\]

\[
= \int_{\mathbb{R}^3} A(x) \cdot (\nabla \times v) \, dx \tag{38}
\]

for 3-dimensional vector field.