Improved results on the nonlinear feedback stabilisation of a rotating body-beam system

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1. Introduction

The stabilisation problem of coupled elastic and rigid parts systems has attracted a huge amount of interest and in particular the rotating disk-beam system which arises in the study of large-scale flexible space structures (Baillieul & Levi, 1987). It consists of a flexible beam (B), which models a flexible robot arm, clamped at one end to the centre of a disk (D) and free at the other end (see Figure 1). Additionally, it is assumed that the centre of mass of the disk is fixed in an inertial frame and rotates in that frame with a non-uniform angular velocity. Under the above assumptions, the motion of the whole structure is governed by the following nonlinear hybrid system (see Baillieul and Levi (1987) for more details):

\[
\begin{align*}
\rho y_t(x, t) + E y_{xxxx}(x, t) &= \rho \omega^2(t) y(x, t), & (x, t) \in (0, \ell), \\
y(0, t) = y_x(0, t) &= 0, & t > 0, \\
y_{xxx}(\ell, t) &= \mathcal{F}(t), & t > 0, \\
y_{xx}(\ell, t) &= \mathcal{M}(t), & t > 0, \\
\frac{d}{dt} \begin{cases}
\omega(t) (I_d + \int_0^\ell \rho y^2(x, t) \, dx)
\end{cases} &= T(t), & t > 0, \\
y(x, 0) = y_0(x), y_1(x, 0) &= y_1(x), & x \in (0, \ell), \\
\omega(0) &= \omega_0 \in \mathbb{R},
\end{align*}
\]

in which \(y\) is the beam’s displacement in the rotating plane at time \(t\) with respect to the spatial variable \(x\) and \(\omega\) is the angular velocity of the disk. Furthermore, \(\ell\) is the length of the beam and the physical parameters \(E, \rho\) and \(I_d\) are respectively the flexural rigidity, the mass per unit length of the beam and the disk’s moment of inertia. Lastly, \(\mathcal{M}(t)\) and \(\mathcal{F}(t)\) are respectively the moment and force control exerted at the free end of the beam, while the torque control \(T(t)\) acts on the disk.

After the pioneer work of Baillieul and Levi (1987), the stabilisation problem of the rotating system has been the object of a considerable mathematical endeavour (see for instance Bloch & Titi, 1990; Chen et al., 2014; Chentouf & Couchouron, 1999; Chentouf, 2006, 2014, 2015, 2016a, 2016b; Chentouf & Wang, 2006, 2015; Coron, 1998; Guo & Wang, 2016; Laouy et al., 1996; Morgül, 1990, 1991a, 1991b, 1994; Xu & Baillieul, 1993; Xu & Sallet, 1992, and the references therein).

Instead of surveying the vast literature on this subject, we are going to highlight only those closely related to the context of our main concern, which is nonlinear stabilisation of the system. In fact, to the best of our knowledge, very few works have been devoted to nonlinear stabilisation of the system (1). Specifically, the authors in Coron (1998) constructed a nonlinear feedback torque control law \(T(t)\), which globally asymptotically stabilises the system. Later, an exponential stabilisation result has been established in Chentouf and Couchouron (1999) via a particular class of nonlinear boundary and torque controls defined below

\[
\begin{align*}
\mathcal{F}(t) &= g(y_1(1, t)), \\
\mathcal{M}(t) &= -f(y_{xx}(1, t)), \\
T(t) &= -\gamma(\omega(t) - \sigma),
\end{align*}
\]

where \(\sigma \in \mathbb{R}\) and \(f, g\) and \(\gamma\) are nonlinear real functions. In fact, the results in Chentouf and Couchouron (1999) are obtained under very restrictive conditions, which we can summarise as follows: \(f\) and \(g\) are almost everywhere linear functions (see...
Chentouf and Couchouron (1999) for more details). We also note that in Guo and Wang (2017), the authors proposed a linear torque control and a nonlinear interior control of type viscous damping to obtain the exponential stability of the closed loop system.

It is worth mentioning that the boundary stabilisation of elastic systems has been the subject of extensive works. For instance, the case of the wave equation with nonlinear boundary feedback has been considered in Lasiecka (1989); Wang and Chen (1989) for the case of the wave equation with nonlinear boundary feedback. The disk-beam system has been considered in Lasiecka (1989); Wang and Chen (1989) for the case of the wave equation with nonlinear boundary feedback. For instance, see, e.g. Alabau-Boussouira (2005), Alabau-Boussouira (2010), Alabau-Boussouira and Ammari (2011), Ammari et al. (2002), Ammari (2003), Ammari et al. (2007), Ammari and Tucsnak (2000), Ammari and Nicaise (2015), Ayadi and Bchatnia (2019), Ayadi et al. (2015), Chen and Delfour (1987), Lasiecka and Tataru (1993), Lions (1988), Nagaya (1995), etc.

The novelty of the present work is to provide a more general approach to the stabilisation problem (1) – (2) under less restrictive conditions on the nonlinear functions $f$ and $g$. Before going into details, we would like to point out that the presence of the moment control $M(t) = -f(y_{xx}(1,t))$ in the feedback law (2) is not required to get the stability result of the closed-loop system but provides more dissipation to the energy of the system (see Lemma 3.1). In other words, one can take $M(t) = 0$ and our stability results will remain valid. This will be made more clear later (see Remark 3.1). In turn, for sake of simplicity and without loss of generality, we shall take $g = -f$ and show that the closed-loop system possesses the very desirable property of exponential stability as long as the angular velocity is bounded (see (5)) and the functions $f$ obeys weaker conditions than those in Chentouf and Couchouron (1999). Indeed, the stability outcome of Chentouf and Couchouron (1999) is established for a very particular case of nonlinear boundary conditions.

Now, let us briefly present an overview of this paper. In Section 2, we formulate the closed-loop system in an abstract first-order evolution equation in an appropriate state space. Section 3 deals with the long time behaviour (in a general context) of a subsystem associated to the closed-loop system. In Section 4, some examples are provided. Section 5 is devoted to the proof of the stability of the closed-loop system. Finally, the paper closes with conclusion and discussion.

2. Preliminaries

First, one can use an appropriate change of variables so that $\ell = 1$. Then, the closed loop system (1) – (2) can be written as follows:

\[
\begin{align*}
\rho y_{tt} + E I y_{xxxx} &= \rho \omega^2(t) y, \\
y(0, t) &= y_x(0, t) = 0, \\
y_{xx}(1, t) &= -f(y_{xx}(1, t)), \\
y_{xxx}(1, t) &= f(y_{t}(1, t)), \\
-\gamma(\omega(t) - \bar{\omega}) &= -y'_{tt}(t) + \int_0^1 y(t) y_t(t) \, dx, \\
\dot{\omega}(t) &= I_d + \rho \int_0^1 y(t)^2 \, dx, \\
y(x, 0) &= y_0(x), y_t(x, 0) = y_1(x), \\
\omega(0) &= \omega_0 \in \mathbb{R}.
\end{align*}
\]

Thereafter, let $z(\cdot, t) = y(\cdot, t)$ and consider the state variable

\[(\phi(t), \omega(t)) = (y(\cdot, t), z(\cdot, t), \omega(t)).\]

Next, for $n \in \mathbb{N}, n \geq 2$, we denote by

\[H^n_l = \{u \in H^n(0, 1); u(0) = u_x(0) = 0\}.\]

Now, consider the state space $\mathcal{X}$ defined by

\[\mathcal{X} = H^2_l \times L^2(0, 1) \times \mathbb{R} = \mathcal{H} \times \mathbb{R},\]

equipped with the following real inner product (the complex case is similar):

\[
\langle (y, z, \omega), (\tilde{y}, \tilde{z}, \tilde{\omega}) \rangle_{\mathcal{H}} = \langle (y, z), (\tilde{y}, \tilde{z}) \rangle_{\mathcal{H}} + \omega \tilde{\omega} + \int_0^1 (E I y_{xx} \tilde{y}_{xx} - \rho \sigma^2 y \tilde{y} + \rho z \tilde{z}) \, dx.
\]

Note that the norm induced by this scalar product is equivalent to the usual one of the Hilbert space $H^2_l(0, 1) \times L^2(0, 1) \times \mathbb{R}$ provided that (see Chentouf and Couchouron (1999))

\[|\sigma| < 3\sqrt{EI}/\rho.\]

Now, we define a nonlinear operator $A$ by

\[
D(A) = \{(y, z) \in H^2_l \times H^2_l; y_{xxx}(1) = f(z(1)), y_{xx}(1) = -f(z_x(1))\},
\]
and
\[
A(y, z) = \left( -z, \frac{EI}{\rho} y_{xxxx} - \sigma^2 y \right). \tag{7}
\]
Whereupon the system (3) can be formulated in \(X\) as follows:
\[
\dot{\phi}(t), \dot{\omega}(t) + (A + B) (\phi(t), \omega(t)) = 0, \tag{8}
\]
in which
\[
\begin{align*}
A(\phi, \omega) &= (A\phi, 0), \text{ with } D(A) = D(A) \times \mathbb{R}, \\
B(\phi, \omega) &= \begin{cases} 
0, (\sigma^2 - \omega^2)y, \\
\gamma (\omega - \sigma) + 2\rho \omega(t) < y, z > L^2(0, 1), \\
I_d + \rho \|y\|^2_{L^2(0, 1)}.
\end{cases}
\end{align*}
\tag{9}
\]

We note that we will adapt here the approach of Lasiou and Tataru (1993) and of Alabau-Boussouira (2005), Alabau-Boussouira (2010) (see also Ammari et al. (2016)). To proceed, we suppose that the functions \(f\) and \(\gamma\) satisfy the following conditions:

**H.I:** \(f: \mathbb{R} \rightarrow \mathbb{R}\) is a non-decreasing differentiable function such that \(f(0) = 0\).

**H.II:** There exists a strictly increasing function \(f_0 \in C([0, +\infty))\) and continuously differentiable in a neighbourhood of \(0\) such that \(f_0(0) = f_0'(0) = 0\) and
\[
\begin{align*}
f_0(|s|) &\leq |f(s)| \leq f_0^{-1}(|s|), \text{ for all } |s| < \sigma, \\
c_1 |s| &\leq |f(s)| \leq c_2 |s|, \text{ for all } |s| \geq \sigma,
\end{align*}
\]
where \(c_i > 0\) for \(i = 1, 2\) and for some \(\sigma > 0\).

**H.III:** The function \(\gamma\) is Lipschitz on each bounded subset of \(\mathbb{R}\) and for some \(k > 0\),
\[
\gamma(s)x \geq 0, \quad |\gamma(x)| \geq \kappa |x|, \quad \forall x \in \mathbb{R}.
\]

Moreover, we define a function \(H\) by
\[
H(x) = \sqrt{xf_0'(\sqrt{x})}, \quad \forall x \geq 0. \tag{10}
\]

Thanks to the assumptions H.I – H.II, the function \(H\) is of class \(C^1\) and is strictly convex on \((0, r^2]\), where \(r > 0\) is a sufficiently small number.

### 3. Exponential stability of a subsystem

In this section, we shall deal with the following subsystem:
\[
\begin{align*}
\rho y_{tt} + E y_{xxxx} - \rho \sigma^2 y &= 0, & 0 < x < 1, \ t \geq 0, \\
y(0, t) = y_t(0, t) &= 0, & t \geq 0, \\
y_{xx}(1, t) &= -f(y_{xx}(1, t)), & t \geq 0, \\
y_{xxx}(1, t) &= f(y(1, t)), & t > 0, \\
y(x, 0) = y_0(x), \ y_t(x, 0) = y_1(x), & 0 < x < 1,
\end{align*}
\tag{11}
\]
or equivalently
\[
\begin{align*}
\dot{\phi}(t) + A\phi(t) &= 0, & \phi(0) = \phi_0, \tag{12}
\end{align*}
\]
where \(\phi = (y, z), \ z = y_t\) and \(A\) is defined by (6) – (7).

We have the following proposition whose proof can be found in Chentouf and Couchouron (1999) (see Lemma 5 and Proposition 3):

**Proposition 3.1:** Assume that the condition (5) holds, that is, \(|\sigma| < 3\sqrt{EI/\rho}\). Then, for any function \(f\) satisfying solely the assumption H.II, the operator \(A\) defined by (6) – (7) is m-accretive in \(\mathcal{H}\) with dense domain. Additionally, we have:

1. For any initial data \(\phi_0 = (y_0, y_1) \in D(A)\), the system (12) admits a unique solution \(\phi(t) = (y(t), z(t)) \in D(A)\) such that
\[
(y, y_t) \in L^\infty(\mathbb{R}^+; D(A)), \quad \text{ and } \frac{d}{dt} (y, y_t) \in L^\infty(\mathbb{R}^+; \mathcal{H})\).
\]

The solution \(\phi = (y, z)\) is given by \(\phi(t) = e^{-At}\phi_0\), for all \(t \geq 0\) where \((e^{-At})_{t \geq 0}\) is the semigroup generated by \(-A\) on \(D(A) = \mathcal{H}\). Moreover, the function \(t \mapsto \|A\phi(t)\|_\mathcal{H}\) is decreasing.

2. For any initial data \(\phi_0 = (y_0, y_1) \in \overline{D(A)} = \mathcal{H}\), Equation (12) admits a unique mild solution \(\phi(t) = e^{-At}\phi_0\) which is bounded on \(\mathbb{R}^+\) by \(\|\phi_0\|_\mathcal{H}\) and
\[
\phi = (y, y_t) \in C^0(\mathbb{R}^+; \mathcal{H}).
\]

3. The semigroup \((e^{-At})_{t \geq 0}\) is asymptotically stable in \(\mathcal{H}\).

The first main result of this article is:

**Theorem 3.1:** Let us suppose that the conditions (5) and H.I – H.II hold, and let \((y_0, y_1) \in \mathcal{H}\). Then, there exist positive constants \(k_1, k_2, k_3\) and \(\varepsilon_0\) such that the energy \(E(t)\) associated to the solution of (11) satisfies
\[
\begin{align*}
E(t) &:= \frac{1}{2} \int_0^1 \left( \rho y_t^2(x, t) + E y_{xx}^2(x, t) - \rho \sigma^2 y^2 \right) dx \\
&\leq k_3 H_1^{-1} (k_1 t + k_2), \quad \forall t \geq 0, \tag{13}
\end{align*}
\]
where
\[
H_1(t) = \int_0^t \frac{1}{H_2(s)} ds, \quad H_2(t) = t H'(\varepsilon_0 t).
\]
Here \(H_1\) is a strictly decreasing and convex function on \((0, 1]\), with \(\lim_{t \to 0} H_1(t) = +\infty\).

Hereby, the remaining task is to provide a proof of the above theorem. To fulfill this objective, we shall establish several lemmas. Note that in view of a standard density argument together with the contraction of the semigroup, it suffices to prove Theorem 3.1 for a strong solution stemmed from a regular initial data in the domain \(D(A)\).

First, we have

**Lemma 3.1:** Let \(y\) be a solution of the system (11). Then, the functional \(E\) satisfies
\[
E'(t) = -E y_{xx}(1, t)f(y_{xx}(1, t)) - E y_t(1, t)f(y(1, t)) \leq 0. \tag{14}
\]

**Proof:** Multiplying the first equation in (11), by \(y_t\), using integrations by parts with respect to \(x\) over \((0, 1]\) along with the boundary conditions and the hypotheses H.I, we obtain (14).
The second lemma is

**Lemma 3.2:** Let \((y, y_1)\) be a solution of the system (11). Then, there exists \(\theta > 0\) such that the functional

\[
F(t) := 2 \int_{0}^{1} xy(t, x)y_1(t, x) \, dx, \quad t \geq 0
\]

verifies the following estimate:

\[
F'(t) \leq y_1^2(1, t) + (1 + 2\theta) \frac{EI}{\rho^2} f^2(y_1(1, t)) + \rho \frac{EI}{\rho^2} f^2(y_1(1, t))
- K\|y(t), y_1(t)\|^2_{L^2}, \text{ a.e. } t \geq 0.
\]  

**Proof:** We differentiate \(F\) with respect to \(t\), we get after a straightforward computation:

\[
F'(t) = -2\frac{EI}{\rho} y_1(1, t) f(y_1(1, t)) - 2\frac{EI}{\rho} y_1(1, t) f(y_1(1, t))
+ \sigma^2 y^2(1, t) + \frac{EI}{\rho} f^2(y_1(1, t)) + y_1^2(1, t)
- \int_{0}^{1} \left[ y_1^2(x, t) + 3\frac{EI}{\rho} y_2^2(x, t) + \sigma^2 y^2(x, t) \right] \, dx,
\text{ a.e. } t \geq 0.
\]  

Using Young’s inequality, we obtain

\[
-2\frac{EI}{\rho} y_1(1, t) f(y_1(1, t)) \leq \frac{EI}{\rho \theta} y_1^2(1, t) + \frac{EI}{\rho} f^2(y_1(1, t)),
\]

and

\[
-2\frac{EI}{\rho} y_1(1, t) f(y_1(1, t)) \leq \frac{EI}{\rho \theta} y_1^2(1, t) + \frac{EI}{\rho} f^2(y_1(1, t)),
\]

for any \(\theta > 0\).

On the other hand, we have

\[
y_2(1, t) = y_2(1, t) - y_2(0, t) = \int_{0}^{1} y_{xx}(x, t) \, dx \leq \|y_{xx}\|_{L^2(0, 1)}.
\]

Consequently, we obtain

\[
y_1^2(1, t) \leq \int_{0}^{1} y_{xx}^2(x, t) \, dx.
\]

Similarly, we use the fact \(y(0, t) = 0\) to get

\[
y_1^2(1, t) \leq \frac{1}{3} \int_{0}^{1} y_{xx}^2(x, t) \, dx.
\]

Combining (17)–(21) yields

\[
F'(t) \leq y_1^2(1, t) + (1 + 2\theta) \frac{EI}{\rho^2} f^2(y_1(1, t)) + \rho \frac{EI}{\rho^2} f^2(y_1(1, t))
+ \frac{1}{\rho} \int_{0}^{1} \left[ -\rho y_1^2(x, t) + \left( \frac{2}{\theta} + \frac{\rho^2}{3EI} - 3 \right) E\rho y_1^2(x, t) \right] \, dx,
\text{ a.e. } t \geq 0.
\]

Recalling the condition (5), one can choose \(\theta\) such that \(\frac{2}{\theta} + \frac{\rho^2}{3EI} - 3 < 0\). Then, we deduce the existence of a positive constant \(K\) such that the inequality (16) is verified.

Now, we are able to achieve the main objective of this section:

**Proof of Theorem 3.1:** First of all, let

\[
E_0(t) := E(t) + \varepsilon F(t) = \frac{1}{2} \|y_1(t)\|^2_{y_1} + \varepsilon F(t)
\]

in which \(\varepsilon\) is a sufficiently small positive constant to be chosen later (see below (27)) and \(F(t)\) is defined by (15).

It is easy to verify that the functional \(E_0\) is equivalent to the energy \(E\).

In fact, we have

\[
|E_0(t) - E(t)| = \varepsilon |F(t)| \leq \varepsilon E(t),
\]

where \(c\) is a positive constant.

On one hand, it follows from (16) and (23) that

\[
E'_0(t) \leq E'(t) + \varepsilon y_1^2(1, t) + \varepsilon (1 + 2\theta) \frac{EI}{\rho} f^2(y_1(1, t))
+ \varepsilon \theta \frac{EI}{\rho} f^2(y_1(1, t))
- \varepsilon KE(t), \quad \text{a.e. } t \geq 0.
\]

Note also that we can suppose that \(\max(r, f_0(r)) < \sigma\), where \(r\) and \(\sigma\) are defined in H.II and (10). Thereafter, we set \(\nu = \min(r, f_0(r))\), and we hence deduce from H.II that

\[
\frac{f_0(\nu)}{\sigma} |s| \leq f_0(|s|) \leq \varepsilon f_0^{-1}(\nu) \leq \frac{f_0^{-1}(\sigma)}{\nu} |s|,
\]

for \(\nu \leq |s| \leq \sigma\). The latter implies that

\[
\left\{ \begin{array}{ll}
\frac{f_0(\nu)}{|s|} \leq \frac{f(\nu)}{f_0^{-1}(\nu)}, & \text{for all } |s| < \nu, \\
\frac{c_0'}{|s|} \leq \frac{f(\nu)}{c_0'}, & \text{for all } |s| \geq \nu,
\end{array} \right.
\]

where \(c_0' > 0\), for \(i = 1, 2\). Additionally, it follows from (10) that

\[
H(s^2) = |s| f_0(|s|) = s f_0(s) \quad \text{and} \quad H(f^2(s)) = f(s) f_0(f(s)),
\]

whenever \(s^2 \geq 0\).

In light of (25), we deduce that \(f_0(|f(\nu)|) \leq |s|, \text{ for } |s| \leq \nu \text{ and hence}

\[
H(f^2(s)) \leq |s| f_0(s) = s f_0(s), \quad \text{for } |s| \leq \nu.
\]

Using the fact that \(H\) is strictly convex on \([0, r^2]\), it follows that

\[
H \left( \frac{1}{2} s^2 + \frac{1}{2} f^2(s) \right)
\]

\[
\leq H(s^2) + H(f^2(s)) \leq s f_0(s), \quad \text{for } |s| \leq \nu.
\]

Since \(H^{-1}\) is strictly increasing in \([0, r^2]\), we conclude

\[
s^2 + f^2(s) \leq 2H^{-1}(s f_0(s)), \quad \text{for } |s| \leq \nu.
\]

On the other hand, by virtue of the assumption H.II, Lemma 3.1 and (26), we obtain

\[
y_1^2(1, t) + \frac{EI}{\rho} f^2(y_1(1, t)) + (1 + 2\theta) \frac{EI}{\rho} f^2(y_1(1, t))
\]

\[
+ \frac{1}{\rho} \int_{0}^{1} \left[ -\rho y_1^2(x, t) + \left( \frac{2}{\theta} + \frac{\rho^2}{3EI} - 3 \right) E\rho y_1^2(x, t) \right] \, dx, \quad \text{a.e. } t \geq 0.
\]
This, together with (24), yields

$$E''_0(t) \leq \left(1 - \varepsilon \varepsilon_2 \right) E'(t) - \varepsilon K E(t)$$

$$+ \varepsilon c_1 H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}}$$

$$+ \varepsilon (1 + 2\theta) \frac{2E}{\rho} H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}}$$

$$\leq -\varepsilon K E(t) + \varepsilon c_1 H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}}$$

$$+ \varepsilon (1 + 2\theta) \frac{2E}{\rho} H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}},$$

a.e. \( t \geq 0, \) (27)

provided that \( \varepsilon \) is small so that \( 1 - \varepsilon \varepsilon_2 > 0. \)

Thereafter, for \( \varepsilon_0 < r^2 \) to be chosen later, we have \( H' \geq 0 \) and \( H'' \geq 0 \) over \((0, r^2)\]. Moreover, \( E' \leq 0. \) Consequently, the functional \( R(t) \) defined by

$$R(t) := H' \left( \frac{E(t)}{E(0)} \right) E'_0(t) + \delta E(t),$$

is equivalent to \( E(t), \) for some \( \delta > 0. \)

In addition, we have \( \varepsilon_0 \frac{E''(t)}{E(0)} H'' \left( \frac{E(t)}{E(0)} \right) E_0(t) \leq 0. \) This, together with (27), allows us to conclude that

$$R'(t) = \varepsilon_0 \frac{E'(t)}{E(0)} H'' \left( \frac{E(t)}{E(0)} \right) E_0(t)$$

$$+ H' \left( \frac{E(t)}{E(0)} \right) E'_0(t) + \delta E'(t)$$

$$\leq -\varepsilon K E(t) H' \left( \frac{E(t)}{E(0)} \right) E_0(t) + \delta E(t).$$

Our objective now is to estimate the second and the third term in the right-hand side of (28). For that purpose, we introduce, as in Alabau-Boussouira (Alabau-Boussouira, 2005), the convex conjugate \( H^* \) of \( H \) defined by

$$H^*(s) = s(H')^{-1}(s) - H((H')^{-1}(s)) \text{ for } s \in (0, H'(r^2)),$$

and \( H^* \) satisfies the following Young inequality:

$$AB \leq H^*(A) + H(B) \quad \text{for } A \in (0, H'(r^2)), \quad B \in (0, r^2). \quad (30)$$

Now, taking first \( A_1 = H' \left( \frac{E(t)}{E(0)} \right) \) and \( B_1 = H^{-1}(y(t, t) f_0(y(t, t))) \) and second \( A_2 = H' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) \) and \( B_2 = H^{-1}(y(t, t) f_0(y(t, t))) \), we obtain

$$R'(t) \leq -\varepsilon K E(t) H' \left( \frac{E(t)}{E(0)} \right)$$

$$+ \varepsilon c_1 H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}}$$

$$+ \varepsilon (1 + 2\theta) \frac{2E}{\rho} H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}}$$

$$\leq -\varepsilon K E(t) H' \left( \frac{E(t)}{E(0)} \right)$$

$$+ \varepsilon c_1 H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}}$$

$$+ \varepsilon (1 + 2\theta) \frac{2E}{\rho} H^{-1}(y(t, t) f_0(y(t, t))) 1_{\{y(t) \leq u\}},$$

a.e. \( t \geq 0, \) (28)

With a suitable choice of \( \varepsilon_0 \) and \( \delta, \) we deduce from the last inequality that

$$R'(t) \leq -\varepsilon \left( KE(0) - \varepsilon_0 \left( 1 + \theta \frac{2E}{\rho} + c_1 \right) \right)$$

$$\times \frac{E(t)}{E(0)} H' \left( \frac{E(t)}{E(0)} \right) \leq -k H_2 \left( \frac{E(t)}{E(0)} \right), \quad (31)$$
where \( k = \varepsilon (KE(0) - \varepsilon_0 (1 + \theta) \frac{2EI}{\rho} + e_1) \) > 0 and \( H_2(s) = sH'(\varepsilon_0 g) \).

Since \( E(t) \) and \( R(t) \) are equivalent, there exist positive constants \( a_1 \) and \( a_2 \) such that
\[
a_1 R(t) \leq E(t) \leq a_2 R(t).
\]
We set now \( S(t) = \frac{a_1 R(t)}{a_2 R(t)}. \) It is clear that \( S(t) \sim E(t) \). Taking into consideration the fact that \( H'(s) > 0 \) over \((0, 1)\) (this is a direct consequence of strict convexity of \( H \) on \((0, r^2)\)) we deduce from (31) that
\[
S'(t) \leq -k_1 H_2(S(t)), \quad \text{for all } t \in \mathbb{R}_+,
\]
with \( k_1 > 0 \).

Recall that \( H_1(t) = \int_t^1 \frac{1}{s} \, ds \) and integrating the last inequality over \([0, t]\), we obtain
\[
H_1(S(0)) \geq H_1(S(t)) + k_1 t.
\]
Finally, since \( H_1^{-1} \) is decreasing (as for \( H_1 \)), we have
\[
S(t) \leq H_1^{-1}(k_1 t + k_2), \quad \text{with } k_2 > 0.
\]
Invoking once again the equivalence of \( E(t) \) and \( R(t) \), we deduce the desired result (13).

**Remark 3.1:** As pointed out in Introduction, the presence of the nonlinear moment control which involves \( y_{\text{ns}}(1, t) \) is not necessary for the result stated in Theorem 3.1. In fact, the reader can easily check that the estimates obtained in the previous section remain valid as long as the nonlinear force control involves the velocity term \( y_1(1, t) \).

### 4. Examples

The objective of this section is to apply the inequality in (13) on some examples in order to show explicit stability results in terms of asymptotic profiles in time. To proceed, we choose the function \( H \) strictly convex near zero.

#### 4.1 Example 1

Let \( f \) be a function that satisfies
\[
c_{3}|s|^p \leq |f(s)| \leq c_4 |s|^q,
\]
with some \( c_3, c_4 > 0 \) and \( p > 1 \).

For \( f_0(s) = e^s \), hypothesis **H.II** is verified. Then \( H(s) = s^{\frac{p+1}{p-1}} \), \( H_2(s) = s^{\frac{p+1}{p-1}} \) and
\[
H_1(t) = \int_0^t \frac{1}{s^{\frac{p+1}{p-1}}} \, ds = \frac{1}{c_2} \left( 1 - t^{-\frac{p}{p-1}} \right),
\]
where \( c_2 > 0 \), for \( i = 1, 2 \). Therefore,
\[
H_1^{-1}(t) = \left( \frac{c_1}{c_2} t + 1 \right)^{-\frac{p}{p-1}}.
\]

Using again (13), we obtain
\[
E_0(t) \leq H_1^{-1}(k_1 t + k_2) = \left( \frac{c_1}{c_2} (k_1 t + k_2) + 1 \right)^{-\frac{p}{p-1}}.
\]

**Remark 4.1:** Note that the exponential decay rate result has been obtained in Chentouf and Couchouron (Chentouf & Couchouron, 1999) for the case \( f(s) = cs \).

#### 4.2 Example 2

Let \( f_0(s) = \frac{1}{2} \exp(-\frac{s}{2}) \). Arguing as in Example 1, one can conclude that the energy of (11) satisfies
\[
E_0(t) \leq \varepsilon \left( \log \left( \frac{k_1 t + k_2 + \varepsilon \exp \left( \frac{1}{\varepsilon_0} \right) }{c} \right) \right)^{-1}.
\]

### 5. Stability of the closed-loop system

Throughout this section, we shall assume without loss of generality that the physical parameters \( E_1, \rho \) are unit. Additionally, we shall suppose that (5), **H.I** and **H.III** are fulfilled. Moreover, the subsystem (11) is exponentially stable for the case \( f_0(s) = cs \) (see Remark 4.1). This implies that there exist positive constants \( M \) and \( \eta \) such that
\[
\| e^{-At} \|_{L^2(\Omega)} \leq Me^{-\eta t}, \quad \forall t \geq 0.
\]

The aim of this section is to show the exponential stability of the closed-loop system (8). We shall use the same arguments as put forth in Chentouf and Couchouron (1999) but with a number of changes born out of necessity.

First of all, thanks to the assumptions (5), **H.I** and **H.III**, it follows from Proposition 3.1 that for any initial data \((\phi, \omega) \in D(A)\), the system has a unique global bounded solution on \((0, \infty)\) (Chentouf & Couchouron, 1999). For sake of completeness, we shall provide a sketch of the proof of this result. First, we know from Proposition 2, p. 526 in Chentouf and Couchouron (1999) that, thanks to the assumption **H.I**, the operator \( A \) defined by (6)-(7) is \( m \)-accretive in \( \mathcal{H} \) with dense domain. This implies that the nonlinear operator \( \tilde{A} \) (see (9)) is also \( m \)-accretive with dense domain in \( \mathcal{X} \). Additionally, the operator \( \mathcal{B} \) defined in (9) is Lipschitz on bounded subsets of \( \mathcal{X} \). Thereby, the closed-loop system (8) has a local solution \((\phi, \omega)(t) = (y(t), z(t), w(t))\), for \( t \in [0, T] \). Next, it suffices to prove that the solution is indeed global. To do so, consider the functional
\[
V(t) = \frac{1}{2} (\omega(t) - \bar{\sigma})^2 \int_0^1 \gamma^2 \, dx + \frac{L_d}{2} (\omega(t) - \bar{\sigma})^2
\]
\[
+ \frac{1}{2} \int_0^1 \left( \gamma^2 + y_2^2 - \sigma^2 y_2^2 \right) \, dx.
\]

Then, one can easily check that \( V \) is a Lyapunov function and satisfies along the regular solutions of (8)
\[
\frac{d}{dt} V(\phi, \omega)(t) = -E \bar{y}_{ts}(1, t)f(y_{ts}(1, t)) - E \bar{y}_{t}(1, t)f(y_{t}(1, t))
\]
\[
- (\omega(t) - \bar{\sigma}) \gamma (\omega(t) - \bar{\sigma}) \leq 0, \quad \forall t \geq 0,
\]
by means of the conditions **H.I** and **H.III**. Therefore, the solution of (8) must be global.

Second, we consider the subsystem
\[
\begin{cases}
\dot{\psi}(t) + A\psi(t) + B(t, \psi(t)) = 0, & t > 0, \\
\psi(0) = \psi_0,
\end{cases}
\]
where \( \psi = (y, z) \), \( \psi_0 \in D(A) \) and
\[ B(t, \psi) = (\sigma^2 - \omega^2(t))P(\psi), \]  

(35)
in which \( P \) is the compact operator on \( \mathcal{H} \) defined by \( P(y, z) = (0, y), \forall (y, z) \in \mathcal{H} \). Moreover, \( \omega(t) \) is the second component of the solution \( (\psi(t), \omega(t)) \) of the global system (8) with initial data \( (\psi_0, \omega_0) \in \mathcal{D}(\mathcal{A}) \). Accordingly, the solution \( \psi(t) \) can be written as follows:

\[ \psi(t) = e^{-(t-T_0)A}\psi(T_0) + \int_{T_0}^{t} e^{-(t-v)A}(\omega^2(v) - \sigma^2)P(\psi(v))\,dv, \]  

(36)

for any \( t \geq T_0 \).

Clearly, it follows from (35) that

\[ \|B(t, \psi(t))\|_{\mathcal{H}} \leq |\sigma^2 - \omega^2(t)|\|\psi(t)\|_{\mathcal{H}}. \]

Furthermore, in the light of (33), (34) and the assumptions H.I–H.III, we have

\[ \sigma - \omega(\cdot) \in L^2([0, \infty]; \mathbb{R}) \cap L^\infty([0, \infty]; \mathbb{R}) \]

and hence \( \lim_{t \to +\infty} \omega(t) = \sigma \) (Chentouf & Couchouron, 1999). Herein, for any \( \epsilon > 0 \), there exists \( T_0 \) sufficiently large such that for each \( t \geq T_0 \),

\[ |\sigma^2(t) - \sigma^2| < \epsilon. \]  

(37)

Combining (37) as well as (32) and (36) and applying Gronwall’s Lemma yields

\[ \|\psi(t)\|_{\mathcal{H}} \leq Me^{-(\gamma - \epsilon M)(t-T_0)}\|\psi(T_0)\|_{\mathcal{H}}, \]  

(38)

for any \( t \geq T_0 \). Thereafter, one can choose \( \epsilon \) so that \( \eta - \epsilon M > 0 \), and thus the solution \( \psi(t) \) of (8) is exponentially stable in \( \mathcal{H} \). Finally, amalgamating these properties and going back to (8) one can show the exponential convergence of \( \omega(t) - \sigma \) in \( \mathbb{R} \) by means of H.III. The proof runs on much the same lines as that of Theorem 2 in Chentouf and Couchouron (1999). Indeed, we deduce from (8) that

\[ \frac{d}{dt} (\omega(t) - \sigma) \]

\[ = \gamma (\omega(t) - \sigma) \|y\|_{L^2(0,1)}^2 - 2Id \omega(t) < y, y_t >_{L^2(0,1)} \frac{Id}{Id + \|y\|_{L^2(0,1)}^2} \]

\[ - \frac{1}{Id} \gamma (\omega(t) - \sigma). \]  

(39)

Multiplying (39) by \( \omega(t) - \sigma \) and using the assumption H.III, we get

\[ \frac{1}{2} \frac{d}{dt} (\omega(t) - \sigma)^2 \]

\[ \leq \frac{2Id \omega(t) (\omega(t) - \sigma) < y, y_t >_{L^2(0,1)}}{Id + \|y\|_{L^2(0,1)}^2} - \frac{\kappa}{Id} (\omega(t) - \sigma)^2. \]  

(40)

Next, solving the above inequality, we get

\[ |\omega(t) - \sigma| \leq ae^{-b(t-T_0)}, \]

for any \( t \geq T_0 \) and for some positive constants \( a \) and \( b \).

To recapitulate, we have proved the second main result:

**Theorem 5.1:** Assume that (5), H.I and H.III are fulfilled. We also suppose that H.II holds with \( f_0(s) = cs \). Then, for each initial data \( (\phi_0, \omega_0) \in \mathcal{D}(\mathcal{A}) \) the solution \( (\phi(t), \omega(t)) \) of the closed-loop system (8) exponentially tends to \( (0, \sigma) \) in \( \mathcal{X} \) as \( t \to +\infty \).

**Remark 5.1:** The authors believe that if the semigroup \( e^{-At} \) has a polynomial decay, then the original system (3) will also be polynomially stable. In fact, if \( f_0(s) = cs^p \), with \( p > 1 \), then the energy \( E_0(t) \) has a rational decay rate (see Example 1 of Section 4.1) and in such a case the system (3) is expected to be polynomially stable. This case is worth investigating in a future work as the problem remains open.

### 6. Conclusion

This note was concerned with the stabilization of the rotating disk-beam system. A feedback law, which consists of a nonlinear torque control exerted on the disk and a nonlinear boundary control applied on the beam, is put forward. Then, it is shown that the closed-loop system is exponentially stable provided that the angular velocity of the disk does not exceed a well-defined value and the nonlinear functions involved in the feedback law obey reasonable conditions. This result improves the previous one in Chentouf and Couchouron (1999), where the functions involved in the boundary control must satisfy strong and restrictive conditions such as almost linear property.

### Acknowledgements

The valuable suggestions and comments from the editor and the anonymous referees are greatly appreciated.

### Disclosure statement

No potential conflict of interest was reported by the author(s).

### References

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