AN OPTIMAL DESIGN PROBLEM WITH NON-STANDARD GROWTH AND NO CONCENTRATION EFFECTS

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Abstract

We obtain an integral representation for certain functionals arising in the context of optimal design and damage evolution problems under non-standard growth conditions and perimeter penalization. Under our hypotheses, the integral representation includes a term which is absolutely continuous with respect to the Lebesgue measure and a perimeter term, but no additional singular term.

We also provide an application to the modelling of thin films.

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1 Introduction

In a recent article [5], we investigated the possibility of obtaining a measure representation, in a suitable sense (cf. Definition 2.7), for two functionals arising in certain relaxation processes for an energy of the type

\[ F(\chi, u) := \int_{\Omega} \chi(x) W_1(\nabla u(x)) + (1-\chi(x)) W_0(\nabla u(x)) \, dx + |D\chi|(\Omega), \]  

(1.1)

where \( \Omega \) a bounded open subset of \( \mathbb{R}^N \), \( \chi \in BV(\Omega; \{0, 1\}) \) and \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \).

This energy has its origin in a problem in optimal design (see [2, 25, 26, 27, 28, 34]) where the perimeter term is added to ensure compactness, and thus existence, of solutions to the corresponding minimization problems. In this setting, the characteristic function \( \chi \) corresponds either to one material, say \( E \subset \Omega \), of a two components sample \( \Omega \), or to one of the phases \( E \) of a single material \( \Omega \). The stored elastic energy or suitable function of the electrostatic potential density of \( E \) is given by \( W_1 \), while \( W_0 \) is the energy associated to the other component or phase, and the term \( |D\chi|(\Omega) \) penalizes the measure of the created interfaces.
Another motivation comes from the modeling of “brutal damage”, we refer to [20] where the first rigorous mathematical description was provided and to [16] for a nonlinear elastic setting in the framework of thin structures. Indeed, damage as an inelastic phenomenon can be described by means of the characteristic function of the damaged region which is a subset of $\Omega$, $\nabla u$ is the deformation strain, the elastic energy is given by the sum of the two contributions in the undamaged and damaged part, $W_0 > W_1$, and a dissipational energy is taken as proportional, via the constant $\kappa > 0$ which represents the material toughness, to the damaged volume. This latter term corresponds to local cost of damaging a healthy part of the sample. We refer to the recent paper [4] and the bibliography therein for an asymptotic analysis, in the linear elastic case, where the damaged zones tend to disappear.

A regularization term is added in the form of the total variation of the characteristic function $\chi$. Among the literature, we refer, for instance, to [15, 29, 31, 35], where a similar term is considered in the case where the damage parameter is assumed to range in the entire set $[0, 1]$. Hence, the total energy contains an extra term with respect to (1.1), and is given by

$$F_d (\chi, u) := \int_{\Omega} \chi(x) W_1(\nabla u(x)) + (1 - \chi(x)) W_0(\nabla u(x)) \, dx + \kappa \int_{\Omega} \chi(x) dx + |D\chi|(\Omega). \quad (1.2)$$

However, we observe that the extra dissipation term, being linear, does not add any particular difficulty to our analysis. Likewise, the possible addition of suitable boundary conditions or the work done by (linear) bulk loads pose no problems and thus are neglected in our subsequent description.

In the theory of shape optimization, where the aim is to find an optimal shape minimizing a cost functional (here the elastic energy), one should either impose directly a volume constraint on the phase (linear) bulk loads pose no problems and thus are neglected in our subsequent description.

Letting $f: \{0, 1\} \times \mathbb{R}^{d \times N} \to \mathbb{R}$ be defined as

$$f(b, \xi) := bW_1(\xi) + (1 - b)W_0(\xi), \quad (1.3)$$

to simplify the notation, the functionals considered in [4] are given by

$$F(\chi, u; A) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}(A; \mathbb{R}^d), \chi_n \in BV(A; \{0, 1\}), \right.$$

$$\left. u_n \to u \text{ in } W^{1,p}(A; \mathbb{R}^d), \chi_n \rightharpoonup^* \chi \text{ in } BV(A; \{0, 1\}) \right\} \quad (1.4)$$

and

$$F_{loc}(\chi, u; A) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; A) : u_n \in W^{1,q}_{loc}(A; \mathbb{R}^d), \chi_n \in BV(A; \{0, 1\}), \right.$$

$$\left. u_n \to u \text{ in } W^{1,p}(A; \mathbb{R}^d), \chi_n \rightharpoonup^* \chi \text{ in } BV(A; \{0, 1\}) \right\}, \quad (1.5)$$

where the exponents $p, q$ satisfy

$$1 < p \leq q < \frac{Np}{N-1} \quad (1.6)$$

(if $N = 1$ we let $1 < p \leq q < +\infty$) and where we consider the localization of (1.4) defined, for every open set $A \subset \Omega$ and every $(\chi, u) \in BV(A; \{0, 1\}) \times W^{1,p}(A; \mathbb{R}^d)$, by

$$F(\chi, u; A) := \int_A f(\chi(x), \nabla u(x)) \, dx + |D\chi|(A). \quad (1.7)$$

The functions $W_i: \mathbb{R}^{d \times N} \to \mathbb{R}$, $i = 0, 1$, in (1.3) are assumed to be continuous and satisfy the following growth condition

$$\exists \beta > 0 : 0 \leq W_i(\xi) \leq \beta (1 + |\xi|^p), \quad \forall \xi \in \mathbb{R}^{d \times N}. \quad (1.8)$$

Under the above hypotheses, in [4] we showed that there exists a non-negative Radon measure $\mu$ defined on the open subsets of $\Omega$ which weakly represents $F(\chi, u; \cdot)$, whereas $F_{loc}(\chi, u; \cdot)$ admits a strong
measure representation (cf. Definition 2.7). Furthermore, assuming convexity of \( f(b, \cdot) \), \( \forall b \in \{0, 1\} \), we proved that, for every open subset \( A \) of \( \Omega \),
\[
F_{\text{loc}}(\chi; u; A) = \int_A f(\chi(x), \nabla u(x)) \, dx + |D\chi|(A) + \nu^*(\chi; u; A),
\]
(1.9)
where \( \nu^* \) is a non-negative Radon measure, singular with respect to the Lebesgue measure (cf. Theorems 4.1 and 4.3 in [5]). This additional singular measure arises since in the above functionals there is a gap between the space of admissible macroscopic fields \( u \in W^{1,p}(A; \mathbb{R}^d) \) and the smaller space \( W^{1,q}(A; \mathbb{R}^d) \) where the growth hypothesis (1.3) ensures boundedness of the energy. Indeed, it is well known that when no such gap is present, i.e. when \( p = q \), and in the case independent of the field \( \chi \), then
\[
F(u; A) = \inf \left\{ \liminf_{n \to +\infty} \int_A f(\nabla u_n(x)) \, dx : u_n \in W^{1,q}(A; \mathbb{R}^d), \, u_n \rightharpoonup u \text{ in } W^{1,p}(A; \mathbb{R}^d) \right\}
\]
\[
= \int_A Qf(\nabla u(x)) \, dx,
\]
where \( Qf \) denotes the quasiconvex envelope of \( f \) (see Definition 2.6).

For the range of exponents considered in (1.6), similar functionals were studied in [1, 6, 18, 32], the case where the integrability exponent \( p(x) \) of the admissible fields depends in a continuous or regular piecewise continuous way on the location in the body was addressed in [12, 33], we also refer to [23] for generalizations of such problems in Orlicz type spaces.

In this paper, we expand on our previous results providing a full characterisation of \( F(\chi; u; \Omega) \) and \( F_{\text{loc}}(\chi; u; \Omega) \), under some hypotheses on \( \chi \). We also assume that the continuous density functions \( W_i : \mathbb{R}^{d \times N} \to \mathbb{R}, \, i = 0, 1 \) in (1.3) satisfy the following stronger growth condition
\[
\exists \beta_1 > 0 : 0 \leq W_1(\xi) \leq \beta_1 (1 + |\xi|^p), \quad \forall \xi \in \mathbb{R}^{d \times N},
\]
(1.10)
\[
\exists \beta_0 > 0 : 0 \leq W_0(\xi) \leq \beta_0 (1 + |\xi|^q), \quad \forall \xi \in \mathbb{R}^{d \times N},
\]
(1.11)
indeed (1.10) is a special case of (1.3), but we relax the condition (1.0) on the exponents \( p \) and \( q \), and we require just that
\[
1 < p \leq q < +\infty.
\]
(1.12)
In the case under consideration, i.e. for suitably chosen \( \chi \) and \( u \), we show that the functionals (1.4) and (1.5) in question, evaluated at \( \Omega \), admit an integral representation comprising a term which is absolutely continuous with respect to the Lebesgue measure, and a perimeter term, but there is no additional singular term.

We point out that one cannot expect this to be true in general. Indeed, an example due to Zhikov [32] shows that some functionals where the integrand \( f \) in (1.3) has the form \( f(b, \xi) := |\xi|^{b+1-b} \), i.e., with a gap in the growth and coercivity exponents, do exhibit concentration effects (cf. 1.10 below and where \( b = \chi_E \) is fixed).

Precisely, under certain structure assumptions on the fields \( \chi \in BV(\Omega; \{0, 1\}) \) and, consequently, in view of (1.10) and (1.11), also on \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \), we prove the following characterisation.

**Theorem 1.1.** Let \( p, q \) satisfy (1.12), let \( f \) be defined as in (1.3), satisfying (1.10), (1.11) and
\[
f(b, \cdot) \text{ is convex for every } b \in \{0, 1\}.
\]
(1.13)
Let \( \chi \) be the characteristic function of an open set \( E \subset \subset \Omega \) that satisfies
\[
\mathcal{H}^{N-1}(\partial E) = P(E; \mathbb{R}^N),
\]
where \( \partial E \) denotes the topological boundary of \( E \) and \( P(E; \mathbb{R}^N) \) is the perimeter of \( E \) in \( \mathbb{R}^N \), and let \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \) be such that \( u \in W^{1,q}(\Omega \setminus \partial E; \mathbb{R}^d) \). Then,
\[
F_{\text{loc}}(\chi; u; \Omega) = \mathcal{F}(\chi; u; \Omega) = \int_{\Omega} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(|\chi|\Omega).
\]
(1.14)
In order to achieve these conclusions we do not invoke the results contained in either Theorem 4.1 or
Theorem 4.3 in \[3\], which provide weak and strong measure representations for \( \mathcal{F} \) and \( \mathcal{F}_{\text{loc}} \), respectively. Instead, we use a direct approach to prove double inequalities starting from (1.4) and (1.5) evaluated at \( \Omega \). For this reason, we allow for the less restrictive range of integrability exponents \( p \) and \( q \) considered in (1.12), as compared with (1.13).

Our proof is based on a result of Schmidt \[36\], which states that under some mild hypotheses on its boundary (see (2.3)), a set \( E \) can be approximated from the inside by smooth sets, in such a way that the perimeters also converge (cf. Theorem 2.3).

In particular, every set with Lipschitz boundary satisfies (2.2). However, if this condition fails to
hold there are known counterexamples that show that the inner approximation by smooth sets may no
longer be possible. We refer to Section 2 for more details.

We point out that, in our context, the set
\[
E := \{(x_1, x_2) \in B_1 : x_1 x_2 > 0\},
\]  

where \( B_1 \subset \Omega \) is the unit ball in \( \mathbb{R}^2 \), introduced by Zhikov in \[37\] page 467, and also considered in \[33\] \( \text{eq. (22)} \) and example 1.15], does not give rise to any concentration effects, since \( E \) lies in the class of Theorem \[33\]. In particular, even if we consider the same energies as in the above mentioned examples, i.e. \( W_0(\xi) = |\xi|^q \) and \( W_1(\xi) = |\xi|^p \), with \( 1 < p < 2 < q \), we have the freedom of approximating \( E \) without increasing the energy, since this approximation can be made from the inside.

Notice also that our integral representation result holds for pairs \((\chi, u)\) under the assumption that \( u \) should be more regular in a certain subset of \( \Omega \), namely \( u \in W^{1,q}(\Omega \setminus E; \mathbb{R}^d) \), where \( \chi \) is the characteristic function of the set \( E \). However, this assumption is not too restrictive since if \( \mathcal{F}(\chi, u; \Omega) < +\infty \), which is the usual condition considered in the literature (cf., for example, \[11\] \[6\] \[18\]), and \( W_0(\xi) = |\xi|^q \), then the additional regularity required of \( u \) follows as a consequence.

Given the motivation stated above and the applications we have in mind, it will also be important
to consider the following related functional, where a volume constraint is imposed. Given \( 0 < \theta < 1 \)
and \( \chi \in BV(\Omega; (0, 1)) \) such that \( \frac{1}{L^N(\Omega)} \int_{\Omega} \chi(x) \, dx = \theta \), we define
\[
\mathcal{F}_{\text{vol}}(\chi, u; \Omega) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; \Omega) : u_n \in W^{1,q}(\Omega \setminus E; \mathbb{R}^d), \chi_n \in BV(\Omega; (0, 1)), \right. \]

\[
\left. u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^d), \chi_n \overset{*}{\rightharpoonup} \chi \text{ in } BV(\Omega; (0, 1)), \frac{1}{L^N(\Omega)} \int_{\Omega} \chi_n(x) \, dx = \theta \right\}. \tag{1.16}
\]

Under the same assumptions of Theorem \[11\] we show that (1.16) admits the same integral representation as (1.14) (cf. Remark 3.2).

We organise the paper as follows. In Section 2 we set the notation and we provide some definitions
and results which will be used throughout. Theorem 1.1 is proved in Section 3, where a similar characteristic result is also shown in the case where the convexity assumption (1.13) is replaced with the weaker assumption that the quasiconvex envelopes, \( QW_1 \) and \( QW_0 \) of \( W_1 \) and \( W_0 \), respectively, are convex. Finally, in Section 4 we give an application to dimension reduction problems.

## 2 Preliminaries

In this section we fix notations and quote some definitions and results that will be used in the sequel.

Throughout the text \( \Omega \subset \mathbb{R}^N \) will denote an open, bounded set.

We will use the following notations:

- \( \mathcal{O}(\Omega) \) is the family of all open subsets of \( \Omega \);
- \( \mathcal{M}(\Omega) \) is the set of finite Radon measures on \( \Omega \);
- \( |\mu| \) stands for the total variation of a measure \( \mu \in \mathcal{M}(\Omega) \);
\(\mathcal{L}^N\) and \(\mathcal{H}^{N-1}\) stand for the \(N\)-dimensional Lebesgue measure and the \((N - 1)\)-dimensional Hausdorff measure in \(\mathbb{R}^N\), respectively;

- the symbol \(dx\) will also be used to denote integration with respect to \(\mathcal{L}^N\);

- \(C\) represents a generic positive constant that may change from line to line.

We start by recalling a well known result due to Ioffe [24 Theorem 1.1].

**Theorem 2.1.** Let \(g : \mathbb{R}^m \times \mathbb{R}^{d \times N} \to [0, +\infty)\) be a Borel integrand such that \(g(b, \cdot)\) is convex for every \(b \in \mathbb{R}^m\). Then the functional

\[
G(v, u) := \int_{\Omega} g(v(x), \nabla u(x)) \, dx
\]

is lower semicontinuous in \(L^1(\Omega; \mathbb{R}^m)^{\text{strong}} \times W^{1,1}(\Omega; \mathbb{R}^d)^{\text{weak}}\).

In the following we give some preliminary notions related with sets of finite perimeter. For a detailed treatment we refer to [3].

To this end, we recall that a function \(w \in L^1(\Omega; \mathbb{R}^d)\) is said to be of bounded variation, and we write \(w \in BV(\Omega; \mathbb{R}^d)\), if all its first order distributional derivatives \(D_j w_i\) belong to \(\mathcal{M}(\Omega)\) for \(1 \leq i \leq d\) and \(1 \leq j \leq N\).

The matrix-valued measure whose entries are \(D_j w_i\) is denoted by \(Dw\) and \(|Dw|\) stands for its total variation. We observe that if \(w \in BV(\Omega; \mathbb{R}^d)\) then \(w \to |Dw|\) is lower semicontinuous in \(BV(\Omega; \mathbb{R}^d)\) with respect to the \(L^1_{\text{loc}}(\Omega; \mathbb{R}^d)\) topology.

**Definition 2.2.** Let \(E\) be an \(\mathcal{L}^N\)-measurable subset of \(\mathbb{R}^N\). For any open set \(\Omega \subset \mathbb{R}^N\) the perimeter of \(E\) in \(\Omega\), denoted by \(P(E; \Omega)\), is given by

\[
P(E; \Omega) := \sup \left\{ \int_E \text{div} \varphi(x) \, dx : \varphi \in C^1_c(\Omega; \mathbb{R}^d), \|\varphi\|_{L^\infty} \leq 1 \right\}.
\]

We say that \(E\) is a set of finite perimeter in \(\Omega\) if \(P(E; \Omega) < +\infty\).

Recalling that if \(\mathcal{L}^N(E \cap \Omega)\) is finite, then \(\chi_E \in L^1(\Omega)\), by [3] Proposition 3.6], it follows that \(E\) has finite perimeter in \(\Omega\) if and only if \(\chi_E \in BV(\Omega)\) and \(P(E; \Omega)\) coincides with \(|D\chi_E| (\Omega)\), the total variation in \(\Omega\) of the distributional derivative of \(\chi_E\). Moreover, a generalized Gauss-Green formula holds:

\[
\int_E \text{div} \varphi(x) \, dx = \int_{\Omega} \langle \nu_E(x), \varphi(x) \rangle \, d|D\chi_E|, \quad \forall \varphi \in C^1_c(\Omega; \mathbb{R}^d),
\]

where \(D\chi_E = \nu_E|D\chi_E|\) is the polar decomposition of \(D\chi_E\).

We also recall that, when dealing with sets of finite measure, a sequence of sets \(E_n\) converges to \(E\) in measure in \(\Omega\) if \(\mathcal{L}^N(E \cap \Omega) \cap (E_n \Delta E)\) converges to 0 as \(n \to +\infty\), where \(\Delta\) stands for the symmetric difference. This convergence is equivalent to \(L^1(\Omega)\) convergence of the characteristic functions of the corresponding sets.

It is well known (cf. [3]) that it is always possible to approximate, in measure, a set \(E\) of finite perimeter in \(\mathbb{R}^N\), with sets \(E_\varepsilon\) with smooth boundary, in such a way that the perimeters also converge. However, an open set of finite perimeter in \(\mathbb{R}^N\) cannot, in general, be approximated strictly from within. In the sequel we rely on the following theorem due to Schmidt [36] which states that, under mild hypotheses on its boundary, the approximation of the set \(E\) is also true with the additional requirement that the smooth sets satisfy \(E_\varepsilon \subset \subset E\).

**Theorem 2.3.** [Strict interior approximation of the perimeter]. Let \(E\) be a bounded open set in \(\mathbb{R}^N\) whose topological boundary \(\partial E\) is well-behaved in the sense that

\[
\mathcal{H}^{N-1}(\partial E) = P(E; \mathbb{R}^N).
\]

Then, for every \(\varepsilon > 0\), there exists an open set \(E_\varepsilon\) with smooth boundary in \(\mathbb{R}^N\) such that

\[
E_\varepsilon \subset \subset E, \ E \setminus E_\varepsilon \subset N_\varepsilon(\partial E) \cap N_\varepsilon(\partial E_\varepsilon), \ P(E_\varepsilon; \mathbb{R}^N) \leq P(E; \mathbb{R}^N) + \varepsilon,
\]

where we have used the notation \(N_\varepsilon(\cdot)\) for \(\varepsilon\)-neighbourhoods of sets in \(\mathbb{R}^N\).
The conditions \( E = \bigcup \varepsilon > 0 E_{\varepsilon} \) and that
\[
\lim_{\varepsilon \to 0^+} \mathcal{L}^N (E_{\varepsilon}) = \mathcal{L}^N (E).
\]
On the other hand, the lower semicontinuity of the perimeter and the fact that \( \partial E_{\varepsilon} \) are smooth, yield
\[
\lim_{\varepsilon \to 0^+} \mathcal{H}^{N-1} (\partial E_{\varepsilon}) = \lim_{\varepsilon \to 0^+} P (E_{\varepsilon}; \mathbb{R}^N) = P (E; \mathbb{R}^N).
\]
In order to achieve the condition \( P (E_{\varepsilon}; \mathbb{R}^N) \leq P (E; \mathbb{R}^N) + \varepsilon \), rather than the weaker bound \( P (E_{\varepsilon}; \mathbb{R}^N) \leq C \mathcal{H}^{N-1} (\partial E) \), for some constant \( C > 1 \), it is not sufficient to cover \( \partial E \) with suitable balls and to construct the approximants \( E_{\varepsilon} \) by removing these balls from \( E \), but instead a covering of \( \partial E \) by suitably flat sets is required.

The conclusions of Theorem 2.3 were already known to hold for bounded Lipschitz domains \( E \) (see the references in [36]). Indeed, every set with Lipschitz boundary satisfies 2.2. However, if this condition fails to hold there are known counterexamples that show that the inner approximation by smooth sets may no longer be possible. Indeed, letting \( E = (0, 1)^{N-1} \times ((0, 1) \setminus \{ \frac{1}{2} \}) \), and applying the lower semicontinuity of the perimeter on both halves of \( E \), one concludes that all approximations \( E_{\varepsilon} \) satisfy
\[
\liminf_{\varepsilon \to 0^+} P (E_{\varepsilon}; \mathbb{R}^N) \geq 2N + 2 > 2N = P (E; \mathbb{R}^N).
\]
Notice that in this example
\[
P (E; \mathbb{R}^N) = 2N < 2N + 1 = \mathcal{H}^{N-1} (\partial E).
\]
We also refer to Example 5.2 in [36] and to Remark 1.27 in [22].

We recall the notions of quasiconvex function and quasiconvex envelope which will be used in Corollary 2.3

**Definition 2.4.** A Borel measurable and locally bounded function \( f : \mathbb{R}^{d \times N} \to \mathbb{R} \) is said to be quasiconvex if
\[
f (\xi) \leq \frac{1}{\mathcal{L}^N (D)} \int_D f (\xi + \nabla \varphi (x)) \, dx,
\]
for every bounded, open set \( D \subset \mathbb{R}^N \), for every \( \xi \in \mathbb{R}^{d \times N} \) and for every \( \varphi \in W^{1, \infty}_0 (D; \mathbb{R}^d) \).

**Remark 2.5.** We recall that if (2.4) holds for a certain set \( D \), then it holds for any bounded, open set in \( \mathbb{R}^N \). Notice also that, in the above definition, the value \(+ \infty\) is excluded from the range of \( f \).

**Definition 2.6.** The quasiconvex envelope of \( f \) is the greatest quasiconvex function that is less than or equal to \( f \).

We conclude this section by recalling the notions of weak and strong representation by means of measures.

**Definition 2.7.** Let \( \mu \) be a Radon measure on \( \Omega \), let \( (\chi, u) \in BV (\Omega; \{0, 1\}) \times W^{1,p} (\Omega; \mathbb{R}^d) \), and \( G (\chi, u; \cdot) \) be a functional defined on \( O (\Omega) \). We say that
a) \( \mu \) (strongly) represents \( G (\chi, u; \cdot) \) if \( \mu (A) = G (\chi, u; A) \) for all open sets \( A \subset \Omega \);
b) \( \mu \) weakly represents \( G (\chi, u; \cdot) \) if \( \mu (A) \leq G (\chi, u; A) \leq \mu (\overline{A}) \) for all open sets \( A \subset \Omega \).

3 Main Result

This section is devoted to the proof of Theorem 1.1. For the readers’ convenience, we restate it here.
Theorem 1.1 Let $p, q$ satisfy (1.12), let $f$ be defined as in (1.3), satisfying (1.10), (1.11) and (1.3). Let $\chi$ be the characteristic function of an open set $E \subset \Omega$ that satisfies (2.2) and let $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ be such that $u \in W^{1,q}(\Omega \setminus \overline{E}; \mathbb{R}^d)$. Then,

$$F_{\text{loc}}(\chi,u;\Omega) = F(\chi,u;\Omega) = \int_{\Omega} f(\chi(x),\nabla u(x)) \, dx + |D\chi|(\Omega).$$

Remarks 3.1. Notice that (1.10) is a special case of (1.8). Hypothesis (1.11), and the requirements placed on $u$, ensure that $F_{\text{loc}}(\chi,u;\Omega) < +\infty$, whereas $E \in \mathcal{O}(\Omega), E \subset \subset \Omega$ and (2.2), yield

$$H^{N-1}(\partial E) = P(E;\Omega) = P(E;\mathbb{R}^N) = |D\chi|(\Omega) = |D\chi|(\mathbb{R}^N).$$

We also point out that, given the nature of the problem (see (1.1), (1.10) and (1.11)), the assumptions made on $u$ depend on the set $E$ so, in the above integral representation result, the fields $\chi$ and $u$ are not independent of each other.

Since for the proof of the upper bound we use recovery sequences which are in $W^{1,q}(\Omega; \mathbb{R}^d)$, we show both that $F(\chi,u;\Omega)$ admits an integral representation, and that it coincides with $F_{\text{loc}}(\chi,u;\Omega)$ for which $\nu^*$ in (1.9) vanishes.

Proof of Theorem 1.1. We obtain the characterisation of $F_{\text{loc}}(\chi,u;\Omega)$ and $F(\chi,u;\Omega)$ directly, by proving a double inequality.

Due to the convexity hypothesis, the proof of the lower bound follows as in the second part of the proof of Theorem 4.3 in [3] by means of Theorem 2.3.

To prove the upper bound we need to construct sequences $u_n \in W^{1,q}(\Omega; \mathbb{R}^d)$, $\chi_n \in BV(\Omega; \{0,1\})$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^d)$, $\chi_n \rightharpoonup \chi$ in $BV(\Omega; \{0,1\})$ and $\operatorname{lim inf}_{n \to +\infty} F(\chi_n,u_n) \leq F(\chi,u)$.

The hypothesis on the set $E$ allows us to apply Theorem 2.3 twice to obtain, for each $\varepsilon > 0$, open sets $E_{2\varepsilon} \subset \subset E \subset \subset E$ such that $\partial E_{2\varepsilon}$ and $\partial E_{\varepsilon}$ are smooth,

$$\lim_{\varepsilon \to 0^+} P(E_{2\varepsilon};\Omega) = \lim_{\varepsilon \to 0^+} P(E_{\varepsilon};\Omega) = P(E;\Omega),$$

and

$$\lim_{\varepsilon \to 0^+} \mathcal{L}^N(E_{2\varepsilon}) = \lim_{\varepsilon \to 0^+} \mathcal{L}^N(E_{\varepsilon}) = \mathcal{L}^N(E).$$

Let $\{\rho_j\}_{j \in \mathbb{N}}$ be the usual sequence of standard mollifiers and, for each $\varepsilon > 0$, denote by $u_{\varepsilon,j} := u * \rho_j$, where the convolution is taken in the set $E_{\varepsilon}$ so that, for $j$ sufficiently large, $\operatorname{supp} u_{\varepsilon,j} \subset E_{\varepsilon} + B(0,\frac{1}{j}) \subset \subset E$. Let $L_{\varepsilon} := E_{\varepsilon} \setminus E_{2\varepsilon}$ and partition $L_{\varepsilon}$ into $T_{\varepsilon,j}$ N pairwise disjoint layers

$$L_{\varepsilon,j} := \{ x \in L_{\varepsilon} : \delta_{i-1} < \operatorname{dist} (x, \partial E_{\varepsilon}) \leq \delta_i \}, \quad i = 1, \ldots, T_{\varepsilon,j},$$

of constant width $\delta_i - \delta_{i-1} = \| u_{\varepsilon,j} - u \|_{L^p(E_{\varepsilon};\mathbb{R}^d)}^{1/p}$, with $\delta_0 = 0$ and $\delta_{T_{\varepsilon,j}} = O(\varepsilon)$, so that

$$T_{\varepsilon,j} \| u_{\varepsilon,j} - u \|_{L^p(E_{\varepsilon};\mathbb{R}^d)}^{1/p} = O(\varepsilon),$$

and, since $u_{\varepsilon,j}$ and $u \in W^{1,p}(\Omega; \mathbb{R}^d)$,

$$\sum_{i=1}^{T_{\varepsilon,j}} \int_{L_{\varepsilon,j}} \left( 1 + \| \nabla u_{\varepsilon,j}(x) - \nabla u(x) \|^p + \| \nabla u(x) \|^p + \frac{|u_{\varepsilon,j}(x) - u(x)|^p}{\| u_{\varepsilon,j} - u \|_{L^p(E_{\varepsilon};\mathbb{R}^d)}} \right) \, dx$$

$$= \int_{L_{\varepsilon}} \left( 1 + \| \nabla u_{\varepsilon,j}(x) - \nabla u(x) \|^p + \| \nabla u(x) \|^p + \frac{|u_{\varepsilon,j}(x) - u(x)|^p}{\| u_{\varepsilon,j} - u \|_{L^p(E_{\varepsilon};\mathbb{R}^d)}} \right) \, dx.$$
Consider cut-off functions $\varphi_{\varepsilon,j} \in C_c^\infty(\Omega; [0, 1])$ such that

$$
\varphi_{\varepsilon,j} = 0 \quad \text{in} \ (E \setminus E_{\varepsilon}) \cup \bigcup_{i=1}^{i_{\varepsilon,j} - 1} L_{x,j}^i \quad \text{and in} \ \Omega \setminus E
$$
$$
\varphi_{\varepsilon,j} = 1 \quad \text{in} \ E_{\varepsilon} \cup \left( \bigcup_{i=i_{\varepsilon,j} + 1}^{i_{E,j}} L_{x,j}^i \right) =: A_{\varepsilon,j},
$$
and

$$
\|\nabla \varphi_{\varepsilon,j}\|_\infty = O\left(\|u_{\varepsilon,j} - u\|_{L^p(E_{\varepsilon}; \mathbb{R}^d)}^{-1/p}\right).
$$

Define

$$
w_{\varepsilon,j}(x) := \varphi_{\varepsilon,j}(x)u_{\varepsilon,j}(x) + (1 - \varphi_{\varepsilon,j}(x))u(x), \ x \in \Omega,
$$

and notice that

$$
\nabla w_{\varepsilon,j} = \varphi_{\varepsilon,j}(\nabla u_{\varepsilon,j} - \nabla u) + \nabla u + (u_{\varepsilon,j} - u) \otimes \nabla \varphi_{\varepsilon,j} \in L^{i_{\varepsilon,j}}.
$$

Then $w_{\varepsilon,j} \in W^{1,q}(\Omega; \mathbb{R}^d)$, and by properties of the convolution, we have that, for each $\varepsilon > 0$,

$$
\lim_{j \to +\infty} \|w_{\varepsilon,j} - u\|_{L^p(\Omega; \mathbb{R}^d)}^p = \lim_{j \to +\infty} \left[ \int_{A_{\varepsilon,j}} |u_{\varepsilon,j}(x) - u(x)|^p \, dx + \int_{L_{x,j}^{i_{\varepsilon,j}}} |u_{\varepsilon,j}(x) - u(x)|^p \, dx \right] = 0.
$$

Furthermore,

$$
\int_{\Omega} \chi_{\varepsilon}(x) W_1(\nabla w_{\varepsilon,j}(x)) + (1 - \chi_{\varepsilon}(x)) W_0(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_{\varepsilon}|(\Omega)
$$
$$
= \int_{\bigcup_{i=1}^{i_{\varepsilon,j} - 1} L_{x,j}^i} W_1(\nabla u(x)) \, dx + \int_{\Omega \setminus E} W_0(\nabla u(x)) \, dx + \int_{A_{\varepsilon,j}} W_1(\nabla u_{\varepsilon,j}(x)) \, dx
$$
$$
+ \int_{L_{x,j}^{i_{\varepsilon,j}}} W_1(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_{\varepsilon}|(\Omega)
$$
$$
\leq \int_{L_{x,j}^{i_{\varepsilon,j}}} W_1(\nabla u(x)) \, dx + \int_{\Omega \setminus E} W_0(\nabla u(x)) \, dx + \int_{E_{\varepsilon}} W_1(\nabla u_{\varepsilon,j}(x)) \, dx
$$
$$
+ \int_{L_{x,j}^{i_{\varepsilon,j}}} W_1(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_{\varepsilon}|(\Omega).
$$

By (3.2),

$$
\lim_{\varepsilon \to 0^+} \int_{L_{x,j}^{i_{\varepsilon,j}}} W_1(\nabla u(x)) \, dx = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus E} W_0(\nabla u(x)) \, dx = \int_{\Omega \setminus E} W_0(\nabla u(x)) \, dx,
$$

and by (3.1)

$$
\lim_{\varepsilon \to 0^+} |D\chi_{\varepsilon}|(\Omega) = |D\chi|(\Omega).
$$

On the other hand, by (3.10), (3.7), (3.1), and (3.3), we have

$$
\liminf_{j \to +\infty} \int_{L_{x,j}^{i_{\varepsilon,j}}} W_1(\nabla w_{\varepsilon,j}(x)) \, dx \leq \liminf_{j \to +\infty} \beta_1 \int_{L_{x,j}^{i_{\varepsilon,j}}} (1 + \|\nabla w_{\varepsilon,j}(x)\|^p) \, dx
$$
$$
\leq C \limsup_{j \to +\infty} \int_{L_{x,j}^{i_{\varepsilon,j}}} \left( 1 + \|\varphi_{\varepsilon,j}(x)\| \|\nabla u_{\varepsilon,j}(x) - \nabla u(x)\|^p + \|\nabla u(x)\|^p + |u_{\varepsilon,j}(x) - u(x)|^p \cdot \|\nabla \varphi_{\varepsilon,j}(x)\|^p \right) \, dx
$$
$$
\leq \limsup_{j \to +\infty} \frac{C}{i_{E,j}} \int_{L_{x,j}^{i_{\varepsilon,j}}} \left( 1 + \|\nabla u_{\varepsilon,j}(x) - \nabla u(x)\|^p + \|\nabla u(x)\|^p + \|u_{\varepsilon,j}(x) - u(x)\|^p \|u_{\varepsilon,j} - u\|_{L^p(E_{\varepsilon}; \mathbb{R}^d)} \right) \, dx
$$
$$
\leq \limsup_{j \to +\infty} \frac{C}{E_{\varepsilon}} \|u_{\varepsilon,j} - u\|_{L^p(E_{\varepsilon}; \mathbb{R}^d)}^p \left[ \int_{L_{x,j}^{i_{\varepsilon,j}}} \left( 1 + \|\nabla u_{\varepsilon,j}(x) - \nabla u(x)\|^p + \|\nabla u(x)\|^p \right) \, dx + \|u_{\varepsilon,j} - u\|_{L^p(E_{\varepsilon}; \mathbb{R}^d)}^{p-1} \right]
$$
$$
= 0.
$$

(3.11)
since the sequence \( u_{\varepsilon,j} \) converges strongly to \( u \) in \( L^p(E_\varepsilon;\mathbb{R}^d) \), as \( j \to +\infty \), and the expression in square brackets is uniformly bounded in \( j \).

Finally, using the convexity of \( W_1 \), and since the measures \( \mu_x^j \) defined by

\[
\langle \mu_x^j, \varphi \rangle := \int_{\mathbb{R}^N} \rho_j(x - y)\varphi(y) \, dy
\]

are probability measures, Jensen’s inequality yields

\[
\liminf_{j \to +\infty} \int_{E_\varepsilon} W_1(\nabla u_{\varepsilon,j}(x)) \, dx = \liminf_{j \to +\infty} \int_{E_\varepsilon} W_1(\nabla(u \ast \rho_j(x))) \, dx
\]

\[
= \liminf_{j \to +\infty} \int_{E_\varepsilon} W_1(\langle \mu_x^j, \nabla u \rangle) \, dx \leq \limsup_{j \to +\infty} \int_{E_\varepsilon} \langle \mu_x^j, W_1(\nabla u) \rangle \, dx
\]

\[
= \int_{E_\varepsilon} W_1(\nabla u(x)) \, dx
\]

and this last integral converges, as \( \varepsilon \to 0^+ \), to \( \int_{E} W_1(\nabla u(x)) \, dx \).

Hence, by \( 3.8, 3.9, 3.10, 3.11 \) and \( 3.12 \), we conclude that

\[
\liminf_{\varepsilon \to 0^+} \liminf_{j \to +\infty} \left[ \int_{\Omega} \chi_\varepsilon(x) W_1(\nabla w_{\varepsilon,j}(x)) + (1 - \chi_\varepsilon(x)) W_0(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_\varepsilon|(|\Omega|) \right]
\]

\[
\leq \limsup_{\varepsilon \to 0^+} \left[ \int_{\Omega} \chi_\varepsilon(x) W_1(\nabla u(x)) + (1 - \chi_\varepsilon(x)) W_0(\nabla u(x)) \, dx + |D\chi_\varepsilon|(|\Omega|) \right]
\]

\[
= \left[ \int_{\Omega} \chi(x) W_1(\nabla u(x)) + (1 - \chi(x)) W_0(\nabla u(x)) \, dx + |D\chi|(|\Omega|) \right] = F(\chi, u).
\]

Thus, the conclusion follows by a standard diagonalisation argument.

\[ \square \]

**Remark 3.2.** The above result also holds if one prescribes the volume fraction of each phase. Precisely, given \( 0 < \theta < 1 \) and \( \chi \in BV(\Omega;\{0,1\}) \) such that \( \frac{1}{\mathcal{L}^N(\Omega)} \int_{\Omega} \chi(x) \, dx = \theta \), we define

\[
\mathcal{F}_{vol}(\chi, u; \Omega) := \inf \left\{ \liminf_{n \to +\infty} F(\chi_n, u_n; \Omega) : u_n \in W^{1,q}(\Omega;\mathbb{R}^d) ; \chi_n \in BV(\Omega;\{0,1\}) , u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^d), \chi_n \rightharpoonup \chi \text{ in } BV(\Omega;\{0,1\}), \frac{1}{\mathcal{L}^N(\Omega)} \int_{\Omega} \chi_n(x) \, dx = \theta \right\}.
\]

Under the hypotheses of Theorem \( 1.1 \) it follows that

\[
\mathcal{F}_{vol}(\chi, u; \Omega) = \int_{\Omega} f(\chi(x), \nabla u(x)) \, dx + |D\chi|(|\Omega|),
\]

for every \( \chi \) characteristic function of an open set \( E \subset \subset \Omega \) that satisfies \( (2.2) \) and \( \mathcal{L}^N(E) = \theta \mathcal{L}^N(\Omega) \), and for every \( u \in W^{1,p}(\Omega;\mathbb{R}^d) \cap W^{1,q}(\Omega \setminus E;\mathbb{R}^d) \). Indeed, in the previous proof, we consider \( B_\varepsilon \) a ball of sufficiently small radius \( \varepsilon \) such that \( B_\varepsilon \subset \subset \Omega \setminus E \) and \( \mathcal{L}^N(B_\varepsilon) = \theta \mathcal{L}^N(\Omega) - \mathcal{L}^N(E_\varepsilon) \), and we let \( \chi_\varepsilon \) be the characteristic function of the disjoint union \( E_\varepsilon \cup B_\varepsilon \). The sequence \( \chi_\varepsilon \) is therefore admissible for \( \mathcal{F}_{vol}(\chi, u; \Omega) \). Estimating the energy \( F(\chi, w_{\varepsilon,j}) \), where \( w_{\varepsilon,j} \) is as in the proof of Theorem \( 1.1 \), two new terms arise, namely \( P(B_\varepsilon; \Omega) \) and \( \int_{B_\varepsilon} W_1(\nabla u(x)) \, dx \), but these converge to zero as \( \varepsilon \to 0^+ \).

**Corollary 3.3.** Let \( p, q \) satisfy \( (1.6) \), let \( f \) be defined as in \( (1.8) \), satisfying \( (1.10) \) and \( (1.11) \). Assume further that \( QW_1 \) and \( QW_0 \), the quasiconvex envelopes of \( W_1 \) and \( W_0 \), respectively, are convex functions. Let \( \chi \) be the characteristic function of an open set \( E \subset \subset \Omega \) that satisfies \( (2.2) \) and let \( u \in W^{1,p}(\Omega;\mathbb{R}^d) \) be such that \( u \in W^{1,q}(\Omega \setminus E;\mathbb{R}^d) \). Then,

\[
\mathcal{F}_{loc}(\chi, u; \Omega) = \mathcal{F}(\chi, u; \Omega) = \int_{\Omega} \chi(x) QW_1(\nabla u(x)) + (1 - \chi(x)) QW_0(\nabla u(x)) \, dx + |D\chi|(|\Omega|).
\]
Proof. Let \( u_n \in W^{1,q}(\Omega; \mathbb{R}^d) \) and \( \chi_n \in BV(\Omega; \{0,1\}) \) be such that \( u_n \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^d) \), and \( \chi_n \rightharpoonup^* \chi \) in \( BV(\Omega; \{0,1\}) \). Then, by the convexity of \( QW_1 \) and \( QW_0 \), Ioffe’s Theorem \[14\] and the lower semicontinuity of the perimeter, we obtain

\[
\int_\Omega \chi(x) QW_1(\nabla u(x)) + (1 - \chi(x)) QW_0(\nabla u(x)) \, dx + |D\chi|(|\Omega|) \\
\leq \liminf_{n \to +\infty} \left( \int_\Omega \chi_n(x) QW_1(\nabla u_n(x)) + (1 - \chi_n(x)) QW_0(\nabla u_n(x)) \, dx + |D\chi_n|(|\Omega|) \right) \\
\leq \liminf_{n \to +\infty} \left( \int_\Omega \chi_n(x) W_1(\nabla u_n(x)) + (1 - \chi_n(x)) W_0(\nabla u_n(x)) \, dx + |D\chi_n|(|\Omega|) \right).
\]

Therefore,

\[
\int_\Omega \chi(x) QW_1(\nabla u(x)) + (1 - \chi(x)) QW_0(\nabla u(x)) \, dx + |D\chi|(|\Omega|) \leq F_{loc}(\chi; u; \Omega) \leq F(\chi; u; \Omega).
\]

To prove the reverse inequality, for each fixed \( \varepsilon > 0 \), let \( \chi_\varepsilon \) be the characteristic function of the set \( E_\varepsilon \) considered in the proof of Theorem \[14\] and let \( w_{\varepsilon,j} \in W^{1,q}(\Omega; \mathbb{R}^d) \) be the sequence constructed therein. By standard relaxation results (cf. \[13, 17\] Theorem 8.4.1]), since \( \chi_\varepsilon \) is fixed, there exists a sequence \( v_{\varepsilon,j,n} \in W^{1,q}(\Omega; \mathbb{R}^d) \) such that \( v_{\varepsilon,j,n} \rightharpoonup w_{\varepsilon,j} \) in \( W^{1,q}(\Omega; \mathbb{R}^d) \) and

\[
\limsup_{n \to +\infty} \left( \int_\Omega f(\chi_\varepsilon(x), \nabla v_{\varepsilon,j,n}(x)) \, dx + |D\chi_\varepsilon|(|\Omega|) \right) = \left[ \int_\Omega \chi_\varepsilon(x) QW_1(\nabla w_{\varepsilon,j}(x)) + (1 - \chi_\varepsilon(x)) QW_0(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_\varepsilon|(|\Omega|) \right]. \tag{3.13}
\]

As in the previous proof, we estimate the expression in \[3.13\] by taking into account the definition of \( w_{\varepsilon,j} \) in each subset of the decomposition of \( \Omega \) given in the proof of Theorem \[14\].

\[
\left[ \int_\Omega \chi_\varepsilon(x) QW_1(\nabla w_{\varepsilon,j}(x)) + (1 - \chi_\varepsilon(x)) QW_0(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_\varepsilon|(|\Omega|) \right] \\
\leq \int_{L_{\varepsilon}} QW_1(\nabla u(x)) \, dx + \int_{\Omega \setminus E_{\varepsilon}} QW_0(\nabla u(x)) \, dx \\
+ \int_{E_{\varepsilon}} QW_1(\nabla u_{\varepsilon,j}(x)) \, dx + \int_{L^*_1} QW_1(\nabla w_{\varepsilon,j}(x)) \, dx + |D\chi_\varepsilon|(|\Omega|).
\]

By \[3.2\],

\[
\lim_{\varepsilon \to 0^+} \int_{L_{\varepsilon}} QW_1(\nabla u(x)) \, dx = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus E_{\varepsilon}} QW_0(\nabla u(x)) \, dx = \int_{\Omega \setminus E} QW_0(\nabla u(x)) \, dx,
\]

and by \[3.1\]

\[
\lim_{\varepsilon \to 0^+} |D\chi_\varepsilon|(|\Omega|) = |D\chi|(|\Omega|).
\]

Since \( QW_1 \leq W_1 \), an argument similar to the one used to obtain \[3.11\], gives

\[
\limsup_{j \to +\infty} \int_{L^*_1} QW_1(\nabla w_{\varepsilon,j}(x)) \, dx = 0,
\]

whereas the convexity of \( QW_1 \), Jensen’s inequality and reasoning as in \[3.12\], lead to

\[
\liminf_{\varepsilon \to 0^+} \liminf_{j \to +\infty} \int_{E_{\varepsilon}} QW_1(\nabla u_{\varepsilon,j}(x)) \, dx = \int_{E} QW_1(\nabla u(x)) \, dx.
\]
Applying once again a standard diagonalization argument, we obtain sequences \( u_n \in W^{1,q}(\Omega; \mathbb{R}^d) \) and \( \chi_n \in BV(\Omega; \{0, 1\}) \) such that \( u_n \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^d) \), \( \chi_n \rightharpoonup^{*} \chi \) in \( BV(\Omega; \{0, 1\}) \) and

\[
\liminf_{n \to +\infty} \left( \int_{\Omega} \chi_n(x) W_1(\nabla u_n(x)) + (1 - \chi_n(x)) W_0(\nabla u_n(x)) \, dx + |D\chi_n|(\Omega) \right)
\leq \int_{\Omega} \chi(x) QW_1(\nabla u(x)) + (1 - \chi(x)) QW_0(\nabla u(x)) \, dx + |D\chi|(\Omega).
\]

Hence,

\[
\mathcal{F}_{\text{loc}}(\chi, u; \Omega) \leq \mathcal{F}(\chi, u; \Omega) \leq \int_{\Omega} \chi(x) QW_1(\nabla u(x)) + (1 - \chi(x)) QW_0(\nabla u(x)) \, dx + |D\chi|(\Omega),
\]

and the proof is complete. \( \square \)

The self-contained argument above was presented for the readers’ convenience but we observe that if \( f \) is as in (4.3), and denoting by \( Qf \) its quasiconvex envelope with respect to the second variable, as in (2.4), this proposition could have been stated and proved in two steps, namely, by showing first that

\[
\mathcal{F}(\chi, u; \Omega) := \inf \left\{ \liminf_{n \to +\infty} \left( \int_{\Omega} Qf(\chi_n(x), \nabla u_n(x)) \, dx + |D\chi_n|(\Omega) \right) : u_n \in W^{1,q}(\Omega; \mathbb{R}^3), \chi_n \in BV(\Omega; \{0, 1\}), u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^3), \chi_n \rightharpoonup^{*} \chi \text{ in } BV(\Omega; \{0, 1\}) \right\},
\]

as in Lemma 4.2 below, and then by applying Theorem 1.1.

### 4 Dimension Reduction

In the sequel we apply the above result to identify the optimal design of plates, in the so-called membranal regime (see e.g. [30] and [10] among a wide literature), by means of dimension reduction, in the spirit of the models described in [8, 10], which also appear in the context of brutal damage evolution. Namely one can deduce, as a rigorous 3D-2D \( \Gamma \)-limit (see [14] for a detailed treatment of the subject) as \( \varepsilon \to 0^+ \), the optimal design of an elastic membrane \( \Omega(\varepsilon) := \omega \times (0, \varepsilon) \), with \( \omega \subset \mathbb{R}^2 \) a bounded open set with Lipschitz boundary constituted by materials with different hyperelastic responses, i.e., which truly exhibit a gap between the growth and coercivity exponents in the set with Lipschitz boundary constituted by materials with different hyperelastic responses, i.e., which

In the following we adopt the standard scaling (see [10] and the references quoted therein) which maps \( x \equiv (x_1, x_2, x_3) \in \Omega(\varepsilon) \to (x_1, x_2, \varepsilon x_3) \in \Omega := \omega \times (0, 1) \), in order to state the problem in a fixed domain (see (4.3) below). We also denote by \( \nabla_{\varepsilon} u \) and \( D_{\varepsilon} \chi \), respectively, the partial derivatives of \( u \) and \( \chi \) with respect to \( x_3 \equiv (x_1, x_2) \), while \( \nabla_3 u \) and \( D_3 \chi \) represent the derivatives with respect to \( x_3 \).

In the model under consideration, the sequence \( \chi_\varepsilon \in BV(\Omega; \{0, 1\}) \) represents the design regions, whereas \( u_\varepsilon \in W^{1,q}(\Omega; \mathbb{R}^3) \) is the sequence of deformations, which might be clamped at the lateral extremities of the membrane. Standard arguments in dimension reduction (see e.g. [30] and [10]) ensure that energy bounded sequences (see the term in square brackets of (4.3)), converge (up to a subsequence), in the relevant topology, to fields \( (\chi, u) \) such that \( D_3 \chi \text{ and } \nabla_3 u \text{ are null, thus they can be identified, with an abuse of notation, with fields } (\chi, u) \in BV(\omega; \{0, 1\}) \times W^{1,p}(\omega; \mathbb{R}^3) \). In what follows we use this notation.

**Proposition 4.1.** Let \( \omega \subset \mathbb{R}^2 \) be a bounded open set and define \( \Omega := \omega \times (0, 1) \). Let \( 1 < p \leq q < +\infty \) and let \( f : \{0, 1\} \times \mathbb{R}^{3 \times 3} \to \mathbb{R} \) be a continuous function as in (1.3), with \( W_1 \) as in (1.10) and (1.11). Assume also that for every \( b \in [0, 1] \)

\[
Qf(b, \cdot) \text{ is convex},
\]

where \( Qf(b, \cdot) \) denotes the quasiconvex envelope of \( f(b, \cdot) \) (see Definition 2.6), and that there exist \( c, c_0 \in \mathbb{R}^+ \) such that

\[
c|\xi|^p - c_0 \leq f(b, \xi),
\]

(4.2)
for every \( b \in \{0, 1\} \) and \( \xi \in \mathbb{R}^{3 \times 3} \).

Let \( \chi \) be the characteristic function of an open set \( E \subset \omega \) that satisfies (4.2) with \( N = 2 \). Let \( u \in W^{1,p}(\omega; \mathbb{R}^3) \) (or \( u \in W^{1,q}(\omega; \mathbb{R}^3) \)), if we assume that the membrane is fixed at the lateral boundary) be such that \( u \in W^{1,q}(\omega \setminus \bar{E}; \mathbb{R}^3) \), and let

\[
\mathcal{F}^{DR}(\chi, u) := \liminf_{\varepsilon \to 0^+} \left[ \int_{\Omega} f(\chi_{\varepsilon}(x), (\nabla_{\alpha} u_{\varepsilon}(x), \frac{1}{\varepsilon} \nabla_{33} u_{\varepsilon}(x))) \, dx + \left| \left( D_{\alpha} \chi_{\varepsilon}, \frac{1}{\varepsilon} D_{33} \chi_{\varepsilon} \right) \right| (\Omega) \right].
\]

(4.3)

\( u_{\varepsilon} \in W^{1,q}(\Omega; \mathbb{R}^3), \chi_{\varepsilon} \in BV(\Omega; \{0, 1\}), u_{\varepsilon} \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^3) \), \( \chi_{\varepsilon} \rightharpoonup \chi \) in \( BV(\Omega; \{0, 1\}) \).

Then

\[
\mathcal{F}^{DR}(\chi, u) = \int_{\omega} Q\hat{f}(\chi(x), \nabla_{\alpha} u(x)) \, dx + |D_{\alpha} \chi|(\omega),
\]

(4.4)

where

\[
\hat{f}(b, \xi_{\alpha}) := \inf_{\xi_{3} \in \mathbb{R}^3} f(b, \xi_{\alpha}, \xi_{3}), \text{ with } b \in \{0, 1\}, (\xi_{\alpha}, \xi_{3}) = \xi \in \mathbb{R}^{3 \times 3},
\]

(4.5)

and \( Q\hat{f}(b, \cdot) \) denotes the quasiconvex envelope of \( \hat{f}(b, \cdot) \) with respect to the second variable.

We point out that the functional \( \mathcal{F}^{DR} \) in (4.3) is defined in full analogy with \( \mathcal{F} \) in (1.4), although it involves an asymptotic process which can be rigorously treated in the framework of \( \Gamma \)-convergence. On the other hand, our proof of the integral representation (4.4) is obtained following the same strategy, based on proving a double inequality, adopted at the end of the previous section, and it is self-contained. Moreover, the sequences \( u_{\varepsilon} \), the limit \( u \) and the related convergence in (4.3) could be taken in \( W^{1,q}(\Omega; \mathbb{R}^3) \) and \( W^{1,p}(\Omega; \mathbb{R}^3) \), respectively, under the assumption that the membrane is fixed at the lateral boundary.

Before addressing the proof of Proposition 4.1 we start by proving a lemma following the ideas presented in [7, Lemma 2.3].

**Lemma 4.2.** Under the conditions of Proposition 4.1 the following holds

\[
\mathcal{F}^{DR}(\chi, u) := \liminf_{\varepsilon \to 0^+} \left[ \int_{\Omega} f(\chi_{\varepsilon}(x), (\nabla_{\alpha} u_{\varepsilon}(x), \frac{1}{\varepsilon} \nabla_{33} u_{\varepsilon}(x))) \, dx + \left| \left( D_{\alpha} \chi_{\varepsilon}, \frac{1}{\varepsilon} D_{33} \chi_{\varepsilon} \right) \right| (\Omega) \right] :=
\]

(4.3)

\( u_{\varepsilon} \in W^{1,q}(\Omega; \mathbb{R}^3), \chi_{\varepsilon} \in BV(\Omega; \{0, 1\}), u_{\varepsilon} \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^3) \), \( \chi_{\varepsilon} \rightharpoonup \chi \) in \( BV(\Omega; \{0, 1\}) \).

**Proof.** As in [7 (2.2)], we have that

\( (Qf)_{\varepsilon}(b, \xi) = Q(f_{\varepsilon})(b, \xi), \)

where for any function \( g : \{0, 1\} \times \mathbb{R}^{3 \times 3} \to [0, +\infty) \),

\[
g_{\varepsilon}(b, \xi_{\alpha}, \xi_{3}) := g(b, \xi_{\alpha}, \frac{1}{\varepsilon} \xi_{3}).
\]

In light of (4.1), \( g_{\varepsilon} \) is convex in the variable \( \xi = (\xi_{\alpha}, \xi_{3}) \).

Let \( \mathcal{F}^{DR}_{\varepsilon}(\chi, u) \) be defined as \( \mathcal{F}^{DR}(\chi, u) \) but replacing \( f \) by \( Qf \). Clearly, since \( Qf \leq f \), it follows that \( \mathcal{F}^{DR}_{\varepsilon} \leq \mathcal{F}^{DR} \) so we only need to prove the opposite inequality. To this end, for every \( \delta > 0 \) and every \( (\chi, u) \) satisfying the hypotheses of Proposition 4.1 let \( (\chi_{\varepsilon}, u_{\varepsilon}) \in BV(\Omega; \{0, 1\}) \times W^{1,q}(\Omega; \mathbb{R}^3) \) be such that \( u_{\varepsilon} \rightharpoonup u \) in \( W^{1,p}(\Omega; \mathbb{R}^3) \), \( \chi_{\varepsilon} \rightharpoonup \chi \) in \( BV(\Omega; \{0, 1\}) \) and

\[
\mathcal{F}^{DR}_{\varepsilon}(\chi, u) \geq \liminf_{\varepsilon \to 0^+} \left[ \int_{\Omega} Qf(\chi_{\varepsilon}(x), (\nabla_{\alpha} u_{\varepsilon}(x), \frac{1}{\varepsilon} \nabla_{33} u_{\varepsilon}(x))) \, dx + \left| \left( D_{\alpha} \chi_{\varepsilon}, \frac{1}{\varepsilon} D_{33} \chi_{\varepsilon} \right) \right| (\Omega) \right] \geq \mathcal{F}^{DR}(\chi, u) - \delta.
\]

Up to the extraction of a subsequence, we may assume that the above lim inf is, in fact, a limit.
By [17] Theorem 8.4.1, there exists \( u_{\varepsilon,k} \in W^{1,q}(\Omega;\mathbb{R}^3) \) such that \( u_{\varepsilon,k} \rightharpoonup u_\varepsilon \) weakly in \( W^{1,q} \), as \( k \to +\infty \), and
\[
\int_\Omega Qf \left( \chi_\varepsilon(x), (\nabla_{\alpha} u_{\varepsilon}(x), \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon}(x)) \right) \, dx + \left| \left( D_\alpha \chi_\varepsilon, \frac{1}{\varepsilon} D_3 \chi_\varepsilon \right) \right| (\Omega) = \lim_{\varepsilon \to 0^+} \lim_{k \to +\infty} \int_\Omega f \left( \chi_\varepsilon(x), (\nabla_{\alpha} u_{\varepsilon,k}(x), \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon,k}(x)) \right) \, dx + \left| \left( D_\alpha \chi_\varepsilon, \frac{1}{\varepsilon} D_3 \chi_\varepsilon \right) \right| (\Omega).
\]
Thus we can say that
\[
F_{Qf}^{DR}(\chi, u) \geq \lim_{\varepsilon \to 0^+} \lim_{k \to +\infty} \int_\Omega f \left( \chi_\varepsilon(x), (\nabla_{\alpha} u_{\varepsilon,k}(x), \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon,k}(x)) \right) \, dx + \left| \left( D_\alpha \chi_\varepsilon, \frac{1}{\varepsilon} D_3 \chi_\varepsilon \right) \right| (\Omega) - \delta, \quad (4.6)
\]
and
\[
\lim_{\varepsilon \to 0^+} \lim_{k \to +\infty} \| u_{\varepsilon,k} - u \|_{L^p} = 0.
\]
The growth from below in (4.2), the convexity of \( b, \xi \) for every \( \varepsilon, q \), \( \xi \), and the fact that the weak topology is metrizable on bounded sets, ensure that there exist a diagonal sequence \( u_{\varepsilon,k} \) and a subsequence \( \chi_{\varepsilon,k} \) such that
\[
(\chi_{\varepsilon,k}, u_{\varepsilon,k}) \rightharpoonup (\chi, u) \text{ in } BV^* \times W^{1,p} \text{-weak, as } k \to +\infty,
\]
the double limit in (4.6) exists, and thus
\[
F_{Qf}^{DR}(\chi, u) \geq \lim_{k \to +\infty} \int_\Omega f \left( \chi_{\varepsilon,k}(x), (\nabla_{\alpha} u_{\varepsilon,k}(x), \frac{1}{\varepsilon} \nabla_3 u_{\varepsilon,k}(x)) \right) \, dx + \left| \left( D_\alpha \chi_{\varepsilon,k}, \frac{1}{\varepsilon} D_3 \chi_{\varepsilon,k} \right) \right| (\Omega) - \delta,
\]
which, in turn, implies that
\[
F_{Qf}^{DR}(\chi, u) \geq F_{Qf}^{DR}(\chi, u) - \delta.
\]
It suffices to let \( \delta \to 0^+ \) to conclude the proof. \( \square \)

**Lemma 4.3.** Assume that \( f \) is as in Proposition [4.1] and \( Qf \), its quasiconvex envelope with respect to the second variable, satisfies [4.11]. Then, for every \( b \in \{0,1\} \),
\[
\widehat{Qf}(b, \cdot) = Q\widehat{f}(b, \cdot), \quad (4.7)
\]
where, for each function \( g : \{0,1\} \times \mathbb{R}^{3 \times 3} \to [0, +\infty) \), \( \widehat{g} : \{0,1\} \times \mathbb{R}^{3 \times 3} \to [0, +\infty) \) is defined as in [4.5].

**Proof.** Rewriting, as in [4.3], \( \xi \) as \( (\xi_\alpha, \xi_3) \in \mathbb{R}^{3 \times 3} \), we observe that \( \widehat{f}(b, \xi_\alpha, \xi_3) \leq f(b, \xi_\alpha, \xi_3) \) for every \( (b, \xi_\alpha, \xi_3) \in \{0,1\} \times \mathbb{R}^{3 \times 3} \), thus
\[
\widehat{Qf}(b, \xi_\alpha, \xi_3) \leq Qf(b, \xi_\alpha, \xi_3) \quad (4.8)
\]
for every \( (b, \xi_\alpha, \xi_3) \in \{0,1\} \times \mathbb{R}^{3 \times 3} \), where, with an abuse of notation, \( \widehat{f} \) and \( \widehat{Qf}(b, \cdot) \) are considered as defined in \( (0,1) \times \mathbb{R}^{3 \times 3} \), assuming that they are independent of \( \xi_3 \), the quasiconvex envelope on the right hand side of (4.8) is taken with respect to the variable \( (\xi_\alpha, \xi_3) \in \mathbb{R}^{3 \times 3} \), and we are taking into account, as in [4.10] Proposition 6, that \( \widehat{Qf}(b, \cdot) \) is quasiconvex as a function of \( (\xi_\alpha, \xi_3) \). Then, applying (4.8) to both sides of (4.8) we have
\[
Q\widehat{f}(b, \xi_\alpha) = \widehat{Qf}(b, \xi_\alpha) \leq \widehat{Qf}(b, \xi_\alpha),
\]
for every \( (b, \xi_\alpha) \in \{0,1\} \times \mathbb{R}^{3 \times 2} \), which proves one inequality.

For what concerns the reverse inequality, since \( Qf(b, \xi) \leq f(b, \xi) \) for every \( (b, \xi) \in \{0,1\} \times \mathbb{R}^{3 \times 3} \), we have
\[
\widehat{Qf}(b, \xi_\alpha) \leq \widehat{f}(b, \xi_\alpha),
\]
for every \( (b, \xi_\alpha) \in \{0,1\} \times \mathbb{R}^{3 \times 2} \). On the other hand, it is easily seen (cf. also [5] (5.10)) that (4.11) entails the convexity of \( \widehat{Qf} \) with respect to the variable \( \xi_\alpha \), thus \( \widehat{Qf} \) is quasiconvex with respect to \( \xi_\alpha \), hence
\[
\widehat{Qf}(b, \xi_\alpha) = Q(\widehat{f})(b, \xi_\alpha) \leq \widehat{Qf}(b, \xi_\alpha)
\]
for every \( (b, \xi_\alpha) \in \{0,1\} \times \mathbb{R}^{3 \times 2} \) which concludes the proof. \( \square \)
**Proof of Proposition 4.4.** The proof of (4.4) is obtained by showing a double inequality. For what concerns the lower bound, it suffices to observe that \( Q \hat{f} \leq f \), and by (4.2)
\[
c|\xi_\alpha|^p - c_0 \leq Q \hat{f}(b, \xi_\alpha),
\]
for every \((b, \xi_\alpha) \in \{0, 1\} \times \mathbb{R}^{3 \times 2}\), and \( Q \hat{W}_0 \) and \( Q \hat{W}_1 \) satisfy (4.10) and (4.11), respectively. On the other hand, we recall that
\[
\text{Theorem 1.1.}
\]
In a similar fashion to (3.6), we define \( \phi \)
\[
\text{Moreover the functional}
\]
is lower semicontinuous with respect to \( BV \)-weak \( \ast \times W^{1,p} \)-weak convergence by Theorem 2.1. Indeed by Lemma 4.3 \( Q \hat{f}(b, \xi_\alpha) = \hat{Q}f(b, \xi_\alpha) \), thus, by (4.1) and (4.5), it is convex in the second variable. Then, the superadditivity of the limit inf, the fact that \([|D_\alpha^\chi_\varepsilon|, \frac{1}{3} D_3^\chi_\varepsilon|](\Omega) \geq |D_\alpha^\chi_\varepsilon|(\Omega)\) and the lower semicontinuity of the total variation, entail that
\[
F^{DR}(\chi, u) \geq \liminf_{\varepsilon \to 0^+} \int \Omega Q \hat{f}(\chi(x_\alpha, x_3), \nabla_\alpha u(x_\alpha, x_3)) \, dx + |D_\alpha^\chi_\varepsilon|(\Omega)
\]
\[
\int \omega \hat{f}(\chi(x_\alpha), \nabla_\alpha u(x_\alpha)) \, dx_\alpha + |D_\alpha^\chi|\omega).
\]
In order to prove the upper bound, we use Lemma 4.2 and replace \( f \) by \( Qf \). We use a two-dimensional version of the proof and the notations of Theorem 1.1.

Let \( \chi \in BV(\omega; \{0, 1\}) \) be the characteristic function of a set \( E \subset \subset \omega \) such that (3.1) holds in \( \omega \), i.e. \( H^1(\partial E) = P(\chi; \omega) = P(\chi; \mathbb{R}^3) \), and let \( u \in W^{1,p}(\omega; \mathbb{R}^q) \cap W^{1,q}(\omega \setminus \overline{E}; \mathbb{R}^3) \). Consider \( \chi_\varepsilon(x_\alpha) \) the characteristic function of the set \( E_\varepsilon \) such that (3.1) and (3.2) hold in \( \omega \), and let \( \omega_{\varepsilon,J}(x_\alpha) \) be defined in (3.6). Given \( \psi \in L^p(\omega; \mathbb{R}^3) \cap L^q(\omega \setminus \overline{E}; \mathbb{R}^3) \), we regularize \( \psi \) in the same way as in the proof of Theorem 1.1. Then, we define \( \psi_{\varepsilon,J}(x_\alpha) = (\psi \ast \rho_j)(x_\alpha) \), where \( \{\rho_j\}_{j \in \mathbb{N}} \) is the usual sequence of standard mollifiers and the convolution is taken in the set \( E_\varepsilon \) so that, for \( j \) sufficiently large, supp \( \psi_{\varepsilon,J} \subset E_\varepsilon + B(0, \frac{1}{j}) \subset \subset E \).

In a similar fashion to (3.6), we define
\[
\psi_{\varepsilon,J}(x_\alpha) := \varphi_{\varepsilon,J}(x_\alpha) \psi_{\varepsilon,J}(x_\alpha) + (1 - \varphi_{\varepsilon,J}(x_\alpha)) \psi(x_\alpha), \quad x_\alpha \in \omega,
\]
where \( \varphi_{\varepsilon,J} \) is the two-dimensional version of the sequence of cut-off functions considered in the proof of Theorem 1.1.

We now define
\[
v_{\varepsilon,J}(x) := \omega_{\varepsilon,J}(x_\alpha) + \varepsilon x_3 \eta_{\varepsilon,J}(x_\alpha),
\]
and, by abuse of notation, consider \( \chi_\varepsilon(x_\alpha) \). Clearly \( \{v_{\varepsilon,J}\} \) and \( \{\chi_\varepsilon\} \) are admissible for \( F^{DR}(\chi, u) \) so we obtain, using Lemma 4.2,
\[
F^{DR}(\chi, u) \leq \liminf_{\varepsilon \to 0^+} \liminf_{j \to +\infty} \left[ \int \Omega Q^f(\chi_\varepsilon(x), (\nabla_\alpha v_{\varepsilon,J}(x), \frac{1}{3} \nabla_3 v_{\varepsilon,J}(x))) \, dx + |D_\alpha^\chi_\varepsilon, \frac{1}{3} D_3^\chi_\varepsilon|((\Omega)) \right]
\]
\[
= \liminf_{\varepsilon \to 0^+} \liminf_{j \to +\infty} \left[ \int \Omega Q^f(\chi_\varepsilon(x), (\nabla_\alpha w_{\varepsilon,J}(x_\alpha), \eta_{\varepsilon,J}(x_\alpha))) \, dx + |D_\alpha^\chi_\varepsilon|((\Omega)) \right]
\]
\[
\leq \int \Omega Q^f(\chi(x_\alpha), (\nabla_\alpha u(x_\alpha), \psi(x_\alpha))) \, dx + |D_\alpha^\chi|((\Omega))
\]
\[
= \int \omega Q^f(\chi(x_\alpha), (\nabla_\alpha u(x_\alpha), \psi(x_\alpha))) \, dx_\alpha + |D_\alpha^\chi|((\omega),
\]
where the last inequality is proved following the estimates provided in the proof of Theorem 1.1. Hence, given the arbitrariness of \( \psi \), we conclude that
\[
F^{DR}(\chi, u) \leq |D_\alpha^\chi|((\omega)) + \inf_{\psi \in L^p(\omega; \mathbb{R}^3) \cap L^q(\omega \setminus \overline{E}; \mathbb{R}^3)} \int \omega Q^f(\chi(x_\alpha), (\nabla_\alpha u(x_\alpha), \psi(x_\alpha))) \, dx_\alpha.
\]
Recalling the continuity and the coercivity of $Qf(b, \cdot)$, as in [12], and using Lemma 4.3, (4.9), and the measurability criterion which provides the existence of $\bar{\psi} \in L^p(\omega; \mathbb{R}^3) \cap L^q(\omega \setminus E; \mathbb{R}^3)$ such that
\[ Q\hat{f}(\chi(x), \nabla_\alpha u(x)) = \hat{Q}f(\chi(x), \nabla_\alpha u(x), \bar{\psi}(x)), \]
it follows that
\[ \mathcal{F}^{DR}(\chi, u) \leq |D_\alpha \chi|_0(\omega) + \int_\omega Q\hat{f}(\chi(x), \nabla_\alpha u(x)) \, dx, \]
which completes the proof. \qed

In order to deal with optimal design problems where the volume fraction of each phase is prescribed, i.e. as in (1.16), it is easily seen that the constraint
\[ \frac{1}{CN(\Omega)} \int_\Omega \chi(x) \, dx = \theta, \theta \in (0, 1), \]
does not affect at all our proof, if we insert it in the form of a Lagrange multiplier into the model, that is, we can add $\kappa \int_\Omega \chi(x) \, dx$, $\kappa > 0$, to the functional $F$ since this is a linear term.

On the other hand this choice allows us to interpret the representation result in Proposition 4.1 in the light of “brutal damage evolution models for thin films” as proposed in (1.12), where, in fact, the linear term describes a dissipation energy.

Another possibility to deal with the volume constraint is to argue as in Remark 3.2, adding in the region with a different energetic behaviour a thin cylinder of sufficiently small width $\varepsilon$ so that the volume fraction of the approximating sequence $\{\chi_\varepsilon\}$ is $\theta$.

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**References**

[1] E. Acerbi, G. Bouchitté & I. Fonseca, Relaxation of convex functionals: the gap problem, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20, 3 (2003), 359-390.

[2] L. Ambrosio & G. Buttazzo, An optimal design problem with perimeter penalization, Calc. Var. Partial Differential Equations 1, (1993), 55-69.

[3] L. Ambrosio, N. Fusco & D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, 2000.

[4] J.-F. Babadjian, F. Iurlano & F. Rindler, Concentration versus oscillation effects in brittle damage, https://arxiv.org/abs/1906.02019.

[5] A.C. Barroso & E. Zappale, Relaxation for optimal design problems with nonstandard growth, Appl. Math. Optim. 80, no. 2, (2019), 515-546.

[6] G. Bouchitté, I. Fonseca & J. Malý, The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent, Proceedings of the Royal Society of Edinburgh, 128 A, (1998), 463-479.

[7] G. Bouchitté, I. Fonseca & M. L. Mascarenhas, Bending moment in membrane theory, J. Elasticity, 73, no. 1-3, (2004), 75-99.

[8] A. Braides, I. Fonseca & G. Francfort, 3D-2D asymptotic analysis for inhomogeneous thin films, Indiana Univ. Math. J. 49, no. 4 (2000), 1367-1404.
[9] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, (2011), xiv+599.

[10] G. Carita & E. Zappale, 3D-2D dimensional reduction for a nonlinear optimal design problem with perimeter penalization, C. R. Math. Acad. Sci. Paris, **350**, no. 23-24, (2012), 1011-1016.

[11] G. Carita & E. Zappale, Relaxation for an optimal design problem with linear growth and perimeter penalization, Proc. Royal Soc. Edinburgh A, **145**, (2015), 223-268.

[12] A. Coscia & D. Mucci, Integral representation and Γ-convergence of variational integrals with $p(x)$-growth, ESAIM Control Optim. Calc. Var. **7**, (2002), 495-519.

[13] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Second Edition, Applied Mathematical Sciences, **78**, Springer, 2008.

[14] G. Dal Maso, *An Introduction to Γ-Convergence*, Progr. Nonlinear Differential Equations Appl., Birkhäuser Boston, Inc., Boston, MA, 1983.

[15] G. Dal Maso & F. Iurlano, Fracture models as Γ-limits of damage models, Commun. Pure Appl. Anal., **12**, (2013), n. 4, 1657-1686.

[16] I. Fonseca & G. Francfort, 3D-2D asymptotic analysis of an optimal design problem for thin films, Journal für die Reine und Angewandte Mathematik, **505**, (1998), 173-202.

[17] I. Fonseca & G. Leoni, *Modern Methods in the Calculus of Variations: $W^{1,p}$ Spaces*, Springer Monographs in Mathematics. Springer, New York, to appear.

[18] I. Fonseca & J. Malý, Relaxation of multiple integrals below the growth exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire, **14**, (1997), 309-338.

[19] I. Fonseca & S. Müller, A-quasiconvexity, lower semicontinuity and Young measures, SIAM J. Math. Anal., **30**, no. 6, (1999), 1355-1390.

[20] G. A. Francfort & J.-J. Marigo, Stable damage evolution in a brittle continuous medium, European J. Mech. A Solids, **12**, (1993), n. 2, 149-189.

[21] W. Gangbo, On the weak lower semicontinuity of energies with polyconvex integrands. J. Math. Pures Appl., **73**, no. 5, (1994), 455-469.

[22] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, **80**, Birkhäuser, 1984.

[23] P. Harjulehto & P. Hästö, Orlicz spaces and generalized Orlicz spaces, Lecture Notes in Mathematics, **2236**, Springer, Cham, 2019 pp. x+167, https://doi.org/10.1007/978-3-030-15100-3.

[24] A. D. Ioffe, On lower semicontinuity of integral functionals I, SIAM Journal of Control and Optimization, **15**, (1977), 521-538.

[25] R.V. Kohn & F. H. Lin, Partial regularity for optimal design problems involving both bulk and surface energies, Chinese Ann. Math. Ser. B, **20**, no. 2, (1999), 137-158.

[26] R.V. Kohn & G. Strang, Optimal design and relaxation of variational problems I, Commun. Pure Appl. Math. **39**, (1986), 113-137.

[27] R.V. Kohn & G. Strang, Optimal design and relaxation of variational problems II, Commun. Pure Appl. Math. **39**, (1986), 139-182.

[28] R.V. Kohn & G. Strang, Optimal design and relaxation of variational problems III, Commun. Pure Appl. Math. **39**, (1986), 353-377.

[29] F. Iurlano, Fracture and plastic models as Γ-limits of damage models under different regimes, Adv. Calc. Var., **6**, (2013), n. 2, 165-189.
[30] H. Le Dret & A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, J. Math. Pures Appl. (9), 74, no. 6, (1995), 549-578.

[31] A. Mielke & T. Roubiček, Rate-independent damage processes in nonlinear elasticity, Math. Models Methods Appl. Sci., 16, (2006) n. 2, 177-209.

[32] G. Mingione & D. Mucci, Integral functionals and the gap problem: sharp bounds for relaxation and energy concentration, SIAM J. Math. Anal., 36, no. 5, (2005), 1540-1579.

[33] D. Mucci, Relaxation of variational functionals with piecewise constant growth conditions, J. Convex Anal., 10, no. 2, (2003), 295-324.

[34] F. Murat & L. Tartar, Calcul des variations et homogénéisation, Homogenization methods: theory and applications in physics (Bréau-sans-Nappe, 1983), Collect. Dir. Études Rech. Élec. France, 57, (1985) 319–369, Eyrolles, Paris.

[35] K. Pham, J.-J. Marigo & C. Maurini, The issues of the uniqueness and the stability of the homogeneous response in uniaxial tests with gradient damage models, J. Mech. Phys. Solids, 59, (2011), n. 6, 1163-1190.

[36] T. Schmidt, Strict interior approximation of sets of finite perimeter and functions of bounded variation, Proc. Amer. Math. Soc., 143, no. 5, (2015), 2069-2084.

[37] V.V. Zhikov, On variational problems and nonlinear elliptic equations with nonstandard growth conditions, Journal of Mathematical Sciences, 173, no. 5, (2011), 463-570.