Evolution of cooperation in an n-player game with opting out

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ABSTRACT

How cooperation could have evolved has been one of the central topics in evolutionary biology. When cooperators are likely to interact with other cooperators, and defectors are likely to interact with other defectors, positive assortment is created, facilitating the evolution of cooperation. Cooperation is observed not only in dyadic interactions but also, sometimes, in sizable groups. Previous studies have found that the opting out rule in which the group is disbanded if and only if the group is heterogeneous, especially facilitates the evolution of cooperation compared to the other opting out rules in multi-player games when the number of rounds is sufficiently large. However, the dynamics between the cooperators and defectors under such an opting out rule have been investigated only in the case where group size is relatively small (e.g., four). In addition, the effect of group size on the evolution of cooperation has not been explored, and considering that humans interact within larger groups, investigation in such large groups is essential. Here, through further algebraic analyses, it is newly revealed that there can be four internal equilibria when the group size is larger than four. In addition, while the effect of group size on cooperation is negative in the case of common goods, it is not straightforward (i.e., can be positive) in the case of public goods.

1. Introduction

How cooperation, which is costly to the actor, could have evolved has been one of the central topics in evolutionary biology (Smith, 1982; Frank, 1998; Hamilton, 1964; Nowak, 2006; McElreath and Boyd, 2007; Sigmund, 2010).

When cooperators are likely to interact with cooperators, and defectors are likely to interact with defectors, positive assortment is created, allowing the possible evolution of cooperation (Aktipis, 2004; Izuquierdo et al., 2010; Izuquierdo et al., 2014; Zhang et al., 2016; Zheng et al., 2017). In a dyadic interaction, it is expected that the evolution of cooperation is most encouraged when the group is disbanded if and only if the group is heterogeneous. Cooperation does not exist only in dyadic interactions, but it can sometimes be observed also in sizable groups. For example, human beings cooperate with others in large real-world communities (e.g., donations of blood or money for charity) (Suzuki and Akiyama, 2005). It is known that the mean group size in apes is approximately 60 and it was predicted that the mean group size in humans would be of approximately 150 individuals (Dunbar, 1993). Krivan and Cressman (2020) have examined which opting out rule most facilitates the evolution of cooperation, not in a pair, but in a group. Let “the best opting out rule” denote an opting out rule which most facilitates the evolution of cooperation among a variety of opting out rules. The results revealed that the opting out rule in which the group is disbanded if and only if the group is heterogeneous, is the best opting out rule in multi-player games, when the number of rounds is sufficiently large.

However, at least two aspects were not examined in the above-mentioned study and they seem worthwhile investigating. Firstly, the analysis of the dynamics under the opting out rule was not conducted for a large group. Specifically, the dynamics were examined in the case where group size was four, revealing that there are at most two internal equilibria. Dunbar stated that the mean group size is approximately 60 individuals in apes. Further, in humans, a mean group size of approximately 150 individuals was estimated by Dunbar. It is unclear whether the dynamics when group size is four and those when it is larger than four (e.g., approximately 60 or 150) are the same. In the present study, analyses were conducted to reveal the dynamics in the case where the group size is > 4. Secondly, Krivan and Cressman (2020) did not explore the effect of group size on the evolution of cooperation. Examining the effect of group size is significant in the field of evolutionary anthropology. The group size effect can depend on whether goods are rivalrous (i.e., common goods (e.g., fish stocks, open ocean fishing, and water)) or non-rivalrous (i.e., public goods (e.g., public information, national defense, and knowledge)); therefore, the group size effect in the case of common goods and that in the case of public goods should be examined.

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respectively.

The contents below are structured as follows: Section 2 describes a mathematical model of the evolution of cooperation in n-player games; Section 3 summarizes the results, illustrating the cooperation dynamics, exploring how the parameters affect them and the positions of internal equilibria; and Section 4 is devoted to the discussion.

2. Model assumption

2.1. Game

In the n-player Prisoner’s Dilemma (PD) game, each of the n players cooperates or defects where

\[ n \geq 2. \]  

(1)

Table 1 summarizes what symbols mean. On one hand, when an individual cooperates, he/she pays cost c and contributes to a public good. In the present study, we dealt with non-excludable goods. Therefore, every n-player including him/her in the group receives an equal benefit b/n. On the other hand, when an individual defects, he/she gives nothing. Let us consider the case where the number of cooperating individuals is n; (i.e., the number of defecting individuals is n – n). This assumption naturally leads to the fact that the payoffs to cooperating (C) and defecting (D) individuals are given by \( b/n \) and \( c \) respectively (Boyd and Richerson, 1988). From the definition of the n-player PD game, the following inequality must be satisfied:

\[ b > c > b/n > 0. \]  

(2)

Rivalry plays a crucial role in the analysis of size effects (Taylor, 1987). On one hand, a benefit can be regarded as rivalrous if the usage of a benefit by an individual reduces its availability to others, and in the above model for a rivalrous good, \( b/c \) does not change with n. On the other hand, a benefit can be regarded as non-rivalrous if the usage of a benefit by an individual does not reduce its availability to others, and in the above model for a non-rivalrous good, \( b/nc \) does not change with n; thus b linearly increases as group size (n) increases. A common good is defined as a rivalrous non-excludable good, and a public good is defined as a non-rivalrous non-excludable good (Taylor, 1987).

2.2. Strategy

Two strategies are here introduced: the player adopting strategy C always cooperates, and the player adopting strategy D always defects. Correspondingly, the game is played by C and D players. \( P_C \) denotes the frequency of strategy i in a population. Here, because an individual is either C or D, it follows the following equation:

\[ P_C + P_D = 1. \]  

(3)

2.3. Group formation

The population size is N and N is infinitely large (N=\( n \)). In the first period, \( n \) individuals are randomly selected and matched, and \( n \) individuals in a group play the n-player PD game. Here, N is a multiple of n and not one group but N/n groups are considered in this game. In later periods, the group composition depends on the outcomes of the previous period. It is assumed that a probability that a group breaks up in a round is determined by the number of cooperators in the group. \( \beta_y \), where \( 0 \leq y \leq n \), denotes the probability that the interaction stops in the next round when \( y \) of \( n \) players are cooperators (i.e., \( n – y \) of \( n \) players are defectors). It is assumed that \( \beta_y \) is either 1 or \( \rho \), where

\[ 0 < \rho < 1. \]  

(4)

The opting out rule can be expressed as \{\( \beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n \). The number of opting out rules is \( 2^{n+1} \). Krivan and Cressman (2009) found that when \( \rho \) is sufficiently small, the best opting out rule, which most facilitates the evolution of cooperation among the \( 2^{n+1} \) opting out rules is:

\[ \beta_y = \left\{ \begin{array}{ll}
\rho & \text{ when } y = 0, n \\
1 & \text{ when } 1 \leq y \leq n - 1
\end{array} \right. \]  

(5)

Under (5), when both cooperators and defectors are present in the group (i.e., the group is heterogeneous), the group is disbanded with probability 1. In contrast, when the group members are all cooperators or defectors (i.e., the group is homogeneous), the interaction stops with probability \( \rho \). The present study investigates the dynamics under this opting out rule (i.e., (5)).

Individuals who did not terminate the interaction with partners in the previous period played the n-player PD game again with the same opponent players in the next period. In contrast, an individual who stopped the interaction in the previous period was paired with \( n-1 \) individuals (who had been randomly selected from similar individuals who stopped interacting with opponents in a previous game). These players were thus paired for the n-player PD game. It was assumed that an individual who stopped interacting with partners could find the next opponents instantly without incurring a searching cost.

\( Q_i(T) \) denotes the frequencies of a group in which \( y \) cooperators and \( n – y \) defectors are present at round T. From the above assumption, the following equation can be derived

\[ Q_i(1) = \left( \begin{array}{c}
\frac{1}{n}
\end{array} \right) P_C P_D^{n-1}. \]  

(6)

The frequency of \( Q_i(T+1) \) in a round for which a group was not dissolved in the previous round is given by \( Q_i(T) (1 - \beta_y) \). The ratio of individuals whose group was dissolved in the previous round to the whole population is \( (\sum_{y=0}^{n-1} \beta_y Q_i(T)) \). The ratio of cooperators whose groups were dissolved to all the individuals whose groups were dissolved in the previous round is given by

\[ \left( \sum_{y=0}^{n-1} \beta_y Q_i(T) \right) \left( \sum_{y=1}^{n-1} \beta_y Q_i(T) \right)^{y - 1} \left( \sum_{y=0}^{n-1} \beta_y Q_i(T) \right)^n. \]

Hence, \( Q_i(T+1) \) is given by
$Q_i(T + 1) = Q_i(T)(1 - b_i) + \sum_{W \in 0}^{n} \beta_i Q_w(T)$

\[
\Delta Q_i(T) = Q_i(T)(1 - b_i) + \sum_{W \in 0}^{n} \beta_i Q_w(T) - Q_i(T).
\]

Here, $\Delta Q_i(T)$ is introduced as

$$\Delta Q_i(T) \equiv Q_i(T + 1) - Q_i(T).$$

$\Delta Q_i(T)$ can be interpreted as the changes of group frequencies between rounds. Based on (7) and (8), the following equation can be obtained:

$$\Delta Q_i(T) = Q_i(T)(1 - b_i) + \sum_{W \in 0}^{n} \beta_i Q_w(T) - \frac{\sum_{W \in 0}^{n} \beta_i Q_w(T)}{\sum_{W \in 0}^{n} \beta_i Q_w(T)}.$$  

(9)

It is assumed that such a group formation repeats infinite times. Under this condition, it is algebraically shown that there is just one equilibrium in the sense of group formation (see supplementary material A for proof). $\bar{Q}$ denotes the group frequencies at an equilibrium. It is numerically indicated that, after infinite repetitions, the group formation approaches the unique equilibrium (see supplementary material B).

More specifically, we have

$$\lim_{t \to \infty} Q_i(T) = \bar{Q}_i.$$  

(10)

From (8) and (10), the following equation can be obtained:

$$\lim_{t \to \infty} \Delta Q_i(T) = 0.$$  

(11)

2.4. Dynamics

It is assumed that the changes in the strategies’ frequencies are much slower than group formation. Therefore, each time there is a change in the strategies’ frequencies, the group formation is at equilibrium. $F_i$ denotes the expected payoff of strategy $i$ from one round, while $\bar{F}_i$ denotes the expected payoff of strategy $i$ from one round when the group formation reaches an equilibrium. Here, it is also assumed that the changes in the strategies’ frequencies in the population can be described by the following replicator equations (Taylor and Jonker, 1978; Hofbauer and Sigmund, 1998):

$$\frac{dF_i}{dt} = P_c(1 - P_c)(\bar{F}_c - \bar{F}_D).$$

(12)

$\varphi_{ij}$ denotes the probability that strategy $i$ belongs to a group in which there are $y$ cooperators and $n - y$ defectors. Here, using $\varphi_{ijk}$, it is possible to describe $F_c$ and $F_D$ as:

$$F_c = \left(-c + \frac{b}{n}\right) + \sum_{W \in 0}^{n} \varphi_{ijk} \frac{b}{n}.$$

(13)

$$F_D = \sum_{W \in 0}^{n} \varphi_{ijk} \frac{b}{n}.$$  

(14)

$\varphi_{ijk}$ denotes the probability that strategy $i$ belongs to a group in which there are $y$ cooperators and $n - y$ defectors when the group formation reaches an equilibrium. Here, using $\varphi_{ijk}$, it is possible to describe $\bar{F}_c$ and $\bar{F}_D$ as:

$$\bar{F}_c = \left(-c + \frac{b}{n}\right) + \sum_{W \in 0}^{n} \bar{\varphi}_{ijk} \frac{b}{n}.$$  

(15)

$$\bar{F}_D = \sum_{W \in 0}^{n} \bar{\varphi}_{ijk} \frac{b}{n}.$$  

(16)

The dynamics between strategies C and D are examined in the present study. By investigating the evolutionary dynamics, we will explore the evolution of cooperation in this paper.

3. Results

The subsections presented below examine the following aspects: 1) how $b/c$ influences the strategies’ dynamics (e.g., when defectors dominate cooperators and when not) and the positions of internal equilibria (subsection 3.1); 2) how $\rho$ influences the strategies’ dynamics (e.g., when defectors dominate cooperators and when not) and the positions of internal equilibria (subsection 3.2); 3) how group size influences the condition under which defectors dominate cooperators and the positions of internal equilibria when $b/c$ is constant (i.e., common good) (subsection 3.3); and 4) how group size influences the condition under which defectors dominate cooperators and the positions of internal equilibria when $b/(nc)$ is constant (i.e., public good) (subsection 3.4).

3.1. Effect of $b/c$ on the evolution of cooperation

3.1.1. The condition under which payoff to cooperators is larger than that to defectors

After algebraic calculations (see supplementary material C for proof), the condition under which cooperators obtain a higher payoff than the defectors do is given by:

$$\frac{b}{c} > h(f).$$

(17)

where

$$h(f) \equiv \frac{\rho q + (1 - \rho)q^{\frac{2}{\rho}}}{\rho + (1 - \rho)[(1 - \rho) + \rho^{\frac{2}{\rho}}]}.$$  

(18)

Here, after algebra (see supplementary material C for proof), it turns out that there exists a unique $q$ so that:

$$P_c \equiv \frac{\rho q + (1 - \rho)q^{\frac{2}{\rho}}}{\rho + (1 - \rho)[(1 - \rho) + \rho^{\frac{2}{\rho}}]}.$$  

(19)

$f$ denotes a quantity that satisfies (19). From (17), (18) and (19), it turns out that once $\frac{n}{c}, n, \rho$, and $P_c$ are determined, it is determined which gets higher payoffs, cooperators or defectors.

Fig. 1 shows the relationship between $P_c$ and the critical value of $b/c$. For a given $P_c$, when and only when the actual value of $b/c$ is larger than the critical value of $b/c$, the payoff to cooperators is larger than that to defectors. There are two kinds of shapes; one kind of shape is unimodal, and the other kind of shape is bimodal (see Fig. 1). In Fig. 1, it is observed that the figures in the case where $n$ is small and (or) $\rho$ is small are unimodal, while the figures in the case where $n$ is large and (or) $\rho$ is large are bimodal. Actually, algebraic calculation proves this observation. More specifically, it turns out that the figures with $\rho(n + 2) + 2^{n - 1}(n - 4) \leq n + 2$ are unimodal and the figures with $\rho(n + 2) + 2^{n - 1}(n - 4) > n + 2$ are bimodal (see supplementary material D for proof).

3.1.2. The effect of parameters on the number of internal equilibria

When $\rho((n + 2) + 2^{n - 1}(n - 4)) > n + 2$, there exists just one $f$ such that $u_i(f) = \rho$, which is presented in supplementary material E in the
range of $0 < f < \frac{1}{2}$ (see supplementary material D for proof that such $f$ is unique). By $f_1$, we denote such $f$. By the definition of $f_1$, we have

$$0 < f_1 < \frac{1}{2}.$$  \hfill (20)

The condition under which there are no internal equilibria is given by

$$\frac{b}{c} < \begin{cases} 1 + \frac{2^n(n-1)\rho}{2n + \left(2^n - 2n\right)\rho} & \text{when } \rho((n + 2) + 2^{n-1}(n - 4)) \leq n + 2 \\ h(f_1) & \text{when } \rho((n + 2) + 2^{n-1}(n - 4)) > n + 2 \end{cases}.$$  \hfill (21)

The condition under which there are two internal equilibria is given by

$$\frac{b}{c} > \begin{cases} 1 + \frac{2^n(n-1)\rho}{2n + \left(2^n - 2n\right)\rho} & \text{when } \rho((n + 2) + 2^{n-1}(n - 4)) \leq n + 2 \\ h(f_1) & \text{when } \rho((n + 2) + 2^{n-1}(n - 4)) > n + 2 \end{cases}.$$  \hfill (22)

The condition under which there are four internal equilibria is given by

$$h(f_1) < \frac{b}{c} < 1 + \frac{2^n(n-1)\rho}{2n + \left(2^n - 2n\right)\rho}.$$  \hfill (23)

and

$$\rho((n + 2) + 2^{n-1}(n - 4)) > n + 2.$$  \hfill (24)

(24) means that when $n \leq 4$, the number of internal equilibria cannot be 4.

From above, it turns out that the condition under which cooperators are not dominated by defectors (note that in such a case, there exist at least two internal equilibria) is given by

$$\frac{b}{c} > \begin{cases} 1 + \frac{2^n(n-1)\rho}{2n + \left(2^n - 2n\right)\rho} & \text{when } \rho((n + 2) + 2^{n-1}(n - 4)) \leq n + 2 \\ h(f_1) & \text{when } \rho((n + 2) + 2^{n-1}(n - 4)) > n + 2 \end{cases}.$$  \hfill (25)

Fig. 2, which is derived from (21), (22), (23), and (24) visually shows how $n$, $b/c$, and $\rho$ affect the number of internal equilibria. We can confirm that when $n = 4$, the number of internal equilibria cannot be 4.

3.1.3. The effect of parameters on the positions of internal equilibria

In the previous subsection (3.1.2.), we investigated the number of internal equilibria. In the current subsection (3.1.3.), we will investigate the positions of internal equilibria. Additionally, we investigate what kind of bifurcation happens when the number of internal equilibria
As shown in Fig. 2, when \( \frac{b}{c} < h(1) \), defectors dominate cooperators. When \( \frac{b}{c} > h(1) \), there are two internal equilibria. Fig. 3 illustrates how the positions of internal equilibria are influenced by \( b/c \). When \( \frac{b}{c} = 1 + \frac{2(n-1)\rho}{2n+2(n-2)\rho} \), a blue-sky bifurcation occurs, and the stable and unstable equilibria emerge (see Fig. 3). The frequencies of cooperators at two internal equilibria are \( P_{C_u} \) and \( P_{C_s} \), respectively. Here, we have
\[
P_{C_{u,s}} = 1 - P_{C_{u,s}}. \tag{27}
\]

In Fig. 3, it is observed that as \( b/c \) increases, \( P_{C_u} \) (i.e., the frequency of cooperators at the unstable equilibrium) decreases, and \( P_{C_s} \) (i.e., that at the stable equilibrium) increases.

Secondly, we examine the case of
\[
\rho((n+2) + 2^{n-1}(n-4)) > n+2. \tag{28}
\]
As shown in Fig. 2, when \( \frac{b}{c} < h(1) \), defectors dominate cooperators. When \( h(f_1) < \frac{b}{c} < 1 + \frac{2(n-1)\rho}{2n+2(n-2)\rho} \), there are just four internal equilibria, at which the frequencies of cooperators are \( P_{C_{u_1}}, P_{C_{u_2}}, P_{C_{s_1}}, \) and \( P_{C_{s_2}} \), respectively. Here, we have
\[
P_{C_{u_1}} = 1 - P_{C_{u_2}}. \tag{29}
\]
When \( \frac{b}{c} > 1 + \frac{2(n-1)\rho}{2n+2(n-2)\rho} \), there are just two internal equilibria, at which the frequencies of cooperators are \( P_{C_{u_j}} \) and \( P_{C_{s_2}} \), respectively.
tution of $n = 5$ into (28) yields $\rho > \frac{1}{2}$. We pick up $\rho = 0.55$ as a parameter satisfying this (Fig. 3). When $\frac{b}{c} = h(f_1)$, two stable and two unstable equilibria emerge (blue-sky bifurcation) (see Fig. 3). When $\frac{b}{c} = 1 + \frac{\rho^2 - 1}{\rho^2 + 2\rho}$, the lower stable equilibrium, at which the frequency of cooperators is $P_{c_s}$, coalesce (fold bifurcation) (see Fig. 3). When $\rho((n+2)+2^{n-1}(n-4)) > n+2$, in Fig. 3, it is observed that as $\frac{b}{c}$ increases, $P_{c_s}$, decreases and $P_{c_h}$, approaches 0 as $\frac{b}{c}$ approaches $n$. Actually, it is shown (see supplementary material F for proof) that

$$\frac{dP_{c_s}}{d(\rho^2)} < 0. \quad (30)$$

In addition, after algebraic calculation (see supplementary material F for proof), we can derive

$$\lim_{\rho \to \infty} P_{c_s} = 0. \quad (31)$$

Then, how does $P_{c_s}$ approach 0? After algebraic calculation (see supplementary material F), we have

$$\lim_{\rho \to \infty} P_{c_s} = \frac{n}{n-1(1-\rho)} \quad (32)$$

When $\rho((n+2)+2^{n-1}(n-4)) > n+2$, in Fig. 3, it is observed that as $\frac{b}{c}$ increases, $P_{c_s}$ increases. Actually, it is shown (see supplementary material F for proof) that

$$\frac{dP_{c_s}}{d(\rho^2)} > 0. \quad (33)$$

3.2. Effect of $\rho$ on the evolution of cooperation

3.2.1. The condition under which payoff to cooperators is larger than that to defectors

Fig. 4 shows the relationship between $P_c$ and the critical value of $\rho$. For a given $P_c$, when and only when the actual value of $\rho$ is smaller than the critical value of $\rho$, the payoff to cooperators is larger than that to defectors. There are two kinds of shapes; one kind of shape is unimodal, and the other kind of shape is bimodal (see Fig. 4). In Fig. 4, it is observed that the figures in the case where $n$ is small and (or) $\frac{b}{c}$ is small are unimodal, while the figures in the case where $n$ is large and (or) $\frac{b}{c}$ is large are bimodal. Actually, algebraic calculation proves this observation. More specifically, it turns out that the figures with $\left(1 - \frac{3}{2}\right) \frac{b}{c} > 2$ are unimodal and the figures with $\left(1 - \frac{3}{2}\right) \frac{b}{c} < 2$ are bimodal.

In addition, after algebraic calculation (see supplementary material F for proof), we can derive

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When $\rho((n+2)+2^{n-1}(n-4)) > n+2$, in Fig. 3, it is observed that as $\frac{b}{c}$ increases, $P_{c_s}$ increases. Actually, it is shown (see supplementary material F for proof) that

$$\frac{dP_{c_s}}{d(\rho^2)} > 0. \quad (33)$$

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When $\rho((n+2)+2^{n-1}(n-4)) > n+2$, in Fig. 3, it is observed that as $\frac{b}{c}$ increases, $P_{c_s}$ increases. Actually, it is shown (see supplementary material F for proof) that

$$\frac{dP_{c_s}}{d(\rho^2)} > 0. \quad (33)$$

3.2. Effect of $\rho$ on the evolution of cooperation

3.2.1. The condition under which payoff to cooperators is larger than that to defectors

Fig. 4 shows the relationship between $P_c$ and the critical value of $\rho$. For a given $P_c$, when and only when the actual value of $\rho$ is smaller than the critical value of $\rho$, the payoff to cooperators is larger than that to defectors. There are two kinds of shapes; one kind of shape is unimodal, and the other kind of shape is bimodal (see Fig. 4). In Fig. 4, it is observed that the figures in the case where $n$ is small and (or) $\frac{b}{c}$ is small are unimodal, while the figures in the case where $n$ is large and (or) $\frac{b}{c}$ is large are bimodal. Actually, algebraic calculation proves this observation. More specifically, it turns out that the figures with $\left(1 - \frac{3}{2}\right) \frac{b}{c} > 2$ are unimodal and the figures with $\left(1 - \frac{3}{2}\right) \frac{b}{c} < 2$ are bimodal.
3.2.2. The effect of parameters on the number of internal equilibria

By $\rho_2$, we denote $\rho$ such that $\frac{1}{\rho_2} = h(f_1)$. The condition under which there are no internal equilibria is given by

$$\rho > \begin{cases} \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)} & \text{when } \left(1 - \frac{2}{n}\right) \frac{b}{c} \leq 2 \\ \rho_2 & \text{when } \left(1 - \frac{2}{n}\right) \frac{b}{c} > 2 \end{cases}$$  \quad (34)$$

The condition under which there are two internal equilibria is given by

$$\rho < \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)}.$$  \quad (35)

The condition under which there are four internal equilibria is given by

$$\frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)} < \rho < \rho_2.$$  \quad (36)

and

$$\left(1 - \frac{2}{n}\right) \frac{b}{c} > 2.$$  \quad (37)

From above, it turns out that the condition under which cooperators are not dominated by defectors (note that in such a case, there exist at least two internal equilibria) is given by

$$\rho < \rho^*,$$  \quad (38)

where

$$\rho^* \equiv \begin{cases} \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)} & \text{when } \left(1 - \frac{2}{n}\right) \frac{b}{c} \leq 2 \\ \rho_2 & \text{when } \left(1 - \frac{2}{n}\right) \frac{b}{c} > 2 \end{cases}.$$  \quad (39)

3.2.3. The effect of parameters on the positions of internal equilibria

In the previous subsection (3.2.2.), the number of internal equilibria was investigated. In the current subsection (3.2.3.), we will investigate the positions of internal equilibria. In addition, we investigate what kind of bifurcation alters the number of internal equilibria.

Firstly, we investigate the case of

$$\left(1 - \frac{2}{n}\right) \frac{b}{c} \leq 2.$$  \quad (40)

Fig. 2 demonstrates that when $\rho < \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)}$, there are two internal equilibria, one unstable and the other stable. When $\rho > \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)}$, cooperators are dominated by defectors, and there are no internal equilibria. Fig. 5 illustrates how the positions of internal equilibria are influenced by $\rho$. We use $n = 5$ when drawing Fig. 5. Substitution of $n = 5$ into (40) yields $\frac{5}{\rho} \leq 0.2$. We pick up $\frac{5}{\rho} = 2.5$ as a parameter satisfying this (Fig. 5). In Fig. 5, it is observed that when $\rho$ reaches the critical value (i.e., $\rho = \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)}$), the stable and unstable internal equilibrium coalesce (fold bifurcation). As $\rho$ increases, the frequency of cooperators at the unstable internal equilibrium ($P_{C_{u2}}$) increases, while that at the stable internal equilibrium ($P_{C_{l1}}$) decreases. Thus, the impact of $\rho$ on the evolution of cooperation is negative.

Secondly, we investigate the case of

$$\left(1 - \frac{2}{n}\right) \frac{b}{c} > 2.$$  \quad (41)

Fig. 2 demonstrates that when $\rho < \frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)}$, there are two internal equilibria, one unstable and the other stable. When $\frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)} < \rho < \rho_2$, four internal equilibria exist. When $\rho > \rho_2$, cooperators are dominated by defectors, and there are no internal equilibria. Substitution of $n = 5$ into (41) yields $\frac{5}{\rho} > \frac{5}{\rho_2}$. We pick $\frac{5}{\rho} = 3.5$ and $\frac{5}{\rho} = 4.5$ as a parameter satisfying this (Fig. 5). In Fig. 5, it is observed that when $\rho$ reaches $\frac{2n}{2^n(n-1) \frac{1}{n} - (2^n - 2n)}$, two additional internal equilibria emerge (blue-sky bifurcation), one unstable and the other stable. When $\rho$ reaches $\rho_2$, the lower unstable and stable internal equilibria, and the higher unstable and stable internal equilibria coalesce (fold bifurcation), respectively. As $\rho$ increases, the frequency of cooperators at the lower unstable equilibrium ($P_{C_{u1}}$) and the frequency of cooperators at the higher unstable equilibrium ($P_{C_{u1}}$) increase, while that at the lower stable equilibrium ($P_{C_{l1}}$) and that at the higher stable equilibrium ($P_{C_{l1}}$) decrease. Thus, the impact of $\rho$ on the evolution of cooperation is negative.

No matter whether $(1 - \frac{2}{n}) \frac{b}{c} \leq 2$ or $(1 - \frac{2}{n}) \frac{b}{c} > 2$, in Fig. 5, it is observed that as $\rho$ approaches 0, $P_{C_{u1}}$ approaches 0. Actually, after algebraic calculation (see supplementary material G), we have

$$\lim_{\rho \to 0} P_{C_{u1}} = 0.$$  \quad (42)

Then, how does $P_{C_{l1}}$ approach 0? Actually, it has been algebraically shown (see supplementary material G) that

$$\lim_{\rho \to 0} P_{C_{l1}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \frac{\frac{b}{c} - 1}{\frac{b}{c} + 1} \end{pmatrix}.$$  \quad (43)

From (43), we have

$$\lim_{\rho \to 0} P_{C_{l1}} = 0.$$  \quad (44)

$P_{C_{l}}$ ($P_{C_{u1}}$) is defined as a frequency of cooperators at which the lower (higher) unstable equilibrium and the lower (higher) stable equilibrium coalesce. Here, we have

$$P_{C_l} = 1 - P_{C_{l1}}.$$  \quad (45)
It is shown that the condition under which cooperators are not dominated by defectors is given by
\[
\lim_{\rho \to 0} \frac{f c \rho}{b} = \lim_{\rho \to 0} \frac{c l(n, \rho)}{b} = \frac{1}{n - l(n, \rho)}.
\] (46)

\[
\lim_{\rho \to 0} \frac{f c \rho}{b} \leq \frac{1}{n} \quad \text{when } \rho \rho(n + 2) = (1 - f^*) \rho^{n-1}(1 - 2f^*) \frac{n(1-f^*)-1}{1-(1-f^*)nf^*},
\] (47)

where
\[
l(n, \rho) = \begin{cases} \frac{1}{1 + \frac{2(n-1)\rho}{2n + (2^2 - 2n)\rho}} & \text{when } \rho \rho(n + 2) + 2^{n-1}(n - 4) \leq n + 2 \\ \frac{1}{\rho(n)} & \text{when } \rho \rho(n + 2) + 2^{n-1}(n - 4) > n + 2 \end{cases}
\] (48)

3.3. Group size effects on cooperation when $b/c$ is constant (i.e., common good)

This subsection explores the effect of group size ($n$) on the evolution of cooperation when $b/c$ is constant (i.e., common good).

3.3.1. From the perspective of the condition under which cooperators are not dominated by defectors

It is shown that the condition under which cooperators are not dominated by defectors is given by
\[
\frac{c}{b} < l(n, \rho),
\] (49)

where
\[
l(n, \rho) = \begin{cases} \frac{1}{1 + \frac{2(n-1)\rho}{2n + (2^2 - 2n)\rho}} & \text{when } \rho \rho(n + 2) + 2^{n-1}(n - 4) \leq n + 2 \\ \frac{1}{\rho(n)} & \text{when } \rho \rho(n + 2) + 2^{n-1}(n - 4) > n + 2 \end{cases}
\] (50)

Fig. 6(a), which is derived from (49) and (50), illustrates when defectors dominate cooperators and when they do not, showing that $l(n, \rho)$ decreases as $\rho$ increases. Fig. 6(a) also shows that the critical value of $c/b$ is smaller when $n$ is larger. Actually, it has been algebraically shown that
\[
\lim_{\rho \to 0} l(n, \rho) = 1,
\] (51)

\[
\lim_{\rho \to 0} \frac{l(n, \rho)}{\rho} = \frac{n - 1}{n} \cdot 2^{n-1}.
\] (52)

As $n$ increases, the right-hand side of (52), which is $\frac{n - 1}{n} \cdot 2^{n-1}$, increases; therefore, the left-hand side of (52), which is $\lim_{\rho \to 0} \frac{l(n, \rho)}{\rho}$, increases. Hence, when $\rho$ is around 0, $l(n, \rho)$ decreases, and consequently, as $n$ increases, the condition under which cooperators are not dominated by defectors becomes more strict.

In addition, it has been algebraically (see supplementary material J for proof) shown that
\[
\lim_{\rho \to 1} l(n, \rho) = \frac{1}{n}.
\] (53)

As $n$ increases, the right-hand side of (53), which is $\frac{1}{n}$, decreases; therefore, the left-hand side of (53), which is $\lim_{\rho \to 1} l(n, \rho)$, decreases. Hence, when $\rho$ is around 1, $l(n, \rho)$ decreases, and consequently, as $n$ increases, the condition under which cooperators are not dominated by defectors becomes tighter.
defectors becomes more strict.

Thus, as the group size increases, the condition under which cooperators are not dominated by defectors becomes more strict. In other words, the evolution of cooperation becomes more difficult as group size increases.

In Fig. 6(a), it is expected that when the group size is very large, defectors dominate cooperators and there are no internal equilibria, irrespective of the values of \( b/c \) and \( \rho \). Actually, it is possible to algebraically (see supplementary material K for proof) show that the following limiting values of \( l(n, \rho) \) exist:

\[
\lim_{n \to \infty} l(n, \rho) = 0.
\]

3.3.2. From the perspective of the positions of internal equilibria

As observed in Fig. 6(a) and suggested by algebraic calculation, the impact of \( n \) on whether cooperators are not dominated by defectors is monotonic. As \( n \) increases, the condition under which cooperators are not dominated by defectors becomes more strict. When \( \frac{b}{c} > \frac{1}{\sqrt[n]{n\rho}} \), defectors dominate cooperators, irrespective of the group size. When \( \frac{b}{c} < \frac{1}{\sqrt[n]{n\rho}} \), cooperators are not dominated by defectors at least when \( n = 2 \). (54) \( \lim_{n \to \infty} l(n, \rho) = 0 \). means that when \( n \) is very large, cooperators are dominated by defectors and there are no internal equilibria. Fig. 6(b) and (c) illustrates how the number of internal equilibria and the positions of internal equilibria are influenced by \( n \).

Fig. 6(a) indicates that the point corresponding to the case where \( b/c \) is large and (or) \( \rho \) is large is above all the critical lines, depending on \( n \). In such cases, defectors dominate cooperators, irrespective of the group size. Fig. 6(a) indicates that the point corresponding to the case where \( b/c \) is small and (or) \( \rho \) is small is below some of the critical lines, depending on \( n \). In such cases, cooperators are not dominated by defectors when the group size is smaller than the critical value.

For some combinations of \( c/b \) and \( \rho \), as the group size increases, the number of internal equilibria turns from 2 to 0. Fig. 6(b) demonstrates that when \( n \) is smaller than the critical value, there are two internal equilibria, one stable and the other unstable. As \( n \) increases, the frequency of cooperators at the unstable equilibrium \( (P_{C, u}) \) increases, while that at the stable equilibrium \( (P_{C, s}) \) decreases. Therefore, as the group size increases, the basin of attractions for the population consisting of only defectors (i.e., \( 0 < P_C < P_{C, u} \)) grows, and the frequency of cooperators in the case where cooperation is established \( (P_{C, s}) \) decreases. When the group size reaches a critical value, the unstable and stable equilibria coalesce (fold bifurcation). When \( n \) is larger than the critical value, cooperators are dominated by defectors, and there are no internal equilibria. Thus, the effect of the group size on the evolution of cooperation is negative.

As observed in the above two cases, the frequency of cooperators at the unstable internal equilibrium \( (P_{C, u}) \) increases as \( n \) increases. Based on (43), it is revealed that when \( \rho \) is around 0, the following relationship holds:

\[
P_{C, u} \approx \rho^{\frac{1}{n}} \left( \frac{b}{c} - \frac{1}{n\rho} \right) \frac{1 - \frac{1}{n\rho}}{1 - \frac{1}{\sqrt[n]{n\rho}}}.
\]

which means that for larger \( n \), \( P_{C, u} \) is larger, when \( \rho \) is around 0.

3.4. Group size effects on cooperation when \( b/(nc) \) is constant (i.e., public good)

This subsection examines the effect of group size \( (n) \) on the evolution of cooperation when \( b/(nc) \) is constant (i.e., public good).

3.4.1. From the perspective of the condition under which cooperators are not dominated by defectors

From (49), the condition under which cooperators are not dominated by defectors is given by

\[
\frac{b}{nc} > j(n, \rho),
\]

where

\[
j(n, \rho) = \frac{1}{n!l(n, \rho)}.
\]

Fig. 7(a), which is derived from (56) and (57), illustrates when defectors dominate cooperators and when they do not, showing that the impact of \( n \) on the critical value of \( b/(nc) \) is not straightforward. Actually, we can get some algebraic results supporting this non-monotonicity in the case of \( \rho \approx 1 \). After algebraic calculation (see supplementary material J), we derive

\[
\lim_{\rho \to 1} j(n, \rho) = 1.
\]

After algebraic calculation (see supplementary material J), we derive

\[
\lim_{\rho \to 1} \frac{1 - j(n, \rho)}{1 - \rho} = \begin{cases} 
\frac{n - 1}{2n^2} & \text{when } n \leq 4 \\
(n - 1)f(1-f^2)(f^{n-2} + (1-f^2)^{n-2}) & \text{when } n \geq 5 
\end{cases}
\]

Here, we denote

\[
g(n) \equiv \lim_{\rho \to 1} \frac{1 - j(n, \rho)}{1 - \rho}.
\]

After algebraic calculation and numerical computation (see Appendix A for details), we have

\[
g(2) = g(3) > g(4) > \lim_{n \to \infty} g(n) > \ldots > g(10) > g(9) > g(8) > g(7) > g(6) > g(5).
\]

Hence, the evolution of cooperation is most likely when \( n = 2 \) or \( n = 3 \), while it is most unlikely when \( n = 5 \). Thus, the impact of group size on the condition under which cooperators are not dominated by defectors is not monotonic.

In Fig. 7(a), it appears that the condition under which cooperators are not dominated by defectors converges as \( n \) increases. Actually, it can
be shown that the limiting values of $j(n, \rho)$ exists. For $\nu > 0$, the following equation can be obtained:

$$u(\nu) \equiv 1 + (e^\nu - v^\nu - 1)\rho. \quad (62)$$

It can be proved (see supplementary material K for proof) that there exists just one $\nu$ satisfying $u(\nu) = 0$ in the range of $\nu > 0$. Here $\nu_1$ denotes $\nu$ satisfying $u(\nu) = 0$. Actually, after algebraic calculation (see supplementary material K), we have

$$\lim_{n \to \infty} j(n, \rho) = \frac{1}{\nu_1}. \quad (63)$$

In addition, after algebraic calculations (see supplementary material L for proof), when $b/(nc)$ is constant, the following equation is obtained:

$$\lim_{n \to \infty} \rho = \frac{1}{1 + e^{-\frac{\rho}{n}}(\frac{b}{nc} - 1)}. \quad (64)$$

Thus, when the group size is sufficiently large, defectors do not dominate cooperators in case of public goods (i.e., $b/(nc)$ is constant), if conditions are satisfied. This result means that the evolution of groupwise cooperation is possible if goods are public goods. This result is in stark contrast with the case of common goods.

### 3.4.2. From the perspective of the positions of internal equilibria

As observed in Fig. 7(a) and confirmed by algebraic calculation, the impact of $n$ on whether cooperators are not dominated by defectors is not monotonic. Besides, (56) means that when $b/(nc)$ is smaller, cooperators are more likely dominated by defectors. In addition, (56) means that when $\rho$ is larger, cooperators are more likely dominated by defectors. Based on the range for group sizes for which cooperators are not dominated by defectors, the combinations for $b/(nc)$ and $\rho$ can be classified into the three scenarios.

Fig. 7(a) indicates that the point corresponding to the case where $b/(nc)$ is small and (or) $\rho$ is large is below all the critical lines, depending on $n$ (i.e., the group size). This means that when $b/(nc)$ is small and (or) $\rho$ is large, cooperators are dominated by defectors and there are no internal equilibria, irrespective of the group size.

Fig. 7(a) indicates that a point, corresponding to a combination for $b/(nc)$ and $\rho$, is below a critical line corresponding to a group size and upper a critical line corresponding to another group size. For a combination for $b/(nc)$ and $\rho$, no internal equilibria exist and cooperators are dominated by defectors for a group size, whereas there exists internal equilibria and cooperators are not dominated by defectors for another group size. Fig. 7(b)(c) illustrates how the positions of internal equilibria are influenced by $n$. Actually, in Fig. 7(b), this can be confirmed. In Fig. 7 (b), it is observed that for some group sizes ($n = 2$ or $6 \leq n \leq 12$), there are no internal equilibria (and defectors dominate cooperators). For some group sizes ($n = 3, 4, 5$), there are two internal equilibria. For other group sizes ($13 \leq n \leq 20$), there are four internal equilibria.

Fig. 7(a) indicates that the point corresponding to the case where $b/(nc)$ is large and (or) $\rho$ is small is upper all the critical lines, depending on $n$ (i.e., the group size). This means that when $b/(nc)$ is large and (or) $\rho$ is small, cooperators are not dominated by defectors, and there are internal equilibria, irrespective of the group size. In Fig. 7(c), it is observed that when group size is smaller than the critical value, there are two internal equilibria, while when it is at the critical value, a blue-sky bifurcation occurs and two additional internal equilibria emerge.

![Fig. 7](image-url)
Thus, when group size is larger than the critical value, there are four internal equilibria based on the frequency of cooperators achieved, as follows: unstable (lowest frequency), stable (second-lowest), unstable (third-lowest), and stable (highest).

In Fig. 7(b)(c), it is observed that when group size increases, the frequencies of cooperators at the lower unstable and stable equilibria ($P_{C_{1,l}}$ and $P_{C_{1,s}}$, respectively) converge to zero, and the frequencies of cooperators at the higher unstable and stable equilibria ($P_{C_{1,h}}$ and $P_{C_{1,s}}$, respectively) converge to one.

Fig. 7(d) illustrates how the positions of internal equilibria and $P_{C_1}$ are influenced by $n$. In Fig. 7(d), it is observed that as $n$ increases, $P_{C_1}$ approaches 0.

Actually, after algebraic calculation (see supplementary material M), we can derive

$$\lim_{n \to \infty} P_{C_{1,l}} = 0. \quad (65)$$

$$\lim_{n \to \infty} P_{C_{1,s}} = 0. \quad (66)$$

$$\lim_{n \to \infty} P_{C_{1,h}} = 0. \quad (67)$$

$$\lim_{n \to \infty} P_{C_1} = 1. \quad (68)$$

How do $P_{C_{1,l}}$, $P_{C_{1,s}}$, and $P_{C_1}$ converge to zero? When

$$\rho < \lim_{n \to \infty} \rho' = \frac{1}{1 + e^{-\frac{1}{n}(\frac{1}{b+c} - \frac{1}{b'c'})}}$$

in the range of $v > 0$, there exist exactly two $v$ so that $\frac{1}{b} - 1 = v\frac{1 - \rho'(v)^{-\frac{1}{\rho}}}{\rho'(v)^{-\frac{1}{\rho}}}$ is satisfied (see supplementary material M for proof). $v_i$ denotes a smaller $v$ satisfying $\frac{1}{b} - 1 = v_i\frac{1 - \rho'(v_i)^{-\frac{1}{\rho}}}{\rho'(v_i)^{-\frac{1}{\rho}}}$, while $v_s$ denotes a larger $v$ satisfying the aforementioned equation. Here, after algebraic calculation (see supplementary material M), we can derive

$$\lim_{n \to \infty} nP_{C_{1,l}} = v_s + 1 - \frac{1}{\rho}, \quad (69)$$

$$\lim_{n \to \infty} nP_{C_{1,s}} = v_s + 1 - \frac{1}{\rho}, \quad (70)$$

$$\lim_{n \to \infty} nP_{C_1} = 1. \quad (71)$$

In Fig. 7(b)(c), it is observed that the impact of group size on $P_{C_1}$ is not monotonic (see Appendix B for algebraic analysis supporting this non-monotonicity). Here, the basin of attraction for the population consisting of only defectors is $0 < P_c < P_{C_1}$. From the non-monotonicity, the group size impact on the basin of attraction for the population consisting of only defectors is not monotonic in the case of public goods.

4. Discussion

Kriván and Cressman (2020) have investigated the evolutionary dynamics of cooperation. Kriván and Cressman (2020) have shown that for sufficiently large number of rounds, the best opting out rule is the one in which the heterogeneous groups are disbanded and the homogeneous groups are not disbanded. The present study investigated the dynamics between the cooperators and defectors under the opting out rule, in more detail. While Kriván and Cressman (2020) studied the finite population, this paper studied the infinite population. In this model setting of infinite population, not only numerical computation but also algebraic analysis is possible. Therefore, we can discuss the case where the parameter values including the group size are general. Owing to this, we noticed that while there are at most two internal equilibria when the group size is equal to or smaller than four, there exists a combination for $\rho$ and $b/c$ that allows the emergence of four internal equilibria when the group size is larger than four. Thus, the cases in which group size is $\leq 4$ or $> 4$ are qualitatively different. Dunbar (1993) stated a mean group size in apes of approximately 60 individuals and predicted a mean group size in humans of approximately 150 individuals. As confirmed in Fig. 2(d) (e) (see also (22)), although there exists a combination for $\rho$ and $b/c$ that allows the emergence of two internal equilibria, such regions are very tiny (almost invisible), when the group size is 60 or 150. Thus, the cases in which group size is $\leq 4$ or $= 60$ or 150 are very different. In the present study, it was revealed that the evolution of cooperation is less likely as the group size increases (i.e., the group size effect is negative) when goods are common, while the impact of the group size on the evolution of cooperation is not straightforward (i.e., can be positive) in the case of public goods.

This study examined the dynamics in an $n$-player game played by cooperators and defectors. The results showed that there are stable coexistences between cooperators and defectors when the cost-to-benefit ratio is sufficiently small and (or) the number of interactions is large enough; however, even in such cases, the population consisting of only defectors is stable against invasions by rare cooperators. Thus, opting out allows cooperation to be maintained once it is established; however, the rule does not facilitate the initial evolution of cooperation, which requires another mechanism.

Kriván and Cressman (2020) revealed that the best opting out rule is the one in which the groups voluntarily stay together between rounds if and only if the groups are homogeneous (i.e., either all cooperators or all defectors) when the number of rounds is sufficient, while the best opting out rule is not so when the number of rounds is insufficient. The present study investigated the dynamics between cooperators and defectors in the above-mentioned opting out rule. However, this rule is the best only when the number of rounds is sufficient, and when this is not the case, another opting out rule, in which heterogeneous groups do not dissolve and the players belonging to such groups continue to interact, can be the best one (Kriván and Cressman, 2020). Moreover, the threshold of the number of rounds tends to increase as the group size increases, at least when the group size is small (two to six) (Kriván and Cressman, 2020). If this tendency is present with a larger group size, it may seem unlikely that this opting out rule would remain the best one when the size is approximately 60 or 150, which is the mean group size among apes or humans, respectively. Further research should be conducted to determine the best opting out rule when group size is large.

Data Availability

No data was used for the research described in the article.

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Appendices

Appendix A Proof for (61)

Based on (59) and (60), the following equation can be obtained:
\[ g(n) = \begin{cases} 
\frac{n - 1}{2^n} & \text{when } n \leq 4 \\
(n - 1)f'(1 - f')(f^{n-2} + (1 - f')^{n-2}) & \text{when } n \geq 5 
\end{cases}. \tag{A.1} \]

Using (A.1), the following equation can be derived:

\[ g(2) = \frac{1}{2} \] \tag{A.2}

\[ g(3) = \frac{1}{2} \] \tag{A.3}

\[ g(4) = \frac{3}{8} \] \tag{A.4}

After algebraic analysis (see supplementary material N for proof), the following are obtained:

\[ g(5) = \frac{1}{3} \] \tag{A.5}

\[ g(6) = \frac{5}{27} (-14 + 5\sqrt{10}) \] \tag{A.6}

\[ g(7) = \frac{2}{45} (-5 + 4\sqrt{10}) \] \tag{A.7}

After algebraic analysis (see supplementary material N for proof), the following is obtained:

\[ \lim_{n \to \infty} g(n) = e^{-1}. \] \tag{A.8}

Using (A.2), (A.3), (A.5), (A.6), (A.7), and (A.8), the following equation can be obtained:

\[ g(2) = g(3) > g(4) > \lim_{n \to \infty} g(n) > g(7) > g(6) > g(5). \] \tag{A.9}

In addition, we derive

\[ \lim_{n \to \infty} (e^{-1} - \lim_{\rho \to 1} \frac{1 - j(n, \rho)}{1 - \rho}) = \frac{1}{2} e^{-1}. \] \tag{A.10}

From (60) and (A.10), the following equation can be obtained:

\[ \lim_{n \to \infty} (e^{-1} - g(n)) = \frac{1}{2} e^{-1}. \] \tag{A.11}

Based on (A.11), it is revealed that when \( n \) is very large, the following relationship holds:

\[ g(n) \approx e^{-1} \left(1 - \frac{1}{2^n}\right). \] \tag{A.12}

which indicates that \( g(n) \) increases as \( n \) increases when \( n \) is very large. Numerical computation (see supplementary material O) also suggests that for \( n \geq 7 \), the following relationship holds:

\[ g(n + 1) > g(n). \] \tag{A.13}

From (A.9) and (A.13), we have (61). This is the end of the proof.

**Appendix B The impact of group size on \( P_{C_{\rho, \lambda}} \)**

Actually, after algebraic calculation, we can find both positive impacts and negative impacts. Firstly, we introduce positive impacts. Based on (43), it is revealed that when \( \rho \) is around 0, the following relationship holds:

\[ P_{C_{\rho, \lambda}} \approx \rho\left(\frac{\frac{1}{2} - \frac{1}{n\lambda}}{1 - \frac{1}{n\lambda}}\right)^{\frac{n}{\lambda}} \] \tag{B.1}

which means that for larger \( n \), \( P_{C_{\rho, \lambda}} \) is larger. Thus, the positive impact of group size on \( P_{C_{\rho, \lambda}} \) is found.

Secondly, we introduce negative impacts. Using (32), the following equation can be obtained:

\[ \lim_{\lambda \to \infty} \frac{P_{C_{\rho, \lambda}}}{1 - \frac{1}{2^n}} = \frac{\rho}{(n - 1)(1 - \rho)} \] \tag{B.2}
From (B.2), it is derived that when $b_{nc}$ is around 1:

$$P_{C_u} \approx \left(1 - \frac{b}{mc}\right)\frac{\rho}{(n-1)(1-\rho)}.$$  \hspace{1cm} (B.3)

which means that for larger $n$, $P_{C_u}$ is smaller. Thus, the negative impact of group size on $P_{C_u}$ is found.

Based on (69), when $n$ is very large, the following relationship holds:

$$P_{C_u} \approx \frac{\nu_s + 1 - \frac{1}{n}}{n}.$$  \hspace{1cm} (B.4)

which means that for larger $n$, $P_{C_u}$ is smaller. Thus, the negative impact of group size on $P_{C_u}$ is found.

Appendix C Supporting information

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.beproc.2022.104754.

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