Entangled quantum state discrimination using a pseudo-Hermitian system

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Abstract
We demonstrate how to discriminate two non-orthogonal, entangled quantum states which are slightly different from each other by using a pseudo-Hermitian system. The positive definite metric operator which makes the pseudo-Hermitian system fully consistent with quantum theory is used for such a state discrimination. We further show that non-orthogonal states can evolve through a suitably constructed pseudo-Hermitian Hamiltonian to orthogonal states. Such an evolution ceases at exceptional points of the pseudo-Hermitian system.

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(Some figures may appear in colour only in the online journal)

Suppose a particular quantum system is described by two states, \(|\psi_1\rangle\) and \(|\psi_2\rangle\), which are non-orthogonal \((\langle\psi_1 | \psi_2\rangle) \neq 0\) and differ slightly \(|\langle\psi_1 | \psi_2\rangle|^2 \approx 1 - O(\epsilon^2)\), \(\epsilon \ll 1\). At any instant of time, the system will be either in state \(|\psi_1\rangle\) or in state \(|\psi_2\rangle\). However, it is not possible to determine the state of such a system with a few measurements as \(|\psi_1\rangle\) and \(|\psi_2\rangle\) differ slightly. This problem of quantum state discrimination is very important in quantum information theory [1]. Recently Bender et al have proposed an alternative approach [2] to discriminate two pure quantum states using the idea of PT-symmetric non-Hermitian quantum theory [3–12, 17–21]. In PT-symmetric non-Hermitian theories, one needs to construct a CPT-inner product [12] to have a fully consistent quantum theory with unitary time evolution [5]. The same inner product has been used to discriminate such states [2]. However, their discussion was restricted to two pure states. In this article we would like to extend the work in [2] for two entangled quantum states using a pseudo-Hermitian system [13–16]. We show that like a PT-symmetric non-Hermitian system, a pseudo-Hermitian quantum system can also be useful for discriminating entangled quantum states. Every pseudo-Hermitian system leads to a fully consistent quantum theory with unitary time evolution in a different Hilbert space endowed with a positive definite inner product [15, 16]. In the same Hilbert space the states \(|\psi_1\rangle\) and \(|\psi_2\rangle\) are shown to become orthogonal. In an alternative approach we show that the non-orthogonal states \(|\psi_1\rangle\) and \(|\psi_2\rangle\) evolve in time in the usual Hilbert space through a
suitably constructed pseudo-Hermitian system to become orthogonal to each other and hence are discriminated. However, one needs to find an appropriate Hamiltonian which is responsible for such an evolution. In this work we show that a pseudo-Hermitian Hamiltonian suitably defined in a Hilbert space can serve the purpose. The states $|\psi_1\rangle$ and $|\psi_2\rangle$ have a unitary time evolution through such a pseudo-Hermitian Hamiltonian to become orthogonal to each other. Such a unitary time evolution not surprisingly breaks at the possible exceptional points [22–30] of the pseudo-Hermitian system. We consider the following entangled states,

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \cos \frac{\theta}{2} |0, 1/2\rangle + \frac{1}{\sqrt{2}} \sin \frac{\theta}{2} |0, -1/2\rangle;$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \cos \frac{\theta + 2\epsilon}{2} |0, 1/2\rangle + \frac{1}{\sqrt{2}} \sin \frac{\theta + 2\epsilon}{2} |0, -1/2\rangle;$$

where $\epsilon$ is a very small quantity and $|\langle \psi_1 | \psi_2 \rangle|^2 \cong 1 - \epsilon^2$. We have used the notation $|n, \frac{1}{2}m_i\rangle$, where $n$ is the eigenvalue for the number operator $a^{\dagger}a$, i.e. $a^{\dagger}a|n, \frac{1}{2}m_i\rangle = n|n, \frac{1}{2}m_i\rangle$ and $m_i = \pm 1$ are the eigenvalues of the operator $\sigma_z$, i.e. $\sigma_z|n, \frac{1}{2}m_i\rangle = m_i|n, \frac{1}{2}m_i\rangle$. For the sake of simplicity we take $\theta = \pi/2 - \epsilon$, so the states can be written as

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \cos \frac{\pi - 2\epsilon}{4} |0, 1/2\rangle + \frac{1}{\sqrt{2}} \sin \frac{\pi - 2\epsilon}{4} |0, -1/2\rangle;$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \cos \frac{\pi + 2\epsilon}{4} |0, 1/2\rangle + \frac{1}{\sqrt{2}} \sin \frac{\pi + 2\epsilon}{4} |0, -1/2\rangle.$$  

These two non-orthogonal states can be made orthogonal with help of the metric operator associated with the pseudo-Hermitian theory described by

$$H = \mu \sigma \cdot B + \hbar a^{\dagger}a + \rho (\sigma_+ a - \sigma_- a^{\dagger}),$$

where $\rho$ is an arbitrary real parameter [14].

This system describes a spin 1/2 particle in the external magnetic field $\mathbf{B}$ coupled to a simple harmonic oscillator through some non-Hermitian interaction. Here $\sigma_i$ are Pauli spin matrices, $\sigma_\pm = 1/2(\sigma_x + i\sigma_y)$ are spin projection operators. $a, a^{\dagger}$ are the usual creation and annihilation operators for the simple harmonic oscillator states. $a|n\rangle = \sqrt{n}|n-1\rangle; a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$ where $|n\rangle$ represents the eigenvectors for a simple harmonic oscillator. Without losing any essential features of the system, we can choose the external magnetic field along the $z$-direction; the Hamiltonian then changes to

$$H = \frac{\epsilon}{2} \sigma_z + \hbar \omega a^{\dagger}a + \rho (\sigma_+ a - \sigma_- a^{\dagger}),$$

with $\epsilon = 2\mu B_z$. It can be checked that this non-Hermitian $(H \not= H^\dagger)$ Hamiltonian is pseudo-Hermitian with respect to two different operators parity ($P$) and $\sigma_z$ i.e. $H^\dagger = \mathcal{P}H\mathcal{P}^{-1}$ and $H^\dagger = \sigma_z H \sigma_z^{-1}$ [14]. The ground state of the system is $|0, -1/2\rangle$ with energy eigenvalue $-\epsilon/2$

$$H|0, -1/2\rangle = -\epsilon/2|0, -1/2\rangle,$$

but $|0, 1/2\rangle$ is not an eigenstate of this system. However, the states $|0, 1/2\rangle$ and $|1, -1/2\rangle$ create an invariant subspace in the space of states as

$$H|0, 1/2\rangle = \epsilon/2|0, 1/2\rangle - \rho|1, -1/2\rangle;$$

$$H|1, -1/2\rangle = \rho|0, 1/2\rangle + (\hbar \omega - \epsilon/2)|1, -1/2\rangle.$$  

(6)

Eigenvalues and eigenfunctions of the corresponding Hamiltonian matrix in this subspace are the eigenvalues and eigenfunctions for the first and second excited states of the system,
respectively. Similarly the third and fourth excited states of this system, which can be obtained from the Hamiltonian matrix in the next invariant subspace, consist of the states |1, 1/2⟩ and |2, −1/2⟩. The action of the Hamiltonian in equation (4) on the state |1, 1/2⟩ is

\[ H|1, 1/2⟩ = (\hbar \omega + \varepsilon/2)|1, 1/2⟩ - \sqrt{2}\rho|2, -1/2⟩. \] (7)

A general invariant subspace consists of |n, 1/2⟩ and |n + 1, −1/2⟩, n = 0, 1, 2... and the Hamiltonian matrix corresponding to this invariant subspace is

\[ H_{n+1} = \begin{pmatrix} \varepsilon/2 + n\hbar \omega & \rho \sqrt{n+1} \\ -\rho \sqrt{n+1} & -\varepsilon/2 + (n+1)\hbar \omega \end{pmatrix}. \] (8)

The eigenvalues of the Hamiltonian matrix are given by

\[ \lambda_{n+1}^+ = \frac{1}{2}(2n + 1)\hbar \omega \pm \sqrt{(\hbar \omega - \varepsilon)^2 - 4\rho^2(n+1)}. \] (9)

These eigenvalues are real provided \((\hbar \omega - \varepsilon) \geq 2\rho\sqrt{n+1}\). Putting \((\hbar \omega - \varepsilon) \sin \alpha = 2\rho\sqrt{n+1}\) the eigenvectors are [14]

\[ \psi_{n+1}^+ = \begin{pmatrix} \sin \alpha/2 \\ \cos \alpha/2 \end{pmatrix}, \]
\[ \psi_{n+1}^- = \begin{pmatrix} \cos \alpha/2 \\ -\sin \alpha/2 \end{pmatrix}. \] (10)

The eigenvectors and eigenvalues of \(H_{n+1}\) coalesce at \(\alpha = \pi/2\), which is the exceptional point of the pseudo-Hermitian system described by the Hamiltonian in equation (4). The positive definite metric operator for this system is then calculated using the spectral method [15] as

\[ \eta = |\psi_{n+1}^+⟩⟨\psi_{n+1}^+| + |\psi_{n+1}^-⟩⟨\psi_{n+1}^-|, \] (11)

where \(\psi_{n+1}^\pm\) are eigenvectors of \(H_{n+1}^\dagger = H_{n+1}\ (\rho \rightarrow -\rho)\)

\[ \psi_{n+1}^+ = \begin{pmatrix} \cos \alpha/2 \\ -\sin \alpha/2 \end{pmatrix}, \]
\[ \psi_{n+1}^- = \begin{pmatrix} -\sin \alpha/2 \\ \cos \alpha/2 \end{pmatrix}. \] (12)

Substituting these in equation (11) we obtain the metric operator as

\[ \eta = \begin{pmatrix} 1 & -\sin \alpha \\ -\sin \alpha & 1 \end{pmatrix}, \] (13)

which can be expressed using \(\sigma_{\pm}\) as

\[ \eta = I - \sigma_+ \sin \alpha - \sigma_- \sin \alpha. \] (14)

Exactly the same metric operator can also be calculated using [16]. Comparison of different methods for calculating \(\eta\) is reported in [31]. The pseudo-Hermitian system (described by the \(H_{n+1}\) given in equation (8)) is a fully consistent quantum theory in the modified Hilbert space endowed with the metric in equation (13). Now we show that the non-orthogonal entangled quantum states |ψ₁⟩ and |ψ₂⟩ become orthogonal in this modified Hilbert space endowed with the positive definite metric \(\eta\). We explicitly calculate the inner product ⟨ψ₁ | ψ₂⟩\(\eta\) in this Hilbert space, which vanishes subject to the condition \(\sin \alpha = \cos \epsilon\). Thus, two arbitrary non-orthogonal entangled quantum states (equation (2)) are discriminated by using a suitably constructed pseudo-Hermitian system (equation (4)).
The normalized eigenvectors dual to $|\psi_1\rangle$ and $|\psi_2\rangle$ in the new Hilbert space endowed with metric operator $\eta$ are given by

$$
\eta|\psi_1\rangle = \frac{1}{\sqrt{2} \sin \epsilon} \left( \cos \frac{\pi - 2\epsilon}{4} - \sin \frac{\pi - 2\epsilon}{4} \cos \epsilon \right) [0, 1/2] + (1, -1/2] \\
+ \frac{1}{\sqrt{2} \sin \epsilon} \left( \sin \frac{\pi - 2\epsilon}{4} - \cos \frac{\pi - 2\epsilon}{4} \cos \epsilon \right) [(0, -1/2] + (1, 1/2]
$$

$$
\eta|\psi_2\rangle = \frac{1}{\sqrt{2} \sin \epsilon} \left( \sin \frac{\pi + 2\epsilon}{4} - \cos \frac{\pi + 2\epsilon}{4} \cos \epsilon \right) [0, 1/2] + (1, -1/2] \\
+ \frac{1}{\sqrt{2} \sin \epsilon} \left( \cos \frac{\pi + 2\epsilon}{4} - \sin \frac{\pi + 2\epsilon}{4} \cos \epsilon \right) [(0, -1/2] + (1, 1/2].
$$

Here we have used equation (14), the relations, $\sigma_+, |n, 1/2\rangle = 0$; $\sigma_- |n, -1/2\rangle = 0$; $\sigma_+ |n, -1/2\rangle = |n, 1/2\rangle$; $\sigma_- |n, 1/2\rangle = |n, -1/2\rangle$ and the orthonormality condition $\langle n_1, \frac{1}{2}m_{n_1} | n_2, \frac{1}{2}m_{n_2}\rangle = \delta_{n_1n_2}\delta_{m_{n_1}m_{n_2}}$, for $n = 0, 1, 2 \ldots$. The normalization factor in the above equation is calculated as $\langle \psi_1 | \psi_1 \rangle = \sin^2 \epsilon = \langle \psi_2 | \psi_2 \rangle$. 

However, to construct the projection operator we need to consider both states of the invariant subspace of the system described by $H_{n+1}$. In this case, each invariant sub-Hilbert space consists of two states. Therefore we consider the other state as

$$
|\psi_3\rangle = \frac{1}{\sqrt{2}} \sin \frac{\theta}{2}[0, 1/2] + (1, -1/2] + \frac{1}{\sqrt{2}} \cos \frac{\theta}{2}[0, -1/2] + (1, 1/2].
$$

The non-orthogonal state which is slightly different from $|\psi_3\rangle$ is

$$
|\psi_4\rangle = \frac{1}{\sqrt{2}} \sin \frac{\theta - 2\epsilon}{2}[0, 1/2] + (1, -1/2] + \frac{1}{\sqrt{2}} \cos \frac{\theta - 2\epsilon}{2}[0, -1/2] + (1, 1/2].
$$

It is interesting to see that these two non-orthogonal entangled states are also discriminated by the same metric operator $\eta$ in equation (13). Now we construct the projection operator $P = \sum_{i=1}^{4} P_i$, where $P_i$ are calculated as

$$
P_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{4 \sin \epsilon} \left[ (1 + \sin \epsilon)(A) + (\sin \epsilon - 1)(B) - (\cos \epsilon)(C) + (\cos \epsilon)(D) \right];
$$

$$
P_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{4 \sin \epsilon} \left[ (\sin \epsilon - 1)(A) + (\sin \epsilon + 1)(B) + (\cos \epsilon)(C) - (\cos \epsilon)(D) \right];
$$

$$
P_3 = |\psi_3\rangle\langle\psi_3| = \frac{1}{4 \sin \epsilon} \left[ (\sin \epsilon - 1)(A) + (\sin \epsilon + 1)(B) + (\cos \epsilon)(C) - (\cos \epsilon)(D) \right];
$$

$$
P_4 = |\psi_4\rangle\langle\psi_4| = \frac{1}{4 \sin \epsilon} \left[ (1 + \sin \epsilon)(A) + (\sin \epsilon - 1)(B) - (\cos \epsilon)(C) + (\cos \epsilon)(D) \right].
$$

where $A, B, C, D$ are denoted as below

$$
A = [0, \frac{1}{2}](0, \frac{1}{2}) + [0, \frac{1}{2}](1, -\frac{1}{2}) + [1, -\frac{1}{2}](0, \frac{1}{2}) + [1, -\frac{1}{2}](1, -\frac{1}{2});
$$

$$
B = [0, -\frac{1}{2}](0, -\frac{1}{2}) + [0, -\frac{1}{2}](1, \frac{1}{2}) + [1, \frac{1}{2}](0, -\frac{1}{2}) + [1, \frac{1}{2}](1, \frac{1}{2});
$$

$$
C = [0, \frac{1}{2}](0, -\frac{1}{2}) + [0, \frac{1}{2}](1, \frac{1}{2}) + [1, -\frac{1}{2}](0, -\frac{1}{2}) + [1, -\frac{1}{2}](1, \frac{1}{2});
$$

$$
D = [0, -\frac{1}{2}](0, \frac{1}{2}) + [0, -\frac{1}{2}](1, -\frac{1}{2}) + [1, \frac{1}{2}](0, \frac{1}{2}) + [1, \frac{1}{2}](1, -\frac{1}{2}).
$$

It is straightforward to check that $P = \sum_{i=1}^{4} P_i = I$. The projection operator $P_1$ (or $P_2$) can be used to discriminate the states $|\psi_1\rangle$ and $|\psi_2\rangle$. On the other hand, application of either $P_3$ or $P_4$ discriminates $|\psi_3\rangle$ from $|\psi_4\rangle$. $P_i$ projects the states $|\psi_i\rangle$ for $i = 1, 2, 3, 4$.

In an alternative method we show that these states can be discriminated through time evolution in the usual Hilbert space. Two non-orthogonal states which differ slightly at
some initial time can evolve through a suitably chosen pseudo-Hermitian Hamiltonian to two orthogonal states at later time. We start with non-orthogonal states \( |\psi_1, \psi_2 \rangle \) or \( |\psi_3, \psi_4 \rangle \) at \( t = 0 \) i.e. \( \langle \psi_1(t = 0) | \psi_2(t = 0) \rangle \neq 0 \neq \langle \psi_3(t = 0) | \psi_4(t = 0) \rangle \). Here, the standard Dirac inner product rule, i.e. complex conjugate and transpose, is used. The pseudo-Hermitian system described by the Hamiltonian \( H \) in equation (4) can be used to discriminate the states \( |\psi_1 \rangle \) and \( |\psi_2 \rangle \) at some later time. The states \( |\psi_1 \rangle \), \( |\psi_2 \rangle \) evolve in the usual Hilbert space through the pseudo-Hermitian \( H \) in equation (8) to two orthogonal states. To show this we write the Hamiltonian matrix in equation (8) with \( n = 0 \) as

\[
H = \frac{1}{2} \hbar \omega \mathbf{I} + \vec{\sigma} \cdot \vec{k}_1,
\]

and the Hermitian conjugate of the above Hamiltonian as

\[
H^\dagger = \frac{1}{2} \hbar \omega \mathbf{I} + \vec{\sigma} \cdot \vec{k}_2
\]

where the components of \( \vec{k}_1 \) and \( \vec{k}_2 \) are \( [0, i \rho, \frac{1}{2}(\epsilon - \hbar \omega)] \) and \( [0, -i \rho, \frac{1}{2}(\epsilon - \hbar \omega)] \), respectively. Now we compute the inner product of the states \( |\psi_1(t) \rangle \) and \( |\psi_2(t) \rangle \) at later time \( t \) as

\[
\langle \psi_1(t) | \psi_2(t) \rangle = \langle \psi_1(t = 0) | e^{iH^\dagger t} e^{-iH t} | \psi_2(t = 0) \rangle
\]

where \( H^\dagger \) and \( H \) are given in equations (20) and (21). Using the identity \( e^{i\vec{\sigma} \cdot \hat{n} \phi} = \cos \phi \mathbf{I} + i(\vec{\sigma} \cdot \hat{n}) \sin \phi \) we calculate

\[
e^{iH^\dagger t} e^{-iH t} \text{ in the matrix form}
\]

\[
\begin{pmatrix}
\cos^2 \beta \cos^2 \alpha + \sin^2 \beta t (1 + \sin^2 \alpha) & \sin 2 \beta t \sin \alpha (-i \cos \alpha - \sin \beta t) \\
\sin 2 \beta t \sin \alpha (i \cos \alpha - \sin \beta t) & \cos^2 \beta \cos^2 \alpha + \sin^2 \beta t (1 + \sin^2 \alpha)
\end{pmatrix}
\]

Figure 1. The time-evolution of the non-orthogonal states with respect to the parameter in the pseudo-Hermitian system for different values of the small parameter \( \epsilon \). The time evolution breaks down at \( \alpha = \pi/2 \), which is the exceptional point of the pseudo-Hermitian theory.
where $\beta = \sqrt{-\rho^2 + \frac{1}{4}(\epsilon - \hbar \omega)^2}$. Substituting equation (23) in the rhs of equation (22) we obtain $\langle \psi_1(t) | \psi_2(t) \rangle = 0$ if
\[
\sin^2 \beta t = \left\{ -4 \cos \alpha (1 - 3 \cos 2\alpha) \cos \epsilon \cot \alpha + (1 - \cos 2\epsilon - 4 \sin \alpha) \cos \alpha \\
+ (-4 \cos^2 \alpha \sin^2 \alpha (\cos \epsilon - \sin \alpha)^2 (\cos 2\alpha + \cos \epsilon \sin \alpha)^2 - 4 \cos^2 \alpha \sin \alpha) \\
+ \frac{1}{16} \left[ (\cos 2\epsilon + 4 \sin \alpha - 1) \sin^2 2\alpha + 2 \cos^2 \alpha \cos \epsilon (3 \sin 3\alpha - 5 \sin \alpha)^2 \right]^2 \right\} \\
\cdot (2(\cos 2\alpha + \cos \epsilon \sin \alpha)^2 - 4 \cos^2 \alpha \sin \alpha))^{-1}.
\]

This equation has a definite solution for $t$ except at $\alpha = \pi/2$ (figure 1), which is the exceptional point in the spectrum of the pseudo-Hermitian system used for time evolution.

Conclusions

Discriminating two non-orthogonal states plays a very important role in quantum information theory. We have shown that two non-orthogonal entangled quantum states which are slightly different from each other can be discriminated by using suitably constructed pseudo-Hermitian systems. The pseudo-Hermitian system becomes a fully consistent quantum theory in a different Hilbert space endowed with a positive definite metric. The same metric operator has been used to discriminate the non-orthogonal entangled states. Alternatively, we have shown that quantum entangled states which are non-orthogonal at $t = 0$ can evolve in the usual Hilbert space through a suitably chosen pseudo-Hermitian Hamiltonian to orthogonal states. Such time evolutions are obstructed by the existence of exceptional points of the pseudo-Hermitian system. We have demonstrated all these features by considering an explicit example.

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