CONVEXITY ESTIMATES FOR MEAN CURVATURE FLOW
WITH FREE BOUNDARY

NICK EDELEN

Abstract. We prove the estimates of [HS99b] and [HS99a] for finite-time singularities of mean-convex, mean curvature flow with free boundary in a barrier $S$. Here $S$ can be any properly embedded, oriented surface in $\mathbb{R}^{n+1}$ of bounded geometry. We also prove the estimate [Hui84] in the case of convex flows and $S = S^n$, which gives an alternative proof to [Sta96a].

1. Introduction

We are interested in immersed, mean-convex, mean curvature flow with free boundary in a surface $S$. We reprove the estimates in [HS99b], and [HS99a] for this class of flows. These provide very direct, general pinching results for limit flows at singularities, and require no embeddedness or curvature assumptions. We further prove the estimates in [Hui84] when $S$ is the sphere.

Consider a smooth, properly embedded, oriented hypersurface $S \subset \mathbb{R}^{n+1}$, with choice of normal $\nu_S$ and bounded geometry. We refer to $S$ as the barrier surface. If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a compact, mean-convex hypersurface with boundary, we say $\Sigma$ meets $S$ orthogonally if $\partial \Sigma \subset S$, and the outer normal of $\partial \Sigma \subset \Sigma$ coincides with $\nu_S$.

Let $\Sigma_0 = \Sigma$ meet the barrier $S$ orthogonally. Then the mean curvature flow of $\Sigma_0$, with free-boundary in $S$, is a family of immersions $F_t : \Sigma_0 \times [0, T) \to \mathbb{R}^{n+1}$ such that

$$\partial_t F_t = -H \nu, \text{ for all } p \in \Sigma, t > 0$$

$$F_t(\Sigma) \text{ meets } S \text{ orthogonally for all } t \geq 0$$

$$F_0 \equiv \text{Id}_{\Sigma_0}.$$ 

Here $H$ is the mean curvature, and $\nu$ the outer normal, oriented so that $H = -H \nu$ is the mean curvature vector. We often write $\Sigma_t = F_t(\Sigma)$, and will equivocate between the surface and its immersion.

It was shown by Stahl [Sta96a] that the mean curvature flow with free-boundary in $S$ always exists on some maximal time interval $[0, T)$, with the property $\max_{\Sigma_t} |A| \to \infty$ as $t \to T$. Here $|A|$ is the norm of the second fundamental form $A$.

Type-I tangent flows of mean curvature flow with free boundary have been classified by Buckland [Buc05]. Our convexity estimates work towards classifying type-II limit flows with free boundary. Stahl [Sta96a] has shown Theorem 1.6 using a different method.

We prove the following theorems concerning the mean curvature flow of $\Sigma_0$ with free-boundary in $S$. Throughout the duration of this paper we assume $\Sigma_0$ is compact, mean-convex.
Theorem 1.1. There are constants \( \alpha = \alpha(S, \Sigma_0) \geq 0 \) and \( C = C(S, \Sigma_0) \) so that
\[
\max_{\Sigma_t} \frac{|A|}{H} \leq C e^{\alpha t}
\]
for all time of existence. In particular, if \( T < \infty \), then \( |A| \leq C(S, \Sigma_0, T)H \).

Definition 1.1.1. Given a vector \( \mu \in \mathbb{R}^n \), and \( k \in \{1, \ldots, n\} \), we let
\[
s_k(\mu) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mu_{i_1} \cdots \mu_{i_k}
\]
be the \( k \)-th symmetric polynomial of \( \mu \). We adopt the convention that \( s_0 \equiv 1 \). If \( s_{k-1}(\mu) \neq 0 \), we let
\[
q_k(\mu) = \frac{s_k(\mu)}{s_{k-1}(\mu)}.
\]

Given a real symmetric \( n \times n \) matrix \( M \), define \( s_k(\mu) = s_k(\mu) \) where \( \mu \in \mathbb{R}^n \) is the vector of eigenvalues of \( M \). Similarly, where possible set \( q_k(M) = q_k(\mu) \). Notice that \( s_k \) is a polynomial in the entries of \( M \).

Given a surface \( \Sigma \), define the smooth function \( S_k \) by
\[
S_k(p) = s_k(A(p)) = s_k(\lambda(p))
\]
where \( \lambda \) the vector of principle curvatures. Similarly where possible set \( Q_k = q_k(A) \). We have that \( H \equiv S_1 \), and \( |A| \equiv S_1^2 - 2S_2 \).

Theorem 1.2 (Convexity pinching). If \( T < \infty \), then for any \( k \in \{1, \ldots, n\} \), \( \eta > 0 \), there is a constant \( C = C(S, \Sigma_0, T, \eta, n) \) such that
\[
S_k \geq -\eta H^k - C
\]
at all points in spacetime.

For \( T < \infty \), by rescaling \( \Sigma_t \) along an essential blow-up sequence (c.f. Section 4 of [HS99b]), we obtain an ancient limit flow \( \tilde{\Sigma}_\tau \) with free boundary in a hyperplane. This can be reflected to a mean curvature flow without boundary. Theorem 4.1 of [HS99a] therefore proves the following Corollary of Theorem 1.2.

Corollary 1.3. If \( T < \infty \), then any limit flow of \( \Sigma_t \) at a type-II singularity is a weakly convex flow \( \Sigma_\tau \) with free boundary in a hyperplane. After reflection to a flow without boundary, either \( \Sigma_\tau \) is a strictly convex translating soliton, or splits as \( \Sigma_\tau = R^{n-k} \times \Sigma^k_\tau \), where \( \Sigma_\tau \) is a strictly convex translating soliton of lower dimension.

Remark 1.4. If \( T = \infty \) then either \( |\Sigma_t| \to 0 \) or by a standard argument \( \Sigma_t \) approaches a minimal surface.

Remark 1.5. Theorems 1.1 and 1.2 also hold in a Riemannian manifold of bounded geometry. In fact the error terms introduced are almost entirely subsumed by the perturbations we already make.

Theorem 1.6 (Umbilic pinching, [Sta96a]). If \( \Sigma_0 \) is convex and \( S = S^n \), then \( \Sigma_t \) shrinks to a point in finite time, and any limit flow at the singularity is umbilic.

Remark 1.7. If \( \Sigma_0 \) is convex and \( S = S^n \), then \( \Sigma_t \) shrinks to a point in finite time, and any limit flow at the singularity is umbilic.
Definition 1.6.1. Writing \( f = O(g) \) means there is a constant \( c = c(n, S) \) such that \( |f| \leq c|g| \).

We outline our approach. The main obstruction to analyzing free boundary behavior in a general barrier \( S \) is obtaining boundary conditions on \( |A| \), or \( S_k \) when \( k > 1 \). We perturb the second fundamental form so that the normal \( \nu_S \) is an eigenvector, which allows us to obtain boundary conditions on the perturbed principle curvatures.

This introduces relatively large error terms into the evolution equations of the perturbed \( |\bar{A}| \) and \( \bar{H} \). The error is too large to naively give exponential behavior of the quantity \( |\bar{A}|/\bar{H} \). To handle this, and to correct the boundary behavior, in proving Theorem 1.1 we must consider instead the evolution of

\[
(2) \quad \frac{|\bar{A}| + a}{\bar{H}} \phi,
\]

for some large constant \( a \), and barrier function \( \phi \).

The evolution equation for (2) will have the right form except for a gradient term resulting from \( \phi \). To control bad gradient terms we observe that by restricting to points where \( |\bar{A}| \geq 2\bar{H} \), we can squeeze a term out of Cauchy’s inequality:

\[
|\nabla \bar{A}|^2 - |\nabla |\bar{A}||^2 \geq \frac{1}{c} |\nabla |\bar{A}||^2 + O(|\bar{A}|^2).
\]

Given Theorem 1.1 we can adapt the Stampacchia iteration scheme used by [HS99a] to prove Theorems 1.2 and 1.6. The key step is proving a trace-like formula for free boundary surfaces. The argument is sufficiently robust to handle without problem the perturbation terms.

I am very grateful to my advisor Simon Brendle for his guidance and encouragement, Brian White for many illuminating discussions, and Otis Chodosh for his support and advice. I also thank Robert Haslhofer and Gerhard Huisken for helpful conversations. This work was partially supported by the Royden fellowship. Some of this work was also completed while visiting Columbia University, and I'm grateful for their hospitality.

2. Michael-Simon with (free) boundary

We adapt the Michael-Simon inequality [MS73] to surfaces with smooth boundary, and surfaces meeting a barrier surface orthogonally.

Lemma 2.1. There is a constant \( c = c(n, S) \) such that for any \( \Sigma \) meeting \( S \) orthogonally, and any \( v \in C^1(\Sigma) \),

\[
(3) \quad \frac{1}{c} \int_{\partial \Sigma} |v| \leq \int_{\Sigma} |\nabla v| + \int_{\Sigma} |Hv| + \int_{\Sigma} |v|.
\]
Proof. Choose (and fix, for the duration of the paper) a smooth vector field $X$ on $\mathbb{R}^{n+1}$ which is 0 outside a neighborhood of $S$, and $X \equiv \nu_S$ on $S$. Then

$$\int_{\partial \Sigma} |v| = \int_{\partial \Sigma} |v|X \cdot \nu$$

$$= \int_{\Sigma} \text{div}_\Sigma(|v|XT)$$

$$= \int_{\Sigma} \nabla|v| \cdot X + |v| \text{div}_\Sigma(X) - |v|X \cdot \nu \mathbb{H}$$

$$\leq \max |X| \int |\nabla v| + n \max |\nabla X| \int |v| + \max |X| \int |v\mathbb{H}| \quad \square$$

**Theorem 2.2.** There is a constant $c = c(n)$ such that for any $v \in C^1(\bar{\Sigma})$, we have

$$\frac{1}{c} \left( \int_{\Sigma} |v|^{\frac{n+1}{n}} \right)^{\frac{n}{n+1}} \leq \int_{\Sigma} |\nabla v| + \int_{\Sigma} |Hv| + \int_{\partial \Sigma} |v|.$$

**Proof.** By replacing $v$ with $|v|$ we can without loss of generality suppose $v \geq 0$.

For $x \in \partial \Sigma$, let $\gamma_x(t)$ be the unit speed geodesic in $\Sigma$ with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) \perp \partial \Sigma$. For sufficiently small $\epsilon$, depending only on the curvatures of $\Sigma$ and $\partial \Sigma \subset \Sigma$, the function $\phi : [0, \epsilon] \times \partial \Sigma \to \Sigma$ mapping $(t, x) \mapsto \gamma_x(t)$ is a diffeomorphism, with its Jacobian bounded like $|J\phi| \in \left[ \frac{1}{2}, 2 \right]$. We deduce, for any $\epsilon$ sufficiently small,

$$\int_{\text{dist}(\cdot, \partial \Sigma) \leq \epsilon} v = \int_0^\epsilon \int_{\partial \Sigma} v|J\phi|$$

$$\leq 2 \int_0^\epsilon \int_{\partial \Sigma} v(0, x) + 2 \int_0^\epsilon \int_{\partial \Sigma} \frac{\partial v}{\partial t}(t^*(x), x)$$

$$\leq 2 \epsilon \int v + \epsilon^2 |\partial \Sigma| \sup_{\Sigma} |\nabla v|$$

(5)

Here $t^*(x) \in (0, \epsilon)$.

Now take $\eta$ a function which is $\equiv 1$ on $\text{dist}(\cdot, \partial \Sigma) \geq \epsilon$ and $\equiv 0$ on $\partial \Sigma$, and such that $|\nabla \eta| \leq 2/\epsilon$. From (3)

$$\int_{\Sigma} ((1 - \eta)v)^{\frac{n+1}{n}} \leq \int_{\text{dist}(\cdot, \partial \Sigma) \leq \epsilon} v^{\frac{n+1}{n}}$$

$$\leq 2 \epsilon \int_{\partial \Sigma} v^{\frac{n+1}{n}} + \epsilon^2 |\partial \Sigma| \sup_{\Sigma} |\nabla (v^{\frac{n+1}{n}})|$$

$$\leq C$$

For $C$ independent of $\epsilon$.

Therefore, using the Michel-Simon inequality and (5) again,

$$||v||^{\frac{n+1}{n}} \leq ||\eta v||^{\frac{n+1}{n}} + ||(1 - \eta)v||^{\frac{n+1}{n}}$$

$$\leq c \int_{\Sigma} \eta |\nabla v| + c \int_{\Sigma} |H| \eta v + c \int_{\Sigma} |\nabla \eta| v + \epsilon^{\frac{n+1}{n}} C$$

$$\leq c \int_{\Sigma} |\nabla v| + c \int_{\Sigma} |H| v + 2c/\epsilon \int_{\text{dist}(\cdot, \partial \Sigma) \leq \epsilon} v + \epsilon^{1/2} C$$

$$\leq c \int_{\Sigma} |\nabla v| + c \int_{\Sigma} |H| v + 4c \int_{\partial \Sigma} v + \epsilon |\partial \Sigma| \sup |\nabla v| + \epsilon^{1/2} C$$
for $c = c(n)$ and all $\epsilon$ sufficiently small. Taking $\epsilon$ to 0 proves the lemma.

\textbf{Theorem 2.3.} If $\Sigma$ meets $S$ orthogonally, and $v \in C^1(\Sigma)$, then for any $p < n,$
\begin{equation}
||v||_p \leq c(||\nabla v||_{p;\Sigma} + ||Hv||_{p;\Sigma} + ||v||_{p;\Sigma})
\end{equation}

where $c = c(n, p, S)$.

\textbf{Proof.} Combine Lemma 2.1 and Theorem 2.2 to obtain the desired inequality with $p = 1$. Then set $v = w^\gamma$ to obtain
\begin{align*}
\left( \int w^{\gamma-1} \right)^{\frac{n-1}{n}} &\leq c \gamma \int w^{\gamma-1} |\nabla w| + c \int w^{\gamma-1} Hw + c \int w^{\gamma-1} w \\
&\leq c \left( w^{\gamma-1} \right)^{\frac{n-1}{n}} (||\nabla w||_p + ||Hw||_p + ||w||_p).
\end{align*}

Now choose $\gamma$ such that
\[\gamma \frac{n}{n-1} = (\gamma - 1) \frac{p}{p-1} \]
\[\square\]

\textbf{Corollary 2.4.} If $n = 2$, then for any $q \in (1, \infty)$,
\begin{equation}
||v||_{2q;\Sigma} \leq c ||\nabla v||_{2;\Sigma} (||\nabla v||_{2;\Sigma} + ||Hv||_{2;\Sigma} + ||v||_{2;\Sigma})
\end{equation}

where $c = c(q, S)$.

\textbf{Proof.} Take $n = 2$ and $p = 2 - \delta$ in Theorem 2.3 for $\delta \in (0, 1)$. Set $q = \frac{2-\delta}{2}$. Then we have for any $r \in (1, \infty)$,
\begin{align*}
||v||_{2q} &\leq c (||\nabla v||_{2-\delta} + ||Hv||_{2-\delta} + ||v||_{2-\delta}) \\
&\leq c ||\nabla v||_{r(2-\delta)} (||\nabla v||_{r(2-\delta)} + ||Hv||_{r(2-\delta)} + ||v||_{r(2-\delta)}).
\end{align*}

Then set $r = \frac{2}{2-\delta}$.
\[\square\]

\textbf{Remark 2.5.} Since $|\nabla v|$ is monotone decreasing (Remark 4.2),
\[c(q, S)|\nabla v|^{1/2q} \leq c(q, S, |\nabla v|).
\]

3. \textbf{General inequalities and Stampacchia iteration}

Each pinching result uses a Stampacchia iteration scheme to obtain pointwise bounds from $L^p$-bounds. All cases can be handled by the following general principle.

Take $(\Sigma_t)_{t \in [0, T]}$ a mean curvature flow with free boundary in $S$, and assume $T < \infty$. Let $f_\alpha$ be some non-negative function on $\Sigma_t$, depending on some parameters $\alpha = \alpha(S, \Sigma_0, T, n)$. Let $\bar{G} \geq 0$ and $\bar{H} > 0$ be functions on $\Sigma_t$ such that
\[H = O(\bar{H}), \quad \nabla \bar{H} = \bar{O}(\bar{G}).\]

Let $f = f_\alpha \bar{H}^\sigma$, and $f_k = (f - k)_+$, where $\sigma > 0$ will be small and $k > 0$ large. Write $A(k) = \{ f \geq k \}$, and $A(k, t) = A(k) \cap \Sigma_t$.

We say $f$ satisfies $(\ast)$ if there are constants $c = c(S, \Sigma_0, T, n, \alpha)$, and $C = C(S, \Sigma_0, T, n, \alpha, p, \sigma)$, such that for any $p > p_0(n, \alpha, c)$, $\sigma < 1/2$, $k > 0$ and $\beta > 0$, the following two equations hold:
\[ \frac{1}{c} \int_{\Sigma_0} f^p \hat{H}^2 \leq (p + p/\beta) \int_{\Sigma_0} f^{p-2} |\nabla f|^2 + (1 + \beta p) \int_{\Sigma_0} \frac{\hat{G}^2}{H^{2-\sigma}} f^{p-1} + \int_{\Sigma_0} f^p \]

\[ + \int_{\partial \Sigma_0} f^{p-1} \hat{H}^\sigma \]

**Evolution-Like**

\[ \partial_t \int_{\Sigma_t} f_k^p \leq -\frac{1}{3} p^2 \int_{\Sigma_t} f_k^{p-2} |\nabla f|^2 - \frac{p}{c} \int_{\Sigma_t} \frac{\hat{G}^2}{H^{2-\sigma}} f_k^{p-1} + cp\sigma \int_{A(k,t)} \hat{H}^2 f_k^p \]

\[ - \frac{1}{5} \int_{\Sigma_t} \hat{H}^2 f_k^p + C \int_{A(k,t)} f^p + C |A(k)| + cp \int_{\partial \Sigma_t} f_k^{p-1} \hat{H}^\sigma \]

This section culminates in proving

**Theorem 3.1.** If \( f \) satisfies (\( \star \)), then for \( p \) sufficiently big, and \( \sigma \) sufficiently small (depending on \( p \)), \( f \) is uniformly bounded in spacetime. The bound will depend on \((S, \Sigma_0, T, n, \alpha, p, \sigma)\).

The following Lemma is the key step in handling the free boundary behavior. We first make a useful observation.

**Remark 3.2.** Let \( g \) be an arbitrary non-negative function on \( \Sigma_t \). If \( r \in (0, 2) \), \( q \in (0, p) \) with \( rp/q < 2 \), then for any \( \mu > 0 \),

\[ \int g^q \hat{H}^r \leq \int f^p \hat{H}^{rp/q} + |\text{spt} \ g| \]

\[ \leq \frac{1}{\mu} \int g^p \hat{H}^2 + C(\mu, r, q, p) \int g^p + |\text{spt} \ g| \]

**Lemma 3.3.** For any \( \mu > 0 \) and \( p > 2 \), we can pick constants \( c = c(n, S) \) and \( C = (n, S, \mu, p) \) such that

\[ \int_{\partial \Sigma_t} f_k^{p-1} \hat{H}^\sigma \leq c \int_{\Sigma_t} |\nabla f|^2 f_k^{p-2} + c\sigma \int_{\Sigma_t} \frac{\hat{G}^2}{H^{2-\sigma}} f_k^{p-1} + \frac{c\mu^2}{\mu} \int_{A(k,t)} f^p \hat{H}^2 \]

\[ + \int_{A(k,t)} f^p + C |A(k)| \]

**Proof.** Using the trace formula of 2.1 and Peter-Paul, we have (all integrals on the right-hand-side are over \( \Sigma_t \))

\[ \int_{\partial \Sigma_t} f_k^{p-1} \hat{H}^\sigma \leq cp \int f_k^{p-2} |\nabla f|^2 + c\sigma \int f_k^{p-1} |\nabla \hat{H}| \]

\[ + c \int f_k^{p-1} \hat{H}^{1+\sigma} + c \int f_k^{p-1} \hat{H}^\sigma \]

\[ \leq c \int f_k^{p-2} |\nabla f|^2 + c\sigma \int f_k^{p-2} \hat{H}^{2+\sigma} + c\sigma \int f_k^{p-1} \hat{G}^2 \]

\[ + c \int f_k^{p-1} (\hat{H}^{1+\sigma} + \hat{H}^\sigma). \]

The Lemma follows by Remark 3.2. \( \square \)
The hard part of Theorem 3.1 is establishing $L^p$ bounds for appropriately large $\sigma$. In particular, we establish spacetime $L^p$ bounds for $\sigma \sim p^{-1/2}$, and thereby have the following wiggle room.

**Lemma 3.4.** Suppose there is a $p_0$ and $c_\sigma$, independent of $p, \sigma$, such that whenever $p > p_0$ and $\sigma < \frac{c_\sigma}{\sqrt{p}}$,

$$\int_0^T \int_{\Sigma_t} f^p < \infty.$$  

Then for $m > 0$,

$$\int_0^T \int_{\Sigma_t} \tilde{H}^m f^p < \infty$$

provided $p > 4m^2/c^2 + p_0$ and $\sigma < \frac{c_\sigma}{\sqrt{p}}$.

**Proof.** Follows directly from

$$\tilde{H}^m f^p = (f \tilde{H}^{\sigma+m/p})^p.$$  

□

**Lemma 3.5.** Given ($\ast$), then

$$\int_0^T \int_{\Sigma_t} f^p < \infty$$

for $p > p_0(c)$, and $\sigma < c_\sigma(c)p^{-1/2}$.

**Proof.** Combining equations (POINCARE-LIKE), (EVOLUTION-LIKE), and Lemma 3.3, we have the following inequalities. We adhere to the convention $c = c(S, \Sigma_0, T, n, \alpha)$.
and $C = C(S, \Sigma_0, T, n, \alpha, p, \sigma, \mu)$. Unless stated otherwise all integrals are on $\Sigma_t$.

$$\partial_t \int \Sigma_t f^p \leq -p^2/3 \int |\nabla f|^2 f^{p-2} - p/c \int \frac{\tilde{G}^2}{H^{2-\sigma}} f^{p-1} + cp\sigma \left[p(1 + 1/\beta) \int |\nabla f|^2 f^{p-2} + (1 + \beta p) \int \frac{\tilde{G}^2}{H^{2-\sigma}} f^{p-1} + \int f^p + \int_{\partial \Sigma_t} f^{p-1}(f + \tilde{H})\right] - 1/5 \int f^p \tilde{H}^2 + C \int f^p + C|\Sigma_t| + cp \int \partial \Sigma_t f^{p-1}(f + \tilde{H}) \]

Choose $\sigma = \frac{1}{2}(c^3 p)^{-1/2}$, $\beta = (cp)^{-1/2}$ and $\mu = 10 cp^3$, then for $p > 12c$ we have that $\int_{\Sigma_t} f^p$ increases at most exponentially.

Now for arbitrary $k$, we can combine equation (EVOLUTION-LIKE) with Lemma 3.5 in an identical manner to obtain

$$\partial_t \int_{\Sigma_t} f_k^p \leq -p^2/12 \int_{\Sigma_t} |\nabla f|^2 f_k^{p-2} + C \int_{A(k,t)} f^p \tilde{H}^2 + C \int_{A(k,t)} f^p + C|A(k,t)|$$

for $\sigma$, and $p$ satisfying the same bounds as Lemma 3.5. Here, as in Lemma 3.5, $c$ and $C$ are both independent of $k$.

The following Theorem will complete the proof of Theorem 3.1.

**Theorem 3.6.** Suppose there is a $p_0$ and $c_\sigma$, independent of $p, \sigma, k$, such that whenever $p > p_0$ and $\sigma < c_\sigma p$, we have

$$\int_0^T \int \Sigma_t f^p < \infty$$

and

$$(8) \quad \partial_t \int_{\Sigma_t} f_k^p + 1/c \int_{\Sigma_t} |\nabla f_k|^{p/2} \leq C \int_{A(k,t)} \tilde{H}^2 f^p + C \int_{A(k,t)} f^p + C|A(k,t)|$$
for any $k > 0$. Here $c$ and $C$ can depend on any quantity except $k$. Then for $p$ sufficiently large, and $\sigma$ sufficiently small, $f^p$ is uniformly bounded in spacetime. The bound will depend on $(S, \Sigma_0, T, n, p, \sigma, \alpha)$.

Proof. By Theorem 2.3 and Corollary 2.4 for each $n \geq 2$ there is a $q > 1$, and $c = c(n, q, |\Sigma_0|)$, such that
\[
\left( \int_S v^{2q} \right)^{1/q} \leq c \int_S |Dv|^2 + c \int_S v^2 H^2 + c \int_S v^2.
\]
So take $v = f_k^{p/2}$ and integrate (8) to obtain (for possibly larger $C$)
\[
\max \left\{ \sup_{[0,T]} \int_{\Sigma_t} f_k^p, \int_0^T \left( \int_{\Sigma_t} f_k^{pq} \right)^{1/q} \right\} \leq C \int_{A(k)} f^p + C \int_{A(k)} \tilde{H}^2 f^p + C|A(k)|.
\]
where all terms on the right are bounded by virtue of Lemma 3.4 and the monotonicity of $|\Sigma_t|$. Therefore
\[
\int_0^T \int f_k^{2p/q} \leq \int_0^T \left( \int f_k^{pq} \right)^{1/q} \left( \int f_k^p \right)^{2q-1} \frac{q}{2q-1}
\]
\[
\leq C \left( \int f^p + \int \tilde{H}^2 f^p + |A(k)| \left( \int f^{pr} \right)^{1/r} + \left( \int \tilde{H}^2 f_k^{pr} \right)^{1/r} + |A(k)|^{1/r} \right)^{2q-1}
\]
\[
\leq C(S, \alpha, p, \sigma, T, c_\sigma, \Sigma_0)|A(k)|^\alpha
\]
for any $r$, provided $p > 16r/c_\sigma^2 + p_0$ and $\sigma < \frac{c_\sigma^2}{2\sqrt{q}}$. If we fix $r$ sufficiently large, then $\alpha = \frac{2q-1}{q}(1 - 1/r) > 1$. Fix $p, \sigma$, then for any $\ell > k$, we have the inequality
\[
|\ell - k|^{\beta} |A(\ell)| \leq C|A(k)|\alpha
\]
where $\beta = \frac{p2q-1}{q} > 0$, and $C$ is independent of $\ell, k$. It follows by a standard argument that $A(k) = 0$ for $k > k_0(\alpha, \beta, C)$, $C$ as in (9). \hfill \qed

4. MEAN CURVATURE FLOW WITH FREE BOUNDARY PRELIMINARIES

Let $(\Sigma_t)_{t \in [0, T)}$ be the mean curvature flow of $\Sigma_0$, with free-boundary in $S$. Here, as always in this paper, $T$ is the maximal time of existence.

Write $g = (g_{ij})$ and $A = (h_{ij})$ for the induced metric and second fundamental form on $\Sigma_t$. We follow the usual convention that $g^{ij}$ is the matrix inverse to $g_{ij}$, and a raised index such as $h^{ij}$ means $\sum_k g^{ik} h_{kj}$. We denote $dV$ the volume form on $\Sigma_t$, and take $N$ for the outward normal of $\partial \Sigma \subset \Sigma$.

We write $\nabla$ for covariant differentiation in $\Sigma$, and $\nabla$ for covariant differentiation in $\mathbb{R}^{n+1}$. We write $(k_{ij})$ for the second fundamental form of the barrier surface $S$.

**Proposition 4.1.** We have the following evolution equations, using summation convention.

\[
\partial_t g_{ij} = -2H h_{ij}
\]
\[
\partial_t h_{ij} = \Delta h_{ij} - 2H h_{lm} h_{ij}^m + |A|^2 h_{ij}
\]
and
\[ \partial_t H = \Delta H + |A|^2 H \]
\[ \partial_t dV = -H^2 dV \]
\[ \partial_t \nu = \nabla H \]

Proof. See [Hui84]. \qed

Remark 4.2. Since the boundary \( \partial \Sigma_t \) is always orthogonal to the direction of motion,
\[ \partial_t |\Sigma_t| = -\int_{\Sigma_t} H^2 dV \leq 0. \]
Specifying other angles of contact would add a boundary term to \( \partial_t |\Sigma_t| \), and could even cause area increase.

Proposition 4.3. We have
\[ N(H) = k_{\nu \nu} H. \]
In particular, positivity of \( H \) is preserved for all time. If \( S \) is convex, then \( H \) is non-decreasing, and in fact must blow up in finite time.

Proof. Differentiate the relation \( \langle N, \nu \rangle = 0 \) in time. Evolution behavior follows from Proposition 4.1. \qed

Remark 4.4. Notice that \( H \) may still decrease. We will show later that \( H \) decreases at worst exponentially in time.

Proposition 4.5. For any \( X \in T_p \partial \Sigma \),
\[ h_{N,X} = -k_{\nu,X}. \]

Proof. Since \( N = \nu_S \) along \( T_p \partial \Sigma \),
\[ h(N, X) = -\langle \nu, \nabla_X \nu_S \rangle = -k(\nu, X) \]

As mentioned in the Introduction the key technical issue in extending the estimates to general barrier surfaces is in calculating \( \nabla_X h_{N,X} = \nabla_X h_{N,X} \), for \( X \in T_p \partial \Sigma \). To avoid the issue we perturb \( h \) so that \( h_{N,X} = 0 \).

Definition 4.5.1. Extend and fix \( k \) and \( \nu_S \) to be defined on \( \mathbb{R}^{n+1} \). Define the perturbed second fundamental form \( \bar{A} \) of \( \Sigma \) to be
\[ h_{ij} = h_{ij} + T_{ij\nu} + D_0 g_{ij} \]
where \( T \) is a 3-tensor defined on the ambient space by
\[ T(X, Y, Z) = k(X, Z)g(Y, \nu_S) + k(Y, Z)g(X, \nu_S). \]

We choose and fix the constant \( D_0 \) so that
\[ T(X, X, \nu) + D_0 \geq 1 \]
for any unit vector \( X \). From henceforth when a constant depends on \( D_0 \) or the extensions of \( k \) or \( \nu_S \), we will only say it depends on the barrier surface \( S \).

Our choice of \( D_0 \) and Proposition 4.3 imply that
\[ \bar{H} \geq H + 1 \geq 1, \quad |\bar{A}| \geq 1. \]
5. Evolution of tensors

Proposition 5.1. Let $T$ be a 3-tensor defined on the ambient space. If $T_{ij\nu}$ is the 2-tensor $T(\cdot,\cdot,\nu)$ restricted to $T\Sigma$, then

\[ \nabla T_{ij\nu} = O(1 + |A|) \]
\[ \nabla^2 T_{ij\nu} = O(1 + |A|^2 + |\nabla A|) \]
\[ (\partial_t - \Delta)T_{ij\nu} = O(1 + |A|^2) \]
\[ (\partial_t - \Delta)T_{ij}^i = O(1 + |A|^2). \]

Proof. Choose orthonormal geodesic coordinates $\partial_i$ at a fixed point $p$. We use the summation convention, excepting of course on $\nu$. We have

\[ \nabla \nabla T_{ij\nu} = \nabla \nabla T_{ij\nu} + h_{ik}T_{ij\nu} - h_{pi}T_{\nu j\nu} - h_{pj}T_{i\nu\nu} \]
\[ = O(1 + |A|) \]

We work towards calculating $\nabla^2 T$ and $\Delta T$. We have

\[ \nabla q (h_{pq}T_{\nu j\nu}) = (\nabla q h_{pq})T_{\nu j\nu} + h_{pi} \nabla q T_{\nu j\nu} \]
\[ = \nabla i h_{pq}T_{\nu j\nu} + h_{pi}(\nabla q T_{\nu j\nu} + h_{qk}T_{\nu jk} + h_{qk}T_{\nu j\nu} - h_{qj}T_{\nu\nu\nu}) \]
\[ = \nabla i h_{pq}T_{\nu j\nu} + O(1 + |A|^2) \]

and

\[ \nabla q (h_{pq}T_{ij\nu}) = \nabla k h_{pq}T_{ij\nu} + h_{pq}(\nabla q T_{ij\nu} - h_{qi}T_{\nu jk} + h_{qi}T_{ij\nu} - h_{qj}T_{i\nu\nu} - h_{qk}T_{ij\nu}) \]
\[ = \nabla k h_{pq}T_{ij\nu} + O(1 + |A|^2) \]

and

\[ \nabla q \nabla p T_{ij\nu} = \nabla^2 q T_{ij\nu} + \nabla q \nabla p T_{ij\nu} + \nabla p T_{ij\nu} + \nabla q \nabla i T_{ij\nu} + \nabla q \nabla j T_{ij\nu} \]
\[ = \nabla^2 q T_{ij\nu} - h_{qp} \nabla q T_{ij\nu} - h_{qi}T_{\nu jk} - h_{qi}T_{ij\nu} + h_{qk} \nabla p T_{ij\nu} \]
\[ = O(1 + |A|). \]

We therefore have

\[ \nabla^2 q T_{ij\nu} = \nabla q (\nabla q T_{ij\nu} + h_{pq}T_{ij\nu} - h_{pi}T_{\nu j\nu} - h_{pj}T_{i\nu\nu}) \]
\[ = \nabla k h_{pq}T_{ij\nu} - \nabla i h_{pq}T_{\nu j\nu} - \nabla j h_{pq}T_{i\nu\nu} + O(1 + |A|^2) \]
\[ = O(1 + |A|^2 + |\nabla A|) \]

and

\[ \Delta T_{ij\nu} = \partial_k HT_{ij\nu} - \partial_i HT_{\nu j\nu} - \partial_j HT_{i\nu\nu} + O(1 + |A|^2) \]
\[ = O(1 + |A|^2 + |\nabla H|). \]
We calculate the time derivative. Here \((\bar{x}^\alpha)\) are standard coordinates in \(R^{n+1}\).

\[
\partial_t T_{ij\nu} = \partial_t \left( T_{\alpha\beta\gamma}(F(x)) \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \nu^\gamma \right)
\]

\[
= \left( \frac{\partial T_{\alpha\beta\gamma}}{\partial x^3} \partial_t F^\gamma \right) \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \nu^\gamma + T_{\alpha\beta\gamma} \left( \frac{\partial}{\partial t} + \frac{\partial F^\alpha}{\partial x^i} \right) \partial_t \nu^\gamma
\]

\[
+ T_{\alpha\beta\gamma} \partial_t F \left( \frac{\partial F^\beta}{\partial x^j} \right) \nu^\gamma + T_{\alpha\beta\gamma} \partial_t F \partial^\beta \nu^\gamma
\]

\[
= -H \nabla_{\nu} T_{ij\nu} + T(\nabla_i(-H\nu), \partial_j, \nu) + T(\partial_i, \nabla_j(-H\nu), \nu) + T(\partial_t, \partial_j, \nabla H)
\]

\[
= -H \nabla_{\nu} T_{ij\nu} - \partial_t H T_{ij\nu} - H h_{ik} T_{k\nu j} - \partial_j H T_{ij\nu} - H h_{jk} T_{i\nu k} + \partial_k H T_{ijk}
\]

\[
= -\partial_t H T_{ij\nu} - \partial_j H T_{i\nu k} + \partial_k H T_{ijk} + O(1 + |A|^2)
\]

which proves the penultimate formula. The last formula follows by observing that \(\partial_t g_{ij} = O(|A|^2)\). \(\square\)

**Corollary 5.2.** We have

\[
1 = O(|\bar{A}|), \quad |A| = O(|\bar{A}|), \quad |\nabla A| = O(|\nabla \bar{A}| + |\bar{A}|).
\]

**Proof.** The first formula follows trivially from \(|\bar{A}| \geq 1\). The second because \(\bar{A} = A + O(1)\). The third since \(\nabla A = \nabla \bar{A} + O(|A| + 1)\). \(\square\)

**Theorem 5.3.** We have the evolution equations

\[
\partial_t \bar{h}_i^j = \Delta \bar{h}_i^j + |\bar{A}|^2 \bar{h}_i^j + O(|\bar{A}|^2)
\]

\[
\partial_t |A|^2 = \Delta |A|^2 + 2|A|^4 - 2|\nabla A|^2 + O(|\bar{A}|^3)
\]

\[
\partial_t H = \Delta H + |\bar{A}|^2 H + O(|\bar{A}|)H
\]

**Proof.** We deduce the first formula by Propositions 4.1 and 5.1

We have

\[
\frac{1}{2} \partial_t - \Delta |\bar{A}|^2 = \frac{1}{2} \partial_t (g^{ik} g^{jl} h_{ij} \bar{h}_{kl}) - \Delta \bar{A}, \bar{A} > -|\nabla \bar{A}|^2
\]

\[
= 2H h^{ik} g^{jl} \bar{h}_{ij} \bar{h}_{kl} + g^{ik} g^{jl}(\partial_t - \Delta)(h_{ij} + T_{ij\nu} + D g_{ij}) \bar{h}_{kl} - |\nabla \bar{A}|^2
\]

\[
= 2H h^{ik} g^{jl}(\bar{h}_{ij} \bar{h}_{kl} - h_{ij} \bar{h}_{kl}) + |A|^2 < A, \bar{A} > + O(1 + |\bar{A}|^3)
\]

\[
- 2D_{ij} H g^{ik} g^{jl} \bar{h}_{ij} \bar{h}_{kl} - |\nabla \bar{A}|^2
\]

\[
= |\bar{A}|^4 - |\nabla \bar{A}|^2 + O(|\bar{A}|^3).
\]

The third formula is an immediate consequence of Proposition 4.1 and Corollary 5.2. \(\square\)

**Lemma 5.4.** Let \(M\) be a symmetric matrix, and \(\eta > 0\). If \(|M| > (1 + \eta)\operatorname{tr}(M)\), then

\[
|M|^2 - \max_i |\lambda_i|^2 \geq \frac{1}{c} |M|^2.
\]

Here \(\{\lambda_i\}\) are the eigenvalues of \(M\), and \(c = c(n, \eta)\).

**Proof.** Otherwise, we can pick a sequence of counterexamples \(M^{(j)}\) with \(|M^{(j)}| = 1\) and

\[
|M^{(j)}|^2 - \max_i |\lambda_i^{(j)}|^2 \leq |M^{(j)}|^2/j = 1/j.
\]
Since each entry lies in the interval $[-1, 1]$, we can pick a subsequence $M^{(j)}$ converging to $M$. Then all but one eigenvalue of $M$ is zero, contradicting $|M| \geq (1 + \eta)\text{tr}(M)$.

**Proposition 5.5.** If $|\vec{A}| > 2\bar{H}$, then

\[
|\nabla \vec{A}|^2 - |\nabla |\vec{A}||^2 \geq \frac{1}{c} |\nabla \vec{A}|^2 + O(|\vec{A}|^2)
\]

where $c = c(n)$.

**Proof.** We have that

\[
\nabla_i \bar{h}_{jk} = \nabla_j \bar{h}_{ik} + O(|\vec{A}|
\]

and therefore, if we pick orthonormal coordinates so that $\partial_t = \nabla |\vec{A}|/|\nabla |\vec{A}|$ at the point in question,

\[
|\vec{A}|^2(|\nabla \vec{A}|^2 - |\nabla |\vec{A}|)^2 = ||\vec{A}|nabla_i \bar{h}_{jk} - \nabla_i |\vec{A}| \bar{h}_{jk}|^2
\]

\[
\geq \frac{1}{4} |\nabla_j |\vec{A}| \bar{h}_{ik} - \nabla_i |\vec{A}| \bar{h}_{jk}|^2 - c|\vec{A}|^3 |\nabla |\vec{A}|
\]

\[
\geq \frac{1}{2} |\nabla |\vec{A}||^2(|\vec{A}|^2 - \sum_k \bar{h}_{ik}^2) - c|\vec{A}|^3 |\nabla |\vec{A}|
\]

\[
\geq \frac{1}{2} |\nabla |\vec{A}||^2(|\vec{A}|^2 - \max_i |\lambda_i|^2) - c|\vec{A}|^3 |\nabla |\vec{A}|.
\]

Here $c = c(n, S)$, and $\lambda_i$ are the eigenvalues of $\vec{A}$.

By Lemma 5.4 there is a $c_n$ depending only on $n$ so that

\[
|\vec{A}|^2 - \max_i |\lambda_i|^2 \geq \frac{1}{c_n} |\vec{A}|^2
\]

and hence by Peter-Paul we deduce that

\[
|\nabla \vec{A}|^2 - |\nabla |\vec{A}||^2 > \frac{1}{2c_n} |\nabla |\vec{A}||^2 - c|\vec{A}|^2.
\]

This can be rearranged to deduce

\[
|\nabla \vec{A}|^2 - |\nabla |\vec{A}||^2 > \frac{1}{2c_n + 1} |\nabla \vec{A}|^2 - c|\vec{A}|^2.
\]

**Corollary 5.6.** Whenever $|\vec{A}| > 2\bar{H}$,

\[
(\partial_t - \Delta)|\vec{A}| \leq |\vec{A}|^3 - \frac{1}{c} \frac{|\nabla \vec{A}|^2}{|\vec{A}|} + O(|\vec{A}|^2)
\]

where $c = c(n)$.

**Proof.** We have (recalling $|\vec{A}| \geq 1$)

\[
(\partial_t - \Delta)|\vec{A}| = (\partial_t - \Delta)\sqrt{|\vec{A}|^2}
\]

\[
= \frac{1}{2} \frac{(\partial_t - \Delta)|\vec{A}|}{|\vec{A}|} + \frac{1}{4} \frac{|\nabla |\vec{A}|^2|^2}{|\vec{A}|^3}
\]

\[
= |\vec{A}|^3 + \frac{|\nabla |\vec{A}|^2|^2 - |\nabla \vec{A}|^2}{|\vec{A}|} + O(|\vec{A}|^2).
\]
Now apply Proposition 5.5.

6. Boundary derivatives

Fix a \( p \in \partial \Sigma \). Choose coordinates so that \( \partial_1 \equiv N \) along \( \partial \Sigma \), \((\partial_i)_{i>1}\) are orthonormal geodesic normal coordinates on \( \partial \Sigma \) at \( p \), and the integral curves of \( \partial_1 \) are geodesics.

**Lemma 6.1.** At \( p \) we have, for \( i,j > 1 \),

\[
\nabla_1 h_{ij} = h_{ij} k_{\nu\nu} + h_{11} k_{ij} - k_{ja} h_{ia} - k_{ja} h_{ia} - \nabla^S_i k_{ij}
\]

\[
\nabla_1 h_{11} = 2(k_{\alpha\beta} h_{\alpha\beta} + h_{11} k_{\nu\nu}) - K h_{11} + \nu(K) - \nabla^S_{\nu\nu} k_{\nu\nu}
\]

where \( \alpha, \beta \) are summed over \( 2, \ldots, n \), and \( K \) is the mean curvature of the barrier \( S \).

**Proof.** We calculate for \( i,j > 1 \)

\[
\partial_1 h_{ij} = -<\partial_1 \partial_j N, \nu> - <\partial_i \partial_j F, \partial_1 \nu> + h_{11} k_{ij}
\]

\[
= k_{ja} h_{ia} - \nabla^S_i k_{ij} - k_{\nu\nu} \nabla^S_i - k_{ij} + h_{11} k_{ij}
\]

and hence

\[
\nabla_1 h_{ij} = \partial_1 h_{ij} - h((\partial_1 \partial_i F)^T, \partial_j) - h(\partial_j, (\partial_i \partial_1)^T) - k_{ja} h_{ia} - \nabla^S_i k_{ij} + h_{ij} k_{\nu\nu} + h_{11} k_{ij}
\]

We calculate, using Proposition 4.3.

\[
N(H) = k_{\nu\nu} H
\]

\[
= \nabla_1 h_{11} + tr_{\partial \Sigma}(\nabla_1 H)
\]

\[
= \nabla_1 h_{11} - 2k_{\alpha\beta} h_{\alpha\beta} - tr_{\partial \Sigma}(\nabla^S_i k_{ij}) + (H - h_{11}) k_{\nu\nu} + h_{11}(K - k_{\nu\nu})
\]

\[
= \nabla_1 h_{11} - 2k_{\alpha\beta} h_{\alpha\beta} - 2h_{11} k_{\nu\nu} - \nu(K) + \nabla^S_{\nu\nu} k_{\nu\nu} + H k_{\nu\nu} + K h_{11}
\]

and the Lemma follows.

**Theorem 6.2.** At \( p \), for \( i,j > 1 \),

\[
\nabla_1 \tilde{h}_{ij} = \tilde{h}_{ij} k_{\nu\nu} + \tilde{h}_{11} k_{ij} - k_{ja} \tilde{h}_{ia} - k_{ja} \tilde{h}_{ia} + O(1)
\]

\[
\nabla_1 \tilde{h}_{11} = 2k_{\alpha\beta} \tilde{h}_{\alpha\beta} - K \tilde{h}_{11} + O(1),
\]

where we sum \( \alpha, \beta \) over \( 2, \ldots, n \).

**Proof.** Follows directly from Lemma 6.1 using Theorem 5.1.

**Remark 6.3.** Writing \( \lambda_i \) for the eigenvalues of \( \overset{\bar{\ }}{A} \),

\[
N \lambda_i = O(|\lambda_i|),
\]

where we interpret the derivative in the sense of \( \lim \inf \) or \( \lim \sup \) of difference quotients. This follows by Theorem 6.2 because \( \tilde{h}_{N,X} = 0 \) for \( X \in T_p \partial \Sigma \).

**Theorem 6.4.** We have that

\[
N|\overset{\bar{\ }}{A}| = O(|\overset{\bar{\ }}{A}|).
\]
Proof. Immediate from Theorem 6.2 and that \( \bar{h}_{N,X} = 0 \) when \( X \in T_p \partial \Sigma \). Or, one can use Remark 6.3 and observe that the derivative actually holds in the usual sense. \( \square \)

7. Controlling \( |\bar{A}| \)

In this section we prove the following Theorem, which will imply Theorem 1.1.

Theorem 7.1. There are constants \( \alpha = \alpha(S, \Sigma_0) \geq 0 \) and \( C = C(S, \Sigma_0) \) so that

\[
\max_{\Sigma_t} \frac{|A|}{H} \leq Ce^{\alpha t}
\]

for all time of existence.

Remark 7.2. If \( S \) is convex, then \( H \) is non-decreasing, and by carefully calculating the normal derivative \( N|\bar{A}| \) one can take \( \alpha = 0 \) in (14).

For arbitrary function \( f \) and \( g \), recall the useful formula

\[
(\partial_t - \Delta) f g = (\partial_t - \Delta) f g - \frac{f}{g^2} (\partial_t - \Delta) g + \frac{2}{g} \nabla f \cdot \nabla g = (\partial_t - \Delta) H g - \frac{f}{g} \nabla f \cdot \nabla g < \frac{f}{g} \nabla f \cdot \nabla g >.
\]

Proof of Theorem 7.1. Recall that

\[
|NH| \leq bH,
\]

\[
|N|\bar{A}| \leq b|\bar{A}|.
\]

Let \( d : R^{n+1} \to [-1,1] \) be a smooth function such that \( d \equiv 0 \) on \( S \), and \( \nu_S(d) \geq 1 \). If a constant depends on \( d \) we will only say it depends on \( S \). Let \( \phi : R^{n+1} \to R_+ \) be the smooth function

\[
\phi(x) = e^{-|\alpha - 2bd|}
\]

so that \( \nu_S(\phi) \leq -2b \).

We have, in geodesic orthonormal coordinates,

\[
(\partial_t - \Delta) \phi = -\alpha \phi + \nabla \phi \cdot (\partial_t F - \Delta F) - \sum_i \nabla^2 \phi (\partial_i F, \partial_i F) = -\alpha \phi + \text{tr}_{\Sigma}(\nabla^2 \phi) = (-\alpha + O(1))\phi.
\]

Choose \( \alpha = \alpha(S, n) \) so that \( (\partial_t - \Delta) \phi < 0 \).

We first show the quantity \( \min_{\Sigma_t} H/\phi \) is non-decreasing. First calculate

\[
N \frac{H}{\phi} \geq b \frac{H}{\phi},
\]

so any spatial minimum is interior. And by our choice of \( \alpha \) we obtain

\[
(\partial_t - \Delta) \frac{H}{\phi} \geq |A|^2 \frac{H}{\phi} + \frac{2}{\phi} < \nabla \frac{H}{\phi} \cdot \nabla \phi >.
\]

In particular, at any spatial minimum \( p \) of \( H/\phi \), we must have

\[
\partial_t \frac{H}{\phi} |_p \geq |A|^2 \frac{H}{\phi} \geq 0.
\]

We now consider the quantity

\[
f = \frac{|\bar{A}| + a}{H/\phi}
\]
for some positive constant $a$ to be determined. We show $\max_{\Sigma} f$ is non-increasing when $f$ is sufficiently big. At the boundary we have by (16)

$$Nf \leq \frac{b|\bar{A}|}{H/\phi} - bH/\phi \frac{|\bar{A}|}{H/\phi} \leq 0.$$  

So any spatial maximum of $f$ is interior.

From Corollary 5.6 and equation (17), wherever $|\bar{A}| > 2\bar{H}$ we have the evolution equations

$$(\partial_t - \Delta) |\bar{A}| \leq |\bar{A}|^3 - \frac{1}{c_n} \frac{|\nabla |\bar{A}||^2}{|\bar{A}|} + c|\bar{A}|^2$$

$$(\partial_t - \Delta) \frac{H}{\phi} \geq |\bar{A}|^2 \frac{H}{\phi} - c|\bar{A}|\frac{H}{\phi} + \frac{2}{\phi} \nabla H \nabla \phi.$$  

Here $c = c(S, n)$ and $c_n = c_n(n)$.

We calculate

$$(\partial_t - \Delta) f \leq \frac{1}{H/\phi} \left( |\bar{A}|^3 - \frac{1}{c_n} \frac{|\nabla |\bar{A}||^2}{|\bar{A}|} + c|\bar{A}|^2 \right) - f(|\bar{A}|^2 - c|\bar{A}|)$$

$$- \frac{2f}{\phi} \nabla f, \nabla \phi > + \frac{2}{\phi} \nabla f, \nabla \frac{H}{\phi} >$$

$$\leq \frac{\phi}{H} \left( |\bar{A}|^3 - \frac{1}{c_n} \frac{|\nabla |\bar{A}||^2}{|\bar{A}|} + c|\bar{A}|^2 - (|\bar{A}| + a)(|\bar{A}|^2 - c|\bar{A}|) \right)$$

$$+ \frac{2}{\phi} |\nabla |\bar{A}|||\nabla \phi| + < \nabla f, \frac{2}{\phi} \nabla \phi > + \frac{2\phi}{H} \nabla \frac{H}{\phi} \nabla \phi >$$

$$\leq \frac{\phi}{H} \left( (2c - a)|\bar{A}|^2 + ac|\bar{A}| - \frac{1}{2c_n} \frac{|\nabla |\bar{A}||^2}{|\bar{A}|} + 2c_n \frac{|\nabla \phi|^2}{|\bar{A}|} \right)$$

$$+ < \nabla f, \frac{2}{\phi} \nabla \phi > + \frac{2\phi}{H} \nabla \frac{H}{\phi} \nabla \phi >.$$  

Notice that $\frac{|\nabla \phi|^2}{|\bar{A}|} = O(1)$. By the above calculations and equation (18), if $f$ attains its spatial maximum at a point $p$, and $|\bar{A}| > 2\bar{H}$ at this point, then

$$\partial_t f|_p \leq \frac{\phi}{H} \left( (c - a)|\bar{A}|^2 + ca|\bar{A}| \right) \leq 0$$

provided we choose $a = 2c$ and ensure $|\bar{A}| > 2c$.

We still need to prove this implies Theorem 7.1. Recall that $\bar{H} = H + O(1)$. Using that $\min_{\Sigma_0} H/\phi$ is non-decreasing, we have

$$\bar{H} \leq H + c \leq \frac{H}{\phi} \left( 1 + \frac{1}{\min_{\Sigma_0} H/\phi} \right).$$

Define the constant

$$C = \frac{4c}{\min_{\Sigma_0} H/\phi} + 2c \left( 1 + \frac{1}{\min_{\Sigma_0} H/\phi} \right).$$
Then if \( f \geq C \), we have
\[
|\bar{A}| \geq C \frac{H}{\phi} - 2c \\
\geq 2c(1 + (\min H/\phi)^{-1}) \frac{H}{\phi} + 4c - 2c \\
\geq 2\bar{H} + 2c.
\]
We deduce that
\[
\frac{|\bar{A}|}{H} \leq \frac{f}{\phi} \leq \phi^{-1} \max\{C, \max f\},
\]
which proves the Theorem. \( \Box \)

8. Convexity pinching

We prove Theorem 1.2. Recall that we wish to show that if \( T < \infty \), then for any \( k \in \{1, \ldots, n\} \) and any \( \eta > 0 \),
\[
S_k \geq -\eta H^k - C
\]
with \( C = C(S, \Sigma_0, T, \eta, n) \). Here \( S_k \) is the \( k \)-th symmetric polynomial of the principle curvatures. We following [HS99a] and prove (19) by induction on \( k \). Notice this is trivially true for \( k = 1 \).

From henceforth assume (19) holds up to a fixed \( k \), i.e. \( S_l \geq -\eta H^l - C \) for every \( l = 1, \ldots, k \). We will now prove (19) for \( k + 1 \). Of course we also from now on assume \( T < \infty \).

In spirit we would like to consider the function
\[
-\frac{S_{k+1}}{H} - \frac{\eta H}{H} - \eta H^k
\]
and show this is bounded above in spacetime. However for general \( k \) we have no positivity control over the denominator \( S_k \). We require a further perturbation of the second fundamental form.

**Definition 8.0.1.** Let \( \tilde{A} = (b_{ij}) \) be the twice-perturbed second fundamental form
\[
b_{ij} = \tilde{h}_{ij} + (\epsilon H + D - D_0)g_{ij}
\]
\[
= h_{ij} + T_{ij\nu} + (\epsilon H + D)g_{ij}.
\]
Here \( D \geq D_0 + 1 \) and \( \epsilon \in (0, \frac{1}{2n}] \) are constants to be fixed later.

We write \( \tilde{\lambda}_i \) for the eigenvalues of \( b_{ij} \), so that if \( \tilde{\lambda}_i \) are the eigenvalues of the first-perturbed \( \tilde{h}_{ij} \), then
\[
\tilde{\lambda}_i = \tilde{\lambda}_i + (\epsilon H + D - D_0).
\]
Correspondingly \( |\tilde{A}| \) is the norm of the twice-perturbed second fundamental form, \( \tilde{H} \) the mean curvature, and \( \tilde{S}_k = s_k(\tilde{\lambda}), \tilde{Q}_k = q_k(\tilde{\lambda}) \) where defined.

Recall we had fixed \( D_0 = D_0(S) \) so that \( T(X, X, \nu) + D_0 \geq 1 \) for any unit vector \( X \). So we still have the conditions
\[
\tilde{H} \geq H + 1 \geq 1, \quad |\tilde{A}| \geq 1
\]
and since \( |\tilde{A}| \leq c(S, \Sigma_0, T) \tilde{H} \), we have
\[
|\tilde{A}| \leq c(S, \Sigma_0, T)\tilde{H}.
\]
Remark 8.1. Since $|\tilde{A}| \geq 1$ and $\epsilon \leq \frac{1}{2n}$, we have

$$1 = O(|\tilde{A}|), \quad |A| = O(|\tilde{A}|), \quad |\nabla A| = O(|\nabla \tilde{A}| + |\tilde{A}|).$$

Lemma 8.2. If $\tilde{h}_{ij} = h_{ij} + O(1)$, and

$$S_l \geq -\theta H^l - C$$

for any $\theta > 0$, then we also have

$$\bar{S}_l \geq -\theta \bar{H}^l - \bar{C}$$

for any $\theta > 0$. Here both $C, \bar{C}$ depend on $S, \Sigma_0, T, \theta, n$.

Proof. Given $\theta > 0$ and the corresponding $C$, we have for $c = c(S, \Sigma_0, T, n, l)$,

$$\bar{S}_l \geq S_l - cH^{l-1} \geq -\theta H^l - C - cH^{l-1} \geq -2\theta \bar{H}^l - C - c \left( \frac{C}{\theta} \right)^{l-1}. \quad \Box$$

Lemma 8.3. Suppose for any $l = 1, \ldots, k$ and any $\theta > 0$, we have

$$S_l \geq -\theta H^l - C.$$

Then for any $\epsilon \in (0, \frac{1}{2n}]$, there is a $D_\epsilon \geq D_0 + 1$ such that

$$\bar{S}_l \geq \frac{\epsilon}{1 + n\epsilon} \cdot \frac{n - k + 1}{k} \bar{S}_{k-1} \tilde{H}$$

whenever $D \geq D_\epsilon$.

Proof. Lemma 8.2 implies the hypothesis holds for $\bar{S}_l (l = 1, \ldots, k)$. Since $b_{ij} = \tilde{h}_{ij} + (\epsilon H + D - D_0)g_{ij}$, by Lemma 2.7 of [HS99a] there exists a $D_1 = D_1(\epsilon, S, \Sigma_0, T)$ such that (22) holds whenever $D - D_0 \geq D_1$. Now set $D_\epsilon = D_1 + D_0 + 1$. \quad \Box

Although we will fix $\epsilon \in (0, \frac{1}{2n}]$ later, for the duration of the paper we take $D = D_\epsilon$ as in Lemma 8.3.

Remark 8.4. Our inductive hypothesis and our choice of $D$ implies that, for $l = 1, \ldots, k$

$$\bar{S}_l \geq c(n)\epsilon \tilde{H} \bar{S}_{l-1} \geq c(n)\epsilon \tilde{H}^{l-1}.$$

Remark 8.5 (Derivatives of $\bar{S}_l$. $\bar{S}_l$ is a homogeneous degree $l$ polynomial in the entries $b_{ij}$. If $\partial$ denotes differentiation in the entries of $b_{ij}$, we have for any $d + s \leq l$ and any $l = 1, \ldots, n$

$$\left| \partial^d \bar{S}_l \right| \leq c(S, T, \Sigma_0, n)\tilde{H}^{l-d}$$

and

$$\left| \nabla^s \partial^d \bar{S}_l \right| \leq c(S, T, \Sigma_0, n)\tilde{H}^{l-d-s}|\nabla \tilde{A}|^s.$$
Definition 8.5.1. For $\eta, \sigma \in (0, 1]$, let
\[
f = \frac{-\tilde{Q}_{k+1} - \eta \tilde{H}}{\tilde{H}^{1+\sigma}}.
\]
We see that $f$ is well-defined by Remark 8.4 and $f \geq 0$ if and only if $\tilde{Q}_{k+1} \leq -\eta \tilde{H}$.

By Remark 8.5 we have that

\[
|\partial_d f| \leq c(S, \Sigma_0, T, \epsilon, n) \tilde{H}^{\sigma - d}.
\]

Lemma 8.6. Suppose for every $\epsilon \in (0, \frac{1}{2n}]$ and $\eta \in (0, 1]$, there exists $\sigma \in (0, 1]$ and $C = C(S, \Sigma_0, T, n, \epsilon, \sigma)$ such that
\[
f_+ < C.
\]
Then for any $\theta > 0$ there is a $\bar{C} = \bar{C}(S, \Sigma_0, T, n, \theta)$ such that
\[
\bar{S}_{k+1} \geq -\theta H^{k+1} - \bar{C}.
\]

Proof. Recall we have fixed $D = D_\epsilon$. The proof of Lemma 2.8 in [HS99a] shows the hypotheses imply that
\[
\bar{S}_{k+1} \geq -\theta H^{k+1} - \bar{C}
\]
for any $\theta > 0$, and $\bar{C} = \bar{C}(S, \Sigma_0, T, n, \theta)$. Now use Lemma 8.2.

We work towards bounding $f_+$, for a given $\eta > 0$. We first calculate the order of boundary derivatives. Choose orthonormal coordinates at a fixed $p \in \partial \Sigma$ such that $\partial_1 \equiv N$.

Theorem 8.7. At $p$ we have, for $i, j > 1$,
\[
\nabla_1 b_{ij} = O(|\bar{A}|)
\]
\[
\nabla_1 b_{11} = O(|\bar{A}|)
\]

Proof. By Theorem 6.2 and Proposition 4.3 we calculate
\[
\nabla_1 b_{11} = \nabla_1 (\bar{h}_{11} + (\epsilon H + D)g_{11})
\]
\[
= O(|\bar{A}| + 1) + \partial_1 H g_{11}
\]
\[
= O(|\bar{A}|)
\]
and the proof for $i, j > 1$ is identical.

Corollary 8.8. Interpreting derivatives in the sense of $\lim \inf$ and $\lim \sup$ of difference quotients, we have
\[
N \bar{\lambda}_i = O(|\bar{A}|).
\]

In particular, for every $l = 1, \ldots, n$,
\[
N \bar{S}_l = O(|\bar{A}|^{l/\epsilon}),
\]
which holds in the usual sense of derivatives.

Proof. The first equation follows from Theorem 8.7 and that $N$ is an eigenvector. The second equation is a consequence of the first, and holds in the usual sense of differentiation because $\bar{S}_l$ is smooth.
Theorem 8.9. We have

\[ |N f| \leq c(S, \Sigma_0, T, n, \epsilon) \tilde{H}^\sigma. \] (26)

Proof. Immediate from Corollary 8.8 and Remark 8.4. \qed

We obtain an (EVOLUTION-LIKE) equation for \( f \).

Proposition 8.10.

(27) \[ \partial_t b_j^i = \Delta b_j^i + |\tilde{A}|^2 b_j^i + O(D|\tilde{A}|^2) \]

Proof. By Propositions 4.1 and 5.1,

\[ \partial_t - \Delta b_j^i = |A|^2(b_j^i + \epsilon Hg_j^i) + O(1 + |A|^2) \]

\[ = |A|^2b_j^i - (D + 2\epsilon \mu)|A|^2 + O(1 + |A|^2) \]

\[ = |\tilde{A}|^2b_j^i + O(D|\tilde{A}|^2) \]

recalling that \( D \geq 1 \). \qed

Lemma 8.11. Let \( B > \eta > 0 \). There are constants \( c_0 = c_0(n, \eta, B) \) and \( c = c(c_0, S, \Sigma_0, T, n, \epsilon) \), such that whenever \( -BS_kS_1 \leq \tilde{S}_{k+1} \leq -\eta S_1S_k \) we have

\[ \frac{\partial^2 \tilde{Q}_{k+1}}{\partial b_{ij} \partial b_{pq}} \nabla_l b_{ij} \nabla_l b_{pq} \leq -\frac{1}{c_0} \frac{|\nabla \tilde{A}|^2}{|A|} + c\tilde{H} \] (28)

Proof. Choose orthonormal coordinates which diagonalize \( b_{ij} \). We have, using the notation of Lemma 2.13 of [HS99a],

\[ \frac{\partial^2 \tilde{Q}_{k+1}}{\partial b_{ij} \partial b_{pq}} \nabla_l b_{ij} \nabla_l b_{pq} = J(\lambda, \nabla_l (b_{ij} - T_{ij\nu}), \epsilon) \]

\[ + 2 \frac{\partial^2 q_{k+1}}{\partial \theta_{ij} \partial \theta_{pq}} (\tilde{A}) \nabla_l (b_{ij} - T_{ij\nu}) \nabla_l T_{ij\nu} \]

\[ + \frac{\partial^2 q_{k+1}}{\partial \theta_{ij} \partial \theta_{pq}} (\tilde{A}) \nabla_l T_{ij\nu} \nabla_l T_{pq\nu} \]

By this same Lemma 2.13,

\[ J(\lambda, \nabla_l (b_{ij} - T_{ij\nu}), \epsilon) \leq -\frac{1}{c_0} \frac{|\nabla (\tilde{A} - T)|^2}{|A|} \]

\[ \leq -\frac{1}{2c_0} \frac{|\nabla \tilde{A}|^2}{|A|} + \frac{1}{c_0} \frac{|\nabla T|^2}{|A|} \]

for \( c_0 = c_0(B, n, \eta) \).

By Theorem 2.5 and Lemma 2.12 of [HS99a], term (31) is non-positive. We bound term (30). Recall that \( |\nabla T| = O(H+1) = O(\tilde{H}) \). Using Remark 8.5

\[ 2 \frac{\partial^2 q_{k+1}}{\partial \theta_{ij} \partial \theta_{pq}} (\tilde{A}) \nabla_l (b_{ij} - T_{ij\nu}) \nabla_l T_{ij\nu} \leq 2 \frac{\partial^2 q_{k+1}}{\partial \theta_{ij} \partial \theta_{pq}} (\tilde{A}) \left( |\nabla \tilde{A}| + |\nabla T| \right) |\nabla T| \]

\[ \leq c |\nabla \tilde{A}| + c\tilde{H} \]

where \( c = c(S, \Sigma_0, T, n, \epsilon) \).

We deduce

\[ \frac{\partial^2 \tilde{Q}_{k+1}}{\partial b_{ij} \partial b_{pq}} \nabla_l b_{ij} \nabla_l b_{pq} \leq -\frac{1}{4c_0} \frac{|\nabla \tilde{A}|^2}{|A|} + c\tilde{H} \]
for \( c = c(S, \Sigma_0, T, n, \epsilon, c_0) \).

Since \( f \) is a homogeneous, degree \( \sigma \), symmetric function of the eigenvalues \( \lambda_i \) of \( b_j \), we obtain that
\[
(\partial_t - \Delta)f = \frac{\partial f}{\partial b_j}(|\tilde{A}|^2 b_j + O(|\tilde{A}|^2 D)) - \frac{\partial^2 f}{\partial b_j^2 \partial b_q} \nabla b_j \nabla b_q \leq -\frac{\partial^2 f}{\partial b_j^2 \partial b_q} \nabla b_j \nabla b_q + \sigma |\tilde{A}|^2 f + cD |\tilde{A}|^2 \sum \left| \frac{\partial f}{\partial b_j} \right| \leq -\frac{\partial^2 f}{\partial b_j^2 \partial b_q} \nabla b_j \nabla b_q + \sigma |\tilde{A}|^2 f + cD \tilde{H}^{1+\sigma}
\]
for \( c = c(S, \Sigma_0, T, n, \epsilon) \). In the last line we used the inequality (\ref{eq:convexity_estimates}).

Lemma 8.12 allows us to crucially obtain a gradient term wherever \( f \) is non-negative: on \( \text{spt} \ f \), we have
\[
(\partial_t - \Delta)f \leq \frac{2(1 - \sigma)}{H} < \tilde{H} f - \sigma \frac{(1 - \sigma)}{H^2} f |\tilde{H}|^2 + \frac{1}{H^{1-\sigma}} \frac{\partial^2 \tilde{Q}_p}{\partial b_{ij} \partial b_{pq}} + \sigma |\tilde{A}|^2 f + cD \tilde{H}^{1+\sigma} \leq \frac{2 |\tilde{H}||\nabla f|}{H} - \frac{1}{c} \frac{|\tilde{A}|^2}{H^{2-\sigma}} + \sigma \tilde{H}^\sigma + \sigma |\tilde{A}|^2 f + cD \tilde{H}^{1+\sigma}.
\]
Lemma 8.12. There are constants \( c = c(S, \Sigma_0, T, n, k, \epsilon, \eta) \) and \( C = C(c, p, \sigma, D) \) such that whenever \( p > p_0(c, n) \), we have
\[
\partial_t \int_{\Sigma_t} f_k^p \leq -p^2/3 \int_{\Sigma_t} f_k^{p-2} |\nabla f|^2 - p/c \int_{\Sigma_t} f_k^{p-1} |\tilde{A}|^2 + c \int_{\partial \Sigma_t} f_k^{p-1} \tilde{H}^\sigma + 2p \sigma \int_{A(k,t)} f^p \tilde{H}^2 + C \int_{A(k,t)} f^p + C |A(k,t)| - 1/5 \int_{\Sigma_t} f_k^p \tilde{H}^2
\]
Proof. We have by equation (\ref{eq:convexity_estimates}) (all integrals over \( \Sigma_t \) unless stated),
\[
\partial_t \int f_k^p = p \int f_k^p \Delta f - \int f_k^p H \leq -p(p - 1) \int f_k^{p-1} |\nabla f|^2 + p \int_{\partial \Sigma_t} f_k^{p-1} |N f| + p^2/3 \int f_k^{p-2} |\nabla f|^2 + 3c \int f_k^{p-1} \frac{|\tilde{H}|^2}{H^{2-\sigma}} - p/c \int f_k^{p-1} |\tilde{A}|^2 + pc \int f_k^{p-1} \tilde{H}^\sigma + p \sigma \int_{A(k,t)} f^p |\tilde{A}|^2 + cD \int f_k^{p-1} \tilde{H}^{1+\sigma} - \int f_k^p H^2
\]
provided \( p > 2c^2n \). Here \( c = c(S, \Sigma_0, T, n, \epsilon, \eta) \) and \( C = C(c, p, \sigma, D) \). The last term results from
\[
H^2 = \left( \frac{1}{1 + nc} \tilde{H} + O(1) \right)^2 \geq \frac{1}{4} \tilde{H}^2 + O(1).
\]
The boundary term is handled by Theorem \ref{thm:boundary_term} and the other terms are handled by Peter-Paul and/or Remark \ref{rem:other_terms}.

We obtain a (POINCARE-LIKE) equation for \( f \).
Lemma 8.13. For $\varepsilon < \varepsilon_0(n, k, \eta)$ we have on spt $f^+$
\[
\frac{\partial \tilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j \tilde{S}_{k+1} \geq \frac{\partial \tilde{S}_k}{\partial b_{ij}} \frac{\partial^2 \tilde{S}_{k+1}}{\partial b_{lm} \partial b_{pq}} \nabla_i b_{lm} \nabla_j b_{pq} + \frac{\partial \tilde{S}_k}{\partial b_{ij}} \frac{\partial \tilde{S}_{k+1}}{\partial b_{lm}} \nabla_i \nabla_m b_{ij}
\]
\[
+ \frac{\varepsilon}{1 + n\varepsilon} \left( (n-k) \frac{\partial \tilde{S}_k}{\partial b_{ij}} - (n-k+1) \frac{\partial \tilde{S}_{k-1}}{\partial b_{ij}} \right) \nabla_i \nabla_j \tilde{H}
\]
\[
+ \frac{1}{2} \eta \tilde{H}^2 \tilde{S}_k^2 - c D |\tilde{A}|^{2k} (D + |\tilde{A}|) - c |\tilde{A}|^{2k-1} |\nabla \tilde{A}|
\]

Proof. We follow the proof of Lemma 2.15 and Corollary 2.16 in [HS99a]. Recall we fixed $D = D_*$. From Proposition 5.1 and Remark 8.1 we have $\nabla_p \nabla_q T_{ij} = O(|\tilde{A}|^2 + |\nabla \tilde{A}|)$. In particular,
\[
\nabla_p \nabla_q \tilde{H} = (1 + n\varepsilon) \nabla_p \nabla_q H + O(|\tilde{A}|^2 + |\nabla \tilde{A}|)
\]
and
\[
\nabla_i \nabla_j \tilde{h}_{lm} - \nabla_i \nabla_m \tilde{h}_{ij} = \tilde{h}_{ij} \tilde{h}_{ri} \tilde{h}_{rm} - \tilde{h}_{im} \tilde{h}_{ir} \tilde{h}_{rj} + \tilde{h}_{im} \tilde{h}_{ir} \tilde{h}_{rj} - \tilde{h}_{ij} \tilde{h}_{mr} \tilde{h}_{ri} + O(|\tilde{A}|^2 + |\nabla \tilde{A}|).
\]
We therefore calculate
\[
\frac{\partial \tilde{S}_k}{\partial b_{ij}} \frac{\partial \tilde{S}_{k+1}}{\partial b_{lm}} (\nabla_i \nabla_j b_{lm} - \nabla_i \nabla_m b_{ij})
\]
\[
= \frac{\partial \tilde{S}_k}{\partial b_{ij}} \frac{\partial \tilde{S}_{k+1}}{\partial b_{lm}} \left[ \nabla_i \nabla_j \tilde{h}_{lm} - \nabla_i \nabla_m \tilde{h}_{ij} \right]
\]
\[
+ \frac{\varepsilon}{1 + n\varepsilon} \left( \delta_{lm} \nabla_i \nabla_j \tilde{H} - \delta_{ij} \nabla_i \nabla_m \tilde{H} + O(|\tilde{A}|^2 + |\nabla \tilde{A}|) \right)
\]
\[
\geq \frac{\partial \tilde{S}_k}{\partial \lambda_i} \frac{\partial \tilde{S}_{k+1}}{\partial \lambda_m} \left[ \nabla_i \nabla_j \tilde{h}_{lm} - \nabla_i \nabla_m \tilde{h}_{ij} \right]
\]
\[
+ \frac{\varepsilon}{1 + n\varepsilon} \left( \delta_{lm} \nabla_i \nabla_j \tilde{H} - \delta_{ij} \nabla_i \nabla_m \tilde{H} + O(|\tilde{A}|^2 + |\nabla \tilde{A}|) \right)
\]
\[
+ \frac{1}{2} \eta \tilde{H}^2 \tilde{S}_k^2 - c |\tilde{A}|^{2k} (D + |\tilde{A}|) - c |\tilde{A}|^{2k-1} |\nabla \tilde{A}|.
\]

Choose a frame which diagonalizes $\tilde{h}_{ij}$, and hence $b_{ij}$, then
\[
\frac{\partial \tilde{S}_k}{\partial \lambda_i} \frac{\partial \tilde{S}_{k+1}}{\partial \lambda_m} (\nabla_i \nabla_j \tilde{h}_{lm} - \nabla_i \nabla_m \tilde{h}_{ij}) = \frac{\partial \tilde{S}_k}{\partial \lambda_i} \frac{\partial \tilde{S}_{k+1}}{\partial \lambda_m} \left[ \tilde{\lambda}_i \tilde{\lambda}_m^2 - \tilde{\lambda}_i^2 \tilde{\lambda}_m + O(|\tilde{A}|^2 + |\nabla \tilde{A}|) \right]
\]
\[
\geq \frac{\partial \tilde{S}_k}{\partial \lambda_i} \frac{\partial \tilde{S}_{k+1}}{\partial \lambda_m} \left[ \tilde{\lambda}_i \tilde{\lambda}_m^2 - \tilde{\lambda}_i^2 \tilde{\lambda}_m + O(|\tilde{A}|^2 + |\nabla \tilde{A}|) \right]
\]
\[
+ \left( \frac{\varepsilon}{1 + n\varepsilon} \tilde{H} + O(D) \right)^2 (\tilde{\lambda}_m - \tilde{\lambda}_i)
\]
\[
+ \left( \frac{\varepsilon}{1 + n\varepsilon} \tilde{H} + O(D) \right) (\tilde{\lambda}_m^2 - \tilde{\lambda}_m^2)
\]
\[
\geq \frac{\partial \tilde{S}_k}{\partial \lambda_i} \frac{\partial \tilde{S}_{k+1}}{\partial \lambda_m} \left[ \tilde{\lambda}_i \tilde{\lambda}_m^2 - \tilde{\lambda}_i^2 \tilde{\lambda}_m 
\right]
\]
\[
+ \left( \frac{\varepsilon \tilde{H}}{1 + n\varepsilon} \right)^2 (\tilde{\lambda}_m - \tilde{\lambda}_i) + \left( \frac{\varepsilon \tilde{H}}{1 + n\varepsilon} \right) (\tilde{\lambda}_i^2 - \tilde{\lambda}_m^2)
\]
\[
- c D |\tilde{A}|^{2k} (D + |\tilde{A}|) - c |\tilde{A}|^{2k-1} |\nabla \tilde{A}|.
Therefore, by precisely the same arguments at in Lemma 2.15 of [HS99a], we have for any \( \epsilon > 0 \)

\[
\frac{\partial \tilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j \tilde{S}_{k+1} \geq \frac{\partial \tilde{S}_k}{\partial b_{ij}} \frac{\partial^2 \tilde{S}_{k+1}}{\partial b_{pq} \partial b_{im} \partial b_{pj}} \nabla_i b_{im} \nabla_j b_{pq} \\
+ \frac{\partial \tilde{S}_k}{\partial b_{ij}} \frac{\partial \tilde{S}_{k+1}}{\partial \nabla_j m, b_{ij}} \\
+ \frac{\epsilon}{1 + n \epsilon} \left( (n - k) \tilde{S}_k \frac{\partial \tilde{S}_k}{\partial b_{ij}} - (n - k + 1) \tilde{S}_{k-1} \frac{\partial \tilde{S}_{k+1}}{\partial b_{ij}} \right) \nabla_i \nabla_j \tilde{H} \\
- \tilde{H} \tilde{S}_k \tilde{S}_{k+1} + (k + 1) \tilde{S}_k^2 + k((k + 1) \tilde{S}_{k+1} - (k + 2) \tilde{S}_k \tilde{S}_{k+2}) \\
+ \left( \frac{c \tilde{H}}{1 + n \epsilon} \right)^2 \left[ (k + 1)(n - k + 1) \tilde{S}_{k+1} \tilde{S}_{k-1} - k(n - k) \tilde{S}_k^2 \right] \\
+ \left( \frac{c \tilde{H}}{1 + n \epsilon} \right) \left[ (n - k) \tilde{S}_k (\tilde{H} \tilde{S}_k - (k + 1) \tilde{S}_{k+1}) \right] \\
+ (n - k + 1) \tilde{S}_{k-1} ((k + 2)(\tilde{S}_{k+2} - \tilde{H} \tilde{S}_{k+1}) \right] \\
- cD|\tilde{A}|^{2k}(D + |\tilde{A}|) - c|\tilde{A}|^{2k-1}|\nabla \tilde{A}|.
\]

And the Lemma follows by the same argument as in Corollary 2.16 of [HS99a]. □

**Lemma 8.14.** There is a constant \( c = c(S, \Sigma_0, T, n, \epsilon, \eta, D) \) such that for any \( p > 2 \) and \( \beta > 0 \), we have

\[
\frac{1}{c} \int_{\Sigma_t} f_+^p \tilde{H}^2 \leq (p + p/\beta) \int_{\Sigma_t} f_+^{p-2} |\nabla f|^2 + (1 + \beta p) \int_{\Sigma_t} f_+^{p-1} \frac{|\nabla \tilde{A}|^2}{\tilde{H}} + \int_{\partial \Sigma_t} f_+^{p-1} \tilde{H}^\sigma
\]

**Proof.** Fix \( \epsilon = \epsilon(\eta, n, k) \) as in Lemma 8.13. Using inequalities of Remarks 8.4 and 8.5, we have for \( c = c(S, \Sigma_0, T, n, k, \epsilon) \),

\[
\frac{\partial \tilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j f \leq -\tilde{H}^{\sigma-1} \tilde{S}_k^{-1} \frac{\partial \tilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j \tilde{S}_{k+1} + c\tilde{H}^{k+\sigma-3}|\nabla \tilde{A}|^2 + c\tilde{H}^{k-2}|\nabla f| |\nabla \tilde{A}| \\
+ \frac{\partial \tilde{S}_k}{\partial b_{ij}} \left[ \tilde{H}^{\sigma-1} \tilde{S}_k^{-2} \tilde{S}_{k+1} \nabla_i \nabla_j \tilde{S}_k + (\eta \tilde{H}^{\sigma-1} - (\sigma - 1) \tilde{H}^{-1} f) \nabla_i \nabla_j \tilde{H} \right]
\]


Multiply by \( f_p \tilde{H}^{-k+1-\sigma} \), integrate, and use Lemma 5.13 to obtain

\[
\frac{\eta}{2c\epsilon(k-1)} \int f_p^\infty \tilde{H}^2 \leq \frac{\eta}{2} \int \tilde{S}_k \tilde{H}^{2-k} f_p^\infty \leq - \int f_p^\infty \tilde{H}^{-k+1-\sigma} \frac{\partial \tilde{S}_k}{\partial b_{ij}} \nabla_i \nabla_j f
\]

\[
- \int f_p^\infty \tilde{H}^{-k} \tilde{S}_k \frac{\partial \tilde{S}_k}{\partial b_{ij}} \left\{ -\tilde{S}_k^{-1} \nabla_i \nabla_j \tilde{S}_k 
\right. 
\]

\[
+ \frac{\partial^2 \tilde{S}_k}{\partial b_{lm} \partial b_{pq}} \nabla_l b_{im} \nabla_j b_{pq} + \frac{\partial \tilde{S}_k+1}{\partial b_{lm}} \nabla_i \nabla_m b_{ij} \}
\]

\[
+ \int f_p^\infty \tilde{H}^{-k} \frac{\partial \tilde{S}_k}{\partial b_{ij}} \left( -\eta - \frac{\epsilon}{1+n\epsilon}(n-k) + (\sigma - 1) \tilde{H}^{-\sigma} f \right) \nabla_i \nabla_j \tilde{H}
\]

\[
+ \frac{\epsilon}{1+n\epsilon}(n-k+1) \int f_p^\infty \tilde{H}^{-k} \tilde{S}_k^{-1} \tilde{S}_k^{-1} \frac{\partial \tilde{S}_k+1}{\partial b_{ij}} \nabla_i \nabla_j \tilde{H}
\]

\[
+ cD \int f_p^\infty \tilde{H} + c \int f_p^\infty \tilde{H}^{-1} |\nabla \tilde{A}| 
\]

\[
+ c \int f_p^\infty \tilde{H}^{-2} |\nabla \tilde{A}|^2 + c \int f_p^\infty \tilde{H}^{-1-\sigma} |\nabla \tilde{A}| |\nabla f|
\]

where \( c = c(S, \Sigma_0, T, n, \epsilon) \). As usual all integrals are over \( \Sigma_t \) unless otherwise stated.

Integrate by parts all double covariant derivatives, using Theorem 8.9 and equation (25) to handle boundary terms. After applying remarks 8.4 and 8.5, we obtain that

\[
\frac{\eta}{c} \int f_p^\infty \tilde{H}^2 \leq \int \frac{f_p^{\infty-1}}{f_p^\infty} f + c \int f_p^{\infty-1} |\nabla \tilde{A}|^2 + c \int \frac{f_p^{\infty-1} |\nabla f| |\nabla \tilde{A}|}{H^{2-\sigma}} + c \int \frac{f_p^{\infty-1} |\nabla f| |\nabla \tilde{A}|}{H^{1+\sigma}}
\]

\[
+ cp \int f_p^{\infty-1} |\nabla f| |\nabla \tilde{A}| + cp \int f_p^{\infty-2} |\nabla f|^2 + c \int f_p^{\infty} |\nabla \tilde{A}|^2 + c \int f_p^{\infty} |\nabla \tilde{A}|^2 + C(c, D, \mu) \int f_p^\infty
\]

where \( \mu > 0 \) is arbitrary. Set \( \mu = 2c/\eta \). Recalling that \( f \leq c\tilde{H}^\sigma \), the Lemma follows by using Peter-Paul on the remaining terms. \( \square \)

In view of Lemma 8.6 and Theorem 6.11 to finish proving Theorem 1.2 we merely need to show \( f_+ \) satisfies (\star) of Section 8. In the language of Section 8, let \( \tilde{H} \) be itself (the twice-perturbed mean curvature), and \( \tilde{G} = |\nabla \tilde{A}| \). Then Lemmas 8.12 and 8.14 imply \( f_+ \) satisfies (\star). We are done.

9. Umbilic pinching when \( S = S^n \)

We consider the case when \( \Sigma_0 \) is strictly convex and \( S \) is the sphere \( S^n \). We prove the umbilic pinching Theorem 1.6. With spherical barriers we can dispense with perturbations. By Proposition 4.3 we know that \( T < \infty \).

**Remark 9.1.** By Theorem 9.7 in [Sta96a], there is an \( \epsilon = \epsilon(\Sigma_0, n) \) such that

\[
h_{ij} \geq \epsilon H g_{ij}
\]

(33)
for all \( t \in [0, T) \). Hence the pointwise estimates of Lemma 2.3 in [Hui84] continue to hold in the spherical-free-boundary case.

Notice in the case \( S = S^n \) that \( h_{N,X} \equiv 0 \) for all \( X \in T_p\partial \Sigma \). In view of this and Remark 9.1, we can work with the unperturbed second fundamental form. By Lemma 6.1

\[
\nabla_N h_{X,Y} = -h_{X,Y} + h_{N,N} < X, Y > \quad (X, Y \in T_p\partial \Sigma)
\]

Hence,

\[
(34) \quad NH = H, \quad N|A| = O(|A|) = O(H).
\]

Define

\[
(35) \quad f = \frac{|A|^2}{H^{2-\sigma}} - \frac{1}{n} \frac{H^2}{H^{2-\sigma}} \frac{2}{n} \sum_{i,j} (\lambda_i - \lambda_j)^2.
\]

Arguing as in [Hui84], to prove Theorem 1.6 it will suffice to show \( f \) is bounded as \( t \to T \). By (34) we have that

\[
(36) \quad Nf = O(H^\sigma)
\]

**Lemma 9.2.** There is a constant \( c = c(n, \epsilon) \) such that for every \( \eta > 0 \) we have

\[
\frac{1}{c} \int_{\Sigma_t} f^p H^2 \leq (\eta p + 1) \int_{\Sigma_t} \frac{\nabla H^2}{H^{2-\sigma}} f^{p-1} + \frac{p}{\eta} \int_{\Sigma_t} |\nabla f|^2 f^{p-2} + \int_{\partial \Sigma} f^{p-1} H^\sigma
\]

**Proof.** We follow the proof of Lemma 5.4 in [Hui84]. In consideration of Remark 9.1, we have

\[
2ne^2 f^p H^2 \leq \frac{2}{H^{2-\sigma}} f^{p-1} Z
\]

\[
\leq f^{p-1} \Delta f
\]

\[
(37) \quad - \frac{2}{H^{2-\sigma}} f^{p-1} < h^0_{ij}, \nabla_i \nabla_j H > + \frac{2(1-\sigma)}{H} f^{p-1} < \nabla H, \nabla f > + \frac{2-\sigma}{H} f^p \Delta H.
\]

Here \( h^0_{ij} \) is the trace-free second fundamental form, and \( Z = H \text{tr}(A^3) - |A|^4 \). We integrate the above relation, and integrate by parts terms (37), (38) and (39). The resulting interior terms are handled by Peter-Paul, and the inequality \( |h^0_{ij}| \leq f H^{2-\sigma} \leq H^2 \). To handle the boundary term use (34) and (36), and that \( h_{N,X} \) vanishes when \( X \in T_p\partial \Sigma \). \( \square \)

Recall that \( f \) satisfies the evolution inequality

\[
(40) \quad \partial_t f \leq \Delta f + \frac{2(1-\sigma)}{H} < \nabla H, \nabla f > - e^2 \frac{|\nabla H|^2}{H^{2-\sigma}} + \sigma |A|^2 f
\]
Lemma 9.3. We have, for $c = c(n, \epsilon)$,

$$\partial_t \int_{\Sigma_t} f_P^p \leq -p^2/3 \int_{\Sigma_t} |\nabla f|^2 f_P^{p-2} - p/c \int_{\Sigma_t} \frac{|\nabla H|^2}{H^{2-\sigma}} f_P^{p-1}$$

$$+ (\sigma p - 1) \int_{\Sigma_t} H^2 f_P^p + cp \int_{\partial \Sigma_t} f_P^{p-1} H^\sigma$$

Proof. Follows directly by (10), and Proposition 4.1. Use Peter-Paul to handle the inner product term, and equation (36) to handle the boundary term obtained upon integration by parts. \qed

In view of Lemmas 9.2 and 9.3, we can take $\tilde{H} = H$ and $\tilde{G} = |\nabla H|$ in Section 3, the result follows by Theorem 3.1.

References

[Buc05] J. Buckland. Mean curvature flow with free boundary on smooth hypersurfaces. J. reine angew. Math., 586:71–90, 2005.

[HS99a] G. Huisken and C. Sinestrari. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. Acta Math, 183:45–70, 1999.

[HS99b] G. Huisken and C. Sinestrari. Mean curvature flow singularities for mean convex surfaces. Calc. Variations & PDE, 8:1–14, 1999.

[Hui84] G. Huisken. Flow by mean curvature of convex surfaces into spheres. Journal of Differential Geometry, 20:237–266, 1984.

[MS73] J. H. Michael and L. M. Simon. Sobolev and mean-value inequalities on generalized submanifolds on $r^n$. Com. on Pure and Applied Math., 26:361–379, 1973.

[Sta96a] A. Stahl. Convergence of solutions to the mean curvature flow with a neumann boundary condition. Calc. Variations & PDE, 4:421–441, 1996.

[Sta96b] A. Stahl. Regularity estimates for solutions to the mean curvature flow with a neumann boundary condition. Calc. Variations & PDE, 4:385–407, 1996.

Department of Mathematics, Stanford University, 450 Serra Mall, Bldg 380, CA 94305

E-mail address: nedelen@math.stanford.edu