Derivatives with respect to the order of the Legendre Polynomials

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Abstract

Expressions for the derivatives of the Legendre polynomials of the first kind with respect to the order of these polynomials i.e. \( P_n(z) = (\partial^n P_\nu(z)/\partial \nu^n)_{\nu=0} \) are given. An explicit form for the fourth derivative is presented.

1 Introduction

Recently, R. Szmytkowski [1] has obtained expressions for the first three derivatives of the Legendre functions of the first kind \( P_\nu(z) \) with respect to order \( \nu \) i.e. \( P_n(z) = (\partial^n P_\nu(z)/\partial \nu^n)_{\nu=0} \) for \( 1 \leq \nu \leq 3 \). That is to say,

\[
\begin{align*}
P_0(z) &= 1, \\
P_1(z) &= \ln\left(\frac{1+z}{1-z}\right) = -Li_1\left(\frac{1-z}{1+z}\right), \\
P_2(z) &= -2Li_2\left(\frac{1-z}{1+z}\right), \\
P_3(z) &= 12Li_3\left(\frac{1-z}{1+z}\right) - 6\ln\left(\frac{1+z}{1-z}\right)Li_2\left(\frac{1-z}{1+z}\right) - \pi^2\ln\left(\frac{1+z}{1-z}\right) - 12\zeta(3),
\end{align*}
\]

where \( Li_\mu(z) \) is the polylogarithm function [2] of order \( \mu \) and \( \zeta(s) \) is the Riemann zeta function. These derivatives arise in studies of tidal hydrodynamics and are part of a recent and continuing interest in the variation of well known polynomials and other higher transcendental functions with respect to their orders [3],[4],[5],[6],[7].

In this note we will give a general expression for the derivatives \( P_n(z) \) and discuss the expected increasing complexity of these expressions as \( n \) increases.

The Legendre functions of the first kind \( P_\nu(z) \) satisfy the differential equation

\[
\left[ \frac{d}{dz} (1-z^2) \frac{d}{dz} + \nu(\nu+1) \right] P_\nu(z) = 0 , \quad -1 \leq z \leq 1.
\]
Differentiating the latter expression with respect to \( \nu \) and evaluating the result at \( \nu = 0 \) we get for \( P_n(z) \) the relation

\[
\frac{d}{dz} \left[ (1 - z^2) \frac{d P_n(z)}{dz} \right] = -n P_{n-1}(z) - n(n-1) P_{n-2}(z).
\]

Expressions for the desired derivatives \( P_n(z) \) can then be reduced to quadratures by

\[
P_n(z) = -n \int \frac{dz}{1 - z^2} \int_z^1 [P_{n-1}(z') + (n-1) P_{n-2}(z')] dz' + C \ln \left( \frac{1 + z}{1 - z} \right),
\]

where \( C \) is a constant of integration. The inner or first integrals in (1) will be discussed further below. With the use of the identity

\[
P_\nu(1) = 1,
\]

from which it follows that for \( n \geq 1 \)

\[
P_n(1) = 0,
\]

the constants of integration which arise in (1) can be evaluated.

### 2 The Expression for \( P_4(z) \)

In this case we have to evaluate the expression

\[
P_4(z) = C \ln \left( \frac{1 + z}{1 - z} \right) - 4 \int \frac{dz}{1 - z^2} \int_z^1 [P_3(z') + 3 P_2(z')] dz',
\]

or more explicitly

\[
P_4(z) = C \ln \left( \frac{1 + z}{1 - z} \right) - 4 \int \frac{I(z)}{1 - z^2} dz,
\]

where

\[
I(z) = \int \left[ 12 Li_3 \left( \frac{1 + z'}{2} \right) - 6 \ln \left( \frac{1 + z'}{2} \right) Li_2 \left( \frac{1 + z'}{2} \right) - \pi^2 \ln \left( \frac{1 + z'}{2} \right) - 12 \zeta(3) - 6 Li_2 \left( \frac{1 - z'}{2} \right) \right] dz'.
\]

The integrals in \( I(z) \) are well known and we find the somewhat remarkable result

\[
I(z) = (z + 1) \left[ 12 Li_3 \left( \frac{1 + z}{2} \right) - 6 \ln \left( \frac{1 + z}{2} \right) Li_2 \left( \frac{1 + z}{2} \right) - \pi^2 \ln \left( \frac{1 + z}{2} \right) - 12 \zeta(3) \right].
\]

In the outer i.e. second integration we have for \( P_4(z) \)

\[
P_4(z) = C \ln \left( \frac{1 + z}{1 - z} \right) - 4 \int \frac{dz}{1 - z^2} [12 Li_3 \left( \frac{1 + z}{2} \right) - 6 \ln \left( \frac{1 + z}{2} \right) Li_2 \left( \frac{1 + z}{2} \right) - \pi^2 \ln \left( \frac{1 + z}{2} \right) - 12 \zeta(3)],
\]

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where \( C \) is a constant of integration. All but one of the integrals are elementary. With a change of variable
\[
t = \frac{1+z}{2}, \quad 0 \leq t \leq 1
\]
we get for \( P_4(z) \)
\[
P_4(z) = C \ln\left(\frac{t}{1-t}\right) + C' + 24 \left[ Li_2(t)^2 + 2 \ln(1-t) \left[ Li_3(t) - \zeta(3) \right] + \frac{z^2}{6} Li_2(1-t) + \mathcal{J}(t) \right],
\]
where
\[
\mathcal{J}(t) = \int \frac{\ln(t) Li_2(t)}{1-t} dt,
\]
and \( C' \) is the second constant of integration.

The integral \( \mathcal{J}(t) \) has been obtained within Mathematica [9]. After some simplification we have
\[
\mathcal{J}(t) = [\ln^2(t) - \ln(t) \ln(1-t)] Li_2(t) - \ln^2(1-t) Li_2(1-t) + \ln^2(\frac{t}{1-t}) Li_2(\frac{t}{1-t}) - \frac{3}{2} Li_2(t)^2
\]
\[
+ 2 \ln(t) Li_3(t) + 2 \ln(1-t) Li_3(1-t) - 2 \ln(\frac{t}{1-t}) Li_3(\frac{t}{1-t})
\]
\[
+ 2 \left[ Li_4(t) - Li_4(1-t) + Li_4(\frac{t}{1-t}) \right] + \ln^2(1-t) [\frac{1}{2} \ln^2(t) - \ln(t) \ln(1-t) + \frac{1}{2} \ln^2(1-t)].
\]

Using the following identities for the di and trilogarithm functions
\[
Li_2(1-z) = - Li_2(z) + \frac{\pi^2}{6} - \ln(z) \ln(1-z),
\]
\[
Li_2\left(\frac{z}{1-z}\right) = - Li_2(z) - \frac{1}{2} \ln^2(1-z),
\]

\[
Li_3\left(\frac{z}{1-z}\right) + Li_3(1-z) + Li_3(z) - \zeta(3) = \frac{\pi^2}{6} \ln(1-z) - \frac{1}{2} \ln(z) \ln^2(1-z) + \frac{1}{6} \ln^3(1-z),
\]

and gathering terms in equations 2 and 3 we get
\[
P_4(z) = \pi^4/15 + 24 \left[ \frac{1}{2} Li_2(t)^2 - \frac{z^2}{6} Li_2(t) + 2 Li_4(t) - 2 Li_4(1-t) + 2 \ln(t) \left\{ Li_3(1-t) - \zeta(3) \right\}
\]
\[
+ \frac{1}{2} \ln^4(1-t) + \frac{z^2}{6} \ln^2(1-t) + 2 Li_4(\frac{t}{1-t})
\]
\[
+ \ln(t) \ln(1-t) \{ Li_2(t) - \frac{z^2}{6} - \frac{1}{2} \ln^2(1-t) + \ln(t) \ln(1-t) \} \right] .
\]

The constants \( C \) and \( C' \) in equation 4 having been evaluated by setting \( P_4(1) = 0 \) with the result that \( C = 0 \) and \( C' = \pi^4/15 \). We note that the expression in equation 4 contains the complicated term \( Li_4\left(\frac{t}{1-t}\right) \). In contrast to the corresponding expressions for the polylogarithms \( Li_2 \) and \( Li_3 \) with the same argument, this term does not appear to be able to be rewritten in terms of the polylogarithm \( Li_4 \) with simpler arguments as noted by Lewin [10]. As a consequence it does not bode well for any likelihood of obtaining explicit analytic expressions for \( P_n(z) \) for \( n \geq 5 \).
Appendix A

Each of the first integrals i.e.
\[
\mathcal{I}_0(z) = \int z P_n(z') \, dz',
\]
appearing in equation 1 occur twice i.e. in successive calculations of the quantities \(P_n(z)\) and \(P_{n+1}(z)\). The first few of these quantities are given below. It should be noted however, that these expressions are more complicated than the latter circumstance may indicate a deeper issue involving the \(I_n(z)\) integrals. We have
\[
\mathcal{I}_0(z) = (1 + z)[\ln \left(\frac{1 + z}{z} \right) - 1],
\]
\[
\mathcal{I}_1(z) = -2 (1 + z)\ln\left(\frac{1 + z}{z} \right) - 1 + 2 \ln(1 - z) Li_2\left(\frac{1}{1 - z} \right),
\]
\[
\mathcal{I}_2(z) = 6 (1 + z)[2 Li_3\left(\frac{1}{1 - z} \right) + \frac{z^2}{6} - 1 + 2 \zeta(3) - \{Li_2\left(\frac{1}{1 - z} \right) + \frac{z^2}{6} - 1\} \ln\left(\frac{1 + z}{z} \right)]
\]
\[
+ 6 (1 - z) \{Li\left(\frac{1}{1 - z} \right) + \ln\left(\frac{1 + z}{z} \right) \ln\left(\frac{1 + z}{z} \right)\},
\]
The expression for \(\mathcal{I}_4(z)\) has been computed using Mathematica and contains in a condensed form seventy-four terms including a fifth order polylogarithm function. That quantity will not be displayed here for the sake of brevity.

Below we include integrals which occur in the calculation of \(\mathcal{I}_4(z)\) but are not directly available within Mathematica i.e.
\[
\int Li_4\left(\frac{1}{1 - t} \right) \, dt = \frac{1}{2} Li_2\left(\frac{1}{1 - t} \right)^2 + t Li_4\left(\frac{1}{1 - t} \right) + \ln(1 - t) Li_3\left(\frac{1}{1 - t} \right),
\]
\[
\int Li_2(t)^2 \, dt = -2 + 6t + 6[1 - t - \frac{z^2}{6} \ln(1 - t) - 2[1 - t - \ln(t)] \ln^2(1 - t)
\]
\[
- 2[t - (1 + t) \ln(1 - t)] Li_2(t) + t Li_2(t)^2 + 4 Li_3(1 - t),
\]
\[
\int \ln^2(t) \ln(1 - t) \, dt =
\]
\[
- 4 + 24x + 12[1 - x] \ln(1 - x) - 2[1 - x] \ln(1 - x)^2 - \frac{1}{2} \ln(1 - x)^4
\]
\[
- 12x \ln(x) - 4[1 - 2x] \ln(1 - x) \ln(x) - 2x \ln(1 - x)^2 \ln(x) + 2 \ln(1 - x)^3 \ln(x)
\]
\[
+ [2 - \ln(1 - x)^2] \ln(x)^2 - (1 - x)[2 - 2 \ln(1 - x) + \ln(1 - x)^2] \ln(x)^2
\]
\[
+ [4 - 4 \ln(1 - x) + 2 \ln(1 - x)^2] Li_2(1 - x) - [4 - 4 \ln(x) + 2 \ln(x)^2] Li_2(x)
\]
\[
- [2 \ln(1 - x)^2 - 4 \ln(1 - x) \ln(x) + 2 \ln(x)^2] Li_2\left(\frac{1}{x} \right)
\]
\[
- 4[1 - \ln(x)] Li_3(x) + 4[1 - \ln(1 - x)] Li_3(1 - x) + 4 \ln(x) - \ln(1 - x)] Li_3\left(\frac{x}{x - 1} \right)
\]
\[
+ 4 Li_4(1 - x) - 4 Li_4(1 - x).
\]
Appendix B

The integral $\mathcal{I}(z)$ is interesting in that it provides a way to obtain a closed form expression for the slowly converging infinite sum $\sum_{k=1}^{\infty} \frac{\Psi'(k)}{k}$ where $\Psi'(k)$ is the trigamma function \[11\]. This sum does not appear to have been previously reported in the literature and is included here. Using the limiting values for $\mathcal{I}(z)$ i.e.

$$\mathcal{I}(1) = -\frac{11}{360} \pi^4,$$
$$\mathcal{I}(0) = -\frac{1}{45} \pi^4,$$

and the infinite series representation for the dilogarithm function which occurs in $\mathcal{I}(z)$ we get

$$\mathcal{I}(1) - \mathcal{I}(0) = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \frac{z^k \ln(z)}{1-z} dz,$$
$$-\frac{\pi^4}{120} = -\sum_{k=1}^{\infty} \frac{\Psi'(k+1)}{k^2}.$$

Expanding the trigamma function with the use of the recurrence relation for $\Psi'$ i.e.

$$\Psi'(k+1) = \Psi'(k) - \frac{1}{k^2},$$

together with the well known sum $\sum_{k=1}^{\infty} 1/k^4 = \pi^4/90$ we get the value of the desired summation i.e.

$$\sum_{k=1}^{\infty} \frac{\Psi'(k)}{k^2} = 7 \frac{\pi^4}{360}.$$

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[10] Ref. 2, p. 184.

[11] Ref. 8, section §6.4.1, Also see wikipedia.org/wiki/Trigamma_function.