DUALITY-SYMMETRIC GRAVITY AND SUPERGRAVITY: TESTING THE PST APPROACH

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ABSTRACT. Drawing an analogy between gravity dynamical equation of motion and that of Maxwell electrodynamics with an electric source we outline a way of appearance of a dual to graviton field. We propose a dimensional reduction ansatz for the field strength of this field which reproduces the correct duality relations between fields arising in the dimensional reduction of D-dimensional gravity action to D-1 dimensions. Modifying the PST approach we construct a new term entering the action of D=11 duality-symmetric gravity and by use of the proposed ansatz we confirm the relevance of such a term to reproduce the correct duality-symmetric structure of the reduced theory. We end up extending the results to the bosonic sector of D=11 supergravity.

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1. INTRODUCTION

Recent progress in studying the hidden symmetry group of Superstring/M-theory \cite{11-15} reveals necessity of taking seriously into account a “graviton dual” field. Dynamics of such a field has been intensively studied in literature in the linearized limit \cite{17, 2, 7, 18, 19} together with studying the actions that lead in the same approximation to the equation of motion for the “graviton dual” field. The attempts to extend the linear theory to a non-linear one faced the troubles that were summarized in the “no-go” theorems \cite{15} which forbid the action for interacting theory solely in terms of the “graviton dual” field. This situation is very similar to that of constructing eleven-dimensional supergravity with a six-index photon field \cite{20, 21} where the simple arguments \cite{22} lead to the conclusion on impossibility of constructing the gauge sector of theory solely in terms of the six-index photon. However a way to go beyond these arguments is to consider a duality-symmetric theory of D=11 supergravity with an original three-index photon field and with the “dual-to-three index photon” field that enter the action on equal footing \cite{23} (see also \cite{24} for an early attempt). Here we will follow the same way and will closely inspect a possibility to construct the non-linear theory of gravity and supergravity with the “graviton dual” field standing on a point of the duality-symmetric formulation. The arguments in favor of such a consideration are as follows. First, we recall

Date: March 27, 2022.

Key words and phrases. Supergravity, duality, dimensional reduction.
the close connection between D=11 supergravity and type IIA theory that follows from the D=11 supergravity by dimensional reduction. Importance of having the “graviton dual” field in D=11 supergravity for D=10 type IIA theory has been pointed out as in the purely algebraic aspect [2] as well as in the context of searching for the complete duality-symmetric version of D=11 supergravity [25] involving the “graviton dual” field. As well as recovering the correct algebra for a non-linear realization of type IIA supergravity requires of having a generator corresponding to the graviton dual partner in its eleven-dimensional counterpart [7] this dual field is required for recovering in a straightforward way the complete duality-symmetric formulation of type IIA supergravity [20]. Second, since the duality-symmetric version of type IIA supergravity was realized on the ground of Pasti-Sorokin-Tonin technique [27], it is naturally to expect the PST-like structure of additional terms in D=11 supergravity action which encode the duality relations between the graviton and its dual partner. As we have known the structure of the duality-symmetric type IIA supergravity [20] we can therefore use of this advantage for deducing an ansatz for the dimensional reduction of the “graviton dual” field. Finally, we can try to apply the ansatz for constructing the additional terms in D=11 supergravity action which after reduction will reproduce the correct duality-symmetric structure of D=10 type IIA supergravity. To this end we shall take into account the observations that have been previously made under construction of duality-symmetric theories for different sub-sectors of maximal higher-dimensional supergravities [23], [28], [29], [26].

It has to be emphasized that as a first step towards completing our task we shall find, without an appeal to a method of constructing the action, a convenient representation of the gravity equation of motion in a way that allows us to present the latter as the Bianchi identity for a dual field. It turns out to be convenient for this purpose to write down the gravity equation of motion in a form which is similar to the dynamics of Maxwell theory with a source. As soon as such a representation is recovered it suggests a way of extracting the dual field and to apply the PST formalism to construct the action from which the duality relations between graviton and its dual partner will follow as equations of motion.

For the sake of simplicity we are getting started our quest of the complete duality-symmetric formulation of D=11 supergravity with pure gravity case extended hereafter the results to the bosonic sector of D=11 supergravity. The scheme we propose does not depend on the space-time dimension $D \geq 4$ where gravity has dynamical degrees of freedom, and since the gravity is the key ingredient of supergravity the results can be applied for all theories of supergravity in diverse dimensions though as we have mentioned above we will end up with the dimension of space-time $D=11$ having in mind mostly the applying to this case.

2. Kaluza-Klein ansatz for graviton dual field

Let us begin with introducing the graviton field described by a vielbein $\hat{e}^{\hat{a}[1]}$ whose index $\hat{a}$ runs from zero to $D - 1$ and takes the value in the tangent space Lorentz group $SO(1, D - 1)$, and with introducing the dual to the vielbein field $\hat{A}^{\hat{a}(D-3)}$. The way in which the latter appears is as follows. The first order action for the pure gravity is

$$S_{EH} = \int_{M^{D}} \hat{R}^{\hat{a}\hat{b}} \wedge \hat{\Sigma}_{\hat{a}\hat{b}};$$

(1)

where $\hat{R}^{\hat{a}\hat{b}} = d\hat{\omega}^{\hat{a}\hat{b}} - \hat{\omega}^{\hat{a}\hat{c}} \wedge \hat{\omega}^{\hat{c}\hat{b}}$ is the curvature 2-form and

$$\hat{\Sigma}_{\hat{a}_{1} \ldots \hat{a}_{n}} = \frac{1}{(D - n)!} \epsilon_{\hat{a}_{1} \ldots \hat{a}_{D}} \hat{e}^{\hat{a}_{n+1}} \wedge \cdots \wedge \hat{e}^{\hat{a}_{D}}$$

(2)
is the \((D-n)\)-form constructed out of vielbeins \(\hat{e}^a\). The equations of motion following from the action are

\[
\hat{\Sigma}_{abc} \wedge \hat{R}^{bc} = 0, \tag{3}
\]

\[
\hat{\Sigma}_{\hat{a}\hat{b}\hat{c}} \wedge \hat{T}^{\hat{c}} = 0, \tag{4}
\]

where we have introduced the torsion 2-form \(\hat{T}^{\hat{a}} = d\hat{e}^{\hat{a}} - \hat{e}^{\hat{b}} \wedge \hat{\omega}^{\hat{a}}_{\hat{b}}\). The latter equation set the torsion to zero that is the algebraic relation expressing the connection \(\hat{\omega}^{\hat{a}}_{\hat{b}}\) through the vielbeins and their derivatives. In the sequel we will be mostly concentrated on the dynamical equation of motion.

It is easy to check that the Einstein equation (3) admits the following representation

\[
d(\hat{\omega}^{bc} \wedge \hat{\Sigma}_{abc}) = \hat{\omega}^{bc} \wedge d\hat{\Sigma}_{abc} + (-)^{D-3} \hat{\omega}^b_d \wedge \hat{\omega}^{dc} \wedge \hat{\Sigma}_{\hat{a}\hat{b}\hat{c}}. \tag{5}
\]

One can rewrite this equation in the form

\[
d\hat{\Sigma}_{\hat{a}}^{[2]} = \hat{\Sigma}_{\hat{a}}^{[1]} \tag{6}
\]

with \(\hat{\Sigma}_{\hat{a}}^{[2]} = \hat{\omega}^{bc} \wedge \hat{\Sigma}_{\hat{a}bc}\), that looks very much like the equation of motion for Maxwell electrodynamics with an electric source. This analogy has been pointing out in literature for a long time (cf. e.g. [30], [31]). However, the difference between the electrodynamics equation of motion and (6) is apparent. The former contains a “bare” potential \(A^{[1]}\) whereas the latter does not, since the r.h.s. and the l.h.s. of (6) depend on the vielbeins and connection. The only vielbein is the true dynamical field, hence one should resolve the connection through the vielbeins using the torsion free constraint (3). The answer is

\[
\hat{\omega}^{\hat{a}\hat{b}}_{\hat{m}} = \frac{1}{2} \hat{e}^{\hat{a}\hat{b}} \partial_{[\hat{m}} \hat{e}^{\hat{b}]\hat{a}} - \frac{1}{2} \hat{e}^{\hat{a}\hat{b}} \partial_{[\hat{m}} \hat{e}^{\hat{b}]\hat{a}} - \frac{1}{2} \hat{e}_{\hat{m}\hat{c}} \hat{e}^{\hat{a}\hat{b}} \hat{e}^{\hat{d}\hat{c}} \partial_{[\hat{m}} \hat{e}^{\hat{d}\hat{c}]}, \tag{7}
\]

where the indices from the second middle of Latin alphabet are the curved ones. Substituting this expression into the l.h.s. of (6) we arrive at

\[
d(\hat{\omega}^{bc} \wedge \hat{\Sigma}_{abc}) = \frac{1}{\alpha_D} d(\hat{\omega}^{bc} \wedge \hat{\Sigma}_{abc}) + d\hat{S}_{\hat{a}}^{[D-2]}, \tag{8}
\]

where we have introduced the form \(\hat{S}_{\hat{a}}^{[D-2]}\) which is a function of vielbeins and their derivatives and the numerical coefficient \(\alpha_D\) which takes \(\pm 1\) in dependence on the space-time signature setting and the Hodge star definition. It is worth mentioning that the r.h.s. of eq. (8) is nothing but the Landau-Lifshitz pseudo-tensor which is a conserved “current” and therefore admits the following representation

\[
\hat{\omega}^{bc} \wedge d\hat{\Sigma}_{\hat{a}bc} + (-)^{D-3} \hat{\omega}^{\hat{a}\hat{b}} \wedge \hat{\omega}^{dc} \wedge \hat{\Sigma}_{\hat{a}\hat{b}\hat{c}} := \hat{\Sigma}_{\hat{a}}^{[1]} = \hat{\Sigma}_{\hat{a}}^{[2]} \tag{9}
\]

Taking into account (8) and (9), eq. (5) can be rewritten as

\[
d(\hat{\omega}^{bc} \wedge \hat{\Sigma}_{abc}) = \frac{1}{\alpha_D} d(\hat{\omega}^{bc} \wedge \hat{\Sigma}_{abc}) + \hat{\Sigma}_{\hat{a}}^{[1]}, \tag{10}
\]

with

\[
\hat{\Sigma}_{\hat{a}}^{[1]} = d\hat{\Sigma}_{\hat{a}}^{[2]}, \tag{11}
\]

where we have denoted \(\hat{\Sigma}_{\hat{a}}^{[2]} = \alpha_D(\hat{S}_{\hat{a}}^{[2]} - \hat{\Sigma}_{\hat{a}}^{[D-2]}). \) Notice that \(\hat{\Sigma}_{\hat{a}}^{[2]}\) is also a function of vielbeins and their derivatives.

One more comment is needed before proceeding further. The Einstein equation (3) is generally covariant while its representations (5), (6), (10) are recorded w.r.t. the usual (non-covariant) derivatives that could lead to the conclusion on their non-covariance. This puzzle can be resolved through the observation (cf. for instance [33]) that the r.h.s. of eqs. (5), (6), (10) is not a true tensor, but a pseudo-tensor, hence the non-covariance of the l.h.s. of the above mentioned eqs. is compensated
by the pseudo-tensor character of the r.h.s. leaving nevertheless eqs. (9), (10) to be covariant. Another way to see that is, for instance, to notice the relation
\[
d(\hat{\star}d\hat{e}_a - \hat{\star}\tilde{G}^{[2]}_a) = \alpha_d (-)^{D-3}\Sigma_{\hat{a}\hat{b}\hat{c}} \wedge \hat{R}^{\hat{b}\hat{c}}.
\]

Now we have the almost complete analogy between gravity and Maxwell theory with an electric source in the sense of representing the former through the only true potentials. The dual to the \( d\hat{e}_a \) field strength is
\[
\hat{F}^{[D-2]}_a = d\hat{A}^{[D-3]}_a + \hat{\star}\tilde{G}^{[2]}_a \equiv \hat{F}^{[D-2]}_a + \hat{\star}\tilde{G}^{[2]}_a,
\]
and the “graviton dual” field has appeared.

Since a part of the dual to the graviton field will become dual fields to a dilaton and a Kaluza-Klein vector field after dimensional reduction from \( D \) to \( D - 1 \), we will use this fact to deduce the reduction ansatz for the field strength of \( \hat{A}^{[D-3]}_a \) that will reproduce the correct structure of the duality-symmetric gravity in \( D - 1 \) space-time dimensions. After establishing the ansatz we will try to apply it for lifting the known \( D - 1 \)-dimensional action up to the \( D \)-dimensional space-time.

Splitting the tangent space index as \( \hat{a} = (a, z) \), \( a = 0, \ldots, D - 2 \) we choose the standard Kaluza-Klein ansatz for the vielbein
\[
\hat{e}^{[1]} = e^{\alpha \phi} e^{\alpha [1]}, \quad \hat{e}^{[1]} = e^{\beta \phi}(dz + A^{[1]}),
\]
where \( \phi \) denotes the dilaton field, \( A^{[1]} \) stands for the Kaluza-Klein vector field and \( z \) is the direction of the reduction. All the fields on the r.h.s. of eq. (13) are independent on the reduction coordinate. The parameters \( \alpha \) and \( \beta \) are related to fixing the space-time dimensions \( D \) and to each other via
\[
\alpha^2 = \frac{1}{2(D - 2)(D - 3)}, \quad \beta = -(D - 3)\alpha,
\]
that corresponds to the Einstein frame after reduction.

The vielbeins’ dimensional reduction ansatz falls into a general class of ansätze which determine the reduction of a (Lorentz-valued) \( n \)-form
\[
\hat{\Omega}^{(a,z)[n]} = \Omega^{(a,z)[n]} + \omega^{(a,z)[n-1]} \wedge (dz + A^{[1]}).
\]
To specify the ansatz one has to determine the forms \( \Omega^{(a,z)[n]} \), \( \omega^{(a,z)[n-1]} \) for each case under consideration. The other relation which will be under the usage in what follows is
\[
\hat{\star} \hat{\Omega}^{(a,z)[n]} = a_{(n)} e^{-2(n-1)\alpha \phi} \Omega^{(a,z)[n]} \wedge (dz + A^{[1]})
+ a_{(n-1)} e^{2(D-n-1)\alpha \phi} \omega^{(a,z)[n-1]}.
\]
The numerical coefficients \( a_{(n)} \), \( a_{(n-1)} \) taking the values \( \pm 1 \) encode the information on the space-time signature choice and the definition of the Hodge operator. To distinguish \( D \) and \( (D - 1) \)-dimensional Hodge stars we have equipped the former with a hat.

Let us now turn to the analysis of gravity equation of motion (10). The equation reads, at least for trivial topology setting,
\[
\hat{\star}d\hat{e}^{[1]} = d\hat{A}^{[D-3]} + \hat{\star}\tilde{G}^{[2]} = \hat{\star}d\hat{a}^{[D-3]} + \hat{\star}\tilde{G}^{[2]}.
\]
Splitting the tangent space index \( \hat{a} \) one arrives at the following relations
\[
\hat{\star}d\hat{e}^{[1]} = d\hat{A}^{[D-3]} + \hat{\star}\tilde{G}^{[2]} = \hat{\star}d\hat{a}^{[D-3]} + \hat{\star}\tilde{G}^{[2]},
\]
\[
\hat{\star}d\hat{e}^{[1]} = d\hat{A}^{[D-3]} + \hat{\star}\tilde{G}^{[2]} = \hat{\star}d\hat{a}^{[D-3]} + \hat{\star}\tilde{G}^{[2]}.
\]
Since the duality relations for the dilaton and the Kaluza-Klein vector field is encoded into the latter equation let us begin our analysis with (19).
Taking into account (13), (15), (16) one arrives at the following relation
\[ a_{(2)}e^{(\beta-2\alpha)\phi} * dA[1] + (dz + A[1]) - a_{(1)}e^{2(D-3)\alpha\phi + \beta \phi} * d\phi = \]
\[ = F[z^{[D-2]}] + F[z^{[D-3]}] \quad (dz + A[1]) \]
\[ + a_{(2)}e^{-2\alpha \phi} * G^[z[2]] \quad (dz + A[1]) + a_{(1)}e^{2(D-3)\alpha \phi} * g^{z[1]}, \]
which contains two independent parts
\[ a_{(2)}e^{(\beta-2\alpha)\phi} * dA[1] = F[z^{[D-3]}] + a_{(2)}e^{-2\alpha \phi} * G^[z[2]], \]
and
\[ -a_{(1)}e^{2(D-3)\alpha \phi + \beta \phi} * d\phi = F[z^{[D-2]}] + a_{(1)}e^{2(D-3)\alpha \phi} * g^{z[1]}. \]
Here we have denoted \( F[z^{[D-2]}] \equiv dA[z^{[D-3]}] = F[z^{[D-2]}] + F[z^{[D-3]}] \quad (dz + A[1]). \)

Since by construction (cf. (8), (10), (11)) \( \hat{G}^[z[2]] \) depends on veilbeins and their derivatives \( G^[z[2]] \) and \( g^{z[1]} \) are completely determined by the reduction ansatz (13).

Hence, our aim is to deduce the reduction ansatz for \( \hat{G}^[z[2]] \), i.e. to determine \( F[z^{[D-2]}] \) and \( F[z^{[D-3]}] \), which will lead to the correct duality relations between the fields.

The ansatz we propose for \( F[z^{[D-3]}] \) has the following form
\[ F[z^{[D-3]}] = -a_{(2)}e^{-\beta \phi} * dA[1] - a_{(2)}e^{-2\alpha \phi} * G^[z[2]]. \]
It is easy to see that the substitution of this ansatz into (21) leads to the correct duality relation between the Kaluza-Klein vector field and its dual partner
\[ e^{2(\beta-\alpha)\phi} * dA[1] = -dA[D-4], \]
which can be extracted from the Kaluza-Klein vector field equation of motion after the dimensional reduction of pure gravity from \( D \) to \( D-1 \).

The other part forming the ansatz for \( \hat{F}[z^{[D-2]}] \) is
\[ \hat{F}[z^{[D-2]}] = a_{(1)}e^{-\beta \phi} *(dA[1] - \frac{3}{4}dA[D-4] \wedge A[1]) + a_{(1)}e^{2(D-3)\alpha \phi + \beta \phi(1 - \beta)} * d\phi \]
\[ -a_{(1)}e^{2(D-3)\alpha \phi} * g^{z[1]}. \]
Substituting the latter into (22) one can find the correct duality relation between the dilaton and its dual partner
\[ *d\phi = -dA[D-3] + \frac{3}{4} dA[D-4] \wedge A[1]. \]

Dealing with eq. (15) is quite a bit complicated. By use of (13), (15), (16) one obtains
\[ a_{(2)}e^{-2\alpha \phi} * d(e^{\alpha \phi} e^a) \wedge (dz + A[1]) = F[\alpha]^{[D-2]} + F[\alpha]^{[D-3]} \quad (dz + A[1]) \]
\[ + a_{(2)}e^{-2\alpha \phi} * G^{[a][2]} \quad (dz + A[1]) + a_{(1)}e^{2(D-3)\alpha \phi} * g^{a[1]}, \]
from which it immediately follows
\[ F[\alpha]^{[D-2]} = -a_{(1)}e^{2(D-3)\alpha \phi} * g^{a[1]}. \]
The other equation that is contained into (27) is
\[ a_{(2)}e^{-\alpha \phi} * (de^a - \alpha d\phi \wedge e^a) \quad F[\alpha]^{[D-3]} + a_{(2)}e^{-2\alpha \phi} * G^{[a][2]}. \]

To find the correct ansatz for \( F[\alpha]^{[D-3]} \) let us introduce a new form \( \hat{S}[\alpha][2] \). This form is defined by
\[ d * \hat{S}[\alpha][2] = \star \hat{S}[\alpha][1] \]
with
\[ \star \hat{S}[\alpha][1] = \omega^{bc} \wedge d\Sigma_{abc} + (-)^{D-4} \omega^b \wedge \omega^{dc} \wedge \Sigma_{abc} - dS[\alpha][D-3] + M[1][D-2]. \]
Here $\omega^{ab}$ is the $(D-1)$-dimensional connection one-form, $\Sigma^{abc}$ stands for the lower-dimensional analog of $[2]$, $S_a^{(D-3)}$ is the $(D-1)$-dimensional counterpart of the form $\hat{S}_a^{(D-2)}$ introduced in [3] that appears after resolving the lower-dimensional torsion free constraint, and $M_a^{(D-2)}$ is the energy-momentum tensor of the dilaton and of the Kaluza-Klein vector field which is a closed form and therefore

$$M_a^{(D-2)} = d\hat{a}_a^{(D-3)},$$

(32)

Taking the ansatz to be

$$F^{(D-3)} = -a(2)e^{-\phi}(dA^{(D-4)} + \ast \hat{G}^{a[2]} + \alpha * (d\phi \wedge e^a) + e^{-\phi} * G^{a[2]}),$$

(33)

one can obtain the duality relation (cf. (17))

$$\ast de^a = -(dA^{(D-4)} + \ast \hat{G}^{a[2]}),$$

(34)

that reproduces the correct equation of motion for the graviton field

$$d(\ast de^a) = -\ast \hat{a}^{[1]},$$

(35)

To summarize, the ansatz we proposed so far is

$$\hat{F}^{(D-2)} = -a(1)e^{2(D-3)\phi} + g^{a[1]}$$

$$- a(2)e^{-\phi}(dA^{(D-4)} + \ast \hat{G}^{a[2]} + \alpha * (d\phi \wedge e^a) + e^{-\phi} * G^{a[2]} \wedge (dz + A^{[1]})),$$

(36)

$$\hat{F}^{(D-2)} = a(1)e^{-\beta\phi}(dA^{(D-3)} - \frac{3}{4}dA^{(D-4)} \wedge A^{[1]})) + a(1)e^{2(D-3)\phi + \beta(1 - \beta)} * d\phi - a(1)e^{2(D-3)\phi} * g^{a[1]} - a(2)e^{-\beta\phi}(dA^{(D-4)} + e^{-2\phi} * G^{(2)} \wedge (dz + A^{[1]}))$$

(37)

Let us close this part of the paper with some remarks on the proposed ansatz structure. First, the exponential factors of leading expressions in [20], [26], [33] are the same as in [13] but are opposite in sign. This is the indication of duality. And second, there are uncommon quantities entering the ansatz such as that of coming from the reduction of $\hat{G}^{a[2]}$. We are forced to include such quantities into the ansatz since we can not overcome the technical problem of evaluating the explicit expression for $\hat{G}^{a[2]}$ which is defined by (11). The same concerns to the $\hat{G}^{a[2]}$. However the ansatz we proposed does not depend on the explicit structure of the above mentioned expressions and is universal, though requires the very well knowledge of the duality relations in lower dimensional theory after reduction.

3. Duality-symmetric action for gravity and complete duality-symmetric action for D=11 supergravity

Having established the reduction ansatz let us turn to the more sophisticated problem of constructing the action for the duality-symmetric gravity. Since that requires more tuning we are fixing the space-time dimensions $D$ to be eleven, and will follow the notation of [20]. In this notation after dimensional reduction we should in particular reproduce from the action the duality relations [24], [26] which now have the following form

$$\mathcal{F}^{[8]} = dA^{[7]} + e^{-\frac{5}{2}\phi} * dA^{[1]} = 0,$$

(38)

$$\mathcal{F}^{[9]} = (dA^{[8]} - \frac{3}{4}dA^{[7]} \wedge A^{[1]})) * d\phi = 0.$$

(39)

Here we have fixed the parameters $\alpha$ and $\beta$ to be

$$\alpha = +\frac{1}{12}, \quad \beta = -\frac{2}{3}.$$  

(40)
The relevant part of the ten-dimensional duality-symmetric action from which these relations come is (cf. \[26\])

\[
S_{d.s.} = \frac{1}{2} \int_{\mathcal{M}^{10}} \sum_{n=1}^{2} \left( F^{[10-n]} \wedge F^{[n]} + i_v F^{[10-n]} \wedge v \wedge F^{[n]} + v \wedge F^{[10-n]} \wedge i_v F^{[n]} \right).
\]

(41)

Our aim is to demonstrate that the action \(41\) follows after dimensional reduction from the part of the gravity duality-symmetric action

\[
S_{PST} = \int_{\mathcal{M}^{11}} \frac{1}{2} \hat{\psi} \wedge \hat{F}^a[9] \wedge i_{\hat{\psi}} \hat{F}^b[2] \eta_{ab},
\]

(42)

where following to the PST approach \[27\] we have introduced a scalar field \(\hat{\psi}\) that enters the action in a non-polynomial way through the one-form \(\hat{\psi}\)

\[
\hat{\psi} = \frac{da(\hat{x})}{\sqrt{-\partial_m a \, g^{mn} \partial_n a}}.
\]

(43)

\(\eta_{ab}\) is the Minkowski metric tensor and

\[
\hat{F}^a[2] = d\hat{e}^a - \hat{\psi} (dA^a[8] + \hat{G}^a[2]),
\]

(44)

\[
\hat{F}^a[9] = -\hat{\psi} \hat{F}^a[2].
\]

(45)

Note that though these generalized field strengths are constructed out of the non-covariant quantities, they enter the generalized field strengths in the covariant combinations. Therefore, \(F^a[2]\) and \(F^a[9]\) are the covariant objects.

Splitting the indices \(\hat{a}, \hat{b}\) we can rewrite \[42\] as

\[
S_{PST} = \int_{\mathcal{M}^{11}} \frac{1}{2} \hat{\psi} \wedge \hat{F}^a[9] \wedge i_{\hat{\psi}} \hat{F}^b[2] \eta_{ab} - \frac{1}{2} \hat{\psi} \wedge \hat{F}^z[9] \wedge i_{\hat{\psi}} \hat{F}^z[2].
\]

(46)

By use of ansätze \[35\], \[37\] (in our notation \(a(1) = a(2) = -1\)) one gets

\[
\hat{F}^a[2] = e^{\frac{i}{2} \phi} \left[ de^a + \ast (dA^a[7] + \ast G^a[2]) \right] \equiv e^{\frac{i}{2} \phi} F^a[2],
\]

(47)

\[
\hat{F}^a[9] = e^{-\frac{i}{2} \phi} F^a[8] \wedge (dz + A[1]), \quad F^a[8] = \ast F^a[2],
\]

(48)

\[
\hat{F}^z[9] = -e^{\frac{i}{2} \phi} (F^0[9] - F^8[8] \wedge (dz + A[1]),
\]

(49)

\[
\hat{F}^z[2] = e^{-\frac{i}{2} \phi} (F^2[2] - F^1[1] \wedge (dz + A[1]), \quad F^0[9] = \ast F[1], \quad F^8[8] = e^{-\frac{i}{2} \phi} \ast F[2],
\]

(50)

where \(F^8[8], F^9[9]\) have been defined in \[35\], \[39\] (they are not equal to zero of course; the latter shall follow from the action as an equation of motion). Supplying the stuff by the following ansatz for \(\hat{\psi}\) \[26\]

\[
\hat{\psi} = e^{\frac{i}{2} \phi} v, \quad i_v (dz + A[1]) = 0, \quad v = \frac{da(\hat{x})}{\sqrt{-\partial_m a \, g^{mn} \partial_n a}}, \quad v \neq \psi(z),
\]

(51)

and using the standard rules of dimensional reduction (see e.g. \[35\] and Refs. therein, or Appendices in \[26\]), after integration over \((dz + A[1])\) one arrives at

\[
S_{PST} = \frac{1}{2} \int_{\mathcal{M}^{10}} v \wedge F^a[8] \wedge i_v F^b[2] \eta_{ab} - v \wedge F^8[8] \wedge i_v F^2[2] - v \wedge F^9[9] \wedge i_v F[1].
\]

(52)
After the standard manipulations one can convince yourself that the reduction of the complete duality-symmetric \(D=11\) gravity action \(S = S_{EH} + S_{PST}\) results in
\[
S = - \int_{\mathcal{M}^{10}} \left( R^{ab} \wedge \Sigma_{ab} + \frac{1}{2} v \wedge F^{8[a] \wedge i_v \hat{F}^{8][b]} \eta_{ab} \right) \\
+ \frac{1}{2} \int_{\mathcal{M}^{10}} \sum_{n=1}^{2} \left( F^{[10-n]} \wedge F^{[n]} + i_v \hat{F}^{[10-n]} \wedge v \wedge F^{[n]} + v \wedge F^{[10-n]} \wedge i_v \hat{F}^{[n]} \right). 
\]
(53)

It is easy to recognize the duality-symmetric structure of \(D=10\) gravity action, and the second line is precisely the action (41) we are looking for. Therefore, we have established the relevance of the proposed \(S_{PST}\) term in the action for the duality-symmetric gravity.

One may wonder why having a new term entering the action we did not modify \(\hat{G}^{[2]}\) though this term is not a topological term and therefore contributes into the energy-momentum tensor. The reason is the same as for another application of the PST approach. The PST terms do not spoil the dynamics of an original theory since their contribution to the equations of motion is a combination of the duality relations which is zero on-shell. The latter is guaranteed by a special symmetry, one of the two PST symmetries [27]. That is for instance why we do not need to take into account the contribution of two last terms of (53) into \(G^{[2]}\). Even leaving the issue of establishing the PST symmetries out of consideration, one could notice an indication of their presence in the eleven-dimensional theory since the special symmetries of the last two terms of the action (53) are the bits of the corresponding eleven-dimensional symmetries.

To give more rigorous arguments in favor of the discussion above, let us consider a variation of the action
\[
S = S_{EH} + S_{PST} 
\]
with \(S_{EH}\) of (41) and \(S_{PST}\) of (42). The variation of the Einstein-Hilbert term results in
\[
\delta S_{EH} = \int_{\mathcal{M}^{11}} \delta \hat{e}^{\hat{a}} \wedge d \hat{F}^{\hat{b}[9]} \eta_{\hat{a}\hat{b}}. 
\]
(55)

What concerns the PST term it is convenient to split its variation into the standard for such an approach part where we will effectively treat the vielbeins on the same footing as another gauge field, i.e. as fields completely independent of metric, and that of varying the metric
\[
\delta S_{PST} = \int_{\mathcal{M}^{11}} \delta_0 \mathcal{L}_{PST} + \delta_\epsilon \mathcal{L}_{PST}. 
\]
(56)

The standard variation ends up with
\[
\delta_0 S_{PST} = \int_{\mathcal{M}^{11}} \left( \delta \hat{A}^{\hat{a}[8]} + \frac{\delta a}{\sqrt{-(\partial a)^2}} i_v \hat{F}^{8}[a] \right) \eta_{\hat{a}\hat{b}} \wedge d(v \wedge i_v \hat{F}^{2}[b]) \\
+ \int_{\mathcal{M}^{11}} \left( \delta \hat{e}^{\hat{a}} + \frac{\delta a}{\sqrt{-(\partial a)^2}} i_v \hat{F}^{2}[a] \right) \eta_{\hat{a}\hat{b}} \wedge d(v \wedge i_v \hat{F}^{9}[b]) \\
- \int_{\mathcal{M}^{11}} \left( \delta \hat{e}^{\hat{a}} \eta_{\hat{a}\hat{b}} \wedge d \hat{F}^{9}[9] + \delta [4 \hat{G}^{2}[2] \eta_{\hat{a}\hat{b}} \wedge v \wedge \hat{F}^{2}[2]] \right). 
\]
(57)

To read off the other part of the variation we shall proceed as in the case of extracting the energy-momentum tensor. In terms of differential forms this variation
is
\[
\delta_s(\Omega^{[n]} \wedge *\Omega^{[n]}) = (\delta_s \Omega^{[n]} + \Omega^{[n]} \wedge \delta_s(*\Omega^{[n]})) = \\
\frac{1}{(n-1)!} \delta e^a_n \wedge e^{a_{n-1}} \wedge \cdots \wedge e^{a_1} \Omega_{a_1 \cdots a_{n-1} a_n} \wedge *\Omega^{[n]} \\
+ \frac{1}{n!} \left( D - n - 1 \right)! \Omega^{[n]} \wedge \delta e^{b_D-n} \wedge e^{b_{D-n-1}} \wedge \cdots \wedge e^{b_1} \epsilon_{b_1 \cdots b_{D-n} a_1 \cdots a_n} \Omega^{[n]}_{a_1 \cdots a_n}.
\] (58)

Following this pattern we arrive at
\[
\delta_s \mathcal{S}^{\text{PST}} = \int_{\mathcal{M}^{11}} \frac{1}{2} \delta \tilde{e}^{\tilde{\alpha}} \left( \frac{1}{8!} \tilde{e}^{\tilde{\alpha}} \wedge \cdots \wedge \tilde{e}^{\tilde{\epsilon_1}} \mathcal{F}^{[9]}_{\tilde{\epsilon_1} \cdots \tilde{\epsilon_8} \tilde{\alpha_9}} \right) \eta_{\tilde{\alpha} \tilde{\beta}} \wedge \tilde{i}_\tilde{e} \mathcal{F}^{[2]}_{\tilde{\beta}} \\
+ \int_{\mathcal{M}^{11}} \frac{1}{2} \delta \tilde{e} \wedge \mathcal{F}^{[9]}_{\tilde{\alpha} \tilde{\beta}} \eta_{\tilde{\alpha} \tilde{\beta}} \wedge \delta \tilde{e} (i_{\tilde{e}} \mathcal{F}^{[2]}_{\tilde{\beta}}) \tilde{\epsilon}.
\] (59)

Taking into account the results of (65), (67), (68), one can derive the following sets of special symmetries
\[
\delta a(\hat{x}) = 0, \quad \delta \hat{\epsilon} \hat{a} = da \wedge \hat{\varphi}^{[0]}, \quad \delta \hat{A}^{[8]} = da \wedge \hat{\varphi}^{[7]} - d^{-1} \delta (\hat{\mathcal{G}}^{[2]}), \\
\delta a(\hat{x}) = \Phi(\hat{x}), \quad \delta \hat{e} \hat{a} = -\frac{\Phi}{\sqrt{-(\partial a)^2}} i_{\hat{e}} \mathcal{F}^{[2]}_{\tilde{\alpha}}, \\
\delta \hat{A}^{[8]} = -\frac{\Phi}{\sqrt{-(\partial a)^2}} i_{\hat{e}} \mathcal{F}^{[9]}_{\tilde{\alpha}} \\
- d^{-1} \left( \delta (\hat{\mathcal{G}}^{[2]} + \frac{1}{2} \delta \hat{e} \hat{\epsilon_0} \wedge \cdots \wedge \hat{e} \hat{\epsilon_8} \mathcal{F}^{[9]}_{\hat{\epsilon_1} \cdots \hat{\epsilon_8} \hat{\alpha_9}} - \frac{1}{2} \delta (\delta \hat{e} \hat{\epsilon} \mathcal{F}^{[2]}_{\tilde{\beta}}) \tilde{\epsilon} \right).
\] (60)

Here we have introduced the inverse to \( d \) operator whose action on an arbitrary form is defined by use of a “Green” function to the equation
\[
d(x) h(x - y) = \delta^{11}(x - y),
\] (62)

where \( \delta^{11}(x) \) is the Dirac delta-function and therefore
\[
d^{-1}(x) \omega^{[p]}(x) = (-)^p \int d^{11} y \ h(x - y) \wedge \omega^{[p]}(y).
\] (63)

To make a sense the latter expression should only deal with the “Green” functions that act on a causality-related region of a space-time.

It should be emphasized that we cannot present the explicit variation of the “graviton dual” field without introducing the non-local expressions since we are not having the explicit local expression for \( \hat{\mathcal{G}}^{[2]} \). But nevertheless, these non-local expressions do not spoil the job of the PST symmetries (60), (61) to extract the duality relations and to establish the pure auxiliary nature of the PST scalar field.

To demonstrate that it has to be noticed that the general solution to the equation of motion of \( \hat{A}^{[8]} \)
\[
d(\hat{e} \wedge i_{\hat{e}} \mathcal{F}^{[2]}_{\tilde{\alpha}}) = 0
\] (64)

has the following form
\[
\hat{e} \wedge i_{\hat{e}} \mathcal{F}^{[2]}_{\tilde{\alpha}} = da \wedge d\hat{\varphi}^{[0]}. 
\] (65)

Under the action of (60) the l.h.s. of the latter expression is transformed as
\[
\hat{e} \wedge i_{\hat{e}} \mathcal{F}^{[2]}_{\tilde{\alpha}} \longrightarrow \hat{e} \wedge i_{\hat{e}} \mathcal{F}^{[2]}_{\tilde{\alpha}} + da \wedge d\hat{\varphi}^{[0]},
\] (66)

and therefore, setting \( \hat{\varphi}^{[0]} = \hat{\varphi}^{[0]} \), one can reduce (65) to
\[
i_{\hat{e}} \mathcal{F}^{[2]}_{\tilde{\alpha}} = 0 \implies \mathcal{F}^{[2]}_{\tilde{\alpha}} = 0.
\] (67)
Taking the latter into account and using the same arguments one can reduce the vielbein equation of motion to

\[ \hat{F}^a[9] = 0. \]  

(68)

It becomes clear that the equation of motion of the PST scalar \( a(x) \) is identically satisfied as a consequence of the equations of motion for the other fields. Therefore, the PST scalar equation of motion is the Noether identity that is a remnant of an additional symmetry which is nothing but the symmetry under the transformations (61).

Let us now extend the results to the bosonic sector of D=11 supergravity. To do that note first that as soon as the three-index photon field \( \hat{A}^{[3]} \) is taken into account eq. (12) following from (10) is replaced with

\[ \hat{F}^a[9] = d\hat{A}^8 + \hat{\phi} \hat{G}^{a[2]} = \hat{\phi} \hat{G}^{a[2]} + \hat{\phi} \hat{G}^{a[2]}, \]  

(69)

where \( \hat{\phi} \hat{G}^{a[2]} \) is defined by \( d\hat{\phi} \hat{G}^{a[2]} = \hat{\phi} \hat{G}^{a[2]} \) with \( \hat{\phi} \hat{G}^{a[2]} = \hat{\phi} \hat{G}^{a[2]} + \hat{\phi} \hat{G}^{a[2]} \). Here as in eq. (51) we have introduced the \( \hat{A}^{[9]} \) energy-momentum form \( \hat{M}^{[10]} \). After such a modification the reduction ansätze (36), (37) shall be modified as follows

\[ \hat{\phi} = e^{\frac{2}{3}} \phi \cdot g{a[1]} \]

\[ + e^{-\frac{4}{3}} \phi (dA^a[7] + \hat{\phi} G^{[a]} + \frac{1}{12} (d \phi \wedge e^a) + e^{-\frac{4}{3}} \phi \cdot G^{[a]} [2]) \wedge (dz + A^1[1]), \]  

(70)

\[ \hat{\phi} = e^{\frac{2}{3}} \phi (dA^a[8] - \frac{3}{4} \hat{F}^a \wedge A^1[1] + \frac{1}{2} \hat{B}^2 \wedge dB^6 + \frac{1}{4} \hat{F}^6 \wedge A^3[3] ) \]

\[ - e^{\frac{2}{3}} \phi (1 + \frac{2}{3}) d \phi + e^{\frac{2}{3}} \phi \cdot g^z[1] \]

\[ + e^{\frac{2}{3}} \phi (dA^a[7] + F^6 \wedge B^2[2] + B^2 \wedge B^2 \wedge dA^3[3] + e^{-\frac{4}{3}} \phi \cdot G^z[2] ) \wedge (dz + A^1[1]). \]  

(71)

To present (70), (71) we have used the reduction rule \( \hat{\phi} \hat{G}^{a[2]} = \hat{\phi} \hat{G}^{a[2]} \cdot A^a[1] \) and \( \hat{\phi} G^{a[2]} \) is the analog of the form defined previously in (60) and extended with the appropriate contribution from the energy-momentum tensor of RR 3-form \( A^3[3] \) and of NS 2-form \( B^2 \). The latter forms come from the reduction of \( \hat{A}^{[9]} \). Finally, \( F^6 \) and \( F^8 \) are the field strengths dual to the \( F^4 = dA^3 - dB^2 \wedge A^1[1] \) and \( F^8 = dA^8 \) (see 20 for their explicit expressions).

Let us end up with the action for the complete duality-symmetric bosonic sector of D=11 supergravity (cf. 23, 26). The action is as follows

\[ S = \int_{M^{11}} \left[ \hat{R}^{ab} \wedge \hat{\Sigma}_{ab} + \frac{1}{2} \hat{F}^{[4]} \wedge \hat{\phi} \hat{F}^{[4]} - \frac{1}{3} \hat{A}^{[3]} \wedge \hat{F}^{[4]} \wedge \hat{F}^{[4]} \right] \]

\[ + \int_{M^{11}} \frac{1}{2} \left[ \hat{\phi} \wedge \hat{F}^{a[9]} \wedge \hat{\phi} \hat{F}^{[2]} \wedge \hat{\phi} \hat{F}^{[7]} \wedge \hat{\phi} \hat{F}^{[4]} \right], \]  

(72)

with

\[ \hat{F}^{a[2]} = d\hat{A}^8 + \hat{\phi} \hat{G}^{a[2]}, \]  

(73)

\[ \hat{F}^{a[9]} = -\hat{\phi} \hat{F}^{a[2]}, \]  

(74)

\[ \hat{F}^{[4]} = d\hat{A}^3, \quad \hat{F}^{[7]} = d\hat{A}^6 + \hat{A}^3 \wedge \hat{F}^{[4]}, \]  

(75)

\[ \hat{F}^{[4]} = \hat{F}^{[4]} - \hat{\phi} \hat{F}^{[7]}, \quad \hat{F}^{[7]} = -\hat{\phi} \hat{F}^{[4]}. \]  

(76)
One can verify that after dimensional reduction with taking into account the ansätze (70), (71) the action (72) reproduces the following action of the completely duality-symmetric type IIA supergravity (cf. eq. (69) in Ref. [26])

\[
S = -\int_{\mathcal{M}^{10}} \left( R^{ab} \wedge \Sigma_{ab} + \frac{1}{2} v \wedge F^{[8]} \wedge i_v F^{[2]} \eta_{ab} \right) + \frac{1}{2} \int_{\mathcal{M}^{10}} \sum_{n=1}^{4} \left( \frac{1}{3^{\left(n-3\right)/4}} F^{[10-n]} \wedge F^{[n]} + i_v F^{[10-n]} \wedge v \wedge F^{[n]} + v \wedge F^{[10-n]} \wedge i_v F^{[n]} \right)
\]

(77)

4. Summary

To summarize, we have exploited a similarity between the gravity equation of motion and that of the Maxwell electrodynamics with an electric-type source. The use of this fact allowed us to write down the second order equation of motion for the vielbein as the first order Bianchi identity for its dual partner. Knowing the structure of the duality-symmetric action that follows from the reduction of gravity action from \( D \) to \( D-1 \) space-time dimensions we have deduced the reduction ansatz for the field strength of the “graviton dual” field that reproduces the correct \( D-1 \)-dimensional duality relations between the dilaton and the Kaluza-Klein vector field and their duals. After establishing the ansatz we have proposed the additional PST-like term entering the action of the duality-symmetric gravity and encoding the duality relation between the graviton and its dual partner, and have verified the relevance of this term by dimensional reduction from eleven to ten comparing the results of reduction with that of previously obtained in studying the duality-symmetric formulation of type IIA supergravity. Finally, we have extended the ansatz to the bosonic sector of \( D=11 \) supergravity and have established after reduction the correct structure of the duality-symmetric type IIA supergravity action. Therefore, we have non-trivially tested the approach by Pasti, Sorokin and Tonin and have confirmed the relevance of the approach to the construction of the duality-symmetric supergravity actions in diverse dimensions.

In the course of our studying it has been also shown that the duality relations for the graviton and its dual partner come as equations of motion from the proposed action of the duality-symmetric (super)gravity and that the PST scalar field entering the action is an auxiliary field and does not spoil the field content of an original theory. To demonstrate that we have established the PST symmetries which are characteristic of the PST approach and one of which shall be used to eliminate from a theory the auxiliary PST scalar field. A feature of these transformations consists in dealing with non-local terms. The same concerns the action we proposed. In general there is not the exact local expression for the Landau-Lifshitz “pre-current” entering the action, so the action is also constructed out of the non-local terms. Having such terms is a feature of the self-interacting theory since the double field approach to the Yang-Mills theory also reveals non-locality [36].

Since we have not explicitly studied the fermionic sector here an extension of the model to involve fermions remains conjectural at this stage, but the experience in an immersion of PST formalism into a supersymmetric theory lends credence to the conjecture.

Acknowledgments

We are very grateful to Igor Bandos and Dmitri Sorokin for valuable comments and encouragement. This work is supported in part by the Grant # F7/336-2001 of the Ukrainian SFFR and by the INTAS Research Project #2000-254.
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