LOCALIZED ASYMPTOTIC BEHAVIOR FOR ALMOST ADDITIVE POTENTIALS

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Abstract. We conduct the multifractal analysis of the level sets of the asymptotic behavior of almost additive continuous potentials \((\phi_n)_{n=1}^\infty\) on a topologically mixing subshift of finite type \(X\) endowed itself with a metric associated with such a potential. We work without additional regularity assumption other than continuity. Our approach differs from those used previously to deal with this question under stronger assumptions on the potentials. As a consequence, it provides a new description of the structure of the spectrum in terms of weak concavity. Also, the lower bound for the spectrum is obtained as a consequence of the study sets of points at which the asymptotic behavior of \(\phi_n(x)\) is localized, i.e. depends on the point \(x\) rather than being equal to a constant. Specifically, we compute the Hausdorff dimension of sets of the form \(\{x \in X : \lim_{n \to \infty} \phi_n(x)/n = \xi(x)\}\), where \(\xi\) is a given continuous function. This has interesting geometric applications to fixed points in the asymptotic average for dynamical systems in \(\mathbb{R}^d\), as well as the fine local behavior of the harmonic measure on conformal planar Cantor sets.

1. Introduction

We say that \((X, T)\) is a topological dynamical system (TDS) if \(X\) is a compact metric space and \(T\) is a continuous mapping from \(X\) to itself. We denote by \(\mathcal{M}(X, T)\) the set of invariant probability measures on \((X, T)\).

We say that \(\Phi = (\phi_n)_{n=1}^\infty\) is almost additive if \(\phi_n\) is continuous from \(X\) to \(\mathbb{R}\) and there is a positive constant \(C(\Phi) > 0\) such that

\[
- C(\Phi) + \phi_n + \phi_p \circ T^n \leq \phi_{n+p} \leq C(\Phi) + \phi_n + \phi_p \circ T^n, \quad \forall n, p \in \mathbb{N}.
\]

Typical examples are the additive potential given by the sequence of Birkhoff sums \((S_n \varphi = \sum_{k=0}^{n-1} \varphi \circ T^k)_{n \geq 1}\) of a continuous function \(\varphi : X \to \mathbb{R}\), and more generally sequences of the form \((\log \|S_n M\|)_{n \geq 1}\), where \((S_n M)_{n \geq 1}\) is the sequence of Birkhoff products \((M \circ T^{n-1}) \cdots (M \circ T) \cdot M\) associated with a continuous function \(M\) from \(X\) to the set of positive square matrices.

By subadditivity, for every \(\mu \in \mathcal{M}(X, T)\), \(\Phi_+(\mu) := \lim_{n \to \infty} \frac{1}{n} \int_X \phi_n \, d\mu\) exists, and we define the compact convex set \(L_\Phi = \{\Phi_+(\mu) : \mu \in \mathcal{M}(X, T)\}\). We denote by \(C_{aa}(X, T)\) the collection of almost additive potentials on \(X\).

The ergodic theorem naturally raises the following question. Given \(\Phi\) an almost additive potential taking values in \(\mathbb{R}^d\) (this means that \(\Phi = (\Phi^1, \cdots, \Phi^d)\) with each \(\Phi^i \in C_{aa}(X, T)\))
and $\xi : X \to \mathbb{R}^d$ a continuous function, what is the Hausdorff dimension of the set

$$E_\Phi(\xi) := \{ x \in X : \lim_{n \to \infty} \frac{\phi_n(x)}{n} = \xi(x) \}.$$ 

When $\xi(x) \equiv \alpha$ is constant, this question has been solved for some $C^{1+\varepsilon}$ conformal dynamical systems, sometimes assuming restrictions on the regularity of $\Phi$, and this problem is known as the multifractal analysis of Birkhoff averages, and more generally almost additive potentials \cite{12, 32, 31, 30, 29, 26, 14, 5, 15, 6, 20, 19, 16, 27, 4}. Moreover, the optimal results are expressed in terms of a variational principle of the following form: $E_\Phi(\alpha) \neq \emptyset$ if and only if $\alpha \in L_\Phi = \{ \Phi_*(\mu) = (\Phi^1_*(\mu), \ldots, \Phi^d_*(\mu)) : \mu \in \mathcal{M}(X,T) \}$ and in this case

$$\dim_H E_\Phi(\alpha) = \max \left\{ \frac{h_\mu(T)}{\int_X \log \| DT \| \, d\mu} : \mu \in \mathcal{M}(X,T), \, \Phi_*(\mu) = \alpha \right\},$$

where $h_\mu(T)$ is the measure theoretic entropy of $\mu$ relative to $T$ (see \cite{1} for such a result in a non-conformal context).

To our best knowledge no result is known for $\dim_H E_\Phi(\xi)$ for non constant $\xi$. We are going to give an answer to this question when $(X,T)$ is a topologically mixing subshift of finite type endowed with a metric associated with a negative almost additive potential, and then transfer our result to geometric realizations on Moran sets like those studied in \cite{2}, the main examples being $C^1$ conformal repellers and $C^1$ conformal iterated function systems (see section 3 for precise definitions and statements). In the setting outlined above, if $d = 1$ and $\xi$ takes its values in $L_\Phi$, we find the natural variational formula

$$\dim_H E_\Phi(\xi) = \max \left\{ \frac{h_\mu(T)}{\int_X \log \| DT \| \, d\mu} : \mu \in \mathcal{M}(X,T), \, \Phi_*(\mu) \in \xi(X) \right\}.$$

As application of this kind of results, we obtain unexpected properties like the following one: Let $d \in \mathbb{N}_+$ and $(m_1, \ldots, m_d)$ be $d$ integers $\geq 2$. Let $T : [0,1]^d \to [0,1]^d$ be the mapping $(x_1, \ldots, x_d) \mapsto (m_1 x_1 \pmod{1}), \ldots, m_d x_d \pmod{1})$. Consider

$$\mathcal{F} := \left\{ x \in [0,1]^d : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k x = x \right\},$$

the set of those points $x$ which are fixed by $T$ in the asymptotic average. Then $\mathcal{F}$ is dense and of full Hausdorff dimension in $[0,1]^d$.

Another application concerns harmonic measure. Let us consider here the special case of the set $J = C \times C \subset \mathbb{R}^2$, where $C$ is the middle third Cantor set. The harmonic measure on $J$ is the probability measure $\omega$ such that for each $x \in J$ and $r > 0$, $\omega(B(x,r))$ is the probability that a planar Brownian motion started at $\infty$ attains $J$ for the first time at a point of $B(x,r)$ (see Section 3.4 for more general examples and a reference). For $x \in J$, one defines the local dimension of $\omega$ at $x$ as $d_\omega(x) = \lim_{r \to 0^+} \log \omega(B(x,r))/\log r$ whenever this limit exists. Let $I$ stand for the set of all possible local dimensions for $\omega$. By using the fact that $\omega$ is a Gibbs measure, we prove that if $\xi : J \to \mathbb{R}_+$ is continuous and $\xi(J) \subset I$, then the set $E_\omega(\xi) = \{ x \in J : d_\omega(x) = \xi(x) \}$ is dense in $J$ and the following variational formula holds:

$$\dim_H E_\omega(\xi) = \sup \{ \dim_H E_\omega(\alpha) : \alpha \in \xi(J) \}, \text{ where } E_\omega(\alpha) = \{ x \in J : d_\omega(x) = \alpha \}.$$

Our approach necessitates to revisit the case where $\xi$ is constant. This brings out an interesting new property of the Hausdorff spectrum $\alpha \mapsto \dim_H E_\Phi(\alpha)$. We call this
property \textit{weak} concavity; it is between concavity and quasi-concavity. This structure turns out to be crucial in establishing our results on fixed points in the asymptotic average.

The paper is organized as follows. In Section 2 we give basic definitions and recalls about thermodynamic formalism, and then state our main results on subshift of finite type. In Section 3 we give the geometric realizations. The other sections are dedicated to the proofs.

2. Definitions, and main results on subshifts of finite type

Section 2.1.1 introduces some additional notions related to almost additive potentials, Section 2.1.2 introduces the metrics we will put on topologically mixing subshifts of finite types, while Section 2.1.3 recalls the variational principle for almost additive potentials. Then Section 2.2 introduces two fundamental dimension functions in the multifractal analysis of almost additive potentials, as well as a notion of weak concavity. Finally Section 2.3 provides our main results on topologically mixing subshifts of finite types.

2.1. Definitions.

2.1.1. Vector-valued almost additive potentials and some associated quantities. Given \( \Phi \in \mathcal{C}_{aa}(X, T) \), define \( \Phi_{\text{max}} := \max(\phi_1 + C(\Phi)) \) and \( \Phi_{\text{min}} := \min(\phi_1 - C(\Phi)) \).

Define two collections of special almost additive potentials on \( X \) as

\[
\mathcal{C}_{aa}^+(X, T) := \{ \Phi \in \mathcal{C}_{aa}(X, T) : \Phi_{\text{min}} > 0 \}
\]

and

\[
\mathcal{C}_{aa}^-(X, T) := \{ \Phi \in \mathcal{C}_{aa}(X, T) : \Phi_{\text{max}} < 0 \}.
\]

These sets contain in particular the sequences of Birkhoff sums of positive continuous functions and negative continuous functions respectively.

For \( \Phi \in \mathcal{C}_{aa}^-(X, T) \) we get

\[
\phi_{n+1}(x) \leq \phi_n(x) + \phi_1(T^n x) + C(\Phi) \leq \phi_n(x) + \Phi_{\text{max}} < \phi_n(x),
\]

So \( \{\phi_n : n \in \mathbb{N}\} \) is a strictly decreasing sequence of functions.

If \( \Phi = (\Phi^1, \ldots, \Phi^d) \) is such that each \( \Phi^j \in \mathcal{C}_{aa}(X, T) \), then we call \( \Phi \) a \textit{vector-valued almost additive potential} and write \( \Phi \in \mathcal{C}_{aa}(X, T, d) \). We have \( \Phi = (\phi_n)_{n=1}^\infty \) with \( \phi_n = (\phi^1_n, \ldots, \phi^d_n) \). If \( \phi : \Sigma_A \to \mathbb{R}^d \) is continuous, we define

\[
\|\phi\|_n := \sup_{x|_n=y|_n} |\phi(x) - \phi(y)|,
\]

where \( |u| \) stands for the euclidean norm of \( u \). For \( \Phi \in \mathcal{C}_{aa}(X, T, d) \), let \( \|\Phi\|_n := \|\phi_n\|_n \).

2.1.2. Weak Gibbs metric on subshifts of finite type. Let \( (\Sigma_A, T) \) be a topologically mixing subshift of finite type over the alphabet \( \{1, \ldots, m\} \), where \( A \) is a \( m \times m \) matrix with entries 0 and 1 such that \( A^{p_0} > 0 \) for some \( p_0 \in \mathbb{N} \) and \( T \) is the shift map. We shall endow \( \Sigma_A \) with a metric \( d_T \) naturally associated with a potential \( \Psi \in \mathcal{C}_{aa}^-(\Sigma_A, T) \). This kind of metrics have been considered in [21] and [23] associated with negative additive potentials in order to transfer to the symbolic side the study of some \( C^1 \) hyperbolic dynamics.

Let \( \Sigma_{A,n} \) be the set of the admissible words of length \( n \) and let \( \Sigma_{A,*} := \bigcup_{n \geq 0} \Sigma_{A,n} \). For \( w \in \Sigma_{A,*} \) and \( w = w_1 \cdots w_n \), we denote the length of \( w \) by \( |w| = n \). Given \( w \in \Sigma_{A,*} \cup \Sigma_A \) with \( |w| \geq n \), we denote \( w_1 \cdots w_n \) by \( w|_n \). Given \( u \in \Sigma_{A,*} \) and \( v \in \Sigma_{A,*} \cup \Sigma_A \), if \( u_j = v_j \) for \( j = 1, \ldots, |u| \), then we say \( u \) is a \textit{prefix} of \( v \) and write \( u \prec v \). For \( u = u_1 \cdots u_n \in \Sigma_{A,n} \),
$u^*$ stands for $u|_{n-1}$. For $x, y \in \Sigma_{A,s} \cup \Sigma_A$ such that $x \neq y$, $x \wedge y$ stands for the common prefix of $x$ and $y$ of maximal length. Given $w \in \Sigma_{A,n}$, the cylinder $[w]$ is defined as

$$[w] := \{ x \in \Sigma_A : x|_n = w \}. $$

Recall that $A^p(i, j) > 0$ for all $1 \leq i, j \leq m$, consequently $A^{p+2}(i, j) > 0$. For each $i, j$ we fix $w(i, j) \in \Sigma_{A, p0}$ such that $iw(i, j)j$ is admissible. Define

$$W := \{ w(i, j) : 1 \leq i, j \leq m \}. $$

For $\Phi \in C_{aa}(\Sigma_A, T)$ and $w \in \Sigma_{A,n}$ we define

$$\Phi[w] := \sup \{ \exp(\phi_n(x)) : x \in [w] \}. $$

Now we fix a $\Psi \in C_{aa}(\Sigma_A, T)$. For $x, y \in \Sigma_A$ define

$$d_\Psi(x, y) := \begin{cases} \Psi[x \wedge y], & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases} $$

**Proposition 1.** $d_\Psi$ is an ultra-metric distance on $\Sigma_A$. If $x \in \Sigma_A$ and $r > 0$, the closed ball $B(x, r)$ is the cylinder $[x|_n]$, where $n$ is the unique integer such that $\Psi[x|_{n-1}] > r$ and $\Psi[x|_n] \leq r$. Each cylinder $[w]$ is a ball with diam([w]) = $\Psi[w]$.

The proof is elementary and we omit it.

For the metric space $(\Sigma_A, d_\Psi)$ we define

$$B_n(\Psi) = \{ w \in \Sigma_{A,s} : [w] \text{ is a closed ball of } \Sigma_A \text{ with radius } e^{-n} \} \quad (n \geq 0).$$

It is clear that $\{ [w] : w \in B_n(\Psi) \}$ is a covering of $\Sigma_A$ for each $n \geq 0$.

If we take $\Psi = (-n \log m)_{n \geq 1}$, it is ready to check that $d_\Psi(x, y) = m^{-|x \wedge y|}$, which is the standard metric on $\Sigma_A$. We denote this special metric by $d_1$.

2.1.3. **Recalls on the thermodynamic formalism.** The thermodynamic formalism for almost additive potentials has been studied in several works [13] [2] [19] [17] [8] [15] [11]. For our purpose, we only need to consider the subshift of finite type case. Let $(\Sigma_A, T)$ be a topologically mixing subshift of finite type. Given $\Phi \in C_{aa}(\Sigma_A, T)$, the topological pressure can be defined as

$$P(T, \Phi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in \Sigma_{A,n}} \exp(\sup_{x \in [w]} \phi_n(x)). $$

The following extension of the classical variational principle valid for additive continuous potentials (see [33]) holds:

**Theorem 2.1.** [3] [4] [11] Let $(\Sigma_A, T)$ be a topologically mixing subshift of finite type. For any $\Phi \in C_{aa}(\Sigma_A, T)$, we have $P(T, \Phi) = \sup \{ h_\mu(T) + \Phi_*(\mu) : \mu \in M(\Sigma_A, T) \}$.

If $\mu \in M(\Sigma_A, T)$ such that $P(T, \Phi) = h_\mu(T) + \Phi_*(\mu)$, then $\mu$ is called an equilibrium state of $\Phi$. It is shown in [3] that every $\Phi \in C_{aa}(\Sigma_A, T)$ has an equilibrium state (in fact the result holds for more general TDS). Define

$$U(\Sigma_A, T) := \{ \Phi \in C_{aa}(\Sigma_A, T) : \Phi \text{ has a unique equilibrium state} \}. $$

For instance, this set contains the sequence of Birkhoff sums of any Hölder continuous function when $\Sigma_A$ is endowed with a metric $d_\Psi$ (see [7]).
For $\Phi^1, \cdots, \Phi^k \in \mathcal{C}_{aa}(\Sigma_A, T)$ and $q = (q_1, \cdots, q_k) \in \mathbb{R}^k$, define

$$F(q) := P(T, q_1\Phi^1 + \cdots + q_k\Phi^k).$$

It is shown in [4] that if $\text{span}\{\Phi^1, \cdots, \Phi^k\} \subset U(\Sigma_A, T)$, then $F(q)$ is convex and in $C^1(\mathbb{R}^k)$.

### 2.2. Two dimension functions; weak concavity.

Let us recall what is the range of those $\alpha$ such that $E_\Phi(\alpha) \neq \emptyset$.

**Proposition 2 (IS).** Let $\Phi \in C_{aa}(\Sigma_A, T, d)$. We have $E_\Phi(\alpha) \neq \emptyset$ if and only if $\alpha \in L_\Phi$.

Now we introduce two functions which will turn out to take the same values on $E(\alpha, \Phi)$ and $E_\Phi(\alpha)$ and variational principle for entropy like [1,2]. The proofs of the propositions stated in this section are given in Section 4.

(1) **Box-counting type function; weakly concave large deviation spectrum:** fix $\Psi \in \mathcal{C}_{aa}(\Sigma_A, T)$ and $\Phi \in \mathcal{C}_{aa}(\Sigma_A, T, d)$. Define $d_\Psi$ and $B_n(\Psi)$ as above. Given $\alpha \in L_\Phi$, $n \geq 1$ and $\epsilon > 0$, define

$$F(\alpha, n, \epsilon, \Phi, \Psi) := \left\{ u \in B_n(\Psi) : \text{there exists } x \in [u] \text{ such that } \frac{\phi_{\alpha}(x)}{|u|} - \alpha < \epsilon \right\}.$$

Let $f(\alpha, n, \epsilon, \Phi, \Psi)$ be the cardinality of $F(\alpha, n, \epsilon, \Phi, \Psi)$.

**Proposition 3.** For any $\Psi \in \mathcal{C}_{aa}(\Sigma_A, T)$, the limit

$$D(\Psi) = \lim_{n \to \infty} \frac{\log \#B_n(\Psi)}{n}$$

exists. Moreover

$$D(\Psi) \leq (1 + 1/|\Psi_{\max}|) \log m.$$

For any $\Phi \in C_{aa}(\Sigma_A, T, d)$ and any $\alpha \in L_\Phi$, we have

$$\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} =: \Lambda^\Psi(\alpha) = \Lambda(\alpha).$$

The function $\Lambda : L_\Phi \to \mathbb{R}$ is upper semi-continuous.

We will prove that $\Lambda(\alpha)$ is the Hausdorff dimension of $E_\Phi(\alpha)$ for all $\alpha \in L_\Phi$. The function $\Lambda$ has more regularity than upper semi-continuity. To make this precise we need several standard notations from convex analysis. Given $A \subset \mathbb{R}^d$, the affine hull of $A$ is the smallest affine subspace of $\mathbb{R}^d$ containing $A$ and is denoted by aff$(A)$. For a convex set $A$, we define $\text{ri}(A)$, the relative interior of $A$ as

$$\text{ri}(A) := \{ x \in \text{aff}(A) : \exists \epsilon > 0, (x + \epsilon B) \cap \text{aff}(A) \subset A \},$$

where $B = B(0, 1) \subset \mathbb{R}^d$ is the unit open ball. Let $A \subset \mathbb{R}^d$ be a convex set and $h : A \to \mathbb{R}$ be a function. If there exists $c \geq 1$ such that for any $\alpha, \beta \in A$, we can find $\gamma_1 = \gamma_1(\alpha, \beta), \gamma_2 = \gamma_2(\alpha, \beta) \in [c^{-1}, c]$ such that for any $\lambda \in [0, 1]$

$$\lambda h(\alpha) + (1 - \lambda)h(\beta) \leq h\left(\frac{\lambda \gamma_1 + (1 - \lambda) \gamma_2}{\lambda \gamma_1 + (1 - \lambda) \gamma_2}\right),$$

then we call $h$ a weakly concave function on $A$. Note that if $c = 1$, we go back to the usual concept of concave function. Also, $h(\gamma) \geq \min(h(\alpha), h(\beta))$ if $\gamma \in [\alpha, \beta] \subset A$, thus $h$ is quasi-concave.
Proposition 4. The function \( \Lambda : L_\Phi \to \mathbb{R} \) is bounded, positive and weakly concave. It is continuous on any closed interval \( I \subset L_\Phi \) and on \( \text{ri}(A) \), where \( A \subset L_\Phi \) is any convex set. Consequently it is continuous on \( \text{ri}(L_\Phi) \). If moreover \( L_\Phi \) is a convex polyhedron, then \( \Lambda \) is continuous on \( L_\Phi \). Assume \( I = [\alpha_0, \alpha_1] \subset L_\Phi \) and \( \alpha_{\max} \in I \) such that \( \Lambda(\alpha_{\max}) = \max(\Lambda(\alpha) : \alpha \in I) \), then \( \Lambda \) is decreasing from \( \alpha_{\max} \) to \( \alpha_j \), \( j = 0, 1 \).

Remark 1. Large deviations spectra for the Hausdorff dimension estimation of sets like \( E_\Phi(\alpha) \) have been considered since the first studies of multifractal properties of Gibbs or weak Gibbs measures and then extended to the study of Birkhoff averages [12, 32, 10, 31, 30, 29, 5, 26, 14, 15, 6, 20]. Until now, in the situations where such a spectrum may be weak Gibbs measures and then extended to the study of Birkhoff averages [12, 32, 10, 31, 30, 29, 5, 26, 14, 15, 6, 20]. Until now, in the situations where such a spectrum may be non-concave [6, 4, 20], no description of its regularity like that of Proposition 4 had been given. Moreover, the methods used in the papers mentioned above seem not adapted to provide this information.

(2) Function associated with a conditional variational principle: For \( \alpha \in L_\Phi \) let
\[
\mathcal{E}(\alpha) = \mathcal{E}_\Phi^\Psi(\alpha) := \sup \left\{ \frac{h_\mu(T)}{-\Psi_*(\mu)} : \mu \in \mathcal{M}(\Sigma_A, T) \text{ such that } \Phi_*(\mu) = \alpha \right\}.
\]

Remark 2. When \( \Phi \) and \( \Psi \) are clear from the context, most of the time we simplify the notations \( \Lambda_\Phi^\Psi, \mathcal{E}_\Phi^\Psi, F(\alpha, n, \epsilon, \Phi, \Psi), f(\alpha, n, \epsilon, \Phi, \Psi) \) to \( \Lambda, \mathcal{E}, F(\alpha, n, \epsilon), f(\alpha, n, \epsilon) \) respectively. We use the full notations only when we want to emphasize the \( \Phi \)- and \( \Psi \)-dependence of the quantities.

2.3. Main results on topologically mixing subshift of finite type.

Throughout this subsection we fix \( \Phi \in C_{aa}(\Sigma_A, d_T) \) and \( \Psi \in C_{aa}(\Sigma_A, T) \). We work on the metric space \( (\Sigma_A, d_T) \). If \( E \subset (\Sigma_A, d_T) \), \( \dim_H E, \dim_P E, \dim_B E \) stand for its Hausdorff, packing and box dimensions respectively. To not assuming additional regularity assumption for \( \Phi \) and \( \Psi \) is natural, since this flexibility on \( \Psi \) makes it possible to describe a larger class of geometric realizations of the next results, and there is no special reason to considers the sets \( E_\Phi(\xi) \) under restrictions on \( \Phi \). However, the proofs will use extensively approximations of almost additive potentials by Hölder potentials.

For convenience we write \( \mathcal{D}(\alpha) = \mathcal{D}_\Phi(\alpha) := \dim_H E_\Phi(\alpha) \).

Theorem 2.2 (Multifractal analysis of the level sets \( E_\Phi(\alpha) \)).

1. \( E_\Phi(\alpha) \neq \emptyset \) if and only if \( \alpha \in L_\Phi \). For \( \alpha \in L_\Phi \) we have
\[
\mathcal{D}(\alpha) = \Lambda(\alpha) = \mathcal{E}(\alpha),
\]
and the function \( \mathcal{D} \) is weakly concave.

2. \( \dim_H \Sigma_A = \dim_B \Sigma_A = \mathcal{D}(\Psi) = \max\{\mathcal{D}(\alpha) : \alpha \in L_\Phi\} \).

Theorem 2.3 (Localized asymptotic behavior). Assume \( \xi : \Sigma_A \to \mathbb{R}^d \) is continuous and \( \xi(\Sigma_A) \subset \text{aff}(L_\Phi) \).

1. \( \dim_H E_\Phi(\xi) \geq \sup\{\mathcal{D}(\alpha) : \alpha \in \xi(\Sigma_A) \cap \text{ri}(L_\Phi)\} \).
2. If \( \xi(\Sigma_A) \subset L_\Phi \) then \( E_\Phi(\xi) \) is dense in \( \Sigma_A \).
3. If \( \sup\{\mathcal{D}(\alpha) : \alpha \in \xi(\Sigma_A) \cap \text{ri}(L_\Phi)\} = \sup\{\mathcal{D}(\alpha) : \alpha \in \xi(\Sigma_A) \cap L_\Phi\} \), then
\[
\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup\{\mathcal{D}(\alpha) : \alpha \in \xi(\Sigma_A) \cap L_\Phi\}.
\]
4. If \( d = 1 \) and \( \xi(\Sigma_A) \subset L_\Phi \), then \( E_\Phi(\xi) \) is dense and
\[
\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup\{\mathcal{D}(\alpha) : \alpha \in \xi(\Sigma_A)\}.
\]
Remark 3. (1) In [6, 4], assuming that
\[
\text{span}\{\Phi^1, \cdots, \Phi^d, \Psi\} \subset U(\Sigma A, T),
\]
the equality \(D(\alpha) = \mathcal{E}(\alpha)\) is shown only for \(\alpha \in \text{int}(L_{\Phi})\), where \(\text{int}(L_{\Phi})\) denotes the interior of \(L_{\Phi}\). The argument is strongly based on the differentiability of the related pressure functions in these cases.

(2) In [20], the authors consider the case of additive potentials \(\Phi\) and \(\Psi\), and work under the assumption that \(\Psi\) corresponds to a Hölder potential. They show \(D(\alpha) = \mathcal{E}(\alpha)\) for all \(\alpha \in L_{\Phi}\). Here we work under weaker assumptions, i.e. both \(\Phi\) and \(\Psi\) are almost additive. Also, we use a different method to compute the function \(D(\alpha)\), namely concatenation of Gibbs measures. Such a method has been used successfully in [23] to deal with the special sets \(E_{\Psi}(\alpha)\) when \(\Psi\) is additive (i.e. taking \(\Phi = \Psi\); notice that in this case the spectrum is always concave). Here, we need to refine such an approach in order to remove some delicate points in our geometric application to attractors of \(C^1\) conformal iterated function systems.

Remark 4. (1) The proof of Theorem 2.3 uses the weak concavity of the spectrum \(D\). It also requires to concatenate Gibbs measures in a more elaborated way than to determine \(D\).

(2) In fact we shall prove a slightly more general result than Theorem 2.3(1):

(1’) Suppose that \(\xi\) is continuous outside a subset \(E\) of \(\Sigma A\), bounded and \(\xi(\Sigma A) \subset \text{aff}(L_{\Phi})\). If \(\dim_H E < \sup\{D(\alpha) : \alpha \in \xi(\Sigma A \setminus E) \cap \text{ri}(L_{\Phi})\}\), then
\[
\dim_H E_{\Phi}(\xi) \geq \sup\{D(\alpha) : \alpha \in \xi(\Sigma A \setminus E) \cap \text{ri}(L_{\Phi})\}\.
\]

(3) An extension of Theorem 2.3(4) is given in the final remark of Section 3.4

3. Geometric results

In this section we show how the main results of the previous section can be applied to multifractal analysis on conformal repellers and on attractors of conformal IFS satisfying the strong open set condition. Such sets fall in the Moran-like geometric constructions considered in [2, 29]. At first we describe this kind of construction (Section 3.1). Then we state the geometric results deduced from Theorems 2.2 and 2.3 (Section 3.2). We give our application to fixed points in the asymptotic average for dynamical systems in \(\mathbb{R}^d\) in Section 3.3. Finally, we give an application to the local scaling properties of weak Gibbs measures in Section 3.4, special example of which is the harmonic measure on planar conformal Cantor sets.

3.1. General setting of geometric realization. Let \((\Sigma A, T)\) be a topologically mixing subshift of finite type over the alphabet \(\{1, \cdots, m\}\) and \(\Psi \in C_{aa}^{-}(\Sigma A, T)\). Let \(X\) be \(\mathbb{R}^d\) or be a connected, \(d\)-dimensional \(C^1\) Riemannian manifold. Consider a family of sets \(\{R_w : w \in \Sigma_{A,n}\}\), where each \(R_w \subset X\) is a compact set with nonempty interior. We assume that this family of compact sets satisfies the following conditions:

(1) \(R_w \subset R_{w'}\) whenever \(w' < w\).

(2) For any integer \(n > 0\), the interiors of distinct \(R_w, w \in \Sigma_{A,n}\) are disjoint.

(3) Each \(R_w\) contains a ball of radius \(\tau_w\) and is contained in a ball of radius \(\tau_w\).
(4) There exists a constant $K > 1$ and a negative sequence $\eta_n = o(n)$ such that for every $w \in \Sigma_{A, \ast}$,
\[(3.1) \quad K^{-1} \exp(\eta_{|w|}) \Psi[w] \leq \tau_{w} \leq K \Psi[w].\]

Notice that $\Psi_{\max} < 0$, then
\[
\mathrm{diam}(R_w) \leq 2\tau_w \leq 2K \Psi[w] \leq 2K \exp(|w|\Psi_{\max}) \to 0, \quad (|w| \to \infty).
\]

Let $J = \bigcap_{n \geq 0} \bigcup_{w \in \Sigma_{A, n}} R_w$. We call $J$ the limit set of the family $\{R_w : w \in \Sigma_{A, \ast}\}$. We can define the coding map $\chi : \Sigma_A \to J$ as $\chi(x) = \bigcap_{n \geq 1} R_{x|n}$, $\forall x \in \Sigma_A$. It is clear that $\chi$ is continuous and surjective when $\Sigma_A$ is endowed with standard metric $d_1$ and $J$ is endowed with the induced metric $\rho$ from $X$.

We say that $J$ is a Moran type geometric realization of $(\Sigma_A, d_\Psi)$.

For this kind of construction we have the following useful observation:

**Proposition 5.** Let $J$ be a Moran type geometric realization of $(\Sigma_A, d_\Psi)$ with $\Psi \in \mathcal{C}_{\infty}(\Sigma_A, T)$, then for any $E \subset J$ we have $\dim_H E = \dim_H(\chi^{-1}(E))$.

In this paper we consider two classes of Moran type geometric realizations of $\Sigma_A$.

1. Topologically mixing $C^1$ conformal repeller $(J, g)$. We refer the book [29] for the definitions and the basic properties related to conformal repellers. It is well known that in this case $(J, g)$ has a Markov partition $\{R_1, \ldots, R_m\}. \quad \text{For each } w = w_1 \cdots w_n, \text{ define}\n R_w := R_{w_1} \cap g^{-1}(R_{w_2}) \cap \cdots \cap g^{-n+1}(R_{w_n})$. Define $\psi(x) = -\log |g'(\chi(x))|$ and $\Psi = (S_n \psi)^\infty_{n=1}$. By the definition of $R_w$ and the property of Markov partition, the condition (1) and (2) are checked directly. (3) and (4) are stated in [29] (Proposition 20.2), except that for (4) we have an additional term $\exp(\eta_{|w|}) = \exp(-\|\Psi\|_{|w|})$ (see Section 5.1 for the definition of $\|\Psi\|_{|w|}$). This is because we only assume $g$ to be continuous rather than Hölder continuous. Thus $J$ is a Moran type geometric realization of $(\Sigma_A, d_\Psi)$ for some primitive matrix $A$ and the potential $\Psi$. Moreover in this case we have $\chi \circ T = g \circ \chi$.

2. Attractors of $C^1$ conformal IFS satisfying the strong open set condition (SOSC) (see [28] for details). Let $\{f_1, \ldots, f_m\}$ be such an IFS and denote by $J$ its attractor. Define $\psi(x) = \log |f_{x_1}'(\chi(Tx))|$ and $\Psi = (S_n \psi)^\infty_{n=1}$. Let $V$ be an open set such that the SOSC holds. For $w = w_1 \cdots w_n$, define $R_w = f_w(V)$, where $f_w := f_{w_1} \circ \cdots \circ f_{w_n}$. Due to the SOSC, (1) and (2) hold for $\{R_w : w \in \Sigma_{A, \ast}\}$. Moreover, arguments similar to those used to prove Proposition 20.2 in [29] show that (3) and (4) also hold. Thus, $\{R_w : w \in \Sigma_{A, \ast}\}$ is a Moran type geometric realization of $(\Sigma_A, d_\Psi)$ with potential $\Psi$. Notice that here $\Sigma_A$ is the full shift $\Sigma_m$. By the uniqueness of the attractor it is easy to verify that the attractor $J$ is the limit set of the family $\{R_w : w \in \Sigma_{A, \ast}\}$.

3.2. Multifractal analysis on Moran type geometric realizations. We are going to conduct multifractal analysis on Moran type geometric realizations, thus we need a dynamics $g$ on $J$ so that $(J, g)$ is a factor of some $(\Sigma_A, T)$. For $C^1$ conformal repellers, there is such a natural dynamic. For the attractor of a $C^1$ conformal IFS, there is no such one in general, the difficulty coming from those points having several codings. However, under the SOSC, we can naturally define such a $g$ by removing a "negligible" part of $J$: 
Let \{f_1, \cdots, f_m\} be a \(C^1\) conformal IFS satisfying the SOSC. Let \(V\) be an open set such that the SOSC holds. By [25], such an open set always exists as soon as the mappings \(f_i\) are \(C^{1+\epsilon}\) and the OSC holds. Define \(\tilde{Z}_\infty := \bigcup_{w \in \Sigma_A} f_w(\partial V)\) and \(Z_\infty := \chi^{-1}(\tilde{Z}_\infty)\). We have the following lemma (proved in Section 8):

**Lemma 3.1.** The set \(\Sigma_A \setminus Z_\infty\) is not empty and \(\chi : \Sigma_A \setminus Z_\infty \to J \setminus \tilde{Z}_\infty\) is a bijection. Moreover \(T(\Sigma_A \setminus Z_\infty) \subset \Sigma_A \setminus Z_\infty\), \(T(Z_\infty) \subset Z_\infty\) and for any Gibbs measure \(\mu\) on \(\Sigma_A\) we have \(\mu(Z_\infty) = 0\).

By the previous lemma we can define the mapping \(\tilde{g} : J \setminus \tilde{Z}_\infty \to J \setminus \tilde{Z}_\infty\) as \(\tilde{g}(x) = \chi \circ T \circ \chi^{-1}\). By construction we have \(\chi \circ T = \tilde{g} \circ \chi\) over \(\Sigma_A \setminus Z_\infty\).

Let \(J\) be a Moran type geometric realization of \((\Sigma_A, d_\Psi)\). We set \(\tilde{J} = J\) when \(J\) is a \(C^1\) conformal repeller and \(\tilde{J} = J \setminus \tilde{Z}_\infty\) when \(J\) is the attractor of a \(C^1\) conformal IFS satisfying the SOSC.

Given a sequence of functions \(\Phi = (\phi_n)_{n=1}^\infty\) from \(\tilde{J}\) to \(\mathbb{R}^d\) and \(\alpha \in \mathbb{R}^d\), we set \(E_\Phi(\alpha) = \left\{ x \in \tilde{J} : \lim_{n \to \infty} \phi_n(x)/n = \alpha \right\}\). We also define \(L_\Phi = \{\alpha \in \mathbb{R}^d : E_\Phi(\alpha) \neq \emptyset\}\). We must redefine these objects because until now they were defined for compact dynamical systems, while \(\tilde{J}\) may be not compact.

When \(J\) is a conformal repeller the system \((J, g)\) is naturally a TDS. For \(\Phi \in C_{aa}(J, g, d)\), if we define \(\tilde{\Phi} := (\phi_n \circ \chi)_{n=1}^\infty\), since \(g \circ \chi = \chi \circ T\), we have \(\tilde{\Phi} \in C_{aa}(\Sigma_A, T, d)\) with \(C(\tilde{\Phi}) = C(\Phi)\). And for \(\alpha \in \mathbb{R}^d\) we have \(E_{\tilde{\Phi}}(\alpha) = \chi(E_\Phi(\alpha))\).

When \(J\) is the attractor of a \(C^1\) conformal IFS satisfying the SOSC, if \(\phi\) is a continuous function from \(J\) to \(\mathbb{R}^d\), it generates the additive potential \(\tilde{\Phi} = (S_n \phi)_{n=1}^\infty\) on \((\Sigma_A, T)\), where \(\tilde{\phi} = \phi \circ \chi\), and it also defines \(\tilde{\Phi} = (S_n \phi)_{n=1}^\infty\) on \((\tilde{J}, \tilde{g})\). Then for \(\alpha \in \mathbb{R}^d\) we have \(E_{\tilde{\Phi}}(\alpha) = \chi(E_{\tilde{\Phi}}(\alpha) \setminus Z_\infty)\).

Write \(D_{\Phi}(\alpha) := \dim_H E_{\Phi}(\alpha)\) for convenience.

**Theorem 3.2.** Let \(J\) be a Moran type geometric realization of \((\Sigma_A, d_\Psi)\). If \(J\) is a \(C^1\) conformal repeller, let \(\Phi \in C_{aa}(J, g, d)\) and define \(\tilde{\Phi}\) as above. If \(J\) is the attractor of a \(C^1\) conformal IFS satisfying the SOSC, let \(\phi\) be a continuous map from \(J\) to \(\mathbb{R}^d\), and define the additive potential \(\tilde{\Phi} = (S_n \phi)_{n=1}^\infty\) on \((\Sigma_A, T)\) with \(\tilde{\phi} = \phi \circ \chi\) and \(\tilde{\Phi} = (S_n \phi)_{n=1}^\infty\) on \((\tilde{J}, \tilde{g})\). Then

\[
(1) \quad L_{\Phi} = L_{\tilde{\Phi}}; \text{ for } \alpha \in L_{\Phi} \text{ we have } D_{\Phi}(\alpha) = \dim_F E_{\Phi}(\alpha) \text{ and } D_{\Phi}(\alpha) = D_{\tilde{\Phi}}(\alpha) = \Lambda_{\tilde{\Phi}}(\alpha) = \mathcal{E}_{\tilde{\Phi}}(\alpha).
\]

\[
(2) \quad \dim_H J = \dim_H \tilde{J} = D(\Psi) = \max\{D_{\Phi}(\alpha) : \alpha \in L_{\Phi}\}.
\]

**Remark 5.** For the case of conformal repellers, the connection between Theorem 3.2 and the other works [6, 20, 4] is similar to that done in Remark 3(1) and (2).

For the set \(E_{\Phi}(\xi)\) we have the following result:

**Theorem 3.3.** Let \(J\) be a Moran type geometric realization of \((\Sigma_A, d_\Psi)\), which is either a \(C^1\) conformal repeller or the attractor of a \(C^1\) conformal IFS satisfying the SOSC. Let \(\Phi\) and \(\tilde{\Phi}\) be the same as in Theorem 3.2. Let \(\xi : J \to \mathbb{R}^d\) be continuous and \(E_{\Phi}(\xi) = \left\{ x \in \tilde{J} : \lim_{n \to \infty} \phi_n(x)/n = \xi(x) \right\}\). If \(\xi(J) \subset \text{aff}(L_{\tilde{\Phi}})\), then
(1) \( \dim_H E_\Phi(\xi) \geq \sup \{ D_\Phi(\alpha) : \alpha \in \xi(J) \cap \text{ri}(L_\Phi) \} \), and \( E_\Phi(\xi) \) is dense if \( \xi(J) \subset L_\Phi \).
(2) If \( \sup \{ D_\Phi(\alpha) : \alpha \in \xi(J) \cap \text{ri}(L_\Phi) \} = \sup \{ D_\Phi(\alpha) : \alpha \in \xi(J) \cap L_\Phi \} \), then
\[
\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup \{ D_\Phi(\alpha) : \alpha \in \xi(J) \cap L_\Phi \}.
\]
(3) If \( d = 1 \) and \( \xi(J) \subset L_\Phi \), then \( E_\Phi(\xi) \) is dense and
\[
\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \sup \{ D_\Phi(\alpha) : \alpha \in \xi(J) \}.
\]

3.3. Application to fixed points in the asymptotic average for dynamical systems in \( \mathbb{R}^d \). Suppose that \((J, g)\) is a dynamical system with \( J \subset \mathbb{R}^d \). We say that \( x \in J \) is a fixed point of \( g \) in the asymptotic average if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g^k x = x.
\]
We are interested in the Hausdorff dimension of the set of all such points:
\[
\mathcal{F}(J, g) = \left\{ x \in J : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g^k x = x \right\}.
\]
If \( \xi \) stands for the identity map on \( J \) and \( \Phi \) stands for the additive potential associated with the potential \( \xi \), in our setting we have \( \mathcal{F}(J, g) = E_\Phi(\xi) \).

The set \( L_\Phi \) is contained in the convex hull of \( J \), and it contains the set of the fixed points of \( g \). An example of trivial situation is provided by the unit circle endowed with dynamic \( g(z) = z^2 \) in \( \mathbb{C} \). There, \( \mathcal{F}(J, g) = \{1\} \). How about general conformal repellers and attractors of conformal IFS? This question is non trivial in general. We are going to describe a class of conformal IFS, namely self-similar generalized Sierpinski carpets, for which the situation is non trivial and we have a complete answer.

We consider a special self-similar IFS \( \{ f_1, \ldots, f_m \} \) on \( \mathbb{R}^d \): \( f_j(x) = \rho_j x + c_j \), \( 0 < \rho_j < 1 \), \( (1 \leq j \leq m) \). We assume further the SOSC fulfills. Let \( x_j \) stand for the unique fixed point of \( f_j \) and let \( J \) be the attractor of this IFS. Notice that the mappings \( f_j \) have no rotation part, thus the convex hull of \( J \) satisfies \( \text{Co}(J) = \text{Co}\{x_1, \ldots, x_m\} =: \Delta \), and is a convex polyhedron. We further assume that \( \text{Co}(J) \) has dimension \( d \) (otherwise we can define this IFS in a smaller affine subspace).

Let \( W \) stand for the open set such that the SOSC holds. It is ready to see that \( V := W \cap \Delta \) is also an open set such that SOSC holds. We can define the dynamics \( \tilde{g} \) on \( \tilde{J} = J \setminus \tilde{Z}_\infty \), where \( \tilde{Z}_\infty \) is defined as in the previous subsection.

Now we have the following result whose proof is given in Section 8.

**Theorem 3.4.** Let \( \Phi = \text{id}_J \). Then \( \mathcal{F}(\tilde{J}, \tilde{g}) \) is dense and \( \dim_H \mathcal{F}(\tilde{J}, \tilde{g}) = \sup \{ D_\Phi(\alpha) : \alpha \in J \} \). Moreover if the point at which \( D_\Phi \) attains its maximum belongs to \( J \), then \( \mathcal{F}(\tilde{J}, \tilde{g}) \) is of full Hausdorff dimension.

We have the following corollary, in which the lower bound for the Hausdorff dimension follows directly from Theorem 3.4 and the upper bound follows from standard estimates based on the bounds provided in Section 5.

**Corollary 1.** Let \( N \in \mathbb{N}_+ \) and let \( d_1, \ldots, d_N \) be \( N \) positive integers. Consider \( N \) self-similar IFS without rotations components \( \{ f_j^{(j)} \}_{1 \leq j \leq N} \), satisfying SOSC and living respectively in \( \mathbb{R}^{d_j} \). Denote by \( J_j, 1 \leq j \leq N \), their respective attractors as well as the corresponding dynamical systems \((\tilde{J}_j, \tilde{g}_j)\). Let \( \tilde{J} = \prod_{j=1}^N \tilde{J}_j \subset \mathbb{R}^{\sum_{j=1}^N d_j} \) be endowed
with the dynamics $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_N)$. We have $\dim_H F(\tilde{J}, \tilde{g}) = \sum_{j=1}^N \dim_H F(\tilde{J}_j, \tilde{g}_j) = \sum_{j=1}^N \sup\{D_{\Phi_j}(\alpha) : \alpha \in J_j\}$, where $\Phi_j = \text{Id}_{\mathbb{R}^d}$.

Both the previous results yield the result presented in the introduction of the paper:

**Theorem 3.5.** Let $d \in \mathbb{N}$ and $(m_1, \ldots, m_d)$ be $d$ integers $\geq 2$. Set $J = [0, 1]^d$ and let $g : J \to J$ be the mapping $(x_1, \ldots, x_d) \mapsto (m_1x_1 \mod 1, \ldots, m_dx_d \mod 1)$. Then $F(J, g)$ is dense and of full Hausdorff dimension in $[0, 1]^d$.

To see this, for a fixed integer $m \geq 2$ let $g_m : [0, 1] \to [0, 1]$ be the mapping $x \mapsto mx \mod 1$.

Let $(\Sigma_m, T)$ be the full shift over alphabet $\{0, \ldots, m-1\}$, where $\Sigma_m$ is endowed with the usual metric $d_\Sigma(x, y) = d_1(x, y) = m^{-\lfloor x/y \rfloor}$. Define a map $\chi : \Sigma_m \to [0, 1]$ as $\chi(x) = \sum_{n=1}^\infty x_n/m^n$. Then $\chi$ is continuous and surjective. Consider the IFS $\{f_j : j = 0, \ldots, m-1\}$ defined as $f_j(x) = (x + j)/m$. It is seen that the SOSC holds with $V = (0, 1)$. Let $\tilde{Z}_\infty = \left\{ \sum_{j=1}^n x_jm^{-j} : n \in \mathbb{N}; x_j = 0, \ldots, m-1 \right\} \cup \{1\}$. Define the dynamics $\tilde{g}$ on $\tilde{J}_m = [0, 1] \setminus \tilde{Z}_\infty$ as in the previous section. Then it is easy to check that $\tilde{g} = g_m|_{\tilde{J}_m}$. Let $\Phi = \text{id}_{[0,1]}$. By theorem 3.4 we get $\dim_H F(\tilde{J}_m, g_m) = \sup\{D_{\Phi}(\alpha) : \alpha \in [0, 1]\}$. By the law of large number applied to the measure of maximal entropy we get $D_{\Phi}(1/2) = 1$. We conclude by noticing that $F(J, g) = \prod_{i=1}^d F([0, 1], g_m)$.

Next we consider concrete examples of carpets in the unit square.

**Heterogeneous carpets in the unit square.** In order to fully illustrate our purpose, we consider an IFS $S_0 = \{f_1, \ldots, f_N\}$ in $\mathbb{R}^2$ made of contractive similitudes without rotations such that the squares $f_1([0, 1]^2)$ form a tiling of $[0, 1]^2$. All these situations have been determined in [3]. In this way, $[0, 1]^2$ can be chosen as the open set such that the SOSC holds, and the boundaries of the sets $f_i([0, 1]^2)$ have big intersections. The picture on the left of Figure 1 give an example of this kind of IFS. This IFS contains 15 dilation maps, and the dynamics on this attractor is highly non-trivial.

Let $\Phi$ denote $\text{Id}_{\mathbb{R}^2}$. For each $\emptyset \neq S \subset S_0$, we denote by $J_S$ the attractor of the IFS $S$. The dynamics $\tilde{g}_S$ defined on $\tilde{J}_S$ is the restriction of $\tilde{g}_{S_0}$ to $\tilde{J}_S$. The set $F(\tilde{J}_{S_0}, \tilde{g}_{S_0})$ is of full Hausdorff dimension, since $J_{S_0} = [0, 1]^2$. If $S \neq S_0$, we have the variational formula $\dim_H F(\tilde{J}_S, \tilde{g}_S) = \sup\{D_{\Phi}(\alpha) : \alpha \in J_S\}$, and in general it is hard to know whether $F(\tilde{J}_S, \tilde{g}_S)$ is of full dimension or not in $J_S$. However, here are two simple examples illustrating both possibilities.

We consider the case of the regular tiling associated with the IFS $S_0 = \left\{ f_{i,j} : x \mapsto \frac{x}{3} + \frac{(i\cdot 2 + j)}{3} : 0 \leq i, j \leq 2 \right\}$. Then, let $S_1 = \{f_{0,0}, f_{0,2}, f_{2,0}, f_{2,2}\}$ and $S_2 = S_1 \cup \{f_{1,1}\}$. We claim that $F(\tilde{J}_{S_1}, \tilde{g}_{S_1})$ is not of full Hausdorff dimension, while $F(\tilde{J}_{S_2}, \tilde{g}_{S_2})$ is.

The simpler situation is that of $S_2$. In this case, $G = (1/2, 1/2)$, the center of symmetry of $J_{S_2}$ is the fixed point of $f_{1,1}$ and it belongs to $L_\Phi$. Moreover, it is obvious that the uniform measure (or Parry measure) on $J_{S_2}$ is carried by the set $E_{\Phi}(G)$. This yields the result by Theorem 3.4 and $\dim_H F(\tilde{J}_{S_2}, \tilde{g}_{S_2}) = \log 5/ \log 3$.

In the case of $S_1$, the point $G$ is still the center of symmetry of $J_{S_1}$, so $D_{\Phi}$ reaches its maximum at $G$. However, $G$ does not belong to $J_{S_1}$. Since $\Phi$ is Hölder continuous and the
tiling is regular, we know that $D_\Phi$ is strictly concave. By using the symmetry, one deduces that the restriction of $D_\Phi$ to $J_{S_{1y}}$ reaches its maximum at any of the four points $(1/3, 1/3)$, $(1/3, 2/3)$, $(2/3, 1/3)$ and $(2/3, 2/3)$. This yields $\dim_H \mathcal{F}(J_{S_{1y}}, g_{S_{1y}}) = D_\Phi((1/3, 1/3)) < \log 4 / \log 3 = \dim_H J_{S_{1y}}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Left: Example of tiling of $[0,1]$ by squares. Middle: IFS $S_1 = \{f_0, f_2, f_2, f_2, f_2, f_2\}$. Right: IFS $S_2 = S_1 \cup \{f_1, f_1\}$.}
\end{figure}

### 3.4. Localized results for weak Gibbs measures

Let $\{f_1, \cdots, f_m\}$ be a homogenous self-similar IFS in $\mathbb{C}$ satisfying the strong separation condition, that is, each function $f_j$ has the form $f_j(z) = a_j z + b_j$ where $0 < \rho = |a_j| < 1$, and there exists a topological closed disk $D$ such that $f_j(D) \subset D$ and the $f_j(D)$ are pairwise disjoint. There is a natural coding map $\chi : \Sigma_m \to J$. Moreover if we define $\psi(x) \equiv \log \rho$ for $x \in \Sigma_m$, and $\Psi = (S_n \psi)_{n=1}^{\infty}$, then $\chi : (\Sigma_m, d_\Psi) \to (J, \rho)$ is a bi-Lipschitz homeomorphism.

Let $\phi : J \to \mathbb{R}$ be continuous and define $\tilde{\phi} = \phi \circ \chi$. By subtracting a constant potential if necessary, we can assume $P(T, \tilde{\phi}) = 0$. There exists a weak Gibbs measure $\tilde{\mu}$ on $\Sigma_m$ (see [22]), i.e. a probability measure such that the exists positive sequence $(C_n)_{n \geq 1}$ such that for all $x \in \Sigma_m$ and $n \geq 1$

$$C_n^{-1} \exp(S_n \phi(x)) \leq \tilde{\mu}([x_n]) \leq C_n \exp(S_n \phi(x)),$$

with $\lim_{n \to \infty} \log(C_n)/n = 0$ (if $\phi$ is Hölder continuous, then $C_n$ is bounded and $\tilde{\mu}$ is a Gibbs measure). In particular,

$$d_{\tilde{\mu}}(x) := \lim_{r \to 0} \frac{\log \tilde{\mu}(B(x, r))}{\log r} = \lim_{n \to \infty} \frac{S_n \phi(x)}{n \log \rho}$$

in the sense that either both the limits do not exist, either they exist and are equal. Define $\mu := \chi_*(\tilde{\mu})$ and $\mu$ is called a weak Gibbs measure associated with $\phi$. By the bi-Lipschitz property of $\chi$ and the strong separate condition, we can easily conclude that $d_\mu(y) = \lim_{n \to \infty} S_n \phi(y)/(n \log \rho)$ for any $y \in J$. Let $\Phi = (S_n \phi)_{n=1}^{\infty}$. If we define $E_\mu(\alpha) = \{y \in J : d_\mu(y) = \alpha\}$, then we get $E_\phi(\alpha) = E_\mu(\alpha / \log \rho)$ for any $\alpha \in L_\phi$.

By applying Theorem 3.3 for $d = 1$, we have the following property regarding the local property of weak Gibbs measure:

**Corollary 2.** Let $\mu$ be the weak Gibbs measure associated with $\phi$. Then the set of all possible local dimension for $\mu$ is the interval $L_\phi / \log \rho$. Assume $\xi : J \to \mathbb{R}$ is continuous
and $\xi(J) \subset L_\Phi / \log \rho$, then
\[
\dim_H \{ x \in J : d_\mu(x) = \xi(x) \} = \sup \{ \dim_H E_\mu(\alpha) : \alpha \in \xi(J) \}.
\]

Now let $\omega$ stand for the harmonic measure on $J$. It is well known that (see for example the survey paper [24]) there exists a Hölder continuous function $\phi : J \to \mathbb{R}$ such that $w \propto \mu$, where $\mu$ is the equilibrium state of $\phi$.

**Corollary 3.** Let $\omega$ be the harmonic measure on $J$ and $I$ is the set of all possible local dimension for $\omega$. Assume $\xi : J \to \mathbb{R}$ is continuous and $\xi(J) \subset I$. Then
\[
\dim_H \{ x \in J : d_\omega(x) = \xi(x) \} = \sup \{ \dim_H E_\omega(\alpha) : \alpha \in \xi(J) \}.
\]

**Final remark.** At least when $d = 1$, it is not difficult to extend the results obtained in this paper by considering $\Upsilon = (\gamma_n)_{n \geq 1} \in C_{ac}(\Sigma_A, T)$ and the more general level sets $E_{\Phi/\Upsilon}(\xi) = \{ x \in \Sigma_A : \lim_{n \to \infty} \phi_n(x)/\gamma_n(x) = \xi(x) \}$; when $\xi$ is constant, such sets have been considered in the contexts examined in [64]. The formula is that if the continuous function $\xi$ takes values in the set $L_{\Phi/\Upsilon} = \{ \Phi_*(\nu)/\Upsilon_*(\nu) : \nu \in \mathcal{M}(\Sigma_A, T) \}$, then $\dim_H (E_{\Phi/\Upsilon}(\xi)) = \sup \{ -h_\nu(T)/\Psi_*(\nu) : \nu \in \mathcal{M}(\Sigma_A, T), \Phi_*(\nu)/\Upsilon_*(\nu) \in \xi(\Sigma_A) \}$. When $\Upsilon = -\Psi$, this can be applied to the local dimension of Gibbs measures associated with Hölder potentials $\varphi$ on any $C^1$ conformal repeller of a map $f$, since in this case we know from [29] that such a measure is doubling so that the local dimension is directly related to the asymptotic behavior of $S_n \varphi / S_n (-\log \|DF\|)$. Consequently, Corollary 3 can be extended to harmonic measure on more general conformal repellors (see [24]).

**Additional definitions and notations.** For $\Phi \in C_{ac}(X, T)$ (see [11] recall that we defined $\Phi_{\text{max}} := \max(\phi_1) + C(\Phi)$ and $\Phi_{\text{min}} := \min(\phi_1) - C(\Phi)$. Then define $\|\Phi\| := |\Phi_{\text{max}}| \vee |\Phi_{\text{min}}|$. By the almost additivity property we easily get
\[
\tag{3.2}
n \Phi_{\text{min}} \leq \phi_n(x) \leq n \Phi_{\text{max}}, \quad \forall \ n \in \mathbb{N}.
\]

Consequently we have $\|\phi_n\|_{\infty} \leq n \|\Phi\|$.

If $\Phi = (\Phi^1, \cdots, \Phi^d) \in C_{ac}(X, T, d)$, we define $\Phi_{\text{max}} := (\Phi_{\text{max}}^1, \cdots, \Phi_{\text{max}}^d)$ and $\Phi_{\text{min}} := (\Phi_{\text{min}}^1, \cdots, \Phi_{\text{min}}^d)$. We also define $\|\Phi\| := \left( \sum_{j=1}^d \|\Phi^j\|^2 \right)^{1/2}$ and $\|\Phi\|_{\text{lim}} := \limsup_{n \to \infty} \frac{\|\phi_n\|_{\infty}}{n}.$

We have $\|\phi_n\|_{\infty} \leq n \|\Phi\|$.

Given $u, v \in \mathbb{R}^d$, we write $[u, v] := \{ tu + (1-t)v : 0 \leq t \leq 1 \}$ to denote the closed interval connecting $u$ and $v$. If $u_i \leq v_i$ for $i = 1, \cdots, d$, then we write $u \leq v$. If $u, v_1, v_2 \in \mathbb{R}^d$ is such that $v_1 \leq u \leq v_2$, it is easy to prove that $|u| \leq |v_1| + |v_2|$. We will use this basic fact several times.

For $\Phi \in C_{ac}(X, T, d)$, after defining $C(\Phi) := (C(\Phi^1), \cdots, C(\Phi^d))$, we also have the following vector almost additivity:
\[
-C(\Phi) + \phi_n + \phi_p \circ T^n \leq \phi_{n+p} \leq C(\Phi) + \phi_n + \phi_p \circ T^n, \quad \forall \ n, p \in \mathbb{N}.
\]

The simplest almost additive potentials are the additive ones: Given $\phi : X \to \mathbb{R}^d$ continuous, define $\phi_n = S_n \phi := \sum_{j=0}^{n-1} \phi \circ T^j$. If $\phi$ is Hölder continuous, we also say that $\Phi = (S_n \phi)_{n=1}^\infty$ is Hölder continuous. The simplest Hölder continuous potentials are the constant potentials $(n \alpha)_{n=1}^\infty, \alpha \in \mathbb{R}^d$, that we also denote as $\alpha$. 

4. Proofs of Propositions 3 and 4

4.1. Proof of Proposition 3. We need some facts gathered in the two following lemmas. We omit their simple proofs based on elementary using of the almost additivity of $\Phi$ and the continuity of the $\phi_n$.

Lemma 4.1. (1) Given $\Phi \in C_{aa}(\Sigma_A, T, d)$ and two constants $C_2 \geq C_1 > 0$, for each $n \in \mathbb{N}$ define

$$\|\Phi\|^*_n := \max\{\|\Phi\| : C_1 n \leq l \leq C_2 n\}. \tag{4.1}$$

Then $\|\Phi\|^*_n/n \to 0$ when $n \to \infty$. Especially $\lim_{n \to \infty} \|\Phi\|^*_n/n = 0$.

Let $\Phi \in C_{aa}(\Sigma_A, T)$ and $C = C(\Phi)$.

(2) For any $u, v \in \Sigma_{A,*}$ such that $uv \in \Sigma_{A,*}$ we have

$$\exp(-C - \|\Phi\|_{|u|})\Phi[u]\Phi[v] \leq \Phi[uv] \leq \exp(C)\Phi[u]\Phi[v]. \tag{4.2}$$

(3) For $w = u_1w_1 \cdots u_nw_nu_{n+1} \in \Sigma_{A,*}$, let $k = \sum_{j=1}^{n+1} |u_j|$. We have

$$\exp(-2nC + k\Phi_{\min}) \prod_{j=1}^{n} \Phi[w_j] \exp(-\|\Phi\|_{|w_j|}) \leq \Phi[w] \leq \exp(2nC + k\Phi_{\max}) \prod_{j=1}^{n} \Phi[w_j]. \tag{4.3}$$

(4) If $\Phi \in C_{aa}(\Sigma_A, T)$, then $\Phi[w] \leq \Phi[u]$ for $u \prec v$.

Lemma 4.2. Let $\Psi \in C_{aa}(\Sigma_A, T)$.

(1) Let $C_1(\Psi) = 1/|\Psi_{\min}|$ and $C_2(\Psi) = 1 + 1/|\Psi_{\max}|$. For any $w \in B_n(\Psi)$ we have

$$C_1(\Psi)n \leq |w| \leq C_2(\Psi)n. \tag{4.4}$$

(2) For any $w \in B_n(\Psi)$ we have

$$\exp(-C(\Psi) - \|\Psi\|_{|w|} + \Psi_{\min})e^{-n} \leq \Psi[w] \leq e^{-n}. \tag{4.5}$$

(3) The balls in $\{|w| : w \in B_n(\Psi)\}$ are pairwise disjoint.

(4) If $u \prec v$ are such that $u \in B_{n_1}(\Psi)$ and $v \in B_{n_2}(\Psi)$, then

$$|v| - |u| \leq \frac{\Psi_{\min} - \|\Psi\|_{|w|} - (n_2 - n_1) - 2C(\Psi)}{\Psi_{\max}}. \tag{4.6}$$

Let us start the proof of Proposition 3. The hard part is (2.7). At first we will show that $\log f(\alpha, n, \epsilon)$, as a sequence of $n$, has a kind of subadditivity property. Due to this subadditivity, by a standard procedure, we get the desired equality of the two limits. The proof is an adaption of that given in \cite{15} (see Proposition 5) and \cite{20} (see Proposition 4.3). However instead of $d_1$ and an additive potential $\phi$ considered in \cite{15}, here we consider $d_\Psi$ and an almost additive potential $\Phi$, so the proof is more involved.

• Subadditivity of $\log f(\alpha, n, \epsilon)$. More precisely we will show that for any $\epsilon > 0$, there exist an $N \in \mathbb{N}$ and positive sequence $\{\beta_n\}$ with $\log \beta_n = o(n)$ such that

$$f(\alpha, n, \epsilon)^p \leq \beta_n^p f(\alpha, (n + \bar{c})p, 2\epsilon)$$

for any $n \geq N$, and any $p \geq 1$, where $\bar{c} = [-p_0\Psi_{\max} - 2C(\Psi)]$. Recall that $p_0$ is a fixed positive integer such that $A_{p_0} > 0$.

To each $(w_1, \cdots, w_p) \in F(\alpha, n, \epsilon)^p$ we can associate an element of $F(\alpha, (n + \bar{c})p, 2\epsilon)$ as follows. Let $w = \overline{w}_1 \cdots \overline{w}_p$, where $\overline{w}_j = w_ju_j$ with $u_j \in W$ such that $w_ju_jw_{j+1}$ is
admissible, where $\mathcal{W}$ is defined in (2.2). Recall (see (2.1)) that for any cylinder $[u]$ and any $x, \bar{x} \in [u]$, we have $|\psi_{|u|}(x) - \psi_{|u|}(\bar{x})| \leq \|\Psi\|_{|u|}$. Thus for any $x \in [u]$,
\begin{equation}
(4.5) \quad \exp(\psi_{|u|}(x)) \geq \Psi[u] \exp(-\|\Psi\|_{|u|}).
\end{equation}

Now for any $x \in [w]$, let $s_0 = 0, s_k = \sum_{j=1}^{k}(|w_j| + p_0)$ $(1 \leq k \leq p)$ and define $x^k = T^{s_{k-1}}x$. We have $|w| = s_p$ and $x^k \in [w_k]$ for $k = 1, \ldots, p$. Then, by using the almost additivity of $\Psi$, (4.5) and Lemma 4.2(2) we can get
\[
\Psi[w] \geq \exp(\psi_{|w|}(x)) \geq \exp\left(\sum_{k=1}^{p} \psi_{|w_k|}(x^k) + p_0 p \Psi_{\min} - (2p - 1)C(\Psi)\right)
\geq \left(\prod_{k=1}^{p} \Psi[w_k]\right) \exp\left(-\sum_{k=1}^{p} \|\Psi\|_{|w_k|} + p_0 \Psi_{\min} - 2C(\Psi)\right) \geq \exp(-p(n + c_1(n))),
\]
where $c_1(n) = -(p_0 + 1)\Psi_{\min} + 3C(\Psi) + 2\|\Psi\|_{\min} > 0$ and $\|\Psi\|_{\min}$ is defined as in (4.1) with the constants $C(\Psi) \geq C(1) > 0$. Lemma 4.1(1) yields $c_1(n)/n \to 0$. By Lemma 4.1(3) and the definition of $\tilde{c}$ we also have
\[
\Psi[w] \leq \exp(2pC(\Psi) + p_0 p \Psi_{\max})\left(\prod_{k=1}^{p} \Psi[w_k]\right) \leq \exp(-p(n - p_0 \Psi_{\max} - 2C(\Psi))) \leq \exp(-p(n + \tilde{c})).
\]
Thus there exists $u \in B_{p(n+\tilde{c})}(\Psi)$ such that $u < w$. Write $w = uw'$.

Claim: $|w'| \leq p(ac_1(n) + b)$ for some constant $a, b > 0$.

Indeed, we have
\[
e^{-p(n + c_1(n))} \leq \Psi[w] \leq e^{C(\Psi)\Psi[u]} \Psi[w'] \leq e^{C(\Psi)} e^{-p(n + \tilde{c})} e^{\|\Psi\|_{\max}}.
\]
Thus $|w'| \leq p(c_1(n) + C(\Psi) - \tilde{c})/(-\Psi_{\max}) \leq p(c_1(n) + 3C(\Psi) + 1)/(-\Psi_{\max})$.

Now since $w_k \in F(\alpha, n, \epsilon)$ we can find $x_k \in [w_k]$ such that $\frac{\phi_{|w_k|}(x_k)}{|w_k|} - \alpha < \epsilon$. Take $x \in [w]$; in particular, $x \in [u]$. Define $s_k$ and $x^k$ as above. We have $|w| = s_p$ and $x^k \in [w_k]$ for $k = 1, \ldots, p$. By almost additivity, we get
\[
\phi_{|w|}(x) + \phi_{|w'|}(T|x|) - C(\Phi) \leq \phi_{|w|}(x) + \phi_{|w'|}(T|x|) + C(\Phi)
\]
(this is a vector inequality). Recall that if $\beta, \beta_1, \beta_2 \in \mathbb{R}^d$ are such that $\beta_1 \leq \beta \leq \beta_2$, then $|\beta| \leq |\beta_1| + |\beta_2|$. Then we conclude that
\[
\left|\phi_{|w|}(x) - \phi_{|w|}(x)\right| \leq \left|\phi_{|w'|}(T|x|) - C(\Phi)\right| + \left|\phi_{|w'|}(T|x|) + C(\Phi)\right|
\leq 2(|w'||\Phi| + |C(\Phi)|).
\]
In other word, $\phi_{|w|}(x) = \phi_{|w|}(x) + \eta_0$ with $|\eta_0| \leq 2|w'||\Phi| + 2|C(\Phi)|$. In a similar fashion we have
\[
\phi_{|w|}(x) = \phi_{|w|}(x) + \eta_0 = \sum_{k=1}^{p} \phi_{|w_k|}(x_k) + \eta_1 + \eta_0
= \sum_{k=1}^{p} \phi_{|w_k|}(x_k) + \eta_2 + \eta_1 + \eta_0 = \sum_{k=1}^{p} \phi_{|w_k|}(x_k) + \eta_3 + \eta_2 + \eta_1 + \eta_0
\]
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where
\[
\begin{align*}
|\eta_0| &\leq 2|w'||\Phi| + 2|C(\Phi)| \leq 2p(ac_1(n) + b)\|\Phi\| + 2|C(\Phi)|; \\
|\eta_1| &\leq 2(p - 1)|C(\Phi)|; \\
|\eta_2| &\leq 2p|\Phi| + |C(\Phi)|; \\
|\eta_3| &\leq \sum_{k=1}^{p} |\Phi|/|w_k| \leq p\|\Phi\|_{\ast}; \\
|\eta_4| &\leq (\sum_{k=1}^{p} |w_k|)\epsilon.
\end{align*}
\]

Since \(s_p = \sum_{k=1}^{p} |w_k| + p_0p\) and \(|w_k| \geq C_1n\), we have \(s_p \geq C_1np\) and
\[
\left| \frac{\phi_{|u|}(x)}{|u|} - \alpha \right| \leq \frac{\left| \left( \sum_{k=1}^{p} |w_k| \right) - |u| \right| \alpha + |\eta_4| + |\eta_3| + |\eta_2| + |\eta_1| + |\eta_0|}{|u|}.
\]

Moreover, \(|u| = s_p - |w'| \geq pC_1n - p(ac_1(n) + b)\) and \(\max(c_1(n)/n, \|\Phi\|_{\ast}/n) \to 0\), so we can choose \(N(\epsilon)\) big enough such that \(|\phi_{|u|}(x)/|u| - \alpha| \leq 2\epsilon\) when \(n \geq N(\epsilon)\). Consequently \(u \in F(\alpha, p(n + \bar{c}), 2\epsilon)\). From this we conclude that
\[
f(\alpha, p(n + \bar{c}), 2\epsilon) \geq f(\alpha, n, \epsilon)^p / m^{p(ac_1(n) + b)}.
\]

We get the desired subadditivity by taking \(\beta_n = m^{ac_1(n) + b}\).

**Coincidence of two limits.** Next we show that
\[
\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log f(\alpha, n, \epsilon)}{n} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon)}{n}.
\]

Note that these limits exist since \(f(\alpha, n, \epsilon)\) is a non-increasing function in the variable \(\epsilon\). Denote by \(\theta\) the left-hand side limit. Then for any \(\delta > 0\), there exists \(\epsilon_0 > 0\) such that
\[
\liminf_{n \to \infty} \log f(\alpha, n, \epsilon_0)/n < \theta + \delta.
\]

Fix \(\delta > 0\) and \(\epsilon_0 > 0\) as above. To show the equality we only need to show that
\[
\limsup_{n \to \infty} \log f(\alpha, n, \epsilon_0/4)/n \leq \theta + \delta.
\]

Take a sequence of integers \(n_k \not\to \infty\) such that \(f(\alpha, n_k, \epsilon_0) < e^{n_k(\theta + \delta)}\) for any \(k \in \mathbb{N}\). Fix \(n \geq N(\epsilon_0/4)\). For each \(k\), write \(n_k = (n + \bar{c})p_k - l_k\) with \(0 \leq l_k < n + \bar{c}\). By the subadditivity property, we have
\[
(f(\alpha, n, \epsilon_0/4))^{p_k} \leq \beta_n^{p_k} f(\alpha, (n + \bar{c})p_k, \epsilon_0/2).
\]

Next we show that there exists a positive integer sequence \(\{\gamma_k\}\) with \(\gamma_k = o(p_k)\) such that
\[
f(\alpha, (n + \gamma_k)p_k, \epsilon_0/2) \leq m^{\gamma_k} f(\alpha, n_k, \epsilon_0).
\]

Assume \(w = w_1w_2\) is such that \(w_1 \in \mathcal{B}_t(\Psi)\) and \(w \in \mathcal{B}_{t+s}(\Psi)\) with \(1 \leq s \leq (n + \bar{c})\), then by Lemma 4.4 we have
\[
|w_2| \leq \frac{\Psi_{\min} - \|\Psi\|_{|w|} - (n + \bar{c}) - 2C(\Psi)}{\Psi_{\max}}.
\]

Thus \(|w_2|/|w| \to 0\) when \(|w| \to \infty\). Choose \(t_0\) large enough so that when \(t \geq t_0\) and \(w_1 \in \mathcal{B}_t(\Psi), w \in \mathcal{B}_{t+s}(\Psi)\) we have
\[
\frac{|w|}{|w_1|} \leq \frac{3}{2}, \quad \frac{|C(\Phi)|}{|w_1|} \leq \frac{\epsilon_0}{16} \quad \text{and} \quad \frac{|w_2|}{|w_1|} \leq \frac{\epsilon_0}{8(2\|\Phi\| + |\alpha|)}.
\]
Let $k_0$ such that \((n + \tilde{c})p_{k_0} \geq t_0 + (n + \tilde{c})\). Let $k \geq k_0$. Fix $w \in F(\alpha, (n + \tilde{c})p_k, \epsilon_0/2)$. There exists $z \in [w]$ such that $|\phi_{|w|}(x) - \phi_{|w|}|(x) - |w|\alpha| \leq |w|\epsilon_0/2$. Note that \((n + \tilde{c})p_k \geq n_k\). Let $w_1 < w$ such that $\Psi[w_1] \leq e^{-n_k}$ and $\Psi[w^*] > e^{-n_k}$ (recall that $w^*$ is obtained by deleting the last letter of $w_1$). Thus $[w_1] \in B_{n_k}(\Psi)$. Write $w = w_1w_2$. By Lemma 4.3, we have

$$\left|\phi_{|w_1|}(x) - |w_1|\alpha\right| \leq \left|\phi_{|w|}(x) - \phi_{|w|}(x) + |\phi_{|w|}(x) - |w|\alpha| + |w_2||\alpha| \leq 2|w_2||\Phi| + 2|C(\Phi)| + |\phi_{|w|}(x) - |w|\alpha| + |w_2||\alpha| \leq |w_1|\epsilon_0,$$

which means that $w_1 \in F(\alpha, n_k, \epsilon_0)$. Write $q_k = [C_2(\Psi)(n + \tilde{c})p_k]$, then $|w| \leq q_k$. Define

$$\gamma_k := \frac{\Psi_{\min} - \|\Psi\|q_k - (n + \tilde{c}) - 2C(\Psi)}{\Psi_{\max}}.$$

It is clear that $\gamma_k = o(p_k)$. Moreover by (4.8), $|w_2| \leq \gamma_k$. From this we conclude (4.7).

Combine (4.6) and (4.7) we get

$$f(\alpha, n, \epsilon_0/4) \leq \beta_n m\gamma_k/p_k f(\alpha, n_k, \epsilon_0)^{1/p_k} \leq \beta_n m\gamma_k/p_k e^{-n_k(\theta + \delta)}/p_k.$$

Letting $k \to \infty$ we get $f(\alpha, n, \epsilon_0/4) \leq \beta_n e^{(n + \tilde{c})(\theta + \delta)}$. Then, letting $n \to \infty$ we have

$$\limsup_{n \to \infty} \log(f(\alpha, n, \epsilon_0/4)) \leq \theta + \delta.$$

- **Upper semi-continuity of $\Lambda(\alpha)$**. Let $\alpha \in L_\Phi$ for any $\eta > 0$ there is $\epsilon > 0$ such that $\liminf_{n \to \infty} \log(f(\alpha, n, \epsilon)/n) < \Lambda(\alpha) + \eta$. Let $\beta \in L_\Phi$ with $|\beta - \alpha| < \epsilon/3$. Given $w \in F(\beta, n, \epsilon/3)$, there exists $x \in [w]$ such that $|\phi_{|w|}(x)/|w| - \beta| \leq \epsilon/3$. Hence $|\phi_{|w|}(x)/|w| - \alpha| \leq |\phi_{|w|}(x)/|w| - \beta| + |\beta - \alpha| < \epsilon$, which means $w \in F(\alpha, n, \epsilon)$. This proves that $F(\beta, n, \epsilon/3) \subset F(\alpha, n, \epsilon)$. It follows that $f(\beta, n, \epsilon/3) \leq f(\alpha, n, \epsilon)$, therefore

$$\Lambda(\beta) \leq \liminf_{n \to \infty} \frac{f(\beta, n, \epsilon/3)}{n} \leq \liminf_{n \to \infty} \frac{f(\alpha, n, \epsilon)}{n} \leq \Lambda(\alpha) + \eta.$$

This establishes the upper semi-continuity of $\Lambda$ at $\alpha$.

- **Results about $D(\Psi)$**. By essentially repeating the same proof as above (in fact it is much easier), we can show

$$\liminf_{n \to \infty} \frac{\log \#B_n(\Psi)}{n} = \limsup_{n \to \infty} \frac{\log \#B_n(\Psi)}{n}.$$

We denote the limit by $D(\Psi)$. By (4.3), for any $w \in B_n(\Psi)$ we have $|w| \leq Cn$, where $C = 1 + 1/|\Psi_{\max}|$. This yields $\#B_n(\Psi) \leq \#\Sigma_{\Lambda, [Cn]}$ and consequently $D(\Psi) \leq C \log m$. □

Now we come to the weak concavity of the function $\Lambda$ on $L_\Phi$.

4.2. **Proof of Proposition 4**. Let $A \subset \mathbb{R}^d$. We say that $x \in A$ is a local cone point, or an $\epsilon$-cone point of $A$, if there exists $\epsilon > 0$ such that for any $y \in A \cap B(x, \epsilon)$ and $y \neq x$, the interval $[x, y] \subset A$, where $y := x + \epsilon(y - x)/|y - x|$.

**Lemma 4.3**. Let $A \subset \mathbb{R}^d$ be a convex set and $h : A \to \mathbb{R}$ be a bounded weakly concave function. Then $h$ is lower semi-continuous at each local cone point of $A$. Especially $h$ is lower semi-continuous on $\text{ri}(A)$ and on any closed interval $I \subset A$. It is lower semi-continuous on $A$ if $A \subset \mathbb{R}^d$ is a convex closed polyhedron.

**Proof**. Let $\beta \in A$ be an $\epsilon$-cone point of $A$ for some $\epsilon > 0$. Suppose that $h$ is not lower semi-continuous at $\beta$. Thus we can find $\eta > 0$ and $\alpha_n \in A \cap B(\beta, \epsilon)$ such that $\alpha_n \to \beta$ and
\( h(\alpha_n) \leq h(\beta) - \eta. \) Define \( \alpha_n' = \beta + \epsilon(\alpha_n - \beta)/|\alpha_n - \beta|, \) then \( \alpha_n' \in A \) since \( \beta \) is a \( \epsilon \)-cone point. Since \( \alpha_n \) is in the open interval \((\alpha_n', \beta)\), there exists a unique \( \lambda_n \in (0, 1) \) such that
\[
\alpha_n = \frac{\lambda_n \gamma_1(\alpha_n', \beta) \alpha_n' + (1 - \lambda_n) \gamma_2(\alpha_n', \beta) \beta}{\lambda_n \gamma_1(\alpha_n', \beta) + (1 - \lambda_n) \gamma_2(\alpha_n', \beta)},
\]
where \( \gamma_1, \gamma_2 \) is from the definition of weak concavity. Since \( \gamma_1, \gamma_2 \in [c_1, c] \) and \( \alpha_n \to \beta \) we conclude that \( \lambda_n \to 0. \) Since \( h \) is bounded, by (2.8) we get \( h(\alpha_n) \geq \lambda_n h(\alpha_n') + (1 - \lambda_n) h(\beta) \to h(\beta) \) (as \( n \to \infty \)), which is in contradiction with the choice of \( \alpha_n. \) So \( h \) is lower semi-continuous at \( \beta. \)

Since each \( x \in E \) is a local cone point of \( E \) when \( E \) is \( ri(A) \), or \( E \) is a closed interval in \( A \), or \( E \) is itself and \( A \) is a convex closed polyhedron, the other results follow. □

Next we prove Proposition 4. The new point in this proposition is the weak concavity. In fact when \( d_\Phi = d_1 \), as shown in [15], the function \( \Lambda \) is indeed concave. When the more general metric \( d_\Psi \) is considered, the length of \( w \in B_n(\Psi) \) has fluctuations (see Lemma 4.2 (1)), which destroy the concavity of \( \Lambda. \) However, these fluctuations are controllable, so that these careful analysis yields the weak concavity of \( \Lambda. \)

At first we show that \( \Lambda \) is bounded and positive. Fix \( \alpha \in L_\Phi. \) By definition \( f(\alpha, n, \epsilon) \leq \#B_n(\Psi) \), consequently \( \Lambda(\alpha) \leq D(\Psi). \) On the other hand since \( \alpha \in L_\Phi \), for any \( \epsilon > 0 \), when \( n \) large enough, \( F(\alpha, n, \epsilon) \neq \emptyset. \) Consequently \( \Lambda(\alpha) \geq 0. \) Thus \( \Lambda(L_\Phi) \subset [0, D(\Psi)]. \)

Next we show that \( \Lambda \) is weakly concave. Let \( \alpha, \beta \in L_\Phi. \) For any \( w_1, \ldots, w_p \in F(\alpha, n, \epsilon) \) and any \( w_{p+1}, \ldots, w_{p+q} \in F(\beta, n, \epsilon), \) let \( w = \overline{w_1} \cdots \overline{w_{p+q}} \) where \( \overline{w_j} = w_j u_j \) with \( u_j \in W \) such that \( w \) is admissible. By the same argument as for Proposition 3, we can show that \( \exp(-(p+q)(n + c_1(n))) \leq \Psi[w] \leq \exp(-(p+q)(n + \overline{c})) \) with the same \( c_1(n) \) and \( \overline{c} \) as in Proposition 3 which means that there exists \( u < w \) such that \( u \in B((p+q)(n + \overline{c}))(\Psi). \)

Write \( w = uu'. \) We also have \( |w'| \leq (p + q)(ac_1(n) + b) \) with the same \( (a, b) \) as in that proposition.

For any \( k \in \mathbb{N} \) define \( F_k(\alpha, n, \epsilon) := \Sigma_{A,k} \cap F(\alpha, n, \epsilon). \) By Lemma 4.2 (1), we have \( F(\alpha, n, \epsilon) = \bigcup_{C_1 n \leq \epsilon \leq C_2 n} F_k(\alpha, n, \epsilon), \) where \( C_i = C_i(\Psi) \) for \( i = 1, 2. \) Define \( f_k(\alpha, n, \epsilon) = \#F_k(\alpha, n, \epsilon). \) Choose \( k_0 \) such that \( f_{k_0}(\alpha, n, \epsilon) = \max_{C_1 n \leq \epsilon \leq C_2 n} f_k(\alpha, n, \epsilon). \) Then \( f_{k_0}(\alpha, n, \epsilon) \geq f(\alpha, n, \epsilon)/(C_2 - C_1)n. \) Write \( k_0 = \gamma_n(\alpha)n, \) thus \( \gamma_n(\alpha) \in [C_1, C_2]. \) Likewise we can find \( \gamma_n(\beta) \in [C_1, C_2] \) such that \( f_{\gamma_n(\beta)n}(\beta, n, \epsilon) \geq f(\beta, n, \epsilon)/(C_2 - C_1)n. \)

Fix a subsequence \( n_k \uparrow \infty \) such that \( \gamma_{nk}(\alpha) \to \gamma(\alpha) \) and \( \gamma_{nk}(\beta) \to \gamma(\beta) \) as \( k \to \infty. \) Take \( w_1, \ldots, w_p \in F_{\gamma_{nk}(\alpha)}(\alpha, n_k, \epsilon) \) and \( w_{p+1}, \ldots, w_{p+q} \in F_{\gamma_{nk}(\beta)}(\beta, n_k, \epsilon). \) Choose \( x_j \in [w_j] \) such that
\[
\begin{cases}
|\phi_{w_j}(x_j)| - |w_j| \alpha \leq |w_j| \epsilon, & \text{if } 1 \leq j \leq p \\
|\phi_{w_j}(x_j)| - |w_j| \beta \leq |w_j| \epsilon, & \text{if } p + 1 \leq j \leq p + q.
\end{cases}
\]

Let \( w = \overline{w_1} \cdots \overline{w_{p+q}} \) and write \( w = uu' \) such that \( u \in B(\overline{p+q}(n_k + \overline{c}))(\Psi). \) Then we know that \( |w| = p(\gamma_{nk}(\alpha)n_k + p_0) + q(\gamma_{nk}(\beta)n_k + p_0) \) and \( |u| = |w| - |w'|. \) Now for any \( x \in [w], \) define \( x^1 = x \) and \( x^j = T_{(\Sigma_{j-1}^{i=1} |w_i| + p_0)x} \) for \( j \geq 2. \) Then we have
\[
\phi_{|w|}(x) = \phi_{|w|}(x) + \eta_0 = \sum_{j=1}^{p+q} \phi_{|w_j|}(x^j) + \eta_1 + \eta_0 = \sum_{j=1}^{p+q} \phi_{|w_j|}(x^j) + \eta_2 + \eta_1 + \eta_0
\]
In particular, if $\alpha$ is large enough, $u \in F((p\gamma(\alpha) + q\gamma(\beta))/\gamma(\alpha) + q\gamma(\beta)), (n_k + c)(p + q), 2\epsilon)$. Thus we conclude that

$$f\left(\frac{p\gamma(\alpha) + q\gamma(\beta)}{p\gamma(\alpha) + q\gamma(\beta)}\right)(n_k + c)(p + q), 2\epsilon)$$

This yields that for $k$ large enough, $u \in F((p\gamma(\alpha) + q\gamma(\beta))/\gamma(\alpha) + q\gamma(\beta)), (n_k + c)(p + q), 2\epsilon)$. Thus we conclude that

$$f\left(\frac{p\gamma(\alpha) + q\gamma(\beta)}{p\gamma(\alpha) + q\gamma(\beta)}\right)(n_k + c)(p + q), 2\epsilon)$$

Combining this with Proposition 3 we get

$$\lambda \Lambda(\alpha) + (1 - \lambda) \Lambda(\beta) \leq \Lambda\left(\frac{\lambda\gamma(\alpha) + (1 - \lambda)\gamma(\beta)\beta}{\gamma(\alpha) + (1 - \lambda)\gamma(\beta)}\right)$$

for any $\lambda = \frac{p}{p+q} \in [0, 1]$. Since $\Lambda$ is upper semi-continuous, we conclude that this formula holds for any $\lambda$, $\Lambda$ is weakly concave.

Assume $A \subset L_\Phi$ is a convex set, and $I \subset L_\Phi$ is a closed interval. By Lemma 4.3, $\Lambda$ is lower semi-continuous on $\text{ri}(A)$ and $I$. Combining this with the upper semi-continuity yields the continuity on $\text{ri}(A)$ and $I$. Taking $A = L_\Phi$, we get the continuity on $\text{ri}(L_\Phi)$.

Now assume $L_\Phi$ is a polyhedron. By Lemma 4.3, $\Lambda$ is lower semi-continuous on $L_\Phi$. This, together with the upper semi-continuity yields the continuity on $L_\Phi$.

Let $I = [\alpha_1, \alpha_2] \subset L_\Phi$ and $\alpha_{\max} \in I$ as defined in the proposition. Assume $\Lambda$ is not decreasing from $\alpha_{\max}$ to $\alpha_1$. Since $\Lambda$ is continuous on $I$, we can find $\beta_1, \beta_2, \beta_3 \in [\alpha_1, \alpha_{\max}]$ such that $\beta_2 \in [\beta_1, \beta_3]$ and $\Lambda(\beta_1) = \Lambda(\beta_3) > \Lambda(\beta_2)$, which is in contradiction with the fact that $\Lambda$ is quasi-concave, since it is weakly concave. Thus $\Lambda$ is decreasing from $\alpha_{\max}$ to $\alpha_1$. The same argument shows that $\Lambda$ is decreasing from $\alpha_{\max}$ to $\alpha_2$. □

5. **Proof of Theorem 2.2 (1)**

By Proposition 2 we have $E_\Phi(\alpha) \neq \emptyset$ if and only if $\alpha \in L_\Phi$. For the next statement, our plan is the following: we show that $D(\alpha) \leq \Lambda(\alpha) \leq \mathcal{E}(\alpha) \leq D(\alpha)$. We divide this into three steps corresponding to the next Sections 5.1, 5.2 and 5.3.

5.1. $D(\alpha) \leq \Lambda(\alpha)$. We prove a slightly more general result for the upper bound. Given $\Phi \in \mathcal{C}_{ad}(\Sigma, T, d)$ and $\Omega \subset L_\Phi$, define $E_\Phi(\Omega) := \bigcup_{\alpha \in \Omega} E_\Phi(\alpha)$.

**Proposition 6.** For any compact set $\Omega \subset L_\Phi$ we have $\dim_F E_\Phi(\Omega) \leq \sup\{\Lambda(\alpha) : \alpha \in \Omega\}$. In particular, if $\alpha \in L_\Phi$ we have $D(\alpha) \leq \dim_F E_\Phi(\alpha) \leq \Lambda(\alpha)$.
We have

Thus Φ

each w

Ψ by two sequences of Hölder potentials. We describe this procedure as follows.

\[ \Lambda(\alpha, \epsilon) := \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon)}{n}, \]

for each α ∈ Ω, there exists ε_α > 0 such that for any 0 < ε ≤ ε_α we have Λ(α, ε) < Λ(α) + η. Since \{B(\alpha, \epsilon_\alpha) : \alpha \in \Omega\} is an open covering of Ω, we can find a finite covering \{B(\alpha_1, \epsilon_1), \ldots, B(\alpha_s, \epsilon_s)\}, where \epsilon_j = \epsilon_{\alpha_j}. For each n ∈ N define

\[ H(n, \eta) := \bigcup_{j=1}^{s} \bigcup_{w \in F(\alpha_j, n, \epsilon_j)} [w] \text{ and } G(k, \eta) := \bigcap_{n \geq k} H(n, \eta) \]

It is standard to prove that \text{dim}_P E_\Phi(\Omega) \subset \bigcup_{k \in \mathbb{N}} G(k, \eta). Consequently

\[ \dim_P E_\Phi(\Omega) \leq \sup_{k \in \mathbb{N}} \dim_P G(k, \eta). \]

By definition, the set \text{dim}_P G(k, \eta) is covered by \{[w] : w \in F(\alpha_j, n, \epsilon_j) ; j = 1, \ldots, s\} for any \ n \geq k. Since each element in \{[w] : w \in F(\alpha_j, n, \epsilon_j)\} is a ball with radius \epsilon^{-n}, we get

\[ \dim_P G(k, \eta) \leq \dim_B G(k, \eta) \leq \limsup_{n \to \infty} \frac{\log \sum_{j=1}^{s} f(\alpha_j, n, \epsilon_j)}{n} \]

\[ \leq \sup_{j=1, \ldots, s} \limsup_{n \to \infty} \frac{\log f(\alpha_j, n, \epsilon_j)}{n} = \sup_{j=1, \ldots, s} \Lambda(\alpha_j, \epsilon_j) \leq \sup \{ \Lambda(\alpha) : \alpha \in \Omega \} + \eta. \]

Combining this with (5.1) we get \text{dim}_P E_\Phi(\Omega) \leq \sup \{ \Lambda(\alpha) : \alpha \in \Omega \} + \eta. \hfill \Box

5.2. \Lambda(\alpha) \leq \mathcal{E}(\alpha). Our approach is inspired by that of [15], which deals with the case that \text{dim}_P \Phi = d_1 and \Phi = (S_n \phi)_{n \geq 1} is additive, where \phi : \Sigma_A \to \mathbb{R}^d \text{ is continuous.}

To show this inequality we need to approximate the almost additive potentials \Phi and \Psi by two sequences of Hölder potentials. We describe this procedure as follows.

Given \Phi ∈ \mathcal{C}_{aa}(\Sigma_A, T, d), for each k ∈ \mathbb{N} we define \Phi(k) ∈ \mathcal{C}_{aa}(\Sigma_A, T, d) as follows. For each \ w ∈ \Sigma_{\alpha, k} choose \ x_w \in [w]. For any \ x \in [w] define \ \tilde{\phi}_k(x) := \phi_k(x_w)/k. Then \ \tilde{\phi}_k \text{ depends only on the first } k \text{ coordinates of } x \in \Sigma_A \text{ and is Hölder continuous. Define}

\[ \Phi(k) = (S_n \tilde{\phi}_k)_{n=1}^{\infty}. \]

Thus \Phi(k) is additive and Hölder continuous.

Lemma 5.1. We have \Phi_{\text{min}} \leq \Phi_{\min}(k) \leq \Phi_{\max}(k) \leq \Phi_{\max}. Moreover

\[ \|\phi_n - S_n \tilde{\phi}_k\| \leq d \left( \frac{n}{k} |C(\Phi)| + 5kn^2 \right). \]

Consequently \\|\Phi - \Phi(k)\|_{\text{lim}} \to 0 \text{ when } k \to \infty.

This lemma will be proved at the end of this section.

Proof of \Lambda(\alpha) \leq \mathcal{E}(\alpha). Given \Phi ∈ \mathcal{C}_{aa}(\Sigma_A, T, d) and \Psi ∈ \mathcal{C}_{aa}^{-}(\Sigma_A, T).

Claim: Given \epsilon > 0 we have

\[ \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} \leq \sup_{|\phi_\nu(\nu) - \phi_\nu| \leq 5\epsilon} \frac{h_\nu}{\Psi_\nu(\nu) + O(\epsilon)}. \]
Let us at first assume the claim holds and finish the proof. Notice that the set of invariant measures \( \nu \) such that \( |\Phi_\ast(\nu) - \alpha| \leq 5\epsilon \) is compact, so by using the upper semi-continuity of \( h_\nu \) and letting \( \epsilon \) tend to 0 we can find an invariant measure \( \nu_0 \) such that \( \Phi_\ast(\nu_0) = \alpha \) and

\[
\Lambda(\alpha) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} \leq \frac{h_{\nu_0}}{-\Psi_\ast(\nu_0)} \leq \mathcal{E}(\alpha).
\]

Next we show that the claim holds. In the following \( C = C(\Psi), C_i = C_i(\Psi) \) for \( i = 1, 2 \). Fix \( \epsilon > 0 \). For any \( k \in \mathbb{N} \) define \( \Phi^{(k)} \) and \( \Psi^{(k)} \) according to (5.2). By Lemma 5.1, we can find \( k \in \mathbb{N} \) such that

\[
\|\Phi - \Phi^{(k)}\|_\text{lim},\|\Psi - \Psi^{(k)}\|_\text{lim} < \epsilon.
\]

Fix this \( k \), then there exists \( N_1 \in \mathbb{N} \) such that when \( n \geq N_1 \)

\[
\|\phi_n - S_n\phi_k\|_\infty < n\epsilon \quad \text{and} \quad \|\psi_n - S_n\psi_k\|_\infty < n\epsilon.
\]

For any \( w \in \mathcal{B}_n(\Psi) \), we have \( C_1 n \leq |w| \leq C_2 n \), thus for \( n \geq N_1/C_1 \), we have \( F(\alpha, n, \epsilon, \Phi, \Psi) \subset F(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi) \), consequently

\[
f(\alpha, n, \epsilon, \Phi, \Psi) \leq f(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi).
\]

Following [15], we introduce a way to classify the words in \( F(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi) \), by which we can estimate the cardinality of it effectively.

For any word \( w \in \Sigma_{A_k} \) such that \( |w| \geq k \), we define the counting function \( \theta_w : \Sigma_{A_k} \to \mathbb{N} \) as \( \theta_w(u) = \#\left\{ j : w_j \cdots w_{j+k-1} = u \right\} \), which counts the numbers of times the word \( u \) appears in \( w \). It is clear that \( h_{\theta_w} := \sum_u \theta_w(u) = |w| - k + 1 \). We call it the \textit{height} of \( \theta_w \).

Let \( \mathcal{P}_k^{(n)} = \{ \theta_w : w \in F(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi) \} \). Then \( \#\mathcal{P}_k^{(n)} \leq (C_2 n)^{m_k} \). For each \( \theta \in \mathcal{P}_k^{(n)} \), let

\[
\mathcal{T}(\theta) = \{ w : w \in F(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi), \theta_w = \theta \}.
\]

Write \( \Gamma(\theta) := \#\mathcal{T}(\theta) \). Then we have

\[
f(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi) = \sum_{\theta \in \mathcal{P}_k^{(n)}} \Gamma(\theta) \leq (C_2 n)^{m_k} \max_{\theta \in \mathcal{P}_k^{(n)}} \Gamma(\theta).
\]

Consequently

\[
(5.5) \quad \frac{\log f(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi)}{n} \leq \max_{\theta \in \mathcal{P}_k^{(n)}} \frac{\log \Gamma(\theta)}{n} + m_k O\left( \frac{\log n}{n} \right).
\]

In the following we estimate \( \log \Gamma(\theta)/n \) for each \( \theta \in \mathcal{P}_k^{(n)} \). Since it is hard to estimate it directly, we turn to the estimations of \( \log \Gamma(\theta)/h_\theta \) and \( n/h_\theta \).

Following [15] we define \( \Delta_{k}^+ \), the set of all positive functions \( p \) on \( \Sigma_{A_k} \) satisfying the following two relations:

\[
\sum_{w \in \Sigma_{A_k}} p(w) = 1; \quad \sum_{w} p(ww_1w_2 \cdots w_{k-1}) = \sum_{w} p(w_1w_2 \cdots w_{k-1}w).
\]

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It is known (see [15]) that for any \( \eta > 0 \), there is a positive integer \( N = N(\eta) \) such that for any \( w \in \Sigma_{A,l+k-1} \) with \( l > N \), there exists a probability vector \( p \in \triangle_k^+ \) such that

\[
\left| \frac{\theta_w(u)}{l} - p(u) \right| < \eta, \quad p(u) > \frac{\eta}{m^{k+1}}.
\]

We discard the trivial case where \( \Phi \equiv 0 \) and fix \( \eta > 0 \) such that \( \eta < \epsilon/(m^k \| \Phi \|) \).

Now take any \( n \geq \max\{ N_1/C_1, 4k \| \Phi \| / C_1 \epsilon \} \) and fix a \( \theta \in \mathcal{P}^{(n)}_k \). Take

\[
w \in F(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi)
\]
such that \( \theta = \theta_w \), then \( |w| = h_\theta + k - 1 \). Fix a \( p \in \triangle_k^+ \) as described above. Consider the Markov measure \( \nu_p \) corresponding to \( p \) (see [15] for the definition and related properties). For any word \( v \in \mathcal{T}(\theta) \) we have

\[
\nu_p([v]) = \frac{p(v|k)}{t(v|k)} \prod_{|u| = k} t(u)^{\theta(u)} \geq \frac{\eta}{m^{k+1}} \prod_{|u| = k} t(u)^{\theta(u)} := \rho,
\]

where

\[
t(a_1 \cdots a_k) = \frac{p(a_1 \cdots a_k)}{\sum_i p(a_1 \cdots a_{k-1} i)}.
\]

Thus \( \rho \cdot \Gamma(\theta) \leq \nu_p(\bigcup_{v \in \mathcal{T}(\theta)[v]} \leq 1 \) and consequently

\[
\Gamma(\theta) \leq \frac{1}{\rho} = \frac{m^{k+1}}{\eta} \prod_{|u| = k} t(u)^{-\theta(u)}.
\]

Since \( C_1 n \leq |w| = h_\theta + k - 1 \leq C_2 n \), we have \( h_\theta \sim n \) when \( n \to \infty \). Notice that \( \eta/m^{k+1} \leq t(u) \leq 1 \), thus

\[
\frac{\log \Gamma(\theta)}{h_\theta} \leq O\left( \frac{k}{n} \right) + O\left( \frac{\log n}{n} \right) - \sum_{|u| = k} \frac{\theta(u)}{h_\theta} \log t(u)
\]

\[
\leq o(1) + O\left( \frac{\log n}{n} \right) - \sum_{|u| = k} p(u) \log t(u) + m^k \eta (\log n + (k + 1) \log m)
\]

\[
= h_{\psi_p} + o(1) + O\left( \frac{\log n}{n} \right) + O(\eta \log n).
\]

Now we estimate \( n/h_\theta \). Let \( x_0 \in [w] \). By (4.4) we have

\[
-n - C(\Psi) - 2\| \Psi \|_\infty + \Psi_{\text{min}} \leq \psi_{|w|}(x_0) \leq \sup_{x \in [w]} \psi_{|w|}(x) \leq -n,
\]

Since \( n \geq N_1/C_1 \), we have \( |w| \geq C_1 n \geq N_1 \). Thus \( \| \psi_{|w|} - S_{|w|} \bigotimes_k \|_\infty \leq |w| \epsilon \). We get

\[
n - C(\Psi) - 2\| \Psi \|_\infty + \Psi_{\text{min}} - C_2 n \epsilon \leq S_{|w|} \bigotimes_k \psi_k(x_0) \leq -n + C_2 n \epsilon.
\]

Notice that \( \tilde{\psi}_k \) is negative and \( |w| = h_\theta + k - 1 \), thus

\[
n - C(\Psi) - 2\| \Psi \|_\infty + \Psi_{\text{min}} - C_2 n \epsilon \leq S_{h_\theta} \bigotimes_k \tilde{\psi}_k(x_0) \leq -n + C_2 n \epsilon + k \| \Psi \|.
\]

On the other hand

\[
\frac{S_{h_\theta} \bigotimes_k \tilde{\psi}_k(x_0)}{h_\theta} = \sum_{|u| = k} \frac{\theta(u)}{h_\theta} \tilde{\psi}_k(x_u) = \sum_{|u| = k} p(u) \tilde{\psi}_k(x_u) + m^k O(\eta)
\]

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Combining this with (5.7) and the fact that \( \|\Psi\|^o/n = o(1) \) we get
\[
\frac{n}{h_\theta} = -\Psi^*(\nu_p) + O(\epsilon) + O(\eta) + o(1).
\]

Combine (5.6) and (5.8) we get
\[
\log \Gamma(\theta) \leq \frac{h_{\nu_p} + o(1) + O(\log \eta / n) + O(\eta \log \eta)}{-\Psi^*(\nu_p) + O(\epsilon) + O(\eta) + o(1)}.
\]

Next we show that \( |\Phi^*(\nu_p) - \alpha| \leq 5\epsilon \). Since \( w \in F(\alpha, n, 2\epsilon, \Phi^{(k)}, \Psi) \), there exists \( y_0 \in [w] \) such that \( |S_{[w]}\bar{\phi}_k(y_0)/|w| - \alpha| \leq 2\epsilon \). Note that \( |w| = h_{\theta w} + k - 1, \) we have
\[
|\Phi^*(\nu_p) - \alpha| \leq |\Phi^{(k)}(\nu_p) - \alpha| + \Phi^*(\nu_p) - \Phi^{(k)}(\nu_p)|
\]
\[
\leq \left| \int \bar{\phi}_k d\nu_p - \alpha \right| + |\sum_{|u|=k} \theta_w(u) \bar{\phi}_k(x_u) - \alpha| + m^k \eta^\alpha \bar{\phi}_k + \epsilon
\]
\[
\leq \left| \frac{S_{[w]}\bar{\phi}_k(x)}{h_{\theta w}} - \alpha \right| + m^k \eta^\alpha \|\Phi\| + \epsilon \quad (\text{for any } x \in [w])
\]
\[
\leq \left| \frac{S_{[w]}\bar{\phi}_k(x) - S_{[w]}\bar{\phi}_k(y_0)}{|w|} \right| + \frac{2k\|\Phi\|}{|w|} \quad (\text{for any } x \in [w])
\]
\[
\leq \left| \frac{S_{[w]}\bar{\phi}_k(x) - S_{[w]}\bar{\phi}_k(y_0)}{|w|} \right| + \frac{2k\|\Phi\|}{|w|} \leq \frac{4k\|\Phi\|}{C_1 n} + m^k \eta^\alpha \|\Phi\| + 3\epsilon.
\]

By our choice of \( \eta \) we have \( m^k \eta^\alpha \|\Phi\| < \epsilon \). Moreover, since \( n \geq 4k\|\Phi\|/(C_1 \epsilon) \) we have \( 4k\|\Phi\|/(C_1 \epsilon) \leq \epsilon \). Thus \( |\Phi^*(\nu_p) - \alpha| \leq 5\epsilon \).

Again by the compactness of the set \( \{\nu : |\Phi^*(\nu) - \alpha| \leq 5\epsilon\} \) and the upper semi-continuity of \( h_\nu \), Combining (5.4), (5.5) and (5.9) we conclude that
\[
\limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n} \leq \sup_{|\Phi^*(\nu) - \alpha| \leq 5\epsilon} \left( \frac{h_\nu + O(\eta \log \eta)}{-\Psi^*(\nu) + O(\epsilon) + O(\eta)} \right).
\]

Let \( \eta \to 0 \) we get (5.3).

**Proof of Lemma 5.1** At first we assume \( \Phi \in C_\infty(S_A, T) \). By (3.2) we get \( \Phi_{\min} \leq \bar{\phi}_k \leq \Phi_{\max} \). Since \( \Phi^{(k)} \) is additive, we have \( \Phi_{\min} \leq \bar{\phi}_{k,\min} = \Phi_{\min}^{(k)} \leq \bar{\phi}_{k,\max} = \Phi_{k,\max} \leq \Phi_{\max} \).

For \( n \in \mathbb{N} \), write \( n = pk + s \) with \( 0 \leq s < k \). Write \( C = C(\Phi) \), by using the almost additivity of \( \Phi \) for each \( 0 \leq j \leq k - 1 \) we have
\[
\phi_n(x) \leq \phi_j(x) + \sum_{l=0}^{p-2} \phi_k(T^{j+lk} x) + \phi_{n-(j+(p-1)k)}(T^{j+(p-1)k} x) + pC
\]

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Similarly we have $\Psi$ and we know that $\Lambda$ by (2.5). Let

$$D$$

consequently (6.1) (1)

Lemma 6.1. Let us prove Theorem 2.2 (2). Suppose first that $\Psi$ is Hölder continuous. Let $\Phi = (\Phi^1, \ldots, \Phi^d) \in C_{aa}(\Sigma_A, T, d)$, applying the result just proven to each component of $\Phi$ we get the result. 

5.3. $E(\alpha) \leq D(\alpha)$. It is contained in Proposition 8 (see Section 7).

6. Proof of Theorem 2.2 (2)

We need to describe the $\Psi$- and $\Phi$- dependence of the function $\Lambda = \Lambda^\Psi_\Phi$. Recall that

$$\Lambda^\Psi_\Phi(\alpha, \epsilon) = \limsup_{n \to \infty} \frac{\log f(\alpha, n, \epsilon, \Phi, \Psi)}{n}$$

and we know that $\Lambda^\Psi_\Phi(\alpha, \epsilon) \searrow \Lambda^\Psi_\Phi(\alpha)$ as $\epsilon \searrow 0$.

Lemma 6.1. (1) Assume $\Psi, \Upsilon \in C^-_{aa}(\Sigma_A, T)$, then we have

$$|D(\Psi) - D(\Upsilon)| \leq 3 \log m \cdot \left(1 + \frac{1}{|\Psi_{\text{max}}|}\right) \left(1 + \frac{1}{|\Upsilon_{\text{max}}|}\right) ||\Psi - \Upsilon||_{\text{lim}}.$$

(2) Assume $\delta_0 := ||\Psi - \Upsilon||_{\text{lim}} \leq 1/(4C_2(\Psi))$. Let $\Phi, \Theta \in C_{aa}(\Sigma_A, T, d)$. Fix $\eta > 0$ and let $\beta \in L_{\Theta}$. Then for any $\alpha \in B(\beta, \eta) \cap L_{\Phi}$ we have

$$\Lambda^\Psi_\Phi(\alpha) \leq \frac{2C_2(\Psi) \log m}{|\Upsilon_{\text{max}}|} \delta_0 + (1 - C_2(\Psi)\delta_0)\Lambda^\Upsilon_\Theta(\beta, a_0 + \kappa\delta_0 + 2\eta),$$

where $a_0 = ||\Phi - \Theta||_{\text{lim}} \alpha$ and $\kappa = \kappa(\Psi, \Upsilon, \Phi) = 18||\Psi||C_2(\Psi)||\Upsilon_{\text{min}}||/|\Upsilon_{\text{max}}|.$

Let us prove Theorem 2.2 (2). Suppose first that $\Psi$ is Hölder continuous. Let $t$ be the solution of the equation $P(t\Psi) = 0$ and $\mu$ be the unique equilibrium state of $t\Psi$, where $P(t\Psi)$ is the topological pressure of $t\Psi$. The measure $\mu$ is ergodic (2.3), and $\dim_H \Sigma_A = \dim_H \mu$ (8). Moreover $t$ is also the box dimension of $(\Sigma_A, d_\Psi)$. Consequently, $t = D(\Psi)$ by (2.5). Let $\alpha = \Phi_* (\mu)$. By the sub-additive ergodic theorem we have $\mu(E_\Phi(\alpha)) = 1$, consequently $D(\alpha) = D(\Psi)$. Thus, when $\Psi$ is a Hölder potential the result holds.

Next we assume $\Psi \in C^-_{aa}(\Sigma_A, T)$. Define $\Psi^{(n)}$ according to (5.2), then we have

$$\lim_{n \to \infty} ||\Psi - \Psi^{(n)}||_{\text{lim}} = 0 \quad \text{and} \quad |\Psi_{\text{max}}| \leq |\Psi_{\text{max}}^{(n)}| \leq |\Psi_{\text{min}}|.$$
and by (6.1) we have \( \lim_{n \to \infty} D(\Psi^{(n)}) = D(\Psi) \). Let \( \mu_n \) be the unique equilibrium state of \( D(\Psi^{(n)}) \cdot \Psi^{(n)} \) and define \( \alpha_n = \Phi_\mu(\mu_n) \). Then \( \alpha_n \in L_\Phi \) and \( \Lambda^{\Psi^{(n)}}_\Phi(\alpha_n) = D(\Psi^{(n)}) \). Let \( \alpha \) be a limit point of the sequence \( \{\alpha_n : n \in \mathbb{N}\} \). Without loss of generality we assume \( \alpha = \lim_{n \to \infty} \alpha_n \). By (6.2) we have

\[
(6.3) \quad \Lambda^{\Psi^{(n)}}_\Phi(\alpha_n) \leq \frac{2C_2(\Psi^{(n)}) \log m}{\|\Psi\|_{\text{max}}} \delta_n + (1 - C_2(\Psi^{(n)})\delta_n) \Lambda^{\Psi}_\Phi(\alpha, \kappa_n \delta_n + 2\eta_n),
\]

where \( \delta_n := \|\Psi - \Psi^{(n)}\|_{\text{lim}} \) and

\[
C_2(\Psi^{(n)}) = 1 + \frac{1}{\|\Psi^{(n)}\|_{\text{max}}}, \quad \kappa_n = \frac{18\|\Phi\|C_2(\Psi^{(n)})\|\Psi\|_{\text{min}}}{\|\Psi\|_{\text{max}}} + \eta_n = |\alpha - \alpha_n|.
\]

By Lemma 5.1 we have \( C_2(\Psi^{(n)}) \leq 1 + 1/|\Psi\|_{\text{max}} \), thus we can rewrite (6.3) as

\[
D(\Psi^{(n)}) \leq d_1 \delta_n + \Lambda^{\Psi}_\Phi(\alpha, d_2 \delta_n + 2\eta_n).
\]

Letting \( n \) tend to \( \infty \) we get \( D(\Psi) \leq \Lambda^{\Psi}_\Phi(\alpha) \). By the definition of box dimension we have \( \dim_B \Sigma_A \leq D(\Psi) \). Thus we have

\[
D(\Psi) \leq \Lambda^{\Psi}_\Phi(\alpha) = \dim_H E_\Phi(\alpha) \leq \dim_H \Sigma_A \leq \dim_B \Sigma_A \leq D(\Psi),
\]

and we get the equality. \( \square \)

**Proof of Lemma 6.1.** (1) Write \( \Psi = (\psi_n)_{n=1}^\infty \) and \( \Upsilon = (\upsilon_n)_{n=1}^\infty \). By the definition of \( \|\cdot\|_{\text{lim}} \), for any \( \delta > \|\Psi - \Upsilon\|_{\text{lim}} \), there exist \( N \in \mathbb{N} \), such that for any \( n \geq N \) we have

\[
\psi_n(x) - n\delta \leq \upsilon_n(x) \leq \psi_n(x) + n\delta,
\]

consequently for any \( w \in \Sigma_A, \) with \( |w| \) large enough we have

\[
\Psi[w]e^{-|w|\delta} \leq \Upsilon[w] \leq \Psi[w]e^{|w|\delta}.
\]

Given \( w \in B_n(\Psi) \), by (4.3) and (4.4) we have

\[
e^{-(1+C(\Psi))\delta} |w| e^{-n(1+C(\Psi))\delta} \leq \Upsilon[w] \leq e^{-(1-C(\Psi))\delta} |w|.
\]

This implies that there exists \( u < w \) such that \( u \in B_{n(1-C(\Psi))\delta}(\Upsilon) \), hence we have

\[
(6.4) \quad \#B_{n(1-C(\Psi))\delta}(\Upsilon) \leq \#B_{n}(\Psi).
\]

Let \( c_1(n) = -\Psi_{\text{min}} + C(\Psi) + \|\Psi\|^*_n \). We have \( c_1(n) > 0 \) and \( c_1(n) = o(n) \). Write \( w = uw' \). The same proof as that of the claim in Proposition 3 yields \( |w'| \leq c_1(n) + 2nC_2(\Psi)\delta + C(\Upsilon)/|\Upsilon|_{\text{max}} \). Thus we can conclude that

\[
(6.5) \quad \#B_{n(1-C(\Psi))\delta}(\Upsilon) \geq \#B_{n}(\Psi)m^{-c_1(n) + 2nC_2(\Psi)\delta + C(\Upsilon)/|\Upsilon|_{\text{max}}}.\]

Combining (6.4), (6.5) and (2.5), we get

\[
(1 - C_2(\Psi)\delta) D(\Upsilon) \leq D(\Psi) \leq (1 - C_2(\Psi)\delta) D(\Upsilon) + 2C_2(\Psi)\delta \log m/|\Upsilon|_{\text{max}}.
\]

By using (2.6) we get \( |D(\Psi) - D(\Upsilon)| \leq a(m, \Psi, \Upsilon)\delta \), where

\[
a(m, \Psi, \Upsilon) = 3C_2(\Psi)C_2(\Upsilon) \log m = 3\left(1 + \frac{1}{\|\Psi\|_{\text{max}}} \right) (1 + \frac{1}{|\Psi|_{\text{max}}}) \log m.
\]

Since \( \delta > \|\Psi - \Upsilon\|_{\text{lim}} \) is arbitrary, we get \( |D(\Psi) - D(\Upsilon)| \leq a(m, \Psi, \Upsilon)\|\Psi - \Upsilon\|_{\text{lim}} \).

(2) Now let \( \Phi, \Theta \in C_{aa}(\Sigma_A, T, d) \). Fix \( 0 < \epsilon < \|\Phi\|, \beta \in L_\Theta, \alpha \in B(\beta, \eta) \cap L_\Phi \), and \( \delta > \|\Psi - \Upsilon\|_{\text{lim}} \).
Let \( w \in F(\alpha, \eta, \Phi, \Psi) \). There exists \( x \in [w] \) such that \( |\phi_{\eta}(x) - \|w\|\alpha| \leq |w|\epsilon \). We have seen in proving (1) that \( w = uw' \) with \( u \in \mathcal{B}_{[h(1-C_2(\Psi)\delta)]}(\mathcal{Y}) \) and
\[
|w'| \leq (c_1(n) + 2nC_2(\Psi)\delta + C(\mathcal{Y}))/|\mathcal{Y}_{\max}|
\]
Notice that \( \text{diam}(L_\Phi) \leq \|\Phi\| \), thus \( |\alpha| \leq \|\Phi\| \). So we have
\[
|\phi_{\eta}(x) - |\alpha| \|w\|\alpha| \leq |\phi_{\eta}(x) - \phi_{\eta}(x)| + |\phi_{\eta}(x) - |w|\alpha| + |w'|\|\alpha| \leq 2|w'| \cdot \|\Phi\| + 2C(\Phi) + |w|\epsilon + |w'|\|\alpha| \leq 4|w'| \cdot \|\Phi\| + 2|C(\Phi)| + |w|\epsilon
\]
\[
= |u| \left( \epsilon + 4|w'| \cdot \|\Phi\| + 2|C(\Phi)| \right).
\]
Since \( 0 < c_1(n) = o(n) \), for large \( n \) we have
\[
4|w'| \cdot \|\Phi\| + 2|C(\Phi)| \leq \frac{9n\|\Phi\|C_2(\Psi)}{|\mathcal{Y}_{\max}|} \delta.
\]
Moreover if \( \delta < 1/(2C_2(\Psi)) \), then
\[
|u| \geq C_1(\mathcal{Y})n(1 - C_2(\Psi)\delta) \geq \frac{n}{2|\mathcal{Y}_{\min}|}.
\]
Thus we get
\[
4|w'| \cdot \|\Phi\| + 2|C(\Phi)| \leq \frac{18\|\Phi\|C_2(\Psi)|\mathcal{Y}_{\min}|}{|\mathcal{Y}_{\max}|} \delta =: \kappa(\Psi, \mathcal{Y}, \Phi)\delta = \kappa\delta.
\]
Fix any \( a > \|\Theta - \Phi\|_{\lim} \). For \( n \) large enough we have
\[
|\theta_{\eta}(x) - |\alpha| \|w\|\alpha| \leq |\theta_{\eta}(x) - \phi_{\eta}(x)| + |\phi_{\eta}(x) - |w|\alpha| + |w|\|\alpha - \beta| \leq a|w| + (\epsilon + \kappa\delta)|u| + \eta|u| = (a + \epsilon + \kappa\delta + \eta)|u|.
\]
As a result \( u \in F(\beta, [n(1 - C_2(\Psi)\delta)], \alpha + \epsilon + \kappa\delta + \eta, \Theta, \mathcal{Y}) \). Thanks to our control of \( |w'| \), we can get
\[
f(\beta, [n(1 - C_2(\Psi)\delta)], \alpha + \epsilon + \kappa\delta + \eta, \Theta, \mathcal{Y}) \geq f(\alpha, \eta, \Phi, \Psi)m^{-\frac{c_1(n)+2nC_2(\Psi)\delta+C(\mathcal{Y})}{|\mathcal{Y}_{\max}|}}.
\]
This yields
\[
A_{\Phi}^\Psi(\alpha, \epsilon) \leq \frac{2C_2(\Psi)\log m}{|\mathcal{Y}_{\max}|} \delta + (1 - C_2(\Psi)\delta)A_{\Theta}^\Psi(\beta, \alpha + \epsilon + \kappa\delta + \eta).
\]
Letting \( \epsilon \downarrow 0 \), then \( a \downarrow a_0 \), and \( \delta \downarrow \delta_0 \) we get
\[
A_{\Phi}^\Psi(\alpha) \leq \frac{2C_2(\Psi)\log m}{|\mathcal{Y}_{\max}|} \delta_0 + (1 - C_2(\Psi)\delta_0)A_{\Theta}^\Psi(\beta, (a_0 + \kappa\delta_0 + \eta) +) \leq \frac{2C_2(\Psi)\log m}{|\mathcal{Y}_{\max}|} \delta_0 + (1 - C_2(\Psi)\delta_0)A_{\Theta}^\Psi(\beta, a_0 + \kappa\delta_0 + 2\eta).
\]

7. Proof of Theorem 2.3

We prove the slightly more general result mentioned in Remark 3.12. Suppose that \( \xi \) is continuous outside a subset \( E \in \Sigma_A \), bounded and \( \xi(\Sigma_A) \in \mathfrak{aff}(L_\Phi) \). Also, suppose that \( \dim_H E < \lambda := \sup \{ D(\alpha) : \alpha \in \xi(\Sigma_A \backslash E) \cap \text{ri}(L_\Phi) \} \). To prepare the proof of our geometric results, we need the following more general result.
Proposition 7. Let \( Z \subset \Sigma_A \) be a closed set such that \( \mu(Z) = 0 \) for any Gibbs measure \( \mu \) fully supported on \( \Sigma_A \). For any \( \delta > 0 \) such that \( \lambda - \delta > \dim_H E \), we can construct a Moran subset \( \Theta \subset \Sigma_A \) such that \( \Theta \setminus E \subset E_\Phi(\xi) \), \( \dim_H \Theta \geq \lambda - \delta \) and there exists an increasing sequence of integers \( (\tilde{g}_j)_{j \geq 1} \) such that \( \Theta \setminus x \not\in Z \) for any \( x \in \Theta \) and any \( j \geq 1 \).

Proof. Fix \( \delta > 0 \) such that \( \lambda - \delta > \dim_H E \). Choose \( \alpha_0 \in \xi(\Sigma_A \setminus E) \cap \text{ri}(L_\Phi) \) such that \( \mathcal{D}(\alpha_0) > \lambda - \delta/2 \). Since \( \mathcal{D} \) is continuous in \( \text{ri}(L_\Phi) \), we can find \( \eta > 0 \) such that \( \tilde{B}_\eta := B(\alpha_0, \eta) \cap \text{aff}(L_\Phi) \subset \text{ri}(L_\Phi) \) and for any \( \alpha \in \tilde{B}_\eta \) we have \( |\mathcal{D}(\alpha) - \mathcal{D}(\alpha_0)| < \delta/2 \). Consequently

\[ \mathcal{D}(\alpha) > \lambda - \delta \quad \text{for all} \quad \alpha \in \tilde{B}_\eta. \]

Now we proceed in four steps. The two first steps provide the scheme of the construction of the set \( \Theta \) and a good measure \( \rho \), a piece of which is supported by \( \Theta \). The next two steps complete the construction to ensure that \( \Theta \) as the required properties and the dimension of \( \rho \) restricted to \( \Theta \) has a Hausdorff dimension larger than or equal to \( \lambda - \delta \).

Step 1: Concatenation of measures. Assume \( L_\Phi \) has dimension \( d_0 \leq d \) and \( \text{aff}(L_\Phi) = \alpha_0 + U(\mathbb{R}^{d_0} \times \{0\}^{d-d_0}) \), where \( U \) is a \( d \times d \) orthogonal matrix. Let \( j_0 \in \mathbb{N} \) such that \( 2^{-j_0} \sqrt{d_0} < \eta \) and define a sequence of sets as follows:

\[ \Delta_j := \tilde{B}_\eta \cap (\alpha_0 + 2^{-j-j_0}U(\mathbb{Z}^{d_0} \times \{0\}^{d-d_0})), \quad j \geq 0. \]

Then \( \Delta_0 \neq \emptyset, \Delta_j \subset \Delta_{j+1} \) for any \( j \geq 0 \) and each \( \Delta_j \) is a finite set. For each \( \alpha \in \bigcup_{j \geq 0} \Delta_j \), we can find a measure \( \mu_\alpha \) such that

\[ \Phi_s(\mu_\alpha) = \alpha \quad \text{and} \quad \mathcal{D}(\alpha) = \mathcal{E}(\alpha) = \frac{h_{\mu_\alpha}}{\gamma_\alpha}, \quad \text{where} \quad \gamma_\alpha = -\Psi_s(\mu_\alpha). \]

Let \( (\varepsilon_j)_{j \geq 1} \in (0,1)^\mathbb{N} \) such that \( \sum_j \varepsilon_j < \infty \). For each \( j \geq 1 \) define

\[ \overline{\Delta}_j = \{ (\alpha, j) : \alpha \in \Delta_j \}. \]

For each \( \sigma = (\alpha, j) \in \overline{\Delta}_j \), we can find a Markov (hence Gibbs) measure \( \mu_\sigma \) such that

\[ \max(|h_{\mu_\sigma} - h_{\mu_\alpha}|, |\beta_\sigma - \alpha|, |\gamma_\sigma - \gamma_\alpha|) < \varepsilon_j, \]

where \( \beta_\sigma = \Phi_s(\mu_\sigma) \) and \( \gamma_\sigma = -\Psi_s(\mu_\sigma) \).

Let \( (\phi^j)_{j \geq 1} \) and \( (\psi^j)_{j \geq 1} \) be two sequences of Hölder potentials defined on \( \Sigma_A \) such that

\[ \|\phi^j - \Phi\|_{\text{lim}} < \varepsilon_j \quad \text{and} \quad \|\psi^j - \Psi\|_{\text{lim}} < \varepsilon_j, \]

where \( \Phi^j = (S_n\phi^j)^{\infty}_{n=1} \) and \( \Psi^j = (S_n\psi^j)^{\infty}_{n=1} \).

For each \( \omega = (\sigma, s) \in \overline{\Delta}_j \times \{1, \cdots, m\} \), we denote by \( \mu_\omega \) the restriction of \( \mu_\sigma \) to \( [s] \) and \( \nu_\omega \) the probability measure \( \mu_\omega/\mu_\omega([s]) \).

Now fix any positive integer sequence \( \{L_j\}_{j \geq 1} \), we will build a concatenated measure \( \rho \) on \( \Sigma_A \) with support contained in a small cylinder \([\vartheta]\). At first we define \( \vartheta \in \Sigma_{A,\ast} \) and inductively a sequence of integers \( \{g_j : j \geq 0\} \) and a sequence of measures \( \{r_j : j \geq 0\} \) such that \( r_j \) is a measure on \( \{[\vartheta], \sigma\{[u] : \vartheta < u \in \Sigma_{A,g_j}\}\} \) for each \( j \geq 0 \), and the measures \( r_j \) are consistent: for each \( j \geq 0 \) the restriction of \( r_{j+1} \) to \( \sigma\{[u] : \vartheta < u \in \Sigma_{A,g_j}\} \) is equal to \( r_j \).

Fix \( \hat{x} \in \Sigma_A \setminus E \) such that \( \xi(\hat{x}) = \alpha_0 \) and write \( \hat{x} = \hat{x}_1 \hat{x}_2 \cdots \). Since \( \xi \) is continuous at \( \hat{x} \), we can choose \( g_0 \in \mathbb{N} \) such that \( \text{Osc}(\xi, [\hat{x}]_{g_0}) \leq 2^{-j_0} \), where \( \text{Osc}(\xi, V) \) stands for the
oscillation of $\xi$ over $V$. Write $\vartheta := \tilde{x}|_{[\vartheta]}$. Define the probability measure $\rho_0$ to be the trivial probability measure on $([\vartheta], \{\vartheta, [\vartheta]\})$. Suppose we have defined $(g_k, \rho_k)_{0 \leq k \leq j}$ for $j \geq 0$ as desired. To obtain $(g_{j+1}, \rho_{j+1})$ from $(g_j, \rho_j)$, define $g_{j+1} := g_j + L_{j+1}$. For every $w \in \Sigma_{A,g_j}$ with $\vartheta \prec w$, choose $x_w \in [w]$. Since $x_w \in [w] \subset [\vartheta]$ we have

$$|\xi(x_w) - \alpha_0| = |\xi(x_w) - \xi(\tilde{x})| \leq 2^{-j_0} \leq \eta.$$ 

Notice that by our assumption $\xi(\Sigma_A) \subset \text{aff}(L_\vartheta)$, thus $\xi(x_w) \in B_\eta$. Take $\alpha_w \in \Delta_{j+1}$ such that $|\xi(x_w) - \alpha_w| \leq 2^{-j-1-j_0}d_0$. Let $\omega = (\alpha_w, j + 1, t_u)$, where for each $u \in \Sigma_{A,s}$, $t_u$ stands for the last letter of $u$, and it is called the type of $u$. Then $\omega \in \Delta_{j+1} \times \{1, \ldots, m\}$. For each $v \in \Sigma_{A,L_{j+1}}$ such that $wv$ is admissible, define

$$\rho_{j+1}([wv]) := \rho_j([w])\nu_\omega([t_u]),$$

By construction the family $\{\rho_j : j \geq 0\}$ is consistent. Denote by $\rho$ the Kolmogorov extension of the sequence $(\rho_j)_{j \geq 0}$ to $([\vartheta], \sigma\{[u] : \vartheta \prec u \in \Sigma_{A,s}\})$. This finishes the construction of the desired measure. Note that $g_j = g_0 + L_1 + \cdots + L_j$ for any $j \geq 1$. Also, by construction, we have the following formula for the $\rho$-mass of any cylinder of generation larger than $g_0$. If $\vartheta \prec u$ and $u \in \Sigma_{A,s}$ with $g_j \leq n < g_{j+1}$, writing $u = \vartheta w^1 \cdots w^j \cdot v$ with $|w^k| = L_k$ and $|v| = n - g_j$, and denoting $\vartheta w^1 \cdots w^k$ by $\tilde{w}_k$, then

$$\rho([u]) = \left( \prod_{k=1}^j \nu_{\omega_k}([t_{w_{k-1}} w^k]) \right) \nu_{\omega_{j+1}}([t_u]),$$

where $\omega_k = (\alpha_{\tilde{w}_{k-1}}, k, t_{w_{k-1}})$ for $k = 1, \ldots, j + 1$.

### Step 2: Construction of the Moran set $\Theta$.

Next we want to specify the integer sequence $\{L_j\}_{j \geq 1}$ and pick out carefully a Moran set $\Theta \subset [\vartheta]$ such that $\rho(\Theta) > 0$ and $\Theta$ has the last property stated in the proposition. We proceed as follows.

Fix $\omega = (\alpha, j, s) \in \Delta_j \times \{1, \ldots, m\}$. For $N \geq 1$ Define

$$E_N(\omega) := \bigcap_{n \geq N} \left\{ x \in [s] : \left| S_n \vartheta^j(Tx) - \alpha \right| - \log \nu_\omega([x]) \right\},$$

Notice that each $\Delta_j$ is a finite set, thus by the ergodicity of each $\mu_{(\alpha,j)}$, (7.3) and (7.4), we can fix an integer $N_j$ such that

$$\nu_\omega(E_N(\omega)) \geq 1 - \varepsilon_j/2, \quad (\forall N \geq N_j, \forall \omega \in \Delta_j \times \{1, \ldots, m\}).$$

Define $V_N := \{ v \in \Sigma_{A,N+1} : [v] \cap Z = \emptyset \}$. By the restriction about $Z$, there exists an integer $\tilde{N}_j$ such that

$$\nu_\omega\left( \bigcup_{v \in V_N} [v] \right) \geq 1 - \varepsilon_j/2, \quad (\forall N \geq \tilde{N}_j, \forall \omega \in \Delta_j \times \{1, \ldots, m\}).$$

Define $V_N(\omega) = \{ v \in V_N, [v] \cap E_{N_j}(\omega) \neq \emptyset \}$. Thus, if $N \geq \max(N_j, \tilde{N}_j)$, by (7.6) and (7.7) we have

$$\nu_\omega\left( \bigcup_{v \in V_N(\omega)} [v] \right) \geq 1 - \varepsilon_j, \quad (\forall \omega \in \Delta_j \times \{1, \ldots, m\}).$$

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Take an integer sequence \( \{L_j\}_{n \geq 1} \) such that \( L_j \geq \max(N_j, \widehat{N}_j) \) for \( j \geq 1 \) and consider the associated measure \( \rho \) constructed in step 1. We define the desired Moran set as
\[
\Theta := \{ x \in [\vartheta] : \forall j \geq 0, T^{g_j-1}x |_{L_{j+1}+1} \in V_{L_{j+1}}(\alpha x_{|g_j}, j + 1, x_{g_j}) \}.
\]
By construction, \( T^{g_j-1}x \notin \mathbb{Z} \) for any \( x \in \Theta \) and any \( j \geq 0 \). Define \( \tilde{g}_j = g_j - 1 \), we checked the last property of \( \Theta \).

Write \( \omega_k := (\alpha x_{|g_j-1}, k, x_{g_j-1}) \in \Delta_k \times \{1, \cdots, m\} \). For each \( j \geq 1 \), by using (7.5) and (7.8) we get
\[
\rho(\{ x \in [\vartheta] : [x|g_j] \cap \Theta \neq \emptyset \}) = \sum_{\vartheta w^1 \cdots w^j \text{ admissible}, \forall 1 \leq k \leq j, x_{g_j-1} w^k \in V_{L_k}(\omega_k)} \rho_j(\vartheta w^1 \cdots w^j) = \sum_{\vartheta w^1 \cdots w^j-1 \text{ admissible}, \forall 1 \leq k \leq j-1, x_{g_j-1} w^k \in V_{L_k}(\omega_k)} \rho_{j-1}(\vartheta w^1 \cdots w^{j-1}) \sum_{x_{g_j-1} w^j \in V_{L_j}(\omega_j)} \nu_{\omega_j}([x_{g_j-1} w^j]) \geq \sum_{\vartheta w^1 \cdots w^{j-1} \text{ admissible}, \forall 1 \leq k \leq j-1, x_{g_j-1} w^k \in V_{L_k}(\omega_k)} \rho_{j-1}(\vartheta w^1 \cdots w^{j-1})(1 - \varepsilon_j) \geq \prod_{k=1}^j (1 - \varepsilon_k).
\]

Since we assumed that \( \varepsilon_j < 1 \) and \( \sum_{j=1}^\infty \varepsilon_j < \infty \) we have
\[
\rho(\Theta) = \lim_{n \to \infty} \rho(\{ x \in [\vartheta] : [x|g_j] \cap \Theta \neq \emptyset \}) \geq \prod_{j=1}^\infty (1 - \varepsilon_j) > 0.
\]

**Step 3:** Conditions on \( (L_j)_{j \geq 1} \) ensuring that \( \Theta \setminus E \subset E \Phi(\xi) \).

For \( \eta \in \{\phi, \psi\} \) and \( j \geq 1 \), let
\[
\text{Var}(\eta^j) = \sup_{n \geq 1} \max_{\nu \in \Sigma_{\Lambda, n}} \max_{x,y \in [\nu]} |S_n \eta^j(x) - S_n \eta^j(y)|.
\]
This number is finite since each \( \eta^j \) is Hölder continuous. By (7.4) we can find integer sequence \( M_j \nearrow \infty \) such that
\[
(7.9) \max(\|S_n \phi^j - \phi_n\|_\infty, \|S_n \psi^j - \psi_n\|_\infty) \leq 2\varepsilon_j n \quad (\forall n \geq M_j).
\]
The sequence \( (L_j)_{j \geq 1} \) can be specified to satisfy the additional properties
\[
L_j \geq M_{j+1} \quad \text{and} \quad \max(K_1(j), K_2(j), K_3(j)) \leq \varepsilon_j g_j,
\]
(recall that \( g_j = g_0 + \sum_{k=1}^j L_k \), where
\[
\begin{align*}
K_1(j) &= \sum_{k=1}^{j+1} (\text{Var}(\phi^k) + \text{Var}(\psi^k)); \\
K_2(j) &= \max_{\alpha \in \Delta_{j+1}} \max_{1 \leq n \leq m} \left( n|\alpha|, \|S_n \phi^{j+1}\|_\infty, \|\log \nu_{(\alpha,j+1,n)}([1,n])\|_\infty, \right) \\
K_3(j) &= \max_{1 \leq n \leq \widehat{N}_{j+1}} \left( \|S_n \psi^{j+1}\|_\infty \right).
\end{align*}
\]

Let us check that \( \Theta \setminus E \subset E \Phi(\xi) \). Let \( x \in \Theta \setminus E \), \( n \geq g_1 \) and \( j \geq 1 \) such that \( g_j \leq n < g_{j+1} \). Since \( n \geq g_j > L_j \geq M_{j+1} \), by (7.4) we have
\[
|\phi_n(x) - n\xi(x)| \leq \|S_n \phi^{j+1} - \phi_n\|_\infty + |S_n \phi^{j+1}(x) - n\xi(x)|.
\]
there exists $\omega \ni x$.

Thus both terms are $(7.11)$

Since $L_0 = 0$ and $L_0 = g_0$

\[
\begin{align*}
|S_{n}\phi^{j+1}(x) - n\xi(x)| & \\
\leq |S_{g_j}\phi^{j+1}(x) - g_j\xi(x)| + |S_{n-g_j}\phi^{j+1}(T^{g_j}x) - (n - g_j)\xi(x)| \\
= |S_{g_j}\phi^{j+1}(x) - \sum_{k=0}^{j} L_k\alpha_k + \sum_{k=0}^{j} L_k\alpha_k - g_j\xi(x)| \\
+ |S_{n-g_j}\phi^{j+1}(T^{g_j}x) - (n - g_j)\xi(x)| \\
\leq \sum_{k=0}^{j} |S_{L_k}\phi^{j+1}(T^{g_k-1}x) - L_k\alpha_k| + \sum_{k=0}^{j} L_k|\alpha_k - \xi(x)| \quad (=: (I) + (II)) \\
+ |S_{n-g_j}\phi^{j+1}(T^{g_j}x) - (n - g_j)\alpha_{j+1}| \quad (=: (III)) \\
+ (n - g_j)|\alpha_{j+1} - \xi(x)| \quad (=: (IV)).
\end{align*}
\]

At first we have $(I) + (III)$

\[
\begin{align*}
\leq \sum_{k=0}^{j} \|S_{L_k}\phi^{j+1} - \phi_{L_k}\|_{\infty} + \sum_{k=0}^{j} \|S_{L_k}\phi^{k} - \phi_{L_k}\|_{\infty} \\
+ \left(\sum_{k=0}^{j} |S_{L_k}\phi^{k}(T^{g_k-1}x) - L_k\alpha_k|\right) + |S_{n-g_j}\phi^{j+1}(T^{g_j}x) - (n - g_j)\alpha_{j+1}|.
\end{align*}
\]

If $L_k \leq M_{j+1}$, then $\|S_{L_k}\phi^{j+1} - \phi_{L_k}\|_{\infty} \leq K_3(j)/(j + 1)$;

if $L_k > M_{j+1}$, then $\|S_{L_k}\phi^{j+1} - \phi_{L_k}\|_{\infty} \leq 2\varepsilon_k L_k$.

Thus we have

\[
(7.10) \quad \sum_{k=0}^{j} \|S_{L_k}\phi^{j+1} - \phi_{L_k}\|_{\infty} \leq K_3(j) + 2 \sum_{k=0}^{j} \varepsilon_k L_k \leq \varepsilon_j g_j + 2 \sum_{k=0}^{j} \varepsilon_k L_k.
\]

Since $L_k \geq M_{k+1} \geq M_k$ we also have

\[
(7.11) \quad \sum_{k=0}^{j} \|S_{L_k}\phi^{k} - \phi_{L_k}\|_{\infty} \leq 2 \sum_{k=0}^{j} \varepsilon_k L_k.
\]

Thus both terms are $o(g_j)$ as $n \to \infty$. Consequently both terms are $o(n)$.

Write $\omega_k = (\alpha_k, k, g_{k-1})$ for $k = 0, \cdots, j$. By the construction of $\Theta$, we have

\[
T^{g_{k-1}-1}x|_{L_{k+1}} = x_{g_{k-1}} \cdot (T^{g_{k-1}-1}x|_{L_k}) \in V_{L_k}(\omega_k),
\]

so $[x_{g_{k-1}} \cdot (T^{g_{k-1}-1}x|_{L_k})] \cap E_{N_k}(\omega_k) \neq \emptyset$. Using the definition of $E_{N_k}(\omega_k)$, since $L_k \geq N_k$, there exists $y \in [T^{g_{k-1}-1}x|_{L_k}]$ such that $|S_{L_k}\phi^{k}(y) - L_k\alpha_k| \leq 2\varepsilon_k L_k$, hence $|S_{L_k}\phi^{k}(T^{g_{k-1}-1}x) - L_k\alpha_k| \leq 2\varepsilon_k L_k + \text{Var}(\phi^k)$.
Similarly if \( n - g(j) \geq N_{j+1} \),
\[
|S_{n-g_j} \phi^{j+1}(T^{g_j}x) - (n - g_j)\alpha_{j+1}| \leq 2\varepsilon_{j+1}(n - g_j) + \text{Var}(\phi^{j+1}).
\]
If \( n - g(j) \leq N_{j+1} \) we trivially have
\[
|S_{n-g_j} \phi^{j+1}(T^{g_j}x) - (n - g_j)\alpha_{j+1}| \leq 2K_2(j).
\]
This yields
\[
\left( \sum_{k=0}^{j} |S_{L_k} \phi^k(T^{g_{k-1}}x) - L_k\alpha_k \right) + |S_{n-g_j} \phi^{j+1}(T^{g_j}x) - (n - g_j)\alpha_{j+1}|
\]
\[
\leq 2 \sum_{k=0}^{j} \varepsilon_k L_k + \sum_{k=0}^{j+1} \text{Var}(\phi^k) + 2\varepsilon_{j+1}(n - g_j) + 2K_2(j)
\]
(7.12) \[
\leq 2 \sum_{k=0}^{j} \varepsilon_k L_k + K_1(j) + 2\varepsilon_{j+1}(n - g_j) + 2K_2(j) = o(g_j) + o(n) = o(n).
\]
Combining (7.10), (7.11) and (7.12) we get (I) + (III) = \( o(n) \).

On the other hand, by construction (recall that \( \alpha_{k+1} = \alpha_{x|g_k} \))
\[
|\xi(x) - \alpha_{k+1}| \leq |\xi(x) - \xi(x_{x|g_k})| + |\xi(x_{x|g_k}) - \alpha_{x|g_k}| \leq \text{Osc}(\xi, [x_{g_k}]) + 2^{-k-j_0} \sqrt{d_0},
\]
where \( x_{x|g_k} \) is the special point in \( [x_{g_k}] \) chosen in the construction of the measure \( \rho \). Since \( x \notin E \), \( \xi \) is continuous at \( x \). We have
\[
\lim_{k \to \infty} \text{Osc}(\xi, [x_{g_k}]) + 2^{-k-j_0} \sqrt{d_0} = 0.
\]
Thus \( \lim_{k \to \infty} \alpha_k = \xi(x) \) and we conclude that (II) + (IV) = \( o(n) \). As a result \( |\phi_n(x) - n\xi(x)| = o(n) \), thus \( x \in E_\Phi(\xi) \). This finishes the proof of \( \Theta \cap E \subset E_\Phi(\xi) \).

**Step 4:** \( \dim_H \Theta \geq \lambda - \delta \).

Let us compute the local lower dimension \( d_\rho(x) \) for any \( x \in \Theta \). By using similar estimates as above, for any \( x \in \Theta \) we can prove that (with \( \sigma_k = (\alpha_{x|g_k}, k) \))
\[
| - \psi_n(x) - \sum_{k=0}^{j} L_k \gamma_{\sigma_k} - (n - g_j)\gamma_{\sigma_{j+1}}| = o(n),
\]
\[
| - \log \rho([x|n]) - \sum_{k=0}^{j} L_k h_{\mu_{\sigma_k}} - (n - g_j)h_{\mu_{\sigma_{j+1}}}| = o(n).
\]
By (7.1), (7.2) and (7.3) we have \( \lim_{j \to \infty} h_{\mu_{\sigma_j}}/\gamma_{\sigma_j} \geq \lambda - \delta \). For any \( y \in [x|n] \) we have \( |\psi_n(y) - \psi_n(x)| = o(n) \), thus we get \( \text{diam}([x|n]) = \Psi[x|n] = \exp(\psi_n(x) + o(n)) \). Combining the above two relations we conclude that
\[
d_\rho(x) = \lim_{n \to \infty} \frac{\log \rho([x|n])}{\log(\text{diam}([x|n]))} \geq \lambda - \delta.
\]
Since \( \rho(\Theta) > 0 \), by the mass distribution principle we get \( \dim_H \Theta \geq \lambda - \delta \) (see [29]). \( \square \)

A slight modification of the above proof with \( \xi \) taken constant yields the following proposition:
Proposition 8. Assume $Z \subset \Sigma_A$ is a closed set such that $\mu(Z) = 0$ for any Gibbs measure $\mu$ fully supported on $\Sigma_A$. For any $\alpha \in L_\Phi$, we can construct a subset $\Theta \subset E_\Phi(\alpha)$ such that $\dim_H \Theta \geq E(\alpha)$ and there exists an integer sequence $g_j \nearrow \infty$ such that $T^{g_j}x \not\in Z$ for any $x \in \Theta$ and any $j \geq 1$. In particular, $E(\alpha) \leq D(\alpha)$.

Proof of Theorem 2.3 (1′) Since $\dim_H E < \lambda - \delta$, by Proposition 7 we have $\dim_H (\Theta \setminus E) = \dim_H \Theta \geq \lambda - \delta$. Consequently $\dim_H E_\Phi(\xi) \geq \dim_H (\Theta \setminus E) \geq \lambda - \delta$. Since $\delta > 0$ is arbitrary, we get $\dim_H E_\Phi(\xi) \geq \lambda$.

Now we assume $\xi$ is continuous everywhere.

(2) If $\xi(\Sigma_A) \subset L_\Phi$, the construction of a Moran subset of $E_\Phi(\xi)$ can be done around any point of $\Sigma_A$, like in the proof of Proposition 7. The only difference is that in this case the dimension of this set is of no importance. Hence, $E_\Phi(\xi)$ is dense.

(3) If
$$
sup\{D(\alpha) : \alpha \in \xi(\Sigma_A) \cap ri(L_\Phi)\} = sup\{D(\alpha) : \alpha \in \xi(\Sigma_A) \cap L_\Phi\} =: \theta,$$
then at first we have $\dim_H E_\Phi(\xi) \geq \theta$. On the other hand by definition we have $E_\Phi(\xi) \subset E_\Phi(\xi(\Sigma_A) \cap L_\Phi)$. Thus by Proposition 6 we have $\dim_P E_\Phi(\xi) \leq \theta$. So we get
$$\dim_H E_\Phi(\xi) = \dim_P E_\Phi(\xi) = \theta.$$

(4) Assume $d = 1$ and $\xi(\Sigma_A) \subset L_\Phi$. Notice that in this case $L_\Phi = [\alpha_1, \alpha_2]$ is an interval. Thus by Proposition 4 and Theorem 2.2 $D$ is continuous on $L_\Phi$. Assume $\alpha_0 \in \xi(\Sigma_A)$ such that $D(\alpha_0) = sup\{D(\alpha) : \alpha \in \xi(\Sigma_A)\}$. If $\alpha_0 \in (\alpha_1, \alpha_2)$, by (3) we conclude. Now assume $\alpha_0 = \alpha_1$. If $\alpha_1$ is not isolated in $\xi(\Sigma_A)$, still by (3) and the continuity of $D$, we get the result. If $\alpha_1$ is isolated in $\xi(\Sigma_A)$, then by the continuity of $\xi$, we can find a cylinder $[w] \subset \Sigma_A$ such that $\xi([w]) = \alpha_1$. From this we get $E_\Phi(\xi) \supset E_\Phi(\alpha_1) \cap [w]$. Thus $\dim_H E_\Phi(\xi) \geq D(\alpha_1)$ and the result holds. If $\alpha_0 = \alpha_2$, the result is the same. \qed

8. Proofs of results in section 3

We will use the following lemma, which is standard and essentially the same as Lemma 5.1 in [21] (the proof is elementary).

Lemma 8.1. Let $X$ and $Y$ be metric spaces and $\chi : X \rightarrow Y$ a surjective mapping with the following property: there exists a function $N : (0, \infty) \rightarrow N$ with $\log N(r) / \log r \rightarrow 0$ when $r \rightarrow 0$ such that for any $r > 0$, the pre-image $\chi^{-1}(B)$ of any $r$-ball in $Y$ can be covered by at most $N(r)$ sets in $X$ of diameter less than $r$. Then for any set $E \subset Y$ we have $\dim_H E \geq \dim_H \chi^{-1}(E)$.

Proof of Proposition 5. Condition (4) implies that $\chi : (\Sigma_A, d_\Psi) \rightarrow (J, d)$ is Lipschitz continuous, thus we have $\dim_H E \leq \dim_H \chi^{-1}(E)$.

For the converse inequality, let us check the condition of the above lemma. Let $B \subset J$ be a ball of radius $r$, let $n \in N$ such that $e^{-n} \leq r < e^{1-n}$. Define
$$G^r_B = \{w \in B_n(\Psi) : R_w \cap B \neq \emptyset\}.$$
One checks that $\{|w| : w \in G^r_B\}$ is an $r$-covering of $\chi^{-1}(B)$. Define $N(r) := \# G^r_B$. Let us estimate the number $\# G^r_B$. Clearly, $\# G^r_B \geq 1$. By condition (4), for each $w \in G^r_B$, $R_w$ is
constructed, the interiors of the sets

\[ R_w \subset B(y, r + 2Ke^{-n}) \subset B(y, (e + 2K)e^{-n}), \]

where \( y \) is the center of \( B \). On the other hand, by Lemma 4.2 (1) there exists \( C > 0 \) such that \( |w| \leq Cn \) for any \( w \in B_n(\Psi) \), thus \( \eta(w) = o(|w|) = o(n) \) for any \( w \in B_n(\Psi) \). By construction, the interiors of the sets \( R_w \), \( w \in G_B^r \), are disjoint and each \( R_w \) contains a ball of radius \( K^{-1} \exp(\eta(w)) = K^{-1}e^{o(n)} \)

by Lemma 4.2 (2). Thus \( #G^r_B \leq K^d(e + 2K)^de^{o(n)} \). So we conclude that \( \log N(r)/\log r = \log #G^r_B/\log r \to 0 \) as \( r \to 0 \). Thus by lemma 8.1 we can conclude that \( \dim_H E \geq \dim_H \chi^{-1}(E) \).

**Proof of Lemma 3.1.** At first we show that \( J \cap V = J \setminus \bar{Z}_\infty \), consequently by the SOSC, \( J \setminus \bar{Z}_\infty \neq \emptyset \) and we get \( \emptyset \neq \chi^{-1}(J \setminus \bar{Z}_\infty) = \Sigma_A \setminus Z_\infty \). In fact

\[
y \in J \setminus \bar{Z}_\infty \iff y \in J \text{ and } \forall n \geq 1 \exists x \in \Sigma_A \text{ s.t. } y \in \text{int}(R_x|_n) = f_x|_n(V) \iff y \in J \text{ and } \forall n \geq 1 \exists x \in \Sigma_A \text{ s.t. } y \in \text{int}(R_x|_n) = f_x|_n(V) \iff y \in J \cap V.
\]

By construction, \( \chi : \Sigma_A \setminus Z_\infty \to J \setminus \bar{Z}_\infty \) is surjective. Since \( J \setminus \bar{Z}_\infty = J \cap V \), it is ready to show that \( \chi \) is also injective.

Next we show that \( T(\Sigma_A \setminus Z_\infty) \subset \Sigma_A \setminus Z_\infty \). Take \( x \in \Sigma_A \setminus Z_\infty \). If \( Tx \in Z_\infty \), then we can find \( n_0 \in \mathbb{N} \) such that \( \chi(Tx) = f_x|_{n_0}(\partial V) \). Consequently \( \chi(x) = f_x(\chi(Tx)) \in f_x_1(f_Tx|_{n_0}(\partial V)) = f_x_1(f_Tx|_{n_0}(\partial V)) = f_x|_{n_0+1}(\partial V) \), which is a contradiction. Next we show that for any Gibbs measure \( \mu \) we have \( \mu(Z_\infty) = 0 \). Define \( \bar{Z}_\infty := \bigcup_{w \in \Sigma_A} \{|w| \leq n \} \) and \( Z_n = \chi^{-1}(\bar{Z}_n) \). The sequence \((Z_n)_{n \geq 1}\) is non decreasing and \( Z_\infty = \bigcup_{n \geq 1} Z_n \). Since the IFS is conformal we can easily get \( T(Z_n) \subset Z_{n-1} \) for \( n \geq 1 \) and \( T(Z_0) \subset Z_0 \). Consequently \( T(Z_n) \subset Z_n \). By the ergodicity we have \( \mu(Z_n) = 0 \) or 1. By the SOSC, \( \Sigma_A \setminus Z_n \) is nonempty and open, thus by the Gibbs property of \( \mu \) we get \( \mu(\Sigma_A \setminus Z_n) > 0 \), hence \( \mu(Z_\infty) = 0 \). Consequently \( \mu(Z_\infty) = 0 \). At last, from \( T(Z_n) \subset Z_n \) we easily get \( T(Z_\infty) \subset Z_\infty \).

**Proof of Theorem 3.2.** (1) At first we notice that by the property (4) assumed in the construction of \( J \) the mapping \( \chi \) is Lipschitz. This is enough to get the desired upper bounds from Theorem 2.2 (1).

Now we deal with the lower bound for dimensions and the equality \( L_\phi = L_\tilde{\phi} \). We notice that the inclusion \( L_\phi \subset L_\tilde{\phi} \) holds by construction.

**Suppose \( J \) is a conformal repellor.** Since we have \( \chi \circ T = g \circ \chi \) on \( \Sigma_A \) and \( \chi \) is surjective, it is seen that \( \chi^{-1}(E_\phi(\alpha)) = E_\tilde{\phi}(\alpha) \) for any \( \alpha \in L_\tilde{\phi} \). Thus \( L_\phi = L_\tilde{\phi} \) and by Proposition 5 we have \( \dim_H E_\phi(\alpha) = \dim_H E_\tilde{\phi}(\alpha) \).

**Suppose \( J \) is the attractor of a conformal IFS with SOSC.** Let \( \alpha \in L_\tilde{\phi} \). Let \( Z = \chi^{-1}(\partial V) \). The set \( Z \) is closed and by Lemma 3.1 \( \mu(Z) = 0 \) for any Gibbs measure \( \mu \). By Proposition 8 we can construct a Moran set \( \Theta \subset E_\tilde{\phi}(\alpha) \) such that \( \dim_H(\Theta) \geq E_\tilde{\phi}(\alpha) = \dim_H E_\tilde{\phi}(\alpha) \) and there exists a sequence \( g_j \uparrow \infty \) such that \( T^j x \notin Z \) for any \( x \in \Theta \) and any \( j \geq 1 \). The last property means that \( \Theta \subset \Sigma_A \setminus Z_\infty \). Since \( \chi \) is a bijection between \( \Sigma_A \setminus Z_\infty \) and \( J \setminus \bar{Z}_\infty \), we conclude that \( \chi^{-1} \circ \chi(\Theta) = \Theta \), thus by Proposition 5 \( \dim_H \chi(\Theta) = \dim_H \Theta \geq \).
dim_H E_\tilde{\Phi}(\alpha). Since we also have \chi \circ T = \tilde{g} \circ \chi on \Sigma_A \setminus Z_\infty, we get that \chi(\Theta) \subset E_\tilde{\Phi}(\alpha). Thus \alpha \in L_{\Phi} and dim_H E_\Phi(\alpha) \geq dim_H E_\tilde{\Phi}(\alpha).

(2) Take E = J in Proposition 3 then use (1) and Theorem 2.2.2.1.

Proof of Theorem 3.3. Define \tilde{\xi} := \xi \circ \chi.

Case 1: J is a conformal repeller. One checks easily that \chi^{-1}(E_\Phi(\xi)) = E_{\tilde{\Phi}}(\tilde{\xi}). Then the result is a consequence of Theorem 3.2, Theorem 2.3 and Proposition 5.

Case 2: J is the attractor of a conformal IFS with SOSC. We conclude by using Proposition 7, Theorem 2.3, Proposition 5 and the same argument as in the proof of Theorem 3.2. \square

Proof of Theorem 3.4. Let \xi = \Phi, then F(\tilde{J}, \tilde{g}) = E_\Phi(\xi). To show the result we need only to check the condition of Theorem 3.3 and the only condition we need to check is that

\[(8.1) \sup \{D_\Phi(\alpha) : \alpha \in \xi(J) \cap \text{ri}(L_{\Phi})\} = \sup \{D_{\tilde{\Phi}}(\alpha) : \alpha \in \xi(J) \cap L_{\Phi}\}.\]

Notice that in this special case we have \xi(J) = J and L_{\Phi} = \text{Co}(J), thus \xi(J) \cap L_{\Phi} = J. Recall that in this case L_{\Phi} is a convex polyhedron, thus by Proposition 4 and Theorem 3.2 D_\Phi is continuous on L_{\Phi}. Thus the supremum in the right hand side of (\ast) can be reached. If the maximum is attained in ri(L_{\Phi}), then the result is obvious. Now suppose that there exists \alpha_0 \in (L_{\Phi} \setminus \text{ri}(L_{\Phi})) \cap J such that D_{\Phi}(\alpha_0) = \sup \{D_{\Phi}(\alpha) : \alpha \in J\}. By the structure of J, it is ready to see that B(\alpha_0, \delta) \cap J \cap \text{ri}(L_{\Phi}) \neq \emptyset for any \delta > 0. By the continuity of D_\Phi, (8.1) holds immediately. \square

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