SPHERICAL T-DUALITY AND
THE SPHERICAL FOURIER-MUKAI TRANSFORM

PETER BOUWKNEGT, JARAH EVSLIN, AND VARGHESE MATHAI

Abstract. In [3], we introduced spherical T-duality, which relates pairs of the form $(P, H)$ consisting of a principal $SU(2)$-bundle $P \to M$ and a 7-cocycle $H$ on $P$. Intuitively, spherical T-duality exchanges $H$ with the second Chern class $c_2(P)$. Unless $\dim(M) \leq 4$, not all pairs admit spherical T-duals and the spherical T-duals are not always unique. In this paper, we define a canonical spherical Poincaré vector bundle $\mathcal{P}$ on $SU(2) \times SU(2)$ and the spherical Fourier-Mukai transform, which implements a degree shifting isomorphism in K-theory on the trivial $SU(2)$-bundle with trivial 7-flux, and then (partially) generalise it to prove that all spherical T-dualities induce a natural degree-shifting isomorphism on the 7-twisted K-theories of the principal $SU(2)$-bundles when $\dim(M) \leq 4$.

1. Introduction

Recall that the renowned Poincaré line bundle $\mathcal{P} \to S^1 \times S^1$ is tautologically defined and comes with a canonical connection whose curvature is the standard symplectic 2-form on $S^1 \times S^1$. More generally, it is defined in the holomorphic context on a polarised abelian variety in Mumford [12], chapters 10-13, where it was used to study fine moduli problems. It was then used by Mukai [11] to give an equivalence of derived categories of coherent sheaves on an abelian variety with its dual abelian variety. In the smooth context, Hori [9] used the Poincaré line bundle to give a (shifted) equivalence of K-theories, and thereby establishing the equivalence of charges in type IIA and type IIB string theories in the absence of background fluxes. In [1, 2] (see also [5]) a deep extension was made for principal torus bundles with nontrivial fluxes, where an equivalence of twisted K-theories was derived but importantly that there was a change in spacetime topology in general for the first time.

In [3, 4], we introduced a new kind of duality for string theory, termed spherical T-duality, for spacetimes that are compactified as principal $SU(2)$-bundles with 7-flux. There we argued that the 7-twisted cohomology and the 7-twisted K-theory which featured in our main theorems classify certain conserved charges in type IIB supergravity. We concluded that spherical T-duality provides a one to one map between conserved charges in certain topologically distinct compactifications and also a novel electromagnetic duality on the fluxes.

2010 Mathematics Subject Classification. Primary 81T30.
Key words and phrases. Spherical T-duality; principal $SU(2)$-bundles; spherical Poincaré vector bundle, spherical Fourier-Mukai transform; twisted K-theory.

JE is supported by NSFC MianShang grant 11375201. PB and VM thank the Australian Research Council for support via ARC Discovery Project grants DP150100008 and DP130103924.
In this paper (section 3) we define a spherical Poincaré vector bundle
\[ \mathcal{P} \rightarrow SU(2) \times SU(2) \]
with connection, for the first time, making our discussion of spherical T-duality almost on par with the torus case. The spherical Poincaré vector bundle represents the diagonal class in K-theory and implements a canonical equivalence of K-theories in the case of trivial \( SU(2) \)-bundles as shown in section 2.

In section 4 we show that the spherical Poincaré vector bundle gives rise to isomorphisms of 7-twisted K-theories for 7-dimensional principal \( SU(2) \)-bundles with 7-fluxes. We also compute the spherical T-duality group.

Contents

1. Introduction
2. Poincaré element and spherical Fourier-Mukai transform in K-theory
3. Spherical Poincaré vector bundle with connection
   3.1. Vector bundle realization of the Poincaré element
   3.2. Smashing spheres
   3.3. The Poincaré bundle on the torus
   3.4. The Poincaré bundle on a product of 3-spheres
4. Spherical T-duality
   4.1. Spherical T-admissibility
   4.2. Spherical T-duality isomorphisms
   4.3. The spherical T-duality group
5. References

2. POINCARÉ ELEMENT AND SPHERICAL FOURIER-MUKAI TRANSFORM IN K-THEORY

The Poincaré element \([\mathcal{P}]\) over \( SU(2) \times \widehat{SU}(2) \), where \( \widehat{SU}(2) = SU(2) \), is the diagonal class in
\[ K^0(SU(2) \times \widehat{SU}(2)) \cong K^0(SU(2)) \otimes K^0(\widehat{SU}(2)) \oplus K^1(SU(2)) \otimes K^1(\widehat{SU}(2)) , \]
that is \([\mathcal{P}] = 1 \otimes \hat{1} + \zeta \otimes \hat{\zeta} \), where \( \zeta \in K^1(SU(2)) \) and \( \hat{\zeta} \in K^1(\widehat{SU}(2)) \) are the generators, represented by degree 1 maps \( SU(2) \hookrightarrow U(N) \), \( N \gg 0 \). Later on, we will describe a canonical vector bundle representative of \([\mathcal{P}]\).

Consider the trivial \( SU(2) \)-bundle \( P = M \times SU(2) \). Consider the commutative diagram
Theorem 2.1. For $E$ a vector bundle over $P$, define the spherical Fourier-Mukai transform as

$$\mathcal{F}[E] = p_*(\hat{p}^*[E] \otimes [P]),$$

giving rise to the spherical Fourier-Mukai transform in K-theory

$$\mathcal{F} : K^i(P) \xrightarrow{\cong} K^{i+1}(\hat{P}).$$

Proof. By the Künneth theorem,

$$K^0(P) \cong K^0(M) \oplus K^1(M) \cong K^1(P),$$

and similarly

$$K^0(\hat{P}) \cong K^0(M) \oplus K^1(M) \cong K^1(\hat{P}).$$

Now if $x \in K^0(P)$, then $x = x_0 \otimes 1 + x_1 \otimes \zeta$ where $x_j \in K^j(M)$, $j = 0, 1$. Then an easy computation shows that

$$\mathcal{F}(x) = \mathcal{F}(x_0 \otimes 1 + x_1 \otimes \zeta) = x_0 \otimes \hat{\zeta} + x_1 \otimes 1.$$ 

It follows that

$$\mathcal{F} : K^0(P) \xrightarrow{\cong} K^1(\hat{P})$$

is an isomorphism.

Similarly, if $x \in K^1(P)$, then $x = x_0 \otimes \zeta + x_1 \otimes 1$ where $x_j \in K^j(M)$, $j = 0, 1$. Then an easy computation shows that

$$\mathcal{F}(x) = \mathcal{F}(x_0 \otimes \zeta + x_1 \otimes 1) = x_0 \otimes 1 + x_1 \otimes \hat{\zeta}.$$ 

It follows that

$$\mathcal{F} : K^1(P) \xrightarrow{\cong} K^0(\hat{P})$$

is also an isomorphism.

Define a commutative, associative products on $K^*(SU(2))$ given by

$$1 \otimes 1 = 1, \quad 1 \star 1 = 0,$$

$$1 \otimes \zeta = \zeta, \quad 1 \star \zeta = 1,$$

$$\zeta \otimes \zeta = 0, \quad \zeta \star \zeta = \zeta,$$
called the tensor product and convolution, respectively. This in turn defines commutative, associative products on $K^\bullet(P)$ and $K^\bullet(\tilde{P})$, both equal to $K^\bullet(M) \otimes K^\bullet(SU(2))$ and one has

**Theorem 2.2.** The Fourier-Mukai transform in $K$-theory

$$\mathcal{F} : K^i(P) \xrightarrow{\cong} K^{i+1}(\tilde{P}),$$

takes the tensor product to convolution and convolution to the tensor product.

**Proof.** $(x \otimes 1) \otimes (y \otimes 1) = (x \otimes y) \otimes 1$ therefore $\mathcal{F}((x \otimes 1) \otimes (y \otimes 1)) = \mathcal{F}((x \otimes y) \otimes 1) = (x \otimes y) \otimes \zeta$.

On the other hand, $\mathcal{F}(x \otimes 1) \star \mathcal{F}(y \otimes 1) = (x \otimes \zeta) \star (y \otimes \zeta) = (x \otimes y) \otimes \zeta$.

$(x \otimes 1) \otimes (y \otimes \zeta) = (x \otimes y) \otimes \zeta$ therefore $\mathcal{F}((x \otimes 1) \otimes (y \otimes \zeta)) = \mathcal{F}((x \otimes y) \otimes \zeta) = (x \otimes y) \otimes 1$.

On the other hand, $\mathcal{F}(x \otimes 1) \star \mathcal{F}(y \otimes \zeta) = (x \otimes \zeta) \star (y \otimes 1) = (x \otimes y) \otimes 1$.

$(x \otimes \zeta) \otimes (y \otimes \zeta) = 0$ therefore $\mathcal{F}((x \otimes 1) \otimes (y \otimes \zeta)) = 0$. On the other hand, $\mathcal{F}(x \otimes \zeta) \star \mathcal{F}(y \otimes \zeta) = (x \otimes 1) \star (y \otimes 1) = 0$.

This shows that $\mathcal{F}$ takes tensor product to convolution.

$(x \otimes 1) \star (y \otimes 1) = 0$ therefore $\mathcal{F}((x \otimes 1) \star (y \otimes 1)) = 0$. On the other hand, $\mathcal{F}(x \otimes 1) \otimes \mathcal{F}(y \otimes 1) = (x \otimes \zeta) \otimes (y \otimes \zeta) = 0$.

$(x \otimes 1) \star (y \otimes \zeta) = (x \otimes y) \otimes 1$ therefore $\mathcal{F}((x \otimes 1) \star (y \otimes \zeta)) = \mathcal{F}((x \otimes y) \otimes 1) = (x \otimes y) \otimes \zeta$.

On the other hand, $\mathcal{F}(x \otimes 1) \otimes \mathcal{F}(y \otimes \zeta) = (x \otimes \zeta) \otimes (y \otimes 1) = (x \otimes y) \otimes \zeta$.

$(x \otimes \zeta) \star (y \otimes \zeta) = (x \otimes y) \otimes \zeta$ therefore $\mathcal{F}((x \otimes 1) \star (y \otimes \zeta)) = \mathcal{F}((x \otimes y) \otimes \zeta) = (x \otimes y) \otimes 1$.

On the other hand, $\mathcal{F}(x \otimes \zeta) \otimes \mathcal{F}(y \otimes \zeta) = (x \otimes 1) \otimes (y \otimes 1) = (x \otimes y) \otimes 1$.

This shows that $\mathcal{F}$ takes convolution to tensor product, completing the proof. □

3. **Spherical Poincaré vector bundle with connection**

3.1. **Vector bundle realization of the Poincaré element.** From the long exact sequence in homotopy for the principal bundle $SU(2) \to SU(3) \to S^6$, we deduce that $\pi_5(SU(3)) \cong \mathbb{Z}$. Let $h : S^5 \to SU(3)$ be a generator, and use it as a clutching function on the equator of $S^6$ to determine a principal $SU(3)$-bundle $P$ over $S^6$. In fact, standard arguments in algebraic topology show that principal $SU(3)$-bundles $P$ over $S^6$ are classified by the third Chern class $c_3(P) \in 2\mathbb{Z}$, cf. [8]. Now $[S^3 \times S^3, S^6] \cong H^6(S^3 \times S^3, \mathbb{Z}) \cong \mathbb{Z}$, so there is a degree 1 map $g : S^3 \times S^3 \to S^6$. Then $g^*(P)$ is a principal $SU(3)$-bundle over $S^3 \times S^3$, whose associated complex vector bundle $P$ of rank 3 represents the Poincaré object. Note that the restriction of $P$ to the submanifolds $S^3 \times \{x\}$ and $\{x\} \times S^3$ are trivializable, similar to the Poincaré line bundle on $S^1 \times S^1$. 

4
When \( P = \text{G}_2 \), that is, \( \text{SU}(3) \to \text{G}_2 \to S^6 \), then \( c_3(P) = 2 \), so that \( P \) is one of the bundles that we are searching for. The associated rank 3 vector bundle \( \mathcal{E} \) over \( S^6 \) is the non-trivial generator of \( K^0(S^6) \). Recall that if \( X \) and \( Y \) are pointed spaces (i.e. topological spaces with distinguished basepoints \( x_0 \) and \( y_0 \)) the wedge sum of \( X \) and \( Y \), denoted \( X \vee Y \), is the quotient space of the disjoint union of \( X \) and \( Y \) by the identification \( x_0 \sim y_0 \). One can think of \( X \) and \( Y \) as sitting inside \( X \times Y \) as the subspaces \( X \times \{y_0\} \) and \( \{x_0\} \times Y \). These subspaces intersect at a single point, \((x_0, y_0)\), the basepoint of \( X \times Y \). So the union of these subspaces can be identified with the wedge sum \( X \vee Y \). Then the smash product of \( X \) and \( Y \), denoted \( X \wedge Y \), is the quotient space \((X \times Y)/X \vee Y \). In particular, \( S^3 \wedge S^3 \) is homeomorphic to \( S^6 \). By the Kervaire-Milnor theorem [10], the smooth structure on any topological \( S^6 \) is unique, therefore \( S^3 \wedge S^3 \) is diffeomorphic to \( S^6 \). Therefore we get a canonical degree 1 smooth projection map \( g : S^3 \times S^3 \to S^6 \), and we can pullback \( \text{G}_2 \) via this projection map, giving rise to a natural principal \( \text{SU}(3) \)-bundle \( g^*(\text{G}_2) \) over \( S^3 \times S^3 \). The associated rank 3 vector bundle \( g^*(\mathcal{E}) \) over \( S^3 \times S^3 \) is the non-trivial generator of \( K^0(S^3 \times S^3) \) and so represents the Poincaré object \([P]\) in K-theory.

### 3.2. Smashing spheres

To construct a Poincaré bundle with connection on \( S^3 \times S^3 \) we will need an explicit formula for the smash product map. In this subsection we will treat the general case \( f : S^n \times S^n \to S^n \wedge S^n \cong S^{2n} \). The Poincaré bundle on \( S^n \times S^n \) is constructed by pulling back a vector bundle with minimal nonzero Euler class from \( S^{2n} \). In the next two subsections we will restrict our attention to the two examples of interest, \( n = 1 \) corresponding to ordinary T-duality and \( n = 3 \) corresponding to spherical T-duality.

We begin by recalling that \( S^n \) is an \( S^{n-1} \) fibration over the interval \( I \) which degenerates to a point at the two endpoints \( \{0, 1\} \in I \). For each point \( r_i \) in the \( i \)th copy of \( S^n \), where \( i = 1 \) or 2, let \( r_i \in I \) and \( v_i \in S^{n-1} \subset \mathbb{R}^n \) be the associated points in \( I \) and the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). Note that when \( r_i = 0 \) and 1, all values of \( v_i \) are equivalent. To write the map \( f \), it will be convenient to embed \( S^{2n} \) as the unit sphere in \( \mathbb{R}^{2n+1} \). The function \( f \) can therefore be decomposed into \( 2n+1 \) functions \( f_i : S^n \times S^n \to \mathbb{R} \) representing the coordinates in \( \mathbb{R}^{2n+1} \).

We will also decompose \( S^{2n+1} \) into an \( S^{2n} \) fibration over the interval, where the interval will be correspond to the last coordinate in \( \mathbb{R}^{2n+1} \). We assert furthermore that the \( n \)-vectors \((f_1, \ldots, f_n)\) and \((f_{n+1}, \ldots, f_{2n})\) are parallel to \( v_1 \) and \( v_2 \) respectively. More precisely, we impose

\[
(f_1, \ldots, f_n) = \alpha_1(r_1, r_2)v_1, \quad (f_{n+1}, \ldots, f_{2n}) = \alpha_2(r_1, r_2)v_2,
\]

where the \( \alpha_i \) are nonnegative functions on \( I \times I \). Similarly we demand that \( f_{2n+1} \) be independent of \( v_i \), and so we will write simply \( f_{2n+1}(r_1, r_2) \) as a function \( I \times I \to [-1, 1] \). The smash product map \( f \) is therefore defined by the three functions \( f_{2n+1}, \alpha_1 \) and \( \alpha_2 \) on \( I \times I \).

By the definition of the smash product, \( f(S^n \vee S^n) \) is a single point, let it be \((0^{2n}, -1)\). Choose the decomposition of \( S^n \) such that \( S^n \vee S^n \) is the subset of \( S^n \times S^n \) such that \( r_1r_2 = 0 \). Then we learn that

\[
f_{2n+1}(0, r_2) = f_{2n+1}(r_1, 0) = -1, \quad \alpha_i(r_1, 0) = \alpha_i(0, r_2) = 0.
\]
As we would like the smash product $f$ to be smooth, we define
\[ f_{2n+1}(r_1, r_2) = -1 + r_1 r_2 \tilde{f}(r_1, r_2), \quad \alpha_i(r_1, r_2) = r_1 r_2 \tilde{\alpha}_i(r_1, r_2). \] (3.2)

The smash product map $f$ must also have degree 1. For this it is sufficient that the preimage of $(0^{2n}, 1)$ contain a single point, which we will fix to be $(r_1, r_2) = (1, 1)$. For this purpose it is sufficient to fix
\[ \tilde{f}(1, 1) = 2, \]
and to demand that $\tilde{f}$ be everywhere nondecreasing in both $r_1$ and $r_2$.

Next, recall that all values of $v_i$ are equivalent when $r_i = 0$ and 1. Therefore $f$ must be independent of $v_i$ when $r_i = 0$ and 1. When $r_i = 0$ this condition is satisfied, as the image is just $(0^{2n}, -1)$. What about $r_i = 1$? Recall that only $(f_1, ..., f_n)$ depends upon $v_1$ and $(f_{n+1}, ..., f_{2n})$ upon $v_2$, as they are parallel. Therefore a necessary and sufficient condition is that each $n$-vector vanishes when the corresponding $r_i = 1$. In other words, we must impose
\[ \tilde{\alpha}_1(1, r_2) = \tilde{\alpha}_2(r_1, 1) = 0. \] (3.3)

Finally, we must impose that the image of $f$ is actually on the unit sphere
\[ 1 = \sum_{i=1}^{2n+1} f_i^2 = f_{2n+1}^2(r_1, r_2) + \alpha_1^2(r_1, r_2) + \alpha_2^2(r_1, r_2), \] (3.4)
and so
\[ \tilde{f}^2 + \tilde{\alpha}_1^2 + \tilde{\alpha}_2^2 - \frac{2\tilde{f}}{r_1 r_2} = 0. \]

Now we are done, any triplet $(\tilde{f}, \tilde{\alpha}_1, \tilde{\alpha}_2)$ of functions on $I \times I$ satisfying the above conditions will induce the smash product $S^n \times S^n \to S^{2n}$.

3.3. The Poincaré bundle on the torus. As an illustration of our current construction, we give another construction of the Poincaré line bundle with connection next. Consider $G = SU(2)$. Using the parametrization
\[ g = e^{i\phi \sigma_1/2} e^{i\theta \sigma_2/2} e^{i\psi \sigma_3/2}, \quad \phi \in [0, 2\pi), \quad \theta \in [0, \pi), \quad \psi \in [0, 4\pi), \]
the Maurer-Cartan form for $G$ can be written as
\[ \omega = g^{-1} dg = \sum_i e^{i} \left( \frac{i \sigma^i}{2} \right), \]
where, in particular,
\[ e^3 = d\psi + \cos \theta \, d\phi. \]

We can use $A = e^3/2$ as a principal connection on the principal U(1)-bundle $S^3$ over $S^2$, where the normalization is chosen such that the integral of $A$ over the fiber is equal to one. Then
\[ F = dA = -\frac{\sin \theta}{2} d\theta \wedge d\phi, \]
and
\[ c_1 = \frac{1}{2\pi} \int_{S^2} F = -1. \]
To obtain the Poincaré bundle on $S^1 \times S^1$, we need to pull this bundle back by the smash product $f : S^1 \times S^1 \rightarrow S^1 \wedge S^1 \cong S^2$. This is the case $n = 1$ of the general construction treated in the previous section. If $\beta \in [0, 2\pi]$ and $\gamma \in [0, 2\pi]$ are the coordinates for the two copies of $S^1$, then we can define the two intervals by the maps

$$r_1 : S^1 \rightarrow I : \theta \mapsto \sin\left(\frac{\beta}{2}\right), \quad r_2 : S^1 \rightarrow I : \phi \mapsto \sin\left(\frac{\gamma}{2}\right).$$

Fibered over each interval is an $S^0$ with coordinates $v_i \in \{-1, 1\}$.

The conditions (3.3) are that when $r_1 = 1$, corresponding to $\theta = \pi$, $\tilde{\alpha}_1 = 0$ and also when $r_2 = 1$, corresponding to $\phi = \pi$, $\tilde{\alpha}_2 = 0$. We satisfy these conditions by choosing

$$\tilde{\alpha}_1 = 2 \left| \cos\left(\frac{\beta}{2}\right) \right|, \quad \tilde{\alpha}_2 = 2 \sin\left(\frac{\beta}{2}\right) \left| \cos\left(\frac{\gamma}{2}\right) \right|. \quad (3.5)$$

Note that the absolute values are multiplied by elements $v_i = \pm 1 \in S^0$ in Eqn. (3.1). The effect of this multiplication is simply to remove the absolute values, resulting in a smooth map. Inserting this into Eq. (3.2) and imposing (3.4) we obtain the smash product map

$$f(\beta, \gamma) = \left( \sin(\beta) \sin\left(\frac{\gamma}{2}\right), \sin^2\left(\frac{\beta}{2}\right) \sin(\gamma), -1 + 2\sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right).$$

In terms of spherical coordinates on the $S^2$ this map is

$$(\theta, \phi) = \left( \arccos(-z), \arctan\left(\frac{y}{x}\right) \right) = \left( \arccos\left(1 - 2\sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right), \arctan\left(\tan\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) \right) \right).$$

To pullback the curvature we will need the derivatives of this map

$$\frac{\partial \theta}{\partial \beta} = -\frac{\cos\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right)}{\sqrt{1 - \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right)}}, \quad \frac{\partial \theta}{\partial \gamma} = -\frac{\sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right)}{\sqrt{1 - \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right)}},$$

$$\frac{\partial \phi}{\partial \beta} = \frac{\cos\left(\frac{\gamma}{2}\right)}{2 \left(1 - \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right)}, \quad \frac{\partial \phi}{\partial \gamma} = -\frac{\sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) \sin\left(\frac{\gamma}{2}\right)}{2 \left(1 - \sin^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\gamma}{2}\right) \right)}.$$

Finally we can compute the curvature on the Poincaré bundle as the pullback of the curvature on the Hopf bundle

$$f^*F = -\frac{\sin(\theta)}{2} \left( \frac{\partial \theta}{\partial \beta} \frac{\partial \phi}{\partial \gamma} - \frac{\partial \theta}{\partial \beta} \frac{\partial \phi}{\partial \gamma} \right) d\beta \wedge d\gamma = -\frac{1}{2} \sin^2\left(\frac{\beta}{2}\right) \cos\left(\frac{\gamma}{2}\right) d\beta \wedge d\gamma.$$

As a consistency check, we can integrate this curvature to obtain the Chern class

$$c_1 = \frac{1}{2\pi} \int_{T^2} f^*F = -1.$$
3.4. The Poincaré bundle on a product of 3-spheres. The subgroup of $\text{SO}(7)$ which preserves a nondegenerate 3-form on $\mathbb{R}^7$ provides a 7-dimensional representation of $G_2$. The restriction to block-diagonal elements $\text{diag}(1, M)$ where $M$ is a $6 \times 6$ matrix yields an $\text{SU}(3)$ subgroup. The right action by the $\text{SU}(3)$ subgroup is free and leaves fixed an $S^6$. The corresponding elements of $S^6$ can easily be read from the leftmost column of a given $G_2$ matrix, which is a unit vector in $\mathbb{R}^7$ and invariant under the $\text{SU}(3)$-action. The free $\text{SU}(3)$-action means that $G_2$ is the total space of the principle $\text{SU}(3)$-bundle

$$\text{SU}(3) \rightarrow G_2 \rightarrow S^6.$$ 

Let $\Theta \in \Omega^1(G_2) \otimes \mathfrak{g}_2$ denote the Maurer-Cartan 1-form on $G_2$ and let $p : C^\infty(G_2, \mathfrak{g}_2) \rightarrow C^\infty(G_2, \mathfrak{su}(3))$ be the invariant projection onto the vertical subbundle of the cotangent bundle of $G_2$. Then $A = p\Theta \in \Omega^1(G_2) \otimes \mathfrak{su}(3)$ is $\text{SU}(3)$-invariant and so provides a connection on the principal bundle $G_2 \rightarrow S^6$.

To construct the Poincaré bundle over $S^3 \times S^3$, we need to pullback $G_2 \rightarrow S^6$ using the smash product map $f : S^3 \times S^3 \rightarrow S^6$. As $S^3$ is the group manifold of $\text{SU}(2)$, we may use the group structure to reexpress the maps used in the general construction above. One realization of the decomposition of $S^3$ into an $S^2$ fibration over an interval is the decomposition of $\text{SU}(2)$ into conjugacy classes corresponding to elements with eigenvalues $e^{\pm i\pi r}$. These conjugacy classes are of topology $S^2$ for $r \in (0, 1)$ and are points, consisting of the elements $\pm 1 \in \text{SU}(2)$, for $r = \{0, 1\}$. More specifically, for each $g \in \text{SU}(2)$ we define $r \in I$ and $v \in S^2 \subset \mathbb{R}^3$ by

$$g = \exp(irv \cdot \sigma),$$

where $\sigma$ are the Pauli matrices such that $i\sigma$ generates the Lie algebra $\mathfrak{su}(2)$. Using this decomposition, to each point $x \in S^3 \times S^3$ we can identify a quadruplet $(r_1, v_1, r_2, v_2)$ where all values of $v_i$ are identified when $r_i = 0$ or $r_i = 1$, as in the general construction in Subsec. 3.2.

To complete the construction, we need to define the pair $\tilde{\alpha}_i$ of functions on $I \times I$. The functions $\tilde{\alpha}_i$ can be defined as in Eqn. (3.5) in the case $n = 1$

$$\tilde{\alpha}_1 = 2\sqrt{1 - r_1^2}, \quad \tilde{\alpha}_2 = 2r_1 \sqrt{1 - r_2^2}.$$

The third function, $\tilde{f}$, is defined by (3.4), choosing the branch which gives a winding number of 1

$$f_{2n+1} = -1 + 2r_1^2 r_2^2,$$

as in the case $n = 1$, thus completing the construction of the smash product $f : S^3 \times S^3 \rightarrow S^6$

$$f(r_1, v_1, r_2, v_2) = \left(2r_1 r_2 \sqrt{1 - r_1^2} v_1, 2r_1^2 r_2 v_1, 2r_1^2 r_2 v_2, -1 + 2r_1^2 r_2^2 \right).$$

The connection on the spherical Poincaré bundle is then $f^*A$. 

8
If we want to calculate this connection explicitly, then we may proceed as in the torus case of the previous subsection. First we construct an arbitrary element of $G_2$ as

$$g = e^{\left(\pi i \arccos(-1+2r^2)\sqrt{1-r_1^2}v_1^1a_1+r_1\sqrt{1-r_2^2}v_2^2a_2 \over \sqrt{1-r_1^2}}\right)}e^{\left(\pi i(-c_8M_3+\sum_{i=1}^7 c_iF_i)\right)},$$

$$a_1 = (M_1, M_2, M_4), \quad a_2 = (M_5, M_6, M_7),$$

where $F_i$ and $M_i$ are generators of $G_2$ defined in Ref. [7], where it was noted that $F_i$ together with $-M_3$ generate an SU(3) subgroup.

As the Maurer-Cartan form is SU(3)-invariant, to obtain the horizontal part of the connection it will be sufficient to restrict our attention to $c_i = 0$, where $g$ is a section of the bundle $G_2 \rightarrow S^6$ restricted to the compliment of the north pole. As in the toroidal case, it will be convenient to work in spherical coordinates. Therefore we define

$$v_i = (\sin(\theta_i)\cos(\phi_i), \sin(\theta_i)\sin(\phi_i), \cos(\theta_i)).$$

Thus we find

$$g = e^{\left(\pi i \arccos(-1+2r^2)\sqrt{1-r_1^2}v_1^1s(\theta_1)c(\phi_1)M_1+s(\theta_1)s(\phi_1)M_2+c(\theta_1)M_4+r_1\sqrt{1-r_2^2}v_2^2s(\theta_2)c(\phi_2)M_5+s(\theta_2)s(\phi_2)M_6+c(\theta_2)M_7 \over \sqrt{1-r_1^2}}\right)},$$

where $s(\theta)$ and $c(\theta)$ represent sin(\theta) and cos(\theta) respectively.

If we define $h$ by $g = e^{ih}$ then we can write the connection as

$$A_k = pg(\partial_k h)g^{-1}.$$

As, by abuse of notation, we have adopted the same notation for coordinates of $S^6$ and $S^3 \times S^3$, the pullback by the smash product acts trivially so this same expression is also the connection of our Poincaré bundle. Finally, the curvature of the Poincaré bundle is

$$F_{jk} = pg[\partial_j h, \partial_k h]g^{-1}.$$

### 4. Spherical T-duality

#### 4.1. Spherical T-admissibility.

In the following sections, we suitably adapt the strategy in [5] Consider the unit sphere $S \subset \mathbb{C}^4 = \mathbb{H}^2$. Let $E := \text{SU}(2) = \text{Sp}(1)$ and $\hat{E} := \text{SU}(2) = \text{Sp}(1)$. Consider the embeddings $i : E \rightarrow S$, $i(z) = (z, 0)$ in quaternionic variables and $\hat{i} : \hat{E} \rightarrow S$, $\hat{i}(\hat{z}) = (0, \hat{z})$ similarly. Let $T := E \times \hat{E}$ and

$$\begin{array}{ccc}
T & \xrightarrow{p} & E \\
\xrightarrow{\hat{p}} & \hat{E}
\end{array}$$

$p : T \rightarrow E$ and $\hat{p} : T \rightarrow \hat{E}$ denote the projections. Define the homotopy $h : I \times T \rightarrow S$ from $i \circ p$ to $\hat{i} \circ \hat{p}$ by

$$h_t(z, \hat{z}) := {1 \over \sqrt{2}}(\sqrt{1-t^2}z, t\hat{z}).$$
Let $\mathcal{K} \in T(S)$ be a higher twist as in [6, 13]. Then $DD(\mathcal{K}) \in H^7(S, \mathbb{Z})$. Choose $\mathcal{K}$ such that $\langle DD(\mathcal{K}), [S] \rangle = 1$. We define $\mathcal{H} := i^* \mathcal{K}$ and $\hat{\mathcal{H}} := \hat{i}^* \mathcal{K}$. The homotopy $h$ induces a unique morphism

$$u : p^* \hat{\mathcal{H}} = p^* \hat{i}^* \mathcal{K} \cong (i \circ \hat{p})^* \mathcal{K} \cong (i \circ p)^* \mathcal{K} \cong p^* i^* \mathcal{K} = p^* \mathcal{H},$$

where $u(h)$, defined in [3], is the uplift of the homotopy $h$ to $T(T)$.

Note that $\hat{p}$ is canonically $K$-oriented since $T\hat{E}$ is canonically trivialized by the $\text{SU}(2)$-action.

We say that 7-twisted $K$-theory is spherical $T$-admissible if

$$\hat{p}_I \circ u(h)^* \circ p^* : K(E, \mathcal{H}) \to K(\hat{E}, \hat{\mathcal{H}})$$

is an isomorphism. Note that the map has degree $-1$.

In fact, following 3.2.4 of Ref. [5], to prove spherical $T$-admissibility it suffices to prove that $\hat{p}_I \circ g^* \circ p^*$ is an isomorphism of twisted $K$-theory with a trivial twist $K(E, 0)$, which of course is isomorphic to untwisted $K$-theory $K(E)$. Here $g$ is a generator of $\mathcal{H}_2(T, \mathbb{Z})$ and $g^*$ acts by shifting the trivialization of the trivial gerbe on $T$ which defines the twist.

**Lemma 4.1.** 7-twisted $K$-theory is spherical $T$-admissible.

**Proof.** Let $l \in K^0(T)$ be the class of the line bundle $\mathcal{P}$ over $T$ representing the Poincaré element $[\mathcal{P}]$ as in section [3, 1] with third Chern class equal to $g \in H^6(T, \mathbb{Z}) \cong \mathbb{Z}$. Then $g^*$ is induced by the cup product with $l$. Let $1 \in K^0(\text{SU}(2))$ and $\zeta \in K^3(\text{SU}(2))$ be the generators. One computes

$$\hat{p}_I \circ g^* \circ p^*(1) = g^* B(\zeta),$$

$$\hat{p}_I \circ g^* \circ p^*(\zeta) = 1,$$

where $B : K^1 \to K^{-1}$ is the Bott periodicity transformation. This is indeed an isomorphism if $g \in \{1, -1\}$. \hfill $\Box$

**4.2. Spherical $T$-duality isomorphisms.** To define $K$-theory on $P$, twisted by a closed 7-form $H_7$ representing $k$ times the generator of $H^7(P, \mathbb{Z})$, we first recall from Corollary 4.7 in [6] that the generator of $H^7(S^7, \mathbb{Z})$ corresponds to the Dixmier-Douady invariant of an algebra bundle $\mathcal{E} \to S^7$ with fibre a stabilized infinite Cuntz $C^*$-algebra $O_\infty \otimes \mathcal{K}$. Now let $f : P \to S^7$ be a degree $k$ continuous map, then $f^*(\mathcal{E}) \to P$ is an algebra bundle with fibre a stabilized infinite Cuntz $C^*$-algebra $O_\infty \otimes \mathcal{K}$ and Dixmier-Douady invariant equal to $k$ times the generator of $H^7(P, \mathbb{Z})$. Then, by [13], the twisted $K$-theory is defined as $K^*(P, \mathcal{H}_7) = K_*(C_0(P, f^*(\mathcal{E})))$, where $C_0(P, f^*(\mathcal{E}))$ denotes continuous sections of $f^*(\mathcal{E})$ vanishing at infinity. This shows that $K^*(P, \mathcal{H}_7)$ is well defined, although we will not use the explicit construction.

We consider two pairs of 7-dimensional manifolds $(P, H_7)$ and $(\hat{P}, \hat{H}_7)$ over $M$ which are spherical $T$-dual to each other. Let $\text{Th} \in H^7(S(V), \mathbb{Z})$ be a Thom class. Choose a twist $\mathcal{K} \in \text{Twist}(S(V))$ such that $DD(\mathcal{K}) = \text{Th}$. Then we define $\mathcal{H} := i^* \mathcal{K} \in \text{Twist}(P)$ and
\( \hat{H} := \hat{i}^* K \in Twist(\hat{P}) \). We have \( DD(H) = H \) and \( DD(\hat{H}) = \hat{H} \). Consider the commutative diagram

\[
\begin{array}{ccc}
P \times_M \hat{P} & \xrightarrow{p} & \hat{P} \\
\downarrow \hat{\pi} & & \downarrow \pi \\
P & \xrightarrow{\pi} & M
\end{array}
\]

This is the parameterized version of the situation considered earlier. In particular, we have a homotopy \( h : I \times P \times_M \hat{P} \to S(V) \) from \( \hat{i} \circ p \) to \( \hat{i} \circ \hat{p} \). It induces the morphism

\[
u : \hat{p}^* \hat{H} = \hat{p}^* \hat{i}^* K \cong (\hat{i} \circ \hat{p})^* K \cong (i \circ p)^* K \cong p^* i^* K = p^* H,
\]

which is natural under pullback of bundles.

We define the spherical T-duality transformation on 7-twisted K-theory on 7-dimensional manifolds as

\[
T := \hat{p}_! \circ u^* \circ p^* : K(P, \mathcal{H}) \to K(\hat{P}, \hat{H})
\]

The main theorem of the present section is the following. Assume that \( M \) is homotopy equivalent to a finite complex.

**Theorem 4.2.** The spherical T-duality transformation \( T \) is an isomorphism.

**Proof.** The proof that spherical T-duality is an isomorphism of 7-twisted K-theory, or in a more general context any spherical T-admissible twisted cohomology theory, is identical to the proof of Th 3.13 in Ref. [5]. \qed

4.3. **The spherical T-duality group.** Consider \( SU(2) \) as the unit quaternions ie \( Sp(1) \). Then quaternionic conjugation is an orientation reversing automorphism of \( SU(2) \). So given a principal \( SU(2) \)-bundle \( P \) over a 4-dimensional manifold \( M \), let \( x, g \) denote the right action of \( g \in SU(2) \) on \( x \in P \). Then \( x, \bar{g} \) also gives a right action of \( SU(2) \) on \( P \), where \( \bar{g} \) is the quaternionic conjugate of \( g \). It is again a free action, so it defines a principal \( SU(2) \)-bundle with the same total space and with 2nd Chern class the negative of \( c_2(P) \). This gives the action of the non-trivial element \(-1 \in GL(1, \mathbb{Z}) \) on spherical T-dualities (with 4D base). Now \(-1 \in GL(1, \mathbb{Z}) \) corresponds to the element \((-1, -1) \in O(1, 1, \mathbb{Z}) \) via the canonical embedding of \( GL(1, \mathbb{Z}) \) in \( O(1, 1, \mathbb{Z}) \). The other generator of \( O(1, 1, \mathbb{Z}) \) is the \( 2 \times 2 \) matrix with 1’s on the off-diagonal and 0’s on the diagonal. This element exchanges the 2nd Chern class and the 7-flux i.e. is the spherical T-duality element. Therefore \( O(1, 1, \mathbb{Z}) \) is the spherical T-duality group.
[1] P. Bouwknegt, J. Evslin and V. Mathai, T-duality: Topology Change from H-flux, Comm. Math. Phys. 249 (2004) 383-415 [arXiv:hep-th/0306062]

[2] P. Bouwknegt, J. Evslin and V. Mathai, On the Topology and Flux of T-Dual Manifolds, Phys. Rev. Lett. 92 (2004) 181601 [arXiv:hep-th/0312052]

[3] P. Bouwknegt, J. Evslin and V. Mathai, Spherical T-duality, Comm. Math. Phys. (to appear), 49 pages, [arXiv:1405.5844 [hep-th]]

[4] P. Bouwknegt, J. Evslin and V. Mathai, Spherical T-duality II: An infinity of spherical T-duals for non-principal SU(2)-bundles, J. Geom. Phys. (to appear), 14 pages, [arXiv:1409.1296 [hep-th]]

[5] U. Bunke and T. Schick, On the topology of T-duality, Rev. Math. Phys. 17 (2005) 77-112 [arXiv:math.GT/0405132]

[6] M. Dadarlat and U. Pennig, A Dixmier-Douady theory for strongly self-absorbing C*-algebras, arXiv:1306.2583 [math.OA]

[7] M. Güneydin and F. Gürsey, Quark Statistics and Octonions, Phys. Rev. D9 (1974) 3387

[8] H. Hamanaka and A. Kono, Homotopy type of gauge groups of SU(3)-bundles over S6, Topology Appl. 154 (2007) 1377-1380

[9] K. Hori, D-branes, T-duality, and index theory, Adv. Theor. Math. Phys. 3 (1999) 281-342, arXiv:hep-th/9902102

[10] M. Kervaire and J. Milnor, Groups of homotopy spheres. I, Ann. of Math. 77 (1963) 504-537

[11] S. Mukai, Duality between D(X) and D(\hat{X}) with its application to Picard sheaves, Nagoya Math. J. 81 (1981) 153-175

[12] D. Mumford, Abelian varieties, Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[13] U. Pennig, A noncommutative model for higher twisted K-Theory, arXiv:1502.02807 [math.KT]

(Peter Bouwknegt) Mathematical Sciences Institute, and Department of Theoretical Physics, Research School of Physics and Engineering, The Australian National University, Canberra, ACT 2601, Australia
E-mail address: peter.bouwknegt@anu.edu.au

(Jarah Evslin) High Energy Nuclear Physics Group, Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou, China
E-mail address: jarah@impcas.ac.cn

(Varghese Mathai) Department of Pure Mathematics, School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
E-mail address: mathai.varghese@adelaide.edu.au