Reconstruction of Polygonal Cavities by Two Boundary Measurements

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Abstract. Reconstruction procedure for identifying an unknown polygonal cavity inside a homogeneous, isotropic conductive medium from two boundary measurements is given. The assumptions of the cavity is that there are at most one concave vertex between two convex vertices and the vertices are well separated. The Neumann data of the boundary measurement can change sign at most once and the sets on which one of them become positive and negative must be disjoint.

Keywords: enclosure method, oscillating-decaying solution

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1. Introduction and Statements of Results

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) with a Lipschitz boundary. \(\Omega\) is considered as a homogeneous isotropic conductive medium with conductivity 1. Suppose that there is an unknown polygonal shaped cavity \(D\) inside \(\Omega\). As a set, we consider \(D\) as an open set. We assume that \(\overline{D} \subset \Omega\) and \(\Omega \setminus \overline{D}\) is connected, which mean that \(D\) is totally inside \(\Omega\) and there is no hole inside \(D\). Furthermore, we assume the following configuration assumption of vertices. That is, there is at most one concave vertex of \(D\) between two convex vertices of \(D\). Here, we call a vertex of \(D\) convex and concave if the inner angles of \(D\) at this vertex are less than \(\pi\) and greater than \(\pi\), respectively.

In this paper we consider the problem of reconstructing \(D\) by means of two current-voltage pairs measured on \(\partial \Omega\). For the precise description of our problem, we state the direct problem and then give the precise formulation of our inverse problem.

As a direct problem, we consider the following Neumann problem \(N[D, g]\):
for any given \(g \in H^{-1/2}_0(\partial \Omega)\) i.e. \(g \in H^{-1/2}(\partial \Omega)\) and \(\int_{\partial \Omega} g \, d\sigma = 0\), find \(u = u(g) \in H^1_0(\Omega \setminus \overline{D})\)

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i.e. \( u \in H^1(\Omega \setminus D) \) and \( \int_{\partial \Omega} u \, d\sigma = 0 \) such that

\[
N[D, g] \begin{cases}
\Delta u = 0 & \text{in } \Omega \setminus \overline{D}, \\
\partial_{\nu}u = 0 & \text{on } \partial D, \\
\partial_{\nu}u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Here and throughout this paper, \( \partial_{\nu} := \nu \cdot \nabla \) with unit normal \( \nu \) directed outside \( \Omega \) and \( D \). The Neumann condition on \( \partial D \) means that \( \partial D \) is insulated. The current-voltage pair on \( \partial \Omega \) is \( (g, u(g)|_{\partial \Omega}) \in H^{-1/2}_{(0)}(\partial \Omega) \times H^{1/2}(\partial \Omega) \). We call this pair a Cauchy data. Also, \( u(g)|_{\partial \Omega} \) and \( g \) are called a Dirichlet data and Neumann data.

The inverse problem to identify a polygonal inclusion is closely related to our problem. There have been some works on this problem. Friedman-Isakov \[5\] proved uniqueness within the class of polygons under the assumption that the polygon \( D \) is away from the boundary \( \partial \Omega \), more precisely,

\[
\text{diam}(D) \leq \text{dist}(D, \partial \Omega). \tag{1}
\]

Seo \[11\] removed the distance condition and proved uniqueness for (convex and non-convex) polygons using two Neumann data satisfying some mild conditions. His method relies on the geometric index theory. Here for the polygonal inclusion \( D \), we have to remark that the configuration assumption of vertices is not assumed.

For identifying cavity, there are uniqueness and conditional stability results by one boundary measurement (\[1\], \[3\], \[4\]). As for reconstructing an unknown polygonal cavity, Ikehata \[7\] proposed a method called enclosure method to reconstruct convex polygonal cavities with the property (1) using exponentially decaying solutions by one Cauchy data such that the Dirichlet data is not identically zero. The numerical implementation of his method was given in \[8\].

The basic scheme of reconstruction in this paper is similar to that of \[7\]. However there are some difficulties arising from the fact that the cavity is non convex.

Let us explain the enclosure method in detail. Let \( L_{\omega} \) be a straight line with a normal unit normal \( \omega \) and \( L_{t,\omega} := L_{\omega} + t\omega \), where \( t \in \mathbb{R} \). Suppose that \( L_{t,\omega} \) has the following property: There is \( t_0 \) such that for all \( t \geq t_0 \) \( D \) lies in the half space \( \{ x | x \cdot \omega < t \} \) and \( \overline{D} \cap L_{t_0,\omega} = \{ P \} \) where \( P \) is a vertex of \( D \). If we can find \( t_0 \) for a given line \( L_{\omega} \), then using two linearly independent lines we can locate the vertex \( P \). The following is a way to find \( t_0 \).

For \( \tau, t \in \mathbb{R} \), let \( v = v_{\tau,t,\omega} \in H^2(\Omega) \) be a solution to \( \Delta v = 0 \) in \( \Omega \). We then define the so called indicator function \( I(\tau,t,\omega) \) by

\[
I(\tau,t,\omega) := \int_{\partial \Omega} (gv - u \partial_{\nu}v) \, d\sigma. \tag{2}
\]

By Green’s formula, one can easily see that

\[
I(\tau,t,\omega) = \int_{\partial D} (u - \mu) \partial_{\nu}v_{\tau,t,\omega} \, d\sigma, \tag{3}
\]

where \( u = u(g) \) is the solution to \( N[D,g] \) and \( \mu \) can be any constant. Suppose that \( v_{\tau,t,\omega} \) has the property that \( v_{\tau,t,\omega} \) is exponentially decreasing on one side of \( L_{t,\omega} \) and oscillating along \( L_{t,\omega} \), then by investigating the behavior of \( I(\tau,t,\omega) \) as \( \tau \to \infty \) for each \( t \), we can determine \( t_0 \). Ikehata used the exponentially growing solution for \( v_{\tau,t,\omega} \):

\[
v_{\tau,t,\omega}(x) := e^{\tau(x \cdot \omega - t + ix \cdot \eta)},
\]

where \( \eta \) is an element in the unit circle \( S^1 \) centered at the origin and it is perpendicular to \( \omega \).
In order to adapt the enclosure method in a way such that we can apply it to detecting a non convex cavity, we need to localize the exponentially growing solution. The localization of the exponentially growing solution has been given in [6] for Schrödinger operator with potential. However, their construction are quite involved and not easy at all. Hence we use the oscillating-decaying solution introduced by Nakamura ([9]) and as a substitute of the exponentially growing solution. This solution can be localized, shares some propeties of the exponentially growing solution. Also, it can be constructed for any elliptic equation whose Dirichlet boundary value problem is well-posed ([10]). Moreover, for the Laplacian the oscillating-decaying solution is very easy to construct. Since the oscillating-decaying solution is not defined in whole $\Omega$, we use the Runge theorem to extend it approximately to an open neighborhood of $\overline{\Omega}$. This approximate extension of the oscillating-decaying solution is the one we really use as the substitute of the exponentially growing solution.

Looking at the form of integral in (3), one can realize that a precise information on the behavior of the solution $u$ near the vertex $x_0$ of $D$ is necessary. For that, we show that $u$ takes, in terms of the polar coordinates $(r, \theta)$ such that $x - x_0 = (r \cos \theta, r \sin \theta)$ around $x_0$, the following form near a corner $P$:

$$u(x_0) + a_2 r^{\lambda_2} \cos \lambda_2 \theta + o(r^{\lambda_2}),$$

where $\lambda_2$ is a number determined by the angle of $D$ at $x_0$. See Lemma 2.1 for the precise statement.

For the reconstruction procedure of this paper to work, it is important to know the coefficient $a_2$ in (4) should not vanish. Thus we seek a condition on the Neumann data $g$ which guarantees that $a_2 \neq 0$. It turns out that the geometric index theory [2] is the most useful tool. We define the class $N_0$ be the class of all functions $g \in C(\partial \Omega) \cap H_{(0)}^{-1/2}(\partial \Omega)$ such that the set $\{x; g(x) \geq 0\}$ is connected. Following the idea of Seo [11], if we use two Neumann data $g_k (k = 1, 2)$, then at each convex vertex, we can guarantee that $a_2$ does not vanish for one of the solutions $u = u(g_k) (k = 1, 2)$ of $N[D, g_k] (k = 1, 2)$. See Theorem 2.2 for the precise statement.

The main result of this paper is as follow.

**Theorem 1.1** Beside all the aforementioned assumptions on $D$, we assume that all the vertices of $D$ are well separated, which means that we know the minimum distance between two different vertices of $D$. Let the Neumann data $g_k \in N_0 (k = 1, 2)$ satisfy the condition $(H)$:

$$c_1 g_1 + c_2 g_2 \in N_0$$

for any $(c_1, c_2) \in \mathbb{R}^2$.

Then, we can reconstruct $D$ from two Cauchy data $\{g_k, u(g_k)|_{\partial \Omega}\} (k = 1, 2)$.

**Remark 1.2** It is easy to see that for $g_k \in N_0 (k = 1, 2)$, if $\{x \in \partial \Omega; g_1(x) > 0\} \cup \{x \in \partial \Omega; g_1(x) < 0\}$ and $\{x \in \partial \Omega; g_2(x) > 0\} \cup \{x \in \partial \Omega; g_2(x) < 0\}$ do not intersect, then the condition $(H)$ is satisfied.

This paper is organized as follows. In section 2, we look for an asymptotic formula (4) for the solution near a convex vertex of $D$ and the question whether $a_2 \neq 0$. In section 3, we review the construction of the oscillating decaying solutions. In the last section, we derive a procedure for reconstruction of $D$. 

2. Behavior of solutions at convex vertex

Let $x_0$ be a convex vertex of $D$. The corner of $D$ at $x_0$ is formed by two segments $\Gamma_p$ and $\Gamma_q$ intersecting at $x_0$. Without loss of generality, we assume that $\Gamma_p$ can be obtained by rotating $\Gamma_q$ in the counter clockwise direction. Let $a$ be the unit direction vector of $\Gamma_p$ directed away from $x_0$. We introduce a polar coordinate $(r, \theta)$ in $\Omega \setminus D$ around $x_0$ where for $x \in \Omega \setminus D$ near $x_0$, $r := |x - x_0|$ and the angle $\theta$ is the angle between the vector $x - x_0$ and $a$.

Then, we have the following lemma for the expansion of the solution $u \in H^1(\Omega \setminus D)$ of $\Delta u = 0$ in $\Omega \setminus D$.

**Lemma 2.1** For a small $\rho > 0$, let $D_\rho := (\Omega \setminus D) \cap B(x_0, \rho)$ with $B(x_0, \rho) := \{x \in \mathbb{R}^2; |x - x_0| < \rho\}$. Then, $u = u(x) = u(r, \theta)$ admits an expansion:

$$u(r, \theta) = \sum_{m=1}^{\infty} \alpha_m r^{\lambda_m} \phi_m(\theta) \quad \text{in} \quad L^2(D_\rho),$$

where $\Theta$ is the outer angle of $D$ at $x_0$ and

$$\lambda_m := (m - 1)\pi/\Theta \quad (m \geq 1),$$

$$\phi_m(\theta) := \begin{cases} 1/\sqrt{\Theta} & (m = 1) \\ \sqrt{2/\Theta} \cos \lambda_m \theta & (m \geq 2). \end{cases}$$

The righthand side of (5) and $r^{1-\lambda_2}$ times its termwise derivative converge absolutely and uniformly in $D_{\rho'}$ for any $0 < \rho' < \rho$.

**Proof.** We just give the idea of the proof. Note that $\phi_m(\theta)$ $(m \geq 1)$ are the normalized eigenfunctions of the operator $-d^2/d\theta^2$ with the boundary condition $\phi_m'(0) = \phi_m'(\Theta) = 0$ realized in $L^2((0, \Theta))$ and each of their eigenvalues is $\lambda_m$. It is well known that these eigenfunctions are complete in $L^2((0, \Theta))$. The expansion (5) can be obtained as a Fourier series expansion of $u(r, \cdot) \in L^2((0, \Theta))$ in terms of $\phi_m(\theta)$ $(m \geq 1)$. The properties of the convergence of (5) can be obtained from

$$\int_0^\rho r |u_m'(r)|^2 dr \leq \|\nabla u_m\|_{L^2(D_\rho)}^2 < \infty \quad (m \geq 1),$$

where

$$u_m(r) := \int_0^\Theta u(r, \theta) \phi_m(\theta) d\theta.$$

See [7] for the detail. \hfill \Box

**Theorem 2.2** Let $D$ be a polygon satisfying the configuration assumption of vertices. If $g_k \in N_0$ $(k = 1, 2)$ satisfy the condition (H), then at each convex vertex $x_0$ of $D$, one of the solutions $u_k := u(g_k) \in H^1_0(\Omega \setminus D)$ $(k = 1, 2)$ of $N[D, g]$ is not in $H^2$ near $x_0$. More precisely, $\alpha_2$ in the expansion (5) is non zero.

This follows from the following lemma which can be easily obtained by stretching the corner to a half disk by a conformal mapping.

**Lemma 2.3** Let $x_0$ be a vertex of $D$ and $\lambda_2$ be the one given by (6).

1. If a solution $u$ of $N[D, g]$ satisfies

$$\lim_{x \to x_0} |x - x_0|^{1-\lambda_2} |\nabla u(x)| = 0,$$

then there are two disjoint finite open sectors $S^+$ and $S^-$ near $x_0$ having $x_0$ as a common vertex such that
(i) \( S^+ \cup S^- \subset (\Omega \setminus \overline{D}) \cap B(x_0, \epsilon) \) for some \( \epsilon > 0 \).
(ii) \( u^+ - u^+(x_0) > 0 \) on \( S^+ \) and \( u^- - u^-(x_0) < 0 \) on \( S^- \), where \( u^\perp \) is a conjugate harmonic of \( u \) such that \( \nabla u \) is obtained by rotating \( \nabla u^\perp \) in the anticlockwise direction by \( \pi/2 \).

(2) If \( u_i, \ i = 1, 2, \) is solution of \( N[D, g_1] \) satisfying (10), then there exist constants \( c_1 \) and \( c_2 \) such that \( u := c_1 u_1 + c_2 u_2 \) has the following property: there are two pairs of disjoint finite open sectors \( S_j^+ \) and \( S_j^- \), \( j = 1, 2, \) near \( x_0 \) having \( x_0 \) as a common vertex such that

(i) they are contained in \( (\Omega \setminus \overline{D}) \cap B(x_0, \epsilon) \) for some \( \epsilon \),
(ii) \( S_1^+ \) and \( S_2^+ \) are separated by one of \( S_1^- \) (\( j = 1, 2 \)), and \( S_1^- \) and \( S_2^- \) are separated by one of \( S_j^+ \) (\( j = 1, 2 \)),
(iii) \( u - u(x_0) > 0 \) on \( S_1^+ \cup S_2^+ \) and \( u - u(x_0) < 0 \) on \( S_1^- \cup S_2^- \).

3. Oscillating-decaying solutions and Runge’s approximation
In this section we first give a brief review of the construction of oscillating-decaying solutions [9], [10]. The Runge approximation for the Laplacian which is an immediate consequence of the weak unique continuation property will be given at the end of this section without proof. We remark that since the Laplacian is a constant coefficient elliptic partial differential operator, the weak unique continuation property for the Laplacian follows from the famous Holmgren uniqueness theorem.

Let \( \Omega, \Omega_1 \subseteq \mathbb{R}^2 \) be bounded domains with smooth boundaries \( \partial \Omega, \partial \Omega_1 \) such that \( \overline{D} \subset \Omega_1 \) and \( \Omega, \Omega_1 \subset \Omega \). Let \( \omega, \eta \in S^1 \) be mutually orthogonal such that \( \omega, \eta \) is positively oriented. For \( t \in \mathbb{R} \) such that \( \Omega_1 \cap \{ x \cdot \omega > t \} \neq \emptyset \), define \( \Sigma_t(\omega) := \Omega_1 \cap \{ x \cdot \omega = t \} \). Let \( \Omega_t(\omega) \) be a bounded domain with smooth boundary such that

\[
\partial \Omega_t(\omega) \cap \Omega_1 = \Sigma_t(\omega)
\]

and

\[
\partial \Omega_t(\omega) \cap \{ x \cdot \omega < t' \} \subset \partial \Omega
\]

for some \( t' < t \). We also define \( \Omega_t(\omega) := \Omega_t(\omega) \cap \Omega \). \( \Omega_t(\omega) \) is the domain in which we construct the oscillating-decaying solution. This domain changes its shape by taking different \( \Omega_1 \) and \( \Omega \).

Let \( \chi_\lambda \in C^\infty(S^1) \) be such that \( \text{supp} \chi_\lambda(x \cdot \eta) \cap \{ x \cdot \omega = t \} \subset \Sigma_t(\omega) \). For a given \( N \in \mathbb{N} \) and a parameter \( \tau \gg 1 \), there exists \( v(x) = v_{\chi_\lambda, \tau, t, N, \omega}(x) \in C^\infty(\Omega_t(\omega)) \) satisfying

\[
\left\{ \begin{array}{l}
\Delta v = 0 \quad \text{in } \Omega_t(\omega), \\
v|_{\Sigma_t(\omega)} = e^{i\tau x \cdot \eta} \chi(x \cdot \eta).
\end{array} \right.
\]

Moreover, \( v_{\chi_\lambda, \tau, t, N, \omega} \) take the form

\[
v_{\chi_\lambda, \tau, t, N, \omega}(x) = \psi_{\chi_\lambda, \tau, t, N, \omega}(x) + r_{\chi_\lambda, \tau, t, N, \omega}(x)
= \chi_\lambda(x \cdot \eta) e^{i\tau x \cdot \eta} e^{-\tau(x \cdot \omega - t)} + \gamma_{\chi_\lambda, \tau, t, N, \omega}(x) + r_{\chi_\lambda, \tau, t, N, \omega}(x),
\]

and \( r \) and \( \gamma \) satisfy

\[
\| \partial^\alpha_x r_{\chi_\lambda, \tau, t, N, \omega} \|_{L^\infty(\Omega_t(\omega))} \leq C_\alpha \tau^{-N+\alpha},
\]

\[
\| \partial^\alpha_x \gamma_{\chi_\lambda, \tau, t, N, \omega} \|_{L^\infty(\Omega_t(\omega))} \leq C_\alpha \tau^{-1} e^{-\tau(s-t)\lambda} \quad (s \geq t)
\]

for any nonnegative integer \( \alpha \), and

\[
\text{supp} \gamma_{\chi_\lambda, \tau, t, N, \omega} \subseteq \text{supp} \chi_\lambda(x \cdot \eta).
\]
Moreover,
\[ v_{\chi',\tau',t,N,\omega} \rightarrow v_{\chi,t,t,N,\omega} \quad \text{in} \quad H^1(\Omega_t(\omega)) \quad \text{as} \quad t' \uparrow t \]
provided that \( \overline{\Omega_t(\omega)} \subset \overline{\Omega}_t'(\omega) \) for \( 0 \leq t' - t << 1 \). Here, we remark that \( \chi_{t'} \) can be taken \( \chi_t \).

We end this section by stating the Runge approximation for the Laplacian.

**Theorem 3.1** (Runge’s approximation) Let \( U \) be an open subset of \( \hat{\Omega} \) such that \( \hat{\Omega} \setminus \overline{U} \) is connected, and \( u \in H^2(U) \) be a solution of \( \Delta u = 0 \) in \( U \). Then, for any open set \( V \) satisfying \( V \subset U \) and \( \varepsilon > 0 \), there exists \( \tilde{u} \in H^2(\hat{\Omega}) \) such that \( \Delta \tilde{u} = 0 \) in \( \hat{\Omega} \) and \( \| \tilde{u} - u \|_{H^2(V)} < \varepsilon \).

### 4. Identification of cavities

Let the unknown cavity \( D \) satisfy the configuration and well separated assumptions for its vertices. Also, let the Neumann data \( g_k \in N_0 \) \((k = 1, 2)\) satisfy the condition (H).

By the Runge approximation, there exist solutions \( \tilde{v}_{\varepsilon,j} \in H^2(\hat{\Omega}) \) \((j \in \mathbb{N})\) to \( \Delta \tilde{v}_{\varepsilon,j} = 0 \) in \( \hat{\Omega} \) such that
\[
\tilde{v}_{\varepsilon,j} \rightarrow v_{\chi_{t+\varepsilon},t+\varepsilon,N,\omega} \quad \text{in} \quad H^2(\Omega_t(\omega)) \quad \text{as} \quad j \to \infty.
\]

Then, we define three indicator functions \( I(\tau,t,\omega) \) and \( J_k(\tau,t,\chi_t,\omega) \) \((k = 1, 2)\) which we use to reconstruct the unknown cavity an adaptive way. They are define as follows.

\[
J_k(\tau,t,\chi_t,\omega) := \lim_{\varepsilon \downarrow 0} \lim_{j \to \infty} \frac{1}{S} \int_{\partial D} (g_k \tilde{v}_{\varepsilon,j} - f_k \partial_v \tilde{v}_{\varepsilon,j}) \, d\sigma \quad (17)
\]

and
\[
I(\tau,t,\omega) := \sup(|J_1(\tau,t,\chi_t,\omega)| + |J_2(\tau,t,\chi_t,\omega)|).
\]

where sup is taken for all \( \chi_t \). Here, remind that for any small \( \varepsilon > 0 \), we can take \( \chi_{t+\varepsilon} = \chi_t \).

If \( D \subset \Omega_t(\omega) \), then by the Green formula, we have the following representation formula for each \( J_k(\tau,t,\chi_t,\omega) \):
\[
J_k(\tau,t,\chi_t,\omega) = \int_{\partial D} (u_k - \mu_k) \partial_v v_{\chi_t,t,N,\omega} \, d\sigma, \quad (18)
\]

where \( \mu_k \in \mathbb{R} \) can be any and we take it as the value of the solution \( u(g_k) \) at a convex vertex.

We first take \( \Sigma(\omega) \) large enough such that \( \Omega \cap \{ x \in \mathbb{R}^2; x \cdot \omega = t \} \subset \Sigma(\omega) \). Then, we can have the following situation. \( \Sigma(\omega) \cap \partial D = \emptyset \) for \( t > t_0 \) and \( \Sigma_{t_0}(\omega) \cap \partial D = V \) where \( V := \{ x_0^{(k)}; 1 \leq k \leq K \} \) is a subset of all the convex vertices of \( D \).

**Lemma 4.1**
(i) If \( t > t_0 \), then
\[
I(\tau,t,\omega) = O(\tau^{-N+1}) \quad (\tau \to \infty).
\]

(ii) There exist \( c > 0 \) and \( \kappa (0 < \kappa < 1) \) such that
\[
I(\tau,t_0,\omega) = c \tau^{-\kappa}(1 + o(1)) \quad (\tau \to \infty).
\]

This follows from the following asymptotic behavior of \( J_k(\tau,t,\chi_t,\omega) \).

**Lemma 4.2** Take \( \eta \in S^1 \) to satisfy \( \det[\eta,\omega] > 0 \). Let (5) be the expansion of the solution \( u = u(g_k) \) to \( N[D,g_k] \) around a convex vertex \( x_0 = x_0^{(k)} \in V \). We denote the angles of \( \Gamma_p \) and \( \Gamma_q \) from the line starting from \( x_0 \) in the direction \( \eta \) by \( p \) and \( q \), respectively. Then, the followings hold.
(i) If \( t > t_0 \), then
\[
J_k(\tau,t,\chi_t,\omega) = O(\tau^{-N+1}) \quad (\tau \to \infty).
\]
(ii) 
\[ J_k(\tau, t_0, \chi_{t_0}, \omega) = -i \sqrt{\frac{2}{\Theta}} \chi_{t_0}(x_0 \cdot \eta)e^{i\tau x_0 \cdot \eta} \tau^{-\lambda_2} \Gamma(1 + \lambda_2)e^{i\frac{\pi}{2} \lambda_2} K_2 + O(\tau^{-\lambda_3}) \quad (\tau \to \infty), \]

where \( K_2 := e^{ip\lambda_2} + e^{iq\lambda_2}, \Gamma \) is the gamma function and \( \alpha_2 \) is that given in the expansion (5).

**Proof.** We only make some remarks on the proof. (i) is almost obvious from the properties of the oscillating-decaying solution. If \( \Sigma_{t_0}(\omega) \) is neither parallel to \( \Gamma_p \) nor \( \Gamma_q \), then (ii) can be proven as in [7]. If this is not the case, it is still possible to prove using asymptotic expansion for the Fourier integral (see Theorem 227 in [12]). Here we remark that by Lemma 2.1 and Theorem 2.2, \( \alpha_2 \) in (5) corresponding to one of \( u = u(g_k) \) is not zero. \( \square \)

By Lemma 4.1, we can recover \( t_0 \) and hence \( \Sigma_{t_0}(\omega) \) on which we have the convex vertices in \( V \). To localize each point \( x^{(k)}_0 \in V \), we use the indicator function \( J_k(\tau, t_0, \chi_{t_0}, \omega) \) \( (k = 1, 2) \) and freedom of choosing \( \chi_{t_0} \), then by Lemma 2.1, Theorem 2.2 and Lemma 4.2, we can almost recover each \( x^{(k)}_0 \). For the complete recovery of each \( x^{(k)}_0 \), we take \( \omega' \) \( (|\omega' - \omega| << 1) \) and use the indicator function \( I(\tau, t, \omega') \) with a small \( \Sigma_t(\omega') \), then likewise before we can recover \( \Sigma_{t_0}^{(k)}(\omega') \) on which \( x^{(k)}_0 \) is located. Then, \( x^{(k)}_0 = \Sigma_{t_0}(\omega) \cap \Sigma_{t_0}^{(k)}(\omega') \). Here we remarked that for the choice of \( \omega' \) we have used the assumption that the convex vertices are well separated.

So far we have recovered all the convex vertices of \( D \) on the boundary of its convex hull. Now to recover the convex vertices inside the convex hull of \( D \), we use the following lemma.

**Lemma 4.3** \( p, q \) can be recovered.

**Proof.** From Lemma 4.1, we know \( \lambda_2 \) and hence \( \Theta \). Then, from (19), we know \( |\alpha_2|, e^{ip\lambda_2} \alpha_2 \) and \( e^{iq\lambda_2} \alpha_2 \). By considering \( (e^{ip\lambda_2} \alpha_2)^4/(e^{iq\lambda_2} \alpha_2)^2 \), we know \( 2p - q \). Combining this with \( p - q = 2\pi - \Theta \), we can recover \( p, q \). \( \square \)

Using Lemma 4.3 and the assumption that the convex vertices are well separated, we can repeat the previous procedure to recover the convex vertices on line and their corners inside the convex hull of \( D \). After obtaining all the convex vertices and their corners, the concave vertices can be obtained as the intersection of the two side lines of near by two convex vertices. Therefore we have reconstructed all the vertices of \( D \) and hence \( D \) itself.

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