Birationally rigid Fano hypersurfaces with isolated singularities

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It is proved that a general Fano hypersurface \( V = V_M \subset \mathbb{P}^M \) of index 1 with isolated singularities of general position is birationally rigid. Therefore it cannot be fibered into uniruled varieties of a smaller dimension by a rational map and any \( \mathbb{Q} \)-Fano variety \( V' \) with Picard number 1 which is birational to \( V \) is actually isomorphic to \( V \). In particular, \( V \) is non-rational. The group of birational self-maps of \( V \) is either \( \{1\} \) or \( \mathbb{Z}/2\mathbb{Z} \), depending on whether \( V \) has a terminal point of the maximal possible multiplicity \( M - 2 \). The proof is based upon the method of maximal singularities and the techniques of hypertangent systems combined with the Shokurov connectedness principle.

0. Introduction
0.1. Birationally rigid varieties
0.2. Regular hypersurfaces
0.3. The main result
0.4. Earlier results on singular Fano varieties
0.5. Acknowledgements
1. Start of the proof
1.1. Maximal singularities
1.2. Isolated singular point
1.3. The crucial fact
2. Infinitely near maximal singularities
2.1. Resolution of a maximal singularity
2.2. Simple examples
2.3. Subvarieties of codimension 2
2.4. The case \( \mu \geq 5 \)
2.5. The harder cases \( \mu = 3 \) and 4
3. The technique of counting multiplicities
3.1. A sequence of blow ups
3.2. The self-intersection of a linear system
3.3. Proof of Proposition 6
Introduction

0.1 Birationally rigid varieties

In this paper we work over the field \( \mathbb{C} \) of complex numbers. Recall that a Fano variety \( X \) of dimension \( \geq 3 \) with \( \mathbb{Q} \)-factorial terminal singularities, \( \text{rk Pic} \ X = 1 \) is said to be birationally superrigid, if for each birational map

\[
\chi : X \dashrightarrow X'
\]
on to a variety \( X' \) of the same dimension, smooth in codimension one, and each linear system \( \Sigma' \) on \( X' \), free in codimension 1 (that is, \( \text{codim Bs} \Sigma' \geq 2 \)), the inequality

\[
c(\Sigma, X) \leq c(\Sigma', X')
\]
holds, where \( \Sigma = (\chi^{-1})_* \Sigma' \) is the proper inverse image of \( \Sigma' \) on \( X \) with respect to \( \chi \), and \( c(\Sigma, X) = c(D, X) \) stands for the threshold of canonical adjunction

\[
c(D, X) = \sup\{b/a | b, a \in \mathbb{Z}_+ \setminus \{0\}, |aD + bK_X| \neq \emptyset\}
\]
\( D \in \Sigma \), and similarly for \( \Sigma', X' \). \( X \) is said to be birationally rigid, if for each \( X', \chi, \Sigma' \) there exists a birational self-map \( \chi^* \in \text{Bir} X \) such that the triple \( X', \chi \circ \chi^*, \Sigma' \) satisfies the condition \( [\Pi] \).

The following fact is well-known.

**Proposition 1.** Assume that \( X \) is rigid. Then:

(i) \( X \) can not be fibered into uniruled varieties by a non-trivial rational map,

(ii) if \( \chi : X \dashrightarrow X' \) is a birational map onto a Fano variety \( X' \) with \( \mathbb{Q} \)-factorial terminal singularities such that \( \text{Pic} X' \otimes \mathbb{Q} = \mathbb{Q}K_{X'} \), then \( X' \) is (biregularly) isomorphic to \( X \). If \( X \) is superrigid, then \( \chi \) itself is a (biregular) isomorphism. In particular, in the superrigid case the groups of birational and biregular self-maps coincide:

\[
\text{Bir} X = \text{Aut} X.
\]

(iii) \( X \) is non-rational.

0.2 Regular hypersurfaces

Let \( W = W_m \subset \mathbb{P}^N \) be a hypersurface of degree \( m \leq N \) in the \( N \)-dimensional complex projective space. For a point \( x \in W \) choose a system of affine coordinates \((z_1, \ldots, z_N)\) on \( \mathbb{C}^N \subset \mathbb{P}^N \) with the origin at \( x \) and write down the equation of the hypersurface \( W \) as

\[
f = q_1 + q_2 + \ldots + q_m,
\]
where \( q_i(z_*) \) are homogeneous polynomials of degree \( i \).

**Definition 1.** The hypersurface \( W \) is regular at a smooth point \( x \in W \), if the sequence

\[
q_1, \ldots, q_k,
\]
\(k = \min\{m, N - 1\}\) is regular in \(\mathcal{O}_{x, \mathbb{P}^N}\), that is, the system of equations
\[q_1 = \ldots = q_k = 0\]
defines in \(\mathbb{P}^N\) an algebraic subset of codimension \(k\).

A dimension count, similar to the arguments of [P3, Sec. 1], shows that a general (in the sense of Zariski topology on \(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^N}(m))\)) hypersurface \(W\) is regular at each point.

Let
\[V = V_M \subset \mathbb{P} = \mathbb{P}^M\]
be a hypersurface of degree \(M\), with at most isolated singularities,
\[f = q_1 + q_2 + \ldots + q_M\]
its equation with respect to a system of affine coordinates \((z_1, \ldots, z_M)\) with the origin at \(x \in V\). Let
\[
\mu = \min\{k \in \mathbb{Z}_+ | q_k \neq 0\} = \text{mult}_x V
\]
be the multiplicity of \(V\) at the point \(x\). Assume that \(M - 2 \geq \mu \geq 2\), that is, \(x \in \text{Sing} V\).

**Definition 2.** The hypersurface \(V\) is regular at the point \(x\), if the following conditions are satisfied:

(i) the sequence \(q_\mu, \ldots, q_M\) is regular in \(\mathcal{O}_{x, \mathbb{P}}\);

(ii) the hypersurface \(T_x V = \{q_\mu = 0\} \subset T = \mathbb{P}(T_x \mathbb{P}) \cong \mathbb{P}^{M-1}\) is smooth and regular at each point \(y \in T_x V\);

(iii) for \(\mu = 3, 4\) and \(M \geq 7\) for any point \(y \in T_x V\) none of the irreducible components of the closed algebraic set
\[
\{q_\mu = q_{\mu+1} = \ldots = q_6 = 0\} \cap T_y(T_x V) \subset T
\]
is contained in the quadric hypersurface
\[T_y(T_y(T_x V) \cap T_x V) \subset T;\] (3)
for \(\mu = 3, M = 6\) it is sufficient that this condition holds with \(q_5\) instead of \(q_6\) in (2).

The condition (iii) should be explained, the more so that we somewhat abuse our notations: the symbol \(T_y(T_x V)\) stands for the hyperplane in \(T\), which is tangent to \(T_x V\) at the point \(y\). Since the hypersurface \(T_x V\) is regular, the intersection
\[T_y(T_x V) \cap T_x V\]
is a hypersurface in the hyperplane \(T_y(T_x V)\) with an isolated singular point of multiplicity 2. The closed set (2) has dimension \(\geq 1\), so that we require that none of its component is contained in the quadric (3). This condition can be formulated in a different way: the intersection of the cycle
\[T_x V \cap T_y(T_y(T_x V) \cap T_x V)\]
with the complete intersection

\[ \{ q_{\mu+1} = \ldots = q_6 = 0 \} \]

is of codimension precisely \(9 - \mu\) in \(T\).

**Proposition 2.** Let \(\mathcal{V}_\mu(x) \subset \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(M)))\) be the space of hypersurfaces of degree \(M \geq 5\), which have a singularity of multiplicity \(\mu\), \(2 \leq \mu \leq M - 2\) at the fixed point \(x \in \mathbb{P}\). The general (in the sense of Zariski topology) hypersurface \(V \in \mathcal{V}_\mu(x)\) is regular at each of its points.

Obviously, for a general \(V \in \mathcal{V}_\mu(x)\) we have \(\text{Sing} V = \{x\}\). Let us point out the following question: for which \(k\)-uples of integers \((\mu_1, \ldots, \mu_k) \in \{2, \ldots, M - 2\}^k\) there exists a hypersurface \(V\), which is regular at each of its points and has \(k\) points \(x_1, \ldots, x_k\) of multiplicities \(\mu_1, \ldots, \mu_k\), respectively? One can show [P5] that for \(\mu_i \equiv 2\) regular hypersurfaces exist for \(k \leq M + 1\), however it seems that the precise limit value of \(k\) is considerably higher.

### 0.3 The main result

The main result of the present paper is the following

**Theorem.** Assume that the hypersurface \(V\) is regular at each point. (i) If for any point \(x \in V\) the estimate \(\text{mult}_x V \leq M - 3\) holds, then \(V\) is a birationally superrigid variety. (ii) If \(x \in V\) is (the only) singular point of multiplicity \(M - 2\), then the projection from this point,

\[ \pi: V \longrightarrow \mathbb{P}^{M-1} \]

is of degree 2 and there exists a birational involution (the Galois involution) \(\tau \in \text{Bir} V\), which permutes the points in the fibers of \(\pi\). The variety \(V\) is birationally rigid and the exact sequence

\[ 1 \rightarrow \text{Aut} V \rightarrow \text{Bir} V \rightarrow \langle \tau \rangle = \mathbb{Z}/2\mathbb{Z} \rightarrow 1. \]

holds. For a general \(V\) obviously \(\text{Aut} V = \{1\}\), so that \(\text{Bir} V = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}\).

### 0.4 Earlier results on singular Fano varieties

The first example of a birationally rigid singular Fano 3-fold was made by the quartic \(V = V_4 \subset \mathbb{P}^4\) with a unique double point of general position \(x \in V\) [P1]. As in the case of arbitrary dimension, the projection

\[ \pi: V \longrightarrow \mathbb{P}^3 \]

from the point \(x\) is of degree 2 and determines the Galois involution \(\tau_x \in \text{Bir} V\). However, in dimension three the group of birational self-maps is much bigger. There are exactly 24 lines through the point \(x \in V\) on \(V\) (in the case of general position), \(L_1, \ldots, L_{24} \subset V\). Let \(L = L_i\) be one of them. The projection

\[ \pi_L: V \longrightarrow \mathbb{P}^2 \]
from this line fibers $V$ into elliptic curves. More exactly, for a general point $p \in \mathbb{P}^2$ the curve $C_p = \pi_L^{-1}(p)$ is a plane cubic, passing through the point $x$. Taking $x$ to be the zero of the group law on $C_p$, we get a birational involution:

$$
\tau_L: \quad V \dashrightarrow V,
\tau_L|_{C_p}: \quad z \mapsto -z.
$$

Set $\tau_0 = \tau_x$, $\tau_i = \tau_{L_i}$. The following fact is true [P1]:

The variety $V$ is birationally rigid. The involutions $\tau_i$, $i = 0, 1, \ldots, 24$, generate in Bir $V$ a subgroup $B(V)$ of finite index, which is their free product. The following exact sequence holds:

$$
1 \rightarrow B(V) = \langle \tau_i \rangle \rightarrow \text{Bir} V \rightarrow \text{Aut} V \rightarrow 1.
$$

Here the action of $\text{Aut} V$ on $B(V)$ is defined in the obvious way.

In [C] Corti essentially simplified the proof, using the Shokurov connectedness theorem [K] for exclusion of an infinitely near maximal singularity over the point $x$. Somewhat later Cheltsov noted that, in its turn, this argument of Corti’s can be simplified, if one applies Shokurov connectedness to the exceptional divisor $E \subset V_0 \rightarrow V$ of the blow up of the point $x$, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Namely, if the point $x$ is not maximal itself, but there is an infinitely near maximal singularity over it, then there is a linear system on $E$, say $\Sigma_E$ (possibly, with fixed components), of curves of type $(m, m)$ such that the log pair $(E, \frac{1}{m} \Sigma_E)$ is not log canonical. But this fact leads to a contradiction. In fact, this has already been proved in [P1], see the proof of the “Graph lemma”.

This way of arguing is used in the present paper when we consider a singular point of the maximal multiplicity $M - 2$.

Furthermore, in [P2] a series of birationally superrigid singular Fano varieties of arbitrary dimension was produced: double spaces of index 1 with a double point of general position. In [P5] singular Fano hypersurfaces $V = V_M \subset \mathbb{P}^M$ with non-degenerate double points were proved to be birationally rigid. Finally, Corti and Mella [CM] considered a larger class of quartic 3-folds with isolated double points.

In the paper [CPR] 95 families of weighted Fano 3-fold hypersurfaces $V_d \subset \mathbb{P}(a_0 = 1, a_1, a_2, a_3, a_4), \quad d = a_1 + \ldots + a_4$ were proved to be birationally rigid (honestly speaking, one should say 94 families, since the family number one in this list is exactly the family of smooth quartics $V \subset \mathbb{P}^4$, which were proved to be superrigid in [IM] 30 years ago, which made the starting point of the whole rigidity theory). The weighted Fano hypersurfaces have terminal factor-singularities. The present paper deals with hypersurface singularities only.

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1 Start of the proof

We prove the theorem by means of the method of maximal singularities, see [IM,P3,P4].

1.1 Maximal singularities

Following the traditional scheme of arguments, let us consider a linear system \( \Sigma \subset |nH| \) on \( V \), where \( H \in \text{Pic} \ V \) is the class of a hyperplane section.

The linear system \( \Sigma \) is assumed to be moving (that is, it has no fixed components).

**Definition 3.** A geometric discrete valuation \( \nu \in \mathcal{N}(V) \) is said to be a *maximal singularity* of the linear system \( \Sigma \), if the *Noether-Fano inequality* holds:

\[ \nu(\Sigma) > n \cdot \text{discrepancy}(\nu). \]

If \( V \) is not superrigid, then there exists a moving linear system with a maximal singularity.

Set \( B = \text{centre}(\nu) \subset V \) to be the centre of the maximal singularity, an irreducible subvariety of \( V \).

**Proposition 3.** \( B = x \in V \) is a singular point of the hypersurface \( V \) of multiplicity \( \mu \geq 3 \).

**Proof.** Assume that \( B \not\subset \text{Sing} \ V \). By [P3, Sec. 3] we can assert that \( \text{codim}_V B \geq 3 \) (otherwise take a projective curve \( C \subset B, C \cap \text{Sing} \ V = \emptyset \). For this curve we have the estimate \( \text{mult}_C \Sigma > n \). Since \( C \) is contained in the smooth part of \( V \), the arguments of [P3] work and give a contradiction.) Consequently, for the cycle

\[ Z = (D_1 \cdot D_2) \]

of scheme-theoretic intersection of general divisors \( D_1, D_2 \in \Sigma \) we get the estimate

\[ \text{mult}_B Z > 4n^2. \]

Now there exists a smooth point \( x \in B \). By the regularity condition the arguments of [P3] give the opposite estimate:

\[ \text{mult}_B Z \leq \frac{4}{M} \deg Z = 4n^2. \]

A contradiction. Therefore, \( B = x \in V \) is a singular point. It was shown in [P5] that the case \( \mu = \text{mult}_x V = 2 \) is impossible. Q.E.D. for the proposition.
1.2 Isolated singular point

Let $x \in V$ be a regular singular point of multiplicity $\mu \geq 3$, $\nu \in \mathcal{N}(V)$ a maximal singularity of the system $\Sigma \subset |nH|$, $x = \text{centre}(\nu)$,

$$\varphi_0: V_0 \to V$$

$$\bigcup E \to x$$

the blow up of the singular point, $E = E_0$ the exceptional divisor. Since $\text{Pic} V_0 = \mathbb{Z}H \oplus \mathbb{Z}E$, for the strict transform of the linear system $\Sigma$ on $V_0$ we get $\Sigma^0 \subset |nH - \nu E|$.

Recall that $E \subset \mathbb{P}^{M-1}$ is a regular hypersurface of degree $\mu \geq 3$.

**Proposition 4.**

(i) For $\mu \leq M - 3$ the divisor $E$ cannot make a maximal singularity.

(ii) For $\mu = M - 2$ the birational involution $\tau \in \text{Bir} V$ is defined by a linear system $|(M - 1)H - ME|$ on $V_0$. If the point $x$ is maximal for the system $\Sigma$, that is, $\nu_0 > n$, then $\nu_0/n \leq M/(M - 1)$ and $\tau_\ast \Sigma$ is a moving linear system on $V$, for which the point $x$ is not maximal.

**Proof.**

(i) Assume the converse: the point $x$ is maximal for the system $\Sigma$. Then $\nu_0 > (M - \mu - 1)n \geq 2n$. Let $D_1, D_2 \in \Sigma$ be two general divisors. For the effective cycle $Z = (D_1 \bullet D_2)$ of codimension 2 on $V$ we have the estimate

$$\text{mult}_x Z \geq (M - \mu - 1)^2 n^2 \mu > M n^2 = \deg Z$$

(since $(M - \mu - 1)^2 \mu > M$), which is impossible. The contradiction proves (i).

(ii) Consider $\tau$ as an element of $\text{Bir} V_0 \cong \text{Bir} V$. It is easy to see that outside an invariant closed subset of codimension 2 the involution $\tau$ is biregular on $V_0$ and its action on $\text{Pic} V_0$ is given by the formulas

$$\tau_\ast H = (M - 1)H - ME,$$

$$\tau_\ast E = \mu H - (\mu + 1)E.$$ If the point $x$ is maximal for $\Sigma$, that is, $\nu_0 > n$, then

$$\tau_\ast |nH - \nu_0 E| \subset |(n(M - 1) - \nu_0 \mu)H - (nM - \nu_0(\mu + 1))E|,$$

where $\mu = M - 2$. Obviousy, $nM - \nu_0(\mu + 1) \leq n(M - 1) - \nu_0 \mu$, so that the point $x$ is no more maximal for the system $\tau_\ast \Sigma$.

Set $T = T_x V \cap V$: this is an irreducible divisor on $V$. Obviously,

$$T \sim (M - 2)H - (M - 1)E.$$ If the system $\Sigma$ is moving then for a general divisor $D \in \Sigma$ the cycle $Z = (T \bullet D)$ is effective, so that we get:

$$\nu_0 \cdot \text{mult}_x T \leq \text{mult}_x Z \leq \deg Z$$

$$\nu_0(M - 1)(M - 2) \quad M(M - 2)n,$$

whence $\nu_0/n \leq M/(M - 1)$, as we claimed it to be. Q.E.D. for the proposition.
1.3 The crucial fact

Now the theorem follows from the following crucial fact.

**Proposition 5.** If the point \( x \in V \) is not maximal for the linear system \( \Sigma \), then there exists no maximal singularity \( \nu \in \mathcal{N}(V) \) such that \( x = \text{centre}(\nu) \).

Indeed, Proposition 5 means that if the point \( x \in V \) is not maximal for the linear system \( \Sigma \), then this system has no maximal singularities at all. And this is exactly birational (super)rigidity.

**Proof of Proposition 5 for** \( \mu = M - 2 \). Since the point \( x \) is not maximal, the existence of a maximal singularity over the point \( x \) implies the existence of a maximal singularity for the system \( \Sigma^0 \): in terms of the log-minimal model program, the pair \((V_0, \frac{1}{n} \Sigma^0)\) is not canonical on \( E \). By Shokurov connectedness theorem [K], the pair \((E, \frac{1}{n} \Sigma^0|_E)\) is not log-canonical. However, this is impossible [Ch]: the set \( Y \subset E \) where the pair \((E, \frac{1}{n} \Sigma^0|_E)\) is not log-canonical, cannot be of positive dimension by [P3] and cannot be purely zero-dimensional by [Ch](recall that the linear system \( \Sigma^0|_E \) is cut out on \( E \) by hypersurfaces of degree \( v_0 \leq n \), since we assumed that the point \( x \) is not maximal, so that the pair

\[
(E, \frac{1}{n} \Sigma^0|_E)
\]

is also not log-canonical). This contradiction completes the proof for \( \mu = M - 2 \).

The arguments of [Ch] extend the arguments of [ChP]k.

2 Infinitely near maximal singularities

2.1 Resolution of a maximal singularity

Recall the standard constructions and notations [P3,P4]. Let

\[
\varphi_{i,i-1} : V_i \to V_{i-1} \\
\bigcup E_i \to B_{i-1},
\]

\( i = 1, \ldots, K \), be the resolution of a valuation \( \nu \in \mathcal{N}(V) \), which is maximal for \( \Sigma \). Here the first \( L \) blow ups correspond to the cycles \( B_{i-1} \) of codimension \( \geq 3 \) (the lower part), whereas the following \( K - L \) blow ups correspond to the cycles \( B_{i-1} \) of codimension 2 (the upper part; it is possible that \( K = L \) and the upper part is empty). Set \( p_i = p_K \), to be the number of paths from \( E_K \) to \( E_i \), \( i = 0, \ldots, K \), in the oriented graph \( \Gamma \) of the valuation \( \nu \) (see [IM,P3,P4]). Set \( \delta_i = \text{codim } B_{i-1} - 1 \), \( i = 1, \ldots, K \), \( \delta_0 = M - \mu - 1 \).

For an irreducible subvariety \( Y \subset V \) of codimension 2 set

\[
m_i(Y) = \text{mult}_x Y, \quad m_{ij}(Y) = \text{mult}_{B_{i-1}} Y^{i-1},
\]

\( i = 1, \ldots, L \), where the upper index \( j \) means that we take the strict transform of the subvariety on \( V_j \). The following statement makes the technical base of the proof.
Proposition 6. If $\nu \in \mathcal{N}(V)$ is a maximal singularity of the system $\Sigma$, centre($\nu$) = $x$, then there exists an irreducible subvariety $Y \subset V$ of codimension 2, which satisfies the following estimate:

$$
\frac{2}{\mu} p_0 m(Y) + \sum_{i=1}^{L} p_i m_i(Y) > \left( \frac{\sum_{i=0}^{K} p_i \delta_i}{\frac{1}{2} p_0 + \sum_{i=1}^{K} p_i} \right)^2 \deg Y.
$$

(4)

Proof is given below in Sec. 3.

Remark. As it was shown in [P3, P4], it is possible to “correct” the coefficients $p_i$ in such a way that the estimate

$$
p_0 \leq \sum_{i=1}^{L} p_i
$$

holds, if only $L \geq 1$. In order to do this, it is sufficient to erase in the graph $\Gamma$ the arrows connecting $E_i$, $i \geq L + 1$, with $E = E_0$, if there are such arrows (otherwise there is nothing to prove). After this operation the Noether-Fano inequality becomes stronger, whereas the proof of Proposition 6 still holds. In what follows, if $L \geq 1$, then we assume that (5) is true without special comments.

Fix an irreducible subvariety $Y \subset V$ of codimension 2, which satisfies the estimate (4). Our aim is to get a contradiction and thus to show that the initial assumption that there is a maximal singularity $\nu \in \mathcal{N}(V)$ with the centre at the point $x$ is wrong. Birational rigidity of $V$ would be an immediate implication of that.

2.2 Simple examples

Proposition 7. $L \geq 1$.

Proof. Assume that $L = 0$. The estimate (4) takes the form of the inequality

$$
m(Y) > \frac{\mu((M - \mu - 1)p_0 + \Sigma_u)^2 \deg Y}{p_0(\frac{1}{2} p_0 + \Sigma_u) M}.
$$

Here for convenience $\Sigma_u = \sum_{i=L+1}^{K} p_i = \sum_{i=1}^{K} p_i$. By the definition of the integers $p_i$ we get an obvious estimate

$$
p_0 \leq \Sigma_u.
$$

It is easy to check that for each $s, t$ the following inequality holds:

$$
\frac{(2s + t)^2}{2s(\frac{2s}{2} + t)} \geq 3,
$$

whence we get the estimate

$$
\text{mult}_x Y > \frac{3\mu}{M} \deg Y.
$$

Let
\[ f_i = q_\mu + \ldots + q_i, \]
\( \mu \leq i \leq M, \) denote the left segment of the equation of the hypersurface \( V. \)

**Definition 4.** The linear system
\[ \Lambda_i = \left| \sum_{j=\mu}^{i} f_j s_{i-j} \right| V, \]
where \( s_k(z_*) \) stands for an arbitrary homogeneous polynomial of degree \( k \) in \( z_* \), is called the \( i \)-th hypertangent linear system at the point \( x \).

Obviously, for any divisor \( D \in \Lambda_i \) we get
\[
\frac{\text{mult}_x D}{\text{deg} D} \geq \frac{i+1}{i} \frac{\mu}{M}. \tag{6}
\]

By the regularity condition
\[
\text{codim}_V \text{Bs} \Lambda_i = i - \mu + 1, \tag{7}
\]
for \( i = \mu, \ldots, M - 1 \). Now let \( D_\mu, D_{\mu+1}, \ldots, D_{M-1} \) be general divisors of the hypertangent linear systems \( \Lambda_\mu, \ldots, \Lambda_{M-1}, \) respectively. It is easy to see that by (7) the set-theoretic intersection
\[
Y \cap D_{\mu+2} \cap D_{\mu+3} \cap \ldots \cap D_{M-1}
\]
is of pure codimension \( M - \mu \) in \( V \). Consider the effective cycle
\[
Y^* = (Y \bullet D_{\mu+2} \bullet D_{\mu+3} \bullet \ldots \bullet D_{M-1})
\]
of the corresponding scheme-theoretic intersection. By (6) we get the estimate
\[
\frac{\text{mult}_x Y^*}{\text{deg} Y^*} \geq \frac{3\mu}{M} \cdot \frac{\mu+3}{\mu+2} \cdot \ldots \cdot \frac{M}{M-1} = \frac{3\mu}{\mu+2} > 1,
\]
which is impossible. This contradiction proves Proposition 7.

Set
\[
R = \{ q_\mu = q_{\mu+1} = 0 \} \cap V.
\]
This is an irreducible cycle of codimension 2 (by the regularity condition). For a general \( V \) its strict transform \( \tilde{R} \) on \( V_0 \) is non-singular in a neighborhood of the exceptional divisor.

**Proposition 8.** \( Y \neq R. \)

**Proof.** The regularity condition implies that
\[
\text{mult}_x R = \frac{\mu+2}{M} \text{deg} R, \quad \text{mult}_{B_{i-1}} R^{i-1} \leq 1.
\]
Now if \( Y = R \), then by (6) we get
\[
2\frac{\mu+2}{\mu} p_0 + \Sigma_l > \frac{(2p_0 + 2\Sigma_l + \Sigma_u)^2}{2p_0 + \Sigma_l + \Sigma_u},
\]
where we set for convenience \( \Sigma_l = \sum_{i=1}^{L} p_i \). Elementary computations show that this inequality is false. Q.E.D. for the proposition.

The two examples, considered above (Propositions 7 and 8) are the simplest cases.

Let us study the general case.

### 2.3 Subvarieties of codimension 2

Let \( y \in E \subset \mathbb{P}^{M-1} \) be an arbitrary point on the exceptional divisor. The regularity condition gives that \( T_y^{(1)} = T_y E \cap E \) is a hypersurface (of degree \( \mu \)) in the hyperplane \( T_y E \cong \mathbb{P}^{M-2} \), with the point \( y \) as an isolated quadric singularity. Set \( T = T_y(T_y^{(1)}) \cap T_y^{(1)} \). This is an irreducible cycle of codimension 2 on \( E \). Obviously, \( \deg T = 2\mu \), \( \text{mult}_y T = 6 \).

**Lemma 1.** Let \( W \neq T \) be an irreducible subvariety of codimension 2 on \( E \). The following estimate holds for \( \mu \geq 4 \):

\[
\frac{\text{mult}_y W}{\deg W} \leq \frac{8}{3\mu}.
\]

**Proof.** Apply the technique of hypertangent systems to \( E \subset \mathbb{P}^{M-1} \). This is possible due to the regularity condition. More precisely, let \( (u_1, \ldots, u_{M-1}) \) be a system of linear coordinates on \( \mathbb{P}^{M-1} \) with the origin at the point \( y \),

\[
e(y) = \xi_1 + \xi_2 + \ldots + \xi_\mu
\]

the equation of the hypersurface \( E \), \( e_i = \xi_1 + \ldots + \xi_i \) its left segment. Here \( \xi_i \) are homogeneous of degree \( i \) in \( u_* \). Set

\[
\Delta_i = |\sum_{j=1}^{i} e_j s_{i-j}|_E,
\]

\( i = 1, \ldots, \mu - 1 \), where \( s_k \) is an arbitrary homogeneous polynomial of degree \( k \). Obviously, by the regularity condition

\[
\text{codim } Bs \Delta_i \geq i,
\]

so that for a general divisor \( D_i \in \Delta_i \) and an arbitrary subvariety \( B \subset E \) of codimension \( i - 1 \) we get \( B \not\subset D_i \). Since \( W \neq T = Bs \Delta_2 \), we get

\[
\text{codim } W \cap D_2 = 3,
\]

so that \( (W \cdot D_2) \) is an effective cycle of codimension 3 on \( E \). Therefore

\[
W \cap D_2 \cap D_4 \cap D_5 \cap \ldots \cap D_{\mu-1}
\]

is of codimension \( \mu - 1 \) on \( E \), so that

\[
W^* = (W \cdot D_2 \cdot D_4 \cdot D_5 \cdot \ldots \cdot D_{\mu-1})
\]
is an effective cycle on $E$. We obtain the estimate

$$1 \geq \frac{\text{mult}_y W^*}{\text{deg} W} \geq \frac{\text{mult}_y W}{\text{deg} W} \cdot \left( \frac{3}{2} \cdot \frac{5}{4} \cdot \ldots \cdot \frac{\mu}{\mu - 1} \right)$$

$$\parallel$$

$$\frac{3\mu}{8},$$

which immediately implies the lemma.

Lemma 2. Let $\mu = 3$. For any irreducible subvariety $W \neq T$ of codimension 2 we get

$$\frac{\text{mult}_y W}{\text{deg} W} \leq \frac{2}{3}.$$

Proof. In the notations of the proof of the previous lemma $\text{codim}(W \cap D_2) = 3$, so that $(W \cdot D_2)$ is an effective cycle of codimension 3 and

$$1 \geq \frac{\text{mult}_y (W \cdot D_2)}{\text{deg} (W \cdot D_2)} \geq \frac{\text{mult}_y W}{\text{deg} W} \cdot \frac{3}{2},$$

which is what we need. Q.E.D.

Let $Y^0 \subset V_0$ be the strict transform of the subvariety $Y$ and $(Y^0 \cdot E) = \mathbb{P}(T_x Y)$ its projectivized tangent cone at $x$. For the effective cycle $(Y^0 \cdot E)$ of codimension 2 on $E$ we get the presentation

$$(Y^0 \cdot E) = aT + W; \quad (8)$$

where $a \in \mathbb{Z}_+$ and the effective cycle $W$ does not contain $T$ as a component.

Lemma 3. $a \geq 1$.

Proof. Assume the converse: $a = 0$. Consider first the case $\mu \geq 4$. By Lemma 1 we get

$$\frac{m_1(Y)}{m(Y)} \leq \frac{\text{mult}_y (Y^0 \cdot E)}{\text{deg}(Y^0 \cdot E)} \leq \frac{8}{3\mu},$$

where $y \in B_0$ is an arbitrary point (since $m(Y) = \text{mult}_x Y = \text{deg}(Y^0 \cdot E)$ and $m_1(Y) \leq \text{mult}_y Y^0 \leq \text{mult}_y (Y^0 \cdot E)$). Thus taking into account the inequality $m_i(Y) \leq m_1(Y)$, we may replace $m_i(Y)$ in $(4)$ by $(8/3\mu)m(Y)$ for $i \geq 1$. Now from $(4)$ we obtain

$$\text{mult}_x Y > \frac{(2p_0 + 2\Sigma_l + \Sigma_u)^2}{(\frac{2}{\mu}p_0 + \frac{8}{3\mu}\Sigma_l)(\frac{1}{2}p_0 + \Sigma_l + \Sigma_u)} \cdot \frac{\text{deg} Y}{M}$$

$$\parallel$$

$$\frac{4p_0^2 + 4p_0(2\Sigma_l + \Sigma_u) + 4\Sigma_l(\Sigma_l + \Sigma_u) + \Sigma_u^2}{p_0^2 + 2p_0(\frac{2}{3}\Sigma_l + \Sigma_u) + \frac{2}{3}\Sigma_l(\Sigma_l + \Sigma_u)},$$

12
whence we get finally that
\[
\frac{\text{mult}_x Y}{\deg Y} > \frac{3}{2M\mu}.
\] (9)

On the other hand, arguing as in the proof of Lemma 1, we see that for general divisors \(D_i \in \Lambda_i\) the set-theoretic intersection
\[
Y \cap D_{\mu+2} \cap D_{\mu+3} \cap \ldots \cap D_{M-1}
\]
is of codimension precisely \(M - \mu\), so that taking the effective cycle
\[
Y^* = (Y \bullet D_{\mu+2} \bullet \ldots \bullet D_{M-1}),
\]
we get the estimate
\[
\frac{\text{mult}_x Y}{\deg Y} \leq \frac{\mu + 2}{M}.
\] (10)
Comparing this inequality with (9), we see that
\[
\frac{3}{2} \mu < \mu + 2,
\]
so that \(\mu < 4\): a contradiction.

Now assume that \(\mu = 3\). By Lemma 2, in this case
\[
\frac{m_1(Y)}{m(Y)} \leq \frac{2}{\mu},
\]
so that, arguing as above, we get the estimate
\[
\frac{\text{mult}_x Y}{\deg Y} > \frac{6}{M},
\]
so that the estimate (10) is true for \(\mu = 3\), either, so that we get a contradiction: \(6 < \mu + 2 = 5\). Q.E.D. for Lemma 3.

**Corollary 1** (from Lemma 3). \(Y \not\subset T_xV\).

**Proof.** Assume the converse: \(Y \subset T_xV\). Then we get \(Y \subset T_xV \cap V\) and therefore \(T_xY \subset T_x(T_xV \cap V)\). However, this is impossible, since \(P(T_xY)\) contains the subvariety \(T\) as a component, whereas
\[
P(T_x(T_xV \cap V)) = \{q_\mu = q_{\mu+1} = 0\} \subset P^{M-1}
\]
does not contain \(T\) by the regularity condition. Q.E.D. for the corollary.
2.4 The case $\mu \geq 5$

Now let us prove Proposition 5 for $\mu \geq 5$. For any irreducible subvariety $W \subset E$ of codimension 2 by Lemma 1 we get

$$\frac{\text{mult}_y W}{\deg W} \leq \frac{3}{\mu},$$

where the equality is attained at $W = T$ only. Arguing as above, we get from (14):

$$\frac{\text{mult}_x Y}{\deg Y} > \mu \frac{(2p_0 + 2\Sigma_l + \Sigma_u)^2}{M(2p_0 + 3\Sigma_l)(\frac{1}{2}p_0 + \Sigma_l + \Sigma_u)} \geq \frac{4}{3M^\mu}. \quad (11)$$

Since $Y \not\subset T_x V$, the intersection $Y \cap D_\mu$ is of codimension 3, so that by the regularity condition

$$Y^* = (Y \bullet D_\mu \bullet D_{\mu+3} \bullet \ldots \bullet D_{M-1})$$

is an effective cycle of codimension $M - \mu$. Now we get

$$1 \geq \frac{\text{mult}_x Y^*}{\deg Y^*} \geq \frac{\text{mult}_x Y}{\deg Y} \cdot \left( \frac{\mu + 1}{\mu} \cdot \frac{\mu + 4}{\mu + 3} \ldots \cdot \frac{M}{M - 1} \right) \parallel \frac{(\mu + 1)M}{\mu(\mu + 3)}, \quad (12)$$

so that combining (11) and (12) we obtain the estimate

$$\frac{\mu(\mu + 3)}{\mu + 1} > \frac{4}{3^\mu},$$

whence $\mu < 5$: a contradiction. Proposition 5 is proved for $\mu \geq 5$.

2.5 The harder cases $\mu = 3$ and 4

There are two hardest cases left: $\mu = 3$ and $\mu = 4$. We will do the second case in full detail. Here one should employ more delicate arguments than those above.

Recall that $T_x Y$ contains $T$ as a non-trivial component and thus $Y \not\subset D_\mu$, as above. The more so $Y \not\subset D_{\mu+1}$ for a general divisor $D_{\mu+1} \in \Lambda_{\mu+1}$. However by the regularity condition one can say more: the intersection

$$T \cap \{ q_{\mu+1} = q_{\mu+2} = 0 \}$$

is of codimension 2 in $T$. In particular, the linear system $\Lambda_{\mu+1}^0|_T$ has no fixed components. Thus none of the components of the closed algebraic set $D_{\mu+1}^0 \cap T$ is contained in the support of the cycle $W$ (8). Set $Y_{\mu+1} = (Y \bullet D_{\mu+1})$. This is an effective cycle of codimension 3 on $V$. We get the following presentation:

$$Y_{\mu+1} = Y_{\mu+1}^y + Y_{\mu+1}^z.$$
where an irreducible component \( X \) of the cycle \( Y^{\mu+1} \) comes into \( Y^{\mu+1} \) (and does not come into \( Y^{\mu+1} \)) when and only when its strict transform \( X^0 \subset V_0 \) contains an irreducible component of the set \( (D_0^{\mu+1} \cap T) \). By what was said above,

\[
(\tilde{Y}_i^{\mu+1} \cdot E) = a^\sharp(T \cdot D_0^{\mu+1}) + (\sharp),
\]

here \( a^\sharp \geq a \geq 1 \). For the cycle \( Y^{\mu+1} \) we get the estimate

\[
\frac{\text{mult}_x Y^{\mu+1}}{\deg Y^{\mu+1}} \leq \frac{\mu + 3}{M}, \tag{13}
\]

which is obtained in the usual way. However, one can say much more about the cycle \( Y^{\mu+1} \): by construction

\[
Y_i^{\mu+1} \not\subset D_1.
\]

Consequently, \((Y_i^{\mu+1} \cdot D_0)\) is an effective cycle of codimension 4, so that we get

\[
\frac{\text{mult}_x Y_i^{\mu+1}}{\deg Y_i^{\mu+1}} \leq \frac{\mu}{\mu + 1} \cdot \frac{\text{mult}_x (Y_i^{\mu+1} \cdot D_0)}{\deg Y_i^{\mu+1}} \leq \frac{\mu(\mu + 4)}{(\mu + 1)M}. \tag{14}
\]

Now set

\[
\begin{align*}
d^\sharp &= \deg Y_i^{\mu+1}, \\
d^+ &= \deg Y^{\mu+1}, \\
b^\sharp &= a \deg T, \\
b^+ &= \deg W,
\end{align*}
\]

\[
\begin{align*}
\deg(\tilde{Y}_i^{\mu+1} \cdot E) &= b^\sharp(\mu + 2) + \delta^\sharp, \\
\deg(Y_i^{\mu+1} \cdot E) &= \delta^+.
\end{align*}
\]

We get a system of inequalities,

\[
(b^\sharp + b^+)(\mu + 2) \leq (\mu + 2)b^\sharp + \delta^\sharp + \delta^+, \leq (\mu + 2)b^\sharp + \delta^\sharp \leq d^\sharp \frac{\mu(\mu + 4)}{M(\mu + 1)},
\]

\[
\delta^+ \leq d^+ \frac{\mu + 3}{M},
\]

where

\[
d^+ + d^\sharp = (\mu + 1) \deg Y,
\]

and

\[
b^+ + b^\sharp = \text{mult}_x Y = m_0.
\]

Note first of all that since the inequality (14) is stronger than (13), we may assume that \( \delta^\sharp = 0 \): otherwise replace \( \delta^+ \) by \( \delta^+ + \delta^\sharp \), \( \delta^\sharp \) by 0, \( d^\sharp \) by

\[
d^\sharp - \delta^\sharp \frac{M(\mu + 1)}{\mu(\mu + 4)}
\]

15
and $d^+$ by
\[ d^+ + \delta^+ \frac{M(\mu + 1)}{\mu(\mu + 4)}. \]

All the inequalities above are still true since
\[ \delta^2 \leq \delta^+ \frac{(\mu + 1)(\mu + 3)}{\mu(\mu + 4)}. \]

Furthermore, by Lemma 1 for $m_1(Y)$ we have the following estimate:
\[ m_1(Y) \leq \frac{3}{\mu} b^2 + \frac{8}{3 \mu} b^+. \]

Taking into account (4), this implies
\[ b^+ \left( \frac{2}{\mu} p_0 + \frac{3}{\mu} \Sigma_l \right) + b^+ \left( \frac{2}{\mu} p_0 + \frac{8}{3 \mu} \Sigma_l \right) > \frac{(2 p_0 + 2 \Sigma_l + \Sigma_u)^2 \deg Y}{(\frac{1}{2} p_0 + \Sigma_l + \Sigma_u) M}. \]

Using the estimates, obtained above, we get now
\[ \frac{\mu(\mu + 4)}{(\mu + 1)(\mu + 2)} d^+ \left( \frac{2}{\mu} p_0 + \frac{3}{\mu} \Sigma_l \right) + \frac{\mu + 3}{\mu + 2} d^+ \left( \frac{2}{\mu} p_0 + \frac{8}{3 \mu} \Sigma_l \right) > \frac{(2 p_0 + 2 \Sigma_l + \Sigma_u)^2 \left( d^+ + d^+ \right)}{(\frac{1}{2} p_0 + \Sigma_l + \Sigma_u) (\mu + 1)}. \]

By linearity, either
\[ \frac{\mu + 4}{\mu + 2} > \frac{(2 p_0 + 2 \Sigma_l + \Sigma_u)^2}{(2 p_0 + 3 \Sigma_l)(\frac{1}{2} p_0 + \Sigma_l + \Sigma_u)} \geq \frac{4}{3}, \]

whence $\mu < 4$ — a contradiction, or
\[ \frac{(\mu + 3)(\mu + 1)}{\mu(\mu + 2)} > \frac{(2 p_0 + 2 \Sigma_l + \Sigma_u)^2}{(2 p_0 + \frac{8}{3} \Sigma_l)(\frac{1}{2} p_0 + \Sigma_l + \Sigma_u)} \geq \frac{3}{2}, \]

whence $\mu^2 < 2 \mu + 6$ — a contradiction again. The case $\mu = 4$ is completed.

Note that the estimates which we obtained above are sufficient to exclude the case of a point of multiplicity $\mu = 3$ on the sextic 5-fold. If $M \geq 7$ and $\mu = 3$, then to prove Proposition 5, one should start with the cycle $(Y \bullet D_{\mu+1})$, then look at those components of this cycle which contain components of the cycle $(T \cap D_0^{\mu+1})$. Then one should intersect these components with $D_{\mu+2}$ and take those components of the intersection which contain components of the cycle $(T \cap D_0^{\mu+1} \cap D_0^{\mu+2})$. Finally, one should intersect them with $D_\mu$ (this is still possible by the regularity condition). The estimates, obtained by means of these manipulations, are already strong enough to exclude the case $\mu = 3$. The corresponding computations are rather tiresome and for this reason we do not give them here in detail.

Q.E.D. for Proposition 5 and for the main theorem.
3 The technique of counting multiplicities

In this section we generalize the technique of counting multiplicities [P3,P4] to certain classes of singularities.

3.1 A sequence of blow ups

Let $x \in X$ be a germ of an isolated terminal $\mathbb{Q}$-factorial singularity and

$$\varphi_{i,i-1} : X_i \to X_{i-1}$$

$$\bigcup \bigcup \ E_i \to B_{i-1}$$

$i = 1, \ldots, K$, a sequence of blow ups with centres $B_{i-1} \subset X_{i-1}$, where $B_0 = x$. Let $E_i = \varphi_{i,i-1}^{-1}(B_{i-1}) \subset X_i$ be the exceptional divisors. We assume that the following conditions hold:

(i) $\varphi_{i,i-1}(B_i) = B_{i-1}$, that is, $B_i \subset E_i$;
(ii) the exceptional divisors $E_i \subset X_i$ are irreducible, reduced and $X_i$ is $\mathbb{Q}$-factorial over a general point of the cycle $B_{i-1}$.

Set $\delta_i = \text{codim } B_{i-1} - 1$. Obviously, we get

$$\left(F_i \cdot (-E_i)^{\delta_i}\right) = \mu_i \geq 1,$$

where

$$\mu_i = \text{mult}_{B_{i-1}} V_{i-1},$$

$F_i$ is a fiber of the morphism $\varphi_{i,i-1} : E_i \to B_{i-1}$.

For a cycle $Y_i \subset X_i$ we denote by the symbol $Y_j \subset X_j$ its strict transform when it is well defined. On the set of exceptional divisors $\{E_i\}$ we define a structure of an oriented graph in the usual way:

$$E_j \to E_i \quad \text{or} \quad i \to j,$$

if $j > i$ and $B_{j-1} \subset E_{j-1}^{-1} [\text{IM,P1-P5}]$. For $j \to i$ we set

$$\beta_{i,j} = \sup_{Y \subset E_j} \frac{\text{mult}_{B_{j-1}} Y^{i-1}}{\deg Y} \in \mathbb{R}_+,$$

where sup is taken over all the prime divisors $Y \subset E_i$, covering $B_{j-1}$ (on the other hand, if $\varphi_{j,j-1}(Y) \neq B_{j-1}$, then $\text{mult}_{B_{j-1}} Y^{i-1} = 0$), and

$$\deg Y = (Y \cdot F_j \cdot (-E_j)^{\delta_j-1})$$

is the “degree” of the intersection $Y \cap F_i$, $F_i = \varphi_{i,i-1}^{-1}(s)$, $s \in B_{i-1}$ is a general point.

For a path $\pi \in P(i,j)$, connecting $i$ with $j$, we define its weight to be

$$\beta(\pi) = \prod_{\alpha=1}^{k} \beta_{i_\alpha,j_{\alpha-1}}.$$
where \( \pi = \{ i = i_k \to i_{k-1} \to \ldots \to i_\alpha \to i_{\alpha-1} \to \ldots \to i_0 = j \} \). We define the coefficients \( w_{i,j} \) by the formula

\[
w_{i,j} = \sum_{\pi \in P(i,j)} \beta(\pi), \quad w_{i,i} = 1.
\]

**Lemma 4.** The following equality holds

\[
w_{i,j} = \sum_{k \to j} w_{i,k} \beta_{k,j}.
\]

**Proof.** Take the disjoint union

\[
P(i, j) = \coprod_{k \to j} P(i, k) \circ \{ k \to j \},
\]

where \( \circ \{ k \to j \} \) means the extension of a path from \( i \) to \( k \) to a path from \( i \) to \( j \) by adding the arrow \( k \to j \). Now the claim of the lemma is obvious by the definition of the numbers \( w_{i,j} \).

### 3.2 The self-intersection of a linear system

Now take linear system \( \Sigma \) on \( X \) without fixed components, and set \( \Sigma^i \) to be its strict transform on \( X_i \), \( D \in \Sigma \) its general divisor. We get

\[
D^i = \varphi^*_i(D^{i-1}) - \nu_i E_i,
\]

so that

\[
D^K = \varphi^*_K(D) - \sum_{i=1}^K \nu_i \varphi^*_K E_i.
\]

Let \( D_1, D_2 \in \Sigma \) be two general divisors. Define a sequence of cycles of codimension two on \( X_i \), setting

\[
D_1 \bullet D_2 = Z_0, \\
D_1^1 \bullet D_2^2 = Z_0^1 + Z_1, \\
\ldots, \\
D_1^i \bullet D_2^i = (D_1^{i-1} \bullet D_2^{i-1})^i + Z_i, \\
\ldots,
\]

where \( Z_i \subset E_i \). From this presentation we get for \( i \leq L \), where \( L \) is defined by the condition \( \text{codim } B_{i-1} \geq 3 \) for \( i \leq L \):

\[
D_1^i \bullet D_2^i = Z_0^i + Z_1^i + \ldots + Z_{i-1}^i + Z_i.
\]

For any \( j > i, j \leq L \) set

\[
m_{i,j} = \text{mult}_{B_{j-1}}(Z_{i}^{j-1}).
\]

Set also

\[
d_i = \deg Z_i = (Z_i \cdot F_i \cdot (-E_i)^{i-1}).
\]
We get the following system of equalities:

\[
\begin{align*}
\mu_1\nu_1^2 + d_1 &= m_{0,1}, \\
\mu_2\nu_2^2 + d_2 &= m_{0,2} + m_{1,2}, \\
&\vdots \\
\mu_i\nu_i^2 + d_i &= m_{0,i} + \ldots + m_{i-1,i}, \\
&\vdots \\
\mu_L\nu_L^2 + d_L &= m_{0,L} + \ldots + m_{L-1,L}.
\end{align*}
\]

Now

\[d_L \geq \sum_{i=L+1}^K \mu_i\nu_i^2.\]

Multiply the \(i\)-th equation by \(w_{L,i}\) and put them all together. In the right-hand part for each \(i \geq 1\) we get the expression

\[\sum_{j=i+1}^L w_{L,j}m_{i,j} = \sum_{j=i}^L w_{L,j}m_{i,j}.\] (15)

However, by the definition of the numbers \(\beta_{j,i}\) we have the estimate

\[m_{i,j} \leq \beta_{j,i}d_i,\]

so the (15) can be bounded from above by the number

\[d_i \sum_{j \rightarrow i}^i w_{L,j}\beta_{j,i} = d_i w_{L,i}.\]

In the left-hand part for each \(i \geq 1\) we see \(d_i w_{L,i}\), so that, throwing away all the \(m_{i,*}, i \geq 1\), from the right-hand part and all the \(d_i, i \geq 1\), from the left-hand part, we get finally:

\[\sum_{j=1}^L w_{L,j}m_{0,j} \geq \sum_{j=1}^L w_{L,j}\mu_j\nu_j^2 + \sum_{i=L+1}^K \mu_i\nu_i^2.\] (16)

### 3.3 Proof of Proposition 6

Let us come back to the singular point \(x \in V\), considered in the present paper. We obviously get

\[w_{i,j} = 1 \quad \text{for} \quad i, j \geq 1,\]

where, in accordance with the notations, which we use in this paper, the sequence of blow ups \(\varphi_{i,i-1}\) starts with \(\varphi_1\), and not with \(\varphi_{1,0}\). For any divisor \(Y \subset E\) and a point \(y \in Y\) we get the estimate

\[\frac{\text{mult}_y Y}{\deg Y} \leq \frac{2}{\mu},\]
where the equality is attained at the divisor $T_y E \cap E$ only. Indeed, by the regularity condition for general divisors $R_i \in \Delta_i$ of the hypertangent linear systems on $E$ we get for $Y \neq R_1 = T_y E \cap E$: the intersection

$$Y \cap R_1 \cap R_3 \cap \ldots \cap R_{\mu - 1}$$

is of codimension precisely $\mu - 1$ on $E$, whence it follows that the cycle

$$Y^* = (Y \cdot R_1 \cdot R_3 \cdot \ldots \cdot R_{\mu - 1})$$

is effective, so that

$$1 \geq \frac{\text{mult}_y Y^*}{\deg Y} \geq \frac{\text{mult}_y Y}{\deg Y} \cdot \left( \frac{2}{1} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{\mu}{\nu - 1} \right)$$

and thus

$$\frac{\text{mult}_y Y}{\deg Y} \leq \frac{3}{2\mu} < \frac{2}{\mu}.$$ 

Consequently, $\beta_{i,0} \leq 2/\mu$ for all $i \to 0$. Now we get from (16):

$$\frac{2}{\mu} p_{L,0} m_0 + \sum_{i=1}^{L} p_{L,i} m_i \geq 2p_0 \nu_0^2 + \sum_{i=1}^{L} p_{L,i} \nu_i^2 + \sum_{i=L+1}^{K} \nu_i^2. \quad (17)$$

By the definition of the integers $p_{i,j}$ the estimate (17) implies the inequality

$$\frac{2}{\mu} p_{0} m_0 + \sum_{i=1}^{K} p_{i} \nu_i \geq 2p_0 \nu_0^2 + \sum_{i=1}^{K} p_{i} \nu_i^2. \quad (18)$$

where $p_i = p_{K,i}$. Let us get a lower bound for the right-hand part of (18). By the Noether-Fano inequality we get

$$\sum_{i=0}^{K} p_{i} \nu_i > n \left( \sum_{i=0}^{K} p_{i} \delta_i \right). \quad (19)$$

Since

$$\sum_{i=0}^{K} \inf_{p_{i} \nu_{i}=C} \{2p_0 \nu_0^2 + \sum_{i=1}^{K} p_{i} \nu_i^2\} = \frac{C^2}{\frac{1}{2}p_0 + \sum_{i=1}^{K} p_{i}},$$

we get finally (taking into consideration that $\deg Z = M n^2$, $Z = (D_1 \cdot D_2)$):

$$\frac{2}{\mu} p_{0} \text{mult}_{x} Z + \sum_{i=1}^{L} p_{i} \text{mult}_{B_{i-1}} Z^{i-1} > \frac{\left( \sum_{i=0}^{K} p_{i} \delta_i \right)^2}{\frac{1}{2}p_0 + \sum_{i=1}^{K} p_{i}} \cdot \frac{\deg Z}{M}.$$ 

It remains to note that this inequality is linear in $Z$. Therefore, there exists an irreducible component $Y$ of this cycle which satisfies (4).

Q.E.D. for Proposition 6.
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