Abstract—The infinite-horizon optimal control problem with stability in the presence of single-input, input-quadratic nonlinear systems is addressed and tackled in this article. In addition, it is shown that similar ideas can be extended to study the property of passivity of the underlying input-quadratic system from a given output. The constructive design of the optimal solution revolves around the interesting fact that the property of optimality of the closed-loop underlying system is shown to be locally equivalent to the property that an input-affine system possesses an $L_2$-gain less than one from a virtual disturbance signal. The global version of the statement requires a technical condition on the graph of the storage function of the latter auxiliary plant, and hence leads to the new notion of graphical storage function. Finally, the theory is corroborated by the application to the optimal control of the movable plane positioning in micromechanical systems actuators.

Index Terms—Backstepping, control Lyapunov functions (CLFs).

I. INTRODUCTION

ALTHOUGH of paramount importance in control theory, the stabilization and regulation tasks of dynamical systems may be significantly complicated by the presence of nonlinearities in the underlying vector fields. To address this aspect, a plethora of techniques and methodologies have been envisioned for the specially structured class of control systems that are nonlinear with respect to the state of the plant though “linear” in the controlled input, namely the so-called input-affine systems (see, e.g., [7], [8], [22] for a survey on such results). Such context has been then typically extended in the literature directly to the class of systems characterized by vector fields that are simultaneously generically nonlinear in the state and in the control input, thus losing the particularly interesting structural insight gained for input-affine systems. On the contrary, the literature concerning the more general with respect to input-affine systems—and practically motivated—class of plants in which the control input appears quadratically in the model is rather limited, due to the additional challenges arising from the input nonlinearity.

In particular, most of the proposed results hinge upon the notion and the use of control Lyapunov functions (CLFs) for the plant, thus partially extending the seminal work of [1] in the general case and [19] in the context of affine nonlinear systems. Specifically, the problem of stabilizing input-quadratic systems via CLFs has been considered in [13], [25] and, in the general case of nonaffine models, in [9], [10], and in [14] by relying on the assumption of Lyapunov stability of the unforced system and on passivity theory of nonaffine systems. Moreover, [15] provided a sufficient condition for the existence of a (piecewise) continuous stabilizing control input for nonaffine systems in terms of certain convexity properties of the underlying CLF. Differently from the previous results based on CLFs, the design proposed in [24] relies on the Immersion and Invariance (I&I) technique. The interest in the class of input-quadratic systems is motivated by several practical applications, encompassing, for instance, magnetic systems [15] and microelectromechanical systems (MEMS), based on electromagnetic or electrostatic actuation forces [17]. Moreover, quadratic inputs may appear in intermediate steps of the popular backstepping stabilizing procedure, see, e.g., [2] for the ball and beam example or [5] for the transient stabilization problem in multimachine power systems.

The main objective of this article is to define and tackle an infinite-horizon optimal control problem, formulated with respect to a quadratic cost functional and a single-input, input-quadratic nonlinear system. Moreover, the optimal control input has to ensure asymptotic stability of the zero equilibrium of the closed-loop system under a natural detectability assumption. The constructive design of the stabilizing control law is based on the fact, interesting per se, that the property of optimality of the underlying closed-loop system is shown to be equivalent to the property that an auxiliary input-affine system possesses $L_2$-gain less than one, hence providing a characterization of the optimal control input in terms of dissipativity theory for input-affine systems, differently from [10] that relies on dissipativity properties of general nonlinear systems. A similar approach is then employed to assess whether the underlying input-quadratic system is passive from a given output function, affine with respect to the control input. The theoretical results are then validated and illustrated by tackling the optimal regulation problem of MEMS.
The rest of the article is organized as follows. In Section II, a few basic definitions and results concerning input-affine dissipative systems are briefly reviewed, while the definition of the problem of interest is given in Section III. The main results of the article—showing the equivalence between the properties of optimality of the original input-quadratic system and of dissipativity of an auxiliary input-affine one—are the topics of Section IV, while the special case of a dynamical system linear in the state and quadratic in the control input together with a quadratic cost functional, i.e., the Linear–Quadratic–Quadratic Regulator (LQQR), is dealt with in Section IV-B. The property of passivity of input-quadratic systems is studied in Section V. Finally, the optimal regulation of the movable plane in MEMS actuators is dealt with in Section VI.

II. NOTATION AND PRELIMINARIES

The aim of this section consists in briefly reviewing a few basic definitions and results that are instrumental for the following derivations. Toward this end, consider a nonlinear, input-affine system described by equations of the form

\[ \dot{x} = f(x) + g(x)v + p(x)w \]  

(1)

where \( x(t) \in \mathcal{X} \subseteq \mathbb{R}^n \) denotes the state of the system, \( v(t) \in \mathcal{V} \subseteq \mathbb{R}^m \) is a control input, and \( w(t) \in \mathcal{W} \subseteq \mathbb{R}^s \) is an exogenous signal. Assume that \( f, g, i = 1, \ldots, m \) and \( p, j = 1, \ldots, s \) are smooth vector fields mapping \( x \mapsto T_x \mathcal{X} \), where \( T_x \mathcal{X} \) denotes the tangent space to \( \mathcal{X} \) at \( x \). Moreover, let \( y(t) \in \mathcal{Y} \subseteq \mathbb{R}^q \) denote a (measured) output of system (1).

Assumption 1: The origin of \( \mathbb{R}^n \), contained in \( \mathcal{X} \), is an equilibrium point of system (1) with \( v(t) = 0 \) and \( w(t) = 0 \), for all \( t \geq 0 \), namely \( f(0) = 0 \). Given a continuous function \( V : \mathbb{R}^n \to \mathbb{R}_+ \), the following definitions and properties are employed in the rest of the article. The notation \( V : \mathbb{R}^n \to \mathbb{R}_+ \) is used to denote a positive definite function around the origin, namely a function such that \( V(0) = 0 \), while \( V(x) \) is strictly positive for any \( x \neq 0 \).

Moreover, the notation \( V : \mathbb{R}^n \to \mathbb{R}_+ \) is used to denote a positive semi-definite function, namely such that \( V(x) \geq 0 \) for any \( x \in \mathbb{R}^n \). Finally, a function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) is proper if for each \( c > 0 \) the set \( \{ x \in \mathbb{R}^n : 0 \leq V(x) \leq c \} \) is compact.

Definition 1: A nonlinear system \( \dot{x} = f(x) + g(x)v \) with output \( y \) is said to be zero-state detectable from the output \( y = k_1(x) \) if for any trajectory such that \( v(t) \equiv 0 \), the condition \( y(t) \equiv 0 \) implies that \( \lim_{t \to \infty} x(t) = 0 \).

Assuming initially that \( v(t) = 0 \), for all \( t \geq 0 \), consider, on the space \( \mathbb{R}^s \times \mathbb{R}^q \) of the external variables of (1), a function \( s : \mathbb{R}^s \times \mathbb{R}^q \to \mathbb{R}_+ \) referred to as the supply rate.

Definition 2: [22] A system (1), with \( v(t) \equiv 0 \), is said to be dissipative with respect to the supply rate \( s \) if there exists a function \( V : \mathbb{R}^n \to \mathbb{R}_+ \), called storage function, such that for all \( x_0 \in \mathcal{X}, T \geq 0 \) and input functions \( w \)

\[ V(x(T)) \leq V(x(0)) + \frac{1}{2} \int_0^T s(w(\tau), y(\tau))d\tau \]  

(2)

with \( x(0) = x_0 \). Moreover, if (2) holds with the equality sign, then system (1) is lossless with respect to \( s \).

As extensively discussed in [22], the dissipation inequality (2) (see also [23]) entails that the stored energy, \( V(x(T)) \), of the system at any future time is upper-bounded by the sum of the initial stored energy, \( V(x_0) \), and that supplied by the function \( s \). Alternative choices of the latter supply rate lead to several interesting properties of (1), encompassing in particular the notions of \( \mathcal{L}_2 \)-gain and that of passivity, as reported below for completeness.

Definition 3: [22] Let \( \gamma > 0 \). System (1), with \( v(t) \equiv 0 \), has \( \mathcal{L}_2 \)-gain less than or equal to \( \gamma \) if it is dissipative with respect to the supply rate \( s(w, y) = \gamma^2 \|w\|^2 - \|y\|^2 \).

By considering \( x_0 = 0 \) and by specializing the dissipation inequality (2) to the supply rate of Definition 3, it follows that

\[ \int_0^T \|w(\tau)\|^2d\tau \leq \gamma^2 \int_0^T \|w(\tau)\|^2d\tau \]  

for any exogenous signal \( w(t) \in \mathbb{R}^s \), with \( t \in [0, T] \), and for which \( \int_0^T \|w(\tau)\|^2d\tau \) is bounded.

Definition 4: [22] System (1), with \( w(t) \equiv 0 \) and \( V = \mathcal{Y} \subseteq \mathbb{R}^q \), is passive if it is dissipative with respect to the supply rate \( s(v, y) = v^Ty \).

The following classical result relates the possibility of imposing via feedback a dissipativity property—namely a desired \( \mathcal{L}_2 \)-gain—to the existence of a solution to a certain first-order quadratic partial differential equation, the so-called Hamilton–Jacobi (HJ) equation, thus also providing a differential characterization of dissipativity as in Definitions 2 and 3.

Proposition 1: [21] Consider the nonlinear system (1) and let \( \gamma > 0 \). Suppose that there exists a smooth solution \( V : \mathbb{R}^n \to \mathbb{R}_+ \) of the HJ equation

\[ 0 = V_x f(x) + \frac{1}{2} (x^c) V_x^c \]

\[ + \frac{1}{\gamma^2} V_x \left( \frac{1}{\gamma^2} p(x)p(x)^\top - g(x)g(x)^\top \right) V_x^x \]

\[ (3) \]

with \( V(0) = 0 \). Then, system (1) in closed loop with \( v = -g(x)^\top V_x^x(x) \) has \( \mathcal{L}_2 \)-gain less than or equal to \( \gamma \) from the input \( w \) to the output \( [y, V] \), with \( V \) as a storage function.

Finally, an infinitesimal characterization also of passivity has been provided in [4], [6] for a system as in (1) with output \( y = k_1(x) \), \( k_1 : \mathbb{R}^n \to \mathbb{R}^q \) continuous function.

Proposition 2: [6] System (1) with output \( y = k_1(x) \), with \( w(t) \equiv 0 \) and \( V = \mathcal{Y} \subseteq \mathbb{R}^q \), is passive with a continuously differentiable storage function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) if and only if it satisfies the Kalman–Yacubovich–Popov (KYP) property, i.e.,

\[ V_x f(x) \leq 0 \]

\[ V_x g(x) = k_1(x)^\top \]

(4)

for any \( x \in \mathcal{X} \).

It is worth stressing, however, that the conditions underlying the KYP property are inevitably related to the affine dependence of system (1) on the control input \( v \) and derive from the requirement that the time derivative of the storage function \( V \) along the trajectories of the system satisfies the inequality

\[ V = V_x f(x) + V_x g(x) y \leq v^T y \]

for any control input \( v(t) \) and any \( t \geq 0 \).

III. PROBLEM FORMULATION AND INTERPRETATION

As mentioned above, the aim of this article consists in providing a comprehensive study of properties similar to those recalled in Section II in the context of input-quadratic nonlinear systems. Therefore, consider a nonlinear system, quadratic in the control variables.
input, described by equations of the form

\[ \dot{x} = f(x) + g(x)u + \frac{h(x)}{2}u^2 \]  (5a)
\[ y = k_1(x) + k_2(x)u \]  (5b)

with \( x(0) = x_0 \), where \( x(t) \in \mathbb{R}^n \) denotes the state of the system, \( u(t) \in \mathbb{R} \) is the control input, and \( y(t) \in \mathbb{R} \) is a measured, and possibly regulated, output of (5a). The vector fields \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( g : \mathbb{R}^n \to \mathbb{R}^n \), and \( h : \mathbb{R}^n \to \mathbb{R}^n \) and the functions \( k_1 : \mathbb{R}^n \to \mathbb{R} \) and \( k_2 : \mathbb{R}^n \to \mathbb{R} \) are assumed to be sufficiently smooth. Throughout the article, we assume that \( f(0) = 0 \), namely that the nonlinear system (5a) possesses an equilibrium at the origin with \( u = 0 \). The general formulation of the control task dealt within this article is provided below together with a few remarks concerning the relation between such general statement and notable properties, such as optimality and passivity for input-quadratic systems.

**Problem 1:** Consider the nonlinear, input-quadratic, system (5) together with the cost functional

\[ J_{x_0}(u) = \frac{1}{2} \int_0^\infty \left[ \begin{array}{c} y(\tau) \\ u(\tau) \end{array} \right]^\top S(x) \left[ \begin{array}{c} y(\tau) \\ u(\tau) \end{array} \right] d\tau \]

\[ = \frac{1}{2} \int_0^\infty \left[ \begin{array}{c} y(\tau) \\ u(\tau) \end{array} \right]^\top \left[ \begin{array}{c} s_{11}(x) \\ s_{12}(x) \\ s_{21}(x) \\ s_{22}(x) \end{array} \right] \left[ \begin{array}{c} y(\tau) \\ u(\tau) \end{array} \right] d\tau \]  (6)

with \( s_{ij} : \mathbb{R}^n \to \mathbb{R} \), \( i = 1, 2 \), \( j = 1, 2 \), continuous functions. The design problem then consists in determining a state-feedback control \( u^* = \alpha(x), \alpha : \mathbb{R}^n \to \mathbb{R} \), with \( \alpha(0) = 0 \), such that \( J_{x_0}(u^*) \leq J_{x_0}(u) \) for any \( u \).

Although it is not explicitly included in the formulation of Problem 1, a desirable, if not unavoidable, property is that of asymptotic stability of the zero equilibrium of the closed-loop (optimal) dynamics. To this end, consider the following standing assumption, which is assumed to hold throughout the entire article.

**Assumption 2:** System (5a) with output (5b) is zero-state detectable.

Interestingly, the general formulation in Problem 1 encompasses several notable properties, as pursued in the following by stating the two alternative sets of assumptions below, which specialize the structure of the matrix \( S(x) \) and which are related to an infinite-horizon optimal control problem and the property of passivity, respectively.

**Assumption 3:** The functions in Problem 1 are such that \( k_2(x) = 0 \), \( s_{11}(x) = s_{22}(x) = 1 \) and \( s_{12}(x) = 0 \) for any \( x \in \mathbb{R}^n \).

**Assumption 4:** The functions in Problem 1 are such that \( k_2(x) > 0 \), \( s_{11}(x) = s_{22}(x) = 0 \) and \( s_{12}(x) = 1 \) for any \( x \in \mathbb{R}^n \).

The fact that Problem 1 together with the structure of the underlying functions dictated by Assumption 3 leads to a classic optimal control task with respect to the running cost \( k_1(x)^2 + u^2 \) is rather straightforward. In this case, we then refer to the function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) defined as \( V(x_0) = \int_0^\infty J_{x_0}(u^*) \), \( \forall x_0 \in \mathbb{R}^n \), as the value function of the optimal control problem. On the other hand, the link between passivity and Problem 1 with Assumption 4 deserves additional attention, and hence it is explored in the following result, which relies on the concept of available storage defined in [23] together with the property that the running cost under Assumption 4 is such that for any \( x_0 \in \mathbb{R}^n \) there exists a control input for which the running cost in (6) is nonpositive.

**Proposition 3:** Consider system (5). Suppose that Problem 1 with Assumption 4 admits a continuous solution \( u^* \) such that

\[ V(x_0) \triangleq -\int_0^\infty r(\phi(\tau, x_0), u^*(\phi(\tau, x_0)))d\tau \]  (7)

for any \( x_0 \in \mathbb{R}^n \), is continuously differentiable, where \( r(x, u) = k_1(x)u + (1/2)k_2(x)u^2 \) and \( \phi(t, x_0) \) denotes the solution to system (5a) with \( u = u^* \) and \( x(0) = x_0 \). Then, system (5) is passive with a continuously differentiable storage function.

The proof of the claim is omitted since it can be derived by straightforward adaptation of the arguments in proofs of Propositions 3.1.11 and 3.1.12 of [22]. The section is concluded by stating a somewhat obvious property of the vector fields \( f, g \), and \( h \) that is instrumental for the derivations in the following sections.

**Fact 1:** Given vector fields \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R}^n \), there exist smooth mappings \( p : \mathbb{R}^n \to \mathbb{R}^{n \times n} \), \( d : \mathbb{R}^n \to \mathbb{R}^{n \times n} \), for some nonnegative integers \( n_p \) and \( n_d \), such that

\[ (f(x)h(x)^\top + h(x)f(x)^\top) = p(x)p(x)^\top - d(x)d(x)^\top \]  (8)

for all \( x \in \mathbb{R}^n \). Furthermore, given a vector field \( g : \mathbb{R}^n \to \mathbb{R}^n \) and \( d \) as obtained in (8), there exists \( l : \mathbb{R}^n \to \mathbb{R}^{n \times n} \), for some nonnegative integer \( n_l \), such that

\[ l(x)l(x)^\top = g(x)g(x)^\top + d(x)d(x)^\top \]  (9)

for all \( x \in \mathbb{R}^n \).

Note that Fact 1 essentially summarizes, in (8), the property of a symmetric matrix-valued function to be decomposed in its positive and negative semi-definite parts, respectively, and, in (9), the property of a positive semi-definite matrix-valued function to be decomposable as the external product of a certain vector field \( l(x) \) of suitable dimensions related to the rank of the matrices on the right-hand side.

**Remark 1:** The decompositions discussed in (8) and (9) may be achieved, for instance, by letting

\[ p(x) = f(x) + h(x), \quad l(x) = [f(x)^\top g(x)^\top h(x)] \]  (10)

which may not be, however, a minimal decomposition in terms of number of columns of the matrix-valued function \( l(x) \) for fixed \( x \in \mathbb{R}^n \), namely with regard to the value of \( n_l \).

Finally, the two following sections characterize the solutions to Problem 1 with Assumption 3 and to Problem 1 with Assumption 4, respectively.

### IV. Optimality of Input-Quadratic Nonlinear Systems

In this section, we explore the relation between the properties of optimality for the underlying input-quadratic system (5) and of dissipativity of an auxiliary input-affine nonlinear system, as discussed below. Throughout this section, we suppose that Assumption 3 holds, hence Problem 1 is equivalent to an optimal control problem for system (5a) with respect to the cost functional

\[ J_{x_0}(u) = \int_0^\infty (k_1(x(\tau))^2 + u(\tau)^2)d\tau \]  (11)

as formalized by the following statement.
Problem 2: Consider the nonlinear, input-quadratic, system (5) together with the cost functional (6). The infinite-horizon optimal control problem consists in solving Problem 1 with Assumption 3.

Remark 2: Similarly to what is discussed in [16, Ch. 13], one may define an additional state $\zeta = u$ with dynamics $\dot{\zeta} = \dot{u}$, $\dot{u}$ being a new control input, and consider a modified (extended) system affine in the input $u$. However, besides the additional smoothness requirements on the (new) control input $\dot{u}$, it must be noted that Problem 2 becomes singular, i.e., the cost functional does not include any penalty on the control input.

To circumvent this issue, one could modify the original cost functional (11) by introducing a cheap-control-like term $\varepsilon u^T u$ to formulate, and possibly solve, a regular problem. Nonetheless, as a consequence, the mentioned approach characterizes the solution to a different problem with respect to Problem 2, which is instead provided below.

Theorem 1: Consider system (5a) together with the cost functional (11) and Assumption 2. Consider the input-affine system

$$
\dot{x} = \bar{f}(x) + l(x)v + p(x)w
$$

with $\bar{f}(x) = f(x) + (1/2)h(x)k_1(x)^2$, with $l(x)$ and $p(x)$ as in Fact 1, for any $x \in \mathbb{R}$, where $w(t) \in \mathbb{R}^n_x$ is a control input and $v(t) \in \mathbb{R}^n_x$ is a disturbance input, and suppose that there exists a smooth solution $V : \mathbb{R}^n \to \mathbb{R}^+, V(0) = 0$, of the HJB equation

$$
0 = V_x f(x) + \frac{1}{2} k_1(x)^2 + \frac{1}{2} k_1(x)^2

+ \frac{1}{2} V_x (p(x)p(x)^T - l(x)l(x)^T)V_x^T
$$

(13)

i.e., system (12) in closed loop with $v = -l(x)^T V_x^T$ has $L_2$-gain less than or equal to one from $w$ to $[y, v]^T$. Then, there exists an open set $U \subset \mathbb{R}^n$, containing the origin, such that

$$
u^*(x) = -(1 + V_x h(x))^{-1} g(x)^T V_x^T
$$

(14)

solves the infinite-horizon optimal control problem with asymptotic stability for the input-affine system (5a) and the cost functional (11), for all $x \in U$.

Proof: Consider the nonlinear, input-quadratic, system (5a) together with the cost functional (11). The corresponding Hamilton–Jacobi–Bellman (HJB) partial differential equation is

$$
0 = \min_u \{ HJB(x, u) \} \triangleq \min_u \left\{ V_x f(x) + V_x g(x)u + \frac{1}{2} V_x h(x)u^2 + \frac{1}{2} k_1(x)^2 + \frac{1}{2} u^2 \right\}

= \min_u \left\{ V_x f(x) + V_x g(x)u + \frac{1}{2} k_1(x)^2 + \frac{1}{2} u^2 \mathcal{M}(x, V_x) \right\}
$$

(15)

with $\mathcal{M}(x, V_x) = 1 + V_x h(x)$, in the unknown $V : \mathbb{R}^n \to \mathbb{R}^+, V \in C^1, V(0) = 0$. Therefore, since $V_x(0) = 0$ and $V_x$ and $h$ are at least continuous vector-valued functions, there exists a nonempty open neighborhood of the origin, $\mathcal{U} \subset \mathbb{R}^n$, such that $\mathcal{M}(x, V_x) > 0$. As a consequence, the minimum with respect to the control input of $HJB(x, u)$ is continuously achieved at (14). Replacing then $u^*(x)$ into (15), the latter reduces to

$$
0 = V_x f(x) + \frac{1}{2} k_1(x)^2 - \frac{1}{2} V_x g(x)M^{-1}(x, V_x)g(x)^T V_x^T
$$

hence

$$
0 = \left( V_x f(x) + \frac{1}{2} k_1(x)^2 \right) (1 + V_x h(x)) - \frac{1}{2} V_x g(x)g(x)^T V_x^T.
$$

(16)

By recalling now the decompositions in (8) and (9), it is straightforward to show that (16) reduces to (13). To conclude the proof, by taking $V(x)$ as a candidate Lyapunov function for the closed-loop system (5), (14), it follows that the corresponding time derivative along the closed-loop trajectories is:

$$
\dot{V} = V_x f(x) + V_x g(x)u^* + (1/2)V_x h(x)(u^*)^2

= V_x f(x) - \frac{V_x g(x)g(x)^T V_x^T}{1 + V_x h(x)} + \frac{1}{2} \frac{V_x h(x)V_x g(x)g(x)^T V_x^T}{(1 + V_x h(x))^2}

= - \frac{1}{2} k_1(x)^2 - \frac{1}{2} (u^*)^2
$$

where the second equation follows by noting that the HJB equation (16) holds, hence implying asymptotic stability by zero-state detectability from the output $y = k_1(x)$. Finally, the neighborhood $U$ is constructed by considering any sublevel set of $V$ entirely contained in $U$.

Remark 3: The proof of Theorem 1 entails that a key feature instrumental to establish the equivalence between the properties of optimality for (5) and dissipativity of the auxiliary system (12) is that, in the single-input case, the adjoint matrix and the determinant of $\mathcal{M}(x, V_x)$ are described by a constant and a linear function, respectively, of the gradient $V_x$. It can be shown that, instead, in the multi-input setting, the former are polynomial functions of $V_x$ with degree higher than one in general. Therefore, in the latter setting, the HJB equation (15) cannot, in general, be exactly rewritten as a quadratic pde in the unknown $V_x$ and a remainder in the equation should be characterized and potentially dealt with. This renders the discussion rather more convoluted than the elegant presentation concerning single-input plants. In particular, few preliminary results toward the extension to the MIMO setting are contained in [18].

Remark 4: From the computational point of view, the interest of the result above lies in the fact that dissipativity of system (12) is related to the existence of a solution to a quadratic first-order partial differential equation, which has been extensively studied in the past decades.

Remark 5: The requirements on the storage function $V$ of the auxiliary input-affine system may be weakened by replacing the equality sign in the HJ equation (13) with an inequality sign ($\geq$). As a consequence, while, on the one hand, system (12) is still guaranteed to be dissipative with respect to the same supply rate (and with the same storage function, hence guaranteeing the property of possessing an $L_2$-gain less than or equal to one), the infinite-horizon optimal control problem, on the other hand, is solved approximately with respect to a running cost augmented by the negativity gap of the relaxed partial differential inequality.

Remark 6: By inspecting the structure of the vector field $l$ as suggested in (10) and its role in the auxiliary input affine
system (12), it appears that—by means of the equivalence established between the two properties above—all the vector fields defined in the original system (5a), including the drift, become available input channels in the auxiliary input affine task, at the price of a matched virtual disturbance to attenuate via the design of the (virtual) input v.

By additional requiring the property of continuity for the state-feedback control law solving Problem 1 with Assumption 3, necessary and sufficient conditions for the equivalence between the properties of optimality and dissipativity can be stated as follows.

**Theorem 2:** Consider system (5a) together with the cost functional (11) and Assumption 2. Then, there exists a (local) continuous stabilizing solution $u^*(x)$ to Problem 2 such that

$$V(x_0) \triangleq \int_0^\infty \left( k_1 (\phi(\tau, x_0))^2 + u^*(\phi(\tau, x_0))^2 \right) d\tau$$

(17)
is continuously differentiable if (and only if) the input-affine system (12) has $L_2$-gain less than one (less than or equal to one) from the input w to the output $[y, v]^\top$ for some smooth feedback v.

**Proof:** $(\Rightarrow)$ To complete the proof of sufficiency based on the arguments employed in the proof of Theorem 1, it remains to show that, first, the property of the closed-loop system (12) of possessing $L_2$-gain less than one implies a similar property for the linearized system by [21, Prop. 22]. The latter in turn guarantees—by applying Theorems 21 and 23 of [21] together with Assumption 2—the existence of a smooth solution to the HJ equation (13), locally around the equilibrium point, from which the proof of Theorem 1 continues.

$(\Leftarrow)$ To prove necessity of the stated conditions, suppose instead that Problem 2 admits a continuous solution $u^*$ such that the corresponding value function defined in (17) is continuously differentiable. As a consequence, the function $V$ in (17) satisfies the HJB equation (15), which expresses necessary and sufficient conditions for optimality, with the additional property that $u^* = \arg \min_v \{ H(x, u, v) \}$, for all x in a neighborhood $U$ of the origin. Continuity of the feedback control law $u^*(x)$ implies that $M(x, V)_{\phi}$ must remain different from zero in $U$, thus allowing to derive the HJ equation (13) from (16), which immediately implies that the $L_2$-gain is less than or equal to one.

The previous characterization in terms of $L_2$-gain of an auxiliary input-affine system is also instrumental for the following statement, which discusses existence and properties of the solution to Problem 2. To provide a concise exposition of the result, let

$$A \triangleq \nabla_x f(x)|_{x=0}, \quad B = g(0), \quad H = h(0)$$

(18)

and $K_1 = \nabla_x k_1(x)|_{x=0}$.

**Proposition 4:** Consider system (5a) together with the cost functional (11). Suppose that the pair $(A, B)$ is controllable and the pair $(A, K_1)$ is observable. Then, Problem 2 admits a (local) continuous solution $u^*$.

**Proof:** To begin with, consider the linearization of (12) around the origin described by the linear system

$$\dot{x} = Ax + [B; H]v + Hw$$

(19)

with the matrices $A$, $B$, and $H$ defined in (18). The algebraic Riccati equation (ARE) associated to the disturbance attenuation problem for system (19) (see also (62) of [21]) then becomes

$$0 = A^T P + PA + K_1^T K_1 + P \left( (\gamma^{-2} - 1) H H^T - B B^T \right) P$$

$$= A^T P + PA + K_1^T K_1 + \gamma^{-1} P B B^T P$$

(20)

where the second equality is obtained by letting $\gamma = 1$. The latter equation, which coincides with the classic ARE associated to a LQR optimal control problem, then by controllability and observability of the pairs $(A, B)$ and $(A, K_1)$, respectively, admits a positive definite solution $P$ such that in addition $\sigma(A - B B^T P) < \mathbb{C}^-$ (compare with (63) of [21]). Therefore, by [21, Thm. 21], system (19) has $L_2$-gain strictly less than $\gamma = 1$, which implies the existence, locally around the origin, of a smooth solution to the HJ equation (13) by [21, Thm. 23].

**A. Geometric Point of View: Graphical Storage Functions and Hamiltonian Systems**

The objective of this section consists in exploring the positivity condition on the function $M(x, V)$, which appears crucial, by inspecting the proof of Theorem 1, to imply global existence of the solution to Problem 2. To this end, in the following, we rely on tools and ideas borrowed from differential geometry (see [7]).

**Definition 5:** Consider the input-affine nonlinear system (12) with output $y = k_1(x)$. Let a smooth vector field $h : X \to T X$ be given. A proper, continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ is said to be a graphical storage function for (12) provided the HJ equation (13) holds and

$$\text{graph } dV \subset H \triangleq \{ (x, v) \in T^* X : \nu h(x) > -1 \}$$

(21)

for any $x \in \mathbb{R}^n$.

The intuition behind the above definition is that a standard storage function for the input-affine (12) is also the value function of an input-quadratic optimal control problem, provided the graph of the gradient satisfies an additional inclusion. Nonetheless, a somewhat different interpretation may be given by ignoring the role played by the input-affine system: the function $V$ resulting from (13) is the value function of Problem 2, provided a certain convexity condition of the HJB equation (15) holds, thus allowing to continuously achieve a unique minimization of the right-hand side of the latter equation.

**Remark 7:** The rationale behind the definition of the term graphical is related to the fact that both requirements associated to its definition may be put into perspective in terms of properties that graph $dV$ must satisfy. In fact, while it is evident that the inclusion in (21) involves a condition on the graph of the storage function $V$, it must be recalled that also the first requirement is related to the property of graph $dV$ of being tangent, at the equilibrium, to the stable eigenspace of the linearized Hamiltonian dynamics associated to the pde (13) (see [21]).

The following two statements explore a few interesting properties of a graphical storage function related to optimality and stability of the input-quadratic system (5a), respectively.

**Theorem 3:** The following properties of a proper, continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{>0}$ are equivalent for all $x \in \mathbb{R}^n$.

$^1T^* X$ denotes the dual space of the linear space $T_x X$, whereas $T_x^* X \triangleq \bigcup_{x \in X} \{ x \} \times T_x^* X$ is referred to as the cotangent bundle over $X$. 


P1) $V$ is a value function, associated to a continuous feedback $w^*$, for the input-quadratic system (5a) with respect to the cost (11).

P2) $V$ is a graphical storage function for the input-affine system (12).

Proof: The implication (P2) ⇒ (P1) has been shown in the proof of Theorem 1 in the local case, while the global statement is obtained by considering that, by definition, graph $dV$ belongs to $\mathcal{H}$, for any $x \in \mathbb{R}^n$, and the function $V$ is proper. To show the inverse implication, namely that (P1) implies (P2), by following arguments similar to those required to show necessity in the proof of Theorem 2, note that the value function of the optimal control problem defined by (5) and (6) satisfies $1 + V x h(x) \neq 0$, for all $x$. Therefore, the scalar function $1 + V x h(x)$ cannot change sign, and since it is positive at $x = 0$, it remains positive for all $x \in \mathbb{R}^n$, implying the inclusion (21). The fact that also equation (13) holds can be shown by arguments similar to those in the Proof of Theorem 1.

The following result characterizes the trajectories of system (5a) in closed loop with the optimal control input (14) in terms of trajectories of a Hamiltonian system. To this end, consider the Hamiltonian function $\mathcal{H} : T^* \mathcal{X} \to \mathbb{R}$ defined as

$$
\mathcal{H}(x, \lambda) \triangleq \frac{1}{2} k_1(x)^2 + \lambda^\top \left( f(x) + f(x) h k_1(x) \right)^2 - \frac{1}{2} \lambda^\top (g(x) g(x)^\top - f(x) f(x)^\top + h f(x) ) \lambda.
$$

(22)

which corresponds to the function naturally associated to the disturbance attenuation problem related to the input-affine auxiliary system (12).

Proposition 5: Consider system (5) and the cost functional (11). Then, Problem 2 is solved by the process $(x^*(t), \lambda^*(t))$ satisfying

$$
\dot{x} = \frac{1}{1 + \lambda^\top h(x)} \nabla_x \mathcal{H}(x, \lambda) - \frac{\mathcal{H}(x, \lambda)}{(1 + \lambda^\top h(x))^2} h(x)
$$

$$
\dot{\lambda} = -\frac{1}{1 + \lambda^\top h(x)} \nabla_x \mathcal{H}(x, \lambda) + \frac{\mathcal{H}(x, \lambda)}{(1 + \lambda^\top h(x))^2} \nabla_x h(x) \lambda.
$$

(23)

with $(x(0), \lambda(0))^\top = (x_0, V_x(x_0))^\top$, with $V \in C^2$ solution of (13), for any $t \geq 0$.

Proof: The proof of the claim is straightforward by considering standard derivations of Hamiltonian dynamics from the corresponding Hamiltonian function and by noting that the Hamiltonian associated to the underlying optimal control problem, denoted by $\mathcal{H}^o(x, \lambda)$, and that corresponding to the (virtual) disturbance attenuation task, provided in (22), are related as

$$
\mathcal{H}^o(x, \lambda) = \mathcal{H}(x, \lambda) \frac{1 + \lambda^\top h(x)}{(1 + \lambda^\top h(x))^2}.
$$

It is worth pointing out that the Hamiltonian system associated to the function $\mathcal{H}(x, \lambda)$ in (22) is described by the equations

$$
\dot{x} = \nabla_x \mathcal{H}(x, \lambda)
$$

$$
\dot{\lambda} = -\nabla_x \mathcal{H}(x, \lambda)
$$

(25)

which appear significantly different from those in (23). In fact, in addition to the nonlinear scaling of the dynamics in (25), i.e., the factor $1/(1 + \lambda^\top h(x))$, the latter contains forcing terms depending on the Hamiltonian function itself matched with the input channel of the quadratic components of the input and with its differential, i.e., $\nabla_x h(x)$, in the dynamics of state and costate, respectively. Nonetheless, an interesting relation between the dynamics in (23) and those in (25) is discussed in the following statement.

Proposition 6: Consider system (5a) together with the cost functional (11). Suppose that the pair $(A, B)$ is stabilizable and the pair $(A, K_1)$ is detectable. Then, there exists a unique submanifold $N \subset T^* X$ that is simultaneously invariant for the Hamiltonian dynamics in (23) and in (25), and such that (23) and (25) restricted to $N$ are asymptotically stable.

Proof: To begin with, the Hamiltonian dynamics (25) linearized around the origin is described by

$$
\begin{bmatrix}
\dot{x} \\
\dot{\lambda}
\end{bmatrix} =
\begin{bmatrix}
A & -BB^\top \\
-K_1^\top K_1 & -A^\top
\end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}
$$

(26)

with the matrices $A, B$, and $K_1$ defined in (18), which immediately shows by stabilizability and detectability of $(A, B)$ and $(A, K_1)$, respectively, that the dynamics in (25) possess a hyperbolic equilibrium point at the origin, i.e., system (26) does not possess eigenvalues on the imaginary axis. Moreover, the stable and unstable eigenspaces are of the form $\text{Im}[I, P^-]^\top$ and $\text{Im}[I, P^+]^\top$, respectively, with $P^-$ and $P^+$ denoting the symmetrical maximal and minimal solutions to (20). By [21, Prop. A.4], this guarantees the existence of an immersed submanifold $N$ invariant for (25), tangent to the stable eigenvalue of (26) and on which (25) is asymptotically stable, whereas [21, Prop. A.7] ensures that $N$ is locally projectable. The latter property implies that $N = \text{graph } dV$ with $V$ such that $\mathcal{H}(x, V_x) = 0$. The claim—showing that the different dynamics (23) and (25) share a common invariant submanifold—is proved by noting that $\mathcal{H}(x, V_x) = 0$ implies $\mathcal{H}^o(x, V_x) = 0$ locally and by [20, Prop. 3]. Finally, local asymptotic stability of zero equilibrium of (23) restricted to $N$ is obtained by noting that, since on $N$, one has that $\mathcal{H}(x, V_x) = 0$ and hence the second term in each equation of (23) is equal to zero, the latter dynamics reduce to (25) apart from a nonlinear, scalar, scaling factor that is positive definite and uniformly bounded in a neighborhood of the origin.

B. Linear–Quadratic–Quadratic Regulator

The objective of this section consists in specializing the results of Section IV-A to the case of a system linear in the state $x$, quadratic in the control input and in the presence of a quadratic cost functional, i.e., the LQQR problem. Toward this end, consider the system

$$
\dot{x} = Ax + Bu + \frac{1}{2} Hu^2
$$

(27)

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, and $H \in \mathbb{R}^n$ constant matrices, and consider the cost functional (11) with $k_1(x) = K_1 x$, $K_1 \in \mathbb{R}^{1 \times n}$.

Proposition 7: Consider system (27) together with the cost functional (11), with $k_1(x) = K_1 x$ and Assumption 2. Consider
the input-affine nonlinear system
\[ \dot{x} = Ax + \frac{1}{2} x^\top K_1^\top K_1 x + [Ax;B;H] v + (Ax + H)w \]
where \( v(t) \in \mathbb{R}^{n_v} \) is a control input and \( w(t) \in \mathbb{R}^{n_w} \) is a disturbance input, and suppose that there exists a smooth solution \( V : \mathbb{R}^n \to \mathbb{R}_{>0} \) such that \( V(0) = 0 \) of the HJ equation
\[ 0 = V_x \left( Ax + \frac{1}{2} x^\top K_1^\top K_1 x H \right) + \frac{1}{2} x^\top K_1^\top K_1 x \]
\[ -\frac{1}{2} V_x (BB^\top - AxH^\top - Hx^\top A^\top) V_x^\top \]
(29)
i.e., (28) in closed loop with \( v = -[Ax;B;H] V_x^\top \) has \( L_2 \)-gain less than or equal to one from the input \( w \) to the output \([y, v]^\top\), such that \( 1 + V_x H > 0 \) for all \( x \in \mathbb{R}^n \). Then, the continuous feedback
\[ u^*(x) = -\frac{B^\top V_x^\top}{1 + V_x H} \]
(30)
solves the LQQR problem.

Remark 8: The structure of the quadratic term in the partial differential equation (29) entails that the LQQR problem cannot be equivalent to a \( H_\infty \) problem for an auxiliary linear system, since the terms \( AxH^\top + Hx^\top A^\top \) cannot yield a constant positive semi-definite term leading to a virtual disturbance \( w \). Moreover, again by inspecting the quadratic term in (29), it is straightforward to show that if \( A = 0 \), then the LQQR problem is equivalent to a nonlinear input-affine optimal control task associated to the system \( \dot{x} = (1/2)x^\top K_1^\top K_1 x + Hx + Bv \) and the quadratic cost (11) with \( r(x, u) = x^\top K_1^\top K_1 x + u^2 \). ▲

1) Energy-Optimal Control of a Quadratic Integrator: To demonstrate the above derivations by means of a few numerical examples, consider first a quadratic integrator described by the equations
\[ \dot{x} = u + u^2 \]
(31)
with state \( x(t) \in \mathbb{R} \) and control input \( u(t) \in \mathbb{R} \). The control objective consists in solving an infinite-horizon optimal control problem with respect to (31) and the cost functional (6) with \( q(x) = x^2 \), namely a LQQR problem as in Section IV-B. It can be easily shown that system (31) is in the form of (5) with \( f(x) = 0, g(x) = 1, \) and \( h(x) = 2 \) for any \( x \in \mathbb{R} \). Therefore, since the requirements in Fact 1 are trivially satisfied, optimality of (31) may be related to dissipativity of the auxiliary system
\[ \dot{x} = x^2 + v \]
(32)
obtained by specializing the dynamics in (12), with respect to the virtual input \( v(t) \in \mathbb{R} \). Note that, since here the vector field \( p(x) \equiv 0 \), it follows that optimality of system (31) is in fact related to optimality of the input-affine system (32), provided the latter value function verifies the graphical inclusion in (21). The HJB associated to system (32) is
\[ 0 = \min_u \left\{ \frac{1}{2} x^2 + \frac{1}{2} u^2 + V_x x^2 + V_x v \right\} \]
\[ = \frac{1}{2} x^2 - \frac{1}{2} (V_x)^2 + V_x x^2 \]
(33)
where the second equality is obtained by replacing the optimal control law \( u^* = -V_x(x) \). The HJB equation (33) is solved by the positive definite function
\[ V(x) = \frac{1}{3}(x^2 + 1)^{3/2} + \frac{1}{3} x^3 - \frac{1}{3} \]
(34)
which satisfies the inclusion in (21), hence it is a graphical storage function and, by Theorem 3, provides also the minimal cost for the original input-quadratic optimal control problem defined with respect to (31). The graph of the function \( V \) is depicted in Fig. 1. The energy-optimal control law for a quadratic single-integrator is defined as the smooth control input
\[ u^*(x) = -\frac{V_x}{1 + 2V_x} = -\frac{x \sqrt{x^2 + 1 + x}}{1 + 2x \sqrt{x^2 + 1 + 2x^2}} \]
(35)
The time histories of the state and control input (top and bottom graphs, respectively) of system (31) in closed loop with the optimal solution (35) are reported in Fig. 2 for \( x(0) = 2 \) (solid line) and \( x(0) = -2 \) (dashed line). Since the graphical storage function \( V \) in (34) constitutes also a control Lyapunov function, the construction proposed in Proposition 8 of [15] can be carried out by letting (according to the notation of [15]) \( a(x) = V_x, b(x) = V_x, c(x) \equiv 0 \) and \( \varphi(x) = V_x / |V_x| \). The resulting, stabilizing, control input is defined as
\[ \ddot{u}(x) = -\frac{V_x + \sqrt{V_x^2 - 4V_x \varphi(x)}}{2V_x} \]
(36)
to be compared with the structure of the left-hand side of (35).
Fig. 3 displays the ratio \( \pi(x) \) between the optimal cost \( J^* = J(u^*) \) and the cost yielded by the control input \( \ddot{u}(x) \), namely \( \pi(x) = J^* / J(\ddot{u}) \) for any \( x \in \mathbb{R} \).

2) Energy-Optimal Control of a Quadratic Double-Integrator: As a second numerical simulation, consider a quadratic double-integrator described by the equations
\[ \dot{x}_1 = x_2 \\
\dot{x}_2 = u + u^2 \]
(37)
with state \( x(t) \in \mathbb{R}^2 \) and scalar control input \( u(t) \in \mathbb{R} \), and the cost functional as in (6) with \( q(x) = \|x\|^2 \). System (37) is in the
Fig. 2. Time histories of the state and control input (top and bottom graphs, respectively) of system (31) in closed loop with the optimal solution (35) for two different initial conditions.

Fig. 3. Ratio $\pi(x)$ between the optimal cost $J^* = J(u^*)$ and the cost yielded by the control input $\tilde{u}(x)$ proposed by [15] and obtained by letting the graphical storage function $V$ in (34) as a CLF.

form of (5) with

$$f(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(x) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$ (38)

By following the approach discussed in Theorem 1, it can be shown that optimality of (37) with respect to (6) is related to dissipativity of the auxiliary input-affine system

$$\dot{x} = \begin{bmatrix} x_2 \\ x_1^2 + x_2^2 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ 1 \end{bmatrix} v + \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} w$$ (39)

with respect to the output $y = [x_1, x_2]^T$. The corresponding HJ equation then becomes

$$0 = \frac{1}{2}(x_1^2 + x_2^2) + V_{x_1} x_1 + V_{x_2} (x_1^2 + x_2^2)$$

$$-\frac{1}{2}(V_{x_1})^2 + 2x_2 V_{x_1} V_{x_2}.$$ (40)

It can be shown that the corresponding Hamiltonian has a hyperbolic equilibrium at the origin (see [21] for the definition and detailed discussions on hyperbolic equilibria of Hamiltonian dynamics), hence the partial differential equation (40) indeed admits a positive semi-definite solution in a neighborhood of the origin. Nonetheless, its explicit computation is a daunting task and it is then approximated here by means of a polynomial function by relying on the methodology introduced in [11] and commented upon in [21]. In particular, in the simulations reported in Fig. 4, the solution to the pde (40) has been approximated with a polynomial function of order four, i.e., $V_4(x)$.

V. PASSIVITY OF INPUT-QUADRATIC NONLINEAR SYSTEMS

The objective of this section consists in characterizing the property of passivity for input-quadratic systems. Therefore, as discussed also in Section II and motivated by the statement of Proposition 3, in the rest of this section, we consider Problem 1 with the structure of the underlying functions dictated by Assumption 4.

**Theorem 4:** Consider system (5) together with Assumption 4. Consider the input-affine system

$$\dot{x} = \tilde{f}(x) + p(x)v + l(x)w$$ (41)

with $\tilde{f}(x) = f(x)k_2(x) - g(x)k_1(x)$ and $p$ and $l$ defined in Fact 1, where $v(t) \in \mathbb{R}^n$ is a control input and $w(t) \in \mathbb{R}^p$ is a disturbance input, and suppose that there exists a smooth solution $V : \mathbb{R}^n \to \mathbb{R}_{>0}, V(0) = 0$, of the HJ equation

$$0 = V_x (f(x)k_2(x) - g(x)k_1(x)) + \frac{1}{2}k_1(x)^2$$

$$+ \frac{1}{2}V_x (l(x)l(x)^T - p(x)p(x)^T)V_x^T.$$ (42)

Then, there exists an open set $U \subset \mathbb{R}^n$, containing the origin, such that the input-quadratic system (5) is passive for any $x_0 \in U$.

**Proof:** Specializing the cost functional introduced in Problem 1 with the functions defined in Assumption 4, one obtains
the HJB equation
\[
0 = \max_u \left\{ V_x f(x) + V_z g(x) u + \frac{1}{2} V_z h(x) u^2 - k_1(x) u \right\} \\
- \frac{1}{2} k_2(x) u^2
\]
\[
= \max_u \left\{ V_x f(x) + (V_z g(x) - k_1(x)) u + \frac{1}{2} (V_z h(x) - k_2(x)) u^2 \right\}.
\]
(43)

Since, by assumption, \(k_2(x) > 0\) for any \(x \in \mathbb{R}^n\), there exists a neighborhood \(\mathcal{U} \subset \mathbb{R}^n\) of the origin such that \(V_z h(x) - k_2(x)\) is negative and the HJB equation (43) admits a local continuous maximizer given by
\[
\hat{u}(x) = -\frac{V_z g(x) - k_1(x)}{V_z h(x) - k_2(x)}.
\]
(44)

Replacing the state-feedback \(\hat{u}\) in (43) yields the partial differential equation
\[
0 = V_x f(x) - \frac{1}{2} (V_z g(x) - k_1(x))^2 / V_z h(x) - k_2(x)
\]
\[
= V_x f(x) + \frac{1}{2} (V_z g(x) - k_1(x))^2 / V_z h(x) - V_z h(x)
\]
(45)

which is equivalent to the partial differential equation (42) since \(k_2(x) - V_z h(x) > 0\) for any \(x \in \mathcal{U}\) and by considering the decompositions in Fact 1. The solution \(V\) to the latter equation together with the input \(\hat{u}(x)\) then verify the conditions of Proposition 3, thus implying passivity of the input-quadratic system (5). □

**Example 1:** Consider a quadratic double-quadratic integrator with output described by the equations
\[
\dot{x} = u + u^2 \tag{46a}
\]
\[
y = x + \frac{k_2(x)}{2} u \tag{46b}
\]
with \(x(t) \in \mathbb{R}, u(t) \in \mathbb{R}, \) and \(y(t) \in \mathbb{R}\). Suppose that Assumption 4 holds, namely \(k_2(x) > 0\) for any \(x \in \mathbb{R}\). System (46) is in the form of (5) with \(f(x) = 0, g(x) = 1, \) and \(h(x) = 2\) for any \(x \in \mathbb{R}\), hence the functions in Fact 1 are defined as \(p(x) = 0\) and \(l(x) = 1\) for any \(x \in \mathbb{R}\). Therefore, by relying on Theorem 4, passivity for any initial condition \(x_0 \in \mathbb{R}\) of the quadratic integrator (46) is implied by the property of the auxiliary system
\[
\dot{x} = -x + w, \quad y = x \tag{47}
\]
of possessing \(L_2\)-gain less than one from the input \(w\) to the output \(y = x\) with a storage function \(V(x)\) such that \(2V_x - k_2(x) < 0\). Note that, since \(p(x) \equiv 0\), the auxiliary control task reduces to a linear analysis problem. It is then straightforward to show that the storage function for the auxiliary \(L_2\)-gain analysis is \(V(x) = (1/2)x^2\), and thus system (46a) is passive provided \(k_2(x)/2\) is superlinear, in addition to \(k_2(x) > 0\) for any \(x \in \mathbb{R}\). For instance, system (46b) is passive from the output \(y = x + (1 + \alpha x^2)u\) for any \(\alpha > 1\) and the corresponding stabilizing feedback is given by \(u(x) = -2x/(3 + \alpha x^2)\).

It is worth pointing out the mirrored roles played by the mappings \(l\) and \(p\) in the auxiliary input-affine systems (12) and (41), the former associated to the virtual control input and the latter to the virtual disturbance, and vice versa, respectively. Interestingly, Theorem 4 suggests that the (analysis) problem of assessing whether the input-quadratic system (5) is passive can be recast into the (synthesis) task of designing a control input such that the input-affine system (41) in closed loop has \(L_2\)-gain less than one from the (virtual) disturbance \(w\) to the output \(y = k_1(x)\). Therefore, by mimicking the rationale behind the statement and the proof of Proposition 4 in the case of infinite-horizon optimal control problems, the following result provides a sufficient condition for passivity of the input-quadratic system (5) that hinges upon the solution to a certain Riccati equation. To this end, in addition to the definition of the matrices in (18), suppose that \(k_2(0) = \alpha > 0\).

**Proposition 8:** Consider system (5) together with Assumption 4. Suppose that there exists a symmetric, positive semidefinite solution \(P = P^T \in \mathbb{R}^{n \times n}\) to
\[
0 = P \left( A - B K_1 \frac{1}{\kappa} \right)^T + \left( A - B K_1 \frac{1}{\kappa} \right) P + \frac{1}{\kappa} K_1^T K_1
\]
\[
+ \frac{1}{\kappa} P B B^T P
\]
(48)
such that \(\sigma(A K - B K_1 + B B^T P) \subseteq \mathbb{C}^+\). Then, there exists an open set \(\mathcal{U} \subset \mathbb{R}^n\), containing the origin, such that the input-quadratic system (5) is passive for any \(x_0 \in \mathcal{U}\). □

**Proof:** Similarly to the proof of Proposition 4, the arguments are based on the study of the attenuation properties of the linearization of system (41), namely
\[
\dot{x} = (A K - B K_1) x + H v + \left[ B^T H \right] w \tag{49}
\]
which has \(L_2\)-gain less than one if and only if equation (48) holds and the matrix \(A K - B K_1 + B B^T P\) is Hurwitz, by [21, Thm. 21].

Finally, the property of passivity is characterized in terms of the existence of an invariant submanifold of certain Hamiltonian dynamics in the following statement, similarly to the well-established characterization of a classic optimal control and disturbance attenuation problems.

**Proposition 9:** Consider system (5) together with Assumption 4. Moreover, consider the Hamiltonian function \(\mathcal{H}_p : T^* \mathcal{X} \rightarrow \mathbb{R}\) defined as
\[
\mathcal{H}_p(x, \lambda) = \frac{1}{2} k_1(x)^2 + (f(x)k_2(x) - g(x)k_1(x))
\]
\[
+ \frac{1}{2} \lambda^T (g(x)g(x)^T - f(x)h(x)^T - h(x)f(x)^T) \lambda
\]
(50)
and let \(X_{\mathcal{H}_p}\) denote the associated Hamiltonian vector field. Suppose that there exists a stable maximal immersed Lagrangian submanifold \(\mathcal{N} \subset T^* \mathcal{X}\) invariant for \(X_{\mathcal{H}_p}\). Then, there exists an open set \(\mathcal{U} \subset \mathbb{R}^n\) such that the input-quadratic system (5) is passive for any \(x_0 \in \mathcal{U}\). □

Note that a sufficient condition for the hypotheses of Proposition 9 to hold is that the dynamics \(X_{\mathcal{H}_p}\) possess an hyperbolic equilibrium at the origin.
VI. OPTIMAL CONTROL OF MEMS ACTUATORS

Due to their extensive use in MEMS, the problem of controlling parallel-plate electrostatic actuators driven by a voltage generator has recently gained particular interest. The control task is challenging from both the practical and the theoretical points of view, since it intrinsically requires nonlinear control design techniques. It is well known, in fact, that a naive static open-loop voltage control scheme guarantees, for the actuated plane, an excursion corresponding to one-third of the underlying full gap, due to instability beyond such a limit; see, for instance, [3] for a survey on the challenges arising in controlling such actuators. Motivated by its practical importance, several feedback control architectures have been proposed in the literature; see e.g., [12], [26] to tackle the control task of extending the moving range of the actuated plane. In [26], a comparison between three nonlinear strategies is carried out by designing a flatness-based control law, by pursuing a control Lyapunov function synthesis and by relying on the backstepping approach. The latter strategy strongly motivates the design suggested in this article, since at an intermediate step of the construction, the virtual control input appears quadratically in the vector field, as shown below in details.

In this section, the optimal set-point regulation of a parallel-plate electrostatic actuators driven by a voltage source is considered, by relying on the results of Section IV. To this end, note that the mechanics of the electrostatic actuator are described by a mass-spring-damper model, i.e., following the notation of [26] as

\[ m \ddot{x} + b \dot{x} + G(t) - G_0 = -\frac{Q(t)^2}{2\varepsilon A} \]  

(51)

with \( G : \mathbb{R} \to \mathbb{R} \) denoting the air gap, \( Q : \mathbb{R} \to \mathbb{R} \) the charge on the device, and \( m, b, k \) denote the mass of the movable plate, the damping coefficient, and the elastic constant, respectively, \( G_0 \) is the zero-voltage gap, \( A \) the plate area, and \( \varepsilon \) the permittivity in the gap. The dynamics of the charge \( Q \) are governed by the equation

\[ \dot{Q}(t) = \frac{1}{R} \left( V - \frac{Q(t)G(t)}{\varepsilon A} \right) \]  

(52)

with \( V \) denoting the voltage source and \( R \) the internal resistance. Following [26], consider the electromechanical model (51), (52) on the modified time-scale \( \tau = \omega_0 t \) and normalized according to

\[ x = 1 - \frac{G}{G_0}, q = \frac{Q}{Q_{pi}}, u = \frac{V}{V_{pi}}, i = \frac{I}{V_{pi}\omega_0 G_0}, r = \omega_0 C_0 R \]  

(53)

where \( C_0 = \varepsilon A/G_0 \) describes the capacitance without applied voltage, \( V_{pi} = \sqrt{8kG_0^2/27C_0} \) and \( Q_{pi} = (3/2)C_0V_{pi} \) are the pull-in voltage and charge, respectively, and finally \( \omega_0 = \sqrt{k/m} \) and \( \zeta = b/2m\omega_0 \) denote the undamped natural frequency and the damping ratio, respectively. In the transformed and scaled coordinates, the dynamics become

\[ \dot{x} = \nu \]  

(54a)

\[ \dot{\nu} = -2\zeta \nu - x + \frac{1}{3} q^2 \]  

(54b)

\[ \dot{q} = -\frac{1}{r} q(1-x) + \frac{2}{3r} u \]  

(54c)

with \((x, \nu, q) \in [0, 1] \times \mathbb{R} \times [0, \infty)\). By inspecting the dynamics (54), it appears evident that the mechanical and the electrical subsystems may be controlled sequentially, by letting \( q \) describe a virtual control input to the dynamics (54a)–(54b) and by mimicking essentially the spirit of the backstepping construction pursued in [26]. Toward this end, suppose that the mechanical part of the actuator has to be regulated to a desired set-point \((x^*, 0, q^*)\), with \( x^* \in [0, 1] \) and \( q^* \) such that \( 0 = -x^* + (1/3)(q^*)^2 \) and consider the change of coordinates \( x_1 = x - x^*, x_2 = \nu \) and \( u_q = q - q^* \). Then, the transformed dynamics become

\[ \dot{x}_1 = x_2 \]  

\[ \dot{x}_2 = -2\zeta x_2 - x_1 + \frac{2}{3} q^* + \frac{1}{3} u_q^2 \]  

(55)

which is in the form of (5a) with

\[ f = \left[ -2\zeta x_2 - x_1 \right], g = \left[ \frac{2}{3} q^* \right], h = \left[ \frac{1}{3} \right]. \]  

(56)

The task of designing a stabilizing control law for the zero-equilibrium of the mechanical subsystem (55) is then approached by formulating an optimal control problem with respect to the input-quadratic system (55) and the cost functional in (11) with \( k_1(x) = \alpha x, \alpha \in \mathbb{R}_+ \). By relying on the results of Theorem 1, it follows that the design of such an optimal control law is equivalent to constructing a feedback policy that assigns, in closed loop, a \( L_2 \)-gain less than or equal to one for the input-affine auxiliary system

\[ \dot{x} = \left[ -x_1 - 2\zeta x_2 + \frac{2}{3} \right] w + \left[ -x_1 - 2\zeta x_2 \right] u \]  

(57)

from the (virtual) disturbance \( w \) to the output \( y = \alpha x \). The latter property, in turn, hinges upon the solution of the HJ partial differential equation

\[ 0 = \frac{\alpha}{2} x_1^2 + V_{x_1} + V_{x_2} \left( -x_1 - 2\zeta x_2 + \frac{2}{3} \right) + \frac{2}{3} V_{x_1} V_{x_2} \left( \frac{4}{3} q^* \right)^2 + \frac{4}{3} x_1 + \frac{8}{3} \zeta x_2 \]  

(58)

Similarly to Section IV-B2, the solution to the partial differential equation (58) is approximated by means of the technique proposed in [11] by a polynomial function \( V_m(x) \) of order \( m \geq 2 \). The resulting (approximate) optimal control law for the input-quadratic system (55) is then

\[ v_m(x_1, x_2) = -\frac{2q^* V_{m,x_1}}{3 + 2V_{m,x_2}}. \]  

(59)

In the following numerical simulations, we consider a desired set-point corresponding to an almost fully extended plate position, i.e., \( x^* = 0.8 \) (80% of the admissible moving range), hence significantly beyond one-third of the admissible range, and we suppose that the mechanical part is only mildly damped, namely \( \zeta = 0.2 \). Fig. 5 displays the comparison between the time histories of the state \( x(t) \) of system (54a)–(54b), considering \( q \) as
a virtual input, in closed loop with the control laws yielded by (59) and with respect to the polynomial functions $V_2 : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$V_2(x_1, x_2) = 3.39x_1^2 + 2.26x_1x_2 + 0.86x_2^2$$

(60)

and $V_4 : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$V_4(x_1, x_2) = 3.39x_1^2 + 2.26x_1x_2 + 0.86x_2^2 + 0.35x_1^4 + 0.44x_2^3$$

$$+ 1.2x_2x_1^2 + 1.13x_1x_2^2 + 0.12x_2^3x_1 - 0.12x_1^3x_2$$

$$+ 0.29x_2x_1 - 0.03x_1^4 + 0.16x_2^3$$

(61)

namely $v_2(x - x^*, \nu)$ and $v_4(x - x^*, \nu)$, dashed and solid lines, respectively. Note that the fourth-order approximation of the solution to the HJ equation (58) results in a reduction of over-shoot in the transient response of more than 60% with respect to that yielded by the second-order approximation, while the resulting optimal costs are such that $V_4(-x^*, 0) = 2.471 < 2.745 = V_2(-x^*, 0)$, as expected since the fourth-order function induces a reduced error with respect to the second-order one and a more accurate approximation of the optimal solution.

Finally, the optimal control input $v_m(x - x^*, \nu)$ for the mechanical subsystem is combined with a control strategy for the electrical part of the actuator, thus yielding a nonlinear feedback control law to tackle the set-point regulation task for system (53). This is achieved by considering the control input

$$u = \frac{3r}{2} \left( \frac{1}{r} q(1 - x) + \dot{v}_m(x - x^*, \nu) - \kappa(q - v_m(x - x^*, \nu) - q^*) \right)$$

(62)

with $\kappa > 0$ sufficiently large. Note that the mechanical subsystem in (54a)–(54b) with $q = 0$ possesses an asymptotically stable equilibrium point at the origin, hence a naive mixed feedback/feedforward strategy for the set-point regulation is given by

$$u_{ff} = \frac{3r}{2} \left( \frac{1}{r} q(1 - x) - \kappa(q - q^*) \right)$$

(63)

which is, however, such that the stabilization of the mechanical component of the actuator relies on the natural damping properties of the device, hence resulting in significant oscillation. The comparison between the closed-loop time histories yielded by the control input $u$ in (62), with $m = 4$, and $u_{ff}$ in (63), is shown in Figs. 6 and 7 for the state $x$ and $\nu$, respectively. Clearly, the optimal stabilization of the mechanical part of the MEMS actuator via the virtual control input $q$, appearing quadratically, yields a drastic improvement of the transient behavior of the device, as it can be appreciated by comparing the solid and dashed lines of Fig. 6.
VII. CONCLUSION

In this article, we have formulated and tackled infinite-horizon optimal control problems in the presence of input-quadratic nonlinear systems. It has been shown that optimality of the input-quadratic plant is equivalent, in a neighborhood of the origin, to the property that an auxiliary input-affine system possesses an $L_2$-gain less than one. The concept of graphical storage function has been introduced and employed to provide a global characterization of the optimal control input. In addition, identical tools permit the study of the property of passivity for input-quadratic nonlinear systems. The theoretical results have been illustrated via the challenging application of optimal regulation of MEMS actuators.

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