RELATIVE BLOCKING IN POSETS

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Abstract. Poset-theoretic generalizations of set-theoretic committee constructions are presented. The structure of the corresponding subposets is described. Sequences of irreducible fractions associated to the principal order ideals of finite bounded posets are considered and those related to the Boolean lattices are explored; it is shown that such sequences inherit all the familiar properties of the Farey sequences.

1. Introduction and preliminaries

Various decision-making, recognition, and voting procedures rely, explicitly or implicitly, on the cardinalities of finite sets and of their mutual intersections. Among mathematical constructions which underlie those procedures are blocking sets (covers, systems of representatives, transversals) (Füredi, 1988, and Chapter 8 of Grötschel et al., 1988), committees (Khachai et al., 2002), and quorum systems (intersecting set systems, intersecting hypergraphs) (Colbourn et al., 2001, Loeb and Conway, 2000, and Naor and Wool, 1998); see also Crama and Hammer (in preparation).

The present paper is devoted to discussing questions concerning mechanisms of blocking in finite posets that go back to set-theoretic committees.

We refer the reader to Chapter 3 of Stanley, 1997, for information and terminology in the theory of posets.

Recall that a set $H$ is called a blocking set for a nonempty family $G = \{G_1, \ldots, G_m\}$ of nonempty subsets of a finite set if it holds $|H \cap G_k| > 0$, for each $k \in \{1, \ldots, m\}$. The family of all inclusion-minimal blocking sets for $G$ is called the blocker of $G$, see, e.g., Chapter 8 of Grötschel et al., 1993. Let $r$ be a rational number such that $0 \leq r < 1$. A set $H$
is called an $r$-committee for $G$ if it holds $|H \cap G_k| > r \cdot |H|$, for each $k \in \{1, \ldots, m\}$, see, e.g., Khachai et al., 2002.

A family of subsets of a finite ground set is called a clutter or a Sperner family if no set from that family contains another. The empty clutter containing no subsets of the ground set, and the clutter whose unique set is the empty subset of the ground set, are called the trivial clutters. The blocker map assigns to a nontrivial clutter its blocker, and this map assigns to a trivial clutter the other trivial clutter, see, e.g., Cordovil et al., 1991.

The set-theoretic blocker constructions are at the foundation of discrete mathematics, see, e.g., Cornuéjols, 2001, and Crama and Hammer (in preparation).

Since the clutters on a ground set are in one-to-one correspondence with the antichains in the Boolean lattice of all subsets of the ground set, the set-theoretic concepts of blocking can be assigned poset-theoretic counterparts. The next natural step consists in a passage from the Boolean lattices to arbitrary finite bounded posets, see Björner et al., 2004, 2005, and Matveev, 2001, 2002, 2003; a poset is called bounded if it has a least and greatest elements.

Throughout the paper, $P$ stands for a finite bounded poset of cardinality greater than one whose least and greatest elements are denoted by $\hat{0}_P$ and $\hat{1}_P$, respectively. $P^a$ denotes the set of all atoms of $P$ (the atoms are the elements covering $\hat{0}_P$). We denote by $\mathcal{I}(A)$ and $\mathcal{F}(A)$ the order ideal and filter of $P$ generated by an antichain $A$, respectively. If $Q$ is a subposet of $P$ then $\min Q$ denotes the set of minimal elements of $Q$.

We call the empty antichain in $P$ and the one-element antichain $\{\hat{0}_P\}$ the trivial antichains in $P$ because they play in our study a role analogous to that played by the trivial clutters in the theory of blocking sets.

We now recall some poset-theoretic blocker constructions. Let $j$ be a nonnegative integer less than $|P^a|$. Given a nontrivial antichain $A$ in $P$, define the antichain

$$b_j(A) := \min \{b \in P : |\mathcal{I}(b) \cap \mathcal{F}(a) \cap P^a| > j \ \forall a \in A\}.$$  
(1.1)

If $A$ is a trivial antichain in $P$ then the antichain $b_j(A)$ by definition is the other trivial antichain.

The antichains $b_j(A)$, defined by (1.1), serve as a poset-theoretic generalization of the notion of set-theoretic blocker of a nontrivial clutter, see Matveev, 2003. From this point of view, the antichain

$$b(A) := b_0(A)$$  
(1.2)
bears a strong resemblance to its set-theoretic predecessor, see Björner and Hultman, 2004, and Matveev, 2001. Antichains (1.1) admit a nice ordering, and some of the structural and combinatorial properties of blockers (1.2) in the Boolean lattices are clarified, see Remark 3.2.

The posets for which

$$\mathbf{b}(\mathbf{b}(A)) = A,$$

for all antichains $A$, are characterized in Björner and Hultman, 2004.

When we deal with construction (1.1) related to a nontrivial antichain $A$, we are interested in the nonemptiness and the cardinalities of the intersections $\mathcal{I}(b) \cap \mathcal{I}(a) \cap P^a$, for $b \in P - \{\hat{0}_P\}$ and $a \in A$, while the cardinalities of the sets $\mathcal{I}(b) \cap P^a$ do not matter. To distinguish the objects we mainly study in the present paper from those similar to (1.1), we say that the antichain $\mathbf{b}_j(A)$ is an example of an absolute poset-theoretic $j$-blocker; a more general definition is given in Section 3.

Let $r$ be a rational number such that $0 \leq r < 1$. A relative counterpart of $\mathbf{b}_j(A)$ is the antichain

$$\min \left\{ b \in P - \{\hat{0}_P\} : \frac{|\mathcal{I}(b) \cap \mathcal{I}(a) \cap P^a|}{|\mathcal{I}(b) \cap P^a|} > r, \quad \forall a \in A \right\}; \quad (1.3)$$

similar constructions form the subject of the present paper.

The study of poset-theoretic generalizations of set-theoretic committees, undertaken in the paper, has been partly motivated by the need for a more detailed analysis of building blocks of decision rules in applied contradictory problems of pattern recognition. See Duda et al., 2001, on the setting of the pattern recognition problem and various methods to solve it.

Consider a finite nonempty collection $\mathcal{H} := \{H_1, \ldots, H_m\}$ of codimension one linear subspaces $H_i := \{x \in \mathbb{R}^n : \langle p_i, x \rangle = 0\}$ in the feature space $\mathbb{R}^n$ with $n \geq 2$, where any two vectors from the rank $n$ set $\{p_i : 1 \leq i \leq m\} \subset \mathbb{R}^n$ are linearly independent; $\langle p_i, x \rangle := \sum_{j=1}^{n} p_{ij}x_j$. The connected components of the complement $\mathbb{R}^n - \bigcup_{1 \leq i \leq m} H_i$ of the hyperplane arrangement $\mathcal{H}$ are called the regions (or chambers) of $\mathcal{H}$, see e.g., Orlik and Terao, 1992.

We call the arrangement of oriented hyperplanes $\mathcal{H}$ (that is the set $\mathcal{H}$ for every hyperplane $H$ of which “positive” and “negative sides” of $H$ are distinguished) a training set, if a partition $\mathcal{H} = \mathcal{A} \cup \mathcal{B}$ of $\mathcal{H}$ into two nonempty training samples $\mathcal{A}$ and $\mathcal{B}$ is given. The hyperplanes from $\mathcal{H}$ are called the training patterns. The training samples $\mathcal{A}$ and $\mathcal{B}$ are thought of as subsets of two disjoint classes $\mathcal{A}$ and $\mathcal{B}$, respectively; these classes, in general, are sets of unknown nature. We say that a pattern $H$ a priori belongs to the class $\mathcal{A}$ and it has the corresponding
label $\lambda(H) := -$, if $H \in A$; the pattern $H$ a priori belongs to the class $B$ and it has the label $\lambda(H) := +$, if $H \in B$.

A region $T$ of $H$ lies on the positive side of a hyperplane $H_i$, if the value $\langle e_i, v \rangle$ is positive for some vector $v \in T$, where the vector $e_i$ is defined by $e_i := -p_i$ for $H_i \in A$, and by $e_i := p_i$ for $H_i \in B$. Denote by $T^+_i$ the set of all regions lying on the positive side of $H_i$.

We say that a subset of regions $K^* := \{R_1, \ldots, R_t\}$ of $H$ is a committee for $H$ if for every $i$, $1 \leq i \leq m$, it holds $|K^* \cap T^+_i| > \frac{1}{2} |K^*|$. In this case a system of representatives $\{w_k \in R_k : 1 \leq k \leq t\}$ is called a committee for the homogeneous system of strict linear inequalities $\{\langle e_i, x \rangle > 0 : 1 \leq i \leq m\}$.

Committees for such inequality systems were apparently first introduced in Ablow and Kaylor, 1965, where it was proved that such very useful collective generalizations of the notion of solution do exist. Those notes laid the foundation of a branch of the theory of pattern recognition; some of the surveys in the committee mathematical methods and their applications are Khachai, 2004, Khachai et al., 2002, Mazurov, 1990, Mazurov et al., 1989, and Mazurov and Khachai, 1999, 2004.

The decision rule $r$ is the mapping $H \rightarrow \{-, +\}$ under which $r : H \mapsto \lambda(H)$; in other words, such a rule must correctly recognize the patterns from the training set.

Given a committee $\{w_k : 1 \leq k \leq t\}$ for the inequality system $\{\langle e_i, x \rangle > 0 : 1 \leq i \leq m\}$, one defines the corresponding committee decision rule $r$ in the following way: if $|\{w_k : \langle p_i, w_k \rangle > 0\}| < \frac{1}{2}$ then $r : H_i \mapsto -$; otherwise, $r : H_i \mapsto +$.

When a new pattern, that is a new oriented hyperplane $G$, is added to the training set $H$, the domain and range of the decision rule $r$, associated to the committee $\{w_k : 1 \leq k \leq t\}$, extend over the sets $H \cup G$ and $\{-, 0, +\}$, respectively. The image of $G$ under $r$ is determined depending on whether a majority of the vectors from $\{w_k : 1 \leq k \leq t\}$ lies on the positive side of $G$. The case $r(G) = 0$ means that the new pattern $G$ is not recognized.

In order to analyze the structural and combinatorial properties of the family of all possible committees for the hyperplane arrangement $H$ in detail, presumably, one may consider the Boolean lattice $P$ of all subsets of the set of regions of $H$. The language of the theory of oriented matroids (which, for example, translates the regions of $H$ to the maximal covectors of a realizable oriented matroid) may be of use; see Björner et al., 1993, on oriented matroids. Recall that the means of computing the rank of $P$, that is the number of regions of $H$, are well-known (Zaslavsky, 1975). Nonempty subsets of regions, regarded
as elements $b$ of $P$, are committees for $\mathcal{H}$ if and only if the inequalities
\[
\frac{|\mathcal{J}(b) \cap \mathcal{J}(a) \cap P^a|}{|\mathcal{J}(b) \cap P^a|} > r
\]
hold for all elements $a$ of the antichain $A := \{\mathcal{T}_1^+, \ldots, \mathcal{T}_m^+\}$ in $P$, under $r := \frac{1}{2}$. From this point of view, the elements of antichain (1.3) are committees (which are inclusion-minimal) of “high quality” for the arrangement $\mathcal{H}$.

In Section 2 of this paper, we introduce and discuss relative blocker constructions that generalize constructions (1.3). In Section 3 we turn to their absolute predecessors going back to blocking sets and set-theoretic blockers similar to (1.1). In Section 4 we remark on a connection between the concepts of absolute and relative blocking in posets. In Section 5 we analyze the structure of relative blocker constructions, and we touch on the subject of enumeration. Our exploration leads us to sequences of irreducible fractions associated to the principal order ideals in posets which are considered in Section 6 and studied, in the Boolean context, in Section 7. It turns out that all the familiar properties of the classical Farey sequences of the theory of numbers are inherited by subsequences of irreducible fractions whose nature is largely poset-theoretic. In Section 8 we apply Farey subsequences to relative blocker constructions in graded posets.

If $Q$ is a subposet of $P$ then, throughout the paper, $\max Q$ stands for the set of maximal elements of $Q$. We denote by $\mathfrak{A}_\Delta(P)$ and $\mathfrak{A}_\triangledown(P)$ distributive lattices of all antichains in $P$ defined in the following way. If $A'$ and $A''$ are antichains in $P$ then we set $A' \leq A''$ in $\mathfrak{A}_\Delta(P)$ if and only if it holds $\mathcal{J}(A') \subseteq \mathcal{J}(A'')$, and we set $A' \leq A''$ in $\mathfrak{A}_\triangledown(P)$ if and only if it holds $\mathcal{J}(A') \subseteq \mathcal{J}(A'')$. We use the notations $\hat{0}_{\mathfrak{A}_\Delta(P)}$ and $\hat{0}_{\mathfrak{A}_\triangledown(P)}$ to denote the least elements of $\mathfrak{A}_\Delta(P)$ and $\mathfrak{A}_\triangledown(P)$, respectively; we use the similar notations $\hat{1}_{\mathfrak{A}_\Delta(P)}$ and $\hat{1}_{\mathfrak{A}_\triangledown(P)}$ to denote the greatest elements. The operations of meet in $\mathfrak{A}_\Delta(P)$ and $\mathfrak{A}_\triangledown(P)$ are denoted by $\wedge_\Delta$ and $\wedge_\triangledown$, respectively; in a similar manner, $\vee_\Delta$ and $\vee_\triangledown$ stand for the operations of join. If $A'$ and $A''$ are antichains in $P$, then we have $A' \wedge_\Delta A'' = \max(\mathcal{J}(A') \cap \mathcal{J}(A''))$, $A' \vee_\Delta A'' = \max(A' \cup A'')$ and, in the dual manner, $A' \wedge_\triangledown A'' = \min(\mathcal{J}(A') \cap \mathcal{J}(A''))$, $A' \vee_\triangledown A'' = \min(A' \cup A'')$.

Recall that in the present paper the least and greatest elements of the lattice $\mathfrak{A}(P)$ are called the trivial antichains in $P$; $\hat{0}_{\mathfrak{A}_\phi(P)}$ is the empty antichain in $P$, and $\hat{1}_{\mathfrak{A}_\phi(P)}$ is the one-element antichain $\{0_P\}$.

$\mathbb{Q}$ denotes rational numbers; $\mathbb{N}$, $\mathbb{P}$, and $\mathbb{Z}$ stand for nonnegative, positive, and all integers, respectively. $i|j$ means that an integer $i$
divides an integer \( j \); \( i \perp j \) means that \( i \) and \( j \) are relatively prime, and \( \gcd(i, j) \) denotes the greatest common divisor of \( i \) and \( j \).

If \( i \) and \( j \) are positive integers then we denote by \([i, j]\) the set \( \{i, i + 1, \ldots, j\} \).

If the poset \( P \) is graded, with the rank function \( \rho : P \to \mathbb{N} \), then we write \( \rho(P) \) instead of \( \rho(\hat{1}_P) \); further, given \( j \in \{0\} \cup [1, \rho(P)] \), we denote by \( P^{(j)} \) the subset \( \{p \in P : \rho(p) = j\} \). The layer \( P^{(1)} =: P^a \) is the set of atoms of \( P \).

Recall that a subposet \( C \) of the poset \( P \) is called convex if the implication \( x, z \in C, y \in P, x \leq y \leq z \) in \( P \implies y \in C \) holds for all elements \( x, y, z \in P \). We regard the empty subposet as a convex one.

The Möbius function (see, e.g., Chapter IV of Aigner, 1979, Björner et al., 1997, Greene, 1982, and Chapter 3 of Stanley, 1997) \( \mu : P \times P \to \mathbb{Z} \) is defined in the following way: \( \mu(x, x) := 1 \), for any \( x \in P \); further, if \( z \in P \) and \( x < z \) in \( P \), then \( \mu_P(x, z) := -\sum_{y \in P : x \leq y < z} \mu_P(x, y) \); finally, if \( x \not\leq z \) in \( P \), then \( \mu_P(x, z) := 0 \).

We denote by \( \mathbb{B}(n) \) the Boolean lattice of finite rank \( n \geq 1 \). \( \mathbb{V}_q(n) \) stands for the lattice of all subspaces of a vector space of finite dimension \( n \geq 1 \) over a finite field of \( q \) elements. \( \binom{j}{i} \) and \( \binom{j}{i}_q \) denote a binomial and \( q \)-binomial coefficient, respectively.

Finally, \( r \) always denotes a rational number such that \( 0 \leq r < 1 \).

2. Relative \( r \)-blockers

Let
\[
\omega : \mathfrak{A}_\lambda(P) \to \{-1\} \cup \mathbb{N}
\]
be a map such that
\[
\hat{0}_{\mathfrak{A}_\lambda(P)} \mapsto -1, \quad \{\hat{0}_P\} \mapsto 0 ;
\]
and for any antichains \( A' \) and \( A'' \) in \( P \) such that \( \{\hat{0}_P\} < A' \leq A'' \) in \( \mathfrak{A}_\lambda(P) \), it holds
\[
0 < \omega(A') \leq \omega(A'') .
\]

From now on, \( \omega \) always means map (2.1) satisfying constraints (2.2) and (2.3). Some relevant examples of \( \omega \) follow:

\begin{itemize}
  \item \( \omega : A \mapsto \rho_\lambda(A) - 1 = |\mathfrak{I}(A)| - 1 \),
  \end{itemize}
where \( \rho_\lambda(A) \) denotes the rank of an element \( A \) in the lattice \( \mathfrak{A}_\lambda(P) \);

\begin{itemize}
  \item \( \omega : A \mapsto \rho_\lambda(A \cap P^a) - 1 = \begin{cases} -1, & \text{if } A = \hat{0}_{\mathfrak{A}_\lambda(P)} ; \\
|\mathfrak{I}(A) \cap P^a|, & \text{if } A \neq \hat{0}_{\mathfrak{A}_\lambda(P)} ; \end{cases} \) \quad (2.4)
\end{itemize}
\begin{align}
\omega : A &\mapsto \begin{cases} 
-1, & \text{if } A = \hat{0}_\mathfrak{A}(P), \\
\max_{a \in A} \rho(a), & \text{if } A \neq \hat{0}_\mathfrak{A}(P),
\end{cases}
\tag{2.5}
\end{align}

if \( P \) is graded, with the rank function \( \rho \).

The maps \( \omega \) defined by (2.1)-(2.3) are sometimes well expressed in terms of incidence functions; see, e.g., Chapter IV of Aigner, 1979, and Chapter 3 of Stanley, 1997, on incidence functions of posets.

Throughout the paper, we write \( \rho \) instead of \( \omega \) when we deal exclusively with map (2.1) defined by (2.5). If \( \{a\} \) is a one-element antichain in \( P \) then we write \( \omega(a) \) instead of \( \omega(\{a\}) \), and we write \( \omega(P) \) instead of \( \omega(\hat{1}_P) = \omega(\hat{1}_{\mathfrak{A}(P)}) \).

**Definition 2.1.** Let \( A \) be a subset of \( P \).

(i) If \( A \) is nonempty and \( A \neq \{\hat{0}_P\} \), then an element \( b \in P - \{\hat{0}_P\} \)

is a relatively \( r \)-blocking element for \( A \) in \( P \) (w.r.t. a map \( \omega \))

if, for every \( a \in A - \{\hat{0}_P\} \), it holds

\[
\frac{\omega(b \wedge a)}{\omega(b)} > r.
\tag{2.6}
\]

(ii) If \( A = \{\hat{0}_P\} \) then \( A \) has no relatively \( r \)-blocking elements in \( P \).

(iii) If \( A \) is empty then every element of \( P \) is a relatively \( r \)-blocking

element for \( A \) in \( P \).

**Remark 2.2.** Let \( A \) be a nonempty subset of \( \mathbb{B}(n) - \{\hat{0}_{\mathbb{B}(n)}\} \). An element \( b \in \mathbb{B}(n) - \{\hat{0}_{\mathbb{B}(n)}\} \)

is a relatively \( r \)-blocking element for \( A \) in \( \mathbb{B}(n) \), w.r.t.

either of the maps \( \omega \) defined by (2.2) and (2.3), if and only if the set \( \mathfrak{I}(b) \cap \mathbb{B}(n)^{(1)} \)
is an \( r \)-committee for the family \( \{\mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)} : a \in A\} \), that is, it holds

\[
|\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)}| > r \cdot |\mathfrak{I}(b) \cap \mathbb{B}(n)^{(1)}|,
\]

for all \( a \in A \).

We denote the subposet of \( P \) consisting of all relatively \( r \)-blocking elements for \( A \), w.r.t. a map \( \omega \), by \( \mathbf{I}_r(P, A; \omega) \). Given \( a \in P \), we write \( \mathbf{I}_r(P, a; \omega) \) instead of \( \mathbf{I}_r(P, \{a\}; \omega) \). If \( k \in [1, \omega(P)] \) then we denote by \( \mathbf{I}_{r,k}(P, A; \omega) \) the subposet \( \{b \in \mathbf{I}_r(P, A; \omega) : \omega(b) = k\} \).

If \( A \) is a nonempty subset of \( P - \{\hat{0}_P\} \) then Definition 2.1 implies
\( \mathbf{I}_r(P, A; \omega) = \mathbf{I}_r(P, \min A; \omega) \); this is the reason why we are primarily interested in relatively \( r \)-blocking elements for antichains.

If \( A \) is a nontrivial antichain in \( \mathbb{B}(n) \) then its order ideal \( \mathfrak{I}(A) \) is assigned the isomorphic **face poset** of the **abstract simplicial complex** whose **facets** are the sets from the family \( \{\mathfrak{I}(a) \cap \mathbb{B}(n)^{(1)} : a \in A\} \). See, e.g., Billera and Björner, 1997, Björner, 1995, Bruns and Herzog,
1998, Buchstaber and Panov, 2004, Hibi, 1992, Miller and Sturmfels, 2004, Stanley, 1996, and Ziegler, 1998, on simplicial complexes.

The following proposition lists some observations.

**Proposition 2.3.** (i) If $A$ is a nontrivial antichain in $P$, then it holds
\[ I_r(P, A; \omega) = \bigcap_{a \in A} I_r(P, a; \omega), \]
for any map $\omega$.

(ii) If $A'$ and $A''$ are antichains in $P$ and $A' \leq A''$ in $\mathcal{A}_\mathcal{V}(P)$, then
\[ I_r(P, A'; \omega) \supseteq I_r(P, A''; \omega), \]
for any map $\omega$.

(iii) Let $r', r'' \in \mathbb{Q}$, $0 \leq r' \leq r'' < 1$. For any antichain $A$ in $P$, and
for any map $\omega$, it holds
\[ I_{r'}(P, A; \omega) \supseteq I_{r''}(P, A; \omega). \]

The minimal elements of the subposets $I_r(P, A; \omega)$ of the poset $P$ are of interest.

**Definition 2.4.** (i) The relative $r$-blocker map on $\mathcal{A}_\mathcal{V}(P)$ (w.r.t. a map $\omega$) is the map $\eta_r : \mathcal{A}_\mathcal{V}(P) \rightarrow \mathcal{A}_\mathcal{V}(P)$, defined by
\[ A \mapsto \min I_r(P, A; \omega) \]
\[ = \min \left\{ b \in P - \{0_P\} : \frac{\omega(\{b\} \triangle \{a\})}{\omega(b)} > r \quad \forall a \in A \right\} \]
if $A$ is nontrivial, and
\[ 0_{\mathcal{A}_\mathcal{V}(P)} \mapsto 1_{\mathcal{A}_\mathcal{V}(P)}, \quad 1_{\mathcal{A}_\mathcal{V}(P)} \mapsto 0_{\mathcal{A}_\mathcal{V}(P)}. \]

(ii) Given an antichain $A$ in $P$, the antichain $\eta_r(A)$ is called the relative $r$-blocker (w.r.t. the map $\omega$) of $A$ in $P$; the elements of $\eta_r(A)$ are called the minimal relatively $r$-blocking elements (w.r.t. the map $\omega$) for $A$ in $P$.

In addition to the minimal relatively $r$-blocking elements, the relatively $r$-blocking elements $b$ for $A$ in $P$ with the minimum value of $\omega(b)$ can be of particular interest.

The following statement is a consequence of Proposition 2.3(ii,iii). It particularly states that the relative $r$-blocker map is order-reversing.

**Corollary 2.5.** Let $r', r'' \in \mathbb{Q}$, $0 \leq r' \leq r'' < 1$. Let $A'$ and $A''$ be antichains in $P$ such that $A' \leq A''$ in $\mathcal{A}_\mathcal{V}(P)$. The relation
\[ \eta_{r''}(A'') \leq \eta_{r'}(A'') \leq \eta_{r'}(A') \]
holds in $\mathcal{A}_\mathcal{V}(P)$. 
Let $A$ be a nontrivial antichain in $P$. If the relative $r$-blocker $\eta_r(A)$ of $A$ in $P$ (w.r.t. a map $\omega$) is not $\hat{0}_{\mathcal{A}_\omega(P)}$, then $A$ is a subset of relatively $r'$-blocking elements for the antichain $\eta_r(A)$, for some $r' \in \mathbb{Q}$. Indeed, for each $a \in A$ and for all $b \in \eta_r(A)$, we by (2.6) have
\[
\frac{\omega(\{a\} \land \{b\})}{\omega(a)} > r \cdot \frac{\omega(b)}{\omega(a)} \geq r \cdot \frac{\min_{p \in \eta_r(A)} \omega(p)}{\max_{p \in A} \omega(p)},
\]
and this observation implies the following statement.

**Proposition 2.6.** If $A$ is a nontrivial antichain in $P$ and $\eta_r(A) \neq \hat{0}_{\mathcal{A}_\omega(P)}$, w.r.t. a map $\omega$, then
\[
A \subseteq I_r(P, \eta_r(A); \omega) ,
\]
where $r' := r \cdot \frac{\min_{p \in \eta_r(A)} \omega(p)}{\max_{p \in A} \omega(p)}$.

3. **Absolute $j$-blockers and convex subposets**

Let $A$ be a nontrivial antichain in $P$. Let $h$ and $k$ be positive integers such that $h \leq k \leq \omega(P)$, for some map $\omega$. In the following sections of the paper we will make use of the auxiliary subposet
\[
\{b \in P : \omega(b) = k, \omega(\{b\} \land \{a\}) \geq h \ \forall a \in A\} . \tag{3.1}
\]
We can consider this subposet, in an equivalent way, as the intersection
\[
\left( \{b \in P : \omega(b) > k - 1\} - \{b \in P : \omega(b) > k\} \right) \cap \{b \in P : \omega(\{b\} \land \{a\}) > h - 1 \ \forall a \in A\} . \tag{3.2}
\]
Each component of expression (3.2) can be described in terms of absolute blocking. Indeed, given a nontrivial antichain $A$ in $P$ and a non-negative integer $j$ less than $\omega(P)$, define the absolute $j$-blocker (w.r.t. the map $\omega$) of $A$ in $P$, denoted by $b_j(A)$, in the following way:
\[
b_j(A) := \min \{b \in P : \omega(\{b\} \land \{a\}) > j \ \forall a \in A\} . \tag{3.3}
\]
For any element $b \in \mathfrak{F}(b_j(A))$, we have $\omega(\{b\} \land \{a\}) > j$, for all $a \in A$. A particular example of absolute $j$-blocker (3.3) is the construction defined by (1.1) and implicitly involving the map $\omega$ defined by (2.4).

We set $b_{\omega(P)}(A) := \hat{0}_{\mathcal{A}_\omega(P)}$. Note that
\[
b_j(A) = \bigwedge_{a \in A} b_j(a) \tag{3.4}
\]
in $\mathcal{A}_\omega(P)$; we write $b_j(a)$ instead of $b_j(\{a\})$. If the trivial antichains in $P$ must be taken into consideration then we set
\[
b_j(\hat{0}_{\mathcal{A}_\omega(P)}) := \hat{1}_{\mathcal{A}_\omega(P)} , \quad b_j(\hat{1}_{\mathcal{A}_\omega(P)}) := \hat{0}_{\mathcal{A}_\omega(P)} . \tag{3.5}
\]
Given an antichain $A$ in $P$ and a map $\omega$, we call the elements of the order filter $\mathfrak{F}(b_j(A))$ the \textit{absolutely j-blocking elements for} $A$ in $P$ (w.r.t. the map $\omega$). The elements of the order filter $\mathfrak{F}(b(A))$, where the antichain $b(A)$ is defined by (1.2), were called in Matveev, 2001, the \textit{intersecters for} $A$ in $P$.

If $P$ is graded, and if the map $\omega$ is defined by (2.5) then, given a nontrivial one-element antichain $\{a\}$ in $P$, we have

$$b_j(a) = \mathfrak{I}(a) \cap P^{(j+1)}.$$  

The \textit{absolute j-blocker map} $b_j : \mathfrak{A}_\tau(P) \to \mathfrak{A}_\tau(P)$ is order-reversing, w.r.t. any map $\omega$. If $A$ is an arbitrary antichain in $P$ then for any nonnegative integers $i$ and $j$ such that $i \leq j < \omega(P)$, the relation

$$b_i(A) \geq b_j(A)$$  

holds in $\mathfrak{A}_\tau(P)$.

If $A$ is a trivial antichain in $P$ then convention (3.5) implies $b_j(b_j(A)) = A$. Now, let $A$ be a nontrivial antichain. If $b_j(A) = 0_{\mathfrak{A}_\tau(P)}$, then we have $b_j(b_j(A)) = 1_{\mathfrak{A}_\tau(P)} > A$ in $\mathfrak{A}_\tau(P)$. Finally, suppose that $b_j(A)$ is a nontrivial antichain in $P$. On the one hand, for each $a \in A$ and for all $b \in b_j(A)$, we have $\omega(\{a\} \triangle \{b\}) > j$. On the other hand, (3.3) implies

$$b_j(b_j(A)) = \min\left\{g \in P : \omega(\{g\} \triangle \{b\}) > j \quad \forall b \in b_j(A)\right\}.$$  

Hence we have

$$b_j(b_j(A)) \geq A$$  

in $\mathfrak{A}_\tau(P)$, for any $A \in \mathfrak{A}_\tau(P)$.

Since $b_j$ is order-reversing and (3.5) holds, the technique of the Galois correspondence (see, e.g., Sections IV.3.B,A of Aigner, 1979) can be applied to the absolute $j$-blocker map $b_j$ on $\mathfrak{A}_\tau(P)$:

\textbf{Proposition 3.1.} Let $b_j : \mathfrak{A}_\tau(P) \to \mathfrak{A}_\tau(P)$ be the absolute $j$-blocker map on $\mathfrak{A}_\tau(P)$, w.r.t. a map $\omega$.

(i) The composite map $b_j \circ b_j$ is a closure operator on $\mathfrak{A}_\tau(P)$.

(ii) The image $b_j(\mathfrak{A}_\tau(P))$ of the lattice $\mathfrak{A}_\tau(P)$ under the map $b_j$ is a self-dual lattice; the restriction of the map $b_j$ to $b_j(\mathfrak{A}_\tau(P))$ is an anti-automorphism of $b_j(\mathfrak{A}_\tau(P))$. As a consequence, for any antichain $B \in b_j(\mathfrak{A}_\tau(P))$ it holds $b_j(b_j(B)) = B$.

The lattice $b_j(\mathfrak{A}_\tau(P))$ is a sub-meet-semilattice of $\mathfrak{A}_\tau(P)$.

(iii) For any $B \in b_j(\mathfrak{A}_\tau(P))$, its preimage $(b_j)^{-1}(B)$ in $\mathfrak{A}_\tau(P)$ under the map $b_j$ is a convex sub-join-semilattice of $\mathfrak{A}_\tau(P)$; the greatest element of $(b_j)^{-1}(B)$ is $b_j(B)$.
Proof. Assertions (i) and (ii) are consequences of Propositions 4.36 and 4.26 of Aigner, 1979.

To prove assertion (iii), pick arbitrary elements \( A', A'' \in (b_j)^{-1}(B) \), where \( B = b_j(A) \), for some \( A \in \mathcal{A}_\mathcal{V}(P) \), and note that \( b_j(A' \vee A'') = b_j(A') \wedge b_j(A'') = B \). Thus, \( (b_j)^{-1}(B) \) is a sub-join-semilattice of \( \mathcal{A}_\mathcal{V}(P) \). If \( B = 0_{\mathcal{A}_\mathcal{V}(P)} \) then \( b_j(B) = 1_{\mathcal{A}_\mathcal{V}(P)} \) is the greatest element of \( (b_j)^{-1}(B) \). If \( B = 1_{\mathcal{A}_\mathcal{V}(P)} \) then \( (b_j)^{-1}(B) \) is the one-element subposet \( \{0_{\mathcal{A}_\mathcal{V}(P)}\} \subset \mathcal{A}_\mathcal{V}(P) \). Finally, if \( B \) is a nontrivial antichain in \( P \) then the element \( b_j(B) = b_j(b_j(A)) \) is by (3.7) the greatest element of \( (b_j)^{-1}(B) \). Since the map \( b_j \) is order-reversing, the subposet \( (b_j)^{-1}(B) \) of \( \mathcal{A}_\mathcal{V}(P) \) is convex. \( \square \\

Remark 3.2. Let \( A \) be an arbitrary antichain in the Boolean lattice \( \mathbb{B}(n) \). The antichain \( b(A) \) defined by (3.1.2) satisfies the equality \( |\mathcal{F}(A)| = |\mathcal{F}(b(A))| = 2^n \). As a consequence, we have \( A = b(A) \) if and only if it holds \( |\mathcal{F}(A)| = 2^{n-1} \). In other words, the layer \( \mathcal{A}_\mathcal{V}(\mathbb{B}(n))^{(2^{n-1})} \) of \( \mathcal{A}_\mathcal{V}(\mathbb{B}(n)) \) is the set of fixed points of the map \( b \). Indeed, we have \( b(\mathcal{A}_\mathcal{V}(\mathbb{B}(n))) = \mathcal{A}_\mathcal{V}(\mathbb{B}(n)) \), and our observations follow immediately from Proposition 3.1(ii).

We now return to consider poset (3.1),(3.2). Note that

\[
\{b \in P : \omega(b) > k - 1\} = \mathcal{F}(b_{k-1}(1_P)), \\
\{b \in P : \omega(b) > k\} = \mathcal{F}(b_k(1_P)),
\]

therefore we obtain

\[
\{b \in P : \omega(b) = k, \omega(\{b \wedge \Delta \{a\}\}) \geq h \quad \forall a \in A\} = \mathcal{F}(b_{h-1}(A)) \;
\]

Since \( b_{k-1}(1_P) \geq b_k(1_P) \) in \( \mathcal{A}_\mathcal{V}(P) \), by (3.6), the second line in expression (3.9) describes an intersection of convex subposets of \( P \); hence the subposet presented in the first line of (3.9) is convex.

Again, let \( h \) and \( k \) be positive integers such that \( h \leq k \leq \omega(P) \). Let \( \{a\} \) be a nontrivial one-element antichain in \( P \). In the following, in addition to subposet (3.1),(3.2),(3.9), we will also need the convex subposet

\[
\{b \in P : \omega(b) = k, \omega(\{b \wedge \Delta \{a\}\}) = h\}
\]

Since \( b_{k-1}(1_P) \geq b_k(1_P) \) in \( \mathcal{A}_\mathcal{V}(P) \), by (3.6), the second line in expression (3.9) describes an intersection of convex subposets of \( P \); hence the subposet presented in the first line of (3.9) is convex.

Again, let \( h \) and \( k \) be positive integers such that \( h \leq k \leq \omega(P) \). Let \( \{a\} \) be a nontrivial one-element antichain in \( P \). In the following, in addition to subposet (3.1),(3.2),(3.9), we will also need the convex subposet

\[
\{b \in P : \omega(b) = k, \omega(\{b \wedge \Delta \{a\}\}) = h\}
\]

Since \( b_{k-1}(1_P) \geq b_k(1_P) \) in \( \mathcal{A}_\mathcal{V}(P) \), by (3.6), the second line in expression (3.9) describes an intersection of convex subposets of \( P \); hence the subposet presented in the first line of (3.9) is convex.

Again, let \( h \) and \( k \) be positive integers such that \( h \leq k \leq \omega(P) \). Let \( \{a\} \) be a nontrivial one-element antichain in \( P \). In the following, in addition to subposet (3.1),(3.2),(3.9), we will also need the convex subposet

\[
\{b \in P : \omega(b) = k, \omega(\{b \wedge \Delta \{a\}\}) = h\}
\]

Since \( b_{k-1}(1_P) \geq b_k(1_P) \) in \( \mathcal{A}_\mathcal{V}(P) \), by (3.6), the second line in expression (3.9) describes an intersection of convex subposets of \( P \); hence the subposet presented in the first line of (3.9) is convex.

Again, let \( h \) and \( k \) be positive integers such that \( h \leq k \leq \omega(P) \). Let \( \{a\} \) be a nontrivial one-element antichain in \( P \). In the following, in addition to subposet (3.1),(3.2),(3.9), we will also need the convex subposet

\[
\{b \in P : \omega(b) = k, \omega(\{b \wedge \Delta \{a\}\}) = h\}
\]

Since \( b_{k-1}(1_P) \geq b_k(1_P) \) in \( \mathcal{A}_\mathcal{V}(P) \), by (3.6), the second line in expression (3.9) describes an intersection of convex subposets of \( P \); hence the subposet presented in the first line of (3.9) is convex.
Remark 3.3. Let \( h, k, m \) and \( n \) be positive integers such that \( m \leq n \) and \( h \leq k \leq n \). Recall that if \( \{a\} \) is a nontrivial one-element antichain in \( \mathbb{V}_q(n) \) with \( \rho(a) =: m \), then we have

\[
\{ b \in \mathbb{V}_q(n) : \rho(b) = k, \rho(b \land a) \geq h \} = \mathfrak{F} \left( \mathfrak{I}(a) \cap \mathbb{V}_q(n)^{(h)} \right) \cap \mathbb{V}_q(n)^{(k)}
\]

and

\[
|\{ b \in \mathbb{V}_q(n) : \rho(b) = k, \rho(b \land a) \geq h \}| = \sum_{j \in [h,k]} \binom{m}{j} \binom{n-m}{k-j} q^{(m-j)(k-j)}.
\]

Similarly, we have

\[
\{ b \in \mathbb{V}_q(n) : \rho(b) = k, \rho(b \land a) = h \}
\]

\[
= \left( \mathfrak{F} \left( \mathfrak{I}(a) \cap \mathbb{V}_q(n)^{(h)} \right) - \mathfrak{F} \left( \mathfrak{I}(a) \cap \mathbb{V}_q(n)^{(h+1)} \right) \right) \cap \mathbb{V}_q(n)^{(k)}
\]

and

\[
|\{ b \in \mathbb{V}_q(n) : \rho(b) = k, \rho(b \land a) = h \}| = \binom{m}{h} \binom{n-m}{k-h} q^{(m-h)(k-h)}.
\]

These expressions for the cardinalities of subposets have a direct connection with the \((q-)\)Vandermonde's convolution, see, e.g., Section 4 of Andrews, 1974.

4. Connection between concepts of absolute and relative blocking

It follows from Definition 2.4 that the relative 0-blocker \( \mathfrak{b}_0(A) \) of a nontrivial antichain \( A \) in \( P \), w.r.t. an arbitrary map \( \omega \), is nothing else than the absolute 0-blocker \( \mathfrak{b}(A) \) of \( A \) in \( P \), defined by (1.2) and considered in Björner et al., 2004, 2005, and Matveev, 2001. Moreover, if \( \mathfrak{b}(A) \subseteq \mathfrak{P}_a \) then \( \bigcap_{a \in A} \mathfrak{I}(a) = \{0_P\} \subseteq \mathfrak{I}_r(P;A;\omega) \) and \( \mathfrak{b}_r(A) = \mathfrak{b}(A) \), for any value of the parameter \( r \).

Again, let \( A \) be a nontrivial antichain in \( P \), and let \( j \in \mathbb{N}, j < \omega(P) \), for some map \( \omega \). If \( \mathfrak{b}_j(A) \neq \mathfrak{0}_{\mathfrak{A}_b(P)} \) then, for all \( b \in \mathfrak{b}_j(A) \) and for all \( a \in A \), we by (3.3) have

\[
\frac{\omega(\{b\} \land_a \{a\})}{\omega(b)} > \frac{j}{\max_{p \in \mathfrak{b}_j(A)} \omega(p)}
\]

\[
\frac{\omega(\{a\} \land_a \{b\})}{\omega(a)} > \frac{j}{\max_{p \in A} \omega(p)}
\]

if \( \mathfrak{b}_r(A) \neq \mathfrak{0}_{\mathfrak{A}_b(P)} \) then, for each \( b \in \mathfrak{b}_r(A) \) and for all \( a \in A \), we by (2.6) have

\[
\omega(\{b\} \land_a \{a\}) > r \cdot \omega(b) \geq r \cdot \min_{p \in \mathfrak{b}_r(A)} \omega(p).
\]
5. Structure and enumeration

We now turn to explore the structure of the subposets of relatively $r$-blocking elements.

For $k \in \mathbb{P}$ such that $k \leq \omega(P)$, define the integer

$$\nu(r \cdot k) := \begin{cases} \lceil r \cdot k \rceil & \text{if } r \cdot k \notin \mathbb{N}, \\ r \cdot k + 1, & \text{if } r \cdot k \in \mathbb{N}. \end{cases} \quad (5.1)$$

If $A$ is a nontrivial antichain in $P$, then it follows from Definition 2.1(i) that it holds

$$I_r(P, A; \omega) = \bigcup_{1 \leq k \leq \omega(P)} \bigcup_{\nu(r \cdot k) \leq h \leq \max_{a \in A} \omega(a)} \{ b \in P : \omega(b) = k, \omega(\{b\} \wedge \{a\}) \geq h \ \forall a \in A \}. \quad (5.2)$$

Recall that for any values of $h$ and $k$ appearing in the above expression, the structure of the poset $\{ b \in P : \omega(b) = k, \omega(\{b\} \wedge \{a\}) \geq h \ \forall a \in A \}$ is described in (3.9). Further, for any $h \geq \nu(r \cdot k)$, we by (3.6) have $\mathfrak{F}(b_{\nu(r \cdot k)}(A)) = \mathfrak{F}(b_{h-1}(A))$, so (5.2) reduces to

$$I_r(P, A; \omega) = \bigcup_{1 \leq k \leq \omega(P)} \{ b \in P : \omega(b) = k, \omega(\{b\} \wedge \{a\}) \geq \nu(r \cdot k) \ \forall a \in A \},$$

and we come to the following conclusion.

**Proposition 5.1.** Let $A$ be a nontrivial antichain in $P$.

(i) For any map $\omega$, it holds

$$I_r(P, A; \omega) = \bigcup_{1 \leq k \leq \omega(P)} \left( \mathfrak{F}(b_{k-1}(\hat{1}_P)) - \mathfrak{F}(b_k(\hat{1}_P)) \right) \cap \mathfrak{F}(b_{\nu(r \cdot k)-1}(A)).$$

(ii) If $P$ is graded, then

$$I_r(P, A; \rho) = \bigcup_{k \in [1, \rho(P)] : \nu(r \cdot k) \leq \min_{a \in A} \rho(a)} \left( P^{(k)} \cap \mathfrak{F}(b_{\nu(r \cdot k)-1}(A)) \right). \quad (5.3)$$

To find the cardinality of subposet (5.3), we can use the combinatorial inclusion-exclusion principle (see, e.g., Chapter IV of Aigner, 1979, ...
and Chapter 2 of Stanley, 1997). Under the hypothesis of Proposition 5.1(ii), we have

\[ |I_r(P, A; \rho)| = \sum_{k \in [1, \rho(P)]: \nu(r \cdot k) \leq \min_{a \in A} \rho(a)} \left( -1 \right)^{|C|-1} \cdot \left| P(k) \cap \mathfrak{I}(C) \cap P(\nu(r \cdot k)) \right| \]

\[ = \sum_{k \in [1, \rho(P)]: \nu(r \cdot k) \leq \min_{a \in A} \rho(a)} \left( \sum_{C \subseteq A: |C| > 0} \left( -1 \right)^{|C|-1} \cdot \left| P(k) \cap \mathfrak{I}(C) \right| \right) \cdot \left( \sum_{E \subseteq P(\nu(r \cdot k)) \cap \mathfrak{J}(A): |E| > 0} \left( -1 \right)^{|E|-1} \cdot \left| n - \rho \left( \bigvee_{e \in E} e \right) \right| \right). \]

For the remainder of the present section, let \( A \) be a nontrivial antichain in a graded lattice \( P \) of rank \( n \), with the property: each interval of length \( k \) in \( P \) contains the same number \( B(k) \) of maximal chains; in other words, we suppose \( P \) to be a principal order ideal of some binomial poset, see Section 3.15 of Stanley, 1997. The function \( B(k) \) is called the binomial function of \( P \); it holds \( B(0) = B(1) = 1 \). The number of elements of rank \( i \) in any interval of length \( j \) is denoted by \( \left[ j \atop i \right] \); it holds \( \left[ j \atop i \right] = \frac{B(j)}{B(i) \cdot B(j-i)} \). If \( P \) is \( B(n) \) or \( V_q(n) \), then \( \left[ j \atop i \right] = \left( \begin{array}{c} j \\ i \end{array} \right) \) or \( \left[ j \atop i \right] = \left( \begin{array}{c} j \\ i \end{array} \right)_q \), respectively.

Given \( k \in [1, n] \) such that \( \nu(r \cdot k) \leq \min_{a \in A} \rho(a) \), we have

\[ |I_{r,k}(P, A; \rho)| = \sum_{C \subseteq A: |C| > 0} \left( -1 \right)^{|C|-1} \cdot \left( \sum_{E \subseteq P(\nu(r \cdot k)) \cap \mathfrak{J}(A): |E| > 0} \left( -1 \right)^{|E|-1} \cdot \left| n - \rho \left( \bigvee_{e \in E} e \right) \right| \right) \cdot \left( \sum_{C \subseteq A: E \subseteq \mathfrak{J}(C)} \left( -1 \right)^{|C|} \cdot \left| n - \rho \left( \bigvee_{e \in E} e \right) \right| \right). \]

Indeed, for example, the sum

\[ \sum_{E \subseteq P(\nu(r \cdot k)) \cap \mathfrak{J}(A): |E| > 0} \left( -1 \right)^{|E|-1} \cdot \left| n - \rho \left( \bigvee_{e \in E} e \right) \right| \]
counts the number of elements of the layer $P^{(k)}$ comparable with, at least, one element of the antichain $P^{(\nu(r-k))} \cap \mathcal{I}(C)$.

To refine expression (5.4) with the help of the technique of the M"{o}bius function, consider some auxiliary lattices which can be associated to the antichain $A$. The first one, denoted by $\mathcal{C}_{r,k}(P, A)$, is the lattice consisting of all sets from the family $\{P^{(\nu(r-k))} \cap \mathcal{I}(C) : C \subseteq A\}$ ordered by inclusion. The greatest element of $\mathcal{C}_{r,k}(P, A)$ is the set $P^{(\nu(r-k))} \cap \mathcal{I}(A)$. The least element of $\mathcal{C}_{r,k}(P, A)$, denoted by $\hat{0}$, is the empty subset of $P^{(\nu(r-k))}$. The remaining lattices, denoted by $\mathcal{E}_{r,k}(P, X)$, where $X$ are nonempty subsets of $P^{(\nu(r-k))} \cap \mathcal{I}(A)$, are defined in the following way. Given an antichain $X \subseteq P^{(\nu(r-k))} \cap \mathcal{I}(A)$, the poset $\mathcal{E}_{r,k}(P, X)$ is the sub-join-semilattice of the lattice $P$ generated by $X$ and augmented with a new least element, denoted by $\hat{0}$ (it is regarded as the empty subset of $P$). The greatest element of $\mathcal{E}_{r,k}(P, X)$ is the join $\bigvee_{x \in X} x$ in $P$. We have

$$\left|\mathbf{I}_{r,k}(P, A; \rho)\right| = \sum_{X \in \mathcal{C}_{r,k}(P, A): \hat{0} < X} \mu_{\mathcal{C}_{r,k}(P, A)}(\hat{0}, X) \cdot \sum_{z \in \mathcal{E}_{r,k}(P, X): \hat{0} < z, \rho(z) \leq k} \mu_{\mathcal{E}_{r,k}(P, X)}(\hat{0}, z) \cdot \left[\frac{n - \rho(z)}{n - k}\right],$$

(5.6)

where $\rho(\cdot)$ means the rank in $P$, and where, for example, the sum

$$- \sum_{z \in \mathcal{E}_{r,k}(P, X): \hat{0} < z, \rho(z) \leq k} \mu_{\mathcal{E}_{r,k}(P, X)}(\hat{0}, z) \cdot \left[\frac{n - \rho(z)}{n - k}\right]$$

is equivalent to sum (5.4) under $X = P^{(\nu(r-k))} \cap \mathcal{I}(C)$.

If $P$ is $\mathbb{B}(n)$ then, in view of Remark 2.2, formulas (5.4) and (5.6) give, for a nontrivial clutter, the number of all its $r$-committees of cardinality $k$.

**Example 5.2.** Figure 1 depicts the Hasse diagram of a Boolean lattice of rank four, its antichain $A := \{a_1, a_2\}$, and lattices $\mathcal{C} := \mathcal{C}_{4,3}(\mathbb{B}(4), A)$ and $\mathcal{E} := \mathcal{E}_{3,3}(\mathbb{B}(4), \mathbb{B}(4)(2) \cap \mathcal{I}(A))$. To compute the number of elements in $\mathbf{I}_{4,3}(\mathbb{B}(4), A; \rho)$, note that $\mu_{\mathcal{C}}(\hat{0}, \{p_1, p_2, p_3\}) = \mu_{\mathcal{C}}(\hat{0}, \{a_2\}) = -1$ and $\mu_{\mathcal{C}}(\hat{0}, \{p_1, p_2, p_3, a_2\}) = 1$. Further, we have $\mu_{\mathcal{E}}(\hat{0}, p_1) = \mu_{\mathcal{E}}(\hat{0}, p_2) = \mu_{\mathcal{E}}(\hat{0}, p_3) = \mu_{\mathcal{E}}(\hat{0}, a_2) = -1$, $\mu_{\mathcal{E}}(\hat{0}, a_1) = 2$, $\mu_{\mathcal{E}}(\hat{0}, p_4) = \mu_{\mathcal{E}}(\hat{0}, p_5) = 1$ and $\mu_{\mathcal{E}}(\hat{1}, \hat{1}) = -1$.

By means of (5.6), we obtain $\left|\mathbf{I}_{4,3}(\mathbb{B}(4), A; \rho)\right| = |\{p_4, p_5\}| = 2$. 
Proposition 5.1(ii) provides us with a general description of the subposets of relatively \( r \)-blocking elements in graded posets. The aim of Section 8 of the present paper is to explore the structure of the above-mentioned subposets in detail; with the help of Theorem 8.4 we will exclude from consideration some layers of graded posets that certainly contain no relatively \( r \)-blocking elements.

6. Principal order ideals and Farey subsequences

Let \( \{a\} \) be a one-element antichain in \( P \). Define the sequence of irreducible fractions

\[
\mathcal{F}(P, a; \omega) := \left\{ \begin{array}{c} 0 \\ 1 \\ \frac{1}{1} \end{array} \right\}
\]

\[
\bigcup \left( \frac{\omega(b)}{\gcd(\omega(b), \omega(b))} : b \in P - \{0_P\} \right)
\]

arranged in ascending order.

Recall that the Farey sequence \( \mathcal{F}_n \) of order \( n \in \mathbb{P} \) is defined to be the ascending sequence of all irreducible fractions between 0 and 1 whose denominators do not exceed \( n \), see, e.g., Chapter 27 of Buchstab, 1967, Chapter 4 of Graham et al., 1994, Chapter III of Hardy and Wright, 1979, and Lagarias and Tresser, 1995. Thus, \( \mathcal{F}(P, a; \omega) \) is a subsequence of the Farey sequence of order \( \omega(P) \).

We always index the fractions from \( \mathcal{F}(P, a; \omega) \) starting with zero:

\[
\mathcal{F}(P, a; \omega) = \left\{ f_0 := 0 < f_1 < f_2 \cdots < f_{|\mathcal{F}(P, a; \omega)|-1} := \frac{1}{1} \right\}
\]
In the present paper, we do not deal with the more general ascending Farey subsequences \( \bigcup_{a \in A} \mathcal{F}(P; a; \omega) \) and \( \bigcap_{a \in A} \mathcal{F}(P; a; \omega) \) associated to nonempty antichains \( A \) in \( P \); such sequences can also be of interest.

Order-preserving maps \( P \to \mathbb{P} \) and \( P \to \mathbb{P}^* \), where \( \mathbb{P}^* \) are positive integers ordered by divisibility, are discussed, e.g., in Smith, 1967, 1969, 1970/1971.

See Pătraşcu and Pătraşcu, 2004, on algorithmic aspects of the Farey sequences.

7. Farey subsequences in Boolean context

In this section we deal almost exclusively with the Boolean lattice \( \mathbb{B}(n) \). Let \( a \) be an arbitrary element of \( \mathbb{B}(n) \), of rank \( m := \rho(a) \).

Consider the Farey subsequence \( \mathcal{F}(\mathbb{B}(n), a; \rho) \) associated to the principal order ideal \( \mathcal{I}(a) \) of \( \mathbb{B}(n) \). The sequences \( \mathcal{F}(\mathbb{B}(n), a; \rho) \) are the same, for all elements \( a \) of rank \( m \) in \( \mathbb{B}(n) \), and we write \( \mathcal{F}(\mathbb{B}(n), m; \rho) \) instead of \( \mathcal{F}(\mathbb{B}(n), a; \rho) \). For any element \( b \in \mathbb{B}(n) - \{0_{\mathbb{B}(n)}\} \), we have \( \rho(a) + \rho(b) - \rho(a \land b) = m + \rho(b) - \rho(a \land b) \leq n \); moreover, \( 0 \leq \rho(a \land b) \leq \rho(a) =: m \), so we are interested in the ascending Farey subsequence

\[
\mathcal{F}(\mathbb{B}(n), m; \rho) = \left\{ \frac{0}{1}, \frac{1}{1} \right\} \cup \left\{ \frac{h}{k} \in \mathcal{F}_n : h \leq m, \ k - h \leq n - m \right\} . \tag{7.1}
\]

**Example 7.1.**

\[
\mathcal{F}_6 = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\mathcal{F}(\mathbb{B}(6), 6; \rho) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\left( \frac{h}{k} \in \mathcal{F}_6 : h \leq 5 \right) = \mathcal{F}_6 ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 5 \right) = \mathcal{F}_6 ,
\]

\[
\mathcal{F}(\mathbb{B}(6), 5; \rho) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\left( \frac{h}{k} \in \mathcal{F}_6 : h \leq 4 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 4 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 3 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 3 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 2 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 2 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 1 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h \leq 1 \right) = \left\{ \frac{0}{1}, \frac{1}{1} < \frac{1}{3} < \frac{1}{2} < \frac{1}{4} < \frac{1}{5} < \frac{1}{6} > \frac{2}{5} > \frac{2}{3} > \frac{3}{4} > \frac{3}{5} > \frac{\bar{2}}{5} > \frac{5}{6} > \frac{1}{1} \right\} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h = 0 \right) = \{ \frac{0}{1} \} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h = 0 \right) = \{ \frac{0}{1} \} ,
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h = 0 \right) = \{ \frac{0}{1} \} .
\]

\[
\frac{h}{k} \in \mathcal{F}_6 : h = 0 \right) = \{ \frac{0}{1} \} .
\]
Remark 7.2. In the sequence $F(\mathbb{B}(n), m; \rho)$ such that $0 < m < n$, we have

$$f_0 = \frac{0}{1}, \quad f_1 = \frac{1}{n-m+1}, \quad f_{|F(\mathbb{B}(n), m; \rho)|-2} = \frac{m}{m+1}, \quad f_{|F(\mathbb{B}(n), m; \rho)|-1} = \frac{1}{1}.$$ 

Let $n \in \mathbb{P}$, and let $S$ be a subset of $[1, n]$. We denote by $\phi(n; S)$ the number of elements from $S$ that are relatively prime to $n$:

$$\phi(n; S) := |\{s \in S : \ n \perp s\} ;$$  \hspace{2cm} (7.2)

thus, $\phi(n; [1, n])$ is the Euler function.

Given a positive integer $i$ such that $i \leq n$, we have $\phi(n; [1, i]) = \sum_{d \in [1,i]}: \ d|n \ \nu(d) \cdot \left\lfloor \frac{i}{d} \right\rfloor$, where $\nu(\cdot)$ stands for the number-theoretic M"obius function: $\nu(1) := 1$; if $p^k|d$, for some prime $p$, then $\nu(d) := 0$; if $d = p_1p_2\ldots p_s$, for distinct primes $p_1, p_2, \ldots, p_s$, then $\nu(d) := (-1)^s$. Thus, given a nonempty subset $\{i'+1, i''\} \subseteq [1, n]$, we have $\phi(n; [i'+1, i'']) = \sum_{d \in [1,i'']}: \ d|n \ \nu(d) \cdot \left(\left\lfloor \frac{i'}{d} \right\rfloor - \left\lfloor \frac{i''}{d} \right\rfloor\right)$.

Proposition 7.3. (i) If $f_t \in F(\mathbb{B}(n), m; \rho) - \{\frac{1}{1}\}$, where $0 < m < n$, then

$$t = \sum_{j \in [1,n]} \phi \left( j; \left\lfloor \max\{1, j + \min\{m, \lfloor j \cdot f_t \rfloor\} - n\} , \min\{m, \lfloor j \cdot f_t \rfloor\} \right\rfloor \right).$$

(ii) The cardinality of the sequence $F(\mathbb{B}(n), m; \rho)$, where $0 < m < n$, equals

$$1 + \sum_{j \in [1,n]} \phi \left( j; \left\lfloor \min\{m, j\} \right\rfloor \right) - \sum_{j \in [n/2+1,n]} \phi \left( j; \left\lfloor 1, 2 \cdot j - n - 1\right\rfloor \right) - \sum_{j \in [n-m+2,n]} \phi \left( j; \left\lfloor 1, j + m - n - 1\right\rfloor \right).$$

Proof. To prove assertion (i), replace $f_t$ with $\frac{i \cdot f_t}{j}$, for every $j \in [1, n]$. According to description (7.1), $t$ equals

$$\sum_{j \in [1,n]} |\{i \in [1, j] : i \perp j, \ \max\{1, j + \min\{m, j \cdot f_t\} - n\} \leq i \leq \min\{m, j \cdot f_t\} \} |,$$

from where the assertion follows, with respect to description (7.2) of the function $\phi(\cdot; \cdot)$. Assertion (i) implies assertion (ii), due to our convention that $\frac{1}{1}$ is the terminal fraction in the sequence $F(\mathbb{B}(n), m; \rho)$,
with the index $|F(\mathbb{B}(n), m; \rho)| - 1$. Indeed, we have

$$|F(\mathbb{B}(n), m; \rho)| - 1 = \sum_{j \in [1, n]} \phi(j; \max\{1, j + \min\{m, j\} - n\}, \min\{m, j\})$$

$$= \sum_{j \in [1, m]} \phi(j; \max\{1, 2j - n\}, j) + \sum_{j \in [m+1, n]} \phi(j; \max\{1, j+m-n\}, m),$$

and assertion (ii) follows. $\Box$

The sum $1 + \sum_{j \in [1, n]} \phi(j; [1, \min\{m, j\}])$ appearing in Proposition 7.3(ii) counts the number of fractions in the sequence $(\frac{a}{b} \in F_n : h \leq m)$, see Remark 7(ii)(b) below.

Description (7.1) of Farey subsequences leads up to the following observation.

**Proposition 7.4.** The map

$$F(\mathbb{B}(n), m; \rho) \to F(\mathbb{B}(n), n - m; \rho), \quad \frac{h}{k} \mapsto \frac{k-h}{k}$$

is order-reversing and bijective, for any $m$, $0 \leq m \leq n$.

We now explore the properties of Farey subsequences (7.1).

**Proposition 7.5.** Let $\frac{h}{k} \in F(\mathbb{B}(n), m; \rho)$, where $0 < m < n$; suppose that $\frac{0}{1} < \frac{h}{k} < \frac{1}{1}$.

(i) Let $x_0$ be the integer such that $kx_0 \equiv -1 \pmod{h}$ and $m - h + 1 \leq x_0 \leq m$. Define integers $y_0$ and $t^*$ by $y_0 := \frac{kx_0+1}{k}$ and $t^* := \left\lfloor \min\left\{\frac{m-x_0}{h}, \frac{n-y_0}{k}, \frac{n-m+x_0-ym}{k-h}\right\rfloor\right\}.$

The fraction $\frac{x_0 + t^*h}{y_0 + t^*k}$ succeeds the fraction $\frac{h}{k}$ in $F(\mathbb{B}(n), m; \rho)$.

(ii) Let $x_0$ be the integer such that $kx_0 \equiv 1 \pmod{h}$ and $m - h + 1 \leq x_0 \leq m$. Define integers $y_0$ and $t^*$ by $y_0 := \frac{kx_0-1}{k}$ and $t^* := \left\lfloor \min\left\{\frac{m-x_0}{h}, \frac{n-y_0}{k}, \frac{n-m+x_0-ym}{k-h}\right\rfloor\right\}.$

The fraction $\frac{x_0 + t^*h}{y_0 + t^*k}$ succeeds the fraction $\frac{h}{k}$ in $F(\mathbb{B}(n), m; \rho)$.

**Sketch of proof.** We sketch the proof of assertion (i).

Since the pair $(x_0, y_0)$ is a solution to the equation $-kx + hy = 1$, the pair $(x_0 + th, y_0 + tk)$ is a solution as well, for any integer $t$. Considering the system of inequalities $0 \leq x_0 + th \leq m$, $1 \leq y_0 + tk \leq n$, $1 \leq y_0 + tk - (x_0 + th) \leq n - m$, where $t$ is an integer variable, we can turn to the solution-equivalent system

$$\begin{align*}
-\frac{x_0}{h} \leq t & \leq \frac{m-x_0}{h}, \\
-\frac{y_0+1}{k} \leq t & \leq \frac{n-y_0}{k}, \\
\frac{x_0-mx_0+1}{k-h} \leq t & \leq \frac{n-m+x_0-ym}{k-h}.
\end{align*}$$

(7.3)
Note that \( \max \{ -\frac{x_0}{k}, -\frac{y_0+1}{k}, \frac{x_0-y_0+1}{k} \} = \frac{x_0-y_0+1}{k-h} \), therefore system (7.3) is solution-equivalent to the inequality
\[
\frac{x_0 - y_0 + 1}{k-h} \leq t \leq \min \left\{ \frac{m-x_0}{h}, \frac{n-y_0}{k}, \frac{n-m+x_0-y_0}{k-h} \right\}.
\] (7.4)

Inequality (7.4) has at least one integer solution, namely \( t = \left\lceil \frac{x_0-y_0+1}{k-h} \right\rceil \). Another observation is that, for any integer solutions \( t' \) and \( t'' \) such that \( t' \leq t'' \), we have \( \frac{0}{t} \leq \frac{x_0+t'h}{y_0+t'k} \leq \frac{x_0+t''h}{y_0+t''k} < \frac{h}{k} \). The proof of assertion (i) is completed by checking that there is no fraction \( \frac{i}{j} \in \mathcal{F}(\mathbb{B}(n), m; \rho) \) such that \( \frac{x_0+t'h}{y_0+t'k} < \frac{i}{j} < \frac{h}{k} \); thus, the fraction \( \frac{x_0+t'h}{y_0+t'k} \) does precede the fraction \( \frac{h}{k} \) in \( \mathcal{F}(\mathbb{B}(n), m; \rho) \).

Assertion (ii) can be proved in an analogous way. \( \square \)

**Remark 7.6.** If \( \frac{1}{k} \in \mathcal{F}(\mathbb{B}(n), m; \rho) \), for some \( k > 1 \), then Proposition 7.5 implies that the fraction
\[
\frac{m+\min \left\{ 0, \left\lceil \frac{n-km-1}{k-1} \right\rceil \right\}}{k \cdot \left( m+\min \left\{ 0, \left\lceil \frac{n-km-1}{k-1} \right\rceil \right\} \right) + 1}
\]
precedes \( \frac{1}{k} \), and the fraction
\[
\frac{m+\min \left\{ 0, \left\lceil \frac{n-km+1}{k-1} \right\rceil \right\}}{k \cdot \left( m+\min \left\{ 0, \left\lceil \frac{n-km+1}{k-1} \right\rceil \right\} \right) - 1}
\]
succeeds \( \frac{1}{k} \) in \( \mathcal{F}(\mathbb{B}(n), m; \rho) \).

**Proposition 7.7.** (i) If \( \frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} \) are two successive fractions of \( \mathcal{F}(\mathbb{B}(n), m; \rho) \), where \( 0 \leq m \leq n \), then
\[
k_j h_{j+1} - h_j k_{j+1} = 1. \quad (7.5)
\]
(ii) If \( \frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}} \) are three successive fractions of \( \mathcal{F}(\mathbb{B}(n), m; \rho) \), where \( 0 < m < n \), then
\[
\frac{h_{j+1}}{k_{j+1}} = \frac{h_j + h_{j+2}}{\gcd(h_j + h_{j+2}, k_j + k_{j+2})} / \frac{k_j + k_{j+2}}{\gcd(h_j + h_{j+2}, k_j + k_{j+2})}. \quad (7.6)
\]

**Proof.** (i) There is nothing to prove if \( m \in \{0, n\} \). If \( 0 < m < n \) then, in terms of Proposition 7.5(i), we have \( h_j = x_0+t^* h_{j+1}, k_j = y_0+t^* k_{j+1} \), and we obtain \( k_j h_{j+1} - k_{j+1} h_j = (y_0+t^* k_{j+1}) h_{j+1} - k_{j+1} (x_0+t^* h_{j+1}) = y_0 h_{j+1} - x_0 k_{j+1} = \frac{x_0 k_{j+1}}{h_{j+1}} h_{j+1} - x_0 k_{j+1} = 1. \)

Proof. (ii)
(ii) First, to see that $k_j h_{j+1} - h_j k_{j+1} = 1$ and $k_{j+1} h_{j+2} - h_{j+1} k_{j+2} = 1$, apply assertion (i) to each of the pairs $\frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}}$ and $\frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}}$. We have $h_j = \frac{k_j h_{j+1} - 1}{k_{j+1}}$, $h_{j+2} = \frac{h_{j+1} k_{j+2} + 1}{k_{j+1}}$, so then $h_j + h_{j+2} = \frac{h_{j+1}}{k_{j+1}} (k_j + k_{j+2})$, and the assertion follows. \( \square \)

The following proposition is a tool of recurrent constructing Farey subsequences \((\ref{eq:farey-seq})\). In practice, such calculations can be performed, for example, based on the successive fractions mentioned in Remarks \((\ref{rem:farey-seq})\) and \((\ref{rem:farey-seq-2})\).

**Proposition 7.8.** Let $\frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}}$ be three successive fractions of $\mathcal{F}(\mathbb{B}(n), m; \rho)$, where $0 < m < n$.

(i) The integers $h_j$ and $k_j$ are computed by

\[
\begin{align*}
h_j &= \min \left\{ \frac{h_{j+2} + m}{h_{j+1}}, \frac{k_{j+2} + n}{k_{j+1}}, \frac{k_{j+2} - h_{j+2} + n - m}{k_{j+1} - h_{j+1}} \right\} \cdot h_{j+1} - h_{j+2}, \\
k_j &= \min \left\{ \frac{h_{j+2} + m}{h_{j+1}}, \frac{k_{j+2} + n}{k_{j+1}}, \frac{k_{j+2} - h_{j+2} + n - m}{k_{j+1} - h_{j+1}} \right\} \cdot k_{j+1} - k_{j+2}.
\end{align*}
\]

(ii) The integers $h_{j+2}$ and $k_{j+2}$ are computed by

\[
\begin{align*}
h_{j+2} &= \min \left\{ \frac{h_j + m}{h_{j+1}}, \frac{k_j + n}{k_{j+1}}, \frac{k_j - h_j + n - m}{k_{j+1} - h_{j+1}} \right\} \cdot h_{j+1} - h_j, \\
k_{j+2} &= \min \left\{ \frac{h_j + m}{h_{j+1}}, \frac{k_j + n}{k_{j+1}}, \frac{k_j - h_j + n - m}{k_{j+1} - h_{j+1}} \right\} \cdot k_{j+1} - k_j.
\end{align*}
\]

**Proof.** To prove assertion (i), note that, with respect to Proposition \((\ref{prop:farey-seq})\) and description \((\ref{eq:farey-seq})\), we have

\[
\begin{align*}
gcd(h_j + h_{j+2}, k_j + k_{j+2}) & \cdot h_{j+1} = h_j + h_{j+2} \leq m + h_{j+2}, \\
gcd(h_j + h_{j+2}, k_j + k_{j+2}) & \cdot k_{j+1} = k_j + k_{j+2} \leq n + k_{j+2}, \\
gcd(h_j + h_{j+2}, k_j + k_{j+2}) & \cdot (k_{j+1} - h_{j+1}) = (k_j - h_j) + (k_{j+2} - h_{j+2}) \leq (n - m) + (k_{j+2} - h_{j+2}),
\end{align*}
\]

from where it follows that

\[
gcd(h_j + h_{j+2}, k_j + k_{j+2}) = \min \left\{ \frac{h_{j+2} + m}{h_{j+1}}, \frac{k_{j+2} + n}{k_{j+1}}, \frac{k_{j+2} - h_{j+2} + n - m}{k_{j+1} - h_{j+1}} \right\}
\]

and we are done.

Assertion (ii) is proved in an analogous way. \( \square \)
Remark 7.9. For all elements \( a \in \mathcal{V}_q(n) \) of rank \( m := \rho(a) \), the Farey subsequences \( \mathcal{F}(\mathcal{V}_q(n), a; \rho) \) are the same, and we write \( \mathcal{F}(\mathcal{V}_q(n), m; \rho) \) instead of \( \mathcal{F}(\mathcal{V}_q(n), a; \rho) \).

We have \( \mathcal{F}(\mathcal{V}_q(n), m; \rho) = \mathcal{F}(\mathcal{B}(n), m; \rho) \), for all \( m, 0 \leq m \leq n \). See also Remark 3.3.

Remark 7.10. In the present paper, we do not deal with the ascending Farey subsequences of the form \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \), where \( 0 < m \leq n \) (see Acketa and Žunić, 1991, and Example 7.1), including the classical Farey sequences \( \mathcal{F}_n \); see Section 6 for some references on \( \mathcal{F}_n \). Nevertheless, such Farey subsequences may be of use for the reader, and we list their basic properties.

(i) In \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \), we have

\[
f_0 = \frac{0}{1}, \quad f_1 = \frac{1}{n}, \quad f_{i+1} = \frac{\min\{m,n-1\}}{\min\{m,n-1\} + 1}, \quad f_{i+2} = \frac{1}{1}.
\]

(ii) If \( f_i \in \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) - \{ \frac{1}{1} \} \) then

\[
t = \sum_{j \in [1,n]} \phi \left( j; \left[ 1, \min\{m,|j|f_i\} \right] \right)
\]

\[
= -1 + \sum_{d \geq 1} \mu(d) \cdot \left( \left\lfloor \frac{n}{d} \right\rfloor + \sum_{j \in [1,\lfloor n/d \rfloor]} \min \left\{ \left\lfloor \frac{m}{d} \right\rfloor, |j|f_i \right\} \right).
\]

(b) The cardinality of the sequence \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \) equals

\[
1 + \sum_{j \in [1,n]} \phi(j; [1,\min\{m,j\}]) = 1 + \sum_{j \in [1,m]} \phi(j; [1,j]) + \sum_{j \in [m+1,n]} \phi(j; [1,m])
\]

\[
= 1 + \sum_{d \geq 1} \mu(d) \cdot \left( \left\lfloor \frac{n}{d} \right\rfloor - \frac{1}{2} \left\lfloor \frac{m}{d} \right\rfloor \right) \cdot \left\lfloor \frac{m}{d} + 1 \right\rfloor .
\]

(iii) Let \( \frac{h}{k} \in \left( \frac{h}{k} \in \mathcal{F}_n : i \leq m \right) \), \( 0 < \frac{h}{k} < \frac{1}{1} \).

(a) Let \( x_0 \) be the integer such that \( kx_0 \equiv -1 \pmod{h} \) and \( m - h + 1 \leq x_0 \leq m \). Define integers \( y_0 \) and \( t^* \) by \( y_0 := \frac{kx_0 + 1}{h} \), and \( t^* := \lfloor \min\{\frac{m-x_0}{h}, \frac{n-y_0}{k}\} \rfloor \). The fraction \( \frac{x_0 + t^* h}{y_0 + t^* k} \) precedes the fraction \( \frac{i}{j} \) in \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \).

(b) Let \( x_0 \) be the integer such that \( kx_0 \equiv 1 \pmod{h} \) and \( m - h + 1 \leq x_0 \leq m \). Define integers \( y_0 \) and \( t^* \) by \( y_0 := \frac{kx_0 - 1}{h} \) and \( t^* := \lfloor \min\{\frac{m-x_0}{h}, \frac{n-y_0}{k}\} \rfloor \). The fraction \( \frac{x_0 + t^* h}{y_0 + t^* k} \) succeeds the fraction \( \frac{h}{k} \) in \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \).

(iv) If \( \frac{h}{k} < \frac{h_{j+1}}{k_{j+1}} \) are two successive fractions of \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \) then (7.5) holds.

(b) If \( \frac{h}{k} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}} \) are three successive fractions of \( \left( \frac{h}{k} \in \mathcal{F}_n : h \leq m \right) \) then (7.6) holds.
Corollary 8.1. Let \( f \) be a nontrivial one-element antichain in \( P \). Given a map \( \omega \), define the map

\[
\omega_{P,a} : \{ r \in \mathbb{Q} : 0 \leq r < 1 \} \rightarrow \mathcal{F}(P,a;\omega)
\]

by

\[
r \mapsto \max \{ f \in \mathcal{F}(P,a;\omega) : f \leq r \} .
\]

Given an element \( a \) of \( \mathbb{B}(n) \), of rank \( m := \rho(a) > 0 \), we write \( f_{\mathbb{B}(n),m,\rho}(r) \) instead of \( f_{\mathbb{B}(n),a,\rho}(r) \).

The following assertion follows immediately from Proposition 7.3(i).

Corollary 8.1. Let \( m \) and \( n \) be positive integers such that \( m < n \). If

\[
f_{\mathbb{B}(n),m,\rho}(r) = f_t \in \mathcal{F}(\mathbb{B}(n),m;\rho),
\]

then

\[
t = \sum_{j \in [1,n]} \phi \left( j ; \left[ \max \{ 1, j + \min \{ m, \lfloor j \cdot r \rfloor \} \} - n, \min \{ m, \lfloor j \cdot r \rfloor \} \} \right) .
\]

The Farey subsequences \( \mathcal{F}(P,a;\omega) \) are, in particular, of use because, given a nontrivial antichain \( A \) in \( P \) and a map \( \omega \), we have

\[
I_r(P,A;\omega) = \bigcap_{a \in A} I_{\omega_{P,a}(r)}(P,a;\omega),
\]

cf. Proposition 2.3(i).

Given a fraction \( f \), we denote by \( \underline{f} \) the numerator of \( f \), and we denote by \( \overline{f} \) its denominator.
Given a nontrivial antichain $A$ in $P$ and a map $\omega$, define a set $D_r(P, A; \omega) \subset P$ in the following way:

$$D_r(P, A; \omega) := \bigcap_{a \in A} \left( \bigcup_{f \in F(P, a; \omega)} \left\{ s \cdot f : 1 \leq s \leq \min \left\{ \left\lfloor \frac{\omega(a)}{f} \right\rfloor, \left\lfloor \frac{\omega(P)}{f} \right\rfloor \right\} \right\} \right) \cup \{ \omega(e) : e \in I(a) - \{ \hat{0} \} \}.$$  

This set of positive integers allows us to give the following comment to Proposition 5.1(i).

**Proposition 8.2.** Let $P$ and $\omega$ satisfy the condition: for any elements $a', a'' \in P$, it holds

$$\omega(\{ a' \} \land_A \{ a'' \}) = \omega(a') \iff a' \leq a''.$$  

Let $A$ be a nontrivial antichain in $P$, and let $k \in [1, \omega(P)]$. Suppose that $| \bigcap_{a \in A} I(a) - \{ \hat{0}_P \} | = 0$. If $k \not\in D_r(P, A; \omega)$ then $| I_{r, k}(P, A; \omega) | = 0$.

**Example 8.3.** If $A$ is an antichain in $B(6)$ such that $\{ \rho(a) : a \in A \} = \{ 2, 3 \}$, then $D_1(B(6), A; \rho) = \{ 1, 2, 3 \}$. Thus, if the set $I_2(B(6), A; \rho)$ is nonempty and if $I_2(B(6), A; \rho) \ni b$, then either $\{ b \} = \bigcap_{a \in A} I(a) - \{ \hat{0}_{B(6)} \}$ and $b$ is of rank one, or $b$ is of rank three.

The concluding statement of the paper is a refinement of Proposition 5.1(ii). Recall that the numbers $\nu(\cdot)$ are defined by (5.1).

**Theorem 8.4.** Let $P$ be a graded poset. If $A$ is a nontrivial antichain in $P$ then, on the one hand,

$$I_r(P, A; \rho) = \left( \bigcap_{a \in A} I(a) - \{ \hat{0}_P \} \right) \cup \bigcup_{k \in D_r(P, A; \rho)} \left( P(k) \cap \tilde{\Phi}(b_{\nu(k-1}(A))) \right).$$  

(8.1)

On the other hand,

$$I_r(P, A; \rho) = \left( \bigcap_{a \in A} I(a) - \{ \hat{0}_P \} \right) \cup \bigcup_{a \in A} \bigcup_{f \in F(P, a; \rho)} \left\{ s \cdot f : 1 \leq s \leq \min \left\{ \left\lfloor \frac{\rho(a)}{f} \right\rfloor, \left\lfloor \frac{\rho(P)}{f} \right\rfloor \right\} \right\} \cup \bigcup_{s \in [1, \min \left\{ \left\lfloor \frac{\rho(a)}{f} \right\rfloor, \left\lfloor \frac{\rho(P)}{f} \right\rfloor \right\}] : s \cdot f \in D_r(P, A; \rho)} \left( P(s) \cap \left( \tilde{\Phi}(b_{s}\cdot f(a)) - \tilde{\Phi}(b_{s}\cdot f(a)) \right) \right).$$  

(8.2)
Proof. First, in both expressions (8.1) and (8.2) we consider the component \( \cap_{a \in A} J(a) - \{0_P\} \); if this component, that corresponds to the terminal fraction \( \frac{1}{r} \) of the Farey subsequences \( F(P, a; \rho) \), \( a \in A \), is nonempty then any element of the component is a relatively \( r \)-blocking element for \( A \) in \( P \). Further, if \( k \not\in D_r(P, A; \rho) \) then the set \( I_{r,k}(P, A; \rho) - (\cap_{a \in A} J(a) - \{0_P\}) \) is empty, see Proposition 8.2. Equality (8.1) now follows from Proposition 5.1(ii).

Let \( a \in A \). We have

\[
I_r(P, a; \rho) = \bigcup_{f \in F(P, a; \rho): \rho(P, a; \rho) < f} \bigcup_{1 \leq s \leq \min\{\lfloor \rho(a)/f \rfloor, \lfloor \rho(P)/f \rfloor \}} \{b \in P: \rho(b) = s \cdot f, \rho(\{b\} \land \{a\}) = s \cdot f\}.
\]

Further, we by (3.10) have

\[
\{b \in P: \rho(b) = s \cdot f\} = P^{(s \cdot f)},
\]

\[
\{b \in P: \rho(\{b\} \land \{a\}) = s \cdot f\} = \mathfrak{A}(b \cdot f, 1) - \mathfrak{A}(b \cdot f, a),
\]

and (8.2) follows, with respect to Proposition 2.3(i). \(\square\)

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CORRIGENDUM:

A.O. Matveev, Relative Blocking in Posets,
Journal of Combinatorial Optimization, 13 (2007), no. 4, 379–403

The use of the set of integers $D_r(P, A; \omega)$ whose definition is given at the top of page 401 may lead to situations where, mistakenly, some relatively $r$-blocking elements from the subposets $I_r(P, A; \omega) \cap \mathcal{J}(A)$ will not be taken into account. We correct the following inaccuracies:

- The definition of the set $D_r(P, A; \omega)$ at the top of page 401 should read:
  $$D_r(P, A; \omega) := \bigcap_{a \in A} \left( \bigcup_{f \in \mathcal{F}(P, a; \omega): f_{P, a; \omega}(r) < f < \frac{1}{r}} \{ s \cdot f: 1 \leq s \leq \min\{\lfloor \omega(a)/f \rfloor, \lfloor \omega(P)/f \rfloor \} \} \right) \cup \{\omega(e): e \in \mathcal{J}(a) - \{\hat{0}\}\}. $$

- The description of the set $D_{\frac{1}{2}}(P, A; \rho)$ in Example 8.3 on page 401 should read: $D_{\frac{1}{2}}(P, A; \rho) = \{1, 2, 3\}$.

- Expression (8.2) of Theorem 8.4 on page 401 should read:
  $$I_r(P, A; \rho) = \left( \bigcap_{a \in A} \mathcal{J}(a) - \{\hat{0}_P\} \right) \bigcup \bigcap_{a \in A} \left( \bigcup_{f \in \mathcal{F}(P, a; \rho): f_{P, a; \rho}(r) < f} \bigcup_{s \in [1, \min\{\lfloor \rho(a)/f \rfloor, \lfloor \rho(P)/f \rfloor \}]} \left( P(s \cdot f) \cap \left( \mathfrak{F}(b_{s \cdot f}(a)) - \mathfrak{F}(b_{s \cdot f - 1}(a)) \right) \right) \right). \quad (8.2)$$