Time Consistent Behavior Portfolio Policy for Dynamic Mean-Variance Formulation

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When we implement a portfolio selection methodology under a mean-risk formulation, it is essential to correctly model investors’ risk aversion which may be time-dependent, or even state-dependent during the investment procedure. In this paper, we propose a behavior risk aversion model, which is a piecewise linear function of the current wealth level with a reference point at a preset investment target. Due to the time inconsistency of the resulting multi-period mean-variance model with an adaptive risk aversion, we investigate in this paper the time consistent behavior portfolio policy by solving a nested mean-variance game formulation. We derive semi-analytical time consistent behavior portfolio policy which takes a piecewise linear feedback form of the current wealth level with respect to the discounted investment target.

Key Words: risk aversion, mean-variance formulation, time consistent behavior portfolio policy.

1 INTRODUCTION

According to the classical investment doctrine in Markowitz (1952), an investor of a mean-variance type needs to strick a balance between maximizing the expected value of the terminal

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wealth, $E[X_1|X_0]$, and minimizing the investment risk measured by the variance of the terminal wealth, $\text{Var}(X_1|X_0)$, by solving the following mean-variance formulation,

$$(MV(\gamma(X_0))) : \min \text{ Var}(X_1|X_0) - \gamma(X_0)E[X_1|X_0],$$

where $X_0$ is the initial wealth level, $X_1$ is the terminal wealth at the end of the (first) time period and $\gamma(X_0) \geq 0$ is the trade-off parameter between the two conflicting objectives. We call $\gamma(X_0)$ risk aversion parameter, which represents the risk aversion attitude of the investor and could depend on the initial wealth level $X_0$. The larger the value of $\gamma(X_0)$, the less the risk aversion of the investor. Mathematically, $(MV(\gamma(X_0)))$ is equivalent to the following formulation,

$$(MV(\omega(X_0))) : \max \text{ E}[X_1|X_0] - \omega(X_0)\text{Var}(X_1|X_0),$$

with risk aversion parameter $\omega(X_0) = \frac{1}{\gamma(X_0)}$.

In a dynamic investment environment, the risk aversion attitude of a mean-variance investor may change from time to time, or could be even state-dependent (i.e., depend on the investor’s current wealth level $X_t$). Björk et al. (2014) and Hu (2013) proposed, respectively, in continuous-time and multi-period settings, that the risk aversion parameter $\omega$ takes the following simple form of the current wealth level $X_t$,

$$\omega(X_t) = \frac{\omega}{X_t}, \quad (\omega \geq 0).$$

Due to the positiveness of the wealth process $X_t$ in the continuous-time setting, $\omega(X_t)$ proposed by Björk et al. (2014) is always nonnegative, which implies that the risk aversion of the mean-variance investor is a decreasing function of the current wealth level. Applying the same model to a multi-period setting as proposed in Hu (2013), however, could encounter some problem, as there is no guarantee for the positiveness of the wealth process in a discrete setting. When the wealth level is negative, $\omega(X_t)$ becomes negative, which implies an irrationality of the investor to maximize both the expected value and the variance of the terminal wealth, which results further in an infinite position on the riskiest asset (See Theorem 7 in Hu (2013)).

In a continuous time setting, Hu et al. (2012) introduced a risk aversion parameter $\gamma$ as a linear function of the current wealth level $X_t$,

$$\gamma(X_t) = \mu_1 X_t + \mu_2, \quad (\mu_1 \geq 0).$$

When the wealth level is less than $-\mu_2/\mu_1$, $\gamma(X_t)$ becomes negative, also leading to an irrationality of the investor, which contradicts the original interests of the investor of a mean-variance type.

In this paper, we propose a behavior risk aversion model as follows,

$$(1) \quad \gamma_t(X_t) = \begin{cases} \gamma_t^+(X_t - \rho_t^{-1}W), & \text{if } X_t \geq \rho_t^{-1}W, \\ -\gamma_t^-(X_t - \rho_t^{-1}W), & \text{if } X_t < \rho_t^{-1}W, \end{cases}$$

where $W$ is the investment target set by the investor at time 0, $\rho_t^{-1}$ is the risk-free discount factor from the current time $t$ to terminal time $T$, and $\gamma_t^+ \geq 0$ and $\gamma_t^- \geq 0$ are constant risk aversion coefficients for the ranges of $X_t$ on the right and left sides of $\rho_t^{-1}W$ respectively. Basically, we consider a piecewise linear state-dependent risk aversion function in our behavior risk aversion model.
This proposed behavior risk aversion model is pretty flexible in incorporating the behavior concern of a mean-variance investor. If the current wealth level is the same as the discounted investment target, the investor becomes fully risk averse and thus invests only in the risk-free asset. If the current wealth level is larger than the discounted investment target, the investor may consider the surplus over the discounted target level as house money and the larger the surplus the less the risk aversion. If the current wealth level is less than the discounted target level, the investor may intend to break-even and the larger the shortage under the discounted target, the stronger the desire to break-even (the less the risk aversion). The magnitude of $\gamma_t^+$ (or $\gamma_t^-$) represents the risk aversion reduction with respect to one unit increase of the surplus (or the shortage). Apparently, different mean-variance investors may have different choices of $\gamma_t^+$ and $\gamma_t^-$. For example, an investor who is eager for breaking-even when facing shortage and feels less sensitive with levels of surplus, may set $\gamma_t^- > \gamma_t^+$.

At time 0, the investor faces the global mean-variance portfolio selection problem,

$$(MV_0(\gamma(X_0))) : \min \ Var(X_T|X_0) - \gamma_0(X_0)E[X_T|X_0],$$

whose pre-committed optimal mean-variance policy is given by Li and Ng (2000) and Zhou and Li (2000) under multi-period setting and continuous time setting, respectively. However, at time $t$, the investor faces a mean-variance portfolio selection problem for a truncated time horizon from $t$ to $T$,

$$(MV_t(\gamma(X_t))) : \min \ Var(X_T|X_t) - \gamma_t(X_t)E[X_T|X_t],$$

whose optimal mean-variance policy is different from the pre-committed optimal policy in general (See Basak and Chabakauri (2010), Cui et al. (2012), Wang and Forsyth (2011)). This phenomenon is called time inconsistency. In the language of dynamic programming, the Bellman's principle of optimality is not applicable to this model formulation, as the global and local objectives are not consistent (See Artzner et al. (2007), Cui et al. (2012)). In the fields of dynamic risk measures and dynamic risk management, time consistency is considered to be a basic requirement (see Rosazza Gianin (2006), Boda and Filari (2006), Artzner et al. (2007) and Jobert and Rogers (2008)).

Strotz (1955-1956) suggested two possible actions to overcome time inconsistency: (1) “He may try to pre commit his future activities either irrevocably or by contriving a penalty for his future self if he should misbehave”, which is termed as the strategy of precommitment; and (2) “He may resign himself to the fact of inter temporal conflict and decide that his ‘optimal’ plan at any date is a will-o’-the-wisp which cannot be attained, and learn to select the present action which will be best in the light of future disobedience”, which is termed the strategy of consistent planning. Strategy of consistent planning is also called time consistent policy in the literature. For dynamic mean-variance model, Basak and Chabakauri (2010) reformulated it as an interpersonal game model where the investor optimally chooses the policy at any time $t$, on the premise that he has already decided his time consistent policies in the future. More specifically, in a framework of time consistency, the investor faces the following nested portfolio selection problem,

$$(NMV_0(\gamma)) : \min_{u_t} Var(X_T|X_t) - \gamma E[X_T|X_t],$$

s.t. $u_j$ solves $(NMV_j(\gamma))$, $t \leq j \leq T,$

with the terminal period problem given as

$$(NMV_{T-1}(\gamma)) : \min_{u_{T-1}} Var(X_T|X_{T-1}) - \gamma E[X_T|X_{T-1}].$$
The time consistent policy is then the equilibrium solution of the above nested problem, which can be derived by backward induction. Björk et al. (2014), Hu et al. (2012) and Hu (2013) extended the results in Basak and Chabakauri (2010) by considering different state-dependent risk aversions mentioned before in this section. In the original setting of dynamic mean-variance portfolio selection with a constant risk aversion, the time inconsistency is caused by the appearance of variance of the terminal wealth in the objective function, which does not satisfy the smoothing property. When we consider more realistic time-varying and wealth dependent risk aversion in this study, it further complicates the degree of time inconsistency, which forces us to consider time consistent policy in this paper. For a general class of continuous-time mean-field linear-quadratic control problems, please refer to Yong (2013).

In this paper, we focus on studying time consistent behavior portfolio policy under the proposed behavior risk aversion model. The remaining parts of this paper are organized as follows: In Section 2, we provide the basic market setting and formulate the nested mean-variance portfolio selection problem. We derive in Section 3 the semi-analytical time consistent behavior portfolio policy, which takes a piecewise linear feedback form of the surplus or the shortage with the discounted wealth target. In Section 4, we extend our main results to cone constrained markets. After we offer in Section 5 numerical analysis to show the trading patterns of investors with different risk aversion coefficients, we conclude the paper in Section 6.

2 MARKET SETTING AND PROBLEM FORMULATIONS

We consider an arbitrage-free capital market consisting of one risk-free asset and $n$ risky assets within a time horizon $T$. Let $s_t$ ($>1$) be a given deterministic return of the risk-free asset at period $t$ and $e_t = [e_1^t, \ldots, e_n^t]'$ be the vector of random returns of the $n$ risky assets at period $t$, which is defined on the probability space $(\Omega, \mathcal{F}_T, P)$. We assume that $e_t$, $t = 0, 1, \ldots, T - 1$, are statistically independent, absolutely integrable continuous random vectors, whose first and second moments, $E[e_t]$ and $E[e_t e_t']$ are known for every $t$ and whose covariance matrixes $\text{Cov}(e_t) = E[e_t e_t'] - E[e_t]E[e_t']$, $t = 0, 1, \ldots, T - 1$, are positive definite. We further define the excess return vector of risky assets $P_t$ as

$$P_t = [P_1^t, P_2^t, \ldots, P_n^t]' = [(e_1^t - s_t), (e_2^t - s_t), \ldots, (e_n^t - s_t)]'.$$

It is not difficult to verify that $\text{Cov}(P_t) > 0$.

An investor invests an initial wealth $X_0$ in financial market at the beginning of period 0. He/she allocates $X_0$ among the risk-free asset and $n$ risky assets at the beginning of period 0 and reallocates his/her wealth at the beginning of each of the following $(T - 1)$ consecutive periods. Let $X_t$ be the wealth of the investor at the beginning of period $t$, and $u_i^t$, $i = 1, 2, \ldots, n$, be the amount invested in the $i$-th risky asset at period $t$. Then, $X_t - \sum_{i=1}^n u_i^t$ is the amount invested in the risk-free asset at period $t$. Then, the wealth at the beginning of period $t+1$ is given as

$$X_{t+1} = s_t \left( X_t - \sum_{i=1}^n u_i^t \right) + e'_t u_t = s_t X_t + P'_t u_t,$$

with $u_t = [u_1^t, u_2^t, \ldots, u_n^t]'$. The information set at the beginning of period $t$ is denoted as

$$\mathcal{F}_t = \sigma(P_0, P_1, \ldots, P_{t-1}).$$

Our main results can be readily extended to situations where random return vectors are correlated. This extension can be achieved based on the concept of the so-called opportunity-neutral measure introduced by Černý and Kallsen (2009).
and \( F_0 \) represents the trivial \( \sigma \)-algebra over \( \Omega \). We confine all admissible investment policies to be \( F_t \)-measurable Markov control, whose realizations are in \( \mathbb{R}^n \). Then, \( P_t \) and \( u_t \) are independent, the controlled wealth process \( \{x_t\} \) is an adapted Markovian process and \( F_t = \sigma(x_t) \).

We now formulate the portfolio decision problem of the investor at time \( t \) as follows,

\[
(MV_t(\gamma_t(X_t))) \min \ \text{Var}_t(X_T) - \gamma_t(X_t)\mathbb{E}_t[X_T],
\]

(2)

s.t. \( X_{j+1} = s_j X_j + P'_j u_j, \quad j = t, t+1, \ldots, T-1, \)

where \( \text{Var}_t(X_T) = \text{Var}(X_T | X_t) \), \( \mathbb{E}_t[X_T] = \mathbb{E}[X_T | X_t] \), \( \rho_t^{-1} = \prod_{j=t}^{T-1} s_j^{-1} \) is the risk-free discount factor with \( \rho_T^{-1} = 1 \) and \( \gamma_t(X_t) \) is given by

\[
\gamma_t(X_t) = \begin{cases} 
\gamma_t^+(X_t - \rho_t^{-1}W), & \text{if } X_t \geq \rho_t^{-1}W; \\
-\gamma_t^-(X_t - \rho_t^{-1}W), & \text{if } X_t < \rho_t^{-1}W.
\end{cases}
\]

Similar to the approach in Basak and Chabakauri (2010), we can further formulate the multi-period mean-variance model into an interpersonal game model in which the investor optimally chooses the policy at any time \( t \), on the premise that he has already decided his time consistent policy in the future. Then the time consistent behavior portfolio policy (or time consistent policy in short) is the optimizer of the following nested mean-variance problem \((NMV)\),

\[
(NMV_t(\gamma_t(X_t))) \min \ \text{Var}_t(X_T) - \gamma_t(X_t)\mathbb{E}_t[X_T],
\]

(3)

s.t. \( X_{t+1} = s_t X_t + P'_t u_t, \)

\[
X_{j+1} = s_j X_j + P'_j u^TC_j, \quad j = t + 1, \ldots, T-1,
\]

\( u^TC_j \) solves problem \((MV_j(\gamma_j(X_j)))\), \( j = t + 1, \ldots, T-1, \)

with terminal period problem given as

\[
(NMV_{T-1}(\gamma_{T-1}(X_{T-1}))) \min \ \text{Var}_{T-1}(X_T) - \gamma_{T-1}(X_{T-1})\mathbb{E}_{T-1}[X_T],
\]

(4)

s.t. \( X_T = s_{T-1} X_{T-1} + P'_{T-1} u_{T-1}, \)

which can be solved by a backward induction. Since the stage-trade off \( \gamma_t(X_t) \) reflects certain behavior pattern in terms of his wealth level, we call the optimal policy to \((NMV)\) time consistent behavior portfolio policy.

3 SEMI-ANALYTICAL TIME CONSISTENT POLICY

In this section, we focus on deriving the time consistent behavior portfolio policy. Before presenting our main results, we introduce several important functions and notations first.
For time $t = 0, 1, \cdots, T - 1$, we define functions, $F_t^+ : \mathbb{R}^n \rightarrow \mathbb{R}$, $F_t^- : \mathbb{R}^n \rightarrow \mathbb{R}$, as follows,

\[ F_t^+(K) = \rho_t^2 K' (E_t[P_t][P_t') - E_t[P_t][P_t]) K + \mathbb{E}_t \left[ b_{t+1}^+ (s_t + P_t'K)^2 1_{\{s_t + P_t'K \geq 0\}} \right] + \mathbb{E}_t \left[ b_{t+1}^- (s_t + P_t'K)^2 1_{\{s_t + P_t'K < 0\}} \right] - \left( \mathbb{E}_t \left[ a_{t+1}^+ (s_t + P_t'K) 1_{\{s_t + P_t'K \geq 0\}} \right] + \mathbb{E}_t \left[ a_{t+1}^- (s_t + P_t'K) 1_{\{s_t + P_t'K < 0\}} \right] \right)^2 + 2 \rho_{t+1} \mathbb{E}_t \left[ a_{t+1}^+ (s_t + P_t'K) 1_{\{s_t + P_t'K \geq 0\}} \right] + a_{t+1}^- (s_t + P_t'K) 1_{\{s_t + P_t'K < 0\}} \right] \]

and

\[ F_t^-(K) = \rho_t^2 K' (E_t[P_t][P_t') - E_t[P_t][P_t]) K + \mathbb{E}_t \left[ b_{t+1}^+ (s_t + P_t'K)^2 1_{\{s_t + P_t'K \geq 0\}} \right] - \left( \mathbb{E}_t \left[ a_{t+1}^+ (s_t + P_t'K) 1_{\{s_t + P_t'K \geq 0\}} \right] + \mathbb{E}_t \left[ a_{t+1}^- (s_t + P_t'K) 1_{\{s_t + P_t'K < 0\}} \right] \right)^2 + 2 \rho_{t+1} \mathbb{E}_t \left[ a_{t+1}^+ (s_t + P_t'K) 1_{\{s_t + P_t'K \geq 0\}} \right] + a_{t+1}^- (s_t + P_t'K) 1_{\{s_t + P_t'K < 0\}} \right] \]

where $a_{t+1}^+, a_{t+1}^-, b_{t+1}^+$ and $b_{t+1}^-$ are deterministic parameters.

The following proposition ensures that the optimizers of $F_t^+(K)$ and $F_t^-(K)$ are finite.

**Proposition 3.1** Suppose that deterministic numbers $a_{t+1}^+, a_{t+1}^-, b_{t+1}^+$ and $b_{t+1}^-$ satisfy

\[ b_{t+1}^+ - (a_{t+1}^+)^2 \geq 0, \quad b_{t+1}^- - (a_{t+1}^-)^2 \geq 0. \]

Then we have

\[ \lim_{\|K\| \rightarrow + \infty} F_t^+(K) = + \infty, \quad \lim_{\|K\| \rightarrow + \infty} F_t^-(K) = + \infty, \]

where $\|K\|$ denotes the Euclidean norm of vector $K$.

Proof. See Appendix A. \qed

According to Proposition 3.1, we denote the finite optimizers of $F_t^+(K)$ and $F_t^-(K)$ as follows,

\[ K_t^+ = \arg \min_{K \in \mathbb{R}^n} F_t^+(K), \quad K_t^- = \arg \min_{K \in \mathbb{R}^n} F_t^-(K), \]
and define the parameters $a_t^+, a_t^-, b_t^+, b_t^-$, for $t = 0, 1, \ldots, T - 1$, by the following backward recursions,

$$
a_t^+ = \rho_{t+1} E_t[P_t]K_t^+ + E_t[a_{t+1}^+(s_t + P'_tK_t^+)1_{\{s_t + P'_tK_t^+ \geq 0\}}]$$

(5)

$$+ E_t \left[a_{t+1}^+(s_t + P'_tK_t^+)1_{\{s_t + P'_tK_t^+ < 0\}}\right];$$

$$a_t^- = \rho_{t+1} E_t[P_t]K_t^- + E_t[a_{t+1}^-(s_t + P'_tK_t^-)1_{\{s_t + P'_tK_t^- \leq 0\}}]$$

(6)

$$+ E_t \left[a_{t+1}^-(s_t + P'_tK_t^-)1_{\{s_t + P'_tK_t^- > 0\}}\right];$$

$$b_t^+ = \rho_{t+1}^2 (K_t^+)'E_t[P_tP'_t]K_t^+ + 2\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P'_tK_t^+)P'_tK_t^+1_{\{s_t + P'_tK_t^+ \geq 0\}}\right]$$

(7)

$$+ 2\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P'_tK_t^+)P'_tK_t^+1_{\{s_t + P'_tK_t^+ < 0\}}\right];$$

$$b_t^- = \rho_{t+1}^2 (K_t^-)'E_t[P_tP'_t]K_t^- + 2\rho_{t+1}E_t\left[a_{t+1}^-(s_t + P'_tK_t^-)P'_tK_t^-1_{\{s_t + P'_tK_t^- \leq 0\}}\right]$$

(8)

$$+ 2\rho_{t+1}E_t\left[a_{t+1}^-(s_t + P'_tK_t^-)P'_tK_t^-1_{\{s_t + P'_tK_t^- > 0\}}\right],$$

with the terminal condition $a_T^+ = a_T^- = 0$ and $b_T^+ = b_T^- = 0$.

**Remark 3.1** In general, functions $F_t^+(K)$ and $F_t^-(K)$ are not convex functions of $K$. However, when $a_{t+1}^+ \geq 0 \geq a_{t+1}^-$, we are able to show that $F_t^+(K)$ and $F_t^-(K)$ are d.c. functions (difference of convex functions) of $K$ (see Horst and Thoai, 1999). In such cases, we can use the existing global search methods in the literature to derive optimal $K_t^+$ and $K_t^-$. 

With the above notations, we show now that the time consistent behavior portfolio policy is a piecewise linear feedback form of the surplus and the shortage of current wealth level in the following theorem.

**Theorem 3.1** The time consistent behavior portfolio policy of $(NMV_t(\gamma_t(X_t)))$ is given as follows for $t = 0, \ldots, T - 1$,

$$u_t^{TC} = K_t^+(X_t - \rho_t^{-1}W)1_{\{X_t \geq \rho_t^{-1}W\}} + K_t^-(X_t - \rho_t^{-1}W)1_{\{X_t < \rho_t^{-1}W\}},$$

(9)

in which the parameters $a_t^+, a_t^-, b_t^+$ and $b_t^-$ defined in (5)-(8) satisfy

$$b_t^+ - (a_t^+)^2 \geq 0, \quad b_t^- - (a_t^-)^2 \geq 0.$$

Furthermore, the mean and variance of the terminal wealth achieved by the time consistent behavior portfolio policy are

$$E_0[X_T]|u^{TC} = \rho_0 X_0 + a_0^+ (X_0 - \rho_0^{-1}W)1_{\{X_0 \geq \rho_0^{-1}W\}} + a_0^- (X_0 - \rho_0^{-1}W)1_{\{X_0 < \rho_0^{-1}W\}};$$

(10)

$$\text{Var}(X_T)|u^{TC} = \left[(b_t^+ - (a_t^+)^2)1_{\{X_0 \geq \rho_t^{-1}W\}} + (b_t^- - (a_t^-)^2)1_{\{X_0 < \rho_t^{-1}W\}}\right](X_0 - \rho_0^{-1}W)^2.$$
Remark 3.2 Proposition 3.1 and Theorem 3.1 have revealed that the nested mean-variance problem \((NMV_t(\gamma_t(X_t)))\) is a well-posed optimization problem in the sense of the existence of a finite subgame Nash equilibrium policy.

Remark 3.3 Under our behavior risk aversion model, the functions \(F_t^+(K)\) and \(F_t^-(K)\) are no longer convex functions of \(K\). However, the optimal investment funds \(K_t^+\) and \(K_t^-\) can be derived off-line via some global search methods, thus reducing the dynamic optimization problem into \(T\) static optimization problems.

Remark 3.4 In the proofs of Proposition 3.1 and Theorem 3.1, the assumption of \(\gamma_t^+ \geq 0\) and \(\gamma_t^- \geq 0\) is not used. Therefore, our main results remain valid for more general cases with \(\gamma_t^+ \in \mathbb{R}\) and \(\gamma_t^- \in \mathbb{R}\).

4 EXTENSION TO CONE CONSTRAINED MARKETS

In real financial markets, realizations of \((\mathcal{F}_t\text{-measurable})\) admissible policy are often confined in a subset of \(\mathbb{R}^n\), instead of the whole space \(\mathbb{R}^n\). In this section, we consider the situation that the realizations of admissible policies are required to be in a cone. Such cone-type constraints have been widely adopted to model regulatory restrictions, for example, restrictions for no-short selling (See Cui et al. (2014) and Li et al. (2001)) or non-tradeable assets. Cone-type constraints are also useful to represent portfolio restrictions, for example, the holding of the real estate stock must be no less than the bank stock. We express the feasible set of the realizations of admissible policies as \(A_t = \{u_t \in \mathbb{R}^n|Au_t \geq 0, A \in \mathbb{R}^{m \times n}\}\) (see Cuoco (1997) and Napp (2003) for more examples). Then, mean-variance investors would face the following constrained nested mean-variance problem,

\[
\begin{align*}
(CNMV_t(\gamma_t(X_t))) \quad & \min_{u_t} \ Var_t(X_T) - \gamma_t(X_t)E_t[X_T], \\
\text{s.t.} \quad & X_{t+1} = s_tX_t + P_t'u_t, \\
& X_{j+1} = s_jX_j + P_{j}'u_{j}^{TC}, \quad j = t + 1, \cdots , T - 1, \\
& u_t \in A_t, \\
& u_{j}^{TC} \text{ solves problem } (MV_j(\gamma_j(X_j))), \quad j = t + 1, \cdots , T - 1,
\end{align*}
\]

with the problem in the last stage given as

\[
\begin{align*}
(CNMV_{T-1}(\gamma_{T-1}(X_{T-1}))) \quad & \min_{u_{T-1}} \ Var_{T-1}(X_T) - \gamma_{T-1}(X_{T-1})E_{T-1}[X_T], \\
\text{s.t.} \quad & X_T = s_{T-1}X_{T-1} + P_{T-1}'u_{T-1}, \\
& u_{T-1} \in A_T.
\end{align*}
\]

Theorem 4.1 The time consistent behavior portfolio policy of \((CNMV_t(\gamma_t(X_t)))\) is given as

\[
u_t^{TC} = \tilde{K}_t^+(X_t - \rho_t^{-1}W)1_{\{X_t \geq \rho_t^{-1}W\}} + \tilde{K}_t^-(X_t - \rho_t^{-1}W)1_{\{X_t < \rho_t^{-1}W\}},
\]

where the optimal investment funds \(\tilde{K}_t^+\) and \(\tilde{K}_t^-\) are

\[
\tilde{K}_t^+ = \arg \min_{K \in A_t} F_t^+(K), \quad \tilde{K}_t^- = \arg \min_{K \in A_t} F_t^-(K),
\]
with \(-\mathcal{A}_t = \{-u_t | u_t \in \mathcal{A}_t\}\) is the negative cone of \(\mathcal{A}_t\), and the parameters \(a_t^+, a_t^-, b_t^+\) and \(b_t^-\) are computed according to recursive functions (5)-(8) with \(K_t^+\) and \(K_t^-\) replaced by \(\tilde{K}_t^+\) and \(\tilde{K}_t^-\), respectively.

Proof. See Appendix C. □

In cone constrained markets, the time consistent behavior portfolio policy remains a piecewise linear feedback form of the current wealth level and the discounted investment target. The only difference from unconstrained markets is that we need to search the optimal investment funds in an \(\mathcal{A}_t\) related cone instead of the entire space.

5 NUMERICAL EXAMPLE

In this section, we study a numerical example to analyze the property of the time consistent behavior portfolio policy proposed in this paper. Consider a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of U.S market and a bank account with annual rate of return equal to 5% \((s_t = 1.05)\). Based on the data provided in Elton et al. (2007), we list the expected values, variances and correlation coefficients of the annual rates of return of these three indices in Table 1.

|                | SP  | EM  | MS  |
|----------------|-----|-----|-----|
| Expected Return| 14% | 16% | 17% |
| Variance       | 18.5% | 30% | 24% |
| Correlation coefficient | 1   | 0.64 | 0.79 |

We assume that the annual rates of return of the three risky indices follow a joint lognormal distribution. An investor with initial wealth \(X_0 = 1\) is facing an investment opportunity of three years \((T = 3)\), with the following behavior risk aversion \(\gamma_t(X_t)\),

\[
\gamma_t(X_t) = \begin{cases} 
\gamma^+(X_t - \rho_t^{-1}W), & \text{if } X_t \geq \rho_t^{-1}W, \\
-\gamma^-(X_t - \rho_t^{-1}W), & \text{if } X_t < \rho_t^{-1}W.
\end{cases}
\]

Based on Remark 3.1, \(F_t^+(K)\) and \(F_t^-(K)\) are d.c. functions of \(K\). By simulating 20,000 sample paths for annual rates of return of the three risky indices and adopting a global search method, we can compute the optimal investment funds \(K_t^+, K_t^-\) and the parameters \(a_t^+, a_t^-, b_t^+\) and \(b_t^-\) backwards. We provide the results for situations of \(\gamma^+ = 1\) in Table 2.

|                | SP  | EM  | MS  |
|----------------|-----|-----|-----|
| Expected Return| 14% | 16% | 17% |
| Variance       | 18.5% | 30% | 24% |
| Correlation coefficient | 1   | 0.64 | 0.79 |

We can find some interesting features from Table 2. First, for given \(\gamma^+\), the larger the value of \(\gamma^-\), the larger the absolute values of \(K_t^-, a_t^-\) and \(b_t^-\). Second, whenever the discounted investment target is less than the current wealth level (i.e., \(\rho_t^{-1}W < X_t\)), the investor chooses to invest a portfolio, which includes almost a fixed proportion \(K_t^+\) with respect to the surplus of current wealth level. Third, when the discounted investment target is larger than the current wealth level (i.e., \(\rho_t^{-1}W > X_t\)), the investor with larger risk aversion coefficient \(\gamma^-\) invests a
larger portfolio, which has larger proportion $K_0^{-}$ with respect to the shortage of the current wealth level. The third feature is quite intuitive. For given $X_t > \rho_t^{-1}W$, the larger the value of $\gamma^-$, the less risk aversion of the investor at time $t$, which may result in larger risky positions.

For the situations of $\gamma^- = 1$, $F_t^+(\mathbf{K})$ and $F_t^-(\mathbf{K})$ are also d.c. functions of $\mathbf{K}$ (see Remark 3.1), and the first and third patterns remain the same as the situation with $\gamma^+ = 1$ (See Table 3). Additionally, for given $\gamma^-$, the larger the value of $\gamma^+$, the less the absolute values of $K_i^{-}$, $a_t^-$, and $b_t^-$). In fact, the same pattern holds for $\gamma^+ = 1$. But the differences are too small to identify in Table 2.

| $\gamma^+$ | $\gamma^-$ | $K_0^+$ | $a_0^+$ | $a_0^+$ | $b_0^+$ | $b_0^+$ |
|------------|------------|---------|---------|---------|---------|---------|
| 0.5 1      | 0.3173    | -0.3610 | -0.6347 | 0.0675  | -0.1349 | 0.214   |
| 1 1        | 0.6347    | -0.7221 | -0.6347 | 0.1349  | -0.1349 | 0.0857  |
| 1.5 1      | 0.5290    | -0.4755 | -0.6347 | 0.2024  | -0.1349 | 0.0857  |
| 2 1        | 1.2694    | -0.4441 | -0.6347 | 0.2698  | -0.1349 | 0.0857  |
| 2.5 1      | 1.5867    | -0.4051 | -0.6347 | 0.3373  | -0.1349 | 0.0857  |

Table 3: Optimal investment portfolios and parameters ($\gamma^+ \geq \gamma^-$)
Next, we analyze the global investment performance of the time consistent behavior portfolio policy proposed in this paper. We assume that all investors choose a very natural investment target \( W = 2 \), which is twice of the value of \( X_0 \) and gives rise to \( \rho_0 \cdot W \geq X_0 \), i.e., the discounted investment target is no less than the initial wealth level. Figures 1(a) and 1(b) show the relationship of Sharpe ratio with respect to \( \gamma^- \) (with \( \gamma^+ = 1 \)) and \( \gamma^+ \) (with \( \gamma^- = 1 \)), respectively. Figures 2(a) and 2(b) show the probability density functions (PDFs) under different settings with different risk aversion coefficients. We can see that different investors may achieve different global investment performances under their different time consistent behavior portfolio policies. However, for the situations of \( \gamma^- = 1 \), all the investors’ time consistent policies are quite similar (see column \( K^-_t \) in Table 3), which result in similar Sharpe ratios and PDFs of the terminal wealth level. In other words, in our setting, the negative risk aversion coefficient \( \gamma^- \) has a higher impact on the model.

![Graphs of Sharpe ratio and PDFs](image)

Figure 1: Relationship between Sharpe ratio and parameter settings

Figure 2: PDFs of terminal wealth level

At last, we analyze our data in a cone constrained market. We present our brief results under a no short selling constraint in Table 4. Due to the presence of the no shorting constraint, the position on risky indices is forced to zero whenever the discounted investment target is larger than the current wealth level, i.e., \( K^-_t = 0 \).
Table 4: Optimal investment portfolios and parameters ($\gamma^+ \geq \gamma^-$)

| $\gamma^+$ | $\gamma^-$ | $K_1^+$ | $K_2^+$ | $a_1^+ \ a_2^+ \ b_1^+ \ b_2^+$ | $a_1^0 \ a_2^0 \ b_1^0 \ b_2^0$ |
|------------|------------|---------|---------|-------------------------------|-------------------------------|
| 1          | 0.5        | [0.6204,0.6594] | [0.0,0] | 0.1346 0 0.0855 0            | 0.3491 0 0.3170 0            |
| 1          | 1          | [0.6204,0.6594] | [0.0,0] | 0.1346 0 0.0855 0            | 0.3491 0 0.3170 0            |
| 1          | 1.5        | [0.6204,0.6594] | [0.0,0] | 0.1346 0 0.0855 0            | 0.3491 0 0.3170 0            |
| 1          | 2          | [0.6204,0.6594] | [0.0,0] | 0.1346 0 0.0855 0            | 0.3491 0 0.3170 0            |
| 0.5        | 1          | [0.3091,0.3324] | [0.0,0] | 0.0675 0 0.0215 0            | 0.3550 0 0.4143 0            |
| 2          | 1          | [1.2430,0.1360] | [0.0,0] | 0.2690 0 0.3415 0            | 0.4851 0 0.6547 0            |

6 CONCLUSIONS

When we implement a portfolio selection methodology under a mean-risk formulation, it is crucial to assess the investor’s subjective trade-off between maximizing the expected terminal wealth and minimizing the investment risk, which in turn requires good understanding of the investor’s risk aversion which is in general an adaptive process of the wealth level. We propose in this paper a behavior risk aversion model to describe the risk attitude of a mean-variance investor, which takes the piecewise linear form of the surplus and the shortage with respect to some preset investment target. Our new risk aversion model is flexible enough to incorporate the features of “house money” and “breaking-even”, thus enriching the modeling power to capture the essence of the investor’s risk attitude.

As the resulting dynamic mean-variance model with adaptive risk aversion is time inconsistent, we focus on its time consistent policy by solving a nested mean-variance game formulation. Fortunately, we obtain the semi-analytical time consistent behavior portfolio policy and reveal its piecewise linear form of the surplus and the shortage with respect to the discounted wealth target. Our numerical analysis sheds light on some prominent features of the time consistent behavior portfolio policy established in our theoretical derivations.
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Proof. Define $\xi = \|K\|$, $L = K\xi^{-1}$ (which implies $\|L\| = 1$) and $y_t = P_t^tL$. Then, for any $L$, we have $M \geq \text{Var}_t(y_t) = L^t\text{Cov}(P_t)L > 0$, where $M$ is the largest eigenvalue of $\text{Cov}(P_t)$.

If $y_t 1_{\{y_t \geq 0\}}$ is zero, (i.e., $y_t \leq 0$ almost surely), we can construct an arbitrage portfolio by shorting $L$ and holding $L'1$ risk-free asset. Similarly, if $y_t 1_{\{y_t < 0\}}$ is zero, (i.e., $y_t \geq 0$ almost surely), we also can construct an arbitrage portfolio by holding $L$ and shorting $L'1$ risk-free asset. Thus, we conclude that $y_t 1_{\{y_t \geq 0\}}$ and $y_t 1_{\{y_t < 0\}}$ are nontrivial random variables with finiteness of the second moment.

Moreover, $P_t$ is absolutely integrable, so do $y_t$, $y_t 1_{\{y_t \geq -\frac{\mu_t}{\sigma_t}\}}$ and $y_t 1_{\{y_t < -\frac{\mu_t}{\sigma_t}\}}$. Then, for given $L$, we have

$$F^+_t(K) = \tilde{F}^+_t(\xi),$$

where

$$\tilde{F}^+_t(\xi) = \rho^2_{t+1} \text{Var}_t(y_t) \xi^2 + \mathbb{E}_t \left[ b^+_{t+1}(s_t + \xi y_t)^2 1_{\{y_t \geq -\frac{\mu_t}{\sigma_t}\}} \right] + \mathbb{E}_t \left[ b^-_{t+1}(s_t + \xi y_t)^2 1_{\{y_t < -\frac{\mu_t}{\sigma_t}\}} \right] - \left( \mathbb{E}_t \left[ a^+_{t+1}(s_t + \xi y_t) 1_{\{y_t \geq -\frac{\mu_t}{\sigma_t}\}} \right] + \mathbb{E}_t \left[ a^-_{t+1}(s_t + \xi y_t) 1_{\{y_t < -\frac{\mu_t}{\sigma_t}\}} \right] \right)^2 + 2\rho_{t+1} \mathbb{E}_t \left[ \left( a^+_{t+1}(s_t + \xi y_t)^2 1_{\{y_t \geq -\frac{\mu_t}{\sigma_t}\}} + a^-_{t+1}(s_t + \xi y_t)^2 1_{\{y_t < -\frac{\mu_t}{\sigma_t}\}} \right) \right] - 2\rho_{t+1} \mathbb{E}_t \left[ \left( a^+_{t+1}(s_t + \xi y_t) 1_{\{y_t \geq -\frac{\mu_t}{\sigma_t}\}} + \mathbb{E}_t \left[ a^-_{t+1}(s_t + \xi y_t) 1_{\{y_t < -\frac{\mu_t}{\sigma_t}\}} \right] \right) (s_t + \mathbb{E}_t[y_t] \xi) \right] - \gamma^+_t \left( \mathbb{E}_t \left[ a^+_{t+1}(s_t + \xi y_t) 1_{\{y_t \geq -\frac{\mu_t}{\sigma_t}\}} \right] + \mathbb{E}_t \left[ a^-_{t+1}(s_t + \xi y_t) 1_{\{y_t < -\frac{\mu_t}{\sigma_t}\}} \right] \right) - \gamma^-_t (s_t + \mathbb{E}_t[y_t] \xi).$$
Furthermore, we have

\[
\hat{F}_t^+ (\xi) \geq \rho_{t+1}^2 \text{Var}(y_t) \xi^2 + (a_{t+1}^+)^2 \mathbb{E}_t \left[ y_{t+1}^2 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] \xi^2 + (a_{t+1}^-)^2 \mathbb{E}_t \left[ y_{t+1}^2 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \xi^2 \\
- \left( a_{t+1}^+ \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] + a_{t+1}^- \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \right)^2 \\
+ 2\rho_{t+1} \left( a_{t+1}^+ \mathbb{E}_t \left[ y_{t+1}^2 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] \xi^2 + a_{t+1}^- \mathbb{E}_t \left[ y_{t+1}^2 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \xi^2 \right) \\
- 2\rho_{t+1} \left( a_{t+1}^+ \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] - a_{t+1}^- \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \right) \mathbb{E}_t[y_t] \xi^2 + O(\xi) \\
= \rho_{t+1}^2 \text{Var}(y_t) \xi^2 + \rho_{t+1}^2 \text{Var}(y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}}) \xi^2 \\
+ 2\rho_{t+1}^2 \left( \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] - \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \right) \xi^2 + (a_{t+1}^+)^2 \text{Var}(y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}}) \xi^2 \\
+ 2a_{t+1}^+ a_{t+1}^- \left( \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] - \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \right) \xi^2 + 2\rho_{t+1} \left( a_{t+1}^+ \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] \xi^2 + a_{t+1}^- \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \xi^2 \right) \\
+ 2\rho_{t+1} a_{t+1}^+ \left( \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] - \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \right) \xi^2 + 2\rho_{t+1} a_{t+1}^- \left( \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} \right] - \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] \right) \xi^2 + O(\xi) \\
= [\rho_{t+1} + a_{t+1}^+, \rho_{t+1} + a_{t+1}^-] \text{Cov}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}}, y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}}, \rho_{t+1} + a_{t+1}^+, \rho_{t+1} + a_{t+1}^- \right] \xi^2 + O(\xi),
\]

where \( O(\xi) \) is the infinity of the same order as \( \xi \) and the second equality holds due to the fact of \( \mathbb{E}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq -\frac{\mu}{\xi}\}} y_{t+1} 1_{\{y_{t+1} < -\frac{\mu}{\xi}\}} \right] = 0 \). Hence,

\[
\lim_{\xi \to +\infty} \hat{F}_t^+ (\xi) = \lim_{\xi \to +\infty} [\rho_{t+1} + a_{t+1}^+, \rho_{t+1} + a_{t+1}^-] \text{Cov}_t \left[ y_{t+1} 1_{\{y_{t+1} \geq 0\}}, y_{t+1} 1_{\{y_{t+1} < 0\}}, \rho_{t+1} + a_{t+1}^+, \rho_{t+1} + a_{t+1}^- \right] \xi^2 + O(\xi) = +\infty.
\]

Based on the discussion for all possible \( L \), we make our conclusion for \( F_t^+(K) \). Similarly we can prove the result of \( F_t^-(K) \).

**Appendix B: The Proof of Theorem 3.1**

Proof. At time \( t (t = 0, 1, \ldots, T) \), the investor faces the following optimization problem,

\[(15) \quad \min_{u_t} J_t(X_t; u_t) = \left( \mathbb{E}_t[X_T^2] - (\mathbb{E}_t[X_T])^2 \right) - \gamma_t(X_t) \mathbb{E}_t[X_T],\]

where the conditional expectations \( \mathbb{E}_t[X_T] = \mathbb{E}[X_T|X_t] \) and \( \mathbb{E}_t[X_T^2] = \mathbb{E}[X_T^2|X_t] \) are computed along the policy \( \{ u_t, u_{t+1}^T, \ldots, u_{T-1}^T \} \).
We now prove by induction that the following two expressions,

\begin{align}
(16) \quad & E_t[X_T] = \rho_t X_t + a^+_t (X_t - \rho_t^{-1} W) 1_{\{X_t \geq \rho_t^{-1} W\}} + a^-_t (X_t - \rho_t^{-1} W) 1_{\{X_t < \rho_t^{-1} W\}}, \\
(17) \quad & E_t[X^2_T] = \rho_t^2 X_t^2 + 2 \rho_t X_t \left[ a^+_t (X_t - \rho_t^{-1} W) 1_{\{X_t \geq \rho_t^{-1} W\}} + a^-_t (X_t - \rho_t^{-1} W) 1_{\{X_t < \rho_t^{-1} W\}} \right] \\
& \quad \quad \quad \quad \quad \quad \quad \quad + b^+_t (X_t - \rho_t^{-1} W)^2 1_{\{X_t \geq \rho_t^{-1} W\}} + b^-_t (X_t - \rho_t^{-1} W)^2 1_{\{X_t < \rho_t^{-1} W\}},
\end{align}

hold along the time consistent policy, \( \{u^{TC}_t, u^{TC}_{t+1}, \ldots, u^{TC}_{T-1}\} \), at time \( t \).

At time \( T \), we have

\[ E_T[X_T] = X_T, \quad E_T[X^2_T] = X^2_T, \]

with \( a^+_T = a^-_T = 0 \) and \( b^+_T = b^-_T = 0 \). Assume that expressions of the first moment and the second moment in (16) and (17), respectively, hold at time \( t + 1 \) along the time consistent policy \( \{u^{TC}_{t+1}, \ldots, u^{TC}_{T-1}\} \). We will prove that these two expressions still hold at time \( t \) and the corresponding time consistent policy is given by (9).

As the wealth dynamic of period \( t \) is

\[ X_{t+1} = s_t X_t + P'_t u_t. \]

It follows from the policy \( \{u_t, u^{TC}_{t+1}, \ldots, u^{TC}_{T-1}\} \) that we have

\begin{align}
(18) \quad & E_t[X_T] = E_t[\mathbb{E}_{t+1}[X_T]] \\
& = E_t[\rho_{t+1} X_{t+1} + a^+_{t+1} (X_{t+1} - \rho_{t+1}^{-1} W) 1_{\{X_{t+1} \geq \rho_{t+1}^{-1} W\}} \\
& \quad \quad \quad + a^-_{t+1} (X_{t+1} - \rho_{t+1}^{-1} W) 1_{\{X_{t+1} < \rho_{t+1}^{-1} W\}}] \\
& = E_t[\rho_{t+1} \left( s_t X_t + P'_t u_t \right)] + E_t \left[ a^+_{t+1} \left( s_t X_t + P'_t u_t - \rho_{t+1}^{-1} W \right) 1_{\{s_t X_t + P'_t u_t \geq \rho_{t+1}^{-1} W\}} \right] \\
& \quad \quad \quad + E_t \left[ a^-_{t+1} \left( s_t X_t + P'_t u_t - \rho_{t+1}^{-1} W \right) 1_{\{s_t X_t + P'_t u_t < \rho_{t+1}^{-1} W\}} \right]
\end{align}

and

\begin{align}
(19) \quad & E_t[X^2_T] = E_t[\mathbb{E}_{t+1}[X^2_T]] \\
& = E_t[\rho_{t+1}^2 X_{t+1}^2 + 2 \rho_{t+1} X_{t+1} a^+_{t+1} (X_{t+1} - \rho_{t+1}^{-1} W) 1_{\{X_{t+1} \geq \rho_{t+1}^{-1} W\}} \\
& \quad \quad \quad + 2 \rho_{t+1} X_{t+1} a^-_{t+1} (X_{t+1} - \rho_{t+1}^{-1} W) 1_{\{X_{t+1} < \rho_{t+1}^{-1} W\}}] \\
& \quad \quad \quad + b^+_{t+1} \left( X_{t+1} - \rho_{t+1}^{-1} W \right)^2 1_{\{X_{t+1} \geq \rho_{t+1}^{-1} W\}} + b^-_{t+1} \left( X_{t+1} - \rho_{t+1}^{-1} W \right)^2 1_{\{X_{t+1} < \rho_{t+1}^{-1} W\}] \\
& = E_t[\rho_{t+1} \left( s_t X_t + P'_t u_t \right)^2] \\
& \quad \quad \quad + 2 \rho_{t+1} E_t \left[ (s_t X_t + P'_t u_t) a^+_{t+1} (s_t X_{t+1} + P'_t u_{t+1} - \rho_{t+1}^{-1} W) 1_{\{s_t X_{t+1} + P'_t u_{t+1} \geq \rho_{t+1}^{-1} W\}} \right] \\
& \quad \quad \quad + 2 \rho_{t+1} E_t \left[ (s_t X_t + P'_t u_t) a^-_{t+1} (s_t X_{t+1} + P'_t u_{t+1} - \rho_{t+1}^{-1} W) 1_{\{s_t X_{t+1} + P'_t u_{t+1} < \rho_{t+1}^{-1} W\}} \right] \\
& \quad \quad \quad + E_t \left[ b^+_{t+1} (s_t X_{t+1} + P'_t u_{t+1} - \rho_{t+1}^{-1} W)^2 1_{\{s_t X_{t+1} + P'_t u_{t+1} \geq \rho_{t+1}^{-1} W\}} \right] \\
& \quad \quad \quad + E_t \left[ b^-_{t+1} (s_t X_{t+1} + P'_t u_{t+1} - \rho_{t+1}^{-1} W)^2 1_{\{s_t X_{t+1} + P'_t u_{t+1} < \rho_{t+1}^{-1} W\}} \right].
\end{align}
For $X_t > \rho_t^{-1} W$, we denote any admissible policy as $u_t = K(X_t - \rho_t^{-1} W)$ with $K \in \mathbb{R}^n$. Then the cost functional can be expressed as

$$J_t(X_t; u_t) = \left( \mathbb{E}_{t}[X_t^2] - \mathbb{E}_{t}[X_T]^2 \right) - \gamma_t^+(X_t - \rho_t^{-1} W)\mathbb{E}_{t}[X_T]$$

$$= \mathbb{E}_t[\rho_{t+1}^2(s_t X_t + P_t' u_t)] - \mathbb{E}_t[\rho_{t+1}(s_t X_t + P_t' u_t)]^2$$

$$+ \mathbb{E}_t[b_{t+1}^+(s_t X_t + P_t' u_t - \rho_t^{-1} W)^21_{s_t X_t + P_t' u_t \geq \rho_t^{-1} W}] + \mathbb{E}_t[b_{t+1}^-(s_t X_t + P_t' u_t - \rho_t^{-1} W)^21_{s_t X_t + P_t' u_t < \rho_t^{-1} W}]$$

$$- \left( \mathbb{E}_t[a_{t+1}^+(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t \geq \rho_t^{-1} W}] + \mathbb{E}_t[a_{t+1}^-(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t < \rho_t^{-1} W}] \right)^2$$

$$+ 2\rho_{t+1}\mathbb{E}_t[(s_t X_t + P_t' u_t - \rho_t^{-1} W)\mathbb{E}_{t}[a_{t+1}^+(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t \geq \rho_t^{-1} W}]$$

$$+ 2\rho_{t+1}\mathbb{E}_t[(s_t X_t + P_t' u_t - \rho_t^{-1} W)\mathbb{E}_{t}[a_{t+1}^-(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t < \rho_t^{-1} W}]$$

$$- 2\rho_{t+1}\left( \mathbb{E}_t[a_{t+1}^+(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t \geq \rho_t^{-1} W}] + \mathbb{E}_t[a_{t+1}^-(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t < \rho_t^{-1} W}] \right)$$

$$- \gamma_t^+(X_t - \rho_t^{-1} W)\mathbb{E}_t[a_{t+1}^+(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t \geq \rho_t^{-1} W}]$$

$$+ \gamma_t^+(X_t - \rho_t^{-1} W)\mathbb{E}_t[a_{t+1}^-(s_t X_t + P_t' u_t - \rho_t^{-1} W)1_{s_t X_t + P_t' u_t < \rho_t^{-1} W}]$$

$$= (X_t - \rho_t^{-1} W)^2\left( \rho_{t+1}^2\mathbb{E}_t[P_t' K] - \mathbb{E}_t[P_t' \mathbb{E}_t[P_t]] \right) K$$

$$+ \mathbb{E}_t\left[b_{t+1}^+(s_t + P_t' K)^21_{s_t + P_t' K \geq 0}\right] + \mathbb{E}_t\left[b_{t+1}^-(s_t + P_t' K)^21_{s_t + P_t' K < 0}\right]$$

$$- \left( \mathbb{E}_t[a_{t+1}^+(s_t + P_t' K)1_{s_t + P_t' K \geq 0}] + \mathbb{E}_t[a_{t+1}^-(s_t + P_t' K)1_{s_t + P_t' K < 0}] \right)^2$$

$$+ 2\rho_{t+1}\mathbb{E}_t\left[a_{t+1}^+(s_t + P_t' K)^21_{s_t + P_t' K \geq 0} + a_{t+1}^-(s_t + P_t' K)^21_{s_t + P_t' K < 0}\right]$$

$$- 2\rho_{t+1}\left( \mathbb{E}_t[a_{t+1}^+(s_t + P_t' K)1_{s_t + P_t' K \geq 0}] + \mathbb{E}_t[a_{t+1}^-(s_t + P_t' K)1_{s_t + P_t' K < 0}] \right)$$

$$- \gamma_t^+(s_t + \mathbb{E}_t[P_t]) - \gamma_t^+(X_t - \rho_t^{-1} W)\rho_t^{-1} W$$

$$= (X_t - \rho_t^{-1} W)^2 F_{t+1}^+(K) - \gamma_t^+(X_t - \rho_t^{-1} W)\rho_t^{-1} W.$$

Applying Proposition 3.1 yields the optimal time consistent policy at time $t$,

$$u_t^{TC} = \arg \min_{u_t \in \mathbb{R}^n} J_t(X_t; u_t) = K_t^+(X_t - \rho_t^{-1} W).$$

Then, substituting the above optimal time consistent policy back into (18) and (19) gives rise
to
\[ E_t[X_T] = \rho_t X_t + (X_t - \rho_t^{-1}W)\left(\rho_{t+1}E_t[P_t|K_t^+] + E_t\left[a_{t+1}^+(s_t + P_t'K_t^+)1_{\{s_t + P_t'K_t^+ \geq 0\}}\right]\right] \\
+ E_t\left[a_{t+1}^+(s_t + P_t'K_t^+)1_{\{s_t + P_t'K_t^+ < 0\}}\right]\right) \\
= \rho_t X_t + a_t^+(X_t - \rho_t^{-1}W) \]
and
\[ E_t[X_T^2] = \rho_t^2 X_t^2 + 2\rho_t X_t (X_t - \rho_t^{-1}W)\left(\rho_{t+1}E_t[P_t|K_t^+] + E_t\left[a_{t+1}^+(s_t + P_t'K_t^+)1_{\{s_t + P_t'K_t^+ \geq 0\}}\right]\right] \\
+ E_t\left[a_{t+1}^+(s_t + P_t'K_t^+)1_{\{s_t + P_t'K_t^+ < 0\}}\right]\right) \\
+ \left(\rho_{t+1}^2(K_t^+)\right)E_t[P_tP_t']_t + 2\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P_t'K_t^+)P_t'K_t^+1_{\{s_t + P_t'K_t^+ \geq 0\}}\right] \\
+ 2\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P_t'K_t^+)P_t'K_t^+1_{\{s_t + P_t'K_t^+ < 0\}}\right] + E_t\left[b_{t+1}^+(s_t + P_t'K_t^+)21_{\{s_t + P_t'K_t^+ \geq 0\}}\right] \\
+ E_t\left[b_{t+1}^+(s_t + P_t'K_t^+)21_{\{s_t + P_t'K_t^+ < 0\}}\right] (X_t - \rho_t^{-1}W)^2 \\
= \rho_t^2 X_t^2 + 2\rho_t X_t a_t^+(X_t - \rho_t^{-1}W) + b_t^+(X_t - \rho_t^{-1}W)^2. \]
Furthermore,
\[ \text{Var}_t(X_T) = E_t[X_T^2] - (E_t[X_T])^2 = (b_t^+ - (a_t^+)^2)(X_t - \rho_t^{-1}W)^2 \geq 0, \]
implies \( b_t^+ - (a_t^+)^2 \geq 0. \)

For \( X_t < \rho_t^{-1}W \), we denote any admissible policy as \( u_t = K(X_t - \rho_t^{-1}W) \) with \( K \in \mathbb{R}^n \). Then the cost function can be expressed as
\[ J_t(X_t; u_t) = (X_t - \rho_t^{-1}W)^2 \left\{ \rho_{t+1}^2 K'(E_t[P_tP'_t] - E_t[P_t']E_t[P_t])K \right\} \\
+ E_t\left[b_{t+1}^+(s_t + P_t'K)^21_{\{s_t + P_t'K \leq 0\}}\right] + E_t\left[b_{t+1}^-(s_t + P_t'K)^21_{\{s_t + P_t'K > 0\}}\right] \\
- \left(\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P_t'K)1_{\{s_t + P_t'K \leq 0\}}\right] + E_t\left[a_{t+1}^-(s_t + P_t'K)1_{\{s_t + P_t'K > 0\}}\right]\right)^2 \\
+ 2\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P_t'K)1_{\{s_t + P_t'K \leq 0\}} + a_{t+1}^-(s_t + P_t'K)1_{\{s_t + P_t'K > 0\}}\right] \\
- 2\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P_t'K)1_{\{s_t + P_t'K \leq 0\}}\right] (s_t + E_t[P_t']K) \\
- 2\rho_{t+1}E_t\left[a_{t+1}^-(s_t + P_t'K)1_{\{s_t + P_t'K > 0\}}\right] (s_t + E_t[P_t']K) \\
+ \gamma_t^+ \left(\rho_{t+1}E_t\left[a_{t+1}^+(s_t + P_t'K)1_{\{s_t + P_t'K \leq 0\}}\right] + E_t\left[a_{t+1}^-(s_t + P_t'K)1_{\{s_t + P_t'K > 0\}}\right]\right) \\
+ \gamma_t^- (s_t + E_t[P_t']K) \]
\[ = (X_t - \rho_t^{-1}W)^2 F_t^- (K) + \gamma_t^- (X_t - \rho_t^{-1}W)\rho_t^{-1}W. \]

Applying Proposition 3.1 yields the optimal time consistent policy at time \( t \),
\[ u_t^{TC} = \arg\min_{u_t \in \mathbb{R}^n} J_t(X_t; u_t) = K_t^- (X_t - \rho_t^{-1}W). \]
Then, substituting the above optimal time consistent policy back into (18) and (19) gives rise to
\[
E_t[X_T] = \rho_t X_t + (X_t - \rho_t^{-1}W)\left(\rho_{t+1} E_t[\mathbf{P}_t]\mathbf{K}_{t}^{-} + E_t \left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_{t}^-)1_{\{s_t + \mathbf{P}_t'\mathbf{K}_{t}^- \leq 0\}}\right]
+ E_t \left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_{t}^-)1_{\{s_t + \mathbf{P}_t'\mathbf{K}_{t}^- > 0\}}\right]\right)
\]
\[
= \rho_t X_t + a_t^- (X_t - \rho_t^{-1}W)
\]

and
\[
E_t[X_T^2] = \rho_t^2 X_t^2 + 2\rho_t X_t(X_t - \rho_t^{-1}W)\left(\rho_{t+1} E_t[\mathbf{P}_t]\mathbf{K}_{t}^{-} + E_t \left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_{t}^-)1_{\{s_t + \mathbf{P}_t'\mathbf{K}_{t}^- \leq 0\}}\right]
+ E_t \left[a_{t+1}^+(s_t + \mathbf{P}_t'\mathbf{K}_{t}^-)1_{\{s_t + \mathbf{P}_t'\mathbf{K}_{t}^- > 0\}}\right]\right) + \rho_t^2 (X_t - \rho_t^{-1}W)^2
\]
\[
= \rho_t^2 X_t^2 + 2\rho_t X_t a_t^- (X_t - \rho_t^{-1}W) + b_t^- (X_t - \rho_t^{-1}W)^2.
\]

Furthermore,
\[
\text{Var}_t(X_T) = E_t[X_T^2] - (E_t[X_T])^2 = (b_t^- - (a_t^-)^2)(X_t - \rho_t^{-1}W)^2 \geq 0,
\]
implies \(b_t^- - (a_t^-)^2 \geq 0\).

For \(X_t = \rho_t^{-1}W\), the cost functional reduces to the conditional variance of the terminal wealth along policy \(\{u_t, u_{t+1}^{TC}, \ldots, u_{T-1}^{TC}\}\), which can be expressed as
\[
J_t(X_t; u_t)
= \rho_{t+1}^2 u_t \left(E_t[\mathbf{P}_t\mathbf{P}'_t] - E_t[\mathbf{P}'_t E_t[\mathbf{P}_t]]u_t \right.
+ E_t \left[b_{t+1}^- (\mathbf{P}_t'u_t)^2 1_{\{\mathbf{P}_t'u_t \geq 0\}}\right] + E_t \left[b_{t+1}^- (\mathbf{P}_t'u_t)^2 1_{\{\mathbf{P}_t'u_t < 0\}}\right]
- \left(E_t \left[a_{t+1}^+ P_t' u_t 1_{\{P_t'u_t \geq 0\}}\right] + E_t \left[a_{t+1}^+ P_t' u_t 1_{\{P_t'u_t < 0\}}\right]\right)^2
+ 2\rho_{t+1} \left(E_t \left[a_{t+1}^+ (\mathbf{P}_t'u_t)^2 1_{\{\mathbf{P}_t'u_t \geq 0\}}\right] + E_t \left[a_{t+1}^+ (\mathbf{P}_t'u_t)^2 1_{\{\mathbf{P}_t'u_t < 0\}}\right]\right)
- 2\rho_{t+1} \left(E_t \left[a_{t+1}^+ P_t' u_t 1_{\{P_t'u_t \geq 0\}}\right] + E_t \left[a_{t+1}^+ P_t' u_t 1_{\{P_t'u_t < 0\}}\right]\right) E_t[\mathbf{P}_t]\mathbf{u}_t
\geq 0.
\]

It is not difficult to conclude that \(u_t^{TC} = \arg \max_{u_t \in \mathbb{R}^n} J_t(X_t; u_t) = 0\).

Therefore, along the time consistent policy \(\{u_t^{TC}, u_{t+1}^{TC}, \ldots, u_{T-1}^{TC}\}\), expressions (16) and (17) hold at time \(t\), which completes our proof.

**Appendix C: The Proof of Theorem 4.1**

Proof. Following the technique in the proof of Theorem 3.1, we can derive the main results directly with the following specifics.
i) For $X_t > \rho_t^{-1}W$, we denote any admissible policy as $u_t = K(X_t - \rho_t^{-1}W)$ with $K \in A_t$.

ii) For $X_t < \rho_t^{-1}W$, we denote any admissible policy as $u_t = K(X_t - \rho_t^{-1}W)$ with $K \in -A_t$, where $-A_t$ is the negative cone of $A_t$.

iii) For $X_t = \rho_t^{-1}W$, we can similarly prove $u_t^{TC} = 0$.

Therefore, we have

$$\tilde{K}_t^+ = \arg \min_{K \in A_t} F_t^+(K), \quad \tilde{K}_t^- = \arg \min_{K \in -A_t} F_t^-(K).$$

This completes the proof. \qed