Spread waves in a viscoelastic cylindrical body of a sector cross section with cutouts

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Abstract. In this article an analysis of well-known works related to wave propagation and dispersion dependences in a cylindrical waveguide are presented. A mathematical model, methodology and algorithm for solving the problem of wave propagation in a cylindrical waveguide, having a sector cut are developed. The obtained equations are solved by the orthogonal sweep method in combination with the Mueller and Gauss methods. The dispersion relation for a viscoelastic cylindrical waveguide, having a sector cut in cross section at an arbitrary angle is obtained. Based on the obtained results, it was found that, there are no waves in the elastic cylinder of a sector section with real parts of the phase velocity. It has been established that, in the case of a wedge-shaped viscoelastic cylindrical panel, for each mode, there are limiting wave propagation velocities and they change with a change in the radius of curvature. The spectral sets of normal waves with an increase in the angular parameter of the sector cut-out, corresponding to lower non-zero frequencies of wave locking, slowly decrease.

1. Introduction
Among the variety of products of the metallurgical, engineering, petroleum and transportation industries, there is a wide range of extended objects, the length of which exceeds the transverse dimensions by more than a hundred times. Such objects include pipes, cylindrical rods or other extended objects having various defects [1-3]. Currently, there is a keen interest in non-destructive testing methods for extended objects based on the use of normal waves - waveguide methods [4-6]. In a number of works, to control linearly extended objects, it is proposed to use a rod wave in the region of minimum velocity dispersion and torsion wave mode in which there is no dispersion [7,8]. As an informative parameter, when the waveguide control of linearly extended objects, as a rule, the reflection coefficient is used. The specified parameter does not allow to detect longitudinal defects [9,10].

One of the main problem of the dynamic theory of elasticity is the study of the propagation of waves in wedge-shaped bodies (or waveguides) [11-14]. The main features of the waveguide are the length in one direction, as well as the limitation and localization of the waves in other directions. These waves everywhere, in their characteristics, are similar to Lamb waves.

Dispersion dependences having a certain number of traveling wave modes in the frequency range were obtained in [15-18].

In [19–23], dynamic behavior and wave effect in mechanical systems were discussed as in the material and paying attention to of the construction features.

These are just some of the works that are devoted to assessing the dynamic behavior and the phenomena of propagation of waves in various structures and systems.
The above review of well-known works shows that, the study of propagation of waves in systems and various structures in different works is carried out differently, and each theory or method used has its advantages and disadvantages.

Therefore, the aim of this work is to study the features of the propagation of normal waves in linearly extended viscoelastic cylindrical bodies with a radial crack in order to increase the information content of the waveguide acoustic control.

One of the problems consist the propagation of waves in an isotropic viscoelastic cylindrical waveguide, with the radial crack, is considered. The dispersion dependences for a waveguide of a sector cross section are investigated. Solution methods and an algorithm have been developed for studying the propagation waves in a cylindrical waveguide of radius R, in the cross section of which have a sector cut of an arbitrary angle. Moreover, the viscoelastic properties of materials are described by the hereditary Boltzmann – Voltaire theory [24-28].

2. Methods
2.1 The statement of the problem

An extended viscoelastic isotropic cylindrical waveguide of radius R is considered, in the cross section of which have a sector cut of an arbitrary angle. In a dimensionless system of cylindrical coordinates \((r, \varphi, z)\) region \(V = \{r_0 < r \leq R, \theta_0 < \varphi \leq 2\pi - \theta_0, -\infty < z < \infty\}\) shown in figure 1.

![Figure 1. Settlement scheme.](image)

Compound path \(\Gamma = \Gamma_0 \cup \Gamma_{0+} \cup \Gamma_{0-}\) the cross section of the waveguide is formed by a superposition of section \(\Gamma_0 = \{r = R, \theta_0 \leq \varphi \leq 2\pi - \theta_0\}\), \(\Gamma_{0+} = \{0 \leq r \leq R, \varphi = \varphi_0\}\), \(\Gamma_{0-} = \{0 \leq r \leq R, \varphi = -\varphi_0\}\).

The waveguide has a collinear direction Oz axis. The problem of analyzing the spectra of normal waves along the waveguide under consideration is formulated using the relations of the spatial linear mathematical model of the dynamic state of stress-strain of bodies paying attention to viscoelastic properties. These relations are formulated for projections of the dimensionless vector of dynamic elastic wave displacements on the axis of the cylindrical coordinate system \([u_r, u_{\theta}, u_z]\), as well as for dimensionless characteristics of the state of stress-strain of the object in question at the main sites of the cylindrical coordinate system \([\sigma_r, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_r, \sigma_{\theta z}, \sigma_{zr}]\) [12].

Deformation and stress are related by equality is as follows [19]

\[
\sigma_{ik} = \tilde{\lambda} \theta \delta_{ik} + 2 \tilde{\mu} \varepsilon_{ik}.
\]

Here \(\sigma_{ik}\) - stress tensor, \(\varepsilon_{ik}\) - tensor strain, \(\theta\) - volumetric deformation, \(\tilde{\lambda}\) and \(\tilde{\mu}\) - operator modulus of elasticity

\[
\tilde{\lambda} f(t) = \lambda_{01} \left[ f(t) - \int_0^t R_s(t - \tau)f(\tau)d\tau \right],
\]

\[
\tilde{\mu} f(t) = \mu_{01} \left[ f(t) - \int_0^t R_s(t - \tau)f(\tau)d\tau \right],
\]

\(f(t)\) - arbitrary function of time; \(R_s(t - \tau)\) \(u R_s(t - \tau)\) - relaxation core and \(\lambda_{01}, \mu_{01}\) - instant modules of elasticity.

The integral terms in relations (2) are assumed to be small, in this case the functions can be represented in the form \(f(t) = \psi(t)e^{-\lambda_0 t}\), where \(\psi(t)\) - is the time’s a slowly changing function, \(\lambda_0\) is a real constant.
Further, applying the freezing procedure [19], we replace relations (2) with approximate forms
\[
\lambda f = \lambda_n \mathbf{I} - \Gamma^+(\omega) - \Gamma^-(\omega) \mathbf{f} \quad \text{and} \quad \eta f = \mu_n \mathbf{I} - \Gamma^+(\omega) - \Gamma^-(\omega) \mathbf{f},
\]
where
\[
\Gamma^+(\omega) = \frac{1}{2 \pi} \int_0^\infty \left[ R_n(t) \cos \omega_r t \, dt \right], \quad \Gamma^-(\omega) = \frac{1}{2 \pi} \int_0^\infty \left[ R_n(t) \sin \omega_r t \, dt \right],
\]
the Fourier images (cosine and sine images) of the core relaxation of the material. On the function of influence \( R(t - \tau) \) requirements for continuity (except), monotony, integrability, sign of certainty:
\[
R > 0, \quad \frac{dR(t)}{dt} \leq 0, \quad 0 < \int \frac{R(t)}{dt} \, dt < 1.
\]
Equations of motion of a viscoelastic cylindrical mechanical waveguide, occupying region \( V \), are defined by the following equations [12]:
\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{zz}}{r} + \frac{1}{r} \frac{\partial \sigma_{\varphi \varphi}}{\partial \varphi} + \frac{\partial \sigma_{\varphi r}}{\partial \varphi} = \rho \frac{\partial^2 u_r}{\partial t^2},
\]
\[
\frac{1}{r} \frac{\partial \sigma_{\varphi \varphi}}{\partial \varphi} + \frac{2 \sigma_{\varphi \varphi}}{r} + \frac{\partial \sigma_{\varphi r}}{\partial \varphi} + \frac{1}{r} \frac{\partial \sigma_{rr}}{\partial \varphi} = \rho \frac{\partial^2 u_\varphi}{\partial t^2},
\]
\[
\frac{\partial \sigma_{rr}}{\partial z} + \frac{\partial \sigma_{\varphi r}}{\partial \varphi} + \frac{1}{r} \frac{\partial \sigma_{\varphi z}}{\partial \varphi} = \rho \frac{\partial^2 u_z}{\partial t^2};
\]
\[
\varepsilon_{rr} = \frac{\partial u_r}{\partial r}; \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}; \quad \varepsilon_{\varphi r} = \frac{1}{r} \frac{\partial u_r}{\partial \varphi} + \frac{u_r}{r};
\]
\[
\varepsilon_{\varphi \varphi} = \frac{1}{2} \left( \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi}{r} \right); \quad \varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right);
\]
\[
\sigma_{rr} = \lambda \left( \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{\partial u_\varphi}{\partial \varphi} \right) + 2 \mu \frac{\partial u_r}{\partial r};
\]
\[
\sigma_{rr} = 2 \mu \varepsilon_{rr} = \mu \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right); \quad \sigma_{\varphi \varphi} = 2 \mu \varepsilon_{\varphi \varphi} = \mu \left( \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi}{r} \right); \quad \sigma_{zz} = 2 \mu \varepsilon_{zz} = \mu \left( \frac{\partial u_z}{\partial z} + \frac{u_z}{r} \right),
\]
where \( \rho \) - material density, \( \sigma_{rr}, \sigma_{\varphi \varphi}, \sigma_{zz}, \sigma_{\varphi r}, \sigma_{\varphi z}, \sigma_{rz} \) - stress tensor components; \( \varepsilon_{rr}, \varepsilon_{\varphi \varphi}, \varepsilon_{zz}, \varepsilon_{\varphi r}, \varepsilon_{rz} \) - components of the strain tensor.

The above equations (4), (5), (6) form a system of differential equations in partial derivatives with complex coefficients. The constructed system of equations is resolved with respect to the first derivatives.
\[
\frac{\partial u_r}{\partial r} = \frac{1}{K_r} \sigma_r - \tilde{\mathcal{X}} \left( \frac{1}{r} \frac{\partial u_r}{\partial \phi} + \frac{u_r}{r} + \frac{\partial u_r}{\partial \zeta} \right),
\]
\[
\frac{\partial u_r}{\partial r} = \frac{1}{\tilde{\mathcal{X}}} \sigma_r - \frac{1}{r} \left( \frac{\partial u_r}{\partial \phi} - u_r \right);
\]
\[
\frac{\partial u_r}{\partial r} = \frac{1}{\tilde{\mathcal{X}}} \sigma_r - \frac{\partial u_r}{\partial \zeta};
\]
\[
\frac{\partial \sigma_r}{\partial r} = \rho \frac{\partial^2 u_r}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{\sigma_r}{r} \frac{\partial u_r}{\partial \phi} \right) - \frac{\partial \sigma_r}{\partial \zeta};
\]
\[
\frac{\partial \sigma_r}{\partial r} = \rho \frac{\partial^2 u_r}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial \phi} \left( \sigma_r - 2\tilde{\mathcal{X}} \left[ \sigma_r - \frac{1}{r} \frac{\partial}{\partial \zeta} B \right] - \frac{\partial}{\partial \zeta} B,\right.
\]

where is introduced the notations:
\[
\tilde{\mathcal{X}} = 2\tilde{\mathcal{X}} \left[ \frac{\partial u_r}{\partial r} - \frac{1}{r} \left( \frac{\partial u_r}{\partial \phi} + u_r \right) \right];
\]
\[
\tilde{B} = \frac{\partial u_r}{\partial \zeta} + \frac{1}{r} \frac{\partial u_r}{\partial \phi}, \quad K_A = \tilde{\mathcal{X}} + 2\tilde{\mathcal{X}}.
\]

The boundary conditions are set in the form:
\[
r = r_0 \rightarrow 0, \quad R:\]
\[
(r_n) = (\sigma_{r_n}) = (\sigma_{\phi_n}) = 0; \quad \phi = \alpha: \quad u_\phi = 0, \quad \sigma_{\phi \phi} = \sigma_{\phi r} = 0; \quad \phi = \alpha^-: \quad u_\phi = 0, \quad \sigma_{\phi \phi} = \sigma_{\phi r} = 0.
\]

2.2. Solution Methods
The geometry of the object and the natural assumption about the nature of the wave motion along the axis \( Oz \) allow us to present the problem’s (7) - (9) solution in a form, that allows separation of variables [9,26]
\[
u_r = \sum_{n=0}^{\infty} w_n (r, \phi) U_r(z) T_r(t); \quad \sigma_r = \sum_{n=0}^{\infty} \sigma_n (r, \phi) U_r(z) T_r(t); \tag{10}
\]
\[
u_\phi = \sum_{n=0}^{\infty} v_n (r, \phi) U_\phi(z) T_r(t); \quad \sigma_{\phi \phi} = \sum_{n=0}^{\infty} \tau_{\phi \phi} (r, \phi) U_\phi(z) T_r(t); \quad \sigma_{\phi r} = \sum_{n=0}^{\infty} \tau_{\phi r} (r, \phi) U_\phi(z) T_r(t);
\]

where \( w_n (r, \phi), v_n (r, \phi), u_n (r, \phi) \) - are the displacement amplitudes, \( \sigma_n (r, \phi), \tau_{\phi \phi} (r, \phi), \tau_{\phi r} (r, \phi) \) - are the stress amplitudes.

The periodicity conditions make it possible to eliminate the dependence of the main unknowns, using the following change of variables:
\[
u_r = \sum_{n=0}^{\infty} w_n (r, \phi) e^{i(\omega - \omega t)}; \quad \sigma_r = \sum_{n=0}^{\infty} \sigma_n (r, \phi) e^{i(\omega - \omega t)}; \tag{11}
\]
\[
u_\phi = \sum_{n=0}^{\infty} v_n (r, \phi) e^{i(\omega - \omega t)}; \quad \sigma_{\phi \phi} = \sum_{n=0}^{\infty} \tau_{\phi \phi} (r, \phi) e^{i(\omega - \omega t)}; \quad \sigma_{\phi r} = \sum_{n=0}^{\infty} \tau_{\phi r} (r, \phi) e^{i(\omega - \omega t)};
\]

where \( \omega = \omega_k + i \omega_t \) is the complex natural frequency, \( \omega_k \) - is the natural wave propagation frequency, \( \omega_t \) - is the damping coefficient, \( C = \omega / k \) - phase velocity, \( k \) - wave number.
Consider the following two cases:
1) $C = C_k + i C_l$, $k = k_c$, then solution (10) has the form of a sinusoid, amplitude of which is decays;
2) $C = C_k, k = k_c + ik_b$, the oscillations are steady, but in $z$, damp.

In further calculations, the index $n$ is skipped.

If we substitute (10) into (7), (8), (9), then the constructed system of differential equations in partial derivatives with complex coefficients reduces to a spectral problem.

To describe the wave process, we use the relations (1), (2), (3) given in the previous paragraph. The resolving system of equations coincides with system (7), the boundary conditions on the surface (8) are preserved without change. The boundary conditions for an arbitrary angle of the wedge, in the case of a free lateral surface, should be written in the form:

$$\varphi = -\frac{\varphi_0}{2} - \frac{\varphi_0}{2} : \quad \sigma_{\varphi \varphi} = \sigma_{r \varphi} = \sigma_{r r} = 0,$$

where $\varphi_0$ - angle at the top of the wedge.

Under condition (12), the separation of the variables $r$ and $\varphi$ is no longer possible; therefore, the direct method is used [29]. In view of (12), the system (7) has the form:

$$w' = \frac{v}{r} - \frac{1}{r} - \frac{1}{r} \left( \frac{\partial w}{\partial \varphi} \right), \quad u' = \frac{v}{r} + kw$$

$$\sigma' = -\sigma \rho w + \frac{1}{r} \left( \frac{\partial \sigma}{\partial \varphi} \right) - k \tau,$$

$$\tau' = -\sigma \rho w - \frac{1}{r} \left( \frac{\partial \tau}{\partial \varphi} \right) + 2 \tau,$$

$$\rho' = -\sigma \rho w - \frac{1}{r} \left( \frac{\partial \rho}{\partial \varphi} \right) + 2 \rho,$$

where

$$A = 2M \left( \frac{1}{r} \left( -\frac{\partial u}{\partial r} \right) + w \right); \quad B = M \left( \frac{1}{r} \left( -\frac{\partial u}{\partial \varphi} \right) - kv \right).$$

The boundary conditions (8) are similarly transformed

$$r = 0, R: \sigma = \tau = \tau = 0.$$

The components of the tensor of stress $\sigma_{\varphi \varphi}, \sigma_{r \varphi}$ and $\sigma_{r r}$ are expressed through the basic unknowns:

$$\sigma_{\varphi \varphi} = \sigma_{rr} + 2M \left( \frac{1}{r} \left( -\frac{\partial u}{\partial r} \right) + \frac{u}{r} - \frac{\partial u}{\partial r} \right),$$

$$\sigma_{rr} = M \left( \frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial \varphi} \right),$$

$$\sigma_{r \varphi} = \sigma_{r \varphi} + 2M \left( \frac{\partial u}{\partial \varphi} - \frac{\partial u}{\partial \varphi} \right).$$

Then, paying attention the first equation of (15), the boundary conditions (14) take the form:

$$\sigma = A + \sigma = a \sigma + b \left( \frac{\partial \varphi}{\partial \varphi} + w \right) + c ku = 0;$$

$$\varphi = -\frac{\varphi_0}{2} - \frac{\varphi_0}{2}; \quad \tau = 0, R = M \left( \frac{\partial u}{\partial \varphi} - k r \right) = 0,$$

where

$$a = 1 + \frac{2M}{k}; \quad b = 2M \left( 1 + \frac{\alpha}{k} \right), \quad c = 2M \frac{\alpha}{k}.$$
The second-order approximating formulas have the form [29]:

\[
\gamma_{i,\varphi} \equiv \frac{\gamma_{i+1} - \gamma_{i}}{2\Delta} \approx \frac{-3\gamma_{i} + 4\gamma_{i+1} - \gamma_{i+2}}{2\Delta} \approx \frac{3\gamma_{i} - 4\gamma_{i+1} + \gamma_{i+2}}{2\Delta},
\]

\[
\gamma''_{i,\varphi} \equiv \frac{\gamma_{i+1} - 2\gamma_{i} + \gamma_{i-1}}{\Delta^2},
\]

where \( i \) varies from 0 to \( N+1 (i = 0, \ldots, N+1) \); \( \Delta \) – coordinate splitting step \( \varphi \).

As a result of sampling, the vector of basic unknowns with a total dimension of \( 6N \) can be written as:

\[
Y = \{(w_i), \{v_i\}, \{u_i\}, \{\sigma_{i+1}\}, \{\tau_{i+1}\}, \{\tau_{i-1}\}\} ^{T}. \quad (i = 1, N)
\]

A derivatives in the right parts of the system of equations (19) is expressed by following formulas:

\[
\begin{align*}
 w_{i,\varphi} &= (w_{i+1} - w_{i-1})/2\Delta; \\
 v_{i,\varphi} &= (v_{i+1} - v_{i-1})/2\Delta; \\
 \tau_{i,\varphi} &= (\tau_{i+1} - \tau_{i-1})/2\Delta; \\
 \sigma_{i,\varphi} &= a(\sigma_{i+1} - \sigma_{i-1})/2\Delta + b/\rho \left( [v_{i+1} - 2v_{i} + v_{i-1}] /\Delta^2 + w_{i,\varphi} \right) + c k u_{i,\varphi}; \\
 B_i &= (u_{i+1} - 2u_i + u_{i-1})/\Delta^2 / k - kv_{i,\varphi}.
\end{align*}
\]

Boundary conditions for \( \varphi = -\theta_0 \), paying to attention in equations, corresponding to straight lines \( i = 1 \).

For basic unknowns outside the boundary conditions \( w_i, v_i, u_i \), right differences are used (16):

\[
\begin{align*}
 w_{i,\varphi} &= (-3w_1 + 4w_2 - w_3)/2\Delta; \\
 v_{i,\varphi} &= (-3v_1 + 4v_2 - v_3)/2\Delta; \\
 u_{i,\varphi} &= (-3u_1 + 4u_2 - u_3)/2\Delta.
\end{align*}
\]

For variable \( \tau_{\varphi} \) conditions (16) paying to attention using the central differences

\[
\tau_{\varphi_i,\varphi} \equiv \left( \tau_{\varphi_{i+1}} - \tau_{\varphi_{i-1}} \right) /2\Delta = -\tau_{\varphi_{i+1}} /2\Delta.
\]

The first and third of conditions (16) paying to attention, approximating the derivatives with respect to \( \varphi \):

\[
\begin{align*}
 \sigma_{i,\varphi} &\equiv (\sigma_{i+1} - \sigma_{i-1})/2\Delta \approx \frac{a\sigma_{i+1} + b/\rho \left( [v_{i+1} - 2v_{i} + v_{i-1}] /2\Delta + w_{i,\varphi} \right) - c k u_{i,\varphi}}{2\Delta}; \\
 B_{i,\varphi} &\equiv (B_{i+1} - B_i)/2\Delta = B_i/2\Delta = \left( u_{i+1} - u_i \right) /\Delta^2 / r - kv_{i,\varphi} /2\Delta.
\end{align*}
\]

Derivatives for the line with the number \( i = N \), paying to attention boundary conditions at \( \varphi = \theta_0 /2 \):

\[
\begin{align*}
 w_{i,\varphi} &= (3w_N - 4w_{N-1} + w_{N-2})/2\Delta; \\
 v_{i,\varphi} &= (3v_N - \ldots)/2\Delta; \\
 U_{i,\varphi} &= (U_{i+1} - U_{i-1})/2\Delta; \\
 \tau_{i,\varphi} &= -\tau_{i+1,\varphi} /2\Delta; \\
 \sigma_{i,\varphi} &= \left( a\sigma_{i+1} + b/\rho \left( [v_{i+1} - 2v_{i} + v_{i-1}] /2\Delta + w_{i,\varphi} \right) + c k u_{i,\varphi} \right) /2\Delta = -\sigma_{i+1} /2\Delta; \\
 B_{i,\varphi} &= -\left( (u_N - u_{N-1}) /2\Delta / r - kv_{N-1} \right) /2\Delta = -B_{N-1} /2\Delta.
\end{align*}
\]

The number of straight lines can be halved by using the conditions of anti symmetry of transverse vibrations of the plate at \( \varphi = 0 \):

\[
w = u = \sigma_{\varphi} = 0.
\]

The corresponding difference relations pay attention conditions (25), can be written in the form:

\[
i = N.
\]
\begin{align*}
w_{i,\varphi} &= -w_{N-1}/2\Delta; \ u_{i,\varphi} = -u_{N-1}/2\Delta; \\
v_{i,\varphi} &= (3v_i - \cdots)/2\Delta; \ \tau_{i,\varphi} = (3\tau_{i,\varphi} - 4\tau_{i-1,\varphi} + \tau_{i+1,\varphi})/2\Delta; \\
\sigma_{i,\varphi} &= -\left(2\sigma_{i,\varphi} + \frac{b}{\tau} \left[(v_N - v_{N-1})/2\Delta + w_{N-1}\right] + cku_{N-1}\right)/2\Delta = -\frac{\sigma_{N-1}}{2\Delta}; \\
B_{i,\varphi} &= -(2u_{\varphi} + u_{N-1})/\Delta^2/r - k\nu_{\varphi,\varphi}.
\end{align*}

And, according to (15) we get the system of differential equations

\begin{align*}
w_i' &= \sigma_i/k - a(ku_i + (w_i + v_{i,\varphi})/R); \\
v_i' &= \tau_{i,\varphi} + (v_i - w_i)/R; \\
u_i' &= \tau_{i,\varphi} + kw_i; \ \sigma_i' = -\omega^2 w_i + \left[2\left(w_i + v_{i,\varphi}\right)/R - w_i\right]/r - k\tau_{i,\varphi}; \\
\tau_i' &= -\omega^2 u_i - \left(B_{i,\varphi} + \tau_i\right)/r + k(\sigma_i + 2(ku_i - w_i)); \\
\tau_{\varphi}' &= -\omega^2 v_i + (\sigma_{\varphi,\varphi} + 2\tau_{i,\varphi})/r - k(u_{i,\varphi}/R - k^2).
\end{align*}

In equations (27), expressions for derivatives $w_{i,\varphi}$, $v_{i,\varphi}$, $u_{i,\varphi}$, $\sigma_{i,\varphi}$, $\tau_{i,\varphi}$ from relations (23) - (26) are selected. Conditions of the free surface are obtained in the form

$$B_i = 0, \ \tau_{i,\varphi} = 0, \ \sigma_{\varphi} = 0. \quad (i=1,N)$$

Thus, the spectral problem (13), (14), (15), using the discretization of the coordinate $\varphi$ by the direct method, reduces to the canonical problem (27), (28), which we solve using the method of orthogonal sweep [30].

### 3. Results and Discussion

#### 3.1 Numerical results

As an example of a model of material, we choose the three parametrical relaxation core:

$$R_c(t) = R_{cs} e^{-\beta t}/t^{\alpha - 1}.\quad$$

Dimensionless quantities are chosen so, that the shear rate $C_s$, density $\rho$, radius $R$ have unit values, and the ratio of Poisson $\nu = 0.25$, kernel parameters:\n
$$A = 0.048; \ \beta = 0.05; \ \alpha = 0.1.$$ 

Table 1 shows the limiting values of the phase velocity, depending on the angle of the wedge (for the first mode of the edge). The found phase velocities within the framework of the described methodology for calculating a wedge are given in columns 5–6 for various boundary conditions. In column 5 represented results calculation options with three internal lines with the boundary conditions (17), in column 6 represented by boundary conditions:

\begin{align*}
\varphi &= -\frac{\theta_0}{2}; \ \sigma_{\varphi,\varphi} = \sigma_{\varphi} = \sigma_{\varphi} = 0; \\
\varphi &= 0; \ u_{\varphi} = u_{\varphi} = \sigma_{\varphi,\varphi} = 0. \quad (29)
\end{align*}

Also, in columns 3 and 4, respectively, the limiting values of the real part of the phase velocity of the first edge mode are given, which was learned in [15] by hypotheses of Kirchhoff – Love and Timoshenko plates. In column 6 shows the calculation results obtained by the formula

$$C_\alpha = C_K \sin(m\varphi), \quad (30)$$

$m\varphi < 90^\circ \ (m = 1, 2, \ldots).$

Table 1 shows the changes (of the real part of the natural frequency) for different ones, for different $\theta_0$, the calculation methods by hypotheses of Kirchhoff - Love, Timoshenko and three-dimensional theory methods agree with each other up to 7% for angles with a base thickness $h_2$ not exceeding 0.5.
Table 1. Limit values of the first boundary mode of the phase velocity with respect of wedge angle

| R   | $\theta_0$ | By the Kirchhoff-Love hypothesis | By Tymoshenko hypothesis | By of the three-dimensional theory, boundary conditions (12) | By of the three-dimensional theory, boundary conditions (29) | According to the formula $[15]$ $R_s = R_p = 0$ |
|------|------------|---------------------------------|--------------------------|----------------------------------------------------------|----------------------------------------------------------|--------------------------------------------------|
| 0.2  | 11$^o$     | 0.2260                          | 0.1962                   | -                                                        | -                                                        | 0.1827                                           |
| 0.3  | 17$^o$     | 0.3802                          | 0.2964                   | 0.2982                                                   | 0.3082                                                   | 0.2763                                           |
| 0.5  | 28$^o$     | 0.5385                          | 0.4429                   | 0.4626                                                   | 0.4755                                                   | 0.4335                                           |
| 0.7  | 38$^o$     | 0.7014                          | 0.5635                   | 0.5923                                                   | 0.6059                                                   | 0.5746                                           |
| 1    | 53$^o$     | 1.0144                          | 0.6918                   | 0.7294                                                   | 0.7414                                                   | 0.7368                                           |
| 2    | 90$^o$     | 2.8637                          | 0.8641                   | 0.8792                                                   | 0.8949                                                   | 0.9212                                           |

The phenomenon should be paying to attention, when studying the dynamic behavior of a cylindrical waveguide of variable thickness. In the case of a cylinder with a sector cross-section, the first complex mode has a frequency cutoff. On locking, the axial displacements are equal to zero and the oscillations of the infinitely viscoelastic cylinder with a sector cross-section occur in a flat deformed state. In the second mode, at the locking frequency, are observed, the ring and radial displacements are equal to zero. Unlike edge waves, in a cylinder of infinite length with a sector cross section, in an acute wedge the waves do not have a final solution for.

4. Conclusions

As a result, the following results were obtained:

1. A mathematical model, technique and algorithm has been developed for solving the problem of propagation of waves in a cylindrical waveguide (viscoelastic), having in the cross section of the body a sector cut.

2. When studying the dispersion dependence for a waveguide with a sector cross-section, it was found that the first complex mode has a frequency cutoff. On locking, the axial displacements are equal to zero and the oscillations of the infinitely viscoelastic cylinder in the sector cross section occur in a flat deformed state.

3. Paying attention to properties of the material makes it possible to evaluate the damping abilities of the system as a whole and reduces the real parts of the wave propagation velocity by 10-15%.

4. It has been established that the spectral sets of normal boundary waves with increasing angular parameter of the sector cut-out, corresponding to the lower non-zero frequency of locking the waves, slowly decrease.

5. It is established that in the wave spectra for the values $\theta_0 \leq 90^o$ there are modes with a nonzero locking frequency that disappear in the spectral families of the problem type under consideration when $\theta_0 \geq 90^o$.

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