General Solution to 2D Steady Navier-Stokes Equation for
Incompressible Flow without vorticity diffusion

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Abstract. The study solves the general solution to 2D steady Navier-Stokes equation for incompressible flow without vorticity diffusion, which is more general than Stokes flow. In order to obtain the general solution, two potential functions are introduced to express the velocity: a vector potential describing the rotational incompressible flow and a scalar potential describing the irrotational incompressible flow. The results show that the vorticity equation expressed with potential functions is a biharmonic function, which means that the potential functions describing the flow field are polynomials of no more than fourth degree. For a steady unidirectional shear flow, the velocity and pressure fields can be described with the vector potential expressed by a polynomial of third degree. For non unidirectional two-dimensional steady shear flow, there may be four independent parameters in the two potential functions.
The study of viscous fluid flow is an ancient topic, which may date back to prehistoric times when humans begin to use streamlined tools like spears and arrows. The attempts to get the exact solution to the problem of the viscous fluid is taken since the Greek philosopher Aristotle (384-322 B.C.) and mathematician Archimedes (287-212 B.C.) [1, 2]. In 1752, Euler first use the partial differential equation to formulate the flow of fluid with negligible viscosity [1, 3]. Subsequently, by adopting the Newton’s definition of friction, Navier and Stokes independently include the viscous forces into the fluid dynamic equation for inviscid fluids and the fluid dynamic equation for viscous fluids is obtained, which is called Navier-Stokes equation now.

Since the Navier-Stokes equation can well describe the particular solutions, such as laminar pipe flows and some boundary layer flows [4, 5], The Navier-Stokes equation are considered to be the fundamental equation for governing fluid motion and dynamics. The Navier-Stokes equation is a nonlinear partial differential equation. Therefore, seeking an analytical solution to the Navier-Stokes equation is a very challenging task, which is considered to be impossible, except for some simple laminar flows. In 2000, the analytical solution to the Navier–Stokes equation was selected to be one of seven Millennium Problems by the Clay Mathematics Institute of Cambridge, U.S. (http://www.claymath.org/millennium-problems). In 2008, the U.S. Defense Advanced Research Projects Agency (DARPA) listed it as one of 23 DARPA Mathematical Challenges — “Mathematical Challenge Four: 21st Century Fluids”. Though the difficulty in seeking an analytical solution to the Navier-Stokes equation is
severe, fluid mechanics researchers never give up their best desire for it [6-10].

The study is one of attempts to find the analytic solution of the Navier-Stokes equation for a special case. In this study, an effort has been taken to solve the general solution to 2D steady Navier-Stokes equation for incompressible flow without vorticity diffusion and the general solution is derived in Cartesian coordinate system.

For incompressible Newtonian fluid (\( \nabla \cdot \mathbf{u} = 0 \)), the Navier-Stokes equation of motion describing the steady flow of viscous fluid is expressed in vector notation as [5]:

$$\rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \eta \nabla^2 \mathbf{u}$$  \hspace{1cm} (1)

where \( \partial / \partial t \) is the material derivative, \( \mathbf{u} \) is the velocity field of fluid, \( \rho \) is the mass density, \( \eta \) is dynamic viscosity, \( p \) is the pressure, \( \mathbf{f} \) is the body force which is a non-curl field. By taking the curl of both sides of the equation of motion, the vorticity equation of steady flow of incompressible Newtonian fluid is obtained:

$$\nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \omega = 0$$  \hspace{1cm} (2)

with \( \omega \) the vorticity (\( \omega = \nabla \times \mathbf{u} \)) and \( \nu \) the kinematic viscosity \( \nu = \eta / \rho \).

For two-dimensional flow without vorticity diffusion (\( \nabla^2 \omega = 0 \)), the vorticity is always perpendicular to the velocity field. Thus, the following relationship always hold:

$$\omega \cdot \nabla \mathbf{u} = 0$$  \hspace{1cm} (3)

In this case, Equation (2) is reduced to the following formula [5]:

$$\mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega = 0$$  \hspace{1cm} (4)

Assuming that the velocity field of the two-dimensional flow can be expressed by two potential functions as:
\[ u = \nabla \times \phi + \nabla \phi \]  \hspace{1cm} (5)

Here, \( \phi \) is the vector potential describing the rotational incompressible flow and \( \phi \) is the scalar potential describing the irrotational incompressible flow \((\nabla \cdot u = \nabla^2 \phi = 0)\). With Equation (5), Equation (4) can be rewritten as:

\[
(\nabla \times \phi \cdot \nabla) \nabla^2 \phi + (\nabla \phi \cdot \nabla) \nabla^2 \phi = \nu \nabla^4 \phi = 0
\]  \hspace{1cm} (6)

When the two-dimensional flow is described in the \( xy \)-plane of Cartesian coordinate system, the vector potential can be expressed as:

\[ \phi = \phi e_z \]  \hspace{1cm} (7)

Here, \( e_z \) is the unit vector of \( z \) axis. Then, Equation (6) can be expressed as:

\[
\left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} \right) \left( \frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2} \right) + \left( \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x} \right) \left( \frac{\partial^3 \phi}{\partial y \partial x^2} + \frac{\partial^3 \phi}{\partial y^3} \right) = \nu \nabla^4 \phi = 0 \]  \hspace{1cm} (8)

It is seen from Equation (8) that \( \phi \) is a polynomial of fourth degree or less than fourth degree. In addition, the vector potential \( \phi \) describes the shear flow part of velocity field, therefore the viscous force should be not zero. This shows that the term less than third degree in \( \phi \) is zero. Under these limitations, \( \phi \) should be expressed as:

\[
\phi = a_1 x^4 + a_2 y^3 + a_3 y^2 x^2 + a_4 y^3 x + a_5 y^4 + b_1 x^3 + b_2 y x^2 + b_3 y^2 x + b_4 y^3 + c_1 x^2 + c_2 y x + c_3 y^2
\]  \hspace{1cm} (9)

With

\[ a_1 + 2a_3 + a_5 = 0 \]  \hspace{1cm} (10)

Here, \( a_i \) and \( b_i \) are undetermined parameters.

When \( \phi \) is a polynomial of third degree, it is obtained by replacing the velocity with the potential function that the viscous force in flow field is a constant:
\[ \mu \nabla^2 \mathbf{u} = \nabla \times \nabla^2 \phi = 2(b_2 + 3b_3) \mathbf{e}_x + 2(3b_1 + b_3) \mathbf{e}_y \]  

(11)

This means that in this case the steady two-dimensional incompressible flow is a steady unidirectional flow and the scalar potential \( \phi \) is unnecessary in describing the velocity field like Poiseuille flow and Couette flow. Equation (1) is also simplified as:

\[ f - \nabla p + \eta \nabla^2 \mathbf{u} = 0 \]  

(12)

When \( \phi \) is a polynomial of fourth degree, the scalar potential \( \phi \) should be expressed as:

\[ \phi = l_1 x^4 + l_2 y^4 - 6l_1 y^2 x^2 - l_2 y^4 x + l_2 y^4 + m_1 x^3 - 3m_2 y x^2 - 3m_1 y^2 x + m_2 y^3 + k_1 x^3 + k_2 y x - k_1 y^2 + h_1 x + h_2 y \]  

(13)

with \( l_i, m_i \) and \( k_i \) undetermined parameters. Submitting Equations (13) and (17) into Equation (8), a system of fifteen linear equations with eight variables \( l_i, m_i \) and \( k_i \) is obtained. Then, the parameters in Equation (17) can be replaced with the parameters in Equation (13) and the relationships among parameters \( (a_i, b_i) \) in Equation (13) can be established. Plus Equation (14), there are sixteen equations to describe the relationships among the twenty variables \( (a_i, b_i, l_i, m_i \) and \( k_i \)). This indicates that there may be only four independent parameters in potential functions describing the non unidirectional two-dimensional shear flow. Further replacing the velocity in Equation (1) with the vector potential \( \varphi \) and the scalar potential \( \phi \), the pressure field can be expressed as:

\[ p = \int \left( f_s + \eta \frac{\partial}{\partial y} \nabla^2 \varphi - \rho \left( \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x} \right) + \left( \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y} \right) \right) \left( \frac{\partial \varphi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial x} \right) dx + p_0 \]  

(14)
\[ p = \int f_y - \eta \frac{\partial}{\partial x} \nabla^2 \varphi - \rho \left( \left( \frac{\partial \varphi}{\partial y} + \frac{\partial \phi}{\partial x} \right) \frac{\partial}{\partial x} + \left( \frac{\partial \phi}{\partial y} - \frac{\partial \varphi}{\partial x} \right) \frac{\partial}{\partial y} + \left( \frac{\partial \phi}{\partial y} - \frac{\partial \varphi}{\partial x} \right) \right) \right) \, dy + p_0 \quad (15) \]

Here, \( p_0 \) is the pressure at a point in the flow field. When the boundary condition is known, the pressure field and potential functions can be solved, and then the velocity field is obtained with potential functions.

In summary, the general solution to 2D steady Navier-Stokes equation for incompressible flow without vorticity diffusion is solved in Cartesian coordinate system, which is more general than Stokes flow. It is found that in this case the vorticity equation expressed with potential functions is a biharmonic function. There is only four independent parameters in the polynomial describing the non unidirectional two-dimensional steady shear flow. For a steady unidirectional shear flow, the velocity and pressure fields can be described with the vector potential \( \varphi \) which is expressed with a polynomial of third degree.

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