Energy-momentum squared symmetric Teleparallel gravity: $f(Q, T_{\mu\nu} T^{\mu\nu})$ gravity

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Abstract: In this work we propose the $f(Q, T_{\mu\nu} T^{\mu\nu})$ gravity as a further extension of the $f(Q)$ and $f(Q, T)$ gravity theories, where $Q$ is the non-metricity and $T_{\mu\nu}$ is the energy-momentum tensor. The action involves an arbitrary function of the non-metricity $Q$ and $T^{2} = T_{\mu\nu} T^{\mu\nu}$ in the gravity Lagrangian. The field equations for the theory are derived in the metric-affine formalism. The theory involves a non-minimal coupling between the geometric and the matter sectors, and hence the covariant divergence of the energy momentum tensor is non-zero, thus implying the non-conservation of the same. The vacuum solutions of the theory are investigated and it is found that the theory perfectly admits a de-Sitter-like evolution of the universe. The cosmological equations are derived and it is found that there are two correction terms arising as modification of the gravity. Two specific toy models of the form $Q + \eta (T^{2})^{n}$ and $f_{0} Q^{m} (T^{2})^{n}$ are explored to gain further insights into the dynamics of the theory. It is seen that the field equations of both the models have similar terms to those arising from the quantum gravity effects and are thus responsible for the avoidance of the singularity. One striking feature of the model is that the non-linear correction terms dominate in the early universe and gradually fade away at later times giving standard FLRW universe. Solutions for the FLRW equations are found wherever possible and the evolution of the scale factor and the matter energy density is plotted. Other cosmological parameters like the equation of state, deceleration parameters and Hubble functions are also studied. Finally the energy conditions are explored in the background of the theory. Using these conditions and some observational data the parameter spaces of the models are considerably constrained. $f(Q, T_{\mu\nu} T^{\mu\nu})$ is a theory that can perfectly explain the cosmological dynamics of both the early and the late universe without resorting to any dark energy.

Keywords: non-metricity, energy-momentum, symmetric teleparallel gravity; cosmology, energy conditions.
1 Introduction

Currently the most comprehensive theory of gravity available to us is the theory of General Relativity (GR) [1] proposed by Albert Einstein in 1916. After a century of extensive research GR has been able to survive quite well with many of its predictions coming true. Some of them being the perihelion precession of mercury, deflection of light by sun [2], gravitational redshift [3], detection of gravitational waves [4] from the mergers of black holes and neutron stars, imaging a black hole shadow in the M87 galaxy by the Event horizon telescope [5], etc. Extensive reviews in experimental and observational tests of GR can be found in [6, 7]. Nevertheless some observations did not go in favour of GR, quite expectedly. The most important one being the discovery of the late cosmic acceleration [8, 9] at the turn of the last century. With this observation to light, GR became incompatible at cosmological distances. One more important observation was that GR was not able to explain the gravitational interaction at the quantum level. Moreover standard cosmology is plagued by problems like the singularity problem, cosmological constant problem, cosmic coincidence problem, etc. So it is clear that we are far from the theory that we need to comprehensively explain our observations. This provided motivation for extensive theoretical research in cosmology over the past century. With the development of quantum mechanics throughout the last century, scientists including Einstein took up the challenge of developing a proper theory of quantum gravity. As a result theories like string theory, loop quantum gravity theory, etc. were proposed, but till date they are far from being comprehensive. To resolve the incompatibility of GR at cosmological scales mainstream research has travelled in two different ways. The first path is the theory of dark
energy (DE) where the matter content of the universe is modelled by an exotic fluid with negative pressure driving the acceleration. Extensive reviews on DE can be found in [10]. The alternative path is the theory of modified gravity, where the gravitational framework of GR has been modified to incorporate the cosmic acceleration. See [11–13] for extensive reviews on modified gravity. In this work we will concentrate on this alternative path of modified gravity theory.

The field equations of GR are basically formulated from the Einstein-Hilbert (EH) action $S = \int (R + L_m) \sqrt{-g} d^4x$, where $R$ is the Ricci scalar, $g$ is the determinant of the metric tensor $(g_{\mu\nu})$ that represents the gravitational field and $L_m$ is the matter Lagrangian. The most fundamental form of modification is imposed by replacing the gravitational Lagrangian $R$ of EH action by an arbitrary function $f(R)$, thus considering a generalized action. This gave rise to $f(R)$ theories [14]. Substantive development in $f(R)$ gravity can be found in [15–20]. Comprehensive reviews on this theory can be found in Refs.[21, 22]. In similar fashion further extensions to the EH action was brought about by replacing the arbitrary function $f(R)$ by $f(R, L_m)$ [23–26]. In ref.[27] the authors modified the EH action by including an arbitrary function $f(R, T)$, where $T$ is the trace of the energy-momentum tensor (EMT). Here a coupling between the matter sector and the gravity sector was considered via the function $f(R, T)$. It was seen that the covariant divergence of the EMT is non-zero for this theory, which means non-conservation of EMT leading to non-geodesic motion of the massive test particles. The reason for this being the coupling effects between matter and geometry which induces extra acceleration on the particles. Further developments in $f(R, T)$ gravity can be found in [28–34]. Katirci and Kavuk in [35] proposed $f(R, T^2)$ theory, where $T^2 = T_{\mu\nu}T^{\mu\nu}$ and $T_{\mu\nu}$ is the EMT. Roshan and Shojai in [36] further developed the theory by investigating the properties of a specific form $R + \eta T^2$ which was termed as energy-momentum squared gravity (EMSG) in the literature. From the field equations of EMSG it was seen that the equations have a flavour of the quantum geometry effects of loop quantum gravity [37], and allowed a maximum energy density $\rho_{\text{max}}$ and a minimum length $\rho_{\text{min}}$ in the early universe. As a result, in a homogeneous and isotropic spacetime this theory admits a cosmological bounce and avoids the existence of the early time singularity. Cosmological models in EMSG were studied by the authors in [38, 39]. An extensive dynamical system analysis in EMSG was presented by the authors of [40]. Observational constraints on EMSG using cosmic chronometers and supernova observations can be found in [41]. Thermodynamics of the apparent horizon in the generalized energy-momentum squared cosmology was studied in [42]. Other substantial developments in EMSG can be found in [43–48].

All the above extensions of GR have one feature common between them, which is the underlying Riemannian geometry [49] (formulated in the Riemann metrical space) lying at the heart of all these classical theories including GR. With the incompatibility of these theories at some scales, naturally a thought arises that if the underlying geometry can be replaced by a far more general geometric structure, then we may be able to remove some of the inconsistencies that has plagued these classical theories over the years. Such a novel attempt was undertaken by Weyl [50], where the main objective was the geometrical unification of gravity and electromagnetism. We know that in Riemannian geometry, the Levi-Civita connection is compatible with the metric and is the basic tool for length comparison between vectors. In Weyl’s theory there is a complete different mechanism where two connections are used, one bearing the information of the length of a vector and the other responsible for the direction of a vector during parallel transport. Physically the length connection is identified with the electromagnetic potential. The most striking feature of the theory being the non-zero covariant divergence of the metric tensor and this property induces a new geometrical quantity known as the non-metricity. Weyl’s geometry is a mathematical masterpiece with corresponding rich physical structure.

Since gravity is identified as the manifestation of the geometrical properties of spacetime, search for other developments in geometry continued. Based on the works [51–54] Cartan proposed an extension of GR known as the Einstein-Cartan theory. A review on these theories may be found in [55–58].
The striking feature of Cartan’s geometry was the introduction of the torsion field, which is interpreted as the spin density from the physical point of view [55]. This torsion field may be introduced in Weyl’s geometric structure to give a natural extension of Weyl-Cartan geometry, which can be an interesting mathematical structure with associated physical implications [56–59]. An extensive review on Riemann-Cartan and Weyl-Cartan geometries can be found in [60]. Another elegant geometrical formalism was given by Weitzenbock known as the Weitzenbock spaces [61], where the manifold is equipped with the properties $R^\alpha_{\beta\gamma\delta} = 0$, $T^\alpha_{\beta\gamma} \neq 0$, $\nabla_{\mu} g_{\alpha\beta} = 0$, where $R^\alpha_{\beta\gamma\delta}$ is the curvature tensor, $T^\alpha_{\beta\gamma}$ is the torsion tensor and $g_{\alpha\beta}$ is the metric tensor of the associated manifold. If the torsion tensor vanishes the Weitzenbock manifold reduces to the Euclidean manifold. Since the Weitzenbock space is curvature-less, the geometry derived from such spaces possesses an important property of absolute parallelism or teleparallelism. This property of Weitzenbock spaces attracted Einstein, who applied this theory in physics by proposing a unified teleparallel theory of gravity and electromagnetism [62]. In teleparallel gravity the basic idea is to replace the metric $g_{\alpha\beta}$ by the tetrad vectors $e^\mu_i$, which generates the torsion that is responsible for the gravitational effects. Since in this theory the concept of torsion exactly replaces the concept of curvature, it is termed as the teleparallel equivalent of general relativity (TEGR) [63–65]. One important thing to note in TEGR is that the spacetime in these theories are flat, due to the absence of curvature. Just like GR was extended to $f(R)$ theories, similarly TEGR was extended to $f(\tau)$ theories [66, 67], where $\tau$ is the scalar torsion. One significant advantage of $f(\tau)$ theories over $f(R)$ theories lies in the fact that the field equations for the former are of the second order in contrast to the fourth order field equations of the latter. More importantly the $f(\tau)$ theories have been successful in explaining various astrophysical processes and specifically the late cosmic acceleration without depending on any forms of exotic matter or dark energy [68–76]. A recent review on Teleparallel gravity can be found in [77]. Further extensions of the teleparallel framework was done via the Weyl-Cartan-Weitzenbock (WCW) theory [78, 79], where the extra condition of the exact cancellation of curvature by the torsion is introduced in the background of the Weyl-Cartan spacetime.

From the above discussions we have seen that there are two alternative formulations of GR: the first one with $R \neq 0$, $\tau = 0$ (curvature formulation), and the second one with $R = 0$, $\tau \neq 0$ (teleparallel formulation). However in both these formulations the non-metricity $Q$ vanishes. Geometrically $Q$ represents the variation of length of a vector in parallel transport. Now in a third equivalent formalism of GR, a non-vanishing non-metricity $Q$ was considered as the basic geometrical variable responsible for all forms of gravitational interactions. This theory was termed the symmetric teleparallel gravity (STG) [80]. Here the Einstein pseudotensor plays the role of the energy-momentum density and in the geometric representation it finally becomes a true tensor. Further research saw the STG being extended to $f(Q)$ gravity [81], which is also known as the coincident general relativity and non-metric gravity. Cosmology of $f(Q)$ gravity and its observational constraints was studied in [82, 83]. Over the past decades there have been various developments in the framework of STG [84–88]. In ref.[88] the authors have proposed an extension of the $f(Q)$ gravity based on the non-minimal coupling between the non-metricity $Q$ and the matter Lagrangian $\mathcal{L}_m$. Quite expectedly the non-minimal coupling between geometry and matter sectors results in the non-conservation of the energy-momentum tensor and appearance of an extra force in the geodesic equation of motion. Xu et. al in [89] developed another extension of the theory namely the $f(Q, T)$ gravity where the gravity Lagrangian is basically an arbitrary function of $Q$ and the trace of the energy-momentum tensor $T$. The field equations were derived and the cosmological evolution of the model was studied. It was seen that in all the considered cases the theory supported an accelerated expansion of the universe ending with a de-Sitter type evolution.

In this work we are motivated to explore another extension of the symmetric teleparallel theory. Our basic aim is to extend $f(Q, T)$ gravity to $f(Q, T^2)$ gravity, where $T^2 = T_{\mu\nu}T^{\mu\nu}$, following the extension of $f(R, T)$ gravity to $f(R, T^2)$ gravity. The novel features of $f(R, T^2)$ gravity, would be
a direct motivation to perform this analogical study. A second motivation would be to develop the $f(R, L_m)$ theory where matter is non-minimally coupled to geometry. Now the matter Lagrangian can be generated via various physical invariants. The most common is the trace of the energy momentum tensor $T = g^{\mu\nu}T_{\mu\nu}$ which led to $f(R, T)$ theory. The next obvious development would be to incorporate some extensions of $T$ and check its evolution in cosmology. This is exactly what we have done by extending to $T$ to $T^2 = T_{\mu\nu}T^{\mu\nu}$, which is perhaps the simplest logical extension, whose evolution is worth studying due to its non linear nature in the matter sector. An important feature of the $f(R, L_m)$ theory and its derivatives is that they have connections with MOND (modified Newtonian dynamics) [90] which is an alternative to the dark matter hypothesis explaining why the dynamics of galaxies do not obey the currently understood laws of physics. This is a huge motivation to study these classes of models.

We will call $f(Q, T^2)$ gravity as the Energy-momentum squared symmetric teleparallel gravity (EMSSTG). The gravitational action will be constituted by an arbitrary function $f(Q, T^2)$ of $Q$ and $T^2$. Then varying the action with respect to the metric we can frame the field equations in a metric-affine formalism. Using those equations we can study the cosmological evolution of the theory in detail. Some specific toy models may help in properly understanding the dynamics of the theory. We also intend to study the energy conditions in the background of the theory to put constraints on the model. The paper is organized as follows: The basic field equations of $f(Q, T_{\mu\nu}T^{\mu\nu})$ gravity is derived in section 2. Cosmological evolution of EMSSTG is explored in section 3. Section 4 is dedicated to studying the dynamics of some specific toy models. In section 5 we investigate the energy conditions in the background of EMSSTG thus constraining the theory. Finally the paper ends with some concluding remarks in section 6.

2 Field equations of $f(Q, T_{\mu\nu}T^{\mu\nu})$ Gravity

In this section we will discuss the various geometrical preliminaries required to frame a relativistic theory of gravitation and go on to derive the field equations of $f(Q, T_{\mu\nu}T^{\mu\nu})$ gravity. We start by considering the action for $f(Q, T_{\mu\nu}T^{\mu\nu})$ gravity as,

$$S = \int \left[ \frac{1}{16\pi} f(Q, T^2) + L_m \right] \sqrt{-g} d^4x \tag{2.1}$$

Here $g \equiv \det(g_{\mu\nu})$ and $T^2 = T_{\mu\nu}T^{\mu\nu}$, where $T_{\mu\nu}$ is the matter energy-momentum tensor. Moreover $f(Q, T^2)$ is a function of the non-metricity $Q$ and $T^2$ and $L_m$ is the matter Lagrangian. The non-metricity tensor $Q_{\lambda\mu\nu}$ is defined as the covariant derivative of the metric tensor with respect to the Weyl-Cartan connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ [55],

$$- \nabla_\lambda g_{\mu\nu} = Q_{\lambda\mu\nu} = - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + g_{\nu\sigma} \tilde{\Gamma}^{\sigma}_{\mu\lambda} + g_{\mu\sigma} \tilde{\Gamma}^{\sigma}_{\nu\lambda} \tag{2.2}$$

The Weyl-Cartan connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ can be written as a combination of three entities: the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$, the contortion tensor $C^{\lambda}_{\mu\nu}$ and the disformation tensor $L^{\lambda}_{\mu\nu}$, in the following way,

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + C^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu} \tag{2.3}$$

In the above expression the first term, i.e. the Christoffel symbol is basically the Levi-Civita connection of the metric $g_{\mu\nu}$ given by,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left( \frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right) \tag{2.4}$$
The contortion tensor $C^\lambda_{\mu\nu}$ is obtained from the torsion tensor $\tilde{\Gamma}^\lambda_{[\mu\nu]}$, which in turn is defined as,

$$\tilde{\Gamma}^\lambda_{[\mu\nu]} = \frac{1}{2} \left( \tilde{\Gamma}^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\nu\mu} \right)$$

(2.5)

Using the above definition the contortion tensor is defined as,

$$C^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{[\mu\nu]} + g^{\lambda\sigma} g_{\mu\kappa} \tilde{\Gamma}^\kappa_{[\nu\sigma]} + g^{\lambda\sigma} g_{\nu\kappa} \tilde{\Gamma}^\kappa_{[\mu\sigma]}$$

(2.6)

The disformation tensor is given by,

$$L^\alpha_{\beta\gamma} = -\frac{1}{2} g^{\alpha\lambda} \left( \nabla_\gamma g_{\beta\lambda} + \nabla_\beta g_{\lambda\gamma} - \nabla_\lambda g_{\beta\gamma} \right) = -\frac{1}{2} g^{\alpha\lambda} \left( Q_{\gamma\beta\lambda} + Q_{\beta\lambda\gamma} - Q_{\lambda\beta\gamma} \right)$$

(2.7)

Using this the non-metricity is given as,

$$Q = -g^{\alpha\mu} \left( L^\alpha_{\beta\mu} L^\beta_{\mu\alpha} - L^\alpha_{\beta\alpha} L^\beta_{\mu\nu} \right)$$

(2.8)

This nonmetricity invariant is by construction equivalent to negative of the Einstein Lagrangian, when the covariant derivative reduces to the partial derivative, i.e., $\nabla_a = \partial_a$. This gauge choice is called the coincident gauge and is consistent with symmetric teleparallel gravity [88]. In the connection given in eqn. (2.3), the contorsion depends on the torsion tensor, the disformation depends on the non metricity tensor and the Levi-Civita connection characterizes the curvature. For nonmetric gravity we have vanishing torsion and curvature, due to which the connection becomes equal to just the disformation in the coincident gauge. Though the Levi-Civita connection is eliminated, curvature continues to play its physical role in the set-up. Now since we will take a trivially connected geometry we should have $\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + L^\lambda_{\mu\nu} = 0$. Furthermore the gravity is realized as a gauge theory of the group of translations. Now since this is a class of generalizations in the trivially connected geometry (vanishing connection) it is actually teleparallelized in the metric affine gauge theory.

Moreover the Weyl-Cartan torsion tensor $\tau^\lambda_{\mu\nu}$ is defined as,

$$\tau^\lambda_{\mu\nu} = \frac{1}{2} \left( \tilde{\Gamma}^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\nu\mu} \right)$$

(2.9)

and the Weyl-Cartan curvature tensor may be defined by,

$$\tilde{R}^\lambda_{\mu\nu\sigma} = \tilde{\Gamma}^\lambda_{\mu\sigma,\nu} - \tilde{\Gamma}^\lambda_{\mu\nu,\sigma} + \tilde{\Gamma}^\alpha_{\mu\sigma} \tilde{\Gamma}^\lambda_{\alpha\nu} - \tilde{\Gamma}^\alpha_{\mu\nu} \tilde{\Gamma}^\lambda_{\alpha\sigma}$$

(2.10)

We know that the trace of the energy-momentum tensor $T_{\mu\nu}$ is given as,

$$T = g^{\mu\nu} T_{\mu\nu} = g_{\mu\nu} T^{\mu\nu}$$

(2.11)

The trace of the non-metricity tensor is given by,

$$Q_\beta = Q^\lambda_{\beta\lambda}, \quad \tilde{Q}_\beta = Q^\lambda_{\beta\lambda}$$

(2.12)

We will bring in the superpotential of the model defined as,

$$P^\alpha_{\mu\nu} = \frac{1}{4} \left[ -Q^\alpha_{\mu\nu} + 2 Q^\alpha_{(\mu\nu)} + Q^\alpha g_{\mu\nu} - \tilde{Q}^\alpha g_{\mu\nu} - \delta^\alpha_{(\mu} Q_{\nu)} \right]$$

(2.13)

Now we can use the non-metricity tensor with the superpotential to obtain the following relation for the non-metricity

$$Q = -Q_{\lambda\alpha\beta} P^{\lambda\alpha\beta} = \frac{1}{4} \left[ -Q^{\lambda\beta\rho} Q_{\lambda\beta\rho} + 2 Q^{\lambda\beta\rho} Q_{\rho\lambda\beta} - 2 Q^{\alpha} \tilde{Q}_{\alpha} + Q^\alpha Q_{\alpha} \right]$$

(2.14)
Now we proceed to derive the field equations. For this, we take the variation of the action $S$ in eqn.\((2.1)\) with respect to the metric tensor and obtain,

\[
\delta S = \int \frac{1}{16\pi} \delta \left[ f(Q, T^2) \sqrt{-g} \right] d^4x + \int \delta \left[ L_m \sqrt{-g} \right] d^4x
\]  

(2.15)

Simplifying the above relation we get,

\[
\delta S = \int \frac{1}{16\pi} \left[ -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} f(Q, T^2) + f_Q \delta Q + f_T \delta T^2 - 8\pi T_{\mu\nu} \delta g^{\mu\nu} \right] \sqrt{-g} d^4x
\]  

(2.16)

where $f_Q = \frac{\partial f}{\partial Q}$ and $f_T = \frac{\partial f}{\partial T}$. Now in the above expression we have made use of the relations,

\[
\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}
\]  

(2.17)

and

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}
\]  

(2.18)

Considering that $L_m$ depends only on the metric tensor and not on its derivatives, we get the relation,

\[
T_{\mu\nu} = g_{\mu\nu} L_m - 2 \frac{\partial L_m}{\partial g^{\mu\nu}}
\]  

(2.19)

Now we see from eqn.\((2.16)\) that we have to calculate the quantities $\delta Q$ and $\delta T^2$. After some rigorous calculations $\delta Q$ is obtained as,

\[
\delta Q = 2 P_{\alpha\beta} \nabla^\alpha \delta g^{\alpha\beta} - (P_{\alpha\beta} Q_{\mu} \alpha\beta - 2Q_{\alpha\beta}^\mu P_{\alpha\beta\nu}) \delta g^{\mu\nu}
\]  

(2.20)

For an explicit calculation of $\delta Q$ see ref.\[89\]. Now let us define a quantity $\theta_{\mu\nu}$ as,

\[
\theta_{\mu\nu} = \frac{\delta (T_{\alpha\beta} T^{\alpha\beta})}{\delta g^{\mu\nu}}
\]  

(2.21)

From the above relation we can write $\delta T^2 = \delta (T_{\alpha\beta} T^{\alpha\beta}) = \theta_{\mu\nu} \delta g^{\mu\nu}$, where an explicit relation for $\theta_{\mu\nu}$ is calculated as,

\[
\theta_{\mu\nu} = 2 T^\alpha_{\mu} T_{\nu\alpha} - 2 L_m \left( T_{\mu\nu} - 2 \frac{1}{2} g_{\mu\nu} T \right) - T^\alpha_{\mu\nu} - 4 T^{\alpha\beta} \frac{\delta^2 L_m}{\delta g^{\mu\nu}} \delta g^{\alpha\beta}
\]  

(2.22)

Using all these in eqn.\((2.16)\) we get the variation of the action as,

\[
\delta S = \int \frac{1}{16\pi} \left[ -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} f(Q, T^2) + f_Q \{2 P_{\alpha\beta} \nabla^\alpha \delta g^{\alpha\beta} - (P_{\alpha\beta} Q_{\mu} \alpha\beta - 2Q_{\alpha\beta}^\mu P_{\alpha\beta\nu}) \delta g^{\mu\nu} \} \right.
\]

\[
+ f_T \theta_{\mu\nu} \delta g^{\mu\nu} - 8\pi T_{\mu\nu} \delta g^{\mu\nu} \right] \sqrt{-g} d^4x
\]  

(2.23)

Now the term $2 f_Q \sqrt{-g} P_{\alpha\mu} \nabla^\alpha \delta g^{\mu\nu}$ on integration and with the use of boundary condition gives $-2 \nabla^\alpha (f_Q \sqrt{-g} P_{\alpha\mu}) \delta g^{\mu\nu}$. Using this and equating the variation of the action to zero we finally obtain the field equations of $f(Q, T^2)$ gravity as,

\[
-\frac{2}{\sqrt{-g}} \nabla^\alpha \left( f_Q \sqrt{-g} P_{\alpha\mu}^\nu \right) - \frac{1}{2} f(Q, T^2) g_{\mu\nu} + f_T \theta_{\mu\nu} - f_Q (P_{\alpha\beta} Q_{\mu} \alpha\beta - 2Q_{\alpha\beta}^\mu P_{\alpha\beta\nu}) = 8\pi T_{\mu\nu}
\]  

(2.24)

For the special case $f(Q, T^2) = f(Q)$, these field equations reduce to those of $f(Q)$ gravity. Moreover for the choice $f(Q, T^2) = Q$ the above field equations reduce to the field equations of the symmetric
teleparallel equivalent of general relativity [84–87, 91]. The (1, 1)- form of the field equations are given by,

\[ f T^\mu_\nu - 8\pi T^\mu_\nu = \frac{2}{\sqrt{-g}} \nabla_\alpha \left( f Q \sqrt{-g} P^\alpha_\nu \right) + \frac{1}{2} f(Q, T^2) \delta^\mu_\nu + f(Q) P^\mu_\alpha Q^\alpha_\nu \delta^\alpha_\nu \]  

(2.25)

Now we consider two constraints, the torsion tensor \( \tau^\alpha_\beta_\gamma = 0 \) and the curvature tensor \( R^\alpha_\beta_\mu_\nu = 0 \), and using the Lagrange multiplier method, we find the variation with respect to the connection. For that we first define the hyper momentum tensor density as [89],

\[ H^\alpha_\beta_\rho \equiv \frac{\sqrt{-g}}{16\pi} f Q T^\alpha_\beta P^\mu_\nu + 4\pi H^\mu_\alpha P^\mu_\nu \]  

(2.26)

where \( \hat{\Gamma}^\rho_\alpha_\beta \) is the Weyl-Cartan connection defined in eq. (2.3).

Now using the above expression and taking the variation of the gravitational action with respect to the connection we get [89],

\[ \nabla_\mu \nabla_\nu \left( \sqrt{-g} f Q P^\mu_\alpha + 4\pi H^\mu_\alpha \right) = 0 \]  

(2.27)

Here we could have used the inertial variation by setting the connection in its pure gauge form in the action. We could have also used a general connection in the action and supplement it with Lagrange multipliers to eliminate the curvature and torsion. We will be using the above quantities in the next section where we investigate the conservation of the matter energy-momentum tensor. In case of minimal coupling of matter there is obviously no violation of the equivalence principle. In that case for the gauge interpretation of the theory we should have \( \partial_\mu \rightarrow \nabla_\mu \), whereas the Weitzenbock teleparallelism will require \( \partial_\mu \rightarrow D_\mu \) [92]. The gauge choices involved in our theory are quite compatible with the cosmological evolution. The trivially connected geometry considered in our theory would mean a universe devoid of torsion and curvature and the gravitational interaction will be fully realized from nonmetricity, at least mathematically. Physically curvature will still have its effects, but due to the vanishing of the connection, all the inertial effects connected with the geometry will be absent. The gravitational effects arising from such a set-up will be interesting. Cosmological compatibility will not be an issue with such a set-up because curvature still plays its role physically as discussed earlier.

Before we go on to study other important features, it will be interesting to discuss the propagating degrees of freedom of the theory. From a thorough Hamiltonian analysis, we come to know that Einstein’s general relativity has two propagating degrees of freedom (DOF) in four dimensions. The metric perturbations can be split into three parts: scalar, vector and tensor components. It is seen that the scalar and the vector components are not propagating degrees of freedom but are mere constraints. Alternatively, we see that in GR we have ten partial differential equations. Four constraints can be obtained from the Bianchi identities and we also have four gauge transformations, which are worth degrees of freedom and are arbitrary. So DOF of GR = 10 - 4 - 4 = 2. The metric \( f(R) \) gravity which is considered as the representative of the modified gravity theory has three DOF [93]. For \( f(\tau) \) gravity (\( \tau \) is torsion scalar) it is found that due to the violation of local Lorentz invariance there are five DOF [94]. It is also shown by the authors in [94] that in general for \( f(\tau) \) gravity there are \((D - 1)\) extra degrees of freedom in \( D \) dimensions when compared to Einstein gravity. For \( f(Q) \) gravity it was shown in [95] that it respects local Lorentz symmetry and harbours no extra degrees of freedom compared to GR. So due to the absence of any extra polarization modes \( f(Q) \) gravity also has two degrees of freedom. Probably this is due to the fact that STG and \( f(Q) \) gravity does not rely on curvature or torsion for its formulation. Instead, the whole theory is developed using non-metricity, which does not produce extra degrees of freedom. Now any modification in the source term (matter sector \( T^\mu_\nu \)) in principle is not expected to bring
about any change in the propagating degrees of freedom even if it is included in the gravitational Lagrangian. This is because it will involve terms comprising of pressure \( p \), matter density \( \rho \) and its derivatives, that do not participate in determining the propagating degrees of freedom. Hence theories like \( f(Q, T) \) should possess similar degrees of freedom as \( f(Q) \) theory. Using similar arguments we can say that \( f(Q, T^2) \) gravity theory will also have two propagating degrees of freedom in four dimensions. In fact we need to perform a detailed metric perturbation analysis to check that unlike GR if any vector perturbation modes become dynamical degrees of freedom for \( f(Q, T^2) \) gravity theory to be sure about our arguments. This can be attempted in a future project.

2.1 Conservation of matter energy-momentum tensor & the momentum conservation equation

The covariant derivative of a (1,1)-form tensor \( w^\mu_{\nu} \) may be expressed as,

\[
\nabla_\mu \! w^\mu_{\nu} = D_\mu w^\mu_{\nu} - \frac{1}{2} Q_\lambda w^\lambda_{\nu} - L^\lambda_{\mu\nu} w^\mu_{\lambda}
\]

(2.28)

where \( D_\mu \) represents the covariant derivative with respect to the Levi-Civita connection \( (\Gamma^\alpha_{\mu\nu}) \) defined as \( \Gamma^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} - L^\alpha_{\mu\nu} \).

Using the above relation we take the covariant derivative of the field equations (2.25) in (1,1)-form and get,

\[
D_\mu [f T^2 \theta^\mu_{\nu} - 8 \pi T^\mu_{\nu}] + \frac{8 \pi}{\sqrt{-g}} \nabla_\alpha \nabla_\mu H^\alpha_{\nu} = \frac{1}{2} f T^2 \partial_\nu T^2 + \frac{1}{\sqrt{-g}} Q_\mu \nabla_\alpha \left( f Q \sqrt{-g} P^{\alpha \nu} \right)
\]

(2.29)

where \( \partial_\nu \) represents partial derivative with respect to the coordinate \( x^\nu \). From the above relation we see that,

\[
D_\mu T^\mu_{\nu} = \frac{1}{8 \pi} \left[ D_\mu (f T^2 \theta^\mu_{\nu}) + \frac{8 \pi}{\sqrt{-g}} \nabla_\alpha \nabla_\mu H^\alpha_{\nu} - \frac{1}{\sqrt{-g}} Q_\mu \nabla_\alpha \left( f Q \sqrt{-g} P^{\alpha \nu} \right) - \frac{1}{2} f T^2 \partial_\nu T^2 \right] \neq 0
\]

(2.30)

So we can see that the matter energy-momentum tensor is not conserved in this case. Now we may introduce a tensor \( A^\nu_{\alpha} \) to solve eqn.(2.27) such that \([89], \)

\[
\nabla_\mu \left( \sqrt{-g} f Q P^{\mu\nu}_{\alpha} + 4 \pi H^\mu_{\alpha \nu} \right) = \sqrt{-g} A^\nu_{\alpha}
\]

(2.31)

After some rigorous calculations we reach the following expression for the covariant derivative of the energy-momentum tensor,

\[
D_\mu T^\mu_{\nu} = \frac{1}{8 \pi} \left[ D_\mu (f T^2 \theta^\mu_{\nu}) + \frac{16 \pi}{\sqrt{-g}} \nabla_\alpha \nabla_\mu H^\alpha_{\nu} - 8 \pi \nabla_\mu \left( \frac{1}{\sqrt{-g}} \nabla_\alpha H^\alpha_{\nu} \right) + 2 \nabla_\mu A^\mu_{\nu} - \frac{1}{2} f T^2 \partial_\nu T^2 \right] = B_\nu \neq 0
\]

(2.32)

As mentioned above the energy-momentum tensor is not conserved for this theory and we get \( D_\mu T^\mu_{\nu} = B_\nu \neq 0 \). The non-conservation vector \( B_\nu \) is a function of the dynamical variables \( Q, T^2 \) and other thermodynamic parameters like energy density and pressure. From eqn.(2.28), it can be seen that the covariant derivative \( \nabla_\mu \) is a function of the covariant derivative on Levi-Civita connection \( D_\mu \), trace of the non-metricity tensor \( Q_\lambda \) and the disformation tensor \( L^\lambda_{\mu\nu} \). So straightforward algebra will tell us that the \( D_\mu \) can be expressed as a function or combination of \( \nabla_\mu, Q_\lambda \) and \( L^\lambda_{\mu\nu} \).
dependence given in eqn. (2.28). Unless there is loss of terms due to the effect of the other two factors in the equation, namely \( Q_\lambda \) and \( L^\lambda_{\mu\nu} \) (which seems unlikely) the quantity \( \nabla_\mu T^\mu_\nu \) will not be conserved because the quantity \( D_\mu T^\mu_\nu \) is not conserved.

On a general note dissipative processes are not compatible with Cosmic microwave background radiation (CMBR) or Large scale structure (LSS). The authors in [96] studied the cosmological and solar system consequences of a class of matter coupling models with geometry. They found that the models considered generally have some inconsistent behaviour when compared with the observational data. There is a fair possibility of this behaviour being transmitted and magnified when we consider cosmology at the galactic and extra-galactic level. It is expected that there will be a fair amount of incompatibility with CMBR or LSS. But it is seen that, this is a purely model dependent phenomenon. By fine tuning the model parameters it is possible to get rid of some or all the inconsistencies as clearly worked out in [96]. At the large scales (galactic and extra galactic levels) there are some implications of the non-minimal matter coupling with geometry. The flattening of the galaxy rotation curves as a dynamically generated effect can be attributed to this non-minimal coupling between matter and geometry [97]. Due to the non-conservation of the energy-momentum tensor, a deviation from the geodesic motion sets in. This accounts for the observed discrepancy between the measured rotation velocity and the classical prediction. It can also be shown that a special type of non-minimal matter coupling with geometry can mimic the dark matter component of the galaxy clusters. For this purpose the authors of [98] investigated the Abell cluster A586, which is a massive nearby relaxed cluster of galaxies in virial equilibrium. Then one can easily extend the dark matter mimicking phenomenon to a large sample of galaxy clusters. Dissipative processes also have their special role in the evolution of radio galaxies as discussed in [99].

If the matter content is described by a perfect fluid with density \( \rho \) and pressure \( p \) and the energy-momentum tensor given by \( T^\mu_\nu = (\rho + p) u_\nu u^\mu + p g^\mu_\nu \), then following [88] we obtain the energy balance equation as,

\[
\dot{\rho} + 3H (\rho + p) = B_\alpha u^\alpha \tag{2.33}
\]

where dot (\( . \)) denotes derivative with respect to time. It can be clearly seen that this is not the standard continuity equation, but contains additional deformation terms on the RHS. Here \( B_\alpha u^\alpha \) is the source term that correlates with the creation or annihilation of energy. If \( B_\alpha u^\alpha = 0 \), then we have an energy conserved gravitational system, otherwise transfer of energy prevails and particle production takes place in the system. The momentum conservation equation representing the motion of massive particles is given by [88, 89],

\[
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_\lambda_\sigma u^\lambda u^\sigma = \frac{\Upsilon^\mu_\nu}{\rho + p} (B_\nu - D_\nu p) = F^\mu \tag{2.34}
\]

where \( \Upsilon^\mu_\nu \) is the projection operator given by \( \Upsilon^\mu_\nu = g^{\mu\nu} + u^\mu u^\nu \). The term \( F^\mu = \frac{\Upsilon^\mu_\nu}{\rho + p} (B_\nu - D_\nu p) \) on the RHS is the deformation term representing additional force coming from the coupling effects of \( Q \) and \( T^2 \). As a result the dynamical evolution of the massive particles is non-geodesic in nature. Due to the presence of the projection operator, this extra force acts orthogonal to the 4-velocity vector such that \( F^\mu u_\mu = 0 \). It is known that the components of the 4-force which are orthogonal to the 4-velocity of a particle can only affect its motion and the path followed. In this sense the extra force \( F^\mu \) derived from the theory is perfectly physical. Now we write the extra force \( F^\mu \) as a combination of three terms as below,

\[
F^\mu = -\frac{\Upsilon^\mu_\nu D_\nu p}{\rho + p} + F^\mu_H + F^\mu_T \tag{2.35}
\]
where \( F^\mu_H \) and \( F^\mu_{T^2} \) are given as,

\[
F^\mu_H = \frac{\Upsilon^\mu_{\nu}}{\rho + p} \left[ 2 \sqrt{-g} \nabla_\alpha \nabla_\beta H^\alpha_\beta - \nabla_\beta \left( \frac{1}{\sqrt{-g}} \nabla_\alpha H^\alpha_\nu \right) + \frac{\nabla_\alpha A^\alpha_\nu}{4\pi} \right]
\]  

(2.36)

\[
F^\mu_{T^2} = \frac{\Upsilon^\mu_{\nu}}{8\pi (\rho + p)} \left[ D_\alpha \left( f_{T^2} \theta^\alpha_\nu \right) - \frac{1}{2} f_{T^2} \partial_\nu T^2 \right]
\]  

(2.37)

In eqn.(2.35) the first term on the right hand side arises from the typical general relativistic contribution of the pressure gradient. The second term on the right hand side \( F^\mu_H \) is the hyper-force whose value is given in eqn.(2.36). The third term \( F^\mu_{T^2} \) is the extra force mainly coming from matter, which is given in eqn.(2.37). If \( f_{T^2} = 0 \), it is evident that the extra force coming from the component \( F^\mu_{T^2} \) vanishes. We discuss the effect of considering a perfect fluid on this extra force in the next subsection.

### 2.2 Matter as perfect fluid

Now we need to consider a specific type of matter component to further simplify the field equations. We assume that the universe is filled with a perfect fluid described by the energy-momentum tensor,

\[
T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}
\]  

(2.38)

where \( \rho \) is the energy density and \( p \) is the pressure. Moreover \( u_\nu \) is the four-velocity of the fluid, which is normalized as \( u_\mu u^\mu = -1 \). Using the above expression we get,

\[
T_{\mu\nu} T^{\mu\nu} = T^2 = \rho^2 + 3p^2
\]  

(2.39)

We consider the matter Lagrangian \( \mathcal{L}_m = p \). From this assumption we see that the final term of \( \theta_{\mu\nu} \) in eqn.(2.22) vanishes because pressure \( p \) does not depend on the metric tensor. Now using eqn.(2.38) in eqn.(2.22) we get,

\[
\theta_{\mu\nu} = - (\rho^2 + 4\rho p + 3p^2) u_\mu u_\nu
\]  

(2.40)

Using the above expressions the field equations take the following form,

\[
- \frac{2}{\sqrt{-g}} \nabla_\alpha \left( f_Q \sqrt{-g} F^\alpha_{\mu\nu} \right) - \frac{1}{2} f(Q, T^2) g_{\mu\nu} - f_Q \left( P_{\alpha\beta} Q^\alpha_\nu - 2Q^{\alpha\beta} \right) P_{\alpha\beta} \nu
\]  

= \left[ 8\pi (\rho + p) + f_{T^2} (\rho^2 + 4\rho p + 3p^2) \right] u_\mu u_\nu + 8\pi pg_{\mu\nu}
\]  

(2.41)

Using eqns.(2.39) and (2.40) in eqn.(2.37) we get the component of extra force for perfect fluid as,

\[
F^\mu_{T^2} = \frac{\Upsilon^\mu_{\nu}}{8\pi (\rho + p)} \left[ D_\alpha \left( -f_{T^2} (\rho^2 + 4\rho p + 3p^2) u^\alpha u_\nu \right) - \frac{1}{2} f_{T^2} \partial_\nu (\rho^2 + 3p^2) \right]
\]  

(2.42)

If we consider the equation of state (EoS) of matter as \( p = k\rho \), where \( k \) is the EoS parameter, then for \( k = -1 \) (LCDM) and \( k = -1/3 \), the first term inside the bracket of the above equation vanishes. For low energy density regime \( (\rho^2 \to 0) \), both the terms inside the bracket vanish and hence the total force component \( F^\mu_{T^2} \) vanishes. This should correspond to very late universe which is highly accelerating in nature and dominated by dark components. But in the early universe \( (\rho^2 \to \infty) \), and this force component \( (F^\mu_{T^2}) \) dominates the first two components in eqn.(2.35). So this force component has its roots in the quantum fluctuations of the early universe. This also shows that the total additional force \( F^\mu \) has significant effect in the evolution of the early universe, but slowly fades away to give general relativistic effects in the late universe.
We know that modified gravity theories are considered equivalent to the exotic matter components or dark energy that is considered responsible for driving the late cosmic acceleration. In our model we are talking of a highly non-standard form of matter coupling with geometry in the action. Since the matter content creates the geometry of the spacetime, it is expected that the matter in such a theory will not be of the usual nature, but will have exotic properties as mentioned above. The major exotic property will be the negative pressure that is possessed by the dark energy models which enables them to drive the cosmic acceleration. It may have other exotic features, which are subject to testing, detection and further research. So here we are dealing with a cosmic fluid with mysterious properties of its own and probably comparison with usual matter will give highly contrasting results. We know that even though dark energy and dark matter have exotic properties, they do satisfy the standard continuity equation. But for our model we can see that due to the non-minimal matter coupling the standard continuity equation is not satisfied. This tells us that both the matter component and the force acting on it in the gravity field are highly non-standard. So it can be stated that, standard matter creates a standard spacetime geometry in GR, a non-standard matter in our theory creates a non-standard geometry of spacetime resulting in non-geodesic motion and extra force.

As we have already seen that in the nonmetric formulation the connection is totally trivialized ($\tilde{\Gamma}_{\alpha \mu \nu} = 0$) by means of a diffeomorphism and as a result the inertial connection vanishes from the nonmetricity sector. This formulation is thus a subtle improvement of GR, since the minimally coupled fermions are still connected metrically [100]. Moreover since the pure gravity sector is now trivially connected, effectively nothing changes but just the higher-derivative boundary term disappears from the action. Non-minimal matter coupling to curvature based theories have been widely studied in literature [101–103]. Energy momentum squared gravity (EMSG) is a class of theories where matter coupling is introduced in the form of $T^2$ with the scalar curvature $R$ [36, 37, 104]. Such theories have a lot of salient and interesting features. However matter coupling to curvature does have its own problems. Due to the higher derivative property of the curvature scalar $R$, these theories are best described as effective theories, which can give rise to problems at certain limits [88]. A good example of this will be, for the density of a canonical scalar field $\phi$ the non-minimal coupling of the form $f(R)L_\phi$ ($L_\phi$ is the Lagrangian corresponding to the scalar field) introduces a kinetic term which does not fit into the viable Horndeski class. However it should be mentioned that such problems are expected to disappear when the coupling is formulated in the metric-affine approach because the field equations remain in the second order [105]. Thus it is worthy to consider nonminimal matter couplings with non-metricity $Q$, because the scalar invariant involves no higher derivatives. So a coupling of the form $f(Q)T^2$ or $f(Q) + T^2$ or any other forms involving $Q$ and $T^2$ results in second order equations of motion. So the models that will be considered here are inspired by the well studied forms of curvature matter couplings in the literature, but here the curvature will be simply replaced by the non-metricity. The motivation is to see whether the subtle improvement of the geometrical formulation of $f(Q)$ gravity, when implemented in the matter sector, would allow more universally consistent and viable realizations of the nonminimal curvature-matter coupling theories [88]. This is the basic advantage of considering non-metricity in place of curvature scalar.

3 Cosmology of EMSSTG

In this section we intend to explore the cosmological applications of the above derived theory. We will assume that the universe is described by the homogeneous, isotropic and spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric given by,

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$  (3.1)
where $a(t)$ is the cosmological scale factor representing the expansion of the universe. The expansion rate may be given by the Hubble parameter $H$ defined as,

$$H \equiv \frac{\dot{a}}{a}$$  \hspace{1cm} (3.2)

where dot (.) represents derivative with respect to time. From some rigorous calculations we see that for the above FLRW metric the non-metricity $Q$ may be given in terms of Hubble parameter as [88, 89],

$$Q = 6H^2$$  \hspace{1cm} (3.3)

Using the metric (3.1) we get the modified Friedmann equations for the $f(Q, T^2)$ gravity as,

$$6f_Q H^2 - \frac{1}{2} \frac{df}{dQ} (Q, T^2) = 8\pi \rho + f_{T^2} (\rho^2 + 4\rho \rho + 3p^2)$$  \hspace{1cm} (3.4)

$$6f_Q H^2 - \frac{1}{2} \frac{df}{dQ} (Q, T^2) - 2 \left( f_Q H + f_Q H \right) = -8\pi p$$  \hspace{1cm} (3.5)

where as mentioned earlier dot (.) represents derivative with respect to time. It should be noted that we have derived the above FLRW equations for a universe filled with perfect fluid with the energy-momentum tensor given by eqn.(2.38). In the first FLRW equation we see that there are two correction terms $1/2 f$ and $f_{T^2} (\rho^2 + 4\rho \rho + 3p^2)$ when compared to standard equations of GR. The first term is present in the other extensions of symmetric teleparallel gravity like the $Q, T$ gravity. But the second term is the one to look out for, because this is the term that is characteristic of $f(Q, T^2)$ theory. Moreover we see that the equations follow perfect correspondence and reduce to those of STG, $f(Q)$ or $f(Q, T)$ under suitable limits. We can write the FLRW equations in the standard form for general relativity as below,

$$3H^2 = 8\pi \rho_{\text{eff}}$$  \hspace{1cm} (3.6)

$$2\dot{H} + 3H^2 = -8\pi p_{\text{eff}}$$  \hspace{1cm} (3.7)

where

$$\rho_{\text{eff}} = \frac{1}{16\pi f_Q} \left[ 8\pi \rho + \frac{1}{2} f(Q, T^2) + f_{T^2} (\rho^2 + 4\rho \rho + 3p^2) \right]$$  \hspace{1cm} (3.8)

and

$$p_{\text{eff}} = \frac{1}{8\pi f_Q} \left[ -8\pi p + \frac{1}{2} f(Q, T^2) - 9f_Q H^2 - 2f_Q H \right]$$  \hspace{1cm} (3.9)

Now the effective energy density and pressure will follow the standard continuity equation given as,

$$\dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0$$  \hspace{1cm} (3.10)

The effective equation of state $\omega_{\text{eff}}$ of the dark energy derived from the theory may be given by,

$$\omega_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{2 \left[ -8\pi p + \frac{1}{2} f(Q, T^2) - 9f_Q H^2 + 2f_Q H \right]}{8\pi \rho + \frac{1}{2} f(Q, T^2) + f_{T^2} (\rho^2 + 4\rho \rho + 3p^2)}$$  \hspace{1cm} (3.11)

To study the late cosmic acceleration of universe we have devised a parameter known as the deceleration parameter $q$, which is defined as,

$$q = -1 - \frac{\dot{H}}{H^2} = \frac{1}{2} \left( 1 + 3\omega_{\text{eff}} \right)$$  \hspace{1cm} (3.12)

Using eqn.(3.11) in eqn.(3.12) we can write the deceleration parameter for this theory as,

$$q = \frac{1}{2} \left[ 1 + \frac{6 \left[ -8\pi p + \frac{1}{2} f(Q, T^2) - 9f_Q H^2 + 2f_Q H \right]}{8\pi \rho + \frac{1}{2} f(Q, T^2) + f_{T^2} (\rho^2 + 4\rho \rho + 3p^2)} \right]$$  \hspace{1cm} (3.13)

A negative value of this parameter corresponds to the accelerated expansion of the universe. We will study this parameter in detail for the specific models.
3.1 The de-Sitter solution

Now let us explore the vacuum solutions, and correspondingly check that whether our theory admits any de-Sitter like solution or not. To perform that we need to constrain our field equations by considering $\rho = p = 0$ and $H = H_0$ (a constant). Using these in eqns.(3.4) and (3.5) we respectively obtain,

$$6f_QH_0^2 - \frac{1}{2}f = 0$$ (3.14)

and

$$6f_QH_0^2 - \frac{1}{2}f - 2H_0\dot{f}_Q = 0$$ (3.15)

Using the above two equations we get $f_Q = f_0$ (a constant), which on integration gives $f = f_0Q + \epsilon$ where $\epsilon$ is the constant of integration. Finally using this result for $f$ in the eqn.(3.14) we get,

$$H_0 = \sqrt{\frac{\epsilon}{6f_0}}$$ (3.16)

which is similar to the results obtained in [88, 89]. For $f_0 = 1$, we get $H_0 = \sqrt{\Lambda}$, which is equivalent to the result obtained for general relativity. We know that for GR, $H_0 \propto \sqrt{\Lambda}$, where $\Lambda$ is the cosmological constant. From this we can see that the constant of integration $\epsilon$ plays the role analogous to the cosmological constant. So it is clear that our EMSSTG theory admits de-Sitter like evolution of the universe when subjected to constraints corresponding to vacuum. Using the relation (3.12) it is quite straightforward to get ideas about the deceleration parameter $q$ and the effective EoS parameter $\omega_{\text{eff}}$ for the de-Sitter universe. Using the derived value of the Hubble parameter from (3.16) in eqn.(3.12), we get $q = -1$, which is the expected value for de-Sitter universe. Similarly the effective EoS parameter is obtained as $\omega_{\text{eff}} = -1$, which mimics the cosmological constant. So the universe in the absence of matter is dominated by vacuum energy that drives the accelerated expansion of the universe. This is in accordance with the known picture of the evolution of the universe and the corresponding literature.

4 Some specific Toy-models

We basically consider two specific toy-models in this section. The models are chosen depending on the nature of coupling between $Q$ and $T^\mu{}^\nu$. In the first models we will consider an additive form and in the second model we will consider a product form. Both the models should have well-defined STG limits, which is important for correspondence. We will derive the FLRW equations and explore the cosmological evolution of the models.

4.1 Model: 1

Now we need to specify a particular form of $f(Q,T^2)$ to further explore the field equations. We will assume the general form

$$f(Q,T^\mu{}^\nu,T_{\mu\nu}^2) = f(Q,T^2) = Q + \eta (T^\mu{}^\nu T_{\mu\nu})^n = Q + \eta (T^2)^n$$ (4.1)

For $\eta \to 0$ we recover the STG limit of the theory. Using eqns.(2.38), (2.39), (2.40) and (4.1) in eqn.(2.24) we get simplified field equations as,

$$\nabla_{\alpha} \left( \sqrt{-g} T^\alpha_{\mu\nu} \right) - \frac{1}{2} Qg_{\mu\nu} - \left( P_{\alpha\beta}Q_{\nu}^\alpha - 2Q^{\alpha\beta}_{\mu} P_{\alpha\beta}^\mu \right)$$

$$= \left[ 8\pi (\rho + p) + n\eta \left( \rho^2 + 3p^2 \right)^n + 1 \right] u_{\mu} u_{\nu} + \left[ 8\pi p + \frac{\eta}{2} \left( \rho^2 + 3p^2 \right)^n \right] g_{\mu\nu}$$ (4.2)
The corresponding Friedmann equations become,
\[ 3H^2 = 8\pi \rho + \eta (\rho^2 + 3p^2)^{n-1} \left( n + \frac{1}{2} \right) (\rho^2 + 3p^2) + 4n \rho p \]  
(4.3)
\[ 3H^2 - 2\dot{H} = \frac{n}{2} (\rho^2 + 3p^2)^n - 8\pi p \]  
(4.4)
From the above equations we see that the correction terms are higher order terms in the density \( \rho \) which should dominate in the high energy density regime as \( \rho \to \infty \), i.e. early universe. In the later universe these terms gradually fade away giving the effects of the standard FLRW model. Looking at the first Friedmann equation we see that the correction terms are somewhat similar in form to those arising from quantum gravity effects in Loop quantum gravity [106]. These equations are also comparable to those arising from the braneworld models [107]. So we are interested in searching for solutions of bouncing cosmology. We consider the following barotropic equation of state for the fluid
\[ p = k \rho \]  
(4.5)
where \( k \) is the barotropic parameter. Using this in eqn.(4.3) we see that the correction terms are quadratic in energy density \( \rho \). At small energy density regime \( \rho^2 \approx 0 \) and we recover the standard Friedmann equation for \( n \geq 1 \). At high energy densities (early universe) we see that for \( n = 1 \), we have two critical points i.e. \( H = 0 \) at,
\[ \rho_c_1 = 0 , \quad \rho_c_2 = - \frac{16\pi}{\eta (9k^2 + 8k + 3)} \]  
(4.6)
The first critical point is quite expected and corresponds to an empty universe, where the expansion is supposed to be driven by vacuum energy. The second critical density is more interesting and makes sense only if \( \eta (9k^2 + 8k + 3) < 0 \Rightarrow \eta < 0 \). So for negative values of \( \eta \) we have a cosmological bounce at a finite non-zero value of energy density. For an early radiation dominated universe i.e. for \( k = 1/3 \), we see that the bounce occurs at \( \rho_c = -\frac{12\pi}{5\eta} \). At this point one can easily check that \( \dot{H} = -\frac{176\pi^2}{25\eta} > 0 \) for \( \eta < 0 \). This indicates that the early universe does not start from an initial singularity but there is a possibility of a cut-off energy density (\( \rho_c \)) at which the universe undergoes bounce. So for this model the cosmological singularity problem can be solved by considering suitable initial conditions. We fortunately get an explicit solution of the eqns.(4.3) and (4.4) for \( n = 1 \) and \( k = -1 \) as below,
\[ a(t) = e^{Ct} , \quad \rho(t) = \frac{\sqrt{2(3C^2(\eta + 8\pi^2)) - 4\pi}}{2\eta} \]  
(4.7)
where \( C \) is the integration constant. This solution resembles with the solution of the de-Sitter scenario but in the presence of a non-vanishing matter energy-momentum tensor. Note that here we have made use of the initial condition \( a(0) = 1 \). Here the energy density obtained is constant which can mimic an empty universe in some suitable limits of the parameters (say \( \eta \to \infty \)). For such a scenario the deceleration parameter will have a constant negative value throughout the evolution of the universe. So this solution does not generate much cosmological interest.

Now from the eqn.(3.10) using eqns.(3.8), (3.9) and (4.1) we get the continuity equation for this model as,
\[ \dot{\rho} = \frac{3H}{2} - \frac{18H^2 + 4\dot{H} + 16 (1 - 2k) \pi \rho + \eta (1 + 3k^2)^{n-1} (3 + 2n + k (9k + 8n + 6kn)) \rho^{2n}}{8\pi + \eta (1 + 3k^2)^{n-1} n (1 + 2n + 8kn + k^2 (3 + 6n)) \rho^{2n-1}} \]  
(4.8)
Further, for \( n = 1 \), the above equation reduces to,
\[
\dot{\rho} + 3H (\rho + p) = \frac{3H \left[ 18H^2 - 4\dot{H} + \rho (48k\pi + \eta (1 + k (14 + k (19 + 18k))) \rho) \right]}{16\pi + 2\eta (3 + k (8 + 9k)) \rho}
\] (4.9)

The non-conservation term can clearly be seen on the RHS of the above equation. For matter energy-momentum tensor conservation we should have
\[
\rho [48k\pi + \eta (1 + k (14 + k (19 + 18k))) \rho] = 4\dot{H} - 18H^2
\] (4.10)

It is obvious looking at the above conservation equations (4.8) and (4.9), it is not straightforward to integrate them and get a general solution. But for \( k = -1 \) and \( n = 1 \) we can integrate the continuity equations and get expressions similar to eqn.(4.7). Using \( k = -1 \) and the energy density given in eqn.(4.7) in the above condition for energy momentum conservation (4.10) we get the following solution for the scale factor,
\[
a(t) = \frac{B}{\cosh \left[ C \left( \frac{9t}{2} - A \right) \right]^{2/3}}
\] (4.11)

where \( A \) and \( B \) are new integration constants and the above expression suggests that if the evolution of the universe follows the above scale factor then the matter energy-momentum tensor is conserved for the model. Obviously it should be kept in mind that this solution is valid only around the \( \Lambda \) CDM regime. Using the expression for scale factor in eqn.(4.11) we get the Hubble parameter as,
\[
H(t) = -C \tanh \left[ C \left( \frac{9t}{2} - A \right) \right]
\] (4.12)

Using the above expression for Hubble parameter in eqn.(3.12) we get the expression for the deceleration parameter as,
\[
q(t) = -1 + \frac{9}{2} \text{Cosech} \left[ C \left( \frac{9t}{2} - A \right) \right]^2
\] (4.13)

In order to obtain a dimensionless form of the evolution parameters we introduce the following transformations [108],
\[
H = H_0h, \quad t = \frac{\tau_p}{H_0}, \quad \rho = 3H_0^2r
\] (4.14)

where \((h, \tau_p, r)\) is a set of dimensionless variables, and \(H_0\) is a fixed value of the Hubble function, which is generally taken as its present value setting the scale of the universe at present time.

In order to get greater insight into the above obtained form of scale factor we generate a plot of \( a \) against the dimensionless time \( \tau_p \) in figure (1). From the figure we see that the scale factor grows with time, which complies with our observation of an expanding universe. So the conservation of matter energy momentum may be a possibility for this model subject to proper initial conditions. Moreover we see that in the late universe the dependency of the scale factor on the initial conditions become just a little more prominent compared to the early universe. In fig.(2) we have generated plots for the above Hubble function \( h \) for different initial conditions against dimensionless time \( \tau_p \) to gain greater insight into the model. From the plot we see that the Hubble function remains at the positive level, which is required for an expanding universe. Gradually it decays over time, especially in the late universe, which is expected. In figure (3) we have plotted the deceleration parameter \( q \) against dimensionless time \( \tau_p \) for different initial conditions. We see that there is a smooth transition from positive to negative values, which shows that lately the universe has entered into a phase of accelerated expansion. Finally all the trajectories asymptotically settle around \( q = -1 \), which is an expected result for non-phantom universes. So this model is very efficient in explaining the evolution of the universe and perfectly incorporates the observational aspects. We also see that
Fig. 1 shows the variation of the scale factor $a$ against the dimensionless time $\tau_p$ for different initial conditions for model-1. We have considered $a(0) \approx 1$.

Fig. 2 shows the variation of the Hubble function $h$ against the dimensionless time $\tau_p$ for different initial conditions for model-1.

Fig. 3 shows the variation of the deceleration parameter $q$ against the dimensionless time $\tau_p$ for different initial conditions for model-1.

the dependency of $q$ on the initial conditions becomes less pronounced in the late universe where all the trajectories nearly converge.

For this model the extra force discussed in eqn.(2.42) can be given as (for $n = 1$),

$$F^\mu_{T^2} = \frac{\eta T^{\mu\nu}}{8\pi (\rho + p)} \left[ D_\alpha \left( - (\rho^2 + 4p\rho + 3p^2) u^\alpha u_\nu \right) - \frac{1}{2} \partial_\nu \left( \rho^2 + 3p^2 \right) \right] \quad (4.15)$$

The qualitative features of the force component are similar to the ones discussed for eqn.(2.42). We see that the factor $\eta$ that appears with the force has a scaling effect on it. In the STG limit $\eta \to 0$, which implies $F^\mu_{T^2} \to 0$, which is in accordance with our expectations.
Looking at the model considered here, it is easy to understand that the model parameter $\eta$ is not dimensionless. Using the FLRW equations for this model we can perform a dimensional analysis to find the dimensions of $\eta$. In this work we have considered Newton’s gravitational constant $G$ and the velocity of light $c$ as unity as can be seen from eqns. (3.6) and (3.7). But for dimensional purposes we want to restore them in Einstein’s constant $\kappa$ to get the exact dimensional results. Generally the constant $\kappa$ in the Einstein’s equations ($G_{\mu\nu} = \kappa T_{\mu\nu}$) is taken as $8\pi G/c^4$, but this is not unique. Note that in the earlier computations for $G = c = 1$, $\kappa$ has been reduced to $8\pi$. Now coming to dimensions of the model parameters, we can make use of the dimensional homogeneity of any physical equation. This implies that in a meaningful equation describing physics, all the terms of an equation will have the same dimensions. For model-1, such equations can be the FLRW equations given by eqns. (4.3) and (4.4). In eqn. (4.4) we will consider dimensional homogeneity of all the four terms and try to determine the dimensions of the model parameter $\eta$ from there.

We take the fundamental dimensions as Mass, Length and Time represented by $[M]$, $[L]$ and $[T]$ respectively. The dimensions of the first term in the LHS is $[H^2] = [T^{-2}]$. This is because the dimensions of Hubble parameter H is $[T^{-1}]$ as can be calculated directly from the Hubble law. The dimensions of the second term in the LHS is $[\dot{H}] = [T^{-2}]$. Traditionally the operator $d/dt$ has the inverse dimensions of time, i.e. $[T^{-1}] [109]$. So $\dot{H} = dH/dt$ should have the dimensions $[T^{-1} \times T^{-1}] = [T^{-2}]$. So both terms in the LHS have the dimensions $[T^{-2}]$. Coming to the second term on the RHS we see that we can treat it in various ways depending on what dimensions we choose to consider for the Einstein’s constant $\kappa$ [109]. From ref. [109] we see that in literature there are various ways to write the Einstein’s constant $\kappa$, which are $\frac{8\pi G}{c^4}$, $\frac{8\pi G}{c^2}$ and $8\pi G$. Although the first one is the most widely used form, we have checked that in the present equations the second form is more suitable. The dimensions of the Newton’s gravitational constant is $[G] = [M^{-1}L^3T^{-2}]$ and that of the velocity of light is $[c] = [LT^{-1}]$. Now using these, we get the dimensions of $\kappa = 8\pi G/c^2$ as $[M^{-1}L]$. The dimensions of pressure is given by $[p] = [\text{Force}]/[\text{area}] = [ML^{-1}T^{-2}]$. Using these we get the dimensions of the second term on the RHS as $[M^{-1}L \times ML^{-1}T^{-2}] = [T^{-2}]$, which matches with the dimensions of the terms in the LHS. Finally we come to the first term in the RHS, which contains the model parameter $\eta$, whose dimensions we are seeking. We consider $n$ as a dimensionless constant since it appears in the exponent. Using eqn. (4.5) in (4.4) the first term in the RHS can be written as, $\frac{2}{3} p^2 n (3 + \frac{1}{M^2 T^2})$. Taking the dimensions of the portion in brackets as 1 or dimensionless, we are left with $\frac{2}{3} p^2 n$. Now the overall dimension of this term must be $[T^{-2}]$ to respect dimensional homogeneity. Therefore we have $[\eta] \times [p^{2n}] = [T^{-2}]$. From this we see that $[\eta] = \frac{[T^{-2}]}{[M^{-2}L^2T^2]}$. Depending on the choice of $n$ we can have different dimensional formulas for $\eta$. For example, if $n = 1$, we have $[\eta] = \frac{[T^{-2}]}{[M^{-2}L^2T^2]} = [M^{-2}L^2T^2]$.

4.2 Model: 2

Here we consider a second model in the product form between the two scalar invariants $Q$ and $T^2$ as below,

$$f(Q,T_{\mu\nu}T^{\mu\nu}) = f(Q,T^2) = f_{0}Q^{n}(T_{\mu\nu}T^{\mu\nu})^{n} = f_{0}Q^{n}(T^2)^n, \quad f_{0} \neq 0 \quad (4.16)$$

For $f_{0} \rightarrow 1$, $n \rightarrow 0$ and $m \rightarrow 1$ we recover the STG limit of the theory. The modified FLRW equations for this model are given by,

$$3H^2 = \left[ f_{0} \left( \frac{2^{3-n} \pi \rho (\rho^2 + 3p^2)^{1-n}}{(6m - 6n - 3)^{1-n} - \frac{1}{2n}} \right) \right]^{1/m} \quad (4.17)$$

$$\dot{H} = \frac{1}{2f_{0}m (6^n H^{2m} + 2^{n+1} \times 3^n H^{2n} (n-1))} \times \left[ H (\rho^2 + 3p^2)^{-n-1} (3H (\rho^2 + 3p^2) \right].$$
\[ \left( 16\pi p + 6^m f_0 H^{2m} (2m - 1) \left( \rho^2 + 3p^2 \right)^n \right) - 2^{n+2} \times 3^n f_0 H^{2n} mn (\rho^2 + 3p^2)^n (\rho \dot{\rho} + 3p\dot{p}) \] (4.18)

We further simplify these equations by considering \( n = m = 1 \) as given below,

\[ 3H^2 = -\frac{8\pi \rho}{f_0 (\rho^2 + 8pp + 3p^2)} \] (4.19)

\[ 2f_0 (\rho^2 + 3p^2) \dot{H} - 3f_0 (\rho^2 + 3p^2) H^2 + 4f_0 (\rho \dot{\rho} + 3p\dot{p}) H = 8\pi p \] (4.20)

From the first FLRW equation we directly see that:

\[ f_0 (\rho^2 + 8pp + 3p^2) < 0 \Rightarrow [f_0 < 0] \text{ OR } [\rho^2 + 8pp + 3p^2 < 0 \Rightarrow -\frac{1}{3} (4 + \sqrt{13}) \rho < p < \frac{1}{3} (-4 + \sqrt{13}) \rho] \]

So we readily get a range of the equation of states for the matter component. This is a very interesting result. We may put the above equations in the standard form as,

\[ 3H^2 = 8\pi \rho_{eff} \] (4.21)

\[ 2\dot{H} + 3H^2 = -8\pi p_{eff} \] (4.22)

where

\[ \rho_{eff} = -\frac{\rho}{f_0 (\rho^2 + 8pp + 3p^2)} \]

\[ p_{eff} = -\frac{1}{8\pi f_0 (\rho^2 + 3p^2)} \left[ 8\pi p - 4f_0 (\rho \dot{\rho} + 3p\dot{p}) H + 6f_0 (\rho^2 + 3p^2) H^2 \right] \] (4.24)

From eqn.(4.19) we see that \( \rho_{c3} = 0 \) is a critical point of the model, which corresponds to the empty universe. Unlike the previous model here we do not get a bouncing scenario from a non-zero finite energy density. One more interesting feature of this model is that there are two singularities corresponding to \( p = \frac{1}{3} (-4 \pm \sqrt{13}) \rho \). Considering barotropic equation of state given in eqn.(4.5) the above conditions reduce to \( k = \frac{1}{4} (-4 \pm \sqrt{13}) \) for a non-empty universe. For this model we are able to get an explicit general solution for any value of \( k \). Using eqn.(4.5) in the eqns.(4.19) and (4.20) we get,

\[ a(t) = C_2 \left[ (k + 1)^2 (3k - 1) t + 2C_1 (3k^2 + 1) \right]^{-2(3k^2+1)/(k+1)^2(3k-1)} \] (4.25)

\[ \rho(t) = \frac{6f_0 (3k^2 + 1)^2 (3k^2 + 8k + 1)}{8\pi \left[ (k + 1)^2 (3k - 1) t + 2C_1 (3k^2 + 1) \right]^2} \] (4.26)

where \( C_1 \) and \( C_2 \) are integration constants. It is seen that the scale factor is undefined at \( k = -1 \) (\( \Lambda \)CDM era) and \( k = 1/3 \) (radiation era). We have generated plots for both scale factor and dimensionless matter energy density in figures (4) and (5) respectively. From fig.(4) we see that scale factor grows with time indicating an expanding universe. We also see that the expansion rate is dominated in the dark energy phase \( k = -2/3 \) compared to the other phases, showing signs of accelerated expansion. Moreover in the late universe the dependency of \( a \) on the initial conditions becomes noteworthy. In fig.(5) it is evident that the energy density decays with time which is expected in an expanding universe. Here also the decay rate is dominated in the exotic phantom phase \( k = -4/3 \) due to the high rate of expansion of the universe. So the trajectories are quite satisfactory and comply with the observations. Here the dependency of \( r \) on the initial conditions is quite prominent in the late times, but not so pronounced in the early universe. We have also
The continuity equation turns out to be,

\[ \dot{\rho} + 3H (\rho + p) = 3H \left[ (1 + k) \rho + \frac{\zeta + f_0 \pi \rho^4 \xi}{\zeta/\rho + 6\pi f_0^2 \xi^2 H \rho^4} \right] \]  

where \( \zeta = \frac{4}{f_0(1 + 8k + 3k^2)} \) and \( \xi = 1 + 3k^2 \). The term on the RHS is the non-conservation term. Setting the non-conservation term equal to zero and solving the corresponding differential equation for the scale factor we have the two following evolutions of the universe,

\[ a(t) = C_3, \quad a(t) = C_3 e^{\xi \sigma, 8\pi(\rho + 3k^2 + k - 1)(2C_1 \xi + 3k^2 (3k - 1) \xi} \]  

where \( \sigma = 1 + 8k + 3k^2, \xi = 1 + 3k^2, \rho = k + 1 \) and \( C_3 \) is the integration constant. The first value suggests that the scale factor is a constant suggesting a static evolution of the universe. Since the scale factor does not grow with time the universe does not expand. This is contrary to our observations. So we ignore this result. The other expression gives an exponential type evolution of the universe. It is quite certain that this will correspond to a de-Sitter like evolution of the universe for suitable initial conditions. So there exists conditions under which this model may satisfy the standard energy momentum conservation relation. The deceleration parameter for this model is plotted in figure (6). From the figure we see that the there is a transition of \( q \) from positive level to negative level at some finite value of the dimensionless time \( \tau_p \). These values of \( \tau_p \) for each trajectory correspond to the redshift value \( z \approx 0.6 \), where the universe enters into the accelerating phase from a decelerating one. This is cosmologically viable with the observations. Here the dependency of the deceleration parameter on the initial conditions grows with time, which is contrary to the result obtained from the previous model.

For this model the extra force discussed in eqn. (2.42) can be given as (for \( n = 1, m = 1 \),

\[ F_{\mu T^2} = \frac{f_0 Q T_{\mu \nu}}{8\pi (\rho + p)} \left[ D_\alpha \left( - (\rho^2 + 4p\rho + 3p^2) u^\alpha u_\nu \right) - \frac{1}{2} \partial_\nu \left( \rho^2 + 3p^2 \right) \right] \]  

\[ Fig.4 \text{ and 5 shows the variation of the scale factor } a \text{ and matter energy density } r \text{ against the dimensionless time } \tau_p \text{ for model-2 for different initial conditions respectively. For Fig.4 we have taken } C_1 = 5, C_2 = -1. \text{ For Fig.5 we have taken } C_1 = -0.5, f_0 = 0.1, 8\pi = 1. \text{ In fig.4 we have considered } a(0) \approx 1. \]
Fig. 6 shows the variation of the deceleration parameter $q$ against the dimensionless time $\tau_p$ for model-2 for different initial conditions. We have taken $C_1 = -0.5$, $C_2 = 0.1$, $f_0 = 0.2$, $8\pi = 1$.

Here we see that along with the constant $f_0$, the non-metricity $Q$ also scales the dynamic force component. Since $Q = 6H^2$, we see that the expansion factor of the universe has a direct influence on the force component which was not the case in the previous model. Moreover we see that with increased expansion rate the term outside the bracket grows and simultaneously the term inside the bracket decays due to decreased matter density (as discussed before). The reverse happens when the expansion rate of universe decreases. So with one factor growing and the other one decaying, the force component is likely to evolve into a constant value with time.

Just like the previous model, here also we are interested in performing a dimensional analysis to determine the dimensions of the model parameter $f_0$. For this we will use the FLRW equation given in eqn.(4.17). Using eqn.(4.5) the FLRW equation may be rewritten as,

\[
(3H^2) f_0 \rho^2 \left\{ (6m - 6n - 3) k^2 + (2m - 2n - 1) - 8nk \right\} = 8\pi \times 2^{1-m} \rho(\rho^2)^{1-n} (1 + 3k^2)^{1-n}
\]

Now using the same expression for the Einstein’s constant used in the previous model $\kappa = 8\pi G/c^2$ we get the dimensions of the term in the RHS as $[M^{-1} L] \times [M L^{-3}]^{3-2n} = [M^{2-2n} L^{6n-8}]$. Here we have used the dimensions of density as $[\rho] = [\text{mass}] / [\text{volume}] = [M L^{-3}]$. In the LHS, considering the dimensions of the term in the bracket as 1 (dimensionless) we are left with $(3H^2)^m f_0 \rho^2$ whose dimensions should be $[M^{2-2n} L^{6n-8}]$ so that dimensional homogeneity is preserved. So we have $[T^{-2m}] \times [f_0] \times [\rho^2] = [M^{2-2n} L^{6n-8}]$, which gives $[f_0] = \left( \frac{(3H^2)^m}{\kappa[M L^{-3}]} \right) = [M^{-n} L^{5n-2} T^{2m}]$, which is the dimensional formula for $f_0$. For a particular case if we consider $m = n = 1$, then the dimensional formula for $f_0$ is $[f_0] = [M^{-2} L^4 T^2]$. Finally we would like to state that the dimensional results derived above (for both the models) is not unique by any means. It depends on various assumptions and choices that we have made from time to time.

5 Energy Conditions

Energy conditions are tools to establish the positiveness of the energy-momentum tensor in the presence of matter. These conditions actually describe the attractive nature of gravity and also take care of the causal and geodesic structure of the spacetime [110]. It is known that the energy conditions are directly linked to GR as they lead to some powerful singularity theorems [111]. Now
the formulation and meaning of energy conditions in the context of modified gravity theories is an extremely delicate issue and has its own implications which is quite contrasting to the implications in GR. Especially the non-standard (fictitious) fluids related to the additional degrees of freedom of modified gravity are supposed to produce interesting results when compiled with the energy conditions, which gives us some ideas about the non-attractive nature of the gravity leading to the cosmic acceleration. This is important considering that we do not yet have a model of cosmology consistent with observations and free from all the cosmological issues. The prime outcomes are that matter may manifest further thermodynamical features and gravity may retain its attractive nature in presence of large negative pressures. On the other hand, we can have repulsive gravity for standard matter. The fact that further degrees of freedom connected with the modified gravity theories, can be dealt under the banner of effective fluids, does allow us to frame consistent energy conditions for large classes of theories. From a cosmological point of view, these considerations are crucial. As an example, we see that the presence of dark energy can be considered a direct violation of energy conditions in the standard sense of GR. However in a generalized approach for modified gravity theory, there is no such violation, but just a reinterpretation of the additional degrees of freedom emerging from the dynamics of the theory [112]. So it is clear that there is a lot to gain in studying energy conditions in modified gravity. The reader may refer to the Refs.[112, 113] for further detailed discussions on energy conditions in modified gravity theories. Moreover energy conditions in the background of various modified gravity theories may be found in [114–118].

In this section we explore the energy conditions that the thermodynamic parameters of the $f(Q, T^2)$ theory need to satisfy and thus put some constraints on the model parameters. We will use a perfect fluid matter distribution. It should be mentioned here that the late cosmic acceleration demands the violation of the strong energy condition, since it requires anti-gravitational effect to play its role. The matter component responsible for this violation may be dark energy. There are basically four energy conditions that can be derived from standard general theory of relativity. Considering isotropic cosmology they are:

(I) Weak Energy Condition (WEC) $\Rightarrow \rho_{eff} \geq 0, \quad \rho_{eff} + p_{eff} \geq 0$

(II) Null Energy Condition (NEC) $\Rightarrow \rho_{eff} + p_{eff} \geq 0$

(III) Dominant Energy condition (DEC) $\Rightarrow \rho_{eff} \geq 0, \quad \rho_{eff} \geq |p_{eff}|$

(IV) Strong Energy condition (SEC) $\Rightarrow \rho_{eff} + 3p_{eff} \geq 0$

Now using eqns.(3.8) and (3.9) in the WEC we get the following inequalities considering $8\pi = 1$,

$$\frac{1}{f_Q} \left[ \rho + \frac{1}{2} f(Q, T^2) + f_{T^2} (\rho^2 + 4\rho \rho + 3p^2) \right] \geq 0 \quad (5.1)$$

$$\frac{1}{f_Q} \left[ 3f(Q, T^2) + 4H \left( 2f_Q - 9H f_Q \right) + 2 (\rho - 2p) + 2 f_{T^2} (\rho^2 + 4\rho \rho + 3p^2) \right] \geq 0 \quad (5.2)$$

The expression (5.2) is the required condition for the satisfaction of NEC. The dominant energy condition $\rho_{eff} \geq |p_{eff}|$ may be modified as $\rho_{eff} \pm p_{eff} \geq 0$. So along with the conditions given in (5.1) and (5.2) we have another condition for DEC given below,

$$\frac{1}{f_Q} \left[ f(Q, T^2) + 4H \left( 2f_Q - 9H f_Q \right) - 2 (\rho + 2p) - 2 f_{T^2} (\rho^2 + 4\rho \rho + 3p^2) \right] \leq 0 \quad (5.3)$$
Finally from the SEC we get,
\[
\frac{1}{f_Q} \left[ 7f(Q, T^2) + 12H \left( 2f_Q - 9H f_Q \right) + 2(\rho - 6p) + 2 f_{TT} \left( \rho^2 + 4\rho + 3p^2 \right) \right] \geq 0 \quad (5.4)
\]
So all the energy conditions finally yield four inequalities given by (5.1), (5.2), (5.3) and (5.4), which can be used to constrain the theory. Now we may use our toy-models discussed above to check how viable and effective these energy conditions are in constraining cosmological models. We discuss them one by one below.

5.1 Model: 1

Here we will use the model given in eqn.(4.1) which has two free parameters \( n \) and \( \eta \). We will also consider the barotropic equation of state \( p = k\rho \). Using the above relations we give the energy conditions for this model as,

- **WEC:** \( 3H_0^2 + \rho_0 + \frac{1}{2} (1 + 3k^2)^{n-1} [1 + 2n + k (3k + 8n + 6kn)] \eta \rho_0^n \geq 0 \)
  
  and
  \[
  -18H_0^2 + 2 (1 - 2k) \rho_0 + (1 + 3k^2)^{n-1} [3 + 2n + k (9k + 8n + 6kn)] \eta \rho_0^n \geq 0 \quad (5.5)
  \]

- **NEC:** \( -18H_0^2 + 2 (1 - 2k) \rho_0 + (1 + 3k^2)^{n-1} [3 + 2n + k (9k + 8n + 6kn)] \eta \rho_0^n \geq 0 \quad (5.6)
  \]

- **DEC:** \( 3H_0^2 + \rho_0 + \frac{1}{2} (1 + 3k^2)^{n-1} [1 + 2n + k (3k + 8n + 6kn)] \eta \rho_0^n \geq 0 \)
  
  and
  \[
  -18H_0^2 + 2 (1 - 2k) \rho_0 + (1 + 3k^2)^{n-1} [3 + 2n + k (9k + 8n + 6kn)] \eta \rho_0^n \geq 0 \quad (5.7)
  \]

- **SEC:** \( 2 (1 - 6k) \rho_0 + (1 + 3k^2)^{n-1} [7 + 2n + k (21k + 8n + 6kn)] \eta \rho_0^n - 66H_0^2 \geq 0 \quad (5.8)
  \]

Since we are interested in constraining the models we have used the present values of the Hubble parameter \( H_0 \) and matter energy density \( \rho_0 \) in the above energy conditions. The present value of the Hubble parameter is estimated to be \( H_0 = 67.9 \, \text{km sec}^{-1} \text{Mpc}^{-1} \) [119, 120] and that of \( \rho_0 \) is \( \rho_0 = 9.9 \times 10^{-30} \text{gm cm}^{-3} \) [121]. Now we may consider various cosmological era for matter by changing the value of \( k \) such as \( k = 1/3, 0, -1/3, -1 \). After putting all these values we will get inequalities connecting only \( \eta \) and \( n \), from where it will be straightforward to put constraints on these two model parameters.

5.1.1 \( k = 1/3 \) (Radiation)

From the WEC conditions we found that \(-0.2 \leq n \leq -0.1\) and \( \eta > 0 \). From further analysis it is evident that the above range of the parameters also satisfy the other conditions NEC, DEC and SEC.
5.1.2 $k = 0$ (Dust)

For this cosmological era it is found that the parameter range obtained from the WEC conditions are $-0.5 \leq n \leq -0.1$ and $\eta > 0$. Using the other conditions the range was reduced to $-0.5 \leq n < -0.1$ and $\eta > 0$.

5.1.3 $k < -1/3$ (Quintessence)

The constraints on the parameter space for this era are found as $n \leq -0.1$ and $\eta > 0$ from the WEC conditions. The other conditions comply with this range with the exception of the SEC condition. This is consistent with the accelerated expansion of the universe.

5.2 Model: 2

In this model we have three free parameters $f_0$, $m$ and $n$. Since for this model we have $\dot{H}$ present in $f_0$ terms, we have used the eqn. (5.12) to define $\dot{H} = -H^2 (1 + q)$. Below we present the energy conditions for this model.

- **WEC:**

$$\frac{8kn + (3k^2 + 1) (1 + 2n)}{m} + \frac{2 \times (6H_0^2)^{-m} \rho_0^{-1-2n}}{f_0 m (1 + 3k^2)^{n-1}} \geq 0,$$

and

$$\frac{1}{m} \left[ 3 + 2\varrho (3\varrho - 2) \xi^{-1}n + 2f_0^{-1}H_0^{-2m} \xi^{-n} (3 - 2\varrho) 6^{-m} \rho_0^{-1-2n} + \frac{2}{3} m \left( 2(H_0^2)^{(n-m)} \left( 2(1-n) q_0 + \frac{\zeta}{q} \right) - 9 \right) \right] \geq 0 \quad (5.9)$$

- **NEC:**

$$\frac{1}{m} \left[ 3 + 2\varrho (3\varrho - 2) \xi^{-1}n + 2f_0^{-1}H_0^{-2m} \xi^{-n} (3 - 2\varrho) 6^{-m} \rho_0^{-1-2n} + \frac{2}{3} m \left( 2(H_0^2)^{(n-m)} \left( 2(1-n) q_0 + \frac{\zeta}{q} \right) - 9 \right) \right] \geq 0 \quad (5.10)$$

- **DEC:**

$$\frac{8kn + (3k^2 + 1) (1 + 2n)}{m} + \frac{2 \times (6H_0^2)^{-m} \rho_0^{-1-2n}}{f_0 m (1 + 3k^2)^{n-1}} \geq 0,$$

$$\frac{1}{m} \left[ 7 + 2\varrho (3\varrho - 2) \xi^{-1}n + 2f_0^{-1}H_0^{-2m} \xi^{-n} (3 - 2\varrho) 6^{-m} \rho_0^{-1-2n} + \frac{2}{3} m \left( 2(H_0^2)^{(n-m)} \left( 2(1-n) q_0 + \frac{\zeta}{q} \right) - 9 \right) \right] \geq 0, \quad (5.11)$$

and

$$\frac{1}{m} \left[ 1 - 2\varrho (3\varrho - 2) \xi^{-1}n - 2f_0^{-1}H_0^{-2m} \xi^{-n} (3 - 2\varrho) 6^{-m} \rho_0^{-1-2n} + \frac{2}{3} m \left( 2(H_0^2)^{(n-m)} \left( 2(1-n) q_0 + \frac{\zeta}{q} \right) - 9 \right) \right] \leq 0 \quad (5.12)$$

- **SEC:**

$$\frac{1}{m} \left[ 3 + 2\varrho (3\varrho - 2) \xi^{-1}n + 2f_0^{-1}H_0^{-2m} \xi^{-n} (7 - 2\varrho) 6^{-m} \rho_0^{-1-2n} + 2m \left( 2(H_0^2)^{(n-m)} \left( 2(1-n) q_0 + \frac{\zeta}{q} \right) - 9 \right) \right] \geq 0 \quad (5.13)$$

In the above expressions $\varrho$, $\xi$ and $\vartheta$ have been defined earlier just after eqn. (4.28) and the expressions for $\zeta$ and $\vartheta$ are given below.
\[ \varsigma = (\xi \rho_0^2)^{-n} \left[ -2f_0 \left( (6H_0^2)^m + 4(6H_0^2)^n (n - 1) \right) \left( 1 - 3(m + 1)n + 2n^2 \right) (\xi \rho_0^2)^n \chi + 6nH_0^2 \left( -6k\rho_0 - f_0 (3(6H_0^2)^m (3m - 1) + 2(6H_0^2)^n m (n - 1)) \right) (\xi \rho_0^2)^n \right] \] (5.14)

\[ \vartheta = f_0 \left[ -2mn(6H_0^2)^{1+n} + \left( (6H_0^2)^m + 2(6H_0^2)^n (n - 1) \right) 2(2n - 1) \chi \right] \] (5.15)

where

\[ \chi = \left( \frac{6 - m(\xi \rho_0^2)^{n-1}}{f_0\rho_0 (\xi (2m - 1) - 2\sigma(3\sigma - 2) n)} \right)^{1/m} \]

Just like the previous model, in addition to the current values \( H_0, \rho_0 \) we also have to use \( q_0 = -0.503 \) [119, 120] for this case, to put constraints on the parameters \( f_0, n \) and \( m \) from the above inequalities (which is pretty straightforward). So the above energy conditions can be used as relations to constrain the free parameters of the theory as done for model-1. Since the conditions for this model are quite complex, we have managed to obtain some general constraints on the model parameters. The results are presented below in a tabular form.

| \( k \) | Range of \( f_0 \) | Constraint on model parameters |
|------|-----------------|--------------------------------|
| 1/3  | • For  \( f_0 > 0 \) | (i) \( n \geq -0.2, \ m > 0 \)  (ii) \( n \leq -0.3, \ m < 0 \) |
|      | • For  \( f_0 < 0 \) | (i) \( n \leq -0.2, \ m < 0 \)  (ii) \( n \geq -0.2, \ m > 0 \) |
| 0    | • For  \( f_0 > 0 \) | (i) \( n \geq -0.4, \ m > 0 \) |
|      | • For  \( f_0 < 0 \) | (i) \( n < 0, \ m < 0 \)  (ii) \( n > 0, \ m \leq -0.1 \) |
| -2/3 | • For  \( f_0 > 0 \) | \( m > 0 \), for all values of \( n \) |
|      | • For  \( f_0 < 0 \) | \( n < 0, \ m > 0 \) |

**Table 1:** Constraints on model parameters (model 2) for different values of EoS parameter \( k \) from the energy conditions.

It should be noted that the parameter values used in the discussion in section 4.1 violate the values obtained from the energy conditions in section 5.1. Energy conditions are basically some mathematically imposed boundary conditions that help us to deduce very powerful and general results regarding the behaviour of strong gravitational fields and cosmological geometries [122]. But of late these conditions have started to look far less secure than once they seemed to be. There can be various reasons behind this. There are subtle quantum effects which are responsible for the violation of the energy conditions. There are also certain classical systems that violate all the energy conditions [122–124]. This directly reflects on the nature of the matter content of the universe and opens up various exotic possibilities such as traversable wormholes, warp drives, time machines, etc. [122]. Over the years energy conditions like the Trace energy condition (TEC) have totally lost
their significance and have now been abandoned. With the discovery of cosmic acceleration and consequent arrival of the concept of dark energy, SEC and NEC have almost been abandoned. So the place of energy conditions in GR and Cosmology needs a radical reassessment.

From the above discussion it is clear that in a late accelerating universe filled with dark energy it is expected that SEC and NEC will be violated. It is known that the early universe was dominated by quantum effects. For the theory we are discussing in this paper, there is clear evidence of quantum gravitational effects in the early universe. These quantum fluctuations are responsible for the violation of the energy conditions in the early universe. Moreover for the inflationary epoch, some of the energy conditions are readily violated. Coming to the period between the inflation and the late cosmic acceleration (we call middle phase), it can be argued that due to a strong quantum gravitational effect in the early universe, there are some reminiscent effects in subsequent eras. This imprint of quantum effects in the system does not allow the energy conditions to hold. Moreover it has been already discussed that the violation of energy conditions is also true for certain classical systems as well [122–124]. So even if there is no direct quantum dominance or dark energy dominance during the middle phase, it is not very strange for the system to violate the energy conditions. For DEC and WEC, \( \rho_{\text{eff}} \geq 0 \) will always hold. But the trouble is with the conditions \( \rho_{\text{eff}} + p_{\text{eff}} \geq 0 \) and \( \rho_{\text{eff}} \geq |p_{\text{eff}}| \). For sufficient negative pressure these two conditions are violated leading to the overall violation of WEC and DEC. Our model seems to violate all the four energy conditions and this is not very unexpected from the above discussion.

6 Discussion & Conclusion

In the present work we have proposed yet another extension of the symmetric teleparallel gravity by generalizing the gravity Lagrangian with an arbitrary function \( f(Q, T_{\mu\nu} T^{\mu\nu}) \). The field equations were derived in a metric-affine formalism. The correction terms introduced by the modified gravity were noted. As expected for any gravity theory involving non-minimal coupling between geometry and matter sectors, the covariant divergence of the energy-momentum tensor was non-zero thus implying the non-conservation of the same. The non-conservation term was derived using the field equations. The momentum conservation equation showed the presence of correction terms implying extra force on the massive particles thus making the motion non-geodesic. The field equations were further simplified by considering perfect fluid as the matter component. Using these field equations we resorted to study the cosmological evolution of the theory. The FLRW equations for a flat homogeneous and isotropic spacetime were derived. It was noted that there were two additional modification terms introduced in the equations in contrast to those of standard GR. The two additional terms were of the form \( f/2 \) and \( f_{T^2} (\rho^2 + 4\rho p + 3p^2) \). The first one came from the coupling between the geometric and matter sector and the second one is completely a source term. These higher order terms dominate in the early universe and gradually fade away at late times giving the effects of the standard FLRW universe. These corrections are totally intrinsic and uniquely describe the modified gravity. Expressions for some cosmological parameters like the equation of state and deceleration parameter were derived. We investigated the vacuum solution of the theory and saw that EMSSTG admits a de-Sitter like solution in its framework. One of the crucial aspect of the theory is the non-conservation of the energy-momentum tensor. Moreover in the momentum conservation equation, an extra force appears which results in non-geodesic motion of massive particles.

To get more insights into the cosmological framework of the theory we studied two specific toy-models models \( Q + \eta T^2 \) and \( f_0 Q^n T^2 \). We saw that both the models had STG as a limiting case and hence we can recover the parent theory from the equations. After deriving the FLRW equations for the first model we saw that the equations had a flavour of the quantum gravity effects of the loop quantum gravity. So solutions for bouncing cosmology was investigated and it
was found that the model indeed supported a cosmological bounce at a finite time, thus avoiding the singularity. Various constraints were imposed on the model from these relations. Although we did not get a general solution of the model, but for $k = -1$ we obtained a solution which resembled the de-Sitter solution. Then we derived the continuity equation for the model and studied the non-conservation term. Using it we were able to trace the evolution of the scale factor for which the non-conservation term will vanish. For this model we plotted the scale factor, the Hubble function and the deceleration parameter to check the viability of the model. Similar studies were undertaken for the second model. In this case we were fortunate enough to get a general solution of the FLRW equations. The obtained scale factor and the matter energy density were plotted and compared to those of the standard $\Lambda$CDM model. We also obtained the plots for the deceleration parameter for this model in a comparative scenario with the $\Lambda$CDM model. The transition from a decelerating to an accelerating universe was clearly evident and the deviation of the trajectories from those of the $\Lambda$CDM model was also noted. All the plots are generated using dimensionless parameters. For both the models a detailed dimensional analysis is performed to determine the dimensions of the model parameters.

It must be stated here that for our theory, the presence of extra force and the corresponding non-geodesic motion of the test particle, implies the violation of the equivalence principle (EP). There are weak and strong forms of the EP. Even it is accepted by many authors that although most of the metric theories of gravity satisfy the weak form of the principle, GR is the only gravity theory in four dimensions that fully incorporates the strong equivalence principle (SEP) [125, 126]. So if we are searching for concepts beyond GR, a promising avenue will be to look for occasions of the violation of the SEP. To complement this we would like to mention that gravity’s rainbow [127], which is an extension of the doubly special relativity [128] to incorporate curvature, is a quantum theory of gravity, where there is a direct violation of the EP. In this theory the path followed by a particle in a gravitational field depends on the energy content of the body and hence there is a modification to the standard EP. Since our theory also has flavours of quantum gravity it is quite expected from analogy that there should be some confrontation with the EP and possible violation. Coming to the tests, there has been no universal acceptance backed by experimental observations of the EP till date. This is evident from the fact that people are continuing to test the principle till date and trying to find ways to prove its validity or disprove it [129, 130]. This shows that, may be the tests are not yet well framed and self consistent or probably our instruments are not yet advanced enough to test the theory, but obviously progress is being made. Also we just cannot rule out the fact that the EP may not be true. We don’t know for the time being and neither can we claim anything. That is why these alternative theories with non-minimal matter coupling have gained importance over the past decade. We need a competing concept to challenge the existing one (at least in the absence of a proof). So for the time being there is place for counter concepts and these are not rare in literature. Our model is nothing new, but just another elegant example of it having very important and impressive properties like quantum gravity. We can also think of modifications to the EP (like gravity’s rainbow) consistent with these non minimal matter coupling theories and try to test them. It has also been reported, from the data of the Abell Cluster A586, that interaction of dark matter and dark energy does imply the violation of the EP [131]. Thus there is a realistic possibility and justification of studying and testing these models with non-minimal matter couplings in the context of the violation of the EP.

Finally we explored the energy conditions in the background of the theory. The basic energy conditions WEC, NEC, DEC and SEC were derived for the theory and also for the two specific toy-models. In the first model we had two free parameters $n$ and $\eta$ after we fixed $H$ and $\rho$ with their current values from the observations. Similarly in the second model we had three free parameters $f_0$, $m$ and $n$ after the fixation from the observations. Basically four constraints were obtained for each model using the energy conditions which are sufficient to put bounds on the free parameters.
of both the models. Thus the models were well-constrained by the energy conditions. It was also found that the energy conditions are violated in the cosmological discussion of the models, which is quite expected given the quantum gravity effects and the exotic nature of the theory. Finally from the study we conclude that the theory is perfectly suitable to describe the cosmological dynamics of both the early and the late universe without resorting to dark energy. There is scope for further development of the theory, which will be undertaken in future projects. The standout feature of EMSSTG is that the field equations contain terms which arise from the quantum gravity effects and thus are responsible for the avoidance of the singularity. So this theory is a singularity free cosmologically viable theory. Moreover the non-linear density terms in the equations dominate in the early universe and gradually fade away at later times. So the quantum effects of modified gravity is predominantly felt in the early universe and it eases out to give the standard FLRW effects at late times.

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