Trakhtenbrot Theorem and First-Order Axiomatic Extensions of MTL

Abstract. In 1950, B.A. Trakhtenbrot showed that the set of first-order tautologies associated to finite models is not recursively enumerable. In 1999, P. Hájek generalized this result to the first-order versions of Lukasiewicz, Gödel and Product logics, w.r.t. their standard algebras. In this paper we extend the analysis to the first-order versions of axiomatic extensions of MTL. Our main result is the following. Let $\mathcal{K}$ be a class of MTL-chains. Then the set of all first-order tautologies associated to the finite models over chains in $\mathcal{K}$, $f\text{TAUT}_\mathcal{K}$, is $\Pi^0_1$-hard. Let $\text{TAUT}_\mathcal{K}$ be the set of propositional tautologies of $\mathcal{K}$. If $\text{TAUT}_\mathcal{K}$ is decidable, we have that $f\text{TAUT}_\mathcal{K}$ is in $\Pi^0_1$. We have similar results also if we expand the language with the $\Delta$ operator.

Keywords: Trakhtenbrot theorem, Many-valued logics, MTL logic, Residuated lattices, Completeness, Arithmetical complexity.

Note by the first author

Sadly, Franco Montagna passed away on 18 February 2015. It is a great loss for the scientific community, and also for the people who personally knew him, including me. I am glad I had the opportunity to work with him, and I remember his kindness, openness in discussions, and his ability in approaching and solving mathematical problems. I was not one of his students, but I can say that he contributed in a significant way to improve my knowledge in the area of mathematical logic.

1. Introduction and Motivations

In [18], Trakhtenbrot showed that the set of first-order tautologies associated to finite models is not recursively enumerable, in classical first-order logic. In particular, it is known that such set is $\Pi^0_1$-complete. In [6, 20] it is shown that the theorem works also with languages containing only predicates, with at least a binary one, and without equality. This result implies the fact that the completeness w.r.t. finite models does not hold in first-order logic. Indeed, the set of theorems of classical predicate logic is $\Sigma^0_1$-complete.

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One can ask if a similar result holds also in non-classical logics, for example many-valued logics. A first answer was given in [12] by Hájek, who generalized Trakhtenbrot theorem to the first-order versions of Łukasiewicz, Gödel and Product logics, with respect to their standard algebras. That paper was published in 1999, and since then a number of larger families of many-valued logics has been introduced, see [8] for details. In this article we focus our attention to the monoidal t-norm based logic MTL and its axiomatic extensions [8,10]. These logics extend the well known full Lambek calculus FL, and they are all algebraizable in the sense of [5]. In particular, the semantics related to each logic forms an algebraic variety.

For every axiomatic extension L of MTL, we have a completeness theorem w.r.t. the class of L-algebras. In the first-order case, however, we need to restrict to totally ordered algebras: indeed, if not, the soundness does not necessarily holds, see [11, Example 5.4] for a counterexample over Gödel logic. This is not by chance, but it is a consequence of the fact that such logics are axiomatized in the way to have the completeness w.r.t. the class of all chains. Such development of first-order logics has many connections with the works of Mostowski and Rasiowa, as explained in [13].

In this article we show a generalized version of Trakhtenbrot theorem, for the first-order versions of axiomatic extensions of MTL.

The paper is structured as follows.

After some basic background in Sect. 2, in Sect. 3 we show the following results. Let $\mathbb{K}$ be a class of MTL-chains. Then the set of all first-order tautologies associated to the finite models over chains in $\mathbb{K}$, $\text{fTAUT}_\mathbb{K}$, is $\Pi^0_1$-hard. Let now $\text{TAUT}_\mathbb{K}$ be the set of propositional tautologies of $\mathbb{K}$. If $\text{TAUT}_\mathbb{K}$ is decidable, we have that $\text{fTAUT}_\mathbb{K}$ is in $\Pi^0_1$. As a consequence, if L is an axiomatic extension of MTL, and $\mathbb{C}$ is the class of all L-chains, then $\text{fTAUT}(L\forall)$ is $\Pi^0_1$-hard. Moreover, if L is decidable, then $\text{fTAUT}(L\forall)$ is $\Pi^0_1$-complete.

Hence the decidability of L is a sufficient condition for the $\Pi^0_1$-completeness of $\text{fTAUT}(L\forall)$. Is it also necessary? In Sect. 4 we tackle this problem, by showing that the answer is positive if L is recursively axiomatizable. However, we have negative results if we expand the language of L with constants, and L is not recursively axiomatizable.

In Sect. 5 we show some negative results about the expansions of MTL with the $\Delta$ operator. Finally, Sect. 6 is devoted to conclusions and open problems.
2. Some Basic Background

We assume that the reader is familiar with monoidal t-norm based logics and its extensions, in the propositional and in the first-order case. For a reference, see [8–10,15].

2.1. Syntax

The language of MTL is based over the connectives \{\&, \&, \to, \perp\}. The formulas are built in the usual inductive way from these connectives, and a denumerable set of variables \text{VAR}=\{x_0, x_1, x_2, \ldots\}.

Useful derived connectives are the following:

\begin{align*}
\text{(negation) } & \quad \neg \phi \overset{\text{def}}{=} \phi \to \perp \\
\text{(disjunction) } & \quad \phi \lor \psi \overset{\text{def}}{=} ((\phi \to \psi) \to \psi) \land ((\psi \to \phi) \to \phi) \\
\text{(biconditional) } & \quad \phi \leftrightarrow \psi \overset{\text{def}}{=} (\phi \to \psi) \land (\psi \to \phi)
\end{align*}

MTL can be axiomatized with a Hilbert style calculus. For the reader’s convenience, we list the axioms of MTL:

\begin{align*}
(A1) \quad & (\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi)) \\
(A2) \quad & (\phi \& \psi) \to \phi \\
(A3) \quad & (\phi \& \psi) \to (\psi \& \phi) \\
(A4) \quad & (\phi \land \psi) \to \phi \\
(A5) \quad & (\phi \land \psi) \to (\psi \land \phi) \\
(A6) \quad & (\phi \&(\phi \to \psi)) \to (\psi \land \phi) \\
(A7a) \quad & (\phi \to (\psi \to \chi)) \to ((\phi \& \psi) \to \chi) \\
(A7b) \quad & ((\phi \& \psi) \to \chi) \to (\phi \to (\psi \to \chi)) \\
(A8) \quad & ((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi) \\
(A9) \quad & \perp \to \phi
\end{align*}

As inference rule we have modus ponens:

\begin{align*}
\text{(MP) } & \quad \phi \quad \phi \to \psi \quad \psi
\end{align*}

An axiomatic extension of MTL is a logic obtained by adding one or more axiom schemata to it. A theory is a set of formulas: the notion of proof and logical consequence are defined as in the classical case.
2.2. Semantics

An MTL-algebra is an algebra $\langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1 \rangle$ such that:

1. $\langle A, \sqcap, \sqcup, 0, 1 \rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1 \rangle$ is a commutative monoid.
3. $\langle *, \Rightarrow \rangle$ forms a residuated pair: $z * x \leq y$ iff $z \leq x \Rightarrow y$ for all $x, y, z \in A$.
4. The following axiom holds, for all $x, y \in A$:

   \[(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1\]

   (Prelinearity)

A totally ordered MTL-algebra is called MTL-chain.

In the rest of the paper $\sim x$ will denote $x \Rightarrow 0$.

Given an MTL-chain $A$, we define $A^+ \overset{\text{def}}{=} \{ x \in A : x > \sim x \}$, the set of the positive elements of $A$; dually $A^- \overset{\text{def}}{=} \{ x \in A : x < \sim x \}$, is the set of the negative elements of $A$. A negation fixpoint is an element such that $x = \sim x$.

An easy check shows that if an MTL-chain has a negation fixpoint, then it is unique.

Let $L$ be an axiomatic extension of MTL. It is known (see [7,17]) that $L$ is algebraisable in the sense of [5], and that the equivalent algebraic semantics forms a subvariety of MTL-algebras, called $L$-algebras. We will denote by $L$ such variety. On the other hand, each subvariety $L$ of MTL is the equivalent algebraic semantics of an axiomatic extension of MTL, that we will denote by $L$.

In particular $L$ is the extension of MTL via a set of axioms $\{ \varphi_i \}_{i \in I}$ if and only if $L$ is the subvariety of MTL-algebras satisfying $\{ \bar{\varphi}_i = 1 \}_{i \in I}$, where $\bar{\varphi}_i$ is obtained from $\varphi_i$ by replacing each occurrence of $\&$, $\to$, $\land$, $\lor$, $\neg$, $\bot$ with $*$, $\Rightarrow$, $\sqcap$, $\sqcup$, $\sim$, $0$, and every formula symbol occurring in $\varphi$ with an individual variable.

Finally, the notions of evaluation, tautology and completeness are defined in the usual way.

2.3. First-Order Case

In this section we briefly present the first-order versions of MTL and its axiomatic extensions. More details can be found in [8,9].

**Definition 1.** A first-order language $L$ is a countable set $P$ of predicate symbols, containing at least a binary one (i.e. we do not consider monadic fragments), without equality.
The notions of term (that coincides with variables, with such a language), formula, closed formula, term substitutable in a formula are defined like in the classical case \[8,9\]; the connectives are those of the propositional level.

**Remark 1.** One can ask the reason for choosing a such language. There are essentially two motivations.

- We need at least a binary predicate in the language, because in classical first-order logic the monadic fragment is decidable, and it has the finite model property. That is, Trakhtenbrot theorem fails to hold in the monadic fragment of classical first-order logic. However, if we have a language with at least a binary predicate, then Trakhtenbrot theorem holds.

- By adding functions symbols, usually one wants also equality: whilst in classical first-order logic there is a “natural” way to define equality (the crisp one), in the (first-order) extensions of MTL the situation is different. Indeed, one can use a crisp equality, but even a fuzzy similarity: in this last case there is no “standard” way to define it.

Let \( L \) be an axiomatic extension of MTL. Then its first-order version, \( L\forall \), is axiomatized as follows:

- The axioms resulting from the axioms of \( L \) by the substitution of the propositional variables by the first-order formulas.

- The following axioms:
  
  \[
  \begin{align*}
  \forall 1 & \quad (\forall x)\varphi(x) \rightarrow \varphi(x/t)( \ t \ \text{substitutable for} \ x \ \text{in} \ \varphi(x)) \\
  \exists 1 & \quad \varphi(x/t) \rightarrow (\exists x)\varphi(x)( \ t \ \text{substitutable for} \ x \ \text{in} \ \varphi(x)) \\
  \forall 2 & \quad (\forall x)(\nu \rightarrow \varphi) \rightarrow (\nu \rightarrow (\forall x)\varphi) \ (x \not\text{free in} \ \nu) \\
  \exists 2 & \quad (\forall x)(\varphi \rightarrow \nu) \rightarrow ((\exists x)\varphi \rightarrow \nu) \ (x \not\text{free in} \ \nu) \\
  \forall 3 & \quad (\forall x)(\varphi \lor \nu) \rightarrow ((\forall x)\varphi \lor \nu) \ (x \not\text{free in} \ \nu)
  \end{align*}
  \]

The rules of \( L\forall \) are: Modus Ponens: \( \varphi \rightarrow \psi \rightarrow \psi \) and Generalization: \( (\forall x)\varphi \).

As regards to semantics, we restrict to \( L \)-chains: given an \( L \)-chain \( \mathcal{A} \), a finite \( A \)-model is a structure \( M = \langle M, \{r_P\}_{P \in \mathcal{P}} \rangle \), where:

- \( M \) is a finite non-empty set.
- For each \( P \in \mathcal{P} \) of arity\(^1\) \( n \), \( r_P : M^n \rightarrow A \).

For each evaluation over variables \( \nu : Var \rightarrow M \), the truth value of a formula \( \varphi (\|\varphi\|_{M,\nu}) \) is defined inductively as follows:

\(^1\)If \( P \) has arity zero, then \( r_P \in A \).
• \[\|P(x_1, \ldots, x_n)\|_{M,v}^A = r_P(v(x_1), \ldots, v(x_n))\].

• The truth value commutes with the connectives of \(L\forall\), i.e.

\[
\begin{align*}
\|\varphi \rightarrow \psi\|_{M,v}^A &= \|\varphi\|_{M,v}^A \Rightarrow \|\psi\|_{M,v}^A \\
\|\varphi \& \psi\|_{M,v}^A &= \|\varphi\|_{M,v}^A \land \|\psi\|_{M,v}^A \\
\|\perp\|_{M,v}^A &= 0 \\
\|\varphi \land \psi\|_{M,v}^A &= \|\varphi\|_{M,v}^A \land \|\psi\|_{M,v}^A
\end{align*}
\]

• \[\|(\forall x)\varphi\|_{M,v}^A = \min\{\|\varphi\|_{M,v'}^A : v' \equiv_x v, \text{ i.e. } v'(y) = v(y) \text{ for all variables except for } x\}\]

• \[\|(\exists x)\varphi\|_{M,v}^A = \max\{\|\varphi\|_{M,v'}^A : v' \equiv_x v, \text{ i.e. } v'(y) = v(y) \text{ for all variables except for } x\}\].

**Remark 2.** • Usually, the last two cases are defined by taking, respectively, inf’s and sup’s of truth values. Since these inf’s and sup’s do not necessarily exist, we have to introduce the notion of safe model, if we drop the requirement that the model is finite. Conversely, every finite model is safe, and in particular it is also witnessed, in the sense of [14]. For this reason we can take min and max.

• Let \(A\) be an MTL-chain, and let \(M\) be a finite \(A\)-model. If \(\varphi\) is a closed formula, then \[\|\varphi\|_{M,v}^A = \|\varphi\|_{M,w}^A\], for every pair of variable evaluations \(v, w\). Hence, in the rest on the article we will use the simpler notation \[\|\varphi\|_{M}^A\], whenever \(\varphi\) is a closed formula.

Let \(P(x_1, \ldots, x_k)\) be an atomic formula, and let \(a_1, \ldots, a_k \in M\). Note that, for every pair of evaluations \(v, w\) over variables such that \(v(x_i) = w(x_i) = a_i\), for \(i \in \{i, \ldots, k\}\), we have that \[\|P(x_1, \ldots, x_k)\|_{M,v}^A = \|P(x_1, \ldots, x_k)\|_{M,w}^A\]. For this reason we will use the simpler notation \[\|P(a_1, \ldots, a_k)\|_{M}^A\].

Let \(L\) be an axiomatic extension of MTL, and \(A\) be an \(L\)-chain. We say that \(L\forall\) is complete w.r.t. the class of finite \(A\)-models, whenever, for every (first-order) formula \(\varphi\):

\[\vdash_{L\forall} \varphi \quad \text{iff} \quad \|\varphi\|_{M,v}^A = 1,\]

for every finite \(A\)-model \(M\), and evaluation \(v\).

**Definition 2.** A family of sets \(\{X_n : n \in \mathbb{N}\}\) is said to be uniformly recursive whenever the relation defined as \(R(x, n)\) if and only if \(x \in X_n\) is recursive.
Given a uniformly recursive family \( \{ X_n : n \in \mathbb{N} \} \), and a set \( Y \), we say that a recursive function \( g(n, x) \) uniformly reduces \( \{ X_n : n \in \mathbb{N} \} \) to \( Y \) if, for every \( n \in \mathbb{N} \) and every \( x, x \in X_n \) if and only if \( g(n, x) \in Y \).

### 3. Trakhtenbrot Theorem and New Results

In this section we extend Trakhtenbrot theorem to the first-order axiomatic extensions of MTL.

We define the following sets: in this article we will study the arithmetical complexity of some of them.

**Definition 3.** Let \( \mathcal{A} \) be an MTL-chain. Then \( \text{fTAUT}^\mathcal{A}_\forall \) is the set of formulas which are valid in all finite models over \( \mathcal{A} \). If \( \mathcal{K} \) is a class of MTL-chains, then \( \text{fTAUT}^\mathcal{K}_\forall \) is defined as \( \bigcap_{A \in \mathcal{K}} \text{fTAUT}^A_\forall \). Let \( L \) be an axiomatic extension of MTL. We define \( \text{fTAUT}(L\forall)_\forall \) as \( \bigcap_{A \in C} \text{fTAUT}^A_\forall \), where \( C \) is the class of \( L \)-chains. Finally, \( \text{TAUT}(\mathcal{A}) \) is the set of propositional formulas which are valid in \( \mathcal{A} \), \( \text{TAUT}(\mathcal{K}) \) is defined as \( \bigcap_{A \in \mathcal{K}} \text{TAUT}(\mathcal{A}) \), and \( \text{TAUT}_L \) is defined as \( \bigcap_{A \in C} \text{TAUT}(\mathcal{A}) \).

We start by recalling the classical Trakhtenbrot theorem. In the rest of the paper, with \( 2 \) we denote the two elements boolean algebra.

**Theorem 1 ([6,18,20]).** The set \( \text{fTAUT}_\forall^\mathcal{2} \) is \( \Pi^0_1 \)-complete.

More recently, Hájek showed that:

**Theorem 2 ([12]).** For \( \mathcal{A} \in \{ [0, 1]_G, [0, 1]_\Pi, [0, 1]_L \} \) \( \text{fTAUT}^\mathcal{A}_\forall \) is \( \Pi^0_1 \)-complete.

Moving to the case of axiomatic extensions of MTL, our main results are the following ones:

**Theorem 3.** (i) If \( \mathcal{A} \) is any MTL-chain and \( \text{TAUT}(\mathcal{A}) \) is decidable, then \( \text{fTAUT}^\mathcal{A}_\forall \) is in \( \Pi^0_1 \). More in general, if \( \mathcal{K} \) is any class of MTL-chains and \( \text{TAUT}(\mathcal{K}) \) is decidable, then \( \text{fTAUT}^\mathcal{K}_\forall \) is in \( \Pi^0_1 \).

(ii) Let \( L \) be an axiomatic extension of MTL. If \( L \) is decidable and is sound and complete with respect to a class of \( L \)-chains \( \mathcal{K} \), that is, if \( \text{TAUT}_L = \text{TAUT}(\mathcal{K}) \), then \( \text{fTAUT}^\mathcal{K}_\forall \) is in \( \Pi^0_1 \).

(iii) For every decidable axiomatic extension \( L \) of MTL, the set \( \text{fTAUT}(L\forall) \) is in \( \Pi^0_1 \).

**Theorem 4.** (i) For every MTL-chain \( \mathcal{A} \), \( \text{fTAUT}^\mathcal{A}_\forall \) is \( \Pi^0_1 \)-hard. More in general, for every class of MTL-chains \( \mathcal{K} \), \( \text{fTAUT}^\mathcal{K}_\forall \) is \( \Pi^0_1 \)-hard.
(ii) Let $L$ be an axiomatic extension of MTL. If $\mathbb{K}$ is a class of $L$-chains and $TAUT_L = TAUT(\mathbb{K})$, then $fTAUT^\mathbb{K}_\forall$ is $\Pi^0_1$-hard.

(iii) For every consistent axiomatic extension $L$ of MTL, the set $fTAUT(L\forall)$ is $\Pi^0_1$-hard.

In the rest of the section we develop the proof of these two theorems.

**Definition 4.** Let $\mathbb{N}^+ \overset{\text{def}}{=} \mathbb{N} \setminus \{0\}$. For every $n \in \mathbb{N}^+$, with $\mathcal{L}^{n\forall}$ we denote the language of MTL\forall expanded with the constants $c_1,\ldots,c_n$. The set of $\mathcal{L}^{n\forall}$ formulas will be called FORM$_n$.

- For every MTL-chain $\mathcal{A}$ and every $n \in \mathbb{N}^+$, with $nTAUT^\mathcal{A}_\forall +$ we will denote the set of all first-order $\mathcal{L}^{n\forall}$ sentences which are valid in every $\mathcal{A}$-model with $n$ elements. For every class of MTL-chains $\mathbb{K}$, $nTAUT^{\mathcal{K}}_{\forall +} \overset{\text{def}}{=} \bigcap_{\mathcal{A} \in \mathbb{K}} nTAUT^\mathcal{A}_\forall +$.

- For every MTL-chain $\mathcal{A}$ and every $n \in \mathbb{N}^+$, with $nTAUT^\mathcal{A}_\forall$ we will denote the set of all first-order MTL\forall sentences (without the additional constants) which are valid in every $\mathcal{A}$-model with $n$ elements. For every class of MTL-chains $\mathbb{K}$, $nTAUT^{\mathcal{K}}_{\forall} \overset{\text{def}}{=} \bigcap_{\mathcal{A} \in \mathbb{K}} nTAUT^\mathcal{A}_\forall$.

**Remark 3.** In the previous definition, working with sentences does not make the results less general. Indeed, for every class $\mathbb{K}$ of MTL-chains, an open first-order formula (in the language of MTL\forall or $\mathcal{L}^{n\forall}$) is valid in all the $\mathbb{K}$-models with $n$ elements iff this holds for the universal closure of $\phi$.

We now present a translation from first-order closed formulas in FORM$_n$ into propositional formulas of MTL.

**Definition 5.** Let $c_n : FORM_n \rightarrow \mathbb{N}$ be a computable map that encodes a first-order formulas into natural numbers. Since we are working with a countable language, this can be done without any problem.

For $n \in \mathbb{N}^+$, we define by induction an interpretation $^*$ from the closed formulas of $\mathcal{L}^\forall$ into propositional formulas of MTL as follows.

- If $\phi$ is atomic, say, $\phi = P(a_1,\ldots,a_k)$, with $a_1,\ldots,a_k$ among $c_1,\ldots,c_n$, then $\phi^*_n = x^n_{c_n(P(a_1,\ldots,a_n))}$. In other terms, every closed atomic formula is mapped into a propositional variable.

- $^*$ commutes with all logical connectives.

- $(\forall x\phi(x))^*_n = \bigwedge_{i=1}^n (\phi(c_i))^*_n$.

- $(\exists x\phi(x))^*_n = \bigvee_{i=1}^n (\phi(c_i))^*_n$. 
Remark 4. • In Definition 5 we are assuming that all the constants in a closed formula $\phi \in \text{FORM}_n$ are among $c_1, \ldots, c_n$. This can be done safely, because by Definition 1 our basic language $\mathcal{L}$ contains only predicates. Hence the only constants are the ones in $\mathcal{L}_n^\forall$.

• Note that $n\text{TAUT}^A_{\forall +}$ and $n\text{TAUT}^A_{\forall}$ have the same computational complexity. Indeed a formula $\phi(c_1, \ldots, c_n)$ is valid in all models with $n$ elements, independently of the interpretation of $c_1, \ldots, c_n$, iff the universal closure of $\phi(x_1, \ldots, x_n)$ is valid in all such models.

• Note that the map $^*_n$ is computable uniformly in $n$.

Lemma 1. Let $\mathcal{A}$ be an MTL-chain, and $\mathcal{K}$ be a class of MTL-chains. For every closed formula $\phi \in \text{FORM}_n$ we have:

(a) $\phi \in n\text{TAUT}^A_{\forall +}$ iff $\phi^*_n \in \text{TAUT}(\mathcal{A})$.
(b) $\phi \in n\text{TAUT}^\mathcal{K}_{\forall +}$ iff $\phi^*_n \in \text{TAUT}(\mathcal{K})$.

Proof.

(a) Let $\mathcal{A}$ be an MTL-chain, and $\phi \in \text{FORM}_n$.

If there is a valuation $v$ in $\mathcal{A}$ such that $v(\phi^*_n) < 1$, we define a model $\mathcal{M}$ with $n$ elements on $\mathcal{A}$ as follows:

– The domain of $\mathcal{M}$ is $\{c_1, \ldots, c_n\}$.
– If $P$ is a $k$-ary predicate symbol and $a_1, \ldots, a_k$ are constants among $c_1, \ldots, c_n$, then set $||P(a_1, \ldots, a_k)||^\mathcal{A}_\mathcal{M} \overset{\text{def}}{=} v(x_{c_n}(P(a_1,\ldots,a_k)))$.

By an easy induction on $\psi$, we see that for every closed formula $\psi \in \text{FORM}_n$, one has $||\psi||^\mathcal{A}_\mathcal{M} = v(\psi^*_n)$. Hence, $||\phi||^\mathcal{A}_\mathcal{M} = v(\phi^*_n) < 1$.

Conversely, if there is an $\mathcal{A}$-model $\mathcal{M}$ with $n$ elements such that $||\phi||^\mathcal{A}_\mathcal{M} < 1$, consider a valuation $v$ of propositional variables on $\mathcal{A}$ such that for every $k$-ary predicate symbol $P$ and for all constants $a_1, \ldots, a_k$ among $c_1, \ldots, c_n$, $v(x_{c_n}(P(a_1,\ldots,a_k))) = ||P(a_1,\ldots,a_k)||^\mathcal{A}_\mathcal{M}$. By an easy induction we see that for every sentence $\psi \in \text{FORM}_n$, $||\psi||^\mathcal{A}_\mathcal{M} = v(\psi^*_n)$. Hence, $v(\phi^*_n) < 1$.

As a consequence, $\phi \notin n\text{TAUT}^A_{\forall +}$ iff $\phi^*_n \notin \text{TAUT}(\mathcal{A})$.

(b) Let $\mathcal{K}$ be a class of MTL-chains. For every $n \in \mathbb{N}^+$, and every $\phi \in \text{FORM}_n$, we have that $\phi \in n\text{TAUT}^\mathcal{K}_{\forall +}$ iff $\phi \in n\text{TAUT}^\mathcal{A}_{\forall +}$, for every $\mathcal{A} \in \mathcal{K}$ iff $\phi^*_n \in \text{TAUT}(\mathcal{A})$ for every $\mathcal{A} \in \mathcal{K}$ iff $\phi^*_n \in \text{TAUT}(\mathcal{K})$.

We are now ready to give the proof of Theorem 3.
Proof of Theorem 3.

(i) Let $\mathcal{A}$ be an MTL-chain. By Lemma 1, for every $\phi \in \text{FORM}_n$, the map $\Gamma$ defined by $\Gamma(\phi, n) \overset{\text{def}}{=} \phi_n^*$ is computable, and reduces, uniformly in $n$, $n\text{TAUT}_\forall^A$ to $\text{TAUT}(\mathcal{A})$. Clearly such equivalence holds even if $\phi$ is a formula in the language of MTL$\forall$, and hence $\Gamma(\phi, n)$ reduces, uniformly in $n$, $n\text{TAUT}_\forall^A$ to $\text{TAUT}(\mathcal{A})$.

Hence, if $\text{TAUT}(\mathcal{A})$ is decidable, then $n\text{TAUT}_\forall^A$ is decidable uniformly in $n$. Since $f\text{TAUT}_\forall^A = \bigcap_{n \in \mathbb{N}^+} n\text{TAUT}_\forall^A$, then $f\text{TAUT}_\forall^A$ is in $\Pi_0^1$.

By Lemma 1(ii), and with an argument similar to the previous one, we can show that, for every class $\mathcal{K}$ of MTL-chains, if $\text{TAUT}(\mathcal{K})$ is decidable, then $f\text{TAUT}_\forall^K$ is in $\Pi_0^1$.

(ii) Immediate by (i) and the hypothesis.

(iii) Immediate by (ii) and the hypothesis. 

We now develop the machinery needed to show Theorem 4.

We use the fact that the set of classical formulas valid in all finite models is $\Pi_1^1$-hard (Trakhtenbrot theorem). A first-order formula is said to be Boolean if its connectives are among $\neg$, $\lor$ and $\land$. Clearly, any first-order formula of classical logic is equivalent (in classical predicate logic) to a Boolean formula.

Definition 6. For each $n$-ary predicate $P$, let us define:

$$\text{PREDEF}(P) \overset{\text{def}}{=} \forall x_1 \ldots \forall x_n \neg(\neg\neg P(x_1, \ldots, x_n) \leftrightarrow \neg P(x_1, \ldots, x_n)).$$

Moreover for every formula $\phi$, $\text{PREDEF}(\phi)$ will denote the lattice conjunction of all formulas $\text{PREDEF}(P)$ such that $P$ is a predicate occurring in $\phi$.

Lemma 2. Let for every MTL-chain $\mathcal{A}$, $\sim(\mathcal{A})$ denote the set $\{\sim b : b \in A\}$. Then, for every finite $\mathcal{A}$-model $\mathbf{M}$:

(a) $\sim(\mathcal{A})$ is closed under negation and under the lattice operations.

(b) Let $P$ be a $k$-ary predicate. Then for every $a_1, \ldots, a_k \in M$, $\parallel \neg P(a_1, \ldots, a_k) \parallel^A_{\mathbf{M}} \in \sim(\mathcal{A})$.

(c) For every sentence $\psi$, $\parallel \text{PREDEF}(\psi) \parallel^A_{\mathbf{M}} \in \sim(\mathcal{A})$.

(d) For every $a \in \sim(\mathcal{A})$, if $\sim a = 0$, then $a = 1$. Moreover, $\sim$ is an order-reversing involution on $\sim(\mathcal{A})$.
PROOF. 
(a) For every \( a \in \sim(A) \) it holds that \( a \in A \), and hence \( \sim a \in \sim(A) \) by definition. Hence \( \sim(A) \) is closed under negation. Since \( A \) is a chain, \( x \sqcup y \in \{x, y\} \) and \( x \sqcap y \in \{x, y\} \). So, if \( x, y \in \sim(A) \), then \( x \sqcup y \in \sim(A) \) and \( x \sqcap y \in \sim(A) \). Hence, \( \sim(A) \) is closed under the lattice operations.

(b) Since \( \sim(A) \) is closed under meet, it suffices to prove that, for every \( x \in A \), \( \sim \sim x \Rightarrow \sim x \in \sim(A) \) and \( \sim x \Rightarrow \sim \sim x \in \sim(A) \). Now using the residuation property, we see that \( \sim \sim x \Rightarrow \sim x = \sim(\sim x \ast x) \), and that \( \sim x \Rightarrow \sim \sim x = \sim(\sim x \ast \sim x) \). Clearly, both elements are negations and hence they belong to \( \sim(A) \).

(c) Since \( \sim(A) \) is closed under finite meets and \( M \) is finite, it suffices to prove that for every \( k \)-ary predicate \( P \) and for every \( a_1, \ldots, a_k \in M \), \( \| \neg \neg P(a_1, \ldots, a_k) \|_M^A \in \sim(A) \). But this is evident, because the formula \( \neg(\neg \neg P(a_1, \ldots, a_k) \leftrightarrow \neg P(a_1, \ldots, a_k)) \) begins with \( \neg \).

(d) If \( a \in \sim(A) \), say, \( a = \sim b \), then \( \sim a = \sim \sim b \). Since \( b \leq \sim b \), if \( \sim a = 0 \), then \( \sim \sim b = 0 \) and \( b = 0 \). Hence, \( a = \sim b = 1 \). That \( \sim \) is an order reversing involution on \( \sim(A) \) follows from the equality \( \sim \sim \sim x = \sim x \).

LEMMA 3. Let \( P \) be a \( k \)-ary predicate. In any first-order finite model \( M \) on an \( MTL \)-chain \( A \), \( PREDEF(P) \) has truth value 0 iff there are \( a_1, \ldots, a_k \in M \) such that \( \| \neg \neg P(a_1, \ldots, a_k) \|_M^A = \| \neg P(a_1, \ldots, a_k) \|_M^A \).

PROOF. It is clear that if there are \( a_1, \ldots, a_k \) such that \( \| \neg \neg P(a_1, \ldots, a_k) \|_M^A = \| \neg P(a_1, \ldots, a_k) \|_M^A \), then \( \| PREDEF(P) \|_M^A = 0 \).

Conversely, if \( \| PREDEF(P) \|_M^A = 0 \), then by the finiteness of \( M \), there are \( a_1, \ldots, a_k \) such that \( \| \neg(\neg \neg P(a_1, \ldots, a_k) \leftrightarrow \neg P(a_1, \ldots, a_k)) \|_M^A = 0 \).

Hence, by Lemma 2, \( \| \neg \neg P(a_1, \ldots, a_k) \|_M^A = \| \neg P(a_1, \ldots, a_k) \|_M^A \). This concludes the proof.

For every Boolean formula \( \phi \), let \( \phi^{\neg\neg} \) be the formula obtained from \( \phi \) replacing every atomic subformula of \( \phi \) by its double negation. Given an \( MTL \)-chain \( A \), and a finite \( A \)-model \( M \), we say that a sentence \( \psi \) is in \( (A, M)^+ \) if \( \| \psi^{\neg\neg} \|_M^A \in A^+ \) and that \( \psi \) is in \( (A, M)^- \) if \( \| \psi^{\neg\neg} \|_M^A \in A^- \).

DEFINITION 7. Let \( A \) be an \( MTL \)-chain. For every finite \( A \)-model \( M \), we define a classical model \( M' \) (over the two-element Boolean algebra 2) as follows:

- The domain of \( M' \) coincides with the domain of \( M \).
• For every \( k \)-ary predicate \( P \) occurring in \( \phi \) and for every \( a_1, \ldots, a_k \in M' \), we set \( M' \models P(a_1, \ldots, a_k) \) iff \( P(a_1, \ldots, a_k) \in (A, M)^+ \).

**Lemma 4.** Let \( \phi \) be a boolean first-order closed formula, \( A \) be an MTL-chain, and let \( M \) be a finite \( A \)-model such that \( \| \text{PREDEF}(\phi) \|_M^A > 0 \).

For every subformula \( \psi \) of \( \phi \), either \( \psi \in (A, M)^+ \) or \( \psi \in (A, M)^- \). Moreover, \( \psi \in (A, M)^+ \) iff \( M' \models \psi \).

**Proof.** First note that since \( \sim (A) \) contains all negated elements and is closed under finite joins and meets, and since, by the finiteness of \( M \), every universal quantifier is interpreted as a finite meet and every existential quantifier is interpreted as a finite join, \( \| \psi \\sim \|_M^A \in \sim (A) \) for every subformula \( \psi \) of \( \phi \).

We now prove Lemma 4 by induction on \( \psi \).

If \( \psi \) is atomic, the first claim of Lemma 4 follows from Lemma 3, and the second claim follows from the definition of \( M' \).

If \( \psi = \neg \gamma \), then by the induction hypothesis, either \( \| \gamma \\sim \|_M^A < \| \neg \gamma \\sim \|_M^A \) and \( M' \models \neg \gamma \), or \( \| \neg \gamma \\sim \|_M^A < \| \gamma \\sim \|_M^A \) and \( M' \models \gamma \). Now since \( \sim \) is an order reversing involution on \( \sim (A) \), we obtain:

\[
\| \neg \psi \\sim \|_M^A < \| \psi \\sim \|_M^A \text{ iff } \| \gamma \\sim \|_M^A < \| \neg \gamma \\sim \|_M^A \text{ iff } M' \models \neg \gamma \text{ iff } M' \models \psi.
\]

This completes the induction step corresponding to negation.

If \( \psi = \gamma \land \delta \), then \( \psi \\sim = \gamma \\sim \land \delta \\sim \), and \( \| \neg \psi \\sim \|_M^A < \| \psi \\sim \|_M^A \text{ iff } \| \gamma \\sim \|_M^A < \| \gamma \\sim \|_M^A \text{ and } \| \neg \delta \\sim \|_M^A < \| \delta \\sim \|_M^A \text{ iff } \text{(by the induction hypothesis) } M' \models \gamma \text{ and } M' \models \delta \text{ iff } M' \models \psi \).

The induction step corresponding to \( \lor \) is similar. Moreover, due to the finiteness of \( M \), quantifiers reduce to finite joins and meets and hence, the induction steps corresponding to quantifiers are similar to the induction steps corresponding to join and meet.

This completes the proof.

**Lemma 5.** Let \( A \) be any non-trivial MTL-chain, \( \mathbb{K} \) be a non-empty class of non-trivial MTL-chains, and let \( \phi \) be a Boolean first-order closed formula. The following are equivalent.

\( (1) \) \( \phi \in f\text{TAUT}^2_\psi \).

\( (2) \) \( \neg \text{PREDEF}(\phi) \lor (\neg \phi \\sim \rightarrow \phi \\sim) \in f\text{TAUT}^A_\psi \).

\( (3) \) \( \neg \text{PREDEF}(\phi) \lor (\neg \phi \\sim \rightarrow \phi \\sim) \in f\text{TAUT}^\mathbb{K}_\psi \).

**Proof.** Suppose \( \phi \) fails in some finite classical model, i.e., in some finite model \( M \) over the two-element Boolean algebra \( 2 \). Then since the model is crisp, \( \| \neg \text{PREDEF}(\phi) \|_M^2 = 0 \), and moreover, \( \| \neg \phi \\sim \rightarrow \phi \\sim \|_M^2 = \| \phi \|_M^2 = 0 \).
Since $2$ is a subalgebra of $\mathcal{A}$, $\neg \text{PREDEF}(\phi) \lor (\neg \phi \rightarrow \phi)$ is not valid in all finite $\mathcal{A}$-models. This shows that (2) implies (1).

Conversely, suppose $\neg \text{PREDEF}(\phi) \lor (\neg \phi \rightarrow \phi)$ fails in some finite $\mathcal{A}$-model $\mathcal{M}$. Then $\| \text{PREDEF}(\phi) \|^\mathcal{A}_\mathcal{M} > 0$, and $\| \neg \phi \rightarrow \phi \|^\mathcal{M} < 1$. As a consequence, we have that $\phi \in (\mathcal{A}, \mathcal{M})^-$: then, by Lemma 4, $\mathcal{M'} |\models \neg \phi$ and $\phi \notin \text{fTAUT}_\mathcal{V}^2$. This concludes the proof of the equivalence of (1) and (2).

We finally arrive to the proof of Theorem 4.

**Proof of Theorem 4.**

(i) By Lemma 5 we have that, for every (non-trivial) MTL-chain $\mathcal{A}$, and every class $\mathcal{K}$ of MTL-chains, $\text{fTAUT}_\mathcal{V}^2$ is recursively reducible to, respectively, $\text{fTAUT}_\mathcal{A}^2$ and $\text{fTAUT}_\mathcal{K}^2$. By Theorem 1 $\text{fTAUT}_\mathcal{V}^2$ is $\Pi^0_1$-complete, and hence both $\text{fTAUT}_\mathcal{A}^2$ and $\text{fTAUT}_\mathcal{K}^2$ are $\Pi^0_1$-hard.

(ii) An immediate consequence of (i).

(iii) An immediate consequence of (ii).

4. **Decidability and $\Pi^0_1$-Completeness**

By Theorem 3 we know that, for every axiomatic extension $L$ of MTL, if $L$ is decidable, then $\text{fTAUT}(L\forall)$ is in $\Pi^0_1$. One can wonder whether the assumption that $L$ is decidable is not only sufficient, but also necessary, in order that $\text{fTAUT}(L\forall)$ is in $\Pi^0_1$.

We start from the following lemma.

**Lemma 6.** If a logic $L$ (thought of as a set of formulas closed under deduction and under substitution) is not in $\Pi^0_1$, then $\text{fTAUT}(L\forall)$ is not in $\Pi^0_1$.

**Proof.** We may interpret propositional variables of $L$ as 0-ary predicates of $L\forall$. Hence, formulas of $L$ are also formulas of $L\forall$. If $\phi$ is a formula of $L$, then $L \vdash \phi$ iff $\phi$ is valid in every $L$-chain $\mathcal{A}$. Note that the truth value of $\phi$ in any $\mathcal{A}$-model $\mathcal{M}$ is independent of $\mathcal{M}$, because $\phi$ is a propositional formula. Hence, $\phi$ is valid in $\mathcal{A}$ iff it is valid in every $\mathcal{A}$-model iff it is valid in every finite $\mathcal{A}$-model, and $L \vdash \phi$ iff $\phi$ is valid in every $L$-chain $\mathcal{A}$ iff it is valid in every finite $\mathcal{A}$-model for every $L$-chain $\mathcal{A}$ iff $\phi \in \text{fTAUT}(L\forall)$. It follows that if $L \notin \Pi^0_1$, then $\text{fTAUT}(L\forall) \notin \Pi^0_1$. ■
From this lemma the next result follows:

**Theorem 5.** Let \( L \) be a recursively axiomatizable consistent propositional logic extending MTL. The following are equivalent.

1. \( L \) is decidable.
2. \( \text{fTAUT}(L\forall) \) is in \( \Pi_1^0 \).
3. \( \text{fTAUT}(L\forall) \) is \( \Pi_0^0 \)-complete.

**Proof.** (1) \( \Rightarrow \) (2) follows from Theorem 3. Moreover, since \( \text{fTAUT}(L\forall) \) is \( \Pi_1^0 \)-hard, it is in \( \Pi_1^0 \) iff it is \( \Pi_0^0 \)-complete. Hence, (2) and (3) are equivalent. In order to prove that (2) implies (1), we argue contrapositively. Suppose that \( L \) is not decidable. Then being recursively axiomatizable, \( L \) is in \( \Sigma_1^0 \), and since it is not decidable, it is not in \( \Pi_1^0 \). Hence, by Lemma 6, \( \text{fTAUT}(L\forall) \) is not in \( \Pi_1^0 \).

We have seen that the decidability of \( L \) is a sufficient condition in order that \( \text{fTAUT}(L\forall) \) is \( \Pi_0^0 \)-complete. Moreover when \( L \) is recursively axiomatizable, this condition is also necessary. We wonder if it is necessary even when \( L \) is not recursively axiomatized. Although we cannot provide a complete answer to this question, we will see that the answer is negative if we extend the language of MTL by adding new constants.

Let \( c_0, \ldots, c_n, \ldots \) be propositional constants, and let \( X \) be a \( \Pi_1^0 \)-complete set. We can represent \( X \) as \( \{ n \in \mathbb{N} : \forall m R(n, m) \} \) for some recursive binary relation \( R \). We can suppose, without loss of generality, that for all \( m < m' \) and for all \( n \), if \( R(n, m') \) then \( R(n, m) \). Indeed, if \( R \) does not satisfy this condition, we may replace it by \( R'(n, m) \) \( \overset{\text{def}}{=} \forall r \leq m R(n, r) \). Then \( R' \) satisfies the above condition and \( X = \{ n \in \mathbb{N} : \forall m R'(n, m) \} \).

Let \( \text{MTL}_C \) be the theory axiomatized over MTL by \( \{ c_n : n \in X \} \) and let as usual \( \text{fTAUT}(\text{MTL}_C\forall) \) be the set of first-order formulas which are valid in every finite model over any \( \text{MTL}_C \)-chain, that is, let \( \text{fTAUT}(\text{MTL}_C\forall) \) be the set of all first-order sentences \( \phi \) such that for every \( \text{MTL}_C \)-chain \( A \) and for every finite \( A \)-model \( M \), \( \| \phi \|_M^A = 1 \).

The main result of the section is the following.

**Theorem 6.** (1) \( \text{fTAUT}(\text{MTL}_C\forall) \) is \( \Pi_0^0 \)-complete.

(2) \( \text{MTL}_C \) is not decidable.

To develop the proof, we need some preliminary results.

Let for \( m \in \mathbb{N} \), \( X_m = \{ n \in \mathbb{N} : R(n, m) \} \). Then \( X_m \) is recursive uniformly in \( m \), and \( X = \bigcap_{m \in \mathbb{N}} X_m \). It follows that if \( m < m' \), then \( X_{m'} \subseteq X_m \); indeed, if \( R(n, m') \) then \( R(n, m) \). As a consequence, if \( n \in X_{m'} \), then \( n \in X_m \).
Let $\text{MTL}_m$ be the theory axiomatized over MTL (enriched by the constants $c_0, \ldots, c_n, \ldots$) by $\{c_n : n \in X_m\}$, and let for $k, m \in \mathbb{N}$, $k\text{TAUT}(\text{MTL}_m \forall)$ be the set of all sentences valid in all models with cardinality $k$ over the class of $\text{MTL}_m$-chains.

Given a propositional formula $\phi$, let $\phi'$ denote the formula obtained from $\phi$ by replacing each constant $c_i$ in $\phi$ such that $i \in X_m$ by 1 and the remaining constants by propositional variables not in $\phi$.

**Lemma 7.** (i) $\text{MTL}_m \vdash \phi$ iff $\text{MTL} \vdash \phi'$.
(ii) $\text{MTL}_m$ is decidable uniformly in $m$.

**Proof.**

(i) Let $\phi''$ be the formula obtained from $\phi$ by replacing every constant $c_i$ with $i \notin X_m$, by a propositional variable not in $\phi$. If $\text{MTL}_m \vdash \phi$, then replacing in any proof of $\phi$ each constant $c_i$ with $i \notin X_m$ by a new propositional variable we obtain a proof in $\text{MTL}_m$ of $\phi''$, (an easy induction on the length of the proof). Now replacing in $\phi''$ each constant $c_i$ with $i \in X_m$ by 1, since $\text{MTL}_m \vdash c_i \leftrightarrow 1$ we obtain a proof of $\phi'$ in $\text{MTL}_m$. But now $\phi'$ does not contain any constant, and since $\text{MTL}_m$ is conservative over MTL (every model of MTL can be extended to a model of $\text{MTL}_m$ by interpreting each $c_i$ into 1), we conclude that $\text{MTL} \vdash \phi'$.

Conversely, if $\text{MTL} \vdash \phi'$, then $\text{MTL}_m \vdash \phi''$, because $\text{MTL}_m$ is closed under substitutions.

Moreover for each $i \in X_m$, $\text{MTL}_m \vdash c_i \leftrightarrow 1$, and hence $\text{MTL}_m \vdash \phi$.

(ii) Since MTL is decidable, by (i) we have that $\text{MTL}_m$ is also decidable.

We can finally present the proof of Theorem 6.

**Proof of Theorem 6.**

(1) The proof of $\Pi^0_1$-hardness is almost a repetition of the proof of Theorem 4. We now prove that $f\text{TAUT}(\text{MTL}_C \forall)$ is in $\Pi^0_1$. Lemma 7 and the argument used in the proof of Theorem 3 give that for every $k, m$, $k\text{TAUT}(\text{MTL}_m \forall)$ is decidable, uniformly in $k, m$. But now for every first-order formula $\psi$ we have:

$$\psi \in f\text{TAUT}(\text{MTL}_C \forall) \iff \forall k \forall m (\psi \in k\text{TAUT}(\text{MTL}_m \forall)).$$

Hence, $f\text{TAUT}(\text{MTL}_C \forall)$ is in $\Pi^0_1$. 
Suppose by contradiction that $\text{MTL}_C$ is decidable. Then \( \{c_i : \text{MTL}_C \vdash c_i\} \) would be also decidable; however this is a contradiction, since an easy check shows that \( \{c_i : \text{MTL}_C \vdash c_i\} \) is recursively isomorphic to $X$, and this last one is $\Pi^0_1$-complete.

5. Axiomatic Extensions of MTL with Baaz Operator $\Delta$

We conclude the paper by analyzing the axiomatic extensions of MTL expanded with the Baaz–Monteiro operator $\Delta$, firstly introduced in [1, 16] (see [7, 8] for other details). For every axiomatic extension $L$ of MTL, we denote with $L_\Delta$ its expansion with an operator $\Delta$ satisfying the following axioms,

\begin{align*}
(\Delta1) & \quad \Delta(\varphi) \lor \neg \Delta(\varphi). \\
(\Delta2) & \quad \Delta(\varphi \lor \psi) \rightarrow ((\Delta(\varphi) \lor \Delta(\psi))). \\
(\Delta3) & \quad \Delta(\varphi) \rightarrow \varphi. \\
(\Delta4) & \quad \Delta(\varphi) \rightarrow \Delta(\Delta(\varphi)). \\
(\Delta5) & \quad \Delta(\varphi \rightarrow \psi) \rightarrow (\Delta(\varphi) \rightarrow \Delta(\psi)).
\end{align*}

and the following additional inference rule: \( \frac{\varphi \Delta}{\Delta \varphi} \).

An $\text{MTL}_\Delta$-chain is an MTL-chain expanded with an operation $\delta$ such that, for every element $x$, $\delta(x) = 1$ if $x = 1$, whilst $\delta(x) = 0$ if $x < 1$. An $\text{MTL}_\Delta$-algebra is a subdirect product of $\text{MTL}_\Delta$-chains: the class of $\text{MTL}_\Delta$-algebras forms an algebraic variety, generated by all the $\text{MTL}_\Delta$-chains.

Let $L$ be an axiomatic extension of $\text{MTL}_\Delta$: as pointed out in [7], $L$ is complete w.r.t. the class of $L$-chains.

**Definition 8.** Let $A$ be an $\text{MTL}_\Delta$-chain. Then $f\text{TAUT}^A_\forall$ is the set of formulas which are valid in all finite models over $A$. If $K$ is a class of $\text{MTL}_\Delta$-chains, then $f\text{TAUT}^K_\forall \overset{\text{def}}{=} \bigcap_{A \in K} f\text{TAUT}^A_\forall$. Let $L$ be an axiomatic extension of $\text{MTL}_\Delta$. We define $f\text{TAUT}(L_\forall) \overset{\text{def}}{=} \bigcap_{A \in C} f\text{TAUT}^A_\forall$, where $C$ is the class of $L$-chains. Finally, $\text{TAUT}(A)$ is the set of propositional formulas which are valid in $A$, $\text{TAUT}(K) \overset{\text{def}}{=} \bigcap_{A \in K} \text{TAUT}(A)$ and $\text{TAUT}_L \overset{\text{def}}{=} \bigcap_{A \in C} \text{TAUT}(A)$.

We now state the main theorem of the section.

**Theorem 7.** (i) If $A$ is any $\text{MTL}_\Delta$-chain and $\text{TAUT}(A)$ is decidable, then $f\text{TAUT}^A_\forall$ is $\Pi^0_1$-complete. More in general, if $K$ is any class of $\text{MTL}_\Delta$-chains and $\text{TAUT}(K)$ is decidable, then $f\text{TAUT}^K_\forall$ is $\Pi^0_1$-complete.
(ii) Let $L$ be an axiomatic extension of $\text{MTL}_\Delta$. If $L$ is decidable and is sound and complete with respect to a class of $L$-chains $\mathbb{K}$, that is, if $\text{TAUT}_L = \text{TAUT}(\mathbb{K})$, then $f\text{TAUT}_\forall^\mathbb{K}$ is $\Pi^0_1$-complete.

(iii) For every decidable axiomatic extension $L$ of $\text{MTL}_\Delta$, the set $f\text{TAUT}(L\forall)$ is $\Pi^0_1$-complete.

**Proof.**

(i) Let $\mathbb{K}$ be a class of $\text{MTL}_\Delta$-chains. For every Boolean formula $\phi$, if $\phi^\Delta$ denotes the formula obtained by replacing each atomic subformula $\gamma$ by $\Delta(\gamma)$, then an easy check shows that $\phi \in f\text{TAUT}_\forall^2$ iff $\phi^\Delta \in f\text{TAUT}_\forall^\mathbb{K}$. Hence $f\text{TAUT}_\forall^\mathbb{K}$ is $\Pi^0_1$-hard.

Finally, imitating the proof of Theorem 3, we can prove that $f\text{TAUT}_\forall^\mathbb{K}$ is in $\Pi^0_1$.

(ii) Immediate from (i).

(iii) Immediate from (ii).

6. Conclusions and Open Problems

In this article we have generalized the well known Trakhtenbrot theorem to the first-order axiomatic extensions of $\text{MTL}$. An immediate consequence of Theorems 3 and 4 is that, for every axiomatic extension $L$ of $\text{MTL}$, $L\forall$ is not complete w.r.t. any class of $L$-chains, if we restrict to the finite models.

However, we have a negative result also from another perspective: indeed, let us consider a first-order logic as the set of first-order tautologies associated to some class of $\text{MTL}$-chains (this is an approach used, for example, for the cases of Gödel and Nilpotent minimum logics, in [2–4]). By Theorems 3 and 4 we have that for every class $\mathbb{K}$ of $\text{MTL}$-chains, $f\text{TAUT}_\forall^\mathbb{K}$ is $\Pi^0_1$-complete, and hence there is no recursively axiomatizable first-order logic $L$ which is complete with respect to the finite models of a class of $\text{MTL}$-chains.

We conclude by mentioning an open problem, pointed out in Sect. 4.

**Problem 1.** Let $L$ be a axiomatic extension of $\text{MTL}$ which is not recursively axiomatizable, and such that $f\text{TAUT}(L\forall)$ is $\Pi^0_1$-complete. Is $\text{TAUT}_L$ decidable?

By Theorem 6 we know that the answer is negative if $L$ is an expansion of $\text{MTL}$, and $L$ is not recursively axiomatizable. However the problem for the axiomatic extensions of $\text{MTL}$ which are not recursively axiomatizable remains open.
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