Accelerated Primal-Dual Proximal Block Coordinate Updating Methods for Constrained Convex Optimization

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Abstract

Block Coordinate Update (BCU) methods enjoy low per-update computational complexity because every time only one or a few block variables would need to be updated among possibly a large number of blocks. They are also easily parallelized and thus have been particularly popular for solving problems involving large-scale dataset and/or variables. In this paper, we propose a primal-dual BCU method for solving linearly constrained convex program in multi-block variables. The method is an accelerated version of a primal-dual algorithm proposed by the authors, which applies randomization in selecting block variables to update and establishes an $O(1/t)$ convergence rate under weak convexity assumption. We show that the rate can be accelerated to $O(1/t^2)$ if the objective is strongly convex. In addition, if one block variable is independent of the others in the objective, we then show that the algorithm can be modified to achieve a linear rate of convergence. The numerical experiments show that the accelerated method performs stably with a single set of parameters while the original method needs to tune the parameters for different datasets in order to achieve a comparable level of performance.

Keywords: primal-dual method, block coordinate update, alternating direction method of multipliers (ADMM), accelerated first-order method.

Mathematics Subject Classification: 90C25, 95C06, 68W20.

1 Introduction

Motivated by the need to solve large-scale optimization problems and increasing capabilities in parallel computing, block coordinate update (BCU) methods have become particularly popular in recent years due to their low per-update computational complexity, low memory requirements, and their potentials in a distributive computing environment. In the context of optimization, BCU first

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appeared in the form of block coordinate descent (BCD) type of algorithms which can be applied
to solve unconstrained smooth problems or those with separable nonsmooth terms in the objective
(possibly with separable constraints). More recently, it has been developed for solving problems
with nonseparable nonsmooth terms and/or constraint in a primal-dual framework.

In this paper, we consider the following linearly constrained multi-block structured optimization
model:

$$\min_x f(x) + \sum_{i=1}^M g_i(x_i), \quad \text{s.t.} \quad \sum_{i=1}^M A_i x_i = b, \quad (1)$$

where $x$ is partitioned into disjoint blocks $(x_1, x_2, \ldots, x_M)$, $f$ is a smooth convex function with
Lipschitz continuous gradient, and each $g_i$ is proper closed convex and possibly non-differentiable.
Note that $g_i$ can include an indicator function of a convex set $X_i$, and thus (1) can implicitly include
certain separable block constraints in addition to the nonseparable linear constraint.

Many applications arising in statistical and machine learning, image processing, and finance can be
formulated in the form of (1) including the basis pursuit [7], constrained regression [23], support
vector machine in its dual form [10], portfolio optimization [28], just to name a few.

Towards finding a solution for (1), we will first present an accelerated proximal Jacobian alternating
direction method of multipliers (Algorithm 1), and then we generalize it to an accelerated random-
ized primal-dual block coordinate update method (Algorithm 2). Assuming strong convexity on the
objective function, we will establish $O(1/t^2)$ convergence rate results of the proposed algorithms by
adaptively setting the parameters, where $t$ is the total number of iterations. In addition, if further
assuming smoothness and the full-rankness we then obtain linear convergence of a modified method
(Algorithm 3).

1.1 Related methods

Our algorithms are closely related to randomized coordinate descent methods, primal-dual coor-
dinate update methods, and accelerated primal-dual methods. In this subsection, let us briefly
review three classes of methods and discuss their relations to our algorithms.

**Randomized coordinate descent methods**

In the absence of linear constraint, Algorithm 2 specializes to randomized coordinate descent (RCD),
which was first proposed in [31] for smooth problems and later generalized in [27,38] to nonsmooth
problems. It was shown that RCD converges sublinearly with rate $O(1/t)$, which can be accelerated
to $O(1/t^2)$ for weakly convex problems and achieves a linear rate for strongly convex problems. By
choosing multiple block variables at each iteration, [37] proposed to parallelize the RCD method
and showed the same convergence results for parallelized RCD. This is similar to setting $m > 1$ in
Algorithm 2, allowing parallel updates on the selected $x$-blocks.
Primal-dual coordinate update methods

In the presence of linear constraints, coordinate descent methods may fail to converge to a solution of the problem because fixing all but one block, the selected block variable may be uniquely determined by the linear constraint. To perform coordinate update to the linearly constrained problem (1), one effective approach is to update both primal and dual variables. Under this framework, the alternating direction method of multipliers (ADMM) is one popular choice. Originally, ADMM [14,17] was proposed for solving two-block structured problems with separable objective (by setting \( f = 0 \) and \( M = 2 \) in (1)), for which its convergence and also convergence rate have been well-established (see e.g. [2,13,22,29]). However, directly extending ADMM to the multi-block setting such as (1) may fail to converge; see [6] for a divergence example of the ADMM even for solving a linear system of equations. Lots of efforts have been made to establish the convergence of multi-block ADMM under stronger assumptions (see e.g. [4,6,16,25,26]) such as strong convexity or orthogonality conditions on the linear constraint. Without additional assumptions, modification is necessary for the ADMM applied to multi-block problems to be convergent; see [12,19,20,40] for example. Very recently, [15] proposed a randomized primal-dual coordinate (RPDC) update method. Applied to (1), RPDC is a special case of Algorithm 2 with fixed parameters. It was shown that RPDC converges with rate \( O(1/t) \) under weak convexity assumption. More general than solving an optimization problem, primal-dual coordinate (PDC) update methods have also appeared in solving fixed-point or monotone inclusion problems [9,34–36]. However, for these problems, the PDC methods are only shown to converge but no convergence rate estimates are known unless additional assumptions are made such as the strong monotonicity condition.

Accelerated primal-dual methods

It is possible to accelerate the rate of convergence from \( O(1/t) \) to \( O(1/t^2) \) for gradient type methods. The first acceleration result was shown by Nesterov [30] for solving smooth unconstrained problems. The technique has been generalized to accelerate gradient-type methods on possibly nonsmooth convex programs [1,32]. Primal-dual methods on solving linearly constrained problems can also be accelerated by similar techniques. Under weak convexity assumption, the augmented Lagrangian method (ALM) is accelerated in [21] from \( O(1/t) \) convergence rate to \( O(1/t^2) \) by using a similar technique as that in [1] to the multiplier update, and [39] accelerates the linearized ALM using a technique similar to that in [32]. Assuming strong convexity on the objective, [18] accelerates the ADMM method, and the assumption is weakened in [39] to assuming the strong convexity for one component of the objective function. On solving bilinear saddle-point problems, various primal-dual methods can be accelerated if either primal or dual problem is strongly convex [3,5,11]. Without strong convexity, partial acceleration is still possible in terms of the rate depending on some other quantities; see e.g. [8,33].
1.2 Contributions of this paper

We accelerate the proximal Jacobian ADMM [12] and also generalize it to an accelerated primal-dual coordinate updating method for linearly constrained multi-block structured convex program, where in the objective there is a nonseparable smooth function. With parameters fixed during all iterations, the generalized method reduces to that in [15] and enjoys $O(1/t)$ convergence rate under mere weak convexity assumption. By adaptively setting the parameters at different iterations, we show that the accelerated method has $O(1/t^2)$ convergence rate if the objective is strongly convex. In addition, if there is one block variable that is independent of all others in the objective (but coupled in the linear constraint) and also the corresponding component function is smooth, we modify the algorithm by treating that independent variable in a different way and establish a linear convergence result. Numerically, we test the accelerated method on quadratic programming and compare it to the (nonaccelerated) RPDC method in [15]. The results demonstrate that the accelerated method performs efficiently and stably with the parameters automatically set in accordance of the analysis, while the RPDC method needs to tune its parameters for different data in order to have a comparable performance.

1.3 Nomenclature and basic facts

Notations. We let $x_S$ denote the subvector of $x$ with blocks indexed by $S$. Namely, if $S = \{i_1, \ldots, i_m\}$, then $x_S = (x_{i_1}, \ldots, x_{i_m})$. Similarly, $A_S$ denotes the submatrix of $A$ with columns indexed by $S$, and $g_S$ denotes the sum of component functions indicated by $S$. We reserve $I$ for the identity matrix and use $\| \cdot \|$ for Euclidean norm. Given a symmetric positive semidefinite (PSD) matrix $W$, for any vector $v$ of appropriate size, we define $\|v\|_W^2 = v^\top W v$. Also, we denote

$$F(x) = f(x) + g(x), \quad \Phi(\hat{x}, x, \lambda) = F(\hat{x}) - F(x) - \langle \lambda, A\hat{x} - b \rangle.$$  \hfill (2)

Preparations. Since there are only linear constraints in (1), a point $x^*$ is a solution to (1) if and only if the Karush-Kuhn-Tucker (KKT) conditions hold, i.e., there exists a $\lambda^*$ such that

$$0 \in \partial F(x^*) - A^\top \lambda^*, \hfill (3a)$$
$$Ax^* - b = 0. \hfill (3b)$$

Together with the convexity of $F$, (3) implies

$$\Phi(x, x^*, \lambda^*) \geq 0, \forall x.$$ \hfill (4)

We will use the following lemmas as basic facts, the proofs of which can be found in [15].

Lemma 1.1 For any vectors $u, v$ and symmetric PSD matrix $W$ of appropriate sizes, it holds that

$$u^\top W v = \frac{1}{2} \left[ \|u\|_W^2 - \|u - v\|_W^2 + \|v\|_W^2 \right]. \hfill (5)$$
Lemma 1.2 Given a function $\phi$, for a given $x$ and a random vector $\hat{x}$, if for any $\lambda$ (that may depend on $\hat{x}$) it holds
\[ \mathbb{E}\Phi(\hat{x}, x, \lambda) \leq \mathbb{E}\phi(\lambda), \] then for any $\gamma > 0$, we have
\[ \mathbb{E}\left[F(\hat{x}) - F(x) + \gamma\|A\hat{x} - b\|\right] \leq \sup_{\|\lambda\| \leq \gamma} \phi(\lambda). \]

Lemma 1.3 Suppose $\mathbb{E}\left[F(\hat{x}) - F(x^*) + \gamma\|A\hat{x} - b\|\right] \leq \epsilon$. Then,
\[ \mathbb{E}\|\hat{x} - b\| \leq \frac{\epsilon}{\gamma - \|\lambda^*\|}, \text{ and } -\frac{\epsilon\|\lambda^*\|}{\gamma - \|\lambda^*\|} \leq \mathbb{E}[F(\hat{x}) - F(x^*)] \leq \epsilon, \]
where $(x^*, \lambda^*)$ satisfies the optimality conditions in (3), and we assume $\|\lambda^*\| < \gamma$.

Outline. The rest of the paper is organized as follows. Section 2 presents the accelerated proximal Jacobian ADMM and its convergence results. In section 3, we propose an accelerated primal-dual block coordinate update method with convergence analysis. Section 4 assumes more structure on the problem (1) and modifies the algorithm in section 3 to have linear convergence. Numerical results are provided in section 5. Finally, section 6 concludes the paper.

2 Accelerated proximal Jacobian ADMM

In this section, we propose an accelerated proximal Jacobian ADMM for solving (1). At each iteration, the algorithm updates all $M$ block variables in parallel by minimizing a linearized proximal approximation of the augmented Lagrangian function, and then it renews the multiplier. Specifically, it iteratively performs the following updates:
\[ x_{i}^{k+1} = \arg\min_{x_{i}} \left\{ \nabla_{i} f(x_{i})^{k} - A_{i}^{\top} (\lambda^{k} - \beta_{k} r_{k})^{k}, x_{i} \right\} + g_{i}(x_{i}) + \frac{1}{2} \|x_{i} - x_{i}^{k}\|_{P_{i}}^{k}, i = 1, \ldots, M, \] (7a)
\[ \lambda^{k+1} = \lambda^{k} - \rho_{k} r_{k+1}^{k}, \] (7b)
where $\beta_{k}$ and $\rho_{k}$ are scalar parameters, $P^{k} = \text{blkdiag}(P^{k}_{1}, \ldots, P^{k}_{M})$ is a block diagonal matrix, and $r_{k}^{k} = A\hat{x}^{k} - b$ denotes the residual. Note that (7a) consists of $M$ independent subproblems, and they can be solved in parallel.

Algorithm 1 summarizes the proposed method. It reduces to the proximal Jacobian ADMM in [12] if $\beta_{k}, \rho_{k}$ and $P^{k}$ are fixed for all $k$ and there is no nonseparable function $f$. We will show that
adapting parameters to the iterations can accelerate the convergence of the algorithm.

Algorithm 1: Accelerated proximal Jacobian ADMM for (1)

1. Initialization: choose $x^1$, set $\lambda^1 = 0$, and let $r^1 = Ax^1 - b$
2. for $k = 1, 2, \ldots$ do
   3. Choose parameters $\beta_k, \rho_k$ and a block diagonal matrix $P^k$
   4. Let $x^{k+1} \leftarrow (7a)$ and $\lambda^{k+1} \leftarrow (7b)$ with $r^{k+1} = Ax^{k+1} - b$.
   5. if a certain stopping criterion satisfied then
      6. Return $(x^{k+1}, \lambda^{k+1})$.

2.1 Technical assumptions

Throughout the analysis in this section, we make the following assumptions.

Assumption 1 There exists $(x^*, \lambda^*)$ satisfying the KKT conditions in (3).

Assumption 2 $\nabla f$ is Lipschitz continuous with modulus $L_f$.

Assumption 3 The functions $f$ and $g$ are convex with moduli $\mu_f \geq 0$ and $\mu_g \geq 0$ respectively, and $\mu = \mu_f + \mu_g > 0$.

The first two assumptions are standard, and the third one is for showing convergence rate of $O(1/t^2)$, where $t$ is the number of iterations. With only weak convexity, Algorithm 1 can be shown to converge in the rate $O(1/t)$ with parameters fixed for all iterations, and the order $1/t$ is optimal as shown in the very recent work [24].

2.2 Convergence results

In this subsection, we show the $O(1/t^2)$ convergence rate result of Algorithm 1. First, we establish a result of running one iteration of Algorithm 1.

Lemma 2.1 (One-iteration analysis) Under Assumptions 2 and 3, let $\{(x^k, \lambda^k)\}$ be the sequence generated from Algorithm 1. Then for any $k$ and any $(x, \lambda)$ such that $Ax = b$, it holds
Then, for any $(x, \lambda)$, there exists a number $k_0$ such that for all $k \geq 1$,

$$
\frac{k + k_0 + 1}{\rho_k} \leq \frac{k + k_0}{\rho_{k-1}},
$$

and there exists a number $k_0$ such that for all $k \geq 1$,

$$
(k + k_0 + 1)(P^k - \beta_k A^\top A + \mu_f I) \preceq (k + k_0)(P^{k-1} - \beta_{k-1} A^\top A + \mu_g I).
$$

Then, for any $(x, \lambda)$ satisfying $Ax = b$, we have

$$
\sum_{k=1}^{t} (k + k_0 + 1)\Phi(x^{k+1}, x, \lambda) + \sum_{k=1}^{t} \frac{k + k_0 + 1}{2} ||x^{k+1}||^2 \\
+ \frac{t + k_0 + 1}{2} ||x^{t+1} - x||_{P^t - \beta_t A^\top A + \mu_f I}^2 \\
\leq \frac{k_0 + 2}{2\rho_1} ||\lambda - \lambda^1||^2 + \frac{k_0 + 2}{2} ||x^1 - x||_{P^1 - \beta_1 A^\top A - \mu_f I}^2.
$$

Using the above lemma, we are able to prove the following theorem.

**Theorem 2.2** Under Assumptions 2 and 3, let $\{(x^k, \lambda^k)\}$ be the sequence generated by Algorithm 1. Suppose that the parameters are set to satisfy

$$
0 < \rho_k \leq 2\beta_k, \quad P^k \succeq \beta_k A^\top A + L_f I, \quad \forall k \geq 1,
$$

and there exists a number $k_0$ such that for all $k \geq 1$,

$$
\frac{k + k_0 + 1}{\rho_k} \leq \frac{k + k_0}{\rho_{k-1}},
$$

and there exists a number $k_0$ such that for all $k \geq 1$,

$$
(k + k_0 + 1)(P^k - \beta_k A^\top A + \mu_f I) \preceq (k + k_0)(P^{k-1} - \beta_{k-1} A^\top A + \mu_g I).
$$

Then, for any $(x, \lambda)$ satisfying $Ax = b$, we have

$$
\sum_{k=1}^{t} (k + k_0 + 1)\Phi(x^{k+1}, x, \lambda) + \sum_{k=1}^{t} \frac{k + k_0 + 1}{2} ||x^{k+1}||^2 \\
+ \frac{t + k_0 + 1}{2} ||x^{t+1} - x||_{P^t - \beta_t A^\top A + \mu_f I}^2 \\
\leq \frac{k_0 + 2}{2\rho_1} ||\lambda - \lambda^1||^2 + \frac{k_0 + 2}{2} ||x^1 - x||_{P^1 - \beta_1 A^\top A - \mu_f I}^2.
$$

In the next theorem, we provide a set of parameters that satisfy the conditions in Theorem 2.2 and establish the $O(1/t^2)$ convergence rate result.

**Theorem 2.3 (Convergence rate of order $1/t^2$)** Under Assumptions 1 through 3, let $\{(x^k, \lambda^k)\}$ be the sequence generated by Algorithm 1 with parameters set to:

$$
\beta_k = \rho_k = (k + 1)\beta, \quad P^k = (k + 1)P + L_f I, \quad \forall k \geq 0,
$$

where $P$ is a block diagonal matrix satisfying $0 < P - \beta A^\top A \preceq \frac{\beta}{2} I$. Then,

$$
\max \left\{ \beta ||x^{t+1}||^2, ||x^{t+1} - x^*||^2_{P - \beta A^\top A} \right\} \leq \frac{2}{(t + 1)(t + k_0 + 1)} \phi(x^*, \lambda^*),
$$

where $\phi(x, \lambda)$ is a block diagonal matrix satisfying $0 < \phi(x, \lambda) \preceq \frac{\beta}{2} I$. Then,
where
\[ k_0 = 1 + \frac{2(L_f - \mu_f)}{\mu}, \tag{15} \]
and
\[ \phi(x, \lambda) = \frac{k_0 + 2}{4\beta} \|\lambda\|^2 + \frac{k_0 + 2}{2} \|x^1 - x\|^2 \| (P - \beta A^\top A + (L_f - \mu_f) I) \]

In addition, letting \( \gamma = \max \{2\|\lambda^*\|, 1 + \|\lambda^*\|\} \) and
\[ T = \frac{t(t + 2k_0 + 3)}{2}, \quad \bar{x}^{t+1} = \frac{\sum_{k=1}^{t}(k + k_0 + 1)x^k}{T}, \]
we have
\[ |F(\bar{x}^{t+1}) - F(x^*)| \leq \frac{1}{T} \max_{\|\lambda\| \leq \gamma} \phi(x^*, \lambda), \tag{16a} \]
\[ \|A\bar{x}^{t+1} - b\| \leq \frac{1}{T \max \{1, \|\lambda^*\|\}} \max_{\|\lambda\| \leq \gamma} \phi(x^*, \lambda). \tag{16b} \]

### 3 Accelerating randomized primal-dual block coordinate updates

In this section, we generalize Algorithm 1 to a randomized setting where the user may choose to update a subset of blocks at each iteration. Instead of updating all \( M \) block variables, we randomly choose a subset of them to renew at each iteration. Depending on the number of processors (nodes, or cores), we can choose a single or multiple block variables for each update.

#### 3.1 The algorithm

Our algorithm is an accelerated version of the randomized primal-dual coordinate update (RPDC) method that is recently proposed in [15]. At each iteration, it performs a block proximal gradient update to a subset of randomly selected primal variables while keeping the remaining ones fixed, followed by an update to the multipliers. Specifically, at iteration \( k \), it selects an index set \( S_k \subset \{1, \ldots, M\} \) with cardinality \( m \) and performs the following updates:

\[ x_i^{k+1} = \begin{cases} \arg \min_{x_i} \langle \nabla_i f(x^k) - A_i^\top (\lambda^k - \rho_k r^k), x_i \rangle + g_i(x_i) + \frac{\eta_k}{2} \|x_i - x_i^k\|^2, & \text{if } i \in S_k, \\ x_i^k, & \text{if } i \not\in S_k \end{cases}, \tag{17a} \]
\[ r^{k+1} = r^k + \sum_{i \in S_k} A_i(x_i^{k+1} - x_i^k), \tag{17b} \]
\[ \lambda^{k+1} = \lambda^k - \rho_k r^{k+1}, \tag{17c} \]

where \( \beta_k, \rho_k \) and \( \eta_k \) are algorithm parameters, and their values will be determined later. Note that we use \( \frac{\eta_k}{2} \|x_i - x_i^k\|^2 \) in (17a) for simplicity. It can be replaced by a PSD matrix weighted norm square term as in (7a), and our convergence results still hold.
Algorithm 2 summarizes the above method. If the parameters $\beta_k, \rho_k$ and $\eta_k$ are fixed during all the iterations, i.e., constant parameters, the algorithm reduces to the RPDC method in [15]. Adapting these parameters to the iterations, we will show that Algorithm 2 enjoys faster convergence rate than RPDC if the problem is strongly convex.

Algorithm 2: Accelerated randomized primal-dual block coordinate update method for (1)

1. **Initialization:** choose $x^1$, set $\lambda^1 = 0$, and let $r^1 = Ax^1 - b$
2. for $k = 1, 2, \ldots$ do
   3. Select $S_k \subset \{1, 2, \ldots, M\}$ uniformly at random with $|S_k| = m$.
   4. Choose parameters $\beta_k, \rho_k$ and $\eta_k$.
   5. Let $x^{k+1} \leftarrow (17a)$ and $\lambda^{k+1} \leftarrow (17c)$.
6. if a certain stopping criterion satisfied then
   7. Return $(x^{k+1}, \lambda^{k+1})$.

3.2 Convergence results

In this subsection, we establish convergence results of Algorithm 2 under Assumptions 1 and 3, and also the following partial gradient Lipschitz continuity assumption.

**Assumption 4** For any $S \subset \{1, \ldots, M\}$ with $|S| = m$, $\nabla_S f$ is Lipschitz continuous with a uniform constant $L_m$.

Note that if $\nabla f$ is Lipschitz continuous with constant $L_f$, then $L_m \leq L_f$.

Similar to the analysis in section 2, we first establish a result of running one iteration of Algorithm 2. Throughout this section, we denote $\theta = \frac{m}{M}$.

**Lemma 3.1 (One iteration analysis)** Under Assumptions 3 and 4, let $\{(x^k, \lambda^k)\}$ be the sequence generated from Algorithm 2. Then for any $x$ such that $Ax = b$, it holds

$$
\mathbb{E} \left[ \Phi(x^{k+1}, x, \lambda^{k+1}) + (\beta_k - \rho_k)\|r^{k+1}\|^2 + \frac{\mu g}{2}\|x^{k+1} - x\|^2 \right] \\
\leq (1 - \theta)\mathbb{E} \left[ \Phi(x^k, x, \lambda^k) + \beta_k\|x^k\|^2 + \frac{\mu g}{2}\|x^k - x\|^2 \right] + \frac{L_m}{2}\mathbb{E}\|x^{k+1} - x^k\|^2 \\
- \frac{\theta \mu f}{2}\mathbb{E}\|x^k - x\|^2 + \frac{\beta_k}{2}\mathbb{E} \left( \|A(x^{k+1} - x)\|^2 - \|A(x^k - x)\|^2 + \|A(x^{k+1} - x^k)\|^2 \right) \\
- \frac{\eta_k}{2}\mathbb{E} \left( \|x^{k+1} - x\|^2 - \|x^k - x\|^2 + \|x^{k+1} - x^k\|^2 \right).
$$

(18)

When $\mu f = \mu g = 0$ (meaning (1) is only weakly convex), Algorithm 2 has $O(1/t)$ convergence rate with fixed $\beta_k, \rho_k, \eta_k$ during all the iterations. This can be shown from (18) and has been established in [15]. For ease of referencing, we give the result below without proof.
Theorem 3.2 (Un-accelerated convergence) Under Assumptions 1 and 4, let \{(x^k, \lambda^k)\} be the sequence generated from Algorithm 2 with \(\beta_k = \beta, \rho_k = \rho, \eta_k = \eta\) for all \(k\), satisfying
\[
0 < \rho \leq \theta \beta, \quad \eta \geq L_m + \beta \|A\|_2^2,
\]
where \(\|A\|_2\) denotes the spectral norm of \(A\). Then
\[
\begin{align*}
|\mathbb{E}[F(\bar{x}^t) - F(x^*)]| & \leq \frac{1}{1 + \theta t} \max_{\|\lambda\| \leq \gamma} \phi(x^*, \lambda), \\
\mathbb{E}\|Ax^t - b\| & \leq \frac{1}{(1 + \theta t) \max\{1, \|\lambda^*\|\} \|\lambda\| \leq \gamma} \max \phi(x^*, \lambda),
\end{align*}
\]
where \((x^*, \lambda^*)\) satisfies the KKT conditions in (3), \(\gamma = \max\{\|2\lambda^*\|, 1 + \|\lambda^*\|\}\), and
\[
\bar{x}^t = \frac{x^t + \theta \sum_{k=1}^{t} x^k}{1 + \theta t}, \quad \phi(x, \lambda) = 1 - \theta(F(x^1) - F(x)) + \frac{\eta}{2} \|x^1 - x\|^2 + \frac{\theta \|\lambda\|^2}{2\rho}.
\]
When \(F\) is strongly convex, the above \(O(1/t)\) convergence rate can be accelerated to \(O(1/t^2)\) by adaptively changing the parameters at each iteration. The following theorem is our main result. It shows \(O(1/t^2)\) convergence result under certain conditions on the parameters. Based on this theorem, we will give a set of parameters that satisfy these conditions and thus provide an implementable way to choose the parameters.

Theorem 3.3 Under Assumptions 1, 3 and 4, let \{(x^k, \lambda^k)\} be the sequence generated from Algorithm 2 with parameters satisfying the following conditions for a certain number \(k_0\):
\[
\begin{align*}
\theta(k + k_0 + 1) & \geq 1, \forall k \geq 2, \\
(\beta_{k-1} - \rho_{k-1})(k + k_0) & \geq (1 - \theta)(k + k_0 + 1)\beta_k, \forall k \geq 2, \\
\frac{\theta(k + k_0 + 1) - 1}{\rho_{k-1}} & \geq \frac{\theta(k + k_0 + 2) - 1}{\rho_k}, \forall 2 \leq k \leq t - 1, \\
\frac{\theta(t + k_0 + 1) - 1}{\rho_{t-1}} & \geq \frac{t + k_0 + 1}{\rho_t}, \\
\beta_k(k + k_0 + 1) & \geq \beta_{k-1}(k + k_0), \forall k \geq 2, \\
(k + k_0 + 1)(\eta_k - L_m)I & \geq \beta_k(k + k_0 + 1)A^\top A, \forall k \geq 1, \\
(k + k_0)(\eta_{k-1} + \mu_g(\theta(k + k_0 + 1) - 1)) & \geq (k + k_0 + 1)(\eta_k - \theta \mu_f), \forall k \geq 2.
\end{align*}
\]
Then for any \((x, \lambda)\) such that \(Ax = b\), we have
\[
\begin{align*}
(t + k_0 + 1) & \mathbb{E}\Phi(x^{t+1}, x, \lambda) + \sum_{k=2}^{t+1} \left(\theta(k + k_0 + 1) - 1\right) \mathbb{E}\Phi(x^k, x, \lambda) \\
\leq (1 - \theta)(k_0 + 2) & \mathbb{E}\left[\Phi(x^1, x, \lambda^1) + \beta_1 \|r^1\|^2 + \frac{\mu_g}{2} \|x^1 - x\|^2\right] + \frac{k_0 + 2}{2}(\eta_1 - \theta \mu_f)\mathbb{E}\|x^1 - x\|^2 \\
& + \frac{(k_0 + 3) - 1}{2\rho_1} \mathbb{E}\|\lambda^1 - \lambda\|^2 - \frac{t + k_0 + 1}{2} \mathbb{E}\|x^{t+1} - x\|_2^2 + \left(\mu_g + \eta_f\right)I - \beta_t A^\top A.
\end{align*}
\]
Specifying the parameters that satisfy (20), we show \(O(1/t^2)\) convergence rate of Algorithm 2.

**Theorem 3.4 (Accelerated convergence)** Under Assumptions 1, 3 and 4, let \(\{(x^k, \lambda^k)\}\) be the sequence generated from Algorithm 2 with parameters taken as:

\[
\rho_k = \begin{cases} 
\frac{\theta \mu (k+2)}{2(6-5\theta)\|A\|_2^2} & \text{for } 1 \leq k \leq t - 1, \\
\frac{t+k_0+1}{\theta}(t+k_0+1)-1 & \text{for } k = t 
\end{cases}
\]

\(\beta_k = \frac{\mu (k+2)}{2\|A\|_2^2}, \forall k \geq 1, \tag{22a}\)

\(\eta_k = \frac{\mu}{2} (k+2) + L_m, \forall k \geq 1, \tag{22b}\)

where \(\rho \geq 1\) and

\(k_0 = \frac{2}{\theta} + \frac{2(L_m + \mu g)}{\theta \mu}. \tag{23}\)

Then

\[
\left| \mathbb{E}[F(\bar{x}^{t+1}) - F(x^*)] \right| \leq \frac{1}{T} \max_{\|\lambda\| \leq \gamma} \phi(x^*, \lambda), \quad \mathbb{E}\|A\bar{x}^{t+1} - b\| \leq \frac{1}{T} \max\{1, \|\lambda^*\|\} \max_{\|\lambda\| \leq \gamma} \phi(x^*, \lambda), \tag{24}\]

where \(\gamma = \max\{2\|\lambda^*\|, 1 + \|\lambda^*\|\},\)

\[
\bar{x}^{t+1} = (t+k_0+1)x^{t+1} + \sum_{k=2}^{t} (\theta(k + k_0 + 1) - 1)x^k,
\]

\[
\phi(x, \lambda) = \left\{ (1 - \theta)(k_0 + 2) \left[ F(x^1) - F(x) + \beta_1 \|r^1\|^2 + \frac{\mu g}{2} \|\lambda^1 - \lambda\|^2 \right] \\
+ \frac{k_0 + 2}{2} (\eta_1 - \theta \mu g) \|x^1 - x\|^2 + \frac{\theta(k_0 + 3) - 1}{2\rho_1} \|\lambda\|^2 \right\}
\]

and

\[
T = (t+k_0+1) + \sum_{k=2}^{t} (\theta(k + k_0 + 1) - 1).
\]

In addition,

\[
\mathbb{E}\|x^{t+1} - x^*\|^2 \leq \frac{2\phi(x^*, \lambda^*)}{(t+k_0+1) \left( \frac{(\rho-1)\mu}{2\rho}(\theta t + \theta + 2) + \mu + \mu_g + L_f \right)}.
\]

### 4 Linearly convergent primal-dual method

In this section, we assume some more structure on (1) and show that a linear rate of convergence is possible. If there is no linear constraint, Algorithm 2 reduces to the RCD method proposed in [31]. It is well-known that RCD converges linearly if the objective is strongly convex. However, with the presence of linear constraints, mere strong convexity of the objective of the primal problem only
ensures the smoothness of its Lagrangian dual function, but not the strong concavity of it. Hence, in general, we do not expect linear convergence by only assuming strong convexity on the primal objective function. To ensure linear convergence on both the primal and dual variables, we need additional assumptions.

Throughout this section, we suppose that there is at least one block variable being absent in the nonseparable part of the objective, namely \( f \). For convenience, we rename this block variable to be \( y \), and the corresponding component function and constraint coefficient matrix as \( h \) and \( B \). Specifically, we consider the following problem

\[
\min_{x,y} f(x_1, \ldots, x_M) + \sum_{i=1}^{M} g_i(x_i) + h(y), \quad \text{s.t.} \quad \sum_{i=1}^{M} A_i x_i + B y = b. \tag{25}
\]

Towards a solution to (25), we modify Algorithm 2 by updating \( y \)-variable after the \( x \)-update. Since there is only a single \( y \)-block, to balance \( x \) and \( y \) updates, we do not renew \( y \) in every iteration but instead update it in probability \( \theta = \frac{m}{M} \). Hence, roughly speaking, \( x \) and \( y \) variables are updated in the same frequency. The method is summarized in Algorithm 3.

**Algorithm 3:** Randomized primal-dual block coordinate update for (25)

1. **Initialization:** choose \((x^1, y^1)\), set \( \lambda^1 = 0 \), and choose parameters \( \beta, \rho, \eta_x, \eta_y, m \).
2. Let \( r^1 = Ax^1 + By^1 - b \) and \( \theta = \frac{m}{M} \).
3. for \( k = 1, 2, \ldots \) do
4.   Select index set \( S_k \subset \{1, \ldots, M\} \) uniformly at random with \( |S_k| = m \).
5.   Keep \( x_i^{k+1} = x_i^k \), \( \forall i \not\in S_k \) and update
6.     \[
x_i^{k+1} = \arg \min_{x_i} \left\{ \nabla_i f(x^k) - A_i^\top (\lambda^k - \beta r^k), x_i \right\} + g_i(x_i) + \frac{\eta_x}{2} \|x_i - x_i^k\|^2, \quad \text{if } i \in S_k. \tag{26}
\]
   Let \( r^{k+\frac{1}{2}} = r^k + \sum_{i \in S_k} A_i (x_i^{k+1} - x_i^k) \).
7.   In probability \( 1 - \theta \) keep \( y^{k+1} = y^k \), and in probability \( \theta \) let \( y^{k+1} = \tilde{y}^{k+1} \), where
8.     \[
\tilde{y}^{k+1} = \arg \min_{y} h(y) - \left\langle B^\top (\lambda^k - \beta r^{k+\frac{1}{2}}), y \right\rangle + \frac{\eta_y}{2} \|y - y^k\|^2. \tag{27}
\]
   Let \( r^{k+1} = r^{k+\frac{1}{2}} + B(y^{k+1} - y^k) \).
9.   Update the multiplier by
10.    \[
\lambda^{k+1} = \lambda^k - \rho r^{k+1}. \tag{28}
\]
   if a certain stopping criterion is satisfied then
11.    Return \((x^{k+1}, y^{k+1}, \lambda^{k+1})\).
4.1 Technical assumptions

Assume $h$ to be differentiable. Then any pair of primal-dual solution $(x^*, y^*, \lambda^*)$ to (25) satisfies the KKT conditions:

$$
0 \in \partial F(x^*) - A^\top \lambda^*, \\
\nabla h(y^*) - B^\top \lambda^* = 0, \\
Ax^* + By^* - b = 0.
$$

(29a) (29b) (29c)

Besides Assumptions 3 and 4, we make two additional assumptions as follows.

**Assumption 5** There exists $(x^*, y^*, \lambda^*)$ satisfying the KKT conditions in (29).

**Assumption 6** The function $h$ is strongly convex with modulus $\nu$, and its gradient $\nabla h$ is Lipschitz continuous with constant $L_h$.

The strong convexity of $F$ and $h$ implies

$$
F(x^{k+1}) - F(x^*) - \langle \nabla F(x^*), x^{k+1} - x^* \rangle \geq \frac{\mu}{2} \|x^{k+1} - x^*\|^2, \\
\langle y^{k+1} - y^*, \nabla h(y^{k+1}) - \nabla h(y^*) \rangle \geq \nu \|y^{k+1} - y^*\|^2,
$$

(30a) (30b)

where $\nabla F(x^*)$ is a subgradient of $F$ at $x^*$.

4.2 Convergence analysis

Similar to Lemma 3.1, we first establish a result of running one iteration of Algorithm 3. It can be proven by similar arguments to those showing Lemma 3.1.

**Lemma 4.1 (One iteration analysis)** Under Assumptions 3, 4, and 6, let $\{(x^k, y^k, \lambda^k)\}$ be the sequence generated from Algorithm 3. Then for any $k$ and any $(x, y)$ such that $Ax + By = b$, it holds

$$
\mathbb{E} \left[ F(x^{k+1}) - F(x) + \langle y^{k+1} - y, \nabla h(y^{k+1}) \rangle - \langle \lambda, A x^{k+1} + B y^{k+1} - b \rangle \right] + (\beta - \rho) \mathbb{E} \|z^{k+1}\|^2 \\
+ \mathbb{E} \left[ \langle x^{k+1} - x, (\eta_x I - \beta A^\top A)(x^{k+1} - x^k) \rangle + \langle y^{k+1} - y, (\eta_y I - \beta B^\top B)(y^{k+1} - y^k) \rangle \right] \\
+ \frac{1}{\rho} \mathbb{E} \langle \lambda^{k+1} - \lambda, \lambda^{k+1} - \lambda^k \rangle - \frac{L_m}{2} \mathbb{E} \|x^{k+1} - x^k\|^2 + \frac{\theta \mu f}{2} \mathbb{E} \|x^{k+1} - x\|^2 + \frac{\mu g}{2} \mathbb{E} \|z^{k+1}\|^2 \\
\leq \left( 1 - \theta \right) \mathbb{E} \left[ F(x^k) - F(x) + \langle y^k - y, \nabla h(y^k) \rangle - \langle \lambda, A x^k + B y^k - b \rangle \right] + \beta (1 - \theta) \mathbb{E} \|r^k\|^2 \\
+ (1 - \theta) \frac{\mu g}{2} \mathbb{E} \|x^{k+1} - x\|^2 + \frac{1 - \theta}{\rho} \mathbb{E} \langle \lambda^k - \lambda, \lambda^k - \lambda^{k-1} \rangle \\
+ \beta \mathbb{E} \langle A(x^{k+1} - x), B(y^{k+1} - y^k) \rangle + \beta (1 - \theta) \mathbb{E} \langle B(y^k - y), A(x^{k+1} - x^k) \rangle.
$$

(31)
For ease of notation, we denote $z = (x, y)$ and

$$P = \eta_y I - \beta A^T A, \quad Q = \eta_y I - \beta B^T B.$$  

Let

$$\Psi(z^k, z^*) = F(x^k) - F(x^*) - \langle \nabla F(x^*), x^k - x^* \rangle + \langle y^k - y^*, \nabla h(y^k) - \nabla h(y^*) \rangle,$$  

and also let

$$\psi(z^k, z^*; P, Q, \beta, \rho, c, \tau) = (1 - \theta)\Psi(z^k, z^*) + \frac{\beta(1 - \theta)}{2}E\|r^k\|^2 + \frac{1}{2}E\|x^k - x^*\|_p^2$$

$$+ \frac{\mu_y - \theta \mu}{2}E\|x^k - x^*\|^2 + \frac{1}{2}E\|y^k - y^*\|_q^2 + \frac{\beta(1 - \theta)}{2\tau}E\|B(y^k - y^*)\|^2$$

$$+ \frac{1}{2\rho}E \left[ \|\lambda^k - \lambda^*\|^2 - (1 - \theta)\|\lambda^{k+1} - \lambda^*\|^2 + \frac{1}{\theta} \|\lambda^{k+1} - \lambda^k\|^2 \right].$$  

(33)

The following theorem is key to establishing linear convergence of Algorithm 3.

**Theorem 4.2** Under Assumptions 3 through 6, let $\{(x^k, y^k, \lambda^k)\}$ be the sequence generated from Algorithm 3 with $\rho = \theta \beta$. Let $0 < \alpha < \theta$ and $\gamma = \max \left\{ \frac{8\|A\|_2}{\alpha \mu}, \frac{8\|B\|_2}{\alpha \nu}\right\}$. Choose $\delta, \kappa > 0$ such that

$$2 \begin{bmatrix} 1 - (1 - \theta)(1 + \delta) & (1 - \theta)(1 + \delta) \\ (1 - \theta)(1 + \delta) & \kappa - (1 - \theta)(1 + \delta) \end{bmatrix} \succeq \begin{bmatrix} \theta & 1 - \theta \\ 1 - \theta & \frac{1}{\theta} - (1 - \theta) \end{bmatrix},$$

(34)

and positive numbers $\eta_x, \eta_y, c, \tau_1, \tau_2, \beta$ such that

$$P \succeq \beta(1 - \theta)\tau_2 A^T A + L_m I$$

(35a)

$$Q \succeq 8cQ^T Q + 4c\rho^2(1 - \theta)(1 + \frac{1}{\delta})B^T BB^T B + \beta \tau_1 B^T B.$$  

(35b)

Then it holds that

$$(1 - \alpha)E\Psi(z^{k+1}, z^*) + \left( \frac{\beta(1 - \theta)}{2} + \frac{1}{\gamma} \right)E\|r^{k+1}\|^2 - c\rho^2 \left( \kappa + 2(1 - \theta)(1 + \frac{1}{\delta}) \right)E\|B^T r^{k+1}\|^2$$

$$- 2c(\beta - \rho)^2E\|B^T r^{k+1}\|^2 + \gamma \left[ \frac{1}{2}E\|x^{k+1} - x^*\|_p^2 + \frac{\alpha \mu}{4}E\|x^{k+1} - x^*\|^2 + \frac{\mu_y}{2}E\|x^{k+1} - x^*\|^2 \right]$$

$$- \frac{\beta}{2\tau_1}E\|A(x^{k+1} - x^*)\|^2 \right] + \gamma \left[ \frac{1}{2}E\|y^{k+1} - y^*\|_q^2 + \frac{3\alpha \nu}{4}E\|y^{k+1} - y^*\|^2 - 4cL_h^2E\|y^{k+1} - y^*\|^2 \right]$$

$$+ \left( \frac{1}{2\rho} + c\sigma_{\min}(BB^T) \right)E \left[ \|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^{k+1} + \lambda^*\|^2 + \frac{1}{\theta} \|\lambda^{k+1} - \lambda^k\|^2 \right]$$

$$\leq \psi(z^k, z^*; P, Q, \beta, \rho, c, \tau_2).$$

(36)

Using Theorem 4.2, a linear convergence rate of Algorithm 3 follows.
Theorem 4.3 (Linear convergence) Under Assumptions 3 through 6, let \( \{(x^k, y^k, \lambda^k)\} \) be the sequence generated from Algorithm 3 with \( \rho = \theta \beta \). Let \( 0 < \alpha < \theta \) and \( \gamma = \max\{\frac{8\|A\|^2_2}{\alpha \mu}, \frac{8\|B\|^2_2}{\alpha \nu}\} \). Assume that \( B \) is full row-rank and \( \max\{\|A\|_2, \|B\|_2\} \leq 1 \). Choose \( \delta, \kappa, \eta, c, \beta, \tau_1, \tau_2 \) satisfying (34) and (35), and in addition,
\[
\frac{\alpha}{2} \mu + \theta \mu > \frac{\beta}{\tau_1}, \tag{37a}
\]
\[
\frac{3\alpha \nu}{4} > 4cL\hat{h}^2 + \frac{\beta(1 - \theta)}{2\tau_2}, \tag{37b}
\]
\[
\frac{1}{\gamma} > c\rho^2 \left( \kappa + 2(1 - \theta)(1 + \frac{1}{\delta}) \right) - 2c(\beta - \rho)^2. \tag{37c}
\]
Then
\[
\psi(z^{k+1}, z^*; P, Q, \beta, \rho, c, \tau_2) \leq \frac{1}{\eta} \psi(z^k, z^*; P, Q, \beta, \rho, c, \tau_2), \tag{38}
\]
where
\[
\eta = \min \left\{ \frac{1 - \alpha}{1 - \theta}, 1 + \frac{2 - 2c \rho^2 (\kappa + 2(1 - \theta)(1 + \frac{1}{\delta}) - 4c(\beta - \rho)^2)}{\beta(1 - \theta)}, 1 + \frac{\frac{3\alpha \nu}{4} - 2cL\hat{h}^2 - \frac{\beta(1 - \theta)}{2\tau_2}}{\eta_\gamma \frac{\gamma}{2} + \frac{\beta(1 - \theta)}{2\tau_2}}, 1 + c\rho \sigma_{\min}(BB^\top) \right\} > 1.
\]

Remark 4.1 We can always rescale \( A, B \) and \( b \) without essentially altering the linear constraints. Hence, the assumption \( \max\{\|A\|_2, \|B\|_2\} \leq 1 \) can be made without losing generality. From (38), it is easy to see that \( (x^k, y^k) \) converges to \( (x^*, y^*) \) \( R \)-linearly in expectation. In addition, note that
\[
\|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^k - \lambda^*\| + \frac{1}{\theta}\|\lambda^{k+1} - \lambda^k\|
\]
\[
= \theta\|\lambda^{k+1} - \lambda^*\|^2 + 2(1 - \theta)(\lambda^{k+1} - \lambda^*, \lambda^{k+1} - \lambda^k) + (\frac{1}{\theta} - 1 + \theta)\|\lambda^{k+1} - \lambda^k\|
\]
\[
\geq \left( \frac{\theta - (1 - \theta)^2}{\frac{1}{\theta} - 1 + \theta} \right)\|\lambda^{k+1} - \lambda^*\|^2
\]
\[
= \frac{\theta}{\frac{1}{\theta} - 1 + \theta}\|\lambda^{k+1} - \lambda^*\|^2.
\]
Hence, (38) also implies an \( R \)-linear convergence from \( \lambda^k \) to \( \lambda^* \) in expectation.

5 Numerical experiments

In this section, we test Algorithm 2 on nonnegative quadratic programming
\[
\min_x F(x) = \frac{1}{2} x^\top Q x + c^\top x, \text{ s.t. } Ax = b, x \geq 0, \tag{39}
\]
and we compare it to the RPDC method proposed in [15]. Note that when applied to (39), RPDC can be regarded as a special case of Algorithm 2 with nonadaptive parameters.

The data was generated randomly as follows. We let $Q = HDH^\top \in \mathbb{R}^{n \times n}$, where $H$ is Gaussian randomly generated orthogonal matrix and $D$ is a diagonal matrix with $d_{ii} = 1 + (i - 1)^{99 \times 100}, i = 1, \ldots, n$. Hence, the smallest and largest singular values of $Q$ are 1 and 100 respectively, and the objective of (39) is strongly convex with modulus 1. The components of $c$ follow standard Gaussian distribution, and those of $b$ follow uniform distribution on $[0, 1]$. We let $A = [B, I] \in \mathbb{R}^{p \times n}$ to guarantee the existence of feasible solutions, where $B$ was generated according to standard Gaussian distribution. In addition, we normalized $A$ so that it has a unit spectral norm.

In the test, we fixed $n = 2000$ and varied $p$ among $\{200, 500\}$. For both Algorithm 2 and RPDC, we evenly partitioned $x$ into 40 blocks, i.e., each block consists of 50 coordinates, and we set $m = 40$, i.e., all blocks are updated at each iteration. The parameters of Algorithm 2 were set according to (22), and those for RPDC were set based on Theorem 3.2 with $\rho = \beta$, $\eta = 100 + \beta$, $\forall k$ where $\beta$ varied among $\{1, 10, 100, 1000\}$. Figures 1 and 2 plot the objective values and feasibility violations by Algorithm 2 and RPDC. From these results, we see that Algorithm 2 performed well for both datasets with a single set of parameters while the performance of RPDC was severely affected by the penalty parameter.

6 Conclusions

In this paper we propose an accelerated proximal Jacobian ADMM method and generalize it to an accelerated randomized primal-dual coordinate updating method for solving linearly constrained
Figure 2: Results by Algorithm 2 and the RPDC method in [15] with different penalty parameter $\beta$ for solving (39) with problem size $n = 2000$ and $p = 500$. Top row: difference of objective value to the optimal value $|F(x^k) - F(x^*)|$; bottom row: violation of feasibility $\|Ax^k - b\|$.

multi-block structured convex programs. We show that if the objective is strongly convex then the methods achieve $O(1/t^2)$ convergence rate where $t$ is the total number of iterations. In addition, if one block variable is independent of others in the objective and its part of the objective function is smooth, we have modified the primal-dual coordinate updating method to achieve linear convergence. Numerical experiments on quadratic programming have shown the efficacy of the newly proposed methods.

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A  Technical proofs: Section 2

In this section, we give the detailed proofs of the lemmas and theorems in section 2.

A.1  Proof of Lemma 2.1

From (7a), we have the optimality conditions

$$0 \in \nabla_i f(x^k) - A_i^\top (\lambda^k - \beta_k r^k) + \partial g_i(x_i^{k+1}) + P_i^k(x_i^{k+1} - x_i^k),$$

or equivalently there is $\tilde{\nabla} g(x^{k+1}) \in \partial g(x^{k+1})$,

$$0 = \nabla f(x^k) - A^\top (\lambda^k - \beta_k r^k) + \tilde{\nabla} g(x^{k+1}) + P^k(x^{k+1} - x^k).$$

Hence, for any $x$ such that $Ax = b$,

$$0 = \left\langle x^{k+1} - x, \nabla f(x^k) - A^\top (\lambda^k - \beta_k r^k) + \tilde{\nabla} g(x^{k+1}) + P^k(x^{k+1} - x^k) \right\rangle$$

$$\geq \left\langle x^{k+1} - x, \nabla f(x^k) \right\rangle + \left\langle x^{k+1} - x, g(x^{k+1}) - g(x) + \frac{\mu_g}{2} \|x^{k+1} - x\|^2 \right\rangle$$

$$\geq \left\langle x^{k+1} - x, g(x^{k+1}) - g(x) + \frac{\mu_g}{2} \|x^{k+1} - x\|^2 \right\rangle$$

$$= \left\langle x^{k+1} - x, g(x^{k+1}) - g(x) + \frac{\mu_g}{2} \|x^{k+1} - x\|^2 \right\rangle$$

where the first inequality uses the strong convexity of $g$, and the second inequality is from $Ax = b$ and the following arguments using gradient Lipschitz continuity and strong convexity of $f$:

$$\left\langle x^{k+1} - x, \nabla f(x^k) \right\rangle = \left\langle x^{k+1} - x, \nabla f(x^k) \right\rangle + \left\langle x^k - x, \nabla f(x^k) \right\rangle$$

$$\geq f(x^{k+1}) - f(x) - \frac{L_f}{2} \|x^{k+1} - x\|^2 + f(x^k) - f(x) + \frac{\mu_f}{2} \|x^k - x\|^2$$

$$= f(x^{k+1}) - f(x) - \frac{L_f}{2} \|x^{k+1} - x\|^2 + \frac{\mu_f}{2} \|x^k - x\|^2.$$

Rearranging (40) gives (by noting $F = f + g$)

$$\Phi(x^{k+1}, x, \lambda) = F(x^{k+1}) - F(x) - \left\langle A x^{k+1} - b, \lambda \right\rangle$$

$$\leq \left\langle A x^{k+1} - b, \lambda^k - \beta_k r^k \right\rangle - \left\langle A x^{k+1} - b, \lambda \right\rangle - \left\langle x^{k+1} - x, P^k(x^{k+1} - x^k) \right\rangle$$

$$\geq \left\langle x^{k+1} - x, \nabla f(x^k) \right\rangle + \frac{\mu_f}{2} \|x^{k+1} - x\|^2 - \frac{L_f}{2} \|x^k - x\|^2.$$  (41)

Using the fact $\lambda^{k+1} = \lambda^k - \rho_k(A x^{k+1} - b)$, we have

$$\left\langle A x^{k+1} - b, \lambda^k - \lambda \right\rangle = \frac{1}{\rho_k} \left\langle \lambda^k - \lambda^{k+1}, \lambda^k - \lambda \right\rangle$$

$$\overset{(5)}{=} \frac{1}{2\rho_k} \left\| \lambda - \lambda^k \right\|^2 - \frac{1}{2\rho_k} \left\| \lambda - \lambda^{k+1} \right\|^2 + \frac{1}{2\rho_k} \left\| \lambda^k - \lambda^{k+1} \right\|^2.$$  (42)
In addition,
\[
\langle Ax^{k+1} - b, -\beta_k r_k \rangle \\
= -\beta_k \langle Ax^{k+1} - b, r^{k+1} + r^k \rangle \\
= -\beta_k \|r^{k+1}\|^2 + \beta_k \langle A(x^{k+1} - x), A(x^{k+1} - x) \rangle \\
\overset{(5)}{=} -\beta_k \|r^{k+1}\|^2 + \beta_k \frac{\beta_k}{2} \left[ \|A(x^{k+1} - x)\|^2 - \|A(x^k - x)\|^2 + \|A(x^{k+1} - x^k)\|^2 \right]
\]
(43)

Substituting (42) and (43) into (41) and also applying (5) to the cross term \(\langle x^{k+1} - x, P^k(x^{k+1} - x)\rangle\) gives the inequality in (8).

### A.2 Proof of Theorem 2.2

First note that
\[
\|\lambda^k - \lambda^{k+1}\|^2 = \rho_k^2 \|r^{k+1}\|^2.
\]  
(44)

Since \(P^k \succeq \beta_k A^T A + L_f I\), it holds that
\[
\|x^{k+1} - x^k\|^2_{p_k} \geq \beta_k \|A(x^{k+1} - x^k)\|^2 + L_f \|x^{k+1} - x^k\|^2.
\]  
(45)

In addition, from the assumption in (10), we have
\[
\sum_{k=1}^{t} \frac{k + k_0 + 1}{2\rho_k} \left[ \|\lambda - \lambda^k\|^2 - \|\lambda - \lambda^{k+1}\|^2 \right] \\
= \frac{k_0 + 2}{2\rho_1} \|\lambda - \lambda^1\|^2 - \frac{t + k_0 + 1}{2\rho_1} \|\lambda - \lambda^{t+1}\|^2 + \sum_{k=2}^{t} \left( \frac{k + k_0 + 1}{2\rho_k} - \frac{k - k_0}{2\rho_{k-1}} \right) \|\lambda - \lambda^k\|^2 \\
\leq \frac{k_0 + 2}{2\rho_1} \|\lambda - \lambda^1\|^2.
\]  
(46)

Finally, note that
\[
\sum_{k=1}^{t} \left( (k + k_0 + 1) \left( \frac{\beta_k}{2} \left[ \|A(x^{k+1} - x)\|^2 - \|A(x^k - x)\|^2 \right] - \frac{\mu_g}{2} \|x^{k+1} - x\|^2 - \frac{\mu_f}{2} \|x^k - x\|^2 \right) \\
= \frac{k_0 + 2}{2} \|x^1 - x\|^2_{p_1 - \beta_1 A^T A - \mu_f I} - \frac{t + k_0 + 1}{2} \|x^{t+1} - x\|^2_{p_t - \beta_t A^T A - \mu_g I} \\
+ \sum_{k=2}^{t} \left( (k + k_0 + 1) \|x^k - x\|^2_{p_k - \beta_k A^T A - \mu_f I} - (k + k_0) \|x^k - x\|^2_{p_{k-1} - \beta_{k-1} A^T A + \mu_g I} \right) \\
\overset{(11)}{\leq} \frac{k_0 + 2}{2} \|x^1 - x\|^2_{p_1 - \beta_1 A^T A - \mu_f I} - \frac{t + k_0 + 1}{2} \|x^{t+1} - x\|^2_{p_t - \beta_t A^T A - \mu_g I}.
\]  
(47)

Now multiplying \(k + k_0 + 1\) to both sides of (8) and adding it over \(k\), we obtain (12) by using (44) through (47).
A.3 Proof of Theorem 2.3

From the choice of \( k_0 \), it follows that
\[
(2k + k_0 + 1)\mu I + (L_f - \mu_f)I = (k + k_0)\mu I, \quad \forall k.
\]
Since \( P - \beta A^\top A \preceq \frac{\mu}{2} I \), the above equation implies
\[
(2k + k_0 + 1)(P - \beta A^\top A) + (L_f - \mu_f)I \preceq (k + k_0)\mu I,
\]
which is equivalent to
\[
(k + k_0 + 1)\left[(k + 1)P - (k + 1)\beta A^\top A + (L_f - \mu_f)I\right] \preceq (k + k_0)\left[kP - k\beta A^\top A + (L_f + \mu_g)I\right].
\]
Hence, the condition in (11) holds. In addition, it is easy to see that all conditions in (9) and (10) also hold. Therefore, we have (12), which, by taking parameters in (13) and \( x = x^* \), reduces to
\[
\sum_{k=1}^{t}(k + k_0 + 1)\Phi(x^{k+1}, x^*, \lambda) + \sum_{k=1}^{t} \frac{(k + k_0 + 1)(k + 1)}{2}\beta\|r^{k+1}\|^2
\]
\[
+ \frac{t + k_0 + 1}{2}\|x^{t+1} - x^*\|^2 \leq \frac{k_0 + 2}{4\beta} \|\lambda\|^2 + \frac{k_0 + 2}{2}\|x^1 - x^*\|^2 \leq \frac{2}{2(P - \beta A^\top A + (L_f - \mu_f)I)},
\]
where we have used the fact \( \lambda^1 = 0 \).

Letting \( \lambda = \lambda^* \), we have from (4) and (48) that (by dropping nonnegative terms on the left hand side):
\[
\frac{(t + k_0 + 1)(t + 1)}{2}\beta\|r^{t+1}\|^2 + \frac{t + k_0 + 1}{2}\|x^{t+1} - x^*\|^2 \leq \frac{k_0 + 2}{4\beta} \|\lambda^*\|^2 + \frac{k_0 + 2}{2}\|x^1 - x^*\|^2 \leq \frac{2}{2(P - \beta A^\top A + (L_f - \mu_f)I)},
\]
which indicates (14). In addition, from the convexity of \( F \), we have from (48) that for any \( \lambda \),
\[
\frac{t(t + 2k_0 + 3)}{2}\Phi(\bar{x}^{t+1}, x^*, \lambda) \leq \frac{k_0 + 2}{4\beta} \|\lambda\|^2 + \frac{k_0 + 2}{2}\|x^1 - x^*\|^2 \leq \frac{2}{2(P - \beta A^\top A + (L_f - \mu_f)I)},
\]
which together with Lemmas 1.2 and 1.3 implies (16).

B Technical proofs: Section 3

In this section, we give the proofs of the lemmas and theorems in section 3.
B.1 Proof of Lemma 3.1

The $x$-update in (17a) can be written more compactly as

$$ x_{S_k}^{k+1} = \arg \min_{x_{S_k}} \langle \nabla S_k f(x^k) - A_{S_k}^T (\lambda^k - \beta_k r^k), x_{S_k} \rangle + g_{S_k}(x_{S_k}) + \frac{\eta_k}{2} \| x_{S_k} - x_{S_k}^k \|^2. \quad (49) $$

We have the optimality condition

$$ 0 = \nabla S_k f(x^k) - A_{S_k}^T (\lambda^k - \beta_k r^k) + \nabla g_{S_k}(x_{S_k}^{k+1}) + \eta_k (x_{S_k}^{k+1} - x_{S_k}^k), $$

where $\nabla g_{S_k}(x_{S_k}^{k+1})$ is one subgradient of $g_{S_k}$ at $x_{S_k}^{k+1}$. Hence, for any $x$, it holds that

$$ \langle x_{S_k}^{k+1} - x_{S_k}, \nabla S_k f(x^k) - A_{S_k}^T (\lambda^k - \beta_k r^k) + \nabla g_{S_k}(x_{S_k}^{k+1}) + \eta_k (x_{S_k}^{k+1} - x_{S_k}^k) \rangle = 0. \quad (50) $$

Recall $\theta = \frac{m}{\eta t}$. Note that $x_i^{k+1} = x_i^k, \forall i \not\in S_k$; therefore,

$$ \begin{align*}
\mathbb{E} \left[ \langle x_{S_k}^{k+1} - x_{S_k}, \nabla S_k f(x^k) - A_{S_k}^T (\lambda^k - \beta_k r^k) \rangle \right] \\
= \mathbb{E} \left[ \langle x_{S_k}^{k+1} - x_{S_k}^k + x_{S_k}^k - x_{S_k}, \nabla S_k f(x^k) - A_{S_k}^T (\lambda^k - \beta_k r^k) \rangle \right] \\
= \mathbb{E} \left[ x_{S_k}^{k+1} - x_{S_k}^k, \nabla f(x^k) - A^T (\lambda^k - \beta_k r^k) \right] + \theta \mathbb{E} \left[ x_{S_k}^{k+1} - x_{S_k}^k, \nabla f(x^k) - A^T (\lambda^k - \beta_k r^k) \right] \\
\geq \mathbb{E} \left[ f(x_{S_k}^{k+1}) - f(x^k) - \frac{L_m}{2} \| x_{S_k}^{k+1} - x_{S_k}^k \|^2 + \langle x_{S_k}^{k+1} - x_{S_k}^k, -A^T (\lambda^k - \beta_k r^k) \rangle \right] \\
+ \theta \mathbb{E} \left[ f(x^k) - f(x) + \frac{H_f}{2} \| x_{S_k}^{k+1} - x_{S_k}^k \|^2 + \langle x_{S_k}^{k+1} - x_{S_k}^k, -A^T (\lambda^k - \beta_k r^k) \rangle \right] \\
= \mathbb{E} \left[ f(x_{S_k}^{k+1}) - f(x) + \langle x_{S_k}^{k+1} - x_{S_k}^k, -A^T (\lambda^k - \beta_k r^k) \rangle \right] - \frac{L_m}{2} \mathbb{E} \| x_{S_k}^{k+1} - x_{S_k}^k \|^2 \\
- (1-\theta) \mathbb{E} \left[ f(x_{S_k}^k) - f(x) + \langle x_{S_k}^{k+1} - x_{S_k}^k, -A^T (\lambda^k - \beta_k r^k) \rangle \right] + \theta \frac{H_f}{2} \mathbb{E} \| x_{S_k}^{k+1} - x_{S_k}^k \|^2 \\
= \mathbb{E} \left[ f(x_{S_k}^{k+1}) - f(x) + \langle A(x_{S_k}^{k+1} - x), \lambda^k \rangle + (\beta_k - \rho_k) \langle A(x_{S_k}^{k+1} - x), r_k^{k+1} \rangle \\
- \beta_k \langle A(x_{S_k}^{k+1} - x), A(x_{S_k}^{k+1} - x) \rangle \right] - \frac{L_m}{2} \mathbb{E} \| x_{S_k}^{k+1} - x_{S_k}^k \|^2 \\
- (1-\theta) \mathbb{E} \left[ f(x_{S_k}^{k+1}) - f(x) - \langle A(x_{S_k}^{k+1} - x), \lambda^k \rangle + \beta_k \langle A(x_{S_k}^{k+1} - x), r_k^{k+1} \rangle \right] + \theta \frac{H_f}{2} \mathbb{E} \| x_{S_k}^{k+1} - x_{S_k}^k \|^2, \quad (51)
\end{align*} $$

where in the inequality, we have used the strong convexity of $f$ and Lipschitz continuity of $\nabla f$, and the last equality follows from the definition of $r^k$ and the update of $\lambda^k$. Also, from the strong convexity of $g$, we have

$$ \begin{align*}
\mathbb{E} \left[ \langle x_{S_k}^{k+1} - x_{S_k}, \tilde{\nabla} g_{S_k}(x_{S_k}^{k+1}) \rangle \right] \\
\geq \mathbb{E} \left[ g_{S_k}(x_{S_k}^{k+1}) - g_{S_k}(x_{S_k}) + \sum_{i \in S_k} \frac{H_g}{2} \| x_{S_k}^{k+1} - x_i \|^2 \right]
\end{align*} $$
In addition, note that

If Proposition B.1

\[ \text{and thus we omit the proofs.} \]

B.2 Proof of Theorem 3.3

\[
\sum_{k=1}^{t} \frac{\beta_k(k + k_0 + 1)}{2} \mathbb{E} \left( \|A(x^{k+1} - x)\|^2 - \|A(x^k - x)\|^2 \right) = \frac{\beta_t(t + k_0 + 1)}{2} \mathbb{E} \|A(x^{t+1} - x)\|^2. \tag{54}
\]

Proposition B.1 If \((20c)\) holds, then

Proposition B.2 If \((20f)\) and \((20g)\) hold, then

\[
\sum_{k=1}^{t} \frac{\beta_k(k + k_0 + 1)}{2} \mathbb{E} \|A(x^{k+1} - x^k)\|^2 - \sum_{k=1}^{t} \frac{k + k_0 + 1}{2} (\eta_k - L_m) \mathbb{E} \|x^{k+1} - x^k\|^2 \\
- \sum_{k=1}^{t} \frac{k + k_0 + 1}{2} \mathbb{E} \left( \eta_k \|x^{k+1} - x\|^2 - (\eta_k - \theta \mu_f) \|x^k - x\|^2 \right) \\
- \frac{\mu_g(t + k_0 + 1)}{2} \mathbb{E} \|x^{t+1} - x\|^2 - \sum_{k=2}^{t} \frac{\mu_g(\theta(k + k_0 + 1) - 1)}{2} \mathbb{E} \|x^k - x\|^2 \\
\leq \frac{(\eta_1 - \theta \mu_f)(k_0 + 2)}{2} \mathbb{E} \|x^1 - x\|^2 - \frac{(\mu_g + \eta_1)(t + k_0 + 1)}{2} \mathbb{E} \|x^{t+1} - x\|^2. \tag{55}
\]

Since \(Ax = b\), the above inequality reduces to (18) by using (5).
Proposition B.3 If (20c) and (20d) hold, then

$$-\frac{t + k_0 + 1}{2\rho_t} \mathbb{E} (||\lambda^{t+1} - \lambda||^2 - ||\lambda^t - \lambda||^2 + ||\lambda^{t+1} - \lambda^t||)$$

$$-\sum_{k=2}^{t} \frac{\theta(k + k_0 + 1) - 1}{2\rho_{k-1}} \mathbb{E} (||\lambda^k - \lambda||^2 - ||\lambda^{k-1} - \lambda||^2 + ||\lambda^k - \lambda^{k-1}||^2)$$

$$\leq \frac{\theta(k_0 + 3) - 1}{2\rho_1} \mathbb{E} ||\lambda^1 - \lambda||^2. \quad (56)$$

Now we are ready to prove Theorem 3.3.

Proof. [Proof of Theorem 3.3]

Multiplying $k + k_0 + 1$ to both sides of (18) and summing it up from $k = 1$ through $t$ gives

$$(t + k_0 + 1) \mathbb{E} \left[ \Phi(x^{t+1}, x, \lambda^{t+1}) + (\beta_t - \rho_t) ||x^{t+1}||^2 + \frac{\mu_t}{2} ||x^{t+1} - x||^2 \right]$$

$$+ \sum_{k=2}^{t} (\theta(k + k_0 + 1) - 1) \mathbb{E} \left[ \Phi(x^k, x, \lambda^k) + \frac{\mu_t}{2} ||x^k - x||^2 \right]$$

$$+ \sum_{k=2}^{t} ((\beta_{k-1} - \rho_{k-1})(k + k_0) - (1 - \theta)(k + k_0 + 1)\beta_k) \mathbb{E} ||r^k||^2$$

$$\leq (1 - \theta)(k_0 + 2) \mathbb{E} \left[ \Phi(x^1, x, \lambda^1) + \beta_1 ||r^1||^2 + \frac{\mu_t}{2} ||x^1 - x||^2 \right]$$

$$+ \sum_{k=2}^{t} \frac{\beta_k(k + k_0 + 1)}{2} \mathbb{E} \left( ||A(x^{k+1} - x)||^2 - ||A(x^k - x)||^2 + ||A(x^{k+1} - x^k)||^2 \right)$$

$$- \sum_{k=1}^{t} \frac{k + k_0 + 1}{2} \mathbb{E} \left( \eta_k||x^{k+1} - x||^2 - (\eta_k - \theta\mu_f)||x^k - x||^2 + (\eta_k - L_m)||x^{k+1} - x^k||^2 \right). \quad (57)$$

From the update of $\lambda$ in (17c), we have

$$\langle \lambda^{k+1} - \lambda, Ax^{k+1} - b \rangle = -\frac{1}{\rho_k} \langle \lambda^{k+1} - \lambda, \lambda^{k+1} - \lambda^k \rangle, \quad (58)$$

and thus

$$(t + k_0 + 1)\langle \lambda^{t+1} - \lambda, Ax^{t+1} - b \rangle + \sum_{k=2}^{t} (\theta(k + k_0 + 1) - 1) \langle \lambda^k - \lambda, Ax^k - b \rangle$$

$$= -\frac{t + k_0 + 1}{\rho_t} \langle \lambda^{t+1} - \lambda, \lambda^{t+1} - \lambda^t \rangle - \sum_{k=2}^{t} \frac{\theta(k + k_0 + 1) - 1}{\rho_{k-1}} \langle \lambda^k - \lambda, \lambda^k - \lambda^{k-1} \rangle$$

$$= -\frac{t + k_0 + 1}{2\rho_t} \left( ||\lambda^{t+1} - \lambda||^2 - ||\lambda^t - \lambda||^2 + ||\lambda^{t+1} - \lambda^t|| \right)$$

$$- \sum_{k=2}^{t} \frac{\theta(k + k_0 + 1) - 1}{2\rho_{k-1}} \left( ||\lambda^k - \lambda||^2 - ||\lambda^{k-1} - \lambda||^2 + ||\lambda^k - \lambda^{k-1}||^2 \right). \quad (59)$$
Adding (59) to (57) and rearranging terms yields

\[
(t + k_0 + 1)\mathbb{E}\Phi(x^{t+1}, x, \lambda) + \sum_{k=2}^{t} (\theta(k + k_0 + 1) - 1)\mathbb{E}\Phi(x^k, x, \lambda) \\
+(t + k_0 + 1)(\beta_t - \mu_t)\mathbb{E}\|x^{t+1}\|^2 + \sum_{k=2}^{t} ((\beta_{k-1} - \rho_{k-1})(k + k_0) - (1 - \theta)(k + k_0 + 1)\beta_k)\mathbb{E}\|r^k\|^2 \\
\leq (1 - \theta)(k_0 + 2)\mathbb{E} \left[ \Phi(x^1, x, \lambda^1) + \beta_1\|r^1\|^2 + \frac{\mu_q}{2}\|x^1 - x\|^2 \right] \\
+ \sum_{k=1}^{t} \frac{\beta_k(k + k_0 + 1)}{2} \mathbb{E} \left( \|A(x^{k+1} - x)\|^2 - \|A(x^k - x)\|^2 + \|A(x^{k+1} - x^k)\|^2 \right) \\
- \sum_{k=1}^{t} \frac{k + k_0 + 1}{2} \mathbb{E} \left( \eta_k\|x^{k+1} - x\|^2 - (\eta_k - \theta\mu_f)\|x^k - x\|^2 + (\eta_k - L_m)\|x^{k+1} - x^k\|^2 \right) \\
- \frac{\mu_g(t + k_0 + 1)}{2} \mathbb{E} \|x^{t+1} - x\|^2 - \sum_{k=2}^{t} \mu_g(\theta(k + k_0 + 1) - 1)\mathbb{E} \|x^k - x\|^2 \\
- \frac{t + k_0 + 1}{2\rho_t} (\|\lambda^{t+1} - \lambda\|^2 - \|\lambda^t - \lambda\|^2 + \|\lambda^{t+1} - \lambda^t\|^2) \\
- \sum_{k=2}^{t} \frac{\theta(k + k_0 + 1) - 1}{2\rho_{k-1}} (\|\lambda^k - \lambda\|^2 - \|\lambda^{k-1} - \lambda\|^2 + \|\lambda^k - \lambda^{k-1}\|^2).
\]

Therefore, we obtain (21) by substituting (54) through (56) into (60) and also noting the summation in the second line of (60) is nonnegative from the condition in (20b).

\[\square\]

\textbf{B.3 Proof of Theorem 3.4}

We first show that the parameters given in (22) satisfy the conditions in (20). Note that (23) implies \(k_0 \geq \frac{4}{\theta}\), and thus (20a) must hold. Also, it is easy to see that (20d) holds with equality from the second equation of (22a). Since \(I \geq \frac{A^\top A}{\|A\|^2}\), we can easily have (20f) by plugging in \(\beta_k\) and \(\eta_k\) defined in (22b) and (22c) respectively.

To verify (20c), we plug in \(\rho_k\) defined in the first equation of (22a), and it is equivalent to requiring for any \(2 \leq k \leq t - 1\) that

\[
\frac{\theta(k + k_0 + 1) - 1}{\theta(k - 1) + 2 + \theta} \geq \frac{\theta(k + k_0 + 2) - 1}{\theta k + 2 + \theta} \iff 1 + \frac{\theta(k_0 + 1) - 3}{\theta k + 2 + \theta} \geq 1 + \frac{\theta(k_0 + 1) - 3}{\theta k + 2 + \theta}.
\]

The second inequality obviously holds, and thus we have (20c).

Plugging in the formula of \(\beta_k\), (20e) is equivalent to

\[
(\theta k + 2 + \theta)(k + k_0 + 1) \geq (\theta k + 2)(k + k_0)
\]

which holds trivially, and thus (20e) follows.
Note that $\rho_k = \frac{\theta}{6-5\theta} \beta_k$ for $k \leq t-1$ from their definition given in (22a) and (22b). Hence, (20b) becomes

$$\frac{6 - 6\theta}{6 - 5\theta} \beta_{k-1}(k + k_0) \geq (1 - \theta)(k + k_0 + 1)\beta_k, \forall k \geq 2$$

$$\iff \frac{6}{6 - 5\theta} (\theta k + 2)(k + k_0) \geq (k + k_0 + 1)(\theta k + 2 + \theta), \forall k \geq 2$$

$$\iff \frac{6}{6 - 5\theta} \geq \frac{(k + k_0 + 1)(\theta k + 2 + \theta)}{(k + k_0)(\theta k + 2)}, \forall k \geq 2$$

$$\iff \frac{6}{6 - 5\theta} \geq \frac{(k_0 + 3)(3\theta + 2)}{(k + k_0 + 2)(2\theta + 2)}$$

$$\iff 12(3 + 2\theta) \geq 3(6 - 5\theta)(3\theta + 2)$$

$$\iff 36 + 24\theta \geq 36 + 24\theta - 45\theta^2,$$

where the sufficient condition in (62) uses the fact $k_0 \geq \frac{3}{\theta}$, and the last inequality apparently holds. Hence, (20b) is satisfied.

Finally, we show (20g). Plugging in $\eta_k$, we have that (20g) is equivalent to

$$(k + k_0)\left(\frac{\mu}{2}(\theta k + 2) + L_m\right) + \mu g \left(\theta (k + k_0 + 1) - 1\right) \geq (k + k_0 + 1)\left(\frac{\mu}{2}(\theta k + 2 + \theta) + L_m - \theta f\right), \forall k \geq 2$$

$$\iff \frac{\mu\theta}{2} (k + k_0 + 1) \geq \frac{\mu}{2} (\theta k + 2) + L_m + \mu g, \forall k \geq 2$$

$$\iff \frac{\mu\theta}{2} (k_0 + 1) \geq \mu + \mu g + L_m$$

$$\iff k_0 + 1 \geq \frac{2}{\theta} + \frac{2(L_m + \mu g)}{\mu\theta},$$

where the last inequality is implied by (23). Therefore, (20g) holds, and we have verified all conditions in (20).

Therefore, we have the inequality in (21) that, as $\lambda^1 = 0$, reduces to

$$(t + k_0 + 1)\mathbb{E}\Phi(x^{t+1}, x, \lambda) + \sum_{k=2}^{t} \theta(k + k_0 + 1) - 1\mathbb{E}\Phi(x^k, x, \lambda)$$

$$\leq (1 - \theta)(k_0 + 2)\mathbb{E}\left[F(x^1) - F(x) + \beta_1||x^1||^2 + \frac{\mu g}{2}||x^1 - x||^2\right] + \frac{k_0 + 2}{2}\mathbb{E}||x^{t+1} - x||^2$$(62)

$$+ (k_0 + 3) - \frac{1}{2\rho_1} \mathbb{E}||\lambda||^2 - \frac{t + k_0 + 1}{2} \mathbb{E}||x^{t+1} - x||^2_{(\mu g + \eta t)^\dagger - \beta_1 A^\top A}.$$
Letting $x = x^*$ and using the convexity of $F$, we have from (62) and the above inequality that
\[
\mathbb{E} \left[ F(\bar{x}^{t+1}) - F(x^*) - \langle \lambda, A\bar{x}^{t+1} - b \rangle \right] \leq \frac{1}{T} \mathbb{E} \phi(x^*, \lambda), \ \forall \lambda, \tag{63}
\]
which together with Lemmas 1.2 and 1.3 with $\gamma = \max(2\|\lambda^*\|, 1 + \|\lambda^*\|)$ indicates (24).

In addition, note
\[
\Phi(x^{t+1}, x^*, \lambda^*) \geq \frac{\mu_2}{2} \|x^{t+1} - x^*\|^2.
\]

Hence, letting $(x, \lambda) = (x^*, \lambda^*)$ in (62) and using (4), we have
\[
\frac{t + k_0 + 1}{2} \left( \frac{(\rho - 1)\mu}{2\rho} (\theta t + \theta + 2) + \mu + \mu_g + L_m \right) \mathbb{E} \|x^{t+1} - x^*\|^2 \leq \phi(x^*, \lambda^*), \tag{64}
\]
and the proof is completed.

C Technical proofs: Section 4

In this section, we provide the proofs of the lemmas and theorems in section 4.

C.1 Proof of Lemma 4.1

Similar to (51), we have
\[
\mathbb{E} \left\langle x_{S_k}^{k+1} - x_{S_k}, \nabla S_k f(x^k) - A_{S_k}^T (\lambda^k - \beta r^k) \right\rangle \\
\geq \mathbb{E} \left[ f(x^{k+1}) - f(x) - \langle A(x^{k+1} - x), \lambda^{k+1} \rangle + (\beta - \rho) \langle A(x^{k+1} - x), r^{k+1} \rangle \right. \\
\left. - \beta \langle A(x^{k+1} - x), A(x^{k+1} - x) + B(y^{k+1} - y^k) \rangle \right] - \frac{L_m}{2} \mathbb{E} \|x^{k+1} - x^k\|^2 \\
- (1 - \theta) \mathbb{E} \left[ f(x^k) - f(x) - \langle A(x^k - x), \lambda^k \rangle + \beta \langle A(x^k - x), r^k \rangle \right] + \frac{\theta \mu_f}{2} \mathbb{E} \|x^k - x\|^2. \tag{65}
\]

From (27), the optimality condition for $\tilde{y}^{k+1}$ is
\[
\nabla h(\tilde{y}^{k+1}) - B^T \lambda^k + \beta B^T r^{k+\frac{1}{2}} + \eta_y(\tilde{y}^{k+1} - y^k) = 0. \tag{66}
\]
Since
\[
\text{Prob}(y^{k+1} = \tilde{y}^{k+1}) = \theta, \quad \text{Prob}(y^{k+1} = y^k) = 1 - \theta,
\]
we have
\[
\mathbb{E} \left\langle y^{k+1} - y, \nabla h(y^{k+1}) - B^T \lambda^k + \beta B^T r^{k+\frac{1}{2}} + \eta_y(y^{k+1} - y^k) \right\rangle \\
= (1 - \theta) \mathbb{E} \left\langle y^k - y, \nabla h(y^k) - B^T \lambda^k + \beta B^T r^{k+\frac{1}{2}} \right\rangle,
\]
or equivalently,
\[
\mathbb{E}\left\langle y^{k+1} - y, \nabla h(y^{k+1}) - B^\top \lambda^{k+1} + (\beta - \rho)B^\top r^{k+1} - \beta B^\top (y^{k+1} - y) + \eta_y(y^{k+1} - y^*)\right\rangle
\]
\[
= (1 - \theta)\mathbb{E}\left\langle y^{k} - y, \nabla h(y^{k}) - B^\top \lambda^{k} + \beta B^\top r^{k}\right\rangle + \beta(1 - \theta)\mathbb{E}\left\langle B(y^k - y), A(x^{k+1} - x^k)\right\rangle. \quad (67)
\]

Plugging (65), (52) and (53) with \(\eta_k = \eta_x\) into (50) with \(\beta_k = \beta\) and summing together with (67), we have, by rearranging the terms, that
\[
\mathbb{E}\left[ F(x^{k+1}) - F(x) + \langle y^{k+1} - y, \nabla h(y^{k+1}) \rangle + \langle x^{k+1} - x, -A^\top \lambda^{k+1} \rangle + \langle y^{k+1} - y, -B^\top \lambda^{k+1} \rangle \right] \\
- \frac{L_m}{2} \mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{\theta \mu_f}{2} \mathbb{E}\|x^k - x\|^2 + \frac{\mu_B}{2} \mathbb{E}\|x^{k+1} - x\|^2 \\
+ (\beta - \rho) \mathbb{E}\left[ \langle x^{k+1} - x, A^\top r^{k+1} \rangle + \langle y^{k+1} - y, B^\top r^{k+1} \rangle \right] \\
+ \mathbb{E}\left\langle x^{k+1} - x, (\eta_y I - \beta A^\top)A(x^{k+1} - x^k) \right\rangle + \mathbb{E}\left\langle y^{k+1} - y, (\eta_y I - \beta B^\top)B(y^{k+1} - y^*) \right\rangle \\
\leq (1 - \theta)\mathbb{E}\left[ F(x^k) - F(x) + \langle y^k - y, \nabla h(y^k) \rangle + \langle x^k - x, -A^\top \lambda^k \rangle + \langle y^k - y, -B^\top \lambda^k \rangle \right] \\
+ \frac{(1 - \theta)\mu_f}{2} \mathbb{E}\|x^k - x\|^2 + \beta(1 - \theta) \mathbb{E}\left[ \langle x^k - x, A^\top r^k \rangle + \langle y^k - y, B^\top r^k \rangle \right] \\
+ \beta \mathbb{E}\left\langle A(x^{k+1} - x), B(y^{k+1} - y^*) \right\rangle + \beta(1 - \theta)\mathbb{E}\left\langle B(y^k - y), A(x^{k+1} - x^k)\right\rangle.
\]

Now use (58) and the fact that \(Ax + By = b\) to have the desired result.

### C.2 Proof of Theorem 4.2

Before proving Theorem 4.2, we establish a few inequalities. First, using Young’s inequality, we have the following results.

**Lemma C.1** For any \(\tau_1, \tau_2 > 0\), it holds that
\[
\langle A(x^{k+1} - x^*), B(y^{k+1} - y^*) \rangle \leq \frac{1}{2\tau_1} \|A(x^{k+1} - x^*)\|^2 + \frac{\tau_1}{2} \|B(y^{k+1} - y^*)\|^2, \quad (68)
\]
\[
\langle B(y^k - y^*), A(x^{k+1} - x^k) \rangle \leq \frac{1}{2\tau_2} \|B(y^k - y^*)\|^2 + \frac{\tau_2}{2} \|A(x^{k+1} - x^k)\|^2. \quad (69)
\]

In addition, we are able to bound the \(\lambda\)-term by \(y\)-term and the residual \(r\). The proofs are given in Appendix C.4 and C.5.

**Lemma C.2** For any \(\delta > 0\), we have
\[
\mathbb{E}\|B^\top(\lambda^{k+1} - \lambda^*)\|^2 - (1 - \theta)(1 + \delta)\mathbb{E}\|B^\top(\lambda^k - \lambda^*)\|^2 \leq 4\mathbb{E}\left[ L_{\eta}^2 \|y^{k+1} - y^*\|^2 + \|Q(y^{k+1} - y^k)\|^2 \right] + 2(\beta - \rho)^2 \mathbb{E}\|B^\top r^{k+1}\|^2 \\
+ 2\rho^2(1 - \theta)(1 + \frac{1}{\delta})\mathbb{E}\|B^\top r^{k+1}\|^2 + \|B^\top B(y^{k+1} - y^k)\|^2. \quad (70)
\]
Proof. Now we are ready to show Theorem 4.2.

Adding (68), (69), and (70) multiplied by the number $\theta$

Lemma C.3 Assume (34). Then

$$\|B^T(\lambda^{k+1} - \lambda^*)\|^2 - (1 - \theta)(1 + \delta)\|B^T(\lambda^k - \lambda^*)\|^2 + \kappa\|B^T(\lambda^{k+1} - \lambda^k)\|^2 \geq \frac{\sigma_{\text{min}}(BB^T)}{2}[\|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^k - \lambda^*\|^2 + \frac{1}{\theta}\|\lambda^{k+1} - \lambda^k\|^2],$$

(71)

where $\sigma_{\text{min}}(BB^T)$ denotes the smallest singular value of $BB^T$.

Now we are ready to show Theorem 4.2.

Proof. [Proof of Theorem 4.2]

Letting $(x, y, \lambda) = (x^*, y^*, \lambda^*)$ in (31), plugging (29) into it, noting $Ax^* + By^* = b$, and using (5), we have

$$\mathbb{E}\Psi(z^{k+1}, z^*) + (\beta - \rho)\mathbb{E}\|r^{k+1}\|^2 + \frac{1}{2}\mathbb{E}\left[\|x^{k+1} - x^*\|^2 + \|x^k - x^*\|^2 + \|x^{k+1} - x^k\|^2\right]$$

$$+ \frac{1}{2}\mathbb{E}\left[\|y^{k+1} - y^*\|^2 - \|y^k - y^*\|^2\right] - \frac{L_m}{2}\mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{\theta\mu_f}{2}\mathbb{E}\|x^k - x^*\|^2$$

$$+ \frac{\mu_g}{2}\mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{1}{2\rho}\mathbb{E}[\|\lambda^{k+1} - \lambda^*\|^2 - \|\lambda^k - \lambda^*\|^2 + \|\lambda^{k+1} - \lambda^k\|^2]$$

$$\leq (1 - \theta)\mathbb{E}\Psi(z^k, z^*) + (1 - \theta)\mathbb{E}\|r^k\|^2 + \frac{1 - \theta}{2\rho}\mathbb{E}[\|\lambda^k - \lambda^*\|^2 - \|\lambda^{k-1} - \lambda^*\|^2 + \|\lambda^k - \lambda^{k-1}\|^2]$$

$$+ \beta\mathbb{E}\langle A(x^{k+1} - x^*), B(y^{k+1} - y^*) \rangle + (1 - \theta)\alpha B\mathbb{E}\langle B(y^k - y^*), A(x^{k+1} - x^*) \rangle + \frac{\mu_g(1 - \theta)}{2}\mathbb{E}\|x^k - x^*\|^2,$$

(72)

where $\Psi$ is defined in (32). Adding $\frac{\theta}{\rho}\mathbb{E}\|\lambda^k - \lambda^*\|^2$ to both sides of (72) and using (28) gives

$$\mathbb{E}\Psi(z^{k+1}, z^*) + (\beta - \rho)\mathbb{E}\|r^{k+1}\|^2 + \frac{1}{2}\mathbb{E}\left[\|x^{k+1} - x^*\|^2 + \|x^k - x^*\|^2 + \|x^{k+1} - x^k\|^2\right]$$

$$- \frac{L_m}{2}\mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{\theta\mu_f}{2}\mathbb{E}\|x^k - x^*\|^2 + \frac{\mu_g}{2}\mathbb{E}\|x^{k+1} - x^k\|^2$$

$$+ \frac{1}{2\rho}\mathbb{E}[\|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^k - \lambda^*\|^2 + \frac{1}{\theta}\|\lambda^{k+1} - \lambda^k\|^2] - \frac{\rho}{2}\mathbb{E}\|r^{k+1}\|^2$$

$$\leq (1 - \theta)\mathbb{E}\Psi(z^k, z^*) + \beta(1 - \theta)\mathbb{E}\|r^k\|^2$$

$$+ \frac{1}{2\rho}\mathbb{E}[\|\lambda^k - \lambda^*\|^2 - (1 - \theta)\|\lambda^{k-1} - \lambda^*\|^2 + \frac{1}{\theta}\|\lambda^k - \lambda^{k-1}\|^2] - \frac{\rho}{2}\mathbb{E}\|r^k\|^2$$

$$+ \beta(1 - \theta)\mathbb{E}\langle A(x^{k+1} - x^*), B(y^{k+1} - y^*) \rangle + \beta(1 - \theta)\mathbb{E}\langle B(y^k - y^*), A(x^{k+1} - x^*) \rangle + \frac{\mu_g(1 - \theta)}{2}\mathbb{E}\|x^k - x^*\|^2,$$

(73)

Adding (68), (69), and (70) multiplied by the number $c > 0$ to (73), and also noting $\rho = \theta\beta$, we
have

\[
\begin{align*}
\mathbb{E}(z^{k+1}, z^*) + \frac{\beta(1 - \theta)}{2}\mathbb{E}\|x^{k+1}\|^2 + \frac{1}{2}\mathbb{E}\|x^{k+1} - x^*\|^2_P - \|x^k - x^*\|^2_P + \|x^{k+1} - x^k\|^2_P
&- \frac{L_n}{2}\mathbb{E}\|x^{k+1} - x^k\|^2 + \frac{\mu_f}{2}\mathbb{E}\|x^k - x^*\|^2 + \frac{\mu_g}{2}\mathbb{E}\|x^{k+1} - x^*\|^2 \\
&+ \frac{1}{2}\mathbb{E}[\|y^{k+1} - y^*\|^2_Q - \|y^k - y^*\|^2_Q + \|y^{k+1} - y^k\|^2_Q] \\
&- 4c\mathbb{E}\left(\left[\frac{L_n^2}{2}\|y^{k+1} - y^*\|^2 + \|Q(y^{k+1} - y^k)\|^2\right] + \frac{\mu^2}{2}(1 - \theta)(1 + \frac{1}{\delta})\|B^\top B(y^{k+1} - y^k)\|^2\right) \\
&+ \frac{1}{2\rho}\mathbb{E}\left[\|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^k - \lambda^*\|^2 + \frac{1}{\delta}\|\lambda^{k+1} - \lambda^k\|^2\right] \\
&+ c\mathbb{E}\left[\|B^\top (\lambda^{k+1} - \lambda^*)\|^2 - (1 - \theta)(1 + \delta)\|B^\top (\lambda^k - \lambda^*)\|^2 + \kappa\|B^\top (\lambda^{k+1} - \lambda^k)\|^2\right] \\
&- c\kappa\mathbb{E}\|B^\top (\lambda^{k+1} - \lambda^k)\|^2 - 2c(\beta - \rho)^2\mathbb{E}\|B^\top r^{k+1}\|^2 - 2c(\beta - \rho)^2\mathbb{E}\|B^\top r^{k+1}\|^2 \\
&\leq (1 - \theta)\Psi(z^k, z^*) + \frac{\beta(1 - \theta - \theta^2)}{2}\mathbb{E}\|x^k\|^2 + \beta\mathbb{E}\left(\frac{1}{2\tau_1}\|A(x^{k+1} - x^*)\|^2 + \frac{\tau_1}{2}\|B(y^{k+1} - y^k)\|^2\right) \\
&+ \frac{1}{2\rho}\mathbb{E}\left[\|\lambda^k - \lambda^*\|^2 - (1 - \theta)\|\lambda^{k-1} - \lambda^*\|^2 + \frac{1}{\delta}\|\lambda^k - \lambda^{k-1}\|^2\right] \\
&+ \beta(1 - \theta)\mathbb{E}\left(\frac{1}{2\tau_2}\|B(y^k - y^*)\|^2 + \frac{\tau_2}{2}\|A(x^{k+1} - x^k)\|^2\right) + \frac{\mu_g(1 - \theta)}{2}\mathbb{E}\|x^k - x^*\|^2.
\end{align*}
\]

From (35a) and (35b), we have

\[
\frac{1}{2}\|x^{k+1} - x^k\|^2_P \geq \beta(1 - \theta)\frac{\tau_2}{2}\|A(x^{k+1} - x^k)\|^2 + \frac{L_n}{2}\|x^{k+1} - x^k\|^2, 
\]

and

\[
\frac{1}{2}\|y^{k+1} - y^k\|^2_Q \geq 4c\|Q(y^{k+1} - y^k)\|^2 + 2c\rho^2(1 - \theta)(1 + \frac{1}{\delta})\|B^\top B(y^{k+1} - y^k)\|^2 + \beta\frac{\tau_1}{2}\|B(y^{k+1} - y^k)\|^2.
\]

and from (30), it follows that

\[
\Psi(z^{k+1}, z^*) \geq (1 - \alpha)\Psi(z^k, z^*) + \frac{\alpha\mu}{2}\|x^{k+1} - x^*\|^2 + \alpha\nu\|y^{k+1} - y^*\|^2.
\]

In addition, note that

\[
\|r^{k+1}\|^2 = \|Ax^{k+1} + By^{k+1} - (Ax^* + By^*)\|^2 \\
\leq 2\|A\|^2\|x^{k+1} - x^*\|^2 + 2\|B\|^2\|y^{k+1} - y^*\|^2 \\
\leq \gamma\left(\frac{\alpha\mu}{4}\|x^{k+1} - x^*\|^2 + \frac{\alpha\nu}{4}\|y^{k+1} - y^*\|^2\right).
\]

Adding (75) through (78) multipliend by c to (74) and rearranging terms gives (36). □
C.3 Proof of Theorem 4.3

First, from $0 < \alpha < \theta$, the full row-rankness of $B$, and (37), it is easy to see that $\eta > 1$. Second, we note the following inequalities:

\[
\Psi\text{-term:} \quad (1 - \alpha)\Psi(z^{k+1}, z^*) \geq \eta(1 - \theta)\Psi(z^{k+1}, z^*),
\]

\[
r\text{-term:} \quad \left(\frac{\beta(1 - \theta)}{2} + \frac{1}{\gamma}\right)\|r^{k+1}\|^2 - \left(c\rho\left(\kappa + 2(1 - \theta)\left(1 + \frac{1}{\delta}\right)\right) + 2c(\beta - \rho)^2\right)\|B^\top r^{k+1}\|^2
\]

\[
\geq \left(\frac{\beta(1 - \theta)}{2} + \frac{1}{\gamma}\right)\|r^{k+1}\|^2 - \left(c\rho\left(\kappa + 2(1 - \theta)\left(1 + \frac{1}{\delta}\right)\right) + 2c(\beta - \rho)^2\right)\|r^{k+1}\|^2
\]

\[
\geq \eta \left(\frac{\beta(1 - \theta)}{2}\right)\|r^{k+1}\|^2,
\]

\[
x\text{-term:} \quad \frac{1}{2}\|x^{k+1} - x^*\|^2_p + \left(\frac{\alpha\mu}{4} + \frac{\mu_g}{2}\right)\|x^{k+1} - x^*\|^2 - \frac{\beta}{2\tau_1}\|A(x^{k+1} - x^*)\|^2
\]

\[
\geq \eta \left(\frac{1}{2}\|x^{k+1} - x^*\|^2_p + \frac{\mu_g - \theta\mu}{2}\|x^{k+1} - x^*\|^2\right),
\]

\[
y\text{-term:} \quad \frac{1}{2}\|y^{k+1} - y^*\|^2_q + \frac{3\alpha\nu}{4}\|y^{k+1} - y^*\|^2 - 4cL_h^2\|y^{k+1} - y^*\|^2
\]

\[
\geq \eta \left(\frac{1}{2}\|y^{k+1} - y^*\|^2_q + \frac{\beta(1 - \theta)}{2\tau_2}\|y^{k+1} - y^*\|^2\right),
\]

\[
\lambda\text{-term:} \quad \left(\frac{1}{2\rho} + \frac{c}{2}\sigma_{\min}(BB^\top)\right)\left[\|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^k - \lambda^*\|^2 + \frac{1}{\theta}\|\lambda^{k+1} - \lambda^k\|^2\right]
\]

\[
\geq \frac{\eta}{2\rho} \left[\|\lambda^{k+1} - \lambda^*\|^2 - (1 - \theta)\|\lambda^k - \lambda^*\|^2 + \frac{1}{\theta}\|\lambda^{k+1} - \lambda^k\|^2\right].
\]

Summing the above inequalities and using (36) gives (38) and completes the proof.

C.4 Proof of Lemma C.2

Let

\[
\tilde{\lambda}^{k+1} = \lambda^k - \rho(Ax^{k+1} + By^{k+1} - b),
\]

which together with (66) implies

\[
B^\top \tilde{\lambda}^{k+1} = \nabla h(y^{k+1}) + Q(y^{k+1} - y^k) + (\beta - \rho)B^\top (Ax^{k+1} + By^{k+1} - b).
\] (79)

Then

\[
\mathbb{E}\|B^\top (\lambda^{k+1} - \lambda^*)\|^2
\]

\[
= \theta\mathbb{E}\|B^\top (\tilde{\lambda}^{k+1} - \lambda^*)\|^2 + (1 - \theta)\mathbb{E}\|B^\top (\lambda^k - \lambda^* - \rho(Ax^{k+1} + By^k - b))\|^2
\]

\[
\leq \theta\mathbb{E}\|B^\top (\tilde{\lambda}^{k+1} - \lambda^*)\|^2 + (1 - \theta)(1 + \delta)\mathbb{E}\|B^\top (\lambda^k - \lambda^*)\|^2 + \rho^2(1 - \theta)(1 + \frac{1}{\delta})\mathbb{E}\|B^\top (Ax^{k+1} + By^k - b)\|^2
\]

33
\[ \begin{align*}
\sum_{k=1}^{\infty} \theta & E \| \nabla h(y^{k+1}) - \nabla h(y^*) + Q(y^{k+1} - y^k) + \beta - \rho \rangle B^T (Ax^{k+1} + By^{k+1} - b) \|^2 \\
& + (1 - \theta)(1 + \delta) E \| B^T (\lambda^k - \lambda^*) \|^2 + \rho^2 (1 - \theta)(1 + \frac{1}{\delta}) E \| B^T (Ax^{k+1} + By^k - b) \|^2 \\
& \leq 2 \theta E \| \nabla h(y^{k+1}) - \nabla h(y^*) + Q(y^{k+1} - y^k) \|^2 + 2 \theta (\beta - \rho)^2 E \| B^T (Ax^{k+1} + By^{k+1} - b) \|^2 \\
& + (1 - \theta)(1 + \delta) E \| B^T (\lambda^k - \lambda^*) \|^2 + \rho^2 (1 - \theta)(1 + \frac{1}{\delta}) E \| B^T (Ax^{k+1} + By^k - b) \|^2 \\
& \leq E \left[ 4L^2 \| y^{k+1} - y^* \|^2 + 4 | Q(y^{k+1} - y^k) |^2 + (1 - \theta)(1 + \delta) E \| B^T (\lambda^k - \lambda^*) \|^2 \\
& + 2 \rho^2 (1 - \theta)(1 + \frac{1}{\delta}) E \| B^T r^{k+1} \|^2 + \| B^T B(y^{k+1} - y^k) \|^2 \right] + 2 (\beta - \rho)^2 E \| B^T r^{k+1} \|^2,
\end{align*} \]
which completes the proof.

### C.5 Proof of Lemma C.3

Note that

\[ \begin{align*}
\| B^T (\lambda^k - \lambda^*) \|^2 & - (1 - \theta)(1 + \delta) \| B^T (\lambda^k - \lambda^*) \|^2 + \kappa \| B^T (\lambda^k - \lambda^*) \|^2 \\
& = (1 - (1 - \theta)(1 + \delta)) \| B^T (\lambda^k - \lambda^*) \|^2 + 2 (1 - \theta)(1 + \delta) \| B^T (\lambda^k - \lambda^*) \| B^T (\lambda^{k+1} - \lambda^k) \\
& + (\kappa - (1 - \theta)(1 + \delta)) \| B^T (\lambda^{k+1} - \lambda^k) \|^2 \\
& = \begin{bmatrix} B^T (\lambda^k - \lambda^*) \\ B^T (\lambda^{k+1} - \lambda^k) \end{bmatrix}^T \begin{bmatrix} (1 - (1 - \theta)(1 + \delta)) I & (1 - \theta)(1 + \delta) I \\
(1 - \theta)(1 + \delta) I & (\kappa - (1 - \theta)(1 + \delta)) I \end{bmatrix} \begin{bmatrix} B^T (\lambda^k - \lambda^*) \\ B^T (\lambda^{k+1} - \lambda^k) \end{bmatrix} \\
& \geq \frac{1}{2} \begin{bmatrix} B^T (\lambda^{k+1} - \lambda^*) \\ B^T (\lambda^{k+1} - \lambda^k) \end{bmatrix}^T \begin{bmatrix} \theta I & (1 - \theta) I \\
(1 - \theta) I & (\frac{1}{\theta} - (1 - \theta)) I \end{bmatrix} \begin{bmatrix} B^T (\lambda^{k+1} - \lambda^*) \\ B^T (\lambda^{k+1} - \lambda^k) \end{bmatrix} \\
& \geq \frac{\sigma_{\min}(BB^T)}{2} \begin{bmatrix} \lambda^{k+1} - \lambda^* \\ \lambda^{k+1} - \lambda^k \end{bmatrix}^T \begin{bmatrix} \theta I & (1 - \theta) I \\
(1 - \theta) I & (\frac{1}{\theta} - (1 - \theta)) I \end{bmatrix} \begin{bmatrix} \lambda^{k+1} - \lambda^* \\ \lambda^{k+1} - \lambda^k \end{bmatrix} \\
& \geq \frac{\sigma_{\min}(BB^T)}{2} \left[ \| \lambda^{k+1} - \lambda^* \|^2 - (1 - \theta)\| \lambda^k - \lambda^* \|^2 + \frac{1}{\theta} \| \lambda^{k+1} - \lambda^k \|^2 \right],
\end{align*} \]

which completes the proof.