MPMICE: A hybrid MPM-CFD model for simulating coupled problems in porous media. Application to earthquake-induced submarine landslides

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Summary
In this paper, we describe a soil-fluid-structure interaction model that combines soil mechanics (saturated sediments), fluid mechanics (seawater or air), and solid mechanics (structures). The formulation combines the Material Point Method, which models large deformation of the porous media and the structure, with the Implicit Continuous-fluid Eulerian, which models complex fluid flows. We validate the model and simulate the whole process of earthquake-induced submarine landslides. We show that this model captures complex interactions between saturated sediment, seawater, and structure, so we can use the model to estimate the impact of potential submarine landslides on offshore structures.

KEYWORDS
material point method, MPMICE, submarine landslide

1 | INTRODUCTION

Many geological natural processes and their interactions with man-made structures are influenced by soil-fluid-structure interactions. The prediction of these processes requires a tool that can capture complex interactions between soil, fluid, and structure, such as the process of submarine landslides. Indeed, The offshore infrastructure as well as coastal communities may be vulnerable to submarine landslides. Submarine landslides contain three stages: triggering, failure, and post-failure. Erosion or earthquakes can trigger slope failures in the first stage. Following the failure, sediments move quickly after the post-failure stage. In other words, solid-like sediments will behave like a fluid after failure. This transition, where the sediment transforms into fluid-like debris and then re-establishes a porous medium, poses a challenging task for simulating submarine landslides.

Due to this transition, submarine landslides can be modeled using either the Eulerian framework or the Lagrangian framework. The Eulerian framework involves the calculation of material response at specific time-space points. For instance, methods within Computational Fluid Dynamics, for example using Finite Volume Methods (FVM) are employed to simulate submarine landslides1-4 by solving governing equations in a full-Eulerian framework. While FVM is capable of handling complex flows, such as turbulent flows, it falls short of accounting for the triggering mechanisms of submarine landslides. This is due to the challenge of incorporating 'constitutive laws' of sediment materials within the Eulerian framework. This is particularly significant because converting material time derivatives into fixed space derivatives involves arduous mathematical tasks, especially for soil materials that rely on nonlinear tensor operations and history-dependent state/internal variables. In contrast, the Lagrangian framework, including various particle-based...
methods, provides a solution to this problem. In the Lagrangian framework, material “particles” are tracked individually through space, and material properties and internal variables are determined at and follow these particles. These methods have been extensively used to simulate landslides, like Material Point Method (MPM), Smooth Particle Hydro Dynamics, Particle Finite Element Method, or Coupled Eulerian-Lagrangian Method. For simplicity, these previous simulations have adopted a total stress analysis, neglecting the pore pressure development which is a key factor triggering slope failures.

Recent advancements in particle-based Lagrangian methods have allowed for the modeling of fluid flows in porous media using sets of Lagrangian particles. Within the MPM family, there is a specific approach known as double-point MPM. In this method, fluid particles and solid particles are overlaid within a single computational grid. However, it is important to note that particle-based methods encounter numerical instability when modeling fluid flows. To address this, various numerical techniques are employed, including the B-bar method, null-space filter, or least square approximation. These methods are necessary, especially when dealing with complex and turbulent fluid flows, such as those seen in submarine landslides. In such scenarios, Eulerian methods like FVM/CFD are preferred due to their computational efficiency, particularly when turbulence occurs at fine resolutions. CFD has even been employed in combination with the Discrete Element Method to study granular grain-fluid interactions, allowing for the examination of micro-scale behavior and realistic grain morphology. However, the computational demands of Discrete Element Methods can be quite challenging when applied to practical scenarios. Therefore, an ideal approach might involve the integration of CFD with particle-based continuum methods. Additionally, MPM can also be coupled with thermal effects, opening up the possibility of capturing hydro-thermal-mechanical coupling. Over the past two decades, more than 50 particle-based methods have been developed to address the simulation of large deformations in solids. Among these, MPM emerges as a strong candidate for coupling with CFD. This is because MPM incorporates a stationary mesh during computation, just like CFD. As such, MPM and CFD can be seamlessly integrated within a unified computational mesh, offering a promising approach for tackling complex fluid-solid interactions.

A numerical method for simulating soil-fluid-structure interaction (Figure 1) involving large deformations, is presented in this work in order to simulate the interaction between sediment (soil), seawater (fluid) and offshore structures (structure) namely MPMICE (Figure 2). In the MPMICE, the Material Point Method (MPM) is coupled with the Implicit Continuous Eulerian (ICE). The MPM method is a particle method that allows the porous soil to undergo arbitrary distortions. The ICE method, on the other hand, is a conservative finite volume technique with all state variables located at the cell center (temperature, velocity, mass, pressure). The ICE method offers certain advantages in comparison to conventional FVM in the realm of flow computation encompassing all velocity ranges. An initial technical report at Los Alamos National Laboratory provided the theoretical and algorithmic foundation for the MPMICE, followed by the MPMICE development and implementation in the high-performance Uintah computational framework for simulating fluid-structure interactions. This paper primarily contributes further to the development of the MPMICE for analyzing the soil-fluid-structure interaction, since sediment should be considered as a porous media (soil) and not as a solid to

![Figure 1: Interaction between soil-fluid-structure.](image-url)
FIGURE 2 Coupling of soil-water-structure interaction using MPMICE.

capture the evolution of the pore water pressure. Baumgarten et al.\textsuperscript{24} made the first attempt at coupling the FVM with the MPM for the simulation of soil-fluid interaction by using an explicit time integration for the single-phase flow. In contrast to the mentioned work, we use implicit time integration for the multi phase flows.

2 | THEORY AND FORMULATION

This section lay out the theoretical framework for the MPMICE model. We use the common notation of the continuum mechanics with vector and tensor denoted simply by using bold font and scalar denoted by using normal font. The notation are shown in Nomenclature.

2.1 | Assumptions

The following assumptions are made for the MPMICE model.

1. Solid phases (MPM) are described in a Lagrangian formulation while fluid phases (ICE) are described in an Eulerian formulation in the framework of continuum mechanics and mixture theory.
2. Solid grains are incompressible while the fluid phases are compressible.
3. There is no mass exchange between solid and fluid phases.
4. Terzaghi’s effective stress is valid.

2.2 | Governing equations

A representative element volume $\Omega$ is decomposed by two domains: solid domains $\Omega_s$ and fluid domains $\Omega_f$. Then, all domains are homogenized into two overlapping continua. Considering the volume fraction of solid $\phi_s = \Omega_s/\Omega$ and fluid $\phi_f = \Omega_f/\Omega$ with the true (or Eulerian) porosity $n = \sum \phi_f$ of the representative element volume, the average density of solid and fluid phases are defined as:

$$\bar{\rho}_s = \phi_s \rho_s, \quad \bar{\rho}_f = \phi_f \rho_f$$

The mass of solid and fluid phases are:

$$m_s = \int_{\Omega_s} \rho_s dV = \bar{\rho}_s V, \quad m_f = \int_{\Omega_f} \rho_f dV = \bar{\rho}_f V$$

(1)
Reviewing the Terzaghi’s effective stress concept for the saturated porous media, the total stress \( \sigma \) is calculated by:

\[
\sigma = \sigma' - p_f I
\]  

The balance equations are derived based on the mixture theory. The representative thermodynamic state of the fluid phases are given by the vector \([m_f, U_f, e_f, T_f, \rho_f] \) which are mass, velocity, internal energy, temperature, specific volume. The representative state of the solid phases are given by the vector \([m_s, U_s, e_s, T_s, \sigma', p_f] \) which are mass, velocity, internal energy, temperature, effective stress and pore water pressure. The derivation is presented in detail in the Appendix.

**Mass conservation**

The mass balance equations for both fluid (e.g., water, air) and solid phases are:

\[
\frac{1}{V} \frac{\partial m_f}{\partial t} + \nabla \cdot (m_f U_f) = 0, \quad \frac{1}{V} \frac{D_s m_s}{Dt} = 0
\]  

Solving the mass balance equation of the solid phase leads to:

\[
\frac{D_s n}{Dt} = \phi_s \nabla \cdot U_s
\]  

**Momentum conservation**

The momentum balance equations for each fluid phases (e.g., water, air) are:

\[
\frac{1}{V} \left[ \frac{\partial (m_f U_f)}{\partial t} + \nabla \cdot (m_f U_f U_f) \right] = -\phi_f \nabla p_f + \nabla \cdot \tau_f + \rho_f b + \sum f_d
\]  

The momentum balance equations for each solid phases are:

\[
\frac{1}{V} \frac{D_s (m_s U_s)}{Dt} = \nabla \cdot (\sigma') - \phi_s \nabla p_f + \rho_s b + \sum f_{fric} - \sum f_d
\]  

**Energy conservation**

The internal energy balance equations for each fluid phases (e.g., water, air) are:

\[
\frac{1}{V} \left[ \frac{\partial (m_f e_f)}{\partial t} + \nabla \cdot (m_f e_f U_f) \right] = -\rho_f \frac{D_f p_f}{Dt} + \tau_f : \nabla U_f + \nabla \cdot q_f + \sum q_{sf}
\]  

The internal energy balance equations for each solid phases are:

\[
\frac{m_s}{V} c_p \frac{D_s (T_s)}{Dt} = \sigma' : \frac{D_s (e_s^p)}{Dt} + \nabla \cdot q_s - \sum q_{sf}
\]  

where \( c_p \) is the specific heat at constant volume of the solid materials.

Closing the systems of equations, the following additional models are needed:

1. A constitutive equation to describe the stress-strain behaviour of solid phase (computing effective stress \( \sigma' \)).
2. Optional turbulent model to compute the viscous shear stress \( \tau_f \).
3. Frictional forces \( f_{fric} \) for the contact for soil-structure interaction between solid/porous materials with the friction coefficient \( \mu_{fric} \).
4. Exchange momentum models (computing drag force \( f_d \)) for interaction between materials.
5. Energy exchange models (computing temperature exchange term \( q_{sf} \)) for interaction between materials.
6. An equation of state to establish relations between thermodynamics variables of each fluid materials \([P_f, \rho_f, \tau_f, T_f, e_f]\).
7. Thermal conduction model to compute thermal flux of solid phase \( q_s \) and liquid phase \( q_s \).
Four thermodynamic relations for the equation of states are:

\[ e_f = e_f(T_f, \rho_f) \]
\[ P_f = P_f(T_f, \rho_f) \]
\[ \phi_f = \alpha_f \bar{\rho}_f \]
\[ 0 = n - \sum_{f=1}^{N_f} \rho_f \bar{\rho}_f \]  \hspace{1cm} (10)

2.2.1 | Constitutive soil model

As a result of the explicit MPM formulation, we can derive the constitutive law in the updated Lagrangian framework of "small strain-large deformation". Therefore, the rotation of the particles (representative element volume) is manipulated by rotating the Cauchy stress tensor. First, the deformation gradient is decomposed into the polar rotation tensor \( R_{n}^{s+1} \) and stretch tensor \( V_{n}^{s+1} \) as:

\[ F_{n}^{s+1} = V_{n}^{s+1} R_{n}^{s+1} \]  \hspace{1cm} (11)

Then, before calling the constitutive model, the stress and strain rate tensor are rotated to the reference configuration as:

\[ \sigma'^{n,s} = (R_{n}^{s+1})^T \sigma^{n,s} R_{n}^{s+1} \]  \hspace{1cm} (12)
\[ \delta e^{n,s} = (R_{n}^{s+1})^T \delta e^{n,s} R_{n}^{s+1} \]  \hspace{1cm} (13)

Using the constitutive model with the input tensors \( \sigma'^{n,s}, \delta e^{n,s} \) to compute the Cauchy stress tensor at the advanced time step \( \sigma'^{n+1,s} \) then rotating it back to current configuration as:

\[ \sigma'^{n+1,s} = R_{n}^{s} \sigma'^{n+1,s} (R_{n}^{s+1})^T \]  \hspace{1cm} (14)

In this paper, we adopt the hyper-elastic Neo-Hookean model for the structure materials and additionally Mohr-Coulomb failure criteria for the soil (porous media) materials. The Cauchy stress of the hyper-elastic Neo-Hookean model can be written as:

\[ \sigma' = \lambda \ln(J) + \mu (FF^T - J) \]  \hspace{1cm} (15)

where \( \lambda \) and \( \mu \) are bulk and shear modulus and \( J \) is the determinant of the deformation gradient \( F \). And the yield function \( f \) and flow potentials \( g \) of the Mohr-Coulomb can be written as:

\[ f = -\sigma'_1 + \sigma'_3 + 2c' \cos(\phi') + (\sigma'_1 + \sigma'_3) \sin(\phi') \]
\[ g = -\sigma'_1 + \sigma'_3 + 2c' \cos(\psi') + (\sigma'_1 + \sigma'_3) \sin(\psi') \]  \hspace{1cm} (16)

In the equations, \( c' \), \( \phi' \), and \( \psi' \) represent the cohesion, friction angle, and dilation angle, respectively. \( \sigma'_1 \) and \( \sigma'_3 \) denote the maximum and minimum principal stresses, with the condition \( \sigma'_1 < \sigma'_3 < 0 \). It is important to note that in our assumptions, stress is considered positive during extension, which means the signs of the stresses in these equations are opposite to those in standard Soil Mechanics' textbooks. The numerical implementation follows the approach described in Clausen et al.\(^{25}\)

2.2.2 | Turbulent model

The turbulent effect is modelled using a statistical approach namely large-eddy simulation. In this approach, the micro-scale turbulent influence in the dynamics of the macro-scale motion is computed through simple models like
Smagorinsky model. In the Smagorinsky model, the residual stress tensor is:

\[ \tau_{ij} = 2\mu_{\text{eff}} \left( S_{ij} - \frac{1}{3} \delta_{ij} S_{kk} \right) + \frac{1}{3} \delta_{ij} \tau_{kk} \]  

(17)

where the strain rate tensor is given by:

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \]  

(18)

and the effective viscosity is sum of molecular viscosity and turbulent viscosity \( \mu_{\text{eff}} = \mu + \mu_t \) in which the turbulent viscosity \( \mu_t \) is calculated by:

\[ \mu_t = (C_s \Delta)^2 \sqrt{2S_{ij}S_{ij}} \]  

(19)

where \( C_s \) is the Smagorinsky constant with the value of 0.1 and \( \Delta = \sqrt{dx dy dz} \) is the grid size that defines the subgrid length scale.

2.2.3  Frictional force for soil-structure interaction

MPMICE includes a contact law for the interaction between soil and structure using the first Coulomb friction contact for MPM presented by Bardenhagen et al. The magnitude of the friction force at the contact depends on the friction coefficient \( \mu_{\text{fric}} \) and the normal force \( f_{\text{norm}} \) computed from the projection of the contact force in the normal direction.

\[ f_{\text{fric}} = \mu_{\text{fric}} f_{\text{norm}} \]  

(20)

The contact determines whether the soil is sliding or sticking to the structure by comparing the friction force with the sticking force \( f_{\text{stick}} \) can be computed from the projection of the contact force in the tangent direction as:

\[ \begin{align*}
  & \text{if } f_{\text{fric}} \geq f_{\text{stick}} \quad \text{no sliding} \\
  & \text{if } f_{\text{fric}} < f_{\text{stick}} \quad \text{sliding occurs}
\end{align*} \]  

(21)

Frictional sliding between solid materials also generates dissipation and the work rate generated from the sliding can be calculated as:

\[ \Delta W_{\text{friction}} = f_{\text{fric}} d \]  

(22)

where \( d \) is the sliding distance which can computed based on the sliding velocity between two materials.

2.2.4  Momentum and energy exchange model

Currently, the energy exchange coefficient \( H_{sf} \) is assumed to be constant for the sake of simplicity. Then the energy exchange can be written as:

\[ q_{sf} = H_{sf}(T_f - T_s) \]  

(23)

On the other hand, the drag force can be calculated as:

\[ f_d = K(U_s - U_f) \]  

(24)
For the momentum exchange between fluid flows and porous media, we assume that the drag force \( f_d \) depends on the average grain size of the grains \( D_p \), the porosity \( n \), the fluid viscosity \( \mu_f \), and is proportional to the relative velocities of soil grains and fluid \( (U_s - U_f) \). Based on recent investigation of CFD simulations of fluid flow around mono- and bi-disperse packing of spheres for \( 0.1 < \phi_s < 0.6 \) and \( Re < 1000 \). The drag force is given by:

\[
\mathbf{f}_d = \frac{18\phi_s(1 - \phi_s)\mu_f}{D_p^2} F(\phi_s, Re)(U_s - U_f)
\]  

(25)

where Reynolds number \( Re \) are computed as:

\[
Re = \frac{n\rho_f D_p}{\mu_f} \left\| (U_s - U_f) \right\|
\]  

(26)

The function \( F(\phi_s, Re) \) can be calculated as:

\[
F(\phi_s, Re) = F(\phi_s, 0) + \frac{0.413Re}{24(1-\phi_s)^2} \left( \frac{(1-\phi_s)^{-1} + 3\phi_s(1-\phi_s) + 8.4Re^{-0.343}}{1 + 10^3\phi_s Re^{-1+4\phi_s/2}} \right)
\]  

(27)

where the low Reynolds coefficient \( F(\phi_s, Re \to 0) \) is:

\[
F(\phi_s, 0) = \frac{10\phi_s}{(1 - \phi_s)^2} + (1 - \phi_s)^2 + 1.5\sqrt{\phi_s}
\]  

(28)

When validating the model with analytical solution, it requires to know the hydraulic conductivity \( K \). In such case, we convert the Equation (25) to Kozeny-Carman formula by assuming \( F(\phi_s, Re) = 10\phi_s/(1 - \phi_s)^2 \), leading to:

\[
\mathbf{f}_d = \frac{180\phi_s^2\mu_f}{D_p^2(1 - \phi_s)} (U_s - U_f)
\]  

(29)

Then, the dragging force following the Darcy law is given by:

\[
\mathbf{f}_d = \frac{n^2\mu_f}{\kappa} (U_s - U_f)
\]  

(30)

where \( \kappa \) being intrinsic permeability of soil which can be written as:

\[
\kappa = \frac{K\mu_f}{\rho_f g}
\]  

(31)

As such, the hydraulic conductivity will be expressed as:

\[
K = \frac{D_p^2(1 - \phi_s)^3}{180\phi_s^2\mu_f} \rho_f g
\]  

(32)

### 2.2.5 Solving momentum and energy exchange with an implicit solver

The derivation of the implicit integration for the momentum exchange is presented in the Appendix’s section 'Momentum and energy exchange with an implicit solver'. The linear equations for multi phases \( i, j = 1 : N \) has the form as:

\[
\begin{bmatrix}
(1 + \beta_{ij}) & -\beta_{ij} \\
-\beta_{ji} & (1 + \beta_{ij})
\end{bmatrix}
\begin{bmatrix}
\Delta U_i \\
\Delta U_j
\end{bmatrix}
= \begin{bmatrix}
\beta_{ij}(U_i^e - U_j^e) \\
\beta_{ji}(U_i^e - U_j^e)
\end{bmatrix}
\]
where the intermediate velocity for fluid phases \( f = 1 : N_f \) and for solid/porous phases \( s = 1 : N_s \) can be calculated by:

\[
U_f^n = U_f^0 + \Delta t \left( -\frac{\nabla P_f^{n+1}}{\rho_f^n} + \frac{\nabla \cdot \tau_f^n}{\rho_f^n} + b \right)
\]

\[
U_s^n = U_s^0 + \Delta t \left( \frac{\nabla \cdot \sigma^n}{\rho_s^n} - \frac{\nabla P_s^{n+1}}{\rho_s^n} + b \right)
\]

(33)

Also, the momentum exchange coefficient can be computed at every time step as

\[
\eta_{12} = K / \rho_f^n \quad \text{and} \quad \eta_{21} = K / \rho_s^n
\]

with the coefficient depending on the different type of interactions (see Figure 3) as for example:

1. The drag force is set to zero in soil-structure interactions, and instead the frictional force is computed.
2. As a result of fluid-structure interaction, the momentum exchange coefficient should be extremely high (1E15) when the solid material points are considered to be zero-porosity/zero-permeability.
3. In the case of soil-fluid interaction, the drag force is calculated using the Equation (25). Considering that air has a much lower viscosity than water, its drag force is much lower than the drag force of water in a pore.
4. A momentum exchange coefficient of 1E5 is applied between multiphase flows. This value is far higher than reality, but it is necessary to have enough numerical stability to conduct simulations in the numerical example.

Similar approach applied for the energy exchange term leading to:

\[
\begin{vmatrix}
1 + \eta_{ij} & -\eta_{ij} \\
-\eta_{ji} & 1 + \eta_{ji}
\end{vmatrix}
\begin{align*}
\Delta T_i \\
\Delta T_j
\end{align*}
= \begin{vmatrix}
\eta_{ij} (T_i^{n} - T_j^{n}) \\
\eta_{ji} (T_j^{n} - T_i^{n})
\end{vmatrix}
\]

with \( \eta \) being the energy exchange coefficient.

2.2.6 Equation of state for fluid phases

The equation of state establishes relations between thermodynamics variables \([P_f, \rho_f, T_f]\). The choice of the equation of state depends on the types of the fluid materials. For example, for the air, it is possible to assume the equation of state for the perfect gas which obeys:

\[
P_f = \rho_f RT_f
\]  

(34)

FIGURE 3 Momentum exchange coefficient between materials.
where $R$ is the gas constant. For the water, a simple linear equation of state is in the following form:

$$P_f = P_{ref} + K_f(\rho_f - \rho_{ref} + \alpha_f(T_f - T_{ref}))$$  \hspace{1cm} (35)

where reference pressure $P_{ref} = 1 \text{ atm} = 101325 \text{ Pa}$, reference temperature $T_{ref} = 10^\circ \text{C}$, reference density $\rho_{ref} = 999.8 \text{ kg/m}^3$, the bulk modulus of water $K_f = 2 \text{ GPa}$, and the water thermal expansion $\alpha_f = 0.18^\circ \text{C}^{-1}$. Equation (35) matches well with the state of the water (see Figure 4).

3 | NUMERICAL IMPLEMENTATION

The fluid phases are discretized in the grid with the state variables stored at the centroid of the cells $[\rho_{f,c}, \textbf{U}_{f,c}, T_{f,c}, c_{f,c}]$ while the solid phase is discretized in the particles with the state variables $[m_p, \textbf{U}_p, T_p, \sigma_{p}']$. In the Material Point Method, we use the generalized interpolation technique\textsuperscript{30} using the weight function as a convolution of a grid shape function $N_i(x)$ in a nodal domain $\Omega_i$ and a characteristic function $\chi_p(x)$ in a particle domain $\Omega_p$ with the volume $V_p(x)$ as follows:

$$S_{ip} = \frac{1}{V_p} \int_{\Omega_i \cap \Omega_p} N_i(x) \chi_p(x) dx$$  \hspace{1cm} (36)

where the volume $V_p(x)$ of the material point $p$ can be calculated as:

$$V_p = \int_{\Omega_p} \chi_p(x) dx$$  \hspace{1cm} (37)

The characteristic function is the Heaviside function as $\chi_p = 1$ if $x \in \Omega_p$, otherwise 0 (see Figure 5). For the interpolation of the centroid of the cell, the linear basis function is used as:

$$S_{ci} = N_i(x_c)$$  \hspace{1cm} (38)

The time discretization is solved using the following steps (summarized in Figure 6).

**Figure 4** Equation of state of water.
3.1 Interpolation from solid particle to grid

The nodal values of the solid state (mass, velocity, temperature, volume) are:

\[ m_{n_{si}} = \sum S_{ip} m_{sp} \]
\[ U_{n_{si}} = \sum S_{ip} \frac{(mU)_{ip}^n}{m_{si}} \]
\[ T_{n_{si}} = \sum S_{ip} \frac{(mT)_{ip}^n}{m_{si}} \]
\[ V_{n_{si}} = \sum S_{ip} \frac{(mV)_{ip}^n}{m_{si}} \]
\[ \sigma_{n_{si}} = \sum S_{ip} \frac{(\sigma V)_{sp}^n}{V_{n_{si}}} \]  

(39)

The nodal internal forces is calculated by:

\[ f_{int,n}^{sl} = -\sum \nabla S_{ip} (\sigma'_{sp})^n V_{n_{sp}}^n \]  

(40)

The nodal external forces \( f_{ext,n}^{sl} \) and the frictional forces from soil-structure interaction \( f_{fric,sl}^n \) from contact between materials are computed here.

Then we compute the solid cell variables as:

\[ m_{sc}^n = \sum S_{ci} m_{si} \]
\[ \rho_{sc}^n = m_{sc}^n \frac{m_{si}}{V} \]
\[ U_{sc}^n = \sum S_{ci} U_{si}^n \]
\[ T_{sc}^n = \sum S_{ci} T_{si}^n \]
\[ V_{sc}^n = \sum S_{ci} V_{si}^n \]
\[ \sigma_{sc}^n = \sum S_{ci} \sigma_{si}^n \]  

(41)
FIGURE 6  Numerical implementation of MPMICE.
3.2 | Compute equation of state for fluid phase

The total fluid material volume of a cell is:

\[ V_{total} = \sum_{f=1}^{N_f} M_f \nu_f \]  \hspace{1cm} (42)

We need to find \( P_n^f \), which allows each fluid materials obey their equation of states \([P_f, \rho_f, \nu_f, T_f, e_f]\) but also allow mass of all fluid materials to fill the entire the pore volume without ongoing compression or expansion following the condition as follows:

\[ 0 = n - \sum_{f=1}^{N_f} \nu_f \bar{\rho}_f \]  \hspace{1cm} (43)

Then, we can use the Newton-Raphson interaction to find the value of \( P_n^f \) which satisfies the Equations (42), (43) and each equation of states of each fluid materials.

3.3 | Compute cell face velocity

Following the derivation in the Appendix: Advanced Fluid Pressure, we first compute the fluid cell face velocity as:

\[ U_{f,FC}^{l}\times = \frac{(\bar{\rho}U)^n_{f,FC}}{\bar{\rho}^n_{f,FC}} + \Delta t \left( -\frac{\nabla F^P_{f,c}}{\bar{\rho}^n_{f,FC}} + \frac{\nabla F^C \cdot \tau^n}{\bar{\rho}^n_{s,FC}} + b \right) \]  \hspace{1cm} (44)

The Equation (44) is discretized in three dimension (noted that \( \nabla F^C \cdot \tau = 0 \)), for example the discretized equation in the \( x \) direction is:

\[ U_{f,c}^{l}\times = \frac{(\bar{\rho}U)^n_{f,c,R} + (\bar{\rho}U)^n_{f,c,L}}{\bar{\rho}^n_{f,c,L} + \bar{\rho}^n_{f,c,R}} + \Delta t \left( -\frac{2(\nu_{f,c,R} - \nu_{f,c,L})}{\nu_{f,c,L} + \nu_{f,c,R}} \frac{P^n_{f,c,R} - P^n_{f,c,L}}{\Delta x} + b \right) \]  \hspace{1cm} (45)

The cell face solid velocity can be calculated as:

\[ U_{s,FC}^{l}\times = \frac{(\bar{\rho}U)^n_{s,FC}}{\bar{\rho}^n_{s,FC}} + \Delta t \left( \frac{\nabla F^C \cdot \sigma^n_c}{\bar{\rho}^n_{s,FC}} - \frac{\nabla F^C P^n_{f,c}}{\rho_s} + b \right) \]  \hspace{1cm} (46)

The Equation (46) is discretized in three dimension (noted that \( \nabla F^C \cdot \sigma_{ij} = 0 \) with \( i \neq j \)), for example the discretized equation in the \( x \) direction is:

\[ U_{s,x}^{l}\times = \frac{(\bar{\rho}U)^n_{s,x,R} + (\bar{\rho}U)^n_{s,x,L}}{\bar{\rho}^n_{s,x,L} + \bar{\rho}^n_{s,x,R}} + \Delta t \left( \frac{2(\sigma_{s,x,R} - \sigma_{s,x,L})}{\sigma_{s,x,L} + \sigma_{s,x,R}} - \frac{P^n_{f,c,x,R} - P^n_{f,c,x,L}}{\Delta x} + b \right) \]  \hspace{1cm} (47)

Then, we compute the modified cell face velocity \( U_{f,FC}^l \) considering the momentum exchange (see the Appendix: Momentum exchange with an implicit solve) as follows:

\[ U_{f,FC}^l = U_{f,FC}^{l}\times + \Delta U_{f,FC} \]
\[ U_{s,FC}^l = U_{s,FC}^{l}\times + \Delta U_{s,FC} \]  \hspace{1cm} (48)
The linear equation below is solved to obtain the increment of temperature due to energy exchange with \( i, j = 1 : N \) as:

\[
\begin{pmatrix}
1 + \beta_{ij} & -\beta_{ij} \\
-\beta_{ij} & (1 + \beta_{ij})
\end{pmatrix}
\begin{pmatrix}
\Delta U_{i, FC} \\
\Delta U_{j, FC}
\end{pmatrix} =
\begin{pmatrix}
\beta_{ij} \left( U_{i, FC}^n - U_{j, FC}^n \right) \\
\beta_{ji} \left( U_{j, FC}^n - U_{i, FC}^n \right)
\end{pmatrix}
\]

### 3.4 Compute cell face temperature

Similar to the velocity, the faced temperature is computed, for example in \( x \) direction, as:

\[
\begin{align*}
\tau_{lx}^l &= \frac{(\bar{\rho}T)^n_{fx,L} + (\bar{\rho}T)^n_{fx,R}}{\bar{\rho}_{lx,R} + \bar{\rho}_{lx,L}} \\
\tau_{lx}^r &= \frac{(\bar{\rho}T)^n_{sx,R} + (\bar{\rho}T)^n_{sx,L}}{\bar{\rho}_{lx,L} + \bar{\rho}_{lx,R}}
\end{align*}
\]

(49)

Then, we compute the modified cell face temperature \( T_{FC}^l \) considering the energy exchange (see the Appendix: Momentum and energy exchange with an implicit solver) as follows:

\[
\begin{align*}
T_{f, FC}^l &= T_{f, FC}^l - \Delta T_{f, FC}^l \\
T_{s, FC}^l &= T_{s, FC}^l - \Delta T_{s, FC}^l
\end{align*}
\]

(50)

The linear equation below is solved to determine the increment of temperature due to energy exchange with \( i, j = 1 : N \) as:

\[
\begin{pmatrix}
1 + \eta_{ij} & -\eta_{ij} \\
-\eta_{ij} & (1 + \eta_{ij})
\end{pmatrix}
\begin{pmatrix}
\Delta T_{i, FC} \\
\Delta T_{j, FC}
\end{pmatrix} =
\begin{pmatrix}
\eta_{ij} \left( \tau_{i, FC}^l - \tau_{j, FC}^l \right) \\
\eta_{ji} \left( \tau_{i, FC}^l - \tau_{j, FC}^l \right)
\end{pmatrix}
\]

### 3.5 Compute fluid pressure (implicit scheme)

For single phase flow, the increment of the fluid pressure can be computed as:

\[
\kappa_f \frac{\Delta P}{\Delta t} = \nabla^c \cdot U_{f, FC}^{n+1}
\]

(51)

For multi-phase flows, the increment of the fluid pressure of the mixture can be computed as:

\[
\kappa \frac{\Delta P}{\Delta t} = \sum_{f=1}^{N_f} \nabla^c \cdot (\phi_{f, FC} U_{f, FC}^n)^{n+1}
\]

(52)

where \( \kappa = \sum_{f=1}^{N_f} (\phi_{f, FC} \kappa_f) / \sum_{f=1}^{N_f} (\phi_{f, FC}) \). Then, the fluid pressure at cell center is:

\[
P_{c}^{n+1} = P_{c}^{n} + \Delta P_{c}^{n}
\]

(53)

Finally, the cell face advanced fluid pressure is:

\[
P_{FC}^{n+1} = \left( \frac{p_{c, L}^{n+1}}{p_{f, L}^{n+1}} + \frac{p_{c, R}^{n+1}}{p_{f, R}^{n+1}} \right) \left( \frac{1}{p_{f, L}^{n+1}} + \frac{1}{p_{f, R}^{n+1}} \right) = \left( \frac{p_{c, L}^{n+1}p_{f, R}^{n+1} + p_{c, R}^{n+1}p_{f, L}^{n+1}}{p_{f, L}^{n+1}p_{f, R}^{n+1}} \right)
\]

(54)
3.6  |  Compute viscous shear stress term of the fluid phase

This part computes the viscous shear stress $\Delta (mU)_{f,c}$ for a single viscous compressible Newtonian fluid and optionally shear stress induced by the turbulent model.

3.7  |  Compute nodal internal temperature of the solid phase

The nodal internal temperature rate is computed based on the heat conduction model as below:

$$dT_{sl}^{L-} = \left( \frac{\Delta W_{sl}^{n} + \Delta W_{fric,i}^{n} + \nabla \cdot q_{sl}^{n}}{m_{sl}^{n} c_{v}} \right)$$

(55)

where $\Delta W_{sl}^{n} = \sigma^{t} : \frac{D(e^{s})}{Dt}$ is the mechanical work rate computed from the constitutive model with $e^{s}$ is the plastic strain, $\Delta W_{fric,i}^{n}$ is the work rate computed from the contact law due to the frictional sliding between solid materials. The heat flux is $q_{sl} = \bar{\rho}_{s} \beta_{s} \nabla T_{s}$ with $\beta_{s}$ being the thermal conductivity of the solid materials.

$$T_{sl}^{L-} = T_{sl}^{n} + dT_{sl}^{L-}$$

(56)

3.8  |  Compute and integrate acceleration of the solid phase

After interpolating from material points to the nodes, the nodal acceleration and velocity are calculated by:

$$a_{sl}^{L-} = \frac{f_{sl}^{int,n} + f_{sl}^{ext,n}}{m_{sl}^{n}} + g$$

(57)

$$U_{sl}^{L-} = U_{sl}^{n} + a_{sl}^{L-} \Delta t$$

(58)

3.9  |  Compute Lagrangian value (mass, momentum and energy)

For the fluid phase, the linear momentum rate, the energy rate are:

$$\Delta (mU)_{f,c} = V n^{e} \nabla p^{e+1} + \Delta (mU)_{f,c} + V \overline{\rho}_{f,c}^{n} \nabla T_{f,c} + \nabla \cdot \overline{\rho}_{f,c}^{n}$$

(59)

$$\Delta (me)_{f,c} = V n^{e} \nabla p^{e+1} \cdot \overline{U}_{f,FC} + \nabla \cdot \overline{\rho}_{f,c}^{n} q_{f,c}^{n}$$

(60)

The heat flux is $q_{f} = \bar{\rho}_{f} \beta_{f} \nabla T_{f}$ with $\beta_{f}$ being the thermal conductivity of the fluid materials. The Lagrangian value of the mass, linear momentum and energy of fluid phases without momentum exchange are:

$$m_{f,c}^{L-} = V \overline{\rho}_{f,c}^{n}$$

(61)

$$mU_{f,c}^{L-} = V \overline{\rho}_{f,c}^{n} U_{f,c}^{n} + \Delta (mU)_{f,c}$$

(62)

$$me_{f,c}^{L-} = V \overline{\rho}_{f,c}^{n} T_{f,c}^{n} c_{v} + \Delta (me)_{f,c}$$

(63)

For the solid phase, the Lagrangian value of the linear momentum and energy of solid phase are:

$$m_{s,c}^{L-} = m_{s,c}^{n}$$

(64)
\[(mU)_{sc}^{l} = \sum S_{i} m_{si}^{n} U_{si}^{l-n} + V(1 - n_{i}^{n}) \nabla^{c} P_{f,c}^{n+1} \]  
\[(me)_{sc}^{l} = \sum S_{i} m_{si}^{n} T_{si}^{l} c_{v} \]  

To consider the momentum exchange, the Lagrangian velocity is modified as:
\[U_{f,c}^{l} = U_{f,c}^{l-n} + \Delta U_{f,c} \]
\[U_{sc}^{l} = U_{sc}^{l-n} + \Delta U_{sc} \]  

where the cell-centered intermediate velocity can be calculated by:
\[U_{f,c}^{l-n} = \frac{(mU)_{f,c}^{l-n}}{m_{f,c}^{l-n}} \]
\[U_{sc}^{l-n} = \frac{(mU)_{sc}^{l-n}}{m_{sc}^{l-n}} \]  

And the increment of the velocity \(U_{f,c}, \Delta U_{sc}\) can be computed by solving the linear equation with \(i,j = 1 : N\) as:
\[
\begin{vmatrix}
1 + \beta_{ij} & -\beta_{ij} \\
-\beta_{ji} & (1 + \beta_{ji})
\end{vmatrix}
\begin{vmatrix}
\Delta U_{i,c} \\
\Delta U_{j,c}
\end{vmatrix} =
\begin{vmatrix}
\beta_{ij} \left( U_{i,c}^{l-n} - U_{j,c}^{l-n} \right) \\
\beta_{ji} \left( U_{j,c}^{l-n} - U_{i,c}^{l-n} \right)
\end{vmatrix}
\]

To consider the energy exchange, the Lagrangian temperature is modified as:
\[T_{f,c}^{l} = T_{f,c}^{l-n} + \Delta T_{f,c} \]
\[T_{sc}^{l} = T_{sc}^{l-n} + \Delta T_{sc} \]  

where the cell-centered intermediate temperature can be calculated by:
\[T_{f,c}^{l-n} = \frac{(mT)_{f,c}^{l-n}}{m_{f,c}^{l-n} c_{v}} \]
\[T_{sc}^{l-n} = \frac{(mT)_{sc}^{l-n}}{m_{sc}^{l-n} c_{v}} \]  

And the increment of the temperature due to energy exchange can be computed by solving the linear equation with \(i,j = 1 : N\) as:
\[
\begin{vmatrix}
1 + \eta_{ij} & -\eta_{ij} \\
-\eta_{ji} & (1 + \eta_{ji})
\end{vmatrix}
\begin{vmatrix}
\Delta T_{i,c} \\
\Delta T_{j,c}
\end{vmatrix} =
\begin{vmatrix}
\eta_{ij} \left( T_{i,c}^{l-n} - T_{j,c}^{l-n} \right) \\
\eta_{ji} \left( T_{j,c}^{l-n} - T_{i,c}^{l-n} \right)
\end{vmatrix}
\]

Finally, we obtain the cell-centered solid acceleration and temperature rate as:
\[dU_{sc}^{l} = \frac{(mU)_{sc}^{l} - (mU)_{sc}^{n}}{m_{sc}^{l} \Delta t} \]
\[dT_{sc}^{l} = \frac{(me)_{sc}^{l} - (me)_{sc}^{n}}{m_{sc}^{l} c_{v} \Delta t} \]
### 3.10 Compute Lagrangian specific volume of the fluid phase

To compute the Lagrangian value of the specific volume of the fluid phase, we need to compute the Lagrangian temperature rate as below:

$$T_{f,c}^{n+1} = \frac{(me)^L_{f,c}}{m_{f,c}^L}$$  \hspace{1cm} (73)

$$\frac{D_f T_{f,c}}{Dt} = \frac{T_{f,c}^{n+1} - T_{f,c}^n}{\Delta t}$$  \hspace{1cm} (74)

As such, the Lagrangian specific volume rate is:

$$\Delta (mv)_{f,c} = V f^\phi_{f,c} \nabla \cdot U + \left( \phi_{f,c} \alpha_{f,c} \frac{D_f T_{f,c}}{Dt} - f^\phi_{f,c} \sum_{n=1}^N \phi_{nc} \alpha_{nc} \frac{D_n T_{n,c}}{Dt} \right)$$  \hspace{1cm} (75)

where $f^\phi_{f} = (\phi_f \kappa_f) / \left( \sum_{n=1}^N \phi_n \kappa_n \right)$ and $U = \nabla \cdot \left( \sum_{n=1}^N \phi_n U_{nc} + \sum_{f=1}^N \phi_f U_{f,c} \right)$. Finally, the Lagrangian specific volume is:

$$(mv)^L_{f,c} = V \mu_{f,c}^{n+1} + \Delta (mv)_{f,c}$$  \hspace{1cm} (76)

### 3.11 Compute advection term and advance in time

The mass, linear momentum, energy and specific volume with advection are:

$$m_{f,c}^{n+1} = m_{f,c}^L - \Delta t \nabla \cdot \left( \tilde{p}^L_{f,c}, \tilde{U}^L_{f,c} \right)$$  \hspace{1cm} (77)

$$(mU)^L_{f,c}^{n+1} = (mU)^L_{f,c} - \Delta t \nabla \cdot \left( \tilde{p} U_{f,c}^L \right) \hspace{1cm} (78)$$

$$(me)^L_{f,c}^{n+1} = (me)^L_{f,c} - \Delta t \nabla \cdot \left( \tilde{p} c_v T^L_{f,c} \right) \hspace{1cm} (79)$$

$$(mv)^L_{f,c}^{n+1} = (mv)^L_{f,c} - \Delta t \nabla \cdot \left( \tilde{p} v^L_{f,c} \right)$$  \hspace{1cm} (80)

Finally, the state variables of the fluid phases of the next time step are:

$$\tilde{p}_{f,c}^{n+1} = \frac{m_{f,c}^{n+1}}{V}$$  \hspace{1cm} (81)

$$U_{f,c}^{n+1} = \frac{(mU)^L_{f,c}}{m_{f,c}^{n+1}}$$  \hspace{1cm} (82)

$$T_{f,c}^{n+1} = \frac{(me)^L_{f,c}}{m_{f,c}^{n+1}}$$  \hspace{1cm} (83)

$$\mu_{f,c}^{n+1} = \frac{(mv)^L_{f,c}}{m_{f,c}^{n+1}}$$  \hspace{1cm} (84)
### 3.12 | Interpolate from cell to node of the solid phase

First we interpolate the acceleration, velocity and temperature rate to the node as below:

\[
a^n_{si} = \sum S_{ci} dU^n_{sc} \tag{85}
\]

\[
U^{n+1}_{si} = \sum S_{ci} dU^n_{sc} \Delta t \tag{86}
\]

\[
dT^n_{si} = \sum S_{ci} dT^n_{sc} \tag{87}
\]

Then the boundary condition and contact forces \(f^n_{si}\) are applied to the nodal velocity, and then accelerations are modified by:

\[
a^n_{si} = \frac{v^{n+1}_{si} - v^n_{si}}{\Delta t} \tag{88}
\]

### 3.13 | Update the particle variables

The state variables of the solid phase \([U_{sp}^{n+1}, x_{sp}^{n+1}, \nabla U_{sp}^{n+1}, T_{sp}^{n+1}, \nabla T_{sp}^{n+1}, F_{sp}^{n+1}, V_{sp}^{n+1}]\) (velocity, position, velocity gradient, temperature, temperature gradient, deformation gradient, volume) are updated as:

\[
U_{sp}^{n+1} = U_{sp}^n + \sum S_{sp} a^n_{si} \Delta t \tag{89}
\]

\[
x_{sp}^{n+1} = x_{sp}^n + \sum S_{sp} U_{si}^{n+1} \Delta t \tag{90}
\]

\[
\nabla U_{sp}^{n+1} = \sum \nabla S_{sp} U_{si}^{n+1} \tag{91}
\]

\[
T_{sp}^{n+1} = T_{sp}^n + \sum S_{sp} dT^n_{si} \Delta t \tag{92}
\]

\[
\nabla T_{sp}^{n+1} = \sum \nabla S_{sp} T_{si}^n \Delta t \tag{93}
\]

\[
F_{sp}^{n+1} = (I + \nabla U_{sp}^{n+1} \Delta t) F_{sp}^n \tag{94}
\]

\[
V_{sp}^{n+1} = \text{det}(F_{sp}^{n+1}) V_{sp}^n \tag{95}
\]

Finally, the effective stress \((\sigma')^{n+1}_f\) is updated from the constitutive model and the pore water pressure is interpolated from the cell as:

\[
p^n_{f} = \sum S_{si} p^n_{c} \tag{96}
\]

### 4 | NUMERICAL VALIDATION

For all simulations, water is characterized by a bulk modulus of 2 GPa, a density of 998 kg/m\(^3\) at a reference temperature of 5 degrees Celsius and a reference pressure of 10,325 Pa (1 atm), and a dynamic viscosity denoted as \(\mu_f\) of 1 mPa s. The air is treated as an ideal gas with a density of 1.17 kg/m\(^3\) at a reference temperature of 5\(^o\)C and a reference pressure of 10,325 Pa (1 atm), and it possesses a dynamic viscosity \(\mu_f\) of \(18.45 \times 10^{-3}\) mPa s.”
4.1 | Fluid flow through isothermal porous media

Fluid flow through porous media is important in many engineering disciplines, like predicting water flow in soil. Fluid flow velocity in one dimension can be calculated from the porous media’s hydraulic conductivity $K$ as:

$$U_f = K \frac{\Delta p_f}{L} \tag{97}$$

If the Carman-Kozeny formula is adopted $F = 10\phi_s/(1 - \phi_s)^2$, the hydraulic conductivity will be expressed as $K = D_p^2(1 - \phi_s)^3 \rho_f g/180\phi_s^2 \mu_f$. Then, the analytical formula of average velocity in one dimension through the porous media is:

$$U_f = \frac{1}{n} D_p^2(1 - \phi_s)^3 \Delta p_f \frac{\phi_s^2}{L} \tag{98}$$

Our numerical model’s validity is confirmed through the simulation of fluid flow in a 1m long porous medium. This porous medium is represented by an elastic material with the following properties: Young’s modulus of 10 MPa, Poisson’s ratio of 0.3, and a density of 2650 kg/m³. The volume fraction of the porous medium, denoted as $\phi_s$, is varied as [0.6, 0.62, 0.66, 0.68, 0.7], while the average grain diameter $d$ is set at 1 mm. The model is discretized into 20 finite elements, with the porous medium represented by 10 finite elements, each containing one material point per element. We apply pressure gradients with three different values: [0.25, 0.5, 1] atm. As depicted in Figure 7, our model demonstrates excellent agreement with theoretical predictions in simulating fluid flow.

4.2 | Isothermal consolidation

A common benchmark for fully saturated porous media is the simulation of one-dimensional consolidation. Using the Carman-Kozeny formula, the time-dependent pressure can be calculated as:

$$p_f = \sum_{m=1}^{\infty} \frac{2F_{ext}}{M} \sin \left( \frac{Mz}{H} \right) e^{-M^2 \tau_v} \text{ with } M = \frac{\pi}{2}(2m + 1) \tag{99}$$

FIGURE 7  Numerical results of the fluid flow through isothermal porous media.
Here, the consolidation rate is defined as $T_v = C_v t / H^2$, the consolidation coefficient as $C_v = E_v n^3 d^2 / (180(1 - n)^2 \mu)$, and the Oedometer modulus as $E_v = E(1 - \nu)/(1 + \nu)/(1 - 2\nu)$.

To validate our numerical model, we simulated the consolidation of a 1 m column of porous media. The porous media is modeled as an elastic material with a Young’s modulus of 10 MPa, a Poisson's ratio of 0.3, and a density of 2650 kg/m$^3$. The volume fraction of porous media $\phi_s$ is set to 0.7, equivalent to a porosity of 0.3, and the average grain diameter $d$ is 1 mm. The model is discretized into 100 finite elements, each with one material point per element. An external pressure of 10 kPa is applied to the top of the column. Figure 8 demonstrates a strong agreement between the predicted fluid flow and theoretical results.

### 4.3 Thermal induced cavity flow

Another benchmark involves the study of thermally-induced cavity flow in porous media. This simulation calculates temperature and velocity distributions within a square, non-deformable, saturated porous medium. The top and bottom walls are insulated, while the left and right walls maintain a fixed temperature gradient of 1$^\circ$, leading to fluid motion in the form of cavity flow due to temperature-induced density variation (See Figure 9). Our numerical model is validated by comparing it with the numerical solution obtained using the finite element method.

![Figure 8](image1.png)

**Figure 8** Comparison between analytical solution and numerical solution.

![Figure 9](image2.png)

**Figure 9** Model schematic.31
The porous medium in this simulation is modeled as a non-deformable material with a density of 2500 kg/m³. The specific heat capacity of the water and porous skeleton is 4181 and 835 J/kg·K, respectively. Thermal conductivity values are 0.598 W/m·K for water and 0.4 W/m·K for the porous skeleton. The volume fraction of porous media \( \phi_s \) is set at 0.6, equivalent to a porosity of 0.4, and the average grain diameter \( d \) is 1 mm. The model is discretized into a 20 × 20 grid of finite elements, with four material points per element. Figure 10 demonstrates that our numerical results align well with the numerical solution obtained using the finite element method.

5 | NUMERICAL EXAMPLES

5.1 | Underwater debris flow

The numerical example is compared to the experimental work of Rzadkiewicz et al. on submarine debris flow. In their experiment, sand within a triangular box is released and slides along a rigid bed inclined at 45° underwater (see Figure 11).

**FIGURE 10** Comparison between MPMICE model and FEM model.

**FIGURE 11** Model schematic.
The material properties in the numerical model are chosen based on Rzadkiewicz et al. experiment. The sand has a saturated density of 1985 kg/m$^3$ and a friction angle of 10°. Young’s modulus, despite its negligible effect on debris flow run-out due to extreme deformation, is set at 50 MPa with a Poisson’s ratio of 0.25. The rigid bed, significantly stiffer, possesses bulk modulus and shear modulus values of 117E$^7$ Pa and 43.8E$^7$ Pa, respectively. The numerical parameters used in this example are outlined in Table 1.

The boundary conditions applied in the numerical model are as follows: all boundary faces have zero velocity ($U = 0$ m/s) and a temperature of 5°C ($T = 5$°C). At the top boundary, pressure has a Neumann boundary condition of $dp/dx = 0$ kPa, and density has a Neumann boundary condition of $d\rho/dx = 0$ kg/m$^3$. The background mesh comprises 700 × 400 cells, resulting in a total of 280,000 cells. Each cell within the debris flow and rigid bed contains 2 × 2 material points.

Figure 12B illustrates snapshots of underwater debris flow sliding, effectively capturing the typical hydroplaning mechanism of the debris flow. Hydroplaning refers to the lifting of the debris flow, causing it to lose contact with the bottom layer. In addition, Figure 13 compares the elevation of the free surface at 0.4 and 0.8 s between our proposed method and other methods, demonstrating the alignment of our computed results with experimental results.

What sets our model apart is its utilization of effective stress analysis instead of total stress analysis. This allows for the analysis of water pressure and temperature within the debris flow. Furthermore, we investigate the differences between underwater debris flow and saturated debris flow in terms of their interaction with obstacles. Figure 12 presents snapshots of simulations of both underwater and saturated debris flow. The saturated debris flow (Figure 12A) exhibits behavior similar to frictional flow, with grains in contact with each other. Conversely, underwater debris flow (Figure 12B) behaves like turbulent flow, with grains separated and showing no contact forces, as reflected by the near-zero effective stress in the turbulence domain.

### 5.2 Validation of soil response to the seismic loading

An experimental study conducted by Hiraoka et al. aimed to investigate the influence of seismic shaking on the deformation of a 0.5 m-high sand slope. The sand used in the experiment was partially saturated, with a moisture content of 10%. The provided soil parameters for the Mohr Coulomb model include the effective friction angle of 23°, apparent cohesion of 0.78 kPa, Young’s modulus of 2.57 MPa, Poisson’s ratio of 0.33, and moist unit weight of 16.5 kN/m$^3$. The soil’s dilatancy angle was assumed to be 0. The experimental setup consisted of a shaking table box with a steel horizontal base and smooth glass vertical sidewalls. Lasers sensors were used to monitor the displacement of the slope’s toe and crest. Figure 14 displays the velocity-time history employed in the experiment.

To simulate the seismic loading in our numerical model, we adopted a method presented by Alsardi et al., which involves specifying the velocity at the corresponding material points representing either the shaking table or the bedrock at the site. In our simulation, we considered the horizontal base to be fully rough and the vertical contact to be fully smooth. The initial stress condition was initiated using gravity, and seismic loading induced the slope failure (see Figure 15).

Previous studies by Bhandari et al., Alsardi et al., and Hiraoka et al. attempted to model this experiment using MPM and SPH models. In this study, we compared our results with those obtained from other particle-based methods (Figure 16). The main difference is that we did not apply 5% numerical damping in our model, unlike the other methods. We found that the final displacement of the slope toe in our MPM model was higher than that observed in the experiment. Nevertheless, the validation of the Mohr-Coulomb model under seismic response demonstrated reasonable soil behavior in terms of displacement.

| Materials          | Bulk module (Pa) | Shear module (Pa) | Density (kg/m$^3$) | Temp (C) | Dynamic viscosity (Pa s) | Friction angle (degrees) |
|--------------------|------------------|-------------------|--------------------|----------|--------------------------|-------------------------|
| Water (at surface) | 2.15e9           | —                 | 999.8              | 5        | 855e−6                   | —                       |
| Air (at top boundary) | —             | —                 | 1.177              | 5        | 18.45e−6                 | —                       |
| Sand (porous media) | 8.33e6         | 20e6              | 1985               | 5        | —                       | 10                      |
| Rigid bed (solid)  | 117e7           | 43.8e7            | 8900               | 5        | —                       | —                       |
In the final example, we conduct a numerical analysis of earthquake-induced submarine landslides. We utilize a plane strain model featuring an underwater slope, as depicted in Figure 17. This model consists of a 20 m high slope with a gradient of 45°, placed within a horizontal and vertical structure formerly used as a shaking table to apply earthquake loading. To simplify the earthquake loading, we simulate ground shaking for 20 s, maintaining a constant ground acceleration of 1g and a consistent frequency of 2 Hz (Figure 18A). This magnitude of earthquake is plausible; for instance, during the 2023 Turkey-Syria Earthquake, significant ground shaking with peak ground acceleration exceeding 1 g was recorded at numerous locations. This real-world example demonstrates the practical occurrence of such high levels of ground acceleration during seismic events. To generate the seismic loading, we employ the same method as presented in the previous numerical example. A non-associated Mohr-Coulomb model is employed to represent the soil in our simulation. The soil
**FIGURE 13**  Evolution of water level in the simulation of underwater debris flow.

**FIGURE 14**  Seismic loading.

**FIGURE 15**  Numerical model of the seismic-induced slope failure with displacement color.
FIGURE 16  Displacement of the toe of the slope.

FIGURE 17  Numerical simulation of the earthquake-induced submarine landslide.

FIGURE 18  Ground acceleration profile, frequency of 2 Hz and magnitude of 1 g.
grains have a density of 2650 kg/m³, a Young’s modulus of 10 kPa, a Poisson’s ratio of 0.3, and zero cohesion. The mobilized friction angle $\phi'_m$ is determined based on the softening curve (as depicted in Figure 19), with a peak friction angle $\phi'_p$ of 45° and a residual friction angle $\phi'_r$ of 10°. The porosity is set to 0.3, and the average grain size of the soil is approximately 0.1 μm to mimic undrained behavior. The mobilized dilatancy angle is calculated using Rowe’s stress dilatancy theory as follows:

$$\sin \psi'_m = \frac{\sin \phi'_m - \sin \phi'_r}{1 - (\sin \phi'_r \sin \phi'_m)}$$  \hspace{1cm} (100)$$

The solid plane is modeled as a rigid body, acting as a shaking table. Frictional contact with a friction coefficient of 0.1 is considered between the horizontal plane and the sand. No artificial damping is applied in the simulation. The contact between the vertical plane and the sand is treated as smooth, with a zero friction coefficient.

Symmetric boundary conditions are imposed on all boundary faces, while Neumann boundary conditions are applied at the top boundary for pressure ($dp/dx = 0$ kPa) and density ($d \rho/dx = 0$ kg/m³). In the context of the simulation, a symmetric boundary condition means that the normal component of the velocity at the boundary face is set to zero, and the tangential component matches the tangential component of the neighboring cells.

The mesh size is set to 0.25 × 0.25 m², resulting in 300,852 element cells and 142,316 material points. The simulation takes several hours to complete 60 s of simulation time, utilizing 4096 CPUs.

We have illustrated the entire process and mechanism of earthquake-induced submarine landslides by presenting the shear strain (Figure 20), pore water pressure in atm (Figure 21), and velocity (Figure 22). The failure mechanism can be characterized as a progressive failure mechanism, and here are some key numerical observations:

1. At the onset of the seismic event, the seismic loading triggers the initial slide at 3 s. By 4 s, the debris starts moving at a maximum speed of around 2–3 m/s, with multiple shear bands developing in the slope. A wave is generated from the submarine slide, propagating with approximately 2–3 m high in the direction of the slide.
2. When the onset of the shear band occurs in the slope (e.g., at 4 and 20 s), negative excess pore water pressure develops along this shear band, with pore water pressure dropping below 1 atm. This behavior is typical of dilatancy when the soil undergoes rapid shearing in an undrained state.
3. As the seismic loading ends at 23 s, the last shear band is mobilized, and the slope quickly reaches its final deposition. There are no further progressive failures in the slope at this stage. A turbulent flow develops due to the interaction between the debris flow and seawater.

In summary, we have presented a comprehensive view of the earthquake-induced submarine landslides, covering (1) the earthquake-triggering mechanism, (2) the initiation of shear bands with the development of negative excess pore water pressure, (3) the progressive failure mechanism, and (4) the generation of submarine landslide-induced waves leading to the final deposition of debris.
**FIGURE 20** Shear strain during the earthquake-induced submarine landslides.

**FIGURE 21** Pore water pressure during the earthquake-induced submarine landslides.
6 | CONCLUSIONS

We have introduced a numerical approach called MPMICE for simulating large deformation soil-fluid-structure interactions, with a specific focus on earthquake-induced submarine landslides. This model leverages two key components: Material Point Method (MPM): MPM is employed to accurately capture the large deformations occurring in iso-thermal porous media and solid structures. Implicit Continuous Eulerian (CFD Formulation): This component is used for modeling the intricate fluid flow, including turbulence, within the system. It adopts a compressible, conservative multi-material CFD formulation. Our model has been implemented within the high-performance Uintah computational framework and rigorously validated through comparisons with analytical solutions and experimental data. Subsequently, we have demonstrated the model’s capabilities in simulating the complete process of earthquake-induced submarine landslides.

NOMENCLATURE

General variables

| Symbol | Description | Unit |
|--------|-------------|------|
| $V$    | Representative volume | $[L^3]$ |
| $n$    | Porosity    |      |
| $\sigma$ | Total stress tensor | $[ML^{-1}t^{-2}]$ |
| $\Delta t$ | Time increment | $[t]$ |
| $\mathbf{b}$ | Body force | $[ML^1t^{-2}]$ |
| $c_v$ | Constant volume specific heat | $[L^2t^{-2}T^{-1}]$ |
| $f_d$ | Drag forces in momentum exchange term | $[ML^1t^{-2}]$ |
| $f_{int}$ | Internal forces | $[MLt^{-2}]$ |
| $f_{ext}$ | External forces | $[MLt^{-2}]$ |
| $q_{hs}$ | Heat exchange term | $[MLt^{-2}]$ |
| $S$ | Weighting function |      |
| $\nabla S$ | Gradient of weighting function |      |

Solid phase

| Symbol | Description | Unit |
|--------|-------------|------|
| $m_s$ | Solid mass | $[M]$ |
| $\rho_s$ | Solid density | $[ML^{-3}]$ |
\( \phi_s \)  
Solid volume fraction

\( \bar{\rho}_s \)  
Bulk solid density, \([ML^{-3}]\)

\( \mathbf{x}_s \)  
Solid position vector, \([L]\)

\( \mathbf{U}_s \)  
Solid velocity vector, \([Lt^{-1}]\)

\( \mathbf{a}_s \)  
Solid acceleration vector, \([Lt^{-2}]\)

\( \mathbf{\sigma}^e \)  
Effective stress tensor, \([ML^{-1}t^{-2}]\)

\( \epsilon \)  
Strain tensor

\( e_s \)  
Solid internal energy per unit mass, \([L^2t^{-2}]\)

\( T_s \)  
Solid temperature, \([T]\)

\( \mathbf{F}_s \)  
Solid deformation gradient

\( V_s \)  
Solid volume, \([L^3]\)

Fluid phase

\( m_f \)  
Fluid mass, \([M]\)

\( \rho_f \)  
Fluid density, \([ML^{-3}]\)

\( \phi_f \)  
Fluid volume fraction

\( \bar{\rho}_f \)  
Bulk fluid density, \([ML^{-3}]\)

\( \mathbf{U}_f \)  
Fluid velocity vector, \([Lt^{-1}]\)

\( \mathbf{\sigma}_f \)  
Fluid stress tensor, \([ML^{-1}t^{-2}]\)

\( p_f \)  
Fluid isotropic pressure, \([ML^{-1}t^{-2}]\)

\( \tau_f \)  
Fluid shear stress tensor, \([ML^{-1}t^{-2}]\)

\( e_f \)  
Fluid internal energy per unit mass, \([L^2t^{-2}]\)

\( T_f \)  
Fluid temperature, \([T]\)

\( \psi_f \)  
Fluid specific volume \( \frac{1}{\rho_f} \), \([L^3/M]\)

\( \alpha_f \)  
Thermal expansion, \([1/T]\)

\( \mu \)  
Fluid viscosity, \([ML^{-1}t^{-1}]\)

\( V_f \)  
Fluid volume, \([L^3]\)

Superscript

\( n \)  
Current time step

\( L \)  
Lagrangian values

\( n + 1 \)  
Next time step

Subscript

\( c \)  
Cell-centered quantity

\( p \)  
Particle quantity

\( i \)  
Node quantity

\( FC \)  
Cell face quantity

\( L, R \)  
Left and right cell faces

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DATA AVAILABILITY STATEMENT

The authors confirm that the data supporting the findings of this study are available within the article. All input files and the analytical calculations in this section are provided in the Github repository (https://github.com/QuocAnh90/Uintah_NTNU) for the reproduction of the numerical results.
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APPENDIX A. EQUATION DERIVATION

Before deriving the governing equation, we define the Lagrangian derivative for a state variable \( f \) as:

\[
\frac{Df}{Dt} = \frac{df}{dt} + \mathbf{U} \cdot \nabla f
\]  

\[
\frac{Df}{Dt} = \frac{df}{dt} + \mathbf{U}_s \cdot \nabla f
\]  

We adopt the following definitions as per:\(^{22}\)

\[
- \frac{1}{V} \left[ \frac{\partial V_f}{\partial p} \right] \equiv \kappa_f \quad \text{isothermal compressibility of fluid} \tag{A3}
\]

\[
\frac{1}{V} \left[ \frac{\partial V_f}{\partial T} \right] \equiv \alpha_f \quad \text{constant pressure thermal expansivity of fluid} \tag{A4}
\]

Then, we calculate the rate of volume within incompressible solid grains as follows:

\[
\frac{1}{V} \frac{D_V V_f}{Dt} = \frac{1}{V} \left( \left[ \frac{\partial V_f}{\partial p} \right] \frac{D_V P_{eq}}{Dt} + \left[ \frac{\partial V_f}{\partial T} \right] \frac{D_V T}{Dt} \right) = \frac{1}{V} \left( -\kappa_f \frac{D_V P_{eq}}{Dt} + \alpha_f \frac{D_V T}{Dt} \right) \tag{A5}
\]

A.1 Evolution of porosity

Solving the solid mass balance Equation (4) with the definition of solid mass in Equation (2), we obtain the rate of porosity as:

\[
\frac{D_s m_s}{Dt} = D_s (\phi_s \rho_s V) = \rho_s V \frac{D_s \phi_s}{Dt} + \phi_s V \frac{D_s \rho_s}{Dt} + \phi_s \rho_s \frac{D_s V}{Dt} = 0 \tag{A6}
\]

Since soil grains are assumed to be incompressible, term 2 on the right-hand side is zero, resulting in:

\[
V \frac{D_s \phi_s}{Dt} + \phi_s \frac{D_s V}{Dt} = 0 \tag{A7}
\]

Dividing all terms by “\( V \)” and using the equation \( \frac{1}{V} \frac{D_V V}{Dt} = \nabla \cdot \mathbf{U}_s \), we get:

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\[
\frac{D_n n}{D_t} = \frac{\partial n}{\partial t} + U_s \cdot \nabla n = \phi_s \nabla \cdot U_s
\]  

(A8)

### A.2 Momentum conservation

The linear momentum balance equations for the fluid phases based on mixture theory are given by:

\[
\frac{1}{V} \frac{D_f(m_f U_f)}{D_t} = \nabla \cdot (-\phi_f p_f I) + \nabla \cdot \tau_f + \bar{\rho}_f \mathbf{b} + \sum f_d + f_b
\]  

(A9)

On the right hand side, the terms include the divergence of partial fluid phases stress, body force, drag force (momentum exchange) and buoyant force as described in Reference 37 for immiscible mixtures, which takes the form:

\[
f_b = \sigma_f \nabla(n)
\]  

(A10)

Hence, the linear momentum balance equations for the fluid phases become:

\[
\frac{1}{V} \frac{D_f(m_f U_f)}{D_t} = -\phi_f p_f \nabla U_f + \sum q_f + (\rho_f b) \cdot U_f + \sum f_d + f_b
\]  

(A11)

The Reynolds stress component can be included in the term \(\tau_f\) to consider the turbulent effects if needed. To derive the linear momentum balance equation for the solid phase, we begin with the linear momentum balance equation for the mixture as:

\[
\frac{1}{V} \frac{D_s(m_s U_s)}{D_t} + \frac{1}{V} \frac{D_f(m_f U_f)}{D_t} = \nabla \cdot (\sigma) + \bar{\rho}_s \mathbf{b}
\]  

(A12)

Combining Terzaghi’s Equation (3) and subtracting both sides with Equation (A11), we obtain the linear momentum balance equations for the solid phase as:

\[
\frac{1}{V} \frac{D_s(m_s U_s)}{D_t} = \nabla \cdot (\sigma') - \phi_s \nabla p_f + \bar{\rho}_s \mathbf{b} - \sum f_d + \sum f_{fri}
\]  

(A13)

Here the \(f_{fri}\) stems from the soil-structure interaction following the contact law between the soil/structure interfaces.

### A.3 Energy conservation

We utilize the general form of the total energy balance equation for porous media from Reference 38. The total energy balance equations for the fluid phases take the following form:

\[
\frac{1}{V} \frac{D_f(m_f(e_f + 0.5U_f^2))}{D_t} = \nabla \cdot (\phi_f p_f I) \cdot U_f + \nabla \cdot q_f + (\bar{\rho}_f \mathbf{b}) \cdot U_f + \sum f_d \cdot U_f + \sum q_{sf}
\]  

(A14)

By applying the product rule \(D(mU^2) = D(mU \cdot U) = 2U \cdot D(mU)\), we can express the left-hand side of Equation (A14) as:

\[
\frac{1}{V} \frac{D_f(m_f(e_f + 0.5U_f^2))}{D_t} = \frac{1}{V} \frac{D_f(m_f e_f)}{D_t} + \frac{1}{V} \frac{D_f(m_f U_f)}{D_t} \cdot U_f
\]  

(A15)

Combining Equations (A11), (A14), and (A15), we derive the final form of the internal energy balance equation for the fluid phases as:

\[
\frac{1}{V} \frac{D_f(m_f e_f)}{D_t} = \frac{1}{V} \left[ \frac{\partial(m_f e_f)}{\partial t} + \nabla \cdot (m_f e_f U_f) \right] = \bar{\rho}_f p_f \frac{D_f e_f}{D_t} + \nabla \cdot q_f + \sum q_{sf}
\]  

(A16)

On the right hand side, the terms include the average pressure-volume work, the average viscous dissipation, the thermal transport and the energy exchange between solid and fluid respectively. The heat flux is \(q_f = \bar{\rho}_f \beta_f \nabla T_f\) with \(\beta_f\) being the
thermal conductivity coefficient. To derive the internal energy balance equation for the solid phase, we introduce the rate of the internal energy for the thermoelastic materials as a function of elastic strain tensor \( \varepsilon^p_s \) and temperature \( T_s \) as follows:

\[
\frac{m_s}{V} \frac{D_s (e_s)}{Dt} = \sigma' : \frac{D_s (e_s)}{Dt} + \frac{m_s}{V} c_v \frac{D_s (T_s)}{Dt} = \sigma' : \frac{D_s (e_s)}{Dt} + c_v \frac{D_s (T_s)}{Dt}
\]  

(A17)

\( c_v \) is the specific heat at the constant volume of the solid materials. The total energy balance equation for the mixture based on Reference 38 can be written as:

\[
\frac{1}{V} \frac{D_f (m_f (e_f + 0.5U_f^2))}{Dt} + \frac{1}{V} \frac{D_s (m_s (e_s + 0.5U_s^2))}{Dt} = \nabla \cdot (\nabla U_f + \nabla U_s + \nabla (\frac{\sigma'}{\rho_f}) : \nabla U_f) + \sum q_{sf} \]

(A18)

Subtracting Equations (A18), (A17) to (A14) and (A13), we obtained the internal energy balance equation for solid phase as:

\[
\sigma' : \frac{D_s (e_s)}{Dt} + \frac{m_s}{V} c_v \frac{D_s (T_s)}{Dt} = \Delta W_s + \Delta W_{friction} + \nabla \cdot q_s - \sum q_{sf}
\]  

(A19)

On the right hand side, the terms include the work rate from frictional sliding between solid materials \( \Delta W_{friction} \), thermal transport and energy exchange between solid and fluid respectively. The heat flux is \( q_s = \bar{\rho}_s \beta s V T_s \) with \( \beta s \) being the thermal conductivity of the solid materials, the mechanical work rate \( \Delta W_s = \sigma' : \frac{D_s (e_s)}{Dt} = \sigma' : \left( \frac{D_s (e_s)}{Dt} + \frac{D_s (e_s)}{Dt} \right) \) computed from the constitutive model with \( e_s \) is the plastic strain tensor. By subtracting the term \( \sigma' : \frac{D_s (e_s)}{Dt} \), we get the final form of the energy balance equation as:

\[
\frac{m_s}{V} c_v \frac{D_s (T_s)}{Dt} = \sigma' : \frac{D_s (e_s)}{Dt} + \Delta W_{friction} + \nabla \cdot q_s - \sum q_{sf}
\]  

(A20)

A.4 Advanced fluid pressure

The discretization of the pressure equation begins with the Lagrangian cell face velocity and the equation for the pressure as:

\[
\bar{\rho}_{f,FC} \frac{U^{n+1}_{f,FC} - U^n_{f,FC}}{\Delta t} = nV^{FC} \left( \frac{p^n_{f,FC}}{\rho^n_{f,FC}} + \frac{\Delta p^n_{f,FC}}{\Delta t} \right) - \frac{\sum q_{sf}}{\bar{\rho}_{f,FC}}
\]  

(A21)

\[
\kappa \frac{dP}{dt} = \nabla \cdot U^{n+1}_{f,FC}
\]  

(A22)

The divergence of the Equation (A21) with \( \nabla \cdot b = 0 \) is:

\[
\nabla \cdot U^{n+1}_{f,FC} - \nabla \cdot U^n_{f,FC} = \nabla \cdot \frac{\Delta t}{\rho^n_{f,FC}} \cdot \nabla^{FC} \left( p^n_{f,FC} + \Delta P^n_{f,FC} \right)
\]  

(A23)

To solve this equation, we define the cell face intermediate velocity \( U^n_{f,FC} \) as:

\[
\bar{\rho}_{f,FC} \frac{U^n_{f,FC} - U^n_{f,FC}}{\Delta t} = nV^{FC} \left( \frac{p^n_{f,FC}}{\rho^n_{f,FC}} + \frac{\Delta p^n_{f,FC}}{\Delta t} \right) - \frac{\sum q_{sf}}{\bar{\rho}_{f,FC}}
\]  

(A24)

The divergence of the Equation (A24) is:

\[
\nabla \cdot U^n_{f,FC} - \nabla \cdot U^n_{f,FC} = \nabla \cdot \frac{\Delta t}{\rho^n_{f,FC}} \cdot \nabla^{FC} p^n_{f,FC}
\]  

(A25)
Combining Equations (A22), (A23), (A25), it leads to:

\[
\left( k - \nabla^c \frac{\Delta t}{\rho^s_{j,FC}} \cdot \nabla^c \right) \Delta P^n_{f, FC} = -\nabla^c \cdot U^r_{f, FC} \tag{A26}
\]

When the fluid is incompressible, \( k \) approaches to zero and the Equation (A26) becomes the Poisson’s equation for the incompressible fluid flow.

A.5 Momentum and energy exchange with an implicit solver

Considering the fluid momentum balance equation as:

\[
(m U^n_{f,FC})^{t+1} = (m U^n_{f,FC}) - \Delta t \left( \nabla \cdot \rho F C \cdot \nu^n_{f, e} + m_f b \right) + VK \Delta t \left( U^{n+1}_{s,FC} - U^{n+1}_{f,FC} \right) \tag{A27}
\]

And assuming \( m^{n+1}_{f,FC} = m^n_{f,FC} \), we get:

\[
U^{n+1}_{f,FC} = U^n_{f,FC} - \Delta t \left( \frac{\nabla F C \cdot \nu^n_{f, FC}}{\rho^n_{f, FC}} + b \right) + \frac{\Delta t K}{\rho^n_{f, FC}} \left( U^{n+1}_{s,FC} - U^{n+1}_{f,FC} \right) \tag{A28}
\]

As defined in the section ‘Advanced Fluid Pressure’, the cell face intermediate fluid velocity \( U^n_{f,FC} = \Delta t (\nabla F C \cdot \nu^n_{f, FC}/\rho^n_{f, FC} + b) \) is computed as:

\[
U^{n+1}_{f,FC} = U^n_{f,FC} - \frac{\Delta t K}{\rho^n_{f, FC}} \left( U^{n+1}_{s,FC} - U^{n+1}_{f,FC} \right) \tag{A29}
\]

Considering the solid momentum balance equation as:

\[
(m U^n_{s,FC})^{t+1} = (m U^n_{s,FC}) - \Delta t \left( \nabla \cdot \rho F C \cdot \nu^n_{s, e} - \nabla (1 - n) \cdot \rho F C \cdot \nu^n_{f, e} + m_s b \right) - VK \Delta t \left( U^{n+1}_{s,FC} - U^{n+1}_{f,FC} \right) \tag{A30}
\]

We define the cell face intermediate solid velocity as \( U^n_{s,FC} = \Delta t (\nabla F C \cdot \nu^n_{s, FC}/\rho^n_{s, FC} - \nabla F C \cdot \nu^n_{f, FC}/\rho^n_{s, FC} + b) \) leading to:

\[
U^{n+1}_{s,FC} = U^n_{s,FC} - \frac{\Delta t K}{\rho^n_{s,FC}} \left( U^{n+1}_{s,FC} - U^{n+1}_{f,FC} \right) \tag{A31}
\]

Combining Equations (A29) and (A31) we get:

\[
U^n_{f,FC} + \Delta U^n_{f,FC} = U^n_{f,FC} + \frac{\Delta t K}{\rho^n_{f,FC}} \left( U^n_{s,FC} + \Delta U^n_{s,FC} - U^n_{f,FC} - \Delta U^n_{f,FC} \right)
\]

\[
U^n_{s,FC} + \Delta U^n_{s,FC} = U^n_{s,FC} - \frac{\Delta t K}{\rho^n_{s,FC}} \left( U^n_{s,FC} + \Delta U^n_{s,FC} - U^n_{f,FC} - \Delta U^n_{f,FC} \right) \tag{A32}
\]

Rearranging the Equation (A32), it leads to the linear system of equations as belows:

\[
\begin{bmatrix}
(1 + \beta_{12,FC}) & -\beta_{12,FC} \\
-\beta_{21,FC} & (1 + \beta_{21,FC})
\end{bmatrix}
\begin{bmatrix}
\Delta U^n_{f,FC} \\
\Delta U^n_{s,FC}
\end{bmatrix}
= \begin{bmatrix}
\beta_{12,FC} (U^n_{s,FC} - U^n_{f,FC}) \\
\beta_{21,FC} (U^n_{s,FC} - U^n_{f,FC})
\end{bmatrix}
\]

Solving this linear equations with \( \beta_{12,FC} = (\Delta t K)/\rho^n_{f,FC} \) and \( \beta_{21,FC} = (\Delta t K)/\rho^n_{s,FC} \) with \( K \) is the momentum exchange coefficient. Similar derivation can be performed to computed the cell-center velocity increment leading to:

\[
\begin{bmatrix}
(1 + \beta_{12c}) & -\beta_{12c} \\
-\beta_{21c} & (1 + \beta_{21c})
\end{bmatrix}
\begin{bmatrix}
\Delta U^n_{f,e} \\
\Delta U^n_{s,c}
\end{bmatrix}
= \begin{bmatrix}
\beta_{12c} (U^n_{s,c} - U^n_{f,e}) \\
\beta_{21c} (U^n_{s,c} - U^n_{f,e})
\end{bmatrix}
\]
with $\beta_{12c} = (\Delta t K) / \bar{\rho}_{f,c}^n$ and $\beta_{21c} = (\Delta t K) / \bar{\rho}_{sc}^n$ and the cell-centered intermediate velocity can be calculated by:

\[
U_{f,c}^* = U_{f,c}^n + \Delta t \left( -\frac{\nabla p_{f,c}^{n+1}}{\bar{\rho}_{f,c}^n} + \frac{\nabla \cdot \mathbf{f}_{f,c}^n}{\bar{\rho}_{f,c}^n} + b \right)
\]

\[
U_{sc}^* = U_{sc}^n + \Delta t \left( \frac{\nabla \cdot \sigma_{n,c}^n}{\bar{\rho}_{sc}^n} - \frac{\nabla p_{sc}^{n+1}}{\bar{\rho}_{sc}^n} + b \right)
\]

(A33)

For generalize multi materials $i,j = 1 : N$, the linear equations is in the form as:

\[
\begin{bmatrix}
(1 + \beta_{ij}) & -\beta_{ij} \\
-\beta_{ji} & (1 + \beta_{ji})
\end{bmatrix}
\begin{bmatrix}
\Delta U_i \\
\Delta U_j
\end{bmatrix}
= \begin{bmatrix}
\beta_{ij} \left(U_i^n - U_j^n\right) \\
\beta_{ji} \left(U_j^n - U_i^n\right)
\end{bmatrix}
\]

Similar approach applied for the energy exchange term leading to:

\[
\begin{bmatrix}
(1 + \eta_{ij}) & -\eta_{ij} \\
-\eta_{ji} & (1 + \eta_{ji})
\end{bmatrix}
\begin{bmatrix}
\Delta T_i \\
\Delta T_j
\end{bmatrix}
= \begin{bmatrix}
\eta_{ij} \left(T_i^n - T_j^n\right) \\
\eta_{ji} \left(T_j^n - T_i^n\right)
\end{bmatrix}
\]

with $\eta$ is the energy exchange coefficient.