CLASSIFICATION RESULTS ON SURFACES IN THE ISOTROPIC 3-SPACE

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Abstract. The isotropic 3-space \( I^3 \) which is one of the Cayley–Klein spaces is obtained from the Euclidean space by substituting the usual Euclidean distance with the isotropic distance. In the present paper, we give several classifications on the surfaces in \( I^3 \) with the constant relative curvature (analogue of the Gaussian curvature) and the constant isotropic mean curvature. In particular, we classify the helicoidal surfaces in \( I^3 \) with constant curvature and analyze some special curves on these.

Keywords: Isotropic space; helicoidal surface; isotropic mean curvature; relative curvature.
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1 Preliminaries

Differential geometry of isotropic spaces have been introduced by K. Strubecker [41], H. Sachs [34]-[36], D. Palman [29] and others. Especially the reader can find a well bibliography for isotropic planes and isotropic 3-spaces in [34, 35].

The isotropic 3-space \( I^3 \) is a Cayley–Klein space defined from a 3-dimensional projective space \( P(\mathbb{R}^3) \) with the absolute figure which is an ordered triple \( (\omega, f_1, f_2) \), where \( \omega \) is a plane in \( P(\mathbb{R}^3) \) and \( f_1, f_2 \) are two complex-conjugate straight lines in \( \omega \) (see [39]). The homogeneous coordinates in \( P(\mathbb{R}^3) \) are introduced in such a way that the absolute plane \( \omega \) is given by \( X_0 = 0 \) and the absolute lines \( f_1, f_2 \) by \( X_0 = X_1 + iX_2 = 0, \ X_0 = X_1 - iX_2 = 0 \). The intersection point \( F(0 : 0 : 0 : 1) \) of these two lines is called the absolute point. The group of motions of \( I^3 \) is a six-parameter group given in the affine coordinates \( x_1 = X_1/X_0, \ x_2 = X_2/X_0, \ x_3 = X_3/X_0 \) by

\[
(x_1, x_2, x_3) \mapsto (x'_1, x'_2, x'_3) : \begin{cases} 
  x'_1 = a + x_1 \cos \phi - x_2 \sin \phi, \\
  x'_2 = b + x_1 \sin \phi + x_2 \cos \phi, \\
  x'_3 = c + dx_1 + ex_2 + x_3,
\end{cases}
\]

(1.1)

where \( a, b, c, d, e, \phi \in \mathbb{R} \).

Such affine transformations are called isotropic congruence transformations or i-motions. It easily seen from (1.1) that i-motions are indeed composed by an Euclidean motion in the \( x_1x_2 \)-plane (i.e. translation and rotation) and an affine shear transformation in \( x_3 \)-direction.

Consider the points \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \). The projection in \( x_3 \)-direction onto \( \mathbb{R}^2, (x_1, x_2, x_3) \mapsto (x_1, x_2, 0) \), is called the top view. The isotropic distance, so-called i-distance of two points \( x \) and \( y \) is defined as the Euclidean distance of their top views, i.e.,

\[
||x - y||_i = \sqrt{\sum_{j=1}^{2} (y_j - x_j)^2}.
\]

(1.2)

The i-metric is degenerate along the lines in \( x_3 \)-direction, and such lines are called isotropic lines.

Planes, circles and spheres. There are two types of planes in \( I^3 \) ([31]-[33]).
(1) **Non-isotropic planes** are planes non-parallel to the $x_3$–direction. In these planes we basically have an Euclidean metric: This is not the one we are used to, since we have to make the usual Euclidean measurements in the top view. An i-circle (of elliptic type) in a non-isotropic plane $P$ is an ellipse, whose top view is an Euclidean circle. Such an i-circle with center $m \in P$ and radius $r$ is the set of all points $x \in P$ with $\|x - m\| = r$.

(2) **Isotropic planes** are planes parallel to the $x_3$–axis. There, $\mathbb{I}^3$ induces an isotropic metric. An i-circle (of parabolic type) is a parabola with $x_3$–parallel axis and thus it lies in an isotropic plane

An i-circle of parabolic type is not the iso-distance set of a fixed point, but it may be seen as a curve with constant isotropic curvature: A curve $c$ in an isotropic plane $P$ (without loss of generality we set $P : x_2 = 0$) which does not possess isotropic tangents can be written as graph $x_3 = f(x_1)$. Then, the i-curvature of $c$ at $x_1 = a_0$ is given by the second derivative $\kappa_i(a_0) = f''(a_0)$. For an i-circle of parabolic type $f$ is quadratic and thus $\kappa_i$ is constant.

There are also two types of isotropic spheres. An i-sphere of the cylindrical type is the set of all points $x \in \mathbb{I}^3$ with $\|x - m\| = r$. Speaking in an Euclidean way, such a sphere is a right circular cylinder with $x_3$–parallel rulings; its top view is the Euclidean circle with center $m$ and radius $r$. The more interesting and important type of spheres are the i-spheres of parabolic type,

$$x_3 = \frac{A}{2} (x_1^2 + x_2^2) + Bx_1 + Cx_2 + D, \quad A \neq 0.$$  

From an Euclidean perspective, they are paraboloids of revolution with $z$–parallel axis. The intersections of these i-spheres with planes $P$ are i-circles. If $P$ is non-isotropic, then the intersection is an i-circle of elliptic type. If $P$ is isotropic, the intersection curve is an i-circle of parabolic type.

*Curvature theory of surfaces.* A surface $M^2$ immersed in $\mathbb{I}^3$ is called admissible if it has no isotropic tangent planes. We restrict our framework to admissible regular surfaces. For such a surface $M^2$, the coefficients $g_{11}, g_{12}, g_{22}$ of its first fundamental form are calculated with respect to the induced metric.

The normal field of $M^2$ is always the isotropic vector $(0, 0, 1)$ since it is perpendicular to all tangent vectors to $M^2$. The coefficients $h_{11}, h_{12}, h_{22}$ of the second fundamental form of $M^2$ are calculated with respect to the normal field of $M^2$ (for details, see [35], p. 155).

The relative curvature (so called isotropic Gaussian curvature) and isotropic mean curvature are defined by

$$K = \frac{\det (h_{ij})}{\det (g_{ij})}, \quad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2 \det (g_{ij})}.$$  

The surface $M^2$ is said to be isotropic flat (resp. isotropic minimal) if $K$ (resp. $H$) vanishes.

The surfaces in the isotropic spaces have been studied by I. Kamenarovic ([21] [22]), B. Pavkovic ([30]) and B. Divjak ([13] [40]) as well as has important applications in several applied sciences, e.g., computer science, Image Processing, Architectural design and microeconomics ([9] [12] [26], [31]–[33]).

Most recently, Z. M. Sipus ([39]) classified the translation surfaces of constant curvature generated by two planar curves in $\mathbb{I}^3$. And then a classification for the ones generated by a space curve and a planar curve with constant curvature was derived in [2].

In [3], the authors established a method to calculate the second fundamental form of the surfaces in the isotropic 4-space $\mathbb{I}^4$ and classified some surfaces in $\mathbb{I}^4$ with vanishing curvatures.

In this paper, the helicoidal surfaces in $\mathbb{I}^3$ with constant isotropic mean and constant relative curvature are classified. Further some special curves on such surfaces are characterized.
2 Classification results on surfaces in \( \mathbb{I}^3 \)

This section is devoted to recall the classification results on surfaces in \( \mathbb{I}^3 \) into separate subsections, such as the translation surfaces, the homothetical surfaces (so-called factorable surfaces), Aminov surfaces, the spherical product surfaces.

2.1 Translation surfaces in \( \mathbb{I}^3 \)

The present author introduced the translation surfaces generated by a space curve and a planar curve as follows (for details, see \([2]\))

\[
    r(u_1, u_2) \mapsto (f_1(u_1), f_2(u_1) + g_2(u_2), f_3(u_1) + g_3(u_2)),
\]

and classified the ones with constant curvature by the following theorems:

Theorem 2.1. \([2]\) Let \( M^2 \) be a translation surface given by (2.1) in \( \mathbb{I}^3 \) with constant relative curvature \( K_0 \). Then it is either a generalized cylinder, i.e. \( K_0 = 0 \), or parametrized by one of the following

(i) \( r(u_1, u_2) = \left(f_1, b_1 f_1 + g_2, a_1 (f_1)^2 + \frac{b_2}{a_1} (g_2)^2 + b_2 f_1 + b_3 g_2\right); \)

(ii) \( r(u_1, u_2) = \left(f_1, a_2 (f_1)^2 + b_4 f_1 + g_2, b_5 (f_1)^2 + \frac{1}{K_0 a_3} (-2K_0 g_2)^3/2 + b_6 f_1 + a_4 g_2\right), \)

where \( a_i \) are nonzero constants and \( b_j \) some constants for \( 1 \leq i \leq 4 \) and \( 1 \leq j \leq 6 \).

Theorem 2.2. \([2]\) Let \( M^2 \) be a translation surface given by (2.1) in \( \mathbb{I}^3 \) with constant isotropic mean curvature \( H_0 \). Then it is determined by one of the following expressions

(i) \( r(u_1, u_2) = \left(f_1, f_2 + g_2, H_0 f_1^2 + b_1 f_2 + b_2 f_1 + b_3 g_2\right), \)

(ii) \( r(u_1, u_2) = \left(f_1, b_4 f_1 + g_2, (H_0 - a_1) (f_1)^2 + a_2 (g_2)^2 + b_5 f_1 + b_6 g_2\right); \)

(iii) \( r(u_1, u_2) = \left(f_1, -\frac{1}{a_3} \ln|\cos(a_3 f_1)| + g_2, H_0 f_1^2 + b_7 f_2 + \frac{1}{a_3} \exp(a_3 g_2) + b_8 f_1 + b_9 g_2\right), \)

where \( a_i \) are nonzero constants and \( b_i \) some constants \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq 9 \).

Remark 2.3. Isotropic minimal translation surfaces can also be classified by Theorem 2.2 as taking \( H_0 = 0 \) in the statements (i)-(iii) of the theorem.

The author and A.O. Ogrenmis \([4]\) introduced the translation hypersurfaces in such a that way a translation hypersurface \( (M^n, F) \) in the isotropic \( (n + 1) \) -space \( \mathbb{I}^{n+1} \), \( n \geq 2 \), is parametrized by

\[
    X : \mathbb{R}^n \rightarrow \mathbb{I}^{n+1}, \ x \mapsto (x, F(x)), \ F(x) := \sum_{j=1}^{n} f_j(x_j), \ x \in \mathbb{R}^n,
\]

where \( f_j \) is a smooth function of one variable for all \( j \in \{1, ..., n\} \). For more details of \( \mathbb{I}^{n+1} \), see \([9, 30, 40]\).

Some classifications were obtained for such hypersurfaces in \( \mathbb{I}^{n+1} \) by the following results:

Theorem 2.4. \([4]\) Let \( (M^n, F) \) be a translation hypersurface in \( \mathbb{I}^{n+1} \) with nonzero constant relative curvature \( K_0 \). Then it has of the form

\[
    X(x) = \left(x, \sum_{j=1}^{n} \alpha_j x_j^2 + \beta_j x_j + \varepsilon\right),
\]
where \( x \in \mathbb{R}^n \), \( \alpha_j \) are nonzero constants and \( \beta_j, \varepsilon \) some constants for all \( j \in \{1, ..., n\} \).

In particular, if \((M^n, F)\) is isotropic flat in \( \mathbb{I}^{n+1} \), then it is congruent to a cylinder from Euclidean perspective.

**Theorem 2.5.** \cite{4} Let \((M^n, F)\) be a translation hypersurface in \( \mathbb{I}^{n+1} \) with constant isotropic mean curvature \( H_0 \). Then it has of the form

\[
X(x) = \left( x, \sum_{j=1}^{n} \alpha_j x_j^2 + \beta_j x_j + \varepsilon \right),
\]

where \( x \in \mathbb{R}^n \) and \( \alpha_j, \beta_j, \varepsilon \) are some constants for all \( j \in \{1, ..., n\} \) such that \( \sum_{j=1}^{n} \alpha_j = \frac{n}{2} H_0 \).

**Remark 2.6.** Isotropic minimal translation hypersurfaces in \( \mathbb{I}^{n+1} \) are also classified by Theorem 2.5 as taking \( H_0 = 0 \).

### 2.2 Homothetical surfaces in \( \mathbb{I}^{n+1} \)

The authors in \cite{4} defined the homothetical hypersurfaces in \( \mathbb{I}^{n+1} \) as follows: A hypersurface \( M^n \) of \( \mathbb{I}^{n+1} \) is called a homothetical hypersurface \((M^n, H)\) if it is the graph of a function of the form:

\[
H(x_1, ..., x_n) := h_1(x_1) \cdot ... \cdot h_n(x_n),
\]

where \( h_1, ..., h_n \) are smooth functions of one real variable.

Next results classify the homothetical hypersurfaces in \( \mathbb{I}^{n+1} \) with constant isotropic mean and relative curvature.

**Theorem 2.7.** \cite{4} Let \((M^n, H)\) be a homothetical hypersurface in \( \mathbb{I}^{n+1} \) with constant isotropic mean curvature \( H_0 \). Then it is isotropic minimal, i.e. \( H_0 = 0 \) and has the following form

\[
X(x) = \left( x, \prod_{j=1}^{n} \left( \gamma_j x_j + \varepsilon_j \right) \right), \quad (3.1)
\]

where \( x \in \mathbb{R}^n \), \( \gamma_j, \varepsilon_j \) some constants.

**Theorem 2.8.** \cite{4} Let \((M^n, H)\) be an isotropic flat homothetical hypersurface in \( \mathbb{I}^{n+1} \). Then it has one of the following forms:

(ii) \( X(x) = \left( x, \gamma \exp \left( \alpha_1 x_1 + \alpha_2 x_2 \right) \prod_{j=3}^{n} h_j(x_j) \right) \) for nonzero constants \( \gamma, \alpha_1, \alpha_2 \);

(ii) \( X(x) = \left( x, \gamma \prod_{j=1}^{n} \left( x_j + \beta_j \right)^{\alpha_j} \right) \), where \( x \in \mathbb{R}^n \), \( \beta_1, ..., \beta_n \) are some constants and \( \gamma, \alpha_1, ..., \alpha_n \) nonzero constants such that \( \sum_{i=1}^{n} \alpha_i = 1 \).

In the particular three dimensional case, the same authors generalized the Theorem 2.8 to the homothetical surfaces with any constant relative curvature

**Theorem 2.9.** \cite{5} Let \( M^2 \) be a homothetical surface in \( \mathbb{I}^3 \) with constant relative curvature \( K_0 \).

(A) If \( K_0 = 0 \), then we have

(A.1) \( H(x, y) = c_1 h_1(x) \) or \( H(x, y) = c_2 h_2(y) \) for nonzero constants \( c_1, c_2 \),

(A.2) \( H(x, y) = c_1 \exp \left( c_2 x + c_3 y \right) \), where \( c_1, c_2, c_3 \) are nonzero constants,
(A.3) \( H(x, y) = (c_1x + d_1)\sqrt{c_2y + d_2} \), where \( c_1, c_2, c_3, c_4 \) are nonzero constants and \( d_1, d_2 \) some constants, \( c_2 \neq 1 \neq c_4 \).

(B) If \( K_0 \neq 0 \), then it is negative (i.e. \( K_0 < 0 \)) and \( h_1, h_2 \) are linear functions.

2.3 Spherical product surfaces and Aminov surfaces in \( \mathbb{I}^4 \)

The present author and I. Mihai (see [3]) established a method to calculate the second fundamental form of the surfaces in the isotropic 4-space \( \mathbb{I}^4 \). Then ones classified the Aminov surfaces, given by

\[
\mathbf{r} : \mathbb{I} \times [0, 2\pi) \rightarrow \mathbb{I}^4, \quad (u, v) \mapsto \mathbf{r}(u, v) := (u, v, r(u) \cos v, r(u) \sin v),
\]

with vanishing curvature as follows:

**Theorem 2.10.** [3] The isotropic flat Aminov surfaces in \( \mathbb{I}^4 \) are only generalized cylinders over circular helices from Euclidean perspective.

**Theorem 2.11.** [3] There does not exist an isotropic minimal Aminov surface in \( \mathbb{I}^4 \).

Furthermore, same authors derived the following classification results for the spherical product surface \( (M^2, c_1 \otimes c_2) \) of two curves \( c_1 \) and \( c_2 \) in \( \mathbb{I}^4 \) which is defined by

\[
\mathbf{r} := c_1 \otimes c_2 : \mathbb{R}^2 \rightarrow \mathbb{I}^4, \quad (u, v) \mapsto (u, f_1(u), f_2(u)v, f_2(u)g(v)),
\]

where the curves \( c_1(u) = (u, f_1(u), f_2(u)) \) and \( c_2(v) = (v, g(v)) \) are called the generating curves of the surface.

**Theorem 2.12.** [3] Let \( (M^2, c_1 \otimes c_2) \) be a isotropic flat spherical product surface in \( \mathbb{I}^4 \). Then either it is a non-isotropic plane or one of the following satisfies

(i) \( c_1 \) is a planar curve in \( \mathbb{I}^3 \) lying in the isotropic plane \( y = \text{const.} \);
(ii) \( c_1 \) is a line in \( \mathbb{I}^3 \);
(iii) \( c_1 \) is a curve in \( \mathbb{I}^3 \) of the form

\[
c_1(u) = \left(u, f_1(u), \lambda \int \sqrt{1 + (f'_1)^2} du + \xi \right), \quad \lambda, \xi \in \mathbb{R}, \lambda \neq 0;
\]

(iv) \( c_2 \) is a line in \( \mathbb{I}^2 \).

**Theorem 2.13.** [3] There does not exist an is isotropic minimal spherical product surface in \( \mathbb{I}^4 \) excepting totally geodesic ones.

3 Helicoidal surfaces in \( \mathbb{I}^3 \)

Rotation surfaces in the Euclidean 3-space \( \mathbb{R}^3 \) with constant mean curvature have been known for a long time [13, 23]. A natural generalization of rotation surfaces in \( \mathbb{R}^3 \) are the helicoidal surfaces that can be defined as the orbit of a plane curve under a screw motion in \( \mathbb{R}^3 \).

Such surfaces in \( \mathbb{R}^3 \) with constant mean and Gaussian curvature have been classified by M. Do Carmo and M. Dajczer in [15]. These classifications were extended to the ones with prescribed mean and Gaussian curvatures by C. Baikoussis and T. Koufogiorgos [7].
The helicoidal surfaces also have been studied by many authors in the Minkowskian 3-space $\mathbb{R}^3_1$, the pseudo-Galilean space $G^1_3$ and several homogeneous spaces as focusing on curvature properties (see [1, 8, 11, 19, 20, 25, 27, 28]).

Moreover, there exist many works related with the helicoidal surfaces satisfying an equation in terms of its position vector and Laplace operator in $\mathbb{R}^3$ and $\mathbb{R}^3_1$. For example see [6, 11, 24, 37, 38].

Now we adapt the above notion to isotropic spaces. Considering the i-motions given by (1.1), the Euclidean rotations in the isotropic space $I^3$ is given by the normal form (in affine coordinates)

\[
\begin{align*}
x_1' &= x_1 \cos \phi - x_2 \sin \phi, \\
x_2' &= x_1 \sin \phi + x_2 \cos \phi, \\
x_3' &= x_3,
\end{align*}
\]

where $\phi \in \mathbb{R}$.

Now let $c$ be a curve lying in the isotropic $x_1x_3$–plane given by $c(u) = (f(u), 0, g(u))$ where $f, g \in C^2$ and $f \neq 0 \neq \frac{df}{du}$. By rotating the curve $c$ around $z$–axis and simultaneously followed by a translation, we obtain that the helicoidal surface of first type in $I^3$ with the profil curve $c$ and pitch $h$ is of the form

\[
r(u, v) = (f(u) \cos v, f(u) \sin v, g(u) + hv), \quad h \in \mathbb{R}.
\] (2.1)

Similarly when the profile curve $c$ lies in the isotropic $x_2x_3$–plane, then the helicoidal surface of second type in $I^3$ with pitch $h$ is given by

\[
r(u, v) = (-f(u) \sin v, f(u) \cos v, g(u) + hv), \quad h \in \mathbb{R}.
\] (2.2)

In case $h = 0$, these reduce to the surfaces of revolution in $I^3$. Also when $g$ is a constant, then $r$ is a helicoid from Euclidean perspective.

**Remark 2.1.** The coordinate functions $f$ and $g$ of the profile curve $c$ are arbitrary functions of class $C^2$ and so one can take $f(u) = u$.

**Remark 2.2.** Since both type of the helicoidal surfaces are locally isometric, we only will focus on the ones of first type.

Let $M^2$ be a helicoidal surface of first type in $I^3$. Then the matrix of the first fundamental form $g$ of $M^2$ is

\[
(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & u^2 \end{bmatrix} \quad \text{and} \quad (g^{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{u^2} \end{bmatrix},
\]

where the prime denotes the derivative with respect to $u$ and $(g^{ij}) = (g_{ij})^{-1}$. Thus the Laplacian of $M^2$ with respect to $g$ is

\[
\Delta = \frac{1}{\sqrt{\det (g_{ij})}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} \left( \sqrt{\det (g_{ij})} g^{ij} \frac{\partial}{\partial u_j} \right)
\]

and by taking $u_1 = u$ and $u_2 = v$, we get

\[
\Delta = \frac{1}{u} \frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2} + \frac{1}{u^2} \frac{\partial^2}{\partial v^2}.
\]
Putting \( r_1(u, v) = u \cos v, r_2(u, v) = u \sin v \) and \( r_3(u, v) = g(u) + hv \), one can easily see that \( \Delta r_i = 0, i = 1, 2 \) and
\[
\Delta r_3 = \frac{1}{u}g' + g''.
\]
Assuming \( \Delta r_3 = \lambda r_3, \lambda \in \mathbb{R} \), we can obtain that \( \lambda \) must be zero and
\[
\frac{1}{u}g' + g'' = 0. \tag{2.3}
\]
After solving (2.3), we derive \( g(u) = \alpha \ln |u| + \beta \) for \( \alpha \in \mathbb{R} \setminus \{0\}, \beta \in \mathbb{R} \).

Thus we have the following result

**Proposition 2.2.** Let \( M^2 \) be a helicoidal surface of first type in \( \mathbb{I}^3 \) satisfying \( \Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R} \). Then it is isotropic minimal and has the form
\[
r(u, v) = (u \cos v, u \sin v, \alpha \ln |u| + hv), \quad \alpha \in \mathbb{R} \setminus \{0\}.
\]

### 4 Helicoidal surfaces with constant curvature in \( \mathbb{I}^3 \)

Let us consider the helicoidal surface of first type \( M^2 \) in \( \mathbb{I}^3 \). Then the components of the second fundamental form are
\[
\mathfrak{h}_{11} = g'', \quad \mathfrak{h}_{12} = -\frac{h}{u}, \quad \mathfrak{h}_{22} = ug'.
\tag{3.1}
\]

Thereby, the relative curvature \( K \) of \( M^2 \) is
\[
K = \frac{u^3 g'g'' - h^2}{u^4}. \tag{3.2}
\]

Assume that \( M^2 \) has constant relative curvature \( K_0 \). We have two cases:

**Case (a).** \( K \) vanishes. It follows from (3.2) that \( g'g'' = \frac{h^2}{u^2} \) or
\[
g' (u) = \left( \alpha - \frac{h^2}{u^2} \right)^{\frac{1}{2}}, \quad \alpha \in \mathbb{R}^+. \tag{3.3}
\]

After solving (3.3), we obtain
\[
g(u) = \sqrt{\alpha u^2 - h^2} + h \arctan \left( \frac{h}{\sqrt{\alpha u^2 - h^2}} \right).
\]

**Case (b).** \( K \) is a nonzero constant \( K_0 \). We can rewrite (3.2) as
\[
g'g'' = K_0 u + \frac{h^2}{u^3}
\]
or
\[
g' (u) = \left( K_0 u^2 - \frac{h^2}{u^2} + \gamma \right)^{\frac{1}{2}}, \quad \gamma \in \mathbb{R}. \tag{3.4}
\]

By solving (3.4), we derive
\[
g(u) = \frac{1}{4} \left( 2a(u) - 2h \arctan \left( \frac{-2h^2 + \gamma u^2}{2a(u)} \right) + \frac{\gamma}{\sqrt{K_0}} \ln \left| \gamma + 2 \left( K_0 u^2 + \sqrt{K_0 a(u)} \right) \right| \right),
\]
where
\[
\gamma \in \mathbb{R}, \quad \text{and} \quad a(u) = \sqrt{K_0 u^4 - h^2 + \gamma u^2}.
\]
Thus, we have the next result

**Theorem 3.1.** Let $M^2$ be a helicoidal surface in $\mathbb{R}^3$ with constant relative curvature $K_0$. Then we have the following items

(i) when $K_0 = 0$, $M^2$ has the form

$$
\begin{align*}
\mathbf{r}(u, v) &= (u \cos v, u \sin v, g(u) + hv), \\
g(u) &= \sqrt{\alpha u^2 - h^2} + h \arctan \left( \frac{h}{\sqrt{\alpha u^2 - h^2}} \right), \quad \alpha \in \mathbb{R}^+, \\
a(u) &= \sqrt{K_0 u^4 - h^2 + \gamma u^2}, \quad \gamma \in \mathbb{R}.
\end{align*}
$$

(ii) otherwise, i.e. $K_0 \neq 0$, it is of the form

$$
\begin{align*}
\mathbf{r}(u, v) &= (u \cos v, u \sin v, g(u) + hv), \\
g(u) &= \frac{1}{4} \left( 2a(u) - 2h \arctan \left( \frac{-2h^2 + \gamma u^2}{2a(u)} \right) + \gamma \sqrt{K_0} \ln |\gamma + 2(K_0 u^2 + \sqrt{K_0} a(u))| \right), \\
a(u) &= \sqrt{K_0 u^4 - h^2 + \gamma u^2}.
\end{align*}
$$

**Example 3.2.** Take $h = 1$, $\alpha = 1$, $u \in [1, 5]$ and $v \in [0, 4\pi]$ in (3.5). Then $M^2$ becomes isotropic flat and can be drawn as in Fig 1.

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**Fig. 1.** A helicoidal surface with $K_0 = 0$, $h = 1$.

The isotropic mean curvature $H$ of $M^2$ is given by

$$
H = \frac{g'}{u} + g''.
$$

Suppose that $M^2$ has constant isotropic mean curvature $H_0$. Then putting $g' = p$, we obtain the following Riccati equation

$$
p' + \frac{p}{u} = H_0.
$$

Solving (3.7), we get

$$
g(u) = \frac{H_0}{4} u^2 + \alpha \ln(u) + \beta
$$

for some constants $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0$.

Therefore we have proved the next result:

**Theorem 3.3.** Let $M^2$ be a helicoidal surface in $\mathbb{R}^3$ with constant isotropic mean curvature $H_0$. Then it has the following form

$$
\begin{align*}
\mathbf{r}(u, v) &= (u \cos v, u \sin v, g(u) + hv), \\
g(u) &= \frac{H_0}{4} u^2 + \alpha \ln(u) + \beta, \quad \alpha \in \mathbb{R} \setminus \{0\}.
\end{align*}
$$
Example 3.4. Let us put \( h = 1.5, H_0 = -\alpha = -1, \beta = 0, u \in [1, 5] \) and \( v \in [-\pi, \pi] \) in (3.8). Then we draw it as in Fig 2.

Fig. 2. A helicoidal surface with \( H_0 = -1, h = 1.5 \).

5 Special curves on the helicoidal surfaces in \( \mathbb{I}^3 \)

For more details of special curves on the surfaces in \( \mathbb{I}^3 \) such as, geodesics, asymptotic lines and lines of curvature, see [35], p. 163-181.

In this section we aim to investigate such curves on a helicoidal surface in \( \mathbb{I}^3 \).

Let \( M^2 \) be a helicoidal surface in \( \mathbb{I}^3 \), then any point of a curve on \( M^2 \) has the position vector

\[
\mathbf{r}(u(s), v(s)) = \mathbf{r}(s) = (u(s) \cos(v(s)), u(s) \sin(v(s)), g(u(s)) + hv(s)),
\]

(4.1)

where \( s \) is arc-length parameter of \( \mathbf{r}(s) \). Denote the derivative with respect to \( s \) by a dot. Then \( \mathbf{t}(s) = \dot{\mathbf{r}} = (t_1(s), t_2(s), t_3(s)) \) is the tangent vector of \( \mathbf{r}(s) \). We can take a side tangential vector \( \sigma = (\sigma_1(s), \sigma_2(s), \sigma_3(s)) \) in the tangent plane of \( M^2 \) such that

\[
\sigma_1^2 + \sigma_2^2 = 1, \quad \sigma_1 t_1 + \sigma_2 t_2 = 0, \quad t_1 \sigma_2 - t_2 \sigma_1 = 1.
\]

Therefore we have an orthonormal triple \( \{\mathbf{t}, \sigma, \mathbf{N} = (0, 0, 1)\} \). The second derivative of \( \mathbf{r}(s) \) with respect to \( s \) has the following decomposition

\[
\ddot{\mathbf{r}} = \kappa_g \sigma + \kappa_n \mathbf{N},
\]

where \( \kappa_g \) and \( \kappa_n \) are respectively called the geodesic curvature and normal curvature of \( \mathbf{r}(s) \) on \( M^2 \). The curve \( \mathbf{r}(s) \) is called geodesic (resp., asymptotic line) if and only if its geodesic curvature \( \kappa_g \) (resp., normal curvature \( \kappa_n \)) vanishes.

Also the first derivative of \( \sigma(s) \) with respect to \( s \) has the decomposition

\[
\dot{\sigma} = -\kappa_g \mathbf{t} + \tau_g \mathbf{N},
\]

in which \( \tau_g \) is called the geodesic torsion of \( \mathbf{r}(s) \) on \( M^2 \).

In terms of the components of the fundamental forms of \( M^2 \), the side tangential vector \( \sigma \) is given by

\[
\sigma = -\frac{1}{\det(g_{ij})} \left[(g_{12}\dot{u} + g_{22}\dot{v}) \mathbf{r}_u - (g_{11}\dot{u} + g_{12}\dot{v}) \mathbf{r}_v\right].
\]

So, the geodesic curvature of \( \mathbf{r}(s) \) on the helicoidal surface \( M^2 \) in \( \mathbb{I}^3 \) is given by

\[
\kappa_g(s) = u^2 (\dot{v})^3 - u\dot{w} \dot{v} - 2(\dot{u})^2 \dot{v} - u\dot{w} \dot{u}.
\]

(4.2)
It is easily from (4.2) that the curves \( v = \text{const.} \) on \( M^2 \) are geodesics of \( M^2 \) but not the curves \( u = \text{const.} \), which implies the next result.

**Theorem 4.1.** The \( v \)--parameter curves on the helicoidal surfaces in \( \mathbb{H}^3 \) are geodesics but not \( u \)--parameter curves.

The normal curvature of \( r(s) \) on \( M^2 \) in \( \mathbb{H}^3 \) is

\[
\kappa_n(s) = g''(\dot{u})^2 - 2\frac{h}{u}(\dot{u}\dot{v}) + ug'(\dot{v})^2.
\]

(4.3)

By (4.3), the curves \( u = \text{const.} \) are asymptotic lines of \( M^2 \) if and only if \( g \) is a constant function. Similarly the curves \( v = \text{const.} \) are asymptotic lines of \( M^2 \) if and only if \( g \) is a linear function.

Hence, we have proved the following

**Theorem 4.2.** (i) The \( u \)--parameter curves on a helicoidal surface in \( \mathbb{H}^3 \) are asymptotic curves if and only if it is a helicoid from Euclidean perspective;

(ii) the \( v \)--parameter curves on the helicoidal surfaces in \( \mathbb{H}^3 \) are asymptotic curves if and only if \( g \) is a linear function.

On the other hand a curve on a surface is called a line of curvature if its geodesic torsion \( \tau_g \) vanishes. The function \( \tau_g \) can be defined as

\[
\tau_g = \frac{1}{\det (g_{ij})} \begin{vmatrix}
    dv^2 & -dudv & du^2 \\
    g_{11} & g_{12} & g_{22} \\
    h_{11} & h_{12} & h_{22}
\end{vmatrix}.
\]

Hence, a curve on \( M^2 \) in \( \mathbb{H}^3 \) is a line of curvature if and only if the following equation satisfies

\[-\left(\frac{h}{u}\right) (\dot{u})^2 + (ug' - u^2g'') \dot{u}\dot{v} + (hu) (\dot{v})^2 = 0.
\]

Therefore we can give the following result.

**Theorem 4.3.** The parameter curves on the helicoidal surfaces in \( \mathbb{H}^3 \) are lines of curvature if and only if those are surfaces of revolution.

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