Fredholm theory of the linearized $\bar{\partial}$-operator and additivity of its index

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1 Introduction

In [L1], we established the compactification of the moduli space of the pseudo-holomorphic maps connecting the two ends of the symplectization of a compact contact manifold. Several new phenomena concerning the behavior of the boundary of the compactification were described there.

The first of them is that each time when a family of connecting pseudo-holomorphic maps bubbles off a bubble, there is a new component of the target lying on the "left" the original one such that the image of the bubble lies on this new component of the target. Moreover, each time when a new top bubble appears, there is also a new principal component lying on the new component of the target. Therefore, even in the simplest case when such a family of connecting maps develops only one bubble, the domain of the limit map necessarily has three components. This is quite different from what happens in Gromov-Floer theory. Note that in the bubbling process above, both the domain and the target of the connecting map split (degenerate) into different components. Furthermore, the rates of the two types of degeneration are independent of each other. Closely related to this is that the $\mathbb{R}$-symmetry of the target splits into a two-dimensional or multi-dimensional symmetries during the bubbling or splitting process.

The second of new phenomena we found is that in order to get a correct definition of a stable map and use it to get the desired compactification, it is necessary to count the $\mathbb{R}^1$-symmetry of a component of the target as many times as the number of the connected components of the stable map in the component of the target.

These new phenomena, once carefully analyzed, lead to a rather different picture on the behavior of the boundary of the compactification of the moduli space of the connecting pseudo-holomorphic maps from what we used to believe. It opens a door to various new constructions in contact geometry, some of which were outlined in [L1].

In fact, the results of [L1] lead to the following alternatives depending on whether or not the additivity of the index formula on the linearized $\bar{\partial}$-operator
is applicable in the contact case. To motivate the work of this paper, we now explain this alternative.

Recall that since in the symplectization the symplectic form is exact, each top bubble in the bubbling necessarily has non-removable singularity at infinity, and along the end at infinity, the bubble is convergent to some closed orbit of the Reeb field (contact field) of the contact manifold. Similar to the usual Floer homology, a family of connecting pseudo-holomorphic maps between two closed orbits at the two ends may split into a family of broken connecting maps. Same as [L1], we are going to assume throughout this paper that the contact structure and the contact 1-form are generic in the sense that along the normal direction of each closed orbit of the contact field, the linearized Poincare returning map does not have any eigen vector of eigen value 1. This implies that the set of unparameterized closed orbits is discrete. As it is well known that the parameterized closed orbits of the contact field can be thought as the critical points for the action functional $a_\lambda : \mathcal{L} \rightarrow \mathbb{R}$ given by the formula

$$a_\lambda (z) = \int (z^* \lambda)(t) dt$$

for any $z \in \mathcal{L}(\mathcal{M})$, where $\mathcal{L}(\mathcal{M})$ is the free loop space of parameterized loops in the contact manifold $\mathcal{M}$ with the contact form $\lambda$. Our assumption implies that the set of critical points is a discrete union of one-dimensional critical manifolds, each of them is a circle of length $c = \int x^* \lambda dt$, where $x(t)$ is a closed orbit in some critical manifold. This brings us to a situation of a particular Bott-type Morse theory with the infinite dimensional action functional $a_\lambda$. The index homology in contact geometry introduced in [L3] is a certain kind of Morse-Floer homology for the action functional.

To define the index homology, we first define its chain complex as follows. Let $\langle \{x\} \rangle$ be the equivalent class of the parameterized closed orbit $x$, where any two of parameterized closed orbits are said to be equivalent if they differ by a linear parameterization. The chain complex is defined to be

$$IC = IC(\lambda) = \oplus_{x \in P} \mathbb{Q}\{x\},$$

where $P$ is the set of parameterized closed orbits of the contact field.

We define the boundary operator $D$ of the chain complex as follows. Given $\{x_+\} \in IC$, $D(\{x_+\}) = \Sigma_{x_- \in P} \langle \{x_+\}, \{x_-\} \rangle$, where $\langle \{x_-\}, \{x_+\} \rangle = \# (\cup_{x_- \in P} \mathcal{M}(x_-, x_+, J))$. Here $\mathcal{M}(x_-, x_+, J)$ is the moduli space of all equivalent classes of $J$-holomorphic maps in the symplectization $\check{M} = M \times \mathbb{R}$ connecting the two closed orbits $x_- \in \{x_-\}$ and $x_+ \in \{x_+\}$ in the two ends $M_-$ and $M_+$ of $\check{M}$. In other words, we define the boundary operator by counting discrete connecting maps. Of course, to get a precise definition one needs to use the corresponding virtual moduli cycles to replace the moduli space used here (for the construction of the virtual moduli cycles in the contact case, see [L2]).

To prove the fact that $D^2 = 0$, it is crucial to show that the virtual co-dimension of the moduli space $\cup_{x \in P} \langle \{x\} \rangle \neq \{x_-\}, \{x_+\}, \mathcal{M}(x_-, x_+, J)$ of broken connecting maps of two elements is one. In fact, a parameterized version
of this plays an important role to prove that the index homology so defined is independent of the choices involved and hence is an invariant of the contact structure. In term of the parameterized moduli space this is equivalent to show that

\[ \dim \tilde{M}(x_-, x_+, J) = \dim \tilde{M}(x_-, x, x_+, J) - 1. \]

Since the virtual dimensions of the above two parameterized moduli spaces can be calculated by the index formula for the corresponding linearized $$\bar{\partial}$$-operators, the desired relation above is equivalent to say that the index for the first moduli space is equal to the index for the second moduli space minus 1.

On the other hand, in the usual Bott-type Floer homology in symplectic case, to prove that the boundary map defined there really gives rise to a homology theory, one needs to consider a similar relation for index formula there. However, in that case, unlike here, the indices of the above two moduli spaces are the same.

As we mentioned in [L1], the index formula above together with the "hidden" symmetries in the target discovered in [L1], implies that bubbling is a codimension two phenomenon and splitting of connecting maps into broken ones of two elements is of codimension one. This will promote various interesting constructions in contact geometry. On the hand, if the index formula is the same as the usual one in the Bott-type Floer homology in symplectic case, we will have a rather different picture. In that case, the splitting of connecting maps will become a co-dimension two phenomenon. This will lead to a quite different alternative for applications. For example, instead of having the index homology outlined in [L1] and various related constructions there, one will have a Floer homology theory and related multiplicative structure in the symplectization by using perturbed connecting maps for some suitable chosen Hamiltonian function on the symplectization. One of the purposes of this paper is to show that the first alternative of the consequences of [L1] can be realized.

The second motivation of this paper is to discuss a special phenomenon concerning the dual features of the transversality of the trivial connecting map. Given a closed orbit $$x$$, let $$x_- = x_+ = x$$, and consider the moduli space $$\mathcal{M}(x_-, x_+, J)$$. In this case, there is only one unparameterized connecting map, which is the trivial map coming from $$x$$. It is well-known that $$\mathcal{M}(x_-, x_+, J)$$ can be realized as the zero set of the $$\bar{\partial}$$-section for some Banach bundle over a suitable chosen Banach manifold $$B$$ containing $$\mathcal{M}(x_-, x_+, J)$$.

In order to prove the invariance of the index homology outlined above, it is crucial to prove that the $$\bar{\partial}$$-section at the trivial map is transversal for the Banach manifold $$B$$ naturally appeared in the situation here. In the Floer homology in symplectic geometry, the corresponding statement looks almost trivial. However, it is a corner stone of the proof of the invariance of Floer homology with respect to the "continuation" of parameters involved in the construction. The reason for this is that it is the only "existence" type statement in the proof.

We have mentioned before that in the simplest case of bubbling off only one bubble, the domain of the limit map will split into three components and target into two. It may happen that the "new" principal component of the limit map
is a trivial component map. In order to prove that the virtual codimension of
the corresponding moduli space, in which the limit map lies, is two, it is crucial
to show that the $\bar{\partial}$-section at the trivial map of the principal component is not
transversal for a suitable choice of Banach manifold $B$.

Therefore, unlike the Floer homology in symplectic geometry, the transvers-
sality of the trivial connecting map has double meanings here. The second main
purpose of this paper is to discuss this special phenomenon.

This paper is organized as follows.

In Section 2, we will derive the formula for the index of the linearized
$\bar{\partial}$-operator and prove the transversality and the non-transversality of the trivial
connecting map.

In Section 3, we will prove the additivity of the index of the $\bar{\partial}$-operator
under the deformation (gluing) of pseudo-holomorphic maps.

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2 Index of linearized $\bar{\partial}$-operator

As this paper is the continuation of [L1], we will assume that the readers are
already familiar with [L1]. We now briefly recall some of the notations used in
[L1].

Let $(M^{2n+1}, \xi)$ be a compact contact manifold, where $\xi$ is a generic $2n$
dimensional subbundle of $TM$. A contact form $\lambda = \lambda_\xi$ associated to $\xi$
is a 1-form such that $\lambda \wedge (d\lambda)^n \neq 0$ and $\xi = ker \lambda$. The 2-form $d\lambda$
is non-degenerate when restricted to $\xi$ and has a 1-dimensional kernel at each tangent space of
$M$. It has a canonic section $X_\lambda$ defined by requiring that $\lambda(X_\lambda) = 1$. The
vector field $X_\lambda$ is called contact field or Reeb field. There is a decomposition
$TM = \xi \oplus RX_\lambda$.

The symplectization of $(M^{2n+1}, \xi, \lambda)$ is defined as follows.

Let $\tilde{M}$ be $M \times \mathbb{R}$ equipped with the exact symplectic form
$\omega = d(e^r \cdot \lambda)$, where $r$ is the coordinate for the $\mathbb{R}$-factor. Since $d\lambda$ is symplectic along $\xi$, there
exists a $d\lambda$-compatible almost complex structure $J$ defined on $\xi$. In fact, the set
of all such $J$’s is contractible. We extend $J$ to an $r$-invariant almost complex structure $\tilde{J}$ by requiring:

$$\tilde{J}(\frac{\partial}{\partial r}) = X_\lambda, \quad \tilde{J}(X_\lambda) = -\frac{\partial}{\partial r}, \quad \text{and} \quad \tilde{J} = J$$

along $\xi$.

• Equation for $\tilde{J}$-holomorphic curves in $\tilde{M}$

Let $\tilde{u} = (u, a) : \Sigma = S^1 \times \mathbb{R} \to \tilde{M}$ be a $\tilde{J}$-holomorphic map where $u : \Sigma \to M$
and $a : \Sigma \to \mathbb{R}$. Then we have

$$\tilde{J}(\tilde{u}) \circ d\tilde{u} = d\tilde{u} \circ i, \quad (\star)$$

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where $i$ is the standard complex structure on $\Sigma$, i.e. $i(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t}$ and $i(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial s}$.

Here $(s,t)$ is the cylindrical coordinate of $\mathbf{R} \times S^1$.

Equation (\star) is equivalent to the following equations:

\[
\begin{cases}
\pi(u)du + J(u)\pi(u)du \circ i = 0 \\
(u^*\lambda) \circ i = da
\end{cases}
\]

Equation (1) is equivalent to :

\[
\pi(u)\frac{\partial u}{\partial s} + J(u)\pi(u)\frac{\partial u}{\partial t} = 0.
\]

\[1'\]

**Banach Space Set-up:**

Local coordinate near $x$: As in [L1], we assume throughout this paper that the constant 1-form $\lambda$ is generic in the sense that 1 is not an eigen-value of the Pincare returning maps along the normal direction of all closed orbits of $\lambda$. This implies that the set of unparameterized closed orbits is discrete.

Given a parameterized closed orbit $x : S^1 = \mathbf{R}/\mathbf{Z} \to M$ of $\lambda$-period $c = \int_{S^1}(x^*\lambda) > 0$, we have $\frac{dx}{dt} = c \cdot X_\lambda(x)$. Here $X_\lambda$ is the contact field of $\lambda$ and $t \in [0,1) = \mathbf{R}/\mathbf{Z}$.

Let $\tau$ be the smallest positive number such that $x(\tau + t) = x(t)$, for any $t \in [0,1]$. Let $T = \tau \cdot c$. For any $z = x(t), t \in [0,\tau)$, on the image of the closed orbit $x$, we define the $\theta$-coordinate of $z$ to be $\theta = c \cdot t, \theta \in [0, T)$. Clearly, along the image of $x$, $\frac{\partial}{\partial \theta} = X_\lambda$.

Let

\[ V_\epsilon = \{ (\theta, y) \mid \theta \in \mathbf{R}/\mathbf{T} \mathbf{Z} = [0, T), y \in \mathbf{R}^{2n}, |y| < \epsilon \}. \]

Define the $(\theta, y)$-coordinate near $x$ by using the exponential map:

\[ Exp : (\theta, y) \to exp_{x(x)}(\Sigma y, e_i). \]

Here $y = (y_1, \cdots, y_{2n}) \in \mathbf{R}^{2n}$ and $\{e_1, \cdots, e_{2n}\}$ is a symplectic framing for the symplectic bundle $\xi_x$.

Let $U_\epsilon(x)$ be the image of $V_\epsilon$ under the exponential map $Exp$.

Now given $x_\pm \in P(\lambda)$ and $\tilde{u} \in \mathcal{M}(x_-, x_+; J)$, we write $\tilde{u}$ as $\tilde{u} = (a,u) = (a, \theta, y)$ when $|s|$ is large enough. Here $P(\lambda)$ is the set of closed orbits of $\lambda$ and $\mathcal{M}(x_-, x_+; J)$ is the moduli space of connecting $J$-holomorphic maps connecting $x_- \in \{x_\pm\}$ and $x_+ \in \{x_\pm\}$.

Recall that in [L1], Section 4, we proved that there exist $N, \delta, C, d_{\pm,1}$ and $d_{\pm,2}$ such that when $|s| > N$, we have

\[
\begin{align*}
|g(s, t)| &< C \cdot e^{-\delta(|s|-N)} \\
|a(s, t) - (c_{\pm} s + d_{\pm,1})| &< C \cdot e^{-\delta(|s|-N)} \\
|\theta(s, t) - (c_{\pm} t + d_{\pm,2})| &< C \cdot e^{-\delta(|s|-N)}.
\end{align*}
\]

Here $c_{\pm} = \int x_{\pm}^* \lambda$. 

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Given such a $\tilde{u} \in \tilde{\mathcal{M}}(x_-,x_+;J)$, we want to find a Banach manifold $\mathcal{B} = B_{L^2,1}^{p} (\tilde{u})$ of certain $L^2$-maps as a neighborhood of $\tilde{u}$ and a Banach bundle $\mathcal{L} \to \mathcal{B}$ such that $\bar{\partial}$ gives rise to a section of the bundle and $\tilde{\mathcal{M}}(x_-,x_+;J) \cap \mathcal{B}$ appears as a zero locus of the $\bar{\partial}$-section.

Given $\epsilon > 0$, let

$$D_{\epsilon}(d) = \{(d'_{\pm,1}, d'_{\pm,2}) \mid \max \{d'_{\pm,1}, d'_{\pm,2}, |d_{\pm,1}|, |d_{\pm,2}| \} < \epsilon \}.$$ 

For any $d' = (d'_{\pm,1}, d'_{\pm,2}) \in D_{\epsilon}(d)$, $N' > N$, we define $\tilde{u}_{d',N'} : \mathbb{R}^1 \times S^1 \to \tilde{\mathcal{M}}$ connecting $x'_- = c_- t + d'_{2,-2} \in \{x_-\}$ and $x'_+ = c_+ t + d'_{2,+2} \in \{x_+\}$ as follows. Write $\tilde{u}_{d',N'} = (a,\theta,y)$, when $|s| > N$.

(i) When $|s| > N' + 1$, define $\tilde{u}_{d',N'}(s,t) = (c_- s + d'_{2,-1}, c_+ t + d'_{2,+2}, 0)$.

(ii) When $|s| < N'$, define $\tilde{u}_{d,N'} = \tilde{u}$.

(iii) When $N' < |s| < N' + 1$, first write the trivial map as $exp_0 \xi$, and then define $\tilde{u}_{d',N'}(s,t) = exp_0 (\beta(s)) \xi(s,t)$, where $\beta(s)$ is a cut-off function such that $|s| < N'$, $\beta(s) \equiv 0$ and $|s| > N' + 1$, $\beta(s) \equiv 1$.

Let $\chi$ be the collection of the parameters, $N',\epsilon > 0, \delta > 0$, we define the Banach neighborhood $\mathcal{B} = B_{L^2,1}^{p}(\tilde{u})$ as follows.

$$\mathcal{B} = \bigcup_{d' \in B_{L^2,1}(d)} \tilde{\mathcal{M}} \ni \tilde{u} = \tilde{u}_{d'} = exp_{\tilde{u}_{d',N'}}(\xi) \mid \xi \in L^p_{1,loc}(\tilde{u}_{d',N'}^{*} \tilde{\mathcal{M}}), \|\xi\|_{1,p,\delta} < \infty.$$ 

Here the weight Sobolev norm is defined to be $\|\xi\|_{1,p,\delta} = \|e^{\delta |s|} \xi\|_{1,p}$. Hence $\mathcal{B}$ is a Banach bundle over $D_{\epsilon}(d)$. Note that here we assume that the exponential map $exp_{\tilde{u}_{d',N'}}\xi$ above is the normal exponential when $|s|$ is large enough. Hence, if we write

$$\xi = \xi_a \frac{\partial}{\partial a} + \xi_0 \frac{\partial}{\partial \theta} + \sum_{i} \xi_i \frac{\partial}{\partial y_i}$$

in terms of the coordinate vector fields, then the exponential map here is just the identity map along $\frac{\partial}{\partial \theta}$ direction and is the identity map shifted by a constant along $\frac{\partial}{\partial y}$-direction under the obvious interpretation. This assumption will simplify some of our calculations. When $\epsilon$ is small enough, let $\pi_{d'}$ be the parallel transport from $\tilde{u}_{d',N'}^{*} \tilde{\mathcal{M}}$ to $\tilde{\mathcal{M}}$ along the shortest geodesic connecting any two points $\tilde{u}_{d',N'}(s,t)$ to $\tilde{u}(s,t)$. We can identify $\mathcal{B}$ with $L^p_{1,\delta}(\tilde{u}_{d',N'}^{*} \mathcal{M}) \times D_{\delta}(d)$ by sending $(\xi,d') \in L^p_{1,\delta}(\tilde{u}_{d',N'}^{*} \tilde{\mathcal{M}}) \times D_{\delta}(d)$ to $\tilde{V} = exp_{\tilde{u}_{d',N'}}(\pi_{d'}(\xi)) \in \mathcal{B}$.

Note that we may assume that the coordinate vector field $(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial a})$ is already parallel along the two trivial maps at two ends $x_{\pm}$. To simplify our presentation, we will assume further that the almost complex structure $J$ on $\xi$ is given by the standard complex structure $J_0$ in terms of the framing $\{e_1, \ldots, e_{2n}\}$ on $U_{\epsilon}(x)$. Here $e_i \in \xi$ such that $\pi_y(e_i) = \frac{\partial}{\partial y_i}$, $i = 1, \ldots, 2n$, where

$$\pi_y : TM \mid U_{\epsilon}(x) = \mathbb{R} \{ \frac{\partial}{\partial \theta} \} \oplus \mathbb{R} \{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{2n}} \} \to \mathbb{R} \{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \}$$

be the bundle projection. Therefore, we have $J(e_i) = e_{i+n}$ and $J(e_{i+n}) = -e_i$, $1 \leq i \leq n$. 

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We define the Banach bundle $\mathcal{L} \to \mathcal{B}$ as follows. For any $\tilde{v} = \tilde{v}_{d'} = \exp_{\tilde{u}_{d',N}^*,\xi} \in \mathcal{B}$, we define $\mathcal{L}_{\tilde{v}} = L^p_{0,d}(\tilde{v}^* \hat{T}\hat{M})$. Then it is clear that $\tilde{\partial} : \mathcal{B} \to \mathcal{L}$ is a smooth section of the bundle.

We now define the local trivialization of the bundle $\mathcal{L} \to \mathcal{B}$. Let $p : \tilde{u}_{d',N}^* T\hat{M} \to \tilde{v}_{d',N}^* T\hat{M}$ be the parallel transport of the bundle induced from the parallel transport along geodesics from $\tilde{v}_{d',N}^*(s,t)$ to $\tilde{v}_{d',N}^*(s,t)$. Here $\tilde{v}_{d',N}^* = \exp_{\tilde{u}_{d',N}^*,\xi}$ as above. We define the trivialization of $\mathcal{L}$ by sending $(\xi, d', \eta) \in L^p_{0,d}(\tilde{u}_{d',N}^* T\hat{M}) \times \mathcal{D}_\epsilon(d') \times L^p_{0,d}(\tilde{u}_{d',N}^* T\hat{M})$ to $p(\pi_{d'}(\eta)) \in \mathcal{L}_{\tilde{v}_{d',N}^*}$. Note that $\pi_{d'} : \tilde{u}_{d',N}^* T\hat{M} \to \tilde{u}_{d',N}^* T\hat{M}$ and $\tilde{v}_{d',N}^* = \exp_{\tilde{u}_{d',N}^*,\xi}$.

**The $\tilde{\partial}$-section in local coordinate and local trivialization:**

In terms of the local coordinate and local trivialization above, $\tilde{\partial}$-section becomes a function:

$$ F : L^p_{1,d}(\tilde{u}_{d',N}^* T\hat{M}) \times \mathcal{D}_\epsilon(d) \to L^p_{0,d}(\tilde{u}_{d',N}^* T\hat{M}). $$

More precisely, for any

$$(\xi, d') \in L^p_{0,d}(\tilde{u}_{d',N}^* T\hat{M}) \times \mathcal{D}_\epsilon(d), \quad F(\xi) = (p \circ \pi_{d'})^{-1}(\tilde{\partial}(\exp_{\tilde{u}_{d',N}^*,\xi} \pi_{d'}(\xi))).$$

For the purpose of this section, it is important to know the formula of $F$ in terms of the local framing defined on $U_\epsilon(x_\pm)$. We will assume that the local framing $\{e_1, \cdots, e_{2n}\}$ for $\xi|_{U_\epsilon(\pm x)}$ is parallel and use it to define the local trivialization of $\mathcal{L}$ above. Note that $e_i|_{\pm x} = \frac{\partial}{\partial s} + x_i$. Then $\{\left(\frac{\partial}{\partial s}, X_\lambda, e_1, \cdots, e_{2n}\right)\}$ forms a local parallel framing on $U_\epsilon(\pm x)$. In $U_\epsilon(\pm x)$, for any given $\eta \in L^p_{0,d}(\tilde{u}_{d',N}^* T\hat{M}$, we write $\eta = \eta|_{U_\epsilon(\pm x)} = \eta_a \frac{\partial}{\partial a} + \eta_0 X_\lambda + \sum_{i=1}^{2n} \eta_i e_i.$

Given $(\xi, d') \in L^p_{1,d}(\tilde{u}_{d',N}^* T\hat{M}) \times \mathcal{D}_\epsilon(d)$, write $\xi = \xi_a \frac{\partial}{\partial a} + \xi_0 \frac{\partial}{\partial \theta} + \sum_{i=1}^{2n} \xi_i \frac{\partial}{\partial \gamma_i}$, where $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial \gamma_i}$ are the coordinate vector field restricted to $x_\pm$.

Write $\exp_{\tilde{u}_{d',N}^*,\xi}(\xi) = (a, \theta, y)$, then by our assumption above, $\exp_{\tilde{u}_{d',N}^*,\xi}(\xi) = (\xi_a + (c_{\pm s} + d_{\pm 1}'), \xi_0 + (c_{\pm t} + d_{\pm 2}'), (\xi_1, \cdots, \xi_{2n})).$

To find $F(\xi)$, we only need to write $\partial \exp_{\tilde{u}_{d',N}^*,\xi}(\xi)$ in terms of the parallel framing $\{\frac{\partial}{\partial s}, X_\lambda, e_1\}$. Now

$$ \frac{\partial}{\partial s}(\exp_{\tilde{u}_{d',N}^*,\xi}(\xi)) = (c_{\pm s} + (\xi_a)_{s} \frac{\partial}{\partial a} + (\xi_0)_{s} \frac{\partial}{\partial \theta} + \sum_{i=1}^{2n} (\xi_i)_{s} \frac{\partial}{\partial \gamma_i}).$$

Denote $\xi_a \frac{\partial}{\partial a} + \sum_{i=1}^{2n} \xi_i \frac{\partial}{\partial \gamma_i}$ by $\eta$, then $\eta_a = \lambda(\eta_a) X_\lambda + \pi(\eta_a)$, where $\pi$ is the projection of $TM = \mathbb{R}X_\lambda \oplus \gamma \to \gamma$, given by $\pi(\eta) = \eta - \lambda(\eta) X_\lambda$, where $\gamma$ is the contact structure. It is proved in [L1] that if we write

$$ \pi(\eta_a) = \eta_a - \lambda(\eta_a) X_\lambda $$

$$ = \sum_{i=1}^{2n} ((\eta_a)_i - \lambda(\eta_a) X_i) \frac{\partial}{\partial y_i} + ((\eta_0)_0 - \lambda(\eta_a) X_0) \frac{\partial}{\partial \theta},$$

then

$$ \pi(\eta_a) = \sum_{i=1}^{2n} ((\eta_a)_i - \lambda(\eta_a) X_i) e_i = \sum_{i=1}^{2n} ((\xi_i)_s - \lambda(\eta_a) X_i) e_i.$$
Hence
\[ \frac{\partial}{\partial s}(\exp \tilde{u}_{a',N'}(\xi)) = (c+ (\xi_0)s) \frac{\partial}{\partial a} + ((\xi_0)s \lambda(\frac{\partial}{\partial \theta}) + \Sigma_{i=1}^{2n}(\xi_i)s\lambda(\frac{\partial}{\partial y_i}))X_\lambda + \Sigma_{i=1}^{2n}((\xi_i)s - \lambda(\theta))X_i e_i. \]

Similarly,
\[ \frac{\partial}{\partial t}(\exp \tilde{u}_{a',N'}(\xi)) = (\xi_0)\frac{\partial}{\partial a} + (c+ (\xi_0)i) \frac{\partial}{\partial \theta} + \Sigma_{i=1}^{2n}((\xi_i)i\lambda(\frac{\partial}{\partial y_i}))X_\lambda \]
and
\[ \frac{\partial}{\partial t}(\exp \tilde{u}_{a',N'}(\xi)) = (\xi_0)i \frac{\partial}{\partial a} + ((c+ (\xi_0)i)\lambda(\frac{\partial}{\partial \theta}) + \Sigma_{i=1}^{2n}((\xi_i)i\lambda(\frac{\partial}{\partial y_i}))X_\lambda + \Sigma_{i=1}^{2n}((\xi_i)i - \lambda(\theta))X_i e_i. \]

Recall that \( \bar{J}(e_i) = e_{i+n}, \tilde{J}(e_{i+n}) = -e_i, \bar{J} (\frac{\partial}{\partial a}) = X_\lambda \) and \( \bar{J}(X_\lambda) = -\frac{\partial}{\partial a}. \)
Hence,
\[ \bar{\partial}(\exp \tilde{u}_{a',N'}(\xi)) = (\frac{\partial}{\partial s} + \bar{J} \frac{\partial}{\partial t})(\exp \tilde{u}_{a',N'}(\xi)) \]
\[ = ((\xi_0)s - (\xi_0)i\lambda(\frac{\partial}{\partial \theta}) + ((\xi_0)i\lambda(\frac{\partial}{\partial \theta}) + (\xi_0)i)X_\lambda + \Sigma_{i=1}^{2n}((\xi_i)s - \lambda(\theta))X_i e_i + \Sigma_{i=1}^{2n}J_0\{((\xi_i)i - \lambda(\theta))X_i e_i \}
\]
\[ = (\Sigma_{i=1}^{2n}(\xi_i)i\lambda(\frac{\partial}{\partial y_i})\frac{\partial}{\partial a} + (\Sigma_{i=1}^{2n}(\xi_i)i\lambda(\frac{\partial}{\partial y_i}))\frac{\partial}{\partial \theta} + \Sigma_{i=1}^{2n}J_0\{((\xi_i)s - \lambda(\theta))X_i e_i). \]

Therefore, in terms of \( \frac{\partial}{\partial a}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial y_i} \) along \( \tilde{u} \),
\[ F(\xi, d') = ((\xi_0)s - (\xi_0)i\lambda(\frac{\partial}{\partial \theta}))\frac{\partial}{\partial a} + ((\xi_0)i\lambda(\frac{\partial}{\partial \theta}) + (\xi_0)i)\frac{\partial}{\partial \theta} - (\Sigma_{i=1}^{2n}(\xi_i)i\lambda(\frac{\partial}{\partial y_i})\frac{\partial}{\partial a} + (\Sigma_{i=1}^{2n}(\xi_i)i\lambda(\frac{\partial}{\partial y_i}))\frac{\partial}{\partial \theta} + \Sigma_{i=1}^{2n}J_0\{((\xi_i)s - \lambda(\theta))X_i e_i \}
\]
\[ + \Sigma_{i=1}^{2n}J_0\{((\xi_i)s - \lambda(\theta))X_i e_i \} \frac{\partial}{\partial y_i}. \]

Note that here \( \lambda(\frac{\partial}{\partial a}), \lambda(\frac{\partial}{\partial \theta}), \lambda(\eta_s) \) and \( \lambda(\theta) \) are evaluated along \( \exp \tilde{u}_{a',N'}(\xi). \)

- **Linearization of the \( \bar{\partial} \)-operator:**
Now
\[
DF(0)(\xi, d') = \lim_{r \to 0} \frac{F(r\xi, d + rd') - F(0, d)}{r}
\]
when \(|s| > N\).

Since for \(|s| > N\), when \(r \to 0\), \(exp_{\tilde{u}_d + d'}(\xi) \to \tilde{u}_d\), near \(x_\pm\), it is easy to see that the \((\partial \partial_a, \partial \partial_\theta)\)-component of
\[
DF(0)(\xi, d') = ((\xi_a)_s - (\xi_0)_t) \partial \partial_a + ((\xi_0)_s + (\xi_a)_t) \partial \partial_\theta.
\]
To find the \(y\)-component of \(DF(0)(\xi, d')\), we use the matrix notation. Let
\[
\xi = \left(\begin{array}{c}
\xi_1 \\
\vdots \\
\xi_{2n}
\end{array}\right) \quad \text{and} \quad Y = \left(\begin{array}{c}
X_1 \\
\vdots \\
X_{2n}
\end{array}\right).
\]
We proved in [L1] that in local \((\theta, y)\)-coordinate,
\[
Y(\theta, y) = \int_0^1 dY(\theta, \tau y) d\tau \left(\begin{array}{c}
y_1 \\
\vdots \\
y_{2n}
\end{array}\right),
\]
where \(dY(\theta, y)\) is a \(2n \times 2n\) matrix whose \((i, j)\)-entry is \(\frac{\partial X_i}{\partial y_j}\).

Denote \(\int_0^1 dY(\theta, \tau y) d\tau\) by \(DY(\theta, y)\).

In our case, for \(exp_{\tilde{u}_d + d'}(\xi)\), \(\theta = (c_\pm t + d + rd') + \xi_0\), and \(y_i = r \cdot \xi_i, i = 1, \cdots, 2n\), the \(y\)-component of
\[
DF(0)(\xi, d') = \xi_s + J_0y = c_\pm dY(c_\pm t + d_{\pm, 2}, 0)\xi.
\]
In [L1], we have assumed that \(c_\pm = 1\) by rescaling. It was proved there that \(S = -c_\pm J_0dY(t, 0)\) is a symmetric matrix and all the eigen values of the self-adjoint elliptic operators \(A : L^2_1(S^1, \mathbb{R}^{2n}) \to L^2_0(S^1; \mathbb{R}^{2n})\) defined by \(A(z) = J_0 \frac{\partial z}{\partial \tau} + S \cdot z\) are non-zero when \(\lambda\) is generic.

Hence the \(y\)-component of \(DF(0)(\xi, d') = \xi_s + A \cdot \xi\).

Write \(\tilde{\xi} = (\xi_0, \xi_0)\) and identify it with \(\xi_0 + \xi_0 i\). Then the \((\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \theta})\)-component of \(DF(0)(\xi, d')\) is given by \(\tilde{\partial} \cdot \xi = (\frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \theta})(\xi_0 + \xi_0 i)\) if we identify \(\{\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \theta}\}\) with \([1, i]\).

Hence when \(|s| > N\),
\[
DF(0)(\xi, d') = \tilde{L}_1(\xi) + \tilde{L}_2(\xi),
\]
when \(\xi = (\tilde{\xi}, \xi), \tilde{L}_1 = \tilde{\partial} = (\frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \theta})\) and \(\tilde{L}_2 = \frac{\partial}{\partial \theta} + A\).
For an arbitrary \( s \), the formula for \( DF(0)(\xi,0) \) is well-known. We have
\[
DF(0)(\xi,0) = \xi + \bar{J}(\bar{u})\xi_t + B(\bar{u}) \cdot \xi,
\]
where \( \bar{J} = \begin{bmatrix} i & 0 \\ 0 & J \end{bmatrix} \) with respect to the decomposition
\[
T\bar{M} = R \{ \frac{\partial}{\partial a} \} \oplus RX_\lambda \oplus \gamma, \text{ and } B(\bar{u}) \text{ is some matrix operator.}
\]

\[
DF(0)(0,d') = \lim_{r \to 0} \frac{1}{r} \{ p^{-1}(\bar{\partial}_{d+r'd',N'}) - p^{-1}(\bar{\partial}_{\bar{u},d,N'}) \}.
\]

Note that when \( |s| > N \), \( \bar{\partial}_{\bar{u},d+r'd',N'} = 0 = \bar{\partial}_{\bar{u},d,N'} \), \( DF(0)(0,d') \equiv 0 \). In general, \( DF(0)(0,d') = C(\bar{u}_{d,N'}) \cdot d' \) for some matrix operator \( C \). Therefore, \( DF(0) \) is an elliptic operator over \( R^1 \times S^1 \), a particular non-compact manifold with cylindrical ends.

In this situation, it is proved in [LM] that the elliptic operator \( DF(0) : L^p_{\delta}(\bar{u}_{d,N'}T\bar{M}) \times R^4 \to L^p_{\delta}(\bar{u}_{d,N'}T\bar{M}) \) is Fredholm if and only if \( \delta \) is not an eigenvalue of the operator \( i\frac{\partial}{\partial t} + A : L^2(S^1, R^{2n+2}) \to L^2(S^1, R^{2n+2}) \).

Write \( B = \begin{pmatrix} B_{\delta,X} & B_1 \\ B_2 & B_3 \end{pmatrix} \). It is easy to see that we can deform \( DF(0) \) to get rid of the \( B_1, B_2 \) and \( B_{\delta,X} \) terms and \( Cd' \) in the middle part but maintain the same asymptotic behavior of \( DF(0) \). Then each operator in the deformation is still Fredholm and therefore has the same index.

Let \( L \) be the resulting operator. Then, \( L = L_1 \oplus L_2 \). Hence,
\[
L_1 : L^p_{\delta}(\bar{u}_{d,N'}(R\{ \frac{\partial}{\partial a} \} \oplus RX_\lambda)) \oplus R^4 \to L^p_{\delta}(R^1 \times S^1, R^2) \oplus R^4 \to L^p_{\delta}(R^1 \times S^1, R^2)
\]
given by \( L_1(\xi,d) = \bar{D}_\xi \) and
\[
L_2 : L^p_{\delta}(\bar{u}_{d,N'}(\xi)) \to L^p_{\delta}(\bar{u}_{d,N'}(\xi))
\]
given by \( L_2(\xi) = (\xi)_s + J(\bar{u}_{d,N'}(\xi)_t + B_3 \xi) \).

Since 0 is not an eigenvalue of \( A \), \( L_2 : L^p_{\delta}(\bar{u}_{d,N'}(\xi)) \to L^p_0(\bar{u}_{d,N'}(\xi)) \) is also Fredholm with the same index as \( L_2 \). We will still use \( L_2 \) to denote \( L_2 \). Note that as far as the computation of index is concerned, we can replace \( p \)-norm, by 2-norm.

We will denote the restriction of \( L_1 \) to \( L^p_{\delta}(R^1 \times S^1, R^2) \) by \( L_{1,\delta} \). For the application of \([L2]\), we need to consider the multi-ends case also.

Given \( l + m \) closed orbits \( x_{-1}, \cdots, x_{-l}, x_{+1}, \cdots, x_{+m} \), with each \( x_{\pm,j} : S^1 \to M \) satisfying the equation \( \frac{dx_{\pm,j}}{dt} = c_{\pm,j} X_\lambda(x_{\pm,j}) \), we denote the collection \( \{ x_{-1}, \cdots, x_{-l} \} \) by \( x_- \) and \( \{ x_{+1}, \cdots, x_{+m} \} \) by \( x_+ \).

Let \( (S^2, z_{-1}, \cdots, z_{-l}, z_{+1}, \cdots, z_{+m}) \in M_{0,l+m} \). We identify a small punctured disc \( D_\nu(z_{-i}) \setminus \{ z_{-i} \} \) of \( S^2 \) with \( R^- \setminus S^1 \), \( i = 1, \cdots, l \) and \( D_\nu(z_{+j}) \setminus \{ z_{+j} \} \) with \( R^+ \setminus S^1 \), \( j = 1, \cdots, m \), so that \( (S^2, z_-, z_+) \) can be thought as a punctured \( S^2 \) with \( l \)-negative cylindrical ends and \( m \) positive ends. Here \( z_- \) stands for \( \{ z_{-1}, \cdots, z_{-l} \} \) and \( z_+ \) for \( \{ z_{+1}, \cdots, z_{+m} \} \). As before, we define
\[
\tilde{M}(x_-, x_+ ; J) = \{ \tilde{u} : S^2 \setminus (z_- , z_+) \to M, \tilde{\partial}_J \tilde{u} = 0, \text{ and } \lim_{y \to z_{\pm}} \tilde{u}(y) = x'_{\pm} \in \{ x_{\pm} \} \},
\]
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where \( \{x_\pm \} \) is the equivalent class of \( x_\pm \). Note that here we allow \( (S^2; z_-, z_+) \) to vary in \( M_{0,l+1} \). As before, we have the exponential decay estimate for \( \tilde{u} \) along each of its cylindrical ends. Given \( N' > N, \epsilon > 0, \) and \( D_\epsilon(d) \), we can define \( \tilde{u}_{d, N'} \), and the Banach neighborhood \( B = B_{1, N'}(\tilde{u}) \) of \( \tilde{u} \) in exactly the same way as before, where \( \chi = (N', \epsilon, \delta) \). Note that \( d' \) and \( d \in \mathbb{R}^{2l+m} \). We define the Banach bundle \( \mathcal{L} \to B \) slightly different from the definition before. For any \( \tilde{v} \in B, \mathcal{L}_{\tilde{v}} = L^p_{0, \delta}(\tilde{v}^*(\wedge^{0,1}(i, J)TM)) \). Then \( \tilde{\partial} \)-operator can be thought as a global section of the bundle \( \mathcal{L} \to B \). Here \( \tilde{\partial} \)-section is defined by

\[
\tilde{\partial}(\tilde{v}) = d\tilde{v} + J(\tilde{v}) \circ d\tilde{v} \circ i.
\]

Arguing the same way as before, we can find a coordinate chart for \( B \) by identifying it with an open set of \( L^p_{1, \delta}(u^*_{d, N'}TM) \times D_\epsilon(d) \) and a trivialization of \( \mathcal{L} \) by identifying it with

\[
L^p_{1, \delta}(u^*_{d, N'}TM) \times D_\epsilon(d) \times L^p_{0, \delta}(u^*_{d, N'}(\wedge^{0,1}(i, J)T(\tilde{M})))
\]

and

\[
DF(0) : L^p_{1, \delta}(u^*_{d, N'}TM) \times D_\epsilon(d) \to L^p_{0, \delta}(u^*_{d, N'}(\wedge^{0,1}(i, J)T(\tilde{M}))).
\]

Finally, we can deform the Fredholm operator \( DF(0) \) into \( L = L_1 \oplus L_2 \) of same index where

\[
L_1 : L^p_{1, \delta}(S^2 \setminus \{z_-, z_+\}, \mathbb{R}^2) \oplus \mathbb{R}^{2l+m} \to L^p_{1, \delta}((\wedge^{0,1}S^2 \setminus \{z_-, z_+\})
\]

given by \( L_1(\tilde{\xi}, d') = \tilde{\partial}\tilde{\xi} \), and

\[
L_2 : L^p_{1, \delta}(u^*_{d, N'}(\tilde{\xi})) \to L^p_{0, \delta}(u^*_{d, N'}(\wedge^{0,1}(i, J)\tilde{\xi}))
\]

given by \( L_2(\tilde{\xi}) = \tilde{\partial}j\tilde{\xi} + \tilde{B} : \tilde{\xi} \). Here \( \tilde{\partial}\tilde{\xi} = (\frac{\partial\tilde{\xi}}{\partial z})dz \) in terms of the local complex coordinate \( z \) of \( S^2 \setminus \{z_-, z_+\} \), \( \tilde{\partial}j\tilde{\xi} = \nabla\tilde{\xi} + J(u_{d, N'} \circ \nabla\tilde{\xi} \circ i) \) and \( \tilde{B} \) is a bundle map \( \tilde{u}^*_{d, N'}(\tilde{\xi}) \to \tilde{u}^*_{d, N'}(\wedge^{0,1}(i, J)\tilde{\xi}) \). Near \( z_\pm \), by using the local coordinate \( z \) of the domain near the ends, we can identify \( \tilde{u}^*_{d, N'}(\wedge^{0,1}(i, J)\tilde{\xi}) = u^*_{d, N'}(\tilde{\xi} \ominus C\{dz\}) \) with \( u^*_{d, N'}(\tilde{\xi}) \). Under this identification, in the local coordinate of \( U_\epsilon(x_\pm) \),

\[
L_2(\tilde{\xi}) = (\tilde{\xi})_t + J(\tilde{u}_{d, N'})(\tilde{\xi})_t + B_\xi^t \tilde{\xi}
\]

as before.

Now the virtual dimension of \( \tilde{\mathcal{M}}(x_-, x_+; J, z_-, z_+) \) is equal to the index of \( L_1 \), which is the same as \( \text{Ind}(L_1) + \text{Ind}(L_2) \). Here \( \mathcal{M}(x_-, x_+; J, z_-, z_+) \) is the subspace of \( \mathcal{M}(x_-, x_+; J) \) whose elements have the fixed domain \( S^2 \setminus \{z_-, z_+\} \). Our goal is to study the behavior of \( \text{Ind}(L) \) under deformation (gluing) of the domain of \( \tilde{u} \). To this end, we will calculate \( \text{Ind}(L_1) \) first.

Recall

\[
L_1 = \tilde{\partial} : L^p_{1, \delta}(S^2 \setminus \{z_-, z_+\}, C) \oplus \mathbb{R}^{2l+m} \to L^p_{0, \delta}(S^2 \setminus \{z_-, z_+\})
\]
and $L_{1,\delta} = L_1$ restricted to $L^p_{1,\delta}(S^2 \setminus \{z_-, z_+\}, C)$. Note that $0 < \delta < 2\pi$ of the firsts eigenvalue of the operator $\frac{i\partial}{\partial t}$ along each ends. Here the weighted Sobolev space $L^p_{1,\delta}(S^2 \setminus \{z_-, z_+\}, C)$ and $L^p_{0,\delta}(\wedge^{0,1}(S^2 \setminus \{z_-, z_+\}))$ are measured with respect to the cylindrical metric along each end of $S^2 \setminus \{z_-, z_+\}$.

**Lemma 2.1** If $\# \{z_-, z_+\} = l + m$, then $\text{Ind}(L_{1,\delta}) = 2 - 2(l + m)$. Therefore, $\text{Ind}(L_1) = 2$.

**Proof:** Start with the case $l = m = 1$. Then $S^2 \setminus \{z_-, z_+\} = R^1 \times S^1$ with cylindrical coordinate $z = (s, t)$. In terms of the global complex coordinate $z$, $dz$ is a global section of $\wedge^{0,1}(R^1 \times S^1)$. Hence $\wedge^{0,1}(R^1 \times S^1) \equiv (R^1 \times S^1) \times C$ of the trivial bundle and under this identification $L_{1,\delta} : L^p_{1,\delta}(R^1 \times S^1, C) \rightarrow L^p_{0,\delta}(\wedge^{0,1}(R^1 \times S^1))$ becomes $\bar{\partial} : L^p_{1,\delta}(R^1 \times S^1; C) \rightarrow L^p_{0,\delta}(R^1 \times S^1; C)$ given by $\xi \rightarrow \frac{\partial}{\partial s} + i\frac{\partial}{\partial t}$.

We have defined $L^p_{1,\delta}(R^1 \times S^1, C)$ by requiring that $\|\xi(s, t) \cdot e^{\delta|s|}\|_{1, p} < \infty$ along both of its ends. We define $L^p_{0,\delta}(R^1 \times S^1, C)$ by requiring $\|\xi(s, t) \cdot e^{\delta|s|}\|_{1, p} < \infty$ for any of its element $\xi$. This is the same as requiring $\|\xi(s, t) \cdot e^{\delta|s|}\|_{1, p} < \infty$ along the positive end $z_+$ and $\|\xi(s, t) \cdot e^{-\delta|s|}\|_{1, p} < \infty$ along the negative end. In general, we can consider multi-ends with arbitrary choices of the sign before $\delta$ and define $L^p_{1,\pm(\delta_{z_-}, \delta_{z_+})}(S^2 \setminus \{z_-, z_+\}, C)$ and $L^p_{0,\pm(\delta_{z_-}, \delta_{z_+})}(\wedge^{0,1}(S^2 \setminus \{z_-, z_+\}))$. Let $L_{1,\pm(\delta_{z_-}, \delta_{z_+})}$ be the corresponding operator.

**Sublemma 2.1** $\text{Ind}(L_{1,\pm(\delta_{z_-}, \delta_{z_+})}) = 0$.

**Proof:** Define the isomorphism:

$$e_0 : L^p_0(R^1 \times S^1, C) \rightarrow L^p_{0,\pm(\delta_{z_-}, \delta_{z_+})}(R^1 \times S^1, C)$$

and

$$e_1 : L^p_1(R^1 \times S^1, C) \rightarrow L^p_{1,\pm(\delta_{z_-}, \delta_{z_+})}(R^1 \times S^1, C)$$

by sending $\xi \rightarrow e^{-\delta s}\xi$.

Consider

$$D = e_0^{-1} \circ L_{1,\pm(\delta_{z_-}, \delta_{z_+})} \circ e_1 = e_0^{-1} \circ \bar{\partial} \circ e_1 : L^p_1(R^1 \times S^1, C) \rightarrow L^p_0(R^1 \times S^1, C).$$

For any $\xi \in L^p_1(R^1 \times S^1, C)$,

$$D(\xi) = e^{\delta s}\bar{\partial}(e^{-\delta s}\xi) = \bar{\partial}\xi - \delta\xi = \frac{\partial}{\partial s}\xi + (\frac{i}{\partial t} - \delta)\xi.$$ 

Clearly $\text{Ind}(L_{1,\pm(\delta_{z_-}, \delta_{z_+})}) = \text{Ind}(D)$.

Now consider the self-adjoint operator

$$\frac{i}{\partial t} + \delta : L^2_1(S^1; C) \rightarrow L^2_0(S^1; C).$$
Since $0 < \delta < 2\pi$, 0 is not an eigenvalue of $i\frac{\partial}{\partial s} - \delta$. Write $D = \frac{\partial}{\partial s} + L$, where $L_s = i\frac{\partial}{\partial s} - \delta$. Define the resolvent $R = (L_s - \lambda i)^{-1} : L^2_0(S^1, C) \to L^2(S^1, C)$ for any $\lambda \in \mathbb{R}$, which is bounded independent of $\lambda$. It is proved in [F] and [LM] that in the situation, if we define $\xi(\lambda) = \int e^{\lambda s}\xi(s)ds$ and $\eta(s) = \frac{1}{2\pi} \int e^{-i\lambda s} R_s \xi(\lambda) d\lambda$ Then $D\eta = \xi$. That is $D$ is invertible. Hence $\text{Ind}(D) = 0$ with respect to $L^2$-norm. Same is true for $L^p$-norm (see [LM]).

It is proved in [LM] that $\text{Ind}(L_{1,(-\delta,\delta)}) - \text{Ind}(L_{1,(-\delta,\delta)}) = i(-\delta_-, \delta_-)$, where $i(-\delta_-, \delta_-)$ is the dimension of the eigen spaces of $i\frac{\partial}{\partial s} : L^2(S^1, C) \to L^2(S^2, C)$ of the asymptotic operator associated to the end $z_-$ with eigenvalue in $(-\delta, \delta)$. It is the same as the dimension of the space of solutions $\xi$ of the form $\xi(s, t) = \exp(\lambda s) \cdot p(s, t)$ with $-\delta < \lambda < \delta$ and $p(s, t)$ is a polynomial in $s$ with coefficient in $C^\infty(S^1; C)$ (See [LM]). This implies that $\text{Ind}(L_{1,(-\delta,\delta)}) = -2$ as $i(-\delta_-, \delta_-) = 2$.

Now we consider the general multi-ends case. Consider first the case $L_{1,(-\delta,\delta)} : L^2(S^1, C) \to L^2_0(-\delta_-, \delta_+) (\bigwedge^1 (S^2 \setminus \{z_-, z_+\}))$.

Let $(s, t)$ be the cylindrical coordinate near an end $p \in \{z_-, z_+\}$ and $w \in C$ the complex coordinate of $S^2$ near $p$ with $w = 0$ at $p$. Then $w = e^{-2\pi(s+i)}$. Given $\xi \in L^2_1(-\delta_-, \delta_+) (S^2 \setminus \{z_-, z_+\}, C)$, we have

$$\int \int_{R^+ \times S^1} |e^{-\delta s}\xi|^2 ds dt < 0.$$  

Write $w = x + iy$, $z = s + it$. Then

$$2 dx \wedge dy = i dw \cdot d\bar{w} = 4\pi^2 \cdot ie^{-4\pi s} dz \wedge d\bar{z} = 4\pi^2 e^{-4\pi s} \cdot 2 ds \wedge dt.$$  

Then

$$\int \int_{R^+ \times S^1} |e^{-\delta s}\xi|^2 ds \wedge dt = \frac{1}{4\pi^2} \int_{D_1 \setminus \{0\}} |e^{(2\pi - \delta) s} \cdot \xi|^2 dx \wedge dy = \frac{1}{4\pi^2} \int_{D_1 \setminus \{0\}} \left| \frac{\xi}{w} \right|^2 dx \wedge dy$$

This implies that $\int_{D_1 \setminus \{0\}} |\xi|^2 dx \wedge dy < \infty$.

Assume that $\xi \in \ker(L_{1,(-\delta,\delta)})$, then $\xi$ is holomorphic on $S^1 \setminus \{z_-, z_+\}$. The elliptic regularity implies that $\xi$ extends over $\{z_-, z_+\}$ and is well-defined on $S^2$. From this, one can easily conclude that $\text{dim}(\ker(L_{1,(-\delta,\delta)})) = 2$.

Let $L^*_1(-\delta_-, \delta_-)$ be the dual of $L_{1,(-\delta,\delta)}$. Then

$$L^*_1(-\delta,\delta) : L^2_{0,(-\delta_-, \delta_+)} (\bigwedge^0 (S^2 \setminus \{z_-, z_+\})) \to L^2_0(-\delta_-, \delta_-) (S^2 \setminus \{z_-, z_+\}, C).$$

$\text{Dim}(\ker(L^*_1(-\delta,\delta))) = \text{dim}(\text{coker} L_{1,(-\delta,\delta)})$. As before, for any $p \in \{z_-, z_+\}$, let $z = s + it$ and $w = e^{-2\pi z}$ be the two types of coordinates. Given $\eta \in L^2_1(-\delta,\delta) (\bigwedge^0 (S^2 \setminus \{z_-, z_+\}))$ near $p$, in terms of the local section $d\bar{z}, \eta = \phi \cdot d\bar{z} = \psi d\bar{w}$ with $\phi \in L^2_1(R^+ \times S^1, C)$ and $\psi \in L^2_1(D_1 \setminus \{0\}, C)$. Then $\phi = \psi \cdot e^{-2\pi z}$. 

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where the cylindrical metric is conformal to the standard metric on $\mathbb{S}^1$. Similarly to the compact case (see [GH] for example), with respect to pairing $\langle, \rangle$ on $\Omega^p_q(\mathbb{S}^2 \setminus \{z_-, z_+\})$, we have shown that in terms of cylindrical coordinate $z, \eta = \phi d\bar{\eta}$ with \( \phi = \psi \cdot e^{-2\pi(s-i\delta)} \). Clearly, for $0 < \delta < 2\pi$,

\[
\| \eta \|_{\mathbb{R}^1 \times \mathbb{S}^1, 2, \delta}^2 = \int_{\mathbb{R}^1 \times \mathbb{S}^1} |e^{\delta s} \eta|^2 ds \wedge dt
\]

\[
= \int_{\mathbb{R}^1 \times \mathbb{S}^1} |e^{\delta s}\phi|^2 ds \wedge dt
\]

\[
= \int_{\mathbb{R}^1 \times \mathbb{S}^1} |e^{-2(\pi - \delta)s} \cdot \psi|^2 ds \wedge dt
\]

\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^1 \setminus \{0\}} e^{4\pi s} \cdot e^{-2(2\pi - \delta)s} |\psi|^2 dx \wedge dy
\]

\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^1 \setminus \{0\}} \left| \frac{\psi}{2\pi} \right|^2 dx \wedge dy < \infty.
\]

Again this implies that $\| \eta \|_{D_1 \setminus \{0\}, 2} < \infty$.

We want to show that if $L^* \eta = 0$, then $\eta$ extends over to $p$ and there is no restriction on the value of $\eta$ at $p$.

To see the second statement, assume that $\eta = \psi d\bar{\eta}$, with $\psi \in C^\infty(D_1, \mathcal{C})$. Then we have shown that in terms of cylindrical coordinate $z, \eta = \phi d\bar{\eta}$ with $\phi = \psi \cdot e^{-2\pi(s-i\delta)}$. Clearly, for $0 < \delta < 2\pi$,

\[
\| \eta \|_{\mathbb{R}^1 \times \mathbb{S}^1, 2, \delta}^2 = \int_{\mathbb{R}^1 \times \mathbb{S}^1} |e^{(\delta - 2\pi)s} \psi|^2 ds \wedge dt
\]

\[
\leq C \int_{\mathbb{R}^1 \setminus \{0\}} e^{2(\delta - 2\pi)s} ds \wedge dt < \infty.
\]

To see the first statement, we write down the explicit expression for $L^*$. As in [LM], we use the cylindrical Hermitian metric on $(\mathbb{S}^2 \setminus \{z_-, z_+\}, i)$ to introduce a Hermitian inner product on the bundle $\bigwedge^{p,q}(\mathbb{S}^2 \setminus \{z_-, z_+\})$, which gives a hermitian product $<,>$ on $\Omega^p_q(\mathbb{S}^2 \setminus \{z_-, z_+\})$ of smooth sections of $\bigwedge^{p,q}(\mathbb{S}^2 \setminus \{z_-, z_+\})$ with compact support. Now $L = L_1,\mathbb{R}^1,\mathbb{S}^1$ is just the $\bar{\partial}$-operator:

\[
C^\infty_0(\mathbb{S}^2 \setminus \{z_-, z_+\}, \mathcal{C}) = \Omega^0_0(\mathbb{S}^2 \setminus \{z_-, z_+\}) \to \Omega^0_1(\mathbb{S}^2 \setminus \{z_-, z_+\}),
\]

Similarly to the compact case (see [GH] for example), with respect to pairing $<,>$,

\[
L^* = \bar{\partial}^* = \ast_z \cdot \bar{\partial} \cdot \ast_z : \Omega^0_1(\mathbb{S}^2 \setminus \{z_-, z_+\}) \to C^\infty_0(\mathbb{S}^2 \setminus \{z_-, z_+\}, \mathcal{C}),
\]

where $\ast_z$ is the Hodge-star operator. Here we use the subscript $z$ to indicate that the $\ast$-operator is taken with respect to the cylindrical metric. Since the cylindrical metric is conformal to the standard metric on $\mathbb{S}^2$ and $\ast$ is conformal invariant on $\bigwedge^0_1$, we have $\ast_z = \ast_w$ of the $\ast$-operator with respect to the standard metric of $\mathbb{S}^2$ on $\bigwedge^0_1$. On the other hand, on the bundle $\bigwedge^1_1$ of top forms, we have $\ast_z = e^{-4\pi s} \cdot \ast_w$, where $\ast_w$ is the dual of $\partial$ with
respect to the standard metric of $S^2$, i.e. the usual dual of $\bar{\partial}$ in Hodge theory. Therefore, if $\eta \in \text{ker}(L^*_{1,(\delta,\delta)}) = \text{ker}(\bar{\partial}^*_w)$, then $\eta \in L^2_{1,(\delta,\delta)}(\wedge^{0,1}(S^2 \setminus \{z_-, z_+\}))$ and $\bar{\partial}^*_w(\eta) = 0$. Since $\bar{\partial}^*_w$ is elliptic and $\|\eta\|_{L^2(\emptyset)} \leq \infty$, $\eta$ extends over all singular points $z_-, z_+$. As we mentioned before that there is no restriction on the values of $\eta$ at $z_-$ and $z_+$, we conclude that $\text{ker}(L_{1,(\delta,\delta)})$ can be identified with $\text{ker}(\bar{\partial}^*_w)$, where $\bar{\partial}^*_w : L^2_1(\wedge^{0,1}(S^2)) \to L^2_0(\wedge^{0,0}(S^2))$. Here the $L^2$-norm is measured with respect to the standard metric of $S^2$. By Hodge theory, $\text{ker}(\bar{\partial}^*_w) = H^0_\delta(S^2) \equiv H^0(S^1, \mathcal{O}(-2)) = 0$. This proves that $\text{Ind}(L_{1,-\delta,-\delta}) = 2$. Hence $\text{Ind}(L_{1,\delta}) = 2 - 2(l + m)$, and $\text{Ind}(L_1) = 2$.

Finally, we deal with the special case where $l = 0, m = 1$. Again, consider $L_{1,-\delta}$ first. Clearly, if $\eta \in \text{ker}(L_{1,-\delta})$, then $\eta$ is holomorphic defined over $\mathbb{C}$ and the ratio of growth of $\eta$ in the cylindrical coordinate $z = s + it$ as $s \to \infty$, is less than $e^{|\delta|s}$. This implies that $\eta \equiv \text{constant}$. Hence $\text{dim(} \text{ker}(L_{1,-\delta})) = 2$. The above argument for general case on the kernel of the dual operator is also applicable here. We have
\[
\text{coker}(L_{1,-\delta}) \equiv \text{ker}(L^*_{1,\delta}) \\
\equiv \text{ker}(\bar{\partial}^*_w) \equiv H^0(S^1, \mathcal{O}(-2)) = 0.
\]
Hence $\text{Ind}(L_{1,\delta}) = \text{Ind}(L_{1,-\delta}) - 2 = 0$ and $\text{Ind}(L_1) = 2$.

\[\text{QED}\]

- **Transversality and non-transversality of the trivial connecting map:**

Now we consider the special case that $u \in \tilde{\mathcal{M}}(x_-, x_+; J)$ with $\{x_-\} = \{x_+\}$. In this case, since $\int_{S^1}(x'_-)^*\lambda = \int_{S^1}(x'_-)\lambda = \epsilon$ for any $x'_+ \in \{x_\pm\}$, the $E_{\lambda}$-energy
\[
E_{\lambda}(\tilde{u}) = \int_{\mathbb{R}^1 \times S^1} \tilde{u}^* d\lambda = 0,
\]
and we have $\tilde{u}(s, t) = (cs + d_1, ct + d_2, 0)$ in the $(a, \theta, y)$ coordinate (for the proof, see, for example, [L1]). Therefore, the two asymptotic limits $\lim_{s \to \pm \infty} u(s, t) = ct + d_2 = x'_+ \in \{x_\pm\}$, which are the same parameterized closed orbits, and $\tilde{u}$ itself is the trivial map.

We want to calculate the index of $L$, which is the linearization $DF$ at $\tilde{u}$. To this end, we only need to know $\text{Ind}(L_2)$ in this case.

Recall that $L_2 : L^p_{1,\delta}(\tilde{u}^* \xi) \to L^p_{0,\delta}(\tilde{u}^* \xi)$ given by sending $\eta$ to $\frac{\partial}{\partial \eta} \eta + J_0 \frac{\partial}{\partial \eta} \eta + B\eta$. Note that the local coordinate $(a, \theta, y)$ is defined over the whole image of $\tilde{u}$. In terms of the coordinate framing, $\tilde{u}^* \xi = \{ \frac{\partial}{\partial \eta_1}, \ldots, \frac{\partial}{\partial \eta_{2n}} \}$. Since $\frac{\partial}{\partial \eta_i} |_{\tilde{u}} = e_i |_{\tilde{u}}$, in terms of $\{ \frac{\partial}{\partial \eta_i} \}$, $\eta = \Sigma_{i=1}^{2n} \eta_i \frac{\partial}{\partial \eta_i}$, and the $d\lambda$-compatible almost complex structure is given by the standard complex structure on $\mathbb{R}^{2n}$. $B = B(s, t)$ is a self-adjoint matrix operator. Hence in terms of the framing $\{ \frac{\partial}{\partial \eta_1}, \ldots, \frac{\partial}{\partial \eta_{2n}} \}$,
\[
L_2 : L^p_1(R^1 \times S^1, R^{2n}) \to L^p_0(R^1 \times S^1, R^{2n}).
\]
Recall that 0 is not an eigenvalue of \( J_0 \frac{\partial}{\partial s} + B : L^2(S^1; \mathbb{R}^{2n}) \to L^2(S^1; \mathbb{R}^{2n}) \).
It follows from [LM] that here we can use the usual Sobolev norm rather than
the \( \delta \)-weighted one. We can also use \( L^2 \)-norm rather than \( L^p \)-norm for
the purpose of calculating the index.

**Lemma 2.2** \( \text{Ind}(L_2) = 0 \). In fact,

\[
L_2 : L^2_1(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^2_0(\mathbb{R} \times S^1, \mathbb{R}^{2n})
\]
given by \( \eta \to \frac{\partial}{\partial s} \eta + J_0 \frac{\partial}{\partial t} \eta + B \eta \) is an isomorphism.

**Proof:**

Write \( L_2 = \frac{\partial}{\partial s} + L_t \), where \( L_t = J_0 \frac{\partial}{\partial t} + B \) is self-adjoint. As before, we
define the resolvent

\[
R_\lambda = (L_t - i\lambda)^{-1} : L^2(S^1; \mathbb{R}^{2n}) \otimes C \to L^2(S^1; \mathbb{R}^{2n}) \otimes C,
\]
which is bounded independent of \( \lambda \in \mathbb{R} \). Define \( \hat{\eta}(\lambda) = \int e^{\lambda s} \eta(s) ds \). Then
\( \gamma(s) = \frac{1}{2\pi} \int e^{-\lambda s} R_\lambda \hat{\eta}(\lambda) ds \) has the property that
\( L_2 \gamma = \eta \). It follows that \( L_2 \) is invertible (See [LM] and [F] for more details).

QED

Therefore, \( \text{Ind}(L) = \text{Ind}(L_1) + \text{Ind}(L - 2) = 2 \).

Recall that we embedded a neighborhood of \( \tilde{u} \) in \( \tilde{\mathcal{M}}(x_-, x_+; J) \) into \( \mathcal{B} = \mathcal{B}^{\mathcal{N}, \mathcal{N}'} \tilde{u} \) and defined \( \mathcal{L} \to \mathcal{B} \) with the \( \partial \)-section. To define \( \mathcal{B} \), we introduced
\( \tilde{u}_{d',\mathcal{N}'} \) before. It is easy to see that in the case \( \{x_-\} = \{x_+\} \), those \( \tilde{u}_{d',\mathcal{N}'} \)’s are
very close to the trivial connecting map \( \tilde{u}_d \) if \( |d - d'| \) is very small. Note also
that the linearized \( \partial \)-section at \( \tilde{u}_d \), \( DF(0) \) is just \( L = L_1 + L_2 \) in this case. We
have shown that \( \ker(L_2) = 0 \), \( \coker(L_2) = 2 \) and \( \text{Ind}(L) = 2 \). Intuitively it is
more or less clear that \( \ker(L) = 2 \). To get a precise prove, we directly calculate the
coker(\( L_1 \)). By the definition of \( DF(0) \), a short computation shows that
\( DF(0)(0, d') = \beta'_+ d'_+ + \beta'_- d'_- \), for any \( d' = (d'_+, d'_-) \in R^4 \) and \( d' \in \mathbb{R}^2 = C \), where \( \beta' \) is the cut-off function we introduced before. This proves that
\( \coker(L_1) = 0 \), and we have the following propose tion on the transversality of
the trivial connecting maps.

**Lemma 2.3** When \( \{x_-\} = \{x_+\} \), \( \tilde{\partial} : \mathcal{B} \to \mathcal{L} \) is a transversal section and its
zero locus \( \tilde{\mathcal{M}}(x_-, x_+; J) \cap \mathcal{B} \) is a smooth manifold of 2-dimensional. Moreover
\( \tilde{\partial} : \mathcal{B} \to \mathcal{L} \) is also a transversal section over the unparameterized moduli space
\( \mathcal{M}(x_-, x_+; J) \). Here \( \mathcal{B} \) is the space of unparameterized map corresponding to \( \mathcal{B} \),
and we obtain \( \mathcal{B} \) from \( \mathcal{B} \) by quotienting out the \( \mathbb{R}^3 \times S^1 \) action of the domains
of its elements.

We non come to the non-transversality of the trivial connecting map. Note
that to define \( \mathcal{B} (\tilde{u}) \), we fixed a \( \mathcal{N}' \) and for any \( d' \in D_{x}(d) \in \mathbb{R}^4 \), we then define
the base connecting map \( \tilde{u}_{d',\mathcal{N}'} \). However in the case that \( \{x_-\} = \{x_+\} \), for
any $\tilde{u} \in \tilde{M}(x_-, x_+; J)$, the two asymptotic ends $x'_-$ and $x'_+$ of $\tilde{u}$ are the same as parameterized closed orbits. This suggests that to define $\tilde{u}_{\delta', N'}$ we only need to consider those $d' \in D_\epsilon(d)$ such that $\theta$-component $d'_{+,2} = d'_{-,2}$. Therefore, we get a subspace $B^0 \hookrightarrow B$ such that locally,

$$B^0 = L^p_{1,\delta}(\tilde{u}_*^0 T(M)) \times D^0(d),$$

where $D^0(d) \hookrightarrow D_\epsilon(d)$ with $d'_{+,2} = d'_{-,2}$ and a Banach bundle $\mathcal{L} \to B^0$. Here the fiber $\mathcal{L}_{\bar{v}}$, $\bar{v} \in B^0$ defined as before, and $\bar{\partial} : B^0 \to B$ is a section such that $\tilde{M}(x_-, x_+; J) \cap B^0$ is the zero locus of $\bar{\partial}$. However the linearized $\bar{\partial}$-operator at $\bar{v}$, $DF(\bar{v}) = L = L_1 \oplus L_2$, has index 1 rather than 2. In fact in this case $L_2$ is still an isomorphism and hence $Ind(L_2) = 0$, but $Ind(L_1) = 1$. Since any element $\bar{v}$ in $\tilde{M}(x_-, x_+; J)$ still has the 2-dimensional symmetries, this implies that $\bar{\partial} : B^0 \to B$ is not transversal along $\tilde{M}(x_-, x_+; J)$.

### 3 Additivity of Fredholm Index

We now come to the question on the additivity of the index under gluing map. Combining with the compactness theorem in [L1] with this formula will give the desired virtual codimension of the boundary components of the moduli space of connecting pseudo-holomorphic maps in the symplectification. In [L2], we will prove that the codimension can be realized in the corresponding virtual moduli cycles. Since we have already written down the precise formula for the index of $L_1$, we only need to find how $L_2$ changes under gluing.

Let $\tilde{M}(x_-, x, x_+; J)$ be the subspace of $\tilde{M}(x_-, x, x_+; J) \times \tilde{M}(x_-, x_+; J)$, which consists of all pairs $(\tilde{u}, \tilde{w})$ such that

$$\tilde{u} : S^2 \setminus \{z_-, z'_-\} \to \tilde{M}, \quad \tilde{w} : S^2 \setminus \{z'_-, z_+\} \to \tilde{M}$$

$$\lim_{s \to z'_-} \tilde{u}(s, t) = \lim_{s \to z'_-} \tilde{w}(s, t) = x'(t), x' \in \{x\}.$$

Given $(\tilde{u}, \tilde{w}) \in \tilde{M}(x_-, x, x_+; J)$ and a gluing parameter $\tau \in \mathbb{R}^+$, we will glue $\bar{u}_N$, and $\bar{w}_{N'}$ together to get a $\bar{v}_{N,\tau} = \bar{u}_N \#_\tau \bar{w}_{N'}$. Here we use $\bar{u}_{N'}$ to denote $\bar{u}_{\delta, N'}$ defined before with $d \in \mathbb{R}^{2(l+m)}$ describing the asymptotic limits of $\tilde{u}$. Similarly for $\bar{w}_{N'}$. Note that since there is no closed loop in the bubble tree, we have assumed in above case that $z'_-$ and $z'_-$ has only one element.

Since both $\bar{u}_N$ and $\bar{w}_{N'}$ are trivial maps when $|s|$ is large enough and they have the same asymptotic limits along the end $z'_+ = z'_-$ as parameterized closed orbit, therefore, given $\tau \in \mathbb{R}^+$, there is an obvious way to define $\bar{v}_{N,\tau} = \bar{u}_N \#_\tau \bar{w}_{N'}$. Namely, we cut off the part of the cylinder $S^2 \setminus \{z_-, z'_-\}$ near $z'_-$ with $s \geq \tau$ and the corresponding part of the cylinder of $S^2 \setminus \{z'_-, z_+\}$ near $z'_+$ with $s \leq -\tau$ and glue the remaining parts together along their boundaries with respect to the same $\theta$-coordinate (the $S^1$-parametrization along $z'_-$ and $z'_+$, see [L1].) To glue the targets of $\tilde{u}$ and $\tilde{w}$, we note that when $|s| > N$ near $z'_-$ and $z'_+$, the $\alpha$-parts of $\tilde{u}$ and $\tilde{w}$ are linear maps only depending on $s$. Let $\bar{M}_u$ and $\bar{M}_w$ be the targets of $\tilde{u}$ and $\tilde{w}$. We get $M = M_u = \bar{M}_u \#_\tau \bar{M}_w$ by
cutting off the parts of $\tilde{M}_u$ with $a > a(\tau)$ and the part of $\tilde{M}_w$ with $a < a(-\tau)$ and glue the remaining parts together.

We then define $\tilde{v}_{N,\tau} : S^2 \setminus \{z_-, z_+\} \to \tilde{M}_v$ by simply applying $\tilde{u}_{N'}$ and $\tilde{w}_{N'}$ to the points in $S^2 \setminus \{z_-, z_+\}$ which are in the domains of $\tilde{u}$ and $\tilde{w}$. Here $S^2 \setminus \{z_-, z_+\}$ is the domain of $\tilde{v}_{N,\tau}$. It is easy to see that when $\tau$ is large enough, $\tilde{v}_{N,\tau}$ is well-defined. To simplify our notation, we denote $\tilde{v}_{N,\tau}$ by $\tilde{v}_\tau$.

Now $\tilde{v}_\tau = \tilde{v}_{N',\tau}$ plays the same rule as $\tilde{u}_{N'}$ and $\tilde{w}_{N'}$ in $\tilde{M}(x_-, x; J)$ and $\tilde{M}(x_+, x; J)$. It is a connecting map between $x_-$ and $x_+$, which is almost $J$-holomorphic and is the trivial map near the two ends $x_-$ and $x_+$. We have three linearized $\partial$-operator $\tilde{L}_{u_{N'}}$, $\tilde{L}_{w_{N'}}$ and $\tilde{L}_{v_\tau}$. Let $D_u$, $D_w$ and $D_{v_\tau}$ be the $L^\infty$ part of these operators. Then

$$D_{u,w} = (D_u, D_v) : L^p_{1,\delta}(\tilde{u}_{N'}^*; \xi) + L^p_{-1,\delta}(\tilde{w}_{N'}^*; \xi) \to L^p_{0,\delta}(\tilde{u}_{N'}^*, \nu_{0,1}^\tau(\xi)) + L^p_{0,\delta}(\tilde{w}_{N'}^*, \nu_{0,1}^\tau(\xi)),$$

and

$$D_{v_\tau} : L^p_{1,\delta}(\tilde{v}_\tau^*; \xi) \to L^p_{0,\delta}(\tilde{v}_\tau^* \nu_{0,1}^\tau(\xi)).$$

**Proposition 3.1** When $\tau$ is large enough,

$$\mathrm{Ind}(D_{v_\tau}) = \mathrm{Ind}(D_{u,w}).$$

**Proof:**

As we mentioned before, for the calculation of the indices here, we can use $L^p$-norm rather than the weighted $L^p$-norm.

In the case that both domains of $\tilde{u}$ and $\tilde{w}$ are the cylinder (hence the domain of $\tilde{v}_\tau$ is the cylinder too), the proposition follows from the fact that in this case the index involved can be calculated by using the spectral flow. The detail of the argument of this type is given in [F] by Floer.

For the general case, the proposition can be proved by using a linear version of the so called ‘gluing’ technique in Floer homology and quantum homology. We will outline this argument here. For details of the non-linear version of this argument, see [F], [L] and [LT].

We assume that $D_{u,w}$ is surjective. Then general case can be reduced to this case (see [LT]). We will show that when $\tau$ is large enough, $D_{v_\tau}$ is also surjective and $\dim(\ker D_{v_\tau}) = \dim D_{u,w}$. For this purpose, we assume that $K_u = \ker(D_u) = \mathbb{R}\{f_1, \cdots, f_m\}$ and $K_w = \ker(D_w) = \mathbb{R}\{g_1, \cdots, g_n\}$. Then $K_{u,w} = \ker(D_{u,w}) = \mathbb{R}\{f_1, \cdots, f_m, g_1, \cdots, g_n\}$. Let $\beta_{\tau,u}$, $\beta_{\tau,w}$ be the cut-off functions specified by

$$\beta_{\tau,u}(s) = \begin{cases} 1 & s < \tau - 1 \\ 0 & \tau < s < \tau + 1 \\ 0 & s > \tau + 1 \end{cases}$$

and

$$\beta_{\tau,w}(s) = \begin{cases} 1 & s > -\tau + 1 \\ 0 & s < -\tau \end{cases}.$$ 

Define $f_{\tau,i} = \beta_{\tau,u} \cdot f_i$ and $g_{\tau,j} = \beta_{\tau,w} \cdot g_j$. Let

$$N_\tau = \mathbb{R}\{f_{\tau,1}, \cdots, f_{\tau,m}, g_{\tau,1}, \cdots, g_{\tau,n}\}$$

be the vector space generated by the asymptotic kernel of $D_{v_\tau}$. Note that each $f_{\tau,i}$ and $g_{\tau,j}$ is defined over $\tilde{v}_\tau$. Let $K_\tau = \ker(D_{v_\tau})$. Let $N_\tau^+$ and $K_\tau^+$ be the $L^2$-complement of $N_\tau$ and $K_\tau$ in $L^p_{1,\delta}(\tilde{v}_\tau^*; \xi)$ respectively.
Sublemma 3.1 There exists a constant $C > 0$ independent of $\tau$ such that for any $\eta \in N^+_\tau$ with $\tau$ large enough,

$$\|\eta\|_{1,p} < \|D_{v_\tau}(\eta)\|_{0,p}.$$  

Proof: Similar statement has been proved in [F], [L] and [LT].

Assume 3.1 is not true. Then there exists $\eta_\tau \in N^\perp_\tau$ such that

(a) $\|\eta_\tau\|_{1,p} = 1$; and
(b) $\|D_{v_\tau}(\eta_\tau)\|_{0,p} \to 0$,

as $\tau \to \infty$.

We want to show that (a) and (b) contradicts to each other.

Let $(\tilde{s}, t)$ be the new cylindrical coordinate for the "neck" part of the domain $S^2_\tau \backslash \{z_-, z_+\}$ of $\tilde{\nu}_\tau$, which starts at the middle of the neck. Then $\tilde{s} \in (-\rho_\tau, \rho_\tau)$, where $\rho_\tau = \tau - N'$. We want to show that in terms of this coordinate, $\|\eta_\tau \|_{[-2,2] \times S^1} \to 0$ as $\tau \to \infty$. To this end, let $\tilde{\eta}_\tau$ be $\eta_\tau|_{(-\rho_\tau, \rho_\tau) \times \tilde{S}^1}$, with respect to $(\tilde{s}, t)$-coordinate. Since $\|\tilde{\eta}_\tau\|_{1,p} < 1$, $\tilde{\eta}_\tau$ is weakly convergent to $\tilde{\eta}_\infty \in L^p_\rho(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ in $L^p_\rho$-norm. Note that here we have used the fact that in the neck part $(-\rho_\tau, \rho_\tau) \times \tilde{S}^1$, $\tilde{\eta}_\tau \in L^p_\rho((-\rho_\tau, \rho_\tau) \times \tilde{S}^1, \mathbb{R}^{2n})$. One can show that (b) implies that $D_\infty \tilde{\eta}_\infty = 0$. Here $D_\infty \equiv \cup \tau D_{v_\tau}$. Note that if $\tau < \tau'$, $D_{v_{\tau'}}((-\rho_{\tau'}, \rho_{\tau'}) \times S^1) = D_{v_\tau}$. However $D_\infty$ is an isomorphism. We have $\tilde{\eta}_\infty = 0$. It follows from Sobolev embedding theorem that on any compact subset $[-R, R] \times S^1$, $\tilde{\eta}_\tau|_{[-R, R] \times S^1} \to 0$ in $C^0$-norm. Since $\|D_{v_\tau}\tilde{\eta}_\tau\|_{0,p} \to 0$, standard elliptic estimate implies that $\|\tilde{\eta}_\tau \|_{[-2,2] \times S^1} \to 0$ as $\tau \to \infty$.

Let

$$\beta(\tilde{s}) = \begin{cases} 1 & -2 < \tilde{s} < 2 \\ 0 & \text{otherwise} \end{cases}$$

be a cut-off function. Write $\eta_\tau = (1 - \beta)\eta_\tau + \beta\eta_\tau$. Then $D_{v_\tau}((1 - \beta)\eta_\tau) = D_{u,\nu}((1 - \beta)\eta)$. Hence

$$\|(1 - \beta)\eta_\tau\|_{1,p} \leq C\|D_{u,\nu}(1 - \beta)\eta_\tau\|_{0,p} + |\pi(1 - \beta)\eta_\tau| = C\|D_{v_\tau}((1 - \beta)\eta_\tau)\|_{0,p} + |\pi(1 - \beta)\eta_\tau| \to 0.$$  

Here $\pi$ is the orthogonal projection map to the kernel of $D_{u,\nu}$. Therefore, $\|\eta_\tau\|_{1,p} \leq \|(1 - \beta)\eta_\tau\|_{1,p} + |\beta\eta_\tau|_{1,p} \to 0$ as $\tau \to \infty$.

Therefore(a) and (b) contradict to each other.

QED

Now define

$$\tilde{D}_\tau : L^p_1(\tilde{\nu}^*_\tau \xi) \to L^0_0(\tilde{\nu}^*_\tau \wedge^{0,1} (\xi)) \oplus N_\tau$$

and

$$\tilde{D}_{u,\nu} : L^p_1(\tilde{u}_{N'}^* \xi) \oplus L^p_1(\tilde{w}_{N'}^* \xi) \
L^0_0(\tilde{u}_{N'}^* \wedge^{0,1} (\xi)) \oplus L^0_0(\tilde{w}_{N'}^* \wedge^{0,1} (\xi)) \oplus K_{u,\nu}$$

as follows.

Define $\tilde{D}_{u,\nu} = D_{u,\nu} \oplus \pi_K$, where $\pi_K$ is the orthogonal projection. Similarly, $\tilde{D}_\tau = D_{v_\tau} \oplus \pi_\tau$, where $\pi_\tau$ is the projection to $N_\tau$.
The previous sublemma shows that $\tilde{D}_\tau|_{\tau^2} = D_{v_\tau}|_{\tau^2}$ is injective. Therefore, $\tilde{D}_\tau$ is injective. Note that $\dim N_\tau = \dim K_{u,w}$. Therefore we only need to show that $\tilde{D}_\tau$ is also surjective when $\tau$ is large enough (note that $\tilde{D}_{u,w}$ is surjective, which follows from our assumption). To this end, it is more convenient to use $L^2_2$-norm for the domain of $D_{u,w}$ and $D_\tau$ before. Observe that since each element $\phi \in \ker_{u,w}$ or $N_\tau$ is in $L^2_2$, the function $\langle \phi, \rangle$ is continuous on $L^2_2$. Therefore, $\pi_K$ and $\pi_\tau$ and hence $\tilde{D}_{u,w}$ and $\tilde{D}_\tau$ are well-defined by replacing $L^2_2$-norm of the domain with $L^2_{2,2}$-norm, and $L^2_2$-norm on the target with $L^2_{2,3}$-norm. Now passing to the dual of $\tilde{D}_\tau^* \tilde{D}_{u,w}$ and note that the domains of the operators have $L^2_3$-norm and the ranges have $L^2_2$-norm.

We now in the position to apply the previous lemma to deduce that the injectivity of $\tilde{D}_{u,w}$ implies the injectivity of $\tilde{D}_\tau^*$ when $\tau$ is large enough.

QED

Now we can compare the virtual dimension of $\tilde{M}(x_-, x, x_+; J)$ near $(\tilde{u}_{N'}, \tilde{w}_{N'})$ with the dimension of $\tilde{M}(x_-, x_+; J)$ near $\tilde{v}_\tau$. We start with two particular but most important cases. The first case is that $\tilde{u}_{N'} \in \tilde{M}(x, J)$ is an almost holomorphic plane with a positive end $x$ and $\tilde{w}_{N'} \in \tilde{M}(x_-, x, x_+; J)$ is a connecting almost holomorphic map of three ends $(x_-, x)$ and $x_+$, where $(x_-, x)$ are negative ends and $x_+$ is the positive one such that their asymptotic limits as parameterized closed orbits agree at $x$. Let $M(\tilde{u}^{-}; J)$ be the moduli space of all such pairs. This corresponding to the case of only one bubble. As we mentioned before, in the simplest case of of bubbling off only one bubble, the domain splits into three components rather than just two as considered here. However, as far as the computation of the index is concerned, we still can consider above case first.

We want to compare the local dimension, $\dim_{loc}(\tilde{M}(x_-, x_+; J))$ near $(\tilde{u}_{N'}, \tilde{w}_{N'})$ with the local dimension $\dim_{loc}(\tilde{M}(x_-, x_+; J))$ near $\tilde{v}_\tau$. Note that $\tilde{M}(x_-, x_+)$ is the moduli space of connecting maps between these two ends $x_-$ and $x_+$.

Now

$$\dim_{loc}(\tilde{M}(x_-, x_+; J))$$

$$= (\Ind(L_{1,u}) + \Ind(L_{1,w}) - 1) + \Ind(D_{u,w})$$

$$= (2 + 2 - 1) + \Ind(D_{u,w}) = 2 + \Ind(D_{u,w}) + 1$$

$$= \Ind(L_{1,v_\tau}) + \Ind(D_{v_\tau}) + 1$$

$$= \dim_{loc,v_\tau}(\tilde{M}(x_-, x_+; J)) + 1.$$

Now the dimension of the symmetries of the domain of $(\tilde{u}_{N'}, \tilde{w}_{N'})$ is 3 while the dimension of the symmetries of the domain of $\tilde{v}_\tau$ is 2. Hence for unparameterized curves, the two moduli spaces have the same dimension. Finally, put the $R$- symmetry in target into consideration, we have:

**Proposition 3.2** $\dim_{loc}(\tilde{M}(x_-, x_+; J)) = \dim_{loc,v_\tau}(\tilde{M}(x_-, x_+; J)) - 1.$

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That is bubbling off only one bubble is a codimension one phenomenon.

The second important case is that both $\tilde{u}_{N'}$ and $\tilde{w}_{N'}$ are connecting maps between two ends. As above, a direct calculation shows the following:

**Proposition 3.3** \( \dim_{\text{loc}}(\mathcal{M}(x_-, x, x_+; J)) = \dim_{\text{loc}}(\mathcal{M}(x_-, x_+; J)) - 1. \) That is splitting a connecting map into broken ones of two elements is also a codimension one phenomenon.

A direct calculation shows that both of above statements are true also for the multi-ends case. We have:

**Proposition 3.4** For a family of connecting pseudo-holomorphic maps with multi-ends, both bubbling off one bubble and splitting off a connecting map of two ends are codimension one phenomenon.

Note that in the case that the above connecting maps split off a trivial connecting map, we have used the reduced Banach space neighborhood \( B^0(u) \) for the trivial map to compute the relevant index here.

**Proposition 3.5** When a family of connecting pseudo-holomorphic map of multi-ends splits into a family of broken connecting pseudo-holomorphic maps of two elements, each being multi-ends, the splitting has codimension two.

**Proof:** Here "multi-ends" means that the number of the ends here is at least three. The proof follows from a direct computation of the relevant indices.

QED

Not that in the last two statements, we allow the marked points to move.

Finally, combine these proposition together, we have the following theorem

**Theorem 3.1** Given a family of connecting pseudo-holomorphic maps, virtually, the bubbling is a co-dimension one phenomenon while the splitting off the connecting maps into the broken ones of two elements is codimension one.

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