Darboux-integrable equations with non-Abelian nonlinearities

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November 6, 2018

Abstract

We introduce a new class of nonlinear equations admitting a representation in terms of Darboux-covariant compatibility conditions. Their special cases are, in particular, (i) the “general” von Neumann equation $i\dot{\rho} = [H, f(\rho)]$, with $[f(\rho), \rho] = 0$, (ii) its generalization involving certain functions $f(\rho)$ which are non-Abelian in the sense that $[f(\rho), \rho] \neq 0$, and (iii) the Nahm equations.

1 Introduction

An investigation of collective phenomena in quantum mechanics leads to various nonlinear evolution equations. Nonlinear equation of a Schrödinger type was derived as a phenomenological equation for the order parameter in superfluid $\text{He}_4$ [1, 2]. Recent experiments on Bose-Einstein condensation [3] significantly raise an interest in nonlinear generalizations of the Schrödinger equation (for a review see [4]). Another kind of nonlinear Schrödinger equations, a by-product of work on classification of groups of diffeomorphisms [5], was recently related to certain aspects of D-brane dynamics [6].

In more realistic situations, where entanglement between interacting particles is properly taken into account, one does not arrive at nonlinear Schrödinger wave equations but rather at their density matrix (von Neumann-type) nonlinear versions [7]

\[ -i\dot{X} = [X, h(X)]. \] (1)
In this case the Hamiltonian $h(X)$ is considered as a “non-Abelian function” of the density operator $X$. As is well known, equations analogous to (1) are often encountered in quantum optics and field theory if one deals with the Heisenberg-picture evolution of observables.

Still another class of nonlinear von Neumann type equations may be derived in dissipative contexts [8, 9] or on the basis of various entropic variational principles [10, 11].

Of some interest is the fact that for a special class of $h(X)$ Eq. (1) can be rewritten as

$$i\dot{X} = [H, f(X)]. \quad (2)$$

Nonlinear equations of this type appeared in the frameworks of nonlinear Nambu-type theories [12] and nonextensive statistics [13]. It should be stressed that $f(X)$ does not always take the usual form known from spectral theory [14].

We will refer to the equations of the general form (1), (2) as nonlinear equations of the von Neumann type. Eq. (2) acquires an additional fundamental flavor if one recalls that for $X = |\psi\rangle\langle\psi|$ and for all functions constructed via the spectral theorem, which satisfy $f(0) = 0$ and $f(1) = 1$, one finds $f(X) = X$ and therefore the dynamics of $X$ is equivalent to the linear Schrödinger equation.

Similar nonlinear equations can be found also in classical theories. The best known physical example is the Euler equation for a freely rotating rigid body

$$\dot{X} = [H, X^2]. \quad (3)$$

The more abstract versions are related to the Euler-Arnold equation for an “$N$-dimensional rigid body” [15], the Lie-Poisson equations occurring in fluid dynamics [15, 16], and the $N$-wave equations for electromagnetic waves in nonlinear media [17].

Particularly interesting and in recent years very intensively investigated class of nonlinear equations are the Nahm equations [18]. Their solutions are used as an intermediate step in construction of non-Abelian monopoles. One family of solutions is in a one-to-one relationship to the Euler rigid-body equations. In this sense the Nahm equations may be regarded as a kind of generalized von Neumann equations.

Finally, quite recently nonlinear equations on free associative algebras, including the ones of the form (1), (2) with $h(X)$ and $f(X)$ being (noncommutative) polynomials, were considered in the framework of the symmetry approach to classification of integrable ordinary differential equations [19]. It was found in particular that Eq. (2), where $f(X) = iX^3$, is symmetry for Eq. (3). A class of equations discussed in [13] was termed “non-$C$-integrable”. Below we show that some of them belong to our class of integrable equations with Darboux covariant Lax representation. It is worth mentioning that we do not assume a polynomial structure of the RHS of Eqs. (1), (2).
As we can see, the reasons for generalizations of the linear von Neumann equation may be different, but they all finally lead to the same fundamental difficulty: The resulting equations involve a large (often infinite) number of degrees of freedom and effective integration procedures are difficult to find. The situation is additionally complicated by constraints typical of density matrices or Hamiltonians.

It was only recently that soliton methods were applied to the density matrix version of \[20, 21, 22\]. The progress was made possible by the observation that there exist Darboux-covariant Lax representations of certain von Neumann-type equations. The technique used in \[20, 21, 22\] is an appropriate modification of the dressing method \[23, 24, 25, 26, 27\] or rather of its analogue constructed via a binary Darboux transformation \[28, 29, 30\]. The technique is called the Darboux transformation since the construction of generalized gauge transformation is performed with the help of additional solutions of the Lax pairs.

The Darboux-type method of integrating the density-matrix analogue of Eq. (3), introduced in \[20\] and further generalized in \[22\], led to discovery of the so-called self-scattering solutions. The process of self-scattering continuously interpolates between two asymptotically linear evolutions. This very characteristic property was found in all the nontrivial solutions of nonlinear von Neumann equation obtained by the above technique.

The paper presents further development of our results. All the equations we derive can be regarded as compatibility conditions for Darboux-covariant Lax pairs. Previously discussed nonlinear von Neumann equations as well as the Nahm equations are particular examples of the class under consideration, but they form just a tip of an iceberg.

Darboux covariance of Lax pairs is proved in detail along the lines of \[31\]. An alternative proof, involving a more general class of Darboux transformations, is given in this volume in \[32\]. The two constructions are based on different mathematical techniques, show different aspects of the same problem, and it is not completely clear whether they are entirely equivalent.

The layout of the paper is as follows. The compatibility-condition representation of a family of nonlinear equations of the von Neumann type is given in Sec. II. The compatibility conditions are brought into a closed form by a special choice of operator coefficients of the Lax pair. The coefficients are defined in terms of additional functions satisfying restrictions following from the compatibility conditions. A wide class of such functions is proposed. Examples of nonlinear equations of the von Neumann type that are generated by some of these functions are given in Sec. III. Darboux covariance of the Lax pairs with the operator-valued coefficients is proved in Sec. IV. In the next section we show that the restrictions carrying the compatibility condition into integrable nonlinear von Neumann equations are Darboux-covariant if the functions introduced in Sec. II are used.
2 Darboux-integrable equations

We begin with the overdetermined system of linear equations (the Lax pair)

\[
\begin{cases}
-\dot{\psi} = \psi A(\lambda) \\
z\lambda \psi = \psi H(\lambda)
\end{cases},
\]

where \(\lambda\) and \(z\lambda\) are complex numbers, \(\psi\) is an element of a linear space \(L\), \(A(\lambda)\) and \(H(\lambda)\) are linear operators \(L \to L\) belonging to an associative ring, the dot denotes a derivative (i.e. an operator satisfying the Leibniz rule). The compatibility condition for the Lax pair is

\[i\dot{H}(\lambda) = [A(\lambda), H(\lambda)].\]  

Assume the operators entering the Lax pair are rational functions of \(\lambda\) with operator coefficients

\[A(\lambda) = \sum_{k=0}^{L} \lambda^k B_k + \sum_{k=1}^{M} \frac{1}{\lambda^k} C_k,\]

\[H(\lambda) = \sum_{k=0}^{N} \lambda^k H_k.\]

The compatibility condition implies two sets of relations between operators \(B_k, C_k\) and \(H_k\)

\[\sum_{k=\max\{0,m-L\}}^{N} [H_k, B_{m-k}] = 0 \quad (N < m \leq L + N),\]

\[\sum_{k=0}^{\min\{N,m+M\}} [H_k, C_{k-m}] = 0 \quad (-M \leq m < 0)\]

and the system of differential equations

\[-i\dot{H}_m = \sum_{k=\max\{0,m-L\}}^{m} [H_k, B_{m-k}] + \sum_{k=m+1}^{\min\{N,m+M\}} [H_k, C_{k-m}]\]

for \((0 \leq m \leq N)\). In order to reduce Eqs. (10) to equations of the von Neumann type one needs to write them in a closed form. In general, Eqs. (3), (4) are inconvenient for defining operators \(B_k\) and \(C_k\) in terms of \(H_k\). Nevertheless, one can express \(B_k\) and \(C_k\) explicitly through operator \(H_k\) by imposing on them some additional relations which obey Eqs. (3), (4). It is clear that not all such additional relations have to be consistent with the requirement of Darboux-covariance of the Lax pair.
Consider

\[
B_k = \frac{1}{(L-k)!} \left( \frac{d^{L-k}}{d\varsigma^{L-k}} f(\varsigma^N H(\varsigma^{-1}), \varsigma^{-1}) \right) \bigg|_{\varsigma=0}, \quad (11)
\]

\[
C_k = \frac{1}{(M-k)!} \left( \frac{d^{M-k}}{d\varepsilon^{M-k}} g(H(\varepsilon), \varepsilon) \right) \bigg|_{\varepsilon=0}, \quad (12)
\]

where \( f(X, \lambda) \) and \( g(X, \lambda) \) are properly defined functions of operator \( X \) and parameter \( \lambda \). The operator at the RHS of the first equation of the Lax pair now reads

\[
A(\lambda) = \sum_{k=0}^{L} \frac{\lambda^{L-k}}{k!} \left( \frac{d^k}{d\varsigma^k} f(\varsigma^N H(\varsigma^{-1}), \varsigma^{-1}) \right) \bigg|_{\varsigma=0}
\]

\[
+ \sum_{k=0}^{M-1} \frac{\lambda^{M-k}}{k!} \left( \frac{d^k}{d\varepsilon^k} g(H(\varepsilon), \varepsilon) \right) \bigg|_{\varepsilon=0} \quad (13)
\]

There exists a large class of functions \( f(X, \lambda) \) and \( g(X, \lambda) \) that results in operators \( B_k \) and \( C_k \) identically satisfying conditions (8) and (9). The class is defined by

\[
[f(X(\lambda), \lambda), X(\lambda)] = [g(X(\lambda), \lambda), X(\lambda)] = 0. \quad (14)
\]

To prove the covariance of Eqs. (11), (12) under the binary Darboux transformation we also assume that these functions possess an additional property, namely they are covariant with respect to the similarity transformation:

\[
f(TX T^{-1}, \lambda) = T f(X, \lambda) T^{-1}, \quad g(TX T^{-1}, \lambda) = T g(X, \lambda) T^{-1}, \quad (15)
\]

where \( T \) is a transformation. The above conditions are satisfied, for example, by polynomials in \( X \) and sums of negative powers of polynomials in \( X \). If \( X \) is selfadjoint, the same is valid for all \( f(X) \) defined via the spectral theorem.

In such a case Eqs. (8) and (9) turn out to be identically fulfilled as a consequence of the trivial identities

\[
\frac{d^n}{d\lambda^n} f(H(\lambda), H(\lambda)) \bigg|_{\lambda=0} \equiv 0, \quad \frac{d^n}{d\lambda^n} g(H(\lambda), H(\lambda)) \bigg|_{\lambda=0} \equiv 0
\]

and Eq. (10) can be written in equivalent form

\[
i \dot{\hat{H}}_m = \sum_{k=m+1}^{N} [H_k, B_{m-k}] + \sum_{k=0}^{m} [H_k, C_{k-m}] \quad (0 \leq m \leq N). \quad (16)
\]

In the next section we will see that there are two representations of the compatibility condition and they correspond to Eqs. (11) and (12).
3 Examples

Below we present a few examples of integrable nonlinear von Neumann-type equations that correspond to different choices of positive integers $N$, $L$, $M$ and functions $f(X, \lambda)$, $g(X, \lambda)$. In what follows we will use the notation

$$H_1 = H, \quad H_0 = \rho.$$  

If $N = 1$ Eqs. (10) imply

$$\dot{H} = 0.$$  

3.1

$N = 1, L = 1, f(X, \lambda) = X^n \ (n \in \mathbb{N}), g(X, \lambda) = 0.$

The compatibility condition gives the equation

$$i\dot{\rho} = \left[ \sum_{k=0}^{n-1} H^{n-k-1}\rho H^k, \rho \right]. \quad (17)$$

The Darboux-covariant Lax pair for this equation was found in [21]. For $n = 2$ Eq. (17) reads

$$i\dot{\rho} = [\rho H + H \rho, \rho] = [H, \rho^2], \quad (18)$$

which is equivalent to Eq. (8). Mutual replacement of $H(\lambda) = \rho + \lambda H$ and $A(\lambda) = H \rho + \rho H + \lambda H^2$ in the corresponding Lax pair results in the compatibility condition

$$i(H \dot{\rho} + \dot{H} \rho) = [H, \rho^2],$$

which is essentially a form of Euler’s top equations given in [15, 33].

3.2

$N = 1, f(X, \lambda) = 0, M = 1, g(X, \lambda) = g(X).$

Here we have

$$i\dot{\rho} = [g(\rho), H].$$

The Lax-pair representation and Darboux covariance properties of this equation have already been established in [22]. It should be stressed that the function $g(X)$ is basically arbitrary. The cases $g(\rho) = i\rho^3$ and $g(\rho) = i\rho^{-1}$ were considered in [22].
3.3

\( N = 1, \, L = 3, \, f(X, \lambda) = \lambda^{-2}(a_0X^2 + (b_0 + \lambda b_1)X^3 + (c_0 + \lambda c_1 + \lambda^2 c_2)X^4), \)
\( g(X, \lambda) = 0. \)

The compatibility condition becomes
\[
i\dot{\rho} = [h(\rho), \rho] = [H, F(\rho)],
\]
where
\[
h(\rho) = a_0(\rho H + H\rho) + b_0(\rho H^2 + H\rho H + H^2\rho) + b_1(\rho^2 H + \rho H\rho + H\rho^2) + c_0(\rho H^3 + H\rho H^2 + H^2\rho H + H^3\rho) + c_1(\rho^3 H^2 + \rho H\rho H + H^2\rho H + H^3\rho) + c_2(\rho^3 H + \rho^2 H\rho + \rho H\rho^2 + H\rho^3),
\]
\[
F(\rho) = a_0\rho^2 + b_0(\rho^2 H + H\rho\rho + H^2\rho^2) + b_1\rho^3 + c_0(\rho^2 H^2 + \rho H\rho H + H^2\rho H + H^3\rho) + c_1(\rho^3 H + \rho^2 H\rho + \rho H\rho^2 + H^3\rho) + c_2\rho^4.
\]
Here \( a_0, b_0, b_1, c_0, c_1, c_2 \) are arbitrary complex parameters independent of \( \lambda \). If the dot is a derivative with respect to a time variable \( t \), they can depend on \( t \). The same is also valid for the next example. Let us note that the map \( \rho \mapsto h(\rho) \) is not a function of \( \rho \) in the standard sense of the spectral theory \( \mathbb{E} \) (such as \( g(\rho) \) of the previous subsection). In particular, \( [h(\rho), \rho] \neq 0 \). We refer to such maps as non-Abelian functions, or non-Abelian nonlinearities.

3.4

\( N = 1, \, f(X, \lambda) = 0, \, M = 2, \)
\( g(X, \lambda) = (a_0 + \lambda a_1)((b_0 + \lambda b_1)I + X)^{-1} + (c_0 + \lambda c_1)((d_0 + \lambda d_1)I + X)^{-1}((e_0 + \lambda e_1)I + X)^{-1}. \)

In this case we obtain
\[
i\dot{\rho} = [H, F(\rho)],
\]
where
\[
F(\rho) = a_0(b_0 I + \rho)^{-1}(d_0 I + H)(e_0 I + \rho)^{-1} - a_1(b_0 I + \rho)^{-1} + c_0((d_0 I + \rho)^{-1}(d_1 I + H)(d_0 I + \rho)^{-1} - (e_0 I + \rho)^{-1}(e_1 I + H)(e_0 I + \rho)^{-1}) - c_1(d_0 I + \rho)^{-1}(e_0 I + \rho)^{-1}.
\]
As opposed to the previous examples \( F(\rho) \) is a non-Abelian nonpolynomial function.
For \( N = 1 \) the nonlinear equations involve only two types of operators: \( \rho \) and \( H \). Increasing \( N \) we can introduce non-Abelian nonlinearities involving an arbitrary number of different operators.

### 3.5

\( N = 2, \) \( L = 2, \) \( f(X, \lambda) = X^2, \) \( g(X, \lambda) = 0. \)

This is the simplest example of \( N = 2 \) nonlinearity. The compatibility conditions are

\[
\begin{align*}
\dot{\rho} &= [H^2, \rho] + [H_2, \rho^2] = [H^2 + H_2\rho + \rho H_2, \rho], \\
\dot{H} &= [H_2, H\rho + \rho H], \\
\dot{H}_2 &= 0.
\end{align*}
\]  

(22)

This system is equivalent to a nonlinear von Neumann equation with two types of nonlinearity: One given in an implicit form and the other of the Euler type.

### 3.6

\( N = 2, \) \( L = 1, \) \( f(X, \lambda) = X, \) \( g(X, \lambda) = 0. \)

The Lax pair is

\[
\begin{align*}
z\lambda \psi_\lambda &= \psi_\lambda (H_0 + \lambda H_1 + \lambda^2 H_2), \\
-\dot{\psi}_\lambda &= \psi_\lambda (H_1 + \lambda H_2)
\end{align*}
\]  

(23) \hspace{1cm} (24)

with the compatibility conditions

\[
\begin{align*}
\dot{H}_2 &= 0, \\
\dot{H}_1 &= [H_2, H_0], \\
\dot{H}_0 &= [H_1, H_0].
\end{align*}
\]  

(25) \hspace{1cm} (26) \hspace{1cm} (27)

Defining

\[
\begin{align*}
F_1 &= (H_0 - H_2)/(2i), \\
F_2 &= (H_0 + H_2)/2, \\
F_3 &= H_1/(2i)
\end{align*}
\]  

(28) \hspace{1cm} (29) \hspace{1cm} (30)

and the connection \( \nabla f = \dot{f} + [f, F_3] \) we can write the compatibility conditions as

\[
\begin{align*}
\nabla F_1 &= i[F_2, F_3], \\
\nabla F_2 &= i[F_3, F_1], \\
\nabla F_3 &= i[F_1, F_2].
\end{align*}
\]  

(31) \hspace{1cm} (32) \hspace{1cm} (33)
The connection can be trivialized if we find an invertible solution $\xi$ of the linear problem
\[ \dot{\xi} = -\xi F_3. \] (34)

Then
\[ f_k = \xi F_k \xi^{-1} \] (35)
satisfies the standard Nahm equations
\[ \dot{f}_1 = i[f_2, f_3], \] (36)
\[ \dot{f}_2 = i[f_3, f_1], \] (37)
\[ \dot{f}_3 = i[f_1, f_2]. \] (38)

4 Binary Darboux transformation

The first step towards extending the technique of Darboux transformations to integrable nonlinear von Neumann-type equations on associative rings is to establish the Darboux covariance of the Lax pair (4) without any additional constraints. In this section we show that an appropriate formulation of the binary Darboux transformation makes the Lax pair covariant.

Assume $\chi$ is a solution of the Lax pair with parameter $\nu$:
\[ \begin{align*}
-\dot{\chi} &= \chi A(\nu) \\
\chi H(\nu) &= \chi H(\nu)
\end{align*} \] (39)

and $\varphi$ is a solution of the dual Lax pair with parameter $\mu$:
\[ \begin{align*}
\dot{\varphi} &= A(\mu) \varphi \\
\varphi H(\mu) &= \varphi H(\mu)
\end{align*} \] (40)

We further suppose that these systems can be related with an operator $P$ satisfying $P^2 = P$ and
\[ -\dot{P} = PA(\nu) P_\perp - P_\perp A(\mu) P, \] (41)

where $P_\perp = 1 - P$. The above assumptions are fulfilled, for example, if $\chi = \langle \chi \mid$ and $\varphi = |\varphi \rangle$ are some “bra” and “ket” associated with a Hilbert space. Then
\[ P = |\varphi \rangle \langle \chi |. \]

Another example is provided by $m \times n$ matrix $\chi$ and $n \times m$ matrix $\varphi$. $P$ is then defined by
\[ P = \varphi (\chi \varphi)^{-1} \chi. \]
Some realizations of Darboux transformations in infinite dimensional cases were given in [34]. Very recently a new construction of the Darboux transformation in terms of Clifford numbers was described in [35].

Defining

$$\psi[1] = \psi D_\lambda,$$

$$D_\lambda = 1 + \frac{\nu - \mu}{\mu - \lambda} P$$

we come to the following

**Theorem 1.** The Lax pair (4) with the coefficients defined by Eqs. (6), (7) is covariant with respect to the binary Darboux transformation \{\psi, A(\lambda), H(\lambda)\} \rightarrow \{\psi[1], A(\lambda)[1], H(\lambda)[1]\}, where

$$A(\lambda)[1] = \sum_{k=0}^{L} \lambda^k B_k[1] + \sum_{k=1}^{M} \frac{1}{\lambda^k} C_k[1],$$

$$B_k[1] = B_k + (\mu - \nu) \sum_{m=k+1}^{L} \left( \mu^{m-k-1} P \perp B_m P - \nu^{m-k-1} P B_m P \perp \right)$$

$$C_k[1] = C_k - (\mu - \nu) \sum_{m=k}^{M} \left( \mu^{k-m-1} P \perp C_m P - \nu^{k-m-1} P C_m P \perp \right)$$

and

$$H(\lambda)[1] = \sum_{k=0}^{N} \lambda^k H_k[1],$$

$$H_k[1] = H_k \quad + (\mu - \nu) \sum_{m=k+1}^{N} \left( \mu^{m-k-1} P \perp H_m P - \nu^{m-k-1} P H_m P \perp \right)$$

**Proof:** The condition of covariance of the second equation of the Lax pair with respect to the transformation yields

$$H(\lambda)[1] = D^{-1}_\lambda H(\lambda) D_\lambda$$

$$= \left( 1 + \frac{\mu - \nu}{\nu - \lambda} P \right) \sum_{k=0}^{N} \lambda^k H_k \left( 1 + \frac{\nu - \mu}{\mu - \lambda} P \right)$$

$$= \sum_{k=0}^{N} \lambda^k H_k + \frac{\mu - \nu}{\nu - \lambda} \sum_{k=0}^{N} \lambda^k P H_k P \perp + \frac{\nu - \mu}{\mu - \lambda} \sum_{k=0}^{N} \lambda^k P \perp H_k P.$$

Taking into account

$$PH(\nu)P_\perp = P_\perp H(\mu)P = 0$$

10
we are able to rewrite the previous expression in the following manner

\[ H(\lambda)[1] = \sum_{k=0}^{N} \lambda^k H_k + \sum_{k=0}^{N} \frac{\mu-\nu}{\nu-\lambda} \sum_{k=0}^{N} (\lambda^k - \nu^k) PH_k P_\perp \]

\[ + \sum_{k=0}^{N} \frac{\nu-\mu}{\mu-\lambda} (\lambda^k - \mu^k) P_\perp H_k P \]

\[ = \sum_{k=0}^{N} \lambda^k H_k + (\nu - \mu) \sum_{k=1}^{N} \sum_{j=0}^{k-1} \lambda^{k-j-1} \nu^j PH_k P_\perp \]

\[ + (\mu - \nu) \sum_{k=1}^{N} \sum_{j=0}^{k-1} \lambda^{k-j-1} \mu^j P_\perp H_k P, \]

which is equivalent to Eq. (47).

From the condition of the Darboux covariance of the first equation of the Lax pair we have

\[ A(\lambda)[1] = D^{-1}_\lambda A(\lambda) D_\lambda - i D^{-1}_\lambda \hat{D}_\lambda. \]

Substitution of Eqs. (41), (43) gives

\[ A(\lambda)[1] = \left(1 + \frac{\mu-\nu}{\nu-\lambda} P\right) A(\lambda) \left(1 + \frac{\nu-\mu}{\mu-\lambda} P\right) \]

\[ + \frac{\nu-\mu}{\mu-\lambda} \left(1 + \frac{\mu-\nu}{\nu-\lambda} P\right) (PA(\nu) P_\perp - P_\perp A(\mu) P) \]

\[ = A(\lambda) + \frac{\mu-\nu}{\nu-\lambda} P(A(\lambda) - A(\nu)) P_\perp + \frac{\nu-\mu}{\mu-\lambda} P_\perp (A(\lambda) - A(\mu)) P \]

\[ = \sum_{k=0}^{L} \lambda^k B_k + \sum_{k=1}^{M} \frac{1}{\lambda^k} C_k \]

\[ + (\mu - \nu) P_\perp \left( \sum_{k=1}^{L} \sum_{j=0}^{k-1} \lambda^{k-j-1} \mu^j B_k - \sum_{k=1}^{M} \sum_{j=0}^{k-1} \lambda^{j-1} \mu^j C_k \right) P \]

\[ + (\nu - \mu) P \left( \sum_{k=1}^{L} \sum_{j=0}^{k-1} \lambda^{k-j-1} \nu^j B_k - \sum_{k=1}^{M} \sum_{j=0}^{k-1} \lambda^{j-1} \nu^j C_k \right) P_\perp. \]

The final expression is (44).

5 The main theorem

Theorem 1 establishes Darboux covariance of Lax pairs involving operators with positive or negative powers of spectral parameters. The compatibility condition
is also covariant: Transformed operators $B_k[1]$, $C_k[1]$ and $H_k[1]$ solve Eqs. (8), (9), and (10). However, if there are additional relations between the operators, they do not have to be Darboux covariant.

**Theorem 2.** The relations (11) and (12) are Darboux covariant if Eqs. (13) are fulfilled.

**Proof:** Let us check the Darboux covariance of Eq. (11), i.e.

$$B_k[1] = \frac{1}{(L-k)!} \left( \frac{d^{L-k}}{d\zeta^{L-k}} f(\zeta^N H(\zeta^{-1})[1], \zeta^{-1}) \right)_{\zeta=0}.$$  

It follows immediately that

$$B_L[1] = B_L$$

and

$$\frac{d^k}{d\zeta^k} D_{1/\zeta} \bigg|_{\zeta=0} = k! (\mu - \nu) \mu^{k-1} P,$$

$$\frac{d^k}{d\zeta^k} D_{-1/\zeta} \bigg|_{\zeta=0} = k! (\nu - \mu) \mu^{k-1} P.$$  

For $k \neq L$ we have, using Eq. (13),

\[
\begin{align*}
\frac{1}{(L-k)!} & \left( \frac{d^{L-k}}{d\zeta^{L-k}} f(\zeta^N H(\zeta^{-1})[1], \zeta^{-1}) \right)_{\zeta=0} \\
& = \frac{1}{(L-k)!} \left( \frac{d^{L-k}}{d\zeta^{L-k}} f(\zeta^N D_{1/\zeta}^{-1} H(\zeta^{-1}) D_{1/\zeta}, \zeta^{-1}) \right)_{\zeta=0} \\
& = \frac{1}{(L-k)!} \left( \frac{d^{L-k}}{d\zeta^{L-k}} \left( D_{1/\zeta}^{-1} f(\zeta^N H(\zeta^{-1}), \zeta^{-1}) D_{1/\zeta} \right) \right)_{\zeta=0} \\
& = \frac{1}{(L-k)!} \sum_{a=0}^{L-k} \sum_{b=0}^{a} \frac{(L-k)!}{(L-k-a)! a! (a-b)! b!} \times \left( \frac{d^{L-k-a}}{d\zeta^{L-k-a}} D_{1/\zeta}^{-1} \right)_{\zeta=0} \left( \frac{d^{a-b}}{d\zeta^{a-b}} f(\zeta^N H(\zeta^{-1}), \zeta^{-1}) \right)_{\zeta=0} \left( \frac{d^b}{d\zeta^b} D_{1/\zeta} \right)_{\zeta=0} \\
& + (\nu - \mu) \mu^{L-k-1} PB_L \\
& = \sum_{a=1}^{L-k} \sum_{b=0}^{L-k-a} \frac{1}{(L-k-a)! b!} \left( \frac{d^{L-k-a}}{d\zeta^{L-k-a}} D_{1/\zeta}^{-1} \right)_{\zeta=0} B_{L+b-a} \left( \frac{d^b}{d\zeta^b} D_{1/\zeta} \right)_{\zeta=0} \\
& + (\nu - \mu) \mu^{L-k-1} PB_L \\
& = \sum_{a=1}^{L-k} \sum_{b=1}^{L-k-a} \frac{(\mu - \nu) \mu^{b-1}}{(L-k-a)! b!} \left( \frac{d^{L-k-a}}{d\zeta^{L-k-a}} D_{1/\zeta}^{-1} \right)_{\zeta=0} B_{L+b-a} P
\end{align*}
\]
\[
L_{k-1} a \sum_{a=1}^{L-k} B_{L-a} + (\nu - \mu) \nu^{L-k-1} P B_L
\]

\[
= \sum_{a=1}^{L-k} \sum_{b=1}^{L-k} \left( \frac{\mu}{L-k-a} \right)^{b-1} \left( \frac{\nu}{(L-k-a)!} \right)^{b-1} B_{L-b-a} + B_1 + (\nu - \mu) \nu^{L-k-1} P B_L
\]

\[
= B_k + (\nu - \mu) \left( \sum_{b=1}^{L-k} \mu^{b-1} B_{k+b} P - \sum_{a=0}^{L-k-1} \nu^{L-k-a-1} P B_{L-a} + (\nu - \mu) \sum_{a=1}^{L-k} \sum_{b=1}^{L-k} \mu^{b-1} \nu^{L-k-a-1} P B_{L-b-a} \right)
\]

\[
= B_k + (\nu - \mu) \left( \sum_{b=1}^{L-k} \mu^{b-1} P \delta_{b} + \sum_{a=0}^{L-k-1} \nu^{L-k-a-1} P B_{L-a} \right)
\]

where

\[
\delta_k = \sum_{b=1}^{L-k} \mu^{b-1} P B_{k+b} P - \sum_{a=0}^{L-k-1} \nu^{L-k-a-1} P B_{L-a} + (\nu - \mu) \sum_{a=1}^{L-k} \sum_{b=1}^{L-k} \mu^{b-1} \nu^{L-k-a-1} P B_{L-b-a}
\]

\[
= \sum_{b=1}^{L-k} \mu^{b-1} P B_{k+b} P - \sum_{a=0}^{L-k-1} \nu^{L-k-a-1} P B_{L-a} + \sum_{a=1}^{L-k} \sum_{b=1}^{L-k} \mu^{b-1} \nu^{L-k-a} P B_{L+b-a}
\]
\[- \sum_{a=1}^{L-k-1} \sum_{b=1}^{a} \mu^b \nu^{L-k-a-1} PB_{L+b-a} P. \]

Combining, respectively, the first and the third, the second and the fourth terms gives

\[
\delta_k = \sum_{a=1}^{L-k} \sum_{b=1}^{a} \mu^b \nu^{L-k-a} PB_{L+b-a} P - \sum_{a=0}^{L-k-1} \sum_{b=0}^{a} \mu^b \nu^{L-k-a-1} PB_{L+b-a} P \equiv 0.
\]

Finally, we obtain

\[
\frac{1}{(L-k)!} \left( \frac{d^{L-k}}{d\xi^{L-k}} f(\xi^N H(\xi^{-1})[1], \xi^{-1}) \right) \bigg|_{\xi=0} = B_k + (\mu - \nu) \left( \sum_{m=1}^{L-k} \mu^{m-1} P \sum_{m=0}^{L-k-1} \nu^{L-k-m-1} P \right) \]

(51)

The last expression coincides with Eq. (45).

Let us prove the Darboux covariance of Eq. (12):

\[
C_k[1] = \frac{1}{(M-k)!} \left( \frac{d^{M-k}}{d\varepsilon^{M-k}} g(H(\varepsilon)[1], \varepsilon) \right) \bigg|_{\varepsilon=0}.
\]

One can show that

\[
\begin{align*}
\frac{d^k}{d\varepsilon^k} D_{\varepsilon} \bigg|_{\varepsilon=0} &= \delta_{k0} 1 + k!(\mu - \nu)\mu^{k-1} P, \\
\frac{d^k}{d\varepsilon^k} D_{\varepsilon}^{-1} \bigg|_{\varepsilon=0} &= \delta_{k0} 1 + k!(\mu - \nu)\nu^{k-1} P.
\end{align*}
\]

(52)

Then

\[
\frac{1}{(M-k)!} \left( \frac{d^{M-k}}{d\varepsilon^{M-k}} g(H(\varepsilon)[1], \varepsilon) \right) \bigg|_{\varepsilon=0} = \frac{1}{(M-k)!} \left( \frac{d^{M-k}}{d\varepsilon^{M-k}} g(D_{\varepsilon}^{-1} g(H(\varepsilon), \varepsilon) D_{\varepsilon}) \right) \bigg|_{\varepsilon=0} = \frac{1}{(M-k)!} \sum_{a=0}^{M-k} \sum_{b=0}^{a} \frac{(M-k)!}{a! (a-b)! b!} \times \left( \frac{d^{M-k-a}}{d\varepsilon^{M-k-a}} D_{\varepsilon}^{-1} \right) \bigg|_{\varepsilon=0} \left( \frac{d^{a-b}}{d\varepsilon^{a-b}} g(H(\varepsilon), \varepsilon) \right) \bigg|_{\varepsilon=0} \left( \frac{d^b}{d\varepsilon^b} D_{\varepsilon} \right) \bigg|_{\varepsilon=0}.
\]
\[ M - k \sum_{a=0}^{M-k} \frac{1}{(M-k-a)!b!} \times \left( \delta_{(M-k-a)_0} 1 + (M-k-a)! (\mu - \nu) \nu^{k+a-M-1}a \right) \times C_{M+b-a} \left( \delta_{b0} 1 + b(\nu - \mu) \mu^{-1}P \right) \]

\[ = \left( 1 + \frac{\mu - \nu}{\nu} P \right) C_k \left( 1 + \frac{\nu - \mu}{\mu} P \right) \]

\[ + (\nu - \mu) \sum_{b=1}^{M-k} \mu^{-b-1} \left( 1 + \frac{\mu - \nu}{\nu} P \right) C_{b+k} P \]

\[ + (\nu - \mu) \sum_{a=0}^{M-k-1} \mu^{k+a-M-1} PC_{M-a} \left( 1 + \frac{\nu - \mu}{\mu} P \right) \]

\[ - (\nu - \mu)^2 \sum_{a=1}^{M-k-1} \mu^{b+a-M-1} \mu^{-b-1} PC_{M+b-a} P \]

\[ = \left( 1 + \frac{\mu - \nu}{\nu} P \right) C_k \left( 1 + \frac{\nu - \mu}{\mu} P \right) \]

\[ + (\nu - \mu) \left( \sum_{a=0}^{M-k-1} \mu^{k+a-M-1} PC_{M-a} P - \sum_{b=1}^{M-k} \mu^{-b-1} P \right) \]

\[ \sum_{b=1}^{M-k} C_{b+k} P + \Delta_k \]

where

\[ \Delta_k = \sum_{a=0}^{M-k-1} \mu^{k+a-M-1} PC_{M-a} P - \sum_{b=1}^{M-k} \mu^{-b-1} P \]

\[ + (\nu - \mu) \sum_{a=1}^{M-k-1} \mu^{b-1} \mu^{k+a-M-1} PC_{M+b-a} P \]

\[ = \sum_{a=0}^{M-k-1} \mu^{k+a-M-1} PC_{M-a} P - \sum_{b=1}^{M-k} \mu^{-b-1} P \]

\[ + \sum_{a=1}^{M-k-1} \mu^{b-1} \mu^{k+a-M} PC_{M+b-a} P \]

\[ - \sum_{a=1}^{M-k-1} \mu^{b-1} \mu^{k+a-M} PC_{M+b-a} P. \]

Combining, respectively, the first and the third, the second and the fourth terms we obtain

\[ \Delta_k = \sum_{a=0}^{M-k-1} \mu^{b-1} \mu^{k+a-M} PC_{M+b-a} P \]
\[-\sum_{a=1}^{M-k} \sum_{b=1}^{a} \mu^{-b} \nu^{a-M-1} PC_{M-a} P_{\perp} \equiv 0.\]

Finally, we have
\[
\frac{1}{(M-k)!} \left( \frac{d^{M-k}}{dz^{M-k}} g(H(z)[1], \varepsilon) \right) \varepsilon=0 \\
= \left( 1 + \frac{\mu - \nu}{\nu} P \right) C_k \left( 1 + \frac{\nu - \mu}{\mu} P \right) \\
+ (\mu - \nu) \left( \sum_{a=0}^{M-k-1} \nu^{k+a-M-1} PC_{M-a} P_{\perp} - \sum_{b=1}^{M-k} \mu^{-b-1} P_{\perp} C_{b+k} P \right) \\
= C_k + \frac{\mu - \nu}{\nu} PC_k P_{\perp} + \frac{\nu - \mu}{\mu} P_{\perp} C_k P \\
+ (\mu - \nu) \left( \sum_{a=0}^{M-k-1} \nu^{k+a-M-1} PC_{M-a} P_{\perp} - \sum_{b=1}^{M-k} \mu^{-b-1} P_{\perp} C_{b+k} P \right) \\
= C_k + (\mu - \nu) \left( \sum_{a=0}^{M-k} \nu^{k+a-M-1} PC_{M-a} P_{\perp} - \sum_{b=0}^{M-k} \mu^{-b-1} P_{\perp} C_{b+k} P \right).
\]

The last expression coincides with Eq. (46).

6 Conclusions

We have established the Darboux-covariance of a large class of nonlinear von Neumann-type equations. The next step is to employ this fact in construction of explicit solutions of such equations. Some classes of solutions have already been found in [20, 22, 36] for nonlinearities
\[
if'(\rho) = [H, f(\rho)] \tag{53}
\]
with \(f(\rho) = \rho^2\) and \(f(\rho) = \rho^q - 2\rho^q - 1\) (\(q\) is an arbitrary real number). Both finite- and infinite-dimensional cases were treated by this technique in [22].

In a forthcoming paper we will describe other classes of solutions of the integrable equations we have introduced. It also seems that Lax pairs that allow us to reduce the compatibility conditions to nonlinear equations in a closed form can be still generalized. We hope in the future work to develop a description of integrable equations of the von Neumann type by taking into consideration the Mikhailov method of automorphisms [27]. This type of generalization is particularly important if reductions characteristic of Nahm-type equations are involved.
Acknowledgments

M.C. is indebted to Jan L. Cieśliński for his comments and, in particular, for the suggestion of using the similarity-transformation form of the Darboux transformation. The work of M.C. was supported by the Alexander von Humboldt Foundation and the KBN Grant 5 P03B 040 20. The work of N.V.U. was supported by Nokia-Poland.

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