KNIZHNIK-ZAMOLODCHIKOV BUNDLES ARE TOPOLOGICALLY TRIVIAL

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Abstract. We prove that the vector bundles at the core of the Knizhnik-Zamolodchikov and quantum constructions of braid groups representations are topologically trivial bundles. We provide partial generalizations of this result to generalized braid groups. A crucial intermediate result is that the representation ring of the symmetric group on \( n \) letters is generated by the alternating powers of its natural \( n \)-dimensional representation.

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1. Introduction

The braid group \( \mathcal{B}_n \) on \( n \) strands can be realized as the fundamental group of \( \mathbb{C}^n/\mathfrak{S}_n \), where \( \mathbb{C}_n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid i \neq j \Rightarrow z_i \neq z_j \} \) and \( \mathfrak{S}_n \) is the symmetric group on \( n \) letters, acting on \( \mathbb{C}^n \) by permutation of coordinates. Representations of the braid groups appear in a number of ways, notably in the study of quantum groups and low-dimensional topology.

A well-known way to obtain and study these representations is to consider them, when possible, as the monodromy of a flat connection on (complex) vector bundles of the form \((\mathbb{C}^n/\mathfrak{S}_n) \times U \rightarrow \mathbb{C}^n/\mathfrak{S}_n\), where \( U \) is a finite-dimensional vector space equipped with a linear action of \( \mathfrak{S}_n \). \( \mathbb{C}_n \times U \rightarrow \mathbb{C}_n \) denotes the corresponding trivial vector bundle over \( \mathbb{C}_n \), and \( \mathfrak{S}_n \) acts on \( \mathbb{C}_n \times U \) by diagonal action. Examples include, but are not restricted to, the case when \( U = A^\otimes n \) is the \( n \)-times tensor power of some Lie algebra representation \( A \) and \( \mathfrak{S}_n \) acts on \( A^\otimes n \) by permuting the tensor factors.

A natural question is whether the quotient vector bundles themselves are or can be topologically trivial. We call such quotient bundles Knizhnik-Zamolodchikov bundles, or KZ-bundles for short. The primary purpose of this work is to answer (positively) to this question.

Theorem 1.1. Every Knizhnik-Zamolodchikov bundle is topologically trivial.

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A consequence of the Kohno-Drinfeld theorem is thus that vector bundles over $\mathbb{C}_n^* / \mathfrak{S}_n$ associated to the braid groups representation afforded by $R$-matrices are also topologically trivial.

The proof of this theorem goes as follows. First note that we may replace $\mathbb{C}_n^* / \mathfrak{S}_n$ by $V$ where $V = \{(z_1, \ldots, z_n) \in \mathbb{C}_n^* \mid z_1 + \cdots + z_n = 0\}$ and the 1-dimensional subspace denoted $\mathbb{C}$ is spanned by the vector $(1, \ldots, 1)$. Step (1) is to explicitly trivialize the bundle when $U = V$. This trivialization had already been observed in [ACC], where the authors used the ‘Vandermonde spiral’ introduced by Arnold in [Ar]. We give a proof that generalizes this fact to arbitrary reflection groups (see below), and provides an explanation of the appearance of the Vandermonde matrix, as a jacobian determinant originating from invariants theory.

Step (2) is the elementary remark that irreducible representations of dimension at least 2 of the symmetric groups $\mathfrak{S}_n$ have actually dimension large enough so that the corresponding vector bundles are in the ‘stable range’, meaning that they are trivial if and only if they are stably trivial. The problem can thus be reduced to the computation of the image of the representation ring $R(\mathfrak{S}_n)$ inside the reduced (complex) $K$-theory of $\mathbb{C}_n^*$.

Step (3) is that $R(\mathfrak{S}_n)$ is generated as a ring by the representations $\Lambda^k V$ (theorem 4.2 below). As far as we know, this is a new result that might be applied elsewhere. Since the alternating powers of a trivial bundle are trivial, this concludes the proof of the theorem.

This problem can be placed in the following larger setting. Let $V$ be a finite-dimensional complex vector space and let $W < \text{GL}(V)$ be a finite group generated by (pseudo-)reflections, i.e. finite-order endomorphisms fixing some hyperplane. Such a group is called a reflection group. For convenience, we assume that $W$ acts irreducibly on $V$. To such a reflection group are associated the arrangement $\mathcal{A}$ of the hyperplanes fixed by the reflections in $W$, an hyperplane complement $X = V \setminus \bigcup \mathcal{A}$, (generalized) pure braid group $P = \pi_1(X)$ and braid group $B = \pi_1(X/W)$. Generalized KZ-systems are then flat connections on the quotient bundles $(X \times U \to X)/W$ where $U$ is a finite-dimensional representation of $W$. This yields the following question.

**Question 1.2.** Are all generalized KZ-bundles topologically trivial ?

Following the same demarch as in the proof of the theorem, we prove that steps (1) and (2) are valid in this more general setting (see propositions 3.3 and 3.6). Complications arise at step (3), because it is no more true in general that the representation ring $R(W)$ is generated by the exterior powers of the reflection representations. However, using several ad-hoc means, we manage to extend the result to several irreducible reflection groups, following their Shephard-Todd classification (see [ST]) :

**Theorem 1.3.** Let $W$ denote an irreducible reflection group. Generalized KZ-bundles are topologically trivial whenever

- $W$ has rank 2
- $W$ is one of the exceptional groups $G_{23} = H_3$, $G_{24}$, $G_{25}$, $G_{26}$, $G_{30} = H_4$, $G_{32}$, $G_{33}$, $G_{35} = E_6$. 
Back to the ordinary braid groups, one interest of this last theorem is that it applies to the exceptional Shephard groups $G_4, G_8, G_{16}, G_{25}, G_{32}$, whose braid groups are ordinary braid groups (see e.g. [BMR]). It follows that the corresponding representations of $B_3, B_4, B_5$ also have trivial image in the reduced K-theory of the group, and more precisely that their associated vector bundles over $\mathbb{C}^n\!/\!S_n$ are topologically trivial.

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2. Topological preliminaries

Let $Y$ denote a connected paracompact topological space which has the homotopy type of a $n$-dimensional CW-complex, $\tilde{Y}$ its universal cover, $B = \pi_1(Y)$. Let $X$ be a finite Galois covering of $Y$ with transformations group $W$, so that $Y = X/W$ and $X = \tilde{Y}/P$, where $P = \text{Ker}(B \to W)$.

All vector bundles considered in this paper are complex ones. We let $\text{Vect}(Y)$ denote the category of vector bundles over $Y$, and $\text{Vect}(y, z)$ denote the semiring of (isomorphism classes of) vector bundles over $Y$. We let $K(Y)$ denote the corresponding K-theoretic ring, and $\tilde{K}(Y)$ the quotient of $K(Y)$ by its subring of trivial bundles. We refer to [Hu] for these classical notions and their basic properties.

Each linear representation $\rho : B \to \text{GL}_N(\mathbb{C})$ defines a vector bundle $\mathcal{V}(\rho) = Y \times_{\rho} \mathbb{C}^N$ to $Y$, defined by $Y \times_{\rho} \mathbb{C}^N = (\tilde{Y} \times \mathbb{C}^N)/B$, where $B$ acts on $\tilde{Y} \times \mathbb{C}^N$ by $g.(y, z) = (g.y, \rho(g).z)$. We let $R^+(B)$ and $R(B)$ denote the representation semiring and representation ring of $B$, respectively. The correspondance $\rho \mapsto \mathcal{V}(\rho)$ defines a semiring morphism $\Phi^+ : R^+(B) \to \text{Vect}(Y)$ and a ring morphism $\Phi : R(B) \to K(Y)$.

The group morphism $B \to W$ induces a ring morphism $R(W) \to R(B)$ and we still denote $\Phi : R(W) \to K(Y)$ the composite morphism. Let $\rho : W \to \text{GL}_N(\mathbb{C})$ be a representation of $W$. Then $\Phi(\rho)$ is the class of $\mathcal{V}(\rho)$, and

$$Y \times_{\rho} \mathbb{C}^N = (\tilde{Y} \times \mathbb{C}^N)/B = ((\tilde{Y} \times \mathbb{C}^N)/P)/W = (X \times \mathbb{C}^N)/W,$$

as $P$ acts trivially on $\mathbb{C}^N$, hence $\mathcal{V}(\rho)$ is the quotient of the trivial vector bundle $X \times \mathbb{C}^N \to X$ by $W$ acting on $X \times \mathbb{C}^N$ through $g.(x, z) = (g.x, \rho(g).z)$.

We state as a proposition a recollection of classical results to be used in the sequel. Recall that to each partition $\lambda$ of some integer $r$ are associated Schur endofunctors $S_\lambda$ of the category of finite-dimensional vector spaces. Being continuous (in the sense of [Hu] §5.6) they extend to endofunctors $S_\lambda$ of $\text{Vect}(Y)$. Our convention is that $S_{[r]}(\xi) = \Lambda^r \xi$. We moreover assume that $B$ is finitely generated, so that the topology of $\text{Hom}(B, \text{GL}_N(\mathbb{C}))$ is naturally defined.

**Proposition 2.1.** Let $\rho$ be a (finite-dimensional) linear representation of $W$.

1. If $N = \dim \rho \geq n/2$, then $\Phi(\rho) \equiv 0$ in $\tilde{K}(Y)$ if and only if $\mathcal{V}(\rho)$ is a trivial vector bundle.
(2) If \( t \mapsto \rho_t \) is a continuous path in \( \text{Hom}(B, \text{GL}_N(\mathbb{C})) \) then \( \Phi^+(\rho_0) = \Phi^+(\rho_1) \).

(3) If \( \mathcal{V}(\rho) \) is a trivial bundle, then so is \( \mathcal{S}_\lambda(\mathcal{V}(\rho)) \simeq \mathcal{V}(\mathcal{S}_\lambda \rho) \).

Proof. The assumption in (1) means that \( \Phi^+(\rho) + \mathbb{1}_Y = \mathbb{1}_Y \) for some \( r \geq 0 \) and \( s = r + \dim \rho \), where \( \mathbb{1}_Y \) denotes the trivial line bundle over \( Y \). Then conclusion that \( \Phi^+(\rho) = \mathbb{1}_{\dim \rho} \) follows from [Hu] §9, theorem 1.5. For fact (2), see [Hu] §3.4. The proof of fact (3) is similar to the more classical special case \( \Lambda^r \mathcal{V}(\rho) \simeq \mathcal{V}(\Lambda^r \rho) \) dealt with in [Hu] §13. \( \square \)

3. Representations of complex reflection groups

In this section we establish general results for arbitrary reflection groups. Let \( V \) be a \( n \)-dimensional vector space, \( W \subset \text{GL}(V) \) be a finite (pseudo-)reflection group, with associated hyperplane arrangement \( \mathcal{A} \). We denote \( X = V \setminus \bigcup \mathcal{A} \), \( P = \pi(X) \), \( B = \pi(X/W) \) the pure braid group and braid group associated to \( W \). To each \( H \in \mathcal{A} \) we associate the order \( e_H \) of the (cyclic) fixer of \( H \) in \( W \), and call distinguished reflection associated to \( H \) the pseudo-reflection with set of fixed points \( H \) and eigenvalue \( \exp(2i\pi/e_H) \).

The spaces \( X \) and \( X/W \) are \( K(\pi, 1) \) (see [Bes]). Moreover, \( X/W \) is the complement of a finite number of hypersurfaces, hence is an affine variety of complex dimension \( n \). By [Mi] theorem 7.2 it follows that the connected space \( Y = X/W \) has the homotopy type of a \( n \)-dimensional CW-complex, and the results of the above section can be applied.

3.1. One-dimensional representations. There is a distinguish class of preimages in \( B \) of distinguished pseudo-reflection that are called braided reflections (see e.g. [Ma8]). We denote \( \omega_H = \text{det} \alpha_H/\alpha_H \) for an arbitrary linear form \( \alpha_H \) with kernel \( H \). If \( s \) is the distinguished reflection associated to \( H \) we may note \( \omega_s = \omega_H \).

Proposition 3.1. If \( \rho_0 \in \text{Irr}(W) \) has dimension 1, then \( \mathcal{V}(\rho_0) \) is a trivial bundle.

Proof. We consider the bundle \( X \times \mathbb{C} \to X \), and the family of 1-form \( \omega_h = h_s \sum s \rho_0(s) \omega_s \in \Omega^1(X) \otimes \text{End}(\mathbb{C}) \), with \( h_s \in \mathbb{C} \) and \( h_s = h_{s'} \) whenever \( s, s' \) are conjugate in \( W \). This 1-form is thus indexed by a tuple \( h \) in \( \mathbb{C}^r \) where \( r \) is the number of conjugacy classes of distinguished reflections. It is easily checked to be integrable and \( W \)-equivariant w.r.t. the diagonal action of \( W \) on \( X \times \mathbb{C} \), where \( W \) acts on \( \mathbb{C} \) through \( \rho_0 \). It thus defines flat connections on the quotient bundle \( (X \times \mathbb{C} \to X)/W \), for which the monodromy is given by \( \rho_0 : B \to \text{GL}(\mathbb{C}) \), with \( \rho_0(\sigma) = \rho_0(s) \exp(2i\pi h_s \rho_0(s)) \), if \( \sigma \) is a braided reflection with \( \pi(\sigma) = s \). There obviously exists a collection of real scalars \( h_s \) such that \( \rho_0(\sigma) = \text{Id} \) for all such distinguished reflections. Since \( B \) is generated by the corresponding braided reflections we get a continuous map \( \mathbb{C}^r \to \text{Hom}(B, \text{GL}(\mathbb{C})) \) connecting the trivial representation and \( \rho_0 \). It follows from proposition 2.1(2) that \( \mathcal{V}(\rho_0) \) is a trivial bundle. \( \square \)

Using more information on the structure of \( B \), it is actually possible to show that any 1-dimensional representation of \( B \) corresponds to a trivial bundle on \( X/W \). Indeed, the abelianization \( B/(B, B) \) is isomorphic to \( \mathbb{Z}^r \).
where \( r \) is the number of classes of hyperplanes in \( \mathcal{A} \) (see \([\text{BMR}]\)). It follows that \( \text{Hom}(B, \text{GL}(\mathbb{C})) \simeq (\mathbb{C}^\times)^r \) is connected, hence any 1-dimensional representation can be deformed to the trivial one.

### 3.2. A remark on holomorphic line bundles.

We emphasize the fact that there is no hope to generalize the results exposed here to arbitrary (non necessarily flat) vector bundles, by noticing the following facts on 1-dimensional holomorphic vector bundles on \( X/W \). First recall that \( X/W \) is a complement of hypersurfaces, hence is affine and in particular is a Stein manifold. Letting \( \mathcal{O} \) denote the sheaf of holomorphic functions on \( X/W \), the exponential exact sequence \( 0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0 \) and Cartan theorem B yields an isomorphism \( H^1(X/W, \mathcal{O}^\times) \to H^2(X/W, \mathbb{Z}) \) given by the first Chern class. On the other hand, \( H^1(X/W, \mathcal{O}^\times) \) classifies rank 1 holomorphic bundles on \( X/W \). In case \( B \) is an ordinary braid group, we have \( H^2(B, \mathbb{Z}) = 0 \) hence all 1-dimensional bundles are holomorphically trivial. This fact however does not extend to arbitrary reflection groups, and the nonvanishing of \( H^2(X/W, \mathbb{Z}) \) in general will illustrate the specificity of the results exposed here.

We first recall the following facts. First of all, \( X/W \) is a \( K(\pi, 1) \), hence \( H^2(X/W, \mathbb{Z}) = H^2(B, \mathbb{Z}) \). Moreover, any such braid group \( B \) is obtained as \( \pi_1(X/W) \) for some reflection group \( W \) with reflections of order 2. Finally, \( H_1(B, \mathbb{Z}) = B/(B, B) \) is torsion-free, hence the universal coefficients exact sequence \( 0 \to \text{Ext}(H_1(B), \mathbb{Z}) \to H^2(B, \mathbb{Z}) \to \text{Hom}(H_j(B), \mathbb{Z}) \to 0 \) implies that \( H^2(B, \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module, of rank \( b_2 = \dim H^2(B, \mathbb{Q}) \). It follows that we only need to compute the second Betti number \( b_2 \) when \( W \) is a 2-reflection group to get \( H^2(X/W, \mathbb{Z}) \) in general.

The Shephard-Todd classification (see \([\text{ST}]\)) divides the irreducible reflection groups into the infinite series \( G(de, e, n) \) and 34 exceptional groups \( G_1, \ldots, G_{37} \). For \( d, e, r \geq 1 \), the group \( G(de, e, r) \) is defined as the group of \( r \times r \) monomial matrices whose nonzero entries lie in \( \mu_{de}(\mathbb{C}) \) and have product in \( \mu_d(\mathbb{C}) \). Inside this infinite series, the groups whose reflections have order 2 are the groups \( W = G(e, e, n) \) and \( W = G(2e, e, n) \).

**Proposition 3.2.** Let \( W \) be an irreducible 2-reflection group with associated braid group \( B \). Then \( H^2(B, \mathbb{Z}) = \mathbb{Z}^{b_2} \) with

- \( b_2 = 2 \) if \( W \) has type \( G_{28} = F_4, G(2e, e, n) \) with \( n \geq 3 \), or \( G(2e, e, 2) \) with \( e \) even.
- \( b_2 = 1 \) if \( W \) has type \( G_{13}, G_{23}, G_{24}, G_{27} \), \( G(e, e, 2) \) with \( e \) even, or \( G(2e, e, 2) \) with \( e \) odd.
- \( b_2 = 0 \) otherwise.

**Proof.** For the infinite series \( G(e, e, n) \) and \( G(2e, e, n) \), we deduce these Betti numbers from \([\text{Le}]\), where they are implicitly computed. More precisely, corollary 6.5 in \([\text{Le}]\) states that the Poincaré polynomial of \( X/W \) is \( 1 + t \) if \( e \) or \( n \) is odd, and \( 1 + t + t^{n-1} + t^n \) is \( e \) and \( n \) are even; for the series \( G(2e, e, n) \), letting \( \hat{Y} = \{ (z_1, \ldots, z_n) \mid z_i \neq 0, z_i/z_j \notin \mu_{2e}(\mathbb{C}) \}/G(2e, 2e, n) \), theorem 6.1 in \([\text{Le}]\) decomposes the rational cohomology groups of \( \hat{Y} \) under the action of \( G(2e, 1, n)/G(2e, 2e, n) \simeq \mu_{2e}(\mathbb{C}) \). In this case \( X/W = \hat{Y}/G(2e, e, n) \) hence \( H^*(X/W, \mathbb{Q}) = H^*(\hat{Y}, \mathbb{Q})^{\mu_{2e}(\mathbb{C})} \) where \( \mu_{2e}(\mathbb{C}) \) is identified
with \( G(2e, e, n)/G(2e, 2e, n) \). The action of \( G(2e, 1, n) \) on \( H^*(\tilde{Y}, \mathbb{Q}) \) factorizes through the projection \( \delta : G(2e, 1, n)/G(2e, 2e, n) = \mu_{2e}(\mathbb{C}) \to \mu_2(\mathbb{C}) \). We have \( \delta(\mu_2(\mathbb{C})) = \{1\} \) if \( e \) is even, and \( \delta(\mu_2(\mathbb{C})) = \mu_2(\mathbb{C}) \) if \( e \) is odd. A consequence of this theorem \( 6.1 \) is thus that the Poincaré polynomial of \( X/W \) is

\[
\begin{cases}
(1 + t)(1 + t + t^2 + \cdots + t^{n-1}) + t^{n-1} + t^n & \text{if } n \text{ and } e \text{ are even} \\
(1 + t)(1 + t + t^2 + \cdots + t^{n-1}) & \text{otherwise.}
\end{cases}
\]

The conclusion follows for the infinite series. For the exceptional groups, using \( H^*(X/W, \mathbb{Q}) = H^*(X, \mathbb{Q})^W \) and the combinatorial description of \( H^*(X, \mathbb{Q}) \), it is shown in [OT] (corollary 6.17) that \( b_2 = \sum_{Z \in T_2} |\mathcal{H}_Z/W_Z| - 1 \) where \( T_2 \) is a system of representatives for the codimension 2 subspaces in the hyperplane arrangement lattice under the action of \( W \) and, for \( Z \in T_2 \), \( \mathcal{H}_Z = \{H \in A \mid H \supset Z\} \) and \( W_Z = \{w \in W \mid w(Z) = Z\} \). Direct computations for all exceptional 2-reflection groups concludes the proof.

3.3. Reflection representations. We consider the diagonal action of \( W \) on \( X \times V \) and the trivial bundle on \( X \times V \to X \) with the corresponding action.

**Proposition 3.3.** The quotient bundle \( (X \times V \to X)/W \) is holomorphically equivalent to the trivial bundle \( (X/W) \times V \to (X/W) \).

**Proof.** Identifying \( V \) with \( \mathbb{C}^n \), the algebra \( \mathbb{C}[V] = S(V^*) \) of polynomial functions on \( V \) is acted upon by \( W \), and the Chevalley-Shephard-Todd theorem (see e.g. [Ben]) states that \( \mathbb{C}[V]^W \) is the algebra of polynomial in \( n \) algebraically independant polynomials \( f_1, \ldots, f_n \). We then define \( f : V \to V \) by \( f = (f_1, \ldots, f_n) \) and consider \( \Psi : V \times V \to V \times V \) defined by \( \Psi(x, v) = (x, d_x f(v)) \), where \( d_x f \in \text{End}(V) \) is the differential of \( f \) at \( x \).

We have \( f(w, x) = x \) for all \( x \in V, w \in W \), hence \( d_{w,x} f(v) = d_x f(w^{-1} v) \).

It follows that \( \Psi(w.(x, v)) = (w.x, d_{w,x} f(w.v)) = (w.x, d_x f(v)) \). On the other hand, \( d_x f \in \text{GL}(V) \) for all \( x \in X \) if and only if the zero locus of \( x \mapsto \det(d_x f) = \det(\frac{\partial f}{\partial x}) \) is contained in \( \bigcup A \). This is well-known to be the case (see e.g. [Ben], §7.3), hence \( \Psi \) defines an holomorphic automorphism of the trivial bundle \( X \times V \to V \) which induces after quotient by \( W \) an holomorphic isomorphism between \( (X \times V \to X)/W \) and \( (X/W) \times V \to (X/W) \). \( \square \)

If \( W \) is the symmetric group \( S_{n+1} \) acting on \( \mathbb{C}^{n+1} = V \times \mathbb{C} \) with \( V = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1 + \cdots + z_{n+1} = 0\} \) then the \( f_1, \ldots, f_n \) can be chosen to be the power sums \( f_k = x_1^k + \cdots + x_{n+1}^k \). In that case, up to a rescaling, the map \( x \mapsto d_x f \) is the Vandermonde spiral of Arnold, and yields the trivialization of \( [ACC] \).

3.4. A lower bound for the dimensions of the representations. We recall that, for \( d, e, r \geq 1 \), the group \( G(de, e, r) \) is defined as the group of \( r \times r \) monomial matrices whose nonzero entries lie in \( \mu_{de}(\mathbb{C}) \) and have product in \( \mu_d(\mathbb{C}) \). In particular \( G(de, e, r) \) contains the symmetric group \( S_r \) of permutation matrices, and we have \( G(de, e, r) = S_r \rtimes D_r \), where \( D_r \) is the subgroup of diagonal matrices in \( G(de, e, r) \). We embed \( G(de, e, r-1) \) in \( G(de, e, r) \) by leaving fixed the last basis vector.
Lemma 3.4. Let $d,e \geq 1$ and $W_r = G(de,e,r)$ with $r \geq 4$. If $p : W_r \to Q$ is a group morphism such that $p(W_{r-1})$ is commutative then $p(W_r)$ is commutative.

Proof. The kernel of $p$ contains the derived subgroup $(\mathfrak{S}_{r-1}, \mathfrak{S}_{r-1})$, which contains a 3-cycle since $r \geq 4$. Since $\text{Ker} p$ is normal it thus contains the alternating group $\mathfrak{A}_r$. In particular $p(\mathfrak{S}_r)$ is commutative, its action on $p(D_r)$ factors through the sign character, and is trivial on $p(D_{r-1})$. Let $\zeta$ be a primitive $de$-th root of 1 and $x = \text{diag}(\zeta^{-1}, 1, \ldots, 1, \zeta)$. It is clear that $D_r$ is generated by $x$ and $D_{r-1}$. But $x$ is conjugate to some element of $D_{r-1}$ by some 3-cycle, hence $p(x) \in p(D_{r-1})$ and $p(D_r) = p(D_{r-1})$. The conclusion follows. □

Lemma 3.5. If $\rho$ is an irreducible representation of $G(de,e,r+1)$ with $\dim \rho > 1$, then its restriction to $G(de,e,r-1)$ is not irreducible.

Proof. By contradiction we assume otherwise. Then its restriction to $G(de,e,r)$ is already irreducible. By [Ma07a] propositions 3.1 and 3.2, in that case $\rho$ can be extended to a representation of $G(de,1,r)$, so we can assume $e = 1$. Since $G(d,1,r+1)$ is a wreath product, $\rho$ is classically associated to a multipartition $(d$-tuple of partitions). By the branching rule for such wreath products, since the restriction to $G(d,1,r)$ is irreducible, the multipartition has only one part, which has the form $[a^b]$ (that is to say, its Young diagram is a rectangle), with $ab = r+1$. Since $\dim \rho > 1$ we have $a,b \geq 2$. But then its restriction to $G(d,1,r-1)$ is not irreducible, because it contains the components labelled by $[a^{b-1}, a-2]$ and $[a^{b-2}, a-1, a-1]$, which are distinct. This contradiction concludes the proof. □

Proposition 3.6. Let $W$ be a finite irreducible pseudo-reflection group of rank $n$. Then every irreducible complex representation $\rho$ of $W$ with $\dim \rho > 1$ has dimension at least $n/2$.

Proof. We use the Shephard-Todd classification and a case-by-case examination to verify that the statement holds for the 34 exceptional groups. We then assume $W = W_r = G(de,e,r)$, and we show that $\dim \rho > 1 \Rightarrow \dim \rho \geq r/2$ by induction on $r$. Since the statement is void for $r \leq 4$ we assume $r \geq 4$, and let $\rho \in \text{Irr}(W_{r+1})$ with $\dim \rho > 1$. This implies that $\rho(W_{r+1})$ is not commutative. By the lemma above the restriction of $\rho$ to $W_{r-1}$ is not irreducible. If one of its component has dimension at least 2, then by the induction hypothesis $\dim \rho \geq 1 + (r-1)/2 = (r+1)/2$. On the other hand, if all the components have dimension 1, then for $r \geq 4$ this would imply that $\rho(W_r)$ and $\rho(W_{r+1})$ are commutative, contradicting $\dim \rho > 1$. This concludes the proof of the proposition. □

4. THE REPRESENTATION RING OF THE SYMMETRIC GROUPS

We denote $\mathfrak{S}_n$ the symmetric group on $n$ letters, naturally embedded on $\mathfrak{S}_{n+1}$ by action on the $n$ first letters. We denote $R_n = R(\mathfrak{S}_n)$. The basis in $R_n$ of irreducible representations is naturally indexed by partitions $\lambda$ of $n$, represented by Young diagrams. We let $V_\lambda$ denote the associated element of $R_n$. 
We let $R_\infty = \bigoplus_{n=1}^{\infty} R_n$, and extend the usual tensor product $\otimes$ on $R_n$ to $R_\infty$ by pointwise multiplication. There is another well-known ring structure on $R_\infty$, given by $V_\lambda \otimes V_\nu = \sum_{\mu} L_{\lambda \mu \nu} V_\mu$, where $L_{\lambda \mu \nu}$ are the well-known Littlewood-Richardson coefficients, and $\nu$ is a partition of arbitrary size. By contrast, the structure constants $V_\lambda \otimes V_\mu = \sum_{\nu} C_{\lambda \mu \nu} V_\nu$, whose study has been initiated by Murnaghan (1938), are notoriously complicated to understand.

A recent and remarkable discovery of Y. Dvir is that these two products are related in the following way. For a partition $\lambda = [\lambda_1, \lambda_2, \ldots]$ with $\lambda_i \geq \lambda_i+1$, of $n$, define the partition $\theta(\lambda) = [\lambda_2, \lambda_3, \ldots]$ of $n-\lambda_1$, and let $d(\lambda) = |\theta(\lambda)| = \lambda_2 + \lambda_3 + \ldots$. It is a classical and elementary fact that $d(\lambda) = \min \{ r \geq 0 \mid V_\lambda \hookrightarrow V^{\otimes r} \}$, where $V = V_{[n-1,1]}$ is the reflection representation of $\mathfrak{S}_n$. In particular $C_{\lambda,\mu,\nu} = 0$ whenever $d(\nu) > d(\lambda) + d(\mu)$. Dvir’s formula can be stated as follows

**Theorem 4.1.** (Dvir [D], theorem 3.3) Let $\lambda, \mu, \nu$ partitions of $n$ such that $d(\lambda) + d(\mu) = d(\nu)$. Then $C_{\lambda,\mu,\nu} = L_{\theta(\lambda),\theta(\mu),\theta(\nu)}$.

A way to interpret this formula is to introduce a filtration $F_n R_\infty$ of the $\mathbb{Z}$-module $R_\infty$ defined by where $F_n R_\infty$ is spanned by all $V_\lambda$ with $d(\lambda) \leq n$. Then $(F_n R_\infty)(F_m R_\infty) \subset F_{n+m} R_\infty$. Considering the induced product on $gr R_\infty$, Dvir formula says that this induced product is basically given by the Littlewood-Richardson rule.

### 4.1. A generating set for $R(\mathfrak{S}_n)$

The ring $(R_\infty, \circ)$ is generated by the elements $V_r$ for $r \geq 0$, which essentially means that the symmetric polynomials are generated by the elementary ones. Dvir formula allows us to derive the following analogous fact.

**Theorem 4.2.** The ring $R(\mathfrak{S}_n)$ is generated by the $\Lambda^k V$ for $0 \leq k \leq n-1$.

**Proof.** Recall that $\Lambda^k V = V_{[n-k,1^k]}$, and notice that $\theta([n-k,1^k]) = [1^k]$. Let $Q$ denote the subring of $R(\mathfrak{S}_n)$ generated by the $\Lambda^k V$. We prove that $V_\lambda \in Q$ for all partition $\lambda$ of $n$ ($\lambda \vdash n$), by induction on $d(\lambda) = |\theta(\lambda)|$. We have $d(\lambda) = 0 \Rightarrow \lambda = [n] \Rightarrow V_\lambda = \Lambda^0 V$ and $d(\lambda) = 0 \Rightarrow \lambda = [n-1,1] \Rightarrow V_\lambda = \Lambda^1 V$, hence $V_\lambda \in Q$ if $d(\lambda) \leq 1$. We thus assume $d(\lambda) \geq 2$.

We need some more notation. For a partition $\alpha$ of $m$, we define the partition $\alpha^0$ by $\alpha_i^0 = \max(0, \alpha_i - 1)$. Clearly, $|\alpha^0| = |\alpha| - \alpha_1'$ hence $|\alpha^0| \leq |\alpha|$, with equality only if $\alpha = 0$.

We let $\alpha = \theta(\lambda)$ and use another induction on $d(\lambda) - \alpha_1' = |\alpha| - \alpha_1'$. The case $d(\lambda) - \alpha_1' = 0$ means $V_\lambda = \Lambda^{\alpha_1'} V \in Q$, so we can assume $d(\lambda) - \alpha_1' \geq 1$.

We let $r = |\alpha| - |\alpha^0|$. Since $d(\lambda) \geq 2$ we have $\theta(\lambda) \neq 0$ and $r \geq 1$. Moreover $\lambda_1 = n - |\alpha| \geq 0$, hence $n - |\alpha^0| \geq 0$. We thus can introduce $\mu = [n-|\alpha^0|, \alpha_1^0, \ldots]$ and consider $M = V_\mu \otimes V' \in R(\mathfrak{S}_n)$. Since $|\alpha^0| < |\alpha|$ we have $d(\mu) < d(\lambda)$ hence $V_\mu \in Q$ and $M \in Q$. Let $\nu \vdash n$ such that $V_\nu \hookrightarrow M$. If $d(\nu) < d(\mu) + r = d(\lambda)$ then $V_\nu \in Q$ by the first induction hypothesis. Otherwise, $\nu_1 = n - |\alpha| = \lambda_1$, and $C_{\mu,[n-r,1-r],\nu} = L_{\alpha_1^0,[1^r],\theta(\nu)}$ by Dvir formula. By the Littlewood-Richardson rule, we have $L_{\alpha_1^0,[1^r],\alpha_1^0} = 1$ and, if $L_{\alpha_1^0,[1^r],\theta(\nu)}$ is nonzero, then either $\theta(\nu_1') > \alpha_1'$, in which case we know that $V_\nu \in Q$ by the second induction hypothesis (as $\nu'_1 = 1 + \theta(\nu)'_1$, or
\theta(\nu) = \alpha. \text{ Hence } M \equiv V_\lambda \text{ modulo } Q, V_\lambda \in Q \text{ and the conclusion follows by induction.} \quad \Box

4.2. Proof of theorem 1.1. The above results are sufficient to give a proof of theorem 1.1 as sketched in the introduction. If \( \rho \) is a representation of \( \mathfrak{g}_n \), it can be split as a direct sum \( \rho_1 \oplus \cdots \oplus \rho_r \) of irreducible representations. As \( \mathcal{V}(\rho) \) is the Whitney sum of the \( \mathcal{V}(\rho_i) \), we can assume that \( \rho \) is irreducible.

If \( \rho \) has dimension 1, the reflection representation \( V_\lambda \), or some alternating power of \( V_\lambda \), then \( \mathcal{V}(\rho) \) is trivial by propositions 3.1, 3.3 and 2.1 (3). In general, \( \Phi(\rho) \) is a polynomial in the \( \Phi(\rho_0) \), for \( \rho_0 \) in the previous list, by theorem 4.2. It follows that \( \Phi(\rho) \) belongs to the subring of trivial bundles. As \( /BV_n^* /S_n \) has the homotopy type of a \((n-1)\)-dimensional CW-complex, if \( \dim \rho_0 \geq (n-1)/2 \), this implies that \( \mathcal{V}(\rho) \) is trivial by 2.1 (1). But this is the case for all irreducible \( \rho \) with \( \dim \rho > 1 \) by proposition 3.6. Since the case \( \dim \rho = 1 \) has been dealt with separately, this concludes the proof of the theorem.

5. Groups of small rank

We let again \( W < \text{GL}(V) \) denote an arbitrary irreducible (pseudo-)reflection group. In this section we prove theorem 1.3. The proof is the same as for theorem 1.1 as soon as we know that \( R(W) \) is generated by a collection of representations \( \rho \) for which \( \mathcal{V}(\rho) \) is trivial. This is the goal of the propositions proved below, which concludes the proof of the theorem.

5.1. Groups of rank 2. The goal of this section is to prove theorem 1.3 for the groups of rank 2, that is \( \dim V = 2 \). We first assume that \( W \) has type \( G(de, e, 2) \). For the classical facts on this group used in the proof, we refer to [AK].

Lemma 5.1. If \( W = G(de, e, 2) \), then \( R(W) \) is generated by \( V \) and the 1-dimensional representations.

Proof. The irreducible representations of \( W \) have dimension at most 2. The ones of dimension 2 can be extended to \( G(de, 1, 2) \), so we can assume without loss of generality that \( e = 1 \). The group \( W \) is generated by \( t = \text{diag}(1, \zeta) \) with \( \zeta = \exp(2i\pi/d) \) and \( s \) the permutation matrix \((1 2)\). Its two-dimensional representations are indexed by couples \((i,j)\) with \( 0 \leq i < j < d \). We extend this notation to \( i, j \in \mathbb{Z} \) with \( j \not\equiv i \mod d \) by taking representatives modulo \( d \) and letting \((i,j) = (j,i)\). A matrix model for the images of \( t \) and \( s \) in the representation \((r, r+k)\) is

\[
 t \mapsto \begin{pmatrix} \zeta^r & 0 \\ 0 & \zeta^{r+k} \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

In particular, \( V = (0,1) \). From these explicit models it is straightforward to check that \((0,1) \otimes (0,1)\) is the sum of \((0,2)\) and 1-dimensional representations, and that \((0,1) \otimes (0,k) = (0,k+1) + (1,k)\). Then we consider the 1-dimensional representation \( \chi_1 : t \mapsto \zeta, s \mapsto 1 \). It is clear that \((i,j) \otimes \chi_1 = (i+1, j+1)\). Letting \( Q \) denote the subring of \( R(W) \) generated by \( V \) and the 1-dimensional representations, through tensoring by \( \chi_1 \) it is enough to show that \((0,k) \in Q\) for all \( 1 \leq k \leq d \). By definition \((0,1) \in Q\), tensoring by \((0,1)\) yields \((0,2) \in Q\), and finally
By proposition 2.1 (2) it follows that
\[
\rho V (0, k) = (0, k + 1) + \chi_1 \otimes (0, k - 1)
\]
proves the result by induction on \(k\).

In the Shephard-Todd classification, exceptional irreducible reflection groups of rank 2 are labelled \(G_n\) for \(4 \leq n \leq 22\). For these exceptional groups, we checked from the character tables that \(R(W)\) is generated by \(V\) and the 1-dimensional representations, using the following naive algorithm: start with \(V\) and the 1-dimensional representations, decompose all possible tensor products, isolate the ones which are the sum of one new irreducible representation (with multiplicity 1) and older ones, add them to the starting list, and repeat the process until all irreducible representations have been obtained. In the sequel, we call this algorithm the \textit{main algorithm}.

Together with the previous lemma, this proves the following.

**Proposition 5.2.** If \(W\) is an irreducible rank 2 reflection group then \(R(W)\) is generated by \(V\) and 1-dimensional representations.

### 5.2. Exceptional groups of higher rank.

Investigating the character tables of higher rank exceptional groups by using the main algorithm yields the following.

**Lemma 5.3.** If \(W\) has type \(G_{24}, G_{25}, G_{26}, G_{33}, G_{35} = E_6\), then \(R(W)\) is generated by the 1-dimensional representations and the alternating powers \(\Lambda^kV\) of the reflection representation.

The case of \(G_{25}\) is specially interesting because, like the rank 2 exceptional groups \(G_4, G_8, G_{16}\), its braid group is an ordinary braid group (here on 4 strands). The remaining one sharing this property is \(G_{32}\). For it, we need an additional trivialization.

We consider first \(W_0 \simeq S_5\) of type Coxeter \(A_4\), and the corresponding spaces \(X_0, X_0/W_0\). We consider the 5-dimensional representation \(\rho_0 : W_0 \to \text{GL}(U)\) labelled by the partition \([3, 2]\), and introduce the 1-parameter family of KZ-connections on \(X \times U \to X\) defined by the integrable 1-form \(\omega = h \sum_s (\text{Id}_U - \rho_0(s)) \omega_s\), choose a basepoint \(\hat{\omega} \in X_0\) and denote \(\rho_h : B_0 \to \text{GL}(U)\) the corresponding representation of the braid group on 5 strands \(B_0 = \pi_1(X_0/W_0)\). We denote \(\sigma_1, \ldots, \sigma_4 \in B_0\) the braided reflections which are the usual Artin generators of \(B_0\). Recall that the \(\sigma_i\) lie in the same conjugacy class, and that transpositions act through \(\rho_0\) on \(U\) by eigenvalues \(1, 1, 1, -1, -1\). For a formal value of the parameter \(h\), it follows that \(\rho_h(\sigma_i)\) is diagonalizable with eigenvalues \(1, 1, 1, -e^{2\pi i h}, -e^{2\pi i h}\) (see e.g. [Ma07b proposition 2.3]). In particular \(\rho_h(\sigma_i)\) is annihilated by \((X - 1)(X + e^{2\pi i h})\) for all \(h \in \mathbb{C}\), hence is semisimple for \(2h - 1 \notin 2\mathbb{Z}\). Moreover its spectrum is determined by the collection \(\text{tr} \rho_h(\sigma^k_i)\) for all \(k\). Since \(\text{tr} \rho_h(\sigma^k_i) = 3 - 2e^{2\pi i h k}\) in \(\mathbb{C}((h))\), the equality also holds for \(h \in \mathbb{C}\).

In particular, for \(h = 5/6\) we get that \(\rho_h(\sigma_i)\) has eigenvalues \(1, 1, 1, j, j\) with \(j = \exp(2i\pi/3)\). The group \(G_{32}\) is a quotient of \(B_0\) by the relation \(\sigma_1^3 = 1\), hence \(\rho_{5/6}\) factorizes through \(G_{32}\). Since \(B_0\) is isomorphic to the braid group \(B\) of \(W = G_{32}\) (see [BMR]), and \(X/W\) as well as \(X_0/W_0\) are \(K(\pi, 1)\), it follows \(X/W\) and \(X_0/W_0\) are homotopically equivalent. By theorem 1.1 the vector bundle \(\mathcal{V}(\rho_0)\) is trivial over \(X_0/W_0\) hence over \(X/W\). By proposition 2.1 (2) it follows that \(\mathcal{V}(\rho_{5/6})\) is also trivial over \(X/W\).
In order to use the character table, we now identify $\rho_{5/6}$ among the representations of $G_{32}$, and denote $\chi_h = \text{tr} \rho_h$. We know $\chi_{5/6}(1) = 5$ and $\chi_{5/6}(s) = 3 + 2j = 2 + \sqrt{-3}$ for $s$ a distinguished reflection. Let now $\beta = (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^5 \in Z(B)$, which corresponds to the loop $t \mapsto xe^{2i\pi t}$. We have $\rho_h(\beta) = \exp(2i\pi h \sum_s (\text{Id} + \rho(s)))$ where the sum runs over all transpositions $s \in S_5$. Since the sum of these transpositions is central in $Z(CS_5)$, one readily gets $\sum_s (\text{Id} + \rho(s))) = 8\text{Id}$ and $\rho_{5/6}(\beta) = j^2$. The irreducible characters of $G_{32}$ of degree at most 5 are either 5-dimensional, 1-dimensional given by $\sigma_i \mapsto j^k$, or 4-dimensional. The value of these 4-dimensional characters on the distinguished reflections of $G_{32}$ belong to $\{-1, -j, -j^2\}$. The only possibility is thus that $\rho_{5/6}$ is irreducible, and there is only one 5-dimensional irreducible character satisfying the above conditions (labelled $\phi_{5,4}$ in the computer system GAP3/CHEVIE).

Launching the main algorithm yields

**Proposition 5.4.** For $W$ of type $G_{32}$, $R(W)$ is generated by the 1-dimensional representations, $V$ and $\rho_{5/6}$.

We now let $W = G_{23} = H_3$. The representation $U = S^2 V - I$ is irreducible, and $S^4 V = I + 2U + X$, with $X$ an irreducible 4-dimensional representation. Using the main algorithm we check

**Proposition 5.5.** For $W = G_{23} = H_3$, $R(W)$ is generated by the 1-dimensional representations, $V$ and $X$.

We let $W = G_{30} = H_4$. Launching the main algorithm on the 1-dimensional representations and the $\Lambda^k V$ provides 25 irreducible representations (among the 34 existing ones) inside the subalgebra $Q$ of $R(W)$ that they generate. We then consider the representations $S_\lambda(V)$ for $\lambda$ a partition of size at most 8, and consider the submodule spanned by their characters and the ones of these 25 representations. Launching the LLL algorithm as implemented in the computer system GAP3 on these characters shows that this submodule actually contains all the 34 irreducible representations of $W$. This proves the following.

**Proposition 5.6.** If $W$ has type $G_{30} = H_4$, then $R(W)$ is generated by the $1$-dimensional representations and the representations $S_\lambda(V)$, for $|\lambda| \leq 8$.

5.3. **Groups of Coxeter type $B$.** We investigate the types $B_n$ for $n$ small, as a good example of complications arising already for close relatives of the symmetric group. The rank 2 groups having been dealt with before, we start with $n = 3$. In that case it is easily checked that $R(W)$ is generated by the 1-dimensional representations and the $\Lambda^k V$, for instance by using the main algorithm.

This is not true anymore for $n = 4$. In that case, it can be checked that the subring generated by these representations has dimension over $Q$ equal to 18, to be compared to the 20 irreducible representations of $W$. A fortiori, $R(W)$ is not generated over $\mathbb{Z}$ by these representations.

We nevertheless show that $V(\rho)$ is a trivial bundle for all representations $\rho$ of $B_4$. For this, we first launch the main algorithm on the 1-dimensional representations and the $\Lambda^k V$. We get 10 irreducible representations (among
The submodule $M$ spanned by them and the $S^kV$ for $2 \leq k \leq 8$ contains 4 additional ones, found by the LLL algorithm. We did not succeed in improving this bound using other algebraic relations and Schur functors. However, we can use a result from [Ma08] saying that the permutation representation of $W$ associated to its action by conjugation on the reflections, denoted $R_0$ in [Ma08], is connected by a path in the representation variety of $B$ to another representation $R_1$. It follows that $\Phi(R_0) = \Phi(R_1)$. By computing the characters we get $R_1 \in M$, and we find that $R(W)$ is spanned by $M$ and $R_1 - R_0$, hence

**Proposition 5.7.** If $W$ is of type $B_n$, $n \leq 4$, then $\Phi(\rho)$ lies inside the subgroup of trivial bundles for any representation $\rho$ of $W$.

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