Focusing properties and focal shift of a vortex cosine-hyperbolic Gaussian beam

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Abstract
In this paper, we investigate the focusing properties of a vortex-cosh-Gaussian (vChG) beam passing through a converging thin lens. Based on the Huygens–Fresnel diffraction integral, we derived the analytical propagation equation as well as the beam width expression of a focused vChGB. It is shown that the focusing properties including the focal shift of the focused vChGB are crucially dependent on the incident beam parameters namely the decentered parameter and the topological charge in addition to the Gaussian Fresnel number $N_F$. From typical numerical examples, it is found that the focused vChGB is transformed into a multi-lobes structure shape at the real focus plane, and the principal maximum intensity of the beam is located away from the axis. The amount of focal shift, which is determined from the minimum spot size criterion, is strongly dependent on the Fresnel number, the decentered parameter and the vortex charge $m$. The obtained results may be useful for the applications of the vChGBs in beam shaping and beam focusing.

Keyword Vortex-cosh-Gaussian beam · Vortex hollow beams · Focal shift · Beam focusing

1 Introduction
In recent years, the beam focusing has received a lot of attention from the lasers researchers for its theoretical and practical interests. It is established that when a light wave is focused by a converging thin lens, the on-axis irradiance maximum or the minimum spot size is not in general located at the geometric focus but is shifted toward the lens; this is the so-called focal shift. This phenomenon plays a crucial role in optics due to the need of determining accurately the real focal plane in practical applications. The focal shift was reported and analyzed for the Gaussian beam by Li and Wolf (1981, 1982), after that, a lot of papers have examined the focal shift in various types of beams focused by unapertured or apertured converging systems (Lü and Huang 1994; Borghi et al. 1998; Green and Hall 1999;...
Hricha et al. 2003; Hricha and Belafhal 2005a, b; Wang et al. 2012; Liu and Pu 2009a). In all the cited papers, it is found that the focal shift is strongly dependent on the Fresnel Number, the aperture size and the parameters of the incident beams.

On the other side, the so-called dark hollow beams have been studied a lot for their significant applications, like in atomic optics, manipulation of particles and wireless communications. During the last few years, various model beams have been presented to describe dark hollow beams (Kuga et al. 1997; Cai et al. 2003; Yin et al. 2003; Wang et al. 2004; Mei and Zhao 2005, 2006a, b, 2008; Cai and Ge 2006; Liu et al. 2014; Zeng et al. 2018a; Yaalou et al. 2019a, b). It is known that a dark hollow beam can give rise to a vortex beam when it passes for instance through a spiral phase plate, the generated beam will possess a helical phase front and may carry the orbital angular momentum. Various vortex beams have been described and their propagation characteristics have been studied (Zhou et al. 2013; Luo et al. 2014; Liu et al. 2016, 2017; Zeng et al. 2018b). The focal shift in a vortex beam depends obviously on the Fresnel number but it may also be affected by the vortex charge of the beam. In laser beams literature, unfortunately, there is little studies on the focal shift effect in vortex hollow beams (Liu and Pu 2009b; Zhao et al. 2018; Lan and Li 2020; Li et al. 2020).

Therefore, the present work is aimed at investigating the focusing characteristics and the focal shift behavior of a vortex cosh-Gaussian beam passing through a converging unper- tured lens. The reminder of the paper is organized as follows: in Sect. 2, the field distribution of the vortex ChGB is described with graphical illustrations. Then, in Sect. 3, the analytical expressions of the vChGB focused by a converging lens as well as the beam width are derived in detail by using the Huygens–Fresnel diffraction integral and the second-order moments definition. The intensity distribution characteristics and the focal shift of the focused vChGB are discussed with numerical examples in Sect. 4. The main results are summarized in the conclusion part.

## 2 Field distribution of vChGBs

In the Cartesian coordinates system, the z-axis is taken to be the propagation direction. The electric field of a vChGB in the source plane \( z = 0 \) can be expressed as

\[
E_m(x_0, y_0, z = 0) = (x_0 + iy_0)^m \frac{\cosh \left( b_x \frac{x_0}{\omega_0} \right)}{\cosh \left( b_y \frac{y_0}{\omega_0} \right)} \exp \left\{ -\left( \frac{x_0^2 + y_0^2}{\omega_0^2} \right) \right\},
\]

(1)

where \((x_0, y_0)\) are the transverse coordinates of an arbitrary point at the source plane, and \(\omega_0\) is the beam waist radius of the Gaussian beam. \(b_x\) and \(b_y\) are the decentered parameters of the \(\cosh(\cdot)\) part along the \(x\)- and \(y\)-directions, respectively. \(m\) being an integer which denotes the topological charge of the vortex.

It is obvious that when \(b_x = b_y\), Eq. (1) gives \(xy\)-symmetrical vortex cosh-hyperbolic Gaussian beam defined firstly in Hricha et al. (2020). Preliminarily illustrations of the vChGB in the initial plane are presented in Fig. 1, where we depicted the intensity distribution of the beam for different values of \(m\) and \((b_x, b_y)\). In all the following calculations, the parameter \(\omega_0\) is taken to be 1 mm.

From the plots of Fig. 1, it is shown the vChGB pattern depends on both the values of \((b_x, b_y)\) and the topological charge \(m\). The field is hollow-Gaussian like (doughnut shape) when the values of \(b_x\) and \(b_y\) are small, say \(b_x, b_y < 1\) (see the top row plots). In this case, the

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dark central region increases with increasing $m$ and so does the beam spot width. For large values of $b_x$ (or $b_y$) the beam possesses two lobes in the x- (or y-) direction. Especially when $b_x$ and $b_y$ are both large, the beam is four petal-like. Besides, one can note (see the bottom row plots) that the lobes are more elongated with increased $m$ and the inter-lobes spacing increases as $b_x$ or $b_y$ are increased.

Fig. 1 Intensity profile of the incident vortex Cosh Gaussian beam at the source plane $z=0$ for different values of $b_x$, $b_y$ and $m$ with $w_0 = 1 \text{mm}$
3 Focusing of a vChG beam by a converging lens

Now, let us consider an incident vChGB passing through an apertureless thin lens, as schematized in Fig. 2. Assuming that the incident beam is located at the lens plane, the transfer matrix for the optical system can be expressed as

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
-\Delta_z/f & f + \Delta_z \\
-1/f & 1
\end{pmatrix},
\]

where \(f\) is the focal length of the lens, and \(z\) is the distance from the lens plane to the output plane. \(\Delta z = z - f\) is the related distance from the geometrical focus to the output plane. \(A, B, C, D\) are the elements of the \(ABCD\) matrix associated with the optical system.

The propagation of the vChGB through a thin lens can be described in the paraxial approximation according to the Huygens Fresnel diffraction integral as (Collins 1970)

\[
E(x, y, z) = \frac{ik}{2\pi B} \exp(-ikz) \int_0^\infty \int_0^\infty E_0(x_0, y_0, 0) \exp \left\{ -\frac{ik}{2B} \left[ A(x_0^2 + y_0^2) - 2(xx_0 + y_0D(x^2 + y^2) \right] \right\} dx_0 dy_0,
\]

where \(E(x, y, z)\) is the field at the receiver plane which is located at the plane \(z\), \(k\) is the wave number related to the wavelength \(\lambda\) by \(k = 2\pi/\lambda\).

On substituting from Eqs. (1) and (2) into (3), and recalling the binomial formula (Hricha et al. 2020)

\[
(x_0 + iy_0)^m = \sum_{l=0}^{m} C_l^m x_0^l y_0^{m-l},
\]

where

\[
C_l^m = \frac{m!}{l!(m-l)!},
\]

then by making some algebraic operations, Eq. (3) can be expressed as

![Fig. 2 Schematic illustration of the focusing system](image)
\[ E(x, y, z) = \frac{ik}{2\pi z} \exp(-ikz) \exp \left\{ \frac{-ik}{2z} \left(x^2 + y^2\right) \right\} \sum_{l=0}^{\infty} C_l^n (i)^{n-l} F_l(x) F_{n-l}(y), \quad (5a) \]

where

\[ F_n(u) = \int_{-\infty}^{+\infty} u^n_0 \cosh \left( b_u \frac{u_0}{\omega_0} \right) \exp \left\{ -\left( \frac{1}{\omega_0^2} + \frac{ik(1 - z/f)}{2z} \right) u_0^2 + \frac{ik u u_0}{z} \right\} du_0, \quad (5b) \]

with \( u = x \) or \( y \) and \( n \) is integer.

Using the definition of \( \cosh(.) \) function, Eq. (5b) can be rewritten as

\[ F_n(u) = \frac{1}{2} \left[ G_n^+(u) + G_n^-(u) \right] \quad (6a) \]

where

\[ G_n^\pm(u) = \int_{-\infty}^{+\infty} u^n_0 \exp \left( -\eta u_0^2 + \left( \frac{ik}{z} u \pm \frac{b_u}{\omega_0} \right) u_0 \right) du_0, \quad (6b) \]

and the auxiliary parameter \( \eta \) is defined as

\[ \eta = \frac{1}{\omega_0^2} + \frac{ik(1 - z/f)}{2z}. \quad (6c) \]

Now, by recalling the integral formula (Gradshteyn and Ryzhik 1994; Belafhal et al. 2020)

\[ \int_{-\infty}^{+\infty} x^n \exp(-px^2 + 2qx) dx = \sqrt{\frac{\pi}{p}} \exp \left( \frac{q^2}{p} \right) \left( \frac{1}{2i\sqrt{p}} \right)^n H_n \left( \frac{iq}{\sqrt{p}} \right), \quad (7) \]

where \( H_n(.) \) is the Hermite polynomial of nth-order, the integral of Eq. (6b) becomes

\[ G_n^\pm(u) = f^\pm(u) \sqrt{\frac{\pi}{\eta}} \left( \frac{1}{2i\sqrt{\eta}} \right)^n H_n \left( \frac{i}{2\sqrt{\eta}} \left( \frac{ik}{2z} u \pm \frac{b_u}{\omega_0} \right) \right), \quad (8a) \]

where

\[ f^\pm(u) = \exp \left\{ \frac{1}{4\eta} \left( \frac{ik}{2z} u \pm \frac{b_u}{\omega_0} \right)^2 \right\}. \quad (8b) \]

On substituting from Eqs. (8a) and (6a) into (5a), we obtain
\[ E(u, v, s) = \frac{i\pi N_F}{2m+2\beta} \left( \frac{s}{\beta} \right)^{m/2} \exp \left\{ \frac{s}{4\beta} \left( \frac{b_x^2 + b_y^2}{2} \right) - \frac{i\pi N_F(1 - i\pi N_F)}{\beta}(u^2 + v^2) \right\} \times \sum_{l=0}^{m} C_l^{(i)}(-l)^{-l} \left[ \exp \left( \frac{i\pi b_y N_F}{\beta} u \right) H_l \left( \frac{1}{2\sqrt{s\beta}} (ib_y - 2\pi N_F u) \right) + \exp \left( \frac{-i\pi b_y N_F}{\beta} v \right) H_{m-l} \left( \frac{1}{2\sqrt{s\beta}} (-ib_y - 2\pi N_F v) \right) \right] \] (9a)

where \( N_F = w_0^2/\lambda f \) is the Gaussian Fresnel number, \( s = z/f \) is the reduced propagation distance, \((u, v)\) denote the normalized coordinates, \( u = x/\alpha_0 \) and \( v = y/\alpha_0 \), and

\[ \beta = s + i\pi N_F(1 - s). \] (9b)

In the limiting case \( m=0 \) (i.e. in the absence of the vortex) and \( b_x = b_y = b \), Eq. (9a) reduces to

\[ E(u, v, s) = \frac{i\pi N_F}{\beta} \exp \left\{ \frac{s}{2\beta} b^2 + Q(u^2 + v^2) \right\} \cosh (Su) \cosh (Sv), \] (10a)

where

\[ S = \frac{i\pi b N_F}{\beta}, \] (10b)

and

\[ Q = -\frac{i\pi N_F(1 - i\pi N_F)}{\beta}. \] (10c)

Equation (10a) is the propagation equation of the focused standard hyperbolic-cosine-Gaussian beam (ChGB) which is consistent with Eq. (7a) of Ref. (Hricha and Belafhal 2005a).

If we take in addition \( b_x = b_y = 0 \) (with \( m=0 \)), Eq. (10a) simplifies to

\[ E(u, v, s) = \frac{i\pi N_F}{\beta} \exp \left( Q(u^2 + v^2) \right), \] (11)

which is the formula of the focused Gaussian beam (Li and Wolf 1981, 1982).

As is well known, the irradiance of the focused vChG beam is expressed as

\[ I(u, v, s) = |E(u, v, s)|^2. \] (12)

So, by substituting from Eq. (9a) into Eq. (12), and after some algebraic operations, we obtain the irradiance expression of the focused vChGB as
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\[ I(u, v, s) = I_0(s) \exp \left\{ \frac{-2 \left( \frac{\pi N_F}{s} \right)^2 (u^2 + v^2)}{1 + \left( \frac{\pi N_F}{s} (s - 1) \right)^2} \right\} \]

\[ \times \sum_{i=0}^{m} C_i^{m} (-i)^{-n} \sum_{n=0}^{m} \left[ \exp \left( 2S_{1, u} H_l(\delta (u - \alpha_s)) H_n(\delta^* (u + \alpha_s)) \right) + \exp \left( -2S_{2, u} H_l(\delta (u + \alpha_s)) H_n(\delta^* (u - \alpha_s)) \right) \right] \times \left[ \exp \left( 2S_{1, v} H_{m-l}(\delta (v - \alpha_s)) H_{m-n}(\delta^* (v + \alpha_s)) \right) + \exp \left( -2S_{2, v} H_{m-l}(\delta (v + \alpha_s)) H_{m-n}(\delta^* (v - \alpha_s)) \right) \right] \]

(13a)

where

\[ I_0(s) = \frac{(\pi/4)^2 s^m (w_0/2)^{2m}}{\left[ (s/N_F)^2 + (\pi(s-1))^2 \right] \left[ (s)^2 + (\pi N_F (s - 1))^2 \right]^{m/2}} \]

\[ \times \exp \left\{ \frac{\left( \frac{b_x^2}{b_y^2} + \frac{b_y^2}{b_x^2} \right)}{2 \left[ 1 + \left( \frac{\pi N_F}{s} (s - 1) \right)^2 \right]} \right\} \]

(13b)

with \( S_{1i}, S_{2i}, \delta, \) and \( \alpha_i \) are given, respectively by

\[ S_{1i} = \frac{\pi^2 b_i (1 - s)}{\left[ (s/N_F)^2 + (\pi(s-1))^2 \right]} \]

(13c)

\[ S_{2i} = \frac{i \pi b_i N_F s}{\left[ s^2 + (\pi N_F (s - 1))^2 \right]} \]

(13d)

\[ \delta = \frac{\pi N_F}{\sqrt{s (s + i \pi N_F (1 - s))}} \]

(13e)

and

\[ \alpha_i = \frac{ib_i}{2\pi N_F}, i = (x or y). \]

(13f)
Equation (13a) expresses the irradiance distribution of the focused vChGB as a function of the beam parameters namely the vortex charge $m$ and the decentered parameters $(b_x, b_y)$.

Thus, the irradiance distribution along $z$-axis is obtained by substituting $(u, v) = (0, 0)$ into Eq. (13a), that is, we get

$$I(0, 0, s) = I_n(s) \times \sum_{l=0}^{m} \sum_{n=0}^{m} C_l^m C_n^m (-i)^{-l-n} H_l(\delta \alpha, s) H_{m-l}(\delta \alpha, s)$$

\[\begin{bmatrix}
(-1)^l H_l(\delta \alpha, s) + (-1)^{l+n} H_n(\delta \alpha, s) \\
+H_n(\delta \alpha, s) + (-1)^n H_m(\delta \alpha, s)
\end{bmatrix}
\]

It is readily seen from Eq. (14) that the intensity at center of the focused vChG beam is not strictly null, i.e. the beam may lose its initial central hole-intensity upon focusing under particular values of $m$ and $(b_x, b_y)$. To further investigating the propagation properties of focused beam, we examine the evolution of the beam width during propagation in the focal region.

As it is well-known, within the framework of the intensity moment theory the root mean square width of a beam is given as (Siegman 1986)

$$W_p = \left( \frac{4}{P_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho^2 |E(x, y, z)|^2 \, dx \, dy \right)^{1/2}$$

(15a)

where $\rho$ denotes $x$ or $y$ transverse coordinate, and $P_0$ denotes the total power of the beam which is given by

$$P_0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |E(x, y, z = 0)|^2 \, dx_0 \, dy_0$$

(15b)

Substituting from Eq. (1) into Eq. (15b) leads to

$$P_0 = \sum_{l=0}^{m} C_l^m I_l(b_x) I_{m-l}(b_y) ,$$

(15c)

where the typical integral expression $F_p(b_i)$ is defined as

$$F_p(b_i) = \int_{-\infty}^{+\infty} u_0^{2p} \cosh^2 \left( \frac{b_i u_0}{\omega_0} \right) \exp \left( -\frac{2u_0^2}{\omega_0^2} \right) \, du_0$$

(15d)

with $i = x$ or $y$.

By applying the integral formula of Eq. (7), and after straightforward algebraic calculations, $F_p(b_i)$ can be expressed as

$$F_p(b_i) = \frac{1}{2} \sqrt{\pi} \left( \frac{1}{2i} \right)^{2p} \left( \frac{w_0}{\sqrt{2}} \right)^{2p+1} \left[ \exp \left( \frac{i b_i}{2} \right) H_{2p} \left( \frac{i b_i}{\sqrt{2}} \right) + H_{2p}(0) \right].$$

(15e)

Because of the symmetry of the vChGB, the second-order moment beam width in $x$- and $y$- directions must be identical. Thus, for convenience, one can only determine the intensity
moments in x-direction. The evaluation of the beam width by using directly Eq. (9a) seems to be too complicated, fortunately, we can achieve this by the ABCD transformation law of the second-order moments which is given by Weber (1992)

\[
\langle x^2 \rangle = A^2 \langle x_0^2 \rangle + 2AB\langle x_0 \theta_{x_0} \rangle + B^2 \langle \theta_{x_0}^2 \rangle
\]  

(16)

where \( x \) and \( x_0 \) are the transverse coordinates in the output plane \( z \) and input plane \( z=0 \), respectively. At the input plane \( z=0 \), the second-order intensity moments are given by Weber (1992)

\[
\langle x_0^2 \rangle = \frac{1}{P_0^4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| E(x_0, y_0, z = 0) \right|^2 \, dx_0 \, dy_0
\]

(17a)

and

\[
\langle x_0 \theta_{x_0} \rangle = \frac{1}{2ikP_0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_0 E(x_0, y_0, z = 0) \frac{\partial E^*(x_0, y_0, z = 0)}{\partial x_0} \, dx_0 \, dy_0 + c.c.,
\]

(17b)

Therefore, for our lens system, we obtain

\[
\langle x^2 \rangle = (s - 1)^2 \langle x_0^2 \rangle + 2 \frac{\nu_0^2}{\lambda N_F} s(1-s)\langle x_0 \theta_{x_0} \rangle + \frac{\nu_0^4}{\lambda^2 N_F^2} s^2 \langle \theta_{x_0}^2 \rangle
\]

(18)

Substituting from Eq. (1) into Eq. (17a), and after some algebraic manipulations, one obtains

\[
\langle x_0^2 \rangle = \frac{1}{P_0^4} \sum_{i=0}^{m} C_i^m F_{i+1}(b_x) F_{m-i}(b_y).
\]

(19)

The application of Eq. (17b) for the incident vChGB gives the value zero, \( \langle x_0 \theta_{x_0} \rangle = 0 \). Substituting from Eq. (1) into Eq. (17c) and after an algebraic rearrangement, the integral expression reads

\[
\langle \theta_{x_0}^2 \rangle = -\frac{1}{P_0^4 k^2} [Q_1 + 2Q_2 + Q_3 + 2Q_4 + 2Q_5 + Q_6]
\]

(20a)

where

\[
Q_1 = \left( \frac{b_x}{a_{y_0}} \right)^2 \sum_{i=0}^{m} C_i^m \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_0^{2i} y_0^{2m-2i} \cosh^2 \left( \frac{y_0}{a_{y_0}} \right) \cosh^2 \left( \frac{y_0}{a_{y_0}} \right) \exp \left( -\frac{2(x_0^2 + y_0^2)}{a_{y_0}^2} \right) \, dx_0 \, dy_0.
\]

(20b)

\[
Q_2 = \left( \frac{b_y}{a_{x_0}} \right)^2 \sum_{i=0}^{m} C_i^m \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_0^{2i+1} y_0^{2m-2i} \cosh \left( \frac{x_0}{a_{x_0}} \right) \sinh \left( \frac{x_0}{a_{x_0}} \right) \cosh \left( \frac{x_0}{a_{x_0}} \right) \exp \left( -\frac{2(x_0^2 + y_0^2)}{a_{x_0}^2} \right) \, dx_0 \, dy_0.
\]

(20c)
\[ Q_3 = \left( \frac{2}{\omega_0^2} \right)^2 \left( \frac{2}{a_0^2} \right)^m \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{+\infty + 2\nu_0 \gamma_{2\nu_0}} \cos^2 \left( b_y \frac{y_0}{a_0} \right) \exp \left( \frac{-2(x_0^2 + y_0^2)}{\omega_0^2} \right) dx_0 dy_0 \]  

\[ Q_4 = m \left( \frac{b_x}{a_0} \right)^{m-1} \sum_{m=0}^{m-1} C_i^{m-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{+\infty + 2\nu_0 \gamma_{2\nu_0}} f(x_0, y_0) \cos^2 \left( b_x \frac{x_0}{a_0} \right) \exp \left( \frac{-2(x_0^2 + y_0^2)}{\omega_0^2} \right) dx_0 dy_0 \]  

\[ Q_5 = e^{-2m} m^{-2} \sum_{m=0}^{m-2} C_i^{m-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{+\infty + 2\nu_0 \gamma_{2\nu_0}} g(x_0, y_0) \cos^2 \left( b_x \frac{x_0}{a_0} \right) \exp \left( \frac{-2(x_0^2 + y_0^2)}{\omega_0^2} \right) dx_0 dy_0 \]  

\[ Q_6 = m(m - 1) \sum_{m=0}^{m-1} C_i^{m-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{+\infty + 2\nu_0 \gamma_{2\nu_0}} h(x_0, y_0) \cos^2 \left( b_x \frac{x_0}{a_0} \right) \exp \left( \frac{-2(x_0^2 + y_0^2)}{\omega_0^2} \right) dx_0 dy_0, \]  

with

\[ f(x_0, y_0) = (x_0 + iy_0) \sinh \left( b_x \frac{x_0}{a_0} \right) \]  

\[ g(x_0, y_0) = (x_0 + iy_0)^2 \cosh \left( b_x \frac{x_0}{a_0} \right) \]  

\[ h(x_0, y_0) = (x_0 + iy_0) \cosh \left( b_x \frac{x_0}{a_0} \right) \]  

Equations 20(b–g) are derived by using the same calculation technique used above, and the results can be expressed as

\[ Q_1 = \left( \frac{b_x}{w_0} \right)^2 P_0, \]  

\[ Q_2 = \left( \frac{-2}{w_0^2} \right) \left( \frac{b_x}{w_0} \right) \sum_{m=0}^{m} C_i^{m} G_i(b_x) F_{m-1}(b_y), \]  

\[ Q_3 = \frac{2P_0}{w_0^2} \left[ \frac{2}{w_0^2} \langle x_0^2 \rangle - 1 \right] \]
\[ Q_4 = \frac{m b_x}{w_0} \sum_{l=0}^{m-1} C_l^{m-1} G_l(b_x) F_{m-l-1}(b_y), \]
\[ Q_5 = \frac{-2m}{w_0^2} \sum_{l=0}^{m-1} C_l^{m-1} F_{l+1}(b_x) F_{m-l-1}(b_y), \]

and

\[ Q_6 = m(m-1) \sum_{l=0}^{m-2} C_l^{m-2} [F_{l+1}(b_x) F_{m-l-2}(b_y) - F_{l}(b_x) F_{m-l-1}(b_y)] \]

where \( F_p(b) \) is given by Eq. (15e) and \( G_p(b_i) \) is given by

\[ G_p(b_i) = \frac{1}{2} \sqrt{\pi} \left( \frac{1}{2i} \right)^{2p+1} \left( \frac{w_0}{\sqrt{2}} \right)^{2p+2} \exp \left( \frac{b_i^2}{2} \right) H_{2p+1} \left( \frac{ib_i}{\sqrt{2}} \right) \]

**Fig. 3** Contour lines of the intensity of focused vChGB along z-axis \( w_0 = 1 \text{ mm} \) \( N_F = 1 \) and \( m=4 \). a \( b_x = b_y = 0.1 \) b \( b_x = 0.1, \ b_y = 4 \) c \( b_x = 4, \ b_y = 0.1 \) and d \( b_x = b_y = 4 \)
4 Numerical analysis and discussion

Based on the analytical expressions obtained above, several numerical calculations were performed to illustrate the focusing characteristics and focal shift of the vChGB as a function of the involved parameters. Figure 3 presents the evolution of the irradiance distribution (contour line) of the beam along z-axis. It is seen from the plots of Fig. 3 that the irradiance distribution is asymmetric about the plane \( z = f \), and the beam power is concentrated slightly before the lens focal plane. It is noteworthy that when the values of \( b_x \) and \( b_y \) are small, the on-axis intensity vanishes (see plots (a)) in contrast with the other cases corresponding to intermediate or large values of \( b_x \) and \( b_y \) when there is a central light spot.

Figure 4 shows the evolutions of the beam width \( W_x(z) \) against relative axial distance \( z/f \) with \( w_0 = 1 \text{ mm} \) and \( N_F = 1 \) for

- (a) \( b_x = b_y = 0.1 \)
- (b) \( b_x = 0.1, b_y = 4 \)
- (c) \( b_x = 4, b_y = 0.1 \)
- (d) \( b_x = b_y = 4 \)

\[ \Delta z = z_m/f - 1 = s - 1 \]

Figure 4c–d shows that for small values of \( b_x \) and \( b_y \), the focal shift \( \Delta z = z_m/f - 1 = s - 1 \) decreases as \( m \) is increased. Whereas for large values of \( b_x \) and \( b_y \), the focal shift almost conceals out for all values of \( m \) (see Fig. 4c–d).
Figure 5 gives the irradiance contour lines of the beams at the real focal plane \( z_m \) with \( w_0 = 1 \text{ mm} \) and \( N_F = 1 \) for different values of \( b_x, b_y \) and \( m \).

As can be seen the irradiance distribution presents multi-lobes structures which are symmetric about the \( x \)- and \( y \)-axes. The irradiance pattern depends strongly on the parameters \( m \) and \((b_x, b_y)\). The on-axis intensity of the focused beam is strictly null when \( b_x \) and \( b_y \) are small, whereas when \( b_x \) and \( b_y \) are large the on-axis intensity can be non-null depending on the value of \( m \). In addition, we can see from the same figure that the principal maximum intensity is located away from the axis, its position depends on the beam parameters.

The dependence of the focal shift on the Fresnel number is depicted in Fig. 6, from which we see that \(|\Delta z|\) decreases as \( N_F \) increases and it vanishes asymptotically for large \( N_F (N_F > 4) \). In addition one can note that the focal shift is larger with smaller \((b_x, b_y)\) or smaller \( m \).
5 Conclusion

In summary, based on the Huygens–Fresnel diffraction integral, we have derived the analytical expressions of the irradiance distribution and the beam width for a focused vChGB. The three-dimensional intensity and the beam width in the focal region are analyzed numerically as a function of the beam parameters and the Fresnel number. It is found that the intensity distribution pattern of the focused vChGB in near focal region is determined by the decentered parameters \((b_x, b_y)\) and the vortex charge, and the amount of the focal shift decreases with increasing the Fresnel number \(N_F\) the vortex charge for small values of \((b_x, b_y)\). For large values of \((b_x, b_y)\), the focal shift is insensitive to the vortex charge value. Our analytical and numerical results may be useful for applications in beam shaping and beam focusing.

Fig. 6 Variation of the relative focal shift \(\Delta z\) against the Fresnel number \(N_F\) for a \(m = 0\), b \(m = 1\), c \(m = 2\) and d \(m = 4\)
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