NEAR OPTIMAL $L^p \to L^q$ ESTIMATES FOR EUCLIDEAN AVERAGES OVER PROTotypical HYpERSURFACES IN $\mathbb{R}^3$

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ABSTRACT. We find the precise range of $(p, q)$ for which local averages along graphs of a class of two-variable polynomials in $\mathbb{R}^3$ are of restricted weak type $(p, q)$, given the hypersurfaces have Euclidean surface measure. We derive these results using non-oscillatory, geometric methods, for a model class of polynomials bearing a strong connection to the general real-analytic case.

1. INTRODUCTION

In this article we establish the sharp range of restricted weak type $(p, q)$ inequalities for averaging operators of the form

$$(1.1) \quad T f(x) := \int_{[-1,1]^2} f(x' - t, x_3 - \varphi(t)) dt, \quad x = (x', x_3) \in \mathbb{R}^3,$$

when $\varphi$ is a mixed homogeneous polynomial, satisfying $\varphi(\sigma^{\kappa_1} t_1, \sigma^{\kappa_2} t_2) = \sigma \varphi(t)$ for all $\sigma > 0$ and some fixed $\kappa_1, \kappa_2 \in (0, \infty)$. When $dt$ is replaced by the affine surface measure, $|\text{det} D^2 \varphi(t)|^{\frac{1}{4}} dt$, the range of $p$ and $q$ is known [12, 7], with a range that is independent of $\varphi$, and those result extends to hypersurfaces in $\mathbb{R}^n$ for every $n$. (See Theorem 2.1.) When the Euclidean surface measure is used instead, as in (1.1), numerous subtleties are introduced.

In the case of a hypersurface with the Euclidean surface measure, only partial results exist for hypersurfaces in $\mathbb{R}^n$, when $n$ is greater than 2. For the Euclidean surface measure, Ferreyra, Godoy, and Urciuolo completed the homogeneous polynomial case for $n = 3$ [6, 15], and the additive case $\varphi(t) = \sum |t_i|^{\kappa_i}$ for every $n$ [5]. In addition, for $n = 3$, Iosevich, Sawyer, and Seeger’s work [10] includes results for convex surfaces, based on multi-type information. More recently, Dendrinos and Zimmerman [4] investigated this question when $n = 3$ in the case that $\varphi$ is a mixed homogeneous polynomial of two variables. Such polynomials form a natural model class, because their members contain all anisotropic scaling limits of polynomials of two variables, and because mixed-homogeneous polynomials are related to the faces of the Newton diagram for all real-analytic functions. Thus, techniques for these polynomials should give insight on how to proceed in a more general case. In many cases, the results in [4] were nearly optimal, missing only the boundary, while in other cases, like $\varphi(t) = t_1^4 + t_1^2 t_2 + \frac{1}{4} t_2^2$, there remained an additional region between the conjecture and the theorem.

In this article we complete the mixed-homogeneous picture, obtaining the full range of $L^p \to L^q$ estimates (albeit restricted weak type at the endpoints). We use an alternate approach to [4], which allows us to obtain the full range of restricted weak type estimates for all such polynomials and which provides some clarification of the relationship between the Euclidean versions of these results and...
the affine versions. More precisely, we completely avoid using oscillatory methods, instead using methods such as Christ’s method of refinements in [1], often complimented by additional techniques such as orthogonality arguments as in [3]. In several places, we modify the method of refinements to find the influence of lower-dimensional curvature information (in the simpler cases, this is instead accomplished with Minkowski’s and Young’s inequalities). Finally, for the cases where regions remain between the Dendrinos-Zimmermann [4] results and conjecture, we alter the method of refinement to exploit the geometric information of the surface.

In Sections 1 and 2, we present two versions of our main theorem, the former written fully in terms of the Newton polytope, and the latter written more in terms of the factorization of $\varphi$, with notation similar to the results in [4]. Then Section 3 and 4 will give further detail into our techniques, along with an outline of our proof.

One additional key purpose of the techniques here will be to clarify the complex relationship between the affine measure results of Oberlin [12] and Grossman [7], restated in Theorem 2.1, and the Euclidean or push-forward measure results proven here.

One hope is that the relationship between mixed-homogeneous polynomials and general real analytic functions will allow for these methods to be greatly generalized. Additionally, some of the concepts demonstrated here can be applied to analyzing the effects of curvature in a much broader class of problems. For example, in [13], which is a joint work of the author and Stovall, these concepts are applied to analyzing the effects of curvature in Fourier restriction.

In this article, we consider the averaging operator

$$T f(x) = \int_{[-1,1]^2} f(x' - t, x_3 - \varphi(t))dt \quad \text{where } x = (x', x_3) \in \mathbb{R}^3.$$  

where $\varphi$ are polynomials satisfying the following definition:

**Definition 1.1.** A polynomial $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is mixed homogeneous if there exists $\kappa_1, \kappa_2 \in (0, \infty)$ such that $\varphi(\sigma^{\kappa_1}t_1, \sigma^{\kappa_2}t_2) = \sigma \varphi(t)$ for all $\sigma > 0$.

To state our main theorem, we need some additional definitions.

**Definition 1.2.** Let $\varphi(z_1, z_2) = \sum_{\alpha_1, \alpha_2 = 0}^M c_{\alpha_1, \alpha_2} z_1^{\alpha_1} z_2^{\alpha_2}$ be a mixed homogeneous polynomial mapping of $\mathbb{R}^2$ into $\mathbb{R}$. We will denote the Taylor support of $\varphi$ at $(0, 0)$, the Newton polyhedron of $\varphi$, and the Newton distance, respectively, to be

$$S(\varphi) := \{(\alpha_1, \alpha_2) \in \mathbb{N}_0^2 : c_{\alpha_1, \alpha_2} \neq 0\};$$

$$N(\varphi) := \text{Conv}\{(\alpha_1, \alpha_2) + \mathbb{R}_+^2 : (\alpha_1, \alpha_2) \in S(\varphi)\};$$

$$d(\varphi) := \inf\{c : (c, c) \in N(\varphi)\};$$

where $\text{Conv}$ denotes the closed convex hull, and where $(d(\varphi), d(\varphi))$ is the intersection of the bisetrix $x_1 = x_2$ with the Newton polyhedron $N(\varphi)$.

Next, define $\varphi_{R_1} = \varphi - \{\text{all terms for which } \alpha_1 = 0\}$, and denote the $z_1$-reduced Taylor support of $\varphi$ as

$$S(\varphi_{R_1}) = \{(\alpha_1, \alpha_2) \in S(\varphi) : \alpha_1 \neq 0\},$$

which removes all $z_2$-axis terms from the Taylor support of $\varphi$. Under this definition, the Newton polytope $N(\varphi_{R_1})$ of $\varphi_{R_1}$ is a simple translate of the Newton polytope
of $\partial_z \varphi$. Let $S(\varphi_{R2})$ be defined similarly, and denote the reduced Newton distance as $d(\varphi_{R}) = \max(d(\varphi_{R1}), d(\varphi_{R2}))$, with $\varphi$ defined accordingly. If $d(\varphi_{R1}) = d(\varphi_{R2})$, then set $\varphi_{R}$ to be $\varphi_{R1}$.

A mixed-homogeneous function $\varphi$ will be said to be linearly adapted if every factor of the form $(z_2 - \lambda z_1)$, some $\lambda \in \mathbb{R}/\{0\}$, has multiplicity less that or equal to $d(\varphi)$. By this definition, if $\varphi$ is mixed homogeneous but not homogeneous, then $\varphi$ is already linearly adapted. This ends up being equivalent to the definition given in [2]. For brevity the proof of this equivalence is omitted.

Finally, let $o(\varphi)$ denote the maximal multiplicity of any real irreducible (over $\mathbb{C}[z_1, z_2]$) factor of $\varphi$, and denote the height of $\varphi$ by $h(\varphi) = \max(d(\varphi), o(\varphi))$.

**Theorem 1.3.** Let $\varphi(z_1, z_2) : \mathbb{R}^2 \to \mathbb{R}$ be a mixed homogeneous polynomial, with $\varphi(0) = 0$, $\nabla \varphi(0) = 0$. If $\varphi$ is homogeneous, we additionally assume that $\varphi$ is linearly adapted and that $\varphi(z_1, z_2) \in C(\lambda_1 z_1 + \lambda_2 z_2)^d$, for any $C \in \mathbb{R}$, $\lambda_1 \in \mathbb{C}$, $J \in \mathbb{N}$. Then $\mathcal{T}$ is bounded from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ in the restricted-weak sense iff $p, q$ satisfy each of the following conditions:

\begin{align*}
(1.2) & \quad \frac{1}{q} \leq \frac{1}{p} \\
(1.3) & \quad \frac{1}{q} \geq \frac{1}{3p} \\
(1.4) & \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1}{d(\varphi)+1} \\
(1.5) & \quad \frac{1}{q} \geq \frac{d(\varphi) + 1}{2d(\varphi) + 1} \frac{1}{p} - \frac{1}{2d(\varphi)+1} \\
(1.6) & \quad \frac{1}{q} \geq \frac{h(\varphi) + 1}{h(\varphi) + 2} \frac{1}{p} - \frac{1}{h(\varphi) + 2} \\
(1.7) & \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1}{h(\varphi)}.
\end{align*}

**Remark 1.4.** Since non-degenerate linear transformations preserve $L^p \to L^q$ norms, up to a constant, the assumptions that $\varphi$ is linearly adapted and that $\varphi(0) = 0$, $\nabla \varphi(0) = 0$ do not weaken the generality of the theorem.

**Remark 1.5.** Equations (1.3), (1.6), and (1.7) only become relevant if $d(\varphi_{R}) > 2d(\varphi)$, $h(\varphi) > d(\varphi) + \frac{1}{2}$, or $h(\varphi) > d(\varphi) + 1$, respectively.

**Remark 1.6.** The two cases not covered by our theorem are when, after a linear transformation, $\varphi \equiv 0$ or $\varphi = z_1^J$, $J \geq 2$. In the former case, it is well known that $\mathcal{T}$ is bounded if and only if $p = q$. In the latter case, it was proved in [6] that $\mathcal{T}$ maps $L^p$ boundedly into $L^q$ precisely when $p, q$ satisfy all of the following:

\begin{align*}
\frac{1}{q} & \leq \frac{1}{p} \\
\frac{1}{q} & \geq \frac{1}{2p} \\
\frac{1}{q} & \geq \frac{2}{p} - 1 \\
\frac{1}{q} & \geq \frac{1}{p} - \frac{1}{2}.
\end{align*}

This can also be shown with a proof similar to that in Subsection 16.4, namely, by interpolating $L^\frac{2}{p} \to L^3$ and $L^\infty \to L^\infty$ bounds after a dyadic decomposition. For brevity this proof is omitted.

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2. Reformulation

In this section, we will state (Theorem 2.4) an alternate formulation of Theorem 1.3 and prove in Proposition 2.5 that these theorems are equivalent. By way of background, we restate a result of Gressman, Theorem 3 of [7]. For \( \varphi : \mathbb{R}^{d-1} \to \mathbb{R} \), and \( x \in \mathbb{R}^d \), define the affine averaging operator \( \mathcal{A} \) as

\[
\mathcal{A}f(x) := \int_{\mathbb{R}^{d-1}} f(x' - t, x_d - \varphi(t)) \left| \det(D^2 \varphi) \right| \frac{1}{|t|} dt,
\]

where \( D^2 \varphi \) is the Hessian.

**Theorem 2.1 (7).** Let \( \varphi : \mathbb{R}^{d-1} \to \mathbb{R} \) be a polynomial. Then \( \mathcal{A} \) extends as a bounded linear operator from \( L^{d+1} \left( \mathbb{R}^d \right) \) to \( L^{d+1} \left( \mathbb{R}^d \right) \), with operator norm bounded by a constant depending only on the dimension \( d \) and the degree of the polynomial \( \varphi \).

In fact, the restricted strong type version of this theorem, due to D. Oberlin (12), suffices for most of our applications. We will use this result extensivel in proving Theorem 1.3.

**Definition 2.2.** Let \( \kappa = (\kappa_1, \kappa_2) \in (0, \infty)^2 \), and let \( \psi : \mathbb{R}^2 \to \mathbb{R} \). We say \( \psi \) is \( \kappa \)-mixed homogeneous if, for every \( \sigma > 0 \), \( \psi(\sigma^{\kappa_1}, \sigma^{\kappa_2}) = \sigma \psi(1, 1) \).

If \( \varphi \) is mixed-homogeneous, there exists a \( \kappa = (\kappa_1, \kappa_2) \in (0, \infty)^2 \) such that \( \varphi \) is \( \kappa \)-mixed homogeneous. If \( \varphi \) is a not a monomial, this \( \kappa \) is unique, while for monomials, uniqueness holds under the additional assumption \( \kappa_1 = \kappa_2 \). We denote this unique \( \kappa \) as \( \kappa_\varphi \). We define homogeneous distance as

\[
d_h := d_h(\varphi) := \frac{1}{\kappa_1 + \kappa_2}\]

Further, there exist unique \( r, s, m \) satisfying \( \gcd\{r, s\} = 1 \) such that \( \kappa_\varphi = (\frac{r}{m}, \frac{s}{m}) \).

The mixed-homogeneous polynomial \( \varphi \) may be expanded as \( \varphi = \sum_{l=0}^{\lfloor \frac{m}{r} \rfloor} c_l z_1^{J-l} z_2^K s^l \),

for some \( J, K, c_l \). We observe that

\[
d_h(\varphi) = \frac{J + Kr}{r + s}.\]

By the Fundamental Theorem of Algebra, we can factor \( \varphi \) as

\[
\varphi(z_1, z_2) = C z_1^{\tilde{m}_2} z_2^{\tilde{m}_2} \prod_{j=3}^{\tilde{m}_2} (z_2^j - \lambda_j z_1^j)^n_j, \quad \text{with} \quad \lambda_j \text{ real iff } j \leq \tilde{m}_1 \leq \tilde{m}_2.
\]

For brevity, we refer to the irreducible factors \( z_1, z_2, z_2^j - \lambda_j z_1^j \), simply as factors. The homogeneous distance \( d_h \) can be rewritten as

\[
d_h = \frac{\tilde{m}_1 + r \tilde{m}_2 + s \sum \tilde{n}_j}{r + s}.
\]

Consider \( \omega := \det(D^2 \varphi) \). As we will show in Lemma 6.3 and (6.2), if \( d_h \neq 1 \), \( \omega \) is \( \frac{\kappa_\varphi}{2-2/d_\varphi} \)-mixed homogeneous, so \( \omega \) is mixed homogeneous and shares the same \( r \) and \( s \) with \( \varphi \), and its mixed homogeneity satisfies

\[
d_\omega := d_h(\omega) = 2d_h - 2.
\]

If \( d_h = 1 \), \( \omega \) is constant (see (6.1)), and we can say \( d_\omega = 0 \). Thus, in either case, we can factor \( \omega \) as

\[
\omega(z_1, z_2) = C z_1^{\tilde{m}_1} z_2^{\tilde{m}_2} \prod_{j=3}^{\tilde{m}_2} (z_2^j - \lambda_j z_1^j)^n_j, \quad \text{with} \quad \lambda_j \text{ real iff } j \leq m_1 \leq m_2.
\]
We will use the following proposition from [8], which follows from simple algebra:

**Proposition 2.3** ([8]). Let \( \psi \) be a mixed homogeneous polynomial as in (2.1) or (2.3), and denote \( d_\psi \) as the homogeneous distance of \( \psi \).

1. If \( \min\{r, s\} > 1 \) then any irreducible factor of \( \psi \) the form \((z_2^r - \lambda z_1^s)\), \( \lambda \neq 0 \), has multiplicity strictly less than \( d_\psi \).
2. If \( \psi \) has a factor of multiplicity greater than \( d_\psi \), then every other factor of \( \psi \) has multiplicity less than \( d_\psi \).

Let \( T \) denote the maximum multiplicity of any real factor of \( \omega \), i.e. \( T = \max\{\{\nu_1, \nu_2\} \cup \{n_j : j \leq m_1\}\} \). As we will see in Proposition 6.6 our assumptions imply that \( \omega \neq 0 \), so \( T \) is well-defined.

Suppose \( T > d_\omega \). By Proposition 2.3 we can define

\[
(2.3) \quad f_T = \text{The unique factor of } \omega, \text{ with multiplicity } T, \text{ when } T > d_\omega.
\]

If \( f_T \) is linear, let \( \nu \) denote its multiplicity in \( \varphi \), and if \( f_T \) is nonlinear, let \( N \) denote its multiplicity in \( \varphi \). If \( f_T \) is linear and \( \nu = 0 \), interchanging indices allows for the assumption \( f_T(z) \neq z_1 \), so we may expand

\[
\varphi(z) = z_1^T + z_1^{T-l} f_T^{l} + O(f_T^{l+s}), \text{ some } l \geq 1,
\]

and in any such case where \( T > d_\omega \), \( f_T \) is linear, and \( \nu = 0 \), we set \( A := l s \).

If we are in any case such that \( \nu \) is not defined in the preceding (likewise \( A, N \)) we set it to be 0. We use the convention that \( \frac{1}{0} = \infty \).

**Theorem 2.4.** Let \( \varphi(z_1, z_2) \) be a mixed homogeneous polynomial, with vanishing gradient at the origin, and not of the form \((\lambda_1 z_1 + \lambda_2 z_2)^J\), \( \lambda_i \in \mathbb{C}, J \in \mathbb{N} \). Then for \( 1 \leq p, q \leq \infty \), \( T \) is of restricted-weak type \((p, q)\) iff the following hold:

\[
(2.4) \quad \frac{1}{q} \leq \frac{1}{p} - \frac{1}{3p} \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_\omega + 1} \\
(2.5) \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1}{N + 1} \\
(2.6) \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1}{\nu + 1} \\
(2.7) \quad \frac{1}{q} \geq \frac{A+1}{2A+1} \frac{1}{p} - \frac{1}{2A+1} \quad \frac{1}{q} \geq \frac{2A+1}{A+1} \frac{1}{p} - 1 \\
(2.8) \quad \frac{1}{q} \geq \frac{N+1}{N+2} \frac{1}{p} - \frac{1}{N+2} \quad \frac{1}{q} \geq \frac{N+1}{N+2} \frac{1}{p} - \frac{2}{N+1} \\
(2.9) \quad \frac{1}{q} \geq \frac{1}{p} - \frac{1}{N}
\]

**Proposition 2.5.** Theorems 1.3 and 2.4 are equivalent.

Proposition 2.5 will be proved in Proposition 6.11.

3. Overview

Our strategy will be as follows: we will use the mixed homogeneity of \( \varphi \) (and by implication that of \( \omega := \det(D^2 \varphi) \), which is comparable to the Gaussian curvature) to decompose \([-1, 1]^2\) into dyadic level-set “strips” on which \( |\omega| \approx 2^{-j} \), and further decompose those strips into dyadic rectangles (after a possible coordinate change). On any such strip or rectangle, we will have three useful restricted weak type \( L^p \to L^q \) estimates: For \( p = q \), Young’s inequality implies \( L^p \to L^q \) bounds with constant equal to the measure of the strip or rectangle. Since \( \omega \) is nearly constant on each strip or rectangle, Theorem 2.4 gives us \( L^{\frac{p}{2}} \to L^{\frac{q}{2}} \) bounds on each dyadic strip. And finally, at \((p, q) = (\frac{3}{2}, 3)\), we will use bounds for averages on curves in \( \mathbb{R}^2 \) ([11] and later [7]), and either a variant of Christ’s method of refinements or,
in simpler cases, Minkowski’s Inequality with Young’s Inequality, to achieve either strong-type or restricted-weak-type versions of $L^\frac{3}{2} \to L^3$ bounds. In some cases, by summing over the dyadic regions, summing over the minimum of the $L^\infty \to L^\infty$, and the $L^\frac{3}{2} \to L^3$ bounds, we can achieve restricted-weak-type bounds for $T$ in the optimal range. In some cases, we also utilize the scaling symmetry of mixed-homogeneous polynomials in this argument, as demonstrated in Proposition 8.4. However, these methods are not always sufficient. Two types of obstructions can arise.

One issue is that degeneracies sometimes lead to a logarithmic factor in the $L^p \to L^p$ bounds on the relevant strips. To remove the logarithmic factor, we use an orthogonality argument incorporated into Christ’s method of refinements, found in [3].

A more delicate issue arises when $\omega = \det(D^2 \phi)$ possesses a factor $f_T$ of high multiplicity $T > d_\omega$, but all components of $\nabla \phi$ lack $f_T$ as a factor. This causes the portion of the surface over our dyadic “rectangles” to have a structure more closely resembling a helix than a parabola or hyperbola. As a consequence, the image of any such dyadic “rectangle” under $\nabla \phi$ is highly non-convex. We can beat the above mentioned interpolation argument by quantifying this non-convexity. We use a variant of Christ’s method of refinements [1], to show that non-convexity makes it impossible for $L^\infty \to L^\infty$ quasi-extremals to also be $L^\frac{3}{2} \to L^3$ quasi-extremals, and we interpolate after quantifying this trade-off. Further detail will be provided in the next section.

4. Outline

We first decompose into the cases $T > d_\omega$ and

*Case: $T < d_\omega$*

and we further break the case $T > d_\omega$ into the following cases: 

*Rectangular Cases*

- Case(ν): $f_T$ linear and $\nu \geq 1$
- Case(A): $f_T$ linear and $\nu = 0$, $A \geq 2$
- Case(N): $f_T$ nonlinear and $N \geq 2$

*Twisted Cases*

- Case(i): $f_T$ linear and $\nu = 0$, $A = 1$
- Case(iia): $f_T$ nonlinear and $N = 0$
- Case(iib): $f_T$ nonlinear and $N = 1$

where $f_T$ was defined in (2.3) to be the unique factor of $\omega = \det(D^2 \phi)$ having multiplicity $T$ when $T > d_\omega$. When $T > d_\omega$ and $f_T$ is linear with $\nu = 0$, $A$ cannot be 0, so this decomposition covers all cases where $T > d_\omega$. As we will see in Corollary 6.4, in cases (ν), (A), and (N), $T$ satisfies $T = 2\nu - 2$, $T = A - 2$, and $T = 2N - 3$, respectively. In these cases, on a local scale, the graph of $\phi$ will resemble a rectangular piece of a paraboloid or a saddle. We classify these three cases as Rectangular Cases. In cases (i), (iia), and (iib), on the other hand, $T$ has no such restrictions, and on a local scale the graph of $\phi$ has a helix-like structure. These cases will be denoted as Twisted Cases.

In Section 6 we prove algebraic lemmas relating $\phi$ and its derivatives, show that the hypotheses of Theorem 2.4 imply that $\omega \neq 0$, show that if $\phi$ is homogeneous and $T > d_\omega$, it can only belong to case(ν), and prove that the two versions of our main theorem, Theorems 1.3 and 2.4, are equivalent. Section 7 is dedicated to proving that every condition stated in Theorem 2.4 is necessary. In Section 8 we
exploit the mixed-homogeneous scaling symmetry of $\varphi$ to shorten the interpolation arguments in numerous later sections.

In Section 9, we make a finite decomposition of the region $[-1, 1]^2$ into neighborhoods of each irreducible subvariety of $\{\omega := det D^2 \varphi = 0\}$. This induces a finite decomposition of the operator. The most troublesome of these neighborhoods, which we denote by $R_T$, is the neighborhood of the subvariety $\{f_T = 0\}$. When $T \leq d_\omega$, we set $R_T = \emptyset$.

We begin Section 10 by showing in Lemmas 10.2 and 10.3 that to prove Theorem 2.4, it suffices to prove that $\mathcal{T}$ is of rwt $(p_{v_1}, q_{v_1})$ and in cases (N) and (A), to additionally prove that $\mathcal{T}$ is of rwt $(p_{v_2}, q_{v_2})$, for a specific $(p_{v_2}, q_{v_2})$ satisfying $p_{v_2} \leq q_{v_2} < 3p_{v_2}$.

And to decompose $\mathcal{T}$, we define $T_{R_f}(x) := \int_R f(x' - t, x_3 - \varphi(t))dt$.

Under these decompositions, our proof of Theorem 2.4 reduces to proving each of the following:

(I) $\mathcal{T}_{[-1,1]^2\setminus R_T}$ is of restricted-weak-type $\left(\frac{2d_\omega + 2}{3}, 2d_\omega + 2\right)$.

(II) In the Rectangular Cases, $T_{R_T}$ is of restricted-weak-type $\left(\frac{T + 1}{3}, T + 4\right)$.

(III) In the Twisted Cases, $T_{R_T}$ is of restricted-weak-type $\left(\frac{2d_\omega + 2}{3}, 2d_\omega + 2\right)$.

(IV) In Cases (N) and (A), $T_{R_T}$ is of restricted-weak-type $(p_{v_2}, q_{v_2})$.

In Section 11, we prove (II) and, when $T \neq d_\omega$, we prove (I). In Sections 11 and 12, we finish the proof of (I) when $T = d_\omega$ by incorporating an orthogonality argument into Christ’s method of refinements, as in [3]. We then prove (III) in Section 11 by analyzing the “twisting” of the graph of $\varphi$ by quantifying the nonconvexity of the image of local regions under $\nabla \varphi$, and using those calculations with a variant of the method of refinements.

Finally, in Section 13, we turn our attention to proving (IV). We first begin by decomposing cases (A) and (N) into subcases, based on the location of $(\frac{1}{q_{v_2}}, \frac{1}{q_{v_1}})$.

In Section 13, we complete the proof of (IV), by going through each subcase and proving that $T_{R_T}$ is of restricted-weak-type $(p_{v_2}, q_{v_2})$. In each subcase, our argument begins by computing restricted-weak-type $(\frac{3}{2}, 3)$ bounds over small regions. In several of the subcases of Case (N), this computation also involves a variant of the method of refinements.

We then proceed to interpolate the restricted-weak-type $(\frac{3}{2}, 3)$ bound with one or both of the restricted-weak-type $(\infty, \infty)$ and restricted-weak-type $(\frac{3}{2}, 4)$ bounds to achieve our result. However, one of the subcases, denoted $(N_{scal}^{q - 2p})$, relies on a more complex argument that involves incorporating an orthogonality argument into the method of refinements, similar to the argument in Section 11.

5. Notation

For the operator $\mathcal{T}$, most of our results will be bounds of restricted-weak-type. $\mathcal{T}$ is of rwt $(p, q)$, with bound $C_{p,q,w}$, iff for all measurable sets $E$ of finite measure,

$$\|T \chi_E\|_{L^{q,w}} \leq C_{p,q,w}\|\chi_E\|_{L^p} = C_{p,q,w}|E|^\frac{1}{p},$$

or, equivalently, if for all $E, F$ finite,

$$\langle T \chi_E, \chi_F \rangle \leq C_{p,q,w}\|\chi_E\|_{L^p}\|\chi_F\|_{L^{p'}} = C_{p,q,w}|E|^\frac{1}{p}|F|^{1 - \frac{1}{p}}.$$
We define \( \pi_1 \) and \( \pi_2 \) to be the projection onto the first and second coordinate in \( \mathbb{R}^2 \), respectively, so that \( \pi_1(z_1, z_2) = z_1 \). And for \( G \in \mathbb{R}^2 \), we denote \( (G) \) to be the span of \( G \), which would be a line, whenever \( G \neq 0 \).

The notation \( \overline{\text{Conv}}(S) \) will denote the closed convex hull of \( S \), while \( \mu(\overline{\text{Conv}}(S)) \) will refer to the Lebesgue measure of the convex hull.

To represent mixed Lebesgue norm spaces, we will use the notation

\[
\|F(u, t)\|_{L^q_{\mu_1} L^r_{\mu_2}((t \in S))} := \|\|F(u, t)\|_{L^q((u_1, u_2) \in \mathbb{R}^2)}\|_{L^r(u_2 \in \mathbb{R})}.
\]

We will sometimes need to decompose the operator \( T \). If \( R \subset \mathbb{R}^2 \) is a measurable set, we define

\[
T_R f(x) := \int_R f(x' - t, x_3 - \varphi(t)) dt.
\]

Also, unless otherwise stated, we will use \( A \lesssim B \) to imply that there exists a constant \( C \), uniform except for a possible \( \varphi \) dependence, such that \( A \leq CB \), and similarly for \( \gtrsim \). If we are incorporating extra dependence, for instance \( c \)-dependence, we will use \( \lesssim_c \). Finally, \( A \approx B \) is equivalent to \( A \lesssim B \) and \( A \gtrsim B \). By \( (A, B) \approx (D, E) \), we mean \( A \approx D \) and \( B \approx E \). Also, \( C \) will be a constant that can change from line to line, which will depend only on \( \varphi \) unless otherwise stated.

## 6. Algebraic Lemmas

Our first few lemmas will connect the form that \( \varphi \) and \( \omega \) take.

**Lemma 6.1.** If \( \varphi = z_1^J (z_2 - z_1^r)^{N'} + \mathcal{O}((z_2 - z_1^r)^{N'+1}) \), some \( J \geq 0 \), \( N' \geq 2 \), and \( r \geq 2 \), then \( \omega = C z_1^{2J+r-2}(z_2 - z_1^r)^{2N'-3} + \mathcal{O}((z_2 - z_1^r)^{2N'-2}) \), some \( C \neq 0 \).

**Lemma 6.2.** If \( \varphi = z_1^J + cz_2^{-1} z_2^l + \mathcal{O}(z_2^{l+1}) \), some \( c \neq 0 \) and \( l \geq 1 \), with \( l \geq 2 \) and \( J \geq 1 \), then \( \omega = C z_1^{2J-l-2} z_2^{-2} + \mathcal{O}(z_2^{l+1}) \), some \( C \neq 0 \).

**Lemma 6.3.** If \( \varphi = z_1^J z_2^J + \mathcal{O}(z_2^{J+1}) \), some \( J, \nu' \geq 1 \), then \( \omega = C z_1^{2J-2} z_2^{2
u'-2} + \mathcal{O}(z_2^{\nu'-1}) \), some \( C \neq 0 \).

**Corollary 6.4.** : In Case(\( N \)), \( T = 2N - 3 \); in Case(\( A \)), \( T = A - 2 \); and in Case(\( \nu \)), \( T = 2\nu - 2 \).

Next, we will prove each of these lemmas.

**Proof of Lemma 6.1.** Making a change of variables, we set \( x = z_1 \) and \( y = z_2 - z_1^r \).

Then \( \partial_x^2 = \partial_x^2 + r^2x^{2r-2}\partial_y^2 - 2rxx^{-1}\partial_x\partial_y - r(r-1)x^{-2}\partial_y^2 \); \( \partial_z^2 = \partial_y^2 \); and \( \partial_{x,z_2} = \partial_{x,y} - xx^{-1}\partial_y^2 \), so

\[
\omega = [\partial_x^2 \varphi \partial_y^2 \varphi - (\partial_x \varphi \partial_y^2 \varphi)] - r(r-1)x^{-2}\partial_y \varphi \partial_y^2 \varphi.
\]

If \( \varphi = x^J y^{N'} + \mathcal{O}(y^{N'+1}) \), with \( N' \geq 2 \), then the terms \( \partial_x^2 \varphi \partial_y^2 \varphi \) and \((\partial_x \varphi \partial_y^2 \varphi)\) are both of \( \mathcal{O}(y^{2N'-2}) \), while the final term can be written as

\[
x^{-2}\partial_y \varphi \partial_y^2 \varphi = (N')^2(N' - 1)x^{2J+r-2}y^{2N'-3} + \mathcal{O}(y^{2N'-2}).
\]

Thus,

\[
\omega = C x^{2J+r-2}y^{2N'-3} + \mathcal{O}(y^{2N'-2}) , \text{ some } C \neq 0 .
\]

\( \square \)
Proof of Lemma 6.2. For \( \varphi = z_1^J + cz_1^{J-1}z_2^s + O(z_2^{s+1}) \),
\[
\omega = \partial_z^2 \varphi \partial_z \varphi - (\partial_z \varphi \partial_z \varphi)^2.
\]
The term \((\partial_z \varphi \partial_z \varphi)^2\) is \(O(z_2^{2ls-2})\), while
\[
\partial_z^2 \varphi \partial_z \varphi = cJ(J-1)ls(1-1)z_1^{2J-1}z_2^s + O(z_2^{s+1}).
\]
Since \(ls \geq 2\), and since \(\nabla \varphi(0) = 0\) implies that \(J \geq 2\),
\[
\omega = Cz_1^{2J-1}z_2^s + O(z_2^{s+1}), \text{ some } C \neq 0. \quad \square
\]

Proof of Lemma 6.3. For \( \varphi = z_1^J z_2^s + O(z_2^{s+1}) \), with \( J, s \geq 1 \),
\[
\omega = \partial_z^2 \varphi \partial_z \varphi - (\partial_z \varphi \partial_z \varphi)^2.
\]
Looking at each term separately,
\[
(\partial_z \varphi \partial_z \varphi)^2 = [J\nu z_1^{J-1}z_2^{s-1}]^2 + O(z_2^{s+1}),
\]
\[
\partial_z^2 \varphi \partial_z \varphi = [J(J-1)z_1^{2J-2}z_2^s] [\nu(\nu-1)z_1^{2J-2}z_2^s] + O(z_2^{s+1}).
\]
Thus,
\[
\omega = [1 - \nu^2] J\nu z_1^{2J-2}z_2^{s-2} + O(z_2^{s+1})
\]
\[
= Cz_1^{2J-2}z_2^{s-2} + O(z_2^{s+1}), \text{ some } C \neq 0. \quad \square
\]

Proof of Corollary 6.4. In Case(N), \( \min\{r, s\} = 1 \) by Proposition 2.8 so after rescaling, \( \varphi \) can be written as in Lemma 6.1 with \( N = N' \geq 2 \), implying that \( T = 2N - 3 \). In Case(A), after rescaling \( \varphi \) can be written as in Lemma 6.2 with \( A = ls \geq 2 \). Additionally, if \( J \) were less than 2, then the hypothesis \( \nabla \varphi(0) = 0 \) of Theorem 2.4 wouldn’t be satisfied. Thus, by Lemma 6.2, \( T = A - 2 \). In Case(\( \nu \)), after rescaling, \( \varphi \) can be written as in Lemma 6.3 with \( \nu = \nu' \geq 1 \). Additionally, if \( J = 0 \), then \( \varphi = z_2^s \), violating the hypothesis \( \nabla \varphi(0) = 0 \) of Theorem 2.4. Thus, by Lemma 6.3, \( T = 2\nu - 2 \). \quad \square

Next, we look at the relationship between the homogeneous distances for \( \varphi \) and its derivatives.

Lemma 6.5. The homogeneous distances of \( \varphi \) and its derivatives obey the following relations:
\( d_\omega = 2d_h - 2 \), \( d_h(\partial_z \varphi) = d_h - \frac{r}{r+s} \), and \( d_h(\partial_{zz} \varphi) = d_h - \frac{r}{r+s} \).

Proof. Since \( \varphi \) is mixed-homogeneous, there exists some \( m \) such that \( \sigma \varphi(z_1, z_2) = \varphi(\sigma^m z_1, \sigma^m z_2) \). Taking derivatives,
\[
\sigma^2 \partial_z^2 \varphi(z_1, z_2) \partial_{zz} \varphi(z_1, z_2) = \sigma^2 \partial_z^2 \varphi \partial_{zz} \varphi(\sigma^m z_1, \sigma^m z_2)
\]
\[
\sigma^2 (\partial_z \varphi \partial_{zz} \varphi(z_1, z_2))^2 = \sigma^2 (\partial_z \varphi \partial_{zz} \varphi(\sigma^m z_1, \sigma^m z_2))^2.
\]
Putting these together, and using \( d_h = \frac{m}{r+s} \):
\[
\sigma^2 d_\omega(z_1, z_2) = d_h(z_1, z_2) \sigma^2 d_h(z_1, z_2).
\]
When \( d_h \neq 1 \), we can change variables using \( \gamma = \sigma^{2-\frac{r}{m}} \) to get
\[
\gamma d_\omega(z_1, z_2) = \omega(\gamma \sigma^{m-2}(z_1, z_2)).
\]
Thus, \( d_\omega = \frac{m(2-2d_h)}{r+s} = d_h(2 - \frac{r}{m}) = 2d_h - 2 \). And when \( d_h = 1 \), \( \omega \) is invariant under scaling, implying that \( \omega \) is constant and \( d_\omega = 0 \). \quad \square
Next, by symmetry, it suffices to show the claim for \( d_h(\partial_z \varphi) = d_h - \frac{s}{r+s} \). Taking a derivative, we get
\[
\sigma \partial_z \varphi(x_1, x_2) = \sigma \frac{d}{dx_1} \partial_z \varphi(\sigma \frac{d}{dx_1} x_1, \sigma \frac{d}{dx_1} x_2),
\]
which becomes, under the change of variables \( r = \sigma^{-1} \frac{d}{dx_1} \), \( \gamma = \sigma^{-1} \frac{d}{dx_1} \),
\[
\gamma \partial_z \varphi(x_1, x_2) = \partial_z \varphi(\gamma x_1, \gamma x_2).
\]
Thus, \( d_h(\partial_z \varphi) = \frac{m}{r+s} d_h - \frac{s}{r+s} \), and therefore by symmetry, \( d_h(\partial_z \varphi) = d_h - \frac{s}{r+s} \).

Next, we want to show that if \( \omega \equiv 0 \), then \( \varphi \) does not satisfy the assumptions of Theorem 2.3.

**Proposition 6.6.** If \( \varphi \) is a mixed homogeneous polynomial with \( \nabla \varphi(0) = 0 \) and \( \omega \equiv 0 \), then \( \varphi \) is a constant multiple of \( z_1^J, z_2^J \), or \( (z_1 + \lambda z_2)^J \), with \( \lambda \in \mathbb{R} \) and \( J \in \mathbb{N} \). Consequently, in Theorem 2.4 \( \omega \) is never identically 0.

**Proof.** Exclude the aforementioned cases. Then after possibly swapping \( z_1 \) and \( z_2 \) and rescaling \( \varphi \), either
1. \( \varphi = z_1^J + \mathcal{O}(z_2^{J+1}) \), some \( J, \nu \geq 1 \), or
2. \( \varphi = z_1^J + c_1 z_1^{J-J_0} z_2 + \mathcal{O}(z_2^{J+1}) \).

In case 1, by Lemma 0.3
\[
\omega = C z_1^{2J-2} z_2^{2\nu-2} + \mathcal{O}(z_2^{2\nu-1}) \neq 0.
\]
In case 2, if \( \max\{l_0, k \} > 1 \), then we can use Lemma 0.2. By symmetry, we need only consider \( l_0 > 1 \). Then for some \( C \neq 0 \),
\[
\omega = C z_1^{2J-J_0} z_2^{l_0-2} + \mathcal{O}(z_2^{l_0-1}) \neq 0.
\]
Lastly, if \( \max\{l_0, k \} = 1 \), then \( \varphi \) is homogeneous and case 2 simplifies to
\[
\varphi = z_1^J + c_1 z_1^{J-J_0} z_2 + \ldots + c_{J-1} z_1 z_2^{J-1} + c_J z_2^J,
\]
with \( c_1 \neq 0 \).

We want to show that if \( \omega \equiv 0 \), then \( \varphi = (z_1 + \lambda z_2)^J \), for some \( \lambda \).

For later convenience, we will prove an even stronger statement here:

**Lemma 6.7.** Suppose that \( \varphi = z_1^J + c_1 z_1^{J-J_0} z_2 + \ldots + c_{J-1} z_1 z_2^{J-1} + c_J z_2^J \), with \( c_1 \neq 0 \), and suppose that \( \omega = \mathcal{O}(z_2^{J-1}) \). Then \( \varphi = (z_1 + \lambda z_2)^J \), with \( \lambda = \frac{c_1}{c_J} \), and consequently \( \omega \equiv 0 \).

**Proof.** Replacing \( c_1 \) with \( J \lambda \), we can write \( \varphi \) as
\[
\varphi = z_1^J + J \lambda z_1^{J-J_0} z_2 + c_2 z_1^{J-2} z_2^2 + \ldots + c_{J-1} z_1 z_2^{J-1} + c_J z_2^J,
\]
with \( \lambda \neq 0 \).

Since \( \tilde{\varphi} = (z_1 + \lambda z_2)^J \) produces such a \( \varphi \), with \( c_2, \ldots, c_J \) fixed in terms of \( \lambda \) and \( J \), and since \( \text{det} D^2 \tilde{\varphi} \equiv 0 \), it suffices to show that:

If \( \omega = \mathcal{O}(z_2^{J-1}) \), then \( c_2, \ldots, c_J \) are uniquely determined by \( \lambda \) and \( J \).

Expanding, there exist functions \( f_{1,K} \) and \( f_{2,K} \), for \( 2 \leq K \leq J \), such that
\[
(\partial_{z_1}^2 \varphi \partial_{z_2}^2 \varphi) = \sum_{K=2}^J \left[ J K (J-1)(K-1) c_K + f_{2,K}(J, \lambda, c_2, \ldots, c_{K-1}) z_1^{2J-K-2} z_2^2 z_2 K-2 + \mathcal{O}(z_2^{J-1}) \right]
\]
and
\[(\partial_{z_1 z_2} \varphi)^2 = \sum_{K=2}^{J} f_{1,K}(J, \lambda, c_2, ..., c_{K-1})z_1^{2J-K-2}z_2^{K-2} + O(z_2^{J-1}),\]
so there exist functions $f_{3,K}$, $2 \leq K \leq J$, such that
\[\omega = \sum_{K=2}^{J} [J K (J-1)(K-1)c_K - f_{3,K}(J, \lambda, c_2, ..., c_{K-1})]z_1^{2J-K-2}z_2^{K-2} + O(z_2^{J-1}).\]

Therefore, $\omega = O(z_2^{J-1})$ implies that, for $K = 2, 3, ..., J$,
\[c_K = \frac{f_{3,K}(J, \lambda, c_2, ..., c_{K-1})}{J K (J-1)(K-1)},\]
and therefore inductively each $c_K$ is uniquely determined by $J$ and $\lambda$, and the Lemma proof is complete.

Then due to Lemma 6.7, the proof of Proposition 6.6 is complete.

**Corollary 6.8.** Theorem 2.4 holds in the case $d_h = 1$, and in the case $d_h < 1$, Theorem 2.4 holds vacuously.

**Proof.** If $d_h = 1$, then $\omega$ is constant by Lemma 6.5. If that constant is nonzero, then Theorem 2.4 applies, while if $\omega \equiv 0$, then Theorem 2.4 holds vacuously by Proposition 6.6. If $d_h < 1$, then $d_\omega < 0$ by Lemma 6.5 implying $\omega \equiv 0$, so again Theorem 2.4 holds vacuously.

Next, we show that homogeneous $\varphi$ can only belong to to a couple cases of Theorem 2.4.

**Proposition 6.9.** If $\varphi$ is homogeneous, and $\varphi$ satisfies the hypotheses of Theorem 2.4, then either $T \leq d_\omega$ or $\varphi$ belongs to case($\nu$). Consequently, in case(A) and twisted case(i), we have $\max\{r, s\} \geq 2$.

**Proof.** If $\varphi$ is homogeneous, then every factor of $\omega$ is linear. Thus, when $T > d_\omega$, $\varphi$ can only belong to case($\nu$), case(A), or twisted case(i).

If $\varphi$ belongs to case(A), then after a linear change of coordinates and rescaling, $\varphi = z_1^{J} + c z_1^{J-A} z_2^{2} + O(z_2^{J+1})$, $A \geq 2$, $c \neq 0$, and by Lemma 6.2 or Corollary 6.3, $T = A - 2$. Also, $d_\omega = 2d_h - 2 = 2\frac{J}{2} - 2 = J - 2$. But since $c \neq 0$ implies that $A \leq J$, then $T > d_\omega$ is impossible, resulting in a contradiction.

If $\varphi$ belongs to case(i), then after a linear change of coordinates and rescaling, $\varphi = z_1^{J} + cz_1^{J-1}z_2 + O(z_2^{2})$, $c \neq 0$, and $\omega = C z_1^{M} z_2^{2} + O(z_2^{J+1})$, some $M$, some $C \neq 0$. Then $T > d_\omega = 2d_h - 2 = 2\frac{J}{2} - 2 = J - 2$, so $T \geq J - 1$. Thus, $\omega = O(z_2^{J-1})$, and by Lemma 6.7, $\omega \equiv 0$ and $\varphi = (z_1 + \frac{\lambda}{z_2})^{J}$, so $\varphi$ violates the hypotheses of Theorem 2.4.

**Remark 6.10.** The argument for case(A) in Proposition 6.9 also implies that if $\varphi = z_1^{J} + cz_1^{J-ls}z_2^{ls} + O(z_2^{J+1})$, with $ls \geq 2$, and $z_2$ has multiplicity $T = d_\omega$ in $\omega$, then $J = ls$ and $\varphi = z_1^{J} + cz_2^{J}$, some $c \neq 0$.

We will now complete the proof of Proposition 6.10.

**Proposition 6.11.** Theorems 1.3 and 2.4 are equivalent.

**Proof.** We begin with three technical lemmas.

**Lemma 6.12.** The Newton distance, $d(\varphi)$, equals $\max\{\nu, d_h\}$.
Proof of Lemma 6.12. First, the Newton diagram of a mixed homogeneous polynomial will have at most 3 edges: a horizontal edge \( y = \nu_2 \) corresponding to the multiplicity of \( z_2 \) in \( \varphi \), a vertical edge \( x = \nu_1 \) corresponding to the multiplicity of \( z_1 \) in \( \varphi \), and an edge corresponding to the mixed homogeneity of \( \varphi \), which if extended into an infinite line, would intersect the bisectrix at \( (d_h, d_h) \). (One can check that \( \varphi(z_1, z_2) + z_1^{d_1} z_2^{d_2} \) is still mixed homogeneous to verify this.) Thus, since the bisectrix intersects the Newton diagram at \( (d(\varphi), d(\varphi)) \), then \( d(\varphi) = \max\{\nu_1, \nu_2, d_h\} \).

If \( \nu_1 > d_h \), then by Lemmas 6.3 and 6.5 \( z_1 \) has multiplicity greater than \( d_\omega \) in \( \omega \), so by our definitions, \( \nu = \nu_1 \). Similarly for \( \nu_2 \). Likewise, if \( \nu > d_h \), then after linearly adapting \( \varphi \) as is assumed by Theorem 1.3, \( \nu \) will be the multiplicity of either \( z_1 \) or \( z_2 \) in \( \varphi \), so \( \nu = \nu_1 \) or \( \nu = \nu_2 \). Thus, \( d(\varphi) = \max\{\nu_1, \nu_2, d_h\} = \max\{\nu, d_h\} \). □

Lemma 6.13. If \( \max\{A, d(\varphi_R)\} > 2d(\varphi) \), then \( A = d(\varphi_R) \).

Proof of Lemma 6.13. Suppose \( d(\varphi_R) > 2d(\varphi) \). Then, since \( d(\varphi_R) > d(\varphi) \), and therefore by Lemma 6.12 \( d(\varphi_R) > d_h \), the bisectrix must intersect the Newton diagram of \( \varphi_R \) on a vertical or horizontal edge, and so we have that \( \varphi = c_1 z_1^j + c_2 z_2^K d(\varphi_R) \), some \( j \geq 1 \), \( K \geq 0 \), \( c_1, c_2 \neq 0 \), up to a swapping of \( z_1 \) and \( z_2 \). Then, by Lemma 6.2 and Lemma 6.3, since \( d(\varphi_R) > 2d_h \), \( z_2 \) will have multiplicity greater than \( d_\omega \) as a factor of \( \omega \), so by the definition of \( A \), \( d(\varphi_R) = A \).

Similarly, suppose \( A > 2d(\varphi) \). Then by Lemma 6.12 \( A > 2d_h \), so by Lemma 6.5 \( A - 2 > d_\omega \). Since \( A \neq 0 \), then after a possible swap of \( z_1 \) and \( z_2 \) and rescaling, \( \varphi(z) = z_1^j + c z_2^K f_T + o(f_T) \), some \( j \geq 1 \), \( K \geq 0 \), \( c \neq 0 \) by the definition of \( A \), where \( f_T \) is not \( z_1 \) linear. Then, after a linear transformation, we can use Lemma 6.2 to find that \( f_T \) has multiplicity greater than \( d_\omega \) in \( \omega \). Then \( \varphi \) belongs to Case(A), so by Lemma 6.9 \( \varphi \) cannot be homogeneous, so \( f_T = z_2 \) up to a constant. Then, \( \varphi_R2 = z_1^K z_2^A + o(z_2^4) \), so since \( A > d_h \), the bisectrix will intersect the Newton diagram of \( \varphi_R2 \) on a vertical or horizontal edge, at \( (A, A) \), implying that \( d(\varphi_{R2}) = A \). Thus, \( d(\varphi_R) \geq A > 2d(\varphi) \), so by the first part of the proof, \( d(\varphi_R) = A \). □

Lemma 6.14. If \( \max\{N, h(\varphi)\} > d(\varphi) + \frac{1}{2} \), then \( N = d(\varphi) \).

Proof of Lemma 6.14. Suppose \( h(\varphi) > d(\varphi) + \frac{1}{2} \). Since \( h(\varphi) = \max\{d(\varphi), o(\varphi)\} \), this implies that \( h(\varphi) = o(\varphi) \), the maximal multiplicity of the real irreducible factors of \( \varphi \). Since all the linear factors of \( \varphi \) have multiplicity not exceeding \( d(\varphi) \) (by our linearly adapted assumption), any factor associated with \( o(\varphi) \) must be nonlinear. Since \( d(\varphi) \geq d_h \) by Lemma 6.12 then by Lemma 6.3 \( h(\varphi) > \frac{d_\omega + 3}{2} \). Since \( \nabla \varphi(0) = 0 \) and \( \varphi(0) = 0 \) imply that \( d(\varphi) > \frac{1}{2} \), then \( o(\varphi) = h(\varphi) > 1 \). Additionally, since \( d(\varphi) \geq d_h \), then \( o(\varphi) > d_h \), so by Proposition 2.3 \( \min\{r, s\} = 1 \). Therefore, Lemma 6.1 can be applied to \( \varphi \) with \( h(\varphi) = N' \), implying that the factor associated with \( h(\varphi) \) will have multiplicity greater than \( d_\omega \) in \( \omega \), so by the definition of \( N \), \( h(\varphi) = o(\varphi) = N \).

Similarly, suppose \( N > d(\varphi) + \frac{1}{2} \). Since \( N > d_h \) by Lemma 6.12 then by Proposition 2.3 \( N \) is the highest multiplicity of any real irreducible factor of \( \varphi \), so \( N = o(\varphi) = h(\varphi) \). □

Now we turn to the proof of Proposition 6.11 by proving Proposition 2.5.

Proof of Proposition 2.5. Inequality (2.4) is equivalent to the validity of (1.2) and (1.3). By Lemma 6.12 the validity of (2.5) and (2.6) is equivalent to (1.4).
If \( \max \{ A, d(\varphi_R) \} \leq 2d(\varphi) \), then by Lemma 6.12, (2.7) follows from (2.4), (2.5), and (2.6), while (1.5) follows from (1.2), (1.3), and (1.4). If \( \max \{ A, d(\varphi_R) \} > 2d(\varphi) \), then by Lemma 6.13, \( A = d(\varphi_R) \), in which case (2.7) is equivalent to (1.5).

If \( \max \{ N, b(\varphi) \} \leq d(\varphi) + \frac{1}{2} \), then inequalities (2.4), (2.5), and (2.6) imply (2.8) and (2.9), while (1.2), (1.3), and (1.4) imply (1.6) and (1.7). By Lemma 6.14 if \( \max \{ N, b(\varphi) \} > d(\varphi) + \frac{1}{2} \), (2.8) is equivalent to (1.6) and (2.9) is equivalent to (1.7).

This completes the proof of Proposition 6.11 and likewise Proposition 2.5.

7. Necessary Conditions

The necessity of each bound in Theorems 2.4 and 1.3 was proven in [4] for the strictly mixed homogeneous cases, with proofs that can also be applied in the homogeneous cases, and by [5] in the homogeneous cases. For completeness, we will include similar proofs here.

Necessity of \( q \geq p \): Let \( E = [-3K, 3K]^3 \) and \( F = [-K, K]^3 \), where \( K \gg 1 \). Then \( |E| \approx |F| \approx K^3 \), and on \( F \), \( T\chi_E \approx 1 \). Then \( T \) being of rwt \((p, q)\) requires that
\[
1 \approx \text{avg}_F T\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx K^{3\left(\frac{1}{p} - \frac{1}{q}\right)},
\]
implying that \( 1 \lesssim K^{3\left(\frac{1}{p} - \frac{1}{q}\right)} \). Since \( K \) can be taken to be arbitrarily large, this implies that \( 0 \leq \frac{1}{p} - \frac{1}{q} \), which simplifies to
\[
q \geq p.
\]

Necessity of \( q \leq 3p \): (By duality the necessity of \( \frac{1}{q} \geq \frac{3}{p} - 2 \) follows.) Let \( E = [-\frac{1}{3}, \frac{1}{3}]^2 \times [\varphi - C\epsilon, \varphi + C\epsilon] \) and \( F = [-\epsilon, \epsilon]^3 \), where \( \epsilon \ll 1 \). Then \( |E| \approx \epsilon \), \( |F| \approx \epsilon^3 \), and since \( |\nabla \varphi| \) is bounded on \([-1, 1]^2 \), there exists a \( C \) large enough, independent of \( \epsilon \), so that on \( F \), \( T\chi_E \approx 1 \). Then \( T \) being of rwt \((p, q)\) requires that
\[
1 \approx \text{avg}_F T\chi_E \lesssim |E|^{\frac{1}{p}}|F|^{-\frac{1}{q}} \approx \epsilon^{\frac{1}{p} - \frac{1}{q}},
\]
implying that \( 1 \lesssim \epsilon^{\frac{1}{p} - \frac{1}{q}} \). Since \( \epsilon \) can be taken to be arbitrarily small, this implies that \( 0 \geq \frac{1}{p} - \frac{1}{q} \), which after simplifying becomes
\[
q \leq 3p.
\]

Necessity of the Scaling line \( \frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_h + 1} \): Let \( S \) be a hypersurface in \( \mathbb{R}^3 \), and \( S_R \) be the portion of the hypersurface over region \( R \in \mathbb{R}^2 \). Denote \( T_R \) as the averaging with hypersurface \( S_R \). By the structure of \( T \), if \( R' \supset R \), then \( T_{R'} \chi_E \geq T_R \chi_E \), so the (strong or restricted-weak-type) bound of \( T \) cannot increase as \( R \) increases.

Denote \( \sigma I \) as \([-\sigma^{\kappa_1}, \sigma^{\kappa_1}] \times [-\sigma^{\kappa_2}, \sigma^{\kappa_2}] \). Then by the upcoming Lemma 8.3 (letting \( \|T\|_{p, q} \) denote either the strong or restricted-weak-type bound at \((\frac{1}{p}, \frac{1}{q})\)),
\[
\|T_{\sigma I}\|_{p, q} = \sigma^{\frac{d_h + 1}{d_h + 1}} \left( \frac{1}{p} - \frac{1}{p + 1} \right) \|T_{[-1, 1]^2}\|_{p, q},
\]
Then, since \( \sigma I \supset [-1, 1]^2 \) for \( \sigma > 1 \), we need \( \sigma^{\frac{d_h + 1}{d_h + 1}} \left( \frac{1}{p} - \frac{1}{p + 1} \right) \geq 1 \) for \( \sigma > 1 \), implying that boundedness at \((\frac{1}{p}, \frac{1}{q})\) requires
\[
\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_h + 1}.
\]

Necessity of \( \frac{1}{q} \geq \frac{1}{p} - \frac{1}{d_h + 1} \) in Case (c): In this case, it suffices to consider \( \varphi = z_1^\nu z_2^\nu + O(z_2^{\nu + 1}) \). Then, choose \( E = [-3, 3] \times [-3\epsilon, 3\epsilon] \times [-3\epsilon^\nu, 3\epsilon^\nu] \) and \( F = \)}
$[-1, 1] \times [-\epsilon, \epsilon] \times [-\epsilon', \epsilon']$, where $\epsilon \ll 1$. Then $|E| \approx |F| \approx \epsilon^{\nu+1}$, and on $F$, $T_{\chi F} \approx \epsilon$. Then $T$ being of rwt $(p, q)$ requires that

$$\epsilon \approx \text{avg}_F T_{\chi F} \lesssim |E|^\frac{1}{p} |F|^{-\frac{1}{q}} \approx \epsilon^{(\nu+1)(\frac{1}{p} - \frac{1}{q})},$$

implying that $\epsilon \lesssim \epsilon^{(\nu+1)(\frac{1}{p} - \frac{1}{q})}$. Since $\epsilon$ can be taken to be arbitrarily small, this implies that $1 \geq (\nu+1)(\frac{1}{p} - \frac{1}{q})$, or after simplifying,

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{\nu+1}.

Necessity of $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{\nu+1}$ in Case$(N)$: In this case, by Proposition 2.3 we can assume $s=1$ and write $\varphi = z_1^T(z_2 - \lambda z_1)^N + O((z_2 - \lambda z_1)^{N+1})$. Then, choose $E = [-3, 3] \times [-3\lambda, 3\lambda] \times [-3e^N, 3e^N]$ and $F = [-1, 1] \times [-\lambda, \lambda] \times [-e^N, e^N]$, where $\epsilon \ll 1$. Then $|E| \approx |F| \approx e^N$, and on $F$, $T_{\chi F} \approx \epsilon$. Then $T$ being of rwt $(p, q)$ requires that

$$\epsilon \approx \text{avg}_F T_{\chi F} \lesssim |E|^\frac{1}{p} |F|^{-\frac{1}{q}} \approx \epsilon^{N(\frac{1}{p} - \frac{1}{q})},$$

implying that $\epsilon \lesssim \epsilon^{N(\frac{1}{p} - \frac{1}{q})}$. Since $\epsilon$ can be taken to be arbitrarily small, this implies that $1 \geq N(\frac{1}{p} - \frac{1}{q})$, or after simplifying,

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{N}.

Necessity of $\frac{1}{q} \geq \frac{N+1}{N+2} - \frac{1}{N+2}$ in Case$(N)$: (By duality the necessity of $\frac{1}{q} \geq \frac{N+1}{N+2} - \frac{2}{N+1}$ follows.) In this case, by Proposition 2.3 we can assume $s=1$ and write $\varphi = z_1^T(z_2 - \lambda z_1)^N + O((z_2 - \lambda z_1)^{N+1})$. Then, choose $E = [\frac{1}{2}, 1] \times [\lambda z_1^T - 3\epsilon, \lambda z_1^T + 3\epsilon] \times [-3e^N, 3e^N]$ and $F = [-\epsilon, \epsilon]^2 \times [-e^N, e^N]$, where $\epsilon \ll 1$. Then $|E| \approx e^{N+1}$, $|F| \approx e^{N+2}$, and on $F$, $T_{\chi F} \approx \epsilon$. Then $T$ being of rwt $(p, q)$ requires that

$$\epsilon \approx \text{avg}_F T_{\chi F} \lesssim |E|^\frac{1}{p} |F|^{-\frac{1}{q}} \approx \epsilon^{(N+1)(\frac{1}{p} - (N+2)\frac{1}{q})},$$

implying that $\epsilon \lesssim \epsilon^{(N+1)(\frac{1}{p} - (N+2)\frac{1}{q})}$. Since $\epsilon$ can be taken to be arbitrarily small, this implies that $1 \geq (N+1)(\frac{1}{p} - (N+2)\frac{1}{q})$, or

$$\frac{1}{q} \geq \frac{N+1}{N+2} - \frac{1}{N+2}.

Necessity of $\frac{1}{q} \geq \frac{A+1}{2A+1} - \frac{1}{2A+1}$ in Case$(A)$: (By duality the necessity of $\frac{1}{q} \geq \frac{A+1}{2A+1} - \frac{1}{2A+1}$ follows.) In this case, it suffices to consider $\varphi = z_1^T + z_1 z_2^A + O(z_2^2)$, $A \geq 2$. Then, choose $E = [\frac{1}{2}, 1] \times [-3\epsilon, 3\epsilon] \times [z_1^T + z_1 z_2^A - 3\epsilon A, z_1^T + z_1 z_2^A + 3\epsilon A]$ and $F = [-\epsilon^A, \epsilon^A] \times [-\epsilon, \epsilon] \times [-\epsilon^A, \epsilon^A]$, where $\epsilon \ll 1$. Then $|E| \approx e^{A+1}$, $|F| \approx e^{2A+1}$, and on $F$, $T_{\chi F} \approx \epsilon$. Then $T$ being of rwt $(p, q)$ requires that

$$\epsilon \approx \text{avg}_F T_{\chi F} \lesssim |E|^\frac{1}{p} |F|^{-\frac{1}{q}} \approx \epsilon^{(A+1)(\frac{1}{p} - (2A+1)\frac{1}{q})},$$

implying that $\epsilon \lesssim \epsilon^{(A+1)(\frac{1}{p} - (2A+1)\frac{1}{q})}$. Since $\epsilon$ can be taken to be arbitrarily small, this implies that $1 \geq (A+1)(\frac{1}{p} - (2A+1)\frac{1}{q})$, or

$$\frac{1}{q} \geq \frac{2A+1}{2A+1} - \frac{1}{2A+1}.

\square
8. Scaling Symmetries

When a vertex of the polygon arising in Theorem 2.3 occurs on the scaling line \( \frac{1}{p} = \frac{1}{q} - \frac{1}{d_k+1} \), the mixed-homogeneity of \( \varphi \) and scale-invariance of \( T \) can be exploited as follows.

Let \( R \subset \mathbb{R}^2 \), \( f : \mathbb{R}^2 \to \mathbb{R} \), \( \kappa = \kappa_\varphi \), and \( \sigma > 0 \). We will use notation

\[
(8.1) \quad R_{\kappa,\sigma} := \{(\sigma^{\kappa_1}z_1, \sigma^{\kappa_2}z_2) | (z_1, z_2) \in R\}, \quad f_\sigma(\cdot, \cdot) := f(\sigma^{\kappa_1} \cdot, \sigma^{\kappa_2} \cdot).
\]

**Definition 8.1.** Let \( \kappa = (\kappa_1, \kappa_2) \in (0, \infty)^2 \), and let \( \psi : \mathbb{R}^2 \to \mathbb{R} \). We say \( \psi \) is \( \kappa \)-mixed homogeneous if, for every \( \sigma > 0 \), \( \psi(\sigma^{\kappa_1}, \sigma^{\kappa_2}) = \sigma \psi(\cdot, \cdot) \).

**Definition 8.2.** \( \mathcal{R} \subset \mathbb{R}^2 \) is \( \kappa \)-scale invariant if \( \mathcal{R} = \mathcal{R}_{\kappa,\sigma} \) for all \( \sigma > 0 \).

**Lemma 8.3.** Let \( R \subset \mathbb{R}^2 \), let \( \kappa = \kappa_\varphi \), and let \( \sigma > 0 \). Let \( \| \cdot \|_{p,q} \) refer to either the strong-type bound \( \| \cdot \|_{L^p \to L^q} \) or the restricted weak-type bound \( \| \cdot \|_{L^{p,1} \to L^{q,\infty}} \). If \( \|T_R\|_{p,q} < \infty \), then \( \|T_{R_{\kappa,\sigma}}\|_{p,q} = \sigma^{(\kappa_1+\kappa_2)}\|T_R\|_{p,q} \).

**Proof.**

\[
T_{R_{\kappa,\sigma}}f := T_R[f(\sigma^{\kappa_1}, \sigma^{\kappa_2}, \cdot)]
\]

\[
= \int_R f(\sigma^{\kappa_1}t_1 - (\sigma^{\kappa_1}t_1), \sigma^{\kappa_2}t_2 - (\sigma^{\kappa_2}t_2), t_3 - \varphi(\sigma^{\kappa_1}t_1, \sigma^{\kappa_2}t_2)) dt_1 dt_2
\]

\[
= \sigma^{-(\kappa_1+\kappa_2)} \int_{R_{\kappa,\sigma}} f(\sigma^{\kappa_1}x_1 - u_1, \sigma^{\kappa_2}x_2 - u_2, x_3 - \varphi(u_1, u_2)) du_1 du_2
\]

\[
= \sigma^{-(\kappa_1+\kappa_2)}(T_{R_{\kappa,\sigma}}f)(\sigma^{\kappa_1}x_1, \sigma^{\kappa_2}x_2, x_3) = \sigma^{-(\kappa_1+\kappa_2)}(T_{R_{\kappa,\sigma}}f)\sigma.
\]

Then, by scaling,

\[
s^{-(\kappa_1+\kappa_2)}s^{-\frac{(\kappa_1+\kappa_2+1)}{q}}\|T_{R_{\kappa,\sigma}}f\|_{L^q(R^3)} = s^{-(\kappa_1+\kappa_2)}\|T_{R_{\kappa,\sigma}}f\|_{L^q(R^3)}
\]

\[
= \|T_R(f)\|_{L^q(R^3)} \leq \|T_R\|_{L^p \to L^q}\|f\|_{L^p(R^3)}
\]

\[
= \sigma^{-(\kappa_1+\kappa_2+1)}\|T_{R_{\kappa,\sigma}}\|_{L^p \to L^q}\|f\|_{L^p(R^3)},
\]

and by looking at \( f \) that are near-extremal, we get, for \( \|T_R\|_{L^p \to L^q} < \infty \),

\[
\|T_{R_{\kappa,\sigma}}\|_{L^p \to L^q} = \sigma^{(\kappa_1+\kappa_2)}\|T_{R_{\kappa,\sigma}}\|_{L^p \to L^q}
\]

\[
= \sigma^{\frac{1}{d_k+1} + \frac{1}{\kappa_1+\kappa_2}}\|T_{R_{\kappa,\sigma}}\|_{L^p \to L^q}
\]

\[
= \sigma^{\frac{d_k+1}{\kappa_1+\kappa_2}}\|T_{R_{\kappa,\sigma}}\|_{L^p \to L^q}.
\]

This proves Lemma 8.3 for strong-type bounds. Replacing \( \|Tf\|_{L^q(L^3)} \) with \( \|Tf\|_{L^\infty(L^3)} \), and \( f \) with \( \chi_E \), we get an identical result for restricted weak-type bounds.

**Proposition 8.4.** Let \( \kappa = \kappa_\varphi \). Let \( \mathcal{R} \) be \( \kappa \)-scale invariant, and let \( \psi \) be \( \frac{\kappa}{D} \)-mixed homogeneous, some \( D > 0 \). Suppose that \( \frac{1}{q_\psi} = \frac{1}{p_\psi} - \frac{1}{d_k+1} \) and that \( \frac{1}{p_{\psi,\frac{1}{q_\psi}}} = (1-\theta)(\frac{1}{p_{r_0}}, \frac{1}{q_{r_0}}) + \theta(\frac{1}{p_{r_1}}, \frac{1}{q_{r_1}}) \), for some \( \theta \in (0,1) \), for some \( (p_0, q_0) \) and \( (p_1, q_1) \) where \( \frac{1}{q_0} \neq \frac{1}{p_0} - \frac{1}{d_k+1} \) and \( \frac{1}{q_1} \neq \frac{1}{p_1} - \frac{1}{d_k+1} \). If \( \mathcal{R} \cap \{|\psi| \approx 1\} \) is of rwt \( (p_0, q_0) \) and \( (p_1, q_1) \), then \( \mathcal{R} \) is of rwt \( (p_{\psi,\frac{1}{q_\psi}}) \).
Thus, by Lemma 8.3,
\[ T_j(E, F) \lesssim \min \{ 2^{j(d_1 + 1)}(\frac{1}{p_0} + \frac{1}{q_0}) |E| \frac{1}{p_0} |F|^{1 - \frac{1}{q_0}}, 2^{j(d_1 + 1)}(\frac{1}{q_1} + \frac{1}{q_1}) |E| \frac{1}{q_1} |F|^{1 - \frac{1}{q_1}} \} \]
Then, interpolating,
\[ T_R(E, F) \lesssim \sum_j \min \{ 2^{j(d_1 + 1)}(\frac{1}{p_0} + \frac{1}{q_0}) |E| \frac{1}{p_0} |F|^{1 - \frac{1}{q_0}}, 2^{j(d_1 + 1)}(\frac{1}{q_1} + \frac{1}{q_1}) |E| \frac{1}{q_1} |F|^{1 - \frac{1}{q_1}} \}
\approx |E|^{\frac{1}{p_0} - \frac{1}{d_1 + 1}} |F|^{1 - \frac{1}{q_0}}.
\]
Thus, \( T_R \) is of rwt \((p_S, q_S)\) for
\[ (\frac{1}{p_S}, \frac{1}{q_S}) = \left( \frac{1}{p_0} - \frac{1}{d_1 + 1}, \frac{1}{q_0} \right) \]
the point on the scaling line \( \frac{1}{q} = \frac{1}{p} - \frac{1}{d_1 + 1} \) directly between \((\frac{1}{p_0}, \frac{1}{q_0})\) and \((\frac{1}{p_1}, \frac{1}{q_1})\). \( \square \)

There are three cases where we will use this result:

\textbf{Case 1}: \((\frac{1}{p_0}, \frac{1}{q_0}) = (\frac{1}{2}, \frac{1}{2})\), and \((\frac{1}{p_1}, \frac{1}{q_1})\) satisfies \( \frac{1}{p_1} = \frac{2}{q_1} \).

In this case, the point \((\frac{1}{p_0}, \frac{1}{q_0})\) is the intersection of the lines \( \frac{1}{p} = \frac{2}{q} \) and scaling line \( \frac{1}{q} = \frac{1}{p} - \frac{1}{d_1 + 1} \), namely \((\frac{1}{p_0}, \frac{1}{q_0}) = (\frac{3}{2d_1 + 2}, \frac{1}{2d_1 + 2}) = (\frac{3}{d_1 + 1}, \frac{1}{d_1 + 1})\), using the fact that \( d_\omega = 2d_1 - 2 \).

\textbf{Case 2}: \((\frac{1}{p_0}, \frac{1}{q_0}) = (\frac{1}{2}, \frac{1}{2})\), and \((\frac{1}{p_1}, \frac{1}{q_1})\) satisfies \( \frac{1}{p_1} = \frac{2}{q_1} \).

In this case, the point \((\frac{1}{p_0}, \frac{1}{q_0})\) is the intersection of the lines \( \frac{1}{p} = \frac{2}{q} \) and scaling line \( \frac{1}{q} = \frac{1}{p} - \frac{1}{d_1 + 1} \), namely \((\frac{1}{p_0}, \frac{1}{q_0}) = (\frac{2}{d_1 + 1}, \frac{1}{d_1 + 1})\).

\textbf{Case 3}: \((\frac{1}{p_0}, \frac{1}{q_0}) = (\frac{1}{2}, \frac{1}{2})\), and \((\frac{1}{p_1}, \frac{1}{q_1})\) satisfies \( \frac{1}{p_1} = \frac{2}{q_1} \).

Here, by simplifying \( \{5.2\} \), \((\frac{1}{p_S}, \frac{1}{q_S})\) simplifies to
\[ \left( \frac{2}{p_S}, \frac{2}{q_S} \right) = \left( 3 - \frac{q_1}{q_1} + \frac{1}{q_1} + \frac{1}{q_1}, 1 - \frac{1}{q_1} + \frac{1}{q_1} \right) \]
Additionally, using identity \( d_\omega = 2d_1 - 2 \), we can rewrite \( \frac{1}{p_S} \) as
\[ \frac{1}{p_S} = \frac{3 - \frac{q_1}{q_1} + \frac{1}{q_1}}{1 - \frac{1}{q_1} + \frac{1}{q_1}(d_\omega + 4)}. \]

9. Decomposition

Our goal in this section will be to decompose \([-1, 1]^2\), and hence our operator \( T_{[-1,1]^2} \), around the factors of the determinant Hessian \( \omega \). Since \( \omega \) is mixed homogeneous, with homogeneous distance \( d_\omega = 2d_1 - 2 \),
\[ \omega(z_1, z_2) = Cz_1^{\nu_1}z_2^{\nu_2}(\sum_{j=3}^{M_2} (z_j^2 - \lambda_j z_1^2)^{\nu_j} \), \]
with \( \lambda_j \) real iff \( j \leq M_1 \), some \( M_1 \leq M_2 \).

Let \( \tilde{\epsilon} > 0 \) be sufficiently small. We use the following covering of \([-1, 1]^2\):
Convexity, denoted Lemma 10.2. The polygon described in Theorem 2.4.

For the polygon described in Theorem 2.4, Lemma 10.3.
The rest of Sections 10-13, we will focus on the first relevant vertex.

This lemma follows by simple algebra with the boundaries in Theorem 2.4. For the rest of Sections 10-13, we will focus on the first relevant vertex.

Lemma 10.2. The polygon described in Theorem 2.4 has exactly one relevant vertex, denoted $(\frac{1}{p_v}, \frac{1}{q_v})$, that lies on the line $q = 3p$. If we are in the twisted cases, case (ν), or if $T \leq d_ω$, this is the only relevant vertex. In Cases (N) and (A), there exists exactly one additional relevant vertex $(\frac{1}{p_v}, \frac{1}{q_v})$, and this additional vertex satisfies $(\frac{1}{p_v}, \frac{1}{q_v}) \in \text{Conv}\{(0,0), (\frac{3}{4}, \frac{1}{4}), (\frac{2}{3}, \frac{1}{3})\}$.

Proof. In the case $T \leq d_ω$ and the twisted cases, $(p_v, q_v)$ lies on the intersection of the curves $q = 3p$ and \( \frac{1}{q} = \frac{1}{p} - \frac{1}{d_ω + 1} \), and since $d_ω = 2d_κ - 2$,

\[
\left(\frac{1}{p_v}, \frac{1}{q_v}\right) = \left(\frac{3}{2d_κ + 2}, \frac{1}{2d_κ + 2}\right) = \left(\frac{3}{d_ω + 4}, \frac{1}{d_ω + 4}\right).
\]

10. First relevant vertex, initial observations

Definition 10.1. A relevant vertex is a vertex $(\frac{1}{p}, \frac{1}{q})$ of the polygon described in Theorem 2.4 that satisfies $q' \leq p < q$.

To prove Theorem 2.4 by Young’s Inequality, duality, and real interpolation, it suffices to prove that $T(E, F) \lesssim |E|^\frac{1}{p} |F|^\frac{1}{q}$ for all relevant vertices.

We recall from Section 4 that we can divide our problem into three broad cases: the case $T \leq d_ω$, the rectangular cases, and the twisted cases, and further decompose the rectangular cases into Cases (ν), (A), and (N).

Lemma 10.2. The polygon described in Theorem 2.4 has exactly one relevant vertex, denoted $(\frac{1}{p_v}, \frac{1}{q_v})$, that lies on the line $q = 3p$. If we are in the twisted cases, case (ν), or if $T \leq d_ω$, this is the only relevant vertex. In Cases (N) and (A), there exists exactly one additional relevant vertex $(\frac{1}{p_v}, \frac{1}{q_v})$, and this additional vertex satisfies $(\frac{1}{p_v}, \frac{1}{q_v}) \in \text{Conv}\{(0,0), (\frac{3}{4}, \frac{1}{4}), (\frac{2}{3}, \frac{1}{3})\}$.

Proof. In the case $T \leq d_ω$ and the twisted cases, $(p_v, q_v)$ lies on the intersection of the curves $q = 3p$ and \( \frac{1}{q} = \frac{1}{p} - \frac{1}{d_ω + 1} \), and since $d_ω = 2d_κ - 2$,

\[
\left(\frac{1}{p_v}, \frac{1}{q_v}\right) = \left(\frac{3}{2d_κ + 2}, \frac{1}{2d_κ + 2}\right) = \left(\frac{3}{d_ω + 4}, \frac{1}{d_ω + 4}\right).
\]
In Case(v), \((p_{v_1}, q_{v_1})\) lies on the intersection of the curves \(q = 3p\) and \(\frac{1}{q} = \frac{1}{p} - \frac{1}{p+1}\), and since \(T = 2\nu - 2\),
\[
\left(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}\right) = \left(\frac{3}{2\nu + 2}, \frac{1}{\nu + 2}\right) = \left(\frac{3}{T + 1}, \frac{1}{T + 1}\right).
\]
In Case(A), \((p_{v_1}, q_{v_1})\) lies on the intersection of the curves \(q = 3p\) and \(\frac{1}{q} = \frac{A + 1}{2A + 1} - \frac{1}{p+1}\), and since \(T = A - 2\),
\[
\left(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}\right) = \left(\frac{3}{A + 2}, \frac{1}{A + 2}\right) = \left(\frac{3}{T + 2}, \frac{1}{T + 2}\right).
\]
In Case(N), \((p_{v_1}, q_{v_1})\) lies on the intersection of the curves \(q = 3p\) and \(\frac{1}{q} = \frac{N+1}{N+2} - \frac{1}{p+1}\), and since \(T = 2N - 3\),
\[
\left(\frac{1}{p_{v_1}}, \frac{1}{q_{v_1}}\right) = \left(\frac{3}{2N+1}, \frac{1}{2N+1}\right) = \left(\frac{3}{T + 1}, \frac{1}{T + 1}\right).
\]

This lemma leads to the following proposition, whose proof will occupy the next 3 sections:

**Proposition 10.4.** The operator \(T_{R_j}\) is of \(rwt\) \((1 + d_{\omega} + 4)\) when \(j = 0\) or when \(n_j\) (likewise \(v_j\)) is less than or equal to \(d_{\omega}\). When \(n_j\) (likewise \(v_j\)) is greater than \(d_{\omega}\), \(T_{R_j}\) is of \(rwt\) \((\frac{T}{3} + 1, T + 4)\).

Together, Proposition [10.4] and Lemma [10.3] imply the following corollary:

**Corollary 10.5.** The operator \(T_{[-1, 1]^2 \setminus R_j}\) is of \(rwt\) \((p, q)\) lying in the polygon of Theorem [2.2]. Additionally, in the rectangular cases, \(T_{R_j}\) is of \(rwt\) \((p_{v_1}, q_{v_1})\).

We will spend the remainder of this section proving Proposition [10.4] in all cases except when \(n_j\) or \(v_j\) equals \(d_{\omega}\). The remaining cases will be handled in Sections [11] and [12].

The following lemma will allow us to use Hölder’s Inequality to compute \(L^\infty \rightarrow L^\infty\) bounds, which will be useful in interpolation.

**Lemma 10.6.** Let \(\mu\) be the standard Lebesgue measure on \(\mathbb{R}^2\). The regions in the dyadic decomposition of \(\omega\) satisfy the following inequalities:

- \(\mu(R_0 \cap \{|\omega| \approx 2^{-m}\}) \lesssim_\epsilon 2^{-\frac{m}{(r+1)}}\).
- \(\mu(R_i \cap \{|\omega| \approx 2^{-m}\}) \lesssim_\epsilon 2^{-\frac{m(\nu_i + d_{\omega})}{p_i}}\) for \(i = 1, 2\), whenever \(\nu_i \neq d_{\omega}\).
- \(\mu(R_i \cap \{|\omega| \approx 2^{-m}\}) \lesssim_\epsilon 2^{-\frac{m(\nu_i + d_{\omega})}{p_i}}\) for \(3 \leq i \leq M_1\), whenever \(n_i \neq d_{\omega}\).

**Proof.** On \(R_0\), \(|z_2 - \lambda_1 z_1| \sim |z_2|^r \sim |z_1|^r \sim |z_1|^r + |z_2|^s\) for all \(j\), so \(|\omega(z_1, z_2)| \gtrsim (|z_1|^r + |z_2|^s)^{\frac{r+m}{r+s}}\). Therefore,
\[
\mu(R_0 \cap \{|\omega| \approx 2^{-m}\}) \lesssim_\epsilon \mu([-1, 1]^2 \cap \{|\omega| \approx 2^{-\frac{m(\nu_i + d_{\omega})}{p_i}}\}) \lesssim_\epsilon 2^{-\frac{m}{(r+1)}}.
\]

Next, on \(R_2\), we have \(|z_2|^s < \epsilon |z_1|^r\), so for \(\epsilon\) sufficiently small, \(|z_2|^s - \lambda_1 |z_1|^r \sim |z_1|^r\) for all \(i\). Thus, \(|\omega| \gtrsim |z_1|^Q|z_2|^\nu_2\), for some \(Q\) satisfying \(Q + \nu_2 = d_{\omega}\). Therefore, after a few elementary calculus calculations,
\[
\mu(R_2 \cap \{|\omega| \approx 2^{-m}\}) \lesssim_\epsilon \mu(R_2 \cap \{|z_2|^Q|z_2|^\nu_2 \lesssim 2^{-m}\})
\]
\[
\lesssim_\epsilon \mu(|z_2| \lesssim \min(2^{\frac{m}{r+s}}|z_1|^r, \epsilon |z_1|^\frac{r}{s}, |z_1| \leq 1))
\]
\[
\lesssim_\epsilon \max(2^{\frac{m}{r+s}}, 2^{-\frac{m(r+s)}{Q(r+s)}}) = 2^{-\frac{m}{d_{\omega} + 2}}
\]
as long as \(\nu_2 \neq d_{\omega}\). The case for \(R_1\) follows similarly.
Finally, on $R_j$, for $j \geq 3$, we have $|z_j^2 - \lambda_j z_1^2| < \epsilon|z_1|^r$, so for $\epsilon$ sufficiently small, $|z_j^2 - \lambda_j z_1^2| \sim |z_j^2| \sim |z_j^2|$ for all $i \neq j$. Thus, $|\omega| \gtrsim |z_1|^{|Q| |z_j^2|^{\nu_j}}$, for some $Q$ satisfying $\frac{Q + n + rs}{r + s} = d_\omega$. Therefore, after a few elementary calculus calculations,

$$
\mu(R_j \cap \{|\omega| \approx 2^{-m}\}) \leq \frac{1}{2} \mu(R_j \cap \{|z_1|^{|Q| |z_j^2|^{\nu_j}} \leq 2^{-m}\})
$$

as long as $n_j \neq d_\omega$.

**Proposition 10.7.** For $\epsilon$ sufficiently small, the operator $T_{R_j}$ is of rwt $(p, q)$ when $(\frac{1}{p}, \frac{1}{q})$ equals:

- $(\frac{3}{d_\omega + 4}, \frac{1}{d_\omega + 4})$ for $j = 0$,
- $(\frac{3}{\max(n_j, d_\omega) + 4}, \frac{1}{\max(n_j, d_\omega) + 4})$ for $j = 1, 2$, and $n_j \neq d_\omega$,
- $(\frac{3}{\max(n_j, d_\omega) + 4}, \frac{1}{\max(n_j, d_\omega) + 4})$ for $3 \leq j \leq M_1$, and $n_j \neq d_\omega$.

**Proof.** Define $T_{j,m} := T_{R_j} \cap \{|\omega| \approx 2^{-m}\}$, and denote $h_j := \max(n_j, d_\omega)$ for $j = 1, 2$, $h_j := \max(n_j, d_\omega)$ for $j \geq 3$, and $h_0 := d_\omega$. We assume that $n_j$ (likewise $\nu_j$) is not equal to $d_\omega$. By Hölder’s Inequality and Lemma 10.6, $\|T_{j,m}\|_{L^\infty} \lesssim 2^{-\frac{3m}{n_j}}$. In addition, by Theorem 2.1, $\|T_{j,m}\|_{4 \rightarrow 4} \lesssim 2^{-\frac{3m}{n_j}}$. Combining these,

$$
T_{R_j}(E, F) \lesssim \sum_m \min(2^{-\frac{3m}{n_j}} |F|, 2^{\frac{3m}{n_j}} |E|) \lesssim |E|^{\frac{4}{n_j + 4}} |F|^{1 - \frac{4}{n_j + 4}}.
$$

Thus, $T_{R_j}$ is of rwt $(\frac{h_j + 4}{3}, \frac{h_j + 4}{3})$.

Proposition 10.7 implies Proposition 10.4 in all cases except when $n_j$ or $\nu_j$ equals $d_\omega$. However, if we extend Lemma 10.6 to the remaining cases, the measures and thereby the $L^\infty \to L^\infty$ bounds will contain logarithmic terms, which are undesirable for the interpolation performed in Proposition 10.7. To avoid this, we will show in Sections 11 and 12 that for $q$ arbitrarily large, the rwt $(\frac{4}{q}, q)$ bounds lack the extra logarithmic term, by proving the following lemma:

**Lemma 10.8.** If $n_j$ or $\nu_j$ equals $d_\omega$, then $T_{R_j \cap \{|\omega| \approx 1\}}$ is of rwt $(\frac{4}{q}, q)$ for all $q$ satisfying $4 \leq q < \infty$.

By Proposition 8.3 (we are in Case 1, as defined after the proof of that proposition), this lemma implies the following corollary:

**Corollary 10.9.** If $n_j$ or $\nu_j$ equals $d_\omega$, then $T_{R_j}$ is of rwt $(\frac{4d_\omega + 4}{3}, d_\omega + 4)$.

Together, Proposition 10.7 and Corollary 10.9 imply Proposition 10.4. In Sections 11 and 12, we will complete our argument by proving Lemma 10.8 with an argument that will have further applications in later sections.

**11. Application of Method of Refinements**

We begin our argument by looking at a general subset $S$ of $\mathbb{R}^2$, which is then subdivided into smaller subsets $\tau_n$. In light of Section 10, for now consider $S$ to be the set $R_j \cap \{|\omega| = 1\}$, where $n_j$ (likewise $\nu_j$) equals $d_\omega$, and consider the sets $\tau_n$ to be some sort of dyadic decomposition of $S$. 
Notation 11.1. Let \( \tau_n \) be subsets of \( \mathbb{R}^2 \), let \( S := \bigcup_{n \in \mathbb{N}} \tau_n \), and denote \( T_n := T_{\tau_n} \). For any \( E, F \subset \mathbb{R}^3 \), to capture the primary contributors to \( T_n(E, F) \), we will define the following refinements:

\[
T_n := \{ u \in E : T_n \chi_E(u) \geq \frac{1}{4} |T_n(E, F)| = \frac{1}{4} \text{avg}_F T_n \chi_E \};
\]

\[
E_n := \{ w \in E : T^*_n \chi_E(w) \geq \frac{1}{4} \frac{T^*_n(E, F_n)}{|F_n|} = \frac{1}{4} \text{avg}_{E,F} T^*_n \chi_{E,F} =: \alpha_{E,F} \};
\]

\[
\check{T}_n := \{ u \in F_n : T_n \chi_{F_n}(u) \geq \frac{1}{4} \frac{T_n(F_n \cap F)}{|F_n|} = \frac{1}{4} \text{avg}_{E,F} T_n \chi_{F_n \cap F} =: \beta_{F_n} \};
\]

\[
E_{kl} := \{ w \in E_k : T^*_n \chi_{F_k \cap F_l}(w) \geq \frac{1}{4} \frac{T^*_n(F_k \cap F_l)}{|F_k|} = \frac{1}{4} \text{avg}_{E_k,F_k} T^*_n \chi_{F_k \cap F_l} =: \beta_{E_k,F_l} \}.
\]

With these refinements,

\[
T_n(E, F) = \int_{E_n} T_n \chi_E + \int_{F_n \setminus E_n} T_n \chi_E \leq T_n(E, F_n) + \int_{F_n \setminus E_n} \frac{1}{4} T_n \leq T_n(E, F_n) + \frac{1}{4} T_n(E, F),
\]

and consequently \( T_n(E, F_n) \geq \frac{1}{4} T_n(E, F) \). Likewise, \( T_n(E_n, F_n) \geq \frac{1}{4} T_n(E, F_n) \) and \( T_n(E, \check{T}_n) \geq \frac{3}{4} T_n(E, F_n) \), so

\[
(11.1) \quad T_n(E_n, \check{T}_n) \approx T_n(E, F).
\]

Therefore, \( E_n \) and \( \check{T}_n \), capture the primary contributions of \( E \) and \( F \) pertaining to \( T_n \). Additionally, let \( b \in \{2, 3\} \) be fixed, and define \( \eta(\epsilon) := \{ n \in \mathbb{N} : T_n(E, F) \approx \epsilon |E|^{\frac{b}{b+1}} |F|^{\frac{1}{b+1}} \} \), for \( \epsilon \in 2\mathbb{Z} \). (Note that \( \eta \) depends implicitly on \( b, E, F \).) Pertaining to Section 10 we will use \( b = 3 \), but in a later section, we will reuse these arguments with \( b = 2 \).

When \( b = 3 \) and when \( S = R_j \cap \{ |\omega| \approx 1 \} \), the following lemma implies Lemma 10.2 if all the hypotheses can be shown to be satisfied.

Lemma 11.2. Let \( \|
\|
\| \infty \rightarrow L^1_{\mu} \leq \Lambda \leq 1 \), some \( \Lambda \) depending only on \( \varphi \), and let \( |\tau_n| \lesssim 1 \) uniformly in \( n \). If

\[
(1) \quad \sum_{n \in \eta(\epsilon)} |E_n| \lesssim \epsilon^{-\delta} |E| \quad \text{and} \quad (2) \quad \sum_{n \in \eta(\epsilon)} |\check{T}_n| \lesssim \epsilon^{-\delta} |F|
\]

for all \( \delta > 0 \), uniformly over \( E, F, \epsilon \), then \( \| T_S \|_{L^\theta_{\mu} \rightarrow L^{\frac{b+1}{b}}_{\mu} \rightarrow L^{\frac{1}{b+1}}_{\mu} \infty} < \infty \) for all \( \theta \in (0, 1] \).

Proof. In what follows, we will introduce an \( a \in [0, 1] \) to be chosen near the end. In the first line, we use \( \| T_n \|_{L^2_{\mu} \rightarrow L^{\frac{b+1}{b}}_{\mu} \rightarrow L^{\frac{1}{b+1}}_{\mu} \infty} \lesssim 1 \) to imply \( \epsilon \lesssim 1 \); the second line is due to (11.1); the definition of \( \eta(\epsilon) \), Young’s Inequality using \( |\tau_n| \lesssim 1 \), and the hypothesis that \( \| T_n \|_{L^2_{\mu} \rightarrow L^{\frac{b+1}{b}}_{\mu} \rightarrow L^{\frac{1}{b+1}}_{\mu} \infty} \lesssim 1 \) give the third line; the fourth line is simple rearrangement; in the fifth line \( a \) is chosen so that \( 1 - a = \frac{b+1}{b+1+b-1} \), and Hölder is applied; for the final line, conditions (1) and (2) give the first inequality, and a geometric sum with \( \delta \) sufficiently small completes the argument:

\[
\langle T_S \chi_E, \chi_F \rangle = \sum_{k \in 2^{-N}} \sum_{n \in \eta(\epsilon)} \langle T_n \chi_E, \chi_F \rangle \approx \sum_{k \in 2^{-N}} \sum_{n \in \eta(\epsilon)} \langle T_n \chi_E, \chi_F \rangle^{a(1-\theta)} (1-\theta) (T_n \chi_{E_n}, \chi_{F_n})^{1-a}
\]
Proof. If Condition 11.3 is satisfied, then (1) and (2) are satisfied.

Lemma 11.4. If \( \sum |\eta| \leq M \), then by Hölder’s inequality,

\[
|\eta| \leq |\eta|^{1/2} = |\eta|^{1/2}. 
\]

To use Lemma 11.2, we will need to show that hypotheses (1) and (2) are satisfied. In Lemma 11.3 we will prove that the following condition will suffice:

Condition 11.3. There exist \( \tilde{M}, B > 0, \xi \in \{-1, 1\} \) independent of \( E,F \) such that, for all \( k, \ell \in \mathbb{N} \) satisfying \( \max\{k, \ell, |k - \ell|\} > M \), we have

\[
|F_k| \geq 2^{(k-\ell)}B^B \alpha_k \beta_{k\ell}, \quad \text{and} \quad |E_k| \geq 2^{(k-\ell)}B^B \alpha_k \beta_{k\ell},
\]

with implicit constants independent of \( k, E, F \).

Lemma 11.4. Let all conditions of Lemma 11.3 be satisfied save for (1) and (2). If Condition 11.3 is satisfied, then (1) and (2) are also satisfied.

Proof. (Previously done in [3].) Since \( \|T_n\|_{L^{1/2} \rightarrow L^{1/2}} \leq \Lambda \) uniformly, then by (11.1),

\[
|E_n| \leq |E|^{1/2},
\]

so \( |E_n| \geq \epsilon |E|^{1/2} \) uniformly. By a similar argument, \( |F_n| \geq \epsilon |F|^{1/2} \).

Suppose condition (1) fails. Then, for \( M > \tilde{M} \) arbitrarily large, there exist \( E,F, \epsilon \leq \Lambda \) such that \( \sum_{n \in \eta(c) \cap (M, \infty)} |E_n| > 10 M \log(10 \Lambda \epsilon^{-1}) |E| \). By the pigeonhole principle, there exists a finite, \( M \log(10 \Lambda \epsilon^{-1}) \)-separated set \( \eta' \subset \eta(c) \cap (M, \infty) \) such that \( \sum_{n \in \eta'} |E_n| > 10 |E| \). Then by Hölder’s inequality,

\[
\sum_{n \in \eta'} |E_n| = \int \sum_{n \in \eta'} \chi_{E_n} \leq |E|^{1/2} \left( \int \left( \sum_{n \in \eta'} \chi_{E_n} \right)^2 \right)^{1/2}
\]

\[
\leq |E|^{1/2} \left( \sum_{n \in \eta'} |E_n| + \sum_{n \neq m \in \eta'} 2 |E_n \cap E_m| \right)^{1/2}.
\]

Since \( \sum_{n \in \eta'} |E_n| \geq \sum_{n \neq m \in \eta'} |E_n \cap E_m| \) would imply \( \sum_{n \in \eta'} |E_n| \leq 3 \frac{|E|}{2} \left( \sum_{n \in \eta'} |E_n| \right)^{1/2} \) and contradict\( \sum_{n \in \eta'} |E_n| > 10 |E| \), then \( \sum_{n \in \eta'} |E_n| < \sum_{n \neq m \in \eta'} |E_n \cap E_m| \), and therefore \( \sum_{n \in \eta'} |E_n| \leq 3 \frac{|E|}{2} \left( \sum_{n \neq m \in \eta'} |E_n \cap E_m| \right)^{1/2} \).

Then, choosing \( n_1, k, \ell \in \eta', k \neq \ell \) so that \( |E_{n_1}| = \min_{n \in \eta'} |E_n| \) and \( |E_k \cap E_\ell| = \max_{n \neq m \in \eta'} |E_n \cap E_m| \), we have \( \frac{|\eta'|}{|E_{n_1}|} |E_{n_1}| \leq 3 \frac{|E|}{2} \left( \frac{\# \eta'}{2} \right)^{1/2} |E_k \cap E_\ell|^{1/2} \), implying

\[
|E_k \cap E_\ell| \geq \frac{|E_{n_1}|^2}{|E_{n_1}|^2} \geq \epsilon \frac{|E|}{2} \frac{|E_k \cap E_\ell|}{|E_{n_1}|} = \epsilon \frac{|E|}{2} |E_{n_1}|^{1/2} |E|. \]

Since \( |k - \ell| > 10 M \log(10 \Lambda \epsilon^{-1}) > M \), then by Condition 11.3

\[
|F_k| \geq (10 \Lambda \epsilon^{-1})^M B_k B_{k\ell} \alpha_k \beta_{k\ell}.
\]
To simplify this, by the definitions of $T(\cdot, \cdot)$ and $E_k$, Eq. (11.1), and Eq. (11.2),

$$T_k(E_k \cap E_l, F_k) = \int_{E_k \cap E_l} T_k \chi_{F_k}(w) dw \gtrsim \int_{E_k \cap E_l} T_k(E, F_k)_{|F_k} \gtrsim \epsilon \frac{\partial \omega}{|b|} T_k(E, F),$$

so $\beta_{F_k} \approx \frac{1}{t} T_k(E_k \cap F_k)_{|F_k} \gtrsim \epsilon \frac{\partial \omega}{|b|} T_k(E, F)$, and $\alpha_{E_l} = \frac{1}{t} T_k(E, F)_{|E_l} \approx T_k(E, F)_{|E_l}$.

Putting these together,

$$|F|^{b-1} \gtrsim (10\Lambda \epsilon^{-1})^{MB-\frac{2 \epsilon}{k+2} T_k(E, F)_{|F_k}} \gtrsim (10\Lambda \epsilon^{-1})^{MB-\frac{2 \epsilon}{k+2} b-1 |E|^b |F|^{b}}$$

Consequently, $|F|^{b-1} |F_n| \gtrsim (10\Lambda \epsilon^{-1})^{MB-\frac{2 \epsilon}{k+2} b-1 |E|^b |F|^{b}}$, which for sufficiently large $M$ implies a contradiction, since $10\Lambda \epsilon^{-1} \geq 10$ and $F_n \subset F$ for all $n$.

And if (2) were false, by replacing each $F_n$ with $E_n$, $E_n$ with $F_n$, and $E_k$, and swapping $E$ and $F$, the proof plays out identically. □

Now, we can finally move on to the main proposition.

**Proposition 11.5.** The operator $TR_j^{\prime} \cap \{w \in [1, 2]\}$ is of rwt $(\frac{2}{3}, q)$ for all $q \in [4, \infty)$ if the following hold:

1. We can decompose $R_j^{\prime} \cap \{w \in [1, 2]\}$ into a finite number of subsets $S \in S$ such that for each $S, K \in S$ is either bounded or the following hold:
   1. $S$ is not bounded and $S$ is of the form $\{w \in [1, 2]\}$, and where $|\tau_n| \lesssim 1$ for each $n$.
   2. On each $\tau_n$, $|\partial_{z_1, 2} \varphi| \approx 2^{nD}$, and $|\partial_{z_2, 2} \varphi| \approx 2^{nK}$, for some fixed $D, K \in \mathbb{R} \setminus \{0\}$, $K \geq D$, that may depend on $S$, where $\{i_1, i_2\}$ is some permutation of $\{1, 2\}$.

**Proof.** If $S$ is bounded, then Young's Inequality, Theorem 2.1 (since $\omega \approx 1$), and interpolation give us the result directly. For $S$ not bounded, we will use the notation of Notation 11.1 and set $b = 3$ (for the purposes of Lemmas 11.2 and 11.3). By our hypotheses, $|\tau_n| \lesssim 1$, and since $\omega \approx 1$ on $S$, then by Theorem 2.1 $\|\tau_n\|_{L^4} \lesssim 1$ uniformly in $n$. Thus, by Lemma 11.2 and Lemma 11.3 it suffices to prove the following version of Condition 11.3.

**Lemma 11.6.** Let $k, l \in \mathbb{N}$, and let $k, l, \text{ and } |k-l|$ be sufficiently large, independent of $E$ and $F$. Furthermore, let $\text{sgn}(k-l) = \text{sgn}(D)$. Then hypotheses 1 and 2 of Proposition 11.3 imply that $|F|^{b-1} \gtrsim 2^{k-l \epsilon D} |E|^{\alpha_3^3} |F_k|^{\beta_{F_k}}$ and $|E|^2 \gtrsim 2^{k-l \epsilon D} |E|^{\alpha_3^3} |F_k|^{\beta_{F_k}}$, with implicit constants independent of $k, l, E, F$.

**Proof.** By symmetry, it will suffice to prove the $F_l$ inequality. Fix $u_0 \in F_k$. Define

$$\Omega_1 := \{ t \in \tau_k : u_0 - (t, \varphi(t)) \in E_k \cap E_l \}.$$

Then $|\Omega_1| = T_k \chi_{E_k \cap E_l}(u_0) \gtrsim \beta_{F_k}$. For $t \in \Omega_1$, define

$$\Omega_2(t) := \{ s \in \tau_l : w(t) + (s, \varphi(s)) \in F_l \}.$$

Then $|\Omega_2(t)| = T_l \chi_{F_l}(w(t)) \gamma_{E_l}$. Finally, we define $\Omega \subset \mathbb{R}^6$ and $\Psi : \Omega \rightarrow \mathbb{R}^6$;

$$\Omega := \{ (t, s_1, s_2) \in \mathbb{R}^6 : t \in \Omega_1, s_i \in \Omega_2(t), i = 1, 2 \};$$

$$\Psi(t, s_1, s_2) := u_0 - (t, \varphi(t)) + (s_1, \varphi(s_1)) \in \Omega.$$

Since $\Psi$ is a polynomial mapping $\Omega \subset \mathbb{R}^6$ into $\mathbb{R}^6$ with $\det D\Psi \neq 0$, this map is $O(1)$-to-one off a set of measure zero. Since $\Psi(\Omega) \subset F_l \times F_l$, then $|F_l|^2 \gtrsim$
\[ \int_{\Omega} |\det D\Psi(t_s, s_1, s_2)| dt_1 ds_1 ds_2. \]

Defining \( G(t, s_i) := \nabla \varphi(t) - \nabla \varphi(s_i) \), we can expand the Jacobian as follows:

\[ |\det D\Psi(t_s, s_1, s_2)| = |\det(G(t, s_1), G(t, s_2))| \approx |G(t, s_1)| |\text{dist}(G(t, s_2), \langle G(t, s_1) \rangle)|, \]

which implies that \( |E_i|^2 \gtrsim \int_{\Omega} |G(t, s_1)| |\text{dist}(G(t, s_2), \langle G(t, s_1) \rangle)| dt_1 ds_1 ds_2. \)

By hypothesis (2) of Proposition 11.5 after a possible reordering we can assume that that \( |\partial_{x_1} \varphi| \approx 2^{m_1} \) and \( |\partial_{x_2} \varphi| \approx 2^{m_2} \) on \( \tau_m \), for each \( m \). Additionally by our hypothesis, \( \text{sgn}(k-l) = \text{sgn}(D) \), so \( 2^{2\gamma} \geq 2^D \) and \( (k-l)D = |k-l||D| \).

Since \( K \neq 0 \) and \( K \geq D \), and since \( k, l, |k-l| \) are all sufficiently large, \( |G(t, s_1)| \approx |\pi_1(G(t, s_1))| \approx \max(2^{2\gamma}, 2^K) \). Here we recall that \( t \in \tau_k \) and \( s_1 \in \tau_l \) for \( i = 1, 2 \).

Fix \( t, s_1 \) and denote \( \overline{G}_{1,t,s_1} := G_1(t, s_1) \), which will be fixed, and \( \overline{G}_{2,t}(s_2) := G_2(t, s_2) \). Then the map \( s_2 \mapsto \overline{G}_{2,t}(s_2) \) has a 1st derivative comparable to \( |\omega| \approx 1 \), so that \( |\overline{G}_{2,t}(\Omega(t))| \gtrsim \alpha_{E_1} \) and

\[ \int_{\Omega_{E_1}} |\text{dist}(G(t, s_2), \langle G(t, s_1) \rangle)| ds_2 \approx \int_{\Omega_{E_1}} |\text{dist}(G_{2,t}, \langle G_{1,t,s_1} \rangle)| ds_2. \]

Since \( \pi_2(\overline{G}_{1,t,s_1} - \overline{G}_{2,t}) = \pi_2(\nabla \varphi(s_2) - \nabla \varphi(s_1)) \gtrsim 2^D \), then \( \overline{G}_{2,t}(\Omega(t)) \subset \mathbb{R} \times \overline{\Gamma}_{1,t,s_1} - C^2 \mathbb{R} \), \( \overline{\Gamma}_{1,t,s_1} + C^2 \mathbb{R} \) for some \( C \) sufficiently large. However, \( \text{slope}(\langle G_{1,t,s_1} \rangle) \approx \frac{2^{2\gamma}}{\max(2^{2\gamma}, 2^k)} \lesssim 1 \), so if we define \( \gamma := \alpha_{E_1} \max\{2^{\gamma}, 2^k\} \), then \( |\text{dist}(\langle G_{1,t,s_1} \rangle)\cap \mathbb{R} \times \overline{\Gamma}_{1,t,s_1} - C^2 \mathbb{R} | \gtrsim \gamma \frac{2^{2\gamma}}{\max(2^{2\gamma}, 2^k)}, \) \( \overline{\Gamma}_{1,t,s_1} + C^2 \mathbb{R} | \gtrsim \gamma \frac{2^{2\gamma}}{\max(2^{2\gamma}, 2^k)}, \) \( \overline{\Gamma}_{1,t,s_1} - C^2 \mathbb{R} | \gtrsim \gamma \frac{2^{2\gamma}}{\max(2^{2\gamma}, 2^k)}. \)

Therefore,

\[ \int_{\overline{\Gamma}_{1,t,s_1}} |\text{dist}(\langle G_{1,t,s_1} \rangle, \overline{G}_{2,t})| ds_2 \gtrsim |\text{dist}(\langle G_{1,t,s_1} \rangle, \overline{G}_{2,t})| \gtrsim \alpha_{E_1} \frac{2^{k+l}}{\max(2^{k+l}, 2^k)}. \]

Thus, \( |E_i|^2 \gtrsim \int_{\Omega_{E_1}} \int_{\Omega_{E_1}} \max\{2^{\gamma}, 2^k\} \frac{2^{k+l}}{\max(2^{k+l}, 2^k)} ds_1 dt \gtrsim 2^{k+l}|\Omega| \alpha_{E_1} \beta_{E_1}. \)

A near-identical argument gives \( |E_i|^2 \gtrsim 2^{k+l}|\Omega| \alpha_{E_1} \beta_{E_1}, \) by swapping \( E_n \) and \( F_n \), and using \( \Psi(t, s_1, s_2) = u_0 + (t, \varphi(t)) - (s_1, \varphi(s_1)) : i = 1, 2 \), with \( u_0 \in E_{\delta_k} \).

This completes the proof of Lemma 11.6.

Hence, by Lemmas 11.2 and 11.3 with \( b = 3 \), the proof of Proposition 11.5 is complete.

\section{Conclusion of case \( T = d_\omega \)}

**Proposition 12.1.** If \( v_j \) (likewise \( n_j \)) equals \( d_\omega \), then \( \varphi \) satisfies the hypotheses of Proposition 11.5.

**Proof.** First, consider the case \( j \geq 3 \). If \( n_j \geq d_\omega \), part (1) of Proposition 11.5 implies that \( \min(r, s) = 1 \). By symmetry, we may then choose \( s = 1 \) and perform the following change of variables over \( R^3_{\gamma} : x = x_1 + y_1 \) and \( y = y_2 \). Then

\begin{align*}
\partial_{x_1} &= \partial_x - \lambda y x^{r-1} \partial_y; & \partial_{x_2} &= \partial_y; \\
\partial_{x_2}^2 &= \partial_y^2; & \partial_{x_1 x_2} &= \partial_{xy} - \lambda y x^{r-1} \partial_y; & \partial_{x_2}^2 &= \partial_y^2 + \lambda^2 y^2 x^{2r-2} \partial_y^2 - 2 \lambda y x^{r-1} \partial_{xy} - \lambda r (r-1) x^{r-2} \partial_y.
\end{align*}
And finally the determinant Hessian $\omega = \det D^2 \varphi$ satisfies

$$
(12.2) \quad \omega = |\partial^2_{x^2} \varphi \partial^2_y \varphi - (\partial_{x^2 y} \varphi)^2| - \lambda_j r (r - 1) x^{r - 2} \partial_y \varphi \partial^2_{y^2} \varphi.
$$

The relation $d_\omega = T$, with $y$ being a factor of $\omega$ of multiplicity $T$, requires that $\omega = C x^T y + O(y^{T+1})$, so in the region $R^c_{\varphi} \cap \{|\omega| \approx 1\}$, $|x| \approx |y|^{-1}$. Similarly, when $j = 1, 2$, $d_\omega = T$ implies $\omega = C z_1^T x_2 + O(z_2^{T+1})$ after possibly swapping $z_1$ and $z_2$, so with the choice $x = z_1$, $y = z_2$, the last sentence also holds for $j = 1, 2$.

By symmetry, it suffices to consider cases with $\lambda_j \geq 0$ and regions with $z_1, z_2 \geq 0$. Additionally, we will focus on $y > 0$, as the proof in the region $y < 0$ is essentially identical.

Next, we decompose $R^c_{\varphi} \cap \{|\omega| \approx 1\} \cap \{z_1, z_2 - \lambda_j z_1 > 0\}$ into $\tau_n = \{x \approx 2^n, y \approx 2^{-n}\}$, with $n = 0, 1, 2, \ldots$. By hypothesis (0) of Proposition 11.5, it suffices to consider only $\tau_n$ where $n$ is sufficiently large, which will allow us to make the asymptotics in the next paragraph work. Each $\tau_n$ satisfies $|\tau_n| \approx 1$, satisfying the first condition of Proposition 11.5.

The partial derivative $\partial_{z_1} \varphi$ satisfies $\partial_{z_1} \varphi = C x^T y + O(y^{T+1}) \approx 2^{(J-L)n}$ for some $J, L \in \mathbb{N}_0$. Thus $\partial_{z_1} \varphi$ satisfies the second condition of Proposition 11.5 as long as $J \neq L$. Suppose $J = L$. Then $d_h(\partial_{z_1} \varphi) = \frac{y + J}{x + r}$, and since $T = d_\omega$, then by Lemma 6.3, $d_h(\partial_{z_1} \varphi) = d_h - \frac{y + J}{x + r} = \frac{y + 2}{x + r} - \frac{y + J}{x + r}$. Combining these equations, $\frac{2s}{r + s} = T - 1 - 2J \in \mathbb{N}$, so for some $m \in \mathbb{N}$, $2s = m(r + s)$, so $(2 - m)s = mr$. This can only be satisfied if $m = 1$ and $r = 1$. A similar argument leads to an identical result for $\partial_{z_2} \varphi$. Hence, except for the homogeneous case where $s = r = 1$, we have $J \neq L$, and the final condition of the Proposition 11.5 is satisfied.

Next, consider the homogeneous case. All factors are linear, so up to a linear transformation, $z_2^T$ is a factor of $\omega$. We can decompose our options for $\varphi$ in the following way:

1. $\varphi = z_1^M z_2 + O(z_2^{T+1})$, with $\nu \geq 2$;
2. $\varphi = z_1^M z_2 + O(z_2)$;
3. $\varphi = z_1^M + O(z_2)$;
4. $\varphi = z_1^M + cz_2^{M-1} z_2 + O(z_2^2)$.

By Lemma 6.3, option 2 would imply $T = 0$, so this case is trivial.

In option 1, $\omega = C z_1^{2M-2, \nu} + O(z_2^{T+1})$ by Lemma 6.3, and since $d_\omega = T$, it follows that $2M - 2 = 2\nu - 2 = T$, implying that $M = \nu$. Then, $\nabla \varphi \approx (z_1^{\nu-1} z_2^{\nu}, z_1^\nu z_2^{\nu-1}) \approx (2^{-n}, 2^n)$ on $\tau_n$, so the lemma conditions are satisfied.

In option 3, by Remark 6.10 following Proposition 6.10, $\varphi = z_1^M + cz_2^M$, some $c \neq 0$. Therefore, $\omega = C z_1^{M-2, \nu} + O(z_2^{T+1})$, and since $\nabla \varphi \approx (z_1^{M-1}, z_2^{M-1}) = (2^{(M-1)n}, 2^{-(M-1)n})$, and since $\nabla \varphi(0, 0) = 0$ and $T > 0$ each require $M \neq 1$, we are done.

Finally, in option 4, $\nabla \varphi \approx (z_1^{M-1}, z_1^{M-1})$, and since $\nabla \varphi(0, 0) = 0$ requires $M \neq 1$, we are done.

Thus, in each case where $d_\omega = T$, the conditions of Proposition 11.5 hold. $
$

Hence, by Proposition 11.5, the proof of Lemma 10.8 and thereby Proposition 10.3 is complete.
13. Twisted Cases

In this section, we will prove that in the twisted cases, the high order vanishing of \( \omega \) has no effect on the \( L^p \) bounds, by proving the following:

**Proposition 13.1.** In the twisted cases (i), (iia), and (iib), \( T_{R_T} \) is of rwt \((p_{v_1}, q_{v_1})\).

A few concrete examples of functions \( \varphi \) falling into the twisted cases are:

\[
\varphi = z_1^4 + z_1^2 z_2 + \frac{1}{2} z_2^2; \quad \varphi = z_1^5 + z_1^3 z_2 + \frac{9}{40} z_1 z_2^2 \quad \text{(Case(i))};
\]

\[
\varphi = z_1^4 + z_1^2 z_2 + z_2^2 \quad \text{(Case(iia))};
\]

\[
\varphi = (z_2 - z_1^3)(z_2 - 2z_1^2) \quad \text{(Case(iib))}.
\]

First, we will discuss the idea of the proof. As before, we will decompose \( R_T \subset [-1,1]^2 \) into suitable dyadic rectangles \( \tau_{j,k} \). In rectangular cases, the dominant factor \( f_T \) of \( \omega \) is also a factor of one or both components of \( \nabla \varphi \), which causes \( \nabla \varphi(\tau_{j,k}) \) to be essentially convex. In the twisted cases, on the other hand, the dominant factor \( f_T \) of \( \omega \) arises from a “fortuitous” cancellation, and not in an obvious way from the structure of \( \varphi \) and \( \nabla \varphi \). This leads to \( \nabla \varphi \) “twisting” near the associated curve \( f_T = 0 \). Because of this, \( \nabla \varphi(\tau_{j,k}) \subset \mathbb{R}^2 \) is highly non-convex, and similar to the neighborhood of a parabola.

In the method of refinements, the Jacobian determinant, which is used to find \( L^{4/3} \to L^4 \) bounds, relies primarily on the convex hull of \( \nabla \varphi(\tau_{j,k}) \), while the \( L^\infty \to L^\infty \) bounds are connected to the measure of \( \nabla \varphi(\tau_{j,k}) \). Hence, \( \nabla \varphi(\tau_{j,k}) \) being highly non-convex causes the classes of \( L^\infty \to L^\infty \) and \( L^{4/3} \to L^4 \) near-extremizers to be disjoint, and quantifying that tradeoff leads to a much better bounds, implying that \( T_{R_T} \) is of rwt \((p_{v_1}, q_{v_1})\), with \((p_{v_1}, q_{v_1})\) lying on the scaling line.

We start our arguments with two lemmas that will be necessary to show that \( \nabla \varphi(\tau_{j,k}) \) is always highly non-convex in the twisted cases.

**Lemma 13.2.** Set \( y := z_2 - z_1^r \), where \( r \geq 2 \). If \( y \) is a factor of \( \omega \) with multiplicity \( T > d_\omega \), and \( y \) is not a factor of \( \varphi \), then \( y \) is not a factor of \( \partial_{z_2} \varphi \).

**Proof.** We express \( \varphi \) as a polynomial in \( x := z_1 \) and \( y := z_2 - z_1^r \). By the hypotheses of Theorem \([\text{23}]\) and the lemma, \( \varphi \neq z_1^j \) and \( y \) does not divide \( \varphi \). Thus, by the mixed homogeneity, after rescaling,

\[
\varphi = x^J + c_1 x^{J-1} y^l + \mathcal{O}(y^{l+1}),
\]

for some \( J \neq 0, c_1 \neq 0 \). Suppose \( l \geq 2 \). From \([\text{122}]\),

\[
\omega = J(J-1)c_1 l(l-1)x^{2J-lr-2}y^{l-2} + \mathcal{O}(y^{l-1}).
\]

Thus, \( y \) is a factor of \( \omega \) with multiplicity \( T = l - 2 \). However, since we require \( T > d_\omega \), and we know \( d_\omega = 2dh - 2 = \frac{2}{r+1} - 2 \), then \( 2J < l(r+1) \). From the \( c_1 \) term of \( \varphi \), we know that \( J \geq lr \geq l \), contradicting \( 2J < l(r+1) \). Thus, \( l = 1 \), in which case \( y \) is not a factor of \( \partial_{y} \varphi = \partial_{z_2} \varphi \).

**Lemma 13.3.** If \( y := z_2 - z_1^r \), \( r \geq 2 \), is a factor of \( \partial_{z_1} \varphi \), and a factor of \( \omega \) with multiplicity \( T > d_\omega \), but not a factor of \( \varphi \), then \( \partial_{z_1} \varphi \equiv 0 \).

**Proof.** Assume \( \partial_{z_1} \varphi \neq 0 \) and let \( M \neq 0 \) denote the multiplicity of \( y \) in \( \partial_{z_1} \varphi \). Defining \( x := z_1 \), by Lemma \([\text{13.2}]\), we can write \( \varphi \) as follows, for some \( c_1 \neq 0 \), after rescaling:

\[
\varphi = x^J + c_1 x^{J-r} y + \mathcal{O}(y^2).
\]
Claim: Let $M$ be the multiplicity of $y$ in $\partial_{z_1} \varphi$, and suppose that $M \neq 0$ and $\partial_{z_1} \varphi \neq 0$. Then $y$ has multiplicity $M - 1$ in $\omega$.

Proof: First, we rewrite $\omega$ in a way that preserves its relationship with $\partial_{z_1} \varphi$, while still taking advantage of the coordinates $x$ and $y$:

$$\omega := \partial_x (\partial_{z_1} \varphi) \partial_{yy} \varphi - \partial_y (\partial_{z_1} \varphi) \partial_{xy} \varphi.$$ 

Since $M \neq 0$, $y$ is a factor of $\partial_y (\partial_{z_1} \varphi)$ of multiplicity $M - 1$, and since $c_1 \neq 0$ (and $J > r$), $y$ is not a factor of $\partial_{xy} \varphi$. Additionally, $y$ is a factor of $\partial_x (\partial_{z_1} \varphi)$ with multiplicity at least $M$. Thus, $y$ is factor of $\omega$ of multiplicity $M - 1$. \quad \blacksquare

Since $d(\partial_{z_1} \varphi) = \frac{dJ + K}{1 + r}$ for some $K$, and by Lemma 6.9 (twice), the assumption $d_\omega < T$, and our above claim,

$$2 \frac{Mr}{1 + r} \leq 2d(\partial_{z_1} \varphi) = 2d_h - \frac{2}{1 + r} = d_\omega + 2 - \frac{2}{1 + r} < T + 2 - \frac{2}{1 + r} = M + 1 - \frac{1}{2 + r}.$$ 

This simplifies to $(M - 1)(r - 1) < 0$, which is impossible by assumptions $M \geq 1$ and $r \geq 1$. Hence $\partial_{z_1} \varphi \equiv 0$. \quad \Box

Next, to shorten the remainder of the argument, we will unify the twisted cases. In Case(i), we can assume that $f_T = z_2$, so that $z_2$ has multiplicity $T$ in $\omega$ and multiplicity $0$ in $\varphi$. Thus, after rescaling, because of mixed homogeneity $\varphi$ can take the following form:

$$(i) \quad \varphi = z_1^r + c_1 z_1^{r-r} z_2 (1 + \mathcal{O}(\frac{z_2}{z_1^{r-1}})) = x^J + c_1 x^{J-r} y (1 + \mathcal{O}(\frac{y}{x^{r-1}})),$$

for some $J \in \mathbb{N}$, where we used the change of coordinates $x := z_1, y := z_2$ for (i). Similarly, in Cases(ii-a) and (ii-b), we can assume after rescaling that $f_T = z_2 - z_1^r$, so that $z_2 - z_1^r$ has multiplicity $T$ in $\omega$ and multiplicity $0$ or $1$ respectively in $\varphi$.

Thus, after rescaling, because of mixed homogeneity, $\varphi$ can take the following form:

$$(ii-a) \quad \varphi = z_1^r (1 + \mathcal{O}(\frac{z_2}{z_1^r})) = x^J (1 + \mathcal{O}(\frac{x}{y^{r-1}}));$$

$$(ii-b) \quad \varphi = z_1^{r-r} (z_2 - z_1^r) (1 + \mathcal{O}(\frac{z_2 - z_1^r}{z_1^r})) = x^{J-r} y (1 + \mathcal{O}(\frac{y}{x^{r-1}})), $$

for some $J$, where we used the change of coordinates $x := z_1, y := z_2 - z_1^r$ for (ii-a-b). Also, we can assume that $r \geq 2$ by Lemma 6.9. Observe that in each case, (13.1)

$$d_h = \frac{1}{r - 1}.$$ 

The following lemma will unify the form of $\varphi$:

Lemma 13.4. In each twisted case, for some $a_0, b_0 \neq 0$, $\nabla \varphi$ can be written as

$$\nabla \varphi = (\partial_{z_1} \varphi, \partial_{z_2} \varphi) = (x^{J-1} (a_0 + \mathcal{O}(\frac{x}{y})), x^{J-r} (b_0 + \mathcal{O}(\frac{x}{y}))).$$

Proof. In case(i), $x = z_1$ and $y = z_2,$ and

$$\nabla \varphi = (J z_1^{J-1} + \mathcal{O}(z_2), c_1 (J - r) z_1^{J-r} + \mathcal{O}(z_2)), $$

where by our definition of case(i), $c_1 \neq 0$. Thus $a_0 = J$ and $b_0 = c_1 (J - r)$ are both nonzero, since if $J = r, \varphi$ would violate the assumption $\nabla \varphi(0) = 0$.

For cases (ii-a) and (ii-b), we will need to rewrite $\partial_{z_1}$ and $\partial_{z_2}$ in terms of $\partial_x$ and $\partial_y$. Doing so, we have

$$\partial_{z_1} = \partial_x - r x^{r-1} \partial_y; \quad \partial_{z_2} = \partial_y.$$ 

In case (ii-a), $\varphi = x^J + c_1 x^{J-r} y + \mathcal{O}(y^2)$, and so

$$\nabla \varphi = (\partial_x \varphi - r x^{r-1} \partial_y \varphi, \partial_y \varphi) = (J - r c_1) x^{J-1} + \mathcal{O}(y), c_1 x^{J-r} + \mathcal{O}(y)).$$
By Lemma 6.3, \( \omega \neq 0 \), so by Lemmas 13.3 and 13.2 respectively, \( a_0, b_0 \neq 0 \).

In this case (iiib), \( \varphi = x^{J-r}y + O(y^T) \), and so

\[
\nabla \varphi = (\partial_x \varphi - rx^{J-1}\partial_y \varphi, \partial_y \varphi) = (-rx^{J-1} + O(y), x^{J-r} + O(y)),
\]

resulting in \( a_0 = -r \) and \( b_0 = 1 \) being both nonzero.

Additionally, we want to consider \( \omega \) itself. There exists some \( Q \) such that

\[
\omega \approx x^{Q}y^T + O(y^{T+1}) = x^{Q+Tr}(\frac{y}{x})^T + O(y^{T+1}),
\]

where \( d_\omega = \frac{Q+Tr}{r+1} = 2d_\mu - 2 = \frac{2J-2r-2}{r+1} \) by (13.1). Thus, \( Q + Tr = 2J - 2r - 2 \), so

\[
(13.2) \quad \omega = x^{Q+Tr}(\frac{y}{x})^T + O(y^{T+1}) = x^{2J-2r-2}(\frac{y}{x})^T + O(y^{T+1}).
\]

Next, we decompose \( R_T = \{ \frac{y}{x} < \epsilon, 0 < x < 1 \} \). Rescaling, it suffices to only consider \( x < \bar{c} \), for some small constant \( \bar{c} \) independent of \( x \) and \( y \). For simplicity, we give details when \( y > 0 \), the case \( y < 0 \) being similar.

We first define, in cases (iia-b), function \( z(x, y) := (x, y + x^T) \), and in case (i), we define \( z \) to be the identity. Thus, in each case, \( z(x, y) = (z_1, z_2) \).

Dyadically decomposing \( R_T \), we define \( m = (m_1, m_2) \in \mathbb{N}^2, x_m := 2^{-m_1}, x_m := 2^{-m_2}, \) and define curved rectangles \( \tau_m := z([x_m, 2x_m] \times [y_m, 2y_m]) \), along with extended rectangles \( \tau^e_m := z([\frac{1}{2}x_m, 3x_m] \times [y_m, 2y_m]) \). Denote \( T_m := T_m \).

To begin, we want to refine \( E \sim E_m \) and \( F \sim F_m \) so that \( T_m(E, F) \approx T_m(E_m, F_m) \). Define

\[
E_m := \{ w \in E : T_m \chi_F(w) \geq \frac{1}{4} \frac{T_m(E, F)}{|E|} \},
\]

\[
F_m := \{ u \in F : T_m \chi_{E_m}(u) \geq \frac{1}{4} \frac{T_m(E_m, F_m)}{|F|} \} =: \alpha_m
\]

\[
E_m := \{ w \in E_m : T_m \chi_{E_m}(w) \geq \frac{1}{4} \frac{T_m(E_m, F_m)}{|E_m|} \} =: \beta_m
\]

Then \( T_m(E_m, F_m) \approx T_m(E, F) \), by an argument similar to the argument leading to (111).

To proceed, we will create a map, and use the size of the Jacobian to acquire the desired bound. Fix \( e_0 \in E_m \), and define:

\[
\Omega_1 := \{ t \in \tau_m \text{ and } e_0 + (t, \varphi(t)) =: e_1(t) \in F \}.
\]

Then \( |\Omega_1| = T_m \chi_{F_m}(e_0) \geq \beta_m \). Next, define, for \( t \in \Omega_1 \),

\[
\Omega_2(t) := \{ s \in \tau_m \text{ and } e_1(t) - (s, \varphi(s)) \in E_m \}.
\]

Then \( |\Omega_2(t)| = T_m \chi_{E_m}(e_1(t)) \geq \alpha_m \). Define the following:

\[
\Omega := \{ (t, s^{(1)}, s^{(2)}) \in \mathbb{R}^6 : t \in \Omega_1, s^{(i)} \in \Omega_2(t), i = 1, 2 \};
\]

\[
\Psi(t; s^{(1)}, s^{(2)}) = (e_0 + (t, \varphi(t)) - (s^{(i)}, \varphi(s^{(i)}))) : i = 1, 2.
\]

Since \( \Psi \) is a polynomial mapping \( \Omega \subset \mathbb{R}^6 \) into \( \mathbb{R}^6 \), with \( \det D\Psi \neq 0 \), it is \( O(1) \)-to-one off a set of measure zero. Then \( \Psi(\Omega) \subset E_m^2 \) implies

\[
|E_m^2| \geq \int_{\Omega} |\det D\Psi(t; s^{(1)}, s^{(2)})| \, dt \, ds^{(1)} \, ds^{(2)}.
\]

Expanding,

\[
|\det D\Psi(t; s^{(1)}, s^{(2)})| = |\det (\nabla \varphi(s^{(1)}) - \nabla \varphi(t), \nabla \varphi(s^{(2)}) - \nabla \varphi(t))| \approx \mu(\text{Conv}\{ \nabla \varphi(s^{(1)}), \nabla \varphi(s^{(2)}), \nabla \varphi(t) \}).
\]
becomes horizontal. We then reflect \( \rho \) as the graph (image. Let \( \tilde{p} \) set of all unit tangent vectors of \( \gamma \) on some subset 

Lemma 13.5. Let \( t \in \Omega_1 \) be fixed. If \( \alpha_m \geq C \frac{\mu m}{x_m y_m} \), then 

\[
\mu(\text{Conv}(\nabla \varphi(t), \nabla \varphi(s^{(1)}), \nabla \varphi(s^{(2)}))) \geq \frac{1}{(\inf \kappa) \alpha^3} \]

on some subset \( \{t\} \times \Omega_2^{(1)}(t) \times \Omega_2^{(2)}(t) \subset \{t\} \times \Omega_2(t) \times \Omega_2(t) \), with \( |\Omega_2^{(i)}(t)| \geq \frac{|\Omega_2(t)|}{\alpha} \).

We will use Lemma 13.6 together with Lemmas 13.7-13.14 to prove Lemma 13.5.

Lemma 13.6. Let \( \gamma \) be a \( C^2 \) curve in \( \mathbb{R}^2 \), with curvature \( \kappa \approx \Theta \), and such that the set of all unit tangent vectors of \( \gamma \) belongs to the same half quadrant, or its mirror image. Let \( p_1, p_2, p_3 \in \mathbb{R}^2 \) lie in a \( \delta \Theta \) neighborhood of \( \gamma \), and satisfy \( \|p_i - p_j\| > a \) for all \( i \neq j \). If \( \alpha > 40 \max(\delta^2 \Theta, \delta(\alpha m) \frac{1}{\alpha}) \), then \( \mu(\text{Conv}(p_1, p_2, p_3)) \geq \Theta a^3 \).

Proof. There exist points \( \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \) on \( \gamma \) closest to \( p_1, p_2, p_3 \), respectively, and since \( a > 40 \delta^2 \Theta \), the points \( \tilde{p}_i \) have separation at least \( \frac{a}{4} \). Since \( \gamma \) cannot be a closed curve by our hypothesis on tangent lines, we can reorder indices such that \( \tilde{p}_2 \) lies between \( \tilde{p}_1 \) and \( \tilde{p}_3 \) on \( \gamma \).

We translate and rotate \( \mathbb{R}^2 \) so that \( \tilde{p}_2 = 0 \) and the tangent line of \( \gamma \) at 0 becomes horizontal. We then reflect \( \mathbb{R}^2 \) if needed so that \( \gamma \) lies in the upper half plane and so that \( \tilde{p}_1 \) lies in the left plain, and \( \tilde{p}_3 \) in the right. We can then write \( \gamma \) as the graph \( (x, g(x)) \) of some function \( g \), where \( g(0) = g'(0) = 0, |g'| \leq 1 \), and \( g'' \in (\frac{1}{2}, \inf \kappa, 4 \sup \kappa) \). Therefore, \( g(x) \geq \frac{1}{4}(\inf \kappa) x^2 \).

We recall that \( \kappa \) projects points in \( \mathbb{R}^2 \) onto their \( i \)-th coordinate. The separation of the \( \tilde{p}_i \) must be at least \( \frac{a}{4} \), and since \( |g'| \leq 1 \), then \( |\kappa(\tilde{p}_1)|, |\kappa(\tilde{p}_2)|, |\kappa(\tilde{p}_3)| \geq \frac{\alpha m}{2\sqrt{2}} \).

Therefore, \( |\kappa(\tilde{p}_1)|, |\kappa(\tilde{p}_2)|, |\kappa(\tilde{p}_3)| \geq \frac{\alpha m}{2\sqrt{2}} \).

Since \( \tilde{p}_i \) is within \( \delta^2 \Theta \) of \( p_i \) for each \( i \), and since \( a > 40 \max(\delta^2 \Theta, \delta(\alpha m) \frac{1}{\alpha}) \), then \( p_1, p_2, p_3 \) form a triangle with base and height greater than \( \frac{\alpha m}{2\sqrt{2}} \) and \( \frac{1}{\alpha^2} \sup \kappa \alpha^2 \), respectively. Hence, the triangle formed by \( p_1, p_2, p_3 \) has area \( \geq \Theta a^3 \).

Using Lemma 13.6 to prove Lemma 13.5 will require us to identify a curve, bound its curvature, restrict the possible unit tangent vectors, establish a minimal separation of points, and show points remain in a neighborhood of the curve. A suitable curve \( \gamma \), or \( \gamma' \), will be defined in Lemmas 13.8-13.10. The curvature will be bounded in Lemma 13.8. Lemma 13.11 will restrict the possible unit tangent vectors. The minimal separation comes from Lemma 13.7 and our lower bound on \( \alpha_m \). And finally, the neighborhood size \( H \) will be bounded in Lemmas 13.11-13.14.
where \( \hat{\Theta} \).

Lemma 13.7. Define new coordinates
\[
\hat{t} := z^{-1}(t); \quad \hat{s}^{(i)} := z^{-1}(s^{(i)}), \quad i = 1, 2,
\]
where we recall that \( z \) was defined so that \( z^{-1}(\tau_m) \) is rectangular. Additionally, define
\[
\Phi \equiv (\Phi_1, \Phi_2) := (\text{sgn}(a_0)(\partial_{z_i} \varphi) \circ z, \text{sgn}(b_0)(\partial_{z_2} \varphi) \circ z).
\]
By Lemma 13.4 \( a_0, b_0 \neq 0 \) and
\[
(13.4) \quad \Phi(x, y) = (x^{-1}[a_0] + \tilde{a}_1 x^r + \mathcal{O}(\frac{x}{z^r}), x^{-1}[b_0] + \tilde{b}_1 x^r + \mathcal{O}(\frac{x}{z^r})),
\]
where \( \tilde{a}_1 \) and \( \tilde{b}_1 \) may be zero. Because \( \Phi \) and \( \nabla \varphi \circ z \) are equal up to coordinate sign changes, we can work purely with \( \Phi \), since
\[
(13.5) \quad \mu(\text{Conv}(\Phi(t), \Phi(\hat{s}^{(1)}), \Phi(\hat{s}^{(2)}))) = \mu(\text{Conv}(\nabla \varphi(t), \nabla \varphi(s^{(1)}), \nabla \varphi(s^{(2)}))).
\]
First, we will show that \( x \) and \( \epsilon_2 = \Phi_2(x, y) \) are strongly related.

Lemma 13.7. Let \( (x, y), (x + \Delta x, y + \Delta y) \in z^{-1}(\tau_m) \), and define \( \delta_\epsilon := x^{-\frac{\nu}{2}} \) and \( \delta_\Phi := x^{J-r} \) as the \( x, \Phi \) uncertainty scales, respectively. Then if either
\[
|\Phi_2(x + \Delta x, y + \Delta y) - \Phi_2(x, y)| \geq C\delta_\Phi, \text{ or } |\Delta x| \geq C\delta_\epsilon,
\]
for a sufficiently large \( C \), then
\[
\text{sgn}(\Delta x)(\Phi_2(x + \Delta x, y + \Delta y) - \Phi_2(x, y)) \approx x^{J-r-1}|\Delta x|.
\]
Proof. By (13.4), recalling that \( b_0 \neq 0 \), we have
\[
|\partial_x \Phi_2| = x^{J-r-1}[|b_0|(J-r) + \mathcal{O}(\frac{x^r}{z^r})] \approx x^{J-r-1}
\]
\[
|\partial_y \Phi_2| = |x^{J-2r}[\tilde{b}_1 + \mathcal{O}(\frac{x^r}{z^r})]| \lesssim x^{J-2r}.
\]
Therefore, since \( |\Delta y| \lesssim y \) in \( z^{-1}(\tau_m) \), a change in \( y \) will only change \( \Phi_2 \) by at most \( C x^{J-2r} \), \( y = C\delta_\epsilon \). Next, consider \( x_2 > x_1 \). Since \( \partial_y \Phi_2 \approx x^{J-r-1} \), then \( \Phi_2(x_2, y_2) - \Phi_2(x_1, y_2) \approx (x_2 - x_1)x^{J-r-1} \). Thus, by the triangle inequality, if \( |\Phi_2(x_2, y_2) - \Phi_2(x_1, y_1)| > C\delta_\epsilon \), or if \( |x_2 - x_1| > C\delta_\epsilon \), for some large \( C > 0 \), then \( \Phi_2(x_2, y_2) - \Phi_2(x_1, y_1) \approx (x_2 - x_1)x^{J-r-1} \). This concludes the proof of Lemma 13.7.

The next lemmas define and analyze the curves that will be used with Lemma 13.6.

Lemma 13.8. For \( y \in [y_m, 2y_m] \), \( \gamma_y : x \mapsto \Phi(x, y) \) has curvature \( \kappa_y \approx x^{-[J-2r+1]} := \hat{\Theta} \).

Proof of Lemma 13.8. By direct computation using (13.4),
\[
\kappa_y(x) = x^{-[J-2r+1]} \frac{|a_0 b_0|(J-1)(J-r)(r-1) + \mathcal{O}(\frac{x}{z^r})}{[x^{2r-3} a_0^2 + b_0^2 + \mathcal{O}(\frac{x}{z^r})]^\frac{3}{2}}.
\]
Then, since \( 0 < x \leq 1 \) and \( a_0, b_0 \neq 0 \),
\[
\kappa_y(x) \approx x^{-[J-2r+1]} \approx x^{-[J-2r+1]} := \hat{\Theta},
\]
concluding the proof of Lemma 13.8.

□
Lemma 13.9. Let $y \in [y_m, 2y_m]$. Then there exists a function $\gamma^*_y : \Phi_2(z^{-1}(\tau_m)) \to \mathbb{R}$ such that

$$
\gamma_y([1/2 x_m, 3x_m]) \cap \{\mathbb{R} \times \Phi_2(z^{-1}(\tau_m))\} = \{(\xi_1, \xi_2) : \xi_1 = \gamma_y^*(\xi_2), \xi_2 \in \Phi_2(z^{-1}(\tau_m))\}.
$$

This lemma is equivalent to the following lemma:

Lemma 13.10. Let $y \in [y_m, 2y_m]$. Then, for each $\xi_2 \in \Phi_2(z^{-1}(\tau_m))$, there exists a unique $\xi_1$ such that $(\xi_1, \xi_2) \in \gamma_y([1/2 x_m, 3x_m])$.

Proof of Lemma \[13.10\]. Recall that $\pi_i$ is projection onto the $i$-th coordinate.

(1) Existence: If $\xi_2 \in \Phi_2(z^{-1}(\tau_m))$, there exists $(\tilde{x}, \tilde{y}) \in z^{-1}(\tau_m)$ such that $\Phi_2(\tilde{x}, \tilde{y}) = \xi_2$. Now consider points in $z^{-1}(\tau_m)$ with minimal and maximal $x$-values: $(\frac{x_m}{2}, y)$ and $(3x_m, y)$. Since $\frac{x_m}{2} \leq \tilde{x} - \frac{x_m}{2} \leq \tilde{x} + c_\delta y$ and $3x_m \geq \tilde{x} + x_m \geq \tilde{x} + c_\delta y$, then

$$
\Phi_2(\frac{x_m}{2}, y) < \xi_2 < \Phi_2(3x_m, y)
$$

by Lemma \[13.7\]. Then, by continuity of $\gamma_y(x)$ and the Intermediate Value Theorem, there exists $x \in [\frac{x_m}{2}, 3x_m]$ such that $\Phi_2(x, y) \equiv \pi_2(\gamma_y(x)) = \xi_2$.

(2) Uniqueness: By \[13.4\], $\pi_1(\frac{dx}{dy} \gamma_y(x)) = \partial_x \Phi_1(x, y) = x^{J-2}(a_0(J-1) + O(\frac{1}{x})) > 0$ for each $(x, y) \in z^{-1}(\tau_m)$, so for fixed $y \in [y_m, 2y_m]$, the map $x \mapsto \pi_1(\gamma_y(x)) \equiv \Phi_1(x, y)$ is one-to-one. Hence, for fixed $y$, $\xi_1$ is mapped to by at most a single unique $\pi_i$, and hence a unique $\tilde{\xi}_2 = \Phi_2(\pi_i, y)$.

This concludes the proof of Lemma \[13.10\]. \qed

Lemma 13.11. For every $y \in [y_m, 2y_m]$, $|\frac{d}{dx} \gamma^*_y(\xi_2)| < \frac{1}{10}$ and $|\frac{d^2}{dx^2} \gamma^*_y(\xi_2)| \approx x_m^{-[J-2r+1]}$ for all $\xi_2 \in \Phi_2(\tau_m)$.

Proof. Since $|\partial_x \Phi_1| \approx x_m^{J-2}$, and $|\partial_x \Phi_2| \approx x_m^{J-r-1}$, then $|\frac{d}{dx} \gamma^*_y(\xi_2)| \approx x_m^{-1} < \frac{1}{10}$, so the curvature of $\xi_2 \mapsto (\gamma^*_y(\xi_2), \xi_2)$ is comparable to $|\frac{d}{dx} \gamma^*_y(\xi_2)|$. Finally, by Lemma \[13.8\], the curve $\xi_2 \mapsto (\gamma^*_y(\xi_2), \xi_2)$ has curvature $\approx x_m^{-[J-2r+1]}$, concluding the proof of Lemma \[13.11\]. \qed

Next, define

$$
\Phi(z^{-1}(\tau_m))^c := \Phi(z^{-1}(\tau_m)) \cap \{\mathbb{R} \times \Phi_2(z^{-1}(\tau_m))\},
$$

and observe that $\Phi(z^{-1}(\tau_m))^c$ satisfies

$$(13.6) \Phi(z^{-1}(\tau_m))^c \subset \Phi(z^{-1}(\tau_m))^c \subset \Phi(z^{-1}(\tau_m)).$$

The expanded image $\Phi(z^{-1}(\tau_m))^c$ essentially rectangularizes our image $\Phi(z^{-1}(\tau_m))$, making it more practical for proving results.

Lemma 13.12. The set $\Phi(z^{-1}(\tau_m))^c$ satisfies

$$
\Phi(z^{-1}(\tau_m))^c = \cup_{\xi_2 \in \Phi_2(z^{-1}(\tau_m))} \min(\gamma^*_y(\xi_2), \gamma^*_y(\xi_2)) \times \{\xi_2\}. \quad \Phi(z^{-1}(\tau_m))^c = \max(\gamma^*_y(\xi_2), \gamma^*_y(\xi_2)) \times \{\xi_2\}.
$$

Proof. Since $|Jac \Phi| \approx |\omega| > 0$ on $\tau_m$, $\Phi$ is an open map by the inverse function theorem, so $\Phi^{-1}(\text{Bdry}\Phi(z^{-1}(\tau_m))) \subset \text{Bdry}z^{-1}(\tau_m)$. By Lemma \[13.7\], for all $y \in [y_m, 2y_m]$, $\Phi_2(x_m, y) \not\subseteq \Phi(z^{-1}(\tau_m))$, so the boundary of $\Phi(z^{-1}(\tau_m))^c$ is a union of two horizontal line segments and subsets of the curves $(\gamma^*_y(\cdot), \cdot)$ and $(\gamma^*_y(\cdot), \cdot)$ over $\Phi_2(z^{-1}(\tau_m))$. \qed
Lemma 13.13. For $\xi_2 \in \Phi_2(z^{-1}(\tau_m))$, $\mu$ the Lebesgue measure on $\mathbb{R}$, $\mu(\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_m))\}) = \text{diam}\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_m))\} = |\gamma_{2y_m}(\xi_2) - \gamma_{y_m}^*(\xi_2)|$.

Proof. Follows directly from Lemma 13.12. □

To proceed, we will use the following definitions:

\[ W_{[\xi_2^{(1)}, \xi_2^{(2)}]} := \frac{1}{\xi_2^{(2)} - \xi_2^{(1)}} \int_{\xi_2^{(1)} \leq \xi_2 \leq \xi_2^{(2)}} \text{diam}\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_m))\} \, d\xi_2; \]

\[ \mathcal{H} := \max_{\xi_2 \in \Phi_2(z^{-1}(\tau_m))} \text{diam}\{\xi_1 : (\xi_1, \xi_2) \in \Phi(z^{-1}(\tau_m))\}. \]

Lemma 13.14. If $\xi_2^{(1)}, \xi_2^{(2)} \in \Phi_2(z^{-1}(\tau_m))$, and $\xi_2^{(2)} - \xi_2^{(1)} > C\delta_\phi$, then

\[ W_{[\xi_2^{(1)}, \xi_2^{(2)}]} \approx x_m^{J-1}(\frac{ym}{x_m})^{T+1}. \]

Proof. By Lemma 13.17, $\Phi([x_1, x_2] \times [ym, 2ym]) \subset \Phi(z^{-1}(\tau_m)) \cap (\mathbb{R} \times [\xi_2^{(1)}, \xi_2^{(2)}]) \subset \Phi([x_1', x_2'] \times [ym, 2ym])$, some $x_1', x_2', x_1, x_2 \in [\frac{1}{2}x_m, 3x_m]$, where $x_2 - x_1 \approx x_2' - x_1' \approx (\frac{1}{x_m})^{J-1}(\xi_2^{(2)} - \xi_2^{(1)})$. Then

\[ W_{[\xi_2^{(1)}, \xi_2^{(2)}]} \approx \frac{|\Phi([x_1, x_2] \times [ym, 2ym])|}{\xi_2^{(2)} - \xi_2^{(1)}} \approx \frac{|\text{Jac}\Phi|ym(x_2 - x_1)}{\xi_2^{(2)} - \xi_2^{(1)}} \approx x_m^{J-2r-2}(ym/x_m)^T y_m x_m^{-1} \approx x_m^{J-1}(ym/x_m)^{T+1} \]

since $|\text{Jac}\Phi| \approx |\omega| \approx x_m^{J-2r-2}(ym/x_m)^T$ by (13.2). □

Lemma 13.15. The following inequality holds: $\mathcal{H} \lesssim \hat{\Theta}\delta^2_\phi$.

Proof. We can consider two cases: $\mathcal{H} \gg W_{\Phi_2(z^{-1}(\tau_m))}$ or $\mathcal{H} \lesssim W_{\Phi_2(z^{-1}(\tau_m))}$.

If $\mathcal{H} \lesssim W_{\Phi_2(z^{-1}(\tau_m))}$, then

\[ \mathcal{H} \leq x_m^{J-1}(ym/x_m)^{T+1} \leq x_m^{J-1}(ym/x_m)^2 = x_m^{J-2r+1}(ym/x_m)^2 = \hat{\Theta}\delta^2_\phi \]

by Lemma 13.14. $T \geq 1$, simple reordering, and the definitions of $\hat{\Theta}$ and $\delta_\phi$.

If $\mathcal{H} \gg W_{\Phi_2(z^{-1}(\tau_m))}$, then by Lemma 13.13 there exist $c_1, \xi_2$ such that $c_1 \in [\xi_2^{(1)}, \xi_2^{(2)} + 3C\delta_\phi] \subset \Phi_2(z^{-1}(\tau_m))$ and

\[ |\gamma_{2y_m}(c_1) - \gamma_{y_m}^*(c_1)| \gtrsim \mathcal{H}. \]

Additionally, by Lemma 13.14

\[ \text{avg}_{[\xi_2^{(1)}, \xi_2^{(2)}]}|\gamma_{2y_m} - \gamma_{y_m}^*| \approx W_{\Phi_2(z^{-1}(\tau_m))} \approx \text{avg}_{[\xi_2^{(1)}, \xi_2^{(2)}] + 3C\delta_\phi}|\gamma_{2y_m} - \gamma_{y_m}^*|, \]

and so, by the Intermediate Value Theorem, there exist $c_2 \in [\xi_2^{(1)} + 2C\delta_\phi, \xi_2^{(2)} + 3C\delta_\phi]$ that

\[ |\gamma_{2y_m}(c_2) - \gamma_{y_m}^*(c_2)| \approx W_{\Phi_2(z^{-1}(\tau_m))}. \]

Then, since $|c_2 - c_3| \approx \delta_\phi$, $|c_1 - c_3| \lesssim \delta_\phi$, and $\mathcal{H} \gg W_{\Phi_2(z^{-1}(\tau_m))}$, by the Mean Value Theorem there exist $c_1, c_3 \in [\xi_2^{(1)}, \xi_2^{(2)} + 3C\delta_\phi]$ such that

\[ |\gamma_{2y_m}(c_4) - \gamma_{y_m}^*(c_4)| \lesssim W_{\Phi_2(z^{-1}(\tau_m))} \text{ and } |\gamma_{2y_m}(c_5) - \gamma_{y_m}^*(c_5)| \lesssim \frac{\mathcal{H}}{\delta_\phi}. \]

Since $\mathcal{H} \gg W_{\Phi_2(z^{-1}(\tau_m))}$ and $|c_4 - c_5| \lesssim \delta_\phi$, by the Mean Value Theorem there exists $c_6 \in [\xi_2^{(1)}, \xi_2^{(2)} + 3C\delta_\phi]$ such that

\[ |\gamma_{2y_m}(c_6) - \gamma_{y_m}^*(c_6)| \gtrsim \frac{\mathcal{H}}{\delta_\phi}. \]
Proposition 13.16.

Lemma 13.5 proof. For fixed $t$, there exist $\Omega_2^{(1)}(t), \Omega_2^{(2)}(t) \subset \Omega_2(t)$ such that $|\Omega_2^{(i)}(t)| \geq \frac{1}{10}|\Omega_2(t)|$, and such that $t_1, s_1^{(1)}$, and $s_1^{(2)}$ (and equivalently $\tilde{t}_1, \tilde{s}_1^{(1)}$, and $\tilde{s}_1^{(2)}$) are separated by at least $\frac{1}{10} \delta \Omega_x m_x = \delta \Omega_x$. Then, by Lemma 13.15, the points $\Phi(\tilde{\Theta})$ and $\Phi(\tilde{s}^{(2)})$ have mutual $\xi_2$ separation of at least $\tilde{C} \delta \Phi$, and we can make $\tilde{C}$ arbitrarily large by choosing the $C$ on the lower bound of $\alpha_m$ sufficiently large. Furthermore, by Lemma 13.15 the points $\Phi(\tilde{t}), \Phi(\tilde{s}^{(i)})$ lie in an $H \approx \tilde{\Theta} \delta \Phi$ neighborhood of a curve, and by Lemmas 13.8 and 13.11, respectively, that curve has curvature $\approx \tilde{\Theta} m \approx \tilde{\Theta} \delta \Phi$, and derivative bounded by $\frac{1}{10}$, which bounds the possible unit tangent vectors.

Thus, by choosing the constant on the lower bound of $\alpha_m$ to be large enough, we can separate $p_1 = \Phi(\tilde{t}), p_2 = \Phi(\tilde{s}^{(1)})$, and $p_3 = \Phi(\tilde{s}^{(2)})$ by $a = \tilde{C} \delta \Phi = \tilde{C} \delta \Phi^{1/2} x_m^{(1)}$ for a sufficiently large $\tilde{C}$. By choosing $\Theta$ and $\delta$ so that $\Theta = \tilde{\Theta}$ and $\Theta \delta^2 = H \approx \tilde{\Theta} \delta \Phi$, the neighborhood $\Theta \delta^2$ will equal $H$ so that $\delta \approx \delta \Phi$. Since $x_m, x_m^{\delta \Phi} < 1$, then $a = C \delta \Phi$ will satisfy $a > 40 \max\{\delta \Theta, \delta^{1/2} \approx \} = C$ if $C$ is sufficiently large (note that $\inf \kappa = \Theta$), completing the hypotheses of Lemma 13.6 and implying that on $\{t\} \times \Omega' \times \Omega''$, 

$\mu(\text{Conv}(\nabla \varphi(t), \nabla \varphi(s^{(1)}), \nabla \varphi(s^{(2)}))) \approx x_m^{[J-2r+1]}(\frac{\alpha_m}{x_m y_m} x_m^{J-r})^3$. $

\Box$

Next, recall our decomposition of $R_T$ into $r_m$. This induces a decomposition $T = \sum_{m} \sum_{m} T_m$. Define $T_{m2} := \sum_{m} T_{m(m), m2}$. Proposition 13.16. There exists $\sigma(T, J, r) > 0$ such that 

$T_{m2}(E, F) \lesssim 2^{-\sigma m2} |E|^{\frac{3}{4}} |F|^{1-\frac{3}{4}}$. 

Proof. By Young’s Inequality, since $|r_m| = x_m y_m$, then $\alpha_m \approx x_m y_m$. First, consider $\alpha_m \geq C x_m y_m$. By (13.3) and Lemma 13.5 

$|E_m|^2 \gtrsim \int_{\Omega} x_m^{[J-2r+1]}(\frac{\alpha_m}{x_m y_m} x_m)^3 dtds_1 ds_2$ 

$\gtrsim x_m^{[J-2r+1]}(\frac{\alpha_m}{x_m y_m})^3 \alpha_m \min(\Omega_2(t)) \approx |\Omega_1| \approx x_m^{[J-2r+1]}(\frac{\alpha_m}{x_m y_m})^3 \alpha_m \beta_m.$ 

We recall that $\beta_m \approx \frac{1}{\alpha_m} T_m(E, F)$ and $\alpha_m \approx \frac{T_m(E, F)}{|E_m|}$, so that 

$|E_m|^2 \gtrsim x_m^{[J-2r+1]}(\frac{\alpha_m}{x_m y_m})^3 \alpha_m \beta_m \approx x_m^{[J-2r+1]}(\frac{\alpha_m}{x_m y_m})^3 \alpha_m \beta_m T_m(E, F)$. 

Then, since $|E| \gtrsim |E_m|$, 

$T_m(E, F) \lesssim (\frac{\alpha_m}{x_m y_m})^{-\frac{1}{2}} x_m^{-\frac{1}{2}} |E_m|^{\frac{3}{4}} |F|^{\frac{3}{4}}$. 

(13.9) 

Also, by the definition of $\alpha_m$, we can write the following $L^\infty \to L^\infty$ bound: 

$(13.10) \quad T_m(E, F) \lesssim \alpha_m |F| = x_m y_m (\frac{\alpha_m}{x_m y_m}) |F| = x_m^{r+1} (\frac{\alpha_m}{x_m y_m}) |F|$. 

Since $\frac{\alpha_m}{x_m y_m} : = 2^{-m2} \lesssim \frac{\alpha_m}{x_m y_m} \lesssim 1$, with $x_m := 2^{-m1}$, define 

$\eta(m) := \{ m : \frac{\alpha_m}{x_m y_m} \approx 2^{-m3} \}$
which induces a function \( m_3 = m_3(m_1, m_2) \). Define \( T_{m_{2,2}} := \sum_{m_1, m_3(m_1, m_2) < m_2 - C} T_m \) and \( T_{m_{2,1}} := \sum_{m_1, m_3(m_1, m_2) \geq m_2 - C} T_m \), for a suitable \( C \geq 0 \). Then, combining (13.9) and (13.10),

\[
T_{m_{2,2}}(E, F) \leq \sum_{m_3 = 0}^{m_2 - C} \sum_{m_1 = 0}^{\infty} \min\{2^{-m_3} 2^{-m_1} |E|^{\frac{4}{3}} |F|^{\frac{4}{3}}, 2^{-m_3} 2^{-m_1} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}\}
\]

\[
\leq \sum_{m_3 = 0}^{m_2 - C} \sum_{m_1 = 0}^{\infty} 2^{-m_3} 2^{-m_1} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}
\]

\[
\leq 2^{-m_3} 2^{-m_1} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}
\]

where we used \( d_h = \frac{J}{r+1} \) from (13.11) and \( d_\omega = 2d_h - 2 \) from Lemma 6.5 in the last line. Next, since \( \omega = Cx^Qy^T + O(y^{T+1}) \) by (13.2), then \( d_\omega = \frac{Q+Tr}{r+1} \geq \frac{2}{3} T \geq \frac{2}{3} \), and the relationship \( d_\omega = 2d_h - 2 \) implies that \( \frac{J}{r+1} = d_h > \frac{4}{3} \). Thus, \( \frac{\min(2J-r-1, J-r-3)}{2(J+r+1)} > 0 \), so \( T_{m_{2,2}} \) satisfies Proposition 13.16.

Finally, consider the case \( \frac{\omega m}{x_m} \leq \frac{\omega m}{x_m} \). By the definition of \( a_{n_m} \),

(13.11) \[
T_m(E, F) \leq x_m^{r-1} \left( \frac{a_m}{x_m} \right)^2 |F|,
\]

and by equation (13.12) and Theorem 2.1

(13.12) \[
T_m(E, F) \leq x_m^{\frac{2}{3}} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}.
\]

Then, combining (13.11) and (13.12),

\[
T_{m_{2,1}}(E, F) \leq \sum_{m_1 = 0}^{\infty} \min\{2^{-m_1} 2^{-m_2} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}\}
\]

\[
\leq 2^{-m_1} 2^{-m_2} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}
\]

\[
= 2^{-m_1} 2^{-m_2} |E|^{\frac{2}{3}} |F|^{\frac{2}{3}}.
\]

where we used the relation \( 2(J-r-1) = Q + Tr \) shown in (13.2), as well as \( d_h = \frac{J}{r+1} \) from (13.11) and \( d_\omega = 2d_h - 2 \) from Lemma 6.5 in the last line. Thus, \( T_{m_{2,1}} \) satisfies Proposition 13.16 so the proof of Proposition 13.16 is complete. □

Then, since \( m_2 \geq 0 \) on \( R_T \), we can sum over all \( m_2 \) to get

\[
T_{R_T}(E, F) \leq \sum_{m_2 = 0}^{\infty} T_{m_{2,1}}(E, F) \leq |E|^{\frac{3}{2}} |F|^{\frac{3}{2}}.
\]

Thus, \( T_{R_T} \) is of \( (\frac{4}{3}, \frac{4}{3}) \), \( d_\omega = 4 \), concluding the proof of Proposition 13.1. □

14. Second Relevant Vertex: Case breakdown

In Lemma 10.2 we found that the polygon arising in Theorem 2.4 has at most two relevant vertices. With Theorem 2.4 now proven for \( \mathcal{T}_{[-1,1]} \setminus R_T \), and proven at the first relevant vertex for \( R_T \), we now turn our attention to proving Theorem 2.3 for \( R_T \) at the second relevant vertex. We recall that the second vertex, \( (\frac{4}{3}, \frac{4}{3}) \), exists only in cases (N) and (A), and belongs in the set \( \overline{\text{Conv}} \{ (0, 0), (\frac{4}{3}, \frac{4}{3}), (\frac{3}{2}, \frac{3}{2}) \} \).
either in the interior or on the boundary. Based on the location of \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})\), we further decompose cases (N) and (A).

**Definition 14.1.** Define \(D := Conv\{(0, 0), (\frac{2}{3}, \frac{1}{3}), (\frac{7}{3}, \frac{1}{3})\}\). Depending on the location of \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})\), we break Cases (N) and (A) into the following cases:

- \((N_{nt}), (A_{nt})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) \in Int(D)\);
- \((N_{p=v}), (A_{p=v})\): \(q_{v_2} = p_{v_2}', (\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) \neq (\frac{2}{3}, \frac{1}{3})\);
- \((N_{scal}, q=2p})\): \(q_{v_2} = 2p_{v_2} \neq 3, \frac{1}{q_{v_2}} = \frac{1}{p_{v_2}} - \frac{1}{d_{h+1}}\);
- \((N_{\#scal}, q=2p})\): \(q_{v_2} = 2p_{v_2} \neq 3, \frac{1}{q_{v_2}} \neq \frac{1}{p_{v_2}} - \frac{1}{d_{h+1}}\).

We will show in the following lemma that the above subcases completely cover cases (N) and (A), and we will furthermore identify qualities of the functions \(\varphi\) that belong to each subcase.

**Lemma 14.2.** Every \(\varphi\) belonging to Cases (N) or (A) belongs to one of the subcases of Definition [14.1]. Additionally, for each subcase, the following hold, where we refer to the line \(\frac{1}{p} = \frac{1}{q} = \frac{1}{d_{h+1}}\) as the scaling line:

- \((A_{nt})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})\) lies on the scaling line, when \(\frac{1}{p_{v_2}} = \frac{(2T+5)-(d_{h+1})}{(d_{h+1})(T+2)}\);
- \((N_{nt})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})\) lies on the scaling line, when \(\frac{1}{p_{v_2}} = \frac{N+1-d_{h}}{d_{h+1}}\). Additionally, in this case \(N < d_{h} + 1\).
- \((N_{scal}, q=2p})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{3}, \frac{1}{N})\), with \(N = d_{h} + 1\), and \(N > 3\).
- \((N_{\#scal}, q=2p})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{3}, \frac{1}{N})\), with \(N > d_{h} + 1\), and \(N > 3\)
- \((N_{p=v})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{2}{3}, \frac{1}{N})\), and up to a rescaling and a reordering of \(z_1\) and \(z_2\), \(\varphi = (z_2 - z_1)^3\).
- \((N_{q=p})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})\) lies where the scaling line intersects the line \(q = p'\), and up to a rescaling and a reordering of \(z_1\) and \(z_2\), \(\varphi = (z_2 - z_1)^3\).
- \((A_{p=v})\): \((\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}})\) lies where the scaling line intersects the line \(q = p'\), and up to a rescaling and a reordering of \(z_1\) and \(z_2\), \(\varphi = z_1^2 \pm z_2^2\), for some \(S \geq 3\).

**Proof.** In case (A), by Corollary [6.4], \(T = A - 2\). Thus, lines \(\frac{1}{q} = \frac{A+1}{2A+1} - \frac{1}{2A+1} = \frac{T+1}{2T+3}p - \frac{1}{2T+3} = \frac{T+1}{2T+3}\) and \(\frac{1}{q} = \frac{1}{p} = \frac{1}{d_{h+1}}\) intersect when

\[
(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = \left(\frac{(2T+5)-(d_{h+1})}{(d_{h+1})(T+2)}, \frac{N+1-d_{h}}{d_{h+1}}\right),
\]

which proves the statement for subcase \((A_{nt})\). Since \(d_{h} > 0\) and \(T > d_{h} = 2d_{h} - 2\), \(2 < \frac{q_{v_2}}{p_{v_2}} < 3\) is always satisfied. Additionally, the inequality \(\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1\) is equivalent to

\[
d_{h} > \frac{2(T+2)}{T+4},
\]

which always holds when \(d_{h} \geq 2\). To find when it fails, since \(\varphi\) is not homogeneous in Case (A) by Proposition [6.9] then by the definition of \(A\), up to a rescaling and an interchanging of \(z_1\) and \(z_2\),

\[
\varphi = z_1^J + c_1 z_2^J z_1^{-l} + o(z_2^J), \text{ some } J, l \geq 1, c_1 \neq 0,
\]

where \(A = ls \geq 2\), and by Corollary [6.4], \(T = ls - 2\). Then, \[(14.1)\] is equivalent to \(d_{h} > \frac{2l}{ls+2}\). Also, from the structure of \(\varphi\), \(d_{h} = \frac{J}{s+t} \) and \(J \geq \min(lr, 2)\) \((J\) cannot...
be 1 since $\nabla \varphi(0,0) = 0)$. When $lr = 1$, we have $l = r = 1$, and
\[ d_h = \frac{ds}{s+r} \geq \frac{2s}{s+r} > \frac{2s}{ls} = \frac{2ls}{ls+2}. \]

When $lr > 2$, then $\frac{r}{s+r} > \frac{2}{ls+2}$, and
\[ d_h = \frac{ds}{s+r} \geq \frac{2r}{s+r} > \frac{2ls}{ls+2}. \]

When $lr = 2$ and $J > 2$, then $\frac{r}{s+r} = \frac{2}{ls+2}$, so
\[ d_h = \frac{ds}{s+r} > \frac{2r}{s+r} = \frac{2ls}{ls+2}. \]

Thus, $q_{v_2}$ is always greater than $p'_{v_2}$ when either $lr \neq 2$ or when $lr = 2$ and $J \neq 2$.

However, if $J = lr = 2$, then $d_h = \frac{2r}{s+r} = \frac{2ls}{ls+2}$, implying that $q_{v_2} = p'_{v_2}$. And since $J = lr = 2$, after rescaling, $\varphi$ can be written as
\[ \varphi = z_1^2 + z_2^S \]
for some $S \geq 3$, as was claimed for subcase $(A_{q=p'})$. This completes the proof for the subcases of Case(A).

For case (N), when $N < d_h + 1$, lines $\frac{1}{p_{v_2}} = \frac{N+1}{N+2} \frac{1}{p} - \frac{1}{N+2}$ and $\frac{1}{q_{v_2}} = \frac{1}{p} - \frac{1}{d_h+1}$ intersect when
\[ \left( \frac{1}{p_{v_2}}, \frac{1}{q_{v_2}} \right) = \left( \frac{(N+2)-(d_h+1)}{d_h+1}, \frac{(N+1)-(d_h+1)}{d_h+1} \right), \]

as stated for subcase $(N_{int})$. Then
\[ \frac{q_{v_2}}{p_{v_2}} = \frac{(N+2)-(d_h+1)}{(N+1)-(d_h+1)} \]
satisfies $\frac{q_{v_2}}{p_{v_2}} < 3$ since $2N - 3 = T > d_\omega = 2d_h - 2$, and $\frac{q_{v_2}}{p_{v_2}} > 2$ is satisfied since $N < d_h + 1$.

Furthermore, since $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} = \frac{(2N+3)-(2d_h+2)}{d_h+1}$, then the inequality $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$ is equivalent to
\[ d_h > \frac{2}{3} N. \]

First, if $N \geq 3$, then $N < d_h + 1$ implies that $d_h > \frac{2}{3} N$. Next, $d_h$ satisfies $d_h = \frac{r(N+k)}{r+1}$ for some $k \geq 0$ in $\mathbb{N} \cup \{0\}$. When $r > 2$ or $k > 0$, then
\[ d_h = \frac{r(N+k)}{r+1} > \frac{2}{3} N. \]

Therefore, when $N < d_h + 1$, if $N \neq 2$, or $k \neq 0$, or $r \neq 2$, then $q_{v_2} > p'_{v_2}$.

However, when $r = 2$, $k = 0$, and $N = 2$, then $N < d_h + 1$ and $d_h = \frac{2}{3} N$, implying that $q_{v_2} = p'_{v_2}$ and that up to rescaling and a swapping of $z_1$ and $z_2$,
\[ \varphi = (z_2 - z_1^2), \]
as was claimed for subcase $(N_{q=p'})$.

Finally, for case (N), with $N \geq d_h + 1$, the lines $\frac{1}{q} = \frac{N+1}{N+2} \frac{1}{p} - \frac{1}{N+2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{d_h+1}$ intersect when $\left( \frac{1}{p_{v_2}}, \frac{1}{q_{v_2}} \right) = \left( \frac{2}{N}, \frac{1}{N} \right)$, as stated for subcases $(N_{scal}^{q=2p})$ and $(N_{q=2p}^{scal})$.

Thus, when $N > d_h + 1$, $\frac{2}{N} = \frac{2}{2} = 1$ is always satisfied.

If $N > 3$, then $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$. Now, for some $k \geq 0$ in $\mathbb{N} \cup \{0\}$, depending on $\varphi$, $d_h = \frac{r(N+k)}{r+1}$. Thus, when either $N > d_h + 1$, or when $N = d_h + 1$ and $l \neq 0$ or $r \neq 2$, $N$ must be larger than 3 implying that $\frac{1}{p_{v_2}} + \frac{1}{q_{v_2}} < 1$. 
Alternatively, when \( N = d_k + 1, \ r = 2, \ k = 0, \) and \( N = 3, \) then \( \left( \frac{1}{p}, \frac{1}{q} \right) = \left( \frac{2}{3}, \frac{1}{3} \right), \) and up to a rescaling and a swapping of \( z_1 \) and \( z_2, \)
\[ \varphi = (z_2 - z_1)^3, \]
as was claimed for subcase \( (N, t_1, t_2). \) This completes the proof for the subcases of Case(N).

15. The rwt \( \left( \frac{2}{3}, 3 \right) \) bound and the second relevant vertex.

In this section, we will go through each subcase from Definition 14.21 proving that \( T_{R_2} \) is of rwt \( (p_{w_2}, q_{w_2}) \) and completing the proof of Theorem 2.4.

15.1. Case \( (A_q, p'). \) By Lemma 14.2 after rescaling, we may assume
\[ \varphi = t_1^2 + t_2^2, \]
where \( S \geq 3. \) By symmetry, it suffices to consider \( \mathcal{T} \) over the set \( \{ t_1, t_2 \geq 0 \}. \)
We decompose \( (\mathbb{R}^2) \) into strips \( S_j = \{ t_2 \approx 2^{-j} \}. \) This induces a decomposition \( \mathcal{T} = \sum T_j. \) We now bound \( \| T_j \|_{L^2 \to L^3} \) using Minkowski’s Inequality, Theorem 2.1 on \( \mathbb{R}^2, \) and Young’s Inequality as follows:

\[
\| T_j f \|_{L^1(\mathbb{R}^3)} = \| f(x - (t, \varphi(t))) \|_{L^2 L^1((t \in S_j))} \\
\lesssim \| f(x - (t, \varphi(t))) \|_{L^2 L^1}\| \varphi \|_{L^\infty L^1((t \in S_j))} \\
\lesssim \| f(y, x_2 - t_2, t_3) \|_{L^2 L^1 L^1_{V_3}((t_2 \approx 2^{-j}))} \\
= \| f(y_1, \cdot, t_3) \|_{L^2 L^1_{V_3}((t_2 \approx 2^{-j}))} \\
\lesssim \| \varphi \|_{L^\infty L^1} \| f \|_{L^2(\mathbb{R}^3)} = 2^{-\frac{3j}{2}} \| f \|_{L^2(\mathbb{R}^3)}.
\]

(15.1)

Since \( |\omega| = |\det D^2 \varphi| \approx t_2^{-2} \approx 2^{-(S-2)j} \) on \( S_j, \) Theorem 2.1 implies

\[
\| T_j f \|_{L^4(\mathbb{R}^3)} \lesssim 2^{-\frac{(S-2)j}{2}} \| f \|_{L^\frac{4}{3}(\mathbb{R}^3)}.
\]

Combining (15.1) and (15.2),
\[ \mathcal{T}_{(\mathbb{R}^2)^2}(E, F) \lesssim \sum_{j = -\infty}^{\infty} \min\{2^{-\frac{3j}{2}}|E|^\frac{3}{2} |F|^\frac{3}{2}, 2^{-\frac{(S-2)j}{2}}|E|^\frac{3}{2} |F|^\frac{3}{2}\} \approx |E|^{\frac{3S+2}{8}} |F|^{\frac{3S+2}{8}}. \]

Thus, \( \mathcal{T}_{R^2} \) is of rwt \( (p_S, q_S) \) for \( \left( \frac{1}{p_S}, \frac{1}{q_S} \right) = \left( \frac{2S+2}{3S+2}, \frac{S}{3S+2} \right). \) As \( \frac{1}{p_S} + \frac{1}{q_S} = 1 \) and \( (p_S, q_S) \) lies on the scaling line of \( \mathcal{T}, \) then \( (p_S, q_S) = (p_{w_2}, q_{w_2}) \) by Lemma 14.2.

15.2. Case \( (N, t_1, t_2). \) By Lemma 14.2 after rescaling, we may assume
\[ \varphi = (t_2 - t_1^2)^2. \]
We decompose \( \mathbb{R}^2 \) into strips \( S_j = \{ |t_2 - t_1^2| \approx 2^{-j} \}, \) which induces a decomposition \( \mathcal{T} = \sum T_j. \) By the change of coordinates \( u_1 := t_1, \ u_2 := t_2 - t_1^2, \) (setting \( \psi(u_1, u_2) := u_2 + u_1^2, \) Minkowski’s Inequality, Theorem 2.1 (on \( \mathbb{R}^2 \)), the change of variables \( v = u_2^2, \) and Young’s Inequality,
\[
\| T_j f \|_{L^3(\mathbb{R}^3)} := \| f(x - (t, \varphi(t))) \|_{L^3_{L^1}((t \in S_j))} \\
\approx \| f(x - (u_1, \psi(u), u_2^2)) \|_{L^3_{L^1}((u_2 \approx 2^{-j}))} \\
\approx \| f(x - (u_1, \psi(u), u_2^2)) \|_{L^3_{L^1}((u_2 \approx 2^{-j}))}.
\]
\[ \begin{align*}
\leq & \|f(x - (u_1, \psi(u), u_2^2))\partial_{u_1}^2 \psi \|^2 \|L^2_{x, u_2} L^3_{x, s_{1, 2}} L^1_{a_1} (\{|u_2| \sim 2^{-j}\}) \\
\lesssim & \|f(y_1, y_2, x_3 - u_2^2)\|^2 \|L^3_{x, u_2} L^1_{a_2} L^2_{y_1 y_2} (\{|u_2| \sim 2^{-j}\}) \\
= & 2^j \|f(y_1, y_2, x - v)\|^2 \|L^3_{x, y_1} L^1_{y_2} L^2_{y_1 y_2} (\{|x| \sim 2^{-j}\}) \\
= & 2^j \|f(y_1, y_2, \cdot)\|^2 \|L^3_{x, y_1} * \chi(|x| \sim 2^{-j})\| L^3_{y_1 y_2}.
\end{align*} \]

(15.3)

Since \(|\omega| := |\text{det}(D^2 \varphi)| \approx |t_2 - t_1^2| \approx 2^{-j}\) on \(S_j\), Theorem 14.1 implies

\[ \|T_j f\|_{L^4(\mathbb{R}^3)} \lesssim 2^j \|f\|_{L^4(\mathbb{R}^3)}. \]

(15.4)

Combining (15.3) and (15.4).

\[ \mathcal{T}_{R^2}(E, F) \lesssim \sum_{j = -\infty}^{\infty} \min(2^{-\frac{b_j}{2}} |E|^\frac{1}{2} |F|^\frac{1}{2}, 2^j |E|^\frac{3}{2} |F|^\frac{3}{2}) \approx |E|^\frac{3}{2} |F|^\frac{3}{2}. \]

Thus, \(T_{R^2}\) is of rwt \((\frac{3}{2}, \frac{3}{2})\). As \(\frac{5}{2} + \frac{3}{2} = 1\), and as \((\frac{3}{2}, \frac{3}{2})\) lies on the scaling line of \(T\), then \((\frac{3}{2}, \frac{3}{2}) = (p_e, q_e)\) by Theorem 14.2.

15.3. Case \((N(\frac{3}{2}, \frac{3}{2}))\). By Lemma 14.2 after rescaling, we may assume
\[ \varphi = (t_2 - t_1^2)^{\frac{3}{2}}. \]

By the change of variable \(u_1 := t_1, u_2 := t_2 - t_1^2\) (setting \(\psi(u_1, u_2) := u_2 + u_1^2\)), Minkowski’s inequality, Theorem 2.1 (on \(\mathbb{R}^2\)), the change of variables \(v = u_2^2\), and Young’s Inequality,

\[ \|T f\|_{L^4(\mathbb{R}^3)} = \|f(x - (t, \varphi(t)))\|_{L^6_{x, t} L^4_{\varphi}} \]
\[ \approx \|f(x - (u_1, \psi(u), u_2^2))\|_{L^6_{x, u_2} L^4_{\psi}} \]
\[ \leq \|f(x - (u_1, \psi(u), u_2^2))\|_{L^{3}_{x, y_1} L^{1}_{y_2} L^{3}_{x, s_{1, 2}} L^{1}_{a_1}} \]
\[ \lesssim \|f(y_1, y_2, x_3 - u_2^2)\|_{L^{3}_{x, y_1} L^{1}_{y_2} L^{2}_{y_1 y_2}} \]
\[ = \|\|f(y_1, y_2, \cdot)\|_{L^{3}_{y_1 y_2}} * \chi(|\cdot| \sim 2^{-j})\| L^{3}_{x, a_1}. \]

Thus, \(T_{R^2}\) has a strong type bound at \((p, q) = (\frac{3}{2}, 3)\), which equals \((p_{e}, q_{e})\) by Lemma 14.2.

15.4. Case \((A_{int})\). In Case(A), we can assume that \(f_T = t_2\), so that \(t_2\) has multiplicity \(T\) in \(\omega\) and multiplicity 0 in \(\varphi\). Thus, after rescaling, because of mixed homogeneity, \(\varphi\) will take the following form:
\[ \varphi = t_1^l \pm t_2^l t_1^{l-r} (1 + \mathcal{O}(\frac{l^2}{t_1^2})) \]
where \(l_s \geq 2, r \geq 1, J \geq 1\),

for some \(l, J\). We recall from the definition of case(A) that \(A = l_s \geq 2\). Then, by Lemma 14.2 \(\omega\) will take the form
\[ \omega = C t_1^{2J-l-2} t_2^{l-2} (1 + \mathcal{O}(\frac{l^2}{t_1^2})), \quad C \neq 0. \]
Thus, \( T = ls - 2 = A - 2 \), and we define \( Q := 2J - lr - 2 \), so that \( |\omega| \approx |t_1|^Q |t_2|^T \) on \( R^*_T = \{|t_2|^{\alpha} \leq \epsilon |t_1|^\gamma\} \). By symmetry, it will suffice to consider only the region where \( t_1, t_2 \geq 0 \). We define \( S := R^*_T \cap \{|\omega| \approx 1\} \cap \{t_1, t_2 \geq 0\} \), which decomposes into regions \( T_j := \{|t_1 | \approx 2^j, t_2 \approx 2^{-j}\} \) and induces the decomposition \( T_S = \sum_j T_j \).

Then, by Minkowski’s Inequality, Theorem 2.11 (on \( \mathbb{R}^2 \)), and Young’s Inequality,

\[
\|T_j f\|_{L^3(\mathbb{R}^3)} = \|f(x - (t, \varphi(t)))\|_{L^3(\mathbb{R}^3)} \\
\approx 2^{-\frac{j}{2}} 2^{(j-2)} \|f(x - (t, \varphi(t)))\|_{L^3(\mathbb{R}^3)} \\
\leq 2^{-\frac{j}{2}} 2^{(j-2)} \|f(x - (t, \varphi(t)))\|_{L^3(\mathbb{R}^3)} \\
\lesssim 2^{-\frac{j}{2}} 2^{(j-2)} \|f\|_{L^3(\mathbb{R}^3)}.
\]

(15.5)

Additionally, since \( |\tau_j | \approx 2^{\frac{j}{2} - \frac{1}{4}} j \), we have

\[
\|T_j\|_{\infty \to \infty} \lesssim 2^{\frac{j}{2} - \frac{1}{4} j}.
\]

Combining (15.5) and (15.6),

\[
\|T_j (E, F)\| \lesssim \min (2^{\frac{j}{2} - \frac{1}{4} j} |E|, 2^{\frac{j}{2} - \frac{1}{4} j} |F|).
\]

Noting that \( T + Q > 0 \) since \( T > d_\omega = \frac{Q + T}{r + s} \), we interpolate as follows:

\[
T_S (E, F) \lesssim \sum_{j = -\infty}^{\infty} \min (2^{\frac{j}{2} - \frac{1}{4} j} |E|, 2^{\frac{j}{2} - \frac{1}{4} j} |F|).
\]

Thus, \( T_{R^*_T \cap \{|\omega| \approx 1\} \) is of rwt \( \left( \frac{2Q}{q}, q_1\right) \) for \( \left( \frac{2}{q_1} , \frac{1}{q_1}\right) = \left( \frac{Q + T}{r + s}, \frac{T - Q}{(r + s)T - Q}\right) \). Using equalities \( Q + T = d_\omega (r + s) \), \( d_h = \frac{L_j}{r + s} \), and \( d_\omega = 2d_h - 2 \), and some messy arithmetic, we can rewrite \( \frac{1}{q_1} \) as

\[
\frac{1}{q_1} = \frac{2(T - d_\omega)}{4(T - d_\omega) + (T + 2d_\omega)}.
\]

Since \( T_{R^*_T \cap \{|\omega| \approx 1\} \) is of stong type \( \left( \frac{1}{2}, 4\right) \) by Theorem 2.11 and of rwt \( \left( p_1, q_1\right) \), where \( q_1 = 2p_1 \), and since \( R^*_T \) is \( \kappa_\omega \)-scale invariant, then by Case 3 of Proposition 8.4 \( T_{R^*_T} \) is of rwt \( \left( p_S, q_2\right) \), when \( \left( p_S, q_2\right) \) lies on the scaling line of \( T \) and

\[
\frac{1}{p_S} = \frac{3 - \frac{2}{q_1} + \frac{1}{q_1} (d_\omega + 4)}{(1 - q_1)(d_\omega + 4)}.
\]

Since \( \left( p_S, q_2\right) \) lies on the scaling line, then by Lemma 14.2 it suffices to verify that \( p_S = p_{uv} \). Using (15.7) and the identity \( d_\omega = 2d_h - 2 \), after some messy arithmetic, we find that

\[
\frac{1}{p_S} = \frac{2^{T-1} d_\omega}{(T+2)(d_h + 1)}.
\]

Thus, by Lemma 14.2 \( p_{uv} = p_S \), and so \( T_{R^*_T} \) is of rwt \( \left( p_{uv}, q_{uv}\right) \). □
15.5. **Cases** \((N_{\text{int}}), (N_{q=2p}^\neq), (N_{q=2p}^=)\). In Cases\((N)\), we can assume after rescaling that \(f_T = z_2 - z_1^T\), so that \(z_2 - z_1^T\) has multiplicity \(T\) in \(\varphi\). Thus, after rescaling, because of mixed homogeneity, \(\varphi\) can take the following form:

\[
\varphi = z_1^J((z_2 - z_1^T)^N(1 + \mathcal{O}(\frac{|z_2 - z_1^T|}{|z_1^T|}))), \quad \text{with } d_h = \frac{J + rN}{N + 1},
\]

for some \(J\). Then, by Lemma \ref{lem:spatial_proofs},

\[
(15.8) \quad \omega = C z_1^{2J + r - 2}(z_2 - z_1^T)^{2N - 3}(1 + \mathcal{O}(\frac{|z_2 - z_1^T|}{|z_1^T|})), \quad C \neq 0.
\]

Thus, \(T = 2N - 3\). We define \(Q := 2J + r - 2\), so that \(|\omega| \approx |z_1|^Q|z_2 - z_1^T|^T\) on \(R_T^T = \{|z_2 - z_1^T| < \tilde{c}|z_1|^T\}\). To simplify the argument, we will demonstrate the proof when \(z_1, z_2 - z_1^T \geq 0\). All other regions follow similarly. We decompose \(R_T^T \cap \{|z_1, z_2 - z_1^T| \geq 0\}\) into sets \(T_{j,k} := \{z_1 \approx 2^{-j} := \pi_j, z_2 - z_1^T \approx 2^{-k} := \gamma_k\}\), which induces a decomposition \(T_{R_T} \cap \{|z_1, z_2 - z_1^T| \geq 0\} = \sum_{j,k} T_{j,k}\). For ease, we perform the change of variables \(x = z_1, y = z_2 - z_1^T\). To find the rwt \((\frac{2}{3}, 3)\) bound in these cases, our previous methods will not work, so we instead use a variant of the method of refinements.

**Lemma 15.1.** The operator \(T_{j,k}\) satisfies the following rwt \((\frac{2}{3}, 3)\) bound:

\[
T_{j,k}(E, F) \lesssim 2^{\frac{2}{3}(J + r - 2)\frac{2}{3}(N - 3)}|E|^\frac{2}{3}|F|^\frac{2}{3} = \left(\frac{z_2}{|z_1^T|}\right)^{2N - 3} |E|^\frac{2}{3}|F|^\frac{2}{3}.
\]

**Proof.** First, we will refine \(E \sim E_{j,k}\) and \(F \sim F_{j,k}\) as follows. Define

\[
F_{j,k} := \{u \in F : T_{j,k}\chi E(u) \geq \frac{1}{4} T_{j,k}(E, F)\}
\]

\[
E_{j,k} := \{w \in E : T_{j,k}\chi F_{j,k}(w) \geq \frac{1}{4} T_{j,k}(E, F)\} =: \alpha_{j,k}
\]

\[
F_{j,k} := \{u \in F_{j,k} : T_{j,k}\chi E_{j,k}(u) \geq \frac{1}{4} T_{j,k}(E, F)\} =: \beta_{j,k}.
\]

Note that \(\alpha_{j,k}\) and \(\beta_{j,k}\) also depend on \(E\) and \(F\). Then \(T_{j,k}(E_{j,k}, F_{j,k}) \approx T_{j,k}(E, F)\) by the short argument leading to \((11.1)\). To proceed, we will construct a map, and use the size of the Jacobian to prove the lemma. First, we define \(\psi(x) := x^r\) and \(z(t) := (t_1, t_2 + \psi(t_1))\). Then, the surface equation has the following equivalent forms: \((z_1, z_2, \varphi(z_1, z_2)) = (x, y + \psi(x), \varphi(x, y + \psi(x)) = (z(x, y), \varphi(z(x, y))).

Fix \(u_0 \in F_{j,k}\) and define

\[
\Omega_1 := \{t \in \mathbb{R}^2 : z(t) \in \tau_{j,k} \text{ and } u_0 - (t_1, t_2 + \psi(t_1), \varphi(z(t))) =: w(t) \in E_{j,k}\}.
\]

Then \(|\Omega_1| = T_{j,k}\chi E_{j,k}(u_0) \geq \beta_{j,k}\). To achieve a lower-dimensional result with the method of refinements, we fix one variable. Rewriting \(|\Omega_1|\),

\[
\frac{|\Omega_1|}{2^{-k}} = \int_{t_2 \approx 2^{-k}} \int_{t_1 \approx 2^{-j}} \chi_{\Omega_1}(t_1, t_2) dt_1 dt_2.
\]

Therefore, by Hölder’s inequality, there exists a fixed \(t_2 \approx 2^{-k}\) such that

\[
|\Omega_{1, t_2}| := \{|t_1 : t \in \Omega_1\}| = \int_{t_1 \approx 2^{-j}} \chi_{\Omega_1}(t_1, t_2) dt_1 \geq \frac{\beta_{j,k}}{2^{-h}} = \frac{\beta_{j,k}}{y^h}.
\]

For \(t_1 \in \Omega_{1, t_2}\), define

\[
\Omega_{2, t_2}(t_1) := \{s \in \mathbb{R}^2 : s \in \tau_{j,k} \text{ and } w(t) + (s_1, s_2 + \psi(s_1), \varphi(z(s))) \in F_{j,k}\}.
\]

Then \(|\Omega_{2, t_2}(t_1)| = T_{j,k}\chi F_{j,k}(w(t)) \geq \alpha_{j,k}.

**Remark.**
Define \( \Omega_2 := \{ (t_1, s_1, s_2) \in \mathbb{R}^3 : t_1 \in \Omega_{1,t_2}, s = (s_1, s_2) \in \Omega_{2,t_2}(t_1) \} \), and
\[
\Psi_{t_2}(t_1, s_1, s_2) := x_0 - (t_1, t_2 + \psi(t_1), \varphi(z(t))) + (s_1, s_2 + \psi(s_1), \varphi(z(s))).
\]
Since \( \Psi_{t_2} \) is a polynomial mapping \( \Omega \subset \mathbb{R}^3 \) into \( \mathbb{R}^3 \), it is \( O(1) \)-to-one off a set of measure zero. Since \( \Psi_{t_2}(\Omega_2) \subset F_{j,k} \), we have \( | \det D\Psi_{t_2}(t_1, s_1, s_2) | \geq \int_{\Omega_{t_2}} | \det D\Psi_{t_2}(t_1, s_1, s_2) | dt_1 ds_1 ds_2 \). Expanding,
\[
| \det D\Psi_{t_2}(t_1, s_1, s_2) | = \left| [ \psi'(t_1) - \psi'(s_1)] \partial_{s_2} \varphi(z(s)) - [\partial_{s_1} \varphi(z(t)) - \partial_{s_1} \varphi(z(s))] \right|
= | \psi'(\tilde{x}) \partial_{s_2} \varphi(z(s))(t_1 - s_1) - (\partial_{s_1} \varphi(z(t))) \cdot (t_2 - s_2) |
= | [ \psi''(\tilde{x}) \partial_{s_2} \varphi(z(s)) - \partial_{s_1} \varphi(z(s))](t_1 - s_1) - \partial_{s_1} \varphi(z(t))(t_2 - s_2) |,
\]
for some \( \tilde{x} \) between \( t_1 \) and \( s_1 \), and some \( v \) lying on the line between \( t \) and \( s \), by the Mean Value Theorem.

Comparing each term, and recalling definition \( x_j := 2^{-j}, y_k := 2^{-k} \):
\[
| \psi''(\tilde{x}) \partial_{s_2} \varphi(z(s)) | \approx \alpha^j k N^{-1} = x_j^{-2} \gamma_k^{-1} = x_j^{J+r-2} \gamma_k^{-1};
\]
\[
| \partial_{s_1} \varphi(z(s)) | \approx \gamma_k^{-1} \approx \alpha^j k \gamma_k^{-1};
\]
\[
| \partial_{s_1} \varphi(z(t)) | \approx \gamma_k^{-1} \approx \alpha^j k \gamma_k^{-1}.
\]
Since \( y_k < \tilde{x}_j \) on \( R_j \), we have \( | \psi''(\tilde{x}) \partial_{s_2} \varphi(z(s)) | \gg | \partial_{s_1} \varphi(z(t)) | \), so we can disregard the \( \partial_{s_1} \varphi(z(t)) \) term. Comparing the remaining terms, as long as
\[
| s_1 - t_1 | | x_j^{-1} \gg | s_2 - t_2 |
\]
the \( \psi''(\tilde{x}) \partial_{s_2} \varphi(z(s))(t_1 - s_1) \) term dominates, and we get
\[
| \det D\Psi_{t_2}(t_1, s_1, s_2) | \approx | s_1 - t_1 | x_j^{J+r-2} y_k^{-1}.
\]
Let \( t_1 \) be fixed. Since \( | \Omega_{t_2}(t_1) | \geq \alpha^t k \) and since \( s \in \Omega_{t_2}(t_1) \) implies that \( s_2 \approx y_k \), then on over half of \( \Omega_{t_2}(t_1) \), we have \( | s_1 - t_1 | \gtrsim \alpha^t k \). Also, since \( s_2, t_2 \approx 2^{-k} = y_k \), we get \( | s_2 - t_2 | \lesssim y_k \).

Thus, based on (15.9), we have two cases: \( \frac{\alpha^t k}{y_k} x_j^{-1} \gg y_k \) and \( \frac{\alpha^t k}{y_k} x_j^{-1} \lesssim y_k \).

In the case \( \frac{\alpha^t k}{y_k} x_j^{-1} \gg y_k \), (15.9) holds on over half of \( \Omega_{t_2}(t_1) \), and
\[
| \det D\Psi_{t_2}(t_1, s_1, s_2) | \approx | s_1 - t_1 | x_j^{J+r-2} y_k^{-1} \gtrsim \frac{\alpha^t k}{y_k} x_j^{J+r-2} y_k^{-1}
\]
on over half of \( \Omega_{t_2}(t_1) \), implying that
\[
| F_{j,k} | \gtrsim \int_{\Omega_{t_2}} \frac{\alpha^t k}{y_k} x_j^{J+r-2} y_k^{-1} dt_1 ds_1 ds_2 \gtrsim \alpha^t k \min(|\Omega_{t_2}(t_1)|,|\Omega_{t_2}|)
\]
\[
\gtrsim \alpha^t k \frac{x_j^{J+r-2} y_k^{-2}}{y_k} \gtrsim \frac{\alpha^t k \beta^t k}{y_k} = x_j^{J+r-2} y_k^{-2} \alpha^t k / \beta^t k.
\]
By definition, \( \beta^t k \approx \frac{T_{j,k}(E,F)}{d_{j,k}} \), and \( \alpha^t k \approx \frac{T_{j,k}(E,F)}{d_{j,k}} \), and so
\[
| F_{j,k} | \gtrsim \frac{x_j^{J+r-2} y_k^{-2}}{y_k} \gtrsim \frac{\alpha^t k \beta^t k}{y_k} \gtrsim x_j^{J+r-2} y_k^{-2} \alpha^t k \beta^t k.
\]
Then, using \( | F_{j,k} | \leq | F | \), \( d_h = \frac{J+N}{r+1} \), \( x_j := 2^{-j} \), and \( y_k := 2^{-k} \), we conclude
\[
T_{j,k}(E,F) \lesssim \frac{(x_j^{J+r-2} y_k^{-2})^{-rac{1}{2}} |E|^{rac{1}{2}} |F|^{rac{1}{2}}}{2^{j(1+d_h)}}
= 2^j \frac{1}{2^{j(1+d_h)}} |E|^{rac{1}{2}} |F|^{rac{1}{2}}
= 2^{-j} 2^{-2} 2^{-4(t+1)(d_h)-2N} |E|^{rac{1}{2}} |F|^{rac{1}{2}},
\]
concluding the first case where $\frac{\alpha_{j,k}}{y_k}x_j^{r-1} \gg y_k$.

For the second case, we combine $\frac{\alpha_{j,k}}{y_k}x_j^{r-1} \lesssim y_k$ and $T_{j,k}(E, F) \lesssim \alpha_{j,k}|E|$ to get the $L^1 \to L^1$ bound
\begin{equation}
T_{j,k}(E, F) \lesssim y_k^2x_j^{-\frac{1}{r-1}}|E|.
\end{equation}
Since $\tau_{j,k}$ has measure $x_jy_k$, Young’s Inequality gives an $L^\infty \to L^\infty$ bound of
\begin{equation}
\tau_{j,k}(E, F) \lesssim x_jy_k|F|.
\end{equation}
Interpolating (15.11) and (15.12), we get the $L^2 \to L^2$ bound
\begin{equation}
\tau_{j,k}(E, F) \lesssim y_k^2x_j^{-\frac{1}{r-1}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.
\end{equation}
Next, by (15.8), we know that on $\tau_{j,k} \subset R_T$, so by Theorem 2.1, we have the
\begin{equation}
|\omega| \approx x_j^{2J+r-2}y_k^{2N-3},
\end{equation}
so by Theorem 2.1 we have the $L^\frac{4}{3} \to L^4$ bound
\begin{equation}
\tau_{j,k}(E, F) \lesssim (x_j^{2J+r-2}y_k^{2N-3})^{-\frac{1}{3}}|E|^{\frac{1}{3}}|F|^{\frac{1}{3}}.
\end{equation}
Finally, interpolating (15.13) and (15.14),
\begin{equation}
\tau_{j,k}(E, F) \lesssim (y_k^2x_j^{-\frac{1}{r-1}}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}})^{\frac{1}{2}}((x_j^{2J+r-2}y_k^{2N-3})^{-\frac{1}{3}}|E|^{\frac{1}{3}}|F|^{\frac{1}{3}})^{\frac{1}{2}}
= (x_j^{2J+r-2}y_k^{2N-3})^{-\frac{1}{6}}|E|^{\frac{1}{6}}|F|^{\frac{1}{6}},
\end{equation}
as in the first case. Thus, the proof of Lemma 15.1 is complete. \qed

15.5.1. Case ($N_{int}$). We start by defining $\tilde{S} := R_T \cap \{ |\omega| \approx 1 \} \cap \{ z_1, z_2 - z_1' \geq 0 \}$, which we can decompose into regions $\tau_{\frac{2J+2}{3}, \frac{4}{3}} := \{ z_1 := x \sim 2^{\frac{2J+2}{3}}, z_2 - z_1' := y \sim 2^{-\frac{4}{3}} \}$, recalling that $T = 2N - 3$ and $Q = 2J + r - 2$ from (15.8). This induces the decomposition $T_{\tilde{S}} = \sum T_j$. From Lemma 15.1, replacing $j$ with $\frac{2J+2}{3}$ and $k$ with $\frac{4}{3}$,
\begin{equation}
\tau_j(E, F) \lesssim 2^{-\frac{2J+2}{3}}|E|^{\frac{1}{3}}|F|^{\frac{1}{3}},
\end{equation}
where $B = (r + 1)[1 + dh] - (rN + 3)$. Since $|\tau_{\frac{2J+2}{3}, \frac{4}{3}}| \approx 2^{\frac{2J+2}{3} - \frac{4}{3}}$, we also have
\begin{equation}
\tau_j(E, F) \lesssim 2^{\frac{2J+2}{3} - \frac{4}{3}}|F|.
\end{equation}
Combining (15.15) and (15.16),
\begin{equation}
\tau_j(E, F) \lesssim \min\{2^{\frac{2J+2}{3} - \frac{4}{3}}|F|, 2^{-\frac{2J+2}{3}}|E|^{\frac{1}{3}}|F|^{\frac{1}{3}}\}.
\end{equation}
Noting that $T - Q > 0$ since $T > d_\omega = \frac{Q+T_1}{r+1}$, we interpolate as follows:
\begin{equation}
T_{\tilde{S}}(E, F) \lesssim \sum \min\{2^{\frac{2J+2}{3} - \frac{4}{3}}|F|, 2^{-\frac{2J+2}{3}}|E|^{\frac{1}{3}}|F|^{\frac{1}{3}}\}
\approx |E|^{\frac{2J+2Q}{3(N+2)T-NQ}}|F|^{1 - \frac{T+Q}{3(N+2)T-NQ}}.
\end{equation}
Thus, $T_{R_T \cap \{ |\omega| \approx 1 \}}$ is of rwt $(\frac{2J+2}{3}, \frac{4}{3})$ for $(\frac{2Q}{T+Q}, \frac{1}{T+Q}) = (\frac{2(T-Q)}{(3+B)T-NQ}, \frac{T-Q}{(3+B)T-NQ})$. 

Since $T_{R_T^2 \cap \{|\omega| \approx 1\}}$ is of strong type $(\frac{3}{2}, 4)$ by Theorem 2.1 and of rwt $(\frac{q_f}{q_i}, q_t)$, and since $R_T^2$ is $\kappa_s$-scale invariant, then by Case 3 of Proposition 8.4 $T_{R_T}$ is of rwt $(p_S, q_S)$, where $(p_S, q_S)$ lies on the scaling line of $T$ and

$$
\left(\frac{p_S}{q_S}, \frac{2}{q_S}\right) = \left(\frac{3 - \frac{4}{q_f} + \frac{4}{q_i} (d_h + 1)}{(1 - \frac{4}{q_f}) (d_h + 1)}, \frac{1 - \frac{4}{q_f} + \frac{4}{q_i} (d_h + 1)}{(1 - \frac{4}{q_f}) (d_h + 1)}\right).
$$

Since $(p_S, q_S)$ lies on the scaling line, then by Lemma 14.2 it suffices to verify that $p_S = p_{v_2}$. Using the identities $Q = 2J + r - 2, B = (r + 1)[1 + d_h] - (rN + 3), T = 2N - 3, d_h = \frac{j + rN}{r + 1}$, and our equation for $\frac{1}{q_i}$, after some messy arithmetic we find that

$$
p_{v_2} = \frac{N + 1 - d_h}{d_h + 1}.
$$

Thus, by Lemma 14.2 $p_{v_2} = p_S$, and so $T_{R_T^n}$ is of rwt $(p_{v_2}, q_{v_2})$.

15.5.2. Case $(N_{q_2=2p}^\text{scal})$. In this case, $(\frac{1}{p_{v_2}}, \frac{1}{q_{v_2}}) = (\frac{3}{2}, \frac{3}{4})$ lies off the scaling line, so we will be using $T_T$ instead of $T_{R_T}$. We decompose $T_T \cap \{z_1, z_2 - z_1^t \geq 0\}$ into regions $\tau_{j,k} = \{x, y = z_2 - z_1^t \geq 2^{-j}\} \cap \{-1, 1\}^{2}$, where $-\infty \leq k \leq \infty$ and $0 \leq j \leq \infty$. This induces the decomposition $T_{R_T^2 \cap \{z_1, z_2 - z_1^t \geq 0\}} = \sum \tau_{j,k}$, $\tau_{j,k}$. Combining the result of Lemma 15.1 with the implications of $(\tau_{j,k}) \leq 2^{2j - 2^{-k}},$ we have

$$
\tau_{j,k}(E, F) \lesssim \min\{2^{2j - 2^{-k}}|E|, 2^{2j - 2^{-k}}2^{\frac{3}{2}[(r+1)(d_h+1)-rN] + 3} |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}}\}.
$$

Defining $\tau_j := \sum \tau_{j,k}$, and interpolating over $k,$

$$
\tau_j(E, F) \lesssim \sum_k \min\{2^{2j - 2^{-k}}|E|, 2^{2j - 2^{-k}}2^{\frac{3}{2}[(r+1)(d_h+1)-rN] + 3} |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}}\}.
$$

Since $N > d_h + 1,$

$$
\tau_{R_T^2 \cap \{z_1, z_2 - z_1^t \geq 0\}}(E, F) \lesssim \sum_{j=0}^{\infty} 2^{\frac{3}{2}[(r+1)(d_h+1)-rN] + 3} |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}} \lesssim |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}},
$$

so $T_{R_T^2}$ is of rwt $(\frac{3}{2}, N)$, which is $(p_{v_2}, q_{v_2})$ by Lemma 14.2.

15.5.3. Case $(N_{q=2p}^\text{scal})$. By Lemma 14.2 it suffices to prove the following proposition:

**Proposition 15.2.** In Case $(N_{q=2p}^\text{scal})$, $T_{R_T^2}(E, F) \lesssim |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}}.$

**Proof.** Our result will quickly follow from the following lemma:

**Lemma 15.3.** In Case $(N_{q=2p}^\text{scal})$, $\|T_{R_T^2 \cap \{|xy| \approx 1\}}\|_{L^\infty, L^{\infty, \infty}} < \infty$ for all $q \in [3, \infty)$.

**Proof.** Recall the change of coordinates $x = z_1, y = z_2 - z_1^t$. It suffices to consider the region with $x, y \geq 0$. We decompose $T_T \cap \{|xy| \approx 1\} \cap \{x, y \geq 0\}$ into regions $\tau_{-n,n} := \{z_1 \approx 2^n, z_2 - z_1^t \approx 2^{-n}\}$ for $n \in \mathbb{N}$, and denote $\tau_n := \tau_{-n,n}$. By Lemma 15.1 using $N = d_h + 1,$

$$
\tau_n(E, F) \lesssim 2^{-\frac{3}{2}[(r+1)N-(rN+3)]+\frac{3}{2}(N-3)} |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}} = |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}}.
$$

We will use the notation of Notation 11.1 with $b = 2$, to refine $E$ and $F$. Since $|\tau_{-n,n}| \lesssim 1$, and since $\tau_n(E, F) \lesssim |E| \frac{1}{\hat{F}} |F| \frac{1}{\hat{F}}$, by Lemma 11.2 and Lemma 11.3 it suffices to prove the following version of Condition 11.3.
Lemma 15.4. Let $k, l \in \mathbb{N}$, and let $k, l$, and $k - l$ be sufficiently large, independent of $E$ and $F$. Then $|F_i| \gtrsim 2^{k - 2l} r \alpha_{E_i}^2 \beta_{F_{kl}}$ and $|E_i| \gtrsim 2^{k - 2l} r \alpha_{E_i}^2 \beta_{E_{kl}}$, with implicit constants independent of $l, k, E, F$.

Proof. By symmetry, it will suffice to prove the $F_i$ inequality. Fix $u_0 \in F_{kl}$. Define

$$\Omega_1 := \{ t \in \mathbb{R}^2 : z(t) \in \tau_{-k,k} \text{ and } w(t) = (t_1, t_2 + t_1', \varphi(z(t))) =: w(t) \in E_k \cap E_l \},$$

recalling that $z(t) := (t_1, t_2 + t_1')$. Then $|\Omega_1| = \mathcal{T}_0 N_{E_k \cap E_l}(u_0) \geq \beta_{F_{kl}}$. To achieve a lower-dimensional result, we fix one variable. Rewriting $|\Omega_1|$

$$\frac{|\Omega_1|}{2^{-k}} = \int_{t_2 \approx 2^{-k}} \int_{t_1 \approx 2^k} \chi_{\Omega_1}(t_1, t_2) dt_1 dt_2.$$ 

Therefore, by Hölder’s inequality, there exists a fixed $t_2 \approx 2^{-k}$ such that

$$|\Omega_1, t_2| := \{ (t_1 : (t_1, t_2) \in \Omega_1) \} = \int_{t_1 \approx 2^k} \chi_{\Omega_1}(t_1, t_2) dt_1 \geq \frac{\beta_{F_{kl}}}{2^{-k}}.$$

For $t_1 \in \Omega_1, t_2$

$$\Omega_2(t_1) := \{ s \in \mathbb{R}^2 : z(s) \in \tau_{-l,l} \text{ and } w(t) = (s_1, s_2, s_1', \varphi(z(s))) \in F_i \}.$$ 

Then $|\Omega_2(t_1)| = T_{t_1} \varphi_{F_i}(w(t)) \geq \alpha_{E_l}$. Finally, define the following:

$$\Omega_2 := \{ (t_1, s_1, s_2) \in \mathbb{R}^3 : t_1 \in \Omega_1, s = (s_1, s_2) \in \Omega_2(t_1) \};$$

$$\Psi_{t_2}(t_1, s_1, s_2) := x_0 - (t_1, t_2, t_1', \varphi(t)) + (s_1, s_2, s_1', \varphi(s_1)).$$

Since $\Psi_{t_2}$ is a polynomial mapping $\Omega_{t_2} \subset \mathbb{R}^3 \to \mathbb{R}^3$, it is $O(1)$-to-one off a set of measure zero. Since $\Psi_{t_2}(\Omega_{t_2}) \subset F_i$, then $|F_i| \gtrsim \int_{\Omega_{t_2}} |\text{det } D\Psi_{t_2}(t_1, s_1, s_2)| d s_1 d s_2.$

Then

$$|\text{det } D\Psi_{t_2}(t_1, s_1, s_2)| = r |t_1' r^{-1} - s_1' r^{-1}| \partial_{s_2} \varphi(z(s)) - |\partial_{t_1}(z(t)) - \partial_{s_1}(z(s))|.$$

Claim: For $t \in z^{-1}(\tau_{-k,k})$, and $s \in z^{-1}(\tau_{-l,l})$, with $k, l, k - l$ sufficiently large, independent of $E, r$, $|t_1' r^{-1} \partial_{s_2} \varphi(z(s))| \gg |s_1' r^{-1} \partial_{s_2} \varphi(z(s))| \gg |\partial_{s_1}(z(s))| \gg |\partial_{t_1}(z(t))|.$

Proof: First, for $k, l, k - l$ sufficiently large, $t_1 \approx 2^k \gg 2^l \approx s_1 \gg 1$.

$$|t_1' r^{-1} \partial_{s_2} \varphi(z(s))| \gg |s_1' r^{-1} \partial_{s_2} \varphi(z(s))|;$$

Since $t_1 \gg s_1$, this is clear.

Next, recall that $\varphi(z(s, x)) = x^t y^N + o(y^N)$. If $J = 0$, then $\partial_{s_2} \varphi(z(s)) = \partial_{t_1} \varphi(z(t)) = 0$, and $|s_1'^{-1} \partial_{s_2} \varphi(z(s))| > 0$, so the rest of the claim holds. Thus, it suffices to consider $J \geq 1$.

$$|s_1'^{-1} \partial_{s_2} \varphi(z(s))| \gg |\partial_{s_1}(z(s))|;$$

First, $s_1 \gg 1$. Also, $\varphi(z(s)) = s_1'^r s_2^N + O(s_2^{N+1})$, with $J \geq 1$, so $|\partial_{s_1} \varphi(z(s))| \approx s_1'^r s_2^N \gg s_1'^{-1} s_2^N \approx |\partial_{s_1} \varphi(z(s))|$, since $s_1 \gg 1$ and $s_2 \ll 1$.

$$|\partial_{s_1} \varphi(z(s))| \gg |\partial_{t_1} \varphi(z(t))|;$$

Since $\varphi(z(s, x)) \approx x^t y^N$, with $N = d_k + 1$, we have the relation $N = 1 = d_k = \frac{d_k + N}{r + 1}$, implying $J = N - r - 1$. Since $s_1 s_2 \approx t_1 t_2 \approx 1$ and $s_2 \gg t_2$, then $|\partial_{s_1} \varphi(z(s))| \approx s_1'^{-1} s_2^N = s_1'^{-1} s_2^N \approx s_2'^{-2} \gg t_2'^{-2} \approx t_1'^{-1} t_2'^{-2} \approx t_1'^{-1} t_2'^{-2} \approx |\partial_{t_1} \varphi(z(t))|$.

Therefore,

$$|\text{det } D\Psi_{t_2}(t_1, s_1, s_2)| \approx |t_1' r^{-1} \partial_{s_2} \varphi(z(s))| \approx t_1'^{-1} s_1'^{-1} s_2^N.$$
scale-invariant, then by Case 2 of Proposition 8.4, we conclude (using □ complete.

Finally, since \(1 = |\tau - t, l| \geq |\Omega_2(t_1)| \geq \alpha E_1\),

\[
|F_l| \gtrsim \int_{\Omega_{t_2}} 2^{k-|l|} r^{-k} dt_1 ds_1 ds_2 \gtrsim 2^{k-|l|} r^{-k} \min(\|\Omega_{t_2}(t_1)\|) |\Omega_{t_2}|.
\]

\[\gtrsim 2^{k-|l|} r^{-k} \alpha E_1 \beta E_{k} \geq 2^{k-|l|} r^{-k} \alpha^2 F_{k,l} \beta_{E_k}.
\]

A near-identical argument gives \(|E_l| \gtrsim 2^{k-|l|} r^{-k} \alpha^2 F_{E_k} \beta_{E_k}\), by swapping \(E_n\) and \(F_n\), and using \(\Psi_{t_2}(t_1, s_2, s_2) = w_0 + (t_1, t_2 + t'_1, \varphi(z(t_1))) - (s_1, s_2 + s'_1, \varphi(z(s)))\), for \(w_0 \in E_{kl}\). This completes the proof of Lemma 15.4.

Hence, by Lemmas 11.2 and 11.4 with \(b = 2\), the proof of Lemma 15.3 is complete.

Finally, since \(|xy|\) is \(\frac{\kappa_\varphi}{D}\)-mixed homogeneous for some \(D > 0\), and since \(R^{e}_{F} = \kappa_\varphi\)-scale-invariant, then by Case 2 of Proposition 8.4 we conclude (using \(d_h + 1 = N\))

\[
\mathcal{T}_{R^{e}_{F}}(E, F) \lesssim |E|^{1/2} |F|^{1-1/2},
\]

so \(\mathcal{T}_{R^{e}_{F}}\) is of rwt \((\frac{\kappa_\varphi}{D}, N)\), which is \((p_{e_2}, q_{e_2})\) by Lemma 14.2.

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