Comment on Ricci dark energy in Chern-Simons modified gravity

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Abstract

We revisit Ricci dark energy in Chern-Simons modified gravity. As far as the cosmological evolution, this is nothing but the Ricci dark energy with a minimally coupled scalar without potential which means that the role of Chern-Simons term is suppressed. Using the equation of state parameter, this model is similar to the modified Chaplygin gas model only when two are around the de Sitter universe deriving by the cosmological constant in the future. However, two past evolutions are different.

Keywords: Ricci dark energy; Chern-Simons modified gravity

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Recently, the authors \cite{1} have investigated the Ricci dark energy model in the dynamic Chern-Simons modified gravity which states that its cosmological evolution is similar to that displayed by the modified Chaplygin gas model.

In this Comment, we wish to draw the reader two important issues: One is that as far as the cosmological evolution, this model is nothing but the Ricci dark energy with a minimally coupled scalar without potential where the role of Chern-Simons term is suppressed totally. The other is that using the equation of state parameter, this model is similar to the modified Chaplygin gas model only when two make turnaround of de Sitter universe deriving by the cosmological constant in the future. In general, two provide different evolutions.

We start with the dynamic Chern-Simons modified gravity action with Ricci dark energy \cite{1}

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{\dot{\theta}}{4} \ast RR - \frac{1}{2} \partial^\mu \dot{\theta} \partial_\mu \dot{\theta} + V(\dot{\theta}) \right] + S_{\text{RDE}},$$  \hspace{1cm} (1)

where \( \ast RR \) is the Pontryagin term, \( \dot{\theta} \) is a dynamical scalar and \( S_{\text{RDE}} \) is the action to give the Ricci dark energy. Here, for simplicity, one chooses \( V(\dot{\theta}) = 0 \). Their equations are given by

$$G_{\mu\nu} + C_{\mu\nu} = 8\pi GT_{\mu\nu},$$  \hspace{1cm} (2)

$$\nabla^2 \ddot{\theta} = -\frac{1}{64\pi} \ast RR,$$  \hspace{1cm} (3)

where \( G_{\mu\nu} \) is the Einstein tensor, \( C_{\mu\nu} \) is the Cotton tensor from the Chern-Simons term \( \ast RR \)-term. The energy-momentum tensor is given by

$$T_{\mu\nu} = T^{\text{RDE}}_{\mu\nu} + T^\ddot{\theta}_{\mu\nu}$$  \hspace{1cm} (4)

with

$$T^{\text{RDE}}_{\mu\nu} = (\rho_{\text{RDE}} + p_{\text{RDE}})u_\mu u_\nu + p_{\text{RDE}} g_{\mu\nu}$$  \hspace{1cm} (5)

and

$$T^\ddot{\theta}_{\mu\nu} = \partial_\mu \dot{\theta} \partial_\nu \dot{\theta} - \frac{1}{2} g_{\mu\nu} \partial_\mu \dot{\theta} \partial_\mu \dot{\theta}.\hspace{1cm} (6)$$

Applying \( \nabla^\mu \) to (2) leads to the conservation-law for \( T^{\mu\nu} \) as

$$\nabla_\mu T^{\mu\nu} = 0,$$  \hspace{1cm} (7)

which plays an important role in the cosmological evolution.
In this work, we consider the flat Friedmann-Robertson-Walker (FRW) spacetimes
\[
    ds^2_{\text{FRW}} = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 \left(dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2\right).
\]
(8)

From (00)-component of (2), we have the Friedmann equation with \( G = 1 \)
\[
    H^2 = \alpha(2H^2 + \dot{H}) + \frac{4\pi}{3} \dot{\tilde{\theta}}.
\]
(9)

In deriving (9), we used
\[
    G_{00} = 3H^2, \quad C_{00} = 0, \quad \rho_{\text{RDE}} = -\frac{\alpha}{16\pi} R = \frac{6\alpha}{16\pi} (2H^2 + \dot{H}), \quad T_{00}^\theta = \frac{1}{2} \dot{\tilde{\theta}}.
\]
(10)

We note here that the Cotton tensor \( C_{\mu\nu} \) vanishes for the FRW metric (8), implying that \("\tilde{\theta} * RR"-term does not derive any cosmological evolution. Therefore, the whole equations reduce to those of the Ricci dark energy model with a minimally coupled scalar \( \tilde{\theta} \).

Because of \(*RR = 0\) for the FRW metric (8), equation (3) leads to the conservation-law for \( \tilde{\theta} \)
\[
    \nabla^2 \tilde{\theta} = 0 \rightarrow \ddot{\tilde{\theta}} + 3H\dot{\tilde{\theta}} = 0,
\]
(11)
whose solution is given by
\[
    \dot{\tilde{\theta}} = \frac{C}{a^3}.
\]
(12)

Finally, the conservation-law (7) provides
\[
    \dot{\rho}_{\text{RDE}} + 3H(\rho_{\text{RDE}} + p_{\text{RDE}}) + \ddot{\tilde{\theta}} + 3H\dot{\tilde{\theta}} = 0,
\]
(13)
while using the conservation-law for \( \tilde{\theta} \) (11), it leads to the conservation-law for the Ricci dark energy
\[
    \dot{\rho}_{\text{RDE}} + 3H(\rho_{\text{RDE}} + p_{\text{RDE}}) = 0.
\]
(14)

Eqs. (3), (11), and (14) state that whole evolution equations amount to the Ricci dark energy with a minimally coupled scalar. This is because the Chern-Simons term of \( \tilde{\theta} * RR \) does not contribute to the cosmological evolution. However, the cosmological perturbation will distinguish between Ricci dark energy in Chern-Simons modified gravity and Ricci dark energy with a minimally coupled scalar \( \tilde{\theta} \). At this stage, we wish to mention that the conservation-law (14) might be not useful to see the cosmological evolution.
because the Friedmann equation (9) does not belong to the standard one due to $\rho_{RDE}$.

Plugging (12) into (9) and then, expressing it in terms of scale factor $a$ leads to [1]

\[
\frac{\ddot{a}}{a} + (\alpha - 1) \left( \frac{\dot{a}}{a} \right)^2 + \frac{\beta}{a^6} = 0
\]

with $\beta = 4\pi C^2/3$. For $\alpha \simeq 1/2$, this was solved for $a(t)$ to be

\[
a(t) = \left( \frac{2\beta}{3c_1} \right)^{1/6} \sinh^{1/3} \left[ 3\sqrt{c_1} t \right],
\]

where $c_1$ is an undetermined integration constant.

The authors [1] insisted that there is a correspondence between the Ricci dark energy in Chern-Simons modified gravity and the modified Chaplygin gas model because the solution (16) was also found in the modified Chaplygin gas model. Aside from the fact that Ricci dark energy in Chern-Simons modified gravity reduces to Ricci dark energy with a minimally coupled scalar, the similarity between two is very restrictive and it is limited to the de Sitter phase derived by the cosmological constant in the future. Therefore, discovering (16) is not sufficient to confirm the correspondence between two models. In order to show this explicitly, we rewrite (9) as the first-order inhomogeneous equation for $H^2$ with $x = \ln a$ instead of scale factor $a$ [2]

\[
\frac{dH^2}{dx} + \left( 4 - \frac{2}{\alpha} \right) H^2 = -\frac{2}{3\alpha} \rho_{RDE}
\]

with

\[
\rho_{RDE} = \rho_{\theta 0} e^{-6x}, \quad \rho_{\theta 0} = \pi C^2 = \frac{3\beta}{4}.
\]

A new variable $x = \ln[\alpha/a_0]$ with $a_0 = 1$ ranges from $-\infty$ to $\infty$ which includes the present $x = 0$ at $a = a_0$. It is important to note that $\rho_{\theta}$ plays a role of the stiff matter with its equation of state $\omega_{\theta} = 1$. Eq. (17) could be integrated to give the standard Friedmann equation with a positive integration constant $\tilde{c}_1$ as

\[
H^2 = \frac{\rho_t}{3}
\]

with

\[
\rho_t = \frac{\rho_{\theta 0} e^{-6x}}{\alpha(\alpha + 1)} + 3\tilde{c}_1 e^{-(4-\frac{2}{\alpha})x}.
\]

The total energy density is divided into two parts as

\[
\rho_t = \rho_{\theta 0} e^{-6x} + \left\{ \frac{1 - \alpha(\alpha + 1)}{\alpha(\alpha + 1)} \rho_{\theta 0} e^{-6x} + 3\tilde{c}_1 e^{-(4-\frac{2}{\alpha})x} \right\}
\]

\equiv \rho_{\theta} + \tilde{\rho}_{RDE},
\]

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where a new scaled Ricci dark energy density is given by

\[ \tilde{\rho}_{\text{RDE}} = \frac{1 - \alpha(\alpha + 1)}{\alpha(\alpha + 1)} \rho_{\tilde{\vartheta}0} e^{-6x} + 3\tilde{c}_1 e^{-(4 - \frac{2}{\alpha})x}. \]  

(23)

For \( \alpha(\alpha + 1) < 1 \), one finds that \( \tilde{\rho}_{\text{RDE}} > 0 \). Without the scalar \( \tilde{\vartheta} \), the pure Ricci dark energy density is given only by the second term in (23) when expressing the standard Friedmann equation like \( \text{(19)} \) \[2\]. In this case, its equation of state is given by

\[ \omega_{\text{RDE}} = \frac{1}{3} \left( 1 - \frac{2}{\alpha} \right) \]  

(24)

which shows that for \( \alpha < 1 \), it describes the dark energy-dominated universe. Also, we note that the energy density \( \rho_{\tilde{\vartheta}} \) in \( \text{(22)} \) satisfies the conservation-law as

\[ \tilde{\rho}_{\tilde{\vartheta}} = -\tilde{\rho}_{\tilde{\vartheta}} - \frac{1}{3} \frac{d\tilde{\rho}_{\tilde{\vartheta}}}{dx} \]  

(25)

for \( \tilde{\rho}_{\tilde{\vartheta}} = \omega_{\tilde{\vartheta}} \rho_{\tilde{\vartheta}} \) with \( \omega_{\tilde{\vartheta}} = 1 \), which indicates that the pure kinetic term of \( \tilde{\vartheta} \) plays a role of the stiff matter.

Substituting \( \tilde{\rho}_{\text{RDE}} \) into the conservation-law,

\[ \tilde{\rho}_{\text{RDE}} = -\tilde{\rho}_{\text{RDE}} - \frac{1}{3} \frac{d\tilde{\rho}_{\text{RDE}}}{dx}, \]  

(26)

we obtain the Ricci dark energy pressure

\[ \tilde{p}_{\text{RDE}} = \frac{1 - \alpha(\alpha + 1)}{\alpha(\alpha + 1)} \rho_{\vartheta0} e^{-6x} + (1 - \frac{2}{\alpha}) \tilde{c}_1 e^{-(4 - \frac{2}{\alpha})x}. \]  

(27)

Importantly, its equation of state takes the form

\[ \tilde{\omega}_{\text{RDE}} = \frac{\tilde{p}_{\text{RDE}}}{\tilde{\rho}_{\text{RDE}}} = \frac{\frac{1 - \alpha(\alpha + 1)}{\alpha(\alpha + 1)} \rho_{\vartheta0} e^{-6x} + (1 - \frac{2}{\alpha}) \tilde{c}_1 e^{-(4 - \frac{2}{\alpha})x}}{\frac{1 - \alpha(\alpha + 1)}{\alpha(\alpha + 1)} \rho_{\vartheta0} e^{-6x} + 3\tilde{c}_1 e^{-(4 - \frac{2}{\alpha})x}}. \]  

(28)

For \( \alpha \approx 1/2 \) and \( x > 0 \), we have an approximate constant equation of state

\[ \tilde{\omega}_{\text{RDE}} \simeq \frac{1}{3} \left( 1 - \frac{2}{\alpha} \right) \rightarrow -1 \]  

(29)

which describes the dark energy-dominated universe deriving by cosmological constant in the future. In order to compare \( \omega_{\text{RDE}} \) with \( \tilde{\omega}_{\text{RDE}} \), see Fig. 1. In this case of \( (4 - 2/\alpha)x \rightarrow \text{const} \), the Friedmann equation \( \text{(19)} \) takes an approximated from

\[ H^2 \approx \tilde{c}_1 \]  

(30)
Figure 1: Two equation of state parameters as functions of $x$ for $\rho_0 = \tilde{c}_1 = 1$ and $\alpha = 1/2$. $x = \ln [a/a_0]$ with $a_0 = 1$ ranges from $-\infty$ to $\infty$ which includes the present $x = 0$ at $a = a_0$. $\omega_{\text{RDE}}$ [dotted line] is always $-1$, whereas $\tilde{\omega}_{\text{RDE}}$ [solid curve] changes from 1 to $-1$ as $x$ increases from the past ($x < 0$) to the future ($x > 0$).

which provides the de Sitter-like solution

$$a(t) \approx e^{\sqrt{\tilde{c}_1} t}. \quad (31)$$

Also, this form could be recovered from (16) for $t \gg 1$ as

$$a(t) \approx e^{\sqrt{\tilde{c}_1} t}. \quad (32)$$

On the other hand, the modified Chaplygin gas model is given by the exotic equation of state of $p = -A/\rho^\alpha$ with $A > 0$ and $0 \leq \alpha \leq 1$. Here we discuss two saturation bounds only. For $\alpha = 1$, it provides the Chaplygin gas model whose energy density is given by

$$\rho_{\alpha=1} = \sqrt{A + \frac{B}{a^6}} = \sqrt{A} \sqrt{1 + \frac{B e^{-6x}}{A}}, \quad (33)$$

where $B$ is the integration constant [4]. For $B e^{-6x}/A \gg 1$, we can approximate $\rho_{\alpha=1}$ like as

$$\rho_{\alpha=1} \approx \frac{\sqrt{B}}{\sqrt{A}} e^{-3x}, \rho_{\alpha=1} \approx 0 \quad (34)$$

which describes the dust matter-dominated universe with $\omega_{\alpha=1} = 0$ in the early stage of the universe. For $B e^{-6x}/A \ll 1$, we have the approximated from

$$\rho_{\alpha=1} \approx \sqrt{A}, \rho_{\alpha=1} \approx -\sqrt{A}, \quad (35)$$
Figure 2: Three energy densities as functions of $x$ for $A = B = A_0 = \rho_{\tilde{g}_0} = \tilde{c}_1 = 1$ and $\alpha = 1/2$. $x = \ln[a/a_0]$ with $a_0 = 1$ ranges from $-\infty$ to $\infty$ which includes the present $x = 0$ at $a = a_0$. On the $\rho$-axis of left-panel, from top to bottom, the curves represent $\rho_t$, $\rho_{\alpha=0}$, and $\rho_{\alpha=1}$, respectively. Even though all curves converge on constants for $x > 0$ [right-panel] which represents de Sitter phase, their past energy densities [left-panel] show different behaviors for $x < 0$. In this choice of parameters, we note that $\rho_{\alpha=0} \approx \rho_{\alpha=1}$.

which describes the dark energy-dominated universe $\omega_{\alpha=1} = -1$ in the future.

For $\alpha = 0$ modified Chaplygin gas model [5], its energy density takes

$$\rho_{\alpha=0} = A + A_0 e^{-3x}$$

which shows a dust matter-dominated phase for $x < 0$, while it indicates de Sitter phase deriving by cosmological constant for $x > 0$.

Let us compare the total energy density (20) with (33) and (36). From Fig. 2, we observe that their past evolutions appear differently for $x < 0$, even though they converge on constants for $x > 0$. The (modified) Chaplygin gas model describes the whole evolution starting from the dust matter-dominated universe with $\omega_{\alpha=0,1} = 0$ to the dark energy-dominated universe with $\omega_{\alpha=0,1} = -1$, while the Ricci dark energy in Chern-Simons modified gravity describes the whole evolution starting from the stiff matter-dominated universe with $\tilde{\omega}_{\text{RDE}} = 1$ to the dark energy-dominated universe with $\tilde{\omega}_{\text{RDE}} = -1$ as is depicted in Fig. 1.

Consequently, the claim of Ref. [1] that there is a correspondence between the Ricci dark energy in Chern-Simons modified gravity and the modified Chaplygin gas model might be led to misleading. Aside from the fact that Ricci dark energy in Chern-Simons modified gravity is nothing but Ricci dark energy with a minimally coupled scalar when choosing the FRW metric, the similarity between two is limited to the de Sitter phase derived by the cosmological constant in the future ($x > 0$). This similarity can be understood partly by reconstructing the Chaplygin gas model in terms of the scalar [4].
The Chern-Simons term will participate in the cosmological evolution when choosing the anisotropic metric instead of the isotropic FRW metric [6].

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