Coherent states for free particles

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The coherent states are reviewed with particular application to the free particle system. The didactic advantages of the formalism is emphasized. Several interesting features, like the relation of the coherent states with the Galilei group and with the Husimi distribution are presented.

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I. INTRODUCTION

Coherent states were discovered by Schrödinger\textsuperscript{1} in 1926 but remained unexploited until Glauber\textsuperscript{2} rediscovered them in 1963 and developed a profound formalism for them with important applications; for instance, the “Glauber states” are fundamental in the description of the electromagnetic field for quantum optics. He was awarded the Nobel price in 2005 for the development of the quantum theory of optical coherence. Besides its important applications in quantum optics, theses states provide an elegant mathematical description of the harmonic oscillator and in many textbooks they are presented as a didactic tool. However, the didactic advantage of the coherent states has been ignored in the treatment of a free particle. It is, in fact, very convenient to present the gaussian states of a free particle, not as is usually done by assuming a gaussian state in momentum space and (through Fourier transformation) obtaining the corresponding gaussian in configuration space with a quite unappealing mathematics, but instead, one can present the coherent

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states for a free particle, and with simple and elegant algebraic methods, derive all properties of the gaussian states. The treatment of the coherent states of a free particle is similar to the corresponding treatment of the harmonic oscillator but there are remarkable differences and therefore it is convenient to present the complete derivation that can be directly used in the teaching of quantum mechanics in the advanced undergraduate or graduate level.

II. THE FREE PARTICLE SYSTEM

Let us consider the simplest quantum system consisting in one free structureless particle moving in a one dimensional space. Let $\mathcal{H}$ be the Hilbert space for the description of the system. It is usual to choose the space of square integrable functions $L_2$ for this space but in principle it can remain abstract. Position $X$ and momentum $P$ are the unique independent observables of this system. Their eigenvectors $\{\varphi_x\}$ and $\{\phi_p\}$, build two mutually unbiased bases and their commutation relation is $[X, P] = i\hbar$. (We assume here an infinite dimension of the Hilbert space; for finite dimension, in a lattice for instance, the commutation relation is more complicated[3]). The hamiltonian of the system is $H = \frac{P^2}{2m}$ and the time evolution operator is given by $U_t = \exp(-\frac{i}{\hbar}Ht)$.

III. THE COHERENT STATES

It is very convenient to introduce a length scale $\lambda$ for the system that will allow us, together with $\hbar$, to make position and momentum dimensionless in order to be able to add them as simple numbers. For this scale we can choose whatever we like, for instance the Compton length of the particle or any measure of delocalization of the particle in space (we will later see that it is convenient to choose $\lambda$ proportional to the width of the space distribution). With this scale we define the operator $A$ and its hermitian adjoint

$$A = \frac{1}{\lambda\sqrt{2}} X + i \frac{\lambda}{\hbar\sqrt{2}} P, \quad (1)$$

$$A^\dagger = \frac{1}{\lambda\sqrt{2}} X - i \frac{\lambda}{\hbar\sqrt{2}} P. \quad (2)$$
These operators are dimensionless and their commutation relation is

$$[A, A^\dagger] = I.$$  (3)

The relations above can be trivially inverted to obtain

$$X = \frac{\lambda}{\sqrt{2}} (A + A^\dagger) ,$$  (4)

$$P = \frac{\hbar}{i\sqrt{2}\lambda} (A - A^\dagger) .$$  (5)

Now we can define the coherent states $\psi_\alpha$ as the eigenvectors of $A$:

$$A\psi_\alpha = \alpha\psi_\alpha .$$  (6)

Since $A$ is not hermitian its eigenvalues are complex: $\alpha \in \mathbb{C}$. It is remarkable that we can very easily calculate these eigenvalues, related with observable properties of an ensemble of particles in this coherent state. Let us find $\langle X \rangle$, $\langle P \rangle$, $\langle X^2 \rangle$, $\langle P^2 \rangle$ and the widths $\Delta_x$ and $\Delta_p$ for the particle in a coherent state $\psi_\alpha$.

$$\langle X \rangle = \frac{\lambda}{\sqrt{2}} (\langle \psi_\alpha, A\psi_\alpha \rangle + \langle \psi_\alpha, A^\dagger\psi_\alpha \rangle) = \frac{\lambda}{\sqrt{2}} (\langle \psi_\alpha, A\psi_\alpha \rangle + \langle A\psi_\alpha, \psi_\alpha \rangle) = \frac{\lambda}{\sqrt{2}} (\alpha + \alpha^* ) ,$$  (7)

and similarly we obtain

$$\langle P \rangle = \frac{\hbar}{i\sqrt{2}\lambda} (\alpha - \alpha^* ) .$$  (8)

Adding them we obtain the eigenvalue $\alpha$ given in terms of the expectation values of position and momentum

$$\alpha = \frac{1}{\sqrt{2}\lambda} \langle X \rangle + i \frac{\lambda}{\sqrt{2}\hbar} \langle P \rangle .$$  (9)

We could have obtained this relation simply by taking the expectation values in Eq.(1). The coherent states are then completely determined by the expectation values of position and momentum. For $\langle X^2 \rangle$ we have

$$\langle X^2 \rangle = \frac{\lambda^2}{2} (\langle A + A^\dagger \rangle^2) = \frac{\lambda^2}{2} (\langle A^2 + A^\dagger A + AA^\dagger + A^\dagger A \rangle)$$

$$= \frac{\lambda^2}{2} (\langle A^2 + A^\dagger A + 2A^\dagger A + 1 \rangle) = \frac{\lambda^2}{2} (\alpha^2 + \alpha^*\alpha + 1) ,$$  (10)

and similarly we get

$$\langle P^2 \rangle = -\frac{\hbar^2}{2\lambda^2} (\alpha^2 + \alpha^*\alpha - 2\alpha^*\alpha + 1) .$$  (11)
The widths of the position and momentum distributions are then

\[ \Delta^2_x = \langle X^2 \rangle - \langle X \rangle^2 = \frac{\lambda^2}{2}, \]  
\[ \Delta^2_p = \langle P^2 \rangle - \langle P \rangle^2 = \frac{\hbar^2}{2\lambda^2}, \]  

where we see that the length scale is related to the width in position and we get
the important result that, in a coherent state, the uncertainty relation is optimized
\[ \Delta_x \Delta_p = \frac{\hbar}{2}. \]  

For this reason, it is often mentioned that the coherent states are the
quantum states closest to classical behaviour. In these coherent states, position
and momentum have the least correlation (the real part of the quantum covariance
function vanishes) in the sense that
\[ \langle XP + PX \rangle - 2\langle X \rangle\langle P \rangle = 0. \]

We have seen that the eigenvectors \( \psi_\alpha \) of the operator \( A \) have very interesting
properties and we may wonder whether the eigenvectors of \( A^\dagger \) may be of some
interest. We can see however that the eigenvalue equation \( A^\dagger \chi_\beta = \beta \chi_\beta \) does not
have solutions because, if \( \chi_\beta \) would exist then we would obtain an absurd result. In
fact, if we assume that \( \chi_\beta \) exists, applying the same techniques as before, we can
calculate \( \langle X \rangle, \langle X^2 \rangle \) and the width \( \Delta^2_x = \langle X^2 \rangle - \langle X \rangle^2 \) for the particle in the state \( \chi_\beta \).
Doing this we obtain \( \Delta^2_x = -\lambda^2/2 \); a negative result! This is of course impossible
because \( \Delta^2_x \) is the expectation value of the positive operator \( (X - \langle X \rangle)^2 \), therefore
there is no \( \chi_\beta \) satisfying \( A^\dagger \chi_\beta = \beta \chi_\beta \).

**IV. TIME EVOLUTION OF COHERENT STATES**

So far, the treatment of the coherent states for a free particle and for the
harmonic oscillator are similar and we can now present some important differences. In
the harmonic oscillator, one defines a “number operator” \( N = AA^\dagger \) having non-
negative integer eigenvalues, with \( A \) and \( A^\dagger \) as shift operators for the eigenvectors
of \( N \). This operator is very important for the harmonic oscillator because it has a
relevant physical meaning; it is, essentially, the hamiltonian operator that controls
the time evolution. Of course we can also define this number operator for the free
particle, and we will do so in the next section, but in this case it is not related
to a physically interesting observable. In the free particle case, the hamiltonian is
given by the momentum squared and the energy eigenvectors are the same as the
momentum eigenvectors \( \phi_p \) (notice that the energy eigenvalue \( E = p^2/2m \) has a
twofold degeneracy with eigenvectors $\phi_p$ and $\phi_{-p}$.

If at time $t = 0$ the free particle is in the coherent state $\psi_\alpha$ then at time $t$ it will be in the state $\Psi(t)$ given by the time evolution operator

$$\Psi(t) = \exp \left( -\frac{i}{\hbar} H t \right) \psi_\alpha = \exp \left( -\frac{it}{2m\hbar} P^2 \right) \psi_\alpha = \exp \left( \frac{i\hbar}{4m\lambda^2} (A - A^\dagger)^2 \right) \psi_\alpha .$$

(14)

Here we can see another important difference with the harmonic oscillator. One can prove that the coherent states of the harmonic oscillator remain coherent through the time evolution whereas in the free particle case the time evolution destroys the coherent states. That is, $\Psi(t)$, for $t \neq 0$, is no longer a coherent state. This is physically understood because, as is well known, the width in the position distribution, $\Delta_x$, increases with time for the free particle and the width in momentum, $\Delta_p$, remains constant (because the momentum distribution is time independent); therefore the uncertainty product $\Delta_x \Delta_p$ will also increase with time and will no longer have the minimal value $\hbar/2$ characteristic of the coherent states. Anyway we will now prove that $\Psi(t)$ is not an eigenvector of $A$ and therefore it is not a coherent state. For this, consider the time evolution applied to the Eq.(6) that results in

$$\exp \left( -\frac{i}{\hbar} H t \right) A \; \exp \left( \frac{i}{\hbar} H t \right) \Psi(t) = \alpha \Psi(t) .$$

(15)

Now, for the operator in the left hand side of this equation we can use the general expression valid for any constant $\gamma$ and any operators $A$ and $B$

$$\exp (-\gamma B) \; A \; \exp (\gamma B) = A + \gamma [B, A] + \frac{\gamma^2}{2!} [B, [B, A]] + \cdots + \frac{\gamma^n}{n!} [B, [B, \cdots, A]] + \cdots$$

(16)

To prove this equation, consider the function $f(\gamma) = \exp (-\gamma B) A \exp (\gamma B)$ and calculate the derivatives with respect to $\gamma$. Now making a Taylor series expansion about $\gamma = 0$ we obtain the relation above. In our case, however, the series is interrupted after the second term:

$$[H, A] = \frac{1}{2m} [P^2, A] = \frac{1}{2\sqrt{2m\lambda}} [P^2, X] = \frac{-i\hbar}{\sqrt{2m\lambda}} P ,$$

(17)

and $[H, [H, A]] = 0$. Therefore Eq.(15) becomes

$$\left( A + \frac{t}{\sqrt{2m\lambda}} P \right) \Psi(t) = \alpha \Psi(t) ,$$

(18)
that is,
\[ A\Psi(t) = \frac{-t}{\sqrt{2m\lambda}} P\Psi(t) + \alpha\Psi(t) \neq \beta\Psi(t). \]  
(19)
The last inequality follows because \( \Psi(t) \) is not an eigenvector of \( P \) (neither is \( \psi_{\alpha} \)) and therefore it is also not an eigenvector of \( A \).

V. THE NUMBER OPERATOR

As was announced at the beginning of last section, we will now study the number operator \( N = AA^\dagger \) that has interesting formal properties although it is not related to any important observable of the free particle physical system, whose most relevant observables are position and momentum. Another operator similar to \( N \) is \( A^\dagger A \) and one may wonder whether it may bring something interesting. However, due to the commutation relation \([A, A^\dagger] = I\), it is clear that this operator is not significantly different from \( N \) and it amounts only to the addition of a constant. The operator \( N \) is hermitian and positive \( N \geq 1 \), that is, it is bounded from below. This is important because one can prove that for such an operator there exist eigenvalues and eigenvectors \((\lambda_n, \chi_n)\). The prove that \( N^\dagger = N \) is trivial and the positivity follows from
\[ \langle \Psi, AA^\dagger \Psi \rangle = \langle A^\dagger \Psi, A^\dagger \Psi \rangle = \|A^\dagger \Psi\|^2 \geq 0, \forall \Psi \in \mathcal{H}. \]  
(20)
Then we have \( N\chi_n = \lambda_n\chi_n \), with \( \lambda_n \geq 0 \). Using the commutation relations
\[ [N, A^\dagger] = A^\dagger \] and \[ [N, A] = -A \]  
(21)
one can easily prove that \( A^\dagger \chi_n \) is another eigenvector of \( N \) but corresponding to the eigenvalue \( \lambda_n + 1 \) and \( A\chi_n \) is another eigenvector of \( N \) but corresponding to the eigenvalue \( \lambda_n - 1 \). Now, if we start with an arbitrary eigenvector \( \chi_n \) and apply \( A \) a sufficient large number of times, then we would get an eigenvector with a negative eigenvalue in contradiction with the positivity of \( N \). The solution of this difficulty is that the eigenvalues must be integer numbers. Then we can set \( \lambda_n = n = 0, 1, 2, \ldots \), and there is a lowest stair \( \chi_0 \) such that \( A\chi_0 = 0 \) from which we can obtain any \( \chi_n \)
by applying \( n \) times the raising operator \( A^\dagger \). Summarizing, we have:

\[
\begin{align*}
N\chi_n &= n\chi_n \text{ with } n = 0, 1, 2, \ldots \quad (22) \\
A^\dagger\chi_n &= \sqrt{n + 1} \chi_{n+1} \quad (23) \\
A\chi_n &= \sqrt{n} \chi_{n-1}, \text{ in particular, } A\chi_0 = 0 \quad (24) \\
\chi_n &= \frac{1}{\sqrt{n!}}(A^\dagger)^n\chi_0. \quad (25)
\end{align*}
\]

Notice that from Eqs. (6 and 24) it follows that the lowest number state coincides with the coherent state for \( \alpha = 0 \), that is, \( \chi_0 = \psi_0 \). We will later see how every coherent state \( \psi_\alpha \) can be expanded in the number states basis \( \{\chi_n\}\).

**VI. THE DRIFT OPERATOR**

An interesting operator for the coherent states formalism is the drift operator defined as

\[
D(\alpha) = \exp \left( \alpha A^\dagger - \alpha^* A \right) \\
= \exp \left( -\frac{|\alpha|^2}{2} \right) \exp (\alpha A^\dagger) \exp (-\alpha^* A) \\
= \exp \left( \frac{|\alpha|^2}{2} \right) \exp (-\alpha^* A) \exp (\alpha A^\dagger). \quad (26)
\]

The last two expressions for \( D(\alpha) \) follow from the Hausdorff-Baker-Campbell (HBC) relation:

\[
\exp(R) \exp(S) = \exp(R + S) \exp \left( \frac{[R, S]}{2} \right), \text{ valid if } [R, [R, S]] = [S, [R, S]] = 0. \quad (27)
\]

Repeated application of this identity results in

\[
\exp(R) \exp(S) = \exp(S) \exp(R) \exp([R, S]), \text{ valid if } [R, [R, S]] = [S, [R, S]] = 0. \quad (28)
\]

With the HBC relation, one can also prove that the drift operator is unitary, \( D^\dagger(\alpha)D(\alpha) = D(\alpha)D^\dagger(\alpha) = I \), therefore \( D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha) \). The set \( \{D(\alpha), \forall \alpha \in \mathbb{C}\} \) is a group whose product can also be calculated with the HBC relation as

\[
D(\alpha)D(\beta) = \exp \left( \frac{\alpha\beta^* - \alpha^* \beta}{2} \right) D(\alpha + \beta). \quad (29)
\]
We will prove that the operator $D(\alpha)$ can be used to relate coherent states corresponding to different values of $\alpha$. More precisely, if $\psi_0$ is the coherent state for $\alpha = 0$ then we have

$$\psi_\alpha = D(\alpha)\psi_0 . \quad (30)$$

In order to prove this, we will need some algebraic properties of the operators $A$ and $A^\dagger$. By mathematical induction one can prove that, from the commutation relation $[A, A^\dagger] = I$, it follows that $[A, A^\dagger^n] = nA^{\dagger(n-1)}$, and from this, we can show that for every function $F$ that can be expanded as a power series it is

$$[A, F(A^\dagger)] = \frac{dF(A^\dagger)}{dA^\dagger} . \quad (31)$$

Applying this to the drift operator we have

$$[A, D(\alpha)] = \alpha D(\alpha) . \quad (32)$$

With this result we can easily see that $D(\alpha)\psi_0$ is an eigenvector of $A$ corresponding to the eigenvalue $\alpha$. In fact, we have:

$$AD(\alpha)\psi_0 = D(\alpha)A\psi_0 + \alpha D(\alpha)\psi_0 = \alpha D(\alpha)\psi_0 . \quad (33)$$

Therefore $D(\alpha)\psi_0 = k\psi_\alpha$, where $k$ is some proportionality constant. In order to determine it, consider the norm of the equation

$$|k|^2 \langle \psi_\alpha, \psi_\alpha \rangle = \langle D(\alpha)\psi_0, D(\alpha)\psi_0 \rangle = \langle \psi_0, D^\dagger(\alpha)D(\alpha)\psi_0 \rangle = \langle \psi_0, \psi_0 \rangle = \|\psi_0\|^2 . \quad (34)$$

Assuming that the coherent states are normalized, $\langle \psi_\alpha, \psi_\alpha \rangle = \|\psi_\alpha\|^2 = 1$, then $|k|^2 = 1$ and $D(\alpha)\psi_0$ and $\psi_\alpha$ can differ by a constant phase that can be set equal to 1 and therefore we obtain Eq.(30). Notice that the algebraic proof given here is different from the usual proof that involves the expansion of the coherent states in the basis of the eigenvectors of the number operator. In this approach we don’t use such a basis because they are physically uninteresting for the free particle system and we prefer to rely only in the algebraic structure of the operators. The drift operator has an intuitive meaning that becomes evident when we write $D(\alpha)$ in terms of the position and momentum operators: Using Eqs.(1, 2, 9, 26), we have

$$D(\alpha) = \exp \left( \frac{i}{\hbar} \langle P \rangle X - \langle X \rangle P \right)$$

$$= \exp \left( -\frac{i}{2\hbar} \langle X \rangle \langle P \rangle \right) \exp \left( \frac{i}{\hbar} \langle P \rangle X \right) \exp \left( -\frac{i}{\hbar} \langle X \rangle P \right)$$

$$= \exp \left( \frac{i}{2\hbar} \langle X \rangle \langle P \rangle \right) \exp \left( -\frac{i}{\hbar} \langle X \rangle P \right) \exp \left( \frac{i}{\hbar} \langle P \rangle X \right) , \quad (35)$$
and recalling that $\exp(-\frac{i}{\hbar}aP)$ is a translation operator that performs $X \rightarrow X + a$ and $\exp(\frac{i}{\hbar}gX)$ changes the momentum $P \rightarrow P + g$, we see that $D(\alpha)$ transforms a coherent state corresponding to a particle at rest, $\langle P \rangle = 0$, located at the origin, $\langle X \rangle = 0$, into a coherent state for the particle at position $\langle X \rangle$ and moving with momentum $\langle P \rangle$. Notice that the different choice of what transformation is performed first, amounts only to a different constant phase. It is then interesting to notice that, effectively, $D(\alpha)$ performs a Galilei transformation on the state $\psi_0$ and therefore the group build with the drift operators $D(\alpha)$ is isomorph with the Galilei group. In fact, the phase appearing in Eq. (29) shows that this group is a projective group (not simple) as is the case with the Galilei group. The emergence of the Galilei group is an additional “bonus” of the formalism of coherent states.

The arguments presented in this section are valid for a free particle as well as for the harmonic oscillator, or for any other potential. The only difference is that, in the harmonic oscillator case, the state $\psi_0$ has the unique feature of being the ground state of the Hamiltonian and one may prefer to choose this state in order to generate all other coherent states. In the free particle case, $\psi_0$ is a state as good as any other because of the Galilei invariance of the free particle system and one might prefer to write Eq. (30) in an unbiased way as $\psi_\beta = D(\beta - \alpha)\psi_\alpha$.

**VII. COHERENT STATES ARE GAUSSIAN**

One of the most beautiful features of the Hilbert space formalism of quantum mechanics is that almost all results can be obtained without specifying a particular representation of the Hilbert space. Everything follows from the algebraic and geometric structure of the Hilbert space. Some features become however easier to grasp in some particular representation of the Hilbert space. For instance if we are interested in the position distribution of a particle it becomes natural to choose the space $L_2$ of square integrable functions of a real variable $x$. As is well known, the position operator $X$ in this space is the multiplication by $x$ and the momentum operator $P$ is the derivative $-i\hbar \frac{d}{dx}$. In order to find the coherent states $\psi_\alpha(x) = \langle \varphi_x, \psi_\alpha \rangle$ in this space we must write and solve Eq. (6) in $L_2$. That is

$$\left[ \frac{1}{\lambda \sqrt{2}} x + i \frac{\lambda}{\hbar \sqrt{2}} (-i\hbar \frac{d}{dx}) \right] \psi_\alpha(x) = \alpha \psi_\alpha(x).$$

(36)
In terms of a dimensionless variable \( y = x / (\sqrt{2}\lambda) \) this becomes

\[
\frac{d\psi_\alpha(y)}{dy} = 2(\alpha - y)\psi_\alpha(y), \tag{37}
\]

that is,

\[
\frac{d\psi_\alpha}{\psi_\alpha} = 2(\alpha - y)dy. \tag{38}
\]

Integrating we have

\[
\ln(\psi_\alpha) = -y^2 + 2\alpha y + \text{Const.}, \tag{39}
\]

that is,

\[
\psi_\alpha(y) = C \exp \left(-y^2 + 2\alpha y\right). \tag{40}
\]

The next step is to complete the square in the exponential and absorb the corresponding factor in the constant \( C \) (that will be anyway determined by normalization). After replacing the original variable \( x \) and \( \alpha \) given by Eq. (9) with \( \langle X \rangle = x_0 \) and \( \langle P \rangle = p_0 \), and replacing the length scale \( \lambda = \sqrt{2}\Delta_x \) we obtain

\[
\psi_\alpha(x) = \left[2\pi\Delta_x^2\right]^{-1/4} \exp \left(-\frac{(x - x_0)^2}{4\Delta_x^2} + \frac{ip_0}{\hbar}x\right). \tag{41}
\]

We see therefore that the coherent states of a free particle are the gaussian states (in order to appreciate the didactic advantages of the coherent states for a free particle, compare this derivation with the presentation of the of gaussian wave packets as is usually done in textbooks).

One last remark in this section is that the coherent states in momentum representation, that is, the functions \( \psi_\alpha(p) = \langle \phi_p, \psi_\alpha \rangle \) are also gaussian. There is in fact a symmetry in the exchange of position and momentum for the coherent states.

**VIII. EXPANSION IN NUMBER STATES AND IN COHERENT STATES**

A very important feature of the Hilbert space formalism is the fact that the eigenvectors of an hermitian operator, that may be related to a physical observable, build a basis. If we expand the state \( \Psi \) of a system in this basis, we may interpret the modulus squared of the expansion coefficients as the probabilities associated with the eigenvalues of the observable.

A relevant question is then, if we can use the set of coherent states \( \{\psi_\alpha\} \) in order to make an expansion of a state \( \Psi \) and interpret the expansion coefficients...
$|\langle \psi_\alpha, \Psi \rangle|^2$ as some probability distribution. The first remark in order to answer this, is that the coherent states are the eigenvectors of an operator that is not hermitian and they do not build an orthonormal basis. They are not even linearly independent (if they were independent we could transform them into a basis by Schmidt orthogonalization procedure). We will show that the coherent states are not orthogonal, that is, $\langle \psi_\alpha, \psi_\beta \rangle \neq 0$.

$$\langle \psi_\alpha, \psi_\beta \rangle = \langle D(\alpha)\psi_0, D(\beta)\psi_0 \rangle .$$  \hspace{1cm} (42)

Using Eq.(26) and considering that $\exp(aA)\psi_0 = \psi_0$ we get

$$\langle \psi_\alpha, \psi_\beta \rangle = \exp \left( -\frac{1}{2} |\alpha|^2 \right) \exp \left( -\frac{1}{2} |\beta|^2 \right) \langle \psi_0, \exp (\alpha A) \exp (\beta A^\dagger) \psi_0 \rangle .$$  \hspace{1cm} (43)

Now, using Eq.(28) we can permute the operators and we get

$$\langle \psi_\alpha, \psi_\beta \rangle = \exp \left( -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \alpha^* \beta \right) \langle \psi_0, \exp (\beta A^\dagger) \exp (\alpha^* A) \psi_0 \rangle .$$

The three operators $X, P, N$ define three bases of the Hilbert space $\{ \varphi_x \}, \{ \phi_p \}$ and $\{ \chi_n \}$. The first two are continuous $x, p \in \mathbb{R}$ and unbiased, that is, $|\langle \varphi_x, \phi_p \rangle|$ is a constant independent of $x$ and $p$. The third basis is numerable and biased with the other two. One can calculate, for instance, that $\langle \varphi_x, \chi_n \rangle = \chi_n(x)$ is related with the Hermite polynomials. We will not reproduce here this calculation that can be found in many quantum mechanics textbooks. Instead of this, we are interested in the expansion of the coherent states in these bases. We have already seen in section VII that the Fourier coefficients of the expansion of the coherent states in the bases of position and momentum, that is $\langle \varphi_x, \psi_\alpha \rangle$ and $\langle \phi_p, \psi_\alpha \rangle$, are gaussian functions. The corresponding coefficients, $\langle \chi_n, \psi_\alpha \rangle$, of the expansion of the coherent states in this new basis $\{ \chi_n \}$ are given by

$$\langle \chi_n, \psi_\alpha \rangle = \left( \frac{1}{\sqrt{n!}} (A^\dagger)^n \chi_0, \psi_\alpha \right) = \frac{1}{\sqrt{n!}} \langle \chi_0, A^n \psi_\alpha \rangle =$$

$$\frac{1}{\sqrt{n!}} \langle \psi_0, \alpha^n \psi_\alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} \langle \psi_0, \psi_\alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} \exp \left( -\frac{|\alpha|^2}{2} \right) .$$  \hspace{1cm} (44)
We have therefore
\[ \psi_\alpha = \sum_n \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{|\alpha|^2}{2}\right) \chi_n. \] (45)

We can now go back to our question of whether it is possible to make an expansion of any state in terms of the coherent states. We have seen that these states are not orthogonal; however they build an (over)complete set of states in the sense that
\[ \pi^{-1} \int_{\alpha \in \mathbb{C}} d^2\alpha \, \psi_\alpha \langle \psi_\alpha, \cdot \rangle = 1 \] (46)

(we use here the correct Hilbert space notation for a projector: \( \Psi \langle \Psi, \cdot \rangle \); for those addict to the Dirac notation this is \( |\Psi\rangle\langle\Psi| \)) and we can use this relation in order to expand any element \( \Psi \) as
\[ \Psi = \mathbb{I} \Psi = \pi^{-1} \int_{\alpha \in \mathbb{C}} d^2\alpha \, \psi_\alpha \langle \psi_\alpha, \Psi \rangle. \] (47)

The proof of the completeness relation in Eq.(46) follows from the completeness of the basis \( \{ \chi_n \} \):
\[
\int_{\alpha \in \mathbb{C}} d^2\alpha \, \psi_\alpha \langle \psi_\alpha, \cdot \rangle = \int_{\alpha \in \mathbb{C}} d^2\alpha \, \sum_n \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_m \frac{\alpha^m}{\sqrt{m!}} \exp\left(-\frac{|\alpha|^2}{2}\right) \chi_n \langle \chi_m, \cdot \rangle
= \sum_{n,m} \chi_n \langle \chi_m, \cdot \rangle \frac{1}{\sqrt{n!m!}} \int_{\alpha \in \mathbb{C}} d^2\alpha \, \alpha^n \alpha^m \exp\left(-|\alpha|^2\right)
= \sum_{n,m} \chi_n \langle \chi_m, \cdot \rangle \frac{1}{\sqrt{n!m!}} \pi n! \delta_{n,m}
= \pi \sum_n \chi_n \langle \chi_n, \cdot \rangle = \pi \mathbb{I}. \] (48)

The integral in the complex plane is performed in polar representation \( \alpha = r \exp(i\theta) \).

The completeness relation in Eq.(46) can be used to expand any Hilbert space element. For instance, we can invert Eq.(45) and give the number states expanded in the coherent states, using Eq.(43):
\[ \chi_n = \mathbb{I} \chi_n = \pi^{-1} \int_{\alpha \in \mathbb{C}} d^2\alpha \, \frac{\alpha^n}{\sqrt{n!}} \exp\left(-\frac{|\alpha|^2}{2}\right) \psi_\alpha. \] (49)

This expression, when written in the Hilbert space of the squared integrable functions, \( \mathcal{L}_2 \), allows an interesting representation of the Hermite polynomials in terms of gaussian functions.

One last comment concerning the expansion in terms of the coherent states is that the set of elements \( \{ \psi_\alpha \} \), as well as the basis elements \( \{ \chi_n \} \), are well defined...
in the Hilbert space. Strictly speaking, the bases associated with position and momentum \( \{ \varphi_x \} \) and \( \{ \phi_p \} \) do not belong to the Hilbert space. In fact, if we choose the Hilbert space of the square integrable functions of some real variable, \( y \), in the position representation where \( \varphi_x(y) = \delta(x - y) \) and \( \phi_p(y) = \frac{1}{\sqrt{2\pi}} \exp(i\frac{\pi}{\hbar} py) \), we can immediately see that they do not belong to the Hilbert space because they are not squared integrable. For this reason, in advanced quantum mechanics books, the Hilbert space is extended to the \textit{rigged Hilbert space} that includes these elements that may be used as bases. This is precisely what we do when we calculate a Fourier transformation. The mathematics to justify the extension of the Hilbert space to the rigged Hilbert space is well explained in Chapter 1 of reference \[6\].

IX. THE HUSIMI DISTRIBUTION

The main interest in the expansion of a state \( \Psi \) of a system in terms of the eigenvector of some observables is that the modulus squared of the expansion coefficient have the physical interpretation as the probability distribution of the corresponding eigenvalues. So, \( \rho(x) = |\langle \varphi_x, \Psi \rangle|^2 \) and \( \varpi(p) = |\langle \phi_p, \Psi \rangle|^2 \) are understood as the probability distributions for position and momentum (although this is perhaps a missnommer\[5\]). Similarly, we may try to interpret \( |\langle \psi_\alpha, \Psi \rangle|^2 \) as some probability distribution for \( \alpha \). From the mathematical point of view, this seems to be possible because, using the over-completeness relation Eq.(46) one can prove that the expectation value of any function \( F(A) \) can be given as

\[
\langle \Psi, F(A)\Psi \rangle = \langle \Psi, F(A)\Psi \rangle = \pi^{-1} \int_{\alpha \in \mathbb{C}} d^2\alpha \ F(\alpha) \ |\langle \psi_\alpha, \Psi \rangle|^2 ,
\]

and therefore \( \pi^{-1}|\langle \psi_\alpha, \Psi \rangle|^2 \) can be interpreted as the probability for the realization of the value \( \alpha \). However, from the physical point of view this does not seem to make sense. The reason for this is that \( A \) is not an hermitian operator and therefore the complex eigenvalues \( \alpha \) can not be associated with the result of a single measurement of some observable. Consequently, it is meaningless to say that the system \textit{has} some value of \( \alpha \) with some probability. In other words, let us consider the complex variable \( \alpha \) as a function of two real variables \( x \) and \( p \) as in Eq.(5) \( \alpha(x, p) = \frac{1}{\sqrt{2\pi}} x + i\frac{1}{\sqrt{2\pi}} p \). Clearly, \( x \) and \( p \) can not be the result of any measurement of position and momentum because these can not be measured simultaneously. Neither is there an observable, with its corresponding hermitian operator, whose single measurement
provides simultaneously the expectation values of position and momentum. These expectation values can only be obtained in a multiplicity of measurements in an ensemble of identically prepared systems but not in some single measurement of one system. More precisely, we can see that given an arbitrary state $\Psi$, that is, any Hilbert space element, with position expectation value $\langle X \rangle = \langle \Psi, X\Psi \rangle$, then there exists no hermitian operator $\tilde{X}$ such that $\tilde{X}\Psi = \langle X \rangle \Psi$. Clearly this is impossible because if such an operator would exist, then all Hilbert space elements would belong to the basis associated with $\tilde{X}$.

Even though $|\langle \psi_\alpha, \Psi \rangle|^2$ is not a probability density, this quantity has been studied in one of the several attempts to establish a compound probability distribution for position and momentum (see for instance Chapter 15 of reference [6]). That is, some function of $x$ and $p$ such that when integrated over $p$ results in the position distribution $\rho(x)$ and when integrated over $x$ delivers the momentum distribution $\varpi(p)$. The most famous attempt is the Wigner distribution that has these two properties but is not everywhere nonnegative. The function $\rho_H(x,p) = \pi^{-1}|\langle \psi_\alpha(x,p), \Psi \rangle|^2$ is known as the Husimi distribution that is everywhere nonnegative but, when integrated over one of its variables, does not provide the correct distribution for the other one. There are interesting properties for this phase space “distribution” that can be found in ref.[6] and a profound comparative study and applications of many phase space distribution functions is found in ref.[7].

**X. CONCLUSIONS**

We have seen that the coherent states for a free particle, very well known for the harmonic oscillator case and in quantum optics, provide a very useful didactic tool that has not been exploited in quantum mechanics textbooks. For instance, the very elegant formal treatment discovered by Glauber allows a much nicer presentation of the gaussian wave packets for a free particle. The similarities and differences appearing in the harmonic oscillator and in the free particles are emphasized providing a global outlook of quantum mechanics. Another advantage of the formalism is that one can easily relate the drift operators with the Galilei group and see that it must be a projective group. The expansion in terms of Hilbert space bases related with observables of position, momentum and number are exposed and compared with the expansion in the over-complete set of coherent states that provide a natural way for
introducing the Husimi distribution. The paper is written with complete calculations, or with sufficient hints for the calculations, in a way that it can be directly handed to students as complement of any textbook.

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