ON THE SIZE OF THE SET OF
NON-DIFFERENTIABILITY POINTS OF MAXIMAL
FUNCTION

HANNES LUIRO

Abstract. We study the size of the set of non-differentiability points for classical Hardy-Littlewood maximal function $Mf$. Our first main result states that $Mf$ is differentiable a.e. if function $f$ is differentiable a.e. Another main theorem is that if $f$ is differentiable (and $Mf \neq \infty$), then for every $0 < \delta < \frac{1}{2}$ the set of non-differentiability points of $Mf$ is included in a countable union of $\delta$-porous sets. This also implies that the Hausdorff-dimension of the non-differentiability points is at most $n - 1$. The results can be also applied to other maximal operators as well as to other important special functions, like convex functions and distance functions.

1. Introduction

The differentiability properties of the classical Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| \, dy = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy,$$

have been actively studied during the past fifteen years. The first step was taken by J. Kinnunen who observed that $M$ is bounded operator in Sobolev-spaces $W^{1,p}(\mathbb{R}^n)$ when $1 < p \leq \infty$ [K]. Some of the extensions and related results can be found e.g. from [AP], [HM], [HO], [KL], [Ko], [KS], [Lu] and [Lu2].

The motivation of this note raised from the following questions:

Question 1.1. Does it hold that $Mf$ is a.e. differentiable if $f$ is a.e. differentiable?

Question 1.2. What can we say about the Hausdorff-dimension of the non-differentiability points of $Mf$ if $f$ is e.g. smooth function.

These questions seem to attract general attention and has not been worked out in any of the previous researches concerning maximal operators. Maybe the most closely related to these issues (especially to question [L1]) is the result from P. Hajłasz and J. Malý who showed that $Mf$ is approximately differentiable if $f$ is approximately differentiable ([HM, Thm. 1]).

It turns out that the solution of the problem (1.1) follows relatively easily by Stepanov’s theorem (see e.g. [Fed, 3.1.8]), which is stated as follows:

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Theorem 1.3. Measurable function \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable a.e. in the set
\[
\left\{ x \in \mathbb{R}^n : \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} < \infty \right\}.
\]

To be more precise, denote the difference quotient of function \( f \) at \( x \), respect to \( h \in \mathbb{R}^n \), by
\[
(2) \quad D^h(x) := \frac{|f(x + h) - f(x)|}{|h|},
\]
and define a 'singular' set of \( f \) by
\[
S_f := \left\{ x \in \mathbb{R}^n : \limsup_{h \to 0} D^h(x) = \infty \right\}.
\]

Observe that Stepanov’s theorem says that \( f \) is differentiable a.e. outside the singular set \( S_f \). In section 2 we prove the following theorem:

Theorem 1.4. Suppose \( f \) is a locally integrable function so that \( Mf(x) \neq \infty \). Then the singular set of \( Mf \) is contained in the singular set of \( f \).

The above theorem, combined with the Stepanov’s Theorem, immediately gives the following corollary:

Corollary 1.5. If \( Mf(x) \neq \infty \) then \( Mf \) is differentiable a.e outside the set \( S_f \). Especially, if \( f \) is differentiable almost everywhere, the same holds for \( Mf \) as well.

The investigation of the another problem turns out to have interesting connections with more general problems. Our original target was to investigate more precise estimates for the size of the nondifferentiability points of \( Mf \) in the case where \( f \) itself is regular enough. Motivation towards this question raises simply from the observation that even for smooth functions the Hausdorff-dimension of the set \( E \) of non-differentiability points of \( Mf \) may be \( n - 1 \) (this is even typical). We will prove that this is the worst case, indeed the dimension of the set can not exceed \( n - 1 \). Actually, even stronger result holds, in terms of porosity of the set. For that, let \( A \subset \mathbb{R}^n, x \in A \), and define (see e.g. [M])
\[
(3) \quad \text{por}(A, x) = \liminf_{r \to 0} \sup\{\delta > 0 : B(y, \delta r) \subset B(x, r) \setminus A \text{ for some } y \in \mathbb{R}^n\}
\]
and
\[
(4) \quad \text{por}(A) = \inf_{x \in A} \text{por}(A, x).
\]

Let us say that set \( A \) is asymptotically \( \sigma - \frac{1}{2} \)-porous if for any \( 0 < \delta < \frac{1}{2} \) set \( A \) is contained in the countable union of sets with porosity at least \( \delta \).

We will show that

Theorem 1.6. If \( f \) is differentiable and \( Mf(x) \neq \infty \), then \( Mf \) is differentiable up to an asymptotically \( \sigma - \frac{1}{2} \)-porous set \( E \). Moreover, \( \dim_H(E) \leq n - 1 \).
It is well known that above the latter claim follows from the first one, see [M, Chapter 11] (or Theorem 3.5).

The proof of Theorem 1.6 is done in section 4 but a remarkable part of the work has been done in section 3. There we will study the general situation where \( \{f_k\}_{k \in \mathbb{N}} \) is a family of functions \( f_k \in C^1(\mathbb{R}^n) \) such that \( \{Df_k\}_{k \in \mathbb{N}} \) is locally uniformly bounded and equicontinuous\(^1\), and \( F \) is defined by

\[
F(x) = \sup_{k \in \mathbb{N}} f_k(x).
\]

This kind of functions are sometimes called in literature as regular upper envelopes or simply maximum-functions, see e.g. [BC, Chapter 4] and references therein. Various special functions in analysis are this kind of, and due to that the differentiability problems for these functions are under constant interest, especially in optimization theory.

We will show in section 3 that Theorem 1.6 holds when replacing \( Mf \) by \( F \), where \( F \) is defined as above (Theorem 3.1). If desired, this theorem can be seen as a sharpened or generalized version for the result [BC] Corollary 4.4, which says that if \( F \) were determined by a finite family \( \{f_k\} \), then it is differentiable up to the countable union of sets \( E_k \) for which \( \text{por}(E_k, x) > 0 \) for every \( x \in E_k \). However, we remark that we were not aware about the preceding theory before our proofs were almost complete. Indeed, our primarily goal in this paper was not to obtain the most elegant generalizations for those previous results but our research around question 1.2 led to the presented version. It should also be pointed out that in [BC] the main goal has obviously been less to derive optimal porosity constants or dimension estimates for Euclidean setting than to work out differentiability questions in more general setting of separable Hilbert spaces.

Dimension estimate in the setting of Theorem 3.1 (as well as in Theorem 1.6) is sharp, as can be easily seen by simply considering the maximum of two smooth functions. Equicontinuity of \( \{Df_k\} \) appears as a natural general assumption and easy examples show that assuming only the uniform boundedness for a family \( \{Df_k\} \) would not be sufficient.

Finally, it is clear that Theorem 3.1 can be applied to the various other maximal functions as well, like fractional maximal function, maximal functions in domain or maximal functions defined via cubes etc. (see e.g. [KS] or [KL]). Moreover, applications are not restricted to the maximal operators but also include other important special functions in analysis (especially in optimization theory), like distance functions or sup/inf-convolutions (see e.g. [CLS]), for which the regularity issues are important and differentiability properties have been widely studied. We introduce some elementary examples of these applications in the end of the section 4. Even these rather simple applications seem to refine the previously published results.

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\(^1\)Equicontinuity implies the uniform boundedness if it is assumed (e.g.) that functions \( f_k \) are positive and \( F \neq \infty \).
2. Almost everywhere differentiability

In this section we concentrate on proving Theorem 1.4. Suppose that \( Mf \neq \infty \) and \( x \in S(Mf) \), thus there exists a sequence of \( h_k \in \mathbb{R}^n \), \( h_k \to 0 \) such that
\[
D^h_k(Mf)(x) = \frac{|Mf(x + h_k) - Mf(x)|}{|h_k|} \to \infty \text{ if } k \to \infty.
\]

Let us prove the claim by contradiction, thus assume that \( x \not\in S(f) \) i.e. there exists constant \( C > 0 \) and \( r_0 > 0 \) such that \( D^h f(x) < C \) when \( |h| < r_0 \). Especially this implies that \( f \) is continuous at \( x \), which in turn implies that \( Mf \) is continuous at \( x \) if \( Mf(x) < \infty \). This holds, since \( Mf(x) = \infty \) would (in this case) imply that \( Mf \equiv \infty \).

Then, let us choose for each \( k \) radius \( r_k \) which almost gives the maximum average at \( x + h_k \). More precisely, choose \( r_k \) such that
\[
Mf(x + h_k) \leq \int_{B(x+h_k,r_k)} |f(y)| \, dy + \frac{|h_k|}{k}.
\]

Now it follows that \( (r_k) \) is bounded. Otherwise one can use consequence (25) of Lemma 4.1 to obtain that
\[
\frac{|Mf(x + h_k) - Mf(x)|}{|h_k|} \to 0 , \text{ when } k \to \infty ,
\]
which contradicts with (8).

Then consider the case when the sequence \( (r_k) \) is bounded from below i.e \( r_k > \lambda > 0 \) for any \( k \). In this case we use the estimate
\[
\frac{Mf(x + h_k) - Mf(x)}{|h_k|} \leq \frac{1}{|h_k|} \left( \int_{B(x+h_k,r_k)} |f(y)| \, dy + \frac{|h_k|}{k} - \int_{B(x,r_k+h_k)} |f(y)| \, dy \right)
\]
\[
\leq \frac{1}{|h_k|} \left( \frac{1}{|B_{r_k}|} - \frac{1}{|B_{r_k+h_k}|} \right) \left( \int_{B(x,r_k+h_k)} |f(y)| \, dy \right) + \frac{1}{k} \leq \frac{C(n)Mf(x)}{\lambda}.
\]

By extracting a subsequence of \( (r_k) \), if needed, we may assume that \( r_k \to r_0 \geq \lambda > 0 \) and it follows (by the continuity of \( Mf \) at \( x \)) that
\[
Mf(x) = \int_{B(x,r_0)} |f(y)| \, dy.
\]

Then one may write the above estimate in reversed direction to obtain that also
\[
\frac{Mf(x) - Mf(x + h_k)}{|h_k|} \leq \frac{C(n)Mf(x)}{\lambda} .
\]

Clearly above estimates imply that \( D^h_k(Mf)(x) \not\to \infty \). The proof is complete if we can show that this holds also in the remaining case where the sequence \( r_k \) is not bounded from below. In this case we may assume, by extracting a subsequence, if needed that \( r_k \to 0 \) as \( k \to \infty \). The first
observation is that by the continuity of \( f \) at \( x \) it follows that \( Mf(x) = f(x) \). Moreover, \( \limsup_{h \to 0} D^h f(x) < \infty \) implies that

\[
(9) \quad f(x + h) = f(x) + u_x(h)|h|,
\]

where \( u_x : \mathbb{R} \to \mathbb{R} \) is finite when \( |h| < r_0 \).

Then the following estimate is valid:

\[
\frac{Mf(x + h_k) - Mf(x)}{|h_k|} \leq \frac{1}{|h_k|} \left( \int_{B(x + h_k, r_k)} |f(y)| \, dy - \int_{B(x, r_k)} |f(y)| \, dy \right) + \frac{1}{k}
\]

\[
= \frac{1}{|h_k|} \left( \int_{B(x + h_k, r_k) \setminus B(x, r_k)} |f(y)| \, dy - \int_{B(x, r_k) \setminus B(x + h_k, r_k)} |f(y)| \, dy \right) + \frac{1}{k}
\]

\[
= \frac{1}{|h_k|} \int_{B(x, r_k) \setminus B(x + h_k, r_k)} \left| f(x) + u_x(y - x) + (y - x)|y - x| \right| \, dy + \frac{1}{k}
\]

\[
\leq \frac{2 \sup_{B(x, r_k) \cup B(x + h_k, r_k)} u_x}{|h_k|} \int_{B(x + h_k, r_k) \cup B(x, r_k) \setminus B(x + h_k, r_k)} |y - x| \, dy + \frac{1}{k}
\]

\[
\leq \frac{2(r_k + h_k)(\sup_{B(x, r_k) \cup B(x + h_k, r_k)} u_x)2 |B(x + h_k, r_k) \setminus B(x, r_k)|}{|h_k|} + \frac{1}{k}.
\]

Now it is easy to check that

\[
|B(x + h_k, r_k) \setminus B(x, r_k)| \leq C(n) \min\{h_k r_k^{-1}, r_k^n\}.
\]

Applying this to the previous estimate in cases \( r_k < h_k \) or \( r_k \geq h_k \) implies that

\[
(10) \quad \frac{Mf(x + h_k) - Mf(x)}{|h_k|} \leq C'(n) \left( \sup_{B(x, r_k) \cup B(x, h_k)} (u_x) \right) + \frac{1}{k}.
\]

Even easier argument shows that also

\[
\frac{Mf(x) - Mf(x + h_k)}{|h_k|} = \frac{|f(x)| - Mf(x + h_k)}{|h_k|}
\]

\[
\leq \frac{1}{|h_k|} \left( |f(x)| - \limsup_{r \to 0} \int_{B(x + h_k, r)} |f(y)| \, dy \right)
\]

\[
\leq C'(n) \left( \sup_{B(x, h_k)} (u_x) \right).
\]

Since \( r_k + h_k < r_0 \) if \( k \) is big enough and \( \sup_{B(x, r_0)} (u_x) < \infty \) it follows that \( D^h_k (Mf)(x) \not\to \infty \). This completes the proof. \( \square \)
3. General pointwise maximum function

In this section our goal is to prove the following Theorem:

**Theorem 3.1.** Suppose that \( F \) is defined as in the introduction and \( F(x) \neq \infty \). Then \( F \) is differentiable up to an asymptotically \( \sigma - \frac{1}{2} \)-porous set \( E \) with \( \dim_H(E) \leq n - 1 \).

Let us begin with some notation. The family of all subsets of set \( X \) is denoted by \( \mathcal{P}(X) \). For \( A \subset \mathbb{R}^n \) and \( x \in \mathbb{R}^n \) we define \( |x - A| = \inf \{|x - y| : y \in A\} \). If \( \delta > 0 \), then the \( \delta \)-extension \( A(\delta) \) of set \( A \subset \mathbb{R}^n \) is defined by

\[
A(\delta) = \{x \in \mathbb{R}^n : |x - A| < \delta\}.
\]

The Hausdorff-distance \( \mathcal{H} \) for two sets \( A \) and \( B \) in \( \mathcal{P}(\mathbb{R}^n) \) is defined by

\[
\mathcal{H}(A, B) = \inf\{\delta > 0 : A \subset B(\delta) \text{ and } B \subset A(\delta)\}.
\]

We continue with some notation and then prove some auxiliary lemmas. In the case where family \( \{f_k\} \) is finite, the corresponding lemmas have been established and can be found from literature (see e.g. [CLSW]).

In the statement of Theorem 3.1 we have a family \( \{f_k\} \) of \( C^1 \)-functions such that \( \{Df_k\} \) is locally uniformly bounded and equicontinuous. To be precise, this means that for any compact \( K \subset \mathbb{R}^n \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|Df_k(x) - Df_k(y)| \leq \varepsilon \text{ if } k \in \mathbb{N}, x, y \in K \text{ and } |x - y| \leq \delta,
\]

and \( 0 < C < \infty \) such that \( |Df_k(x)| \leq C \) for every \( x \in K \) and \( k \in \mathbb{N} \).

**Remark.** Local equicontinuity of functions \( \{Df_k\} \) implies the local uniform boundedness if it is assumed (e.g.) that functions \( f_k \) are locally uniformly bounded from below, especially, if functions \( f_k \) are positive.

Now it is easy to check that our assumptions imply that \( F \) is continuous function on \( \mathbb{R}^n \) and \( F(x) < \infty \) everywhere.

Let us attach for each \( x \) the set of sequences of natural numbers by

\[
K(x) = \{(a_i)_{i=1}^\infty : a_i \in \mathbb{N} \text{, } F(x) = \lim_{i \to \infty} f_{a_i}(x)\},
\]

and also define

\[
D_x = \{D \in \mathbb{R}^n : \lim_{i \to \infty} Df_{a_i}(x) = D \text{ for some } (a_i) \in K(x)\}.
\]

Set \( D_x \) turns out to be a subset of subdifferentials for \( F \) at point \( x \), as it is proven in Lemma 3.3. Since \( \{Df_k\} \) is locally uniformly bounded, it follows that \( D_x \) is non-empty for every \( x \).

**Lemma 3.2.** For every \( x \in \mathbb{R}^n \)

\[
\inf_{D \in D_x} |D - D_{x+h}| \to 0 \text{ as } h \to 0.
\]

**Proof.** Fix point \( x \in \mathbb{R}^n \) and assume, on the contrary, that there exist sequence \( h_l \to 0 \) such that

\[
\inf_{D \in D_x} |D - D_{x+h_l}| > \lambda > 0 \text{ for every } l \in \mathbb{N}.
\]
Then we find $D_l \in D_{x+h_l}$ so that $|D_l - D_x| > \lambda$ for every $l$ and by extracting a subsequence, if needed, we may assume that $D_l \to D' \in \mathbb{R}^n$. Thus, it follows that $|D' - D_x| \geq \lambda$. However, it follows from the definition of $D_{x+h_l}$ that there exist a sequence $(a^l_i)$ so that $D_l = \lim_{i \to \infty} Df_{a^l_i}(x + h_i)$ and $F(x + h_l) = \lim_{i \to \infty} f_{a^l_i}(x)$. Then one can easily check by using the equicontinuity of $\{Df_k\}$ and continuity of $F$ that actually $(a^l_{g(i)}) \in K(x)$ (for suitably chosen $g$) and also
\[
D' = \lim_{l \to \infty} D_l = \lim_{i \to \infty} Df_{a^l_{g(i)}}(x),
\]
which in turn implies that $D' \in D_x$. This gives us the desired contradiction since we had $|D' - D_x| > \lambda$.

**Lemma 3.3.** For every compact $K \subset \mathbb{R}^n$ there exists positive function $\varepsilon_K$ on $(0, \infty)$ such that $\varepsilon_K(h) \to 0$ as $h \to 0$ and
\[
F(x + h) \geq F(x) + D \cdot h - |h|\varepsilon_K(h),
\]
for every $x \in K$ and $D \in D_x$.

**Proof.** Let $K \subset \mathbb{R}^n$ compact and define
\[
\varepsilon_K(h) = \sup\{|Df_k(x + t) - Df_k(x)| : k \in \mathbb{N}, x \in K \text{ and } |t| \leq h\}.
\]
Now it is rather easy to check that $\varepsilon_K$ satisfies the requirements above. By the equicontinuity of $\{Df_k\}$ it follows that $\varepsilon_K(h) \to 0$ as $h \to 0$.

Suppose then that $x \in K$ and $D \in D_x$, thus there exist sequence $(a_i) \in K(x)$ such that $Df_{a_i}(x) \to D$. Then
\[
F(x + h) \geq \lim_{i \to \infty} f_{a_i}(x + h) = \lim_{i \to \infty} \left( f_{a_i}(x) + \int_0^1 Df_{a_i}(x + th) \cdot h \, dt \right)
\]
\[
= \lim_{i \to \infty} \left( f_{a_i}(x) + Df_{a_i}(x) \cdot h + \int_0^1 \left( Df_{a_i}(x + th) - Df_{a_i}(x) \right) \, dt \right)
\]
\[
\geq \lim_{i \to \infty} \left( f_{a_i}(x) + Df_{a_i}(x) \cdot h - |h|\varepsilon_K(h) \right)
\]
\[
= F(x) + D \cdot h - |h|\varepsilon_K(h).
\]

\[\square\]

Similar computations as in the proof of the previous Lemma also show that for above defined function $\varepsilon_K$ it holds that
\[
|f_k(x + h) - f_k(x) - Df_k(x) \cdot h| \leq \varepsilon_K(h)|h| \tag{15}
\]
if $x, x + h \in K$ and $k \in \mathbb{N}$.

**Remark.** In what follows in this section and section 4 the notation $\varepsilon_x$ or $\varepsilon_K$ should be understood like it is defined in the previous Lemma. Especially, we write down for which particular set $K$ we define $\varepsilon_K$ only in the case when there is possibility of misunderstanding.

**Lemma 3.4.** Function $F$ is differentiable at $x$ if and only if $\#D_x = 1$ ($\#$ denotes the counting measure).
Proof. Observe first that if \( \sharp \mathcal{D}_x > 1 \), then the non-differentiability of \( F \) at \( x \) follows immediately from the previous Lemma \ref{lem:non-diff}.

Suppose then that \( x \) is such that \( \mathcal{D}_x =: \{ D \}, D \in \mathbb{R}^n \) and let us then show that \( F \) is differentiable at \( x \) and \( DF(x) = D \). Again using Lemmas \ref{lem:porous} and \ref{lem:non-diff} we get

\[
F(x + h) \geq F(x) + D \cdot h - |h|\varepsilon_K(h),
\]

if \( D \in \mathcal{D}_x \) and

\[
F(x) = F(x + h + (-h)) \geq F(x + h) + D'_h \cdot (-h) - |h|\varepsilon_K(h),
\]

if \( D'_h \in \mathcal{D}_{x+h} \). Then the claim follows since \( D'_h \rightarrow D \) if \( h \rightarrow 0 \).

As we mentioned in the introduction, the claim \( \dim_H(E) \leq n-1 \) in theorems \ref{thm:dimH} and \ref{thm:non-diff} follows from the first claim, that is \( E \) is asymptotically \( \sigma - \frac{1}{2} \)-porous, by the following well known result (see \cite{M} Chapter 11) relating to the connection between the Hausdorff-dimension and the porosity of the set.

**Theorem 3.5.** There exists function (decreasing and continuous) \( g : [0, \frac{1}{2}] \rightarrow [0,1] \) such that \( g(t) \rightarrow 0 \) if \( t \rightarrow \frac{1}{2} \) and

\[
\dim_H(A) \leq n - 1 + g(\text{por}(A)).
\]

**Proof of Theorem 3.5.** Let \( 0 < \delta < \frac{1}{2} \). By the previous lemmas our claim is equivalent with the proposition that set \( E \),

\[
E = \{ x \in \mathbb{R}^n : \sharp \mathcal{D}_x \geq 2 \},
\]

is contained in a countable union of sets with porosity at least \( \delta \). Since

\[
E = \bigcup_{i=1}^{\infty} E_i := \bigcup_{i=1}^{\infty} \{ x \in B(0, l) : \exists D, D' \in \mathcal{D}_x \text{ s.t. } |D - D'| > \frac{1}{l} \},
\]

it suffices to prove that the claim holds for each \( E_i \). Continuing the same lines we again divide sets \( E_i \) into smaller parts. Indeed, choose first \( R_l > 0 \) so that \( |D_{fk}(x)| < R_l \) if \( x \in B(0, l) \) and \( k \in \mathbb{N} \). Then, let \( \varepsilon > 0 \) be small, depending on \( l \) and \( \delta \) on a suitable way that will be determined later. Define

\[
G = \{ m\varepsilon : m \in \mathbb{Z} \cap \left[ -\frac{R_l}{\varepsilon}, \frac{R_l}{\varepsilon} \right] \}^n \text{ and } \mathcal{G} := \mathcal{P}(G) =: \{ P_1, P_2, \ldots, P_k \}.
\]

Then observe that \( \mathcal{G} \) is \( \varepsilon \)-dense in \( \mathcal{P}(B(0, R_l)) \) respect to the Hausdorff-distance, i.e. for any \( S \subset B(0, R_l) \) there exists \( P_{k_0} \in \mathcal{G} \) so that \( \mathcal{H}(S, P_{k_0}) < \varepsilon \). This implies that

\[
E_k = \bigcup_{i=1}^{k} \{ x \in E_i : \mathcal{H}(\mathcal{D}_x, P_i) < \varepsilon \} =: \bigcup_{i=1}^{k} E^i_k.
\]

We are going to show that by choosing \( \varepsilon \) above small enough it follows that each \( E^i_k \) is \( \delta \)-porous. For this, suppose that \( x \in E^i_k \), \( D \in \mathcal{D}_x \) and \( D' \in \mathcal{D}_x \) such that \( |D - D'| \geq \frac{1}{k} \) and choose \( \theta = \frac{|D - D'|}{|D - D'|} \). Let \( r > 0 \) and consider the set

\[
B(x + \frac{r\theta}{2}, \delta r) \bigcap E^i_k =: B' \bigcap E^i_k.
\]
Observe first that if \( x + h \) lies in \( B' \cap E_1^3 \), then

\[
(D - D') \cdot h \geq \frac{|h|c_6}{l}.
\]

This is geometrically evident. Moreover, it follows that

\[
\mathcal{H}(D_2, D_{x+h}) \leq \mathcal{H}(D_2, P) + \mathcal{H}(P, D_{x+h}) \leq 2\varepsilon.
\]

Using this estimate we find \( D'' \in D_{x+h} \) such that \( |D'' - D'| \leq 2\varepsilon \). Combining this with (17) yields that

\[
(D - D'') \cdot h \geq |h|c_6 \left( \frac{1}{l} - 2\varepsilon \right) \geq |h|c_6 \frac{1}{2l},
\]

if \( \varepsilon \) is small enough. Then we simply use twice the estimate in Lemma 3.3 to obtain (let \( K = \overline{B}(0, 2l) \))

\[
F(x + h) \geq F(x) + D \cdot h - |h|\varepsilon_K(|h|) = F(x + h - h) + D \cdot h - |h|\varepsilon_K(|h|)
\geq F(x + h) + D'' \cdot (-h) - |h|\varepsilon_K(|h|) + D \cdot h - |h|\varepsilon_K(|h|)
= F(x + h) + (D - D'') \cdot h - 2|h|\varepsilon_K(|h|)
\]

implying that

\[
(D - D'') \cdot h \leq 2|h|\varepsilon_K(|h|) \leq 2|h|\varepsilon_K(r).
\]

However, this contradicts with (18) if \( r \) is chosen to be small enough. Indeed, we have shown that \( B' \cap E_1^3 = \emptyset \) and thus each \( E_1^3 \) is \( \delta \)-porous (if \( \varepsilon = \varepsilon(t, \delta) \) is small enough). Summing up, this implies that \( E \) is asymptotically \( \sigma - \frac{1}{2} \)-porous and the latter claim follows from Theorem 3.5. \( \square \)

4. Applications to the maximal function

In this section we apply the results from the previous section to Hardy-Littlewood maximal functions. The main result in this section will be Theorem 4.1. Let us begin with assuming that \( f \) is locally integrable function, let \( \{q_k\} \) be the positive rationals and define functions \( f_k \) by

\[
f_k(x) = \int_{B(x, q_k)} |f(y)| \, dy.
\]

To apply the results in section 3 we need functions \( f_k \) to be \( C^1 \)-functions. For this property the local integrability itself is clearly not sufficient. However, one can check by straightforward calculation that if \( f \) is continuous then each \( f_k \) is \( C^1 \)-function and

\[
D_i f_k(x) = C_n \frac{r}{r} \int_{\partial B(x, q_k)} |f(y)| \frac{y_i - x_i}{r} \, d\sigma(y),
\]

where \( \sigma \) denotes the \( n-1 \)-dimensional Hausdorff-measure. It is also clear that here continuity of \( f \) is not necessary. For example, assuming \( f \in W^{1,1}_loc(\mathbb{R}^n)^2 \)

\( W^{1,1}_loc(\mathbb{R}^n) \) includes locally integrable functions \( f \) whose weak partial derivatives \( D_i f \) are locally integrable functions.
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would yield that

\begin{equation}
D_i f_k(x) = \int_{B(x,q_k)} D_i |f(y)| \, dy
\end{equation}

and $C^1$-property for functions $f_k$ would follow from the absolute continuity of the integral. Remark that (21) may be also written in the form of (20) using Trace theorem for Sobolev-functions.

Let then $0 \leq a < b \leq \infty$ and consider the equicontinuity of the family \{Df_k\}$_{k \in \mathbb{N}, a < q_k < b}$. Remark that functions $f_k$ are non-negative implying that local uniform boundedness would directly follow from equicontinuity if it is assumed that $Mf \not\equiv \infty$. If $0 < a < b < \infty$ and $f$ is continuous then one can deduce the equicontinuity simply from (20) and the uniform continuity of $f$ on compact sets. In the case $f \in W^{1,1}_k(\mathbb{R}^n)$ equicontinuity in turn follows from (21) and the continuity of function $g$, defined on $\mathbb{R}^n \times \mathbb{R}$ by

\begin{equation}
g(x,r) = \int_{B(x,r)} D_i |f(y)| \, dy.
\end{equation}

As expected, for the equicontinuity of \{Df_k\} the cases $a = 0$ or $b = \infty$ appear as the most delicate. Let us first consider the case $a > 0, b = \infty$. Problems in this case may occur if $f$ is too large in infinity. One can easily check by using (20) that additional assumption

\begin{equation}
\sup_{x \in K,q_k \geq \lambda} \frac{1}{q_k} \int_{\partial B(x,q_k)} |f(y)| \, d\sigma(y) \xrightarrow{\lambda \to \infty} 0,
\end{equation}

for every compact set $K$, guarantees the equicontinuity of \{Df_k\}$_{q_k \geq a}$. Standard computations show that this assumption is automatically fulfilled if $f \in W^{1,p}_k(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$. However, in Theorem 1.6 we can avoid assuming (23), thanks to the following Lemma:

**Lemma 4.1.** Suppose that $Mf(x_0) < \infty$ and $x_0 + h_k \to x_0$ so that

\begin{equation}
Mf(x_0 + h_k) \leq \int_{B(x_0 + h_k, r_k)} |f(y)| \, dy + \frac{|h_k|}{k}
\end{equation}

and $r_k \to \infty$. Then $Mf(x_0)$ is the global minimum of $Mf$ and

\begin{equation}
Mf(x_0) \geq Mf(x_0 + h_k) - a_k |h_k|,
\end{equation}

where $a_k$ are positive real numbers and $a_k \to 0$ as $k \to \infty$. 
Proof. The claim follows by rather standard estimate for Hardy-Littlewood maximal function. Let us first verify (25):

\[ Mf(x_0) \geq \int_{B(x_0,r_k+|h_k|)} |f(y)| \, dy \geq \left( \frac{r_k}{r_k + |h_k|} \right)^n \int_{B(x_0+h_k,r_k)} |f(y)| \, dy \]

\[ \geq (1 - \frac{|h_k|}{r_k + |h_k|})^n (Mf(x_0 + h_k) - \frac{|h_k|}{k}) \]

\[ \geq (1 - C_n \frac{|h_k|}{r_k}) (Mf(x_0 + h_k)) - \frac{|h_k|}{k} \]

\[ \geq Mf(x_0 + h_k) - \frac{C_n|h_k|2Mf(x_0)}{r_k} - \frac{|h_k|}{k}. \]

Exactly the same calculation implies that \( Mf(x_0) \) is the global minimum, one only has to first fix \( x \neq x_0 \) and use the facts

\[ \infty > Mf(x_0) = \lim_{k \to \infty} Mf(x_0 + h_k) \text{ and } r_k \to \infty. \]

\[ \square \]

**Proof of Theorem 1.6** Our assumptions in Theorem 1.6 were that \( f \) is differentiable and \( Mf(x) < \infty \) for some \( x \in \mathbb{R}^n \). Let then \( E \) denote the set of points where \( Mf \) is not differentiable. We have to prove that for any \( 0 < \delta < \frac{1}{2} \) set \( E \) is contained in the countable union of \( \delta \)-porous sets. Thus, let \( 0 < \delta < \frac{1}{2} \) and remark first that if \( Mf(x) = \infty \) at some point \( x \in \mathbb{R}^n \), then one can show that \( Mf \equiv \infty \). Thus, condition \( Mf(x) < \infty \) for some \( x \) implies that \( Mf(x) < \infty \) everywhere.

Our assumptions does not need to imply that \( \{Df_q\}_{0 < q \leq \infty} \) is equicontinuous. To make this hold we should assume e.g. that \( f \) is bounded \( C^1 \)-function, which assumption would make the proof much shorter (by Theorem 3.1). However, additional difficulties can be treated using Lemma 4.1 (for the case \( q_k \to \infty \)) and estimates from section 2 (for the case \( q_k \to 0 \)).

First define

(26) \[ A = \{ x : \forall R > 0 \text{ and } \delta > 0 \exists y \in B(x, \delta) \text{ s.t. } Mf(y) > \sup_{q_k \leq R} f_{q_k}(y) \}, \]

which will be the 'bad' set where Lemma 4.1 has to be used. Then we divide set \( E \) into three parts so that \( E = E_1 \cup E_2 \cup E_3 \), where

\[ E_1 = E \cap \{ Mf(x) > |f(x)| \} \cap A^c, \]

\[ E_2 = E \cap \{ Mf(x) > |f(x)| \} \cap A, \]

\[ E_3 = E \cap \{ Mf(x) = |f(x)| \}. \]

The easiest case now (after Theorem 3.1) is set \( E_1 \): It follows from the definition of set \( E_1 \) that for every \( x \in E_1 \) there exists \( r_x > 0 \) and \( \lambda_x > 0 \) such that

\[ Mf(y) = \sup_{\frac{1}{\lambda_x} \leq q_k \leq \lambda_x} \int_{B(y,q_k)} |f(z)| \, dz =: \sup_{\frac{1}{\lambda_x} \leq q_k \leq \lambda_x} f_{q_k}(y) \]
for every \( y \in B(x, r_x) \). Since \( \{ Df_{q_k} \}_{k \leq q_k \leq \lambda} \) is equicontinuous (as it was established above) we get by Theorem \([3.1]\) that \( E \cap B(x, r_x) \) is contained in a countable union of \( \delta \)-porous sets. This implies the same property also for \( E_1 \), since it is a subset of a countable union of sets of type \( E \cap B(x_k, r_{x_k}) \), \( x_k \in E_1 \).

Let us then show the claim for \( E_3 \), which turns out to be the most technical case, while in this case the result is even stronger than in the case of \( E_1 \), namely we show that \( \text{por}(E_3, x) = \frac{1}{4} \) for every \( x \in E_3 \). Thus, let \( x \in E_3 \) and observe that we may assume that \( |f(x)| > 0 \), because \( |f(x)| = 0 = Mf(x) \) implies \( Mf(x) \equiv 0 \). Since \( |f(x)| > 0 \) we know that \( f \) is differentiable at \( x \).

Then it is clear that \((Mf \geq |f|)\)

\[
Mf(x + h) \geq Mf(x) + D|f|(x) \cdot h - \varepsilon_x(h)|h|
\]

(27)

Because \( Mf \) is not differentiable at \( x \) it follows that there exists sequence of points \( x + h_k \to x \) and \( c > 0 \) such that

\[
Mf(x + h_k) \geq Mf(x) + D|f|(x) \cdot h_k + c|h_k|
\]

(28)

for every \( k \in \mathbb{N} \). Let again \( q_k > 0 \) be such that

\[
Mf(x + h_k) \leq \int_{B(x, q_k)} |f(y)| dy + \frac{h_k}{k}.
\]

Let us then show that \( q_k > \lambda > 0 \) for every \( k \in \mathbb{N} \). This follows since otherwise we may assume that \( q_k \to 0 \) (by extracting a subsequence, if needed) and then by repeating the argument in the proof of Theorem \([3.4]\) we get that (below the only equality follows by elementary calculations, details are left to the reader)

\[
Mf(x + h_k) - Mf(x)
\]

\[
\leq \frac{1}{|B(x, q_k)|} \left( \int_{B(x + h_k, q_k) \setminus B(x, q_k)} |f(x)| + D|f|(x) \cdot (y - x) + \varepsilon_x(y - x)|y - x| dy 
\]

\[
- \int_{B(x, q_k) \setminus B(x + h_k, q_k)} |f(x)| + D|f|(x) \cdot (y - x) + \varepsilon_x(y - x)|y - x| dy
\]

\[
= D|f|(x) \cdot h_k + \frac{1}{|B(x, q_k)|} \left( \int_{B(x + h_k, q_k) \setminus B(x, q_k)} \varepsilon_x(y - x)|y - x| dy 
\]

\[
- \int_{B(x, q_k) \setminus B(x + h_k, q_k)} \varepsilon_x(y - x)|y - x| dy
\]

\[
\leq D|f|(x) \cdot h_k + \frac{C_n |B(x + h_k, q_k) \setminus B(x, q_k)|}{q_k^n} \left( \sup_{|a| \leq q_k + h_k} \varepsilon_x(a) \right)(q_k + |h_k|) + \frac{|h_k|}{k}.
\]

By treating separately the cases \( q_k < h_k \) or \( q_k > h_k \) one can verify that

\[
|B(x + h_k, q_k) \setminus B(x, q_k)|(q_k + |h_k|)
\]

(29)
and finally we get that

\[ Mf(x + h_k) - Mf(x) \leq D[f](x) \cdot h_k + |h_k|C'_n\left(\sup_{|a| \leq q_k + |h_k|} \varepsilon_x(a)\right) + \frac{|h_k|}{k}. \]

Because \( \varepsilon_x(h) \to 0 \) as \( h \to 0 \) and \( q_k + |h_k| \to 0 \) as \( k \to \infty \), above inequality contradicts with (28).

Now we have verified that \( q_k \geq \lambda > 0 \) for every \( k \in \mathbb{N} \). Suppose then that \( (q_k) \) is also bounded from above. As before, in that case family \( \{Df_{q_k}\} \) is equicontinuous. Suppose then that \( F \) and sets \( D_x \) are defined respect to family \( \{f_{q_k}\} \) as we did previously. Assume also that \( D_k \in D_{x+h_k} \) (recalling that sets \( D_x \) are non-empty for every \( x \)). Then we use Lemma 3.3 to get that

\[ Mf(x) = F(x) = F(x + h_k - h_k) \geq F(x + h_k) - D_k \cdot h_k - \varepsilon_K(-h_k)|h_k| \]

\[ \geq Mf(x + h_k) - \frac{|h_k|}{k} - D_k \cdot h_k - \varepsilon_K(-h_k)|h_k| \]

\[ \geq Mf(x) + Df(x) \cdot h_k + c|h_k| - \frac{|h_k|}{k} = D_k \cdot h_k - \varepsilon_K(-h_k)|h_k|. \]

Since \( \varepsilon_K(-h_k) \to 0 \) as \( k \to \infty \) we conclude that for large \( k \)

\[ (D_k - D[f](x)) \cdot h_k \geq \frac{c}{2}|h_k|. \]

Because \( \{D_k\} \) is bounded sequence, we may assume (extract a subsequence, if needed) that \( D_k \to D' \) as \( k \to \infty \). Then (30) implies that \( D' \neq D[f](x) \) and moreover, it follows from Lemma 3.2 that \( D' \in D_x \). Using again Lemma 3.3 we get that

\[ Mf(x + h) \geq F(x + h) \geq F(x) + D' \cdot h - \varepsilon_K(h)|h| \]

\[ = Mf(x) + D' \cdot h - \varepsilon_K(h)|h|. \]

Now, since \( D' \neq D[f](x) \) it is easy to check that (31) and (27) imply that

\[ \text{por}(\{y : Mf(y) = |f(y)|\}, x) = \frac{1}{2}. \]

Thus, we have proved that \( \text{por}(E_3, x) = \frac{1}{2} \) if \( (q_k) \) is bounded. Still, we have to treat the remaining case when \( (q_k) \) is not bounded. In that case Lemma 4.1 says that \( Mf(x) \) is the global minimum for \( Mf \). Therefore it follows that if \( D[f](x) \neq 0 \), then we again easily obtain (32). On the other hand, case \( D[f](x) = 0 \) is impossible, since in that case another consequence of Lemma 4.1 (25) contradicts with (28). Summing up, we have shown that \( \text{por}(E_3) = \frac{1}{2} \).

Let us then attack the final case, set \( E_2 \). Suppose that \( x \in E_2 \) and recall that \( x \in A \) implies that \( Mf(x) \) is the global minimum of \( Mf \) (by Lemma 4.1). Combining this with the fact that \( Mf \) is not differentiable at \( x \) it follows that there exists \( c > 0 \) and a sequence of points \( x + h_k \to x \) such that

\[ Mf(x + h_k) \geq Mf(x) + c|h_k| \]

for every \( k \in \mathbb{N} \). Let us again choose sequence \( q_k \) such that

\[ Mf(x + h_k) \leq \int_{B(x,q_k)} |f(y)| \, dy + \frac{|h_k|}{k}. \]
Then sequence \((q_k)\) has to be bounded, since otherwise \((25)\) in Lemma 4.1 contradicts with \((33)\). On the other hand, since \(Mf(x) > |f(x)|\), it follows that \((q_k)\) is also bounded from below. From this on, we continue exactly in the same way as before in the case of set \(E_3\); \(\{Df_{q_k}\}\) is equicontinuous and let \(F\) and sets \(D_x\) be defined respect to family \(\{f_{q_k}\}\). The same lines as above may be repeated, the only change is that \(Df(x)\) above may be replaced with \(0\). We conclude with exactly same conclusion, thus we find \(D' \in D_x\) such that \(D' \neq 0\) and \((31)\) is valid for \(Mf\). Thus, we have shown that \(\text{por}(E_2) = \text{por}(E_3) = \frac{1}{2}\). This completes the proof. 

If the assumptions in Theorem 1.6 are weakened such that \(f\) is assumed to be only continuous function, then the natural version of Theorem 1.6 is the following:

**Corollary 4.2.** Let \(f\) be continuous function so that \(Mf \not\equiv \infty\) and denote by \(E\) the set of the non-differentiability points of \(Mf\). Then set
\[
E \cap \{x : Mf(x) > |f(x)|\}
\]
is asymptotically \(\sigma - \frac{1}{2}\)-porous.

In the case \(f \in W^{1,p}(\mathbb{R}^n), 1 \leq p \leq \infty\) the corresponding result is that in the set
\[
\{x : \exists r_x > 0 \text{ and } \lambda_x > 0 \text{ s.t. } Mf(y) = \sup_{q_k > \lambda_x} f_k(y) \text{ in } B(x, r_x)\}
\]
\(Mf\) is differentiable up to an asymptotically \(\sigma - \frac{1}{2}\)-porous set.

**Other examples of applications.** It is clear that Theorem 3.1 can be applied to various other maximal operators as well (maybe with some additional components as in Theorem 1.6). For example, if balls \(B(x, r)\) in the definition of \(Mf\) are replaced with \(n\)-dimensional cubes, centered at \(x\), then Theorem 1.6 stays valid. The same applies in the case of non-centered maximal operator, where balls \(B(x, r)\) in \((1)\) are replaced with all balls containing point \(x\).

Third example is the fractional maximal operator \(M^\alpha\), where \(0 < \alpha < n\), defined by
\[
M^\alpha f(x) = \sup_{r > 0} r^\alpha \int_{B(x, r)} |f(y)| \, dy.
\]
This operator is important tool in potential theory and in the theory of partial differential equations. In this case it is easy to check that Theorem 1.6 holds true for \(M^\alpha\) even in the case where \(f\) is only locally bounded function. Differentiability or continuity of \(f\) is not needed since for every compact set \(K\) there exists \(r_0 > 0\) such that for every \(x \in K\) the maximal average in \((36)\) is achieved for some \(r\) such that \(r \geq r_0\).

Let us also mention a couple of applications outside the maximal functions. The results below may be known, yet we managed to find only some
weaker versions in literature. In any case, our point here is to emphasize the generality and feasibility of Theorem 3.1.

Firstly, let us consider, as a toy-example, convex functions. It is clear that every convex function $f : \mathbb{R}^n \to \mathbb{R}$ can be expressed as a supremum over countably many linear functions (with locally uniformly bounded derivatives). Then Theorem 3.1 immediately implies that $f$ is differentiable up to an asymptotically $\sigma - \frac{1}{2}$-porous set $E$ with dimension at most $n - 1$ (even this result seems to improve [BC, Corollary 4.6]).

Furthermore, let $A \subset \mathbb{R}^n$ be closed set and consider the distance function $d$ on $\mathbb{R}^n \setminus A$, 

$$d(x) = \inf_{a \in A} |x - a| = - \sup_{a \in A} (-|x - a|).$$

Moreover, let $u$ be continuous function, $f \in C^1_b(\mathbb{R}^n \times \mathbb{R}^n)$ (with convenient growth conditions) and define

$$g(x) = \sup_{y \in \mathbb{R}^n} \left( u(y) + f(x, y) \right).$$

Here $g$ is so called sup-convolution of $u$ and $f$, appearing as standard tool in optimization theory or in the theory of partial differential equations (see e.g. [CIL]). Now it is rather easy to see that Theorem 3.1 can be applied to show that both $d$ and $g$ are differentiable up to an asymptotically $\sigma - \frac{1}{2}$-porous set $E$ with $\dim_H(E) \leq n - 1$ (the result for distance function refines [BC, Corollary 4.8]).

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Department of Mathematics and Statistics, University of Jyväskylä, P.O.Box 35 (MaD), 40014 University of Jyväskylä, Finland

E-mail address: haluiro@maths.jyu.fi