ALMOST SURE LOCAL WELL-POSEDNESS FOR CUBIC NONLINEAR SCHRÖDINGER EQUATION WITH HIGHER ORDER OPERATORS

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Abstract. In this paper, we study the local well-posedness of the cubic Schrödinger equation:
\[(i∂_t - L)u = \pm |u|^2 u \text{ on } I \times \mathbb{R}^d,\]
with randomized initial data, and \(L\) being an operator of degree \(\sigma \geq 2\). Using estimates in directional spaces, we improve and extend known results for the standard Schrödinger equation (i.e. \(L = \Delta\)) to any dimension and obtain results under natural assumptions for general \(L\).

1. Introduction

In this paper, we investigate the local well-posedness of the cubic nonlinear Schrödinger equation
\[
\begin{cases}
(i∂_t - L)u = \pm |u|^2 u & \text{on } I \times \mathbb{R}^d, \\
u(0) = f \in H^S_x(\mathbb{R}^d)
\end{cases}
\]
with a general operator \(L\) and randomized initial conditions, see (1.3) below.

First, we illustrate our results for the classical cubic Schrödinger equation, that is, for (1.1) with \(L = -\Delta\).

Theorem 1.1. Fix \(d \geq 3\) and
\[
S > \frac{d - 2}{2} \times \begin{cases}
\frac{4}{d - 3} & \text{if } d = 3, \\
\frac{d - 2}{d - 1} & \text{if } d \geq 4,
\end{cases}
\]
and assume \(f \in H^S_x(\mathbb{R}^d)\). If \(f^\omega\) is the randomization of \(f\) as in (1.3), then almost surely there exists an open interval \(0 \in I\) and a unique solution
\[
u(t) \in e^{it\Delta} f^\omega + C(I; H^{\frac{d - 2}{2}}(\mathbb{R}^d))
\]
to
\[
\begin{cases}
(i∂_t + \Delta)u = \pm |u|^2 u & \text{on } I \times \mathbb{R}^d, \\
u(0) = f^\omega \in H^S_x(\mathbb{R}^d).
\end{cases}
\]

As detailed below, Theorem 1.1 improves known results in dimensions \(d \geq 5\); furthermore, our general Theorem 1.2 extends Theorem 1.1 to a large class of operators \(L\), improving on existing results in all dimensions. We begin by briefly reviewing background and known results for (1.1) with fixed deterministic initial condition \(f\). The main operators of interest are \(L = -\Delta\) yielding classical Schrödinger equation,
and \( L = \Delta^2 - \mu \Delta \), \( \mu \in \{-1, 0, 1\} \) leading to the fourth order Schrödinger equation with mixed dispersion introduced by Karpman and Shagalov [KS00] (see also [Kar96]).

Let \( S_{\text{crit}} := \frac{d-\sigma}{2} \) be the special value called the energy critical exponent; for initial data \( f \in H^S(\mathbb{R}^d) \), we say that the Cauchy problem (1.1) is
\[
\begin{aligned}
\text{subcritical} & \quad \text{if } S > S_{\text{crit}}, \\
\text{critical} & \quad \text{if } S = S_{\text{crit}}, \\
\text{supercritical} & \quad \text{if } S < S_{\text{crit}}.
\end{aligned}
\]
The relevance of \( S_{\text{crit}} \) can be seen by neglecting lower order terms, that is, by assuming that \( L = (-\Delta)^{\sigma/2} \). Then, (1.1) possesses a natural scaling symmetry: if \( u \) satisfies the equation in (1.1), then
\[
\dot{u}_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x), \quad \lambda > 0,
\]
also satisfies the same equation. In addition,
\[
\| u_\lambda(0) \|_{H^S} = \lambda^{S-S_{\text{crit}}} \| u(0) \|_{H^S} = \lambda^{S-S_{\text{crit}}} \| f \|_{H^S},
\]
where \( \| \cdot \|_{H^S} \) denotes the homogeneous Sobolev norm (see Section 1.1 for the definition).

In the subcritical or critical regime, local in time solutions can be constructed using Strichartz estimates and a classical fixed point argument. We refer to [Caz03; CW90; CKSTT08; RV07] for results concerning the classical nonlinear Schrödinger equation and to [Pau07; PS10] for results on the fourth order one. On the other hand, in the supercritical regime, (1.1) is ill-posed by a result of Christ, Colliander, and Tao [CCT03].

From a practical perspective, ill-posedness is observable only if it does not vanish after an introduction of small fluctuations. Unlike the Schrödinger equation, the initial conditions often originate in measurements, which are naturally susceptible to errors that are inherently random. We use a standard randomization of initial data based on a unit-scale decomposition of frequency space. The unit scale in frequency can be thought of as characteristic scale of our measurements or of the data based on a unit-scale decomposition of frequency space. The unit scale is centered at 0, such that
\[
\sum_{k \in \mathbb{Z}^d} \psi(\xi - k) = 1
\]
for all \( \xi \in \mathbb{R}^d \). Then, for \( f \in H^S_x(\mathbb{R}^d) \), we define the randomization of \( f \) by
\[
f^\omega = \sum_{k \in \mathbb{Z}^d} g_k(\omega) Q_k f.
\]
Here \( (g_k)_{k \in \mathbb{Z}^d} \) is a sequence of i.i.d zero-mean complex random variables with finite moments of all orders on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) (for example \( \mathcal{N}(0, 1; \mathbb{C}) \) Gaussian variables). The operators \( Q_k \) are unit scale frequency approximate projection operators given on the frequency side by
\[
\mathcal{F}(Q_k f)(\xi) = \psi(\xi - k) \mathcal{F}(f)(\xi), \quad \text{for } \xi \in \mathbb{R}^d,
\]
where \( \mathcal{F}(f) \) stands for the Fourier transform of \( f \).

The randomization (1.3) does not improve the differentiability properties of \( f \): if \( f \in H^S(\mathbb{R}^d) \setminus H^{S+\varepsilon}(\mathbb{R}^d) \) for some \( \varepsilon > 0 \), then \( f^\omega \in H^S(\mathbb{R}^d) \setminus H^{S+\varepsilon}(\mathbb{R}^d) \) almost surely, see [BT08b]. In particular, if the problem was super-critical for \( f \), then it stays almost surely super-critical for the initial condition \( f^\omega \). However, since \( Q_k f \)
is localized on a set of bounded diameter in Fourier space, the Bernstein inequality (see Lemma 2.1 below) implies for any \( 1 \leq r_1 \leq r_2 \leq \infty \) and any \( k \in \mathbb{Z}^d \) that
\[
\| Q_k f \|_{L^2_r(\mathbb{R}^d)} \leq C_{r_1, r_2} \| Q_k f \|_{L^{r_2}_x(\mathbb{R}^d)}
\]
with a constant \( C \) independent of \( k \). We exploit (1.5) to show that \( f^\omega \) and \( e^{-it\mathcal{L}} f^\omega \) possess better local integrability properties than \( f \) and \( e^{-it\mathcal{L}} f \), respectively (see Lemma 4.3).

The first results on the probabilistic well-posedness were proved by Bourgain [Bou94; Bou96] and McKean [McK95], who showed that a suitable randomization of the initial data can be used to construct local or even global solutions in the supercritical regime. Specifically, they proved an almost sure local existence of solutions of (1.2) on torus. Also, with a help of invariant (Gibbs) measures the local solutions were extended to global ones (see also [LRS88; Sy21] for other results in this direction). There is a number of results for the randomized nonlinear Schrödinger equation on torus, (see survey [Nah15] and references therein) but the techniques are very different compared to the non-compact case of \( \mathbb{R}^d \). For example, local smoothing obtained in Lemma 2.5 is not expected to hold true on torus. Randomization techniques on the torus were used for other equations such as Navier-Stokes equation [ZF12], nonlinear wave equation [BT08a], or Hartree NLS [DNY21].

The literature contains several well-posedness results for (1.2) on \( \mathbb{R}^d \) using our randomization. Bényi, Oh, and Pocovnicu [BOP15] proved the almost sure local well-posedness of (1.1) for \( d \geq 3 \) and \( S > \frac{d-4}{2} \) in the following sense: there exist \( c, C, \gamma > 0 \) such that for each \( 0 < T \ll 1 \), there exists a set \( \Omega_T \subset \Omega \) with the following properties:

- \( \mathbb{P}(\Omega_T^c) < C \exp(-\gamma \frac{T^{d+4}}{T^{d+4}}) \).
- For almost each \( \omega \in \Omega_T \), there exists a unique solution \( u \) to (1.1) with \( u(0) = f^\omega \) in the class
  \[ e^{-it\mathcal{L}} f^\omega + C([-T,T]; H^{S,m}(\mathbb{R}^d)) \subset C([-T,T]; H^S(\mathbb{R}^d)). \]

Later, Brereton [Bre18] obtained analogous results for \( \mathcal{L} = -\Delta \) and quintic non-linearity. When \( d = 3 \), Shen, Soffer, and Wu [SSW23] recently, obtained the local well-posedness of (1.2) for \( S > \frac{1}{4} \) improving [BOP15]. All described results rely on a fixed point argument for operators on variants of the \( X^{s,b} \) spaces adapted to the variation spaces \( V^p \) and \( U^p \) introduced by Koch, Tataru, and collaborators [HHK09; HTT11; KTV14]. The result of [BOP15] was also improved by Dodson, Lührmann, and Mendelson [DLM19] when \( d = 4 \), which corresponds to the energy-critical Schrödinger equation. More precisely, they proved the local well-posedness of (1.2) when \( d = 4 \) and \( S > \frac{1}{4} \). It is important to note, that instead of using variants of \( X^{s,b} \), [DLM19] used a directional norm denoted by \( L^ {a,b}_e \), \( e \in \mathbb{S}^{d-1} \subset \mathbb{R}^d \), introduced by Ionescu and Kenig [IK06; IK07] to prove well-posedness for the Schrödinger map equation.

The only local well-posedness result for general (1.1) with a higher order operators \( \mathcal{L} \) and randomized data was obtained in [DD21] for \( \mathcal{L} = |\Delta|^{2} - \mu \Delta, \mu \geq 0 \), \( d \geq 5 \) under the assumption \( S > \max \left\{ \frac{(d-1)(d-4)}{2(d+5)}, \frac{d-4}{4} \right\} \).

The idea of the mentioned results is to subtract the linear evolution of the initial condition given by \( e^{-it\mathcal{L}} f^\omega \) which presumably has the worst regularity. Then, the regularity level \( S \) is chosen such that the remainder is smooth enough to belong to a sub-critical space, where the fixed point argument can be used.

By using iterative procedure based on a partial power expansion, one can subtract higher order terms as in Bényi, Oh, and Pocovnicu [BOP19] to obtain local well-posedness for any \( S > \frac{1}{4} \). Note that the condition \( S > \frac{1}{4} \) was improved (by
including the endpoint) to \( S \geq \frac{d}{4} \) in the mentioned result [SSW23] without need for iterations.

Theorem 1.1, the special case of our general Theorem 1.2, recovers or improves upon the existing results in any dimension with a unified approach. In particular, we obtain the optimal condition from [DLM19] for \( d = 4 \), we reproduce the result in [SSW23] when \( d = 3 \) except for the endpoint regularity case, and we improve [BOP15] (or any other existing result) for \( d \geq 5 \). In addition, Theorem 1.1 allows for generalizations to other operators, as detailed below, and in a forthcoming work [JBGJ] we show how to use our framework to further lower the regularity requirements by including higher order expansions. We remark that our general Theorem 1.2 improves all existing results for higher order operators in any dimensions (see for example [DD21]).

Next, we specify our assumptions on \( \mathcal{L} \) and formulate our general result. We abuse notation and denote by \( \mathcal{L} \) both the differential operator and its symbol so that

\[
\mathcal{L} f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \xi} \mathcal{L}^\prime(2\pi i \xi) \hat{f}(\xi) d\xi.
\]

We assume that the symbol is real-valued, and there is a real \( \sigma \geq 2 \) such that for all \( \xi \) large enough one has (assumptions are on the symbol)

\[
|\partial^\alpha \mathcal{L}(\xi)| \lesssim |\xi|^{\sigma - |\alpha|} \quad \text{for all } |\alpha| \leq \left\lfloor \frac{d}{\sigma} \right\rfloor + 2, \\
|\nabla \mathcal{L}(\xi)| \gtrsim |\xi|^{\sigma - 1} \\
|\xi|^{d(\sigma - 2)} \lesssim |\det \mathcal{D}^2 \mathcal{L}(\xi)| \lesssim |\xi|^{d(\sigma - 2)},
\]

(1.6)  
(1.7)

where \( \mathcal{D}^2 \mathcal{L}(\xi) \) is the Hessian of \( \mathcal{L}(\xi) \) and \( |z| \) denotes the integer part of \( z \), that is, the largest integer smaller than \( z \). These conditions are trivially satisfied if \( \mathcal{L} = |\Delta|^\sigma/2 + \mathcal{L}^1 \) for \( \sigma \geq 2 \) and \( \mathcal{L}^2 \) is a lower order operator with real, smooth symbol.

For the differentiability \( S \) of initial data, we show, quite interestingly, that there are two different regimes:

\[
S_{\min}(\sigma, d) := d - \sigma \times \begin{cases} 
\frac{1}{2} & \text{if } \frac{d+2}{3} \leq \sigma, \\
\frac{d+1-2\sigma}{d-1} & \text{if } \sigma \leq \frac{d+2}{3}.
\end{cases}
\]

(1.8)

Our main result reads as follows:

**Theorem 1.2.** Let \( d > \sigma \geq 2 \), and let \( \mathcal{L} \) be a differential operator whose Fourier symbol \( \mathcal{L} \) is smooth, real, and satisfies (1.6) and (1.7). For any \( S > S_{\min}(\sigma, d) \) with \( S_{\min} \) as in (1.8) assume \( f \in H^S(\mathbb{R}^d) \) and let \( f^\omega \) be the randomization of \( f \) as in (1.3). Then, for a.e. \( \omega \in \Omega \), there exists an open interval \( I \in \mathbb{R} \) and a unique solution

\[
u(t) \in e^{-\mu t} \mathcal{L} f^\omega + C(1; H^{S_{\min}}(\mathbb{R}^d))
\]

to

\[
\begin{cases} 
(i\partial_t - \mathcal{L})u = \pm |u|^2 u & \text{on } I \times \mathbb{R}^d, \\
u(0) = f^\omega.
\end{cases}
\]

(1.9)

The special case \( \mathcal{L} = -\Delta \) was formulated in Theorem 1.1 and when \( \mathcal{L} = \Delta^2 \pm \mu \Delta \), we obtain local well-posedness if

\[
S_{\min}(4, d) = \frac{d - 4}{2} \times \begin{cases} 
\frac{1}{3} & \text{if } 4 < d \leq 10, \\
\frac{d-7}{d-1} & \text{if } d \geq 10.
\end{cases}
\]

(1.10)

Let us briefly comment on assumptions of Theorem 1.2. The condition \( d > \sigma \) ensures that we are in the super-critical regime (we do not consider \( S < 0 \), because
there the functions in $H^S$ are not defined point-wise and the non-linearly $|u|^2u$ has to be interpreted differently). If $d \leq \sigma$, that corresponds to any $S > 0$ being energy subcritical, the well-posedness results can be obtained using softer techniques. Our assumptions on $\mathcal{L}$ are satisfied, for example, by the operator $\mathcal{L} = \Delta^2 \pm \Delta$, or by symbols that may fail to be convex. Furthermore, our conditions are stable with respect to perturbations by lower-order terms.

A careful inspection of our methods gives explicit estimates from below on the time of existence of solutions, similar to ones mentioned above and in [BOP15]. There are several techniques that extend the local well-posedness theory to global well-posedness with high probability for small initial data. These techniques and results are closely related to scattering. There are, however, obstacles to such approaches for the class of operators $\mathcal{L}$ we consider. First, since we work in the energy supercritical regime, solutions do not satisfy a priori global energy estimates. Second, there are no global in time dispersive estimates even for the linear evolution. In particular, if the symbol of $\mathcal{L}$ has vanishing curvature on non-trivial sets scattering behavior is unlikely. Overall, for the clarity and length of the manuscript we decided not to include global-in-time existence results, which would require us to restrict the class of operators considered. We also omit these explicit considerations on the time of existence for local solutions.

The proof of Theorem 1.2 starts by subtracting the free (random) evolution $F = e^{-it\mathcal{L}}f\omega$, which, heuristically, has the lowest regularity. This allows us to transform (1.9) into a forced cubic NLS equation for the remainder term $v = u - F$:

$$\begin{aligned}
(i\partial_t - \mathcal{L})v &= \pm |F + v|^2(F + v) := H(t, x), \\
v(0) &= 0.
\end{aligned}$$

Due to stochastic cancellation effects, $F$ has almost surely better space-time integrability properties (but not smoothness) than its deterministic counterpart. In fact, the problem becomes sub-critical allowing one to find $v$ by finding the fixed point by the Banach fixed point theorem.

Inspired by a functional framework of [DLM19], we use directional spaces and prove the contraction of an appropriate map in a sub-critical space denoted by $X^{s+\tilde{\varepsilon},\varepsilon}$. The space $X^{s+\tilde{\varepsilon},\varepsilon}$ contains classical Strichartz, directional maximal type, and directional local smoothing type components. The central idea relies on the observation, that if $v \in X^{s+\tilde{\varepsilon},\varepsilon}$, then the forcing term $H$ belongs to the dual space $(X^{s+\tilde{\varepsilon},\varepsilon})^*$. This differs from the approach of [DLM19], where the local smoothing term is absent from the norm of the space $X^{s+\tilde{\varepsilon},\varepsilon}$, and $H$ needs to be controlled in an space denoted $G$ (see [DLM19]). Controlling the forcing term $H$, in a dual space of $X^{s+\tilde{\varepsilon},\varepsilon}$ rather than $G$ proves more natural and appropriate for generalizations to higher dimensions. Finally, the extension to more general operators $\mathcal{L}$ requires a finer analysis of the oscillatory integrals occurring in the study of the free evolution of the Schrödinger equation; some of these arguments have a micro-local flavor (see Lemma 2.5).

The proof of Theorem 1.2 can be summarized by the following steps.

**Step 1:** We control the linear evolution $F$ in the norm of the space $Y^{S,\varepsilon}$, which has regularity $S > S_{\text{min}}$, and in particular $Y^{S,\varepsilon}$ is super-critical. The norm of $Y^{S,\varepsilon}$ is based on a dyadic decomposition of Fourier space and on each dyadic annulus in frequency, the norm of $Y^S$ is an appropriately weighted combination of classical space-time norms $L^p_tL^q_x$ and directional norms $L^p_{\alpha,b}$. The cancellations stemming from the randomization of initial conditions, allow us to use a higher integrability exponent in the spatial directions compared to the classical Strichartz norms.
Step 2: The existence and uniqueness of $v$ is shown by the fixed point argument in the space $X^{s,\varepsilon}(I)$. The parameter $s \geq S_{cr} = \frac{d-\sigma}{2}$ indicates differentiability and $\varepsilon > 0$ is a small parameter that allow us to avoid working in endpoint spaces such as $L^\infty$. The space $X^{s,\varepsilon}$ is endowed with a norm based, again, on a dyadic composition of Fourier space; on each dyadic annulus in frequency, the norm is a combination of Strichartz admissible $L_t^qL_x^p$ norms and appropriately weighed directional norms $L_{x,\varepsilon}^{q,h}$. The key step is the estimate of $v$ in the $X^{s,\varepsilon}(I)$ norm, by the $(X^{s+\varepsilon,\varepsilon})^*$ norm of $H$ (see (1.10) for the definition of $H$).

Step 3: Based on the results described in Step 2, one needs to control the forcing term $H$ in the $(X^{s+\varepsilon,\varepsilon})^*$ norm. More specifically, if $F \in Y^{S,\varepsilon}$ (established in Step 1) and $v \in X^{s,\varepsilon}$ (postulated in Step 2), then we show that $H$ belongs to $X^{s+\varepsilon,\varepsilon}(I)^*$. Since $H$ can be viewed as sum of cubic monomials $C$ in the variables $F, v, \overline{F}, \overline{\pi}$ controlling any $C$ in the norm $(X^{s+\varepsilon,\varepsilon}(I))^*$ requires, by duality, testing $C$ against a function with bounded $X^{s,\varepsilon}$ norm. Thus, the estimate on $C$ is equivalent to suitable quadrilinear estimates, which we factor through two bilinear estimates mapping into $L_t^2L_x^2$. Our bilinear estimates implicitly contain the bilinear Strichartz estimates of [Bou98; OT98]. In this step, we differ from the functional framework of [DLM19], where a different space appears instead of $(X^{s+\varepsilon,\varepsilon})^*$, making generalizations less efficient.

Remark 1.3. As observed in [BOP19], one can use higher order multilinear expansions to obtain a solution to (1.9). More precisely, one can consider solutions to (1.9) of the form

$$u = F_1 + F_3 + F_5 + \ldots + F_{2k+1} + v_k$$

with $v \in C(I; \dot{H}_x^\alpha(\mathbb{R}^d))$ for some $\alpha > \frac{d-\sigma}{2}$, where $F_1 = e^{-it\Delta} f^\omega$ and

$$F_{2k+1} := -i \int_0^t e^{-i(t-t')\Delta} |F_1 + \ldots + F_{2k-1}|^2 (F_1 + \ldots + F_{2k-1}) \, dt'.$$

The functions $F_j$ are chosen so that they cancel out higher order terms on the right-hand side of (1.1), which are independent of $v$ and $\overline{\pi}$.

Our forthcoming paper [JBHJG] uses such higher order expansion for $\mathcal{L} = \Delta$ in $d \in \{3, 4\}$ and significantly improves the requirements on $S$. Thus, the functional framework developed in this paper partly serves as the foundation for and efficient treatment of the higher order expansions.

When $d \gg \sigma$, that is, when the nonlinearity is ‘very supercritical’, then the monomials $C$ including $v$ become dominant, and further expansion is less obvious.

Remark 1.4. The $X^{s+\varepsilon,\varepsilon}(I)$ norm, used for the fixed point theorem, is a combination of directional as well as classical Strichartz norms. The classical Strichartz norms are included for convenience, because, with minor adjustments, the contraction only needs bounds in the directional norms.

Remark 1.5. Our approach crucially relies on gain of derivatives in the directional local smoothing estimates (see Lemma 2.5). These estimates are not expected to hold on compact domains.

Remark 1.6. Our results split naturally into a deterministic analysis of (1.1) in directional norms and probabilistic estimates that yields improved bound on the free evolution in the directional norms. The former analysis may be of interest in the general theory of Schrödinger operators, outside the stochastic setting. We remark
that a directional analysis of Schrödinger equation was done in [BBFGI18], however, to our best knowledge, the results were not applied to non-linear Schrödinger equation.

The paper is organized as follows.

In Section 2, we recall generalities of Fourier analysis and the Littlewood-Paley theory (dyadic annuli decomposition in Fourier space). We state classical Strichartz estimates for free evolution operator. Then we introduce directional norms $L^b_a$ and prove directional maximal and local smoothing estimates for the free evolution operator. In Section 3, we use results of Section 2 to obtain estimates on solutions of (1.10) with a generic forcing term $H$. The bounds proved in Section 3 are sufficient to prove all ideas in Step 2, above. In Section 4, we establish probabilistic estimates for the linear evolution of the randomized initial data. We also recall several properties of sums of Gaussian random variables. Section 5 contains trilinear estimates that control interactions in the cubic non-linearity. Finally, in Section 6 we establish Theorem 1.2 by using a fixed point argument.

Theorem 1.2 requires $d > \sigma \geq 2$ which we implicitly assume henceforth, and we will not state it explicitly in the statements below.

1.1. Notation.

- We define $\mathbb{N} = \{0, 1, \cdots \}$ the set of non-negative integers.
- In a $d$-dimensional space $\mathbb{R}^d$, we denote $\{e_1, \cdots, e_d\}$ to be the standard basis.
- The Fourier transform of the function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is
  \[ \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi x} \, dx, \]
  where the dimension $d$ is deduced from the context. By the Fourier inversion formula:
  \[ f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i \xi x} \, dx \]

- For two expressions $G$ and $H$ we write $G \lesssim H$ if there exists a constant $C > 0$ depending only on the fixed parameters of the problem such that $G \leq C H$. In particular, we typically assume that $C$ is independent of $N$. If $C$ depends on a variable $\varepsilon$, we use $G \lesssim_{\varepsilon} H$.
- We write $G \approx H$, if $G \lesssim H$ and $H \lesssim G$.
- The symbol $O(\varepsilon)$ stands for any function $[0, 1) \rightarrow \mathbb{R}$ such that
  \[ |O(\varepsilon)| \lesssim \varepsilon \]
  for all $\varepsilon \in (0, 1]$. The specific function denoted by $O(\varepsilon)$ can change from line to line.
- The open ball with radius $r$ and center $x$ is denoted $B_r(x)$; if $x = 0$ we simply write $B_r$. The dimension of the ball is to be understood from the context.
- For $p > 1$, $p'$ stands for the dual of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. If $p = 0$, then we set $p' = \infty$.
- The Japanese bracket $\langle \cdot \rangle$ is defined as $\langle N \rangle := (1 + N^2)^{\frac{1}{2}}$.
- For $S > 0$, we denote $\langle \Delta \rangle^{S/2}$ the operator with the Fourier multiplier $(1 + 4\pi^2 \xi^2)^{S/2}$, that is, $\mathcal{F}(\langle \Delta \rangle^{S/2} f)(\xi) = \langle 4\pi^2 \xi^2 \rangle^{S/2} \hat{f}(\xi)$. Then, $H^S(\mathbb{R}^d)$ denotes the Sobolev space endowed with the semi-norm
  \[ \|u\|_{H^S(\mathbb{R}^d)}^2 = \|\langle \Delta \rangle^{S/2} u(x)\|_{L^2(\mathbb{R}^d)}. \]
• We denote by $\mathbb{1}_A$ the characteristic function of a set $A$, that is, $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise. In addition, if for example $x > y$, then we write $\mathbb{1}_{x>y}$ to indicate the function that is equal to 1 when $x > y$ and vanishes otherwise. The variable of the function is to be deduced from the context.
• We denote by $\text{spt} f := \{x \in \mathbb{R}^d : |f(x)| > 0\}$ the support of the function $f$, where $\text{cl}(A)$ denotes the closure of a set $A$. Similarly, we define $\text{spt} \hat{f}$.
• The diameter of a set $A \subset \mathbb{R}^d$ is $\text{diam}(A) := \sup_{x,y \in A} |x - y|$.

2. Generalities

In this section, we recall Littlewood-Paley projections, Strichartz estimates, and we review directional norms $L^{a,b}_c$ introduced by Ionescu and Kenig [IK06; IK07]. Then, we prove new maximal function estimate (2.5) and a local smoothing estimate (2.6) for $L^{a,b}_c$. Below, the estimate (2.6) allows us to ‘gain’ $\frac{d}{d-\sigma}$ derivatives in our estimates of the nonlinear terms.

We begin by defining the Littlewood-Paley projections $P_N$ for $N \in 2\mathbb{N}$. For a fixed smooth cutoff function $\varphi \in C^\infty(\mathbb{R})$, that is, a function such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| > 1 + 2^{-100}$, we set
\[
\varphi_N(\xi) = \begin{cases} 
\varphi(\xi) & \text{if } N = 2^0, \\
\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{\xi}{N^{1/2}}\right) & \text{if } N \in 2^k, n \in \mathbb{N} \setminus \{0\}.
\end{cases}
\]
Observe that if $N > 1$, the function $\varphi_N$ is supported on $\{\xi : N/2 \leq |\xi| \leq N(1 + 2^{-100})\}$ and $\varphi_N(\xi) = 1$ when $(1 + 2^{-100})N/2 < |\xi| < N$. We define
\[
P_Nf(\xi) = \varphi_N(|\xi|) \hat{f}(\xi).
\]
Note that this projection is different from the one introduced in (1.4). Next, we recall the classical Bernstein estimates.

**Lemma 2.1.** For any $1 \leq r_1 \leq r_2 \leq \infty$ it holds that
\[
\|f\|_{L^{r_2}_x(\mathbb{R}^d)} \lesssim \text{diam} \left(\text{spt} \hat{f}\right)^{d \left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \|f\|_{L^{r_1}_x(\mathbb{R}^d)}.
\]
In particular, since $\text{diam} \left(\text{spt} \mathcal{F}(Q_k f)\right) \approx 1$, (1.5) holds. We remark that for $P_N$ as in (2.2), $\text{diam} \left(\text{spt} P_N \hat{f}\right) \approx N$.

Next, we recall the Strichartz estimates for general self-adjoint operator $\mathcal{L}$ of order $s$ with constant coefficients. We say that a pair $(p, q)$ is $\mathcal{L}$-admissible if
\[
\frac{\sigma}{p} + \frac{d}{q} = \frac{d}{2} \quad \text{and} \quad q \in \left[2, \frac{2d}{d-\sigma}\right).
\]

**Lemma 2.2** ([COX11; Din18; GV92; KT98]). Fix $d > \sigma \geq 2$ and let $\mathcal{L}$ satisfy (1.6) and (1.7). Then, there exists $T_0 > 0$ such that for any open interval $I \subset \mathbb{R}$ with $|I| \leq T_0$, we have
\[
\|e^{-it \mathcal{L}} f\|_{L^p_x L^q_t(I \times \mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.
\]

**Remark 2.3.** Since the evolution operator $e^{-it \mathcal{L}}$ commutes with projections $P_N$ for any $N \in 2\mathbb{N}$ and with $Q_n$ for any $n \in \mathbb{Z}^d$ (all are Fourier multipliers), (2.4) holds with $f$ replaced by $P_N f$ or by $Q_n f$ on both sides of the inequality.

In the rest of the paper, we assume that all time intervals have length less than $T_0$ given in Lemma 2.2, and therefore the Strichartz estimates hold. Since we are only interested in the local existence, this assumption does not influence our main results.
To introduce the directional norms, decompose $x \in \mathbb{R}^d$ for any $l \in \{1, \ldots, d\}$ as

$$x = x_1 e_1 + \sum_{i=1, i \neq l}^{d} x_i e_i =: x_t e_l + x'_t$$

and, if there is no possible confusion, we write $x' := x'_t$. Fix $I \subset \mathbb{R}$ and $l \in \{1, \ldots, d\}$, and for $1 \leq a, b < \infty$ define

$$||h||_{L^a_t L^b_x(I \times \mathbb{R}^d)} = \left( \int_I \left( \int_{\mathbb{R}^{d-1}} |h(t, x_t e_l + x'_t)|^b dx_t' dt \right) \right)^{\frac{1}{b}},$$

where $h : I \times \mathbb{R}^d \to \mathbb{C}$ is such that the right-hand side is finite. When $a = \infty$ or $b = \infty$, we use the standard modifications by the supremum norm. Next, we establish a maximal and a local smoothing estimates for the directional norms.

**Lemma 2.4.** Fix $d > \sigma \geq 2$ and $\mathcal{L}$ satisfying (1.6) and (1.7). There exists $T_0 > 0$ such that for any open interval $I \subset \mathbb{R}$ with $|I| \leq T_0$, any $l \in \{1, \ldots, d\}$, $N \in 2^\mathbb{N}$, and $f \in L^2(\mathbb{R}^d)$ we have

$$N^{-\frac{d-1}{2}} \| e^{-it \mathcal{L}} P_N f \|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} \lesssim \| P_N f \|_{L^2(\mathbb{R}^d)}, \quad (2.5)$$

**Lemma 2.5.** Fix $d > \sigma \geq 2$ and $\mathcal{L}$ satisfying (1.6) and (1.7). There exists $T_0 > 0$ such that for any open interval $I \subset \mathbb{R}$ with $|I| \leq T_0$, any $l \in \{1, \ldots, d\}$, $N \in 2^\mathbb{N}$, and $f \in L^2(\mathbb{R}^d)$ we have

$$N^{-\frac{d-1}{2}} \| e^{-it \mathcal{L}} P_N \mathcal{U}_{e_l} f \|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} \lesssim \| P_N f \|_{L^2(\mathbb{R}^d)}, \quad (2.6)$$

where $\mathcal{U}_{e_l}$ is a frequency projection operators given by $(\mathcal{U}_{e_l} f)(\xi) := \mathbb{1}_{\mathcal{U}_{e_l}}(\xi) \hat{f}(\xi)$ with

$$\mathcal{U}_{e_l} := \left\{ \xi \in \mathbb{R}^d : |\nabla \mathcal{L}(\xi) \cdot e_l| > \frac{|\nabla \mathcal{L}(\xi)|}{2\sqrt{d}} \right\} \setminus \bigcup_{l'=1}^{l-1} \left\{ \xi \in \mathbb{R}^d : |\nabla \mathcal{L}(\xi) \cdot e_{l'}| > \frac{|\nabla \mathcal{L}(\xi)|}{2\sqrt{d}} \right\}. \quad (2.7)$$

**Remark 2.6.** Notice that the scaling power $N^{-\frac{d-1}{2}}$ in (2.5) does not depend on $\mathcal{L}$ but only on the dimension $d$, whereas $N^{-\frac{d-1}{2}}$ in (2.6) only depends on the order of $\mathcal{L}$ but not on the dimension $d$.

**Proof of Lemma 2.4.** We use a $TT^*$ argument for the operator $T$ given by

$$T f(t, x) := \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} e^{-it \mathcal{L}(\xi)} \chi_N(\xi) \hat{f}(\xi) d\xi, \quad (2.8)$$

where

$$\chi_N(\xi) := \sum_{|N' - N| \leq 2} \varphi_{N'}(|\xi|)$$

so that $\chi_N(\xi) \varphi_N(|\xi|) = \varphi_N(|\xi|)$. Bound (2.5) can be rewritten as

$$\| T f \|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} \lesssim N^{d-1} \| f \|_{L^2(\mathbb{R}^d)},$$

or equivalently by duality $\| TT^* g \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{d-1} \| g \|_{L^2_t L^2_x(I \times \mathbb{R}^d)}$. We claim that it suffices to show that

$$\| TT^* g \|_{L^2_x(\mathbb{R}^d)} \lesssim N^{d-1} \| g \|_{L^2_t L^2_x(I \times \mathbb{R}^d)}. \quad (2.9)$$

Indeed, if (2.9) holds, then

$$\| TT^* g \|_{L^2_x(\mathbb{R}^d)}^2 = \langle T^* g, T^* g \rangle = \langle TT^* g, g \rangle \lesssim \| TT^* g \|_{L^2_t L^\infty_x} \| g \|_{L^2_t L^2_x} \lesssim N^{d-1} \| g \|_{L^2_t L^2_x(I \times \mathbb{R}^d)}^2.$$

as desired. Direct computations show that

\begin{equation}
TT^* g(t, x) = \int_{\mathbb{R} \times \mathbb{R}^d} K_N(t - s, x - y) g(s, y) \, ds \, dy,
\end{equation}

\begin{equation}
K_N(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{2\pi i \xi x - it \mathcal{L}(\xi)} \chi_N^2(\xi) \, d\xi,
\end{equation}

reducing our proof, by Young convolution inequality, to

\[ \|K_N\|_{L^1_{\xi} \to L^\infty} \lesssim N^{d-1}. \]

For simplicity, we henceforth suppose \( e_1 = e_1 \). Since \( \chi_N \) is supported on a set of measure of order \( N^d \), by interchanging the integral and absolute value, we have

\begin{equation}
\|K_N\|_{L^1_{\xi} \to L^\infty} \lesssim N^d.
\end{equation}

In addition, for \( x_1 \neq 0 \), an integration by parts, and an oscillation of the linear phase yield a decay in \( x_1 \):

\begin{equation}
|K_N(t, x)| = \frac{1}{4\pi^2} \left| \int_{\mathbb{R}^d} \frac{1}{x_1^2} \partial_{\xi_1}^2 \left( e^{2\pi i x_1 \xi_1} \right) e^{2\pi i \xi \cdot x' - it \mathcal{L}(\xi)} \chi_N^2(\xi) \, d\xi \right|
\lesssim \frac{N^d}{|x_1|^2} \left| e^{-it \mathcal{L}(\xi)} \chi_N^2(\xi) \right|_{C^2}.
\end{equation}

Fix \( N_0 \in 2^N \) large enough, depending on \( \mathcal{L} \), so that (1.6) and (1.7) hold for any \( \xi \in \text{spt}(\chi_N) \) and \( N > N_0 \). Since \( \chi_N \) and \( \mathcal{L} \) are smooth and \( \chi_N \) is compactly supported, we obtain (2.11) from (2.12), and therefore (2.5) holds for all \( N \leq N_0 \).

Thus, it remains to consider \( N > N_0 \) and in particular (1.6) and (1.7) hold. Then, by [HHZ17, Lemma 2.1] we obtain the bound

\begin{equation}
K_N(t, x_1, x') \lesssim |t|^{-\frac{d}{2}}.
\end{equation}

Hence, if \( |x_1| \lesssim (N)^{-\sigma - 1} |t| \), then (2.11) and (2.13) imply

\[ |K_N(t, x_1, x')| \lesssim \min \left( N^d, |t|^{-\frac{d}{2}} \right) \lesssim \min \left( N^d, N^{d - \frac{d}{2}} |x_1|^{-\frac{d}{2}} \right) \lesssim N^d (N x_1)^{-\frac{d}{2}} \]

and since \( d > \sigma \)

\begin{equation}
\left\| \mathds{1}_{|x_1| \leq N^{-\sigma - 1} |t|} K_N(t, x_1, x') \right\|_{L^1_{x_1} \to L^\infty} \leq \int_{\mathbb{R}} N^d (N x_1)^{-\frac{d}{2}} \, dx_1 \lesssim N^{d-1}
\end{equation}

and (2.5) follows. Finally, we restrict to \( |x_1| \gtrsim N^{-\sigma - 1} |t| \), where we obtain a lower bound on the derivative of the phase

\begin{equation}
\partial_{\xi_1} \left( 2\pi (x_1 \xi_1 + x' \cdot \xi) - t \mathcal{L}(\xi) \right) = 2\pi x_1 - t \partial_{\xi_1} \mathcal{L}(\xi).
\end{equation}

Indeed, by (1.6), for \( \xi \in \text{spt}(\chi_N) \) we have \( |\partial_{\xi_1} \mathcal{L}(\xi)| \lesssim |\xi|^{-\sigma - 1} \lesssim N^{-\sigma - 1} \). In particular, for \( |x_1| \gtrsim N^{-\sigma - 1} |t| \) one has

\begin{equation}
|2\pi x_1 - t \partial_{\xi_1} \mathcal{L}(\xi)| \gtrsim |x_1|.
\end{equation}

To show a decay of \( K_N \) in \( x_1 \), we exploit oscillations of the phase. By integrating by parts twice we obtain

\begin{equation}
K_N(t, x_1, x') = -\int_{\mathbb{R} \times \mathbb{R}^{d-1}} \chi_N^2(\xi) \left( \frac{1}{2\pi x_1 - t \partial_{\xi_1} \mathcal{L}(\xi)} \partial_{\xi_1} \right)^2 e^{2\pi i x_1 \xi \cdot x' - it \mathcal{L}(\xi)} \, d\xi_1 \, d\xi'.
\end{equation}

\begin{equation}
= -\int_{\mathbb{R} \times \mathbb{R}^{d-1}} e^{2\pi i x_1 \xi \cdot x' - it \mathcal{L}(\xi)} \partial_{\xi_1} \left( \frac{1}{2\pi x_1 - t \partial_{\xi_1} \mathcal{L}(\xi)} \partial_{\xi_1} \left( \frac{\chi_N^2(\xi)}{2\pi x_1 - t \partial_{\xi_1} \mathcal{L}(\xi)} \right) \right) \, d\xi_1 \, d\xi'.
\end{equation}
By rescaling,
\[ |\partial_t^k \chi_N^2(\xi)| \lesssim N^{-k} \| \hat{g} \|_{l^\infty} \]
while by (1.6) and (2.15)
\[ \left| \frac{1}{2\pi x_1 - t \partial_{x_1}} \mathcal{L}(\xi) \right| \lesssim \left| \frac{t \partial_{x_1}^2 \mathcal{L}(\xi)}{t} \right| \lesssim \frac{t|\xi|^\sigma - 2}{|x_1|^2} \lesssim \frac{1}{N |x_1|} \]
and similarly
\[ \left| \frac{1}{2\pi x_1 - t \partial_{x_1}} \mathcal{L}(\xi) \partial_t (\frac{\chi_N^2(\xi)}{2\pi x_1 - t \partial_{x_1}} \mathcal{L}(\xi)) \right| \lesssim \frac{1}{N^2 |x_1|^2} \| \hat{g} \|_{l^\infty} \].

Combining these estimates we obtain
\[ \left| \frac{1}{2\pi x_1 - t \partial_{x_1}} \mathcal{L}(\xi) \partial_t (\frac{\chi_N^2(\xi)}{2\pi x_1 - t \partial_{x_1}} \mathcal{L}(\xi)) \right| \lesssim \frac{1}{N^2 |x_1|^2} \| \hat{g} \|_{l^\infty} \]
and after integration over \( \xi \),
\[ \| |x_1|^{2N^\sigma - 1} |K_N(t, x_1, x')| \|_{L^1_t L^\infty_x} \lesssim N^{d-2} |x_1|^{-2} \]
Since by (2.11) one has \( |K_N(t, x_1, x')| < N^d \), we obtain
\[ \| |x_1|^{2N^\sigma - 1} |K_N(t, x_1, x')| \|_{L^1_t L^\infty_x} \lesssim N^d (N x_1)^{-2} \]
and consequently
\[ \| |x_1|^{2N^\sigma - 1} K_N(t, x_1, x') \|_{L^1_t L^\infty_x} \lesssim \int N^d (N x_1)^{-2} \, dx_1 \leq N^{-d-1}, \]
as required. \( \square \)

**Proof of Lemma 2.5.** Since the \( U^\mathcal{L}_{t_1} \) is a bounded projection on \( L^2 \), it commutes with the evolution \( e^{it \mathcal{L}} \) and with the Littlewood-Paley projections \( P_N \). Without loss of generality assume \( e_1 = e_1 \) and let \( T \) be defined analogously to (2.8) as
\[ Tf(t, x) := \int e^{2\pi i \xi \cdot x - it \mathcal{L}(\xi)} \chi_N(\xi) \| \mathcal{L}_{t_1} \|_{L^2_t L^\infty_x} \xi \| \mathcal{L}_{t_1} \|_{L^2_t L^\infty_x} \xi \hat{f}(\xi) \, d\xi. \]
The assertion of the lemma is equivalent to
\[ \sup_{x_1 \in \mathbb{R}} \| T f(t, x_1, x') \|_{L^1_t L^\infty_x} \lesssim N^{-\frac{d-1}{2}} \| f \|_{L^2(\mathbb{R}^d)}. \]
Using the Plancherel's identity in \( x' \) gives that
\[ \| T f(t, x_1, x') \|_{L^1_t L^\infty_x} \lesssim N^{-\frac{d-1}{2}} \| f \|_{L^2(\mathbb{R}^d)}. \]
Thus, let us fix \( x_1 \in \mathbb{R} \) and \( \xi' \in \mathbb{R}^{d-1} \) and show that
\[ \left( 2.16 \right) \int_{I} \left| \int_{\mathbb{R}} e^{2\pi i \xi_1 x_1 - it \mathcal{L}(\xi)} \chi_N(\xi) \| \mathcal{L}_{t_1} \|_{L^2_t L^\infty_x} \xi \| \mathcal{L}_{t_1} \|_{L^2_t L^\infty_x} \xi \hat{f}(\xi, \xi') \, d\xi_1 \right|^2 \, dt \lesssim N^{-(\sigma - 1)} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi_1, \]
where, as usual, \( \xi = (\xi_1, \xi') \). Choose \( N_0 \in 2^\mathbb{N} \) large enough that (1.6) and (1.7) hold for any \( (\xi_0, \xi') \) with \( |\xi_0|, |\xi'| \geq N \).

If \( N \leq N_0 \), by Cauchy-Schwarz inequality and \( |I| \lesssim T_0 \) one has that the left-hand side of (2.16) is bounded by \( N |I| \| \mathcal{L}_{t_1} \|_{L^2_t L^\infty_x}^2 \) and our claim follows.
Assume $N > N_0$. Since $\xi_1 \to L(\xi_1, \xi')$ is smooth, the set $\{\xi_1 \in \Omega_{\epsilon'} : |\nabla L(\xi_1, \xi') \cdot e_1| \neq 0\} \subset \mathbb{R}$ is open, and it can be represented as a countable union of disjoint open intervals:

$$\{\xi_1 \in \Omega_{\epsilon'} : |\nabla L(\xi_1, \xi') \cdot e_1| \neq 0\} = \bigcup_{W \in \mathcal{I}} W.$$  

The integrand on the left-hand side of (2.16) vanishes unless $(\xi_1, \xi') \in \mathcal{U}_{\epsilon_1}$ or unless

$$\left| \nabla L(\xi_1, \xi') \cdot e_1 \right| > \frac{|\nabla L(\xi_1, \xi')|}{2\sqrt{d}}.$$

Let $\mathcal{I}_0 \subset \mathcal{I}$ be the sub-collection of intervals such that $W \in \mathcal{I}_0$ if and only if $W \cap \{\xi_1 : (\xi_1, \xi') \in \mathcal{U}_{\epsilon_1}\} \neq \emptyset$. We claim that $\mathcal{I}_0$ is finite with cardinality independent of $N$. Indeed, for fixed $(\xi_1^*, \xi') := W \in \mathcal{I}_0$ we can without loss of generality assume $W \subset \text{spt}(\chi_N(\cdot, \xi'))$, because there are at most four intervals in $\mathcal{I}$ that contain an endpoint (one of at most four) of $\text{spt}(\chi_N(\cdot, \xi'))$. Fix $\xi_1^0 \in W \cap \mathcal{U}_{\epsilon_1}$, and in particular

$$\left| \nabla L(\xi_1^0, \xi') \cdot e_1 \right| > \frac{|\nabla L(\xi_1^0, \xi')|}{2\sqrt{d}}.$$  

Since $(\xi_1^0, \xi') \in \text{spt}(\chi_N)$, then $|\xi_1^0, \xi')| \geq N \geq N_0$, and $|\nabla L(\xi_1^0, \xi')| \geq N^{\sigma - 1}$ by (1.6). In particular, $|\nabla L(\xi_1^0, \xi') \cdot e_1| \geq N^{\sigma - 1}$ by (2.17). On the other hand, $\nabla L(\xi_1^0, \xi') \cdot e_1 = 0$. The bound on the second derivative in (1.6) yields $|\partial_\xi L(\nabla L(\xi_1^0, \xi') \cdot e_1)| = |\partial_{\xi_1}^2 L(\xi_1^0, \xi')| \leq N^{\sigma - 2}$. Then, by the mean value theorem, $|W| \geq N$. Since the intervals $W \in \mathcal{I}_0$ are pairwise disjoint and $W \subset \text{spt}(\chi_N(\cdot, \xi'))$, the claimed bound on the cardinality of $\mathcal{I}_0$ follows.

Observe that $\xi_1 \to \partial_\xi L(\xi_1, \xi')$ does not change sign on $W$. Hence, in (2.16) restricted to $W$, we use the change of variables $\xi_1 \to \theta = L(\xi_1, \xi')$, apply Plancherel identity in the variable $t$, and change back to the variable $\xi_1$ and obtain after a use of (2.15) on $\mathcal{U}_{\epsilon_1}$ and (1.6) on $\text{spt}(\chi_N(\cdot, \xi'))$

$$\int_I \left| \int_W e^{2\pi i \xi \cdot x - it} \chi_N(\xi) \mathbb{I}_{\mathcal{U}_{\epsilon_1}}(\xi) \hat{f}(\xi_1, \xi') d\xi_1 \right|^2 dt \leq \int_{\mathbb{R} \theta(W)} \left| \int_{\mathbb{R} \theta(W)} e^{2\pi i \xi \cdot \xi_1(\theta) - it\theta} \chi_N(\xi_1(\theta), \xi') \mathbb{I}_{\mathcal{U}_{\epsilon_1}}(\xi_1(\theta), \xi') \hat{f}(\xi_1(\theta), \xi') \frac{d\theta}{\partial_{\xi_1} L(\xi_1(\theta), \xi')} \right|^2 dt \leq \int_{\mathbb{R} \theta(W)} \chi_N(\xi_1, \xi') \mathbb{I}_{\mathcal{U}_{\epsilon_1}}(\xi_1, \xi') \left| \frac{1}{\xi_1} \partial_{\xi_1} L(\xi_1, \xi') \right|^2 |\partial_{\xi_1} L(\xi_1, \xi')| |d\xi_1| \leq N^{-(\sigma - 1)} \int_{\mathbb{R}} |\hat{f}(\xi_1, \xi')|^2 d\xi_1,$$

as desired.

3. Linear non-homogeneous estimates

This section contains estimates for the solution $v$ of the non-homogeneous Schrödinger equation

$$\begin{cases}
(i\partial_t - L)v = h & \text{on } I \times \mathbb{R}^d, \\
v(t_0) = 0 & \text{on } I \times \mathbb{R}^d.
\end{cases}$$  

(3.1)
in Besov-type spaces. Specifically, for \( \varepsilon \in [0, 1) \) and \( s > 0 \) we set
\[
\|v\|_{X^{s,\varepsilon}(I)} := \left( \sum_{N \in 2^\mathbb{N}} N^{2s} \|P_N v\|^2_{X^0_N(I)} \right)^{\frac{1}{2}},
\]
\[
\|v\|_{X^s(I)} := \|v\|_{L^\infty(L^2) \times (I \times \mathbb{R}^d)} + \sum_{l=1}^d \|v\|_{L^\infty_{x_l} L^2_{x_\leq l,\varepsilon}(I \times \mathbb{R}^d)} + \sum_{l=1}^d \|v\|_{L^\infty_{x_l} L^2_{x_\leq l,\varepsilon}(I \times \mathbb{R}^d)},
\]
where \( U^\varepsilon_{x_l} \) is as in Lemma 2.5. Observe that the norm of \( X^{s,\varepsilon} \) contains Strichartz-type component \( L^\infty_t L^2_x L^\infty_{x_\leq l,\varepsilon}(I \times \mathbb{R}^d) \), which is admissible, and \( L^\infty_t L^2_x(I \times \mathbb{R}^d) \) which is close to an admissible space \( L^\infty_t L^2_x(I \times \mathbb{R}^d) \) if \( \varepsilon > 0 \) is small, see Lemma 2.2. The other two components are close to directional maximal (see Lemma 2.4), and directional local smoothing (see Lemma 2.5) spaces.

We assume that the non-homogeneous term \( h \) is controlled in a space dual to \( X^{s,\varepsilon}(I) \). We set
\[
\|h\|_{X^{s,\varepsilon}(I)^*} := \left( \sum_{N \in 2^\mathbb{N}} N^{2s} \|P_N h\|^2_{X^0_N(I)^*} \right)^{\frac{1}{2}},
\]
\[
\|h\|_{X^s(I)^*} := \sup \left\{ \left| \int_{I \times \mathbb{R}^d} h(x,t) u_\varepsilon(x,t) \, dt \, dx \right| : \|u_\varepsilon\|_{X^s(I)} \leq 1 \right\}.
\]
The main result of this section is the following a priori bound on \( v \) in terms of \( h \).

**Proposition 3.1.** The solution \( v : I \times \mathbb{R}^d \to C \) to
\[
\begin{cases}
(i\partial_t - \mathcal{L}) v = h & \text{on } I \times \mathbb{R}^d, \\
v(0) = 0
\end{cases}
\]
satisfies for any \( \varepsilon \in (0, \frac{1}{d + \sigma}) \) and any \( N \in 2^\mathbb{N} \) the bound
\[
\|P_N v\|_{X^s(I)} \lesssim N^{O(\varepsilon)} \|P_N h\|_{X^s(I)^*},
\]
where the function \( \varepsilon \mapsto |O(\varepsilon)| \) depends only on \( \mathcal{L}, d, \sigma \), and \( s > 0 \). As a consequence,
\[
\|v\|_{X^{s,\varepsilon}(I)} \lesssim \|h\|_{X^{s,|O(\varepsilon)|,\varepsilon}(I)^*}
\]
for any \( s > 0 \).

The proof Proposition 3.1 relies on a sequence of lemmata that combine interpolation, dual versions of Lemma 2.2, Lemma 2.4, and Lemma 2.5 and the “Christ-Kiselev” lemma [CK01]. The solution to (3.1) is given by the Duhamel formula:
\[
v(t,x) = \int_0^t e^{-i(t-s)\mathcal{L}} h(s,x) \, ds.
\]
The results of this section can be seen as properties of the mapping \( h \mapsto v \) between spaces \( X^{s,\varepsilon}(I) \) and \( X^{s,\varepsilon}(I) \).

**Lemma 3.2.** Fix a time interval \( I \subset \mathbb{R} \) with \( 0 \in I \), \( |I| < T_0 \) and let \( J, J' \subset I \) be two disjoint sub-intervals. For any \( N \in 2^\mathbb{N} \) it holds that
\[
\left\| \int_J e^{-i(t-s)\mathcal{L}} P_N h(s) \, ds \right\|_{X^s_N(J)} \lesssim N^{O(\varepsilon)} \|P_N h\|_{X^s_N(J)^*}.
\]
We stress that the implicit constant do not depend on $J$, $J'$, or $N$.

**Lemma 3.3.** Let $I \subset \mathbb{R}$ be a time interval with $0 \notin I$, $|I| < T_0$. For any $\varepsilon \in (0, \frac{\pi}{2})$ it holds that

$$\left\| \int_0^t e^{-i(t-s)\mathcal{L}} P_N h(s) \, ds \right\|_{X^s_N(I)} \lesssim_\varepsilon N^{\mathcal{O}(\varepsilon)} \| P_N h(s) \|_{X^s_N(I)},$$

for any $N \in 2^\mathbb{N}$. The implicit constants do not depend on $N$.

**Proof of Lemma 3.3.** Note that Lemma 2.2, Bernstein inequality (Lemma 2.1), and Fubini theorem imply

$$\left\| e^{-is\mathcal{L}} P_N u \right\|_{L^\infty_t L^{\infty_x}(I \times \mathbb{R}^d)} \leq \left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{\infty}_t L^{\infty_x}(I \times \mathbb{R}^d)} \lesssim N^{\frac{d}{2}} \left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{2}_t L^{2_x}(I \times \mathbb{R}^d)} \lesssim N^{\mathcal{O}(\varepsilon)} \| P_N u \|_{L^2(\mathbb{R}^d)}.$$

Then, interpolating Lemma 2.2, Lemma 2.4, and Lemma 2.5 and using the Hölder inequality implies

$$\left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{2}_t L^{2_x}(I \times \mathbb{R}^d)} \lesssim \left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{1-\varepsilon}_t L^{2_x}(I \times \mathbb{R}^d)} \left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{\infty}_t L^{\infty_x}(I \times \mathbb{R}^d)} \lesssim N^{\mathcal{O}(\varepsilon)} \| P_N u \|_{L^2(\mathbb{R}^d)},$$

and

$$\left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{2}_t L^{\infty_x}(I \times \mathbb{R}^d)} \lesssim \left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{1-\varepsilon}_t L^{2_x}(I \times \mathbb{R}^d)} \left\| e^{-is\mathcal{L}} P_N u \right\|_{L^{\infty}_t L^{\infty_x}(I \times \mathbb{R}^d)} \lesssim N^{\mathcal{O}(\varepsilon)} \| P_N u \|_{L^2(\mathbb{R}^d)},$$

and

$$\left\| e^{-is\mathcal{L}} P_N U^\varepsilon_{\text{el}} u \right\|_{L^\infty_t L^{2_x}(I \times \mathbb{R}^d)} \lesssim \left\| e^{-is\mathcal{L}} P_N U^\varepsilon_{\text{el}} u \right\|_{L^{1-\varepsilon}_t L^{2_x}(I \times \mathbb{R}^d)} \left\| e^{-is\mathcal{L}} P_N U^\varepsilon_{\text{el}} u \right\|_{L^{\infty}_t L^{\infty_x}(I \times \mathbb{R}^d)} \lesssim N^{-\mathcal{O}(\varepsilon)} \| P_N U^\varepsilon_{\text{el}} u \|_{L^2(\mathbb{R}^d)}.$$

In summary, for $I = J$

$$\left\| e^{-is\mathcal{L}} P_N u \right\|_{X^s_N(J)} \lesssim N^{\mathcal{O}(\varepsilon)} \| P_N u \|_{L^2(\mathbb{R}^d)}$$

and by duality for $I = J'$

$$\left\| \int_{J'} e^{is\mathcal{L}} P_N v ds \right\|_{L^2(\mathbb{R}^d)} \lesssim N^{\mathcal{O}(\varepsilon)} \| P_N v \|_{X^s_N(J')}.$$

Consequently,

$$\left\| \int_{J'} e^{-i(t-s)\mathcal{L}} P_N h(s) \, ds \right\|_{X^s_N(J')} \lesssim N^{\mathcal{O}(\varepsilon)} \left\| \int_{J'} e^{is\mathcal{L}} P_N h(s) \, ds \right\|_{L^2(\mathbb{R}^d)} \lesssim N^{\mathcal{O}(\varepsilon)} \| P_N h \|_{X^s_N(J')}.$$

as desired. □

**Proof of Lemma 3.3.** Let $v_1, v_2 : I \times \mathbb{R}^d \to \mathbb{C}$ are supported on disjoint time intervals $J_1, J_2 \subset I$, then we claim that there are norms $X^s_N(I)$ equivalent to $X^s_N(I)$ with some constant independent of $N$, such that

$$\left\| v_1 + v_2 \right\|_{X^s_N(I)} \leq \left\| v_1 \right\|_{X^s_N(J_1)} + \left\| v_2 \right\|_{X^s_N(J_2)}$$

(3.5)
and if \( h_1, h_2: I \times \mathbb{R}^d \to \mathbb{C} \) are supported on disjoint time intervals \( J_1, J_2 \subset I \), then

\[
\|h_1 + h_2\|_{X^s_{\dot{H}}(I)} \leq \|h_1\|_{X^s_{\dot{H}}(J_1)} + \|h_2\|_{X^s_{\dot{H}}(J_2)},
\]

where the norm \( X^s_{\dot{H}}(I) \) is the dual norm to \( \dot{X}^s_{\dot{H}}(I) \) (cf (3.3)). Consequently, by induction

\[
\left\| \sum_i v_i \right\|_{\dot{X}^s_{\dot{H}}(I)} \leq \sum_i \|v_i\|_{\dot{X}^s_{\dot{H}}(J_i)}
\]

and

\[
\left\| \sum_i h_i \right\|_{X^s_{\dot{H}}(I)} \geq \sum_i \|h_i\|_{X^s_{\dot{H}}(J_i)}.
\]

Fix \( h: I \times \mathbb{R}^d \to \mathbb{C} \) and without loss of generality suppose \( I = [0, T_0] \). Then, there is a sequence of interval \( \{I^n_k = [t^n_k, t^n_{k+1})\}_{n \in \mathbb{N}, \, k \in \{0, \ldots, 2^n - 1\}} \) such that \( t^n_k \leq t^n_{k+1} \), the intervals \( \{I^n_k\}_{k \in \{0, \ldots, 2^n - 1\}} \) partition \( I \) and \( I^n_k = I^n_{2k} \cup I^n_{2k+1} \). Furthermore, the intervals are constructed so that

\[
\|h\|_{\dot{X}^s_{\dot{H}}(I^n_k)} \leq 2^{-\frac{n-1}{2}} \|h\|_{\dot{X}^s_{\dot{H}}(I)}.
\]

We postpone the construction of such sequence till the end of the proof.

Using the constructed intervals, we have for any \( t \in I \)

\[
\mathbb{I}_{[0,t)}(s) = \sum_{n=1}^{\infty} \sum_{k=0}^{2^n - 1} \mathbb{I}_{I^n_k}(s) \mathbb{I}_{I^n_{k+1}}(t)
\]

and by triangle inequality and (3.7) for any regular \((t,s,x) \mapsto F(t,s,x)\)

\[
\left\| \int_0^t F(t,s) \, ds \right\|_{\dot{X}^s_{\dot{H}}(I)} = \left\| \int_I \mathbb{I}_{[0,t)}(s) F(t,s) \, ds \right\|_{\dot{X}^s_{\dot{H}}(I)}
\]

\[
\leq \sum_{n=1}^{\infty} \left( \sum_{k=0}^{2^n - 1} \left\| \int_I \mathbb{I}_{I^n_k}(s) \mathbb{I}_{I^n_{k+1}}(t) F(t,s) \, ds \right\|_{\dot{X}^s_{\dot{H}}(I)} \right)^{\frac{1-s}{2}}
\]

where we suppressed the dependence of \( F \) on \( x \). Then, since norms \( X^s_{\dot{H}}(I) \) and \( \dot{X}^s_{\dot{H}}(I) \) are equivalent and \( I^n_k \) and \( I^n_{k+1} \) are disjoint, it follows from Lemma 3.2 with
We choose \( v \) which maximize \( \{ \int_{J_1 \times \mathbb{R}^d} h_1 v_1 + \int_{J_2 \times \mathbb{R}^d} h_2 v_2 \} \) and

\[
\sup_{v_1, v_2} \left( \frac{\int_{J_1 \times \mathbb{R}^d} h_1 v_1 + \int_{J_2 \times \mathbb{R}^d} h_2 v_2}{\| v_1 + v_2 \|_{\tilde{X}^s_n(I)}} \right)
\]

where the supremum is taken over \( v_1, v_2 \in \tilde{X}^s_n(I) \) supported on \( J_1 \) and \( J_2 \) respectively. We choose \( v \) which maximize \( v \mapsto \int_{J_1 \times \mathbb{R}^d} h_j v \) over \( v \) with \( \| v \|_{\tilde{X}^s_n(J_j)} = \| h_j \|_{\tilde{X}^s_n(J_j)} \). By duality between the spaces \( \tilde{X}^s_n(J) \) and \( \tilde{X}^s_n(J)^* \), this supremum is exactly \( \| h_1 \|_{\tilde{X}^s_n(J)} \) and therefore

\[
\| h_1 + h_2 \|_{\tilde{X}^s_n(I)} \geq \frac{\| h_1 \|_{\tilde{X}^s_n(J_1)} + \| h_2 \|_{\tilde{X}^s_n(J_2)}}{\| h_1 \|_{\tilde{X}^s_n(J_1)} + \| h_2 \|_{\tilde{X}^s_n(J_2)}}
\]

as desired. To prove (3.5), it suffices to set

\[
\| v \|_{\tilde{X}^s_n(I)} := \| v \|_{L^\infty_t L^{\frac{d}{d-1}}_x (I \times \mathbb{R}^d)} + \| v \|_{L^{\frac{d}{d-1}}_t L^{\frac{d}{d-s}}_x (I \times \mathbb{R}^d)}
\]

and note that norms of \( X^s_n(I) \) and \( \tilde{X}^s_n(I) \) are equivalent with constants independent of \( N \), possibly depending on \( \varepsilon \). To show (3.5) let us first focus on directional norms. Note that, it suffices to prove the following, more general estimate for \( p, q \in [1, \infty] \) and \( r \leq \min\{ p, q \} \):

\[
\| v_1 + v_2 \|_{L^r_t L^q_x (I \times \mathbb{R}^d)} \leq \| v_1 \|_{L^p_t L^q_x (I \times \mathbb{R}^d)} + \| v_2 \|_{L^p_t L^q_x (I \times \mathbb{R}^d)}.
\]
Indeed, if \( p \leq q \), then \( r \leq p \), and since \( v_1 \) and \( v_2 \) have disjoint supports and \((a+b)^s \leq a^s + b^s\) if \( s \in [0,1] \), we have

\[
\|v_1 + v_2\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} = \left( \int \left( \|v_1(x)\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} + \|v_2(x)\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} \right)^p \, dx \right)^{1/p}
\]

\[
= \left( \int \left( \|v_1(x)\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} + \|v_2(x)\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} \right)^q \, dx \right)^{1/q}
\]

\[
\leq \left( \|v_1\|_{L^q_t L^p_x(I \times \mathbb{R}^d)}^p + \|v_2\|_{L^q_t L^p_x(I \times \mathbb{R}^d)}^p \right)^{1/p}
\]

If \( p \geq q \), then \( r \leq q \), and by the triangle inequality

\[
\|v_1 + v_2\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} = \left( \int \left( \|v_1(x)\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} + \|v_2(x)\|_{L^q_t L^p_x(I \times \mathbb{R}^d)} \right)^p \, dx \right)^{1/p}
\]

\[
\leq \left( \|v_1\|_{L^q_t L^p_x(I \times \mathbb{R}^d)}^p + \|v_2\|_{L^q_t L^p_x(I \times \mathbb{R}^d)}^p \right)^{1/p}
\]

For isotropic norms we have for \( r \leq p \)

\[
\|v_1 + v_2\|_{L^r_t L^2_x(I \times \mathbb{R}^d)}^r \leq \left( \int \|v_1(x)\|_{L^2_x}^r + \|v_2(x)\|_{L^2_x}^r \, dx \right)^{r/p}
\]

\[
= \left( \int \|v_1(t)\|_{L^2_x}^p + \|v_2(t)\|_{L^2_x}^p \, dt \right)^{r/p}
\]

\[
\leq \left( \int \|v_1(t)\|_{L^2_x}^p \, dt \right)^{r/p} + \left( \int \|v_2(t)\|_{L^2_x}^p \, dt \right)^{r/p}
\]

Finally, let us inductively construct the dyadic grid. One can start by setting \( t_0^n = I \). Assume that \( t_{k+1}^n = [t_k^n, t_{k+1}^n] \) was already constructed and set \( t_{2k+1}^{n+1} = t_{k+1}^n \). Since \( t \mapsto \|h_1\|_{X^{s \varepsilon}_N(t_k^n, t_{k+1}^n)} \) is continuous, there is \( t_{2k+1}^{n+1} \in [t_k^n, t_{k+1}^n] \) so that

\[
\|h_1\|_{X^{s \varepsilon}_N(t_{2k+1}^{n+1}, t_{2k+1}^{n+1})} = \frac{1}{2} \|h_1\|_{X^{s \varepsilon}_N(t_k^n, t_{k+1}^n)} - \frac{1}{2} \|h_1\|_{X^{s \varepsilon}_N(t_{2k+1}^{n+1}, t_{2k+1}^{n+1})}.
\]

By (3.6) it holds that

\[
\|h_1\|_{X^{s \varepsilon}_N(t_{2k+1}^{n+1}, t_{2k+1}^{n+1})} \leq \frac{1}{2} \|h_1\|_{X^{s \varepsilon}_N(t_{2k+1}^{n+1}, t_{2k+1}^{n+1})} - \frac{1}{2} \|h_1\|_{X^{s \varepsilon}_N(t_{2k+1}^{n+1}, t_{2k+1}^{n+1})} = \frac{1}{2} \|h_1\|_{X^{s \varepsilon}_N(t_{2k+1}^{n+1}, t_{2k+1}^{n+1})},
\]

and the iterations yield required required.

\[ \square \]

**Proof of Proposition 3.1.** The solution \( v \) is given by the Duhamel formula (3.4). The first claimed bound is the statement of Lemma 3.3. The latter bound follows by adding up the estimates over \( N \in 2^N \) as per (3.2). \[ \square \]

4. Probabilistic Estimates

The aim in this subsection is to establish an almost sure bound of the free evolution with random initial data. The solution \( e^{it \mathcal{L}} f^x \) is measured in a Besov - type norm \( Y^{S,\varepsilon}(I) \) with regularity index \( S > 0 \) (see (4.1) below), which as the norm \( X^{s,\varepsilon}(I) \), contains components of Strichartz -type, directional maximal components (see Lemma 4.2), and directional local smoothing (see Lemma 2.5) components.
However, because of the random nature of $f^\omega$ we prove a gain of derivatives in the estimates compared to the deterministic counterparts. In fact, we do not have any loss of derivatives for the Strichartz-type norms with high spatial integrability, and the directional maximal function estimate for unit-scale frequency localized data. Note that the estimates compared to the deterministic counterparts. In fact, we do not have any loss of derivatives for deterministic evolution in the norm $X^{s,c}(I)$.

For $\varepsilon \in [0, 1]$ we set

\begin{equation}
\|F\|_{Y^{s,c}(I)} := \left( \sum_{N \in 2^\mathbb{N}} N^{2s} \|P_N F\|_{Y^{s,c}_N(I)}^2 \right)^{1/2},
\end{equation}

\begin{equation}
\|F\|_{Y^{s}_N(I)} := \|F\|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} + \|F\|_{L^{2+\delta}_{t,x} L^{1-\delta}_{s} (I \times \mathbb{R}^d)} + \sum_{l=1}^d N^{-\frac{2\sigma}{d-l}} \|F\|_{L^{2+\delta}_{t,x} L^{\frac{4\sigma}{d-l}} (I \times \mathbb{R}^d)}.
\end{equation}

The main result of this section is the following.

**Proposition 4.1.** Let $S > 0$ and $c \in (0, \frac{1}{2})$. Given $f \in H^{S+|O(\varepsilon)|}(\mathbb{R}^d)$ denote by $f^\omega$ the randomization (1.3) of $f$. There exist constants $C > 0$ and $c > 0$ such that for any $\lambda > 0$ it holds that

$$P\left( \{ \omega \in \Omega : \|e^{-it \mathcal{L}} f^\omega\|_{Y^{s,c}(I)} > \lambda \} \right) \leq C e^{-\frac{\text{const}}{n^{2+|O(\varepsilon)|}(\mathbb{R}^d)}}.$$

In particular, almost surely we have

$$\|e^{-it \mathcal{L}} f^\omega\|_{Y^{s,c}(I)} < \infty.$$

The proof of the above statement is provided at the end of this section. We begin with a version of Lemma 2.4 for unit frequency scale operators $Q_n$ defined in (1.4), where we already prove the gain of regularity. The proof is based on a maximal function estimate for unit-scale frequency localized data. Note that the improvement in the local smoothing estimate Lemma 2.5 is not expected.

**Lemma 4.2.** Let $d > \sigma \geq 2$, let $\mathcal{L}$ satisfy (1.6) and (1.7). Then for all $l \in \{1, \ldots, d\}$ and all $n \in \mathbb{Z}^d$ we have

\begin{equation}
\langle n \rangle^{-\frac{1}{d-l}} \|e^{-it \mathcal{L}} Q_n f\|_{L^{2+\delta}_{t,x} L^{1-\delta}_{s} (I \times \mathbb{R}^d)} \lesssim \|Q_n f\|_{L^2_x(\mathbb{R}^d)}.
\end{equation}

**Proof.** We mimic the proof of Lemma 2.4, using a $TT^*$ argument. Define

$$\chi_n(\xi) := \sum_{k \in \mathbb{Z}^d \atop |k-n| \leq 10d} \psi(\xi - k)$$

so that $\psi(\xi - n) \chi_n(\xi) = \psi(\xi - n)$. Analogously to (2.8), let

$$Tf(t, x) := \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x - it \mathcal{L}} \chi_n(\xi) \hat{f}(\xi) d\xi$$

and as in (2.8) we have

$$TT^* g(t, x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x - it \mathcal{L}} \chi_n(\xi) d\xi K_N(t - s, x - y) g(s, y) ds dy.$$

For $\varepsilon \in [0, 1]$ we set

$$\|F\|_{Y^{s,c}_N(I)} := \|F\|_{L^2_t L^\infty_x(I \times \mathbb{R}^d)} + \|F\|_{L^{2+\delta}_{t,x} L^{1-\delta}_{s} (I \times \mathbb{R}^d)} + \sum_{l=1}^d N^{-\frac{2\sigma}{d-l}} \|F\|_{L^{2+\delta}_{t,x} L^{\frac{4\sigma}{d-l}} (I \times \mathbb{R}^d)}.$$
reducing our proof to showing that
\[ \|K_n\|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \langle n \rangle^{\sigma - 1}. \]
Without loss of generality suppose \( e_1 = e_1 \). Since \( \chi_n \) is supported on a set of measure of order 1 (instead of \( N^d \) as was the case of Lemma 2.4) we have that
\[ |K_n(t, x)| \lesssim 1. \]
The required bounds for \( |n| \leq N_0 \) for a large \( N_0 \in 2^N \) are obtained analogously to Lemma 2.4. Let us concentrate on the case \( |n| > N_0 \) for \( N_0 \) large enough so that (1.6) and (1.7) hold for \( \xi \in \text{spt} \chi_n \).
As in (2.14) we obtain the desired result when \( |x_1| \lesssim |n|^{\sigma - 1}|t| \):
\[ \|\mathbb{1}_{|x_1| \leq |n|^{\sigma - 1}|t|} K_n(t, x_1, x')\|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \min \left\{ 1, \frac{|n|^{\sigma - 1} \phi}{|x_1|^2} \right\} \|K_n\|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim \langle n \rangle^{\sigma - 1}. \]
Next, we focus on the regime \( |x_1| \gtrsim |n|^{\sigma - 1}|t| \). For \( \xi \in \text{spt}(\chi_n) \), \( |\xi| \approx |n| \), and by (1.6) we have \( |\partial_{\xi_1} \mathcal{L}(\xi)| \lesssim |\xi|^{\sigma - 1} \approx |n|^{\sigma - 1} \), and therefore \( |t\partial_{\xi_1} \mathcal{L}(\xi)| \lesssim |t||n|^{\sigma - 1} \lesssim |x_1| \). Consequently, for the derivative of the phase we have
\[ |\partial_{\xi_1} \left( 2\pi x_1 \xi + x' \cdot \xi \right) \approx |2\pi x_1 - t\partial_{\xi_1} \mathcal{L}(\xi)| \approx |x_1|. \]
Also, (1.6) implies \( |\partial_{\xi_1, \xi_2} \mathcal{L}(\xi)| \lesssim |\xi|^{\sigma - 2} \approx |n|^{\sigma - 2} \), and we deduce that
\[ |\partial_{\xi_1} \left( \frac{2\pi x_1 - t\partial_{\xi_1} \mathcal{L}(\xi)}{|2\pi x_1 - t\partial_{\xi_1} \mathcal{L}(\xi)|^2} \right) | \lesssim \frac{|t||n|^{\sigma - 2}}{|x_1|^2}, \]
and similarly since \( |\partial_{\xi_1, \xi_2} \mathcal{L}(\xi)| \lesssim |\xi|^{\sigma - 3} \approx |n|^{\sigma - 3} \) we deduce that
\[ \left| \frac{1}{2\pi x_1 - t\partial_{\xi_1} \mathcal{L}(\xi)} \right| \lesssim \frac{|t||n|^{\sigma - 3}}{|x_1|^3} + \frac{|t^2||n|^{2\sigma - 4}}{|x_1|^3}. \]
Combining these estimates with \( |x_1| \gtrsim |n|^{\sigma - 1}|t| \), \( |n| \gg 1 \), we obtain
\[ \left| \frac{1}{2\pi x_1 - t\partial_{\xi_1} \mathcal{L}(\xi)} \right| \lesssim \frac{1}{|x_1|^2} \left( 1 + \frac{|t||n|^{\sigma - 2}}{|x_1|^2} + \frac{|x_1|^3}{|t||n|^{\sigma - 3}} + \frac{|x_1|^3}{|t^2||n|^{2\sigma - 4}} \right) \lesssim \frac{1}{|x_1|^2} \chi_N(x). \]
And integration in \( \xi \) gives
\[ \mathbb{1}_{|x_1| \geq |n|^{\sigma - 1}|t|} |K_n(t, x_1, x')| \lesssim |x_1|^{-2}. \]
Combined with the fact that \( |K_n(t, x_1, x')| < 1 \) we obtain that
\[ \mathbb{1}_{|x_1| \geq |n|^{\sigma - 1}|t|} |K_n(t, x_1, x')| \lesssim |x_1|^{-2} \]
and thus
\[ \|\mathbb{1}_{|x_1| \geq |n|^{\sigma - 1}|t|} K_n(t, x_1, x')\|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R}^d)} \lesssim 1. \]
This completes the required assertion. \( \square \)

The improved unit scale directional maximal bounds of Lemma 4.2, the directional local smoothing estimates of Lemma 2.5, and the Strichartz estimates of Lemma 2.2 together with the unit scale Bernstein inequality (1.5) provide us with almost sure boundedness of \( \|e^{-it\mathcal{L}} f\|_{L^{\infty}_{T \times \mathbb{R}^d}} \) thanks to the following large deviation estimates proved in [BT08b, Lemma 3.1]. For \( \gamma > 0 \), we write \( \|F\|_{L^\gamma_T} \) to denote \( (\mathbb{E}|F|^\gamma)^{1/\gamma} \). We remark that the following lemmas provide a bridge between the stochastic nature of the initial condition and deterministic estimates.
Lemma 4.3. Let \( \{g_n\}_{n=1}^{\infty} \) be a sequence of real valued, independent, zero mean, random variables associated with the distributions \( \{\mu_n\}_{n=1}^{\infty} \) on a probability space \((\Omega, \mathfrak{A}, P)\). Assume that there exists \( c > 0 \) such that
\[
\left| \int_{-\infty}^{\infty} e^{\gamma x} d\mu_n(x) \right| \leq e^{\gamma^2}, \quad \text{for all} \; \gamma \in \mathbb{R} \; \text{and all} \; n \in \mathbb{N}.
\]

Then, there exists \( \alpha > 0 \) such that for any \( \lambda > 0 \) and any sequence \( \{c_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}; \mathbb{C}) \),
\[
P\left( \left\{ \omega : \left| \sum_{n=1}^{\infty} c_n g_n(\omega) \right| > \lambda \right\} \right) \leq 2 e^{-\alpha \frac{\lambda^2}{\sum_n |c_n|^2}}.
\]

Hence, there exists \( C > 0 \) such that for \( 2 \leq \gamma < \infty \) and every \( \{c_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}; \mathbb{C}) \)
\[
\left\| \sum_{n=1}^{\infty} c_n g_n(\omega) \right\|_{L_\gamma(\Omega)} \leq C \sqrt{\gamma} \left( \sum_{n=1}^{\infty} |c_n|^2 \right)^{\frac{1}{2}}.
\]

The proof of the next lemma is a slight modification of the proof in [Tzw09, Lemma 4.5].

Lemma 4.4. Let \( F \) be a real valued measurable function on a probability space \((\Omega, \mathfrak{A}, P)\). Suppose that there exists \( C_0 > 0 \), \( K > 0 \), and \( p_0 \geq 1 \) such that for any \( \gamma \geq \gamma_0 \) we have
\[
\| F \|_{L_\gamma(\Omega)} \leq \sqrt{\gamma} C_0 K.
\]

Then, there exist \( c > 0 \) and \( C_1 > 0 \) depending on \( C_0 \) and \( p_0 \), but independent of \( K \), such that for every \( \lambda > 0 \),
\[
P(\{ \omega \in \Omega : |F(\omega)| > \lambda \}) \leq C_1 e^{-c \lambda^2/K^2}.
\]

In particular, we have
\[
P(\{ \omega \in \Omega : |F(\omega)| < \infty \}) = 1.
\]

Proof of Proposition 4.1. We estimate each of the Littlewood-Paley pieces \( Y^{\delta, \epsilon}_\delta(\mathbb{R}) \) of \( Y^{S, \epsilon}(\mathbb{R}) \) separately. By Minkowski’s inequality, for any function \( F \) and any \( \gamma \geq 2 \) it holds that
\[
\| F \|_{Y^{S, \epsilon}(I)} \equiv \left\| \left( \sum_{N \in 2^\mathbb{N}} N^{2S} \| P_N F \|_{\hat{Y}^\epsilon_{\gamma}(I)}^2 \right)^{\frac{1}{2}} \right\|_{L_\gamma} \leq \left( \sum_{N \in 2^\mathbb{N}} N^{2S} \| P_N F \|_{Y^{\delta, \epsilon}_{\gamma}(I)}^2 \right)^{\frac{1}{2}}.
\]

In the following we substitute \( F = e^{-it \mathcal{L}} f^\omega \). First, since \((\infty, 2)\) is an admissible pair, by Lemma 2.2, Lemma 4.3, and Jensen inequality
\[
\left\| P_N e^{-it \mathcal{L}} f^\omega \right\|_{L^2(I \times \mathbb{R}^d)} \leq \left\| P_N f^\omega \right\|_{L^2(\mathbb{R}^d)} \leq \left\| P_N f \right\|_{L^2(\mathbb{R}^d)}.
\]
Next, we bound the moments of the $L_t^{\frac{4+\sigma}{2}} L_x^{\frac{2d+\sigma}{d}}$ norm. For any $\gamma > \frac{2}{d} > 2 \frac{d+\sigma}{\sigma}$, by Minkowski’s inequality and Lemma 4.3

\[
\| P_N e^{-it \mathcal{L}} f \|_{L_t^\gamma L_x^{\frac{2d+\sigma}{d}} (\mathbb{R} \times \mathbb{R}^d)} \leq \left\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) e^{-it \mathcal{L}} P_N Q_n f \right\|_{L_t^\gamma L_x^{\frac{2d+\sigma}{d}} (\mathbb{R} \times \mathbb{R}^d)} \\
\lesssim \sqrt{\gamma} \left( \sum_{n \in \mathbb{Z}^d} \| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{\gamma} (R \times R^d)}^2 \right)^{\frac{1}{2}} \left\| \sum_{n \in \mathbb{Z}^d} |e^{-it \mathcal{L}} P_N Q_n f|^2 \right\|_{L_t^{\gamma} L_x^{\frac{2d+\sigma}{d}} (\mathbb{R} \times \mathbb{R}^d)}^{\frac{1}{2}}.
\]

Note that $\left( \frac{2d+\sigma}{\sigma}, \frac{2d(d+\sigma)}{d(d+\sigma)-\sigma^2} \right)$ is an $\mathcal{L}$-admissible pair of exponents and

\[
2 \frac{d + \sigma}{\sigma} > \frac{2d(d + \sigma)}{d(d + \sigma) - \sigma^2}.
\]

so we can use the unit scale Bernstein estimate (1.5) coupled with the Strichartz estimate (2.4) to obtain

\[
\| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{\gamma} L_x^{\frac{2d+\sigma}{d}} (\mathbb{R} \times \mathbb{R}^d)} \lesssim \| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{\gamma} L_x^{\frac{2d+\sigma}{d}} (\mathbb{R} \times \mathbb{R}^d)} \lesssim \| P_N Q_n f \|_{L_t^{\gamma} (\mathbb{R}^d)^n}.
\]

Next, we estimate the $L_t^{2, \frac{\gamma}{2}}$ component of the $Y_\xi^\gamma (I)$ norm. As above, Lemma 4.3 together with Minkowski’s inequality yields for any $\gamma > \frac{2}{d}$ that

\[
\left\| \sum_{n=1}^{d} N^{-\frac{2d+\sigma}{d}} \left\| e^{-it \mathcal{L}} P_N Q_n f \right\|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(I \times \mathbb{R}^d)} \right\|_{L_t^{\gamma}} \\
\lesssim \sum_{n=1}^{d} N^{-\frac{2d+\sigma}{d}} \left\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) e^{-it \mathcal{L}} P_N Q_n f \right\|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(\mathbb{R} \times \mathbb{R}^d)} \\
\lesssim \sqrt{\gamma} \left( \sum_{n \in \mathbb{Z}^d} \left| e^{-it \mathcal{L}} P_N Q_n f \right|^2 \right)^{\frac{1}{2}} \left\| \sum_{n \in \mathbb{Z}^d} \left| e^{-it \mathcal{L}} P_N Q_n f \right|^2 \right\|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(\mathbb{R} \times \mathbb{R}^d)}^{\frac{1}{2}}.
\]

From the Hölder inequality and Lemma 2.2 follows

\[
\| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(I \times \mathbb{R}^d)} = \| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(\mathbb{R} \times \mathbb{R}^d)} \leq |T_0|^\frac{1}{2} \| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(\mathbb{R} \times \mathbb{R}^d)} \\
\lesssim |T_0|^\frac{1}{2} \| P_N Q_n f \|_{L^{2} (\mathbb{R}^d)}
\]

and an interpolation with Lemma 4.2 gives us

\[
\langle n \rangle^{-\frac{2d+\sigma}{d}} \| e^{-it \mathcal{L}} P_N Q_n f \|_{L_t^{2} L_x^{\frac{2d+\sigma}{d}}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \langle n \rangle^{O(\varepsilon)} \| P_N Q_n f \|_{L^{2} (\mathbb{R}^d)}
\]
and since $P_NQ_n$ is nonzero only if $|n| \approx N$,
\[
\left\| \sum_{l=1}^{d} N^{-\frac{d-1}{2}} e^{-itX} P_N f^\omega \right\|_{L^2_t(I \times \mathbb{R}^d)} \lesssim \sqrt{\gamma} \sum_{l=1}^{d} N^{(O(\varepsilon))} \left( \sum_{n \in \mathbb{Z}^d} \| P_NQ_n f \|_{L^2_\omega(\mathbb{R}^d)} \right)^{\frac{1}{2}} \lesssim \sqrt{\gamma} \sum_{l=1}^{d} N^{(O(\varepsilon))} \| P_N f \|_{L^2_\omega(\mathbb{R}^d)}.
\]

Similarly,
\[
\| P_N e^{-itX} f^\omega \|_{L^2_\omega(I \times \mathbb{R}^d)} = \| P_N e^{-itX} f^\omega \|_{L^2_\omega L^2_t(I \times \mathbb{R}^d)} \leq |T|^{\frac{1}{2}} \| P_N e^{-itX} f^\omega \|_{L^2_\omega L^2_t(I \times \mathbb{R}^d)} \lesssim \| P_N f \|_{L^2_\omega(\mathbb{R}^d)}
\]
and an interpolation with (2.6) yields
\[
\| P_N e^{-itX} f^\omega \|_{L^2_\omega(I \times \mathbb{R}^d)} \lesssim N^{-\frac{d-1}{2}+O(\varepsilon)} \| P_N f \|_{L^2_\omega(\mathbb{R}^d)}.
\]

A combination of previous estimates and the definition of the Sobolev space $H^{S+|O(\varepsilon)|}$ give us
\[
\left\| e^{-itX} f^\omega \right\|_{Y^S(\mathbb{R})} \lesssim \sqrt{\gamma} \| f \|_{H^{S+O(\varepsilon)}(\mathbb{R}^d)},
\]
and we conclude by using Lemma 4.4. \hfill \square

5. Nonlinear estimates

In this section we estimate the expression
\[
|F + v|^2(F + v)
\]
in the norm $X^{s,c}(I)^*$, where $v$ is a fixed function with finite $X^s(I)$ norm and $F$ is controlled in the norm $Y^{S,c}(I)$. In the next section, we set $F := e^{-itX} f^\omega$.

A Littlewood-Paley decomposition of both $F$ and $v$ yields
\[
F = \sum_{N \in 2^\mathbb{N}} P_N F, \quad v = \sum_{N \in 2^\mathbb{N}} P_N v,
\]
so that
\[
|F + v|^2(F + v) = \sum_{N_1,N_2,N_3 \in 2^\mathbb{N}} (P_{N_1} F + P_{N_1} v)(P_{N_2} F + P_{N_2} v)(P_{N_3} F + P_{N_3} v)
= \sum_{h_1,h_2,h_3 \in \{F,v\}, N_1,N_2,N_3 \in 2^\mathbb{N}} P_{N_1} P_{N_2} h_2 P_{N_3} h_3
= \sum_{h_1,h_2,h_3 \in \{F,v\}, N_1,N_2,N_3,N_4 \in 2^\mathbb{N}} P_{N_4}(P_{N_1} h_1 P_{N_2} h_2 P_{N_3} h_3).
\]

For every assignment $h_1,h_2,h_3 \in \{F,v\}$ we estimate $P_{N_4}(P_{N_1} h_1 P_{N_2} h_2 P_{N_3} h_3)$ in $X^{s,c}(I)^*$. By duality is suffices to bound
\[
\left| \int_{I \times \mathbb{R}^d} (P_{N_1} h_1)(P_{N_2} h_2)(P_{N_3} h_3)(P_{N_4} v) \, dt \, dx \right|
\]
for any \( v_* \) with \( \| v_* \|_{X^s_t(I)} = 1 \). To unify the notation, we define a number \( \omega_j \) and the norm \( \| \cdot \|_{N_j, \epsilon} \) as

\[
\omega_j = S \quad \text{and} \quad \| P_N h_j \|_{N_j, \epsilon} = \| P_N F \|_{Y^s_t(I)} \quad \text{if} \quad h_j = F
\]

\[
\omega_j = S \quad \text{and} \quad \| P_N h_j \|_{N_j, \epsilon} = \| P_N v \|_{X^s_t(I)} \quad \text{if} \quad h_j = v
\]

\[
\omega_j = -S \quad \text{and} \quad \| P_N h_j \|_{N_j, \epsilon} = \| P_N v_* \|_{X^s_t(I)} \quad \text{if} \quad h_j = v_*.
\]

The main result of this section is the bound

\[
\int_I \sum_{N_1, N_2, N_3, N_4} N^{\omega_j} \| P_N (h_1 \bar{h}_2 h_3) \|_{X^s_t(I)}^2 \lesssim \tilde{\epsilon} \max(N_1, N_2, N_3, N_4)^{-\frac{3}{2}} \prod_{j=1}^4 N_j^{\omega_j} \| P_N h_j \|_{N_j, \epsilon},
\]

for any small enough positive \( \tilde{\epsilon} > 0 \) and any \( h_1, h_2, h_3, h_4 \in \{ F, v, v_* \} \) with \( h_j = v_* \) for exactly \( j \in \{ 1, \cdots, 4 \} \). Before proceeding, we state an application of (5.2).

**Corollary 5.1.** Let \( h_1, h_2, h_3 \in \{ F, v \} \) and suppose that (5.2) holds for any small \( \tilde{\epsilon} > 0 \) and any small \( \epsilon > 0 \), possibly depending on \( \tilde{\epsilon} \). Then,

\[
\| h_1 \bar{h}_2 h_3 \|_{X^{s, \epsilon}^t(I)^*} \lesssim \prod_{j=1}^3 \| h_j \|_{\omega_j, \epsilon},
\]

where

\[
\| h_j \|_{\omega_j, \epsilon} = \begin{cases} \| F \|_{Y^{s, \epsilon}^t(I)} & \text{if } h_j = F \\ \| v \|_{X^{s, \epsilon}^t(I)} & \text{if } h_j = v \end{cases}
\]

**Proof.** By the definition of \( X^{s, \epsilon}^t(I)^* \) and \( X^s_t(I)^* \)

\[
\| h_1 \bar{h}_2 h_3 \|_{X^{s, \epsilon}^t(I)^*}^2 = \sum_{N \in 2^\mathbb{N}} N^{2s+\frac{d}{2}} \| P_N (h_1 \bar{h}_2 h_3) \|_{X^s_t(I)^*}^2
\]

and

\[
\| P_N (h_1 \bar{h}_2 h_3) \|_{X^s_t(I)^*} = \sup_{v_*} \left| \int_I \sum_{N_1, N_2, N_3} (P_N h_1)(P_N h_2)(P_N h_3) P_N v_* \, dt \, dx \right|
\]

\[
= \sup_{v_*} \left| \int_I h_1 \bar{h}_2 h_3 P_N v_* \, dt \, dx \right|
\]

where the supremum is taken over \( v_* \) with \( \| v_* \|_{X^s_t(I)} \leq 1 \). Then, by (5.2) and \( \max(N_1, N_2, N_3, N_4)^{-\frac{3}{2}} \leq (N_1 N_2 N_3 N_4)^{-\frac{3}{2}} \) one has

\[
\| P_N (h_1 \bar{h}_2 h_3) \|_{X^s_t(I)^*} \lesssim N^{-s-\frac{3}{2}} \sum_{N_1, N_2, N_3} \prod_{j=1}^3 N_j^{\omega_j} \| P_N h_j \|_{N_j, \epsilon},
\]

and therefore by Cauchy-Schwarz inequality and by the summability of \( \sum_{N \in 2^\mathbb{N}} N^{2s+\frac{d}{2}} \) we obtain

\[
\| h_1 \bar{h}_2 h_3 \|_{X^{s, \epsilon}^t(I)^*} \lesssim \sum_{N \in 2^\mathbb{N}} N^{-\frac{3}{2}} \prod_{j=1}^3 \left( \sum_{N_j} N_j^{\omega_j} \| P_N h_j \|_{N_j, \epsilon} \right)^2
\]

\[
\lesssim \sum_{j=1}^3 \| P_N h_j \|_{N_j, \epsilon}^2 \lesssim \prod_{j=1}^3 \| h_j \|_{\omega_j, \epsilon}^2,
\]
as desired.

\begin{proposition}
If
\[
\frac{d - \sigma}{2} \geq S > S_{\text{min}} := \frac{(d - \sigma)}{2} \left\{ 1 + \frac{2\sigma}{(d - 1)} \right\},
\]
then there exists \( s > \frac{d - \sigma}{2} \) and \( \varepsilon > 0 \) sufficiently small such that bound \((5.2)\) holds.
\end{proposition}

The proof of Proposition \ref{proposition:5.2} relies on the following bilinear estimates.

\begin{lemma}
Let \( N_+ \), \( N_- \) \( \in 2^\mathbb{N} \) with \( N_+ \geq N_- \). The following bilinear estimates hold for any two functions \( h_+, h_- : I \times \mathbb{R}^d \to \mathbb{C} \):
\begin{equation}
\| P_{N_+} h_+ P_{N_-} h_- \|_{L^2(I \times \mathbb{R}^d)} \leq N_+^{(\theta)} N_-^{(\theta)} \| P_{N_+} h_+ \|_{X_{N_+}^I(I)} \| P_{N_-} h_- \|_{X_{N_-}^I(I)}, \tag{5.4}
\end{equation}
\begin{equation}
\| P_{N_+} h_+ P_{N_-} h_- \|_{L^2(I \times \mathbb{R}^d)} \leq N_+^{\sigma} N_-^{\sigma} \| P_{N_+} h_+ \|_{Y_{N_+}^I(I)} \| P_{N_-} h_- \|_{Y_{N_-}^I(I)}, \tag{5.5}
\end{equation}
\begin{equation}
\| P_{N_+} h_+ P_{N_-} h_- \|_{L^2(I \times \mathbb{R}^d)} \leq N_+^{\sigma} N_-^{\sigma} \| P_{N_+} h_+ \|_{Y_{N_+}^I(I)} \| P_{N_-} h_- \|_{Y_{N_-}^I(I)}, \tag{5.6}
\end{equation}
and
\begin{equation}
\| P_{N_+} h_+ P_{N_-} h_- \|_{L^2(I \times \mathbb{R}^d)} \leq N_+^{\sigma} N_-^{\sigma} \| P_{N_+} h_+ \|_{Y_{N_+}^I(I)} \| P_{N_-} h_- \|_{Y_{N_-}^I(I)} \tag{5.7}
\end{equation}
for any \( \theta \in [0, 1] \).
\end{lemma}

\begin{proof}
The proof of all the bounds follows from the Hölder inequality and the definition of the norms \( X_{N_\pm}^I \) and \( Y_{N_\pm}^I \) and Lemma \ref{lemma:2.1}. To prove \((5.4)\), notice that
\[
\| P_{N_+} h_+ P_{N_-} h_- \|_{L^2(I \times \mathbb{R}^d)} \leq \sum_{i=1}^d \left\| P_{N_+} U_{E_i} h_+ P_{N_-} h_- \right\|_{L^2(I \times \mathbb{R}^d)} \leq \sum_{i=1}^d \left\| P_{N_+} U_{E_i} h_+ P_{N_-} h_- \right\|_{L^2(I \times \mathbb{R}^d)} \leq \sum_{i=1}^d \left\| P_{N_+} U_{E_i} h_+ \right\|_{L^\infty(I \times \mathbb{R}^d)} \\| P_{N_-} h_- \|_{L^\infty(I \times \mathbb{R}^d)} \leq N_+ \| \right\| \| P_{N_+} h_+ \|_{X_{N_+}^I(I)} \| P_{N_-} h_- \|_{X_{N_-}^I(I)}
\]
and \((5.4)\) follows after an interpolation with
\[
\| P_{N_+} h_+ P_{N_-} h_- \|_{L^2(I \times \mathbb{R}^d)} \leq \| P_{N_+} h_+ \|_{L^1(I \times \mathbb{R}^d)} \| P_{N_-} h_- \|_{L^1(I \times \mathbb{R}^d)} \leq \| P_{N_+} h_+ \|_{L^\infty(I \times \mathbb{R}^d)} \| P_{N_-} h_- \|_{L^\infty(I \times \mathbb{R}^d)} \leq N_+ \| \right\| \| P_{N_+} h_+ \|_{X_{N_+}^I(I)} \| P_{N_-} h_- \|_{X_{N_-}^I(I)}
\]
\end{proof}
where we used Bernstein inequality, Lemma 2.1. To prove (5.5), we use
\[ \| P_{N_i} h + P_{N_-} h - \|_{L_t^4 L_x^2} (I \times \mathbb{R}^d) \leq \sum_{l=1}^d \| P_{N_i} U_{\xi_l}^l h + P_{N_-} h - \|_{L_t^4 L_x^2} (I \times \mathbb{R}^d) \]
\[ = \sum_{l=1}^d \| P_{N_i} U_{\xi_l}^l h + P_{N_-} h - \|_{L_t^4 L_x^2} (I \times \mathbb{R}^d) \leq \sum_{l=1}^d N_i^{-\frac{\sigma}{d}} N_-^{-\frac{\sigma}{d}} \| P_{N_i} h + \|_{Y_{N_i}^*(I)} \| P_{N_-} h - \|_{Y_{N_-}^*(I)} , \]
interpolated with
\[ \| P_{N_i} h + P_{N_-} h - \|_{L_t^\infty L_x^\infty} (I \times \mathbb{R}^d) \leq \| P_{N_i} h + \|_{L_t^\infty L_x^\infty} (I \times \mathbb{R}^d) \| P_{N_-} h - \|_{L_t^\infty L_x^\infty} (I \times \mathbb{R}^d) \leq N_i^{-\frac{\sigma}{d}} N_-^{-\frac{\sigma}{d}} \| P_{N_i} h + \|_{Y_{N_i}^*(I)} \| P_{N_-} h - \|_{Y_{N_-}^*(I)} , \]
where we again used Lemma 2.1. Next, (5.6) follows from
\[ \| P_{N_i} h + P_{N_-} h - \|_{L_t^2 L_x^2} (I \times \mathbb{R}^d) \leq \sum_{l=1}^d \| P_{N_i} U_{\xi_l}^l h + P_{N_-} h - \|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}} (I \times \mathbb{R}^d) \leq \sum_{l=1}^d N_i^{-\frac{\sigma}{d}} N_-^{-\frac{\sigma}{d}} \| P_{N_i} h + \|_{X_{N_i}^*(I)} \| P_{N_-} h - \|_{X_{N_-}^*(I)} . \]
Finally, (5.7) is a consequence of an interpolation between
\[ \| P_{N_i} h + P_{N_-} h - \|_{L_t^3 L_x^3} (I \times \mathbb{R}^d) \leq \| P_{N_i} h + \|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}} (I \times \mathbb{R}^d) \| P_{N_-} h - \|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}} (I \times \mathbb{R}^d) \leq N_i^{-\frac{\sigma}{d}} N_-^{-\frac{\sigma}{d}} \| P_{N_i} h + \|_{X_{N_i}^*(I)} \| P_{N_-} h - \|_{X_{N_-}^*(I)} \]
and
\[ \| P_{N_+} h_+ P_{N_-} h_- \|_{L_t^2 L_x^2} (I \times \mathbb{R}^d) \leq \| P_{N_+} h_+ \|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}} (I \times \mathbb{R}^d) \| P_{N_-} h_- \|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}} (I \times \mathbb{R}^d) \leq \| P_{N_+} h_+ \|_{X_{N_+}^*(I)} \| P_{N_-} h_- \|_{X_{N_-}^*(I)} . \]

**Proof of Proposition 5.2.** For ease of notation we drop the complex conjugation, as it does not influence our estimates. Furthermore, by symmetry, we assume that
\[ N_1 \geq N_2 \geq N_3 \geq N_4 . \]
Note that unless \( N_1 \lesssim N_1 \) one has \( N_2 + N_3 + N_4 < 2^{-10} N_1 \) and thus
\[ (P_{N_1} h_1)(P_{N_2} h_2)(P_{N_3} h_3)(P_{N_4} h_4) \]
is supported on \( |\xi| < 2^{-2} N_1 \) in Fourier space. This implies
\[ \left| \int_{I \times \mathbb{R}^d} (P_{N_1} h_1)(P_{N_2} h_2)(P_{N_3} h_3)(P_{N_4} h_4) \, dt \, dx \right| = 0 , \]
and therefore we restrict to proving bound (5.2) only when
\[ N_1 \approx N_2 \geq N_3 \geq N_4 . \]
We consider all the cases \( (h_1, h_2, h_3, h_4) \in \{F, v, v_*\}^4 \) with \( v_* \) appearing exactly once. Henceforth, we assume that \( \|P_{h_j}f\|_{N_j, \varepsilon} = N_j^{-\alpha_j}, \ j \in \{1, 2, 3, 4\} \), set
\[
\sigma = \frac{d - \sigma}{2} + \tilde{c} + |O(\varepsilon)|,
\]
and prove the bound
\[
\Lambda(h_1, h_2, h_3, h_4) := \left| \int_{I \times \mathbb{R}^d} (P_{h_1}f_1)(P_{h_2}f_2)(P_{h_3}f_3)(P_{h_4}f_4) \, dt \, dx \right| \lesssim N_1^{-\varepsilon}.
\]

In general, by Cauchy-Schwarz inequality
\[
\Lambda(h_1, h_2, h_3, h_4) \leq \|P_{h_1}f_1P_{h_2}f_2\|_{L^2(I \times \mathbb{R}^d)} \|P_{h_3}f_3P_{h_4}f_4\|_{L^2(I \times \mathbb{R}^d)}
\]
and for each term on the right-hand side we use \( N_1 \approx N_2 \) and Lemma 5.3 to obtain
\[
(5.8) \quad \Lambda(h_1, h_2, h_3, h_4) \leq \prod_{j=1}^{4} N_j^{\beta_j - \alpha_j} N_k^{-\alpha_k} \lesssim N_1^{O(\varepsilon)} N_1^{\beta_1 + \beta_2 - \alpha_1 - \alpha_2} N_3^{\beta_3 - \alpha_3} N_4^{\beta_4 - \alpha_4},
\]
where \( \alpha_j \) was defined in (5.1) and \( \|P_{h_j}f\|_{N_j, \varepsilon} = N_j^{-\alpha_j} \), and \( \beta_j \) is an appropriate exponent originating in estimates in Lemma 5.3. Let us provide details for various cases.

**Case 1.** Assume there exists \( v \) with higher frequency than \( v_* \). More precisely, assume there is \( j < k \) such that \( h_j = v \) and \( h_k = v_* \). Of course by the symmetry in (5.8) and \( N_1 \approx N_2 \) we also allow \( j = 2 \) and \( k = 1 \).

After an exchange of \( h_1 \) for \( h_2 \) if necessary, we can assume \((j, k) \not\in \{(1, 4), (2, 3)\}\). Then, \( \alpha_j = \sigma \) and \( \alpha_k = -\sigma \), which implies \( \alpha_j + \alpha_k = 0 \). Observe that if \( k \in \{1, 2\} \), then \( j \in \{1, 2\} \) and by \( N_1 \approx N_2 \) and \( \alpha_j + \alpha_k = 0 \), one has
\[
\Lambda(h_1, h_2, h_3, h_4) > 0, \text{ and therefore } \beta_k - \alpha_k > 0. \text{ Since } N_j \geq N_k \text{ and } \alpha_j + \alpha_k = 0, \text{ one has}
\]
\[
(5.9) \quad N_j^{\beta_j - \alpha_j} N_k^{-\alpha_k} \lesssim N_1^{\beta_j + \beta_k},
\]
where in the last inequality we used that either \( j \in \{1, 2\} \) or \( j > 2 \) which implies \( \beta_j, \beta_k \geq 0 \).

If \((h_1, h_4) \not\in \{(v, v), (v_*, v), (v, v_*), (v_*, v_*)\}\), then the exponents of \( N_+ \) and \( N_- \) in Lemma 5.3 add up to zero (we take \( \theta = 0 \) in (5.7)), which in our notation means
\[
(5.11) \quad \beta_1 + \beta_4 = 0.
\]
Similarly, (5.11) holds for indices \((1, 4)\) replaced by \((2, 3)\). For any \( l \) with \( l \neq j, k \), we have \(-\alpha_l \leq 0 < \tilde{c} + |O(\varepsilon)|\). If \( l \geq 3 \), \( \beta_l > 0 \), and we choose small \( \varepsilon, \tilde{c}, \varepsilon > 0 \) such that \( \beta_l - \tilde{c} - |O(\varepsilon)| > 0 \). If \( l \leq 2 \), we choose any small \( \varepsilon, \tilde{c}, \varepsilon > 0 \) so that \( \alpha_l \geq \tilde{c} + |O(\varepsilon)| \). Then, \( N_j^{\beta_j - \alpha_j} \lesssim N_1^{\beta_j - \tilde{c} - |O(\varepsilon)|} \) and after using \( \beta_1 + \cdots + \beta_4 = 0 \) we obtain
\[
(5.12) \quad \Lambda(h_1, h_2, h_3, h_4) \leq N_1^{O(\varepsilon)} N_1^{\sum_{l \neq j, k} \beta_l - \tilde{c} - |O(\varepsilon)|} \lesssim N_1^{-\varepsilon}
\]
as desired.

If \((h_1, h_4) = (v, v)\), then after set \( j = 1 \) (higher frequency than \( v_* \)) and by Lemma 5.3, \( \beta_1 = -\frac{d}{2} + \sigma \) and \( \beta_4 = \frac{d}{2} - \sigma \leq \frac{d}{2} - \tilde{c} - |O(\varepsilon)| \). Thus, we transformed the problem to \((h_1, h_4) = (v, F)\) from above (after \( A_1 \) was replaced by \( \tilde{c} + |O(\varepsilon)| \)).

Assume \((h_1, h_4) = (v, v_*)\). If \( h_2 = F \), we exchange \( h_1 \) with \( h_2 \) (recall \( N_1 \approx N_2 \)), to obtain \((h_1, h_4) \neq (v, v_*)\). Thus, we can assume \( h_2 = v \). Then, by (5.9) and (5.10) with \((j, k) = (1, 4)\) and \( N_3 \leq N_1 \) we have
\[
(5.12) \quad N_1^{\beta_1 - \beta_4} N_3^{\beta_3 - \beta_3} N_4^{\beta_4 - \beta_4} \lesssim N_1^{\beta_1 + \beta_2 + \beta_4 - \sigma + \max(\beta_3 - \beta_3, 0)}.
\]
In addition,

\[ (5.13) \quad B_1 + B_4 - s < -\frac{\sigma - 1}{2} + \frac{d - 1}{2} - \frac{d - s}{2} = 0 \]

and if \( h_3 = F \), then \( B_3 - \sigma h_3 = \frac{\sigma - 1}{2} - S \leq \frac{\sigma - 1}{2} \) and if \( h_3 = v \), then \( B_3 - \sigma h_3 = \frac{d - 1}{2} - s < \frac{\sigma - 1}{2} \). Since \( B_2 = -\frac{\sigma - 1}{2} \), we obtain

\[ (5.14) \quad N_1^{B_1 - \sigma h_1 + B_2 - \sigma h_2} N_3^{B_3 - \sigma h_3} N_4^{B_4 - \sigma h_4} \lesssim N_1^{-\varepsilon - |O(x)|} , \]

for any small \( \varepsilon, \tilde{\varepsilon} > 0 \) such that \( \tilde{\varepsilon} + |O(x)| < s - \frac{d - s}{2} \).

Finally, if \( (h_1, h_4) = (u, v) \), then there is \( v \) with higher frequency than \( u \), implying \( h_2 = v \). As above, for \( l \in \{3, 4\} \), we have \( B_l - \sigma h_l < \frac{\sigma - 1}{2} \), and therefore

\[ (5.15) \quad N_1^{B_1 - \sigma h_1 + B_2 - \sigma h_2} N_3^{B_3 - \sigma h_3} N_4^{B_4 - \sigma h_4} \lesssim N_1^{B_1 + B_2 + \sigma - 1} \]

and the assertion follows as above, because \( B_1 + B_2 = -\frac{\sigma - 1}{2} \).

Case 2. Assume that among \( (h_j)_{j=1}^4 \) are two functions \( v \). Specifically, assume \( (h_1, h_2, h_3, h_4) = (u, F, v, v) \) or \( (F, u, F, v, v) \). Since we can interchange \( N_1 \) and \( N_2 \), we can just treat \( (h_1, h_2, h_3, h_4) = (u, F, v, v) \). By (5.8), Lemma 5.3, and \( s < \frac{d - 1}{2} \) with \( N_3 \geq N_4 \) one has for any \( \theta \in [0, 1] \)

\[ |\Lambda(u, F, v, v)| \lesssim N_1^{O(|\varepsilon|) + \varepsilon - S} \lesssim N_1^{-\varepsilon} \]

as long as \( S > |O(x)| + 2\varepsilon \). If \( \frac{d + 1}{d} > \sigma \), then set \( \theta = \frac{d + 1 - 2\varepsilon}{2\varepsilon} \) (note that \( \theta \in (0, 1) \)) to obtain

\[ |\Lambda(u, F, v, v)| \lesssim N_1^{O(|\varepsilon|) + \varepsilon + \frac{d + 1 - 2\varepsilon}{2\varepsilon} - S - \frac{\sigma - 1}{2} - \frac{d + 1 - 2\varepsilon}{2\varepsilon} \lesssim N_1^{-\varepsilon} \]

as long as

\[ (5.16) \quad S > |O(x)| + 2\varepsilon + \left( \frac{d + 1}{2} - \sigma \right) \left( \frac{d - \sigma}{d - 1} \right) \]

Case 3. Assume there is exactly one \( j \) with \( h_j = v \). We distinguish two cases.

Case 3a. Suppose \( (h_1, h_2, h_3, h_4) \in \{(u, F, F, v), (F, v, F, v), (F, F, v, u)\} \), that is, \( h_4 = v \). Since we can interchange \( N_1 \) and \( N_2 \), we only treat option \( (u, F, F, v) \) and \( (F, F, v, u) \). To treat \( (u, F, v, v) \), we apply (5.7) with parameter \( \theta \in [0, 1] \) to be determined below and use \( N_1 \approx N_2 \) to have

\[ |\Lambda(u, F, F, v)| \lesssim N_1^{O(|\varepsilon|) - S - \theta \frac{\sigma - 1}{2} - \frac{\sigma - 1}{2} - \frac{d - \sigma}{d - 1} - \frac{\sigma - 1}{2} - \frac{d - \sigma}{d - 1} \]

Set \( \theta = \frac{d - \sigma}{2} \) and since \( N_3, N_4 \leq N_1 \), and \( s > \frac{d - \sigma}{2} \),

\[ |\Lambda(u, F, F, v)| \lesssim N_1^{O(|\varepsilon|) + \varepsilon - S - \frac{d - \sigma}{2} - \frac{\sigma - 1}{2} - \frac{d - \sigma}{2} - \frac{\sigma - 1}{2} - \max(-S + \frac{\sigma - 1}{2}, 0) \lesssim N_1^{-\varepsilon} \]

as long as

\[ S > \frac{d - \sigma}{2} - \frac{d - \sigma - 1}{2} - \frac{\sigma - 1}{2} + 2\varepsilon + |O(x)| \]

and

\[ S > \frac{1}{2} \left( \frac{d - \sigma}{2} - \frac{(d - \sigma)(\sigma - 1)}{4(d - 1)} \right) + 2\varepsilon + |O(x)| = \frac{(d - \sigma)^2}{4(d - 1)} + 2\varepsilon + |O(x)| \]
If \((h_1, h_2, h_3, h_4) = (F, F, v_*, v_*)\), then similarly
\[
|\Lambda(F, F, v_*, v_*)| \lesssim N_1^{O(\varepsilon)} |S - \theta|^{\frac{s-1}{2}} |\sigma|^{\frac{s-1}{2}} N_3^{-\theta + \theta_1^{\frac{s-1}{2}} N_4^{\frac{s-1}{2}}}
\]
and by choosing \(\theta_1 = 0\) and \(\theta_2 = \frac{d}{2 - \sigma}\), and using \(N_3 \leq N_1\) we have
\[
|\Lambda(F, F, v_*, v_*)| \lesssim N_1^{O(\varepsilon)} |S - \theta|^{\frac{s-1}{2}} |\sigma|^{\frac{s-1}{2}} N_3^{-\theta + \theta_1^{\frac{s-1}{2}} N_4^{\frac{s-1}{2}}} + \max\{S + \frac{\theta_1}{2}, 0\}
\]
and the assertion follows from the previous case.

**Case 3b.** Suppose \((h_1, h_2, h_3, h_4) \in \{(v_*, F, F, v), (F, v_*, v, v), (F, F, v_*, v), (F, F, v_*, F)\}\), that is, \(h_3 = v\).
Since, we can interchange \(N_1\) and \(N_2\), we only treat \((v_*, F, v, F)\). We apply (5.7) with parameter \(\theta \in [0, 1]\) to be determined below and use \(N_1 \approx N_2\) to have
\[
|\Lambda(v_*, F, v, F)| \lesssim N_1^{O(\varepsilon)} |S - \theta|^{\frac{s-1}{2}} |\sigma|^{\frac{s-1}{2}} N_3^{-\theta + \theta_1^{\frac{s-1}{2}} N_4^{\frac{s-1}{2}}} + \max\{S + \frac{\theta_1}{2}, 0\},
\]
and we conclude as in Case 3a.

**Case 4.** Assume
\((h_1, h_2, h_3, h_4) \in \{(v_*, F, F, F), (F, v_*, F, F), (F, F, v_*, F), (F, F, v_*, v)\}\),
that is, there is no \(v\) in the product. Again, since we can interchange \(N_1\) and \(N_2\), the first two options are equivalent, and we show that the latter two follow from the first one.

It holds that
\[
|\Lambda(v_*, F, F, F)| \lesssim N_1^{O(\varepsilon)} |S|^{\frac{s-1}{2}} |\sigma|^{\frac{s-1}{2}} N_3^{-\theta + \theta_1^{\frac{s-1}{2}} N_4^{\frac{s-1}{2}}} + \max\{S + \frac{\theta_1}{2}, 0\}
\]
where the last inequality holds as long as
\[
S + 2 \min \left( S - \frac{\sigma - 1}{2}, 0 \right) > \frac{d - \sigma}{2} - (\sigma - 1) + 2\bar{\varepsilon} + |O(\varepsilon)|,
\]
and, in particular, if
\[
\begin{aligned}
S > \frac{d - \sigma}{2} + 2\bar{\varepsilon} + |O(\varepsilon)|, \\
S > \left( \frac{d + 1}{2} - \sigma \right) - \frac{\theta_1}{2} + 2\bar{\varepsilon} + |O(\varepsilon)|.
\end{aligned}
\]

Also,
\[
|\Lambda(F, F, v_*, v_*)| \lesssim N_1^{O(\varepsilon)} |S - \theta|^{\frac{s-1}{2}} |\sigma|^{\frac{s-1}{2}} N_3^{-\theta + \theta_1^{\frac{s-1}{2}} N_4^{\frac{s-1}{2}}} + \max\{S + \frac{\theta_1}{2}, 0\}
\]
and the assertion follows from the previous one. Moreover,
\[
|\Lambda(F, F, v_*, v_*)| \lesssim N_1^{O(\varepsilon)} |S|^{\frac{s-1}{2}} |\sigma|^{\frac{s-1}{2}} N_3^{-\theta + \theta_1^{\frac{s-1}{2}} N_4^{\frac{s-1}{2}}} + \max\{S + \frac{\theta_1}{2}, 0\},
\]
and the assertion follows as above.

Let us summarize the restrictions on \(S\).
1.2 4.1 is an immediate conse-
sequence of the local well-posedness for a forced cubic equation and the bound on

\[ \parallel Y_{S,\varepsilon} \parallel \leq \sigma - \frac{1}{2} + 2\varepsilon + O(\varepsilon) \]

and

\[ \frac{d - \sigma}{d - 1} \left( \frac{d + 1}{2} - \sigma \right) \geq \frac{d - \sigma}{d - 1} - \frac{d - \sigma - 1}{2} \]

are equivalent after standard algebraic manipulations to \( \sigma \geq 1 \). Also,

\[ \frac{(d - \sigma)^2}{4(d - 1)} = \frac{3d - \sigma}{4} + \frac{1}{4} \left( \frac{d - \sigma}{d - 1} \left( \frac{d + 1}{2} - \sigma \right) \right) \leq \max \left\{ \frac{d - \sigma}{6}, \frac{d - \sigma}{d - 1} \left( \frac{d + 1}{2} - \sigma \right) \right\}, \]

as claimed.

To compare the two largest bounds, we note that

\[ \frac{d - \sigma}{d - \sigma_0} \geq \left( \frac{d - \sigma}{d - \sigma_0} \right) \left( \frac{d + 1}{2} - \sigma \right) \]

and

\[ \sigma = 1 \text{ if and only if } \sigma \geq \frac{d + 2}{4} \]

and the assertion follows. \( \square \)

6. Almost sure local well-posedness of cubic NLS

This section is devoted to the proof of Theorem 1.2, that is, to the local almost sure well-posedness of the cubic NLS. Theorem 1.2 is an immediate consequence of the local well-posedness for a forced cubic equation and the bound on

\[ \| e^{-itP} f \|_{Y_{S,\varepsilon}(I)} \]

established in Proposition 4.1. Specifically, we consider the problem

\[ \begin{cases} (i\partial_t - \mathcal{L})v = \pm |F + v|^2(F + v), \\
v(0) = 0 \end{cases} \tag{6.1} \]

for some \( F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) such that \( \| F \|_{Y_{S,\varepsilon}(I)} < \infty \). We have the following local well-posedness result.

**Proposition 6.1.** Let \( I \) be an open time interval containing 0 such that \( |I| < T_0 \) for some small fixed \( T_0 > 0 \) depending only on \( \mathcal{L} \) and \( d \) defined in Lemma 2.2. Fix \( S, \varepsilon > 0 \). Then, there exists \( 0 < \delta \ll 1 \) such that if \( F \in Y_{S,\varepsilon}(I) \) satisfies

\[ \| F \|_{Y_{S,\varepsilon}(I)} \leq \delta, \]

then there exists a unique solution

\[ v \in C(I; H_{x\sigma}^{d_{\sigma}(\mathbb{R}^d)}) \cap X^{\sigma,\varepsilon}(I) \]

to (6.1) on \( I \times \mathbb{R}^d \).

**Proof.** Fix small \( \delta > 0 \) determined below, and use a fixed point argument to construct our solution. We define

\[ \mathcal{B}_\delta = \left\{ v \in X(I) : \| v \|_{X^{\sigma,\varepsilon}(I)} \leq 2\delta \right\} \]
and the map
\[ \Phi(v)(t) = \pm \int_0^t e^{-i(t-s)\mathcal{L}} [F + v]^2 (F + v)(s) \, ds, \]
which, by Duhamel’s formula, is a solution of (6.1). Next, Proposition 3.1 and Corollary 5.1 imply for any sufficiently small \( \varepsilon, \tilde{\varepsilon}, \delta > 0 \) and any \( v \in \mathcal{B}_\delta \),
\[ \| \Phi(v) \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)} \lesssim \| F + v \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)}^2 \lesssim \| v \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)}^3 + \| F \|_{Y^{S, \varepsilon}(I)}^2 \leq 2\delta, \]
and in addition, for any \( v_1, v_2 \in \mathcal{B} \),
\[ \| \Phi(v_1) - \Phi(v_2) \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)} \lesssim \| F + v_1 \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)} - \| F + v_2 \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)} \lesssim \frac{\| v_1 - v_2 \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)}^3}{2}. \]
So \( \Phi : \mathcal{B}_\delta \rightarrow \mathcal{B}_\delta \) is a contraction with respect to the \( X^{S, \varepsilon}(I) \) norm for sufficiently small \( \varepsilon > 0 \). Thus, there exists a unique solution to (6.1).

Next, we state a continuity property for the norms \( X^{S, \varepsilon}(I) \) and \( Y^{S, \varepsilon}(I) \) with respect to the interval which follows from the dominated convergence theorem.

**Lemma 6.2.** Let \( I \subset \mathbb{R} \) be a closed interval. Fix \( S, \varepsilon > 0 \). Assume that \( \| v \|_{X^{\tilde{\varepsilon}, \varepsilon}(I)} < \infty \) and \( \| F \|_{Y^{S, \varepsilon}(I)} < \infty \). Then the mappings
\[ t \in I \rightarrow \| v \|_{X^{\tilde{\varepsilon}, \varepsilon}([t, t])}, \quad t \in I \rightarrow \| F \|_{Y^{S, \varepsilon}([t, t])} \]
and
\[ t \in I \rightarrow \| v \|_{X^{\tilde{\varepsilon}, \varepsilon}([t, \sup I])}, \quad t \in I \rightarrow \| F \|_{Y^{S, \varepsilon}([t, \sup I])} \]
are continuous. We can also allow half-open and open intervals \( I \).

We are now in position to prove Theorem (1.2).

**Proof of Theorem 1.2.** We are looking for a solution to (1.9) of the form
\[ v(t) = e^{-it\mathcal{L}} f^\omega + v(t), \]
where \( v \) is a solution to
\[ \begin{cases} (i\partial_t - \mathcal{L}) v = \pm [e^{it\mathcal{L}} f^\omega + v]^2 (e^{-it\mathcal{L}} f^\omega + v) & \text{on } I \times \mathbb{R}^d, \\ v(0) = 0. \end{cases} \]
(6.2)
Since by Proposition 4.1 one has \( \| e^{it\mathcal{L}} f^\omega \|_{Y^{S, \varepsilon}(\mathbb{R})} < \infty \) for a.e. \( \omega \in \Omega \), then by Lemma 6.2, we can find an interval \( I^\omega \subset \mathbb{R} \) with \( 0 \in I^\omega \) such that
\[ \| e^{-it\mathcal{L}} f^\omega \|_{Y^{S, \varepsilon}(I^\omega)} \leq \delta, \]
where \( 0 < \delta \ll 1 \) is the constant given in Proposition 6.1, and consequently there exists a unique solution \( v \in C(I^\omega; \dot{H}^S_0(\mathbb{R}^d)) \cap X^{S, \varepsilon}(I^\omega) \) to (6.2) for a.e. \( \omega \in \Omega \). To show global uniqueness, assume that \( w \) is a solution of (6.2) belonging to \( v \in C(I; \dot{H}^{\frac{\varepsilon}{2}}(\mathbb{R}^d)) \cap X^{S, \varepsilon}(I) \). Then, from the continuity and \( w(0) = 0 \) follows that \( w \) is small for short times, and therefore by the uniqueness of small solutions, \( wy = v \).

Iterating this procedure, we obtain uniqueness on \( I^\omega \).
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