Geometric Local Hidden State Model for Some Two-qubit States

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Adopting the geometric description of steering assemblages and local hidden states (LHS) model, we propose a geometric LHS model for some two-qubit states under continuous projective measurements of the steering side. We show that the model is the optimal LHS model for these states, and obtain a sufficient steering criterion for all two-qubit states. Then we demonstrate asymmetric steering using the results we get.

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After quantum nonlocality was introduced by Einstein, Rosen and Podolsky [1], the concept of quantum steering was given by Schrödinger [2]. Consider two distant observers, Alice and Bob, sharing a pair of entangled particles. Quantum steering describes the phenomenon that the measurement performed by one side changes the state of the other. Recently quantum steering was recognized as a type of quantum nonlocality intermediate between entanglement and Bell nonlocality [3, 4], and it is intrinsically asymmetric [5].

Let Alice be the steering side and Bob be the steered side, that is, Alice would be the one who performs the measurements, then Bob would check if his local system is genuinely influenced by Alice’s measurements. Let $\rho$ be the bipartite state held by Alice and Bob, $\mathcal{M}_A$ be the set of measurements Alice is able to perform, $A$ be a measurement in $\mathcal{M}_A$ and $a$ be an arbitrary outcome of $A$. To make sure that his system is genuinely influenced by the Alice’s measurement instead of some preexisting local hidden states (LHS), Bob must exclude the LHS model:

$$\tilde{\rho}_A^a = \int p(a|A, \lambda) \rho_A q(\lambda) d\lambda,$$

where $\tilde{\rho}_A^a = \text{Tr}_A(A_a \otimes I_B \rho)$ is the unnormalized conditioned state on Bob’s side after Alice obtains outcome $a$ in measurement $A$. The set $\{\tilde{\rho}_A^a\}$ is referred to as a measurement assemblage [6]. The local hidden variable $\lambda$ is distributed with density $q(\lambda)$. The probability distributions in Eq. (1) must satisfy

$$\sum_a p(a|A, \lambda) = 1, \int q(\lambda) d\lambda = 1,$n

$$\int p(a|A, \lambda) q(\lambda) d\lambda = p(a|A).$$

A bipartite state is steerable from Alice to Bob if and only if there is no LHS model $\{p(a|A, \lambda), q(\lambda), \rho_A\}$ such that both Eqs. (1) and (2) hold for all $a$ and $A \in \mathcal{M}_A$. If we want to bound the precise set of steerable states under certain measurement set, we must construct the optimal LHS model for the measurement assemblages. Here the optimal LHS model means that if such LHS model do not satisfy Eqs. (1) and (2) for an assemblage, no other LHS model satisfies them [3, 4]. In a former work [7] we introduced a geometric characterization of steering assemblages of two-qubit states and the LHS models for them. Also we proposed a sufficient and necessary steering criterion and defined the optimal LHS models.

To characterize a measurement assemblage, we put the shrinked Bloch vectors $s_A^a$ of the unnormalized conditioned states

$$\tilde{\rho}_A^a = \rho(a|A) \rho_A = \frac{1}{2} [p(a|A) I + s_A^a \cdot \sigma],$$

into a unit sphere $\hat{B}$ which we called probability Bloch sphere.

A two-qubit state $\rho$ can be written in Pauli bases as $\rho = \frac{1}{4} \sum_{u,v} G_{uv} \sigma_u \otimes \sigma_v$, where $G_{uv}$ is the element of real matrix $G = \left( \begin{array}{cc} 1/b & b^T \alpha \\ \alpha & T \end{array} \right)$, $\alpha$ and $b$ are Bloch vectors, $T$ is a $3 \times 3$ matrix and superscript $T$ means transposition [8]. When Alice’s side is projected onto a pure state $A_a = \frac{1}{2} (I + x_A^a \cdot \sigma)$, Bob’s conditioned state becomes

$$\tilde{\rho}_A^a = \frac{1}{4} [(1 + x_A^a \cdot \alpha) I + (b + T^T x_A^a) \cdot \sigma].$$

By comparing Eqs. (3) and (4), we can obtain that $p(a|A) = \frac{1}{2} (1 + x_A^a \cdot \alpha)$ and $s_A^a = \frac{1}{2} (b + T^T x_A^a)$. The geometric figure Bob obtains in $\hat{B}$ under projective measurements by Alice is shaped by $\frac{1}{2} T^T x_A^a$, translated by $\frac{1}{2} b$ and independent of $\alpha$. The shrinked Bloch vectors obtained by POVM are inside the figures, since any POVM operator can be written as a mixture of some projectors. We denote such steering figures for an measurement assemblage with $S$.

To characterize the LHS model for the assemblage (satisfying Eqs. (1) and (2)), we propose a geometric model $\mathcal{G}$ for a
steering figure is a set of nonnegative distributions \( \{ q(\xi), p(a|A, \xi) \} \) satisfying

(1) Equations

\[
\sum_a p(a|A, \xi) = 1, \\
\int_{C} q(\xi)p(a|A, \xi)d\sigma = p(a|A) \int_{NC} q(\xi)d\sigma,
\]

(5)

hold for all \( a \) and \( A \), where \( NC = N \cup C \) is the combination area of the surface \( N \) and the center \( C \) of \( B \). \( \{ \xi \} \) is the set containing unit vectors that generate surface \( N \) and zero vector at \( C \) distributed with \( q(\xi) \).

Strictly speaking, the probability of \( \xi \) at \( C \) should be a discrete \( p(0) \), but for convenience we still write it in the integral form, which satisfies \( \int_C q(0)d\xi = p(0) \) and \( \int_C p(a|A, 0)q(0)d\xi = p(a|A, 0)p(0) \).

(2) Equations

\[
s^a_A = \int_{NC} p(a|A, \xi)q(\xi)\xi d\sigma
\]

hold for all \( a \) and \( A \).

We also define a quantity \( \mathbb{S} \), which represents the integral \( \int_{NC} q(\xi)d\sigma \) for a g-model \( G \). Usually there are different g-models \( \{ G_i \} \) for one steering figure \( S \), the optimal g-model \( G_o \), among them is the one with \( S_o = \min_i \{ S_i \} \), where \( S_i \) is the steering quantity of some g-model \( G_i \). In a former work, a method was given to quantify steerability \[7\]. In ref. \[7\] we propose \( S_o \) quantifies the steerability of the steering assemblages for two-qubit states. A two-qubit state is unsteerable from Alice to Bob if and only if \( S_o \leq 1 \) for Bob, and the LHS model corresponding to \( G_o \) is the optimal LHS model \[7\].

Different from LHS model, g-model can be constructed for any assemblage. When quantity \( \mathbb{S} \) is greater than 1 for a g-model, there is no corresponding LHS model for it. So we set aside the normalization condition \( \int q(\lambda)d\lambda = 1 \) of LHS model in Eq. (2), and obtain a modified state model that can be one-to-one mapped to g-models. We call such state models modified LHS (MLHS) models. To construct an LHS (MLHS) model for a two-qubit state, we just need to construct a g-model for its steering figure \[7\].

In this work we propose a specific g-model for some two-qubit states of high symmetry under projective measurements, and generalize the result to obtain a lower bound of quantity \( \mathbb{S} \) for any steering figure, as well as a practical sufficient steering criterion for two-qubit states. Then we show the how to reveal the asymmetry of steering using this model by an example.

Consider the states with \( G = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \), which are called T states \[10\]. Since steerability is unchanged under local unitaries, the steerability of T states can be completely described by the states with diagonal T matrices without loss of generality \[7\]. Such states can be written as \( \rho = \frac{1}{2}(I \otimes I + \sum_{i=1}^{3} T_i \sigma_i \otimes \sigma_i) \). They’re also called Bell diagonal states since they can be obtained by convex combinations of Bell states.

For Bell diagonal states, the steering figures under projective measurements are central symmetric about the center \( C \) of sphere \( B \). Let \( S_D \) denote such figures. \( S_D \) could be a dot (1D), an ellipse (2D) and the surface of an ellipsoid (3D) depending on the rank of matrix \( T \). All of them could be called steering ellipsoids in a general sense \[8\]. Werner states are also a special type of Bell diagonal states of which steering ellipsoids are spheres for both steering directions.

Now we are going to construct a specific conditioned distribution \( p_G(a|A, \xi) \) for all \( S_D \). Then we will prove that if there is nonnegative distribution \( q_G(\xi) \) which vanishes at \( \xi = 0 \) and satisfies Eqs. (5) and (6) under \( p_G(a|A, \xi), \{ q_G(\xi), p_G(a|A, \xi) \} \) would be the optimal g-model. We will denote such g-model with \( \mathcal{G} \). We will show that \( \mathcal{G} \) exists as the optimal g-model for the steering ellipsoids of Bell diagonal states, then propose a quantity that gives a lower bound of quantity \( \mathbb{S} \) for all steering ellipsoids.

**Optimal g-models for Bell diagonal states.**— When we construct a g-model for Bell diagonal states under projective measurements, we just need to consider the \( s^a_A \) that generates the surface areas of \( S_D \), since the models for the \( s^a_A \) inside can be obtained by directly mixing the models of the ones on the surface. Demonstration would be performed from 1D case to 3D case. Note that we always let \( q_G(0) = 0 \) and \( p_G(a|A, 0) = \frac{1}{2} \) for all cases, therefore the \( q_G(\xi) \) and \( p_G(a|A, \xi) \) we discuss hereafter are distributions for unit vectors \( \{ \xi \} = 1 \).

In 1D ellipsoid case, (i) when \( S_D \) are dots at \( C \), we let \( p_G(a|A, \xi) = \frac{1}{2} \) and \( q_G(\xi) = 0 \). \( S_G = 0 \) for this case, so \( \mathcal{G} \) is the optimal g-model for \( S_D \); (ii) when \( S_D \) are segments of length \( L \leq 1 \) and symmetric about the center \( C \), its surface are two opposite vectors, each with modulus \( \frac{L}{2} \). We use \( s^{a_+}_{A_0} \) and \( s^{a_-}_{A_0} \) to denote these two vectors, where \( A_0 \) is the corresponding measurement and \( a_{\pm} \) are its two outcomes. Let \( p_G(a_{\pm}|A_0, \xi) = \frac{1}{2}(1 + \xi \cdot s^{a_{\pm}}_{A_0}) \) and \( q_G(\xi) = \frac{L}{2} \cdot \delta(\xi - s^{a_+}_{A_0}) + \delta(\xi - s^{a_-}_{A_0}) \), where \( s^{a_{\pm}}_{A_0} = \frac{s^{a_+}_{A_0}}{|s^{a_+}_{A_0}|}, \delta(\xi) \) is the Dirac delta function. Such model reproduces \( s^{a_{\pm}}_{A_0} \) while satisfying Eqs. (5) and (6), thus reproducing all \( s^a_A \) on the segment. For \( \mathcal{G} \), quantity \( S_G = L \), being equal to the length of the segment. This implies that all Bell diagonal states with an 1D steering ellipsoid are unsteerable. We will prove the optimality of \( \mathcal{G} \) later.

For 2D ellipsoid (ellipse) case, we let

\[
p_G(a|A, \xi) = \begin{cases} 
1, & \xi \in \tilde{E}^A, \\
0, & \xi \in \mathbb{E}/\tilde{E}^A, \\
\frac{1}{2}, & \xi \notin \mathbb{E},
\end{cases}
\]

where \( \tilde{E}^A \) is the semicircle consisting of the unit vectors.
that value $S_G$ for any 2D $S_D$ equals half of the circumference of the ellipse.

Later we'll show that $\mathcal{S}$ is the optimal g-model for 1D and 2D ellipsoids, before that we consider the 3D ellipsoid case. Let $\mathbf{n}_A^n$ denote the outer normal vector of $S_D$ corresponding to $\mathbf{s}_A^n$, and region $R_A^n$ be the hemisphere consisting of unit vectors $\mathbf{v}$ satisfying $\mathbf{v} \cdot \mathbf{n}_A^n \geq 0$ on $\mathcal{B}$. For every 3D $S_D$, the conditioned distribution we construct is

$$p_G(a|A, \xi) = \begin{cases} 1, & \xi \in R_A^n, \\ 0, & \text{otherwise}. \end{cases}$$

**Theorem 1.** If distribution $q_G(\xi)$ exists for a 3D $S_D$, model $\mathcal{S} = \{q_G(\xi), p_G(a|A, \xi)\}$ would be an optimal g-model for it.

**Proof.** By projecting both sides of equation (6) onto the corresponding $\mathbf{n}_A^n$, a new equation is obtained

$$r(\mathbf{n}_A^n) = \int_N p(a|A, \xi)q(\xi)|\mathbf{n}_A^n| d\sigma,$$

where $r(\mathbf{n}_A^n) = s_A^n \cdot \mathbf{n}_A^n$; note that we omit the area $C$ since $\xi \cdot \mathbf{n}_A^n$ vanishes when $\xi = 0$.

Let $c(A, \xi)$ denote the expression $p(a_{\xi_+}|A, \xi) - p(a_{\xi_-}|A, \xi)$, where $a_{\xi_{\pm}}$ are the two outcomes of measurement $A$. The subscripts $\xi_{\pm}$ indicate the relation between the outcomes $a$ and $\xi$, the outcome $a$ that satisfies $\mathbf{n}_A^n \cdot \xi \geq 0$ is chosen to be $a_{\xi_+}$, and the other one is $a_{\xi_-}$.

Since all $\mathbf{n}_A^n$ are unit vectors, we also place them in the sphere $\mathcal{B}$. Then, integrating both sides of Eq. (13) with respect to $\mathbf{n}_A^n$ over surface $N$, we have

$$\int_N r(\mathbf{n}_A^n)d\sigma_n = \int_N q(\xi)\int_{N_\xi} c(A, \xi)|\mathbf{n}_A^n|d\sigma_n d\sigma_\xi$$

where $N_\xi$ is the hemisphere consisting of unit vectors $\mathbf{u}$ satisfying $\mathbf{u} \cdot \xi \geq 0$, and $d\sigma_n$ is the infinitesimal area on $N$ corresponding to $\mathbf{n}_A^n$. The inner integral with $d\sigma_n$ is with respect to $\mathbf{n}_A^n$ and the other one with $d\sigma_\xi$ is with respect to $\xi$.

For $\mathcal{S}$, $c(A, \xi) = 1$. Under g-model $\mathcal{S}$, Eq. (14) could be simplified as

$$\int_N r(\mathbf{n}_A^n)d\sigma_n = \int_N q_G(\xi)\int_{N_\xi} \mathbf{n}_A^n d\sigma_n d\sigma_\xi.$$ 

Let $I_3$ denote the integral $\int_{N_\xi} \mathbf{n}_A^n d\sigma_n$. Its value is $I_3 = \pi$, independent of $\xi$. Then equation (15) becomes

$$\int_N r(\mathbf{n}_A^n)d\sigma_n = I_3 \cdot \int_N q_G(\xi)d\sigma_\xi = I_3 \cdot S_G.$$ 

From (16) we obtain

$$S_G = \frac{\int_N r(\mathbf{n}_A^n)d\sigma_n}{I_3}.$$
Consider another g-model $G_x$. For any $G_x$ there is $c_x(a|A, \xi) \leq 1$. Then equation (14) becomes

$$\int_N r(n_A^x) d\sigma_n = \int_N q_x(\xi) \int_{N_\xi} c_x(A, \xi) \cdot n_A^x d\sigma_n d\xi$$

$$\leq \int_N q_x(\xi) \int_{N_\xi} \xi \cdot n_A^x d\sigma_n d\xi$$

$$= I_1 \cdot \int_N q_x(\xi) d\sigma_x. \quad (18)$$

Since $S_x = \int_N q_x(\xi) d\sigma_x \geq \int_N q_x(\xi) d\sigma_x$, inequality (18) indicates that $S_x \geq \frac{\int_N r(n_A^x) d\sigma_n}{I_2} = S_G$, thus theorem 1 is proved.

The existence of $q_G(\xi)$ for 3D Bell diagonal states is proved in some former works [11][12], so we can use Eq. (17) to calculate the optimal steering quantity $S_G$ even without calculating $q_G(\xi)$.

**Theorem 2.** $\xi$ is the optimal g-model for 2D and 1D ellipsoids.

**Proof.** For 2D case, by projecting both sides of equation (6) for 2D ellipsoids onto corresponding $n_A^3$, we get an equation similar to Eq. (13). Integrating both sides of the equation with respect to $n_A^3$ over $\hat{E}$, we have

$$\int_{\hat{E}} r(n_A^3) d\theta_n = \int_N q(\xi) c(A, \xi) \cdot n_A^3 d\theta_n d\sigma_\xi, \quad (19)$$

where $d\theta_n$ is the infinitesimal angle corresponding to varying $n_A^3$, $\hat{E}_\xi$ is the semicircle generated by the intersection of $N_\xi$ and circle $\hat{E}$. Similar to 3D case, for model $\xi$, $c(A, \xi) = 1$. Then we obtain from Eq. (19) that

$$S_G = \frac{\int_{\hat{E}} r(n_A^3) d\theta_n}{I_2}, \quad (20)$$

where $I_2$ denotes $\int_{\hat{E}_\xi} \xi \cdot n_A^3 d\theta_n$ and $I_2 = 2$. For any g-model $G_x$ of which $c_x(A, \xi) \leq 1$, or $q(\xi) \neq 0$ when $\xi \notin \hat{E}$, the value of the inner integral in Eq. (19) would be not more than $I_2$, thus $S_x \geq S_G$.

We can also perform similar procedure for 1D case and get an equation

$$r(n_A^{*3}) + r(n_A^{-3}) = \int_N q(\xi) c(A, \xi) \cdot n_A^3 d\sigma_\xi, \quad (21)$$

where $n_A^{*3}$ are two opposite unit vectors parallel to the steering segments, $r(n_A^{*3}) = \frac{1}{2}$. Using Eq. (21) we can prove in a similar way that the model $\xi$ we proposed earlier for 1D case is the optimal one.

Also we can summarize an equation for $S_G$ in all cases

$$S_G = \frac{V_d}{I_d}, \quad (22)$$

where $d$ represents the dimension of the steering ellipsoid. $V_1 = r(n_A^{*3}) + r(n_A^{-3}) = L$, $I_1 = 1$, $V_d$ and $I_d$ for higher dimension cases have been given in Eqs. (17) and (20). We can see that $V_d$ depends only on the steering ellipsoid and is theoretically calculable. For 1D and 2D cases, $V_d$ equals to the length and circumference of the steering ellipsoids respectively.

Note that the optimal quantity $S_o = S_G$ for a 2D $S_D$, with the value $\frac{1}{2} T(a + b)$, where $a$ and $b$ are the length of the two semi-axes of the ellipse, $T$ is the elliptic coefficient. When $b \rightarrow 0$, the 2D $S_D$ becomes an 1D $S_D$. And at the same time, $T \rightarrow 4$, the quantity $S_o \rightarrow 2a = L$. This shows that $\frac{1}{2} T(a + b)$ can be the common expression of quantity $S_o$ for 1D and 2D cases. Then, we may wonder if the quantity $S_o$ of all $S_D$ can be written as $K(a + b + c)$, where $K$ is a coefficient depends only on the shape, but not the size of the ellipse $S_D$. This question is left for further study.

Some examples. — Now we calculate the $S_G$ for some states that have a g-model $\xi$.

(i) Werner state

two-qubit Werner states [14] can be written as

$$W(p) = p|\psi\rangle\langle\psi| + (1 - p)I/4 \quad (23)$$

where $|\psi\rangle\langle\psi|$ is the singlet state and $I$ the identity.

The steering ellipsoid $S_D$ of $W(p)$ is a sphere of radius $p/2$. Distribution $q_G(\xi)$ for $S_D$ exists as a constant depending only on $p$. Using Eq. (17) we obtain that $S_G = 2p$ for $W(p)$, so the largest $p$ that admits an LHS model is 1/2.

(ii) 2D Bell diagonal states

For these states, quantity $S_G$ equals half of the circumference of steering ellipses, that is $\frac{1}{2} T(a + b)$.

Consider the polytope of separable Bell diagonal states, for the 2D Bell diagonal states at the edge of this polytope, $S_G$ varies from 0.785 to 1. This means all 2D Bell diagonal states are unsteerable. But this is not a trivial result since with all $S_G$ of them calculated, we can obtain a large set of unsteerable states by mixing these states with other states whose $S_G > 1 [7]$.

Obtaining a sufficient steering criterion. — Using quantity $S_G$, a sufficient steering criterion for all two-qubit states can be obtained. Earlier we showed that for a two-qubit state

$$\rho = \frac{1}{4}(I + a \cdot \sigma \otimes I_B + I_A \otimes \sigma \cdot b + \sum_{u,v=2}^4 T_{uv} \sigma_u \otimes \sigma_v), \quad (24)$$

which can also be represented by a coefficient matrix $G = \begin{pmatrix} 1 & b \\ a & T \end{pmatrix}$, its steering ellipsoid is shaped by matrix $T$ and translated by $b/2$. We call all the ellipsoids shaped by the same $T$ in $B$ (with different $a$ and $b$) congruent ellipsoids, and we call the Bell diagonal state with ellipsoid $S_D$ the basic state of all states whose steering ellipsoids are congruent to $S_D$.

**Lemma 1.** $G_i$ is an arbitrary g-model for the steering
ellipsoid $S$ of a two-qubit states under projective measurements, with a steering quantity $S_i$. The steering ellipsoid of the basic state of this two-qubit state is denoted as $S_D$, with a $S_G$ calculable by former method. Now we have: $S_i \geq S_G$.

**Proof.**—Let $\{q_i(\xi), p_i(a\mid A, \xi)\}$ denote g-model $G_i$. For 3D ellipsoids $S$, the proof of theorem 1 can be directly used for Lemma 1, by substituting the vectors of $S$ and distributions of $G_i$ into the both sides of Eq. (18). Note that the integral $\int_N \hat{r}(s_i^A)\hat{d}s_n$ depends only on the shape and the size of the steering ellipsoid, it keeps unchanged upon translations of the steering figures, even when some $\hat{r}(s_i^A)$ becomes negative.

For 2D ellipsoids, the proof of theorem 2 can be directly used for congruent ellipses in the planes that contain the center $C$. For those congruent ellipses in the planes that do not contain $C$ (we denote these planes with $P$), we project center $C$, vectors $\xi$ and $s_1^A$ onto $P$, and denote their projections with $\overline{C}$, $\overline{\xi}$ and $\overline{s_1^A}$ respectively. The unit circle centered at $\overline{C}$ is denoted with $\overline{E}$. An projected equation of Eq. (6) can also be obtained as

$$\overline{s_1^A} = \int_{NC} p_i(a\mid A, \xi)q_i(\xi)\overline{\xi}d\sigma.$$

(25)

Then we obtain the outer normal vectors $\overline{n_1^A}$ of the ellipses on plane $\overline{P}$. By projecting Eq. (25) on corresponding $\overline{n_1^A}$ and integrating the projected equation with respect to $\overline{n_1^A}$, we can obtain an equation on $\overline{P}$

$$\int_{\overline{P}} \frac{r(s_1^A)\overline{d}\sigma}{\overline{n_1^A}} = \int_N q_i(\xi)\int_{E\xi} c_i(A, \xi)\overline{\xi} \cdot \overline{n_1^A}d\overline{n}d\sigma \xi,$$

where lines over the terms indicate that they are in plane $\overline{P}$. $r(s_1^A) = \overline{n_1^A} \cdot \overline{s_1^A}$. $E\xi$ is the semicircle consisting of unit vectors $\overline{u}$, satisfying $\overline{u} \cdot \overline{\xi} \geq 0$, $\overline{d}\overline{n}$ are infinitesimal angles corresponding to $\overline{n_1^A}$. The left side of (26) equals $\int_{\overline{P}} \frac{r(s_1^A)\overline{d}\sigma}{\overline{n_1^A}}$ of the $S_D$ which is congruent to $S$. Then we let $q_i(\xi) = q_i(\xi) \cdot \overline{\xi}$, $\overline{\xi} = \overline{\xi}/|\overline{\xi}|$, and for the right side we have

$$\left\{ \begin{array}{c}
\int_{E\xi} \overline{\xi} \cdot \overline{n_1^A}d\overline{n} = \int_{E\xi} \overline{\xi} \cdot \overline{n_1^A}d\overline{n} = I_2.
\end{array} \right.$$

Then we have

$$\int_N q_i(\xi)d\sigma \xi > \int_N q_i(\xi)d\sigma \xi = \frac{\int_{\overline{P}} \frac{r(s_1^A)\overline{d}\sigma}{\overline{n_1^A}}}{I_2}, \tag{28}$$

which proves that value $S_i$ is larger than $S_G$. The proof of 1D ellipsoid case is similar to the 2D case. }

Lemma 1 shows that $S_G$ for $S_D$ is a lower bound of quantity $S$ for all the congruent ellipsoids of $S_D$. Using lemma 1 a criterion for steering can be directly obtained.

**Criterion 1.** A two-qubit state $W$ is steerable for both directions if $S_G > 1$ for the steering figure of its basic state.

**Demonstration of asymmetric steering.**—In Ref. 2, a state which exhibits asymmetric one-way steering under all projective measurements was proposed as

$$\rho = 1/2 |\psi^-\rangle \langle |\psi^-\rangle + 1/5 |0\rangle \langle 0 | \otimes |I_B \rangle \langle I_B | + 3/10 |I_A \rangle \langle I_A | \otimes |1\rangle \langle 1|,$$

(29)

where $|\psi^-\rangle$ is the density matrix of the singlet state $\{|01\rangle - |10\rangle\}/\sqrt{2}$. State $\rho$ is unsteerable from Bob to Alice but steerable from Alice to Bob. Using g-models and the results above we can also demonstrate that states

$$\varphi(p) = 1/2 |\psi^-\rangle \langle \rho \otimes |I_B \rangle \langle I_B | + 3/2 |p\rangle \langle I_A | \otimes |\phi^+\rangle + (1 - 5/2) |I_D \rangle \langle I_B |$$

(0 < p ≤ 1/2) exhibits asymmetric steering under projective measurements, where $\phi$ is a qubit pure state and $\phi^+$ is the pure state orthogonal to $\phi$, $I$ is the two-qubit identity. Before giving the detailed demonstration, we state that we always let $q(0)$ vanish in all g-models hereinafter, thus vector $\xi$ denote unit vectors only. This would simplify the process without influencing the result.

The steering ellipsoids of $\varphi(p)$ are spheres with radius $\frac{1}{2}$, which are congruent to the steering figure of Werner state $W(\frac{1}{2})$ (we denote it as $S_W$). The value $S_G$ for $S_W$ is 1, from lemma 1 we know that any g-models $G_i$ for $\varphi(p)$ would have quantity $S_i \geq 1$.

Let $u_X$ denote the Bloch vector of $\phi$ and $u_Y$ denote the one of $\phi^+$, we have $u_Y = -u_X$. Let $S_Y$ denote the steering ellipsoid for Bob under projective measurements by Alice and $S_X$ denote the ellipsoid for Alice under projective measurements by Bob. Using former results we know that $S_X$ is a sphere with radius $\frac{1}{2}$, translated by $\frac{p}{2}u_X$, and

$$p(b|B) = \frac{1}{2} + \frac{3}{4} p \cos \beta,$$

(31)

where $p(b|B)$ is the probability that Bob gets result $b$ under projective measurement $B$, $\beta$ is the angle between $u_Y$ and the Bloch vector $X_B$ of projector $B_b$.

1. **Unsteerability from Bob to Alice.**

For $S_X$, we propose a model $\{q_X(\xi), p_X(b|B, \xi)\}$ (we'll denote it with $G_X$)

$$p_X(b|B, \xi) = \begin{cases} 1, & \xi \cdot n_B^b \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$q_X(\xi) = \frac{1}{4\pi} (1 + 3p \cos \beta')$$

(32)

where $\beta'$ is the angle between $\xi$ and $u_X$, $n_B^b$ is the outer normal vector of $S_X$ at $s_B^b$, $s_B^b$ are the shrinked Bloch vectors that constitute $S_X$, corresponding to the assemblage $\{P_B^b\}$.

Note that $p_X(b|B, \xi)$ is actually the same as $p_G(b|B, \xi)$ of the basic Bell diagonal state of $\varphi(p)$, which means we
just change distribution $q_G(\xi)$ into $q_X(\xi)$, then we obtain above model for $S_X$ from $\mathcal{G}$. Actually, for the g-model $\mathcal{G}$ of an arbitrary $S_W$, any distributions $\{q'(\xi), p'(a|A, \xi)\}$ that satisfy

$$
p'(a|A, \xi) = p_G(a|A, \xi),
q'(\xi) + q'(-\xi) = q_G(\xi) + q_G(-\xi),
$$

(33)

under the same set of projectors $\{A_a\}$ would have

$$
s_A' = s_W^a + t,
$$

(34)

where $s_A'$ and $s_W^\nu$ are vectors obtained by substituting $\{q'(\xi), p'(a|A, \xi)\}$ and $\{q_G(\xi), p_G(a|A, \xi)\}$ into Eq. (6) respectively. Equation (34) shows that the steering figure obtained by $\{q'(\xi), p'(a|A, \xi)\}$ is a congruent ellipsoid of $S_W$ with a translation $t$. We can see that $\mathcal{G}_X$ is such a model, the steering figure it generates by (6) is a translated sphere with radius $\frac{1}{4}$, and its translation vector $t_X$ is

$$
t_X = \int_{R_{u_a}^+} \xi \cdot \frac{3}{4\pi} p \cos \beta' d\sigma = \frac{p}{2} \cdot u_X,
$$

(35)

where $R_{u_a}^+$ is the hemisphere consisting of unit vectors $v$ satisfying $v \cdot u_X \geq 0$. Also we calculate the probability $p(b|B)$ that this model produces according to equations (5),

$$
p(b|B) = \int_{R_B^+} q_X(\xi) d\sigma = \frac{1}{2} + \frac{3}{4\pi} \int_{R_B^+} \cos \beta' d\sigma,
$$

(36)

where $R_B^+$ is the hemisphere consisting of unit vectors $v$ satisfying $v \cdot n_B^b \geq 0$ (remember that $x_B^b$ is the Bloch vector of projector $B_b$). Using the method of changing reference frame in $[5]$, we obtain that

$$
\int_{R_B^+} \cos \beta' d\sigma = \pi \cos \beta.
$$

(37)

Then we have

$$
p(b|B) = \frac{1}{2} + \frac{3}{4} p \cos \beta.
$$

(38)

This means that model $\mathcal{G}_X$ is a g-model for $S_X$. From (33) we know that value $S_a = S_G = 1$, thus the LHS model corresponding to $\mathcal{G}_X$ is an LHS model for $\phi(p)$ from Bob to Alice, and $\phi(p)$ are unsteerable from Bob to Alice. Using lemma 1 we know that $\mathcal{G}_X$ is the optimal g-model for $S_X$, thus the LHS model is also the optimal one.

2. Steerability from Alice to Bob.

Similarly, $S_Y$ is a sphere with radius $\frac{1}{4}$, translated by $\frac{3}{4} p \cdot u_Y$, and

$$
p(a|A) = \frac{1}{2} + \frac{1}{2} p \cos \alpha,
$$

(39)

where $\alpha$ is the angle between $u_X$ and the Bloch vector $x_B^a$ of projector $A_a$.

According to lemma 1, if there is an LHS model for $\rho$ from Alice to Bob, there is a g-model $\{q_Y(\xi), p_Y(a|A, \xi)\}$ (we denote as $\mathcal{G}_Y$) for $S_Y$ with $S_b = S_G = 1$. This g-model must satisfy probability condition

$$
p_Y(a|A, \xi) = p_G(a|A, \xi),
p_Y(a|A) = p(a|A),
$$

(40)

and its translation vector $t_Y$ should be

$$
t_Y = \int_R q_Y(\xi) \cdot d\sigma = \frac{3p}{4} \cdot u_Y,
$$

(41)

where $\{q_G(\xi), p_G(a|A, \xi)\}$ is the model $\mathcal{G}$ for the ellipsoid $S_W$ of the Werner state $W(\frac{1}{2})$, $R$ is an arbitrary hemisphere on $B$. We propose a $q_Y(\xi)$ of the form

$$
q_Y(\xi) = \frac{1}{4\pi} (1 + 2p \cos \alpha'),
$$

(42)

where $\alpha'$ is the angle between $\xi$ and $u_Y$. Similar to former result, this $q_Y(\xi)$ can realize the conditioned probability $p(a|A)$.

However, the translation $t_Y'$ under the proposed $q_Y(\xi)$ is

$$
t_Y' = \int_{R_{u_b}^+} \xi \cdot \frac{1}{2\pi} p \cos \alpha' d\sigma = \frac{p}{3} \cdot u_Y,
$$

(43)

which is not equal to $\frac{3}{4} p \cdot u_Y$.

In appendix, we prove that any distributions $\{q(\xi), p(a|A, \xi)\}$ that satisfy (40) have the same translation vectors $t$. This means that there is not a g-model that satisfies (40) and (41) simultaneously for $S_Y$. For any other g-model of $S_Y$, its quantity $S > 1$. Therefore there is not an LHS model for $\phi(p)$ from Alice to Bob, $\phi(p)$ are steerable in this direction.

Discussion.— We have proposed a specific g-model $\mathcal{G}$, and shown that the model is the optimal g-model for Bell diagonal states. Also we have provided a way to calculate the steerability $S_G$ of $\mathcal{G}$ without calculating its distribution $q_G(\xi)$. The quantity $S_G$ corresponding to a steering ellipsoid $S_D$ of Bell diagonal state provides a lower bound of value $S$ for all two-qubit states which have ellipsoids congruent to $S_D$. Using this result we obtained a sufficient criterion for steering and demonstrated asymmetric steering.

Finding the optimal g-model for more two-qubit states is very useful since it not only provides a necessary and sufficient criterion of steering, but also can be used to find more states that demonstrates asymmetric steering. It is also likely to generate the method to propose specific g-model for bipartite states of higher dimensions. Although it’s hard to picture the steering figure of arbitrary higher dimensional states, it’s worth trying to do it for some highly symmetric states such as isotropic states.
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Appendix: Proof that translation vectors are the same in Alice to Bob case in the main text.

Suppose there are two g-models \( \{ q(\xi), p(a|A, \xi) \} \) and \( \{ q'(\xi), p'(a|A, \xi) \} \) satisfying

\[
p(a|A, \xi) = p'(a|A, \xi) = p_G(a|A, \xi),
\]

\[
p(a|A) = p'(a|A), \tag{A1}
\]

and

\[
s'_A = s_A^w + t,
\]

\[
s''_A = s_A^w + t', \tag{A2}
\]

where \( \{ q_G(\xi), p_G(a|A, \xi) \} \) is the g-model \( G \) for the steering figure \( S_W \) of an arbitrary Werner state in \( B \), \( s_a^w \) are the vectors that generate \( S_W \). In this appendix we'll prove that \( t = t' \), and therefore all g-models that realize (A1) and (A2) (corresponding to the Alice to Bob steering case in the main text) have the same translation vector \( t \).

Let \( y(\xi) \) denote the difference \( q(\xi) - q(-\xi) \). First we prove that \( y(\xi) = y'(\xi) \).\( y(\xi) \) we do it by proving that distribution \( y(\xi) \) is unique for every \( \{ q(\xi), p(a|A, \xi) \} \) which satisfies (A1) and (A2). We choose an arbitrary measurement \( A_0 \) and one of its results \( a_0 \), let \( s_{A_0}^a \) denote its shrunk Bloch vector and \( n_{A_0}^a \) denote the outer normal vector of the steering figure at \( s_{A_0}^a \). Then we choose a set of projectors \( \{ P_{a'}^A \} \) (we'll denote them with their Bloch vectors \( \{ x_{a'}^A \} \)), in which every \( x_{a'}^A \) has a small angle \( d\theta \) between \( x_{a_0}^A \). Then we have a set of vectors \( \{ s_{a'}^A \} \) and corresponding outer normal vectors \( \{ n_{A_0}^a \} \), note that each \( n_{A_0}^a \) also has an angle \( d\theta \) between \( n_{A_0}^a \) in this case.

Since

\[
p(a|A, \xi) = \begin{cases} 1, & \xi \cdot n_{A_0}^a \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{A3}
\]

together with (5) we have

\[
p(a|A) = \int_{R_{a'}^A} q(\xi)d\sigma, \tag{A4}
\]

where \( R_{a'}^A \) is the hemisphere consisting of unit vectors \( v \) satisfying \( v \cdot n_{A_0}^a \geq 0 \). Now we do the subtraction \( p(a'|A') - p(a_0|A_0) \) for all \( p(a'|A') \), and for each \( p(a'|A') \) there is

\[
p(a'|A') - p(a_0|A_0) = \int_{D_{a'}^A} q(\xi)d\sigma - \int_{\bar{D}_{a'}^A} q(\xi)d\sigma, \tag{A5}
\]

where \( D_{a'}^A \) and \( \bar{D}_{a'}^A \) are the non-intersecting areas of \( R_{a'}^A \), and \( R_{a_0}^a \). We plot a figure and construct a reference frame to illustrate the case more clearly. (FIG. A1)

Let \( g(\phi) \) denote \( p(a'|A') - p(a_0|A_0) \), where \( \phi \) is the coordinate of \( n_{A_0}^a \), and \( D_{a'}^A \) denote the area \( d_{a'}^A \cup \bar{d}_{a'}^A \). From (A5) there is

\[
g(\phi) = \int_{D_{a'}^A} q(\xi(\phi, \phi')) \cos(\phi' - \phi)d\phi'. \tag{A6}
\]

\( g(\phi) \) is a continuous function on \( \phi \in [0, 2\pi] \) determined by \( p(a|A) \) and the choice of \( x_{A_0}^a \). For each \( \phi \) there is an equation of (A6) type.

Let \( f_\phi(\phi') \) denote the function

\[
f_\phi(\phi') = \begin{cases} \cos(\phi - \phi'), & \phi' \in [\phi - \frac{\pi}{2}, \phi + \frac{\pi}{2}], \\ 0, & \text{otherwise}, \end{cases} \tag{A7}
\]

then equation (A6) can also be written as

\[
g(\phi) = \int_0^{2\pi} [g(\phi') - q(-\phi')]f_\phi(\phi')d\phi', \tag{A8}
\]

where \( q(\xi) \) is written as \( q(\phi') \) since each \( \xi \) can be represented by \( (\frac{\pi}{2}, \phi') \) in the reference frame. Here we would
also write \( y(\xi') \) as \( y(\phi') \), denoting the term \( q(\phi') - q(-\phi') \).

Let \( T(l) \) denote the translation operator in the circle area \( \theta = \frac{\pi}{2} \), satisfying \( T(l) f(\phi) = f(\phi - l) \) for any functions on the circle. The set of all functions \( f_\phi(\phi') \) can be generated by performing \( T(l') \) with \( l' \in [0, 2\pi) \) on one arbitrary function in the set. The eigenfunctions of \( T(l') \) in the circle area \( \phi' \in [0, 2\pi] \) are \( \{\exp(in\phi')| n = Z \} \) with the eigenvalues \( \exp(-in\theta') \), where \( Z \) is the set of integers. This set of eigenvectors is also a set of basis of the circle area.

Now we choose an \( f_{\phi_0}(\phi') \) where \( \phi_0 = 0 \) and expand it using the basis, that is \( f_{\phi_0}(\phi') = \sum_n F_{\phi_0}(n) \exp(in\phi') \), then other \( f_\phi(\phi') \) can be written

\[
f_\phi(\phi') = T(l') f_{\phi_0}(\phi') = \sum_n \exp(-inl') F_{\phi_0}(n) \exp(in\phi'),
\]

where \( l' = \phi - \phi_0 = \phi \) here.

Function \( g(\phi) \) can also be expanded as \( g(\phi) = \sum_n G(n) \exp(in\phi) \), combining (A8) and (A9) we have

\[
G(n) = F_{\phi_0}(n) \int_0^{2\pi} y(\phi') \exp(in\phi') d\phi'.
\]

where the integral part is just \( Y(-n) \), the expansion coefficient of \( y(\phi') \) under the basis (constant factor ignored). So there is \( Y(n) = k \cdot \frac{G(n)}{F_{\phi_0}(n)} \) \((k \text{ is a constant})\), which shows that if solution \( y(\phi) \) \((Y(n)) \) exists for a \( g(\phi) \), it’s an unique one. By above process we have proved that \( y(\xi) \) is unique under (A1) on area \( \theta = \frac{\pi}{2} \). Since \( A_0 \) and \( a_0 \) are arbitrarily chosen, we can repeat the process for all projectors and prove that \( y(\xi) \) is unique on the whole surface of sphere \( B \).

Then we prove that \( t = t' \) in such case. We choose an arbitrary measurement \( A_1 \), and denote its two results with \( a_+ \) and \( a_- \). Substituting (5) and (6) in the main text into equations (A1) and (A2), after simple calculations we can obtain that

\[
t - t' = \int_{R_+} [q(\xi) - q'(\xi)] \cdot \xi d\sigma,
\]

where \( R_+ \) is the hemisphere satisfying \( p(a_+ | A_1, \xi) = 1 \). We can also obtain that

\[
t - t' = \int_{R_-} [q(\xi) - q'(\xi)] \cdot \xi d\sigma,
\]

where \( R_- \) is the hemisphere satisfying \( p(a_- | A_1, \xi) = 1 \). Using former result we have

\[
q(\xi) - q(-\xi) = q'(\xi) - q'(-\xi) = y(\xi),
\]

so we can change the integrating area of (A12) into \( R_+ \), and (A12) becomes

\[
t - t' = \int_{R_+} [q(\xi) - q'(\xi)] \cdot (-\xi) d\sigma.
\]

Comparing (A11) and (A14) we have \( t - t' = 0 \), which implies that any models which realize probability conditions (A1) and (A2) have the same translation vectors \( t \).