THE QUOTIENT OF A KAUFFMAN BRACKET SKEIN ALGEBRA BY THE SQUARE OF AN AUGMENTATION IDEAL

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Abstract. We give an explicit basis \( B \) of the quotient of the Kauffman bracket skein algebra \( S(\Sigma) \) on a surface \( \Sigma \) by the square of an augmentation ideal. As an application, it induces two kinds of finite type invariants of links in a handle body in the sense of Le [6]. Moreover, we construct an embedding of the mapping class group of a compact connected surface of genus 0 into the Kauffman bracket skein algebra on the surface completed with respect to a filtration coming from the augmentation ideal.

1. Introduction

The discovery of the Jones polynomial made a remarkable development in knot theory. L. Kauffman gave an effective method to compute the Jones polynomial by the Kauffman bracket, which is an invariant of a framed unoriented link assigning a Laurent polynomial in a variable \( A \). The Kauffman bracket is computed by some skein relation. The Kauffman bracket module in a handle body defined by this skein relation is useful in the study of knots and links in the handle body. For example, see Lickorish [7], Prztycki [10] and our paper [11] section 5. Furthermore, recently, we find a new relationship between the study of the mapping class group of a surface and the Kauffman bracket skein modules in the product of the surface and the closed interval \([0,1]\). In particular, we obtain a formula of the action of an Dehn twist \( t_c \) on the completed skein module \( \hat{S}(\Sigma,J) \) in terms of the inverse function of hyperbolic cosine function

\[
\exp\left(\frac{-A + A^{-1}}{4\log(-A)} \left(\text{arccosh}\left(-\frac{c}{2}\right)\right)^2\right) = t_c(\cdot) : \hat{S}(\Sigma,J) \rightarrow \hat{S}(\Sigma,J).
\]

[11] Theorem 4.5. This formula is an analogy of a formula for the action of \( t_c \) on the completed group ring of the fundamental group of the surface [4], [5], [8].

To describe the relationship more precisely, we need to clarify the Kauffman bracket skein module. In this paper, we construct an embedding of the mapping class group of a compact connected oriented surface of genus 0 into the completed Kauffman bracket skein algebra on the surface in Theorem 6.3. In our succeeding paper [12], we also construct an embedding of the Torelli group of a compact connected oriented surface with non-empty connected boundary into the completed Kauffman bracket skein algebra on the surface.

In [11], we introduce a filtration of the skein module on a surface and its completion, in order to define the logarithm of a Dehn twist, the exponential of the action

2010 Mathematics Subject Classification. 57N05, 57M27.
Key words and phrases. Mapping class group, Kauffman bracket, braid group, Skein algebra.
of the skein algebra on the surface and the formula \([1]\). In this paper, we introduce another filtration of the skein algebra in order to consider the logarithm of other elements of the mapping class group of the surface. In particular, using this new filtration, we can define the logarithm of any element of the mapping class group of a surface of genus 0. See Theorem 6.3. In our subsequent papers, we consider the logarithm of any element of the Torelli group of a surface with non-empty connected boundary. In order to define the filtration, we give an explicit basis of the quotient of the Kauffman bracket skein algebra by the square of an augmentation ideal.

Let \(\Sigma\) be a compact connected oriented surface, \(I\) the closed interval \([0,1]\), \(S(\Sigma)\) the Kauffman bracket skein algebra on \(\Sigma\) and \(\epsilon\) the augmentation map defined by \(\epsilon(A + 1) = 0\) and \(\epsilon([L] - (-2)^{|L|}) = 0\). Here \(|L|\) is the number of the components of \(L\). For a link \(L\) in \(\Sigma \times I\), the element \((-2)^{-|L|}L \in S(\Sigma)/\ker(\epsilon)^n\) is a finite invariant of order \(n\) in the sense of Le \([6]\) (3.2). By \([11]\) Lemma 5.3, we actually have

\[
\sum_{L' \subset L} (-1)^{|L'|}(-2)^{-|L'|}|L'| \in (\ker \epsilon)^n
\]

for a link \(L\) having components more than \(n\), where the sum is over all sublinks \(L' \subset L\) including the empty link.

Now we assume \(\Sigma\) has a non-empty boundary. Then we introduce a family \(B\) of elements of \(S(\Sigma)/\ker(\epsilon)^2\) by

\[
B \overset{\text{def}}{=} \{1\} \cup \{A + 1\} \cup \{(x_i, x_j)|i \leq j\} \cup \{(x_i, x_j, x_k)|i < j < k\}
\]

where the fundamental group \(\pi_1(\Sigma)\) is freely generated by \(x_1, \ldots, x_M\). For details, see Lemma 3.6. In section 3, we prove the set \(B\) generates \(S(\Sigma)/\ker(\epsilon)^2\) as \(\mathbb{Q}\) vector space. In section 4, we prove the \(\mathbb{Q}\)-linear independence of the set \(B\). To prove the independence, we define a bilinear form of \(\vartheta((\cdot,\cdot)) : S(\Sigma) \times S(\Sigma) \to \mathbb{Q}[A^{\pm 1}]\). This bilinear form is non-degenerate, in other words, for any \(x \in S(\Sigma)\{0\}\), there exists \(y \in S(\Sigma)\) satisfying \(\vartheta(xy) \neq 0\). Furthermore, we have \(\vartheta((\ker \epsilon)^n(\ker \epsilon)^m) \in ((A + 1)^n)^{n+m}\). Using this basis, we obtain an explicit map

\[
\mathbb{Q}T(\Sigma) \to S(\Sigma)/\ker(\epsilon)^2 \simeq \mathbb{Q}B, L \mapsto (-2)^{-|L|}|L|
\]

which is a finite type invariant of order 2 for links in \(\Sigma \times I\), where \(T(\Sigma)\) is the set of unoriented framed link in \(\Sigma \times I\).

As an application of the basis, we introduce a new filtration \(\{F^nS(\Sigma)\}_{n \geq 0}\) satisfying \(F^{2n}S(\Sigma) = (\ker \epsilon)^n\), \(F^3S(\Sigma)/F^4S(\Sigma) = \mathbb{Q}\{(x_i, x_j, x_k)|i < j < k\}\) and \(F^{2n+1}S(\Sigma) = F^3S(\Sigma)F^{2n-2}S(\Sigma)\). We remark that

\[
F^2S(\Sigma)/F^3S(\Sigma) = \mathbb{Q} \oplus S^2(H_1(\Sigma, \mathbb{Q})),
\]

\[
F^3S(\Sigma)/F^4S(\Sigma) = \wedge^3(H_1(\Sigma, \mathbb{Q})),
\]

where we denote the second symmetric tensor of \(H_1(\Sigma, \mathbb{Q})\) by \(S^2(H_1(\Sigma, \mathbb{Q}))\) and the third exterior power of \(H_1(\Sigma, \mathbb{Q})\) by \(\wedge^3(H_1(\Sigma, \mathbb{Q}))\). This filtration is finer than \(\{(\ker \epsilon)^n\}_{n \geq 0}\). In fact, by Proposition 5.8 and Proposition 5.15, we have

\[
F^{2n+1}S(\Sigma)F^{2m+1}S(\Sigma) \subset F^{2n+2m+2}S(\Sigma),
\]

\[
[F^{2n+1}S(\Sigma), F^{2m+1}S(\Sigma)] \subset F^{2n+2m}S(\Sigma),
\]
The bracket $[ , ]$ is defined by $[x, y] = \frac{1}{A + 1} (xy - yx)$. Furthermore, by Corollary 5.7, we have
\begin{equation}
\vartheta(F^n S(\Sigma)F^n S(\Sigma)) \subset ((A + 1)^{\frac{n + m + 1}{2}}).
\end{equation}
where $\lfloor x \rfloor$ is the largest integer not greater than $x$ for $x \in \mathbb{Q}$. By the equation, $\vartheta$ induces $\vartheta_n : F^n S(\Sigma)/F^{n+1} S(\Sigma) \to ((A + 1)^n)/(A + 1)^{n+1} \simeq \mathbb{Q}$. By the proof of the independence of $S(\Sigma)/(\ker \epsilon)^2$, $\vartheta_2$ and $\vartheta_3$ are non-degenerate.

We define an evaluation map $ev : \mathbb{Q}^T(\Sigma) \to \text{Hom}_{\mathbb{Q}}(F^{2n-1} S(\Sigma), \mathbb{Q}[A^\pm])$ by
\[
(ev(L))(x) = \frac{1}{(A + 1)^n} \vartheta((-2)^{-L}[L], x).
\]
Using the equation (2), the $\mathbb{Q}$-linear map $ev$ induces a finite type invariant $ev : \mathbb{Q}^T(\Sigma) \to \text{Hom}_{\mathbb{Q}}(F^{2n-1} S(\Sigma), \mathbb{Q}[A^\pm]/(A + 1)^m)$ of order $m$.

Let $\Sigma$ be a compact connected oriented of genus 0. By Lemma 6.1, we can consider the Baker-Campbell-Hausdorff series on $\hat{S}(\Sigma)$. In section 6, as an application of this filtration, we construct an embedding $\zeta$ of the mapping class group $\mathcal{M}(\Sigma)$ of $\Sigma$ into $\hat{S}(\Sigma)$ with respect to a group law using the Baker-Campbell-Hausdorff series. Furthermore we have
\[
\xi(\cdot) = \exp(\sigma(\xi(\cdot))) : \hat{S}(\Sigma, J) \to S(\Sigma, J)
\]
for any $\xi \in \mathcal{M}(\Sigma)$ and $J$ the finite subset of $\partial \Sigma$.

Acknowledgment

The author would like to thank his adviser, Nariya Kawazumi, for helpful discussion and encouragement. He is also grateful to Kazuo Habiro, Gwénaël Massuyeau, Jun Murakami, Tomotada Ohtsuki and Masatoshi Sato for helpful comments. In particular, the author thanks to Jun Murakami for valuable comments about the theory of 3-dimensional manifolds, knots and links. This work was supported by JSPS KAKENHI Grant Number 15J05288 and the Leading Graduate Course for Frontiers of Mathematical Sciences and Physic.

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2. Definition and Review

We review some definitions and facts about the Kauffman bracket skein algebra on a surface, for details, see \[11\].

Through this section, let $\Sigma$ be a compact connected surface and $I$ the closed interval $[0, 1]$.

2.1. Kauffman bracket skein algebras and modules. Let $J$ be a finite subset of $\partial \Sigma$. We denote by $T(\Sigma, J)$ the set of unoriented framed tangles in $\Sigma \times I$ with base point set $J$ \[11\] Definition 2.1. For a tangle diagram $d$, we denote by $T(d)$ the tangle presented by $d$. The Kauffman bracket skein module $S(\Sigma, J)$ is the quotient of $\mathbb{Q}[A, A^{-1}]T(\Sigma, J)$ by the skein relation and the trivial knot relation \[11\] Definition 3.2. The skein relation is

$$T(d_1) = AT(d_\infty) + A^{-1}T(d_0)$$

where $d_1$, $d_\infty$, and $d_0$ are differ only in an open disk shown in Figure 1, Figure 3 and Figure 4, respectively. The trivial knot relation is

$$T(d) = (-A^2 - A^{-2})T(d')$$

where $d$ and $d'$ are differ only in an open disk shown a boundary of a disk and empty, respectively. We remark that we don’t assume ”the boundary skein relation” and ”the value of a contractible arc” in Muller \[9\]. We write simply $S(\Sigma)$ $\overset{\text{def}}{=} S(\Sigma, \emptyset)$. The element of $S(\Sigma, J)$ represented by $T \in T(\Sigma, J)$ is written $[T]$. We remark that we have

$$[T(d_1)] - [T(d_2)] = (A - A^{-1})([T(d_\infty)] - [T(d_0)]),$$

where $d_1$, $d_2$, $d_\infty$, and $d_0$ are differ only in an open disk shown in Figure 1, Figure 2, Figure 3 and Figure 4, respectively.

![Fig 1. $d_1$](image1.png) ![Fig 2. $d_2$](image2.png) ![Fig 3. $d_\infty$](image3.png) ![Fig 4. $d_0$](image4.png)

We denote by $\mathcal{M}(\Sigma)$ the mapping class group of $\Sigma$ fixing the boundary pointwise. There is a natural action of $\mathcal{M}(\Sigma)$ on $S(\Sigma, J)$ \[11\] section 2. The product of $S(\Sigma)$ and the right and left actions of $S(\Sigma)$ on $S(\Sigma, J)$ are defined by Figure 5 for details,
3.1. The Lie bracket \([,] : S(\Sigma) \times S(\Sigma) \to S(\Sigma)\) is defined by \([x, y] \overset{\text{def.}}{=} \frac{1}{A + A^{-1}}(xy - yx)\). Furthermore, the action \(\sigma() : S(\Sigma) \times S(\Sigma, J) \to S(\Sigma, J)\) defined by \(\sigma(x)(z) \overset{\text{def.}}{=} \frac{1}{A + A^{-1}}(xz - zx)\) making \(\mathcal{S}(\Sigma, J)\) a \((S(\Sigma), [,])\)-module with \(\sigma\). For details, see [11] 3.2.

By [11] Theorem 3.3, we have the following proposition.

**Proposition 2.1.** Let \(J\) be a finite subset of \(\partial \Sigma\) and \(\partial_1, \cdots, \partial_b\) the connected components of \(\Sigma\). If \(b \geq 1\) and \(\partial_i \cap J \neq \emptyset\) for each \(i \in \{1, 2, \cdots, b\}\), then \(\mathcal{M}(\Sigma) \to \text{Aut}(\mathcal{S}(\Sigma, J))\) is injective.

2.2. **Filtrations and completions of \(S(\Sigma)\) and \(S(\Sigma, J)\).** The augmentation map \(\epsilon : S(\Sigma) \to \mathbb{Q}\) is defined by \(A \mapsto -1\) and \([L] \mapsto (-2)^{|L|}\) for \(L \in \mathcal{T}(\Sigma)\) where \(|L|\) is the number of components of \(L\). The augmentation map \(\epsilon\) is well-defined by [11] Proposition 3.10. We consider the topology on \(S(\Sigma)\) induced by the filtration \(\{(\ker \epsilon^n)\}_{n \geq 0}\) and denote its completion by \(\overline{S(\Sigma)} \overset{\text{def.}}{=} \varprojlim_{i \to \infty} S(\Sigma)/\left(\ker \epsilon\right)^i\).

Let \(J\) be a finite subset of \(\partial \Sigma\). We also consider the topology on \(S(\Sigma, J)\) induced by the filtration \(\{(\ker \epsilon^n S(\Sigma, J))\}_{n \geq 0}\) and denote its completion by \(\overline{S(\Sigma, J)} \overset{\text{def.}}{=} \varprojlim_{i \to \infty} S(\Sigma, J)/\left(\ker \epsilon\right)^i S(\Sigma, J)\). The product of \(S(\Sigma)\), the right action and the left action of \(S(\Sigma)\) on \(S(\Sigma, J)\), the bracket \([,\) of \(S(\Sigma)\) and the action \(\sigma\) of \(S(\Sigma)\) on \(S(\Sigma, J)\) are continuous. For details, see [11] Theorem 3.12.

**Proposition 2.2** ([11] Theorem 5.1). Let \(\Sigma\) and \(\Sigma'\) be two compact connected oriented surfaces, \(J\) and \(J'\) finite subsets of \(\partial \Sigma\) and \(\partial \Sigma'\), respectively. We assume there exists an orientation preserving diffeomorphism \(\mathcal{X} : (\Sigma \times I, J \times I) \to (\Sigma' \times I, J' \times I)\). Then we have \(\mathcal{X}(\ker \epsilon^n S(\Sigma, J)) = (\ker \epsilon^n S(\Sigma', J'))\) for each \(n\).

**Proposition 2.3** ([11] Theorem 5.5). Let \(\Sigma\) be a compact connected oriented surface and \(J\) a finite subset of \(\partial \Sigma\). If \(\partial \Sigma \neq \emptyset\), then the natural homomorphism \(S(\Sigma, J) \to \overline{S(\Sigma, J)}\) is injective.

By this theorem and Proposition [2.1], we have the following.

**Corollary 2.4.** Let \(J\) be a finite subset of \(\partial \Sigma\) and \(\partial_1, \cdots, \partial_b\) the connected components of \(\partial \Sigma\). If \(b \geq 1\) and \(\partial_i \cap J \neq \emptyset\) for each \(i \in \{1, 2, \cdots, b\}\), then \(\mathcal{M}(\Sigma) \to \text{Aut}(\mathcal{S}(\Sigma, J))\) is injective.

For a simple closed curve \(c\), we denote

\[
L(c) \overset{\text{def.}}{=} \frac{-A + A^{-1}}{4\log(-A)}(\text{arccosh}(-\frac{c}{2}))^2
\]
where \( c \) is also denoted by the element of \( S(\Sigma) \) represented by the knot presented by the simple closed curve \( c \).

**Theorem 2.5** ([11] Theorem 4.1). Let \( J \) be a finite subset of \( \partial \Sigma \), \( c \) a simple closed curve and \( t_c \) the Dehn twist along \( c \). Then we have

\[
t_c(\cdot) = \exp(\sigma(L(c))) \overset{\text{def}}{=} \sum_{i=0}^{\infty} \frac{1}{\Pi} (\sigma(L(c)))^i \in \text{Aut}(S(\Sigma, J)).
\]

3. The calculations of \( S(\Sigma)/(\ker \epsilon)^2 \)

Let \( \Sigma \) be a compact connected oriented surface as in section [2]

3.1. the unoriented Goldman Lie algebra. We recall the Goldman Lie algebra.

We denote by \( \hat{\pi}(\Sigma) = [S^1, \Sigma] \) the homotopy set of oriented free loops on \( \Sigma \). In other words, \( \hat{\pi}(\Sigma) \) is the set of conjugacy classes of \( \pi_1(\Sigma) \).

Let \( \Sigma \) be a compact connected oriented surface. Let \( \alpha \) and \( \beta \) be oriented im-

mersed loops on \( \Sigma \) such that their intersections consist of transverse double points.

For each \( p \in \alpha \cap \beta \), let \( \alpha_p \beta_p \in \pi_1(\Sigma, p) \) be the loop going first along the loop \( \alpha \) based at \( p \), then going along \( \beta \) based at \( p \). Also, let \( \epsilon(\alpha, \beta) \in \{1, -1\} \) be the local intersection number of \( \alpha \) and \( \beta \) at \( p \). The Goldman bracket of \( \alpha \) and \( \beta \) is defined as

\[
[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \epsilon(\alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Q} \hat{\pi}(\Sigma).
\]

Here we denote by \(|\cdot| : \pi_1(\Sigma) \to \hat{\pi}(\Sigma) \) the natural projection, and we also denote by \(|\cdot| : \mathbb{Q}\pi_1(\Sigma) \to \mathbb{Q}\hat{\pi}(\Sigma) \) its \( \mathbb{Q} \)-linear extension. The free \( \mathbb{Q} \)-vector space \( \mathbb{Q}\hat{\pi}(\Sigma) \) spanned by the set \( \hat{\pi}(\Sigma) \) equipped with this bracket is a Lie algebra. See [2].

According to [2] there is a Lie algebra similar to \( \mathbb{Q}\hat{\pi}(\Sigma) \) but based on homotopy classes of unoriented loops. The map \( \hat{\pi}(\Sigma) \to \hat{\pi}(\Sigma), |a| \to |a^{-1}| \) which reverses the orientation of oriented loops extends to a Lie algebra automorphism \( \mathbb{Q}\hat{\pi}(\Sigma) \to \mathbb{Q}\hat{\pi}(\Sigma) \) denoted by \( \nu \). Clearly, \((\nu + \text{id})(\mathbb{Q}\hat{\pi}(\Sigma))\) is a Lie subalgebra of \( \mathbb{Q}\hat{\pi}(\Sigma) \) which is a free module over the set of all \( |a| + \nu(|a|) \overset{\text{def}}{=} (a) \bigcirc, a \in \pi_1(\Sigma) \). The following formula given in [2] computes the bracket of generators \((a) \bigcirc \) and \((b) \bigcirc \)

\[
[(a) \bigcirc, (b) \bigcirc] = \sum_{p \in |a| \cap |b|} \epsilon(\nu, |b|(a_p b_p) \bigcirc - (a_p b_p^{-1}), (b) \bigcirc).
\]

We denote simply \( \{r \bigcirc | r \in \pi_1(\Sigma)\} \) and \((\nu + \text{id})(\mathbb{Q}\hat{\pi}(\Sigma))\) by \( \mathbb{Q}\pi(\Sigma) \) and \( \mathbb{Q}\pi(\Sigma) \).

3.2. The homomorphism \( \kappa : \mathbb{Q}\pi(\Sigma) \to (\ker \epsilon)/(\ker \epsilon)^2 \). In this subsection, we construct a Lie algebra homomorphism \( \kappa : \mathbb{Q}\pi(\Sigma) \to (\ker \epsilon)/(\ker \epsilon)^2 \). This is an analogy of [13] Theorem 3.3.

For \( x \in \pi_1(\Sigma) \), we define \( x \in (\ker \epsilon)/(\ker \epsilon)^2 \) by \( [L_x] + 2 - 3w(L_x)(A - A^{-1}) \) using \( L_x \in T(\Sigma) \) with \( p_1(L_x) = |x| \) where the writhe \( w(L_x) \) is the sum of the signs of the crossing of a diagram presenting \( L_x \).

**Lemma 3.1.** The map \( \langle \cdot \rangle : \pi_1(\Sigma) \to (\ker \epsilon)/(\ker \epsilon)^2 \) is well-defined.

**Proof.** Let \( d_1, d_2, d_3 \) and \( d_4 \) be four link diagrams in \( \Sigma \) which are differ only in an open disk shown in Figure [1] Figure [2] Figure [3] and Figure [4] respectively such
that $|T(d_1)| = |T(d_2)| = |T(d_\infty)| = 1$ and $|T(d_0)| = 2$. We denote $L_i \overset{\text{def.}}{=} T(d_i)$ for $i \in \{1, 2, 0, \infty\}$. We have $w(L_1) - w(L_2) = 2$. By the equation \[3\], we have
\[
([L_1] + 2 - 3w(L_1)(A - A^{-1})) - ([L_2] + 2 - 3w(L_2)(A - A^{-1}))
= (A - A^{-1})([L_\infty] - [L_0]) - 3(w(L_1) - w(L_2))(A - A^{-1})
= (A - A^{-1})([L_\infty] - [L_0] - 6).
\]
Since $\epsilon(A - A^{-1}) = \epsilon([L_\infty] - [L_0] - 6) = 0$, we have $([L_1] + 2 - 3w(L_1)(A - A^{-1})) - ([L_2] + 2 - 3w(L_2)(A - A^{-1})) \in (\ker \epsilon)^2$.

Let $d$ and $d'$ be two knot diagrams in $\Sigma$ which are differ only in an open disk shown in the figure. We denote $L \overset{\text{def.}}{=} T(d)$ and $L' \overset{\text{def.}}{=} T(d')$.

Then $w(L) - w(L') = -1$, and so
\[
([L] + 2 - 3w(L)(A - A^{-1})) - ([L'] + 2 - 3w(L')(A - A^{-1}))
= (A^3 + 1)([L'] - 3(w(L_1) - w(L_2))(A - A^{-1})
= (A + 1)((A^2 - A + 1)[L'] + 3(1 - A^{-1})).
\]
Since $\epsilon(A + 1) = \epsilon((A^2 - A + 1)[L'] + 3(1 - A^{-1})) = 0$, we have $([L] + 2 - 3w(L)(A - A^{-1})) - ([L'] + 2 - 3w(L')(A - A^{-1})) \in (\ker \epsilon)^2$.

This finishes the proof.

Here we remark that $(1) = 0$ for the identity $1 \in \pi_1(\Sigma)$ and $\langle x \rangle = \langle yxy^{-1} \rangle$ for $x, y \in \pi_1(\Sigma)$. We also denote the $\mathbb{Q}$-linear extension of the map $\langle \cdot \rangle$ by $\langle \cdot \rangle : \mathbb{Q}\pi_1(\Sigma) \rightarrow \ker \epsilon/(\ker \epsilon)^2$.

Since $[\ker \epsilon, (\ker \epsilon)^2] \subset (\ker \epsilon)^2$ and $[(\ker \epsilon)^2, \ker \epsilon] \subset (\ker \epsilon)^2$ \[11\] Lemma 3.11, $[] : \mathcal{S}(\Sigma) \times \mathcal{S}(\Sigma) \rightarrow \mathcal{S}(\Sigma)$ induces $[] : (\ker \epsilon)/(\ker \epsilon)^2 \times (\ker \epsilon)/(\ker \epsilon)^2 \rightarrow (\ker \epsilon)/(\ker \epsilon)^2$.

**Theorem 3.2.** This $\mathbb{Q}$-linear map $\kappa : \mathbb{Q}\pi_1(\Sigma) \rightarrow (\ker \epsilon)/(\ker \epsilon)^2, (x) \mapsto -\langle x \rangle$ is a Lie algebra homomorphism.

**Proof.** Let $d_a$ and $d_b$ be two knot diagrams in $\Sigma$ whose intersections consist of transverse double points $P_1, P_2, \cdots, P_m$. We fix orientations of $d_a$ and $d_b$. Let $\alpha$ and $\beta$ be two elements of $[S^1, \Sigma]$ such that $\alpha = p_1(d_a)$ and $\beta = p_1(d_b)$. We write simply $\epsilon_i \overset{\text{def.}}{=} \epsilon(P_i, d_a, d_b)$ for $i = 1, 2, \ldots, m$. Let $\alpha_s$ and $\beta_s$ be two elements of $\pi_1(\Sigma)$ such that $|\alpha_s| = \alpha$ and $|\beta_s| = \beta$.

For $i = 1, 2, \cdots, m$, let $d(1, i)$ and $d(-1, i)$ be two tangle diagrams satisfying the following conditions.

- The two tangle diagrams $d(1, i)$ and $d(-1, i)$ equal $d_a \cup d_b$ with the same height-information as $d_a$ and $d_b$ except for the neighborhoods of the intersections of $d_a$ and $d_b$.
- The branches of $d(1, i)$ and $d(-1, i)$ in the neighborhood of $P_j$ belonging to $d_a$ are over crossings for $j = 1, \cdots, i - 1$. 

\[\begin{align*}
L & = T(d) = \langle d \rangle - \langle d \rangle \\
L' & = T(d') = \langle d' \rangle - \langle d' \rangle \end{align*}\]
• The branches of \(d(1,i)\) and \(d(-1,i)\) in the neighborhood of \(P_j\) belonging to \(d_a\) are over crossings for \(j = i + 1, \cdots, m\).

• The two tangle diagrams \(d(1,i)\) and \(d(-1,i)\) are as shown in Figure 6 and Figure 7, respectively, in the neighborhood of \(P_i\).

\[
\begin{align*}
\text{Fig 6.} & \quad \text{Fig 7.}
\end{align*}
\]

We denote \(T(1,i) \overset{\text{def}}{=} T(d(1,i))\) and \(T(-1,i) \overset{\text{def}}{=} T(d(-1,i))\) for \(i = 1, \cdots, m\). Using the equation (3), we have

\[
[[T_a], [T_b]] = -\sum_{i=1}^{m} ([T(1,i)] - [T(-1,i)]).
\]

If \(\epsilon_i = 1\), we have

\[
\begin{align*}
\langle \alpha_{P_i} \beta_{P_i} \rangle &= [T(1,i)] + 2 - 3(w(T) + w(T') - \epsilon_1 - \cdots - \epsilon_{i-1} + \epsilon_{i+1} + \cdots + \epsilon_m)(A - A^{-1}), \\
\langle \alpha_{P_i} \beta_{P_i}^{-1} \rangle &= [T(-1,i)] + 2 - 3(w(T) + w(T') + \epsilon_1 + \cdots + \epsilon_{i-1} - \epsilon_{i+1} - \cdots - \epsilon_m)(A - A^{-1}).
\end{align*}
\]

If \(\epsilon_i = -1\), we have

\[
\begin{align*}
\langle \alpha_{P_i} \beta_{P_i}^{-1} \rangle &= [T(1,i)] + 2 - 3(w(T) + w(T') + \epsilon_1 + \cdots + \epsilon_{i-1} - \epsilon_{i+1} - \cdots - \epsilon_m), \\
\langle \alpha_{P_i} \beta_{P_i} \rangle &= [T(-1,i)] + 2 - 3(w(T) + w(T') - \epsilon_1 - \cdots - \epsilon_{i-1} + \epsilon_{i+1} + \cdots + \epsilon_m).
\end{align*}
\]
Hence, we have
\[
-\langle \alpha_*, \beta_* \rangle
= [T_a + 2 - 3w(T_a)(A - A^{-1}), T_b + 2 - 3w(T_b)(A - A^{-1})]
= [T_a, T_b]
= -\sum_{i=1}^{m} ([T(1, i)] - [T(-1, i)])
\]
\[
= -\sum_{i=1}^{m} \epsilon_i((\alpha_{P_i} \beta_{P_i}) - (\alpha_{P_i} \beta_{P_i}^{-1}) + 6(-\epsilon_1 - \cdots - \epsilon_{i-1} + \epsilon_{i+1} + \cdots + \epsilon_m)(A - A^{-1}))
\]
\[
= -\sum_{i=1}^{m} \epsilon_i((\alpha_{P_i} \beta_{P_i}) - (\alpha_{P_i} \beta_{P_i}^{-1})) + 6(A - A^{-1}) \sum_{i<j} (\epsilon_i \epsilon_j - \epsilon_j \epsilon_i)
\]
\[
= -\sum_{i=1}^{m} \epsilon_i((\alpha_{P_i} \beta_{P_i}) - (\alpha_{P_i} \beta_{P_i}^{-1})).
\]
This finishes the proof. \(\square\)

The following proposition plays a key role in our calculations of \(\mathcal{S}(\Sigma) / (\ker \epsilon)^2\).

**Proposition 3.3.** For \(x \) and \(y \) \(\in \pi_1(\Sigma)\), we have
\[
2\langle x \rangle + 2\langle y \rangle = \langle xy \rangle + \langle xy^{-1} \rangle.
\]

**Proof.** We fix the base point \(* \) of \(\pi_1(\Sigma)\) in \(\Sigma \setminus \partial \Sigma\). Let \(d_a\) and \(d_b\) be knot diagrams in \(\Sigma\) whose intersections consist of transverse double points and which transverse at \(*\) as shown in Figure 8.

Let \(d_{ab}\) and \(d_{ab}^{-1}\) be two knot diagrams satisfying the following conditions.

- The two knot diagrams \(d_{ab}\) and \(d_{ab}^{-1}\) equal \(d_a \cup d_b\) with the same height-information as \(d_a\) and \(d_b\) except for the neighborhood of the intersections of \(d_a\) and \(d_b\).
- The branches of \(d_{ab}\) and \(d_{ab}^{-1}\) in the neighborhood of any point of \(d_a \cap d_b \setminus \{\ast\}\) belonging to \(d_a\) are over crossings.
- The two tangle diagrams \(d_{ab}\) and \(d_{ab}^{-1}\) are as shown in Figure 9 and Figure 10, respectively.

We denote \(T_i \overset{\text{def}}{=} T(d_i)\) for \(i \in \{a, b, ab, ab^{-1}\}\). We remark that \(w(T_{ab}) + w(T_{ab}^{-1}) = 2(w(T_a) + w(T_b))\).

Hence we have
\[( [T_a] + 2 ) ( [T_b] + 2 ) \]
\[= [T_a][T_b] + 2[T_a] + 2[T_b] + 4 \]
\[= A[T_{ab}] + A^{-1}[T_{ab^{-1}}] + 2[T_a] + 2[T_b] + 4 \]
\[= (A + 1)([T_{ab}] + A^{-1}[T_{ab^{-1}}]) - [T_{ab}] - [T_{ab^{-1}}] + 2[T_a] + 2[T_b] + 4 \]
\[= (A + 1)([T_{ab}] + A^{-1}[T_{ab^{-1}}]) \]
\[= ([T_{ab}] + 2 - 3(A - A^{-1})w(T_{ab})) - ([T_{ab^{-1}}] + 2 - 3(A - A^{-1})w(T_{ab^{-1}})) \]
\[+ 2([T_a] + 2 - 3(A - A^{-1})w(T_a)) + 2([T_b] + 2 - 3(A - A^{-1})w(T_b)) \]

This proves the proposition. \[\square\]

### 3.3. The calculation lemma
In this subsection, we check some equations.

**Lemma 3.4.** For \( a, b, c \) and \( d \in \pi_1(\Sigma) \), we have

\[
\langle (a - 1)(b - 1)(c - 1) \rangle = \langle -(b - 1)(a - 1)(c - 1) \rangle,
\]
\[
\langle (a - 1)(b - 1)(c - 1)(d - 1) \rangle = 0,
\]
\[
\langle (a - 1)(b - 1)(c - 1) \rangle = 0,
\]
\[
\langle (a - 1)(b - 1)(c - 1)(d - 1) \rangle = 0.
\]

**Proof.** We have

\[
\langle abc \rangle = \langle -b^{-1}a^{-1}c + 2ab + 2c \rangle = \langle b^{-1}ac - 2b^{-1}c - 2a + 2ab + 2c \rangle = \langle -bac + 2ac + 2b + 2bc - 4b - 4c - 2a + 2ab + 2c \rangle = \langle -bac + 2ab + 2bc + 2ca - 2a - 2b - 2c \rangle.
\]

Hence we have \( \langle (a - 1)(b - 1)(c - 1) \rangle = \langle -(b - 1)(a - 1)(c - 1) \rangle \). This proves the equation (5). By this equation, we have

\[
\langle (a - 1)(b - 1)(c - 1)(d - 1) \rangle = -\langle (a - 1)(b - 1)(c - 1)(d - 1) \rangle = \langle (a - 1)(b - 1)(c - 1)(d - 1) \rangle = -\langle (a - 1)(b - 1)(c - 1)(d - 1) \rangle.
\]

This proves the equation (6). By equation (5), we have

\[
\langle (a - 1)(b - 1)(b - 1) \rangle = -\langle (a - 1)(b - 1)(b - 1) \rangle.
\]

This proves the equation (7). By the equation (5), the equation (6) and the equation (7), we have
\[ \langle [a,b]c - c \rangle = \langle (ab - ba)(a^{-1}b^{-1}c - 1) + ab - ba \rangle = \langle (a - 1)(b - 1) - (b - 1)(a - 1)(a^{-1}b^{-1}c - 1) \rangle = 2(a - 1)(b - 1)(a^{-1}b^{-1}c - 1) \]

This proves the equation (8). We have

\[ \langle a^2 - 2a + 1 \rangle = \langle -1 + 2a + 2a - 2a + 1 \rangle = 2\langle a \rangle. \]

This proves the equation (9). This finishes the proof. \hfill \Box

We denote \( H_1 \overset{\text{def}}{=} H_1(\Sigma, \mathbb{Q}) = \mathbb{Q} \otimes \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)] \).

**Definition 3.5.** A \( \mathbb{Q} \)-linear map \( \lambda : H_1 \wedge H_1 \wedge H_1 \to \ker \epsilon/(\ker \epsilon)^2 \) is defined by

\[ [a] \wedge [b] \wedge [c] \overset{\text{def}}{=} \langle (a - 1)(b - 1)(c - 1) \rangle \]

for \( a, b, c \in \pi_1(\Sigma) \) where we denote the third exterior of \( H_1 \) by \( H_1 \wedge H_1 \wedge H_1 \).

By the equation (5) and the equation (6), we have \( \lambda \) is well-defined.

Since the \( \mathbb{Q} \) vector space \( S(\Sigma)/(\ker \epsilon)^2 \) is generated by \( \{1, A + 1\} \cup \kappa(\mathbb{Q} \boxtimes (\Sigma)) \) as a \( \mathbb{Q} \) vector space, we have the following by teh equation (5), the equation (6), the equation (8), and the equation (9).

**Lemma 3.6.** We assume \( \pi_1(\Sigma) \) is generated by \( \{x_1, x_2, \cdots, x_M\} \). Then \( S(\Sigma)/(\ker \epsilon)^2 \) is generated by

\[ \{1\} \cup \{A + 1\} \cup \{\langle x_i, x_j \rangle|i \leq j\} \cup \{\langle x_i, x_j, x_k \rangle|i < j < k\} \]

as a \( \mathbb{Q} \) vector space. Here we denote \( \langle x_i, x_j \rangle \overset{\text{def}}{=} \langle (x_i - 1)(x_j - 1) \rangle \) and \( \langle x_i, x_j, x_k \rangle \overset{\text{def}}{=} \langle (x_i - 1)(x_j - 1)(x_k - 1) \rangle \).

4. A Basis of \( S(\Sigma)/(\ker \epsilon)^2 \)

In this section, we will prove the following.

**Theorem 4.1.** Let \( \Sigma \) be a connected compact oriented surface with non-empty boundary. We assume \( \pi_1(\Sigma) \) is freely generated by \( \{x_1, x_2, \cdots, x_M\} \). Then

\[ \{1\} \cup \{A + 1\} \cup \{\langle x_i, x_j \rangle|i \leq j\} \cup \{\langle x_i, x_j, x_k \rangle|i < j < k\} \]

is a basis of \( S(\Sigma)/(\ker \epsilon)^2 \) as a \( \mathbb{Q} \) vector space.

To prove this theorem, we introduce a bilinear form of the Kauffman bracket skein algebra. Let \( \xi \) be an element of the mapping class group \( \mathcal{M}(\Sigma) \) represented by a diffeomorphism \( \chi \) such that \( S^3 \simeq \Sigma \times I/\sim_\xi \), where the equivalence relation \( \sim_\xi \) is generated by \( (x, 0) \sim_\xi (\chi(x), 1) \) for \( x \in \Sigma \) and \( (x, t) \sim_\xi (x, 1 - t) \) for \( x \in \partial \Sigma \) and \( t \in I \). We remark that for any compact connected oriented surface \( \Sigma \) with non-empty boundary there exists an element of \( \mathcal{M}(\Sigma) \) satisfying the above condition.
The embedding $e_\xi : \Sigma \times I \to S^3 : (x,t) \mapsto (x, \frac{t}{2})$ induces a $\mathbb{Q}[A^{\pm 1}]$-module homomorphism $\vartheta_\xi : S(\Sigma) \to \mathbb{Q}[A^{\pm 1}] : [L] \mapsto K(e_\xi(L))$ where $K$ is Kauffman bracket. In this paper, the Kauffman bracket is defined by $[L] = K(L)[\emptyset]$ for $L \in \{\text{the set of unoriented framed links in } S^3\} = T(I \times I)$. We remark that $\vartheta_\xi(xy) = \vartheta_\xi(xy)$. By [11] Lemma 5.7 and Lemma 5.8, we have the followings.

**Proposition 4.2.** For any $x \in S(\Sigma) \setminus \{0\}$, there exists $y$ such that $\vartheta_\xi(xy) \neq 0$.

**Lemma 4.3.** We have $\vartheta_\xi((\ker \epsilon)^N) \subset ((A + 1)^N)\mathbb{Q}[A, A^{-1}]$ for any $N$.

Let $\Sigma$ be a connected compact oriented surface of genus 0 with $b + 1$ boundary components. Let $r_i$ be an element of $\pi_1(\Sigma)$ as in Figure 12 for $1 \leq i \leq b$ and $c_{i_1 i_2 \cdots i_j}$ a simple closed curve presenting by $[r_{i_1} \cdots r_{i_j}]$ for $1 \leq i_1 < i_2 < \cdots < i_j \leq b$. We remark that $\Sigma \times I/ \sim_{t_{i_1} t_{i_2} \cdots t_{i_j}} \simeq S^3$ and that the embedding $e_{t_{i_1} t_{i_2} \cdots t_{i_j}}$ is as in Figure 11. We simply denote $\vartheta \overset{\text{def}}{=} \vartheta_{t_{i_1} t_{i_2} \cdots t_{i_j}}$. We remark that $\vartheta(xy) = \vartheta(yx)$.

We denote by $c$ an element represented by a tangle presented by simple closed curve $c \in \{c_{i_1 i_2 \cdots i_j} \mid 1 \leq i_1 < i_2 < \cdots < i_j \leq b\}$, $\langle i, j \rangle \overset{\text{def}}{=} 2(c_i - c_\emptyset)$ for $1 \leq i \leq b$, $\langle i, j \rangle \overset{\text{def}}{=} 2(c_{i_1 i_2 \cdots i_j} - 1)$ for $1 \leq i_1 < i_2 < \cdots < i_j \leq b$. 

![Fig 11. $e : \Sigma \times I \to S^3$](image1)

![Fig 12. $r_i$](image2)
for \(1 \leq i < j \leq b\) and \(\langle i, j, k \rangle \) def \( = c_{ijk} - c_{ij} - c_{jk} - c_{ik} + c_i + c_j + c_k - c_0\) for \(1 \leq i < j < k \leq b\) where \(c_0 \) def \(= -A^2 - A^{-2}\). We need some calculations of \(\vartheta\).

Lemma 4.4. \(1\) Let \(i_1, i_2\) be elements of \(\{1, \cdots, b\}\). We have

\[
\vartheta((i_1, i_1)\langle i_2, i_2\rangle)
\begin{cases}
4(A^{12} + A^8 + 2A^7 + 2A^4 + 4A^3 + 3 + 2A^{-1} + A^{-4}) & i_1 = i_2 \\
4(A^3 + 1)^2(A^2 + A^{-2})^2 & i_1 \neq i_2,
\end{cases}
\]

and

\[
\vartheta((i_1, i_1)\langle i_2, i_2\rangle) = \begin{cases}
240(A + 1)^2 \mod ((A + 1)^3) & i_1 = i_2 \\
144(A + 1)^2 \mod ((A + 1)^3) & i_1 \neq i_2.
\end{cases}
\]

\(2\) Let \(i_1, j_1, i_2\) be elements of \(\{1, \cdots, b\}\) satisfying \(i_1 < j_1\). We have

\[
\vartheta((i_1, j_1)\langle i_2, i_2\rangle)
\begin{cases}
-2(A^3 + 1)(A^{12} + A^8 + 2A^7 + 2A^4 + 4A^3 + 3 + 2A^{-1} + A^{-4}) & i_2 \in \{i_1, j_1\} \\
-2(A^3 + 1)^3(A^2 + A^{-2})^2 & i_2 \notin \{i_1, j_1\},
\end{cases}
\]

and

\[
\vartheta((i_1, j_1)\langle i_2, i_2\rangle) = 0 \mod ((A + 1)^3).
\]

\(3\) Let \(i_1, j_1, k_1, i_2\) be elements of \(\{1, \cdots, b\}\) satisfying \(i_1 < j_1 < k_1\). We have

\[
\vartheta((i_1, j_1)\langle i_2, i_2\rangle)
\begin{cases}
2(A^3 + 1)^2(A^{12} + A^8 + 2A^7 + 2A^4 + 4A^3 + 3 + 2A^{-1} + A^{-4}) & i_2 \in \{i_1, j_1, k_1\} \\
2(A^3 + 1)^4(A^2 + A^{-2})^2 & i_2 \notin \{i_1, j_1, k_1\},
\end{cases}
\]

and

\[
\vartheta((i_1, j_1)\langle i_2, i_2\rangle) = 0 \mod ((A + 1)^4).
\]

\(4\) Let \(i_1, j_1, i_2, j_2\) be elements of \(\{1, \cdots, b\}\) satisfying \(i_1 < j_1\) and \(i_2 < j_2\). We have

\[
\vartheta((i_1, j_1)\langle i_2, i_2\rangle)
\begin{cases}
(A^3 + 1)^4(A^2 + A^{-2})^2 & \sharp((i_1, j_1) \cap \{i_2, j_2\}) = 0 \\
(A^3 + 1)^2(A^{12} + A^8 + 2A^7 + 2A^4 + 4A^3 + 3 + 2A^{-1} + A^{-4}) & \sharp((i_1, j_1) \cap \{i_2, j_2\}) = 1 \\
A^{20} + A^{16} + 4A^{15} + 3A^{12} + 4A^{11} + 4A^{10} + 2A^8 + 8A^7 + 8A^6 + 3A^4 + 12A^3 + 4A^2 + 5 + 4A^{-1} + A^{-4} & \sharp((i_1, j_1) \cap \{i_2, j_2\}) = 2,
\end{cases}
\]

and

\[
\vartheta((i_1, i_1)\langle i_2, i_2\rangle) = \begin{cases}
0 \mod ((A + 1)^3) & \sharp((i_1, j_1) \cap \{i_2, j_2\}) = 0, 1 \\
48(A + 1)^2 \mod ((A + 1)^3) & \sharp((i_1, j_1) \cap \{i_2, j_2\}) = 2.
\end{cases}
\]
(5) Let \(i_1, j_1, k_1, i_2, j_2\) be elements of \(\{1, \ldots, b\}\) satisfying \(i_1 < j_1 < k_1\) and \(i_2 < j_2\). We have

\[
\vartheta((i_1, j_1, k_1) \langle i_2, j_2 \rangle) = (-A^3 + 1)^5(A^2 + A^{-2})^2 \mod ((A + 1)^3)
\]

\[
\vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 0 \mod ((A + 1)^3)
\]

(6) Let \(i_1, j_1, k_1, i_2, j_2, k_2\) be elements of \(\{1, \ldots, b\}\) satisfying \(i_1 < j_1 < k_1\) and \(i_2 < j_2 < k_2\). We have

\[
\vartheta((i_1, j_1, k_1) \langle i_2, j_2, k_2 \rangle) = \begin{cases} 
(A^3 + 1)^6(A^2 + A^{-2})^2 & \vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 0 \\
(A^3 + 1)^6(A^{12} + A^8 + 2A^7 + 2A^4 + 4A^3 + 3 + 2A^{-1} + A^{-4}) & \vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 1 \\
(A^3 + 1)^6(A^{20} + A^{16} + 4A^{15} + 3A^{12} + 4A^{11} + 4A^{10} + 2A^8 + 8A^7 + 8A^6 + 3A^4 + 12A^3 + 4A^2 + 5 + 4A^{-1} + A^{-4}) & \vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 2 \\
A^{28} + A^{24} + 6A^{23} + 4A^{20} + 6A^{19} + 12A^{18} + 3A^{16} + 18A^{15} + 12A^{14} + 8A^{13} + 6A^{12} + 12A^{11} + 24A^{10} + 16A^9 + 3A^8 + 18A^7 + 36A^6 + 8A^5 + 4A^4 + 30A^3 + 12A^2 + 9 + 6A^{-1} + A^{-4} & \vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 3,
\end{cases}
\]

and

\[
\vartheta((i_1, j_1) \langle i_2, j_2 \rangle) = \begin{cases} 
0 \mod ((A + 1)^4) & \vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 0, 1, 2 \\
192(A + 1)^3 \mod ((A + 1)^4) & \vartheta((i_1, j_1, k_1) \cap \{i_2, j_2, k_2\}) = 3.
\end{cases}
\]

Proof. Let \(L_n\) be the link in \(S^3\) as in Figure 13. We remark that \(\mathcal{K}(L_n) = A^{2n}(A^4 + 1 + A^{-4}) + A^{-6n}\). Let \(i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_m\) be elements of \(\{1, \ldots, b\}\) satisfying \(i_1 < i_2 < \cdots < i_n, j_1 < j_2 < \cdots < j_m\) and \(\vartheta(\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_m\}) = k\). We have \(\vartheta((i_1, i_2, \ldots, i_n) \langle j_1, j_2, \ldots, j_m \rangle) = (-A^3)^{n+m-k}\mathcal{K}(L_k)\). Using this formula, we obtain the above equations. This finishes the proof.

By Theorem 2.2 Theorem 4.1 follows from the following lemma.

Lemma 4.5. The set

\[
\{1\} \cup \{A + 1\} \cup \{(i, j)|i \leq j\} \cup \{(i, j, k)|i < j < k\}
\]

is a basis of \(S(\Sigma)/(\ker \epsilon)^2\) as a \(\mathbb{Q}\) vector space.

Proof. It is enough to show that if \(X = Q + q(A + 1) + \sum_i q_i(i, i) - 12(A + 1) + \sum_{i<j} q_{ij}(i, j) + \sum_{i<k} q_{ik}(i, j, k) \in (\ker \epsilon)^2\), we have \(Q = q = 0, q_i = 0\) for \(1 \leq i \leq b, q_{ij} = 0\) for \(1 \leq i < j \leq b\) and \(q_{ijk} = 0\) for \(1 \leq i < j < k \leq b\). We assume \(X \in (\ker \epsilon)^2\). Since \(X \in \ker \epsilon\), we have \(Q = 0\). By Lemma 4.4 (1) (2) (3) (4) (5), we have

\[
\vartheta(X^2) = (q^2 + \sum_i 96q_i^2 + \sum_{i<j} 48q_{ij}^2)(A + 1)^2 \mod ((A + 1)^3).
\]
Since $\vartheta(X^2) \in ((A + 1)^4)$, we obtain $q = 0$, $q_i = 0$ for any $i$ and $q_{ij} = 0$ for any $i$, $j$. By Lemma 4.4 (6), we have

$$\vartheta(X^2) = (\sum_{i<j<k} 192 q_{ijk}^2)(A + 1)^3 \mod ((A + 1)^4).$$

Since $\vartheta(X^2) \in ((A + 1)^4)$, we obtain $q_{ijk} = 0$ for any $i$, $j$, $k$. This proves the lemma.

As a corollary of Theorem 4.1, we have the following.

**Corollary 4.6.** If $\partial \Sigma \neq \emptyset$, then $\mathbb{Q}$-linear map $\lambda : H_1 \wedge H_1 \wedge H_1 \rightarrow \ker \epsilon / (\ker \epsilon)^2$ is injective.

**Proof.** We have $\{\{x_i, x_j, x_k\}|i < j < k\} = \{\lambda([x_i] \wedge [x_j] \wedge [x_k])|i < j < k\}$. This proves the corollary.

**Corollary 4.7 ([11] Remark 3.13).** Let $\Sigma$ be a compact connected surface of genus $g$ with $b+1$ boundary components. We assume $0 \leq b$. We have $\dim_{\mathbb{Q}}(\ker \epsilon)/(\ker \epsilon)^2 = 1 + \frac{N(N+1)}{2} + \frac{N(N-1)(N-2)}{6}$, where $N = b+2g$. Furthermore $\dim_{\mathbb{Q}}(\ker \epsilon)^m/(\ker \epsilon)^{m+1} < \infty$ for any $m$.

5. **Filtrations**

In this section, Let $\Sigma$ be a connected compact oriented surface with non-empty boundary. In this section, we introduce a new filtration of the Kauffman bracket skein modules of $\Sigma$. We denote by $\varpi$ the natural quotient map $\ker \epsilon \rightarrow (\ker \epsilon / (\ker \epsilon)^2)/\text{im}\lambda$.

**Definition 5.1.** The filtration $\{F^n S(\Sigma)\}_{n \geq 0}$ of $S(\Sigma)$ is defined by

- $F^0 \overset{\text{def}}{=} S(\Sigma)$,
- $F^1 S(\Sigma) = F^2 S(\Sigma) \overset{\text{def}}{=} \ker \epsilon$,
- $F^3 S(\Sigma) \overset{\text{def}}{=} \ker \varpi$,
- $F^n S(\Sigma) \overset{\text{def}}{=} \ker \epsilon F^{n-2} S(\Sigma)$ (for $4 \leq n$).
We remark that $F^nS(\Sigma)$ is an ideal of $S(\Sigma)$ for any $n$. By definition, the filtrations $\{\ker c\}^n$ and $\{F^nS(\Sigma)\}$ induce the same topology of $S(\Sigma)$. More precisely, we have $F^2S(\Sigma) = (\ker c)^n$.

5.1. The filtrations depend only on the underlying 3-manifold. We define another filtration $\{F^nS(\Sigma)\}$ of $S(\Sigma)$ which depends only on the underlying 3-manifold.

Let $\Sigma_{0,b+1}$ be a connected compact oriented surface of genus 0 with $b+1$ boundary components. We denote $\eta \triangleq c_{123} - c_{12} - c_{23} - c_{13} + c_1 + c_2 + c_3 + A^2 + A^{-2} \in S(\Sigma_{0,4})$ and $\nu \triangleq c_1 + A^2 + A^{-2}$. Any embedding $\iota : \Sigma_{0,2} \times I \times \{1, 2, \cdots , n\} \coprod \Sigma_{0,4} \times I \times \{1, 2, \cdots , m\} \to \Sigma \times I$ induces $\iota_* : (\oplus^n S(\Sigma_{0,2})) \oplus (\oplus^m S(\Sigma_{0,4})) \to S(\Sigma)$.

**Definition 5.2.** The filtration $\{F^nS(\Sigma)\}$ is defined as follows.

- $F^0S(\Sigma) \triangleq S(\Sigma)$ and $F^1S(\Sigma) \triangleq F^2S(\Sigma)$.
- If $n \geq 1$, $F^{2n}S(\Sigma)$ is the $\mathbb{Q}[A, A^{-1}]$-submodule of $S(\Sigma)$ generated by
  $$\{\iota_*((\oplus^n \nu) \oplus (\oplus^{N-n} \iota) : \Sigma_{0,2} \times I \times \{1, 2, \cdots , N\} \to \Sigma \times I \text{ embedding} \}$$

and
  $$(A + 1)F^{2n-2}S(\Sigma).$$

- If $n \geq 1$, $F^{2n+1}S(\Sigma)$ is the $\mathbb{Q}[A, A^{-1}]$-submodule of $S(\Sigma)$ generated by
  $$\{\iota_*((\oplus^n \nu) \oplus (\oplus^{N-n} \iota) \oplus \eta) : \Sigma_{0,2} \times I \times \{1, 2, \cdots , N\} \coprod \Sigma_{0,4} \times I \to \Sigma \times I \text{ embedding}, \}$$
  $$\{\iota_*((\oplus^{n+1} \nu) \oplus (\oplus^{N-n-1} \iota) : \Sigma_{0,2} \times I \times \{1, 2, \cdots , N\} \to \Sigma \times I \text{ embedding} \}$$

and
  $$(A + 1)F^{2n-1}S(\Sigma).$$

**Lemma 5.3.** (1) For an embedding $\iota : \Sigma_{0,4} \times I \to \Sigma \times I$, we have $\iota_* (\eta) \in \ker \varpi = F^3S(\Sigma)$.

(2) We have $F^3S(\Sigma) = F^3S(\Sigma)$.

**Proof.** We choose $\gamma_1, \gamma_2$ and $\gamma_3 \in \pi_1(\Sigma)$ satisfying $|p_1(\iota(r_i))| = |\gamma_i|$ for $i \in \{1, 2, 3\}$. We remark that

$$w(\iota(c_{123})) - w(\iota(c_{12})) - w(\iota(c_{23})) - w(\iota(c_1)) + w(\iota(c_2)) + w(\iota(c_3)) = 0.$$ Here we also denote by $c \in \mathcal{T}(\Sigma_{0,4})$ an knot presented by $c$ for $c \in \{c_1, c_2, c_3, c_{12}, c_{13}, c_{23}, c_{123}\}$. Hence we have

$$\iota_* (\eta) = \kappa((\gamma_1 - 1)(\gamma_2 - 1)(\gamma_3 - 1)) \mod (\ker c)^2.$$ This proves the lemma (1).

For any element $x \in F^3S(\Sigma)$, we prove $x \in F^3S(\Sigma)$. We choose $\gamma_1, \gamma_2$ and $\gamma_3 \in \pi_1(\Sigma)$ and an embedding $\iota : \Sigma_{0,4} \times I \to \Sigma \times I$ satisfying $\kappa((\gamma_1 - 1)(\gamma_2 - 1)(\gamma_3 - 1)) \mod (\ker c)^2$, and $|p_1(\iota(r_i))| = |\gamma_i|$ for $i \in \{1, 2, 3\}$. Since $x = \iota_* (\eta) = 0 \mod F^3S(\Sigma) \subset (\ker c)^2$, we have $x \in F^3S(\Sigma)$.

In order to prove $F^3S(\Sigma) \subset F^3S(\Sigma)$, it is enough to prove $\iota_* ((\oplus^N \iota) : F^3S(\Sigma)$ for any embedding $\iota : \Sigma_{0,2} \times I \times \{1, 2, \cdots , N\} \coprod \Sigma_{0,4} \times I \to \Sigma \times I$. We have $\iota_* ((\oplus^N \iota) \oplus (\eta) = (-2)^N \iota_* (\eta) \mod F^3S(\Sigma)$. By this lemma (1), $\iota_* (\eta) \in F^3S(\Sigma)$.

Hence we obtain $F^3S(\Sigma) = F^3S(\Sigma)$. This proves the lemma (2). □
Let \( c_1, c_2, c_3, c_{12}, c_{23}, c_{13}, c_{123} \) and \( c_* \) be simple closed curves in \( \Sigma_{1,3} \) in Figure 14, 15 and 16. We also denote by \( c \) an element of \( S(\Sigma_{1,3}) \) represented a knot presented by \( c \) for \( c \in \{ c_1, c_2, c_3, c_{12}, c_{23}, c_{13}, c_{123}, c_* \} \). We obtain the lemma by a straightforward calculation.

**Lemma 5.4.** We have

\[
\langle 1, 2, 3 \rangle c_* - c_* \langle 1, 2, 3 \rangle = (-A + A^{-1})(t_c, ((1, 2, 3)) - t_c^{-1}((1, 2, 3)))
\]

where \( (1, 2, 3) = c_{123} - c_{12} - c_{23} - c_{13} + c_1 + c_2 + c_3 + A^2 + A^{-2} \) and \( t_c \) is a Dehn twist along \( c_* \).

Using this lemma, we have the following.

**Lemma 5.5.** We have \( F^n S(\Sigma) = F^{*n} S(\Sigma) \).

**Proof.** By Lemma 5.3., we have \( F^{2n} S(\Sigma) = F^{*2n} S(\Sigma) = (\ker \epsilon)^n \) for any \( n \). It is enough to show \( F^{2n+1} S(\Sigma) = F^{*2n+1} S(\Sigma) \) for \( n \in \mathbb{Z}_{\geq 1} \). To prove it, we use the induction on \( n \). If \( n = 0 \), the claim follows from \( F^{*1} S(\Sigma) = F^{*2} S(\Sigma) = \ker \epsilon = F^2 S(\Sigma) = F^1 S(\Sigma) \). If \( n = 1 \), the claim follows from Lemma 5.3 (2). We assume \( F^{2n+1} S(\Sigma) = F^{*2n+1} S(\Sigma) \). The embeddings \( \iota, \iota' : \Sigma_{0,2} \times I \times \{ 1, 2, \cdots, N \} \coprod \Sigma_{0,4} \times \)
$I \to \Sigma \times I$ are only differ in an open ball in $\Sigma \times I$ shown in Figure 17 and Figure 18 respectively. By Lemma 5.4, we have

$$I^*((\oplus^{n+1}\nu) \oplus (\oplus^{N-n-1}c_1) \oplus \eta) - I^*((\oplus^{n+1}\nu) \oplus (\oplus^{N-n-1}c_1) \oplus \eta)$$

$$\in (-A + A^{-1}) F^{*2n+1} S(\Sigma) = (A + 1) F^{2n+1} S(\Sigma).$$

Using this equation repeatedly, we have

$$F^{*2n+3} S(\Sigma)$$

$$= (-A + A^{-1}) F^{*2n+1} S(\Sigma) + F^{*2n} S(\Sigma) F^{*3} S(\Sigma)$$

$$= (-A + A^{-1}) F^{2n+1} S(\Sigma) + F^{2n} S(\Sigma) F^{3} S(\Sigma)$$

$$= F^{2n+3} S(\Sigma).$$

This proves the lemma.

Using this lemma, we have the following.
Theorem 5.6. Let $\Sigma$ and $\Sigma'$ be two oriented compact connected surfaces such that there exists a diffeomorphism $X : (\Sigma \times I) \to (\Sigma' \times I)$. Then we have $\mathcal{X}(F^n_S(\Sigma)) = F^n_S(\Sigma')$ for $n \geq 0$.

Proof. By definition, we have $\mathcal{X}(F^\ast n_S(\Sigma)) = F^\ast n_S(\Sigma')$. By Lemma 5.5, we have $F^\ast n_S(\Sigma) = F^n_S(\Sigma)$ and $F^\ast n_S(\Sigma') = F^n_S(\Sigma')$. Hence we have $\mathcal{X}(F^n_S(\Sigma)) = F^n_S(\Sigma')$. This proves the theorem.

\[\square\]

Corollary 5.7. We have $F^\ast n_S(D^2) = (\ker \epsilon)^{\lfloor \frac{n+1}{2} \rfloor} = (A + 1)^{\lfloor \frac{n+1}{2} \rfloor} \mathbb{Q}[[\pm A]]$. Furthermore, in the situation of section 4, we have $\theta_\xi(F^n_S(\Sigma)) \subset (A + 1)^{\lfloor \frac{n+1}{2} \rfloor} \mathbb{Q}[[\pm A]]$.

5.2. The product and the filtration. In this subsection, we prove the following.

Proposition 5.8. For $n$ and $m \in \mathbb{Z}_{\geq 0}$, we have $F^n_S(\Sigma)F^m_S(\Sigma) \subset F^{n+m}_S(\Sigma)$.

For $N \in \mathbb{Z}_{\geq 2}$ and $1 \leq i_1 < i_2 < \cdots < i_j \leq N$, we denote

$$\langle i_1, i_2, \cdots, i_j \rangle \overset{\text{def.}}{=} \sum_{\{k_1, \cdots, k_l\} \subset \{i_1, \cdots, i_j\}, k_1 < k_2 < \cdots < k_l} (-1)^{j-l} c_{k_1 \cdots k_l} \in S(\Sigma_{0,N+1}).$$

Here we define $c_\emptyset \overset{\text{def.}}{=} -A^2 - A^{-2}$.

Lemma 5.9. We have $\langle 1, 2, 4 \rangle \langle 3, 5, 6 \rangle \in (\ker \epsilon)^3 = F^6_S(\Sigma_{0,7})$.

Proof. By a straight calculation, we have

$$\langle 1, 3 \rangle \langle 2, 4 \rangle + \langle 2, 4 \rangle \langle 1, 3 \rangle =$$

$$\langle A^2 + A^{-2} \rangle \langle 1, 2, 3, 4 \rangle + \langle A^4 + A^{-4} \rangle \langle 1, 2 \rangle \langle 3, 4 \rangle + \langle 1, 4 \rangle \langle 2, 3 \rangle$$

$$+ \langle A^2 + A^{-2} \rangle \langle 1, 2 \rangle \langle 3 \rangle \langle 4 \rangle + \langle 2, 3 \rangle \langle 4 \rangle \langle 1 \rangle + \langle 1, 4 \rangle \langle 2 \rangle \langle 3 \rangle + \langle 3, 4 \rangle \langle 1 \rangle \langle 2 \rangle$$

$$+ 2 \langle 1 \rangle \langle 2, 3, 4 \rangle + \langle 2 \rangle \langle 1, 3, 4 \rangle + \langle 3 \rangle \langle 1, 2, 4 \rangle + \langle 4 \rangle \langle 1, 2, 3 \rangle$$

$$+ 2 \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 4 \rangle,$$
and
\[
\langle 1, 3 \rangle \langle 2, 4 \rangle - \langle 2, 4 \rangle \langle 1, 3 \rangle = (A^2 - A^{-2})
\]
\[
\left( \frac{1}{2} (2 \langle 1, 4 \rangle - \langle 1 \rangle \langle 4 \rangle + (A - A^{-1})^2((1) + (4))) (2 \langle 2, 3 \rangle - \langle 2 \rangle \langle 3 \rangle + (A - A^{-1})^2((2) + (3)))
\right.
\]
\[
- \left( \frac{1}{2} (2 \langle 1, 2 \rangle - \langle 1 \rangle \langle 2 \rangle + (A - A^{-1})^2((1) + (2))) (2 \langle 3, 4 \rangle - \langle 3 \rangle \langle 4 \rangle + (A - A^{-1})^2((3) + (4)))
\right)
\]
\[
+ (A - A^{-1})^2(c_{12}c_{34} - c_{14}c_{23})
\].

For \( y_1, y_2, y_3, y_4 \in \pi_1(\Sigma_{0,7}) \), we denote by \( \iota(y_1, y_2, y_3, y_4) \) the embedding \( \Sigma_{0,4} \to \Sigma_{0,7} \) inducing \( \pi_1(\Sigma_{0,4}) \to \pi_1(\Sigma_{0,7}) \) such that \( \gamma_i \mapsto y_i \) for \( i = 1, 2, 3, 4 \). We also denote by \( \iota(y_1, y_2, y_3, y_4) \) the homeomorphism \( S(\Sigma_{0,4}) \to S(\Sigma_{0,7}) \) induced by the embedding \( \iota(y_1, y_2, y_3, y_4) \).

Using the equations stated above and the two \( \mathbb{Q}[A, A^{-1}] \)-module homomorphisms
\[
\sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}} (-1)^{\epsilon_1 + \epsilon_2 + \epsilon_3} \iota(\gamma_1^{\epsilon_1} \gamma_2^{\epsilon_2} \gamma_3^{\epsilon_3}, \gamma_4, \gamma_5, \gamma_6)
\]
\[
\sum_{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{0, 1\}} (-1)^{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4} \iota(\gamma_1^{\epsilon_1} \gamma_2^{\epsilon_2} \gamma_3^{\epsilon_3} \gamma_4^{\epsilon_4}, \gamma_5, \gamma_6)
\]
we have
\[
(A^2 + A^{-2})(1, 2, 3, 4, 5, 6)
\]
\[
+ (A^4 + A^{-4})(\langle 1, 2, 3 \rangle \langle 4, 5, 6 \rangle + \langle 1, 2, 5, 6 \rangle \langle 3, 4 \rangle) - (\langle 1, 2 \rangle \langle 3, 5, 6 \rangle - \langle 3, 5, 6 \rangle \langle 1, 2, 4 \rangle)
\]
\[
+ (A^2 + A^{-2})(\langle 1, 2, 3 \rangle \langle 4, 5, 6 \rangle + \langle 3, 4, 5, 6 \rangle \langle 1, 2 \rangle + (A - A^{-1})^2(\langle 3 \rangle \langle 4 \rangle + (1, 2) \langle 3 \rangle \langle 4, 5, 6 \rangle)
\]
\[
+ 2(\langle 1, 2 \rangle \langle 3, 4, 5, 6 \rangle + (\langle 1, 2, 4, 5, 6 \rangle + (\langle 1, 2, 3, 5, 6 \rangle + (\langle 4 \rangle \langle 1, 2, 3, 4 \rangle + (\langle 5 \rangle \langle 1, 2, 3, 4, 5 \rangle + (\langle 5 \rangle \langle 1, 2, 3, 4, 6 \rangle)
\]
\[
= 0,
\]
and
\[
\langle 1, 2, 4 \rangle \langle 3, 5, 6 \rangle - \langle 3, 5, 6 \rangle \langle 1, 2, 4 \rangle
\]
\[
= (A^2 - A^{-2})
\]
\[
\left( \frac{1}{2} (2 \langle 1, 2, 5, 6 \rangle - \langle 1, 2 \rangle \langle 5, 6 \rangle + (A - A^{-1})^2(\langle 1, 2 \rangle + (5, 6))) (2 \langle 3, 4 \rangle - \langle 3 \rangle \langle 4 \rangle + (A - A^{-1})^2(\langle 3 \rangle + (4)))
\right.
\]
\[
- \left( \frac{1}{2} (2 \langle 1, 2, 3 \rangle - \langle 1, 2 \rangle \langle 3 \rangle + (A - A^{-1})^2(\langle 1, 2 \rangle + (3))) (2 \langle 4, 5, 6 \rangle - \langle 4 \rangle \langle 5, 6 \rangle + (A - A^{-1})^2(\langle 4 \rangle + (5, 6)))
\right)
\]
\[
+ (A - A^{-1})^2((c_{123} - c_{13} - c_{23})(c_{456} - c_{46} - c_{56})
\]
\[
(c_{1256} - c_{125} - c_{126} - c_{156} - c_{256} + c_{15} + c_{16} + c_{25} + c_{26})c_{34})
\).
By these equations, we have $(1, 2, 4)\langle 3, 5, 6 \rangle \in (\ker \epsilon)^3 = F^6S(\Sigma_{0, 7})$ and $(1, 2, 4)\langle 3, 5, 6 \rangle - (1, 2, 4)\langle 1, 2, 4 \rangle \in (\ker \epsilon)^3 = F^8S(\Sigma_{0, 7})$. This finishes the proof. □

**Corollary 5.10.** We have $(1, 2, 3, 4) \in (\ker \epsilon)^2 = F^4S(\Sigma_{0, 5})$. Furthermore, we have $(1, 2, 3, 4) = \frac{1}{2}(-1, 2)\langle 3, 4 \rangle - (1, 4)\langle 2, 3 \rangle + (1, 3)\langle 2, 4 \rangle$ mod $F^8S(\Sigma_{0, 5})$.

**Corollary 5.11.** We have $2(1, 2, 4)\langle 3, 5, 6 \rangle = (1, 3)\langle 2, 5 \rangle + (1, 5)\langle 2, 6 \rangle + (1, 6)\langle 2, 3 \rangle \langle 4, 5 \rangle - (1, 3)\langle 2, 6 \rangle\langle 4, 5 \rangle - (2, 5)\langle 1, 6 \rangle\langle 3, 4 \rangle - (1, 5)\langle 2, 3 \rangle\langle 4, 6 \rangle$ mod $F^7S(\Sigma_{0, 7})$.

**Proof.** By the proof of Lemma [5.9] we have

$$\begin{align*}
&- (1, 2)\langle 5, 6 \rangle\langle 3, 4 \rangle - (1, 6)\langle 2, 5 \rangle\langle 3, 4 \rangle + (1, 5)\langle 2, 6 \rangle\langle 3, 4 \rangle \\
&- 2(1, 2, 4)\langle 3, 5, 6 \rangle + 2(1, 2)\langle 3, 4 \rangle\langle 5, 6 \rangle \\
&- (1, 2)\langle 3, 4 \rangle\langle 5, 6 \rangle - (1, 2)\langle 3, 6 \rangle\langle 4, 5 \rangle + (1, 2)\langle 3, 5 \rangle\langle 4, 6 \rangle \\
&= - (1, 2)\langle 3, 6 \rangle\langle 4, 5 \rangle - (1, 6)\langle 2, 3 \rangle\langle 4, 5 \rangle + (1, 3)\langle 2, 6 \rangle\langle 4, 5 \rangle \\
&+ (1, 2)\langle 3, 5 \rangle\langle 4, 6 \rangle + (1, 5)\langle 2, 3 \rangle\langle 4, 6 \rangle - (1, 3)\langle 2, 5 \rangle\langle 4, 6 \rangle \\
&\text{mod } F^7S(\Sigma_{0, 7}).
\end{align*}$$

This proves the corollary. □

**Proof of Proposition [5.8]** By Lemma [5.5] we have

$$F^2S(\Sigma)F^3S(\Sigma) = F^3S(\Sigma)F^2S(\Sigma) = F^5S(\Sigma) = F^6S(\Sigma).$$

Using this equation, if one of $n, m \in \mathbb{Z}_{\geq 0}$ is even number, we have

$$F^nS(\Sigma)F^mS(\Sigma) \subset F^{n+m}S(\Sigma).$$

It is enough to show

$$F^{2n+1}S(\Sigma)F^{2m+1}S(\Sigma) \subset F^{2n+2m+2}S(\Sigma).$$

We choose two embedding

$$\begin{align*}
t_1 : \Sigma_{0, 2} \times I \times \{1, 2, \ldots, N\} &\to \Sigma \times \frac{1}{2}, 1], \\
t_2 : \Sigma_{0, 2} \times I \times \{1, 2, \ldots, M\} &\to \Sigma \times \frac{1}{2} \to [0, 1].
\end{align*}$$

Since $F^{2n+2m+2}S(\Sigma) = F^{2n+2m+2}S(\Sigma)$, it is enough to show

$$t_1(\oplus^{n-1}N N^{-n+1}C_1 \oplus \eta) t_2(\oplus^{m-1}N -n+1) C_1 \oplus \eta) \in F^{2n+2m+2}.$$ 

Choose an embedding

$$t : \Sigma_{0, 2} \times \{1, 2, \ldots, N + M\} \to \Sigma \times I$$

satisfying

$$\begin{align*}
t|\Sigma_{0, 2} \times I \times \{n\} &= t_1|\Sigma_{0, 2} \times I \times \{n\}, \\
t|\Sigma_{0, 2} \times I \times \{n+m\} &= t_2|\Sigma_{0, 2} \times I \times \{m\}
\end{align*}$$

for any $n \in \{1, 2, \ldots, N\}$ and $m \in \{1, 2, \ldots, M\}$ and

$$\begin{align*}
t_1(\oplus^{n-1}N N^{-n+1}C_1 \oplus \eta) t_2(\oplus^{m-1}N N^{-n+1}C_1) \oplus \eta) \\
= t((\oplus^{n-1}N N^{-n+1}C_1) \oplus (\oplus^{m-1}N M^{-m+1}N)- (\{1, 2, 4\langle 3, 5, 6 \}))
\end{align*}$$

By Lemma [5.9] we have

$$t_1(\oplus^{n-1}N N^{-n+1}C_1) \oplus \eta) t_2(\oplus^{m-1}N M^{-m+1}N) \in F^{2n+2m+2}$$
5.3. The Goldman Lie algebra and its filtration. We first review some classical facts about the Goldman Lie algebra of $\Sigma$ and the group ring of $\pi_1(\Sigma)$. Fix a base point $*$, $*$' of $\partial \Sigma$. We denote by $\pi_1(\Sigma, *)$ the fundamental group of $\Sigma$ and by $\hat{\pi}(\Sigma)$ the set of conjugacy classes of $\pi_1(\Sigma, *)$. Furthermore, we denote by $\pi_1(\Sigma, *, *')$ the fundamental groupoid from $*$ to $*'$.

Let $|\cdot| : \pi_1(\Sigma, *) \to \hat{\pi}(\Sigma)$ be the quotient map.

We consider the action $\sigma_\pi : \mathbb{Q}\hat{\pi}(\Sigma) \times \mathbb{Q}\pi(\Sigma, *, *') \to \mathbb{Q}\pi(\Sigma, *, *')$ defined by

$$
\sigma_\pi(|x|)(r) \overset{\text{def}}{=} \sum_{p \in x^r} c(p, x, r) r_{p, x} r_{p, x^2}
$$

for $|x| \in \hat{\pi}(\Sigma)$ and $r \in \pi(\Sigma, *, *')$ in general position. For details, see [4] Definition 3.2.1. We remark $|x, |r| = |\sigma(x)(r)|$ for $x \in \mathbb{Q}\hat{\pi}$ and $y \in \mathbb{Q}\pi_1(\Sigma, *)$.

We denote by $\epsilon_\pi : \mathbb{Q}\pi_1(\Sigma, *) \to \mathbb{Q}$ the augmentation map defined by $x \in \pi_1(\Sigma, *) \mapsto 1$.

**Proposition 5.12** ([4] Theorem 4.1.2). We have

$$
\sigma(|(\ker \epsilon_\pi)^n|)(|(\ker \epsilon_\pi)^m|) \mathbb{Q}\pi_1(\Sigma, *, *)) \subset (\ker \epsilon_\pi)^n + m - 2(\mathbb{Q}\pi_1(\Sigma, *, *))
$$

$$
|(|(\ker \epsilon_\pi)^n|, |(\ker \epsilon_\pi)^m|) \subset |(\ker \epsilon_\pi)^n + m - 2|
$$

for any $n$ and $m$. Furthermore, we have

$$
\sigma(|(\ker \epsilon_\pi)^n|)(\mathbb{Q}\pi_1(\Sigma, *, *)) \subset (\ker \epsilon_\pi)^n - 1(\mathbb{Q}\pi_1(\Sigma, *, *))
$$

for any $n$ and $m$.

We denote by $H_1 \overset{\text{def}}{=} H_1(\Sigma, \mathbb{Q})H$. Using a Magnus expansion, we have the following. See, for example, [3] [5].

**Proposition 5.13.** The following $\mathbb{Q}$ linear map

$$
R(n) : (\ker \epsilon_\pi)^n \mathbb{Q}\pi_1(\Sigma, *, *)/(\ker \epsilon_\pi)^{n+1} \mathbb{Q}\pi_1(\Sigma, *, *) \to H_1^{\otimes n}, (x_1 - 1) \cdots (x_n - 1) \mapsto [x_1] \otimes \cdots \otimes [x_n]
$$

$$
C(n) : |(\ker \epsilon_\pi)^n|/|(\ker \epsilon_\pi)^{n+1}| \to c(H_1^{\otimes n}), (x_1 - 1) \cdots (x_n - 1) \mapsto c([x_1] \otimes \cdots \otimes [x_n])
$$

are well defined and isomorphisms where $c : H_1^{\otimes n} \to H_1^{\otimes n}$ is defined by $c([x_1] \otimes \cdots \otimes [x_n]) = \sum_{i=1}^n [x_i] \otimes [x_{i+1}] \otimes \cdots \otimes [x_n] \otimes [x_1] \otimes [x_2] \otimes \cdots \otimes [x_{i-1}]$.

Let $\mu : H_1 \times H_1 \to \mathbb{Q}$ be the intersection form. By the above proposition, the action $\sigma_\pi$ induces the action $\sigma_{\pi, n, m} : |(\ker \epsilon_\pi)^n|/|(\ker \epsilon_\pi)^{n+1}| \times (\ker \epsilon_\pi)^m/(\ker \epsilon_\pi)^{m+1} \to (\ker \epsilon_\pi)^{n+m-2}/(\ker \epsilon_\pi)^{n+m-1}$ and the bracket $[\cdot]$ of $\mathbb{Q}\pi(\Sigma)$ induces the bracket

$$
b_{\pi, n, m} : |(\ker \epsilon_\pi)^n|/|(\ker \epsilon_\pi)^{n+1}| \times |(\ker \epsilon_\pi)^m|/|(\ker \epsilon_\pi)^{m+1}| \rightarrow |(\ker \epsilon_\pi)^{n+m-2}|/|(\ker \epsilon_\pi)^{n+m-1}|.
$$

We denote $\sigma'_{\pi, n, m} \overset{\text{def}}{=} R(n + m - 2) \circ \sigma_{\pi, n, m} \circ (C(n)^{-1} \times R(m)^{-1})$ and $b'_{\pi, n, m} \overset{\text{def}}{=} C(n + m - 2) \circ b_{\pi, n, m} \circ (C(n)^{-1} \times C(m)^{-1})$.

This proves the theorem. □
Proposition 5.14 ([3], [5]). We have
\[
\sigma^\prime_{\pi,n,m}(c(a_1 \otimes \cdots \otimes a_n))(b_1 \otimes \cdots \otimes b_m)
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(a_i, b_j)b_1 \otimes \cdots \otimes b_{j-1} \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes b_{j+1} \otimes \cdots \otimes b_m
\]
\[
\sigma^\prime_{\pi,n,m}(c(a_1 \otimes \cdots \otimes a_n))(c(b_1 \otimes \cdots \otimes b_m))
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(a_i, b_j)c(b_1 \otimes \cdots \otimes b_{j-1} \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes b_{j+1} \otimes \cdots \otimes b_m).
\]

5.4. The bracket and the filtration. In this subsection, we prove the proposition.

Proposition 5.15. We have \([F^n\mathcal{S}(\Sigma), F^m\mathcal{S}(\Sigma)] \subset F^{n+m-2}\mathcal{S}(\Sigma)\).

In order to prove this proposition, by the Leibniz rule, it is enough to show Lemma 5.17.

By Theorem [3.2], the equation [6] and Proposition 5.12 ([5] Theorem 4.1.2), we have the following.

Proposition 5.16. (1) The Lie algebra homomorphism \(\kappa : |Q\pi|_\square \rightarrow \ker \epsilon/(\ker \epsilon)^2\) induces \(\kappa : |Q\pi|_\square/(|\ker \epsilon\rangle)^2 \rightarrow \ker \epsilon/\ker \varpi\).

(2) The Lie algebra homomorphism \(\kappa : |Q\pi|_\square \rightarrow \ker \epsilon/(\ker \epsilon)^2\) induces \(\kappa : |Q\pi|_\square/(|\ker \epsilon\rangle)^2 \rightarrow \ker \epsilon/\ker \epsilon\).

In order to prove Proposition 5.16, we need the lemma.

Lemma 5.17. (1) We have \([\ker \epsilon, \ker \epsilon] \subset \ker \epsilon\).

(2) We have \([\ker \epsilon, \ker \varpi] \subset \ker \varpi\).

(3) We have \([\ker \varpi, \ker \varpi] \subset (\ker \epsilon)^2\).

Proof. The claim (1) is [[11] Lemma 3.11]. By [[5] Theorem 4.1.2], we have
\[
[\ker \epsilon, \im \lambda] = \kappa([[\epsilon\pi])^2|_\square, |[(\epsilon\pi)^3]|_\square] \subset \kappa([[\epsilon\pi])^2|_\square, |[(\epsilon\pi)^3]|_\square) = \im \lambda \mod (\ker \epsilon)^2,
\]
\[
[\im \lambda, \im \lambda] = \kappa([[\epsilon\pi])^3|_\square, |[(\epsilon\pi)^3]|_\square] \subset \kappa([[\epsilon\pi])^3|_\square, |[(\epsilon\pi)^3]|_\square) = 0 \mod (\ker \epsilon)^2.
\]

This proves the lemma.

Definition 5.18. The \(\mathbb{Q}\)-linear map \(\rho : \mathbb{Q} \oplus H_1 \cdot H_1 \rightarrow F^2\mathcal{S}(\Sigma)/F^3\mathcal{S}(\Sigma)\) is defined by \(1 \in \mathbb{Q} \mapsto (A+1)\) and \([a] \cdot [b] \mapsto \langle ab \rangle - \langle a \rangle - \langle b \rangle\) for \(a\) and \(b\) in \(\pi_1(\Sigma)\) where \(H_1 \cdot H_1\) is the symmetric tensor of \(H_1\).

By Theorem 4.1, \(\rho\) is bijection.

By Proposition 5.16 and Proposition 5.14 we have following.

Corollary 5.19. (1) We have
\[
[\rho(\alpha_1 \cdot \beta_2), \rho(\beta_1 \cdot \beta_2)] = \sum_{(i_1, i_2)=(1, 2), (2, 1), (j_1, j_2)=(1, 2), (2, 1)} -2\mu(\alpha_{i_1}, \beta_{j_1})\rho(\alpha_{i_2} \cdot \beta_{j_2}) \mod F^3\mathcal{S}(\Sigma).
\]
We have
\[
[rho(a_1 \cdot a_2), \lambda(\beta_1 \land \beta_2 \land \beta_3)]
= \sum_{(i_1, i_2) = (1, 2), (2, 1), (i_1, j_2, j_3) = (1, 2, 3), (2, 1, 3)} -2m(\alpha_{i_1}, \beta_{j_1}) \lambda(\alpha_{i_2} \land \beta_{j_2} \land \beta_{j_3}) \pmod{F^4S(\Sigma)}.
\]

**Corollary 5.20.** Let $V_1$ and $V_2 \subset V_1$ be $\mathbb{Q}$-linear subspaces of $H$ satisfying $\mu(v, v') = 0$ for any $v \in V_2$ and $v' \in V_1$. We denote
\[
S = \{ x \in \widehat{S(\Sigma)} | \rho^{-1}(x \mod F^3S(\Sigma)) \in V_1 \cdot V_2 \}.
\]
Then we have
\[
\sigma(s_1) \circ \sigma(s_2) \circ \cdots \circ \sigma(s_{2i-1})(F^{i-1}S(\Sigma)) \subset F^iS(\Sigma)
\]
for any $i \in \mathbb{Z}_{\geq 1}$ and $s_1, \cdots, s_{2i-1} \in S$.

**Proof.** We denote $H_1$ by $V_0$. Let $V(2i)_j$ be the submodule of $F^{2i}S(\Sigma)/F^{2i+1}S(\Sigma)$ defined by
\[
V(2i)_j \overset{\text{def}}{=} \sum_i \sum_{\ell = j} h^{i-\ell} \rho(V_j, V_{j+2i}) \cdots \rho(V_{j+\ell}, V_{j+2i}).
\]
Let $V(2i+3)_j$ be the submodule of $F^{2i+3}S(\Sigma)/F^{2i+4}S(\Sigma)$ defined by
\[
V(2i+3)_j \overset{\text{def}}{=} \sum_i \sum_{\ell = j} h^{i-\ell} \rho(V_j, V_{j+2i}) \cdots \rho(V_{j+\ell}, V_{j+2i}) \lambda(V_j, V_{j+2i} \cdots V_{j+2i}).
\]
For any $s \in S$, $i$ and $j \in \{0, 1, 2, \cdots, 2i - 1\}$, we have
\[
\begin{align*}
\sigma(s)(V(i)_j) & \subset V(i)_{j+1}, \\
\sigma(s)(V(i)_{2i}) & = 0.
\end{align*}
\]
This proves the corollary. \(\square\)

**5.5. Skein module and \(\rho\).** The aim of this subsection is to prove the following.

**Proposition 5.21.** Let $V_1$ and $V_2 \subset V_1$ be $\mathbb{Q}$-linear subspaces of $H$ satisfying $\mu(v, v') = 0$ for any $v \in V_2$ and $v' \in V_1$. We denote
\[
S = \{ x \in \widehat{S(\Sigma)} | \rho^{-1}(x \mod F^3S(\Sigma)) \in V_1 \cdot V_2 \}.
\]
Then, for any $i \in \mathbb{Z}_{\geq 1}$ and any finite subset $J \subset \partial \Sigma$, there exists $j_i \in \mathbb{Z}_{\geq 1}$ such that
\[
\sigma(s_1) \circ \sigma(s_2) \circ \cdots \circ \sigma(s_{s_j})(F^{i-1}S(\Sigma, J)) \subset (\ker \epsilon)^iS(\Sigma, J)
\]
for any $s_1, \cdots, s_{s_j} \in S$.

By Leibniz rule and Corollary 5.20, the following lemma induces the above proposition.

**Lemma 5.22.** We have
\[
\sigma(s_1) \circ \sigma(s_2) \circ \cdots \circ \sigma(s_{s_j})(S(\Sigma, J)) \subset \ker \epsilon S(\Sigma, J)
\]
for any $J = \{s_1, s_2\} \subset \partial \Sigma$ and $s_1, \cdots, s_{s_j} \in S$. 
We consider $S(\Sigma, J)/\ker \epsilon S(\Sigma, J)$. For $r \in \pi(\Sigma, \ast_1, \ast_2) \overset{\text{def}}{=} [(I, \{0\}, \{1\}), (\Sigma, \{\ast_1\}, \{\ast_2\})]$, let $\langle r \rangle$ be the element $S(\Sigma, J)/\ker \epsilon S(\Sigma, J)$ of presented by $r$. By equation (3), $\langle \cdot \rangle : \pi(\Sigma, \ast_1, \ast_2) \to S(\Sigma, J)/\ker \epsilon S(\Sigma, J)$ is well-defined. We also denoted $\langle \cdot \rangle : \mathbb{Q}\pi(\Sigma, \ast_1, \ast_2) \to S(\Sigma, J)/\ker \epsilon S(\Sigma, J)$ by its $\mathbb{Q}$-linear extension. By the skein relation of $S(\Sigma, J)$, we have

$$2\langle yr \rangle = \langle yxr \rangle + \langle yx^{-1}r \rangle$$

for $x, y \in \pi_1(\Sigma, \ast_1)$ and $r \in \pi(\Sigma, \ast_1, \ast_2)$. Using this equation, we have the following calculations.

**Lemma 5.23.** For $x, y, z \in \pi_1(\Sigma, \ast_1)$ and $r \in \pi(\Sigma, \ast_1, \ast_2)$, we have

$$\langle (x-1)(y-1)r \rangle = -\langle (y-1)(x-1)r \rangle,
\langle (x-1)(y-1)(z-1)r \rangle = 0.$$

**Proof.** We have

\[
\begin{align*}
\langle (x-1)(y-1)r \rangle &= \langle -y^{-1}(x-1)(y-1)r \rangle
= \langle y^{-1}x^{-1}r + 2r - xr - yr + r \rangle
= \langle y^{-1}xr - 2y^{-1}r + 2r - xr - yr + r \rangle
= \langle -yxr + 2xr + 2yr - 4r + 2r - xr - yr + r \rangle
= \langle (y-1)(x-1)r \rangle.
\end{align*}
\]

Using this equation, we have

\[
\begin{align*}
\langle (x-1)(y-1)(z-1)r \rangle &= \langle (y-1)(x-1)(z-1)r \rangle
= -\langle (y-1)(z-1)(x-1)r \rangle
= \langle (y-1)(x-1)(z-1)r \rangle
= -\langle (x-1)(y-1)(z-1)r \rangle.
\end{align*}
\]

This proves the lemma. \qed

We consider the action $\sigma_\pi : \mathbb{Q}\pi_\square \times \mathbb{Q}\pi(\Sigma, \ast_1, \ast_2) \to \mathbb{Q}\pi(\Sigma, \ast_1, \ast_2)$ defined by $\sigma_\pi([x]|(r)) = \sum_{p \in \pi \cap r} \epsilon(p, x, r)(r_{*1}p_{x}r_{p}r_{*2} - r_{*1}p_{x}r_{p}^{-1}r_{p+2})$ for $[x] \in \pi_\square$ and $r \in \pi(\Sigma, \ast_1, \ast_2)$ in general position. For details, see [4] Definition 3.2.1.

Since $\sigma((\ker \epsilon)^2(\Sigma(\Sigma, J)) \subset \ker \epsilon S(\Sigma, J)$ and $\sigma(S(\Sigma)(\ker \epsilon S(\Sigma, J)) \subset \ker \epsilon S(\Sigma, J)$, the action $\sigma : S(\Sigma) \times S(\Sigma, J) \to S(\Sigma)$ induces $\sigma : S(\Sigma)/((\ker \epsilon)^2 \times S(\Sigma, J)/\ker \epsilon S(\Sigma, J) \to S(\Sigma, J)/\ker \epsilon S(\Sigma, J)$. By equation (3), we have

$$\sigma([x])(\langle r \rangle) = -\langle \sigma_\pi([x]|(r)) \rangle.$$

Using this equation, it is enough to show the following in order to prove Lemma 5.22.

**Proposition 5.24.** Let $V_1$ and $V_2 \subset V_1$ be $\mathbb{Q}$-linear subspaces of $H$ satisfying $\mu(v, v') = 0$ for any $v \in V_2$ and $v' \in V_1$. We denote

$$S = \{ x \in [(\ker \epsilon)^2]|C(2)|(x \mod [(\ker \epsilon)^3]) \in c(V_1 \otimes V_2) \}.$$

Then we have the following.
(1) We have $\sigma(s)(\mathbb{Q}\pi_1(\Sigma,*,*)) \subset \ker \epsilon_2 \pi_1(\Sigma,*,*)$ for any $s \in S$.

(2) We have $\sigma(s_1) \circ \sigma(s_2) \circ \sigma(s_3) (\ker \epsilon_2 \pi_1(\Sigma,*,*)) \subset (\ker \epsilon_2)^2 \pi_1(\Sigma,*,*)$ for any $s_1, s_2, s_3 \in S$.

(3) We have $\sigma(s_1) \circ \sigma(s_2) \circ \sigma(s_3) \circ \sigma(s_4) ((\ker \epsilon_2)^2 \pi_1(\Sigma,*,*)) \subset (\ker \epsilon_2)^3 \pi_1(\Sigma,*,*)$ for any $s_1, s_2, s_3, s_4, s_5 \in S$.

Proof. The first claim is obvious. Let $V_0$ be $H_1$. We have $\sigma'_i(2)(s))((V_i)) \subset V_{i+1}$ for $i = 0, 1$ for any $s \in S$. Furthermore, we have $\sigma'(c(2)(s))(V_2) = 0$ for any $s \in S$. This proves the second claim. We define

$$V(2)_j = \sum_{j_1 + j_2 = j} c(V_{j_1} \otimes V_{j_2})$$

for $j = 0, 1, 2, 3, 4$. We have $\sigma''(2)(s))(V(2)_i) \subset V(2)_{i+1}$ for $i = 0, 1, 2, 3$. Furthermore, we have $\sigma''(2)(s))(V(2)_4) = 0$. This proves the third claim. □

6. Framed pure braid group

In this section, let $\Sigma$ be a compact connected surface of genus 0 and $b+1$ boundary components. We remark the mapping class group $\mathcal{M}(\Sigma)$ of $\Sigma$ is isomorphic to the famed pure braid group with $b$ strings.

The completed Kauffman bracket skein module $\widehat{S}(\Sigma)$ has a filtration $\{F^n \widehat{S}(\Sigma)\}_{n \geq 0}$ satisfying $\widehat{S}(\Sigma)/F^n \widehat{S}(\Sigma) \cong \widehat{S}(\Sigma)/F^n \widehat{S}(\Sigma)$ for $n \in \mathbb{Z}_{\geq 0}$.

6.1. The Baker-Campbell-Hausdorff series. By Corollary 5.19 we have the following.

Lemma 6.1. We have

$$[\widehat{S}(\Sigma), F^n \widehat{S}(\Sigma)] \subset F^{n+1} \widehat{S}(\Sigma).$$

In this paper, we define the Baker-Campbell-Hausdorff series $\text{bch}$ by

$$\text{bch}(a_1, a_2, \cdots, a_m) \overset{\text{def}}{=} (-A + A^{-1}) \log \left( \prod_{i=1}^m \exp \left( \frac{a_i}{-A + A^{-1}} \right) \right)$$

for $a_1, a_2, \cdots, a_m \in \widehat{S}(\Sigma)$. As elements of the associated Lie algebra $(\widehat{S}(\Sigma), [ , ])$, it has a usual expression. For example, $\text{bch}(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots$.

By Lemma 6.1, the Baker-Campbell-Hausdorff series is well-defined. The Baker-Campbell-Hausdorff series satisfies

$$\text{bch}(a, -a) = 0,$$

$$\text{bch}(0, a) = \text{bch}(a, 0) = a,$$

$$\text{bch}(a, \text{bch}(b, c)) = \text{bch}(\text{bch}(a, b), c),$$

$$\text{bch}(a, b, -a) = \exp(\sigma(a))(b).$$

Hence $(\widehat{S}(\Sigma), \text{bch})$ is a group whose identity is 0. By Proposition 5.21 we define $\exp(\sigma(s))(: \in \text{Aut}(\widehat{S}(\Sigma), J))$ is well-defined for any $s \in \widehat{S}(\Sigma)$ and any finite subset $J \subset \partial \Sigma$. Furthermore, $\exp : (\widehat{S}(\Sigma), \text{bch}) \rightarrow \text{Aut}(\widehat{S}(\Sigma), J)$ is a group homomorphism, i.e, $\exp(\sigma(\text{bch}(a, b))) = \exp(\sigma(a)) \circ \exp(\sigma(b))$ for $a, b \in \widehat{S}(\Sigma)$. 

The Baker-Campbell-Hausdorff series \( \text{bch} \) is defined by

\[
\text{bch}(a_1, a_2, \cdots , a_m) \overset{\text{def}}{=} (-A + A^{-1}) \log(\prod_{i=1}^{m} \exp(\frac{a_i}{-A + A^{-1}}))
\]

for \( a_1, a_2, \cdots , a_m \in \widehat{S(\Sigma)} \). For example,

\[
\text{bch}(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots.
\]

The Baker-Campbell-Hausdorff series satisfies

\[
\text{bch}(a, -a) = 0, \\
\text{bch}(0, a) = \text{bch}(a, 0) = a, \\
\text{bch}(a, \text{bch}(b, c)) = \text{bch}(\text{bch}(a, b), c), \\
\text{bch}(a, b, -a) = \exp(\sigma(a))(b).
\]

Hence \( \widehat{S(\Sigma)}, \text{bch} \) is a group whose identity is 0. By Proposition 5.21, \( \exp(\sigma(s))(\cdot) \in \text{Aut}(\widehat{S(\Sigma)}, \text{bch}) \) is well-defined for any \( s \in \widehat{S(\Sigma)} \) and any finite subset \( J \subset \partial \Sigma \). Furthermore, \( \exp : (\widehat{S(\Sigma)}, \text{bch}) \rightarrow \text{Aut}(\widehat{S(\Sigma)}, J) \) is a group homomorphism, i.e., \( \exp(\sigma(\text{bch}(a, b))) = \exp(\sigma(a) \circ \exp(\sigma(b)) \) for \( a, b \in \widehat{S(\Sigma)} \).

6.2. **The group homomorphism** \( \zeta : \mathcal{M}(\Sigma) \rightarrow (\widehat{S(\Sigma)}, \text{bch}) \). The mapping class group \( \mathcal{M}(\Sigma) \) is generated by \( \{t_{ij}|1 \leq i < j \leq b\} \cup \{t_i|1 \leq i \leq b\} \), where \( t_{ij} \overset{\text{def}}{=} t_{c_{ij}} \) and \( t_i \overset{\text{def}}{=} t_{c_i} \). Furthermore \( \mathcal{M}(\Sigma) \) is presented by the relations

\[
\text{ad}(t_i)(t_j) = t_j, \\
\text{ad}(t_{s})(t_{ij}) = t_{ij}, \\
\text{ad}(t_{rs})(t_{ij}) = t_{ij} \quad \text{if} \quad r < s < i < j, \\
\text{ad}(t_{rs})(t_{ij}) = t_{ij} \quad \text{if} \quad i < r < s < j, \\
\text{ad}(t_{rs})(t_{ij}) = t_{ij} \quad \text{if} \quad r < s < i < j, \\
\text{ad}(t_{rs})(t_{ij}) = t_{ij} \quad \text{if} \quad i = r < s < j, \\
\text{ad}(t_{rs})(t_{ij}) = t_{ij} \quad \text{if} \quad r < i < s < j,
\]

where \( \text{ad}(a)(b) \overset{\text{def}}{=} aba^{-1} \). See, for example, [1] p.20 Lemma 1.8.2.

**Definition 6.2.** The group homomorphism \( \zeta : \mathcal{M}(\Sigma) \rightarrow (\widehat{S(\Sigma)}, \text{bch}) \) is defined by \( t_i \rightarrow L(c_i) \) and \( t_{ij} \rightarrow L(c_{ij}) \), where \( L(c) \overset{\text{def}}{=} \frac{-A + A^{-1}}{4 \log(\text{arccosh}(-\xi))} (\text{arccosh}(\frac{-\xi}{2}))^2 - (-A + A^{-1}) \log(-A) \).

**Theorem 6.3.** The group homomorphism \( \zeta \) is well-defined and injective.

**Proof.** Let \( c, c'_1, c'_2, \cdots , c'_k \) be elements of \( \{c_i, c_{ij}\} \) in \( \Sigma \) and \( c_1, \cdots , c_k \) be elements of \( \{\pm 1\} \) satisfying \( t_{c_1}^{+1} t_{c_2}^{-1} t_{c_3}^{+1} \cdots t_{c_k}^{-1}(c) = c \). It is enough to check

\[
\text{bch}(c_1 L(c'_1), \cdots , c_k L(c'_k), L(c), -c_k L(c'_k), \cdots , -c_1 L(c'_1)) = L(c).
\]
By Theorem 2.5, we have
\[
\text{bch}(\epsilon_1 L(c_1'), \cdots, \epsilon_2 L(c_2'), \cdots, \epsilon_k L(c_k'), L(c), -\epsilon_k L(c_k'), \cdots, -\epsilon_1 L(c_1'))
\]
\[
= \exp(\sigma(\epsilon_1 L(c_1'))) \circ \cdots \circ \exp(\sigma(\epsilon_k L(c_k')))(L(c))
\]
\[
= t_{c_1'} t_{c_2'} \cdots t_{c_k'}(L(c)) = L(c).
\]
This finishes the proof of well-definedness of \(\zeta\).

By definition of \(\text{bch}\), we have \(\xi(\cdot) = \exp(\sigma(\zeta(\xi)))\)(\cdot) : \(\widehat{\mathcal{S}(\Sigma,J)} \rightarrow \widehat{\mathcal{S}(\Sigma,J)}\) for any \(\xi \in \mathcal{M}(\Sigma)\) and any finite subset \(J\) of \(\partial \Sigma\). Using Corollary 2.4, we have \(\xi = \text{id}_\Sigma\) if and only if \(\zeta(\xi) = 0\). This finishes the proof of injectivity of \(\zeta\). \(\square\)

**Remark 6.4.** Using the lantern relation
\[
\text{bch}(L(c_{123}), -L(c_{12}), -L(c_{23}), -L(c_{13}), L(c_1), L(c_2), L(c_3)) = 0,
\]
we have
\[
\zeta(t_c) = L(c)
\]
for any simple closed curve \(c\) in \(\Sigma_{0,h+1}\). Details will appear in [12].

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