Starlikeness for Certain Close-to-Star Functions

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Abstract. We find the radius of starlikeness of order \( \alpha \), \( 0 \leq \alpha < 1 \), of normalized analytic functions \( f \) on the unit disk satisfying either \( \text{Re}(f(z)/g(z)) > 0 \) or \( |(f(z)/g(z)) - 1| < 1 \) for some close-to-star function \( g \) with \( \text{Re}(g(z)/(z + z^2/2)) > 0 \) as well as of the class of close-to-star functions \( f \) satisfying \( \text{Re}(f(z)/(z + z^2/2)) > 0 \). Several other radii such as radius of univalence and parabolic starlikeness are shown to be the same as the radius of starlikeness of appropriate order.

1. Introduction

Let \( \mathbb{D}_r := \{ z \in \mathbb{C} : |z| < r \} \) and \( \mathbb{D} := \mathbb{D}_1 \). Let \( \mathcal{A} := \{ f : \mathbb{D} \to \mathbb{C} | f \text{ is analytic}, f(0) = f'(0) - 1 = 0 \} \) and let \( \mathcal{S} \) be its subclass consisting of univalent functions. For each \( f \in \mathcal{S} \), we associate the function \( s_f : \mathbb{D} \to \mathbb{C} \) defined by \( s_f(z) = zf'(z)/f(z) \). For \( 0 \leq \alpha < 1 \), the class \( \mathcal{S}^*(\alpha) \) of functions starlike of order \( \alpha \) is the subclass of \( \mathcal{S} \) consisting of functions \( f \) satisfying the inequality \( \text{Re}(s_f(z)) > \alpha \). The class \( \mathcal{S}^* := \mathcal{S}^*(0) \) is the usual class of starlike functions. The image \( k(\mathbb{D}) \) of the Koebe function \( k(z) = z/(1 - z)^2 \) is not convex but the image \( k(\mathbb{D}_{2 - \sqrt{3}}) \) is convex and \( 2 - \sqrt{3} \) is the largest such radius. This radius is known as the radius of convexity of the Koebe function. More generally, given a class of functions \( \mathcal{F} \) and another class \( \mathcal{G} \) characterised by a property \( P \), the largest number \( R_G(\mathcal{F}) \) with \( 0 \leq R_G(\mathcal{F}) \leq 1 \) such that every function in \( \mathcal{F} \) has the property \( P \), in each disk \( \mathbb{D}_r \) for each \( r \) with \( 0 < r < R_G(\mathcal{F}) \) is called the \( G \) radius of \( \mathcal{F} \). Kaplan [10] introduced the class of close-to-convex functions \( f \) satisfying \( \text{Re}(f'(z)/g'(z)) > 0 \) for some convex function \( g \). In [16, 15], MacGregor found the radius of starlikeness for the class of functions \( f \) satisfying either \( \text{Re}(f(z)/g(z)) > 0 \) or \( |(f(z)/g(z)) - 1| < 1 \) for some \( g \in \mathcal{S} \); related radius problems were discussed in [2, 3, 4, 5, 6, 9, 11, 12, 14, 20, 23]. Reade [21] defined a function \( f \in \mathcal{A} \), with \( f(z) \neq 0 \) for \( z \in \mathbb{D} \setminus \{0\} \), to be close-to-star if there exists a starlike function \( g \) (not necessarily normalized) satisfying \( \text{Re}(f(z)/g(z)) > 0 \). The function \( f(z) = z + z^2/2 \) maps \( \mathbb{D} \) onto the domain bounded by the cardioid \( u + 1/2 = \cos t(1 + \cos t) \) and \( v = \sin t(1 + \sin t) \) and therefore starlike in \( \mathbb{D} \). This function \( f \) also satisfies the inequality \( |f'(z) - 1| < 1 \) (which also implies univalence of \( f \)). Using this starlike function, we introduce the following three classes:

\[ \mathcal{F}_1 := \{ f \in \mathcal{A} : \text{Re}(f(z)/g(z)) > 0, \quad \text{Re}(g(z)/(z + z^2/2)) > 0 \quad \text{for some } g \in \mathcal{A} \}, \]

\[ \mathcal{F}_2 := \{ f \in \mathcal{A} : |(f(z)/g(z)) - 1| < 1, \quad \text{Re}(g(z)/(z + z^2/2)) > 0 \quad \text{for some } g \in \mathcal{A} \}, \]

and

\[ \mathcal{F}_3 := \{ f \in \mathcal{A} : \text{Re}(f(z)/(z + z^2/2)) > 0 \}. \]

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These classes \( \mathcal{F}_1, \mathcal{F}_2 \) and \( \mathcal{F}_3 \) are nonempty. Indeed, if the functions \( f_i : \mathbb{D} \to \mathbb{C}, i = 1, 2, 3 \), are defined by

\[
f_1(z) = \frac{(1+z)^2(z+z^2/2)}{(1-z)^2}, \quad f_2(z) = \frac{(1+z)^2(z+z^2/2)}{(1-z)}
\]

and

\[
f_3(z) = \frac{(1+z)(z+z^2/2)}{(1-z)}
\]

then it follows that \( f_i \) belongs to the class \( \mathcal{F}_i \); the functions \( f_1 \) and \( f_2 \) satisfy the respective condition with \( g = f_3 \). It is also clear that the class \( \mathcal{F}_3 \) is a subclass of close-to-star functions while the classes \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are not. The functions in these classes are also not necessarily univalent. Indeed, the radius of univalence \( R_S(\mathcal{F}_1) \approx 0.210756, R_S(\mathcal{F}_2) \approx 0.248032, R_S(\mathcal{F}_3) \approx 0.347296 \) are respectively the smallest positive zero of the polynomials \( P_i \) given by

\[
P_1(r) = 1 - 5r + r^2 + r^3, \quad P_2(r) = 2 - 8r - r^2 + 3r^3,
\]

and

\[
P_3(r) = 1 - 3r + r^2.
\]

These radius are in fact the radius of starlikeness of the respective classes (see Theorems 2.1, 3.1 and 4.1). The sharpness of these radii follows as the derivative of \( f_1, f_2 \) and \( f_3 \), given by

\[
f'_1(z) = \frac{(1+z)(1+5z+z^2-z^3)}{(1-z)^3}, \quad f'_2(z) = \frac{(1+z)(2+8z-z^2-3z^3)}{2(1-z)^2},
\]

and

\[
f'_3(z) = \frac{1+3z-z^3}{(1-z)^2},
\]

clearly vanishes at \( z = -R_S(\mathcal{F}_i) \) for \( i = 1, 2, 3 \) respectively.

Several subclasses of starlike functions are defined through subordination. An analytic function \( f \) is subordinate to the analytic function \( g \), written \( f \prec g \), if there exists an analytic function \( \omega : \mathbb{D} \to \mathbb{D} \) with \( \omega(0) = 0 \) and \( f(z) = g(\omega(z)) \) for all \( z \in \mathbb{D} \). For univalent superordinate function \( g \), we have \( f \prec g \) if \( f(\mathbb{D}) \subset g(\mathbb{D}) \) and \( f(0) = g(0) \). Consider the functions \( \varphi_i : \mathbb{D} \to \mathbb{C} \) defined by \( \varphi_1(z) := \sqrt{z+1}, \varphi_2(z) := e^z, \varphi_3(z) := 1+(4/3)z+(2/3)z^2, \varphi_4(z) := 1+\sin z, \varphi_5(z) := z+\sqrt{1+z^2}, \varphi_6(z) := 1+((zk+z^2)/(k^2-kz)) \) where \( k = \sqrt{2}+1 \) and \( \varphi_7(z) := 1+(2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2/\pi^2) \). For \( \varphi = \varphi_i, (i = 1, 2, \ldots, 7) \) the class \( S^*(\varphi) := \{ f \in \mathcal{A} : s_f \prec \varphi \} \) respectively becomes \( S^*_z, S^*_e, S^*_z, S^*_{\infty}, S^*_q, S^*_R \) and \( S^*_p \); these classes were studied in [7, 13, 17, 18, 22, 24, 27]. For these \( \varphi_i \), we study the \( S^*(\varphi) \) radius of the classes \( \mathcal{F}_i, i = 1, 2, 3 \) introduced above. For example, for the class \( \mathcal{F}_1 \), we have shown that the radius of starlikeness of order \( \alpha, 0 \leq \alpha < 1 \), is the smallest positive root in \((0,1)\) of the equation

\[(\alpha - 2)r^3 - (2\alpha + 2)r^2 + (10 - \alpha)r + 2\alpha - 2 = 0.
\]

In addition to finding radius of lemniscate starlikeness, we have also shown that \( R_S = R_{S^*}, R_{S^*(1/2)} = R_{S^*_p}, R_{S^*(1/\sqrt{2})} = R_{S^*_e}, R_{S^*(1-\sin 1)} = R_{S^*_{\sin}}, R_{S^*(\sqrt{2}-1)} = R_{S^*_{\sqrt{2}-1}} = R_{S^*_R} \), and \( R_{S^*(1/3)} = R_{S^*_q} \). Similar results have been proved for the other two classes.
2. Radius Problem for $\mathcal{F}_1$

For the function $f \in \mathcal{F}_1$, we first determine the disk containing the image of the disk $\mathbb{D}_r$ under the mapping $zf''(z)/f(z)$. This is done by associating the function $f$ with suitable functions with positive real part and then applying the inequality (see [25, Lemma 2])
\[
\left| \frac{z p'(z)}{p(z)} \right| \leq \frac{2(1 - \alpha)|z|}{(1 - |z|)(1 + (1 - 2\alpha)|z|)} \quad (z \in \mathbb{D})
\] (2.1)
for the function $p$ in the class $\mathcal{P}(\alpha)$ of all analytic function $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\text{Re} p(z) > \alpha$. We also need to know the image of the disk $\mathbb{D}_r$ under the transform $w(z) = (z + 1)/(z + 2)$. This is a linear fractional transformation and it maps the disk $\mathbb{D}_r$ onto the disk
\[
\left| w(z) - \frac{2 - r^2}{4 - r^2} \right| \leq \frac{r}{4 - r^2}. \quad (2.2)
\]

Since $f \in \mathcal{F}_1$, there is a function $g \in \mathcal{A}$ such that $\text{Re}(f(z)/g(z)) > 0$ and $\text{Re}(g(z)/(z + z^2/2)) > 0$ for all $z \in \mathbb{D}$. Thus, the functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ defined by $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/(z + z^2/2)$ are functions in $\mathcal{P}(0)$ and
\[
f(z) = p_1(z)p_2(z) \left( z + \frac{z^2}{2} \right) \quad (z \in \mathbb{D}).
\] (2.3)
From (2.3), it follows that
\[
\frac{zf''(z)}{f(z)} = \frac{zp'_1(z)}{p_1(z)} + \frac{zp'_2(z)}{p_2(z)} + \frac{2(z + 1)}{z + 2}. \quad (2.4)
\]
Using (2.1) (with $\alpha = 0$) and (2.2) in (2.4), we see that the image of the disk $\mathbb{D}_r$ under the mapping $zf''(z)/f(z)$ is contained in the disk
\[
\left| \frac{zf''(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| \leq \frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)}. \quad (2.5)
\]
From (2.3), it readily follows that
\[
\text{Re} \left( \frac{zf''(z)}{f(z)} \right) \geq \frac{4 - 2r^2}{4 - r^2} - \frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)} \quad (|z| \leq r). \quad (2.6)
\]

Let $R_{S^*} \approx 0.2108$ be the unique zero in $(0,1)$ of the polynomial $1 - 5r + r^2 + r^3$. Then, for every function $f \in \mathcal{F}_1$, the inequality (2.6) shows that $\text{Re}(s_f(z)) > 0$ in each disk $\mathbb{D}_r$, for $0 \leq r < R_{S^*}$. For the function $f_1$ defined in (1.1), we have
\[
{s_{f_1}(z) = \frac{zf'_1(z)}{f_1(z)} = \frac{2(1 + 5z + z^2 - z^3)}{(2 + z)(1 - z^2)} \quad (2.7)}
\]
and hence $\text{Re}(s_{f_1}(z))$ vanishes at $z = -R_{S^*}$. Thus, the radius of starlikeness $R_{S^*}$ for the class $\mathcal{F}_1$ is the unique positive zero in $(0,1)$ of the polynomial $P_1$ defined in (1.3) and is the same as the radius of univalence $R_{S}$. Using the inequality (2.5), we now determine $S^*(\alpha)$, $S_{\ell^*}^*, S_{r^*}^*, S_{s^*}^*, S_{\sin}^*$, $S_{Q}^*$ and $S_{R}^*$ radii for the class $\mathcal{F}_1$.

**Theorem 2.1.** The following sharp radii results hold for the class $\mathcal{F}_1$:

(i) For any $0 \leq \alpha < 1$, the radius $R_{S^*(\alpha)}$ is the smallest positive root of the equation
\[
(\alpha - 2)r^3 - (2\alpha + 2)r^2 + (10 - \alpha)r + 2\alpha - 2 = 0. \quad (2.8)
\]
(ii) The radius \( R_{S-}^c \) (≈ 0.0918) is the smallest positive root of the equation
\[
(2 - \sqrt{2})r^3 - (2 + 2\sqrt{2})r^2 + (\sqrt{2} - 10)r + 2\sqrt{2} - 2 = 0.
\] (2.9)

(iii) The radius \( R_{S_+}^c \) (≈ 0.1092) is the same as \( R_{S_s(1/2)}^c \).

(iv) The radius \( R_{S_{\rm{sin}}}^c \) (≈ 0.1370) is the same as \( R_{S_s(1/\epsilon)}^c \).

(v) The radius \( R_{S_{\rm{sin}}}^c \) (≈ 0.17969) is the same as \( R_{S_s(1-\sin 1)}^c \).

(vi) The radius \( R_{S_{\rm{sin}}}^c \) (≈ 0.12734) is the same as \( R_{S_s(\sqrt{2}-1)}^c \).

(vii) The radius \( R_{S_R}^c \) (≈ 0.0380) is the same as \( R_{S_{s(2(\sqrt{7}-1))}}^c \).

(viii) The radius \( R_{S_{\rm{en}}}^c \) (≈ 0.14418) is the same as \( R_{S_s(1/3)}^c \).

\textbf{Proof.} (i) Let the function \( f \in F_1 \) and \( \alpha \in [0, 1] \). Let \( R := R_{S_s(\alpha)}^c \) be the smallest positive root of the equation (2.8) so that
\[
2(1 - 5R + R^2) = \alpha(2 - R)(1 - R^2).
\] (2.10)

The function
\[
h(r) = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)}
\]
is decreasing in \([0, 1]\) and hence, for \( 0 \leq r < R \), we have, using (2.6) and (2.10),
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)} > \frac{2(1 - 5R + R^2 + R^3)}{(2 - R)(1 - R^2)} = \alpha.
\]
This proves that the function \( f \) is starlike of order \( \alpha \) in each disk \( \mathbb{D}_r \) for \( 0 \leq r < R \). At the point \( z = -R \), it can be seen, using (2.7) and (2.10), that the function \( f_1 \) defined in (1.11) satisfies
\[
\text{Re} \left( \frac{zf'_1(z)}{f_1(z)} \right) = \frac{2(1 - 5R + R^2 - R^3)}{(2 - R)(1 - R^2)} = \alpha.
\]

This shows that the radius \( R \) is the sharp radius of starlikeness of order \( \alpha \) of the class \( F_1 \).

(ii) Let \( R := R_{S-}^c \) be the smallest positive root of the equation (2.9) so that
\[
2(1 + 5R + R^2) = \sqrt{2}(1 - R^2)(2 + R).
\] (2.11)

Since the function
\[
h(r) := \frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 + 5r + r^2 - r^3)}{(2 + r)(1 - r^2)}
\]
is an increasing function of \( r \) in \([0, 1]\), it follows that, for \( 0 \leq r < R \), \( h(r) < h(R) = \sqrt{2} \) and hence, for \( 0 \leq r < R \), we have
\[
\frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} < \sqrt{2} - \frac{4 - 2r^2}{4 - r^2}. \tag{2.12}
\]

From (2.5) and (2.12), we obtain
\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \sqrt{2} - \frac{4 - 2r^2}{4 - r^2} \quad (|z| \leq r).
\]

For \( 0 \leq r < R \), the center of the above disk \( c(r) = (4 - 2r^2)/(4 - r^2) \) (being a decreasing function of \( r \) on \([0, 1]\)) lies in the interval \([c(R), 1] \subset (c(0.1), 1] \approx (0.997494, 1] \subset
[2\sqrt{2}/3, \sqrt{2})]. When \(a \in [2\sqrt{2}/3, \sqrt{2})\), by [1] Lemma 2.2, the disk \(|w - a| < \sqrt{2} - a\) is contained in the lemniscate region \(|w^3 - 1| < 1\) and hence, for \(0 \leq r < R\), we have
\[
\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1.
\]
Thus, the radius of lemniscate starlikeness of the class \(\mathcal{F}_1\) is at least \(R\). To show that the radius \(R\) is sharp, using (2.7) and (2.11), we see that the function \(f_1\) defined in (1.1) satisfies, at \(z = R\),
\[
\frac{zf_1'(z)}{f_1(z)} = \frac{2(1 + 5R + R^2 - R^3)}{(2 + R)(1 - R^2)} = \sqrt{2}
\]
and therefore
\[
\left| \left( \frac{zf_1'(z)}{f_1(z)} \right)^2 - 1 \right| = 1. \tag{2.13}
\]
(iii) The number \(R := R_{\mathcal{S}_p}\) is the smallest positive root of the equation
\[
2(1 - 5R + R^2 + R^3) = (1/2)(2 - R)(1 - R^2). \tag{2.14}
\]
Since the function
\[
h(r) := \frac{6r(r^2 - 3)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)}
\]
is decreasing function of \(r \in (0,1)\), it follows that, \(h(r) > h(R) = 1/2\) for \(0 \leq r < R\), and hence, for \(0 \leq r < R\), we have
\[
\frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{2}. \tag{2.15}
\]
From (2.5) and (2.15), we get
\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{2} \quad (|z| \leq r). \tag{2.16}
\]
For \(0 \leq r < R\), the center of the above disk \(c(r) = (4 - 2r^2)/(4 - r^2)\) (being a decreasing function of \(r\) on \([0,1]\)) lies in the interval \([c(R), 1] \subset (c(0.2), 1] \approx (.989899, 1] \subset (1/2, 3/2)\). When \(a \in (1/2, 3/2)\), by [26, Lemma 2.2], the disk \(|w - a| < a - (1/2)\) is contained in the parabolic region \(|w - 1| < \text{Re}(w)\) and hence, for \(0 \leq r < R\), we have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \text{Re} \left( \frac{zf'(z)}{f(z)} \right) \quad (|z| \leq r). \tag{2.17}
\]
Thus, the radius of parabolic starlikeness of the class \(\mathcal{F}_1\) is at least \(R\). To show that the radius \(R\) is sharp, using (2.7) and (2.14), we see that the function \(f_1\) defined in (1.1) satisfies, at \(z = -R\),
\[
\frac{zf_1'(z)}{f_1(z)} = \frac{2(1 - 5R + R^2 + R^3)}{(2 - R)(1 - R^2)} = \frac{1}{2}
\]
and therefore
\[
\left| \frac{zf_1'(z)}{f_1(z)} - 1 \right| = \frac{1}{2} = \text{Re} \left( \frac{zf_1'(z)}{f_1(z)} \right). \tag{2.18}
\]
(iv) The number $R := R_{S^*}$ is the smallest positive root of the equation
\begin{equation}
2(1 - 5R + R^2 + R^3) = (1/e)(2 - R)(1 - R^2).
\end{equation}
Since the function
\begin{equation}
h(r) := \frac{6r(r^2 - 3)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)}
\end{equation}
is decreasing function of $r$ in $[0,1)$, it follows that $h(r) > h(R) = 1/e$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have
\begin{equation}
\frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{e}.
\end{equation}
From (2.5) and (2.20), we get
\begin{equation}
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{e} \quad (|z| \leq r).
\end{equation}
For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (0.989899, 1] \subset (e^{-1}, (e + e^{-1})/2]$. When $a \in (e^{-1}, (e + e^{-1})/2]$, by [17 Lemma 2.2], the disk $|w - a| < a - e^{-1}$ is contained in the region $|\log w| < 1$ and hence, for $0 \leq r < R$,
\begin{equation}
|\log \left( \frac{zf'(z)}{f(z)} \right)| < 1 \quad (|z| \leq r).
\end{equation}
Thus, the radius of exponential starlikeness of the class $F_1$ is at least $R$. To show that the radius $R$ is sharp, using (2.7) and (2.19), we see that the function $f_1$ defined in (1.1) satisfies, at $z = -R$,
\begin{equation}
\frac{zf'_1(z)}{f_1(z)} = \frac{2(1 - 5R + R^2 + R^3)}{(2 + R)(1 - R^2)} = \frac{1}{e}
\end{equation}
and therefore
\begin{equation}
\left| \log \left( \frac{zf'_1(z)}{f_1(z)} \right) \right| = 1.
\end{equation}
(v) The number $R := R_{S^*}$ is the smallest positive root of the equation
\begin{equation}
2(1 - 5R + R^2 + R^3) = (1 - \sin 1)(2 - R)(1 - R^2).
\end{equation}
Since the function
\begin{equation}
h(r) := \frac{6r(r^2 - 3)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)} > 1 - \sin 1
\end{equation}
is decreasing function of $r$ in $[0,1)$, it follows that, $h(r) > h(R) = 1 - \sin 1$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have
\begin{equation}
\frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} + \sin 1 - 1.
\end{equation}
From (2.5) and (2.24), we get
\begin{equation}
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \sin 1 - 1 + \frac{4 - 2r^2}{4 - r^2} \quad (|z| \leq r).
For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (0.989899, 1] \subset (-1 - \sin 1, 1 - \sin 1)$. When $a \in (-1 - \sin 1, 1 + \sin 1)$, by (11), the disk $|w - a| < |a - 1|$ is contained in the region $\varphi_{c}(\mathbb{D})$, where $\varphi_{c}(z) = 1 + \sin z$ and hence, for $0 \leq r < R$, $sf(\mathbb{D}) \subset \varphi_{c}(\mathbb{D})$. Thus, the radius of sine starlikeness of the class $F$ is contained in the region $r < R$.

From (2.5) and (2.26), we get

$$
\frac{zf'(z)}{f(z)} = \frac{2(1 - 5R + R^2 + R^3)}{(2 - R)(1 - R^2)} = 1 - \sin 1 = \varphi_{c}(-1) \in \partial\varphi_{c}(\mathbb{D}).
$$

(vi) The number $R := R_{\mathcal{S}_{\lambda}}$ is the smallest positive root of the equation

$$
2(1 - 5R + R^2 + R^3) = (\sqrt{2} - 1)(2 - R)(1 - R^2). \tag{2.25}
$$

Since the function

$$
h(r) := \frac{6r^2 - 3}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 5r^2 + r^3)}{(2 - r)(1 - r^2)}
$$

is decreasing function of $r$ in $[0,1)$, it follows that $h(r) > h(R) = \sqrt{2} - 1$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$
\frac{6r^3 - 2r^2}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} + 1 - \sqrt{2}. \tag{2.26}
$$

From (2.5) and (2.26), we get

$$
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} + 1 - \sqrt{2} \quad (|z| \leq r). \tag{2.27}
$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (0.989899, 1] \subset (\sqrt{2} - 1, \sqrt{2} + 1)$. When $a \in (\sqrt{2} - 1, \sqrt{2} + 1)$, by (8) Lemma 2.1, the disk $|w - a| < 1 - |\sqrt{2} - a|$ is contained in the region $|w^2 - 1| < 2|w|$ and hence, for $0 \leq r < R$,

$$
\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 2 \left| \left( \frac{zf'(z)}{f(z)} \right) \right| \quad (|z| \leq r). \tag{2.27}
$$

Thus, the radius of lune starlikeness of the class $F$ is at least $R$. To show that the radius $R$ is sharp, using (2.7) and (2.25), we see that the function $f_1$ defined in (11) satisfies

$$
\frac{zf_1'(z)}{f_1(z)} = \frac{2(1 - 5R + R^2 + R^3)}{(2 - R)(1 - R^2)} = \sqrt{2} - 1
$$

at $z = -R$ and therefore

$$
\left| \left( \frac{zf_1'(z)}{f_1(z)} \right)^2 - 1 \right| = 2 \left| \left( \frac{zf_1'(z)}{f_1(z)} \right) \right|. \tag{2.28}
$$

(vii) The number $R := R_{\mathcal{S}_{\lambda}}$ is the smallest positive root of the equation

$$
2(1 - 5R + R^2 + R^3) = 2(\sqrt{2} - 1)(2 - R)(1 - R^2). \tag{2.29}
$$
Since the function
\[ h(r) := \frac{6r(r^2 - 3)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)} \]
is decreasing function of \( r \) in \([0,1]\), it follows that \( h(r) > h(R) = 2(\sqrt{2} - 1) \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have
\[ \frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - 2(\sqrt{2} - 1). \]
(2.30)
From (2.3) and (2.30), we get
\[ \left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - 2(\sqrt{2} - 1) \quad (|z| \leq r). \]
For \( 0 \leq r < R \), the center of the above disk \( c(r) = (4 - 2r^2)/(4 - r^2) \) (being a decreasing function of \( r \) on \([0,1]\)) lies in the interval \([c(R), 1] \subset (c(0.1), 1] \approx (.99741, 1] \subset (2(\sqrt{2} - 1), \sqrt{2}]. \) When \( a \in (2(\sqrt{2} - 1), \sqrt{2}], \) by [13] Lemma 2.2, the disk \(|w - a| < a - 2(\sqrt{2} - 1)\) is contained in the region \( \varphi_2(\mathbb{D}) \), where \( \varphi_2(z) := 1 + (zk + z^2/(k^2 - kz)) \) and \( k = \sqrt{2} + 1. \) Hence, for \( 0 \leq r < R \), \( s_f(\mathbb{D}_r) \subset \varphi_2(\mathbb{D}) \). Thus, the radius of the class \( \mathcal{F}_1 \) is at least \( R. \)
To show that the radius \( R \) is sharp, using (2.7) and (2.29), we see that the function \( f_1 \) defined in (1.1) satisfies, at \( z = -R, \)
\[ \frac{zf'(z)}{f_1(z)} = \frac{2(1 - 5R + R^2 + R^3)}{(2 + R)(1 - R^2)} = 2(\sqrt{2} - 1) = \varphi_2(-1) \in \partial \varphi_2(\mathbb{D}). \]
(viii) The number \( R := R_{S_{\mathbb{C}}} \) is the smallest positive root of the equation
\[ 2(1 - 5R + R^2 + R^3) = (1/3)(2 - R)(1 - R^2). \]
(2.31)
Since the function
\[ h(r) := \frac{6r(r^2 - 3)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 5r + r^2 + r^3)}{(2 - r)(1 - r^2)} \]
is decreasing function of \( r \) in \([0,1]\), it follows that \( h(r) > h(R) = 1/3 \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have
\[ \frac{6r(3 - r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{3}. \]
(2.32)
From (2.5) and (2.32), we get
\[ \left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{3} \quad (|z| \leq r). \]
For \( 0 \leq r < R \), the center of the above disk \( c(r) = (4 - 2r^2)/(4 - r^2) \) (being a decreasing function of \( r \) on \([0,1]\)) lies in the interval \([c(R), 1] \subset (c(0.1), 1] \approx (.989899, 1] \subset (1/3, 5/3). \) When \( a \in (1/3, 5/3), \) by [24] Lemma 2.5, the disk \(|w - a| < a - 1/3\) is lies in the cardioid region \( \varphi_3(\mathbb{D}) \). Hence, for \( 0 \leq r < R \), \( s_f(\mathbb{D}_r) \subset \varphi_3(\mathbb{D}) \). Thus, the radius of cardioid starlikeness of the class \( \mathcal{F}_1 \) is at least \( R. \) To show that the radius \( R \) is sharp, using (2.7) and (2.31), we see that the function \( f_1 \) defined in (1.1) satisfies, at \( z = -R, \)
\[ \frac{zf'(z)}{f_1(z)} = \frac{2(1 - 5R + R^2 + R^3)}{(2 - R)(1 - R^2)} = 1/3 = \varphi_3(-1) \in \partial \varphi_3(\mathbb{D}). \]
3. Radius Problem for $\mathcal{F}_2$

For the function $f \in \mathcal{F}_2$, there is a function $g \in \mathcal{A}$ such that $\text{Re}(g(z)/f(z)) > 1/2$ and $\text{Re}(2g(z)/(z^2 + 2)) > 0$. The functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ defined by $p_1(z) = g(z)/f(z)$, $p_2(z) = g(z)/(z + z^2/2)$ are the functions in $\mathcal{P}(1/2)$ and $\mathcal{P}(0)$ respectively and

$$f(z) = (p_2(z)/p_1(z))(z + z^2/2) \quad (z \in \mathbb{D}).$$

(3.1)

From (3.1), it follows that

$$zf'(z) = zp'_2(z) - \frac{zp'_1(z)}{p_1(z)} + \frac{2(z + 1)}{z + 2}.$$  

(3.2)

Using (2.1) and (2.2) in (3.2), we see that the image of the disk $\mathbb{D}_r$ under the mapping $zf'(z)/f(z)$ is contained in the disk

$$\left|zf'(z)/f(z) - \frac{4 - 2r^2}{4 - r^2}\right| \leq \frac{r(14 + 4r - 5r^2 + r^3)}{(1 - r^2)(4 - r^2)},$$

(3.3)

From (3.3), it readily follows that

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{4 - 2r^2}{4 - r^2} - \frac{r(14 + 4r - 5r^2 - r^3)}{(1 - r^2)(4 - r^2)} = \frac{2 - 8r - r^2 + 3r^3}{2(1 - r^2)(1 - r^2)} \quad (|z| \leq r).$$

(3.4)

Let $R_{S^*} \approx 0.248$ be the zero in $(0, 1)$ of the polynomial $3r^3 - r^2 - 8r + 2$. Then, for every function $f \in \mathcal{F}_2$, the inequality (3.4) shows that $\text{Re}(s_f(z)) > 0$ in each disk $\mathbb{D}_r$, for $0 \leq r < R_{S^*}$. For the function $f_2$ defined in (1.1), we have

$$s_{f_2}(z) = \frac{zf'_2(z)}{f_2(z)} = \frac{2 + 8z - z^2 - 3z^3}{(2 + z)(1 - z^2)}$$

(3.5)

and hence $\text{Re}(s_{f_2}(z))$ vanishes at $z = -R_{S^*}$. Thus, the radius of starlikeness $R_{S^*}$ for the class $\mathcal{F}_2$ is the smallest positive zero in $(0, 1)$ of the polynomial $P_2$ defined in (1.3) and is the same as the radius of univalence $R_S$. Using the inequality (3.3), we now determine $S^*(\alpha)$, $S^*_P$, $S^*_e^*$, $S^*_c^*$, $S^*_sin$, $S^*_q$ and $S^*_R$ radii for the class $\mathcal{F}_2$.

**Theorem 3.1.** The following sharp radii results hold for the class $\mathcal{F}_2$:

(i) For any $0 \leq \alpha < 1$, the radius $R_{S^*(\alpha)}$ is the smallest positive root of the polynomial

$$3(1 - \alpha)r^3 + (2\alpha - 1)r^2 - (8 - \alpha)r + 2 - 2\alpha = 0.$$  

(3.6)

(ii) The radius $R_{P^*}$ ($\approx 0.1341$) is the same as $R_{S^*(1/2)}$.

(iii) The radius $R_{S^*_e}$ ($\approx 0.16628$) is the same as $R_{S^*(1/\sqrt{2})}$.

(iv) The radius $R_{S^*_sin}$ ($\approx 0.2142$) is the same as $R_{S^*(1 - \sin 1)}$.

(v) The radius $R_{S^*_q}$ ($\approx 0.1551$) is the same as $R_{S^*(\sqrt{2} - 1)}$.

(vi) The radius $R_{S^*_R}$ ($\approx 0.0481$) is the same as $R_{S^*(2(\sqrt{2} - 1))}$.

(vii) The radius $R_{S^*_c}$ ($\approx 0.1744$) is the same as $R_{S^*(1/\sqrt{3})}$.

**Proof.** (i) Let the function $f \in \mathcal{F}_2$ and $\alpha \in [0, 1]$. Let $R := R_{S^*(\alpha)}$ be the smallest positive root of the equation (3.6) so that

$$2 - 8r - R^2 + 3R^3 = \alpha(2 - R)(1 - R^2).$$

(3.7)

The function

$$h(r) = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}$$

is the same as
is decreasing in \([0, 1]\) and hence, for \(0 \leq r < R\), we have, using (3.4) and (3.7),

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)} > \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = \alpha \quad (0 \leq r < R).
\]

This proves that the function \(f\) is starlike of order \(\alpha\) in each disk \(\mathbb{D}_r\), for \(0 \leq r < R\). At the point \(z = -R\), it can be seen, using (3.5) and (3.7), that the function \(f_2\) defined in (1.1) satisfies

\[
\text{Re} \left( \frac{zf'_2(z)}{f_2(z)} \right) = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = \alpha.
\]

This shows that the radius \(R\) is the radius of starlikeness of order \(\alpha\) of the class \(F\).

(ii) The number \(R := R_{S_F}\) is the smallest positive root of the equation

\[
2 - 8R - R^2 + 3R^3 = (1/2)(2 - R)(1 - R^2).
\]

Since the function

\[
h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}
\]

is decreasing function of \(r\) in \([0, 1]\), it follows that \(h(r) > h(R) = 1/2\) for \(0 \leq r < R\) and hence, for \(0 \leq r < R\), we have

\[
\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > \frac{1}{2}.
\]

From (3.3) and (3.9), we get

\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| \leq \frac{4 - 2r^2}{4 - r^2} - \frac{1}{2} \quad (|z| \leq r).
\]

For \(0 \leq r < R\), the center of the above disk \(c(r) = (4 - 2r^2)/(4 - r^2)\) (being a decreasing function of \(r\) on \([0, 1]\)) lies in the interval \([c(R), 1] \subset (c(0.2), 1) \approx (.9898991, 1] \subset (1/2, 3/2)\). When \(a \in (1/2, 3/2)\), by [26] Lemma 2.2, the disk \(|w - a| < a - (1/2)|\) is contained in the parabolic region \(|w - 1| < \text{Re}(w)|\) and hence, for \(0 \leq r < R\), we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \text{Re} \left( \frac{zf'(z)}{f(z)} \right) \quad (|z| \leq r).
\]

Thus, the radius of parabolic starlikeness of the class \(F_2\) is at least \(R\). To show that the radius \(R\) is sharp, using (3.5) and (3.8), we see that the function \(f_2\) defined in (1.1) satisfies

\[
\left| \frac{zf'_2(z)}{f_2(z)} - 1 \right| = \text{Re} \left( \frac{zf'_2(z)}{f_2(z)} \right).
\]

(iii) The number \(R := R_{S_F}\) is the smallest positive root of the equation

\[
2 - 8R - R^2 + 3R^3 = (1/e)(2 - R)(1 - R^2).
\]

Since the function

\[
h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}
\]
is decreasing function of \( r \) in \([0,1]\), it follows that \( h(r) > h(R) = 1/e \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have

\[
\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > \frac{1}{e}. \tag{3.13}
\]

From (3.3) and (3.13), we get

\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{e} \quad (|z| \leq r).
\]

For \( 0 \leq r < R \), the center of the above disk \( c(r) = (4 - 2r^2)/(4 - r^2) \) (being a decreasing function of \( r \) on \([0,1]\)) lies in the interval \([c(R), 1] \subset (c(0.2), 1] \approx (.9898991, 1] \subset (e^{-1}, (e + e^{-1})/2]. \) When \( a \in (e^{-1}, (e + e^{-1})/2] \), by [17] Lemma 2.2, the disk \( |w - a| < a - e^{-1} \) is contained in the region \(|\log w| < 1\) and hence, for \( 0 \leq r < R \), we have

\[
\left| \log \left( \frac{zf'(z)}{f(z)} \right) \right| < 1 \quad (|z| \leq r). \tag{3.14}
\]

Thus, the radius of exponential starlikeness of the class \( \mathcal{F}_2 \) is at least \( R \). To show that the radius \( R \) is sharp, using (3.15) and (3.12), we see that the function \( f_2 \) defined in (1.1) satisfies

\[
\frac{zf_2'(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = \frac{1}{e}
\]

at \( z = -R \) and therefore

\[
\left| \log \left( \frac{zf_2'(z)}{f_2(z)} \right) \right| = 1. \tag{3.15}
\]

(iv) The number \( R := R_{\sin}^{\varphi} \) is the smallest positive root of the equation

\[
2 - 8R - R^2 + 3R^3 = (1 - \sin 1)(2 - R)(1 - R^2). \tag{3.16}
\]

Since the function

\[
h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}
\]

is decreasing function of \( r \) in \([0,1]\), it follows that \( h(r) > h(R) = 1 - \sin 1 \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have

\[
\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > 1 - \sin 1. \tag{3.17}
\]

From (3.3) and (3.17), we get

\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| \leq \frac{4 - 2r^2}{4 - r^2} + \sin 1 - 1 \quad (|z| \leq r).
\]

For \( 0 \leq r < R \), the center of the above disk \( c(r) = (4 - 2r^2)/(4 - r^2) \) (being a decreasing function of \( r \) on \([0,1]\)) lies in the interval \([c(R), 1] \subset (c(0.3), 1] \approx (.976982, 1] \subset (-1 - \sin 1, 1 - \sin 1). \) When \( a \in (-1 - \sin 1, 1 + \sin 1) \), by [17] Lemma 3.3, the disk \( |w - a| < \sin 1 - |a - 1| \) is contained in the region \( \varphi_4(\mathbb{D}) \), where \( \varphi_4(z) = 1 + \sin z \) and hence, for \( 0 \leq r < R \), \( s_f(\mathbb{D}_r) \subset \varphi_4(\mathbb{D}) \). Thus, the radius of sine starlikeness of the class \( \mathcal{F}_2 \) is at
least \( R \). To show that the radius \( R \) is sharp, using (3.5) and (3.10), we see that the function \( f_2 \) defined in (1.1) satisfies, at \( z = -R \),

\[
\frac{zf'_2(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = 1 - \sin 1 = \varphi_4(-1) \in \partial \varphi_4(D).
\]

(v) The number \( R := R_{S \phi}^\ast \) is the smallest positive root of the equation

\[
2 - 8R - R^2 + 3R^3 = (\sqrt{2} - 1)(2 - R)(1 - R^2).
\]  

(3.18)

Since the function

\[
h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}
\]

is decreasing function of \( r \) in \([0,1]\), it follows that \( h(r) > h(R) = \sqrt{2} - 1 \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have

\[
\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > \sqrt{2} - 1.
\]  

(3.19)

From (3.3) and (3.19), we get

\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} + 1 - \sqrt{2} \quad (|z| \leq r).
\]

For \( 0 \leq r < R \), the center of the above disk \( c(r) = (4 - 2r^2)/(4 - r^2) \) (being a decreasing function of \( r \) on \([0,1]\)) lies in the interval \([c(R), 1] \subset (c(0.2), 1) \approx (.9898991, 1] \subset (\sqrt{2} - 1, \sqrt{2} + 1)\). When \( a \in (\sqrt{2} - 1, \sqrt{2} + 1) \), by [8, Lemma 2.1], the disk \( |w - a| < 1 - |\sqrt{2} - a| \) is contained in the region \( |w^2 - 1| < 2|w| \) and hence, for \( 0 \leq r < R \), we have

\[
\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 2 \left| \frac{zf'(z)}{f(z)} \right| \quad (|z| \leq r).
\]  

(3.20)

Thus, the radius of lune starlikeness of the class \( F_2 \) is at least \( R \). To show that the radius \( R \) is sharp, using (3.5) and (3.18), we see that the function \( f_2 \) defined in (1.1) satisfies

\[
\frac{zf'_2(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = \sqrt{2} - 1
\]

at \( z = -R \) and therefore

\[
\left| \left( \frac{zf'_2(z)}{f_2(z)} \right)^2 - 1 \right| = 2 \left| \frac{zf'_2(z)}{f_2(z)} \right|.
\]  

(3.21)

(vi) The number \( R := R_{S \phi}^\ast \) is the smallest positive root of the equation

\[
2 - 8R - R^2 + 3R^3 = 2(\sqrt{2} - 1)(2 - R)(1 - R^2).
\]  

(3.22)

Since the function

\[
h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}
\]
is decreasing function of $r$ in $[0,1]$, it follows that $h(r) > h(R) = 2(\sqrt{2} - 1)$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$
\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > 2(\sqrt{2} - 1) \quad (0 \leq r < R).
$$

(3.23)

From (3.3) and (3.23), we get

$$
\frac{|zf'(z)|}{f(z)} - \frac{4 - 2r^2}{4 - r^2} < \frac{4 - 2r^2}{4 - r^2} - 2(\sqrt{2} - 1) \quad (|z| \leq r).
$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.1), 1] \approx (0.99741, 1] \subset (2(\sqrt{2} - 1), \sqrt{2}]$.

When $a \in (2(\sqrt{2} - 1), \sqrt{2}]$, by [13] Lemma 2.2, the disk $|w - a| < a - 2(\sqrt{2} - 1)$ is contained in the region $\varphi_6(\mathbb{D})$, where $\varphi_6(z) := 1 + ((zk + z^2)/(k^2 - k))$ and $k = \sqrt{2} + 1$. Hence, for $0 \leq r < R$, $s_f(\mathbb{D}_r) \subset \varphi_6(\mathbb{D})$. Thus, the $S^*_R$ radius of the class $\mathcal{F}_2$ is at least $R$.

To show that the radius $R$ is sharp, using (3.5) and (3.22), we see that the function $f_2$ defined in (1.1) satisfies, at $z = -R$,

$$
\frac{zf'(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = 2(\sqrt{2} - 1) = \varphi_6(-1) \in \partial \varphi_6(\mathbb{D}).
$$

(vii) The number $R := R_{S^*_C}$ is the smallest positive root of the equation

$$
2 - 8R - R^2 + 3R^3 = (1/3)(2 - R)(1 - R^2).
$$

(3.24)

Since the function

$$
h(r) := \frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2 - 8r - r^2 + 3r^3}{(2 - r)(1 - r^2)}
$$

is decreasing function of $r$ in $[0,1]$, it follows that $h(r) > h(R) = 1/3$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$
\frac{r(r^3 + 5r^2 - 4r - 14)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} > \frac{1}{3}.
$$

(3.25)

From (3.3) and (3.25), we get

$$
\frac{|zf'(z)|}{f(z)} - \frac{4 - 2r^2}{4 - r^2} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{3} \quad (|z| \leq r).
$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.2), 1] \approx (0.989899, 1] \subset (1/3, 5/3)$. When $a \in (1/3, 5/3)$, by [24] Lemma 2.5, the disk $|w - a| < a - 1/3$ is lies in the cardioid region $\varphi_3(\mathbb{D})$. Hence, for $0 \leq r < R$, $s_f(\mathbb{D}_r) \subset \varphi_3(\mathbb{D})$. Thus, the $S^*_C$ radius of the class $\mathcal{F}_2$ is at least $R$. To show that the radius $R$ is sharp, using (3.5) and (3.24), we see that the function $f_2$ defined in (1.1) satisfies, at $z = -R$,

$$
\frac{zf'(z)}{f_2(z)} = \frac{2 - 8R - R^2 + 3R^3}{(2 - R)(1 - R^2)} = 1/3 = \varphi_3(-1) \in \partial \varphi_3(\mathbb{D}).
$$
4. Radius Problem for $F_3$

If the function $f \in F_3$, then the function $p : \mathbb{D} \to \mathbb{C}$ defined by $p(z) = f(z)/(z + z^2/2)$ is a function in the class $S(0)$ and

$$f(z) = p(z)(z + z^2/2) \quad (z \in \mathbb{D}). \quad (4.1)$$

From $(4.1)$, it follows that

$$\frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} + \frac{2(z + 1)}{z + 2}. \quad (4.2)$$

Using $(2.1)$ (with $\alpha = 0$) and $(2.2)$ in $(4.2)$, we see that the image of the disk $\mathbb{D}$, under the mapping $zf'(z)/f(z)$ is contained in the disk

$$\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| \leq \frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} \quad (|z| \leq r). \quad (4.3)$$

From $(4.3)$, it readily follows that

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq \frac{4 - 2r^2}{4 - r^2} - \frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} = \frac{2(1 - 3r + r^3)}{(2 - r)(1 - r^2)} \quad (|z| \leq r). \quad (4.4)$$

Let $R_{S^*} \approx 0.3473$ be the unique zero in $(0,1)$ of the polynomial $r^3 - 3r + 1$. Then, for every function $f \in F_3$, the inequality $(3.4)$ shows that $\operatorname{Re}(s_f(z)) > 0$ in each disk $\mathbb{D}$, for $0 \leq r < R_{S^*}$. For the function $f_3$ defined in $(1.2)$, we have

$$s_{f_3}(z) = \frac{zf_3'(z)}{f_3(z)} = \frac{2(1 + 3z - z^3)}{(2 + z)(1 - z^2)} \quad (4.5)$$

and hence $\operatorname{Re}(s_{f_3}(z))$ vanishes at $z = -R_{S^*}$. Thus, the radius of starlikeness $R_{S^*}$ for the class $F_3$ is the unique zero in $(0,1)$ of the polynomial $P_3$ defined in $(1.4)$ and is the same as the radius of univalence $R_S$. Using the inequality $(4.3)$, we now determine $S^*(\alpha), S^*_z, S^*_p, S^*_e, S^*_c, S^*_{\sin}, S^*_{\cos}$ and $S^*_R$ radii for the class $F_3$.

**Theorem 4.1.** The following sharp radii results hold for the class of function $F_3$:

- **(i)** For any $0 \leq \alpha < 1$, the radius $R_{S^*(\alpha)}$ is the smallest positive root of the polynomial

  $$(2 - \alpha)r^3 + (2\alpha)r^2 + (\alpha - 6)r + 2 - 2\alpha = 0. \quad (4.6)$$

- **(ii)** The radius $R_{S^*_{z}} (\approx 0.1645)$ is the smallest positive root of the polynomial

  $$(\sqrt{2} - 2)r^3 + (2\sqrt{2})r^2 + (6 - 2\sqrt{2})r + 2 - 2\sqrt{2} = 0. \quad (4.7)$$

- **(iii)** The radius $R_{S^*_p} (\approx 0.19028)$ is the same as $R_{S^*(1/2)}$.

- **(iv)** The radius $R_{S^*_e} (\approx 0.2355)$ is the same as $R_{S^*(1/e)}$.

- **(v)** The radius $R_{S^*_{\sin}} (\approx 0.3017)$ is the same as $R_{S^*(1-\sin 1)}$.

- **(vi)** The radius $R_{S^*_{\cos}} (\approx 0.2199)$ is the same as $R_{S^*(\sqrt{2}-1)}$.

- **(vii)** The radius $R_{S^*_R} (\approx 0.0679)$ is the same as $R_{S^*(2(\sqrt{2}-1))}$.

- **(viii)** The radius $R_{S^*_z} (\approx 0.2469)$ is the same as $R_{S^*(1/3)}$.

**Proof.** (i) Let the function $f \in F_3$ and $\alpha$ in $[0,1]$. The root $R := R_{S^*(\alpha)}$ be the smallest positive root of the equation $(4.6)$ so that

$$2(1 - 3R + R^3) = \alpha(2 - R)(1 - R^2). \quad (4.8)$$
The function
\[ h(r) := \frac{2(1 - 3r + r^3)}{(2 - r)(1 - r^2)} \]
is decreasing in \([0, 1]\) and hence, for \(0 \leq r < R\), we have, using (4.4) and (4.8),
\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq \frac{2(1 - 3r + r^3)}{(2 - r)(1 - r^2)} > \frac{2(1 - 3R + R^3)}{(2 - R)(1 - R^2)} = \alpha. \]
This proves that the function \(f\) is starlike of order \(\alpha\) in each disk \(D_r\) for \(0 \leq r < R\). At the point \(z = -R\), it can be seen, using (4.5) and (4.8), that the function \(f_3\) defined in (1.2) satisfies
\[ \text{Re} \left( \frac{zf'_3(z)}{f_3(z)} \right) = \frac{2(1 - 3R - R^3)}{(2 - R)(1 - R^2)} = \alpha. \]
This shows that the radius \(R\) is the radius of starlikeness of order \(\alpha\) of the class \(F_3\).

(ii) Let \(R := R_{F_3}\) be the smallest positive root of the equation (4.7) so that
\[ 2(1 + 3R - R^3) = \sqrt{2}(2 + R)(1 - R^2). \]
Since the function
\[ h(r) := \frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 + 3r - r^3)}{(1 - r^2)(2 + r)} \]
is an increasing function of \(r\) in \([0, 1]\), it follows that \(h(r) < h(R) = \sqrt{2}\) for \(0 \leq r < R\) and hence, for \(0 \leq r < R\), we have
\[ \frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} < \sqrt{2} - \frac{4 - 2r^2}{4 - r^2}. \]
From (4.3) and (4.10), we get
\[ \left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \sqrt{2} - \frac{4 - 2r^2}{4 - r^2} \quad (|z| \leq r). \]
For \(0 \leq r < R\), the center of the above disk \(c(r) = (4 - 2r^2)/(4 - r^2)\) (being a decreasing function of \(r\) on \([0, 1]\)) lies in the interval \([c(R), 1] \subset (c(0.1), 1] \approx (0.997494, 1] \subset [2\sqrt{2}/3, \sqrt{2}]\). When \(a \in [2\sqrt{2}/3, \sqrt{2}]\), by [11 Lemma 2.2], the disk \(|w - a| < \sqrt{2} - a\) is contained in the lemniscate region \(|w^2 - 1| < 1\) and hence, for \(0 \leq r < R\), we have
\[ \left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1. \]
Thus, the radius of lemniscate starlikeness of the class \(F_3\) is at least \(R\). To show that the radius \(R\) is sharp, using (4.5) and (4.9), we see that the function \(f_3\) defined in (1.2) satisfies
\[ \frac{zf'_3(z)}{f_3(z)} = \frac{2(1 + 3R - R^3)}{(1 - R^2)(2 + R)} = \sqrt{2} \]
at \(z = -R\) and therefore
\[ \left| \left( \frac{zf'_3(z)}{f_3(z)} \right)^2 - 1 \right| = 1. \]
(iii) The number \( R := R_{S_p} \) is the smallest positive root of the equation
\[
2(1 - 3R + R^3) = (1/2)(2 - R)(1 - R^2).
\] (4.13)

Since the function
\[
h(r) := \frac{2r(2r^2 - 5)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 3r + r^3)}{(1 - r^2)(2 - r)}
\]
is decreasing function of \( r \) in \([0,1]\), it follows that, \( h(r) > h(R) = 1/2 \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have
\[
\frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{2}.
\] (4.14)

From (4.3) and (4.14), we get
\[
\text{For } 0 \leq r < R, \text{ the center of the above disk } c(r) = (4 - 2r^2)/(4 - r^2) \text{ (being a decreasing function of } r \text{ on } [0,1]) \text{ lies in the interval } [c(R), 1] \subset (c(0.2), 1] \approx (0.9898991, 1] \subset (1/2, 3/2).\]

When \( a \in (1/2, 3/2) \), by [26] Lemma 2.2, the disk \( |w - a| < a - (1/2) \) is contained in the parabolic region \( |w - 1| < \text{Re}(w) \) and hence, for \( 0 \leq r < R \), we have
\[
\left| \frac{zf''(z)}{f'(z)} - 1 \right| < \text{Re}\left( \frac{zf''(z)}{f(z)} \right) \quad (|z| \leq r).
\] (4.15)

Thus, the radius of parabolic starlikeness of the class \( \mathcal{F}_3 \) is at least \( R \). To show that the radius \( R \) is sharp, using (4.5) and (4.13), we see that the function \( f_3 \) defined in (1.2) satisfies
\[
\frac{zf_3''(z)}{f_3(z)} = \frac{2(R^3 - 3R + 1)}{(1 - R^2)(2 - R)} = \frac{1}{2}
\]
at \( z = -R \) and therefore
\[
\left| \frac{zf_3''(z)}{f_3(z)} - 1 \right| = \frac{1}{2} = \text{Re}\left( \frac{zf_3''(z)}{f_3(z)} \right).
\] (4.16)

(iv) The number \( R := R_{S^*_p} \) is the smallest positive root of the equation
\[
2(1 - 3R + R^3) = (1/e)(2 - R)(1 - R^2).
\] (4.17)

Since the function
\[
h(r) := \frac{2r(2r^2 - 5)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 3r + r^3)}{(1 - r^2)(2 - r)}
\]
is decreasing function of \( r \) in \([0,1]\), it follows that \( h(r) > h(R) = 1/e \) for \( 0 \leq r < R \) and hence, for \( 0 \leq r < R \), we have
\[
\frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{e}.
\] (4.18)

From (4.3) and (4.18), we get
\[
\left| \frac{zf''(z)}{f(z)} - 1 \right| < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{e} \quad (|z| \leq r).
\]
For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (.976982, 1] \subset (e^{-1}, (e + e^{-1})/2]$. When $a \in (e^{-1}, (e + e^{-1})/2]$, by [17] Lemma 2.2, the disk $|w - a| < a - e^{-1}$ is contained in the region $|\log w| < 1$ and hence, for $0 \leq r < R$, we have
\[
\left| \log \left( \frac{zf'(z)}{f(z)} \right) \right| < 1 \quad (|z| \leq r).
\] (4.19)

Thus, the radius of exponential starlikeness of the class $\mathcal{F}_3$ is at least $R$. To show that the radius $R$ is sharp, using (4.5) and (4.17), we see that the function $f_3$ defined in (1.2) satisfies
\[
zf'_3(z) = \frac{2(1 - 3R + R^3)}{(1 - R^2)(2 - R)} = \frac{1}{e}
\]

at $z = -R$ and therefore
\[
\left| \log \left( \frac{zf'_3(z)}{f_3(z)} \right) \right| = 1. \tag{4.20}
\]

(v) The number $R := R_{\text{sin}}^*$ is the smallest positive root of the equation
\[
2(1 - 3R + R^3) = (1 - \sin 1)(2 - R)(1 - R^2). \tag{4.21}
\]

Since the function
\[
h(r) := \frac{2r(2r^2 - 5)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 3r + r^3)}{(1 - r^2)(2 - r)}
\]
is decreasing function of $r$ in $[0,1)$, it follows that $h(r) > h(R) = 1 - \sin 1$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have
\[
\frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} + \sin 1 - 1. \tag{4.22}
\]

From (4.3) and (4.22), get
\[
\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \sin 1 - 1 + \frac{4 - 2r^2}{4 - r^2} \quad (|z| \leq r).
\]

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (.995833, 1] \subset (-1 - \sin 1, 1 - \sin 1)$. When $a \in (-1 - \sin 1, 1 + \sin 1)$, by [7] Lemma 3.3, the disk $|w - a| < \sin 1 - |a - 1|$ is contained in the region $\varphi_4(\mathbb{D})$, where $\varphi_4(z) = 1 + \sin z$ and hence, for $0 \leq r < R$, $s_f(\mathbb{D}_r) \subset \varphi_4(\mathbb{D})$. Thus, the radius of sine starlikeness of the class $\mathcal{F}_3$ is at least $R$. To show that the radius $R$ is sharp, using (1.5) and (1.21), we see that the function $f_3$ defined in (1.2) satisfies, at $z = -R$,
\[
zf'_3(z) = \frac{2(1 - 3R + R^3)}{(2 + R)(1 - R^2)} = 1 - \sin 1 = \varphi_4(-1) \in \partial \varphi_4(\mathbb{D}).
\]

(vi) The number $R := R_{\text{sin}}^*$ is the smallest positive root of the equation
\[
2(1 - 3R + R^3) = (\sqrt{2} - 1)(2 - R)(1 - R^2). \tag{4.23}
\]

Since the function
\[
h(r) := \frac{2r(2r^2 - 5)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 3r + r^3)}{(1 - r^2)(2 - r)}
\]
is a decreasing function of $r$ in $[0,1)$, it follows that $h(r) > h(R) = \sqrt{2} - 1$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$\frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} + 1 - \sqrt{2}. \quad (4.24)$$

From (4.3) and (4.24), we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} + 1 - \sqrt{2} \quad (|z| \leq r).$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (.976982, 1] \subset (\sqrt{2} - 1, \sqrt{2} + 1)$. When $a \in (\sqrt{2} - 1, \sqrt{2} + 1)$, by [8 Lemma 2.1], the disk $|w - a| < 1 - |\sqrt{2} - a|$ is contained in the region $|w^2 - 1| < 2|w|$ and hence, for $0 \leq r < R$, we have

$$\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} + 1 - \sqrt{2} \quad (|z| \leq r). \quad (4.25)$$

Thus, the radius of lune starlikeness of the class $\mathcal{F}_3$ is at least $R$. To show that the radius $R$ is sharp, using (4.5) and (4.23), we see that the function $f_3$ defined in (1.2) satisfies

$$\frac{zf'_3(z)}{f_3(z)} = \frac{2(1 - 3R + R^3)}{(1 - R^2)(2 - R)} = \sqrt{2} - 1$$

at $z = -R$ and therefore

$$\left| \frac{(zf'_3(z))^2}{f_3(z)} - 1 \right| = 2 \left| \frac{zf'_3(z)}{f_3(z)} \right|. \quad (4.26)$$

(vii) The number $R := R_{S'_R}$ is the smallest positive root of the equation

$$2(1 - 3R + R^3) = 2(\sqrt{2} - 1)(2 - R)(1 - R^2). \quad (4.27)$$

Since the function

$$h(r) := \frac{2r(2r^2 - 5)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 3r + r^3)}{(1 - r^2)(2 - r)}$$

is a decreasing function of $r$ in $[0,1)$, it follows that $h(r) > h(R) = 2(\sqrt{2} - 1)$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have

$$\frac{2r(5 - 2r^2)}{(1 - r^2)(4 - r^2)} < \frac{4 - 2r^2}{4 - r^2} - 2(\sqrt{2} - 1). \quad (4.28)$$

From (4.3) and (4.28), we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{4 - 2r^2}{4 - r^2} \right| < \frac{4 - 2r^2}{4 - r^2} - 2(\sqrt{2} - 1) \quad (|z| \leq r).$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0,1]$) lies in the interval $[c(R), 1] \subset (c(0.1), 1] \approx (.99741, 1] \subset (2(\sqrt{2} - 1), \sqrt{2})$. When $a \in (2(\sqrt{2} - 1), \sqrt{2})$, by [13 Lemma 2.2], the disk $|w - a| \neq a - 2(\sqrt{2} - 1)$ is contained in the region $\varphi_0(\mathbb{D})$, where $\varphi_0(z) := 1 + ((zk + z^2)/(k^2 - k))$ and $k = \sqrt{2} + 1$. Hence, for $0 \leq r < R$, $s_f(\mathbb{D}_r) \subset \varphi_0(\mathbb{D})$. Thus, the $S'_R$ radius of the class $\mathcal{F}_3$ is at least $R$. 

To show that the radius $R$ is sharp, using (4.15) and (4.23), we see that the function $f_3$ defined in (1.2) satisfies, at $z = -R$, $$rac{zf_3'(z)}{f_3(z)} = 2(1 - 3R + R^3) = 2(\sqrt{2} - 1) = \varphi_6(-1) \in \partial\varphi_6(\mathbb{D}).$$

(viii) The number $R := R_{S^c_3}$ is the smallest positive root of the equation $$2(1 - 3R + R^3) = (1/3)(2 - R)(1 - R^2).$$

Since the function $$h(r) := \frac{2r(2r^2 - 5)}{(1 - r^2)(4 - r^2)} + \frac{4 - 2r^2}{4 - r^2} = \frac{2(1 - 3r + r^3)}{(1 - r^2)(2 + r)}$$
is decreasing function of $r$ in $[0, 1]$, it follows that $h(r) > h(R) = 1/3$ for $0 \leq r < R$ and hence, for $0 \leq r < R$, we have $$\frac{2r(5 - 2r^2)}{4 - r^2} < \frac{4 - 2r^2}{4 - r^2} - \frac{1}{3}. \quad (4.30)$$

From (4.3) and (4.30), we get $$|zf'(z)| \leq \frac{4 - 2r^2}{4 - r^2} - \frac{1}{3} \quad (|z| \leq r).$$

For $0 \leq r < R$, the center of the above disk $c(r) = (4 - 2r^2)/(4 - r^2)$ (being a decreasing function of $r$ on $[0, 1]$) lies in the interval $[c(R), 1] \subset (c(0.3), 1] \approx (0.976982, 1] \subset (1/3, 5/3)$. When $a \in (1/3, 5/3)$, by [24, Lemma 2.5], the disk $|w - a| < a - 1/3$ is lies in the cardioid region $\varphi_3(\mathbb{D})$. Hence, for $0 \leq r < R$, $s_f(D_r) \subset \varphi_3(\mathbb{D})$. Thus, the radius of the class $\mathcal{F}_3$ is at least $R$. To show that the radius $R$ is sharp, using (4.5) and (4.29), we see that the function $f_3$ defined in (1.1) satisfies, at $z = -R$, $$\frac{zf_3'(z)}{f_3(z)} = \frac{2(1 - 3R + R^3)}{(2 - R)(1 - R^2)} = 1/3 = \varphi_3(-1) \in \partial\varphi_3(\mathbb{D}).$$

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