Finding Small Multi-Demand Set Covers with Ubiquitous Elements and Large Sets is Fixed-Parameter Tractable

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Abstract

We study a variant of \textsc{Set Cover} where each element of the universe has some demand that determines how many times the element needs to be covered. Moreover, we examine two generalizations of this problem where a set can be included multiple times and where sets have different prices. We prove that all three problems are fixed-parameter tractable with respect to the combined parameter budget, the maximum number of elements missing in one of the sets, and the maximum number of sets in which one of the elements does not occur. Lastly, we point out how our fixed-parameter tractability algorithm can be applied in the context of bribery for the (collective-decision) group identification problem.

We consider the following variant of the traditional \textsc{Set Cover} problem:

\textbf{Set Cover with Demands}

\textbf{Input:} Universe $U = [n]$, a list of demands $d_1, \ldots, d_n \in [m]$, a family of subsets $F = \{F_1, \ldots, F_m\}$ over $U$, and an integer $k \in \mathbb{N}$.

\textbf{Question:} Does there exist a subset $S \subseteq F$ with $|S| = k$ such that each element $i \in U$ is included in at least $d_i$ sets from $S$?

We start by introducing some notation. A family of subsets over some universe $U$ is called covering system. Given some covering system $\mathcal{F}$ and an element $i \in U$, we denote by $\mathcal{F}(i)$ the subfamily of sets containing $i$. Analogously, for a subset of elements $F \subseteq U$, we denote the subfamily of sets from $\mathcal{F}$ containing any element from $F$ as $\mathcal{F}(F)$, i.e., $\mathcal{F}(F) = \bigcup_{i \in F} \mathcal{F}(i)$.

Let $s_{\text{min}} = \min_{i \in m} |F_i|$ be the minimum size of a set from the covering system and $o_{\text{min}} = \min_{i \in n} |\mathcal{F}(i)|$ the minimum number of occurrences of an element in sets from the covering system. Symmetrically, let $s_{\text{max}} = \max_{i \in m} |F_i|$ be the maximum size of a set from the covering system and $o_{\text{max}} = \max_{i \in n} |\mathcal{F}(i)|$ the maximum number of occurrences of an element in sets from the covering system.

We consider \textsc{Set Cover with Demands} parameterized by the number of sets to be selected, the maximum number of elements missing in a set, and the maximum number of sets in which an element does not appear. This means that we develop
an algorithm for situations in which the budget is small, each set contains almost all elements, and each element is contained in almost all sets. In the following, we show that Set Cover with Demands parameterized by \( k + (n - s_{\text{min}}) + (m - o_{\text{min}}) \) is fixed-parameter tractable. To this end, we show fixed-parameter tractability of the “complement” problem:

**Set Cover with Capacities**

**Input:** Universe \( U = [n] \), a list of capacities \( c_1, \ldots, c_n \in [m] \), a family of subsets \( \mathcal{F} = \{F_1, \ldots, F_m\} \) over \( U \), and an integer \( k \in \mathbb{N} \).

**Question:** Does there exist a subset \( S \subseteq \mathcal{F} \) with \(|S| = k\) such that each element \( i \in U \) is included in at most \( c_i \) sets from \( S \)?

Note that each instance \( I' \) of Set Cover with Demands with parameter \( s'_{\text{min}} \) and \( o'_{\text{min}} \) can be easily transformed into an instance \( I \) of Set Cover with Capacities by replacing each set from the covering system by its complement and setting \( c_i = k - d_i \) for all \( i \in U \). By replacing all sets by their complement, it holds that \( s_{\text{max}} = n - s'_{\text{min}} \) and \( o_{\text{max}} = m - o'_{\text{min}} \). Thus, to show our initial claim, it is enough to show the following theorem.

**Theorem 1.** Parameterized by \( k + s_{\text{max}} + o_{\text{max}} \), Set Cover with Capacities is solvable in \( \mathcal{O}\left((n + m) \cdot (s_{\text{max}} \cdot o_{\text{max}})^k\right) \) time.

**Proof.** The algorithm for this problem follows a branch-and-bound approach and constructs a solution \( S \) by adding a set to the solution at each level. To do so, the algorithm keeps track of the elements \( U' \) that still have free capacity left and the collection of sets \( \mathcal{F}' \) where all elements in the set have still free capacity. These are the sets that can still be added to \( S \). At each level, we select one set \( F^* \) from \( \mathcal{F}' \) and branch over adding to \( S \) this set or a set that overlaps with \( F^* \) in at least one element. We will see that if there exists a solution not using any additional of the sets overlapping with \( F^* \), then there needs to exist a solution containing \( F^* \), as it is possible to replace one of the sets in the solution without \( F^* \) by \( F^* \) without violating the capacity constraints of any element.

This reasoning gives rise to the recursive algorithm presented in Algorithm 1, which can be called to solve the problem as CalcCover(\( U, \mathcal{F}, U', \mathcal{F}', \emptyset, k, c_1, \ldots, c_m \)).

Every solution returned by the algorithm is also an actual solution of the problem, as we only add sets where all elements in the set still have capacity left. Let \( Z \subseteq 2^\mathcal{F} \) be the set of all solutions. In the following, assuming that \( Z \neq \emptyset \), we prove that Algorithm 1 calculates a solution by proving via induction that at each depth \( i \in [1, k] \) there exists at least one branch which has calculated an \( i \)-subset of a solution from \( Z \).

We start by examining the initial call of the algorithm, i.e. depth 1. Let \( F^* \) and \( \mathcal{F}' \) be the values of the respective variables after line 8. We claim that there exists a solution \( Z \in Z \) of which either the selected set \( F^* \) or at least one set \( F \in \mathcal{F}' \cap \mathcal{F}(F^*) \) is part of. Let \( Z' \in Z \) be a solution where this is not the case. As \( F^* \in \mathcal{F}' \), it holds that \( c_i > 0 \) for all \( i \in F^* \). Moreover, as no set from \( \mathcal{F} \setminus \mathcal{F}' \) can be part of \( Z' \) and neither \( F^* \) nor a set from \( \mathcal{F}' \cap \mathcal{F}(F^*) \) is part of \( Z' \), all elements from \( F^* \) are not covered at all by \( Z' \). Consequently, it is possible to replace an arbitrary set in \( Z' \) by \( F^* \).
Corollary 1. Parameterized by $k + (n - s_{\min}) + (m - o_{\min})$, Set Cover with Demands is solvable in $O \left( (n \cdot m) \cdot (n - s_{\min}) \cdot (m - o_{\min})^k \right)$ time.

In the following, we will explain how the algorithm from above can be adapted to also work for two variants of the Set Cover with Demands and Set Cover with Capacities problems.

However, before we do so, we want to remark that if the only goal is to prove that the two problems are fixed-parameter tractable with respect to the three considered parameters and the exact running time of the algorithm is not important, it is also possible to employ the following simpler algorithm for Set Cover with Capacities.

We start by deleting all elements with zero capacity from the instance and all sets that contain one of these elements. For the resulting instance, we distinguish
two different cases: In case that the size $m$ of the covering system $\mathcal{F}$ is smaller than or equal to $k \cdot s_{\text{max}} \cdot o_{\text{max}}$, we brute force over all $k$-subsets of $\mathcal{F}$ and check for each of them whether it is a valid solution. Otherwise, it holds that $m > k \cdot s_{\text{max}} \cdot o_{\text{max}}$. This directly implies that it is always possible to construct a valid solution as follows. We start by picking an arbitrary set from $\mathcal{F}$ and include it in the solution $\mathcal{S}$. Subsequently, we delete all sets from $\mathcal{F}$ in which an element $i$ occurs for which the number of sets including $i$ in $\mathcal{S}$ is equal to $c_i$. We repeat the two steps from above until $\mathcal{S}$ contains $k$ sets. As each set contains at most $s_{\text{max}}$ elements, in each step, only the capacity of at most $s_{\text{max}}$ different elements can get full. Moreover, as each element is contained in at most $o_{\text{max}}$ different sets, in each step, only at most $s_{\text{max}} \cdot o_{\text{max}}$ sets can get deleted from the covering system $\mathcal{F}$. Since it holds that $m > k \cdot s_{\text{max}} \cdot o_{\text{max}}$, it is always possible to construct a solution of size $k$, as there will always be a set left in $\mathcal{F}$ that one can pick during the construction of the solution.

Note that this algorithm has a running time of $O(2^{k \cdot s_{\text{max}} \cdot o_{\text{max}}})$. The approach can be extended if multiplicities are allowed but does not longer work in the presence of prices.

**Introducing Multiplicities**

So far, we required that each set from the covering system can be only included once in the solution. However, it is also possible to allow that a set is allowed to be included an arbitrary number of times in the solution arriving at the following adapted versions of our two problems:

**Set Cover with Demands (Capacities) and Multiplicities**

**Input:** Universe $U = [n]$, a list of demands $d_1, \ldots, d_n \in [m]$ (capacities $c_1, \ldots, c_n \in [m]$), a family of subsets $\mathcal{F} = \{F_1, \ldots, F_m\}$ over $U$, and an integer $k \in \mathbb{N}$.

**Question:** Do there exist integer multiplicities $\ell_1 + \cdots + \ell_m = k$ such that for each element $i \in U$ it holds that $\sum_{j \in F(i)} \ell_j$ is at least $d_i$ (at most $c_i$)?

Algorithm 1 needs to be only slightly adapted to solve these problems. Firstly, $\mathcal{S}$ needs to be a multiset of sets and secondly, line 10 of Algorithm 1 needs to be modified as follows. Instead of excluding the selected set $F$ from the collection of sets $\mathcal{F}'$ that are still in question to be used, $F$ remains in $\mathcal{F}'$. Thereby, each set can be selected multiple times. In fact, all arguments from the proof of Theorem 1 still apply here. The only small modification that needs to be made is that in the induction step we assume that $\mathcal{Z}' \in \mathcal{Z}$ with $\mathcal{S} \subseteq \mathcal{Z}'$ is a solution where neither $F^*$ nor some $F \in \mathcal{F}' \cap \mathcal{F}(F^*)$ is part of $\mathcal{Z}' \setminus \mathcal{S}$. This proves the following corollary:

**Corollary 2.** Parameterized by $k + s_{\text{max}} + o_{\text{max}}$, Set Cover with Capacities and Multiplicities is solvable in $O\left((n \cdot m) \cdot (s_{\text{max}} \cdot o_{\text{max}})^k\right)$ time.

Parameterized by $k + (n - s_{\text{min}}) + (m - o_{\text{min}})$, Set Cover with Demands and Multiplicities is solvable in $O\left((n \cdot m) \cdot ((n - s_{\text{min}}) \cdot (m - o_{\text{min}}))^k\right)$ time.
Introducing Prices

It is also possible to consider a generalized version of the two considered problems where the different sets have different prices:

**Set Cover with Demands (Capacities) and Prices**

**Input:** Universe $U = [n]$, list of demands $d_1, \ldots, d_n \in [m]$ (capacities $c_1, \ldots, c_n \in [m]$), a list of prices $p_1, \ldots, p_n \in \mathbb{N}$, a family of subsets $\mathcal{F} = \{F_1, \ldots, F_m\}$ over $U$, and two integers $k \in \mathbb{N}$ and $t \in \mathbb{N}$.

**Question:** Does there exist a subset $S \subseteq \mathcal{F}$ with $|S| = k$ and $\sum_{F_i \in S} p_i \leq t$ such that each element $i \in U$ is included in at least $d_i$ (at most $c_i$) sets from $S$?

Note that, in principle, it is also possible to drop the constraint that exactly $k$ sets need to be selected here. However, the resulting variant with capacities would become trivial then. Because of this and to ensure that the two problems can be still directly related, we selected the formulation from above. Nevertheless, our algorithm for the problems from above is also applicable to different variants of the problems without $k$, since as a first step it is possible to guess the value of $k$ which is guaranteed to lie between 1 and $t$.

Algorithm $[1]$ can be again slightly adjusted to solve Set Cover with Capacities and Prices, and, thus, also the problem with demands. To do so, we need to provide the prices of the sets and the budget $t$ as part of the input of the algorithm. Moreover, in line 6, we also reject if the price of the constructed solution $S$ exceeds $t$. Lastly, in line 8, we always pick the set with the lowest price from $\mathcal{F}'$. Again, the argumentation from the proof of Theorem $[1]$ still applies here. The only additional observation one needs is that $F^*$ is always the cheapest set from $\mathcal{F}'$. Thus, if there exists a solution $Z'$ with $S \subseteq Z'$ not containing $F^*$ or some set from $\mathcal{F}' \cap \mathcal{F}(F^*)$, it is always possible to replace one set from $Z' \setminus S$ by $F^*$, as all sets from $Z' \setminus S$ need to be part of $\mathcal{F}'$, all elements from $F^*$ need to have free capacity and $F^*$ is guaranteed to be not more expensive than all sets from $Z' \setminus S$.

**Corollary 3.** Parameterized by $k + s_{\text{max}} + o_{\text{max}}$, Set Cover with Capacities and Prices is solvable in $O\left((n \cdot m) \cdot (s_{\text{max}} \cdot o_{\text{max}})^k\right)$ time.

Parameterized by $k + (n - s_{\text{min}}) + (m - o_{\text{min}})$, Set Cover with Demands and Prices is solvable in $O\left((n \cdot m) \cdot ((n - s_{\text{min}}) \cdot (m - o_{\text{min}}))^k\right)$ time.

**Application: Bribery in Group Identification**

We now describe an application where the Set Cover with Capacities problem naturally arises and our FPT algorithm is directly applicable: In group identification, we are given a set $A = \{a_1, \ldots, a_n\}$ of agents and the task is to identify a so-called socially qualified subgroup of the agents $[3]$. To do so, we are given a qualification profile $\varphi: A \times A \rightarrow \{-1, 1\}$ that denotes for each agent $a$ which of the other agents $a$ deems qualified, i.e., agent $a$ qualifies $a'$ if $\varphi(a, a') = 1$ and disqualifies $a'$ if $\varphi(a, a') = -1$. For an agent $a \in A$, let $Q^+_{\varphi}(a) = \{a' \in A \mid \varphi(a', a) = 1\}$ denote the set of agents qualifying $a$ and $Q^-_{\varphi}(a) = \{a' \in A \mid \varphi(a', a) = -1\}$ the set of agents disqualifying $a$.

To decide given a set of agents and a qualification profile which agents are socially qualified, different social rules have been proposed. One popular rule parameterized
by two integers $s$ and $t$ with $s + t \leq n + 2$ is the consent rule, denoted $f^{(s,t)}$ \[4\]. Under the consent rule, an agent $a \in A$ with $\varphi(a,a) = 1$ is socially qualified if and only if at least $s$ agents (including $a$ itself) qualify $a$. Similarly, an agent $a \in A$ with $\varphi(a,a) = -1$ is socially disqualified if and only if at least $t$ agents (including $a$ itself) disqualify $a$.

Recently, Erdelyi et al. \cite{erdelyi2020} initiated the study of the computational complexity of bribery in the context of group identification, among others, asking the following question:

**Constructive-$f^{(s,t)}$ Agent Bribery**

**Input:** Set $A$ of agents, qualification profile $\varphi$, subset $A^+ \subseteq A$ of agents to be made socially qualified, and budget $\ell$.

**Question:** Is it possible to modify the opinion of at most $\ell$ agents such that after the modifications all agents from $A^+$ are socially qualified under the consent rule $f^{(s,t)}$?

Erdelyi et al. \cite{erdelyi2020} proved that this problem is NP-hard even for $s = 1$ and $t = 2$. Subsequently, Boehmer et al. \cite{boehmer2021} conducted a detailed study of the parameterized complexity of this question considering the parameters $\ell$, $s$, $t$, and $|A^+|$. Among others, they proved that Constructive-$f^{(s,t)}$ Agent Bribery is $W[1]$-hard with respect to $s + \ell$ even if $t = 1$.

Let $\Delta$ be the maximum number of agents from $A^+$ that an agent $a \in A$ qualifies, i.e., $\Delta := \max_{a \in A} |\{a' \in A^+ \mid \varphi(a,a') = 1\}|$. We now prove that our algorithm for Set Cover with Demands can be used to prove that Constructive-$f^{(s,t)}$ Agent Bribery with $t = 1$ is fixed-parameter tractable with respect to $\Delta + s$.

**Theorem 2.** Constructive-$f^{(s,t)}$ Agent Bribery is solvable in $O(n^2 \cdot (\Delta \cdot s)^s)$ time for $t = 1$.

**Proof.** First of all, as $t = 1$ implies that every agent that disqualifies itself is also socially disqualified, we bribe all agents in $A^+$ who do not qualify themselves to qualify everyone and adjust the budget $\ell$ accordingly. We delete from $A^+$ all agents who are already socially qualified after this bribery, while keeping them in the set of agents.

If we have $\ell \geq s$ for the resulting budget, we are done as we can simply pick $\ell$ agents and make them qualify everyone, which results in all agents from $A^+$ being socially qualified.

Consequently, we are left with the situation where $\ell < s$. We now reduce the problem to an instance of Set Cover with Demands as follows. We set the universe $U = A^+$ and for each $a \in A^+$ its demand to $s - |Q^+(a)|$ (the number of additional qualification $a$ needs to get by the bribery to become socially qualified). For each agent $a \in A$ who does not qualify all other agents, we add a set $F_a$ to our covering system $F$ containing all agents from $A^+$ which $a$ does not qualify, i.e., $F_a := \{a' \in A^+ \mid \varphi(a,a') = -1\}$. Finally, we set $k := \ell$. Bribing an agent $a \in A$, which results in all agents from $F_a$ getting an additional qualification, corresponds to including $F_a$ in the cover. It is easy to see that there exists a successful bribery if and only if there exists a solution to the constructed Set Cover with Demands instance. Note that in the constructed instance $k$ is bounded by $s$. Moreover, for each $a \in A$, it holds that $n - |F_a|$ is equal to the number of agents $a$ qualifies before
the bribery and, thus, $\Delta \geq |A^+| - \min_{F \in \mathcal{F}} |F|$. Lastly, note that as each agent $a \in A^+$ can only be approved by at most $s - 1$ agents before the bribery, $a$ needs to appear in all but at most $s - 1$ sets. Thus, applying the algorithm from Theorem 1, we can solve the problem in $O(n \cdot (\Delta \cdot s)^s)$ time.

It is even possible to extend this result to a fixed-parameter tractable algorithm for the parameters $s + t + \ell + \Delta$ (note that as proven by Boehmer et al. [1] CONSTRUCTIVE-\textit{f}(s,t) AGENT BRIbery is W[1]-hard with respect to $s + t + \ell$):

**Corollary 4.** CONSTRUCTIVE-\textit{f}(s,t) AGENT BRIbery is solvable in $O\left(n^2 \cdot \left((\ell + t)^\ell \cdot (\Delta \cdot s)^s\right)\right)$ time.

**Proof.** If we bribe an agent, we always make him qualify all agents. For all agents $a \in A^+$ with $\phi(a, a) = -1$ and $|Q^-(a)| \geq \ell + t$, $a$ must be bribed; so we bribe $a$. Thus, we can assume $|Q^-(a)| < \ell + t$ for all $a \in A^+$ with $\phi(a, a) = -1$. Now, as long as there exists an $a \in A^+$ with $\phi(a, a) = -1$, we branch on bribing $a$ or bribing $|Q^-(a)| - (t - 1)$ agents from $Q^-(a)$ and update $\phi$, $\ell$, and $A^+$ accordingly (we delete agents from $A^+$ if they became socially qualified). We reject the current branch if $\ell < 0$. For each non-rejected branch, it remains to consider agents from $A^+$ who qualify themselves. This problem is similar to the case when $t = 1$ and we can apply Theorem 2.

As the branching factor in each step is bounded by $\ell + t$ and the depth is bounded by $\ell$, the algorithm from Theorem 2 is employed at most $(\ell + t)^\ell$ times, which results in an overall running time of $O\left(n^2 \cdot \left((\ell + t)^\ell \cdot (\Delta \cdot s)^s\right)\right)$.

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