HAMiltonian Reduction of suPER Osp(1,2) AND SL(2,1) Kac-Moody alGebras

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ABSTRACT

We present the Wakimoto construction of the super OSp(1,2) and SL(2,1) Kac-Moody algebras and the free field representation of the corresponding WZW models. After imposing suitable constraints, we can lead the Feigin-Fuchs representation of Virasoro algebras and coadjoint actions of the N=1 and N=2 conformal symmetries. This formulation corresponds to a supercovariant extension of the Drinfeld and Sokolov Hamiltonian reduction.

1. Introduction

There have recently been remarkable developments in the study of two dimensional systems, for example, conformal field theories, integrable models, two dimensional gravity and topological field theories. These theories are closely related to each other. In particular, there are various connections between Wess-Zumino-Witten(WZW) models and these theories. WZW models have a symmetry associated with the Kac-Moody(KM) algebras and provide us with a useful method for the characterization of the chiral algebraic structure of rational conformal field theories.

It is well-known that there are two methods of constructing extended conformal symmetries, for example, super Virasoro algebras and W algebras. One is the coset construction and the other is the Hamiltonian reduction of KM algebras[1]. Both of the methods can be studied by realizing them as gauged WZW models, which give a Lagrangian approach and a geometric interpretation of symmetries[2][3][4]. Drinfeld and Sokolov first succeeded in giving the relation among the W algebra,

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the generalized KdV equations and the Toda field theories by means of the Hamiltonian reduction. There are several attempts extending the Hamiltonian reduction to supersymmetric ones[5][6][7]. Komata et al. considered the series of Lie superalgebra which lead to the supersymmetric Toda field theories[8][9]. They investigated two series of the extended superconformal algebras associated with the affine Lie superalgebras SL(n + 1, n)(n ≥ 1) and OSp(2n ± 1, 2n)(n ≥ 1) and gave the Feigin-Fuchs construction of these algebras. On the other hand, the free field realization is also a powerful method to investigate two-dimensional systems systematically. The Feigin-Fuchs construction provides us with means to describe extended conformal algebras in the Fock space of free fields and to calculate characters and correlation functions by means of BRST cohomology. This realization can be applied to KM algebras with an arbitrary central charge, called Wakimoto construction[10]. The currents of algebras are expressed by free bosons corresponding to the Cartan part and ghosts for roots.

It is transparent to perform the Hamiltonian reduction based on the free field realization. Bershadsky and Ooguri first studied the Hamiltonian reduction of Osp(1, 2) KM algebra in terms of free fields[11]. Ito[12] has extended their method to Lie superalgebras. He showed that the quantum Hamiltonian reduction of the affine Lie superalgebras SL(n + 1, n)(n ≥ 1) has a W algebra structure with the N=2 superconformal symmetry and leads to the N=2 CP_{n} = SU(n + 1)/SU(n) × U(1) coset model constructed by Kazama and Suzuki. Inami and Izawa proposed the Hamiltonian reduction of WZW models based on Lie supergroups with a fermionic simple root system[13]. They obtained actions of the super-Toda theories and also geometric actions of the superconformal groups. In those papers, however it was inevitably necessary to introduce extra fermions by hand. Kuramoto pointed out that the Hamiltonian reduction of super OSp(1, 2)KM algebras does not need fermionic auxiliary fields because the superpartners of the bosonic currents play the role of them[14]. Nevertheless, his formulation is not supercovariant and may not probably be adequate to general Lie superalgebras.

In this paper we shall present explicitly the Wakimoto construction of the super OSp(1, 2) and SL(2, 1)[14] KM algebras, in other words, the free boson representation of the super WZW models and perform the Hamiltonian reduction in a supercovariant way. After setting supercovariant constraint, the N=1 and N=2 super Virasoro algebras appear in both the forms of the Feigin-Fuchs construction and of the superschwarziann derivatives. In this approach, it is regarded that the supersymmetric Miura transformation is generated by a nilpotent sub-supergroup which preserves the first-class constraints. We also show the actions of WZW models become to the coadjoint actions of superconformal symmetry or those of super Liouville actions by choosing suitable constraints, respectively. In spite of results of this paper being not quit new, I believe the formulation of superfields may be useful for considering properties of geometry and topology for superconformal theories.

The paper is organized as follows. Sect.2 is devoted to the free realization of the super KM algebras and WZW models. In sect.3 we explain the method of the
Hamiltonian reduction of the super OSp(1,2) KM algebra. Sect.4 is devoted to the application for the super SL(2,1) KM algebra. Discussions are contained in sect.5.

2. Wakimoto Construction of Super KM Algebras

We begin with the free field realization of KM algebras based on Lie superalgebras discussed by Ito[11]. A root system of a Lie superalgebra g is expressed as the sum of the set of even roots $\Delta^0$ and that of odd roots $\Delta^1$. A group element $g$ may be decomposed into the direct sum of $g_0$ which is generated by the Cartan part and the even roots and $g_1$ spanned by the odd roots. A Kac-Moody algebra based on a Lie superalgebra of rank $r$ is generated by the fermionic currents $j_\alpha(x)(\alpha \in \Delta^1)$, the bosonic currents $J_\alpha(x)(\alpha \in \Delta^0)$ and $H_i(x)(i = 1, \cdots, r)$ corresponding to the Cartan part.

In this section we extend these KM algebras to the super KM algebra s, which have supersymmetry of two-dimensional spacetime We concretely construct the super OSp(1,2) KM algebra[14]. For the first step, we construct the lagrangian of the super OSp(1,2) WZW model.

In general, the action of the supersymmetric OSp(1,2) WZW model is given by the following form:

$$S = \frac{k}{16\pi} \left\{ \int dx_+ dx_- < g^{-1} \hat{D}_- g, g^{-1} \hat{D}_+ g > \right. \\
- \left. \int dx_+ dx_- d\tau < g^{-1} \partial_\tau g, [g^{-1} \hat{D}_+ g, g^{-1} \hat{D}_- g]_+ > \right\}. \quad (2.1)$$

Here $x_+$ and $x_-$ are light cone coordinates and $g(x_+, x_-, \theta_+, \theta_-)$ is a group element of the super OSp(1,2) and $< A, B >$ denotes supertrace of supermatrices $A$ and $B$, and $[,]_+$ indicates an anticommutation relation. Further, $\hat{D}_+$ and $\hat{D}_-^{-1}$ are covariant derivatives accompanied with $d\theta_+$ and $d\theta_-$, respectively:

$$\hat{D}_+ = d\theta_+ D_+ = d\theta_+ \left( \frac{\partial}{\partial \theta_+} + \theta_+ \frac{\partial}{\partial x_+} \right), \quad \hat{D}_- = d\theta_- D_- = d\theta_- \left( \frac{\partial}{\partial \theta_-} + \theta_- \frac{\partial}{\partial x_-} \right). \quad (2.2)$$

Now we are going to express the model in terms of free fields as the same way as in the case of the SL(2) WZW model[15]. The Gauss expansion of a group element $g$ is given by

$$g(x_+, x_-, \theta_+, \theta_-) = g_1 g_2 g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ C & 1 & -\Psi & 0 \\ 0 & e^{\Phi} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & F & \xi \\ 0 & 1 & 0 \\ 0 & \xi & 1 \end{pmatrix}, \quad (2.3)$$

This notation is convenient for actual calculations, because we can treat as in the bosonic case. The chiralities of $\theta_+$ and $\theta_-$ are opposite to an ordinary definition, but there is no problem in our discussions.
where $\Phi, C$ and $F$ are grassmann even scalar superfields and $\Psi$ and $\xi$ are grassmann odd ones[11]. Using the Polyakov-Wiegmann relation, the action (2.1) is rewritten in terms of the fields $\Phi, C, F, \Psi$ and $\xi$ as

$$S \sim \int dx^+ dx^- \left\{-2k \hat{D}_{-} \Phi \hat{D}_{+} \Phi + e^{2\Phi} (\hat{D}_{-} + \Psi \hat{D}_{-} \Psi) (\hat{D}_{+} + F + \xi \hat{D}_{+} \xi) - 2e^{\Phi} \hat{D}_{-} \Psi \hat{D}_{+} \xi \right\}$$

(2.4)

$$= \int dx^+ dx^- d\theta^+ d\theta^- \left\{2k D_{-} \Phi D_{+} \Phi + BD_{-} C + 2\Psi^\dagger D_{-} \Psi \right\},$$

where we define fields $B$ and $\Psi^\dagger$ as

$$B = ke^{2\Phi} (D_{+} F - \xi D_{+} \xi),$$

$$\Psi^\dagger = ke^{\Phi} D_{+} \xi + \frac{1}{2} B \Psi.$$  

(2.5)

This form tells the field $\Phi$ is a free scalar superfield and the fields $(B, C)$ and $(\Psi^\dagger, \Psi)$ are superghosts. The equations of motion are derived from invariance of the action under an infinitesimal variation $g \to (1 + \epsilon)g$:

$$D_{+} (g^{-1} D_{-} g) = 0, \quad D_{-} (D_{+} g g^{-1}) = 0.$$  

(2.6)

This equations indicate conservation of left currents $J_{+} = k D_{+} g g^{-1}$ and right ones $J_{-} = k g^{-1} D_{-} g$. Substituting $g$ into the currents $J_{+}$, we obtain the Wakimoto currents in terms of the fields $(\Phi, C, B, \Psi, \Psi^\dagger)$:

$$J = k D_{+} g g^{-1} = \begin{pmatrix} H & J^- & j^- \\ J^+ & -H & -j^+ \\ j^+ & j^- & 0 \end{pmatrix},$$

(2.7)

where

$$\begin{cases} 
  j^- = \Psi^\dagger + \frac{1}{2} B \Psi, \\
  j^+ = -\Psi^\dagger C + \frac{1}{2} \Psi C B - k \Psi D \Phi + k D \Psi, \\
  H = -\Psi \Psi^\dagger C B + k D \Phi, \\
  J^- = B, \\
  J^+ = -BC^2 + 2k C D \Phi + k DC - 2\Psi \Psi^\dagger C + k D \Psi \Psi.
\end{cases}$$

It is reasonable to set the following Poisson brackets among the fields $(\Phi, C, B, \Psi, \Psi^\dagger)$ from the action(2.4):

$$\left\{ \begin{array}{c} D_{1} \Phi_{1}, \Phi_{2} \end{array} \right\} = \frac{1}{2k} \theta_{12} \delta(x_{12}),$$

$$\left\{ B_{1}, C_{2} \right\} = -\theta_{12} \delta(x_{12}),$$

$$\left\{ \Psi_{1}^\dagger, \Psi_{2} \right\} = \frac{1}{2} \theta_{12} \delta(x_{12}).$$

(2.8)

2)We drop the sign $+$ which indicates a left component. If signs of left and right components are dropped in equations, they are regarded as left components.
where \( \theta_{12} = \theta_1 - \theta_2 \), \( x_{12} = x_1 - x_2 - \theta_1 \theta_2 \).

From the definition of the currents, \( H, J^+ \) and \( J^- \) are Grassmann odd currents which generate the super SL(2) Kac-Moody algebra. \( j^+ \) and \( j^- \) are Grassmann even ones.

The classical Kac-Moody algebras can be obtained from the Poisson brackets (2.8):

\[
\begin{aligned}
\{H_1, H_2\} &= \frac{k}{2} \delta(x_{12}), \\
\{H_1, J^+_2\} &= \pm \theta_{12} J^+_2 \delta(x_{12}), \\
\{J^+_1, J^-_2\} &= 2 \theta_{12} H_2 \delta(x_{12}) - k \delta(x_{12}), \\
\{j^+_1, j^-_2\} &= - \theta_{12} H_2 \delta(x_{12}) + \frac{k}{2} \delta(x_{12}), \\
\{H_1, j^+_2\} &= \pm \frac{1}{2} \theta_{12} j^+_2 \delta(x_{12}), \\
\{J^+_1, j^+_2\} &= \mp \frac{1}{2} \theta_{12} J^+_2 \delta(x_{12}), \\
\{J^+_1, j^-_2\} &= \mp \frac{1}{2} \theta_{12} J^-_2 \delta(x_{12}).
\end{aligned}
\]

In order to obtain the currents of quantum version, one has to consider the quantum effect which derived from reparametrization of the measure of a functional integral, namely, changing the fields \( (\Phi, C, \Psi, F, \xi) \) to ones \( (\Phi, B, C, \Psi, \Psi^\dagger) \). This quantum effect shifts the normalization of the scalar fields in the non-super case. Explicit expressions of the OSp(1, 2) currents are written in the papers[14]. On the other hand, this shift does not appear in the super case because there exists two-dimensional spacetime supersymmetry. Then there is no change in the currents of quantum version. Canonical quantization can be done by setting the following operator product expansions:

\[
\begin{aligned}
D_1 \Phi(x_1, \theta_1) \Phi(x_2, \theta_2) &\sim \frac{1}{2k} \frac{\theta_{12}}{z_{12}}, \\
B(x_1, \theta_1) C(x_2, \theta_2) &\sim - \frac{\theta_{12}}{z_{12}}, \\
\Psi^\dagger(x_1, \theta_1) \Psi(x_2, \theta_2) &\sim - \frac{\theta_{12}}{2z_{12}}.
\end{aligned}
\]

3. Hamiltonian Reduction of Super OSp(1,2) KM Algebra

Now we are in a position to apply the Drinfeld and Sokolov reduction to the super OSp(1, 2) KM algebra realized in terms of free fields. Bershadsky and Ooguri showed the \( N=1 \) super Virasoro algebra is derived from the bosonic OSp(1, 2) KM algebra by introducing a free fermion in addition to free fields of the Wakimoto currents[11].

Kuramoto first indicated that starting from the manifestly super OSp(1, 2) KM algebra, the \( N=1 \) super Virasoro algebra can be derived without a additional fermion[14]. His approach, however, is not explicitly supercovariant and not probably suitable for extended conformal symmetries.
In this section we will formulate the Hamiltonian reduction of the super $\text{OSp}(1, 2)$ KM algebra in a super covariant way. Let us begin with the following first-class constraints.

\[ J^-(x_+, x_-, \theta_+, \theta_-) = 0, \quad j^-(x_+, x_-, \theta_+, \theta_-) = 1. \]  

(3.1)

These constraints preserve two-dimensional spacetime supersymmetry. The components of these constraints coincide with those in the paper[14]. The form of the residual gauge transformation is written as

\[ \Omega = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & -\beta \\ \beta & 0 & 1 \end{pmatrix}, \]  

(3.2)

which corresponds to a Borel sub-supergroup. Under this transformation the differential operator $\mathcal{L} = kD - J$ is transformed as

\[ \mathcal{L} \rightarrow \hat{\Omega} \mathcal{L} \Omega^{-1}, \]  

(3.3)

where $\hat{\Omega}$ indicates changing signs of the odd fields of $\Omega$. There exist two kinds of gauge choices, in which the phase space of the transformed currents commute with the currents $J^-$ and $j^-$. One of these choices is called the Drinfeld and Sokolov gauge given by

\[ \Omega_1 = \begin{pmatrix} 1 & 0 & 0 \\ j^+ - kH & 1 & -H \\ H & 0 & 1 \end{pmatrix}. \]  

(3.4)

$\mathcal{L}$ is transformed as

\[ \mathcal{L}_1 = \hat{\Omega}_1 \mathcal{L} \Omega_1^{-1} = kD - \begin{pmatrix} 0 & 0 & 1 \\ T & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]  

(3.5)

where

\[ T = J^+ + 2j^+ H + kH D H + kD (j^+ + kD H). \]  

(3.6)

Substituting the Wakimoto currents constrained with eq.(3.1) into $T$, it is rewritten as

\[ T = k^3 (D\Phi \partial\Phi + \partial D\Phi). \]  

(3.7)

This form is nothing but the Feigin-Fuchs construction of the N=1 Virasoro algebra. Because of the constraint(3.1),

\[ ke^{2\Phi} D\xi = 1, \]  

(3.8)

3) This phase space is nothing but a quotient space divided by the residual symmetries.
T is also expressed by the superschwarzian derivative of $\xi$:

$$T = -k^3 \left( \frac{\partial^2 \xi}{D \xi} - 2 \frac{\partial \xi (\partial D \xi)}{(D \xi)^2} \right). \quad (3.9)$$

We can obtain the following the classical N=1 Virasoro algebra from the Poisson bracket of the field $\Phi$:

$$\{T_1, T_2\} = \frac{k^2}{2} \delta''(x_{12}) - \frac{3}{2} \theta_{12}T_2 \delta'(x_{12}) + \frac{1}{2} D_2 \delta(x_{12}) + \theta_{12}T_2 \delta(x_{12}). \quad (3.10)$$

The other gauge choice is a diagonal gauge given by

$$\Omega_2 = \begin{pmatrix} 1 & 0 & 0 \\ -C & 1 & \Psi \\ -\Psi & 0 & 1 \end{pmatrix}. \quad (3.11)$$

Under this gauge $g$ is transformed as

$$g \rightarrow \Omega_2 g = \begin{pmatrix} e^\Phi & 0 & 0 \\ 0 & e^{-\Phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & F & \xi \\ 0 & 1 & 0 \\ 0 & \xi & 1 \end{pmatrix}. \quad (3.12)$$

Then we have the transformed currents by setting the fields $C$ and $\Psi$ equal zero in the currents(2.7):

$$L_2 = \Omega_2 \mathcal{L} \Omega_2^{-1} = kD - \begin{pmatrix} D\Phi & 0 & 0 \\ 0 & -D\Phi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.13)$$

The phase space of this current is one of the super U(1) KM algebra. It is the super Miura transformation to connect the field $T$ with the U(1) KM algebra. We can express the Miura map in terms of the gauge transformation of the Borel sub-supergroup.

$$\mathcal{L}_1 = (\tilde{\Omega}_1 \Omega_2^{-1}) \mathcal{L}_2 (\Omega_1 \Omega_2^{-1})^{-1}. \quad (3.14)$$

The action (2.4) becomes a coadjoint action of the N=1 super Virasoro group by imposing the first-class constraints(3.1) with the diagonal gauge as the second constraint:

$$S_{super\ Virasoro} \sim \int dx_+ dx_- d\theta_+ d\theta_- D_+ \frac{D_+ \partial_+ \xi}{(D_+ \xi)^2}. \quad (3.15)$$

This is the same form as a geometric action for the theory of the coadjoint orbits of the superconformal group[16][17] and its components correspond with the result in the paper[11][13]. Imposing the similar first-class constraints for the right-handed parts as same as eq.(3.1), one can obtain the N=1 super Liouville action from the WZW action:

$$S_{super\ Liouville} \sim \int dx_+ dx_- d\theta_+ d\theta_- \left( 2D_- \Phi D_+ \Phi + ke^{-\Phi} \right). \quad (3.16)$$
Supersymmetry is inherited from the structure of the Lie superalgebra in the case of the bosonic $\text{OSp}(1,2)$ KM algebra. In our case, starting from the manifestly supersymmetric KM algebra and imposing the supercovariant constraints, the N=1 supersymmetry is derived from two-dimensional spacetime.

4. Hamiltonian Reduction of Super $\text{SL}(2,1)$ KM Algebra

It has been shown that the Hamiltonian reduction of WZW model based on the Lie superalgebras $\text{SL}(n+1,n)(n \geq 1)$ gives the models which have W algebra structures with the N=2 superconformal symmetry. It is also understood that these models are the N=2 coset models $\text{CP}_n = \text{SU}(N+1)/\text{SU}(N) \times \text{U}(1)$ constructed by Kazama and Suzuki. In this section we will discuss the simplest case of the super $\text{SL}(2,1)$ WZW model. We show that N=2 superconformal symmetry can be obtained by imposing the constraints which do not break N=1 supersymmetry.

The Gauss decomposition of a group element is written as follows:

$$g(x_+, x_-, \theta_+, \theta_-) = \begin{pmatrix} 1 & 0 & 0 \\ C & 1 & -\Psi_2 \\ \Psi_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\Phi_1} & 0 & 0 \\ 0 & e^{-\Phi_2} & 0 \\ 0 & 0 & e^{\Phi_1 - \Phi_2} \end{pmatrix} \begin{pmatrix} 1 & F & \xi_2 \\ 0 & 1 & 0 \\ 0 & \xi_1 & 1 \end{pmatrix}.$$  \hspace{1cm} (4.1)

where $\Phi_1, \Phi_2, C$ and $F$ are grassmann even superfields and $\Psi_1, \Psi_2, \xi_1$ and $\xi_2$ are grassmann odd ones. The Wakimoto currents are also given by

$$J = kDgg^{-1} = \begin{pmatrix} H_1 & J^- \\ J^+ & -H_2 \end{pmatrix} \begin{pmatrix} j_1 & j_2^- \\ j_2^+ & \tilde{j}_2 \end{pmatrix},$$  \hspace{1cm} (4.2)

where

\[
\begin{align*}
H_1 &= -BC - \Psi_1^\dagger \Psi_1 + kD\Phi_1, \\
H_2 &= -BC - \Psi_2^\dagger \Psi_2 + kD\Phi_2, \\
J^- &= B, \\
J^+ &= -BC^2 - C(\Psi_1^\dagger \Psi_1 + \Psi_2^\dagger \Psi_2) + kDC, \\
&\quad + kC(D\Phi_1 + D\Phi_2) + kD\Phi_1 \Psi_2 \Psi_1 + kD\Psi_2 \Psi_1, \\
j_1^- &= \Psi_1^\dagger, \\
j_2^- &= \Psi_2^\dagger + B\Psi_1, \\
j_1^+ &= \Psi_1 BC - \Psi_2^\dagger C - \Psi_2^\dagger \Psi_2 \Psi_1 + kD\Phi_2 \Psi_1 + kD\Psi_1, \\
j_2^+ &= -C\Psi_1^\dagger - k\Psi_2 D\Phi_1 + kD\Psi_2.
\end{align*}
\]  \hspace{1cm} (4.3)

Here $B$, $\Psi_1^\dagger$ and $\Psi_2^\dagger$ are defined as

$$B = ke^{\Phi_1} e^{\Phi_2} (DF - D\xi_1 \xi_2), \quad \Psi_1^\dagger = ke^{\Phi_2} D\xi_1 + B\Psi_2, \quad \Psi_2^\dagger = ke^{\Phi_1} D\xi_2.$$  \hspace{1cm} (4.4)

The action is written as
\[ S \sim \int dx_+dx_-d\theta_+d\theta_- \left( kD_+\Phi_1D_+\Phi_2 + kD_-\Phi_2D_+\Phi_1 
+ BD_-C + D_-\Psi_1\Psi_\dagger_1 + D_-\Psi_2\Psi_\dagger_2 \right). \tag{4.5} \]

From this form it is natural to give the following Poisson brackets for the fields \( \Phi_i, (B, C) \) and \((\Psi_\dagger_i, \Psi_i)\):

\[
\begin{align*}
\{D\Phi_{i,1}, \Phi_{j,2}\} &= \frac{1}{k} \eta_{ij}\theta_{12}\delta(x_{12}), \\
\{\Psi_{\dagger_i,1}, \Psi_{j,2}\} &= \delta_{ij}\theta_{12}\delta(x_{12}), \\
\{B_1, C_2\} &= -\theta_{12}\delta(x_{12}).
\end{align*} \tag{4.6}
\]

where \( \eta_{ij} \) are elements of a matrix of which off-diagonal elements equal one and the others equal zero. The classical Kac-Moody algebra can be calculated as

\[
\begin{align*}
\{H_{i,1}, H_{j,2}\} &= \frac{1}{k} \eta_{ij}\delta(x_{12}), \\
\{J^+_1, J^-_2\} &= \theta_{12}(H_1 + H_2)\delta(x_{12}) - k\delta(x_{12}), \\
\{H_{i,1}, j^-_{j,2}\} &= -\delta_{ij}\theta_{12}j^-_{j,2}\delta(x_{12}), \\
\{H_{i,1}, j^+_{j,2}\} &= \delta_{ij}\theta_{12}j^+_{j,2}\delta(x_{12}), \\
\{j^+_{i,1}, j^-_{j,2}\} &= -\eta_{ij}\theta_{12}H_{j,2}\delta(x_{12}) + k\delta(x_{12}).
\end{align*} \tag{4.7}
\]

The other Poisson brackets have the similar form as the case of OSp(1,2).

Let us perform the Hamiltonian reduction. First-class constraints are given by

\[
J^-(x_+, x_-, \theta_+, \theta_-) = 0, \quad j^-_1(x_+, x_-, \theta_+, \theta_-) = 1, \quad j^-_2(x_+, x_-, \theta_+, \theta_-) = 1. \tag{4.8}
\]

It is a nilpotent sub-supergroup that preserves the constraints:

\[
\Omega = \begin{pmatrix}
1 & 0 & 0 \\
A & 1 & \alpha \\
-\beta & 0 & 1
\end{pmatrix}, \tag{4.9}
\]

Here we choose the Drinfeld and Sokolov gauge:

\[
\Omega_1 = \begin{pmatrix}
\frac{1}{2}\{j^+_1 - j^-_2 - H_1H_2 + kD(H_1 + H_2)\} & 0 & 0 \\
H_1 & 1 & -H_2 \\
0 & 1 & 0
\end{pmatrix}. \tag{4.10}
\]

In this gauge the differential operator \( \mathcal{L} \) transforms to the form:

\[
\mathcal{L} \rightarrow \mathcal{L}_1 = \hat{\Omega}_1\mathcal{L}\Omega_1^{-1} = \begin{pmatrix}
0 & 0 & 1 \\
T & 0 & I \\
I & 1 & 0
\end{pmatrix}, \tag{4.11}
\]
where
\[
\begin{align*}
I &= \frac{1}{2} \{ j_1^+ - j_2^+ - H_1 H_2 + k D (H_1 - H_2) \}, \\
T &= J^+ + H_2 j_1^+ + H_1 j_2^+ + \frac{k}{2} D \{ j_1^+ - j_2^+ - H_1 H_2 + k D (H_1 + H_2) \} + k D H_2 H_1.
\end{align*}
\]
(4.12)
Substituting the Wakimoto currents with the constraints (4.8), one can obtain the Feigin-Fuchs construction of the N=2 super Virasoro algebra
\[
\begin{align*}
T &= \frac{k^3}{2} \{ \partial \Phi_1 D \Phi_2 + \partial \Phi_2 D \Phi_1 + \partial D (\Phi_1 + \Phi_2) \}, \\
I &= \frac{k^2}{2} \{ D \Phi_1 D \Phi_2 + \partial \Phi_1 + \partial \Phi_2 \}.
\end{align*}
\]
(4.13)
The fields T and I satisfy the following the N=2 super Virasoro algebra:
\[
\begin{align*}
\{ T_1, T_2 \} &= + \frac{k^2}{2} \delta''(x_{12}) - \frac{3}{2} \theta_{12} T_2 \delta'(x_{12}) + \frac{1}{2} D_2 T_2 \delta(x_{12}) + \theta_{12} T_2 \delta(x_{12}), \\
\{ T_1, I_2 \} &= - \theta_{12} I_2 \delta'(x_{12}) + \frac{1}{2} D_2 I_2 \delta(x_{12}) + \theta_{12} I_2 \delta(x_{12}), \\
\{ I_1, I_2 \} &= \frac{k^3}{2} \delta'(x_{12}) - \frac{1}{2} \theta_{12} T_2 \delta(x_{12})
\end{align*}
\]
(4.14)
From the constraints (4.8)
\[
ke^{\Phi_1} D \xi_2 = ke^{\Phi_2} D \xi_1 = 1.
\]
(4.15)
T and I are also expressed by superschwarzian derivatives of the fields \( \xi_1 \) and \( \xi_2 \):
\[
\begin{align*}
T &= \frac{k^3}{2} \left( \frac{\partial \xi_1 D \partial \xi_2}{D \xi_1 D \xi_2} + \frac{\partial \xi_2 D \partial \xi_1}{D \xi_2 D \xi_1} + \frac{\partial \xi_2 D \partial \xi_1}{D \xi_1 D \xi_2} + \frac{\partial^2 \xi_1}{D \xi_1} + \frac{\partial^2 \xi_2}{D \xi_2} \right), \\
I &= \frac{k^2}{2} \left( \frac{\partial \xi_1 D \xi_2}{D \xi_1 D \xi_2} + \frac{D \partial \xi_2}{D \xi_1} - \frac{D \partial \xi_2}{D \xi_2} \right).
\end{align*}
\]
(4.16)
From the form of (4.11) the N=2 supersymmetry comes from the structure of the Lie superalgebra and one of two-dimensional spacetime supersymmetry. We can also obtain the coadjoint action of the N=2 super conformal group by taking the first-class constraints (4.8) with the diagonal gauge:
\[
S_{\text{super Virasoro}} \sim \int dx^+ dx^- d\theta^+ d\theta^- \left( \int \left[ \frac{\partial_+ D_+ \xi_2}{D_+ \xi_1 D_+ \xi_2} - \frac{\partial_+ \xi_2 D_+ \xi_1 D_+ \xi_2}{(D_+ \xi_1 D_+ \xi_2)^2} \right] D_- \xi_1 + \left[ \frac{\partial_+ D_+ \xi_1}{D_+ \xi_1 D_+ \xi_2} - \frac{\partial_+ \xi_1 D_+ \xi_2 D_+ \xi_1}{(D_+ \xi_1 D_+ \xi_2)^2} \right] D_- \xi_2 \right).
\]
(4.17)
As the same way as in the case of OSp(1,2), the N=2 super Liouville action is obtained as
\[
S_{\text{super Liouville}} \sim \int dx^+ dx^- d\theta^+ d\theta^- \left( D_- \Phi_1 D_+ \Phi_2 + D_- \Phi_2 D_+ \Phi_1 + e^{-\Phi_1} + e^{-\Phi_2} \right).
\]
(4.18)
5. Discussions

We have presented an explicit construction of the super OSp(1, 2) and SL(2, 1) KM algebras in terms of scalar superfields and superghosts systems. We could show the Feigin-Fuchs representations and the coadjoint actions of the N=1 and N=2 super conformal symmetries by restricting the phase space with constraints which preserve two-dimensional spacetime supersymmetry. In general, super-Toda theories can be obtained by use of special Lie superalgebras with a completely fermionic simple root system[13][18]. We may obtain super W algebras and coadjoint actions of them by imposing the constraints such as currents of simple roots equal constant. Ito has already study the quantum Hamiltonian reduction of super SL(n + 1, n)(n ≥ 1) algebras by means of an operator formulation. Our approach needs a path integral quantization and it is necessary to realize these algebras as gauged super WZW models. We have to calculate determinants derived from gauge fixing conditions. These are under investigation. The N=2 superconformal symmetry is especially important in both the points of studying the compactification of superstrings and the connection with topological field theories. So it is interesting to apply our formulation to these theories[19].
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