Abstract

Generalized Linear Bandits (GLBs) are powerful extensions to the Linear Bandit (LB) setting, broadening the benefits of reward parametrization beyond linearity. In this paper we study GLBs in non-stationary environments, characterized by a general metric of non-stationarity known as the variation-budget or parameter-drift, denoted $B_T$. While previous attempts have been made to extend LB algorithms to this setting, they overlook a salient feature of GLBs which flaws their results. In this work, we introduce a new algorithm that addresses this difficulty. We prove that under a geometric assumption on the action set, our approach enjoys a $\tilde{O}(B_T^{1/3}T^{2/3})$ regret bound. In the general case, we show that it suffers at most a $\tilde{O}(B_T^{1/5}T^{4/5})$ regret. At the core of our contribution is a generalization of the projection step introduced in Filippi et al. (2010), adapted to the non-stationary nature of the problem. Our analysis sheds light on central mechanisms inherited from the setting by explicitly splitting the treatment of the learning and tracking aspects of the problem.

Keywords: Stochastic Bandits, Generalized Linear Model, Non-Stationarity.

1. Introduction

Linear Bandits and non-stationarity. The Linear Bandit (LB) framework has proven to be an important paradigm for sequential decision making under uncertainty. It notably extends the Multi-Arm Bandit (MAB) framework to address the exploration-exploitation dilemma when the arm-set is large (potentially infinite) or changing over time. While the LB has now been extensively studied (Dani et al., 2008; Rusmevichientong and Tsitsiklis, 2010; Abbasi-Yadkori et al., 2011; Abeille and Lazaric, 2017) in its original formulation, a recent strand of research studies its adaptation to non-stationary environments. Notable are the contributions of Cheung et al. (2019b); Russac et al. (2019); Zhao et al. (2020) which prove that under appropriate algorithmic changes, existing LB concepts can be leveraged to handle a drift of the reward model. Aside their theoretical interests, these results further anchor the spectrum of potential applications of the LB framework to real-world problems, where non-stationarity is commonplace.

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Extensions to Generalized Linear Bandits. Perhaps the main limitation of LB resides in its inability to model specific (e.g. binary, discrete) rewards. One axis of research to operate beyond linearity was initiated with the introduction of Generalized Linear Bandit (GLBs) by Filippi et al. (2010). This framework allows to handle rewards which (in expectation) can be expressed as a generalized linear model. Notable members of this family are the logistic and Poisson models. Given the remarkable importance and widespread use of such models in practice, ensuring their resilience to non-stationarity stands as a crucial missing piece. At first glance, as the analysis of GLBs mainly relies on tools from the LB literature, one could expect this demonstration to be straightforward, and almost anecdotal. As a matter of fact, the treatment of GLBs in non-stationary environments was already proposed as a direct extension of non-stationary LB algorithms ((Cheung et al., 2019a, Section 8.3) and (Zhao et al., 2020, Section 5.2)). However, as recently pointed out by Russac et al. (2020), some crucial subtleties of the GLBs flaw the analysis and negate the validity of such extensions. An answer to this issue was brought by Russac et al. (2020, 2021) who proposed a valid analysis for GLBs in non-stationary environments. However, their investigation is restricted to a specific kind of non-stationarity known as abrupt changes, leaving the treatment of the superior parameter-drift case for future work. To the best of our knowledge, a correct derivation of GLBs’ behavior under this more general description of non-stationarity is still missing.

Scope and contributions. We focus in this paper on closing this gap. Our main contribution is (1) the design of BVD-GLM-UCB (Algorithm 1), the first GLB algorithm resilient to parameter-drift and matching the known minimax rates - though only for some action sets (Theorem 1). For more general configurations, we still provide a sub-linear regret bound, slightly lagging behind the known rates for non-stationary LBs. Our result relies on (2) a generalization of the projection step of Filippi et al. (2010) to non-stationary environments, of similar complexity than its stationary counterpart (Proposition 1). Our analysis (3) sheds light on some salient mechanisms of non-stationary bandits.

2. Preliminaries

We consider in this work the stochastic contextual bandit setting under parameter-drift. The environment starts by picking a sequence of parameters \(\{\theta_t^*\}_{t=1}^{\infty}\). A repeated game then begins between the environment and an agent. At each round \(t\), the environment presents the agent with a set of actions \(\mathcal{X}_t\) (potentially contextual, large or even infinite). The agent selects an action \(x_t \in \mathcal{X}_t\) and receives a (stochastic) reward \(r_{t+1}\). In this paper we work under the fundamental assumption that there exists a structural relationship between actions and their associated reward in the form of:

\[
\mathbb{E}[r_{t+1} \mid \mathcal{F}_t, x_t] = \mu(\langle x_t, \theta_t^* \rangle).
\]  

The filtration \(\mathcal{F}_t := \sigma(\{x_s, r_{s+1}\}_{s=1}^{t-1})\) represents the information acquired at round \(t\), and \(\mu\) is a strictly increasing, continuously differentiable real-valued function most often referred to as the inverse link function. Notable instances of such a problem include the logistic bandit and the Poisson bandit. The goal of the agent is to minimize the cumulative pseudo-regret:

\[
R_T := \sum_{t=1}^{T} \mu(\langle x_t^*, \theta_t^* \rangle) - \mu(\langle x_t, \theta_t^* \rangle) \quad \text{where} \quad x_t^* = \arg \max_{x \in \mathcal{X}_t} \mu(\langle x, \theta_t^* \rangle).
\]

We make the following assumption common in the study of parametric bandits:
We will denote
\[ \Theta = \{ \theta : \| \theta \|_2 \leq S \} \]
which is well-defined and unique as the minimizer of a strictly convex and coercive function. Further:
\[ \text{quasi-maximum} \]
As in the stationary setting, learning can be canonically performed through the
principle, albeit with adequate modifications. Let
\[ b(\langle x, \theta \rangle) = \sum_{s=1}^{t-1} \gamma^{t-1-s} b(\langle x_s, \theta \rangle) - r_{s+1}(\langle x_s, \theta \rangle) + \frac{\lambda c_{\mu}}{2} \| \theta \|_2^2, \]
which is well-defined and unique as the minimizer of a strictly convex and coercive function. Further:
\[ g_t(\theta) := \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle x_s, \theta \rangle) x_s + \lambda c_{\mu} \theta. \]
Finally, we will use
\[ V_T := \sum_{s=1}^{t-1} \gamma^{t-1-s} x_s x_s^T + \lambda I_d \text{ and } \tilde{V}_T := \sum_{s=1}^{t-1} \gamma^{2(t-1-s)} x_s x_s^T + \lambda I_d. \]
Some of our results requires the following assumption on the arm-sets \( X_t \). We will discuss the reasons
behind this hypothesis, as well as its main implications in the following section.

**Assumption 4 (Orthogonal arm-set)** Let \( \{ e_i \}_{i=1}^{d} \) an orthonormal basis of \( \mathbb{R}^d \). We call a collection of arm-sets \( \{ X_t \} \), orthogonal if for all \( t \geq 1 \) and any \( x \in X_t \), there exists \( \alpha \) and \( i \) such that
\[ x = \alpha e_i. \]
3. Related work: limitations and challenges

3.1. GLBs and non-stationary LB

GLBs were first introduced by Filippi et al. (2010) who studied optimistic algorithms which enjoy a $\tilde{O}(R_d \sqrt{d/T})$ regret upper-bound, later refined for $K$-arms problem to $\tilde{O}(R_\mu \sqrt{d \log(K T)})$ (Li et al., 2017). These findings were extended to randomized algorithms, both in the frequentist (Abeille and Lazaric, 2017) and Bayesian setting (Russo and Van Roy, 2014; Dong and Van Roy, 2018). GLBs also received an increasing attention targeted at improving their practical implementations (Jun et al., 2017; Dumitrascu et al., 2018).

Non-stationarity in bandits was first studied in the MAB framework under the specific assumption of abruptly-changing environments (also known as switching or piece-wise stationary bandits) by Garivier and Moulines (2011). They introduce an algorithm for which they prove $\tilde{O}(\sqrt{\Gamma_T T})$ regret bounds, where $\Gamma_T$ is an upper bound on the number of switches. The effects of the more general parameter-drift were first studied in the MAB setting by Besbes et al. (2014) who for $K$-arm MAB achieved a dynamic regret bound of $\tilde{O}(K^{1/3} B_T^{1/3} T^{2/3})$. Such results were recently extended to the stochastic LB: Cheung et al. (2019b) developed dynamic policies by resorting to a sliding-window, Russac et al. (2019) introduced a similar approach based on an exponential moving average, and Zhao et al. (2020) advocated for a simpler restart-based solution. All three aforementioned approaches claim regret bounds of the form $\tilde{O}(d^{2/3} B_T^{1/3} T^{2/3})$, henceforth matching the lower-bound of Cheung et al. (2019a) up to logarithmic factors. Unfortunately, an error in their analysis was recently pointed out by Touati and Vincent (2021). It turns out that a correct analysis yields degraded regret bounds, scaling as $\tilde{O}(d^{7/8} B_T^{1/4} T^{3/4})$. This can be improved when the arm sets are orthogonal (Assumption 4) to retrieve the aforementioned minimax-optimal rates. Note that although this is a rather strong requirement, it does not reduce to MAB as it still allows for infinite and changing arm-sets.

3.2. Toward non-stationary GLBs: limitations

On the limits of piece-wise stationarity. To the best of our knowledge, the first valid analysis of non-stationary GLBs was conducted by Russac et al. (2020, 2021). However, their work is restricted to piece-wise stationary environments, characterized by the number $\Gamma_T$ of switches of the reward signal. On the practical side, this drastically narrows down the non-stationary scenarios that can be efficiently addressed, as the measure $\Gamma_T$ can grossly overestimate the importance of the non-stationarity. In such case, any algorithm based on this measure will be sub-optimal and discard too fast previous data, quickly judged uninformative since the level of non-stationarity is expected to be high. This is typically the case in environments with many switches of small amplitude, characteristic of smooth drifts (e.g. user-fatigue in recommender systems). On the theoretical side, this approach tells us little about the difficulties and challenges brought by the non-stationarity, as it relies on the fact that far enough from a switch, the environment is stationary. On the contrary, the variation-budget metric $B_T$ introduced and discussed in Besbes et al. (2014, Section 2), allows for much finer considerations. It stands as a powerful characterization of the non-stationarity, measuring the number of switches and their amplitude jointly. As a result, it can efficiently cover different scenarios, from drifting to piece-wise stationary environments. An adequate treatment of GLBs under this superior metric is therefore a crucial missing piece, and requires a sensibly different analysis and an appropriate algorithmic design.
Parameter-drift and GLBs: flaws of previous approaches. Most of the existing non-stationary LB algorithms address the parameter-drift setting and their extension to GLBs was at first considered as relatively straight-forward (Cheung et al., 2019a; Zhao et al., 2020). Unfortunately, existing analyses suffer from important caveats because they overlook a crucial feature of GLBs. Following Filippi et al. (2010), they rely on a linearization of the reward function around \( \hat{\theta}_t \). Naturally, the linear approximation must accurately describe the effective behavior of the reward signal (characterized by the ground-truth \( \theta^*_t \)). From Assumption 2, this translates in the structural constraint \( \hat{\theta}_t \in \Theta \), which is implicitly assumed to hold in previous attempts. Unfortunately, there exists no proof guaranteeing that \( \hat{\theta}_t \in \Theta \) could hold. Even worse, existing deviation bounds (Abbasi-Yadkori et al., 2011, Theorem 1) rather suggest that in some directions, even in the stationary case without degrading neither the learning nor the tracking guarantees. This rules out the projection step was made possible thanks to their piece-wise stationarity assumption. 

\[ \bar{\text{The later (tracking) measures the deviation of the two sources of deviation (i.e learning and tracking).} \]

\[ \{ \text{a tension in the design of the projection as this requires to incorporate the knowledge inherited from the drifting nature of the sequence one would have obtained if one could have averaged an infinite number of realizations of the trajectory).} \]

\[ \hat{\theta}_t \in \Theta \text{ is not merely a theoretical construction inherited from potentially loose deviation bounds. As highlighted in Figure 2(b), we can see in our numerical simulations that this often happens in practice when the environment is non-stationary.} \]

3.3. Non-stationary GLBs: challenges

In their seminal work, Filippi et al. (2010) countered the aforementioned difficulty by introducing a projection step, mapping \( \hat{\theta}_t \) back to an admissible parameter \( \hat{\theta}_t \in \Theta \). Formally, they compute:

\[ \tilde{\theta}_t = \arg \min_{\theta \in \Theta} \left\| g_t(\theta) - g_t(\hat{\theta}_t) \right\| \| \varphi_t^{-1} \tag{P0} \]

and use \( \tilde{\theta}_t \) to predict the performance of the available actions. The projection step (P0) essentially incorporates the prior knowledge \( \theta_* \in \Theta \) (Assumption 2) without degrading the learning guarantees of the maximum likelihood estimator. This strategy was also leveraged by Russac et al. (2020), which was made possible thanks to their piece-wise stationarity assumption.

The situation is different in our setting, as the parameter-drift framework allows the sequence \( \{ \theta^*_t \} \) to change at every round. This introduces (1) the need to characterize two phenomena of different nature that we will designate as learning and tracking. The former (learning) is linked to the deviation of the maximum-likelihood estimator \( \tilde{\theta}_t \) from its noiseless counterpart \( \hat{\theta}_t \) (the estimator that one would have obtained if one could have averaged an infinite number of realization of the trajectory). The later (tracking) measures the deviation of \( \theta_t \) from the current \( \theta^*_t \), due to an incompressible error inherited from the drifting nature of the sequence \( \{ \theta^*_s \}_{s=1}^I \). The learning and tracking mechanisms are both sources of deviation of \( \tilde{\theta}_t \) away from \( \Theta \), each under a different metric. This leads to (2) a tension in the design of the projection as this requires to incorporate the knowledge \( \{ \theta^*_t \} \in \Theta \), without degrading neither the learning nor the tracking guarantees. This rules out the projection step (P0), oblivious to the tracking aspect of the problem and which needs to be generalized to adapt to the two sources of deviation (i.e learning and tracking).
4. Algorithm and regret bound

4.1. Algorithm

This section is dedicated to the description of the design of our new algorithm BVD–GLM–UCB. It operates in two steps: (Step 1) the computation of an appropriate admissible parameter \( \tilde{\theta}_t \in \Theta \) (to be used for predicting the rewards associated with the actions \( x \in X_t \) available at round \( t \)) and (Step 2) the construction of a suitable exploration bonus to compensate for prediction errors.

The first step builds on the following set, linked to the deviation incurred through the learning process:

\[
\mathcal{E}_t^\delta(\theta) := \left\{ \theta' \in \mathbb{R}^d \text{ s.t. } \| g_t(\theta') - g_t(\tilde{\theta}_t) \|_{V_t^{-1}} \leq \beta_t(\delta) \right\}, \tag{3}
\]

where \( \beta_t(\delta) \) is a slowly-increasing function of time (to be defined later) and \( \delta \in (0, 1] \).

**Step 1.** We start by identifying an intermediary parameter \( \theta_t^p \), solution of the following constrained optimization program (ties can be broken arbitrarily):

\[
\theta_t^p \in \arg \min_{\theta \in \mathbb{R}^d} \left\{ \| g_t(\theta) - g_t(\tilde{\theta}_t) \|_{V_t^{-2}} \text{ s.t. } \Theta \cap \mathcal{E}_t^\delta(\theta) \neq \emptyset \right\}. \tag{P1}
\]

The optimization program (P1) is well-posed as it consists in minimizing a smooth function over a non-empty compact set\(^2\). Once \( \theta_t^p \) is computed, the algorithm simply chooses any parameter \( \hat{\theta}_t \in \Theta \cap \mathcal{E}_t^\delta(\theta_t^p) \). An efficient procedure to find such a parameter is detailed in Section 4.3. The different parameters of interest for BVD–GLM–UCB are illustrated in Figure 1.

\(^2\) Notice that \( \{ \theta \text{ s.t. } \Theta \cap \mathcal{E}_t^\delta(\theta) \neq \emptyset \} \) always contains \( 0_d \), while the compactness is inherited from \( \Theta \).
We provide in Theorem 1 a high-probability bound on the regret of BVD-GLM-UCB under general arm-set geometry and Assumptions 1-2-3, setting $\gamma = 1$.

Without this assumption, the upper-bound suffers a small lag behind the LB rates, from $T^{3/4}$ to $T^{4/5}$. Second, one can notice the presence in the bound of the ratio $R_\mu$, typical of the linearization approach performed to analyze GLBs. The bounds presented in

**Remark 2** Notice the difference with the projection step used in the stationary case. In our case it is possible that $E^\delta_t(\tilde{\theta}_t)$ (which is the confidence set centered at $\tilde{\theta}_t$) does not intersect the admissible set $\Theta$. Our strategy for finding $\tilde{\theta}_t$ is then to compute an appropriate **vibration** $E^\delta_t(\tilde{\theta}_t^0)$ of $E^\delta_t(\tilde{\theta}_t)$ which does intersect $\Theta$, while minimizing the deviation between $\tilde{\theta}_t^0$ and $\tilde{\theta}_t$ according to a metric related to the tracking error (through the map $g_t$ and the squared inverse of the design matrix).

**Step 2.** The exploration bonus at round $t$ for a given arm $x \in \mathcal{X}$ is defined as $b_t(x) = 2R_\mu \beta_t(\delta) ||x||_{V_t^{-1}}$, where $\delta \in (0,1]$ and:

$$\beta_t(\delta) = \sqrt{\lambda c_{\mu} S + \sigma \sqrt{2 \log(1/\delta) + d \log \left(1 + \frac{L^2(1-\gamma^2t)}{\lambda d(1-\gamma^2)}\right)}}.$$

BVD-GLM-UCB then follows an optimistic strategy, boosting the predicted reward associated with $\tilde{\theta}_t$ by $b_t$ and plays $x_t \in \arg \max_{x \in \mathcal{X}} \mu(\langle x, \tilde{\theta}_t \rangle) + b_t(x)$. The pseudo-code is summarized in Algorithm 1.

**Algorithm 1** BVD-GLM-UCB

| Input | $\lambda$, regularization, $\delta$, inverse link function $\mu$, weight $\gamma$, constants $S$, $L$ and $\sigma$. |
|-------|--------------------------------------------------|
| Initialization | Compute $R_\mu$, let $V_1 \leftarrow \lambda I_d$ and $\tilde{\theta}_1 \leftarrow 0_d$. |
| for $t \geq 1$ do | |
| Find $\tilde{\theta}_t^0$ by solving (P1) and select $\tilde{\theta}_t \in \Theta \cap \mathcal{E}_t^\delta(\tilde{\theta}_t^0)$. |
| Play $x_t \leftarrow \arg \max_{x \in \mathcal{X}} \mu(\langle x, \tilde{\theta}_t \rangle) + 2R_\mu \beta_t(\delta) ||x||_{V_t^{-1}}$. |
| Observe reward $r_{t+1}$, update $\tilde{\theta}_{t+1}$ by solving Equation (2). |
| Update design matrix: $V_{t+1} \leftarrow \gamma V_t + x_t x_t^T + (1-\gamma)\lambda I_d$. |
| end for |

4.2. Regret bound

We provide in Theorem 1 a high-probability bound on the regret of BVD-GLM-UCB.

**Theorem 1** Under Assumptions 1-2-3 and 4, setting $\gamma = 1 - (B_T/(dT))^{2/3}$ ensures that the regret of BVD-GLM-UCB satisfies:

$$R_T = \tilde{O} \left(R_\mu d^{2/3} B_t^{1/3} T^{2/3}\right) \quad \text{w.h.p}$$

Under general arm-set geometry and Assumptions 1-2-3, setting $\gamma = 1 - (B_T/(\sqrt{d}T))^{2/5}$ ensures that the regret of BVD-GLM-UCB satisfies:

$$R_T = \tilde{O} \left(R_\mu d^{9/10} B_t^{1/5} T^{4/5}\right) \quad \text{w.h.p}$$

A few comments are in order. First, we note that as in the linear case, under Assumption 4 the upper-bound on $R_T$ matches the asymptotic rates of the LB lower-bound under parameter drift (Cheung et al., 2019a, Theorem 1). Without this assumption, the upper-bound suffers a small lag behind the LB rates, from $T^{3/4}$ to $T^{4/5}$. Second, one can notice the presence in the bound of the ratio $R_\mu$, typical of the linearization approach performed to analyze GLBs. The bounds presented in
Theorem 1 are therefore quite natural and extends the work of Filippi et al. (2010) to non-stationary worlds. We emphasize that if the result seems unsurprising, it required a substantially different machinery, both for the design of the algorithm and its analysis. We highlight this last point in Section 5, dedicated at providing a comprehensive sketch of proof for Theorem 1. The complete and detailed proof is deferred to Section B in the supplementary material.

4.3. Solving the projection step

The optimization program (P1) and the subsequent search of a valid parameter $\tilde{\theta}_t$ can raise some legitimate concerns regarding the ease of practical implementation. Indeed, the feasible set of (P1) is given by $\{\theta \text{ s.t. } \Theta \cap E_{\delta_t}(\theta) \neq \emptyset\}$, where $E_{\delta_t}(\theta)$ is defined in (3). Hence, the associated constraint is implicit as it involves an additional non-convex minimization program. As a result, it makes the constraint uneasy to manipulate and even hard to check. The same difficulty arises when searching for $\tilde{\theta}_t \in \Theta \cap E_{\delta_t}(\theta_p)$ where $\theta_p$ is a solution of (P1), due to the non-convexity of the set $E_{\delta_t}(\theta_p)$. The following proposition provides an alternative that avoids those difficulties.

**Proposition 1** Let $\tilde{\theta}_t$ be such that:

$$\begin{align*}
\left(\tilde{\theta}_t, \eta_p\right) &\in \arg \min_{\theta' \in \mathbb{R}^d, \eta \in \mathbb{R}^d} \left\{ \|g_t(\theta') + \beta_\delta(\delta)\sqrt{V_t} \eta - g_t(\hat{\theta}_t)\|_{V^{-1}_t} \text{ s.t. } \|\theta'\|_2 \leq S, \|\eta\|_2 \leq 1 \right\}.
\end{align*}$$

It exists $\theta_p$ solution of (P1) such that $\tilde{\theta}_t \in \Theta \cap E_{\delta_t}(\theta_p)$.

Proposition 1 shows that a valid $\tilde{\theta}_t$ can be found by solving (P2), bypassing the need to compute $\theta_p$. Essentially, the initial two-steps procedure to find $\tilde{\theta}_t$ (through the intermediary program (P1)) is replaced by a single minimization program augmented with a slack variable $\eta$. The attentive reader may notice that (P2) is now similar to (P0), the projection step employed in Filippi et al. (2010). As a result, BVD-GLM-UCB is comparable to the original algorithm GLM-UCB in terms of computational burden. The proof of Proposition 1 is given in Section C in the appendix.

4.4. Online estimation of the variation-budget

**Motivation.** The attentive reader may notice that the minimax-optimality of BVD-GLM-UCB is conditioned on the knowledge of an upper-bound $B_T$ for the true parameter-drift $B_T,\ast$. Naturally, the tighter this upper-bound, the better the performance. Yet, whether such a knowledge is available in real-life problems is, to say the least, questionable. This issue is not specific to our approach but is shared with all non-stationary parametric bandit methods - see for instance (Cheung et al., 2019b; Zhao et al., 2020). For linear bandits, previous approaches circumvented this drawback with a Bandit-over-Bandit strategy (Cheung et al., 2019a, Section 7), where $B_T,\ast$ is learned online by a master algorithm. This guarantees sub-linear regret without having the knowledge of $B_T,\ast$. We however note that this technique was specialized for linear bandits and for the sliding-window strategy. As hinted in the introduction one could easily design a sliding-window approach of BVD-GLM-UCB (using very similar arguments as the ones displayed in this paper) and extend the Bandit-over-Bandit of Cheung et al. (2019a) to the GLB framework. Here, we follow a different path and introduce an equivalent method for the exponential-weighting strategy. To the best of our knowledge, this technique was missing in the non-stationary parametric bandit literature. It notably proves that the online learning of $B_T,\ast$ can be efficiently performed under discounted strategies.
Bandit-over-Bandit for discounted strategies. For the sake of simplicity, we describe the Bandit-over-bandit approach adopted when Assumption 4 holds. A similar reasoning holds in general but naturally yields different rates. Notice that naive bounding gives $B_{T, \star} \in (0, 2ST]$. The main idea for learning $B_{T, \star}$ online is to grid on a log-scale the interval $(0, 2ST]$ with $N$ values $\{B_{T, j}\}_{j=1}^N$. We then create $N$ instances of BVD-GLM-UCB, each set with a different discount factor:

$$\gamma_j = 1 - \left(\frac{B_{T, j}}{dT}\right)^{2/3} = 1 - \frac{2^{j-1}2^{2j-1}d^{2/3}T^{2/3}}{25/3d^{2/3}T^{2/3}}.$$ 

These instances will be our experts. We then deploy a master algorithm - a version of EXP3 (Auer et al., 2002), which acts repeatedly as follows: 1. it chooses an expert $j$ (i.e. a new instance of BVD-GLM-UCB with parameter $\gamma_j$) to interact with the environment during a time frame of length $H$ ($H$ is a positive integer). 2. The master algorithm then observes the cumulative reward (aggregated on the time frame) of the expert $j$. We give the pseudo-algorithm of this procedure in Algorithm 2.

**Algorithm 2** BOB-BVD-GLM-UCB (a more detailed version is deferred to Appendix E.2).

**Input.** Length $H$, time horizon $T$, regularization $\lambda$, confidence $\delta$, inverse link function $\mu$, constants $S, L$ and $\sigma$.

**Initialization.** Let $N \leftarrow \lceil 2 \log_2(2ST^{3/2}) \rceil$ and $\mathcal{H} \leftarrow \{\gamma_j = 1 - \frac{2^{j-1}2^{2j-1}d^{2/3}T^{2/3}}{25/3d^{2/3}T^{2/3}}\}_{j=1}^N$, initialize EXP3 with action set indexed by $\mathcal{H}$.

for $i = 1, \ldots, \lceil T/H \rceil$ do
  $j \leftarrow$ action selected by EXP3.
  Initialize a sub-routine BVD-GLM-UCB with parameter $\gamma_j$.
  for $t = 1, \ldots, H$ do
    Play with BVD-GLM-UCB with parameter $\gamma_j$, observe reward $r_{t+1}$.
  end for
  Update EXP3 with reward $\sum_{t=1}^H r_{t+1}$.
end for

Informally, the idea is that EXP3 will learn to select the best performing $\gamma_j$ associated with the best estimate $B_{T, j}$ of $B_{T, \star}$. Intuitively, this should guarantee small regret as EXP3 will mostly play instances of BVD-GLM-UCB which nearly capture the true magnitude of the non-stationarity. This intuition is made rigorous in Theorem 2, whose proof is deferred to Section E in the appendix.

**Theorem 2** Under Assumptions 1-2 and 4, the regret of BOB-BVD-GLM-UCB when setting $H = \lfloor d\sqrt{T} \rfloor$ satisfies:

$$\mathbb{E}[R_T] = \tilde{O} \left(R_\mu d^{2/3}T^{2/3} \max \left(B_{T, \star}, d^{-1/2}T^{1/4}\right)^{1/3}\right).$$

Essentially, we obtain a regret bound which is identical to the ones of the Bandit-over-Bandit algorithms of Cheung et al. (2019a) and Zhao et al. (2020). The conclusions are therefore of similar nature: namely, when $B_{T, \star} \geq d^{-1/2}T^{1/4}$ we obtain a minimax rate, without knowing $B_{T, \star}$. Again, note here the presence of the problem-dependant constant $R_\mu$, inherited from the non-linear reward structure imposed in GLBs.
5. Proof sketch

In this section, we detail the key steps of the proof of Theorem 1. In particular, we shed light on the tension between the learning and tracking aspects of the problem and their role in the choice of the estimator \( \bar{\theta}_t \), through the use of an appropriate projection step. For simplicity we assume that Assumption 4 holds, although the spirit of the proof is almost identical in the general case.

**Learning versus tracking.** A crucial feature of non-stationary GLBs lies in the singular nature of the deviation of \( \hat{\theta}_t \) from \( \theta^*_t \). This arises from two fundamentally different mechanisms: learning and tracking. We introduce the following estimator, which allows for a clean-cut distinction between the two phenomenons:

\[
\bar{\theta}_t := \arg \min_{\theta \in \mathbb{R}^d} \left\{ \sum_{s=1}^{t-1} \gamma^{t-s} b(s, \langle x_s, \theta \rangle) - \mu (\langle x_s, \theta^*_s \rangle \langle x_s, \theta \rangle) + \frac{\lambda c \mu}{2} \| \theta - \theta^*_s \|^2 \right\}.
\] (4)

The parameter \( \bar{\theta}_t \) is the minimizer of a strictly convex and coercive function, thus is well-defined and unique. Intuitively, \( \bar{\theta}_t \) would be the estimator obtained under a perfect (e.g. noiseless) observation of the reward\(^3\). As a result, the deviation between \( \hat{\theta}_t \) and \( \bar{\theta}_t \) is solely due to the stochastic nature of the problem ("learning"). On the other hand, the deviation between \( \bar{\theta}_t \) and \( \theta^*_t \) is a consequence of the unpredictable changes of the sequence \( \{\theta^*_s\}_s \) ("tracking"). The introduction of the reference point \( \bar{\theta}_t \) allows us to characterize both deviations separately in Lemma 1 and Lemma 2.

**Lemma 1** [Learning] Let \( \delta \in (0, 1] \). With probability at least \( 1 - \delta \):

\[
\text{for all } t \geq 1, \quad \bar{\theta}_t \in \mathcal{E}_t^{\delta}(\hat{\theta}_t) = \left\{ \theta \in \mathbb{R}^d \text{ s.t. } \| g_t(\theta) - g_t(\bar{\theta}_t) \|_{\bar{V}^{-1}_t} \leq \beta_t(\delta) \right\}.
\]

Lemma 1 ensures that with high probability the set \( \mathcal{E}_t^{\delta}(\hat{\theta}_t) \) is a confidence set for \( \bar{\theta}_t \). A complete proof of this result is deferred to Section A.1 in the supplementary material.

**Lemma 2** [Tracking with orthogonal action sets] Let \( D \in \mathbb{N}^* \). The following holds:

\[
\| g_t(\bar{\theta}_t) - g_t(\theta^*_t) \|_{\bar{V}^{-1}_t} \leq \frac{2k \mu L^2 S}{\lambda} \frac{\gamma^D_D}{1 - \gamma} + k \mu \sum_{s=t-D}^{t-1} \| \theta^*_s - \theta^*_s+1 \|_2.
\]

Lemma 2 effectively links the deviation of \( \bar{\theta}_t \) from \( \theta^*_t \) to the variation-budget \( B_T \) through the drift \( \sum_{s=t-D}^{t-1} \| \theta^*_s - \theta^*_s+1 \|_2 \). The proof of this result borrows tools from Russac et al. (2019) and is deferred to Section A.5 in the supplementary material. The integer \( D \) appearing in Lemma 2 is introduced for the sake of the analysis only. It allows to treat separately old and recent observations. We provide its optimal value later in this section.

**Remark 3** Behind the statement of Lemma 1 and Lemma 2 hides the main reason why the projection step of Filippi et al. (2010) needs to be generalized. Indeed, it appears that the deviations \( (\hat{\theta}_t \leftrightarrow \bar{\theta}_t) \) and \( (\bar{\theta}_t \leftrightarrow \theta^*_t) \) are controlled through different metrics (\( \bar{V}^{-1}_t \) and \( V^{-2}_t \), respectively). Projecting according to the first metric would corrupt the control of the second deviation, and conversely.

\[^3\text{Note the difference between } \hat{\theta}_t \text{ and } \bar{\theta}_t, \text{ where the rewards } r_{t+1} \text{ are replaced by their conditional expected values } \mu (\langle x_s, \theta^*_s \rangle) \]
Regret decomposition and prediction error. To bound the instantaneous regret at round \( t \), we rely on the prediction error \( \Delta_t \) defined as follows for any arm \( x \in \mathcal{X}_t \):

\[
\Delta_t(x) := \left| \mu \left( \langle x, \hat{\theta}_t \rangle \right) - \mu \left( \langle x, \theta^*_t \rangle \right) \right|
\]

The next Lemma ties the cumulative pseudo-regret to the sum of prediction errors. This derivation is classical and the proof is deferred to Section B.1 in the supplementary material.

**Lemma 3** The following holds:

\[
R_T \leq 2 R_{\mu} \sum_{t=1}^{T} \beta_t(\delta) \left[ \|x_t\|_{V_t^{-1}} - \|x^*_t\|_{V_t^{-1}} \right] + \sum_{t=1}^{T} \left[ \Delta_t(x_t) + \Delta_t(x^*_t) \right].
\]

Thanks to Lemma 3 we are left to characterize the prediction error \( \Delta_t(x) \) for any \( x \in \mathcal{X}_t \). Following Filippi et al. (2010), we rely on the mean-value theorem to ensure that it exists \( \hat{\theta}_t \in [\bar{\theta}_t, \theta^*_t] \) such that\(^4\):

\[
\Delta_t(x) \leq k_{\mu} \left< x, H_t(\hat{\theta}_t) \left( g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right) \right>,
\]

where \( H_t(\theta) := \sum_{x=1}^{t-1} \mu(\langle x, \theta \rangle) x x^T + \lambda c_{\mu} I_d \). Since \( \hat{\theta}_t, \theta^*_t \in \Theta \), we obtain by convexity that \( \hat{\theta}_t \in \Theta \) and we can use the lower bound \( H_t(\hat{\theta}_t) \succeq c_{\mu} V_t \).

**Remark 4** In this last inequality resides the mistake that was made in previous extension of Filippi et al. (2010) to the non-stationary setting (Cheung et al., 2019a; Zhao et al., 2020). Indeed, if the prediction error is measured at \( \hat{\theta}_t \), we are left with \( \theta_t \in [\theta^*_t, \hat{\theta}_t] \), and \( \hat{\theta}_t \) can lie outside of the admissible set \( \Theta \) (since \( \hat{\theta}_t \) can). The lower-bound linking \( H_t(\hat{\theta}_t) \) and \( V_t \) would therefore not hold. More precisely, and as detailed in Section 3.2, when \( \hat{\theta}_t \in [\theta^*_t, \hat{\theta}_t] \) not much can be said on the link between \( H_t(\hat{\theta}_t) \) and \( V_t \) without severely degrading the final regret guarantees.

Adding and removing \( g_t(\hat{\theta}_t) + g_t(\theta^*_t) + g_t(\hat{\theta}_t) \) inside the inner-product in Equation (5), followed by easy manipulations yields:

\[
\Delta_t(x) \leq R_{\mu} \|x\|_{V_t^{-1}} \left( \left\| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t^{-1}} + \left\| g_t(\hat{\theta}_t) - g_t(\hat{\theta}_t) \right\|_{V_t^{-1}} \right) := \Delta_t^{\text{learn}}(x)
\]

\[
+ R_{\mu} \|x\|_2 \left( \left\| g_t(\theta^*_t) - g_t(\hat{\theta}_t) \right\|_{V_t^{-2}} + \left\| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t^{-2}} \right) := \Delta_t^{\text{track}}(x).
\]

**Leveraging the projection step** We can now bound the terms \( \Delta_t^{\text{learn}}(x) \) and \( \Delta_t^{\text{track}}(x) \) separately. Lemma 1 along with the design \( \hat{\theta}_t \in \mathcal{E}_t^B(\theta^*_t) \) leads to:

\[
\Delta_t^{\text{learn}}(x) \leq 2 R_{\mu} \|x\|_{V_t^{-1}} \beta_t(\delta) \quad \text{w.h.p} \quad (6)
\]

The first term in \( \Delta_t^{\text{track}}(x) \) is kept under control by the specific design of the projection step (P1). This is formalized in the following Lemma, whose proof is deferred to Section A.4 in the appendix.

\[^4\text{Formally,} \hat{\theta}_t \in [\hat{\theta}_t, \theta^*_t] \text{ means that there exists } v \in [0, 1] \text{ such that } \hat{\theta}_t = v \hat{\theta}_t + (1 - v) \theta^*_t.\]
Regret Bounds for Generalized Linear Bandits under Parameter Drift

**Lemma 4** Under the event \( \{ \hat{\theta}_t \in \mathcal{E}_t^k(\hat{\theta}_t) \} \) the following holds:

\[
\| g_t(\theta_t^\mu) - g_t(\hat{\theta}_t) \|_{V_t^{-2}} \leq \| g_t(\hat{\theta}_t) - g_t(\theta_t^\mu) \|_{V_t^{-2}}.
\]

As a result, bounding \( \Delta_t^{\text{track}}(x) \) reduces to bounding \( \| g_t(\hat{\theta}_t) - g_t(\theta_t^\mu) \|_{V_t^{-2}} \). Combined with Lemma 2, this result states that the deviation between \( \theta_t^\mu \) and \( \hat{\theta}_t \) is characterized by \( B_t \), the parameter-drift up to round \( t \), as illustrated in Figure 1. This leads to:

\[
\Delta_t^{\text{track}}(x) \leq 2R_\mu \| x \|_2 \left( \frac{2k_\mu L^2 S}{\lambda} \gamma D \frac{1 - \gamma}{1 - \gamma} + k_\mu \sum_{s=t-D}^{t-1} \| \theta_s^* - \theta_{s+1}^* \|_2 \right) \quad \text{w.h.p} \tag{7}
\]

**Putting everything together.** Combining Equations (6) and (7) with Lemma 3 and the Elliptical Lemma (Lemma 8 in the supplementary material) yields:

\[
R_T \leq C_1 R_\mu dT \log(1/\gamma) + C_2 R_\mu \gamma D T/(1 - \gamma) + C_3 R_\mu DB_T \quad \text{w.h.p}
\]

where the constants \( C_1, C_2 \) and \( C_3 \) hide \( \log(T) \) multiplicative dependencies. A detailed proof of this result is deferred to Section B.2 in the supplementary material. Setting the hyper-parameters \( D = \log(T)/(1 - \gamma) \) and \( \gamma = 1 - (\frac{2R_T}{dT})^{2/3} \) concludes the proof of Theorem 1.

6. Experiments

We illustrate in Figure 2 the behavior and performance of BVD-GLM-UCB with numerical simulations in a two-dimensional non-stationary logistic environment. Formally, we let \( r_{t+1} \sim \text{Bernoulli}(\mu(\langle x_t, \theta_t^s \rangle)) \) where \( \mu(z) = (1 + e^{-z})^{-1} \) is the logistic function. The sequence \( \{\theta_t^s\}_{t \geq 1} \) evolves as follows: we let \( \theta_t^s = (0, 1) \) for \( t \in [1, T/3] \). Between \( t = T/3 \) and \( t = 2T/3 \) we smoothly rotate \( \theta_t^s \) from \( (0, 1) \) to \( (1, 0) \). Finally we let \( \theta_t^s = (0, 1) \) for \( t \in [2T/3, T] \). A thorough description of the experimental setting can be found in Appendix F. We compare in Figure 2(a) the four following algorithms: OFUL (Abbasi-Yadkori et al., 2011) (stationary, here misspecified), GLM-UCB (Filippi et al., 2010) (stationary, here well-specified), D-LinUCB (Russac et al., 2019) (an exponentially weighted LB algorithm, non-stationary but here misspecified) and BVD-GLM-UCB (non-stationary, well-specified). For D-LinUCB and BVD-GLM-UCB we use the value of \( \gamma \) recommended by the asymptotic analysis. This figure highlights the necessity to employ algorithms that are well-specified; both GLM-UCB and BVD-GLM-UCB outperform their linear counterparts (OFUL and D-LinUCB, respectively). Note that an appropriate treatment of non-stationarity is also crucial to obtain small regret as for the considered horizon the two best performing algorithms are D-LinUCB and BVD-GLM-UCB. The latter being well-specified and resilient to non-stationary, it naturally performs best. In Figure 2(b) we highlight the fact that the projection step is necessary as, in this non-stationary setting, \( \hat{\theta}_t \) regularly leaves the admissible set \( \Theta \).

**Conclusion and future work**

We highlight in this paper a central difficulty in the theoretical treatment of non-stationary GLBs, overlooked in existing approaches and intimately linked to the non-linear nature of the reward function. To overcome this difficulty, we introduce a generalization of the projection step from (Filippi et al., 2010), which allows to simultaneously track the non-stationary ground-truth while preserving the
(a) Regret bounds of different stochastic bandit algorithms under parameter-drift. The grey region indicates a smooth drift of $\theta^*_t$.

(b) Evolution of the parameters of interest $(\theta^*_t, \hat{\theta}_t, \tilde{\theta}_t)$ for BVD-GLM-UCB. Note that in this non-stationary setting $\theta_t \notin \Theta$ is frequent.

Figure 2: Numerical simulations in a non-stationary logistic setting. For the first figure, results are average over 50 independent runs and shaded areas represent one standard-deviation variation.

learning guarantees of weighted maximum-likelihood strategies. This novel algorithmic design along with a careful analysis proves that an order-optimal (w.r.t $d$, $T$ and $B_T$) regret-bound can be achieved for GLBs under parameter-drift, although up to a rather restrictive assumption on the arm set’s geometry. The nature of the minimax-rates in the general case is open, as in both LB (c.f Touati and Vincent, 2021) and GLB setting (this work) we observe a mismatch between existing upper-bounds and the lower-bound of (Cheung et al., 2019b).

We underlined in Section 3.2 the problematic scaling of the problem-dependent constant $R_{\mu}$. Consequent research efforts have recently been deployed to reduce its impact on regret-bounds, both in the stationary (Faury et al., 2020; Abeille et al., 2020; Jun et al., 2020) and piece-wise stationary (Russac et al., 2021) settings. What is the optimal dependency w.r.t $R_{\mu}$ in the more general parameter-drift setting, and how it can be achieved are exciting open questions that we here leave for future work.

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Organization of the appendix

The appendix is organized as follows:

- In Section A we provide some concentration results, along with a bound on the prediction error $\Delta_t$ inherited from the design of the projection step.
- In Section B we link the prediction error $\Delta_t$ to the regret $R_T$ of $\text{BVD-GLM-UCB}$. We then proceed to prove the bound on $R_T$ announced in Theorem 1.
- In Section C we provide a proof for the equivalence of the optimization programs $(\text{P1})$ (along with the computation of $\bar{\theta}_t$) and $(\text{P2})$.
- Section D contains some secondary lemmas needed for the analysis, such as a version of the Elliptical Lemma for weighted matrices.
- In Section E we provide a proof for the regret upper-bound of $\text{BOB-BVD-GLM-UCB}$ claimed in Theorem 2.
- Finally, in Section F we provide some details on our numerical simulations.

Appendix A. Concentration and predictions bound

A.1. Confidence sets

**Lemma 1** [Learning] Let $\delta \in (0, 1]$. With probability at least $1 - \delta$:

$$\text{for all } t \geq 1, \quad \bar{\theta}_t \in \mathcal{E}_t^\delta(\bar{\theta}_t) = \left\{ \theta \in \mathbb{R}^d \text{ s.t. } \|g_t(\theta) - g_t(\bar{\theta}_t)\|_{\tilde{V}_t^{-1}} \leq \beta_t(\delta) \right\}.$$ 

**Proof** Recall that:

$$\mathcal{E}_t^\delta(\bar{\theta}_t) = \left\{ \theta \in \mathbb{R}^d \text{ s.t. } \|g_t(\theta) - g_t(\bar{\theta}_t)\|_{\tilde{V}_t^{-1}} \leq \beta_t(\delta) \right\},$$

where

$$\beta_t(\delta) = \sqrt{\lambda c_{\mu} S + \sigma \sqrt{2 \log(1/\delta) + d \log \left( 1 + \frac{L^2(1 - \gamma^2 t)}{\lambda d (1 - \gamma^2 t)} \right)}}.$$ 

Also, from the definition of $\bar{\theta}_t$ in Equation (4), by setting to 0 the differential of the convex objective minimized by $\bar{\theta}_t$ we obtain that:

$$g_t(\bar{\theta}_t) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle \theta_{s-1}^*, x_s \rangle) x_s + \lambda c_{\mu} \theta_{s-1}^t.$$ 

(8)

Further, for all $s \geq 1$, define

$$\epsilon_{s+1} = r_{s+1} - \mu(\langle \theta_s^*, x_s \rangle).$$ 

(9)
Let \( \hat{F}_s = \sigma(x_1, r_2, \ldots, x_{s-1}, r_s, x_s) \), which compared to \( F_s \) includes the arm \( x_s \). Note that:

\[
\begin{align*}
\mathbb{E} \left[ \epsilon_{s+1} \big| \hat{F}_s \right] &= 0 \quad \text{ (Equation (1))} \\
-\mu(\langle \theta^*_s, x_s \rangle) &\leq \epsilon_{s+1} \leq 2\sigma + \mu(\langle \theta^*_s, x_s \rangle) \quad \text{a.s \ (Assumption 2)}
\end{align*}
\]

Therefore \( \epsilon_{s+1} \) is \( \sigma \)-subGaussian conditionally on \( \hat{F}_s \). Furthermore, by optimality of \( \hat{t}_t \), differentiating the objective function in Equation (2) yields:

\[
\sum_{s=1}^{t-1} \gamma^{t-1-s} \left[ \mu(\langle \hat{t}_t, x_s \rangle) - r_{s+1} \right] x_s + \lambda c_\mu \hat{t}_t = 0
\]

\[
\Rightarrow g_t(\hat{t}_t) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle \theta^*_s, x_s \rangle) x_s + \sum_{s=1}^{t-1} \gamma^{t-1-s} \epsilon_{s+1} x_s \quad \text{(Equation (9))}
\]

\[
\Rightarrow g_t(\hat{t}_t) = g_t(\tilde{t}_t) + \sum_{s=1}^{t-1} \gamma^{t-1-s} \epsilon_{s+1} x_s - \lambda c_\mu \theta^*_s \quad \text{(Equation (8))}
\]

\[
\Rightarrow \|g_t(\tilde{t}_t) - g_t(\hat{t}_t)\|_{\tilde{V}^{-1}} = \left\| \sum_{s=1}^{t-1} \gamma^{t-1-s} \epsilon_{s+1} x_s - \lambda c_\mu \theta^*_s \right\|_{\tilde{V}^{-1}}.
\]

Therefore since \( \theta^*_s \in \Theta \) and \( \tilde{V}_t \succeq \lambda I_d \) we obtain:

\[
\|g_t(\tilde{t}_t) - g_t(\hat{t}_t)\|_{\tilde{V}^{-1}} \leq \sqrt{\lambda c_\mu S} + \left\| \sum_{s=1}^{t-1} \gamma^{t-1-s} \epsilon_{s+1} x_s \right\|_{\tilde{V}^{-1}}.
\]

Simplifying the factors \( \gamma^{t-1} \) in the most right term and applying Proposition 1 of Russac et al. (2019) proves that with probability at least \( 1 - \delta \), for all \( t \geq 1 \):

\[
\|g_t(\tilde{t}_t) - g_t(\hat{t}_t)\|_{\tilde{V}^{-1}} \leq \sqrt{\lambda c_\mu S} + \sigma \sqrt{2 \log(1/\delta) + d \log \left( 1 + \frac{L^2 (1 - \gamma^2 t)}{\lambda d (1 - \gamma^2)} \right)} = \beta_t(\delta),
\]

hence proving the desired result. \( \blacksquare \)

A.2. Bounding the prediction error

**Lemma 5** Let \( \delta \in (0, 1] \) and \( D \in \mathbb{N}^* \). With probability at least \( 1 - \delta \): for all \( t \geq 1 \), for all \( x \in \mathcal{X}_t \), under Assumption 4 the following holds.

\[
\Delta_t(x) \leq \frac{2k_\mu L}{c_\mu} \beta_t(\delta) \| x \|_{V_t^{-1}} + \frac{4k_\mu^2 L^3 S}{c_\mu \lambda} \gamma^D \left( \frac{\gamma}{1-\gamma} \right) + \frac{2k_\mu L}{c_\mu} \sum_{s=t-D}^{t-1} \| \theta^*_s - \theta^*_s + 1 \|_2.
\]

Without Assumption 4, under general arm-set geometry, the following holds.

\[
\Delta_t(x) \leq \frac{2k_\mu L}{c_\mu} \beta_t(\delta) \| x \|_{V_t^{-1}}
\]

\[
+ \frac{2k_\mu L}{c_\mu} \sqrt{1 + \frac{L^2}{\lambda (1-\gamma)}} \left( \frac{2k_\mu S L^2}{\lambda} \frac{\gamma^D}{1-\gamma} + k_\mu \sqrt{d \frac{d}{\lambda (1-\gamma)}} \sum_{s=t-D}^{t-1} \| \theta^*_s - \theta^*_s + 1 \|_2 \right).
\]
Proof In the following, we assume that the event \( E_\delta = \{ \hat{\theta}_t \in \mathcal{E}_t^\delta(\hat{\theta}_t) \text{ for all } t \geq 1 \} \) holds, which happens with probability at least \( 1 - \delta \) (Lemma 1). From the definition of the prediction error:

\[
\Delta_t(x) = |\mu(\langle x, \hat{\theta}_t \rangle) - \mu(\langle x, \hat{\theta}_t \rangle)| \\
\leq \left( \sup_{x \in \mathcal{X}, \theta \in \Theta} \hat{\mu}(\langle x, \theta \rangle) \right) \left| \langle x, \hat{\theta}_t - \hat{\theta}_t \rangle \right| \quad (x \in \mathcal{X}, \theta_t \in \Theta, \hat{\theta}_t \in \Theta) \\
\leq k_\mu \left| \langle x, \hat{\theta}_t - \hat{\theta}_t \rangle \right| . 
\] 

(by definition of \( k_\mu \)) (11)

Further, thanks to the mean value theorem:

\[
g_t(\hat{\theta}_t) - g_t(\theta^*_t) = \sum_{s=1}^{t-1} \gamma^{t-s} \left[ \mu(\langle \hat{\theta}_t, x_s \rangle) - \mu(\langle \theta^*_t, x_s \rangle) \right] + \lambda c_\mu (\hat{\theta}_t - \theta^*_t) \\
= \sum_{s=1}^{t-1} \gamma^{t-s} \left[ \int_0^1 \hat{\mu} \left( \langle x_s, (1-v)\theta^*_t + v\hat{\theta}_t \rangle \right) dv \right] x_s x_s^T (\hat{\theta}_t - \theta^*_t) + \lambda c_\mu (\hat{\theta}_t - \theta^*_t) \\
= G_t \cdot (\hat{\theta}_t - \theta^*_t) , 
\] 

(12)

where:

\[
G_t := \sum_{s=1}^{t-1} \gamma^{t-s} \left[ \int_0^1 \hat{\mu} \left( \langle x_s, (1-v)\theta^*_t + v\hat{\theta}_t \rangle \right) dv \right] x_s x_s^T + \lambda c_\mu I_d \geq c_\mu V_t . 
\]

Note that because \( x_s \in \mathcal{X} \) for all \( s \in [t-1] \) and \( \hat{\theta}_t, \theta^*_t \in \Theta \) we have \( G_t \geq c_\mu V_t \). Assembling together Equations (11) and (12) we get:

\[
\Delta_t(x) \leq k_\mu \left| \langle x, G_t^{-1}(g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \rangle \right| \\
\leq k_\mu \left| \langle x, G_t^{-1}(g_t(\hat{\theta}_t) - g_t(\theta^*_t) + g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \rangle \right| \\
\leq k_\mu \left| \langle x, G_t^{-1}(g_t(\hat{\theta}_t) - g_t(\theta^*_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \rangle \right| \\
:= \Delta_t^{\text{learn}}(x) \\
\begin{aligned}
&+ k_\mu \left| \langle x, G_t^{-1}(g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \rangle \right| \\
&:= \Delta_t^{\text{track}}(x) \\
&\leq \Delta_t^{\text{learn}}(x) + \Delta_t^{\text{track}}(x) .
\end{aligned} 
\] 

(13)

This decomposition brings out the contribution of two different phenomena \((\text{learning and tracking})\) which will be handled separately. Starting with the learning:
\[ \Delta_t^{\text{learn}}(x) = k_\mu \left| \langle x, G_t^{-1}(g_t(\hat{\theta}_t) - g_t(\theta^*_t)) + g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t) \rangle \right| \\
= k_\mu \left| \langle \tilde{V}^{1/2}_t G_t^{-1} x, \tilde{V}^{1/2}_t (g_t(\hat{\theta}_t) - g_t(\theta^*_t)) + g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t) \rangle \right| \\
\leq k_\mu \|x\| \|G_t^{-1} \tilde{V}_t G_t^{-1} (g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \| + \|g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t)\| \|\tilde{V}_t^{-1}\| \\
\leq k_\mu \|x\| \|G_t^{-1} \tilde{V}_t G_t^{-1} (g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \| + \|g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t)\| \|\tilde{V}_t^{-1}\| \quad \text{(Cauchy-Schwarz)} \\
\leq \frac{k_\mu}{\sqrt{c_\mu}} \|x\| \|V_t^{-1} (g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \| + \|g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t)\| \|V_t^{-1}\| \quad \text{($G_t^{-1} \leq c_\mu^{-1} V_t^{-1}$)} \\
\leq \frac{k_\mu}{c_\mu} \|x\| \|V_t^{-1} (g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \| + \|g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t)\| \|V_t^{-1}\| \quad \text{($\hat{\theta}_t \in E_t^\delta(\theta^*_t)$)} \\
\leq \frac{k_\mu}{c_\mu} \|x\| \|V_t^{-1} (\beta_t(\delta) + \beta_t(\delta)) \| \quad \text{($E_\delta$ holds)} \\
\]

We used $\tilde{V}_t \leq V_t$ which is a consequence of $\gamma \in (0, 1)$. As a result:

\[ \Delta_t^{\text{learn}}(x) \leq \frac{2k_\mu}{c_\mu} \beta_t(\delta) \|x\| \|V_t^{-1}\|. \quad (14) \]

The tracking term is bounded differently when the action set satisfies Assumption 4 or for general arm-set geometry. The bound on the tracking term is reported in Lemma 6 and its proof is reported in Section A.3.

**Lemma 6** Let $D \in \mathbb{N}^*$. When Assumption 4 holds, we have the following:

\[ \Delta_t^{\text{track}}(x) \leq \frac{4k_\mu^2 L^3 S}{c_\mu \lambda} \frac{\gamma D}{(1 - \gamma)} + \frac{2k_\mu^2 L}{c_\mu} \sum_{s=t-D}^{t-1} \|\theta_s^* - \theta_{s+1}^*\|_2 \quad (15) \]

For general arm-set geometry, we have the following:

\[ \Delta_t^{\text{track}}(x) \leq \frac{2k_\mu L}{c_\mu} \sqrt{1 + \frac{L^2}{\lambda(1 - \gamma)}} \left( \frac{2k_\mu S L^2}{\lambda} \frac{\gamma D}{1 - \gamma} + k_\mu \sqrt{\frac{d}{\lambda(1 - \gamma)}} \sum_{s=t-D}^{t-1} \|\theta_s^* - \theta_{s+1}^*\|_2 \right) \]

Assembling Equations (13), (14) and the two different inequalities from Lemma 6 gives the two statements of the proof.

**A.3. Proof of Lemma 6**

**Lemma 6** Let $D \in \mathbb{N}^*$. When Assumption 4 holds, we have the following:

\[ \Delta_t^{\text{track}}(x) \leq \frac{4k_\mu^2 L^3 S}{c_\mu \lambda} \frac{\gamma D}{(1 - \gamma)} + \frac{2k_\mu^2 L}{c_\mu} \sum_{s=t-D}^{t-1} \|\theta_s^* - \theta_{s+1}^*\|_2 \quad (15) \]
For general arm-set geometry, we have the following
\[
\Delta^\text{track}_t(x) \leq \frac{2k \mu L}{c \mu} \sqrt{1 + \frac{L^2}{\lambda(1 - \gamma)}} \left( \frac{2k \mu S L^2}{\lambda} \gamma^D + k \mu \sqrt{\frac{d}{\lambda(1 - \gamma)}} \sum_{s=t-D}^{t-1} \left\| \theta^*_s - \theta^*_s + 1 \right\|_2 \right)
\]

**Proof** Throughout the proof, we will use the following lemma, proven in Section A.4.

**Lemma 4** Under the event \{\( \hat{\theta}_t \in \mathcal{E}_t^1(\hat{\theta}_t) \)\} the following holds:
\[
\left\| g_t(\theta^*_t) - g_t(\hat{\theta}_t) \right\|_{V_t}^2 \leq \left\| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2.
\]

With Assumption 4. In this first part of the proof, we assume that Assumption 4 holds. We have the following:
\[
\Delta^\text{track}_t(x) = k \mu \left\langle x, \mathbf{G}_t^{-1}(g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \right\rangle
\]
\[
\leq \frac{2k \mu L}{c \mu} \left\| g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2 (\text{Cauchy-Schwarz})
\]
\[
\leq \frac{k \mu L}{c \mu} \left( \left\| g_t(\theta^*_t) - g_t(\hat{\theta}_t) \right\|_{V_t}^2 + \left\| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2 \right) \quad (\text{Triangle inequality})
\]
\[
\leq \frac{2k \mu L}{c \mu} \left\| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2 (\text{Lemma 4})
\]

where the third inequality can be obtained only because when Assumption 4 holds \( \mathbf{G}_t \) and \( V_t \) commute. The final is obtained using Lemma 2 reported here and established in Section 2.

**Lemma 2** [Tracking with orthogonal action sets] Let \( D \in \mathbb{N}^* \). The following holds:
\[
\left\| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2 \leq \frac{k \mu L^2}{c \mu} \gamma^D + k \mu \sum_{s=t-D}^{t-1} \left\| \theta^*_s - \theta^*_s + 1 \right\|_2.
\]

Without Assumption 4. We now explain how to extend the analysis with general arm-set geometry.
\[
\Delta^\text{track}_t(x) = k \mu \left\langle x, \mathbf{G}_t^{-1}(g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t)) \right\rangle
\]
\[
= \frac{2k \mu L}{c \mu} \left\| g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2 (\text{Cauchy-Schwarz})
\]
\[
\leq \frac{k \mu L}{\sqrt{\lambda_{\text{max}}(V_t)}} \left\| g_t(\theta^*_t) - g_t(\hat{\theta}_t) + g_t(\hat{\theta}_t) - g_t(\theta^*_t) \right\|_{V_t}^2 (\text{Lemma 4})
\]
We then use:
\[ \lambda_{\text{max}}(V_t) \leq \frac{L^2}{1 - \gamma} + \lambda. \quad (16) \]

That can be obtained by computing the operator norm of the matrix $V_t$. Combining this with Lemma 7 reported here and proved in Section A.6 achieves the proof.

**Lemma 7** [Tracking with general action sets] Let $D \in \mathbb{N}^*$. The following holds:
\[ \|g_t(\bar{\theta}_t) - g_t(\theta^*_*)\|_{V_t^{-2}} \leq \frac{2k_\mu L^2 S}{\lambda} \gamma^D \frac{\sqrt{d}}{1 - \gamma} \sqrt{1 - \gamma} + \lambda \sum_{s=1}^{t-1} \|\theta^*_s - \theta^*_{s+1}\|_2. \]

\[ \text{Lemma 4} \]

Under the event $\{\bar{\theta}_t \in \mathcal{C}_t(\hat{\theta}_t)\}$ the following holds:
\[ \|g_t(\theta^*_t) - g_t(\bar{\theta}_t)\|_{V_t^{-2}} \leq \|g_t(\bar{\theta}_t) - g_t(\hat{\theta}_t)\|_{V_t^{-2}}. \]

**Proof** We prove this result by contradiction. Assume that:
\[ \|g_t(\theta^*_t) - g_t(\bar{\theta}_t)\|_{V_t^{-2}} > \|g_t(\bar{\theta}_t) - g_t(\hat{\theta}_t)\|_{V_t^{-2}}, \quad (17) \]

For all $s \geq 1$ define:
\[ \tilde{r}_{s+1} := \mu(\langle x_s, \theta^*_s \rangle) + \epsilon_{s+1}, \quad (18) \]

where $\{\epsilon_s\}_s$ is defined in Equation (9). Further, let:
\[ \theta_c := \arg \min_{\theta \in \mathbb{R}^d} \sum_{s=1}^{t-1} \gamma^{t-1-s} [\mu(\langle \theta, x_s \rangle) - \tilde{r}_{s+1} \langle \theta, x_s \rangle)] + \frac{\lambda_c \mu}{2} \|\theta\|_2^2, \]

which is well-defined as the minimizer of a strictly convex, coercive function. Upon differentiating we get:
\[ g_t(\theta_c) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \tilde{r}_{s+1} x_s \]
\[ = \sum_{s=1}^{t-1} \gamma^{t-1-s} \epsilon_{s+1} x_s + \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle x_s, \theta^*_s \rangle) x_s \quad \text{(Equation (18))} \]
\[ = g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t) + \lambda_c \mu \theta^*_t + \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle x_s, \theta^*_s \rangle) x_s \quad \text{(Equation (10))} \]
\[ = g_t(\hat{\theta}_t) - g_t(\bar{\theta}_t) + g_t(\theta^*_t). \quad (19) \]
Therefore:

\[
\| g_t(\theta_c) - g_t(\hat{\theta}_t) \|_{\mathbf{V}_t^{-2}}^2 = \| g_t(\hat{\theta}_t) - g_t(\theta^*_t) \|_{\mathbf{V}_t^{-2}}^2 \\
< \| g_t(\theta^*_t) - g_t(\hat{\theta}_t) \|_{\mathbf{V}_t^{-2}}^2 .
\]  
(Equation 17)

Further from Equation (19) we get:

\[
\| g_t(\theta_c) - g_t(\theta^*_t) \|_{\mathbf{V}_t^{-2}} = \| g_t(\hat{\theta}_t) - g_t(\hat{\theta}_t) \|_{\mathbf{V}_t^{-1}} \\
\leq \beta_t(\delta) (\hat{\theta}_t \in \mathcal{E}_t^\delta(\hat{\theta}_t)) \\
\Leftrightarrow \theta^*_t \in \mathcal{E}_t^\delta(\theta_c).
\]

To sum-up, we have \( \| g_t(\theta_c) - g_t(\hat{\theta}_t) \|_{\mathbf{V}_t^{-2}} < \| g_t(\theta^*_t) - g_t(\hat{\theta}_t) \|_{\mathbf{V}_t^{-2}} \) and \( \mathcal{E}_t^\delta(\theta_c) \cap \Theta \neq \emptyset \) since \( \theta^*_t \in \Theta \cap \mathcal{E}_t^\delta(\theta_c) \). This contradicts the definition of \( \theta^*_t \) (in \( \textbf{P1} \)) and therefore Equation (17) must be wrong, which proves the announced result. \( \blacksquare \)

**A.5. Proof of Lemma 2**

**Lemma 2** *Tracking with orthogonal action sets* Let \( D \in \mathbb{N}^+ \). The following holds:

\[
\| g_t(\bar{\theta}_t) - g_t(\theta^*_t) \|_{\mathbf{V}_t^{-2}} \leq \frac{2k_\mu L^2 S}{\lambda} \frac{\gamma^D}{1 - \gamma} + k_\mu \sum_{s=t-D}^{t-1} \| \theta^*_s - \theta^*_s+1 \|_2 .
\]

**Proof** Thanks to Equation (8) we have:

\[
g_t(\bar{\theta}_t) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle x_s, \theta^*_s \rangle) x_s + \lambda \epsilon_t \theta^*_t \\
\Leftrightarrow g_t(\bar{\theta}_t) - g_t(\theta^*_t) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \left[ \mu(\langle x_s, \theta^*_s \rangle) - \mu(\langle x_s, \theta^*_s \rangle) \right] x_s
\]

\[
\Leftrightarrow g_t(\bar{\theta}_t) - g_t(\theta^*_t) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \int_{v=0}^{1} \mu \left( x_s, v \theta^*_s + (1-v)\theta^*_s \right) dv \right] x_s x_s^T (\theta^*_s - \theta^*_t) \text{ (mean-value theorem)}
\]

\[
\Leftrightarrow g_t(\bar{\theta}_t) - g_t(\theta^*_t) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta^*_s - \theta^*_t)
\]

where we defined:

\[
\alpha_s := \int_{v=0}^{1} \mu \left( x_s, v \theta^*_s + (1-v)\theta^*_s \right) dv \in [c_\mu, k_\mu] .
\]

Therefore:

\[
\| g_t(\bar{\theta}_t) - g_t(\theta^*_t) \|_{\mathbf{V}_t^{-2}} = \left\| \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta^*_s - \theta^*_t) \right\|_{\mathbf{V}_t^{-2}} .
\]  
(20)
The rest of the proof follows the strategy of Russac et al. (2019) to yield the announced result. Let \( D \in \mathbb{N}^* \) and notice that:

\[
\| \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^s - \theta_*^t) \|_{\mathbf{V}_t^{-2}} \leq \underbrace{\| \sum_{s=1}^{t-D-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^s - \theta_*^t) \|_{\mathbf{V}_t^{-2}}}_{=: d_1} + \underbrace{\| \sum_{s=t-D}^{t-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^s - \theta_*^t) \|_{\mathbf{V}_t^{-2}}}_{=: d_2}.
\]

Both terms are bounded separately; starting with \( d_1 \):

\[
d_1 \leq \lambda^{-1} \| \sum_{s=1}^{t-D-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^s - \theta_*^t) \|_{\mathbf{V}_t^{-2}} \leq \lambda^{-1} \sum_{s=1}^{t-D-1} \gamma^{t-1-s} \| \alpha_s \| \| x_s x_s^T (\theta_*^s - \theta_*^t) \|_{\mathbf{V}_t^{-2}} \leq 2k_p \lambda^{-1} SL^2 \sum_{s=1}^{t-D-1} \gamma^{t-1-s} \| x_s \|_2 \leq \lambda, \theta_*^s, \theta_*^t \in \Theta, \| \alpha_s \| \leq k_p \]

\[
\leq 2k_p \lambda^{-1} SL^2 \gamma D (1 - \gamma)^{-1}.
\]

For \( d_2 \) a careful analysis is required.

\[
d_2 = \| \mathbf{V}_t^{-1} \sum_{s=t-D}^{t-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^s - \theta_*^t) \| \]

\[
= \| \sum_{s=t-D}^{t-1} \mathbf{V}_t^{-1} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^s - \theta_*^t) \| \]

\[
= \| \sum_{s=t-D}^{t-1} \mathbf{V}_t^{-1} \gamma^{t-1-s} \alpha_s x_s x_s^T \sum_{p=s}^{t-1} (\theta_*^p - \theta_*^{p+1}) \| \text{ (Telescopic sum)}
\]

\[
\leq \| \sum_{p=t-D}^{t-1} \mathbf{V}_t^{-1} \sum_{s=t-D}^{p} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^p - \theta_*^{p+1}) \| \text{ (Re-arranging)}
\]

\[
\leq \| \sum_{p=t-D}^{t-1} \mathbf{V}_t^{-1} \sum_{s=t-D}^{p} \gamma^{t-1-s} \alpha_s x_s x_s^T (\theta_*^p - \theta_*^{p+1}) \| \text{ (Triangle inequality)}
\]

At this point, Assumption 4 can be used to upper-bound the operator norm of the matrix \( \mathbf{V}_t^{-1} \sum_{s=t-D}^{p} \gamma^{t-1-s} \alpha_s x_s x_s^T \). Under Assumption 4, the following holds:

\[
\mathbf{V}_t^{-1} \sum_{s=t-D}^{p} \gamma^{t-1-s} \alpha_s x_s x_s^T = \mathbf{V}_t^{-1/2} \sum_{s=t-D}^{p} \gamma^{t-1-s} \alpha_s x_s x_s^T \mathbf{V}_t^{-1/2} := \mathbf{M}_t
\]

(21)
The advantage, now is that the matrix on the right-hand side of Equation (21) is symmetric and we can use the relation \( \| Mx \| \leq \| M \| \| x \|_2 \) that holds for all symmetric matrix \( M \) and where \( \| M \| \) denotes the operator norm of \( M \). The final step consists in upper-bounding the operator norm of \( M_t \).

We have,

\[
\forall x, \| x \|_2 \leq 1, \quad x^T M_t x = x^T V_t^{-1/2} \sum_{s=t-D}^{p} \gamma^{t-1-s} x_s x_s^T V_t^{-1/2} x
\]

\[
= \sum_{s=t-D}^{p} \alpha_s x^T V_t^{-1/2} x_s x_s^T V_t^{-1/2} x
\]

\[
= \sum_{s=t-D}^{p} \gamma^{t-1-s} \alpha_s \left( x_s^T V_t^{-1/2} x \right)^2
\]

\[
\leq k_\mu \sum_{s=t-D}^{p} \gamma^{t-1-s} \left( x_s^T V_t^{-1/2} x \right)^2
\]

\[
\leq k_\mu x^T V_t^{-1/2} \sum_{s=t-D}^{p} \gamma^{t-1-s} x_s x_s^T V_t^{-1/2} x .
\]

Furthermore,

\[
\forall x, \| x \|_2 \leq 1, \quad x^T V_t^{-1/2} \sum_{s=t-D}^{p} \gamma^{t-1-s} x_s x_s^T V_t^{-1/2} x \leq x^T V_t^{-1/2} \left( \sum_{s=1}^{t-1} \gamma^{t-1-s} x_s x_s^T \right) V_t^{-1/2} x
\]

\[
\leq x^T x \leq 1.
\]

Combining the two inequalities ensures that

\[
\| M_t \| \leq k_\mu
\]

Finally,

\[
d_2 \leq k_\mu \sum_{p=t-D}^{t-1} \| \theta^p - \theta^{p+1}_* \| .
\]

A.6. Proof of Lemma 7

**Lemma 7** [Tracking with general action sets] Let \( D \in \mathbb{N}^* \). The following holds:

\[
\| g_t(\theta) - g_t(\theta^* \| V_t^{-2} \leq \frac{2k_\mu L^2 S}{\lambda} \frac{\gamma^D}{1 - \gamma} + \frac{k_\mu}{\sqrt{\lambda}} \frac{\sqrt{d}}{\sqrt{1 - \gamma}} \sum_{s=t-D}^{t-1} \| \theta_s^* - \theta_s^{*+1} \|_2 .
\]
Proof Following the proof of Lemma 2, one has:

\[ \|g_t(\bar{\theta}_t) - g_t(\theta^*_t)\|_{V_t}^2 = \left\| V_t^{-1} \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s \theta^*_s - \theta^*_t \right\|_2 \]

We follow the line of proof from (Touati and Vincent, 2021, Appendix C) where the only difference is the \( \alpha_s \) term. We use

\[ \left\| V_t^{-1} \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right\|_2 = \max_{x: \|x\|_2 = 1} \left| x^T V_t^{-1} \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right| \quad (23) \]

Let \( x \in \mathbb{R}^d \) such that \( \|x\|_2 = 1 \), we have

\[ \left| x^T V_t^{-1} \sum_{s=1}^{t-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right| \leq \left| x^T V_t^{-1} \sum_{s=1}^{t-D-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right| + \left| x^T V_t^{-1} \sum_{s=t-D}^{t-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right| \]

For the first term, using Cauchy-Schwarz, we obtain the term \( d_1 \) from the proof of Lemma 2. Hence,

\[ \left| x^T V_t^{-1} \sum_{s=1}^{t-D-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right| \leq \|x\|^2 \frac{2k_\mu^2 S L^2}{\lambda} \frac{\gamma^{D}}{1 - \gamma} \]

Let \( b = \left| x^T V_t^{-1} \sum_{s=t-D}^{t-1} \gamma^{t-1-s} \alpha_s x_s x^T_s (\theta^*_s - \theta^*_t) \right| \). One has,

\[ b = \sum_{s=t-D}^{t-1} \gamma^{t-1-s} |x^T V_t^{-1} x_s| \|\alpha_s\| \|x^T_s\| (\theta^*_s - \theta^*_t) \| \quad \text{(Triangle inequality)} \]

\[ \leq k_\mu \sum_{s=t-D}^{t-1} \|x^T V_t^{-1} x_s\| \|\alpha_s\| \left| \sum_{p=s}^{t-1} (\theta^*_p - \theta^*_p) \right| \]

\[ \leq k_\mu L \sum_{s=t-D}^{t-1} \|x^T V_t^{-1} x_s\| \left| \sum_{p=s}^{t-1} (\theta^*_p - \theta^*_p) \right| \| \quad \text{(Cauchy-Schwarz, \( \|x_s\|_2 \leq L \))} \]

\[ \leq \sum_{p=t-D}^{t-1} \sum_{s=t-D}^{t-1} \gamma^{t-1-s} |x^T V_t^{-1} x_s| \|\theta^*_p - \theta^*_p \| \|_2 \]

\[ \leq \sum_{p=t-D}^{t-1} \sum_{s=t-D}^{t-1} \gamma^{t-1-s} \sqrt{x^T V_t^{-1} x_s \sqrt{x^T V_t^{-1} x_s}} \|\theta^*_p - \theta^*_p \| \|_2 \quad \text{(Cauchy-Schwarz)} \]

\[ \leq \sum_{p=t-D}^{t-1} \sum_{s=t-D}^{t-1} \gamma^{t-1-s} x^T V_t^{-1} x_s \left| \sum_{s=t-D}^{t-1} \gamma^{t-1-s} x^T V_t^{-1} x_s \|\theta^*_p - \theta^*_p \| \|_2 \right. \quad \text{(Cauchy-Schwarz)} \]
Now, \[
\sqrt{\sum_{s=t-D}^{t-1} \gamma^{t-1-s} x_s^T V_t^{-1} x_s} \leq \sqrt{\text{tr} \left( \sum_{s=t-D}^{t-1} \gamma^{t-1-s} x_s^T V_t^{-1} x_s \right)} \leq \sqrt{\text{tr}(I_d)} = \sqrt{d}
\]

Further, \[
\sqrt{\sum_{s=t-D}^{t-1} \gamma^{t-1-s} x^T V_t^{-1} x} \leq \frac{1}{\sqrt{\lambda}} \|x\|_2 \frac{1}{\sqrt{1-\gamma}}
\]

Bringing things together, yields the announced result.

\[\square\]

Appendix B. Regret bound

B.1. Regret decomposition

Lemma 3 The following holds:

\[R_T \leq \frac{2k \mu}{c \mu} \sum_{t=1}^{T} \beta_t(\delta) \left[ \|x_t\|_{V_t^{-1}} - \|x_t^t\|_{V_t^{-1}} \right] + \sum_{t=1}^{T} \left[ \Delta_t(x_t) + \Delta_t(x_t^t) \right].\]

Proof We recall that \(x_t^t = \arg \max_{x \in X_t} \mu(\langle \theta_t^t, x \rangle).\) Note that:

\[
R_T = \sum_{t=1}^{T} \mu(\langle x_t^*, \theta_t^* \rangle) - \mu(\langle x_t, \theta_t^* \rangle)
= \sum_{t=1}^{T} \left[ \mu(\langle x_t^*, \theta_t^* \rangle) - \mu(\langle x_t^t, \tilde{\theta}_t \rangle) + \mu(\langle x_t^t, \tilde{\theta}_t \rangle) - \mu(\langle x_t, \tilde{\theta}_t \rangle) + \mu(\langle x_t, \tilde{\theta}_t \rangle) - \mu(\langle x_t, \theta_t^* \rangle) \right]
= \sum_{t=1}^{T} \left[ \mu(\langle x_t^t, \tilde{\theta}_t \rangle) - \mu(\langle x_t, \tilde{\theta}_t \rangle) \right] + \sum_{t=1}^{T} \left[ \mu(\langle x_t^t, \tilde{\theta}_t \rangle) - \mu(\langle x_t^t, \tilde{\theta}^t_t \rangle) \right] + \sum_{t=1}^{T} \left[ \mu(\langle x_t, \tilde{\theta}_t \rangle) - \mu(\langle x_t, \theta_t^* \rangle) \right]
\leq \frac{2k \mu}{c \mu} \sum_{t=1}^{T} \beta_t(\delta) \left[ \|x_t\|_{V_t^{-1}} - \|x_t^t\|_{V_t^{-1}} \right]
+ \sum_{t=1}^{T} \left[ \mu(\langle x_t^t, \theta_t^* \rangle) - \mu(\langle x_t^t, \tilde{\theta}_t \rangle) \right] + \sum_{t=1}^{T} \left[ \mu(\langle x_t, \tilde{\theta}_t \rangle) - \mu(\langle x_t, \theta_t^* \rangle) \right].
\]

In the last inequality, we used the fact that \(x_t = \arg \max_{x \in X_t} \left\{ \mu(\langle x, \tilde{\theta}_t \rangle) + \frac{2k \mu}{c \mu} \beta_t(\delta) \|x\|_{V_t^{-1}} \right\}.
\]

Using the definition of \(\Delta_t(x)\) we conclude that:

\[R_T \leq \frac{2k \mu}{c \mu} \sum_{t=1}^{T} \beta_t(\delta) \left[ \|x_t\|_{V_t^{-1}} - \|x_t^t\|_{V_t^{-1}} \right] + \sum_{t=1}^{T} \left[ \Delta_t(x_t) + \Delta_t(x_t^t) \right].\]

\[\square\]
B.2. Regret bound

We now claim Theorem 1, bounding the regret of BVD-GLM-UCB.

**Theorem 1** Let $\delta \in (0, 1]$ and $D \in \mathbb{N}^*$. Under Assumptions 1-2-3 and Assumption 4 with probability at least $1 - \delta$:

$$R_T \leq C_1 R_\mu \beta_T(\delta) \sqrt{T \log(1/\gamma)} + \log \left( 1 + \frac{L^2(1 - \gamma^T)}{\lambda d(1 - \gamma)} \right) + C_2 R_\mu \frac{\gamma^D}{1 - \gamma} T + C_3 R_\mu DB_T$$

Further, setting $\gamma = 1 - (B_T/(dT))^{2/3}$ ensures:

$$R_T = \tilde{O}\left( \frac{k_{\mu} d^{2/3} B_T^{1/3} T^{2/3}}{c_\mu} \right) \quad \text{w.h.p.}$$

Under general arm-set geometry and Assumptions 1-2-3, with probability at least $1 - \delta$

$$R_T \leq C_1 R_\mu \beta_T(\delta) \sqrt{T \log(1/\gamma)} + \log \left( 1 + \frac{L^2(1 - \gamma^T)}{\lambda d(1 - \gamma)} \right) + C_4 R_\mu \frac{\gamma^D}{1 - \gamma} T + C_5 k_{\mu} R_\mu \frac{\gamma^D}{1 - \gamma} T + C_6 k_{\mu} R_\mu \sqrt{d} \frac{d T}{1 - \gamma} + C_7 k_{\mu} R_\mu \frac{\sqrt{d} T}{1 - \gamma} B_T$$

Further, setting $\gamma = 1 - \frac{B_T^{2/5}}{d(\sqrt{5} T)^{2/5}}$ ensures:

$$R_T = \tilde{O}\left( \frac{k_{\mu} d^{4/10} B_T^{1/5} T^{4/5}}{c_\mu} \right) \quad \text{w.h.p.}$$

**Proof** In the following, we assume that the event $\{\hat{\theta}_t \in E^{\delta}(\hat{\theta}_t), \forall t \geq 1\}$ holds, which happens with probability at least $1 - \delta$ (Lemma 1). Thanks to Lemma 5, when Assumption 4 the following holds:

$$\Delta_t(x_t) + \frac{2k_{\mu}}{c_\mu} \beta_T(\delta) \|x_t\|_V^{-1} \leq \frac{4k_{\mu}}{c_\mu} \beta_T(\delta) \|x_t\|_V^{-1} + \frac{4k_{\mu}^2 L^3 S}{c_\mu \lambda(1 - \gamma)} \gamma^D + \frac{2k_{\mu}^2 L}{c_\mu} \sum_{s=t-D}^{t-1} \|\theta_s^* - \theta_s^{*+1}\|_2$$

$$\Delta_t(x_t^l) - \frac{2k_{\mu}}{c_\mu} \beta_T(\delta) \|x_t^l\|_V^{-1} \leq \frac{4k_{\mu}^2 L^3 S}{c_\mu \lambda(1 - \gamma)} \gamma^D + \frac{2k_{\mu}^2 L}{c_\mu} \sum_{s=t-D}^{t-1} \|\theta_s^* - \theta_s^{*+1}\|_2$$

Assembling this result with Lemma 3 yields:

$$R_T \leq \sum_{t=1}^{T} \frac{4k_{\mu}}{c_\mu} \beta_T(\delta) \|x_t\|_V^{-1} + \sum_{t=1}^{T} \left[ \frac{8k_{\mu}^2 L^3 S}{c_\mu \lambda(1 - \gamma)} \gamma^D + \frac{4k_{\mu}^2 L}{c_\mu} \sum_{s=t-D}^{t-1} \|\theta_s^* - \theta_s^{*+1}\|_2 \right].$$

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We now bound each term separately. Starting with $R_{T}^{\text{learn}}$:

$$R_{T}^{\text{learn}} \leq \frac{4k_\mu}{c_\mu} \beta_T(\delta) \sum_{t=1}^{T} \|x_t\|_{V_{t}^{-1}}$$  \quad (t \rightarrow \beta_t(\delta) \text{ increasing})

$$\leq \frac{4k_\mu}{c_\mu} \beta_T(\delta) \sqrt{T} \sqrt{\mathbb{E}_{t=1}^{T} \|x_t\|^2_{V_{t}^{-1}}} \quad \text{(Cauchy-Schwarz)}$$

$$\leq \frac{4k_\mu}{c_\mu} \beta_T(\delta) \sqrt{2T \max(1, L^2/\lambda)} \sqrt{dT \log(1/\gamma) + \log \left( \frac{\det V_{T+1}}{\lambda^d} \right)} \quad \text{(Lemma 8)}$$

$$\leq \frac{4k_\mu}{c_\mu} \beta_T(\delta) \sqrt{2dT \max(1, L^2/\lambda)} \sqrt{T \log(1/\gamma) + \log \left( 1 + \frac{L^2(1 - \gamma)^T}{\lambda d(1 - \gamma)} \right)} \quad \text{(Lemma 9)}$$

The bounding of the tracking term is straight-forward:

$$R_{T}^{\text{track}} = \frac{8k_\mu^2 L^3 S}{c_\mu \lambda(1 - \gamma)} \gamma^D T + \frac{4k_\mu^2 L}{c_\mu \lambda(1 - \gamma)} \gamma^D T + \frac{\sum_{t=1}^{T} \sum_{t'-t}^{t-1} \|\theta_{t'}^s - \theta_{t'+1}^s\|_2}{2}$$

$$\leq \frac{8k_\mu^2 L^3 S}{c_\mu \lambda(1 - \gamma)} \gamma^D T + \frac{4k_\mu^2 L}{c_\mu \lambda(1 - \gamma)} DB_T.$$  

Assembling this two bounds ($R_{T}^{\text{learn}}$ and $R_{T}^{\text{track}}$) yields the first announced result, with the following constants:

$$C_1 = \sqrt{32 \max(1, L^2/\lambda)}.$$  

$$C_2 = \frac{8k_\mu^2 L^3 S}{\lambda}.$$  

$$C_3 = 4k_\mu L.$$  

The last part of the proof follows the asymptotic argument of Russac et al. (2019). We assume that $B_T$ is sub-linear and let:

$$D = \frac{\log T}{1 - \gamma}, \quad \gamma = 1 - \left( \frac{B_T}{dT} \right)^{2/3}.$$  

We therefore have the following asymptotic equivalences (omitting logarithmic dependencies):

$$\beta_T(\delta) \sqrt{dT \sqrt{T \log(1/\gamma)}} \sim dT \cdot \left( \frac{B_T}{dT} \right)^{1/3} = d^{2/3} B_T^{1/3} T^{-2/3}$$

$$\gamma^D T/(1 - \gamma) \sim \exp(- \log T) T \left( \frac{B_T}{dT} \right)^{-2/3} = d^{2/3} B_T^{-2/3} T^{2/3}$$

$$DB_T \sim B_T \left( \frac{B_T}{dT} \right)^{-2/3} = d^{2/3} B_T^{1/3} T^{2/3}$$

Merged with the regret-bound we just proved, this yields the announced result.
Without Assumption 4 similar results can be obtained. The main difference consists in using the upper-bound for the tracking term under general arm-set geometry which is slightly more complicated. Plugging the bound from Lemma 5 and upper bounding \( \sqrt{1 + \frac{L^2}{\lambda(1-\gamma)}} \) by \( 1 + \frac{L}{\sqrt{\lambda}(1-\gamma)} \) gives the announced regret decomposition. Let:

\[
\gamma = 1 - \frac{B_T^{2/5}}{d^{1/5}T^{2/5}}
\]

\[
\beta_T(\delta)\sqrt{dT\sqrt{T\log(1/\gamma)}} \sim dT \cdot \frac{B_T^{1/5}d^{-1/10}}{T^{1/5}} = d^{9/10}B_T^{1/5}T^{4/5}
\]

\[
\gamma^D T/(1 - \gamma)^{3/2} \sim \exp(-\log T)T \left( \frac{d^{1/5}T^{2/5}}{B_T^{2/5}} \right)^{3/2} = d^{3/10}B_T^{-3/5}T^{3/5}
\]

\[
\frac{\sqrt{d}}{1 - \gamma} DB_T \sim d^{1/2}B_T \left( \frac{d^{1/5}T^{2/5}}{B_T^{2/5}} \right)^2 = d^{9/10}B_T^{1/5}T^{4/5}
\]

\[
\frac{\sqrt{d}}{\gamma D T/(1 - \gamma)} \sim \exp(-\log T)T \left( \frac{d^{1/5}T^{2/5}}{B_T^{2/5}} \right)^{3/2} = d^{3/10}B_T^{-3/5}T^{3/5}
\]

\[
\frac{\sqrt{d}}{1 - \gamma} DB_T \sim d^{1/2}B_T \left( \frac{d^{1/5}T^{2/5}}{B_T^{2/5}} \right)^2 = d^{9/10}B_T^{1/5}T^{4/5}
\]

Appendix C. On the projection step

C.1. Equivalent minimization program

Recall the original minimization program for finding \( \theta^p_t \):

\[
\theta^p_t \in \arg \min_{\theta \in \mathbb{R}^d} \left\{ \left\| g_t(\theta) - g_t(\hat{\theta}_t) \right\|_{V_t^{-2}} \text{ s.t } \Theta \cap \mathcal{E}_t^\delta(\theta) \neq \emptyset \right\} . \quad (P1)
\]

Note that this minimum exists (0.d is feasible) and is indeed attained (the feasible set is compact and the objective smooth). The following reformulation is motivated by the fact that only \( \tilde{\theta}_t \in \Theta \cap \mathcal{E}_t^\delta(\theta^p_t) \) is needed for the algorithm. To this end, we explicitly introduce \( \tilde{\theta}_t \) in the program via a slack variable. Formally, we study:

\[
\left( \tilde{\theta}_t, \theta^p_t \right) \in \arg \min_{\theta \in \mathbb{R}^d, \theta \in \mathbb{R}^d} \left\{ \left\| g_t(\theta) - g_t(\hat{\theta}_t) \right\|_{V_t^{-2}} \text{ s.t } \theta' \in \mathcal{E}_t^\delta(\theta) \cap \Theta \right\} . \quad (P1')
\]

We also introduce the following program:

\[
\left( \tilde{\theta}_t, \eta \right) \in \arg \min_{\theta' \in \mathbb{R}^d, \eta \in \mathbb{R}^d} \left\{ \left\| g_t(\theta') + \beta_t(\delta)\tilde{V}_t^{1/2}\eta - g_t(\hat{\theta}_t) \right\|_{V_t^{-2}} \text{ s.t } \left\| \theta' \right\|_2 \leq S, \left\| \eta \right\|_2 \leq 1 \right\} . \quad (P2)
\]

We claim and prove the following result, which is an equivalent reformulation of Proposition 1.

**Proposition 2** The programs (P1’) and (P2) are equivalent.
Proof The proof consists in building a bijection between the solutions of $(P1')$ and $(P2)$. Let us introduce the mapping:

$$f : \Theta \times \mathbb{R}^d \rightarrow \Theta \times \mathbb{R}^d$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} f_1(x) \\ f_2(x, y) \end{pmatrix} = \left( \beta_t^{-1}(\delta) V_t^{-1/2} \left( g_t(y) - g_t(x) \right) \right)$$

We now claim the following Lemma, which proof is deferred to Section C.2.

**Lemma 1** The function:

$$g_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$$

$$\theta \rightarrow \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle \theta, x_s \rangle)x_s + \lambda c_p \theta$$

is a bijection.

A straightforward implication of this Lemma is the bijectivity of $f$. Let $(\tilde{\theta}_1^1, \theta^p)$ be a solution of $(P1')$ and let:

$$\begin{pmatrix} \tilde{\theta}_2^1 \\ \eta^p \end{pmatrix} = f \left( \begin{pmatrix} \tilde{\theta}_1^1 \\ \theta^p \end{pmatrix} \right).$$

We are going to show that $(\tilde{\theta}_2^1, \eta^p)$ is a solution of $(P2)$. Because $(\tilde{\theta}_1^1, \theta^p)$ is optimal for $(P1')$, we have that:

$$\|g_t(\theta^p) - g_t(\tilde{\theta}_1^1)\|_{V_t^{-2}} \leq \|g_t(\theta) - g_t(\tilde{\theta}_1^1)\|_{V_t^{-2}}$$

$$\forall (\theta', \theta) \in \Theta \times \mathbb{R}^d \text{ s.t } \theta' \in \mathcal{E}_t^\delta(\theta)$$

$$\iff \|g_t(\theta^p) - g_t(\tilde{\theta}_1^1)\|_{V_t^{-2}} \leq \|g_t(\theta) - g_t(\tilde{\theta}_1^1)\|_{V_t^{-2}}$$

$$\forall (\theta', \theta) \in \Theta \times \mathbb{R}^d \text{ s.t } \|g_t(\theta') - g_t(\theta)\|_{\tilde{V}_t} \leq \beta_t(\delta)$$

$$\iff \|g_t(\theta^p) - g_t(\tilde{\theta}_1^1)\|_{V_t^{-2}} \leq \|g_t(\theta) - g_t(\tilde{\theta}_1^1)\|_{V_t^{-2}}$$

$$\forall (\theta', \theta) \in \Theta \times \mathbb{R}^d \text{ s.t } \|f_2(\theta', \theta)\|_2 \leq 1$$
Noticing that for all \((x, y) \in \Theta \times \mathbb{R}^d\) we have \(g_t(y) = g_t(x) + \beta_t(\delta) V^{1/2}_t f_2(x, y)\) we therefore obtain:

\[
\|g_t(\tilde{\theta}^1) + \beta_t(\delta) \tilde{V}^{1/2}_t f_2(\tilde{\theta}^1, \theta^p) - g_t(\tilde{\theta}_t)\|_{V^{-2}} \leq \|g_t(\theta') + \beta_t(\delta) \tilde{V}^{1/2}_t f_2(\theta', \theta) - g_t(\tilde{\theta}_t)\|_{V^{-2}}
\]

\[
\quad \forall (\theta', \theta) \in \Theta \times \mathbb{R}^d \text{ s.t. } \|f_2(\theta', \theta)\|_2 \leq 1
\]

\[
\Leftrightarrow \|g_t(\tilde{\theta}^1) + \beta_t(\delta) \tilde{V}^{1/2}_t \eta^p - g_t(\tilde{\theta}_t)\|_{V^{-2}} \leq \|g_t(\theta') + \beta_t(\delta) \tilde{V}^{1/2}_t f_2(\theta', \theta) - g_t(\tilde{\theta}_t)\|_{V^{-2}}
\]

\[
\quad \forall (\theta', \theta) \in \Theta \times \mathbb{R}^d \text{ s.t. } \|f_2(\theta', \theta)\|_2 \leq 1 \quad (\tilde{\theta}^1 = \tilde{\theta}^2)
\]

\[
\Leftrightarrow \|g_t(\tilde{\theta}^2) + \beta_t(\delta) \tilde{V}^{1/2}_t \eta^p - g_t(\tilde{\theta}_t)\|_{V^{-2}} \leq \|g_t(\theta') + \beta_t(\delta) \tilde{V}^{1/2}_t f_2(\theta', \theta) - g_t(\tilde{\theta}_t)\|_{V^{-2}}
\]

\[
\quad \forall (\theta', \theta) \text{ s.t. } \|f_2(\theta', \theta)\|_2 \leq 1, \|\theta'\|_2 \leq S
\]

\[
\Leftrightarrow \|g_t(\tilde{\theta}^2) + \beta_t(\delta) \tilde{V}^{1/2}_t \eta^p - g_t(\tilde{\theta}_t)\|_{V^{-2}} \leq \|g_t(\theta') + \beta_t(\delta) \tilde{V}^{1/2}_t \eta - g_t(\tilde{\theta}_t)\|_{V^{-2}}
\]

\[
\quad \forall (\theta', \eta) \text{ s.t. } \|\eta\|_2 \leq 1, \|\theta'\|_2 \leq S
\]

where we last used the fact that \(f_2\) spans \(\mathbb{R}^d\) (surjectivity). Finally, we have that:

\[
\|\tilde{\theta}^2\|_2 \leq S \quad (\tilde{\theta}^2 = \tilde{\theta}^1 \in \Theta)
\]

\[
\|\eta^p\|_2 = \beta_t^{-1}(\delta) \left\| g_t(\theta') - g_t(\tilde{\theta}^1) \right\| \leq 1 \quad (\tilde{\theta}^1 \in E^\delta(\theta^p))
\]

Combining the last two results proves that \((\tilde{\theta}^2, \eta^p)\) is feasible for \((P2)\), and optimal within the feasible set. As a consequence, \((\tilde{\theta}^2, \eta^p)\) is a solution of \((P2)\). Therefore, \(f\) is a bijection between the minimizers of \((P1')\) and \((P2)\), which concludes the proof.

**C.2. Bijectivity of \(g_t\)**

**Lemma 2** The function:

\[
g_t : \mathbb{R}^d \rightarrow \mathbb{R}^d
\]

\[
\theta \rightarrow \sum_{s=1}^{t-1} \gamma^{t-1-s} \mu(\langle \theta, x_s \rangle) x_s + \lambda c \mu \theta
\]

is a bijection.

**Proof** **Injectivity.** Notice that \(\forall \theta \in \mathbb{R}^d:\)

\[
\nabla_{\theta} g(\theta) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \hat{\mu}(\langle \theta, x_s \rangle) x_s x_s^T + \lambda c \mu I_d > 0.
\]

Hence \(\nabla_{\theta} g\) is P.S.D, and a simple integral Taylor expansion is enough to prove injectivity.
Surjectivity Let \( z \in \mathbb{R}^d \). Let \( A = \text{Span}(x_1, \ldots, x_{t-1}) \) be the vectorial space spanned by \( \{x_s\}_{s=1}^{t-1} \). Let \( z_\perp \) be the orthogonal projection of \( z \) on \( A \) and \( z_\parallel = z - z_\perp \). Since \( z_\perp \in A \), there exists \( \{\alpha_s\}_{s=1}^{t-1} \in \mathbb{R}^{t-1} \) such that:
\[
z_\perp = \sum_{s=1}^{t-1} \alpha_s x_s.
\]
Recall that \( b(\cdot) \) is a primitive of \( \mu \), which is convex since \( \mu \) is strictly increasing. Define:
\[
L(\theta) = \sum_{s=1}^{t-1} \gamma^{t-1-s} \left[ b(\langle \theta, x_s \rangle) - \frac{\alpha_s}{\gamma^{t-1-s}} \langle \theta, x_s \rangle \right] + \frac{\lambda c_\mu}{2} \left\| \theta - \frac{z_\parallel}{\lambda c_\mu} \right\|^2.
\]
which is a strictly convex, coercive function. Its minimum \( \theta_z \) (which therefore exists and is uniquely defined) checks:
\[
\nabla_\theta L(\theta_z) = 0 \\
\Leftrightarrow \sum_{s=1}^{t-1} \gamma^{t-1-s} \left[ \mu(\langle \theta_z, x_s \rangle) - \frac{\alpha_s}{\gamma^{t-1-s}} \right] x_s + \lambda c_\mu \left( \theta_z - \frac{z_\parallel}{\lambda c_\mu} \right) = 0 \\
\Leftrightarrow g(\theta_z) = \sum_{s=1}^{t-1} \alpha_s x_s + z_\parallel \\
\Leftrightarrow g(\theta_z) = z_\perp + z_\parallel = z.
\]
which proves surjectivity.

Appendix D. Useful lemmas

The following Lemma is a version of the Elliptical Potential Lemma for weighted sums, similar to Proposition 4 of Russac et al. (2019).

**Lemma 8** Let \( \{x_s\}_{s=1}^\infty \) a sequence in \( \mathbb{R}^d \) such that \( \|x_s\|_2 \leq L \) for all \( s \in \mathbb{N}^* \), and let \( \lambda \) be a non-negative scalar. For \( t \geq 1 \) define \( V_t := \sum_{s=1}^{t-1} \gamma^{t-1-s} x_s x_s^T + \lambda I_d \). The following inequality holds:
\[
\sum_{t=1}^{T} \|x_t\|_V^2 \leq 2 \max(1, L^2/\lambda) \left( dT \log(1/\gamma) + \log \left( \frac{\det V_{T+1}}{\lambda^d} \right) \right).
\]
Proof For all $t \geq 1$, by definition:

$$V_{t+1} = \sum_{s=1}^{t} \gamma^{t-s} x_s x_s^T + \lambda I_d$$

$$= \gamma \sum_{s=1}^{t-1} \gamma^{t-1-s} x_s x_s^T + x_t x_t^T + \lambda I_d$$

$$\geq \gamma \left( \sum_{s=1}^{t-1} \gamma^{t-1-s} x_s x_s^T + x_t x_t^T + \lambda I_d \right)$$

$$(\gamma \leq 1)$$

$$\geq \gamma \left( V_t + x_t x_t^T \right)$$

$$\geq \gamma V_t \left( I_d + V_t^{-1/2} x_t x_t^T V_t^{-1/2} \right) ,$$

which after some easy manipulations yields:

$$d \log(1/\gamma) + \log \det V_{t+1} - \log \det V_t \geq \log \left( 1 + \|x_t\|_{V_t^{-1}}^2 \right).$$

After summing from $t = 1$ to $t = T$ and telescoping we obtain:

$$dT \log(1/\gamma) + \log \left( \frac{\det V_{T+1}}{\lambda^d} \right) \geq \sum_{t=1}^{T} \log \left( 1 + \|x_t\|_{V_t^{-1}}^2 \right)$$

$$\geq \sum_{t=1}^{T} \log \left( 1 + \frac{1}{\max(1, L^2/\lambda)} \|x_t\|_{V_t^{-1}}^2 \right).$$

Finally, noticing that $\frac{1}{\max(1, L^2/\lambda)} \|x_t\|_{V_t^{-1}}^2 \leq 1$ and using the fact that for all $x \in (0, 1]$ we have $\log(1 + x) \geq x/2$ we obtain:

$$dT \log(1/\gamma) + \log \left( \frac{\det V_{T+1}}{\lambda^d} \right) \geq \frac{1}{2 \max(1, L^2/\lambda)} \sum_{t=1}^{T} \|x_t\|_{V_t^{-1}}^2 ,$$

which in turn yields:

$$\sum_{t=1}^{T} \|x_t\|_{V_t^{-1}}^2 \leq 2 \max(1, L^2/\lambda) \left( dT \log(1/\gamma) + \log \left( \frac{\det V_{T+1}}{\lambda^d} \right) \right) ,$$

which is the announced result.

We also remind here the determinant-trace inequality for the weighted design matrix which can be extracted from Proposition 2 of Russac et al. (2019).

Lemma 9 Let $\{x_s\}_{s=1}^\infty$ a sequence in $\mathbb{R}^d$ such that $\|x_s\|_2 \leq L$ for all $s \in \mathbb{N}^*$, and let $\lambda$ be a non-negative scalar. For $t \geq 1$ define $V_t := \sum_{s=1}^{t-1} \gamma^{t-1-s} x_s x_s^T + \lambda I_d$. The following inequality holds:

$$\det(V_{t+1}) \leq \left( \lambda + \frac{L^2(1-\gamma^t)}{d(1-\gamma)} \right)^d .$$
Appendix E. BVD–GLM–UCB algorithm

E.1. High-level ideas

In this part of the appendix, we denote $\gamma^\star$ as follows:

$$\gamma^\star = 1 - \frac{1}{2} \left( \frac{B_{T,\star}}{dT2S} \right)^{2/3}. \quad (24)$$

**Remark 3** $\gamma^\star$ as defined in Equation (24) has a different expression than the discount factor proposed in Theorem 1. This slight modification is to ensure that $\gamma^\star$ is larger than 1/2 and simplifies the finite time analysis of the regret. Yet, it has no consequence on the asymptotic bound.

$B_{T,\star}$ being unknown, we cannot compute the optimal discount factor that depends on the parameter drift. The general idea is to use a set of different values for the discount factor (respectively the $B_{T,\star}$ values) called $H$, covering the $[1/2, 1)$ space (respectively the $[0, 2ST)$ space). Then, we divide the time horizon $T$ into different blocks of length $H$. Every $H$ steps, we create a new instance of BVD–GLM–UCB with a $\gamma$ that is chosen by a master algorithm: the EXP3 algorithm from Auer et al. (2002). At the end of each block, this master algorithm receives the cumulative rewards from the instantiated worker and updates its probability distribution over the set $H$. The objective of the master algorithm is to learn the most suitable value of $\gamma$ so as to maximise the cumulative rewards in accordance with the dynamics of the environment. On the other side, the different workers algorithms act exactly as if the BVD–GLM–UCB algorithm was launched on a $H$-steps experiment. This setting is similar to the one presented in Cheung et al. (2019a) (respectively Zhao et al. (2020)) with discount factors instead of sliding windows (respectively restart parameters). This framework is called Bandit-over-Bandit (BOB) precisely because of this two-stage structure between the master and the workers algorithms.

E.2. Algorithm

The coverage $H$ with the different discount factors is defined in the following way:

$$H = \{ \gamma_i = 1 - \mu_i | i = 1, \ldots, N \} \quad (25)$$

with $N = \left\lceil \frac{2}{3} \log_2 \left( 2ST^{3/2} \right) \right\rceil + 1$ and $\mu_i = \frac{1}{2} \frac{2^{i-1}}{d^{2/3}T(2S)^{2/3}}. \quad (26)$

The main algorithm is an instance of the EXP3 algorithm from Auer et al. (2002) where the different arms correspond to the different discount factors. Following EXP3 analysis (Auer et al., 2002), the probability of drawing $\gamma_j$ for the block $i$ is

$$p_{ij}^\gamma = (1 - \alpha) \frac{s_j^\gamma}{\sum_j s_j^\gamma} + \frac{\alpha}{N}, \quad \forall j = 1, 2, \ldots, N, \quad (27)$$

where $\alpha$ is defined as

$$\alpha = \min \left\{ 1, \sqrt[3]{\frac{N \log(N)}{(e - 1)|T/H|}} \right\} \quad (28)$$
and $s_i^{\gamma_j}$ is initialised at 1 and is updated at the end of each block when selected with

$$s_i^{\gamma_j+1} = s_i^{\gamma_j} \exp \left( \frac{\alpha \sum_{t=(i-1)H+1}^{\min\{iH,T\}} r_{t+1}}{N p_i^{\gamma_j}} \right).$$

(29)

Note that in Equation (29), $r_{t+1}$ is the noisy reward obtained when the action $x_t$ is selected with the BVD–GLM–UCB algorithm with parameter $\gamma_j$. Equation (27), (28) and (29) are the same as in Auer et al. (2002) except for the rescaling of the cumulative rewards on a block that is required to ensure that they lie in $[0, 1]$. Details on this rescaling part can be found in Proposition 4.

**Algorithm 3 BOB–BVD–GLM–UCB (detailed)**

**Input.** Length $H$, time horizon $T$, regularization $\lambda$, confidence $\delta$, inverse link function $\mu$, constants $S, L$ and $\sigma$.

**Initialization.** Create the covering space $\mathcal{H}$ as defined in Eq. (25), set $s_i^{\gamma_i} = 1, \forall \gamma_i \in \mathcal{H}$.

for $i = 1, \ldots, \lceil T/H \rceil$ do

$\gamma_j \sim p_i^{\gamma_j}$, the probability vector defined in Eq. (27).

Start a BVD–GLM–UCB subroutine with parameter $\gamma_j$

for $t = (i - 1)H + 1, \ldots, \min\{iH, T\}$ do

Receive the action set $\mathcal{X}_t$.

Select $x_t(\gamma_j) \in \mathcal{X}_t$ with BVD–GLM–UCB.

Observe reward $r_{t+1}$.

end for

Update $s_i^{\gamma_j+1}$ according to Equation (29).

Update $s_i^{\gamma_j+1} = s_i^{\gamma_j}, \forall \gamma \neq \gamma_j$.

end for

**Remark 4** We denote $x_t(\gamma)$ the action chosen with the BVD–GLM–UCB algorithm with a discount factor $\gamma$. 

**E.3. Regret guarantees**

In this section, we give an upper-bound for the expected dynamic regret of BOB–BVD–GLM–UCB.

By construction, it is natural to decompose the regret into two sources of errors. First the master error committed by the EXP3 algorithm by not choosing the best possible discount factor. Second the worker error inherent to the BVD–GLM–UCB algorithm. Note that there are two independent sources of randomness: the stochasticity of the rewards (whose expectation is denoted $E_N$) and the randomness of the EXP3 algorithm (denoted $E_{\text{EXP3}}$). Bringing things together,
\[ \mathbb{E}[R_T] = \mathbb{E}_N \left[ \sum_{t=1}^{T} \mu(\langle x_t^*, \theta_t^* \rangle) - \mathbb{E}_{\text{EXP3}}[r_{t+1}] \right] \]
\[ = \mathbb{E}_N \left[ \sum_{t=1}^{[T/H]} \mu(\langle x_t^*, \theta_t^* \rangle) - \sum_{i=1}^{\min\{iH,T\}} \sum_{t=(i-1)H+1}^{t} \mu(\langle x_t(\bar{\gamma}), \theta_t^* \rangle) \right] \]
\[ + \mathbb{E}_N \left[ \sum_{i=1}^{\min\{iH,T\}} \sum_{t=(i-1)H+1}^{t} \mu(\langle x_t(\bar{\gamma}), \theta_t^* \rangle) - \mathbb{E}_{\text{EXP3}}[r_{t+1}] \right]. \]

(30)

The next step consists in upper-bounding the worker error and the master error from Eq. (30) respectively.

**Lemma 10** With pavement \( \mathcal{H} \) defined in Equation (25) for any unknown \( B_{T,*} > 0 \), setting \( k = \lfloor \frac{2}{3} \log_2(B_{T,*} T^{1/2}) \rfloor + 1 \) yields

\[ \gamma_{k+1} \leq \gamma^* \leq \gamma_k. \]

**Proof** With assumption 1, we have \( B_{T,*} \leq 2ST \). Using this, \( k \) (as defined in the statement of the lemma) is smaller than \( N \). We have,

\[ k - 1 \leq \frac{2}{3} \log_2(B_{T,*} T^{1/2}) \leq k \]

\[ \Leftrightarrow - \frac{1}{2} \frac{2^{k-1}}{d^{2/3} T (2S)^{2/3}} \geq - \frac{1}{2} \left( \frac{B_{T,*}}{dT2S} \right)^{2/3} \geq - \frac{1}{2} \frac{2^k}{d^{2/3} T (2S)^{2/3}}. \]

Adding one for the different terms gives the result. \( \blacksquare \)

For the rest of the section, we set \( \bar{\gamma} = \gamma_k \) with \( k \) defined in Lemma 10. We denote \( B_{i,*} = \sum_{t=(i-1)H+1}^{iH-1} \| \theta_{i+1}^* - \theta_i^* \|_2 \) and

\[ \beta_{H}^* = \sqrt{\lambda S} + \sigma \sqrt{\frac{2 \log(T) + d \log \left( 1 + \frac{2L^2}{\lambda d(1 - \gamma^*)} \right)}{\log(2)}}. \]

(31)

**Proposition 3** The worker error can be upper-bounded in the following way:

\[ \text{worker} \leq 2\sigma \frac{T}{H} + C_1 R_{\mu} \beta_{H}^* \sqrt{dT} \sqrt{2T(1 - \gamma^*)} + \frac{T}{H} \log \left( 1 + \frac{2L^2}{d\lambda(1 - \gamma^*)} \right) \]

\[ + \frac{2C_2 R_{\mu} \sqrt{T}}{1 - \gamma^*} + \frac{3C_3 R_{\mu} B_{T,*} \log(T)}{\log(2)} \frac{1}{1 - \gamma^*}, \]

with \( C_1, C_2, C_3 \) constant terms from Theorem 1 and \( \beta_{H}^* \) defined in Equation (31).

**Proof** First, note that our objective here is to bound the expected regret whereas Theorem 1 bounds the pseudo-regret and gives a high probability upper-bound. We denote \( E_t^i = \{ \theta_t \in E_t^i(\bar{\theta}_t) \text{ for } t \text{ s.t. } (i -
1) \( H + 1 \leq t \leq \min\{iH, T\} \). This event holds with probability higher than \( 1 - \delta \). When \( E_\delta^i \) does not hold, the maximum regret could theoretically be suffered for all time instants.

As explained in the algorithm mechanism, a new instance of BVD–GLM–UCB will be launched every \( H \) steps with a discount factor selected by the EXP3 algorithm. Restarting a new algorithm and forgetting previous information comes at a cost in terms of regret. This is made explicit in the following decomposition of \( \text{worker} \).

\[
\begin{align*}
\text{worker} &= \mathbb{E}_N \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH, T\}} \mu(x_t^i, \theta_*^i) - \mu(x_t(\gamma), \theta_*^i) \right] \\
&= \mathbb{E}_N \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH, T\}} \mu(x_t^i, \theta_*^i) - \mu(x_t(\gamma), \theta_*^i) \right] \mathbb{P} \left( \bigcap_{i=1}^{[T/H]} E_\delta^i \right) \\
&\quad + \mathbb{E}_N \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH, T\}} \mu(x_t^i, \theta_*^i) - \mu(x_t(\gamma), \theta_*^i) \right] \mathbb{P} \left( \bigcup_{i=1}^{[T/H]} (E_\delta^i)^c \right)
\end{align*}
\]

Thanks to Lemma 1, \( E_\delta^i \) holds with probability higher than \( 1-\delta \). By setting \( \delta = 1/T \), we have

\[
\mathbb{P} \left( \bigcup_{i=1}^{[T/H]} (E_\delta^i)^c \right) \leq \frac{[T/H]}{1/T}.
\] (32)

Under the event \( \bigcup_{i=1}^{[T/H]} (E_\delta^i)^c \) not much can be said. The maximum regret \( r_{\text{max}} = 2\sigma \) can be suffered at every time step. Therefore, using the upper-bound from Eq. (32), we obtain

\[
\begin{align*}
\text{worker}_2 &= \mathbb{E}_N \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH, T\}} \mu(x_t^i, \theta_*^i) - \mu(x_t(\gamma), \theta_*^i) \right] \mathbb{P} \left( \bigcup_{i=1}^{[T/H]} (E_\delta^i)^c \right) \\
&\leq r_{\text{max}} \frac{[T/H]}{}.
\end{align*}
\]

This term is related to the number of restarts of the algorithm. In the BOB framework, whatever the worker algorithm (sliding window, restart factor) a cost of order \( T/H \) will be paid due to the restarting of the \( \text{worker} \) at the beginning of each block.

On the contrary, under the event \( \bigcap_{i=1}^{[T/H]} E_\delta^i \), using the assumption that the blocks are independent, we can follow the line of proof from Lemma 3 and Theorem 1 for every block. We introduce

\[
\beta_H = \sqrt{\lambda S} + \sigma \sqrt{2 \log(T) + d \log \left( 1 + \frac{L^2(1 - \gamma^2_H)}{\lambda d(1 - \gamma^2_k)} \right)}.
\] (33)
worker_1 = E_N \left[ \sum_{i=1}^{[T/H]} \min_{t=(i-1)H+1}^{t=iH+1} \mu_i (x_i, \theta_i^*) - \mu_i (x_i \hat{\gamma}, \theta_i^*) \right] \left\{ \bigcap_{i=1}^{[T/H]} E_i^\delta \right\} \\
\leq E_N \left[ \sum_{i=1}^{[T/H]} \min_{t=(i-1)H+1}^{t=iH+1} \mu_i (x_i, \theta_i^*) - \mu_i (x_i \hat{\gamma}, \theta_i^*) \right] \left\{ \bigcap_{i=1}^{[T/H]} E_i^\delta \right\} \\
\leq \sum_{i=1}^{[T/H]} \left( C_1 \beta_H \sqrt{dH} \sqrt{H \log(1/\hat{\gamma}) + \log \left( 1 + \frac{L^2}{d\lambda(1-\hat{\gamma})} \right)} + C_2 \frac{\gamma^D}{1-\gamma} H + C_3 B_t^* D \right) \\
\leq C_1 \beta_H \sqrt{dT} \left( T \log(1/\hat{\gamma}) + \frac{T}{H} \log \left( 1 + \frac{L^2}{d\lambda(1-\hat{\gamma})} \right) + C_2 \frac{\gamma^D}{1-\gamma} T + C_3 B_t^* D \right),

where the second inequality is a consequence of Theorem 1. We set,

\[ D = \frac{3/2 \log(T)}{\log(1/\hat{\gamma})}. \] (34)

Hence,

\[ C_3 B_t^* D \leq \frac{3 C_3 B_t^* \log(T)}{2 \log(1/\hat{\gamma})} \]

\[ \leq \frac{3 C_3}{2 \log(2)} B_t^* \log(T) \frac{\hat{\gamma}}{1-\hat{\gamma}} \] (Using \( \log(x) \geq \log(2)(x-1) \) for \( x \in [1, 2] \))

\[ \leq \frac{3 C_3}{2 \log(2)} \frac{B_t^* \log(T)}{1-\gamma_k} \] (\( \hat{\gamma} \leq 1 \))

\[ \leq \frac{3 C_3}{2 \log(2)} \frac{B_t^* \log(T)}{1-\gamma_k+1} \] (Definition of \( H \))

\[ \leq \frac{3 C_3}{2 \log(2)} \frac{B_t^* \log(T)}{1-\gamma^*} \] (Lemma 10).

We also have,

\[ C_2 \frac{\gamma^D}{1-\gamma} T \leq C_2 \frac{1}{\sqrt{T}} \frac{1}{1-\gamma} \] (Equation (34))

\[ \leq 2C_2 \frac{1}{\sqrt{T}} \frac{2}{1-\gamma_k+1} \] (Definition of \( H \))

\[ \leq 2C_2 \frac{1}{\sqrt{T}} \frac{1}{1-\gamma^*} \] (Lemma 10).

Finally, using \( x \mapsto \log(x) \leq x - 1 \) for \( x > 1 \) and Lemma 10, one has:

\[ T \log(1/\hat{\gamma}) + \frac{T}{H} \log \left( 1 + \frac{L^2}{d\lambda(1-\hat{\gamma})} \right) \leq \frac{T}{H} \frac{1-\gamma}{\gamma} + \frac{T}{H} \log \left( 1 + \frac{2L^2}{d\lambda(1-\gamma^*)} \right) \]

\[ \leq 2T(1-\gamma^*) + \frac{T}{H} \log \left( 1 + \frac{2L^2}{d\lambda(1-\gamma^*)} \right). \]
Following similar steps, we can upper-bound $\beta_H$ from Equation (33) by

$$\beta_H \leq \beta_H^*.$$  

Bringing things together, we have shown that under the event $\{\gamma_i^{[T/H]} \in \mathcal{E}_i\}$ all the terms depending on $\hat{\gamma}$ can be replaced by terms depending only on $\gamma^*$ at the cost of multiplicative constant independent of $T$. Finally, one has

$$\text{worker} \leq 2\sigma \frac{T}{H} + C_1 R_\mu \beta^*_H \sqrt{dT} \sqrt{2T(1 - \gamma^*) + \frac{T}{H} \log \left( 1 + \frac{2L^2}{d\lambda(1 - \gamma^*)} \right)}$$

$$+ \ 2C_2 R_\mu \frac{1}{\sqrt{T}} \frac{1}{1 - \gamma^*} + \frac{3C_3 R_\mu B_{T_\star} \log(T)}{\log(2)} \frac{1}{1 - \gamma^*}.$$  

The above proposition bounds the regret incurred if the same discount factor $\hat{\gamma}$ is used for each block. To successfully upper bound $\text{BVD-GLM-UCB}$'s regret, we need to upper bound the second part $\text{master}$ which is the error due to the use of the EXP3 algorithm. This part can be controlled thanks to the analysis proposed in Auer et al. (2002). Yet, two issues need to be overcome. (1) The rewards received at the end of a block does not lie in $[0, 1]$ which is required to use the result from Auer et al. (2002). (2) We are in a stochastic environment with noisy rewards.

In the next proposition, we upper-bound the term of interest and explain how to deal with the two issues. The big picture is the following: using the assumption on the bounded rewards we can obtain an upper-bound for the maximum reward on a single block.

**Proposition 4** The regret due to the master algorithm can be bounded in the following way,

$$\mathbb{E}_N \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH,T\}} \mu(\langle x_t(\hat{\gamma}), \theta^*_t \rangle) - \mathbb{E}_{\text{EXP3}} \left[ r_{t+1} \right] \right] \leq 4\sigma H \sqrt{e - 1} \sqrt{\frac{T}{H} \text{card}(\mathcal{H}) \log(\text{card}(\mathcal{H}))}$$

**Proof** We denote $\gamma_i$ the discount factor chosen by the EXP3 algorithm in the $i$-th block. The regret due to the use of the EXP3 $\text{main}$ algorithm can be written as follows:

$$\text{master} = \mathbb{E}_N \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH,T\}} \mu(\langle x_t(\hat{\gamma}), \theta^*_t \rangle) - \mathbb{E}_{\text{EXP3}} \left[ \sum_{i=1}^{[T/H]} \sum_{t=(i-1)H+1}^{\min\{iH,T\}} r_{t+1} \right] \right].$$

We introduce $Q_i(\gamma_j) = \min_{t=(i-1)H+1}^{\min\{iH,T\}} r_{t+1}(\gamma_j) = \min_{t=(i-1)H+1}^{\min\{iH,T\}} \mu(\langle x_t(\gamma_j), \theta^*_t \rangle) + \epsilon_{t+1}$, using Equation (9). This quantity corresponds to the reward obtained on the $i$-th block when using $\text{BVD-GLM-UCB}$ with the discount factor $\gamma_j$. We also use $Q_i = \max_{\gamma \in \mathcal{H}} Q_i(\gamma)$.

Contrarily to existing works in the linear setting (e.g (Cheung et al., 2019b, Lemma3)) our assumption on the bounded rewards is sufficient to solve both problems. We have, $|Q_t| \leq 2\sigma H$ almost surely using $r_t \leq 2\sigma$ for all time instants.

Let $\mathcal{U} = \{\forall t \leq T, 0 \leq r_t \leq 2\sigma\}$. Thanks to Assumption 2, we have $\mathbb{P}(\mathcal{U}) = 1$.  

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One has,

\[
\text{master} \leq \mathbb{E}_N \left[ \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma_k) - \max_{\gamma \in \mathcal{H}} \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma) + \max_{\gamma \in \mathcal{H}} \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma) - \mathbb{E}_{\text{EXP3}} \left[ \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma_i) \right] \right]
\]

\[
\leq \mathbb{E}_N \left[ \max_{\gamma \in \mathcal{H}} \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma) - \mathbb{E}_{\text{EXP3}} \left[ \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma_i) \right] \right]
\]

\[
\leq \mathbb{E}_N \left[ \max_{\gamma \in \mathcal{H}} \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma) - \mathbb{E}_{\text{EXP3}} \left[ \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Q_i(\gamma_i) \right] \right] \mathbb{P}(\mathcal{U}).
\]

We introduce

\[
Y_i(\gamma_j) = \frac{Q_i(\gamma_j)}{2\sigma H}.
\]

For all \( \gamma \in \mathcal{H} \), \( Y_i(\gamma) \) lies in \([0, 1]\). Therefore,

\[
\text{master} \leq 2\sigma H \mathbb{E}_N \left[ \max_{\gamma \in \mathcal{H}} \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Y_i(\gamma) - \mathbb{E}_{\text{EXP3}} \left[ \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Y_i(\gamma_i) \right] \right] \mathbb{P}(\mathcal{U}).
\]

The last step consists in using \((\text{Auer et al., 2002, Corollary 3.2})\). We have,

\[
\max_{\gamma \in \mathcal{H}} \sum_{i=1}^{\left\lfloor T/H \right\rfloor} Y_i(\gamma) \leq T / H.
\]

All the conditions of Corollary 3.2 in \(\text{Auer et al. (2002)}\) are met and we obtain:

\[
\text{master} \leq 4\sigma H \sqrt{e - 1} \sqrt{\frac{T}{H} \text{card}(\mathcal{H}) \log(\text{card}(\mathcal{H}))}.
\]

The two parts of regret in Equation (30) are bounded in Proposition 3 and Proposition 4 respectively. Combining them, we get our main result below:

**Theorem 2** Under Assumptions 1-2 and 4, the regret of \(\text{BOB-BVD-GLM-UCB}\) when setting \(H = \lfloor d\sqrt{T} \rfloor\) satisfies:

\[
\mathbb{E}[R_T] = \tilde{O}\left( R_{\mu} d^{2/3} T^{2/3} \max\left( B_{T,*}, d^{-1/2} T^{1/4} \right)^{1/3} \right).
\]

**Remark 5** This theorem establishes an upper-bound for the expected regret in the Generalized Linear Bandits framework when the variational budget is unknown. When \(B_{T,*} \geq d^{-1/2} T^{1/4}\) the obtained bound can not be improved. Yet, there is still a gap with the lower bound when the variation budget is small. This can be explained by the frequent restarts in the \(\text{BOB}\) framework.
**Proof** Using Proposition 4 and Proposition 3, we obtain:

\[
\mathbb{E}[R_T] \leq 2\sigma \frac{T}{H} + C_1 R_\mu \beta^*_H \sqrt{T} \left[ 2T(1 - \gamma^*) + \frac{T}{H} \log \left( 1 + \frac{2L^2}{d\lambda(1 - \gamma^*)} \right) \right] \\
+ C_2 R_\mu \frac{2}{\sqrt{T}} \frac{1}{1 - \gamma^*} + \frac{3C_3 R_\mu}{\log(2)} \frac{B_{T,*} \log(T)}{1 - \gamma^*} + 4\sigma H \sqrt{e - 1} \sqrt{\frac{T}{H} \text{card}(\mathcal{H}) \log(\text{card}(\mathcal{H}))}
\]

First note that \(\text{card}(\mathcal{H}) = N\) defined in Equation (26) scales as \(\log(T)\) and \(\beta^*_H\) scales as \(\sqrt{d \log(T)}\). By plugging \(H = [d \sqrt{T}]\) in the upper-bound we obtain:

\[
\frac{T}{H} = O(d^{-1/2} \sqrt{T}).
\]

\[
\beta^*_H \sqrt{dT} \left[ 2T(1 - \gamma^*) + \frac{T}{H} \log \left( 1 + \frac{2L^2}{d\lambda(1 - \gamma^*)} \right) \right] = \tilde{O} \left( d\sqrt{T} \max \left( \frac{TB_{T,*}^{2/3}}{d^{2/3} T^{2/3}}, \frac{T}{d\sqrt{T}} \right) \right) = d^{2/3} T^{2/3} \max(B_{T,*}^{1/3}, d^{-1/6} T^{1/4}) = d^{2/3} T^{2/3} (\max(B_{T,*}, d^{-1/2} T^{1/4}))^{1/3}.
\]

\[
\frac{1}{\sqrt{T}} \frac{1}{1 - \gamma^*} = O \left( \frac{T^{1/6}}{d^{2/3} B_{T,*}^{2/3}} \right).
\]

\[
\frac{B_{T,*}}{1 - \gamma^*} = O \left( d^{2/3} B_{T,*}^{1/3} T^{2/3} \right).
\]

\[
H \sqrt{\frac{T}{H} \text{card}(\mathcal{H}) \log(\text{card}(\mathcal{H}))} = \tilde{O} \left( d^{1/2} T^{3/4} \right).
\]

To conclude we notice that when \(B_{T,*} \leq d^{-1/2} T^{1/4}\),

\[
d^{1/2} T^{3/4} = d^{2/3} T^{2/3} (\max(B_{T,*}, d^{-1/2} T^{1/4}))^{1/3}.
\]

On the contrary, when \(B_{T,*} \geq d^{-1/2} T^{1/4}\),

\[
d^{1/2} T^{3/4} \leq d^{2/3} T^{2/3} (\max(B_{T,*}, d^{-1/2} T^{1/4}))^{1/3}.
\]

Finally, keeping the highest order term yields the announced result. \(\blacksquare\)
Appendix F. Experimental set-up

This section is dedicated to providing useful details about the illustrative experiments presented in Section 6. The logistic setting at hand is characterized by the constants $S = L = 1$. At each round, the environment randomly draws 10 news arms, presented to the agent. All algorithms use the same $\ell_2$ regularization parameter $\lambda = 1$. The sequence $\theta^*_t$ evolves as follows: we let $\theta^*_t = (0, 1)$ for $t \in [1, T/3]$. Between $t = T/3$ and $t = 2T/3$ we smoothly rotate $\theta^*_t$ from $(0, 1)$ to $(1, 0)$. Finally we let $\theta^*_t = (0, 1)$ for $t \in [2T/3, T]$. Easy computations show that the total variation budget is

$$B_T = (2T/3) \sin \left( \frac{3\pi}{4T} \right) \simeq 1.5.$$

We used the optimal value of $\gamma$ recommended by the asymptotic analysis for D-LinUCB and BVD-GLM-UCB. We solve the projection step of GLM-UCB and BVD-GLM-UCB by (constrained) gradient-based methods, thanks to the SLSQP solver of scipy.

Remark In our experiments, we did not report performances of the algorithms from Russac et al. (2020, 2021) (which use a similar projection step as in Filippi et al. (2010)). Because such algorithms are based on discrete switches of the reward signal, their behavior in this slowly-varying environment is largely sub-optimal. Indeed, in our experiment the number of abrupt-changes is $\Gamma_T = 1000$. For exponentially weighted algorithms, the recommended asymptotic value for the weights becomes $\gamma \simeq 0.70$, which in turns leads to algorithms that over-estimate the non-stationary nature of the problem, and perform poorly in practice.