The generalizations of fuzzy monoids and vague monoids

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Abstract
Since the unit interval with a $t$-norm or a $t$-conorm is a special monoid, a natural question is to consider the fuzzification of monoid. In this paper, we mainly present the fuzzy monoid and vague monoid with the help of aggregation functions. Firstly, the fuzzy submonoid about aggregation function is proposed, which can further be fuzzified to the fuzzy $t$-subnorm and $t$-subconorm to deal with imprecision. In particular, the corresponding concepts and properties can be obtained directly when aggregation function takes uninorm and nullnorm, respectively. Further, the concept of fuzzy submonoid is extended to lattice structure. Finally, the notion of vague monoid about aggregation operator is constructed and further the special cases under uninorms and nullnorms are considered.

Keywords Aggregation function · $t$-norm · Uninorm · Nullnorm · Fuzzy monoid · Vague monoid

1 Introduction

1.1 Brief review of aggregation function
Aggregation function (Grabisch et al. 2009, 2011; Mesiar et al. 2008) can be seen as a mathematical model that combines information from different sources into one representative value, which plays imperative applications in many fields such as mathematics (Beliakov 2009; Bustince et al. 2009; Komornikova and Mesiar 2011; Grabisch et al. 2011), computer and engineering sciences (Scott and Antonsson 1998; Paternain et al. 2015). In order to fit the needs of different fields, there is a rich variety of aggregation functions, which are mainly divided into four categories: conjunctive types (e.g., triangular norm (short for $t$-norm), semi-triangular norm (short for semi-$t$-norm), and Copulas) (Klement et al. 2013; Nelsen 2007; Klement et al. 1996), disjunctive types (e.g., triangular conorm (short for $t$-conorm), semi-triangular conorm (short for semi-$t$-conorm)) (Klement et al. 2013, 1996), homogeneous types (e.g., OWA operators, integral operators operator) (Yager 1988) and mixed types (uninorm, nullnorm) (Yager and Rybalov 1996; Calvo et al. 2001). In terms of application, it is a challenge for many scholars to choose the appropriate aggregation function to solve the relevant application problems. In the theoretical aspect, the properties of aggregation functions and their interrelationships are complex and interesting.

As a generalization of the classical trigonometric inequalities in statistical measure spaces, $t$-norm and $t$-conorm were introduced by Menger (2003), and their definitions were given by Schweizer and Sklar in Schweizer and Sklar (1960), i.e., $t$-norm (resp. $t$-conorm) is a binary function on $[0, 1]^2$ satisfying commutativity, associativity, monotonic increasing property and the identity element 1 (resp. 0). However, $t$-norm and $t$-conorm take 1 and 0 as the identity element, respectively, which are too strict to aggregate general information in practical applications.

Therefore, the concept of uninorm was proposed by Yager and Rybalov (1996) as a generalized form of $t$-norm and $t$-conorm, which has a wide range of application prospects (Fodor et al. 1997; De Baets and Fodor 1999; Yager 2001; Moodley et al. 2019; Su et al. 2015), such as fuzzy logic, fuzzy decision making, image processing and expert system. A uninorm is a binary function on $[0, 1]^2$ that satisfies commutativity, associativity, monotonic increasing property and the identity element $e \in [0, 1]$. In particular, uninorm degenerates to a $t$-norm when $e = 1$ and it degenerates to a $t$-conorm when $e = 0$. In addition to generalizing the concept of $t$-norm and $t$-conorm from the perspective of identity element, Frank (1979) proposed Frank’s equation:

$$T(x, y) + S(x, y) = x + y,$$

for any $x, y \in [0, 1]$. 

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Further, Calvo et al. (2001) replaced one of the $t$-norm and $t$-conorm in the Frank’s equation with a uninorm, and the concept of nullnorm was introduced in the solution process, which can be considered as a binary function on $[0, 1]^2$ with commutativity, associativity, monotonic increasing property and has absorbing elements $k \in [0, 1]$.

### 1.2 A review of fuzzy monoid and vague monoid

The fuzzification or relaxation of logical connectives can be effective in solving the practical problems with imprecision, lack of accuracy or noise. Further, the fuzzy algebraic structures, such as groups, rings, actions (Boixader and Recasens 2018; Rosenfeld 1971; Addis et al. 2018; Mordeson et al. 2012; Oh 2019; Demirci and Recasens 2004) have been explored by many scholars. In addition, the extension of above fuzzy algebraic structures to lattice structures has become a hot topic of research, for example, Kim (1996) introduced the concepts of fuzzy subgroupoid, fuzzy sub-monoid and fuzzy subgroups on lattice structure, and Gerla proposed $L$-fuzzy sub-monoid and $L$-preorders on a complete residuated lattice. In particular, Boixader and Recasens (2022) explored the fuzzification of $t$-norm and $t$-conorm from two perspectives: one is fuzzifying the concept of sub-monoid and another is constructing the vague $t$-norm and $t$-conorm. More specifically, the concept of sub-monoid is fuzzified with the help of $t$-norm and $t$-conorm, and then the $T$-fuzzy $t$-subnorm and $T$-fuzzy $t$-subconorm are obtained. On the other end of the spectrum, the vague monoids are explored by the concept of $T$-indistinguishability operator.

### 1.3 The motivation of this paper

The above conclusions are put forward on the basis of $t$-norm and $t$-conorm. Due to the stronger restrictive conditions of $t$-norm and $t$-conorm, it is not conducive to the generalizing of fuzzy monoid and vague monoid. Therefore, we mainly extend $T$-fuzzy monoid and $T$-vague monoid to $A$-fuzzy monoid and $A$-vague monoid based on the idea of general aggregation function.

Similar to the method of fuzzy monoid and vague monoid in Boixader and Recasens (2022), we will study the fuzzification of sub-monoid by generalizing the $t$-norm $T$ to aggregation function $A$, named $A$-fuzzy sub-monoid of set $M$. In particular, when $M$ takes $t$-norm and $t$-conorm, we get $A$-fuzzy sub-monoid of $([0, 1], T)$ and $A$-fuzzy sub-monoid of $([0, 1], S)$, respectively. In addition, the corresponding conclusions are given when aggregation function $A$ takes uninorm $U$ and nullnorm $F$. Further, the concepts of $A$-vague binary operation and $A$-vague monoid are introduced and the homomorphich relationship between $A$-vague monoids and $A$-fuzzy monoids is constructed. Finally, the above conclusions are extended to the special case of uninorm and nullnorm.

As a class of extension studies, our paper further generalizes the study of fuzzy monoid and vague monoid to enrich the relevant theoretical knowledge.

The rest of this paper is arranged as follows. In Sect. 2, we review some basic concepts that are necessary to understand this paper. Section 3 introduces the definition of fuzzy sub-monoid which generated by aggregation functions and their corresponding conclusions. Further, the above concepts and properties are extended to the lattice structure. In Sect. 4, the vague binary operators and vague monoids are introduced, meanwhile, the concept of homomorphic mapping between vague monoids and fuzzy monoids are shown. Next, we obtain the corresponding conclusions when aggregation function $A$ takes the uninorm $U$ and nullnorm $F$. Finally, conclusions on our research are given in Sect. 5.

### 2 Preliminaries

In this section, we briefly recall some fundamental concepts about aggregation function such as $t$-norm, $t$-conorm, uni-norm and nullnorm, which shall be needed in the sequel.

**Definition 1** (Beliakov et al. 2007; Gómez and Montero 2004; Komorníková 2001; Mayor and Trillas 1986) A function $H : [0, 1]^n \rightarrow [0, 1]$ is a $n$-ary aggregation function, which satisfies the following conditions:

1. $A(0, 0, \ldots, 0) = 0$ and $A(1, 1, \ldots, 1) = 1$;
2. If $x_i \leq y$ for all $i \in \{1, 2, \ldots, n\}$, then

   
   $A(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\
   \leq A(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n).$

For more details about aggregation functions, please refer to (Beliakov et al. 2007; Gómez and Montero 2004; Komorníková 2001; Mayor and Trillas 1986).

**Definition 2** (Klement et al. 2013) A binary function $T : [0, 1]^2 \rightarrow [0, 1]$ (resp. $S : [0, 1]^2 \rightarrow [0, 1]$) is called a $t$-norm (resp. $t$-conorm), if it verifies the property of commutative, associative and non-decreasing in each variable with an identity element $1$ (resp. $0$).

**Example 1** The common $t$-norms and $t$-conorms are shown in Table 1.

**Definition 3** (Klement et al. 2013) An element $x \in [0, 1]$ is called an idempotent element for a $t$-norm $T$ (resp. $t$-conorm $S$) if and only if $T(x, x) = x$ (resp. $S(x, x) = x$), $0$ and $1$ which are idempotent elements for $T$ and $S$, respectively, called trivial idempotent elements of $T$ and $S$, the rest are non-trivial idempotent element of $T$ and $S$. The set of idempotent elements of $T$ (resp. $S$) is noted as $Ide(T)$ (resp. $Ide(S)$).
Remark 1 A continuous $t$-norm $T$ (resp. $t$-conorm $S$) is Archimedean if and only if $\text{Ide}(T) = \{0, 1\}$ (resp. $\text{Ide}(S) = \{0, 1\}$).

Example 2 Notice that $T_L$, $T_P$ are Archimedean $t$-norms, and $T_M$ is not.

The concepts of additive generators about $t$-norm and $t$-conorm are given as follows.

**Proposition 1** (Klement et al. 2013; Ouyang et al. 2008) A continuous $t$-norm $T$ is Archimedean if and only if there exists a continuous and strictly decreasing function $f : [0, 1] \rightarrow [0, \infty)$ with $f(1) = 0$ such that

$$T(x, y) = f^{-1}(f(x) + f(y)),$$

where $f^{-1}$ is the pseudo inverse of $f$, denoted by

$$f^{-1}(x) = \begin{cases} f^{-1}(x), & \text{if } x \in [0, f(0)], \\ 0, & \text{otherwise.} \end{cases}$$

$f$ is an additive generator of $T$ and two additive generators of the same $t$-norm differ only by a positive constant multiple.

**Proposition 2** (Klement et al. 2013) A continuous $t$-conorm $S$ is Archimedean if and only if there exists a continuous and strictly decreasing function $g : [0, 1] \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$S(x, y) = g^{-1}(g(x) + g(y)),$$

where $g^{-1}$ is the pseudo inverse of $g$, denoted by

$$g^{-1}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0, g(1)], \\ 1, & \text{otherwise.} \end{cases}$$

g is an additive generator of $S$ and two additive generators of the same $t$-conorm differ only by a positive constant multiple.

Next, the notions of uninorm and nullnorm are given.

**Definition 4** (Yager and Rybalov 1996; De Baets and Fodor 1999) A uninorm is a binary function $U : [0, 1]^2 \rightarrow [0, 1]$, for all $x, y, z \in [0, 1]$, which satisfies the following conditions:

1. $(U1)$ $U(x, y) = U(y, x)$;
2. $(U2)$ $U(U(x, y), z) = U(x, U(y, z))$;
3. $(U3)$ $U$ is non-decreasing in each argument;
4. $(U4)$ $U(x, e) = x, e \in [0, 1]$.

Remark 2 In particular, a uninorm is a $t$-norm when $e = 1$ and a $t$-conorm when $e = 0$. For any $e \in (0, 1)$, uninorm is equivalent to a $t$-norm in $[0, e]^2$, a $t$-conorm in $[e, 1]^2$, and its values in $A(e) = [0, e] \times (e, 1] \cup (e, 1] \times [0, e)$ have the following form:

$$\min(x, y) \leq U(x, y) \leq \max(x, y).$$

Remark 3 A uninorm $U$ is called conjunctive if $U(1, 0) = 0$ and disjunctive if $U(1, 0) = 1$. A conjunctive (resp. disjunctive) uninorm $U$ is said to be locally internal on the boundary if it satisfies $U(1, x) \in \{1, x\}$ (resp. $U(0, x) \in \{0, x\}$) for all $x \in [0, 1]$. In addition, several common classes of uninorms are shown below.

- $\mathcal{U}_{\min}$, $\mathcal{U}_{\max}$: those given by minimum and maximum in $A(e)$, respectively.
- $\mathcal{U}_{\text{ide}}$: $U(x, x) = x$ for all $x \in [0, 1]$.
- $\mathcal{U}_{\text{rep}}$: those that have an additive generator.
- $\mathcal{U}_{\cos}$: continuous in the open square $(0, 1)^2$.

**Theorem 3** (Fodor et al. 1997) Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm with identity element $e \in (0, 1)$. Then the sections $x \mapsto U(x, 1)$ and $x \mapsto U(x, 0)$ are continuous in each point except perhaps for $e$ if and only if $U$ is given by one of the following formulas.
(1) If \( U(0, 1) = 0 \), then

\[
U(x, y) = \begin{cases} 
  eT \left( \frac{x}{e}, \frac{y}{e} \right), & \text{if } (x, y) \in [0, e], \\
  e + (1 - e)S \left( \frac{x - e}{1 - e}, \frac{y - e}{1 - e} \right), & \text{if } (x, y) \in [e, 1]^2, \\
  \min(x, y), & \text{if } (x, y) \in A(e).
\end{cases}
\]

The set of uninorms as above will be denoted by \( \mathcal{U}_{\min} \).

(2) If \( U(0, 1) = 1 \), then \( \mathcal{U}_{\max} \) has the same structure by changing minimum by maximum in \( A(e) \).

**Theorem 4** (Ruiz-Aguilera et al. 2010; De Baets 1999; Çayli and Drygaś 2018) *U* is an idempotent uninorm with identity element \( e \in [0, 1] \) if and only if there exists a non-increasing function \( g : [0, 1] \rightarrow [0, 1] \), symmetric with respect to the main diagonal, with \( g(e) = e \), such that

\[
U(x, y) = \begin{cases} 
  \min(x, y), & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g(g(x))), \\
  \max(x, y), & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g(g(x))), \\
  \min(x, y) \text{ or } \max(x, y), & \text{if } y = g(x) \text{ and } x = g(g(x)).
\end{cases}
\]

being commutative in the points \( (x, y) \) such that \( y = g(x) \) with \( x = g(g(x)) \).

**Theorem 5** (Fodor et al. 1997; He and Wang 2021; Petrík and Mesiar 2014) A binary operation \( U : [0, 1]^2 \rightarrow [0, 1] \) is a representable uninorm if and only if there exists a continuous strictly increasing function \( h : [0, 1] \rightarrow [-\infty, +\infty] \) with \( h(0) = -\infty, h(e) = 0 (e \in (0, 1)) \) and \( h(1) = +\infty \) such that

\[
U(x, y) = h^{-1}(h(x) + h(y)),
\]

for all \( (x, y) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\} \) and \( U(0, 1) = U(1, 0) \in [0, 1] \). The function \( h \) is usually called an additive generator of \( U \).

**Theorem 6** (Hu and Li 2001; Ruiz and Torrens 2006; Li and Liu 2021) Let \( U \) be a uninorm continuous in \((0, 1)^2 \) with identity element \( e \in (0, 1) \). Then either one of the following cases is satisfied.

(a) There exist \( u \in [0, e], \lambda \in [0, u], \) two continuous \( t \)-norms \( T_1 \) and \( T_2 \) and a representable uninorm \( R \) such that \( U \) can be represented as

\[
U(x, y) = \begin{cases} 
  \lambda T_1 \left( \frac{x}{\lambda}, \frac{y}{\lambda} \right), & \text{if } x, y \in [0, \lambda], \\
  \lambda + (u - \lambda)T_2 \left( \frac{x - \lambda}{u - \lambda}, \frac{y - \lambda}{u - \lambda} \right), & \text{if } x, y \in [\lambda, u], \\
  u + (1 - u)R \left( \frac{x - u}{1 - u}, \frac{y - u}{1 - u} \right), & \text{if } x, y \in [u, 1], \\
  1, & \text{if } \min(x, y) \in (\lambda, 1) \text{ and } \max(x, y) = 1, \\
  \lambda \text{ or } 1, & \text{if } (x, y) \in \{ (\lambda, 1), (1, \lambda) \}, \\
  \min(x, y), & \text{otherwise}.
\end{cases}
\]

(b) There exist \( v \in (e, 1], \omega \in [v, 1], \) two continuous \( t \)-conorms \( S_1 \) and \( S_2 \) and a representable uninorm \( R \) such that \( U \) can be represented as
Further, \( \mathcal{U}_{\cos} \) is used to denote the class of all uninorms continuous in \((0, 1)^2\). A uninorm as in (1) will be defined by \( U \equiv (T, \lambda, T_2, u, (R, e))_{\cos, \min} \), \( \mathcal{U}_{\cos, \min} \) for short. Similarly, a uninorm as in (2) will be defined by \( U \equiv ((R, e), v, S_1, \omega, S_2)_{\cos, \max} \), \( \mathcal{U}_{\cos, \max} \) for short.

Definition 5 (Calvo et al. 2001; Mas et al. 1999) A nullnorm is a binary function \( F : [0, 1]^2 \rightarrow [0, 1] \), for all \( x, y, z \in [0, 1] \), which satisfies the following conditions:

\begin{enumerate}
  \item[(F1)] \( F(x, y) = F(y, x) \);
  \item[(F2)] \( F(F(x, y), z) = F(x, F(y, z)) \);
  \item[(F3)] \( F \) is non-decreasing in each place;
  \item[(F4)] there exists an absorbing element \( k \in [0, 1] \), \( F(k, x) = k \) and the following statements hold.
    \begin{enumerate}
    \item[(i)] \( F(0, x) = x \) for all \( x \leq k \);
    \item[(ii)] \( F(1, x) = x \) for all \( x \geq k \).
    \end{enumerate}
\end{enumerate}

In general, \( k \) is always given by \( F(0, 1) \).

In the following, some concepts related to fuzzy subset and monoid are introduced. Denote \( X \) as the universe, then a fuzzy subset \( A \) of \( X \) can be represented as \( \mu_A : X \rightarrow [0, 1] \), where for any \( x \in X \), \( \mu_A(x) \) is denoted as the membership function of \( x \) in fuzzy set \( A \). The family of all fuzzy sets of \( X \) will be defined as \( \mathcal{F}(X) \).

Definition 6 (Boixader and Recasens 2022) A monoid \( (H, \ast) \) consists of a set \( H \) with a binary operation \( \ast : H^2 \rightarrow H \), which has identity element and associativity.

Definition 7 (Boixader and Recasens 2022; Rosenfeld 1971) Let \( (H, \ast) \) be a monoid with identity element \( e \), \( T \) a \( T \)-norm and \( \sigma \) a fuzzy subset of \( H \). Then \( \sigma \) is a \( T \)-fuzzy submonoid of \( H \) which is equivalent to the following conditions.

\begin{itemize}
  \item \( T(\sigma(a), \sigma(b)) \leq \sigma(a \ast b) \) for any \( a, b \in H \).
  \item \( \sigma(e) = 1 \).
\end{itemize}

3 The generalization of fuzzy submonoid

In this section, we further generalize the fuzzy submonoid (Boixader and Recasens 2022) and its corresponding concepts, and give the \( A \)-fuzzy submonoid based on aggregation function \( A \). The details are listed below.

3.1 \( A \)-fuzzy submonoid

Similar to Definition 7, the concept of fuzzy submonoid based on aggregation function \( A \) is proposed.

Definition 8 Let \( A \) be an aggregation function, \((M, \circ)\) a monoid with identity element \( e \) and \( \sigma \) a fuzzy subset of \( M \). Then \( \sigma \) is an \( A \)-fuzzy submonoid of \( M \) if and only if \( \sigma \) satisfies the following two conditions:

\begin{enumerate}
  \item [(1)] \( A(\sigma(x_1), \ldots, \sigma(x_n)) \leq \sigma(x_1 \circ \ldots \circ x_n) \), for any \( x_1, \ldots, x_n \in M \);
  \item [(2)] \( \sigma(e) = 1 \).
\end{enumerate}

Proposition 7 Let \((M, \circ)\) be a monoid and \( \sigma \) an \( A \)-fuzzy submonoid of \( M \). Then the core \( H \) of \( \sigma \) (i.e., the set of element \( x \) in \( M \) such that \( \sigma(x) = 1 \)) is a submonoid of \( M \).

Proof The identity element of \( H \) obviously exists and the associativity is inherited.

For all \( x, y \in H \), we have \( 1 = A(\sigma(x), \sigma(y), \sigma(e), \ldots, \sigma(e)) \). Since \( \sigma \) is an \( A \)-fuzzy submonoid of \( M \), we have that

\begin{align*}
A(\sigma(x), \sigma(y), \sigma(e), \ldots, \sigma(e)) & \leq \sigma(x \circ y \circ e \circ \ldots \circ e) \\
& = \sigma(x \circ y).
\end{align*}

Hence, \( \sigma(x \circ y) = 1 \), that is, \( x \circ y \in H \). \( \square \)

Example 3 Let \( A_{\min}(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n) \), fuzzy subset \( \sigma_1(x) = x \), which is the \( A_{\min} \)-fuzzy submonoid of \((0, 1), T_M \). Similarly, fuzzy subset \( \sigma_2(x) = 1 - x \), which is the \( A_{\min} \)-fuzzy submonoid of \((0, 1), S_M \).
In fuzzy logic, the unit interval with a \(t\)-norm or a \(t\)-conorm is the most important monoid. Thus, we will consider fuzzy submonoids of a given \(t\)-norm or \(t\)-conorm.

**Definition 9** Let \(A\) and \(T\) be an aggregation function and a \(t\)-norm, respectively. An \(A\)-fuzzy submonoid of \(([0, 1], T)\) will be called an \(A\)-fuzzy \(t\)-subnorm of \(T\).

More specifically, it should be pointed out that a fuzzy subset \(\sigma\) of \([0, 1]\) is a \(A\)-fuzzy \(t\)-subnorm of \(T\) when \(\sigma(0) = 1\) and \(A(\sigma(x), \sigma(y)) \leq \sigma(T(x, y))\) for any \(x, y \in [0, 1]\).

**Definition 10** Let \(A\) and \(S\) be an aggregation function and a \(t\)-conorm, respectively. An \(A\)-fuzzy submonoid of \(([0, 1], S)\) will be called an \(A\)-fuzzy \(t\)-subconorm of \(S\).

In other words, a fuzzy subset \(\sigma\) of \([0, 1]\) is a \(A\)-fuzzy \(t\)-subconorm of \(S\) when \(\sigma(1) = 1\) and \(A(\sigma(x), \sigma(y)) \leq \sigma(S(x, y))\) for any \(x, y \in [0, 1]\). The following propositions help us better understand the concepts of \(A\)-fuzzy \(t\)-subnorm and \(A\)-fuzzy \(t\)-subconorm.

**Proposition 8** If \(A(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)\), then \(\sigma\) is an \(A\)-fuzzy \(t\)-subnorm of \(T_M\) if and only if \(\sigma(1) = 1\).

**Proof** For all fuzzy subset \(\sigma\) of \([0, 1]\), one can conclude that
\[
\min(\sigma(x_1), \ldots, \sigma(x_n)) \leq \sigma(\min(x_1, \ldots, x_n)),
\]
where \(x_1, \ldots, x_n \in [0, 1]\). \(\Box\)

**Proposition 9** If \(A(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)\), then \(\sigma\) is an \(A\)-fuzzy \(t\)-subconorm of \(S_M\) if and only if \(\sigma(1) = 1\).

**Proof** For all fuzzy subset \(\sigma\) of \([0, 1]\), one can conclude that
\[
\min(\sigma(x_1), \ldots, \sigma(x_n)) \leq \sigma(\max(x_1, \ldots, x_n)),
\]
where \(x_1, \ldots, x_n \in [0, 1]\). \(\Box\)

### 3.2 Fuzzy submonoid of special aggregation functions

As two special types of binary aggregation functions, uninorm and nullnorm further generalize the concepts of \(t\)-norm and \(t\)-conorm, which have a wider application prospect in both theory and application.

#### 3.2.1 \(U\)-fuzzy submonoid

In particular, when aggregation functions take uninorms, we have the following conclusions.

**Definition 11** Let \(U\) be a uninorm, \((M, \circ)\) a monoid with identity element \(e\) and \(\sigma\) a fuzzy subset of \(M\). Then \(\sigma\) is an \(U\)-fuzzy submonoid of \(M\) if and only if \(\sigma\) satisfies the following conditions:

1. \(U(\sigma(x), \sigma(y)) \leq \sigma(x \circ y)\), for any \(x, y \in M\);
2. \(\sigma(e) = 1\).

**Proposition 10** Let \(U\) be a uninorm, \((M, \circ)\) a monoid and \(\sigma\) a \(U\)-fuzzy submonoid of \(M\). Then the core \(H\) of \(\sigma\) (i.e., the set of elements \(x\) in \(M\) such that \(\sigma(x) = 1\)) is a submonoid of \(M\).

**Proof** The identity element of \(H\) obviously exists and the associativity is inherited. For all \(x, y \in H\), we have that
\[
1 = U(\sigma(x), \sigma(y)) \leq \sigma(x \circ y).
\]

Therefore, \(x \circ y \in H\). \(\Box\)

**Definition 12** A discrete uninorm is a submonoid of a uninorm containing 0 and 1.

**Example 4** Let \(U_L = (T_L, e, S_L)_{\text{min}}, L_{n,m} = \{0, \frac{e}{n}, \ldots, e, e + \frac{1-e}{m}, \ldots, 1\}\). Then \(L_{n,m}\) is a discrete uninorm of \(U_L\).

**Definition 13** Let \(U\) and \(T\) be a uninorm and a \(t\)-norm, respectively. A \(U\)-fuzzy submonoid of \(([0, 1], T)\) will be called a \(U\)-fuzzy \(t\)-subnorm of \(T\).

**Definition 14** Let \(U\) and \(S\) be a uninorm and a \(t\)-conorm, respectively. A \(U\)-fuzzy submonoid of \(([0, 1], S)\) will be called a \(U\)-fuzzy \(t\)-subconorm of \(S\).

The following theorem gives the necessary and sufficient conditions for \(\sigma\) to be a \(U\)-fuzzy submonoid of \((M, \circ)\).

**Theorem 11** For a monoid \(M\) with identity element \(e\), if a uninorm \(U\) is disjunctive, then \(\sigma\) is \(U\)-fuzzy submonoid if and only if \(\sigma \equiv 1\).

**Proof** Since \(U\) is disjunctive, i.e., \(U(0, 1) = 1\), it holds that \(U(x, 1) = 1\) for all \(x \in [0, 1]\). If \(\sigma\) is a \(U\)-fuzzy submonoid of \(([0, 1], M)\), then we get that

- \(\sigma(e) = 1\),
- \(U(\sigma(x), \sigma(y)) \leq \sigma(x \circ y)\).

Take \(y = e\) for any \(x \in [0, 1]\), then \(1 = U(\sigma(x), 1) \leq \sigma(x)\).

On the other hand, if \(\sigma \equiv 1\), then \(\sigma\) is obviously a \(U\)-fuzzy submonoid of \(([0, 1], M)\). \(\Box\)

Further, if we take monoid as \(t\)-norm and \(t\)-conorm, the following corollaries hold.

**Corollary 12** For a \(t\)-norm \(T\), if uninorm \(U\) is disjunctive, then \(\sigma\) is \(U\)-fuzzy \(t\)-subnorm of \(([0, 1], T)\) if and only if \(\sigma \equiv 1\).

**Corollary 13** For a \(t\)-conorm \(S\), if uninorm \(U\) is disjunctive, then \(\sigma\) is \(U\)-fuzzy \(t\)-subconorm of \(([0, 1], S)\) if and only if \(\sigma \equiv 1\).
Example 5 In light of Theorem 11, for any uninorm $U \in \mathcal{U}_{\text{max}}$, if fuzzy subset $\sigma$ is $U$-fuzzy submonoid of $M$, then $\sigma \equiv 1$. In particular, $U$ is the fuzzy $t$-subnorm of $([0, 1], T)$ and fuzzy $t$-subconorm of $([0, 1], S)$.

Proposition 14 Let $U$ be a uninorm and $B$ be a set, denoted as $B = \{x \in [0, 1] | \sigma(x) \in [e, 1]\}$ and

$$U(x, y) = \begin{cases}
edad T \left( \frac{x - y}{e}, \frac{y}{e} \right), & (x, y) \in [0, e]^2, \\
(1 - e)S_M \left( \frac{x - y}{1 - e}, \frac{y - y}{1 - e} \right), & (x, y) \in (e, 1]^2, \\
\min(x, y), & \text{otherwise},
\end{cases}$$

where $\sigma$ is fuzzy subset. Then $\sigma$ is $U$-fuzzy $t$-subnorm of $([0, 1], T)$ if and only if $\sigma$ is decreasing on $B$ and $\sigma(1) = 1$.

Proof (1) If $U(\sigma(x), \sigma(y)) \in [0, e]^2$, then

$$U(\sigma(x), \sigma(y)) = eT \left( \frac{\sigma(x)}{e}, \frac{\sigma(y)}{e} \right) \leq \min(\sigma(x), \sigma(y)) \leq \sigma(\min(x, y)).$$

(2) If $U(\sigma(x), \sigma(y)) \in [e, 1]^2$, then

$$U(\sigma(x), \sigma(y)) = \max(\sigma(x), \sigma(y)) \leq \sigma(\min(x, y)),$$

if and only if $\sigma(y) \leq \sigma(x)$ with $x \leq y$.

(3) If $U(\sigma(x), \sigma(y)) \in [0, e] \times [e, 1]$ (resp. $U(\sigma(x), \sigma(y)) \in [e, 1] \times [0, e]$), one concludes that

$$U(\sigma(x), \sigma(y)) = \min(\sigma(x), \sigma(y)) \leq \sigma(\min(x, y)).$$

Then the dual conclusions about $U$-fuzzy $t$-subnorm of $([0, 1], S_M)$ can be obtained.

Proposition 15 Let $U$ be a uninorm and $B$ be a set, denoted as $B = \{x \in [0, 1] | \sigma(x) \in [e, 1]\}$ and

$$U(x, y) = \begin{cases}
edad T \left( \frac{x - y}{e}, \frac{y}{e} \right), & (x, y) \in [0, e]^2, \\
(1 - e)S_M \left( \frac{x - y}{1 - e}, \frac{y - y}{1 - e} \right), & (x, y) \in (e, 1]^2, \\
\min(x, y), & \text{otherwise},
\end{cases}$$

where $\sigma$ is a fuzzy subset. Then $\sigma$ is a $U$-fuzzy $t$-subconorm of $([0, 1], S_M)$ if and only if $\sigma$ is increasing on $B$ and $\sigma(0) = 1$.

Proof It can be proven in a similar way as Proposition 14. □

Example 6 In particular, the fuzzy subset $\sigma$ is denoted as:

$$\sigma(x) = \begin{cases}x, & 0 \leq x < e, \\
e, & e \leq x \leq 1.
\end{cases}$$

According to Proposition 14, $B = \{x \in [e, 1] | \sigma(x) = 1\}$ and $\sigma$ is decreasing on $B$, then $\sigma$ is $U$-fuzzy $t$-subnorm of $([0, 1], T_M)$.

Not all fuzzy subset $\sigma$ of monoid $M$ can find corresponding uninorm $U$ such that $\sigma$ is a $U$-fuzzy submonoid of $M$. Then the following propositions hold.

Proposition 16 Let $\sigma(x) = x$ be a fuzzy subset, $T$ be a t-norm. There is no uninorm $U$ such that $\sigma$ is a $U$-fuzzy $t$-subnorm of $([0, 1], T)$.

Proof Assume that there exists uninorm $U$ with identity element $e$ that $\sigma$ is a $U$-fuzzy $t$-subnorm of $([0, 1], T)$, then

$$U(\sigma(x), \sigma(y)) \leq \sigma(T(x, y)).$$

Furthermore, it holds that

$$U(x, y) \leq T(x, y) \leq \min(x, y),$$

which contradicts with the case of $x = e, y > e$. □

Proposition 17 Let $\sigma(x) = 1 - x$ be a fuzzy subset, $S$ be a t-conorm. There is no uninorm $U$ such that $\sigma$ is a $U$-fuzzy $t$-subconorm of $([0, 1], S)$.

Proof Assume that there exists uninorm $U$ with identity element $e$ that $\sigma$ is a $U$-fuzzy $t$-subconorm of $([0, 1], S)$, then

$$U(\sigma(x), \sigma(y)) \leq \sigma(S(x, y)).$$

Furthermore, it holds that

$$U(1 - x, 1 - y) \leq 1 - S(x, y) \leq 1 - \max(x, y),$$

which contradicts with the case of $x = 1 - e, y < 1 - e$. □

Next, a sufficient and necessary condition for fuzzy subset $\sigma$ to be $U$-fuzzy $t$-subnorm of $([0, 1], T)$ (resp. $U$-fuzzy $t$-subconorm of $([0, 1], S)$) is explored based on representable uninorm and continuous Archimedean $t$-norm (resp. $t$-conorm).

Proposition 18 Let $U$ be an representable uninorm and $T$ be a continuous Archimedean $t$-norm. $h$ and $t$ are additive generators of $U$ and $T$, respectively. A fuzzy subset $\sigma$ on $[0, 1]$ is a $U$-fuzzy $t$-subnorm of $([0, 1], T)$ if and only if the mapping $f : [-\infty, \infty] \rightarrow [-\infty, \infty]$ with $f = (h \circ \sigma \circ t^{-1})$ is subadditive.

Proof According to Definition 13, $\sigma$ is a $U$-fuzzy $t$-subnorm of $([0, 1], T)$ if and only if for all $x, y \in [0, 1]$,

$$U(\sigma(x), \sigma(y)) \leq \sigma(T(x, y)).$$
Further, it can be denoted as
\[ h^{-1}(h(\sigma(x)) + h(\sigma(y))) \leq \sigma \left( t^{-1}(t(x) + t(y)) \right). \]

Since \((-h)^{-1}((-h)(\sigma(x)) + (-h)(\sigma(y))) = h^{-1}(h(\sigma(x)) + h(\sigma(y))),\) and \((-h)^{-1}((-h)(\sigma(x)) + (-h)(\sigma(y))) \leq \sigma \left( t^{-1}(t(x) + t(y)) \right),\) which is equivalent to
\[ (-h)(\sigma(x)) + (-h)(\sigma(y)) \geq (-h) \left( \sigma \left( t^{-1}(t(x) + t(y)) \right) \right). \]

Let \(t(x) = a\) and \(t(y) = b,\)
\[ (-h) \left( \sigma \left( t^{-1}(a) \right) \right) + (-h) \left( \sigma \left( t^{-1}(b) \right) \right) \geq (-h) \left( \sigma \left( t^{-1}(a + b) \right) \right). \]

\(\square\)

**Corollary 19** Let \(f : [0, \infty) \rightarrow [-\infty, \infty]\) be a subadditive mapping and representable uninorm \(U\) and continuous Archimedean t-norm \(T\) have additive generators \(h\) and \(t,\) respectively. Then \((-h)^{-1} \circ f \circ t\) is a \(U\)-fuzzy t-subnorm of \([0, 1], T).\)

**Example 7** Let \(T_L\) be the Łukasiewicz t-norm with additive generator \(t,\) where \(t(x) = 1 - x.\) The subadditive mapping \(f(x) = \sqrt{x}\) and \(U_p\) is a uninorm with additive generator \(h,\)
\[ h(x) = \begin{cases} \ln(2x), & 0 \leq x < \frac{1}{2}, \\ -\ln(-2x + 2), & \frac{1}{2} \leq x \leq 1. \end{cases} \]
which can be shown in Fig. 1. At this moment, \(u(x) = (h^{-1}) \circ f \circ t\) is the \(U_p,\)-fuzzy t-subnorm of \(T_L\).

**Example 8** Similarly, if we substitute the addition generator of \(U_p\) in Example 7 with the following form,
\[ h'(x) = \begin{cases} 1 - \frac{1}{2x}, & 0 \leq x \leq \frac{1}{2}, \\ -\frac{1}{2(x - 1)}, & \frac{1}{2} < x \leq 1. \end{cases} \]
Then \(u'(x) = (h'^{-1}) \circ f \circ t\) is the \(U'_p,\)-fuzzy t-subnorm of \(T_L\) and \(U_p\) as shown in Fig. 2.

**Proposition 20** Let \(U\) be a representable uninorm and \(S\) be a continuous Archimedean t-conorm. \(h\) and \(s\) are additive generators of \(U\) and \(S,\) respectively. A fuzzy subset \(\sigma\) of \([0, 1]\) is a \(U\)-fuzzy t-subnorm of \([0, 1], S)\) if and only if the mapping \(f : [0, \infty) \rightarrow [-\infty, \infty] with f = h \circ \sigma \circ s^{-1}\) is subadditive.

**Corollary 21** Let \(f : [0, \infty) \rightarrow [-\infty, \infty]\) be a subadditive mapping, representable uninorms \(U\) and continuous Archimedean t-conorms \(S\) have additive generators \(h\) and \(s,\) respectively. Then \((-h)^{-1} \circ f \circ s\) is a \(U\)-fuzzy t-subconorm of \([0, 1], S).\)

### 3.2.2 F-fuzzy submonoid

Similar to Sect. 3.2.1, when aggregation functions take null-norms, we have the following conclusions.

**Definition 15** Let \(F\) be a nullnorm, \((M, \circ)\) a monoid with identity element \(e\) and \(\sigma\) a fuzzy subset of \(M.\) \(\sigma\) is an \(F\)-fuzzy submonoid of \(M\) if and only if \(\sigma\) satisfies the following conditions:
nullnorm with absorbing element $k$ and $T$ is a $t$-norm, then $\sigma(x) = 1$ is a submonoid of $M$. 

**Corollary 23**

The identity element of $H$ obviously exists and the associativity is inherited.

For any $x, y \in H$, $1 = F(\sigma(x), \sigma(y)) \subseteq \sigma(x \circ y)$. Therefore, $x \circ y \in H$. 

**Definition 17**

A discrete nullnorm is a submonoid of a nullnorm containing $0$ and $1$.

Let $S$ and $T$ be a $t$-conorm and a $t$-norm, respectively. The nullnorm $F = (S, k, T)$ with absorbing element $k$ can be shown as follows.

$$F(x, y) = \begin{cases} F(\sigma(x), \sigma(y)) & (x, y) \in [0, k]^2, \\ k + (1 - k)T & (x, y) \in (k, 1)^2, \\ k, & \text{otherwise}. \end{cases}$$

**Example 9**

Let $F_L$ be a nullnorm where $F_L = (S_L, k, T_L)$. $L_{n,m} = [0, \frac{k}{n}, \ldots, \frac{k}{n}, \frac{1-k}{m}, \ldots, 1]$. Then $L_{n,m}$ is a discrete uninnor of $F_L$.

**Definition 18**

Let $F$ and $T$ be a nullnorm and a $t$-norm, respectively. An $F$-fuzzy submonoid of $(0, 1, T)$ will be called an $F$-fuzzy $t$-subnorm of $T$.

**Definition 19**

Let $F$ and $S$ be a uninnorm and a $t$-conorm, respectively. An $F$-fuzzy submonoid of $(0, 1, S)$ will be called an $F$-fuzzy $t$-subnorm of $S$.

**Proposition 22**

If $\sigma$ is an $F$-fuzzy submonoid of $M$ where $F$ is a nullnorm with absorbing element $k$ and $M$ is a monoid with identity element $e$, then $\sigma(x) \geq \max \{e, k\}$ for any $x \in [0, 1]$.

**Proof**

If $\sigma$ is an $F$-fuzzy submonoid of $M$, then we obtain that

$$F(\sigma(x), \sigma(y)) \leq \sigma(x \circ y) \text{ and } \sigma(e) = 1.$$ 

Let $y = e$, it holds that

$$k = F(\sigma(x), k) \leq F(\sigma(x), \sigma(e)) \leq \sigma(x).$$

It is easy to get the following corollaries when $M$ takes the $t$-norm $T$ and $t$-conorm $S$.

**Corollary 23**

If $\sigma$ is an $F$-fuzzy $t$-subnorm of $T$ where $F$ is a nullnorm with absorbing element $k$ and $T$ is a $t$-norm, then $\sigma(x) \geq \max \{e, k\}$ for any $x \in [0, 1]$.

**Corollary 24**

If $\sigma$ is an $F$-fuzzy $t$-subconorm of $S$ where $F$ is a nullnorm with absorbing element $k$ and $S$ is a $t$-conorm, then $\sigma(x) \geq \max \{e, k\}$ for any $x \in [0, 1]$.

**Example 10**

Let $F$ be the unique idempotent nullnorm with absorbing element $k$, which can be represented as follows:

$$F(x, y) = \begin{cases} \max \{x, y\} & (x, y) \in [0, k]^2, \\ \min \{x, y\} & (x, y) \in (k, 1]^2, \\ k, & \text{otherwise}. \end{cases}$$

and fuzzy subset $\sigma(x) = 1$ for any $x \in [0, 1]$. Then it is easy to verify that $\sigma$ is an $F$-fuzzy subconorm of $M$ with $\sigma \geq k$ for any $x \in [0, 1]$. In particular, $F$ is an $F$-fuzzy $t$-subnorm of $(0, 1, T)$ and $F$-fuzzy $t$-subconorm of $(0, 1, S)$.

Let $S$ and $T$ be a $t$-conorm and a $t$-norm, $F_M$ a nullnorm with absorbing element $k$, which can be shown as follows.

$$F_M(x, y) = \begin{cases} F(\sigma(x), \sigma(y)) & (x, y) \in [0, k]^2, \\ k + (1 - k)T_M & (x, y) \in (k, 1]^2, \\ k, & \text{otherwise}.\end{cases}$$

**Proposition 25**

A fuzzy subset $\sigma$ is an $F_M$-fuzzy $t$-subnorm of $(0, 1, T_M)$ where $F_M$ is a nullnorm with absorbing element $k$ if and only if $\sigma(1) = 1, \sigma(x) \geq k$ for any $x \in [0, 1]$.

**Proof**

Firstly, if $\sigma$ is an $F_M$-fuzzy $t$-subnorm of $(0, 1, T_{\min})$, then we obviously have $\sigma(1) = 1$ and $\sigma(x) \geq k$ for any $x \in [0, 1]$.

Conversely, let $\sigma(1) = 1$ and $\sigma(x) \geq k$ for any $x \in [0, 1]$. Then the discussion will be divided into three cases:

1. If $(x, y) \in [0, k]^2$, then

$$F_M(\sigma(x), \sigma(y)) = kS(\frac{\sigma(x)}{k}, \frac{\sigma(y)}{k}) \leq k \leq \sigma(T_M(x, y)).$$

2. If $(x, y) \in (k, 1]^2$, then

$$F_M(\sigma(x), \sigma(y)) = k + (1 - k)T_M \left( \frac{\sigma(x) - k}{1 - k}, \frac{\sigma(y) - k}{1 - k} \right) \leq \min(\sigma(x), \sigma(y)) \leq \sigma(T_M(x, y)).$$

3. In the remaining case, $F_M(\sigma(x), \sigma(y)) \leq \sigma(\sigma(x), \sigma(y))$ clearly holds.
Proposition 26 A fuzzy subset $\sigma$ is an $F_M$-fuzzy $t$-subconorm of $([0, 1], S_M)$ where $F_M$ is a nullnorm with absorbing element $k$ if and only if $\sigma(0) = 1$, $\sigma(x) \geq k$ for any $x \in [0, 1]$.

Proof It can be proven in a similar way as Proposition 25. \Box

3.3 Lattice value fuzzy submonoid

Let $L$ be a nonempty poset, $(L, \leq)$ is called lattice if any two elements have infimum and supremum. Further, a bounded lattice $(L, \leq, 0_L, 1_L)$ is a lattice which has top element $1_L$ and bottom element $0_L$, i.e., there exist two elements $1_L, 0_L \in L$ such that $0_L \leq x \leq 1_L$, for all $x \in L$. A map $A : M \rightarrow L$ will be called $L$-fuzzy set of $M$ and the family of all $L$-fuzzy sets of $M$ is denoted $\mathcal{F}_L(M)$. Next, the concept of fuzzy subset on $L$ will be given.

Definition 20 (Lowen 2012; Goguen 1967) Let $(L, \leq)$ be a lattice. A fuzzy subset $A$ of $L$ can be represented as follows:

$$\mu_A : L \rightarrow [0, 1],$$

where for any $x \in L$, $\mu_A(x)$ is denoted as the membership function of $x$ in fuzzy set $A$. The family of all fuzzy sets of $L$ will be defined as $\mathcal{F}(L)$.

Definition 21 Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $(M, \circ)$ a monoid with identity element $e$ and $\sigma$ a $L$-fuzzy set of $M$. Then $\sigma$ is a $\wedge$-fuzzy submonoid of $M$ if and only if $\sigma$ satisfies the following conditions:

1. $(\sigma(x) \wedge \sigma(y)) \leq \sigma(x \circ y)$, for all $x, y \in M$;
2. $\sigma(e) = 1_L$.

Proposition 27 Let $(M, \circ)$ be a monoid and $\sigma$ a $\wedge$-fuzzy submonoid. The core $H$ of $\sigma$ (i.e., the set of element $x$ in $M$ such that $\sigma(x) = 1_L$) is a submonoid of $M$.

Proof The identity element of $H$ obviously exists and the associativity is inherited.

For all $x, y \in H$,

$$1_L \leq (\sigma(x) \wedge \sigma(y)) \leq \sigma(x \circ y).$$

Hence, $x \circ y \in H$. \Box

Further, we can generalize the definition of fuzzy $t$-subnorm and fuzzy $t$-subconorm to lattice structure.

Definition 22 Let $(L, \leq, 0_L, 1_L)$ and $T$ be a bounded lattice and $t$-norm, respectively. A $\wedge$-fuzzy submonoid of $([0, 1], T)$ is also called $\wedge$-fuzzy $t$-subnorm of $([0, 1], T)$.

Definition 23 Let $(L, \leq, 0_L, 1_L)$ and $S$ be a bounded lattice and $t$-conorm, respectively. A $\wedge$-fuzzy submonoid of $([0, 1], S)$ is also called $\wedge$-fuzzy $t$-subconorm of $([0, 1], S)$.

In light of the above generalization method, the binary operator $\vee$ has similar definitions and propositions.

4 Vague monoid

To explore the homomorphisms between $A$-vague monoid and $A$-fuzzy monoid, the concepts of $A$-vague binary operation and $A$-vague monoid are put forward below.

4.1 Vague monoid by aggregation function

The fuzzification of equivalence relation is the fuzzy relation called $T$-fuzzy equivalence or $T$-indistinguishability operator, which is essential in fuzzy logic and has been widely studied in (Recasens 2010; Demirci and Recasens 2004; Jayaram and Mesiar 2009; Valverde 1985). In the following, we extend the definition of $T$-fuzzy equivalence to binary aggregation function.

Definition 24 Let $A$ be a binary aggregation function and $X$ a set. A fuzzy relation $E : X \times X \rightarrow [0, 1]$ is an $A$-indistinguishability operator, which satisfies the following conditions for all $x, y, z \in X$:

1. reflexivity: $E(x, x) = 1$;
2. symmetry: $E(x, y) = E(y, x)$;
3. $A$-transitivity: $A(E(x, y), E(y, z)) \leq E(x, z)$.

If $E(x, y) = 1$ implies $x = y$, then it is said that $E$ separates points.

Definition 25 A fuzzy binary operation on a set $M$ is a mapping $\delta : M \times M \times M \rightarrow [0, 1]$, where $\delta(x, y, z)$ is interpreted as the degree in which $z$ is $x \circ y$.

Similar to the method of (Boixader and Recasens 2022), the vague binary operation based on aggregation function is given below.

Definition 26 Let $E$ be an $A$-indistinguishability operator on $M$. An $A$-vague binary operation is a fuzzy binary operation on $M$ when it satisfies the following conditions for all $x, y, z, x', y', z' \in M$:

1. $A(\delta(x, y, z), E(x, x'), E(y, y'), E(z, z')) \leq \delta(x', y', z')$;
2. $A(\delta(x, y, z), \delta(x, y, z')) \leq E(z, z')$;
3. for all $x, y \in M$, there exists $z'' \in M$ such that $\delta(x, y, z'') = 1$.

Proposition 28 Let $E$ be an $A$-indistinguishability operator on $M$ separating points and $\delta$ a vague binary operation on $M$. Then $z$ in Definition 26 (3) is unique.
**Proof** Let \( z, z' \in M \) satisfy \( \bar{\circ}(x, y, z) = 1 \) and \( \bar{\circ}(x, y, z') = 1 \). From Definition 26, one can conclude that

\[
1 = A(\bar{\circ}(x, y, z), \bar{\circ}(x, y, z')) \leq E(z, z').
\]

Hence, \( E(z, z') = 1 \). Since \( E \) separates points, we have that \( z = z' \). \( \square \)

**Definition 27** Let \( \bar{\circ} \) be an \( A \)-vague binary operation on \( M \) with respect to an \( A \)-indistinguishability operator \( E \) on \( M \). Then \( (M, \bar{\circ}) \) is an \( A \)-vague monoid if and only if it satisfies the following properties.

1. **Associativity:**

\[
A(\bar{\circ}(y, z, d), \bar{\circ}(x, d, m), \bar{\circ}(x, y, q), \bar{\circ}(q, z, w)) \leq E(m, w),
\]

where \( x, y, z, d, m, q, w \in M \).

2. **Identity:** there exists identity element \( e \in M \) such that

\[
\bar{\circ}(e, x, x) = 1 \quad \text{and} \quad \bar{\circ}(x, e, x) = 1 \quad \text{for all} \quad x \in M.
\]

**Definition 28** An \( A \)-vague monoid is commutative if and only if \( A(\bar{\circ}(x, y, m), \bar{\circ}(y, x, w)) \leq E(m, w) \) for any \( x, y, m, w \in M \).

Next, we give a sufficient condition for the uniqueness of identity element of \( A \)-vague monoid \( (M, \bar{\circ}) \).

**Proposition 29** Let \( A \) be an aggregation function, \( E \) an \( A \)-indistinguishability operator separating points on set \( M \) and \( (M, \bar{\circ}) \) an \( A \)-vague monoid. Then the identity element is unique.

**Proof** Let \( e \) and \( e' \) be two identity elements of \( M \). Then, since \( e \) is an identity element, it holds that

\[
\bar{\circ}(e, e', e') = 1.
\]

Further, if \( e' \) is an identity element, then we have that

\[
\bar{\circ}(e, e', e) = 1.
\]

According to Definition 26, \( 1 = A(\bar{\circ}(e, e', e'), \bar{\circ}(e, e', e)) \leq E(e, e') \), so \( E(e, e') = 1 \) and \( e = e' \) because \( E \) separates points. \( \square \)

**Definition 29** Let \( \circ \) be a binary operation on \( M \) and \( E \) an \( A \)-indistinguishability operator on \( M \). Then \( E \) is regular with respect to \( \circ \) if and only if for any \( x, y, z \in M \),

\[
E(x, y) \leq E(x \circ z, y \circ z)
\]

and

\[
E(x, y) \leq E(z \circ x, z \circ y).
\]

Under the condition of regular \( A \)-indistinguishability operator, the definition of the fuzzy mapping \( \bar{\circ} \) can be given and further the \( A \)-vague monoid is obtained.

**Proposition 30** Let \( E \) be a regular \( A \)-indistinguishability operator on \( M \) with respect to a binary operation \( \circ \) on \( M \).

1. The fuzzy mapping \( \bar{\circ} : M \times M \times M \rightarrow [0, 1] \) defined by

\[
\bar{\circ}(x, y, z) = E(x \circ y, z), \quad \text{for any} \quad x, y, z \in M,
\]

is an \( A \)-vague binary operation on \( M \).

2. If \( (M, \circ) \) is a monoid, then \( (M, \bar{\circ}) \) is an \( A \)-vague monoid.

**Proof** (1) Firstly, we prove that \( \bar{\circ} \) satisfies the properties of Definition 26 for any \( x, y, z, x', y', z' \in M \). Firstly, it holds that

\[
A(\bar{\circ}(x, y, z), E(x, x'), E(y, y'), E(z, z'))
\]

\[
= A(E(x \circ y, z), E(x, x'), E(y, y'), E(z, z'))
\]

\[
\leq A(E(x \circ y, z'), E(x, x'), E(y, y))
\]

\[
\leq A(E(x \circ y, z), E(x \circ y, x' \circ y), E(y, y'))
\]

\[
\leq A(E(x' \circ y, z'), E(y, y'))
\]

\[
\leq A(E(x' \circ y, x', E(x' \circ y, x' \circ y'))
\]

\[
\leq E(x' \circ y, z')
\]

\[
= \bar{\circ}(x', y, z').
\]

Then, we can obtain that

\[
A(\bar{\circ}(x, y, z), \bar{\circ}(x, y, z'))
\]

\[
= A(E(x \circ y, z), E(x \circ y, z')) \leq E(z, z').
\]

Furthermore, if \( z = x \circ y \), then it follows that

\[
\bar{\circ}(x, y, z) = E(x \circ y, x \circ y) = 1.
\]

(2) In the following, the associativity and identity element of \( (M, \bar{\circ}) \) are verified. Since the associativity of \( \circ \) and the regularity of \( E \), for any \( x, y, z, d, m, q, w \in M \), it holds that

\[
A(\bar{\circ}(y, z, d), \bar{\circ}(x, d, m), \bar{\circ}(x, y, q), \bar{\circ}(q, z, w))
\]

\[
= A(E(y \circ z, d), E(x \circ d, m), E(x \circ y, q), E(q \circ z, w))
\]

\[
\leq A(E(x \circ (y \circ z), x \circ d),
\]

\[
E(x \circ d, m), \quad E((x \circ y) \circ z, q \circ z), E(q \circ z, w))
\]

\[
\leq A(E(x \circ (y \circ z), m), E((x \circ y) \circ z, w))
\]

\[
\leq E(m, w).
\]
On the other side, due to the identity element $e$ of $M$, one concludes that
\[
\tilde{\circ}(x, e, x) = E(x \circ e, x) = E(x, x) = 1,
\]
\[
\tilde{\circ}(e, x, x) = E(e \circ x, x) = E(x, x) = 1,
\]
that is, $e$ is the identity element of $\tilde{\circ}$. Hence, $(M, \tilde{\circ})$ is an $A$-vague monoid.

Conversely, a monoid $(M, \circ)$ can be obtained from a vague monoid $(M, \tilde{\circ})$.

**Proposition 31** Let $(M, \tilde{\circ})$ be a vague monoid with respect to an $A$-indistinguishability operator $E$ separating points. Then $(M, \circ)$ is a monoid where $x \circ y$ is the unique $z \in M$ such that $\tilde{\circ}(x, y, z) = 1$.

**Proof** First to verify the associativity,

\[
1 = A(\tilde{\circ}(y, z, x \circ z), \tilde{\circ}(x, y \circ z, x \circ (y \circ z))),\tilde{\circ}(x, y, x \circ y),
\]
\[
\tilde{\circ}(x \circ y, z, (x \circ y) \circ z))) \leq E(x \circ (y \circ z), (x \circ y) \circ z).
\]

Therefore, $x \circ (y \circ z) = (x \circ y) \circ z$ holds.

Further, we can obtain that

\[
1 = A(\tilde{\circ}(x, e, x), \tilde{\circ}(x, e, x \circ e))) \leq E(x \circ e, x),
\]
\[
1 = A(\tilde{\circ}(e, x, x), \tilde{\circ}(e, x, e \circ x)) \leq E(e \circ x, x).
\]

Hence, $E(x \circ e, x) = 1$ and $E(e \circ x, x) = 1$, that is, $x \circ e = x$ and $e \circ x = x$, $e$ is the identity element of $(M, \circ)$.

**Definition 30** Let $(M, \tilde{\circ})$ be a vague monoid with respect to an $A$-indistinguishability operator $E$ separating points. Then $(M, \circ)$ is the monoid associated to the vague monoid $(M, \tilde{\circ})$.

Further, if $A$ is an aggregation function and $(M, \circ)$ is a monoid, then there exist bijective maps between their $A$-vague monoids and regular $A$-indistinguishability operators such that they can represent each other:

- $\tilde{\circ}(x, y, z) = E(x \circ y, z);$  
- $E(x, y) = \tilde{\circ}(x, e, y).$

Regarding commutativity, the connection between $(M, \tilde{\circ})$ and $(M, \circ)$ is given below.

**Proposition 32** Let $E$ be an $A$-indistinguishability operator separating points, $(M, \tilde{\circ})$ an $A$-vague monoid and $(M, \circ)$ its associated monoids $(x \circ y = z$ if and only if $\tilde{\circ}(x, y, z) = 1).$ Then $(M, \tilde{\circ})$ is commutative if and only if $(M, \circ)$ is commutative.

**Proof** Firstly, if $(M, \tilde{\circ})$ is commutative, we have that

\[
1 = A(\tilde{\circ}(x, y, x \circ y), \tilde{\circ}(y, y, y \circ x)) \leq E(x \circ y, y \circ x).
\]

Since $E$ is a separating point and $E(x \circ y, y \circ x) = 1$, it holds that $x \circ y = y \circ x$.

Inversely, if $(M, \circ)$ is commutative, we get that

\[
A(\tilde{\circ}(x, y, z), \tilde{\circ}(y, x, z')) = A(E(x \circ y, z), E(y \circ x, z')) = A(E \circ y, z), E(x \circ y, z')) \leq E(z, z').
\]

Hence, $(M, \tilde{\circ})$ is commutative.

The relationship between $T$-vague monoid and $T$-fuzzy monoid has been studied in (Boixader and Recasens 2022). In the following, the homomorphisms between them is extended to $A$-vague monoid and $A$-fuzzy monoid.

**Definition 31** Let $(M, \tilde{\circ})$ and $(N, \tilde{\bullet})$ be two $A$-vague monoids with respect to the $A$-indistinguishability operators $E$ and $F$, respectively. A map $f : M \rightarrow N$ is a homomorphism from $M$ onto $N$ if and only if $\tilde{\circ}(x, y, z) \leq \tilde{\bullet}(f(x), f(y), f(z))$ for all $x, y, z \in M$.

**Lemma 1** Let $(M, \tilde{\circ})$ and $(N, \tilde{\bullet})$ be two $A$-vague monoids with respect to $A$-indistinguishability operators $E$ and $F$ respectively and $f : M \rightarrow N$ a homomorphism from $M$ onto $N$. If $e$ is the identity element of $M$, then $f(e)$ is the identity element of $N$.

**Proof** According to the known, the following equations hold.

\[
1 = \tilde{\circ}(x, e, x) \leq \tilde{\bullet}(f(x), f(e), f(x)),
\]
\[
1 = \tilde{\circ}(e, x, x) \leq \tilde{\bullet}(f(e), f(x), f(x)).
\]

Hence, $\tilde{\bullet}(f(x), f(e), f(x)) = \tilde{\circ}(f(e), f(x), f(x)) = 1$ and $f(e)$ is the identity element of $N$.

**Proposition 33** Let $(M, \tilde{\circ})$ and $(N, \tilde{\bullet})$ be two $A$-vague monoids with respect to $A$-indistinguishability operators $E$ and $F$ with $E \leq F$. Then the identity map $id : M \rightarrow M$ is a homomorphism from $(M, \tilde{\circ})$ onto $(M, \tilde{\bullet})$.

**Proof** It is easy to obtain that $\tilde{\circ}(x, y, z) = E(x \circ y, z) \leq F(x \circ y, z) \leq \tilde{\bullet}(x, y, z)$.

A crisp monoid $(M, \circ)$ is an $A$-vague monoid, if $x \circ y = z$ for any $x, y, z \in M$ then $\circ(x, y, z) = 1$; otherwise, $\circ(x, y, z) = 0$.

**Corollary 34** Let $(M, \tilde{\circ})$ be an $A$-vague monoid with respect to an $A$-indistinguishability operator $E$ separating points. Then the identity map $id : M \rightarrow M$ is a homomorphism from $(M, \circ)$ onto $(M, \tilde{\circ})$. 

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Definition 32 Let \( f : M \rightarrow N \) be a homomorphism from \((M, \circ)\) onto \((N, \circ)\). The kernel of \( f \) is the fuzzy subset \( \sigma \) of \( M \) defined by \( \sigma(x) = E(f(x), e') \) for all \( x \in M \) where \( e' \) is the identity element of \((N, \circ)\).

Proposition 35 Let \((M, \circ)\) be an \( A \)-vague monoid with respect to an \( A \)-indistinguishability operator \( E \) separating points. The kernel of the identity map \( \text{id} : M \rightarrow M \) is an \( A \)-fuzzy submonoid of \((M, \circ)\).

Proof Firstly, we have that \( \ker \text{id}(e) = E(e, e) = 1 \). On the other side, it holds that

\[
A(\ker \text{id}(x), \ker \text{id}(y)) = A(E(x, e), E(y, e)) \\
\leq A(E(x \circ y, y), E(y, e)) \\
\leq E(x \circ y, e) \\
= \ker \text{id}(x \circ y).
\]

\[\square\]

4.2 Vague monoid of special aggregation function

Since uninorm and nullnorm can be considered as two special types of aggregation functions, further, similar conclusions can be drawn by bringing them into vague monoid.

4.2.1 Uninorm vague monoid

Definition 33 Let \( U \) be a uninorm and \( A \) a set. A fuzzy relation \( E : X \times X \rightarrow [0, 1] \) is a \( U \)-indistinguishability operator, which satisfies the following conditions for all \( x, y, z \in X \):

1. reflexivity: \( E(x, x) = 1 \);
2. symmetry: \( E(x, y) = E(y, x) \);
3. \( U \)-transitivity: \( U(E(x, y), E(y, z)) \leq E(x, z) \).

If \( E(x, y) = 1 \) implies \( x = y \), then it is said that \( E \) separates points.

Definition 34 Let \( E \) be a \( U \)-indistinguishability operator on \( M \). A \( U \)-vague binary operation \( \tilde{\circ} : M \times M \times M \rightarrow [0, 1] \) is a fuzzy binary operation on \( M \) when it satisfies the following conditions for any \( x, y, z, x', y', z' \in M \):

1. \( U(\tilde{\circ}(x, y, z), E(x, x'), E(y, y'), E(z, z')) \leq \tilde{\circ}(x', y', z') \);
2. \( U(\tilde{\circ}(x, y, z), \tilde{\circ}(x, y, z')) \leq E(z, z') \);
3. For all \( x, y \in M \), there exists \( z'' \in M \) such that \( \tilde{\circ}(x, y, z'') = 1 \).

Proposition 36 Let \( E \) be a \( U \)-indistinguishability operator on \( M \) separating points and \( \tilde{\circ} \) a vague binary operation on \( M \). Then the \( z \) in Definition 26 (3) is unique.

Definition 35 Let \( \tilde{\circ} \) be a \( U \)-vague binary operation on \( M \) with respect to a \( U \)-indistinguishability operator \( E \) on \( M \). Then \((M, \tilde{\circ})\) is a \( U \)-vague monoid if and only if it satisfies the following properties.

1. Associativity:
   \[
   U(\tilde{\circ}(y, z, d), \tilde{\circ}(x, d, m), \tilde{\circ}(x, y, q), \tilde{\circ}(q, z, w)) \leq E(m, w),
   \]
   where \( x, y, z, d, m, q, w \in M \);
2. Identity: there exists identity element \( e \in M \) such that \( \tilde{\circ}(e, x, x) = 1 \) and \( \tilde{\circ}(x, e, x) = 1 \) for all \( x \in M \).

Definition 36 A \( U \)-vague monoid is commutative if and only if for any \( x, y, m, w \in M \),

\[
U(\tilde{\circ}(x, y, m), \tilde{\circ}(y, x, w)) \leq E(m, w).
\]

Next, we give a sufficient condition for the uniqueness of identity element of \( U \)-vague monoid \((M, \tilde{\circ})\).

Proposition 37 Let \( U \) be a uninorm, \( E \) a \( U \)-indistinguishability operator separating points on set \( M \) and \((M, \tilde{\circ})\) a \( U \)-vague monoid. Then the identity element is unique.

Definition 37 Let \( \circ \) be a binary function on \( M \), and \( E \) a \( U \)-indistinguishability operator on \( M \). \( E \) is regular with respect to \( \circ \) if and only if for any \( x, y, z \in M \),

\[
E(x, y) \leq E(x \circ z, y \circ z)
\]

and

\[
E(x, y) \leq E(z \circ x, z \circ y).
\]

Under the condition of regular \( U \)-indistinguishability operator, the definition of fuzzy mapping \( \tilde{\circ} \) can be given and further the \( U \)-vague monoid is obtained.

Proposition 38 Let \( E \) be a regular \( U \)-indistinguishability operator on \( M \) with respect to a binary operation \( \circ \) on \( M \).

1. The fuzzy mapping \( \tilde{\circ} : M \times M \times M \rightarrow [0, 1] \) defined by
   \[
   \tilde{\circ}(x, y, z) = E(x \circ y, z),
   \]
   is a \( U \)-vague binary operation on \( M \) for any \( x, y, z \in M \).
2. If \((M, \circ)\) is a monoid, then \((M, \tilde{\circ})\) is a \( U \)-vague monoid.

Conversely, a monoid \((M, \circ)\) can be obtained from a vague monoid \((M, \tilde{\circ})\).
Proposition 39 Let \((M, \tilde{\circ})\) be a vague monoid with respect to a \(U\)-indistinguishability operator \(E\) separating points. Then \((M, \circ)\) is a monoid where \(x \circ y = z\) if and only if \(\tilde{\circ}(x, y, z) = 1\).

Definition 38 Let \((M, \tilde{\circ})\) be a vague monoid with respect to a \(U\)-indistinguishability operator \(E\) separating points. Then \((M, \circ)\) is the monoid associated to the vague monoid \((M, \tilde{\circ})\).

Further, if \(U\) is a uninorm and \((M, \circ)\) is a monoid, then there exist bijective maps between their \(U\)-vague monoids and regular \(U\)-indistinguishability operators such that they can represent each other:

- \(\tilde{\circ}(x, y, z) = E(x \circ y, z)\);
- \(E(x, y) = \tilde{\circ}(x, e, y)\).

Regarding commutativity, the connection between \((M, \tilde{\circ})\) and \((M, \circ)\) is given below.

Proposition 40 Let \(E\) be a \(U\)-indistinguishability operator separating points, \((M, \tilde{\circ})\) a \(U\)-vague operation and \((M, \circ)\) its associated operation \(x \circ y = z\) if and only if \(\tilde{\circ}(x, y, z) = 1\). Then \((M, \tilde{\circ})\) is commutative if and only if \((M, \circ)\) is commutative.

4.2.2 Nullnorm vague monoid

Definition 39 Let \(F\) be a nullnorm and \(X\) a set. A fuzzy relation \(E : X \times X \rightarrow [0, 1]\) is an \(F\)-indistinguishability operator, which satisfies the following conditions for all \(x, y, z \in X\):

1. reflexivity: \(E(x, x) = 1\);
2. symmetry: \(E(x, y) = E(y, x)\);
3. \(F\)-transitivity: \(F(E(x, y), E(y, z)) \leq E(x, z)\).

If \(E(x, y) = 1\) implies \(x = y\), then it is said that \(E\) separates points.

Definition 40 Let \(E\) be an \(F\)-indistinguishability operator on \(M\). An \(F\)-vague binary operation \(\tilde{\circ} : M \times M \times M \rightarrow [0, 1]\) is a fuzzy binary operation on \(M\) when it satisfies the following conditions for all \(x, y, z, x', y', z' \in M\):

1. \(F(\tilde{\circ}(x, y, z), E(x, x'), E(y, y'), E(z, z')) \leq \tilde{\circ}(x', y', z')\);
2. \(F(\tilde{\circ}(x, y, z), \tilde{\circ}(x, y, z')) \leq E(z, z')\);
3. for all \(x, y \in M\) there exists \(z \in M\) such that \(\tilde{\circ}(x, y, z) = 1\).

Proposition 41 Let \(E\) be an \(F\)-indistinguishability operator on \(M\) separating points and \(\tilde{\circ}\) a vague binary operation on \(M\). Then the \(z\) of Definition 26 is unique.

Definition 41 Let \(\tilde{\circ}\) be an \(F\)-vague binary operation on \(M\) with respect to an \(F\)-indistinguishability operator \(E\) on \(M\). Then \((M, \tilde{\circ})\) is an \(F\)-vague monoid if and only if it satisfies the following properties.

1. Associativity:
   \[F(\tilde{\circ}(y, z, d), \tilde{\circ}(x, d, m), \tilde{\circ}(x, y, q), \tilde{\circ}(q, z, w)) \leq E(m, w),\]
   where \(x, y, z, d, m, q, w \in M\).
2. Identity: there exists identity element \(e \in M\) such that
   \[\tilde{\circ}(e, x, x) = 1\] and \(\tilde{\circ}(x, e, x) = 1\) for all \(x \in M\).

Definition 42 An \(F\)-vague monoid is commutative if and only if \(F(\tilde{\circ}(x, y, m), \tilde{\circ}(y, x, w)) \leq E(m, w))\) for any \(x, y, m, w \in M\).

Next, we give a sufficient condition for the uniqueness of identity element of \(F\)-vague monoid \((M, \tilde{\circ})\).

Proposition 42 Let \(F\) be nullnorm, \(E\) an \(F\)-indistinguishability operator separating points on set \(M\) and \((M, \tilde{\circ})\) an \(F\)-vague monoid. Then the identity element is unique.

Definition 43 Let \(\circ\) be a binary operation on \(M\), and \(E\) an \(F\)-indistinguishability operator on \(M\). \(E\) is regular with respect to \(\circ\) if and only if for all \(x, y, z \in M\),

\[E(x, y) \leq E(x \circ z, y \circ z)\]

and

\[E(x, y) \leq E(z \circ x, z \circ y)\]

Under the condition of regular \(F\)-indistinguishability operator, the definition of the fuzzy mapping \(\tilde{\circ}\) can be given and further the \(F\)-vague monoid is obtained.

Proposition 43 Let \(E\) be a regular \(F\)-indistinguishability operator on \(M\) with respect to a binary operation \(\circ\) on \(M\).

1. The fuzzy mapping \(\tilde{\circ} : M \times M \times M \rightarrow [0, 1]\) defined by
   \[\tilde{\circ}(x, y, z) = E(x \circ y, z),\]
   is an \(F\)-vague binary operation on \(M\) for any \(x, y, z \in M\).
2. If \((M, \circ)\) is a monoid, then \((M, \tilde{\circ})\) is an \(F\)-vague monoid.

Conversely, a monoid \((M, \circ)\) can be obtained from a vague monoid on \((M, \tilde{\circ})\).
Proposition 44 Let \((M, \delta)\) be a vague monoid with respect to an \(F\)-indistinguishability operator \(E\) separating points. Then \((M, \circ)\) is a monoid where \(x \circ y = z\) if and only if \(\delta(x, y, z) = 1\).

Definition 44 Let \((M, \delta)\) be a vague monoid with respect to an \(F\)-indistinguishability operator \(E\) separating points. Then \((M, \circ)\) is the monoid associated to the vague monoid \((M, \delta)\).

Further, if \(F\) is a nullnorm and \((M, \circ)\) is a monoid, then there exist bijective maps between their \(F\)-vague monoids and regular \(F\)-indistinguishability operators and they can represent each other:

- \(\delta(x, y, z) = E(x \circ y, z);\)
- \(E(x, y) = \delta(x, e, y).\)

Regarding commutativity, the connection between \((M, \delta)\) and \((M, \circ)\) is given below.

Proposition 45 Let \(E\) be an \(F\)-indistinguishability operator separating points, \((M, \delta)\) an \(F\)-vague operation and \((M, \circ)\) its associated monoid \((x \circ y = z\) if and only if \(\delta(x, y, z) = 1\)). Then \((M, \circ)\) is commutative if and only if \((M, \circ)\) is commutative.

5 Conclusion

Compared to \(t\)-norm, the diversity of aggregation functions gives us more flexibility in dealing with related problems. In this paper, we mainly generalize the fuzzy monoid and vague monoid based on \(t\)-norm to aggregation function, details are shown as follows.

- Notice that \(t\)-norm can be regarded as a special aggregation function, the \(A\)-fuzzy-submonoid is introduced based on aggregation function and related properties are generalized. In the meantime, special aggregation functions such as uninorm and nullnorm are brought in to enrich the scope of our research. Further, the concepts of \(\land\)-fuzzy submonoid and \(\lor\)-fuzzy submonoid are established on bounded lattice.
- On the other end of the spectrum, \(A\)-vague monoid based on aggregation function is proposed, which defined by \(A\)-vague binary operation. In a similar way, the special \(U\)-vague monoid and \(F\)-vague monoid are explored.
- In future works, the fuzzification of monoid with the help of other fuzzy logical operators such as implication (Deschrijver and Kerre 2005; Bandler and Kohout 1980; Mas et al. 2007; Fodor 1991) should be considered.

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