Abstract

Prices generated by automated price experimentation algorithms often display wild fluctuations, leading to unfavorable customer perceptions and violations of individual fairness [1]: e.g., the price seen by a customer can be significantly higher than what was seen by her predecessors, only to fall once again later. To address this concern, we propose demand learning under a monotonicity constraint on the sequence of prices, within the framework of stochastic convex optimization with bandit feedback.

Our main contribution is the design of the first sublinear-regret algorithms for monotonic price experimentation for smooth and strongly concave revenue functions under noisy as well as noiseless bandit feedback. The monotonicity constraint presents a unique challenge: since any increase (or decrease) in the decision-levels is final, an algorithm needs to be cautious in its exploration to avoid over-shooting the optimum. At the same time, minimizing regret requires that progress be made towards the optimum at a sufficient pace. Balancing these two goals is particularly challenging under noisy feedback, where obtaining sufficiently accurate gradient estimates is expensive. Our key innovation is to utilize conservative gradient estimates to adaptively tailor the degree of caution to local gradient information, being aggressive far from the optimum and being increasingly cautious as the prices approach the optimum. Importantly, we show that our algorithms guarantee the same regret rates (up to logarithmic factors) as the best achievable rates of regret without the monotonicity requirement.

1 Introduction

In the recent years, a wide range of industries have begun deploying sophisticated data-driven algorithms for price experimentation to optimize their revenue in the face of unknown and uncertain demand. Although these algorithms enjoy attractive theoretical guarantees in terms of their impact on the firm’s revenue, the impact of their pricing decisions on customer perceptions is often overlooked. A key component of these algorithms is exploration, which typically leads to haphazard price movements that may be perceived unfavorably or even as unfair by customers [2]. Indeed, it may happen that a customer gets offered a price that is higher than that offered to a recent customer, only to have the price fall again after the customer makes the purchase. Thus motivated, we consider the design of data-driven decision-making algorithms for applications such as pricing that output decisions that are monotonic over time.
There are several reasons to study this problem from a fairness standpoint. In particular, there
has been extensive research in behavioral sciences investigating consumer perception of price fairness
[3, 4, 5, 6, 7]. The central idea in these works is that notions of price fairness essentially stem from
comparison: without explicit explanations, customers think they are similar to other customers
buying the same item, and thus should pay equal prices [3, 8]. As a result, consumers are likely to
regard interpersonal price disparity as unfair [6, 7]. The idea that all customers must be treated
similarly is also an instance of a general notion of fairness proposed in the burgeoning literature on
algorithmic fairness, called “Individual Fairness,” which essentially says that “similar” individuals
must receive “similar” decisions under an algorithm [1].

However, it has been shown recently in [9] that imposing individual fairness in data-driven online
decision making may hinder the learning of optimal decisions [1]. This is a trivial observation in the
context of pricing: if all customers must receive the same price or very similar prices across time,
there is no hope for an algorithm to experiment and learn the optimal price. Gupta and Kamble
instead proposes a relaxation of individual fairness called “individual fairness in hindsight,” which
only guarantees that at the time of the decision, an agent’s treatment is no less conducive than that
seen by similar individuals that came before her [9]. In the context of price experimentation, this
notion boils down to the requirement that the prices only decrease over time, thus ensuring that
the price offered to a customer is the lowest price anyone has been offered so far. This notion is
particularly suitable for pricing products whose desirability is expected to diminish over time, which
is the case for most consumer goods such as electronics, apparel, packaged foods, etc. The authors
further show that this relaxation can indeed allow data-driven algorithms to learn optimal decisions
over time in stationary environments [9].

In contrast, in some contexts, it is desirable to ensure that prices only increase over time. This is
the case for sale of items like fine wine or club memberships, whose perceived value is expected to
increase over time (e.g., palatability of wine improves as it ages, clubs become more desirable as
they get more members). In these cases, falling prices may displease customers who bought early at
high prices. This can be seen as a counterpart to individual fairness in hindsight, that may be called
“individual fairness after the decision,” requiring that an agent’s treatment is no less conducive than
that seen by similar individuals who arrive after her [9]. Finally, apart from fairness considerations,
such monotonicity could be demanded by other business constraints: for example, in some contexts,
it may be necessary to ensure that prices only increase over time to avoid the high operational
overhead and the associated revenue loss when customers who bought at high prices are allowed to
return the items to purchase again at a lower price.

The main challenge in our pursuit is the preservation of the attractive theoretical properties of
these algorithms in terms of utility to the firms, while respecting such monotonicity constraints. We
consider algorithm design for one-dimensional decision-making within well-known stochastic convex
optimization framework. In this framework, a decision-maker iteratively makes decisions to minimize
an unknown convex cost function. We assume a bandit feedback model: i.e., the decision-maker
only observes the function-value at the decision it makes, potentially corrupted with some noise.
Under this setup, an algorithm for choosing decisions seeks to minimize regret, which is the difference
between the total cost achieved by the algorithm and the cost yielded by the best fixed decision
in hindsight. Several algorithms have been designed in the literature, which achieve no regret, or
vanishing (average) regret in various settings, i.e., their regret is sublinear in the duration of the
decision-making horizon. However, as we discuss in Section 2, none of these algorithms can be easily
adapted to satisfy monotonicity of decisions over time.

\footnote{In a recent work, Blum and Thodoris show that imposing group notions of fairness also leads to certain impossibilities
in the context of online learning [10].}
We restrict ourselves to settings where the unknown cost function is known to be smooth and strongly convex. In this case, algorithms exist that attain a regret of $O(1)$ in the noiseless bandit feedback model [11], and a regret of $O(\sqrt{T})$ in the noisy bandit feedback model [12], where $T$ is the decision-making horizon. Both these rates are known to be the best attainable [12, 13]. Our main contribution is to design algorithms that achieve these optimal rates of regret, while ensuring the monotonicity of decisions over time.

At their core, our algorithms rely on gradient estimates constructed from monotonic two-point function evaluations (i.e., “secant” information) to improve decisions. Due to the monotonicity constraint, the challenge is to ensure that sufficient progress is continually made towards reaching the optimal decision using gradient information, while avoiding overshooting the optimum (since backtracking is disallowed). The key idea that ensures the latter is to take a conservatively defined gradient step from the “lagged” point of the two points used to construct the gradient estimate. However, as the decisions approach the optimum and the gradient becomes smaller, it becomes increasingly difficult to make progress while moving from a lagged point. This is not a challenge in the noiseless bandit setting, since the “lag” can be as small as desired without hindering the ability to construct accurate gradient estimates. However, with noisy feedback, small lags result in an increased number of samples required to construct sufficiently accurate gradient estimates, resulting in high regret. We tackle this design challenge by decreasing lag sizes adaptively based on local gradient information in anticipation of the optimum solution, while ensuring that this adaptive choice procedure respects monotonicity. Our contributions are summarized in Table 1.

The paper is organized as follows. We discuss relevant related work in Section 2 and problem formulations with useful preliminaries in Section 3. We first consider the noiseless setting with bandit feedback in Section 4 and show $O(1)$ regret bound using our novel algorithm LAGGED GRADIENT DESCENT. In Section 5.1, we next adapt LAGGED GRADIENT DESCENT to the noisy setting so that it moves with estimated secants, but this gives us $O(T^{2/3})$ regret. The analysis of the algorithm is presented to highlight the complexity in balancing three competing objectives: accuracy in gradient estimates, accumulation of regret while gradient is estimated to sufficient accuracy, and exploration regret due to monotonicity of iterates. We next present our novel algorithm, ADAPTIVE LAGGED GRADIENT DESCENT, with the main result of this work: an $\tilde{O}(T^{1/2})$ regret algorithm, in Section 5.2. We conclude with open questions stemming from this work in Section 6.

| Information | Monotonicity | Algorithm          | Bound              | Idea                                 |
|-------------|--------------|--------------------|--------------------|--------------------------------------|
| non-noisy   | no           | Golden-Section     | $O(1)$             | Divide and conquer                    |
|             | yes          | LGD Algorithm [1]  | $O(1)$             | Lagged secant descent                |
| noisy       | no           | Kiefer-Wolfowitz   | $O(T^{1/2})$       | Secant descent                       |
|             | yes          | Static LGD         | $O(T^{2/3})$       | Lagged secant descent with a fixed lag size |
|             | yes          | Adaptive LGD       | $\tilde{O}(T^{1/2})$ | Lagged secant descent with adaptive lag sizes |

Table 1: Best known bounds for minimizing a strongly convex and smooth function over a compact one-dimensional interval $\mathcal{P}$, under the assumption that the true optimum is in $\mathcal{P}$. 
2 Related Work

**Fairness in online learning.** There is growing literature exploring the impact of various fairness-motivated constraints on the performance of sequential decision-making algorithms in uncertain environments \[14\]. In particular, various works have explored the impact of imposing individual fairness notions in the context of learning with bandit feedback \[15, 16, 17, 18, 9, 19\]. The model considered in \[15, 17\] and \[18\], is slightly different from our work, wherein they require individual fairness to be satisfied across agents arriving within a single period of a decision-making horizon and not across periods. These works show that sublinear regret guarantees can be achieved in these settings. The closest in terms of modeling approach to our work is \[9\], which defines the notion of individual fairness in hindsight and shows that this definition, unlike individual fairness, allows algorithms to attain sublinear regret guarantees. However, the analysis in the paper is restricted to cases where the decision-maker tries to distinguish between a finite set of utility models. Whereas, we consider the problem of optimizing an unknown strongly convex and smooth function and hence the algorithm in \[9\] is inapplicable in our setting.

A related notion of fairness is **group fairness**, in which a decision-maker must ensure that certain average performance characteristics of the decisions must be equalized over multiple groups of people. This constraint has been studied in online learning settings where people arrive sequentially in time and must be mapped to decisions; these works have explored the challenges arising in these settings due to adversarial nature of group arrivals \[20\], the requirement of satisfying group fairness across groups with overlapping memberships \[10\], etc. Zhang and Liu offer an excellent survey of many of these recently proposed models and tradeoffs \[14\].

**Stochastic convex optimization.** One-dimensional stochastic convex optimization with bandit feedback is a well-studied problem. In the noiseless bandit feedback setting, Kiefer gave an $O(1)$-regret algorithm, now well known as **Golden-Section Search**, for minimizing a one-dimensional convex function \[11\]. This algorithm iteratively uses three-point function evaluations to “zoom-in” to the optimum, by eliminating a point and sampling a new point in each round. Its mechanics renders it infeasible to implement it in a fashion that respects monotonicity of decisions. For higher dimensions, a $O(1)$-regret algorithm has been designed by Nesterov and Spokoiny \[21\]. This algorithm is based on gradient-descent using a one-point gradient estimate constructed by sampling uniformly in a ball around the current point. This key idea recurringly appears in several works on convex optimization with bandit feedback. However, due to the randomness in the direction chosen to estimate the gradient, such an approach does not satisfy monotonicity.

For the case of noisy bandit feedback, Cope showed a lower bound of $\Omega(T^{1/2})$ in the single-dimensional setting that holds for smooth and strongly convex functions, and showed that an appropriately tuned version of the well-known Kiefer-Wolfowitz \[22\] stochastic approximation algorithm achieves this rate \[12\]. This algorithm uses two-point function evaluations to construct gradient estimates, which are then utilized to perform gradient-descent. Our algorithms are the most related to this approach, where we tackle the significant additional challenge that the two-point evaluations need to consistently satisfy monotonicity over time, while attaining the same regret rate (up to logarithmic factors). As we discussed in Section \[1\] new design tradeoffs arise in tackling this challenge.

Also for the case of noisy bandit feedback, Agrawal et al designed an algorithm that achieves the $\Omega(\sqrt{T})$ bound for any convex function defined over a compact set \[23\]. In the one-dimensional case, their approach is the most related to the golden-section search procedure of Kiefer \[11\], and as such, is infeasible to implement in a monotonic fashion. Yu and Mannor used a similar approach to show that an $\Omega(\sqrt{T})$ bound can be achieved for unimodal functions in the one dimensional setting \[24\]. Auer et al showed that a $O(\sqrt{T})$ regret can be achieved in a one-dimensional non-convex problem.
with a finite number of optima \cite{25}. Their approach relies on discretizing the decision-space and utilizing a finite-armed bandit algorithm – such an approach pulls different “arms” in a haphazard fashion and hence does not satisfy monotonicity over time. Finally, many sub-linear regret algorithms have been designed for more general bandit problems with a continuum of arms, using a related adaptive discretization-then-exploration procedure \cite{26, 27, 28}.

**Online convex optimization.** A more difficult setting is that of online convex optimization with bandit feedback, where the function that a decision-maker faces in each period may change adversarially over time. This area has seen a lot of algorithmic development in the recent years \cite{29, 30, 31, 32}, with the most state of the art algorithms achieving a $\tilde{O}(\sqrt{T})$ regret for minimizing general convex functions in a constrained set \cite{33, 34, 35}; however, none of these algorithms are designed to satisfy monotonicity. We refer the reader to \cite{36, 37} for a survey of these results.

### 3 Problem Formulation and Preliminaries

**Problem Formulation.** Given an underlying unknown smooth and strongly concave revenue curve $\text{rev}(\cdot)$, our goal is to find the optimal price in a feasible set $\mathcal{P} = [p_{\min}, p_{\max}]$ via monotonic price exploration; that is, prices set by the algorithm cannot oscillate over time. For the sake of congruence with the optimization literature, we translate this maximization problem to a minimization problem:

$$\min_{x \in \mathcal{P}} f(x),$$

where $f(\cdot) = -\text{rev}(\cdot)$ is smooth and strongly convex with optimum $x^* = \arg \min_{x \in \mathbb{R}} f(x) \in (p_{\min}, p_{\max})$.

At each time period $t$, the algorithm produces a price $x_t \in \mathcal{P}$ and observes bandit feedback (i.e., feedback regarding the function value). In the noiseless setting (Section 4), the feedback observed is the function value itself, $f(x_t)$. In the noisy setting (Section 5), the feedback observed is a random variable $f(x_t) + \varepsilon_t$, where the random variables $\varepsilon_1, \ldots, \varepsilon_T$ are independent, have mean zero, and have bounded support: $\max_t \text{diam} \left( \text{supp}(\varepsilon_t) \right) \leq 1$. This assumption can be relaxed to allow for any zero-mean sub-Gaussian noise distribution.

This problem is online, in the sense that the decision $x_t$ made at time $t$ is made using information from—and constrained by—the past. In particular, if $Y_s$ is the feedback received regarding $x_s$ at time $s$, then the decision $x_t$ at time $t$ is made using the information $\mathcal{H}_{t-1} = (x_1, Y_1, \ldots, x_{t-1}, Y_{t-1})$, constrained by monotonicity: $x_t \geq \max\{x_s : s < t\}$.

**Preliminaries.** We begin by defining strong convexity and smoothness.

**Definition 1** (\(\alpha\)-strong convexity). Let $D$ be a convex set. A function $f : D \to \mathbb{R}$ is called $\alpha$-strongly convex if

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\alpha}{2} \|y - x\|^2,$$

for all $x, y \in D$.

Strong convexity lower bounds how quickly the gradient can change. Analogously, smoothness upper bounds how quickly the gradient can change:

**Definition 2** (\(\beta\)-smoothness). Let $D$ be a convex set. A function $f : D \to \mathbb{R}$ is called $\beta$-smooth if

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{\beta}{2} \|y - x\|^2,$$

for all $x, y \in D$. 


Below, we present some lemmas which will be used in the analysis of our algorithms. The first lemma states that if \( \nabla f(x) \) and \( \nabla f(y) \) are close, where \( f \) is strongly convex, then \( x \) and \( y \) are close as well. This will allow us to argue that points with small gradient incur small instantaneous regret. We include a proof in Appendix A.

**Lemma 1.** Let \( f : \mathcal{K} \rightarrow \mathbb{R} \) be an \( \alpha \)-strongly convex function. Then

\[
\|y - x\| \leq \frac{1}{\alpha} \|\nabla f(y) - \nabla f(x)\|, \quad \text{for all } x, y \in \mathcal{K}.
\]

As \( \beta \)-smoothness is equivalent to \( \beta \)-Lipschitzness of the gradient (for \( L_2 \) norm), Lemma 1 implies that for an \( \alpha \)-strongly convex and \( \beta \)-smooth function \( f \), \( \alpha \|y - x\| \leq \|\nabla f(y) - \nabla f(x)\| \leq \beta \|y - x\| 
\) for all \( x, y \) in the domain of \( f \).

Next, we state a lemma regarding the zeroth-order estimation of gradients of a smooth and strongly convex function with noisy evaluations. In particular, the lemma states that given enough samples at \( x \) and \( y \), with \( x < y \), one can obtain an upper bound on \( \nabla f(x) \) and a lower bound on \( \nabla f(y) \) with large probability.

**Lemma 2.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be an \( \alpha \)-strongly convex and \( \beta \)-smooth function, and let \( x < y \). Suppose that querying \( f \) at any point \( z \) yields a noisy observation \( f(z) + \varepsilon \), where \( \varepsilon \) is noise, and suppose that the noise variables \( \varepsilon_1, \varepsilon_2, \ldots \) of successive queries are independent, identically distributed, have mean zero, and have bounded support: \( \sup_i \text{diam} \left( \text{supp}(\varepsilon_i) \right) \leq 1 \).

Now fix \( p \in (0, 1) \), and let \( \bar{f}(x), \bar{f}(y) \) be the averages of \( \frac{32\log \frac{3}{\alpha^2(y-x)}}{\alpha^2(y-x)} \) samples at \( x \) and \( y \), respectively. Then the estimated secant \( g = \frac{\bar{f}(y) - \bar{f}(x) + \frac{\alpha(y-x)^2}{y-x}}{y-x} \) satisfies

\[
\nabla f(x) \leq \frac{\bar{f}(y) - \bar{f}(x)}{y-x} \leq g \leq \nabla f(y),
\]

with probability at least \( (1 - p)^2 \).

The proof of Lemma 2 is deferred to Appendix A.

### 4 The Noiseless Case

We first consider algorithms in the noiseless bandit setting: a decision \( x_t \in \mathcal{P} \) is made at each time \( t \in [T] \), and for each decision \( x_t \), the algorithm observes \( f(x_t) \). The decisions made by the algorithm are constrained to be monotone: \( x_1 \leq x_2 \leq \cdots \leq x_T \).

We present a monotonic algorithm (Algorithm 1) for this setting, which is a variation on classical gradient descent. At each round, we use two queries (one at \( x_t \) and one at the “lagged” point \( x'_t = x_t - \delta \)) to estimate the gradient. Since we need to ensure monotonicity of iterates, we need to sample first at the lagged point \( x'_t \) and next at \( x_t \) to get an estimate of the gradient. The size of \( \delta \) will depend on the time horizon \( T \) and be optimized for minimizing the regret of the overall scheme. We then move by an amount proportional to the estimated gradient.

While moving, even in this non-noisy case, we need to be careful about not overshooting the optimum point, which could result in high regret due to the monotonicity constraint. To ensure we do not overshoot, we move proportional to the gradient from the lagged point \( x'_t \) instead of from \( x_t \). Since we jump from \( x'_t \) instead of \( x_t \), a small jump may violate monotonicity. To avoid this, we only jump forward if the gradient is steep enough; in particular, if the magnitude of the estimated gradient is at least \( \beta(1 + \gamma)\delta \), for some constant \( \gamma \), which will be optimized in Theorem 1.
Algorithm 1: Lagged Gradient Descent (noiseless bandit)

**Input:** str. convexity parameter $\alpha$, smth. parameter $\beta$, time horizon $T$, feasible domain $[p_{\text{min}}, p_{\text{max}}]

1. Set $x_1' = p_{\text{min}}$ and $x_1 = x_1' + \delta$
2. Observe $f(x_1')$ and $f(x_1)$
3. for $t = 1, \ldots, T/2$
   4. Let $\tilde{\nabla}_t \leftarrow \frac{f(x_t) - f(x_t')}{x_t - x_t'}$
   5. if $-\frac{1}{\beta} \tilde{\nabla}_t \geq (1 + \gamma)\delta$
      6. Sample $f(\cdot)$ at $x_{t+1}' = x_t' - \frac{1}{\beta} \tilde{\nabla}_t - \delta_{t+1}$
      7. Sample $f(\cdot)$ at $x_{(t+1)}' = x_{t+1}' + \delta_{t+1}$
   8. else
      9. Exit from loop and stabilize at $x_t$
10. return $x_{T/2}$

Since the estimated gradient $\tilde{\nabla}_t$ in Algorithm 1 is less steep than the true gradient at $x_t'$, the smoothness of $f$ allows us to ensure that we never overshoot. Additionally, we show that the convergence rate attained by Algorithm 1 is exponential. Lemma 3 makes these claims precise.

**Lemma 3.** Let $f : \mathcal{P} \to \mathbb{R}$ be an $\alpha$-strongly convex and $\beta$-smooth function. Let $x_1, \ldots, x_{T/2}$ be the points generated by Lagged Gradient Descent (Algorithm 1), and assume that $x_1 \leq x^* = \arg \min_{x \in \mathbb{R}} f(x)$. Then, for $\gamma > 1$, the following hold:

1. Points increase monotonically toward the optimum: $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{T/2} \leq x^*$;
2. The convergence rate is exponential up to halting: $h_{t+1} \leq h_1 \exp \left( -2\alpha c t \right)$, where $c = \frac{1}{2\beta} - \frac{1}{(1 + \gamma)^2}$.

**Proof of Lemma 3.** We begin by proving “1.” To show that we never overshoot, we will exploit smoothness. In particular, for any $t$ such that $x_t \leq x^*$, we have

$$x_{t+1}' - x_t' = -\frac{1}{\beta} \tilde{\nabla}_t$$

$$= -\frac{1}{\beta} \nabla f(x_t)$$

$$= \frac{1}{\beta} \| \nabla f(x_t) \|$$

$$\leq x^* - x_t$$

assuming $\nabla f(x^*) = 0$

$$\leq x^* - x_t'$$.

This proves “1.”

To show the convergence (“2”), first note that by $\alpha$-strong convexity, we have that $f(y) \geq f(x) - \frac{1}{2\alpha} \| \nabla f(x) \|^2$. For $y = x^*$, this becomes

$$\| \nabla f(x) \|^2 \geq 2\alpha \left[ f(x) - f(x^*) \right].$$ (1)
Now we wish to bound the gap \( h_t = f(x_t) - f(x^*) \). For any \( t \geq 2 \), we have

\[
 h_{t+1} - h_t = f(x_{t+1}) - f(x_t) \\
\leq \nabla_t^T(x_{t+1} - x_t) + \frac{\beta}{2}(x_{t+1} - x_t)^2 \\
\leq (\nabla_t^T + \beta\delta)(x_{t+1} - x_t) + \frac{\beta}{2}(x_{t+1} - x_t)^2 \\
= -\frac{1}{2\beta}\|\nabla_t\|^2 - \delta \nabla_t - \frac{\beta}{2}\delta^2 \\
\leq -\frac{1}{2\beta}\|\nabla_t\|^2 - \left(-\frac{\nabla_t}{(1+\gamma)\beta}\right)\nabla_t - \frac{\beta}{2}\delta^2 \\
= -\left(\frac{1}{2\beta} - \frac{1}{(1+\gamma)\beta}\right)\|\nabla_t\|^2 - \frac{\beta}{2}\delta^2 \\
\leq -\left(\frac{1}{2\beta} - \frac{1}{(1+\gamma)\beta}\right)\|\nabla_t\|^2.
\]

By the mean value theorem, there is some \( \bar{x}_t \in [x'_t, x_t] \subset [x_{t-1}, x_t] \) such that \( \nabla f(\bar{x}_t) = \nabla_t \). This allows us to apply (1):

\[
 h_{t+1} - h_t \leq -c\|\nabla_t\|^2 = -c\|\nabla f(\bar{x}_t)\|^2 \\
\leq -2\alpha c[f(\bar{x}_t) - f(x^*)] \\
\leq -2\alpha ch_t.
\]

Note that for \( \gamma > 1 \) as specified in the lemma, \( 2\alpha c \in (0, 1) \). So,

\[
 h_{t+1} \leq (1 - 2\alpha c)h_t \leq \ldots \leq (1 - 2\alpha c)^t h_1 \leq h_1 \exp(-2\alpha ct) .
\]

With this established convergence rate, we can calculate a regret bound for Algorithm 1. We can break the regret of this algorithm into three categories: regret during exploration, regret due to stopping (i.e., regret incurred after the “for loop” has ended), and regret due to potential overshooting (which is 0 by Lemma 3). We balance these three to obtain an \( O(1) \) regret bound.

**Theorem 1.** Assume that \( x^* = \arg \min_{x \in \mathbb{R}} f(x) \in (p_{\min}, p_{\max}) \), and fix lag size \( \delta = T^{-1/2} \) and \( \gamma = 1 + \frac{1}{\log T} \). Then LAGGED GRADIENT DESCENT (Alg. 1) is a \( O(1) \)-regret algorithm for non-contextual dynamic pricing with smooth and strongly concave revenue in the noiseless bandit setting.

**Proof of Theorem 1.** As stated in the theorem, fix \( \delta = T^{-1/2} \) as the lag size. Since overshooting never occurs (by Lemma 3), we need only calculate the exploration and stopping regret. By Lemma 3, the exploration regret is bounded by \( 2 \sum_{t=1}^{\infty} h_1 \exp(-2\alpha c(t - 1)) \), which is constant.

Now we analyze the stopping regret. If the algorithm stops at some time \( t \), then it must be that

\[
 -\frac{1}{\beta} \nabla_t \leq (1 + \gamma)\delta ,
\]
which allows us to bound the gradient:
\[ \|\nabla_t\| \leq \|\tilde{\nabla}_t\| \leq (1 + \gamma)\delta. \] (2)

In other words, if we stop at time \( t \), then \( \|\nabla_t\| \in O(\delta) \), for \( \gamma = 1 + \frac{1}{\log T} \). When \( \|\nabla_t\| \leq 3\delta \), Lemma 1 tells us that \( \|x_t - x^*\| \leq 3\delta/\alpha \), and so \( f(x_t) - f(x^*) \in O(\delta^2) \) by \( \beta \)-smoothness. Hence we get a regret of
\[
\text{regret} \leq 2 \sum_{t=1}^{\infty} h_1 \exp \left( -2\alpha c(t - 1) \right) + \frac{T\delta^2}{\in O(1)}. 
\]

So, we get a regret of \( O(1) \).  

Theorem 1 shows that imposing monotonicity has no (asymptotic) effect on the hardness of the noiseless setting: a non-monotonic constant-regret algorithm was known beforehand [11], and Algorithm 1 is a monotonic constant-regret algorithm.

One important aspect of the noiseless setting is that gradient estimation is easy: not only can it be done using a mere two samples, but the gap \( \delta \) between the samples can be arbitrarily small, resulting in arbitrarily accurate gradients. As we will discuss in the next section, when noise is introduced, there is a trade-off between the size of \( \delta \) and the number of samples required to accurately estimate the gradient. This tension ultimately results in a higher regret bound in the noisy setting.

5 The Noisy Case

In the noisy bandit setting, upon querying the \( t \)th point \( x_t \), the algorithm observes \( f(x_t) + \varepsilon_t \). Recall that \( \varepsilon_1, \ldots, \varepsilon_T \) are independent, identically distributed, mean 0, and satisfy \( \text{diam}(\text{supp}(\varepsilon_t)) \leq 1 \).

In this section, we will present two algorithms for the noisy bandit setting, both based on the Lagged Gradient Descent algorithm (Algorithm 1). Due to the noise in function value observations, jumps are made based on estimated secants (instead of true secants), averaged over numerous samples. In Section 5.1, we present such an algorithm with constant lag sizes (similar to Algorithm 1), and in Section 5.2, we show how adaptive lag sizes yield an improved regret bound.

5.1 Static Lag Sizes in the Noisy Case

We adapt the noiseless Lagged Gradient Descent algorithm (Algorithm 1) to the noisy setting by using an estimate of the slope of the secant line from a lagged point \( x_t' \) to \( x_t \) (instead of the true slope of the secant line). This estimate is obtained by querying the function until the secant can be estimated to a prescribed degree of certainty. When the observed gradient is too small, one may violate monotonicity by taking a further step, so the algorithm cannot proceed further in the same manner. To illustrate how this idea of gradient descent with estimated secants can achieve sublinear regret in the noisy setting, we present Static Lagged Gradient Descent in Algorithm 2.

As with its noiseless counterpart (Algorithm 1), Algorithm 2 achieves an exponential convergence rate up to halting, does not overshoot (with high probability), and is monotonic. These claims are stated precisely in Lemma 4.

Lemma 4. Let \( f : [p_{\min}, p_{\max}] \to \mathbb{R} \) be an \( \alpha \)-strongly convex and \( \beta \)-smooth function, and assume the noise is iid with support of diameter at most 1. Let \( x_1, x_2, \ldots \) be the points generated by Algorithm 2 and assume that for each point \( x \) queried by the algorithm,
\[ |\tilde{f}(x) - f(x)| \leq \frac{\varepsilon}{2}, \] (3)
Algorithm 2: Static Lagged Gradient Descent (noisy bandit)

\begin{algorithm}
\textbf{input:} str. convexity parameter $\alpha$, smth. parameter $\beta$, time horizon $T$, feasible set $\mathcal{P} = [p_{\min}, p_{\max}]$, sample size $n$, function value precision $\varepsilon$, lag size $\delta$, stopping parameter $\gamma > 1$
\begin{algorithmic}[1]
\State $\tilde{f}(x'_1), \tilde{f}(x_1) \leftarrow$ average of $n$ samples at $x'_1 = p_{\min}, x_1 = p_{\min} + \delta$, resp. $f(x'_1), f(x_1)$ \hfill \text{// estimate } f(x'_1), f(x_1)$
\For {$t = 1, \ldots, \lfloor T/2n \rfloor$}
\State Let $\tilde{\nabla}_t \leftarrow \frac{\tilde{f}(x_t)-\tilde{f}(x_s)+\varepsilon}{x_t-x'_s}$ \hfill \text{// compute the approximate secant}
\If {$\frac{1}{\beta} \tilde{\nabla}_t \geq (1 + \gamma) \delta$}
\State $\tilde{f}(x'_{t+1}) \leftarrow$ average of $n$ samples at $x'_{t+1} = x_t - \frac{1}{\beta} \tilde{\nabla}_t - 2\delta$ \hfill \text{// estimate } f(x'_{t+1})$
\State $\tilde{f}(x_{t+1}) \leftarrow$ average of $n$ samples at $x_{t+1} = x'_{t+1} + \delta = x_t - \frac{1}{\beta} \tilde{\nabla}_t - \delta$ \hfill \text{// estimate } f(x_{t+1})$
\Else
\State Exit from loop and set $x_s \leftarrow x_t$ for $s > t$
\EndIf
\EndFor
\end{algorithmic}
\end{algorithm}

for some $\varepsilon \leq \frac{\alpha \beta^2}{4}$, where $\tilde{f}(x)$ is the average of the samples at $x$. Additionally assume that the stopping parameter $\gamma$ satisfies $\gamma > 1$ (as noted in the algorithm).

Then, assuming $x_1 \leq x^* = \arg \min_{x \in \mathbb{R}} f(x) \in (p_{\min}, p_{\max})$, the following hold:

1. Points increase monotonically toward the optimum: $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x^*$;

2. The convergence rate is exponential up to halting: $h_{t+1} \leq h_1 \exp \left(-2\alpha c t \right)$, where $c = \frac{1}{2\beta} - \frac{1}{\beta(1 + \gamma)}$.

\textbf{Proof sketch.} By estimating function values to an error of $\varepsilon$, we obtain a gradient estimate $\tilde{\nabla}_t$ to an error of $\approx \varepsilon/\delta$. By choosing $\varepsilon$ to be small enough (i.e., $\mathcal{O}(\delta^2)$), we can ensure (with high probability) that $\tilde{\nabla}_t \in [\nabla f(x'_t), \nabla f(x_1)]$. Once this is established, the proof is similar to that of Lemma 3.

As with Algorithm 1 we can break down the regret into three categories: exploration regret, regret from stopping, and regret from overshooting. In this noisy setting, there is a tradeoff between the accuracy in gradient estimation and the exploration regret, since each additional query contributes to the regret. There is likewise a tradeoff between the risk of overshooting and the convergence rate, as a more cautious step size reduces the chance of overshooting.

\textbf{Theorem 2.} Assume that $x^* = \arg \min_{x \in \mathbb{R}} f(x) \in (p_{\min}, p_{\max})$ and assume the noise is i.i.d. with mean 0 and support of diameter at most 1. Then the regret of Static Lagged Gradient Descent (Algorithm 2) is $\tilde{\mathcal{O}}(T^{2/3})$ for one-dimensional monotonic minimization of an $\alpha$-strongly convex $\beta$-smooth function over a compact interval.

\textbf{Proof sketch.} Since overshooting is avoided with high probability, we need only consider exploration regret (regret incurred before stopping) and stopping regret (regret incurred after stopping). Since the convergence is exponential, the exploration regret is of order $n$, where $n$ is the number of samples needed at any given point. Since we seek a function-value accuracy of $\varepsilon$, we require $n \in \tilde{\mathcal{O}}(\varepsilon^{-2})$ samples at each of these points by Hoeffding’s inequality. This results in a total exploration regret is of order $\tilde{\mathcal{O}}(\varepsilon^{-2})$.  

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The stopping regret is $O(T^2)$ since $\|\nabla f(x_t)\| \in O(\delta)$ at stopping. In total, the regret is of order $\frac{1}{\varepsilon} + T\delta^2$, where $\varepsilon \in O(\delta^2)$ to ensure good gradient estimates (by Lemma 2). Setting $\delta = \Theta(T^{-1/6})$ and $\varepsilon \in \Theta(T^{-1/3})$, we get the desired bound.

We have thus shown that a monotonic variant of gradient descent with cautious (“lagged”) jumps results in sublinear regret. Algorithm 2, however, is not optimal, and we now discuss how to improve it. The regret of Algorithm 2 comes from three places: exploration, stopping, and overshooting. The regret from overshooting is constant if the function is sampled well enough to obtain accurate gradient estimates with high probability (in particular, obtaining the necessary high probability guarantee increases the number of samples needed by only a logarithmic factor), so we only focus on exploration and stopping regret.

Since the convergence to the optimum is fast, the exploration regret is dominated by the number of samples required to estimate gradients, which is $\tilde{O}(\delta^{-4})$. Additionally, since the algorithm stops jumping when the estimated gradient is of order $\delta$, we incur a stopping regret of order $T\delta^2$. Intuitively, we want to choose a large $\delta$ so that the exploration regret is small, and a small $\delta$ so that the stopping regret is small. Of these two terms, the stopping regret seems to be a key bottleneck in the analysis of Algorithm 2 since, while attaining accurate gradient estimates is important for obtaining exponential convergence, the stopping condition is somewhat artificial.

It seems that we need to find a way to continue to make progress after the stopping condition is met. One possible idea is to switch to a smaller fixed step size $\eta$ and continue moving forward iteratively if the function value at the current point is estimated to be lower than that at the previous point. However, this two-phase approach simply makes the stopping criterion of this second phase the bottleneck, where the stopping regret is now $O(T\eta^2)$. Although overall it does yield a lower regret bound of $\tilde{O}(T^{10/17})$ (see Theorem 4 in Appendix B), it seems that we can do better for two reasons: (a) we are ignoring gradient information, utilizing which can yield faster convergence to optimum, and (b) one need not stop after two phases: such exploration can be allowed to continue in a perpetual sequence of phases. We explore this in the next section.

5.2 Adaptive Lag Sizes in the Noisy Case

As discussed in Section 5.1, one bottleneck in the analysis of Static Lagged Gradient Descent (Algorithm 2) is that early stopping can result in high regret. This observation points to potential improvement: if we run a gradient-based algorithm in phases, adaptively adjusting the stopping criterion to continue to make progress toward the optimum, this could potentially eliminate the stopping regret altogether.

Recall that Algorithms 1 and 2 stop jumping whenever $-\frac{1}{2}g < (1 + \gamma)\delta$, where $g$ is the estimated secant at the current point. This condition ensures that monotonicity is maintained while jumping from the lagged point. So, if the stopping condition has been met and we have not overshot yet, for an appropriate reduction in $\delta$, the algorithm can proceed without breaking monotonicity. In this fashion, by reducing lag sizes when appropriate, we can ensure continual progress toward the optimum, thus eliminating the stopping regret. Keeping in mind the fact that smaller lags require more sampling, a key design challenge is to determine how the lag sizes should be reduced. Additionally, the reduction of the lag size presents new challenges in balancing the two objectives of maintaining monotonicity and avoiding overshooting, as we discuss below.

To preface our proposed algorithm Adaptive Lagged Gradient Descent (Algorithm 3), we first discuss when and how to reduce the lag sizes. Suppose our current lag size is $\delta_i$, and we are currently sampling at $x_t - \delta_i$ (see Figure 1). If we ultimately decide to switch to a smaller lag size, say $\delta_{i+1} = q\delta_i$, then this decision should be made before jumping to $x_t$ (otherwise, monotonicity would prevent sampling at $x_t - \delta_{i+1}$). To get around this, we pre-emptively sample at $x_t - \delta_{i+1}$,
which has the added benefit of providing a gradient estimate at $x_t - \delta_i$. This estimate in turn gives us an estimate of the gradient at $x_t$, which can be used in deciding whether or not the lag size should be reduced. While this process increases exploration regret, the overall regret is reduced to $\mathcal{O}(T^{1/2})$.

In order to explain the high-level ideas of the algorithm, we introduce some notation. In our algorithm there are “non-lagged” iterates, denoted as $(x_t)_{t \in \mathbb{N}}$, and “lagged” iterates denoted in relation to the non-lagged iterates, e.g., $x_t - \delta_i$ for some specified $i$. For any non-lagged iterate $x_t$ such that $x_{t+1} = x_t - \frac{\nabla f_t}{\beta} - \delta_i$, we say that $\delta_i$ is the lag size of $x_t$. Iterates with the same lag size form a phase, and phases are numbered chronologically: Phase 1 is the phase containing $x_1$, Phase 2 is the next phase, and so on. The lag size associated to Phase $i$ is denoted $\delta_{n_i}$. Based on these phases, we can uniquely associate to each $x_t$ a pair $(s, i)$ such that $x_t$ is the $s$th iterate in the $i$th phase; we denote such an iterate as $x_t = y_s^{(i)}$. If $x_t = y_s^{(i)}$ and the gradient estimated using $x_{t+1} - \delta_{n_i}$ and $x_{t+1} - \delta_{n_i+1}$ is not steep enough, then the lag size is reduced to $\delta_{n_i+1}$. Since multiple lag sizes can be skipped between $x_t$ and $x_{t+1}$, it may be the case that $n_{i+1} > n_i + 1$. For example, see Figure 2, where the first jump is taken with a lag size of $\delta_2 = q\delta_1$, and accordingly, $y_1^{(1)} = x_1$. While sampling at $x_{t+1} - \delta_2$ and $x_{t+1} - \delta_3$, since the estimated gradient is not steep enough, the algorithm begins sampling at $x_{t+1} - \delta_4$. The algorithm decreases the lag size twice more before deciding that $\delta_5$ is an appropriate lag size. In this case, $x_{t+1} - \delta_5$ is the “chosen” lagged point, and $x_{t+1} = y_{1}^{(2)}$.

Theorem 3. Assume that $x^* = \arg \min_{x \in \mathbb{R}} f(x) \in (p_{\min}, p_{\max})$, and assume the noise is 0-mean and i.i.d. with support of diameter at most 1. Then Adaptive Lagged Gradient Descent (Algorithm 3) on input of $\delta_1 = 1/\log T$, $\gamma = 1 + \frac{1}{\log T}$, any $q \in (0, 1)$, and $p = T^{-2}$, incurs regret of order $\mathcal{O}((\log T)^2T^{1/2})$.

We will begin the proof of Theorem 3 by showing that all gradient estimates are “good” with high probability, using tail estimates as we did for Theorem 2. With this established, we will show that we achieve exponential convergence to the optimum, iterates are monotonic, and overshooting does not occur (again, with high probability) across all iterates, regardless of lag size. The non-trivial part will be bounding the regret from non-lagged and lagged iterates, which we will address separately. For each source of regret, we will provide a phase-dependent regret bound that is inversely proportional to the local gradient in that phase; this bounds the regret for the early phases. The regret from the later phases is bounded due to improving proximity to the optimum. Balancing regret between early and late phases will allow us to show the $\mathcal{O}(T^{1/2})$ regret bound.

Proof of Theorem 3 We break the proof into several claims. We refer to the estimated gradient at the $\delta_i$-lagged point: $q_t^{(i)} = \frac{f(x_{t-\delta_{i+1}}) - f(x_{t-\delta_i}) + s(\xi_{\delta_i})}{\xi_{\delta_i}}$; and the probability parameter of the algorithm: $p = T^{-2}$.
$p_{\min} = x_1 - \delta_1$

$x_2 = x_1 - \delta_2 - \frac{1}{\beta} \tilde{\nabla}_1(x_1 - \delta_2, x_1)$

$x_3 = x_2 - \delta_5 - \frac{1}{\beta} \tilde{\nabla}_2(x_2 - \delta_5, x_2)$

$x_4 = x_3 - \delta_5 - \frac{1}{\beta} \tilde{\nabla}_3(x_3 - \delta_5, x_3)$

$x_5 = x_4 - \delta_8 - \frac{1}{\beta} \tilde{\nabla}_4(x_4 - \delta_8, x_4)$

Figure 2: Illustration of the points: the algorithm starts exploring at $p_{\min} = x_1 - \delta_1$ followed by $x_1 - \delta_2$. Following our notation of phases, in this example, we see that $x_1 = y_1^{(1)}$, $x_2 = y_1^{(2)}$, $x_3 = y_2^{(2)}$, $x_4 = y_1^{(3)}$, and $x_5 = y_1^{(4)}$. This means that Phase 1 consists of $x_1$, Phase 2 consists of $x_2$ and $x_3$, Phase 3 consists of $x_4$, and Phase 4 consists of $x_5$; the indices of the phases are $n_1 = 2$, $n_2 = 5$, $n_3 = 8$, and $n_4 = 9$. If the gradient condition is met using samples at $x_t - \delta_i$ and $x_t - \delta_{i+1}$, then no more samples are collected and the algorithm moves to $x_{t+1}$. Note that the point $x_{t+1}$ is computed when the algorithm reaches $x_t$, but the algorithm moves to $x_{t+1}$ slowly by exploring intermediate $\{x_{t+1} - \delta_i\}$ points. The computation of $x_{t+1}$ is equivalent to movement by approximate gradient from the chosen lagged point, as depicted by the dotted lines, using the estimate $\tilde{\nabla}_i(x_t - \delta_i, x_t)$ obtained by sampling at $x_t - \delta_i$ and $x_t$ as depicted in the figure. We first bound the regret of the iterates in the shaded gray boxes, followed by the regret of the remaining iterates, in Claim 5.
Algorithm 3: Adaptive Lagged Gradient Descent (noisy bandit)

\textbf{input:} str. convexity parameter \( \alpha \), smth. parameter \( \beta \), time horizon \( T \), feasible set \( \mathcal{P} = [p_{\text{min}}, p_{\text{max}}] \), initial lag size \( \delta_1 \), stopping parameter \( \gamma > 1 \), \( q \in (0, 1) \)

1. \( \delta_i \leftarrow q^{-1}\delta_1 \) for \( i \geq 2 \)
2. \( x_1 \leftarrow p_{\text{min}} + \delta_0, t \leftarrow 1, i \leftarrow 0 \)
3. \textbf{repeat}
   4. \( i \leftarrow i - 1 \)
   5. \textbf{repeat}
      6. \( \hat{f}(x_t - \delta_i) \leftarrow \text{average of} \ \frac{2\log^2 \frac{\epsilon}{\delta_i}}{2\epsilon^2} \ \text{samples at} \ x_t - \delta_i \quad \text{// estimate} \ f(x_t - \delta_i) \)
      7. \( \tilde{f}(x_t - \delta_{i+1}) \leftarrow \text{average of} \ \frac{2\log^2 \frac{\epsilon}{\delta_i}}{2\epsilon^2} \ \text{samples at} \ x_t - \delta_{i+1} \quad \text{// estimate} \ f(x_t - \delta_{i+1}) \)
      8. \( g_t^{(i)} \leftarrow \frac{\tilde{f}(x_t - \delta_{i+1}) - \tilde{f}(x_t - \delta_i) + \epsilon(\delta_i)}{\epsilon(\delta_i)} \quad \text{// compute the approximate secant} \)
   9. \textbf{until} \( -\frac{1}{\beta} g_t^{(i)} \geq (2 + \gamma)\delta_i \)
10. \( \bar{f}(x_t) \leftarrow \text{average of} \ \frac{2\log^2 \frac{\epsilon}{\delta_i}}{2\epsilon^2} \ \text{samples at} \ x_t \quad \text{// estimate} \ f(x_t) \)
11. \( \bar{\nabla}_t \leftarrow \frac{\bar{f}(x_t) - \bar{f}(x_t - \delta_i) + \epsilon(\delta_i)}{\delta_i} \quad \text{// compute the approximate secant} \)
12. Compute \( x_{t+1} \leftarrow x_t - \frac{1}{\beta} \bar{\nabla}_t - \delta_i \)
13. \( t \leftarrow t + 1 \)
14. \textbf{until} \( T \) samples have been taken

Claim 1 (gradient accuracy). Let \( \bar{\nabla}_t = \frac{\bar{f}(x_t) - \bar{f}(x_t - \delta_i) + \epsilon(\delta_i)}{\delta_i} \) be the estimated secant at epoch \( t \), and let \( g_t^{(i)} = \frac{\tilde{f}(x_t - \delta_{i+1}) - \tilde{f}(x_t - \delta_i) + \epsilon(\delta_i)}{\epsilon(\delta_i)} \), where \( \delta_i = q^{-1}\delta_1 \) is the \( i \)th lag size and \( \xi = 1 - q \). Then these gradient estimates of the algorithm satisfy

\[
\bar{\nabla}_t \in [\nabla f(x_t - \delta_i), \nabla f(x_t)] \quad \text{and} \quad g_t^{(i)} \in [\nabla f(x_t - \delta_i), \nabla f(x_t - \delta_{i+1})].
\]

(4) each with probability at least \( (1 - p)^2 \), where \( p = T^{-2} \).

Note that for this choice of \( p \), we have that \( (1 - p)^T \to 1 \). This claim follows immediately from Lemma 2 since each of the estimates is constructed by sampling \( \frac{2\log^2 \frac{\epsilon}{d}}{\epsilon(d)^2} \) times, where \( \epsilon(d) = \alpha d^2/4 \) and \( d \) is the gap between the two points at which we are sampling (i.e., \( d = \delta_i \) in the case of \( \bar{\nabla}_t \) and \( d = \xi\delta_i \) in the case of \( g_t^{(i)} \)).

Claim 2 (overshooting). Assuming (4) holds for all estimated gradients\(^2\) and that \( x_1 = p_{\text{min}} + \delta_1 < x^* \), then all the iterates \( x_1, x_2, \ldots, x_k \) (for \( k \leq T \)) generated by the lagged secant movements in the outer loop of the algorithm do not overshoot the optimum; that is, \( x_t \leq x^* \) for all \( t \leq k \). Note that this implies that all lagged points sampled by the algorithm (i.e., those sampled between \( x_t \) and \( x_{t+1} \), for some \( t \)) also do not overshoot.

\(^2\)This happens with probability at least \( (1 - p)^T \).
PROOF OF CLAIM 3. We show this by induction, where the base case follows from assumption that \(x_1 \leq x^*\). Suppose \(x_t \leq x^*\), and that \(x_{t+1}\) was chosen based on a lag size of \(\delta_i\). In other words, \(x_{t+1} = x_t - \frac{1}{\beta} \tilde{\nabla} - \delta_i\), where \(\tilde{\nabla}_t = \overline{f(x_t)} - \overline{f(x_t - \delta_i)} + \varepsilon_\delta\) and \(\varepsilon_\delta = \alpha \delta^2 / 4\). By (4) and the mean value theorem, \(\tilde{\nabla}_t = \nabla f(\overline{x}_t)\) for some \(\overline{x}_t \in [x_t - \delta_i, x_t]\). So,

\[
x_{t+1} - (x_t - \delta_i) = -\frac{1}{\beta} \tilde{\nabla}_t = -\frac{1}{\beta} \nabla f(\overline{x}_t)
\]

for some \(\overline{x}_t \in [x_t - \delta_i, x_t]\). Since \(x_t \leq x^*\),

\[
\geq x^* - \overline{x}_t
\]

since \(\nabla f(x^*) = 0\) and \(f\) is smooth.

This proves Claim 3. \(\square\)

Claim 3 (monotonicity). Assuming (4) holds for all estimated gradients, and that \(x_1 < x^*\), samples taken by the algorithm (including lagged and non-lagged iterates) are non-decreasing.

PROOF OF CLAIM 3. Again, suppose that \(x_{t+1}\) was chosen based on a lag size of \(\delta_i\). In other words, \(x_{t+1} = x_t - \frac{1}{\beta} \tilde{\nabla} - \delta_i\), where \(\tilde{\nabla}_t = \overline{f(x_t)} - \overline{f(x_t - \delta_i)} + \varepsilon_\delta\) and \(g^{(i)}_t = \overline{f(x_t - \delta_i)} - \overline{f(x_t - \delta_i) + \varepsilon_\delta}\), where \(\delta_i = q^{-1} \delta_1\) is the \(i\)th lag size and \(\xi = 1 - q\). Note that \(\xi \delta_i = (x_t - \delta_{i+1}) - (x_t - \delta_i)\) is the domain gap between \(x_t - \delta_i\) and \(x_t - \delta_{i+1}\). To show monotonicity, we need to show that the next lagged point exceeds the current point; i.e., we must show that \(x_t \leq x_{t+1} - \delta_i\).

Note that the lagged step is taken only for the first \(\delta_i\) that achieves \(-\frac{1}{\beta} g^{(i)}_t \geq (2 + \gamma)\delta_i\). So we have that:

\[
-\frac{1}{\beta} \tilde{\nabla}_t \geq -\frac{1}{\beta} \nabla f(x_t)
\]

by grad. bounds (4)

\[
\geq -\frac{1}{\beta} (\nabla f(x_t - \delta_i) + \beta \delta_i)
\]

by \(\beta\)-smoothness

\[
\geq -\frac{1}{\beta} (g^{(i)}_t + \beta \delta_i)
\]

by grad. bounds (4)

\[
\geq (1 + \gamma)\delta_i
\]

by assumption.

Since \(\gamma > 1\) by assumption, we have that \((x_{t+1} - \delta_i) - x_t = -\frac{1}{\beta} \tilde{\nabla}_t - 2\delta_i \geq (1 + \gamma)\delta_i - 2\delta_i > 0\). In other words, we do not break monotonicity. This proves Claim 3. \(\square\)

Claim 4 (convergence rate). Assume (4) holds for all estimated gradients, and that \(x_1 < x^*\). Now let \(h_t = f(x_t) - f(x^*)\) be the instantaneous regret at \(x_t\). Then \(h_{t+1} \leq h_t \exp(-2\alpha c t)\), where \(c = \frac{1}{2\beta^2} - \frac{1}{(1 + \gamma)^2}\). In particular, letting \(h^{(i)}_t = f(y^{(i)}_t) - f(x^*)\) be the instantaneous regret at the \(i\)th point of the \(i\)th phase, we have that

\[
h^{(i)}_{t+1} \leq h^{(i)}_t \exp(-2\alpha c i) \quad (\text{Phase } i \text{ convergence}),
\]

when at least \(t + 1\) distinct, non-lagged points are sampled in Phase \(i\). Moreover, across non-trivial phases, we get a cumulative contraction:

\[
h^{(i)}_t \leq h^{(i)}_1 \exp(-2\alpha c k_i) \quad \text{when } k_i \text{ distinct non-lagged points are sampled up to Phase } i.
\]
Consequently, since each phase contains at least one non-lagged point,

\[ h_1^{(i)} \leq h_1^{(1)} \exp \left( -2\alpha c (i - 1) \right) \quad \text{(inter-phase convergence)}. \] (7)

**Proof of Claim 4.** To show the exponential convergence rate in Claim 4, we will show that the improvement \( h_t - h_{t+1} \) at time \( t \) is of order at least (approximately) \( |\nabla f(x_t)|^2 \). This will imply, by the strong convexity assumption, a contraction in \( h_t \), which gives us an exponential rate of convergence.

To that end, suppose we jump from \( x_t \) using a lag size of \( \delta_i \); in other words, suppose \( x_{t+1} = x_t - \frac{1}{\beta} \nabla f(x_t) - \delta_i \). Then the following holds:

\[ h_{t+1} - h_t = f(x_{t+1}) - f(x_t) \] (8)
\[ \leq \nabla_t^\top (x_{t+1} - x_t) + \frac{\beta}{2} (x_{t+1} - x_t)^2 \quad \text{\( \beta \)-smooth} \] (9)
\[ \leq (\nabla_t^\top + \beta \delta_i) (x_{t+1} - x_t) + \frac{\beta}{2} (x_{t+1} - x_t)^2 \quad \text{by (4)} \] (10)
\[ = (\nabla_t^\top + \beta \delta_i) (-\nabla_t - \delta_i) + \frac{\beta}{2} (-\nabla_t - \delta_i)^2 \] (11)
\[ = -\frac{1}{2\beta} \|\nabla_t\|^2 - \delta_i \nabla_t - \frac{\beta \delta_i^2}{2} \] (12)
\[ \leq -\frac{1}{2\beta} \|\nabla_t\|^2 - \left( -\frac{\nabla_t}{(1 + \gamma)\beta} \right) \nabla_t - \frac{\beta \delta_i^2}{2} \] (13)
\[ = -\left( \frac{1}{2\beta} - \frac{1}{(1 + \gamma)\beta} \right) \|\nabla_t\|^2 - \frac{\beta \delta_i^2}{2} \] (14)
\[ \leq -\left( \frac{1}{2\beta} - \frac{1}{(1 + \gamma)\beta} \right) \|\nabla_t\|^2. \] (15)

By (4) and the mean value theorem, there is some \( \bar{x}_t \in [x_t - \delta_i, x_t] \subset [x_{t-1}, x_t] \) such that \( \nabla f(\bar{x}_t) = \nabla_t \). This allows us to apply (1):

\[ h_{t+1} - h_t \leq -c \|\nabla_t\|^2 = -c \|\nabla f(\bar{x}_t)\|^2 \]
\[ \leq -2\alpha c [f(\bar{x}_t) - f(x^*)] \quad \text{by strong convexity inequality (1)} \]
\[ \leq -2\alpha c h_t \quad \text{since we do not overshoot.} \] (16)

Note that for \( \gamma > 1 \) (as specified in the algorithm), \( 2\alpha c = \frac{2\alpha}{2\beta} - \frac{2\alpha}{(1 + \gamma)\beta} \leq \frac{\alpha}{\beta} \in (0, 1) \) (since \( \alpha \leq \beta \)). So,

\[ h_{t+1} \leq (1 - 2\alpha c) h_t \leq \cdots \leq (1 - 2\alpha c)^t h_1 \]
\[ \leq h_1 \exp \left( -2\alpha c t \right). \]

The contractions in (5)-(6) follow immediately, and (7) follows from the fact that if \( y_1^{(i)} = x_t \) and \( y_1^{(i+1)} = x_s \), then \( t < s \). \( \square \) **Claim 4**
Claim 5 (regret from gradient descent jumps). Assume \( \delta \) holds for all estimated gradients, and that \( x_1 < x^* \). Then, letting regret\(_{\text{LGD}}\)(\(k\)) be the total regret incurred during the first \( k \) phases on the points \( x_t \), as well as the two lagged points just after \( x_t \) (see the boxed points in Figure 2), i.e.:

- the \( \frac{32 \log 2}{\alpha^2 \delta_{n_i}^4} \) samples at \( x_s = y_t^{(i)} \),
- the \( \frac{32 \log 2}{\alpha^2 \xi^4 \delta_{n_i}^4} \) samples at \( x_{s+1} - \delta_{n_i} \), and
- the first \( \frac{32 \log 2}{\alpha^2 \xi^4 \delta_{n_i}^4} \) samples at \( x_{s+1} - \delta_{n_i+1} \),

for phases \( i \in [k] \) (for any \( k \)) and their respective \( t^{th} \) points, then regret\(_{\text{LGD}}\)(\(k\)) during the first \( k \) non-trivial phases is of order \( O \left( \frac{h \log \frac{1}{\alpha c}}{\delta_1} + \log \frac{1}{\rho} \sum_{i=1}^{k} \frac{1}{|\nabla f(y_{i+1})|^2} \right) \).

**Proof of Claim 5.** As we progress toward the optimum, we potentially use smaller and smaller lag sizes. This results in cumbersome sampling when close to the optimum. To prove Claim 5, we will use the exponential decay of instantaneous regret to allay some of this regret. Additionally, we will show that the instantaneous regret during Phase \( i \) is of order \( \delta_{n_i}^2 \); since the gradient at \( y_t^{(i)} \) is of order \( \delta_{n_i} \), this fact will ultimately allow us to bound regret\(_{\text{LGD}}\) in terms of gradients.

Recall that we are bounding regret incurred during the first \( k \) phases only, as stated in the claim. Let \( q \) be the contraction in the lag sizes, i.e., \( \delta_{i+1} = q \delta_i \). We will bound the regret at \( x_t, x_{t+1} - \delta_i, x_{t+1} - \delta_{i+1} \); among these three points, the instantaneous regret is highest at \( x_t \), so we simply bound the instantaneous regret at each of these points by \( h_t \). For any \( i \), let \( T^{(i)} \) be the index of the last point in Phase \( i \); i.e., \( y_{T^{(i)}}^{(i)} \) is the last point in Phase \( i \). The number of samples at each of these three points is bounded by \( \frac{32 \log 2}{\alpha^2 \delta_{n_i}^4} \), so this in turn gives us the following upper bound, where \( h = f(p_{\text{min}}) - f(x^*) \):

\[
\text{regret}_{\text{LGD}} \leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta_{1}^4} h + \sum_{i=1}^{k} \frac{96 \log 2}{\alpha^2 \delta_{n_i}^4} \sum_{t=1}^{T^{(i)}} h_t^{(i)} 
\]

\[
= \frac{64 \log 2}{\alpha^2 \xi^4 \delta_{1}^4} h + \sum_{i=1}^{k} \frac{96 \log 2}{\alpha^2 \delta_{n_i}^4} \sum_{t=1}^{T^{(i)}} h_t^{(i)} 
\]

\[
= \frac{64 \log 2}{\alpha^2 \xi^4 \delta_{1}^4} h + \frac{96 \log 2}{\alpha^2 \delta_{1}^4} \sum_{i=1}^{k} \frac{1}{q^m_i} \sum_{t=1}^{T^{(i)}} h_t^{(i)} 
\]

\[
\leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta_{1}^4} h + \frac{96 \log 2}{\alpha^2 \delta_{1}^4} \sum_{i=1}^{k} \frac{1}{q^m_i} \sum_{t=1}^{T^{(i)}} h_t^{(i)} \exp \left( -2\alpha c(t - 1) \right) \quad \text{by (5)}
\]

\[
= \frac{64 \log 2}{\alpha^2 \xi^4 \delta_{1}^4} h + \frac{96 \log 2}{\alpha^2 \delta_{1}^4} \sum_{i=1}^{k} \frac{1}{q^m_i} \sum_{t=1}^{T^{(i)}} h_t^{(i)} \exp \left( -2\alpha c(t - 1) \right) 
\]

\[
\leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta_{1}^4} h + \frac{96 \log 2}{\alpha^2 \delta_{1}^4} \sum_{i=1}^{k} \frac{1}{q^m_i} h_t^{(i)} .
\]
To continue this analysis, we bound $h^{(i)}_1$. To that end note that $|\nabla f(y^{(i)}_1)| < \beta(2 + \gamma)\delta^{-1}$, since a previous secant $g$ must have satisfied $-\beta g < (2 + \gamma)\delta^{-1}$. So, assuming the gradient $g$

$$
\beta(2 + \gamma)\delta^{-1} > |\nabla f(y^{(i)}_1)| = |\nabla f(y^{(i)}_1) - \nabla f(x^*)| \geq \alpha|x^* - y^{(i)}_1|,
$$

where (a) follows from the fact that $|\nabla f(y^{(i)}_1)| < |\nabla f(y^{(i)}_1 - \delta_j)| \leq |g|$ for some $j$ (by (41)), and (b) follows from strong-convexity. Finally, by smoothness, we have

$$
h^{(i)}_1 \leq \beta 2(x^* - y^{(ni)})^2 \leq \frac{\beta \gamma}{2}(2 + \gamma)^2\delta^{-2}.
$$

again where (c) follows from smoothness (assuming $\nabla f(x^*) = 0$).

Continuing the analysis from above, we have

$$
\text{regret}_{LGD} \leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta^4} h + \frac{96 \log 2}{\alpha^2 \xi^4 \delta^4 (1 - e^{-2\alpha c})^2} \sum_{i=1}^{k} \frac{1}{q^{4i}} h^{(i)}_1 \leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta^4} h + \frac{48 \beta \gamma^2 (2 + \gamma)^2 \log 2}{\alpha^4 \xi^4 \delta^2} \sum_{i=1}^{k} \frac{\delta^2}{q^{4i}} = \frac{64 \log 2}{\alpha^2 \xi^4 \delta^4} h + \frac{48 \beta \gamma^2 (2 + \gamma)^2 \log 2}{\alpha^4 \xi^4 \delta^2} \sum_{i=1}^{k} \frac{1}{q^{2ni}}.
$$

At this point, for convenience, we will replace $\frac{1}{q^{2ni}}$ with a gradient estimate. In particular, let $x_{t_i} = y^{(i+1)}_1$ be the iterate at which we jump using $\delta_{n_{i+1}}$, for the first time. Then it must be that $|\nabla f(x_{t_i})| < \beta(2 + \gamma)\delta_{n_i}$, or in other words, $q^{ni} > \frac{|\nabla f(x_{t_i})|}{\beta(2 + \gamma)\delta_{n_i}}$. So, continuing the inequalities above,

$$
\text{regret}_{LGD} \leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta^4} h + \frac{48 \beta \gamma^2 (2 + \gamma)^2 \log 2}{\alpha^4 \xi^4 \delta^2} \sum_{i=1}^{k} \frac{1}{q^{2ni}} \leq \frac{64 \log 2}{\alpha^2 \xi^4 \delta^4} h + \frac{48 \beta \gamma^2 (2 + \gamma)^4 \log 2}{\alpha^4 \xi^4} \sum_{i=1}^{k} \frac{1}{|\nabla f(x_{t_i})|^2}
$$

Claim 6 (regret from transitioning lag sizes). Assume [4] holds for all estimated gradients, and that $x_1 < x^*$. Here we bound the regret which is unaccounted for in Claim 5, i.e., we bound the regret incurred at exploratory lagged points. More precisely, let regret_{\delta, transition}(k) be the regret incurred from sampling at $y^{(i+1)}_1 - \delta_{n_{i+1}}, \ldots, y^{(i+1)}_1 - \delta_{n_{i+1}}$ for $1 \leq i \leq k$. Then, regret_{\delta, transition}(k) \in O\left(\log \frac{1}{\sum_{i=1}^{k} |\nabla f(y^{(i+1)}_1)|^2}\right), for any $k \leq T$.

Proof of Claim 6. Here we bound the regret incurred while transitioning lag sizes (e.g., the regret from the samples taken at points which do not appear in boxes in Figure 2). In order to bound the regret resulting from \delta-transitions, we argue that whenever we transition from phase $n_i$, you can provide the specific number of transitions or any other relevant details if needed.
Above by (31) holds. This allows us to bound the number of lagged points sampled during any transitions from \( \delta_{n_i} \) to \( \delta_{n_i+1} \), which in turn gives us a bound on the total regret resulting from these transitions.

Let \( t_1, \ldots, t_k \) be such that \( x_{t_i} = y^{(i+1)}_1 \). To bound the regret from transitioning lag sizes, let us consider the regret incurred transitioning from \( \delta_n \) to \( \delta_{n+1} \), which happens at time \( t_i \) (note that \( n_i \) and \( n_{i+1} \) are not consecutive if the algorithm goes down multiple lag sizes during the same round). The regret incurred from transitioning to \( \delta_{n+1} \) is the regret incurred at \( x_{t_i} - \delta_{n+1} \), \( x_{t_i} - \delta_{n+2} \), \ldots \( x_{t_i} - \delta_{n+1+1} \) (see Figure 3). Let us begin by bounding the number of lag sizes that can be passed in one round. Suppose we sample at \( x_{t_i} - \delta_{n+i} \) and decide to decrease the lag size to \( \delta_{n+i} \). In other words, we observe that

\[
\frac{1}{\beta} \cdot \frac{f(x_{t_i} - \delta_{n+i}) - f(x_{t_i} - \delta_{n+i+1}) + \epsilon(\xi \delta_{n+i+1})}{\xi \delta_{n+i+1}} < (2 + \gamma) \delta_{n+i+1} \tag{31}
\]

and therefore need to begin sampling at \( x_{t_i} - \delta_{n+i+1} \). In this case, observe that

\[
|\nabla f(x_{t_i})| \leq |g| \leq \beta(2 + \gamma) \delta_{n+i+1} \leq \beta(2 + \gamma) q^{-1} \delta_{n_i}.
\]

It follows that for \( J_i = \max \left\{ 2, \left\lfloor \frac{1}{\log q} \log \left( \frac{|\nabla f(x_{t_i})|}{\beta(2 + \gamma) \delta_{n_i}} \right) + 1 \right\rfloor \right\} \), we will not transition; i.e., \( n_{i+1} < n_i + J_i \) and we will not sample at \( x_{t_i} - \delta_{n_i + J_i} \).

Next, we will bound the instantaneous regret incurred by sampling at \( x_{t_i} - \delta_{n+i} \). Suppose we have just transitioned to \( \delta_{n+i} \), and must now sample at \( x_{t_i} - \delta_{n+i} \) and \( x_{t_i} - \delta_{n+i+1} \) in order to determine whether or not to reduce the lag size further. Since we have just transitioned to \( \delta_{n+i} \), we know that (31) holds. This in turn implies that

\[
|\nabla f(x_{t_i} - \delta_{n+i})| \leq |g| < \beta(2 + \gamma) q^{-1} \delta_{n_i}.
\]

So, the regret we get from each sample (whether at \( x_{t_i} - \delta_{n+i} \) or \( x_{t_i} - \delta_{n+i+1} \)), is bounded above by

\[
f(x_{t_i} - \delta_{n+i}) - f(x^*) \leq \frac{\beta}{2\alpha} \left[ \beta(2 + \gamma) q^{-1} \delta_{n_i} \right]^2 = \frac{\beta^3(2 + \gamma)^2 q^{2(j-1)} \delta_{n_i}^2}{2\alpha} \tag{32}
\]
We can now calculate our regret from lag transitions as follows:

\[
\text{regret}_{\delta}\text{-transition} \leq \sum_{i=1}^{k} \sum_{j=1}^{J_i} \frac{64 \log \frac{2}{p}}{\alpha^2 \xi \delta^4_{n_i+j}} \cdot \left[ f(x_{t_i} - \delta_{n_i+j}) - f(x^*) \right] \tag{33}
\]

\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{J_i} \frac{64 \log \frac{2}{p}}{\alpha^2 \xi \delta^4_{n_i+j}} \cdot \frac{\beta^3 (2 + \gamma)^2 q^{2(\gamma-1)} \delta^2_{n_i}}{2\alpha} \quad \text{by (32)} \tag{34}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \sum_{j=1}^{J_i} \frac{1}{\delta^2_{n_i}} \sum_{j=1}^{J_i-j} \frac{1}{q^{2j}} \tag{35}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} \frac{1}{q^{2(J_i+1)} - 1} \tag{36}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} \frac{1}{q^{2(J_i-1)} - 1} \tag{37}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} e^{(\log q - 2)(J_i-2)} \tag{38}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} e^{(\log q - 2)(\frac{1}{\log q} \log (\frac{|\nabla f(x_{t_i})|}{\sqrt{f(x_{t_i})}}) - 1)} \tag{39}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} e^{(\log q - 2)(\frac{1}{\log q} \log (\frac{|\nabla f(x_{t_i})|}{\sqrt{f(x_{t_i})}}))} \tag{40}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} e^{(\log q - 2)(\frac{1}{\log q} \log (\frac{|\nabla f(x_{t_i})|}{f(x_{t_i})^{\frac{1}{2}}}))} \tag{41}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} e^{(\log q - 2)2 \log (\frac{\beta(2 + \gamma)\delta_{n_i}}{\sqrt{f(x_{t_i})}})} \tag{42}
\]

\[
= \frac{32 \beta^3 (2 + \gamma)^2 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} \left( \frac{\beta(2 + \gamma)\delta_{n_i}}{|\nabla f(x_{t_i})|} \right)^2 \tag{43}
\]

\[
= \frac{32 \beta^5 (2 + \gamma)^4 \log \frac{2}{p}}{\alpha^3 \xi^4 q^2} \sum_{i=1}^{k} \frac{1}{\delta^2_{n_i}} \left( \frac{1}{|\nabla f(x_{t_i})|} \right)^2. \tag{44}
\]
Claim 7 (total regret). Assume \( \frac{2}{\alpha} \) holds for all estimated gradients, and that \( x_1 < x^* \). Then the regret is of order \( (\log T)^2 T^{1/2} \), for \( \delta_1 = 1/\log T \), \( p = T^{-\lambda} \) for \( \lambda > 3/2 \), and \( \gamma = 1 + \frac{1}{\log T} \).

Proof of Claim 7. The expressions of regret in Claims 5 and 6 grow rapidly as \( k \) increases, so we only use these expressions to bound regret in early phases. For later phases, we argue that the gradient is small, which gives us a bound on the late-phase regret. Balancing the regret between early and late stages ultimately achieves the claimed \( \Theta(T^{1/2}) \) regret bound.

In particular, we will break the regret from the \( \delta \)-transition into two categories: the regret when the gradient is small (\( |\nabla f(x_t)| \leq T^{-d} \)) and the regret when the gradient is large (\( |\nabla f(x_t)| > T^{-d} \)). If \( k_d \) is the number of \( \delta \)-transitions until \( |\nabla f(x_t)| \leq T^{-d} \), then \( k_d \) is of order \( \log T \). To show this precisely, observe that

\[
|\nabla f(x_t)|^2 \leq \frac{2\beta^2}{\alpha} (f(x_t) - f(x^*)) \\ \leq \frac{2\beta^2 h_1^{(1)}}{\alpha} e^{-2\alpha c(i-1)}.
\]

So, if \( i > \frac{1}{\alpha_c} \left( \frac{1}{2} \log \frac{2\beta^2 h_1^{(1)}}{\alpha} + d \log T \right) + 1 \), then \( |\nabla f(x_t)| \leq T^{-d} \). In turn, if \( |\nabla f(x_t)| \leq T^{-d} \), then

\[
f(x_t) - f(x^*) \leq \frac{\beta}{2\alpha^2} T^{-2d},
\]

by smoothness and strong convexity. Now, define regret of \( \delta \)-transition to be the total regret from transitioning lag sizes over all iterations; that is, if \( m \) is the last phase of the algorithm (which is well-defined since number of iterations is bounded by \( T \)), then regret of \( \delta \)-transition = regret of \( \delta \)-transition\( (m) \). Then we can bound regret of \( \delta \)-transition as

\[
\text{regret of } \delta \text{-transition} = \text{regret of } \delta \text{-transition}(k_d) + \left[ \text{regret of } \delta \text{-transition} - \text{regret of } \delta \text{-transition}(k_d) \right] \\ \leq \frac{32\beta^5(2 + \gamma)^4 \log 2}{\alpha^3 \xi^4 q^8} k_d T^{2d} + \frac{\beta}{2\alpha^2} T^{1-2d}.
\]

Setting \( d = 1/4 \) and bounding \( k_d \) by \( \left[ \frac{1}{\alpha_c} \left( \frac{1}{2} \log \frac{2\beta^2 h_1^{(1)}}{\alpha} + d \log T \right) \right] + 2 \), we get that

\[
\text{regret of } \delta \text{-transition} \in \mathcal{O} \left( (\log T)^2 T^{1/2} \right).
\]

Since regret of \( \text{LGd} \) is of the same order as regret of \( \delta \)-transition (ignoring the lower-order term \( \frac{h \log 1}{\delta^4} \)), we have

\[
\text{regret} \leq (1 - p)^T \left[ \text{regret of } \text{LGd} + \text{regret of } \delta \text{-transition} \right] + \left( 1 - (1 - p)^T \right) T \\
\in \mathcal{O} \left( (1 - p)^T (\log T)^2 T^{1/2} + (1 - (1 - p)^T) T \right) \\
= \mathcal{O} \left( (1 - o(1))(\log T)^2 T^{1/2} + o(T^{1/2}) \right),
\]

for \( p = T^{-\lambda} \), \( \lambda > 3/2 \). This gives us a regret bound of \( \mathcal{O} \left( (\log T)^2 T^{1/2} \right) \).
6 Conclusion and Open Questions

Motivated by applications to fair pricing, we presented the first order-optimal algorithms for minimizing regret in stochastic optimization of strongly convex and smooth functions using bandit feedback, which satisfy monotonicity of decisions over time. There is a flurry of open questions naturally resulting from our work. The most natural open question is whether the $\Theta(\sqrt{T})$ regret bound from the unconstrained setting can be achievable for general convex functions [12, 23]. This is unclear, since our present algorithms critically depend on both the strong convexity as well as the smoothness assumptions to achieve their regret guarantees. Another important question is whether optimal unconstrained rates can be achieved in higher dimensions with a component-wise monotonicity constraint over time. More generally, it would be useful to understand the impact of imposing such monotonicity constraints on the regret rates achievable in the adversarial setting of online convex optimization. Additionally, it would also be interesting to consider this question in stochastic, non-stationary settings where the revenue curves drift over time: for example, in situations where the utility of a product improves or deteriorates over time.

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A Useful lemmas

**Proof of Lemma** [1] Let $x, y \in K$. If $x = y$, then the above inequality holds. So, we may assume that $x \neq y$. By strong convexity, we have

\[
\begin{align*}
    f(y) &\geq f(x) + \nabla f(x)^\top (y - x) + \frac{\alpha}{2} \|y - x\|^2, \text{ and} \\
    f(x) &\geq f(y) + \nabla f(y)^\top (x - y) + \frac{\alpha}{2} \|x - y\|^2.
\end{align*}
\]

Adding these two inequalities gives that

\[
\|y - x\|^2 \leq \frac{1}{\alpha} \langle \nabla f(x) - \nabla f(x), y - x \rangle \leq \frac{1}{\alpha} \|\nabla f(y) - \nabla f(x)\| \|y - x\|,
\]

where the last inequality follows from Cauchy-Schwarz. Finally, rearranging gives the desired result. \hfill \Box

**Proof of Lemma** [2] Let $\varepsilon = \alpha(y - x)^2/4$. Then by Hoeffding’s inequality\(^3\) we have that

\[
|\overline{f}(x) - f(x)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad |\overline{f}(y) - f(y)| \leq \frac{\varepsilon}{2}
\] \hfill (47)

with probability at least $(1 - p)^2$. For the remainder of the proof, we assume that (47) holds.

\(^3\)Hoeffding’s inequality states that for independent random variables $X_1, \ldots, X_n$, each with support in $[0, 1]$, we have that $\mathbb{P}(|\overline{X} - \frac{1}{n} \sum_{i=1}^n X_i| \geq s) \leq 2e^{-2ns^2}$, where $\overline{X} = \frac{1}{n} \sum_{i=1}^n X_i$ is the sample mean.
We first bound \( g - f(y) - f(x) - f(x) \) from above and below:

\[
g - f(y) - f(x) = \frac{\overline{f}(y) - \overline{f}(x)}{y - x} - f(y) - f(x)
\]

by defn. of \( g \)

\[
= \frac{1}{y - x} \left[ (\overline{f}(y) - f(y)) + (f(x) - \overline{f}(x)) + \varepsilon \right] \in \left[ 0, \frac{2\varepsilon}{y - x} \right].
\]

By definition of \( \varepsilon \), this gives us:

\[
0 \leq g - f(y) - f(x) \leq \frac{\alpha}{2} (y - x) \quad \text{with probability at least } (1 - p)^2. \tag{48}
\]

Since we have just shown that \( g \) is close to the secant \( \frac{f(y) - f(x)}{y - x} \), the only thing remaining is to show that \( \nabla f(y) \) is sufficiently far away from the secant. We can do this by using strong convexity:

\[
\frac{f(y) - f(x)}{y - x} \leq \frac{\nabla f(y)(y - x) - \frac{\alpha}{2} (y - x)^2}{y - x} \quad \text{by } \alpha\text{-strong convexity}
\]

\[
= \nabla f(y) - \frac{\alpha}{2} (y - x).
\]

We’ve thus shown that

\[
0 \leq g - \frac{f(y) - f(x)}{y - x} \leq \frac{\alpha}{2} (y - x) \leq \nabla f(y) - \frac{f(y) - f(x)}{y - x}.
\]

It follows that the estimated gradient \( g \) satisfies

\[
\nabla f(x) \leq \frac{f(y) - f(x)}{y - x} \leq g \leq \nabla f(y),
\]

where the first inequality follows from convexity, thus proving the lemma. \( \square \)

### B A sub-optimal algorithm for the noisy case

**Theorem 4 (LGD to Constant Steps).** For appropriate \( \delta, \eta, \iota, \) and \( p \), Algorithm 4 has regret \( \tilde{O}(T^{10/17}) \).

**Proof sketch.** Note that Algorithm 4 essentially runs Algorithm 2 and transitions to a constant step size \( \eta \) when the stopping condition is met. The constant-step phase operates by sampling at each point to an accuracy of \( O(\iota) \), and moving forward when the current point is deemed lower than the previous point (with probability \( (1 - p)^2 \)).

We incur a regret of order \( \frac{1}{\varepsilon^2} \) due to the lagged gradient descent exploration, where \( \varepsilon \in O(\delta^2) \) (as discussed in the proof of Theorem 2). Our exploration regret from the constant-step phase is

\[
\sum_{t=1}^{\delta/\eta^2} \frac{t^2 \eta^2}{\iota^2} = \frac{\eta^2}{\iota^2} \sum_{t=1}^{\delta/\eta^2} t^2 \approx \frac{\eta^2}{\iota^2} \left( \frac{\delta^3}{\eta^3} \right) = \frac{\delta^3}{\eta^2}.
\]

Finally, note that with high probability, the constant-step phase stops at \( x_T' \), where \( |f(x_{T' - 1}) - f(x_T')| < \iota \), resulting in an exploitation regret of \( T \frac{\eta^2}{\iota^2} \) (from estimation) and \( T \iota^2 \) (from the discretization of \( P \)). In total, our regret is

\[
\frac{1}{\varepsilon^2} + \frac{\delta^3}{\eta^2} + T \frac{\eta^2}{\iota^2} + T \eta^2.
\]
\begin{algorithm}
\textbf{input} : str. convexity parameter $\alpha$, smth. parameter $\beta$, time horizon $T$, feasible set $\mathcal{P} = [p_{\min}, p_{\max}]$, sample size $n$, function value precision $\varepsilon$, lag size $\delta$, stopping parameter $\gamma > 1$

1. $\bar{f}(x'_1), \bar{f}(x_1) \leftarrow \text{average of n samples at } x'_1 = p_{\min}, x_1 = p_{\min} + \delta$, resp. \hspace{1cm} // estimate $f(x'_1), f(x_1)$

2. for $t = 1, \ldots, \lfloor T/2n \rfloor$ do
   3. \hspace{0.5cm} Let $\hat{\nabla}_t \leftarrow \bar{f}(x_t) - \bar{f}(x_t') + \varepsilon \frac{x_t - x_t'}{x_t - x_t'}$
   4. \hspace{0.5cm} if $-\frac{1}{\beta} \hat{\nabla}_t \geq (1 + \gamma) \delta$ then
      5. \hspace{1cm} $\bar{f}(x'_{t+1}) \leftarrow \text{average of n samples at } x'_{t+1} = x_t - \frac{1}{\beta} \hat{\nabla}_t - 2\delta \hspace{0.5cm} // \text{estimate } f(x'_{t+1})$
      6. \hspace{1cm} $\bar{f}(x_{t+1}) \leftarrow \text{average of n samples at } x_{t+1} = x'_{t+1} + \delta = x_t - \frac{1}{\beta} \hat{\nabla}_t - \delta \hspace{0.5cm} // \text{estimate } f(x_{t+1})$
   else
      7. \hspace{1cm} Exit from the loop

9. $T' \leftarrow \text{current time period}$

10. for $t = T', \ldots$ do
    11. \hspace{0.5cm} $\bar{f}(x_t) \leftarrow \text{average of } \frac{2\log \frac{2}{\eta^2}}{T^2} \text{ samples at } x_t$ \hspace{0.5cm} // estimate $f(x_t)$
    12. \hspace{0.5cm} $\bar{f}(x_t + \eta) \leftarrow \text{average of } \frac{2\log \frac{2}{\eta^2}}{T^2} \text{ samples at } x_t + \eta \hspace{0.5cm} // \text{estimate } f(x_t + \eta)$
    13. \hspace{0.5cm} if $\bar{f}(x_t) - \frac{\eta}{2} > \bar{f}(x_t + \eta) + \frac{\eta}{2}$ then
        14. \hspace{1cm} $x_{t+1} \leftarrow x_t + \eta$
    15. \hspace{0.5cm} else
        16. \hspace{1cm} Exit from the loop and set $x_s \leftarrow x_t$ for $s > t$

\end{algorithm}

Setting $\varepsilon = \delta^2$, this becomes
\[ \frac{1}{\delta^4} + \frac{\delta^3}{\eta^2} + T \frac{\delta^2}{\eta^2} + T \eta^2. \]

Setting $\delta = T^{-5/34}$, $\eta = T^{-7/34}$, and $t = T^{-7/17}$, we get a regret of $\tilde{O} (T^{10/17})$. \qed

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