Construction of optimal codes in deletion and insertion metric

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Abstract

We improve Levenshtein’s upper bound for the cardinality of a code of length four that is capable of correcting single deletions over an alphabet of even size. We also illustrate that the new upper bound is sharp. Furthermore we construct an optimal perfect code that is capable of correcting single deletions for the same parameters.

1 Introduction

Let $B_q = \{0, 1, \ldots, q-1\}$ be a set with $q$ elements. $B_q$ is referred to as an alphabet and its elements are referred to as letters. A sequence $x = (x_1, \ldots, x_n)$ of $n$ letters of $B_q$ is called a word, and the number $n$ is called its length. Together with writing $x = (x_1, \ldots, x_n)$, we will also use the notation $x = x_1 \cdots x_n$. Let $B_n^q$ be the set of words over $B_q$ of length $n$, and define $B_*^q = \bigcup_{n=0}^{\infty} B_n^q$.

The deletion and insertion distance $\rho(x, y)$, which was first introduced by Levenshtein \cite{Levenshtein}, between two words $x$ and $y$ in $B_*^q$ is defined by the minimum number of deletions and insertions of letters required to transform $x$ into $y$. For example, let $x = 12243$ and $y = 14223$ be two words in $B_5^5$. Deleting the fourth letter of $x$ and the second of $y$ yields the identical words $x' = 1223$ and $y' = 1223$, respectively. Hence $\rho(x, y) = 2$. For a code $C \subseteq B_n^q$ with $|C| \geq 2$, we define $\rho(C) = \min \{ \rho(x, y) \mid x, y \in C, x \neq y \}$.

Let $N(n, q, d) = \max \{ |C| \mid C \subseteq B_n^q, \rho(C) > 2d \}$. A code $C$ in $B_n^q$ is called an optimal code if $|C| = N(n, q, 1)$. Levenshtein \cite{Levenshtein} estimated an upper bound of $N(n, q, 1)$ for any $n \geq 2$ and $q \geq 2$, namely,

$$N(n, q, 1) \leq \left\lfloor \frac{q^{n-1} + (n-2)q^{n-2} + q}{n} \right\rfloor. \quad (1)$$

The upper bound in (1) is sharp when $n = 3$. Indeed, a code that is capable of correcting single deletions whose cardinality meets the upper bound in (1) was constructed in \cite{Levenshtein} when $n = 3$ and $q \geq 2$ is an arbitrary integer. However this bound is not sharp when $n \geq 4$. In this paper, we improve Levenshtein’s upper bound for the case in which $n = 4$ and $q$ is even, and prove that the new upper bound is sharp.

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In Section 2, we will derive a new upper bound for the cardinality of a code that is capable of correcting single deletions when \( n = 4 \) and \( q \) is even by analyzing the deletion map.

In Section 3, we will construct a code in \( B_q^4 \) for \( q \equiv 2 \) or \( 4 \) (mod 6) that is capable of correcting single deletions whose cardinality meets the new upper bound derived in Section 2. We first introduce the concept of the step property for a Steiner quadruple system, and prove that there is a code in \( B_q^4 \) that is capable of correcting single deletions whose cardinality meets the upper bound derived in Section 2, under the assumption that there is a Steiner quadruple system on \( B_q \) that satisfies the step property. Then we show that there is a Steiner quadruple system on \( B_q \) that satisfies the step property for \( q \equiv 2 \) or \( 4 \) (mod 6).

In Section 4, we will construct a code in \( B_q^4 \) for \( q \equiv 0 \) (mod 6) that is capable of correcting single deletions whose cardinality meets the upper bound derived in Section 2. Since there does not exist a Steiner quadruple system on an alphabet of this size, we will use a group divisible system. We divide our construction into two steps. In the first step, we prove that an optimal code exists for an alphabet of size \( q = 6m \) where \( m \) is odd. In the next step, we prove that an optimal code exists for an alphabet of size 2\( q \) under the assumption that an optimal code exists for an alphabet of size \( q \).

In Section 5, we modify our construction of optimal codes slightly, and construct an optimal perfect code in \( B_q^4 \) when \( q \) is even.

2 New upper bound of \( N(4,q,1) \) for \( q \) even

In this section, we improve Levenshtein’s upper bound for the case in which \( n = 4 \) and \( q \) is even. Our result follows from an analysis of the deletion map as follows.

We begin with a simple observation. For any word \( x \) over \( B_q \) and any positive integer \( s \), denote by \( \lfloor x \rfloor_s \) the set of words obtained from \( x \) by deleting \( s \) of its letters. The multi-map \( \lfloor \rfloor_s : B_n^q \to B_{n-s}^q \) which sends \( x \) to \( \lfloor x \rfloor_s \) will be called an \( s \)-deletion map or simply a deletion map. For a subset \( C \subseteq B_n^q \), denote by \( \lfloor C \rfloor_s \) the set \( \bigcup_{x \in C} \lfloor x \rfloor_s \). A set \( C \subseteq B_n^q \) is called a code that is capable of correcting \( s \) deletions if all sets \( \lfloor x \rfloor_s \) \((x \in C)\) do not pairwise intersect. It follows from the definition that a code \( C \) in \( B_n^q \) satisfies \( \rho(C) > 2s \) if and only if \( C \) is capable of correcting \( s \) deletions.

The following lemma is an immediate consequence of this observation.

**Lemma 2.1 (Substitution lemma).** Let \( C \) be a code in \( B_q^n \) such that \( \rho(C) > 2s \). If \( x \in C, y \in B_q^n, \) and \( \lfloor y \rfloor_s \subseteq \lfloor x \rfloor_s \), then \( \rho((C \setminus \{x\}) \cup \{y\}) > 2s \). More generally, if \( A \subseteq C, B \subseteq B_q^n, \lfloor B \rfloor_s \subseteq \lfloor A \rfloor_s, \) and \( \rho(B) > 2s \), then \( \rho((C \setminus A) \cup B) > 2s \).

From now on, we will use \( C \) to denote a code in \( B_q^4 \) with \( \rho(C) > 2 \) and we define

\[
C_i = \{ x \in C \mid |\lfloor x \rfloor_1| = i \} \ (1 \leq i \leq 4).
\]

Let \( a, b, c, \) and \( d \) be distinct elements in \( B_q \). By the substitution lemma, we
may assume without changing the cardinality of \( C \) that
\[
\begin{align*}
    x \in C_1 & \Rightarrow x \text{ is of the type } (a,a,a,a), \\
    x \in C_2 & \Rightarrow x \text{ is of the type } (a,a,b,b), \\
    x \in C_3 & \Rightarrow x \text{ is of the type } (a,b,b,a), (a,a,b,c), (b,c,a,a), \text{ or } (a,b,b,c), \\
    x \in C_4 & \Rightarrow x \text{ is of the type } (a,b,c,d), (a,b,a,c), (a,b,c,a), \text{ or } (b,a,c,a).
\end{align*}
\]

In the case of \( 3 \leq i \leq 4 \), we analyze further and define
\[
\begin{align*}
    C_{3,1} &= \{ x \in C_3 \mid x \text{ is of the type } (a,a,b,c), (b,c,a,a), \text{ or } (a,b,b,c) \}, \\
    C_{3,2} &= \{ x \in C_3 \mid x \text{ is of the type } (a,b,b,a) \}, \\
    C_{4,1} &= \{ x \in C_4 \mid x \text{ is of the type } (a,b,c,d) \}, \\
    C_{4,2} &= \{ x \in C_4 \mid x \text{ is of the type } (a,b,c,a) \}, \\
    C_{4,3} &= \{ x \in C_4 \mid x \text{ is of the type } (a,b,a,c) \text{ or } (b,a,c,a) \}.
\end{align*}
\]

Consider the following subsets of \( B_q^3 \):
\[
\begin{align*}
    U &= \{ y \in B_q^3 \mid y \text{ is of the type } (a,a,a) \}, \\
    V &= \{ y \in B_q^3 \mid y \text{ is of the type } (a,a,b) \text{ or } (a,b,b) \}, \\
    W &= \{ y \in B_q^3 \mid y \text{ is of the type } (a,b,a) \}, \\
    Z &= \{ y \in B_q^3 \mid y \text{ is of the type } (a,b,c) \}.
\end{align*}
\]

Note that
\[
\begin{align*}
    |U| &= q, |V| = 2q(q - 1), \\
    |W| &= q(q - 1), |Z| = q(q - 1)(q - 2).
\end{align*}
\]

Table 1 counts the contribution of each codeword in \( C_i \) (or \( C_{i,j} \)) to \( U, V, W, \) and \( Z \) under the deletion map \( [ ] : B_q^4 \to B_q^3 \).

| \( C_1 \) | \( C_2 \) | \( C_{3,1} \) | \( C_{3,2} \) | \( C_{4,1} \) | \( C_{4,2} \) | \( C_{4,3} \) |
|---|---|---|---|---|---|---|
| \( U \) | 1 | 0 | 0 | 0 | 0 | 0 |
| \( V \) | 0 | 2 | 2 | 0 | 0 | 1 |
| \( W \) | 0 | 0 | 1 | 0 | 2 | 1 |
| \( Z \) | 0 | 0 | 1 | 0 | 4 | 2 | 2 |

Table 1: The contribution of each codeword in \( C_i \) (or \( C_{i,j} \)) to \( U, V, W, \) and \( Z \)

Let \( X \in \{ U, V, W, Z \} \) and define \( X_1 = X \cap [C_i]_1 \) and \( X_{i,j} = X \cap [C_{i,j}]_1 \) (for example \( V_{3,2} = V \cap [C_{3,2}]_1 \)). The following relations can be easily obtained from (2) and Table 1:
\[
\begin{align*}
    |U_1| \leq |U| &= q, \\
    |V_2| + |V_{3,1}| + |V_{3,2}| + |V_{4,3}| \leq |V| = 2q(q - 1), \\
    |W_{3,2}| + |W_{4,2}| + |W_{4,3}| \leq |W| = q(q - 1), \\
    |Z_{3,1}| + |Z_{4,1}| + |Z_{4,2}| + |Z_{4,3}| \leq |Z| = q(q - 1)(q - 2), \\
    |C_1| = |U_1|, |C_2| = \frac{1}{2}|V_2|, |C_{3,1}| = \frac{1}{2}|V_{3,1}| = |Z_{3,1}|, |C_{3,2}| = \frac{1}{2}|V_{3,2}| = |W_{3,2}|, \\
    |C_{4,1}| = \frac{1}{2}|Z_{4,1}|, |C_{4,2}| = \frac{1}{2}|W_{4,2}| = \frac{1}{2}|Z_{4,2}|, |C_{4,3}| = |V_{4,3}| = |W_{4,3}| = \frac{1}{2}|Z_{4,3}|.
\end{align*}
\]

This information allows derivation of the first main result.

**Theorem 2.2.** Let \( C \) be a code in \( B_q^4 \) with an even \( q \) and \( \rho(C) > 2 \). Then
\[
|C| \leq \frac{q^2(q + 2)}{4}.
\]

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Proof. It follows from the previous calculation that

\[ |C| = |C_1| + |C_2| + |C_{3,1}| + |C_{3,2}| + |C_{4,1}| + |C_{4,2}| + |C_{4,3}| \]

\[ = |U_1| + \frac{1}{2}|V_2| + \frac{1}{2}|V_{3,1}| + \frac{1}{2}|V_{3,2}| + \frac{1}{2}|Z_{4,1}| + \frac{1}{2}|Z_{4,2}| + \frac{1}{2}|Z_{4,3}| \]

\[ = |U_1| + \frac{1}{2}(|V_2| + |V_{3,1}| + |V_{3,2}| + |V_{4,1}|) - \frac{1}{2}|V_{4,2}| \]

\[ + \frac{1}{2}(|Z_{3,1}| + |Z_{4,1}| + |Z_{4,2}| + |Z_{4,3}|) - \frac{1}{2}|Z_{3,1}| + \frac{1}{2}|Z_{4,2}| + \frac{1}{2}|Z_{4,3}| \]

\[ = |U_1| + \frac{1}{2}(|V_2| + |V_{3,1}| + |V_{3,2}| + |V_{4,1}|) + \frac{1}{2}(|Z_{3,1}| + |Z_{4,1}| + |Z_{4,2}| + |Z_{4,3}|) \]

\[ + \frac{1}{2}|C_{4,2}| - \frac{1}{4}|C_{3,1}| \]

\[ \leq |U| + \frac{1}{2}|V| + \frac{1}{2}|Z| + \frac{1}{2}|C_{4,2}| - \frac{1}{4}|C_{3,1}|. \]

Suppose that \((a, b, c, a) \in C_{4,2}, (a, b, c, a) \in C_{4,1}, (a, a, b, c) \neq (a, b, c, a).\) From the condition \(\rho(C) > 2,\) it follows that \(\{b_1, c_1\} = \emptyset.\) Therefore

\[ |C_{4,2}| \leq q \left\lfloor \frac{q - 1}{2} \right\rfloor = \frac{q(q - 2)}{2}. \]

Since \(|C_{3,1}| \geq 0,\) the result follows.

\[ \square \]

Remark 1. Theorem 2.2 improves the bound in (1) when \(q\) is even and it gives the same bound when \(q\) is odd. But this bound is not sharp when \(q\) is odd. Obtaining a sharp bound when \(q\) is odd seems to be a very difficult problem.

Remark 2. The proof of Theorem 2.2 can be used in the construction of optimal codes in the following way. Let \(C\) be a code in \(B^4_q\) with \(|C| = \frac{q^2(q + 2)}{4}.\) Then we should have

\[ |U_1| = |U| = q, |V_2| = |V_{3,1}| + |V_{3,2}| + |V_{4,1}| + |V_{4,2}| = |V| = 2q(q - 1), |C_{4,1}| = 0, \]

\[ |Z_{3,1}| + |Z_{4,1}| + |Z_{4,2}| + |Z_{4,3}| = |Z| = q(q - 1)(q - 2), |C_{4,2}| = \frac{q(q - 2)}{2}. \]

Since \(|C_{3,1}| = 0,\) we have \(|V_{3,1}| = |Z_{3,1}| = 0.\) From the fact that \(|C_{4,2}| = \frac{q(q - 2)}{2},\) we may assume that \(|C_{3,2}| = |C_{4,3}| = 0.\) Then \(|V_{3,2}| = |V_{4,3}| = |Z_{4,3}| = 0.\) In this case \(|V_2| = |V|.\) Hence

\[ |C_1| = q, |C_2| = q(q - 1), |C_3| = 0, \]

\[ |C_{4,1}| = \frac{q(q - 1)(q - 2) - q(q - 2)}{4}, |C_{4,2}| = \frac{q(q - 2)}{2}. \]

This means that we can obtain an optimal code if we include every word of the types \((a, a, a, a)\) and \((a, a, b, b),\) the maximal number of words of the type \((a, b, c, a),\) and generate words of the type \((a, b, c, b)\) using all remaining words of length 3 of the type \((a, b, c).\) Since \(|U_1| = |U|\) and \(|V_2| = |V|\) are always possible, to construct an optimal code we only need to consider combinations of elements in \(W \cup Z\) that satisfy

\[ |C_4| = \frac{q(q - 1)(q - 2) - q(q - 2)}{4} + \frac{q(q - 2)}{2}. \]

From these considerations, we can construct optimal codes in \(B^4_q\) (Table 2) and \(B^4_q\) (Table 3) which coincide with the construction in [4].

In the next two sections, we will construct an optimal code in \(B^4_q\) when \(q\) is even. Let \(x\) be a word of length 4 which consists of pairwise distinct letters. The basic ingredients in our construction are the codes \(\langle x \rangle_{A^4_q}\) generated by \(x\) in \(A^4_q\) and \(\langle x \rangle_{B^4_q}\) generated by \(x\) in \(B^4_q\). These codes will be defined below. The code \(\langle x \rangle_{A^4_q}\) was already used in [6]. The essence of our construction is to
Table 2: An optimal code in $B_4^4$

|          | 0000 | 1111 | 2222 | 3333 |
|----------|------|------|------|------|
| 0011     | 0022 | 0033 | 1100 |
| 1122     | 1133 | 2200 | 2211 |
| 2233     | 3300 | 3311 | 3322 |
| 0230     | 1231 | 2012 | 3013 |
| 0321     | 2103 | 1302 | 3120 |

Table 3: The $C_4$ of an optimal code in $B_6^4$

|          | 0230 | 1231 | 2012 | 3013 | 4014 | 5015 |
|----------|------|------|------|------|------|------|
| 0450     | 1451 | 2452 | 3453 | 4234 | 5235 |
| 0251     | 1304 | 2053 | 3105 | 4035 | 5102 |
| 0342     | 1325 | 2140 | 3124 | 4120 | 5143 |
| 0431     | 1503 | 2413 | 3520 | 4215 | 5321 |
| 0524     | 1542 | 2504 | 3541 | 4302 | 5340 |

replace $\langle x \rangle_{A_4^q}$ by $\langle x \rangle_{B_4^q}$ for some words such that our construction satisfies the conditions of Remark 2.

Let $A_4^q$ be the set of all words in $B_4^n$ that have pairwise distinct letters. For a word $x = (a_1, a_2, a_3, a_4)$ in $A_4^q$, the codes $\langle x \rangle_{A_4^q}$ and $\langle x \rangle_{B_4^q}$ are defined as follows:

$\langle x \rangle_{A_4^q} = \{(a_1, a_2, a_3, a_4), (a_1, a_4, a_3, a_2), (a_2, a_4, a_1, a_3), (a_3, a_4, a_1, a_2), (a_4, a_1, a_2, a_3), (a_1, a_3, a_4, a_2), (a_2, a_3, a_4, a_2), (a_3, a_1, a_2, a_3), (a_4, a_1, a_2, a_4)\}$

$\langle x \rangle_{B_4^q} = \{(x)_{A_4^q} \setminus \{(a_1, a_2, a_3, a_4), (a_3, a_4, a_1, a_2)\}\}$

The following lemma which describes the basic properties of these codes under the deletion map can be easily verified.

**Lemma 2.3.** Let $x = (a_1, a_2, a_3, a_4)$ and $y = (b_1, b_2, b_3, b_4)$ be distinct words of $A_4^q$, and $L(x)$ be the set of letters in $x$. Then

(i) $\langle x \rangle_{A_4^q}$ and $\langle x \rangle_{B_4^q}$ are codes that are capable of correcting single deletions,

(ii) if $|L(x) \cap L(y)| \leq 2$, then $\langle x \rangle_{A_4^q} \cup \langle y \rangle_{B_4^q}$ is a code that is capable of correcting single deletions,

(iii) if $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$ or $\{a_3, a_4\} \cap \{b_3, b_4\} = \emptyset$, then $\langle x \rangle_{B_4^q} \cup \langle y \rangle_{B_4^q}$ is a code that is capable of correcting single deletions.

3 Construction of optimal codes in $B_4^4$ for $q \equiv 2 \text{ or } 4 \pmod 6$

In this section, we will construct a code in $B_4^4$ for $q \equiv 2 \text{ or } 4 \pmod 6$ that is capable of correcting single deletions whose cardinality meets the upper bound that was established in Theorem 2.2. Our construction consists of two steps.
In the first step, we introduce the concept of the step property for a Steiner quadruple system, and prove that there is a code in $B_q^4$ that is capable of correcting single deletions whose cardinality meets the upper bound in Theorem 2.2 under the assumption that there is a Steiner quadruple system on $B_q$ that satisfies the step property. In the next step, we follow the construction of Hanani to show that there is a Steiner quadruple system on $B_q$ that satisfies the step property for $q \equiv 2$ or $4 \pmod{6}$.

When $q = 2$, we construct a code $C$ in $B_2^4$ with codewords

$$C = \{(0,0,0,0), (1,1,1,1), (0,0,1,1), (1,1,0,0)\},$$

by following the method in Remark 2. Since $|C| = 4$, the code $C$ should be an optimal code.

Let $SQS(q)$ denote a Steiner quadruple system on $B_q$. Recall that an $SQS(q)$ is a set of 4-element subsets of $B_q$, called quadruples, with the property that every 3-element subset of $B_q$ is a subset of exactly one quadruple in the set. It is well known that there is a Steiner quadruple system on $B_q$ if and only if $q \equiv 2$ or $4 \pmod{6}$. From now on, we assume that $q \equiv 2$ or $4 \pmod{6}$ and that $q \geq 4$.

To construct optimal codes, we define the step property.

**Definition 3.1 (The step property).** Let $SQS(q)$ be a Steiner quadruple system on $B_q$ and $L_0 < L_1 < \cdots < L_{q-1}$ be a total order on $B_q = \{L_0, L_1, \ldots, L_{q-1}\}$. Let $\{L_{2t}, L_{2t+1}, L_a, L_b\}$ be a quadruple in $SQS(q)$. We say that $\{L_{2t}, L_{2t+1}, L_a, L_b\}$ satisfies the step property with respect to the given order if either $a < 2t$, $b < 2t$ or $2t + 1 < a$, $2t + 1 < b$. We also say that $SQS(q)$ satisfies the step property if every quadruple of the form $\{L_{2t}, L_{2t+1}, L_a, L_b\}$ satisfies the step property with respect to the given total order.

If there is a Steiner quadruple system that has the step property, optimal codes can be constructed using the following theorem.

**Theorem 3.2.** Suppose that there is an $SQS(q)$ that satisfies the step property. Then

$$N(4,q,1) = \frac{q^2(q + 2)}{4}.$$

**Proof.** Suppose that there is an $SQS(q)$ that satisfies the step property. Without loss of generality, we may assume that $0 < 1 < \cdots < q - 1$ is the total order which admits the step property for $SQS(q)$. Let $\phi : SQS(q) \to B_q^4$ be the map defined by $\phi([x,y,z,w]) = (x,y,z,w)$ where $x < y < z < w$, and

$$S(q) = \{\{2t, 2t + 1, a, b\} \in SQS(q) \mid 2t + 1 < a, 2t + 1 < b\}.$$

It follows from the definition of a Steiner quadruple system and Lemma 2.2 that the code

$$M = \bigcup_{x \in \phi(SQS(q) \setminus S(q))} \langle x \rangle_{A_t^4} \bigcup_{x \in \phi(S(q))} \langle x \rangle_{B_t^4}$$

is capable of correcting single deletions. $|S(q)| = \frac{2(q-2)}{8}$, so

$$|M| = 6 \cdot \left\{ \frac{q(q-1)(q-2)}{24} - \frac{q(q-2)}{8} \right\} + 8 \frac{q(q-2)}{8} = \frac{q^2(q-2)}{4}.$$
We obtain a code $C$ from $M$ by adding all words of the types $(a, a, a, a)$, and $(a, a, b, b)$ ($a \neq b$). It is easy to see that $C$ is capable of correcting single deletions. Finally we have

$$|C| = q + q(q - 1) + \frac{q^2(q - 2)}{4} = \frac{q^2(q + 2)}{4}.$$ 

Hanani [2] inductively constructed an $SQS(q)$ for $q \equiv 2$ or $4 \pmod{6}$. Using Hanani’s construction, we will prove that there exists an $SQS(q)$ with the step property for all $q \equiv 2$ or $4 \pmod{6}$.

In subsequent sections, the following definitions and notations will be used. For a natural number $n$, $B_n$ (resp. $\bar{B}_n$) denotes the set $\{0, 1, \ldots, n - 1\}$ (resp. $\{1, 2, \ldots, n\}$). We introduce two partitions of unordered pairs of the set $B_{2m}$.

We decompose the $m(2m - 1)$ pairs $\{r, s\}$ from the set $B_{2m}$ into $2m - 1$ systems $P_\alpha(m)$ ($\alpha \in B_{2m-1}$), each containing $m$ mutually disjoint pairs.

For $m \equiv 0 \pmod{2}$ we form the systems $P_\alpha(m)$ as follows:

$$P_{2\beta}(m) = \{\{2a, 2a + 2\beta + 1\} \mid a \in B_m\} \quad (\beta \in B_{\frac{q-1}{2}}),$$
$$P_{2\beta+1}(m) = \{\{2a, 2a - 2\beta - 1\} \mid a \in B_m\} \quad (\beta \in B_{\frac{q-1}{2}}),$$
$$P_{m+\gamma}(m) = \left\{ \begin{array}{l}
\{b, 2\gamma - b\} \mid b \in B_{\gamma} \\
\{c, 2m + 2\gamma - c - 2\} \mid 2\gamma + 1 \leq c \leq m + \gamma - 2 \\
\{2m - \frac{3}{2} - (-1)^{\frac{\gamma}{2}}, \gamma\} \\
\{2m - \frac{3}{2} + (-1)^{\frac{\gamma}{2}} m + \gamma - 1\}
\end{array} \right\} \quad (\gamma \in B_{m-1}).$$

For $m \equiv 1 \pmod{2}$ we let:

$$P_{2\beta}(m) = \{\{2a, 2a + 2\beta + 1\} \mid a \in B_m\} \quad (\beta \in B_{\frac{q-1}{2}}),$$
$$P_{2\beta+1} = \{\{2a, 2a - 2\beta - 1\} \mid a \in B_m\} \quad (\beta \in B_{\frac{q-1}{2}}),$$
$$P_{m-1+\gamma}(m) = \left\{ \begin{array}{l}
\{b, 2\gamma - b\} \mid b \in B_{\gamma} \\
\{c, 2m + 2\gamma - c\} \mid 2\gamma + 1 \leq c \leq m + \gamma - 1 \\
\{\gamma, m + \gamma\}
\end{array} \right\} \quad (\gamma \in B_m).$$

It can be easily verified that the pairs in every system are mutually disjoint and no pair appears twice. Because the number of pairs in the systems is $m(2m - 1)$, it follows that every pair appears in some system.

We shall also need another decomposition of the $m(2m - 1)$ pairs from $B_{2m}$ into $2m$ systems $P_\xi(m)$ ($\xi \in B_{2m}$) such that each of the $m$ systems $P_\eta(m)$ ($\eta \in B_m$) should contain $m - 1$ mutually disjoint pairs not containing the elements $2\eta$ and $2\eta + 1$, and each of the other $m$ systems should contain $m$ mutually disjoint pairs. We shall form the system $P_\xi(m)$ using the systems $P_\alpha(m)$.

If $m \equiv 0 \pmod{2}$, it can easily be seen that

$$\left\{ \begin{array}{l}
\{2\mu, 4\mu + 1\} \in P_{2\mu}(m) \quad (\mu \in B_{\frac{q-1}{2}}), \\
\{2m - 2 - 2\mu, 2m - 1 - 2\mu\} \in P_{2\mu-1}(m) \quad (\mu \in B_{\frac{q-1}{2}}), \\
\{2m - 2, 0\} \in P_m(m), \\
\{2m - 1, 1\} \in P_{m+1}(m).
\end{array} \right\}$$
Clearly, these pairs are mutually disjoint. We remove them from their respective systems and use them to form a new system.

Performing the following permutation of the elements

\[
\begin{pmatrix}
2\mu & 4\mu + 1 & 2m - 2 - 2\mu & 2m - 1 - 4\mu & 2m - 2 & 0 & 2m - 1 & 1 \\
4\mu & 4\mu + 1 & 4\mu - 2 & 4\mu - 1 & 1 & 0 & 2m - 1 & 2m - 2
\end{pmatrix}
(\mu \in \bar{B}_{\frac{m-1}{2}}),
\]

we obtain the new systems \(\bar{P}_x(m)\) by a suitable reordering of the systems.

If \(m \equiv 1 \pmod{2}\),

\[
\begin{aligned}
\{2\mu, 4\mu + 1\} & \in P_{2\mu}(m) \quad (\mu \in \bar{B}_{m-1}) , \\
\{2m - 2 - 2\mu, 2m - 3 - 4\mu\} & \in P_{2\mu+1}(m) \quad (\mu \in \bar{B}_{m-1}) , \\
\{m - 1, 2m - 1\} & \in P_{2m-2}(m).
\end{aligned}
\]

These pairs are again mutually disjoint.

By the permutation

\[
\begin{pmatrix}
2\mu & 4\mu + 1 & 2m - 2 - 2\mu & 2m - 3 - 4\mu & m - 1 & 2m - 1 & 2m - 2 & 2m - 1
\end{pmatrix}
(\mu \in \bar{B}_{\frac{m-1}{2}})
\]

of the elements and using the same procedure as in the case \(m \equiv 0 \pmod{2}\) we obtain the systems \(\bar{P}_x(m)\).

If a system \(SQS(f)\) exists, we say that a quadruple system can be formed from \(B_f\) and write \(f \in S\). Similarly, if a system \(SQS(f)\) with the step property exists, we denote the condition as \(f \in SP\).

If \(f \in S\), we shall use \(\{x, y, z, t\} \in B_f\) to denote any quadruple in \(B_f\), that is, an element of \(SQS(f)\). If \(f + 1 \in S\), then we assume that \(SQS(f + 1)\) is formed by the alphabet \(B_f \cup \{A\}\) where \(A\) is an additional element. The quadruples that contain \(A\) will be denoted by \(\{A, u, v, w\}\).

The following was inductively shown by Hanani [2].

**Theorem 3.3.** If \(q \equiv 2 \text{ or } 4 \pmod{6}\), then \(q \in S\).

Using a similar argument as in [2], we will prove the following.

**Theorem 3.4.** Suppose that \(q \geq 4\). If \(q \equiv 2 \text{ or } 4 \pmod{6}\), then \(q \in SP\).

**Proof.** We will proceed by induction on \(q\). Clearly, \(SQS(4)\) has the step property, hence \(4 \in SP\). As an induction step, we will prove the following: Let \(q \equiv 2 \text{ or } 4 \pmod{6}\). If \(f \in SP\) for every \(f < q\) satisfying \(f \equiv 2 \text{ or } 4 \pmod{6}\), then \(q \in SP\). The proof will be given separately for each of the following cases which evidently exhaust all the possibilities:

\[
\begin{aligned}
q & \equiv 4 \text{ or } 8 \pmod{12}, \\
q & \equiv 4 \text{ or } 10 \pmod{18}, \\
q & \equiv 34 \pmod{36}, \\
q & \equiv 26 \pmod{36}, \\
q & \equiv 2 \text{ or } 10 \pmod{24} \quad (q > 2), \\
q & \equiv 14 \text{ or } 38 \pmod{72}.
\end{aligned}
\]

**Case I:** \(q \equiv 4 \text{ or } 8 \pmod{12}\).

Let \(q = 2f\) where \(f \equiv 2 \text{ or } 4 \pmod{6}\). Since \(f \in SP\), there is an \(SQS(f)\) with
the step property. Without loss of generality, we may assume that $0 < 1 < \cdots < f - 1$ is the total order on $B_f$ which admits the step property for $SQS(f)$. Let $N = \{(i, j) \mid i \in B_2, j \in B_f\}$ and \{$x, y, z, t$\} be any element in $SQS(f)$. The following quadruples in $N$ form an $SQS(q)$ \[ \text{(2)} \]:

\[
\begin{align*}
Q_1 & : \{(a_1, x), (a_2, y), (a_3, z), (a_4, t)\} \quad (a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{2}), \\
Q_2 & : \{(0, j), (0, j'), (1, j), (1, j')\} \quad (j \neq j').
\end{align*}
\]

We rename the letters of $N$ as follows: for $j \in B_f$,

\[
(0, j) \rightarrow (j, 1, j) \rightarrow f + j.
\]

Note that the solutions of $a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{2}$ are

\[
(a_1, a_2, a_3, a_4) = \begin{cases}
(0, 0, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), \\
(1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0), (1, 1, 1, 1).
\end{cases}
\]

To investigate the step property on $SQS(q)$, we only need to consider the following quadruples in each type: for $t \in B_{\frac{3}{2}}$,

\[
\begin{align*}
Q_1 & : \{(0, 2t), (0, 2t + 1), (0, x), (0, y)\}, \{(0, 2t), (0, 2t + 1), (1, x), (1, y)\}, \\
& \quad \{(1, 2t), (1, 2t + 1), (0, x), (0, y)\}, \{(1, 2t), (1, 2t + 1), (1, x), (1, y)\}, \\
Q_2 & : \{(0, 2t), (0, 2t + 1), (1, 2t), (1, 2t + 1)\}.
\end{align*}
\]

Note that \{(2t, 2t + 1, x, y)\} is a quadruple in $SQS(2)$. Since $SQS(f)$ satisfies the step property with respect to the order $0 < 1 < \cdots < f - 1$, it follows from \[ \text{(3)} \] that $SQS(q)$ satisfies the step property. Therefore $q \in SP$.

**Case II:** $q \equiv 4 \text{ or } 10 \pmod{18}$.

Let $q = 3f + 1$ where $f \equiv 1 \text{ or } 3 \pmod{6}$. Since $f + 1 \equiv 2 \text{ or } 4 \pmod{6}$, we obtain $f + 1 \in SP$ by induction. Hence we may assume that $SQS(f + 1)$ satisfies the step property with respect to the order $0 < 1 < \cdots < f - 1 < A$. In accordance with the notation in \[ \text{(2)} \], we define $N = \{(i, j), A \mid i \in B_3, j \in B_f\}$. Also, let \{$x, y, z, t$\} and \{$A, u, v, w$\} be quadruples in $B_f$ and $B_f \cup \{A\}$, respectively. Note that both of them are elements of $SQS(f + 1)$. According to \[ \text{(2)} \], the following quadruples in $N$ form an $SQS(q)$:

\[
\begin{align*}
Q_1 & : \{(a_1, x), (a_2, y), (a_3, z), (a_4, t)\} \quad (a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{3}), \\
Q_2 & : \{(A, (b_1, u), (b_2, v), (b_3, w))\} \quad (b_1 + b_2 + b_3 \equiv 0 \pmod{3}), \\
Q_3 & : \{(i, u), (i, v), (i + 1, w), (i + 2, w)\}, \\
Q_4 & : \{(i, j), (i, j'), (i + 1, j), (i + 1, j')\} \quad (j \neq j'), \\
Q_5 & : \{(A, (0, j), (1, j), (2, j))\}.
\end{align*}
\]

In cases $Q_1$ and $Q_4$, calculations are performed modulo 3.

We then rename letters in $N$ as follows: for $t \in B_{\frac{1}{2}}$,

\[
\begin{align*}
(0, 2t) & \rightarrow 6t, \quad (0, 2t + 1) \rightarrow 6t + 1, \\
(1, 2t) & \rightarrow 6t + 2, \quad (1, 2t + 1) \rightarrow 6t + 3, \\
(2, 2t) & \rightarrow 6t + 4, \quad (2, 2t + 1) \rightarrow 6t + 5, \\
(0, f - 1) & \rightarrow 3f - 3, \quad (1, f - 1) \rightarrow 3f - 2, \\
(2, f - 1) & \rightarrow 3f - 1, \quad A \rightarrow 3f.
\end{align*}
\]
To investigate the step property, we only need to consider the following quadruples of each type: for $t \in B_{f-1}$ and $i \in B_k$,

\[
\begin{align*}
Q_1 &= \{ (0, 2t), (0, 2t+1), (0, x), (0, y) \}, \{ (0, 2t), (0, 2t+1), (1, x), (2, y) \}, \{ (1, 2t), (1, 2t+1), (0, x), (1, y) \}, \{ (1, 2t), (1, 2t+1), (2, x), (2, y) \}, \\
Q_2 &= \{ (1, 2t), (2, 2t+1), (0, x), (2, y) \}, \{ (2, 2t), (2, 2t+1), (1, x), (1, y) \}, \\
Q_3 &= \{ (0, 2t), (0, 2t+1), (1, x), (2, x) \}, \{ (1, 2t), (1, 2t+1), (2, x), (0, x) \}, \\
Q_4 &= \{ (1, 2t), (2, 2t+1), (0, x), (1, x) \}, \{ (2, 2t), (2, 2t+1), (0, f-1), (1, f-1) \}, \\
Q_5 &= \{ (i, 2t), (i+1, 2t), (i+1, 2t+1) \},
\end{align*}
\]

Note that $\{2t, 2t+1, a, b\}$, $\{2t, 2t+1, a, A\}$, and $\{a, b, f-1, A\}$ are quadruples in $SQS(f+1)$. Since $SQS(f+1)$ on $B_f \cup \{A\}$ satisfies the step property with respect to the order $0 < 1 < \cdots < f-1 < A$, it follows from [4] that $SQS(q)$ satisfies the step property. Therefore $q \in SP$.

**Case III**: $q \equiv 34 \pmod{36}$.

Let $q = 3f+4$ where $f \equiv 10 \pmod{12}$, and denote $f = 12k+10$. By induction, we may assume that $f+4 \in SP$. In accordance with the notation in [2], we can assume that there is an $SQS(f+4)$ on the alphabet $B_f \cup \{(A, 0), (A, 1), (A, 2), (A, 3)\}$. Defining $L_t = 2t, L_{t+1} = 2t+1 (t \in B_k)$, we also assume that given $SQS(f+4)$ satisfies the step property with respect to the order

$$L_0 < L_1 < \cdots < L_{f-1} < (A, 0) < (A, 1) < (A, 2) < (A, 3).$$

Let $N = \{(i, j, (A, h) | i \in B_3, j \in B_f, h \in B_4\}$ and $\{x, y, z, t\}$ be any quadruple in $SQS(f+4)$. The following quadruples in $N$ form an $SQS(q)$ [2]:

\[
\begin{align*}
Q_1 &= \{ (A, 0), (A, 1), (A, 2), (A, 3) \}, \\
Q_2 &= \{ (i, x), (i, y), (i, z), (i, t) \} \text{(the quadruple in $Q_1$ excluded)}, \\
Q_3 &= \{ (A, a_1), (0, a_2), (1, a_3), (2, a_4) \} \{a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{f}\}, \\
Q_4 &= \{ (i + 2, b_3), (i, b_1 + 2k + 1 + i(4k + 2) - d), \}
\]

\[
\begin{align*}
&\quad (i, b_1 + 2k + 2 + i(4k + 2) + d), (i + 1, b_2) \} \\
&\quad \text{where } b_1 + b_2 + b_3 \equiv 0 \pmod{f} \text{ and } d \in B_{2k+1}, \\
Q_5 &= \{ (i, r_a), (i, s_a), (i + 1, r'_a), (i + 1, s'_a) \} \text{ where } r_a, s_a \text{ and } r'_a, s'_a
\end{align*}
\]

are (equal or different) pairs in $P_5(6k+5) \{4k + 2 \leq \alpha \leq 12k + 8\}$.

In case $Q_2$, we define $(i, (A, h)) = (A, h)$ for all $i \in B_3$ and $h \in B_4$. For both cases $Q_4$ and $Q_5$, calculations are conducted modulo 3 and $f$ for the first and second coordinates, respectively.

We rename letters in $N$ as follows:

\[
\begin{align*}
(i, L_t) &\rightarrow if + t \ (i \in B_3, t \in B_f), \\
(A, h) &\rightarrow 3f + h \ (h \in B_4).
\end{align*}
\]

To investigate the step property, we first consider the following quadruples of each type except $Q_5$: for $t \in B_{k+1}$, $i \in B_3$, and $h \in B_2$,

\[
\begin{align*}
Q_1 &= \{ (A, 0), (A, 1), (A, 2), (A, 3) \}, \\
Q_2 &= \{ (i, L_{2i}), (i, L_{2i+1}), (i, x), (i, y) \}, \{ (i, x'), (i, y'), (A, 2h), (A, 2h+1) \}, \\
Q_3, Q_4 &= \text{no quadruples to consider}.
\end{align*}
\]
Note that \{(A, 0), (A, 1), (A, 2), (A, 3)\}, \{L_{2t}, L_{2t+1}, x, y\}, and \{x', y', (A, 2h), (A, 2h+1)\} are quadruples in $SQS(f + 4)$. Since $SQS(f + 4)$ satisfies the step property with respect to the given order, it can be easily checked by (5) that the quadruples above satisfy the step property.

In $Q_5$, the following quadruples should be considered: for \(i \in B_3\),
\[
\{(i, L_{2i}), (i, L_{2i+1}), (i+1, x), (i+1, y)\}, \{(i, x'), (i, y'), (i+1, L_{2i}), (i+1, L_{2i+1})\},
\]
where \(\{2t, 2t+1\}\) and \(\{x, y\}\) are pairs in $P_0(6k + 5)$ \((4k + 2 < \alpha < 12k + 8)\). These quadruples also satisfy the step property. Therefore \(q \in \mathcal{S}P\).

**Case IV: \(q \equiv 26 \pmod{36}\).**
This case is similar to Case III. Let \(q = 3f + 2\) where \(f \equiv 8 \pmod{12}\), and denote \(f = 12k + 8\). By induction, we have \(f + 2 \in \mathcal{S}P\). We may assume that there is an $SQS(f + 2)$ on the alphabet $B_{f} \cup \{(A, 0), (A, 1)\}$. Defining \(L_t = 2t, L_{t+1} = 2t + 1\) \((t \in B^+_2)\), we also assume that the given $SQS(f + 2)$ satisfies the step property with respect to the order
\[
L_0 < L_1 < \cdots < L_{f-1} < (A, 0) < (A, 1).
\]

Let \(N = \{(i, j), (A, h) \mid i \in B_3, j \in B_f, h \in B_2\}\) and \(\{x, y, z, t\}\) be any quadruple in $SQS(f + 2)$. The following quadruples in \(N\) form an $SQS(q)$ [2]:
\[
\begin{align*}
Q_1 & : \{(i, x), (i, y), (i, z), (i, t)\}, \\
Q_2 & : \{(A, a_1), (0, a_2), (1, a_3), (2, a_4)\} \ (a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{f}), \\
Q_3 & : \{(i + 2 + b_3), (i, b_1 + 2k + 1 + i(4k + 2) - d), \\
& \quad (i, b_1 + 2k + 2 + i(4k + 2) + d), (i + 1, b_2)\} \\
& \quad \text{where } b_1 + b_2 + b_3 \equiv 0 \pmod{f} \text{ and } d \in B_{2k+1}, \\
Q_4 & : \{(i, r_0), (i, s_0), (i, s_0'), (i + 1, r_0')\} \ \text{where } \{r_0, s_0\} \text{ and } \{r_0', s_0'\} \\
& \quad \text{are (equal or different) pairs in } P_0(6k + 4) \ (4k + 2 < \alpha < 12k + 6).
\end{align*}
\]
In case $Q_1$, we define \((i, (A, h)) = (A, h) \text{ for all } i \in B_3 \text{ and } h \in B_2\). For both cases $Q_3$ and $Q_4$, calculations are performed modulo 3 and \(f\) for the first and second coordinates, respectively.

We again rename letters in $N$ as follows:
\[
\begin{align*}
(i, L_i) & \rightarrow if + t \ (i \in B_3, t \in B_f), \\
(A, h) & \rightarrow 3f + h \ (h \in B_2).
\end{align*}
\]
To investigate the step property, we will first consider the following quadruples of each type except $Q_4$ : for \(t \in B^+_f\) and \(i \in B_3\),
\[
\begin{align*}
Q_1 & : \{(i, L_{2i}), (i, L_{2i+1}), (i, a), (i, b)\}, \{(i, a'), (i, b'), (A, 0), (A, 1)\}, \\
Q_2, Q_3 & : \text{no quadruples to consider.}
\end{align*}
\]
Note that \(\{L_{2t}, L_{2t+1}, a, b\}\) and \(\{a', b', (A, 0), (A, 1)\}\) are quadruples in $SQS(f + 2)$. Since $SQS(f + 2)$ satisfies the step property with respect to the order above, it can be easily checked by (6) that the quadruples above satisfy the step property.

In case $Q_4$, the following quadruples should be considered: for \(i \in B_3\),
\[
\{(i, L_{2i}), (i, L_{2i+1}), (i + 1, a), (i + 1, b)\}, \{(i, a), (i, b), (i + 1, L_{2i}), (i + 1, L_{2i+1})\},
\]

where \( \{a, b\} \) are pairs in \( P_\alpha(6k + 4) \) \((4k + 2 \leq \alpha \leq 12k + 6)\). These quadruples also satisfy the step property. Therefore \( q \in \mathcal{SP} \).

**Case V**: \( q \equiv 2 \) or \( 10 \pmod{24} \) \((q > 2)\).

Let \( q = 4f \) + 2 where \( f \equiv 0 \) or \( 2 \pmod{6} \) \((f > 0)\), and denote \( f = 2k \). By induction, we have \( f + 2 \in \mathcal{SP} \). Hence we may assume that \( SQS(f + 2) \) on \( B_f \cup \{(A, 0), (A, 1)\} \) satisfies the step property with respect to the order

\[ 0 < 1 \cdots < f - 1 < (A, 0) < (A, 1) \]

Let \( N = \{(h, i, j), (A, l) \mid h \in B_2, i \in B_2, j \in B_f, l \in B_2 \} \) and \( \{x, y, z, t\} \) be any quadruple in \( SQS(f + 2) \). The following quadruples in \( N \) form an \( SQS(q) \) assuming that \( c_1 + c_2 + c_3 \equiv 0 \pmod{k} \) and \( \epsilon \in B_2 \),

\[
\begin{align*}
Q_1 &= \{(h, i, x), (h, i, y), (h, i, z), (h, i, t)\}, \\
Q_2 &= \{(A, l), (0, 0, 2c_1), (0, 1, 2c_2 - \epsilon), (1, \epsilon, 2c_3 + l)\}, \\
Q_3 &= \{(A, l), (0, 0, 2c_1 + 1), (0, 1, 2c_2 - 1 - \epsilon), (1, \epsilon, 2c_3 + 1 - l)\}, \\
Q_4 &= \{(A, l), (0, 0, 2c_1) + 1, (1, 1, 2c_2 - \epsilon), (0, \epsilon, 2c_3 + 1 - l)\}, \\
Q_5 &= \{(A, l), (1, 0, 2c_1 + 1), (1, 1, 2c_2 - 1 - \epsilon), (0, \epsilon, 2c_3 + 1 - l)\}, \\
Q_6 &= \{(h, 0, 2c_1 + \epsilon), (h, 1, 2c_2 - \epsilon), (h + 1, 0, 0)\}, \\
Q_7 &= \{(h, 0, 2c_1 - 1 + \epsilon), (h, 1, 2c_2 - \epsilon), (h + 1, 1, 0)\}, \\
Q_8 &= \{(h, 0, 2c_1 + \epsilon), (h, 1, 2c_2 - \epsilon), (h + 1, 1, 0)\}, \\
Q_9 &= \{(h, 0, 2c_1 - 1 + \epsilon), (h, 1, 2c_2 - \epsilon), (h + 1, 1, 0)\}, \\
Q_{10} &= \{(h, 0, r_\alpha), (h, 0, s_\alpha), (h, 1, r'_\alpha), (h, 1, s'_\alpha)\}
\end{align*}
\]

Note that if a quadruple in \( Q_1 \) contains an element of the form \( (h, i, (A, l)) \), we simply denote it by \( (A, l) \). For each case, calculations are conducted modulo 2 and \( f \) for the first and third coordinates, respectively.

We rename letters in \( N \) as follows: for \( j \in B_f \),

\[
\begin{align*}
(0, 0, j) &\to j, & (0, 1, j) &\to f + j, \\
(1, 0, j) &\to 2f + j, & (1, 1, j) &\to 3f + j, \\
(A, 0) &\to 4f, & (A, 1) &\to 4f + 1.
\end{align*}
\]

To investigate the step property, the following quadruples of each type should be considered, except \( Q_6 - Q_{10} \) for \( t \in B_2 \), \( h \in B_2 \), and \( i \in B_2 \),

\[
\begin{align*}
Q_1 &= \{(h, i, 2t), (h, i, 2t + 1), (h, i, a), (h, i, b)\}, \{(A, 0), (A, 1), (h, i, a), (h, i, b)\}, \\
Q_2 - Q_5 &= \text{no quadruples to consider}.
\end{align*}
\]

In case \( Q_1 \), the quadruples \( \{2t, 2t + 1, a, b\} \) and \( \{(A, 0), (A, 1), a, b\} \) are in \( SQS(f + 2) \). Since \( SQS(f + 2) \) satisfies the step property with respect to the given order, it can be easily checked by (7) that the quadruples above satisfy the step property.

In \( Q_6 \), the following quadruples are to be considered: for \( h \in B_2 \) and \( \epsilon \in B_2 \),

\[
\{(h, 0, 2c_1 + \epsilon), (h, 1, 2c_2 - \epsilon), (h + 1, 0, 2t), (h + 1, 0, 2t + 1)\}.
\]

These quadruples also satisfy the step property. Cases \( Q_7 - Q_{10} \) are similar to \( Q_6 \). As a result, \( q \in \mathcal{SP} \).
Case VI: $q \equiv 14$ or 38 (mod 72).

Let $q = 12f + 2$ where $f \equiv 1$ or 3 (mod 6). As the first step, we will prove that $14 \in \mathcal{S}$ and $38 \in \mathcal{S}$. According to [1], we can construct an $SQS(14)$ (Table 4).

Defining $N' = \{(i, j), (A, h) \mid i \in B_3, j \in B_{12}, h \in B_2\}$ to be a set of 38 elements, we will show that $38 \in \mathcal{S}$. Let $\{x', y', z', t'\}$ be any quadruple in $SQS(14)$. The following quadruples in $N'$ form an $SQS(38)$ [2]: assuming that $b_1 + b_2 + b_3 \equiv 0 \pmod{12}$, $e \in B_2$, $g \in B_9$, and $c \in B_4$,

\[
\begin{align*}
Q_1 &= \{(i, x'), (i, y'), (i, z'), (i, t')\}, \\
Q_2 &= \{(A, h), (A, b_1), (1, b_2), (2, b_3 + 3h)\}, \\
Q_3 &= \{(i, b_1 + 4 + i), (i, b_1 + 7 + i), (i + 1, b_2), (i + 2, b_3)\}, \\
Q_4 &= \{(i, j), (i + 1, j + 6e), (i + 2, 6e - 2j + 1), (i + 2, 6e - 2j - 1)\}, \\
Q_5 &= \{(i, j + 1, j + 6e), (i + 2, 6e - 2j + 2), (i + 2, 6e - 2j - 2)\}, \\
Q_6 &= \{(i, j + 1, j + 6e - 3), (i + 2, 6e - 2j + 1), (i + 2, 6e - 2j + 2)\}, \\
Q_7 &= \{(i, j + 1, j + 6e + 3), (i + 2, 6e - 2j - 1), (i + 2, 6e - 2j - 2)\}, \\
Q_8 &= \{(i, j), (i, j + 6), (i + 1, j + 3e), (i + 1, j + 6 + 3e)\}, \\
Q_9 &= \{(i, 2g + 3e), (i, 2g + 6 + 3e), (i', 2g + 1), (i', 2g + 5)\} (i' \neq i), \\
Q_{10} &= \{(i, 2g + 3e), (i, 2g + 6 + 3e), (i', 2g + 2), (i', 2g + 4)\}, \\
Q_{11} &= \{(i, j), (i, j + 1), (i + 1, j + 3e), (i + 1, j + 3e + 1)\}, \\
Q_{12} &= \{(i, j), (i, j + 2), (i + 1, j + 3e), (i + 1, j + 3e + 2)\}, \\
Q_{13} &= \{(i, j), (i, j + 4), (i + 1, j + 3e), (i + 1, j + 3e + 4)\}, \\
Q_{14} &= \{(i, r_a), (i, r_a), (i', r_a'), (i', r_a')\} \text{ where } \{r_a, s_a\} \text{ and } \{r_a', s_a'\}
\end{align*}
\]

are (equal or different) pairs in $P_\alpha(6)$ ($4 \leq \alpha \leq 5$).

In case $Q_1$, we define $(i, (A, h)) = (A, h)$ for all $i \in B_3$ and $h \in B_2$. For each case, calculations are conducted modulo 3 and 12 for the first and second coordinates, respectively.

Now we construct $SQS(q)$ for $q \equiv 14$ or 38 (mod 72), that is, $f \equiv 1$ or 3 (mod 6). Since $f \equiv 1$ or 3 (mod 6), we can assume that $f + 1 \in \mathcal{S}$. In this case, the alphabet of $SQS(f + 1)$ is $B_f \cup \{A\}$ as we assumed. Let $N = \{(i, j), (A, h) \mid i \in B_f, j \in B_{12}, h \in B_2\}$ be a set of $12f + 2$ elements and $\{A, u, v, w\}$ be any quadruple in $SQS(f + 1)$ that contains $A$. The following quadruples from $N$
form an $SQS(q)$:

$$R_1 : \{(i, x'), (i, y'), (i, z'), (i, t')\}, \quad R_2 : \{(i, a_1), (i, a_2), (w, a_3), (w, a_4)\}.$$ 

$$\{[i, z], (i, \beta_2), (i', \beta_2), (i, \beta_4)\} \quad \{i', i\} \subset \{u, v, w\} \text{ and } i' \neq i.$$ 

In case $R_2$, $\alpha_3$ and $\beta_3$ ($\nu \in \mathcal{B}_4$) are to be replaced by the second indices of $Q_3 - Q_{14}$, corresponding to the first indices $0, 1, 2$ for $u, v, w$, respectively. Note that $i$ and $i'$ define uniquely a $\{u, v, w\}$ in which they are contained. Therefore they may be considered as two indices from $\{u, v, w\}$.

$$R_3 : \{(x, a_1), (y, a_2), (z, a_3), (t, a_4)\} \quad (a_1 + a_2 + a_3 + a_4 \equiv 0 \text{ (mod 12)}).$$

From now on, we will show that $q \in \mathcal{S} \mathcal{P}$ for $q \equiv 14$ or 38 (mod 72). In the case that $q = 14$, we rename the letters of $B_{14}$ in Table 4 as follows:

$$\begin{cases} 
t \rightarrow L_{2t} \ (t \in \mathcal{B}_6), \\
t \rightarrow L_{2t-13} \ (7 \leq t \leq 12), \\
6 + 7i \rightarrow L_{i+12} = (A, i) \ (i \in \mathcal{B}_2) 
\end{cases}$$

Assuming $L_0 < L_1 < \cdots < L_{13}$, the following quadruples satisfy the step property:

$$\left\{0,7,1,8\right\}, \left\{0,7,2,9\right\}, \left\{0,7,3,10\right\}, \left\{0,7,4,11\right\}, \left\{0,7,5,12\right\}, \left\{0,7,6,13\right\}, \left\{1,8,0,7\right\}, \left\{1,8,2,9\right\}, \left\{1,8,3,10\right\}, \left\{1,8,4,11\right\}, \left\{1,8,5,12\right\}, \left\{1,8,6,13\right\}, \left\{2,9,0,7\right\}, \left\{2,9,1,8\right\}, \left\{2,9,2,9\right\}, \left\{2,9,4,11\right\}, \left\{2,9,5,12\right\}, \left\{2,9,6,13\right\}, \left\{3,10,0,7\right\}, \left\{3,10,1,8\right\}, \left\{3,10,2,9\right\}, \left\{3,10,4,11\right\}, \left\{3,10,5,12\right\}, \left\{3,10,6,13\right\}, \left\{4,11,0,7\right\}, \left\{4,11,1,8\right\}, \left\{4,11,2,9\right\}, \left\{4,11,3,10\right\}, \left\{4,11,5,12\right\}, \left\{4,11,6,13\right\}, \left\{5,12,0,7\right\}, \left\{5,12,1,8\right\}, \left\{5,12,2,9\right\}, \left\{5,12,3,10\right\}, \left\{5,12,4,11\right\}, \left\{5,12,6,13\right\}, \left\{6,13,0,7\right\}, \left\{6,13,1,8\right\}, \left\{6,13,2,9\right\}, \left\{6,13,3,10\right\}, \left\{6,13,4,11\right\}, \left\{6,13,5,12\right\}.$$

This shows that $14 \in \mathcal{S} \mathcal{P}$.

In the case that $q = 38$, we rename letters of $N'$ as follows:

$$\begin{cases} 
(i, L_t) \rightarrow 12i + t \ (i \in B_{13}, \ t \in B_{12}), \\
(A, h) \rightarrow 36 + h \ (h \in \mathcal{B}_2). 
\end{cases}$$

To investigate the step property of $SQS(38)$, we will consider quadruples of each type.

In case $Q_1$, the following quadruples should be considered: for $i \in B_{13},$

$$\{(i, L_d), (i, L_{d+1}), (i, a), (i, b)\}, \{(i, a), (i, b), (A, 0), (A, 1)\}.$$ 

Since $\{L_{2d}, L_{2d+1}, a, b\}$ and $\{a, b, (A, 0), (A, 1)\}$ are in $SQS(14)$ and satisfy the step property, these quadruples also satisfy the step property by $[5]$.

In cases $Q_2 = Q_{13}$, there are no quadruples to consider.

In case $Q_{14}$, the following quadruples should be considered:

$$\{(i, L_d), (i, L_{d+1}), (i', a), (i', b)\}.$$ 

By $[5]$, these quadruples satisfy the step property. Therefore $38 \in \mathcal{S} \mathcal{P}$.

Finally, consider general cases. First, rename letters in $N$ as follows:

$$\begin{cases} 
(i, L_t) \rightarrow 12i + t \ (i \in B_{13}, \ t \in B_{12}), \\
(A, h) \rightarrow 12f + h \ (h \in \mathcal{B}_2). 
\end{cases}$$

From now on, we will investigate the step property of quadruples of each type.
R1) \{(i, L_{2t}), (i, L_{2t+1}), (i, a), (i, b)\}, \{(i, a), (i, b), (A, 0), (A, 1)\}.
Since \{L_{2t}, L_{2t+1}, a, b\} and \{a, b, (A, 0), (A, 1)\} satisfy the step property, these quadruples also satisfy the step property by (9).

R2) \{(i, L_{2t}), (i, L_{2t+1}), (i', a), (i', b)\}.
By the step property of the constructed SQS(38), these quadruples satisfy the step property by (9).

R3) In this case, there are no quadruples to consider.
Therefore \(q \in SP\)

We constructed an optimal code in \(B_2^4\), so if we use Theorems 3.2 and 3.4, we can deduce the following theorem.

Theorem 3.5. If \(q \equiv 2\) or \(4 \pmod{6}\), then \(N(4, q, 1) = q^2(q+2)/4\).

Remark 3. The authors of [10] informed us that every Steiner quadruple system in [10] satisfies the step property. It is not known whether every Steiner quadruple system satisfies the step property.

4 Construction of optimal codes in \(B_q^4\) for \(q \equiv 0 \pmod{6}\)

In this section, we will construct a code in \(B_q^4\) for \(q \equiv 0 \pmod{6}\) that is capable of correcting single deletions whose cardinality meets the upper bound that was established in Theorem 3.2. Since there dose not exist a Steiner quadruple system for any alphabet of this size, we will use a group divisible system. We divide our construction into two steps. In the first step, we prove that an optimal code exists over an alphabet of size \(q = 6m\), where \(m\) is odd. In the next step, we prove that an optimal code exists over an alphabet of size \(2q\) under the assumption that an optimal code exists over an alphabet of size \(q\).

We begin with the definition of a group divisible system. By an \(r\)-subset of a set \(X\), we mean a subset of \(X\) with \(r\) elements.

Definition 4.1 ([8]). Let \(m\) and \(r\) be positive integers. Let \(\mathcal{T} = \{T_1, T_2, \ldots, T_m\}\) be a collection of disjoint \(r\)-sets whose union is \(T\). An \((m, r, k, b)\) group divisible system or a \(G(m, r, k, b)\) system on \(T\) is a collection \(\mathcal{B} = \{K_1, K_2, \ldots, K_u\}\) of \(k\)-subsets of \(T\) such that every \(b\)-subset in \(T\) is either contained in \(T_i\) for some \(i \in \mathcal{B}\) or it is contained in a unique \(k\)-subset in \(\mathcal{B}\) but not both.

By a simple counting argument,

\[|G(m, r, k, b)| = \binom{mr}{k} - m\binom{r}{k}.\]

The following result was proved by Mills [8].

Theorem 4.2. A \(G(m, 6, 4, 3)\) system exists for every positive integer \(m\).
Because we will use a subfamily $A_1$ of a $G(m, 6, 4, 3)$ system in our construction of an optimal code, we briefly review the proof of Theorem 4.2 in [8], which introduces $A_1$ when $m$ is odd. Let $m$ be odd. For $i \in B_m$, let $T_i$ be the set of ordered pairs $(i, \alpha)$ with $\alpha \in B_6$ and $T$ be the union of these $T_i$. We partition the elements of $B_6$ into pairs in three ways:

\[
\begin{align*}
P_1 &= \{(0, 3), (1, 5), (2, 4)\}, \\
P_2 &= \{(1, 4), (2, 0), (3, 5)\}, \\
P_3 &= \{(2, 5), (3, 1), (4, 0)\}.
\end{align*}
\]

For any ordered pair $(w, x)$ ($w < x$) in $B_m$ and any $\lambda \in \bar{B}_3$, we form the nine quadruples

\[
\{(w, \alpha), (w, \beta), (x, \gamma), (x, \delta)\},
\]

where $(\alpha, \beta) \in P_\lambda$ and $(\gamma, \delta) \in P_\lambda$. This gives a collection of $\frac{27m(m-1)}{2}$ quadruples, denoted by $A$, which is contained in $G(m, 6, 4, 3)$. Among the quadruples in $A$, choose those with $\lambda = 1$ and denote them by $A_1$. Note that $|A_1| = 9\binom{m}{2}$.

**Theorem 4.3.** $N(4, q, 1) = \frac{q^2(q+2)}{2}$ for $q = 6m$ where $m$ is odd.

**Proof.** We retain the notation used in the preceding discussion throughout this proof. Let $m$ be odd and consider $G(m, 6, 4, 3)$ on $T = \{T_1, T_2, \ldots, T_m\}$. Recall that our goal is to construct an optimal code in $B_q^4$ where $q = |T| = 6m$. Define an order on $T$ as follows:

\[
\begin{align*}
(i, \alpha) &< (j, \alpha) \ (i < j), \\
(i, \alpha) &< (j, \beta) \ (\alpha < \beta).
\end{align*}
\]

To each quadruple $\{x, y, z, w\}$ in $G(m, 6, 4, 3)$ with $x < y < z < w$, we associate the word $(x, y, z, w)$. From now on, we consider the quadruples in $G(m, 6, 4, 3)$ as words of length 4 defined as above.

Let

\[
M = \left( \bigcup_{x \in A_1} \langle x \rangle_{B_q^4} \right) \bigcup \left( \bigcup_{x \in G(m, 6, 4, 3) \setminus A_1} \langle x \rangle_{A_1^4} \right).
\]

Note that for any two distinct elements $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ of $A_1$, if $|L(a) \cap L(b)| = 2$, then either $\{a_1, a_2\} = \{b_1, b_2\}$ or $\{a_3, a_4\} = \{b_1, b_4\}$. From the structure of $A_1$, the definition of $G(m, 6, 4, 3)$, and Lemma 2.3 we deduce that $M$ is a code in $B_q^4$ that is capable of correcting single deletions. Since $|A_1| = 9\binom{m}{2}$ and $|G(m, 6, 4, 3)| = \frac{6m(6m-1)(6m-2)}{24} - 5m$, it follows that

\[
|M| = 8|A_1| + 6|G(m, 6, 4, 3) \setminus A_1| = 54m^3 - 18m^2 - 36m.
\]

Since $N(4, 6, 1) = 72$ (cf. Table 3) and $|T_i| = 6$ for each $i \in B_m$, we can choose an optimal code $C_i$ over $T_i$ of cardinality 72 that is capable of correcting single deletions. Let

\[
C = \left( \bigcup_{i=1}^{m} C_i \right) \bigcup M \bigcup \{(a, a, b, b) \mid a \in T_i, b \in T_j, i \neq j\}.
\]
It is easy to check that \( C \) is a code in \( B_q^4 \) that is capable of correcting single deletions. From a simple calculation

\[
|C| = m|C_1| + |M| + 2 \times 36 \left( \frac{m}{2} \right)
\]

\[
= 72m + 54m^3 - 18m^2 - 36m + 36m(m - 1)
\]

\[
= 54m^3 + 18m^2 = \frac{q^2(q + 2)}{4}.
\]

This proves the theorem.

The following lemma, originally due to Reiss [3], constructs the systems that are equivalent to the systems \( P_\alpha(m) \). Because we need some terminology used in [3] to construct optimal codes, we borrow a sketch of the proof from [3].

**Lemma 4.4 (Reiss).** The \( n(2n - 1) \) pairs of \( 2n \) elements can be partitioned into \( 2n - 1 \) sets \( S_1, S_2, \ldots, S_{2n-1} \) such that each set contains \( n \) disjoint pairs.

**Proof.** Let \( l_{ij} = i + j - 1 \pmod{2n - 1} \) where \( i \in \bar{B}_{2n-1}, j \in \bar{B}_{2n-1} \), and \( l_{ij} \in \bar{B}_{2n-1}, \bar{B}_{2n-1} \). Take

\[
S_q = \{(i, j) \mid i < j \text{ and } l_{ij} = q\} \quad (q \in \bar{B}_{2n-1}).
\]

Then these sets \( S_q \) (\( q \in \bar{B}_{2n-1} \)) have the desired property.

**Theorem 4.5.** If \( N(4, q, 1) = \frac{2^q(q+2)}{4} \) for \( q = 6m \), then \( N(4, 2q, 1) = (2^q)^2(2q+2) \).

**Proof.** Suppose that \( N(4, q, 1) = \frac{2^q(q+2)}{4} \) for \( q = 6m \). Let \( \bar{B}_q(\alpha) = \{(a, \alpha) \mid a \in \bar{B}_q\} \) (\( \alpha \in \bar{B}_2 \)). We will construct a code in \( \bar{B}_q(1) \cup \bar{B}_q(2) \) that is capable of correcting single deletions with cardinality \( \frac{(2^q)^2(2q+2)}{4} \). By applying Lemma 4.4 with \( a = 3m \), the pairs of elements of \( \bar{B}_q(\alpha) \) can be partitioned into the sets \( S_1^\alpha, S_2^\alpha, \ldots, S_{6m-1}^\alpha \), namely

the set of pairs of \( \bar{B}_q(\alpha) = S_1^\alpha \cup S_2^\alpha \cup \cdots \cup S_{6m-1}^\alpha \) (\( \alpha \in \bar{B}_2 \)),

where \( S_j^\alpha = \{(a, \alpha) \mid a < b, a + b - 1 = l \pmod{6m - 1}\} \) for \( \alpha \in \bar{B}_2 \) and \( l \in \bar{B}_{6m-1} \). Note that \( |S_j^\alpha| = 3m = \frac{q}{2} \) for all \( \alpha \) and \( l \). Consider the set \( S_{1,2} \) of ordered pairs defined as follows:

\[
S_{1,2} = \{(x, y) \mid x \in S_1^i, y \in S_2^i, i \in \bar{B}_{6m-1}\} = \bigcup_{i=1}^{6m-1} (S_1^i \times S_2^i).
\]

Since \( x \) and \( y \) are pairs of elements in \( \bar{B}_q(\alpha) \) (\( \alpha \in \bar{B}_2 \)), \( S_{1,2} \) is a set of quadruples of elements in \( \bar{B}_q(1) \cup \bar{B}_q(2) \) whose cardinality is \( (6m - 1)|S_1^i||S_2^i| = 9m^2(6m - 1) = \frac{q^2(q-1)}{4} \). Furthermore, \( S_{1,2} \) has the following property: every triple \( \{(c_1, i), (c_2, j), (c_3, k)\} \) with elements from \( \bar{B}_q(1) \cup \bar{B}_q(2) \), except triples with element from only one of \( \bar{B}_q(1) \) or \( \bar{B}_q(2) \), belongs to a unique quadruple in \( S_{1,2} \).

We define an order on \( \bar{B}_q(1) \cup \bar{B}_q(2) \) as follows:

\[
\begin{cases}
(a, \alpha) < (b, \alpha) \text{ if and only if } a < b, \\
(a, 1) < (b, 2) \text{ for all } a, b.
\end{cases}
\]
To each quadruple $x = \{(a, 1), (b, 1), (c, 2), (d, 2)\}$ in $S_{1}^{1.2}$ where $((a, 1), (b, 1)) \in S_{1}^{1}$ and $((c, 2), (d, 2)) \in S_{2}^{2}$, we associate a word $(a_1, a_2, a_3, a_4)$ of length 4 such that $a_1 < a_2 < a_3 < a_4$ and $a_i \in \{(a, 1), (b, 1), (c, 2), (d, 2)\}$. From now on, we consider the quadruples in $S_{1}^{1.2}$ as words of length 4 over $B_{q}(1) \cup B_{q}(2)$ defined as above. Let

$$M = \left( \bigcup_{x \in S_{1}^{1} \times S_{2}^{2}} \langle x \rangle_{B_{q}^{2}} \right) \bigcup \left( \bigcup_{x \in S_{1.1}^{2} \times S_{2}^{2}} \langle x \rangle_{A_{q}^{2}} \right).$$

From the construction of $S_{1}^{1.2}$ and Lemma 2.3, $M$ is a code in $B_{q}(1) \cup B_{q}(2)$ that is capable of correcting single deletions. Note that

$$|M| = 8 |S_{1}^{1}||S_{2}^{2}| + 6 (|S_{1}^{1.2}| - |S_{1}^{1}||S_{2}^{2}|) = \frac{2q^{2}(3q - 2)}{4}.$$

Since $N(4, q, 1) = \frac{q^{2}(q+2)}{4}$ for $q = 6m$, there exists a code $C_{a}$ in $B_{q}(\alpha)$ ($\alpha \in B_{2}$) that is capable of correcting single deletions with cardinality $|C_{a}| = N(4, q, 1) = \frac{q^{2}(q+2)}{4}$ ($\alpha \in B_{2}$).

Finally, let

$$C = C_{1} \cup C_{2} \cup M \cup \{(a, a, b, b), (b, b, a, a) \mid a \in B_{q}(1), b \in B_{q}(2)\}.$$

It is easy to check that $C$ is a code in $B_{q}(1) \cup B_{q}(2)$ that is capable of correcting single deletions with $|C| = \frac{2q^{2}(2q+2)}{4}$. Because $|B_{q}(1) \cup B_{q}(2)| = |B_{2q}|$ and $B_{q}(1) \cap B_{q}(2) = \emptyset$, this proves the theorem.

Combining Theorem 4.3 and 4.5 yields the following theorem.

**Theorem 4.6.** There exists a code $C$ in $B_{q}^{4}$ that is capable of correcting single deletions with $|C| = N(4, q, 1) = \frac{q^{2}(q+2)}{4}$ for $q = 6m$ ($m \geq 1$).

Combining Theorems 3.5 and 4.6, we finally obtain the following.

**Theorem 4.7.** For any even $q$, $N(4, q, 1) = \frac{q^{2}(q+2)}{4}$.

## 5 Perfect codes

In this section, we modify our construction of optimal codes slightly, and construct an optimal perfect code in $B_{q}^{4}$ when $q$ is even.

We start with simple definitions. Recall that a code $C$ in $B_{q}^{n}$ is an $s$-covering of $B_{q}^{n}$ from below if $[C]_s = [B_{q}^{n}]_s$ [8]. A code $C$ in $B_{q}^{n}$ that is capable of correcting $s$ deletions is called a perfect code that is capable of correcting $s$ deletions if $C$ is an $s$-covering of $B_{q}^{n}$ from below. For brevity a perfect code in $B_{q}^{n}$ that is capable of correcting single deletions will be referred to as a perfect code.

Levenshtein [6] showed that there exists a perfect code in $B_{q}^{4}$ of cardinality $\frac{q^{4}+q^{2}+2q}{4}$ for any even $q$. Note that this code is not optimal. In previous sections, we have constructed an optimal code $C$ in $B_{q}^{4}$ for any even $q$. By counting the cardinality of $[C]_1$, one can show that it cannot be a perfect code (for example,
the optimal code in Table 2 is not perfect). However, the construction of an optimal code can be modified to obtain an optimal perfect code as follows.

Suppose that $C$ is an optimal code in $B_4^q$ for an even $q$ which is constructed using the method in Section 2. Decompose $C$ into subcodes $C_1, C_2, \text{ and } C_4$, where $C_i = \{x \in C \mid \| [x]_1 \| = i\}$ for $i \in B_2$ or $i = 4$. From the structure of $C$, the codes $C_1$ and $C_2$ are as follows:

\[
C_1 = \{(a, a, a, a) \mid a \in B_q\},
\]

\[
C_2 = \{(a, a, b, b) \mid \{a, b\} \subset B_q \text{ and } a \neq b\}.
\]

Note that $|C_1| = q, |C_2| = q(q - 1)$, and that $|C_4| = \frac{2^q(q+2)}{4} - (q + q(q - 1))$.

Now we modify the subcode $C_2$ to make $C_2'$ as follows:

\[
C_2' = \left(C_2 \setminus \{(2t, 2t, 2t + 1, 2t + 1) \mid t \in B_{q/2} - 1\}\right) \cup \{(2t, 2t + 1, 2t + 1) \mid t \in B_{q/2} - 1\}.
\]

It can be easily verified that $C' = C_1 \cup C_2' \cup C_4$ is a code that is capable of correcting single deletions with $|C| = |C'|$. Note that $|C_2'| = |C_2|$ and $|C_2' \setminus C_2| = \frac{q}{2}$.

For each $x \in C'$, the following relations arise after single deletions:

\[
\begin{align*}
| [x]_1 | &= 1 \text{ (} x \in C_1 \text{),} \\
| [x]_1 | &= 2 \text{ (} x \in C_2 \text{),} \\
| [x]_1 | &= 4 \text{ (} x \in C_2' \setminus C_2 \text{),} \\
| [x]_1 | &= 4 \text{ (} x \in C_4 \text{).}
\end{align*}
\]

Computing the cardinality of $|C'|_1$ yields:

\[
|C'|_1 = |C_1|_1 + |C_2'|_1 + |C_4|_1 = q + 2(q(q - 1) - \frac{q}{2}) + 4\frac{q}{2} + 4\left(\frac{2^q(q+2)}{4} - q - q(q - 1)\right) = q^3.
\]

Hence $C'$ is an optimal perfect code.

Therefore we obtain the following theorem.

**Theorem 5.1.** For any even $q$, we can construct an optimal perfect code in $B_4^q$ that is capable of correcting single deletions.

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