Limits of the trivial bundle on a curve

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Abstract. We attempt to describe the vector bundles on a curve $C$ which are specializations of $O_C^2$. We get a complete classification when $C$ is Brill-Noether-Petri general, or when it is hyperelliptic; in both cases all limit vector bundles are decomposable. We give examples of indecomposable limit bundles for some special curves.

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Titre. Limites du fibré trivial sur une courbe

Résumé. Nous essayons de décrire les fibrés vectoriels qui sont des spécialisations de $O_C^2$. Nous obtenons une classification complète lorsque $C$ est générale au sens de Brill-Noether-Petri, ou lorsque $C$ est hyperelliptique; les fibrés limites sont décomposables dans chacune des deux situations. Nous donnons également des exemples de fibrés limites indécomposables sur certaines courbe spéciales.
1. Introduction

Let $C$ be a smooth complex projective curve, and $E$ a vector bundle on $C$, of rank $r$. We will say that $E$ is a limit of $O_C^r$ if there exists an algebraic family $(E_b)_{b \in B}$ of vector bundles on $C$, parametrized by an algebraic curve $B$, and a point $o \in B$, such that $E_o = E$ and $E_b \cong O_C^r$ for $b \neq o$. Can we classify all these vector bundles? If $E$ is a limit of $O_C^2$ clearly $E \oplus O_C^{r-2}$ is a limit of $O_C^r$, so it seems reasonable to start in rank 2.

We get a complete classification in two extreme cases: when $C$ is generic (in the sense of Brill-Noether theory), and when it is hyperelliptic. In both cases the limit vector bundles are of the form $L \oplus L^{-1}$, with some precise conditions on $L$. However for large families of curves, for instance for plane curves, some limits of $O_C^2$ are indecomposable, and those seem hard to classify.

2. Generic curves

Throughout the paper we denote by $C$ a smooth connected projective curve of genus $g$ over $\mathbb{C}$.

**Proposition 1.** Let $L$ be a line bundle on $C$ which is a limit of globally generated line bundles (in particular, any line bundle of degree $\geq g + 1$). Then $L \oplus L^{-1}$ is a limit of $O_C^2$.

*Proof.* By hypothesis there exist a curve $B$, a point $o \in B$ and a line bundle $L$ on $C \times B$ such that $L_{|C \times \{o\}} \cong L$ and $L_{|C \times \{b\}}$ is globally generated for $b \neq o$. We may assume that $B$ is affine and that $o$ is defined by $f = 0$ for a global function $f$ on $B$; we put $B' := B \setminus \{o\}$.

We choose two general sections $s, t$ of $L$ on $C \times B'$; reducing $B'$ if necessary, we may assume that they generate $L$. Thus we have an exact sequence on $C \times B'$

$$0 \to L^{-1} \xrightarrow{(t,-s)} O^2_{C \times B'} \xrightarrow{(s,t)} L \to 0$$

which corresponds to an extension class $e \in H^1(C \times B', L^{-2})$. For $n$ large enough, $f^ne$ comes from a class in $H^1(C \times B, L^{-2})$ which vanishes along $C \times \{o\}$; this class gives rise to an extension

$$0 \to L^{-1} \to E \to L \to 0$$

with $E_{|C \times \{b\}} \cong O^2_C$ for $b \neq o$, and $E_{|C \times \{o\}} \cong L \oplus L^{-1}$. \hfill $\square$

**Remark 1.** Let $E$ be a vector bundle limit of $O_C^r$. We have $\det E = O_C$, and $h^0(E) \geq 2$ by semi-continuity. If $E$ is semi-stable this implies $E \cong O_C^r$; otherwise $E$ is unstable. Let $L$ be the maximal destabilizing sub-line bundle of $E$; we have an extension $0 \to L \to E \to L^{-1} \to 0$, with $h^0(L) \geq 2$. Note that this extension is trivial (so that $E = L \oplus L^{-1}$) if $H^1(L^2) = 0$, in particular if $\deg(L) \geq g$.

**Proposition 2.** Assume that $C$ is Brill-Noether-Petri general. The following conditions are equivalent:

(i) $E$ is a limit of $O_C^2$;
(ii) $h^0(E) \geq 2$ and $\det E = \mathcal{O}_C$;

(iii) $E = L \oplus L^{-1}$ for some line bundle $L$ on $C$ with $h^0(L) \geq 2$ or $L = \mathcal{O}_C$.

**Proof.** We have seen that (i) implies (ii) (Remark 1). Assume (ii) holds, with $E \not\cong \mathcal{O}_C^2$. Then $E$ is unstable, and we have an extension $0 \to L \to E \to L^{-1} \to 0$ with $h^0(L) \geq 2$. Since $C$ is Brill-Noether-Petri general we have $H^0(C, K_C \otimes L^{-2}) = 0$ [ACG, Ch. 21, Proposition 6.7], hence $H^1(C, L^2) = 0$. Therefore the above extension is trivial, and we get (iii).

Assume that (iii) holds. Brill-Noether theory implies that any line bundle $L$ with $h^0(L) \geq 2$ is a limit of globally generated ones \footnote{Indeed, the subvariety $W^r_d(C)$ parametrizing line bundles $L$ with $h^0(L) \geq r+1$ is equidimensional, of dimension $g - (r+1)(r+g-d)$; the line bundles which are not globally generated belong to the subvariety $W^r_{d-1}C$, which has codimension $r$.}. So (i) follows from Proposition 1.

### 3. Hyperelliptic curves

**Proposition 3.** Assume that $C$ is hyperelliptic, and let $H$ be the line bundle on $C$ with $h^0(H) = \deg(H) = 2$. The limits of $\mathcal{O}_C^2$ are the decomposable bundles $L \oplus L^{-1}$, with $\deg(L) \geq g + 1$ or $L = H^k$ for $k \geq 0$.

**Proof.** Let $\pi : C \to \mathbb{P}^1$ be the two-sheeted covering defined by $|H|$. Let us say that an effective divisor $D$ on $C$ is simple if it does not contain a divisor of the form $\pi^* p$ for $p \in \mathbb{P}^1$. We will need the following well-known lemma:

**Lemma 1.** Let $L$ be a line bundle on $C$.

1) If $L = H^k(D)$ with $D$ simple and $\deg(D) + k \leq g$, we have $h^0(L) = h^0(H^k) = k + 1$.

2) If $\deg(L) \leq g$, $L$ can be written in a unique way $H^k(D)$ with $D$ simple. If $L$ is globally generated, it is a power of $H$.

**Proof of Lemma 1.** 1) Put $\ell := g - 1 - k$ and $d := \deg(D)$. Recall that $K_C \cong H^{g-1}$. Thus by Riemann-Roch, the first assertion is equivalent to $h^0(H^\ell(-D)) = h^0(H^\ell) - d$. We have $H^0(C, H^\ell) = \pi^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\ell))$; since $D$ is simple of degree $\leq \ell + 1$, it imposes $d$ independent conditions on $H^0(C, H^\ell)$, hence our claim.

2) Let $k$ be the greatest integer such that $h^0(L \otimes H^{-k}) > 0$; then $L = H^k(D)$ for some effective divisor $D$, which is simple since $k$ is maximal. By 1) $D$ is the fixed part of $|L|$, hence is uniquely determined, and so is $k$. In particular the only globally generated line bundles on $C$ of degree $\leq g$ are the powers of $H$.

**Proof of the Proposition:** Let $E$ be a vector bundle on $C$ limit of $\mathcal{O}_C^2$. Consider the exact sequence

$$0 \to L \to E \to L^{-1} \to 0,$$

where we can assume $\deg(L) \leq g$ (Remark 1). By Lemma 1 we have $L = H^k(D)$ with $D$ simple of degree $\leq g - 2k$. After tensor product with $H^k$, the corresponding cohomology exact sequence reads

$$0 \to H^0(C, H^{2k}(D)) \to H^0(C, E \otimes H^k) \to H^0(C, \mathcal{O}_C(-D)) \xrightarrow{\partial} H^1(C, H^{2k}(D))$$

which implies $h^0(E \otimes H^k) = h^0(H^{2k}(D)) + \dim \ker \partial = 2k + 1 + \dim \ker \partial$ by Lemma 1.

By semi-continuity we have $h^0(E \otimes H^k) \geq 2h^0(H^k) = 2k + 2$; the only possibility is $D = 0$ and $\partial = 0$. But $\partial(1)$ is the class of the extension (l), which must therefore be trivial; hence $E = H^k \otimes H^{-k}$.
4. Examples of indecomposable limits

To prove that some limits of $O_C^2$ are indecomposable we will need the following easy lemma:

**Lemma 2.** Let $L$ be a line bundle of positive degree on $C$, and let

$$0 \to L \to E \to L^{-1} \to 0$$

be an exact sequence. The following conditions are equivalent:

(i) $E$ is indecomposable;

(ii) The extension (2) is nontrivial;

(iii) $h^0(E \otimes L) = h^0(L^2)$.

**Proof.** The implication (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) : After tensor product with $L$, the cohomology exact sequence associated to (2) gives

$$0 \to H^0(L^2) \overset{i}{\to} H^0(E \otimes L) \to H^0(O_C) \overset{\partial}{\to} H^1(L^2),$$

where $\partial$ maps $1 \in H^0(O_C)$ to the extension class of (2). Thus (ii) implies that $i$ is an isomorphism, hence (iii).

(iii) $\Rightarrow$ (i): If $E$ is decomposable, it must be equal to $L \oplus L^{-1}$ by unicity of the destabilizing bundle. But this implies $h^0(E \otimes L) = h^0(L^2) + 1.$

The following construction was suggested by N. Mohan Kumar:

**Proposition 4.** Let $C \subset \mathbb{P}^2$ be a smooth plane curve, of degree $d$. For $0 < k < \frac{d}{4}$, there exist extensions

$$0 \to O_C(k) \to E \to O_C(-k) \to 0$$

such that $E$ is indecomposable and is a limit of $O_C^2$.

**Proof.** Let $Z$ be a finite subset of $\mathbb{P}^2$ which is the complete intersection of two curves of degree $k$, and such that $C \cap Z = \emptyset$. By [S, Remark 4.6], for a general extension

$$0 \to O_{\mathbb{P}^2}(k) \to E \to I_Z(-k) \to 0,$$

the vector bundle $E$ is a limit of $O_{\mathbb{P}^2}^2$; therefore $E|_C$ is a limit of $O_C^2$.

The extension (3) restricts to an exact sequence

$$0 \to O_C(k) \to E|_C \to O_C(-k) \to 0.$$

To prove that $E|_C$ is indecomposable, it suffices by Lemma 2 to prove that $h^0(E|_C(k)) = h^0(O_C(2k)).$ Since $2k < d$ we have $h^0(O_C(2k)) = h^0(O_{\mathbb{P}^2}(2k)) = h^0(E(k))$, so in view of the exact sequence

$$0 \to E(k-d) \to E(k) \to E|_C(k) \to 0$$

it suffices to prove $H^1(E(k-d)) = 0$, or by Serre duality $H^1(E(d-k-3)) = 0$.

The exact sequence (3) gives an injective map $H^1(E(d-k-3)) \hookrightarrow H^1(I_Z(d-2k-3))$. Now since $Z$ is a complete intersection we have an exact sequence

$$0 \to O_{\mathbb{P}^2}(-2k) \to O_{\mathbb{P}^2}(-k)^2 \to I_Z \to 0;$$

since $4k < d$ we have $H^2(O_{\mathbb{P}^2}(d-4k-3)) = 0$, hence $H^1(I_Z(d-2k-3)) = 0$, and finally $H^1(E(d-k-3)) = 0$ as asserted. \hfill $\square$
We can also perform the Strømme construction directly on the curve $C$, as follows. Let $L$ be a base point free line bundle on $C$. We choose sections $s, t \in H^0(L)$ with no common zero. This gives rise to a Koszul extension
\[ 0 \rightarrow L^{-1} \xrightarrow{i} O_C^2 \xrightarrow{p} L \rightarrow 0 \quad \text{with} \quad i = (-t, s), \ p = (s, t). \tag{4} \]
We fix a nonzero section $u \in H^0(L^2)$. Let $\mathcal{L}$ be the pull-back of $L$ on $C \times \mathbb{A}^1$. We consider the complex ("monad")
\[ L^{-1} \xrightarrow{\alpha} L^{-1} \oplus O_C \oplus \mathcal{L} \xrightarrow{\beta} \mathcal{L}, \quad \alpha = (\lambda, i, u), \ \beta = (u, p, -\lambda), \]
where $\lambda$ is the coordinate on $\mathbb{A}^1$. Let $\mathcal{E} := \text{Ker} \beta/\text{Im} \alpha$, and let $E := \mathcal{E}_{|C \times \{0\}}$.

**Lemma 3.** $E$ is a rank 2 vector bundle, limit of $O_C^2$. There is an exact sequence $0 \rightarrow L \rightarrow E \rightarrow L^{-1} \rightarrow 0$; the corresponding extension class in $H^1(L^2)$ is the product by $u^2 \in H^0(L^4)$ of the class $e \in H^1(L^{-2})$ of the Koszul extension (4).

*Proof.* The proof is essentially the same as in [S]; we give the details for completeness.

For $\lambda \neq 0$, we get easily $\mathcal{E}_{|C \times \{\lambda\}} \cong O_C^2$; we will show that $E$ is a rank 2 vector bundle. This implies that $\mathcal{E}$ is a vector bundle on $C \times \mathbb{A}^1$, and therefore that $E$ is a limit of $O_C^2$.

Let us denote by $\alpha_0, \beta_0$ the restrictions of $\alpha$ and $\beta$ to $C \times \{0\}$. We have $\text{Ker} \beta_0 = L \oplus N$, where $N$ is the kernel of $(u, p) : L^{-1} \oplus O_C^2 \rightarrow L$. Applying the snake lemma to the commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & L^{-1} \\
\downarrow & & \downarrow i \\
0 & \rightarrow & O_C^2 \\
\downarrow & & \downarrow p \\
0 & \rightarrow & L \\
\end{array}
\]
we get an exact sequence
\[ 0 \rightarrow L^{-1} \rightarrow N \rightarrow L^{-1} \rightarrow 0, \tag{5} \]
which fits into a commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & L^{-1} \\
\downarrow & & \downarrow i \\
0 & \rightarrow & O_C^2 \\
\downarrow & & \downarrow p \\
0 & \rightarrow & L \\
\end{array}
\]
this means that the extension (5) is the pull-back by $\times u : L^{-1} \rightarrow L$ of the Koszul extension (4).

Now since $E$ is the cokernel of the map $L^{-1} \rightarrow L \oplus N$ induced by $\alpha_0$, we have a commutative diagram
\[
\begin{array}{ccc}
0 & \rightarrow & L^{-1} \\
\downarrow & & \downarrow i \\
0 & \rightarrow & N \\
\downarrow & & \downarrow p \\
0 & \rightarrow & E \\
\end{array}
\]
so that the extension $L \rightarrow E \rightarrow L^{-1}$ is the push-forward by $\times u$ of (5). This implies the Lemma. \hfill \square

Unfortunately it seems difficult in general to decide whether the extension $L \rightarrow E \rightarrow L^{-1}$ nontrivial. Here is a case where we can conclude:

**Proposition 5.** Assume that $C$ is non-hyperelliptic. Let $L$ be a globally generated line bundle on $C$ such that $L^2 \cong K_C$. Let $0 \rightarrow L \rightarrow E \rightarrow L^{-1} \rightarrow 0$ be the unique nontrivial extension of $L^{-1}$ by $L$. Then $E$ is indecomposable, and is a limit of $O_C^2$.
Proof. We choose \( s, t \) in \( H^0(L) \) without common zero, and use the previous construction. It suffices to prove that we can choose \( u \in H^0(K_C) \) so that \( u^2 e \neq 0 \): since \( H^1(K_C) \cong \mathbb{C} \), the vector bundle \( E \) will be the unique nontrivial extension of \( L^{-1} \) by \( L \), and indecomposable by Lemma 2.

Suppose that \( u^2 e = 0 \) for all \( u \) in \( H^0(K_C) \); by bilinearity this implies \( uv e = 0 \) for all \( u, v \) in \( H^0(K_C) \). Since \( C \) is not hyperelliptic, the multiplication map \( S^2 H^0(K_C) \to H^0(K_C^2) \) is surjective, so we have \( w e = 0 \) for all \( w \in H^0(K_C^2) \). But the pairing

\[
H^1(K_C^{-1}) \otimes H^0(K_C^2) \to H^1(K_C) \cong \mathbb{C}
\]

is perfect by Serre duality, hence our hypothesis implies \( e = 0 \), a contradiction.

\( \square \)

Remark 2. In the moduli space \( \mathcal{M}_g \) of curves of genus \( g \geq 3 \), the curves \( C \) admitting a line bundle \( L \) with \( L^2 \cong K_C \) and \( h^0(L) \geq 2 \) form an irreducible divisor \([T2]\); for a general curve \( C \) in this divisor, the line bundle \( L \) is unique, globally generated, and satisfies \( h^0(L) = 2 \) \([T1]\). Thus Proposition 5 provides for \( g \geq 4 \) a codimension 1 family of curves in \( \mathcal{M}_g \) admitting an indecomposable vector bundle limit of \( O_C^2 \).

Remark 3. Let \( \pi : C \to B \) be a finite morphism of smooth projective curves. If \( E \) is a vector bundle limit of \( O_B^2 \), then clearly \( \pi^* E \) is a limit of \( O_C^2 \). Now if \( E \) is indecomposable, \( \pi^* E \) is also indecomposable. Consider the nontrivial extension \( 0 \to L \to E \to L^{-1} \to 0 \) (Remark 1); by Lemma 2 it suffices to show that the class \( e \in H^1(B, L^2) \) of this extension remains nonzero in \( H^1(C, \pi^* L^2) \). But the pull-back homomorphism \( \pi^* : H^1(B, L^2) \to H^1(C, \pi^* L^2) \) can be identified with the homomorphism \( H^1(B, L^2) \to H^1(B, \pi_* \pi^* L^2) \) deduced from the linear map \( L^2 \to \pi_* \pi^* L^2 \), and the latter is an isomorphism onto a direct factor; hence \( \pi^* \) is injective and \( \pi^* e \neq 0 \), so \( E \) is indecomposable.

Thus any curve dominating one of the curves considered in Propositions 4 and 5 carries an indecomposable vector bundle which is a limit of \( O_C^2 \).

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