Using a local gyrokinetic code to study global ion temperature gradient modes in tokamaks

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Abstract

In this paper the global eigenmode structures of linear ion temperature gradient (ITG) modes in tokamak plasmas are obtained using a novel technique which combines results from the local gyrokinetic code GS2 with analytical theory to reconstruct global properties. Local gyrokinetic calculations are performed for a range of radial flux surfaces, \(x\), and ballooning phase angles, \(p\), to map out the local complex mode frequency, \(\Omega \left( x, p \right) = \omega \left( x, p \right) + i \gamma \left( x, p \right)\) for a single toroidal mode number, \(n\). Taylor expanding \(\Omega \) about a reference surface at \(x = 0\), and employing the Fourier-ballooning representation leads to a second order ODE for the amplitude envelope, \(A(p)\), which describes how the local results are combined to form the global mode. The equilibrium profiles impact on the variation of \(\Omega \left( x, p \right)\) and hence influence the global mode structure. The simulations presented here are based upon a global extension to the CYCLONE base case and employ the circular Miller equilibrium model. In an equilibrium with radially varying profiles of \(a/L_T\) and \(a/L_n\), peaked at \(x = 0\), and with all other equilibrium profiles held constant, including \(\eta_i = L_i/L_n\), \(\Omega \left( x, p \right)\) is found to have a stationary point. The reconstructed global mode sits at the outboard mid-plane of the tokamak, with global growth rate, \(\gamma \sim \text{Max}[\gamma_0]\). Including the radial variation of other equilibrium profiles like safety factor and magnetic shear, leads to a mode that peaks away from the outboard mid-plane, with a reduced global growth rate. Finally, the influence of toroidal flow shear has also been investigated through the introduction of a Doppler shift, \(\omega \rightarrow \omega - n \Omega_\phi \), where \(\Omega_\phi\) is the equilibrium toroidal flow, and a prime denotes the radial derivative. The equilibrium profile variations introduce an asymmetry into the global growth rate spectrum with respect to the sign of \(\Omega_\phi\), such that the maximum growth rate is achieved with non-zero shearing, consistent with recent global gyrokinetic calculations.

Keywords: local gyrokinetics, profile shearing, flow shear, global gyrokinetics

(Some figures may appear in colour only in the online journal)

1. Introduction

Tokamaks [1] provide one of the most stable and promising configurations for magnetic confinement fusion. However, confinement in tokamaks is not perfect; there are a number of mechanisms by which energy and particles can be transported across the magnetic flux surfaces from the core confinement region to the plasma edge. The main contribution is typically due to turbulent transport, which is widely believed to originate primarily from microinstabilities driven by density and temperature gradients. The so-called ‘drift’ modes, with low frequency compared to the cyclotron frequency, are the dominant tokamak microinstabilities [1, 2], and the turbulence they drive influences the minimum size of magnetic confinement fusion reactors, such as ITER and DEMO [3]. It is therefore important to
Understand these drift modes in order to seek ways to reduce their impact. Previous theoretical and numerical studies have shown that flow shear can control the stability of drift waves, providing a mechanism to suppress or even stabilise them completely [4–9]. While microinstabilities must ultimately be saturated by nonlinear physics (e.g. zonal flows), it nevertheless remains important to understand the structure of linear instabilities and the threshold gradients associated with their onset: linear physics lies close to the heart of many simplified plasma models that are of considerable value, e.g. the quasi-linear based TGLF model of anomalous transport in tokamaks [10]. Microinstabilities are often investigated via numerical solutions of the gyrokinetic equation [11–13] and several different approaches are typically used, as we now discuss.

For high toroidal mode numbers, n, the rational flux surfaces (i.e. those where the safety factor, q, is rational) are closely packed so equilibrium quantities vary only weakly from one surface to the next. Two length scales can then be identified: the equilibrium length scale, characterised by the minor radius, a, and the distance between rational surfaces, \( \Delta = \left( \frac{d \theta}{d r} \right) \), where the derivative is with respect to radius, r. The separation between these length scales is exploited in ballooning theory [14–17] to perform an expansion in the small parameter, \( \Delta a \). The rational surfaces spanned by a mode are then equivalent to leading order. Local gyrokinetic codes, like GS2 [16, 18], exploit this ‘ballooning symmetry’ to reduce the intrinsic 2D problem (in radius, r, and poloidal angle, \( \theta \)) to 1D in the extended ballooning coordinate, \( \eta \), which is aligned with magnetic field lines. Local gyrokinetic codes only provide the structure along the magnetic field line, together with the local eigenvalue, \( \Omega_0(x, p) \) (where \( x = (r - r_0)/a \) is the normalised radial distance from the reference surface, \( r_0 \), and the ballooning phase angle, \( p \), is the value of \( \eta \) where the radial derivative of the eigenfunction is zero). Local codes, however, provide neither the radial mode structure nor the global eigenvalue \( \Omega \), as both \( x \) and \( p \) are free parameters at this lowest order in the ballooning expansion where local gyrokinetics is valid. The radial mode structure and global eigenvalue, \( \Omega \), are determined at the next order in the \( \Delta a \) expansion, where the eigenfunction’s dependence on \( x \) and \( p \) becomes constrained.

Exploiting the higher order theory to solve for the full global eigenmode has been demonstrated previously for a simplified fluid model of ion temperature gradient (ITG) modes [19]. In this paper, we build on [19] to show how this formalism can be extended to more realistic full gyrokinetic plasma models. Our approach exploits the GS2 code [16, 18] to provide the local mode structure, \( \xi(x, p, \eta) \), and \( \Omega_0(x, p) \), from which the higher order theory provides the global mode structure and \( \Omega \).

It has previously been demonstrated that under the very special conditions where \( \Omega_0(x, p) \) has a stationary point, a so-called isolated or conventional ballooning mode is obtained from the higher order analysis [14]. This type of mode, originally studied in the context of ideal MHD theory [20, 21], is usually located at the outboard mid-plane, where the poloidal angle \( \theta = 0 \) [19]. The higher order theory for such an isolated mode provides a global complex mode frequency \( \Omega \), which includes an \( O(1/n) \) correction to the local eigenvalue, \( \Omega_0(x, p) \), evaluated at the maximally unstable \( x \) and \( p \). This maximally unstable ‘isolated mode’ requires a stationary point in the local complex mode frequency \( \Omega_0(x, p) \), which is a very special situation. More usually a second class of mode, known as the ‘general mode’, would be expected. The higher order theory for these general modes predicts that they peak away from the outboard mid-plane and have a reduced growth rate relative to the isolated mode. Local gyrokinetic simulations alone cannot distinguish between these two types of mode, but evidence for both can be found in global gyrokinetic simulations that include the radial variation of equilibrium profiles. For example, electrostatic ITG modes are found to be shifted relative to the outboard mid-plane in linear global gyrokinetic simulations of ASDEX Upgrade plasmas [22]. Such up–down asymmetries are important as they could provide a mechanism for flow generation in tokamaks [23–26], which may be important in a machine like ITER for which the external torque is small. While nonlinear simulations are likely necessary for a complete understanding of turbulence and flows in plasmas, linear theory provides a picture of the important physical mechanisms. The technique we present here is an alternative approach to full global simulations that uses an efficient formalism to shed light on the key linear physics.

The paper is organised as follows. In section 2 we review the theoretical formalism on which this paper is based. Results from applying this technique to the so-called CYCLONE base case [27, 28], are presented in section 3. Employing this standard test case enables us to compare with previously published global simulations [29]. Finally, section 4 summarises our conclusions, and plans for future work.

2. The technique: from local to global gyrokinetic calculations

We employ the initial value local gyrokinetic code, GS2 [16, 18], to solve the linearised gyrokinetic equation [11, 12] numerically. GS2 provides the local eigenvalue, \( \Omega_0(x, p) \) as well as the mode structure along a given magnetic field line \( \xi(x, p, \eta) \) in the infinite domain in \( \eta \) with a periodic dependence on \( p \). To reconstruct the linear global mode properties from these local modes, we employ the Fourier–Ballooning (FB) representation [30]:

\[
\phi(x, \theta, t) = \int_{-\infty}^{\infty} \xi(x, \theta, t) \exp(-i n q_0 \theta) \exp(-i n q \phi(x(\theta - p)) d \phi d p
\]

where \( \phi(x, \theta, t) \) is the global mode structure for fluctuations in the electrostatic potential, \( x \) is the radial variable, \( q_0 \) is the safety factor at \( x = 0 \), and the local mode structure \( \xi(x, p, \theta, t) \) is obtained from GS2. Note that, the \( x \)-dependence in \( \xi \) is due to a slow dependence of the equilibrium on \( x \), and is a parameter at this order. For the linear global modes studied in this paper, \( \phi(x, \theta, t) \) and \( \xi(x, p, \theta, t) \) have a separable time dependence of the form \( \propto e^{-\mu t} \) and \( e^{-\mu |\theta|} e^{-\mu |p|} \) where \( \Omega \) and \( \Omega_0(x, p) \)

\[\bot\]
are the global and local complex linear eigenmode frequencies, respectively. In the rest of the paper, we will frequently drop the explicit $t$ dependence, and use $\phi(x, \theta)$ and $\xi(x, p, \theta)$ to denote the separated spatial dependent part of the linear eigenmode structure. The mapping in equation (1) from the infinite domain in $\eta$ to the periodic poloidal angle $\theta$, is possible because of the symmetry property $\xi(x, p + 2m \pi, \theta + 2m \pi) = \xi(x, p, \theta)$, for any integer $m$.

In order to employ equation (1) to find the global mode structure, $\phi(x, \theta, t)$, we must evaluate the envelope $A(p)$, which can be obtained from the higher order theory as follows. We seek a global eigenmode satisfying

$$\frac{\partial \phi(x, \theta, t)}{\partial t} = -i \Omega \phi(x, \theta, t)$$

$$\frac{\partial \xi(x, p, \theta, t)}{\partial t} = -i \Omega \xi(x, p, \theta, t)$$

(2)

where $\Omega = \omega + i \gamma$, with $\omega$ and $\gamma$ corresponding to the global frequency and growth rate, respectively. Differentiating equation 1 with respect to $t$ and using equation 1a allows us to write:

$$\Omega \phi(x, \theta, t) = \int \Omega \phi(x, \theta, t) \xi(x, p, \theta, t)$$

$$\times \exp(-i \eta \phi) \exp(-i \eta \chi(x - p)) |A(p)| dp$$

(3)

Using equation (1) for $\phi$, we can then write our eigenmode condition in the form:

$$\int \left[ \Omega - \Omega \phi(x, \theta, t) \right] \xi(x, p, \theta, t)$$

$$\times \exp(-i \eta \phi) \exp(-i \eta \chi(x - p)) |A(p)| dp = 0$$

(4)

Here, the local complex mode frequency, $\Omega \phi(x, p)$, and $\xi$ are mapped out by running the local gyrokinetic code, GS2, several times spanning the required range of $x$ and over $-\pi < p < \pi$. This process is trivially parallelised, and therefore will usually be more efficient than a full global simulation. Anticipating radially localised solutions in $x$ we may Taylor expand $\Omega \phi$ to second order in $x$ about $x = 0$ and Fourier expand in $p$ to write:

$$\Omega \phi(x, p) = \sum_{k=0}^{N} a_k^{(m)} x^m \cos(kp)$$

(5)

where $N$ is the number of Fourier modes retained. The coefficients $a_k^{(m)}$ are complex numbers, obtained by fitting the expansion to the full $\Omega \phi(x, p)$ results which are obtained from GS2. Now substituting equation (5) into (4), we obtain

$$\int \left[ \Omega - \sum_{k=0}^{N} \left( a_k^{(0)} + a_k^{(1)} x + a_k^{(2)} x^2 \right) \cos(kp) \right] A(p)$$

$$\times \xi(x, p, \theta, t) \exp(-i \eta \phi) \exp(-i \eta \chi(x - p)) dp = 0.$$  

We note that $nq/p$ is a radial wave number. Therefore, following standard Fourier transform procedures and assuming $A(p)$ varies much more rapidly than $\xi$ with $p$, allows us to replace $x^m A(p)$ with $(-i nq/p)^m A dp dp^m$, to give:

$$\int \left[ \Omega A - \sum_{k=0}^{N} \left( a_k^{(0)} \cos(kp) - \frac{ia_k^{(1)}}{nq} \cos(kp) \right) \frac{d}{dp} \right] A(p)$$

$$\times \xi(x, p, \theta, t) \exp(-i \eta \phi) \exp(-i \eta \chi(x - p)) dp = 0.$$  

(6)

Equation (6) must hold for all $x$ and $\theta$, which then provides our final equation for $A(p)$:

Table 1. CYCLONE equilibrium parameters.

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| $\xi$     | 0.8   | $r_0(m)$  | 0.313 |
| $q_0$     | 1.4   | $\beta$   | 0.0   |
| $a/l_x$   | 2.54  | $nq^2$    | 144   |
| $a/l_w$   | 0.81  | $\nu_{dal/c}^2$ | 0.28 |
| $k_{l_\rho_1}$ | 0.58 | $T_i/T_e$ | 1.0 |
| $a(m)$    | 0.625 | $\rho_{l_\rho_1}$ | 0.003384 |
| $R(m)$    | 1.70  | $\rho_{l_\rho_1}/a$ | 0.005415 |
Section 2 to reconstruct the global structure for the most unstable mode.

Before considering the global calculations, we first describe the local ballooning analysis. Our focus is on ITG modes, believed to be one of the common causes for turbulent transport in many tokamak plasmas. Nevertheless, we note that our technique applies to all classes of high $n$ toroidal drift modes. We adopt the CYCLONE parameters [27, 28] and for convenience we have limited ourselves to electrostatic calculations with adiabatic electrons, which allows us to make comparisons with previously published results from global simulations. The model parameters are given in table 1 for a Miller equilibrium with circular flux surfaces.

Figure 1 shows the local real frequency, $\omega$, and growth rate, $\gamma$, obtained from GS2 as functions of normalised binormal wave number, $k_p\rho_i$, for the dominant modes with $p = 0$, together with the local mode structure, $\xi$ at $x = 0$. The results for a normalised ion–ion collision frequency $\nu_{\text{ii}} = 0.28$ are found to be very similar to those for a collisionless plasma for $p = 0$. We use this finite collision rate in the remaining calculations as it helps to suppress unphysical modes found by GS2 at values of $p$ close to marginal stability, and yet gives local eigenmode results similar to the collisionless case. The most unstable mode is found at $k_p\rho_i = 0.58$ which corresponds to toroidal mode number $n = 39$. Now we apply the technique presented in section 2 to reconstruct the global structure for the most unstable mode.
Table 2. The model coefficients, \( a_{k}^{(m)} \), with ten Fourier modes. The real and imaginary components contribute to the real frequency, \( \omega_{0} \), and linear growth rate, \( \gamma_{0} \), respectively. Note that coefficients with \( m = 1 \) are all zero for this special case.

| \( k \) | \( m = 0 \) | \( m = 2 \) |
|------|-------|-------|
| 0    | 0.1177 − 0.0680 i | −1.5689 − 1.9352 i |
| 1    | 0.1804 + 0.1221 i | 1.3347 − 2.2466 i |
| 2    | 0.0462 + 0.0461 i | 0.0825 + 0.8734 i |
| 3    | 0.0229 + 0.0231 i | −0.0015 + 0.2477 i |
| 4    | 0.0068 + 0.0098 i | −0.1007 − 0.0227 i |
| 5    | 0.0012 + 0.0078 i | −0.1090 + 0.1078 i |
| 6    | −0.0022 + 0.0045 i | −0.1134 + 0.1240 i |
| 7    | −0.0033 + 0.0023 i | −0.0861 + 0.0856 i |
| 8    | −0.0035 − 0.0000 i | −0.0461 + 0.0105 i |
| 9    | −0.0026 − 0.0018 i | 0.0111 − 0.0525 i |

3 Results including the equilibrium variation shown in figures 2(a) and (b) will be considered in the next subsection.

The radial mode width, \( \Delta \omega_{r} \), and its variation with the toroidal mode number, \( n \) (or equivalently \( 1/p_{\phi} \)), has also been calculated and is shown in figure 5. Here the width is defined as the full width half maximum of a Gaussian fit to the magnitude of \( \phi \) at \( \theta = p_{\theta} \), where \( p_{\theta} \) is the ballooning phase angle at which the global mode peaks poloidally. We find that the radial width of the mode scales inversely with the square root of \( n, \Delta \omega_{r} \propto n^{-0.49} \) as expected for isolated modes [20].

Changing the toroidal mode number (and \( p_{\phi} \) to keep \( k_{B} p_{\phi} \) fixed) also affects the global mode frequency, \( \Omega \). Figure 6 shows that both the real frequency, \( \omega_{r} \), and the linear growth rate, \( \gamma_{0} \), scale with \( n \) according to: \( \omega_{r} \propto 0.364 − 0.072 n^{−0.49} \) and \( \gamma_{0} \propto 0.147 − 0.492 n^{−1.02} \), respectively. This indicates that the finite \( n \) correction scales inversely with the toroidal mode number as expected from conventional ballooning theory [20]. In the limit \( n \to \infty \), the global complex mode frequency \( \Omega \) converges to the local complex mode frequency at \( p = 0 \), where \( \Omega_{0}(0, 0) = \omega_{0}(0, 0) + i \gamma_{0}(0, 0) = 0.364 + 0.147 i \). This confirms that our results are consistent with analytic theory and higher order ballooning calculations presented in [14, 19, 31].

To summarise, this mode has all the characteristics of the isolated mode identified in [14, 15, 19]. It exists in this particular case because our choice of profiles ensure that \( \Omega_{0}(x, p) \)
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has a stationary point at $x = 0, p = 0$, as is clear from figure 3. However, as we shall see in the following subsection, taking into account the radial variation of other equilibrium profiles introduces a small, but significant, deviation from conventional ballooning (or isolated) modes.

3.2. Global calculations with profile variations

Now we introduce the other radially varying profiles given in figure 2, and repeat the analysis of section 3.1. These profiles vary over the radial scale of the instability and give rise to a significant linear radial variation in $\Omega_0(x, p)$, such that the $a_k^{(1)}$ terms in equation (5) become significant. These extra linear terms impact $\omega_0(x, p)$ and $\gamma_0(x, p)$ differently, which influences the global mode structure. Specifically, the positions of the maxima in $\omega_0(x, p)$ and $\gamma_0(x, p)$ occur at different $x$, which affects the stability and poloidal position of the global mode.

Figure 7 shows the reconstructed global mode structure for a number of cases where we introduce different profiles. The effect of variation in radius $x$ itself is presented in panel (a). The mode shifts slightly downward with respect to the outboard mid-plane. Panel (b) shows the effect of $q$ and $\hat{\xi}$ profile variations and results in a mode with an upwards poloidal shift. Panel (c) shows the combined influence of both (a) and (b) and demonstrates that for this particular case they are competing and almost cancel. The result is slightly in favour of the $x$ variation, leading to a slight net downward poloidal shift with respect to the outboard mid-plane. Finally, in panel (d), in addition to the profiles considered in (c), we include radial variations in the $T$ and $n_e$ profiles. The reconstructed global mode is then shifted poloidally downward with respect to the outboard mid-plane.

For the various profile variations considered above, the up–down poloidal symmetry is broken. This moves the mode slightly off the mid-plane and towards the good curvature region, which reduces the growth rate compared to the flat $\eta_i$ profile case considered in section 3.1. Specifically, for case (d) with full profile variation we find $\Omega = 0.334 + 0.119i$. As we can see, the profile variations tilt the structure on the outboard mid-plane. This finding agrees qualitatively with other published results for full global simulations of linear ITG modes [29]. Our interpretation is that this is a consequence of profile variations in the equilibrium.

3.3. Calculations with profile variations and sheared toroidal flow

We now consider the influence of a small toroidal flow shear on stability and the reconstructed mode structures. This has been achieved by moving into the frame rotating toroidally with the plasma surface at $x = 0$, and by introducing the following Doppler shift into $\Omega_0^3$:

$$\Omega_0(x, p) \rightarrow \Omega_0(x, p) - n\Omega_R x = \left[\omega_0(x, p) - n\Omega_R x\right] + i\eta_0(x, p),$$

where $\Omega_R$ is constant and sets the flow shearing rate. Thus $\Omega_R x$ represents the toroidal rotation frequency (normalised

4 While this method can in principle handle an arbitrary experimental profile of toroidal rotation, for the simple example here we assume that the toroidal flow varies linearly in $x$. 

Figure 4. (a) presents the numerical solution of the envelope function, $A(p)$, obtained from equation (7), as a function of ballooning phase angle, $p$, and (b) shows the reconstructed electrostatic potential global mode structure, $\phi(x, \theta)$, in the poloidal plane. The profile variations other than $L_T$ and $L_n$ are excluded from this calculation, and the mode structure, centred on the outboard mid-plane, is aligned radially where $\theta = 0$. The solid black lines indicate the radial domain of the calculation.

Figure 5. The radial mode width, $\Delta w_r$, as a function of toroidal mode number, $n$. It scales, approximately, inversely with square root of $n$ according to: $\Delta w_r = 0.687n^{-0.49}$, as expected for an isolated mode.
to \(c_\ell/a\) of the magnetic flux surfaces measured relative to the mode rational surface at \(x = 0\) where the reconstructed global mode sits. We have considered \(-0.06 \leq \Omega_\ell^k \leq +0.06\). Note that we assume a purely toroidal flow here, as poloidal flows are expected to be damped in a tokamak plasma. From equation (8) it is clear that the flow shear influences the \(a_k^{(1)}\) coefficients of equation (5). Flow shear will therefore influence global modes in a similar way to the variations in other profiles, which we will now quantify.

Depending on the sign of the flow shear the reconstructed global mode shifts upward or downward relative to the outboard mid-plane with a structure in the poloidal plane that is very similar to that in figure 7. For a mode peaking at \(\theta = 0\) in the absence of flow shear, increasing flow shear tilts the mode structure on the outboard mid-plane and lowers the linear growth rate. Figure 8 shows how the growth rate varies with flow shear for the different profile variations considered earlier in figures 4 and 7. Keeping only the symmetric \(L_T, L_n\) profiles the growth rate curve is symmetric about \(\Omega_\ell^k = 0\), shown by curve (e) of figure 8. However, when other profile variations are taken into account, an asymmetry is introduced into the dependence of growth rate on the sign of \(\Omega_\ell^k\), as shown in figure 8 (for curves a–d).

Only in the case where we have purely symmetric profiles about \(x = 0\) is the growth rate maximised at \(\Omega_\ell^k = 0\). In the other cases the peak in growth rate occurs at non-zero flow shear. We can explain this as follows. We expect this maximally unstable isolated mode to exist when both \(\omega_0(x, p)\) and \(\gamma_0(x, p)\) are stationary at the same position. From symmetry we anticipate \(\partial \omega_0/\partial p \bigg|_p = 0 = \partial \gamma_0/\partial p \bigg|_p = 0\), so let

\[
\begin{align*}
\omega(x, \theta) &= \omega_0(x) + \gamma_0(x) \theta, \\
\gamma(x, \theta) &= \gamma_0(x) + \omega_0(x) \theta.
\end{align*}
\]

Figure 6. (a) The global real frequency \(\omega\) and (b) growth rate \(\gamma\) as functions of toroidal mode number, \(n\), for fixed \(k, \rho_i = 0.58\). The \(\omega\) and \(\gamma\) are measured in units of \((c_s/a)\). They scale with \(n\) as: \(\omega = 0.364 - 0.072n^{-1.12}\), \(\gamma = 0.147 - 0.492n^{-1.02}\).

Figure 7. The reconstructed electrostatic potential global mode structure, \(\phi(x, \theta)\), for \(n = 39\), in the poloidal plane for different radial profile variations taken from figure 2: (a) \(L_T, L_n\) and \(x\) vary, here both temperature \(T\) and density \(n_e\) are assumed to be constant, (b) \(L_T, L_n, q\) and \(\delta\) vary (c) \(L_T, L_n, x, q\) and \(\delta\) vary and finally (d) full profile variation in which \(L_T, T, L_n, n_e, x, q\) and \(\delta\) all vary.
us consider \( p = 0 \). Then Doppler-shifting our expression for \( \Omega_0(x, p) \) given in equation (5) and differentiating with respect to \( x \), we find

\[
\frac{\partial \Omega_0}{\partial x} \bigg |_{p=0} = a_r^{(1)} - n\Omega_0' + 2a_r^{(2)}x
\]

and

\[
\frac{\partial \eta}{\partial x} \bigg |_{p=0} = a_i^{(1)} + 2a_i^{(2)}x
\]

where \( a_{r,i}^{(m)} = \sum_k a_{k(r,i)}^{(m)} \) and the subscripts \( r \) and \( i \) indicate the real and imaginary component, respectively. We require \( \frac{\partial \eta}{\partial x} = 0 \), which then provides an equation for the mode’s radial position, \( x = x_0 = -a_i^{(1)}/2a_i^{(2)} \). Substituting this into equation (9) we find the critical shearing rate, \( \Omega_{\phi_m} \) for which \( \partial \omega_0/\partial x = 0 \) at this same value of \( x \):

\[
\Omega_{\phi_m}' = -\frac{1}{n} \left[ a_r^{(2)}/a_i^{(2)} - a_r^{(1)}/a_i^{(1)} \right]
\]

Thus, we expect the maximally unstable isolated mode to exist for this critical shearing rate, \( \Omega_{\phi_m}' \), but centred on \( x = x_0 \) rather than \( x = 0 \). Considering the full profile variation case (curve-d), our numerical results show the growth rate peaks at \( \Omega_{\phi_m}' \approx -0.017 \) with a value of \( \gamma = 0.136 \), which is \( \sim 10\% \) higher than the growth rate for zero flow shear (\( \gamma = 0.119 \)). For this case \( a^{(1)} = -0.668 + 0.269i \) and \( a^{(2)} = 0.361 - 6.68i \). Substituting these values into equation (11) provides \( \Omega_{\phi_m}' = -0.0168 \), which is in good agreement with curve-d of figure 8.

We can compare our flow shear results with the global simulations performed in [29]. This is complicated as we employ a toroidal flow, while the simulations of [29] employ an \( E \times B \) flow, which is almost poloidal. Nevertheless, if parallel flows have a negligible impact, the two can be related by a geometric factor. We can avoid needing this geometric factor by considering the ratio of the flow shear which maximises the growth rate to the value required to stabilise the mode. Our result of \( \phi = -0.38 \) then agrees very well with that of [29], which is \( \phi = -0.39 \).

Figure 9 presents the scaling of \( \Omega_{\phi_m}' \) and offset in the poloidal angle \( p_0 \) with toroidal mode number \( n \). As expected
from equation (11) $\Omega_{p\ell m}$ scales with $1/n$ (or $\rho_s$). Furthermore, we find at large $n$ that $|\gamma_0| = a + bn^{-\alpha}$, where for our profile choice $a = 0.041$, $b = 1.88$ and $\alpha = 0.925$. We have also investigated the effect of asymmetry in the growth rate spectrum on the reconstructed global mode structure, and this is illustrated in figure 10. For $\Omega_\phi = 0$ the structure is already tilted, but increasing flow shear in the negative direction acts to realign the mode radially and for a critical value of flow shear, $\Omega_\phi = \Omega_{p\ell m} \approx -0.017$, the effect of the profile variation is completely compensated, allowing an isolated mode again to form with largest growth rate, $\gamma = \max[\gamma_0(x, p)]$. Note the mode is radially shifted slightly relative to $x = 0$ at this value of $\Omega_\phi$. This is consistent with the above analysis. Increasing flow shear even further, beyond the critical value, tilts the mode structure in the opposite direction and lowers its linear growth rate again. These results, obtained purely from solutions of GS2 and the higher order theory, are again in good qualitative agreement with global calculations of linear electrostatic ITG modes presented in [29].

To summarise, for realistic and experimentally relevant cases where we take the profile variations into account, we do not in general expect to find pure isolated modes. Isolated modes can form only in special radial locations where the equilibrium profiles produce a stationary point in $\Omega_\phi(x, p)$. However, making adjustments to one equilibrium profile while the others are fixed, can produce the required stationary point and lead to the onset of the isolated mode, as arises in the above example for a critical toroidal flow shear equal to $\Omega_{p\ell m}$.

4. Conclusion

In this work we have reconstructed the 2D global mode structure and the global growth rate for linear electrostatic ITG modes, using local solutions from a gyrokinetic code (GS2) and higher order ballooning theory. This approach, which is solid provided that equilibrium quantities vary slowly across rational surfaces, has provided additional insight into the physics of global simulations of linear microinstabilities in tokamak plasmas. Our first investigations used radial profiles for the mode drives that were peaked and symmetric about $x = 0$, and we held all other equilibrium profiles constant; this results in the local complex mode frequency, $\Omega_\phi(x, p)$, having a stationary point at $x = 0$. This condition produces a special class of mode, known as the ‘isolated mode’, that peaks at the outboard mid-plane with a large growth rate, $\gamma \sim \max[\gamma_0(x, p)]$. These results are in very good qualitative agreement with the simplified fluid model of ITG modes presented in [19]. Introducing radial variation into other equilibrium profiles, we show that the radial position of the stationary points in both local frequency, $\Omega_\phi(x, p)$, and growth rate, $\gamma_0$, become shifted with respect to each other. In this case, the reconstructed global mode becomes less unstable and shifts poloidally away from the outboard mid-plane.

Toroidal flow shear, introduced as a Doppler shift in the real frequency, also influences the global mode. Starting from the conditions of an isolated mode, with drive profiles peaked and symmetric about $x = 0$ and with no other profile variations, adding a constant flow shear is always found to be stabilising. When other profile variations are included, flow shear can be destabilising when the flow shear counteracts the tilting of the mode structure at the outboard mid-plane that is induced by the other profile variations. This results in an asymmetry in the growth rate as a function of flow shear about $\Omega_\phi = 0$, which is in qualitative agreement with previous global gyrokinetic calculations [29]. Moreover, flow shear is also found to shift the mode radially. For a critical flow shear (or a critical toroidal mode number for a given flow shear—see equation (11)) the isolated mode can exist even with arbitrary profiles.
In this paper we have focussed mainly on studying special surfaces where fast growing isolated modes arise because of a peaked drive that results in a stationary point in $\Omega_d(x, p)$. For an arbitrary equilibrium the drive for a particular mode number may indeed have local maxima on some special surfaces, which can be located using local gyrokinetic solutions to find stationary points in $\Omega_d(x, p)$. It is this strongly growing isolated mode that will typically be captured by global, initial value gyrokinetic codes, providing such surfaces exist. Nevertheless, away from these special surfaces we do not expect to find high growth rate isolated modes, but slower growing general modes will predominate. (We note that local codes cannot distinguish between isolated and general modes without including higher order corrections.) This might have important implications for quasilinear transport models, and for flow generation mechanisms, though other effects associated with nonlinearities and turbulence will also be important: fully nonlinear global simulations are needed to assess the relative importance of the linear physics that is described here. In this paper we have exploited the higher order analysis to obtain the global mode structure and frequency of the stronger isolated mode. Future work will assess the impact of the slower growing general modes on transport in other regions of the plasma away from stationary points in $\Omega_d(x, p)$.

Finally, we point out that the procedure used in this work is quite general and can be used to explore more realistic tokamak equilibria and more complicated instabilities, including the effect of shaping (e.g. elongation and triangularity) and electromagnetic modes with kinetic electrons. Future work will extend our study to explore these effects.

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References

[1] Wesson J 2004 Tokamaks (Oxford: Clarendon)
[2] Horton W 1999 Rev. Mod. Phys. 71 735–78
[3] ITER Physics Basis 1999 Nucl. Fusion 39 2137–638
[4] Connor J W and Martin T J 2007 Plasma Phys. Control. Fusion 49 1497–507
[5] Terry P 2000 Rev. Mod. Phys. 72 109
[6] Kishimoto Y et al 1999 Plasma Phys. Control. Fusion 41 A663
[7] Roach C M et al 2009 Plasma Phys. Control. Fusion 51 124020
[8] Biglari H, Diamond P H and Terry P W 1990 Phys. Fluids B 2 1
[9] Waltz R E, Kerbel G D and Milovich J 1994 Phys. Plasmas 1 1229
[10] Staebler G M, Kinsey J E and Waltz R E 2007 Phys. Plasmas 51 055909
[11] Rutherford P H and Frieman E A 1968 Phys. Fluids 11 569–85
[12] Frieman E A and Chen L 1982 Phys. Fluids 25 502–8
[13] Taylor J B and Hastie R J 1968 Plasma Phys. 10 479
[14] Taylor J B, Wilson H R and Connor J W 1996 Plasma Phys. Control. Fusion 38 243–50
[15] Dewar R L 1997 Plasma Phys. Control. Fusion 39 453–70
[16] Kotschenreuther M, Rewoldt G and Tang W M 1995 Comput. Phys. Commun. 88 128
[17] Connor J W, Taylor J B and Wilson H R 1993 Phys. Rev. Lett. 70 1803
[18] Dorland W, Jenko F, Kotschenreuther M and Rogers B N 2000 Phys. Rev. Lett. 85 5579
[19] Dickinson D, Roach C M, Skipper J M and Wilson H R 2014 Phys. Plasmas 21 010702
[20] Taylor J B, Connor J W and Hastie R J 1979 Proc. R. Soc. Lond. A: Math. Phys. Sci. 365 1–7
[21] Connor J W, Hastie R J and Taylor J B 1978 Phys. Rev. Lett. 40 396
[22] Bottino A et al 2004 Phys. Plasmas 11 198–2006
[23] Camenen Y, Idomura Y, Jolliet S and Peeters A G 2011 Nucl. Fusion 51 073039
[24] Peeters A G et al 2011 Nucl. Fusion 51 094027
[25] Diamond P H et al 2013 Nucl. Fusion 53 104019
[26] Parra F I and Barnes M 2014 arXiv:1407.1286
[27] Dimits A M et al 2000 Phys. Plasmas 7 969
[28] Falchetto G L et al 2008 Plasma Phys. Control. Fusion 50 124015
[29] Hill P et al 2012 Plasma Phys. Control. Fusion 54 065011
[30] Zhang Y Z and Mahajan S M 1991 Phys. Lett. A 157 133
[31] Romanelli F and Zonca F 1993 Phys. Fluids B 5 4081