UNIFORM PARAMETRIZATION IN PSEUDO-COMPLEX HYPERBOLIC $\mathbb{C}^n$ SPACE

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ABSTRACT. The parametrization theorem is derived in a flat $nD$ pseudo-complex affine space. The pseudo-complex hyperbolic space accommodates $n$-number of uncompactified time-like extra dimensions with signature $(s, r)$, where $s$ and $r$ are the numbers of minus and plus signs associated with the diagonalized metric matrix. The main result of the theorem suggests a uniform parametrization for both time-like and space-like dimensions. The uniformization requirement preserves complex-hyperbolic inner product associated with the space. As application, the elements of the space is shown to be invariant under linear transformation.

1. INTRODUCTION

The standard approach in dealing with higher dimensional theories is to consider almost exclusively space-like extra dimensions\cite{1}. Large extra-dimensions have been used to address the hierarchy problem, whereas Higgs mass is proven to be finite\cite{2}. The effects of the extra space-like dimensions have been examined in the context of 4D superspace formalism\cite{3}. However, there is no prior reason why extra time-like dimensions cannot exist. Time-like extra dimensions have been ignored due to serious conflicts with causality and unitarity\cite{4}\cite{5}\cite{6}. Time-like extra dimensions have been used within the framework of brane world models, as an alternative in reconciling the mass hierarchy problem\cite{7}. It has been shown that extra time-like dimensional theories can yield tachyons-free modes\cite{8}\cite{9}. The main result of this paper is a generalization of some of the results obtained in\cite{10}, specifically, the constraint of the time-like dimensions. The complex nature of the pseudo-complex hyperbolic affine $\mathbb{C}^n$ space and metric function lead to the uniform parametrization theorem.

In the next section, the hyperbolic parametrization of elements, $p^A(\psi) \in M \subset \mathbb{C}^n$ space and $\tilde{p}^A(\psi) \in W \subset T_{p^A(\psi)}(M)$, where $M$ and $W$ are some open subsets of $\mathbb{R}^{s+r}$ space and its associated tangent space $T_{p^A(\psi)}(M)$, are derived and contained in the lemma. The results from the lemma are used in one of the inductive steps associated with the uniformization theorem.

Key words and phrases. Uniformization, Tangent Bundle, Pseudo-Hyperbolic Space.
1.1. **Mathematics.** This section serves as a review on higher dimensional manifold\[11\]. Consider a smooth \( n \)-dimensional differential manifold \( M \) equipped with an atlas of charts \( (U_A, p_A) \) and

\[
p_A : U_A \to \mathbb{R}^n,
\]

where \( U_A \subset M \) is an open contractible subset. Equivalently, a point \( p^A \) in \( M \) is be parametrized by \( \psi \in \mathbb{R} \)

\[
p^A : \mathbb{R} \to M.
\]

Since \( U_A \) is an open contractible subset of \( M \), then there is a restriction on the diffeomorphism from \( T(U_A) \) to \( U_A \times \mathbb{R}^n \) to just linear isomorphism from \( T_{p^A(\psi)}(U_A) \) to \( \{p^A(\psi)\} \times \mathbb{R}^n \). The tangent bundle \( T(M) \) of \( M \) defined as the disjoint union of the tangent spaces of \( M \)

\[
T(M) = \bigsqcup_{p^A(\psi) \in M} T_{p^A(\psi)}(M) = \bigcup_{p^A(\psi) \in M} \{p^A(\psi)\} \times T_{p^A(\psi)}(M),
\]

where \( T_{p^A(\psi)}(M) \) denotes the tangent space to \( M \) at the point \( p^A(\psi) \). By inspection, the dimensionality of the tangent bundle \( T(M) \) is twice the dimension of the differential manifold \( M \). The primary purpose of the tangent bundle \( T(M) \) is to provide a domain and range for the derivative of a smooth function, i.e.

\[
f : M \to W,
\]

where \( M \) and \( W \) are some smooth differential manifolds. The derivative of the function \( f, Df \) is also a smooth function

\[
Df : T(M) \to T(W).
\]

Elements of \( T(M) \) are pairs of the forms \( (p^A(\psi), \dot{p}^A(\psi)) \), where

\[
\dot{p}^A(\psi) = \frac{dp^A(\psi)}{d\psi}.
\]

The elements or points on a smooth differential manifold \( M \) can be obtained via a natural projection map \( \pi \) defined by

\[
\pi : T(M) \to M,
\]

i.e.

\[
\pi(p^A(\psi), \dot{p}^B(\psi)) = p^A(\psi).
\]

The mapping

\[
\tilde{p}_A : \pi^{-1}(U_A) \to \mathbb{R}^{2n},
\]

defined by

\[
\tilde{p}_A(p^A(\psi), v^B B) = (p^A(\psi), v^B),
\]

where \( \tilde{A} = 1, \ldots, 2n \) and \( A, B = 1, \ldots, n \). The tangent bundle of \( M \) is itself a manifold of dimensionality of \( 2n \), and provides the domain and range through the derivative map

\[
Df : T(M) \to T(W).
\]
The higher-order tangent bundle can be recursively defined by the relation
\[ T^p(M) = T(T^{p-1}(M)) , \]
where \( T^p \) is \( p \)-order tangent bundle, provides the domain and range through the \( p \)-derivative map
\[ D^p f : T^p(M) \to T^p(W) . \]
The dimensionality of higher-order tangent bundle can be obtained via equation (1.12). The second-order tangent bundle
\[ T^2(M) = \bigcup_{p^A(\psi) \in T^p(M)} \{ p^A(\psi) \} \times T_{p^A(\psi)}(T^p(M)) , \]
has dimensionality of \( 4n \). It is straightforward to see that the \( p \)-order tangent bundle
\[ T^p(M) = T(T^{p-1}(M)) = \bigcup_{p^A(\psi) \in T^{p-1}(M)} \{ p^A(\psi) \} \times T_{p^A(\psi)}(T^{p-1}(M)) , \]
has dimensionality of \( 2^p \cdot n \).

**Lemma 1.** Show that the hyperbolic parametrized forms of \( p_{A,l+1-s}(\psi) \) and \( \dot{p}_{A,l+1-s}(\psi) \) are given by
\[ p_{A,l+1-s}(\psi) = \sqrt{r} \sum_{i=1}^{s} \sinh(\sqrt{sr}\psi) \hat{t}_i + R_{eff} \sum_{j=s+1}^{k+1} \cosh(\sqrt{sr}\psi) \hat{x}_j , \]
and
\[ \dot{p}_{A,l+1-s}(\psi) = r R_{eff} \sum_{i=1}^{s} \cosh(\sqrt{sr}\psi) \hat{t}_i + \sqrt{sr} R_{eff} \sum_{j=s+1}^{k+1} \sinh(\sqrt{sr}\psi) \hat{x}_j , \]
subjected to the following initial condition
\[ p_{A,n}(0) = (t_i(0), x_j(0)) = \left( 0, ..., 0, \underbrace{R_{eff}, ..., R_{eff}}_{s}, \underbrace{0, ..., 0}_{k+1-s} \right) , \]
where \( i \in \{1, ..., s\} \) and \( j \in \{s+1, ..., k+1\} \).

**Proof.** The \( nD \) flat pseudo-complex affine hyperbolic space \( H^{s,r} \) is a subset of \( \mathbb{C}^n \), and is defined as
\[ H^{s,r} = \left\{ (t_i, x_j) : -\sum_{i=1}^{s} t_i^2 + \sum_{j=s+1}^{k+1} x_j^2 = R^2 \right\} \]
where \( R_{eff} = f(r)R \), \( f(r) \) is a scaling function of \( r \), \( R_{eff} \) is the effective positive constant curvature of \( H^{s,r} \) affine space. For the case of one extra time-like dimension, the space defined in [10] can be shown to be equivalent to a 5D AdS space. The hyperbolic parametrized \( p_{A,l+1-s}(\psi) \) and \( \dot{p}_{A,l+1-s}(\psi) \) can be obtained by solving
the following systems of differential equations

\[
\begin{align*}
\dot{x}_j &= \sum_{i=1}^s t_i, \\
\dot{t}_i &= \sum_{j=s+1}^{k+1} x_j \\
p \otimes \dot{p} &= -\sum_{i=1}^s t_i t_i + \sum_{j=s+1}^{k+1} x_j \dot{x}_j \\
p \otimes \dot{p} &= -\sum_{j=s+1}^{k+1} \sum_{i=1}^s t_i x_j + \sum_{j=s+1}^{k+1} \sum_{i=1}^s x_j t_i = 0
\end{align*}
\]
(1.20)

By inspection, the systems of differential equations satisfy the following constrained relations:

\[
\begin{align*}
p_{A,t+1-s} (\psi) &= (t_i, x_j), \\
p_{A,t+1-s} (\psi) \@ p_{A,t+1-s} (\psi) &= -\sum_{i=1}^s t_i^2 + \sum_{j=s+1}^{k+1} x_j^2 = R^2, \\
p_{A,t+1-s} (\psi) \@ p_{A,t+1-s} (\psi) &= -\sum_{i=1}^s \sum_{j=1}^s t_i t_j \delta_{ij}, \\
p_{A,t+1-s} (\psi) \@ \dot{p}_{A,t+1-s} (\psi) &= -\sum_{i=1}^s t_i^2 + \sum_{j=s+1}^{k+1} x_j \dot{x}_j = 0,
\end{align*}
\]
(1.22)

where \(i \in \{1, \ldots, s\}\) and \(j \in \{s+1, \ldots, k+1\}\). The vectors \(p_A^\prime\) and \(\dot{p}_A\) are said to be \(H^{a^r}\) perpendiculars w.r.t. each other. By inspection, the hyperbolic parametrization for elements of \(p^A\) and \(p_A\) can be obtained by solving the following systems of differential equations,

\[
\begin{align*}
\dot{x}_j (\psi) &= t_1 (\psi) + \cdots + t_s (\psi) = \sum_{i=1}^s t_i \\
\dot{x}_j (\psi) &= \sum_{i=1}^s t_i (\psi),
\end{align*}
\]
(1.24)

and

\[
\dot{t}_i (\psi) = \sum_{j=s+1}^{k+1} x_j (\psi).
\]
(1.25)

Taking the derivative of equation (1.24) and using the constrained equation (1.25), we have

\[
\begin{align*}
\ddot{x}_j (\psi) &= \sum_{i=1}^s \dot{t}_i (\psi) \\
\ddot{x}_j (\psi) &= \sum_{i=1}^s \left( \sum_{j=s+1}^{k+1} x_j (\psi) \right) \\
\ddot{x}_j (\psi) &= s \left( \sum_{j=s+1}^{k+1} x_j (\psi) \right) \\
\ddot{x}_j (\psi) &= s r x_j (\psi)
\end{align*}
\]
(1.26)

The solution to equation (1.26) takes a general form of

\[
\dot{x}_j (\psi) = A_j e^{\sqrt{s} \tau \psi} + B_j e^{-\sqrt{s} \tau \psi} \quad \forall j \in \{s+1, \ldots, k+1\},
\]
(1.27)

where \(A_j\) and \(B_j\) are arbitrary constants and will be determined by the initial condition. To determine the constants, take the derivative of equation (1.27), and substituting in for equation (1.24)

\[
\dot{x}_j (\psi) = \sqrt{s} r A_j e^{\sqrt{s} \tau \psi} - \sqrt{s} r B_j e^{-\sqrt{s} \tau \psi} = \sum_{i=1}^s t_i (\psi).
\]
(1.28)
Imposing the temporal initial condition (1.18) to equation (1.28), yields
\[ \begin{align*}
\dot{x}_j(0) &= \sqrt{sr} A_j e^{\sqrt{sr} \psi} - \sqrt{sr} B_j e^{-\sqrt{sr} \psi} \\
&= \sum_{i=1}^{s} t_i(0) = 0 \\
\Rightarrow &\quad A_j - B_j = 0 \quad \forall j \in \{s + 1, ... k + 1\}.
\end{align*} \tag{1.29} \]

The spatial solution (1.27) becomes
\[ x_j(\psi) = A_j \left( e^{\sqrt{sr} \psi} + e^{-\sqrt{sr} \psi} \right). \tag{1.30} \]

Imposing the spatial initial condition on equation (1.30)
\[ x_j(0) = A_j \left( e^{\sqrt{sr} \psi} + e^{-\sqrt{sr} \psi} \right) = R_{\text{eff}} \Rightarrow 2A_j = R_{\text{eff}} \Rightarrow A_j = \frac{R_{\text{eff}}}{2} \quad \forall j \in \{s + 1, ... k + 1\}. \tag{1.31} \]

The particular spatial solution yields
\[ x_j(\psi) = R_{\text{eff}} \left( e^{\sqrt{sr} \psi} + e^{-\sqrt{sr} \psi} \right) / 2 \tag{1.32} \]

Using the spatial solution (1.32), the s-number of temporal solutions contained in equation (1.25) can now be obtained from
\[ \begin{align*}
t_1(\psi) &= (k + 1 - s) x_j(\psi) \\
&= r x_j(\psi) = x_j(\psi) = r R_{\text{eff}} \cosh \left( \sqrt{sr} \psi \right)
\end{align*} \tag{1.33} \]

where \( r = (k + 1 - s) \). For each \( i = 1, ..., s \), we have the following system of differential equations
\[ \begin{align*}
\frac{d t_1(\psi)}{d\psi} &= r R_{\text{eff}} \cosh \left( \sqrt{sr} \psi \right) \Rightarrow t_1(\psi) = \frac{r R_{\text{eff}}}{\sqrt{sr}} \sinh \left( \sqrt{sr} \psi \right) + C_1, \\
\frac{d t_2(\psi)}{d\psi} &= r R_{\text{eff}} \cosh \left( \sqrt{sr} \psi \right) \Rightarrow t_2(\psi) = \frac{r R_{\text{eff}}}{\sqrt{sr}} \sinh \left( \sqrt{sr} \psi \right) + C_2, \\
\frac{d t_3(\psi)}{d\psi} &= r R_{\text{eff}} \cosh \left( \sqrt{sr} \psi \right) \Rightarrow t_3(\psi) = \frac{r R_{\text{eff}}}{\sqrt{sr}} \sinh \left( \sqrt{sr} \psi \right) + C_s.
\end{align*} \tag{1.34, 1.35, 1.36} \]

Imposing the temporal initial condition on solutions on the above system of differential equations, yields vanishing constants \( C_i, \forall i \in \{1, ..., s\} \). The temporal solutions become
\[ t_i(\psi) = \sqrt{\frac{r}{s}} R_{\text{eff}} \sinh \left( \sqrt{sr} \psi \right), \quad \forall i \in \{1, ..., s\}. \tag{1.37} \]

Hyperbolically parameterized by \( \psi \), the temporal and spatial components of \( p^A \in M \subset H^{n, r} \) can finally be prescribed as
\[ p_{A,i+1-s}(\psi) = \sqrt{\frac{r}{s}} R_{\text{eff}} \sum_{i=1}^{s} \sinh \left( \sqrt{sr} \psi \right) \hat{t}_i + R_{\text{eff}} \sum_{j=s+1}^{k+1} \cosh \left( \sqrt{sr} \psi \right) \hat{x}_j. \tag{1.38} \]
Similarly, taking the derivative of equation (1.38) yields the perpendicular to \( p^A \)

\[
p_{A,t+1-s}(\psi) = \sqrt{\frac{n}{s}} \sqrt{\text{sr}} R_{\text{eff}} \sum_{i=1}^s \cosh(\sqrt{\text{sr}}) \hat{t}_i + \sqrt{\text{sr}} R_{\text{eff}} \sum_{j=s+1}^{k+1} \sinh(\sqrt{\text{sr}}) \hat{x}_j,
\]

(1.39)

where \( \hat{t}_i \) and \( \hat{x}_j \) are unit vectors pointing in the time-like and space-like directions, respectively.

\[\square\]

**Theorem 1.** Uniform Parametrization Theorem:

Consider \( p_{A,R}(\psi) \in M \subset H^{s,r} \subset \mathbb{C}^n \) be a smooth mapping, where \( M \) is some open subset of \( H^{s,r} \) and \( \psi \in \mathbb{R} \). The pseudo-complex hyperbolic affine space is defined as \( H^{s,n-s} = \{(t_i, x_j) : -\sum_{i=1}^s t_i^2 + \sum_{j=s+1}^n x_j^2 = R^2\} \), where \( s \) and \( r = n - s \) are the numbers of minuses and pluses dictated by the metric function. The \( p_{A,R} \) and \( \dot{p}_{A,R} = \frac{dp_{A,R}}{\text{d}t} \) are elements of \( H^{s,n-s} \) space and its associated tangent space \( T_{p_{A,R}}(M) \), respectively, along with the initial condition (1.13). The elements of the respective spaces are defined by \( p_{A,R}(\psi) = (t_i(\psi), x_j(\psi)) \) and \( \dot{p}_{A,R}(\psi) = (\dot{t}_i(\psi), \dot{x}_j(\psi)) \), where \( i \in \{1, \ldots, s\} \) and \( j \in \{s+1, \ldots, n\} \). Then elements of \( H^{s,n-s} \) and its tangent space \( T_{p_{A,R}}(M) \) must be uniformly parametrized by parametrization parameter \( \psi \).

**Proof of the Main Theorem.** Proof is by induction. Let us redefine the index of the mapping by \( n = k + 1 \), the element of \( H^{s,k+1-s} \) becomes \( p_{A,k+1}(\psi) \in M \subset H^{s,k+1-s} \subset \mathbb{C}^{k+1} \). For \( k = 1 \), we have \( p_{A,1+1-1}(\psi) = (t_1(\psi), x_2(\psi)) \) and \( \dot{p}_{A,2}(\psi) = (\dot{t}_1(\psi), \dot{x}_2(\psi)) \), where \( i = 1, j = 2 \) and \( A = 1, 2 \). With the given metric function, the invariant squared of \( p_{A,2}(\psi) \) yields

\[
p^2 = p_{A,2}(\psi) \circ p_{A,2}(\psi) = -t_1^2(\psi) + x_2^2(\psi) = R^2.
\]

Equation (1.41) is equivalent to taking the \( H^{1,1} \) inner product of \( p_{A,2} \) and \( \dot{p}_{A,2} \), thus \( p_{A,2}(\psi) \circ \dot{p}_{A,2}(\psi) = -t_1 t_1 + x_2 \dot{x}_2 = 0 \). Equation (1.41) is satisfied by using the following differential equations, \( x_2 = t_1 \) and \( t_1 = x_2 \). Solving these two dfe’s for temporal and spatial components, and imposing the initial condition, \( p_{A,2}(0) = (t_1(0), x_2(0)) = (0, R_{\text{eff}}) \), yields \( t_1 = R_{\text{eff}} \sinh \psi \) and \( x_2 = R_{\text{eff}} \cosh \psi \). Hence, the elements \( p_{A,2} \) and \( \dot{p}_{A,2} \) take the following hyperbolic parametrized forms, \( p_{A,2}(\psi) = (R_{\text{eff}} \sinh \psi, R_{\text{eff}} \cosh \psi) \) and \( \dot{p}_{A,2}(\psi) = (R_{\text{eff}} \cosh \psi, R_{\text{eff}} \sinh \psi) \). The elements of \( H^{1,1} \) and its associated tangent space \( T_{p_{A,2}}(M) \) can easily be shown to satisfy equations (1.40) and (1.41). Hence, true for \( k = 1 \).
Assume true for \( k = l \), we have the following relations:

\begin{align}
(1.42) \quad p_{A,l+1-s}(\psi) &= (t_i, x_j), \\
(1.43) \quad p_{A,l+1-s}(\psi) \circledast p_{A,l+1-s}(\psi) &= -\sum_{i=1}^{s} t_i^2 + \sum_{j=s+1}^{k+1} x_j^2 = R^2, \\
\quad p_{A,l+1-s}(\psi) \circledast \hat{p}_{A,l+1-s}(\psi) &= -\sum_{i=1}^{s} \sum_{i'=1}^{s} t_i t_{i'} \delta_{j,j'} \\
&\quad + \sum_{j=s+1}^{k+1} \sum_{j'=s+1}^{k+1} x_j x_{j'} \delta_{j,j'} \\
(1.44) \quad p_{A,l+1-s}(\psi) \circledast \hat{p}_{A,l+1-s}(\psi) &= -\sum_{i=1}^{s} t_i + \sum_{j=s+1}^{k+1} x_j x_j = 0,
\end{align}

where \( i \in \{1, \ldots, s\} \) and \( j \in \{s+1, \ldots, k+1\} \). From the lemma, the assumed hyperbolic parametrized forms are given as

\begin{align}
(1.45) \quad p_{A,l+1-s}(\psi) &= \sqrt{\frac{r}{s}} R_{eff} \sum_{i=1}^{s} \sinh (\sqrt{s} \hat{r}) \hat{t}_i \\
&\quad + R_{eff} \sum_{j=s+1}^{k+1} \cosh (\sqrt{s} \hat{r}) \hat{x}_j,
\end{align}

and

\begin{align}
(1.46) \quad \hat{p}_{A,l+1-s}(\psi) &= r R_{eff} \sum_{i=1}^{s} \cosh (\sqrt{s} \hat{r}) \hat{t}_i \\
&\quad + \sqrt{s} R_{eff} \sum_{j=s+1}^{k+1} \sinh (\sqrt{s} \hat{r}) \hat{x}_j,
\end{align}

where \( R_{eff} = \frac{1}{\sqrt{r}} R \). We could also assume the following systems of differential equations

\begin{align}
(1.47) \quad \dot{x}_j &= \sum_{i=1}^{s} t_i, \quad \text{and} \\
(1.48) \quad \dot{t}_i &= \sum_{j=s+1}^{k+1} x_j \\
\Rightarrow \quad p_{A,l+1-s}(\psi) \circledast \hat{p}_{A,l+1-s}(\psi) &= -\sum_{i=1}^{s} t_i \dot{t}_i + \sum_{j=s+1}^{k+1} x_j \dot{x}_j \\
&\quad = -\sum_{i=1}^{s} \sum_{i=1}^{s} t_i x_j \dot{x}_j + \sum_{j=s+1}^{k+1} \sum_{i=1}^{s} x_j t_i \dot{t}_i = 0
\end{align}

We need to show true for \( k = l + 1 \),

\[ p_{A,l+2-s}(\psi) = (t_i, x_j), \quad i \in \{1, \ldots, s, s+1\} \quad \text{and} \quad j \in \{s+1, \ldots, k+1, k+2\}. \]

The \( H^{s+1,l+2-s} \) inner product of \( p_{A,l+2-s}(\psi) \), yields

\begin{align}
(1.49) \quad p_{A,l+2-s}(\psi) \circledast p_{A,l+2-s}(\psi) &= -\sum_{i=1}^{s+1} t_i^2 + \sum_{j=s+2}^{k+2} x_j^2 = R^2 \\
(1.50) &= -\sum_{i=1}^{s} t_i^2 + \sum_{j=s+2}^{k+2} x_j^2 - t_{s+1}^2.
\end{align}

We shift the index \( j' = j - 1 \),

\begin{align}
(1.51) \quad p_{A,l+2-s}(\psi) \circledast p_{A,l+2-s}(\psi) &= -\sum_{i=1}^{s} t_i^2 + \sum_{j=s+2}^{k+1} x_j^2 - t_{s+1}^2.
\end{align}

To show \( t^2_{s+1} = 0 \), we add \( t_{s+1} \) to both sides of equation \( (1.47) \),

\begin{align}
\dot{x}_j + t_{s+1} &= \sum_{i=1}^{s} t_i + t_{s+1}, \\
\dot{t}_{s+1} &= \sum_{i=1}^{s+1} t_i - x_j.
\end{align}
Thus equation (1.51) yields an invariant quantity

\[ \text{Adding } (1.55) \]

\[ t_{s+1}^2 = t_{s+1} \sum_{i=1}^{s} t_i + 2t_{s+1} \sum_{i=1}^{s} t_i = t_{s+1} \sum_{i=1}^{s} t_i = 0. \]

Thus equation (1.50) yields an invariant quantity

\[ p_{A,l+2-s}(\psi) \odot p_{A,l+2-s}(\psi) = - \sum_{i=1}^{s} t_i^2 + \sum_{j=1}^{k+1} x_j^2 - t_{s+1}^2 \]

Taking the derivative of equation (1.49) and using equations (1.47) and (1.48), we have

\[ p \odot p = - \sum_{i=1}^{s+1} t_i t_i + \sum_{j=s+2}^{k+2} x_j x_j \]

\[ = - \sum_{i=1}^{s+1} t_i \left( \sum_{j=s+2}^{k+2} x_j + \sum_{j=s+2}^{k+2} x_j \sum_{i=1}^{s+1} t_i \right) \]

\[ = - \sum_{i=1}^{s+1} t_i \sum_{j=s+2}^{k+2} x_j + t_{s+1} \sum_{j=s+2}^{k+2} x_j \sum_{i=1}^{s+1} t_i \]

\[ - \sum_{i=1}^{s} t_i \sum_{j=s+2}^{k+2} x_j + \sum_{j=s+2}^{k+2} x_j \sum_{i=1}^{s} t_i = 0. \]

Adding \( t_{s+1} \) to both sides of equation (1.47), we have

\[ x_j + t_{s+1} = \sum_{i=1}^{s} t_i + t_{s+1}, \]

\[ t_{s+1} = \sum_{i=1}^{s+1} t_i - x_j. \]

Squaring equation (1.50), yields

\[ t_{s+1}^2 = \left( \sum_{i=1}^{s+1} t_i - x_j \right) \left( \sum_{s+1}^{k+1} t_i - x_j \right) \]

\[ = \sum_{i=1}^{s+1} t_i \sum_{j=s+2}^{k+2} x_j - \sum_{i=1}^{s+1} t_i \sum_{j=s+2}^{k+2} x_j + \sum_{i=1}^{s+1} t_i \sum_{j=s+2}^{k+2} x_j + \sum_{i=1}^{s+1} t_i \sum_{j=s+2}^{k+2} x_j = 0. \]

Thus equation (1.51) yields an invariant quantity

\[ p_{A,l+2-s}(\psi) \odot p_{A,l+2-s}(\psi) = - \sum_{i=1}^{s} t_i^2 + \sum_{j=1}^{k+1} x_j^2 - t_{s+1}^2 \]

\[ p_{A,l+2-s}(\psi) \odot p_{A,l+2-s}(\psi) = R^2. \]

Therefore true for \( l = k + 1. \)
REFERENCES

[1] I. Antoniadis, Phys. Lett. B246 (1990) 377.
[2] A. Pomarol and M.Quiros, hep-ph/9806626; I. Antoniadis, S. Dimopoulos, A. Pomarol and M. Quiros, hep-ph/9810410; A. Delgado, A. Pomarol and M. Quiros, hep-ph/9812489; R. Barbieri, L. J. Hall and Y. Nomura, hep-ph/0011311; N. Arkani-Hamed, L. J. Hall, Y. Nomura, D. R. Smith and N. Weiner, hep-ph/0102090; A. Delgado and M. Quiros, hep-ph/0103058; A. Delgado, G. von Gersdorff, P. John and M. Quiros, hep-ph/0104112; R. Contino and L. Pilo, hep-ph/0104130; H. D. Kim, hep-th/0109101; V. Di Clemente, S. F. King and D. A. J. Rayner, hep-ph/0107290

[3] M. Truong, Physical Review D74 (2006).
[4] F. J. Yndurain, Phys. Lett. B256 (1991) 15-16.
[5] G. Dvali, G. Gabadadze, and G. Senjanovic, (hep-ph/9910207)
[6] R. Erdem, and C. S. Un, Eur. Phys. J. C47 (2006) 845-850 (hep-ph/0510207)
[7] M. Chaichian, and A. B. Kobakhidze, Phys. Lett. B488 (2000) 117-122 (hep-th/0003269)
[8] I. Quiros, arXiv:0707.0714v1 (2007)
[9] I. Quiros, arXiv:0706.2400v3 (2007)
[10] M. Truong, arXiv:1001.3821 hep-th/1001.3821
[11] B. O’Neill, “Elementary Differential Geometry” 2nd edition, Academic Press, 1977

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