The boundary F-theorem
for free fields

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The boundary free energy, as defined by Gaiotto, is further analysed for free scalars on a hemisphere and shown to be the same as the N-D determinant that earlier occurred in a treatment of GJMS operators. It is also shown to be identical, up to spin degeneracy, to the free energy for a spin–half field on the hemisphere boundary. This is also true if the hemisphere is replaced by a lune. The calculations are carried out in arbitrary dimensions.
1. Introduction

Gaiotto, [1], has introduced a ‘boundary free energy’, $F_\partial$, on the (four–dimensional) hemisphere $(H S^4)$ for a 4–d conformal field theory (CFT) by the definition

$$F_\partial \equiv \frac{1}{2} \log Z_{S^4} - \log Z_{H S^4}$$

$$= F_{H S} - \frac{1}{2} F_S$$

in terms of the indicated partition functions, with any remaining infinities removed. This gives that part of the free energy on a hemisphere due to the existence of its boundary.

The conjecture (the boundary F–theorem) is that $F_\partial$ is monotonic along boundary renormalisation group flows from one conformal boundary condition to another (for the same CFT).

As a simple example, Gaiotto considers Dirichlet (D) and Neumann (N) boundary conditions for a conformally coupled free scalar field $\phi$ in four dimensions and shows that $F_\partial^N > F_\partial^D$ i.e. Neumann conditions flow to Dirichlet under a particular boundary perturbation. In this communication I wish to relate $F_\partial$ to a previously computed quantity and, thereby, to extend the notion to all even dimensional hemisphere CFTs.

2. $N \cup D$

In a previous work, [2], concerned with the conformal anomaly and effective action (free energy) for free scalar fields propagating via the GJMS conformally invariant higher Laplacian operator, $P_{2k}$, on spheres, the spectrum on the sphere was obtained as the union of the D and N hemisphere spectra. This had also been found convenient in several earlier analyses, in various contexts, for the full sphere, and it also gave access to the individual hemisphere quantities, [3,4].

For present needs, the higher rank GJMS operators are not required and $k$ could be set to unity since $P_2$ is the standard Penrose–Yamabe conformally invariant Laplacian. However, as the evaluation applies for all allowable $k$, I will retain $P_{2k}$ for a while. I also sometimes refer to the free energy as half the ‘logdet’ of the appropriate raw operator, with boundary conditions.

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2 If the boundary has non–zero extrinsic curvature, then it is necessary to use a Robin rather than a pure Neumann condition.
It was shown in [2] that the conformal and multiplicative anomalies \((P_{2k}\) is a product) on the D and N hemispheres have the same sign in even and the opposite sign in odd dimensions. (See Section 5.) Importantly, the same sign association holds also for the infinities because in \(\zeta\)-function renormalisation, \(e.g.,\) the only infinities, and corresponding scale log terms, are driven by the conformal anomaly.

Now, the spectral union of D and N shows that (1) can be written, for all dimensions
\[
F^N_\partial = -F^D_\partial = \frac{1}{2} \left( F^N_{HS} - F^D_{HS} \right) \quad (2)
\]
since
\[
F_S = F^N_{HS} + F^D_{HS}.
\]

Hence, for odd dimensions it is \(F_S\) which is universal \(i.e.\) no infinities, or consequences thereof, while for, even dimensions, \(F_\partial\) provides a universal candidate. The first statement has already appeared in the context of entanglement entropy.

The computation of the effective action, \(F_S\), for \(P_2\) on spheres, even and odd, is very old and will not be recounted here. For \(P_{2k}\) it is given in [2] and references found there. The computation of \(F_\partial\), expressed as (2), for even dimensions was also undertaken in [2] and, since it is now relevant, I recount some of the results.

2. The N-D determinant ratio for even dimensions

The method of finding spherical logdets used in [2] is that employed earlier in [3,5] and results in expressions involving Barnes \(\zeta\)-functions, in particular multiple \(\Gamma\)-functions.

For even \(d\), as already said, subtracting the N and D \(d\)–hemisphere log determinants removes all anomalies. This difference is easily found from a general expression, [2], which it not worthwhile to explicate, and is,
\[
F^N_{HS} - F^D_{HS} = \frac{1}{2} \log \frac{\Gamma_d(d/2 - k)}{\Gamma_d(d/2 + k)}
= \frac{1}{2} \log \frac{\Sin_{d+1}(d/2 + k)}{\Sin_{d+1}(d/2 - k)}, \quad d \text{ even},
\]
in terms of Kurokawa’s multiple sine function. (This is of exactly the same form as the odd \(d\)–sphere \(F^D_{HS} + F^N_{HS}\).)

The explicit calculation, [2], yields, in the light of (2),
\[
\delta F_\partial \equiv F^N_\partial - F^D_\partial = \frac{1}{2(d-1)!} \int_0^k dz \pi z \cot \pi z \prod_{j=1}^{d/2-1} (z^2 - j^2), \quad k < d/2. \quad (3)
\]
It is easy to show that the right-hand side alternates in sign as the dimension, \(d\), increases, being positive at \(d = 4\) for \(k = 1\), which is the value of most interest here. For the purposes of the flow, only the sign is relevant and one sees that the flow alternates between \(N \rightarrow D\) and \(D \rightarrow N\) as the dimension increases from \(d = 4\). If one always wishes to flow from \(N\) down to \(D\), then one should take \((-1)^{d/2} F_\partial\).

The integral can be taken further analytically and can be expressed in terms of Riemann \(\zeta\)-functions, after expansion of the product. (See section 6). If only a number is required, quadrature is quite efficient, accuracy to 14 places being rapidly achieved.

4. Alternative treatment

The main ‘physical’ point of this note has already been expounded, but no calculational details have been given. I now wish to give an analytical alternative which will allow me to introduce some of the basic ingredients and, at the same time, permit a generalization of the geometry and propagation equation. This is done through an alternative representation of the boundary free energy to (3). For simplicity, I restrict the discussion to the usual Laplacian, \(k = 1\).

Again, the relevant formulae have been mostly given before, [6], but it is necessary now to present some technical details, and first some generalities.

The main calculational tool is the \(\zeta\)-function associated with the propagation operator. I denote this, generically, by \(\zeta(s)\). Then, according to one version of \(\zeta\)-function regularisation, the effective action (free energy), including divergences, is given by

\[
F = -\frac{1}{2} \lim_{s \to 0} \frac{\zeta(s)}{s} = -\frac{1}{2} \left( \lim_{s \to 0} \frac{\zeta(0)}{s} + \zeta'(0) \right).
\]

(4)

For simplicity, the scaling length, \(L\), has been set to unity. (An equivalent role is played by the first (divergent) term, but \(L\) could easily be restored.) For convenience, I will refer to \(\zeta(0)\) as the conformal anomaly.

It is sufficient to introduce the specific \(\zeta\)-function,

\[
\zeta(s, a, \alpha | \omega) = \sum_{m=0}^{\infty} \frac{1}{((a + m\omega)^2 - \alpha^2)^s},
\]

(5)

because the eigenvalues of the improved Laplacian, \(P_2\), on the \(d\)-hemisphere are

\[
\lambda_{HS} = (a + m\omega)^2 - \frac{1}{4},
\]

(6)
where \( a = a_N \equiv (d - 1)/2 \) and \( a = a_D = a_N + 1 \) for Neumann and Dirichlet conditions on the hemisphere rim. The \( d \)-vector \( \omega \) here has all components equal to one \( \equiv 1_d \). I have introduced \( \omega \) because for \( \omega = (q, 1_{d-1}) \) \( (q \in \mathbb{Z}) \), the hemisphere is replaced by a lune \((L)\), of apex angle \( \pi/q \) and I might as well do this now to save repetition later. In this case the parameter \( a_D \) is modified to \( a_N + q \). I will also keep \( \alpha \) general.

5. Cancellation of the conformal anomaly

Because it is important, I repeat the demonstration that the N and D conformal anomalies, i.e. \( \zeta(0) \), for the \( \zeta \)-function (5) have the requisite sign behaviour mentioned earlier.

It is shown in [3] that this value is given in terms of Barnes’ \( \zeta \)-function, \( \zeta_d \), by, in the present case,

\[
\zeta(0) = \frac{1}{2} \left( \zeta_d(0, a - \alpha|q, 1) + \zeta_d(0, a + \alpha|q, 1) \right),
\]

where \( \alpha = 1/2 \) for conformal coupling in \( d \) dimensions and the relevant, N or D, value for parameter \( a \) has to be chosen.

Then using Barnes’ results, [7],

\[
\begin{align*}
\zeta^N(0) &= \frac{1}{2q d!} \left( B_d^{(d)}((d - 1)/2 - \alpha|q, 1) + B_d^{(d)}((d - 1)/2 + \alpha|q, 1) \right), \\
\zeta^D(0) &= \frac{1}{2q d!} \left( B_d^{(d)}((d - 1)/2 + \alpha + q|q, 1) + B_d^{(d)}((d - 1)/2 - \alpha + q|q, 1) \right),
\end{align*}
\]

(7)
in terms of higher Bernoulli polynomials.

The symmetry,

\[
B_n^{(d)}((d - 1)/2 + q - \alpha|q, 1) = (-1)^n B_n^{(d)}((d - 1)/2 + \alpha|q, 1).
\]
gives the sought for answer,

\[
\zeta^N(0) = (-1)^d \zeta^D(0),
\]
and shows that it is the combination, on the lune,

\[
F^N - (-1)^d F^D = -\frac{1}{2} \left( \zeta'^N(0) - (-1)^d \zeta'^D(0) \right),
\]
(8)
that should be taken as universal. The \( F\)s can thus be identified with the \( F\)s of section 2 (if the hemisphere is replaced by the lune).
6. The calculation

The individual \(\zeta\)-function derivatives in (8) have been evaluated in [6] where they were added to give the full (or periodic) lune in odd dimensions. Here I subtract them and work in even dimensions.

As mentioned in [4], following Candelas and Weinberg, [8], and Minakshisundaram, [9], I can employ the Bessel function form for the \(\zeta\)-function (5),

\[
\zeta(s, a, \alpha | \omega) = \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \frac{d\tau \exp(-a\tau)}{\prod_{i=1}^d (1 - \exp(-\omega_i \tau))} \left(\frac{\tau}{2\alpha}\right)^{s-1/2} I_{s-1/2}(\alpha\tau). \tag{9}
\]

Subtracting the N and D parts of the \(\zeta\)-function yields the expression,

\[
\zeta_{N-D}(s, \alpha | \omega) \equiv \frac{2\sqrt{\pi}}{\Gamma(s)} \int_0^\infty \frac{d\tau \sinh \left((\omega - a_N)\tau\right)}{\prod_{i=1}^d 2 \sinh(\omega_i \tau/2)} \left(\frac{\tau}{2\alpha}\right)^{s-1/2} I_{s-1/2}(\alpha\tau)
\equiv \frac{I(s, \alpha)}{\Gamma(s)}, \tag{10}
\]

where the parameter \(\omega = \frac{1}{2} \sum_i \omega_i\).

The object is to evaluate \(\zeta_{N-D}(0, \alpha | \omega)\) which necessitates a continuation of (10) to around \(s = 0\). I adopt the procedure of Candelas and Weinberg, [8], as used by Chodos and Myers, [10]. First note that, because the conformal anomaly, \(\zeta_{N-D}(0, \alpha | \omega)\), is zero, the integral, \(I(s, \alpha)\), is well behaved and, therefore, the derivative at zero is just \(I(0, \alpha)\).

The continuation to \(s = 0\) is effected by a contour method which is slightly modified from [8], [10] because the parity of the integrand has been reversed being now of the form \(\tau^{2s-d} f(\tau^2)\) so that \(I(s, \alpha)\) can be continued via the complex integral,

\[
I(s, \alpha) = \frac{2\sqrt{\pi}}{1 + e^{\pi i (2s-d)}} \int_{-\infty+iy}^{\infty+iy} \frac{d\tau \sinh \left((\omega - a_N)\tau\right)}{\prod_{i=1}^d 2 \sinh(\omega_i \tau/2)} \left(\frac{\tau}{2\alpha}\right)^{s-1/2} I_{s-1/2}(\alpha\tau), \tag{11}
\]

in which \(s\) can be set to zero, so long as \(d\) is even and \(y\) sits between zero and the first zero of the denominator (which lies on the imaginary axis) i.e. \(y < 2\pi/\max \omega_i\).

Setting \(s\) to zero and putting in the appropriate values of \(a_N\) and \(\omega\) produces,

\[
I(0, \alpha) = \frac{\sqrt{\pi}}{2^{d-1}} \int_C \frac{d\tau \cosh(\alpha\tau)}{\sinh^{d-1}(\tau/2)} \left(\frac{\tau}{2\alpha}\right)^{1/2} I_{-1/2}(\alpha\tau)
= \frac{1}{2^{d-1}} \int_C \frac{d\tau \cosh(\alpha\tau)}{\tau \sinh^{d-1}(\tau/2)} \tag{12}
\]
and one sees that the $q$ dependence has dropped out. This is to be expected since this quantity is associated with the boundary and the boundary of any lune is always the same sphere, intrinsically.

The conformal value is,

$$I(0,1/2) = \frac{1}{2^{d-1}} \int_C \frac{d\tau \cosh(\tau/2)}{\tau \sinh^{d-1}(\tau/2)}.$$  \hfill (13)

The integral, (12), has been encountered previously, [11]. It is, to a spin factor, (minus) the logdet of the spin–half operator $(\nabla^2 - \alpha^2)$ on the $d$–lune boundary, which is a $(d-1)$–sphere (odd dimensional). Hence the bulk computed quantity, $F_\partial$, associated with the boundary of the lune (in particular with a hemisphere rim), does have an intrinsic boundary spectral significance. A direct, spectral proof of this is given in the Appendix.

In the conformal case, this can also be seen by comparing the ‘Plancherel’ form, (3) with the corresponding equation (equation (29)) in [11]. Using this form, a holographic linkage was also encountered in [2].

The integral (13) provides a numerical alternative to (3). Otherwise, an explicit expression in terms of Riemann $\zeta$–functions can be found by residues and is given in [11]. For example, from the particular values given there, one has for $\delta F_\partial$ of (3).

$$-\frac{\zeta(3)}{8\pi^2} = 0.01522422, \quad d = 4$$

$$-\frac{\zeta(3)}{96\pi^2} - \frac{\zeta(5)}{32\pi^4} = -0.00160134, \quad d = 6$$

and the sign alternates, although this is not apparent from the general form.

The first value was obtained by Gaiotto using eigenvalues and also appears in Klebanov, Pufu and Safdi, [12].

7. Comments

It has been noted that the scalar boundary free energy defined by Gaiotto has appeared in another context and it has been extended to any (even) dimension. It is shown to be proportional to the free energy of a spin–half operator on the boundary, up to spin degeneracy.

The quantities (14) are the logdets of the spherical spin–half GJMS operator,

$$\frac{\Gamma(|\nabla| + k + 1/2)}{\Gamma(|\nabla| - k + 1/2)},$$

at the value $k = 1$. The entire development of section 4 onwards can be extended to GJMS operators.

6
Appendix

Here I show by a direct spectrum argument, as encoded in the $\zeta$–functions, that the intrinsic boundary interpretation of $F_\partial$ is the free energy of a Dirac–like operator on the boundary.

Referring to the $\zeta$–function, (5), and the eigenvalues, (6), I remark firstly that for spin–half the parameter $a$ equals $d/2$ for both types of boundary condition. (Hence the values on the full sphere are obtained by multiplying the hemisphere value by two.) Secondly, one might expect from factorising the eigenvalues that immediately, for the $\zeta$–function (5),

$$\zeta'(0, a, \alpha | \omega) = \zeta'_d(0, a + \alpha | \omega) + \zeta'_d(0, a - \alpha | \omega) + MA$$

(16)

in terms of the Barnes $\zeta$–function $\zeta_d$. Here $MA$ stands for the multiplicative anomaly. Setting, just for now, $\Xi(a) = \zeta'_d(0, a + \alpha | \omega)$, one requires the N–D combination,

$$\Xi(a_N + \alpha) + \Xi(a_N - \alpha) - \Xi(a_D(a + \alpha)) - \Xi(a_D - \alpha)$$

$$= \Xi((d - 1)/2 + \alpha) + \Xi((d - 1)/2 - \alpha) -$$

$$\Xi((d - 1)/2 + q + \alpha) - \Xi((d - 1)/2 + q - \alpha).$$

(17)

The corresponding combination of multiplicative anomalies can be shown to vanish, cf [2,11].

The standard recursion of the Barnes $\zeta$–function, [7], allows one to combine the first term in (17) with the third, and the second with the fourth to give,

$$\zeta'_{d-1}(0, (d - 1)/2 + \alpha | 1_{d-1}) + \zeta'_{d-1}(0, (d - 1)/2 - \alpha | 1_{d-1})$$

which equals $\zeta'(0, (d - 1)/2, \alpha | 1_{d-1})$ by the reverse of the argument that led to (16). (There are no multiplicative anomalies in odd dimensions.) This last quantity is recognised as the $-\log\det$ of the Dirac operator $\nabla^2 - \alpha^2$ on the $(d - 1)$–sphere.

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