Hilbert schemes, commuting matrices and hyperkähler geometry

Roger Bielawski  |  Carolin Peternell

Institut für Differentialgeometrie, Gottfried Wilhelm Leibniz Universität Hannover, Hannover, Germany

Correspondence
Roger Bielawski, Institut für Differentialgeometrie, Gottfried Wilhelm Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany.
Email: bielawski@math.uni-hannover.de

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Abstract
We represent algebraic curves via commuting matrix polynomials. This allows us to show that the Hilbert scheme of cohomologically stable non-planar curves of genus 0 and degree \( d \) in \( \mathbb{P}^3 \setminus \mathbb{P}^1 \) is isomorphic to a complexified hyperkähler quotient of an open subset of a vector space by a non-reductive Lie group.

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It has been observed in [4] that the Hilbert scheme of real cohomologically stable (that is, satisfying \( h^0(\mathcal{N}(-2)) = 0 \)) non-planar curves of fixed genus and degree in \( \mathbb{P}^3 \), not intersecting a fixed real line, carries a natural pseudo-hyperkähler structure. Here ‘real’ means invariant under a fixed point free antilinear involution of \( \mathbb{P}^3 \). In the case of \( g = 0 \) and \( d = 3 \) this pseudo-hyperkähler structure was shown in [4] to be flat, and, in fact, the manifold of cohomologically stable pure-dimensional Cohen–Macaulay non-planar curves in \( \mathbb{P}^3 \setminus \mathbb{P}^1 \) with Hilbert polynomial \( 3n + 1 \) with its natural complexified hyperkähler structure was shown there to be isomorphic to \( \mathbb{C}^{12} \cong \mathbb{C}^3 \otimes \text{Mat}_{2 \times 2}(\mathbb{C}) \). The proof of this relies on such curves being ACM and so clearly different methods are needed to study the pseudo-hyperkähler geometry of the corresponding open subset of \( \text{Hilb}_{d,g} \) for other values of \( d \) and \( g \).

In the present article we present such a method via a correspondence between algebraic curves equipped with a flat projection onto \( \mathbb{P}^1 \) and commuting matrix polynomials. This correspondence allows us to describe the locus of cohomologically stable non-planar curves of arithmetic genus 0 and degree \( d \) in \( \mathbb{P}^3 \setminus \mathbb{P}^1 \) as a complexified hyperkähler quotient of (an open subset of) a vector space by a non-reductive Lie group (Theorem 5.6). Formally, our moment map equations are very similar to the complex ADHM equations used by Frenkel and Jardim [10] in their construction of admissible torsion-free sheaves on \( \mathbb{P}^3 \). The main difference is that the Lie group acting on solutions is no longer reductive.
Restricting this description to real curves, we obtain the above pseudo-hyperkähler structure for $g = 0$ and any odd $d$ (for even $d$ there are no real rational curves in the above sense) as a hyperkähler quotient of an open subset of a flat quaternionic vector space by a non-reductive Lie group. There is also an analogous description of the natural hypersymplectic structure on cohomologically stable non-planar curves of genus 0 and any degree which are invariant under an antilinear involution of $\mathbb{R}P^3$, the fixed point set of which is $\mathbb{R}P^3$.

We briefly describe the structure and the content of the paper. In the next section we provide alternative (to the one given by Nakajima [22]) descriptions of the Hilbert scheme $(\mathbb{C}^2)^*[n]$ and of its open subset of non-collinear points. In §2 we discuss the above-mentioned correspondence between algebraic curves and commuting matrix polynomials. In §3 we restrict our attention to space curves, and show how to construct a twistor space $Z_d \to \mathbb{P}^1$, the sections of which can be identified with either (a) algebraic curves of fixed degree, or (b) equivalence classes of certain commuting pairs of polynomials. Section 4 discusses complexified hyperkähler and hypercomplex structures and their quotients. Finally, in §5 we apply these ideas and results to genus 0 space curves and show that in this case the description (b) of sections of $Z_d$ is equivalent to a complexified hyperkähler quotient.

Remark 0.1. Throughout the paper a curve means a projective Cohen–Macaulay scheme of pure dimension 1.

1 | HILBERT SCHEMES OF POINTS IN $\mathbb{C}^2$ AND COMMUTING MATRICES

We begin with the following easy observation (cf. [22] if $k = 2$ and [16, Theorem 2.5] for general $k$).

Proposition 1.1. There exists a natural set-theoretic bijection between the Hilbert scheme $(\mathbb{C}^k)^*[n]$ of $n$ points in $\mathbb{C}^k$ and $GL(n, \mathbb{C})$-orbits of $k$-tuples $(A_1, \ldots, A_k)$ of $n \times n$ matrices such that $\mathbb{C}[A_1, \ldots, A_k]$ is an $n$-dimensional commutative algebra which is conjugate to its image in $\text{End}(\mathbb{C}^n)$ under the regular representation.

Proof. A point in the Hilbert scheme $(\mathbb{C}^k)^*[n]$ is defined by an ideal $I \subset \mathbb{C}[z_1, \ldots, z_k]$ of length $n$, that is, such that $\dim \mathbb{C}[z_1, \ldots, z_k]/I = n$. Multiplication by $z_i, i = 1, \ldots, k$, defines an endomorphism $A_i$ of $\mathbb{C}[z_1, \ldots, z_k]/I$. Clearly the $A_i$ commute and $\dim \mathbb{C}[A_1, \ldots, A_k] = \dim \mathbb{C}[z_1, \ldots, z_k]/I = n$. Moreover, directly from the construction, $\mathbb{C}[A_1, \ldots, A_k]$ is conjugate to its image under the regular representation. Conversely, given $k$ commuting matrices $A_1, \ldots, A_k$ with $\dim \mathbb{C}[A_1, \ldots, A_k] = n$ we define a homomorphism $\mathbb{C}[z_1, \ldots, z_k] \to \mathbb{C}^n \cong \mathbb{C}[A_1, \ldots, A_k]$ via $p(z_1, \ldots, z_k) \mapsto p(A_1, \ldots, A_k)$. Clearly it is surjective and its kernel is an ideal of length $n$. Moreover this ideal does not change under simultaneous conjugation of $A_1, \ldots, A_k$, and so we obtain a well-defined point of $(\mathbb{C}^k)^*[n]$. Since we assume that $\mathbb{C}[A_1, \ldots, A_k]$ is conjugate to its image under the regular representation, the two maps are inverse to each other.

Remark 1.2. The condition that the algebra $A = \mathbb{C}[A_1, \ldots, A_k]$ is conjugate to its image under the regular representation is equivalent to the existence of a cyclic vector for $A$. This is the same argument as in [22, p. 8]. Also, the set of $k$-tuples $(A_1, \ldots, A_k)$ of commuting $n \times n$ matrices with $\dim \mathbb{C}[A_1, \ldots, A_k] = n$ and having a cyclic vector is open in the set of all commuting $k$-tuples. Indeed, due to the Cayley–Hamilton theorem, any element of $\mathbb{C}[A_1, \ldots, A_k]$ belongs to the linear...
span of $A_1^{i_1} \ldots A_k^{i_k}$ for $i_1, \ldots, i_k \leq n - 1$. Thus the condition of not having cyclic vector is equivalent to $\dim \langle A_1^{i_1} \ldots A_k^{i_k}v; i_1, \ldots, i_k \leq n - 1 \rangle \leq n - 1$ for any $v$. This is a closed condition.

Henri and Jardim show that the quotient of the variety of $k$ commuting matrices together with a choice of a cyclic vector by $GL(n, \mathbb{C})$ is a geometric quotient [16, §2], and that the resulting scheme is isomorphic to $(\mathbb{C}^k)[n]$ [16, Corollary 4.9]. For $k = 2$, these results are due to Nakajima [22, §1.2].

We shall now modify this description of $(\mathbb{C}^2)[n]$ in several ways. First of all, we shall want to eliminate the dependence on a cyclic vector. We recall the following theorem of Neubauer and Saltman [23].

**Theorem 1.3** (Neubauer–Saltman). Let $A$ and $B$ be two commuting $n \times n$ matrices. Then the following conditions are equivalent.

1. $\dim \mathbb{C}[A, B] = n$.
2. $\dim Z(A, B) = n$, where $Z(A, B)$ is the centraliser of the pair $(A, B)$.
3. $(A, B)$ is a non-singular point of the variety of commuting matrices.

The equivalence between (1) and (2) will be used repeatedly throughout the paper.

Let us write $M_n$ for the variety of pairs of commuting $n \times n$ matrices and define

$$M_n^0 = \{(A, B); A, B \in \text{Mat}_{n \times n}(\mathbb{C}), [A, B] = 0, \dim Z(A, B) = n\},$$

and its open subset $M_n^{\text{reg}}$ consisting of those $(A, B) \in M_n^0$ for which $\mathbb{C}[A, B]$ is conjugate to its image in $\text{Mat}_{n \times n}(\mathbb{C})$ under the regular representation.

**Proposition 1.4.** $(\mathbb{C}^2)[n]$ is the geometric quotient of $M_n^{\text{reg}}$ by $GL(n, \mathbb{C})$.

**Proof.** It is the matter of checking the conditions that a geometric quotient has to satisfy [21, Definition 0.6]. The only one which perhaps requires an argument is that the structure sheaf of $(\mathbb{C}^2)[n]$ is equal to the $GL(n, \mathbb{C})$-invariant part of the structure sheaf of $M_n^{\text{reg}}$. Nakajima [22] shows that the Hilbert scheme of $n$ points in $\mathbb{C}^2$ is isomorphic to the geometric quotient of the (smooth) variety

$$\tilde{H}_n = \{(A, B, v); [A, B] = 0, v \text{ is a cyclic vector for } (A, B)\}$$

by the (free) action of $GL(n, \mathbb{C})$. Hence $\mathcal{O}_{(\mathbb{C}^2)[n]} = (\mathcal{O}_{\tilde{H}_n})^{GL(n, \mathbb{C})}$.

We have the natural forgetful map $p : \tilde{H}_n \to M_n^{\text{reg}^0}, (A, B, v) \mapsto (A, B)$, and the proof of [22, Theorem 1.9] shows that $p$ is a submersion. Observe now that the fibre $p^{-1}(A, B)$ is isomorphic to the stabiliser of $(A, B)$ in $GL(n, \mathbb{C})$. Indeed, the stabiliser is $n$-dimensional and it acts freely on the fibre. Since, owing to Proposition 1.1, the projection $\tilde{H}_n \to (\mathbb{C}^2)[n]$ factors (set-theoretically) through $M_n^{\text{reg}^0}$, the action of the stabiliser on the fibre must be transitive. Therefore, for any open subset $U \subset M_n^{\text{reg}^0}$, $\mathcal{O}(U)^{GL(n, \mathbb{C})} = \mathcal{O}(p^{-1}(U))^{GL(n, \mathbb{C})}$, and so $\mathcal{O}_{M_n^{\text{reg}^0}} = (\mathcal{O}_{\tilde{H}_n})^{GL(n, \mathbb{C})}$. □

We shall now present another description of $(\mathbb{C}^2)[n]$: as a symplectic quotient of an open dense subset of pairs of matrices by a non-reductive group. The relevant subset $K_n$ consists of pairs $(A, B)$ of $n \times n$ matrices, such that the vector $e_1 = (1, 0, \ldots, 0)^T$ is cyclic for the pair $A, B$. The group $G$ is
the subgroup of $GL(n, \mathbb{C})$ preserving the cyclic vector $e_1$, that is, the first column of elements of $G$ is equal to $e_1$. The symplectic form on $K_n$ is $\text{tr} \, dA \wedge dB$, and the corresponding moment map $\mu : K_n \to \mathfrak{g}^*$ is the projection of $[A, B]$ onto the last $n - 1$ rows.

**Theorem 1.5.** The action of $G$ on $\mu^{-1}(0) \subset K_n$ is free and proper, and the symplectic quotient $\mu^{-1}(0)/G$ is biholomorphic to $(\mathbb{C}^2)^{[n]}$.

**Proof.** First of all, we claim that $\mu(A, B) = 0$ implies $[A, B] = 0$. Indeed, $\mu(A, B) = 0$ means that $[A, B]$ can have non-zero entries only in the first row. Suppose that $[A, B] \neq 0$. Then $\text{Im}[A, B] = \langle e_1 \rangle$ is a 1-dimensional subspace which is cyclic for $(A, B)$. On the other hand, according to [13] (see also [9, Lemma 12.7] for a simple proof) $A$ and $B$ can be simultaneously conjugated into upper-triangular matrices. But then any vector in $\text{Im}[A, B]$ has the last coordinate equal to zero and since no such vector can be cyclic for a pair of upper-triangular matrices, we obtain a contradiction. Hence $[A, B] = 0$.

The action of $G$ on $\mu^{-1}(0)$ is free, since the action of $GL(n, \mathbb{C})$ on $\mathcal{H}_n$ is free. This implies, in particular, that $\mu^{-1}(0)$ is smooth. To show that the action of $G$ on $\mu^{-1}(0)$ is proper, it is enough to show that the action of $GL(n, \mathbb{C})$ on $\mathcal{H}_n$ is proper, since $K_n$ is closed in $\mathcal{H}_n$ and $G$ is closed in $GL(n, \mathbb{C})$. The properness of the $GL(n, \mathbb{C})$ on $\mathcal{H}_n$ is equivalent to $\mathcal{H}_n$ being a principal $GL(n, \mathbb{C})$-bundle over $(\mathbb{C}^2)^{[n]}$. Nakajima [22, Theorem 3.24 & Corollary 3.42] shows that $(\mathbb{C}^2)^{[n]}$ is a (real) symplectic quotient $\nu^{-1}(c)/U(n)$ of $\mathcal{H}_n$, where $\nu$ denotes the moment map. Since $U(n)$ is compact, $\nu^{-1}(c)$ is a principal $U(n)$-bundle over $(\mathbb{C}^2)^{[n]}$. This means that $\nu^{-1}(c) \to (\mathbb{C}^2)^{[n]}$ admits local sections. Such a local section gives a local section of $\mathcal{H}_n \to (\mathbb{C}^2)^{[n]}$, and, consequently, $\mathcal{H}_n$ is a principal $GL(n, \mathbb{C})$-bundle over $(\mathbb{C}^2)^{[n]}$. As explained above, this implies that the action of $G$ on $\mu^{-1}(0)$ is proper. It follows that $\mu^{-1}(0)/G$ is a complex manifold [7, Ch. III, Proposition 10], biholomorphic to $\mathcal{H}_n/GL(n, \mathbb{C})$. □

### 1.1 $(\mathbb{C}^2)^{[n]}$ and torsion-free sheaves on $\mathbb{P}^2$

We shall now give a description of an open subset of $(\mathbb{C}^2)^{[n]}$ consisting of 0-dimensional subschemes not contained in any line (thus, necessarily, $n \geq 3$). It is closely related to the description of the moduli space $\mathcal{M}(2, n - 1)$ of framed torsion-free sheaves on $\mathbb{P}^2$ of rank 2 and $c_2 = n - 1$ in terms of the ADHM equations [22, Theorem 2.1]. In order to simplify the notation, set $k = n - 1$. The moduli space $\mathcal{M}(2, k)$ is biholomorphic to the $GL(k, \mathbb{C})$-quotient of the set $U$ of stable solutions to the equation

$$[X, Y] + ij = 0, \quad X, Y \in \text{Mat}_{k,k}(\mathbb{C}), \ i \in \text{Mat}_{k,2}(\mathbb{C}), \ j \in \text{Mat}_{2,k}(\mathbb{C}).$$

Here a quadruple $(X, Y, i, j)$ is called *stable* if there is no proper subspace $S$ of $\mathbb{C}^k$ such that $\text{Im} \, i \subset S, XS \subset S, YS \subset S$, and the framing of a sheaf $F$ is a trivialisation on the line $l_\infty \subset \mathbb{P}^2$ (in particular $c_1 = 0$).

**Lemma 1.6.** Let $F \in \mathcal{M}(2, k)$. The following conditions are equivalent:

(i) the rank of $i$ is equal to 1;

(ii) $H^0(\mathbb{P}^2, F) \neq 0$;
(iii) \( F \) is an extension of the form \( 0 \to O_{\mathbb{P}^2} \to F \to I_Z \to 0 \), where \( I_Z \) is the ideal sheaf of \( Z \in (\mathbb{P}^2 \setminus \{l_\infty\})^{[k]} \).

**Proof.** Given \( c_2(F) = k \), the conditions (ii) and (iii) are clearly equivalent. We prove the equivalence of (i) and (iii). If \( i \) has rank 1, then \( j = i_0(\alpha_1 j_1 + \alpha_2 j_2) \) for some \( i_0 \in \mathbb{C}^k \), \( \alpha_1, \alpha_2 \in \mathbb{C} \), where \( j_1, j_2 \) are the two rows of \( j \). Equation (1.1) implies that \([X, Y] \) has rank 1, and therefore \( \alpha_1 j_1 + \alpha_2 j_2 = 0 \) [22, Proposition 2.8]. Thus \( j \) has also rank 1 and we can write \( \mathbb{C}^2 \cong W_1 \oplus W_2 \), where \( W_1 = \text{Im} j = \text{Ker} i \). The sheaf \( F \) is then isomorphic to \( \text{Ker} b / \text{Im} a \), where \((a, b)\) is the monad given in [22, p. 23]. The embedding \( W_1 \otimes O_{\mathbb{P}^2} \hookrightarrow \text{Ker} b \) induces an injective morphism \( O_{\mathbb{P}^2} \to F \). Its cokernel is the sheaf obtained from the monad with \( W \) replaced by \( W_2 \) and \( j = 0 \).

This is a rank one framed torsion-free sheaf with \( c_2 = k \), that is, the ideal sheaf of \( Z \in (\mathbb{P}^2 \setminus \{l_\infty\})^{[k]} \).

Conversely, given \( F \) as in the statement, we obtain from \( I_Z \) a stable solution \((X, Y, i_0, 0)\) to the ADHM equation with \( i_0 \in \mathbb{C}^k \). Let \( j^T_0 \) be the extension class of \( F \) in \( \text{Ext}^1(I_Z, O_{\mathbb{P}^2}) \cong \mathbb{C}^k \). Then \((X, Y, i \oplus 0, 0 \oplus j)\) is a solution to the ADHM equation which yields (the isomorphism class of) \( F \).

\[ \square \]

We denote by \( \mathcal{M}(2, k)^0 \) the complement of the (isomorphism classes of) sheaves described in this lemma. If \((X, Y, i, j)\) is the ADHM-data corresponding to a point in \( \mathcal{M}(2, k)^0 \), then \( i \) has rank 2 and we can use the action of \( GL(k, \mathbb{C}) \) to fix \( i \) to be

\[ i = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \end{pmatrix}^T. \tag{1.2} \]

The stabiliser \( G_0 \) of \( i \) consists of matrices of the form

\[ \begin{pmatrix} 1_{2 \times 2} & * \\ 0 & * \end{pmatrix}, \]

and \( \mathcal{M}(2, k)^0 \) is also biholomorphic to the quotient of the space \( U_0 \) of such \((X, Y, i, j)\) by \( G_0 \). Here ‘quotient’ means that \( U_0 \) is a principal \( G_0 \)-bundle over \( \mathcal{M}(2, k)^0 \). Indeed, a well-known description of \( \mathcal{M}(2, k) \) as a hyperkähler quotient means that \( U \) is a principal \( GL(k, \mathbb{C}) \)-bundle over \( \mathcal{M}(2, k) \) (cf. the proof of Theorem 1.5) and, consequently, the action of \( GL(k, \mathbb{C}) \) on \( U \) is free and proper. This implies that the action of \( G_0 \) on \( U_0 \) is also free and proper, so that \( U_0 \) is a principal \( G_0 \)-bundle over \( \mathcal{M}(2, k)^0 \). In fact, \( j \) is now superfluous: it is simply given by the first two rows of \([X, Y]\).

Equation (1.1) is now equivalent to the last \( k - 2 \) rows of \([X, Y]\) being identically zero. Since the projection onto the last \( k - 2 \) rows is precisely the moment map \( \mu \) for the action of \( G_0 \) on pairs of matrices \( X, Y \) with respect to the symplectic form \( \text{tr} dX \wedge dY \), we conclude the following.

**Proposition 1.7.** Let \( V^s \) be the set of pairs \((X, Y) \in \text{Mat}_{k, k}(\mathbb{C})^2 \) such that \((X, Y, i)\) is stable,\(^\dagger\) where \( i \) is given by (1.2). The group \( G_0 \) acts freely and properly on \( \mu^{-1}(0) \subset V^s \) and the symplectic quotient \( \mu^{-1}(0)/G_0 \) is biholomorphic to \( \mathcal{M}(2, k)^0 \).

**Remark 1.8.** Clearly, there is an analogous description of an open dense subset of \( \mathcal{M}(r, k) \) for sheaves of higher rank \( r \).

\(^\dagger\) Stability does not depend on \( j \).
Given a quadruple \((X, Y, i, j)\) (not necessarily satisfying (1.1) or (1.2)), we define a pair of \((k + 1) \times (k + 1)\)-matrices as follows (cf. [5]):

\[
\begin{pmatrix}
0 & -j_2 \\
i_1 & X
\end{pmatrix}, \quad
\begin{pmatrix}
0 & j_1 \\
i_2 & Y
\end{pmatrix},
\]

where \(i_s\) (respectively, \(j_s\)) denotes the \(s\)th column (respectively, \(s\)th row) of \(i\) (respectively, of \(j\)).

**Lemma 1.9.** \((X, Y, i, j)\) satisfies (1.1) if and only if \([\hat{X}, \hat{Y}]\) has non-zero entries only in the first row or the first column. Moreover \((X, Y, i, j)\) is stable if and only if \(e_1\) is cyclic for \((\hat{X}, \hat{Y})\).

**Proof.** The first statement is obvious. For the second one, observe that if \(S\) is a destabilising subspace for \((X, Y, i, j)\), then \(S \oplus \langle e_1 \rangle\) is invariant for \((\hat{X}, \hat{Y})\). Conversely, suppose that \(\hat{S}\) contains \(e_1\) and is invariant for \((\hat{X}, \hat{Y})\). Let \(p : \mathbb{C}^{k+1} \to \mathbb{C}^k = \mathbb{C}^{k+1} / \langle e_1 \rangle\) be the projection and set \(S = p(\hat{S})\). Then \(i_1 = p(\hat{X}e_1), i_2 = p(\hat{Y}e_1)\) belong to \(S\). Moreover, if \(v \in S\), then there is an \(\alpha \in \mathbb{C}\) such that \(\alpha e_1 + v \in \hat{S}\). Since

\[
\hat{X}(\alpha e_1 + v) = \alpha i_1 + Xv \mod \langle e_1 \rangle
\]

and similarly for \(Y\), it follows that \(X S \subset S\) and \(Y S \subset S\). Therefore \(S\) is a destabilising subspace for \((X, Y, i, j)\). \(\square\)

We now fix \(i\) to be (1.2). Let \(\hat{G}_0\) denote the subgroup of \(GL(k + 1, \mathbb{C})\), consisting of matrices of the form

\[
\begin{pmatrix}
1 & u \\
0 & h
\end{pmatrix}, \quad h \in G_0, \quad u_1 = u_2 = 0,
\]

that is, matrices, the first 3 columns of which are \((e_1, e_2, e_3)\). \(\hat{G}_0\) is a semidirect product of \(G_0\) and \(\mathbb{C}^{k-2}\). Its action by conjugation on \(\hat{X}, \hat{Y}\) defined in (1.3) is given by

\[
\hat{X} \mapsto \begin{pmatrix} 0 & -j_2 h^{-1} + uXh^{-1} \\
i_1 & -i_1 uh^{-1} + hXh^{-1} \end{pmatrix}, \quad
\hat{Y} \mapsto \begin{pmatrix} 0 & j_1 h^{-1} + uYh^{-1} \\
i_2 & -i_2 uh^{-1} + hYh^{-1} \end{pmatrix}.
\]

This means that \(\hat{G}_0\) acts on pairs of matrices \(X, Y\) via

\[
(X, Y) \mapsto (-i_1 uh^{-1} + hXh^{-1}, -i_2 uh^{-1} + hYh^{-1}).
\]

(1.4)

Owing to Lemma 1.9, \(\hat{G}_0\) preserves \(V^s\) of Proposition 1.7.

We consider the following codimension 2 subset of \(V^s\):

\[
V^s_0 = \{(X, Y) \in V^s; X_{12} = Y_{11}, X_{22} = Y_{21}\}.
\]

It is \(\hat{G}_0\)-invariant and the restriction of the symplectic form \(\text{tr} \, dX \wedge dY\) to \(V^s_0\) is non-degenerate.
Theorem 1.10. Let $\hat{\mu}$ denote the moment map for the action of $\hat{G}^0$ on $V_s^0$. The action of $\hat{G}^0$ on $\hat{\mu}^{-1}(0)$ is free and proper and the symplectic quotient $\hat{\mu}^{-1}(0)/\hat{G}_0$ is biholomorphic to the open subset of $(\mathbb{C}^2)^{[k+1]}$ consisting of 0-dimensional subschemes not contained in any line.

Proof. The moment map for the action of $\hat{G}_0 \simeq G_0 \ltimes \mathbb{C}^{k-2}$ is equal to $\hat{\mu}(X, Y) = (\mu(X, Y), \alpha(X, Y))$, where

$$\alpha(X, Y) = (X_{32} - Y_{31}, \ldots, X_{k2} - Y_{k1})^T.$$ 

Thus, given the definition of $V_s^0$, the condition $\alpha(X, Y) = 0$ means that the first column of $Y$ is equal to the second column of $X$. This in turn implies that the first column of $[\hat{X}, \hat{Y}]$ is equal to 0. On the other hand $\mu(X, Y) = 0$ means that the last $k - 2$ rows of $[X, Y]$ are zero. If we now take $j_1$ to be the first row of $[Y, X]$ and $j_2$ the second row of $[X, Y]$, then $[\hat{X}, \hat{Y}]$ has non-zero entries only in the first row. The argument in the proof of Theorem 1.5 together with Lemma 1.9 imply that $[\hat{X}, \hat{Y}]=0$. Let $K_{k+1}$ be the set of pairs $(A, B) \in \text{Mat}_{k+1, k+1}(\mathbb{C})^2$ such that $e_1$ is cyclic for $(A, B)$, the first column of $A$ is equal to $e_2$, and the first column of $B$ is equal to $e_3$. The above discussion shows that the map $V_s^0 \to K_{k+1}$, sending $(X, Y)$ to the matrices $[\hat{X}, \hat{Y}]$ defined above, restricts to a $\hat{G}_0$-equivariant isomorphism between $\hat{\mu}^{-1}(0)$ and $T = \{(A, B) \in K_{k+1}; [A, B] = 0\}$. The action of $\hat{G}_0$ on $T$ is free and proper, owing to the same argument as in the proof of Theorem 1.5. It remains to show that $T/\hat{G}_0$ is biholomorphic to the open subset of $(\mathbb{C}^2)^{[k+1]}$ consisting of 0-dimensional subschemes not contained in any line.

Let $H^0 \subset (\mathbb{C}^2)^{[k+1]}$ be the set of 0-dimensional subschemes not contained in any line. If $D \in H^0$ has ideal $I$, then $1, z_1, z_2$ are linearly independent elements of $\mathbb{C}[z_1, z_2]/I$. Therefore we can choose a basis $\mathbb{C}[z_1, z_2]/I$ such that $1, z_1, z_2$ are its first 3 elements. This means that the resulting commuting matrices $A, B$ belong to $K_{k+1}$. The group preserving $K_{k+1} \subset K_n$ is precisely $\hat{G}_0$, so that $H^0 \subset T/\hat{G}_0$. Conversely, let $D$ be a 0-dimensional subscheme of $\mathbb{C}^2$ represented by an element of $T/\hat{G}_0$. The ideal of $D$ is given by polynomials $p(z_1, z_2)$ vanishing on $(A, B)$. Given the form of the first column of $A$ and of $B$, no linear polynomial vanishes on $(A, B)$. \square

2 CURVES AND COMMUTING MATRICES

We now wish to describe a correspondence between finite coverings of $\mathbb{P}^1$ and certain commuting matrix polynomials.

2.1 Flat projections

Let $C$ be a connected curve (cf. Remark 0.1) of arithmetic genus $g$ and $\pi : C \to \mathbb{P}^1$ a flat projection of degree $d$. We consider the sheaf of algebras $\pi_* \mathcal{O}_C$. As a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules it is locally free, that is, a vector bundle of rank $d$, which we denote by $E_\pi$. It has a trivial summand and all remaining summands have a negative degree. Moreover the degree of $E_\pi$ is equal to $-(d + g - 1)$ (since $\chi(E_\pi) = \chi(\mathcal{O}_C) = 1 - g$). Consider now the regular representation of $\pi_* \mathcal{O}_C$ on itself. It gives a global injective morphism $\pi_* \mathcal{O}_C \to \mathcal{E}nd(E_\pi)$, that is, a global section of $\mathcal{E}nd(E_\pi) \otimes E_\pi^*$. Suppose that

$$E_\pi \simeq \bigoplus_{i=1}^d \mathcal{O}_{\mathbb{P}^1}(-k_i), \quad (2.1)$$
where \(0 = k_1 < k_2 \leq \cdots \leq k_{d-1}\). Then a global section of \(\mathcal{E}nd(E_\pi) \otimes E_\pi^*\) corresponds to \(d \times d\) matrices \(A_1, \ldots, A_d\) of polynomials in one variable with \(A_1 = 1\) and the \((i, j)\) entry of \(A_i\) having degree \(k_j - k_i + k_i\). Moreover these matrices commute, since \(\mathcal{O}_C\) is commutative. Finally, since each stalk of \(\pi_*\mathcal{O}_C\) is a \(d\)-dimensional algebra, we have \(\dim \mathbb{C}[A_2(t), \ldots, A_d(t)] = d\) for each \(t\). Clearly the freedom in choosing the matrices \(A_i\) is equivalent to the choice of the isomorphism (2.1).

Conversely, let \(R\) be such a \(d\)-dimensional commutative subalgebra of \(\text{Mat}_{d \times d}(\mathbb{C}[t])\) with identity. Then \(R\) is integral over \(\mathbb{C}[t]\). Consider the matrices \(A_i'\) over \(\mathbb{C}[1/t]\) obtained by conjugating \(A_i\) by \(\text{diag}(t^{k_1}, \ldots, t^{k_d})\) and dividing by \(t^{k_i}\). We obtain a commutative ring \(R'\), integral over \(\mathbb{C}[1/t]\). The affine curves Spec \(R\) and Spec \(R'\) have an obvious gluing and we obtain a curve \(C\) together with a projection \(C \to \mathbb{P}^1 = \text{Spec } \mathbb{C}[t] \cup \text{Spec } \mathbb{C}[1/t]\). Each fibre \(C_t\) is a 0-dimensional scheme of length \(d\) and so the projection \(C \to \mathbb{P}^1\) is flat. It is clear that the constructions are inverse to each other.

**Proposition 2.1.** There exists a natural 1–1 correspondence between:

(A) connected curves \(C\) with a flat projection \(\pi\) of degree \(d\) onto \(\mathbb{P}^1\) and a fixed isomorphism

\[
\pi_*\mathcal{O}_C \simeq \mathcal{O}_{\mathbb{P}1} \oplus \bigoplus_{i=2}^d \mathcal{O}_{\mathbb{P}1}(-k_i),
\]

(B) \(d-1\)-tuples of commuting \(d \times d\) matrix polynomials \(A_2(t), \ldots, A_d(t)\) such that

(i) the first column of \(A_i\) is the constant vector \(e_i\);

(ii) the \((i, j)\) entry of \(A_i\) has degree \(k_j - k_i + k_i\);

(iii) \(\dim \mathbb{C}[A_2(t), \ldots, A_d(t)] = d\) for any \(t \in \mathbb{P}^1\).

### 2.2 Curves in projective spaces

Suppose now that \(C\) is a connected curve of degree \(d\) in \(\mathbb{P}^r \setminus \mathbb{P}^{r-2}\), not contained in any hyperplane, and let the projection \(\pi\) be the restriction of the projection

\[
[z_1, \ldots, z_{r+1}] \mapsto [z_r, z_{r+1}].
\]

Each \(z_i, i = 1, \ldots, r-1\), defines a direct summand of \(\pi_*\mathcal{O}_C\), isomorphic to \(\mathcal{O}_{\mathbb{P}1}(-1)\). Moreover, since \(C\) is not contained in a hyperplane, these summands are linearly independent, and, consequently, \(k_2 = \cdots = k_r = 1\) in (2.1). Since \(\mathcal{O}_{C_t}\) is generated by \(z_1, \ldots, z_{r-1}\), the matrices \(A_2(t), \ldots, A_d(t)\) generate \(\mathbb{C}[A_2(t), \ldots, A_d(t)]\) for each \(t\). We conclude the following.

**Proposition 2.2.** There exists a natural 1–1 correspondence between:

(A) pairs \((C, \phi)\), where \(C\) is a connected curve \(C\) of degree \(d\) in \(\mathbb{P}^r \setminus \mathbb{P}^{r-2}\) not contained in any hyperplane and such that the projection \(\pi\) onto the complementary \(\mathbb{P}^1\) is flat, while \(\phi\) is a fixed isomorphism

\[
\pi_*\mathcal{O}_C \simeq \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus r-1} \oplus \bigoplus_{i=r+1}^d \mathcal{O}(-k_i),
\]

and
(B) \((r - 1)\)-tuples of commuting \(d \times d\) matrix polynomials \(A_2(t), \ldots, A_r(t)\) such that

(i) the first column of \(A_l\) is the constant vector \(e_l, l = 2, \ldots, r\);
(ii) the \((i,j)\)-entry of \(A_l\) has degree \(k_j - k_i + k_l\) (here \(k_1 = 0\) and \(k_2 = \cdots = k_r = 1\));
(iii) for any \(t \in \mathbb{P}^1\), \(\mathbb{C}[A_2(t), \ldots, A_r(t)]\) has dimension \(d\) and is conjugate to its image in \(\text{End}(\mathbb{C}^d)\) under the regular representation.

Example 2.3. Let \(C\) be a smooth curve of degree \(r\) in \(\mathbb{P}^r \setminus \mathbb{P}^{r-2}\). Such a curve is cut out by the \(2 \times 2\) minors of a \(2 \times r\) matrix of linear forms in homogeneous coordinates \(z_1, \ldots, z_{r+1}\). Thus we obtain \(\binom{r}{2}\) quadratic equations. Since we assume that the projection onto \([z_r, z_{r+1}]\) is flat, the matrix of coefficients of \(z_i z_j, i, j \leq r - 1\), is invertible and we can write the equations in affine coordinates \(t = z_r/z_{r+1}, x_i = z_i/z_{r+1}, i \leq r - 1\), as

\[
x_i x_j = \sum_{k=1}^{r-1} a_{ij}^k(t)x_k + b_{ij}(t), \quad i, j = 1, \ldots, r - 1,
\]

where \(a_{ij}\) are linear in \(t\) and \(b_{ij}\) is quadratic in \(t\). It follows that the commuting matrices \(A_2(t), \ldots, A_r(t)\) are given by

\[
A_l(t) = \begin{pmatrix}
0 & b_{l1}(t) & \cdots & b_{l,r-1}(t) \\
0 & a_{l1}(t) & \cdots & a_{l,r-1}(t) \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_{l1}(t) & \cdots & a_{l,r-1}(t) \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{l,r-1}(t) & \cdots & a_{l,r-1}(t)
\end{pmatrix}.
\]

Example 2.4. The correspondence in Proposition 2.2 commutes with projections. Thus, if \(C\) is a curve of degree \(d\) in \(\mathbb{P}^n \setminus \mathbb{P}^{n-2}\) obtained by projection onto the first \(r\) coordinates from a curve \(\hat{C}\) in \(\mathbb{P}^n \setminus \mathbb{P}^{n-2}\), then the matrix polynomials \(A_2(t), \ldots, A_r(t)\) corresponding to the curve \(C\) are simply the first \(r - 1\) matrix polynomials corresponding to the curve \(\hat{C}\).

Example 2.5. Let \(C\) be a rational curve of degree 4 in \(\mathbb{P}^3\) parametrised by \([u^3 v, v^3 u, u^4, v^4]\). We can either use the previous example or proceed directly as follows: if we set \(x = z_1/z_4, y = z_2/z_4, t = z_3/z_4\), then the ideal of \(C\) is generated by

\[
xy - t, \quad t^2 y - x^3, \quad ty^2 - x^2, \quad y^3 - x.
\]

It follows that \(\mathcal{O}(C_t)\) is spanned by \(1, x, y, y^2\) for \(t \neq \infty\) and by \(1, x, y, x^2\) for \(t \neq 0\). Computing the endomorphisms \(x \cdot (\ )\) and \(y \cdot (\ )\) gives:

\[
A = \begin{pmatrix}
0 & 0 & t & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & t & 0 \\
0 & t & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & t & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
Example 2.6. Let $C$ a complete intersection of two quadrics in $\mathbb{P}^3$, given by equations $Q_1(x, y, t) = 0$, $Q_2(x, y, t) = 0$. Let $\tilde{Q}_1(x, y), \tilde{Q}_2(x, y)$ be the parts involving only $x, y$ and let $\tilde{Q}_3(x, y)$ be the quadratic polynomial in $x, y$ independent of $\tilde{Q}_1, \tilde{Q}_2$. Then $\mathcal{O}(C_t)$ is spanned by $1, x, y, \tilde{Q}_3$, and hence $E_\pi \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. In particular, the arithmetic genus of $C$ is 1. Generically, such a $C$ is a smooth elliptic curve, but if $Q_1$ and $Q_2$ are two of the three quadrics defining a twisted cubic, then $C$ is the union of this cubic and a line, intersecting in two points.

3 | SPACE CURVES

We consider the fibrewise Hilbert scheme $Z_d$ of $d$ points for the projection $\pi : \mathbb{P}^3 \setminus \mathbb{P}^1 \to \mathbb{P}^1$ (see [1, Ch. 1, §7] for a definition and properties of relative Hilbert schemes). Locally, over an open subset $U$ of $\mathbb{P}^1$, it is just $U \times (\mathbb{C}^2)^{[d]}$. It is therefore a $(2d + 1)$-dimensional complex manifold with a holomorphic projection $p : Z_d \to \mathbb{P}^1$.

$Z_d$ satisfies the necessary conditions to be the twistor space of a pseudo-hyperkähler manifold (that is, a pseudo-Riemannian manifold $M$ with a fibrewise action of quaternions on $T_M$, parallel with respect to the Levi-Civita connection):

(i) it has an antiholomorphic involution $\sigma$ covering the antipodal map on $\mathbb{P}^1$, induced from the standard antiholomorphic involution on the total space of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathbb{P}^3 \setminus \mathbb{P}^1$;
(ii) it has an $\mathbb{O}_{\mathbb{P}^1}(2)$-valued symplectic form $\omega$ along the fibres of $\pi$, again induced$^\dagger$ from the $\mathcal{O}_{\mathbb{P}^1}(2)$-valued fibrewise symplectic form on the total space of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

The pseudo-hyperkähler manifold is then the Kodaira moduli space of $\sigma$-invariant sections of $p$, the normal bundle of which splits as $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2d}$ [18, §3(F)].

3.1 | Normal sheaf of space curves

Proposition 3.1. There exists a natural isomorphism between an open subset of the Hilbert scheme of $\mathbb{P}^3 \setminus \mathbb{P}^1$ consisting of degree $d$ curves which are flat over $\mathbb{P}^1$, and the Hilbert scheme of sections of $p : Z_d \to \mathbb{P}^1$.

Remark 3.2. The Hilbert scheme of curves of degree $d$ is the union of all components of $\text{Hilb}(\mathbb{P}^3 \setminus \mathbb{P}^1)$ which have the Hilbert polynomial of the form $h(n) = dn + c$, $c \in \mathbb{Z}$. The Hilbert scheme of sections of $p : Z_d \to \mathbb{P}^1$ is an open subscheme of $\text{Hilb}(Z_d)$.

Proof. If $C$ is a degree $d$ curve in $\mathbb{P}^3 \setminus \mathbb{P}^1$, flat over $\mathbb{P}^1$, then its scheme-theoretic intersection with each fibre of $\mathbb{P}^3 \setminus \mathbb{P}^1 \to \mathbb{P}^1$ yields a section of $Z_d$. The inverse map is defined as follows: given a section of $Z_d$, pullback the universal family over the relative Hilbert scheme $Z_d$ to $\mathbb{P}^1$. A flat family of curves $W$ in $\mathbb{P}^3 \setminus \mathbb{P}^1$ parameterised by a scheme $T$ is sent to a flat family of sections of $p$ (viewed as a flat family of functions of $\mathbb{P}^1$ in $Z_d$). Similarly, the inverse map sends a flat family of sections to a flat family of degree $d$ curves. The functorial interpretation of Hilbert schemes implies that both maps are morphisms of schemes.

$^\dagger$ Recall [2] that the Hilbert scheme of $d$ points on a symplectic surface has a canonical symplectic structure.
We can relate the normal bundle $N_s/Z_d$ of a section to the normal sheaf $\mathcal{N}_{C/p^3}$ of the corresponding curve as follows.

**Lemma 3.3.** $N_s/Z_d \cong \pi_\ast \mathcal{N}_{C/p^3}$.

**Proof.** The normal bundle of a section is isomorphic to the restriction of the vertical tangent bundle $\text{Ker } d\pi$ of $Z_d$ to the section. Since $Z_d$ is the fibrewise Hilbert scheme of points, the fibre of the vertical tangent bundle at $D \in Z_d$ is $H^0(D, \mathcal{N}_D/F)$, where $F \cong \mathbb{C}^2$ is the fibre containing $D$. On the other hand, the ideal of $D$ in $F$ is just $J_C \otimes F$, where $J_C$ is the ideal of $C$ in $\mathbb{P}^3$. Thus $\mathcal{N}_D/F = \mathcal{N}_C/p^3 \otimes F$, and hence the function $\mathbb{P}^1 \ni z \mapsto h^0(\pi^{-1}(z), \mathcal{N}_{C/p^3}|_{\pi^{-1}(z)})$ is constant. It follows from a result of Grauert [14, Corollary III.12.9] that $\pi_\ast \mathcal{N}_{C/p^3}$ is locally free. Therefore both $\pi_\ast \mathcal{N}_{C/p^3}$ and $N_s/Z_d$ are vector bundles and the natural map $\pi_\ast \mathcal{N}_{C/p^3} \to N_s/Z_d$ is an isomorphism on fibres. $\square$

**Corollary 3.4.** The normal sheaf of a Cohen–Macaulay curve in $\mathbb{P}^3$ is torsion-free.

**Proof.** We can find a projective line $l \subset \mathbb{P}^3$ such that the projection $\pi : C \to l$ is flat. Were $\mathcal{N}_{C/p^3}$ not torsion free, neither would be $\pi_\ast \mathcal{N}_{C/p^3}$, contradicting the above lemma. $\square$

Lemma 3.3 implies that the normal bundle of a section splits as $\mathbb{S} \mathbb{P}^1(1) \oplus 2d$ if and only if $H^0(C, \mathcal{N}_{C/p^3}(-2)) = 0$. As mentioned at the beginning of the section, the parameter space of curves satisfying this condition has a natural complexified pseudo-hyperkähler structure and on its $\sigma$-invariant part a genuine pseudo-hyperkähler structure [4]. We now want to investigate this parameter space via commuting matrix polynomials.

If $E$ is a rank $d$ vector bundle on $\mathbb{P}^1$, we denote by $\mathcal{K}(E)$ the subsheaf of $(\mathcal{E}nd(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2}$ defined by

$$\mathcal{K}(E)(U) = \{(A(t), B(t)) \in (\mathcal{E}nd(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2}(U); \forall t \in U \ (A(t), B(t)) \in M_{d}^{\text{reg}} \}.$$ 

We then denote by $K(E)$ the total space of $\mathcal{K}(E)$, that is, the subset of the total space of $(\mathcal{E}nd(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2}$ consisting of points through which passes a local section of $\mathcal{K}(E)$. Since $Z_d$ is the relative Hilbert scheme for the projection $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1$, Proposition 1.4 implies that $Z_d$ is biholomorphic to the quotient of $K(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2d})$ by the fibrewise action of $GL(d, \mathbb{C})$. In fact we have the following lemma.

**Lemma 3.5.** Let $E$ be a vector bundle of rank $d$ on $\mathbb{P}^1$. Then the quotient of $K(E)$ by the fibrewise action of $GL(d, \mathbb{C})$ is biholomorphic to $Z_d$.

**Proof.** Let $E \cong \bigoplus \mathcal{O}_{\mathbb{P}^1}(\lambda_i)$. Then $\mathcal{E}nd(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ is isomorphic to two copies $\mathbb{C} \times \text{Mat}_{d \times d}(\mathbb{C})$ glued over $t \neq 0, \infty$ by $(\tilde{t}, \tilde{A}) = (1/t, t^4 A^r \lambda)$, where $\lambda = \text{diag}(\lambda_1, ..., \lambda_d)$. Thus, after taking the fibrewise quotient by $GL(d, \mathbb{C})$, we obtain the same complex manifold, independently of the choice of $E$. $\square$

From now on $d \geq 3$. Let $C$ be a connected non-planar curve of degree $d$ in $\mathbb{P}^3 \setminus \mathbb{P}^1$, flat over $\mathbb{P}^1$. If $C$ satisfies $\pi_\ast \mathcal{O}_C \cong E$, then the corresponding section of $p : Z_d \to \mathbb{P}^1$ arises as the projection of a global section of $\mathcal{K}(E) \to \mathbb{P}^1$, that is, it can be represented by a pair of commuting matrix
polynomials, the degrees of which satisfy the constraints of part B in Proposition 2.2. We should like to remark that this gives constraints for possible degrees of commuting pairs of polynomials $A(t), B(t)$, such that $\dim Z(A(t), B(t)) = d$ for each $t$.

Let $Z$ denote the bundle of centralisers of $(A, B)$, that is, a subbundle of $End(E)$, the fibre of which over each $t$ is spanned by $Z(A(t), B(t))$. Let $T$ denote the kernel of the homomorphism

$$D : (\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^\oplus_2 \to \text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(2), \quad (a, b) \to [A, b] + [a, B]. \quad (3.1)$$

In other words $T$ is the fibrewise tangent bundle to $K(E)$. We obtain a short exact sequence of locally free sheaves on $\mathbb{P}^1$:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(d) \to T \to N \to 0, \quad (3.2)$$

where $N$ is the normal bundle of the section in $Z_d$ and the first map is given by $\rho \mapsto ([\rho, A], [\rho, B])$, that is, its image consists of fundamental vector fields for the action of $GL(d, \mathbb{C})$.

**Lemma 3.6.** $Z \simeq E$ and if we write $E \simeq \mathcal{O}_{\mathbb{P}^1} \oplus E'$, then the embedding $Z \hookrightarrow \text{End}(E) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus E' \oplus (E')^* \oplus \text{End}(E')$ is the isomorphism with the direct sum of the first two summands.

**Proof.** Let $C$ be the curve in $\mathbb{P}^3 \setminus \mathbb{P}^1$ determined by $A(t), B(t)$. We know that $\pi_\ast \mathcal{O}_C \simeq E$. Let $E' \simeq \bigoplus_{i=2}^d \mathcal{O}_{\mathbb{P}^1}(\lambda_i)$. Each summand defines a section $A_i(t)$ of $\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(-\lambda_i)$ and these matrix polynomials commute. Moreover the first column of each $A_i(t)$ is just $e_i$. The bundle $Z$ is spanned over each $t$ by 1 and the $A_i(t)$, that is, it is isomorphic to the first column of $\text{End}(E)$. The claim follows. $\square$

The image sheaf of $D$ is also locally free and can be identified with the annihilator of $Z$ in $\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$ (as vector bundles):

$$\rho \in \text{Im} D \iff \text{tr} \rho z = 0 \quad \forall z \in Z.$$

Let us write $\mathcal{L} = (E')^* \oplus \text{End}(E') \simeq \text{End}(E)/Z$. As a matrix of endomorphisms of $E$, it has the first column equal to 0. From the above characterisation of $\text{Im} D$ it follows that $\text{Im} D \simeq \mathcal{L}^*(2)$, and, consequently, we can write the two short exact sequences (that is, $(3.2)$ and the sequence induced by $(3.1)$) as:

$$0 \to \mathcal{L} \to T \to N \to 0, \quad (3.3)$$

$$0 \to T \to (\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^\oplus_2 \to \mathcal{L}^*(2) \to 0. \quad (3.4)$$

These sequence describe $Z_d$, at least infinitesimally, as a fibrewise symplectic quotient of the total space of $(\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^\oplus_2$ by a Lie group, the Lie algebra of which is $\mathcal{L}$. Our idea, on how to study the hyperkähler structure of the Hilbert scheme of curves in $\mathbb{P}^3 \setminus \mathbb{P}^1$, is essentially to describe the hyperkähler manifold of sections of $Z_d$ as a ‘hyperkähler quotient’ of the (open dense subset of) vector space $H^0(\mathbb{P}^1, (\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^\oplus_2)$ by a Lie group, the Lie algebra of which is $H^0(\mathbb{P}^1, \mathcal{L})$. This cannot quite work in this form for various reasons, the most obvious of which is that the sections of $(\text{End}(E) \otimes \mathcal{O}_{\mathbb{P}^1}(1))^\oplus_2$ have the wrong normal bundle in order to form a hyperkähler
In §5 we shall show how to modify and implement this idea for curves of genus 0. For higher genera, one probably has to work with the $P$-structures of Gindikin [12], and with their quotients.

**Remark 3.7.** The above sequences provide numerical restrictions on curves which are cohomologically stable. Indeed, tensor (3.3) and (3.4) with $\mathcal{O}(-2)$ and put:

$$a = h^0((\mathcal{L}(-2))) = h^1(\mathcal{L}^\ast), \quad b = h^1((\mathcal{L}(-2))) = h^0(\mathcal{L}^\ast),$$

$$c = h^0(\mathcal{E}nd(E)(-1)) = h^1(\mathcal{E}nd(E)(-1)).$$

Then $h^0(T(-2)) \geq 2c - b$ and, consequently, if $C$ is cohomologically stable, that is, $h^0(N(-2)) = h^1(N(-2)) = 0$, then $a \geq 2c - b$. Let

$$\pi_\ast \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^1}(-i)^{m_i}.$$ 

Then $a = \sum_{j \geq i+2} (j-i-1)m_j + \sum_{j \geq 2} (i-1)m_i$, $b = \sum_{j \geq i} (j-i+1)m_j$, $c = \sum_{j \geq i+1} (j-i)m_j + \sum i^2 m_i$. Thus

$$0 \geq 2c - a - b = \sum_i (i+1)m_i - \sum m_i^2.$$ 

For example, a curve such that $m_i \leq i + 1$ for all $i$ and $m_i \leq i$ for at least one $i$ cannot be cohomologically stable (cf. Example 2.6).

### 3.2 Twistor space revisited

Given the constraints of Proposition 2.2 on the matrices $A, B$ representing a space curve, we can replace $Z_d$ for $d \geq 3$ by its open dense subset $Z'_d$, the fibres of which consist of 0-dimensional subschemes of $\mathbb{C}^2$ not contained in any line. We have an analogue of Proposition 3.1, the proof of which is the same.

**Proposition 3.8.** Let $d \geq 3$. There exists a natural isomorphism between an open subset of the Hilbert scheme of $\mathbb{P}^3 \setminus \mathbb{P}^1$ consisting of degree $d$ curves which are flat over $\mathbb{P}^1$, and the Hilbert scheme of sections of $p : Z'_d \to \mathbb{P}^1$.

Theorem 1.10 tells us how to construct $Z'_d$ from a vector bundle on $\mathbb{P}^1$. Let $E'$ be a vector bundle of rank $d - 1$ over $\mathbb{P}^1$ of the form $E' \cong \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^1}(-k_i)$ with $k_1 = k_2 = 1$ and all $k_i$ positive. Define $\mathcal{K}^0(E')$ to be the subsheaf of $(\mathcal{E}nd(E') \otimes \mathcal{O}_{\mathbb{P}^1}(1))^\bigotimes 2$, the local section of which are pairs $(X(t), Y(t))$ such that for all $t$:

(i) the first column of $Y(t)$ is equal to the second column of $X(t)$;
(ii) there is no subspace $S$ of $\mathbb{C}^{d-1}$ such that $e_1, e_2 \in S$, $X(t)S \subset S$, $Y(t)S \subset S$;
(iii) $[X(t), Y(t)]$ has non-zero entries only in the first two rows.
We denote by $K^0(E)$ the total space of $\mathcal{K}^0(E)$ (defined as for $K(E)$). We define $Z(E')$ to be the fibre-wise quotient of $K^0(E')$ by the action (1.4) of the group $\hat{G}^0$ defined in §1.1. Theorem 1.10 implies that $Z(E') \cong Z'_d$ and that is a smooth manifold fibreing over $\mathbb{P}^1$ and we can conclude the following.

**Proposition 3.9.** $Z(E') \cong Z'_d$.

**Proof.** This is the same argument as in the proof of Lemma 3.5. □

Consequently, sections of $Z'_d$ corresponding to curves with $\pi_*\mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus E'$ arise as projections of global sections of $K(E')$, and we can represent them by pairs $(X(t), Y(t))$ of matrix polynomials satisfying the above constraints.

## 4 INTERLUDE: COMPLEXIFIED HYPERKÄHLER STRUCTURES

The complexified algebra of quaternions is isomorphic to $\text{Mat}_{2,2}(\mathbb{C})$. Consequently, a complexification of a real manifold with a geometry based on quaternions (such as hyperkähler, hypercomplex and quaternionic) or on split quaternions (hypersymplectic geometry) will possess a geometry based on this algebra (called *algebra of biquaternions* by Hamilton). Such geometries have been considered in the past, in particular by Jardim and Verbitsky in [19]. They call a complexified hyperkähler structure a *trisymplectic structure generating an $SL(2, \mathbb{C})$-web*. Although there are good reasons for this terminology, we prefer the shorter name $\mathbb{C}$-hyperkähler.

**Definition 4.1.** A complex manifold $M$ is called *almost $\mathbb{C}$-hypercomplex* if its holomorphic tangent bundle $TM$ decomposes as $TM \cong E \otimes \mathbb{C}^2$, where $E$ is a holomorphic vector bundle. It is $\mathbb{C}$-hypercomplex if, in addition, for any $v \in \mathbb{C}^2$ the subbundle $E \otimes v$ defines an integrable distribution on $M$.

**Definition 4.2.** A $\mathbb{C}$-hypercomplex manifold is called *$\mathbb{C}$-hyperkähler* if it is equipped with a holomorphic section $g$ of $S^2T^*M$, such that:

(i) at any $m \in M$, $g_m$ is non-degenerate,
(ii) for any $A \in \text{Mat}_{2,2}(\mathbb{C})$, $g(A \cdot \cdot) = g(\cdot, A_{\text{adj}} \cdot)$, and
(iii) for any $A \in \mathfrak{sl}(2, \mathbb{C})$, the holomorphic 2-form $\omega_A = g(A \cdot, \cdot)$ is closed.

Given a $\mathbb{C}$-hypercomplex manifold we can define an integrable distribution $D$ on $M \times \mathbb{P}^1$ by $D|_{M \times \{z\}} = E \otimes h$, where $h$ is the highest weight vector for the maximal torus in $SL(2, \mathbb{C})$ corresponding to $z \in \mathbb{P}^1$. If $D$ is simple,\(^1\) then the leaf space $Z$ is the twistor space of $M$, and points of $M$ correspond to sections of $Z \to \mathbb{P}^1$ with normal bundles splitting as $\bigoplus \mathcal{O}_{\mathbb{P}^1}(1)$. If the distribution $D$ is not simple, we need to view $Z$ in terms of foliated geometry.

If $M$ is $\mathbb{C}$-hyperkähler, then its twistor space is equipped with a fibrewise $\mathcal{O}_{\mathbb{P}^1}(2)$-valued complex symplectic form.

We shall now discuss $\mathbb{C}$-hypercomplex and $\mathbb{C}$-hyperkähler quotients. In the case of a reductive Lie group, $\mathbb{C}$-hyperkähler quotients have been introduced and studied by Jardim and Verbitsky.

\(^1\) We call a distribution $D$ on a manifold $Y$ *simple*, if the leaf space $X$ of the corresponding foliation is a manifold and the map $Y \to X$ is a submersion.
under the name ‘trisymplectic reduction’. For us the case of non-reductive groups will be of paramount importance.

**Definition 4.3.** Let $M$ be a $\mathbb{C}$-hypercomplex manifold equipped with a holomorphic action of a complex Lie group $G$ preserving the $\mathbb{C}$-hypercomplex structure. A $\mathbb{C}$-**hypercomplex moment map** is a $G$-equivariant holomorphic map $\mu : M \to g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$, such that there exists a holomorphic $g^*$-valued 1-form $\phi$ with the property that $\Phi = \phi \cdot 1_{2 \times 2} + d\mu : TM \to g^* \otimes \text{Mat}_{2,2}(\mathbb{C})^*$ satisfies $\langle \Phi(Av), B \rangle = \langle \Phi(v), A_{adj}B \rangle$ for any $A \in \text{Mat}_{2,2}(\mathbb{C})$.

**Definition 4.4.** Let $M$ be a $\mathbb{C}$-hyperkähler manifold equipped with a holomorphic action of a complex Lie group $G$ preserving the $\mathbb{C}$-hyperkähler structure. A $\mathbb{C}$-**hyperkähler moment map** is a $G$-equivariant holomorphic map $\mu : M \to g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$, such that, for any $A \in \mathfrak{s}\mathfrak{l}(2, \mathbb{C})$ and any fundamental vector field $X_\rho$, $\rho \in \mathfrak{g}$, $\langle d\mu(\cdot), \rho \otimes A \rangle = \omega_A(X_\rho, \cdot)$.

**Remark 4.5.** A $\mathbb{C}$-hyperkähler moment map is also a $\mathbb{C}$-hypercomplex moment map, with $\langle \phi(v), \rho \rangle = g(X_\rho, v)$, $\rho \in \mathfrak{g}, v \in TM$.

The $\mathbb{C}$-hypercomplex or $\mathbb{C}$-hyperkähler reduction proceeds now along the usual lines: given $G$ and a moment map $\mu : M \to g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$, choose a $G$-invariant element $c \in g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$. Unlike in the hyperkähler case, the freeness of the action of $G$ on $\mu^{-1}(c)$ does not imply that $\mu^{-1}(c)$ is smooth. Moreover, even if $\mu^{-1}(c)$ is smooth and the action of $G$ on $\mu^{-1}(c)$ is free and proper, then although $\mu^{-1}(c)/G$ is a complex manifold [7, Chapter III, Proposition 10], it is not necessarily a $\mathbb{C}$-hypercomplex or $\mathbb{C}$-hyperkähler manifold. As observed by several authors in related settings (especially [20, §3], [17], [8, §4]), both smoothness of $\mu^{-1}(c)$ and the existence of induced geometry on $\mu^{-1}(c)/G$ are guaranteed by a single non-degeneracy condition. Namely, we have the following theorem.

**Theorem 4.6.** Let $\mu : M \to g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$ be a $\mathbb{C}$-hypercomplex (respectively, $\mathbb{C}$-hyperkähler) moment map and let $c \in g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$ be $G$-invariant. Suppose that the action of $G$ on $\mu^{-1}(c)$ is free and proper and that, at any $m \in \mu^{-1}(c)$, $\phi_m(X_\rho) \neq 0$ for any $\rho \in \mathfrak{g}$ (respectively, the restriction of $g$ to the subspace of $T_mM$ generated by fundamental vector fields is non-degenerate). Then $\mu^{-1}(c)/G$ is a $\mathbb{C}$-hypercomplex (respectively, $\mathbb{C}$-hyperkähler) manifold.

However, the assumptions in Theorem 4.6 are not the most general ones, under which one obtains a non-degenerate $\mathbb{C}$-hyperkähler quotient.

**Proposition 4.7.** Let $\mu : M \to g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$ be a $\mathbb{C}$-hyperkähler moment map and let $c \in g^* \otimes \mathfrak{s}\mathfrak{l}(2, \mathbb{C})^*$ be $G$-invariant. Suppose that $\mu^{-1}(c)$ is smooth and the action of $G$ on $\mu^{-1}(c)$ is free and proper. Then $\mu^{-1}(c)/G$ is a $\mathbb{C}$-hyperkähler manifold if and only if $\mathfrak{f} \cap \mathfrak{f}^\perp$ is $\text{Mat}_{2,2}(\mathbb{C})$-invariant along $\mu^{-1}(c)$, where $\mathfrak{f}$ denotes the subspace of $T_mM$ generated by fundamental vector fields, and $\mathfrak{f}^\perp = \{v; g(X_\rho, v) = 0 \ \forall \rho \in \mathfrak{g}\}$.

**Proof.** Let $I, J, K \in \mathfrak{s}\mathfrak{l}(2, \mathbb{C})$ denote the standard basis of quaternions. Hitchin [17] (see also [8, p.102]) shows that along $\mu^{-1}(c)$,

$$\text{Ker } \omega_I = \mathfrak{g} + J\mathfrak{g} \cap \mathfrak{g}^\perp + K\mathfrak{g} \cap \mathfrak{g}^\perp,$$
and cyclically in $I, J, K$. Thus $\omega_I$ descends to a non-degenerate symplectic form on $\mu^{-1}(c)/G$ if and only if $J\hat{g} \cap \hat{g}^\perp + K\hat{g} \cap \hat{g}^\perp \subset \hat{g}$. Therefore $\mu^{-1}(c)/G$ is a $C$-hyperkähler manifold if and only if $\hat{g} \cap \hat{g}^\perp$ is $\text{Mat}_{2,2}(C)$-invariant.

Proposition 4.8. Suppose that the conditions of the above proposition are satisfied. Then $\dim \hat{g} \cap \hat{g}^\perp = k$ is constant along $\mu^{-1}(c)$ and

$$\dim \mu^{-1}(c)/G = \dim M - 4 \dim G + 2k.$$ 

Proof. The smoothness of $\mu^{-1}(c)$ implies that $(\text{Ker } d\mu)^\perp = I\hat{g} + J\hat{g} + K\hat{g}$ has constant dimension along $\mu^{-1}(c)$, where $\perp$ denotes the orthogonal ‘complement’ with respect to the form $\mu$. Suppose that $IX_{\rho_1} + JX_{\rho_2} + KX_{\rho_3} = 0$. Taking the scalar products (for the form $\mu$) with $I X_{\rho}, J X_{\rho}, K X_{\rho}$ shows that $X_{\rho_1}, X_{\rho_2}, X_{\rho_3} \in \hat{g} \cap \hat{g}^\perp$. Therefore $\dim (\text{Ker } d\mu)^\perp = 3 \dim G - 2 \dim \hat{g} \cap \hat{g}^\perp$ and it follows that $\dim \hat{g} \cap \hat{g}^\perp$ must be constant along $\mu^{-1}(c)$. 

We shall use the above results in the following situation.

Corollary 4.9. Let $\mu = \mu_H \oplus \mu_L : M \to g^* \otimes \mathfrak{sl}(2, C)^*$ be a $C$-hyperkähler moment map for a semidirect product $G \simeq H \ltimes L$ and let $c = (c_H, c_L) \in g^* \otimes \mathfrak{sl}(2, C)^*$ be $G$-invariant. Suppose that the following conditions are satisfied:

(i) the action of $G$ on $\mu^{-1}(c)$ is free and proper;
(ii) $\mu_L^{-1}(c_L)$ is smooth and the action of $L$ on $\mu_L^{-1}(c_L)$ is free and proper;
(iii) $\hat{I}$ is $\text{Mat}_{2,2}(C)$-invariant along $\mu_L^{-1}(c_L)$;
(iv) $\hat{I} \subset \hat{g} \cap \hat{g}^\perp$ along $\mu^{-1}(c)$, with equality holding generically.

Then:

(a) $\mu^{-1}(c)/G$ is smooth if and only if $\dim I\hat{h} + J\hat{h} + K\hat{h} = 3 \dim H$ along $\mu^{-1}(c)$;
(b) $\mu^{-1}(c)/G$ is a $C$-hyperkähler manifold if and only if $\hat{g} \cap \hat{g}^\perp = \hat{I}$ along $\mu^{-1}(c)$.

In both (a) and (b), the dimension of $\mu^{-1}(c)/G$ is $\dim M - 4 \dim H - 2 \dim L$.

Proof. Conditions (ii) and (iii) together with Proposition 4.7 imply that $M_L = \mu_L^{-1}(c_L)/L$ is a $C$-hyperkähler manifold (of dimension $\dim M - 2 \dim L$). Since $L$ is normal in $G$, we obtain a $C$-hyperkähler action of $H$ on $M_L$ and a $C$-hyperkähler moment map $\bar{\mu}_H : M_L \to h^* \otimes \mathfrak{sl}(2, C)^*$. The $C$-hyperkähler quotient of $M_L$ by $H$ is isomorphic to the $C$-hyperkähler quotient of $M$ by $G$. Statement (a) follows, since $\dim I\hat{h} + J\hat{h} + K\hat{h} = 3 \dim H$ is equivalent to $\bar{\mu}_H$ being a submersion at points of $\bar{\mu}_H^{-1}(c_H)$. On the other hand, the condition in (b) means that the triple $(M_L, H, \bar{\mu}_H)$ satisfies conditions of Theorem 4.6, and is therefore sufficient. Conversely, if $\mu^{-1}(c)/G$ is a $C$-hyperkähler manifold, then, owing to (a), $\dim I\hat{h} + J\hat{h} + K\hat{h} = 3 \dim H$, which means that $\bar{\mu}_H^{-1}(c_H)$ is smooth. The necessity of $\hat{g} \cap \hat{g}^\perp = \hat{I}$ follows now from Proposition 4.7.

5 | GENUS ZERO SPACE CURVES

We return to the situation discussed in §3 and consider in detail the case of genus 0 space curves. We denote by $R_d$ an open subset of the Hilbert scheme $\text{Hilb}_{d,0}$ of subschemes in $\mathbb{P}^3$ with Hilbert polynomial $h(n) = dn + 1$, consisting of those $C \in \text{Hilb}_{d,0}$ which satisfy
(i) $C$ is contained in $\mathbb{P}^3 \setminus \{[z_0, z_1, 0, 0]\}$;
(ii) the projection $\pi$ of $C$ onto $\{0, 0, z_2, z_3\}$ is flat;
(iii) $C$ is non-planar.

Such a $C$ is automatically Cohen–Macaulay and connected. We denote by $R_d^i$, $i = 0, 1, 2$, an open subset of $R_d$ consisting of those $C \in \text{Hilb}_{d,0}$ which satisfy, in addition, $h^1(N_{C/\mathbb{P}^3}(-i)) = 0$. We have $R_d^2 \subset R_d^1 \subset R_d^0$ and $R_d^0$ is precisely the smooth locus of $R_d$. Moreover, the results of Ghione and Sacchiero [11, Corollary 1.4] imply that any immersed rational curve in $R_d$ belongs to $R_d^1$.

For a $C \in R_d$ we have $\pi_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(d-1)}$ (since $\deg \pi_* \mathcal{O}_C = -(d + g - 1)$, $g = 0$, the degrees of the summands are non-positive, and $h^0(\mathcal{O}_C) = 1$) and thus curves in $R_d$ correspond to sections of $p : Z_d \to \mathbb{P}^1$ which arise as projections of sections of $K(E) \to \mathbb{P}^1$ with $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(d-1)}$ (and the map associating a section to a curve is a biholomorphism between $R_d$ and the corresponding open subset of the Kodaira moduli space of sections of $p$, cf. Proposition 3.1).

Equivalently, owing to Proposition 3.9, curves in $R_d$ can be obtained as sections of $Z(E')$ with $E' \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(d-1)}$.

If $C \in R_d^2$, then the normal bundle of the corresponding section of $Z_d$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2d}$ and, hence, $R_d^2$ comes equipped with a $\mathbb{C}$-hyperkähler structure. Moreover, given a real structure $\sigma$ on $|\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)|$, covering a real structure on $\mathbb{P}^1$, we obtain a pseudo-hyperkähler or hypersymplectic structure on $(R_d^2)^\sigma$ (depending on whether $\sigma$ is fixed-point free or not).

**Example 5.1.** Let $C$ be a twisted rational curve of degree 3 not meeting the $\mathbb{P}^1$. Its ideal is generated by the minors of the $3 \times 2$ matrix

$$
\begin{pmatrix}
x & 0 \\
y & x \\
0 & y
\end{pmatrix} - C(t),
$$

where the entries of $C$ are constant or linear in $t$. Thus

$$
x^2 = (c_{11} + c_{22})x - c_{12}y - C_3, \quad xy = c_{32}x + c_{11}y - C_2, \quad y^2 = (c_{21} + c_{32})y - c_{31}x - C_1,
$$

where $C_i$ denotes the determinant of a $2 \times 2$ matrix obtained by deleting the $i$th row from $C$. Thus the matrices $A$ and $B$ are

$$
A(t) = \begin{pmatrix}
0 & -C_3 \\
1 & c_{11} + c_{22} \\
0 & -c_{12}
\end{pmatrix}, \quad B(t) = \begin{pmatrix}
0 & -C_2 \\
0 & c_{32} \\
1 & c_{11} + c_{21} + c_{32}
\end{pmatrix}.
$$

The metric is given as the coefficient of $t$ in power series expansion of

$$
\text{tr} \, dA \wedge dB = d(c_{11} + c_{22}) \wedge dc_{32} + dc_{32} \wedge dc_{11} + dc_{12} \wedge dc_{31} + da_{11} \wedge d(c_{21} + c_{32}) =
$$

$$
= d(c_{11} + c_{22}) \wedge d(c_{32} + c_{21}) - dc_{22} \wedge dc_{21} + dc_{12} \wedge dc_{31}.
$$

The antiholomorphic involution $\sigma$, covering the antipodal map on $\mathbb{P}^1$, acts on linear polynomials $c_{ij}(t)$ as

$$
\begin{pmatrix}
c_{11}(t) & c_{12}(t) \\
c_{21}(t) & c_{22}(t) \\
c_{31}(t) & c_{32}(t)
\end{pmatrix} \mapsto t \begin{pmatrix}
c_{32}(-1/\bar{t}) & c_{31}(-1/\bar{t}) \\
-c_{22}(-1/\bar{t}) & c_{21}(-1/\bar{t}) \\
-c_{12}(-1/\bar{t}) & -c_{11}(-1/\bar{t})
\end{pmatrix}.
$$
Restricting the above formula for $\text{tr} \, dA \wedge dB$ to $\sigma$-invariant sections shows that the hyperkähler metric on the space of real twisted cubics is flat with signature $(8,4)$.

We shall now describe $R^2_d$, $d \geq 4$, as a $\mathbb{C}$-hyperkähler quotient of a flat $\mathbb{C}$-hyperkähler manifold. Let $M = \text{Mat}_{d-1,d-1}(\mathbb{C}) \otimes \mathbb{C}^4$ and write its elements as $(X_0, X_1, Y_0, Y_1)$. $M$ is a $\mathbb{C}$-hyperkähler manifold with the action of $\text{Mat}_{2,2}(\mathbb{C})$ on $TM$ given by the left multiplication on $\text{Mat}_{2,2}(\mathbb{C}) \simeq \mathbb{C}^4$, that is,

$$
\begin{pmatrix}
  X_0 & Y_0 \\
  X_1 & Y_1
\end{pmatrix}
\mapsto
\begin{pmatrix}
  aI & bI \\
  cI & dI
\end{pmatrix}
\begin{pmatrix}
  X_0 & Y_0 \\
  X_1 & Y_1
\end{pmatrix}.  \tag{5.1}
$$

The symmetric holomorphic $(2,0)$-tensor $g$ is

$$
g = \text{tr} \, (dX_1 dY_0 - dX_0 dY_1). \tag{5.2}
$$

The twistor space $Z$ of $M$ is the total space of $\text{Mat}_{d-1,d-1}(\mathbb{C}) \otimes \mathbb{P}^1(1)^{\otimes 2}$, and an element $(X_0, X_1, Y_0, Y_1)$ of $M$ is identified with the section

$$(X(t), Y(t)) = (X_0 + tX_1, Y_0 + tY_1)$$

of $Z \to \mathbb{P}^1$. The twisted fibrewise symplectic form on $Z$ is $\text{tr} \, d(X_0 + tX_1) \wedge d(Y_0 + tY_1)$.

We define a $\mathbb{C}$-hyperkähler submanifold $M_d$ of $M$ to consist of $(X_0, X_1, Y_0, Y_1)$ such that:

(i) $\forall t \in \mathbb{P}^1 (X(t), Y(t), (e_1, e_2))$ is stable.

(ii) $\forall t \in \mathbb{P}^1 X_{12}(t) = Y_{11}(t), \ X_{22}(t) = Y_{21}(t)$.

Next we define the relevant group $\hat{G}_d$ acting on $M_d$. It consists of invertible $d \times d$ matrices of the form

$$
g(t) =
\begin{pmatrix}
  1 & 0 & 0 & u_1(t) & \ldots & u_{d-3}(t) \\
  0 & 1 & 0 & * & \ldots & * \\
  0 & 0 & 1 & * & \ldots & * \\
  0 & 0 & 0 & * & \ldots & * \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & * & \ldots & *
\end{pmatrix},  \tag{5.3}
$$

where asterisks denote complex numbers, and $u_i(t)$, $i = 1, \ldots, d - 3$, are linear polynomials. The group structure is given by matrix multiplication. Observe that if we fix $t$, we obtain a group $\hat{G}_d(t)$ isomorphic to the group $\hat{G}_0$ of §1.1. We have $\hat{G}_d \simeq G_0 \ltimes L$, where $G_0$ is obtained by deleting the first row and the first column of (5.3) and $L \simeq \mathbb{C}^{2(d-3)}$ is the additive group of $d - 3$ linear polynomials. If we write an element of $L$ as $u(t) = (0, 0, u_1(t), \ldots, u_{d-3}(t))$, then the action of $\hat{G}_d$ on pairs of linear matrix polynomials $(X(t), Y(t))$ can be written as (cf. (1.4)):

$$
(g_0, u(t)).(X(t), Y(t)) = (g_0 X(t)g_0^{-1} - e_1 u(t)g_0^{-1}, \ g_0 Y(t)g_0^{-1} - e_2 u(t)g_0^{-1}).  \tag{5.4}
$$

This action preserves the $\mathbb{C}$-hyperkähler structure of $M_d$ and it lifts to an action on the twistor space of $M_d$: an element $g(t)$ acts as $g(t_0)$ on the fibre over $t_0$. We easily compute the $\mathbb{C}$-hyperkähler
moment map $\mu : M_d \to \hat{\mathfrak{g}}_d^* \otimes \mathfrak{sl}(2, \mathbb{C}) \simeq (\mathfrak{g}_0^* \oplus \mathfrak{I}^*) \otimes \mathfrak{sl}(2, \mathbb{C})$:

$$
\mu_{ij} = \begin{cases} 
\frac{1}{2} (\pi([X_0, Y_1] + [X_1, Y_0]), (X_1 + tX_0)e_2 - (Y_1 + tY_0)e_1) & \text{if } i=j=11, \\
(\pi([X_0, Y_0]), X_0e_2 - Y_0e_1) & \text{if } i=j=21, \\
(\pi([X_1, Y_1]), t(X_1e_2 - Y_1e_1)) & \text{if } i=j=12,
\end{cases}
$$

(5.5)

where $\pi : \text{Mat}_{d-1, d-1} \to \mathfrak{g}_0^*$ is the projection onto the last $d-3$ rows. We conclude:

**Lemma 5.2.** An element $(X_0, X_1, Y_0, Y_1)$ of $M_d$ belongs to $\mu^{-1}(0)$ if and only if, for every $t \in \mathbb{P}^1$, $X(t)e_2 = Y(t)e_1$ and $[X(t), Y(t)]$ has non-zero entries only in the first two rows.

**Remark 5.3.** We should like to remark that the moment map equations for $\hat{\mathcal{G}}_d$ are formally similar to the complex ADHM equations considered in [10]. The difference is that the $((d-1) \times 2, 2 \times (d-1))$-part $(i, j)$ (cf. §1.1) of the ADHM-datum is no longer linear in $t$; $i$ is now constant, while $j$ is quadratic.

**Lemma 5.4.** The action of $\hat{\mathcal{G}}_d$ on $\mu^{-1}(0)$ is free and proper.

**Proof.** If $g \in \hat{\mathcal{G}}_d$ fixes a point $(X_0, X_1, Y_0, Y_1)$ of $M_d$, then $g(t_0)$ fixes $(X(t_0), Y(t_0))$, for every $t_0$. Choosing a $t_0$ such that $g(t_0) \neq 1$ contradicts Theorem 1.10. Therefore $\hat{\mathcal{G}}_d$ acts freely. The moment map for the twisted symplectic form $\text{tr} dX(t) \wedge dY(t)$ is $\mu(t) = \mu_{21} + 2\mu_{11}t + \mu_{12}t^2$. Setting $X_0 = \mu(0)^{-1}(0)$, $X_1 = \mu(1)^{-1}(0)$, $X_2 = \mu(\infty)^{-1}(0)$, we have $\mu^{-1}(0) = X_0 \cap X_1 \cap X_2$. For every $t_0$, the group $\hat{\mathcal{G}}_d(t_0)$ (obtained by evaluating (5.3) at $t_0$) acts properly on $\mu(t_0)^{-1}(0)$, owing to Theorem 1.10. It follows that, if $m_i \in \mu^{-1}(0)$, $g_i = (g_i^0, u_i(t)) \in \hat{\mathcal{G}}_d$, $i \in \mathbb{N}$, and $m_i \to m$, $g_im_i \to m'$, then both $(g_i^0, u_i(0))$ and $(g_i^0, u_i(\infty))$ have convergent subsequences. Therefore $(g_i)$ has a convergent subsequence and $\hat{\mathcal{G}}_d$ acts properly on $\mu^{-1}(0)$. □

This lemma implies that $\mu^{-1}(0)/\hat{\mathcal{G}}_d$ is a Hausdorff topological space, and that the smooth part of $\mu^{-1}(0)$ is a principal $\hat{\mathcal{G}}_d$-bundle over a complex manifold. Observe now that the fundamental vector field corresponding to $u_0 + tu_1 \in \mathbb{I}$ is

$$
X_u = (-e_1u_0, -e_1u_1, -e_2u_0, -e_2u_1),
$$

and, consequently, the subspace of these vector fields is $\text{Mat}_{2, 2}(\mathbb{C})$-invariant, owing to (5.1). Observe also that $g(X_\rho, X_\rho) = 0$ at points of $\mu^{-1}(0)$, for every $\rho \in \mathfrak{g}_d$. Corollary 4.9 implies the following proposition.

**Proposition 5.5.** Let $m = (X_0, X_1, Y_0, Y_1) \in \mu^{-1}(0)$ and let $\bar{m}$ denote its image in $\mu^{-1}(0)/\hat{\mathcal{G}}_d$.

(i) $\bar{m}$ is a smooth point of $\mu^{-1}(0)/\hat{\mathcal{G}}_d$ if and only if the dimension of the subspace of $T_m M_d$ spanned by $AX_\rho$, $A \in \mathfrak{sl}(2, \mathbb{C})$, $\rho \in \mathfrak{g}_0$, equals $3 \dim \mathfrak{g}_0$;

(ii) $\bar{m}$ is a smooth point of $\mu^{-1}(0)/\hat{\mathcal{G}}_d$ and the $\mathbb{C}$-hyperkähler structure is non-degenerate at $\bar{m}$ if and only if the quadratic form (5.2) is non-degenerate on the subspace generated by $X_\rho$, $\rho \in \mathfrak{g}_0$. 

Let us denote by $M^0_d$ (respectively, $M^2_d$) the subset of $M_d$ where the condition (i) (respectively, (ii)) is satisfied. Thus $\mu^{-1}(0) \cap M^0_d/\hat{G}_d$ is a complex manifold and $\mu^{-1}(0) \cap M^2_d/\hat{G}_d$ is a $\mathbb{C}$-hyperkähler manifold. We can now state and prove our main result.

**Theorem 5.6.** There is a natural bijection between $R_d$ and $\mu^{-1}(0)/\hat{G}_d$. This bijection is a biholomorphism between $R_0^d$ and $(\mu^{-1}(0) \cap M^0_d)/\hat{G}_d$ and a $\mathbb{C}$-hyperkähler isomorphism between $R_2^d$ and $(\mu^{-1}(0) \cap M^2_d)/\hat{G}_d$.

**Proof.** Propositions 3.8 and 3.9 show that there is a natural isomorphism between $R_d$ and sections of $Z(E')$ with $E' \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (d-1)}$. From the construction, $Z(E')$ is the twistor space of $\mu^{-1}(0)/\hat{G}_d$. Sections of $Z(E')$ are projections of sections of $K(E')$, which in turn correspond to points of $\mu^{-1}(0)$ (as shown in Lemma 5.2). The smooth locus of the (component of) Hilbert scheme of sections of $Z(E')$ is then the smooth locus of $\mu^{-1}(0)/\hat{G}_d$, that is, $\mu^{-1}(0) \cap M^0_d/\hat{G}_d$, and the locus of sections, the normal bundle of which splits as $\mathbb{C} \oplus 2d \mathbb{P}^1$, is $(\mu^{-1}(0) \cap M^2_d)/\hat{G}_d$. The result follows.

**Remark 5.7.** The bijection in the first statement of this theorem is an isomorphism if we give $\mu^{-1}(0)/\hat{G}_d$ the scheme structure of the Hilbert scheme of sections of $Z(E')$. A natural question is whether this scheme structure can be obtained via non-reductive invariant theory [3].

**Remark 5.8.** One can compute the restriction of (5.2) to the subspace generated by fundamental vector fields, and thus characterise $M^2_d$ as consisting of points $m$ such that certain linear operator $L_m : \mathfrak{g}_0 \to \mathfrak{g}_0^*$ is invertible.

### 5.1 Real structures

Up to the action of $PGL(4, \mathbb{C})$, $\mathbb{P}^3$ has two antilinear involutions:

\[
\sigma[z_0, z_1, z_2, z_3] = [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2],
\]

\[
\sigma'[z_0, z_1, z_2, z_3] = [\bar{z}_1, \bar{z}_0, \bar{z}_3, \bar{z}_2].
\]

Both of them preserve $\mathbb{P}^3 = \{[z_0, z_1, 0, 0]\}$, and the space of invariant lines in $\mathbb{P}^3 \setminus \mathbb{P}^1$ is diffeomorphic to $\mathbb{R}^4$. In the case of $\sigma$, the $\mathbb{C}$-hyperkähler structure on the space of lines restricts to a flat hyperkähler structure on $\mathbb{R}^4$, while in the case of $\sigma'$, it restricts to a flat hypersymplectic structure [8]. The involutions $\sigma, \sigma'$ act on curves in $\mathbb{P}^3 \setminus \mathbb{P}^1$ and we obtain a pseudo-hyperkähler (respectively, hypersymplectic) structure on manifolds of $\sigma$-invariant (respectively, $\sigma'$-invariant) cohomologically stable curves of fixed genus and degree. We want to describe these manifolds in the case of genus 0 curves, that is, $(R^2_d)^\sigma$ and $(R^2_d)^{\sigma'}$. First of all we have the following.

**Proposition 5.9.** $R^2_d$ is empty if $d$ is even.

**Proof.** For $C$ in $R_d$ and the corresponding $\pi : C \to \mathbb{P}^1$ we have, as observed before, $\pi_*\mathcal{O}_C \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (d-1)}$. The involution $\sigma$ induces an antilinear involution on $W = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (d-1)}$, which covers the antipodal map on $\mathbb{P}^1$. This, in turn, induces an antilinear involution on
\( \Lambda^{d-1} W^* \simeq \mathcal{O}_{\mathbb{P}^1}(d - 1) \), which covers the antipodal map. Since this involution has no fixed points, the number of zeros of any section of \( \Lambda^{d-1} W^* \) must be even.

If \( d \) is odd, then there is an induced involution \( \sigma \) on \( E' \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus (d-1)} \). Modulo conjugation we can write \( \sigma \) in the standard trivialisation over \( t \neq \infty \) as

\[
\sigma(t; f_1, \ldots, f_{d-1}) = (-1/i; -\bar{t}f_2, \bar{t}f_1, -\bar{t}f_4, \bar{t}f_3, \ldots, -\bar{t}f_{d-1}, \bar{t}f_{d-2}).
\]

This, in turn, yields an antiholomorphic involution on \( \text{End}(E') \):

\[
M \mapsto \sigma M \sigma
\]

and, finally, on \( M_d \):

\[
(X(t), Y(t)) \mapsto (-\sigma Y(t)\sigma / \bar{t}, \sigma X(t)\sigma / \bar{t}).
\]

It follows that:

\[
(X_0, X_1, Y_0, Y_1) \mapsto (-\tau Y_1, \tau Y_0, \tau X_1, -\tau X_0),
\]

where

\[
\tau_{ij} = \begin{cases} 
-1 & \text{if } j = i + 1 \text{ and } i \text{ is odd}, \\
1 & \text{if } j = i - 1 \text{ and } i \text{ is even}, \\
0 & \text{otherwise}.
\end{cases}
\]

Consequently:

\[
M_d^\sigma = \{(X_0, X_1, Y_0, Y_1) \in M_d; Y_0 = \tau X_1, Y_1 = -\tau X_0\}.
\]

An easy computation shows that the quadratic form (5.2) restricted to \( M_d^\sigma \) is a pseudo-Riemannian metric of signature \((2(d - 1)^2 + 2(d - 3), 2(d - 1)^2 - 2(d - 1))\). The subgroup \( \hat{G}_d^\sigma \) commuting with \( \sigma \) consists of elements \((h, u) \in G_0 \ltimes L\), where \( h = \tau \bar{h} \tau^{-1} \) and \( u_{2i-1}(-1/\bar{t}) = -u_{2i}(t)/\bar{t}, i = 1, \ldots, (d - 3)/2 \). We conclude from Theorem 5.6.

**Theorem 5.10.** Let \( d \) be odd. The pseudo-hyperkähler manifold \( (R_2^d)^{\hat{G}_d^\sigma} \) of cohomologically stable connected \( \sigma \)-invariant Cohen-Macaulay curves of arithmetic genus 0 in \( \mathbb{P}^3 \setminus \mathbb{P}^1 \) is isomorphic to the hyperkähler quotient of \( (M_d^2)^{\hat{G}_d^\sigma} \) by the group \( \hat{G}_d^\sigma \).

**Remark 5.11.** It follows from the main result of [6] that the signature of the pseudo-hyperkähler metric on \( (R_2^d)^{\hat{G}_d^\sigma} \) is \((2d + 2, 2d - 2)\).

There is an analogous description of the hypersymplectic manifold \( (R_2^d)^{\hat{G}_d^{\hat{L}}} \). Essentially, we just have to remove all the minus signs in the above formulae (in particular, \( d \) can be arbitrary). We leave the details to the interested reader.

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