Abstract. Let $\mathcal{R}$ be an epireflective category of $\textbf{Top}$ and let $F_{\mathcal{R}}$ be the epireflective functor associated with $\mathcal{R}$. If $\mathcal{A}$ denotes a (semi)topological algebraic subcategory of $\textbf{Top}$, we study when $F_{\mathcal{R}}(A)$ is an epireflective subcategory of $\mathcal{A}$. We prove that this is always the case for semi-topological structures and we find some sufficient conditions for topological algebraic structures. We also study when the epireflective functor preserves products, subspaces and other properties. In particular, we solve an open question about the coincidence of epireflections proposed by Echi and Lazar in [10, Question 1.6] and repeated in [11, Question 1.9]. Finally, we apply our results in different specific topological algebraic structures.

1. Introduction

In this paper we deal with some applications of epireflective functors in the investigation of topological algebraic structures. In particular, we are interested in the following question: Let $\mathcal{R}$ be an epireflective subcategory (i.e., productive and hereditary) of $\textbf{Top}$ containing the 1-point space, and let $F_{\mathcal{R}}$ be the epireflective functor associated with $\mathcal{R}$. If $\mathcal{A}$ denotes a (semi)topological algebraic subcategory of $\textbf{Top}$, we study when $F_{\mathcal{R}}(A)$ is an epireflective subcategory of $\mathcal{A}$. From a different viewpoint, this question has attracted the interest of many workers recently (cf. [1, 14, 15, 17, 20, 18, 19, 22]) and, to some extent, our motivation for this research has been to give a unified approach to this topic. First, we recall some definitions and basic facts.
A full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is *reflective* if the canonical embedding of $\mathcal{A}$ in $\mathcal{B}$ has a left adjoint $F : \mathcal{B} \to \mathcal{A}$ (called *reflection*). Thus for each $\mathcal{B}$-object $B$ there exists an $\mathcal{A}$-object $F(B)$ and and a $\mathcal{B}$-morphism $r_B : B \to F(B)$ such that for each $\mathcal{B}$-morphism $f : B \to A$ to an $\mathcal{A}$-object $A$, there exists a unique $\mathcal{A}$-morphism $\overline{f} : F(B) \to A$ such that the following diagram commutes

$$
\begin{array}{ccc}
B & \xrightarrow{r_B} & F(B) \\
\Big/ f & & \Big/ \overline{f} \\
A & \Big/ & 
\end{array}
$$

The pair $(F(B), r_B)$ is called the *$\mathcal{A}$-reflection* of $B$ and the morphism $r_B$ is called the *$\mathcal{A}$-reflection arrow*. If all $\mathcal{A}$-reflection arrows are epimorphisms, then the subcategory $\mathcal{A}$ is said to be *epireflective*. The functor $F : \mathcal{B} \to \mathcal{A}$, which is called the *reflector*, assigns to each $\mathcal{B}$-morphism $f : X \to Y$, the $\mathcal{A}$-morphism $F(f)$ that is determined by the following commuting diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\Big/ r_X & & \Big/ r_Y \\
F(X) & \xrightarrow{F(f)} & F(Y).
\end{array}
$$

2. Basic facts

We collect in this section some known facts about epireflective categories that will be used along the paper. Here, we look at the category $\text{Top}$ of topological spaces and continuous functions. Following Kennison [3], by a topological property $\mathcal{P}$, we
mean a full subcategory of $\mathbf{Top}$ which is closed under the formation of equivalent ( = homeomorphic) objects. (In general such subcategories are called replete.) A topological property $\mathcal{P}$ is hereditary (resp. divisible, productive, or coproductive) if the objects of $\mathcal{P}$ are closed under the formation of relative subspaces (resp. quotient spaces, product spaces, or coproduct spaces.) Here, the terms “quotient space” and “relative subspace” are used in their topological sense, while the topological product is equivalent to the category product. In particular, Kennison proved that a full subcategory $\mathcal{P}$ of $\mathbf{Top}$ is epireflective if and only if $\mathcal{P}$ is hereditary and productive (cf. [3]). Well known examples of reflections in $\mathbf{Top}$ are: the Stone-Cech compactification, the Hewitt realcompactification, the classes of all $T_0$, $T_1$, $T_2$, and $T_3$ spaces, the class of all regular spaces, the completely regular spaces, and the class of all totally disconnected spaces (cf. [3]).

Let $\mathcal{C}$ denote an epireflective subcategory of $\mathbf{Top}$. That is, for each topological space $X$, there exists an associated topological space $\mathcal{C}(X) \in \mathcal{C}$ and a surjective continuous function $\varphi_{\mathcal{C}(X)}: X \rightarrow \mathcal{C}(X)$ such that for every continuous function $f: X \rightarrow Y$, with $Y \in \mathcal{C}$, there exists a continuous function $\tilde{f}: \mathcal{C}(X) \rightarrow Y$ such that the following diagram commutes

$$
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \varphi_{\mathcal{C}(X)} \quad \downarrow \tilde{f} \\
\mathcal{C}(X)
\end{array}
$$

Furthermore, the space $\mathcal{C}(X)$ is uniquely determined up to homeomorphism. It is well known that for every subcategory $\mathcal{A}$ of $\mathbf{Top}$ there exists a smallest epi-reflective subcategory $\mathcal{E}(\mathcal{A})$ in $\mathbf{Top}$ containing $\mathcal{A}$. It is said that $\mathcal{A}$ generates $\mathcal{E}(\mathcal{A})$. In case
there is a single space $X$ with $E(\{X\}) = C$, we say that $X$ generates $C$ and $C$ is called simply generated (cf. [6, 7]). For example the class $\text{Top}_0$ of $T_0$ spaces is generated by the Sierpinski space. However, the class $\text{Top}_1$ of $T_1$ spaces is not simply generated. In fact, the class $\text{Top}_1$ is generated by the family of cofinite spaces. Furthermore, given a cardinal $\kappa$, the space $\kappa \text{cof}$, which is the set $\kappa$ equipped with the cofinite topology, simply generates the $T_1$-reflection for all spaces in $\text{Top}$ of cardinality less than or equal to $\kappa$ (see [5, 6]).

For self-completeness, we recall below a realization of the epireflection associated to an epireflective subcategory $E(A)$ that is generated by a subcategory $A$ of $\text{Top}$.

Let $F(X, A)$ denote the class of all continuous functions of $X$ onto spaces in $A$. We set the following equivalence relation: for $f : X \to Y$ and $g : X \to Z$ in $F(X, A)$, it is said that $f$ and $g$ are equivalent, $f \sim g$, if there is a homeomorphism $\psi : Y \to Z$ such that $g = \psi \circ f$. Sea $\kappa = |X|$. Since every continuous image of $X$ can be identified as a subset of $\kappa$, the family of equivalence classes $E(X, A) = F(X, A)/\sim$ defines a set. Let $E(X, A)$ be the set defined by selecting a fixed element in each equivalence class in $E(X, A)$ and let $\varphi_{A(X)} = \Delta_E X : X \to \prod_{f \in E} f(X)$ be the diagonal map of $X$ into the product $\Pi_E X = \prod_{f \in E} f(X)$. We have that $\varphi_{A(X)}$ is a continuous function from $X$ into $\Pi_E X$ and, since $\Pi_E X \in E(A)$, it follows that $\varphi_{A(X)}(X) \in E(A)$. It is easy to check that $(A(X), \varphi_{A(X)})$ satisfies the universal property of a reflection. Indeed, let $h : X \to Y$ be a continuous function from $X$ into $Y \in A$. Then, there exists $f \in F(X, A)$, say $f : X \to Z$, such that $f \sim h$. Let $\psi : Z \to Y$ be a homeomorphism with $h = \psi \circ f$ and let $\pi_f$ be the canonical projection of $\Pi_E X$ in $f(X)$. We have $f = \pi_f \circ \varphi_{A(X)}$, which yields $h = \psi \circ (\pi_f \circ \varphi_{A(X)}) = (\psi \circ \pi_f) \circ \varphi_{A(X)}$. The general case, when $Y \in E(A)$, follows easily observing that $Y$ is a subspace of a product of spaces in $A$. 
The following facts are easily verified.

**Proposition 2.1.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be epireflective subcategories in $\text{Top}$ such that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Then the pair $(\mathcal{C}_2(\mathcal{C}_1(X)), \varphi_{\mathcal{C}_2(\mathcal{C}_1(X))})$ is a realization of the $\mathcal{C}_2$-reflection of $X$ in $\text{Top}$.

**Definition 2.2.** A class $\mathcal{C}$ in $\text{Top}$ is closed under supertopologies if whenever $(X, \tau) \in \mathcal{C}$ and $\rho$ is a topology on $X$ finer than $\tau$, it follows that $(X, \rho) \in \mathcal{C}$.

The following result, whose proof is folklore, clarifies the action the epi-reflection functor for subcategories closed under supertopologies. (see [7] for the proof, which is straightforward anyway).

**Theorem 2.3.** A class $\mathcal{C}$ in $\text{Top}$ is closed under supertopologies if and only if the reflection arrow $\varphi_X$ is a quotient mapping.

**Corollary 2.4.** The reflection arrow $\varphi_X$ is a quotient mapping for each of the following subcategories of $\text{Top}$ defined by the separating axioms: $T_0$, $T_1$, $T_2$, functionally Hausdorff, and Urysohn.

Next result gives a general realization of the reflection functor for categories whose reflections arrows are quotients. We omit its easy proof here.

**Proposition 2.5.** Let $X$ be a topological space and let $\mathcal{C}$ denote an epireflective category whose reflection arrow is a quotient. If $R_\mathcal{C}$ is the intersection of all equivalence relations $R$ on $X$ such that $X/R \in \mathcal{C}$, then $\mathcal{C}(X) = X/R_\mathcal{C}$. 

By Theorem 2.3, the proposition above applies to the epireflections defined by the separating axioms: \( T_0, T_1, T_2, \) functionally Hausdorff, and Urysohn. In particular, if \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) denote the subcategories defined by \( T_1 \) and \( T_2 \), we have the following characterization, whose proof is folklore.

**Proposition 2.6.** Let \( X \) be a topological space and let \( R \) be an equivalence relation on \( X \). The following assertions are fulfilled:

1. \( X/R \) is \( T_1 \) if and only if every equivalence class in \( R \) is closed in \( X \).
2. Assuming that the quotient map \( \varphi: X \to X/R \) is open, then \( X/R \) is Hausdorff if and only if \( R \) is a closed subset in \( X \times X \).

**Corollary 2.7.** Let \( X \) be a topological space. Then

1. \( \mathcal{C}_1(X) = X/R_{c_1} \), where \( R_{c_1} \) is the intersection of all equivalence relations whose equivalence classes are closed in \( X \).
2. If \( \varphi_{c_2(X)} \) is an open map, then \( \mathcal{C}_2(X) = X/R_{c_2} \) where \( R_{c_2} \) is the intersection of all equivalence relations \( R \) such that the quotient map \( \varphi: X \to X/R \) is open and \( R \) is closed in \( X \times X \).

**Proof.** The verification of (1) is clear. As for the proof of (2), it suffices to observe that, since \( \varphi_{c_2(X)} \) is an open map, if \( R \) is an equivalence relation in \( X \) such that \( X/R \) is Hausdorff, the the quotient map \( \varphi: X \to X/R \) is automatically open. \( \square \)

3. **Epireflective categories in topological and semitopological algebraic structures**

So far, only categories of topological spaces have been considered. However, our main interest lies on topological algebraic categories. Taking the terminology of Hart and
Kunen [9], in what follows an algebraic system \( \mathcal{L} \) is a set (possibly empty or infinite) of symbols of constants, symbols of functions (every function symbol has arity \( \geq 1 \)) and a set of equations \( \Sigma_\mathcal{L} \) that the functions and constants in \( \mathcal{L} \) must satisfy.

A structure \( \mathcal{U} \) for \( \mathcal{L} \) is a non empty set \( A \) (the domain) together with elements (of) and functions (defined on) \( A \) corresponding to the symbols in \( \mathcal{L} \) that satisfy the equations established in \( \Sigma_\mathcal{L} \). E.g., when we talk about groups, it is understood that \( \mathcal{L} = \{\cdot, i, 1, \Sigma_\mathcal{L}\} \) (symbols of the product, inverse element, identity and \( \Sigma_\mathcal{L} \) denotes the equations that define a group). In general, groups (and other algebraic systems) are displayed as \( \mathcal{U} = (A;\cdot,i,1) \), avoiding the use of the corresponding set of equations \( \Sigma_\mathcal{L} \) for short. Here, only algebraic systems that are specified by a set of equations are considered (cf. [9]).

Let \( \mathcal{U} \) be a structure for \( \mathcal{L} \) and \( f : A \rightarrow X \). If \( \Phi \in \mathcal{L} \) is an \( n \)-ary function symbol, then \( f(\Phi_\mathcal{U}) \) denotes \( \{(f(a_1,\ldots,f(a_n),f(b)) : (a_1,\ldots,a_n,b) \in \Phi_\mathcal{U}\} \). Here \( \Phi_\mathcal{U} \) is identified to the graph of \( \Phi \). We have that \( f(\Phi_\mathcal{U}) \subset X^{n+1} \) but is not necessary the graph of an \( n \)-ary function.

A topological structure (resp. semitopological structure) for \( \mathcal{L} \) is a pair \((\mathcal{U}, \tau)\) where \( \mathcal{U} \) is a structure for \( \mathcal{L} \), and \( \tau \) is a topology on \( A \) making all functions in \( \mathcal{U} \) continuous (resp. separately continuous). We write \( \mathcal{U} \) for \((\mathcal{U}, \tau)\) if the topology is understood.

Let \( \mathcal{U} \) and \( \mathcal{V} \) be two (semi)topological structures of \( \mathcal{L} \), and \( f : A \rightarrow B \). The map \( f \) is a \( \mathcal{L} \) homomorphism from \( \mathcal{U} \) to \( \mathcal{V} \) iff \( f \) is continuous, \( f(\Phi_\mathcal{U}) \subset \Phi_\mathcal{V} \) for each function symbol \( \Phi \) of \( \mathcal{L} \), and \( f(c_\mathcal{U}) = c_\mathcal{V} \) for each constant symbol \( c \) of \( \mathcal{L} \). The class consisting of \( \mathcal{L} \)-topological (resp. \( \mathcal{L} \)-semitopological) structures and (continuous) \( \mathcal{L} \)-morphisms defines a subcategory of \( \textbf{Top} \) that will be denoted by \( \textbf{Top}\mathcal{L} \) (resp. \( \textbf{STop}\mathcal{L} \)). For
example, the category of topological groups $\text{TopGrp}$ is specified by $\mathcal{L} = (\cdot, i, 1)$ with arities $(2, 1, 0)$.

**Definition 3.1.** Let $\Phi$ be a $n$-ary function on $X$. A $\Phi$-congruence in $X$ is an equivalence relation $R$ in $X$ such that if $x_i, y_i \in X$, $i = 1, \ldots, n$ and $(x_i, y_i) \in R$ for $i = 1, \ldots, n$, then $(\Phi(x_1, \ldots, x_n), \Phi(y_1, \ldots, y_n)) \in R$.

Let $\mathcal{L}$ be an algebraic system and let $\mathcal{U}$ be a structure for $\mathcal{L}$. If $X$ is the domain of $\mathcal{U}$ and $R$ is an equivalence relation on $X$ that is a $\Phi_{\mathcal{U}}$-congruence for all function symbol $\Phi \in \mathcal{L}$, then we say that $R$ is an $\mathcal{L}$-congruence.

The following proposition is easily verified.

**Proposition 3.2.** Let $\Phi$ be a $n$-ary function on $X$ and let $R$ be a $\Phi$-congruence. If $\pi: X \to X/R$ is the quotient map, then there is an $n$-ary map $\Phi_R: (X/R)^n \to X/R$ defined by $\Phi_R(\pi(x_1), \ldots, \pi(x_n)) = \pi(\Phi(x_1, \ldots, x_n))$.

**Corollary 3.3.** Let $\mathcal{L}$ be an algebraic system and let $\mathcal{U}$ be a structure for $\mathcal{L}$. If $X$ is the domain of $\mathcal{U}$ and $R$ is an $\mathcal{L}$-congruence on $X$, then $X/R$ is the domain for a structure $\mathcal{U}/R$ for $\mathcal{L}$.

**Proposition 3.4.** Let $\Phi$ and $\Psi$ $n$-ary maps on $X$ and $Y$, respectively. If $f: X \to Y$ is a map such that $f(\Phi(x_1, \ldots, x_n)) = \Psi(f(x_1), \ldots, f(x_n))$ for all $(x_1, \ldots, x_n) \in X^n$. Then there is a $\Phi$-congruence $R$ on $X$ and a map $\tilde{f}: X/R \to Y$ that makes the following diagram commutative.
In particular $\bar{f}(\Phi_R(\pi(x_1),\ldots,\pi(x_n))) = \Psi(f(x_1),\ldots,f(x_n))$.

Proof. It suffices to consider the congruence $R$ in $X$ defined by $(x_1,x_2) \in R$ if $f(x) = f(y)$. $\square$

The following result is a generalization of the first isomorphism theorem for arbitrary $\mathcal{L}$-structures.

**Theorem 3.5.** Let $f: X \to Y$ be an $\mathcal{L}$-homomorphism from $\mathcal{U}$ to $\mathcal{V}$. Then there is a $\mathcal{L}$-congruence $R$ on $X$ and a map $\bar{f}: X/R \to Y$ that makes the following diagram commutative

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\pi \downarrow & & \downarrow \bar{f} \\
X/R & \xrightarrow{\bar{f}} & \mathcal{V}
\end{array}
$$

where $\bar{f}$ is an $\mathcal{L}$-homomorphism from $\mathcal{U}/R$ to $\mathcal{V}$.

Even though the proof of next proposition is standard, we include its proof here since it will become essential in many subsequent results.

**Proposition 3.6.** Let $\{X_1,\ldots,X_n\}$ be a finite number of topological spaces and let $f: X_1 \times \ldots, X_n \to Y$ be a separately continuous map into a topological space $Y$. If $\mathcal{E}$ is an epireflective subcategory of Top, then there is a (necessarily unique) separately
continuous map $\bar{f}: \mathcal{C}(X_1) \times \ldots, \mathcal{C}(X_n) \rightarrow \mathcal{C}(Y)$ such that $\bar{f}(\varphi_{X_1}(x_1), \ldots, \varphi_{X_n}(x_n)) = \varphi_Y(f(x_1, \ldots, x_n))$.

Proof. In order to simplify the notation, we treat the case $n = 2$ only, as this is representative for the general case. The proof for $n > 2$ is obtained proceeding by induction.

For each fixed point $c \in X_2$, the map $f_c: X_1 \rightarrow Y$, defined by $f_c(x_1) := f(x_1, c)$, is continuous. Accordingly, there exists a continuous function $\tilde{f}_c: \mathcal{C}(X_1) \rightarrow \mathcal{C}(Y)$ such that $\tilde{f}_c(\varphi_{X_1}(x_1)) = \varphi_Y(f_c(x_1)) = \varphi_Y(f(x_1, c))$.

Remark that the equality above implies that the map $\tilde{f}_c: \mathcal{C}(X_1) \times X_2 \rightarrow Y$, defined by $\tilde{f}_c(x_1, c) := \varphi_Y(f_c(x_1)) = \varphi_Y(f(x_1, c))$, is separately continuous. Repeating the argument for $\tilde{f}_c: X_2 \rightarrow Y$, defined by $\tilde{f}_c(x_2) := \tilde{f}(c, x_2)$ with $c \in \mathcal{C}(X_1)$, we can extend it to a continuous map $\bar{f}: \mathcal{C}(X_2) \rightarrow \mathcal{C}(Y)$ for all $c \in \mathcal{C}(X_1)$. It is now clear that the map $\bar{f}: \mathcal{C}(X_1) \times \mathcal{C}(X_2) \rightarrow \mathcal{C}(Y)$ defined by $\bar{f}(\varphi_{X_1}(x_1), \varphi_{X_2}(x_2)) = \varphi_Y(f(x_1, x_2))$ is separately continuous. □

As a consequence of the previous result, it follows that epirelections respect semitopological structures.

**Proposition 3.7.** Let $\textbf{STop}\mathcal{L}$ be a category of semitopological structures. If $\mathcal{C}$ is an epireflective subcategory of $\textbf{Top}$, then $\mathcal{C}(\textbf{STop}\mathcal{L}) \subseteq \textbf{STop}\mathcal{L}$. Furthermore the reflection map $\varphi$ is a homomorphism in $\textbf{STop}\mathcal{L}$.

Proof. Let $(X, \tau)$ be a semitopological structure for $\textbf{STop}\mathcal{L}$. We equip $\mathcal{C}(X)$ with the algebraic structure built by taking the constants $c\mathcal{C}(X) := \varphi_X(c_X)$ for all constants $c \in \mathcal{L}$ and, if $\Phi \in \mathcal{L}$ is a separately continuous $n$-ary function symbol, then we apply Proposition 3.6 in order to define $\Phi_{\mathcal{C}(X)}(\varphi_X(x_1), \ldots, \varphi_X(x_n)) = \varphi_X(\Phi_X(x_1, \ldots, x_n))$. 

This definition implies that $\Phi_{\mathcal{C}(X)}$ is a separately continuous $n$-ary function symbol for all $\Phi \in \mathcal{L}$ that equips $\mathcal{C}(X)$ with the semitopological $\mathcal{L}$-structured inherited from the $\mathcal{L}$-structure in $X$. Furthermore, it is also clear that $\varphi_X$ is a continuous $\mathcal{L}$-homomorphism.

We are now in position of establishing the main result in this section.

**Theorem 3.8.** Let $\text{STop}\mathcal{L}$ be a category of semitopological structures. If $\mathcal{C}$ is an epireflective subcategory of $\text{Top}$, then $\mathcal{C}(\text{STop}\mathcal{L})$ is an epireflective subcategory of $\text{STop}\mathcal{L}$.

**Proof.** By Proposition 3.7, we know that $\mathcal{C}(\text{STop}\mathcal{L})$ is equipped with a semicontinuous $\mathcal{L}$-structure, where the reflection maps $\varphi_e$ are epimorphisms. Thus it will suffice to show that $\mathcal{C}$ preserves $\mathcal{L}$-morphisms.

Let $(X, \tau)$ and $(Y, \tau')$ be two semitopological structures for $\text{STop}\mathcal{L}$ and let $f: X \to Y$ be a continuous $\mathcal{L}$-morphism. If $\Phi \in \mathcal{L}$ is a separately continuous $n$-ary function symbol, by the commutativity of the diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \varphi_X \quad \downarrow \varphi_Y \\
\mathcal{C}(X) \underset{e(f)}{\longrightarrow} \mathcal{C}(Y)
\end{array}
\]

it follows
\( \mathcal{C}(f)(\Phi_{\mathcal{C}}(\varphi_X(x_1), \ldots, \varphi_X(x_n)) = \mathcal{C}(f)(\varphi_X(\Phi_X((x_1, \ldots, x_n)))) \\
= \varphi_Y(f(\Phi_X(x_1, \ldots, x_n))) \\
= \varphi_Y(\Phi_Y(f(x_1) \cdots, f(x_n))) \\
= \Phi_{\mathcal{E}(Y)}(\varphi_Y(\varphi_Y(f(x_1)), \ldots, \varphi_Y(f(x_n))) \\
= \Phi_{\mathcal{E}(Y)}(\mathcal{E}(f)(\varphi_X(x_1)) \cdots, \mathcal{E}(f)(\varphi_X(x_n))) \\
\)

Theorem 3.8 allows us to obtain a neat realization of epireflections whose reflection arrows are quotient maps.

**Corollary 3.9.** Let \( \text{STop}\mathcal{L} \) be a category of semitopological structures. If \( \mathcal{C} \) is an epireflective subcategory of \( \text{Top} \) whose reflection arrows are a quotient maps then \( \mathcal{C}(X) = X/R_{\mathcal{E}} \) for all \( X \in \text{STop}\mathcal{L} \), where \( R_{\mathcal{E}} \) coincides with the intersection of all \( \mathcal{L} \)-congruences such that \( X/R_{\mathcal{E}} \in \mathcal{C} \).

**Proof.** Let \( \pi: X \rightarrow X/R_{\mathcal{E}} \) be the canonical quotient map. Since the reflection arrow \( \varphi_{\mathcal{E}(X)} \) is morphism in \( \text{STop}\mathcal{L} \) and also a quotient, it follows that \( \mathcal{E}(X) = X/L \), being \( L \) an \( \mathcal{L} \)-congruence in \( X \). Now, since \( X/R_{\mathcal{E}} \in \mathcal{C} \), there is a continuous \( \mathcal{L} \)-morphism \( f: \mathcal{E}(X) \rightarrow X/R_{\mathcal{E}} \) such that \( f \circ \varphi_{\mathcal{E}(X)} = \pi \). Since \( R_{\mathcal{E}} \subseteq L \), we obtain that \( R_{\mathcal{E}} = L \), which completes the proof. \( \square \)

Using Corollary 2.7, we obtain

**Corollary 3.10.** Let \( \text{STop}\mathcal{L} \) be a category of semitopological structures and let \( X \) be a space in \( \text{STop}\mathcal{L} \). Then
(1) \( C_1(X) = X/R_{e_1} \) where \( R_{e_1} \) is the intersection of all \( L \)-conguences whose equivalence classes are closed in \( X \).

(2) If \( \varphi \) is an open map, then \( C_2(X) = X/R_{e_2} \) where \( R_{e_2} \) is the intersection of all \( L \)-conguences \( L \) such that the quotient map \( \varphi: X \to X/L \) is open and \( L \) is closed in \( X \times X \).

The following result was established by Tkachenko [20, Theorem 3.4] for semitopological groups. Here, we give a slightly more general formulation.

**Corollary 3.11.** Let \( L \) denote the algebraic system defined by groups and left (resp. right) translations and let \( \text{STop}L \) be the corresponding category of left (resp. right) semitopological groups. Then \( C_1(G) = G/H \) for all \( G \in \text{STop}L \), where \( H \) is the intersection of all closed subgroups of \( G \).

**Proof.** It suffices to observe that if \( R \) is an \( L \)-conguence defined on \( G \), the the equivalence classes are cosets of the subgroup \( H \), which is the equivalence class that contains the neutral element. \( \square \)

4. Products

In this section we deal with the epireflections that preserve products, which is a crucial fact in order to study the preservation of topological structures.

Let \( C \) be an epireflective category in \( \text{Top} \) and let \( \{X_i\} \) a set (resp. finite set) of topological spaces. Then there is a canonical continuous map \( \mu_e: C(\prod X_i) \to \prod C(X_i) \) defined uniquely by the condition that the following diagram commutes for every \( j \)
\[
\begin{array}{ccc}
\mathcal{C}(\prod X_i) & \xrightarrow{\mu_e} & \prod \mathcal{C}(X_i) \\
\downarrow \mathcal{C}(\pi_{X_j}) & & \downarrow \mathcal{C}(\pi_{X_j}) \\
\mathcal{C}(X_j) & & \mathcal{C}(X_j)
\end{array}
\]

here \(\pi\) denotes the canonical projection. If \(\mu_e\) is a homeomorphism onto \(\prod \mathcal{C}(X_i)\) for every family of topological spaces (resp. finite family of topological spaces) then we say that \(\mathcal{C}\) preserves products (resp. \(\mathcal{C}\) preserves finite products).

For example, from [8, Cor. 2], it follows the following necessary condition for the preservation of finite products.

**Proposition 4.1.** Let \(\mathcal{C}\) be an epireflective category in \(\textbf{Top}\) and let \(\{X_i : i \in I\}\) set of topological spaces such that \(\varphi_{X_i}\) is open for all \(i \in I\). Then \(\mathcal{C}\) preserves the product \(\prod_{i \in I} X_i\).

**Proof.** It is easily seen that the canonical map \(\mu_e\) defined above is a bijection. Moreover, since every reflection arrow \(\varphi_{X_i}\) is open, it follows that the reflection arrow for the product \(\prod_{i \in I} \varphi_{X_i}\) is also open. Therefore, we have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}(\prod X_i) & \xrightarrow{\varphi_{\prod X_i}} & \prod \mathcal{C}(X_i) \\
\downarrow \mathcal{C}(\varphi_{X_i}) & & \downarrow \mathcal{C}(\varphi_{X_i}) \\
\mathcal{C}(X_j) & & \mathcal{C}(X_j)
\end{array}
\]

Since both \(\varphi_{\prod X_i}\) and \(\prod \varphi_{X_i}\) are open, it follows that \(\mu_e\) is a homeomorphism. \(\square\)

It is a well-known fact that the \(T_0\)-reflection in \(\textbf{Top}\) preserves finite products but the \(T_1\), \(T_2\) and \(T_3\) reflections do not preserve all finite products.
The following result improves Proposition 3.6 for epireflections that preserve finite products.

**Proposition 4.2.** Let \( \{X_1, \ldots, X_n\} \) be a finite number of topological spaces and let \( f: X_1 \times \ldots, X_n \rightarrow Y \) be a continuous map into a topological space \( Y \). If \( \mathcal{C} \) is an epireflective subcategory of \( \text{Top} \) that preserves finite products then there is a (necessarily unique) continuous map \( \overline{f}: \mathcal{C}(X_1) \times \ldots, \mathcal{C}(X_n) \rightarrow \mathcal{C}(Y) \) such that

\[ \overline{f}(\varphi_{X_1}(x_1), \ldots, \varphi_{X_n}(x_n)) = \varphi_Y(f(x_1, \ldots, x_n)). \]

**Proof.** Again, in order to simplify the notation, we treat the case \( n = 2 \) only, as this is representative for the general case. The proof for \( n > 2 \) is obtained proceeding by induction.

We have already verified that \( \overline{f}: \mathcal{C}(X_1) \times \mathcal{C}(X_2) \rightarrow \mathcal{C}(Y) \) is separately continuous in Proposition 3.6. By hypothesis, there is a canonical homeomorphism

\[ \mu_\mathcal{C}: \mathcal{C}(X_1 \times X_2) \rightarrow \mathcal{C}(X_1) \times \mathcal{C}(X_2) \]

which implies that the inverse mapping defined by \( \mu_\mathcal{C}^{-1}(\varphi_{X_1}(x_1), \varphi_{X_2}(x_2)) = \varphi_{(X_1 \times X_2)}(x_1, x_2) \) is continuous. Therefore, we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{C}(X_1 \times X_2) & \xrightarrow{\mu_\mathcal{C}} & \mathcal{C}(X_1) \times \mathcal{C}(X_2) \\
\varphi_{(X_1 \times X_2)} \downarrow & & \downarrow \varphi_Y \\
X_1 \times X_2 & \xrightarrow{f} & Y \\
\end{array}
\]

\[ \overline{f} = \mathcal{C}(f) \circ \mu_\mathcal{C}^{-1} \]

Hence \( \overline{f} = \mathcal{C}(f) \circ \mu_\mathcal{C}^{-1} \) is a continuous map. □
The following results improves Theorem 3.8 when the reflection functor preserves products.

**Theorem 4.3.** Let $\text{Top}L$ be a category of topological structures. If $C$ is an epireflective subcategory of $\text{Top}$ that preserves products, then $C(\text{Top}L)$ is an epireflective subcategory of $\text{Top}L$.

**Proof.** By Theorem 3.8 we know that $C(\text{Top}L)$ is an epireflective subcategory of $\text{STop}L$. Thus, it suffices to observe that $\Phi_{C(X)}$ is continuous for every $n$-ary function symbol $\Phi \in L$ as a consequence of Proposition 4.2. □

Ravsky proved in [15] that the semiregularization of a paratopological group is a paratopological group. Next corollary extends this result to arbitrary topological structures.

**Corollary 4.4.** Let $\text{Top}L$ be a category of topological structures. If $C$ denotes either the class of $T_0$ or semiregular spaces. Then $C(\text{Top}L)$ is an epireflective subcategory of $\text{Top}L$.

**Proof.** The classes of $T_0$ and semiregular spaces are both epireflective and preserve products in $\text{Top}$. □

**Remark 4.5.** Let $\text{TopGrp}$ denote the category of topological groups and let $C_0$ denote the epireflective subcategory of $T_0$ spaces in $\text{Top}$. Then $C_0(\text{TopGrp})$ is an epireflective subcategory of $\text{TopGrp}$. Furthermore, since every $T_0$ topological group is Tychonov, it follows that every member in $C_0(\text{TopGrp})$ in Tychonov. In other words, for topological groups, the $T_0$ reflection and the Tychonov reflection coincide.
5. Subspaces

In this section we study when epireflective functors preserve subspaces in topological algebraic structures. It is easy to show that epireflections do not preserve subspaces in general. In order to see this, consider the epireflective subcategories defined by the separating axioms $T_0$, $T_1$, $T_2$, $T_3$, Funcionalmente Hausdorff, $T_{3,5}$, Regular and Tychonoff, that we denote by $C_{0}$, $C_{1}$, $C_{2}$, $C_{3}$, $C_{fh}$, $C_{3,5}$, $C_{r}$ and $C_{t}$, respectively. Take a set $X$ of arbitrary infinite cardinality and an ideal point $p \notin X$. Set $X^* \overset{\text{def}}{=} X \cup \{p\}$ and consider the following two topologies on $X^*$

\[
\tau_p = \{U \subseteq X^* : p \notin U\} \cup \{X^*\}
\]

\[
\tau^p = \{U \subseteq X^* : p \in U\} \cup \{\emptyset\}.
\]

Pick a point $x$ in $X$. Since every neighborhood of $p$ in $(X^*, \tau_p)$ contains $x$, we have $\varphi_{e_i(X^*)}(x) = \varphi_{e_i(X^*)}(p)$ for $i \in \{1, 2, fh, u, r, t\}$. That is to say $\varphi_{e_i(X^*)}$ is a single-valued map. Now, take $X$, which is a discrete, dense, open subset of $(X^*, \tau_p)$. We have that $C_i(X) = X \neq \varphi_{e_i(X^*)}(X) = \varphi_{e_i(X^*)}(p)$.

As for the $C_3$ reflection, remark that no closed subset of $X^*$ is contained in a proper open subset. Therefore $C_3(X^*)$ is the indiscrete space and again $C_3(X) = X$, which yields $\varphi_{e_i(X^*)}(X) \neq C_3(X)$. This completes the proof for open subsets. For closed subsets, it suffices to take the space $(X^*, \tau^p)$.

Definition 5.1. Let $X$ be a topological space and let $\mathcal{A}$ be a subset of Top. A subset $A$ of $X$ is said $\mathcal{A}$-oset if there is a space $Y \in \mathcal{A}$ and a continuous map $f : X \rightarrow Y$ such that $A = f^{-1}(U)$ for some open subset $U$ of $Y$. It is cleat that the family of all
A-sets forms a subbase for the initial topology $\tau_A$ in $X$. The subsets $A \in \tau_A$ are called $A$-open. A subset $F$ of $X$ is said $A$-closed, if $X \setminus F$ is $A$-open.

In case $A = C_i$ for some $i \in \{1, 2, 3, r, fh, t\}$, we will use the symbolism $T_i$-open for short.

**Lemma 5.2.** Let $X$ be a topological space and let $\mathcal{C} = \mathcal{C}(A)$ be an epireflective subcategory of $\textbf{Top}$ that is generated by a family of spaces $A \subseteq \textbf{Top}$. Given a subset $A$ of $X$, the following assertions are equivalent:

(a) $A$ is $\mathcal{C}$-open (res. $\mathcal{C}$-closed).

(b) $A = \varphi_{e(X)}^{-1}(U)$, for some open (resp. closed) subset $U$ of $\mathcal{C}(X)$.

(c) $A$ is $A$-open (res. $A$-closed)

**Proof.** (a) $\Rightarrow$ (b). Let $A$ be a $\mathcal{C}$-open subset of $X$. Then there are $Y \in \mathcal{C}$, $V$ open in $Y$ and a continuous map $f : X \to Y$ such that $f^{-1}(V) = A$. By the functorial definition of epireflections, there exists a continuous map $g : \mathcal{C}(X) \to Y$ such that $g \circ \varphi_{e(X)} = f$. Therefore $A = f^{-1}(V) = \varphi_{e(X)}^{-1}(g^{-1}(V))$ and it suffices to take $U = g^{-1}(V)$.

(b) $\Rightarrow$ (c). We have seen in Section 2 that the space $\mathcal{C}(X)$ can be realized as the diagonal of a product $\Pi_{E \subseteq X} Y_f$ where $Y_f \in A$ for all $f \in E$ and $f$ stands for a continuous map $f : X \to Y_f$. Thus, the family $\{\pi_f^{-1}(W) \cap \mathcal{C}(X) : W \text{ open in } Y_f, f \in E\}$ form a open subbase in $\mathcal{C}(X)$ and, as a consequence, the topologies $\tau_e$ and $\tau_A$ both coincide. Therefore, if $A = \varphi_{e(X)}^{-1}(U)$ for some open subset $U$ of $\mathcal{E}(X)$, it follows that $U$ is $A \in \tau_A$.

(c) $\Rightarrow$ (a) is obvious. \qed

The following result characterizes when an epireflective functor preserves subspaces.
Proposition 5.3. Let $\mathcal{C}$ be an epireflective subcategory of $\textbf{Top}$ and let $X$ be a topological space. If $A$ is a subspace of $X$ we have that $\mathcal{C}(A) = \varphi_{e(X)}(A)$ if and only if the following two properties are satisfied:

(1) For all $a_1, a_2$ in $A$ such that $\varphi_{e(A)}(a_1) \neq \varphi_{e(A)}(a_2)$, we have $\varphi_{e(X)}(a_1) \neq \varphi_{e(X)}(a_2)$;

(2) For every $\mathcal{C}$-closed (resp. $\mathcal{C}$-open) subset $F$ of $A$, there is a $\mathcal{C}$-closed (resp. $\mathcal{C}$-open) subset $E$ of $X$ such that $E \cap A = F$.

Proof. Suppose that $\mathcal{C}(A) = \varphi_{e(X)}(A)$. Then (1) is obviously satisfied. As for (2) Let $U$ be a $\mathcal{C}$-open subset of $A$. By Lemma 5.2 there is an open subset $V$ in $\mathcal{C}(A)$ such that $U = \varphi_{e(A)}^{-1}(V)$. Furthermore, by hypothesis, we may assume that there is an open set $W$ in $\mathcal{C}(X)$ such that $U = \varphi_{e(A)}^{-1}(W \cap \varphi_{e(X)}(A)) \cap A$. Thus $\varphi_{e(X)}^{-1}(W)$ is $\mathcal{C}$-open and $U = \varphi_{e(X)}^{-1}(W) \cap A$.

Conversely, suppose that (1) and (2) are satisfied. Let $g: \mathcal{C}(A) \rightarrow \varphi_{e(X)}(A)$ continuous onto mapping that makes commutative the following diagram

$$
\begin{array}{ccc}
A & \xrightarrow{j_A} & X \\
\varphi_{e(A)} \downarrow & & \varphi_{e(X)} \downarrow \\
\mathcal{C}(A) & \xrightarrow{g} & \mathcal{C}(X)
\end{array}
$$

It is clear, by (1), that $g$ is one-to-one. We will show that $g$ is also open.

Let $V$ be an open subset in $\mathcal{C}(A)$. Then $\varphi_{e(A)}^{-1}(V)$ is $\mathcal{C}$-open in $A$. By (2), there is a $\mathcal{C}$-open subset $U$ of $X$ such that $\varphi_{e(A)}^{-1}(V) = U \cap A$. By the commutativity of the diagram and taking into account that $U$ is the inverse image of an open set in $\mathcal{C}(X)$,
it follows that \( g(V) = \varphi_{e(x)}(U) \cap \varphi_{e(x)}(A) \), which is open in \( C(A) \). This completes the proof. \(\square\)

**Definition 5.4.** Let \( X \) be a topological space and let \( \mathcal{A} \) denote a family of topological spaces, we say that a subset \( A \) of \( X \) is \( \mathcal{A} \)-embedded when for every continuous map \( f: A \to Z \), with \( Z \in \mathcal{A} \), there exists \( Y \in \mathcal{A} \), with \( Z \subseteq Y \), a continuous map \( \overline{f}: X \to Y \) such that \( \overline{f}|_A = f \). In other words, every continuous map on \( A \) taking values in a space in \( \mathcal{A} \) can be extended to a continuous map on \( X \) taking values in a possibly different space in \( \mathcal{A} \).

**Corollary 5.5.** Let \( X \) be a topological space and let \( \mathcal{C} = C(A) \) be an epireflective subcategory of \( \text{Top} \) that is generated by a \( \mathcal{A} \). If \( A \) is a \( \mathcal{A} \)-embedded subspace of \( X \), then \( \mathcal{C}(A) = \varphi_{e(x)}(A) \).

**Proof.** Use Proposition 5.3, where assertion (1) is obviously fulfilled. As for assertion (2), it suffices to observe that the collection of \( \mathcal{A} \)-osets forms an open subbase for the topology \( \tau_A = \tau_C \). \(\square\)

**Example 5.6.** We have already discussed how the (sub)category \( \text{Top}_0 \) is generated by the Sierpinsky space. Using this fact and Corollary 5.5, it is easily seen that the \( \mathcal{C}_0 \)-reflection preserves subspaces.

Even though the \( \mathcal{C}_1 \)-reflection does not preserve subspaces, the results above provide a neat characterization for this property. First we need the following lemma.

**Lemma 5.7.** Let \( A \) be a subset of a topological space \( X \). Then \( A \) is \( T_1 \)-closed if and only if there is a continuous mapping \( f: X \to Y_{cof} \), where \( Y_{cof} \) is a set equipped with the cofinite topology, such that \( A = f^{-1}(p) \) for a singleton \( p \in Y \).
Proof. Suppose that \( A \) is \( T_1 \)-closed. Then there is a \( T_1 \)-space \( Z \) and a continuous mapping \( g: X \to Z \) such that \( A = g^{-1}(B) \) for a closed subset \( B \) in \( Z \). If we identify \( B \) with a singleton, say \( p_B \), and define \( Y \) as \( (Y \setminus B) \cup \{p_B\} \), then the map \( f: X \to Y_{cof} \), defined by \( f(x) = g(x) \) if \( x \notin g^{-1}(B) \) and \( f(x) = a_B \) if \( x \in g^{-1}(B) \), is continuous and \( A = f^{-1}(p_B) \). The reverse implication is obvious.

Since the category \( \textbf{Top}_1 \) of \( T_1 \)-spaces is generated by the spaces equipped with the cofinite topology, we say that a subset \( A \) of a topological space \( X \) is \( T_1 \)-embedded if for every continuous map \( f: A \to Z_{cof} \), where \( Z \) is a set equipped with the cofinite topology, there exists a set \( Y \), with \( Z \subseteq Y \), a continuous map \( \tilde{f}: X \to Y_{cof} \) such that \( \tilde{f}|_A = f \).

**Lemma 5.8.** Every \( T_1 \)-closed subspace \( A \) of a topological space \( X \) is \( T_1 \)-embedded.

Proof. Let \( f: A \to Z_{cof}^1 \) be a continuous map defined on \( A \). By Lemma 5.7, there is a continuous map \( g: X \to Z_{cof}^2 \) and a point \( p \in Z^2 \) such that \( A = g^{-1}(p) \). Set \( Y \) as \( Z^1 \sqcup Z^2 \), the disjoint union of \( Z^1 \) and \( Z^2 \), and define \( \tilde{f}: X \to Y_{cof} \) by \( \tilde{f}(x) = f(x) \) if \( x \in A \) and \( \tilde{f}(x) = g(x) \) if \( x \notin A \). It is clear that inverse image by \( \tilde{f} \) of every singleton in \( Y \) is closed in \( X \), which yields the continuity of the map.

**Theorem 5.9.** If \( A \) is a \( T_1 \)-closed subspace of a topological space \( X \), then \( \mathcal{C}(A) = \varphi(c,x)(A) \)

Proof. Apply Corollary 5.5 and Lemma 5.8.

In [19, Lemma 3.7] M. Tkachenko proved that the \( T_1 \)-reflection preserves closed subgroups in the category of semitopological groups. Next Corollary is a variant of Tkachenko’s result. Again, our formulation is somewhat more general.
Corollary 5.10. Let $G$ be a left (resp. right) topological semigroup and let $H$ be closed subgroup of $G$. Then \( \mathcal{C}(H) = \varphi_{\mathcal{C}(G)}(H) \)

Proof. Set $X \overset{\text{def}}{=} G/H$ and remark that the canonical quotient $\pi : G \to X_{\text{cof}}$ is continuous. Furthermore, we have that $H = \pi^{-1}(\pi(H))$, which implies that $H$ is $T_1$-closed in $G$. \( \square \)

6. Coincidence of epireflections

The following general question is dealt with in this section: Let \( \mathcal{C} \) and \( \mathcal{E} \) be two epireflective subcategories of \textbf{Top} such that \( \mathcal{C} \supseteq \mathcal{E} \). Characterize the spaces $X$ such that $\mathcal{C}(X) = \mathcal{E}(X)$. This topic has been studied in \cite{10, 11} where it is left as an specific open question the characterization of the spaces $X$ for which $\mathcal{C}_1(X) = \mathcal{C}_t(X)$, being $\mathcal{C}_1$ and $\mathcal{C}_t$ the epireflective functors associated to $\mathcal{C}_1$ and $\mathcal{C}_t$ the subcategories of $T_1$-spaces and Tychonoff spaces, respectively.

Our approach is based in the notion of $\mathcal{C}$-open subset that has been introduced previously.

Theorem 6.1. Let \( \mathcal{C} \) and \( \mathcal{E} \) be two epireflective subcategories of \textbf{Top}_0 such that \( \mathcal{C} \supseteq \mathcal{E} \) and let $X$ be a topological space. Then $\mathcal{C}(X) = \mathcal{E}(X)$ if and only if every $\mathcal{C}$-open subset of $X$ is $\mathcal{E}$-open.

Proof. Consider the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_{\mathcal{E}(X)}} & \mathcal{E}(X) \\
\downarrow{\varphi_{\mathcal{C}(X)}} & & \downarrow{\varphi_{\mathcal{E}(X)}} \\
\mathcal{C}(X) & \xrightarrow{r_X} & \mathcal{E}(X)
\end{array}
\]
where \( r_X \) is the continuous map canonically defined since \( \mathcal{C} \supseteq \mathcal{E} \). It will suffice to verify that \( r_X \) is 1-to-1 and open. Suppose first that \( x, y \) belong to \( X \) and \( \varphi_{e(X)}(x) \neq \varphi_{e(X)}(y) \). By our initial assumption \( \mathcal{C} \) is included in \( \text{Top}_0 \). Thus there is an open subset \( W \) in \( \mathcal{C}(X) \) that contains only one of these points. Assume wlog that \( \varphi_{e(X)}(X)(x) \not\in W \), which yields \( x \in \varphi_{e(X)}^{-1}(W) \). By hypothesis, there must be an open subset \( V \) of \( \mathcal{E}(X) \) such that \( \varphi_{e(X)}^{-1}(W) = \varphi_{e(X)}^{-1}(V) \). Thus \( \varphi_{e(X)}(x) \neq \varphi_{e(X)}(y) \), which proves the injectivity of \( r_X \).

Now, let \( W \) be an arbitrary open subset of \( \mathcal{C}(X) \). Again, there must be an open subset \( V \) of \( \mathcal{E}(X) \) such that \( \varphi_{e(X)}^{-1}(W) = \varphi_{e(X)}^{-1}(V) \). Furthermore, the commutativity of the diagram above implies that \( \varphi_{e(X)}^{-1}(W) = \varphi_{e(X)}^{-1}(r_X^{-1}(V)) \). This implies that \( W = r_X^{-1}((V)) \) and, as a consequence that \( r_X(W) = V \). This completes the proof.

**Definition 6.2.** Let \( X \) be a topological space. The ring of all real valued continuous functions defined on \( X \) will be denoted by \( C(X) \). Two subsets \( A \) and \( B \) are said to be completely separated in \( X \) if there exists a mapping \( f \in C(X) \) such that \( f(a) = 0 \) for all \( a \in A \) and \( f(b) = 1 \) for all \( b \in B \).

A space \( X \) is said to be completely regular if every closed set \( F \) of \( X \) is completely separated from any point \( x \not\in F \). A completely regular \( T_1 \)-space is called a Tychonov space.

The following result answers Question 1.6 in [10], repeated in [11, Question 1.9].

**Corollary 6.3.** Let \( X \) be a topological space. Then \( \mathcal{C}_1(X) = \mathcal{C}_t(X) \) if and only if every \( T_1 \)-closed subset \( F \) of \( X \) is completely separated from any point \( x \not\in F \).

**Proof.** Necessity: Suppose that \( \mathcal{C}_1(X) = \mathcal{C}_t(X) \). By Theorem 6.1 every \( T_1 \)-closed subset \( F \) of \( X \) is \( \mathcal{C}_t \)-closed. Therefore, there is a closed subset \( E \subseteq \mathcal{C}_t(X) \) such that
\[ F = \varphi^{-1}_{e(X)}(E). \] Thus, \( \varphi_{e(X)}(x) \notin E \) for all \( x \notin F \). Being \( \mathcal{C}_t(X) \) a Tychonov space, this implies that \( F \) is completely separated from any point \( x \notin F \).

Sufficiency: Let \( U \) be a \( T_1 \)-open subset of \( X \), we must verify that \( U \) is \( \mathcal{C}_t \)-open in order to apply Theorem 6.1. By hypothesis \( X \setminus U \) is completely separated any point \( x \in U \). Hence \( X \setminus U = \varphi^{-1}_{e(X)}(\varphi_{e(X)}(X \setminus U)) \), which implies that \( X \setminus U \) is \( \mathcal{C}_t \)-closed and therefore \( U \) must be a \( \mathcal{C}_t \)-open subset of \( X \), which completes the proof. \( \square \)

7. Mal’tsev spaces

Definition 7.1. A Mal’tsev operation on a space \( X \) is a map \( f: X^3 \to X \) satisfying the identity \( f(x,x,y) = f(y,x,x) = y \) for all \( x,y \in X \). A space is a (topological) Mal’tsev space if it admits a continuous Mal’tsev operation. For example, if \( G \) is a topological group, then the map \( (x,y,z) \mapsto xy^{-1}z \) is a Mal’tsev operation on \( G \). Hence every topological group is a Mal’tsev space.

We define a left topological Mal’tsev (resp. right topological Mal’tsev) space if the map \( x \mapsto f(a,b,x) \) (resp. \( x \mapsto f(x,a,b) \)) is continuous for all \( a,b \in X \).

The classes \( \mathcal{M} \), \( \mathcal{LM} \) (resp. \( \mathcal{RM} \)) of topological Mal’tsev spaces, left topological (resp. right topological) Mal’tsev spaces are algebraic subcategories in \( \text{Top} \) with the continuous maps that respect these algebraic structures as arrows.

Mal’tsev spaces were defined by Uspenskij in [21] and have subsequently been studied by several authors. In this section we deal with these spaces and our main motivation is to transfer much of the behavior of topological groups to Mal’tsev spaces. We will see that most epireflective functors that preserve the topological group structure also respect the categories of Mal’tsev spaces.
The following result is attributed to Mal’tsev [12] by Reznichenko and Uspenskij [16]. Our formulation is somewhat more general.

**Lemma 7.2.** Let \((X, \Phi)\) be a left or right Mal’tsev space and let \(R\) be a \(\Phi\)-congruence in \(X\). Then the quotient map \(\pi: X \rightarrow X/R\) is open.

**Proposition 7.3.** Let \((X, \Phi)\) be a left or right Mal’tsev space and let \(\mathcal{C}\) be an epi-reflective class in \(\text{Top}\) closed under supertopologies. Then the reflection arrow \(\varphi_X\) is an open map.

**Proof.** By Theorem 3.8 we know that \((\mathcal{C}(X), \mathcal{C}(\Phi))\) is a left or right Mal’tsev space. Furthermore, the reflection arrow \(\varphi_X\) is a \(\Phi\)-homomorphism. By Theorem 3.5 there is a \(\Phi\)-congruence \(R\) such that the following diagram commutes

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_X} & \mathcal{C}(X) \\
\downarrow{\pi} & & \downarrow{\varphi_X} \\
X/R & &
\end{array}
\]

By Theorem 2.3 is a quotient \(\Phi\)-homomorphism. Since \(\pi\) is open by Lemma 7.2 and \(\widetilde{\varphi}_X\) is one-to-one, it follows that \(\varphi_X\) is open. \(\square\)

From Proposition 4.1 and Proposition 7.3 we obtain.

**Theorem 7.4.** Let \(\mathcal{C}\) be an epi-reflective category in \(\text{Top}\) that is closed under supertopologies. If \(\{X_i\}\) is a family of left or right Mal’tsev spaces, then \(\mathcal{C}\) preserves the product of \(\{X_i\}\).
Corollary 7.5. If $\mathcal{C}$ denotes the epi-reflective category of $\text{Top}$ defined by any of the following separating axioms: $T_0$, $T_1$, $T_2$ and functionally Hausdorff, then $\mathcal{C}$ preserves arbitrary products of left or right Mal’tsev spaces.

Next result is an application of the techniques developed in this paper. It shows that the modification of a Mal’tsev space by most separating axioms is a Mal’tsev space.

Theorem 7.6. If $\mathcal{C}$ denotes the epi-reflective category of $\text{Top}$ defined by any of the following separating axioms: $T_0$, $T_1$, $T_2$ and functionally Hausdorff, and $\text{TopM}$ denotes the subcategory of Mal’tsev spaces in $\text{Top}$. Then $\mathcal{C}(\text{TopM})$ is an epi-reflective subcategory of $\text{TopM}$.

Proof. Use Corollary 7.5 and Theorem 4.3.

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