Quantization of Liénard’s nonlinear harmonic oscillator and its solutions in the framework of supersymmetric quantum mechanics

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Abstract
Liénard-type nonlinear one dimensional oscillator is quantized using van Roos symmetric ordering recipe for the kinetic-like part of the new derived Hamiltonian. The corresponding Schrödinger equation is exactly solved in momentum space via the approach of supersymmetric quantum mechanics (SUSYQM). The bound-states energy spectra and corresponding wave functions are given explicitly in terms of the ambiguity parameters. The limiting case of no deformation agrees exactly with the eigenenergies and eigenfunctions of the ordinary quantum harmonic oscillator.

Keywords: Liénard’s differential equation, one dimensional nonlinear harmonic oscillator, SUSYQM

1. Introduction

The study of quantum mechanical systems with position-dependent effective mass (PDEM) [1] has witnessed a certain degree of importance due to their importance in describing the physics of many microstructures and mesoscopic structures of current interest [2]. However, a new formulation in momentum-space to common problems starts to gain interest in quantum mechanics [3–5].

Recently an attempt to quantify the nonlinear Liénard-type one dimensional differential equation

\[ \ddot{x} + k\dot{x} + \frac{k^2}{9}x^3 + \omega^2x = 0; \quad \left(\frac{d}{dt}\right), \]

defined on the real axis (\(x \in \mathbb{R}\)), where \(k\) and \(\omega\) are, a priori, arbitrary positive parameters, has been carried out in that space [4, 5]. This equation, which may be seen as a deformation of the linear harmonic oscillator (lho) equation, namely \(\ddot{x} + \omega^2x = 0\), by \(k\)-dependent terms, admits a periodic solution [6]

\[ x(t) = \frac{A \sin(\omega t + \delta)}{1 - \frac{4A}{3\omega} \cos(\omega t + \delta)}, \quad 0 \leq A < \frac{3\omega}{k}, \]

that converges to the latter for \(k = 0\); i.e. in the absence of deformation. For this reason, and in addition to the conditions that must satisfy any physical solution of the Schrödinger equation, the quantum version of equation (1) must coincide with the quantum harmonic oscillator in the absence of deformation (\(k \to 0\)). A condition that is not satisfied neither by the spectrum nor by the wave functions expressions derived in [5]. Indeed, though the chosen Lagrangian in [5] leads to equation (1), it does not reduce to the lho in the limit \(k \to 0\). Consequently, the obtained results do not fulfill the required condition in the absence of deformation.

Our goal in this work is to propose a suitable Lagrangian for equation (1) that is not affected by this inconsistency, to deduce the corresponding quantum Hamiltonian and to give the exact solution of the corresponding Schrödinger equation for the bound-states.
In section 2, we use Jacobi Last Multiplier (JLM) method and its relationship to the Lagrangian description of second order differential equations [7, 8]. Thus, a suitable Lagrangian with the corresponding classical Hamiltonian of equation (1) are deduced. In section 3, the classical Hamiltonian is quantized using von Roos recipe. In section 4, the corresponding Schrödinger equation is solved for bound-states in momentum-space using the approach of SUSYQM after a brief review of the latter. Then the bound-states spectrum is deduced algebraically and the corresponding wave functions are determined. Finally, a special case concerning the results in the limit of no deformation is discussed and the conclusion is established.

2. Lagrangian and Hamiltonian formulation of the Liénard-type equation

The Lagrangian functions corresponding to equation (1) may be determined by means of the JLM method [5, 7, 8]. Indeed, one can show that for the general second order differential equation
\[ \ddot{x} + s f(x) + g(x) = 0, \]
particular (basic) Lagrangians are obtained as
\[ \mathcal{L} = \frac{1}{(2 - \sigma^{-1})(1 - \sigma^{-1})} \left( \dot{x} + \frac{g(x)}{\sigma f(x)} \right)^2 - \frac{\dot{x}}{\sigma f(x)}, \]
provided that \( f(x) \) and \( g(x) \) satisfy the condition
\[ \frac{d}{dx} \left( \frac{g(x)}{f(x)} \right) = \sigma (1 - \sigma) f(x) \quad \text{with} \quad \sigma \neq 0, \frac{1}{2}, 1. \]

Substituting in equation (5) \( f(x) = px \) and \( g(x) = \frac{k^2}{9} x^3 + \omega^2 x \), corresponding to equation (1), it follows that \( \sigma \) takes two values: \( \sigma = \frac{1}{3}, \frac{2}{3} \), such that two basic Lagrangians are possible. Using the usual properties of Lagrange’s functions [9], for each value of \( \sigma \), more general Lagrangians are given by
\[ L = \frac{K}{(2 - \sigma^{-1})(1 - \sigma^{-1})} \left( \dot{x} + \frac{g(x)}{\sigma f(x)} \right)^2 - \frac{\dot{x}}{\sigma f(x)} + \frac{d}{dt} G(x, t), \]
where \( K \) is an arbitrary nonzero constant and \( G(x, t) \) is an arbitrary function. However, to obtain a suitable Lagrangian, which is reduced to the lho in the limit \( k \to 0 \) (in the absence of the deformation), \( K \) and \( G(x, t) \) have to be chosen judiciously.

In this work, we will treat the case with \( \sigma = \frac{2}{3} \) and set
\[ K = \frac{3}{\sqrt{2k^2} x^2} \quad \text{and} \quad G(x, t) = \frac{9x^4}{k^2} t + \frac{3\omega^2 x}{k}, \]
so that our Lagrangian reads
\[ L(x, \dot{x}) = \frac{9\omega^2 x}{k^2} \left[ 1 + \frac{2k}{3\omega^2} \dot{x} + \frac{k^2 x^2}{9\omega^2} \right]^{1/2} + \frac{k}{3\omega^2} \dot{x} + 1 \]
and is reduced in the limit \( k \to 0 \) to the lho, as it should be, namely \( \lim_{k \to 0} L(x, \dot{x}) = L_{\text{lho}}(x, \dot{x}) = \frac{1}{2} (\dot{x}^2 - \omega^2 x^2) \).

A straightforward calculation shows that for \( L(x, \dot{x}) \) to remain a real function and that the resulting Euler–Lagrange equation is equivalent to equation (1), one must impose the following constraint on the phase variables,
\[ 1 + \frac{2k}{3\omega^2} + \frac{k^2 x^2}{9\omega^2} > 0. \]

The conjugate momentum \( p \) resulting from \( L(x, \dot{x}) \) is given by
\[ p = \frac{\partial L(x, \dot{x})}{\partial \dot{x}} = 3\omega^2 \left[ \frac{k}{2} \left( 1 + \frac{2k}{3\omega^2} \dot{x} + \frac{k^2 x^2}{9\omega^2} \right)^{-1/2} \right], \]
which, by virtue of equation (8), must satisfy the constraint
\[ p \leq \frac{3\omega^2}{k}. \]

Thus, the classical Hamiltonian, \( H(x, p) = px - L(x, \dot{x}) \), associated with equation (1), can be written as a function of the canonical variables \( x \) and \( p \) as
\[ H(x, p) = \frac{1}{2\left( 1 - \frac{k}{3\omega^2} p \right)} p^2 + \frac{1}{2}\left( 1 - \frac{k}{3\omega^2} p \right) \omega^2 x^2, \]
which is defined for \( p \leq \frac{3\omega^2}{k} \). As it should be, it is reduced in the limit \( k \to 0 \) to the classical Hamiltonian of the lho, namely \( \lim_{k \to 0} H(x, p) = H_{\text{lho}}(x, p) = \frac{1}{2} (p^2 + \omega^2 x^2) \).

3. Hamiltonian quantization according to von Roos recipe

The Hamiltonian (11) is of nonstandard-type. However, it may be written in a suitable form similar to that of a standard position-dependent mass Hamiltonian:
\[ H(x, p) = \frac{x^2}{2m(p)} + U(p), \]
where
\[ m(p) = \frac{1}{\omega^2 \left( 1 - \frac{k}{3\omega^2} p \right)}, \]
and
\[ U(p) = \frac{p^2}{2 \left( 1 - \frac{k}{3\omega^2} p \right)}. \]

Hence, by interchanging the roles of the canonical variables \( x \) and \( p \), \( U(p) \) may be seen as the potential energy and \( \frac{\dot{x}^2}{2m(p)} \) as the kinetic-like part with a \( p \)-dependent mass function \( m(p) \). So, \( H(x, p) \) looks like a Hamiltonian with position-dependent mass, defined on the interval \( p \leq \frac{3\omega^2}{k} \).

In order to quantize the Hamiltonian (11) and obtain a Hermitian operator, we first make use of von Roos recipe [10] for the kinetic-like part and write it in the following
symmetrical form

\[
\left\{ \frac{x^2}{2m(p)} \right\} = \frac{1}{4} [m^\alpha(p)x m^\beta(p)x m^\gamma(p) \\
+ m^\gamma(p)x m^\beta(p)x m^\alpha(p)],
\]

where \( \alpha, \beta, \) and \( \gamma, \) called ambiguity parameters, are real and satisfy the condition \( \alpha + \beta + \gamma = -1. \) Hence, considering the quantization in \( p \)-representation, \( [\hat{x}, \hat{p}] = i\hbar, \) with \( \hat{p} \equiv p \) and \( \hat{x} = i\hbar \frac{d}{dp}, \) and after some algebraic manipulations, the quantized Hamiltonian associated to the classical form (equation (11)) may be put in the form

\[
H = -\frac{\hbar^2}{2} \frac{d^2}{dp^2} + V(p),
\]

where \( V(p), \) that we call the effective potential, expresses in terms of two free ambiguity parameters \( (\alpha \) and \( \gamma \)) and also of some mass terms in the form

\[
V(p) = U(p) + \frac{\hbar^2}{2} \left[ \alpha\gamma \left( \frac{m'(p)^2}{m(p)} \right)^2 + (\alpha + \gamma) \left( \frac{m'(p)^2}{m^2(p)} \right) - \frac{m''(p)}{m^2(p)} \right]
\]

(17)

with \( (') \) denotes differentiation with respect to \( p \left( \frac{d}{dp} \right) \).

Substituting equations (13) and (14) into equations (17) and (16), one obtains

\[
V(p) = \frac{1}{2 \left( 1 - k \frac{x^2}{p} \right)} \left[ p^2 + \alpha\gamma \left( \frac{\hbar k}{3\omega} \right)^2 \right] \text{ for } p \leq \frac{3\omega^2}{k},
\]

and consequently,

\[
H = \left( \frac{\hbar\omega}{2} \right)^2 \frac{d}{dp} \left( 1 - k \frac{x^2}{p} \right) \frac{d}{dp} \\
+ \frac{1}{2 \left( 1 - k \frac{x^2}{p} \right)} \left[ p^2 + \alpha\gamma \left( \frac{\hbar k}{3\omega} \right)^2 \right] \text{ for } p \leq \frac{3\omega^2}{k}.
\]

This is our proposed quantum Hermitian Hamiltonian in momentum representation corresponding to equation (1).

4. SUSYQM approach and exact solution of the Schrödinger equation

4.1. Brief review of SUSYQM approach for PDEM Hamiltonians

In connection to the approach of SUSYQM for constant mass supersymmetric Hamiltonians [11–13], the formalism can be extended to PDEM Hamiltonians [14]. Considering the general Hermitian Hamiltonian (16), one defines two associated intertwined partner Hamiltonians, \( H_- \) and \( H_+ \), as

\[
H_+ = AA^+,
\]

and

\[
H_- = AA^-,
\]

(21)

where the adjoint operators \( A \) and \( A^+ = A' \) are defined in terms of the mass function \( m(p) \) and the superpotential \( W(p) \) as

\[
A = \frac{\hbar}{\sqrt{2}} \frac{1}{\sqrt{m(p)}} \frac{d}{dp} + W(p),
\]

\[
A^+ = -\frac{\hbar}{\sqrt{2}} \frac{d}{dp} \frac{1}{\sqrt{m(p)}} + W(p).
\]

(22)

The partner Hamiltonians \( H_\pm \) may be put in the following forms

\[
H_- = -\frac{\hbar^2}{2} \frac{d^2}{dp^2} + V_- (p),
\]

and

\[
H_+ = -\frac{\hbar^2}{2} \frac{d^2}{dp^2} + V_+ (p),
\]

where the effective partner potentials \( V_\pm (p) \) are given by

\[
V_\pm (p) = W^2 (p) - \frac{\hbar}{\sqrt{2}} \left( \frac{W(p)}{m^{3/2}(p)} \right)',
\]

and

\[
V_- (p) = W^2 (p) + \frac{\hbar}{\sqrt{2}} \left( \frac{W(p)'}{m^{3/2}(p)} + \frac{W(p)m'(p)}{2m^{3/2}(p)} \right)
- \frac{\hbar^2}{2} \left( \frac{3m'^2(p)}{4m^2(p)} - \frac{1}{2m^2(p)} \right).
\]

(26)

Denoting by \( \psi^{+}_n \) and \( \psi^{-}_n \) respectively the bound-states eigenvalues and eigenfunctions of \( H_\pm \), namely

\[
H_- \psi^{-}_n (p) = A^+ A \psi^{-}_n (p) = \epsilon^{-}_n \psi^{-}_n (p),
\]

and

\[
H_+ \psi^{+}_n (p) = AA^+ \psi^{+}_n (p) = \epsilon^{+}_n \psi^{+}_n (p),
\]

(28)

it follows that the spectra are semi-positive definite, i.e. \( \epsilon^{\pm}_n \geq 0 \). Furthermore, by setting \( \epsilon^{-}_n = 0 \), the ground-state eigenfunction \( \psi^{-}_0 (p) \) is given in terms of the superpotential \( W(p) \) by

\[
\psi^{-}_0 (p) = N_0 \exp \left( -\frac{\sqrt{2}}{\hbar} \int^p \frac{W(p')}{m(p')} dp' \right),
\]

(29)

where \( N_0 \) is the normalization constant. When the symmetry is not spontaneously broken, i.e. if \( \psi^{-}_0 (p) \) is square integrable on the domain of \( p \), the eigenvalues and the corresponding normalized eigenfunctions for all physical states of the partner Hamiltonians are linked by

\[
\epsilon^{+}_n = \epsilon^{-}_{n+1} > 0, \text{ for } n = 0, 1, 2, \cdots,
\]

(30)

and

\[
\psi^{+}_n (p) = \frac{1}{\sqrt{\epsilon^{+}_{n+1}}} A \psi^{-}_n (p), \text{ with } n = 0, 1, 2, \cdots.
\]

(31)
The partner potentials $V_{\pm}(p)$ are said to be shape-invariant potentials if they satisfy [15]
\begin{equation}
V_{\pm}(p; \{\tilde{a}_1\}) = V(p; \{\tilde{a}_2\}) + R(\{\tilde{a}_1\}),
\end{equation}
where $\{\tilde{a}_1\}$ and $\{\tilde{a}_2\}$ are two sets of real parameters related by a certain function $f(\{\tilde{a}_1\})$ and the remainder function $R(\{\tilde{a}_1\})$ is independent of $p$.

If the requirement (32) is satisfied, the full energy spectrum $\varepsilon_n$ can be deduced algebraically [12, 13, 15]:
\begin{equation}
\varepsilon_0 = 0, \quad \varepsilon_n = \sum_{i=1}^{n} R(\{\tilde{a}_i\}) \quad \text{for} \quad n = 1, 2, \ldots, \tag{33}
\end{equation}
where $\{\tilde{a}_i\} = \{a_1 \circ a_2 \circ \ldots \circ a_i(\{\tilde{a}_1\})\}$.

In addition, the normalized excited states eigenfunctions are given by the recurrence formula [16]:
\begin{equation}
\psi_n(p; \{\tilde{a}_1\}) = A^+(p; \{\tilde{a}_1\})\psi_{n-1}(p; \{\tilde{a}_2\}) \quad \text{for} \quad n = 1, 2, \ldots, \tag{34}
\end{equation}

4.2. Energy spectrum and corresponding wave functions

Our goal is to solve the time-independent Schrödinger equation associated to the derived Hamiltonian (19) for bound-states, using SUSYQM approach, where
\begin{equation}
H_0(p) = \varepsilon_n \psi_n(p). \tag{35}
\end{equation}

Then, the challenge is to find the superpotential $W(p)$ so that the Hamiltonians $H$ and $H_0$, given respectively by equations (19) and (23), are related by
\begin{equation}
H - \varepsilon_0 = H_0, \tag{36}
\end{equation}
where $\varepsilon_0$ is the ground-state energy of $H$. Otherwise, $H$ and $H_0$ share the same eigenfunctions, $\psi_n(p) = \psi_n(p)$, and the energy spectra are related by
\begin{equation}
\varepsilon_n = \varepsilon_0 + \varepsilon_n. \tag{37}
\end{equation}

Inserting equations (16), (23) and (25) into equation (36), the superpotential $W(p)$ must satisfy the following Riccati-like nonlinear differential equation
\begin{equation}
W^2(p) - \frac{\hbar}{\sqrt{2}} \left( \frac{W(p)}{\sqrt{m(p)}} \right)' = V(p) - \varepsilon_0, \tag{38}
\end{equation}
where $V(p)$ is given by equation (18). We suggest the superpotential in the form
\begin{equation}
W(p; a, b) = \frac{ap + b}{\sqrt{1 - \frac{k}{\sqrt{2}}p}}, \tag{39}
\end{equation}
depending on two real parameters $a$ and $b$, to be fixed so as to satisfy equation (38) and so that the resulting normalized ground-state eigenfunction (equation (29)):
\begin{equation}
\psi_0(p; a, b) = N_0 \left( 1 - \frac{k}{3\omega^2 p} \right)^{-\frac{1}{2}\frac{k}{\omega^2}} \exp \left( \frac{3\sqrt{2}\omega a}{\hbar k} p \right), \tag{40}
\end{equation}
is square integrable in the interval $p \leq \frac{3\omega^2}{k}$.

To satisfy the latter requirement, $\psi_0(p; a, b)$ must verify the boundary conditions $\psi_0(-\infty; a, b) = \psi_0(\frac{3\omega^2}{k}; a, b) = 0$, which require that $a > 0$ and $b > \frac{\omega^2}{T}a$. On the other hand, substituting equations (39) and (18) into equation (38), it is straightforward to deduce that the parameters $a$ and $b$ are given by
\begin{equation}
a = \frac{1}{\sqrt{2}}, \quad b = \frac{\hbar k}{3\sqrt{2}\omega}(\lambda - a), \tag{41}
\end{equation}
where we use the notations
\begin{equation}
\lambda = a^2 + \alpha \gamma \quad \text{and} \quad a = \left( \frac{9\omega^3}{\hbar k^2} \right), \tag{42}
\end{equation}
with the following constraint on the ambiguity parameters
\begin{equation}
\alpha \gamma \geq -a^2. \tag{43}
\end{equation}

Consequently, the resulting ground-state energy, $\varepsilon_0$, that we obtain from equation (38), is given by
\begin{equation}
\varepsilon_0 = \left( \frac{1}{2} + \lambda - a \right)\hbar \omega. \tag{44}
\end{equation}

4.2.1. Energy spectrum $\varepsilon_n$. Inserting equations (13) and (39) into equations (25) and (26), and after some algebra, one can show that $V_{\pm}(p; a, b)$ may be put in the compact forms
\begin{equation}
V_{\pm}(p; a, b) = \frac{(ap + b)^2}{\left( 1 - \frac{k}{3\omega^2 p} \right)^2} - \frac{\hbar^2 \omega a}{\sqrt{2}}, \tag{45}
\end{equation}
and
\begin{equation}
V_{\pm}(p; a, b) = \frac{(ap + b + \frac{\hbar k}{6\sqrt{2}\omega})^2}{\left( 1 - \frac{k}{3\omega^2 p} \right)^2} + \frac{\hbar^2 \omega a}{\sqrt{2}}. \tag{46}
\end{equation}

Thus, taking $a_1 = a_2 = a$, $b_1 = b$ and $b_2 = b_1 + \frac{\hbar k}{6\sqrt{2}\omega}$, one can be easily convinced that the shape invariance condition (32) is satisfied for the partner potentials $V_{\pm}(p; a, b)$ and that the remainder function depends only on the parameter $a$ as
\begin{equation}
R(a_1, b_1) = \sqrt{2}a_1\hbar \omega. \tag{47}
\end{equation}
Thus, combining equations (33), (37), (41) and (44) the spectrum is given by
\begin{equation}
\varepsilon_n = \left( n + \frac{1}{2} + \lambda - a \right)\hbar \omega \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{48}
\end{equation}

Note that the eigenenergies $\varepsilon_n$ differ from those of the quantum harmonic oscillator only by a constant shift $\lambda - a$ that is depending explicitly on deformation and ambiguity parameters. Obviously, this shift is not important since it can be absorbed in the definition of the quantum Hamiltonian. What is important physically is that the energy levels are equidistant with a width
\begin{equation}
\Delta \varepsilon_n = \varepsilon_{n+1} - \varepsilon_n = \hbar \omega, \tag{49}
\end{equation}
just like the nondeformed linear quantum oscillator. This of course means that neither deformation nor von Roos
symmetrisation modifies the physical character of the problem, irrespective of the choice of deformation and ambiguity parameters.

In addition, the shift is zero in case \( k \rightarrow 0 \) (no deformation) whatever the choice of the ambiguity parameters \( \alpha, \gamma \) and is also zero for \( \alpha \gamma = 0 \) whatever the choice of the deformation parameter \( k \):

\[
\lim_{k \to 0} \varepsilon_n = \varepsilon_n|_{\gamma=0} = \left( n + \frac{1}{2} \right) \hbar \omega. \tag{50}
\]

### 4.2.2. Wave functions \( \psi_n(p) \)

While the eigenfunctions \( \psi_n(p) \) can be deduced from the recurrence formula (equation (34)), we shall deduce them by direct calculation. By using the variable change \( y = 2a \left( 1 - \frac{1}{\sqrt{\hbar \omega}} \right) \) where \( 0 \leq y < \infty \), the Hamiltonian (19) may be written in a compact form as

\[
H = -\hbar \omega \left( \frac{d^2}{dy^2} + \frac{d}{dy} - \frac{\lambda^2}{y} - \frac{y}{4} + a \right), \tag{51}
\]

with \( a \) and \( \lambda \) were defined previously. Now, setting the normalized wave functions in the form

\[
\psi_n(p) = N_n y^\lambda e^{-\frac{y}{2}} \varphi_n(y), \tag{52}
\]

where \( N_n \) are the normalization constants, and making use of the energy spectrum expression (48), it follows from the Schrödinger equation (35), after some straightforward algebra, that the new functions \( \varphi_n(y) \) satisfy the differential equation

\[
y \varphi_n''(y) + (1 + 2\lambda - y) \varphi_n'(y) + n \varphi_n(y) = 0, \tag{53}
\]

which is only the differential equation of the associated Laguerre polynomials, \( L_n^{\lambda}(y) \) [17]. Consequently, the corresponding normalized wave functions are given by

\[
\psi_n(p; a, \lambda) = N_n \left( a \left( 1 - \frac{p}{\sqrt{\hbar \omega}} \right) \right)^{\lambda} \times \exp \left( -a \left( 1 - \frac{p}{\sqrt{\hbar \omega}} \right) \right) L_n^{\lambda} \left( 2a \left( 1 - \frac{p}{\sqrt{\hbar \omega}} \right) \right), \tag{54}
\]

where \( N_n \) may be straightforwardly evaluated [18].

\[
N_n = \frac{\sqrt{a}}{\sqrt{\hbar \omega}} \frac{2n!}{\Gamma(2\lambda + n + 1)}. \tag{55}
\]

The eigenfunctions (54) explicitly depend on the deformation and ambiguity parameters as expected. However, they express in terms of Laguerre polynomials, which suggests that the original problem is related to the isotonic oscillator. However, for the latter, though the energy levels are also equidistant, their width is twice that of the linear oscillator, namely [19–21]

\[
\Delta \varepsilon_n|_{\text{isotonic}} = 2\hbar \omega.
\]

In fact, the isotonic character of the eigenfunctions comes from the representation used to express them. Since for \( k = 0 \) the momentum \( p \) varies on a half axis \( \left( p \leq \frac{3\sqrt{\pi}}{k} \right) \) we can not expect the corresponding corresponding eigenfunctions to be expressed in terms of Hermite polynomials, which are defined on the whole axis. It turns out that in momentum representation, the eigenfunctions are expressed in terms of Laguerre polynomials which are well defined on a half-axis. However, as we shall see later, in the limit \( k \rightarrow 0 \), i.e. in the absence of deformation, the range of variation of the momentum \( p \) will extend to the entire axis and consequently the eigenfunctions are expressed in terms of Hermite polynomials.

### 4.3. Special case

We have shown that in the limit \( k \rightarrow 0 \), the eigenvalues \( \varepsilon_n \) coincide with those of the quantum harmonic oscillator, equation (50). Let us see what it is about the corresponding eigenfunctions. We have \( \lim_{k \to 0} \lambda \approx a \to \infty \), such that in that limit the eigenfunctions are independent of the ambiguity parameters and read

\[
\psi_n^\lim(p) = \lim_{a \to \infty} \psi_n(p; a, \lambda) = \psi_n(p; \lambda, \alpha) = \psi_n(p; \lambda) = \psi_n(p), \tag{56}
\]

In order to evaluate \( \lim_{a \to \infty} \psi_n(p; a, \lambda) \), we have to proceed as follows:

(i) Fixing \( n \) while \( a \to \infty \), using the Gamma function functional equation (see [17]), we have by recurrence:

\[
\Gamma(2a + n + 1) \approx \Gamma(2a). \tag{57}
\]

In addition, \( \Gamma(2a) \) may be expressed in the asymptotic region as [17]:

\[
\ln \Gamma(2a) = \left( 2a - \frac{1}{2} \right) \ln(2a) - 2a + \frac{1}{2} \ln 2\pi + O(\sim(2a)^{-2i+1}) \text{ for } i = 0, 1, 2, \cdots, \tag{58}
\]

where \( O(\sim(2a)^{-2i+1}) \) are correction terms that tend to zero as \( a \to \infty \). Hence, equation (57) leads to

\[
\Gamma(2a + n + 1) \approx \frac{2n!}{n!} \Gamma(2a + n + 1) \Gamma(2a) \approx \frac{2n!}{n!} \Gamma(2a)^{n+1} \tag{59}
\]

(ii) Next, the second term in equation (56) may be written in a more compact form as

\[
\left( 2a \left( 1 - \frac{p}{\sqrt{\hbar \omega}} \right) \right)^{\lambda} \exp \left( -a \left( 1 - \frac{p}{\sqrt{\hbar \omega}} \right) \right) \tag{60}
\]

\[
= (2a)^n e^{-\frac{p}{\sqrt{\hbar \omega}}} \exp \left( -\frac{p}{\sqrt{\hbar \omega}} \right)^{\lambda}. \tag{60}
\]
Also, for $a \to \infty$, one can write
\[
1 - \frac{p}{\sqrt{a/\hbar \omega}} = \exp\left(\ln\left(1 - \frac{p}{\sqrt{a/\hbar \omega}}\right)\right).
\]
\[
\approx \exp\left(-\frac{p}{\sqrt{a/\hbar \omega}} - \frac{p^2}{2a/\hbar \omega}\right).
\]
\[
1 - \frac{p}{\sqrt{a/\hbar \omega}} \approx \exp\left(-\frac{p^2}{2a/\hbar \omega}\right),
\]
such that
\[
\frac{1}{\exp(-\frac{p}{\sqrt{a/\hbar \omega}})} \approx \exp\left(-\frac{p^2}{2a/\hbar \omega}\right),
\]
and consequently
\[
\left(2a\left(1 - \frac{p}{\sqrt{a/\hbar \omega}}\right)\right)^{a} \exp\left(-a\left(1 - \frac{p}{\sqrt{a/\hbar \omega}}\right)\right) \approx (2a)^a e^{-a} \exp\left(-\frac{p^2}{2a/\hbar \omega}\right).
\]

(iii) Finally, an asymptotic calculation [18] shows that the Laguerre polynomials $T_n^{2a} \left(2a - 2\sqrt{a} \left(\frac{p}{\hbar \omega}\right)\right)$ may be reduced to the Hermite polynomials $H_n\left(\frac{p}{\sqrt{\hbar \omega}}\right)$ via:
\[
\lim_{a \to \infty} (2\sqrt{a})^n T_n^{2a} \left(2a - 2\sqrt{a} \left(\frac{p}{\hbar \omega}\right)\right) = \frac{1}{2^n n!} H_n\left(\frac{p}{\sqrt{\hbar \omega}}\right),
\]
where now, according to equation (10), $p \in [-\infty, \infty]$.

Hence, combining equations (59), (63) and (64), we deduce that $\psi_n^{\text{bo}}(p)$ in equation (56) is given by
\[
\psi_n^{\text{bo}}(p) = \frac{1}{\sqrt{2^n n! \sqrt{\pi} \hbar \omega}} \exp\left(-\frac{p^2}{2\hbar \omega}\right) H_n\left(\frac{p}{\sqrt{\hbar \omega}}\right),
\]
which is, obviously, the eigenfunction of the quantum harmonic oscillator corresponding to the eigenenergy (50).

5. Conclusion

In this paper, we have investigated the bound-states solutions of the quantum version of the Liénard-type nonlinear oscillator. We have first used JLM method in order to obtain a suitable Lagrangian for the problem. It turned out then that the corresponding Hamiltonian is not of the standard type since it requires the inversion of the roles of the canonical variables to put it in a form typical to the position-dependent mass Hamiltonians. Thus, its quantization required the use of von Roos recipe to write it in Hermitian form, depending on two ambiguity parameters.

To solve the corresponding Schrödinger equation, we have used the SUSYQM approach which allowed us to deduce the spectrum in an elegant algebraic way. The corresponding eigenfunctions are obtained directly using a direct resolution method based on an appropriate point transformation. It turned out then that even if the spectrum depends explicitly on the ambiguity parameters, the latter do not modify the physics of the problem, which remains equivalent to the quantum harmonic oscillator.

What follows from the investigation of this problem is that from the energetic point of view, Liénard’s nonlinear oscillator remains equivalent to the ordinary quantum harmonic oscillator since they share relatively the same spectrum. However, its eigenfunctions in momentum representation mimic those of the isotonic quantum oscillator.

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