Automorphism Group of Marked Interval Graphs in FPT

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Abstract

An interval graph is the intersection graph of a set of intervals on the real line. By a classical result of [Colbourn and Booth], the automorphism group of an interval graph can be computed in linear time using so-called PQ-trees. We consider the following generalization; some cliques of our interval graph are marked with colors, and we restrict to only such automorphisms that map every marked clique to a marked clique of the same color. We call such automorphisms marking preserving, and we are in fact interested in the action of their group on the marked sets. While this problem is at least as hard as general graph isomorphism (i.e., GI-hard), we give an FPT-time algorithm with respect to the parameter equal to the largest number of marked sets incomparable by inclusion. This result is useful, e.g., for the isomorphism problem of chordal graphs of bounded leafage.

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Introduction

The graph isomorphism problem is to determine whether the two given graphs are isomorphic, denoted by $G \cong H$; i.e., to decide whether there exists a bijection between the vertex sets $V(G)$ and $V(H)$ such that it preserves the edges. All isomorphisms of a graph $G$ onto itself form a permutation group called the automorphism group of $G$.

Graph isomorphism is in a sense a quite special problem in computer science; on one hand, under some widely-believed complexity-theoretic assumptions, it can be shown that graph isomorphism is not an NP-hard problem, while on the other hand, a polynomial-time algorithm for graph isomorphism is still elusive (and not everybody expects existence of such algorithm). It has actually defined its own complexity class GI of the problems which are reducible in polynomial time to graph isomorphism. The current state of the art is a quasi-polynomial algorithm of Babai [5]. Nevertheless, the problem has been shown to be solvable efficiently for various natural graph classes such as trees, planar and interval graphs [1,6,9].

A graph $G$ is an interval graph if the vertex set of $G$ can be mapped into some set of intervals on the real line such that two vertices of $G$ are adjacent if and only if the corresponding intervals intersect. The recognition and the isomorphism problems, as well as computing the automorphism group, for interval graphs can be solved in linear time [5,7].

While studying the isomorphism problem for a certain generalization of interval graphs – for so-called T-graphs [3,2], we have identified the following core subproblem which generalizes computation of the automorphism group of an interval graph (and we think it deserves treatment on its own).

Definition 1 (Action of the marking-preserving automorphism group of an interval graph). The problem AUTOMMARKEDINT$(G; A^1, \ldots, A^m)$ is defined as follows:
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Figure 1 (a) An interval graph (horizontally) depicted with black intervals, and the families \( A^1, \ldots, A^7 \) of marked sets (subsets of black intervals) which are used to “mark” those intervals which “branch” from the base interval line at depicted branching points. The marked sets belonging to a particular family \( A^i \) are depicted by the colored horizontal strips covering the black intervals belonging to those sets (e.g., the family \( A^1 \) consists of three distinct sets, among which one occurs with multiplicity three), and each one forms a clique.

Input An interval graph \( G \), and families \( A^1, \ldots, A^m \) of nonempty subsets of \( V(G) \) such that every set \( A \in A^i \) where \( i \in \{1, \ldots, m\} \) induces a clique of \( G \). The sets \( A \in A^1 \cup \ldots \cup A^m \) are called the marked sets of \( G \), and specially, \( A \in A^i \) is called a marked set of color \( i \).

Task Compute the group \( \Gamma \) of such permutations of \( A := A^1 \cup \ldots \cup A^m \) that are the actions of the \( (A^1, \ldots, A^m) \)-preserving automorphisms of \( G \). In other words, a permutation \( \tau \) of \( A \) belongs to \( \Gamma \), if and only if there exists an automorphism \( \varrho \) of \( G \) such that, for every \( i \in \{1, \ldots, m\} \) and all \( A \in A^i \), we have \( \tau(A) \in A^i \) and \( \tau(A) = \varrho(A) \).

To stay on the more general side, we consider set families in the multiset setting, meaning that the same set \( A \) may occur in a family \( A^i \) multiple times. In regard of the above stated condition, a permutation \( \tau \) of \( A \) then obviously has to map \( A \in A^i \) to a set \( \tau(A) = A' \in A \) such that the multiplicities of \( A \) and of \( A' \) in each of \( A^1, \ldots, A^m \) are the same.

Note that the problem in Definition 1 is generally GI-hard since \( G \) may be chosen as a clique and \( A = A^1 \) as the edge set of an arbitrary graph \( H \) (where \( V(H) = V(G) \)); then, a solution is equivalent to computing the automorphism group of \( H \). In our approach, we focus on such instances in which the maximum number of inclusion-incomparable sets in \( A \), i.e., the antichain size of \( A \), is bounded.

To informally explain the importance of the problem in Definition 1, let us briefly mention that \( T \)-graphs are intersection graphs of “intervals” which are connected subsets of a topological set homeomorphic to the fixed tree \( T \) (hence classical interval graphs are \( K_2 \)-graphs). When approaching the isomorphism problem of \( T \)-graphs via the tools for interval graphs, we additionally want to capture the subsets (actually subcliques) of “intervals” which “branch away” in a particular branching point of a \( T \)-representation. This is informally illustrated in Figure 1 and we refer to [3] for full details.

Regarding computational complexity, it is important to mention what is the input size of an AUTOMARKEDINT\((G; A^1, \ldots, A^m)\) instance. If \( G \) is an \( n \)-vertex graph, the number of sets in \( A \) may be up to exponential in \( n \). However, considering our parameter \( a \) equal to the maximum antichain size of \( A \), we easily get that there are at most \( an \) distinct sets in \( A \) (at

\footnote{This more general multiset view will actually make the algorithmic solution described in Section 4 slightly easier to formulate, and it is as well more useful for the original \( T \)-graph isomorphism problem.}
Figure 2 (a) An interval representation of a graph $G$, and (b) its unique (up to equivalence transformations) PQ-tree. P-nodes are triangle-shaped and Q-nodes are rectangle-shaped. Note also the three levels of gray used to depict the intervals in (a) that determine to which level of nodes in (b) the intervals are assigned as inner vertices (cf. Section 2).

most $a$ of each cardinality between 1 and $n$). Hence we can always upper-bound the input size by $|V(G)| + |E(G)| + \sum_{A \in A} |A| \in \mathcal{O}(a \cdot |V(G)|^2)$.

Our main result (as used further in [3]) is the following one:

▶ Theorem 2. The problem $\text{AUTOMMARKEDINT}(G; A_1, \ldots, A_m)$ is solvable in FPT-time with respect to the parameter $a$ which is the maximum antichain size of $A := A_1 \cup \ldots \cup A_m$.

In the course of proving Theorem 2 we use classical PQ-trees for capturing the internal structure of interval graphs [6], and a colored extension of the algorithm for computing the automorphism group of a set family of bounded antichain size from [2].

▶ Remark 3. Note that our algorithm solving Theorem 2 does not compute the $(A_1, \ldots, A_m)$-preserving automorphism group of $G$, not even implicitly; there is a reason for that. However, the proof in Section 4 can be extended to compute also the latter in FPT time with respect to $a$.

2 Interval Graphs and PQ-trees

A clique in a graph is a set of its vertices which are pairwise adjacent. A clique is maximal if it can not be extended by adding another vertex. Interval graphs have linearly many maximal cliques which can easily be listed in linear time using the simplicial-vertex elimination procedure (a simplicial vertex is one whose neighbors induce a clique).

To recognize and test the isomorphism of interval graphs, Booth and Lueker [6] invented PQ-trees (see Figure 2), ordered rooted trees which have the maximal cliques of an interval graph in its leaves, and every internal node is either one of the following:

- A P-node: the order of its children can be permuted arbitrarily.
- A Q-node: the order of its children can be reversed (but not changed otherwise).

The above permissible reorderings at P- and Q-nodes are called equivalence transformations of a PQ-tree, and the following is well-known:

▶ Theorem 4 (Booth and Lueker [6]). For every interval graph $G$ one can in linear time construct a PQ-tree $T$, such that the following hold:

- This PQ-tree $T$ of $G$ is unique up to equivalence transformations.
- Every possible interval representation of $G$ corresponds to a PQ-tree $T'$ of $G$ (i.e., one equivalent to $T$). The correspondence is that the linear order of the maximal cliques in the representation of $G$ is the same as the one given by the linear order of the leaves of $T'$.

In particular, the latter point means that every automorphism of $G$ can be represented as an equivalence transformation of a PQ-tree of $G$ (though, not the other way round without
further information associated with the tree). One may go further this way. We say that an assignment of PQ-trees to interval graphs is canonical if, whenever we take isomorphic graphs $G \simeq G'$ and their canonical PQ-trees $T$ and $T'$, then $T$ and $T'$ are isomorphic respecting the order of the trees (one may say “the same”). The following fact is crucial for us:

> **Corollary 5** (Colbourn and Booth [7], noted already in [6]). For every interval graph $G$, one can in linear time compute the automorphism group of $G$ and a PQ-tree of $G$ which is canonical.

While the definition of a PQ-tree (of an interval graph) explicitly refers only to the maximal cliques of $G$ and not directly to its vertices, but it will be useful to clearly understand the relation of PQ-tree nodes to the particular vertices of $G$. Every node $p$ of a PQ-tree $T$ of $G$ can be associated with a subgraph of $G$ formed by the union of all cliques of the descendant leaves of $p$ – this subgraph is said to belong to $p$. Then for a node $p$ we define the inner vertices assigned to $p$ as those vertices of $G$ which belong to $p$ and, if $p$ is not a leaf, they belong to at least two child nodes of $p$, but they do not belong to any node which is not an ancestor or a descendant of $p$. (See the illustration in Figure 2, and further in Figure 3.)

Note that every vertex of $G$ is an inner vertex of precisely one node of its PQ-tree $T$. Moreover, by the ‘consecutive-ones’ property of a PQ-tree, the following holds: if a vertex $v$ of $G$ is an inner vertex of a node $p$ of $T$, and $v$ also belongs to a son $p_1$ of $p$, then $v$ belongs to every descendant of $p_1$ in $T$. This illustrates the unique nature of the node $p$ (to which $v$ is an inner vertex) for the vertex $v$ within the PQ-tree $T$.

In the case of a P-node, the inner vertices assigned to $p$ belong to all child nodes of $p$, but this is generally not true for Q-nodes. We thus additionally define the ranking of inner vertices; this is trivial for P-nodes (all inner vertices of the same rank). For a Q-node $q$ of $T$, we index the sons of $q$ from left to right in a palindromic way (i.e., as 1, 2, 3, 2, 1 or 1, 2, 1, 2, 3, 2, 1 depending on parity), which is invariant upon reversal. The rank of every inner vertex $w$ assigned to $q$ is then the multiset of indices of the sons of $q$ that $w$ belongs to (obviously, this must be a consecutive section of the index sequence). Observe that since two inner vertices of the same node and of the same rank are in the same collection of maximal cliques of $G$, they are mutually symmetric in the automorphism group of $G$.

### 3 Automorphisms of Set Families of Bounded Antichain Size

In this section we give the main technical tool of this paper – a procedure efficiently computing the automorphism group of a set family under the assumption of bounded antichain size, which builds on ideas used already in our past paper [2]. Here we formulate those ideas in an extended form as the standalone result in Theorem [8]. Again, to stay on the more general side, we consider set families in the multiset setting, meaning that the same set may occur in a family multiple times (but this does not pose any additional difficulties in the coming arguments besides having to observe the multiplicity as a label on a set).

Note that the problem of computing the automorphism group of a colored set family is actually a special case of the problem AUTOMMARKEDINT($G; A^1, \ldots, A^m$) where $G$ is a clique, but, at the same time, we are going to prove in the next Section 4 that the AUTOMMARKEDINT reduces to the problem solved here.

> **Definition 6** (Automorphism group of a colored set family).

The problem AUTOMSET($X; U^1, \ldots, U^m$) is defined as follows:

**Input** For a finite ground set $X$, a finite set family $U \subseteq 2^X$ partitioned into $m \geq 1$ color classes $U = U^1 \cup \ldots \cup U^m$ (allowing the same set to occur in a family multiple times).
Task Compute the group $\Gamma$ of the color-preserving permutations of $\mathcal{U}$ which come from a permutation of the ground set $X$. Precisely, a permutation $\sigma$ of $\mathcal{U}$ belongs to $\Gamma$ if and only if there exists a permutation $\tau$ of $X$ such that, for every $i \in \{1, \ldots, m\}$ and all $B \in \mathcal{U}^i$, we have $\tau(B) \in \mathcal{U}^i$ and $\tau(B) = \sigma(B)$ (this trivially gives $|B| = |\tau(B)|$).

Note that the latter condition also immediately implies that the (possible) multiplicities of $\mathcal{B}$ and of $\sigma(B)$ in each family $\mathcal{U}^i$ are equal.

The problem $\text{AUTOMSIMPLESET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ is the same as $\text{AUTOMSET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ with the following condition on the input: For every set $B \in \mathcal{U}$, there is exactly one index $i \in \{1, \ldots, m\}$ such that $B \in \mathcal{U}^i$, and the multiplicity of $B$ in $\mathcal{U}^i$ equals one.

We again, as with Theorem 2 above, estimate the input size here by the same simple argument. If the maximum antichain size in the family $\mathcal{U} = \mathcal{U}^1 \cup \ldots \cup \mathcal{U}^m \subseteq 2^X$ equals $a$, then the input size of an instance of $\text{AUTOMSET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ is at most $\sum_{B \in \mathcal{U}} |B| \in O(a \cdot |X|^2)$. (Possible multiplicities of sets in the family $\mathcal{U}$ are negligible in this regard since they are encoded as integer labels of the multiple sets.)

We start with a simple observation that will simplify the next theorem:

► Proposition 7. The problem $\text{AUTOMSET}(X; \mathcal{V}^1, \ldots, \mathcal{V}^n)$ reduces in linear time to the problem $\text{AUTOMSIMPLESET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ for suitable $\mathcal{U}^1, \ldots, \mathcal{U}^m$.

Proof. For every set $B \in \mathcal{V} = \mathcal{V}^1 \cup \ldots \cup \mathcal{V}^n$, we record the integer vector $m_B := (b_1, \ldots, b_n)$ where $b_i$ is the multiplicity of $B$ in the family $\mathcal{V}^i$. As noted already in Definition 6 any permutation $\tau$ in the solution of $\text{AUTOMSET}(X; \mathcal{V}^1, \ldots, \mathcal{V}^n)$ must preserve this multiplicity vector; $m_B = m_{\tau(B)}$. Let $\mathcal{U}$ be the simplification of the (multi)family $\mathcal{V}$, i.e., without repetition of multiple member sets. We hence define a partition $(\mathcal{U}^1, \ldots, \mathcal{U}^m)$ as the partition of $\mathcal{U}$ given by the equality of the vectors $m_B$. Then $\tau$ projects to a permutation in $\text{AUTOMSIMPLESET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$.

Conversely, for any permutation $\sigma$ in the solution of $\text{AUTOMSIMPLESET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$, we expand $\sigma$ to permutations $\tau$ of $\mathcal{V}$ as follows. Every functional assignment $B \mapsto \sigma(B)$ is lifted, for each $i \in \{1, \ldots, m\}$, to all bijections of the set (possibly empty) of multiple copies of $B$ in $\mathcal{V}^i$ to the set of multiple copies of $\sigma(B)$ in $\mathcal{V}^i$. Then every such $\tau$ belongs to the solution group of $\text{AUTOMSET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$.

► Theorem 8. The problem $\text{AUTOMSET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$ is solvable in FPT-time with respect to the parameter $a$ which is the maximum antichain size of $\mathcal{U}$.

► Remark 9. We do not explicitly state the runtime in Theorem 8 partly since it is not really useful, and partly since neither [8] which we use states explicit runtime in the paper.

The rest of the section is devoted to the proof of Theorem 8 which, in view of Proposition 7, is sufficient to prove for the problem $\text{AUTOMSIMPLESET}(X; \mathcal{U}^1, \ldots, \mathcal{U}^m)$.

Cardinality Venn diagrams. First of all, we review what Definition 4 specifically the words “there exists a permutation $\sigma$ of $X$”, mean for us. Since it is not much efficient to deal with the many permutations of $X$, we now show a simple fact that it will be enough to observe certain cardinalities to decide the existence of such permutation $\sigma$ as required in the definition. For a set family $\mathcal{U}$, we call a cardinality Venn diagram of $\mathcal{U}$ the integer vector $(t_{\mathcal{U},i} : \emptyset \neq i \subseteq \mathcal{U})$ such that $t_{\mathcal{U},i} := |L_{\mathcal{U},i}|$ where $L_{\mathcal{U},i} = \bigcap_{B \in \mathcal{U}^i} A \setminus \bigcup_{B \in \mathcal{U}^i \setminus B} B$. That is, informally, we record the cardinality of every internal cell of the Venn diagram of $\mathcal{U}$. See an illustration in Figure 3.

Consider now a permutation $\varphi$ of $\mathcal{U}$ as above. For $\mathcal{U}_i \subseteq \mathcal{U}$, let naturally $\varphi(\mathcal{U}_i) = \{\varphi(B) : B \in \mathcal{U}_i\}$. We have easily got:
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Lemma 10. Let \( g \) be a permutation of \( U \) over \( X \). There exists a permutation \( \sigma \) of \( X \) such that, for every \( B \in U \), we have \( g(B) = \sigma(B) \), if and only if the cardinality Venn diagrams of \( U \) and of \( \sigma(U) \) are the same (equal), meaning that \( \ell_{U, U_1} = \ell_{U, \sigma(U_1)} \) for all \( \emptyset \neq U_1 \subseteq U \).

Furthermore, for \( U' \subseteq U \) such that \( g(U') = U' \), one can in \( O(|X| + \sum_{U \in U'} |U|) \) time test the condition as above, i.e., whether the equalities \( \ell_{U', U_1} = \ell_{U', \sigma(U_1)} \) hold for all \( \emptyset \neq U_1 \subseteq U' \).

Proof. \( \Rightarrow \) Suppose that there exists such a permutation \( \sigma \) of \( X \). Then, for all \( \emptyset \neq U_1 \subseteq U \), every element of \( L_{U,U_1} \) is mapped by \( \sigma \) into \( L_{U,\sigma(U_1)} \), and so the claim follows since \( \sigma \) is a permutation.

\( \Leftarrow \) For any \( \emptyset \neq U_1 \subseteq U \), we have that \( |L_{U,U_1}| = |L_{U,\sigma(U_1)}| \) which implies an existence of a bijection from \( L_{U,U_1} \) to \( L_{U,\sigma(U_1)} \). Since \( L_{U,U_1} \cap L_{U,U_2} = \emptyset \) for \( U_1 \neq U_2 \), the composition of these bijections is sound and results in a permutation \( \sigma \) of \( X \). Picking any \( B \in U \), we have \( B = \bigcup \{ L_{U,U_1} : \{B\} \subseteq U_1 \subseteq U \} \), and hence \( \sigma(B) = \bigcup \{ \sigma(L_{U,U_1}) : \{B\} \subseteq U_1 \subseteq U \} = \bigcup \{ L_{U,\sigma(U_1)} : \{B\} \subseteq U_1 \subseteq U \} = g(B) \).

As for testing the condition \( \ell_{U',U_1} = \ell_{U',\sigma(U_1)} \) hold for all \( \emptyset \neq U_1 \subseteq U' \), we loop through all elements \( x \in X \), and for each \( x \) we record in \( O(n) \) time to which of the sets in \( U' \) this \( x \) belongs to. Summing the obtained records at the end precisely gives the \( O(|X|) \) nonzero values \( \ell_{U',U_1} \) over \( \emptyset \neq U_1 \subseteq U' \). We analogously compute the values \( \ell_{U',\sigma(U_1)} \) over \( \emptyset \neq U_1 \subseteq U' \), and then compare.

Our strategy for the proof of Theorem 8 is as follows.

\( \square \)

Let, for \( n = |X| \) and every \( j \in \{1, \ldots, m\} \), \( U^j = \bigcup_{c=1}^n U^j_c \) be a partition of \( U \) into subfamilies of sets of cardinality \( c \in \{1, \ldots, n\} \). We make the initial group \( \Gamma' \) (of permutations of \( U = \bigcup_{c=1}^n U^j \)) as the direct product of the symmetric groups (those of all permutations) on all nonempty families \( U^j_c \) over \( j \in \{1, \ldots, m\} \) and \( c \in \{1, \ldots, n\} \).

We compute the subgroup \( \Gamma \subseteq \Gamma' \) of those permutations which fulfill the condition \( \ell_{U,\sigma(U_1)} = \ell_{U,\sigma(U_1)} \) for all \( \emptyset \neq U_1 \subseteq U' \) of Lemma 10. Then \( \Gamma \) will be, by Lemma 10, the solution to the AUTOMSIMPLESET \((X; U^1, \ldots, U^m)\) problem. At this point it will be quite important that each of the subfamilies \( U^j_c \) (as an antichain) has bounded cardinality.
The latter point, however, is not an easy task, and we will employ classical Babai’s tower-of-groups procedure to “gradually refine” \( \Gamma' \) into \( \Gamma \), ensuring that the conditions of Lemma 10 hold, in a sense, for more and more combinations of the subfamilies \( \mathcal{U}' \) until all are satisfied.

Before getting into the group-computing tools, we introduce one more technical result which will be crucial in the gradual refinement of \( \Gamma' \) into \( \Gamma \). In the setting of Lemma 10 we say that \( \mathcal{U}' \subseteq \mathcal{U} \) is Venn-good with \( \varphi \) if \( \ell_{\mathcal{U}'} : \mathcal{U} \to \mathbb{R} \) holds true for all \( \emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}' \), and we call such \( \mathcal{U}_1 \) a witness (of \( \mathcal{U}' \) not being Venn-good) if \( \ell_{\mathcal{U}'} : \emptyset \neq \mathcal{U}_1 \subseteq \mathcal{U}' \).

Lemma 11. Let \( \varphi \) be a permutation of a set family \( \mathcal{U} \), and \( \mathcal{U}' \subseteq \mathcal{U} \) be such that \( \varphi(\mathcal{U}') = \mathcal{U}' \).

If \( \mathcal{U}' \) is not Venn-good with \( \varphi \), then there exist \( \mathcal{U}_2, \mathcal{U}_3 \subseteq \mathcal{U}' \) such that \( |\mathcal{U}_2| \leq 2 \) or \( \mathcal{U}_2 \) is an antichain in the inclusion, \( \emptyset \neq \mathcal{U}_3 \subseteq \mathcal{U}_2 \) and \( \ell_{\mathcal{U}_2, \mathcal{U}_3} \neq \ell_{\varphi(\mathcal{U}_2), \varphi(\mathcal{U}_3)} \) (i.e., \( \mathcal{U}_2 \) is not Venn-good).

Proof. Choose \( \mathcal{U}_2 \subseteq \mathcal{U}' \) such that \( \mathcal{U}_2 \) is not Venn-good with \( \varphi \) and it is minimal such by inclusion, and assume (for a contradiction) that there are \( A_1, A_2 \in \mathcal{U}_2 \) such that \( A_1 \subseteq A_2 \). If \( \varphi(A_1) \not\subseteq \varphi(A_2) \), then already \( \mathcal{U}_2 := \{ A_1, A_2 \} \) is not Venn-good (with a witness \( \{ A_1 \} \)), and so let \( \varphi(A_1) \subseteq \varphi(A_2) \). Let \( \mathcal{U}_3 \) be a witness of \( \mathcal{U}_2 \) not being Venn-good, and for \( j = 2, 3 \) denote:

\[ V_j^0 := \mathcal{U}_j \setminus \{ A_1, A_2 \}, \quad V_j^1 := \{ (U_1 \cup \{ A_1 \}) \setminus \{ A_1 \}, V_j^2 := \mathcal{U}_j \setminus \{ A_1 \} \}, \quad V_j^3 := \mathcal{U}_j \setminus \{ A_1, A_2 \}. \]

Since \( A_1 \subseteq A_2 \) and \( \varphi(A_1) \subseteq \varphi(A_2) \) (and so \( \ell_{\mathcal{U}_2, \mathcal{U}_3} = \emptyset \)), we easily derive

\[ \ell_{\mathcal{U}_2, V_3^3} = 0 = \ell_{\varphi(\mathcal{U}_2), \varphi(V_3^3)} \]

and, since by our minimality assumption, all three subfamilies \( V_j^0 \), \( V_j^1 \) and \( V_j^2 \) are Venn-good,

\[ \ell_{\mathcal{U}_2, V_j^2} = \ell_{\mathcal{U}_2 \setminus \{ A_1 \}, V_j^2} = \ell_{\mathcal{U}_2 \setminus \{ A_2 \}, V_j^2} = \ell_{\varphi(\mathcal{U}_2), \varphi(V_j^2)} \]

Then, using the previous equalities, and the trivial observation \( \ell_{\mathcal{U}_2, V_j^3} = \ell_{\mathcal{U}_2, \mathcal{U}_3} + \ell_{\mathcal{U}_2, V_3^3} \) with the analogous equality under \( \varphi \), we conclude

\[ \ell_{\mathcal{U}_2, V_j^3} = \ell_{\mathcal{U}_2, V_j^3} \ell_{\mathcal{U}_2, \mathcal{U}_3} + \ell_{\mathcal{U}_2, \mathcal{U}_3} = \ell_{\varphi(\mathcal{U}_2), \varphi(V_j^3)} \ell_{\varphi(\mathcal{U}_2), \varphi(\mathcal{U}_3)} + \ell_{\varphi(\mathcal{U}_2), \varphi(\mathcal{U}_3)} = \ell_{\varphi(\mathcal{U}_2), \varphi(V_j^3)} \]

However, \( \mathcal{U}_3 \subset \{ V_j^0, V_j^1, V_j^2, V_j^3 \} \), and so one of the latter four derived equalities contradicts the assumption that \( \mathcal{U}_3 \) witnessed \( \mathcal{U}_2 \) not being Venn-good with \( \varphi \).

Since we deal with set families of bounded-size antichains, Lemma 11 essentially tells us that either a family is Venn-good, or it contains a “small” subfamily not being Venn-good.

Babai’s tower-of-groups [4]. The famous classical paper of Babai [4] is not just a standalone algorithm, but more an outline of how to efficiently compute a subgroup \( \Gamma' \) of a given group \( \Gamma \) in a setting in which the conditions defining the subgroup \( \Gamma' \) can be stepwise refined in sufficiently small steps. Here, by “computing a group” we mean to output a set of its generators which is at most polynomially large by [8]. Note that this task, in general, cannot be done directly by processing all members (even though we have an efficient membership test at hand) of the group \( \Gamma' \) which can be exponentially large.

Babai’s “tower-of-groups” approach informally works as follows: we iteratively compute a chain of subgroups \( \Gamma' = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_h = \Gamma \), where each \( \Gamma_{h+1} \) consists of those members of \( \Gamma_h \) which satisfy a suitably chosen additional condition, until \( \Gamma_h = \Gamma \) satisfies all the defining conditions, and so it is the desired outcome. The important ingredient which makes this procedure work efficiently is that the ratio of orders (sizes) of consequent groups \( \Gamma_i \) and \( \Gamma_{i+1} \) in the chain is always bounded. The number \( h \) of steps should also not be too large. In this setting, each of the refinement steps can be done using another classical result:
Algorithm 1 One (i-th) step of the computation of the subgroup Γ ⊆ Γ′

Require: a set family \( \mathcal{U} \) of maximum antichain size \( a \in \mathbb{N} \), a partition \( \mathcal{W} \subseteq 2^\mathcal{U} \) of \( \mathcal{U} \) into parts of size \( \leq a \), and a group \( \Gamma_{i-1} \) (via a generator set) of permutations of \( \mathcal{U} \) which set-wise stabilizes every part in \( \mathcal{W} \).

Ensure: either a certificate that \( \mathcal{U} \) is Venn-good with every generator of \( \Gamma_{i-1} \); or a subgroup \( \Gamma_i \subseteq \Gamma_{i-1} \) (via a generator set) such that, for \( \mathcal{T} \subseteq \mathcal{U} \) which is the union of some at most \( a \) parts of \( \mathcal{W} \), \( \mathcal{T} \) is Venn-good precisely with every member of \( \Gamma_i \) (and not with members of \( \Gamma_{i-1} \setminus \Gamma_i \)).

1: Let \( \mathcal{W} = \{ \mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k \} \)
2: \( \mathcal{T} \leftarrow \emptyset \)
3: repeat for \( p := 1, 2, \ldots, a \):
4: repeat for \( q := 1, 2, \ldots, k + 1 \):
5: if \( q > k \) then return “\( \mathcal{U} \) is Venn-good with all generators of \( \Gamma_{i-1} \)”;
6: \( \mathcal{T}_1 \leftarrow (\mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_q) \cup \mathcal{T} \)
7: until \( \mathcal{T}_1 \) is not Venn-good (cf. Lemma 10) with some generator of \( \Gamma_{i-1} \);
8: \( j_p \leftarrow q \);
9: \( \mathcal{T} \leftarrow \mathcal{W}_{j_p} \cup \mathcal{W}_{j_p+1} \cup \cdots \cup \mathcal{W}_{j_p} \)
10: until \( j_p = 1 \) or \( p = a \);
11: Call the algorithm of Theorem 12 to compute the subgroup \( \Gamma_i \subseteq \Gamma_{i-1} \), such that the membership test of \( g \in \Gamma_1 \) checks whether \( \mathcal{T} \) is Venn-good with \( g \) (again Lemma 10);
12: return \( \Gamma_i \)

▶ Theorem 12 (Furst, Hopcroft and Luks [8, Cor. 1]).
Let \( \Pi \) be a permutation group given by its generators, and \( \Pi_1 \) be any subgroup of \( \Pi \) such that one can test in polynomial time whether \( \pi \in \Pi_1 \) for any \( \pi \in \Pi \) (membership test). If the ratio \( |\Pi|/|\Pi_1| \) is bounded by a function of a parameter \( d \), then a set of generators of \( \Pi_1 \) can be computed in \( FPT \)-time (with respect to \( d \)).

The collected ingredients show a clear road to computing the subgroup \( \Gamma \subseteq \Gamma' \) from the above outlined proof strategy for Theorem 8. In every intermediate step of a chain \( \Gamma' = \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_h = \Gamma \), we assume that the whole family \( \mathcal{U} \) is not Venn-good (with some generator of \( \Gamma_i \)), and we use Lemma 11 to argue that there exists a small subfamily \( \mathcal{U}_2 \) of \( \mathcal{U} \) which is not Venn-good with some generator of \( \Gamma_{i-1} \). Such \( \mathcal{U}_2 \) can be found efficiently with a little trick (note that a brute-force approach would not give an \( FPT \)-time algorithm here), and then we define and compute the next group \( \Gamma_i \) as that of permutations for which \( \mathcal{U}_2 \) is Venn-good with Theorem 12. In this way we eventually arrive at the group \( \Gamma \) such that \( \mathcal{U} \) is Venn-good with every generator of \( \Gamma \). The details follow.

Proof of Theorem 8. Recall that, for an instance of AUTOMSIMPLESET(\( X; \mathcal{U}_1, \ldots, \mathcal{U}_m \)) over an \( n \)-element set \( X \) and for every \( j \in \{1, \ldots, m\} \), we have defined a refined partition \( \mathcal{U}' = \bigcup_{j=1}^m \mathcal{U}_j \) where \( \mathcal{U}_j \) consists of the sets from \( \mathcal{U}_j \) of cardinality \( c \). Since the maximum antichain size of \( \mathcal{U} \) is \( a \), we have that each \( \mathcal{U}_j \) contains at most \( a \) distinct sets. For simplicity, we may assume \( a \geq 2 \) since the case of \( a = 1 \) is trivial. Let \( \mathcal{W} := \{ \mathcal{U}_j^k : 1 \leq j \leq m, 1 \leq k \leq n, \mathcal{U}_j^k \neq \emptyset \} \) be a system of all these nonempty families.

To solve the given instance, we start with the group \( \Gamma' = \Gamma_0 \) of permutations of \( \mathcal{U} \) which is the direct product of the symmetric groups on all parts of \( \mathcal{W} \). For \( i := 1, 2, \ldots \), we iteratively call Algorithm 1 with \( \mathcal{U} \), \( \mathcal{W} \) and \( \Gamma_{i-1} \), until the outcome (in \( h \)-th step) is that \( \mathcal{U} \) is Venn-good with all generators of \( \Gamma_h \). Note that the latter immediately gives that
\( \mathcal{U} \) is Venn-good with all permutations in \( \Gamma_h \). By the assurance of Algorithm 1, \( \Gamma = \Gamma_h \) contains all permutations for which \( \mathcal{U} \) is Venn-good and indeed is the desired solution of \textsc{AutomSimpleSet}(\( X; \mathcal{U}^1, \ldots, \mathcal{U}^m \)).

So, it remains to finish two things: analyze and prove one call to Algorithm 1, and prove that the number \( h \) of steps is finite and not “too large”. We start with the former. If \( \mathcal{U} \) is Venn-good for every generator of \( \Gamma_{i-1} \), then we find this already in the first iteration of \( p = 1 \), on line 5. Hence we may further assume that \( \mathcal{U} \) is not Venn-good. By Lemma 1 there exists a subfamily \( \mathcal{U}_2 \subseteq \mathcal{U} \) which is also not Venn-good, and \( |\mathcal{U}_2| \leq \max(a, 2) \leq a \) since \( \mathcal{U}_2 \) is an antichain. Note that \( \mathcal{U}_2 \) thus intersects at most \( a \) parts of \( \mathcal{W} \), which implies that there exist at most \( a \) parts of \( \mathcal{W} \) whose union is not Venn-good.

Let \( k \geq j'_1 > j'_2 > \cdots > j'_p \geq 1 \) be an index sequence of length \( r \leq a \) such that the subfamily \( \mathcal{W}_{j'_1} \cup \mathcal{W}_{j'_2} \cup \cdots \cup \mathcal{W}_{j'_r} \) is not Venn-good with some generator of \( \Gamma_{i-1} \), and the vector \( (j'_1, j'_2, \ldots, j'_r) \) is lexicographically minimal of these properties. Then one can straightforwardly verify that \( (j'_1, j'_2, \ldots, j'_r) \) is a prefix of (or equal to) the vector \( (j_1, j_2, \ldots, j_p) \) computed by Algorithm 1. Consequently, the collection \( \mathcal{T} \), when leaving the cycle on line 10 is not Venn-good with some generator of \( \Gamma_{i-1} \). We look at the subset \( \Gamma_i \subseteq \Gamma_{i-1} \) defined as on line 11. In particular, \( \Gamma_i \subseteq \Gamma_{i-1} \). The important point is that, as \( \Gamma_{i-1} \) set-wise stabilizes every part of \( \mathcal{W} \), \( \Gamma_i \) is closed under composition of permutations, and so it forms a subgroup as expected by the algorithm.

Next, we verify the fulfillment of the assumptions of Theorem 12. Generators of \( \Gamma_{i-1} = \Pi \) have been given to Algorithm 1. The ratio \( |\Pi|/|\Pi_i| \), where \( \Pi_i = \Gamma_i \) in our case, can be bounded as follows (despite we do not know \( \Gamma_i \) yet): by standard algebraic arguments, \(|\Gamma_{i-1}|/|\Pi_i|\) equals the number of distinct cosets of the subgroup \( \Gamma_i \) in \( \Gamma_{i-1} \). If we consider two automorphisms \( \alpha, \beta \in \Gamma_{i-1} \) which are equal when restricted to \( \mathcal{T} \) (recall that they set-wise stabilize \( \mathcal{T} \)), then the automorphism \( \alpha^{-1} \beta \) determines a permutation of \( \mathcal{U} \) which is identical on \( \mathcal{T} \) (so it is Venn-good with \( \alpha^{-1} \beta \)), and hence \( \alpha^{-1} \beta \in \Gamma_i \). The latter means that \( \alpha \) and \( \beta \) belong to the same coset of \( \Gamma_i \), and consequently, the number of distinct cosets is at most the number of distinct subpermutations on \( \mathcal{T} \) possibly induced by \( \Gamma_{i-1} \), that is at most \( (a!)^a \) (at most the symmetric group on each of at most \( a \) parts of \( \mathcal{W} \) which form \( \mathcal{T} \)). Therefore, we can finish one iteration on line 11 in \textbf{FPT}-time with respect to \( d = a \) by Theorem 12.

Lastly, we estimate the number of steps \( h \) in the refinement process \( \Gamma' = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots \supseteq \Gamma_h = \Gamma \). By Lagrange’s group theorem, \( |\Gamma_i| \) divides \( |\Gamma_{i-1}| \), and since \( \Gamma_{i-1} \neq \Gamma_i \), we have \( |\Gamma_i| \leq \frac{1}{2} |\Gamma_{i-1}| \). Hence the number of strict refinement steps in our chain of subgroups is \( h \leq \log_2 |\Gamma'| \). Since trivially \( |\Gamma'| \leq (a!)^a \) where \( n = |\mathcal{X}| \), we get \( h = O(a) \). Therefore, the overall computation of the resulting group \( \Gamma \) of the problem instance of \textsc{AutomSimpleSet}(\( X; \mathcal{U}^1, \ldots, \mathcal{U}^m \)) is finished in \textbf{FPT}-time with respect to the parameter \( a \), the maximum antichain size of the set family \( \mathcal{U} \). We remark that the only steps in our algorithm which require \textbf{FPT}-time (i.e., are possibly not of polynomial-time) are the calls to the algorithm of Theorem 12.

\section{Handling PQ-trees of marked interval graphs}

In this section, we provide the proof – an \textbf{FPT} algorithm, for Theorem 2. For this purpose we revisit the PQ-trees of Section 2 from the point of view of marked interval graphs of Definition 1. While efficient handling of PQ-trees with an arbitrary marking seems basically infeasible, we deal with the special assumption of bounded antichains which appears very helpful within PQ-trees. Informally, we can say that the marked sets under this assumptions affect only very small part of the whole PQ-tree of our marked graph \( G \).
Recall that we have got an interval graph $G$, families $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of nonempty (marked) subsets of $V(G)$ such that every set $A \in \mathcal{A}_i$ where $i \in \{1, \ldots, m\}$ induces a clique of $G$, and $\mathcal{A} := \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m$. Let the maximum antichain size in $\mathcal{A}$ be $a$. Let $T$ be a PQ-tree of $G$, and recall what are inner vertices of $G$ assigned to nodes of $T$ and their rank from Section 2. We call a node $p$ of $T$ clean if the inner vertices assigned to $p$ are disjoint from $\bigcup \mathcal{A}$ (that is, not belonging to any marked set). The subtree rooted at $p$ is then clean if $p$ and all descendants of $p$ in $T$ are clean. It can be shown that the number of “incomparable” non-clean subtrees is bounded by the antichain size, but we skip the partial details since we actually prove much more in Lemma 13. See an illustration in Figure 4.

Our answer to handling PQ-trees with marked sets is to precompute the complete isomorphism types of all clean subtrees in $T$, and then to introduce new marked sets (in addition to $\mathcal{A}$) which encode the (few) non-clean subtrees of $T$. Importantly, the latter can be done without increasing the maximum antichain size of the marked sets. Altogether, this is formulated in detail as follows:

**Lemma 13.** Assume an instance of the problem $\text{AUTOMMARKEDINT}(G; \mathcal{A}_1, \ldots, \mathcal{A}_m)$, where the maximum antichain size of $\mathcal{A} := \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_m$ equals $a$. Then there are families...
\( \mathcal{B}^1 \cup \ldots \cup \mathcal{B}^k = \mathcal{A} \) of subsets of \( V(G) \) where \((\mathcal{B}^1, \ldots, \mathcal{B}^k)\) is a partition of \( \mathcal{A} \) refining \((A^1, \ldots, A^m)\), and another family \( \mathcal{C} \subseteq 2^{V(G)} \) of nonempty subsets of vertices where \( \mathcal{C} \) is partitioned into families \((C^1, \ldots, C^\ell)\), such that the following holds. If a group \( \Gamma \) is the solution to the problem \( \text{AUTOMSET}(V(G); \mathcal{B}^1, \ldots, \mathcal{B}^k, C^1, \ldots, C^\ell) \), then \( \Gamma \) restricted to \( \mathcal{A} \) is the sought solution of \( \text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m) \). Furthermore, the maximum antichain size in \( \mathcal{A} \cup C^1 \cup \ldots \cup C^\ell \) is also \( a \), and the families \( \mathcal{B}^1, \ldots, \mathcal{B}^k \) and \( C^1, \ldots, C^\ell \) can be computed in linear time.

**Proof.** Let \( T \) be a PQ-tree of our interval graph \( G \). We first show how to refine the partition \((A^1, \ldots, A^m)\) of \( \mathcal{A} \), using an auxiliary annotation assigned to the sets of \( \mathcal{A} \). Observe that since every set \( A \in \mathcal{A} \) induces a clique in \( G \), the set \( A \) cannot intersect the inner vertices of two nodes of \( T \) which are incomparable in the tree order. This holds since \( A \) must be a subset of some maximal clique of \( G \), and so if \( A \) intersects some inner vertex assigned to a node \( q \), the max clique containing \( A \) must be among the descendant leaves of \( q \) in \( T \) (it may be useful to imagine this fact in an actual interval representation of \( G \)).

Consequently, the nodes with assigned inner vertices from \( A \in \mathcal{A} \) lie on a root-to-leaf path of \( T \), and so it is sound (and automorphism-invariant) to annotate \( A \) with the numbers of its elements which are the inner vertices assigned to nodes at each level of \( T \) from the root. For \( Q \)-nodes of \( T \), we additionally annotate \( A \) with the numbers of elements which are inner vertices of each rank at every level. Clearly, by uniqueness of a PQ-tree up to transformations, if an automorphism of \( G \) maps \( A \in \mathcal{A} \) into \( A' \in \mathcal{A} \), then the annotations of \( A \) and of \( A' \) are the same.

Then we "reduce" the PQ-tree \( T \) of \( G \) to a tree \( T' \subseteq T \) by discarding all clean subtrees (with all their nodes) from it. See Figure 4(c). Moreover, for every node \( q \) of \( T' \), we denote by \( T_q \) the subtree of \( T \) rooted at \( q \) and formed by all clean subtrees of \( q \). We compute by Corollary 5 the canonical PQ-tree \( T_q' \) of the subgraph \( G_q \) represented by \( T_q \), and store \( T_q' \) with the implied presentation of the (unlabelled) graph \( G_q \) as an annotation of the node \( q \) in \( T' \). If \( q \) is a \( Q \)-node, we additionally store in the annotation of \( q \) the appropriate positions of the non-clean subtrees of \( q \) in \( T \) within the order of the sons of \( q \) in \( T_q' \).

For \( q \in V(T') \), we now define the set \( C_q \) as the vertices of \( G \) which belong to \( q \) in \( T \), i.e., \( C_q \) equals the union of all maximal cliques of \( G \) being in the descendant leaves of \( q \). Since \( q \) belongs also to \( T' \), it is not clean, and so there is a set \( A \in \mathcal{A} \) intersecting the inner vertices assigned to \( q \). Observe that, whenever \( A \in \mathcal{A} \) contains some of the inner vertices assigned to \( q \), then \( A \subseteq C_q \). This holds, as above, since \( A \) is a subset of some maximal clique of \( G \), and this clique containing \( A \) must be among the descendant leaves of \( q \) in \( T \).

Let \( C := \{ C_q : q \in V(T') \} \). We claim that the maximum antichain size in \( \mathcal{A} \cup C \) is \( a \) (as in \( \mathcal{A} \) itself). To prove this claim, let \( \mathcal{D} \subseteq \mathcal{A} \cup C \) be an antichain such that \( |\mathcal{D} \cap C| \) is the least possible among antichains of the same size. If \( \mathcal{D} \subseteq \mathcal{A} \), we are done, and so let \( C_q \in \mathcal{D} \cap C \) for some node \( q \in V(T') \). By the previous, there exists \( A_q \in \mathcal{A} \) such that \( A_q \) intersects some of the inner vertices of \( G \) assigned to \( q \) and \( A_q \subseteq C_q \). Since \( \mathcal{D} \cap C_q \) is an antichain, no set from \( \mathcal{D} \) is contained in \( A_q \). Assume that there is some \( A \in \mathcal{D} \cap \mathcal{A} \) such that \( A \supseteq A_q \). Then \( A \) contains an inner vertex of \( q \), and hence \( A \subseteq C_q \) which contradicts \( \mathcal{D} \) being an antichain. Finally, assume that there is \( C_{q'} \in \mathcal{D} \cap C \), where \( q' \in V(T') \), such that \( C_{q'} \supseteq A_q \). Then \( q' \) is not a descendant of \( q \) since \( C_{q'} \not\subseteq C_q \) and, likewise, \( q' \) is not an ancestor of \( q \). However, \( C_{q'} \) contains an inner vertex of \( A \) assigned to \( q \) which contradicts the definition of inner vertices. Therefore, \( (\mathcal{D} \setminus \{ C_q \}) \cup \{ A_q \} \) is an antichain again. Overall, we get that indeed \( \mathcal{D} \subseteq \mathcal{A} \), and so \( |\mathcal{D}| \leq a \).

We refine the partition \((A^1, \ldots, A^m)\) of \( \mathcal{A} \) into wanted \((\mathcal{B}^1, \ldots, \mathcal{B}^k)\) (for appropriate \( k \geq m \)) according to the above annotation; for \( i = 1, \ldots, m \), two sets \( A_1, A_2 \in \mathcal{A}^i \) fall into
the same part – color, of \((B^1, \ldots, B^k)\) if and only if the annotations of \(A_1\) and of \(A_2\) in \(T\) are the same. Likewise, we partition \(C\) into subfamilies \((C^1, \ldots, C^\ell)\) (with an arbitrary number \(\ell\) of parts) as follows: two sets \(C_q\) and \(C_{q'}\) fall into the same part – color, if and only if the nodes \(q\) and \(q'\) have received in \(T'\) above the same annotation (in particular, the same canonical PQ-tree of the clean subtrees of \(q\) or \(q'\)).

We now verify the claimed properties of the defined partitions of vertex set families. Recall the permutation group \(\Gamma\) is the solution to \(\text{AUTOMSET}(V(G); B^1, \ldots, B^k, C^1, \ldots, C^\ell)\). Let a permutation group \(\Delta\) be the solution of \(\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)\).

For every permutation \(\alpha \in \Delta\) of \(\mathcal{A}\), there exists (by Definition 1) an underlying \((A^1, \ldots, A^m)\)-preserving automorphism \(\beta\) of \(G\). Then \(\beta\) induces a permutation on the maximal cliques of \(G\) which, in turn, gives an automorphism \(\beta'\) of the PQ-tree \(T\). Note that in this context, naturally, an automorphism of a PQ-tree not only preserves the underlying rooted tree, but also the order of every Q-node up to reversal. As noted above, since \(\beta\) is an automorphism, \(\alpha\) must preserve the annotations of the sets of \(\mathcal{A}\), i.e., the partition \((B^1, \ldots, B^k)\). Since clean subtrees are also preserved by \(\beta'\), the restriction of \(\beta'\) gives an annotation-preserving automorphism of the tree \(T'\) by the definition of a canonical PQ-tree. Hence \(\beta'\) induces a permutation of \(C\) respecting the partition \((C^1, \ldots, C^\ell)\), which composed together with \(\alpha\) on \(\mathcal{A}\) gives a unique permutation \(\gamma\in \Gamma\).

Conversely, let \(\gamma \in \Gamma\) be a permutation on \(\mathcal{A} \cup C\) respecting both partitions \((B^1, \ldots, B^k)\) and \((C^1, \ldots, C^\ell)\). In particular, every set \(C_q \in C^i \subseteq C\), where \(q \in V(T')\) and \(1 \leq i \leq \ell\), is mapped into \(C_r \supseteq C_q\), where \(r \in V(T')\). Henceforth, \(\gamma\) induces (as \(\gamma: q \mapsto r\)) an annotation-preserving permutation \(\beta_0\) of \(V(T')\). Since we have, for any two nodes \(p, p' \in V(T')\), that \(p\) is an ancestor of \(p'\) if and only if \(C_p \supseteq C_{p'}\), and \(\gamma\) preserves the inclusion relation, we conclude that \(\beta_0\) is an annotation-preserving automorphism of \(T'\). Since our annotation at every node of \(T'\) includes the canonical PQ-tree of the clean subtrees, and the ordering of the non-clean subtrees under Q-nodes, \(\beta_0\) extends to a permutation \(\beta_1\) of whole \(V(T')\) which hence is an automorphism of the whole PQ-tree \(T\). This gives an underlying automorphism \(\beta\) of the graph \(G\). Since, moreover, \(\gamma\) preserves the annotations of the sets in \(\mathcal{A}\), the automorphism \(\beta\) can be chosen such that it agrees with the permutation \(\gamma\) on \(\mathcal{A}\). We have got the restriction of \(\gamma\) to \(\mathcal{A}\) in the group \(\Delta\).

It remains to analyze the runtime of the described reduction, i.e., of the computation of the families \(B^1, \ldots, B^k\) and \(C^1, \ldots, C^\ell\). For that we first compute a PQ-tree \(T\) of our interval graph \(G\) in linear time by Theorem 4. From \(T\), we straightforwardly compute the annotations of the sets of \(\mathcal{A}\), as specified above. Then we easily in linear time identify all clean subtrees of \(T\), and hence get the tree \(T'\). The annotations of the nodes of \(T'\) are computed again in linear time using Corollary 5 applied to their clean subtrees, and since the considered clean subtrees in this computation are pairwise disjoint, the overall runtime is linear in the size of \(G\). Knowing the annotations, we then easily output the families \(B^1, \ldots, B^k\) and \(C^1, \ldots, C^\ell\). \(\blacksquare\)

**Proof of Theorem 2.** We first apply Lemma 13 in this way we transform an instance of \(\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)\) into an instance of \(\text{AUTOMSET}(V(G); B^1, \ldots, B^k, C^1, \ldots, C^\ell)\) of the same parameter value \(a\). Then, we use Theorem 8 to solve the latter in FPT time with respect to \(a\), and straightforwardly output the corresponding restricted solution of \(\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)\). \(\blacksquare\)

5 Conclusions

We have finished with our auxiliary problem, solving \(\text{AUTOMMARKEDINT}(G; A^1, \ldots, A^m)\) in FPT time. We hope that this self-contained exposition of its solution could be interesting.
and useful on its own, not only as a technical tool to solve the $T$-graph isomorphism problem in \cite{AGAOGLU2021}.

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