NUMERICAL AND ANALYTICAL SOLUTIONS OF A MODEL OF POPULATION DYNAMICS

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ABSTRACT. The study presents numerical and approximate analytical approximations to a model of population dynamics with unbounded mortality function. The mathematical model involves a nonlocal boundary condition. A finite difference method is implemented for the numerical solution while the homotopy analysis method (HAM) is applied to obtain the approximate series solution. The HAM solution contains an auxiliary parameter which provides a convenient way of controlling the convergence regions of series solution. Results are presented for typical test problem provided in literature. Comparison of the results of both methods show validity and efficiency of the methods.

Keywords: Population dynamics; approximate solution to system of PDEs and ODEs; homotopy analysis method; nonlocal boundary value problems.

1. Introduction. In this, we consider the following linear initial-boundary value problem introduced by Lotka - MC Kedrick [7]

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mu(x)u = 0, \quad 0 < x < x_c, \quad 0 < t < T
\]

\[
u(0, t) = \int_{0}^{x_c} \beta(x)u(x, t)dx, \quad 0 < t < T
\]

\[u(x, 0) = u_0(x), \quad 0 \leq x \leq x_c
\]

where, \(u(x, t)\) represents the age-specific density of individuals of age \(x\) at time \(t\). The demographic parameters \(\mu(x) \geq 0\) and \(\beta(x) \geq 0\) denotes the age-specific mortality and natality rates, respectively. This system describes the evolution of the age density \(u(x, t)\) of a population for the maximum age \(x \leq x_c\). The parameters \(\mu(x)\) and \(\beta(x)\) regulates the growth rates.

The numerical study of proposed model has been an interesting topic in literature. For instance the work presented in [6, 9, 10, 12, 14, 15, 17, 18] provide a good review to the reader. Milner [18] presented a standard Galerkin finite element in age by using backward Euler in time. Authors in [11] have obtained the numerical solutions by employing a discontinuous Galerkin (DG) to the model of population dynamics, which is known to be optimal DG method.

In this paper, we have solved the presented model by homotopy analysis method (HAM) and a novel finite difference method. Special attention is devoted to the treatment of non-local boundary condition. The work presents the series analytic solutions for the model (1). Appropriate analytical solutions are obtained by implementation of homotopy analysis method, originally proposed by Liao [15]. The detailed description on
The formulation of the method is presented in [6]. A remarkable number of applications are available in the literature. These areas include linear, non-linear, homogeneous and non-homogeneous systems of differential equations. These systems result from modelling physical phenomena from science and engineering. Non-linearity and implementation of physical boundary conditions imposes a challenging task for solving such models mathematically. The homotopy analysis method (HAM) is applied for this purpose. The HAM contains a certain auxiliary parameter $h$ which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. In this paper, an alternative approach based on HAM is presented to approximate the solution on nonlinear coupled system of PDEs.

The article is organized as follows: Section II presents a brief introduction to the homotopy analysis method. In section III, the series analytic solution is presented for the system. The finite difference method is formulated in section IV. Section IV comprises of the comparison of analytical and numerical solutions for some test problems. A concluding remark is given in the final section.

2. Basic Ideas of HAM. Let us consider the differential equations given as,

$$N_i[z_i(x,t)] = 0, \quad i = 1, 2, \ldots, n,$$  \hspace{1cm} (2)

where $N_i$ stands for nonlinear operators of the whole equations, $x$ and $t$ denote the independent variables and $z_i(x,t)$ are the unknown functions in variables $x$ and $t$ respectively. Adapting the generalized homotopy method, Liao [13], the zero-order deformation equations are obtained as follows,

$$(1-q)L[\Phi_i(x,t;q) - z_{i,0}(x,t)] = qh_i N_i[\Phi_i(x,t;q)],$$ \hspace{1cm} (3)

where $q \in [0,1]$ is an embedding parameter, $h_i$ are nonzero auxiliary functions, $L$ is an auxiliary linear operator, $z_{i,0}(x,t)$ are initial guesses of $z_i(x,t)$ and $\Phi_i(x,t;q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects in HAM. Obviously, when $q = 0$ and $q = 1$, both $\Phi_i(x,t;0) = z_{i,0}(x,t)$ and $\Phi_i(x,t;1) = z_i(x,t)$ hold. Hence the approximate solution approaches the exact solution as $q$ increases from 0 to 1. Taylor series expansion of $\Phi_i(x,t;q)$ with respect to $q$, is

$$\Phi_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t),$$ \hspace{1cm} (4)

where

$$z_{i,m} = \frac{i}{m!} \frac{\partial^m \Phi_i(x,t;q)}{\partial q^m} \bigg|_{q=0}$$ \hspace{1cm} (5)

The series equation [4] converges at $q = 1$ subject to suitable choices of the auxiliary linear operator, the initial guess, the auxiliary parameters and the auxiliary functions and the auxiliary functions. So proceed as follows under this assumption of suitable choice and we have

$$\Phi_i(x,t;1) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t),$$ \hspace{1cm} (6)

as one of solutions of the original nonlinear equations, as proved by Liao [2]. As $h_i = -1$, Eq [3] becomes
\[(1 - q)L[\phi_i(x, t; q) - z_i,0(x, t)] = qN_i[\phi_i(x, t; q)], \tag{7}\]

which are frequently used in the homotopy-perturbation method \[29\].

According to \[5\], the governing equations can be deduced from the zero-order deformation equations \[3\]. Define the vectors,

\[Z_{i,n} = \{z_{i,0}(x, t), z_{i,1}(x, t), \ldots, z_{i,n}(x, t)\}. \tag{8}\]

Differentiating \[3\] \(m\) times with respect to the embedding parameter \(q\) and then setting \(q = 0\) and finally dividing them by \(m!\), we have the so-called \(m\)th-order deformation equations,

\[L[z_{i,m}(x, t) - \chi_m z_{i,m}(x, t)] = h R_{i,m}(z_{i,m-1}), \tag{9}\]

Where

\[R_{i,m}(z_{i,m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N_i}[\phi_i(x, t; q)]}{\partial q^{m-1}} \bigg|_{q=0} \tag{10}\]

and

\[\chi_m = 0 \text{ for } m \leq 1 \text{ while, } \chi_m = 1 \text{ for } m > 1.\]

3. Finite Difference System. For the numerical solution of system \[1\] using finite difference method a uniform mesh is considered. The interval \([0, x]\) is discretized into uniform intervals each having length \(h\). Let \(h = \Delta x = l / M\) and \(k = \Delta k = T / N\) be the space and time discretizations and take \(x_i = ih\) and, \(t_n = nk\) as a grid point in the domain \([0, 1] \times [0, t]\). Here \(M\) and \(N\) are the total number of intervals along the space and time axis respectively. Define \(u_{i,n} = u(x_i, t_n)\) as the discrete analogue of the continuous density function \(u(x, t)\). The forward difference approximation is used for time derivative discretization and a backward difference approximation is employed for the space derivative discretization. For the treatment of nonlocal boundary condition, the trapezoidal rule has been incorporated. This yield a finite difference system which is presented in the matrix form as follows.

\[u_{i+1}^{k+1} = A^k u_i^k + b_i^k, \tag{11}\]

where

\[A^k = \begin{bmatrix} a^k & 0 & 0 & 0 & \cdots & 0 & 0 \\ a^k & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^k & 0 & 0 & 0 & \cdots & 0 \\ a^k & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \]
Is a matrix of order \((M - 1) \times (N - 1)\). The column vector \(b_i^k\) corresponds to the constant values from initial and boundary conditions. Here, \(\theta^k = -r + r\frac{B(k)}{2} \Delta t, \mu(x_i) + 1\), \(r = \frac{\Delta t}{\Delta x}\) and \(\theta_0 = u_0^0 \beta_0 / (2 - 2\beta(x_i))\) and \(\delta_i = -r + 1 - \Delta t, \mu(x_i)\). A matlab code is designed for the numerical implementation of these methods. The results are shown in Figure 1 and Figure 2.

4. Numerical Tests

4.1. Problem 1:
Consider the following differential system:

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \mu(x)u = 0, \quad t < T < 0 \quad 0 < x \leq 1
\]

\[
\mu(x) = 10e^{-100(1-x)},
\]

\[
u(0,t) = 2 - e^{-t}, \quad t < T < 0
\]

\[
u(x,0) = w_0 e^{ax}, \quad \beta(x) = 20x(1-x), \quad 0 < x \leq 1.
\]

In order to get solution of [13] by homotopy analysis method, the initial approximations are chosen as

\[
u_0(x,t) = w_0 e^{ax},
\]

while the linear operator is taken as follows

\[
L[\phi_i(x,t;q)] = \frac{\partial \phi_i(x,t;q)}{\partial t},
\]

with \(i = 1\) and the property

\[
L[c_i] = 0,
\]

where \(c_i\) are the integral constants. According to these definitions, the zeroth-order deformation equations are

\[
(1 - q)\frac{\partial \phi_i(x,t;q)}{\partial t} = \frac{\partial \phi_i(x,t;q)}{\partial t}, \quad i = 1
\]

For \(q = 0\) and \(q = 1\), we have

\[
u(x,t;0) = z_{i,0}(x,t) = u_0(x,t), \quad \phi_i(x,t;1) = \phi_i(x,t).
\]

Thus as \(q\) from 0 to 1, the solution varies from the initial guess to the exact solutions. Expanding in Taylor series with respect to \(q\), one has

\[
\phi_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{\infty} z_{i,m}(x,t)q^m,
\]

where

\[
z_{i,m} = \left. \frac{i}{m!} \frac{\partial^m \phi_i(x,t;q)}{\partial q^m} \right|_{q=0}.
\]

For suitable choices of the auxiliary linear operator, the initial guess, the auxiliary parameters and the auxiliary
functions, the series equation [19] converges at \( q = 1 \) and we have

\[
\mathbf{u}(x,t) = z_{i,0}(x,t) + \sum_{m=2}^{\infty} z_{i,m}(x,t),
\]

(21)

Which must be one of the solutions of the system. Consider a vector which is defined as

\[
Z_{i,m} = \{z_{i,0}(x,t), z_{i,1}(x,t), \ldots, z_{i,m}(x,t)\}.
\]

(22)

So the \( m \)th-order deformation equations are

\[
L[z_{i,m}(x,t) - \chi_m z_{i,m}(x,t)] = h R_{i,m}(z_{i,m-1}),
\]

(23)

With the initial conditions

\[
z_{i,m}(x,0) = 0.
\]

And

\[
R_{i,m}(z_{i,m-1}) = (z_{i,m-1}), + \mu(x,t)(z_{i,m-1}).
\]

(24)

The solution of the \( m \)th-order deformation equation [23] is given as

\[
z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) + h \int_0^t R_{i,m}(z_{i,m-1}) d\tau + c_i,
\]

(25)

Where the integration constant \( c_i \) are obtained by the initial conditions given in [13]. The series solution by HAM can be written in the form as follows

\[
\mathbf{u}(x,t) = z_{i,0}(x,t) + z_{i,1}(x,t) + z_{i,2}(x,t) + z_{i,3}(x,t) + \cdots.
\]

(26)

For \( h = -1 \), the approximated series solutions are computed as follows:

\[
\mathbf{u}(x,t) = w_0 e^{ax} + 2h [w_0 e^{ax} + w_0(1-x)e^{ax}].
\]

(27)

\[
S = h^2 [w_0(1-x)e^{ax} + w_0(1-x)e^{ax} + w_0(1-x)e^{ax} + 2 + w_0(1-x)e^{ax} + 2(1+a)].
\]

(28)

A plot is shown below for \( \mathbf{u}(x,t) \) at \( t = 0.3 \) and \( h = -1 \).

4.2. Problem 2.

To demonstrate the application of HAM for a larger class of reaction functions, a more general system is taken as follows:

\[
\frac{\partial\mathbf{u}}{\partial t} + \frac{\partial \mu(x)\mathbf{u}}{\partial x} = 0, \quad t < T < 0 \quad 0 < x \leq 1
\]

(29)

\[
\mu(x) = 1/(1-x)
\]

\[
\mathbf{u}(0,t) = (1-x)e^{-x}, \quad 0 < t < T
\]

\[
\mathbf{u}(x,0) = w_0 e^{ax}, \quad \beta(x) = 20(1-x), \quad 0 < x \leq 1.
\]

The series solution is computed by HAM presented as:

\[
\mathbf{u}(x,t) = (1-x)e^{-x} + 3h t x e^{-x} + h^2 [x e^{-x} + x t^2 e^{-x} / 2 + t^2 x e^{-x} / (2 = 2x)] + \cdots.
\]

(30)

It is important to note that the series solutions are valid with the assumption that series [30] converges for \( q = 1 \). The auxiliary parameter plays a basic role for this assumption. For different values of \( h \), the curves are plotted and checked the region of \( h \) for which solution series is convergent. For that proper values of \( h \), the solution is best approximation of the exact solution of the problem.

The plots shown in Figure 1 and Figure 2 shows the solution obtained by HAM and finite difference method.
The time is taken to be \( t = 0.3 \).

5. Conclusion. This study is based upon the approximate series solutions and numerical solutions for some problems from literature. Series solutions are obtained by the well known homotopy analysis method, while the numeric results are computed by implementation of finite difference algorithm. Both of these methods have applied first time in the literature for the presented problems. The plotted results agree with those already available. The techniques encures an efficient and alternative way for mathematical analysis of the problems from different aspects. The numerical methods gives a stable solution that is checked by stability analysis of the scheme, while approximate solutions obtained by HAM are convergent series solution which is confirmed by the appropriate convergence region of the auxiliary parameter. Future study is in progress for developing non-standard finite difference algorithms and their stability analysis.

![Figure 1. Result of Problem 1.](image1)

![Figure 2. Result of Problem 2.](image2)

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