1. INTRODUCTION

The Hilbert space of the quantum state of a composite system of coupled particles is the tensor product of the Hilbert spaces of these particles. That is to say, a quantum state of a composite system is in general a superposition of all the products of the bases states of the constituent particles. This leads to the situation that the state of a composite system may not be factorized to be a direct product of the states of the constituent particles. This so-called quantum entanglement has been a central attention in foundations of quantum mechanics ever since the early days [1–4]. Moreover, in recent years, it was found that superposition and entanglement of quantum states give rise to powerful quantum information processing. For example, a quantum computer is much more efficient than a classical computer in solving certain problems such as factoring large numbers [5–8]. It is, however, highly challenging to physically build a quantum computer, a major obstacle being decoherence, i.e. the fragility of a superposed or entangled state towards coupling with the environment, although the quantum error-correction code diminishes this difficulty [9]. Stimulated by both its power and the challenge in its physical implementations, the field of quantum information is attracting more and more attention.

Here we extend the consideration to coupled many-particle systems, each of which is composed of identical particles, but the coupled systems are distinguishable with each other. Specifically, we consider coupled Bose condensates, each of which is in a double-well potential or has two Josephson-coupled internal states (note that we refer to Josephson-coupled condensates as one condensate, since in this case, all the particles, even though from different sources originally, are overlapping identical particles [1]). Because of coupling between each particle in one condensate and each particle in the other condensate, we consider the mean-field many-body state which is a product of copies of a same wavefunction of inter-condensate pair of particles, with symmetrization. With the flexibility of manipulation, this situation may be realizable in the context of Bose-Einstein condensation of trapped atoms [10–12].

A Bose condensate is well described by the condensate wavefunction, which, in a mean field theory, is the single particle wavefunction in which the condensation occurs. It can also be defined as the spontaneous gauge symmetry breaking (SGSB) of the field operator, or through the off-diagonal long-range order (ODLRO) of the one-particle reduced density matrix. It is a superposition for a condensate in a double-well potential or with two Josephson-coupled internal states. In our consideration of the coupled condensates, the wavefunction of each inter-condensate pair of particles becomes the joint condensate wavefunction, which can entangle the condensate wavefunctions of individual condensates.

With superposition and entanglement, it may be possible to use condensate wavefunction to implement a qubit in quantum computation. The first possible advantage is the intrinsic robustness and stability. It may be viewed as a natural realization of a fault tolerance prescription based on symmetrization [13]. To have the best accuracy, each Bose system had better be non-interacting. In principle, this is possible since the interaction can be tuned in the case of atomic condensate [14]. On the other hand, it was noted that incorporating nonlinear evolution in quantum computation leads to computational power of solving NP-complete and #P problems in polynomial time [15]. This result has remained as an academic curiosity, since quantum mechanics is linear. Now, for a weakly interacting Bose system, the condensate wavefunction satisfies a nonlinear Schrödinger equation, often called Gross-Pitaevskii
equation [16]. Thus Bose condensates may make it possible to implement nonlinear quantum computation, for the sake of special power. Therefore Bose statistics may be a resource of both fault tolerance and computational power. We will illustrate the ideas in terms of Bose-Einstein condensation of trapped atoms, mainly using a condensate in a double-well potential, with the two bits represented by the condensate wavefunctions localized at the two wells. We also discuss a condensate with two Josephson-coupled internal states, which might encode the bits. But there are difficulties due to coupling with the motional degree of freedom and the nonlinearity. However, the physical problems are still interesting.

The paper is organized in the following way. Section 2 is an overview of the concept of condensate wavefunction. In Appendix A, we give a derivation of Gross-Pitaevskii equation from ODLRO. Bose-Einstein condensation in a double-well potential is a prototype in discussions from Secs. 3 to 5. As a prelude, superposition of condensate wavefunction is discussed in Subsec. 3.1. In Subsec. 3.2, we consider coupling two different condensates, and discuss possible entanglement between them. Many-body Hamiltonian and the equation of motion of the joint condensate wavefunction are given in Sec. 4. Section 5 contains illustrative schemes of one-bit and two-bit operations in terms of condensates in double-well potentials. In Sec. 6, we discuss, in parallel to Secs. 3 to 5, the case of spinor condensate, which has two components with different internal states. We suggest that coupling two spinor condensates gives rise to a four-component condensate wavefunction. Section 7 contains some additional remarks. A summary is given in Sec. 8.

2. INTRODUCING CONDENSATE WAVEFUNCTION

In the following, we review three approaches to the condensate wavefunction.

(i) In the mean field theory, which becomes exact in the absence of the interaction, the Bose-condensed state is in a Hatree form,

$$\Psi(r_1, \cdots, r_N) = \phi(r_1) \cdots \phi(r_N),$$

where $\phi(r_i)$ is the single particle state, which turns out to be the condensate wavefunction. One can define

$$\Phi(r) = \sqrt{N} \phi(r).$$

(ii) In the approach of SGSB [17–19],

$$\Phi(r) = \langle \hat{\psi}(r) \rangle,$$

where $\hat{\psi}(r)$ is the boson field operator. The particle number is not conserved. One may use (2) to define $\phi(r)$, with $N$ now being mean particle number.

(iii) The general criterion for Bose-Einstein condensation is the ODLRO of the one-particle reduced density matrix [20,21],

$$\langle r' | \hat{\rho}_1 | r \rangle = \langle \hat{\psi}(r') \hat{\psi}^\dagger(r) \rangle = Tr[\hat{\rho} \hat{\psi}(r') \hat{\psi}^\dagger(r)] = 0,$$

where $\hat{\rho}$ is the density matrix. ODLRO can be expressed as

$$\langle r' | \hat{\rho}_1 | r \rangle \to \Phi(r') \Phi^*(r) \neq 0$$

as $|r - r'| \to \infty$. This applies to both particle number conservation and non-conservation cases. In case of particle number conservation, $\Phi(r) = \sqrt{\lambda} \phi_0(r)$, where $\phi_0(r)$ is the eigenfunction of $\hat{\rho}_1$ with the largest eigenvalue $\lambda$, of the order $o(N)$. In case of particle number non-conservation, $\Phi(r) = \langle \hat{\psi}(r) \rangle$. When $N \to \infty$, there is no practical difference in using the particle number non-conserved state or the particle number conserved state.

The many-body Hamiltonian is

$$\hat{H} = \sum_{i=1}^N \hat{h}(r_i) + \sum_{i<j} U(r_i - r_j),$$

where
\[ \hat{H}(r) = -\frac{\hbar^2 \nabla^2}{2m} + V(r) \]  

is the single particle Hamiltonian, \( U(r_i - r_j) \) is the particle-particle interaction, \( m \) is the particle mass, \( V(r) \) is the external potential, e.g. the trapping potential in case of the trapped atoms. In terms of the field operator, the many-body Hamiltonian is

\[ \mathcal{H} = \int dr \hat{\psi}^\dagger(r) \hat{h}(r) \hat{\psi}(r) + \frac{1}{2} \int dr_1 dr_2 \hat{\psi}^\dagger(r_1) \hat{U}(r_1 - r_2) \hat{\psi}(r_2) \hat{\psi}(r_1), \]  

which leads to the equation of motion of \( \hat{\psi}(r) \),

\[ i\hbar \frac{\partial \hat{\psi}(r,t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V(r) \right] \hat{\psi}(r,t) \]  

With weak interaction, one can use s-wave approximation \( U(r - r') = g\delta(r - r') \), with \( g = 4\pi \hbar^2 \eta/m \), where \( \eta \) is the s-wave scattering length. Under the SGSB ansatz, one replaces \( \hat{\psi}(r,t) \) with \( \sqrt{N} \phi \) to obtain the Gross-Pitaevskii Equation for the condensate wavefunction,

\[ i\hbar \frac{\partial \phi(r,t)}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V(r) + gN|\phi(r,t)|^2 \right) \phi(r,t). \]

It was shown that the ground state energy and density given by Gross-Pitaevskii equation become exact as the particle number \( N \) tends to infinity while \( N\eta \) is fixed \([24]\), and the error is about 1% under current experimental conditions \([23]\).

Gross-Pitaevskii equation can also be obtained by using the Hatree wavefunction \([1]\) in the many-body Schrödinger equation, with the Hamiltonian \([10]\) \([\[10\]]\). It can also be obtained phenomenologically from a Ginzburg-Landau functional \([11]\). In Appendix A, we give a derivation from ODLRO.

3. SUPERPOSITION AND ENTANGLEMENT OF CONDENSATE WAVEFUNCTIONS

3.1. Superposition of condensate wavefunctions

In discussing Josephson-like effect, e.g. a condensate in a symmetric double-well potential, a widely used ansatz is that the total condensate wavefunction is a superposition of two bases wavefunctions, with time-dependent coefficients \([24,25]\). As a prelude to the next subsection, here we make some discussions based on the construction of the field operator and then imposing SGSB or ODLRO.

The basis set of the single particle wavefunctions is \( \{ \phi_{\alpha,n}(r) \} \), where \( \alpha \) denotes the energy level, \( n = 0,1 \) denotes the two wells. Because of finiteness of the barrier, \( |\int \phi^{\dagger}_{\alpha,0}(r)\phi_{\alpha,1}(r)dr|^2 = \epsilon << 1 \), i.e. the two states are nearly orthogonal though not exactly so. The field operator can be constructed as

\[ \hat{\psi}(r,t) = \sum_{n=0,1} \sum_{\alpha} \phi_{\alpha,n}(r) \hat{a}_{\alpha,n}(t), \]

where \( \hat{a}_{\alpha,n}(t) \) is the annihilation operator corresponding to the single particle state \( \phi_{\alpha,n}(r) \). Therefore

\[ \hat{\psi}(r,t) = \hat{\psi}_0(r,t) + \hat{\psi}_1(r,t), \]

where

\[ \hat{\psi}_n(r,t) = \sum_{\alpha} \phi_{\alpha,n}(r) \hat{a}_{\alpha,n}(t), \]

with \( n = 0,1 \). By making SGSB average of \([12]\), we have

\[ \Phi(r,t) = \Phi_0(r,t) + \Phi_1(r,t), \]

with \( \Phi(r,t) = \langle \hat{\psi}(r,t) \rangle \), \( \Phi_n(r,t) = \langle \hat{\psi}_n(r,t) \rangle \). Therefore, a general condensate wavefunction is a superposition of the condensate wavefunctions corresponding to the two wells \([20]\).
A justification can also be made in terms of ODLRO. With \[\Phi_0(r,t) = \Phi_0(r,t) + \Phi_1(r,t)\], the one-particle reduced density matrix is
\[
\langle r' | \hat{\rho}_1 | r \rangle = \langle r', 0 | \hat{\rho}_1 | r, 0 \rangle + \langle r', 0 | \hat{\rho}_1 | r, 1 \rangle + \langle r', 1 | \hat{\rho}_1 | r, 0 \rangle + \langle r', 1 | \hat{\rho}_1 | r, 1 \rangle,
\]
where \[\langle r', n' | \hat{\rho}_1 | r, n \rangle = \langle r', n' | \hat{\psi}_a(r', t) \hat{\psi}_b(r, t) \rangle\], with \(n = 0, 1, n' = 0, 1\). The existence of ODLRO implies
\[
\langle r', n' | \hat{\rho}_1 | r, n \rangle \rightarrow \Phi_{n'}(r', t)\Phi_n(r, t).
\]
Hence
\[
\langle r' | \hat{\rho}_1 | r \rangle \rightarrow (\Phi_0(r', t) + \Phi_1(r', t))(\Phi_0(r, t) + \Phi_1(r, t))^*,
\]
which indicates the existence of the superposed condensate wavefunction \(\Phi(r, t) = \Phi_0(r, t) + \Phi_1(r, t)\).

In a mean field theory, each particle occupies the single particle ground state, hence the condensate wavefunction is given by the superposition of the two single particle ground states at the two wells. This is called two-mode approximation in [14]. So
\[
\phi(r, t) = \frac{\Phi(r, t)}{\sqrt{N}} = c_0(t)\phi_{a_0, 0}(r) + c_1(t)\phi_{a_1, 1}(r),
\]
where \(\phi_{a_0,n}(r)\) is the single particle ground state at well \(n\). The many-body ground state is [19]
\[
\Psi(r_1, \cdots, r_N, t) = \phi(r_1, t) \cdots \phi(r_N, t).
\]
For brevity, we write [16] as
\[
\phi(r, t) = c_0(t)\phi_0(r) + c_1(t)\phi_1(r).
\]

### 3.2. Entanglement of condensate wavefunctions

Consider two coupled Bose systems \(a\) and \(b\), say, in a double-well potential (Fig. 1). For simplicity but without lose of generality, suppose each consists of \(N\) identical particles. For system \(a\), the wells are denoted as \(n^a\), while for system \(b\), the wells are denoted as \(n^b\). These two Bose systems should be non-overlapping if they are composed of a same kind of particles, but can be mixed if they are composed of two different kinds of particles.

The existence of a joint order parameter can be seen in general by considering the product of the two field operators.
\[
\hat{\psi}(r^a, t)\hat{\psi}(r^b, t) = \sum_{n^a} \sum_{n^b} \hat{\psi}_{n^a}(r^a, t)\hat{\psi}_{n^b}(r^b, t),
\]
Imposing SGSB, one has
\[
\Phi(r^a, r^b, t) = \sum_{n^a} \sum_{n^b} \Phi_{n^a, n^b}(r^a, r^b, t),
\]
where \(\Phi(r^a, r^b, t) \equiv \langle \hat{\psi}(r^a, t)\hat{\psi}(r^b, t) \rangle\), \(\Phi_{n^a, n^b}(r^a, r^b, t) \equiv \langle \hat{\psi}_{n^a}(r^a, t)\hat{\psi}_{n^b}(r^b, t) \rangle\). Because of coupling between \(a\) and \(b\) in general the many-body state cannot be factorized to be a direct product of that of \(a\) and that of \(b\), and there is interaction energy between \(a\) and \(b\). Consequently, \(\Phi(r^a, r^b, t)\) cannot be factorized to be a product of a condensate wavefunction of system \(a\) and a condensate wavefunction of system \(b\).

The justification can also be made in terms of ODLRO. One can define a one-particle-pair reduced density matrix,
\[
\langle r'^a, r'^b | \hat{\rho}_1 | r^a, r^b \rangle \equiv \langle \hat{\psi}(r'^a, t)\hat{\psi}(r'^b, t)\hat{\psi}^\dagger(r^b, t)\hat{\psi}^\dagger(r^a, t) \rangle
\]
\[
= \sum_{n^a'} \sum_{n^b'} \sum_{n^a} \sum_{n^b} \langle r'^a, n^a', r'^b, n^b' | \hat{\rho}_1 | r^a, n^a, r^b, n^b \rangle,
\]
where \(\langle r'^a, n^a', r'^b, n^b' | \hat{\rho}_1 | r^a, n^a, r^b, n^b \rangle \equiv \langle \hat{\psi}_{n^a'}(r'^a, t)\hat{\psi}_{n^b'}(r'^b, t)\hat{\psi}_{n^b}^\dagger(r^b, t)\hat{\psi}_{n^a}^\dagger(r^a, t) \rangle\). With ODLRO,
\[
\langle \hat{\psi}_{n^a'}(r'^a, t)\hat{\psi}_{n^b'}(r'^b, t)\hat{\psi}_{n^b}^\dagger(r^b, t)\hat{\psi}_{n^a}^\dagger(r^a, t) \rangle \rightarrow \Phi_{n^a', n^b'}(r'^a, r'^b, t)\Phi_{n^a, n^b}^\dagger(r^a, r^b, t).
\]
Consequently, (21) approaches
\[
\left[\sum_{n^a} \sum_{n^b} \phi_{n^a,n^b}(r_{a}^{n^a}, r_{b}^{n^b}, t)\right] \left[\sum_{n^a} \sum_{n^b} \phi_{n^a,n^b}(r_{a}^{n^a}, r_{b}^{n^b}, t)\right]^*,
\]
implies that there exists a joint condensate wavefunction \( \sum_{n^a,n^b} \Phi_{n^a,n^b}(r_{a}^{n^a}, r_{b}^{n^b}, t) \).

For convenience, we may write (22) as
\[
\phi(r_{a}^{a}, r_{b}^{b}, t) = \sum_{n^a} \sum_{n^b} \phi_{n^a,n^b}(r_{a}^{n^a}, r_{b}^{n^b}, t),
\]
with \( \phi(r_{a}^{a}, r_{b}^{b}, t) = \Phi(r_{a}^{a}, r_{b}^{b}, t)/N, \phi_{n^a,n^b}(r_{a}^{n^a}, r_{b}^{n^b}, t) = \Phi_{n^a,n^b}(r_{a}^{n^a}, r_{b}^{n^b}, t)/N. \)

Consider the following Bose-condensed state,
\[
\Psi(r_{1}^{a}, \ldots, r_{N}^{a}; r_{1}^{b}, \ldots, r_{N}^{b}; t) = \frac{1}{\sqrt{N!}} \sum_{P} \left[ \phi_{\alpha_{0}^{a},\alpha_{0}^{b}}(r_{1}^{a}, r_{P_{1}}^{b}, t) \cdots \phi_{\alpha_{0}^{a},\alpha_{0}^{b}}(r_{N}^{a}, r_{P_{N}}^{b}, t) \right],
\]
where
\[
\phi_{\alpha_{0}^{a},\alpha_{0}^{b}}(r_{a}^{a}, r_{b}^{b}, t) = \sum_{n^a} \sum_{n^b} C_{n^a,n^b}(t) \phi_{\alpha_{n^a},\alpha_{n^b}}(r_{a}^{n^a}) \phi_{\alpha_{n^a},\alpha_{n^b}}(r_{b}^{n^b}).
\]

The symmetrization in state (24) represents all possible ways of pairing. The systems Bose-condense into a coupled single particle pair state, and (23) can be used as the joint condensate wavefunction. In this state, particles in a same system occupy a same ground state, while maintain coupling with particles in the other system. It is a natural generalization of the ansatz \( \Phi \). However, it remains to be rigorously examined whether this state is indeed the many-body ground state.

Like using two-mode approximation for a single condensate, one may make two-mode approximation for each condensate, and consider the coupling between them. Then, in consistent with the above mean-field many-body state, the joint condensate wavefunction is
\[
\phi(r_{a}^{a}, r_{b}^{b}, t) \equiv \frac{\langle \hat{\psi}(r_{a}^{a}, t) \hat{\psi}(r_{b}^{b}, t) \rangle}{N^2} = \phi_{\alpha_{0}^{a},\alpha_{0}^{b}}(r_{a}^{a}, r_{b}^{b}, t)
\]
for brevity, we write
\[
\phi(r_{a}^{a}, r_{b}^{b}, t) = \sum_{n^a} \sum_{n^b} C_{n^a,n^b}(t) \phi_{\alpha_{n^a},\alpha_{n^b}}(r_{a}^{n^a}) \phi_{\alpha_{n^a},\alpha_{n^b}}(r_{b}^{n^b}).
\]

4. MANY-BODY HAMILTONIAN AND EQUATION OF MOTION

For two coupled Bose systems \( a \) and \( b \), the general form of the Hamiltonian is
\[
\hat{H} = \sum_{i=1}^{N} \hat{h}(r_{i}^{a}) + \sum_{i<j}^{N} U(r_{i}^{a} - r_{j}^{a}) + \sum_{k=1}^{N} \hat{h}(r_{k}^{b}) + \sum_{k<l}^{N} U(r_{k}^{b} - r_{l}^{b}) + \sum_{i,k} W(r_{i}^{a} - r_{k}^{b}),
\]
where \( \hat{h}(r_{i}^{a}) \) and \( \hat{h}(r_{k}^{b}) \) are single particle Hamiltonians of a particle in \( a \) and \( b \), respectively, as given in (7). \( U(r_{i}^{a} - r_{j}^{a}) \) is the particle-particle interaction within \( a \), while \( U(r_{k}^{b} - r_{l}^{b}) \) is the interaction within \( b \). \( W(r_{i}^{a} - r_{k}^{b}) \) is the interaction between a particle in \( a \) and a particle in \( b \). The field theoretic Hamiltonian can be written as
\[
\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5,
\]
with
\[
\mathcal{H}_1 = \int \! \! dr^{a} \hat{\psi}^\dagger(r^{a}) \hat{h}(r^{a}) \hat{\psi}(r^{a})
\]
(30)
\( \mathcal{H}_2 = \int dr^b \hat{\psi}^\dagger(r^b) \hat{h}(r^b) \hat{\psi}(r^b), \) \hfill (31)

\[ \mathcal{H}_3 = \frac{1}{2} \int dr^a dr^b \hat{\psi}^\dagger(r^a) \hat{r}^b \hat{\psi}(r^a, r^b) U(r^a - r^b) \hat{\psi}(r^a) \hat{\psi}(r^b) \hfill (32) \]

\[ \mathcal{H}_4 = \frac{1}{2} \int dr^a dr^b \hat{\psi}^\dagger(r^a) \hat{r}^b \hat{\psi}(r^a, r^b) U(r^a - r^b) \hat{\psi}(r^a) \hat{\psi}(r^b) \hfill (33) \]

\[ \mathcal{H}_5 = \int dr^a dr^b \hat{\psi}^\dagger(r^a) \hat{r}^b \hat{\psi}(r^a, r^b) W(r^a - r^b) \hat{\psi}(r^a) \hat{\psi}(r^b). \hfill (34) \]

From the Hamiltonian \([29]\), the equation of motion of \( \hat{\psi}(r^a) \hat{\psi}(r^b) \) is obtained as

\[ i\hbar \frac{\partial \psi(r^a, r^b, t)}{\partial t} = \left[ \hat{h}(r^a) + \hat{h}(r^b) + \int dr^{a'} \hat{\psi}^\dagger(r^{a'}, t) U(r^a - r^{a'}) \hat{\psi}(r^{a'}, t) \right. \]

\[ + \int dr^{b'} \hat{\psi}^\dagger(r^{b'}, t) U(r^b - r^{b'}) \hat{\psi}(r^{b'}, t) \]

\[ + \int dr^{a'} dr^{b'} \psi(r^{a'}, r^{b'}, t) W(r^a - r^{a'}) \psi(r^{a'}, r^{b'}, t) \]

\[ + \int dr^{a'} dr^{b'} \hat{\psi}^\dagger(r^{a'}, r^{b'}, t) W(r^a - r^{a'}) \hat{\psi}(r^{a'}, r^{b'}, t) \psi(r^a, r^b, t), \hfill (35) \]

from which, using \( \int dr^a \hat{\psi}^\dagger(r^a, t) \hat{\psi}(r^a, t) = \int dr^b \hat{\psi}^\dagger(r^b, t) \hat{\psi}(r^b, t) = N \), one obtains the equation of motion for the joint condensate wavefunction \( \phi(r^a, r^b, t) \), as a generalization of the Gross-Pitaevskii equation,

\[ i\hbar \frac{\partial \phi(r^a, r^b, t)}{\partial t} = \left[ \hat{h}(r^a) + \hat{h}(r^b) + N \int dr^{a'} \int dr^{b'} \phi(r^{a'}, r^{b'}, t)^2 U(r^a - r^{a'}) \right. \]

\[ + N \int dr^{a'} \int dr^{b'} \phi(r^{a'}, r^{b'}, t)^2 U(r^b - r^{b'}) \]

\[ + \int dr^{a'} dr^{b'} \phi(r^{a'}, r^{b'}, t)^2 W(r^a - r^{a'}) \]

\[ + \int dr^{a'} dr^{b'} \phi(r^{a'}, r^{b'}, t)^2 W(r^a - r^{b'}) \phi(r^a, r^b, t). \hfill (36) \]

This equation may also be obtained directly from the many-body Schrödinger equation, using \([24]\) and \([28]\).

5. QUANTUM COMPUTATION WITH DOUBLE-WELL CONDENSATES

5.1. Possible robustness

If a qubit is implemented as a Bose condensate, then there is possible robustness and stability due to the macroscopic occupation of a same single particle state. For example, consider an excited state where only one particle is away from the ground state, for distinguishable particles there are \( N \) possibilities, while there is only one possibility when all the particles are identical \([27]\). For a single condensate, the Bose-condensed state is a product of the same single particle state. For two coupled condensates, the Bose-condensed state is now suggested to be a product of the same single particle pair state, with symmetrization over all possible ways of pairing. In general, symmetrization always needs to be made on the many-body state. This reduces the error probability. Symmetrization has been studied as a way of reducing error in quantum computation, which was found to suppress the error probability by \( 1/N \) \([3\text{]}\]. Bose condensation can be viewed as a natural realization of this prescription. The error reduction due to symmetrization may help one to understand why the single particle state emerges out as a macroscopic wavefunction. The robustness of condensate wavefunction was demonstrated in the interference experiments \([28]\). Nevertheless, the condensate wavefunction brings the issue of phase diffusion, which may cause error.

5.2. NP-complete problems and Nonlinear Quantum Computation

The class of NP-complete problems is a foundation of the computational complexity theory. It includes thousands of practically interesting problems, such as travelling salesman, satisfiability, etc. NP stands for “non-deterministic polynomial time”. NP-complete problems are those for which a potential solution can be verified in polynomial time, yet finding a solution appears to require exponential time in the worst case. The completeness means that if an
efficient, i.e. polynomial-time, algorithm could be found for solving one of these problems, one would immediately have an efficient algorithm for all NP-complete problems. A fundamental conjecture in classical computation is that no such an efficient algorithm exists. Abrams and Lloyd found that with nonlinearity, a quantum computer can solve NP-complete problems by efficiently determining if there exists an \( x \) for which \( f(x) = 1 \), and can solve \#P problems by efficiently determining the number of solutions \([15]\). Their algorithm is based on one or two one-bit nonlinear gates, together with linear gates. However, it is an experimental fact that fundamental quantum mechanics is linear to the available accuracy, while nonlinear fundamental quantum theory \([29]\) usually violates the second law of thermodynamics \([30]\) and the theory of relativity \([31]\).

Now, because the condensate wavefunction is nonlinear when there is interaction between particles, while it has been shown that the condensate wavefunctions of different Bose condensates can be entangled, we propose that condensate qubit may be used to realize the nonlinear quantum computing, and thus deal with NP-complete and \#P problems. The nonlinearity of Gross-Pitaevskii equation is just of Weinberg-type \([29]\), which is used in the algorithms in \([15]\).

As explained in Sec. 2, in general the condensate wavefunction is not the total pure state of a closed system, hence of course its nonlinearity has nothing to do with, and does not share the problems of, nonlinear quantum mechanics. In the absence of the interaction, the condensate wavefunction reduces to the pure state of a single particle, and, consistently, the nonlinearity disappears.

### 5.3. One-bit operation

We illustrate how to implement the one-bit gates, in terms of a Bose condensate of trapped atoms in a symmetric double-well trapping potential \( V(r) \) (Fig. 1). This is essentially the Josephson-like effect investigated previously \([24]\). We may represent bits \(|0\rangle\) and \(|1\rangle\) as the condensate wavefunctions at the two wells, respectively. Thus

\[
|n\rangle = \int \phi_n(r)|r\rangle dr,
\]

where \( n = 0, 1 \), \( \phi_n(r) = u(r - r_n) \) is the condensate wavefunction corresponding to the local potential \( \tilde{V}(r - r_n) \), which may be parabolic, at the vicinity of the bottom \( r_n \). In accordance with Eq. \([18]\), a qubit \(|q(t)\rangle\) is in general a superposition of \(|0\rangle\) and \(|1\rangle\),

\[
|q(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle \doteq \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix},
\]

where \( \doteq \) denotes the matrix representation, with

\[
|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Gross-Pitaevskii equation leads to

\[
i\hbar \frac{\partial}{\partial t}|q(t)\rangle = \hat{H}|q(t)\rangle,
\]

with

\[
\hat{H} = E\hat{I} + \Omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \kappa N \begin{pmatrix} |c_0|^2 & 0 \\ 0 & |c_1|^2 \end{pmatrix},
\]

where

\[
\Omega = \int dr \phi_0^*(r)[V(r) - \tilde{V}(r - r_0)]\phi_1(r)
\]

represents the Josephson-like tunnelling effect.
\[ \kappa = g \int dr |\phi_0(r)|^4, \]  

(E is the energy of the basis |0\rangle and |1\rangle). Therefore, when the s-wave interaction is turned off, \( \kappa = 0 \), we may have an arbitrary one-bit linear transformation, depending on the time span \( \tau \):

\[ |q(\tau)\rangle \rightarrow \exp\left[-\frac{i}{\hbar}(E\hat{I} + \Omega\hat{\sigma}_x)\tau\right]|q(0)\rangle. \]

Thus one may construct one-bit linear gates. When the s-wave interaction is turned on, \( \kappa \neq 0 \), there is a twisting rotation in the state space spanned by |0\rangle and |1\rangle. By choosing appropriate time span, this may be used to construct one-bit nonlinear gates.

We mention that when the wavefunction \( \phi_n(r) \) is real, \( \phi_0(r) \pm \phi_1(r) \) are orthogonal, and might be used as qubit.

### 5.4. Two-bit Operations

The nonlinear one-bit gates of Bose-Einstein condensates may be integrated with the linear gates of other qubit carriers, so that the algorithms in \[ \text{[3]} \] can be implemented, since only one-bit nonlinear gates are needed there. A network of condensates is also possible. In the following, we investigate the evolution of two coupled condensates based on a direct interaction, as described in Section 5. We consider trapped atoms in a double-well potential. A long range interaction, denoted as \( W(\mathbf{r} - \mathbf{r}') \), such as dipole-dipole interaction, is a possible basis of the inter-condensate interaction. It is considerable either for the magnetic moments of trapped atoms with high magnetic moments \[ \text{[2]} \], or for the electric dipoles induced by strong dc fields \[ \text{[3]} \]. It would be ideal if the kind of interaction between atoms in different condensates, for the purpose of coupling, is absent or somehow canceled out between atoms in a same condensate. However, we shall discuss the general case, since it is interesting no matter whether it is used for quantum computation. The discussion is formal, without detailed consideration of the suitable physical conditions.

In the presence of a long range interaction \( W(\mathbf{r} - \mathbf{r}') \) in addition to the s-wave interaction, one should substitute \( U(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}') + W(\mathbf{r} - \mathbf{r}') \) in the equation of motion. One-bit Hamiltonian \[ \text{[4]} \] is then modified to

\[ \hat{H} = E \hat{I} + \Omega \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) + N \left( \begin{array}{ccc} |c_0|^2\kappa & |c_0|^2\mu_1 & |c_1|^2\mu_2 \\ |c_0|^2\mu_1 & 0 & |c_1|^2\mu_2 \\ |c_1|^2\mu_2 & |c_1|^2\mu_2 & 0 \end{array} \right), \]

where \( \mu_1 = \int W(\mathbf{r} - \mathbf{r}')|\phi_n(\mathbf{r})|^2|\phi_n(\mathbf{r}')|^2 d\mathbf{r} d\mathbf{r}' \) is due to \( W \) within a same well, while \( \mu_2 = \int W(\mathbf{r} - \mathbf{r}')|\phi_n(\mathbf{r})|^2|\phi_{n'}(\mathbf{r}')|^2 d\mathbf{r} d\mathbf{r}' \), with \( \pi = 1 - n \), is due to \( W \) within different wells. \( \mu_1 \gg \mu_2 \), since \( W \) between atoms within a same well is much larger than that between atoms in different wells.

A two-bit gate may be constructed by putting together two double-wells, each of which confines a condensate. They are close to each other in a face-to-face way (Fig. 1), i.e., |0\rangle\_a is close to |0\rangle\_b, and |1\rangle\_a is close to |1\rangle\_b. Therefore for atoms in different condensates, \( W \) between, say, an atom in |0\rangle\_a and an atom in |0\rangle\_b is much larger than that between an atom in |0\rangle\_a and an atom in |1\rangle\_b. This is the origin of the conditional dynamics.

Substituting Eq. \[ \text{(27)} \] to the equation of motion of the total condensate wavefunction, Eq. \[ \text{(36)} \], with \( U(r^i - r'^i) = g^i\delta(r^i - r'^i) + W(r^i - r'^i), \) \( (i = a, b) \), we obtain

\[ i\hbar \frac{\partial C_{n^a,n^b}}{\partial t} = \sum_{n^a_1,n^b_1} C_{n^a_1,n^b_1}(I + II + III + IV + V + VI + VII + VIII), \]

with

\[ I = \int \int dr^a dr^b \phi^*_n(\mathbf{r}^a)\phi^*_n(\mathbf{r}^b)\hat{h}(\mathbf{r}^a)\phi_n(\mathbf{r}^a)\phi_n(\mathbf{r}^b) \]

\[ = \delta_{n^a,n^b}\left(E^a\delta_{n^a,n^a} + \Omega^a\delta_{n^a,n^a}^\pi\right), \]

\[ II = \int \int dr^a dr^b \phi^*_n(\mathbf{r}^a)\phi^*_n(\mathbf{r}^b)\hat{h}(\mathbf{r}^b)\phi_n(\mathbf{r}^a)\phi_n(\mathbf{r}^b) \]

\[ = \delta_{n^a,n^b}\left(E^b\delta_{n^b,n^b} + \Omega^b\delta_{n^b,n^b}^\pi\right), \]

\[ III = g^aN \int \int dr^a dr^b dr^a' dr^b' \phi^*_n(\mathbf{r}^a)\phi^*_n(\mathbf{r}^b)\phi_n(\mathbf{r}^a')\phi_n(\mathbf{r}^b')|\phi(\mathbf{r}^a', \mathbf{r}^b')|^2 \delta(\mathbf{r}^a - \mathbf{r}^a') \]

\[ = \delta_{n^a,n^a}\delta_{n^b,n^b}g^aN|C_{n^a,n^b}|^2 \int |\phi_n(\mathbf{r}^a)|^2 dr^a \int |\phi_n(\mathbf{r}^b)|^2 dr^b \]

\[ = \delta_{n^a,n^a}\delta_{n^b,n^b}g^aN|C_{n^a,n^b}|^2 \kappa^a\kappa^b, \]
\[ \begin{align*}
IV & = N \int \int \int dr^a dr^b dr^{a'} dr^{b'} \phi_n^a(r^a) \phi_n^b(r^b) \phi_\nu^{a'}(r^{a'}) \phi_\nu^{b'}(r^{b'}) \rho^{(r^a, r^{a'})} \rho^{(r^b, r^{b'})} W(r^a - r^{a'}) \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} N \sum_{n^{a'}, n^{b'}} |C_{n^{a'}, n^{b'}}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^a - r^{a'}) dr^a dr^{a'} \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} g^a N |C_{n^a, n}|^2 \int |\phi_n(r^a)|^2 dr^a \int |\phi_\nu(r^b)|^2 dr^b, \\
V & = g^n N \int \int \int dr^a dr^b dr^{a'} dr^{b'} \phi_n^a(r^a) \phi_n^b(r^b) \phi_\nu^{a'}(r^{a'}) \phi_\nu^{b'}(r^{b'}) \rho^{(r^a, r^{a'})} \rho^{(r^b, r^{b'})} W(r^b - r^{b'}) \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} \delta_{n^5, n^6} g^n N |C_{n^a, n}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^b - r^{b'}) dr^b dr^{b'} dr^{a'} dr^{a''} \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} \delta_{n^5, n^6} g^n N |C_{n^a, n}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^b - r^{b'}) dr^b dr^{b'} dr^{a'} dr^{a''}, \\
VII & = N \int \int \int dr^a dr^b dr^{a'} dr^{b'} \phi_n^a(r^a) \phi_n^b(r^b) \phi_\nu^{a'}(r^{a'}) \phi_\nu^{b'}(r^{b'}) \rho^{(r^a, r^{a'})} \rho^{(r^b, r^{b'})} W(r^{a'} - r^a) \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} \delta_{n^5, n^6} N \sum_{n^{a'}, n^{b'}} |C_{n^{a'}, n^{b'}}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^{a'} - r^a) dr^a dr^{a'} dr^{b'} dr^{b''} \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} \delta_{n^5, n^6} N \sum_{n^{a'}, n^{b'}} |C_{n^{a'}, n^{b'}}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^{a'} - r^a) dr^a dr^{a'} dr^{b'} dr^{b''}, \\
VIII & = N \int \int \int dr^a dr^b dr^{a'} dr^{b'} \phi_n^a(r^a) \phi_n^b(r^b) \phi_\nu^{a'}(r^{a'}) \phi_\nu^{b'}(r^{b'}) \rho^{(r^a, r^{a'})} \rho^{(r^b, r^{b'})} W(r^{b'} - r^b) \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} \delta_{n^5, n^6} N \sum_{n^{a'}, n^{b'}} |C_{n^{a'}, n^{b'}}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^{b'} - r^b) dr^b dr^{b'} dr^{a'} dr^{a''} \\
& = \delta_{n^1, n^2} \delta_{n^3, n^4} \delta_{n^5, n^6} N \sum_{n^{a'}, n^{b'}} |C_{n^{a'}, n^{b'}}|^2 \int |\phi_n(r^a)|^2 |\phi_n(r^b)|^2 |\phi_\nu^{a'}(r^{a'})|^2 |\phi_\nu^{b'}(r^{b'})|^2 W(r^{b'} - r^b) dr^b dr^{b'} dr^{a'} dr^{a''},
\end{align*} \]

where \( E^i, \Omega^i, \) and \( \nu^i, (i = a, b) \) have the same meanings, respectively, as those quantities without the superscripts, defined above for a single condensate. \( \overline{\mu} = 1 - n^i. \) For simplicity, we may set \( E^a = E^b, \) \( \Omega^a = \Omega^b, \) \( g^a = g^b, \) \( \kappa^a = \kappa^b. \) \( \mu_{n^i, n^j} = \int |\phi_n(r)|^2 |\phi_\nu(r)|^2 W(r^i - r^j) dr^i dr^j \) is due to the interaction \( W \) within a single condensate qubit. \( \nu_{n^i, n^j} = \int |\phi_n(r)|^2 |\phi_\nu(r)|^2 W(r^i - r^j) dr^i dr^j \) is due to the interaction \( W \) between different condensate qubits. With symmetry, \( \mu_{n^i, n^j} = \mu^1_i \) for \( n^i = n^j \) while \( \mu_{n^i, n^j} = \mu^2_i \) for \( n^i \neq n^j. \) \( \mu^1_i > \mu^2_i. \) Similarly, \( \nu_{n^i, n^j} = \nu_1 \) for \( n^i = n^j \) while \( \nu_{n^i, n^j} = \nu_2 \) for \( n^i \neq n^j. \) \( \nu_1 > \nu_2. \) Then Eq. \( (55) \) can be written as a matrix equation,

\[
\frac{i\hbar}{\partial t} \begin{pmatrix}
C_{00} \\
C_{01} \\
C_{10} \\
C_{11}
\end{pmatrix} = \begin{pmatrix}
G_{00} + N F_{00} & \Omega^{b} & 0 & 0 \\
\Omega^{b} & G_{01} + N F_{01} & 0 & \Omega^{a} \\
0 & 0 & G_{10} + N F_{10} & \Omega^{b} \\
0 & \Omega^{a} & \Omega^{b} & G_{11} + N F_{11}
\end{pmatrix} \begin{pmatrix}
C_{00} \\
C_{01} \\
C_{10} \\
C_{11}
\end{pmatrix},
\]

where

\[
G_{n^a n^b} = E^n + E^b + (g^a + g^b) N \kappa^a \kappa^b |C_{n^a n^b}|^2,
\]

and

\[
F_{00} = (\mu_1 + \nu_1)(2 |C_{00}|^2 + |C_{01}|^2 + |C_{10}|^2 + |C_{11}|^2) + (\mu_2 + \nu_2)(|C_{00}|^2 + |C_{11}|^2 + |C_{01}|^2 + |C_{10}|^2),
\]

\[
F_{01} = (\mu_1 + \nu_2)(|C_{00}|^2 + 2 |C_{01}|^2 + |C_{11}|^2 + |C_{10}|^2) + (\mu_2 + \nu_1)(|C_{00}|^2 + 2 |C_{11}|^2 + |C_{01}|^2 + |C_{10}|^2),
\]

\[
F_{01} = (\mu_1 + \nu_2)(|C_{00}|^2 + 2 |C_{10}|^2 + |C_{11}|^2) + (\mu_2 + \nu_1)(|C_{00}|^2 + 2 |C_{01}|^2 + |C_{11}|^2),
\]

\[
F_{11} = (\mu_1 + \nu_1)(|C_{00}|^2 + |C_{01}|^2 + |C_{10}|^2 + |C_{11}|^2) + (\mu_2 + \nu_2)(2 |C_{00}|^2 + |C_{01}|^2 + |C_{10}|^2).
\]

We have set \( \mu^1_i = \mu^1_1 = \mu_1, \mu^2_i = \mu^2_2 = \mu_2. \)

In principle, \( (47) \) is a basis for two-bit operations. For the sake of quantum computation, there are some questions worthy of investigations, for example, whether \( (47) \) can be used to realize universal two-bit gates, whether there is universality for nonlinear gates, how to construct algorithms for NP-complete and \#P problems based directly on \( (39) \) and \( (44), \) how to realize linear and simpler two-bit operations for the Bose-Einstein condensates, whether swapping operation \( (54) \) can be constructed, etc.
6. SPINOR CONDENSATES

6.1. Spinor condensate wavefunction

Up to now, the internal state is irrelevant. In this section, in parallel to the above discussions on a condensate in a double-well potential, we discuss the Josephson-coupled internal states of a condensate. In this case, the field operator and the condensate wavefunction are spinors. We use a two-component condensate in an atom trap [35] as the prototype for discussions.

Suppose that the two internal states are |0⟩ and |1⟩. Then together with the motional degree of freedom, the single particle basis state can be written as φ_{a,n}(r)|n⟩, (n = 0, 1). The field operator can be defined as

\[ \hat{\psi}(r, t) = \sum_{n=0, 1} \phi_{a,n}(r)|n⟩\hat{a}_{a,n}(t) \]

= \hat{\psi}_0(r, t)|0⟩ + \hat{\psi}_1(r, t)|1⟩,

\[ \hat{\psi}_n(r, t) = \sum_{a} \phi_{a,n}(r)\hat{a}_a(t). \]

where \( \hat{\psi}_n(r, t) = \sum_{a} \phi_{a,n}(r)\hat{a}_a(t) \). Imposing SGSB on (60) leads to the spinor condensate wavefunction

\[ \Phi(r, t) = \Phi_0(r, t)|0⟩ + \Phi_1(r, t)|1⟩ \]

with \( \Phi(r, t) = \hat{\psi}(r, t) \), (61) can be written as

\[ \phi(r, t) = \phi_0(r, t)|0⟩ + \phi_1(r, t)|1⟩, \]

with \( \phi(r, t) = \Phi(r, t) / \sqrt{N} \), \( \phi_0(r, t) = \Phi_n(r, t) / \sqrt{N} \). We have used notations similar to those for the double-well condensate. But note the differences in meaning.

The justification in terms of ODLRO can also be made. The one-particle reduced density matrix is

\[ \langle r' | \hat{\rho}_{1} | r ⟩ = |0⟩⟨0| | r' , 0 | \hat{\rho}_{1} | r , 0 ⟩ + |1⟩⟨1| | r' , 0 | \hat{\rho}_{1} | r , 1 ⟩ + |1⟩⟨0| | r' , 1 | \hat{\rho}_{1} | r , 0 ⟩ + |1⟩⟨1| | r' , 1 | \hat{\rho}_{1} | r , 1 ⟩, \]

where \( \langle r' , n' | \hat{\rho}_{1} | r , n ⟩ = \langle \hat{\psi}_{n'}(r', t) \hat{\psi}_{n}(r, t) \rangle \). The existence of ODLRO implies \( \langle r' , n' | \hat{\rho}_{1} | r , n ⟩ \rightarrow \Phi_{n'}(r', t) \Phi^{*}_{n}(r, t) \), and thus

\[ \langle r' | \hat{\rho}_{1} | r ⟩ \rightarrow (\Phi_0(r', t)|0⟩ + \Phi_1(r', t)|1⟩)(\Phi_0(r, t)|0⟩ + \Phi_1(r, t)|1⟩)^*, \]

which indicates the existence of the spinor condensate wavefunction \( \Phi(r) \), as in (61).

6.2. Four-component spinor condensate wavefunction of two coupled condensates

For two coupled two-component Bose systems \( a \) and \( b \),

\[ \hat{\psi}(r^a, t)\hat{\psi}(r^b, t) = \sum_{n^a} \sum_{n^b} \sum_{n^a} \phi_{\alpha^a, n^a}(r^a)\phi_{\alpha^b, n^b}(r^b)|n^a⟩|n^b⟩\hat{a}_{\alpha^a, n^a}(t)\hat{a}_{\alpha^b, n^b}(t) \]

= \sum_{n^a} \sum_{n^b} \hat{\psi}_{n^a}(r^a, t)\hat{\psi}_{n^b}(r^b, t)|n^a⟩|n^b⟩,

\[ \hat{\psi}_{n^a}(r^a, t)\hat{\psi}_{n^b}(r^b, t), \]

\[ \hat{\psi}_0(r^a, t)\hat{\psi}_0(r^b, t), \]

\[ \hat{\psi}_0(r^a, t)\hat{\psi}_1(r^b, t), \]

\[ \hat{\psi}_1(r^a, t)\hat{\psi}_0(r^b, t), \]

\[ \hat{\psi}_1(r^a, t)\hat{\psi}_1(r^b, t) \]

\[ \hat{\psi}_0(r^a, t)\hat{\psi}_0(r^b, t), \]

\[ \hat{\psi}_0(r^a, t)\hat{\psi}_1(r^b, t), \]

\[ \hat{\psi}_1(r^a, t)\hat{\psi}_0(r^b, t), \]

\[ \hat{\psi}_1(r^a, t)\hat{\psi}_1(r^b, t) \]

\( \hat{\psi}_0(r^a, t)\hat{\psi}_0(r^b, t), \)

\( \hat{\psi}_0(r^a, t)\hat{\psi}_1(r^b, t), \)

\( \hat{\psi}_1(r^a, t)\hat{\psi}_0(r^b, t), \)

\( \hat{\psi}_1(r^a, t)\hat{\psi}_1(r^b, t) \)

\( \hat{\psi}_0(r^a, t)\hat{\psi}_0(r^b, t), \)

\( \hat{\psi}_0(r^a, t)\hat{\psi}_1(r^b, t), \)

\( \hat{\psi}_1(r^a, t)\hat{\psi}_0(r^b, t), \)

\( \hat{\psi}_1(r^a, t)\hat{\psi}_1(r^b, t) \)

Imposing SGSB on (63), one obtains
where, with the number of particles in each internal state conserved, the potential implying the existence of the joint condensate wavefunction as a four component spinor, as given in (66).

Consequently, (67) approaches

$$\Phi(r^a, r^b, t) = \sum_{n^a=0,1} \sum_{n^b=0,1} \Phi_{n^a, n^b}(r^a, r^b, t)|n^a\rangle|n^b\rangle \equiv \left( \begin{array}{c} \Phi_{00}(r^a, r^b, t) \\ \Phi_{01}(r^a, r^b, t) \\ \Phi_{10}(r^a, r^b, t) \\ \Phi_{11}(r^a, r^b, t) \end{array} \right),$$

(66)

where $\Phi(r^a, r^b, t) \equiv \langle \psi(r^a, r^b, t) \rangle$, $\Phi_{n^a, n^b}(r^a, r^b, t) \equiv \langle \psi_{n^a, n^b}(r^a, r^b, t) \rangle$. Therefore for two coupled two-component condensates, the total condensate wavefunction is a four-component spinor.

The reasoning can also be cast in the form of ODLRO. The one-particle-pair reduced density matrix is

$$\langle r^{a\prime}, r^{b\prime}| \hat{\rho}_1 |r^a, r^b\rangle \equiv \langle \psi(r^{a\prime}, t) \psi(r^{b\prime}, t) \psi(r^b, t) \psi(r^a, t) \rangle = \sum_{n^a, n^b, n^a, n^b} \langle r^{a\prime}, n^a, r^{b\prime}, n^b| \hat{\rho}_1 |r^a, n^a, r^b, n^b\rangle|n^a\rangle|n^b\rangle,$$

(67)

where

$$\langle r^{a\prime}, n^a, r^{b\prime}, n^b| \hat{\rho}_1 |r^a, n^a, r^b, n^b\rangle = \langle \psi_{n^a, n^b}(r^{a\prime}, t) \psi_{n^b, n^a}(r^{b\prime}, t) \psi_{n^a, n^b}(r^b, t) \psi_{n^b, n^a}(r^a, t) \rangle.$$

(68)

With ODLRO,

$$\langle \hat{\psi}_{n^a, n^b}(r^{a\prime}, t) \hat{\psi}_{n^a, n^b}(r^{b\prime}, t) \hat{\psi}_{n^b, n^a}(r^b, t) \hat{\psi}_{n^a, n^b}(r^a, t) \rangle \rightarrow \Phi_{n^a, n^b}(r^{a\prime}, r^{b\prime}, t) \Phi_{n^a, n^b}(r^a, r^b, t).$$

(69)

Consequently, (67) approaches

$$\left[ \sum_{n^a=0,1} \sum_{n^b=0,1} \Phi_{n^a, n^b}(r^{a\prime}, r^{b\prime}, t)|n^a\rangle|n^b\rangle \right] \left[ \sum_{n^a=0,1} \sum_{n^b=0,1} \langle n^a| \langle n^b| \Phi_{n^a, n^b}(r^a, r^b, t),$$

implying the existence of the joint condensate wavefunction as a four component spinor, as given in (66).

We may write (67) as

$$\phi(r^a, r^b, t) = \sum_{n^a=0,1} \sum_{n^b=0,1} \phi_{n^a, n^b}(r^a, r^b, t)|n^a\rangle|n^b\rangle,$$

(70)

with $\phi(r^a, r^b, t) = \Phi(r^a, r^b, t)/N$, $\phi_{n^a, n^b}(r^a, r^b, t) = \Phi_{n^a, n^b}(r^a, r^b, t)/N$.

The many-body states are similar to the cases in Sec. 3, only with each single particle wavefunction $\phi_{a,n}(r)$ replaced as the combination of motional state and the internal state $\phi_{a,n}(r)|n\rangle$.

6.3. Hamiltonians and equations of motion

When there is no coupling between the internal state and the motional state, $\phi_{a,n}(r)$ in (68) is independent of $n$. $\phi_n(r, t)$ in (68) can be written as $c_n(t)\phi(r, t)$, where $\phi(r, t)$ is independent of $n$, while $c_n(t)$ is independent of $r$. With an electromagnetic field, the Hamiltonian for the internal state is

$$\hat{H}_n = \frac{\omega}{2} \hat{\sigma}_x + \frac{\delta}{2} \hat{\sigma}_z,$$

(71)

with $\hat{\sigma}_z|n\rangle = (2n - 1)|n\rangle$. $\omega$ is the Rabi frequency, while $\delta$ is the detuning.

With the coupling between the motional and internal states, the total Hamiltonian for a two-component condensate is

$$\hat{H} = \sum_{i=1}^N [\hbar(r_i) \otimes \hat{1}_n + \hat{1} \otimes \hat{H}_n(i)] + \sum_{i<j} U(r_i - r_j),$$

(72)

where, with the number of particles in each internal state conserved, the potential $U$ is

$$U \doteq \begin{pmatrix} U_{00} & 0 & 0 & 0 \\ 0 & \frac{i}{2} U_{01} & \frac{i}{2} U_{01} & 0 \\ 0 & \frac{i}{2} U_{01} & \frac{i}{2} U_{01} & 0 \\ 0 & 0 & 0 & U_{11} \end{pmatrix},$$

(73)
in the basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$. The field theoretical Hamiltonian is

$$\mathcal{H} = \int dr \dot{\psi}^\dagger(r) [\hat{h}(r) \otimes \hat{1}_{in}] \dot{\psi}(r) + \int dr \dot{\psi}^\dagger(r) [\hat{1} \otimes \hat{H}_{in}] \dot{\psi}(r) + \frac{1}{2} \int dr_1 dr_2 \dot{\psi}^\dagger(r_1) \hat{V}(r_2) \dot{\psi}(r_2) \dot{\psi}(r_1),$$

(74)

where the field operator $\dot{\psi}(r)$ is as defined in (20). The equation of motion of $\dot{\psi}(r,t)$ is

$$i\hbar \frac{\partial \dot{\psi}(r,t)}{\partial t} = \left[ (\frac{-\hbar^2 \nabla^2 r}{2m} + V(r)) \otimes \hat{1}_{in} + \hat{1} \otimes \hat{H}_{in} + \int \dot{\psi}^\dagger(r',t) U(r-r') \dot{\psi}(r',t) \right] \dot{\psi}(r,t),$$

(75)

where a $2 \times 2$ unit matrix is omitted in front of $\dot{\psi}^\dagger(r',t)$ (similar is the following). The vertical position of the trapping potential depends on the internal state $|\text{3}\rangle$,

$$V(r) = \text{diag}[V_0(r), V_1(r)],$$

(76)

where $V_n(r) = \omega_p r^2 / 2 + \omega_z (z - z_n)^2 / 2$. Therefore we have

$$i\hbar \frac{\partial \phi_0(r,t)}{\partial t} = \left( \begin{array}{cc} H_{00} + \delta/2 & \omega/2 \\ \omega/2 & H_{11} - \delta/2 \end{array} \right) \left( \begin{array}{c} \phi_0(r,t) \\ \phi_1(r,t) \end{array} \right),$$

(77)

where

$$H_{nn} = \frac{-\hbar^2 \nabla^2 r}{2m} + V_n(r) + \sum_{n=0,1} \int |\psi_{n'}(r',t)|^2 U_{nn'}(r-r') dr'.$$

(78)

where $U_{10} \equiv U_{01}$. Consequently, through SGSB or ODLRO, one obtains the coupled Gross-Pitaevskii Equations for the spinor condensate wavefunction $\phi_n = \langle \psi_n(r) \rangle / \sqrt{N}$,

$$i\hbar \frac{\partial \phi_0(r,t)}{\partial t} = \left( \begin{array}{cc} H_{00} + \delta/2 & \omega/2 \\ \omega/2 & H_{11} - \delta/2 \end{array} \right) \left( \begin{array}{c} \phi_0(r,t) \\ \phi_1(r,t) \end{array} \right),$$

(79)

$$\mathcal{H}_{nn} = \frac{-\hbar^2 \nabla^2 r}{2m} + V_n(r) + N \sum_{n'=0,1} \int |\psi_{n'}(r',t)|^2 U_{nn'}(r-r') dr'.$$

(80)

If the interaction is s-wave interaction, $U_{nn'}(r-r') = g_{nn} \delta(r-r')$.

For two coupled Bose systems $a$ and $b$, the general Hamiltonian is similar to Eq. (25), with each term now changed to a matrix in the internal state space, in a way similar to the above case of a single Bose system. In terms of the field operators $\psi(r^a, t)$ and $\psi(r^b, t)$, formally the field theoretical Hamiltonian can be expressed as (25), while the equation of motion is in the form of (33). However, each potential or interaction energy operator depends in the appropriate way on the internal states. In the matrix representation,

$$V(r^i) = \text{diag}[V(r^i), V_1(r^i)],$$

(81)

in the basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, $(i = a, b)$, while it is assumed that

$$W(r^i - r^j) = \text{diag}[W_{00}(r^i - r^j), W_{01}(r^i - r^j), W_{10}(r^i - r^j), W_{11}(r^i - r^j)]$$

(82)

in the basis of $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, $(i, j = a, b, i \neq j)$. The long-range interaction within a same condensate can be neglected, compared with $U$. Then it is easy to write Eqs. (30) to (35) in the matrix form. The following is the equation of motion for $\psi_n(r^a, t) \psi_n(r^b, t)$.

$$i\hbar \frac{\partial \psi_n(r^a, t) \psi_n(r^b, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 r^a + V_{na}(r^a) - \frac{\hbar^2}{2m} \nabla^2 r^b + V_{nb}(r^b) + \int dr' \psi_n^\dagger(r'^a, t) U_{nn'}(r^a - r'^a) \psi_{n'}(r'^a, t) + \int dr' \psi_n^\dagger(r'^b, t) U_{nn'}(r^b - r'^b) \psi_{n'}(r'^b, t) + \int dr' \psi_n^\dagger(r'^a, t) W_{nn'}(r^a - r'^a) \psi_{n'}(r'^a, t) + \int dr' \psi_n^\dagger(r'^b, t) W_{nn'}(r^b - r'^b) \psi_{n'}(r'^b, t) \right] \psi_n(r^a, t) \psi_n(r^b, t),$$

(83)
from which one obtains the equation of motion for the condensate wavefunction component $\phi_{n^a n^b}(r^a, r^b)$,

$$i\hbar \frac{\partial \phi_{n^a n^b}(r^a, r^b, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2_r + V_{nc}(r^a) - \frac{\hbar^2}{2m} \nabla^2_{r^b} + V_{nb}(r^b) + N \int d^{d-1}r' d^{d-1}r'' \langle |\phi_{n^a=0}(r^a, r'^a, t)|^2 \rangle \right] \phi_{n^a n^b}(r^a, r^b, t) + N \int d^{d-1}r' d^{d-1}r'' \langle \delta(r^a - r'^a) \delta(r^b - r''^b) \rangle \phi_{n^a n^b}(r^a, r^b, t).$$

In principle, the internal state may encode qubit. A simple implementation of one-bit linear operations can be made if there is no coupling with the motional state. Then applying an electromagnetic field can realize one-bit operations, based on Eq. (71). There is no nonlinearity here, since nonlinearity appears only when the motional degree of freedom is involved.

Because for different bases internal states of a condensate in a trap, vertical positions are different, the inter-condensate interaction $W_{n^a n^b}(r^a - r^b)$ depends on $n^a$ and $n^b$. It can be arranged in such a way that $W_{00}(r^a - r^b) = W_{11}(r^a - r^b)$, $W_{01}(r^a - r^b) = W_{01}(r^a - r^b)$. Then conditional dynamics described above may be realized. However, it is difficult to use this to realize two-bit operations for the purpose of quantum computation. One might be tempted to realize conditional phases coming from the motional degree of freedom, in a way similar to some proposals for individual atoms and ions [41,42]. However, the nonlinearity makes this scheme hard to be realized.

7. Additional remarks

[1] The effect of nonlinearity on the interference [38] needs more investigations, both from the perspective of physics and the perspective of quantum computation. In our scheme of entangling condensates, nonlinearity appears in the two-bit evolution. One might find an alternative way, which does not introduce two-bit nonlinearity. On the other hand, it is interesting to study computational issues in presence of nonlinearity. Even though two-bit nonlinearity is a disadvantage for quantum computing, the physics problem itself is still interesting.

[2] In the previously studied two-species condensate [39], each condensate has a single condensate wavefunction, so the issue of entanglement is out of question. Moreover, if the two species are two different internal states of a same kind atom, they can be Josephson-coupled. A possible way of realizing a mixture with entanglement or a four-component spinor condensate, as discussed here, is to mix two different kinds of atoms, each with the two internal states coupled through an electromagnetic field. The equation of motion of the joint field can still be written as Eq. (83), but the main interactions are all due to s-wave scattering, hence $W_{n^a n^b} = U_{n^a n^b}, U_{n^a n^b}(r^i - r^j) = g_{n^a n^b}\delta(r^i - r^j), (i, j = a, b)$.

[3] An extreme case of the coupled many-particle systems considered here is a system of identical composite particles. Then the proposed situation of quantum computation becomes Bose condensation of identical quantum computers.

[4] Superposition of condensate wavefunctions might be useful for quantum computation no matter whether the entanglement is realized and used.

[5] If a swapping operation can be realized, then probably by using an optical lattice trapping many condensates, an architect similar to the one in [37] may be constructed, where a head qubit mediates operations between distant qubits.

[6] Many studies on Josephson junction between superconductors are based on quantizing a macroscopic Hamiltonian with macroscopic variables such as the charge or particle number. This is equivalent to the approach of condensate wavefunction [10]. Therefore it is also a kind of condensate qubit that is used in the Josephson-junction quantum computation [11,12]. Nevertheless, in these proposals, it is either the particle number or the phase of the condensate wavefunction that encodes the qubit. In our proposal, it is the two branches of the condensate wavefunction that encode the bits. Josephson-junction qubits, since they are charged, were proposed to be coupled via an electric circuit.

8. SUMMARY

Coherent properties of Bose-Einstein condensation are well described by condensate wavefunction, which, in a mean field theory, is the single particle state in which the condensation occurs. The many-body state is the product of this same state occupied by all the particles. In Josephson-like effect, the condensate wavefunction is the superposition of two bases wavefunctions. Here we go a step further to suggest that for two such condensates interacting with each
other, there is a joint condensate wavefunction which is a superposition of the products of the bases wavefunctions of the two condensate, hence there can be entanglement. The many-body state is suggested to be a product of copies of this joint condensate wavefunction, with symmetrization.

With superposition and entanglement, condensate wavefunction, or macroscopic wavefunction, may be used to implement quantum computation. That many identical particles occupy a single particle state may lead to some intrinsic robustness and stability, although there is an issue of phase diffusion of the condensate wavefunction. For better accuracy, one may try to realize linear evolution of the the condensate wavefunction. On the other hand, the nonlinearity due to particle-particle interaction may turn out to be a resource of computational power, as indicated in a nonlinear quantum algorithm for NP-complete problems \[15\]. We have illustrated the ideas by using Bose condensation of trapped atoms, especially in double-well potentials. We have also discussed the existence of a four-component condensate wavefunction, as a result of the coupling between two two-component condensates, each of which has two Josephson-coupled internal states.

With these justifications, future researches may include detailed calculations under realistic physical conditions and the open issues mentioned above. It is also interesting to extend the consideration to other macroscopic quantum coherent systems. It seems that our work in the meantime is the first discussion on quantum entanglement between “second quantized” many-particle systems or quantum fields, with off-diagonal long-range order or spontaneous symmetry breaking. Through this work, it is also seen that new physics emergent on a new level of complexity may lead to new properties of computation. Indeed, “more is different” \[13\].

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**APPENDIX**

Here we derive Gross-Pitaevskii equation from ODLRO.

The one-particle reduced density matrix can be written as

\[
\langle r'_{1}|\hat{\rho}_{1}(t)| r_{1}\rangle = \frac{1}{(N-1)!} \int \cdots \int dr_{2} \cdots dr_{N} \langle r'_{1}, r_{2}, \ldots, r_{N}|\hat{\rho}(t)| r_{1}, r_{2}, \ldots, r_{N}\rangle,
\]

from which we obtain

\[
i\hbar \frac{\partial}{\partial t} \langle r'_{1}|\hat{\rho}_{1}(t)| r_{1}\rangle = \frac{1}{(N-1)!} \int \cdots \int dr_{2} \cdots dr_{N} \langle r'_{1}, r_{2}, \ldots, r_{N}|(\hat{H}\hat{\rho} - \hat{\rho}\hat{H})| r_{1}, r_{2}, \ldots, r_{N}\rangle.
\]

One can obtain the equation of motion of \(\hat{\rho}_{1}\) as

\[
i\hbar \frac{\partial}{\partial t} \langle r'_{1}|\hat{\rho}_{1}(t)| r_{1}\rangle = \hat{h}(r'_{1})\langle r'_{1}|\hat{\rho}_{1}(t)| r_{1}\rangle - \langle r'_{1}|\hat{\rho}_{1}(t)| r_{1}\rangle \hat{h}(r_{1}) + \int dr_{2}[V(r'_{1} - r_{2})(\langle r'_{1}, r_{2}|\rho_{2}| r_{1}, r_{2}\rangle - \langle r'_{1}, r_{2}|\rho_{2}| r_{1}, r_{2}\rangle V(r_{1} - r_{2})],
\]

where \(\langle r'_{1}, r_{2}|\hat{\rho}_{2}| r_{1}, r_{2}\rangle\) is the two-particle reduced density matrix, to which the major contribution comes from \(\langle r'_{1}|\hat{\rho}_{1}| r_{1}\rangle\). Assuming

\[
\langle r'_{1}, r_{2}|\hat{\rho}_{2}| r_{1}, r_{2}\rangle = \langle r'_{1}|\hat{\rho}_{1}| r_{1}\rangle \langle r_{2}|\hat{\rho}_{1}| r_{2}\rangle,
\]

and considering \(V(r_{1} - r_{2}) = g\delta(r_{1} - r_{2})\), and ODLRO as given in equation \[3\], one obtains

\[
i\hbar \frac{\partial \phi(r'_{1}, t)}{\partial t} = \hat{h}(r'_{1})\phi(r'_{1}, t) + \frac{\partial \phi^{*}(r'_{1}, t)}{\partial t}\phi(r'_{1}, t) - \hat{h}(r_{1})\phi^{*}(r_{1}, t)\phi(r'_{1}, t) + gN|\phi(r'_{1}, t)|^{2}\phi^{*}(r_{1}, t) - gN|\phi(r_{1}, t)|^{2}\phi^{*}(r'_{1}, t),
\]

which leads to the Gross-Pitaevskii equation.
Figure captions:

Fig. 1. Two interacting Bose condensates. Each condensate is trapped in a double-well potential, hence may represent a qubit. $|0\rangle$ is represented by the condensate wavefunction at one well, while $|1\rangle$ is represented by the condensate wavefunction at the other well. A two-bit operation between qubits $a$ and $b$ may be based on an interaction between atoms in different condensates.

Fig2. Two interacting two-component Bose condensates. Each condensate has two Josephson-coupled internal states. The trapping potentials for $|0\rangle$ and $|1\rangle$ are displaced between each other, hence the interaction between $a$ and $b$, based on the long-range interaction between atoms in different condensates, depends on the internal states of $a$ and $b$. Each internal state may be regarded as a bit. However, due to coupling with the motional states and the nonlinearity, it is difficult to use this situation to realize a two-bit phase gate.
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