Multivariable versions of a lemma of Kaluza’s

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Abstract
Let $d \in \mathbb{N}$ and $f(z) = \sum_{\alpha \in \mathbb{N}^d_0} c_{\alpha} z^\alpha$ be a convergent multivariable power series in $z = (z_1, \ldots, z_d)$. In this paper, we present two conditions on the positive coefficients $c_{\alpha}$ that imply that $f(z) = \frac{1}{1 - \sum_{\alpha \in \mathbb{N}^d_0} q_{\alpha} z^\alpha}$ for nonnegative coefficients $q_{\alpha}$. If $d = 1$, then both of our results reduce to a lemma of Kaluza’s. For $d > 1$, we present examples to show that our two conditions are independent of one another. It turns out that functions of the type

$$f(z) = \int_{[0,1]^d} \frac{1}{1 - \sum_{j=1}^d t_j z_j} d\mu(t)$$

satisfy one of our conditions, whenever $d\mu(t) = d\mu_1(t_1) \times \cdots \times d\mu_d(t_d)$ is a product of probability measures $\mu_j$ on $[0,1]$. Our results have applications to the theory of Nevanlinna–Pick kernels.

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1 | INTRODUCTION

In 1928, Theodor Kaluza published the following fact about power series of one complex variable [16].

Theorem 1.1 (Kaluza’s lemma). Let $M > 0$ and let $\{c_n\}_{n \geq 0}$ be a sequence of positive real numbers with $c_0 = 1$. Define a sequence of real numbers $\{r_n\}_{n \geq 1}$ by $r_n := \frac{c_n}{c_{n-1}}$ for each $n \in \mathbb{N}$.
If \( \{r_n\}_{n \geq 1} \) is a nondecreasing sequence that is bounded above by \( M \), then \( f(z) := \sum_{n=0}^{\infty} c_n z^n \) converges for all \( z \in B(1/M) = \{ z \in \mathbb{C} : |z| < 1/M \} \) and the coefficients \( \{q_n\}_{n \geq 1} \) defined by

\[
f(z) = \frac{1}{1 - \sum_{n=1}^{\infty} q_n z^n}
\]

are all nonnegative.

Of course, another way of saying that \( \{r_n\} \) is nondecreasing is to say that \( \{c_n\} \) is logarithmically convex. But because of Theorem 1.1, such sequences have also been called Kaluza sequences in the literature, see, for example, [13, 17, 19, 23], or [7, 11, 12]. Sequences that satisfy the conclusion of Theorem 1.1 have come up in the theory of renewal processes (see, e.g., [18, 21]), in considerations about convergence rates for mean ergodic theorems of Banach space operators ([11, 12]), and in the theory of complete Nevanlinna–Pick reproducing kernels (see, e.g., [1, 2]).

A scaling argument can be used to reduce the statement in Theorem 1.1 to the special case where \( M = 1 \). Then the boundedness of \( r_n \) by 1 easily implies that \( f(z) \) is defined and analytic in the unit disc \( D \), and it satisfies \( f(0) = 1 \). Hence, \( f = \frac{1}{1-g} \) for some function \( g \) that is analytic in a neighborhood of 0 with \( g(0) = 0 \). Thus, the coefficient sequence \( \{q_n\}_{n \geq 0} \) exists with \( q_0 = 0 \), and it is the nondecreasing property of \( r_n \) that implies that each \( q_n \) is nonnegative. If one knows that the power series converge, then by multiplying with the denominator and adding a term, one sees that the relationship between the sequences \( c = \{c_n\}_{n \geq 0} \) and \( q = \{q_n\}_{n \geq 0} \) is given by

\[
f(z) = \sum_{n=0}^{\infty} c_n z^n = 1 + \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} c_{n-k} q_k \right) z^n \quad \text{or} \quad c_n = \delta_{n,0} + \sum_{k=0}^{n} c_{n-k} q_k \quad \text{for all} \quad n \geq 0.
\]

This last equation can be written as the discrete renewal equation

\[
c = \delta_0 + c * q, \tag{1.1}
\]

where \( \delta_0 = \{\delta_{n,0}\}_{n \geq 0} \). In Section 3, we will consider a renewal equation on certain strongly graded monoids (see Equation (2.2)). We will establish a version of Kaluza’s lemma in that context, see Theorem 3.1. The monoids include \( \mathbb{F}_d \), the free semigroup on \( d \)-generators, and the multi-indices \( \mathbb{N}_d^0 \), see Corollaries 3.2 and 3.3. By use of a symmetrization technique, the \( \mathbb{F}_d \)-Corollary will then be used to derive a second version of Kaluza’s lemma on Monoids (Corollary 5.3)

These results lead to two versions of Kaluza’s lemma for functions on the \( \ell_1 \)-balls in \( \mathbb{C}^d \), which will be stated now. Naturally, in order to state our theorems, it is most convenient to use multi-index notation. Let \( d \in \mathbb{N} \) and let \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_d^0 \) be a multi-index. Particular examples of multi-indices are \( \mathbf{0} = (0, \ldots, 0) \), and for \( 1 \leq j \leq d \), we use \( e_j \) for the multi-index that has 1 in the \( j \)th component and 0’s elsewhere.

Furthermore, \( |\alpha| = \alpha_1 + \cdots + \alpha_d \) and \( \alpha! = \alpha_1! \cdots \alpha_d! \). For \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \), we write \( z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d} \), and with this notation for each \( n \in \mathbb{N} \), the multinomial formula says that

\[
\left( \sum_{j=1}^{d} z_j \right)^n = \sum_{|\alpha| = n} \frac{|\alpha|!}{\alpha!} z^\alpha.
\]
Addition and subtraction of multi-indices are defined componentwise. We also define a partial order on the multi-indices. For \(\alpha, \beta \in \mathbb{N}_0^d\), we say \(\alpha \leq \beta\) if and only if \(\alpha_j \leq \beta_j\) for each integer \(j\) with \(1 \leq j \leq d\). We say that a sequence of real numbers \(\{r_\alpha\}_{\alpha \in \mathbb{N}_0^d}\) is nondecreasing if and only if \(r_\alpha \leq r_\beta\) for all pairs of multi-indices with \(\alpha \leq \beta\).

For \(\alpha \in \mathbb{N}_0^d, \alpha \neq 0\), we define the immediate predecessor set

\[
P_\alpha = \{\alpha - e_k : 1 \leq k \leq d\ \text{and} \ \alpha_k > 0\}.
\]

Finally, the \(\ell_1\)-ball of \(\mathbb{C}^d\) is denoted by

\[
\mathbb{B}_{\ell_1}^d = \{z = (z_1, \ldots, z_d) : \|z\|_1 := |z_1| + |z_2| + \cdots + |z_d| < 1\}.
\]

**Theorem 1.2.** Let \(\{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}\) satisfy \(c_0 = 1\) and \(c_\alpha > 0\) for all \(\alpha \in \mathbb{N}_0^d, \alpha \neq 0\). Define \(\{r_\alpha\}_{\alpha \in \mathbb{N}_0^d}\) by \(r_0 = 0\) and

\[
r_\alpha = \frac{c_\alpha}{\sum_{\beta \in P_\alpha} c_\beta}, \quad \alpha \neq 0.
\]

If \(\{r_\alpha\}_{\alpha \in \mathbb{N}_0^d}\) is nondecreasing and bounded above by 1, then \(f(z) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha z^\alpha\) converges absolutely on \(\mathbb{B}_{\ell_1}^d\) such that for all \(z \in \mathbb{B}_{\ell_1}^d\),

\[
\frac{1}{2} + \frac{1 - \|z\|_1}{1 + \|z\|_1} \leq \text{Re}(f(z)) \leq \frac{1}{1 - \|z\|_1}.
\]

Furthermore, there is a sequence of nonnegative real numbers \(\{q_\gamma\}_{\gamma \in \mathbb{N}_0^d}\) such that

\[
\sum_{\gamma \in \mathbb{N}_0^d} q_\gamma |z_\gamma| \leq \|z\|_1 \quad \text{and} \quad f(z) = \frac{1}{1 - \sum_{\gamma \in \mathbb{N}_0^d} q_\gamma z_\gamma^\gamma} \quad \text{on} \ \mathbb{B}_{\ell_1}^d.
\]

We note for \(\alpha \neq 0\), we have \(P_\alpha \neq \emptyset\), and hence, the denominator of \(r_\alpha\) is strictly positive. Thus, \(r_\alpha\) has been well defined.

**Theorem 1.3.** Suppose that \(\{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}\) satisfies \(c_0 = 1\) and \(c_\alpha > 0\) for all other \(\alpha \in \mathbb{N}_0^d\). If for all \(\alpha \in \mathbb{N}_0^d\) and all \(i, j \in \mathbb{N}\) with \(1 \leq i, j \leq d\), we have \(c_\alpha \leq |\alpha|!\) and

\[
\frac{c_{\alpha + e_i} c_{\alpha + e_j}}{c_\alpha c_{\alpha + e_i + e_j}} \leq \begin{cases} 
\frac{|\alpha| + 1}{\alpha!} & \text{if } i \neq j \\
\frac{(|\alpha| + 1)(\alpha_i + 2)}{(|\alpha| + 2)(\alpha_i + 1)} & \text{if } i = j,
\end{cases}
\]

then \(f(z) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha z^\alpha\) converges absolutely on \(\mathbb{B}_{\ell_1}^d\) such that for all \(z \in \mathbb{B}_{\ell_1}^d\),

\[
\frac{1}{2} + \frac{1 - \|z\|_1}{1 + \|z\|_1} \leq \text{Re}(f(z)) \leq \frac{1}{1 - \|z\|_1}.
\]
Furthermore, there is a sequence of nonnegative real numbers \( \{q_\gamma\}_{\gamma \in \mathbb{N}_0^d} \) such that
\[
\sum_{\gamma \in \mathbb{N}_0^d} q_\gamma |z^\gamma| \leq \|z\|_1 \quad \text{and} \quad f(z) = \frac{1}{1 - \sum_{\gamma \in \mathbb{N}_0^d} q_\gamma z^\gamma} \quad \text{on } \mathbb{B}^1_0.
\]

It appears that the hypothesis in Theorem 1.2 will be difficult to check for many examples, as the definition of the auxiliary sequence \( \{r_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) is fairly complicated. Nevertheless, if \( \{r_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) is any nondecreasing sequence that is bounded above by 1 and satisfies \( r(\mathbf{0}) = 0 \), then one can use the conditions \( c_0 = 1 \) and \( c_\alpha = r_\alpha(\sum_{\beta \in P_\alpha} c_\beta) \) for \( \alpha \neq \mathbf{0} \) to inductively define \( c_\alpha \). That is, because for \( |\alpha| = n \), we have \( |\beta| = n - 1 \) for all \( \beta \in P_\alpha \). This leads to many examples, where the conclusion of the theorem may be nonobvious. Furthermore, in Section 4, we will use this observation to construct an example that satisfies the hypothesis of Theorem 1.2, but not the one of Theorem 1.3. In Section 6, we will show that
\[
f(z) = \int_{[0,1]^d} \frac{1}{1 - \sum_{j=1}^d t_j z_j} d\mu(t)
\]
satisfies the conditions of Theorem 1.3, whenever \( d\mu(t) = d\mu_1(t_1) \times \cdots \times d\mu_d(t_d) \) is a product of probability measures \( \mu_j \) on \([0,1]\). We will also see that for \( d\mu(s,t) = dsdt \), the function \( f \) does not satisfy the hypothesis of Theorem 1.2. Thus, Theorems 1.2 and 1.3 are independent of one another.

In Section 7, we will note that our theorems lead to sufficient conditions for reproducing kernels to be so-called complete Nevanlinna–Pick kernels. We will then give an example to illustrate what the underlying Hilbert function spaces may look like.

After the work on our main theorems was completed, we learned of the papers [24, 25], and [26] in which the author also establishes sufficient conditions on the \( c_\alpha \)'s of Theorems 1.2 and 1.3 that imply that the \( q_\gamma \)'s are nonnegative (in those papers, Kaluza’s lemma is wrongfully attributed to Hardy). The papers do not contain any examples, but the established conditions are different from ours. In fact, one can check easily that neither the function \( f(z) = \frac{1}{1 - (z_1 + \cdots + z_d)} \) nor two further examples that we discuss in the current paper (Example 4.4 and the example at the end of Section 6) satisfy the hypothesis of the main result of [24].

We would like to thank Yuri Tomilov for bringing several references about Kaluza sequences to our attention. Furthermore, we would also like to thank the referee for insightful comments that improved the quality of this paper. In particular, it was the referee’s suggestion to verify that the function \( f \) in Theorems 1.2 and 1.3 is not zero in \( \mathbb{B}^1_0 \), and this lead us to include the stated lower bound on \( \text{Re } f(z) \).

## 2 Preliminaries on Graded Monoids

Recall that a semigroup is a set \( M \) with an operation \( M \times M \to M, (x, y) \to xy \) that satisfies the associative law. A monoid is a semigroup \( M \) with identity \( e \) satisfying \( ex = xe = x \) for all \( x \in M \). We will say that a monoid \( M \) is strongly graded (by \( \mathbb{N}_0 \)), if there is a collection of mutually disjoint subsets \( \{M_n\}_{n \in \mathbb{N}_0} \) such that \( M = \bigcup_{n=0}^{\infty} M_n \) and \( M_n M_m = M_{n+m} \) for all \( n, m \in \mathbb{N}_0 \). On strongly graded monoids, we can define the length function \( | \cdot | : M \to \mathbb{N}_0 \) by \( |w| = n \), if \( w \in M_n \). We note that the condition \( M_n M_m = M_{n+m} \) implies that this function is a homomorphism.
We will now assume that $M$ is a strongly graded monoid satisfying two further conditions:

**TK:** the length function has a trivial kernel: $|w| = 0$, if and only if $w = e$ and

**RC:** the following (right) cancelation law holds: if $x, y \in M_1$ and $w \in M$ such that $xw = yw$, then $x = y$.

By definition condition, TK is equivalent to saying that $M_0 = \{e\}$, and it implies for strongly graded monoids that whenever $w \in M$ and $n \in \mathbb{N}_0$, then $w \in M_n$ if and only if there are $x_1, \ldots, x_n \in M_1$ such that $w = x_1 \cdots x_n$. Of course, in general, such a representation may not be unique.

**Example 2.1.** $\mathbb{N}_0^d$, the multi-indices. In this case, we take $e = 0$, the semigroup operation is componentwise addition of multi-indices, and for each $n \in \mathbb{N}_0$, we can take $M_n = \{\alpha \in \mathbb{N}_0^d : |\alpha| = n\}$, where $|\cdot|$ is the usual length function of multi-indices. It is easily checked that this defines a strong grading that satisfies conditions TK and RC as well.

**Example 2.2.** $M = \mathbb{N}_0^{\mathbb{N}_0^d}$, the multi-indices. In this case, we take $e = 0$, the semigroup operation is componentwise addition of multi-indices, and for each $n \in \mathbb{N}_0$, we can take $M_n = \{\alpha \in \mathbb{N}_0^{\mathbb{N}_0^d} : |\alpha| = n\}$, where $|\cdot|$ is the usual length function of multi-indices. It is easily checked that this defines a strong grading that satisfies conditions TK and RC as well.

**Example 2.3.** Let $A$ be a nonempty set, and let $F(A)$ be the free monoid over the alphabet $A$. That is, $F(A)$ is the collection of all words of finite length that can be formed with letters from $A$ together with the empty word $0$. Thus, $F(A) = \{0\} \cup \{w_1w_2\ldots w_n : n \in \mathbb{N} \text{ and } w_1, \ldots, w_n \in A\}$. The semigroup operation is concatenation of words $(u, v) \rightarrow uv$ and $e = 0$. If $w \in F(A)$, then it has a unique representation of the type $w = w_1w_2\ldots w_n$, where $w_1, \ldots, w_n \in A$ and we can define the length of $w$ by $|w| = n$. Then, we set $M_n = \{w \in F(A) : |w| = n\}$ and it is now easy to see that $\{M_n\}_{n \in \mathbb{N}_0}$ induces a strong grading on $F(A)$ that satisfies conditions TK and RC.

If the cardinality of $A$ equals $d \in \mathbb{N}$, then $F(A)$ is isomorphic to $F_d = F(\{1, 2, \ldots, d\})$, the free monoid on $d$ generators.

**Example 2.4.** If $M$ and $N$ are strongly graded monoids that satisfy TK and RC, then so is $M \times N$. In this case, the semigroup operation is defined by $(x, v)(y, w) = (xy, vw)$, the unit is $e = (e_M, e_N)$, and the grading is given by $$(M \times N)_n = \bigcup_{k=0}^n M_k \times N_{n-k}.$$

**Proof.** It is easy to see that $M \times N$ is strongly graded with $(M \times N)_0 = \{e\}$. We check the right cancelation law RC:

Suppose $(x, v) \in M \times N$ and $(y_1, w_1), (y_2, w_2) \in (M \times N)_1$ such that $(y_1, w_1)(x, v) = (y_2, w_2)(x, v)$. Then $y_1x = y_2x$ and $w_1v = w_2v$. Since $(y_1, w_1) \in (M \times N)_1 = (M_1 \times \{e_N\}) \cup (\{e_M\} \times N_1)$, we have $y_1 = e_M$ or $w_1 = e_N$.

If $y_1 = e_M$, then $w_1 \in N_1$ and hence $1 + |v|_N = |w_1v|_N = |w_2|_N + |v|_N$. Hence, $|w_2|_N = 1$ and by the right cancelation law in $N$, we have $w_1 = w_2$. Furthermore, $(y_2, w_2) \in (M \times N)_1$ now implies $y_2 = e_M$. Hence, $(y_1, w_1) = (y_2, w_2)$. The case $w_1 = e_N$ follows by symmetry. \qed
If $M$ is a strongly graded monoid, then the universal property of the free monoid $\mathbb{F}(M_1)$ implies that there exists a homomorphism

$$\varphi : \mathbb{F}(M_1) \to M$$

such that $\varphi(x) = x$ for each $x \in M_1$. The strong grading of $M$ implies that $\varphi$ is onto, and hence, $M$ is isomorphic to the quotient $\mathbb{F}(M_1)/\sim$, where the equivalence relation $\sim$ is defined by $w \sim v$, if and only if $\varphi(w) = \varphi(v)$. We note that it is easy to check that $\varphi$ (and hence $\sim$) preserves the length functions, $|\varphi(w)|_M = |w|_{\mathbb{F}(M_1)}$ for all $w \in \mathbb{F}(M_1)$. If $M = \mathbb{N}_0^d$, then one can take $\varphi : \mathbb{F}_d \to \mathbb{N}_0^d$ is given by the usual correspondence between words $w = w_1w_2...w_n \in \mathbb{F}_d$ and multi-indices $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$: $\varphi(w) = \alpha$, if $\alpha_k$ is the number of times the letter $k$ is used in the word $w$.

We will use the map $\varphi$ in Section 5 in order to prove Theorem 1.3.

We define a (right) partial order on such an $M$ by saying $x \leq y$, if there is $z \in M$ such that $xz = y$. Furthermore, we say that a function $r : M \to \mathbb{R}$ is (right) nondecreasing, if $r(x) \leq r(y)$, whenever $x \leq y$. For later reference, we record a simple lemma. We omit the easy inductive proof.

**Lemma 2.5.** If $r : M \to \mathbb{R}$ satisfies $r(e) = 0$, then $r$ is right nondecreasing, if and only if $r(xv) \leq r(xyv)$ for all $x, y \in M_1$ and all $v \in M$.

For $w \in M$, we write $D_w = \{(u, v) \in M \times M : uv = w\}$, the decompositions of $w$, and $P_w = \{v \in M : xv = w \text{ for some } x \in M_1\}$, the immediate predecessors of $w$.

**Lemma 2.6.** If $x, y, w \in M$, then $xy \in P_w$, if and only if there is $u \in M$ such that $uy = w$ and $x \in P_u$. Such a $u$ is necessarily unique and it satisfies $u \leq w$.

Hence, for $w \in M$, we have

$$A_w = \{(x, y) : xy \in P_w\} = \{(x, y) : \exists! u \leq w \text{ such that } (u, y) \in D_w \text{ and } x \in P_u\}.$$

**Proof.** If $xy \in P_w$, then there is a $v \in M_1$ such that $vxy = w$. Setting $u = vx$, we have $x \in P_u$ and $uy = w$. Conversely, if $u \in M$ with $uy = w$ and $x \in P_u$, then there is $v \in M_1$ such that $vx = u$ and hence $vxu = w$. This implies $xy \in P_w$.

It is clear that any such $u$ satisfies $u \leq w$. Finally, assume that $u_1x = w = u_2x$ and $y \in P_{u_1} \cap P_{u_2}$, then there are $v_1, v_2 \in M_1$ such that $v_jy = u_j$ for $j = 1, 2$ and hence, $v_1yx = u_1x = u_2x = v_2yx$. Thus, the cancelation law implies that $v_1 = v_2$ and hence $u_1 = u_2$. □

We say that $f : M \to \mathbb{C}$ is locally summable, if $\sum_{|x| \leq n} |f(x)| < \infty$ for each $n \in \mathbb{N}_0$. Of course, if $M_1$ is finite, then each $M_n$ must be finite, and hence, every function is locally summable. If $f, g : M \to \mathbb{C}$ are locally summable, then for each $w \in M$, we let $n = |w|$ and we have

$$\sum_{(u, v) \in D_w} |f(u)g(v)| \leq \sum_{|u| \leq n} \sum_{|v| \leq n} |f(u)g(v)| < \infty.$$

Hence, we can define the convolution of $f$ and $g$ by

$$(f \ast g)(w) = \sum_{(u, v) \in D_w} f(u)g(v).$$

Let $\delta_e : M \to \mathbb{C}$ be defined by $\delta_e(x) = 1$, if $x = e$ and $\delta_e(x) = 0$ otherwise.
Given a function $f : M \to \mathbb{R}$, we are interested in solutions $g$ of the renewal equation

$$f = \delta_e + f \ast g. \quad (2.2)$$

**Lemma 2.7.** For every locally summable $f : M \to \mathbb{R}$ with $f(e) = 1$, the renewal equation (2.2) has a unique locally summable solution $g : M \to \mathbb{R}$. The solution satisfies $g(e) = 0$ and

$$g(w) = f(w) - \sum_{(u,v) \in D_w, v \neq w} f(u)g(v) \quad \text{if} \ w \neq e. \quad (2.3)$$

**Proof.** First, suppose that $g$ is a locally summable solution to the renewal equation. Since $D_e = \{(e, e)\}$, we have $1 = f(e) = \delta_e(e) + (f \ast g)(e) = 1 + f(e)g(e) = 1 + g(e)$ and hence $g(e) = 0$. Note that if $uw = w$, then condition TK implies that $u = e$. Thus, if $w \neq e$, then $D_w = \{(u,v) : uv = w, v \neq w\} \cup \{(e,w)\}$ is a disjoint union, and hence,

$$f(w) = 0 + (f \ast g)(w) = \sum_{(u,v) \in D_w, v \neq w} f(u)g(v) + g(w).$$

This implies that $g$ satisfies (2.3). It also shows that there is at most one solution. Indeed, we note that if $(u,v) \in D_w$ with $v \neq w$, then $|w| = |u| + |v| > |v|$. Thus, if $w \in M_n$ for some $n \in \mathbb{N}$, then all $v \neq w$ such that there is $u \in M$ with $(u,v) \in D_w$ must satisfy $v \in \bigcup_{j=0}^{n-1} M_j$ and the identity (2.3) can be used to define $g$ inductively (starting with $g(e) = 0$). Uniqueness follows, as does existence, we just need to make sure that $g$ turns out to be locally summable (in order for the convolution to be well defined).

If $w \in M_1$, then $D_w = \{(e,w),(w,e)\}$, hence we have

$$g(w) = f(w) - f(w)g(e) = f(w).$$

This together with $g(e) = 0$ and the hypothesis that $f$ is locally summable implies $\sum_{|w| \leq n} |g(w)| < \infty$.

Suppose $n \geq 1$, and for all $w \in M$ with $|w| \leq n$, we have already found $g(w) \in \mathbb{R}$ that satisfy

$$\sum_{|w| \leq n} |g(w)| < \infty$$

and such that $g(w)$ satisfies (2.3). Then,

$$\sum_{|w| = n+1} |f(w) - \sum_{(u,v) \in D_w, v \neq w} f(u)g(v)|$$

$$\leq \sum_{|w| = n+1} |f(w)| + \sum_{|u| \leq n+1, |v| \leq n} |f(u)g(v)| < \infty.$$

Thus, for each $w \in M_{n+1}$, we can use (2.3) to define $g(w) \in \mathbb{R}$, and this will satisfy

$$\sum_{|w| = n+1} |g(w)| < \infty.$$

This concludes the inductive step to show that we can define a locally summable function $g$ that satisfies the renewal equation. \qed
3  KALUZA’S LEMMA FOR STRONGLY GRADED MONOIDS

Theorem 3.1. Let $M$ be a strongly graded monoid that satisfies the conditions TK and RC of Section 2. Let $f : M \to (0, \infty)$ be locally summable with $f(e) = 1$. Define a function $r : M \to [0, \infty)$ by $r(e) = 0$ and

$$r(w) = \frac{f(w)}{\sum_{v \in P_w} f(v)} \text{ if } w \neq e.$$ 

If $r$ is right nondecreasing, then the solution $g$ of the renewal equation (2.2) is nonnegative.

Note that if $w \neq e$, then $P_w \neq \emptyset$. Hence, the positivity of $f$ implies that $r$ is well defined in all cases.

Proof. We assume that $r$ is right nondecreasing. Since $M = \bigcup_{n=0}^{\infty} M_n$, we can use induction on $n$ and the inductive formula from Lemma 2.7 to show that $g$ is nonnegative. The case $n = 0$ follows immediately since $M_0 = \{e\}$ and $g(e) = 0$. We also check the case $n = 1$. If $w \in M_1$, then we saw in the proof of Lemma 2.7 that $g(w) = f(w) > 0$.

Now assume that $n \in \mathbb{N}$ and that $g(v) \geq 0$ for all $v \in M$ with $|v| \leq n$. Let $|w| = n + 1$. We have to show that $g(w) \geq 0$.

Note for later that for all $u \in M, u \neq e$, we have

$$f(u) = r(u) \sum_{x \in P_u} f(x).$$

For $u \in P_w$, set $p(u) = \frac{f(u)}{\sum_{v \in P_w} f(v)}$. Then, $\sum_{u \in P_w} p(u) = 1$ and $r(w)f(v) = f(w)p(v)$.

If $v \in P_w$, then since $|w| \geq 2$, we have $v \neq e$, and hence, we can use the renewal equation to conclude that

$$f(v) = \sum_{(x,y) \in D_v} f(x)g(y).$$

Hence, by Lemma 2.7,

$$g(w) = f(w) - \sum_{(u,y) \in D_w, y \neq w} f(u)g(y)$$

$$= \sum_{v \in P_w} f(w)p(v) - \sum_{(u,y) \in D_w, y \neq w} f(u)g(y)$$

$$= \sum_{v \in P_w} r(w)f(v) - \sum_{(u,y) \in D_w, y \neq w} r(u) \sum_{x \in P_u} f(x)g(y) \text{ by (3.1)}$$

$$= \sum_{(x,y) : xy \in P_w} r(w)f(x)g(y) - \sum_{(u,y) \in D_w, y \neq w} r(u) \sum_{x \in P_u} f(x)g(y) \text{ by (3.2)}$$

$$= \sum_{(x,y) : xy \in P_w} r(w)f(x)g(y) - \sum_{(u,y) \in D_w, y \neq w, x \in P_u} r(u)f(x)g(y).$$

Note that both terms in this difference are positive term sums that are finite by the identities used. Hence, there is no issue with rearranging the sums. Let $A_w$ denote the set from Lemma 2.6, and
note that if \((x, y) \in A_w\), then \(y \neq w\). Hence, Lemma 2.6 implies that

\[
g(w) = \sum_{(x, y) \in A_w} (r(w) - r(u))f(x)g(y),
\]

where we have written \(u = u(x, y)\) for the element in \(M\) whose existence and uniqueness was proven in Lemma 2.6. Hence, \(g(w) \geq 0\) by the inductive hypothesis and the fact that \(r\) is right nondecreasing. □

We collect two corollaries for the special cases where \(M = \mathbb{F}_d\) and \(M = \mathbb{N}_0^d\).

**Corollary 3.2.** Let \(d \in \mathbb{N}\), \(f : \mathbb{F}_d \to (0, \infty)\) with \(f(0) = 1\). If also

\[
f(av) \leq f(vb) \quad \text{for all } a, b, v \in \mathbb{F}_d, |a| = |b| = 1,
\]

then the solution \(g : \mathbb{F}_d \to \mathbb{R}\) to the renewal equation (2.2) is nonnegative.

**Proof.** By Theorem 3.1 and Lemma 2.5, we have to show that

\[
\frac{f(av)}{\sum_{u \in P_{av}} f(u)} \leq \frac{f(vb)}{\sum_{u \in P_{avb}} f(u)}
\]

for all \(a, b, v \in \mathbb{F}_d, |a| = |b| = 1\). But in \(\mathbb{F}_d\), we have \(P_{av} = \{v\}\) and \(P_{avb} = \{vb\}\). The result follows. □

**Corollary 3.3.** Let \(d \in \mathbb{N}\), \(f : \mathbb{N}_0^d \to (0, \infty)\) with \(f(0) = 1\). Define \(r : \mathbb{N}_0^d \to [0, \infty)\) by \(r(0) = 0\) and

\[
r(\alpha) = \frac{f(\alpha)}{\sum_{1 \leq k \leq d, \alpha_k > 0} f(\alpha-e_k)} \quad \text{if } \alpha \neq 0.
\]

If \(r(\alpha) \leq r(\beta)\) for all \(\alpha, \beta \in \mathbb{N}_0^d\) with \(\alpha_j \leq \beta_j\) for all \(j = 1, \ldots, d\), then the solution \(g : \mathbb{N}_0^d \to \mathbb{R}\) to the renewal equation (2.2) is nonnegative.

**Proof.** This also follows from Theorem 3.1 by noting that for \(\alpha \in \mathbb{N}_0^d\), we have \(P_{\alpha} = \{\alpha - e_k : 1 \leq k \leq d\) and \(\alpha_k > 0\}\), and that the partial order that we defined on the multi-indices agrees with the partial (right) ordering that we defined on strongly graded monoids. □

### 4 | THE PROOF OF THEOREM 1.2

In this section, we assume that \(\{c_\alpha\}_{\alpha \in \mathbb{N}_0^d}\) is a given sequence of positive reals with \(c_0 = 1\), and that \(r_\alpha\) is defined as in Theorem 1.2.

**Lemma 4.1.** If \(n \geq 2\) and \(z = (z_1, \ldots, z_d) \in C^n\), then

\[
\left(1 - \sum_{j=1}^d z_j\right) \sum_{|\alpha| = n} c_\alpha z^\alpha = 1 - \sum_{0 < |\alpha| \leq n} (1 - r_\alpha) \sum_{\beta \in P_\alpha} c_\beta - \sum_{|\alpha| = n+1} z^\alpha \sum_{\beta \in P_\alpha} c_\beta.
\]
**Proof.** We have
\[
\sum_{|\alpha| \leq n} c_{\alpha} z^\alpha = 1 + \sum_{0 < |\alpha| \leq n} r_{\alpha} z^\alpha \sum_{\beta \in P_{\alpha}} c_{\beta} \tag{4.1}
\]
and
\[
\left(\sum_{j=1}^{d} z_j \right) \sum_{|\alpha| \leq n} c_{\alpha} z^\alpha = \sum_{|\beta| \leq n} \sum_{j=1}^{d} c_{\beta} z^{\beta + e_j} = \sum_{0 < |\alpha| \leq n+1} z^\alpha \sum_{\beta \in P_{\alpha}} c_{\beta}. \tag{4.2}
\]
The proof now follows by simple algebra. \(\square\)

**Lemma 4.2.** If \(r_{\alpha} \leq 1\) for all \(\alpha \in \mathbb{N}_0^d\), then
\[
\sum_{\alpha \in \mathbb{N}_0^d} |c_{\alpha} z^\alpha| \leq \frac{1}{1 - \|z\|_1} < \infty
\]
for all \(z \in \mathbb{B}_d^{\ell_1}\). Furthermore, the nonnegative coefficients \(b_{\alpha}\) defined for \(\alpha \neq \mathbf{0}\) by \(b_{\alpha} = \frac{c_{\alpha}}{r_{\alpha}} (1 - r_{\alpha}) = (1 - r_{\alpha}) \sum_{\beta \in P_{\alpha}} c_{\beta}\) satisfy
\[
\sum_{\alpha \in \mathbb{N}_0^d} c_{\alpha} z^\alpha = \frac{1 - \sum_{\alpha \in \mathbb{N}_0^d, \alpha \neq \mathbf{0}} b_{\alpha} z^\alpha}{1 - \sum_{j=1}^{d} z_j} \quad \text{on } \mathbb{B}_d^{\ell_1}. \tag{4.3}
\]

**Proof.** Let \(z \in \mathbb{B}_d^{\ell_1}\) and write \(x = (|z_1|, |z_2|, \ldots, |z_d|)\), then \(\|z\|_1 = \|x\|_1 = \sum_{j=1}^{d} x_j < 1\). Thus, for \(n \in \mathbb{N}\), we have by the hypothesis that \(1 - r_{\alpha} \geq 0\) and by Lemma 4.1 applied with the point \(x\) that
\[
\sum_{|\alpha| \leq n} c_{\alpha} x^\alpha \leq \frac{1}{1 - \|z\|_1} < \infty.
\]
Hence, the series converges absolutely on \(\mathbb{B}_d^{\ell_1}\). Thus, we can let \(n \to \infty\) in Equations (4.1) and (4.2) of the proof of Lemma 4.1 and obtain
\[
\sum_{\alpha \in \mathbb{N}_0^d} c_{\alpha} z^\alpha = 1 + \sum_{\alpha \in \mathbb{N}_0^d, \alpha \neq \mathbf{0}} r_{\alpha} z^\alpha \sum_{\beta \in P_{\alpha}} c_{\beta}
\]
and
\[
\left(\sum_{j=1}^{d} z_j \right) \sum_{\alpha \in \mathbb{N}_0^d} c_{\alpha} z^\alpha = \sum_{\alpha \in \mathbb{N}_0^d, \alpha \neq \mathbf{0}} z^\alpha \sum_{\beta \in P_{\alpha}} c_{\beta}.
\]
The identity (4.3) follows by subtracting the two equations and dividing by \((1 - \sum_{j=1}^{d} z_j)\). \(\square\)

**Lemma 4.3.** If \(\sum_{\alpha \in \mathbb{N}_0^d} c_{\alpha} |z^\alpha| \leq \frac{1}{1 - \|z\|_1}\) for all \(z \in \mathbb{B}_d^{\ell_1}\), then by absolute convergence, the function \(f(z) = \sum_{\alpha \in \mathbb{N}_0^d} c_{\alpha} z^\alpha\) is analytic in \(\mathbb{B}_d^{\ell_1}\) with \(f(0) = c_0 = 1\). In a neighborhood of \(0\), define the analytic function \(g\) by \(f = \frac{1}{1 - g} \).
If $g(z) = \sum_{\alpha \in \mathbb{N}^d_0} q_{\alpha} z^\alpha$ for nonnegative coefficients $q_{\alpha}$, then the power series for $g$ converges absolutely in $\mathbb{B}^\ell_1$, and for all $z \in \mathbb{B}^\ell_1$, we have

$$|g(z)| \leq \|z\|_1 \quad \text{and} \quad \frac{1}{2} + \frac{1 - \|z\|_1}{1 + \|z\|_1} \leq \text{Re} f(z).$$

**Proof.** The definition of $g$ implies that $f = 1 + f g$ in a neighborhood of 0. Then, the coefficient sequences $c = \{c_{\alpha}\}$ and $q = \{q_{\beta}\}$ of $f$ and $g$ satisfy $c = \delta_0 + c \ast q$. Hence, for each $\alpha \in \mathbb{N}^d_0$, we have $c_{\alpha} = \delta_0(\alpha) + \sum_{\beta + \gamma = \alpha} c_{\beta} q_{\gamma}$.

Now let $z \in \mathbb{B}^\ell_1$ and set $x = (|z_1|, \ldots, |z_d|) \in \mathbb{B}^\ell_1$. The nonnegativity of the coefficients implies that

$$1 + f(x)g(x) = 1 + \sum_{\alpha} \left( \sum_{\beta + \gamma = \alpha} c_{\beta} q_{\gamma} \right) x^\alpha = \sum_{\alpha} c_{\alpha} x^\alpha = f(x) < \infty.$$  

Since $f(x) > 0$, this implies that the power series for $g$ converges absolutely at $z$, and

$$|g(z)| \leq g(x) = 1 - \frac{1}{f(x)} \leq \|z\|_1$$

by the original assumption that $f(x) \leq \frac{1}{1 - \|z\|_1}$.

This implies

$$\text{Re} f(z) = \frac{1}{2} + \frac{1 - |g(z)|^2}{|1 - g(z)|^2} \geq \frac{1}{2} + \frac{1 - |g(z)|^2}{(1 + |g(z)|)^2} \geq \frac{1}{2} + \frac{1 - \|z\|_1}{1 + \|z\|_1}. \quad \square$$

**Proof of Theorem 1.2.** We now assume that $r_{\alpha} \leq 1$ for all $\alpha \in \mathbb{N}^d_0$ and that $\{r_{\alpha}\}_{\alpha \in \mathbb{N}^d_0}$ is nondecreasing. By Lemma 4.2 and the general theory of holomorphic functions of several complex variables, it follows that $f(z) = \sum_{\alpha \in \mathbb{N}^d_0} c_{\alpha} z^\alpha$ is holomorphic in $\mathbb{B}^\ell_1$. It satisfies $f(0) = 1$ and $|f(z)| \leq \frac{1}{1 - \|z\|_1}$.

Then, $g(z) = 1 - 1/f(z)$ is holomorphic in a neighborhood of 0, and hence, it has a power series representation $g(z) = \sum_{\beta \in \mathbb{N}^d_0} q_{\beta} z^\beta$. The functions $f$ and $g$ satisfy the identity

$$f(z) = 1 + (f g)(z).$$

If we write $c = \{c_{\alpha}\}_{\alpha \in \mathbb{N}^d_0}$ and $q = \{q_{\beta}\}_{\beta \in \mathbb{N}^d_0}$, then the sequence of power series coefficients of the function $f g$ is given by $c \ast q$. Thus, we see that the coefficient sequences satisfy the renewal equation $c = \delta_0 + c \ast q$. Hence, the nonnegativity of the terms of the sequence $q$ follows from Corollary 3.3.

The lower bound on $\text{Re} f(z)$ and the upper bound on $\sum_{\alpha} q_{\alpha} |z^\alpha|$ now follow from Lemma 4.3. \quad \square

We will now construct the promised example to show that for $d \geq 2$, Theorem 1.2 does not follow from Theorem 1.3.

**Example 4.4.** Let $d \in \mathbb{N}$, $d \geq 2$, and $0 < a, b < 1$ and define $\{r_{\alpha}\}_{\alpha \in \mathbb{N}^d_0}$ by $r_0 = 0, r_{\epsilon_1} = a, r_{\epsilon_2} = b$, and $r_{\alpha} = 1$ for all other $\alpha \in \mathbb{N}^d_0$. Then, $r_{\alpha} \leq 1$ for all $\alpha$ and $\{r_{\alpha}\}_{\alpha \in \mathbb{N}^d_0}$ is nondecreasing.
Inductively, define \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) by \( c_0 = 1 \) and if \( |\alpha| \neq 0 \) set \( c_\alpha = r_\alpha \sum_{\beta \in P_\alpha} c_\beta \). Then, \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) satisfies the hypothesis of Theorem 1.2.

Next compute

\[
c_{e_1} = r_{e_1} c_0 = a, \quad c_{e_2} = r_{e_2} c_0 = b, \quad c_{2e_1} = r_{2e_1} c_{e_1} = a \quad \text{and} \quad c_{e_1 + e_2} = r_{e_1 + e_2} (c_{e_1} + c_{e_2}) = a + b, \quad c_{2e_1 + e_2} = r_{2e_1 + e_2} (c_{e_1} + e_2 + c_{2e_1}) = 2a + b.
\]

The sequence \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) does not satisfy the hypothesis of Theorem 1.3 whenever \( a \neq b \). Without loss of generality, we assume \( a < b \). Then we take \( \alpha = e_i, \ i = 1, j = 2 \) in the hypothesis of Theorem 1.3:

\[
\frac{c_\alpha + e_1 c_\alpha + e_2}{c_\alpha c_{e_1 + e_2}} = \frac{c_{2e_1} c_{e_1 + e_2}}{c_{e_1} c_{2e_1 + e_2}} = \frac{a + b}{2a + b} > \frac{2}{3} = \frac{|\alpha| + 1}{|\alpha| + 2}.
\]

This shows that the hypothesis of Theorem 1.3 is not met.

We can also calculate the coefficients \( b_\alpha \) of Lemma 4.2. Indeed, whenever \( r_\alpha = 1 \), then \( b_\alpha = 0 \). Hence, we only calculate \( b_{e_1} = 1 - a \) and \( b_{e_2} = 1 - b \), and from Lemma 4.2, we obtain

\[
f(z) = \sum_\alpha c_\alpha z^\alpha = \frac{1 - (1 - a)z_1 - (1 - b)z_2}{1 - \sum_{j=1}^d z_j}.
\]

The calculation just done highlights that it may sometimes be useful to start with the sequence \( b = \{b_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) of nonnegative coefficients such that \( b_0 = 0 \) and \( \sum_{\alpha \in \mathbb{N}_0^d} b_\alpha |z^\alpha| < 1 \) for all \( z \in B_{d_1}^1 \), and then, one may want to compute the sequence \( c = \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) satisfying

\[
f(z) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha z^\alpha = \frac{1 - \sum_{\alpha} b_\alpha z^\alpha}{1 - \sum_{j=1}^d z_j}.
\]

We note that one can express \( c \) in terms of \( b \) by

\[
c_\alpha = \frac{|\alpha|!}{\alpha!} - \sum_{\beta \in \alpha} \frac{|\alpha - \beta|!}{(\alpha - \beta)!} b_\beta.
\tag{4.4}
\]

In order to prove this, write \( B(z) = \sum_{\alpha \in \mathbb{N}_0^d} b_\alpha z^\alpha \) and \( D = \{d_\alpha\}_{\alpha \in \mathbb{N}_0^d}, \ d_\alpha = \frac{|\alpha|!}{\alpha!} \). By the multinomial theorem, we have

\[
D(z) = \frac{1}{1 - \sum_{j=1}^d z_j} = \sum_{n=0}^\infty \left( \sum_{j=1}^d z_j \right)^n = \sum_{n=0}^\infty \sum_{|\alpha|=n} d_\alpha z^\alpha.
\]

The hypothesis can be written as \( f = (1 - B)D = D - DB \), and thus, the sequences of power series coefficients satisfy \( c = d - d = b \), which is exactly (4.4).
THE PROOF OF THEOREM 1.3

Suppose now that $M$ is a strongly graded monoid that satisfies the conditions TK and RC. Let $e$ be its identity. We will now use the homomorphism $\varphi : F(M_1) \to M$ that established that $M$ is a quotient of $F(M_1)$ (see (2.1)). As a matter of convenience for notation, we will use letters like $u, v, w$ to denote elements in $F(M_1)$ (words) and $\alpha, \beta, \gamma$ for elements in $M$ (think, e.g., multi-indices).

Define a function on $M$ by

$$N(\alpha) = \text{card } \varphi^{-1}(\{\alpha\}), \ \alpha \in M.$$

We will now assume that $M$ satisfies the additional hypothesis that $N$ only takes finite values. For example, that will always be the case, if $M_1$ is finite, say $\text{card } M_1 = d$. Since $\varphi$ preserves the length function, we have $|w| = |\alpha|$ for every $w \in \varphi^{-1}(\{\alpha\})$ and hence $N(\alpha) \leq d^{||\alpha||}$. It is also true for $M = \mathbb{N}^\omega$. In that case, $\varphi(w)$ equals the multi-index associated with the string $w$, that is, if $w = x_1x_2 \ldots x_n$ for $x_k \in \mathbb{N}$, then $\varphi(w)_j$ is the number of times that the integer $j$ shows up among the integers $x_1, x_2, \ldots, x_n$. Thus, if $\alpha \in \mathbb{N}^\omega$ is supported in $\{1, \ldots, d\}$, then as above $N(\alpha) \leq d^{||\alpha||}$.

For $f : F(M_1) \to \mathbb{R}$, define the "symmetrization of $f$" by $Sf : M \to \mathbb{R}$,

$$Sf(\alpha) = \sum_{w \in \varphi^{-1}(\{\alpha\})} f(w).$$

Lemma 5.1. Let $f, g : F(M_1) \to \mathbb{R}$ be locally summable. Then, $Sf$ and $Sg$ are locally summable and

$$S(f * g) = (Sf) * (Sg).$$

Proof. The fact that $\varphi$ preserves the length functions easily implies that $Sf$ is locally summable, whenever $f$ is.

Let $\alpha \in M$, then

$$(Sf * Sg)(\alpha) = \sum_{(\beta, \gamma) \in D_\alpha} (Sf)(\beta)(Sg)(\gamma)$$

$$= \sum_{(\beta, \gamma) \in D_\alpha} \sum_{\varphi(u) = \beta \varphi(v) = \gamma} f(u)g(v)$$

$$= \sum_{\varphi(u)\varphi(v) = \alpha} f(u)g(v)$$

$$= \sum_{uv \in \varphi^{-1}(\{\alpha\})} f(u)g(v).$$

Furthermore, if $w \in \varphi^{-1}(\{\alpha\})$, then $(f * g)(w) = \sum_{(u, v) \in D_w} f(u)g(v)$, and hence,

$$S(f * g)(\alpha) = \sum_{w \in \varphi^{-1}(\{\alpha\})} \sum_{(u, v) \in D_w} f(u)g(v) = \sum_{uv \in \varphi^{-1}(\{\alpha\})} f(u)g(v) = (Sf * Sg)(\alpha). \quad \Box$$

Theorem 5.2. Let $c : M \to \mathbb{R}$ be locally summable with $c(e) = 1$ and define $f : F(M_1) \to \mathbb{R}$ by

$$f(w) = \frac{c(\varphi(w))}{N(\varphi(w))}. \quad \text{Note that } f(0) = 1. \quad \text{Let } g : F(M_1) \to \mathbb{R} \text{ be the unique solution to the } F(M_1)-\text{renewal equation } f = \delta_0 + f * g.$$

Then $q = Sg$ satisfies the $M$-renewal equation $c = \delta_e + c * q$. 

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Proof. Note that the definition of \( f \) implies that for each \( \alpha \in M, f \) is constant on \( \varphi^{-1}(\{\alpha\}) \). Thus, it is easily seen that \( Sf = c \). Hence, by the linearity of \( S \) and the previous lemma,

\[
c = Sf = S\delta_0 + S(f \ast g) = \delta_e + Sf \ast Sg = \delta_e + c \ast q.
\]

From the definition of \( q = Sg \), it follows that \( q \) is nonnegative, whenever \( g \) is nonnegative. Thus, if \( \text{card} M_1 = d < \infty \), we can use Corollary 3.2 to give a sufficient condition for \( q \) to be nonnegative. We write that out in terms of the function \( c \) and obtain in the following.

**Corollary 5.3.** Let \( M \) be a strongly graded monoid that satisfies conditions TK and RC and assume that \( \text{card} M_1 = d < \infty \). Let \( c : M \to (0, \infty) \) be a function with \( c(e) = 1 \). If for all \( \alpha, \beta, \gamma \in M \) with \( \alpha, \gamma \in M_1 \), we have

\[
\frac{c(\alpha\beta)c(\beta\gamma)}{c(\beta)c(\alpha\beta\gamma)} \leq \frac{N(\alpha\beta)N(\beta\gamma)}{N(\beta)N(\alpha\beta\gamma)},
\]

then the unique solution \( q : M \to \mathbb{R} \) to the renewal equation \( c = \delta_e + c \ast q \) is nonnegative.

We note that there is a little bit of room to play in this setup. In Theorem 5.2, we used the most natural way to define \( f : \mathcal{F}(M_1) \to \mathbb{C} \) from a given \( c : M \to \mathbb{C} \) so that \( c = Sf \). A different choice of \( f \) could lead to a different version of Corollary 5.3 and of Theorem 1.3.

**Proof of Theorem 1.3.** We start by showing that \( c_\alpha \leq \frac{||\alpha||}{\alpha!} \) implies the absolute convergence of the power series \( \sum_\alpha c_\alpha z^\alpha \) in \( B_{d, 1} \). The proof is similar to the earlier argument. If \( z \in B_{d, 1} \), then we set \( x = (|z_1|, \ldots, |z_d|) \) and observe \( ||z||_1 = \sum_{j=1}^d x_j \) and

\[
\sum_\alpha |c_\alpha z^\alpha| = \sum_\alpha c_\alpha x^\alpha \leq \sum_\alpha \frac{||\alpha||}{\alpha!} x^\alpha = \frac{1}{1 - ||z||_1} < \infty.
\]

Thus, the power series for \( f \) converges with \( |f(z)| \leq \frac{1}{1 - ||z||_1} \). Hence, as in the proof of Theorem 1.2, the coefficients \( q_\alpha \) exist and are given by the solution to the renewal equation, and we will establish their nonnegativity by showing that the second condition in Theorem 1.3 reduces to the hypothesis of Corollary 5.3 when \( M = \mathbb{N}_0^d \).

In that case for a multi-index \( \alpha \), the quantity \( N(\alpha) \) equals the number of words in \( \mathcal{F}_d \) that give rise to the multi-index \( \alpha \), that is, \( N(\alpha) = \frac{||\alpha||}{\alpha!} \). The set \( M_1 \) equals \( \{e_1, \ldots, e_d\} \) and the semigroup operation is addition, and hence, by Corollary 5.3, the condition equals

\[
\frac{c_{\beta+e_i}c_{\beta+e_j}}{c_{\beta}c_{\beta+e_i+e_j}} \leq \frac{N(\beta + e_i)N(\beta + e_j)}{N(\beta)N(\beta + e_i + e_j)} \quad \text{for all} \quad 1 \leq i, j \leq d, \beta \in \mathbb{N}_0^d,
\]

and is easily seen to equal the expression in Theorem 1.3.

As in the proof of Theorem 1.2, the lower bound on \( \text{Re} f(z) \) and the upper bound on \( \sum_\alpha q_\alpha |z^\alpha| \) now follow from Lemma 4.3. \( \square \)
6  |  EXAMPLES FOR THEOREM 1.3

In this section, we will present a large class of examples that satisfy the hypothesis of Theorem 1.3, and we will see that not all of them satisfy the hypothesis of Theorem 1.2. We will start very general and then move toward making it more concrete.

Suppose that for \( i = 1, \ldots, d \), we are given sequences \( S_i = \{s_{i,n}\}_{n \in \mathbb{N}_0} \) that are Kaluza sequences and satisfy the hypothesis of Theorem 1.1 with \( M = 1 \). Then, define \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) by

\[
c_\alpha = \frac{|\alpha|!}{\alpha!} \prod_{i=1}^d s_{i,\alpha_i},
\]

(6.1)

We claim that any such \( \{c_\alpha\}_{\alpha \in \mathbb{N}_0^d} \) satisfies the hypothesis of Theorem 1.3.

If \( d = 1 \), then the hypothesis of Theorem 1.3 is seen to be equivalent to the hypothesis of Theorem 1.1 with \( M = 1 \), so there is nothing to show. Assume now that \( d \geq 2 \). It is immediately clear that \( c_0 = 1 \) and that \( 0 < c_\alpha \leq \frac{|\alpha|!}{\alpha!} \).

To help us with the computations that are required to check the remaining condition, we introduce the notation

\[
P_i := \prod_{k=1 \atop k \neq i}^d s_{k,\alpha_k}, \quad Q_{i,j} := \prod_{k=1 \atop k \neq i,j}^d s_{k,\alpha_k},
\]

where the latter is only defined for \( i \neq j \). With this, we have for any \( 1 \leq i, j \leq d, \ i \neq j \):

\[
c_\alpha + e_i = \frac{(|\alpha| + 2)!}{\alpha!(\alpha_i + 1)(\alpha_i + 2)} s_{i,\alpha_i + 1} P_i
\]

and we have shown that \( \{c_\alpha\} \) satisfies the hypothesis of Theorem 1.3.

Now let \( \mu_1, \ldots, \mu_d \) be probability measures on \([0,1]\) and define a measure on \([0,1]^d\) by \( \mu = \mu_1 \times \cdots \times \mu_d \). Then, we claim that the coefficients of the function

\[
f(z) = \int_{[0,1]^d} \frac{1}{1 - \sum_{j=1}^d t_j z_j} d\mu(t)
\]

(6.2)
are of the type as considered above. Indeed, we have

\[
\int_{[0,1]^d} \frac{1}{1 - \sum_{j=1}^d t_j z_j} \, d\mu(t) = \sum_{n=0}^{\infty} \int_{[0,1]^d} \left( \sum_{j=1}^d t_j z_j \right)^n \, d\mu(t)
\]

\[
= \sum_{n=0}^{\infty} \sum_{|\alpha| = n} \frac{|\alpha|!}{\alpha!} z^\alpha \int_{[0,1]^d} t^\alpha \, d\mu(t) \quad \text{by the Multinomial Theorem}
\]

\[
= \sum_{n=0}^{\infty} \sum_{|\alpha| = n} \frac{|\alpha|!}{\alpha!} z^\alpha \prod_{i=1}^d \int_{[0,1]} t_i^\alpha \, d\mu_i(t_i)
\]

\[
= \sum_{\alpha \in \mathbb{N}_0^d} \frac{|\alpha|!}{\alpha!} z^\alpha \prod_{i=1}^d s_{i,\alpha_i}, \quad \text{where } s_{i,k} = \int_{[0,1]} t^k \, d\mu_i(t)
\]

\[
= \sum_{\alpha \in \mathbb{N}_0^d} c_{\alpha} z^\alpha.
\]

It remains to note that \(\{\int_{[0,1]} t^n \, d\sigma(t)\}_{n \in \mathbb{N}_0}\) is a Kaluza sequence for each probability measure \(\sigma\) on \([0,1]\). In fact, this final step is well known to be true. Namely, it follows easily from Hölder’s inequality that \(F(s) := \int_0^1 t^s \, d\sigma(t)\) defines a log-convex function on \([0,\infty)\) (also see [3], Lemma 5.1).

By taking \(d = 2\) and both \(\mu_1\) and \(\mu_2\) to be Lebesgue measure on \([0,1]\), we can show that the resulting \(\{c_{\alpha}\}\) do not satisfy the conditions of Theorem 1.2, namely, that the sequence \(r_{\alpha}\) is not nondecreasing. Let us write \((m, n)\) for arbitrary \(\alpha \in \mathbb{N}_0^2\). We claim that \(r_{(0,2)} > r_{(1,2)}\). To compute these quantities, we note that \(s_{i,n} = \frac{1}{n+1}\) for \(i = 1\) and \(2\), and subsequently that \(c_{(m,n)} = \frac{(m+n)!}{(m+1)!(n+1)!}\). We use this to calculate \(c_{(0,1)} = c_{(1,1)} = c_{(1,2)} = 1/2\) and \(c_{(2,0)} = 1/3\). From there, all we need to do employ the definition of \(r_{(m,n)}\) to see:

\[
r_{(0,2)} = \frac{c_{(0,2)}}{c_{(0,1)}} = 2/3 > 3/5 = \frac{c_{(1,2)}}{c_{(0,2)} + c_{(1,1)}} = r_{(1,2)}.
\]

Note that there are probability measures on \([0,1]^2\) such that the function \(f\) in (6.2) does not satisfy the conclusion of Theorems 1.2 and 1.3. Let \(0 < a < 1\), and for \(w \in \mathbb{B}_2\), write \(\delta_w\) for the unit point mass at \(w\). Then, \(\mu = \frac{1}{2} (\delta_{(a,0)} + \delta_{(0,a)})\) is a probability measure, and

\[
f(z, w) = \int_{[0,1]^2} \frac{1}{1 - sz - tw} \, d\mu(s, t) = \frac{1}{2} \left( \frac{1}{1 - az} + \frac{1}{1 - aw} \right) = \frac{1 - \frac{a}{2}(z + w)}{(1 - az)(1 - aw)}.
\]

Then,

\[
q(z, w) = 1 - \frac{1}{f(z, w)} = \frac{1 - \frac{a}{2}(z + w) - (1 - az)(1 - aw)}{1 - \frac{a}{2}(z + w)}
\]
\[
\frac{a(z + w) - a^2zw}{1 - \frac{a}{2}(z + w)} = \sum_{n=0}^{\infty} \left( \frac{a}{2} \right)^{n+1} (z + w)^{n+1} - \sum_{n=0}^{\infty} a^2zw \left( \frac{a}{2} \right)^n (z + w)^n.
\]

Thus, we determine the coefficient of \(z^2w\) to be \(q_{(2,1)} = \frac{a^3}{8} - \frac{a^3}{2} < 0\).

### 7 APPLICATIONS TO REPRODUCING KERNELS

For \(d \in \mathbb{N}\), let \(\mathcal{B}_d\) denote the \(\ell_2\)-ball of \(\mathbb{C}^d\), \(\mathcal{B}_d = \{z \in \mathbb{C}^d : \sum_{j=1}^{d} |z_j|^2 < 1\}\). A Hilbert function space over \(\mathcal{B}_d\) is a Hilbert space of functions \(\mathcal{B}_d \to \mathbb{C}\) such that point evaluations for all \(z \in \mathcal{B}_d\) are continuous. Each Hilbert function space has a reproducing kernel \(k = \{k_w(z)\}\) that satisfies \(k_w \in \mathcal{H}\) for each \(w \in \mathcal{B}_d\) and \(f(w) = \langle f, k_w \rangle\) for each \(f \in \mathcal{H}\) and \(w \in \mathcal{B}_d\).

**Definition 7.1.** Let \(k\) be a reproducing kernel on \(\mathcal{B}_d\).

(a) \(k\) is called a normalized complete Nevanlinna–Pick kernel, if there is an auxiliary Hilbert space \(\mathcal{K}\) and a function \(Q : \mathcal{B}_d \to \mathcal{K}\) such that \(Q(0) = 0\) and \(k_w(z) = \frac{1}{1 - \langle Q(z), Q(w) \rangle}\) for all \(z, w \in \mathcal{B}_d\).

(b) \(k\) is called a normalized de Branges–Rovnyak kernel, if there is an auxiliary Hilbert space \(\mathcal{K}\) and a function \(B : \mathcal{B}_d \to \mathcal{K}\) such that \(B(0) = 0\) and \(k_w(z) = \frac{1 - \langle B(z), B(w) \rangle}{1 - \langle z, w \rangle}\) for all \(z, w \in \mathcal{B}_d\).

A Hilbert function space is called a normalized complete Nevanlinna–Pick space (normalized de Branges–Rovnyak space) if its reproducing kernel is a normalized complete Nevanlinna–Pick kernel (normalized de Branges–Rovnyak kernel). Both of these types of kernels and spaces have been studied extensively in the literature. For de Branges–Rovnyak spaces, the results are most complete if \(d = 1\) and \(\mathcal{K}\) is one dimensional, see, for example, [8–10, 22]. If \(d = 1\) and \(\mathcal{K}\) is finite dimensional, then [4] and [20] contain some further results. Multivariable versions of de Branges spaces have been studied for their use in the so-called transfer function realizations of analytic functions in \(\mathcal{B}_d\), see, for example, [5, 6], but they have also been studied in their own right, [14, 15]. For complete Nevanlinna–Pick kernels, many results are available for all \(d \in \mathbb{N}\) and all separable auxiliary Hilbert spaces \(\mathcal{K}\), we refer to [1] and [2] for the basics. However, these types of reproducing kernels are the subject of ongoing research, and there are many recent publications on the topic. The intuition is that normalized complete Nevanlinna–Pick spaces share many properties with the Hardy space \(H^2\) of the unit disc and its multiplier algebra \(H^\infty\). For example, there is a Pick interpolation theorem, a commutant lifting theorem, a Beurling theorem, a representation of the contractive multipliers as transfer functions of unitary colligations, and a theorem saying that every function in a normalized complete Nevanlinna–Pick space can be written as a ratio of multipliers of the space. It is thus of interest to determine whether a given reproducing kernel is a complete Nevanlinna–Pick kernel or whether a given Hilbert function space is a complete Nevanlinna–Pick space.

If \(\mathcal{H} \subseteq \text{Hol}(\mathcal{B}_d)\) is a Hilbert function space such that the monomials \(z^\alpha\) are contained in \(\mathcal{H}\) and are mutually orthogonal, then for \(f = \sum_{\alpha \in \mathbb{N}_0^d} \hat{f}(\alpha)z^\alpha\), we have

\[
\|f\|^2 = \sum_{\alpha \in \mathbb{N}_0^d} |\hat{f}(\alpha)|^2 \|z^\alpha\|^2.
\]
Since $z^\alpha \in H$ and since point evaluations are bounded, we must have $\|z^\alpha\| \neq 0$ and we see that the reproducing kernel is of the form

$$k_w(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \overline{w}^\alpha z^\alpha, \quad c_\alpha = 1/\|z^\alpha\|^2.$$  

Thus, Theorems 1.2 and 1.3 give sufficient conditions for such a kernel to be a normalized complete Nevanlinna–Pick kernel, and Lemma 4.2 and the discussion at the end of Section 4 show that the conditions of Theorem 1.2 also are sufficient for $k_w(z)$ to be a normalized de Branges–Rovnyak kernel.

We give an example to illustrate what the norms may look like when the kernel is a normalized complete Nevanlinna–Pick kernel of the form

$$k_w(z) = \int_{[0,1]^d} \frac{1}{1 - \sum_{j=1}^d t_j z_j \overline{w}_j} d\mu(t),$$

where $\mu = \mu_1 \times \ldots \mu_d$ for probability measures on $[0,1]^d$. In [3], conditions on a probability measure $\mu$ on $[0,1]$ were given that implied that the norm on the space with reproducing kernel

$$k_w(z) = \int_{[0,1]} \frac{1}{1 - t \langle z, w \rangle} d\mu(t)$$

is equivalent to a weighted Besov-norm of the type $\|f\|^2 = |f(0)|^2 + \int_{\mathbb{B}_d} |R^f|^2 \omega dV$. Here, $R$ is the radial derivative operator, $\gamma$ is a possibly fractional order, $dV$ is Lebesgue measure on $\mathbb{B}_d$, and $\omega$ is a radial weight on $\mathbb{B}_d$ ([3], Lemma 5.1 and Theorem 5.2). Results and methods from [3] can be used to substantially generalize the following example, if one is willing to settle for equivalence rather than equality of norms.

We take $d = 2$ and $\mu_1 = \mu_2 =$ Lebesgue measure on $[0,1]$. Then, in Section 6, we calculated

$$c_{(m,n)} = \frac{(m+n)!}{(m+1)!(n+1)!}.$$ 

If $F$ is a function of the complex variables $w = (w_1, w_2)$, then we write

$$D_{w_j}F = \frac{\partial}{\partial w_j} w_j F.$$

If $f \in \text{Hol}(\mathbb{B}_2)$, then define $Tf : \mathbb{D} \times \mathbb{B}_2 \times \mathbb{D}^2 \to \mathbb{C}$ by $Tf(\lambda, z, w) = f(\lambda z_1 w_1, \lambda z_2 w_2)$ and

$$\|f\|^2 = \int_{\mathbb{D}} \int_{\partial \mathbb{B}_2} \int_{\mathbb{D}^2} |D_\lambda D_{w_1} D_{w_2} T f(\lambda, z, w)|^2 \frac{dA(w_1)}{\pi} \frac{dA(w_2)}{\pi} d\sigma(z) d\lambda.$$  

Here, $dA$ denotes Lebesgue measure on $\mathbb{D}$ and $d\sigma$ is the rotationally invariant probability measure on $\partial \mathbb{B}_2$. It is easy to see that the monomials are orthogonal to one another in the inner product that comes from this norm. If $f(z) = z^{(m,n)}$, then $Tf(\lambda, z, w) = \lambda^{m+n} z^{(m,n)} w^{(m,n)}$ and

$$D_\lambda D_{w_1} D_{w_2} T f = (m + n + 1)(m + 1)(n + 1) \lambda^{m+n} z^{(m,n)} w^{(m,n)}.$$ 

Now note that

$$\int_{\mathbb{D}} |\lambda^{n+m}|^2 \frac{dA(\lambda)}{\pi} = \frac{1}{m + n + 1},$$
\[ \int_{\partial \mathbb{B}_2} |z^{(m,n)}|^2 d\sigma(z) = \frac{m!n!}{(m+n+1)!}, \quad \text{and} \]
\[ \int_{\mathbb{D}} \int_{\mathbb{D}} |w^{(m,n)}|^2 \frac{dA(w_1)}{\pi} \frac{dA(w_2)}{\pi} = \frac{1}{(m+1)(n+1)}. \]

Then,
\[ \|z^{(m,n)}\| = \frac{(m+n+1)^2(m+1)^2(n+1)^2}{m+n+1} \frac{m!n!}{(m+n+1)!} \frac{1}{(m+1)(n+1)} \]
\[ = \frac{(m+1)!(n+1)!}{(m+n)!} \]
\[ = \frac{1}{c_{(m,n)}}. \]

Thus, if \( \mathcal{H} \) is the space of all holomorphic functions \( f \) on \( \mathbb{B}_2 \), where the norm is given by (7.1), then the reproducing kernel is
\[ k_w(z) = \int_0^1 \int_0^1 \frac{1}{1 - tz_1 \bar{w}_1 - sz_2 \bar{w}_2} dt ds, \]
and this is a complete Nevanlinna–Pick kernel.

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