On fully real eigenconfigurations of tensors

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Abstract. We construct generic real symmetric tensors with only real eigenvectors or, equivalently, real homogeneous polynomials with the maximum possible finite number of critical points on the sphere.

Introduction

In 2005 Lim [7] and Qi [11] independently initiated the spectral theory of high order tensors. They introduced several generalizations of the classical concept of an eigenvector of a matrix. Our work concerns $l^2$-eigenvectors of Lim or $Z$-eigenvectors of Qi. Moreover, we only consider real symmetric tensors, a direct generalization of symmetric matrices.

An $n$-dimensional degree $d$ tensor (hereinafter $n^d$-tensor) $A = (a_{i_1...i_d})$ is said to be symmetric if $a_{i_{σ_1}...i_{σ_d}} = a_{i_1...i_d}$ for any permutation $σ ∈ S_d$. A non-zero vector $x ∈ \mathbb{C}^n$ is called an eigenvector of $A$ if

$$Ax_{d-1} ∧ x = 0, \quad Ax_{d-1} := d \left( \sum_{i_2,...,i_d=1}^{n} a_{i_2...i_d} x_{i_2} ... x_{i_d}, ... , \sum_{i_2,...,i_d=1}^{n} a_{ni_2...i_d} x_{i_2} ... x_{i_d} \right).$$

For $d = 2$ one recovers the definition of an eigenvector of an $n \times n$ matrix. The point $[x] ∈ \mathbb{C}P^{n-1}$ defined by an eigenvector $x ∈ \mathbb{C}^n \setminus \{0\}$ is called an eigenpoint and the set of all eigenpoints is called an eigenconfiguration. Cartwright and Sturmfels [2] proved that the number of eigenpoints of a generic symmetric $n^d$-tensor is equal to

$$m_{d,n} := \frac{(d - 1)^n - 1}{d - 2} = (d - 1)^{n-1} + \cdots + (d - 1) + 1$$

But unlike to the case of symmetric matrices ($d = 2$) not all eigenvectors of a general symmetric tensor of degree $d ≥ 3$ are real. In fact, “most” of symmetric tensors have eigenpoints in $\mathbb{C}P^{n-1} \setminus \mathbb{R}P^{n-1}$. Abo, Seigal and Sturmfels [11, Conj. 6.5] conjectured that for any $n ≥ 2, d ≥ 1$ there exists a generic symmetric $n^d$-tensor having only real eigenvectors and proved it for $n = 3, d ≥ 1$ and for $n = d = 4$. The cases $n = 2, d ≥ 1$ and $n ≥ 2, d = 2$ are elementary, the case of general $n, d$ was unknown (see for example [10]). In the present work we cover the case of arbitrary $n$ and $d$ and prove the following theorem.

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1As it often happens in real algebraic geometry problems the objects of “maximal complexity” are rare and “numerically invisible”.

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Theorem 1. For any \( n \geq 2 \) and \( d \geq 1 \) there exists a generic symmetric \( n^d \)-tensor whose all \( m_{d,n} \) eigenpoints are real. Equivalently (see section 1.2), there exists a homogeneous polynomial of degree \( d \) in \( n \) real variables whose restriction to the sphere \( S^{n-1} \subset \mathbb{R}^n \) has the maximum possible finite number of critical points that is equal to \( 2m_{d,n} \). Moreover, such a symmetric tensor (homogeneous polynomial) exists among traceless tensors (harmonic polynomials).

In [5] Gichev constructed for any \( d \geq 1 \) a homogeneous harmonic polynomial of degree \( d \) in 3 variables having \( 2m_{d,3} = 2(d^2 - d + 1) \) critical points on the sphere \( S^2 \). The idea of the proof of Theorem 1 is based on the construction of Gichev.

We can also realize any admissible number of real eigenpoints by an appropriate symmetric tensor.

Theorem 2. Let \( n \geq 2 \).

(i) If \( d \geq 2 \) is even then for every \( i = 0, \ldots, \frac{1}{2}(m_{d,n} - n) \) there exists a generic symmetric \( n^d \)-tensor with \( n + 2i \) real eigenpoints.

(ii) If \( d \geq 1 \) is odd then for every \( i = 0, \ldots, \frac{1}{2}(m_{d,n} - 1) \) there exists a generic symmetric \( n^d \)-tensor with \( 1 + 2i \) real eigenpoints.

Furthermore, these are the only possibilities for the number of eigenpoints of a generic symmetric tensor.

An equivalent statement holds for the possible number of critical points of a homogeneous polynomial on the sphere.

1 Preliminaries

1.1 Critical points and typical bifurcations of smooth functions

Let \( f : M \to \mathbb{R} \) be a smooth function on a smooth \( n \)-dimensional manifold \( M \).

- The differential \( df \) of \( f \) at a point \( x \in M \) is a linear form on \( T_x M \), it sends \( v = \gamma'(0) \in T_x M \) to \( df(v) := (f \circ \gamma)'(0) \), where \( \gamma = \gamma(t) \subset M \) is a smooth curve passing through \( x = \gamma(0) \) with the tangent vector \( v \) at \( t = 0 \).

- \( x_* \in M \) is a critical point of \( f \) if \( dx_* f = 0 \).

- The second differential \( df^2 \) of \( f \) at a critical point \( x_* \in M \) is a quadratic form on \( T_{x_*} M \), it sends \( v = \gamma'(0) \in T_{x_*} M \) to \( df^2 f(v) := (f \circ \gamma)''(0) \). In local coordinates \( x = (x_1, \ldots, x_n) \) near \( x_* \) one has

\[
    df^2 f(v) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (x_*) v_i v_j
\]

The matrix \( \left( \frac{\partial^2 f}{\partial x_i \partial x_j} (x_*) \right) \) is called the Hessian matrix.
A critical point \( x_\ast \in M \) is said to be (non-)degenerate if \( d^2_{x_\ast} f \) is a (non-)degenerate quadratic form. In some local coordinates \( x = (x_1, \ldots, x_n) \) near a non-degenerate critical point \( x_\ast \) the function \( f \) takes the form
\[
f(x) = f(x_\ast) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2
\]
where the number \( k \) is called the index of \( f \) at \( x_\ast \) and is equal to the dimension of a maximal subspace of \( T_{x_\ast} M \) on which \( d^2_{x_\ast} f \) is negative definite.

The third differential \( d^3_{x_\ast} f \) of \( f \) at a degenerate critical point \( x_\ast \in M \) is a cubic form on \( \text{Ker} \, d^2_{x_\ast} f \subset T_{x_\ast} M \), it sends \( v = \gamma'(0) \in \text{Ker} \, d^2_{x_\ast} f \) to \( d^3_{x_\ast} f(v) := (f \circ \gamma)'''(0) \).

In local coordinates \( x = (x_1, \ldots, x_n) \) near \( x_\ast \) one has
\[
d^3_{x_\ast} f(v) = \sum_{i,j,k=1}^n \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(x_\ast)v_i v_j v_k
\]

A degenerate critical point \( x_\ast \in M \) is called a birth-death singularity if the corank of \( d^2_{x_\ast} f \) is one and \( d^3_{x_\ast} f \) is non-zero. In some local coordinates \( x = (x_1, \ldots, x_n) \) near such \( x_\ast \) the function \( f \) takes the form
\[
f(x) = f(x_\ast) - x_1^3 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2
\]

Both non-degenerate and birth-death singularities are isolated. A smooth function is called Morse if it has only non-degenerate critical points. The set of Morse functions is open and dense in the space \( C^\infty M \) of smooth functions on \( M \) endowed with the \( C^\infty \)-topology. Its complement is described as follows.

**Theorem 3** (Cerf [3]). The set \( \Delta \) of non-Morse functions on \( M \) is stratified and its smooth stratum \( \Delta^{(1)} \) of codimension one consists of functions having only one degenerate critical point which is a birth-death singularity. A generic path \( f_t \) connecting two Morse functions \( f_0 \) and \( f_1 \) crosses \( \Delta^{(1)} \) transversely at finitely many times \( t_1, \ldots, t_l \in (0, 1) \) and the number of critical points of \( f_t \) changes by two when passing each \( t_i \).

For a Morse function \( f : M \to \mathbb{R} \) on a compact \( n \)-dimensional manifold \( M \) denote by \( \mu_k(f) \) the number of critical points of \( f \) of index \( k = 0, \ldots, n \). The numbers \( \mu_k(f) \) satisfy the Morse inequalities:
\[
\mu_k(f) - \mu_{k-1}(f) + \cdots + (-1)^k \mu_0(f) \geq b_k - b_{k-1} + \cdots + (-1)^k b_0, \quad k = 0, \ldots, n
\]
where \( b_k = b_k(M) \) denotes the \( k \)-th Betti number of \( M \).

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2In [3] Cerf describes the complement of the set of excellent Morse functions, which are Morse and have at most one critical point on each level set \( f^{-1}(c), c \in \mathbb{R} \). For our purposes it is enough to deal just with Morse functions.
1.2 From symmetric tensors to homogeneous polynomials

A generic symmetric $n \times n$ matrix has $n$ real simple eigenvalues and $n$ corresponding eigenpoints. Moreover, in the space of all symmetric $n \times n$ matrices those which have repeated eigenvalues form a real algebraic subvariety, called the discriminant, and generic matrices belong, by definition, to its complement. The codimension of the discriminant is two and this justifies the fact that the number of real eigenpairs is the same for all generic matrices.

Let $A = (a_{i_1...i_d})$ be an $n$-dimensional symmetric tensor of order $d$. A complex number $\lambda \in \mathbb{C}$ is called an eigenvalue corresponding to an eigenvector $x \in \mathbb{C}^n$ if $Ax^{d-1} = \lambda x$. In this case the pair $(x, \lambda) \in \mathbb{C}^n \setminus \{0\} \times \mathbb{C}$ is called an eigenpair of $A$. Two eigenpairs $(x, \lambda)$ and $(x', \lambda')$ of $A$ are said to be equivalent if they define the same eigenpoint $[x] = [x'] \in \mathbb{CP}^{n-1}$. Theorem 1.2 in [2] asserts that the number of eigenpoints (equivalence classes of eigenpairs) of a generic symmetric $n^d$-tensor is equal to $m_{d,n} = ((d-1)^n - 1)/(d-2) = (d-1)^{n-1} + \cdots + (d-1) + 1$. Non-generic tensors are cut out by an algebraic hypersurface, called the eigendiscriminant [1], and the number of eigenpoints of a non-generic tensor is not equal to the expected $m_{d,n}$. On each connected component of the complement of the eigendiscriminant the number of real eigenpoints (equivalence classes of real eigenpairs) is constant.

There is an obvious one-to-one correspondence between the set $\mathcal{P}_{d,n}$ of real homogeneous polynomials of degree $d$ in $n$ variables and the set of real symmetric $n^d$-tensors:

$$f_A = \sum_{i_1,...,i_d=1}^{n} a_{i_1...i_d} x_{i_1} \cdots x_{i_d} \leftrightarrow A = (a_{i_1...i_d})$$

The critical points of the restriction $f_A|_{S^{n-1}}$ of a homogeneous polynomial $f_A$ to the unit sphere are precisely unit real eigenvectors of the corresponding symmetric tensor $A$. Indeed, by the method of Lagrange multipliers, if $x \in S^{n-1}$ then

$$d_x f_A |_{S^{n-1}} = 0 \iff d_x f_A = \lambda d_x \left( \|x\|^2 - 1 \right) \iff Ax^{d-1} = \lambda x$$

Note that the Lagrange multiplier $\lambda$ is the eigenvalue corresponding to $x$. In the terminology of Lim [7] and Qi [11] unit real eigenvectors are $l^2$-eigenvectors and Z-eigenvectors respectively. Theorem [2] Thm. 1.2] gives an upper bound on the number of critical points of the restriction of a homogeneous polynomial to the sphere.

**Lemma 1.** If a polynomial $f \in \mathcal{P}_{d,n}$ defines a Morse function $f|_{S^{n-1}}$ on the sphere then the number of critical points of $f|_{S^{n-1}}$ is bounded by $2m_{d,n} = 2((d-1)^n - 1)/(d-2)$.

**Proof.** If $f_A|_{S^{n-1}}$ is a Morse function and the tensor $A$ is generic then $A$ has $m_{d,n}$ eigenpoints in $\mathbb{CP}^{n-1}$ which implies that the number of unit real eigenvectors is bounded by $2((d-1)^n - 1)/(d-2)$. If $f_A|_{S^{n-1}}$ is a Morse function but $A$ is not generic, then it is possible to find a perturbation $\tilde{A}$ of $A$ such that $f_{\tilde{A}}|_{S^{n-1}}$ and $f_{\tilde{A}}|_{S^{n-1}}$ have the same number of critical points and the tensor $\tilde{A}$ is generic. 

\[\square\]
1.3 Spherical harmonics

Consider the space
$$H_{d,n} = \left\{ f |_{S^{n-1}} : f \in \mathcal{P}_{d,n}, \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \right\}$$

of restrictions to the sphere $S^{n-1}$ of homogeneous harmonic polynomials of degree $d$.

Note that a polynomial $f_A \in \mathcal{P}_{d,n}$ is harmonic if and only if the symmetric tensor $A$ is traceless, i.e.
$$\sum_{i_1, \ldots, i_d = 1}^n a_{i_1i_2\ldots i_d} = 0 \quad \forall i_3, \ldots, i_d = 1, \ldots, n$$

It is well-known that $H_{d,n}$ is the eigenspace of the spherical Laplace operator $\Delta_{S^{n-1}}$ corresponding to the eigenvalue $-d(d + n - 2)$. Functions in $H_{d,n}$ are called spherical harmonics of degree $d$ and the dimension of $H_{d,n}$ is equal to
$$\dim H_{d,n} = \binom{n + d - 1}{d} - \binom{n + d - 3}{d - 2} \quad \text{if} \quad d \geq 2 \quad \text{and} \quad \dim H_{0,n} = 1, \quad \dim H_{1,n} = n$$

For any point $y \in S^{n-1}$ and any $d$ there exists a spherical harmonic $Z_{d}^y \in H_{d,n}$, called zonal, which is invariant under rotations preserving $y$:
$$Z_{d}^y(Rx) = Z_{d}^y(x), \quad R \in \text{SO}(n), \ R y = y$$

The function $Z_{d}^y(x)$ is determined uniquely up to a constant and is proportional to $G_{d,n}(\langle x, y \rangle)$ where $\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n$ is the standard scalar product in $\mathbb{R}^n$ and $G_{d,n}$ is the Gegenbauer polynomial of degree $d$ and parameter $\frac{n-2}{2}$. The polynomials $\{G_{d,n}\}_{d \geq 0}$ are defined by the recurrence relation
$$\begin{align*}
G_{0,n}(x) &= 1, \\
G_{1,n}(x) &= (n - 2)x, \\
G_{d,n}(x) &= \frac{1}{d} \left[ 2x(d + \frac{n}{2} - 2)G_{d-1,n}(x) - (d + n - 4)G_{d-2,n}(x) \right]
\end{align*}$$

and form an orthogonal family on the interval $[-1, 1]$ with respect to the measure $(1 - z^2)^{\frac{n-3}{2}} dz$:
$$\int_{-1}^{1} G_{d_1,n}(z) G_{d_2,n}(z) (1 - z^2)^{\frac{n-3}{2}} dz = 0, \quad d_1 \neq d_2$$

Therefore by [4, Prop. I.1.1] $G_{d,n}$ has $d$ simple real roots in $(-1, 1)$ and hence its derivative $G_{d,n}'$ has $d-1$ roots in $(-1, 1)$ which we denote by $\alpha_{d,1}, \ldots, \alpha_{d,d-1}$. The following lemma characterizes the critical points of a zonal spherical harmonic.

\[3\]According to the usual definition [9, page 143] a zonal harmonic $Z_{d}^y$ is determined uniquely by some normalization condition. Since a normalization is unimportant for our purposes we abuse the terminology and call zonal any spherical harmonic with the mentioned invariance property.
Lemma 2. The set of critical points of $Z_d^n$ consists of $y, -y$ and $d - 1$ affine hyperplane sections of the sphere $\{x \in S^{n-1} : \langle x, y \rangle = \alpha_d\}$, $i = 1, \ldots, d - 1$. The critical points $y$ and $-y$ are non-degenerate.

Proof. A point $x \in S^{n-1}$ is critical for $G_{d,n}(\langle x, y \rangle)$ if and only if $G_{d,n}(\langle x, y \rangle)y$ is proportional to $x$. This is possible either if $\langle x, y \rangle$ is a root of $G_{d,n}$ or $x = \pm y$. To prove the non-degeneracy of $x = \pm y$ we assume without loss of generality that $y = (0, \ldots, 0, 1) \in S^{n-1}$ and then in local coordinates

$$(x_1, \ldots, x_{n-1}) \mapsto \left(x_1, \ldots, x_{n-1}, \pm \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}\right) \in S^{n-1}$$

around $x = \pm y$ our function $G_{d,n}(\langle x, y \rangle)$ takes the form $G_{d,n}\left(\pm \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}\right)$. One can easily verify that its Hessian matrix at $(x_1, \ldots, x_{n-1}) = (0, \ldots, 0)$ is non-degenerate.

The inclusion map

$$i : \mathcal{P}_{d,n} \hookrightarrow \mathcal{P}_{d,n+1}$$

$$f \mapsto i(f)(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n)$$

induces the linear inclusion

$$\hat{\mathcal{H}}_{d,n} \hookrightarrow \hat{\mathcal{H}}_{d,n+1}$$

$$h = f|_{S^{n-1}} \mapsto \hat{h} = i(f)|_{S^n}$$

The critical points of $\hat{h}$ are described as follows.

Lemma 3. Assume that $d, n \geq 2$ and $h \in \mathcal{H}_{d,n}$.

(i) If the zero locus $\{h = 0\} \subset S^{n-1}$ is regular then the set of critical points of

$h \in \mathcal{H}_{d,n+1}$ consists of $\pm(0, \ldots, 0, 1) \in S^n$ and the points $(x_1, \ldots, x_n, 0)$, where

$(x_1, \ldots, x_n) \in S^{n-1}$ is critical for $h$. Moreover, for $d \geq 3$ the points $\pm(0, \ldots, 0, 1) \in S^n$ are always degenerate.

(ii) If $\{h = 0\}$ is singular then, additionally, for each singular point $(x_1, \ldots, x_n) \in \{h = 0\}$ the great circle $\{(tx_1, \ldots, tx_n, \pm \sqrt{1 - t^2}) : 0 \leq t \leq 1\} \subset S^n$ consists of critical points of $\hat{h}$.

Proof. If $h = f|_{S^{n-1}}$ for some harmonic polynomial $f \in \mathcal{P}_{d,n}$ the critical points of

$\hat{h} = i(f)|_{S^n} \in \mathcal{H}_{d,n+1}$ are characterized by

$$\frac{\partial f}{\partial x_1} = \lambda x_1, \quad \ldots \quad \frac{\partial f}{\partial x_n} = \lambda x_n, \quad \frac{\partial f}{\partial x_{n+1}} = 0 = \lambda x_{n+1} \quad (3)$$

Obviously $(x_1, \ldots, x_n, 0) \in S^n$ is a critical point of $\hat{h}$ if $(x_1, \ldots, x_n) \in S^{n-1}$ is critical for $h$. Now if $\lambda = 0$ and $\{h = 0\} \subset S^{n-1}$ is regular then $x_1 = \cdots = x_n = 0$ and $x_{n+1} = \pm 1$.

If, instead, $\{h = 0\}$ is singular and $(x_1, \ldots, x_n) \in \{h = 0\}$ is a solution of $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$ then due to the homogeneity of $f$ any point $(tx_1, \ldots, tx_n, \pm \sqrt{1 - t^2})$, $0 \leq t \leq 1$ is a solution of the system $(3)$ with $\lambda = 0$. 

\[\Box\]
2 Proof of the main results

Denote by \( Z_{d,n} \) a zonal spherical harmonic \( Z_d^m(x) = G_{d,n}(\langle x, y \rangle) = G_{d,n}(x_n) \in \mathcal{H}_{d,n} \) corresponding to the point \( y = (0, \ldots, 0, 1) \in S^{n-1} \) and let \( M_{d,n} \in \mathcal{H}_{d,n} \) be a Morse spherical harmonic of degree \( d \) with the maximum possible number of critical points. By Lemma 1 this number is bounded by \( 2m_{d,n} = 2((d-1)^n - 1)/(d-2) \). In dimension \( n = 2 \), any spherical harmonic \( h \in \mathcal{H}_{d,2} \) is just a trigonometric polynomial

\[
h = a \cos(d\theta) + b \sin(d\theta), \quad a, b \in \mathbb{R}, \quad \theta \in [0, 2\pi)
\]

and therefore it has \( 2m_{d,2} = 2d \) critical points. For \( n \geq 3, d \geq 3 \) the number of critical points of a general spherical harmonic \( h \in \mathcal{H}_{d,n} \) is not anymore a constant and depends significantly on the choice of \( h \). However, there exist spherical harmonics in \( \mathcal{H}_{d,n} \) with \( 2m_{d,n} \) critical points.

**Proposition 1.** For any \( d, n \geq 2 \) and a sufficiently small \( \varepsilon > 0 \) the spherical harmonic \( Z_{d,n+1} + \varepsilon \tilde{M}_{d,n} \in \mathcal{H}_{d,n+1} \) is a Morse function on \( S^n \) with \( 2m_{d,n+1} \) critical points.

**Proof.** As observed above one can take \( M_{d,2} = a \cos(d\theta) + b \sin(d\theta) \) which is obviously a Morse function having \( 2d \) critical points on \( S^1 \). Suppose that for some \( n \geq 2, M_{d,n} \) has \( 2m_{d,n} \) critical points on \( S^{n-1} \). By Lemmas 2 and 3 we have that the points \( \pm (0, \ldots, 0, 1) \) are critical for both \( Z_{d,n+1} \) and \( \tilde{M}_{d,n} \) and hence also for the perturbation \( Z_{d,n+1} + \varepsilon \tilde{M}_{d,n} \). Since the points \( \pm (0, \ldots, 0, 1) \) are non-degenerate for \( Z_{d,n+1} \) they remain non-degenerate for the perturbation for small enough \( \varepsilon > 0 \).

We prove that each of the \( d-1 \) critical circles \( \{ x \in S^n : \langle x, y \rangle = \alpha_{d,i} \}, i = 1, \ldots, d-1 \) of \( Z_{d,n+1} \) breaks into \( 2m_{d,n} \) non-degenerate critical points when \( Z_{d,n+1} \) is slightly perturbed by \( \tilde{M}_{d,n} \). The idea is shown in figure 2, where the red/purple color represents positive/negative values of functions. In spherical coordinates

\[
x_1 = \sin \theta_n \cdot \tilde{x}_1 = \sin \theta_n \sin \theta_{n-1} \cdots \sin \theta_2 \sin \theta_1 \\
x_2 = \sin \theta_n \cdot \tilde{x}_2 = \sin \theta_n \sin \theta_{n-1} \cdots \sin \theta_2 \cos \theta_1 \\
x_3 = \sin \theta_n \cdot \tilde{x}_3 = \sin \theta_n \sin \theta_{n-1} \cdots \cos \theta_2 \\
\vdots \\
x_n = \sin \theta_n \cdot \tilde{x}_n = \sin \theta_n \cos \theta_n \\
x_{n+1} = \cos \theta_n
\]

on \( S^n \), where \( \langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle \in S^{n-1} = \{ x \in S^n : x_{n+1} = 0 \} \), we have

\[
Z_{d,n+1}(x_1, \ldots, x_{n+1}) = G_{d,n+1}(x_{n+1}) = G_{d,n+1}(\cos \theta_n) \\
\tilde{M}_{d,n}(x_1, \ldots, x_{n+1}) = \sin^d \theta_n \tilde{M}_{d,n}(\tilde{x}_1, \ldots, \tilde{x}_n)
\]
and hence the critical points of $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$ are described by the equations

$$
\begin{align*}
\varepsilon \sin^d \theta_n \frac{\partial}{\partial \theta_1} M_{d,n}(\bar{x}_1, \ldots, \bar{x}_n) &= 0 \\
&\vdots \\
\varepsilon \sin^d \theta_n \frac{\partial}{\partial \theta_{n-1}} M_{d,n}(\bar{x}_1, \ldots, \bar{x}_n) &= 0 \\
\frac{\partial}{\partial \theta_n} \left[ G_{d,n+1}(\cos \theta_n) + \varepsilon \sin^d \theta_n M_{d,n}(\bar{x}_1, \ldots, \bar{x}_n) \right] &= 0
\end{align*}
$$

(4)

Since the $d - 1$ zeroes of $G_{d,n+1}'$ are non-degenerate, then for a fixed $(\bar{x}_1, \ldots, \bar{x}_n) \in S^{n-1}$ the equation (4) has $d - 1$ non-degenerate solutions provided that $\varepsilon$ is small enough. It follows that each critical point $(\bar{x}_1, \ldots, \bar{x}_n) \in S^{n-1}$ of $M_{d,n}$ gives rise to $d - 1$ critical points of $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$. In spherical coordinates the Hessian matrix of $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$ computed at a critical point $\theta = (\theta_1, \ldots, \theta_{n-1}, \theta_n)$ takes the block-diagonal form:

$$
\begin{pmatrix}
\varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_1^2}(\theta) & \ldots & \varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_1 \partial \theta_{n-1}}(\theta) & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_{n-1} \partial \theta_n}(\theta) & \ldots & \varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_{n-1}^2}(\theta) & 0 \\
0 & \ldots & 0 & \frac{\partial^2}{\partial \theta_n^2} \left[ G_{d,n+1}(\cos \theta_n) + \varepsilon \sin^d \theta_n M_{d,n}(\bar{x}_1, \ldots, \bar{x}_n) \right]
\end{pmatrix}
$$
It is non-degenerate since the function $M_{d,n}$ is, by definition, Morse and for a small $\varepsilon$ the solutions of \([4]\) are non-degenerate. Thus, the function $Z_{d,n+1} + \varepsilon M_{d,n}$ has $2 + (d-1) \cdot 2 ((d-1)^n - 1)/(d-2) = 2 ((d-1)^{n+1} - 1)/(d-2) = 2m_{d,n+1}$ non-degenerate critical points.

Theorem 11 follows from the proposition.

By Theorem 3 smooth functions on $S^{n-1}$ having a birth-death singularity as its only degenerate critical point are the most general among all non-Morse functions on $S^{n-1}$. We show that this still holds (up to the antipodal symmetry) in the space $\mathcal{P}_{d,n}|_{S^{n-1}}$ of polynomials restricted to the sphere.

Denote by $\Delta_{d,n}$ the set of $f \in \mathcal{P}_{d,n}$ whose restriction $f|_{S^{n-1}}$ is not Morse and denote by $\Delta^{(1)}_{d,n} \subset \Delta_{d,n}$ the subset of polynomials having only two (antipodal) degenerate critical points on $S^{n-1}$ which are birth-death singularities.

**Lemma 4.** The sets $\Delta^{(1)}_{d,n}$, $\Delta_{d,n} \setminus \Delta^{(1)}_{d,n}$ in $\mathcal{P}_{d,n}$ are semialgebraic of codimension one and two respectively and $\Delta^{(1)}_{d,n}$ is smooth.

**Proof.** Under the double covering $S^{n-1} \overset{2:1}{\to} \mathbb{R}P^{n-1}$ the set

$$
\mathcal{F} := \bigcup_{k=1}^{n} \left\{ (x_1, \ldots, x_k, 0, \ldots, 0) \in S^{n-1} : x_k < 0 \right\}
$$

is mapped bijectively onto $\mathbb{R}P^{n-1}$. In the stereographic coordinates

$$
x_i = \frac{4y_i}{y_1^2 + \cdots + y_{n-1}^2 + 4}, \quad i = 1, \ldots, n - 1,
\quad x_n = \frac{y_1^2 + \cdots + y_{n-1}^2 - 4}{y_1^2 + \cdots + y_{n-1}^2 + 4}, \quad y \in \mathbb{R}^{n-1}
$$
on $S^{n-1} \setminus \{(0, \ldots, 0, 1)\}$ we have

$$
\mathcal{F} = \left\{ (y_1, \ldots, y_{n-1}) : \sum_{i=1}^{n-1} y_i^2 < 4 \right\} \cup \bigcup_{k=1}^{n-1} \left\{ (y_1, \ldots, y_k, 0, \ldots, 0) : \sum_{i=1}^{k} y_i^2 = 4, y_k < 0 \right\}
$$

Under the first projection $\pi_1 : \mathcal{P}_{d,n} \times \mathbb{R}^{n-1} \to \mathcal{P}_{d,n}$ the semialgebraic set

$$
\tilde{\Delta}_{d,n} := \left\{ (f, y) \in \mathcal{P}_{d,n} \times \mathcal{F} : \frac{\partial f}{\partial y_1}(y) = \cdots = \frac{\partial f}{\partial y_{n-1}}(y) = 0, \det \left( \frac{\partial^2 f}{\partial y_i \partial y_j}(y) \right) = 0 \right\}
$$

maps onto $\Delta_{d,n}$. Hence $\Delta_{d,n}$ is semialgebraic and $\dim \Delta_{d,n} \leq \dim \tilde{\Delta}_{d,n} \leq \dim \mathcal{P}_{d,n} - 1$.

Denote by $\Delta_{d,n}^{(2,1)}$ the set of polynomials $f \in \Delta_{d,n}$ such that $\text{Ker} \, d^2 f|_{S^{n-1}}$ is one-dimensional and $d^2 f|_{S^{n-1}} : \text{Ker} \, d^2 f|_{S^{n-1}} \to \mathbb{R}$ is trivial and denote by $\Delta_{d,n}^{(2,2)}$ the set of polynomials $f \in \Delta_{d,n}$ such that $\text{Ker} \, d^2 f|_{S^{n-1}}$ is at least two-dimensional, $x \in S^{n-1}$ is
some degenerate critical point of $f|_{S^{n-1}}$. The sets

$$
\tilde{\Delta}^{(2,1)}_{d,n} := \left\{ (f, y, [v]) \in \tilde{\Delta}_{d,n} \times \mathbb{R}P^{n-2} : \text{corank} \left( \frac{\partial^2 f}{\partial y_i \partial y_j}(y) \right) = 1, \right.
$$

$$
\sum_{j=1}^{n-1} \frac{\partial^2 f}{\partial y_i \partial y_j}(y) v_j = 0, i = 1, \ldots, n - 1, \sum_{i,j,k=1}^{n-1} \frac{\partial^3 f}{\partial y_i \partial y_j \partial y_k}(y) v_i v_j v_k = 0 \right\}
$$

(6)

$$
\tilde{\Delta}^{(2,2)}_{d,n} := \left\{ (f, y) \in \tilde{\Delta}_{d,n} : \text{corank} \left( \frac{\partial^2 f}{\partial y_i \partial y_j}(y) \right) \geq 2 \right\}
$$

are semialgebraic in $\mathcal{P}_{d,n} \times \mathcal{F} \times \mathbb{R}^{n-1}$ and $\mathcal{P}_{d,n} \times \mathcal{F}$ respectively and they map onto $\Delta^{(2,1)}_{d,n}$ and $\Delta^{(2,2)}_{d,n}$ under the projections onto the first factor. Therefore, $\Delta^{(2,1)}_{d,n}$ and $\Delta^{(2,2)}_{d,n}$ are semialgebraic subsets of $\mathcal{P}_{d,n}$ of codimension at least two. Denote by $\Delta^{(2,3)}_{d,n}$ the set of polynomials $f \in \Delta_{d,n}$ such that $f|_{S^{n-1}}$ has more than one pair of (antipodal) degenerate critical points. Using standard techniques from semialgebraic geometry one can show that $\Delta^{(2,3)}_{d,n}$ is a semialgebraic subset of $\mathcal{P}_{d,n}$ of codimension at least two.

Now let’s show that the semialgebraic set $\Delta^{(1)}_{d,n} = \Delta_{d,n} \setminus \left( \Delta^{(2,1)}_{d,n} \cup \Delta^{(2,2)}_{d,n} \cup \Delta^{(2,3)}_{d,n} \right)$ is smooth and has codimension one in $\mathcal{P}_{d,n}$. First note that $\Delta^{(1)}_{d,n}$ is fibered over $P(T\mathbb{R}P^{n-1})$, the projectivized tangent bundle to the projective space, in the following way. Consider the map

$$
\pi : \Delta^{(1)}_{d,n} \to P(T\mathbb{R}P^{n-1}) := \{(x, v) : x \in \mathbb{R}P^{n-1}, v \in P(T_x\mathbb{R}P^{n-1})\}
$$

sending a polynomial $f \in \Delta^{(1)}_{d,n}$ to the pair $([x], [v])$, where $x \in S^{n-1}$ is the unique (up to symmetry) degenerate critical point of $f|_{S^{n-1}}$ and the vector $v \in T_xS^{n-1} \simeq T_x\mathbb{R}P^{n-1}$ spans $\ker d^2f|_{S^{n-1}}$. From the definition of $\Delta^{(1)}_{d,n}$, it follows that the fibers of $\pi$ do not intersect. The orthogonal group $SO(n)$ acts transitively on $P(T\mathbb{R}P^{n-1})$ by sending for each $g \in SO(n)$ a pair $([x], [v]) \in P(T_x\mathbb{R}P^{n-1})$ to $g([x], [v]) := ([gx], [gv]) \in P(T_{[gx]}\mathbb{R}P^{n-1})$. It also acts on $\Delta^{(1)}_{d,n}$ by change of variables: $g \in SO(n)$ sends a function $f \in \Delta^{(1)}_{d,n}$ to $(g^{-1})^* f := f \circ g^{-1}$. The projection $\pi$ is an equivariant map with respect to these two actions: $\pi \circ (g^{-1})^* = g \pi$. Consequently, $\pi : \Delta^{(1)}_{d,n} \to P(T\mathbb{R}P^{n-1})$ is a smooth fiber bundle. It can be shown (combining (5) and (3)) that the dimension of any fiber $\pi^{-1}([x], [v])$ equals $\dim \mathcal{P}_{d,n} - \dim P(T\mathbb{R}P^{n-1}) - 1$. Therefore, the dimension of the total space $\Delta^{(1)}_{d,n}$ of the fibre bundle $\pi$ equals $\dim \pi^{-1}([x], [v]) + \dim P(T\mathbb{R}P^{n-1}) = \dim \mathcal{P}_{d,n} - 1$.

Finally we derive Theorem 2.

**Proof of Theorem 3** Note first that a continuous function on a compact manifold has at least one minimum and one maximum. Among homogeneous polynomials of odd degree
“the most simple” is the one given by a linear function $\langle l, x \rangle\|x\|^{2d}$, $l \in \mathbb{R}^n \setminus \{0\}$. It defines a Morse function on $S^{n-1}$ with one maximum $l/\|l\|$ and one minimum $-l/\|l\|$.

Let’s consider a continuous path $\gamma : [0, 1] \to P_{2d+1,n}$ that connects $\langle l, x \rangle\|x\|^{2d}$ with $M_{2d+1,n}$ constructed in Proposition 11. By Thom transversality theorem [6, Thm. 2.1] and Lemma 4, the path $\gamma$ can be chosen in such a way that it intersects $\Delta^{(1)}_{2d+1,n}$ transversely finitely many times $t_1, \ldots, t_l \in (0, 1)$ and does not intersect $\Delta^{(1)}_{2d+1,n} \setminus \Delta^{(1)}_{2d+1,n}$. For each $t_i, i=1, \ldots, l$ and a birth-death singularity $x^i \in S^{n-1}$ of $\gamma(t_i)$ there exist local coordinates $(\varepsilon, y_1, \ldots, y_{n-1})$ on a small neighbourhood of $(t_i, x^i) \in (0, 1) \times S^{n-1}$ in which $(t_i, x^i) = 0$ and $\gamma(\varepsilon, y) = \gamma(0) + \varepsilon y_1 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_{n-1}^2$. From this it follows that when passing $t_i$ the number of critical points of $\gamma(t)|_{S^{n-1}}$ either increases or decreases by 4 (2 for each of the two antipodal birth-death singularities). Therefore, along $\gamma$ we meet one by one generic symmetric tensors with $1 + 2l$ eigenpoints, $i = 0, \ldots, \frac{l}{2}(m_{2d+1,n} - 1)$.

Note that a polynomial $f$ of even degree $d \geq 2$ defines an even function on the sphere. Thus if $x \in S^{n-1}$ is a non-degenerate critical point of $f|_{S^{n-1}}$ of index $k = 0, \ldots, n-1$ then $-x$ is also a non-degenerate critical point of $f|_{S^{n-1}}$ of the same index $k$ and therefore the numbers $\mu_k = \mu_k(f|_{S^{n-1}}), k = 0, \ldots, n-1$ are all even. Combining this with (2) we have

\[
\begin{align*}
\mu_0 &\geq 2 \\
\mu_{2k-1} - \mu_{2k-2} + \cdots - \mu_0 &\geq 0 \\
\mu_{2k} - \mu_{2k-1} + \cdots + \mu_0 &\geq 2 \\
\mu_{2k+1} - \mu_{2k} + \cdots - \mu_0 &\geq 0
\end{align*}
\]

which implies that $\mu_k \geq 2, k = 0, \ldots, n-1$, and therefore a generic homogeneous polynomial (symmetric tensor) of even degree must have at least $2n$ critical points on the sphere ($n$ real eigenpoints). For a symmetric $n \times n$ matrix $A$ with simple eigenvalues the associated quadratic form $f_A(x) = x^tAx$ defines a Morse function on the sphere with $2n$ critical points. Connecting $f_A(x)\|x\|^{2d}$ with $M_{2d+2,n}$ by a continuous path and again invoking Thom transversality theorem [6, Thm. 2.1] and Lemma 4 we can realize any number of real eigenpoints given by the formula in the statement of Theorem 2.

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