SINGULARITIES OF NORMAL QUARTIC SURFACES
III (CHAR=2, NON-SUPERSINGULAR)

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Abstract. We show, in this third part, that the maximal number of singular points of a normal quartic surface \( X \subset \mathbb{P}_K^3 \) defined over an algebraically closed field \( K \) of characteristic 2 is at most 12, if the minimal resolution of \( X \) is not a supersingular K3 surface. We also provide a family of explicit examples, valid in any characteristic.

1. Introduction

This paper continues our study of normal quartic surfaces \( X \subset \mathbb{P}_K^3 \) defined over an algebraically closed field \( K \) of characteristic 2. This was started in [Cat21b] and continued in [CS21]. In detail, we proved:

**Theorem 1** ([CS21, Thm. 1]). A normal quartic surface \( X \subset \mathbb{P}_K^3 \) contains at most 14 singular points. If the maximum number of 14 singularities is attained, then all singularities are nodes and the minimal resolution of \( X \) is a supersingular K3 surface. The variety of quartics with 14 nodes contains an irreducible component, of dimension 24.

Recall that a K3 surface is supersingular if it has maximum Picard number \( \rho = 22 \) ([Art74]). The present paper concerns the non-supersingular case. In this case, Theorem 1 implies that \( X \) contains at most 13 singular points. Our main result improves this to the following sharp bound:

**Theorem 2.** Let \( S \) be the minimal resolution of a normal quartic surface \( X \). If \( S \) is not a supersingular K3 surface, then \( X \) contains at most 12 singular points. If there are 12 singular points, then they all have types \( A_1 \) or \( A_2 \), and there are at most 3 \( A_2 \)’s. The variety of quartics with 12 nodes contains an irreducible component, of dimension 22, such that generically \( S \) is a non-supersingular K3 surface.

Naturally Theorem 2 leads to the question about what is true for other quasi-polarized K3 surfaces in characteristic 2. We will prove some partial results in this direction, for instance the following.

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Theorem 3. Let $X$ be a K3 surface in characteristic 2 with Picard number $18 \leq \rho(X) \leq 20$. Then $X$ contains at most 12 disjoint smooth rational curves.

Along the way to prove these results, we also develop a subtle characteristic-free relation between disjoint smooth rational curves and fibre components of elliptic fibrations which may be of independent interest (Proposition 24).

Convention: We work over an algebraically closed field $K$, mostly of characteristic 2, though many results may also be stated over non-closed fields.

2. Elliptic fibrations

This section reviews parts of the theory of elliptic fibrations on K3 surfaces used in the proofs of Theorems 2 and 3. Usually, in characteristics 2 and 3, this encompasses quasi-elliptic fibrations as well, but since we presently restrict to non-supersingular K3 surfaces, this will not be necessary for us (since quasi-elliptic over $\mathbb{P}^1$ implies unirational which in turn implies supersingular).

Let $X \subset \mathbb{P}^3$ be a normal quartic with only rational double points as singularities (we proved in [CS21, Prop. 14] that if $X$ has at least 13 singularities, then they are all rational double points), of which we fix $P$ and $Q$. Denote the minimal resolution of $X$ by $S$. Then the pencil of hyperplanes containing $P$ and $Q$ endows $S$ with a genus one fibration $S \to \mathbb{P}^1$.

All other singular points give disjoint fibre components (or, rather, disjoint ADE-configurations contained in the fibres of $S$). For singular points not collinear with $P$ and $Q$, this is obvious; for the other case, observe that, if the line $L = \overline{PQ}$ contains a third singular point, then it is contained in $X$ and it is a multiple component of some plane through $P, Q$, and its strict transform is contained in the corresponding fibre.

Our interest in normal quartics with many singular points thus leads us to study elliptic fibrations on K3 surfaces with many disjoint smooth rational fibre components. Outside characteristic 2, there can be as many as 16 disjoint smooth rational fibre components (realized on Kummer surfaces isogenous to a product). In characteristic 2, however, this is prevented by the wild ramification at additive fibres.

The wild ramification $\delta_v$ measures the discrepancy between the Euler number $e(F_v)$ of the fibre $F_v$ and the local multiplicity of the discriminant (which can be computed on the Jacobian fibration). The following table, reproduced from [CS21], lists standard information on the fibres, given both in terms of Kodaira’s types and Dynkin types, namely the number of irreducible components $m_v$ and the Euler number $e(F_v)$. It
also gives bounds for $\delta_v$ from [SS13, Prop. 5.1] and the maximal number $N_v$ of disjoint (-2)-fibre components which can be inferred directly from inspecting the corresponding extended Dynkin diagrams.

| fibre type | $I_n$ | II | III | IV | $I_n^*$ ($n \neq 1$) | $I_1^*$ | IV$^*$ | III$^*$ | II$^*$ |
|------------|------|----|-----|----|-------------------|--------|-------|--------|-------|
| Dynkin type | $A_{n-1}$ | $A_0$ | $A_1$ | $A_2$ | $D_{n+4}$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ |
| $m_v$      | $n$  | 1  | 2   | 3   | $n + 5$ | 6     | 7     | 8     | 9     |
| $\delta_v$ | 0    | $\geq 2$ | $\geq 1$ | 0 | $\geq 2$ | 1     | 0     | $\geq 1$ | $\geq 1$ |
| $e(F_v)$   | $n$  | 2  | 3   | 4   | $n + 6$ | 7     | 8     | 9     | 10    |
| $N_v$      | $\left\lfloor \frac{n}{2} \right\rfloor$ | 0    | 1   | 1   | $4 + \left\lfloor \frac{n}{2} \right\rfloor$ | 4     | 4     | 5     | 5     |

Table 1. Singular fibre data in characteristic 2

Note that, by inspection of the table,

\[(1) \quad N_v \leq \frac{1}{2} (e(F_v) + \delta_v).\]

Summing over all singular fibres, one obtains the following strong restrictions:

**Proposition 4 ([CS21 Prop. 22, Cor. 24 & 25]).**

(i) In characteristic 2, on an elliptic K3 surface the singular fibres contain at most 12 disjoint (-2)-curves.

(ii) If the fibres of an elliptic K3 surface in characteristic 2 contain 12 disjoint (-2)-curves, then the only possible singular fibre types are (with minimum possible $\delta_v$ each)

$I_{2n}$ ($n > 0$), $I_{2n}^*$ ($n \geq 0$), $I_1^*$, IV$^*$, III$^*$.

(iii) If the fibres of an elliptic K3 surface in characteristic 2 support 12 disjoint ADE-configurations of smooth rational curves, then each has type $A_1$.

Using explicit calculations with the Weierstrass form of the Jacobian fibration (which has the same configuration of singular fibres), we then established the following characterization of elliptic K3 surfaces whose fibres contain 12 disjoint smooth rational curves:

**Proposition 5 ([CS21 Prop. 26]).** Let $X$ be an elliptic K3 surface such that there are 12 disjoint (-2)-curves contained in the fibres. Then $X$ is supersingular or there are two additive fibres.

Note that the additive fibres have the same types as those in Proposition 4 (ii) different from $I_{2n}$, in particular, they are non-reduced. This will be instrumental for the proof of Theorem 2.

3. **Bounding the number of singular points**

Throughout this section we assume that $X \subset \mathbb{P}^3$ is a normal quartic containing 13 singular points. By [CS21 Proposition 15], all singularities are rational double points, so the minimal resolution $S$ is a K3
surface which we assume to be non-supersingular, i.e. $\rho(S) \leq 20$. Note that, in particular, this implies that $S$ and $X$ cannot be unirational which rules out many cases from [CS21] – especially the quasi-elliptic fibrations. Hence all genus one fibrations on $S$ are elliptic. By Theorem 1 we can restrict to the case where there are exactly 13 singular points. We first draw some consequences valid for all configurations of singularities.

3.1. Non-reduced fibre. As in Section 2, consider an elliptic fibration

\[ S \to \mathbb{P}^1 \]

induced by two singular points on $X$. Recall that this has at least 11 disjoint smooth rational fibre components.

**Lemma 6.** The fibration \((2)\) has a non-reduced fibre.

**Proof.** If there are at least 12 disjoint smooth rational fibre components, the statement follows from Proposition 5. Otherwise, 11 disjoint smooth rational fibre components leave a little more room for fibre types compared to Proposition \(4\) (ii). Namely, if we exclude non-reduced fibres, then Table \(1\) and \(1\) imply that all fibres have types $I_{2n}$ for varying $n$ (as in Proposition \(4\) (ii)) except that

(i) either there are 2 $I_n$ fibres with odd $n$ (ii-III) or there is one fibre of type III (with minimal wild ramification $\delta = 1$)

(ii-IV) or there is one fibre of type IV.

To rule out all cases, we switch to the Jacobian of \((2)\) (with the same singular fibres etc) and consider its Weierstrass form

\[ (3) \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

Here $a_i \in k[t]$, and in order for $S$ (and its Jacobian) to be a K3 surface, one needs $\text{deg}(a_i) \leq 2i$ for each $i$.

A priori, the zeros of $a_1$ give either supersingular fibres (if $a_3 \neq 0$ at this point) or additive fibres (if $a_3 = 0$). In case (i), there remains only the first alternative because there cannot be an additive fibre, and then we can derive a contradiction as follows. Locate the supersingular fibre at $t = \infty$ and the two multiplicative fibres with odd number of components at $\alpha, \beta$. By [RS18b, Lemma 6.4], the discriminant $\Delta$ has vanishing $t^{23}$-coefficient, but spelling this out we get

\[ \Delta = (t + \alpha)^{n_1}(t + \beta)^{n_2}(...)^2 = t^{24} + (\alpha + \beta)t^{23} + \ldots. \]

By assumption, $\alpha \neq \beta$, so this gives the required contradiction. In cases (ii-III) and (ii-IV), we can argue using the fact that the discriminant $\Delta$ is a square, here

\[ \Delta = a_3^4 + a_1^2 a_3^2 + a_4^2 + a_4^2 a_2 a_3^2 + a_5^2 a_3 a_4 + a_6^3 a_6. \]
This holds since the vanishing order of $\Delta$ equals the contributions to the Euler number at the fibres $I_{2n}$, and these are even; moreover, for type III with $\delta = 1$ and for type IV, the vanishing order is 4. Hence all vanishing orders are even, so $\Delta$ is a square.

Since there are no non-reduced fibres by assumption, there is only one additive fibre which we locate at $t = 0$. Arguing exactly as in the proof of Proposition 26 in [CS21] (around formula (9)) [which we cited here as Proposition 5], we can reduce to the case $a_1 = t^2$. Tate’s algorithm gives the normal form

$$y^2 + t^2xy + ta'_3y = x^3 + ta'_2x^2 + ta'_4x + t^2a'_6.$$ 

Resolving the singularity shows that we have type III if and only if $t \nmid a'_4$, and type IV if and only if $t \mid a'_4$, but $t \nmid a'_3$ (cf. [Sil94, Steps 4 & 5]).

Expand now $\Delta$ as

$$\Delta = t^4(a'_3^4 + t^5a'_3^3 + t^6 \ldots).$$

Thus $\Delta$ could only be square if $t \mid a'_3$ which is ruled out for type IV and leads to ramification index $\delta_0 > 2$ for type III, but this was supposed to be minimal, i.e. $\delta_0 = 1$. This gives the required contradiction.

3.2. Nodes as singularities. We first cover the case where all singular points are nodes. Since the exceptional curves above nodes appear with multiplicity one in the fibres, we can improve the above results in case $P$ and $Q$ are collinear with another node $R \in X$.

**Lemma 7.** If there are 3 collinear nodes, then the induced fibration has exactly one non-reduced fibre, and this has type $I^{\ast}_0$. The fibre is induced by a plane which intersects $X$ in two double lines, each containing 3 nodes.

**Proof.** The 3 collinear nodes are contained in a line $L \subset X$. The fibres of [2] are thus induced by the cubics residual to $L$ in the planes $H$ containing $L$. Unless $H$ contains $L$ with higher multiplicity, the residual cubic meets $L$ in 3 different points and is thus reduced. Since the exceptional curves above the nodes also appear with multiplicity one in the fibres, the first half of the claim of the lemma follows. More precisely, using Lemma 5, there is exactly one non-reduced fibre, corresponding to the unique plane $H_0$ containing $L$ with higher multiplicity. Since the nodes on $L$ induce sections while a non-reduced fibre has at least 5 components, there have to be 4 components provided by the conic $Q$ residual to $L$ in $H_0$ and by the nodes it contains. Then $Q$ must be non-reduced, hence $Q$ is a double line containing 3 nodes as stated, and we conclude that we have a fibre of type $I^{\ast}_0$.

We take the lemma as the first step to compute all possible non-reduced fibre types for all settings.
**Lemma 8.** A fibre of \((2)\) is non-reduced if and only if the underlying hyperplane section is an irreducible double conic (with 6 nodes) or splits into two double lines (with 3 nodes each). If the conic is irreducible, we get a fibre of type \(I_0^*\).

**Proof.** If the fibration \((2)\) is induced by 3 collinear nodes, then we have seen this already in Lemma 7 (in fact, only the second alternative).

In general, since exceptional curves appear with multiplicity one in the fibres, a fibre corresponding to some plane \(H\) through nodes \(P, P'\) can only be non-reduced if \(H \cap X\) is non-reduced. If it is an irreducible double conic given by \(\{q = z = 0\}\), then the equation of \(X\),

\[
F = q^2 + zg + z^2 \ldots,
\]

(5)
directly reveals the 6 nodes given by \(\{z = q = g = 0\}\) (under the assumption that \(X\) is normal and all singular points are nodes). This verifies the first alternative of the lemma.

Otherwise \(H\) splits into a double line \(L = \{y = z = 0\} \subset X\) and a residual conic given by \(\{q = z = 0\}\). As before, the equation

\[
F = y^2q + zg + z^2 \ldots,
\]

(6)
reveals three nodes on \(L\) given by \(\{z = y = g = 0\}\). Now consider the elliptic fibration induced by \(L\) and conclude by applying Lemma 7. □

Note that the irreducible double conic in Lemma 8 only arises if no two of the 6 nodes are collinear with another node (by Bézout’s theorem), and that it gives a fibre of type \(I_0^*\), while the other configuration may arise for both set-ups (giving type \(I_0^*\) and three sections if the two specified nodes are collinear with a third node as in Lemma 7, resp. type \(I_1^*\) and two bisections otherwise). We emphasize that, as shown in the proof, these configurations do not involve a line \(PQ \subset X\) for two nodes \(P, Q \in X\) unless the line contains a third node.

**Remark 9.** The two previous lemmata and their proofs (esp. equations (5), (6)) also show that any non-reduced plane is automatically everywhere non-reduced, falling into the two alternatives from Lemma 8. We will thus only refer to non-reduced planes in what follows.

### 3.3. Non-reduced planes.

In this section, we piece together the information about non-reduced fibres to prove the bound of Theorem 2 in the case of nodes. We first limit the possible intersections of non-reduced planes.

**Lemma 10.** The intersection of two non-reduced planes cannot contain three nodes.

**Proof.** If the intersection of the non-reduced planes \(H_1, H_2\), the line \(L\), say, were to contain 3 nodes, then \(L \subset X\) and the fibration induced
by $L$ would have exactly one non-reduced fibre by Lemma 7. This contradicts the fact that both $H_1$ and $H_2$ give non-reduced fibres. □

By Lemma 8, each pair of nodes is contained (together with 4 other nodes) in a non-reduced plane as above. This implies that any node $P$ is contained in at least 3 distinct non-reduced planes, say $H_1, H_2, H_3$.

**Claim 11.** $H_1$ and $H_2$ meet in exactly two nodes.

**Proof.** Each plane $H_i$ contains $P$ and 5 other nodes. Since there are 12 nodes other than $P$ in total, there have to be at least 3 duplicate points. By Lemma 10, this amounts to exactly one duplicate (other than $P$) for each pair $(H_i, H_j)$. □

**Claim 12.** Each node is contained in exactly 3 non-reduced planes.

**Proof.** If some node $P$ were contained in 4 non-reduced planes, then each plane would contain $P$ together with 5 other nodes, so in total we count

$$1 + 4 \cdot 5 - \binom{4}{2} = 15$$

nodes by Claim 11, which is absurd. □

### 3.4. Census for 13 nodes.

Let $m$ be the number of non-reduced planes. Each contains 6 nodes by Lemma 8. On the other hand, each node is contained in exactly 3 non-reduced planes by Claim 12. Hence

$$6m = 13 \cdot 3.$$

This shows that a non-supersingular normal quartic cannot contain 13 nodes (proving a substantial part of Theorem 2). □

### 3.5. Higher singularities.

We shall now assume that one of the 13 singular points is not a node (but of ADE-type by [CS21, Prop. 15]). We will derive a contradiction in 3 steps:

1. limit the possible configurations of singularities (Lemma 13);
2. prove that certain fibrations admit suitable sections (Lemma 16);
3. play this off against non-reduced fibres and the restrictions on disjoint smooth rational fibre component (Section 3.7).

**Lemma 13.** The only options for non-nodes are

1. one $A_3$-singularity and 12 nodes, or
2. at most three $A_2$-singularity and all other singularities nodes.

**Remark 14.** In Proposition 31 we will prove by a different argument that with 12 or more singular points, a non-supersingular quartic can only admit $A_1$ and $A_2$ singularities. More precisely, by Proposition 32 there can be at most 3 $A_2$'s.
Proof. Assume that there is a singular point \( P \) which is not of type \( A_1 \) or \( A_2 \). Then \( A_3 \) embeds into the corresponding Dynkin diagram. If \( P \) lies in a fibre of some elliptic fibration induced by two or three other singularities, then the exceptional curves furnish it with 2 disjoint \((-2)\)-curves (corresponding to the embedding \( A^2_1 \hookrightarrow A_3 \)) – also disjoint to the 10 exceptional curves above the other singular points on the fibres (and the connecting line if there are three collinear base points). By Proposition\[\text{(iii)}, \]the corresponding 12 orthogonal root lattices of type \( A_1 \) supported on the fibres cannot be extended while staying disjoint. Hence \( P \) has type \( A_3 \) and the other nine or ten singular points on the fibres are nodes. By symmetry, this implies that all singular points other than \( P \) are nodes as stated in (1).

It remains to bound the number of \( A_2 \)-singularities. For this purpose, we introduce the following notation for a fibre \( F_v \):

\[
N_v^{(i)} = \max \left\{ \left\{ r \mid \exists \text{disjoint ADE-configurations } C_1, \ldots, C_r \text{ supported on } F_v \text{ such that } i \text{ of the } C_j \text{'s contain } A_2 \text{'s} \right\} \right\}.
\]

One might expect that all other ADE-configurations in the above set-up will have type \( A_1 \), but then one notices that except for type III, on all fibre types not attaining equality in (1) for the minimal \( \delta_v \), one can replace one \( A_1 \) by \( A_2 \) while preserving orthogonality. In fact, this is the first step towards proving the following:

**Fact 15.** \( N_v^{(i)} \leq \frac{1}{2}(e(F_v) + \delta_v - i) \) unless \( i = 3 \) and \( F_v \) has type \( IV^* \) where \( N_v^{(3)} = 3 \).

\[\text{Proof.}\] This can be verified directly by going through all Kodaira types. Note that the bound is sharp for small \( i \), but ceases to stay sharp for larger \( i \). \(\square\)

Seeking for a contradiction, we assume that \( X \) contains 13 singular points, among them four \( A_2 \) singularities. Pick an elliptic fibration given by two or three nodes such that 4 \( A_2 \)'s contribute to the fibres - together with 7 disjoint \( A_1 \)'s (or higher singularities). If there is a fibre of type \( IV^* \) supporting three \( A_2 \)'s, say at \( v = \infty \), then there is one other fibre containing an \( A_2 \). For the number of disjoint ADE-configurations including the 4 \( A_2 \)'s supported on the fibres, we thus obtain, by Fact \[15\]

\[
11 \leq 3 + \sum_{v \neq \infty} N_v^{(i_v)} \leq 3 + \frac{1}{2} \sum_{v \neq \infty} (e(F_v) + \delta_v) - \frac{1}{2} \leq 3 + \frac{16 - 1}{2},
\]

yielding the desired contradiction. If there are no 3 \( A_2 \)'s supported on a fibre of type \( IV^* \), then we can even rule out 3 \( A_2 \) singularities on the fibres since, by Fact \[15\]

\[
11 \leq \sum_v N_v^{(i_v)} \leq \frac{1}{2} \sum_v (e(F_v) + \delta_v - i_v) \leq \frac{24 - 3}{2}
\]
gives again a contradiction. □

3.6. Fibrations with sections.

**Lemma 16.** For every fibration induced by a singular point $P$ which is not a node, and by one or two nodes, there is a section orthogonal to 11 disjoint $(-2)$-curves supported on the fibres.

**Proof.** We consider first the case where the fibration is induced by $P$ and a node $P'$ which are not collinear with a third singular point. Consider the configuration $C$ of $(-2)$-curves given by the exceptional curves lying above $P$ and $P'$, plus the strict transform of the connecting line in case it is contained in $X$. The configuration of these $(-2)$-curves is a priori among the types $A_1 + A_2$, $A_1 + A_3$, $A_4$, $A_5$, $D_5$. Except for the $D_5$ case, the fibration $|F|$ is obtained from the pull-back of the hyperplane divisor $H$ by subtracting the fundamental cycle(s) of the configuration $C$ (which is reduced).

Then one of the two outer components of the exceptional divisor above $P$ is an outer component of the fundamental cycle: hence it induces the claimed section (since it intersects $F$ with multiplicity 1). In the $D_5$ case, in order to obtain the fibration one has to subtract all the curves in the configuration $C$ with multiplicity one, except that the exceptional curve $E$ which has three adjacent curves in the configuration has to be subtracted with multiplicity 2. In fact, the line $L := P P'$ must be contained inside $X$, and since its strict transform intersects $E$, this line $L$ is the intersection of the two planes which form the tangent cone at $P$, which is a singularity of type $A_3$, $xy = z^4$: then each plane containing $L$ contains the sum of the exceptional divisor over $P$ with $E$, as a local calculation shows.

In particular, $E$ provides a section. In all cases, the section is disjoint from the remaining 11 singular points, so the claim follows.

If $P$ is collinear with nodes $P'$, $P''$, then the exceptional curves and the strict transform of the line (which meets three other components) may form the configurations $D_5$, $D_6$ or $\tilde{D}_5$. The last case is excluded by Proposition 1(i), since then we have a Kodaira fibre of type $I_1^*$ (of some elliptic fibration) which contains 4 disjoint $(-2)$-curves and is disjoint from the remaining 10 singularities. In the first two cases, the fibration $|F|$ is obtained again by subtracting the reduced divisor supported on the configuration $C$. All exterior components of the configuration provide sections while the interior components give fibre components (since they have zero intersection with $F$). In particular, each of these sections is disjoint from 11 $(-2)$-curves supported on the fibres, given by one interior component of the configuration and the exceptional curves above the 10 remaining singular points. □
3.7. Sections vs. non-reduced fibres. To complete the argument for higher singularities, consider a fibration induced by $P$ and some nodes as in Lemma 16. By Lemma 6 there is a non-reduced fibre $F_v$, say with extended Dynkin diagram $\tilde{V}$. By Lemma 16, there is a section $O$ and the disjoint ADE-configuration supported on $\tilde{V}$ is already supported on the Dynkin diagram of type $V$ obtained by omitting the simple fibre component met by $O$. Denote by $N'_v$ the maximal number of disjoint $(-2)$-curves supported on $V$. A case-by-case inspection teaches us that this is one less than $N_v$:

**Fact 17.** For a non-reduced fibre, one has

$$N'_v \leq \frac{1}{2} (e(F_v) + \delta_v) - 1.$$ 

**Remark 18.** The same inequality holds true when we omit any other odd multiplicity component of a non-reduced fibre.

For the fibres to contain 11 disjoint $(-2)$-curves, this implies that all inequalities in (1) are in fact inequalities, and the classification of possible fibre types in Proposition 4 (ii) is still valid. In particular, this implies that the fibration supports indeed 12 disjoint $(-2)$-curves. (This can also be seen by adding the component met by $O$, and maybe moving one component to a simple component, or shortening an $A_2$ configuration to $A_1$ disjoint from the component added.) Since $S$ is not supersingular by assumption, Proposition 5 and the ensuing remark imply that there are in fact two non-reduced fibres. Hence Fact 17 applies to both of them, and there can be at most 10 disjoint $(-2)$-curves which are supported on the fibres while being orthogonal to $O$. This contradiction completes the proof that a normal quartic whose resolution is not supersingular cannot contain 13 (or more) singular points.

3.8. A 22-dimensional family of non-supersingular quartics with 12 nodes.

**Example 19.** To exhibit a quartic with 12 nodes over a field $k$ of arbitrary characteristic, in fact, consider 4 linear forms $l_1, \ldots, l_4$ and a quadric $q \in k[x_1, x_2, x_3, x_4]$. Then the quartic

$$l_1 \cdots l_4 + q^2 = 0$$

has generally 12 nodes. To see that it is generally not supersingular, it suffices to specialize to the case where $l_i = x_i, q = \lambda(x_1 + \ldots + x_4)^2$. Then the pencil of quartics is the classical Dwork pencil, where each quartic has 6 $A_3$-singularities, (over $\mathbb{C}$, it would be the mirror of the Dwork pencil, cf. [EIS08], and at $\lambda = -1/81$ [EIS08, section 11] we get a K3 surface $S$ with $\rho = 20$, both over $\mathbb{Q}$ (or $\mathbb{C}$) and in characteristic 2. To see this, we use that $S$ contains the twisted cubic $C$ parametrized by

$$\mathbb{P}^1 \ni t \mapsto (-t^3, 1 - t, t, (t - 1)^3).$$
Together with the exceptional curves and the 4 obvious lines in the plane \( \{ x_1 + \ldots + x_4 = 0 \} \), the twisted cubic generates \( \text{Pic}(S_{\mathbb{Q}}) = \text{Pic}(S_{\mathbb{C}}) \), a hyperbolic lattice of rank 20 and determinant \(-7\). It follows that the Picard number of the reduction \( S_p = S \otimes \overline{\mathbb{F}}_p \) at a prime \( p \neq 7 \) is controlled by the field \( \mathbb{Q}(\sqrt{-7}) \) as follows:

\[
\rho(S_p) = \begin{cases} 
20 & \text{if } p \text{ splits in } \mathbb{Q}(\sqrt{-7}); \\
22 & \text{if } p \text{ is inert in } \mathbb{Q}(\sqrt{-7}) 
\end{cases}
\]

(cf. [Sch10] Rem. 13 for instance). Since 2 splits in \( \mathbb{Q}(\sqrt{-7}) \), we infer that \( \rho(S_2) = 20 \) as claimed.

**Remark 20.** In [Cat21b Thm. 14], there is given a 3-dimensional family of \( S_4 \)-invariant quartics with 12 nodes, of the form

\[
a_1 \sigma_1(x)^4 + a_2 \sigma_1(x)^2 \sigma_2(x) + a_3 \sigma_1(x) \sigma_3(x) + a_4 \sigma_4(x) + \beta \sigma_2(x)^2 = 0,
\]

where \( a_2 a_4 = a_3^2 \).

Setting \( a_2 = a_3 = 0 \), we get two-dimensional subfamily of \( (7) \) with an \( S_4 \)-symmetry (that is, the linear forms \( l_i \) become the coordinates and we let \( q \) to be a symmetric polynomial).

Using the one-dimensional subgroup of \( \text{PGL}(4) \) preserving the symmetry \( (x_i \mapsto \lambda x_i + \mu \sigma_1(x)) \), we see that the surfaces in the 3-dimensional family are projectively equivalent to the surfaces in the subfamily \( a_2 = a_3 = 0 \); indeed this follows by [Cat21b Proposition 15] since any point of the form \((1, 1, b, c)\) is transformed to a point of the form \((0, 0, 1, b')\) by such a transformation with \( \lambda + \mu (b + c) = 0 \).

**Remark 21.** Using deformation theory, for instance starting from the above two-dimensional family with an \( S_4 \)-symmetry, one can show that there are non-supersingular quartics in \( \mathbb{P}^3 \) with any number of nodes up to 12 (in any characteristic). Indeed, the semiuniversal deformation of nodes of equation

\[
z^2 + uv
\]

is given by

\[
z^2 + uv + a + bz.
\]

We emphasize the following dichotomy: while \( z^2 + uv + a \) yields a smoothing in characteristic different from 2 for \( a \neq 0 \), this is equisingular in characteristic 2; in turn, \( z^2 + uv + bz \) is equisingular in characteristic different from 2, but yields a smoothing in characteristic 2 for \( b \neq 0 \). In either case, this can be made to work at the 12 nodes of the above shape to prove the claim.

4. Generalizations

Proposition 4 applies to any fibre class \( F \) of a genus one fibration perpendicular to more than 12 \((-2)\)-curves. Here we will extend this result to any non-trivial isotropic vector \( E \) (replacing \( F \); the argument
is surprisingly subtle – despite claims to the contrary, see the discussion in [RS18a], especially Remark 2.10.

A key ingredient is provided by the following result from [Sch18] about divisibilities among \((-2)\)-curves which, over \(\mathbb{C}\), can be proved (and was already known before for some cases) by topological methods.

**Theorem 22.** Let \(R \subset \text{Pic}(X)\) be a root lattice generated by \((-2)\)-curves on a K3 surface \(X\). Denote the primitive closure by 
\[
R' = (R \otimes \mathbb{Q}) \cap \text{Pic}(X)
\]
and let \(D \in R' \setminus R\).

Then \(D\) is neither effective nor anti-effective. In particular, \(D^2 \leq -4\).

Let us now state the generalization of Proposition 4 (i), the main characteristic 2 result for this section.

**Proposition 23.** Let \(X\) be a K3 surface in characteristic 2, endowed with a non-zero isotropic vector \(E \in \text{Pic}(X)\) and at least 13 disjoint \((-2)\)-curves orthogonal to \(E\). Then the genus one fibration induced by \(E\) is quasi-elliptic.

**Proof.** By standard arguments (see e.g. [CS21, §5]), we may assume that \(E\) is effective and primitive, and we can apply a composition of reflections \(\sigma\) in \((-2)\)-curves such that \(E' = \sigma(E)\) has no base locus, i.e. it is the fibre class \(F\) of a genus one fibration on \(X\).

Denote the \((-2)\)-curves perpendicular to \(E\) by \(C_1, \ldots, C_s (s > 12)\). The reflections map these curves to \((-2)\)-divisors 
\[
B_i = \sigma(C_i) \perp F
\]
which are effective or anti-effective by Riemann–Roch and thus supported on the fibres of \(|F|\) – or more precisely on a single fibre \(F_v\) each. Proposition 23 will follow at once as soon as we know that the number of mutually orthogonal \((-2)\)-divisors \(B_i\) obtained as above and supported on a single fibre \(F_v\) of an elliptic fibration still satisfies (1).

This follows in greater generality from the next instrumental proposition. 

The next auxiliary result arises naturally from the problem of embedding \((-2)\)-curves into reducible fibres via reflections. We emphasize that it does not depend on the characteristic.

**Proposition 24.** Let \(X\) be a K3 surface in arbitrary characteristic endowed with a genus one fibration. Assume that there are mutually orthogonal \((-2)\)-divisors \(B_1, \ldots, B_r\) supported on a single fibre \(F_v\) which have arisen from disjoint \((-2)\)-curves by way of reflections as above. Then \(r \leq N_v\) for \(N_v\) from Table 1 which we reproduce here for the convenience of the reader.

**Remark 25.** The entries of the table are compatible with the bounds in characteristic 2 in [17] which are due to the wild ramification.
Remark 26. The statement of Proposition 24 is not valid without the assumption that the $B_i$’s arise from disjoint $(-2)$-curves by way of reflections. In fact, each of the lattices $D_{2n}$ ($n \geq 2$) admits $A_1^{2n}$ as a finite index sublattice (as we shall exploit in 4.5.1), so $A_1^{2n}$ embeds into the corresponding fibre, but for $n > 2$ this is not compatible with the assumptions in the proposition. (For $n = 2$, compare Example 27.)

Proof. Assume that the $(-2)$-divisors $B_1, \ldots, B_r$ are supported on a single fibre $F_v$. In particular, each $B_i$ embeds into the negative-semidefinite root lattice of $F_v$. Here classical arguments (cf. e.g. [Nis96]) imply that the embedding always is unique up to reflections in fibre components – and up to fibre multiples (part of the subtlety announced, see Example 27). Hence we will continue to apply the appropriate reflections without introducing new notation for $\sigma$ and for the $B_i$’s.

4.1. Type $\tilde{A}_n$. We start with Dynkin type $\tilde{A}_n$ ($n > 0$) (corresponding to Kodaira type $I_{n+1}$). If $n = 1$, we’re done. Else we get (e.g. by [Nis96, Cor. 4.4])

$$B_1^\perp = \mathbb{Z}F \oplus L \quad \text{with} \quad L_{\text{root}} = A_{n-2}.$$

Inductively we can thus read off that $\tilde{A}_n$ supports no more than $(n + 1)/2$ orthogonal $(-2)$-divisors, confirming the claim.

4.2. Additive fibre types. Turning to the remaining Dynkin types $V = D_n, E_n$, the fibre corresponds to the extended Dynkin type $\tilde{V} = \tilde{D}_n, \tilde{E}_n$. We fix an embedding

$$\iota : \ V \hookrightarrow \tilde{V}$$

(usually obtained by omitting any fixed simple fibre component of $F$).

In the converse direction, there is a surjection

$$\pi : \tilde{V} \twoheadrightarrow V$$

obtained by considering divisors modulo $F$, which is a left inverse for $\iota$. In particular, $\pi$ is injective on the $\mathbb{Z}$-span $\tilde{M} = \langle B_1, \ldots, B_r \rangle_\mathbb{Z} \subset \tilde{V}$ of the orthogonal $(-2)$-classes $B_1, \ldots, B_r$:

$$\pi|_{\tilde{M}} : \tilde{M} \hookrightarrow V.$$

Then we can uniquely (up to the given choice of fibre component) identify

$$v \in \pi(\tilde{M}) =: M \quad \text{with a vector} \quad v + m_vF \in \tilde{M} \quad (m_v \in \mathbb{Z}).$$

It is this fibre multiple which makes things quite subtle as illustrated by the next example.
Example 27. For fibre type \( \tilde{D}_4 \), there is nothing to prove in Proposition 24 as there are obviously four \((-2)\)-curves disjoint embeddings into such a fibre (and no more). Moreover, the embedding

\[ \tilde{M} = 4A_1 \hookrightarrow \tilde{D}_4 \]

is primitive, but this ceases to hold true modulo the fibre. Indeed, then \( M = 4A_1 \) has index two inside \( D_4 \), and this is explained by the fact that the sum of the curves and the fibre becomes 2-divisible.

Note that, if Proposition 24 were not to hold for Dynkin type \( \tilde{D}_5 \), then the above arguments would give an embedding \( A_5 \hookrightarrow D_5 \). Since the square classes of the discriminants of the two lattices do not agree, this is impossible. Hence we will assume that \( n \geq 6 \) in what follows.

4.3. Proof strategy. Consider \( \tilde{\mathbb{V}} = \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_n \) \((n \geq 6)\) and assume that the \((-2)\)-divisors \( B_1, \ldots, B_r \) embed orthogonally into \( \tilde{\mathbb{V}} \), with \( r \) exceeding the bound given in Proposition 24 by one. We shall now consider

- the lattice \( M = \langle \pi(B_1), \ldots, \pi(B_r) \rangle_\mathbb{Z} \cong A_r \subset \mathbb{V} \),
- its primitive closure \( M' = (M \otimes \mathbb{Q}) \cap \mathbb{V} \), and
- its orthogonal complement \( M^\perp \subset \mathbb{V} \).

We proceed by comparing their 2-lengths \( l_2(M) \), i.e. the lengths (defined as the minimum number of generators) of the 2-parts of their discriminant groups \( A_M = M^\vee/M \) etc (using the theory laid out by Nikulin [Nik80]). The proof has 3 steps:

1. prove that \([M' : M] \geq 4\) (Lemma 28);
2. show that \( M'/M \) contains a subgroup of size 4 whose elements are represented by roots (Lemma 29);
3. derive a contradiction using Theorem 22.

4.4. Index \([M' : M]\).

Lemma 28. In the above set-up, we have the index \([M' : M] \geq 4\).

Proof. The next table collects all data relevant to prove the lemma. The first 2 rows are immediate; for the third row, compare [SS19, Table 2.4, p. 33]. For the 4th row, we use that, with \( M' \oplus M^\perp \) embedding into \( \mathbb{V} \) (with summands embedding primitively by definition), the discriminant groups of \( M' \) and \( M^\perp \) have to be compatible, i.e. each contains the cokernel \( V/(M' \oplus M^\perp) \) as a subgroup. In practice, this can be quite complicated, but at any rate it gives the bound

\[ l_2(M') \leq l_2(V) + l_2(M^\perp), \]

as we are now going to show.

Indeed, write \( N := M' \), so that \( N, N' := N^\perp = M^\perp \) are primitively embedded into \( V \), that is, we can write \( V = N \oplus T = T' \oplus N' \), which is a direct sum, but not orthogonal.
The roles of $N, N'$ are symmetric, and we observe that we have a series of inclusions

$$0 \subset N \oplus N' \subset V \subset V^{\vee} \subset N^{\vee} \oplus (N')^{\vee}$$

where some inclusions are given by the quadratic form. Let us first take the quotient by $N$: we get then

$$0 \subset N' \subset T \subset N^{\vee}/N \oplus T^{\vee} \subset N^{\vee}/N \oplus (N')^{\vee},$$

then we take the quotient by $N'$, yielding

$$0 \subset T/N' \subset N^{\vee}/N \oplus T^{\vee}/N' \subset N^{\vee}/N \oplus (N')^{\vee}/N'.$$

To obtain a system of generators of the discriminant group $N^{\vee}/N$ it suffices to lift a system of generators of $V^{\vee}/V$, the quotient of the second by the first piece of the filtration, and then to lift a system of generators of $(N')^{\vee}/N'$, which contains $T/N'$: then the submodule generated by above elements surjects onto $(N)^{\vee}/N$. The same argument applies for the generators of the binary parts.

Since the length is trivially bounded by the rank, i.e. $l_2(M^{\perp}) \leq \text{rk}M^{\perp} = n - r$, adding the entries in the second and third rows gives the bounds in the fourth row.

| Dynkin type $V$ | $\text{D}_2m$ ($m > 2$) | $\text{D}_{2m+1}$ ($m > 2$) | $E_6$ | $E_7$ | $E_8$ |
|-----------------|-----------------|-----------------|-------------|-------------|-------------|
| $r = \text{rk}M$ | $m + 3$ | $m + 3$ | 5 | 6 | 6 |
| $n - r = \text{rk}M^{\perp}$ | $m - 3$ | $m - 2$ | 1 | 1 | 2 |
| $l_2(V)$ | 2 | 1 | 0 | 1 | 0 |
| $l_2(M') \leq$ | $m - 1$ | $m - 1$ | 1 | 2 | 2 |

Finally we compare the 2-lengths of $M$ and $M'$. Letting $\mu = l_2(M) - l_2(M')$, we infer that $2^{\mu} \det(M') | \det(M)$, so standard formulas yields

$$|M'/M| \geq 2^{\lceil \frac{\mu}{2} \rceil}.$$  

Since $l_2(M) = r$, we get $\mu \geq 4$ for each Dynkin type. This implies the lemma.

**4.5. Roots in $M' \setminus M$.**

**Lemma 29.** There is a subgroup $H \subset M'/M$ of size 4 whose elements are represented by roots.

**Proof.** The overlattice $M'$ is encoded in some isotropic subgroup of $A_M$ (of size at least 4 by Lemma 28). Since $M \cong A_1^r$, we have $A_M \cong (\mathbb{Z}/2\mathbb{Z})^r$, and all isotropic vectors in $A_M$ are represented by roots if $r < 8$ (precisely $(a_1 + \ldots + a_4)/2$ and its permutations). This settles the lemma for $n \leq 9$ and leaves types $\text{D}_n$ for $n \geq 10$. We shall use the standard fact (cf. e.g. [Nis96 Cor. 4.4]) that

$$A_1^r \cong A_1 \oplus D_{n-2} \subset D_n \quad \forall n \geq 4$$

Finally, we compare the 2-lengths of $M$ and $M'$. Letting $\mu = l_2(M) - l_2(M')$, we infer that $2^{\mu} \det(M') | \det(M)$, so standard formulas yields

$$|M'/M| \geq 2^{\lceil \frac{\mu}{2} \rceil}.$$  

Since $l_2(M) = r$, we get $\mu \geq 4$ for each Dynkin type. This implies the lemma.
(with the convention that $D_2 = A_1^2$ and $D_3 = A_3$).

Geometrically, we can see this from the diagram

\[
\begin{array}{cccccc}
(D_n) & \cdots & d_{n-2} & d_{n-1} & d_n \\
d_1 & d_2 & \cdots \\
\end{array}
\]

by taking the original $A_1$ to be generated by $d_1$. Then its orthogonal complement visibly contains $\langle d_3, \ldots, d_n \rangle \cong D_{n-2}$, but also another orthogonal summand generated by the fundamental cycle $\gamma_n = d_1 + 2(d_2 + \ldots + d_{n-2}) + d_{n-1} + d_n$.

4.5.1. Type $\tilde{D}_{2m}$ ($m > 4$). Applied successively to each summand of $M = A_{r1}$ ($r = m + 3$) embedding into $D_{2m}$, equation [9] implies that $M^\perp$ is an overlattice of $A_{m-3}^1$ (since each $D_{2k}$ is an overlattice of $A_{2k}^1$). Apply now [9] to the finite index sublattice $A_{m-3}^1 \subset M^\perp$ to find the finite index inclusions

\[ M' = (M^\perp)^\perp \supseteq D_6 \oplus A_{m-3}^1 \supseteq M. \]

The rightmost inclusion identifies the desired subgroup $H \subset M'/M$ as $H \cong D_6/A_6^1 \cong (\mathbb{Z}/2\mathbb{Z})^2$ whose elements are represented by roots (by the same argument used to reduce to $n \geq 10$).

4.5.2. Type $\tilde{D}_{2m+1}$ ($m > 4$). This case is reduced to the previous one by the following general lemma:

**Lemma 30.** For all $r, m > 0$, any embedding $A_1^r \hookrightarrow D_{2m+1}$ factors through $D_{2m}$ (primitively embedded in $D_{2m+1}$).

**Proof.** The statement about the primitive embedding is obvious since otherwise the primitive closure $D'$ of $D_{2m}$ inside $D_{2m+1}$ would be a unimodular lattice (by inspection of its discriminant group), its orthogonal complement would be a rank one summand $T$, and the sum $D' \oplus T$ would be equal to $D_{2m+1}$, since the quotient $D_{2m+1}/(D' \oplus T)$ embeds into the discriminant group $A_D$ which is trivial. For determinant reasons, we have $T \cong \langle -4 \rangle$ contradicting the fact that $D_{2m+1}$ is generated by roots.

We continue to prove the main statement of the lemma by induction on $m$.

For $m = 1$, we have, by our convention, $D_3 = A_3$ which only allows for $r = 1, 2$. That is, $A_1$ and $A_1^2 = D_2$ embed into $D_3 = A_3$, and the claim is already there.

For the induction step from $m - 1$ to $m$, we first embed one copy of $A_1$ into $D_{2m+1}$. By [9] the remaining copies $A_1^{r-1}$ embed as follows:

\[ A_1^{r-1} \hookrightarrow A_1 \oplus D_{2m-1}. \]
There are two cases. If the embedding (10) involves the first orthogonal summand of the target lattice, then

\[ A_1^{r-2} \hookrightarrow D_{2m-1} \quad \Longrightarrow \quad A_1^{r-2} \hookrightarrow D_{2m-2} \hookrightarrow D_{2m-1} \]

by the induction hypothesis. By \cite{Nis96} [Lemma 4.2 (ii)], \( D_{2m-2} \) embeds uniquely, up to isometries, as \( \langle d_2, \ldots, d_{2m-1} \rangle \) in the notation of the previous figure. Hence

\[ (11) \quad A_1' \hookrightarrow A_1^2 \oplus D_{2m-2} \hookrightarrow A_1^2 \oplus D_{2m-1} \hookrightarrow D_{2m+1}, \]

where the middle embedding is primitive (by what we have argued before) and the primitive closure of \( A_1^2 \oplus D_{2m-1} \) inside \( D_{2m+1} \) is given, in terms of the previous embedding with image \( \langle d_1 \rangle \oplus \langle \gamma_{2m+1} \rangle \oplus \langle d_3, \ldots, d_{2m+1} \rangle \), by adjoining the root \( \delta = (d_1 + \gamma_{2m+1} + d_2m + d_{2m+1})/2 \). In turn, the primitive closure of \( A_1^2 \oplus D_{2m-2} = \langle d_1 \rangle \oplus \langle \gamma_{2m+1} \rangle \oplus \langle d_4, \ldots, d_{2m+1} \rangle \) inside \( D_{2m+1} \) amounts to adjoining \( \delta \), too (equivalently, one adjoins the root \( d_3 + d_4 \) to \( A_1^2 \oplus D_{2m-2} \)). Hence, the primitive closure of \( A_1^2 \oplus D_{2m-2} \) inside \( D_{2m+1} \) is isometric to \( D_{2m} \), and the claim of the lemma follows in this case.

If the embedding (10) does not involve the first orthogonal summand of the target lattice, then along the same lines

\[ A_1^{r-1} \hookrightarrow D_{2m-1} \quad \Longrightarrow \quad A_1^{r-1} \hookrightarrow D_{2m-2} \hookrightarrow D_{2m-1}. \]

The chain of embeddings (11) has to be modified to

\[ (12) \quad A_1' \hookrightarrow A_1 + D_{2m-2} \hookrightarrow A_1^2 + D_{2m-2} \hookrightarrow D_{2m+1}, \]

but still the second rightmost lattice has primitive closure \( D_{2m} \) inside \( D_{2m+1} \) as claimed.

With all Dynkin types covered, the proof of Lemma 29 is complete.

\[ \square \]

4.6. Conclusion of the proof of Proposition 24. Lemma 29 furnishes us with an isotropic subgroup \( H \subset M' \setminus M \) of size 4 whose non-zero elements are represented by roots \( v_j \in M' \setminus M \). Since \( H \cong (\mathbb{Z}/2\mathbb{Z})^2 \) (because \( H \) is a subgroup of \( A_M \cong (\mathbb{Z}/2\mathbb{Z})^r \)), there is a relation

\[ (13) \quad v_1 + v_2 + v_3 = 0 \quad \text{mod} \quad M. \]

For each \( j \), we have \( 2v_j \in M \), so we can consider the unique pre-images of these vectors in \( \tilde{M} \):

\[ w_j = 2v_j + m_j F \in \tilde{M}. \]

Since \( \pi|_{\tilde{M}} \) is injective, (13) implies that

\[ w_1 + w_2 + w_3 \in 2\tilde{M}. \]

Hence \( (m_1 + m_2 + m_3)F \in 2\tilde{M} \), and since no multiple of \( F \) is in \( \tilde{M} \) (because \( \tilde{M} \) is negative-definite), we infer that \( m_1 + m_2 + m_3 = 0 \). In particular, one of the \( m_j \) is even and the corresponding vector \( w_j \in M \).
is 2-divisible in \( \tilde{V} \). Applying the inverse reflections \( \sigma^{-1} \), we obtain a linear combination \( \sigma^{-1}(w_j) \) of the \( C_i \) such that
\[
\sigma^{-1}(w_j)/2 \in \text{Pic}(X) \setminus \langle C_1, \ldots, C_r \rangle \mathbb{Z}
\]
is a root. This contradicts Theorem 22 and thus completes the proof of Proposition 23.

\[ \square \]

4.7. Application to higher singularities.

**Proposition 31.** Let \( X \subset \mathbb{P}^3 \) be quartic surface with 12 singular points such that the minimal resolution \( S \) is not supersingular. Then all singularities have types \( A_1 \) or \( A_2 \).

**Proof.** Assume there is a singular point of type \( A_n \) (\( n \geq 3 \)), \( D_k \) or \( E_l \). Then \( S \) contains an \( A_3 \) configuration of smooth rational curves \( C, C', C'' \) lying above this singular point and 11 smooth rational curves \( C_1, \ldots, C_{11} \) lying above the other singularities. Hence there is an isotropic vector
\[
E = H - (C + 2C' + C'').
\]
Note that there are 13 disjoint smooth rational curves orthogonal to \( E \), namely \( C_1, \ldots, C_{11}, C, C'' \). Hence, Proposition 23 gives the contradiction that \( S \) is supersingular.

\[ \square \]

**Proposition 32.** Let \( X \subset \mathbb{P}^3 \) be quartic surface with 12 singular points such that the minimal resolution \( S \) is not supersingular. Then there are at most 3 singularities of type \( A_2 \).

**Proof.** Assume that there are 4 singular points of type \( A_2 \) (or higher). Denote the exceptional curves in \( S \) lying above 3 of them by
\[
C_1, C_2, \quad C'_1, C'_2, \quad C''_1, C''_2.
\]
Consider the isotropic vector
\[
E = 3H + (C_1 + 2C_2) + (C'_1 + 2C'_2) + 2(C''_1 + 2C''_2).
\]
Then \( |E| \) induces an elliptic fibration on \( S \), and by the proof of Proposition 24, the fibres contain orthogonal configurations of smooth rational curves of types \( A_2 \) (lying above the fourth \( A_2 \) singularity) and \( A_{11} \) (\( C_1, C'_1, C''_1 \) and exceptional curves above the remaining 8 singular points). But this is excluded by Proposition 4 (iii).

\[ \square \]

4.8. **Proof of Theorem 2** By Theorem 11 a normal quartic surface \( X \subset \mathbb{P}^3 \) whose minimal resolution is not a supersingular K3 surface contains at most 13 singular points. We ruled out 13 nodes in 3.4 and other configurations of rational double points in 3.7. This proves that \( X \) contains at most 12 singular points. By Proposition 31 they have types \( A_1 \) or \( A_2 \), and by Proposition 32 there are at most 3 \( A_2 \)'s.

The generic member of the family in Example 19 contains 12 nodes; its minimal resolution is a K3 surface which is not supersingular. The
family has dimension 22 in the space of quartics. This completes the proof of Theorem 2.

5. General results

Let us draw some consequences of Proposition 23. The first one is very much in line with Theorem 1.

Corollary 33. Let $X$ be a K3 surface in characteristic 2. Fix a positive vector $h \in \text{Pic}(X)$ with square $h^2 = 2n > 0$ and assume that $n$ is the sum of $r$ squares. Then $X$ contains no more than $12 + r$ $(−2)$-curves orthogonal to $h$ unless $X$ is supersingular.

Proof. Assume that $X$ contains $m = 13 + r$ $(−2)$-curves $C_1, \ldots, C_m$. Writing $n = a_1^2 + \ldots + a_r^2$, we obtain the isotropic vector 

$$E = h - a_1C_1 - \ldots - a_rC_r.$$ 

This is perpendicular to the 13 $(−2)$-curves $C_{r+1}, \ldots, C_m$, so the genus one fibration induced by $|E|$ is quasi-elliptic by Proposition 23. Equivalently, by [Ru-Sha79], $X$ is supersingular.

In comparison with Theorem 1, Corollary 33 affords for much more flexibility as $h$ is not required to give a quasi-polarization; also the $(−2)$-curves need not correspond to simple nodes (e.g., we could also take two curves from an $A_3$-configuration or three curves from a $D_4$-configuration).

5.1. Proof of Theorem 3. As a second corollary, we indicate the proof of Theorem 3. Assume that $X$ contains 13 $(−2)$-curves $C_1, \ldots, C_{13}$. Their orthogonal complement in $\text{Pic}(X)$ is a hyperbolic lattice of rank at least 5, thus it represents zero by some non-trivial vector $E$ by Meyer’s theorem [Ser70, IV.3.2, cor. 2 to th. 8]). By Proposition 23, the genus one fibration induced by $|E|$ is quasi-elliptic. But then $X$ is unirational, so $\rho(X) = 22$, contradiction.

Remark 34. The result is sharp by virtue of the fibrations from [Shio74a].

We can also prove that the 12 disjoint $(−2)$-curves in Theorem 3 cannot be extended:

Proposition 35. Let $X$ be a K3 surface in characteristic 2 with Picard number $18 \leq \rho(X) \leq 20$. If $X$ contains 12 disjoint ADE configurations, then each has type $A_1$.

Proof. Assume that at least one configuration has two components and consider a subconfiguration of type $A_1^1 + A_2$. As in the proof of Theorem 3, its orthogonal complement has rank at least 5, so the subconfiguration is supported on the singular fibres of some elliptic fibration. However, this is impossible by Proposition 3 (iii).
**Remark 36.** Note the following consequence of Proposition 35: if $S$ in Theorem 2 has Picard number 18, 19 or 20, then we can strengthen the statement to the extent that all 12 singular points of $X$ necessarily are nodes.

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