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A correction of the decomposability result in a paper by Meyer-Neutsch

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Abstract

In their paper of 1993, Meyer and Neutsch established the existence of a 48-dimensional associative subalgebra in the Griess algebra $G$. By exhibiting an explicit counter example, the present paper shows a gap in the proof of one of the key results in Meyer and Neutsch’s paper, which states that an idempotent $a$ in the Griess algebra is indecomposable if and only if its Peirce 1-eigenspace (i.e. the 1-eigenspace of the linear transformation $L_a : x \mapsto ax$) is one-dimensional. The present paper fixes this gap, and shows a more general result: let $V$ be a real commutative nonassociative algebra with an associative inner product, and let $c$ be a nonzero idempotent of $V$ such that its Peirce 1-eigenspace is a subalgebra; then, $c$ is indecomposable if and only if its Peirce 1-eigenspace is one-dimensional. The proof of this result is based on a general variational argument for real commutative metrised algebras with inner product.

Keywords: Commutative nonassociative algebras, Griess algebra, Idempotents, Associative bilinear form, Metrised algebras

1. Introduction

In their 1993 paper [5], Meyer and Neutsch established the existence of a 48-dimensional associative subalgebra in the Griess algebra $G$ and conjectured that 48 is the largest possible dimension of an associative subalgebra of $G$. This conjecture has been proved by Miyamoto [6].

One of the key results claimed in [5], Theorem 11 asserts that an idempotent $a \in G$ is indecomposable if and only if the Peirce eigenspace $G_a(1)$ with the eigenvalue 1 is at most one-dimensional. This result was used by Miyamoto (see Theorem 6.8 in [6]) and in a later research on associative subalgebras of low-dimensional Majorana algebras, see for instance p. 576 in [2] and section 3.1 in [4].

Unfortunately, there is a gap in the argument in the proof of Theorem 11 in [5]. Recall the statement of the theorem.

**Theorem 1.1** (Theorem 11, in [5]). An idempotent $a \in G$ is indecomposable if and only if the Peirce eigenspace $G_a(1) = \{ x \in G : ax = x \}$ is at most one-dimensional.

**Remark 1.2.** Note also that $G_a(1)$ is a subalgebra in $G$, see [5].
More precisely, in their proof of the implication “dim \( \theta_c(1) \geq 2 \Rightarrow c \) is decomposable”, Meyer and Neutsch use an erroneous argument that a (normalized) cubic form

\[
\varphi(x) := \frac{\langle x, x^2 \rangle}{\langle x, x \rangle}
\]

provides at least two linearly independent idempotents. Let us explain what is a gap here. It is well known (and easy to see) that \( x \neq 0 \) is a stationary point of \( \varphi(x) \) precisely when \( x^2 = \lambda x \) for some \( \lambda \in \mathbb{R} \). If \( \lambda \neq 0 \) then \( x \) gives rise to a (nonzero) idempotent by normalizing: indeed \( c = x/\lambda \) implies \( c^2 = c \). In order to get two distinct (linearly independent) idempotents, Meyer and Neutsch suggest that a minimum and a maximum point of \( \varphi \) would provide us with two linearly independent idempotents (see p. 15 in [5]). But, since \( \varphi \) is an odd function, this may give us two anti-collinear, and therefore linearly dependent, elements. In particular, it is clear from (1) that \( x \) is a local maximum of \( \varphi \) if and only if \( -x \) is a local minimum of \( \varphi \).

**Remark 1.3.** We will point out that the variational argument used by the authors in the proof of Lemma 2 and Theorem 8 in [5] causes no problems because one uses there an even function (based on a quartic form). This, together with nonzero lower estimate (76), allows one to guarantee at least two linear independent stationary points (corresponding to a local maximum and minimum).

The variational argument used in [5] would be easily fixed if one could prove that a general cubic form always has at least 2 linearly independent stationary points with nonzero gradient\(^1\). However, this property fails in general, as a counterexample in Section 3 below shows.

Thus, the method suggested in [5] does not work in general. Nevertheless, the claim of Theorem 11 holds true. More precisely, we have the following result which contains Theorem 11 in [5] as a corollary.

**Theorem 1.4.** Let \( V \) be a commutative nonassociative algebra over \( \mathbb{R} \) with an associative inner product \( \langle , \rangle \). Suppose \( c \neq 0 \) is an idempotent of \( V \) and the Peirce eigenspace \( V_c(1) \) is a subalgebra of \( V \). Then \( \dim V_c(1) = 1 \) if and only if \( c \) is indecomposable.

The theorem easily follows from an auxiliary result (Lemma 2.3 below) which has an independent interest. The proof of our results uses the notion of metrised algebra. In the next section we recall some standard definitions and introduce the so-called algebras of cubic forms.

### 2. Algebras of cubic forms

Let \( V \) be a vector space over a field \( k \) of characteristic not 2 or 3. A multiplication on \( V \) is a bilinear map (denoted by juxtaposition)

\[
V \times V \ni (x, y) \mapsto xy \in V.
\]

A vector space with a multiplication is called an algebra over \( k \).

A symmetric \( k \)-bilinear form \( Q(x, y) : V \times V \to k \) is called nonsingular if \( Q(x, y) = 0 \) for all \( y \in V \) implies \( x = 0 \) [3]. In what follows, we use the standard inner product notation for \( Q \):

\[
Q(x, y) = \langle x, y \rangle
\]

\(^1\)Note that the vanishing of gradient implies rather a 2-nilpotent, i.e. \( x^2 = 0 \), than idempotent. Therefore one assumes that \( \lambda \neq 0 \).
and the squared norm notation $|x|^2 = \langle x, x \rangle$. Note that $|x|$ may be not defined in general.

If $V$ is an algebra then the bilinear form $\langle \cdot, \cdot \rangle$ is called associative [9], [8] if
\[ \langle xy, z \rangle = \langle x, yz \rangle, \quad \forall x, y, z \in V. \tag{2} \]

An algebra carrying a non-singular symmetric bilinear form is called metrised, cf. [1]. The standard examples are Lie algebras with the Killing form and Jordan algebras with the generic trace bilinear form. The most striking corollary of the existence of an associative bilinear form on a commutative algebra is that the operator of left (=right) multiplication
\[ L_x : y \to xy, \]
is self-adjoint, i.e.
\[ \langle L_y z, x \rangle = \langle y, L_z x \rangle, \quad \forall x, y, z \in V. \tag{3} \]

A function $u : V \to k$ is called a cubic form if its full linearization
\[ u(x, y, z) = u(x + y + z) - u(x + y) - u(x + z) + u(x) + u(y) + u(z) \]
is a trilinear form. Note that thus defined linearization is obviously symmetric and the cubic form is recovered by
\[ 6u(x) = \langle x, x, x \rangle. \tag{4} \]
The vector space of all cubic forms on $V$ will be denoted by $C_3(V)$.

The following is an immediate corollary of the definitions.

**Proposition 2.1.** Given a vector space $V$ with a non-singular symmetric bilinear form $\langle \cdot, \cdot \rangle$, there exists a canonical bijection between $C_3(V)$ and commutative metrised algebras $(V, \langle \cdot, \cdot \rangle)$, where the multiplication $(x, y) \to xy$ is uniquely determined by
\[ xy := \text{the unique element satisfying } \langle xy, z \rangle = u(x, y, z) \text{ for all } z \in V \tag{5} \]

Given a cubic form $u$ on $V$, the commutative nonassociative algebra obtained according to Proposition 2.1 is denoted by $V(u)$ and is called the algebra of the cubic form $u$. Conversely, given a commutative metrised algebra $A = (V, \langle \cdot, \cdot \rangle)$, define the cubic form
\[ u_A(x) := \frac{1}{6} \langle xx, x \rangle. \tag{6} \]
The factor $\frac{1}{6}$ is chosen to agree with (4). Then $A$ is isomorphic to $V(u_A)$.

We point out that as a corollary of the symmetry of $u(x, y, z)$, the algebra $V(u)$ is always commutative but may be nonassociative. It is easy to see that the following is true.

**Proposition 2.2.** $V(u)$ is a zero algebra if and only if $u \equiv 0$.

Now we consider the specialization of the above definitions when $V$ is a vector space over $k = \mathbb{R}$ or $\mathbb{C}$. Let $\langle \cdot, \cdot \rangle$ be some non-singular symmetric bilinear form on $V$ and let us fix a cubic form $u$. Then it readily follows from (6) and (5) that the gradient of $u(x)$ is essentially the square of the element $x$ (in $V(u)$):
\[ Du(x) = \frac{1}{2} xx = \frac{1}{2} x^2, \tag{7} \]
and, furthermore, the multiplication in $V(u)$ is explicitly recovered by

$$\langle D^2u(x) y, z \rangle = \langle x, z \rangle,$$

where $D^2u(x)$ is the Hessian matrix of $u$ at $x$. This observation is an important ingredient in the proof of the main results of [5], in particular of Theorem 11. The relations (7) and (8) are also important tools in recent applications of nonassociative algebras to global analysis, see [7, Ch. 6], [10]. Note also that if $k$ is an arbitrary field, then (7) and (8) can be interpreted as formal definitions of the gradient and the Hessian of $u$.

The case $k = \mathbb{R}$ has a particular interest as the following property shows.

**Lemma 2.3.** Let $(V, \langle \cdot, \cdot \rangle)$ be a (nonzero) commutative metrized algebra over $\mathbb{R}$, $\langle \cdot, \cdot \rangle$ being positive definite. The set $E$ of constrained stationary points of the variational problem

$$\langle x, x^2 \rangle \to \max \quad \text{subject to a constraint} \quad \langle x, x \rangle = 1$$

is nonempty and the maximum is attained. Denote by $E_0 \subset E$ the (nonempty) set of local maxima in (9). Then for any $x \in E$, either $x^2 = 0$ or $c := x / \langle x^2, x \rangle$ is an idempotent in $V$. If additionally $x \in E_0$ then $\langle x^2, x \rangle > 0$ and the corresponding idempotent $c = x / \langle x^2, x \rangle$ satisfies the extremal property

$$L_c \leq \frac{1}{2} \quad \text{on } c^\perp := \{ x \in V : \langle x, c \rangle = 0 \}.$$  

In particular, the eigenvalue 1 of $L_c$ is simple.

**Proof.** By the positive definiteness of $\langle \cdot, \cdot \rangle$, the unit sphere $S = \{ x \in V : \langle x, x \rangle = 1 \}$ is compact in the standard Euclidean topology on $V$ induced by $|x|^2 = \langle x, x \rangle$. By the assumption and Proposition 2.2, the cubic form $u(x) = \frac{1}{2} \langle x^2, x \rangle$ is not 0. Note that the algebra $V(u)$ is isomorphic to $V$.

Since $u$ is continuous as a function on $S$, it attains its maximum value there. In particular both $E_0$ and $E$ are nonempty. Let $x \in E$ be a stationary point, then using Lagrange’s undetermined multipliers and (7) we obtain for the directional derivative

$$0 = \partial_x u(x) = \frac{1}{2} \langle x^2, y \rangle \quad \text{for any } y \in V \text{ and } \langle x, y \rangle = 0,$$

which implies that $x$ and $x^2$ are parallel, i.e. $x^2 = \lambda x$ where $\lambda = \langle x^2, x \rangle$. Thus, $x^2 = 0$ if and only if $\langle x^2, x \rangle = 0$, otherwise scaling appropriately we have that $c := x / \langle x^2, x \rangle$ is a nonzero idempotent in $V$.

To prove the second claim of the lemma, note that the function $f(x) = \frac{\langle x^2, x \rangle}{|x|}$ is homogeneous of degree zero and smooth outside the origin. We have

$$\partial_x f(x) = \frac{3 \langle x^2, y \rangle |x|^2 - \langle x^2, x \rangle \langle x, y \rangle}{|x|^4},$$

implying by duality the expression for the gradient

$$\frac{1}{2} Df(x) = \frac{x^2 |x|^2 - \langle x^2, x \rangle x}{|x|^4}.$$  

Arguing similarly, one obtains

$$\frac{1}{2} \partial_x Df(x) = \frac{2 |x|^4 xz - 3 |x|^2 (x^2 \langle x, z \rangle + \langle x^2, z \rangle x) - \langle x^2, x \rangle |x|^2 z + 5 \langle x, z \rangle \langle x^2, x \rangle x}{|x|^7}.$$
hence the Hessian is given by
\[ \frac{1}{3} D^2 f(x) = \frac{2|x|^4 L - (x^2, x)|x|^2 - 3|x|^2(x \otimes x^2 + x^2 \otimes x) + 5(x^2, x)x \otimes x}{|x|^5}. \]  (12)

Now let \( x \in E_0 \). Then \( x \) is a local maximum of \( f \) and obviously \( f(x) = (x^2, x) > 0 \) (because \( \langle x^2, x \rangle \neq 0 \) on \( V \)). By the above \( c = x/(x, x) \) is an idempotent, and by the homogeneity \( f(x) = f(c) \), hence \( c \) is a local maximum of \( f \) too. Therefore the maximum condition implies that the Hessian matrix is non-positive definite
\[ 0 \geq \frac{1}{3} D^2 f(c) = \frac{(2L - 1)|c|^2 - c \otimes c}{|c|^3}. \]

Since \( c \otimes c = 0 \) on \( c^\perp \) we conclude that \( 2L - 1 \leq 0 \) on \( c^\perp \), thus, the desired conclusion follows.

**Definition 2.4.** An idempotent \( c \in (V, Q) \) corresponding to any local maximum in (9) is said to be extremal.

Recall that an idempotent is called decomposable if it may be expressed as a sum of at least two nonzero idempotents; otherwise, the idempotent is called indecomposable. An important corollary of Lemma 2.3 is the following

**Corollary 2.5.** Let a nonzero unital commutative algebra \( V \) over \( \mathbb{R} \) possess a symmetric positive definite associative bilinear form and \( \dim V \geq 2 \). Then there exists an idempotent in \( V \) distinct from the unit. In particular, the unit is decomposable and \( V \) has at least three distinct nonzero idempotents.

**Proof.** By Lemma 2.3 there exists a (nonzero) extremal idempotent \( c \in V \). By the extremal property, one also has \( L_c \leq \frac{1}{2} \) on \( c^\perp \). Note that the subspace \( c^\perp \) is nontrivial because \( \dim c^\perp = \dim V - 1 \geq 1 \). Let \( e \in V \) be the algebra unit. Then \( e \) is an idempotent and \( e \neq c \) because \( L_c \equiv 1 \) on \( V \). Then \( (e - c)^2 = e - 2ec + c = e - c \), hence \( e - c \) is also an idempotent. The claim follows.

Now we are able to finish the proof of the main result.

**Proof of Theorem 1.4.** Let \( c \) be a decomposable idempotent, i.e. \( c = c_1 + c_2 \) where \( c_1, c_2 \) are nonzero idempotents. Then \( c_1c_2 = 0 \) and \( c_1, c_2 \) are obviously linearly independent. One has therefore \( c_1c_2 = c_1 \), hence \( c_1 \in V_c(1) \), implying \( \dim V_c(1) \geq 2 \).

In the converse direction, suppose that \( c \neq 0 \) is an idempotent in \( V \) with \( \dim V_c(1) \geq 2 \) and \( V_c(1) \) is a subalgebra of \( V \). Note that \( c \) acts on the subalgebra \( V_c(1) \) as a unit. Therefore by Corollary 2.5, \( c \) is decomposable.

**Remark 2.6.** Some final remarks are also in order. In the proof of Lemma 2.3, we made an essential use of the condition that the associative form is positive definite. It is unclear, however, if Theorem 1.4 remains true if one drops the definiteness condition and consider a weaker assumption, for example, that \( \langle \cdot, \cdot \rangle \) is merely non-singular.
3. A counterexample

Here we construct an example of a cubic form whose stationary points with nonzero gradient lie on a line. On the nonassociative algebra level, this shows that for any \( n \geq 2 \) there exist nontrivial \( n \)-dimensional algebras over \( \mathbb{R} \) with associative bilinear form having only one nonzero idempotent.

Let \( n \geq 2 \) and consider a cubic form \( u(x) = \frac{1}{2}x_1(x_1^2 + 3a_2x_2^2 + \ldots + 3a_nx_n^2) \) in \( \mathbb{R}^n \) equipped with the standard Euclidean scalar product. Suppose that all \( a_i \) are different and \( 0 < a_i < \frac{1}{2} \). We claim that any stationary point of \( \varphi(x) = u(x)/|x|^3 \) with nonzero gradient is parallel to \( \xi := (1, 0, \ldots, 0) \).

Indeed, given a stationary point \( x \) of \( \varphi(x) \), the stationary point condition readily yields \( Du(x) = \lambda x \) for some real \( \lambda \). By the assumption \( Du(x) \neq 0 \), hence \( \lambda \neq 0 \), in particular \( x \neq 0 \). Since \( u(-x) = -u(x) \), one has \( Du(x) = Du(-x) \), thus interchanging \( x \) and \( -x \), one may assume without loss of generality that \( \lambda > 0 \), and, also by the homogeneity of \( u \) that \( \lambda = 1 \). This yields

\[
x_1^2 + (a_2x_2^2 + \ldots + a_nx_n^2) = x_1, \\
2a_kx_kx_1 = x_k, \quad \text{for } k = 2, \ldots, n.
\]

(13)

(14)

If \( x_k = 0 \) for \( 2 \leq k \leq n \) then by (13) \( x_1^2 = x_1 \), thus \( x_1 = 1 \), i.e. \( x = \xi \). If at least two \( x_i \) and \( x_j \) are nonzero for \( 2 \leq i, j \leq n \) then (14) implies \( x_1 = \frac{1}{2a_k} = \frac{1}{2a_j} \), a contradiction with \( a_k \neq a_j \). Therefore there can exist only one nonzero \( x_i \neq 0 \) for some \( 2 \leq i \leq n \). But in that case, \( x_1 = \frac{1}{2a_i} \) and (13) yields by the assumptions on \( a_i \) that \( x_1^2 = \frac{2a_i - 1}{4a_i^3} < 0 \), a contradiction again. This proves the claim.

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