Dynamical protection of quantum computation from decoherence in laser-driven cold-ion and cold-atom systems

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\textbf{Abstract.} We show that by dynamically modulating the gate fields so as to minimize the spectral overlap of the decoherence and modulation spectra specifically for each qubit, and concurrently controlling all qubits, one can significantly increase the fidelity of quantum computation. Notably, cross-decoherence, introduced by two-qubit entanglement, can be eliminated by appropriate dynamical control. In this scheme, contrary to traditional schemes, one can increase the gate duration, while, simultaneously, increasing the total gate fidelity. Experimental scenarios involving laser-driven cold-ions and cold-atoms are shown to benefit from this counterintuitive scheme.
1. Introduction

The experimental realization of a universal quantum computer is widely recognized to be
difficult due to decoherence effects, particularly dephasing, which entail loss of quantum
information \([1]–[4]\) through randomization of the relative phase of quantum states \([5]\).
The entanglement of qubits via two-qubit gates \([6]–[8]\), which is at the heart of quantum
computation, enhances the effects of decoherence and thus causes a faster decay of
computational fidelity \([9, 10]\).

The decoherence of a single qubit has been extensively investigated theoretically \([11]\)
and experimentally \([12]\). Suggestions have been made to combat the dephasing only during
the storage stage, by applying sufficiently frequent, fast and strong pulses \([13]\). Multipartite
decoherence and its dynamical control have also been studied \([14]–[16]\). Furthermore, the
use of coherent control techniques has resulted in better gate pulses in complex experimental
set-ups \([17]\). However, the adverse effects of qubit entanglement, which introduces cross-
dephasing, and the design of single- and two-qubit gate pulses aimed at reducing the dephasing,
have not been addressed as yet \([1, 14, 18, 19]\).

In this paper, we describe a universal dynamical control approach to the dephasing
problem during all the stages of quantum computations, namely (i) storage, wherein the
quantum information is preserved in between gate operations; (ii) single-qubit gates, wherein
individual qubits are manipulated, without changing their mutual entanglement; and (iii) two-
qubit gates, that introduce controlled entanglement. We show that in terms of reducing the
effects of dephasing, it is advantageous to concurrently and specifically control all the qubits
of the system, whether they undergo quantum gate operations or not. Our approach consists of
specifically tailoring each dynamical quantum gate, with the aim of suppressing the dephasing, thereby greatly increasing the gate fidelity. In the course of two-qubit entangling gates, we show that cross-dephasing can be completely eliminated by introducing additional control fields. Most significantly, we show that one can increase the gate duration, while simultaneously reducing the effects of dephasing, resulting in a total increase in gate fidelity. This is at odds with the conventional approaches, whereby one tries to either reduce the gate duration, or increase the coherence time.

The paper is organized as follows: section 2 describes the multi-qubit system, its Hamiltonian and the solution for its evolution in the presence of control fields, to second order in the random frequency-shifts (noise). In section 3, the resulting fidelities are derived for single- and two-qubit gates under concurrent dephasing and control. A detailed numerical example of three-qubit computation is given in section 4, followed by suggested experimental implementations for trapped ions and cold atoms in optical lattices in section 5. We conclude with a discussion in section 6.

2. Control of multi-qubit dephased system

At the outset of our treatment we introduce the multi-qubit system with its gates implementation. We then arrive at a general solution to the problem of its dynamically controlled dephasing. Our system comprises $N$ qubits, with ground and excited states $|g\rangle_j, |e\rangle_j$, respectively, and identical excitation energy $\hbar \omega_a$. Each qubit’s excited state experiences random fluctuations, $\hbar \delta_j(t)$, thus introducing random dephasing. The total Hamiltonian is given by:

$$\hat{H} = \hat{H}^{(0)}(t) + \hat{H}^{(1)}(t) + \hat{H}^{(2)}(t),$$

where the superscript on each term denotes the manipulation type (e.g. 1 and 2 for one- and two-qubit manipulation, respectively). Using subscripts to denote the qubits that are subject to manipulation, these terms are:

$$\hat{H}^{(0)}(t) = \hbar \sum_{j=1}^{N} [\omega_a + \delta_j(t)] |e\rangle_j\langle e| \bigotimes_{k \neq j} I_k,$$

$$\hat{H}^{(1)}(t) = \hbar \sum_{j=1}^{N} \left(V^{(1)}_j(t)|e\rangle_j\langle g| + \text{H.c.} \right) \bigotimes_{k \neq j} I_k,$$

$$\hat{H}^{(2)}(t) = \hbar \sum_{j=1}^{N} \sum_{k=j+1}^{N} \left(V^{(2)g}_{jk}(t)|ge\rangle_{jk}\langle eg| + V^{(2)e}_{jk}(t)|ee\rangle_{jk}\langle gg| + \text{H.c.} \right) \bigotimes_{l \neq j,k} I_l.$$

Here, $V^{(1)}_j(t)$ is the time-dependent local single-qubit driving field of the $j$th qubit, $V^{(2)g,2}e_{jk}(t)$ are two possible time-dependent non-local two-qubit driving fields, acting on qubits $j$ and $k$, where the notation is derived from their diagonalization basis, i.e. the Bell-states basis,

$$|\Psi_\pm\rangle = 1/\sqrt{2}(|eg\rangle \pm |ge\rangle), \quad |\Phi_\pm\rangle = 1/\sqrt{2}(e^{-i2\omega_0t}|ee\rangle \pm |gg\rangle).$$

Also, $I$ is the identity matrix and H.c. is Hermitian conjugate.
We phenomenologically treat the random dephasing or noise, specific to each qubit, as a stochastic process. If this noise is Gaussian, it has the following first and second ensemble-average-moments,

$$\overline{\delta_j(t)} = 0, \quad \Phi_{jk}(t) = \overline{\delta_j(t)\delta_k(0)}.$$  \hspace{1cm} (6)

In what follows we shall adopt the rotating-wave approximation and assume that the driving fields of the single-qubit gates are resonant, with a time-dependent real envelope, and the driving fields of the two-qubit gates are resonant on their transition, with a time-dependent real envelope,

$$V^{(1)}_j(t) = \Omega^{(1)}_j(t)e^{-i\omega_0t} + \text{c.c.},$$  \hspace{1cm} (7)

$$V^{(2)_y}_{jk}(t) = \Omega^{(2)_y}_{jk}(t) + \text{c.c.},$$  \hspace{1cm} (8)

$$V^{(2)_x}_{jk}(t) = \Omega^{(2)_x}_{jk}(t)e^{-i2\omega_0t} + \text{c.c.}$$  \hspace{1cm} (9)

The rotating-wave approximation is used here.

We shall separately analyze three generic types of gate operations, namely, (i) only single-qubit gates; (ii) only two-qubit gates; and (iii) either single- or two-qubit gates, but never both at once, on a given qubit.

2.1. Single-qubit gates

The case [14] of single-qubit manipulation (\(\hat{H}^{(2)}(t) = 0\)), can be solved by transforming to the interaction picture, and diagonalizing \(\hat{H}^{(1)}(t)\). Since \(\hat{H}^{(1)}(t)\) is separable, the latter step can be made by diagonalizing each qubit, where the diagonalizing basis for the driven qubit is

$$|\pm\rangle_j = 1/\sqrt{2}(e^{-i\omega_0t}|e\rangle + |g\rangle_j).$$  \hspace{1cm} (10)

The diagonalizing basis for the entire Hamiltonian is then given by \(2^N\) basis states \(|\Psi^{(1)}_j\rangle\) and the total wavefunction \(|\psi\rangle\) is given by:

$$|\psi\rangle = \sum_{j=1}^{2^N} \beta_j(t)|\psi^{(1)}_j\rangle, \quad |\psi^{(1)}_j\rangle = \bigotimes_{j=1}^N |b_j^l\rangle_j,$$  \hspace{1cm} (11)

where \(l = 0, \ldots, 2^N - 1\), \(|b_j^l\rangle\) is the binary representation of \(l\), meaning \(l = b_1^l b_2^l \ldots, b_N^l\), with \(b_j^l = 0, 1\) corresponding to \(|\pm\rangle_j\), respectively.

The solution, to second order in \(\delta_j(t)\), for the density matrix of the ensemble, \(\overline{\rho(t)} = |\psi\rangle\langle\psi|\), is then found to be:

$$\overline{\rho(t)} = \rho(0) - \int_0^t dt' \int_0^{t'} dt'' \overline{[\hat{W}^{(1)}(t'), \hat{W}^{(1)}(t''), \rho(0)]}.$$  \hspace{1cm} (12)

Here, \(\rho(0)\) is the initial density matrix in the diagonalized basis and the decoherence matrix \(\hat{W}^{(1)}(t)\) has the elements:

$$w^{(1)}_{lm}(t) = w^{(1)*}_{ml}(t) = \begin{cases} \delta_j(t)\epsilon^{(1)*}_j(t) & b_j^l = 0, b_j^m = 1, \\ b_k^l = b_k^m \forall k \neq j, \\ 0 & \text{otherwise,} \end{cases}$$  \hspace{1cm} (13)
then found to be:

$$\Omega_1^{(1)}$$

In this case, there is an even number of qubits, and only two-qubit gates are applied, i.e. \(\tilde{H}^{(1)}(t) = 0\). Each qubit is driven by one two-qubit field, meaning that there are \(N/2\) gate fields, \(\Omega_{pj}^{(2)}(t)\), acting on different qubit pairs \(p_j = \{k, k'\}, j = 1, \ldots, N/2\). This case can be solved by transforming to the interaction picture and diagonalizing \(\tilde{H}^{(2)}(t)\). The latter is separable into contributions of qubit pairs, where the diagonalizing basis of each driven pair is the Bell basis. The diagonalizing basis for the entire Hamiltonian is then given by \(2^N\) basis states \(|\Psi_i^{(2)}\rangle\) and the total wavefunction \(|\psi\rangle\) is given by:

$$|\psi\rangle = \sum_{j=1}^{2^N} \beta_j(t)|\Psi_i^{(2)}\rangle, \quad |\Psi_i^{(2)}\rangle = \bigotimes_{j=1}^{N/2} |c_j^i\rangle_{p_j},$$

where \(l = 0, \ldots, 2^N - 1\), \(|c_j^i\rangle\) is the quarternary representation of \(l\), meaning \(l = c_1^i c_2^i \ldots c_N^i\), with \(c_j^i = 0, 1, 2, 3\) corresponding to

$$|\Phi_{\pm}\rangle_{kk'} = \frac{1}{\sqrt{2}} (|e\rangle_k|e\rangle_{k'} \pm |g\rangle_k|g\rangle_{k'}),$$

respectively. The solution for the density matrix of the ensemble, to second order in \(\delta_j(t)\), is then found to be:

$$\overline{\rho}(t) = \rho(0) - \int_0^t dt' \int_0^{t'} dt'' [\tilde{W}^{(2)}(t'), [\tilde{W}^{(2)}(t''), \rho(0)]]$$

Here, again \(\rho(0)\) is the initial density matrix in the appropriate diagonalized basis and the decoherence matrix \(\tilde{W}^{(2)}(t)\) has the form:

$$\tilde{W}^{(2)}(t) = \sum_{l,m=1}^{2^N} w_{lm}^{(2)}(t)|\Psi_j^{(2)}\rangle\langle\Psi_m^{(2)}|,$$

with

$$\epsilon_j^{(1)}(t) = e^{i\phi_j^{(1)}(t)},$$

$$\phi_j^{(1)}(t) = \int_0^t dt' \Omega_j^{(1)}(t'),$$

\(\phi_j^{(1)}(t)\) being the accumulated phase.

\[ \text{2.2. Two-qubit gates} \]

In this case, there is an even number of qubits, and only two-qubit gates are applied, i.e. \(\tilde{H}^{(1)}(t) = 0\). Each qubit is driven by one two-qubit field, meaning that there are \(N/2\) gate fields, \(\Omega_{pj}^{(2)}(t)\), acting on different qubit pairs \(p_j = \{k, k'\}, j = 1, \ldots, N/2\). This case can be solved by transforming to the interaction picture and diagonalizing \(\tilde{H}^{(2)}(t)\). The latter is separable into contributions of qubit pairs, where the diagonalizing basis of each driven pair is the Bell basis. The diagonalizing basis for the entire Hamiltonian is then given by \(2^N\) basis states \(|\Psi_i^{(2)}\rangle\) and the total wavefunction \(|\psi\rangle\) is given by:

$$|\psi\rangle = \sum_{j=1}^{2^N} \beta_j(t)|\Psi_i^{(2)}\rangle, \quad |\Psi_i^{(2)}\rangle = \bigotimes_{j=1}^{N/2} |c_j^i\rangle_{p_j},$$

where \(l = 0, \ldots, 2^N - 1\), \(|c_j^i\rangle\) is the quarternary representation of \(l\), meaning \(l = c_1^i c_2^i \ldots c_N^i\), with \(c_j^i = 0, 1, 2, 3\) corresponding to

$$|\Phi_{\pm}\rangle_{kk'} = \frac{1}{\sqrt{2}} (|e\rangle_k|e\rangle_{k'} \pm |g\rangle_k|g\rangle_{k'}),$$

respectively. The solution for the density matrix of the ensemble, to second order in \(\delta_j(t)\), is then found to be:

$$\overline{\rho}(t) = \rho(0) - \int_0^t dt' \int_0^{t'} dt'' [\tilde{W}^{(2)}(t'), [\tilde{W}^{(2)}(t''), \rho(0)]]$$

Here, again \(\rho(0)\) is the initial density matrix in the appropriate diagonalized basis and the decoherence matrix \(\tilde{W}^{(2)}(t)\) has the form:

$$\tilde{W}^{(2)}(t) = \sum_{l,m=1}^{2^N} w_{lm}^{(2)}(t)|\Psi_j^{(2)}\rangle\langle\Psi_m^{(2)}|,$$

with

$$\epsilon_j^{(1)}(t) = e^{i\phi_j^{(1)}(t)},$$

$$\phi_j^{(1)}(t) = \int_0^t dt' \Omega_j^{(1)}(t'),$$

\(\phi_j^{(1)}(t)\) being the accumulated phase.
with
\[ \delta_j(t) = \delta_k(t) \pm \delta_k'(t), \quad (22) \]
\[ \epsilon_j^{(2\phi,2\phi)}(t) = e^{i\phi_{p_j}(t)}, \quad (23) \]
\[ \phi_{\rho_j}^{(2\phi,2\phi)}(t) = \int_0^t \! dt' \Omega_{\rho_j}^{(2\phi,2\phi)}(t'), \quad (24) \]
the latter being accumulated phases due to dynamical control.

2.3. General quantum computation

A more general case, which satisfies all the requirements of quantum computation, is that of single- and two-qubit gates, applied simultaneously on different qubits, where each qubit is subject to a single gate, either as an individual single-qubit gate, or as part of a two-qubit gate pair. This case is solvable by combining the two results above, where the total Hamiltonian is separable into terms of qubit-pairs subject to two-qubit gates, and individual qubit terms subject to single-qubit gates. The solution is given in the form of equations \((12)\) and \((19)\), where the general interaction operator \(\hat{W}(t)\) is a combination of equations \((13)\) and \((20)\). Unless otherwise stated, this general case is assumed throughout the rest of the paper.

In order to evaluate and understand these results, we inspect each of the three stages of quantum computation. During the storage stage one requires the quantum information to be preserved. However, the dephasing results in loss of that information and its rate strongly depends on the initial state. By dynamically driving the qubits, either by the single-qubit resonant field or the two-qubit fields, one can reduce the dephasing. However, there is a restriction on the driving fields, namely, that the state to be preserved does not change due to the gates. Thus, the overall phase accumulated by the state due to the application of the gate fields at the end of the storage stage, at time \(t = T\), should be \(\phi^{(q)}(T) = 2\pi M, M = 0, \pm 1, \ldots\).

During the single-qubit gate stage, one can apply several gate operations on different qubits. The gate operations are defined by the accumulated phase of the qubits, e.g. \(\phi_j^{(1)}(T) = \pi/4\) is the Hadamard gate applied to the \(j\)th qubit \([20]\). During the two-qubit gate stage, one can apply several gate operations on different qubit pairs. These gate operations are also defined by the accumulated phase of the two qubits, where the \(\Omega_{kk'}^{(2\phi)}(t)\) change the phase in the \(|\Phi_\pm\rangle\) plane and \(\Omega_{kk'}^{(2\phi)}(t)\) change the phase in the \(|\Psi_\pm\rangle\) plane of qubits \(k\) and \(k'\). For example, \(\phi_{kk'}^{(2\phi)}(T) = \pi/2\) is a SWAP gate between qubits \(k\) and \(k'\) \([1]\). However, during these stages, the other qubits that do not participate in the gate operations are in the storage stage, meaning that one can still apply the fields described above, with the appropriate restrictions.

3. Dynamically-controlled gate fidelities

One needs a measure of the efficiency of the control schemes during the quantum gate operations. We use fidelity, defined as
\[ F(T) = \text{Tr}(\rho_{\text{target}}^{1/2} \hat{P}(T) \rho_{\text{target}}^{1/2}), \quad (25) \]
where \(\rho_{\text{target}}\) is the target density matrix after the quantum computation, e.g. \(\rho_{\text{target}} = \rho(0)\) for the storage stage. The error of the gate operation is then
\[ E(T) = 1 - F(T). \quad (26) \]
However, since quantum computations presume lack of knowledge of the initial qubits’ state, we shall also use the average fidelity,

\[ F_{\text{avg}}(T) = \langle F(T) \rangle, \]

where \( \langle \cdots \rangle \) is the average over all possible initial pure states.

Armed with the general solutions and efficiency measures, we proceed to analyze in detail the case of quantum computation of two qubits experiencing random dephasing.

### 3.1. Dynamically-controlled fidelity of single-qubit gate operations

First, we apply local single-qubit gates on each one of the qubits, where the accumulated phase determines the gate type, e.g. \( \phi^{(1)}_j(T) = 2\pi M \) means storage of the first qubit. The average fidelity of this scheme is given by:

\[ F_{\text{avg}}(T) = 1 - \frac{5}{12} \left( J_{11}^{(1)}(T) + J_{22}^{(1)}(T) \right), \]

where \( J_{jk}^{(q)}(t) \) is the modified dephasing function due to fields \( \Omega_{jk}^{(q)}, q = 1, 2, \Phi \). Here, \( G_{jk}(\omega) \) is the dephasing spectrum, and \( \epsilon_{j,t}^{(q)}(\omega) \) is the finite-time Fourier transform of the modulation (for a thorough analysis of the modified dephasing function, see [11]).

Equations (28)–(32) reveal the dependence of the fidelity on the spectral characteristics of the fields and the dephasing, and suggest how to tailor specific gate and control pulses: the tailoring should be aimed at reducing the spectral overlap of the dephasing and modulation spectra, according to our universal prescription [11]. Furthermore, they show that single-qubit gate fields do not cause cross-dephasing, since equation (28) depends only on single-qubit dephasing, \( \Phi_{jj}(t) \). This comes about from the averaging over all initial qubits: for each initial entangled state that loses fidelity from cross-dephasing there is another entangled state that gains fidelity from cross-dephasing. Thus, for the triplet, \( |\Phi_\rangle \),

\[ F(T, |\Phi_\rangle) = 1 - 1/2(J_{11}^{(1)} + J_{22}^{(1)} + J_{12}^{(1)} + J_{21}^{(1)}), \]

while for the singlet, \( |\Psi_\rangle \),

\[ F(T, |\Psi_\rangle) = 1 - 1/2(J_{11}^{(1)} + J_{22}^{(1)} - J_{12}^{(1)} - J_{21}^{(1)}). \]

Equation (28) also shows that the same modified dephasing function appears regardless of the accumulated phase, meaning that if one applies a gate field on one qubit, one can still benefit from applying a control field on the other, stored, qubit.
3.2. Dynamically-controlled fidelity of two-qubit gate operations

Next, we explore non-local two-qubit gate operations in the presence of control and dephasing. The average fidelity for this scenario is found to be:

\[
F_{\text{avg}}(T) = 1 - \frac{5}{24} \sum_{j,k=1,2} (J_{jk}^{(2\theta)}(T) + (-1)^{j+k} J_{jk}^{(2\phi)}(T)).
\]  

(35)

Here we see that cross-dephasing does not cancel due to averaging, but has opposite signs for the different two-qubit fields acting on the \( |\Phi_+\rangle_{jk} \leftrightarrow |\Phi_-\rangle_{jk} \) transition and \( |\Psi_+\rangle_{jk} \leftrightarrow |\Psi_-\rangle_{jk} \) transition, respectively (equations (17) and (18)). Hence, for example, a \( \sqrt{\text{SWAP}} \) gate [1], which is an entangling gate and in our notation is given by \( \phi_{jk}^{(2\phi)}(T) = \pi/4 \), may benefit from cross-dephasing.

Furthermore, we see that applying both two-qubit gate fields can reduce dephasing, even if only one field is needed for the actual gate operation. The other two-qubit field, used for storage, with \( \phi_{1,2}^{(2\theta)}(T) = 2\pi M \), \( M = 1, 2, \ldots \), can reduce dephasing, if used along with, e.g., a \( \sqrt{\text{SWAP}} \) gate.

This novel approach, consisting in applying an auxiliary (control) field, concurrently with the gate field that performs the logic operation, may require longer gate operations, due to limitations such as the maximal achievable peak-power. Yet, if the dephasing is reduced by the control fields despite the longer gate duration, the overall gate fidelity may increase, contrary to traditional schemes.

4. Dynamical control of gate operations in three-qubit systems

A combined application of single- and two-qubit control may occur in a system of three qubits subject to dephasing and control. We shall describe the dephasing by a set of random fluctuations with correlation-functions \( \Phi_{jk}(t) = (\gamma/t_{\text{c,jk}}) e^{-t/t_{\text{c,jk}}} \xi(|r_{jk}|) \). Here, \( \gamma \) is the long-time (Markovian) dephasing rate, \( t_{\text{c}} \) is the common correlation-time of these dephasings (without control fields, we have \( J(t \gg t_{\text{c}}) = \gamma t \)), and \( \xi(|r_{jk}|) \) is the cross-dephasing factor, with \( \xi = 0 \) (1) denoting no (maximal) cross-dephasing, depending on the inter-qubit distance \( r_{jk} \). The gate fields will be described by Gaussian pulses, with a restriction on their minimal duration and maximal peak-power, figure 1. The additional restrictions on the gate phases, e.g. \( \phi_{jk}^{(q)}(T) = 2\pi M_j \), leave only one free parameter per field, namely \( M_j \).

4.1. Dynamical control of single-qubit gates in three-qubit systems

The initial state of the system is picked to be \( |\psi(0)\rangle = |\uparrow\rangle_1 |e_2\rangle_3 \), where \( |\uparrow\rangle (|\downarrow\rangle) = \frac{1}{\sqrt{2}} (i|1\rangle \pm |0\rangle) \). Three single-qubit gates are applied, one per qubit, for time \( T_j \). The resulting gate error (equation (12)) is given by:

\[
E(T_1) = \frac{1}{2} \left( J_{11}^{(1),\text{Re}}(T_1) + J_{22}^{(1),\text{Im}}(T_1) + J_{33}^{(1),\text{Re}}(T_1) \right),
\]  

(36)

where we have introduced the following dephasing functions associated with the real and imaginary parts of the modulation functions, respectively:

\[
J_{jk}^{(q),\text{Re}}(t) = \int_0^t dt' \int_0^{t'} dt'' \Phi_{jk}(t'-t'') \text{Re} \epsilon_j^{(q)}(t') \text{Re} \epsilon_k^{(q)}(t'').
\]  

(37)
Figure 1. Dynamical gate fields in time (left) and frequency (right) domains. The gate fields (green lines) parameters for the first (solid), second (dotted) and third (dashed) qubits are $\phi_{1,2,3}^{(1)}(T_1) = \{0, \pi/4, 7\pi/4\}$, $\{2\pi, \pi/4, 7\pi/4\}$ and $\{4\pi, \pi/4, 7\pi/4\}$ for sequences 1–3, respectively. The gate durations are chosen such that the peak-power is the same for all sequences. The bath coupling spectra (red lines) are also shown for correlation time $t_c = 0.1$ (solid) and $t_c = 1.0$ (dashed). Here $\gamma = 0.1$.

From this expression, one can deduce several important characteristics of the dephasing control. Firstly, it depends on the initial state, as seen from the different dephasing functions of qubits 1, 2 and 3. Secondly, as in the two-qubit case analyzed above, one can observe that the fidelity does not depend on cross-dephasing, e.g. $\Phi_{12}(t)$, due to the fact that the state, and its manipulation, are separable and do not introduce entanglement.

One possible single-qubit gate sequence is to store the first qubit, and apply gates on the other two, with the restrictions on the gate pulses being $\phi_1^{(1)}(T_1) = 2\pi M$; $\phi_2^{(1)}(T_1) = \pi/4$; and $\phi_3^{(1)}(T_1) = 7\pi/4$. The desired (target) state at the end of this stage is $\psi_{\text{target}}(T_1) = i|\uparrow_1\rangle|\uparrow_2\rangle|g_3\rangle$.

In order to demonstrate the advantageous effects of these complex dynamical gate sequences, we compare our proposed approach (figures 1(c)–(f)) and the conventional approach, whereby minimal-duration Gaussian pulses are applied to achieve the required gates (figure 1(a)). By definition, our proposed approach requires longer gates, for the same peak-power. Comparing the proposed sequences to the conventional sequence (figure 2), demonstrates the trade-off between the beneficial effects of controlling the stored qubit and the detrimental effects of longer duration.
Examination of the spectral domain (figures 1(b), (d) and (f)) reveals that the longer durations shift the modulation spectrum peaks to higher frequencies. For short correlation times, i.e. wide bath coupling spectra, this shift in frequency has little effect, and the fidelity can decrease due to longer durations of the gate pulse sequence. However, for long non-Markovian correlation times, which characterize several experimental set-ups [21], the overlap between the narrow bath-coupling spectrum and the shifted modulation spectrum is reduced.

4.2. Dynamical control of two-qubit gates in three-qubit systems

The next stage is to apply a two-qubit gate between the second and third qubits, and store the first one. We assume that \( \rho(0) = |\psi_{\text{target}}(T_1)\rangle \langle \psi_{\text{target}}(T_1)| \). The resulting fidelity is (equations (12) and (19)):

\[
F(T_2) = 1 - \frac{1}{4} \text{Re}(2J_{11}^{(1)}, \text{Re}(T_2) + \text{J}_{22}^{(2)}(T_2) + \text{J}_{33}^{(2)}(T_2) + \text{J}_{23}^{(2)}(T_2)) \quad (39)
\]

\[
- \text{J}_{23}^{(2)}(T_2) - \text{J}_{32}^{(2)}(T_2) + \text{J}_{23}^{(2)}(T_2) + \text{J}_{32}^{(2)}(T_2) \quad (40)
\]

One possible gate sequence is to apply the \( \Omega_{23}^{(2)} \) two-qubit gate on the second and third qubits, and use the other two gates to control the dephasing, resulting in gate pulse restrictions, \( \phi_1^{(1)}(T_2) = 2\pi M, \phi_2^{(2)}(T_2) = 3\pi /2, \phi_2^{(2)}(T_2) = 2\pi M \). While the desired state at the end of this stage, \( |\psi_{\text{target}}(T_1 + T_2)\rangle = -i|1\rangle g_2 |\rangle \), is not entangled, the results presented are qualitatively similar to the case of an entangled target state: \( \Omega_{23}^{(2)} \) is an entangling gate, while \( \phi_2^{(2)}(T_2) \) determines whether the final state is entangled or not.

At this stage, the entangling gate has no effect on the dephasing of the first, ‘idle’ qubit. At the same time, this gate introduces modified cross-dephasing terms, e.g. \( J_{23} \). Whereas the modified dephasing of each individual qubit (i.e. \( J_{jj} \)) reduces the fidelity, the cross-dephasing can have an advantageous effect. In this example, there is a delicate interplay between two conflicting effects of the \( \Omega_{23}^{(2)} \) gate field: on the one hand, it increases the fidelity by reducing the dephasing of the individual qubits, as per equation (40), but on the other hand it decreases...
Figure 3. Gate errors at the end of the second stage, $E(T_2)$, as a function of cross-dephasing overlap, $\xi$, for different dynamical gate fields. The gate fields parameters are $\{\phi^{(1)}_1, \phi^{(2)}_{23}, \phi^{(2)}_{23}\} = \{0, 3\pi/2, 0\}$ (solid), $\{0, 3\pi/2, 2\pi\}$ (dotted), $\{2\pi, 3\pi/2, 0\}$ (dashed) and $\{2\pi, 3\pi/2, 2\pi\}$ (dash-dot). The gate durations are chosen such that the peak-power is the same for all sequences. Here $\gamma = 0.1$, and the results were obtained after averaging over 1000 realizations.

the fidelity by reducing the cross-dephasing of the two qubits, as per equation (41). Hence, one must appropriately design this control field in order to optimize the gate sequence, as discussed below.

Figure 3 illustrates the effects of adding control fields concurrently with the desired SWAP gate. The application of the gate field reduces the modified dephasing function, $J^{(2)\phi}_{23}$, making the other functions more dominant. Thus, one can observe the cross-dephasing (overlap-dependent) increase in gate error. A non-local control field for the second two-qubit gate eliminates this cross-dephasing dependence, resulting in a tradeoff between longer gate-field duration and cross-dephasing reduction. By contrast, local single-qubit control reduces the error, but leaves the cross-dephasing intact. Combining the two control schemes results in an even greater decrease in error, without cross-dephasing, as seen in figure 3. In this example cross-dephasing can be combated by non-local two-qubit fields, whereas local fields can only reduce single-qubit dephasing.

5. Possible experimental implementations

This section discusses possible implementations of the proposed scheme in current experimental designs for quantum computations. We first verify that each implementation conforms to our Hamiltonian, and then discuss the available dynamical dephasing control. Most implementations have limitations on addressability [22] or dynamical control [23], and thus cannot fully benefit from the proposed scheme. On the other hand, other implementations [24, 25] that may have full addressability and dynamical control can greatly improve their gate fidelities by using our scheme.

We start with implementation in trapped ions as an example of a fully dynamically controlled and addressable system. We continue with optically trapped atoms that have dynamical control but lack addressability at present. We conclude with remarks on NMR quantum computing, that may allow for only single-qubit dynamical control.
5.1. Implementation in trapped ions

In a string of ions in a linear trap [24, 25], the qubits are encoded by two internal states of each ion ($|g(e)\rangle_j$). A ‘bus-mode’ qubit is encoded by the ground and first excited common vibrational levels ($|0(1)\rangle_N$). The gates are realized by applying individually-addressing laser pulses: single-qubit gates on the ‘carrier’ ($\Omega_1^{(1)}(t), |g\rangle \leftrightarrow |e\rangle$), (figure 4(a)), and two-qubit gates involving the $j$th and ‘bus’ mode qubits on the ‘blue-sideband’ ($\Omega_1^{(2)x}(t), |g\rangle_j|0\rangle_N \leftrightarrow |e\rangle_j|1\rangle_N$) and ‘red-sideband’ ($\Omega_1^{(2)y}(t), |g\rangle_j|1\rangle_N \leftrightarrow |e\rangle_j|0\rangle_N$) frequencies of the electronic quadrupole transition, (figure 4(b)). Dephasing in the ion trap system can appear due to fluctuating Zeeman shifts in the qubit levels.

The difficulty with our protocol in a harmonic trap driven by control fields [25] discussed in section 3, is that the blue- (red-) sideband also couples to higher excitation levels, e.g. $|g\rangle_j|1\rangle_N \leftrightarrow |e\rangle_j|2\rangle_N$ ($|e\rangle_j|1\rangle_N \leftrightarrow |g\rangle_j|2\rangle_N$) and thus complicates the concurrent application of both two-qubit gates. This difficulty can be circumvented by imposing anharmonicity on the
linear trap: transitions between the higher vibrational levels $|1\rangle_N \leftrightarrow |2\rangle_N$ are then sufficiently far detuned to be excluded from our protocol.

A SWAP gate of two ions, involving the first two common vibrational levels and excluding others by anharmonicity, has been simulated to examine the ability of control and gate fields to combat dephasing (figure 4(c)). The conventional pulse sequence (figure 4(c)(1)), yielded $F_{\text{avg}}(t = 500 \mu s) = 0.93$, whereas our proposed sequence (figure 4(c)(2)), results in $F_{\text{avg}}(t = 600 \mu s) = 0.97$. This demonstrates that gate fidelity may be improved by the proposed gate operation, despite its longer duration.

5.2. Implementation in optically trapped atoms

Another possible implementation of quantum computing involves ultracold atoms trapped in optical lattices [22]. Loading an optical lattice with a Bose–Einstein condensate and imposing a Mott insulating state, creates the situation where each lattice site has exactly one atom. The logical qubits are encoded in the spin of the atoms.

Controlled entanglement between atoms on different lattice sites can be realized with the help of spin-dependent lattice potentials. In such spin-dependent potentials, atoms in different internal states experience different lattice potentials. These lattices can be moved relative to each other such that two initially separated atoms can be brought into controlled contact. First, the atoms on different lattice sites are placed in a coherent superposition of the two internal states. Then spin-dependent potentials are used to split each atom such that it simultaneously moves to the right and to the left and is brought into contact with the neighboring atoms. There both atoms interact and controlled phase shift $\phi$ is introduced. After such controlled collision, the atoms are again moved back to their original lattice sites. Random dephasing in this set-up can occur due to magnetic field fluctuations or collisions with hot atoms, resulting in dephasing time of $\approx 2 \text{ ms}$.

One can reformulate the experimental scenario described in [22] using our notation. Setting $\omega_a = 0$ in equation (2) results in having a degenerate qubit, where $|g\rangle = |0\rangle$ and $|e\rangle = |1\rangle$. Starting with the state $|0\rangle_j |0\rangle_{j+1}$, where the subscript denotes the atom’s position in the lattice, the experimental sequence [22] is composed of (i) $\pi/4$ global single-qubit phase shift by a pulse affecting all the qubits; (ii) fast spin-dependent transport, followed by collisional interaction over a fixed ‘hold’ time and then an inverse spin-dependent transport; and (iii) another global $\pi/4$ single-qubit phase shift. The final entangled state is $(|0\rangle |\bar{\rangle} + |1\rangle |+\rangle)/2$.

This entanglement scheme can be modified according to our effective Hamiltonian in section 2 and the solutions in sections 3 and 4 by the following sequence: (i) a single-qubit gate stage combined with control fields, such that the accumulated phase satisfies the requirements

$$
\phi_{j}^{(1)}(T_1) = \pi/4, \quad \phi_{j}^{(1)}(T_1) = 2\pi M;
$$

(ii) a two-qubit gate stage combined with control fields such that the accumulated entanglement phase satisfy the requirements

$$
\phi_{j,j+1}^{(2)\text{e}}(T_2) = \pi/4, \quad \phi_{j,j+1}^{(2)\text{e}}(T_2) = \pi/4.
$$

The implementation of our protocol using controlled collisions between optically trapped atoms has at present several limitations. Firstly, there is no single-site addressability, and thus one cannot implement a single-qubit local gate operation or control [15]. Secondly, although in our effective Hamiltonian two independent two-qubit fields are applied, here they are correlated,
because only one collisional interaction occurs in the present experimental design. Thus, one cannot control them independently for the time being.

The most straightforward extension of current experiments towards our control scheme is as follows. Whereas in the experiment in [22], the collisional interaction was constant during the time both atoms occupied the same site (i.e. $\Omega_1(t) = \Omega_2(t) = \text{const.}$), one can dynamically control this coupling by inducing more elaborate spin-dependent transport. Thus, one can repeatedly transport the split atomic wave packets back and forth over variable time intervals, so that they intermittently occupy the same site, subject to the requirement of accumulating the same desired phase (figure 5). This can be an important tool in combatting the short dephasing time.

Future extensions of this experimental set-up may allow the full implementation of our protocol: (i) single-site addressability would require the addition of position-sensitive control fields, using near-field techniques, (ii) two different collisional interactions for the control two-qubit gates may be achieved by laser-induced dipole–dipole interactions (LIDDI) between atoms at the same site [26]–[29]. LIDDI can be effected by a laser that is detuned well outside the natural line-width of an optical transition in the atoms. We have recently shown that atoms in different hyperfine states may exhibit LIDDI with suppressed Rayleigh-scattering losses, allowing lifetimes compatible with the ‘hold’ time required for the spin-dependent collision [22].

5.3. NMR quantum computing

In another possible implementation, such as NMR quantum computing [23], the two-qubit interaction is always on and cannot be controlled. The analyzed Hamiltonian, equations (1)–(4), is a general phenomenological dephasing Hamiltonian, allowing for arbitrary modulations. The modulations we apply are general enough to describe the NMR scenario, by setting the time-dependent two-qubit coupling field to be constant. This, obviously, restricts the amount of control on the system. Nevertheless, this case is fully described by our analysis.
6. Conclusions

One can deduce certain general control guidelines from the foregoing results. First, the temporal shape of the correlation function and the dynamics of the quantum gates may have a large effect on the modified dephasing of the quantum system. Thus, one can tailor specific gate fields that optimize fidelity [19]. However, in contrast to our previous works [16], in which dynamical modulation was aimed at protecting a given initial state, i.e. state-preservation, the scheme presented here is mainly concerned with dynamical modulation aimed at protecting gate operations in multi-qubit systems. Hence, our requirements are far more complex than those concerned only with storage, which requires operating a single qubit at a fixed optimum point [30, 31].

The second control guideline is that one benefits from applying control fields to all the qubits at all times, according to the function of each qubit at that time, which can pertain to one of the three stages of quantum computation. However, if there are limitations on the gate-fields’ duration and peak-power, there is a tradeoff between the duration of the computational gate and the benefit from the control of the other qubits. Hence, one must optimize the duration and shape of the gate pulses, so as to benefit from the dynamics of the gates and the control of the qubits in the storage stage.

In the case of two-qubit entangling gates, one must also consider the effects of the gate fields on the cross-dephasing between the qubits. Here the tradeoff is between eliminating the cross-dephasing, which can be either detrimental or beneficial [15] depending on the gate, and the gate-field duration.

The analytical results in equations (28)–(35) for the case of two qubits can be generalized to more qubits. The effects of applying control fields (concurrently with the gate operations) are enhanced with the increase in the number of qubits. They show the advantages of our general scheme to the control of all qubits, whether they are manipulated by gates or kept in storage. To this end, one must be able to control all the qubits at all times, which may be experimentally difficult. This can be overcome by applying a global modulation scheme for all the stored qubits, as opposed to local and selective fields [14, 15], for the gate operations.

To conclude, we have formulated a universal protocol for dynamical dephasing control during all stages of quantum information processing, namely, storage, single- and two-qubit gate operations. It amounts to controlling all the qubits, whether they participate in the computation or not, and tailoring specific gate and control fields that reduce the dephasing. We have specifically shown that two-qubit entangling control fields can eliminate the cross-dephasing dependence. This counter-intuitive protocol has a great advantage over others and can be of great experimental value in that it increases the fidelity of the operation required, whether storage or computation, despite its longer duration. The setback is that it cannot be readily implemented in existing experimental set-ups, such as ion traps or optical lattices, but rather requires their redesign.

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