Multi-Sequential Word Relations

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Abstract. Rational relations are binary relations of finite words that are realised by non-deterministic finite state transducers (NFT). A particular kind of rational relations is the sequential functions. Sequential functions are the functions that can be realised by input-deterministic transducers. Some rational functions are not sequential. However, based on a property on transducers called the twinning property, it is decidable in $\text{PTime}$ whether a rational function given by an NFT is sequential. In this paper, we investigate the generalisation of this result to multi-sequential relations, i.e. relations that are equal to a finite union of sequential functions. We show that given an NFT, it is decidable in $\text{PTime}$ whether the relation it defines is multi-sequential, based on a property called the weak twinning property. If the weak twinning property is satisfied, we give a procedure that effectively constructs a finite set of input-deterministic transducers whose union defines the relation. This procedure generalises to arbitrary NFT the determinisation procedure of functional NFT.

Finite transducers extend finite automata with output words on transitions. Any successful computation (called run) of a transducer defines an output word obtained by concatenating, from left to right, the words occurring along the transitions of that computation. Since transitions are non-deterministic in general, there might be several successful runs on the same input word $u$, and hence several output words associated with $u$. Therefore, finite transducers can define binary relations of finite words, the so-called class of rational relations $[7,4]$. Unlike finite automata, the equivalence problem, i.e. whether two transducers define the same relation, is undecidable $[9]$. This has motivated the study of different subclasses of rational relations, and their associated definability problems: given a finite transducer $T$, does the relation $\mathcal{R}(T)$ it defines belong to a given class $\mathcal{C}$ of relations? We survey the most important known subclasses of rational relations.

Rational Functions. An important subclass of rational relations is the class of rational functions. It enjoys decidable equivalence and moreover, it is decidable whether a transducer is functional, i.e. defines a function. This latter result was first shown by Schützenberger with polynomial space complexity $[14]$ and the complexity has been refined to polynomial time in $[10,9]$.

A subclass of rational functions which enjoys good algorithmic properties with respect to evaluation is the class of sequential functions. Sequential functions are those functions defined by finite transducers whose underlying input automaton is deterministic (called sequential transducers). Some rational functions are not sequential. E.g., over the alphabet $\Sigma = \{a, b\}$, the function $f_{\text{swap}}$
mapping any word of the form $u\sigma$ to $\sigma u$, where $u \in \Sigma^*$ and $\sigma \in \Sigma$, is rational but not sequential, because finite transducers process input words from left-to-right, and therefore any transducer implementing that function should guess non-deterministically the last symbol of $u\sigma$. Given a functional transducer, it is decidable whether it defines a sequential function \[5\], based on a structural property of finite transducers called the twinning property. This property can be decided in \textit{PTIME} \[3\] and therefore, deciding whether a functional transducer defines a sequential function is in \textit{PTIME}. If the twinning property holds, one can “determinise” the original transducer into an equivalent sequential transducer.

It turns out that many examples of rational functions from the literature which are not sequential are \textit{almost} sequential, in the sense that they are equal to a finite union of sequential functions. Such functions are called \textit{multi-sequential}. For instance, the function $f_{\text{swap}}$ is multi-sequential, as $f_{\text{swap}} = f_a \cup f_b$, where $f_a$ is the partial sequential function mapping all words of the form $ua$ to $au$ (and similarly for $f_b$). Some rational functions are not multi-sequential, such as functions that are iterations of non-sequential functions. E.g., the function mapping $u_1#u_2#\ldots#u_n$ to $f_{\text{swap}}(u_1)\ldots f_{\text{swap}}(u_n)$ for some separator #, is not multi-sequential. Multi-sequential functions have been considered by Choffrut and Schützenberger in \[6\], where it is shown that multi-sequentiality for functional transducers is a decidable problem.

\textbf{Contribution} In this paper, we investigate multi-sequential relations, i.e. relations that are equal to a finite union of sequential functions. Our main result shows that, given a finite transducer, it is decidable in \textit{PTIME} whether the relation it defines is multi-sequential. Our procedure is based on a simple characterisation of multi-sequential relations by means of a structural property, called the weak twinning property (WTP), on finite transducers. We show that a finite transducer defines a multi-sequential relation if it satisfies the WTP. We define a “determinisation” procedure of finite transducers satisfying the WTP, that decomposes them into finite unions of sequential transducers. Finally, we also investigate the computational properties of multi-sequential relations and show, that, for a natural computational model for word relations, multi-sequential relations correspond to the relations that can be evaluated with constant memory.

\textbf{Related Works} As already mentioned, multi-sequential functions were considered in \[6\]. Our weak twinning property is close to the characterisation of multi-sequential functions of \[6\], which is based on analysing families of branching paths in transducers. Thanks to the notion of delay between output words, our property is simpler and can be decided in \textit{PTIME}, for arbitrary (not necessarily functional) transducers. Compared to \[6\], our decomposition procedure is more constructive. It extends the known determinisation procedure of functional transducers, to multi-sequential relations, and applies directly on arbitrary finite state transducers (in \[6\], the functional transducers are assumed to be unambiguous, but removing ambiguity is worst-case exponential).

Finite-valued rational relations are rational relations such that any input word has at most $k$ images by the relation, for a fixed constant $k$. Finite-
Table 1. Definability problems for rational relations given by finite transducers

| Property          | Determinacy | Definability | Functionality | Multi-sequentiality | Finite-valuedness |
|-------------------|-------------|--------------|---------------|---------------------|-------------------|
| Decidable         | PSpace [5]  | PSpace [5]   | PSpace [14]   | PTime [this paper]  | PTime [13,15]    |
| PTime             | Decidable [6] | PT (this paper) | PTime [10]   | PTime [this paper]  | PTime [13,15]    |

For rational relations, their existence is decidable in PSpace [5]. Definability for functions is PSpace [5] and for multi-sequential relations, it is PSpace [14]. Functionality is PSpace [14], and for multi-sequential relations, it is PTime [this paper]. Finite-valuedness is PTime [13,15].

Finally, finitely-sequential relations have been considered in [1]. They correspond to relations that can be realized by an input-deterministic transducer whose accepting states can, at the end of the run, output additional words from a finite set. Such relations are much weaker than multi-sequential relations.

The known and new results of this paper are summarised in Table 1.

1 Rational Word Relations

We denote by \( \mathbb{N} \) the set of natural numbers \( \{0, 1, \ldots\} \), and by \( \mathcal{P}(A) \) the set of subsets of a set \( A \), and by \( \mathcal{P}_f(A) \) the set of finite subsets of \( A \).

**Words and delays** Let \( \Sigma \) be a (finite) alphabet of symbols. The elements of the free monoid \( \Sigma^* \) are called words over \( \Sigma \). The length of a word \( w \) is written \( |w| \). The free monoid \( \Sigma^* \) is partially ordered by the prefix relation \( \leq \). We denote by \( \Sigma^{-1} \) the set of symbols \( \sigma^{-1} \) for all \( \sigma \in \Sigma \). Any word \( u \in (\Sigma \cup \Sigma^{-1})^* \) can be reduced into an irreducible word \( \pi \) by the equations \( \sigma \sigma^{-1} = \sigma^{-1} \sigma = \epsilon \) for all \( \sigma \in \Sigma \). Let \( G_\Sigma \) be the set of irreducible words over \( \Sigma \cup \Sigma^{-1} \). The set \( G_\Sigma \) equipped with concatenation \( u \cdot v = \bar{w} \) is a group, called the free group over \( \Sigma \). We denote by \( u^{-1} \) the inverse of \( u \), E.g., \( (a^{-1}bc)^{-1} = c^{-1}b^{-1}a \). The delay between two words \( v \) and \( w \) is the element \( \Delta(v, w) := v^{-1}w \in G_\Sigma \). E.g., \( \Delta(ab, acd) = b^{-1}cd \).

**Finite automata** A (finite) automaton over a monoid \( M \) is a tuple \( \mathcal{A} = (Q, E, I, T) \) where \( Q \) is the finite set of states, \( I \subset Q \) is the set of initial states, \( T \subset Q \) is the set of final states, and \( E \subset Q \times M \times Q \) is the finite set of edges, or transitions, labelled by elements of the monoid \( M \).

For all transitions \( e = (q_1, m, q_2) \in E \), \( q_1 \) is called the source of \( e \), \( q_2 \) its target and \( m \) its label. A run of an automaton is a sequence of transitions \( r = e_1 \ldots e_n \) such that for every \( 1 \leq i < n \), the target of \( e_i \) is equal to the source of \( e_{i+1} \). We write \( p \xrightarrow{m} q \) (or just \( p \xrightarrow{m} q \) when it is clear from the context) to mean that there is a run \( e_1 \ldots e_n \) such that \( p \) is the source of \( e_1 \), \( q \) the target of \( e_n \), and \( m \) is the product of the labels of the \( e_i \). A run is called accepting if its source is an initial state and its target is a final state. An automaton is called trim if each of its states occurs in at least one accepting run. It is well-known that any automaton can be trimmed in polynomial time. The language recognised
by an automaton over a monoid $M$ is the set of elements of $M$ labelling its accepting runs. A $\Sigma$-automaton is an automaton over the free monoid $\Sigma^*$ such that each edge is labelled by a single element $\sigma$ of $\Sigma$. A $\Sigma$-automaton is called deterministic if it has a single initial state, and for all $q \in Q$ and $\sigma \in \Sigma$, there exists at most one transition labelled $\sigma$ of source $q$.

Given two automata $A_1$ and $A_2$ over a monoid $M$, their disjoint union $A_1 \cup A_2$ is defined as the disjoint union of their set of states, initial and final states, and transitions. It recognises the union of their respective languages.

Finite transducers

Let $\Sigma$ and $\Gamma$ two alphabets. A (finite) transducer $T$ from $\Sigma^*$ to $\Gamma^*$ is a tuple $(Q,E,I,T,f_T)$ such that $(Q,E,I,T)$ is a finite automaton over the monoid $\Sigma^* \times \Gamma^*$, called the underlying automaton of $T$, such that $E \subseteq Q \times \Sigma \times \Gamma^* \times Q$, and $f_T : T \rightarrow \Gamma^*$ is the final output function. In this paper, the input and output alphabets are always denoted by $\Sigma$ and $\Gamma$. Hence, we just use the terminology transducer instead of transducer from $\Sigma^*$ to $\Gamma^*$.

A run (resp. accepting run) of $T$ is a run (resp. accepting run) of its underlying automaton. We write $p \xrightarrow{u|v} q$ instead of $p \xrightarrow{(u,v)} q$. The relation recognised by $T$ is the set $\llbracket T \rrbracket$ of pairs $(v,w)\llbracket t \rrbracket$ such that $i \xrightarrow{v|w} t$ for $i \in I$ and $t \in T$. A transducer $T$ is functional if $\llbracket T \rrbracket$ is a function. It is called trim if its underlying automaton is trim. The input automaton of a transducer is the $\Sigma$-automaton obtained by projecting the labels of its underlying automaton on their first component. Given a transducer $T$, we denote by $M_T$ the maximal integer $|v|$, $v \in \Gamma^*$, such that $v$ labels a transition of $T$ or $v = f_T(q)$ for some $q \in Q$.

A rational transducer is defined as a transducer, except that its transitions are labeled in $\Sigma^* \times \Gamma^*$. Rational transducers are strictly more expressive than transducers and define the class of rational relations. E.g., by using loops $q \xrightarrow{\epsilon|w} q$, a word can have infinitely many images by a rational relation. If there is no such loop, it is easily shown that rational transducers are equivalent to transducers.

2 Multi-Sequential Relations

In this section, we define multi-sequential relations, and give a decidable property on transducers that characterise them.

2.1 From sequential functions to multi-sequential relations

A transducer $T = (Q,E,I,T,f_T)$ is sequential if its input automaton is deterministic. A function $f : A^* \rightarrow B^*$ is sequential if it can be realised by a sequential transducer $T$, i.e. $f = \llbracket T \rrbracket$ for some sequential transducer $T$.

Let $\Sigma = \Gamma = \{a,b\}$. Fig. 1 depicts transducers that implement sequential and non-sequential functions. All states which is the target of a source-less arrow

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1 Transducers are sometimes called real-time in the literature, and rational transducers just transducers. To avoid unnecessary technical difficulties, we establish our main result for (real-time) transducers, but, as shown in Remark[8] it still holds for rational transducers.

2 These functions are sometimes called subsequential in the literature. We follow the terminology of [12].
are initial, and those which are the source of an arrow without target, or whose target is a word, are accepting. The function $f_a$ that maps any word of the form $a^n$, $n > 0$, to $a^{n-1}b$, is sequential. It is realised by the transducer of Fig. 1(a).

The function $f_{\text{blank}}$ replaces each block of consecutive $b$ by a single $b$, but the last one. E.g. $f_{\text{blank}}(abaab) = aba$. It is sequential and defined by the sequential transducer of Fig. 1(b). The function $f_{\text{swap}}$ maps any word of the form $a^n\sigma$ to $\sigma a^n$, for $\sigma \in \Sigma$. It is not sequential, because the transducer has to guess the last symbol $\sigma$. It can be defined by the transducer of Fig. 1(c).

Sequential functions have been characterised by a structural property of the transducers defining them, called the **twinning property**. Precisely, a trim transducers with initial state $q_0$ is twinned iff for all states $q_1, q_2$, all words $u, v \in \Sigma^*$ and $u_1, v_1, u_2, v_2 \in \Gamma^*$, if $q_0 \xrightarrow{u|u_1} q_1 \xrightarrow{v|v_1} q_1$ and $q_0 \xrightarrow{u|u_2} q_2 \xrightarrow{v|v_2} q_2$, then $\Delta(u_1, u_2) = \Delta(u_1 u_2, v_1 v_2)$. E.g., by taking $u = v = a$, $u_1 = aa$, $u_2 = ba$ and $v_1 = v_2 = a$, it is easy to see that the transducer of Fig. 1(c) is not twinned.

**Theorem 1** ([53]). Let $T$ be a trim transducer.

1. $T$ is twinned iff $\|T\|$ is sequential.
2. It is decidable in PTIME whether a trim transducer is twinned.

The following result is a folklore result that we show in Appendix for the sake of completeness. It states that the difference (the delay) between the outputs of two input words is linearly bounded by the difference of their input words.

**Proposition 2.** Let $D$ be a sequential transducer. For all pairs $(u_1, v_1), (u_2, v_2) \in \|D\|$, $|\Delta(v_1, v_2)| \leq M_D(|\Delta(u_1, u_2)| + 2)$.

**Multi-sequential relations** The function $f_{\text{swap}}$ is not sequential, but it is multi-sequential, in the sense that it is the union of two sequential functions $f_1, f_2$ such that $f_1$ is the restriction of $f_{\text{swap}}$ to words in $a^+a$ and $f_2$ its restriction to words in $a^+b$. Precisely:

**Definition 3 (Multi-sequential relations).** A relation $R \subseteq \Sigma^* \times \Gamma^*$ is multi-sequential if there exist $k$ sequential functions $f_1, \ldots, f_k$ such that $R = \bigcup_{i=1}^{k} f_i$.

The multi-sequentiality problem asks, given a transducer $T$, whether $\|T\|$ is multi-sequential. It should be clear that the answer to this problem is not
always positive. Indeed, even some rational functions are not multi-sequential.

It is the case for instance for the function \( f_{\text{swap}} \) that maps any word of the form \( u_1 \# u_2 \# \ldots \# u_n \) to \( f_{\text{swap}}(u_1) \# f_{\text{swap}}(u_2) \# \ldots \# f_{\text{swap}}(u_n) \), where \( u_i \in \Sigma^* \) and \( \# \not\in \Sigma \) is a fresh symbol. This function is rational, as it can be defined by the transducer of Fig. 1(d). In this paper, we investigate the intrinsic reasons making a rational relation like \( f_{\text{swap}} \) multi-sequential and a rational relation like \( f_{\text{swap}}^* \) not. In particular, we define a weaker variant of the twinning property that characterises the multi-sequential relations by structural properties of the transducers which define them.

2.2 Weak Twinning Property

Definition 4. Let \( T \) be trim transducer and \( q_1, q_2 \) be two states of \( T \). We say that \( q_1 \) is weakly twinned to \( q_2 \) if for all words \( u, v \in \Sigma^* \) and all words \( u_1, u_2, v_1, v_2 \in \Gamma^* \), if \( q_1 \xrightarrow{u|v_1} q_1 \xrightarrow{u|u_2} q_2 \xrightarrow{v|v_2} q_2 \), or graphically

\[
\begin{array}{c}
\text{q}_1 \\
\xrightarrow{u|v_1} \\
\text{q}_2 \\
\end{array}
\begin{array}{c}
\xrightarrow{u|u_2} \\
\text{q}_1 \\
\xrightarrow{v|v_2} \\
\text{q}_2 \\
\end{array}
\]

then \( \Delta(u_1, u_2) = \Delta(u_1 v_1, u_2 v_2) \).

\( T \) satisfies the weak twinning property (WTP) if for any two states \( q_1, q_2 \) of \( T \), \( q_1 \) is weakly twinned to \( q_2 \). \( T \) is weakly twinned if it satisfies the WTP.

Remark 5. A transducer satisfying the twinning property also satisfies the weak twinning property. Indeed, suppose that \( T \) satisfies the twinning property. We show that in any pattern depicted in Definition 4 we immediately get \( \Delta(u_1, u_2) = \Delta(u_1 v_1, u_2 v_2) \). Indeed, since \( T \) is trim, there exist words \( (x, x') \in \Sigma^* \times \Gamma^* \) such that \( q_0 \xrightarrow{x|x'} q_1 \), where \( q_0 \) is the initial state of \( T \). Then, we have:

\[
\begin{array}{c}
q_0 \\
\xrightarrow{xu|x'u_1} \\
q_1 \\
\xrightarrow{u|v_1} \\
\text{q}_2 \\
\end{array}
\begin{array}{c}
q_0 \\
\xrightarrow{xu|x'u_2} \\
q_2 \\
\xrightarrow{v|v_2} \\
\text{q}_2 \\
\end{array}
\]

Since \( T \) satisfies the twinning property, then \( \Delta(x'u_1, x'u_2) = \Delta(x'u_1 v_1, x'u_2 v_2) \), and therefore \( \Delta(u_1, u_2) = \Delta(u_1 v_1, u_2 v_2) \).

2.3 Main Result

We show that the weak twinning property characterises the transducers that define multi-sequential relations, and that it is decidable in polynomial time.

Theorem 6 (Main Result). Let \( T \) be a trim transducer.

1. \( T \) is weakly twinned iff \( \square[T] \) is multi-sequential.
2. It is decidable in PTIME whether a trim transducer is weakly twinned.

Deciding the WTP is done with a reversal-bounded counter machine, whose emptiness is known to be decidable in PTIME [11] (see Appendix). The proof of Theorem 6.1 is done via two lemmas, Lemma 9 and 11, that are shown in the rest of this paper. An immediate consequence of this theorem and the fact that any transducer can be trimmed in polynomial time, is the following corollary:

Corollary 7. It is decidable in PTIME whether a transducer defines a multi-sequential relation.
Remark 8. Theorem 6 is also true when $T$ is a rational transducer. Indeed, if $T$ is weakly twinned, then there is no loop of the form $q \xrightarrow{i|w} q$, otherwise by taking $q_1 = q_2 = q$, $u = v = u_1 = v_1 = \epsilon$ and $u_2 = v_2 = w$ in the definition of the WTP, one would raise a contradiction. It is easily shown that $T$ can be transformed into an equivalent real-time transducer, while preserving the WTP. Conversely, if $\llbracket T \rrbracket$ is multi-sequential, then it is finite-valued, and therefore there is no loop of the form $q \xrightarrow{i|w} q$. As before, one can transform $T$ into a real-time transducer while preserving the WTP.

Lemma 9. Let $T$ be a trim transducer. If $T$ is weakly twinned, then $\llbracket T \rrbracket$ is multi-sequential.

Proof. The proof of this Lemma is the goal of Sec. 3 which provides a procedure that decomposes a transducer $T$ into a union of sequential transducers. This procedure generalises to relations the determinisation procedure of functional transducers. In particular, it is based on a subset construction extended with delays, and a careful analysis of the strongly connected components of $T$. □

The following lemma is a key result to prove the other direction of Theorem 7. It states that the WTP is preserved by transducer inclusion and equivalence, and therefore, is independent from the transducer that realises the relation.

Lemma 10. Let $T_1, T_2$ be two trim transducers.

1. If $\llbracket T_1 \rrbracket \subseteq \llbracket T_2 \rrbracket$ and $T_2$ is weakly twinned, then $T_1$ is weakly twinned.

2. If $\llbracket T_1 \rrbracket = \llbracket T_2 \rrbracket$, then $T_1$ is weakly twinned iff $T_2$ is weakly twinned.

Proof. Clearly, 2 is a consequence of 1. Let us prove 1 by contradiction. Suppose that $T_1$ is not weakly twinned. Hence, it can be shown that there exist two states $q_1$ and $q_2$ of $T_1$, an accepting run $q_0 \xrightarrow{u_1} q_1 \xrightarrow{u_2} q_2 \xrightarrow{y} q_f$ and loops $q_1 \xrightarrow{w_1} q_1$, $q_1 \xrightarrow{v_1} q_1$ and $q_2 \xrightarrow{w_2} q_2$ such that for every $n \in \mathbb{N}$, $|\Delta(u_1 v_1 \ldots u_n v_n)| \geq n$ (see Lemma 20 in appendix).

Since $T_2$ is weakly twinned, by Lemma 9 there exist sequential transducers $D_1, \ldots, D_k$ such that $\llbracket T_2 \rrbracket = \bigcup_{i=1}^{k} \llbracket D_i \rrbracket$. Let $M = \max \{|u_1|, |v_1|, |y|\}$. Let $b$ be the maximal $M_{D_i}$, $1 \leq i \leq k$. Let $r = 4b(k + 1)(M + 1)$. For every $1 \leq i \leq k + 1$, consider the pair $(w_i, w'_i)$ in $\llbracket T_1 \rrbracket$ defined by

\[
  w_i = xv^{-i}uv^{i+1} \ldots uw^{i}v^{i+1}y, \quad w'_i = x'v^{-i}u_1v_1^{-i-1} \ldots u_nv_{n-1}v_{n+1}^i u_2v_2y'.
\]

Since $\llbracket T_1 \rrbracket \subseteq \llbracket T_2 \rrbracket$, and $(w_i, w'_i) \in \llbracket T_1 \rrbracket$, we have $(w_i, w'_i) \in \llbracket T_2 \rrbracket$, hence there exists $1 \leq j \leq k$ such that $(w_i, w'_i) \in \llbracket D_j \rrbracket$. As there are $k$ different $D_j$ and $k + 1$ pairs $(w_i, w'_i)$, there exist $k \geq i_1 > i_2 \geq 0$ such that $(w_{i_1}, w'_{i_1}), (w_{i_2}, w'_{i_2}) \in \llbracket D_j \rrbracket$.

By Proposition 2, $|\Delta(w_{i_1}, w'_{i_2})| \leq b(|\Delta(w_{i_1}, w'_{i_2})| + 2)$. However,

\[
\begin{align*}
  b(|\Delta(w_{i_1}, w'_{i_2})| + 2) &\leq b(|w_1v_1^{-1} \ldots u_nv_nv_{n+1}^i u_2v_2y'| + 2b) \\
  &\leq b(|u_1v_1^{-1} \ldots u_nv_nv_{n+1}^i| + 2b) \\
  &= b(k+1)(M+1) + 2b(M+1) + 2b \\
  \leq 2r^i - r^{i+1} &< \frac{r^i}{r^{i+1}}
\end{align*}
\]

which is a contradiction. □
Lemma 11. Let $T$ be a trim transducer. If $\left\langle T \right\rangle$ is multi-sequential, then $T$ is weakly twinned.

Proof. If $\left\langle T \right\rangle$ is multi-sequential, then $T$ is equivalent to a transducer $T'$ given as a union of $k$ sequential transducers $D_i$ for some $k \geq 0$ with disjoint sets of states. Clearly, if each $D_i$ is weakly twinned, then so is $T'$. Since the $D_i$ are sequential, they satisfy the twinning property, and therefore the weak twinning property by Remark 5. Hence, $T'$ is weakly twinned. By Lemma 10 and since $T'$ and $T$ are equivalent, $T$ is also weakly twinned. □

The following result implies that, in order to show that a rational relation is not multi-sequential, it suffices to exhibit a function contained in that relation, which is not multi-sequential.

Corollary 12. Let $R$ be a rational relation, and $f$ a rational function such that $f \subseteq R$ and $f$ is not multi-sequential. Then $R$ is not multi-sequential.

Proof. We assume that $R$ and $f$ are defined by transducers $T$ and $T_f$. The result still holds for rational transducers, for the same reasons as the one explained in Remark 8. Since $f$ is not multi-sequential, by Theorem 9, $T_f$ is not weakly twinned. Since $f \subseteq R$, by Lemma 10 it implies that $T$ is not weakly twinned, and hence not multi-sequential, again by Theorem 9. □

3 Decomposition Procedure

In this section, we show how to decompose a transducer into a union of sequential transducers, via a series of constructions, whenever the weak twinning property is satisfied. For simplicity, we sometimes consider multi-transducers, i.e. transducers such that the function $f_T$ maps any final state to a finite set of output words. Let $T = (Q, E, I, T, f_T)$ be a transducer. Let $\sim \subseteq Q^2$ defined by $q_1 \sim q_2$ if $q_1$ and $q_2$ are strongly connected, i.e. if there exist a run from $q_1$ to $q_2$ and a run from $q_2$ to $q_1$. The equivalence classes of $\sim$ are called the strongly connected components (SCC) of $T$. An edge of $T$ is called transient if its source and target are in distinct SCC, or equivalently, if there exist no run from its target to its source. The condensation of $T$ is the directed acyclic graph $\Psi(T)$ whose vertices are the SCC of $T$ and whose edges are the transient edges of $T$. A transducer is called separable if it has a single initial state and any two edges of same source and same input symbol are transient.

Split Let $T = (Q, E, I, T, f_T)$ be a transducer. Let $P$ be the paths of the condensation $\Psi(T)$ starting in an SCC containing an initial state. Note that $P$ is finite as $\Psi(T)$ is a DAG. For each path $p \in P$, let $T_p$ be the subtransducer of $T$ obtained by removing all the transient edges of $T$ but the ones occurring in $p$. The split of $T$ is the transducer $\text{split}(T) = \bigcup_{p \in P} T_p$. Clearly,

Lemma 13. The transducer $\text{split}(T)$ is equivalent to $T$, i.e. $\left\langle \text{split}(T) \right\rangle = \left\langle T \right\rangle$. 

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If \( \mathcal{T} \) is separable, then \( \text{split}(\mathcal{T}) \) is a decomposition of \( \mathcal{T} \) into sequential transducers. Since any multi-transducer can be transformed into an equivalent union of transducers over the same underlying automaton while preserving separability, we get the following result (fully proved in Appendix):

**Lemma 14.** Let \( \mathcal{T} \) be a separable multi-transducer with a single initial state. Then \( \llbracket \mathcal{T} \rrbracket \) is multi-sequential.

**Determinisation** We recall the determinisation procedure for transducers, for instance presented in [2]. It extends the subset construction with delays between output words, and outputs the longest common prefix of all the output instance presented in [2]. It extends the subset construction with delays between output words, and outputs the longest common prefix of all the output words produced on transitions on the same input symbol, and keep the remaining suffixes (delays) in the states. Precisely, let \( \mathcal{T} = (Q, E, I, T, f_T) \) be a trim transducer. For every \( U \in \mathcal{P}_f(Q \times \Gamma^*) \), for every \( \sigma \in \Sigma \), let

\[
R_{U,\sigma} = \{(q, w) \in Q \times \Gamma^* | \exists (p, u) \in U, \exists (p, \sigma|v, q) \in E \text{ s.t. } w = uv\},
\]

\( w_{U,\sigma} \) be the largest common prefix of the words \( \{w|q \in Q \text{ s.t. } (q, w) \in R_{U,\sigma}\} \),

\[
P_{U,\sigma} = \{(q, w)|(q, w_{U,\sigma}w) \in R_{U,\sigma}\}.
\]

Let \( \bar{\mathcal{D}}(\mathcal{T}) \) be the infinite-state multi-transducer over the set of states \( \mathcal{P}_f(Q \times \Gamma^*) \), with set of edges \( \{(U, \sigma|w_{U,\sigma}, P_{U,\sigma})|U \in \mathcal{P}_f(Q \times \Gamma^*), \sigma \in \Sigma\} \), initial state \( U_0 = I \times \{\epsilon\} \), set of final states \( \{U \in \mathcal{P}_f(Q \times \Gamma^*)|U \cap (T \times \Gamma^*) \neq \emptyset\} \), and final output relation that maps each final state \( U \) to \( \{w_{f_T}(q)|q \in T \text{ and } (q, w) \in U\} \). Note that \( \bar{\mathcal{D}}(\mathcal{T}) \) has a deterministic (potentially infinite) input-automaton.

The determinisation of \( \mathcal{T} \), written \( \mathcal{D}(\mathcal{T}) \), is the trim part of \( \bar{\mathcal{D}}(\mathcal{T}) \). The transducer \( \mathcal{D}(\mathcal{T}) \) is equivalent to \( \mathcal{T} \) (see corollary [2] appendix). It is well-known that \( \mathcal{D}(\mathcal{T}) \) is a (finite) sequential transducer if \( \mathcal{T} \) satisfies the twinning property.

Fig. 2(a) depicts a transducer that satisfies the weak twinning property, but not the twinning property. As a consequence, \( \mathcal{D}(\mathcal{T}) \) is infinite (a part of \( \mathcal{D}(\mathcal{T}) \) can be seen on Fig. 2(b)). The non satisfaction of the twinning property is witnessed by the two runs \( q_0 \xrightarrow{aaaa|aaaa} q_4 \xrightarrow{a_1a} q_4 \) and \( q_0 \xrightarrow{aaaa|aaaa} q_4 \xrightarrow{a_1a} q_4 \). Note that these runs do not harm the weak twinning property. The idea of the next construction, called the weak determinisation, is to keep some, well-chosen, non-deterministic transitions, and reset the determinisation whenever it definitively leaves an SCC (the SCC \( \{q_0, q_1, q_2\} \) in this example). We explain this procedure when there is a single initial state, as any transducer can be easily transformed into a finite union of transducers with single initial states.

**Weak determinisation** Let \( \mathcal{T} = (Q, E, I, T, f_T) \) be a trim transducer with a single initial state. For every \( U \in \mathcal{P}_f(Q \times \Gamma^*) \), let the rank \( n_U \) of \( U \) be the set containing all the SCC of \( \mathcal{T} \) accessible from the states \( q \in U \). The multi-transducer \( \bar{\mathcal{W}}(\mathcal{T}) \) is obtained from \( \mathcal{D}(\mathcal{T}) \) by splitting the edges that do not preserve the rank, as follows. If \( (U, u|v, U') \) is an edge of \( \mathcal{D}(\mathcal{T}) \) such that \( n_{U'} \) is strictly included in \( n_U \), it is removed, and replaced by the set of edges \( \{(U, u|w, \{(q, \epsilon)\})|(q, w) \in U'\} \). It is easily shown that any pair of distinct edges of the form \( U \xrightarrow{a_{i_1}} U_1 \) and \( U \xrightarrow{a_{i_2}} U_2 \) in \( \bar{\mathcal{W}}(\mathcal{T}) \) have necessarily been created

[9]
by this transformation (because without this transformation everything stays input-deterministic). Therefore, since the rank strictly decreases \((n_{U_2} \subsetneq n_{U_1}\) and \(n_{U_1} \subsetneq n_{U_2}\)) and can never increase in \(\bar{W}(T)\), there is no run from \(U_2\) to \(U\), nor from \(U_1\) to \(U\) in \(\bar{W}(T)\), and the two edges are transient. As a consequence,

**Lemma 15.** The infinite transducer \(\bar{W}(T)\) is separable.

The weak determinisation of \(T\), written \(W(T)\), is the trim part of \(\bar{W}(T)\).

**Proposition 16.** \(W(T)\) and \(T\) are equivalent. Moreover, if \(T\) is weakly twinned, \(W(T)\) is finite, and it is a multi-transducer.

The main idea is to prove that, as long as the weak twinning property is satisfied, the length of the words present in the states of \(W(T)\) can be bounded. The proof can be found in Appendix.

**Example 17.** Let us illustrate the weak determinisation on the transducer \(T\) of Fig. 2(a). Consider the determinisation \(D(T)\) of Fig. 2(b). When it is in state \(U_1 = \{(q_2, ab), (q_3, ba)\}\), on input \(a\), it moves to state \(U_2 = \{(q_3, aba), (q_4, baa)\}\), definitely leaving the SCC \(\{q_0, q_1, q_2\}\) of \(T\) (the rank \(n_{U_2}\) of \(U_2\) is strictly included in the rank \(n_{U_1}\) of \(U_1\)). As a result, this transition is removed from \(\bar{D}(T)\), and replaced by the transitions \(U_2 \xrightarrow{a|aba} \{(q_3, \epsilon)\}\) and \(U_2 \xrightarrow{a|baa} \{(q_4, \epsilon)\}\). The resulting transducer \(\bar{W}(T)\) is depicted on Fig. 3(a) (where the new transitions are dotted). Fig. 3(b) shows how the latter transducer is split into a union.

**Proof of Lemma 9** We can finally prove that the every weakly twinned transducer is multi-sequential. Let \(\bar{T} = (Q, E, I, \bar{T}, f_{\bar{T}})\) be a weakly twinned transducer. Then \(\bar{T}\) is equivalent to \(\bigcup_{i \in I} \bar{T}_i\), where \(\bar{T}_i\) is the transducer obtained by keeping only \(i\) as initial state. Given \(i \in I\), as we just saw, \(W(\bar{T}_i)\) is a transducer equivalent to \(\bar{T}_i\). Moreover, as \(\bar{W}(\bar{T}_i)\) is separable, so is \(\bar{W}(\bar{T}_i)\), hence, by Lemma 14, it is multi-sequential. The desired result follows.

### 4 Application to Multi-Output Streamability Problem

Sequential functions have the advantage of being efficiently computable. They are exactly the word functions that can be evaluated with constant memory in a sequential, left-to-right, manner. This computability notion have been defined
Formally in [8] with the model of Turing transducers. Informally, a Turing transducer has three tapes: a read-only left-to-right input tape, a working tape, and a write-only left-to-right output tape. The amount of memory is measured only on the working tape. For any sequential function $f$, there exists a Turing transducer $M$ and constant $K \in \mathbb{N}$ such that for all words $u \in \text{dom}(f)$, $f(u)$ can be computed by $M$ while using at most $K$ cells of the working tape. This model is a streaming model in the sense that the input tape is left-to-right, and therefore one can think of receiving the input word $u$ as a stream. The converse also holds true: any word function computable with constant memory by a Turing transducer is sequential. Therefore, the following problem, called the streamability problem, is decidable in $\text{PTime}$, based on the twinning property: given a functional transducer, does it define a function that can be evaluated with constant memory? In this section, we establish a similar result for relations.

We extend the model of Turing transducer to a model for computing relations. We rather explain this model in words, avoiding a tedious and technical definition of intuitive concepts. This model can be thought of as a streaming model where the input word $u$ is a stream, and the outputs words are produced on-the-fly, while processing $u$, and sent through different channels (represented as output tapes in the model). More precisely, let $k \in \mathbb{N}$. A $k$-output Turing transducer $M$ is a deterministic Turing machine with (1) a read-only left-to-right input tape, (2) a two-way working tape, and (3) $k$ write-only left-to-right output tapes without stay transitions (therefore a cell cannot be rewritten). Additionally, the machine can disable/enable some output tapes. A tape configuration of such a machine is a tuple $(u, v, W)$ where $u$ is the content of the input tape, $v$ is the content of the working tape, and $W$ is the set of contents of the enabled output tapes. By content we mean the sequence of symbols to the first blank symbol.

To define the class of constant memory computable relations, we will allow some preprocessing of the input stream, i.e., the computation of a constant amount of information that can be then exploited by the machine to compute the output words. In the setting of sequential function, this information is implicitly present in the definition of constant memory computability: it is assumed that the input stream belongs to the domain of the function, otherwise it could start producing some output word and realise later on that the word was not in the domain. The information of being in the domain or not (a 0/1 bit) is typically some preprocessing computation that can be performed on the input word, by some other application. For instance, if the stream is generated by another ap-
plication, this application could also send the information on whether the stream belong to the domain of the function or not.

We come to the definition of constant memory computability for relations. Let $R \subseteq \Sigma^* \times \Gamma^*$. We say that $R$ is constant memory computable if there exist two constants $k, K \in \mathbb{N}$, a computable function $f : \Sigma^* \rightarrow \{0,1\}^*$ such that for all $u \in \Sigma^*$, $|f(u)| \leq k$, and a $k$-output Turing transducer $M$ over the alphabet $\Sigma \cup \Gamma \cup \{0,1,#,\}$ (where $\#$ stand for a blank symbol) such that for all $u \in \Sigma^*$, on initial tape configuration $(f(u)\#u,\epsilon,\emptyset)$ (all output tapes are initially disabled), the machine uses at most $K$ cells of the working tape along its computation, and halts in a tape configuration $(f(u)\#u,\alpha,W)$ such that $W = R(u)$.

**Theorem 18.** Let $R \subseteq \Sigma^* \times \Gamma^*$. Then $R$ is constant memory computable iff it is multi-sequential. Moreover, if $R$ is given by a transducer, it is decidable in PTime whether it is constant memory computable.

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Appendix

Proofs of section 2

Lemma 19. Let $\mathcal{T} = (Q, E, I, T, f_T)$ be a transducer that does not satisfy the weak twining property. Then there exists a run

$$q_1 \xrightarrow{u_1} q_1 \xrightarrow{v_1} q_1 \xrightarrow{u_2} q_2 \xrightarrow{v_2} q_2$$

such that either $|v_1| \neq |v_2|$, or $v_1, v_2 \neq \epsilon$ and there is a mismatch between $u_1$ and $u_2$.

Proof. Suppose that $\mathcal{T}$ is not weakly twinned. Then there exists a run

$$q_1 \xrightarrow{u_1} q_1 \xrightarrow{v_1} q_1 \xrightarrow{u_2} q_2 \xrightarrow{v_2} q_2$$

such that $\Delta(u_1, u_2) \neq \Delta(u_1v_1, u_2v_2)$. If $|v_1| \neq |v_2|$, we are done. If $|v_1| = |v_2|$, then they are both distinct from $\epsilon$, otherwise $\Delta(u_1, u_2)$ would be equal to $\Delta(u_1v_1, u_2v_2)$. We have to consider three possibilities.

If there is a mismatch between $u_1$ and $u_2$, we are done.

If $u_1$ is a prefix of $u_2$, suppose ab absurdo that for every $n \in \mathbb{N}$, $u_1v^n_1$ is a prefix of $u_2v^n_2$. In particular, this means that there exist $k, k' \in \mathbb{N}$ such that $u_2 = u_1v^k_1v'_{1}$, where $v'_1$ is a prefix of $v_1$, and $u_2v_2 = u_1v^k_1v'_1v''_1$, where $v''_1$ is a prefix of $v_1$. However, as $|v_1| = |v_2|$ by supposition, $k' = k + 1$ and $|v'_1| = |v''_1|$, hence $v'_1 = v''_1$.

Therefore

$$\Delta(u_1v_1, u_2v_2) = \Delta(u_1v_1, u_1v^k_1v'_1) = \Delta(\epsilon, v^k_1v'_1) = \Delta(u_1, u_1v^k_1v'_{1}) = \Delta(u_1, u_2),$$

which is a contradiction.

Therefore there exists $n \in \mathbb{N}$ such that $u_1v^n_1$ is not a prefix of $u_2v^n_2$. As $|u_1v^n_1| \leq |u_2v^n_2|$, there is a mismatch between $u_1v^n_1$ and $u_2v^n_2$, hence the run

$$q_1 \xrightarrow{uv^n} q_1 \xrightarrow{v_1} q_1 \xrightarrow{uv^n} q_2 \xrightarrow{v_2} q_2$$

satisfies the desired conditions.

Finally, if $u_2$ is a prefix of $u_1$, the desired result can be proved in a similar way.

Lemma 20. Let $\mathcal{T} = (Q, E, I, T, f_T)$ be a transducer that does not satisfy the weak twining property. Then there exists a run

$$q_1 \xrightarrow{u_1} q_1 \xrightarrow{v_1} q_1 \xrightarrow{u_2} q_2 \xrightarrow{v_2} q_2$$

such that for every $n \in \mathbb{N}$, $|\Delta(u_1v^n_1, u_2v^n_2)| \geq n$.

Proof. By the preceding lemma, there exists a run

$$q_1 \xrightarrow{u_1} q_1 \xrightarrow{v_1} q_1 \xrightarrow{u_2} q_2 \xrightarrow{v_2} q_2$$
such that either $|v_1| \neq |v_2|$, or $v_1, v_2 \neq \epsilon$ and there is a mismatch between $u_1$ and $u_2$.

If $|v_1| < |v_2|$, we have to consider two possibilities. If $|u_1| \leq |u_2|$, $|\Delta(u_1 v_1^n, u_2 v_2^n)| \geq |u_2 v_2^n| - |u_1 v_1^n| \geq n$. If $|u_1| > |u_2|$, by considering the run

$$q_1 \xrightarrow{u_1} q_1 \xrightarrow{u_2 | u_1 - |u_2| |u_1| - |u_2|} q_1 \xrightarrow{u_2 | u_1| - |u_2| |u_2| - |u_2|} q_2 \xrightarrow{v_1} q_2,$$

we are back in the first situation.

If $|v_1| > |v_2|$, we can proceed in a similar way.

Finally, if $u_1, u_2 \neq \epsilon$ and there is a mismatch between $u_1$ and $u_2$, the delay grows at each step by $|v_1| + |v_2| > 1$, and the result follows.

**Proof of Proposition 2**

Proof. Let $p(u_1, u_2)$ denote the largest common prefix of $u_1$ and $u_2$, and let $d(u_1, u_2)$ and $d(u_2, u_1)$ be the words such that $u_1 = p(u_1, u_2)d(u_1, u_2)$ and $u_2 = p(u_1, u_2)d(u_2, u_1)$. We use similar notations for $v_1$ and $v_2$.

As $(u_1, v_1)$ and $(u_2, v_2)$ are recognised by $D$, there exist two accepting runs $p_1 : \bar{q} \xrightarrow{u_1 | v'_1} q_1$ and $p_2 : \bar{q} \xrightarrow{u_2 | v'_2} q_2$, where $v'_1 f_T(q_1) = v_1$ and $v'_2 f_T(q_2) = v_2$. As the input automaton is deterministic, each run can be decomposed in two parts as follows.

$$p_1 : \bar{q} \xrightarrow{p(u_1, u_2) | v} q_1 \xrightarrow{d(u_1, u_2) | v'_1} q_1,$$

$$p_2 : \bar{q} \xrightarrow{p(u_1, u_2) | v} q_1 \xrightarrow{d(u_2, u_1) | v'_2} q_2.$$

Therefore, $v$ is a prefix of both $v_1$ and $v_2$, hence it is a prefix of $p(v_1, v_2)$. Finally,

$$|\Delta(v_1, v_2)| = |d(v_1, v_2)| + |d(v_2, v_1)|$$

$$= |v_1| + |v_2| - 2|p(v_1, v_2)|$$

$$= 2|v_1| + |v'_1| + |v'_2| + |f_T(q_1)| + |f_T(q_2)| - 2|p(v_1, v_2)|$$

$$\leq |v'_1| + |v'_2| + |f_T(q_1)| + |f_T(q_2)|$$

$$\leq MD|d(u_1, u_2)| + MD|d(u_2, u_1)| + 2MD$$

$$= MD(|\Delta(u_1, u_2)| + 2).$$

**Proof of Theorem 6.2**

Proof. Given a trim transducer $T = (Q, E, I, T, f_T)$, we construct a counter machine $M$ whose counters are reversal-bounded (they alternate between increasing and decreasing modes only a bounded number of times), such that $L(M)$ is empty if $T$ is weakly twinned. The result will follow since emptiness of reversal-bounded counter machine is decidable in $\text{PTIME}$ [11]. We base our construction on checking the conditions of Lemma 19.

Let $\#$ be a fresh symbol that does not belong to $\Sigma$. The machine $M$ will be the union of two machines $M_1$ and $M_2$ that respectively accept words of the form $u\#v$ such that there exist $q_1, q_2 \in Q$ and $v_1, v_2, u_1, u_2 \in \Gamma^*$ such that

$$q_1 \xrightarrow{u_1 | u_4} q_1 \xrightarrow{v_1 | u_1} q_1 \xrightarrow{u_2 | v_2} q_2 \xrightarrow{v_2} q_2$$

and
− \(|v_1| \neq |v_2|\) (for \(M_1\))
− \(v_1, v_2 \neq \epsilon\) and there is a mismatch between \(u_1\) and \(u_2\) (for \(M_2\))

Let us explain how to construct \(M_1\). First, to check the property about runs, \(M_1\) non-deterministically guesses \(q_1\) and \(q_2\), and will simulate two runs of \(T\) in parallel on \(u\) and \(v\). While reading \(u\), \(M_1\) simulates the product of \(T\) with itself, by guessing two runs of \(T\) from the pair of state \((q_1, q_1)\) to the pair of states \((q_1, q_2)\). Similarly, it will simulate two runs of \(T\) from the pair \((q_1, q_2)\) to the pair \((q_1, q_2)\), on \(v\). By simulating these runs, \(M_1\) will non-deterministically trigger transitions of \(T\), and therefore will guess output words \(u_1, u_2, v_1, v_2\). In order to check that \(|v_1| \neq |v_2|\), the machine \(M_1\) uses two counters \(c_1, c_2\), both initialised, non-deterministically, to some value \(\iota\) (by using a loop that increments both \(c_1\) and \(c_2\), and by non-deterministically deciding to leave that loop). When reading \(v\), if \(M\) simulates two transitions of \(T\) of the form \(q \xrightarrow{aw} q'\) and \(p \xrightarrow{aw'} p'\) in parallel, the machine \(M_1\) decreases the counters \(c_1\) and \(c_2\) from respectively the value \(|w|\) and \(|w'|\). It accepts only if it ends in the pair of states \((q_1, q_2)\), \(c_1 = 0\), and \(c_2 > 0\) (meaning that the guessed value \(\iota\) equals \(|v_1|\), and \(|v_2| > |v_1|\)), or if \(c_2 = 0\) and \(c_1 > 0\) (meaning \(|v_1| > |v_2| = \iota\)). Counter machines can only test for zero, but this can be easily turned into more complicated tests, such as \(c_1 = 0\) and \(c_2 > 0\), with polynomial space.

The machine \(M_2\) is constructed similarly. It can easily check that \(v_1, v_2 \neq \epsilon\) (if the transitions of \(T\) it simulates in parallel only produce empty words, then the machine stays in a non-accepting state). To check the existence of a mismatch between \(u_1\) and \(u_2\), the machine again uses two counters \(c_1, c_2\) both initialised to some value \(\iota\). It will check that \(\iota \leq |u_1|, |u_2|\), and that \(u_1[\iota] \neq u_2[\iota]\). Like for \(M_1\), this is done by decreasing in parallel \(c_1\) and \(c_2\) by the length of the respective words produced on the simulated transitions of \(T\) when reading \(u\). If one of the two counters reaches 0, say \(c_1\), then the symbol \(u_1[\iota]\) is stored in the state of \(M_2\), and \(c_2\) still continues to be decreased, until it reaches 0. At that point, it suffices to check that the symbol \(u_2[\iota]\) is different from the stored symbol, and to accept only in that case.

The two machines \(M_1\) and \(M_2\) can be constructed in polynomial time from \(T\), and therefore have polynomial size. Their counters are reversal-bounded: there is only one change of polarity. Therefore, the emptiness of \(M_1\) and \(M_2\) can be checked in polynomial time (in the size of \(T\)).

**Proofs of section 3**

**Proof of lemma [4]**

Proof. First, we prove the result for transducers. Let \(T = (Q, E, I, T, f_T)\) be a separable transducer. We shall show that the trim part of \(\text{split}(T)\) is a decomposition of \(T\) into sequential transducers. Using the same notations as in the definition, we have \(\text{split}(T) = \bigcup_{p \in P} T_p\). Then for every \(p\) in \(P\), the input automaton \(A_p\) of \(T_p\) has a single initial state. Suppose ab absurdo that \(A_p\) admits two distinct edges of same source and same label. By supposition, both are transient edges of \(T\), hence, by definition of \(T_p\), they are part of the path \(p\). However,
as they share the same source, the first one encountered by \( p \) is not transient, which is a contradiction. Therefore \( \mathcal{A}_p \) is deterministic, hence \( \mathcal{T}_p \) is sequential, which proves the desired result.

The extension to multi-transducers uses the fact that any multi-transducer can be transformed in an equivalent finite union of transducers over the same underlying automaton, preserving separability.

**Lemma 21.** Given \( U \in \mathcal{P}_f(Q, \Gamma^* \times \Gamma^*) \) and \( v \in \Gamma^* \), let

\[
R_{U,v} = \{(q, w) \in Q \times \Gamma^* | \exists (p, u) \in U, v' \in \Gamma^* \text{ s.t. } p \xrightarrow{v|v'} q \text{ and } w = wv'\},
\]

\( w_{U,v} \) be the largest common prefix of the words \( \{w|\exists q \in Q \text{ s.t. } (q, w) \in R_{U,v}\} \),

\( P_{U,v} = \{(q, w)|q \in Q \text{ and } w \in w_{U,v}\} \).

Then \( r : U \xrightarrow{v|w_{U,v}} P_{U,v} \) is the only run of source \( U \) and input \( v \) in \( \bar{\mathcal{D}}(\mathcal{T}) \).

**Proof.** This is proved by induction on the length of \( v \).

If \( v = \sigma \in \Sigma \), this follows immediately from the definition of the edges of \( \bar{\mathcal{D}}(\mathcal{T}) \).

Otherwise, \( v = v_0\sigma \) for some non-empty word \( v_0 \) and some letter \( \sigma \in \Sigma \).

Suppose that the result is true for \( v_0 \). Then the only run of source \( U \) and input \( v_0 \) in \( \bar{\mathcal{D}}(\mathcal{T}) \) is

\[
U \xrightarrow{v_0|w_{U,v_0}} P_{U,v_0}.
\]

By definition of the edges of \( \bar{\mathcal{D}}(\mathcal{T}) \), the only edge of source \( P_{U,v_0} \) and input \( \sigma \) is

\[
P_{U,v_0} \xrightarrow{\sigma|w_{P_{U,v_0},\sigma}} P_{P_{U,v_0},\sigma}.
\]

Hence the only run of source \( U \) and input \( v \) in \( \bar{\mathcal{D}}(\mathcal{T}) \) is

\[
U \xrightarrow{v|w_{U,v_0} w_{P_{U,v_0},\sigma}} P_{P_{U,v_0},\sigma}.
\]

However, by definition, \( R_{U,v} = \{(q, w_{U,v_0}, v')|(q, v') \in R_{P_{U,v_0},\sigma}\} \), Therefore \( w_{U,v} = w_{U,v_0} w_{P_{U,v_0},\sigma} \) and \( P_{U,v} = P_{P_{U,v_0},\sigma} \), which proves the desired result.

**Corollary 22.** The transducer \( \mathcal{T} \) is equivalent to \( \mathcal{D}(\mathcal{T}) \).

**Proof.** Let \( (u, v) \in \Sigma^* \times \Gamma^* \). Then

\[
(u, v) \in \mathcal{T} \iff \text{there exists an accepting run } q_0 \xrightarrow{u|w} q \text{ s.t. } v = w f_T(q)
\]

\[
\iff \exists q \in T, \exists w \in \Gamma^* \text{ s.t. } (q, w) \in R_{U_0,u} \text{ and } v = w f_T(q)
\]

\[
\iff \exists q \in T, \exists w' \in \Gamma^* \text{ s.t. } (q, w) \in P_{U_0,u} \text{ and } v = w_{U_0,u} w' f_T(q)
\]

\[
\iff (u, v) \in \mathcal{D}(\mathcal{T})
\]

where the last equivalence follows from lemma 21.

**Corollary 23.** The transducer \( \mathcal{T} \) is equivalent to \( \mathcal{W}(\mathcal{T}) \).
Proof. Using the definition of $W(\mathcal{T})$, and the fact that, by lemma 21, for every $U \in \mathcal{T}(Q \times \Gamma^*)$, for every $v \in \Sigma^*$,

\[
\{ (q, w_U, w) | (q, w) \in P_{U,v} \} = \bigcup_{(q, w) \in U} \{ (q', w_{(q,v)}, w') | (q', w') \in P_{(q,v),v} \},
\]

one can prove that $W(\mathcal{T})$ is equivalent to $D(\mathcal{T})$. The result then follows from the previous corollary.

**Corollary 24.** Let $U \in \mathcal{P}_f(Q, \Gamma^*)$ be the target of a non empty run in $\mathcal{W}(\mathcal{T})$. Then the largest common prefix of the words $\{ w | \exists q \in Q. (q, w) \in R_{U,v} \}$ is equal to $\epsilon$.

Proof. By definition of $\mathcal{W}(\mathcal{T})$, either $U = \{ (q, \epsilon) \}$ for some $q \in Q$, and the results follows immediately, or $U$ is the target of a run in $\mathcal{D}(\mathcal{T})$, and the result follows from lemma 21.

**Proof of Proposition 16**

Proof. By corollary 23, $W(\mathcal{T})$ and $\mathcal{T}$ are equivalent. Suppose that $\mathcal{T}$ is weakly twinned. We shall show that the subset $Q'$ of $\mathcal{P}_f(Q \times \Gamma^*)$ accessible from $U_0$ in $\mathcal{W}(\mathcal{T})$ is finite.

Let $Q_b$ be the subset of $Q'$ composed of the elements $U \in Q'$ that contain at least two pairs $(q_1, v_1), (q_2, v_2)$ such that $|\Delta(v_1, v_2)| \geq 2M_\mathcal{T}|Q|^3$. We shall prove that $Q_b$ is empty, which proves the desired result, as it bounds the size of the words present in the sets $U \in Q'$ : if there were an element $U$ of $Q'$ containing a word of size greater than $2M_\mathcal{T}|Q|^3$, then it would be in $Q'$, by corollary 24.

Suppose ab absurdo that $Q_b$ is not empty. Let $r : U_0 \xrightarrow{u|v} V \in Q_b$ be a run in $\mathcal{W}(\mathcal{T})$ such that for every $U \in Q_b$, for every run $r_U : U_0 \xrightarrow{x|y} U$ in $\mathcal{W}(\mathcal{T})$, the length of $r$ is shorter than or equal to the length of $r_U$. The run $r$ can be decomposed into two parts $r : U_0 \xrightarrow{u|v} V_0 \xrightarrow{u|v} V$ where $V_0$ is the first state such that the part of the run following it preserves the rank. Therefore, the second part of the run is a run of $\mathcal{D}(\mathcal{T})$. Note that either $V_0 = U_0$, or its rank is different than the one of its predecessor. In both cases, by definition of $\mathcal{W}(\mathcal{T})$, $V_0 = \{ (p, \epsilon) \}$ for some $p \in Q$. As $n_{V_0} = n_V$, there exists a pair $(p_0, v_0)$ in $V$ such that there exists a run from $p_0$ to $q$ in $\mathcal{T}$. Moreover, by supposition, $V$ contains two pairs $(p_1, v_1), (p_2, v_2)$ such that $|\Delta(v_1, v_2)| \geq M_\mathcal{T}|Q|^3$. Then, by lemma 21, $\mathcal{T}$ admits three runs

\[
\begin{align*}
r_0 : p = p_0 & \xrightarrow{u_1|v_0} p_1 \xrightarrow{u_2|v_2} \ldots \xrightarrow{u_n|v_n} p_0 = p_0, \\
r_1 : p = p_1 & \xrightarrow{u_1|v_1} p_1 \xrightarrow{u_2|v_2} \ldots \xrightarrow{u_n|v_n} p_1 = p_1, \\
r_2 : p = p_2 & \xrightarrow{u_1|v_2} p_2 \xrightarrow{u_2|v_2} \ldots \xrightarrow{u_n|v_n} p_2 = p_2,
\end{align*}
\]

such that $u_1 \ldots u_n = \bar{u}$, and for every $0 \leq i < n$, for every $0 \leq j \leq 2$, $p_j \xrightarrow{u_{i+1}|v_{i+1}} p_{j+1}$ is a transition of $\mathcal{T}$ and $v_1 \ldots v_n = \bar{v}v_j$. Then $n \geq |Q|^3$, because $|v_1 \ldots v_n| +
\[ |v_2^1 \ldots v_2^n| \geq \Delta(v_1^1, v_2^2) \geq 2M_T|Q|^3 \] by supposition, and for every \(0 \leq i \leq n, |v_1^i| + |v_2^i| \leq 2M_T\) by definition of \(M_T\). Therefore there exist \(0 \leq i_1 < i_2 \leq n\) such that for every \(0 \leq j \leq 2, p_j^{i_1} = p_j^{i_2}.
If \(\Delta(v_1^1 \ldots v_1^{i_1}, v_2^1 \ldots v_2^{i_1}) = \Delta(v_1^1 \ldots v_1^{i_2}, v_2^1 \ldots v_2^{i_2})\), then
\[ \Delta(v_1^1 \ldots v_1^{i_1}v_2^{i_1} \ldots v_2^{i_1}, v_1^1 \ldots v_2^{i_2} \ldots v_2^{i_2}) = \Delta(v_1^1 \ldots v_2^n), v_2^1 \ldots v_2^n) \geq 2M_T|Q|^3, \]

hence the run in \(W(T)\) corresponding to the runs of \(T\) obtained by removing the part between \(i_1\) and \(i_2\) of \(r_1\) and \(r_2\) is a run strictly shorter than \(r\) between \(U_0\) and an element of \(Q_k\), which contradicts the minimality of \(r\).

If \(\Delta(v_1^1 \ldots v_1^{i_1}, v_2^1 \ldots v_2^{i_1}) \neq \Delta(v_1^1 \ldots v_1^{i_2}, v_2^1 \ldots v_2^{i_2})\), then \(\Delta(v_1^1 \ldots v_2^n, v_1^{i_1} \ldots v_1^{i_2}) \neq \Delta(v_1^1 \ldots v_2^n, v_2^{i_1} \ldots v_2^{i_2})\), for \(j = 1\) or \(j = 2\). Moreover, by choice of \(p_0\), there exists a run \(p_0 \xrightarrow{u_0, p_0} p\). Therefore there is a contradiction with the fact that \(T\) is weakly twinned, exposed by taking the runs

\[ p_0 \xrightarrow{u_0, p_0} p \xrightarrow{u_1^1 \ldots u_1^i, v_2^{i_1} \ldots v_1^{i_1}} p_j^{i_1}, \quad p_0 \xrightarrow{u_1^2 \ldots u_2^i, v_2^{i_2} \ldots v_2^{i_2}} p_j^{i_2}, \]

\[ p_0 \xrightarrow{u_1^1 \ldots u_1^i, v_2^{i_1} \ldots v_2^{i_1}} p_0, \quad p_0 \xrightarrow{u_1^2 \ldots u_2^i, v_2^{i_2} \ldots v_2^{i_2}} p_0. \]

This concludes the proof.

**Proofs of section 4**

**Proof of lemma 18**

Proof. If \(R\) is multi-sequential, then \(R = \bigcup_{i=1}^{k} [D_i]\) for \(k\) sequential transducers \(D_1, \ldots, D_k\). Let us show that \(R\) is constant memory computable, i.e., let us construct \(f, M, K\) as in the definition of constant memory computability. The function \(f\) associates with \(u\) a word \(b_1 \ldots b_k \in \{0,1\}^*\) such that \(b_i = 1\) iff \(u \in dom(D_i)\). The machine \(M\) simulates all the \(D_i\) such that \(b_i = 1\) successively. It uses at most \(k\) output tapes to write the output words \([D_i](u)\). It does not need to use the working tape (and therefore we can take \(K = 0\)) because it can simulate the states of \(D_i\) by using its internal states. A state of \(M\) is therefore a tuple of states from distinct \(D_i\). \(M\) uses \(f(u)\) to initialise the first tuple of states to \((q_0^i)_{i \in \{j \mid b_i=1\}}\), where \(q_0^i\) is the initial state of \(D_i\). \(M\) stops when it reads the first blank symbol on the input tape.

Conversely, let us show that any constant memory computable relation is multi-sequential. Let \(f, k, M, K\) as in the definition of constant memory computability. Let \(P = \{ f(u) \mid u \in \Sigma^* \}. \) Since for all \(u \in \Sigma^*, |f(u)| \leq k\), the set \(P\) is finite. Given \(v \in P\), the machine \(M\) reading input of the form \(v\#u\) can be seen as the union of \(k\) sequential transducers \(D_{1,v}, \ldots, D_{k,v}\). Informally, since the memory used on the working tape is bounded by \(K\), there are only a constant number of possible configurations on that tape. The states of \(D_{i,v}\) are these configurations. Then, the transducer \(D_{i,v}\) simulates \(M\) and only produces the symbols produced by \(M\) on the \(i^{th}\) output tape. The accepting states of \(D_{i,v}\) are the configurations where \(M\) halts. Note that since \(M\) can perform
several transitions without reading any input symbol, the resulting transducers \( D_{i,v} \) may have \( \epsilon \)-transitions. It is easily shown that the \( D_{i,v} \) can be turned into proper transducers without \( \epsilon \)-transitions, as shown in [8] for constant memory computable functions. Finally, we have \( R = \bigcup_{v \in P} \bigcup_{i=1}^k \|D_{i,v}\| \), and therefore \( R \) is multi-sequential.

\[ \square \]

Examples

Here are two more examples of weakly twinned transducers that are not twinned.

The figure 4 presents the transducer \( \mathcal{T}_1 \) along with its determinisation and weak determinisation. It is not twinned, as exposed by the runs

\[
\begin{align*}
q_0 & \xrightarrow{aaa|a} q_5 \xrightarrow{ba|a} q_5, \\
q_0 & \xrightarrow{aaa|b} q_5 \xrightarrow{ba|a} q_5.
\end{align*}
\]

Therefore, the determinisation is infinite. However, the weak determinisation is finite. The dotted edge of \( \mathcal{D}(\mathcal{T}_1) \), definitely leaving the SCC \( \{q_0, q_1, q_2\} \) of \( \mathcal{T}_1 \), is split into the three dotted edges of \( \mathcal{W}(\mathcal{T}_1) \). The figure 5 exposes the decomposition of \( \mathcal{T}_1 \) into a union of sequential transducers.

The figure 6 present the transducer \( \mathcal{T}_2 \) along with its determinisation and weak determinisation. It is not twinned, as exposed by the runs

\[
\begin{align*}
q_0 & \xrightarrow{aaaa|aaaa} q_6 \xrightarrow{a|a} q_6, \\
q_0 & \xrightarrow{aaaa|abaaa} q_6 \xrightarrow{a|a} q_6.
\end{align*}
\]

Therefore, the determinisation is infinite. However, the weak determinisation is finite. The dotted edge of \( \mathcal{D}(\mathcal{T}_2) \), definitely leaving the SCC \( \{q_0, q_1, q_2, q_3\} \) of \( \mathcal{T}_1 \), is split into the three dotted edges of \( \mathcal{W}(\mathcal{T}_2) \). The figure 7 exposes the decomposition of \( \mathcal{T}_2 \) into a union of sequential transducers.
Fig. 4. Multisequential transducer that is not sequential.
Fig. 5. The transducer \( \text{trim(split}(\mathcal{W}(T_1))) \)
Fig. 6. Multisequential transducer that is not sequential.
Fig. 7. The transducer $\text{trim}(\text{split}(W(T_2)))$