Rogue Waves as Self-Similar Solutions on a Background: A Direct Calculation

C. B. Ward\textsuperscript{1} and P. G. Kevrekidis\textsuperscript{1}

\textsuperscript{1}Department of Mathematics and Statistics, University of Massachusetts, Amherst MA 01003-4515, USA

In the present work, we explore the possibility of developing rogue waves as exact solutions of some nonlinear dispersive equations, such as the nonlinear Schrödinger equation, but also, in a similar vein, the Hirota, Davey-Stewartson, and Zakharov models. The solutions that we find are ones previously identified through different methods. Nevertheless, they highlight an important aspect of these structures, namely their self-similarity. They thus offer an alternative tool in the very sparse (outside of the inverse scattering method) toolbox of attempting to identify analytically (or computationally) rogue wave solutions. This methodology is importantly independent of the notion of integrability. An additional nontrivial motivation for such a formulation is that it offers a frame in which the rogue waves are stationary. It is conceivable that in this frame one could perform a proper stability analysis of the structures.

\section{INTRODUCTION: MOTIVATION \& APPROACH}

The study of waves that are characterized by extreme amplitudes, and which are often referred to as freak or rogue structures has gained considerable traction over the last decade. This can, arguably, to a large degree be attributed to the exploration of the relevant waveforms in a variety of experimental realizations spanning diverse areas of physics, starting from the traditional field of hydrodynamics \cite{1-4}, but also extending to numerous other areas. These include, but are not limited to, nonlinear optics \cite{5-9}, superfluid helium \cite{10}, as well as plasmas \cite{11}. These multifaceted experimental studies have, in turn, triggered a wide range of theoretical explorations which by now have been summarized in a series of reviews \cite{12-14}, but also importantly in a series of books on this research theme \cite{15-18}.

The nonlinear Schrödinger equation (NLS) \cite{19,20} has, arguably, been a principal vehicle of choice for the exploration of rogue waves, with the so-called Peregrine soliton \cite{21}, the Akhmediev breather (AB) \cite{22}, and the Kuznetsov-Ma \cite{23,24} (KM) soliton playing a crucial role in the corresponding studies as the prototypical structures of relevance; in this context, the work of Dysthe-Trulsen is also worth mentioning \cite{25}. Nevertheless, these waveforms have emerged from toolboxes associated with integrability theory and inverse scattering. This naturally prompts the question of accessibility of relevant solutions beyond the integrable limit and of their persistence under non-integrable perturbations which are especially important in numerous, among the above mentioned, applications. In this vein, analytical tools based on perturbative (exact or approximate) solutions \cite{26}, as well as on persistence criteria for such waveforms \cite{27,28} have been proposed. In a recent study \cite{29}, we developed a complementary approach to the above analytical ones attempting to identify rogue waves via a numerically developed toolbox which traces the existence of such structures as a steady-state problem in space-time (i.e., considering time as a space-like variable and performing a conjugate-gradient minimization to obtain the wave pattern). Our methodology was successful in establishing the persistence of Peregrine-like patterns, which are localized in both space and time, in generalized NLS models, for instance featuring a different than cubic nonlinearity.

Here, we would like to take a different path in the vein of offering a different kind of theoretical (but possibly also computational) tool towards the identification of rogue wave structures. We are motivated by two facts: on the one hand, the availability of analytical techniques for finding rogue waves is largely limited to methods stemming from inverse scattering transform (IST) and integrable systems. Nevertheless, in many of the realistic non-integrable systems where such extreme waves appear the IST formulation is not available or applicable. Moreover, from a structural perspective, rogue waves are self-similar in their functional form, even though, contrary to what is the case for other such solutions \cite{19}, they do not blow up in finite time. In light of these developments, we offer an unprecedented approach on the basis of self-similarity to capture such solutions. We should caution the reader from the get go that our aim at least in the present work is not to identify solutions that have not appeared before in the literature. All the solutions that we will present have been previously found via methods related, in some way or other, to integrability. Rather, we offer a different perspective of these solutions that is not bound by integrability and can, in principle, be generalized to models that possess freak waves, yet are not known or expected to be integrable. Moreover, as we will argue in our Future Challenges section, this aspect prompts the potential consideration of the stability of such solutions in the self-similar frame.

\section{SELF-SIMILAR CALCULATION FOR THE NLS CASE}

Let us start with the prototypical NLS model of the form:

\begin{equation}
    i u_t = -\frac{1}{2} u_{xx} - |u|^2 u + u,
\end{equation}

\textsuperscript{1}Department of Mathematics and Statistics, University of Massachusetts, Amherst MA 01003-4515, USA
whereby we have already set the background at unity without loss of generality. Now let us recall that the Peregrine waveform emerges against the backdrop of a constant background $u = 1 + w$. Then, the equation for the new variable $w(x, t)$ reads:

$$iw_t = -\frac{1}{2}w_{xx} - |w|^2w - (w + w^*) - (2|w|^2 + w^2).$$

(2)

The star here denotes complex conjugation. We now follow the so-called MN-dynamics self-similar framework which has been detailed in a number of different publications [30–32] (and is also tantamount to methods generally applied for obtaining self-similar solutions; see, e.g., [33]). More specifically, we use $w(x, t) = A(t)F\left(\frac{x}{L(t)}, t\right)$. This leads to the equation:

$$i \left(AF + AF^* - A\xi \xi F \frac{L_t}{L}\right) = -\frac{1}{2L^2}F_{\xi\xi} - |A|^2|F|^2F$$

$$- (AF + A^*F^*) - (2|A|^2|F|^2 + A^2F^2).$$

(3)

In the above equation, $\xi = x/L(t)$. We will now seek self-similar solutions, by making two important assumptions. First, we will assume that the solutions are *stationary in the self-similar frame*. This is not a restricting assumption: it simply demands that we have self-similar solutions as such. The second assumption is that we will assume $F$ to be a real profile. A priori, this is not mandated, nevertheless an understanding of the inner workings of the method suggests that it would not be possible to scale terms like the ones of the second line of Eq. (3) barring such a restriction. Effectively, we assign all the complex phase.

For self-similarity to work here in the corresponding self-similar frame, we need the time dependent terms to cancel and independently the time-independent terms to do the same. This leads to the choice

$$R^2 = \frac{1}{L^2} = 2R \cos(\theta), \quad 2 \cos^2(\theta) = \theta_t.$$

(6)

The latter, in turn, yields $\tan(\theta) = 2(C + t)$, where $C$ shifting the origin of time can be set to 0 without loss of generality. Then $\cos^2(\theta) = 1/(1 + 4t^2)$ leading to

$$L = \frac{1}{R} = \frac{\sqrt{1 + 4t^2}}{2} \Rightarrow A = \frac{2}{1 - 2it}.$$  

(7)

Having found the temporal prefactor, the resulting real equation for the wave profile now reads:

$$-\frac{1}{2}F_{\xi\xi} - F^3 - \frac{3}{2}F^2 = 0.$$

(8)

Using the above Eqs. (6)-(7) to simplify the imaginary part of the Eq. (4) canceling from all the terms a factor of $\sin(2\theta)$, we retrieve a first-order equation for the same profile, namely:

$$\xi F_{\xi} = -2F - F^2.$$

(9)

We can solve this first order ODE to obtain the solution

$$F(\xi) = \frac{2}{1 + D\xi^2};$$

(10)

now a direct substitution of this waveform in Eq. (8) yields straightforwardly that the solution is valid only for $D = 1$, hence we obtain $F = -2/(1 + \xi^2)$. We now reconstruct our solution:

$$u = 1 + W = 1 + \frac{2}{1 - 2it} \left(\frac{2}{1 + \xi^2}\right) = 1 - 4 \frac{1 + 2it}{1 + 4x^2 + 4t^2},$$

(11)

which naturally retrieves the well-known Peregrine structure [21]. It is important to make some remarks here:
• The expressions of Eq. (11) bring forth the self-similar nature of the Peregrine, which, in our view, seems to have been overlooked in the literature. Factoring out the time dependence as a complex factor (∝ 1/(1 − 2it)), one is left with an effective Lorentzian self-similar waveform which is at the heart of the self-similarity-based calculation of the structure as a steady state solution of the relevant formulation.

• Nevertheless, there are some nontrivial differences of this calculation from other similar calculations, e.g., in the above mentioned references such as [30–32]. Here, when separating, for instance, the real part of the solution, there are both time-dependent and time-independent terms and these need to be balanced out between them separately (the former on their own, and the latter on their own). Perhaps even more importantly, the real and imaginary parts yield 2 distinct differential equations that have a particular solution in common that needs to be singled out upon a suitable selection of a compatibility constant.

Nevertheless, the above procedure can be utilized whenever it may be believed (e.g., motivated from numerical or physical experiments) that extreme waves may exist in a certain system. The methodology only hinges on identifying such a self-similar solution on top of a background and in no way utilizes the integrable structure of the model. In a sense, it is an analogous calculation (for rogue waves) to the reduction of the NLS, when looking for its standing waves, to a Duffing oscillator whose homoclinic or heteroclinic connections correspond to the bright or dark solitons respectively. To the best of our knowledge such as an ODE-reduction-based calculation has not previously appeared in the context of rogue waves. To illustrate that this viewpoint for the consideration of rogue wave patterns can be used in other (admittedly related) examples beyond “just NLS”, we consider now a series of generalizing cases, including the Hirota model in 1+1 dimensions and the Davey-Stewartson and Zakharov models in higher (i.e., 2+1) dimensional settings. The Hirota model incorporates effects such as the third order dispersion and the time-delay correction to the cubic nonlinearity, as discussed, e.g., in [34]. The Davey-Stewartson model is a relevant one for the examination of the evolution of a wave-packet in a (2+1)-dimensional setting for water of finite depth [35]; here, we consider the setting of large surface tension in the form of the so-called DSI model. Finally, the Zakharov equation was derived in [36] as a prototypical integrable model in (2+1)-dimensions.

III. GOING BEYOND NLS: OTHER MODELS

A. Hirota Equation

The Hirota equation:

\[ iu_t + \frac{1}{2} u_{xx} + |u|^2 u - \alpha i u_{xxx} - 6 \alpha i |u|^2 u_x = 0, \]

(12)

where \( \alpha \) is an arbitrary constant, has a rogue wave solution of the form [37]

\[ u(x, t) = e^{it} \left( 1 - \frac{4(1 + 2it)}{1 + 4(x + 6\alpha t)^2 + 4t^2} \right) \]

(13)

Again, by factoring out 1 + 4t^2 from the bottom, we may write

\[ u(x, t) = e^{it} \left( 1 + \frac{2(1 + 2It)}{1 + 4t^2} \cdot \frac{-2}{1 + (\frac{2(x + 6\alpha t)}{\sqrt{1 + 4t^2}})^2} \right). \]

(14)

where

\[ \xi = \frac{2(x + 6\alpha t)}{\sqrt{1 + 4t^2}}. \]

We then see that the rogue wave of the Hirota equation is, in a very natural way, a self-similar, Peregrine-like structure in a co-traveling reference frame where the coherent structure travels with speed 6\( \alpha \). Hence, it can be retrieved by a similar calculation as the one above.

B. Davey-Stewartson I

The (2 + 1)-dimensional Davey-Stewartson I (DSI) equation
\[ iu_t = u_{xx} + u_{yy} + |u|^2 u - 2Qu \]  
\[ Q_{xx} - Q_{yy} = (|u|^2)_{xx}, \]  
\( iu_t = u_{xx} + u_{yy} + |u|^2 u - 2Qu \) \( (15) \)
\[ Q_{xx} - Q_{yy} = (|u|^2)_{xx}, \]  
\( (16) \)

has a rogue wave solution \( [38] \) of the form:

\[ u(x, y, t) = \sqrt{2} \left( 1 - \frac{4(1 - 2i\omega t)}{1 + (k_1 x + k_2 y)^2 + 4\omega^2 t^2} \right) \]  
\[ (17) \]
\[ Q(x, y, t) = 1 - 4k_1^2 \frac{1 - (k_1 x + k_2 y)^2 + 4\omega^2 t^2}{(1 + (k_1 x + k_2 y)^2 + 4\omega^2 t^2)^2}, \]  
\( (18) \)

where \( k_1 = p - \frac{1}{p}, k_2 = p + \frac{1}{p}, \omega = \frac{1}{\sqrt{p}}, \) and \( p \) is an arbitrary constant. We note here that this is a line rogue wave, resembling the Peregrine structure extended along a line in the \( xy \)-plane. We can write this in the self-similar form

\[ u(x, y, t) = \sqrt{2} \left( 1 + 2 \left( 1 - \frac{2i\omega t}{1 + 4\omega^2 t^2} \right) \right) \]  
\[ (19) \]
\[ Q(x, y, t) = 1 + \frac{4k_1^2}{1 + 4\omega^2 t^2} \cdot \frac{\xi^2 - 1}{(\xi^2 + 1)^2}, \]  
\( (20) \)

where

\[ \xi = \frac{k_1 x + k_2 y}{\sqrt{1 + 4\omega^2 t^2}}. \]

Hence, in this case too, the structure can be thought of as being stationary in a suitable self-similar frame of reference.

\section{C. Zakharov Equation}

The (2+1)-dimensional Zakharov equation assumes the form \( [36] \):

\[ iu_t = u_{xy} + Qu \]  
\[ Q_y = 2(|u|^2)_x. \]  
\( (21) \)
\( (22) \)

This model also admits line-type rogue waves \( [39] \), which we will simply give in self-similar form:

\[ u(x, y, t) = 1 + \frac{2(1 + 4it)}{1 + 16t^2} \cdot \frac{-2}{1 + \xi^2} \]  
\[ (23) \]
\[ Q(x, y, t) = \frac{16}{1 + 16t^2} \cdot \frac{\xi^2 - 1}{(\xi^2 + 1)^2} \]  
\( (24) \)

where

\[ \xi = \frac{2(x - y)}{\sqrt{1 + 16t^2}}. \]

Lastly, we remark that a change in the time scale will change the \( \sqrt{1 + 16t^2} \) to \( \sqrt{1 + 4t^2} \), as in the other examples. This is due to a rescaling of time imposed effectively by the prefactor within Eq. \( (22) \).

\section{IV. CONCLUSIONS & FUTURE WORK}

In the present work we have revisited the examination of rogue wave structures in the context of dispersive nonlinear models. We have argued that while the IST and related techniques (including e.g. the Darboux transformation etc.) provide valuable tools for identifying such solutions in the realm of integrable models, this methodology is limited in comparison to more realistic
models that bear non-integrable perturbations and for which experimental or numerical observations suggest that the structures may persist. A perturbative framework either analytically [26–28] or numerically [29] may be useful in such scenarios. Nevertheless, we argue that a potentially valuable complementary perspective is that of looking at rogue wave patterns as self-similar solutions that are associated with a (potentially complex) time-dependent prefactor and a self-similar (e.g., in the NLS case, Lorentzian) profile, arising against the backdrop of a constant, non-vanishing background. Seeking these solutions through a self-similar type of methodology, as was done for some prototypical case examples herein, enables a way to tackle such solutions that is not bound by the limitations of integrable models and can instead be applied to a wider range of systems.

That being said, in the present proof of principle exposition we have only retrieved case examples where the existence of such waveforms was already identified by integrable structure means, in order to illustrate the ability of the method to capture such waveforms. However, it would be of particular interest to attempt a similar search in model examples where rogue structures are expected to exist, yet the absence of integrability does not allow for their identification. This is one of the key challenges of the method towards future work. A related challenge lies in the potential for consideration of stability features in the self-similar frame. This was done, e.g., in [31], but also, importantly, in a series of works of [40, 41]; see also [42]. These efforts have not only identified the spectra of self-similar waveforms (which are steady in the self-similar frame); they have importantly made a substantial effort to “reinterpret” the spectra of the latter setting into the original frame. An important example of this class is that certain symmetries (such as, e.g., the potential shift of the collapse time in non-autonomous systems) may amount to eigendirections appearing as unstable, which, yet, are not so due to the existence of the corresponding symmetry (in the original frame). This is yet to be done for the Peregrine soliton of the NLS and related waveforms. We have attempted this and have been hindered by technical complications having to do with the nature of the emerging terms in Eq. [3]. This is the same complication that we encountered previously in that some of the terms are autonomous and some are not. The potential existence of a systematic way to bypass this complication would pave a systematic way for identifying (and subsequently reinterpreting in the spirit of [40, 41]) the spectra of extreme wave events, a feature crucial for formulating a more precise notion of their stability. Up to now the latter has been explored in either a somewhat empirical (and often mathematically not suitably substantiated) way or in the form of a limiting procedure of, e.g., periodic states; see the relevant discussion of [43]. Nevertheless, such a direct approach as proposed here would be fundamentally superior to the current state of the art, in our view, and hence constitutes a particularly worthwhile topic for future study.

[1] A. Chabchoub, N. P. Hoffmann, and N. Akhmediev, Phys. Rev. Lett. 106, 204502 (2011).
[2] A. Chabchoub, N. Hoffmann, M. Onorato, and N. Akhmediev, Phys. Rev. X 2, 011015 (2012).
[3] A. Chabchoub and M. Fink, Phys. Rev. Lett. 112, 124101 (2014).
[4] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, Nature 450, 1054 (2007).
[5] B. Kibler et al., Nature Phys. 6, 790 (2010).
[6] B. Kibler et al., Sci. Rep. 2, 463 (2012).
[7] J. M. Dudley, F. Dias, M. Erkintalo, and G. Genty, Nat. Photon. 8, 755 (2014).
[8] B. Frisquet et al., Sci. Rep. 6, 20785 (2016).
[9] C. Lecaplain, Ph. Grelu, J. M. Soto-Crespo, and N. Akhmediev, Phys. Rev. Lett. 108, 233901 (2012).
[10] A. N. Ganshin, V. B. Efimov, G. V. Kolmakov, L. P. Mezhov-Deglin, and P. V. E. McClintock, Phys. Rev. Lett. 101, 065303 (2008).
[11] H. Baiting, S. K. Sharma, and Y. Nakamura, Phys. Rev. Lett. 107, 255005 (2011).
[12] M. Onorato, S. Residori, U. Bortolozzo, A. Montinad, and F. T. Arecchi, Phys. Rep. 528, 47 (2013).
[13] P. T. S. DeVore, D. R. Solli, D. Borsaalg, C. Ropers, and B. Jalali, J. Opt. 15, 064031 (2013).
[14] Z. Yan, J. Phys. Conf. Ser. 400, 012084 (2012).
[15] E. Pelinovsky and C. Kharif (eds.), Extreme Ocean Waves (Springer, NY, 2008).
[16] C. Kharif, E. Pelinovsky, and A. Slunyaev, Rogue Waves in the Ocean (Springer, NY, 2009).
[17] A. R. Osborne, Nonlinear Ocean Waves and the Inverse Scattering Transform (Academic Press, Amsterdam, 2010).
[18] M. Onorato, S. Residori, and F. Baronio, Rogue and Shock Waves in Nonlinear Dispersive Media, Springer-Verlag (Heidelberg, 2016).
[19] C. Sulem and P.L. Sulem, The Nonlinear Schrödinger Equation, Springer-Verlag (New York, 1999).
[20] M.J. Ablowitz, B. Prinari and A.D. Trubatch, Discrete and Continuous Nonlinear Schrödinger Systems, Cambridge University Press (Cambridge, 2004).
[21] D. H. Peregrine, J. Austral. Math. Soc. B 25, 16 (1983).
[22] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Theor. Math. Phys. 72, 809 (1987).
[23] E. A. Kuznetsov, Sov. Phys.-Dokl. 22, 507 (1977).
[24] Ya. C. Ma, Stud. Appl. Math. 60, 43 (1979).
[25] K. B. Dysthe and K. Trulsen, Phys. Scr. T82, 48 (1999).
[26] A Ankiewicz, N Devine, N Akhmediev, Physics Letters A 373 (2009) 3997
[27] A. Calini and M. R. Schober, Phys. Rev. Lett. 105, 014501 (2010).
[28] A. Calini, M. R. Schober, Phys. Rev. E 83, 065303 (2011).
[29] C. B. Ward, P. G. Kevrekidis, N. Whitaker, [arXiv:1712.03292]
[30] D.G. Aronson, S.I. Betelu, I. G. Kevrekidis, [arXiv:calin/0111055]
[31] C.I. Siettos, I.G. Kevrekidis, P.G. Kevrekidis, Nonlinearity 16, 497 (2003).
[32] P.G. Kevrekidis, C.I. Siettos, Y.G. Kevrekidis, Nature Comms. 8, 1562 (2017).
[33] G.I. Barenblatt, Scaling, self-similarity and intermediate asymptotics, Cambridge University Press (Cambridge, 1996).
[34] Yu.V. Sedletsky, J. Exp. Theor. Phys. 97, 180 (2003)
[35] M. J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform (SIAM, Philadelphia, 1981).
[36] V. E. Zakharov, Solitons in Topics Curr. Phys., Vol. 17, R. K. Bullough and P. J. Caudrey (Eds.) Springer-Verlag (Berlin 1980).
[37] A. Ankiewicz, J. M. Soto-Crespo, and N. Akhmediev, Physical Review E 81, 046602 (2010).
[38] Y. Ohta and J. K. Yang, Physical Review E 86, 036604 (2012).
[39] J. Rao, L. Wang, W. Liu, and J. He, Theoretical and Mathematical Physics 193, 1783 (2017)
[40] A.J. Bernoff, T.P. Witelski, Appl. Math. Lett. 15, 599 (2002).
[41] A.J. Bernoff, T.P. Witelski, J. Eng. Math 66, 11 (2010).
[42] S. Ray, R.C. Viesca, J. Geophys. Res.: Solid Earth 122, 8214 (2017).
[43] J. Cuevas-Maraver, P. G. Kevrekidis, D. J. Frantzeskakis, N. I. Karachalios, M. Haragus, and G. James Phys. Rev. E 96, 012202 (2017)