The exponential map of a $C^{1,1}$-metric

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Abstract

Given a pseudo-Riemannian metric of regularity $C^{1,1}$ on a smooth manifold, we prove that the corresponding exponential map is a bi-Lipschitz homeomorphism locally around any point. We also establish the existence of totally normal neighborhoods in an appropriate sense. The proofs are based on regularization, combined with methods from comparison geometry.

Keywords: Exponential map, low regularity, (totally) normal neighborhoods

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1 Introduction

In (smooth) pseudo-Riemannian geometry, the fact that the exponential map is a diffeomorphism locally around 0 is of central importance for many fundamental constructions. As examples we mention the existence of normal neighborhoods, Riemannian normal coordinates, geodesic convexity, injectivity and convexity radius, comparison methods, or also, in the Lorentzian case, causality theory.

The standard way of proving this result rests on an application of the inverse function theorem. It is therefore applicable to $C^2$ pseudo-Riemannian metrics, where the exponential map itself is still $C^1$. On the other hand, the lowest regularity for which the geodesic equation still has unique solutions in general is $C^{1,1}$ (continuously differentiable with Lipschitz derivatives). In the literature it is generally held that $C^{1,1}$ delimits the regularity where one can still reasonably expect the ‘standard’ results to remain valid. However, for $C^{1,1}$-metrics the exponential map is only Lipschitz, so the inverse function theorem is no longer applicable. In [3], which provides a careful analysis of causality theory in minimal regularity, P. Chruściel indicates that certain inverse function theorems for Lipschitz functions might shed light on this problem. To our knowledge, however, so far there is no proof available in the literature that the exponential map of a $C^{1,1}$-metric retains the maximal possible regularity, namely that it is a bi-Lipschitz homeomorphism around 0. In this work we supply a proof of this result.

In analogy to the smooth case one may call the image of a (star-shaped) open neighborhood of 0 in $T_p M$ where $\exp_p$ is a bi-Lipschitz homeomorphism a normal neighborhood of $p$. We show that in fact there always exist totally normal neighborhoods around any point of the manifold, i.e., open sets that are normal neighborhoods of each of their elements.

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Our method of proof consists in regularizing the metric locally via convolution with a mollifier to obtain a net $g_\varepsilon$ of smooth metrics of the same signature. We then use methods from comparison geometry to obtain sufficiently strong estimates on the exponential maps of the regularized metrics to be able to carry the bi-Lipschitz property through the limit $\varepsilon \to 0$. More precisely, we rely on new comparison methods, developed only recently by B. L. Chen and P. LeFloch in their studies on the injectivity radius of Lorentzian metrics (2). In the Riemannian case, one may alternatively use the Rauch comparison theorem, as well as injectivity radius estimates due to Cheeger, Gromov and Taylor, as will be pointed out in Section 3. To show existence of totally normal neighborhoods we use the uniform estimates derived above to adapt the standard proof of the smooth case.

Our notation is standard, cf., e.g., [8, 10]. By $B_h(p, r)$ we denote the open ball around the point $p$ of radius $r$ with respect to the Riemannian metric $h$. To distinguish exponential maps stemming from various metrics we will use a superscript, as in $\exp^{g_\varepsilon}$.

To conclude this introductory section, we recall what is known about the exponential map of $C^{1,1}$ pseudo-Riemannian metric in general. In J. H. C. Whitehead’s classical paper [11], a path is a solution of a system of ODEs of the form

$$\frac{d^2c_k}{dt^2} + \Gamma^k_{ij}(c(t)) \frac{dc_i}{dt} \frac{dc_j}{dt} = 0,$$

where the $\Gamma^k_{ij}$ are merely supposed to be Lipschitz and symmetric in $i, j$ (but are not necessarily the Christoffel symbols of some metric). In [11, Sec. 3] it is proved that under these mild assumptions every point $p \in M$ has a neighborhood $S$ which is a simple region, in the sense that any two points in $S$ can be connected by at most one path. In particular, this result applies to the geodesics of a $C^{1,1}$ pseudo-Riemannian metric $g$ of arbitrary signature. It follows that $\exp^g_p : (\exp^g_p)^{-1}(S) \to S$ is continuous and bijective, hence a homeomorphism by invariance of domain, and one has:

**Theorem 1.1.** Let $M$ be a smooth manifold with a $C^{1,1}$ pseudo-Riemannian metric $g$ and let $p \in M$. Then there exist open neighborhoods $U$ of $0 \in T_p M$ and $V$ of $p$ in $M$ such that

$$\exp^g_p : U \to V$$

is a homeomorphism.

Our aim thus is to strengthen this result by additionally establishing the bi-Lipschitz property of $\exp^g_p$. We note, however, that our proof is self-contained and will not pre-suppose Th. 1.1. Rather, it implicitly provides an alternative proof for this result.

## 2 The main result

The aim of this section is to prove the following result:

**Theorem 2.1.** Let $M$ be a smooth manifold with a $C^{1,1}$ pseudo-Riemannian metric $g$ and let $p \in M$. Then there exist open neighborhoods $U$ of $0 \in T_p M$ and $V$ of $p$ in $M$ such that

$$\exp^g_p : U \to V$$

is a bi-Lipschitz homeomorphism.
Since the result is local, we may assume $M = \mathbb{R}^n$ and $p = 0$. By $g_E$ or $(\cdot, \cdot)_E$ we denote the standard Euclidean metric on $\mathbb{R}^n$, and we write $\| \cdot \|_E$ for the corresponding standard Euclidean norm, as well as for mapping norms induced by the Euclidean norm.

As was already indicated in the introduction, our strategy of proof will be to approximate $g$ by a net $g_\varepsilon$ of smooth pseudo-Riemannian metrics and then use comparison results to control the relevant geometrical quantities derived from the $g_\varepsilon$ uniformly in $\varepsilon$ so as to preserve the bi-Lipschitz property as $\varepsilon \to 0$.

Thus take $\rho \in \mathcal{D}(\mathbb{R}^n)$ with unit integral and define the standard mollifier $\rho_\varepsilon := \varepsilon^{-n}\rho(\varepsilon \cdot)$ ($\varepsilon > 0$). We set $g_\varepsilon := g * \rho_\varepsilon$ (componentwise convolution).

Remark 2.2. For later reference, we note the following properties of the approximating net $g_\varepsilon$.

(i) $g_\varepsilon \to g$ in $C^1(M)$ and the second derivatives of $g_\varepsilon$ are bounded, uniformly in $\varepsilon$, on compact sets.

(ii) On any compact subset of $M$, for $\varepsilon$ sufficiently small the $g_\varepsilon$ are a family of pseudo-Riemannian metrics of the same signature as $g$ whose Riemannian curvature tensors $R_\varepsilon$ are bounded uniformly in $\varepsilon$.

In order to proceed we need to determine a neighborhood of 0 in $T_pM$ that is a common domain for all $\exp_{g_\varepsilon}^p$ for $\varepsilon$ sufficiently small. Here, and in several places later on, we will make use of the following consequence of a standard result on the comparison of solutions to ODE [4, 10.5.6, 10.5.6.1]:

**Lemma 2.3.** Let $F, G \in C(H, X)$ where $H$ is a convex open subset of a Banach space $X$. Suppose:

\[
\sup_{x \in H} \| F(x) - G(x) \| \leq \alpha,
\]

$G$ is Lipschitz continuous on $H$ with Lipschitz constant $\leq k$, and $F$ is locally Lipschitz on $H$. For $\mu > 0$, define

\[
\varphi(\xi) := \mu e^{\xi k} + \alpha (e^{\xi k} - 1) \frac{1}{k}, \quad \xi \geq 0.
\]

Let $x_0 \in H$, $t_0 \in \mathbb{R}$ and let $u$ be a solution of $x' = G(x)$ with $u(t_0) = x_0$ defined on $J := (t_0 - b, t_0 + b)$ such that $\forall t \in J$, $\overline{B(u(t), \varphi(|t - t_0|))} \subseteq H$. Then for every $\tilde{x} \in H$ with $\| \tilde{x} - x_0 \| \leq \mu$ there exists a unique solution $v$ of $x' = F(x)$ with $v(t_0) = \tilde{x}$ on $J$ with values in $H$. Moreover, $\| u(t) - v(t) \| \leq \varphi(|t - t_0|)$ for $t \in J$.

We rewrite the geodesic equation for the metric $g$ as a first order system:

\[
\begin{align*}
\frac{dc^k}{dt} &= y^k(t) \\
\frac{dy^k}{dt} &= -\Gamma^k_{ij}(c(t))y^i(t)y^j(t)
\end{align*}
\]

and analogously for the metrics $g_\varepsilon$. Hence, $\exp_p^g(v) = c(1)$ where $c(0) = p$, $y(0) = v$. Let $t_0 = 0$ and $x_0 = (p, 0)$. We fix $b > 1$ and set $J = (-b, b)$. Now take $u$ to be the constant

\[\text{Nevertheless we will write } p \text{ below to distinguish considerations in } T_p M \text{ from those in } M.\]
solution of (1) with initial condition \( x_0 = (p,0) \), let \( \delta > 0 \) and set \( H := B(x_0, 2\delta) \subseteq \mathbb{R}^{2n} \). The Christoffel symbols \( \Gamma_g \) are Lipschitz functions on \( H \), and by Remark 2.2 (i) it follows that there is a common Lipschitz constant \( k \) for \( \Gamma_g \) and the \( \Gamma_{g_0} \) on \( H \). Choose \( \alpha > 0 \), \( \mu > 0 \) such that

\[
\varphi(b) = \mu e^{bk} + \frac{\alpha}{k}(e^{bk} - 1) < \delta.
\]

and choose \( \varepsilon_0 > 0 \) such that \( \forall \varepsilon < \varepsilon_0 \) we have \( \sup_H \| \Gamma_g - \Gamma_{g_0} \| < \alpha \). Then \( B(u(t), \varphi(|t|)) \subseteq H, \forall t \in J \). By Lemma 2.3 for all \( \tilde{x} = (p, w) \in H \) with \( \| \tilde{x} - x_0 \| = \| w \| \leq \mu \), there exists a unique solution \( u_\varepsilon \) on \((-b,b)\) of

\[
\begin{align*}
\frac{dc^k}{dt} &= y^k(t), \\
\frac{dy^k}{dt} &= -\Gamma_{g_0,ij}(c(t))y^i(t)y^j(t),
\end{align*}
\]

with values in \( H \) and \( u_\varepsilon(0) = p \) as well as a unique solution to (1) with these initial conditions. Therefore a common domain of \( \exp_g^p \) and all \( \exp_{g_0}^p (\varepsilon < \varepsilon_0) \) is given by \( \{ w \in \mathbb{R}^n | \| w \| \leq \varepsilon \} := B_0(0, \mu) \).

**Remark 2.4.** From Remark 2.2 we obtain that for some \( \varepsilon_0 > 0 \) we have:

(i) There exists a constant \( K_1 > 0 \) such that, for \( \varepsilon < \varepsilon_0 \), \( \| R_\varepsilon \|_E \leq K_1 \) uniformly on \( B_0(0, \mu) \).

(ii) For some \( K_2 > 0 \) and \( \varepsilon < \varepsilon_0 \),

\[
\| \Gamma_{g_0} \|_E \leq K_2,
\]

uniformly on \( B_0(0, \mu) \).

**Lemma 2.5.** Let \( r_1 < \min\left( \frac{1}{2K_2}, \frac{1}{16\mu} \right) \). Then for all \( \varepsilon < \varepsilon_0 \),

\[
\exp_{g_0}^p (B_E(0, r_1)) \subseteq B_E(p, \mu).
\]

**Proof.** Let \( \gamma : [0, r_1] \to M \) be a \( g_0 \)-geodesic with \( \gamma(0) = p \) and \( \| \gamma'(0) \|_E = 1 \) and set \( s_0 := \sup\{ s \in [0, r_1] | \gamma|_{[0,s]} \subseteq B_E(p, \mu) \} \). Then \( s_0 > 0 \) and for \( s \in [0, s_0) \) we have

\[
\left| \frac{d}{ds}(\gamma'(s), \gamma'(s))_E \right| = 2 \left| \langle \Gamma_{g_0}(\gamma'(s), \gamma'(s)), \gamma'(s) \rangle_E \right| \leq 2K_2\| \gamma'(s) \|^2_E,
\]

and therefore \( \| \gamma'(s) \|_E^{-1} \leq K_2 \). From this, setting \( f(s) := \| \gamma'(s) \|_E \), for \( s \in [0, s_0) \) we obtain

\[
\frac{f(0)}{f(s)} = \int_0^s f(0) \frac{d}{d\tau} \left( \frac{1}{f(\tau)} \right) d\tau + 1 \in [1/2, 3/2],
\]

so

\[
\frac{1}{2} \| \gamma'(0) \|_E \leq \| \gamma'(s) \|_E \leq 2\| \gamma'(0) \|_E \quad (s \in [0, s_0)).
\]

Therefore,

\[
L_E(\gamma|_{[0,s_0)}) = \int_0^{s_0} \| \gamma'(s) \|_E ds \leq 2r_1\| \gamma'(0) \|_E < \mu,
\]

implying that \( s_0 = r_1 \).
We next want to determine a ball around \(0 \in T_p M\) on which each \(\exp^{g_e}_p\) is a local diffeomorphism. To achieve this, we first need to derive estimates on Jacobi fields along geodesics, based on [2] Sec. 4).

**Lemma 2.6.** Set \(C_1 := 2K_2, \ C_2 := 4K_1\) and let

\[
\frac{1}{C_1} \log \left(\frac{C_1 + C_2}{C_1/2 + C_2}\right), \ (2 + C_1)^{-1}
\]

Then for \(\varepsilon < \varepsilon_0\), any \(g_e\)-geodesic \(\gamma : [0, r_2] \to M\) with \(\gamma(0) = p\) and \(\|\gamma'(0)\|_E = 1\) lies entirely in \(B_E(p, \mu)\). Moreover, if \(J\) is a \(g_e\)-Jacobi field along \(\gamma\) with \(J(0) = 0\) and \(\|\nabla_{g_e, \gamma'} J(0)\|_E = 1\) then \(\|J(s)\|_E \leq 1\) and \(\frac{1}{2} \leq \|\nabla_{g_e, \gamma'} J\|_E \leq 2\) for all \(s \in [0, r_2]\).

**Proof.** By Lemma 2.5 \(\gamma\) lies in \(B_E(p, \mu)\). Also, (2) implies

\[
\max_{s \in [0, r_2]} \|\gamma'(s)\|_E \leq 2.
\]

Suppose that \(s_0 := \sup\{s \in [0, r_2] \mid \|J(t)\|_E \leq 1 \forall t \in [0, s]\} < r_2\). By assumption, \(J\) satisfies

\[
\nabla_{g_e, \gamma'} \nabla_{g_e, \gamma'} J(s) = -R_e(J(s), \gamma'(s))\gamma'(s) \quad J(0) = 0, \quad \|J'(0)\|_E = 1.
\]

Thus by Remark 2.4 and (3), on \([0, s_0]\) we obtain

\[
\frac{d}{ds}\langle J, J \rangle_E = 2\langle \nabla_{g_e, \gamma'} J, \nabla_{g_e, \gamma'} J \rangle_E - \langle \Gamma_{g_e}(\nabla_{g_e, \gamma'} J, \gamma'), \nabla_{g_e, \gamma'} J \rangle_E \\
\leq 8K_1\|\nabla_{g_e, \gamma'} J\|_E + 4K_2\|\nabla_{g_e, \gamma'} J\|_E^2,
\]

so that

\[
\frac{d}{ds}\|\nabla_{g_e, \gamma'} J\|_E \leq 4K_1 + 2K_2\|\nabla_{g_e, \gamma'} J\|_E = C_1\|\nabla_{g_e, \gamma'} J\|_E + C_2.
\]

Taking into account that \(\|\nabla_{g_e, \gamma'} J(0)\|_E = 1\) by assumption, integration of (4) leads to

\[
-\frac{C_2}{C_1} + \left(1 + \frac{C_2}{C_1}\right)e^{-C_1s} \leq \|\nabla_{g_e, \gamma'} J(s)\|_E \leq -\frac{C_2}{C_1} + \left(1 + \frac{C_2}{C_1}\right)e^{C_1s}.
\]

Due to our choice of \(r_2\), this entails

\[
\frac{1}{2} \leq \|\nabla_{g_e, \gamma'} J\|_E \leq 2
\]

on \([0, s_0]\). From this, we get

\[
\frac{d}{ds}\|J(s)\|_E = \frac{1}{\|J(s)\|_E} \left|\langle \nabla_{g_e, \gamma'} J(s), J(s) \rangle_E - \langle \Gamma_{g_e}(J(s), \gamma'(s)), J(s) \rangle_E\right| \leq 2 + 2K_2.
\]

Therefore,

\[
\|J(s)\|_E \leq (2 + 2K_2)s < s/r_2 < 1
\]

for \(s \in [0, s_0]\). For \(s = s_0\), this gives a contradiction to the definition of \(s_0\). □
Lemma 2.7. There exists some $0 < r_3 < r_2$ such that, for all $\varepsilon < \varepsilon_0$, $\exp_p^{\varepsilon}$ is a local diffeomorphism on $B_E(0, r_3)$.

Proof. For any Jacobi field $J$ as in Lemma 2.6 we have:

\[
\frac{d}{ds} \langle \nabla g_{\gamma'} \nabla g_{\gamma'} J, J \rangle_E(s) = \langle \nabla g_{\gamma'} \nabla g_{\gamma'} J, J \rangle_E(s) - \langle \Gamma_{g_{\gamma'}}(\nabla g_{\gamma'} J, \gamma'), J \rangle_E(s) \\
+ \langle \nabla g_{\gamma'} J, \nabla g_{\gamma'} J \rangle_E(s) - \langle \nabla g_{\gamma'} J, \Gamma_{g_{\gamma'}}(J, \gamma') \rangle_E(s)
\]

Of these four terms, the third one is bounded from below by 1 due to (5). For the others, employing Lemma 2.6 (see (3), (4), (6)), we obtain for $s \in [0, r_2]$:

\[
\|\langle \nabla g_{\gamma'} \nabla g_{\gamma'} J, J \rangle_E(s)\| \leq K_1\|\gamma'(s)\|_E^2\|J(s)\|_E^2 \leq 4K_1 s^2
\]

\[
\|\langle \Gamma_{g_{\gamma'}}(\nabla g_{\gamma'} J, \gamma'), J \rangle_E(s)\| \leq K_2\|\gamma'(s)\|_E\|\nabla g_{\gamma'} J(s)\|_E\|J(s)\|_E \leq 4K_2 s
\]

\[
\|\langle \nabla g_{\gamma'} J, \Gamma_{g_{\gamma'}}(J, \gamma') \rangle_E(s)\| \leq K_2\|\gamma'(s)\|_E\|J(s)\|_E\|\nabla g_{\gamma'} J(s)\|_E \leq 4K_2 s
\]

From this we obtain an $r_3 = r_2(r_2, K_1, K_2) < r_2$ such that on $[0, r_3]$, $\frac{d}{ds} \langle \nabla g_{\gamma'} J, J \rangle_E$ is bounded from below by a positive constant. By the same estimates and (5) again, it is also bounded from above. Hence for some $c_1 > 0$, any $\varepsilon < \varepsilon_0$ and $s \in [0, r_3]$ we obtain:

\[
e^{-c_1} \leq \frac{d}{ds} \langle \nabla g_{\gamma'} J, J \rangle_E(s) \leq e^{c_1},
\]

and therefore

\[
e^{-c_1} s \leq \langle \nabla g_{\gamma'} J, J \rangle_E(s) \leq e^{c_1} s.
\]

Combined with (5) and (6), this entails:

\[
\frac{1}{r_2^2} \geq \frac{\|J(s)\|_E}{\|\langle \nabla g_{\gamma'} J, J \rangle(s)\|_E} \geq \frac{e^{-c_1}}{2}s.
\]

Altogether, we find $c_2 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $s \in [0, r_3]$:

\[
e^{-c_2} s \leq \|J(s)\|_E \leq e^{c_2} s.
\]

In terms of the exponential map, any Jacobi field as in Lemma 2.6 is of the form $J(s) = T_{s\gamma'(0)} \exp_p^{\varepsilon}(s \cdot w)$, with $w \in T_p M$, $\|w\|_E = 1$. Thus

\[
e^{-c_2} \leq \|T_{s\gamma'(0)} \exp_p^{\varepsilon}(w)\|_E \leq e^{c_2} \quad (s \in [0, r_3]).
\]

Since $\|\gamma'(0)\|_E = 1$ we conclude that $\forall \varepsilon < \varepsilon_0, \forall v \in B_E(0, r_3), \forall w \in T_p M$:

\[
e^{-c_2}\|w\|_E \leq \|T_v \exp_p^{\varepsilon}(w)\|_E \leq e^{c_2}\|w\|_E.
\]

(7)

In particular, $\exp_p^{\varepsilon}$ is a local diffeomorphism on $B_E(0, r_3)$.

We note that (7) can equivalently be formulated as

\[
e^{-2c_2} g_E \leq (\exp_p^{\varepsilon})^* g_E \leq e^{2c_2} g_E
\]

for $\varepsilon < \varepsilon_0$ on $B_E(0, r_3)$. 

\[\square\]
Lemma 2.8. For \( r_4 < e^{-c_2} r_3, r_5 < e^{-c_2} r_4 \) and \( \tilde{r} := e^{c_2} r_4 \) we have, \( \forall \varepsilon < \varepsilon_0: \)

\[
\exp^a_p(B_E(0, r_5)) \subseteq B_E(p, r_4) \subseteq \exp^a_p(B_E(0, \tilde{r})) \subseteq \exp^a_p(B_E(0, r_3)).
\]

Proof. For \( q \in B_E(p, r_4) \), let \( \alpha : [0, a] \rightarrow M \) be a piecewise smooth curve from \( p \) to \( q \) in \( B_E(p, r_4) \) of Euclidean length less than \( r_4 \). Since \( \exp^a_p \) is a local diffeomorphism on \( B_E(0, r_3) \), for \( b > 0 \) sufficiently small there exists a unique \( \exp^a_p \)-lift \( \hat{\alpha} : [0, b] \rightarrow B_E(0, r_3) \) of \( \alpha_{|[0,b]} \) starting at \( 0 \). We claim that \( \alpha' := \sup \{ b < a | \hat{\alpha} \text{ exists on } [0, b] \} = a \). Indeed, suppose that \( \alpha' < a \). Then

\[
L(\exp^a_p)^{-1}_{|E}(\alpha_{|[0,a']}) = L_{g_E}(\alpha_{|[0,a']}) = \int_0^{\alpha'} \| \alpha'(t) \|_E dt \leq r_4.
\]

Hence by (8) we obtain \( L_{g_E}(\alpha_{|[0,a']}(|a,a'}) \leq \tilde{r} \). Now let \( \alpha_n \not\rightarrow \alpha' \). Then \( \alpha(a_n) \in B_E(0, \tilde{r}) \), so some subsequence \( (\tilde{\alpha}(a_{n_k})) \) converges to a point \( v \in B_E(0, \tilde{r}) \). Since \( \exp^a_p \) is a diffeomorphism on a neighborhood of \( v \) and \( \exp^a_p(v) = \lim \alpha(a_{n_k}) = \alpha(a') \), this shows that \( \alpha \) can be extended past \( \alpha' \), a contradiction. Thus \( q = \exp^a_p(\hat{\alpha}(a')) \in \exp^a_p(B_E(0, \tilde{r})) \).

For the first inclusion, take \( v \in T_{p,q}M \) with \( \|v\|_E \leq r_5 \) and set \( q := \exp^a_p(v) \). Then for the radial geodesic \( \gamma : [0, 1] \rightarrow M, t \mapsto \exp^a_p(tv) \) from \( p \) to \( q \), by (7) we obtain for \( s \) small:

\[
L_E(\gamma_{|[0,s]}) = \int_0^s \| T_{tv} \exp^a_p(v) \|_E dt \leq e^{c_2} \|v\|_E < r_4
\]

From this we conclude that \( \sup \{ s \in [0,1] | \gamma_{|[0,s]} \subseteq B_E(p, r_4) \} = 1 \), so \( q \in B_E(p, r_4) \).

Note that \( \exp^a_p : B_E(0, r_3) \rightarrow B_E(p, r_4) \) is a surjective local homeomorphism between compact Hausdorff spaces, hence is a covering map. Using this, we obtain:

Lemma 2.9. For any \( \varepsilon < \varepsilon_0 \), \( \exp^a_p \) is injective (hence a diffeomorphism) on \( B_E(0, r_5) \).

Proof. Suppose to the contrary that there exist \( v_0, v_1 \in B_E(0, r_5) \), \( v_0 \not= v_1 \), and \( \varepsilon < \varepsilon_0 \) such that \( \exp^a_p(v_0) = q = \exp^a_p(v_1) \). Hence, \( \gamma_i(t) := \exp^a_p(tv_i), i = 0, 1 \), are two distinct geodesics starting at \( p \) which intersect at the point \( q \). Then \( \gamma_0(t) = s_0 \gamma_1(t) + (1 - s_0) \gamma_1(0) \) is a fixed endpoint homotopy connecting \( \gamma_0 \) and \( \gamma_1 \) in the ball \( B_E(p, \tilde{r}) \). Since \( \exp^a_p \) is a covering map, and using Lemma 2.8 we can lift this homotopy to \( B_E(0, \tilde{r}) \). But the lifts of \( \gamma_0 \) and \( \gamma_1 \) are \( t \mapsto tv_i, i = 0, 1 \), which obviously are not fixed endpoint homotopic in \( B_E(0, \tilde{r}) \), a contradiction.

From (7) we obtain a uniform Lipschitz constant for all \( \exp^a_p \) with \( \varepsilon < \varepsilon_0: \exists c_3 > 0 \) such that

\[
\| \exp^a_p(u) - \exp^a_p(v) \|_E \leq c_3 \| u - v \|_E.
\]

For the corresponding estimate from below we use the following result that provides a mean value estimate for \( C^1 \)-functions on not necessarily convex domains (cf. [6, 3.2.47]).

Lemma 2.10. Let \( \Omega \subseteq \mathbb{R}^n, \Omega' \subseteq \mathbb{R}^m \) be open, \( f \in C^1(\Omega, \Omega') \) and suppose that \( K \subseteq \Omega \). Then there exists \( C > 0 \), such that

\[
\| f(x) - f(y) \| \leq C \| x - y \|, \forall x, y \in K.
\]

C can be chosen as \( C_1 \cdot \sup_{x \in L} \|f(x)\| + \|Df(x)\| \) for any fixed compact neighborhood \( L \) of \( K \) in \( \Omega \), where \( C_1 \) depends only on \( L \).
Using Lemma 2.8 we now pick \( r_7 < r_6 =: e^{-c_2} \hat{r} < \hat{r} < r_5 \) such that for \( \varepsilon < \varepsilon_0 \) we have
\[
\exp_p^g (B_E(0, r_7)) \subseteq B_E(p, r_6) \subseteq \exp_p^g (B_E(0, \hat{r})) \subseteq \exp_p^g (B_E(0, r_5)).
\]
Again by (7), we have \( \forall \varepsilon < \varepsilon_0 \),
\[
e^{-c_2} \| \xi \|_E \leq \| T_q (\exp_q^g)^{-1}(\xi) \|_E \leq e^{c_2} \| \xi \|_E,
\]
\( \forall q \in B_E(p, r_6), \forall \xi \in T_q M \). Thus by Lemma 2.10 there exists some \( c_4 > 0 \) such that
\[
\| (\exp_q^g)^{-1}(q_1) - (\exp_q^g)^{-1}(q_2) \|_E \leq c_4^{-1} \| q_1 - q_2 \|_E
\]
\( \forall \varepsilon < \varepsilon_0, \forall q_1, q_2 \in \exp_p^g (B_E(0, r_7)) \).
Summing up, for all \( \varepsilon < \varepsilon_0 \) and all \( u, v \in B_E(0, r_7) \) we have
\[
c_4 \| u - v \|_E \leq \| \exp_p^g(u) - \exp_p^g(v) \|_E \leq c_3 \| u - v \|_E.
\]
Finally, let \( \varepsilon \to 0 \). Then for all \( u, v \in B_E(0, r_7) \) we get
\[
c_4 \| u - v \|_E \leq \| \exp_p^g(u) - \exp_p^g(v) \|_E \leq c_3 \| u - v \|_E.
\]
Thus, \( \exp_p^g \) is a bi-Lipschitz homeomorphism on \( U := B_E(0, r_7) \subseteq T_p M \). In particular, \( V = \exp_p^g(U) \) is open in \( M \) (invariance of domain). This concludes the proof of Theorem 2.1

3 The Riemannian case

In the special case where \( g \) is a \( C^{1,1} \) Riemannian metric, in this section we point out some alternatives to the reasoning given in the previous section. We start out from the following version of the Rauch comparison theorem, cf., e.g., [8 Cor. 4.6.1].

**Theorem 3.1.** Let \((M, h)\) be a smooth Riemannian manifold and suppose that \( \exp_h \) is defined on a ball \( B_{h_p}(0, R) \), for some \( R > 0 \), and that there exist \( \rho \leq 0 \), \( \kappa > 0 \) such that the sectional curvature \( K \) of \( M \) satisfies \( \rho \leq K \leq \kappa \) on some open set which contains \( \exp_h(B_{h_p}(0, R)) \). Then for all \( v \in T_p M \) with \( \| v \|_{h_p} = 1 \), all \( w \in T_p M \), and all \( 0 < t < \min(R, \frac{\pi}{\sqrt{\kappa}}) \),
\[
\frac{\sin_t(t)}{t} \| w \| \leq \| (T_v \exp_h)(w) \| \leq \frac{\sin_t(t)}{t} \| w \|.
\]
Here, for \( \alpha \in \mathbb{R} \),
\[
\sin_\alpha(t) := \begin{cases} 
\frac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha}t) & \text{for } \alpha > 0 \\
t & \text{for } \alpha = 0 \\
\frac{1}{\sqrt{-\alpha}} \sinh(\sqrt{-\alpha}t) & \text{for } \alpha < 0
\end{cases}
\]
As an immediate consequence, we obtain that for any \( 0 < r < \min(R, \frac{\pi}{\sqrt{\kappa}}) \), there exists some \( c > 0 \) such that \( \forall v \in B_{h_p}(0, r), \forall w \in T_p M \)
\[
e^{-c} \| w \| \leq \| (T_v \exp_h)(w) \| \leq e^c \| w \|. \quad (9)
\]
Remark 3.2. For $(M, g)$ a smooth Riemannian manifold, its sectional curvature $K$ is a smooth function on the 2-Grassmannian bundle $G(2, TM)$. Since the fibers of $G(2, TM)$ are compact, local bounds on the Riemann curvature tensor $R$ imply local bounds on $K$ on any relatively compact subset of $M$. However, an analogous argument is not possible in the Lorentzian (or general pseudo-Riemannian) setting since in that case $K$ is only defined on non-degenerate 2-planes, forming an open subbundle of $G(2, TM)$. Indeed, a Lorentzian manifold has bounded sectional curvature $K$ only in the trivial case where $K$ is constant ([9], cf. also [7]).

Now let $g$ be a $C^{1,1}$ Riemannian metric $M$, and let $g_\varepsilon$ be approximating smooth metrics as in Section 2. Then we may fix some $r' > 0$ and some $\varepsilon_0 > 0$ such that $\exp_p^g$ and $\exp_p^{g_\varepsilon} (\varepsilon < \varepsilon_0)$ are defined on $B_g(p, r')$ and such that (by locally uniform convergence of $\exp_p^{g_\varepsilon}$ to $\exp_p^g$) there exists an open, relatively compact subset $W \subseteq M$ with $\bigcup_{\varepsilon < \varepsilon_0} \exp_p^{g_\varepsilon}(B_g(p, r')) \subseteq W$. On $W$, by Remarks 2.2 (ii) and 3.2 we obtain uniform bounds on the sectional curvatures $K_\varepsilon$ of $g_\varepsilon$, i.e.,

$$\exists \rho \leq 0, \kappa > 0 : \forall \varepsilon < \varepsilon_0 \; \rho \leq K_\varepsilon \leq \kappa.$$ 

Thus by (9), for any $r < \min(r', \sqrt{\frac{\varepsilon}{\kappa}})$, there exists some $c > 0$ depending only on $\rho$ and $\kappa$ such that for all $\varepsilon < \varepsilon_0$

$$e^{-c}\|w\|_{g_\varepsilon} \leq \|(T_v \exp_p^{g_\varepsilon})(w)\|_{g_\varepsilon} \leq e^{c}\|w\|_{g_\varepsilon}, \tag{10}$$

$\forall v \in B_g(p, 0, r), \forall w \in T_p M$. In particular, by the inverse function theorem every $\exp_p^{g_\varepsilon}$ is a local diffeomorphism on $B_g(p, 0, r)$. Thus we may rewrite (10) equivalently as

$$e^{-2c}g_{\varepsilon, p} \leq (\exp_p^{g_\varepsilon})^*g_\varepsilon \leq e^{2c}g_{\varepsilon, p},$$

on $B_g(p, 0, r)$. Since $g_\varepsilon \to g$ locally uniformly, by increasing $c$ we obtain (5) on a suitable Euclidean ball and can proceed as in Section 2.

Finally, we note that to obtain a common domain (and injectivity) of the approximating exponential maps $\exp_p^{g_\varepsilon}$ one may alternatively employ the following result of Cheeger, Gromov and Taylor ([11], the formulation below is taken from [2]), which provides a lower bound on the injectivity radii $\text{Inj}_{g_\varepsilon}(M, p)$.

**Theorem 3.3.** Let $M$ be a $C^\infty$ $n$-manifold with a smooth Riemannian metric $g$. Suppose that $B_g(p, 1) \subseteq M$ for some point $p$ in $M$. Then for any $K, \nu > 0$ there exists some $i = i(K, \nu, n) > 0$ such that if

$$\|R_g\|_{L^\infty(B(p, 1))} \leq K, \; \text{Vol}_g(B(p, 1)) \geq \nu,$$

then the injectivity radius $\text{Inj}_g(M, p)$ at $p$ is bounded from below by $i$,

$$\text{Inj}_g(M, p) \geq i.$$ 

Since the distance function $d_g$ of the $C^{1,1}$-metric $g$ induces the manifold topology, $B_g(p, 2r)$ is an open, relatively compact subset of $M$ for $r > 0$ sufficiently small. Thus for $\varepsilon$ small, $B_{g_\varepsilon}(p, r) \subseteq B_g(p, 2r)$ is relatively compact and

$$\text{Vol}_{g_\varepsilon}(B_{g_\varepsilon}(p, r)) \geq \text{Vol}_{g_\varepsilon}(B_g(p, r/2)) \geq \frac{1}{2} \text{Vol}_g(B_g(p, r/2)) > 0.$$ 

By Theorem 3.3 there exists some $r_0$ such that

$$\text{Inj}(g_\varepsilon, p) \geq r_0, \; \forall \varepsilon \leq \varepsilon_0,$$

9
so $\exp^g_p$ is a diffeomorphism on $B_{g_p}(p, r_0) \forall \varepsilon \leq \varepsilon_0$. Since $B_g(p, \frac{\varepsilon_0}{2}) \subseteq B_{g_p}(p, r_0)$ for $\varepsilon$ small, it follows that $\exp^g_p$ is a diffeomorphism on $B_g(p, \frac{\varepsilon_0}{2})$. From here, using Theorem 3.4 we may proceed as in the argument following Lemma 2.9 to conclude that $\exp^g_p$ is a bi-Lipschitz homeomorphism on some neighborhood of $0 \in T_pM$.

4 Totally normal neighborhoods

For a smooth pseudo-Riemannian metric $g$ on a manifold $M$, a neighborhood $U$ of $p \in M$ is called a normal neighborhood of $p$ if $\exp^g_p$ is a diffeomorphism from a starshaped open neighborhood $\bar{U}$ of $0 \in T_pM$ onto $U$. $U$ is called totally normal if it is a normal neighborhood of each of its points. This terminology is in line with [5] while, e.g., in [10] such sets are called geodesically convex.

Analogously, if $g$ is a $C^{1,1}$-pseudo-Riemannian metric on a smooth manifold $M$ we call a neighborhood of a point $p \in M$ normal if there exists a starshaped open neighborhood $\bar{U}$ of $0 \in T_pM$ such that $\exp^g_p$ is a bi-Lipschitz homeomorphism from $\bar{U}$ onto $U$. $U$ is called totally normal if it is a normal neighborhood of each of its points.

In what follows we adapt the standard proof for the existence of totally normal neighborhoods, cf., e.g., [10] Prop. 5.7 (tracing back to [11] Sec. 4) to the $C^{1,1}$-situation.

**Theorem 4.1.** Let $M$ be a smooth manifold with a $C^{1,1}$ pseudo-Riemannian metric $g$. Then each point $p \in M$ possesses a basis of totally normal neighborhoods.

**Proof.** The main point to note is that the explicit bounds derived in Section 2 on the radius of the ball in $T_pM$ where $\exp^g_p$ is a bi-Lipschitz homeomorphism depend only on quantities that can be uniformly controlled on compact sets. Therefore, for any $p \in M$ there exists a neighborhood $V'$ of $p$ and some $r > 0$ such that, $\forall q \in V'$,

$$\exp^g_q : B_{h,q}(0, r) \rightarrow \exp^g_q(B_{h,q}(0, r))$$

is a bi-Lipschitz homeomorphism. Here, $h$ is any background Riemannian metric.

Now define $S := \{ v \in TM \mid \pi(v) \in V', \|v\|_h < r \}$, with $\pi$ the natural projection of $TM$ onto $M$. Let $E : TM \rightarrow M \times M, E(v) = (\pi(v), \exp^g(v))$. Then by (11)

$E : S \rightarrow E(S) =: W$ is a continuous bijection, hence a homeomorphism by invariance of domain. Let $(\psi = (x^1, ..., x^n), V)$ be a coordinate system centered at $p$ (in the smooth case $\psi$ is usually taken to be a normal coordinate system, which is not available to us, but this is in fact not needed).

Define the $(0, 2)$-tensor field $B$ on $V$ by

$$B_{ij}(q) := \delta_{ij} - \sum_k \Gamma^{k}_{ij}(q)x^k(q).$$

Since $\psi(p) = 0$ we may assume $V$ small enough that $B$ is positive definite on $V$. In addition, we may suppose that $W \subseteq V \times V$. Set $N(q) := \sum_{i=1}^n (x^i(q))^2$, and let $V(\delta) := \{ q \in V \mid N(q) < \delta \}$. Then if $\delta$ is so small that $V(\delta) \times V(\delta) \subseteq W$, $E$ is a homeomorphism from $U_\delta := E^{-1}(V(\delta) \times V(\delta))$ onto $V(\delta) \times V(\delta)$ and $\exp^g([0, 1] \cdot U_\delta) \subseteq \exp^g(S) \subseteq V$.

We will show that $V(\delta)$ is totally normal. For $q \in V(\delta)$ and $U_q := U_\delta \cap T_qM$, $\exp^g_q = E|_{U_q} : U_q \rightarrow V(\delta)$ is a homeomorphism, so it is left to show that $U_q$ is starshaped. Let $v \in U_q$. Then
$\sigma : [0, 1] \to M$, $\sigma(t) = \exp_q^q(tv)$ is a geodesic from $q$ to $\sigma(1) =: \tilde{q} \in V(\delta)$ that lies entirely in $V$.

If $\sigma$ is contained in $V(\delta)$ then $tv \in U_q$, $\forall t \in [0, 1]$: suppose to the contrary that $\bar{t} := \sup\{t \in [0, 1] | 0, t \cdot v \in U_q\} < 1$. Then $\bar{tv} \in \partial U_q$ and since $(\exp_q^q|_{U_q})^{-1}(\sigma([0, 1])) \subset U_q$, there exists some $t_1 < \bar{t}$ such that $U_q \ni \bar{tv} \notin (\exp_q^q|_{U_q})^{-1}(\sigma([0, 1]))$, a contradiction. Hence the entire segment $\{tv | t \in [0, 1]\}$ lies in $U_q$, so $U_q$ is starshaped. It remains to show that $\sigma$ cannot leave $V(\delta)$. If it did, there would exist $t_0 \in [0, 1]$ such that $N(\sigma(t_0)) \geq \delta$. Since $N(q), N(\tilde{q}) < \delta$, the function $t \mapsto N \circ \sigma$ has a maximum at some point $\bar{t} \in (0, 1)$. However,

$$\frac{d^2(N \circ \sigma)}{dt^2}(\bar{t}) = 2B_{\sigma(\bar{t})}((\psi \circ \sigma)'(\bar{t})), (\psi \circ \sigma)'(\bar{t})) > 0,$$

a contradiction.

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