HIGHER DERIVATIVE EXTENSIONS OF 3d CHERN-SIMONS MODELS: CONSERVATION LAWS AND STABILITY

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Abstract. We consider the class of higher derivative 3d vector field models with the field equation operator being a polynomial of the Chern-Simons operator. For n-th order theory of this type, we provide a general receipt for constructing n-parameter family of conserved second rank tensors. The family includes the canonical energy-momentum tensor, which is unbounded, while there are bounded conserved tensors that provide classical stability of the system for certain combinations of the parameters in the Lagrangian. We also demonstrate the examples of consistent interactions which are compatible with the requirement of stability.

INTRODUCTION

In this paper we consider a class of 1-form field $A = A\mu dx^\mu$ models on 3d Minkowski space with the action

$$S = \frac{m^2}{2} \int A \wedge (-a_0 \ast A + a_1 \frac{2}{m} dA + a_2 \frac{4}{m^2} d \ast dA + a_3 \frac{8}{m^3} d \ast d \ast dA + a_4 \frac{16}{m^4} d \ast d \ast d \ast dA + \ldots),$$  \hspace{1cm} (1)

where $m$ is a constant with dimension of mass, $a_0, a_1, a_2, a_3, \ldots$ are some real dimensionless coefficients, $\ast$ is Hodge conjugation, and the signature is $(+,-,\ldots)$. The coefficient $a_0 m^2$ corresponds to the usual mass term, $a_1 m$ is the Chern-Simons mass, $a_2$ is a coefficient at Maxwell’s Lagrangian, $a_3$ corresponds to the extended Chern-Simons Lagrangian [1] and the fourth order term appears in the Podolsky’s electrodynamics Lagrangian [2]. With appropriate choice of the coefficients $a_k, k = 0, 1, 2, \ldots$, this action reproduces various known 3d models, including the Chern-Simons-Proca [3, 4], Maxwell-Chern-Simons [5, 6], Maxwell-Chern-Simons-Proca [7, 8] and the other previously studied higher derivative models [9, 10].

In any dimension, inclusion of the higher derivative terms results in the unbounded canonical energy, so classical stability becomes the issue. It is also known that the ghost poles can emerge in the propagator once higher derivatives are included in the action.

The specifics of higher order terms in three dimensions is that they can be viewed as derived from the Chern-Simons term by the repeated shift of the field by its strength: $A \mapsto A + 2m^{-1} \ast dA$. As a result, the operator of field equations is a polynomial in first order operator $W = 2m^{-1} \ast dA$. This special structure allows us to make some conclusions concerning conservation laws and stability. The observation is that the n-th order theory of the class [1] admits n parametric family of conserved second rank tensors whenever $a_0 \neq 0$. Once $a_0 = 0$ (the theory is gauge invariant in this case), there exists an $n-1$ parametric family of conserved tensors. The canonical energy-momentum is included in the family in every instance. We provide the general receipt for constructing these conservation laws, and related symmetries. The construction in fact applies to any system (of
any field $A$, not necessarily 1-form) with the operator of field equations being a polynomial in another operator

$$MA = 0, \quad M = m^2 \sum_{k=0}^{n} a_k W^k,$$

where $W$ can be any self-adjoint differential operator, $a_k$ are real constants, and $a_n \neq 0$. We term the models of the type (2) derived from the theory with equations $WA = 0$. For the case (1), when $W = 2m^{-1} * d$, we apply the general procedure to explicitly deduce the conserved tensors for the third order actions of this class. As we see, the bounded conserved tensors are contained in the family, once the polynomial $M(2m^{-1} * d)$ has only simple roots, or at most one double zero root. In this generic case, the theory is classically stable even though the canonical energy-momentum is unbounded. As we explain, these models can admit certain interactions such that the stability survives at nonlinear level. The case of multiple roots is special. It also admits a family of $n$ conserved tensors, including the canonical energy-momentum, though there are no bounded conserved quantities in this family. As we see, the corresponding representations of the Poincaré group are non-unitary, while in the generic case, the representation decomposes into unitary ones.

The article is organized in the following way. In the next section we describe the general structure of field equations in the higher derivative models that fall into the class of derived theories (2). For the generic derived system of order $n$ we suggest a procedure of constructing $n$-parametric family of conserved tensors whose structure depends on the coefficients $a_k$ in the field equations (2). In Section 2, we explicitly construct the families of conserved tensors for the theory (1) involving terms up to third order. As we see, four different cases are possible from the viewpoint of existence the bounded representative in the family of conserved quantities. These cases are distinguished by the structure of roots in the polynomial (2). Once the positive conserved quantity exists, the theory is stable at classical level, even though the canonical energy is unbounded. In Section 3, we demonstrate the example of the self-interaction such that the nonlinear theory remains stable. In conclusion, we summarize the results and comment on stability of the theory (1) at quantum level.

1. Derived theories, higher symmetries and conservation laws

In this section, we consider the field equations of general structure (2). We demonstrate that combining the space-time translations with the powers of operator $W$, one can construct non-trivial higher order symmetries and find related conserved tensors. The construction is quite general, it applies to any system of the form (2). The explicit details for the extension Chern-Simons theory (1) are provided in Section 2.

1.1. Derived theories. Consider a set of fields $A^J(x)$ on $d$–dimensional Minkowski space with local coordinates $x^\mu$. The multi-index $J$ accommodates all the tensor, spinor, isotopic indices labeling the field components. Here, we suppose that the theory admits appropriate constant metrics that can be used to rise and lower the

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1The conjugation rule is explained in the next section.
multi-indices. In this setting, any local linear system of field equations can be represented in the following form:

\[ M_{IJ}(\partial)A^J = 0, \quad (3) \]

where \( M_{IJ}(\partial) \) is a square matrix whose entries are polynomials in the formal variables \( \partial_\mu \). If \( \partial_\mu \) are understood as the partial derivatives in Minkowski space coordinates \( x_\mu \), (3) will be a linear PDE system. The formal adjoint to the operator \( M \) is defined by

\[ M^\dagger_{IJ}(\partial) = M_{JI}(-\partial). \quad (4) \]

The field equations (3) are variational whenever \( M = M^\dagger \), in which case the action reads

\[ S = \int d^d x L, \quad L = \frac{1}{2} A^I M_{IJ}(\partial)A^J. \quad (5) \]

Let us further suppose that the self-adjoint linear differential operator \( W_{IJ}(\partial) \) exists (Cf. (2)) such that the operator of field equations is polynomial in \( W \):

\[ M(W) = m^2 \sum_{k=0}^n a_k W^k = a_m m^2 \prod_{i=1}^r (W - \lambda_i)^{p_i} \prod_{j=1}^s (W^2 - (\omega_j + \bar{\omega}_j)W + \omega_j\bar{\omega}_j)^{q_j}. \quad (6) \]

The real numbers \( \lambda_i \) and complex conjugate numbers \( \omega_j, \bar{\omega}_j \) are the roots of the polynomial \( M(W) \) with multiplicities \( p_i \) and \( q_j \), respectively. The multiplicity of roots is connected with the total degree of the polynomial

\[ \sum_{i=1}^r p_i + 2 \sum_{j=1}^s q_j = n. \]

If \( W \) is a differential operator of finite order \( n_W \), the order of PDE system (3) will not exceed \( n \times n_W \).

Once the field equation operator \( M(\partial) \) is a polynomial of another self-adjoint operator \( W(\partial) \), we say that the theory is a derived model. In (11), the special case of the factorization (6) was studied, where \( M \) has two different simple real roots in \( W \). This simple assumption has far-reaching consequences. In particular, each of the factors defines its own Lagrangian theory whose order is lower than that of the derived theory. Let us mention some of these consequences noticed in (11). Once the two lower order theories are translation invariant, the derived higher derivative theory has a two-parameter family of independent conserved tensors. This family includes the canonical energy-momentum tensor of the derived theory. The canonical energy is unbounded in general, as it should be in the higher derivative system, while some other conserved quantities can be bounded in this family. The existence of the bounded conserved quantities guarantees the classical stability of dynamics. As it was demonstrated in the paper (11), every conserved tensor in this family can be connected to translation
invariance of the system by appropriate Lagrange anchor. As we will see in this section, for any derived system \( W \), one can construct \( n \)-parameter family of conserved tensors, where \( n \) is the order of polynomial \( M(W) \).

We consider two ways of constructing conserved tensors in the derived theories. At first, we make notice that the symmetry algebra of the derived theory includes higher order symmetries generated by the operator \( W \), and translations, once \( W \) is translation invariant. Then, we derive the conserved tensors from these symmetries by the Noether theorem. Another option employs the procedure of reducing the order of higher derivative theory \( 2 \) by assigning a lower order system to every irreducible factor in decomposition of the polynomial \( 6 \). Then, making use of the canonical conserved tensors for the lower order systems, we get the family of the conserved tensors for the original theory \( 2 \). Although the Noether theorem provides a uniform way for deducing conservation laws from given symmetries, the conserved tensors obtained from the lower order equivalent system appear in a more convenient form in this case, and we will use them for further analysis of stability.

1.2. Higher order symmetries and conservation laws. Provided the operator \( W \) is translation invariant, the action \( 5 \) admits the following symmetry transformations:

\[
\delta_\varepsilon A^J = -\varepsilon^\alpha \partial_\alpha (W^k A)^J, \quad k = 0, \ldots, n - 1. \tag{7}
\]

The space-time translations correspond to \( k = 0 \). The higher order transformations with \( k = n, n + 1, \ldots \) are equivalent to the lower order ones with account of the equations of motion \( 3 \), while for \( k < n \) one has independent symmetries. By the Noether theorem one can link the symmetries \( 7 \) with the conserved tensors

\[
(T^k)_\mu^\nu(A), \quad \partial_\mu (T^k)_\mu^\nu = -\left( \partial_\nu (W^k A)^J \right) (MA)_J, \quad k = 0, 1, \ldots, n - 1. \tag{8}
\]

Here, \( k = 0 \) corresponds to the usual energy-momentum tensor. There are \( n \) independent tensors in the set \( 8 \).

1.3. Conservation laws by the reduction of order. Consider the polynomial \( 5 \). Denote the cofactors to the real roots \( \lambda_i \) and complex roots \( \omega_j \) by \( \Lambda_i \) and \( \Omega_j \), respectively,

\[
\Lambda_i = \prod_{k \neq i} (W - \lambda_k)^{p_k} \prod_{j=1}^s (W^2 - (\omega_j + \overline{\omega}_j)W + \omega_j \overline{\omega}_j)^{q_j},
\]

\[
\Omega_j = \prod_{i=1}^r (W - \lambda_i)^{p_i} \prod_{k \neq j} (W^2 - (\omega_k + \overline{\omega}_k)W + \omega_k \overline{\omega}_k)^{q_k}. \tag{9}
\]

By definition, the polynomials \( \Lambda_i(W) \) and \( \Omega_j(W) \) are coprime. Obviously,

\[
M = a_n m^2 (W - \lambda)^{p_i} \Lambda_i = a_n m^2 (W^2 - (\omega_j + \overline{\omega}_j)W + \omega_j \overline{\omega}_j)^{q_i} \Omega_j \quad \text{(no summation in } i, j). \tag{2}
\]

The notion of the Lagrange anchor was introduced in the work [12] in relation to the path-integral quantization of not necessarily Lagrangian systems. Later it was shown that every Lagrange anchor admitted by the equations of motion maps the conserved quantity to the symmetry of equations [13]. In the paper [11] it was noticed that once the operator \( M \) decomposes into two self-adjoint independent factors, the equations admit a two-parameter family of Lagrange anchors such that map any representative of the family of conserved tensors to the space-time translation. In this sense, any of these tensors can be understood as energy-momentum of the theory.
For each cofactor, we introduce the new set of fields,

\[(\xi_i)^J = (\Lambda_i A)^J, \quad i = 1, \ldots, r, \quad (\zeta_j)^J = (\Omega_j A)^J, \quad j = 1, \ldots, s,\]  

(10)
called components. Once the original fields \(A\) are subject to the original field equations (9), the components satisfy the lower order derived equations

\[a_n m^2 (W - \lambda_i)^p \xi_i = 0, \quad a_n m^2 (W^2 - (\omega_j + \overline{\omega} j) W + \omega_j \overline{\omega} j)^q \zeta_j = 0,\]

(11)
where \(p_i, q_j\) are the multiplicities of the roots \(\lambda_i, \omega_j\) in the operator of the original equations (6).

It is easy to see the one-to-one correspondence between solutions of these equations and the original system (9). The inverse transformation to (10) is established by the relations

\[A^J = \sum_{i=1}^r (B_i \xi_i)^J + \sum_{j=1}^s (C_j \zeta_j)^J, \quad B_i = \sum_{p=0}^{p_i-1} b_i^p W^p, \quad C_j = \sum_{q=0}^{2q_j-1} c_j^q W^q,\]

(12)
where the polynomials \(B_i(W)\) and \(C_j(W)\) can be found by the method of undetermined coefficients. The coefficients \(b_i^p, c_j^q\) are defined by the relation

\[\sum_{i=1}^r B_i \Lambda_i + \sum_{j=1}^s C_j \Omega_j = 1.\]

(13)
The last equality is just Bezout’s identity for the coprime univariate polynomials \(\Lambda_i(W)\) and \(\Omega_j(W)\).

Whenever the equivalent formulation (11) is known, the conserved tensors can be obtained by applying the relation (10) separately to every component and then summarizing the results. We denote the conserved tensors for the components by

\[(\tau_i^p)^\mu_\nu(\xi_i), \quad p = 0, \ldots, p_i - 1, \quad (\sigma_j^q)^\mu_\nu(\zeta_j), \quad q = 0, \ldots, 2q_j - 1,\]

(14)
where the indices \(i, j\) label the corresponding components (10) while \(p_i, q_j\) are the multiplicities of corresponding roots (9). The conserved tensors of original derived theory are obtained by substitution (10):

\[(T_i^p)^\mu_\nu(\Lambda_i A) = (\tau_i^p)^\mu_\nu(\xi_i) \bigg|_{\xi_i = \Lambda_i A}, \quad (U_j^q)^\mu_\nu(\Omega_j A) = (\sigma_j^q)^\mu_\nu(\zeta_j) \bigg|_{\zeta_j = \Omega_j A}.\]

(15)
By construction,

\[\partial_\mu (T_i^p)^\mu_\nu(\Lambda_i A) = - (\partial_\mu (W^p \Lambda_i A)^J)(M A)_J, \quad \partial_\mu (U_j^q)^\mu_\nu = - (\partial_\mu (W^q \Omega_j A)^J)(M A)_J.\]

(16)
There are \(n\) conserved tensors (10). The relationship between “new” and “old” conserved tensors is established by comparing their divergences (9) and (10). In particular, the canonical energy-momentum tensor of the derived theory (2) has the following representation:

\[(T^0)^\mu_\nu(A) = \sum_{i=1}^r \sum_{p=0}^{p_i-1} b_i^p(T_i^p)^\mu_\nu(\Lambda_i A) + \sum_{j=1}^s \sum_{q=0}^{2q_j-1} c_j^q(U_j^q)^\mu_\nu(\Omega_j A),\]

with the coefficients of linear combination being defined by Rel. (13).
Notice that some combinations of the conserved tensors \( T_\mu^\nu \) or \( U_\mu^\nu \) may be trivial. A conserved tensor is said to be trivial if it is given by the divergence of an antisymmetric tensor modulo equations of motion, i.e.,

\[
T_\mu^\nu(A) = \partial_\alpha \Sigma_\alpha^\mu \nu, \quad \Sigma_\alpha^\mu \nu = -\Sigma_\alpha^\nu \mu.
\]

The trivial conserved tensors do not result in any conserved quantity and have to be systematically ignored. However, we provide the expressions for the conserved tensors modulo divergence terms, but keep the contributions from the equations of motion. Consistency of the computations can then be verified by taking the divergence, see \( (8) \) and \( (16) \).

As the issue of stability is concerned, the positive conserved tensors are relevant. By positive tensor we mean the one whose 00-component is positive for any solution which is not a pure gauge. We consider the ansatz for the general conserved tensor of the derived theory \( (3) \) in the form

\[
T_\mu^\nu(A) = \sum_{i=1}^r \sum_{p=0}^{p_i-1} \beta_i^p (T_i^p)^\mu \nu (\Lambda_i A) + \sum_{j=1}^s \sum_{q=0}^{2q_j-1} \gamma_j^q (U_j^q)^\mu \nu (\Omega_j A). \tag{17}
\]

The ansatz means that we consider the conserved tensors being additive in the contributions from bilinear combinations of \( \Lambda_i A \) and \( \Omega_j A \), where \( \Lambda_i, \Omega_j \) are the cofactors \( (3) \) to the real roots \( \lambda_i \) and complex roots \( \omega_j \) in the decomposition \( (9) \). The quadratic forms \( (T_i^p) \) and \( (U_j^q) \) are defined by Rel. \( (14), (15) \). In fact, they represent in terms of the original field \( A \) the conserved tensors \( (5) \) of the component fields \( \xi_i, \zeta_j \) subject to equations \( (11) \). Here, \( (T_i^0), (U_j^0) \) are just the energy-momentum tensors for the component fields \( \xi_i, \zeta_j \) expressed in terms of \( A \) by substitution \( (10) \), while \( p, q > 0 \) correspond to the higher order symmetries \( (7) \) of component fields.

As far as the components \( (10) \) are independent, the conserved tensor \( (17) \) is positive if and only if so are the tensors

\[
\sum_{p=0}^{p_i} \beta_i^p (T_i^p)^\mu \nu (\xi_i), \quad \sum_{q=0}^{q_j} \gamma_j^q (U_j^q)^\mu \nu (\zeta_j).
\]

In the other words, the derived theory \( (3) \) is stable if and only if all the components \( (10) \) are stable.

Below, we examine the third order extension of the Chern-Simons theory from the viewpoint of existence of bounded 00-components of conserved tensors we found above.

2. Conserved tensors in the third order extension of the Chern-Simons theory

The field equations of higher derivative extension of the Chern-Simons model \( (1) \) fall into the class of derived theories \( (2) \), with \( W \) being composition of the Hodge and de Rham operators:

\[
(W)^\mu_\nu = (2m^{-1} * d)^\mu_\nu, \quad W_\mu^\nu A_\nu = m^{-1} \varepsilon_\mu^\alpha \nu \partial_\alpha A_\nu, \quad \varepsilon_{012} = \varepsilon^{012} = 1. \tag{18}
\]

The \( n \)-th order theory \( (11) \) has \( n \) degrees of freedom if there are no zero roots in the polynomial \( (6) \). If the zero root exists of any multiplicity (including simple zero root) one degree of freedom is gauged out by transformation \( \delta \chi A = d \chi(x) \), so the theory has \( n - 1 \) DoF. The theory \( (11) \) describes a (decomposable) representation of the
The notation is used to refer to the self-duality equation proposed in [3, 4], it has one physical polarization. On the generalities of the Poincaré group unitary irreducible representations in \( d = 3 \) we refer to [13, 15, 16].

Applying this formula one can express all the conserved tensors in terms of the energy-momentum tensor of the component, while the others are connected to the higher order symmetries of the components:

\[
\delta \xi_{i} = -\varepsilon^{\alpha} \partial_{\alpha} (W^{p} \xi_{i}), \quad p = 1, \ldots, p_{i} - 1, \quad \delta \zeta_{j} = -\varepsilon^{\alpha} \partial_{\alpha} (W^{q} \zeta_{j}), \quad q = 1, \ldots, 2q_{j} - 1, \]

where \( p_{i}, q_{j} \) are multiplicities of real and complex roots. Below we will observe that equations do not have positive conserved quantities in the family [19] once they involve tachyon or non-unitary representations (that corresponds to complex, double or higher multiplicity nonzero real or triple or higher multiplicity zero roots).

The models leading to the unitary representations (that corresponds to simple roots, or at most one double zero root in [6]) admit the conserved tensors with bounded 00-component even though the canonical energy is unbounded in all the instances.

The conserved tensors [3] and [15] of higher derivative extension of the Chern-Simons model are given by

\[
(T^{k})_{\nu} = -\frac{m^{2}}{2} \left\{ \frac{1}{m^{2}} \delta_{\nu}^{\mu} (W^{k} A)_{\alpha}^{\alpha} (MA)_{\alpha} + \sum_{l=0}^{n} a_{l} (t^{k,l})_{\mu}^{\alpha} (A) \right\},
\]

\[
(U^{q})_{\nu} = -\frac{m^{2}}{2} \left\{ \frac{1}{m^{2}} \delta_{\nu}^{\mu} (W^{q} A)_{\alpha}^{\alpha} (MA)_{\alpha} + \sum_{l=0}^{n} a_{n} \sum_{q_{j} \geq 1} \frac{q_{j}^{l} (\omega_{q_{j}}) A_{\nu}^{q_{j} - l - k}}{l! k! (q_{j} - l - k)!} (p^{q_{j} - l + k})_{\mu}^{\alpha} (\Omega_{A} A) \right\},
\]

\[
(k,l)_{\nu} = \frac{1}{m} \varepsilon^{\alpha \beta} \left[ \sum_{s=1}^{l-k} (W^{k+s-1} A)_{\alpha}^{\alpha} \partial_{\nu} (W^{l-s} A)_{\beta} - \sum_{s=1}^{k-1} (W^{k-s} A)_{\alpha} \partial_{\nu} (W^{l+s-1} A)_{\beta} \right], \quad (k,l)_{\mu}^{\nu} \big|_{l=k} = 0.
\]

The expressions for the conserved tensors [19] can be simplified making use of the identity

\[
\frac{1}{m} \varepsilon^{\alpha \beta} (W^{k} A)_{\alpha}^{\alpha} \partial_{\nu} (W^{l} A)_{\beta} = (W^{k+1} A)_{\nu}^{\mu} (W^{l+1} A)_{\mu}^{\mu} (W^{k})_{\nu}^{\nu} - \delta_{\nu}^{\mu} (W^{l+1} A)_{\alpha}^{\alpha} (W^{k} A)_{\alpha} - \frac{1}{m} \delta_{\nu}^{\alpha} (W^{l+1} A)_{\nu}^{\nu} (W^{k} A)_{\beta}^{\beta}, \quad k, l \geq 0.
\]

Applying this formula one can express all the conserved tensors in terms of \( W^{k} A, k = 0, \ldots, n - 1 \).
For $a_0 = 0$, only $n - 1$ of $n$ conserved tensors \(^{(2)}\) are non-trivial. The trivial conserved tensor reads
\[
(T_i^{p_i-1})_{\mu} = \sum_{k=p_i}^{n} a_k (T^{k-1})_{\mu} = -(MA)^{\mu}(W^{p_i-1}\Lambda_i A)_{\nu} + \frac{m}{2} \partial_\alpha (\varepsilon^{\mu\alpha\beta}(W^{p_i-1}\Lambda_i A)_{\nu}(W^{p_i-1}\Lambda_i A)_{\beta}),
\]
with $\lambda_i = 0$. The simplest example of that kind is provided by the energy-momentum tensor for the Chern-Simons theory, where $n = 1$, $\lambda_1 = 0$, $p_1 = 1$.

With account of (21), we consider the following ansatz for the general conserved tensor of the derived theory (2):
\[
T^{\mu}_{\nu}(W^{n-1}A, \ldots, W A, A) = \sum_{r=1}^{\tilde{p}_i-1} \sum_{s=0}^{\beta} (T^p)_{\mu}^r (\Lambda_i A) + \sum_{j=1}^{s} \sum_{q=0}^{\gamma} (U^q)_{\mu}^r (\Omega_j A),
\]
where $\tilde{p}_i = p_i$ if $\lambda_i \neq 0$ and $\tilde{p}_i = p_i - 1$ otherwise. Here, its 00-component is given by the quadratic form in $W^k A, k = 0, \ldots, n - 1$ ($k = 1, \ldots, n - 1$ in case $a_0 = 0$). Identification of the range of the parameters $\beta$ and $\gamma$ that satisfy positivity condition is a well-known problem of linear algebra. It can be always solved in various ways, for example, by the Silvester criterion.

Let us turn to the case when $a_k = 0$ for $k > 3$ and $a_3 = 1$. This is the most general case of the third order derived theory. The equations of motion (2) read
\[
M^{\mu}_{\nu} A_{\nu} = 0, \quad M^{\mu}_{\nu} = m^2 (W^{\mu}_{\alpha} W^{\alpha}_{\beta} W^{\nu}_{\beta} + a_2 W^{\beta}_{\mu} W^{\nu}_{\beta} + a_1 W^{\mu}_{\nu} + a_0 \delta^{\mu}_{\nu}) =
\]
\[
= -\frac{1}{m} \Box \varepsilon^{\mu}_{\nu} \partial_{\nu} - a_2 (\Box \delta^{\mu}_{\nu} - \partial_{\mu} \partial^{\nu}) + a_1 m \varepsilon^{\mu}_{\nu} \partial_{\nu} + a_0 m^2 \delta^{\mu}_{\nu}.
\]

This model has a three-parameter family of conserved tensors if $a_0 \neq 0$ and a two-parameter family if $a_0 = 0$. Depending on the structure of roots in the decomposition (20) for the third order equations (23), the four different cases are seen with different behavior of 00-component of the conserved tensors:

Case A: Three different real roots. The family of conserved tensors includes the one with positive 00-component.

Case B: Simple real root and real root of multiplicity 2. The conserved tensor exists with the positive 00-component if the double root is zero, otherwise the conserved quantity is unbounded.

Case C: Simple real root and pair of complex conjugate roots. The conserved tensor with the positive 00-component does not exist.

Case D: Real root of multiplicity 3. The conserved tensor with the positive 00-component does not exist.

Below we elaborate on each case separately.

2.1. **Case A.** The coefficients $a_0, a_1, a_2$ are defined by three real roots $\lambda_1 < \lambda_2 < \lambda_3$ of the polynomial (20),
\[
a_2 = -(\lambda_1 + \lambda_2 + \lambda_3), \quad a_1 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3, \quad a_0 = -\lambda_1 \lambda_2 \lambda_3.
\]

The factorization (20) for the equations of motion (23) reads
\[
M = m^2 \prod_{i=1}^{3} (W - \lambda_i),
\]
that corresponds to \( r = 3, s = 0, p_i = 1 \).

The general solution to the theory (23) is decomposed into three components (10),

\[
\xi_i = \Lambda_i A, \quad \Lambda_i = \prod_{j \neq i} (W - \lambda_j), \quad i = 1, 2, 3,
\]

that satisfy the Chern-Simons-Proca equations

\[
m^2(W - \lambda_i)\xi_i = 0.
\]  (25)

Each of the equations describes the massive vector field with the mass \( m|\lambda_i| \). Thus, the third-order theory describes a collection of three massive fields with different masses. At the level of propagator, the decomposition into irreducible components has been noticed in already in the original paper [1], where the third order extension was proposed for the Chern-Simons theory. In this paper we see the decomposition at the level of solutions to the equations of motion and elaborate on conserved tensors. In case of second-order theory, \( n = 2 \), the decomposition into components was noticed [17]. The solution (12) to the original theory (23) is reconstructed by the formula

\[
A = \sum_{i=1}^{3} B_i \xi_i, \quad B_i \equiv b^0_i = \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1}.
\]  (26)

The conserved tensors (14) are labelled by the indices \( i = 1, 2, 3, p = 0 \) and have the form

\[
(T^0)_{\mu\nu}(\Lambda_i A) = -\frac{m^2}{2} \left\{ 2\lambda_i (\Lambda_i A)^{\mu}(\Lambda_i A)_{\nu} - \lambda_i \delta^{\mu}_{\nu}(\Lambda_i A)^{\alpha}(\Lambda_i A)_{\alpha} \right\} - (MA)^{\mu}(\Lambda_i A)_{\nu}.
\]  (27)

The sign of the corresponding 00-component coincides with the sign of \(-\lambda_i\),

\[
(T^0)_{00}(\Lambda_i A) = -\frac{m^2}{2} \lambda_i (\Lambda_i A)_{0}(\Lambda_i A)_{0} - (MA)^0(\Lambda_i A)_{0}.
\]  (28)

Here, the Euclidean scalar product is used,

\[
(\Lambda_i A, \Lambda_i A) = ((\Lambda_i A)_0)^2 + ((\Lambda_i A)_1)^2 + ((\Lambda_i A)_2)^2 > 0.
\]

The conserved tensors (27) can be combined into the tensor

\[
T^\mu_{\nu}(A) = \sum_{i=1}^{3} \beta^0_i (T^0)_{\mu\nu}(\Lambda_i A)
\]  (29)

with the positive 00-component if and only if \(-\beta^0_i \lambda_i > 0\). This result admits simple physical interpretation. Each of the tensors (27) has the sense of the energy-momentum tensor of the component \( \xi_i \). The 00-component of the general conserved tensor (29) is bounded if the contributions of all the components have the same sign. In contrast, the 00-component of the canonical energy-momentum tensor with \( \beta^0_i = b^0_i \) is always unbounded because the different components contribute with different signs.

Finally, there is an option when one of the roots is zero. In this case, the corresponding conserved tensor becomes trivial and the positivity of the 00-component of the general conserved tensor (22) is ensured by imposing condition \(-\beta^0_i \lambda_i > 0\) for the nonzero roots. The 00-component of the canonical energy-momentum tensor is again unbounded.
2.2. Cases B and C. We deduce the explicit expressions for the conserved quantities in the Case C. Corresponding expressions for Case B follow from the ones of the Case C by setting the imaginary part of complex root to zero.

The polynomial \( P_2 \) has the simple real root \( \lambda_1 \) and the simple complex root \( \omega_1 \), i.e.,
\[
M = m^2(W - \lambda_1)(W^2 - (\omega_1 + \bar{\omega}_1)W + \omega_1\bar{\omega}_1).
\]
Here, \( r = 1 \) and \( s = 1 \), so the indices \( i, j \) numerating real and complex roots take a single value \( i = j = 1 \). The parametrization for the coefficients \( a_0, a_1, a_2 \) of the polynomial \( P_2 \) reads
\[
a_2 = - (\lambda_1 + \omega_1 + \bar{\omega}_1), \quad a_1 = \lambda_1(\omega_1 + \bar{\omega}_1) + \omega_1\bar{\omega}_1, \quad a_0 = -\lambda_1\omega_1\bar{\omega}_1.
\]

The general solution to the theory \( P_3 \) decomposes into the pair of components \( P_1 \)
\[
\xi_1 = \Lambda_1 A, \quad \zeta_1 = \Omega_1 A, \quad \Lambda_1 = W^2 - (\omega_1 + \bar{\omega}_1)W + \omega_1\bar{\omega}_1, \quad \Omega_1 = W - \lambda_1
\]
that satisfy the first-order and the second-order equations \( \Xi_1 \),
\[
m^2(W - \lambda_1)\xi_1 = 0, \quad m^2(W^2 - (\omega_1 + \bar{\omega}_1)W + \omega_1\bar{\omega}_1)\zeta_1 = 0,
\]
respectively. The equations for the \( \xi \)-component correspond to the Chern-Simons-Proca theory \( [3, 4] \) with mass \( m|\lambda_1| \). The \( \zeta \)-field satisfies the (tachyon) Maxwell-Chern-Simons-Proca equations \( \zeta_1 \). The solution \( \Xi_1 \) to the original theory \( \Xi_0 \) is reconstructed by the formula
\[
A = \frac{1}{(\lambda_1 - \omega_1)(\lambda_1 - \bar{\omega}_1)}\xi_1 + \left[ \frac{W - \omega_1}{(\bar{\omega}_1 - \lambda_1)(\bar{\omega}_1 - \omega_1)} + \frac{W - \bar{\omega}_1}{(\omega_1 - \lambda_1)(\omega_1 - \bar{\omega}_1)} \right] \zeta_1.
\]
The conserved tensors \( \Xi_0 \) of the theory are parameterized by the indices \( p = 0 \) and \( q = 0, 1 \). The expressions for the tensors have the form
\[
(T_0^0)_{\nu} = - \frac{m^2}{2} \left\{ 2\lambda_1(A_1 A)^\mu(A_1 A)_\nu - \lambda_1 \delta_\nu^\mu (A_1 A)^{\alpha}(A_1 A)_\alpha \right\} - (MA)^\mu(A_1 A)_\nu,
\]
\[
(U_0^0)_{\nu} = - \frac{m^2}{2} \left\{ (W\Omega_1 A)^\mu(W\Omega_1 A)_\nu - 2\omega_1\bar{\omega}_1(\Omega_1 A)^\mu(\Omega_1 A)_\nu - \delta_\nu^\mu[(W\Omega_1 A)^\alpha(W\Omega_1 A)_\alpha - 

\quad - \omega_1\bar{\omega}_1(\Omega_1 A)^\alpha(\Omega_1 A)_\alpha] \right\} - (MA)^\mu(\Omega_1 A)_\nu,
\]
\[
(U_1^0)_{\nu} = - \frac{m^2}{2} \left\{ 2(\omega_1 + \bar{\omega}_1)(W\Omega_1 A)^\mu(W\Omega_1 A)_\nu - 2\omega_1\bar{\omega}_1((\Omega_1 A)^\mu(W\Omega_1 A)_\nu + (W\Omega_1 A)^\mu(\Omega_1 A)_\nu) - 

\quad - \delta_\nu^\mu[(\omega_1 + \bar{\omega}_1)(W\Omega_1 A)^\alpha(W\Omega_1 A)_\alpha - 2\omega_1\bar{\omega}_1(W\Omega_1 A)^\alpha(\Omega_1 A)_\alpha] \right\} - (MA)^\mu(W\Omega_1 A)_\nu.
\]
The 00-components read
\[
(T_0^0)_0 = - \frac{m^2}{2} \lambda_1(A_1 A, A_1 A) - (MA)^0(A_1 A)_0,
\]
\[
(U_0^0)_0 = - \frac{m^2}{2} \left\{ (W\Omega_1 A, W\Omega_1 A) - \omega_1\bar{\omega}_1(\Omega_1 A, \Omega_1 A) \right\} - (MA)^0(\Omega_1 A)_0,
\]
\[
(U_1^0)_0 = - \frac{m^2}{2} \left\{ (\omega_1 + \bar{\omega}_1)(W\Omega_1 A, W\Omega_1 A) - 2\omega_1\bar{\omega}_1(W\Omega_1 A, \Omega_1 A) \right\} - (MA)^0(W\Omega_1 A)_0.
\]
The sign of \((T^0_1)^0\) coincides with the sign of \(-\lambda_1\), see (28). The linear combination of \((U^0_1)\) and \((U^1_1)\) does not give rise to a positive conserved tensor unless \(\omega_1 w_1 = 0\) (Case B, \(\lambda_2 = 0\)). Thus, Cases B and C of theory (8) are unstable unless the decomposition (6) has one simple nonzero root and double zero root. Degrees of freedom of stable theory include one massive and one massless vector mode.

2.3. Case D. The polynomial (6) has the simple real root \(\lambda_1\) multiplicity 3, i.e.,

\[ M = m^2(W - \lambda_1)^3 = m^2(W^3 - 3\lambda_1 W^2 + 3\lambda_1^2 W - \lambda_1^3). \]

The comparison with (10) brings us to the identification \(s = 0\) and \(p_1 = 3\). The parametrization for the coefficients \(a_0, a_1, a_2\) of the polynomial (6) reads

\[ a_2 = -3\lambda_1, \quad a_1 = 3\lambda_1^2, \quad a_0 = -\lambda_1^3. \]

The general solution to the theory (28) consists of one component. The new variables (10) are not introduced.

The conserved tensors are constructed by the general rule (19) and parameterized by the indices \(i = 1\) and \(p = 0, 1, 2\). The expressions for the tensors have the form

\[
(T^0_1)^{\mu}_{\nu} = -\frac{m^2}{2}\left\{2(w^2 A)^{\mu}(WA)_{\nu} + 2(WA)^{\mu}(w^2 A)_{\nu} + 2\lambda_1 (wA)^{\mu}(wA)_{\nu} - \delta^{\mu}_{\nu}\right[2(w^2 A)^{\alpha}(WA)_{\alpha} + \lambda_1 (wA)^{\alpha}(wA)_{\alpha}\right]\} - (MA)^{\mu}(WA)_{\nu},
\]

\[
(T^1_1)^{\mu}_{\nu} = -\frac{m^2}{2}\left\{2(wWA)^{\mu}(wWA)_{\nu} + 2\lambda_1 (w^2 A)^{\mu}(WA)_{\nu} + 2\lambda_1 (WA)^{\mu}(w^2 A)_{\nu} - \delta^{\mu}_{\nu}\right[2(wWA)^{\alpha}(wWA)_{\alpha} + 2\lambda_1 (w^2 A)^{\alpha}(wA)_{\alpha}\right]\} - (MA)^{\mu}(WA)_{\nu},
\]

\[
(T^2_1)^{\mu}_{\nu} = -\frac{m^2}{2}\left\{2\lambda_1 (w^2 A)^{\mu}(w^2 A)_{\nu} + 4\lambda_1 (wWA)^{\mu}(wWA)_{\nu} + 2\lambda_1^2 (w^2 A)^{\mu}(WA)_{\nu} + 2\lambda_1^3 (w^2 A)^{\mu}(w^2 A)_{\nu} - \delta^{\mu}_{\nu}\right[\lambda_1 (w^2 A)^{\alpha}(w^2 A)_{\alpha} + 2\lambda_1 (wWA)^{\alpha}(wA)_{\alpha}\right]\} - (MA)^{\mu}(W^2 A)_{\nu},
\]

where the notation \(w = W - \lambda_1\) is used.

The 00-components read

\[
(T^0_1)^0 = -\frac{m^2}{2}\left\{2(WA, W^2 A) - 3\lambda_1 (WA, WA) + \lambda_1^2(A, A)\right\} - (MA)^0 A_0,
\]

\[
(T^1_1)^0 = -\frac{m^2}{2}\left\{(W^2 A, W^2 A) - 3\lambda_1^2 (WA, WA) + 2\lambda_1^3 (WA, A)\right\} - (MA)^0(WA)_{0},
\]

\[
(T^2_1)^0 = -\frac{m^2}{2}\lambda_1\left\{3(W^2 A, W^2 A) - 6\lambda_1 (W^2 A, WA) + \lambda_1^2(WA, WA) + 2\lambda_1^3(W^2 A, A)\right\} - (MA)^0(W^2 A)_{0}.
\]

One can check that the quantities (33) are not combined into a positive tensor. This result also applies to the case \(\lambda_1 = 0\). The theory with root of multiplicity three has to be considered as unstable anyway.
3. An example of stable self-interactions

As we have seen, some of the higher derivative extensions of the Chern-Simons theory admit positive conserved tensors at free level. In this section, we provide an example of interaction in the Case A such that the theory still has positive conserved tensor and remains therefore classically stable. The equations of motion read

$$\mathcal{M} A \equiv m^2 (W - \lambda_1)(W - \lambda_2)(W - \lambda_3)A - U'(\xi^\alpha \xi_\alpha)\xi = 0, \quad \xi = \sum_{i=1}^{3} \beta_i^0 \Lambda_i A, \quad (35)$$

where $U(s)$ can be any scalar function, $U'(s) = \frac{dU(s)}{ds}$ and $\beta_i^0$ are treated as the parameters of interactions. The interaction could be constructed by the factorization method of the papers [11, 18] that ensures survival of selected conservation law of free theory at interacting level. Here, we do not elaborate on the procedure for constructing the interaction, we just examine consistency and stability of interacting model.

The interaction could be constructed by the factorization method of the papers [11, 18] that ensures survival of selected conservation law of free theory at interacting level. Here, we do not elaborate on the procedure for constructing the interaction, we just examine consistency and stability of interacting model.

The theory admits the conserved tensor

$$T^\mu_\nu(A) = \sum_{i=1}^{3} \beta_i^0 (T_i^0)^\mu_\nu (\Lambda_i A) + \frac{1}{2} \delta^\mu_\nu U(\xi^\alpha \xi_\alpha), \quad \partial_\mu T^\mu_\nu = -\partial_\nu \xi^\alpha (\mathcal{M} A)_\alpha. \quad (36)$$

With account of equations of motion it can be rewritten as

$$T^\mu_\nu = -\sum_{i=1}^{3} \frac{m^2 \beta_i^0 \lambda_i}{2} \{2(\Lambda_i A)^\mu(\Lambda_i A)_\nu - \delta^\mu_\nu (\Lambda_i A)^\alpha(\Lambda_i A)_\alpha\} - U'(\xi^\alpha \xi_\alpha)\xi^\mu \xi_\nu + \frac{1}{2} \delta^\mu_\nu U(\xi^\alpha \xi_\alpha) - (\mathcal{M} A)^\mu_\nu \xi_\nu. \quad (37)$$

The consistent inclusion of interactions should not change the degree of freedom number. The interaction (35) is consistent. This fact can be seen from decomposition of solution into components (24). The equations of motion for the components take the form

$$\mathcal{M}_i \xi_i \equiv m^2 (W - \lambda_i)\xi_i - U'(\xi^\alpha \xi_\alpha)\xi = 0, \quad \xi = \sum_{j=1}^{3} \beta_j^0 \xi_j, \quad i = 1, 2, 3. \quad (38)$$

At the free level, elimination of longitudinal degree of freedom is ensured by the transversality conditions $\partial^\mu(\xi_i)_\mu = 0, i = 1, 2, 3$. In non-linear theory, the transversality conditions are modified but still remain the first-order constraints,

$$\partial^\mu (\mathcal{M}_i \xi_i)_\mu = \partial^\mu \{m^2 \lambda_i (\xi_i)_\mu + U'(\xi^\alpha \xi_\alpha)(\xi_\mu)\} = 0. \quad (39)$$

The degree of freedom number can be also covariantly computed without depressing the order, e.g. by bringing the original higher derivative equations into the involutive form as is it explained in [19]. Anyway, the equations (38) still describe three degrees of freedom, so the interaction (35) is stable (if $U > 0, U' < 0$) and consistent.

Rare examples are known of stable interactions in the higher derivative systems. The best known example is $f(R)$-gravity [20, 21] where the canonical energy is bounded at linearized level. This exceptional phenomenon happens because the theory is strongly constrained. In the paper [22], the stability of some interactions is demonstrated for the Pais-Uhlenbeck oscillator (whose canonical energy is unbounded) by numerical simulations.
The stable interactions were recently proposed for the Podolsky electrodynamics \cite{11} and for the higher order Pais-Uhlenbeck oscillator \cite{18}. The example of this section extends the limited list of known stable interactions in higher derivative models.

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