Dual Resonance Model Solves the Yang-Baxter Equation

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Abstract

The duality of dual resonance models is shown to imply that the four point string correlation function solves the Yang-Baxter equation. A reduction of transfer matrices to $A_l$ symmetry is described by a restriction of the KP $\tau$ function to Toda molecules.
By the old duality I mean the \textit{stu} duality embodied in the dual resonance model \cite{1} developed in the late 1960’s and 1970’s as a model which describes hadron scattering processes. The purpose of this article is to supply an argument which clarifies the link among three independent subjects in physics, i.e., string models in particle physics, solvable lattice models in statistical physics, and soliton theory. In particular I will show in the following that the old duality assures a four point string correlation function to solve the Yang-Baxter equation.

The correspondence between the soliton theory and the string models is rather straight-forward \cite{2}. The string correlation functions solve the Hirota bilinear difference equation (HBDE), defined as \cite{3}

$$\alpha f(k_1 + 1, k_2, k_3)f(k_1, k_2 + 1, k_3 + 1)$$
$$+ \beta f(k_1, k_2 + 1, k_3)f(k_1 + 1, k_2, k_3 + 1)$$
$$+ \gamma f(k_1, k_2, k_3 + 1)f(k_1 + 1, k_2 + 1, k_3) = 0 \quad (1)$$

with $\alpha + \beta + \gamma = 0$. This single equation is equivalent to the KP-hierarchy in soliton theory \cite{5}. A solution of this equation is called the tau function \cite{3}.

The soliton theory and the solvable lattice models, on the other hand, share the common structure of integrability, called the quantum inverse scattering method \cite{6}. Very recently a new light was cast into this connection through the papers \cite{7}, \cite{8} in which was pointed out that the algebraic relation satisfied by the transfer matrix of the solvable lattice model with $A_1$ symmetry is nothing but HBDE \cite{3}. Moreover in \cite{7} the authors showed that the linear Bäcklund transformation of HBDE \cite{9} generates a series of Bethe ansatz solutions.

These results can be summarized such that HBDE unifies the three problems in our consideration. The same solution, however, is interpreted quite differently from each other, a correlation function of strings, the tau function of soliton theory, and a transfer matrix of the solvable lattice models. From mathematical point of view it is apparent that HBDE embodies a very large symmetry which guarantees integrability of systems with infinite degrees of freedom irrespective to their physical interpretations. Such a mathematical unification of different models, however, does not mean understanding them from the side of physics.

In the case of our concern the link between the string models and the solvable lattice models is most obscure since their relation is indirect. Apart from the fact that the correlation functions of strings and the transfer matrix of solvable models are governed by the same equation, their connection is not manifest at all. I like to fill this gap by showing that the lattice models, whose Boltzmann weight is given by the four point correlation function of the string models, are solvable. The proof is achieved by noticing that the \textit{stu} duality of the dual resonance models \cite{1} guarantees the Yang-Baxter equation to be solved.

To begin with let us reformulate briefly the string correlation functions in a way suitable for our
discussion in the following [10, 2]. We consider \( N \) external strings interacting each other through the world sheet specified by the ground state \( |G\rangle \). It is given by

\[
F_G(K_1, K_2, \ldots, K_N) := \langle 0| W(K_1, g_1) W(K_2, g_2) \cdots W(K_N, g_N) |G\rangle.
\] (2)

Here the \( j \)-th string is supposed to have momentum \( K_j^\mu (z) \) distributed along a path \( g_j(z) \) in the world sheet. The path is assumed to close a contour as the local coordinate \( z \) of the world sheet moves around a circle. The interaction takes place via the vertex operator [11]

\[
W(K_j, g_j) = \exp \left[ \frac{1}{2\pi} \oint \frac{dz}{z} K_j^\mu (z) X_+^\mu (g_j(z)) \right] \times \exp \left[ \frac{1}{2\pi} \oint \frac{dz}{z} K_j^\mu (z) X_-^\mu (g_j(z)) \right].
\] (3)

Here the string coordinate \( X^\mu (z) = X_+^\mu (z) + X_-^\mu (z) \) is defined by the following expansion:

\[
X_-^\mu (z) = x^\mu + \sum_{n=1}^{\infty} \frac{a_n^\mu}{\sqrt{n}} z^n,
\]

\[
X_+^\mu (z) = ip^\mu \ln z + \sum_{n=1}^{\infty} \frac{a_n^\mu}{\sqrt{n}} z^{-n}
\] (4)

whose components satisfy

\[
[x^\mu, p^\nu] = i\delta^{\mu\nu}, \quad [a_m^\mu, a_n^\nu] = \delta^{\mu\nu} \delta_{mn}, \quad m, n \in \mathbb{Z}_{\geq 1}.
\] (5)

If \( K_j^\mu \) is a constant vector \( k_j^\mu \), the vertex operator \( W(K_j, g_j) \) turns to the ordinary vertex operator for the external ground state particle of momentum \( k_j^\mu \)

\[
V(k_j, z_j) = e^{ik_j^\mu X_+^\mu (z_j)} e^{ik_j^\mu X_-^\mu (z_j)}
\] (6)

where \( z_j = g_j(0) \). Therefore \( k_j^\mu \) is the barycentric momentum of the \( j \)-th string. To make simpler the expressions of formulae the space-time indices \( \mu, \nu, \cdots \) will be suppressed in what follows.

The empty state \( |0\rangle \) is defined by

\[
p|0\rangle = a_n|0\rangle = 0, \quad n = 1, 2, \cdots,
\]

while the ground state is defined by \( |G\rangle = G(X)|0\rangle \), where [12]

\[
G(X) = \theta \left( \zeta - \frac{1}{2\pi} \oint dX(z) \oint \omega \right) \times \exp \left[ \frac{1}{8\pi^2} \oint dX(x) \oint dX(y) \ln \frac{E(x,y)}{x-y} \right].
\] (7)

\( \theta, \omega, E(x,y), \zeta \) are the Riemann theta function, the Abel differential, the prime form, and an arbitrary vector, respectively, all defined on the world sheet of definite number of genus.
The connection of the correlation function (2) with the tau function of the KP hierarchy was shown to follow to

\[ F_G(K_1, K_2, \cdots, K_n) = F_0(K_1, K_2, \cdots, K_n) = \tau(t). \]

Here \( F_0 \) is given by (2) with \( |G\rangle \) replaced by \( |0\rangle \), and \( t \) denotes the collection of the soliton coordinates \( \{t_1, t_2, \cdots\} \) which are related with the string variables by

\[ t_n = \frac{1}{n} \sum_{j=1}^{N} \frac{1}{2\pi} \oint \frac{dz}{z} K_j(z) g^n_j(z), \quad n = 1, 2, \cdots. \]

Note that, when \( K^\mu_j(z) = k^\mu_j \), this reduces to the Miwa transformation [4]. The proof that (8) satisfies (1) is exactly the same as in ref. [2]. The variables in (1) are any three chosen out of the constant components of \( K_j(z) \)'s.

We now consider the link between the string model and the solvable lattice model. The main step toward this problem is to define the Boltzmann weight properly, so that the Yang-Baxter equation is solved. Here I propose a two dimensional lattice model whose links are specified by string momenta \( K(z) \)'s and Boltzmann weight is given by the four point string correlation function

\[ R^{K_1''', K_1''}(g, g', g'', \bar{g}'') = (0|W(K, g)W(K', g')\bar{W}(K'', g'')\bar{W}(K''', g''')|0). \]

Here \( \bar{W} \) is the operator whose in-state and out-state are reversed and \( \bar{g}(z) = g(\frac{1}{z}) \).

Using this Boltzmann weight the transfer matrix of the model is defined by

\[ T_{K_1, K_2, \cdots, K_M}^{K_1'', K_2'', \cdots, K_M''} = \sum_{\{K_j''\}} R_{K_1''', K_1''}^{K_1'', K_1'} R_{K_2''', K_2''}^{K_2'', K_2'} \cdots R_{K_M''', K_M''}^{K_M'', K_M'}. \]

The summation over \( K_j'' \) means the functional integration over all possible paths of strings \( K_j'' \). Hence it turns out to be given explicitly by the 2M point string correlation function, defined on a torus world sheet associated with the ground state \( |G_1\rangle \):

\[ = (0|W(K_1, g_1)\bar{W}(K_1', g_1')W(K_2, g_2)\cdots \bar{W}(K_M', g_M')|G_1) \]

\[ = F_G_1(K_1, K_1', K_2, K_2', \cdots, K_M, K_M'). \]
From the transfer matrix (11) we define an operator $\mathcal{T}$ simply by omitting the summation over $K''_1$.

According to the standard argument \cite{13} two of $\mathcal{T}$ operators with different spectral parameters commute with each other when the $R$ matrix satisfies the Yang-Baxter equation:

$$
\sum_{K,K',K''} R_{K_6,K'_3} R_{K_1,K_2} R_{K_4,K_5} R_{K,K''} = \\
\sum_{K,K',K''} R_{K_5,K'_3} R_{K_6,K_2} R_{K,K'} R_{K,K''}
$$

(13)

Hence, if (13) holds, the lattice model is solvable.

To prove (13), I claim that it is nothing but the duality relation. In fact we first notice that in- and out-states of the string $K(z)$ is exchanged by the Hermite conjugation of it:

$$
K(z) \rightarrow K(z) = K \left( \frac{1}{z} \right),
$$

from which follows

$$
\tilde{W}(K,g) = W(K,\bar{g}).
$$

(14)

Using this and the fact that a string correlation function is symmetric under the cyclic permutation the four point function can be rewritten as

$$
\langle 0 | W(K,g) W(K',g') \tilde{W}(K'',g'') \tilde{W}(K''',g''') | 0 \rangle \\
= \langle 0 | W(K'',g''') W(K,g) \tilde{W}(K',g') \tilde{W}(K'',g'') | 0 \rangle \\
= \langle 0 | W(K'',g''') W(K',g') \tilde{W}(K,g) \tilde{W}(K',g') | 0 \rangle
$$

(15)

Application of these identities yields for the lhs of (13)

$$
\sum_{K,K',K''} R_{K_6,K'_3} R_{K_1,K_2} R_{K_4,K_5} R_{K,K''} = \\
\sum_{K,K',K''} R_{K_5,K'_3} R_{K_6,K_2} R_{K,K'} R_{K,K''}
$$

(16)
and for the rhs

\[
\sum_{K,K',K''} R_{K_6,K_3} R_{K_1,K_2} R_{K''} = \sum_{K,K',K''} R_{K_6,K_3} R_{K_1,K_2} R_{K''}.
\]

Both of these represent six point string functions defined on the world sheet of torus. In fact they are identical due to the duality and is given by

\[
\langle 0 | \bar{W}(K_6,g_6) W(K_5,g_5) \bar{W}(K_4,g_4) \times W(K_3,g_3) \bar{W}(K_2,g_2) W(K_1,g_1) | G_1 \rangle.
\]

This justifies the claim.

We have just established the direct link between the string models and the solvable lattice models. In the rest of this letter I like to demonstrate that through some reduction we can obtain a familiar solvable lattice model. It will, at the same time, explain the mysterious relation between the Yang-Baxter equation and HBDE recently discussed in \[7\],\[8\].

The transfer matrix \( T^{(\mu)}(\lambda) \) of the solvable lattice model associated with \( A_l \) symmetry was shown \[7\],\[8\] to solve HBDE (1). The variables \( \mu, \nu, \lambda \) of \( T^{(\mu)}(\lambda) \) denote the size of the \( \mu \times \nu \) rectangular Young tableaux and the spectral parameter which specifies the Boltzmann weight. They are related with the variables \( k_1, k_2, k_3 \) of HBDE according to \( \mu = k_2 + k_3 - 1, \nu = k_3 + k_1 - 1, \lambda = k_1 + k_2 - 1 \). This correspondence sounds rather artificial because \( \mu \) and \( \nu \) have meaning of size of Young tableaux and range a certain finite intervals, while \( k_1, k_2, k_3 \) range all integers or periodic boundary conditions are imposed.

In order to resolve this unnaturalness I first write HBDE in terms of the new variables

\[
\alpha g(\lambda + 1, \mu, \nu)g(\lambda - 1, \mu, \nu) + \beta g(\lambda, \mu + 1, \nu)g(\lambda, \mu - 1, \nu) + \gamma g(\lambda, \mu, \nu + 1)g(\lambda, \mu, \nu - 1) = 0,
\]

where \( g(\lambda, \mu, \nu) = f(k_1, k_2, k_3) \). In the following I remark the results quoted from our recent work \[14\] in a slightly different form, appropriate in our discussion.

**Remark :**

Let \( g(\lambda, \mu, \nu) \) be a solution of HBDE (19), and \( A(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \) an octahedron consisting of the nearest neighbours of the point at \( (\lambda, \mu, \nu) = (\bar{\lambda}, \bar{\mu}, \bar{\nu}) \) in the lattice space \( \mathbb{Z}^3 \), then

\[
\bar{g}(\lambda, \mu, \nu) = \begin{cases} 
g(\lambda, \mu, \nu), & (\lambda, \mu, \nu) \in A(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \\
0 & \text{otherwise} \end{cases}
\]

is also a solution to HBDE (19).
This is the smallest piece of Toda lattice which is shown in Fig. a. The proof of (20) is simple. Namely consider another octahedron $A'$ which shares at least one point of $A$. Since $g(\lambda, \mu, \nu) = 0$ on every lattice point surrounding $A$, $g(\lambda, \mu, \nu)'s$ on the octahedron $A'$ automatically satisfy (19).

Fig. a

The generalization of (20) to an arbitrary size of piece of Toda lattice is straightforward. Let us call such a piece a Toda molecule according to ref. [15]. Then the smallest unit (20) should be called a Toda atom. A Toda molecule must be rectangular when it is sliced perpendicular to each axis of the lattice, for it being a solution of HBDE. [13]. We can consider a collection of Toda molecules if they are disjoined with each other. An example of a slice of such a collection is given in Fig. b. Note that each piece can be an independent solution of HBDE.

Fig. b A slice of Toda molecules

Now we go back to our lattice model whose Boltzmann weight is given by $R_{K,K'}^{K''}$ of [10] but $K'$ and $K''$ are reduced to their barycentric momenta taking only integral numbers. Using this Boltzmann weight we construct the transfer matrix which carries only integral numbers in its legs. According to the above remarks we can consider any size of Toda molecules in equal basis. From this point of view we are allowed to think of these numbering of the legs as specifying the size of the rectangular slice of a molecule instead of address on the lattice space.

Adopting this convention we consider a solvable lattice model with the $A_4$ symmetry. We identify the transfer matrix associated with the $\mu \times \nu = (k_2 + k_3 - 1) \times (k_3 + k_1 - 1)$ rectangular type Young tableux
with the same size of Toda molecule in the \((\mu, \nu)\) plane. Further identification of the spectral parameter \(\lambda\) with \(k_1 + k_2 - 1\) completes the desired correspondence.

Before closing this note I like to mention a few comments. The partition function of our lattice model itself is a correlation function \(\langle \mathcal{A} \rangle\) of strings. Hence it is a solution of HBDE. This form of general string correlation function was calculated \[16\] explicitly to reproduce \(\langle \mathcal{A} \rangle\) using the vertex operator \(W\) in \(\langle \mathcal{A} \rangle\) as a building block. From this point of view the solvable lattice model is nothing but a special case of analogue (or fish net) models \[17\] discussed in connection with hadron scattering processes. Therefore this model has been shown integrable in two folds. Namely it satisfies HBDE \[3\] and also satisfies the Yang-Baxter equation as it was shown in this paper.

The braid of strings was discussed in \[18\]. There the vertex operator \(W\) was regarded as representing a state of string, and a braid of strings was caused through an exchange of the order of \(W\)’s. The exchange matrix was derived assuming that states of the strings were not changed under their exchange of order because of the duality. Hence it is included as a special case of present work.

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The term ‘Toda molecule’ is often used in a little different sense. We use this name to what we defined in the remark. I would like to thank Professor Hirota for the communication.

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