Global well-posedness for 2-D Boussinesq system with the temperature-dependent viscosity and supercritical dissipation

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Abstract

The present paper is dedicated to the global well-posedness issue for the Boussinesq system with the temperature-dependent viscosity in $\mathbb{R}^2$. We aim at extending the work by Abidi and Zhang (Adv. Math. 2017 (305) 1202–1249) to a supercritical dissipation for temperature.

Key Words: Global well-posedness; Boussinesq system; Littlewood-Paley theory

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1. Introduction and the main result

In this paper, we mainly study the Cauchy problem of the Boussinesq system with the temperature-dependent viscosity in $\mathbb{R}^2$:

\begin{equation}
\begin{aligned}
\partial_t \theta + u \cdot \nabla \theta + \kappa |D|^s \theta &= 0, \\
\partial_t u + u \cdot \nabla u - \text{div}(2\mu(\theta)d(u)) + \nabla \Pi &= \theta e_2, \\
\text{div} u &= 0,
\end{aligned}
\end{equation}

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ denotes the velocity vector field, $d(u) = (\nabla u + \nabla u^T)/2$ denotes the deformation matrix, $\Pi = \Pi(x,t)$ is the scalar pressure, the scalar function $\theta = \theta(x,t)$ is the temperature, $e_2$ is the unit vector in $\mathbb{R}^2$, the thermal conductivity coefficient $\kappa \geq 0$, the kinematic viscous coefficient $\mu(\theta)$ is a smooth, positive and non-decreasing function on $[0,\infty)$. Furthermore, in all that follows, we shall always denote $|D|^s$ to be the Fourier multiplier

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with symbol $|\zeta|^s$. In the whole paper, we also assume that $\kappa = 1$ and

$$0 < 1 \leq \mu(\theta), \quad \mu(\cdot) \in W^{2,\infty}(\mathbb{R}^+), \quad \mu(0) = 1. \tag{1.2}$$

The Boussinesq system arises from a zeroth order approximation to the coupling between Navier-Stokes equations and the thermodynamic equations. It can be used as a model to describe many geophysical phenomena [24]. If we consider the more general Boussinesq system with the temperature-dependent viscosity and thermal diffusivity to take the following form:

$$\begin{cases}
\partial_t \theta + u \cdot \nabla \theta - \text{div}(\kappa(\theta) \nabla \theta) = 0, \\
\partial_t u + u \cdot \nabla u - \text{div}(\mu(\theta) \nabla u) + \nabla \Pi = \theta e_d, \\
\text{div} u = 0, \\
(\theta, u)|_{t=0} = (\theta_0, u_0),
\end{cases} \tag{1.3}$$

the problem becomes much more complicated. Lorca and Boldrin in [22] proved the global existence of strong solution for small data, and the global existence of weak solution and the local existence and uniqueness of strong solution for general data in [21]. Recently, Wang and Zhang in [28] mainly used De-Giorgi method and Harmonic analysis tools to get the global existence of smooth solutions in $\mathbb{R}^2$. Sun and Zhang in [26] extended the result in [28] to the case of bounded domain. More precisely, the authors in [26] got the global existence of strong solution to the initial-boundary value problem of the 2-D Boussinesq system and 3-D infinite Prandtl number model with viscosity and thermal conductivity depending on the temperature. Li and Xu in [20] also generalized the result in [28] to the inviscid case (that is $\mu(\theta) = 0$). They got the global strong solution for arbitrarily large initial data in Sobolev spaces $H^s(\mathbb{R}^2), s > 2$. Francesco in [11] obtained the global existence of weak solutions to the system (1.3) in $\mathbb{R}^d$, with viscosity dependent on temperature. The initial temperature in [11] is only supposed to be bounded, while the initial velocity belongs to some critical Besov Space, invariant to the scaling of this system. Jiu and Liu in [16] obtained the global well-posedness of anisotropic nonlinear Boussinesq equations with horizontal temperature-dependent viscosity and vertical thermal diffusivity in $\mathbb{R}^2$. Using $\kappa|D|\theta$ instead of $\text{div}(\kappa(\theta) \nabla \theta)$ in system (1.3), Abidi and Zhang in [3] got the global solution in $\mathbb{R}^2$ provided the viscosity coefficient is sufficiently close to some positive constant in $L^\infty$ norm.
When $\kappa(\theta)$ and $\mu(\theta)$ are two positive constants which do not depend on the temperature, Cannon and DiBenedetto in [6] used the classical method to get the global solutions in $\mathbb{R}^2$. Recently, more and more researchers (see [5], [7], [10], [12], [13], [16], [17], [19], [29], [30], [32]) pay much more attentions to the following model:

\[
\begin{aligned}
&\partial_t \theta + u \cdot \nabla \theta + \kappa |D|^\alpha \theta = 0, \\
&\partial_t u + u \cdot \nabla u + \mu |D|^\beta \theta + \nabla \Pi = \theta \varepsilon_d, \\
&\text{div} \ u = 0,
\end{aligned}
\tag{1.4}
\]

where $\mu \geq 0$, $\kappa \geq 0$, $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$ are real parameters. The fractional diffusion operators considered here in appear naturally in the study in hydrodynamics as well as anomalous diffusion in semiconductor growth. Mathematically, the problem for global regularity of (1.4) is an interesting and a subtle one. Intuitively, the lower the values of $\alpha, \beta$ are, the harder it is to prove that solutions emanating from sufficiently smooth and localized data persist globally. In particular, the problem with no dissipation (i.e. $\mu = \kappa = 0$) remains open. This is very similar to the Euler equation in two and three spatial dimensions and in fact numerous studies explore the possibility of finite time blow up.

Our goal here is to relax the dissipation needed in [3] for global well-posedness in $\mathbb{R}^2$. More precisely, we get the following theorem:

**Theorem 1.1.** For any $2/3 < \alpha \leq 1$, $8/(3\alpha - 2) < p < 1/C_1 \mu(\cdot) - 1 \|L^\infty_\theta - 1\|_{L^\infty_\theta}$, $\alpha/(2\alpha - 1) < q < \min \{2, 4\alpha / 3(2\alpha - 1)\}$ and $3 - 2\alpha < s_0 < 4\alpha / q - 8\alpha + 6$. Assume $\theta_0 \in (L^q \cap \dot{H}^{-s_0} \cap H^{\alpha/2}) \cap B^{\alpha/2}_{p,\infty}(\mathbb{R}^2)$ and $u_0 \in B^{-1}_{\infty,1} \cap H^1(\mathbb{R}^2)$ be a solenoidal vector field. Then there exists some sufficiently small $\varepsilon_0$ so that if we assume

\[
\|\mu(\cdot) - 1\|_{L^\infty_\theta} \leq \varepsilon_0,
\tag{1.5}
\]

(1.1) has a unique global solution $(u, \theta)$ so that

\[
\begin{aligned}
&u \in C([0, \infty); H^1) \cap \dot{L}^2_{2/\alpha}(\mathbb{R}^+; B^{\alpha/2}_{2,\infty} \cap L^1 \cap B^{-1}_{\infty,1}), \quad \partial_t u \in L^2(\mathbb{R}^+; L^2), \\
&\theta \in C([0, \infty); L^q \cap \dot{H}^{-s_0} \cap H^{\alpha/2}) \cap L^\infty(\mathbb{R}^+; B^{\alpha/2}_{p,\infty} \cap L^2(\mathbb{R}^+; H^\alpha) \cap \dot{L}^1_{\text{loc}}(\mathbb{R}^+; B^{3\alpha/2}_{p,\infty}).
\end{aligned}
\tag{1.7}
\]
Moreover, we have

\[ \| \theta (t) \|_{L^2} \leq CE_0(t)^{-\sigma_0/\alpha}, \]  

(1.8)

with

\[ E_0 = \mathcal{E}_0(1 + \mathcal{E}_0), \quad \mathcal{E}_0 = \| \theta_0 \|_{H^{-\sigma_0}} + \| \theta_0 \|_{L^2} + (\| u_0 \|_{L^2} + \| \theta_0 \|_{L^q})(1 + \| \theta_0 \|_{L^q}). \]  

(1.9)

**Remark 1.** The proof about this theorem shares the same ideas as the case \( \alpha = 1 \) treated in [3] but with much more technical difficulties.

The paper is organized as follows. In Section 2, we recall the Littlewood-Paley theory and give some useful lemmas. In Section 3, we take several steps to give the key a priori estimates. In Section 4, we complete the proof of our main theorem.

Let us complete this section by describing the notations which will be used in the sequel.

**Notations** : Let \( A, B \) be two operators, we denote \( [A, B] = AB - BA \), the commutator between \( A \) and \( B \). For \( a \lesssim b \), we mean that there is a uniform constant \( C \), which may be different on different lines, such that \( a \leq Cb \). For \( X \) a Banach space and \( I \) an interval of \( \mathbb{R} \), we denote by \( C(I; X) \) the set of continuous functions on \( I \) with values in \( X \). For \( q \in [1, +\infty] \), the notation \( L^q(I; X) \) stands for the set of measurable functions on \( I \) with values in \( X \), such that \( t \to \| f(t) \|_X \) belongs to \( L^q(I) \). We always let \( (d_j)_{j \in \mathbb{Z}} \) (resp. \( (c_j)_{j \in \mathbb{Z}} \)) be a generic elements of \( \ell^1(\mathbb{Z}) \) (resp. \( \ell^2(\mathbb{Z}) \)) so that \( \sum_{j \in \mathbb{Z}} d_j = 1 \) (resp. \( \sum_{j \in \mathbb{Z}} c_j^2 = 1 \)).

**2. Preliminaries**

In this section, we recall some basic facts on Littlewood-Paley theory (see [4] for instance). Let \( \chi, \varphi \) be two smooth radial functions valued in the interval \([0,1]\), the support of \( \chi \) be the ball \( \mathcal{B} = \{ \xi \in \mathbb{R}^d : |\xi| \leq 4/3 \} \), the support of \( \varphi \) be the annulus \( \mathcal{C} = \{ \xi \in \mathbb{R}^d : 3/4 \leq |\xi| \leq 8/3 \} \), so that

\[ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \]

\[ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d. \]
Let \( h = \mathcal{F}^{-1} \varphi \) and \( \tilde{h} = \mathcal{F}^{-1} \chi \), the inhomogeneous dyadic blocks \( \Delta_j \) are defined as follows:

\[
\text{if } j = -1, \quad \Delta_j f = \Delta_{-1} f = \int_{\mathbb{R}^d} \tilde{h}(y) f(x-y) dy,
\]

\[
\text{if } j \geq 0, \quad \Delta_j f = 2^j \int_{\mathbb{R}^d} h(2^j y) f(x-y) dy, \quad \text{ if } j \leq -2, \quad \Delta_j f = 0.
\]

The inhomogeneous low-frequency cut-off operator \( S_j \) is defined by \( S_j f = \sum_{j' \leq j-1} \Delta_{j'} f \).

**Definition 2.1.** Let \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \). The inhomogeneous Besov space \( \mathcal{B}_{p,r}^s(\mathbb{R}^d) \) consists of all the distributions \( u \) in \( \mathcal{S}'(\mathbb{R}^d) \) such that

\[
\| u \|_{\mathcal{B}_{p,r}^s} \overset{\text{def}}{=} \left\| \left( 2^j \| \Delta_j u \|_{L^r} \right)_{j \geq s} \right\|_{\ell^p} < \infty.
\]

**Remark 2.** Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \) and \( u \in \mathcal{S}'(\mathbb{R}^d) \). Then there exists a positive constant \( C \) such that \( u \) belongs to \( \mathcal{B}_{p,r}^s(\mathbb{R}^d) \) if and only if there exists \( \{c_{j,r}\}_{j \geq 1} \) such that \( c_{j,r} \geq 0, \| c_{j,r} \|_{\ell^p} = 1 \) and

\[
\| \Delta_j u \|_{L^r} \leq C c_{j,r} 2^{-js} \| u \|_{\mathcal{B}_{p,r}^s}, \quad \forall j \geq -1.
\]

If \( r = 1 \), we denote by \( d_j \overset{\text{def}}{=} c_{j,1} \).

We also need to use Chemin-Lerner type Besov spaces introduced in (see [4]).

**Definition 2.2.** Let \( s \in \mathbb{R} \) and \( 0 < T \leq +\infty \). We define

\[
\| u \|_{\mathcal{L}_s^r(\mathcal{B}_{p,r})} \overset{\text{def}}{=} \left( \sum_{j \geq 1} 2^{js} \left( \int_0^T \| \Delta_j u(t) \|_{L^r}^r dt \right)^{r/\sigma} \right)^{1/r}
\]

for \( p \in [1, \infty], r, \sigma \in [1, \infty) \), and with the standard modification for \( r = \infty \) or \( \sigma = \infty \).

**Remark 3.** It is easy to observe that for \( 0 < s_1 < s_2, \theta \in [0, 1], p, r, \lambda, \lambda_1, \lambda_2 \in [1, +\infty] \), we have the following interpolation inequality in the Chemin-Lerner space (see [4]):

\[
\| u \|_{\mathcal{L}_s^r(\mathcal{B}_{p,r})} \leq \| u \|_{\mathcal{L}_{s_1}^{r_1}(\mathcal{B}_{p,r_1})}^{\theta} \| u \|_{\mathcal{L}_{s_2}^{r_2}(\mathcal{B}_{p,r_2})}^{1-\theta}
\]

with \( 1/\lambda = \theta/\lambda_1 + (1-\theta)/\lambda_2 \) and \( s = \theta s_1 + (1-\theta) s_2 \).

Let us emphasize that, according to the Minkowski inequality, we have

\[
\| f \|_{\mathcal{L}_s^r(\mathcal{B}_{p,r})} \leq \| f \|_{\mathcal{L}_{s_1}^{r_1}(\mathcal{B}_{p,r_1})} \quad \text{if } \lambda \leq r, \quad \| f \|_{\mathcal{L}_s^r(\mathcal{B}_{p,r})} \geq \| f \|_{\mathcal{L}_{s_2}^{r_2}(\mathcal{B}_{p,r_2})} \quad \text{if } \lambda \geq r.
\]

The following Bernstein’s lemma will be repeatedly used throughout this paper.
Lemma 2.3. Let \( B \) be a ball and \( C \) a ring of \( \mathbb{R}^d \). A constant \( C \) exists so that for any positive real number \( \lambda \), any non-negative integer \( k \), any smooth homogeneous function \( \sigma \) of degree \( m \), and any couple of real numbers \( (a,b) \) with \( 1 \leq a \leq b \), there hold

\[
\text{Supp } \tilde{u} \subset \lambda B \Rightarrow \sup_{|x|=k} \| \partial^a u \|_{L^p} \leq C^{k+1} \lambda^{k+d(1/a-1/b)} \| u \|_{L^p},
\]

\[
\text{Supp } \tilde{u} \subset \lambda C \Rightarrow \| \partial^a u \|_{L^p} \leq \sup_{|x|=k} \| \partial^a u \|_{L^p} \leq C^{k+1} \lambda^k \| u \|_{L^p},
\]

\[
\text{Supp } \tilde{u} \subset \lambda C \Rightarrow \| \sigma(D)u \|_{L^p} \leq C_{\sigma,m} \lambda^{m+d(1/a-1/b)} \| u \|_{L^p}.
\]

Lemma 2.4. (see \cite{18}) Let \( s > 0 \), \( 1 \leq p, r \leq \infty \), \( f, g \in L^\infty \cap B^s_{p,r} (\mathbb{R}^d) \), then

\[
\| fg \|_{B^s_{p,r}} \leq C \left( \| f \|_{B^s_{p,r}} \| g \|_{L^\infty} + \| g \|_{B^s_{p,r}} \| f \|_{L^\infty} \right).
\]

The action of smooth functions on the space \( B^s_{p,r} (\mathbb{R}^d) \) can be stated as follows:

Lemma 2.5. (see \cite{14}) Let \( I \) be an open interval of \( \mathbb{R} \) and \( F : I \to \mathbb{R} \). Let \( s > 0 \) and \( \sigma \) be the smallest integer such that \( \sigma \geq s \), and \((p,r) \in [1, \infty]^2 \). Assume that \( F(0) = 0 \) and that \( F'' \) belongs to \( W^{s,\infty} (I; \mathbb{R}) \). Let \( u, v \in B^s_{p,r} (\mathbb{R}^d) \cap L^\infty (\mathbb{R}^d) \) have values in \( I \subset I \). There exists a constant \( C = C(s, I, J, N) \) such that

\[
\| F(u) \|_{B^s_{p,r}} \leq C (1 + \| u \|_{L^\infty})^{s'} \| F'' \|_{W^{s,\infty}(I)} \| u \|_{B^s_{p,r}}
\]

and

\[
\| F \circ u - F \circ v \|_{B^s_{p,r}} \leq C (1 + \| u \|_{L^\infty} + \| v \|_{L^\infty})^{s'} \| F'' \|_{W^{s,\infty}(I)}
\]

\[
\times (\| u - v \|_{B^s_{p,r}} \sup_{\tau \in [0,1]} \| v + \tau (u - v) \|_{L^\infty} + \| u - v \|_{L^\infty} \sup_{\tau \in [0,1]} \| v + \tau (u - v) \|_{B^s_{p,r}}).
\]

We shall also use the following commutator’s lemma to prove our theorem:

Lemma 2.6. (Lemma 2.100 in \cite{14}). Let \( \sigma \in \mathbb{R}, 1 \leq r \leq \infty, 1 \leq p \leq p_1 \leq \infty \) and \( v \) be a vector field over \( \mathbb{R}^d \). Assume that

\[
\sigma > -d \min \left\{ \frac{1}{p_1}, \frac{1}{p'} \right\} \quad \text{or} \quad \sigma > -1 - d \min \left\{ \frac{1}{p_1}, \frac{1}{p'} \right\} \quad \text{if} \quad \text{div} v = 0.
\]

Define \( R_j \overset{\text{def}}{=} [v \cdot \nabla, \Delta_j] f \) (or \( R_j \overset{\text{def}}{=} \text{div} [v, \Delta_j] f, \) if \( \text{div} v = 0 \)). There exists a constant \( C \) depending continuously on \( p, p_1, \sigma, \) and \( d \), such that

\[
\| (2^{j \sigma} |R_j|_{L^r}) f \|_{L^p} \leq C \| \nabla v \|_{L^\infty} \| f \|_{B^s_{p_1, \infty}} \quad \text{if} \quad \sigma < 1 + \frac{d}{p_1}.
\]
Further, if \( \sigma > 0 \) (or \( \sigma > -1 \), if \( \text{div} \nu = 0 \)) and \( \frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1} \), then
\[
\|(2^j \| R_j \|_{L^p})_j\|_{L^p} \leq C(\| \nabla \nu \|_{L^\infty} \| f \|_{B_{p_1}^{\sigma}} + \| \nabla \nu \|_{B_{p_1}^{\sigma-1}} \| \nabla f \|_{L^p}).
\]
Especially, when \( \sigma > 1 + \frac{d}{p_1} \) (or \( \sigma = 1 + \frac{d}{p_1} \) and \( r = 1 \)), the above inequality ensures that
\[
\|(2^j \| R_j \|_{L^p})_j\|_{L^p} \leq C \| \nabla \nu \|_{B_{p_1}^{\sigma-1}} \| f \|_{B_{p_1}^{\sigma}}.
\]
In the limit case \( \sigma = -\min(\frac{d}{p_1}, \frac{d'}{p'}) \) (or \( \sigma = -1 - \min(\frac{d}{p_1}, \frac{d'}{p'}) \) if \( \text{div} \nu = 0 \)), we have
\[
\sup_{j \geq -1} 2^j \| R_j \|_{L^p} \leq C \| \nabla \nu \|_{B_{p_1}^{\sigma-1}} \| f \|_{B_{p_1}^{\sigma}}.
\]
We will also use the following Osgood’s Lemma:

**Lemma 2.7.** (see [4]) Let \( g \geq 0 \) be a measurable function, \( \gamma \) be a locally integrable function and \( \Lambda \) be a positive, continuous and nondecreasing function. \( a \) be a positive real number and assume that \( g \) satisfy the inequality
\[
g(t) \leq a + \int_0^t \gamma(s) \Lambda(g(s)) ds.
\]
If \( a > 0 \), then we have
\[
-\Omega(g(t)) + \Omega(a) \leq \int_0^t \gamma(s) ds,
\]
where
\[
\Omega(x) = \int_x^1 \frac{dr}{\Lambda(r)}.
\]
If \( a = 0 \) and \( \Lambda \) satisfies
\[
\int_0^1 \frac{dr}{\Lambda(r)} = +\infty,
\]
then the function \( g \equiv 0 \).

Finally, we give the \( L^p \) estimate for the transport (-diffusion) equation.

**Lemma 2.8.** (see [9]) Let \( u \) be a smooth divergence-free vector field in \( \mathbb{R}^d (d \geq 2) \) and \( \theta \) be a smooth solution of the following transport (-diffusion) equation
\[
\partial_t \theta + u \cdot \nabla \theta + \kappa |D|^\alpha \theta = f, \quad \text{div} \: u = 0, \quad \theta_{|_{t=0}} = \theta_0, \quad \alpha \in (0, 2),
\]
with \( \kappa \geq 0 \). Then for any \( t \in \mathbb{R}^+ \) and \( 1 \leq p \leq \infty \), there holds:
\[
\|\theta\|_{L^p} \leq \|\theta_0\|_{L^p} + \int_0^t \|f\|_{L^p} \: d\tau.
\]
3. The key a priori estimates

In this section, we will use several steps to give the key a priori estimates. Firstly, we present the basic energy estimate for $\theta$ and $u$. Secondly, we give the derivative and improved derivative energy estimates for $\theta$ and $u$ respectively. In the last step, we get $\|u\|_{L^q_{t,x}([0,T^*)}}$ and $\|\theta\|_{L^q_{t,x}([0,T^*)}}$.

3.1. The basic energy estimate for $\theta$ and $u$

In order to explain the index we will be used more essentially in the following, we will generalize our’s argument to a $d$ dimension. More precisely, we get the following proposition:

**Proposition 3.1.** Let $(\theta, u)$ be a smooth enough solution of the system \((1.1)\) on $[0, T^*)$. Assume $\theta_0 \in (L^3 \cap L^2 \cap H^{-s_0})(\mathbb{R}^d)$ and $u_0 \in L^2(\mathbb{R}^d)$. For any $0 < \alpha \leq 1$, $s_0 \in (ad/q + (d + 2)/2 - \alpha(d + 4)/2, 2ad/q - ad - 6\alpha + 3 + 3d/2)$ for some $q \in (2\alpha d/(6\alpha + ad - d - 2), 2), \text{ then there}\ hold\ holds$

$$\|\theta(t)\|_{L^2} \leq E_0 < \alpha/s_0, \text{ for } \forall t < T^*.$$  \hspace{1cm} (3.1)

Especially, If $d = 2$, for any $2/3 < \alpha \leq 1$, $s_0 \in (\alpha, 4\alpha/q - 8\alpha + 6), \alpha/(2\alpha - 1) < q < \min \{2, 4\alpha/(3(3\alpha - 2))\}$, there hold (3.1) and

$$\|u\|_{L^q(\mathbb{R}^d)} + \|\theta\|_{L^q(\mathbb{R}^d)} + \|\nabla u\|_{L^q(\mathbb{R}^d)} + \|\theta\|_{L^q(\mathbb{H}^{n/2})} \leq CE_0.$$  

**Proof.** The key part to prove this proposition is to derive the decay of $\|\theta(t)\|_{L^2}$. We will follow Schonbek’s strategy in \([25]\) (or Proposition 4.1 in \([3]\)) to obtain this decay.

On one hand, we get by taking a standard $L^2$ energy estimates to the $u$ equation of \((1.1)\) that

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} \mu(\theta) d(u) : d(u) dx = \int_{\mathbb{R}^d} \theta u_u dx. \hspace{1cm} (3.2)$$

Thanks to the Hölder inequality, interpolation inequality and Young inequality, we infer from \((1.2)\) and \((3.2)\) that

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \leq \|\theta\|_{L^q} \|u\|_{L^q}^{\alpha/\theta} \hspace{1cm} (3.2)$$

$$\leq \|\theta\|_{L^q} \|u\|_{L^2}^{(\frac{q}{2} - \frac{1}{2})d} \|\nabla u\|_{L^2}^{(\frac{q}{2} - \frac{1}{2})d} \hspace{1cm} (3.2)$$

$$\leq \|\theta\|_{L^q} \frac{4q}{(d+2)(d-2d)} \|u\|_{L^2}^{\frac{2(q+2d)}{(d+2)(d-2d)}} + \frac{1}{2} \|\nabla u\|_{L^2}^2.$$
Applying Osgood’s Lemma 2.7 to the above inequality gives
\[
\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \left( \int_0^t \|\theta(t')\|_{L^4}^{4g/(d+4g-2d)} \, dt' \right)^{(d+4g-2d)/8g}.
\]
which implies
\[
\|u\|_{L^\infty_t(L^2)}^2 + \|\nabla u\|_{L^2_t(L^2)}^2 \lesssim \|u_0\|_{L^2}^2 + \|\theta\|_{L^4_t(L^4)}^{4g/(d+4g-2d)} \cdot
\]
Then by virtue of Lemma 2.8, we have
\[
\|u\|_{L^\infty_t(L^2)}^2 + \|\nabla u\|_{L^2_t(L^2)}^2 \lesssim \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^4_t(L^4)}^{(d+4g-2d)/8g}.
\] (3.3)

On the other hand, we get, by taking $L^2$ inner product of the temperature equation in (1.1) with $\theta$, that
\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{L^2}^2 + \|\theta(t)\|_{H^{2/2}}^2 = 0.
\] (3.4)

Motivated by Schonbek’s strategy for the classical Navier-Stokes system in [25] (see also [3]), we split the phase-space $\mathbb{R}^d$ into two time-dependent regions $S(t) \triangleq \{ \xi \in \mathbb{R}^d, |\xi| \leq g(t) \}$ and $S^c(t)$, the complement of the set $S(t)$ in $\mathbb{R}^d$, for some $g(t) \sim \langle t \rangle^{-1/\alpha}$ to be determined hereafter. A simple computation can help us get from (3.4) that
\[
\frac{d}{dt} \|\theta(t)\|_{L^2}^2 + 2(g(t))^{\alpha} \|\theta(t)\|_{L^2}^2 \leq 2(g(t))^{\alpha} \int_{S(t)} |\widehat{\theta}(t, \xi)|^2 \, d\xi.
\] (3.5)

We have to deal with the term on the right hand side of (3.5). According to Duhamel’s formula, one can deduce from the first equation of (1.1) that
\[
\widehat{\theta}(t, \xi) = e^{-t|\xi|^\alpha} \widehat{\theta}_0(\xi) - \int_0^t e^{-(t-t')|\xi|^\alpha} \xi \cdot \mathcal{F}_\xi (\theta u)(t', \xi) \, dt'.
\] (3.6)

As $\theta_0 \in H^{-s_0}(\mathbb{R}^d)$, we have
\[
\int_{S(t)} |e^{-(t-t')|\xi|^\alpha} \widehat{\theta}_0(\xi)|^2 \, d\xi \lesssim (g(t))^{2s_0} \|\theta_0\|_{H^{-s_0}}^2.
\] (3.7)

On one hand, it follows from Young’s inequality that
\[
\int_{S(t)} \int_0^t e^{-(t-t')|\xi|^\alpha} \xi \cdot \mathcal{F}_\xi (\theta u)(t', \xi) \, dt' \, d\xi \lesssim (g(t))^{d+2} \left( \int_0^t \|\theta u\|_{L^1} \, dt' \right)^2 \lesssim (g(t))^{d+2} \|\theta\|_{L^\infty_t(L^2)}^2 \|u\|_{L^\infty_t(L^1)}^2.
\] (3.8)
On the other hand, we can infer from (3.3) that

\[ \|u\|_{L^q_t(L^r)} \lesssim (t)^{\frac{(d+4)q-2d}{4q}} \|u\|_{L^2_t(L^r)} \]

\[ \lesssim (t)^{\frac{(d+4)q-2d}{4q}} \|u\|_{L^2_t(L^r)} \|\nabla u\|_{L^2_t(L^r)} \lesssim (t)^{\frac{(d+4)q-2d}{4q}} (\|u_0\|_{L^2} + \|\theta_0\|_{L^q}). \quad (3.9) \]

Plugging the estimate (3.9) into (3.8) gives

\[ \int_{S(t)} |\hat{\theta}(t, \xi)|^2 d\xi \lesssim (g(t))^{2s_0} \|\theta_0\|^2_{H^{-s_0}} + (g(t))^{a+d+2}(t)^{\frac{(d+4)q-2d}{4q}} \|u_0\|_{L^2}^2 \|\theta_0\|^2_{L^q}, \quad (3.10) \]

from which and estimate (3.7), we finally infer that

\[ \int_{S(t)} |\hat{\theta}(t, \xi)|^2 d\xi \lesssim (g(t))^{2s_0} \|\theta_0\|^2_{H^{-s_0}} + (g(t))^{a+d+2}(t)^{\frac{(d+4)q-2d}{4q}} \|u_0\|_{L^2}^2 \|\theta_0\|^2_{L^q}. \quad (3.10) \]

Inserting the above estimate (3.10) into (3.5), choosing \( g(t) \sim (t)^{-1/\alpha} \) and using the assumption that \( s_0 \geq ad/q + (d + 2)/2 - a(d + 4)/2 \), we obtain

\[
\frac{d}{dt} \|\theta(t)\|^2_{L^2} + 2(g(t))^\alpha \|\theta(t)\|^2_{L^2} \\
\leq (g(t))^{a+2s_0} \|\theta_0\|^2_{H^{-s_0}} + (g(t))^{a+d+2}(t)^{\frac{(d+4)q-2d}{4q}} \|u_0\|_{L^2}^2 \|\theta_0\|^2_{L^q} \\
\leq (t)^{-\frac{a+d+2}{\alpha}} (t)^{\frac{(d+4)q-2d}{4q}} \mathcal{E}_0^2\cdot \quad (3.11)
\]

with \( \mathcal{E}_0 \triangleq \|\theta_0\|_{H^{-s_0}} + \|\theta_0\|_{L^2} + (\|u_0\|_{L^2} + \|\theta_0\|_{L^q})(1 + \|\theta_0\|_{L^q}) \).

Multiplying by \( \exp(2\int_0^t (g(t'))^\alpha dt') \) on both hand sides of (3.11) leads to

\[
\frac{d}{dt} \left( \|\theta(t)\|^2_{L^2} \exp(2\int_0^t (g(t'))^\alpha dt') \right) \leq C\mathcal{E}_0^2(t)^{-\frac{d+2}{\alpha} + \frac{(d+4)q-2d}{4q}} \exp(2\int_0^t (g(t'))^\alpha dt').
\]

Let us choose \( g(t) = (\beta(t))^{-1/\alpha} \) for \( \beta > d + 2/(2\alpha) - ((d + 4)q - 2d)/2q \) in the above inequality to get

\[
\langle t \rangle^{2\beta} \|\theta(t)\|^2_{L^2} \lesssim \|\theta_0\|^2_{L^2} + \mathcal{E}_0^2(t)^{2\beta - \frac{d+2}{\alpha} + \frac{(d+4)q-2d}{4q}},
\]

which implies for any \( t \in (0, T^*) \)

\[
\|\theta(t)\|_{L^2} \lesssim \mathcal{E}_0(t)^{-\frac{d+2}{\alpha} + \frac{(d+4)q-2d}{4q}}. \quad (3.12)
\]
Combining with estimates (3.2) and (3.12), we get for any $2a \alpha / (6\alpha + ad - d - 2) < q < 2$ that
\[
\|u(t)\|_{L^2} \lesssim \|u_0\|_{L^2} + \|\theta\|_{L^2} \lesssim \|u_0\|_{L^2} + E_0(t) \frac{1 - \frac{d+2}{2\alpha} + \frac{2(d+4)q - 2d}{q}}{2} \lesssim E_0(t) \frac{1 - \frac{d+2}{2\alpha} + \frac{2(d+4)q - 2d}{q}}{2}.
\] (3.13)

Thanks to (3.12), (3.13), we get, by a similar derivation of (3.10), that
\[
\int_{S(t)} |\tilde{\theta}(t, \xi)|^2 d\xi \lesssim E_0^2(t)^{-\frac{2s_0}{\alpha}} + (g(t))^d + 2 \left( \int_0^t \|u(t')\|_{L^2} \|\theta(t')\|_{L^2} dt' \right)^2 
\lesssim E_0^2(t)^{-\frac{2s_0}{\alpha}} + E_0^4(g(t))^d + 2 \left( \int_0^t \langle t' \rangle^{1 - \frac{d+2}{2\alpha} + \frac{2(d+4)q - 2d}{q}} dt' \right)^2 
\lesssim E_0^2(t)^{-\frac{2s_0}{\alpha}} + E_0^4(t)^{-\frac{d+2}{2\alpha} + \frac{2(d+4)q - 2d}{q}} 
\lesssim E_0^2(t)^{-\frac{2s_0}{\alpha}} + E_0^4(t)^{-\frac{3(d+2)}{2\alpha} + \frac{2(d+4)(q - 4d)}{q}} 
\lesssim E_0^2(t)^{-\frac{2s_0}{\alpha}} (1 + E_0^3(t))^{-\frac{2s_0}{\alpha}}.
\] (3.14)

in which we have let $s_0 \leq \frac{2ad}{q - ad} - 6\alpha + 3 + 3d/2$.

Inserting the estimate (3.14) into (3.5) gives
\[
\frac{d}{dt} \|\theta(t)\|_{L^2}^2 + 2(g(t))^\alpha \|\theta(t)\|_{L^2}^2 \lesssim E_0^2(t)^{2\beta} \langle t \rangle^{-\frac{2s_0}{\alpha}} \lesssim E_0^2(t)^{2\beta} \langle t \rangle^{-\frac{2s_0}{\alpha}}.
\]

Thus taking $g(t) = (\beta(t))^{-1/\alpha}$ for $\beta > s_0/\alpha$ in the above inequality, we get, by using a similar derivation of (3.12), that
\[
\langle t \rangle^\beta \|\theta(t)\|_{L^2}^2 \lesssim \|\theta_0\|_{L^2}^2 + E_0^2(t)^{2\beta - \frac{2s_0}{\alpha}}.
\]

Divided this inequality by $\langle t \rangle^{2\beta}$ leads to
\[
\|\theta(t)\|_{L^2} \lesssim E_0 \langle t \rangle^{-\frac{s_0}{\alpha}}, \quad \text{for } \forall t < T^*.
\]

If $d = 2$, for any $2/3 < \alpha \leq 1$, $\alpha < s_0 < 4\alpha/q - 8\alpha + 6$ for some $q$ satisfies $\alpha/(2\alpha - 1) < q < \min \{2, 4\alpha/(3(3\alpha - 2))\}$, we finally get that
\[
\|\theta\|_{L^2_t(L^2)} \leq CE_0,
\]
from which and (3.4), (3.14) we infer that
\[
\|u\|_{L^\infty_t(L^2)} + \|\theta\|_{L^\infty_t(L^2)} + \|\nabla u\|_{L^2_t(L^2)} + \|\theta\|_{L^2_t(H^{6/2})} \leq \|u_0\|_{L^2} + C\|\theta\|_{L^2_t(L^2)} \leq CE_0.
\]

This completes the proof of Proposition 3.1. \square
3.2. The derivative energy estimates for \( \theta \) and \( u \)

In this subsection, we will follow the method in \([3]\) to get the derivative energy estimates for \( \theta \) and \( u \) in \( \mathbb{R}^2 \). The first important estimate is to get the energy inequality of \((1.1)\). In fact, when \( d = 2 \), under the assumptions of Theorem \((1.1)\) we can deduce from Proposition \((3.1)\) that

\[
\| \theta(t) \|_{L^2} \leq C E_0(t)^{-s_0/\alpha}, \tag{3.15}
\]

\[
\| u(t) \|_{L^2}^2 + \| \theta(t) \|_{L^2}^2 + 2 \int_0^t (\| \nabla u \|_{L^2}^2 + \| \theta \|_{H^{n/2}}^2) \, d\tau \leq E_0^2, \tag{3.16}
\]

where \( E_0 \) is given in \((1.9)\).

In the following, we continue to prove the \( H^1 \) energy estimates for \( u \). More precisely, we obtain the following proposition:

**Proposition 3.2.** Let \((\theta, u)\) be a smooth enough solution of \((1.1)\) on \([0, T^*)\). Then under the assumptions of Theorem \((1.1)\) for any \( 4/\alpha < p < 1/C \| \mu(\cdot) - 1 \|_{L^\infty} \) and any \( t < T^* \), we have

\[
\| \nabla u \|_{L^p_t(L^2)}^2 + \| \partial_t u \|_{L^p_t(L^2)}^2 \leq C (\| \nabla u_0 \|_{L^2}^2 + \| \theta \|_{L^2_t(L^2)}^2) \times \exp \left\{ C \int_0^t (1 + \| u \|_{L^2}^2) \| \nabla u \|_{L^2}^2 + \| D^\alpha \theta \|_{L^2_t(L^2)}^2 \right\} \right\}. \tag{3.17}
\]

**Proof.** This proposition can be obtained similarly to Lemma 4.3 in \([3]\) and we only need to make fully use of the renormalized equation \( \partial_t \mu(\theta) + u \cdot \nabla \mu(\theta) + \kappa |D|^\alpha \mu(\theta) = 0 \). For simplicity, we omit the details here.

From \((3.15)\), we can easily deduce that \( \| \theta \|_{L^2_t(L^2)}^2 \leq E_0^2 \), thus, combining with \((3.16)\), in order to get the \( H^1 \) energy estimates for \( u \), we have to estimate \( \| \theta \|_{L^2_t(H^1)}^2 \). In fact, we get the following proposition:

**Proposition 3.3.** Let \((\theta, u)\) be a smooth enough solution of \((1.1)\) on \([0, T^*)\). Then under the assumptions of Theorem \((1.1)\) for any \( t < T^* \), we have

\[
\| \theta \|_{L^p_t(H^{n/2})}^2 + \| \theta \|_{L^p_t(H^1)}^2 \leq \| \theta_0 \|_{H^{n/2}}^2 + C \| \nabla u \|_{L^2_t(L^2)}^2 \left( 1 + \| \theta_0 \|_{L^\infty}^2 + \| \theta_0 \|_{L^2}^2 + \| \theta_0 \|_{L^\infty} \ln(e + \| \theta_0 \|_{B_{p,\infty}^\alpha} + \| \theta_0 \|_{L^\infty} \| \nabla u \|_{L^2_t(L^p)}) \right). \tag{3.18}
\]

**Proof.** We first deduce from the first equation of \((1.1)\) and the following commutator’s estimate which the proof can be founded in \([14]\)

\[
\| [\Delta_j, u \cdot \nabla] \theta \|_{L^p} \leq C \| \nabla u \|_{L^p} \| \theta \|_{B_{p,\infty}^\alpha}, \quad \forall \ 1 \leq p \leq \infty \text{ and } \forall \ j \geq -1,
\]
that
\[ \|\Delta_j \theta(t)\|_{L^p} \leq \|\Delta_j \theta(0)\|_{L^p} e^{-c\alpha^2} + C \int_0^t e^{-c(t-t')2^{j\alpha}} \|\nabla u\|_{L^p} \|\theta\|_{L^\infty} dt', \] (3.19)
from which and Lemma 2.8 we have
\[ \|\Delta_j \theta\|_{L^\infty(L^p)} + 2^{j\alpha/2} \|\Delta_j \theta\|_{L^2(L^p)} \leq C(\|\Delta_j \theta(0)\|_{L^p} + 2^{-j\alpha/2} \|\theta\|_{L^\infty} \|\nabla u\|_{L^2(L^p)}). \]

Multiplying by $2^{j\alpha/2}$ on both hands of the above inequality and then taking supremum about $j$ that
\[ \|\theta\|_{L^p(B_{p,\infty}^{j\alpha/2})} + \|\theta\|_{L^2(B_{p,\infty}^{j\alpha/2})} \leq \|\theta(0)\|_{B_{p,\infty}^{j\alpha/2}} + \|\theta\|_{L^\infty} \|\nabla u\|_{L^2(L^p)}. \] (3.20)

In the following, applying $\Delta_j$ to the first equation of (1.1) and then taking the $L^2$ inner product of the resulting equation with $\Delta_j \theta$ that
\[ \frac{1}{2} \frac{d}{dt} \|\Delta_j \theta\|_{L^2}^2 + \int_{\mathbb{R}^2} \Delta_j (|D|^{\alpha} \theta) \cdot \Delta_j \theta \, dx = -\int_{\mathbb{R}^2} \Delta_j (u \cdot \nabla \theta) \cdot \Delta_j \theta \, dx. \] (3.21)
The Bony's decomposition will be applied to estimate the term on the right hand side of (3.21) that
\[ u \cdot \nabla \theta = \hat{T}_u \nabla \theta + \hat{\nabla}_\theta u + \hat{R}(u, \nabla \theta). \] (3.22)

By Lemma 2.3 we have
\[ \|\Delta_j (\hat{T}_u \nabla \theta)\|_{L^2} \lesssim \sum_{|j-j'| \leq 4} \|\hat{S}_{j-1} \nabla \theta\|_{L^\infty} \|\hat{\Delta}_{j'} u\|_{L^2} \lesssim \sum_{|j-j'| \leq 4} \|\hat{S}_{j-1} \theta\|_{L^\infty} \|\hat{\Delta}_j \nabla u\|_{L^2} \lesssim c_j(t) \|\theta\|_{L^\infty} \|\nabla u\|_{L^2}. \] (3.23)

Similarly, using the fact that $\text{div} \ u = 0$ implies
\[ \|\Delta_j (\hat{R}(u, \nabla \theta))\|_{L^2} \lesssim 2^j \|\Delta_j (\hat{R}(u, \theta))\|_{L^2} \lesssim 2^j \sum_{j' \geq j-3} \|\hat{\Delta}_{j'} u\|_{L^2} \|\hat{\Delta}_j \theta\|_{L^\infty} \lesssim 2^j \sum_{j' \geq j-3} c_j(t) 2^{-j'\alpha} \|\nabla u\|_{L^2} \|\theta\|_{L^\infty} \lesssim c_j(t) \|\nabla u\|_{L^2} \|\theta\|_{L^\infty}. \] (3.24)

The last term in (3.22) will be estimated through the following commutator’s argument:
\[ \int_{\mathbb{R}^2} \Delta_j (\hat{T}_u \nabla \theta) \cdot \Delta_j \theta \, dx = \int_{\mathbb{R}^2} \sum_{|j-j'| \leq 5} (\hat{S}_{j-1} u - \hat{S}_{j-1} u) \Delta_j \Delta_j \nabla \theta \, dx \]
\[ + \int_{\mathbb{R}^2} \sum_{|j-j'| \leq 5} [\Delta_j, \hat{S}_{j-1} u] \Delta_j \nabla \theta \cdot \Delta_j \theta \, dx \]
\[ \triangleq I_1 + I_2. \] (3.25)
Thanks to Lemma 2.3 and the commutator’s estimate in [4], we obtain

\[ I_1 \lesssim 2^{-j\alpha} c_j^2(t) \|
abla u\|_{L^2} \|	heta\|_{H^s} \|	heta\|_{L^\infty}. \]

\[ I_2 \lesssim 2^{-j} \sum_{|j-j'| \leq 5} \|S_{j-j'} \nabla u\|_{L^2} \|\Delta_j \nabla \theta\|_{L^\infty} \|\Delta_j \theta\|_{L^2} \lesssim 2^{-j\alpha} c_j^2(t) \|
abla u\|_{L^2} \|	heta\|_{H^s} \|	heta\|_{B^0_{p,2}}. \quad (3.26) \]

Inserting the estimates about (3.23), (3.24), (3.26) into (3.21) and summing up about \( j \) give

\[ \|\theta\|_{L^p(H^{s/2})}^2 + \|\theta\|_{L^2(H^s)}^2 \leq \|\theta_0\|_{H^{s/2}}^2 + \frac{t}{0} \|\theta\|_{H^s} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\theta\|_{B^0_{p,2}} \|\nabla \theta\|_{L^2} \] \[ \leq \|\theta_0\|_{H^{s/2}}^2 + \varepsilon \|\theta\|_{L^2(H^s)}^2 + C \left( \|\theta\|_{L^p(H^s)}^2 + \|\theta\|_{L^p(B^0_{p,2})}^2 \right) \|\nabla u\|_{L^2}). \]

Choosing \( \varepsilon \) small enough in the above inequality implies

\[ \|\theta\|_{L^p(H^{s/2})}^2 + \|\theta\|_{L^2(H^s)}^2 \leq \|\theta_0\|_{H^{s/2}}^2 + C \left( \|\theta\|_{L^p(H^s)}^2 + \|\theta\|_{L^p(B^0_{p,2})}^2 \right) \|\nabla u\|_{L^2}). \quad (3.27) \]

Note that for any positive integer \( N \) and \( p > 4/\alpha \), we have

\[ \|\theta\|_{L^p(B^0_{p,2})} \leq \|\theta\|_{L^p(L^2)} + \left( \sum_{1 \leq j \leq N} \|\Delta_j \theta\|_{L^p(L^\infty)}^2 \right)^{1/2} \left( \sum_{j > N} \|\Delta_j \theta\|_{L^p(L^\infty)}^2 \right)^{1/2} \]

\[ \leq \|\theta_0\|_{L^2} + \|\theta_0\|_{L^\infty} N^{-1/2} + 2^{(2/p - \alpha/2)N} \|\theta\|_{L^p(B^0_{p,2})}^2. \]

Choosing \( N \) in the above inequality such that \( 2^{(a/2 - 2/p)N} \approx \|\theta\|_{L^p(B^0_{p,2})} \), we have

\[ \|\theta\|_{L^p(B^0_{p,2})} \leq C \left( 1 + \|\theta_0\|_{L^2} + \|\theta_0\|_{L^\infty} \ln^4(e + \|\theta\|_{L^p(B^0_{p,2})}) \right). \quad (3.28) \]

Taking estimate (3.20) into the above estimate (3.28) and then inserting the resulting inequality into (3.27) give (3.18).

Inserting the estimate about \( \|\theta\|_{L^2(H^s)}^2 \) in Proposition 3.3 into (3.17), one can deduce from (3.16) and estimate \( \|\theta\|_{L^2(H^s)}^2 \lesssim E_0^2 \) that

\[ \|\nabla u\|_{L^p(L^2)}^2 + \|\partial_t u\|_{L^2(L^2)}^2 \leq C(\|\nabla u_0\|_{L^2}^2 + \|\theta\|_{L^2(L^2)}^2) \exp \left\{ C(1 + \|u\|_{L^2(L^2)}^2) \|\nabla u\|_{L^2(L^2)} \right\} \]

\[ \times \exp \left\{ C\|\theta_0\|_{H^{s/2}}^2 + C\|\nabla u\|_{L^2(L^2)}^2 \left( 1 + \|\theta_0\|_{L^\infty}^2 + \|\theta_0\|_{L^2}^2 \right) \right. \]

\[ + \|\theta_0\|_{L^\infty} \ln(e + \|\theta_0\|_{B^0_{p,2}} + \|\theta_0\|_{L^\infty} \|\nabla u\|_{L^2(L^p)}) \right) \}

\[ \leq C(\|\nabla u_0\|_{L^2}^2 + E_0^2) \exp \left\{ C(1 + E_0^2) E_0^2 \right\} \]

\[ \times \exp \left\{ C\|\theta_0\|_{H^{s/2}}^2 + CE_0^2 \left( 1 + \|\theta_0\|_{L^\infty}^2 + \|\theta_0\|_{L^2}^2 \right) \right\} \]

\[ \times \left( e + \|\theta_0\|_{B^0_{p,2}} + \|\theta_0\|_{L^\infty} \|\nabla u\|_{L^2(L^p)} \right) \exp \left\{ C\|\theta_0\|_{L^2}^2 \|\nabla u\|_{L^2(L^2)} \right\} \]

\[ \triangle CM_0 \left( e + \|\theta_0\|_{B^0_{p,2}} + \|\theta_0\|_{L^\infty} \|\nabla u\|_{L^2(L^p)} \right) \exp \left\{ C\|\theta_0\|_{L^2}^2 \|\nabla u\|_{L^2(L^2)} \right\}. \quad (3.29) \]
In the following, we have to estimate \( \| \nabla u \|_{L_t^2(L^p)} \).

Thanks to the fact
\[
\nabla u = \nabla (-\Delta)^{-1} P \text{div}(\mu(\theta) - 1) \nabla u) - \nabla (-\Delta)^{-1} P \text{div}(\mu(\theta) \nabla u),
\]
and the interpolation inequality
\[
\| f \|_{L^r(R^2)} \leq C \sqrt{r} \| f \|_{L^2(R^2)}^{2/r} \| \nabla f \|_{L^2(R^2)}^{1-2/r}, \quad 2 \leq r < \infty,
\]
we can deduce for any \( p \in [2, \infty) \) that
\[
\| \nabla u \|_{L^p} \leq C_0 \sqrt{p} \| \mu(\theta) - 1 \|_{L^\infty} \| \nabla u \|_{L^p} + C \sqrt{p} \| \nabla u \|_{L^2}^{2/p} \| \text{div}(\mu(\theta) \nabla u) \|_{L^1}^{1-2/p}
\]
with \( C_0 > 0 \) being a universal constant.

Using the second equation of (1.1) and taking \( \varepsilon_0 \) sufficiently small in (1.5), we obtain for \( 2 \leq p \leq 1/(2C_0 \| \mu(\theta) - 1 \|_{L^\infty}) \) that
\[
\| \nabla u \|_{L^p} \leq C \sqrt{p} \| \nabla u \|_{L^2}^{2/p} \| \partial_t u + u \cdot \nabla u - \theta e_2 \|_{L^2}^{1-2/p}
\leq C \sqrt{p} \| \nabla u \|_{L^2}^{2/p} \left( \| \partial_t u \|_{L^2}^{1-2/p} + \| u \|_{L^4}^{1-2/p} \| \nabla u \|_{L^4}^{1-2/p} + \| \theta \|_{L^2}^{1-2/p} \right).
\]

Especially, taking \( p = 4 \) in the above inequality (3.30), one has
\[
\| \nabla u \|_{L_t^2(L^4)} \leq C \left( \| \nabla u \|_{L_t^2(L^2)}^{1/2} \| \partial_t u \|_{L_t^2(L^2)}^{1/2}
+ \| u \|_{L_t^2(L^2)}^{1/2} \| \nabla u \|_{L_t^2(L^2)}^{1/2} \| \nabla u \|_{L_t^2(L^2)}^{1/2} + \| \nabla u \|_{L_t^\infty(L^2)}^{1/2} \right),
\]
from which and (3.30), we infer
\[
\| \nabla u \|_{L_t^2(L^p)} \leq C \sqrt{p} \| \nabla u \|_{L_t^2(L^2)}^{2/p} \left( \| \partial_t u \|_{L_t^2(L^2)}^{1-2/p} + \| u \|_{L_t^2(L^2)}^{1-2/p} \| \nabla u \|_{L_t^2(L^2)}^{1-2/p} + \| \theta \|_{L_t^2(L^2)}^{1-2/p} \right)
\leq C \sqrt{p} E_0 (1 + E_0)^{1-2/p} \left( 1 + \| \partial_t u \|_{L_t^2(L^2)} + \| \nabla u \|_{L_t^\infty(L^2)} \right).
\]

Substituting the above inequality into (3.17) gives
\[
\| \nabla u \|_{L_t^p(L^\infty)}^2 + \| \partial_t u \|_{L_t^2(L^2)}^2 \leq CM_0 \left\{ e + \| \theta_0 \|_{B_{p/2}^{1/2}}^2 + C \sqrt{p} E_0 (1 + E_0)^{1-2/p} \| \theta_0 \|_{L^\infty} (1 + \| \partial_t u \|_{L_t^2(L^2)} + \| \nabla u \|_{L_t^\infty(L^2)}) \right\}^{C \| \theta_0 \|_{L_t^\infty} \| \nabla u \|_{L_t^2(L^2)}^2},
\]
where \( M_0 \) is defined in (3.29).

To close the \( H^1 \) energy estimate about \( u \), we also follow the method in [3] to prove the non-concentration of energy in the time variable. More precisely, we need the following lemma:
Lemma 3.4. (see [3]) Let \((\theta, u)\) be a smooth enough solution of (1.1) on \([0, T^*)\). Then under the assumptions of Theorem 1.1 for any \(t < T^*\), we have

\[
\|u\|_{L_t^\infty(L^2)} \leq CE_0(1 + E_0)
\]

for \(E_0\) given by (1.9). If moreover, there holds (1.5), then for any small enough constant \(v > 0\), there exists \(\lambda > 0\) such that if \(0 \leq t_1 < t_2 < T^*\) and \(t_2 - t_1 \leq \lambda\), there holds

\[
\|\nabla u\|_{L_t^2([t_1, t_2];L^2)} \leq v.
\]

With estimate (3.33) and Lemma 3.4 in hand, we can also use the same boot-strap argument to get the global in time estimate of \(\|\nabla u\|_{L_t^\infty(L^2)}\) and \(\|\nabla u\|_{L_t^2(L^p)}\). The whole process can be obtained similarly to Proposition 4.2 in [3] without any difficulties. Here, we omit the details for convenience. Yet, we still use the same notations as in [3] in our further estimates. In fact, we obtain the following proposition:

Proposition 3.5. Let \((\theta, u)\) be a smooth solution of (1.1) on \([0, T^*)\). Then under the assumptions of Theorem 1.1 and for some sufficiently small \(\varepsilon\), for any \(t < T^*\), we have

\[
\|\theta\|_{L_t^{\infty}(H^{\alpha/2})} + \ln(1 + \|\theta\|_{L_t^{\infty}(B^{1/2}_{p,\infty})}^2) + \|\theta\|_{L_t^{2}((H^\alpha)}^2
\]

\[
\leq C^2 + C\|\theta_0\|_{L_t^{\infty}}^2 E_0^3 (A + B) + \|\theta_0\|_{L_t^{2}}^2 + \ln(1 + \|\theta_0\|_{L_t^{2}}^2) \leq G_1,
\]

(3.34)

\[
\|\nabla u\|_{L_t^{\infty}(L^2)} + \|\partial_t u\|_{L_t^{2}}^2 \leq C(\|
abla u_0\|_{L^2}^2 + E_0^3) \exp(C E_0^3 (1 + E_0^3) + G_1) \leq G_2,
\]

(3.35)

\[
\|\nabla u\|_{L_t^{2}(L^p)} \leq C\sqrt{\tau} E_0 (1 + E_0)^{1 - 2\alpha} (1 + \sqrt{G_2}) \leq G_3,
\]

(3.36)

where \(A \equiv CE_0^3 (1 + E_0^3 + \|\theta_0\|_{L_t^{\infty}}^2), \ B \equiv A + E_0^3 \|\theta_0\|_{L_t^{\infty}}^2 \ln(1 + CE_0 (1 + E_0) \|\theta_0\|_{L_t^{\infty}})\).

3.3. The improved derivative energy estimates for \(\theta\) and \(u\)

With the \(\dot{H}^1\) energy estimates for \(u\) and \(\dot{H}^{\alpha/2}\) for \(\theta\) in hand in the last subsection, the most important thing in what follows is to get \(\|u\|_{L_t^{1} (B^{1/2}_{p,\infty})}\) and \(\|\theta\|_{L_t^{1} (B^{1/2}_{p,\infty})}\). More precisely, we get the following proposition:

Proposition 3.6. Let \((\theta, u)\) be a smooth enough solution of (1.1) on \([0, T^*)\). Then under the assumptions of Theorem 1.1 and for any \(t < T^*\), if we assume moreover that \(u_0 \in \dot{B}^{-1}_{\infty, 1}\), there holds

\[
\|u\|_{L_t^{1} (B^{1/2}_{p,\infty})} \leq \|u_0\|_{B^{1/2}_{\infty, 1}} + t^{1/2} \|\theta_0\|_{L_t^{1}}^{1/2} \|\theta_0\|_{L_t^{\infty}}^{1/2} + E_0^2 (1 + E_0) + G_3 (t^{1/2} \|\theta_0\|_{L_t^{1}}) \frac{\ln (1 + E_0) \|\theta_0\|_{L_t^{1}}}{\ln (1 + E_0)} \leq G_4 \frac{E_0^{1/2}}{\ln (1 + E_0)}
\]

(3.37)
where $G_4 \triangleq \| \theta_0 \|_{B^{q/2}_{p,\infty}} + C \| \theta_0 \|_{L^\infty} G_3$.

**Proof.** Using (3.36) we can obtain similarly to the first estimate in Proposition 3.3 that

$$
\| \theta \|_{L^\infty_t(B^{q/2}_{p,\infty})} + \| \theta \|_{L^2_t(B^{0}_{p,\infty})} \leq \| \theta_0 \|_{B^{-1}_{p,\infty}} + \| \theta \|_{L^\infty_t L^2} \leq \| \theta_0 \|_{B^{0}_{p,\infty}} + C \| \theta_0 \|_{L^\infty} G_3 \triangleq G_4. \tag{3.38}
$$

From equation (1.1), one can easily deduce that

$$
\Delta u(t) = e^{t\Delta} \Delta u_0 + \int_0^t e^{-(t-t')\Delta} \Delta \{ \text{div}(2(\mu(\theta) - 1)d(u)) - u \cdot \nabla u + \theta e_2 \} (t') \, dt'. \tag{3.39}
$$

A standard energy estimate gives

$$
\| u \|_{L^1_t(B^{1/2}_{p,\infty})} \lesssim \| u_0 \|_{B^{-1/2}_{p,\infty}} + \| u \cdot \nabla u \|_{L^1_t(B^{1/2}_{p,\infty})} + \| (\mu(\theta) - 1) \nabla u \|_{L^1_t(B^{0}_{p,\infty})} + \| \theta \|_{L^1_t(B^{0}_{p,\infty})}
$$

$$
\lesssim \| u_0 \|_{B^{-1/2}_{p,\infty}} + \epsilon \| u \|_{L^1_t(B^{0}_{p,\infty})} + \| \nabla u \|_{L^2_t(L^p)} + \| u \|_{L^\infty_t(L^2)} \| \nabla u \|_{L^2_t(L^p)} + \| \mu(\theta) - 1 \|_{L^\infty} \| u \|_{L^1_t(B^{0}_{p,\infty})} + \| \theta \|_{L^1_t(B^{0}_{p,\infty})}
$$

$$
\lesssim \| u_0 \|_{B^{-1/2}_{p,\infty}} + \epsilon \| u \|_{L^1_t(B^{0}_{p,\infty})} + \| \nabla u \|_{L^2_t(L^p)} + \| u \|_{L^\infty_t(L^2)} \| \nabla u \|_{L^2_t(L^p)} + \| \mu(\theta) - 1 \|_{L^\infty} \| u \|_{L^1_t(B^{0}_{p,\infty})} + \| \theta \|_{L^1_t(B^{0}_{p,\infty})}
$$

$$
\lesssim \| u_0 \|_{B^{-1/2}_{p,\infty}} + \epsilon \| u \|_{L^1_t(B^{0}_{p,\infty})} + \| \nabla u \|_{L^2_t(L^p)} + \| u \|_{L^\infty_t(L^2)} \| \nabla u \|_{L^2_t(L^p)} + \| \mu(\theta) - 1 \|_{L^\infty} \| u \|_{L^1_t(B^{0}_{p,\infty})} + \| \theta \|_{L^1_t(B^{0}_{p,\infty})}
$$

where we have used the following two estimates which can be proved similarly as in \[3\]

$$
\| u \cdot \nabla u \|_{B^{1/2}_{p,\infty}} \lesssim \| \nabla u \|_{L^2} + \| u \|_{L^{1/2}} \| \nabla u \|^{1/2}_{L^2} \| u \|^{1/2}_{B^{1/2}_{p,\infty}},
$$

$$
\| (\mu(\theta) - 1) \nabla u \|_{B^{0}_{p,\infty}} \lesssim \| \nabla u \|_{L^p} \| \theta \|_{B^{1/2}_{p,\infty}} + \| (\mu(\theta) - 1) \|_{L^{\infty}} \| u \|_{B^{0}_{p,\infty}}.
$$

In the above inequality (3.40), choosing $\epsilon$ small enough and $L, N$ such that

$$
\| \theta_0 \|_{L^2} 2^{(2/q-1)L} \sim 2^{-L} \| \theta_0 \|_{L^\infty}, \quad t^{1/2} \| \theta_0 \|_{L^2} 2^{(1+2/p)N} \sim \| \theta \|_{L^2_t(B^{q/2}_{p,\infty})} 2^{-(\alpha-4/p)N},
$$

we can obtain

$$
\| u \|_{L^1_t(B^{1/2}_{p,\infty})} \lesssim \| u_0 \|_{B^{-1/2}_{p,\infty}} + \| \nabla u \|_{L^2_t(L^2)} + \| u \|_{L^\infty_t(L^2)} \| \nabla u \|_{L^2_t(L^2)}
$$

$$
+ \| \nabla u \|_{L^2_t(L^p)} (t^{1/2} \| \theta_0 \|_{L^2}) \| \theta \|_{L^2_t(B^{q/2}_{p,\infty})} + t \| \theta_0 \|_{L^2_t} \| \theta_0 \|_{L^\infty}^{1/2},
$$

thus, using (3.16), (3.36), (3.38), we have

$$
\| u \|_{L^1_t(B^{1/2}_{p,\infty})} \lesssim \| u_0 \|_{B^{-1/2}_{p,\infty}} + E_0^2 (1 + E_0) + G_3 (t^{1/2} \| \theta_0 \|_{L^2}) \| \theta \|_{L^2_t(B^{q/2}_{p,\infty})} + t \| \theta_0 \|_{L^2_t} \| \theta_0 \|_{L^\infty}^{1-\alpha/2}. \tag{3.41}
$$

$\square$
With estimate $\|u\|_{L_t^1(B_{r0}^{1/2})}$ in hand, we can use the following commutator’s estimate

$$\|[\Delta_j, u \cdot \nabla] \theta\|_{L^p} \lesssim 2^{-j/2} \|\nabla u\|_{L^\infty} \|\theta\|_{B_{r0}^{3/2}}$$

which the proof can be easily obtained by using Bony’s decomposition that

$$\|\theta\|_{L_t^\infty(B_{r0}^{3/2})} + \|\theta\|_{L_t^1(B_{r0}^{3/2})} \leq \|\theta_0\|_{B_{r0}^{3/2}} \exp(CH(t)).$$

From estimate (3.42), we can obtain the following corollary about Theorem 1.1 and for any $t < T^*$, there holds

$$\|u\|_{L_t^2(B_{2r0}^{3/2})} \lesssim \|u_0\|_{H^1} + E_0 \left( G_1^{1/2} (1 + E_0^{1/2} G_2^{1/4} + E_0^{1/2} (1 + G_2)^{1/4}) + \exp(CG_1) \right) \equiv G_5.$$

**Proof.** From equation (3.39), we can get by a similar derivation of (3.40) that

$$\|u\|_{L_t^2(B_{2r0}^{3/2})} \lesssim \|u_0\|_{B_{2r0}^{3/2}} + \|\theta\|_{L_t^1(B_{2r0}^{3/2})} \|\nabla u\|_{L_t^2(L^4)} \|u\|_{L_t^\infty(L^2)} + \|(\mu(\theta) - 1) \nabla u\|_{L_t^2(B_{2r0}^{3/2})}.$$  (3.43)

By using Bony’s decomposition, para-product estimates and interpolation inequality, we can obtain for any $p > 4/\alpha$ that

$$\|(\mu(\theta) - 1) \nabla u\|_{L_t^2(B_{2r0}^{3/2})} \lesssim \|\mu(\theta) - 1\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^2(B_{2r0}^{3/2})} + \|\theta\|_{L_t^1(B_{r0}^{3/2})} \|\nabla u\|_{L_t^2(B_{r0}^{1-\alpha/2})}$$

$$\lesssim \|\mu(\theta) - 1\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^2(B_{r0}^{3/2})} + \|\theta\|_{L_t^1(B_{r0}^{3/2})} \|\nabla u\|_{L_t^2(B_{r0}^{3/2})}^{a/4/p} \|u\|_{L_t^2(B_{r0}^{3/2})}^{1-a/4/p}$$

$$\lesssim \|\mu(\theta) - 1\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^2(B_{r0}^{3/2})} + \|\theta\|_{L_t^1(B_{r0}^{3/2})} \|\nabla u\|_{L_t^2(B_{r0}^{3/2})}^{a/4/p} \|u\|_{L_t^2(B_{r0}^{3/2})}^{1-a/4/p}$$

Taking (1.5) into consideration in the above estimate, we have

$$\|(\mu(\theta) - 1) \nabla u\|_{L_t^2(B_{2r0}^{3/2})} \lesssim \|\mu(\theta) - 1\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^2(B_{2r0}^{3/2})} + \|\theta\|_{L_t^1(B_{r0}^{3/2})}^{p/(ap-4)} \|\nabla u\|_{L_t^2(L^2)}. \quad (3.44)$$

Substituting (3.44) into (3.43) and choosing $\epsilon$ small enough imply

$$\|u\|_{L_t^2(B_{2r0}^{3/2})} \lesssim \|u_0\|_{B_{2r0}^{3/2}} + \|\theta\|_{L_t^2(L^4)} \|u\|_{L_t^2(L^2)} + \|\theta\|_{L_t^1(B_{r0}^{3/2})}^{p/(ap-4)} \|\nabla u\|_{L_t^2(L^2)}. \quad (3.45)$$

On one hand, from estimates (3.31) and (3.35) we have

$$\|\nabla u\|_{L_t^2(L^4)} \lesssim (1 + E_0) E_0^{1/2} (1 + G_2)^{1/2}. \quad (3.46)$$
On the other hand, it’s easy to get from (3.15) that
\[ \|\theta\|_{L_t^4(L^2)} \lesssim E_0. \] (3.47)

Thus, taking estimates (3.46), (3.47) into (3.45) and using (3.16), (3.34), we have
\[ \|u\|_{\tilde{L}_t^2(\dot{B}^{3/2}_{2,\infty})} \lesssim \|u_0\|_{\dot{H}^{1-\sigma}} + E_0 + (1 + E_0)E_0^{1/2}(1 + G_2)^{1/2} + \|\theta\|_{L_t^\infty(\dot{B}^{p/2}_{p,\infty})} E_0 \]
\[ \lesssim \|u_0\|_{\dot{H}^{1-\sigma}} + E_0 + (1 + E_0)E_0^{1/2}(1 + G_2)^{1/2} + \exp \left( \frac{(p/2)(\alpha p - 4)}{1 + G_2} \right) G_1 E_0. \]

Consequently, we complete the proof of this corollary. \[ \square \]

4. Proof of Theorem 1.1

4.1. The existence of Theorem 1.1

Before giving the existence of Theorem 1.1, we will present the following lemma about the propagation of low regularities for the temperature function \( \theta \).

Lemma 4.1. Let \((u, \theta)\) be a smooth solution of system (1.1) on \([0, T^*]\). Then under the assumptions of Theorem 1.1, we have for any \( t < T^* \)
\[ \|\theta\|_{L_t^\infty(\dot{H}^{-\sigma})} \leq CE_0(1 + E_0(1 + E_0 + G_1)) \triangleq G_6 \] (4.1)
for \( E_0 \) and \( G_1 \) given by (1.9) and (3.34) respectively.

Proof. We first get by a similar derivation of (3.19) that
\[ \|\dot{\Delta}_j \theta(t)\|_{L^2} \leq e^{-ct^{2/\alpha}} \|\dot{\Delta}_j \theta_0\|_{L^2} + C \int_0^t e^{-c(t-t')^{2/\alpha}} \|\dot{\Delta}_j [u \cdot \nabla] \theta\|_{L^2} dt'. \] (4.2)

To continue our argument, we will use the following commutator’s estimate which the proof can be obtained as Lemma 3.3 in [3]:
\[ \sum_{j \in \mathbb{Z}} 2^{-js} \|\dot{\Delta}_j [u \cdot \nabla] \theta\|_{L^2} \leq C \|\theta\|_{\dot{H}^{1-\sigma}} \|\nabla u\|_{L^2}, \quad -1 < s < 2. \]

Thus, a simple computation helps us get from (4.2) and the above estimate that
\[ \|\theta\|_{L_t^\infty(\dot{H}^{-\sigma})} \lesssim \|\theta_0\|_{\dot{H}^{-\sigma}} + C \int_0^t \sum_{j \in \mathbb{Z}} 2^{-js} \|\dot{\Delta}_j [u \cdot \nabla] \theta\|_{L^2} dt' \]
\[ \leq \|\theta_0\|_{\dot{H}^{-\sigma}} + C \|\theta\|_{L_t^2(\dot{H}^{-\sigma})} \|\nabla u\|_{L_t^2(L^2)}. \] (4.3)
By the same manner, we have

\[ \| \theta \|_{L^2_t(H^1-\theta)} \leq \| \theta_0 \|_{H^{1-s_0-\alpha/2}} + C \int_0^t \sum_{j \in \mathbb{Z}} 2^{j(1-s_0-\alpha/2)} \| [\Delta_j, u \cdot \nabla \theta] \|_{L^2} \, dt' \]

\[ \leq \| \theta_0 \|_{H^{1-s_0-\alpha/2}} + C \| \theta \|_{L^2_t(H^{2-s_0-\alpha/2})} \| \nabla u \|_{L^2_t(L^2)} \]

\[ \leq \| \theta_0 \|_{H^{1-s_0-\alpha/2}} + C (\| \theta_0 \|_{H^{2-s_0-\alpha}} + \| \theta \|_{L^2_t(H^{3-s_0-\alpha})}) \| \nabla u \|_{L^2_t(L^2)}. \quad (4.4) \]

As \(2/3 < \alpha \leq 1\) and \(3 - 2\alpha < s_0 < 4\alpha / q - 8\alpha + 6\), one can infer \(0 < 3 - s_0 - \alpha \leq \alpha\), thus,

\[ \| \theta_0 \|_{H^{1-s_0-\alpha/2}} \leq C \| \theta_0 \|_{H^{-s_0\cap L^2}}, \quad \| \theta_0 \|_{H^{2-s_0-\alpha}} \leq C \| \theta_0 \|_{H^{-s_0\cap H^{1/2}}, \quad (4.5) \}

and

\[ \| \theta \|_{L^2_t(H^{3-s_0-\alpha})} \leq C \| \theta \|_{L^2_t(L^2) \cap L^2_t(H^s)}. \quad (4.6) \]

Inserting the estimates (4.5), (4.6) into (4.4), we can deduce from (4.3) that

\[ \| \theta \|_{L^2_t(H^{-s_0})} \leq \| \theta_0 \|_{H^{-s_0}} \]

\[ + (\| \theta_0 \|_{H^{-s_0\cap L^2}} + \| \theta_0 \|_{H^{-s_0\cap H^{1/2}} + \| \theta \|_{L^2_t(L^2) \cap L^2_t(H^{1/2})}) \| \nabla u \|_{L^2_t(L^2)}) \| \nabla u \|_{L^2_t(L^2)}. \quad (4.7) \]

From decay estimate (3.15) and estimates (3.16), (3.34), we have

\[ \| \nabla u \|_{L^2_t(L^2)} \leq C E_0, \quad \| \theta \|_{L^2_t(L^2) \cap L^2_t(H^s)} \leq C E_0 + G_1, \]

thus, taking the above estimates into (4.7), we can finally arrive at (4.1).

We are in a position to prove the existence part of Theorem 1.1. The strategy first is to solve an appropriate approximate of (1.1) and then prove the uniform bounds for such approximate solutions, and the last step consists in proving the convergence of such approximate solutions to a solution of the original system. One can check similar argument from page 1239 to page 1240 of [3] for details, here, we omit it.

4.2. The uniqueness of Theorem 1.1

In this subsection, we will present the uniqueness of Theorem 1.1. As the \( \theta \) equation has a supercritical regularity, thus, there will be more complicated discussion than [3]. Let \( u^i, \theta^i \) (with \( i = 1, 2 \)) be two solutions of the system (1.1) which satisfy (1.6), (1.7).
Denote $(\delta u, \delta \theta, \nabla \delta \Pi) \triangleq (u^2 - u^1, \theta^2 - \theta^1, \nabla \Pi^2 - \nabla \Pi^1)$. Then $(\delta u, \delta \theta, \nabla \delta \Pi)$ solves

$$
\begin{align*}
\partial_t \delta \theta + u^2 \cdot \nabla \delta \theta + |D|^\alpha \delta \theta &= -\delta u \cdot \nabla \theta^1, \\
\partial_t \delta u + u^2 \cdot \nabla \delta u - \text{div}(\mu(\theta^2)\delta (\delta u)) + \nabla \delta \Pi - \delta \theta \varepsilon_2 &= \text{div}((\mu(\theta^2) - \mu(\theta^1))\delta (u^1)) - \delta u \cdot \nabla u^1, \quad (4.8) \\
\text{div} \delta u &= 0, \\
(\delta \theta, \delta u)|_{t=0} &= (0, 0).
\end{align*}
$$

Taking $L^2$ inner product $\delta u$ with the $\delta u$ equation, $\delta \theta$ with the $\delta \theta$ equation in the above equation, using the Hölder inequality and Young inequality, we can finally get that

\[
\frac{1}{2} \frac{d}{dt} \left( \|\delta u\|^2_{L^2} + \|\delta \theta\|^2_{L^2} + \|\nabla \delta u\|^2_{L^2} + \|\delta \theta\|^2_{L^2} \right)_{\Omega_{t/2}} \lesssim \|\delta \theta\|_{L^{4p/(4 + 4p - 3\alpha)p}} \|\nabla \delta u\|_{L^{2}} \\
+ (1 + \|\nabla \theta^1\|_{L^\infty} + \|\nabla u^1\|^2_{L^2}) \left( \|\delta u\|^2 + \|\delta \theta\|^2_{L^2} \right) \\
\lesssim \epsilon \|\nabla \delta u\|^2_{L^2} + \|\delta \theta\|^2_{L^{4p/(4 + 4p - 3\alpha)p}} \|\nabla u^1\|^2_{L^{4p/(3\alpha p - 4 - 2p)}} \\
+ (1 + \|\nabla \theta^1\|_{L^\infty} + \|\nabla u^1\|^2_{L^2}) \left( \|\delta u\|^2 + \|\delta \theta\|^2_{L^2} \right). \quad (4.9)
\]

In the following, we will use the following lemma of which the proof can be obtained similarly to Proposition 3.1 in [3] (with a small modification) to control the term $\|\delta \theta\|^2_{L^{4p/(4 + 4p - 3\alpha)p}}$ in (4.9).

**Lemma 4.2.** Denote $\gamma \triangleq 4p/(4 + 4p - 3\alpha p)$. Assume $p > 4/(3\alpha - 2)$ and $\theta_0 \in B_{T, \infty}^0$, let $v \in L^2_T(\mathcal{W}^{1,4/\alpha})$ be a solenoidal vector field and $f \in L^2_T(B_{T, \infty}^{-\alpha/2})$. Then the equation below

$$
\partial_t \theta + u \cdot \nabla \theta + |D|^\alpha \theta = f \quad \text{and} \quad \theta|_{t=0} = \theta_0,
$$

has a unique solution $\theta$ so that for $t \leq T$

$$
\|\theta\|_{L^p_T(B_r^0, \infty)} \lesssim (\|\theta_0\|_{B_r^0, \infty} + \|f\|_{L^2_T(B_{r/2}^{-\alpha/2})}) \exp(C\|\nabla v\|_{L^2_T(L^4/\alpha)}). \quad (4.10)
$$

Applying the first equation in (4.8) to the above Lemma 4.2 yields

\[
\|\delta \theta\|^2_{L^2_T(B_r^0, \infty)} \leq C \|\delta u \cdot \nabla \theta^1\|^2_{L^2_T(B_{r/2}^{-\alpha/2})} \exp(C\|\nabla u^2\|_{L^2_T(L^4/\alpha)}) \\
\leq C \exp(C\|\nabla u^2\|_{L^2_T(L^4/\alpha)}) \int_0^t \|\delta u\|^2_{L^2} \|\theta^1\|^2_{B_r^0, \infty} dt', \quad (4.11)
\]

where the following estimate has been used, which the proof will be given later

$$
\|\delta u \cdot \nabla \theta^1\|_{L^2_T(B_r^{-\alpha/2})} \leq C \|\delta u\|_{L^2} \|\theta^1\|_{B_r^0}. \quad (4.12)
$$

\[\text{21}\]
Notice that for any positive integer $N$ and $p > 4/(3\alpha - 2)$, we have
\[
\|\delta \theta\|_{L^1_t} \lesssim \|\delta \theta\|_{B^0_t} \lesssim \|\delta \theta\|_{L^2} + \sqrt{N}\|\delta \theta\|_{B^0_t} + 2^{-N(\alpha/2 + 1/\gamma - 1)}\|\delta \theta\|_{H^{3/2}}
\]  
(4.13)
taking $N$ in the above inequality such that
\[
2^{N(\alpha/2 + 1/\gamma - 1)} \sim \|\delta \theta\|_{H^{3/2}}/\|\delta \theta\|_{B^0_t}.
\]
then we have
\[
\|\delta \theta\|_{L^p/(4 + 4p - 3\alpha)(L^2)} = \|\delta \theta\|_{L^p_\theta} \lesssim \|\delta \theta\|_{L^2} + \ln^{1/2} \left( e + (\|\theta_1\|_{H^{3/2}} + \|\theta_2\|_{H^{3/2}})/\|\delta \theta\|_{B^0_t}\right).
\]  
(4.14)
Taking (4.14) into (4.9) and choosing $\epsilon$ small enough, we have
\[
\|\delta u\|^2_{L^p_t(L^2)} + \|\delta \theta\|^2_{L^p_\theta} \lesssim \|\nabla \delta u\|^2_{L^2_t(L^2)} + \|\nabla \delta \theta\|^2_{H^{3/2}}
\]  
\[
\lesssim C \exp(C\|\nabla u\|^2_{L^2_t(L^{4/3})}) \int_0^t \|\delta u\|^2_{L^2_t} \|\theta^1\|^2_{B^0_t} \, dt'
\]
\[
\quad + \int_0^t (1 + \|\nabla \theta^1\|_{L^\infty} + \|\nabla u\|^2_{L^2}) (\|\delta u\|^2_{L^2} + \|\delta \theta\|^2_{L^2}) \, dt'
\]
\[
\quad + \int_0^t \|\nabla u\|^2_{L^4_p/(3\alpha - 2p)} \|\theta^1\|^2_{B^0_t} \ln \left( e + (\|\theta_1\|_{H^{3/2}} + \|\theta_2\|_{H^{3/2}})/\|\delta \theta\|_{B^0_t}\right) \, dt'.
\]  
(4.15)
Denote
\[
Y(t) \triangleq \|\delta u\|^2_{L^p_t(L^2)} + \|\delta \theta\|^2_{L^p_\theta} + \|\delta \theta\|^2_{B^0_t}.
\]
One can deduce from (4.15) that
\[
Y(t) \lesssim C \exp(C\|\nabla u\|^2_{L^2_t(L^{4/3})}) \int_0^t Y(t') \|\theta^1\|^2_{B^0_t} \, dt'
\]
\[
\quad + C \int_0^t (1 + \|\nabla \theta^1\|_{L^\infty} + \|\nabla u\|^2_{L^2}) (\|\delta u\|^2_{L^2} + \|\delta \theta\|^2_{L^2}) Y(t') \, dt'
\]
\[
\quad + C \int_0^t \|\nabla u\|^2_{L^4_p/(3\alpha - 2p)} Y(t') \ln \left( e + (\|\theta_1\|_{H^{3/2}} + \|\theta_2\|_{H^{3/2}})/Y(t')\right) \, dt'.
\]  
(4.16)
For any $8/(3\alpha - 2) \leq p < 1/C\|\mu(\cdot) - 1\|_{L^\infty}$ and $2/3 < \alpha \leq 1$, by using the interpolation inequality, we have
\[
\|\nabla \theta^1\|_{L^2_t(L^\infty)} \lesssim \|\theta^1\|_{B^0_t}^{3\alpha - 2p - 4} \|\theta^1\|_{H^{3/2}}^{2p - 13} \|\theta^1\|_{B^0_t}^{3\alpha - 2p - 4} \|\theta^1\|_{H^{3/2}}^{2p - 13}.
\]
\[
\|\nabla u^2\|_{L^p_\theta(L^2)} \lesssim \|\nabla u^2\|_{L^p_\theta(L^2)}^{4p-4} \|\nabla u^2\|_{L^p_\theta(L^2)}^{2p-2} \|\nabla u^2\|_{L^p_\theta(L^2)}^{2p-2} \|\nabla u^2\|_{L^p_\theta(L^2)}^{2p-2} \|\nabla u^2\|_{L^p_\theta(L^2)}^{2p-2}.
\]
Thus, applying Osgood’s Lemma 2.7 to (4.16), we can infer that $Y(t) = 0$. This complete the uniqueness of Theorem 1.1.

Consequently, we have completed the proof of our’s main Theorem 1.1.
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