A Multiparametric Quon Algebra

Hery Randriamaro *

May 17, 2019

Abstract

The quon algebra is an approach to particle statistics introduced by Greenberg in order to provide a theory in which the Pauli exclusion principle and Bose statistics are violated by a small amount. In this article, we generalize these models by introducing a deformation of the quon algebra generated by a collection of operators $a_i$, $i \in \mathbb{N}^*$, on an infinite dimensional module satisfying the $q_{i,j}$-mutator relations $a_i a_j^\dagger - q_{i,j} a_j^\dagger a_i = \delta_{i,j}$. The realizability of our model is proved by means of the Aguiar-Mahajan bilinear form on the chambers of hyperplane arrangements. We show that, for suitable values of $q_{i,j}$, the module generated by the particle states obtained by applying combinations of $a_i$'s and $a_i^\dagger$'s to a vacuum state $|0\rangle$ is an indefinite-Hilbert module. Furthermore, studying the matrix of that bilinear form permits us to establish the conjecture of Zagier.

Keywords: Quon Algebra, Indefinite-Hilbert Module, Hyperplane Arrangement

MSC Number: 05E15, 81R10

1 Introduction

Denote by $\mathbb{C}[q_{i,j}]$ the polynomial ring $\mathbb{C}[q_{i,j} \mid i, j \in \mathbb{N}^*]$ with variables $q_{i,j}$. The quons are particles whose annihilation and creation operators obey the quon algebra which interpolates between fermions and bosons. By multiparametric quon algebra $A$, we mean the free algebra $\mathbb{C}[q_{i,j}]\{a_i \mid i \in \mathbb{N}^*\}$ subject to the anti-involution $\dagger$ exchanging $a_i$ with $a_i^\dagger$, and to the commutation relations

$$a_i a_j^\dagger = q_{i,j} a_j^\dagger a_i + \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta.

That algebra is a generalization of the deformed quon algebra studied by Meljanac and Svrtan subject to the restriction $q_{i,j} = \tilde{q}_{i,j}$ [5, § 1.1]. Their algebra is in turn a generalization of the quon algebra introduced by Greenberg [4] which is subject to the commutation relations $a_i a_j^\dagger = q a_j^\dagger a_i + \delta_{i,j}$ obeyed by the annihilation and creation operators of the quon particles, and generating a model of infinite statistics. Finally, the quon algebra is a generalization of the classical Bose and Fermi algebras corresponding to the restrictions $q = 1$ and $q = -1$ respectively, as well as of the intermediate case $q = 0$ suggested by Hegstrom and investigated by Greenberg [3].

*Lot II B 32 bis Faravohitra, 101 Antananarivo, Madagascar
E-mail: hery.randriamaro@gmail.com
In a Fock-like representation, the generators of $A$ are the linear operators $a_i, a_i^\dagger : V \to V$ on an infinite dimensional $\mathbb{C}[q_{i,j}]$-vector module $V$ satisfying the commutation relations

$$a_i a_j^\dagger = q_{i,j} a_j^\dagger a_i + \delta_{i,j},$$

and the relations

$$a_i |0\rangle = 0,$$

where $a_i^\dagger$ is the adjoint of $a_i$, and $|0\rangle$ is a nonzero distinguished vector of $V$. The $a_i$'s are the annihilation operators and the $a_i^\dagger$'s the creation operators.

Define the $q_{i,j}$-conjugate $\tilde{P}$ of a monomial $P = \mu \prod_{u \in [n]} q_{i_u,j_u} \in \mathbb{C}[q_{i,j}]$, where $\mu \in \mathbb{C}$, by

$$\tilde{P} := \tilde{\mu} \prod_{s \in [n]} q_{i_s,j_s} \quad \text{with} \quad \tilde{\mu} = \bar{\mu}, \quad \text{and} \quad \tilde{q}_{i_s,j_s} = q_{j_s,i_s},$$

and the $q_{i,j}$-conjugate of a monomial sum $Q = P_1 + \cdots + P_k \in \mathbb{C}[q_{i,j}]$ by $\tilde{Q} = \tilde{P}_1 + \cdots + \tilde{P}_k$.

Define an indefinite inner product on $V$ by a map $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}[q_{i,j}]$ such that, for $\mu \in \mathbb{C}$, and $u, v, w \in V$, we have

- $(\mu u, v) = \mu (u, v)$ and $(u + v, w) = (u, w) + (v, w)$,
- $(u, v) = \overline{(v, u)}$,
- and, if $u \neq 0$, $(u, u) \neq 0$.

Let $H$ be the submodule of $V$ generated by the particle states obtained by applying combinations of $a_i$'s and $a_i^\dagger$'s to $|0\rangle$, that is $H := \{ a|0\rangle \mid a \in A \}$. The aim of this article is to prove the realizability of that model through the following theorem.

**Theorem 1.1.** Under the condition $|q_{i,j}| < 1$, the module $H$ is an indefinite-Hilbert module for the map $\langle \cdot , \cdot \rangle : H \times H \to \mathbb{C}[q_{i,j}]$ defined, for $\mu, \nu \in \mathbb{C}[q_{i,j}]$, and $a, b \in A$, by

$$\langle \mu a|0\rangle , \nu b|0\rangle \rangle := \mu \nu \langle 0|a b^\dagger|0\rangle \quad \text{with} \quad \langle 0|0\rangle = 1,$$

and where the usual bra-ket product $\langle 0|a b^\dagger|0\rangle$ is subject to the defining relations of $A$.

The indefinite inner product of Theorem 1.1 becomes an inner product when the matrix representing $(\cdot , \cdot )$ is diagonalizable. Theorem 1.1 is particularly a generalization of the realizability of the deformed quon algebra model established by Meljanac and Svrtan [5, Theorem 1.9.4], which in turn is a generalization of the realizability of the quon algebra model established by Zagier [7, Theorem 1].

To prove Theorem 1.1, we first show with Lemma 3.1 that

$$B := \{ a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle \mid (i_1, \ldots , i_n) \in (\mathbb{N}^+)^n, \ n \in \mathbb{N} \}$$

is a basis of $H$, so that we can assume

$$H = \left\{ \sum_{i=1}^{n} \mu_i b_i \mid n \in \mathbb{N}^*, \mu_i \in \mathbb{C}[q_{i,j}], b_i \in B \right\}.$$
The infinite matrix associated to the map of Theorem 1.1 is \( M := ((b, a))_{a,b \in B} \).

Let \( \left\{ \frac{N^*}{n} \right\} \) be the set of multisets of \( n \) elements in \( N^* \). We prove with Lemma 3.2 that

\[
M = \bigoplus_{n \in N^*} \bigoplus_{I \in \left\{ \frac{N^*}{n} \right\}} M_I \quad \text{with} \quad M_I = \left( \langle 0 | a^{\dagger}(n) \cdots a^{\dagger}(1) a^{\dagger} \cdots a^{\dagger}(n) | 0 \rangle \right) \sigma^{\dagger} \sigma^t \in \mathcal{S}_I,
\]

where \( \mathcal{S}_I \) is the permutation set of the multiset \( I \). For example,

\[
M_{[3]} = \begin{pmatrix}
1 & q_{3,2} & q_{2,1} & q_{2,1} q_{3,1} & q_{3,1} q_{2,1} & q_{3,1} q_{2,1} q_{3,1} \\
q_{2,3} & 1 & q_{2,1} q_{3,1} & q_{2,1} q_{3,1} q_{2,3} & q_{3,1} & q_{3,1} q_{2,1} \\
q_{1,2} & q_{1,2} q_{3,2} & 1 & q_{3,1} & q_{3,2} q_{1,2} q_{3,1} & q_{3,2} q_{3,1} \\
q_{1,2} q_{3,2} & q_{1,3} & q_{2,3} q_{1,1} q_{2,3} & q_{2,3} q_{1,2} & 1 & q_{2,1} \\
q_{1,3} q_{2,3} & q_{1,3} q_{2,3} & q_{2,3} q_{2,1} & q_{2,3} q_{1,2} & q_{2,3} & 1
\end{pmatrix}.
\]

Proposition 2.1 and Lemma 3.3 permits us to deduce that, if \( J \in \left( \frac{N^*}{n} \right) \subseteq \left\{ \frac{N^*}{n} \right\} \), then

\[
\det M_J = \prod_{K \in \mathbb{C}^I} \left( 1 - \prod_{\{s,t\} \in (K)} q_{s,t} \right)^{(\# K - 2)! (n - \# K + 1)!}.
\]

For example, \( \det M_{[3]} = (1 - q_{1,2} q_{2,1})^2 (1 - q_{1,3} q_{3,1})^2 (1 - q_{2,3} q_{3,2})^2 (1 - q_{1,2} q_{2,1} q_{3,1} q_{2,3} q_{3,2}) \).

That determinant was independently computed by Meljanac and Svrtan for the specialization \( q_{i,j} = \bar{q}_{i,j} \) [5 Theorem 1.9.2], by Duchamp et al. for the specialization \( q_{i,j} = q_{j,i} \) [2 § 6.4.1], and by Zagier for the specialization \( q_{i,j} = \bar{q}_{j,i} \) [7 Theorem 2].

Moreover, consider the multiset \( I = \{ i_1, \ldots, i_1, i_2, \ldots, i_2, \ldots, k_1, \ldots, k_k \} \in \left\{ \frac{N^*}{n} \right\} \). For \( s \in [n] \), let \( \tilde{s} := i_j \) if \( s \in [p_j + p_j - 1 + \cdots + p_1] \setminus [p_j - 1 + \cdots + p_1] \). Suppose that the generators of \( A \) satisfy the commutation relations \( a_s a_t^\dagger = q_{s,t} a_t a_s + \delta_{s,t} \). In that case, if we regard \( M_{[n]} \) as the matrix representing a linear map \( \alpha : M \to M \) on a module \( M \), then, we prove with Proposition 2.2 and Lemma 3.3 that \( M_I \) is the matrix representing \( \alpha \) restricted to a submodule \( N \subseteq M \) such that \( \alpha(N) = N \).

Therefore, we can infer that, for every \( I \in \left\{ \frac{N^*}{n} \right\} \), \( M_I \) is nonsingular for \( |q_{i,j}| < 1, i, j \in N^* \).

When, for special values of the \( q_{i,j} \)'s, \( M_{[n]} \) is diagonalizable, then \( M_I \) becomes positive definite. Indeed, as \( M_I \) is the identity matrix if \( q_{i,j} = 0 \), for every \( i, j \in N^* \), we deduce by continuity that \( M_I \) is positive definite. For these suitable values of \( q_{i,j} \), \( M \) becomes a positive definite matrix or, in other terms, the map in Theorem 1.1 becomes an inner product on \( H \). It is the case of the algebras investigated by Meljanac and Svrtan, and Zagier since, with their models, \( M_{[n]} \) is a hermitian matrix, that is consequently diagonalizable.

To finish, we establish the conjecture of Zagier [7 § 1] in Section 4:

**Proposition 1.2.** Let \( n \in N^* \), and assume that the generators of \( A \) satisfy the commutation relations \( a_s a_t^\dagger = q a_t^\dagger a_s + \delta_{s,t} \). Then, each entry of \( M_{[n]}^{-1} \) is an element of

\[
\prod_{i \in [n-1]} \frac{\mathbb{C}[q]}{(1 - q^{2+i})}.
\]

3
2 Hyperplane Arrangements

We establish two results we need concerning the hyperplane arrangement associated to the permutation group of $n$ elements to prove Theorem 1.1.

Recall that a hyperplane in the space $\mathbb{R}^n$ is a $(n-1)$-dimensional linear subspace, and a hyperplane arrangement is a finite set of hyperplanes. For a hyperplane $H$, denote its two associated open half-spaces by $H^+$ and $H^-$, and let $H^0 := H$. A face of a hyperplane arrangement $\mathcal{A}$ is a subset of $\mathbb{R}^n$ having the form

$$F := \bigcap_{H \in \mathcal{A}} H^{\epsilon_H(F)}$$

with $\epsilon_H(F) \in \{+,0,-\}$.

A chamber of $\mathcal{A}$ is a face in $F_{\mathcal{A}}$ whose sign sequence contains no 0. Denote the set of $\mathcal{A}$-chambers by $C_{\mathcal{A}}$. For two chambers $C, D \in C_{\mathcal{A}}$, the set of half-spaces containing $C$ but not $D$ is $H_{C,D} := \{H^{\epsilon_H(C)} \mid H \in \mathcal{A}, \epsilon_H(C) = -\epsilon_H(D)\}$. Assign a variable $h^\varepsilon_H$, $\varepsilon \in \{+,-\}$, to every half-space $H^\varepsilon$. We work with the polynomial ring $R_\mathcal{A} := \mathbb{Z}[h^\varepsilon_H \mid H \in \mathcal{A}, \varepsilon \in \{+,-\}]$.

The module of $R_\mathcal{A}$-linear combinations of chambers is $M_{\mathcal{A}} := \left\{ \sum_{C \in C_{\mathcal{A}}} x_CC \mid x_C \in R_\mathcal{A}\right\}$.

Define the bilinear form $v : M_{\mathcal{A}} \times M_{\mathcal{A}} \to R_\mathcal{A}$ for chambers $C, D \in C_{\mathcal{A}}$ by

$$v(C, C) = 1 \quad \text{and} \quad v(C, D) = \prod_{H^\varepsilon \in H_{C,D}} h^\varepsilon_H \quad \text{if} \ C \neq D.$$

That bilinear form derives from the distance function on the chambers of hyperplane arrangements introduced by Aguiar and Mahajan [1, § 8.1.1].

From $v$, we define the linear map $\gamma : M_{\mathcal{A}} \to M_{\mathcal{A}}$, for a chamber $D \in C_{\mathcal{A}}$, by

$$\gamma(D) := \sum_{C \in C_{\mathcal{A}}} v(D, C) C.$$

Let $x = (x_1, \ldots, x_n)$ be a variable of $\mathbb{R}^n$. We mainly investigate the hyperplane arrangement associated to the permutation group $S_n$ of $n$ elements

$$\mathcal{A}_n = \{ H_{i,j} \mid i, j \in [n], i < j \} \quad \text{with} \quad H_{i,j} = \{ x \in \mathbb{R}^n \mid x_i = x_j \}.$$  

The set of $\mathcal{A}_n$-chambers is

$$C_{\mathcal{A}_n} = \{ C_{\sigma} \mid \sigma \in S_n \} \quad \text{with} \quad C_{\sigma} := \{ x \in \mathbb{R}^n \mid x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)} \}.$$

For $i, j \in [n]$ with $i \neq j$, assign the variable $q_{i,j}$ to the half-space $\{ x \in \mathbb{R}^n \mid x_i < x_j \}$. On the hyperplane arrangement $\mathcal{A}_n$, we work with the polynomial ring $R_{\mathcal{A}_n} := \mathbb{Z}[q_{i,j} \mid i, j \in [n]]$, and the module of $R_{\mathcal{A}_n}$-linear combinations of chambers $M_{\mathcal{A}_n} := \left\{ \sum_{\sigma \in S_n} x_\sigma C_{\sigma} \mid x_\sigma \in R_{\mathcal{A}_n}\right\}$.

Restricted on $\mathcal{A}_n$, $v$ becomes the bilinear form $v_n : M_{\mathcal{A}_n} \times M_{\mathcal{A}_n} \to R_{\mathcal{A}_n}$ defined, for chambers $C_{\sigma}, C_\tau \in C_{\mathcal{A}_n}$, by

$$v_n(C_{\sigma}, C_\tau) = \prod_{\{i,j\} \in {[n]\choose 2}, i < j} q_{\sigma(i),\sigma(j)}^{-1} \tau^{-1} \sigma(i) > \tau^{-1} \sigma(j) \tau^{-1} \sigma(j)$$

and $\gamma$ the linear map $\gamma_n : M_{\mathcal{A}_n} \to M_{\mathcal{A}_n}$ defined, for a chamber $C_\tau \in C_{\mathcal{A}_n}$, by

$$\gamma_n(C_\tau) := \sum_{\sigma \in S_n} v_n(C_\tau, C_{\sigma}) C_{\sigma}.$$
Proposition 2.1. For an integer \( n \geq 2 \), we have
\[
\det \gamma_n = \prod_{I \in \binom{\{s,t\}}{2}} \left( 1 - \prod_{\{i,j\} \in \binom{\{s,t\}}{2}} q_{i,j}q_{j,i} \right)^{(\#I - 2)! \left( n - \#I + 1 \right)!}.
\]

Proof. We first discuss about the general case of hyperplane arrangements. A flat of \( \mathcal{A} \) is an intersection of \( \mathcal{A} \)-hyperplanes. Denote the set of \( \mathcal{A} \)-flats by \( L_{\mathcal{A}} \). The weight of a flat \( E \in L_{\mathcal{A}} \) is the monomial
\[
b_E := \prod_{H \in \mathcal{A}} h_H^+, h_H^-, \]
and, if we choose a hyperplane \( H \) containing \( E \), the multiplicity \( \beta_E \) of \( E \) is half the number of chambers \( C \in \mathcal{C}_H \) which have the property that \( E \) is the minimal edge containing \( C \cap H \). Aguiar and Mahajan proved that [1, Theorem 8.11]
\[
\det \gamma = \prod_{E \in L_{\mathcal{A}}} (1 - b_E)^{\beta_E}.
\]

Now, concerning \( \mathcal{A}_n \), let \( L'_{\mathcal{A}_n} = \{ E \in L_{\mathcal{A}_n} \mid \beta_E \neq 0 \} \). For a subset \( I \subseteq [n] \) with \(#I \geq 2\), denote by \( E_I \) the edge \( \bigcap_{\{i,j\} \in \binom{I}{2}} H_{i,j} \). Randriamaro proved that [6, Lemma 3.2 - 3.3]
\[
L'_{\mathcal{A}_n} = \{ E_I \mid I \subseteq [n], \#I \geq 2 \} \quad \text{and} \quad \beta_{E_I} = (\#I - 2)! (n - \#I + 1)!.
\]

\[
\]

Take a partition \( \lambda = (p_1, \ldots, p_k) \in \text{Par}(n) \) of \( n \). Denote by \( \mathcal{S}_\lambda \) the subgroup \( \prod_{i \in [k]} \mathcal{S}_{\lambda_i} \) of \( \mathcal{S}_n \), where \( \mathcal{S}_{\lambda_i} \) is the permutation group of the set \([p_i + p_{i-1} + \cdots + p_1] \setminus [p_i + p_{i-1} + \cdots + p_1] \).

Consider the multiset \( I_\lambda = \{1, \ldots, 1, 2, \ldots, 2, \ldots, k, \ldots, k\} \). Denote by \( \mathcal{S}_{I_\lambda} \) the permutation set of the multiset \( I_\lambda \). For \( s \in [n] \), define \( \hat{s} := i \) if \( s \in [p_i + p_{i-1} + \cdots + p_1] \setminus [p_i + p_{i-1} + \cdots + p_1] \).

Let \( p : \mathcal{S}_n \to \mathcal{S}_{I_\lambda} \) be the projection \( p(\sigma) := \sigma(1) \sigma(2) \ldots \sigma(n) \). For \( \hat{\sigma} \in \mathcal{S}_{I_\lambda} \), define the element \( C_\hat{\sigma} := \sum_{\sigma \in p^{-1}(\hat{\sigma})} C_\sigma \in M_{\mathcal{A}_n} \). Denote by \( M_{\mathcal{A}_n}^\lambda \) the submodule
\[
M_{\mathcal{A}_n}^\lambda := \left\{ \sum_{\hat{\sigma} \in \mathcal{S}_{I_\lambda}} x_{C_\hat{\sigma}} C_\hat{\sigma} \mid x_{C_\hat{\sigma}} \in R_{\mathcal{A}_n} \right\}.
\]

For \( s, t \in [n] \) with \( s \neq t \), assign the variable \( q_{s,t} \) to the half-space \( \{ x \in \mathbb{R}^n \mid x_s < x_t \} \).

Proposition 2.2. Let \( n \in \mathbb{N}^+ \), and \( \lambda \in \text{Par}(n) \). Then, \( \gamma_n(M_{\mathcal{A}_n}^\lambda) = M_{\mathcal{A}_n}^\lambda \).

Proof. If \( \sigma \in \mathcal{S}_n \) such that \( p(\sigma) = \hat{\sigma} \in \mathcal{S}_{I_\lambda} \), then \( p^{-1}(\hat{\sigma}) = \mathcal{S}_{I_\lambda} \). Let \( \nu \sigma \in \mathcal{S}_{I_\lambda} \), and \( \hat{\tau} \in \mathcal{S}_{I_\lambda} \).
If \( p(\tau) = \dot{\tau} \),

\[
 v_n(C_{\dot{\tau}}, C_{\sigma}) = \sum_{\varphi \in S} v_n(C_{\varphi \tau}, C_{\sigma})
\]

\[
 = \sum_{\varphi \in S} \prod_{\{i,j\} \in \binom{[n]}{2}} q_{\varphi \tau}(i), q_{\varphi \tau}(j)
\]

\[
 = \sum_{\varphi \in S} \prod_{\{i,j\} \in \binom{[n]}{2}} q_{\tau}(i), q_{\tau}(j)
\]

\[
 = \sum_{\varphi \in S} \prod_{\{i,j\} \in \binom{[n]}{2}} q_{\tau}(i), q_{\tau}(j)
\]

\[
 = \sum_{\varphi \in S} \prod_{\{i,j\} \in \binom{[n]}{2}} q_{\tau}(i), q_{\tau}(j)
\]

Hence,

\[
 \gamma_n(C_{\dot{\tau}}) = \sum_{\sigma \in S_n} v_n(C_{\dot{\tau}}, C_{\sigma}) C_{\sigma}
\]

\[
 = \sum_{\sigma \in S_{I\lambda}} \sum_{\varphi \in S} v_n(C_{\dot{\tau}}, C_{\sigma}) C_{\sigma}
\]

\[
 = \sum_{\sigma \in S_{I\lambda}} \sum_{\varphi \in S} v_n(C_{\dot{\tau}}, C_{\sigma}) C_{\sigma}
\]

\[
 = \sum_{\sigma \in S_{I\lambda}} v_n(C_{\dot{\tau}}, C_{\sigma}) \sum_{\varphi \in S} C_{\sigma}
\]

\[
 = \sum_{\sigma \in S_{I\lambda}} v_n(C_{\dot{\tau}}, C_{\sigma}) C_{\sigma}.
\]

\[\square\]

3 The Bra-Ket Product on \( \mathbf{H} \)

We prove some useful properties of the map in Theorem 1.1. We particularly connect it to the bilinear form \( v_n \) on \( M_{A_n} \).

**Lemma 3.1.** The vector space generated by our particle states is

\[
 \mathbf{H} = \left\{ \sum_{i=1}^{n} \mu_i b_i \mid n \in \mathbb{N}^*, \mu_i \in \mathbb{C}[q_{i,j}], b_i \in \mathbf{B} \right\}.
\]

**Proof.** Let \( i \in \mathbb{N}^* \). We have,

\[
 a_i a_j^\dagger \ldots a_l^\dagger = q_{i,j_1} \ldots q_{i,j_l} a_j^\dagger \ldots a_l^\dagger a_i
\]

\[
 + \sum_{u \in [l]} q_{i,j_1} \ldots q_{i,j_{u-1}} a_j^\dagger \ldots a_{j_{u-1}}^\dagger a_{j_{u-1}} \ldots a_j^\dagger
\]
where the hat over the \( u \)th term of the product indicates that this term is omitted. So
\[
\mathbf{a}_i \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger |0\rangle = \sum_{u \in [t]} \sum_{j_u = 1} q_{i,j_1} \cdots q_{i,j_{u-1}} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_u}^\dagger \cdots \mathbf{a}_{j_t}^\dagger |0\rangle.
\]
Thus, one can recursively remove every annihilation operator \( \mathbf{a}_i \) of an element \( \mathbf{a}(0) \in \mathbf{H} \).

**Lemma 3.2.** Let \((i_1, \ldots, i_s) \in (\mathbb{N}^+)^s \) and \((j_1, \ldots, j_t) \in (\mathbb{N}^+)^t \). If, as multisets, \( \{i_1, \ldots, i_s\} \) is different from \( \{j_1, \ldots, j_t\} \), then \( \langle 0 | \mathbf{a}_{i_1} \cdots \mathbf{a}_{i_s} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger |0\rangle = 0 \).

**Proof.** Suppose that \( v \) is the smallest integer in \([s]\) such that \( i_v \notin \{j_1, \ldots, j_t\} \cup \{i_1, \ldots, i_{v-1}\} \). Then
\[
\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_s} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger = P \mathbf{a}_{i_{v-1}} + Q \mathbf{a}_{i_v}
\]with \( P, Q \in \mathbf{A} \).

We deduce that \( \mathbf{a}_{i_1} \cdots \mathbf{a}_{i_s} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger \) is a multiset permutation of \( (i_1, \ldots, i_s) \setminus \{j_1, \ldots, j_u\} \). Then
\[
\mathbf{a}_{i_1} \cdots \mathbf{a}_{i_s} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger = \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_u}^\dagger P' + \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger Q'
\]with \( P', Q' \in \mathbf{A} \).

And \( \langle 0 | \mathbf{a}_{i_1} \cdots \mathbf{a}_{i_s} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger \rangle = \langle 0 | \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_u}^\dagger P' + \langle 0 | \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_t}^\dagger Q' = 0 \).

Therefore, we just need to investigate the product \( \langle 0 | \mathbf{a}_{i_1} \cdots \mathbf{a}_{i_s} \mathbf{a}_{j_1}^\dagger \cdots \mathbf{a}_{j_n}^\dagger |0\rangle \) where \((j_1, \ldots, j_n)\) is a multiset permutation of \((i_1, \ldots, i_n)\).

**Lemma 3.3.** Let \( \sigma, \tau \in \mathfrak{S}_n \), and \( C_\sigma, C_\tau \in C_{A_n} \) their associated chambers. Then,
\[
\langle 0 | \mathbf{a}_{\sigma(1)} \cdots \mathbf{a}_{\sigma(n)} \mathbf{a}_{\tau(1)}^\dagger \cdots \mathbf{a}_{\tau(n)}^\dagger |0\rangle = v_n(C_\sigma, C_\tau).
\]

**Proof.** We have
\[
\langle 0 | \mathbf{a}_{\sigma(1)} \cdots \mathbf{a}_{\sigma(n)} \mathbf{a}_{\tau(1)}^\dagger \cdots \mathbf{a}_{\tau(n)}^\dagger |0\rangle = \prod_{s \in [n]} \prod_{\tau^{-1} \circ \sigma(s) > \tau^{-1} \circ \sigma(t)} q_{\sigma(s), \sigma(t)}
\]
\[
= \prod_{\{s,t\} \in \binom{n}{2}} q_{\sigma(s), \sigma(t)}
\]
\[
= v_n(C_\sigma, C_\tau).
\]

For \( s, t \in [n] \) with \( s \neq t \), assign the variable \( q_{s,t} \) to the half-space \( \{x \in \mathbb{R}^n \mid x_s < x_t\} \).

**Lemma 3.4.** Let \( \lambda \in \text{Par}(n) \), and \( \bar{\sigma}, \bar{\tau} \in \mathfrak{S}_{I_\lambda} \). Then, for every \( \tau \in \text{p}^{-1}(\bar{\tau}) \),
\[
\langle 0 | \mathbf{a}_{\bar{\sigma}(1)} \cdots \mathbf{a}_{\bar{\sigma}(n)} \mathbf{a}_{\tau(1)}^\dagger \cdots \mathbf{a}_{\tau(n)}^\dagger |0\rangle = v_n(C_\bar{\sigma}, C_\bar{\tau}).
\]

7
Proof. We have

\[
\langle 0 \mid a_{\sigma(n)} \cdots a_{\sigma(1)} \, a_{\tau(1)} \cdots a_{\tau(n)} \, 0 \rangle = \sum_{\sigma \in S_n} \prod_{s \in [n]} \prod_{t \in [n] \setminus [s]} q_{\sigma(s), \tau_{\sigma(t)}} \\
= \sum_{\sigma \in S_n} \prod_{\{s,t\} \in \binom{[n]}{2}} \tau_{\sigma(s), \tau_{\sigma(t)}}.
\]

For every \( \sigma \in p^{-1}(\hat{\sigma}) \), and \( \tau \in p^{-1}(\hat{\tau}) \), we have, on one side,

\[\hat{\sigma} = \hat{\tau} \circ v \iff \mathcal{G}_\lambda \sigma = \mathcal{G}_\lambda \tau v \iff v \in \tau^{-1} \mathcal{G}_\lambda \sigma.\]

On the other side, if id is the identity permutation, there exists \( \varphi \in \mathcal{G}_\lambda \) such that \( v = \tau^{-1} \varphi \sigma \), and

\[\hat{\tau} \circ v(t) = \hat{\tau} \circ \tau^{-1} \varphi \sigma(t) = \text{id} \circ \varphi \sigma(t) = \text{id} \circ \sigma(t) = \hat{\sigma}(t).\]

Then,

\[
\langle 0 \mid a_{\sigma(n)} \cdots a_{\sigma(1)} \, a_{\tau(1)} \cdots a_{\tau(n)} \, 0 \rangle = \sum_{\sigma \in \mathcal{G}_\lambda} \prod_{\{s,t\} \in \binom{[n]}{2}} q_{\sigma(s), \sigma(t)} \\
= \nu_n(C_\sigma, C_\tau)
\]

\[\square\]

4 The Conjecture of Zagier

To prove the conjecture of Zagier, we first have to come back to the general case of hyperplane arrangements. The set \( F_A \) of \( A \)-faces forms the Tits monoid with the product \( FG \) defined, for \( F, G \in F_A \), by

\[\epsilon_H(FG) = \begin{cases} 
\epsilon_H(F) & \text{if } \epsilon_H(F) \neq 0, \\
\epsilon_H(G) & \text{otherwise}.
\end{cases}\]

It is also a meet-semilattice with partial order \( \preceq \) defined, for \( F, G \in F_A \), by

\[F \preceq G \iff \epsilon_H(F) = \epsilon_H(G) \text{ whenever } \epsilon_H(F) \neq 0.\]

Denote by \( O \) the face in \( F_A \) such that, for every \( H \in A \), \( \epsilon_H(O) = 0 \). The rank of a face \( F \in F_A \) is \( \text{rk } F := \dim \bigcap_{H \in A, F \subseteq H} - \dim O \). A nested face is a pair \((F, G)\) of faces in \( F_A \) such that \( F \prec G \).

For a nested face \((F, G)\), a flag from \( F \) to \( G \) is a sequence \((F = F_0 \prec F_1 \cdots \prec F_k = G)\) of faces in \( F_A(F,G) \) such that, for \( i \in [k] \), we have \( \text{rk } F_i = \text{rk } F_{i-1} + 1 \). Denote the set of flags from the face \( F \) to the face \( G \) by \( F_A^{(F,G)} \).

For a face \( F \in F_A \), define the monomial \( b_F := \prod_{H \subseteq F \subseteq H} h^+_H h^-_H \), in particular \( b_F = 0 \) if \( F \in C_A \).
For a flag $K = (F_0 \prec F_1 \prec \cdots \prec F_k) \in F_{\mathcal{A}}^{[F,G]}$, define the polynomial $\Delta_K := \prod_{i \in [k]} (1 - b_{F_i})$, and the set of polynomial fractions $\text{Frac}(F,G) := \left\{ \frac{p}{\Delta_K} \ \middle| \ p \in R_{\mathcal{A}}, \ K \in F_{\mathcal{A}}^{[F,G]} \right\}$.

**Proposition 4.1.** Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Each entry of $\gamma^{-1} : M_{\mathcal{A}} \to M_{\mathcal{A}}$ is an element of $\bigcup_{C \in C_{\mathcal{A}}} \text{Frac}(O,C)$.

**Proof.** As $\det \gamma$ is a polynomial in $R_{\mathcal{A}}$ with constant term 1, $\gamma$ is consequently invertible. For a chamber $D \in C_{\mathcal{A}}$, and a nested face $(A,D)$, define $m(A,D) := \sum_{C \in C_{\mathcal{A}}} v(D,C) C$.

We prove by backward induction that $m(A,D) = \sum_{C \in C_{\mathcal{A}}} x_C \gamma(C)$ with $x_C \in \text{Frac}(A,D)$.

We have $m(D,D) = \gamma(D) = \frac{\gamma(D)}{1 - b_D}$.

The opposite of a face $F \in F_{\mathcal{A}}$ is the face $\tilde{F}$ such that, for every $H \in \mathcal{A}$, $\epsilon_H(\tilde{F}) = -\epsilon_H(F)$. For a nested face $(F,G)$, define the set of faces $F_{\mathcal{A}}^{(F,G)} := \{ K \in F_{\mathcal{A}} \mid F \preceq K \preceq G \}$. If we assign a variable $x_C$ to each chamber $C \in C_{\mathcal{A}}$, a more general formulation of the Witt identity states that [1, Proposition 7.30]

$$\sum_{F \in F_{\mathcal{A}}^{(A,D)}} (-1)^{rk F} \sum_{C \in C_{\mathcal{A}}} x_C = (-1)^{rk D} \sum_{C \in C_{\mathcal{A}}} x_C.$$ 

That formulation applied to $x_C = v(D,C) C$ yields

$$\sum_{F \in F_{\mathcal{A}}^{(A,D)}} (-1)^{rk F} m(F,D) = (-1)^{rk D} \sum_{C \in C_{\mathcal{A}}} v(D,C) C.$$ 

Since $v(D,C) = v(D,AC) v(AC,C) = v(D,AD) v(AD,C)$, then

$$\sum_{F \in F_{\mathcal{A}}^{(A,D)}} (-1)^{rk F} m(F,D) = (-1)^{rk D} v(D,AD) \sum_{C \in C_{\mathcal{A}}} v(AD,C) C = (-1)^{rk D} v(D,AD) m(A,AD).$$ 

So, $m(A,D) - (-1)^{rk D - rk A} v(D,AD) m(A,AD) = \sum_{F \in F_{\mathcal{A}}^{(A,D)} \setminus \{A\}} (-1)^{rk F - rk A + 1} m(F,D)$. By induction hypothesis, for every $C \in C_{\mathcal{A}}$, there exists $a_C \in \bigcup_{F \in F_{\mathcal{A}}^{(A,D)} \setminus \{A\}} \text{Frac}(F,D)$, such that

$$\sum_{F \in F_{\mathcal{A}}^{(A,D)} \setminus \{A\}} (-1)^{rk F - rk A + 1} m(F,D) = \sum_{C \in C_{\mathcal{A}}} a_C \gamma(C).$$
Remark that, for every $F \in F_A^{(A,D)}$, we have $b_F = b_{A,F}$, which means that

$$\bigcup_{F \in F_A^{(A,D)} \setminus \{A\}} \text{Frac}(F, A\hat{D}) = \bigcup_{F \in F_A^{(A,D)} \setminus \{A\}} \text{Frac}(F, D).$$

So, since $A \leq A\hat{D}$ and $A(A\hat{D}) = D$, by remplacing $D$ with $A\hat{D}$, we note that, for every $C \in C_A$, there exists also $e_C \in \bigcup_{F \in F_A^{(A,D)} \setminus \{A\}} \text{Frac}(F, D)$, such that

$$m(A, A\hat{D}) - (-1)^{rk A\hat{D}} v(A\hat{D}, D) m(A, D) = \sum_{C \in C_A} e_C \gamma(C).$$

Therefore,

$$m(A, D) - (-1)^{rk D} v(D, A\hat{D}) m(A, A\hat{D}) = \sum_{C \in C_A} a_C \gamma(C)$$

$$m(A, D) - v(D, A\hat{D}) v(A\hat{D}, D) m(A, D) = \sum_{C \in C_A} (a_C + (-1)^{rk D} v(D, A\hat{D}) e_C) \gamma(C)$$

$$m(A, D) = \sum_{C \in C_A} a_C + (-1)^{rk D} v(D, A\hat{D}) e_C \gamma(C),$$

with $\frac{a_C + (-1)^{rk D} v(D, A\hat{D}) e_C}{1 - b_A} \in \bigcup_{F \in F_A^{(A,D)}} \text{Frac}(F, D) = \text{Frac}(A, D)$.

For every chamber $D \in C_A$, we have $m(O, D) = D$. So, there exist $x_C \in \text{Frac}(O, D)$ such that $D = \sum_{C \in C_A} x_C \gamma(C)$, and $\gamma^{-1}(D) = \sum_{C \in C_A} x_C C$. Therefore, each entry of $\gamma^{-1}$ is an element of

$$\bigcup_{C \in C_A} \text{Frac}(O, C).$$

We can deduce the conjecture of Zagier.

**Corollary 4.2.** Let $n \geq 2$, and suppose that $q_{i,j} = q_{j,i} = q$. Then, each entry of $\gamma_n^{-1}$ is an element of

$$\prod_{i \in [n-1]} (1 - q^{2i+1}).$$

**Proof.** Let $(O = F_0 \prec F_1 \prec \cdots \prec F_{n-1} = C_o) \in F_{A_n}^{(O[C_o])}$, $i + 1 \in [n]$, and $j \in [n]$ such that $j \leq i + 1$. The face $F_i$ has the form

$$F_i = \{ x \in \mathbb{R}^n \mid x_{\sigma(1)} < \cdots < x_{\sigma(j)} = x_{\sigma(j+1)} = \cdots = x_{\sigma(j+n-i-1)} < \cdots < x_{\sigma(n)} \}.$$ 

Then, $b_{F_i} = \prod_{\{k,l\} \in \binom{\{1,2\}}{2}} q_{k,l} q_{l,k}$ with $I_i = \{ \sigma(j), \sigma(j+1), \ldots, \sigma(j+n-i-1) \}$. If $q = q_{k,l} = q_{l,k}$, then $b_{F_i} = q^{(n-i-1)^2 + (n-i-1)}$, and each entry of $\gamma_n^{-1}$ is an element of

$$\prod_{i \in [n-1]} (1 - q^{2i+1}).$$

$$\square$$
References

[1] M. Aguiar, S. Mahajan, *Topics in Hyperplane Arrangements*, Mathematical Surveys and Monographs 226, 2017.

[2] G. Duchamp, A. Klyachko, D. Krob, J.-Y. Thibon, *Noncommutative Symmetric Functions III: Deformations of Cauchy and Convolution Algebras*, Discrete Math. Theor. Comput. Sci. (1) (1997), 159–216.

[3] O. Greenberg, *Example of Infinite Statistics*, Phys. Rev. Lett. (64) 7 (1990), 705–708.

[4] O. Greenberg, *Particles with small Violations of Fermi or Bose Statistics*, Phys. Rev. D (43) 12 (1991), 4111–4120.

[5] S. Meljanac, D. Svrtan, *Study of Gram Matrices in Fock Representation of Multiparametric Canonical Commutation Relations, extended Zagier’s Conjecture, Hyperplane Arrangements and Quantum Groups*, Math. Commun. (1) 1 (1996).

[6] H. Randriamaro, *Computing the Varchenko Determinant of a Bilinear Form*, Irish Math. Soc. Bulletin (82) (2018), 79–90.

[7] D. Zagier, *Realizability of a Model in Infinite Statistics*, Comm. Math. Phys. (147) (1992), 199–210.