Gravitational Collapse of Perfect Fluid in N-Dimensional Spherically Symmetric Spacetimes

Jaime F. Villas da Rocha *

Departmento de Astronomia Galática e Extra-Galática, Observatório Nacional – CNPq, Rua General José Cristino 77, São Cristóvão, 20921-400 Rio de Janeiro – RJ, Brazil

Anzhong Wang †

Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, Maracanã, 20550-013 Rio de Janeiro – RJ, Brazil

(November 12, 1999)

The Riemann, Ricci and Einstein tensors for N-dimensional spherically symmetric spacetimes in various systems of coordinates are studied, and the general metric for conformally flat spacetimes is given. As an application, all the Friedmann-Robertson-Walker-like solutions for a perfect fluid with an equation of state \( p = k \rho \) are found. Then, these solutions are used to model the gravitational collapse of a compact ball, by first cutting them along a timelike hypersurface and then joining them with asymptotically flat Vaidya solutions. It is found that when the collapse has continuous self-similarity, the formation of black holes always starts with zero mass, and when the collapse has no such symmetry, the formation of black holes always starts with a finite non-zero mass.

I. INTRODUCTION

Critical phenomena in gravitational collapse have attracted much attention \( \|$ since the pioneering work of Choptuik \( \|$ . From the known results obtained so far, the following emerges \( \|$ : Critical collapse of a fixed matter field in general can be divided into three different classes according to the self-similarities that the critical solution possesses. If the critical solution has no self-similarity, continuous or discrete, the formation of black holes always starts with a mass gap (Type I collapse), otherwise it will start with zero mass (Type II collapse), and the mass of black holes takes the scaling form

\[
M_{\text{BH}} \propto (P - P^*)^\gamma,
\]

where \( P \) characterizes the strength of the initial data. In the latter case, the collapse can be further divided into two subclasses according to whether the critical solution has continuous self-similarity (CSS) or discrete self-similarity (DSS). Because of this difference, the exponent \( \gamma \) is usually also different. Whether the critical solution is CSS, DSS, or none of them, depending on both the matter field and the regions of the initial data space \( \|$ . The co-existence of Type I and Type II collapse was first found in the SU(2) Einstein-Yang-Mills case \( \|$ , and later extended to both the Einstein-scalar case \( \|$ and the Einstein-Skyme case \( \|$ , while the co-existence of CSS and DSS critical solutions was found in the Brans-Dicke theory \( \|$ .

The uniqueness of the exponent \( \gamma \) in Type II collapse is well understood in terms of perturbations \( \|$ , and is closely related to the fact that the critical solution has only one unstable mode. This property now is considered as the main criterion for a solution to be critical \( \|$ . While the uniqueness of the exponent \( \gamma \) crucially depends on the numbers of the unstable modes of the critical solution, that \emph{whether or not the formation of black holes starts with a mass gap seemingly only depends on whether the spacetime has self-similarity or not}. Thus, even the collapse is not critical, if a spacetime has CSS or DSS, the formation of black holes may still turn on with zero mass. To study this problem in its general term, it is found difficult. Recently, we studied it for gravitational collapse of massless scalar field and radiation fluid \( \|$ and lately extended it to the case of perfect fluid \( \|$ , and found that when solutions have CSS, the formation of black holes indeed starts with zero-mass, while when solutions have no such symmetry it starts with a mass gap.

*E-mail: roch@on.br
†E-mail: wang@dft.if.uerj.br
Lately, there has been interest in studying critical collapse in higher dimensional spacetimes \[11\]. In particular, it was found that the exponent \(\gamma\) depends on the dimensions of the spacetimes considered \[12\]. This is similar to the critical phenomena in Quantum Field Theory and Statistic Mechanics \[13\].

In this paper, we shall generalize our previous studies to the case of perfect fluid in \(N\)-dimensional spacetimes. Specifically, in Sec. II we shall derive the general metric for conformally flat spherically symmetric spacetimes. As an application, all the Friedmann-Robertson-Walker-like (FRW) solutions for a perfect fluid with a state equation \(p = k\rho\) are found, where \(\rho\) is the energy density of the fluid, \(p\) the pressure, and \(k\) an arbitrary constant. In Sec. III the main properties of these solutions are studied in the context of gravitational collapse, and it is found that some of these solutions represent the formation of black holes, due to the gravitational collapse of the perfect fluid. However, the mass of such formed black holes is usually infinitely large. To remed the shortage, in Sec. IV, the spacetimes are cut along a timelike hypersurface, and then joined to an asymptotically flat Vaidya solution in \(N\)-dimensional spacetimes, so the resulted black holes have finite masses. Sec. V contains our main conclusions, while in Appendix A all the physical quantities, such as, the Christoffel symbols, the Riemann, Ricci, and Einstein tensors, are given in terms of the two metric, \(g_{ab}\), which is orthogonal to the \((N-2)\)-dimensional unit sphere. In Appendix B, the Christoffel symbols, the Riemann, Ricci and Einstein tensors in the \((1+1)\)-dimensional spacetimes, \(g_{ab}\), and the extrinsic curvature of a timelike hypersurface, are given in the three usually used systems of coordinates, the Schwarzschild-like coordinates, Eddington-Finkelstein-like coordinates, and the double null coordinates.

II. THE GENERAL CONFORMALLY FLAT METRIC AND THE FRW SOLUTIONS IN \(N\)-DIMENSIONAL SPACETIMES

The general metric for \(N\)-dimensional spacetimes with spherical symmetry can be split into two blocks,

\[ds^2 = g_{ab}(x^0, x^1)dx^a dx^b - C^2(x^0, x^1)d\Omega^2_{N-2}, \quad (a, b = 0, 1),\]

where \(\{x^\mu\} \equiv \{x^0, x^1, \theta^2, \theta^3, ..., \theta^{N-1}\} \quad (\mu = 0, 1, 2, ..., N-1)\) are the usual \(N\)-dimensional spherical coordinates, \(d\Omega^2_{N-2}\) is the line element on the unit \((N-2)\)-sphere, given by

\[d\Omega^2_{N-2} = (d\theta^2)^2 + \sin^2(\theta^2)(d\theta^3)^2 + \sin^2(\theta^2)\sin^2(\theta^3)(d\theta^4)^2 + \ldots + \sin^2(\theta^2)\sin^2(\theta^3)\sin^2(\theta^{N-2})(d\theta^{N-1})^2\]

\[= \sum_{i=2}^{N-1} \prod_{j=2}^{i-1} \sin^2(\theta^j)(d\theta^i)^2,\]

The corresponding physical quantities, such as, the Christoffel symbols, Riemann, Ricci, and Einstein tensors, are given in Appendix A in terms of the \((1+1)\)-metric, \(g_{ab}\).

It can be shown that with a perfect fluid as source, the Einstein field equations, \(G_{\mu\nu} = \kappa [(\rho + p)u_\mu u_\nu - pg_{\mu\nu}]\), in \(N\)-dimensional spacetimes can be written as

\[(G^0_0 - G^2_2) (G^1_1 - G^2_2) - G^0_0G^1_1 = 0,\]

\[\rho = \kappa^{-1} (G^0_0 + G^1_1 - G^2_2),\]

\[p = -\kappa^{-1} G^2_2,\]

\[u_0^2 = \frac{g_{00}(G^0_0 - G^2_2)}{G^0_0 + G^1_1 - 2G^2_2},\]

\[u_i = 0, \quad (i = 2, 3, 4, ..., N - 1),\]

where \(\kappa \equiv 8\pi G/c^4\) is the Einstein constant, \(u_\mu\) is the velocity of the fluid. Once \(u_0\) is known, the component \(u_1\) can be obtained from the condition \(u_ju^j = 1\). It is interesting to note that these equations were first found by Walker in four-dimensional spacetime \[14\]. However, the above shows that they are valid even in \(N\)-dimensions, and the dimensional dependence of the Einstein field equations appear explicitly only when they are written in terms of the Ricci tensor,

\[(R^0_0 - R^2_2)(R^1_1 - R^2_2) - R^0_0R^1_1 = 0,\]

\[\rho = \frac{1}{2\kappa} (R^0_0 + R^1_1 - NR^2_2),\]
Making the coordinate transformations, \( x^0 = x^0(t, r), \ x^1 = x^1(t, r) \), the metric \( \Box \) can be brought into its isotropic form,

\[
ds^2 = G(t, r)dt^2 - K(t, r) \left( dv^2 + r^2d\Omega_{N-2}^2 \right).
\]

Then, it can be shown that the conformally flat condition \( C_{\mu\nu\lambda\delta} = 0 \), where \( C_{\mu\nu\lambda\delta} \) denotes the Weyl tensor, reduces to a single equation

\[
D_{rr} - \frac{D_r}{r} = 0,
\]

where \( D \equiv (G/K)^{1/2} \), and \((.)_r \equiv \partial(\)/\partial r\), etc. The above equation has the general solution

\[
D(t, r) = f_1(t) + f_2(t)r^2,
\]

where \( f_1 \) and \( f_2 \) are two arbitrary functions of \( t \). Hence, there exist three possibilities,

\[
i) \ f_1(t) \neq 0, f_2(t) = 0, \quad ii) \ f_1(t) = 0, f_2(t) \neq 0, \quad iii) \ f_1(t) \neq 0, f_2(t) \neq 0.
\]

In case i), by introducing a new coordinate \( \tilde{t} = \int f_1(t)dt \) we can bring the metric to a form that is conformally flat to the Minkowski metric. Thus, without loss of generality, in this case we can set \( f_1(t) = 1 \). By a similar argument, we can set \( f_2(t) = 1 \) in cases ii) and iii). Once this is done, cases i) and ii) are not independent. In fact, by a coordinate transform \( r = 1/\tilde{r} \), the metric of case ii) will reduce to that of case i). Therefore, it is concluded that the general conformally flat \( N \)-dimensional metric with spherically symmetry takes the form

\[
ds^2 = G(t, r) \left[ dt^2 - h^2(t, r) \left( dv^2 + r^2d\Omega_{N-2}^2 \right) \right],
\]

where

\[
h(t, r) = \begin{cases} 1, & f_1(t) = 0, \\ \left[ f_1(t) + r^2 \right]^{-1}, & f_1(t) \neq 0. \end{cases}
\]

with \( f_1(t) \neq 0 \). In the following, we shall refer solutions with \( h(t, r) = 1 \) as Type A solutions, and solutions with \( h(t, r) = f_1(t) + r^2 \) as Type B solutions. When \( f_1(t) = \text{Const.} \), say, \( f_1 \), we can introduce a new radial coordinate \( \tilde{r} \) via the relation

\[
\tilde{r} = \frac{r}{f_1 + r^2},
\]

then the metric \( \Box \) becomes

\[
ds^2 = G(t, r) \left( dt^2 - \frac{d\tilde{r}^2}{1 - 4f_1\tilde{r}^2} - \tilde{r}^2d\Omega_{N-2}^2 \right), \quad (f_1(t) = \text{Const.}).
\]
\[ \xi = \frac{2}{(N - 3) + (N - 1)k}. \]  

(22)

The corresponding energy density and velocity of the fluid are given, respectively, by

\[ p = k\rho = 3k\xi^2 P^2 (1 - Pt)^{-2(\xi+1)}, \]
\[ u_0 = (1 - Pt)^{\xi}, \quad u_1 = 0. \]

(23)

When \( k = 0, (N - 1)^{-1} \), the above solutions reduce, respectively, to the one for a dust and radiation fluid, studied in the context of higher dimensional cosmology [15]. Except for these two particular cases, the solutions, to our knowledge, are new. In the following, we shall use them to model the gravitational collapse of a compact ball, and leave the study of their cosmological implications, together with the one of Type B solutions to be given below, be considered in [16].

**Type B solutions.** In this case, the condition \( G(t, r) = G(t), f_1(t) = \text{Const.} \) leads to the following solutions,

\[ h(t, r) = \frac{1}{f_1 + r^2}, \quad G(t) = [A\cosh(\omega t) + B\sinh(\omega t)]^{2\xi}, \]

(24)

where \( \omega \equiv 2\sqrt{-f_1}/\xi \), \( A \) and \( B \) are integration constants, and \( \xi \) is defined by (21). The energy density and velocity of the fluid now are given by

\[ p = k\rho = 12kf_1(A^2 - B^2) [A\cosh(\omega t) + B\sinh(\omega t)]^{-2(1+\xi)}, \]
\[ u_0 = [A\cosh(\omega t) + B\sinh(\omega t)]^\xi, \quad u_1 = 0. \]

(25)

Clearly, this class of solutions also belongs to the FRW family, but with the curvature of the (N-2)-unit sphere different from zero. In fact, when \( f_1 > 0 \), the curvature is positive, and the spacetime is close, and when \( f_1 < 0 \), the curvature is negative, and the spacetime is open. As far as we know, these solutions are new.

It should be noted that the above solutions are valid for any constant \( k \). However, in the rest of the paper we shall consider only the case where \( 0 \leq k \leq 1 \), so that the dominant energy condition is satisfied [17]. When \( N = 4 \), the solutions reduce to the FRW solutions, which have been studied in the context of gravitational collapse in [9,10]. Therefore, in the following we shall assume that \( N \neq 4 \).

**III. GRAVITATIONAL COLLAPSE OF PERFECT FLUID IN N-DIMENSIONAL SPACETIMES**

To study the above solutions in the context of gravitational collapse, we need first to define the local mass function. Recently, Chatterjee and Bhui generalized the Cahill and Macvittie mass function in four-dimensional spacetimes [18] to N-dimensional spacetimes [19],

\[ m_{CB}(t, r) = \frac{(N - 3)r_{ph}^{N-3}}{2} R_{3232}, \]

(26)

where \( r_{ph} \) is the physical radius defined by \( r_{ph} = rhG^{1/2} \) for the metric (2). It can be shown that this definition is consistent with, but not equal to, the following one,

\[ 1 - \frac{2m(t, r)}{B_N r_{ph}^{N-4}} = -g^\mu\nu r_{ph,\mu} r_{ph,\nu}, \]

(27)

where

\[ B_N = \frac{\kappa \Gamma \left( \frac{N-1}{2} \right)}{2(N-2)\pi^{(N-1)/2}}, \]

(28)

with \( \Gamma \) denoting the gamma function. Clearly, when \( N = 4 \), it reduces to the one usually used in four-dimensional spacetimes [20], and when the spacetime is static, it will give the correct mass of black holes in N-dimensions [21]. Thus, in this paper we shall use Eq. (27) as the definition for the mass function, from which we can immediately localize the apparent horizons, which are given by

\[ g^\mu\nu r_{ph,\mu} r_{ph,\nu} = 0. \]

(29)
Then, the mass function on the apparent horizon is given by

$$M_{AH} = \frac{B_N}{2} r_{AH}^{N-3},$$  \hspace{1cm} (30)$$

which is usually taken as the mass of black holes in gravitational collapse. With the above definition for the mass function, let us study the main properties of the above two types of solutions separately.

A. Type A solutions

The mass function defined by Eq.(27) in this case takes the form

$$m(t, r) = \frac{B_N}{2} \xi^2 P^2 r^{N-1} \left( \frac{1}{1 - Pt} \right)^{2-\xi(N-3)},$$  \hspace{1cm} (31)$$

while Eq.(29) has the solution

$$r_{AH} = \frac{1}{\xi} \left| \frac{1 - Pt}{P} \right|^{-1},$$  \hspace{1cm} (32)$$

which represents the location of the apparent horizon of the solutions. When $\xi = 1$, the apparent horizon represents a null surface in the $(t, r)$-plane, and when $0 \leq \xi < 1$, the apparent horizon is spacelike, while when $\xi > 1$, it is timelike.

The spacetime is singular when $t = 1/P$. This can be seen, for example, from the Kretschmann scalar, which now is given by

$$R \equiv R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 6 \xi^2 P^4 \left\{ N - 1 + \xi^2 \left[ N - 2 + \sum_{A=1}^{N-3} (N - 3 - A) \right] \right\} (1 - Pt)^{-4(1+\xi)}. $$  \hspace{1cm} (33)$$

When $P > 0$, it can be shown that the singularity always hides behind the apparent horizon, and when $P < 0$, the singularity is naked. In the latter case, the solutions can be considered as representing cosmological models, while in the former the solutions as representing the formation of black holes due to the gravitational collapse of the perfect fluid. Substituting Eq.(22) into Eq.(31) we find that, as $t \to +\infty$, the mass of the black hole becomes infinitely large.

To remedie this shortage, in the next section we shall cut the spacetime along a timelike hypersurface, say, $r = r_0$, and then join the part with $r < r_0$ with an asymptotically flat Vaidya solution in N-dimensional spacetimes.

It is interesting to note that this class of solutions admits a homothetic Killing vector,

$$\zeta_0 = \frac{1 - Pt}{(1 + \xi)P}, \quad \zeta_1 = \frac{r}{1 + \xi},$$  \hspace{1cm} (34)$$

which satisfies the conformal Killing equation,

$$\zeta_{\mu;\nu} + \zeta_{\nu;\mu} = 2g_{\mu\nu}. $$  \hspace{1cm} (35)$$

Introducing two new coordinates via the relations,

$$\bar{t} = \frac{(1 - Pt)\xi + 1}{(1 + \xi)P}, \quad \bar{r} = r^{1+\xi},$$  \hspace{1cm} (36)$$

we find that the metric can be written in an explicit self-similar form,

$$ds^2 = d\bar{t}^2 - \left[ (\xi + 1)^{-1/\xi} Px \right]^{2\xi} d\bar{r}^2 - \left[ (\xi + 1)Px \right]^{2\xi} \bar{r}^2 d\Omega_{N-2}^2,$$  \hspace{1cm} (37)$$

where $x \equiv \bar{t}/\bar{r}$.

It is well-known that an irrotational “stiff” fluid ($k = 1$) in four-dimensional spacetimes is energetically equal to a massless scalar field. It can be shown that this also the case for N-dimensional spacetimes. In particular, for the above solutions with $k = 1$, the corresponding massless scalar field $\phi$ is given by

$$\phi = \pm \left[ \frac{N - 1}{\kappa(N - 2)} \right]^{1/2} \ln (1 - Pt) + \phi_0,$$  \hspace{1cm} (38)$$

where $\phi_0$ is a constant.
B. Type B solutions

The solutions in this case are given by (24). When \( f_1 > 0 \), the spacetime is close, and to have the metric real the constant \( B \) has to be imaginary. The spacetimes are singular when

\[
|t|_{f_1 > 0} = \frac{1}{|\omega|} \arctan \left( \frac{|B|}{A} \right) + 2n\pi,
\]

where \( n \) is an integer. When \( f_1 < 0 \), the spacetime is singular only when

\[
|t|_{f_1 < 0} = \frac{1}{\omega} \arctanh \left( \frac{B}{A} \right).
\]

Therefore, in the following we shall consider only the case where \( f_1 < 0 \). In this case, to have the energy density of the fluid be non-negative, we need to impose the condition \( B^2 \geq A^2 \). Then, the metric coefficient \( G(t) \) can be written as

\[
G = (B^2 - A^2)^{\frac{\epsilon}{2}} \sinh^{2\xi} \left[ \omega(t_0 - \epsilon t) \right],
\]

where \( \epsilon = \text{sign}(B) \), and \( t_0 \) is defined as

\[
\sinh(\omega t_0) = \frac{A}{(B^2 - A^2)^{1/2}}.
\]

Clearly, the conformal factor \((B^2 - A^2)^{\frac{\epsilon}{2}} \) does not play any significant role, without loss of generality, in the following we shall set it to be one. If we further introduce a new radial coordinate via the relation,

\[
\bar{r} = -\int h(t,r)dr = \frac{1}{a} \ln \left| \frac{a + r}{a - r} \right|,
\]

where \( a \equiv (-f_1)^{1/2} \), the corresponding metric takes the form,

\[
ds^2 = \sinh^{2\xi} \left[ \frac{2a}{\xi} (t_0 - \epsilon t) \right] \left\{ dt^2 - dr^2 - \frac{\sinh^2(2r)}{a^2} d\Omega^2_{N-2} \right\}.
\]

Note that in writing the above expression, we had dropped the bars from \( r \). Then, the mass function and Kretschmann scalar are given, respectively, by

\[
m(r,t) = \frac{B_N}{2N-2} \sinh^{N-1}(2r) \sinh^{\xi(N-3)-2} \left[ 2\xi^{-1}(t_0 - \epsilon t) \right],
\]

\[
R = \frac{96}{\xi^2} \left\{ (N - 1) + \xi^2 \left[ N - 2 + \sum_{A=1}^{N-3} (N - 2 - A) \right] \right\} \times \sinh^{-4(\xi+1)} \left[ 2\xi^{-1}(t_0 - \epsilon t) \right],
\]

while the apparent horizon is located at

\[
r = r_{AH} \equiv \xi^{-1}(t_0 - \epsilon t).
\]

From the above equations, we can see that the solutions are singular on the hypersurface \( t = \epsilon t_0 \). When \( \epsilon = +1 \), the singularity is hidden behind the apparent horizon, and the solutions represent the formation of black holes from the gravitational collapse of the fluid. When \( \epsilon = -1 \), the singularity is naked, the solutions can be considered as representing cosmological models or white holes. As in the type A case, the mass of such formed black holes also diverges at the limit \( t \to +\infty \). Thus, to have finite masses of black holes, we also need to make a “surgery” to the spacetimes. This will be considered, together with the Type A case, in the next section.

Before closing this section, we note that, when \( k = 1 \), the corresponding massless scalar field is given by

\[
\phi = \pm \left[ \frac{N - 1}{\kappa(N - 2)} \right]^{1/2} \ln \left\{ \tanh[(N - 2)(t_0 - \epsilon t)] \right\} + \phi_0, \ (k = 1).
\]
IV. MATCHING THE SOLUTIONS WITH AN OUTGOING RADIATION FLUID

In order to have the black hole mass finite, one way is to cut the spacetime along a timelike hypersurface, say, \( r = r_0(t) \), and then join the internal part with an asymptotically flat spacetime. From Eqs. (23) and (25) we can see that the fluid is comoving in both of the two cases. Thus, the timelike hypersurface now should be given by

\[
r = r_0 = \text{Const.}
\]

(47)

Then, from Eq. (35) we find that the extrinsic curvature is given by

\[
K_{\tau\tau}^- = 0, \quad K_{22}^- = \left\{ \prod_{k=2}^{i-1} \sin^2 (\theta_k) \right\}^{-1} K_{ii}^- = J'(r_0)J(r_0)F^\xi(t), \quad (i = 2, 3, 4, \ldots, N-1),
\]

(48)

where a prime denotes the ordinary differentiation, and

\[
F(t) = \begin{cases} 
1 - Pt, & \text{Type A}, \\
\sinh[2a\xi^{-1}(t_0 - \epsilon t)], & \text{Type B},
\end{cases}
\]

\[
J(r) = \begin{cases} 
r, & \text{Type A}, \\
\frac{1}{2} \sinh(2r), & \text{Type B}.
\end{cases}
\]

(49)

There are various possibilities to choose the solutions outside the hypersurface \( r = r_0 \). In this paper we shall choose the out-going Vaidya solutions in N-dimensions \[25\],

\[
ds^2 = \left[ 1 - \frac{2m(v)}{B_N R^{N-1}} \right] dv^2 + 2dvdR^2 - R^2 d\Omega_{N-2}^2,
\]

(50)

which is a particular case of Eq. (B.10). The hypersurface \( r = r_0 \) in these coordinates is given by \( R = R_0(v) \), or

\[
R = R(\tau), \quad v = v(\tau),
\]

(51)

where \( \tau \) is defined via the relation

\[
d\tau = \left[ 1 - \frac{2M(\tau)}{B_N R^{N-1}} + 2 \frac{dR}{dv} \right]^{1/2} dv,
\]

(52)

where \( M(\tau) \equiv m(v(\tau)) \). This equation can be also written as,

\[
M(\tau) = \frac{B_N R^{N-3}}{2 \dot{v}^2} \left( \dot{v}^2 + 2\ddot{v}R - 1 \right).
\]

(53)

Then, from Eq. (B.17) we find that the extrinsic curvature in these coordinates is given by

\[
K_{\tau\tau}^+ = \frac{\dot{v}}{v} - \frac{\dot{v} M(\tau)}{R^{N-2}},
\]

\[
K_{22}^+ = \prod_{k=2}^{i-1} \sin^2 (\theta_k) \left\{ \prod_{k=2}^{i-1} \sin^2 (\theta_k) \right\}^{-1} K_{ii}^+ = R \left\{ \dot{v} \left[ 1 - \frac{2M(\tau)}{(N-3)R^{N-3}} \right] + \ddot{R} \right\}, \quad (i = 2, 3, 4, \ldots, N-1).
\]

(54)

Hence, the surface energy-momentum tensor, defined by \[22\],

\[
\tau_{AB} = -\frac{1}{\kappa} \left\{ [K_{AB}] - g_{AB} [K] \right\}, \quad (A, B = 1, 2, \ldots, N-1),
\]

(55)

where \( [K_{AB}] \equiv K_{AB}^+ - K_{AB}^- \), \( [K] = g^{AB}[K_{AB}] \), now can be written in the form,

\[
\tau_{AB} = \sigma w_A w_B + \eta \left[ \sum_{i=2}^{N-1} \theta_{(i)A} \theta_{(i)B} \right],
\]

(56)

where
\[
\sigma = \frac{2}{\kappa} \left\{ \dot{R} - \frac{1}{\nu} + J'(r_0) \right\},
\]
\[
\eta = \frac{1}{2\kappa R\nu} \left\{ (N-3)\dot{v}^2 + (N-4)\dot{R} - 2\dot{v}R - 2\dot{v}J'(r_0) + (5-N) \right\},
\]
\[
w_A = \delta_A^i \sin^2(\theta^i) \delta_A^j, \quad \theta_{(i)A} = \frac{1}{\sin^2(\theta)} \sum_{j=2}^{N-1} \sin^2(\theta^j) \delta_A^j, \quad (i = 2, 3, \ldots, N-1).
\]

To fix the motion of the shell, we need to specify the equation of state of the shell, which will be taken as \( \eta = \alpha \sigma \), where \( \alpha \) is an arbitrary constant. Then, Eq. (57) yields,
\[
(N-3)\dot{v}^2 - 2R\dot{v} + [N-4(1+\alpha)]\dot{v}\dot{R} - 2(1+2\alpha)J'(r_0)\dot{v} + (5+4\alpha-N) = 0.
\]

Unfortunately, we are not able to solve the above equation to get \( \nu(\tau) \), except for the cases where
\[
\alpha = -\frac{N-4}{4}.
\]

In order to have the shell satisfy the dominant energy condition \([17]\), \( \alpha \) is restricted to \( 0 \leq \alpha \leq 1 \). Then, from the above expression we can see that this is the case only when \( 4 \leq N \leq 8 \). Since the case where \( N = 4 \) has been already considered in \([11]\), in the following we shall exclude this case.

### A. Type A solutions

In this case, it can be shown that Eqs. (58) and (59) have the integral,
\[
\dot{v}(\tau) = \frac{1 - Y}{1 - (N-3)Y} v_0, \quad (60)
\]
where
\[
x = \frac{[\xi + 1](\tau_0 - \tau)]^{1/\xi}, \quad R_0 \equiv r_0 P^{1/\xi}, \quad Y \equiv e^{[(N-4)(2v_0 R_0 - \xi)]/(2\alpha R_0)},
\]
and \( v_0 \) and \( \tau_0 \) are integration constants. Substituting the above expressions into Eq. (53), we find that
\[
M(x) = \frac{2B_N R_0^{N-3}}{2} \left\{ [2(N-4) - ((N-3)^2 - 1)Y] xY - 2(1 - Y)[1 - (N-3)Y] \xi R_0 \right\} (1 - Y)^{-2}x^{(N-3)-1}.
\]

At the moment \( \tau = \tau_{AH} \) (or \( x = x_{AH} = \xi R_0 \)), the shell collapses inside the apparent horizon. Consequently, the total mass of such formed black hole is given by
\[
M_{BH} \equiv M(x_{AH}) = \frac{2B_N}{2} \left\{ 4(N-3) + [2 - (N-2)^2] Y_{ah} - 2 \right\} \xi r_0^{(1+\xi)} Y_{ah}^{-(N-3)} P^{(N-3)\xi},
\]

where \( Y_{ah} \equiv e^{(N-4)(2v_0 - \xi)/2} \). Clearly, this mass is finite and can be positive by a proper choice of the parameter \( v_0 \) for any given \( \xi \). The contributions of the fluid and the shell to this mass are, respectively, given by,
\[
m^f_{BH} \equiv m^f_{AH}(\tau_{AH}) = \frac{4B_N}{2} \xi r_0^{(N-3)(1+\xi)} P^{(N-3)},
\]
\[
m^{shell}_{BH} \equiv 4\pi B_N R^{N-2}(\tau_{AH})\sigma(\tau_{AH}) = \frac{8\pi B_N}{\kappa} \left( \xi r_0^{(1+\xi)} \right)^{N-3} Y_{ah} P^{(N-3)\xi} (Y_{ah} - 1).
\]

From the above expressions we can see that all the masses are proportional to \( P \), the parameter that characterizes the strength of the initial data of the collapsing ball. Thus, when the initial data is very weak (\( P \to 0 \)), the mass of the formed black hole is very small (\( M_{BH} \to 0 \)). In principle, by properly tuning the parameter \( P \) we can make it as small as wanted. Recall that now the solutions have CSS.

Note that although the mass of black holes takes a scaling form in terms of \( P \), the exponent \( \gamma \) is not uniquely defined. This is because in the present case the “critical” solution (\( P = 0 \)) separates black holes from white holes, and the latter is not the result of gravitational collapse. Thus, the solutions considered here do not really represent the critical collapse. As a result, we can replace \( P \) by any function \( P(\bar{P}) \), and for each of such replacements, we will have a different \( \gamma \) \([23]\). However, such replacements do not change the fact that by properly tuning the parameter we can make black holes with masses as small as wanted. It should be also noted that the definition of the total mass of a thin shell is not unique. It is possible to use equally other definitions, such as, \( m^{shell}_{BH} \equiv M_{BH} - m^f_{BH} \), but our final conclusions will not depend on them.
B. Type B solutions

In this case, the integration of Eq. (58) yields,

\[
\dot{v} = \frac{1}{2(N-3)} \left\{ n_0 - n_1 \tanh \left( t_1 + \frac{n_1}{2 \sinh(2r_0)} \right) \right\},
\]

where \( n_0 \equiv (N-2) \cosh(2r_0), \quad n_1 \equiv \left[ (N-2)^2 \cosh^2(2r_0) - 4(N-3) \right]^{1/2}, \)

and \( t_1 \) is an integration constant. This solution reduces to the ones studied in [9,10] when \( N = 4 \). It can be shown that now the total mass of the black hole is given by

\[
M_{BH} = \frac{B_N \sinh^{N-3}(2r_0) \left\{ \sinh \left[ 2 \xi^{-1}(t_0 - t) \right] \right\}^{(N-3)-1}}{2N-2n_4^2} \left\{ n_3^2 \sinh \left[ 2 \xi^{-1}(t_0 - t) \right] - 4(N-3)n_4 \cosh \left[ t_1 + n_3t \right] \sinh(2r_0) \cos^2 \left[ t_1 + n_3t \right] \left[ 2 \xi^{-1}(t_0 - t) \right] \right\},
\]

where

\[
n_3 \equiv \frac{n_1}{2 \sinh(2r_0)}, \quad n_4 \equiv n_0 \cosh \left[ t_1 + n_3t \right] - n_1 \cosh \left[ t_1 + n_3t \right].
\]

The contributions of the collapsing fluid and shell to the total mass of black hole are given, respectively, by

\[
M_{BH}^f = m_{AH}^f(\tau_{AH}) = \frac{B_N}{2N-2} \sinh^{N-3}(\xi+1)(2r_0),
\]

\[
M_{BH}^{shell} = 4\pi B_N R^{N-2}(\tau_{AH}) \sigma(\tau_{AH}) = \frac{\pi B_N \sinh^{(N-3)+1}(2r_0)}{\kappa \eta 4h 2^{N-1}} \times \left\{ n_4h \cosh(2r_0) - 2(N-3) \cosh \left[ t_1 + n_3(t_0 - \xi r_0) \right] \right\},
\]

where \( n_4h = n_0 \cosh[t_1 + n_3(t_0 - \xi r_0)] \). From the above expressions we can see that for any given \( r_0 \), \( M_{BH}, M_{BH}^f \) and \( M_{BH}^{shell} \) are always finite and non-zero. Thus, in the present case black holes start to form with a mass gap.

V. CONCLUDING REMARKS

The N-dimensional spherically symmetric spacetimes have been studied, and the Christoffel symbols, the Riemann, Ricci and Einstein tensors have been given explicitly in the three usually used systems of coordinates, the Schwarzschild-like coordinates, Eddington-Finkelstein-like coordinates, and the double null coordinates. We wish that this would simplify the studies of these spacetimes. The general form of the metric for conformally flat spacetimes has been found. As an application of it, all the Friedmann-Robertson-Walker-like solutions for a perfect fluid with an equation of state \( p = \kappa \rho \) are found. Then, these solutions have been used to model the gravitational collapse of a compact ball, by first cutting them along a timelike hypersurface and then joining them with asymptotically flat Vaidya solutions in N-dimensional spacetimes [24]. It has been shown that when the collapse has continuous self-similarity, the formation of black holes always starts with zero mass, and when the collapse has no such a symmetry, the formation of black holes always starts with a finite non-zero mass. This is consistent with our previous results obtained in four-dimensional spacetimes [9,10]. Thus, they provide further evidences to support the speculation that the formation of black holes always starts with zero mass or not is closely related to the symmetries of the collapse (CSS or DSS), rather than to that whether the collapse is critical or not.
APPENDIX A: THE CHRISTOFFEL SYMBOLS, THE RIEMANN, RICCI AND EINSTEIN TENSORS IN N-DIMENSIONAL SPHERICALLY SYMMETRIC SPACETIMES

The general metric for N-dimensional spacetimes with spherical symmetry takes the form,

$$ds^2 = g_{ab}(x^0, x^1)dx^a dx^b - C^2(x^0, x^1)dx^2_{N-2}, (a, b = 0, 1).$$  \(\text{(A.1)}\)

In this paper, we shall follow the conventions of d’Inverno [26], except for the following: Greek indices run from 0 to \(N - 1\), lower-case latin, \(a, b, c, d, ...,\) from 0 to 1, lower-case latin, \(i, j, k, l, ...,\) from 2 to \(N - 1\), and upper-case latin, \(A, B, C, D, ...,\) from 1 to \(N - 1\).

It can be shown that the non-vanishing Christoffel symbols, defined by

$$N\Gamma_{\mu\lambda}^\nu = \frac{1}{2}N g^{\mu\sigma} \left( N g_{\sigma\lambda,\nu} + N g_{\nu\sigma,\lambda} - N g_{\nu\lambda,\sigma} \right), \ (\mu, \nu, \lambda = 0, 1, 2, ..., N - 1),$$  \(\text{(A.2)}\)

are given by

$$N\Gamma^a_{bc} = \Gamma^{a}_{bc}, \quad N\Gamma^{a}_{a2} = N\Gamma^{a}_{ai} = \frac{C_a}{C},$$

$$N\Gamma^{a}_{22} = \left[ \prod_{k=3}^{i-1} \sin^2 \left( \theta^k \right) \right]^{-1} N\Gamma^{a}_{ii} = C C^a,$$

$$N\Gamma^{a}_{33} = \left[ \prod_{k=3}^{i-1} \sin^2 \left( \theta^k \right) \right]^{-1} N\Gamma^{a}_{ii} = -\sin(\theta^2) \cos(\theta^2),$$

$$N\Gamma^{i}_{jj} (j > i) = -\cos(\theta^i) \sin(\theta^i) \prod_{k=i+1}^{j-1} \sin^2 \left( \theta^k \right),$$

$$N\Gamma^{i}_{2i} = \cotan(\theta^2), \quad N\Gamma^{i}_{ij} (j > i) = \cotan(\theta^i),$$  \(\text{(A.3)}\)

where \((\_)_a \equiv \partial(\_)/\partial x^a\), \(i, j = 2, 3, 4, ..., N - 1\), \(a, b, c = 0, 1\), and \(\Gamma^{a}_{bc}\) denote the Christoffel symbols calculated from the two metric \(g_{ab}\). Note that in Eq.\((\text{A.3)}\) the repeating indices, one is up and the other is down, do not represent sum. The same for the cases where the two same indices are all down.

The Riemann tensor, defined by,

$$N R^{a}_{\nu\sigma\lambda} = N\Gamma^{a}_{\nu\sigma,\lambda} - N\Gamma^{a}_{\nu\lambda,\sigma} + N\Gamma^{a}_{\delta\lambda} \Gamma^{\delta}_{\nu\sigma} - N\Gamma^{a}_{\delta\sigma} \Gamma^{\delta}_{\nu\lambda},$$  \(\text{(A.4)}\)

has the following non-vanishing components,

$$N R_{abcd} = R_{abcd}, \quad N R_{2323} = -\sin \left( \theta^2 \right)^2 C^2 (1 + C^a C^a),$$

$$N R_{aiab} = \left[ \prod_{k=2}^{i-1} \sin^2 \left( \theta^k \right) \right] CC_{ab},$$

$$N R_{i22i} = -\left[ \prod_{k=2}^{i-1} \sin^2 \left( \theta^k \right) \right] C^2 (1 + C^a C^a),$$

$$N R_{ijij} (j > i) = -\left[ \prod_{k=2}^{j-1} \sin^2 \left( \theta^k \right) \right] \left[ \prod_{l=2}^{i-1} \sin^2 \left( \theta^l \right) \right] C^2 (1 + C^a C^a),$$  \(\text{(A.5)}\)

where \(C_{ab}\) denotes the covariant derivative with respect to the two metric \(g_{ab}\), and \(R_{abcd}\) denotes the Riemann tensor calculated from this two metric, which has only one independent component, say, \(R_{0101}\),

$$R_{abcd} = R_{0101} \left( \delta^0_0 \delta^1_0 \delta^0_0 \delta^1_0 - \delta^0_0 \delta^1_0 \delta^1_0 \delta^0_0 - \delta^1_0 \delta^0_0 \delta^0_0 \delta^1_0 + \delta^1_0 \delta^0_0 \delta^1_0 \delta^0_0 \right).$$  \(\text{(A.6)}\)

Defined the Ricci tensor as,

$$N R_{\mu\nu} = N R_{\mu\lambda\nu},$$  \(\text{(A.7)}\)
we find that it has the following non-vanishing components,

\[ N \mathcal{R}_{ab} = R_{ab} - (N - 2) \frac{C_{,ab}}{C}, \quad N \mathcal{R}_{22} = C \mathcal{C} + (N - 3)(1 + C_{,a}C_{,a}), \]

\[ N \mathcal{R}_{ii} = \left[ \prod_{k=2}^{i-1} \sin^2 (\theta^k) \right] N \mathcal{R}_{22}, \quad (i = 2, 3, 4, ..., N - 1), \]  
\[ (A.8) \]

where \( \mathcal{C} \equiv g^{ab}C_{ab} \), and the 2-dimensional Ricci tensor \( R_{ab} \) in terms of \( R_{0101} \) is given by,

\[ R_{ab} = R^c_{\ acb} = R_{0101} \left[ g^{00} \delta^1_0 \delta^1_b - g^{01} (\delta^1_0 \delta^1_b + \delta^0_0 \delta^1_b) + g^{11} \delta^0_0 \delta^1_b \right]. \]  
\[ (A.9) \]

Then, the Ricci scalar reads

\[ N \mathcal{R} = R - \frac{N - 2}{C^2} \left[ (2\mathcal{C} \mathcal{C} + (N - 3)(1 + C_{,a}C_{,a}) \right], \]  
\[ (A.10) \]

where \( R \equiv g^{ab}R_{ab} \), while the Einstein tensor, defined as \( N \mathcal{G}_{\mu\nu} = N^{-1} R_{\mu\nu} - \frac{1}{2} N g_{\mu\nu} N \mathcal{R} \), has the non-vanishing components,

\[ N \mathcal{G}_{ab} = \frac{N - 2}{2C^2} \left\{ g_{ab} \left[ 2\mathcal{C} \mathcal{C} + (N - 3)(1 + C_{,a}C_{,a}) \right] - 2C_{,ab} \right\}, \]

\[ N \mathcal{G}_{22} = -\frac{1}{2} \left\{ C^2 R + (N - 3) \left[ 2\mathcal{C} \mathcal{C} + (N - 4)(1 + C_{,a}C_{,a}) \right] \right\}, \]

\[ N \mathcal{G}_{ii} = \left[ \prod_{k=2}^{i-1} \sin^2 (\theta^k) \right] N \mathcal{G}_{22}, \quad (i = 2, 3, 4, ..., N - 1). \]  
\[ (A.11) \]

It can be shown that when \( N = 4 \) the above expressions are consistent with the corresponding ones given in [20].

**APPENDIX B: THE EXTRINSIC CURVATURE OF A TIMELIKE HYPERSURFACE IN N-DIMENSIONAL SPHERICAL SPACETIMES**

In this appendix, we shall give the extrinsic curvature of a timelike hypersurface in three different systems of coordinates, which are used very often in the literature. They are the Schwarzschild-like coordinates, Eddington-Finkelstein-like coordinates, and the double null coordinates. In the following let us consider them separately.

**A. The Schwarzschild-like coordinates**

In these coordinates, the metric can be cast in the form,

\[ ds^2 = A^2(r, t) dt^2 - B^2(r, t) dr^2 - C^2(r, t) d\Omega_{N-2}^2. \]  
\[ (B.1) \]

Then, the non-vanishing two-dimensional Christoffel symbols are given by

\[ \Gamma^{t}_{tt} = \frac{A_t}{A}, \quad \Gamma^r_{rr} = \frac{B_r}{B}, \quad \Gamma^t_{tr} = \frac{A_r}{A}, \]

\[ \Gamma^r_{tr} = \frac{B_t}{B}, \quad \Gamma^t_{rr} = \frac{B}{A^2 B_t}, \quad \Gamma^{tt} = \frac{A}{B^2 A_r}. \]  
\[ (B.2) \]

We also have

\[ C_{,a}C_{,a} = \frac{C^2}{A^2} - \frac{C_{,r}^2}{B^2}, \]

\[ C_{;ab} = \left( C_{,tt} - \frac{A_t C_{,t}}{A} - \frac{A A_{,r} C_{,r}}{B^2} \right) \delta^0_0 \delta^0_b \]

\[ + \left( C_{,rr} - \frac{A_r C_{,r}}{A} - \frac{B_t C_{,r}}{B} \right) (\delta^0_0 \delta^1_b + \delta^1_0 \delta^0_b) \]
where the intrinsic coordinates are chosen as 

\[ \{ \omega \} \]

the normal vector is given by

\[ n^\omega = \frac{B^2}{B^2_0} \left( \delta^\omega + \delta^\tau \right) \]

Then, the extrinsic curvature, defined by

\[ R_{0101} = AB \left\{ \left( \frac{B_0}{A} \right)_{,t} - \left( \frac{A_r}{B} \right)_{,r} \right\} \]

Substituting these expressions into Eq. (A.11) we find that the resulting expressions for the Einstein tensor are consistent with those given in [15], except for the one of \( G^0_0 \), given by Eq. (5) in [13], where the third term \( \omega''/2 \) should be replaced by \( n\omega'/2 \).

For a timelike hypersurface,

\[ r = r_0(t) \]

the normal vector is given by

\[ n_\alpha = \frac{ABC}{A^2 - r_0(t)B^2} \left( -r_0(t) \delta^\tau + \delta^\tau \right) \]

where \( r_0(t) \equiv dr_0/dt \). On the surface, the metric [3.1] reduces to

\[ ds^2|_{r=r_0(t)} = g_{\alpha\beta} d\xi^\alpha d\xi^\beta = dr^2 - C^2(t, r_0(t)) d\Omega^2_{N-2}, \]

where the intrinsic coordinates are chosen as \( \{ \xi^A \} = \{ \tau, \theta^2, \theta^3, ..., \theta^N \} \), with \( \tau \) being given by

\[ d\tau = \left[ A^2 - r_0(t)^2 B^2 \right]^{1/2} dt. \]

Then, the extrinsic curvature, defined by,

\[ K_{\alpha\beta} = -n_\alpha \left[ \frac{\partial^2 x^\alpha}{\partial \xi^A \partial \xi^B} + \Gamma^\alpha_{\beta\gamma} \frac{\partial x^\beta}{\partial \xi^A} \frac{\partial x^\gamma}{\partial \xi^B} \right] \]

has the following non-vanishing components,

\[ K_{\tau\tau} = \frac{AB}{A^2 - r_0(t)^2 B^2} \left\{ r_0(t)^3 B B_{,t} + r_0(t)^2 \left[ 2 \frac{A_r}{A} - \frac{B_r}{B} \right] \right\} \]

\[ K_{22} = \frac{ABC}{\sqrt{A^2 - r_0(t)^2 B^2}} \left[ \frac{r_0(t) C_{,t} + C_t}{A^2} \right] \]

\[ K_{ii} = \prod_{k=2}^{i-1} \sin^2 (\theta^k) K_{22}, \quad (i = 2, 3, 4, ..., N - 1). \]

(12)
In this case, the metric can be written in the form

$$ds^2 = e^{\psi(v,r)} dv \left[ f(v,r)e^{\psi(v,r)} dv + 2\epsilon dr \right] - C^2(v,r)d\Omega^2_{N-2},$$

where $\epsilon = \pm 1$. When $\epsilon = +1$, the radial coordinate $r$ increases toward the future along a ray $v = \text{Const.}$, i.e., the light cone $v = \text{Const.}$ is expanding. When $\epsilon = -1$, the radial coordinate $r$ decreases toward the future along a ray $v = \text{Const.}$, and the light cone $v = \text{Const.}$ is contracting.

The two-dimensional non-vanishing Christoffel symbols in this case are given by

$$
\Gamma_{\psi \psi}^r = \frac{\epsilon}{2} e^{-\psi} (e^{2\psi} f)_{,r}, \quad \Gamma_{\psi r}^r = \psi_{,r},
$$

$$
\Gamma_{r r}^r = \frac{\epsilon}{2} e^{-\psi} (e^{2\psi} f)_{,r}, \quad \Gamma_{\psi v}^v = \frac{1}{2} \left[ f (e^{2\psi} f)_{,r} + \epsilon e^{\psi} f_{,r} \right].
$$

Combining Eq.(A.3) and Eq.(B.11) given above with Eq.(4) in [19], we find that some non-vanishing terms of the Christoffel symbols are missing in [19].

We also have

$$C^aC_a = 2e^{-\psi}C_{,r}C_{,r} - fC_{,r}^2,$$

$$C_{ab} = \left\{ C_{,v}v - \left[ \psi_{,v} - \frac{\epsilon}{2} e^{-\psi} (e^{2\psi} f)_{,r} \right] C_{,v} - \frac{1}{2} \left[ f (e^{2\psi} f)_{,r} + \epsilon e^{\psi} f_{,r} \right] C_{,r} \right\} \delta_a^v \delta_b^v + \left[ C_{,r,v} - \frac{\epsilon}{2} e^{-\psi} (e^{2\psi} f)_{,r} C_{,r} \right] \left( \delta_a^v \delta_b^v + \delta_a^r \delta_b^r \right) + (C_{,r,v} - \psi_{,r}C_{,r}) \delta_a^v \delta_b^r,$$

$$\Box C = 2\epsilon e^{-\psi}C_{,r,v} - e^{-\psi}(e^{\psi} fC_{,r})_{,r},$$

$$R_{0101} = e^{2\psi} \left\{ e^{-\psi} \psi_{,rr} - f \psi_{,rr} - f\psi_{,r}^2 - \frac{3}{2} \psi_{,rr}f_{,r} - \frac{1}{2} f_{,rr} \right\}.$$

Inserting it into Eqs.(A.3) and (A.11), we find that the resulting expressions for the Riemann and Einstein tensors are consistent with the corresponding ones given in [19].

A timelike hypersurface in these coordinates can be written as

$$r = r_0(v),$$

or

$$r = r(\tau), \quad v = v(\tau).$$

On this hypersurface, the metric will reduce to the one given by Eq.(B.6) with $\tau$ being defined by

$$d\tau = e^{\psi/2} \left( fe^{\psi} + 2\epsilon \frac{dr_0(v)}{dv} \right)^{1/2} dv,$$

while its normal vector is given by

$$n_\alpha = e^{\psi} (-\dot{r} \delta_\alpha^v + \dot{v} \delta_\alpha^r),$$

where $\dot{r} \equiv dr/d\tau$, etc. Then, it can be shown that the extrinsic curvature tensor has the following non-vanishing components,

$$K_{rr} = \frac{\dot{v}}{\dot{v}} + \epsilon \dot{\psi}_{,r} - \frac{\dot{v}}{2} e^{\psi} f_{,r} - \epsilon e^{\psi} f_{,r},$$

$$K_{22} = C \left[ \epsilon (\dot{r}C_{,r} - \dot{v}C_{,v}) + e^{\psi} f_{,r} \right],$$

$$= C \left[ \frac{-\dot{v}}{\dot{v}} \left( 1 + e^{2\psi} f^2 \right) C_{,r} - \epsilon e^{\psi} f_{,r} \right],$$

$$K_{ii} = \prod_{j=2}^{i-1} \sin^2 (\theta^j) K_{22}, \quad (i = 2, 3, 4, ..., N - 1).$$
C. The Double Null Coordinates

In these coordinates, the metric can be written as,

\[ ds^2 = 2e^{2\sigma(u,v)} du dv - C^2(u,v) d\Omega^2_{N-2}. \]  

(B.18)

Then, the two-dimensional non-vanishing Christoffel symbols are given by

\[ \Gamma^v_{uv} = 2\sigma_{,v}, \quad \Gamma^u_{uu} = 2\sigma_{,u}, \]  

(B.19)

while

\[
\begin{align*}
C^{,a}C_{,a} &= 2e^{-2\sigma} C_{,u} C_{,v}, \\
C_{;ab} &= (C_{,v,v} - 2\sigma_{,v} C_{,v}) \delta^v_a \delta^v_b + C_{,u,v} (\delta^v_a \delta^v_b + \delta^v_b \delta^v_a) + (C_{,u,u} - 2\sigma_{,u} C_{,u}) \delta^v_a \delta^v_b, \\
\square C &= 2e^{-2\sigma} C_{,u,v}, \\
R_{0101} &= 2e^{2\sigma} \sigma_{,uv}.
\end{align*}
\]  

(B.20)

Substituting these expressions into Eqs. (A.3) - (A.11), we find that the resulting expressions are consistent with the corresponding ones given in [27] for the case \( N = 4 \).

A timelike hypersurface in these coordinates is given by,

\[ u = u_0(v), \]  

(B.21)

or

\[ u = u(\tau), \quad v = v(\tau). \]  

(B.22)

On this hypersurface, the metric (B.18) also reduces to the one given by Eq. (B.6) but now with \( \tau \) being defined as,

\[ d\tau = \left[ 2e^{2\sigma} \frac{du_0(v)}{dv} \right]^{1/2} dv, \]  

(B.23)

and its normal vector is given by

\[ n_\alpha = \frac{1}{2u} \left( -2e^{2\sigma} u^2 \delta^v_\alpha + \delta^u_\alpha \right). \]  

(B.24)

It can be shown that in these coordinates the extrinsic curvature tensor of the hypersurface has the following non-vanishing components,

\[ K_{\tau\tau} = - \left( 2\sigma_{,u} \dot{u} + \frac{\ddot{u}}{u} \right), \]

\[ K_{22} = \frac{C}{2u} \left( 2C_{,u} u^2 - e^{-2\sigma} C_{,u} \right), \]

\[ K_{ii} = \prod_{k=2}^{i-1} \sin^2(\theta_k) K_{22}, \quad (i = 2, 3, 4, ..., N - 1). \]  

(B.25)

ACKNOWLEDGMENTS

We would like to express our gratitude to S.K. Chatterjee for valuable discussions. The financial assistance from CAPES (JFVR), CNPq (AW) and FAPERJ (AW) is gratefully acknowledged.
[1] C. Gundlach, Adv. Theor. Math. Phys. 2, 1 (1998), gr-qc/9712084 (1997); M.W. Choptuik, “The (unstable) threshold of black hole formation,” gr-qc/9803075, preprint (1998).
[2] M.W. Choptuik, Phys. Rev. Lett. 70, 9 (1993).
[3] A.Z. Wang and H.P. de Oliveira, Phys. Rev. D56, 753 (1997).
[4] M.W. Choptuik, T. Chmaj, and P. Bizoń, Phys. Rev. Lett. 77, 424 (1996); C. Gundlach, Phys. Rev. D55, 6002 (1997).
[5] M.W. Choptuik, S.L. Liebling, and E.W. Hirschmann, Phys. Rev. D55, 6014 (1997); P. Brady, C.M. Chambers, and S.M.C.V. Goncalves, ibid. D56, 6057 (1997).
[6] P. Bizoń and T. Chmaj, Phys. Rev. D58, 041501 (1998); ibid., D59, 104003 (1999).
[7] S.L. Liebling and M.W. Choptuik, Phys. Rev. Lett. 77, 1424 (1996); E.W. Hirschmann and D.M. Eardley, Phys. Rev. D56, 4696 (1997).
[8] T. Hara, T. Koike, and S. Adachi, “Renormalization group and critical behavior in gravitational collapse,” gr-qc/9607010, preprint (1996).
[9] J.F. Villas da Rocha, A.Z. Wang, and N.O. Santos, Phys. Lett. A255, 213 (1999).
[10] A.Z. Wang, J.F. Villas da Rocha, and N.O. Santos, Phys. Rev. D56, 7692 (1997).
[11] J. Soda and K. Hirata, Phys. Lett. B387, 271 (1996); A.V. Frolov, Class. Quantum Grav. 16, 407 (1999).
[12] D. Garfinkle, C. Cutler, and G.C. Duncan, “Choptuik Scaling in Six Dimensions,” gr-qc/9908044, preprint (1999).
[13] N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group,” (Addison-Wesley Publishing Company, New York, 1992).
[14] A.G. Walker, Quart. J. Math. 6, 81 (1935); W.B. Bonnor and H. Knutsen, Inter. J. Theor. Phys. 32, 1061 (1993).
[15] S. Chatterjee and B. Bhui, Mon. Not. R. astr. Soc. 247, 57 (1990).
[16] J.F. Villas da Rocha and A.Z. Wang, in preparation.
[17] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, 1973) pp. 88-96.
[18] M. E. Cahill and G. C. McVittie J. Math. Phys., 11 1382 (1970).
[19] S. Chatterjee and B. Bhui, Inter. J. Theor. Phys. 32, 671 (1993).
[20] E. Poisson and W. Israel, Phys. Rev. D41, 1796 (1990).
[21] R.C. Myers and M.J. Perry, Ann. Phys. (N.Y.), 172, 304 (1986).
[22] R. Tabensky and A.H. Taub, Common. Math. Phys. 29, 61 (1973).
[23] S. Chatterjee B. Bhui, and A. Banerjee J. Math. Phys. 31, 2208 (1990).
[24] W. Israel, Nuovo Cimento, B44, 1 (1966); ibid., B48, 463(E) (1967).
[25] C. Gundlach, “Critical Phenomena in Gravitational collapse” gr-qc/9606023, preprint (1996).
[26] R. d’Inverno, Introducing Einstein’s Relativity (Clarendon Press, Oxford, 1996).
[27] B. Waugh and K. Lake, Phys. Rev. D34, 2978 (1986).