Magnetic energy stored in relativistic force-free magnetosphere

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Abstract. A magnetar is one of variable sources, whose activity is supplied by huge magnetic energy. Magnetar magnetosphere is gradually twisted due to shearing motion at the surface. At the same time, the energy is stored there. When it exceeds a threshold, the energy is abruptly released on a dynamical timescale as observed in energetic flares. Here, static and axially symmetric force-free magnetospheres are calculated in the exterior of non-rotating relativistic star. The magnetic energy increases along a sequence characterized by the twist degree. In a highly twisted case, a magnetic flux rope, in which large amount of toroidal field is confined, is detached in vicinity of the star. It is found that larger amount of energy is capable to be stored in the relativistic magnetosphere. In an extreme case, the magnetic energy in presence of current flow increases up to a few times larger than that of current-free dipole. There is an upper limit in any model, and a catastrophic event like a giant flare would occur, when the field is further twisted.

1. Introduction
A diversity of neutron stars has been gradually disclosed in 50 years since the first discovery of a radio pulsar. Magnetars belong to a peculiar class. Their strong magnetic fields cause a remarkable difference, although there are some exceptional sources such as ‘low-dipole-field’ magnetars. So, the problem is not simple. The activity of magnetars is widely believed to be supplied by the strong magnetic field of \( B_0 = 10^{14} - 10^{15} \) G. Enormous energy (\( \sim 10^{44} - 10^{46} \) erg) instantaneously released in giant flares is a part of magnetic energy \( 10^{48} (B_0 / 10^{15} G)^2 (R/10 \text{km})^3 \) erg. Such strong field may be amplified by dynamo action at the birth, or the progenitor itself may be different. See also [1]. Stellar properties of magnetars are not well-known. For example, suppose that their masses have a bias toward massive one (\( M \approx 2M_\odot \)), then such information would be a key to their origin. Information of stellar radius \( R \) or a ratio \( M/R \) is also equivalently valuable, since their relation is inferred by nuclear equation of state. However, direct observational evidences will be rarely found. Theoretical study, which predicts a particular property through different observed events, might be helpful.

A giant flare model on magnetars is discussed in an analogy of solar flares. In the model, the energy is stored in the magnetospheres by twisting the magnetic field lines. Here, the magnetospheres are calculated by taking into account general relativistic effect (\( M/R = 0.25 \) in geometrical units of \( c = G_N = 1 \)). Results are compared with those obtained in a flat spacetime. It is found that the general relativistic effect can not be neglected, and is therefore necessary for an appreciate model construction.
2. Relativistic force-free magnetosphere
We consider the static magnetic configuration in curved spacetime outside a non-rotating compact object with mass $M$. The metric is given by

$$ds^2 = -\alpha^2 dt^2 + \alpha^{-2} dr^2 + r^2 d\theta^2 + \varpi^2 d\phi^2,$$

where

$$\alpha = \left(1 - \frac{2M}{r}\right)^{1/2}, \quad \varpi = r \sin \theta.$$ (2)

The magnetic field for the axially symmetric case is generally given by

$$\vec{B} = \vec{\nabla} \times \left(\frac{G}{\varpi} \vec{e}_\phi\right) + \frac{S}{\alpha \varpi} \vec{e}_\phi = \frac{\vec{\nabla} \times \vec{e}_\phi}{\varpi} + \frac{S}{\alpha \varpi} \vec{e}_\phi,$$ (3)

where $\vec{e}_\phi = \varpi^{-1} \partial_\phi$ is a unit vector in the azimuthal direction. The force-free magnetic field $\vec{j} \times \vec{B} = 0$ means that current flows along the magnetic field lines. Poloidal component $(\vec{j} \times \vec{B})_\phi = 0$ is reduced to $S = S(G)$, and $4\pi \alpha \vec{j} = S' \vec{B}$. Azimuthal component of the Biot-Savart law is expressed as

$$\alpha^2 \frac{\partial}{\partial r} \left(\alpha^2 \frac{\partial G}{\partial r}\right) + \frac{\alpha^2 \sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial G}{\partial \theta}\right) = -SS'.$$ (4)

We adopt a power-law form for the current model with strength parameter $\gamma$ and index $n$, so that the right hand side is $SS' = \gamma G^n$. When $S$ is a linear function of $G$, i.e., $n = 1$, eq.(4) is a wave equation with angular frequency $\gamma^{1/2}$. General relativistic effect appears by a factor $\alpha^2$ of eq.(4). When $\alpha^2 = 1$, we have models in flat spacetime. The correction factor $\alpha^2$ on typical neutron star surface is $\alpha^2 \approx 0.5$ at most, so that the effect might be expected to be small. However, actual results given in next section show that the relativistic correction is not small, since we consider a non-linear system. Qualitative difference due to $\alpha^2$ becomes small with decreasing index $n$. In this paper, we concentrate on $n = 7$ model. See [2] for details of other models.

Magnetic energy is given by integrating over 3-dimensional volume

$$E_{EM} = \int \frac{\alpha B^2}{8\pi} \sqrt{g_3} d^3 x = \frac{1}{4} \left[ \left(\frac{\partial G}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial G}{\partial \theta}\right)^2 + \left(\frac{S}{\alpha}\right)^2 \right] dr d\theta \sin \theta,$$ (5)

where $\sqrt{g_3} = \alpha^{-1} r^2 \sin \theta$ is a determinant of 3-dimensional space metrics. Total relative helicity $H_R$ defined by the integration range $r \geq R$ is

$$H_R = \int \vec{A} \cdot \vec{B} \sqrt{g_3} d^3 x = 4\pi \int \frac{GS}{\alpha^2} dr d\theta \sin \theta,$$ (6)

where $\vec{A}$ is a vector potential of $\vec{B}$.

3. Results
We first construct a magnetosphere model with a dipole in vacuum, and gradually increase the toroidal component for a given boundary condition. It is known from previous calculations[3] in flat spacetime that a free parameter $\gamma$ is not suitable for specifying the sequence of models, because there are multiple solutions concerning $\gamma$. Instead, the azimuthal magnetic flux or helicity is used as the degree of twist, and the constant $\gamma$ is posteriorly determined. We follow
the same method, and calculate the energy and the relative helicity along the sequence of solutions.

Results are shown in Figure 1. The left panel shows a model in flat spacetime $M/R = 0$, while the right a relativistic model with $M/R = 0.25$. Increment of magnetic energy $\Delta E$ from the potential dipole field, which is discriminated by poloidal $\Delta E_p$ or toroidal components $\Delta E_t$, is given as a function of $\gamma$. Relative helicity $H_R$ is also given. Each symbols in the figure represents one force-free solution. There is a maximum in $H_R$, and it is impossible to calculate equilibrium model beyond it. A possible evolutionary sequence is constructed by connecting models in order of $H_R$. That is, magnetosphere is gradually distorted by helicity injection from initial potential vacuum field. The structure of magnetic field lines also changes, and two examples are displayed: potential dipole ($\Delta E = 0$ and $\gamma = 0$) at the bottom, and highly twisted state at the top. Contour lines of magnetic function $G$ are quite different. In the presence of toroidal component, magnetic field lines are outwardly stretched. In an extreme case of relativistic models, a loop structure is formed in the vicinity of the surface. The maximum of magnetic function $G$ is located at $(r/R, \theta) \approx (1.5, \pi/2)$. Three-dimensional structure of it is axially symmetric torus or rope. Both magnetic field components are strong there, since $B_0 = (\alpha \omega)^{-1} S \propto G^4$. Strong currents ($\vec{j} \propto \alpha^{-1} S' \vec{B}$) also flow there. Thus, flux rope is highly non-linear equilibrium structure.

We consider how magnetic energy stored in the magnetosphere increases by twisting fields. The dipolar potential field energy $E_0$ is expressed as $E_0 = 0.33B_0^2R^3$ in a flat spacetime, and $E_0 = 0.71B_0^2R^3$ in a curved spacetime with $M/R = 0.25[2]$. Here we used typical strength $B_0(\equiv \mu R^{-3})$, which is determined by magnetic dipole moment $\mu$ asymptotically at infinity. The difference in $E_0$ between relativistic and non-relativistic models comes from the normalization. For example, the surface field strength in relativistic model is a few times larger. It is therefore

![Figure 1](https://example.com/f1.png)

**Figure 1.** Comparison of models in flat (left panel) and curved spacetime (right panel). Increase in magnetic energy $\Delta E/(B_0^2R^3)$ from the potential dipole field is shown in the left scale and total relative helicity $H_R/(4\pi B_0^2R^4)$ is shown in the right scale, where typical field strength $B_0$ is defined by magnetic dipole moment $\mu$ and the surface radius $R$, $B_0 \equiv \mu R^{-3}$. The poloidal component of the energy is denoted by crosses, the toroidal component by asterisks, and helicity by squares. The horizontal axis denotes the dimensionless value $\gamma(B_0R^2)^{1/2}$. Contour lines of magnetic function $G$ in the $r-\theta$ plane are also displayed nearby: the potential dipolar field at the bottom and, the highly twisted state at the top.
better to compare maximum increment $\Delta E$, in a ratio to $E_0$: $\Delta E/E_0 \approx 0.2$ for a model in a flat spacetime, $\Delta E/E_0 \approx 2$ for a relativistic model. There is still a big difference, which comes from the formation of the flux rope, a braid of poloidal and toroidal magnetic fields. A larger amount of energy is capable of being stored there.

4. Discussion

A family of axially symmetric static force-free magnetosphere models is calculated. Total magnetic energy increases with increasing helicity. Numerical calculations suggest that there is a maximum of the helicity deposited in the magnetosphere. Beyond this, we could not find any static solutions.

We have also compared models in flat and curved spacetimes. General relativistic effects are significant. A larger amount of energy is capable of being stored in the relativistic model. In extreme cases, the magnetic energy increases up to a few times larger than that of a current-free dipole. This marks a contrast with the models in flat spacetime, in which the increment of the magnetic energy is less than 20 percent. This big difference is related to the formation of an axially symmetric magnetic rope in the vicinity of the surface. The rope contains a large amount of magnetic energy constructed by poloidal and toroidal components. The flux rope is not inherent in relativistic magnetosphere, but possibly originates from some nonlinearities. Actually, this kind of structure is also realized in flat spacetime, but the condition to the power-law index $n$ is severe. Namely, the structure is evident only in the model with stronger nonlinearity, $n > 7[3]$. Curvature due to general relativity helps the confinement. See also numerical models[4] obtained by solving relativistic MHD equations with a low-density plasma, in which an evident flux rope can be seen in their magnetosphere. Thus, relativistic model is preferable for larger energy events. At moment, it is impossible to determine the relativistic factor $M/R$ of magnetar only by magnetosphere model. Further improvement of the model is necessary.

The existence of a maximum energy or helicity stored in the magnetosphere seems to be reasonable, and is important for a catastrophic event such as a flare. When the total helicity injected into the magnetosphere exceeds a certain capacity, a transient event may occur subsequently. The transition of the magnetosphere can be calculated by resistive dynamical simulation e.g., [5, 6]. After a burst, magnetic field lines are open with the same boundary condition at surface, after some energy transfers to mass ejection, particle acceleration and the radiation. The transition is therefore allowed, only when open field energy $E_{\text{open}}$ is larger than the maximum energy $E_{\text{max}}$ before the burst. The work to compare $E_{\text{max}}$ with $E_{\text{open}}$ is in preparation.

Acknowledgments
This work was supported by JSPS KAKENHI Grant Number JP26400276.

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