Exclusion sets in eigenvalue inclusion sets for tensors

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Abstract

By excluding some sets, which don’t include any eigenvalue of a tensor, from some existing eigenvalue inclusion sets, two new sets are given to locate all eigenvalues of a tensor. And it is shown that these two sets are contained in the Gershgorin eigenvalue inclusion set of tensors provide by Qi (Journal of Symbolic Computation 2005; 40:1302-1324) and the Brauer-type eigenvalue inclusion set provide by Li et al. (Numer. Linear Algebra Appl. 2014; 21:39-50) respectively. Two sufficient conditions such that the determinant of a tensor is not zero are also provided.

Keywords: Tensor eigenvalue; Exclusion set; Gershgorin set; Brauer-type set

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1. Introduction

We call $\mathcal{A} = (a_{i_1 \ldots i_m})$ a complex (real) tensor of order $m$ dimension $n$, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]} \ (\mathcal{A} \in \mathbb{R}^{[m,n]})$, if

$$a_{i_1 \ldots i_m} \in \mathbb{C} \ (\mathbb{R}),$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. A real tensor $\mathcal{A} = (a_{i_1 \ldots i_m})$ is called symmetric if

$$a_{i_1 \ldots i_m} = a_{\pi(i_1 \ldots i_m)}, \forall \pi \in \Pi_m,$$

where $\Pi_m$ is the permutation group of $m$ indices. Furthermore, a complex number $\lambda$ is called an eigenvalue of $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$ and a nonzero complex vector $x$ an eigenvector of $\mathcal{A}$ associated with $\lambda$ if $\lambda$ and $x$ satisfy

$$\mathcal{A}x^{m-1} = \lambda x^{m-1},$$

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where

\[(A x^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \quad x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T\]

and \(N = \{1, 2, \ldots, n\}\). This definition was introduced by Qi in [16] where he assumed that \(A\) is an order \(m\) dimension \(n\) symmetric tensor and \(m\) is even. Independently, in [13], Lim gave such a definition but restricted \(x\) to be a real vector and \(\lambda\) to be a real number. In this case, we call \(\lambda\) an H-eigenvalue of \(A\) and \(x\) an H-eigenvector of \(A\) associated with \(\lambda\) [15, 16]. Note that there are other definitions of eigenvalue and eigenvectors, such as, \(D\)-eigenvalue and \(Z\)-eigenvalue; see [4, 8, 17, 18, 19, 20, 23].

One of the important problems on eigenvalues of a tensor is to locate all its eigenvalues, i.e., to give a set including all its eigenvalues in the complex plane. The first work owes to Liqun Qi. He in [16] gave an eigenvalue inclusion set for real symmetric tensors, which is a generalization of the well-known Geršgorin set of matrices [5, 21, 22]. Subsequently, Li et al. provided some Brauer-type eigenvalue inclusion sets for general tensors [3, 10, 11], and shown that the Brauer-type eigenvalue inclusion sets capture all eigenvalues of a tensor precisely than the set given by Qi. Very recently, Bu et al. extend the Brualdi set of matrices to higher order tensors. In addition, another eigenvalue inclusion sets were also given, for details, see [2, 3, 7, 12].

When constructing these existing eigenvalue inclusion sets, one didn’t consider the problem that whether or not there is some proper subset of these sets in which each eigenvalue of a tensor is not included. In this paper, by using (1) and the eigenvector corresponding an eigenvalue of a tensor, we give some such sets, and exclude them respectively from the Geršgorin set of tensors in [16] and the Brauer-type eigenvalue inclusion set in [9] to give two new sets including all eigenvalues of a tensor. As applications, two sufficient conditions such that the determinant of a tensor is not zero are also provided.

2. Exclusion sets in the Geršgorin set for tensors

In [16], Qi extended the well-known Geršgorin’s eigenvalue inclusion theorem [5, 21, 22] of matrices to real symmetric tensors. This result can be easily generalized to general tensors [24] (see Theorem 1).

**Theorem 1.** Let \(A = (a_{i_2 \ldots i_m}) \in \mathbb{C}^{[m,n]}\). Then

\[\sigma(A) \subseteq \Gamma(A) = \bigcup_{i \in N} \Gamma_i(A),\]

where \(\sigma(A)\) is the set of all the eigenvalues of \(A\),

\[
\Gamma_i(A) = \{z \in \mathbb{C} : |z - a_{i_2 \ldots i_m}| \leq r_i(A)\}, \quad r_i(A) = \sum_{i_2 \ldots i_m \in N, \delta_{i_2 \ldots i_m} = 0} |a_{i_2 \ldots i_m}|,
\]
and
\[ \delta_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases} \]

\( \Gamma_i(A) \) is a disk in the complex plane with \( a_{i_1} \) as its center and \( r_i(A) \) as their radii. Obviously, \( \Gamma(A) \) consists of \( n \) disks. The proof of Theorem 1 relies on (1), and is listed as follows, which is useful for getting some exclusion sets.

**The proof of Theorem 1** Suppose that \( \lambda \in \sigma(A) \) with a corresponding eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \). Let
\[ |x_t| = \max_{i \in N} |x_i|. \]
Consider the \( t \)th equation of (1). We have
\[ (\lambda - a_{i_1 \cdots i_m})x_t^{m-1} = \sum_{\delta_{i_1 \cdots i_m} = 0} a_{i_1 \cdots i_m}x_{i_1} \cdots x_{i_m}. \]
Taking absolute values on both sides and using the triangle inequality yields
\[ |\lambda - a_{i_1 \cdots i_m}| |x_t|^{m-1} = \sum_{\delta_{i_1 \cdots i_m} = 0} |a_{i_1 \cdots i_m}| |x_{i_1}| \cdots |x_{i_m}| \leq \sum_{\delta_{i_1 \cdots i_m} = 0} |a_{i_1 \cdots i_m}| |x_t|^{m-1} = r_i(A)|x_t|^{m-1}. \]
Hence,
\[ |\lambda - a_{i_1 \cdots i_m}| \leq r_i(A), \]
that is,
\[ \lambda \in \Gamma_i(A). \]

We do not know which \( t \) each eigenvalue corresponds to, hence we have \( \lambda \in \bigcup_{i \in N} \Gamma_i(A) \), consequently, \( \sigma(A) \subseteq \Gamma(A) \). \( \square \)

It is easy to see that we only use the largest modulus \( |x_i| \) of the eigenvector \( x \) and the \( t \)-th equation of (1) in the proof of Theorem 1. However, the other components of the eigenvector \( x \) and the other equations of (1) are not considered, which may result in losing some informations on \( \Gamma(A) \). Next, by considering the other components \( x_j \) of the eigenvector \( x \) with \( j \neq t \), we give an improvement of \( \Gamma(A) \).

**Theorem 2.** Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{C}^{[m,n]} \). Then
\[ \sigma(A) \subseteq \Omega(A) = \bigcup_{i \in N} \Omega_i(A), \]
where \( \Omega_i(A) = \Gamma_i(A) \setminus \Delta_i(A) \),
\[ \Delta_i(A) = \bigcup_{j \neq i} \Delta_{ij}(A) \]
and
\[ \Delta_{ij}(A) = \{ z \in \mathbb{C} : |z - a_{j_1 \cdots j} - 2 |a_{ji} \cdots |a_{ji}| - r_j(A) \}. \]
Furthermore, \( \Omega(A) \subseteq \Gamma(A) \).
Proof. Suppose that \( \lambda \in \sigma(A) \) with a corresponding eigenvector \( x = (x_1, x_2, \ldots, x_n)^T \). According to the proof of Theorem 1, (2) holds. Furthermore, for any \( j \in N \) and \( j \neq t \), we have

\[
|x_j| \leq |x_t|,
\]

and

\[
(\lambda - a_{j\ldots t})x_j^{m-1} = \sum_{\delta_{ji2\ldots im}=0, \delta_{\bar{i}2\ldots im}=0} a_{ji2\ldots im} x_i \cdots x_m + a_{jt\ldots t} x_t^{m-1}.
\]

Hence,

\[
a_{jt\ldots t} x_t^{m-1} = (\lambda - a_{j\ldots t})x_j^{m-1} - \sum_{\delta_{ji2\ldots im}=0, \delta_{\bar{i}2\ldots im}=0} a_{ji2\ldots im} x_i \cdots x_m
\]

and

\[
|a_{jt\ldots t}||x_t|^{m-1} \leq |\lambda - a_{j\ldots t}||x_j|^{m-1} + \sum_{\delta_{ji2\ldots im}=0, \delta_{\bar{i}2\ldots im}=0} |a_{ji2\ldots im}||x_i| \cdots |x_m|
\]

\[
\leq |\lambda - a_{j\ldots t}||x_t|^{m-1} + \sum_{\delta_{ji2\ldots im}=0, \delta_{\bar{i}2\ldots im}=0} |a_{ji2\ldots im}||x_t|^{m-1},
\]

which implies

\[
|a_{jt\ldots t}| \leq |\lambda - a_{j\ldots t}| + \sum_{\delta_{ji2\ldots im}=0, \delta_{\bar{i}2\ldots im}=0} |a_{ji2\ldots im}|
\]

and

\[
|\lambda - a_{j\ldots t}| \geq |a_{jt\ldots t}| - \sum_{\delta_{ji2\ldots im}=0, \delta_{\bar{i}2\ldots im}=0} |a_{ji2\ldots im}| = 2|a_{jt\ldots t}| - r_j(A),
\]

i.e.,

\[
\lambda \notin \Delta_{tj}(A).
\]

Note that (3) holds for any \( j \neq t \). Then

\[
\lambda \notin \bigcup_{j \neq t} \Delta_{tj}(A) = \Delta_t(A).
\]

Combining (2) and (4) gives

\[
\lambda \in \Gamma_t(A) \setminus \Delta_t(A) = \Omega_t(A),
\]

consequently, \( \lambda \in \bigcup_{i \in N} \Omega_i(A) = \Omega(A) \) and \( \sigma(A) \subseteq \Omega(A) \).

Moreover, from

\[
\Omega_i(A) = \Gamma_i(A) \setminus \Delta_i(A) \subseteq \Gamma_i(A)
\]

we can easily obtain \( \Omega(A) \subseteq \Gamma(A) \). The proof is completed. \( \square \)
Remark 1. Note that $|a_{ji...i}| \leq r_j(A)$ and $2|a_{ji...i}| - r_j(A) \leq r_j(A)$. Hence,

$$\Delta_{ij}(A) \subseteq \Gamma_j(A), j \neq i, j \in N$$

and

$$\Delta_i(A) = \bigcup_{j \neq i} \Delta_{ij}(A) \subseteq \Gamma(A).$$

On the other hand, it is shown by Theorem 2 that $\Delta_i(A)$ does not include any eigenvalues of a tensor $A$, and $\Omega(A)$ is obtained by excluding some proper subsets $\Delta_i(A)$ from the Geršgorin set $\Gamma(A)$. And hence $\Delta_i(A)$ is a so-called exclusion set for the Geršgorin set $\Gamma(A)$.

Consider the tensor $A = (a_{ijk}) \in \mathbb{C}[3,4]$, where

- $a_{111} = 12, a_{222} = 14, a_{333} = 8 + i, a_{444} = 11,$
- $a_{122} = 4 + i, a_{144} = 15 - i, a_{233} = 5 - i, a_{211} = -2 - i,$
- $a_{322} = 6, a_{344} = 4, a_{411} = 16, a_{422} = 2,$

and other $a_{ijk} = 0$. The sets $\Omega_1(A), \Omega_2(A), \Omega_3(A)$ and $\Omega_4(A)$ are drawn in Figure 1. And their union $\Omega(A)$ are drawn in Figure 2. The exact eigenvalues of $A$ are plotted with asterisks, which are computed by the MATLAB code `solve`.

The determinant of a tensor $A \in \mathbb{C}^{m,n}$, denoted by $\det(A)$, is the resultant of the ordered system of homogeneous equations $A x^{m-1} = 0$ and is closely related to the eigenvalue inclusion set of a tensor. Next, based on Theorem 2 and the fact that $\det(A) = 0$ if and only if $0 \in \sigma(A)$ for a tensor $A$, we can easily obtain the following condition such that $\det(A) \neq 0$.

**Corollary 1.** Let $A = (a_{i_1i_2...i_n}) \in \mathbb{C}^{m,n}$. If for each $i \in N$, either

- $|a_{i...i}| > r_i(A)$

or

- $|a_{j...j}| < 2|a_{ji...i}| - r_j(A)$ for some $j \neq i$, then $\det(A) \neq 0$.

A matrix is a tensor of order 2. Hence, when $m = 2$, Theorem 2 reduces to the following result.

**Corollary 2.** Let $A = (a_{ij})$ be a complex matrix. Then

$$\sigma(A) \subseteq \Omega(A) = \bigcup_{i \in N} \Omega_i(A),$$
where $\Omega_i(A) = \Gamma_i(A) \setminus \Delta_i(A)$,
$$\Delta_i(A) = \bigcup_{j \neq i} \Delta_{ij}(A)$$
and
$$\Delta_{ij}(A) = \{ z \in \mathbb{C} : |z - a_{jj}| < 2|a_{ji}| - r_j(A) \}.$$  
Furthermore, $\Omega(A) \subseteq \Gamma(A)$.

Remark here that the set $\Omega(A)$ in Corollary 2 is a correction of the eigenvalue inclusion set
$$\bigcup_{i \in \mathbb{N}} \left( \Gamma_i(A) \setminus \bigcup_{j \neq i} \Delta'_{ij}(A) \right)$$
for matrices in [14], where
$$\Delta'_{ij}(A) = \{ z \in \mathbb{C} : |z - a_{jj}| \geq 2|a_{ji}| - r_j(A) \}.$$  

3. Exclusion sets for the Brauer-type set for tensors

Another well-known eigenvalue inclusion set for matrices are provided by Brauer in [1]. In [9] Li et al. gave an example to show that this set cannot be extended to higher order tensors, and gave a Brauer-type set to locate all eigenvalues of a tensor as follows.

**Theorem 3.** Let $A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$. Then
$$\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{i,j \in \mathbb{N}, j \neq i} \mathcal{K}_{ij}(A),$$
where
$$\mathcal{K}_{ij}(A) = \{ z \in \mathbb{C} : \left( |z - a_{i_1 \ldots i_m}| - r_j^i(A) \right) |z - a_{j_1 \ldots j_m}| \leq |a_{ij_1 \ldots j_m}|r_j(A) \}$$
and
$$r_j^i(A) = \sum_{\delta_{i_1 \ldots i_m} = 0, \delta_{j_1 \ldots j_m} = 0} |a_{i_1 \ldots i_m}| = r_i(A) - |a_{ij_1 \ldots j_m}|.$$

Next we try to find some proper subsets of $\mathcal{K}(A)$ in which there is not any eigenvalue of a tensor $A$.

**Theorem 4.** Let $A = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$ with $n \geq 2$. Then
$$\sigma(A) \subseteq \Theta(A) = \bigcup_{i,j \in \mathbb{N}, j \neq i} \Theta_{ij}(A),$$
where $\Theta_{ij}(A) = \mathcal{K}_{ij}(A) \setminus \Lambda_i(A)$, $\Lambda_i(A) = \bigcup_{p \neq i} \Lambda_{ip}(A)$ and
$$\Lambda_{ip}(A) = \{ z \in \mathbb{C} : (|z - a_{i_1 \ldots i}| + r_p^i(A))|z - a_{p_1 \ldots p_m}| < |a_{ip_1 \ldots p_m}|(2|a_{p_1 \ldots i}| - r_p(A)) \}.$$  
Furthermore, $\Theta(A) \subseteq \mathcal{K}(A)$.  

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Proof. For any \( \lambda \in \sigma(A) \), let \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n \setminus \{0\} \) be an associated eigenvector, i.e.,

\[
A x^{m-1} = \lambda x^{m-1}.
\]

Let

\[
|x_t| \geq |x_s| \geq \max\{|x_k| : k \in N, k \neq s, k \neq t\}
\]

(where the last term above is defined to be zero if \( n = 2 \)). Obviously, \( |x_t| > 0 \).

From (4), we have

\[
(\lambda - a_{t \cdots t}) x_t^{m-1} = \sum_{\delta_{t_2 \cdots t_m} = 0, \delta_{s_2 \cdots s_m} = 0} a_{t_2 \cdots t_m} x_{t_2} \cdots x_{t_m} + a_{t s \cdots s} x_s^{m-1}.
\]

Taking modulus in the above equation and using the triangle inequality gives

\[
|\lambda - a_{t \cdots t}| |x_t|^{m-1} \leq \sum_{\delta_{t_2 \cdots t_m} = 0, \delta_{s_2 \cdots s_m} = 0} |a_{t_2 \cdots t_m}| |x_{t_2}| \cdots |x_{t_m}| + |a_{t s \cdots s}| |x_s|^{m-1}
\]

\[
\leq \sum_{\delta_{t_2 \cdots t_m} = 0, \delta_{s_2 \cdots s_m} = 0} |a_{t_2 \cdots t_m}| |x_{t_2}| \cdots |x_{t_m}| + |a_{t s \cdots s}| |x_s|^{m-1}
\]

\[
= r^s(A) |x_t|^{m-1} + |a_{t s \cdots s}| |x_s|^{m-1},
\]

equivalently,

\[
(|\lambda - a_{t \cdots t}| - r^s(A)) |x_t|^{m-1} \leq |a_{t s \cdots s}| |x_s|^{m-1}.
\]

(5)

If \( |x_s| = 0 \), then \( |\lambda - a_{t \cdots t}| - r^s(A) \leq 0 \) as \( |x_t| > 0 \), and it is obvious that \( \lambda \in K_{t s}(A) \subseteq K(A) \). Otherwise, \( |x_s| > 0 \). Moreover, from (4), we similarly get

\[
|\lambda - a_{s \cdots s}| |x_s|^{m-1} \leq r_s(A) |x_t|^{m-1}.
\]

(6)

Multiplying Inequality (5) with Inequality (4), we have

\[
(|\lambda - a_{t \cdots t}| - r^s(A)) |\lambda - a_{s \cdots s}| |x_t|^{m-1} |x_s|^{m-1} \leq |a_{t s \cdots s}| r_s(A) |x_t|^{m-1} |x_s|^{m-1}.
\]

Note that \( |x_t|^{m-1} |x_s|^{m-1} > 0 \). Then

\[
(|\lambda - a_{t \cdots t}| - r^s(A)) |\lambda - a_{s \cdots s}| \leq |a_{t s \cdots s}| r_s(A),
\]

which implies

\[
\lambda \in K_{t s}(A).
\]

(7)

By the \( p \)-th equation of (1) for each \( p \neq t \), we have

\[
(\lambda - a_{p \cdots p}) x_p^{m-1} = \sum_{\delta_{p_2 \cdots p_m} = 0, \delta_{t_2 \cdots t_m} = 0} a_{p_2 \cdots p_m} x_{p_2} \cdots x_{p_m} = a_{p t \cdots t} x_t^{m-1}
\]

\[7\]
and
\[ |a_{pt \ldots t}|x_t|^{m-1} \leq |\lambda - a_{p \ldots p}|x_t|^{m-1} + \sum_{\delta_{pi_2 \ldots i_m} = 0, \delta_{ti_2 \ldots i_m} = 0} |a_{pi_2 \ldots i_m}|x_t|^{m-1}, \]
equivalently,
\[ (2|a_{pt \ldots t}| - r_p(A))|x_t|^{m-1} \leq |\lambda - a_{p \ldots p}|x_t|^{m-1}. \] (8)
Similarly, by the \( t \)th equation of (11), we have
\[ (\lambda - a_{t \ldots t})x_t^{m-1} - \sum_{\delta_{ti_2 \ldots i_m} = 0, \delta_{pi_2 \ldots i_m} = 0} a_{ti_2 \ldots i_m}x_{i_2} \cdot \cdot \cdot x_{i_m} = a_{tp \ldots p}x_p^{m-1} \]
and
\[ |a_{tp \ldots p}|x_p|^{m-1} \leq |\lambda - a_{t \ldots t}|x_t|^{m-1} + \sum_{\delta_{ti_2 \ldots i_m} = 0, \delta_{pi_2 \ldots i_m} = 0} |a_{ti_2 \ldots i_m}|x_t|^{m-1}, \]
equivalently,
\[ |a_{tp \ldots p}|x_p|^{m-1} \leq (|\lambda - a_{t \ldots t}| + r_t^p(A))|x_t|^{m-1}. \] (9)
If \( |x_p| > 0 \), then multiplying Inequality (8) with Inequality (9) gives
\[ (2|a_{pt \ldots t}| - r_p(A))|a_{tp \ldots p}|x_p|^{m-1}|x_t|^{m-1} \leq |\lambda - a_{p \ldots p}|(|\lambda - a_{t \ldots t}| + r_t^p(A))|x_t|^{m-1}|x_p|^{m-1} \]
and
\[ (2|a_{pt \ldots t}| - r_p(A))|a_{tp \ldots p}| \leq |\lambda - a_{p \ldots p}|(|\lambda - a_{t \ldots t}| + r_t^p(A)), \] (10)
i.e.,
\[ \lambda \notin \Lambda_{tp}(A). \] (11)
If \( |x_p| = 0 \), then \( 2|a_{pt \ldots t}| - r_p(A) \leq 0 \) holds from (8), and (10) also holds, consequently, (11) holds. Note that (11) holds for any \( p \neq t \). Hence,
\[ \lambda \notin \bigcup_{p \neq t} \Lambda_{tp}(A). \] (12)
By (7) and (12) we have
\[ \lambda \in K_{ts}(A) \setminus \left( \bigcup_{p \neq t} \Lambda_{tp}(A) \right), \]
this implies
\[ \lambda \in \left( \bigcup_{i,j \in N, j \neq t} K_{ij}(A) \setminus \bigcup_{p \neq i} \Lambda_{ip}(A) \right) = \bigcup_{i,j \in N, j \neq t} K_{ij}(A) \setminus \Lambda_i(A) = \bigcup_{i,j \in N, j \neq t} \Theta_{ij}(A) = \Theta(A). \]
Furthermore, from \( \Theta_{ij}(A) = K_{ij}(A) \setminus \Lambda_i(A) \subseteq K_{ij}(A) \), we have \( \Theta(A) \subseteq K(A) \). The conclusion follows. \( \square \)
From Remak [1], we have that for any $p \neq i$, $2|a_{pi-i}| - r_p(A) \leq r_p(A)$ and then

$$\Lambda_{ip}(A) \subseteq K_{ip}(A).$$

However, $\lambda \notin \Lambda_{ip}(A)$ for each $\lambda \in \sigma(A)$. Hence, $\Lambda_{ip}(A)$ is a so-called exclusion set for the Brauer-type set $K(A)$. Consider again the tensor $A$ in Remak [1]. The sets $\Theta_{ij}(A)$, $j \neq i$ are drawn in Figure 3, and their union $\Theta(A)$ are drawn in Figure 4. It is easy to see that $\sigma(A) \subseteq \Theta(A)$ and $\Theta(A) \subseteq K(A)$. In addition, by the relationship of $K(A)$ and $\Gamma(A)$, that is, $K(A) \subseteq \Gamma(A)$, we have

$$\Theta(A) \subseteq \sigma(A) \subseteq \Gamma(A).$$

However, $\Theta(A) \subseteq \Omega(A)$ may not hold in general, which can be shown by Figure 2 and Figure 4.

Similarly to Corollary [1] and Corollary [2], we can obtain the following results from Theorem [3].

**Corollary 3.** Let $A = (a_{ij}) \in \mathbb{C}^{m,n}$. If for any $i, j \in N$ and $j \neq i$, either

$$\left(|a_{i\cdots i}| - r_j^p(A)\right)|a_{j\cdots j}| > |a_{ij}|r_j(A)$$

or

$$\left(|a_{i\cdots i}| + r_j^p(A)\right)|a_{p\cdots p}| < |a_{ip\cdots p}|(2|a_{pi\cdots i}| - r_p(A)) \text{ for some } p \neq i,$$

then $\det(A) \neq 0$.

**Corollary 4.** Let $A = (a_{ij})$ be a complex matrix. Then

$$\sigma(A) \subseteq \Theta(A) = \bigcup_{i \in N, j \neq i} \Theta_{ij}(A),$$

where $\Theta_{ij}(A) = K_{ij}(A) \setminus \Lambda_i(A)$, $\Lambda_i(A) = \bigcup_{p \neq i} \Lambda_{ip}(A)$ and

$$\Lambda_{ip}(A) = \{z \in \mathbb{C} : (|\lambda - a_{ii}| + r_j^p(A))|\lambda - a_{pp}| < |a_{ip}|(2|a_{pi}| - r_p(A))\}.$$

4. Conclusions

In this paper, we exclude some proper subsets respectively, which do not include any eigenvalues of a tensor, from the Geršgorin eigenvalue inclusion set $\Gamma(A)$ of tensors in [10] and the Brauer-type eigenvalue inclusion set $K(A)$ in [9] to give two new eigenvalue inclusion sets $\Omega(A)$ and $\Theta(A)$ with

$$\Omega(A) \subseteq \Gamma(A), \text{ and } \Theta(A) \subseteq K(A).$$

Besides the sets $\Gamma(A)$ and $K(A)$, there are another eigenvalue inclusion sets, such as the sets in [2, 3, 7, 10, 11, 12]. Hence, for these sets it is interesting to find their proper subsets which do not include any eigenvalue of a tensor to exclude them.
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References

[1] Brauer A. Limits for the characteristic roots of a matrix II. *Duke Mathematical Journal* 1947; **14**:21-26.

[2] Bu CJ, Wei YP, Sun LZ, Zhou J. Brualdi-type eigenvalue inclusion sets of tensors. *Linear Algebra and its Applications*. 2015; **480**:168-175.

[3] Bu CJ, Jin XQ, Li HF, Deng Cl. Brauer-type eigenvalue inclusion sets and the spectral radius of tensors. *Linear Algebra and its Applications* 2017; **512**: 234-248.

[4] Cartwright D, Sturmfels B. The number of eigenvalues of a tensor. *Linear Algebra and its Applications* 2013; **438**: 942-952.

[5] Geršgorin S. Über die Abgrenzung der Eigenwerte einer Matrix. *Izvestija Akademii Nauk SSSR, Serija Matematika* 1931; **7**(3):749-754.

[6] Hu SL, Huang ZH, Ling C, Qi LQ. On determinants and eigenvalue theory of tensors, *J. Symbolic Comput.* 2013; **50**: 508-531.

[7] Huang ZG, Wang LG, Xu Z, Cui, JJ. A new S-type eigenvalue inclusion set for tensors and its applications. *Journal of Inequalities and Applications*, 2016; **2016**: 254.

[8] Kolda TG, Mayo JR. Shifted power method for computing tensor eigen-pairs. *SIAM Journal on Matrix Analysis and Applications*, 2011; **32**: 1095-1124.

[9] Li CQ, Li YT, Kou X. New eigenvalue inclusion sets for tensors, *Numerical Linear Algebra and its Applications*, 2014; **21**:39-50.

[10] Li CQ, Zhou JJ, Li YT.A new Brauer-type eigenvalue localization set for tensors, *Linear and Multilinear Algebra*, 2016; **64**: 727-736.

[11] Li CQ, Li YT. An eigenvalue localization set for tensors with applications to determine the positive (semi-) definiteness of tensors, *Linear and Multilinear Algebra*, 2016; **64**: 587-601.

[12] Li CQ, Jiao AQ, Li YT. An S-type eigenvalue localization set for tensors, *Linear Algebra and its Applications*, 2016; **493**:469-483.
[13] Lim LH, Singular values and eigenvalues of tensors: A variational approach. in CAMSAP’05: Proceeding of the IEEE International Workshop on Computational Advances in MultiSensor Adaptive Processing. 2005; 129-132.

[14] Melman A. Gershgorin Disk Fragments, Mathematics Magazine 2010: 83:123-129.

[15] Qi LQ. Eigenvalues of a supersymmetric tensor and positive definiteness of an even degree multivariate form, Department of Applied Mathematics, The Hong Kong Polytechnic University, 2004.

[16] Qi LQ. Eigenvalues of a real supersymmetric tensor. Journal of Symbolic Computation 2005; 40:1302-1324.

[17] Qi LQ. Eigenvalues and invariants of tensors. Journal of Mathematical Analysis and Applications 2007; 325:1363-1377.

[18] Qi LQ, Sun W, Wang Y. Numerical multilinear algebra and its applications. Frontiers of Mathematics in China 2007; 2:501-526.

[19] Qi LQ, Wang F, Wang Y. Z-eigenvalue methods for a global polynomial optimization problem. Mathematical Programming 2009; 118:301-316.

[20] Qi LQ, Wang Y, Wu EX. D-eigenvalues of diffusion kurtosis tensors. Journal of Computational and Applied Mathematics 2008; 221:150-157.

[21] Varga RS. Geršgorin and his circles. Springer-Verlag, Berlin, 2004.

[22] Varga RS, Krautstengl A. On Geršgorin-type problems and ovals of cassini. Electronic Transactions on Numerical Analysis 1999; 8:15-20.

[23] Wang Y, Qi LQ, Zhang X. A practical method for computing the largest M-eigenvalue of a fourth-order partially symmetric tensor. Numerical Linear Algebra with Applications 2009; 16:589-601.

[24] Yang YN, Yang QZ. Further results for Perron-Frobenius Theorem for non-negative tensors, SIAM. J. Matrix Anal. Appl. 2010; 31 2517-2530.
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