Cosymmetries and Nijenhuis recursion operators for difference equations

Alexander V Mikhailov, Jing Ping Wang and Pavlos Xenitidis

1 Department of Applied Mathematics, University of Leeds, UK
2 School of Mathematics, Statistics and Actuarial Science, University of Kent, UK

Received 13 September 2010, in final form 12 May 2011
Published 6 June 2011
Online at stacks.iop.org/Non/24/2079

Abstract
In this paper we discuss the concept of cosymmetries and co-recursion operators for difference equations and present a co-recursion operator for the Viallet equation. We also discover a new type of factorization for the recursion operators of difference equations. This factorization enables us to give an elegant proof that the pseudo-difference operator \( R \) presented in Mikhailov et al 2011 Theor. Math. Phys. 167 421–43 is a recursion operator for the Viallet equation. Moreover, we show that the operator \( R \) is Nijenhuis and thus generates infinitely many commuting local symmetries. The recursion operator \( R \) and its factorization into Hamiltonian and symplectic operators have natural applications to Yamilov’s discretization of the Krichever–Novikov equation.

Mathematics Subject Classification: 37K05, 37K10, 35Q53

1. Introduction

The existence of hierarchies of infinitely many symmetries and/or conservation laws is one of the characteristic properties of integrable equations. It has been successfully used in the classification of given families of partial differential and differential–difference equations [1–10].

The infinite hierarchies of symmetries can be generated by recursion operators mapping symmetries to symmetries [11–13]. Most of the known examples of recursion operators have the property of being Nijenhuis operators. This property was independently studied by Fuchssteiner–Fokas [14–16] and Magri [17], who named such operators as hereditary symmetries. It guarantees that one can generate infinitely many local commuting symmetry flows starting from some seed symmetries. Furthermore, the conjugated recursion operators can generate infinitely many local covariants (the variational derivatives of conserved densities). For Hamiltonian systems recursion operators can be factorized into symplectic and Hamiltonian operators.

The most well known example is the famous Korteweg–de Vries (KdV) equation

\[
u_t = K = u_{xxx} + 6uu_x,
\]
which possesses a recursion operator

$$\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1},$$

where $D_x^{-1}$ stands for the inverse of $D_x$. The operator $\mathcal{R}$ satisfying

$$D_{u_{-K}} \mathcal{R} = \mathcal{R} D_{u_{-K}},$$

where $D_{u_{-K}}$ is the Fréchet derivative of the KdV equation, is a Nijenhuis operator and generates the local commuting KdV hierarchy of symmetries

$$u_j = \mathcal{R}^j (u_0), \quad j = 0, 1, 2, \ldots,$$

the first member of which, i.e. a seed symmetry, is $u_0 = u_x$. The next member of the hierarchy is the KdV equation itself.

The operator conjugated to the recursion operator $\mathcal{R}$ for the KdV equation is

$$\mathcal{R}^* = D_x^2 + 4u - 2D_x^{-1} u_x$$

and satisfies the operator equation

$$D_{u_{-K}} \mathcal{R}^* = \mathcal{R}^* D_{u_{-K}}.$$

It recursively produces infinitely many local covariants $G_n = \mathcal{R}^* (G_{n-1}), \; n \in \mathbb{N}$, with $G_0 = 1$, which are also cosymmetries of the KdV equation. Indeed, they satisfy $D_{u_{-K}}^* (G_n) = 0$.

Thus, for the KdV equation the conjugate operator $\mathcal{R}^*$ is a co-recursion operator mapping cosymmetries to cosymmetries.

The recursion operator $\mathcal{R}$ can be represented as the composition of a symplectic operator $\mathcal{I}$ and a Hamiltonian operator $\mathcal{H}_1$, i.e.

$$\mathcal{R} = \mathcal{I} \cdot \mathcal{H}_1, \quad \mathcal{I} = D_x^{-1}, \quad \mathcal{H}_1 = D_x^2 + 4u D_x + 2u_x.$$

The KdV equation is a multi-Hamiltonian system with infinitely many Hamiltonian operators $\mathcal{H}_n = \mathcal{R}^n \cdot \mathcal{H}_0, \; n \in \mathbb{N}$, where $\mathcal{H}_0 = \mathcal{I}^{-1} = D_x$. It is not difficult to show that all Hamiltonian operators are weakly non-local. The concept of weak non-locality was discussed in detail in [18].

Difference equations may possess nontrivial generalized symmetries. Similar to the continuous case, the existence of an infinite algebra of commuting symmetries reflects integrability. In our recent work [19], we defined a pseudo-difference operator, which maps symmetries of difference equations into symmetries. Similar to the continuous case, we called it a recursion operator for difference equations. If a recursion operator exists and satisfies the Nijenhuis property, it generates an infinite hierarchy of symmetries of the difference equation.

For the Viallet equation [22]

$$Q := a_1 u_{0,0} u_{1,0} u_{0,1} u_{1,1} + a_2 (u_{0,0} u_{0,1} u_{1,0} + u_{1,0} u_{0,1} u_{0,0} + u_{0,1} u_{0,0} u_{1,1}) + a_3 (u_{0,0} u_{0,1} + u_{0,1} u_{1,0} + u_{0,0} u_{1,1}) + a_4 (u_{1,0} u_{0,0} + u_{0,0} u_{0,1}) + a_5 (u_{0,0} u_{0,1} + u_{1,0} u_{1,1}) + a_6 (u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) + a_7 = 0,$$

where $a_i$ are free complex parameters, we proposed in [19] a pseudo-difference operator $\mathcal{R}$ (see also (25)) and claimed (theorem 5 in [19]) that $\mathcal{R}$ is a recursion operator for equation (1).

Equation (1) is a quite general and useful example of an integrable difference equation. By a point fractional-linear transformation this equation with a generic choice of parameters can

---

3 Here we would like to clarify that the recursion operator defined in our paper [19] and the operator constructed by Levi–Martina [21] (also called a recursion operator by the authors) are two unrelated objects. While our operator generates symmetries for the difference equations, the operator constructed by Levi–Martina connects difference equations sharing the same Lax operator.
be reduced to Adler’s equation, also referred to as the Q4 equation in the ABS classification [23, 24]. All of the ABS equations can be obtained from the Viallet equation by a simple specialization of the parameters. Thus, the results obtained for the Viallet equation can be easily applied to every equation in the ABS list.

In this paper, we introduce and discuss the concepts of cosymmetries and co-recursion operators for difference equations. We give a complete proof of theorem 5 formulated in [19], namely that the operator \( R \) presented in the theorem is indeed a recursion operator of the Viallet equation (1) and it is a composition of Hamiltonian and symplectic operators. Moreover, this operator possesses the Nijenhuis property and thus produces infinitely many commuting local symmetries starting from two seed symmetries. Equation (1) possesses an infinite hierarchy of cosymmetries which can be generated by a co-recursion operator. Here we would like to stress that the cosymmetries of equation (1) do not coincide with its covariants and that the co-recursion operator is not conjugated to the recursion operator. Since the first symmetry of the Viallet equation (1) coincides with Yamilov’s discretization of the Krichever–Novikov equation (YdKN) [10], as a by-product, our theory immediately provides multi-Hamiltonian structures, a recursion operator and an infinite hierarchy of commuting symmetries for the YdKN equation.

In order to be self-contained we introduce our notation, sketch the framework and give basic definitions in section 2. More detailed description of the framework and motivations for the definitions the reader can find in our previous paper [19]. In section 2.1 we define the field of fractions \( \mathcal{F}_Q \) corresponding to a difference equation \( Q = 0 \), the elimination map \( \mathcal{E} \) and the set of dynamical variables. In the next subsection we give the definitions of symmetries and cosymmetries illustrated by useful examples for equation (1).

In section 3 we discuss difference and pseudo-difference operators. In particular we define weakly non-local pseudo-difference operators. They are difference analogues of weakly non-local pseudo-differential operators in the theory of Poisson brackets (see [18]). We give the definitions of recursion and co-recursion operators for difference equations and show that operator \( R \) (25) is a recursion operator for the Viallet equation (1). On the course of the proof we discover a new factorization property of the recursion operator (25). Namely, there is a constant \( \mu \) (33) such that the operator \( R - \mu \) can be factorized over the field \( \mathcal{F}_Q \) into a difference and a weakly non-local pseudo-difference operator. Interestingly enough, the coefficients of \( R \) are elements of the subfield \( \mathcal{F}_a \subset \mathcal{F}_Q \) (of rational functions of variables \( \{u_n\}_{n \in \mathbb{Z}} \), see section 2.1) while the coefficients of the factors are in \( \mathcal{F}_Q \) and cannot be restricted to \( \mathcal{F}_a \). A co-recursion operator for equation (1) is given explicitly in theorem 2.

In section 4 we discuss Hamiltonian and symplectic operators related to difference equations and show that the recursion operator (25) is hereditary (Nijenhuis). The main result of this section is theorem 3, which states that symmetries of equation (1) generated by the recursion operator \( R \) (25) are all local and commuting, despite of \( R \) being a weakly non-local pseudo-difference operator. The definitions of Hamiltonian and symplectic operators, the hereditary property of a recursion operator as well as the proofs of assertions are not different from the continuous case if we reformulate the problem in terms of a variational complex. It enables us easily to adapt the theory developed for the continuous case [7, 25–27] to the difference one.

2. Framework and basic definitions

To make the paper self-contained, we give the basic definitions of the elimination map, dynamical variables, symmetries and cosymmetries of difference equations. Motivations for these definitions have been discussed in detail in our paper [19].
2.1. Difference equations and dynamical variables

Difference equations on $\mathbb{Z}^2$ can be regarded as a difference analogous of partial differential equations. Let us denote by $u = u(n, m)$ a complex-valued function $u : \mathbb{Z}^2 \mapsto \mathbb{C}$, where $n$ and $m$ are ‘independent variables’ and $u$ will play the role of a ‘dependent’ variable in a difference equation. Instead of partial derivatives we have two commuting shift maps $S$ and $T$ defined as

$$S : u \mapsto u_{1,0} = u(n + 1, m),$$

$$T : u \mapsto u_{0,1} = u(n, m + 1).$$

For uniformity of the notation, it is convenient to denote the ‘unshifted’ function $u$ as $u_{0,0}$. In the theory of difference equations we shall treat $u_{p,q}$ as variables. The set of all shifts of the variable $u$ will be denoted by $U = \{u_{p,q} | (p, q) \in \mathbb{Z}^2\}$. For a function $f = f(u_{p,q}, \ldots, u_{p,q})$ of variables $u_{p,q}$ the action of the operators $S, T$ is defined as

$$S^i T^j (f) = f_{i,j} = f(u_{p+i,q+j}, \ldots, u_{p+i,q+j}).$$

In this paper, we consider quadrilateral difference equations of the form

$$Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0,$$  \hspace{1cm} (2)

where $Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1})$ is an irreducible polynomial of the ‘dependent variable’ $u = u_{0,0}$ and its shifts. Irreducibility means that $Q$ cannot be factorized and presented as a product of two polynomials. It is assumed that equation (2) is valid at every point $(n, m) \in \mathbb{Z}^2$ and thus (2) represents the infinite set of equations

$$Q_{p,q} = Q(u_{p,q}, u_{p+q}, u_{p+q+1}, u_{p+q+1}) = 0, \quad (p, q) \in \mathbb{Z}^2.$$  \hspace{1cm} (3)

Moreover, we assume that $Q$ is an irreducible affine-linear polynomial which depends non-trivially on all variables, i.e.

$$\partial_{u_{i,j}} Q \neq 0, \quad \partial_{u_{i,j}}^2 Q = 0, \quad i, j \in \{0, 1\}, \quad Q \in \mathbb{C}[u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}].$$  \hspace{1cm} (4)

Here and later in the text $\partial_{u_{i,j}} f$ refers to the partial derivative of $f$ with respect to $u_{i,j}$ which we also often denote as $f_{u_{i,j}}$.

Let $\mathbb{C}[U]$ be the ring of polynomials of the variables $U$. Maps $S$ and $T$ are automorphisms of $\mathbb{C}[U]$ and thus $\mathbb{C}[U]$ is a difference ring. We let $\bar{I}_Q = \langle \{Q_{p,q}\} \rangle$ denote the ideal generated by the difference equation and all its shifts (3). For any $a \in \bar{I}_Q$, we have $S(a) \in \bar{I}_Q$ and $T(a) \in \bar{I}_Q$. Therefore, $\bar{I}_Q$ is a difference ideal. If the ideal $\bar{I}_Q$ is prime, then we omit the hat and write $I_Q$; otherwise, we take its algebraic saturation $J_Q = \text{sat}(\bar{I}_Q)$ (the definition of the algebraic saturation $\text{sat}(\bar{I}_Q)$ can be found in [20]). It follows from Theorem 22 in [20] that for irreducible affine-linear equations, the difference ideal $\text{sat}(\bar{I}_Q)$ is prime, and the quotient difference ring $\mathbb{C}[U]/J_Q$ is therefore an integral domain. The corresponding field of fractions we denote as

$$\mathcal{F}_Q = [\{a\}/\{b\}]a, b \in \mathbb{C}[U], \quad \{a\}, \{b\} \not\subseteq J_Q,$$

where $[\{a\}/\{b\}]a, b \in \mathbb{C}[U], \quad \{a\}, \{b\} \not\subseteq J_Q$,

where $[\{a\}/\{b\}]a, b \in \mathbb{C}[U], \quad \{a\}, \{b\} \not\subseteq J_Q$, the two rational functions $a/b$ and $c/d$ represent the same element of $\mathcal{F}_Q$ if $ad - bc \in J_Q$. The field $\mathcal{F}_Q$ is the main object in our theory.

The fields of rational functions of variables

$$U_s = \{u_{n,0} | n \in \mathbb{Z}\}, \quad U_t = \{u_{0,n} | n \in \mathbb{Z}\}, \quad U_0 = U_s \cup U_t$$

are denoted, respectively, as

$$\mathcal{F}_s = \mathbb{C}(U_s), \quad \mathcal{F}_t = \mathbb{C}(U_t), \quad \mathcal{F}_0 = \mathbb{C}(U_0).$$
Since the ideal $J_Q$ is generated by affine-linear polynomials $Q_{p,q}$, we can define the elimination map $E$ which simplifies drastically computations modulo the ideal. Since $Q$ is an affine-linear polynomial, one can uniquely resolve equation $Q = 0$ with respect to each variable

\[ u_{0,0} = F(u_{1,0}, u_{0,1}, u_{1,1}), \quad u_{1,0} = G(u_{0,0}, u_{0,1}, u_{1,1}), \]

\[ u_{0,1} = H(u_{0,0}, u_{1,0}, u_{1,1}), \quad u_{1,1} = M(u_{0,0}, u_{1,0}, u_{0,1}) \]  

and functions $F, G, H$ and $M$ are rational functions of their arguments. Equations (5) enable us recursively and uniquely to express any variable $u_{p,q}$ in terms of the variables $U_0 = U_a \cup U_t$.

**Definition 1.** For elements of $U$ the elimination map $E : U \mapsto \mathbb{C}(U_0)$ is defined recursively:

\[ \forall p \in \mathbb{Z}, \quad E(u_{0,p}) = u_{0,p}, \quad E(u_{p,0}) = u_{p,0}. \]

if $p > 0, q > 0$,
\[ E(u_{p,q}) = M(E(u_{p-1,q-1}), E(u_{p,q-1}), E(u_{p-1,q})), \]

if $p < 0, q > 0$,
\[ E(u_{p,q}) = H(E(u_{p,q-1}), E(u_{p+1,q-1}), E(u_{p+1,q})), \]

if $p > 0, q < 0$,
\[ E(u_{p,q}) = G(E(u_{p-1,q}), E(u_{p-1,q+1}), E(u_{p,q+1})), \]

if $p < 0, q < 0$,
\[ E(u_{p,q}) = F(E(u_{p+1,q}), E(u_{p,q+1}), E(u_{p+1,q+1})). \]

For polynomials $f(u_{p_1,q_1}, \ldots, u_{p_n,q_n}) \in \mathbb{C}[U]$ the elimination map $E : \mathbb{C}[U] \mapsto \mathbb{C}(U_0)$ is defined as

\[ E : f(u_{p_1,q_1}, \ldots, u_{p_n,q_n}) \mapsto f(E(u_{p_1,q_1}), \ldots, E(u_{p_n,q_n})) \in \mathbb{C}(U_0). \]

For rational functions $a/b, a, b \in \mathbb{C}[U], b \notin J_Q$ the elimination map $E$ is defined as

\[ E : a/b \mapsto E(a)/E(b). \]

It follows from the above definition that $E : J_Q \mapsto 0$. The elimination map is the difference field isomorphism $E : F_Q \mapsto F_0$. Any element of $F_Q$ can be uniquely represented by a rational function of variables $U_0$. Variables $U_0$ we shall call the dynamical variables.

### 2.2. Symmetries and cosymmetries of difference equations

Symmetries and conservation laws of difference equations have been discussed in detail in our paper [19]. In this section we recall the definition of a continuous symmetry of a difference equation and give the definition of a cosymmetry. For difference equations cosymmetries do not coincide with covariances (the variational derivatives of conserved densities) as in the case of evolutionary equations. Cosymmetries are new objects in the theory of difference equations, and they will play important role in our construction of recursion and co-recursion operators.

**Definition 2.** Let $Q = 0$ be a difference equation. Then $K \in F_Q$ is called a generator of an infinitesimal symmetry (or simply, a symmetry) of the difference equation if

\[ D_Q(K) = 0. \]  

(7)

Here $D_Q$ is the Fréchet derivative of $Q$ defined as

\[ D_Q = \sum_{i,j \in \mathbb{Z}} R_{a,i} S_j T^j, \quad R_{a,i} = \partial_{a,i} Q. \]  

(8)

For a quadrilateral equation the sum in (8) has only four terms. In (7) $D_Q(K)$ is equal to zero as an element of $F_Q$. The way to check this is to apply the elimination map and therefore $K$ is a symmetry if $E(D_Q(K)) = 0$. If the difference equation $Q = 0$ admits symmetries, then they form a Lie algebra denoted as $\mathfrak{g}_Q$ [19].

Similarly, we define cosymmetries:
**Definition 3.** Let \( Q = 0 \) be a difference equation. Then \( \omega \in \mathcal{F}_Q \) is called a cosymmetry (or a characteristic of a conservation law) of the difference equation if

\[
D_Q^*(\omega) = 0,
\]

where \( D_Q^* \) is a formally adjoint operator of the Fréchet derivative which is defined as

\[
D_Q^* = \sum_{i,j \in \mathbb{Z}} S^{-1} T^{-j} Q_{u_{i,j}}.
\]

Cosymmetries of equation \( Q = 0 \) form a linear space over \( \mathbb{C} \) which we denote as \( \Omega_Q \).

We shall see that equation (1) has infinite hierarchies of symmetries and cosymmetries which can be naturally expressed in terms of the discriminants of \( Q \)

\[
h(u_{0,0}, u_{1,0}) := Q \partial_{u_{k,1}} \partial_{u_{1,1}} Q - \partial_{u_{k,1}} Q \partial_{u_{1,1}} Q;
\]

\[
g(u_{1,0}, u_{0,1}) := Q \partial_{u_{k,1}} \partial_{u_{1,1}} Q - \partial_{u_{k,1}} Q \partial_{u_{1,1}} Q;
\]

and the function

\[
w = \frac{1}{u_{1,0} - u_{-1,0}}.
\]

For any \( f \in \mathcal{F}_Q \) we shall use the abbreviated notation \( f_k = S^k(f) \). In particular

\[
h_k = S^h(u_{0,0}, u_{1,0}), \quad g_k = S^h(g(u_{1,0}, u_{0,1})), \quad w_k = S^h w.
\]

In this notation, the first two symmetries of equation (1) (see [28, 29]) can be written in the following form

\[
K^{(1)} = h w - \frac{1}{2} \partial_{u_{1,1}} h = h_{-1} w + \frac{1}{2} \partial_{u_{-1,1}} h_{-1};
\]

\[
K^{(2)} = h h_{-1} w^2 (w_1 + w_{-1}).
\]

In [29, 30] it was shown that the above symmetries commute

\[
[K^{(1)}, K^{(2)}] = D_{K^{(1)}}(K^{(1)}) - D_{K^{(2)}}(K^{(2)}) = 0.
\]

It is much less known about cosymmetries. For partial differential equations cosymmetries were introduced and studied in [7]. The simplest cosymmetry for equation (1), which we have found by direct solution of equation (9), can be written in the form

\[
\omega^{(1)} = \frac{w_1}{Q_{u_{1,0}}} - \frac{w}{Q_{u_{1,0}}} + \frac{\partial_{u_{1,1}} g}{2g Q_{u_{1,0}}} + \frac{\partial_{u_{1,1}} h}{2h Q_{u_{1,0}}}
\]

It is easy to check that \( \omega^{(1)} \) is a cosymmetry. Indeed, using the elimination map one can show that

\[
S^{-1}(Q_{u_{1,0}} \omega^{(1)}) + \omega^{(1)} Q_{u_{1,0}} = -T^{-1} S^{-1}(Q_{u_{1,1}} \omega^{(1)}) - T^{-1}(\omega^{(1)} Q_{u_{1,1}}) = \frac{K^{(1)}}{h} - \frac{K^{(1)}}{h_{-1}}.
\]

This immediately leads to \( D_Q^* \omega^{(1)} = 0 \) and proves that \( \omega^{(1)} \) is a cosymmetry of (1) according to definition 3. The next cosymmetry \( \omega^{(2)} \) (cf (39)) looks considerably more complicated and it will be derived in the next section.

Symmetries \( K^{(1)}, K^{(2)} \) are elements of \( \mathcal{F}_s \), they do not depend on variables \( u_{k,0} \) with \( p \neq 0 \) and thus it is relatively easy to find them starting from definition 2. Cosymmetries \( \omega^{(1)}, \omega^{(2)} \) are elements of \( \mathcal{F}_s \) which cannot be reduced to elements of \( \mathcal{F}_s \) or \( \mathcal{F}_t \) by the elimination map.

Equation (1) has an infinite hierarchy of commuting symmetries as well as an infinite hierarchy of cosymmetries. In the next sections we shall show that the hierarchy of symmetries can be generated by successive applications of a recursion operator to the seed symmetries \( K^{(1)}, K^{(2)} \), while the hierarchy of cosymmetries can be obtained by applying successively a co-recursion operator to the seed cosymmetries \( \omega^{(1)}, \omega^{(2)} \).
3. Recursion and co-recursion operators of difference equations

In this section we prove that equation (1) possesses weakly non-local pseudo-difference recursion and co-recursion operators.

We recall that difference operators are finite sums of the form
\[ A = a_n S^n + a_{n-1} S^{n-1} + \cdots + a_0, \quad a_k \in \mathcal{F}_Q. \] (19)

They form a non-commutative ring \( \mathcal{F}_Q[S] \) of polynomials in \( S \) with coefficients from \( \mathcal{F}_Q \). Difference operators we also call local operators since the action \( A : a \mapsto A(a) \in \mathcal{F}_Q \) of a difference operator \( A \) is naturally defined for any element \( a \in \mathcal{F}_Q \).

For our theory the operator of a finite difference \( \Delta = S - 1 \) is of particular importance. This is a difference analogue of the derivation \( D_t \) in a differential field in the continuous case. The corresponding inverse pseudo-difference operator \( \Delta^{-1} \) is the analogue of \( D_t^{-1} \) or integration. The action of the operator \( \Delta^{-1} \) is not defined for all elements of \( \mathcal{F}_Q \), but only on the elements of image space \( \text{Im} \Delta = \Delta \mathcal{F}_Q \). In general the kernel space \( \text{Ker} \Delta \subset \mathcal{F}_Q \) could be rather nontrivial (see discussion in [19]), but in the case of equation (1), where \( Q \) is an irreducible polynomial satisfying (4), one can show that \( \text{Ker} \Delta = \mathbb{C} \). Thus for \( a = b_1 - b, \ b \in \mathcal{F}_Q \) we have \( \Delta^{-1}(a) = b + \alpha, \ \alpha \in \mathbb{C} \). In what follows we shall ignore ‘the constant of integration’ \( \alpha \), since its effect on results is inessential similar to the continuous case [2].

We define weakly non-local pseudo-difference operators as finite sums of the form
\[ B = B_0 + a_1 \Delta^{-1} \circ b_1 + a_2 \Delta^{-1} \circ b_2 + \cdots + a_m \Delta^{-1} \circ b_m, \quad a_k, b_k \in \mathcal{F}_Q, \ B_0 \in \mathcal{F}_Q[S]. \] (20)

It is a difference analogue of weakly non-local pseudo-differential operators which play important role in the theory of multi-Hamiltonian partial differential equations [18]. A composition of two weakly non-local pseudo-difference operators is a pseudo-difference operator, but not necessarily weakly non-local.

Formally conjugated difference (19) and weakly non-local pseudo-difference (20) operators are defined as
\[ A^* = S^{-n} \circ a_n + S^{1-n} \circ a_{n-1} + \cdots + S^{-m} \circ a_m, \]
\[ B^* = B_0^* - b_1 \Delta^{-1} \circ a_1 - b_2 \Delta^{-1} \circ a_2 - \cdots - b_m \Delta^{-1} \circ a_m. \]

The action of a pseudo-difference operator is not defined for all elements of \( \mathcal{F}_Q \). The subset of \( \mathcal{F}_Q \) for which the action of \( B \) is defined \( \text{Dom}(B) = \{ a \in \mathcal{F}_Q | B(a) \in \mathcal{F}_Q \} \) is called the domain of \( B \). For example \( \text{Dom}(\Delta^{-1}) = \text{Im} \Delta \) and for \( A \in \mathcal{F}_Q[S] \) we have \( \text{Dom}(A) = \mathcal{F}_Q \).

By a recursion operator of a difference equation (1) we shall understand a pseudo-difference operator \( \mathcal{R} \) such that
\[ \mathcal{R} : \text{Dom}(\mathcal{R}) \cap \mathfrak{A}_Q \mapsto \mathfrak{A}_Q, \]
where \( \mathfrak{A}_Q \) is the linear space of symmetries of this difference equation. In other words, if the action of \( \mathcal{R} \) on a symmetry \( K \in \mathcal{F}_Q \) is defined, i.e. \( \mathcal{R}(K) \in \mathcal{F}_Q \), then \( \mathcal{R}(K) \) is a symmetry of the same difference equation. The operator of multiplication by a constant is a trivial recursion operator. It is easy to see that a pseudo-difference operator \( \mathcal{R} \) is a recursion operator for a difference equation \( Q = 0 \) if there exists a pseudo-difference operator \( \mathcal{P} \) such that \( \mathcal{R} \) and \( \mathcal{P} \) satisfy the following operator equation
\[ D_Q \circ \mathcal{R} = \mathcal{P} \circ D_Q. \] (21)

Indeed, if \( K \) is a symmetry of this difference equation and \( \hat{K} = \mathcal{R}(K) \in \mathcal{F}_Q \), then it follows from (21) that \( D_Q \hat{K} = 0 \), and thus \( \hat{K} \) is also a symmetry.
The equality in (21) has to be understood in the sense of the field $\mathcal{F}_Q$, or in the usual sense after application of the elimination map to the coefficients of the pseudo-difference operators. In [19] it was shown that the operator equation (21) is equivalent to two equations

$$\mathfrak{P} \circ (Q_{u_1} S + Q_{u_2}) = (Q_{u_1} S + Q_{u_2}) \circ \mathfrak{R},$$

and

$$\mathfrak{P} \circ (Q_{u_1} S + Q_{u_2}) = (Q_{u_1} S + Q_{u_2}) \circ \mathfrak{T}(\mathfrak{R}).$$

We say a pseudo-difference operator $\mathfrak{P}$ is a co-recursion operator of the difference equation $Q = 0$ if it maps cosymmetries to cosymmetries $\mathfrak{P} : \text{Dom}(\mathfrak{P}) \cap \Omega_Q \mapsto \Omega_Q$. If the difference equation possesses a recursion operator $\mathfrak{R}$ satisfying equation (21) then $\mathfrak{P} = \mathfrak{P}^\ast$. Indeed, conjugating (21) we obtain

$$D_Q^\ast \circ \mathfrak{P}^\ast = \mathfrak{R}^\ast \circ D_Q^\ast$$

and thus if $\omega$ is a cosymmetry and $\hat{\omega} = \mathfrak{P}(\omega) \in \text{Dom}(\mathfrak{P})$, then $D_Q^\ast(\hat{\omega}) = 0$.

In the rest of this section, we show that with the operator $\mathfrak{R}$ presented in paper [19] there exists (and will be given explicitly) a pseudo-difference operator $\mathfrak{P}$ satisfying equation (21). It implies that $\mathfrak{R}$ is a recursion operator of the difference equation (1) and $\mathfrak{P} = \mathfrak{P}^\ast$ is a co-recursion operator.

We rewrite operator $\mathfrak{R}$ in [19] in the form

$$\mathfrak{R} = hh_{-1} w^2 u_1^2 S^2 + hh_{-1} w^2 u_{-1}^2 S^{-2} + 2K^{(1)} K^{(2)} \left(S^{-1} + \frac{1}{h} \right)$$

$$- w^2 (h_{-1} h_1 u_1^2 + h_{-1} h_2 u_{-1}^2) + \frac{2}{h_{-1}} \left(K^{(1)} K_{-1}^{(2)} + K^{(2)} K_{-1}^{(1)} \right)$$

$$+ 2K^{(1)} \Delta^{-1} \circ \left(\frac{K^{(2)} h_{-1}}{h_{-1}} - K_{-1}^{(2) h_{-1}} \right) + 2K^{(2)} \Delta^{-1} \circ \left(\frac{K_{-1}^{(1)} h_{-1}}{h_{-1}} - K_{-1}^{(1) h_{-1}} \right),$$

where $h, w, K^{(1)}$ and $K^{(2)}$ are given by (11), (13), (14) and (15), respectively. It is a weakly non-local pseudo-difference operator and can be written as the product of the following operators

$$\mathcal{H} = \frac{h_{-1} h_1}{(u_{1,0} - u_{-1,0})^2 (u_{2,0} - u_{0,0})^2} S - \frac{h_{-1} h_1}{(u_{1,0} - u_{-1,0})^2 (u_{2,0} - u_{0,0})^2} S^{-1}$$

$$+ 2K^{(1)} \Delta^{-1} \circ K^{(2)} + 2K^{(2)} \Delta^{-1} \circ K^{(1)},$$

$$\mathcal{I} = \frac{1}{h} S - \frac{1}{h},$$

that is, $\mathfrak{R} = \mathcal{H} \circ \mathcal{I}$. In section 4, we are going to show that operator $\mathcal{H}$ is Hamiltonian and operator $\mathcal{I}$ is symplectic.

**Theorem 1.** Operator $\mathfrak{R}$ given by (25) is a recursion operator of equation (1).

Before we prove this theorem, we first give a few lemmas.

**Lemma 1.** If a pseudo-difference operator

$$A = a^{(2)} S^2 + a^{(0)} S + a^{(0)} S^{-1} + a^{(2)} S^{-2} + b^{(1)} \Delta^{-1} b^{(1)} + b^{(2)} \Delta^{-1} b^{(2)}$$

can be factorized as the product of the following two operators

$$C = c^{(1)} S + c^{(0)} S + c^{(0)} S^{-1} + c^{(2)} S^{-2} + b^{(1)} \Delta^{-1} d^{(1)} + b^{(2)} \Delta^{-1} d^{(2)},$$

$$E = e^{(1)} S + e^{(0)},$$

then

$$\mathfrak{P} \circ (Q_{u_1} S + Q_{u_2}) = (Q_{u_1} S + Q_{u_2}) \circ \mathfrak{R},$$

and

$$\mathfrak{P} \circ (Q_{u_1} S + Q_{u_2}) = (Q_{u_1} S + Q_{u_2}) \circ \mathfrak{T}(\mathfrak{R}).$$
that is, $A = C \circ E$, then the coefficients of the operator $C$ satisfy the following conditions

\begin{align}
    a^{(2)} &= c^{(1)} e_1^{(1)}, & a^{(-2)} &= c^{(-2)} e_2^{(0)}, \\
    a^{(1)} &= c^{(0)} e_1^{(1)} + c^{(1)} e_1^{(0)}, & a^{(-1)} &= c^{(-2)} e_2^{(0)} + c^{(-1)} e_{-1}^{(0)}, \\
    b_r^{(1)} &= (d_r^{(1)} e_1^{(1)})_{-1} + d_r^{(1)} e_1^{(0)}, & b_r^{(2)} &= (d_r^{(2)} e_1^{(1)})_{-1} + d_r^{(2)} e_1^{(0)}, \\
    a^{(0)} &= c^{(-1)} e_{-1}^{(1)} + b_r^{(1)} (d_r^{(1)} e_1^{(1)})_{-1} + b_r^{(2)} (d_r^{(2)} e_1^{(1)})_{-1} + c^{(0)} e_0^{(0)}. \\
\end{align}

**Proof.** Expanding the composition of operators $C$ and $E$ and using the identity $\Delta^{-1} S = 1 + \Delta^{-1}$ we obtain

\begin{align}
    C \circ E &= c^{(1)} e_1^{(1)} S^2 + (c^{(0)} e_1^{(1)} + c^{(1)} e_1^{(0)}) S + (c^{(-2)} e_{-2}^{(1)} + c^{(-1)} e_{-1}^{(0)}) S^{-1} + c^{(-2)} e_{-2}^{(0)} S^{-2} \\
    &+ (c^{(-1)} e_{-1}^{(1)} + b_r^{(1)} (d_r^{(1)} e_1^{(1)})_{-1} + b_r^{(2)} (d_r^{(2)} e_1^{(1)})_{-1} + c^{(0)} e_0^{(0)}) \\
    &+ b_r^{(1)} \Delta^{-1} ((a_r^{(1)} e_1^{(1)})_{-1} + d_r^{(1)} e_1^{(0)}) + b_r^{(2)} \Delta^{-1} ((a_r^{(2)} e_1^{(1)})_{-1} + d_r^{(2)} e_1^{(0)}).
\end{align}

Comparing the coefficients of the operator $A$ with the above operator we obtain (28)–(32).

**Lemma 2.** Let $Q$ and $R$ be given by (1) and (25), respectively. Then for the constant

\begin{equation}
    \mu = (a_4 - a_3)(a_3 a_4 a_5 - a_3 a_5^2 - 2a_1 a_6^2 + 4a_5 a_6 - 2a_1 a_6^2 + a_4 a_1 a_6 - a_5 a_6^2),
\end{equation}

the operator $R - \mu$ can be factorized over $F_Q$ as

\begin{equation}
    R - \mu = M \cdot (Q_{u,0} S + Q_{u,0}),
\end{equation}

where

\begin{equation}
    M = c^{(1)} S + c^{(0)} + c^{(-1)} S^{-1} + c^{(-2)} S^{-2} + 2K^{(1)} \Delta^{-1} \omega^{(2)} - 2K^{(2)} \Delta^{-1} \omega^{(1)}
\end{equation}

with coefficients

\begin{align}
    c^{(1)} &= \frac{h h_{-1} w^2 w_1^2}{S(Q_{u,0})}; & c^{(-2)} &= \frac{h h_{-1} w^2 w_1^2}{S^{-2}(Q_{u,0})}; \\
    c^{(0)} &= \frac{1}{h Q_{u,0}} \left( 2K^{(1)} K^{(2)} - \frac{h h_{-1} w^2 w_1^2 S(Q_{u,0})}{S(Q_{u,0})} \right); \\
    c^{(-1)} &= \frac{1}{h S^{-1}(Q_{u,0})} \left( 2K^{(1)} K^{(2)} - \frac{h h_{-1} w^2 w_1^2 S^{-2}(Q_{u,0})}{S^{-2}(Q_{u,0})} \right); \\
    \omega^{(2)} &= \frac{K^{(2)}_1}{h Q_{u,0}} - \frac{K^{(2)}_1}{h Q_{u,0}} = \frac{K^{(2)}_1}{h Q_{u,0}} S(Q_{u,0}) + \frac{K^{(2)}_1}{h K^{(1)}_1} \omega^{(1)} + \frac{K^{(2)}_1 K^{(1)}_1}{h K^{(1)}_1 Q_{u,0}} \left( \frac{\mu}{2K^{(1)}_1 Q_{u,0}} + \frac{h w^2 w_1^2}{2K^{(1)}_1 Q_{u,0}} \right) \frac{h S^{-1}(Q_{u,0})}{S(Q_{u,0})} - \frac{h}{Q_{u,0}} \right) \\
    &+ \frac{h w^2 w_1^2}{2K^{(1)}_1 Q_{u,0}} \left( \frac{h S(Q_{u,0})}{S(Q_{u,0})} - \frac{h}{Q_{u,0}} \right) \omega^{(2)} + \frac{h w^2 w_1^2}{2K^{(1)}_1 Q_{u,0}} \frac{h S(Q_{u,0})}{S(Q_{u,0})} - h_2 \right),
\end{align}

and $\omega^{(1)}$ given by (17). Moreover, $\omega^{(2)}$ is a cosymmetry of equation (1).
Therefore, operator $R$ pseudo-difference operator $M = \hat{\omega}(1)$ satisfies relation (30). Using condition (32), we obtain

$$\omega^{(2)} = \frac{1}{2K^{(1)} Q_{u,1,0}} \left( -w_1^2 \left( hh_1 w_2^2 + h_\sigma h_1 w_2^2 \right) + \frac{2}{h} \left( K^{(1)} K^{(2)} + K^{(1)} K^{(1)} \right) - \mu \right)$$

$$- \frac{1}{2K^{(1)} Q_{u,0,0}} \left( \frac{2K^{(1)} K^{(2)}}{h} - \frac{hh_1 w_2^2 w_1^2 S^{-1}(Q_{u,1,0})}{S^{-1}(Q_{u,1,0})} \right) + K^{(2)} \omega^{(1)}$$

$$- \frac{1}{2K^{(1)} Q_{u,0,0}} S(Q_{u,2,0}) \left( \frac{2K^{(1)} K^{(2)}}{h} - \frac{hh_1 w_2^2 w_1^2 S^2(Q_{u,2,0})}{S^2(Q_{u,2,0})} \right),$$

which can be rewritten as in (39). Substituting the latter into condition (31), which now becomes

$$\frac{K^{(2)}}{h} = \frac{K^{(2)}}{h} = S^{-1}(Q_{u,1,0}) \omega^{(2)} + \omega^{(2)} Q_{u,1,0},$$

we obtain the value of $\mu$. Using the elimination map and definition 3 one can directly check that $\omega^{(2)}$ given by (39) is a cosymmetry of equation (1).

Factorizations (34) is quite remarkable. Indeed, the coefficients of the weakly non-local pseudo-difference operator $R - \mu$ are in the field $\mathcal{F}_{\sigma}$, but the coefficients of both of its factors depend on the variable $u_{0,1} \notin \mathcal{F}_{\sigma}$. It is indeed a factorization over $\mathcal{F}_{Q}$.

Similarly, we obtain the following factorization for operator $T(R) - \mu$:

**Lemma 3.** For operator $R$ given by (25) and the constant $\mu$ given by (33), the following factorization holds for all solutions of equation (1)

$$T(R) - \mu = \hat{\mathcal{M}} \cdot (Q_{u,1,0} S + Q_{u,2,0}), \quad (40)$$

where

$$\hat{\mathcal{M}} = \hat{c}^{(1)} S + \hat{c}^{(0)} + \hat{c}^{(0)} S^{-1} + \hat{c}^{(0)} S^{-2} - 2K^{(1)} \Delta^{-1} \omega^{(2)} + 2K^{(2)} \Delta^{-1} \omega^{(2)}.$$  \hspace{1cm} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
From the proof of theorem 1, it follows that
\[ D_Q \circ (\mathcal{R} - \mu) = \mathcal{P} \circ D_Q \] (45)
for all solutions of equation (1). This leads to the following result:

**Theorem 2.** Operator \( \mathcal{M} = \mathcal{P}^* \) is a co-recursion operator of equation (1), where
\[ \mathcal{P} = (Q_{u,0}S + Q_{m,0})M \]
and operator \( M \) is defined in lemma 2.

4. Locality of symmetry hierarchies

In this section, we prove that recursion operator (25) is a Nijenhuis operator. Despite being weakly non-local, it generates an infinite hierarchy of local commuting symmetry flows. The main result of this section is a re-statement of the theorem proven in the continuous case [27] with the adjustments to the difference variational complex.

4.1. Difference variational complex and Lie derivatives

Note that all coefficients of operator \( \mathcal{R} \) are elements in \( F_s \). In this section we sketch the difference variational complex over the ring \( F_s \) [31, 32] in the same spirit as the variational complex [7, 25, 26, 33]. We also adapt the formula of the Lie derivatives along evolutionary vector fields.

Recall that \( F_s \) is the field of rational functions of variables \( U_s = \{u_n|n \in \mathbb{Z}\} \). Without causing any confusion, we denote \( u_{n,0} \) by \( u_n \) for simplicity in what follows.

The field \( F_s \) is also a linear space over \( \mathbb{C} \). Let us consider the extended linear space \( L_s = F_s \oplus \text{Span}_\mathbb{C}\{\log a|a \in F_s\} \), so that elements of \( L_s \) are linear combinations with complex coefficients of elements in \( F_s \) and logarithms of elements in \( F_s \). For any element \( g \in L_s \), we define an equivalence class (or a functional) \( \int g \) by saying that two elements \( g, h \in L_s \) are equivalent if \( g - h \in \Delta(L_s) \). The space of functionals will be denoted by \( F'_s \), it is a linear space over \( \mathbb{C} \) and it does not inherit a ring or field structure of \( F_s \).

The evolutionary vector fields over the ring \( F_s \)
\[ X_P = \sum_{n \in \mathbb{Z}} (S^n P) \frac{\partial}{\partial u_n} \]
form a Lie algebra denoted by \( \mathfrak{h} \). The action of any element \( P \in \mathfrak{h} \) on \( \int g \in F'_s \) can be defined as
\[ P \circ \int g = \int X_P(g) = \int \sum_{n \in \mathbb{Z}} (S^n P) \frac{\partial g}{\partial u_n} = \int D_g[P]. \] (46)
This is a representation of the Lie algebra \( \mathfrak{h} \). We build up a Lie algebra complex associated with it. This complex is called the difference variational complex. Here we give the first few steps.

We denote the space of functional \( n \)-forms by \( \Omega^n \) starting with \( \Omega^0 = F'_s \). We now consider the space \( \Omega^1 \). For any vertical 1-form on the ring \( F_s \), i.e. \( \omega = \sum_n h(n) du_n \), there is a natural non-degenerate pairing with an element \( P \in \mathfrak{h} \):
\[ \langle \omega, P \rangle = \int \sum_{n \in \mathbb{Z}} h(n) S^n P = \int \left( \sum_{n \in \mathbb{Z}} S^{-n} h(n) \right) P. \] (47)
Thus any element of \( \Omega^1 \) is completely defined by \( \xi du_0 = \sum_n S^{-n} h(n) du_0 \). We simply say \( \xi \in \Omega^1 \).
The pairing (47) allows us to give the definition of (formal) adjoint operators to linear (pseudo-) difference operators [31, 33].

**Definition 4.** Given a linear operator \( A : \mathfrak{h} \to \Omega^1 \), we call the operator \( A^* : \Omega^1 \to \mathfrak{h} \) the adjoint operator of \( A \) if \( \langle AP_1, P_2 \rangle = \langle A^* P_2, P_1 \rangle \), where \( P_i \in \mathfrak{h} \) for \( i = 1, 2 \).

Similarly, we can define the adjoint operator for an operator mapping from \( \Omega^1 \) to \( \mathfrak{h} \), from \( \mathfrak{h} \) to \( \mathfrak{h} \) or from \( \Omega^1 \) to \( \Omega^1 \).

The difference variational derivative (Euler operator) of each functional \( \int g \in \mathcal{F}'_s \) denoted by \( \delta u_0(\int g) \in \Omega^1_1 \) is defined so that

\[
\delta u_0(\int g) \cdot P = \left( d \int g \right)(P) = \int \left( \sum_{n \in \mathbb{Z}} S^{-n} \frac{\partial g}{\partial u_n} \right) P = \left( \sum_{n \in \mathbb{Z}} S^{-n} \frac{\partial g}{\partial u_n} \right) \in \Omega^1, \tag{48}
\]

where \( d : \Omega^n \to \Omega^{n+1} \) is a coboundary operator. Due to the non-degeneracy of the pairing (47), we have

\[
\delta u_0(\int g) = D_0^*(1) = \sum_{n \in \mathbb{Z}} S^{-n} \frac{\partial g}{\partial u_n} = \frac{\partial}{\partial u_0} \left( \sum_{n \in \mathbb{Z}} S^{-n} g \right) \in \Omega^1.
\]

For any \( \xi \in \Omega^1 \), it can be shown that \( d \xi = D_0 - D_0^* \) [26, 31]. Thus \( d \xi = 0 \) is equivalent to \( D_0 = D_0^* \). We say that the 1-form \( \xi \) is closed if \( d \xi = 0 \).

We can define the Lie derivative of a given object in the complex along vector field \( K \in \mathfrak{h} \). It can be expressed explicitly in terms of Fréchet derivatives as follows [26]:

\[
L_K \int g = \int D_0 \left[ K \right] \quad \text{for } \int g \in \mathcal{F}'_s; \\
L_K h = [K, h] \quad \text{for } h \in \mathfrak{h}; \\
L_K \xi = D_0 \left[ K \right] + D_0^* (\xi) \quad \text{for } \xi \in \Omega^1; \\
L_K \mathfrak{R} = D_0 \left[ K \right] - D_0 \mathfrak{R} + \mathfrak{R} D_0 \quad \text{for } \mathfrak{R} : \mathfrak{h} \to \mathfrak{h}; \\
L_K \mathcal{H} = D_0 \left[ K \right] - D_0 \mathcal{H} + \mathcal{H} D_0^* \quad \text{for } \mathcal{H} : \Omega^1 \to \mathfrak{h}; \\
L_K \mathcal{I} = D_0 \left[ K \right] + D_0^* \mathcal{I} + \mathcal{I} D_0 \quad \text{for } \mathcal{I} : \mathfrak{h} \to \Omega^1. \tag{49}
\]

For a given object \( \sigma \) and \( K \in \mathfrak{h} \), if \( L_K \sigma = 0 \) we say \( \sigma \) is conserved (or invariant) along the vector field \( K \) and the vector field \( K \) is a symmetry of the object \( \sigma \). This complex plays an important role in the study of differential–difference equations or symmetry flows of difference equations since we can associate equations (flows) with vector fields. Hence we can define interesting objects such as conserved densities, symmetries and recursion operators as invariants along vector fields.

### 4.2. Symplectic, Hamiltonian and Nijenhuis operators

In the context of difference variational complex, we now define symplectic and Hamiltonian operators. Such definitions are the same as in the continuous case, where we define them in the context of the complex of variational calculus. All the results to determine whether a given (pseudo-) differential operator is Hamiltonian or symplectic are still valid. We shall use them later on without redeveloping them.

**Definition 5.** A linear operator \( A : \mathfrak{h} \to \Omega^1 \) (or \( \Omega^1 \to \mathfrak{h} \)) is anti-symmetric if \( A = -A^* \).
Given an anti-symmetric operator \( I : \mathfrak{h} \to \Omega^1 \), there is an anti-symmetric 2-form associated with it. Namely,

\[
\omega(P, G) = \langle I(P), G \rangle = -\langle I(G), P \rangle = -\omega(G, P), \quad P, G \in \mathfrak{h}.
\] (50)

Here the functional 2-form \( \omega \) has the canonical form [33]

\[
\omega = \frac{1}{2} \int du_0 \wedge \mathcal{I} du_0.
\] (51)

**Definition 6.** An anti-symmetric operator \( I : \mathfrak{h} \to \Omega^1 \) is called symplectic if its associated 2-form (51) is closed, i.e. \( d\omega = 0 \).

**Proposition 1.** For any function \( f \in \mathcal{F} \) depending only on \( u_0 \) and \( u_1 \), the operator \( f S - S^{-1} f \) is symplectic.

**Proof.** This operator is obviously anti-symmetric and its associated canonical 2-form is

\[
\omega = \frac{1}{2} \int (fdu_0 \wedge du_1 - du_0 \wedge S^{-1} (fdu_0)) = \int f du_0 \wedge du_1.
\]

Now we have

\[
d\omega = \int \left( \frac{\partial f}{\partial u_0} du_0 \wedge du_0 \wedge du_1 + \frac{\partial f}{\partial u_1} du_1 \wedge du_0 \wedge du_1 \right) = 0.
\]

Therefore \( \omega \) is a closed 2-form and thus the given operator is symplectic. \( \blacksquare \)

It immediately follows that operator \( I \) defined by (27) is symplectic since the difference polynomial \( h \) given by (11) depends only on \( u_0 \) and \( u_1 \).

For an anti-symmetric operator \( H : \Omega^1 \to \mathfrak{h} \), we define a bracket of two functionals as

\[
\{\int f, \int g\} = \left\langle \delta u_0 \left( \int f \right), H \delta u_0 \left( \int g \right) \right\rangle.
\] (52)

**Definition 7.** The operator \( H \) is Hamiltonian if the bracket defined by (52) is Poisson, that is, anti-symmetric and satisfying the Jacobi identity

\[
\{\{\int f, \int g\}, \int h\} + \{\{\int g, \int h\}, \int f\} + \{\{\int h, \int f\}, \int g\} = 0.
\]

The Jacobi identity is abstractly the same as in the continuous case when \( H \) is a differential operator. Thus the results to determine whether a given difference operator is Hamiltonian are still valid. In [33] (see p 443), it is formulated as the vanishing of the functional tri-vector:

\[
X_{\mathcal{H}(\Theta)}(\Theta) = 0, \quad \text{where} \quad \Theta = \frac{1}{2} \int \Theta \wedge \mathcal{H}(\Theta).
\] (53)

It is clear that any anti-symmetric constant operator is Hamiltonian. For example, operators \( S = S^{-1} \), \((S + 1)(S - 1)^{-1}\) and \((S - 1)(S + 1)^{-1}\) are all Hamiltonian operators.

**Proposition 2.** Anti-symmetric operator \( \mathcal{H} \) defined by (26) is a Hamiltonian operator.

**Proof.** According to (53), we check \( X_{\mathcal{H}(\Theta)}(\Theta) = 0 \), where the associated bi-vector of \( \mathcal{H} \) is

\[
\Theta_{\mathcal{H}} = \frac{1}{2} \int \Theta \wedge \mathcal{H}(\Theta) = \int \Theta \wedge (h_{-1} h h_{1} w^2 w'_{1} \Theta_{1} + 2 K^{(2)} \Delta^{-1} K^{(1)} \Theta).
\]
Instead of writing out the full calculation, we only demonstrate the method by picking out terms of \( \theta \wedge \Delta^{-1} K^{(1)} \theta \wedge \Delta^{-1} K^{(2)} \theta \). The relevant terms in \( X_{\mathcal{H}(\theta)}(\Theta_\mathcal{H}) \) are

\[
\int (\theta \wedge X_{\mathcal{H}(\theta)}(K^{(2)}) \wedge \Delta^{-1} K^{(1)} \theta + K^{(2)} \theta \wedge \Delta^{-1}(X_{\mathcal{H}(\theta)}(K^{(1)}) \wedge \theta)
= \int (\theta \wedge X_{\mathcal{H}(\theta)}(K^{(2)}) \wedge \Delta^{-1} K^{(1)} \theta + \int (S^{-1} - 1)^{-1} K^{(2)} \theta \wedge X_{\mathcal{H}(\theta)}(K^{(1)}) \wedge \theta
= \int \Delta^{-1} K^{(2)} \theta \wedge X_{\mathcal{H}(\theta)}(K^{(1)}) \wedge \theta
= \int (\theta \wedge X_{\mathcal{H}(\theta)}(K^{(2)}) \wedge \Delta^{-1} K^{(1)} \theta + \int (\theta \wedge X_{\mathcal{H}(\theta)}(K^{(1)}) \wedge \Delta^{-1} K^{(2)} \theta
= \int \Delta^{-1} K^{(2)} \theta \wedge X_{\mathcal{H}(\theta)}(K^{(1)}) \wedge \theta + \int \Delta^{-1} K^{(1)} \theta \wedge X_{\mathcal{H}(\theta)}(K^{(2)}) \wedge \theta
+ \int \Theta_1 \cdot \sum_i S'(\mathcal{H}(\theta)) \partial_{\theta_i} \mathcal{H}^{(k)} \wedge \Delta^{-1} K^{(1)} \theta.
\] (54)

Note that the terms involving either \( \Delta^{-1} K^{(1)} \theta \) or \( \Delta^{-1} K^{(2)} \theta \) in \( S'(\mathcal{H}(\theta)) \), \( i \in \mathbb{Z} \), equal to

\[ S'(\mathcal{H}(\theta)) : 2K^{(1)} \Delta^{-1} K^{(2)} + 2K^{(2)} \Delta^{-1} K^{(1)} \theta. \]

Substituting into (54), we obtain that the coefficient of \( 2 \theta \wedge \Delta^{-1} K^{(2)} \theta \wedge \Delta^{-1} K^{(1)} \theta \) is

\[ D_{\mathcal{H}^{(i)}}[K^{(1)}] - D_{\mathcal{H}^{(i)}}[K^{(2)}] = [K^{(1)}, K^{(2)}] = 0. \]

By working out other terms, we can show that the tri-vector \( X_{\mathcal{H}(\theta)}(\Theta_\mathcal{H}) \) vanishes, which implies that \( \mathcal{H} \) is a Hamiltonian operator.

The Jacobi identity is a quadratic relation for the operator \( \mathcal{H} \). In general, the linear combination of two Hamiltonian operators is no longer Hamiltonian. If it is, we say that these two Hamiltonian operators form a Hamiltonian pair. Hamiltonian pairs play an important role in the theory of integrability. They naturally generate Nijenhuis operators.

**Definition 8.** A linear operator \( \mathcal{R} : \mathfrak{h} \to \mathfrak{h} \) is called a Nijenhuis operator if it satisfies

\[ [\mathcal{R} P, \mathcal{R} G] - \mathcal{R} [P, \mathcal{R} G] - \mathcal{R} [P, \mathcal{R} G] + \mathcal{R}^2 [P, G] = 0, \quad P, G \in \mathfrak{h}. \] (55)

Using formula (49), this identity is equivalent to

\[ L_{\mathcal{R} G} \mathcal{R} = \mathcal{R} L_G \mathcal{R}, \quad G \in \mathfrak{h}. \] (56)

Alternatively, \( D_{\mathfrak{h} G} \mathcal{R} G \) must be symmetric with respect to \( P \) and \( G \) [16]. It can be used to check directly whether a given operator is Nijenhuis or not [27].

The properties of Nijenhuis operators [26] provide us with the explanation of how the infinitely many commuting symmetries and conservation laws of integrable equations arise. In applications, there are non-local terms in Nijenhuis operators. For pseudo-differential operators, a lot of work has been done to find sufficient conditions for Nijenhuis operators to produce local objects [27, 34, 35].

**Proposition 3.** Operator \( \mathcal{R} \) defined by (25) is a Nijenhuis operator.

**Proof.** We need to check that the expression \( H := D_{\mathfrak{h} G} \mathcal{R} G - \mathcal{R} D_{\mathfrak{h} G} \mathcal{R} [P, G] \) is symmetric with respect to \( P \) and \( G \). The calculation is straightforward but rather long and not suitable for a presentation in a journal article. Here we show only one step by picking out the terms in \( H \) involving either \( \mathfrak{h} G \) or \( P \). We use the notation \( P_i = S^i P \) and \( G_j = S^j G \). Since \( \mathcal{R} \) is...
a second order operator, we compute the terms containing either $G_2$ or $P_2$ ($P_3$ for non-local terms) in expression $D_9[P](G)$. These terms are

\[-2hh_1w^3h^3_1(P_1 - P_3)G_2 - 2hh_1w^2h^3_1(P_2 - P)G_2 + D_9[P]h_1w^2h^3_1G_2 + 2K^{(1)}h_1w^2h^3_1P_2G_1 - 2K^{(1)}h_1w^2h^3_1P_2G_1 - 2h_1w^2h^3_1P_2G_3
\]
\[+2h_1w^2h^3_1h^2_1P_2G - 2K^{(1)}h_1w^2h^3_1P_2G + 2K^{(1)}\Delta^{-1}(h_1w^2h^3_1P_2G)
\]
\[= -2w^2hh_1w^2h^2_1\Delta^{-1}(K^{(1)}G/h - K^{(1)}G/h).
\]

(57)

Now the term $hh_1w^2h^2_1S^2$ in $\mathfrak{R}$ acting on the above expression leads to the terms with either $G_4$ or $P_3$ in $\mathfrak{R}D_9[P](G)$, denoted by $H^{1,4}$. We obtain

\[H^{1,4} = -2hh_1w^3h^3_1w^2h^2_1w^3_1w^3_1w^3_1w^3_1(P_1 - P_3)G_2 - 2hh_1w^2h^2_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1(P_2 - P)G_2 + D_9[P]h_1w^2h^2_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1G_2 + 2K^{(1)}h_1w^2h^2_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_2G_1
\]
\[+2K^{(1)}h_1w^2h^2_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_2G_1 - 2w^2hh_1w^2h^2_1\Delta^{-1}(K^{(1)}G/h - K^{(1)}G/h).
\]

(57)

We then collect the terms with either $G_1$ or $P_1$ in $D_9[\mathfrak{R}P](G)$, denoted by $H^{2,4}$. These terms can be obtained from the terms containing $P_2$ in (57). Simply replacing $P_2$ by $S^2(\mathfrak{R}P)$, we obtain

\[-2hh_1w^3w^3_1(\mathfrak{R}P)G_2 - 2K^{(1)}h_1w^2h^3_1(\mathfrak{R}P)G_2 + 2K^{(1)}h_1w^2h^3_1(\mathfrak{R}P)G_2 - 2w^2h_1w^3_1(\mathfrak{R}P)G - 2K^{(1)}h_1w^2h^3_1(\mathfrak{R}P)G
\]
\[+2K^{(1)}\Delta^{-1}(h_1w^2h^3_1(\mathfrak{R}P)G).
\]

It follows that

\[H^{2,4} = -2hh_1w^3w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2 - 2K^{(1)}h_1w^2h^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2 - 2hh_1w^3w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2 - 2K^{(1)}h_1w^2h^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2 - 2K^{(1)}h_1w^2h^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2
\]
\[+2K^{(1)}h_1w^2h^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2 - 2K^{(1)}h_1w^2h^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1w^3_1P_3G_2.
\]

Using relation (14), it is easy to see that $H^{1,4} = H^{2,4}$ is symmetric with respect to $P$ and $G$.

In a similar way, one can check the symmetric property of the remaining terms.  

\[\square\]
4.3. Locality of symmetries generated by recursion operators

The sufficient conditions for Nijenhuis pseudo-differential operators to produce local objects formulated in [27, 34, 35] are valid for pseudo-difference operators over the field $\mathcal{F}_s$. In a recent paper [27], we proved that Nijenhuis operators, which are the product of weakly non-local Hamiltonian and symplectic operators [18], generate hierarchies of commuting local symmetries and conserved densities in involution under some easily verified conditions. To be self-contained, we restate this result in [27] for Hamiltonian operator $\mathcal{H}$ (26) and symplectic operator (27).

Consider a Hamiltonian operator $\mathcal{H}$ of the form (26) and a symplectic operator (27) such that $\mathfrak{g} = \mathcal{H} \cdot \mathcal{I}$ is a Nijenhuis operator (proposition 3). Assume that

$$L_{K^{(i)}} K^{(2)} = L_{K^{(i)}} I = L_{K^{(i)}} \mathcal{H} = L_{K^{(i)}} \mathcal{I} = 0.$$

Then the vector fields $p^{(i,j)} = (\mathcal{H} I)^j K^{(i)} \in \mathfrak{h}$ commute and $\omega^{(i,j)} = I p^{(i,j)} \in \Omega^1$ are closed 1-forms for $i = 1, 2$ and $j = 0, 1, 2, \ldots$

**Theorem 3.** For operator $\mathfrak{g}$ defined by (25), all $\mathfrak{g}/K^{(i)}$ commute and all 1-forms $I \mathfrak{g}/K^{(i)}$ are closed for $i = 1, 2$ and $j = 0, 1, 2, \ldots$

**Proof.** We know from (16) that $K^{(1)}$ and $K^{(2)}$ are commuting generalized symmetries. Therefore, we need to verify that $L_{K^{(i)}} \mathcal{I} = L_{K^{(i)}} \mathcal{H} = 0$ while we shall skip the proof of the same properties for $K^{(2)}$.

We can write anti-symmetric operators $\mathcal{I}$ and $\mathcal{H}$ as $\mathcal{I} = Y - Y^*$, where $Y = \frac{1}{2} S$, and $\mathcal{H} = Z - Z^*$, where

$$Z = h_{-1} h_1 w^2 w_1^2 S + 2 K^{(2)} \Delta^{-1} \circ K^{(1)},$$

respectively. We denote operator $D_Y[K^{(1)}] + D_Y^*[Y + Y D_Y^{(1)}]$ by $F$. It follows from (49) that $L_{K^{(i)}} \mathcal{I} = F - F^*$. Similarly, denoting operator $D_Z[K^{(1)}] - D_Z^*[Z - Z D_Z^{(1)}]$ by $G$, we have that $L_{K^{(i)}} \mathcal{H} = G - G^*$.

We now compute operators $F$ and $G$. The Fréchet derivative of $K^{(1)}$ and its adjoint operator are given by

$$D_{K^{(1)}} = -h_{-1} w^2 S + \partial_{u_0} K^{(1)} + h u^2 S^{-1}$$

and

$$D_{K^{(1)}}^* = h_1 w_1^2 S + \partial_{u_0} K^{(1)} - h_{-2} w_{-1}^2 S^{-1},$$

respectively. Substituting them into the expression for $F$, we obtain

$$F = -h^{-2} (K^{(1)}_{1} \partial_{u_1} h + K^{(1)} \partial_{u_0} h) S + h^{-1} \partial_{u_0} K^{(1)} S - h_{-1} h_{-2} w_{-1}^2 + h^{-1} \partial_{u_1} K^{(1)} S + h^{-1} h_1 w_1^2 = -h^{-1} h_{-1} h_{-2} w_{-1}^2 + h^{-1} h_1 w_1^2,$$

where we also used $K^{(1)} = h w - \frac{1}{2} \partial_{u_1} h$ and $K^{(1)} = h w_1 + \frac{1}{2} \partial_{u_0} h$ from (14). Hence

$$L_{K^{(i)}} \mathcal{I} = F - F^* = 0. \hspace{1cm} (58)$$

By a similar calculation, we find operator $G$ being a symmetric difference operator. It leads to

$$L_{K^{(i)}} \mathcal{H} = G - G^* = 0. \hspace{1cm} (59)$$

Thus we proved the statement.
4.4. Yamilov’s discretization of the Krichever–Novikov equation

It follows from (58) and (59) that

\[ L_K^\dagger \mathcal{R} = (L_K^\dagger \mathcal{H}) I + \mathcal{H} L_K^\dagger I = 0. \]

This implies that operator \( \mathcal{R} \) is a recursion operator of differential–difference equation \( u_\nu = K^{(1)} \). Substituting \( Q(1) \) into \( h \) defined by (11) and then \( h \) into (14) we can rewrite the latter equation in the form

\[ u_\nu = \frac{R(u_1, u, u_{-1})}{u_1 - u_{-1}}, \quad (60) \]

where \( R \) is the polynomial

\[ R(u, v, q) = (\alpha v^2 + 2\beta v + \gamma)uq + (\beta v^2 + \lambda v + \delta)(u + q) + \gamma v^2 + 2\delta v + \epsilon \]

with

\[ \alpha = 2(a_1a_3 - a_2^2); \quad \beta = a_2a_3 + a_3a_6 - a_2a_5 - a_2a_4; \quad \gamma = 2(a_2a_6 - a_4a_5); \]
\[ \lambda = a_2^2 + a_1a_7 - a_2^2 - a_5^2; \quad \delta = a_3a_6 + a_2a_7 - a_4a_6 - a_5a_6; \quad \epsilon = 2(a_3a_7 - a_5^2), \]

(61)

where \( a_i \) are constant parameters for the Viallet equation (1). Equation (60) can be identified as Yamilov’s discretization of the Krichever–Novikov equation (YdKN) [9], cf equation (V4) when \( \nu = 0 \) in [10]. Such relations for all the ABS equations and their generalizations introduced in [28] were established in [36].

It is straightforward to check that \( R(u_1, u, u_{-1}) \) is a symmetric and bi-quadratic polynomial and is related to \( h \) defined by (11) as follows

\[ h(u, u_{-1}) = \frac{1}{2} R(u_1, u, u_{-1}). \]

Thus we can express \( K^{(2)} (15), \) recursion operator \( \mathcal{R} (25), \) Hamiltonian operator (26) and symplectic operator (27) in terms of polynomial \( R(u, v, q) \). For example

\[ u_\nu = K^{(2)} = \frac{1}{4} \frac{R(u_1, u, u_1) R(u, u_{-1}, u)}{(u_1 - u_{-1})^2} \left( \frac{1}{u_2 - u} + \frac{1}{u - u_{-2}} \right), \quad (62) \]

is the next member in the hierarchy of commuting symmetries of the YdKN equation (60), which was first given in [30]. Higher symmetries can be obtained by application of the recursion operator to the seeds (60) and (62). The locality and commutativity of these symmetries are guaranteed by theorem 3. Cosymmetries of equation (60) coincide with its covariants, i.e. the variational derivatives of the conserved densities. Conserved densities can be obtained as residues of the powers of the recursion operator. They coincide with the conserved densities for the Viallet equation (1) [19]. Obviously, equations (60) and (62) and every member of the hierarchy are multi-Hamiltonian systems. For example equation (60) can be written in a Hamiltonian form

\[ I(u_\nu) = \frac{\delta H_0}{\delta u}, \]

where the symplectic operator is of the form

\[ I = \frac{2}{R(u_1, u, u_1)} S - S^{-1} \frac{2}{R(u_1, u, u_1)}, \]

and the Hamiltonian \( H_0 = \ln(u_1 - u_{-1}) - \frac{1}{2} \ln R(u, u_{-1}, u) = -\frac{1}{2} \text{res} \ln \mathcal{R} \) follows from theorem 3 that \( I_n = 2^n \mathcal{R}^n \) are symplectic operators. The compatibility of these Hamiltonian structures follows from the Nijenhuis property of the recursion operator.
Acknowledgments

AVM and PX would like to thank the University of Kent for its hospitality during their visits. JPW is grateful to the University of Kent for granting the study leave. PX is supported by the Newton International Fellowship grant NF082473 entitled ‘Symmetries and integrability of lattice equations and related partial differential equations’.

References

[1] Žiber A V and Shabat A B 1979 The Klein-Gordon equation with nontrivial group Dokl. Akad. Nauk SSSR 247 1103–7
[2] Sokolov V V and Shabat A B 1984 Classification of integrable evolution equations Mathematical Physics Reviews (Soviet Science Review Section C: Mathematical Physics Review vol 4) (Chur: Harwood Academic) pp 221–80
[3] Mikhailov A V, Shabat A B and Yamilov R I 1988 Extension of the module of invertible transformations Classification of integrable systems Commun. Math. Phys. 115 1–19
[4] Mikhailov A V, Shabat A B and Yamilov R I 1987 A symmetry approach to the classification of nonlinear equations Complete lists of integrable systems Usp. Mat. Nauk 42 3–53
[5] Mikhailov A V, Shabat A B and Sokolov V V 1991 The symmetry approach to classification of integrable equations What is Integrability? (Springer Series Nonlinear Dynamics) (Berlin: Springer) pp 115–84
[6] Sanders J A and Wang J P 1998 On the integrability of homogeneous scalar evolution equations J. Diff. Eqns 147 410–34
[7] Wang J P 1998 Symmetries and Conservation Laws of Evolution Equations PhD Thesis Vrije Universiteit/Thomas Stieltjes Institute, Amsterdam
[8] Adler V E, Shabat A B and Yamilov R I 2000 Symmetry approach to the integrability problem Theor. Math. Phys. 125 1603–61
[9] Yamilov R I 1983 Classification of discrete evolution equations Ups. Mat. Nauk 38 155–6
[10] Yamilov R I 2006 Symmetries as integrability criteria for differential difference equations J. Phys. A: Math. Gen. 39 R541–623
[11] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 Inverse scattering transform-fourier analysis for nonlinear problems Stud. Appl. Math. 53 249–315
[12] Olver P J 1977 Evolution equations possessing infinitely many symmetries J. Math. Phys. 18 1212–5
[13] Hereman W, Sanders J A, Sayers J and Wang J P 2005 Symbolic computation of polynomial conserved densities, generalized symmetries, and recursion operators for nonlinear differential–difference equations Group Theory and Numerical Analysis (CRM Proceedings and Lecture Notes vol 39) (Providence, RI: American Mathematical Society) pp 133–148
[14] Fuchssteiner B 1979 Application of hereditary symmetries to nonlinear evolution equations Nonlinear Anal., Theory, Methods Appl. 3 849–62
[15] Fokas A S and Fuchssteiner B 1980 On the structure of symplectic operators and hereditary symmetries Lett. Nuovo Cimento (2) 28 299–303
[16] Fuchssteiner B and Fokas A S 1981 Symplectic structures, their Bäcklund transformations and hereditary symmetries Physica D 4 47–66
[17] Magri F 1980 A Geometrical Approach to the Nonlinear Solvable Equations (Lecture Notes in Physics vol 120) (Berlin: Springer) pp 233–63
[18] Maltsev A Ya and Novikov S P 2001 On the local systems Hamiltonian in the weakly nonlocal Poisson brackets Physica D: Nonlinear Phenom. 156 53–80
[19] Mikhailov A V, Wang J P and Xenitidis P 2011 Recursion operators, conservation laws and integrability conditions for difference equations Theor. Math. Phys. 167 421–43 (arXiv:1004.5346)
[20] Zhang G L and Gao X S 2008 Properties of ascending chains for partial difference polynomial systems Asian Symp. on Computer Mathematics (ASCM 2007), ed D Kaput Proc. ASCM 2007 (Singapore, December 2007) (LNAI vol 5081) pp 307–21 (Berlin: Springer) ISSN 978-3-540-87827-8
[21] Levi D and Martina L 2001 Integrable hierarchies of nonlinear difference–difference equations and symmetries J. Phys. A: Math. Gen. 34 10357–68
[22] Viallet C 2009 Integrable lattice maps: Q_V, a rational version of Q_d Glasgow Math. J. A 51 157–63
[23] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs. The consistency approach Commun. Math. Phys. 233 513–43
Cosymmetries and Nijenhuis recursion operators for difference equations

2097

[24] Adler V E, Bobenko A I and Suris Yu B 2009 Discrete nonlinear hyperbolic equations. Classification of integrable cases Funct. Anal. Appl. 43 3–21

[25] Gel’fand I M and Dorfman I Ya 1979 Hamiltonian operators and algebraic structures related to them Funct. Anal. Appl. 13 248–62

[26] Dorfman I 1993 Dirac Structures and Integrability of Nonlinear Evolution Equations (Chichester: Wiley)

[27] Wang J P 2009 Lenard scheme for two-dimensional periodic Volterra chain J. Math. Phys. 50 023506

[28] Tongas A, Tsoubelis D, and Xenitidis P 2007 Affine linear and $D_4$ symmetric lattice equations: symmetry analysis and reductions J. Phys. A: Math. Gen. 40 13353–84

[29] Xenitidis P 2009 Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case Proc. 4th Workshop ‘Group Analysis of Differential Equations and Integrable Systems’ (26–30 October 2008, Protaras, Cyprus) (University of Cyprus) pp 226–42 (arXiv:0902.3954)

[30] Shabat A B and Yamilov R I 1988 Lattice representations of integrable systems Phys. Lett. A 130 271–5

[31] Kupershmidt B A 1985 Discrete Lax equations and differential–difference calculus Asterisque 123 (Société Mathématique de France)

[32] Mansfield E L and Hydon P E 2001 On a variational complex for difference equations Contemp. Math. 285 121–29

[33] O’liver P J 1993 Applications of Lie Groups to Differential Equations (Graduate Texts in Mathematics vol 107) 2nd edn (New York: Springer)

[34] Sanders J A and Wang J P 2001 Integrable systems and their recursion operators Nonlinear Anal. 47 5213–40

[35] Sergyeyev A 2005 Why nonlocal recursion operators produce local symmetries: new results and applications J. Phys. A: Math. Gen. 38 3397–407

[36] Levi D, Petrera M, Scimiterna C and Yamilov R I 2008 On Miura transformations and Volterra-type equations associated with the Adler–Bobenko–Suris equations Symmetry, Integr. Geom.: Methods Appl. 4 077