LOCAL WELL-POSEDNESS OF THE HALL-MHD SYSTEM IN $H^s(\mathbb{R}^n)$ WITH $s > \frac{n}{2}$

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Abstract. We establish local well-posedness of the Hall-magneto-hydrodynamics (Hall-MHD) system in the Sobolev space $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2}$. The previously known local well-posedness space was $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2} + 1$. Thus the result presented here is an improvement.

KEY WORDS: Hall-magneto-hydrodynamics; local well-posedness; Littlewood-Paley theory.
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1. Introduction

Considered here is the incompressible Hall-magneto-hydrodynamics (Hall-MHD) system with fractional magnetic diffusion:

\[
\begin{align*}
    u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p &= \nu \Delta u, \\
    b_t + u \cdot \nabla b - b \cdot \nabla u + \eta \nabla \times ((\nabla \times b) \times b) &= -\mu (-\Delta)^\alpha b, \\
    \nabla \cdot u &= 0,
\end{align*}
\]

with $(x, t) \in \mathbb{R}^n \times [0, \infty)$, $n \geq 2$, and initial conditions

\[
\begin{align*}
    u(x, 0) &= u_0(x), & b(x, 0) &= b_0(x), & \nabla \cdot u_0 = \nabla \cdot b_0 &= 0.
\end{align*}
\]

Here $u$ is the fluid velocity, $p$ pressure and $b$ the magnetic field. The constants $\nu$, $\mu$ and $\eta$ denote the kinematic viscosity, the reciprocal of the magnetic Reynolds number and the Hall effect coefficient, respectively. We assume $\nu > 0$, $\mu > 0$ and $\alpha > \frac{1}{2}$. The Hall term $\nabla \times ((\nabla \times b) \times b)$ is the only difference between the Hall-MHD and the usual MHD system. For mathematical study on this model, we refer to [1, 3, 4, 5, 7, 8, 9, 10] and reference therein.

The purpose of this paper is to find the largest possible Sobolev spaces where the Hall-MHD system is locally well-posed. Previously, it was shown in [7] that system (1.1) with $\alpha = 1$ is locally well-posed in $(H^s(\mathbb{R}^3))^2$ with $s > \frac{5}{2}$. Later, in the case of $\frac{1}{2} < \alpha < 1$, local well-posedness was obtained in $(H^s(\mathbb{R}^n))^2$ with $s > \frac{n}{2} + 1$. We aim to improve the aforementioned findings and establish the main result below.

Theorem 1.1. Let $\nu, \mu > 0$ and $\alpha > \frac{1}{2}$. Assume $(u_0, b_0) \in (H^s(\mathbb{R}^n))^2$ with $s > 2 - 2\alpha + \frac{n}{2}$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. There exists a time $T = T(\|u_0\|_{H^s}, \|b_0\|_{H^s}) > 0$ and a unique solution $(u, b)$ of (1.1) on $[0, T]$ such that

\[(u, b) \in (C([0, T]; H^s(\mathbb{R}^n)))^2.\]

Remark 1.2. Notice that $s > 2 - 2\alpha + \frac{n}{2} = \frac{n}{2}$ for $\alpha = 1$; and $2 - 2\alpha + \frac{n}{2} < \frac{n}{2} + 1$ for $\frac{1}{2} < \alpha < 1$. Thus for the regular Hall-MHD system, that is (1.1) with $\alpha = 1$,
we obtain the local well-posedness in \((H^s(\mathbb{R}^n))^2\) with \(s > \frac{n}{2}\), which is a larger than \((H^{5/2}(\mathbb{R}^3))^2\) for \(n = 3\).

The techniques involved are based on the Littlewood-Paley decomposition theory and the frequency-localization approach.

**Notation.** For the sake of brevity, we denote by: \(A \lesssim B\) an estimate of the form \(A \leq CB\) with an absolute constant \(C\); \(A \sim B\) an estimate of the form \(C_1B \leq A \leq C_2B\) with absolute constants \(C_1, C_2\); \(\|\cdot\|_p\) the norm of space \(L^p\); and \((\cdot, \cdot)\) the \(L^2\)-inner product. The notations associated with Littlewood-Paley decomposition theory and related concepts are introduced in Appendix.

## 2. A priori estimate

The core of the proof of local well-posedness is the a priori estimate satisfied by smooth solutions in \(H^s\) with \(s > 2 - 2\alpha + \frac{n}{2}\), which is the content of this section. The local existence of smooth solutions will then follow from certain traditional approximating and limiting process. The uniqueness and continuous dependance on initial data can be also obtained through standard arguments. Thus, we only show

**Theorem 2.1.** Let \((u_0, b_0) \in (H^s(\mathbb{R}^n))^2\) with \(s > 2 - 2\alpha + \frac{n}{2}\) and \((u, b)\) be a smooth solution of (1.1) starting from the data \((u_0, b_0)\). There exists a time \(T = T(\|u_0\|_{H^s}, \|b_0\|_{H^s}) > 0\), such that, for every \(t \in [0, T]\) we have

\[
\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^s}^2 \leq C (\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2),
\]

where the constant \(C\) depends on \(T, \nu, \mu, \|u_0\|_{H^s}\), and \(\|b_0\|_{H^s}\).

**Proof:** Multiplying the first equation of (1.1) by \(\lambda_q^{2s} \Delta_q^2 u\) and the second one by \(\lambda_q^{2s} \Delta_q^2 b\), and taking summation for all \(q \geq -1\) gives us

\[
\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} (\lambda_q^{2s} \|u_q\|_2^2 + \lambda_q^{2s} \|b_q\|_2^2) \\
\leq -\nu \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|_2^2 - \mu \sum_{q \geq -1} \lambda_q^{2s+2\alpha} \|b_q\|_2^2 + I_1 + I_2 + I_3 + I_4 + I_5,
\]

with

\[
I_1 = -\sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx, \quad I_2 = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla b) \cdot u_q \, dx, \\
I_3 = -\sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla b) \cdot b_q \, dx, \quad I_4 = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla u) \cdot b_q \, dx, \\
I_5 = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q ((\nabla \times b) \times b) \cdot \nabla \times b_q \, dx.
\]

As expected, the estimate of \(I_1, I_2, I_3,\) and \(I_4\) are less challenging than that of \(I_5\). On the other hand, due to the similarity of \(I_1\) and \(I_3\) with \(I_2\) and \(I_4\), we are eligible to only show the details of handling \(I_3\) and \(I_2\), not \(I_1\) and \(I_4\).
We first decompose $I_3$ by adapting Bony’s paraproduct \[I_3 = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(u_{p-2} \cdot \nabla b_p) \cdot b_q \, dx
\]
and then by commutator (4.19) to rewrite $I_3$ as
\[I_3 = I_{31} + I_{32} + I_{33};\]
and by commutator (4.19) to rewrite $I_{31}$
\[I_{31} = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, u_{p-2} \cdot \nabla] b_p \cdot b_q \, dx
\]

Since $\sum_{|p-q| \leq 2} \Delta_q b_p = b_q$ and $\nabla \cdot u_{p-2} = 0$, one can infer $I_{312} = 0$.

To estimate $I_{311}$, it follows from the commutator estimate in Lemma 4.2, Hölder’s inequality, and Bernstein’s inequality that
\[|I_{311}| \leq \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2s} \|\nabla u_{p-2}\|_{\infty} \|b_p\|_2 \|b_q\|_2
\]
\[\lesssim \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2 \sum_{p \leq q} \lambda_p^{1+\frac{2}{n}} \|u_p\|_2
\]
\[\lesssim \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{\alpha} \lambda_p^{1+\frac{2}{n} - \delta - \delta \alpha - s} (\lambda_q^{s+1} \|b_q\|_2)^{\delta} \left(\lambda_q^{s} \|b_q\|_2\right)^{2-\delta}
\]
\[\cdot \left(\lambda_p^{s+1} \|u_p\|_2\right)^{\delta} \left(\lambda_p^{s} \|u_p\|_2\right)^{1-\delta}
\]
\[\lesssim \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{\alpha} (\lambda_q^{s+1} \|b_q\|_2)^{\delta} \left(\lambda_q^{s} \|b_q\|_2\right)^{2-\delta} \left(\lambda_p^{s+1} \|u_p\|_2\right)^{\delta} \left(\lambda_p^{s} \|u_p\|_2\right)^{1-\delta}
\]
for some parameter $0 < \delta < 1$ satisfying
\[s \geq 1 + \frac{n}{2} - \delta - \delta \alpha.
\]
We continue the estimate of $I_{311}$ by using Young’s inequality with parameters satisfying
\[\frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3} + \frac{1}{\delta_4} = \delta \alpha, \quad 0 < \delta_1, \delta_2, \delta_3, \delta_4 < 1
\]
and
\[\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} + \frac{1}{\theta_4} = 1, \quad \theta_1 = \theta_3 = \frac{2}{\delta}, \quad 1 < \theta_2, \theta_4 < \infty.
\]
We pause to analyze the parameters. In view of (2.4) and (2.5), we obtain that

\[ |I_{311}| \leq \frac{\mu}{16} \sum_{q \geq 1, p \leq q} \lambda_p \theta_q |b_q|^2 + C_{\nu, \mu} \sum_{q \geq 1, p \leq q} \lambda_p \theta_q (\lambda_q^s |b_q|_2)^{2-\delta} \theta_q \\
+ \frac{\nu}{16} \sum_{q \geq 1, p \leq q} \lambda_p \theta_q \lambda_{q+2} |b_{q+2}|_2 + C_{\nu, \mu} \sum_{q \geq 1, p \leq q} \lambda_p \theta_q (\lambda_q^s |u_p|_2)^{2-\delta} \theta_q \\
\leq \frac{\mu}{16} \sum_{q \geq 1} \lambda_q^{2s+2 \alpha} |b_q|^2 + \frac{\nu}{16} \sum_{q \geq 1} \lambda_q^{2s+2} |u_q|^2 \\
+ C_{\nu, \mu} \left( \sum_{q \geq 1} \lambda_q^{2s} |b_q|^2 \right)^{\gamma_1} + C_{\nu, \mu} \left( \sum_{q \geq 1} \lambda_q^{2s} |u_q|^2 \right)^{\gamma_2},
\]

with various constants \( C_{\nu, \mu} \) that depend on \( \nu, \mu \) and tend to infinity as \( \nu, \mu \to 0 \).

We pause to analyze the parameters. In view of (2.4) and (2.5), we obtain that

\[ |I_3| \leq \frac{\mu}{8} \sum_{q \geq 1} \lambda_q^{2s+2 \alpha} |b_q|^2 + \frac{\nu}{8} \sum_{q \geq 1} \lambda_q^{2s+2} |u_q|^2 \\
+ C_{\nu, \mu} \left( \sum_{q \geq 1} \lambda_q^{2s} |b_q|^2 \right)^{\gamma_1} + C_{\nu, \mu} \left( \sum_{q \geq 1} \lambda_q^{2s} |u_q|^2 \right)^{\gamma_2},
\]

with certain constants \( \gamma_1, \gamma_2 > 1 \).

Adapting the same decomposition strategy of using Bony’s paraproduct and commutator, we deconstruct \( I_2 \) and \( I_4 \) as follows

\[ I_2 = \sum_{q \geq 1, |q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_{q-p} \cdot \nabla b_p) \cdot u_q \, dx \\
+ \sum_{q \geq 1, |q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla b_{q-p}) \cdot u_q \, dx \\
+ \sum_{q \geq 1, p \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla b) \cdot u_q \, dx \\
= I_{21} + I_{22} + I_{23},
\]

with

\[ I_{21} = \sum_{q \geq 1, |q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_{q-p} \cdot \nabla b_p) \cdot u_q \, dx \\
+ \sum_{q \geq 1, |q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{q-p} \cdot \nabla \Delta_q b_p) \cdot u_q \, dx \\
+ \sum_{q \geq 1, p \geq 0} \lambda_q^{2s} \int_{\mathbb{R}^3} ((b_{q-p} - b_{q-p}) \cdot \nabla \Delta_q b_p) \cdot u_q \, dx \\
= I_{211} + I_{212} + I_{213};
\]
and

\[ I_4 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_{\leq p-2} \cdot \nabla u_p) \cdot b_q \, dx \]
\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla u_{\leq p-2}) \cdot b_q \, dx \]
\[ + \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\tilde{b}_p \cdot \nabla u_p) \cdot b_q \, dx \]
\[ = I_{41} + I_{42} + I_{43}, \]

with

\[ I_{41} = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{\leq p-2} \cdot \nabla] u_p \cdot b_q \, dx \]
\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \nabla \Delta u_p) \cdot b_q \, dx \]
\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((b_{\leq p-2} - b_{\leq q-2}) \cdot \nabla \Delta u_p) \cdot b_q \, dx \]
\[ = I_{411} + I_{412} + I_{413}. \]

We claim that \( I_{212} + I_{412} = 0 \). Indeed, we have

\[ I_{212} + I_{412} = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \nabla \Delta u_p) \cdot b_q \, dx \]
\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \nabla \Delta u_p) \cdot b_q \, dx \]
\[ = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \nabla b_q) \cdot u_q \, dx + \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \nabla u_q) \cdot b_q \, dx \]
\[ = 0. \]

The fact \( \sum_{|p-q| \leq 2} \Delta_q b_p = b_q \) and \( \sum_{|p-q| \leq 2} \Delta_q u_p = u_q \) justifies the second equality above.

The rest terms in \( I_2 + I_4 \) are relatively simple. We only choose one representative term, \( I_{211} \), to carry out the details of estimating. Applying Hölder’s inequality and
Bernstein’s inequality leads to

\[
|I_{211}| \leq \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{2s} \|b_{q-p-2}\|_\infty \|b_p\|_2 \|u_q\|_2
\]

\[
\lesssim \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2 \|u_q\|_2 \sum_{p \leq q} \lambda_p^{1+\frac{d}{p}} \|b_p\|_2
\]

\[
= \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{\delta_1,\alpha+\delta_2} \lambda_p^{1+\frac{d}{p} - \delta_1,\alpha-\delta_1 - \delta_2 - s} (\lambda_q^{s+\alpha} \|b_q\|_2)^{\delta_1} (\lambda_q^{s+\alpha} \|b_q\|_2)^{1-\delta_1}
\]

\[
\cdot (\lambda_q^{s+\alpha} \|u_q\|_2)^{\delta_2} (\lambda_q^{s+\alpha} \|b_q\|_2)^{1-\delta_2} (\lambda_p^{s+\alpha} \|b_p\|_2)^{\delta_3} (\lambda_p^{s+\alpha} \|b_p\|_2)^{1-\delta_3}
\]

\[
\leq C \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{\delta_1,\alpha+\delta_2} (\lambda_q^{s+\alpha} \|b_q\|_2)^{\delta_1} (\lambda_q^{s+\alpha} \|b_q\|_2)^{1-\delta_1}
\]

\[
\cdot (\lambda_q^{s+\alpha} \|u_q\|_2)^{\delta_2} (\lambda_q^{s+\alpha} \|b_q\|_2)^{\delta_3} (\lambda_q^{s+\alpha} \|b_q\|_2)^{1-\delta_3}
\]

for parameters \(0 < \delta_1, \delta_2, \delta_3 < 1, \delta_2 = (2 - \delta_1 - \delta_2)\alpha\), and

\[
(2.8) \quad s \geq 1 + \frac{n}{2} - \delta_1 \alpha - \delta_3 \alpha - \delta_2.
\]

Adapting Young’s inequality with parameters \(\zeta_i, 1 \leq i \leq 6\), such that

\[
\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 = \delta_1 \alpha + \delta_2, \quad \zeta_1, \ldots, \zeta_6 > 0
\]

\[
(2.9) \quad \theta_1 = \frac{1}{\delta_1}, \quad \theta_3 = \frac{1}{\delta_3}, \quad \theta_5 = \frac{1}{\delta_5}, \quad 1 < \theta_2, \theta_4, \theta_6 < \infty
\]

we have

\[
|I_{211}| \leq \frac{\mu}{8} \sum_{g \geq -1} \lambda_g^{2s+2\alpha} \|b_g\|_2^2 + \frac{\mu}{16} \sum_{q \geq -1} \lambda_q^{2s+2\alpha} \|u_q\|_2^2 + C_{q,\mu} \sum_{g \geq -1} (\lambda_g^s \|b_g\|_2)^{(1-\delta_1)\theta_2}
\]

\[
+ C_{q,\mu} \sum_{g \geq -1} (\lambda_g^s \|b_g\|_2)^{(1-\delta_3)\theta_6} + C_{q,\mu} \sum_{g \geq -1} (\lambda_g^s \|u_q\|_2)^{(1-\delta_3)\theta_4}.
\]

Again, the parameter constraints \(2.8\) and \(2.9\) imply that

\[
s \geq 1 + \frac{n}{2} - 2\alpha + (\alpha - 1)\delta_2 + 2\alpha \left(\frac{1}{\theta_2} + \frac{1}{\theta_4} + \frac{1}{\theta_6}\right)
\]

\[
= 1 + \frac{n}{2} - 2\alpha + (\alpha - 1)\delta_2 + \epsilon
\]

for large enough \(\theta_2, \theta_4, \text{ and } \theta_6\). Notice that \(s \geq \frac{n}{2} - 1 + \epsilon\) for \(\alpha = 1\). In general for \(\delta_2\) close enough to 1, we have

\[
(2.10) \quad s \geq \frac{n}{2} - \alpha + \epsilon.
\]

To conclude, we expect to have for \(s\) satisfying \(2.10\)

\[
|I_2| \leq \frac{\mu}{8} \sum_{g \geq -1} \lambda_g^{2s+2\alpha} \|b_g\|_2^2 + \frac{\mu}{16} \sum_{q \geq -1} \lambda_q^{2s+2\alpha} \|u_q\|_2^2
\]

\[
+ C_{q,\mu} \left(\sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^{\gamma_1}\right) + C_{q,\mu} \left(\sum_{q \geq -1} \lambda_q^{2s} \|u_q\|_2^{\gamma_2}\right),
\]

for some constants \(\gamma_1, \gamma_2\).
Now we are left to estimate $I_5$. By Bony’s paraproduct and commutator (4.21), the routine decomposition procedure yields

$$I_5 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_{\leq p-2} \times (\nabla \times b_p)) \cdot \nabla \times b_q \, dx$$

$$+ \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \times (\nabla \times b_{\leq p-2})) \cdot \nabla \times b_q \, dx$$

$$+ \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \times (\nabla \times b_p)) \cdot \nabla \times b_q \, dx$$

$$= I_{51} + I_{52} + I_{53};$$

with

$$I_{51} = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{\leq p-2}] \cdot (\nabla \times b_p) \cdot \nabla \times b_q \, dx$$

$$+ \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} b_{\leq p-2} \times (\nabla \times b_q) \cdot \nabla \times b_q \, dx$$

$$+ \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq p-2} - b_{\leq q-2}) \times (\nabla \times (b_p)_q) \cdot \nabla \times b_q \, dx$$

$$= I_{511} + I_{512} + I_{513}.$$ 

The cross product property implies immediately that $I_{512} = 0$. We deduce from the commutator estimate in Lemma [4.3] that

$$|I_{511}| \lesssim \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2s+1} \|
abla b_{\leq p-2}\|_\infty \|b_p\|_2 \|b_q\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2s+1} \|b_q\|_2 \sum_{p \leq q} \lambda_p^{1+\frac{s}{2}} \|b_p\|_2$$

$$= \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{s+\alpha-1} \lambda_p^{2+\frac{s}{2} - \delta_1 \alpha - \delta_2 \alpha - s} (\lambda_q^{s+\alpha} \|b_q\|_2)^{\delta_1} (\lambda_q^{s} \|b_q\|_2)^{2-\delta_1}$$

$$\cdot (\lambda_p^{s+\alpha} \|b_p\|_2)^{\delta_2} (\lambda_p^{s} \|b_p\|_2)^{1-\delta_2}$$

$$\leq C \sum_{q \geq -1} \sum_{p \leq q} \lambda_q^{s+\alpha-1} (\lambda_q^{s+\alpha} \|b_q\|_2)^{\delta_1} (\lambda_q^{s} \|b_q\|_2)^{2-\delta_1}$$

$$\cdot (\lambda_p^{s+\alpha} \|b_p\|_2)^{\delta_2} (\lambda_p^{s} \|b_p\|_2)^{1-\delta_2}$$

for parameters satisfying $\frac{1}{2} < \delta_1 < 2$, $0 < \delta_2 = 1$, and

$$s \geq 2 + \frac{n}{2} - \delta_1 \alpha - \delta_2 \alpha.$$ 

By Young’s inequality we have for the parameters

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = \delta_1 \alpha - 1, \quad \zeta_1, \ldots, \zeta_4 > 0$$

$$\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3} + \frac{1}{\theta_4} = 1, \quad \theta_1 = \frac{2}{\delta_1}, \quad \theta_3 = \frac{2}{\delta_2}, \quad 1 < \theta_2, \theta_4 < \infty,$$
such that
\[ |I_{511}| \leq \mu \sum_{q \geq 1} \sum_{p \geq q} \lambda_{q}^{2s} \int_{\mathbb{R}^3} |(b_{p} - b_{q})|^2 \cdot \nabla \times |b_{q}| \, dx \]
\[ = \sum_{q \geq 1} \sum_{p \geq q} \lambda_{q}^{2s} \int_{\mathbb{R}^3} \Delta \nabla \times (b_{p} - b_{q}) \cdot \nabla \times b_{q} \, dx \]
\[ = I_{521} + I_{522} + I_{523}. \]
We will only show the estimate of \( I_{522} \), since \( I_{521} \) enjoys the same estimate as \( I_{511} \) due to the commutator estimate in Lemma 4.3 and \( I_{523} \) can be estimated as \( I_{513} \).
Integration by parts, identity (4.20) along with the fact that $\nabla \cdot b = 0$ infers

$$I_{522} = \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \nabla \times (\nabla \times b_{\leq q-2} \times b_q) \cdot b_q \, dx$$

$$= \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} [(b_q \cdot \nabla)\nabla \times b_{\leq q-2} - (\nabla \cdot \nabla \times b_{\leq q-2})b_q] \cdot b_q \, dx$$

$$- \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (\nabla \times b_{\leq q-2} \cdot \nabla) b_q \cdot b_q \, dx.$$  

Since $\nabla \cdot (\nabla \times b_{\leq q-2}) = 0$, it is obvious the last integral vanishes. Thus we have

$$|I_{522}| \leq \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} |(b_q \cdot \nabla)\nabla \times b_{\leq q-2} - (\nabla \cdot \nabla \times b_{\leq q-2})b_q| \cdot b_q| \, dx$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2s} \|\Delta b_{\leq q-2}\|_\infty \|b_q\|_2^2$$

which share the same estimate of $I_{511}$.

The last term $I_{53}$ is treated as

$$|I_{53}| \leq \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} |\Delta_q(b_p \times \nabla \times \hat{b}_p) \cdot \nabla \times b_q| \, dx$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2s} \|\Delta b_q\|_\infty \sum_{p \geq q-3} \|b_p\|_2 \|\nabla b_p\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2s+1+\frac{\mu}{2}} \|b_q\|_2 \sum_{p \geq q-3} \lambda_p \|b_p\|_2^2$$

$$\lesssim \sum_{p \geq -1} \lambda_p \|b_p\|_2^2 \sum_{q \leq p+3} \lambda_q^{2s+1+\frac{\mu}{2}} \|b_q\|_2$$

which turns out to be similar as $I_{511}$ again. Summarizing the analysis above, we obtain

(2.15)

$$|I_5| \lesssim \frac{\nu}{8} \sum_{q \geq -1} \lambda_q^{2s+2\alpha} \|b_q\|_2^2 + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2 \right)^{\gamma_1} + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2 \right)^{\gamma_2}$$
for some $\gamma_1, \gamma_2 > 1$. Putting together of (2.3), (2.7), (2.11), and (2.15), there exist constants $C_\nu$, $C_\mu$, and $C_{\nu, \mu}$ such that

$$
\frac{d}{dt} \left( \|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right) + \nu \sum_{q \geq -1} \lambda_q^{2s+2} \|u_q\|^2_T + \mu \sum_{q \geq -1} \lambda_q^{2r+2\alpha} \|b_q\|^2_T
$$

\begin{equation}
\leq C_\nu \left( \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|^2_T \right)^{\gamma_1} + C_\nu \left( \sum_{q \geq -1} \lambda_q^{2s} \|u_q\|^2_T \right)^{\gamma_2} + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|^2_T \right)^{\gamma_1} + C_\mu \left( \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|^2_T \right)^{\gamma_2}
\end{equation}

\begin{equation}
\leq C_{\nu, \mu} \left( \|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right)^{\gamma_1} + C_{\nu, \mu} \left( \|u\|_{H^s}^2 + \|b\|_{H^r}^2 \right)^{\gamma_2}
\end{equation}

Notice that $\gamma_1, \gamma_2 > 1$ and hence the energy inequality (2.16) is in the type of Riccati. It follows that, there exists a time $T > 0$ which depends on $\nu, \mu$ and $\|u_0\|_{H^s}, \|b_0\|_{H^r}$, such that

$$
\|u(t)\|_{H^s}^2 + \|b(t)\|_{H^r}^2 \leq C(\nu, \mu, T, \|u_0\|_{H^s}, \|b_0\|_{H^r}) \left( \|u_0\|_{H^s}^2 + \|b_0\|_{H^r}^2 \right)
$$

for $0 \leq t < T$, and a constant $C$ depending on $\nu, \mu, T$ and $\|u_0\|_{H^s}, \|b_0\|_{H^r}$.

### 3. Convergence of the Hall-MHD to the MHD System

In this section, we show that solutions $(u^n, b^n, p^n)$ of (1.1) with $\alpha = 1$ in $H^{\frac{3}{2}}$ converges to a solution $(u, b, p)$ of the MHD system, as $\eta \to 0$. Namely, we prove

**Theorem 3.1.** Let $(u^n, b^n, p^n)$ be a solution to (1.1) with $\alpha = 1$ obtained in Theorem 1.1 associated with initial data $(u_0, b_0)$. Let $(u, b, p)$ be a solution to (1.1) with $\eta = 0$ and $\alpha = 1$ under the same initial data. Then we have

$$
\lim_{\eta \to 0} \left( \|u^n - u\|_2 + \|b^n - b\|_2 \right) = 0.
$$

**Proof:** Take the difference $U = u^n - u$, $B = b^n - b$ and $\pi = p^n - p$, which satisfy the equations:

\begin{align}
U_t + u \cdot \nabla U - b \cdot \nabla B + U \cdot \nabla u^n - B \cdot \nabla b^n + \nabla \pi &= \nu \Delta U, \\
B_t + u \cdot \nabla B - b \cdot \nabla U + U \cdot \nabla b^n - B \cdot \nabla u^n - \eta \nabla \times ((\nabla \times b^n) \times b^n) &= \mu \Delta B,
\end{align}

\begin{align*}
\nabla \cdot U &= 0, \quad \nabla \cdot B = 0.
\end{align*}

Multiplying the first equation by $U$ and the second by $B$, we obtain (formally)

\begin{align}
\frac{1}{2} \frac{d}{dt} \|U\|_2^2 + \nu \|\nabla U\|_2^2 &\leq \int_{\mathbb{R}^3} b \cdot \nabla B \cdot U \, dx - \int_{\mathbb{R}^3} U \cdot \nabla u^n \cdot U \, dx + \int_{\mathbb{R}^3} B \cdot \nabla b^n \cdot U \, dx, \\
\frac{1}{2} \frac{d}{dt} \|B\|_2^2 + \mu \|\nabla B\|_2^2 &\geq \int_{\mathbb{R}^3} b \cdot \nabla U \cdot B \, dx - \int_{\mathbb{R}^3} U \cdot \nabla b^n \cdot B \, dx + \int_{\mathbb{R}^3} B \cdot \nabla u^n \cdot B \, dx \\
&\quad + \eta \int_{\mathbb{R}^3} \nabla \times ((\nabla \times b^n) \times b^n) \cdot B \, dx.
\end{align}
Adding the two yields, provided that \((u^\eta, b^\eta, p^\eta)\) and \((u, b, p)\) are regular enough,

\[
\frac{1}{2} \frac{d}{dt} (\|U\|^2 + \|B\|^2) + \nu \|\nabla U\|^2 + \mu \|\nabla B\|^2
\]

\[
= - \int_{\mathbb{R}^3} U \cdot \nabla u^\eta \cdot U \, dx + \int_{\mathbb{R}^3} B \cdot \nabla b^\eta \cdot U \, dx - \int_{\mathbb{R}^3} U \cdot \nabla b^\eta \cdot B \, dx
\]

\[
+ \int_{\mathbb{R}^3} B \cdot \nabla u^\eta \cdot B \, dx + \eta \int_{\mathbb{R}^3} \nabla \times ((\nabla \times b^\eta) \times b^\eta) \cdot B \, dx
\]

\[= I_1 + I_2 + I_3 + I_4 + I_5.\]

It is straightforward to notice that

\[|I_1 + I_2 + I_3 + I_4| \leq C (\|\nabla u^\eta\|_\infty + \|\nabla b^\eta\|_\infty) (\|U\|^2 + \|B\|^2);\]

and also

\[|I_1 + I_2 + I_3 + I_4| \leq C (\nu^{-1} + \mu^{-1}) \left(\|u^\eta\|_\infty + \|b^\eta\|_\infty\right) (\|U\|^2 + \|B\|^2)
\]

\[
+ \frac{1}{4} \nu \|\nabla U\|^2 + \frac{1}{4} \mu \|\nabla B\|^2.
\]

We estimate \(I_5\) as

\[|I_5| = \left| \eta \int_{\mathbb{R}^3} ((\nabla \times b^\eta) \times b^\eta) \cdot \nabla \times B \, dx \right|
\]

\[\leq C \eta \|\nabla b^\eta\|_\infty \|b^\eta\|_2 \|\nabla B\|_2
\]

\[\leq C \eta^2 \mu^{-1} \|\nabla b^\eta\|_\infty \|b^\eta\|^2_2 + \frac{1}{4} \mu \|\nabla B\|^2_2\]

or as

\[|I_5| = \left| \eta \int_{\mathbb{R}^3} ((\nabla \times b^\eta) \times b^\eta) \cdot \nabla \times B \, dx \right|
\]

\[\leq C \eta \|b^\eta\|_\infty \|\nabla b^\eta\|_2 \|\nabla B\|_2
\]

\[\leq C \eta^2 \mu^{-1} \|b^\eta\|^2_\infty \|\nabla b^\eta\|^2_2 + \frac{1}{4} \mu \|\nabla B\|^2_2\]

Combining the above estimates leads to, for \(s > \frac{n}{2}\)

\[
\frac{d}{dt} (\|U\|^2 + \|B\|^2) \leq C (\|U\|^2 + \|B\|^2) + C \eta^2 \mu^{-1} \|\nabla b^\eta\|^2_2,
\]

from which Grönwall’s inequality implies that

\[\|U(t)\|^2 + \|B(t)\|^2 \leq C \eta^2 \mu^{-1} + (\|U(0)\|^2 + \|B(0)\|^2 + C \eta^2 \mu^{-1}) e^{Ct}.
\]

Note that \(U(0) = B(0) = 0\). Thus

\[\lim_{\eta \to 0} (\|U(t)\|^2 + \|B(t)\|^2) = 0,
\]

and the convergence rate is \(O(\eta^2)\). \(\square\)
4. Appendix

4.1. Littlewood-Paley decomposition. Our analysis is built on the Littlewood-Paley decomposition theory. Basic languages and concepts are introduced briefly below.

We choose a nonnegative radial function \( \chi \in C^\infty_0(\mathbb{R}^n) \) satisfying

\[
\chi(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq \frac{3}{4} \\
0, & \text{for } |\xi| \geq 1.
\end{cases}
\]

Denote \( \lambda_q = 2^q \) for integers \( q \). A sequence of cut-off functions are defined,

\[
\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi), \quad \varphi_q(\xi) = \begin{cases} 
\varphi(\lambda_q^{-1} \xi) & \text{for } q \geq 0, \\
\chi(\xi) & \text{for } q = -1.
\end{cases}
\]

For a tempered distribution vector field \( u \) we define the Littlewood-Paley projection

\[
h = \mathcal{F}^{-1} \varphi, \quad \tilde{h} = \mathcal{F}^{-1} \chi,
\]

\[
u_q := \Delta_q u = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1} \xi) \mathcal{F} u) = \lambda_q^n \int h(\lambda_q y) u(x - y) dy, \quad \text{for } q \geq 0,
\]

\[
u_{-1} = \mathcal{F}^{-1}(\chi(\xi) \mathcal{F} u) = \int \tilde{h}(y) u(x - y) dy,
\]

where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transform and inverse Fourier transform, respectively. Due to the Littlewood-Paley theory, the identity

\[
u = \sum_{q=-1}^{\infty} \nu_q
\]

holds in the sense of distribution, which is the fundamental idea of shell decomposition. We also denote the various summation terms simply by

\[
u_{\leq Q} = \sum_{q=-1}^{Q} \nu_q, \quad \nu_{(Q,N)} = \sum_{p=Q+1}^{N} \nu_p, \quad \tilde{\nu}_q = \sum_{|p-q| \leq 1} \nu_p.
\]

We can adapt the norm of Sobolev space \( \dot{H}^s \) as

\[
\|u\|_{\dot{H}^s} \sim \left( \sum_{q=-1}^{\infty} \lambda_q^{2s} \|\nu_q\|_2^2 \right)^{1/2}, \quad s \in \mathbb{R}.
\]

Bernstein’s inequality satisfied by the dyadic blocks \( \nu_q \) is introduced below.

**Lemma 4.1.** Let \( n \) be the space dimension and \( r \geq s \geq 1 \). Then for all tempered distributions \( u \), we have

\[
\|u_q\|_r \lesssim \lambda_q^{n(\frac{1}{r} - \frac{1}{s})} \|u_q\|_s.
\]
4.2. Bony’s paraproduct and commutators. We adapt the following version of Bony’s paraproduct

\[
\Delta_q(u \cdot \nabla v) = \sum_{|q-p| \leq 2} \Delta_q(u_{\leq p-2} \cdot \nabla v_p) + \sum_{|q-p| \leq 2} \Delta_q(u_p \cdot \nabla v_{\leq p-2}) \\
+ \sum_{p \geq q-2} \Delta_q(\tilde{u}_p \cdot \nabla v_p),
\]

which is used through the paper to decompose the nonlinear terms. We introduce a commutator as

\[
[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p = \Delta_q(u_{\leq p-2} \cdot \nabla v_p) - u_{\leq p-2} \cdot \nabla \Delta_q v_p.
\]

Lemma 4.2. The following estimate holds, for any \(1 < r < \infty\)

\[
\|[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p\|_r \lesssim \|\nabla u_{\leq p-2}\|_\infty \|v_p\|_r.
\]

To treat the Hall term, we recall a fundamental identity for vector valued functions \(F\) and \(G\),

\[
\nabla \times (F \times G) = [(G \cdot \nabla)F - (\nabla \cdot F)G] - [(F \cdot \nabla)G - (\nabla \cdot G)F].
\]

In addition, two more commutators are defined

\[
[\Delta_q, F \times \nabla \times G] = \Delta_q(F \times (\nabla \times G)) - F \times (\nabla \times G),
\]

\[
[\Delta_q, (\nabla \times F) \times G] = \Delta_q((\nabla \times F) \times G) - (\nabla \times F) \times G.
\]

They satisfy the estimates below.

Lemma 4.3. Assume \(\nabla \cdot F = 0\) and \(F, G\) vanish at large \(|x| \in \mathbb{R}^3\). For any \(1 \leq r \leq \infty\), we have

\[
\|[\Delta_q, F \times \nabla \times G]\|_r \lesssim \|\nabla F\|_\infty \|G\|_r;
\]

\[
\|\Delta_q, (\nabla \times F) \times G\|_r \lesssim \|\nabla F\|_\infty \|G\|_r.
\]

Lemma 4.4. Assume the vector valued functions \(F, G\) and \(H\) vanish at large \(|x| \in \mathbb{R}^3\). For any \(1 \leq r_1, r_2 \leq \infty\) with \(1/r_1 + 1/r_2 = 1\), we have

\[
\left| \int_{\mathbb{R}^3} [\Delta_q, (\nabla \times F) \times G \cdot \nabla H \ dx] \right| \lesssim \|\nabla^2 F\|_\infty \|G\|_{r_1} \|H\|_{r_2}.
\]

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