Twisted derived equivalences for affine schemes

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Abstract
We show how work of Rickard and Toën completely resolves the question of when two twisted affine schemes are derived equivalent.

1 Introduction

The question of when $D^b(X)$ is equivalent as a $k$-linear triangulated category to $D^b(Y)$ for two $k$-varieties $X$ and $Y$ has been extensively studied since Mukai proved that $D^b(\hat{A}) \cong D^b(A)$ for an abelian variety $A$ and its dual $\hat{A}$ [5]. Since in general $A$ and $\hat{A}$ are not isomorphic, derived equivalence is a weaker condition than isomorphism. However, derived equivalence nevertheless does preserve a great deal of information: derived equivalent varieties have the same dimension, the same algebraic $K$-theory, and the same Hochschild homology.

The cohomological Brauer group of a scheme $X$ is $Br'(X) = H^2_{et}(X, G_m)$ tors. When $X$ is quasi-compact, there is an inclusion $Br(X) \subseteq Br'(X)$, where $Br(X)$ denotes the Brauer group of $X$, which classifies Brauer equivalence classes of Azumaya algebras on $X$. In many cases of interest, $Br(X) = Br'(X)$. Examples include all quasi-projective schemes over affine schemes [3]. The Brauer group comes into play because in many problems on moduli of vector bundles, there is an obstruction, living in the Brauer group of the coarse moduli space, to the existence of a universal vector bundle. Another way to say this is that this class in the Brauer group is the obstruction to the coarse moduli space being fine. At times one then obtains an equivalence $D^b(X) \cong D^b(Y, \beta)$, where $D^b(Y, \beta)$ is the derived category of $\beta$-twisted coherent sheaves. Particular cases of this arise in the study of K3 surfaces for example.

The systematic study of when $D^b(X, \alpha) \cong D^b(Y, \beta)$ began with Căldăraru’s thesis [1]. In this short note, we are interested in the following two problems.

**Problem 1.1.** Let $R$ be a commutative ring, let $X$ and $Y$ be two locally noetherian $R$-schemes, and fix $\alpha \in Br'(X)$ and $\beta \in Br'(Y)$. Determine when there exists an $R$-linear equivalence of triangulated categories $D^b(X, \alpha) \cong D^b(Y, \beta)$.

**Problem 1.2.** Let $R$ be a commutative ring, let $X$ and $Y$ be two quasi-compact and quasi-separated $R$-schemes, and fix $\alpha \in Br'(X)$ and $\beta \in Br'(Y)$. Determine when there exists an $R$-linear equivalence of triangulated categories $D_{perf}(X, \alpha) \cong D_{perf}(Y, \beta)$.
Here, $D^b(X, \alpha)$ denotes the bounded derived category of $\alpha$-twisted coherent sheaves, while $D_{\text{perf}}(X, \alpha)$ is the triangulated category of perfect complexes of $\alpha$-twisted $\mathcal{O}_X$-modules. When $X$ is regular and noetherian, the existence locally of finite-length finitely generated locally free resolutions implies that the natural map $D_{\text{perf}}(X, \alpha) \to D^b(X, \alpha)$ is an equivalence of $R$-linear triangulated categories.

**Question 1.3.** Are Problems 1.1 and 1.2 equivalent for $X$ and $Y$ noetherian and quasi-separated?

The contents of our paper are as follows. In Section 2, we give some background on twisted derived categories and equivalences between them. Then, in Section 3, the affine case of Problems 1.1 and 1.2 is solved completely, and we explain how work of Rickard shows that these two problems are equivalent for affine schemes. We do not claim that this result is new, but rather that it is not as well-known as it should be.

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## 2 Twisted derived categories

Let $X$ be a scheme, and take $\alpha \in H^1_c(X, \mathbb{G}_m)$. Then, $\alpha$ is represented by a $\mathbb{G}_m$-gerbe $\mathcal{X} \to X$. There is a good notion of quasi-coherent sheaf on $\mathcal{X}$, or of coherent sheaf when $X$ is locally noetherian. An $\mathcal{O}_X$-module $\mathcal{F}$ comes naturally with a left action of the sheaf $\mathcal{G}_m, X$. But, there is a second, inertial action, which can be described as saying that a section $u \in \mathcal{G}_m(U)$ over $U \to X$ acts on $\mathcal{F}(U)$ via the isomorphism $u^*\mathcal{F}|_U \to \mathcal{F}|_U$, which induces an isomorphism $u^* : \mathcal{F}(U) \to \mathcal{F}(U)$. There is an associated left action of the inertial action. An $\alpha$-twisted $\mathcal{O}_X$-module is by definition an $\mathcal{O}_X$-module $\mathcal{F}$ for which these two left actions agree. It is shown in [4, Proposition 2.1.3.3] that this agrees with the definition of $\alpha$-twisted sheaf given by Căldăraru.

If $X$ is a scheme and $\alpha \in H^1_c(X, \mathbb{G}_m)$, write $D_{\text{perf}}(X, \alpha)$ for the derived category of complexes of $\alpha$-twisted sheaves that are étale locally quasi-isomorphic to finite-length complexes of vector bundles. This is naturally a full subcategory of $D_{\text{qc}}(X, \alpha)$ of complexes of $\alpha$-twisted sheaves with ($\alpha$-twisted) quasi-coherent cohomology sheaves. If $X$ is regular and noetherian, then $D_{\text{perf}}(X, \alpha) \cong D^b(X, \alpha)$, the bounded derived category of $\alpha$-twisted coherent $\mathcal{O}_X$-modules.

Let $A$ be an Azumaya algebra on $X$ with class $\alpha$. A complex of right $A$-modules $P$ (in the abelian category $\text{Mod}_{\mathcal{O}_X}$) is perfect if there is an open affine cover $\{U_i\}_{i \in I}$ of $X$ such that $P_{U_i}$ is quasi-isomorphic to a bounded complex of finitely generated projective right $\mathcal{O}_X(U_i, A)$-modules. The derived category of perfect complexes of right $A$-modules will be denoted $D_{\text{perf}}(X, A)$. Then, as explained in [1], $D_{\text{perf}}(X, \alpha) \cong D_{\text{perf}}(X, A)$. In the same way, there is a big derived category of all all complexes of right $A$-modules with quasi-coherent cohomology sheaves $D_{\text{qc}}(X, A)$ and an equivalence $D_{\text{qc}}(X, \alpha) \cong D_{\text{qc}}(X, A)$.

In the next section, we will need dg enhancements of these categories. Write $\text{Perf}(X, \alpha)$ and $\text{QC}(X, \alpha)$ for dg enhancements of $D_{\text{perf}}(X, \alpha)$ and $D_{\text{qc}}(X, \alpha)$, respectively.
These are pretriangulated dg categories. The big dg category $\mathcal{QC}(X, \alpha)$ is constructed for example in Toën [9]. The small dg category $\mathcal{Perf}(X, \alpha)$ can then be taken to be the dg category of compact objects in $\mathcal{QC}(X, \alpha)$.

3 Twisted derived equivalences over affine schemes

Many of us first learned of twisted derived categories from Căldăraru’s thesis [1] and the paper [2]. In that paper, Căldăraru cites a private communication from Yekuteli giving the following theorem [1, Theorem 6.2].

Theorem 3.1. Suppose that $R$ is a commutative local ring and that $A$ and $B$ are Azumaya $R$-algebras with classes $\alpha, \beta \in \text{Br}(R)$. Then, the following are equivalent:

1. $\alpha = \beta$ in $\text{Br}(R)$;

2. $A$ and $B$ are derived Morita equivalent over $R$—that is, there is an $R$-linear equivalence of triangulated categories $D^b(A) \cong D^b(B)$.

It is the purpose of our paper to advertise the fact that the condition that $R$ be local is unnecessary.

Theorem 3.2. Suppose that $R$ is a commutative ring and that $\alpha, \beta \in \text{Br}(R)$. Then, the following are equivalent:

1. $\alpha = \beta$ in $\text{Br}(R)$;

2. there is an $R$-linear equivalence of triangulated categories $D_{\text{perf}}(R, \alpha) \cong D_{\text{perf}}(R, \beta)$.

Moreover, if $R$ is noetherian, these are equivalent to:

3. there is an $R$-linear equivalence of triangulated categories $D^b(R, \alpha) \cong D^b(R, \beta)$.

Proof. Since $\alpha, \beta \in \text{Br}(R)$, we can assume that $\alpha$ is represented by an Azumaya $R$-algebra $A$, and that $\beta$ is represented by an Azumaya $R$-algebra $B$. In this case, $D_{\text{perf}}(X, \alpha) \cong D_{\text{perf}}(X, A) \cong D^b(\text{proj}_A)$, where $D^b(\text{proj}_A)$ is the bounded derived category of finitely generated projective $A$-modules. The second equivalence follows because on an affine scheme, every perfect complex is quasi-isomorphic to a bounded complex of finitely generated projectives (see [8, Proposition 2.3.1(d)]), and this generalizes in a straightforward way to Azumaya algebras on an affine scheme.

When $\alpha = \beta$, the Azumaya algebras $A$ and $B$ are Brauer equivalent. This means that there exist finitely generated projective $R$-modules $M$ and $N$ and an $R$-algebra isomorphism

$$A \otimes_R \text{End}_R(M) \cong B \otimes_R \text{End}_R(N).$$

It follows from classical Morita theory that there is an equivalence $\text{Mod}_A \cong \text{Mod}_B$ of abelian categories of right $A$ and $B$-modules. From this it follows immediately that $D^b(\text{proj}_A) \cong D^b(\text{proj}_B)$. This proves that (1) implies (2).

So, suppose that $D_{\text{perf}}(R, \alpha) \cong D_{\text{perf}}(R, \beta)$, or in other words that $D^b(\text{proj}_A) \cong D^b(\text{proj}_B)$. Rickard’s theorem [6, Theorem 6.4] as refined in [7] implies that there is
a tilting complex inducing an $R$-linear equivalence $D^b(\text{proj}_A) \simeq D^b(\text{proj}_B)$. (Rickard’s theorem does not imply that this is the equivalence we began with, but it is still $R$-linear.) The existence of the tilting complex implies that there is an equivalence of $R$-linear dg categories $\text{Perf}(R, \alpha) \simeq \text{Perf}(R, \beta)$, which is then a derived Morita equivalence. That is, there is an equivalence of the “big” $R$-linear dg categories $\text{QC}(R, \alpha) \simeq \text{QC}(R, \beta)$. These are locally presentable dg categories with descent in the language of [9]. Now, the derived Brauer group of $R$, denoted $\text{dBr}(R)$, classifies locally presentable dg categories with descent over $R$ that are étale locally equivalent to $\text{QC}(R)$. Since $\text{Spec} \ R$ is affine, the $R$-linear equivalence $\text{QC}(R, \alpha) \simeq \text{QC}(R, \beta)$ means that $\alpha$ and $\beta$ define the same element of $\text{dBr}(R)$ (see [9, Section 3]). But, $\text{Br}(R) \cong \text{H}^2_{\text{et}}(\text{Spec} \ R, \mathbb{Z}) \times \text{H}^2_{\text{et}}(\text{Spec} \ R, \mathbb{G}_m)$ by [9, Theorem 1.1]. Since $\text{Br}(R) \subseteq \text{dBr}(R)$, it follows that $\alpha = \beta$, and so (2) implies (1).

Finally, the fact that (2) and (3) are equivalent follows from [6, Propositions 8.1, 8.2]. This completes the proof.

\textbf{Remark 3.3.} By [3], $\text{Br}(R) = \text{Br}^r(R) = \text{H}^2_{\text{et}}(\text{Spec} \ R, \mathbb{G}_m)_{\text{tors}}$.

We expand briefly on the philosophy of the proof. Write $\Omega \mathcal{C}$ for the étale stack of locally presentable dg categories with dg category of sections over $f : Y \to X$ the locally presentable dg category $\text{QC}(Y)$, which is a dg category enhancement of the derived category $\text{D}_{\text{qc}}(Y)$ of complexes of $\mathcal{O}_Y$-modules with quasi-coherent cohomology sheaves. The derived Brauer group $\text{dBr}(X)$ of a scheme classifies stacks of locally presentable dg categories that are étale locally equivalent to $\Omega \mathcal{C}$ up to equivalence of stacks.

\textbf{Motto 3.4.} The Brauer group classifies Azumaya algebras $\mathcal{A}$ up to derived Morita equivalence of stacks of dg categories of complexes of $\mathcal{A}$-modules.

For $\alpha \in \text{dBr}(X)$, write $\Omega \mathcal{C}(\alpha)$ for the associated stack. For instance, if $\alpha$ is the Brauer class of an Azumaya algebra $\mathcal{A}$ over $X$ then the dg category of sections over $f : Y \to X$ of $\Omega \mathcal{C}(\alpha)$ is $\text{QC}(Y, f^* \mathcal{A})$, which is a dg category enhancement of $\text{D}_{\text{qc}}(Y, \mathcal{A}) \simeq \text{D}_{\text{qc}}(Y, \alpha)$. The key point in the proof of the theorem was that over an affine scheme $\text{Spec} \ R$, giving an equivalence of stacks $\Omega \mathcal{C}(\alpha) \simeq \Omega \mathcal{C}(\beta)$ is equivalent to giving an $R$-linear equivalence of the global sections $\text{QC}(R, \alpha) \simeq \text{QC}(R, \beta)$.

On non-affine schemes, giving an equivalence of global sections is, not surprisingly, insufficient. The following example is due to Căldăraru [1, Example 1.3.16]. Let $X$ be a smooth projective K3 surface over the complex numbers given as a double cover of $\mathbb{P}^2$ branched along a smooth sextic curve. The involution $\phi$ of $X$ given by interchanging the sheets of the cover has the property that $\phi^* \alpha = -\alpha$ for $\alpha \in \text{Br}(X)$. Clearly $\phi$ induces an equivalence $\text{D}^b(X, \alpha) \simeq \text{D}^b(X, -\alpha)$. But, since $\text{Br}(X)$ contains non-zero $p$-torsion for every prime $p$, there is a class $\alpha \in \text{Br}(X)$ such that $\alpha \neq -\alpha$. Thus, the theorem fails in the non-affine case. The problem is that the equivalence does not respect restriction to open subsets of $X$.

We now prove the conjecture suggested by Căldăraru after [2, Theorem 6.2].

\textbf{Corollary 3.5.} Suppose that $R$ and $S$ are commutative rings and that there is an equivalence of triangulated categories $\text{D}_{\text{perf}}(R, \alpha) \simeq \text{D}_{\text{perf}}(S, \beta)$ for $\alpha \in \text{Br}(R)$ and $\beta \in \text{Br}(S)$. Then, there exists a ring isomorphism $\phi : R \to S$ such that $\phi^*(\alpha) = \beta$ in $\text{Br}(S)$.
Proof. Let $A$ be an Azumaya algebra with class $\alpha$ over $R$, and let $B$ an be Azumaya over $S$ with class $\beta$. Then, our hypotheses say that $D^b(\text{proj}_A) \simeq D^b(\text{proj}_B)$. By Rickard [6, Proposition 9.2], the centers of $A$ and $B$ are isomorphic. Thus, there is an isomorphism $\phi: R \to S$, and there are equivalences $D_{\text{perf}}(S, \beta) \simeq D_{\text{perf}}(R, \alpha) \simeq D_{\text{perf}}(S, \phi^*(\alpha))$. The composition induces a ring automorphism $\sigma: S \to S$. So, by composing on the $\phi$ on the right with $\sigma^{-1}$, we can assume that $D_{\text{perf}}(S, \beta) \simeq D_{\text{perf}}(S, \phi^*(\alpha))$ is $S$-linear. The corollary follows now from the theorem.

We end by observing that the condition of $R$-linearity in Theorem 3.2 is necessary.

Remark 3.6. Consider the field $k = \mathbb{C}(w, x, y, z)$ and the quaternion division algebras $(w, x)$ and $(y, z)$ over $k$. Then, these algebras are evidently derived Morita equivalent over $\mathbb{C}$ (they are even isomorphic over $\mathbb{C}$). However, $[(w, x)] \neq [(y, z)]$ in $\text{Br}(k)$.

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