SPECTRAL COINCIDENCE OF TRANSITION OPERATORS, AUTOMATA GROUPS
AND BBS IN TROPICAL GEOMETRY

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Abstract. We give the automata which describe time evolution rules of the box-ball system (BBS) with a carrier. It can be shown by use of tropical geometry, such systems are ultradiscrete analogues of KdV equation. We discuss their relation with the lamplighter group generated by an automaton. We present spectral analysis of the stochastic matrices induced by these automata, and verify their spectral coincidence.

Contents

1. Introduction 1
2. Automata groups 3
3. Discrete KDV equation 4
4. Tropical geometry 5
5. BBS with carrier capacity 7
6. Stochastic matrices 9
7. Spectral computation 9
8. Ergodicity of the Markov operators 11
9. Conjugacy by permutation 12
References 18

1. Introduction

From the viewpoint of dynamical systems, automata constitute semi-group actions on trees which play the essential roles in two different subjects, where one is theory of automata groups and the other is discrete integrable systems.

Both subjects have been developed from the point of view of dynamical scale transform called tropical geometry or ultradiscretization (they are essentially the same but the original sources have been different, where the former arose in real algebraic geometry and the latter from discretization of infinite integrable systems). It provides with a correspondence between automata and real rational dynamics, which by taking scaling limits of parameters, allows us to study two dynamical systems at the same time, whose dynamical natures are very different from each other. Particularly it eliminates detailed activities in rational dynamics and extracts framework of their structure in automata, which allows us to induce some uniform analytic estimates.

From the computational interests, many of the integrable systems have been discretized. In particular KdV equation is a fundamental equation in the infinite integrable systems, and its discretized equation has been extensively studied [5], as a rational dynamical systems. In [6], tropical transform has been applied to the discrete KdV equation and obtained the ultradiscrete KdV equation, which is the so-called box-ball systems (BBS). It is given by a direct limit of the Mealy type automata BBS_k for k ≥ 1, and each BBS_k is described by the automaton diagram.

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In [7], we have verified that the automaton is recursive if and only if the associated rational dynamics is quasi-recursive. Quasi-recursiveness allows errors from periodicity, but within uniform estimates which are independent of the choice of initial values.

As an extension of the above property, we have applied tropical geometry to theory of automata groups to analyze global behaviour of real rational dynamical systems. A discrete group is called an automata group, if it is realized by actions on the rooted trees, which are represented by a Mealy automaton. Automata groups is a quite important class in group theory, which have given answers to many important questions. Of particular interests for us are, counter-example to the Milnor’s conjecture, solution to the Burnside problem on the existence of finitely generated infinite torsion groups, non-uniform exponential growth groups, etc. These applications are described in [18] and [19]. As an application of tropical geometry to the construction of the Burnside group, we have verified that there exists a rational dynamical systems of Mealy type which satisfy infinite quasi-recursiveness [8]. This property again allows error from recursiveness which corresponds to infinite torsion, while rationality corresponds to finite generation.

We want to investigate BBS systems via spectra of some operators associated to them, as it is common in non-commutative geometry.

In case of finitely generated groups one can consider as the space $l^2$ functions on groups and as the operator the sum of translations by chosen generators and their inverses. The study of spectra of such operators was initiated by Kesten and the normalized operators are called random walk or Markov operators.

In general it is a very difficult problem to compute spectra of these operators. In recent years important progress was achieved by studying different approximations of such operators using the representations of the group, in particular their actions on finite sets. For instance in [1] the spectrum of the random walk operator on the Heisenberg group was computed using as approximations Harper operators via theory of rotational algebras. In case of groups generated by automata one can study their actions on finite sequences as it was done in [2]. The simplest case when one obtains an interesting spectral information is the automaton on two states which generated the lamplighter group. All other two state automata lead to very elementary cases as was proven in [2].

In the case of BBS we do not deal with invertible transformations which would define groups. However we can still define the operators similar to random walk operators and consider their action on finite sequences. This enables us to compute the limit spectral measures for such sequences as was done for automata in [2].

Rational dynamics can be regarded as approximations of the corresponding evolutional systems in partial differential equations. From the view point of dynamical scaling limits, automata can be regarded as frame-dynamics which play the roles of underlying mechanics for PDE. Integrable systems focus on symmetry in study of PDE systems, and equip with many conservation quantities. So one may think that the frame structure of their dynamics should posses high symmetry. So our basic and general question is, whether the frame dynamics of integrable systems share their structural similarity with geometric properties of discrete groups.

In this paper we describe our discovery on the spectral coincidence between KdV in mathematical physics and automata groups in geometric group theory passing through tropical geometry. One of the author(T) found a cell diagram of the BBS, which was the starting point of our study.

Our main theorem is the following:

**Theorem** (Theorem 7.1). (i) The spectra of the Markov operators coincide with each other between the lamplighter group as an automata group and BBS with carrier capacity 1. It is totally discrete and dense in $[-1, 1]$.

Since the eigenvalue distributions coincide, both Markov operators are mutually conjugate by some orthogonal matrices. It would be natural to ask whether the conjugation might be chosen from tree automorphisms. Recall that both the lamplighter group and BBS act on the rooted binary tree $T$.

We have the negative answer:

**Proposition** (Proposition 8.5). There are no automorphisms of $T$ which conjugate between $M_L$ and $M_B$.

On the other hand, one might still ask whether it comes from permutations, or from an automorphism of the one sided shift. We have the affirmative answer, which gives the complete answer to the conjugations.
Let $M_B^{(n)}, M_L^{(n)} \in \text{Mat}(2^n \times 2^n; \mathbb{R})$. Let us denote the set of indices as $I_n = \{0, 1, \ldots, 2^n - 1\}$.

**Theorem** (Theorem 9.1). There exists a permutation matrix $\sigma_n$, such that

$$\sigma_n^* M_B^{(n)} \sigma_n = M_L^{(n)}$$

holds.

We have the explicit recurrence formulas for $\sigma_n$ which involves the Sierpinski gasket pattern.

If one reduces an integrable system to an automaton by extracting its dynamical framework, then it should possess high symmetry, which will have some structural similarity with finitely generated groups.

It would be interesting to investigate further coincidence between spectra of automata associated to integrable system and the one associated to automata groups.

## 2. Automata Groups

An *automaton* is defined by finite rules which can create quite complicated state dynamics over the sequences of alphabets.

Let $Q$ and $S$ be finite sets, and consider the set of all infinite sequences:

$$S^\mathbb{N} = \{(s_0, s_1, \ldots) : s_i \in S\}.$$

A *Mealy automaton* $A$ is given by a pair of functions:

$$\varphi : Q \times S \to Q, \quad \psi : Q \times S \to S$$

which gives rise to the state dynamics on $S^\mathbb{N}$ as follows. Let us choose any $q \in Q$ and $\bar{s} = (s_0, s_1, \ldots) \in S^\mathbb{N}$. Then:

$$A_q : S^\mathbb{N} \to S^\mathbb{N}$$

$A_q(\bar{s}) = (s'_0, s'_1, \ldots)$ is determined inductively by:

$$s'_i = \psi(q, s_i), \quad q_{i+1} = \varphi(q, s_i) \quad (q_0 = q).$$

Besides the dynamics over $S^\mathbb{N}$, the change of the state sets play important roles in a hidden dynamics.

Any sequences $\bar{q} = (q^0, \ldots, q^j) \in Q^{j+1}$ give dynamics by compositions:

$$A_{\bar{q}} = A_{q^j} \circ \cdots \circ A_{q^0} : S^\mathbb{N} \to S^\mathbb{N}.$$

It can happen that different automata give the same state dynamics. In such a case, the dynamics of $A_q$ are the same, but the systems of change of state sets can be very different. Such two automata are called *equivalent*.

Suppose

$$\psi : (q, \_ : S \to S$$

are permutations for all $q \in Q$. If we identify $S^\mathbb{N}$ with the rooted tree, then the Mealy dynamics give the group actions on the tree, since the actions can be restricted level-setwisely. The group generated by these states is called the *automata group* given by the automaton $(\varphi, \psi)$.

### 2.1. Lamplighter group.

The lamplighter group:

$$(\oplus \mathbb{Z}_2) \rtimes \mathbb{Z}$$

is generated by canonical generators, which are $u$, one copy of $\mathbb{Z}_2$, and $v$, the generator of $\mathbb{Z}$.

The corresponding automaton can be represented by the following picture:

```
0|1
\node (a0) at (0,0) {a0};
\node (a1) at (1,0) {a1};
\draw (a0) -- (a1);
\draw (a0) -- (0,1);
\draw (a1) -- (1,1);
```

In this case, we have two operator recursions

$$a_0 = \begin{pmatrix} 0 & a_1 \\ a_0 & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix},$$

where $a_0$ corresponds to $u^{-1}v$ and $a_1$ to $v ([GZ])$. 

3. Discrete KDV equation

The KdV (Korteweg-de Vries) equation

\[
\frac{\partial u}{\partial x_3}(x_1, x_3) + 6u \frac{\partial u}{\partial x_1}(x_1, x_3) + \frac{\partial^3 u}{\partial x_1^3}(x_1, x_3) = 0
\]

has particular solutions

\[
u = 2\frac{\partial^2}{\partial x_1^2} \left( \log \det(\delta_{x_1}^{j=1} \phi_k)_{1 \leq j, k \leq N} \right), \quad \phi_k = \alpha_k e^{\lambda_k x_1 - 4\lambda_k^2 x_3} + \beta_k e^{-\lambda_k x_1 + 4\lambda_k^2 x_3},
\]

where \(\alpha_k, \beta_k, \lambda_k (k = 1, 2, \ldots, N)\). We call these particular solutions “soliton solutions” which exhibit interactions among multiple solitons (N-soliton).

These type of solutions induce the Hirota’s \(\tau\)-function as

\[
u = 2\frac{\partial^2}{\partial x_1^2} \log \tau
\]

which enable us to derive the famous bilinear equation of the KdV equation,

\[
(D_{x_3}D_{x_1} + D_{x_1}^4)\tau \cdot \tau = 0,
\]

which can be rewritten as

\[
((\partial_{x_3} - \partial_{x_1}^2) \partial_{x_1} - \partial_{x_1}^4) + (\partial_{x_1} - \partial_{x_1}^2)^4 \tau(x_1, x_3) \tau(x_1, x_3)^3 \tau(x_1, x_3)^3) = 0.
\]

A discrete analogue of (3) is given by

\[
(1 + \delta)\tau_{n+1}^{(t+1)} \tau_n^{(t-1)} = \tau_n^{(t)} \tau_{n+1}^{(t)} + \delta \tau_n^{(t+1)} \tau_{n+1}^{(t-1)}
\]

which tends to (3) in the continuous limit \(\varepsilon \to 0\) under the following parametrizations:

\[
\delta = -\varepsilon^3/(1 + \varepsilon^3), \quad x_3 = \varepsilon^5 n/3, \quad x_1 = -\varepsilon(n - t) - 2\varepsilon^4 n.
\]

It is important to note that (4) is also integrable, since it can be derived from the discrete KP (Kadomtsev-Petviashvili) equation that is one of the master equation of the integrable system. By taking

\[
u^{(t)}_n = \frac{\tau_n^{(t)} \tau_{n+1}^{(t-1)}}{\tau_n^{(t+1)} \tau_{n+1}^{(t)}},
\]

as a discrete analogue of the dependent variable \(\nu\) of the KdV equation, (4) can be transformed to

\[
\frac{1}{\nu_n^{(t+1)} - 1} - 1 \nu_n^{(t)} = \frac{\delta}{1 + \delta} \left( \nu_n^{(t)} - \nu_n^{(t+1)} \right),
\]

which we call the discrete KdV equation.

Next we consider the modified KdV (mKdV) equation

\[
\frac{\partial \tilde{v}}{\partial x_3}(x_1, x_3) + 6a\varepsilon^2 \frac{\partial \tilde{v}}{\partial x_1}(x_1, x_3) + \frac{1}{4a} \frac{\partial^3 \tilde{v}}{\partial x_1^3}(x_1, x_3) = 0,
\]

where \(a\) is a constant, which is derived from the Bäcklund transformation of the KdV equation. Let us start from the two bilinear equations:

\[
(1 + \delta + \gamma) f_{n+1}^{(t+1)} g_n^{(t)} = (1 + \gamma) f_{n+1}^{(t)} g_n^{(t+1)} + \delta f_n^{(t+1)} g_{n+1}^{(t),}
\]

\[
(1 + \delta + \gamma) f_n^{(t)} g_{n+1}^{(t+1)} = (1 + \delta) f_{n+1}^{(t)} g_n^{(t+1)} + \gamma f_{n+1}^{(t+1)} g_{n+1}^{(t)},
\]

which is the system of the 2-reduced non-autonomous discrete KP equation. By taking the dependent variable

\[
u_n^{(t)} = \frac{f_{n+1}^{(t+1)} g_n^{(t+1)}}{f_{n+1}^{(t)} g_n^{(t)}},
\]
one can obtain
\[
(1 + \delta + \gamma) \frac{f_{n+1}^{(t+1)} g_{n}^{(t)}}{f_{n}^{(t+1)} g_{n+1}^{(t)}} = (1 + \gamma) w_{n}^{(t)} + \delta, \\
(1 + \delta + \gamma) \frac{f_{n}^{(t+1)} g_{n+1}^{(t)}}{f_{n+1}^{(t+1)} g_{n}^{(t)}} = (1 + \delta) w_{n}^{(t)} + \gamma.
\]

Then we have the discrete mKdV equation
\[
w_{n+1}^{(t+1)} (1 + \gamma) w_{n}^{(t+1)} + \delta = w_{n}^{(t)} (1 + \gamma) w_{n+1}^{(t)} + \delta, \\
(1 + \delta) w_{n}^{(t+1)} + \gamma = (1 + \delta) w_{n+1}^{(t)} + \gamma,
\]
which reduces to the mKdV equation under some suitable continuous limit.

4. Tropical geometry

4.1. Tropical transform. Tropical geometry covers wide class of relative (max, +)-functions, but we will rather restrict those functions we treat in this article, which is enough to apply Mealy automata.

A relative (max, +)-function \( \varphi \) is a piecewise linear function equipped with its presentation of the form:
\[
\varphi(x) = \max(\alpha_1 + \bar{a}_1 x, \alpha_2 + \bar{a}_2 x) - \max(\beta_1 + \bar{b}_1 x, \beta_2 + \bar{b}_2 x).
\]

Tropical geometry associates the parametrized rational function by a kind of scaling transform (see [Mi]), which we call a relative elementary function:
\[
f_t(z) = \frac{t^{a_1} z^{a_1} + t^{a_2} z^{a_2}}{t^{b_1} z^{b_1} + t^{b_2} z^{b_2}}
\]
which have one to one correspondence of their presentations to \( \varphi \). They take positive real numbers if the inputs are also positive.

A crucial property is that \( f_t \) converge to \( \varphi \) by letting \( t \to \infty \). Among various scaling parameters, tropical geometry behaves quite nicely, which allow us to obtain several uniform estimates by comparisons of both dynamical behaviours at the same time.

The tropical transform is also known as “ultradiscrete limit”,
\[
\lim_{\varepsilon \to +0} \varepsilon \log[1 + \exp(Z/\varepsilon)] = \max(0, Z).
\]

It is quite characteristic of (max, +)-functions that different presentations can give the same functions. For example:
\[
\varphi(y, x) = \max(x, -x) - y, \quad \psi(x, y) = \max(\varphi(x, y), -y)
\]
are the same functions but have different presentations. The corresponding rational functions are mutually \( f_t(w, z) = w^{-1}(z + z^{-1}) \) and \( h_t(w, z) = w^{-1}(z + z^{-1} + 1) \), which are different even as functions. This motivates us to introduce a notion of tropical equivalence between such \( f_t \) and \( h_t \).

4.2. Automata and Tropical geometry. Let us take finite sets \( S, Q \subset \mathbb{Z} \subset \mathbb{R} \), and consider a Mealy automaton \( A \) given by:
\[
\varphi : Q \times S \to Q, \quad \psi : Q \times S \to S.
\]
One can extend these functions over the real number by piecewise linear functions which admit their presentations by the relative (max, +) functions:
\[
\varphi(\bar{r}) = \max(\alpha_1 + \bar{a}_1 \bar{r}, \alpha_2 + \bar{a}_2 \bar{r}) - \max(\beta_1 + \bar{b}_1 \bar{r}, \beta_2 + \bar{b}_2 \bar{r}), \\
\psi(\bar{l}) = \max(\gamma_1 + \bar{c}_1 \bar{l}, \gamma_2 + \bar{c}_2 \bar{l}) - \max(\delta_1 + \bar{d}_1 \bar{l}, \delta_2 + \bar{d}_2 \bar{l})
\]
We say that the extensions are stable over \((Q, S)\), if there are some \( 0 < \delta < 1 \) and \( 0 \leq \mu < 1 \) so that the Lipschitz constants of the pair \((\varphi, \psi)\) is bounded by \( \mu \) on \( \delta \) neighbourhoods of \( Q, S \subset \mathbb{R} \) respectively. We have verified that stable extensions always exist for Mealy automata ([K3]).
The Mealy dynamics with respect to the pair \((\varphi, \psi)\) is given by the discrete dynamics inductively defined by the iterations:

\[
x_{i}^{j+1} = \varphi(y_{i}^{j}, x_{i}^{j}), \quad y_{i+1}^{j} = \psi(y_{i}^{j}, x_{i}^{j})
\]

where \(x_{i}^{0} = x_{i}\) and \(y_{0}^{j} = y_{j}\) are the initial values for \(i, j \geq 0\).

Let us visualize this dynamics as follows. \(y = y^{0}\) determines a map:

\[
A_{y} : \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}, \quad A_{y}(x_{0}, x_{1}, \ldots) = (x_{0}^{1}, x_{1}^{1}, x_{2}^{1}, \ldots).
\]

So by composition, finite sequences \((y^{0}, y^{1}, \ldots, y^{l})\) give the maps:

\[
A_{y^{0}, y^{1}, \ldots, y^{l}} = A_{y^{l}} \circ \cdots \circ A_{y^{0}} : \mathbb{R}^{N} \rightarrow \mathbb{R}^{N},
\]

\[
A_{y^{0}, y^{1}, \ldots, y^{l}}(x_{0}, x_{1}, \ldots) = (x_{0}^{j+1}, x_{1}^{j+1}, x_{2}^{j+1}, \ldots)
\]

The Mealy automaton creates the automaton semi group, and the above dynamics exactly extend the semi group actions on the trees.

Let us consider the corresponding parametrized rational functions \(f_{l} \) and \(g_{l} \) with respect to \(\psi\) and \(\varphi\) respectively.

The rational dynamics of Mealy type is given by the corresponding rational dynamics:

\[
z_{i}^{j+1} = f_{l}(w_{i}^{j}, z_{i}^{j}), \quad w_{i+1}^{j} = g_{l}(w_{i}^{j}, z_{i}^{j})
\]

where the initial values are given by \(z_{0}^{i} = z_{i}, w_{0}^{j} = w^{j} > 0\).

Now let \((\varphi^{2}, \psi^{2})\) be the initial values of soliton solutions: \(\varphi^{2}, \psi^{2}\) provide the defining equation:

\[
\text{relative elementary functions. Assume that both pairs are tropically equivalent:}
\]

\[
(g_{1}^{l}, f_{1}^{l}) \sim (g_{2}^{l}, f_{2}^{l}).
\]

For \(l = 1, 2\), let \(\{w^{j}(l)\}_{j \geq 0}\) and \(\{z_{i}(l)\}_{i \geq 0}\) be the initial sequences of positive numbers, and denote the solutions by \((z_{i}^{j}(l), w_{i}^{j}(l))\) to the state systems of the rational dynamics:

\[
z_{i}^{j+1}(l) = f_{l}(w_{i}^{j}(l), z_{i}^{j}(l)), \quad w_{i+1}^{j}(l) = g_{l}(w_{i}^{j}(l), z_{i}^{j}(l))
\]

with the initial values \(z_{0}^{i}(l) = z_{i}(l)\) and \(w_{0}^{j}(l) = w^{j}(l)\).

If two automata are equivalent, then the corresponding rational dynamics also show mutual structural similarity on the large scale. In fact we have verified that two rational dynamics of the stable extensions corresponding to two equivalent Mealy automata admit uniform estimates between their orbits \(\{z_{i}^{j}(l)\}_{i, j}\) and \(\{w_{i}^{j}(l)\}_{i, j}\). Quasi-recursive rational dynamics in the introduction is obtained by applying it to the Mealy automaton by Aleshin-Grigorchuk of Burnside group.

4.3. From discrete KdV to BBS. Let us recall the discrete KdV equation \[5\):

\[
\frac{1}{u_{n+1}^{(t+1)}} - \frac{1}{u_{n}^{(t)}} = \frac{\delta}{1 + \delta} \left( u_{n+1}^{(t)} - u_{n}^{(t+1)} \right).
\]

This presentation can not be applied the tropical geometry, since it contains minus. So let us rewrite it by the subtraction free form:

\[
1/u_{n}^{(t)} + \delta = (1 + \delta)/(u_{n}^{(t)}u_{n-1}^{(t-1)}),
\]

\[
v_{n+1}^{(t)}u_{n+1}^{(t+1)} = v_{n}^{(t)}u_{n}^{(t)},
\]

which provide the defining equation:

\[
v_{n}^{(t)} = \frac{u_{n}^{(t)}}{u_{n-1}^{(t+1)}}v_{n-1}^{(t)} = \cdots = \prod_{j=-\infty}^{n-1} \frac{u_{j+1}^{(t)}}{u_{j}^{(t+1)}},
\]

under the boundary conditions of soliton solutions: \(|u_{n}^{(t)}| \rightarrow 1 (n \rightarrow \infty)\).

Suppose that \(u_{n}^{(t)}, \delta\) take positive values, then we introduce following dependent variable transformations:

\[
u_{n}^{(t)} = \exp(B_{n}^{(t)}/\varepsilon), \quad \delta = \exp(-1/\varepsilon).
\]
By making use of the ultradiscrete limit (7), the subtraction free form of the discrete KdV equation (5) is reduced to the piecewise-linear system,

(8) \[ B_{n}^{(t+1)} = \min(1 - B_{n}^{(t)}, \sum_{j=-\infty}^{n-1} (B_{j}^{(t)} - B_{j}^{(t+1)})) \]

When we restrict the values of \( B_{j}^{(t)} \) to be \{0,1\}, then the evolution rule is essentially identical to the BBS. The BBS is one of the ultradiscrete integrable systems. The BBS is composed of an array of infinitely many boxes, finite number of balls in the boxes, and a carrier of balls. Each box can contain only one ball and the carrier can hold arbitrary number of balls. The evolution rule from time \( j \) to time \( j+1 \) is defined as follows. The carrier moves from left to right and passes each box. When the carrier passes a box containing a ball, the carrier gets the ball; when the carrier passes an empty box, if the carrier holds balls, the carrier puts one ball into the box.

Let us describe BBS with carrier capacity \( k \). In this case the carrier can hold at most \( k \) balls. The only difference with the previous situation is that when the carrier holds \( k \) balls and passes a box containing a ball, the carrier does nothing.

Similar to the case of KdV, the BBS with carrier capacity \( k \) can be obtained from the discrete mKdV equation [13]. Let us introduce the dependent variable \( y_{n}^{(t)} \) and the auxiliary variable \( z_{n}^{(t)} \), defined by

\[ y_{n}^{(t)} = \frac{\sum_{j=-\infty}^{j=n} (\tilde{B}_{j}^{(t)} - \tilde{B}_{j}^{(t+1)})}{\tilde{B}_{n}^{(t)} - \tilde{B}_{n+1}^{(t)}}, \quad z_{n}^{(t)} = \frac{\sum_{j=-\infty}^{j=n} (\tilde{B}_{j}^{(t)} - \tilde{B}_{j}^{(t+1)})}{\tilde{B}_{n+1}^{(t)} - \tilde{B}_{n+2}^{(t)}}, \]

Direct calculations show that

\[ w_{n}^{(t)} = y_{n+1}^{(t+1)} z_{n+1}^{(t+1)} = y_{n+1}^{(t)} z_{n+1}^{(t)} \]

The ultradiscrete mKdV equation is presented by

\[ \tilde{B}_{n}^{(t+1)} = \min \left( 1 - \tilde{B}_{n}^{(t)}, \sum_{j=-\infty}^{j=n-1} (\tilde{B}_{j}^{(t)} - \tilde{B}_{j+1}^{(t+1)}) \right) + \max \left( 0, \sum_{j=-\infty}^{j=n} (\tilde{B}_{j}^{(t)} - \tilde{B}_{j+1}^{(t+1)}) - k \right), \]

which can be derived from the discrete mKdV equation (9) by introducing the dependent variables:

\[ y_{n}^{(t)} = e^{-\tilde{B}_{n}^{(t)}/\varepsilon}, \quad \delta = e^{-1/\varepsilon}, \quad \gamma = e^{-k/\varepsilon} \]

and taking the limit \( \varepsilon \to +0 \).

5. BBS with carrier capacity

**Lemma 5.1.** The BBS with carrier capacity \( k \) can be depicted by

The (simple) BBS is obtained as the limiting case of the above automaton with \( k \to \infty \).
Proof. The state \( a_i \) corresponds to the situation when the carrier holds \( i \) balls. Thus we start at the state \( a_0 \). If we have 1 as the input we go from the state \( a_i \) to \( a_{i+1} \) if \( i < k \) and we change 1 to 0. This corresponds to the fact the carrier picks up the ball if the number of balls it already holds is \( i < k \). If we have 0 as the input we go from the state \( a_i \) to \( a_{i-1} \) if \( i > 0 \) and change 1 to 0. This corresponds to the fact the carrier puts the ball if the number of balls it already holds is at least 1. It remains to check the situation for \( a_0 \) with the input 0 and for \( a_k \) with the input 1. The first one corresponds to the carrier with 0 balls passing an empty box (it does nothing and still holds no balls) and the last one to the carrier with \( k \) balls passing a box with a ball (it does nothing and still holds \( k \) balls).

5.1. **BBS with carrier capacity** \( k = 1 \).

We have two operator recursions

\[
\begin{align*}
a_0 &= a_{0;k=1} = \begin{pmatrix} a_0 & a_1 \\ 0 & 0 \end{pmatrix}, \\
a_1 &= a_{1;k=1} = \begin{pmatrix} 0 & 0 \\ a_0 & a_1 \end{pmatrix}.
\end{align*}
\]

We can describe the action of \( a_0 \) and \( a_1 \) on the binary sequences of length \( n \) by the matrices \( a^{(n)}_0 \) and \( a^{(n)}_1 \). From the definition of our automaton, they satisfy the following recurrence relations:

\[
\begin{align*}
a^{(n+1)}_0 &= \begin{pmatrix} a^{(n)}_0 & a^{(n)}_1 \\ 0 & 0 \end{pmatrix}, \\
a^{(n+1)}_1 &= \begin{pmatrix} 0 & 0 \\ a^{(n)}_0 & a^{(n)}_1 \end{pmatrix}.
\end{align*}
\]

We can consider the following transition operator:

\[
M^{(n)}_{k=1} = \frac{1}{4} \left( a^{(n)}_0 + a^{(n)*}_0 + a^{(n)}_1 + a^{(n)*}_1 \right)
\]

In the next section, we verify that these transition operators are stochastic.

5.2. **BBS with carrier capacity** \( k = 2 \). In analogy to \( k = 1 \) case, for \( k = 2 \), we can consider the following operators.

We have three operator recursions

\[
\begin{align*}
a_0 &= a_{0;k=2} = \begin{pmatrix} a_0 & a_1 \\ 0 & 0 \end{pmatrix}, \\
a_1 &= a_{1;k=2} = \begin{pmatrix} 0 & a_2 \\ a_0 & 0 \end{pmatrix}, \\
a_2 &= a_{2;k=2} = \begin{pmatrix} 0 & 0 \\ a_1 & a_2 \end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
a^{(n+1)}_0 &= \begin{pmatrix} a^{(n)}_0 & a^{(n)}_1 \\ 0 & 0 \end{pmatrix}, \\
a^{(n+1)}_1 &= \begin{pmatrix} 0 & a^{(n)}_2 \\ a^{(n)}_0 & 0 \end{pmatrix}, \\
a^{(n+1)}_2 &= \begin{pmatrix} 0 & 0 \\ a^{(n)}_1 & a^{(n)}_2 \end{pmatrix}.
\end{align*}
\]

\[
M^{(n)}_{k=2} = \frac{1}{6} \left( a^{(n)}_0 + a^{(n)*}_0 + a^{(n)}_1 + a^{(n)*}_1 + a^{(n)}_2 + a^{(n)*}_2 \right)
\]

In the next sections, we will analyze in detail the spectral properties of the transition operator for \( k = 1 \).
6. Stochastic matrices

We define a sequence of $k + 1$ matrices $(a_0^{(n)}, \ldots, a_k^{(n)})$ of dimension $2^n$, for $n = 0, 1, \ldots$ by the following matrix recursion ($0$ represents here $2^n \times 2^n$ null matrix).

$$a_0^{(n+1)} = \begin{pmatrix} a_0^{(n)} & a_1^{(n)} \\ 0 & 0 \end{pmatrix}$$

For $i = 1, \ldots, k - 1$

$$a_i^{(n+1)} = \begin{pmatrix} 0 & a_i^{(n)} \\ a_{i-1} & 0 \end{pmatrix}$$

and

$$a_k^{(n+1)} = \begin{pmatrix} 0 & 0 \\ a_{k-1} & a_k \end{pmatrix}$$

with the initial data $a_i^{(0)} = 1$ for all $i = 0, \ldots, k$.

We consider the following $2^n \times 2^n$ matrix $M_k^{(n)}$

$$M_k^{(n)} = \frac{1}{2k + 2}(a_0^{(n)} + (a_0^{(n)})^* + \ldots + a_k^{(n)} + (a_k^{(n)})^*).$$

**Proposition 6.1.** The matrix $M_k^{(n)}$ is double stochastic for all $k \geq 1$, $n \geq 0$, i.e. the sum of each row and each column is equal to 1.

**Proof.** The matrix $M_k^{(n)}$ is symmetric and therefore it suffices to prove that the sum of columns is constant.

Clearly the recursive relations for $a_0^{(n+1)}, \ldots, a_k^{(n+1)}$ show that the matrix we obtain from each of them is the matrix with constant sum of columns (equal to $k$).

Thus it is enough to show that $(a_0^{(n)})^* + \ldots + (a_k^{(n)})^*$ has constant column sum. Let us prove this by induction. It is clear for $n = 0$. Then using recursion formula

$$(a_0^{(n+1)})^* + \ldots + (a_k^{(n+1)})^* = \begin{pmatrix} (a_0^{(n)})^* & (a_0^{(n)})^* + \ldots + (a_k^{(n)})^* \\ (a_1^{(n)})^* + \ldots + (a_{k-1}^{(n)})^* & (a_k^{(n)})^* \end{pmatrix}$$

and thus the sum of the left matrix blocks and right matrix blocks is equal to

$$(a_0^{(n)})^* + \ldots + (a_k^{(n)})^*.$$

Therefore the statement follows by induction. \hfill \square

7. Spectral computation

We consider the case $k = 1$ and provide the computation of eigenvalues of $M_1^{(n)}$. Let:

$$M_1^{(n)} = \text{Sp} \left( \frac{1}{2(k + 1)} \sum_{j=0}^{k} (a_{j,k}^{(n)} + a_{j;k}^{(n)}) \right)$$

for $n \in \mathbb{N}$. Define the counting spectral measures of $M_1^{(n)}$, i.e. $\sigma^{(n)}_k : [0, 1] \rightarrow [0, 1]$ and for $x \in [0, 1]$ by:

$$\sigma^{(n)}_k(x) = \frac{\sharp \left\{ \lambda \in \text{Sp}(M_1^{(n)}) \mid \lambda \leq 2(k + 1) \cos(\pi x) \right\}}{\sharp \left\{ \lambda \in \text{Sp}(M_1^{(n)}) \right\}}.$$

Denote the multiplicity of eigenvalue $\lambda$ in $\text{Sp}(M_1^{(n)})$ by $m^{(n;k)}(\lambda) = \sharp \left\{ \lambda \in \text{Sp}(M_1^{(n)}) \right\}$.

Our computation on the spectra verify the following:
Theorem 7.1.

$$Sp \left( M^{(n)}_{k=1} \right) = Sp \left( \frac{1}{4} \sum_{j=0}^{1} \left( a_{j,k=1}^{(n)} + a_{j,k=1}^{(n)^*} \right) \right) = \left\{ 1 \cup \cos \left( \frac{p\pi}{q} \right) \mid p, q \in \mathbb{N}, 1 \leq p < q \leq n + 1 \right\}$$

If $p$ and $q$ are mutually prime, then the multiplicity of eigenvalue $\cos \left( \frac{pq^{-1}\pi}{q} \right)$, denoted by $m^{(n)}_{p,q}$, is given by

$$m^{(n)}_{p,q} = \begin{cases} 2^n \left( \frac{2^n - 2^{-q \left( \frac{n}{2} \right) + 1}}{1 - 2^{-q}} \right) & \end{cases}$$

$$\lim_{n \to \infty} Sp \left( M^{(n)}_{k=1} \right) = \lim_{n \to \infty} \left\{ \lambda \in Sp \left( M^{(n)}_{k=1} \right) \mid m^{(1;j)}(\lambda) \leq \cdots \leq m^{(n-1;j)}(\lambda) \leq m^{(n;j)}(\lambda) \right\} \text{ for } j \in \mathbb{N}.$$  

The figures 2 and 3 present the histogram of the spectra for $k = 1, n = 7$, and $k = 2, n = 14$ respectively.

Lemma 7.2. For every $n$

$$a_n a_n^* + b_n b_n^* = 2\text{Id}_{2^n}.$$  

Proof. We have

$$a_{n+1}a_{n+1}^* = \begin{pmatrix} a_n & b_n \end{pmatrix} \begin{pmatrix} a_n^* & 0 \\ b_n^* & 0 \end{pmatrix} = \begin{pmatrix} a_n a_n^* + b_n b_n^* & 0 \\ 0 & 0 \end{pmatrix}$$

and the statement follows by induction. □

Proof of theorem 7.1: In order to simplify the notation we define $a_n = a_0^{(n)}$ and $b_n = a_1^{(n)}$. Let us put:

$$\Phi_n(\lambda, \mu) = \det(a_n + a_n^* + b_n + b_n^* - \frac{1}{2} \mu(a_n b_n^* + b_n a_n^*) - \lambda \text{Id}_{2^n})$$

Then by use of lemma 7.2, we have the equalities:

$$\Phi_{n+1}(\lambda, \mu) = \det(a_{n+1} + a_{n+1}^* + b_{n+1} + b_{n+1}^* - \frac{1}{2} \mu(a_{n+1} b_{n+1}^* + b_{n+1} a_{n+1}^*) - \lambda \text{Id}_{2^{n+1}})$$

$$= \det \left( \begin{array}{cc} a_n + a_n^* - \lambda & b_n + a_n^* - \frac{1}{2} \mu(a_n a_n^* + b_n b_n^*) \\ a_n + b_n^* - \lambda & b_n + b_n^* - \lambda \end{array} \right)$$

$$= \det \left( \begin{array}{cc} a_n + a_n^* - \lambda & b_n + a_n^* - \mu \\ a_n + b_n^* - \mu & b_n + b_n^* - \lambda \end{array} \right)$$

$$= \det \left( \begin{array}{cc} 2\mu - 2\lambda & a_n^* - b_n - \mu + \lambda \\ a_n - b_n + \lambda - \mu & b_n + b_n^* - \lambda \end{array} \right)$$

Using the fact that

$$\det \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \det(AD - CB)$$

provided that $A$ commutes with $C$ we get

$$\Phi_{n+1}(\lambda, \mu) = \det((2\mu - 2\lambda)(b_n + b_n^* - \lambda) - (a_n - b_n + \lambda - \mu)(b_n + b_n^* - \lambda))$$

$$= \det((\mu - \lambda)(a_n + a_n^* + b_n + b_n^*) - \frac{1}{2} \mu(a_n b_n^* + b_n a_n^*) + (\lambda^2 - \mu^2)\text{Id}_{2^n})$$

Therefore

$$\Phi_{n+1}(\lambda, \mu) = (\mu - \lambda)^{2^n} \Phi_n \left( \frac{2 - \lambda^2 + \mu^2}{\mu - \lambda}, \frac{-2}{\mu - \lambda} \right).$$

This is exactly the formula from [2] which leads to the explicit computation of all eigenvalues.
8. Ergodicity of the Markov operators

8.1. Ergodicity on the boundary of the binary tree. Let \( \{ M^{(n)}_{k=1} \}_{n=1,2,...} \) be the family of Markov operators for lamplighter or BBS$_{k=1}$ automata. We have verified that those are stochastic \( 2^n \times 2^n \) matrices equipped with the canonical maps:

\[ \cdots \rightarrow M^{(n+1)}_{k=1} \rightarrow M^{(n)}_{k=1} \rightarrow \cdots \]

**Definition 8.1.** Let \( M \) be a stochastic \( k \times k \) matrix. \( M \) is ergodic, if there is \( s_0 \geq 1 \) and \( \alpha > 0 \) so that inequalities:

\[ m_{i,j}^{(s_0)} \geq \alpha \]

hold for all \( i,j \), where \( M^s = (m_{i,j}^{(s)})_{1 \leq i,j \leq k} \).

For stochastic matrix, if the above property is satisfied for some \( s_0 \), then the same property holds for all \( s \geq s_0 \).

Recall the fundamental result on ergodicity:

**Theorem 8.1.** Let \( M \) be a stochastic \( k \times k \) matrix, and consider the associated Markov chain on the space \( X = \{1, \ldots, k\} \). If \( M \) is ergodic, then there is a unique probability distribution \( \pi \) on \( X \) which satisfies two properties (1) \( \pi M = \pi \), and (2) \( \lim_{s \rightarrow \infty} m_{i,j}^{(s)} = \pi_j \).

The unique probability distribution \( \pi = (\pi_1, \ldots, \pi_k) \) is called the stationary distribution with respect to \( M \).

**Lemma 8.2.** \( M \) is ergodic, if and only if the spectrum of \( M \) satisfies

(1) the multiplicity of the eigenvalue 1 is just 1, and
(2) it does not contain −1.

**Proof.** Suppose \( M \) is ergodic. Let \( v_1 \) and \( v_2 \) be two orthogonal eigenvectors with eigenvalue 1. Then \( v_1 M = v_1 \) hold, and so:

\[ \langle v_1 M^2, v_2 \rangle = \langle v_1 M^s, v_2 M^s \rangle = \langle v_1, v_2 \rangle = 0 \]

must hold. Let \( a_i \) be the sum of coordinates of \( v_i \). Then \( a_i \) can not be zero, since \( v_i = \lim_{s \rightarrow \infty} v_i M^s = a_i \pi \) hold by Theorem [8.1]. By letting \( s \rightarrow \infty \) in the above equalities, it follows \( \pi = 0 \) is zero vector, which is a contradiction, since \( \pi_i \geq \alpha > 0 \). So the multiplicity of the eigenvalue 1 must be less than or equal to 1. It is at least 1 because constant vectors have eigenvalue 1.

As we noticed that the limit exists:

\[ wM^s \equiv (x_1, \ldots, x_n)M^s \rightarrow (a \pi_1, \ldots, a \pi_n) \]

by Theorem [8.1] where \( a = \sum_{i=1}^n x_i \). But if \( w \) is an eigenvector with eigenvalue −1, then \( wM^s \in \{ w, -w \} \) oscillates, which is a contradiction.
Suppose the above two properties hold. Let \( \{v_1, \ldots, v_k\} \) be the orthogonal eigenvectors such that \( v_1 \) corresponds to the eigenvalue 1. Then for any \( \mathbf{v} = \sum_{i=1}^{k} a_i v_i \),
\[
\lim_{s \to \infty} v M^s = a_1 v_1 + \lim_{s \to \infty} \sum_{i=2}^{k} \lambda_i^s a_i v_i = a_1 v_1
\]
hold, since \(-1 < \lambda_i < 1\) hold for \( i \geq 2 \).

Suppose \( M \) is not ergodic, i.e. for every \( s \), there exist \( i,j \) such that \( m_{i,j}^{(s)} = 0 \) hold. Let \( \delta_i = (0, \ldots, 0, 1, 0, \ldots) \). Then \( \langle \delta_i M^s, \delta_j \rangle = m_{i,j}^{(s)} = 0 \) holds. It follows that there exist \( i,j \) such that \( \langle \delta_i M^l, \delta_j \rangle = 0 \) hold for infinitely many \( l \). So it also holds for \( l \to \infty \). It follows that \( \delta_i \) or \( \delta_j \) is orthogonal to \( v_1 \). Since \( M \) is stochastic, we can put \( v_1 = (1, \ldots, 1) \), and so this is a contradiction. This completes the proof. \( \square \)

**Remark 8.3.** For the stochastic matrix, the property (1) is equivalent to connectivity, and property (2) is to non bi-partiteness of the associated graph.

**Corollary 8.4.** Let \( M_L^{(n)} \) and \( M_B^{(n)} \) be the Markov operators for the lamplighter and BBS\(_{k=1}\) automata, respectively. Then they are all ergodic.

**Proof.** The result follows from our computation of their spectra in theorem 7.1 with Lemma 8.2. \( \square \)

### 8.2. On automorphisms of the tree

Let \( T \) be the binary tree, and \( T_k \) be \( k \)-th level set. Then the Markov operators satisfy:
\[
M^{(k+1)}|_{T_k} = M^{(k)}.
\]

Let us consider the canonical maps:
\[
\cdots \to M^{(n+1)} \to M^{(n)} \to \cdots
\]
and take the projective limit:
\[
M \equiv \lim_{n \to \infty} M^{(n)}.
\]

\( M \) gives an ergodic Markov chain on \( \partial T \), if \( M^{(n)} \) are ergodic.

**Proposition 8.5.** There are no automorphisms of \( T \) which conjugate between \( M_L^{(k)} \) and \( M_B^{(k)} \).

**Proof.** If there were an automorphism of the tree which would conjugate two operators on some level \( n \) it would also conjugate these operators on the previous levels. Thus it is enough to prove the statement for the level \( n = 2 \). For this level the operator corresponding to the BBS system has \((2, 0, 0, 2)\) on the diagonal and the operator corresponding to the lamplighter has \((0, 0, 2, 2)\) on the diagonal. The last one under the tree automorphism can be transformed to itself or \((2, 2, 0, 0)\) only. \( \square \)

### 9. Conjugacy by permutation

Let \( M_B^{(n)}, M_L^{(n)} \in \text{Mat}(2^n \times 2^n, \mathbb{Z}) \). Let us denote the set of indices as \( I_n = \{0, 1, \ldots, 2^n - 1\} \). We denote the concatenation of two vectors \( u \in \mathbb{C}^n \) and \( v \in \mathbb{C}^m \) by \((u, v) \in \mathbb{C}^{n+m} \). For \( c \in I_n \), we denote \([c] \in \mathbb{Z}_2\), and consider the binary expansion:
\[
c = \sum_{j=1}^{n} [c_j] 2^{n-j} \in I_n
\]
which we denote as:
\[(c_1, c_2 \cdots, c_n)_2.\]

In this section, we verify the following:
Theorem 9.1. There exists a family of the transformation matrices \( \sigma_n \) such that
\[
\sigma^n_M^{(n)} \sigma_n = M^{(n)}_L
\]
hold, where \( \sigma_n \) is determined by the permutation vector \( \hat{\sigma}_n = \left( \mu_0^{(n)}, \mu_1^{(n)}, \ldots, \mu_{2^n-1}^{(n)} \right) \) by
\[
\sigma_n e_j = \mu_j^{(n)}
\]
for any \( j \in I_n \), where \( e_j = \{0, \ldots, 0, 1, 0, \ldots, 0\} \).

The permutation vectors \( \hat{\sigma}_n = (\mu_0^{(n)}, \mu_1^{(n)}, \ldots, \mu_{2^n-1}^{(n)}) \) are uniquely determined by
\[
\begin{align*}
&\text{• } \hat{\sigma}_1 = (\mu_0^{(1)}, \mu_1^{(1)}) = (0, 1), \\
&\text{• } \text{there exists a binary sequence } \nu^{(n)} = (\nu_0^{(n)}, \nu_1^{(n)}, \ldots, \nu_{2^n-1}^{(n)}) \in \{0, 1\}^{2^n-1} \text{ such that } \\
&\hat{\sigma}_n = (\hat{\sigma}_{n-1}, \hat{\sigma}_{n-1}) + 2^{n-1}(1 - \nu_0^{(n)}, \nu_0^{(n)}, \ldots, 1 - \nu_{2^n-1}^{(n)}, \nu_{2^n-1}^{(n)}), \\
&\text{• the binary sequences } \nu^{(n)} \in \{0, 1\}^{2^n-1} \text{ are determined by use of the binary pattern } g \text{ of the Sierpinski gasket as follows:}
\end{align*}
\]
for \( n \geq 2 \). The operator \( T_\alpha \) is defined by
\[
T_\alpha(s_1, s_2, \ldots, s_m) = \begin{cases} 
(s_1, s_2, \ldots, s_m, s_1, s_2, \ldots, s_m) & \text{if } \alpha = 0 \\
(s_1, s_2, \ldots, s_m, 1 - s_1, 1 - s_2, \ldots, 1 - s_m) & \text{if } \alpha = 1
\end{cases}
\]
Here the binary pattern \( g \) of the Sierpinski gasket is given by \( g_1^{(n)} = g_0^{(n)} = 1 \) and
\[
g_m^{(n)} = g_{m-1}^{(n-1)} + g_m^{(n-1)} \mod 2
\]
for \( m = 2, 3, \ldots, n - 1 \) and \( n = 1, 2, \ldots \).

Remark 9.2. (1) Let us see the orbit of \( g \):
\[
\begin{align*}
g^{(1)} &= (g_1^{(1)}) = (1), \\
g^{(2)} &= (g_2^{(2)}, g_2^{(2)}) = (1, 1), \\
g^{(3)} &= (g_3^{(3)}, g_2^{(3)}, g_3^{(3)}) = (1, 0, 1), \\
g^{(4)} &= (g_4^{(4)}, g_2^{(4)}, g_3^{(4)}, g_4^{(4)}) = (1, 1, 1, 1), \\
g^{(5)} &= (g_5^{(5)}, g_2^{(5)}, g_3^{(5)}, g_4^{(5)} g_5^{(5)}) = (1, 0, 0, 0, 1), \\
g^{(6)} &= (g_6^{(6)}, g_2^{(6)}, g_3^{(6)}, g_4^{(6)}, g_5^{(6)}, g_6^{(6)}) = (1, 1, 0, 0, 1, 1), \\
g^{(7)} &= (g_7^{(7)}, g_2^{(7)}, g_3^{(7)}, g_4^{(7)}, g_5^{(7)}, g_6^{(7)}, g_7^{(7)}) = (1, 0, 1, 0, 1, 0, 1),
\end{align*}
\]
which gives the pattern of the Sierpinski gasket.

(2) Another formula of \( g \) is given by:
\[
g_m^{(n)} = \frac{(n - 1)!}{(m - 1)! (n - m)!} \mod 2
\]

Corollary 9.3. The formulas hold for all \( k \in I_{n-1} \):
\[
\mu_2^{(n)} + \mu_2^{(n)} = 2^n - 1, \quad \mu_2^{(n)} \in 2I_{n-1}.
\]

Proof. We proceed by induction. Suppose the conclusion holds up to \( n - 1 \). It follows from (11) that
\[
\begin{align*}
\mu_2^{(n)} + \mu_2^{(n)} &= \mu_2^{(n-1)} + \mu_2^{(n-1)} + 2^{n-1} \\
&= 2^n - 1.
\end{align*}
\]
The latter formula follows immediately. \( \square \)
Lemma 9.4. (i) \( \nu^{(n)} \) is given by
\[
\nu^{(n)}_k = \nu^{(n)}_{(k_1, k_2, \ldots, k_{n-1})_2} = \sum_{j=1}^{n-1} k_j g^{(n)}_j.
\]

(ii) \( \tilde{\sigma}_n = (\mu^{(n)}_0, \mu^{(n)}_1, \ldots, \mu^{(n)}_{2^n - 1}) \) is a permutation vector of \( I_n \), that is,
\[
\mu^{(n)}_j \in I_n, \quad \mu^{(n)}_j \neq \mu^{(n)}_j' \text{ for all distinct pairs } j \neq j'.
\]

Proof. (i) Let us rewrite \( T_n \) as:
\[
T_n(s_1, s_2, \ldots, s_m) = (s_1, s_2, \ldots, s_m, s_1 + \alpha, s_2 + \alpha, \ldots, s_m + \alpha) \pmod{2}.
\]
For example we see the case of \( \nu^{(4)} \),
\[
\nu^{(4)}(g_1(0), g_2(0), g_3(0), g_4(0)) = (0, g_2(4), g_2(4), g_1(4), g_1(4) + g_3(4) + g_2(4) + g_4(4)) \pmod{2},
\]
and
\[
\nu^{(4)}_{(0,0,0,0)} = [0 + 0 + 0], \quad \nu^{(4)}_{(0,0,0,1)} = [0 + 0 + g_3(4)], \quad \nu^{(4)}_{(0,0,1,0)} = [0 + g_2(4) + 0],
\]
\[
\nu^{(4)}_{(0,1,1,0)} = [0 + g_2(4) + g_3(4)], \quad \nu^{(4)}_{(1,0,0,0)} = [g_1(4) + 0 + 0], \quad \nu^{(4)}_{(1,0,1,0)} = [g_1(4) + 0 + g_3(4) + g_4(4)]
\]
Let us consider the general case. For any \( h_j \in \{0, 1\} \), let us define
\[
v = (v_0, \ldots, v_{2^n - 1}) = (T_{h_{n-2}} \circ T_{h_{n-3}} \cdots \circ T_{h_1})(0)
\]
\[
\tilde{v} = (\bar{v}_0, \ldots, \bar{v}_{2^n - 1}) = (T_{\bar{h}_n} \circ T_{h_{n-2}} \circ T_{h_{n-3}} \cdots \circ T_{h_1})(0)
\]
\[
= (v, v + h_n) \pmod{2}.
\]
If \( v_k = v_{(k_1, \ldots, k_{n-1})_2} = \sum_{j=1}^{n-1} k_j h_{n-j} \), then
\[
\tilde{v}_k = \tilde{v}_{(k_1, \ldots, k_n)_2} = [v_k + \tilde{k}_1 h_n] = \left[ \sum_{j=1}^{n} \tilde{k}_j h_{n-j+1} \right]
\]
for \( \tilde{k} \in \{0, \ldots, 2^n - 1\} \), since \( k_1 = \tilde{k}_2, \ldots, k_{n-1} = \tilde{k}_n \) hold.

If we insert \( g^{(n)}_1 \) into \( h_{n-1} \) in \( v \), then we obtain \( \nu^{(n)} \), that is
\[
\nu^{(n)}_k = \sum_{j=1}^{n-1} k_j g^{(n)}_j.
\]
(ii) We proceed by induction. For \( n = 1 \), \( \tilde{\sigma}_1 = (0, 1) \) corresponds to the identity over \( I_1 = \{0, 1\} \).
Suppose that the conclusion holds up to \( n - 1 \) so that \( \tilde{\sigma}_{n-1} \) be a permutation vector of \( I_{n-1} \). It follows from the expression (i) that for any \( k_2, \ldots, k_{n-1} \in \{0, 1\} \), the equalities hold:
\[
[g^{(n)}_1 + 2 \sum_{j=2}^{n-1} k_j g^{(n)}_j] = g^{(n)}_1 = 1.
\]
In particular \( \nu^{(n)}_{(0, k_2, \ldots, k_{n-1})_2} \neq \nu^{(n)}_{(1, k_2, \ldots, k_{n-1})_2} \), and hence
\[
|\mu^{(n)}_j - \mu^{(n)}_{j + 2^{n-1}}| = g^{(n)}_1 2^{n-1} = 2^{n-1}
\]
hold for any \( j \in I_{n-1} \).

By the assumption, \( \tilde{\sigma}_{n-1} \) is a permutation vector on \( I_{n-1} \) so that \( \mu^{(n)}_j \neq \mu^{(n)}_l \) hold for any distinct pair \( 0 \leq j, l \leq 2^{n-1} - 1 \). Since the value of \( \mu^{(n)}_j \) does not exceed \( 2^n \), it follows that \( \mu^{(n)}_j \neq \mu^{(n)}_l \) hold for any distinct pair \( 0 \leq j, l \leq 2^n - 1 \), and hence \( \tilde{\sigma}_n \) must be a permutation vector of \( I_n \). \( \square \)
Proposition 9.5. \( \hat{\sigma}_n^2 = \text{id} \) hold on \( I_n = \{0, 1, \ldots, 2^n - 1\} \).

Proof. Let \( k \in I_n \). For \( k = (k_1, k_2, \ldots, k_n) \), let us denote the corresponding binary expansions:

\[
\hat{\sigma}_n(k) = \hat{\sigma}_n((k_1, k_2, \ldots, k_n)_2) = (k'_1, k'_2, \ldots, k'_n)_2,
\]

\[
\hat{\sigma}_n^2(k) = \hat{\sigma}_n((k'_1, k'_2, \ldots, k'_n)_2) = (k''_1, k''_2, \ldots, k''_n)_2,
\]

respectively. Firstly let us verify the formulas:

\[
(12) \quad \hat{\sigma}_n(k) = \left( \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2}, 2, \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2}, \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2}, \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2} \right)_2
\]

where \( \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2} \in \{0, 1\} \) is defined by

\[
\hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2} \equiv [1 + \nu_{(k_1, k_2, \ldots, k_{n-1})_2} + k_n] = [1 + \sum_{i=1}^{n-\kappa+1} k_{\kappa+i-1} g_i^{(n-\kappa+1)}].
\]

Since \( \hat{\sigma}_1 = (0, 1) \), one can see that \( \hat{\sigma}_1(0) = 0 \) and \( \hat{\sigma}_1(1) = 1 \).

Notice that \( \nu^{(2)} = T_{g_1^{(2)}}(0) = T_1(0) = (0, 1) \) and hence

\[
\hat{\sigma}_2 = (\hat{\sigma}_1, \hat{\sigma}_1) + 2(1 - \nu^{(2)}_0, \nu^{(2)}_0, 1 - \nu^{(2)}_1, \nu^{(2)}_1)
\]

\[= (0, 1, 0, 1) + 2(1 - 0, 0, 1 - 1, 1) = (2, 1, 0, 3).\]

It can be presented as

\[
(\hat{\nu}_{k_1}, \nu_{k_2})_2 = (1 + k_1 g_1^{(2)} + k_2, k_2)_2 = 2[1 + k_1 + k_2] + k_2,
\]

for any \( k \in I_2 \). In fact the equalities follow from direct computations:

\[
\hat{\sigma}_2(0) = 2[1 + 0 + 0] + 0 = 2, \quad \hat{\sigma}_2(1) = 2[1 + 0 + 1] + 1 = 1,
\]

\[
\hat{\sigma}_2(2) = 2[1 + 1 + 0] + 0 = 0, \quad \hat{\sigma}_2(3) = 2[1 + 1 + 1] + 1 = 3.
\]

Suppose the formula holds up to \( n - 1 \). Then we have the equalities:

\[
(13) \quad \hat{\sigma}_n(k) = \hat{\sigma}_n((k_1, k_2, \ldots, k_n)_2)
\]

\[
= \hat{\sigma}_{n-1}((k_2, k_3, \ldots, k_n)_2) + \begin{cases} 2^{n-1} \nu^{(n)}_{(k_1, k_2, \ldots, k_{n-1})_2} & \text{if } k_n = 1 \\ 2^{n-1}(1 - \nu^{(n)}_{(k_1, k_2, \ldots, k_{n-1})_2}) & \text{if } k_n = 0 \end{cases}
\]

\[
= (0, \hat{\nu}_{(k_2, k_3, \ldots, k_n)_2}, \ldots, \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2}, k_n)_2 + (1 + \nu^{(n)}_{(k_1, k_2, \ldots, k_{n-1})_2} + k_n, 0, \ldots, 0)_2
\]

\[
= \left( \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2}, \hat{\nu}_{(k_2, k_3, \ldots, k_n)_2}, \ldots, \hat{\nu}_{(k_1, k_2, \ldots, k_{n-1})_2} \right)_2.
\]

So it holds for \( n \).

Next by use of (12), we obtain the equalities:

\[
k'_n = [1 + \sum_{i=1}^{n-\kappa+1} k_{\kappa+i-1} g_i^{(n-\kappa+1)}], \quad k''_n = k_n,
\]

\[
k''_n = [1 + \sum_{i=1}^{n-\kappa+1} k'_n g_i^{(n-\kappa+1)}], \quad k''_n = k_n.
\]
where \( \kappa = 1, 2, \ldots, n - 1 \). For \( n - \kappa \in \{1, 2, \ldots, n - 1\} \),

\[
k'_n^{\kappa} = \left[ 1 + \sum_{j=1}^{\kappa+1} k'_{n-\kappa+j-1} g^{(\kappa+1)}_j \right]
\]

\[
= \left[ 1 + \sum_{j=1}^{\kappa} (1 + \sum_{l=1}^{\kappa-j+2} k_{n-\kappa+j+l-2} g^{(\kappa-j+2)}_l g^{(\kappa+1)}_j + k'_{n+1} g^{(\kappa+1)}_{k+1} \right]
\]

\[
= \left[ 1 + \sum_{j=1}^{\kappa} \left( g^{(\kappa+1)}_j \right) + k_{n-\kappa} g^{(\kappa+1)}_1 g^{(\kappa+1)}_1 \right.
\]

\[
+ \sum_{j=1}^{\kappa} \left( g^{(\kappa+1)}_j \right) + k_{n-\kappa} g^{(\kappa+1)}_1 g^{(\kappa+1)}_1
\]

\[
\left. + \sum_{j=2}^{\kappa+1} k_{n-\kappa+j-1} \left( \sum_{l=1}^{j} g^{(\kappa-l+2)}_{l-1} g^{(\kappa+1)}_l \right) \right] = k_{n-\kappa},
\]

where we have used the equalities \( g^{(1)}_1 = g^{(\kappa+1)}_1 = g^{(\kappa+1)}_{k+1} = 1, \left[ 1 + \sum_{j=1}^{\kappa} \left( g^{(\kappa+1)}_j \right) \right] = 0, \)

\[
g^{(\kappa-l+2)}_{l-1} g^{(\kappa+1)}_l = \frac{\kappa!}{(\kappa-j+1)! (j-l)! (l-1)!} = g^{(\kappa-j-l+2)}_{(j-l+1)+1} g^{(\kappa+1)}_{j-l-1}, \]

\[
g^{(\kappa-m+1)}_{m+1} g^{(\kappa+1)}_{m+1} = \frac{\kappa!}{(\kappa-m+1)! m! m!} = \frac{\kappa!}{(\kappa-2m)! m! (2m)!} = 2 \left( \begin{array}{c} \kappa+1 \\ m \end{array} \right), \]

and

\[
\left( \sum_{l=1}^{j} g^{(\kappa-l+2)}_{l-1} g^{(\kappa+1)}_l \right) = \left( \begin{array}{c} \kappa+1 \\ j \end{array} \right) = \left( \begin{array}{c} \kappa+1 \\ 2m \end{array} \right), \quad \left( \begin{array}{c} \kappa+1 \\ j \end{array} \right) = \left( \begin{array}{c} \kappa+1 \\ 2m+1 \end{array} \right) = 0.
\]

Hence \( \tilde{\sigma}^2_n(k) = (k_1, k_2, \ldots, k_n)_2 = k \) holds.

**Proof of Theorem 7.7** Let \( j, k \in I_n \), and denote the binary expansions:

\[
\begin{align*}
  j &= (j_1, j_2, \ldots, j_n)_2, \quad k = (k_1, k_2, \ldots, k_n)_2, \\
  \tilde{\sigma}_n(j) &= (j'_1, j'_2, \ldots, j'_n)_2, \quad \tilde{\sigma}_n(k) = (k'_1, k'_2, \ldots, k'_n)_2.
\end{align*}
\]

Let us denote the two generating elements \( a_0, a_1 \) of the dynamics for the lamplighter operators (11) and the BBS (11) by:

\[
\begin{align*}
  a^{(0)}_L, a^{(1)}_L, a^{(0)}_B, a^{(1)}_B
\end{align*}
\]

respectively. These operators satisfy the following recursive relations for \( \varepsilon = 0, 1 \):

\[
\begin{align*}
  a^{(\varepsilon)}_L = \left( j + k + \varepsilon \right) a^{(k)}_L |_{0 \leq j, k \leq 1}, \quad a^{(\varepsilon)}_B = \left( j + 1 + \varepsilon \right) a^{(k)}_B |_{0 \leq j, k \leq 1}.
\end{align*}
\]
By applying these formulas repeatedly, we obtain four matrices of the size $2^n$ by $2^n$ given by

$$a^{(0; n)}_B = \begin{pmatrix} \alpha^{(0)}_{j, k} \\ \beta^{(0)}_{j, k} \end{pmatrix}_{0 \leq j, k < 2^n} = \begin{pmatrix} (j_1 + 1 + \varepsilon)(j_2 + 1 + k_1) \cdots (j_n + 1 + k_{n-1}) \end{pmatrix}_{0 \leq j, k < 2^n},$$

$$a^{(1; n)}_B = \begin{pmatrix} \alpha^{(1)}_{j, k} \\ \beta^{(1)}_{j, k} \end{pmatrix}_{0 \leq j, k < 2^n} = \begin{pmatrix} (j_1 + k_1 + \varepsilon)(j_2 + k_2 + k_1) \cdots (j_n + k_n + k_{n-1}) \end{pmatrix}_{0 \leq j, k < 2^n}.$$

We shall verify the stronger formulas:

$$a^{(0; n)}_B + a^{(1; n)}_B = \sigma_n \left( a^{(0; n)}_L + a^{(1; n)}_L \right) \sigma_n^{-1}.$$  

This is enough to conclude proposition:

$$M^{(n)}_B = a^{(0; n)}_B + a^{(1; n)}_B + a^{(0; n)*}_B + a^{(1; n)*}_B = \sigma_n \left( a^{(0; n)}_L + a^{(1; n)}_L + a^{(0; n)*}_L + a^{(1; n)*}_L \right) \sigma_n^{-1} = \sigma_n M^{(n)}_L \sigma_n^{-1}.$$

Since $\tilde{\sigma}_n$ is a permutation vector (Proposition 9.4) and $\tilde{\sigma}_n^{-1} = \tilde{\sigma}_n$ (Proposition 9.5), equation (10) is equivalent to the equalities:

$$a^{(0)}_{j, k} + a^{(1)}_{j, k} = \beta^{(0)}_{\tilde{\sigma}_n(j), \tilde{\sigma}_n(k)} + \beta^{(1)}_{\tilde{\sigma}_n(j), \tilde{\sigma}_n(k)}$$

for all $j, k \in I_n$.

Let us compute both sides, and divide into two cases on $(j_2 + 1 + k_1)(j_3 + 1 + k_2) \cdots (j_n + 1 + k_{n-1})$, where:

(i) all the factors $j_{i+1} + 1 + k_i$ $(i = 1, 2, \ldots, n - 1)$ take odd integer values,

(ii) otherwise, that is, there exists an integer $\kappa$ such that $j_{i+1} + 1 + k_i$ is even.

To treat both cases, we claim the following formula: suppose for some $1 \leq \kappa \leq n$,

$$j_{i+1} + k_i + 1$$

is odd for $\kappa + 1 \leq i \leq n - 1$, then

$$[j_i + k_i + k_i' + 1] = [j_i + k_i + 1]$$

for $\kappa + 1 \leq i \leq n$.

In fact we have the equalities:

$$[j_i' + k_i' + k_i' - 1] = [3 + \sum_{l=1}^{n-i+1} (j_{i+l-1} + k_{i+l-1})g_l^{(n-i+1)} + \sum_{l=1}^{n-i+2} k_{i+l-2}g_l^{(n-i+2)}]$$

$$= [3 + j_i + k_i - 1 + \sum_{l=2}^{n-i+1} (j_{i+l-1} + k_{i+l-2})g_l^{(n-i+1)} + 2 \sum_{l=2}^{n-i+1} k_{i+l-2}g_l^{(n-i+1)} + 2k_n]$$

$$= [j_i + k_i + 1]$$

where we used the defining relation $g_m^{(n)} = g_m^{(n-1)} + g_m^{(n-1)}$. This verifies the claim.

Case (i): From the assumption, we obtain $[j_i + k_i + 1] = 1$ for $i \in \{1, 2, \ldots, n - 1\}$. Thus one can show the equalities:

$$\alpha^{(0)}_{j, k} + \alpha^{(1)}_{j, k} = \beta^{(0)}_{\tilde{\sigma}_n(j), \tilde{\sigma}_n(k)} + \beta^{(1)}_{\tilde{\sigma}_n(j), \tilde{\sigma}_n(k)} = [j_1 + 1] + [j_1 + 2] = 1.$$  

By use of (18), we obtain the equalities:

$$\beta^{(0)}_{j', k'} + \beta^{(1)}_{j', k'} = [\beta^{(0)}_{j', k'} + \beta^{(1)}_{j', k' - 1}] = (j_i' + k_i' + 1) + [j_2 + k_1 + [j_3 + k_2 + 1] \cdots [j_n + k_{n-1} + 1]$$

$$= [j_i' + k_i'] + [j_i' + k_i' + 1] = 1.$$  

Thus (17) is proven in this case.
Case (ii): In this case there exists the largest $\kappa$ such that $j_{n+1} + 1 + k_{\kappa}$ is even, then $\alpha_{j,k}^{(0)}$ and $\alpha_{j,k}^{(1)}$ are equal to 0. Hence

$$\alpha_{j,k}^{(0)} + \alpha_{j,k}^{(1)} = 0.$$ 

Thus we obtain the equalities:

$$\beta_{j,k}^{(0)} + \beta_{j,k}^{(1)} = [\beta_{j,k}^{(0)}] + [\beta_{j,k}^{(1)}]$$

$$= ([j_1 + k_1] + [j_1 + k_1 + 1]) ([j_2 + k_2 + k_1] \cdots [j_{\kappa} + k_{\kappa} + k_{\kappa-1}]$$

$$\times [j_{n+1} + k_{n+1}] \cdots [j_n + k_n + 1]$$

$$= 0$$

since $[j_{n+1} + k_{n+1}] = 0$. This completes the proof. □

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