Electromagnetic and Gravitational Invariants

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Abstract. The curvature invariants have been subject of recent interest in the context of the experimental detection of the gravitomagnetic field, namely due to the debate concerning the notions of “extrinsic” and “intrinsic” gravitomagnetism. In this work we explore the physical meaning of the curvature invariants, dissecting their relationship with the gravitomagnetic effects.

1. Introduction

There is an ongoing debate concerning the question if there is a fundamental difference between the “translational” gravitomagnetism claimed to have been detected with Lunar Laser Raging [1], and (to an high accuracy) in the observations [2] of the binary pulsar PSR 1913 +16, and the gravitomagnetic field produced by the rotation of the Earth, detected in the LAGEOS Satellites data [3] (whose detection was also the primary goal of the Gravity Probe-B mission). These phenomena have been dubbed, respectively, “extrinsic” and “intrinsic” gravitomagnetism, such distinction being based [4, 5] on the quadratic curvature invariants \( R \cdot R \), \( \star R \cdot R \), and driven by their formal analogy with the invariants \( F \cdot F \), \( \star F \cdot F \) of the Maxwell tensor. In this work [6], starting from previous knowledge on the classification of purely electric/magnetic spacetimes [7, 8], and using insight from the formal analogy \( \{ F \cdot F, \star F \cdot F \} \leftrightarrow \{ R \cdot R, \star R \cdot R \} \), we explain the invariant structure of the relevant gravitational fields. Then, using the gravito-electromagnetic analogy based on tidal tensors [9] as a physical guiding principle, we clarify the physical meaning of the curvature invariants and the implications on the motion of test particles.

2. Electromagnetic and gravitational scalar invariants — algebraic meaning

Wrt any unit timelike 4-vector \( u^\alpha \), the Maxwell tensor splits irreducibly into the two spatial vectors \( (E^u)^\alpha = F^{\alpha \beta} u_\beta \) and \( (B^u)^\alpha = \star F^{\alpha \beta} u_\beta \) (c.f. Eq. (16) of [10]), which are, respectively, the electric and magnetic fields as measured by an observer of 4-velocity \( u^\alpha \). Both \( (E^u)^\alpha \) and \( (B^u)^\alpha \) depend on the observer 4-velocity \( u^\alpha \), but combining them...
one can construct the two quadratic (i.e., second order) scalar invariants (e.g. [11]):

\[ E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} = -\frac{1}{2} \star F \cdot F; \quad E^\alpha B_\alpha = -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta} = -\frac{1}{4} \star F \cdot F. \quad (1) \]

They have the following physical interpretation: (i) if \( \star F \cdot F \neq 0 \) then for all observers \( u^\alpha \) one has \((E^\alpha) u^\alpha \neq 0 \neq (B^\alpha) u^\alpha \); (ii) if \( \star F \cdot F = 0 \) and \( F \cdot F > 0 \) \((< 0)\) then observers \( u^\alpha \) do exist wrt \((B^\alpha)u^\alpha = 0 \((E^\alpha)u^\alpha = 0 \); (iii) the case \( F \cdot F = \star F \cdot F = 0 \) (but \( F^{\alpha\beta} \neq 0 \)) corresponds to pure electromagnetic radiation.

On the other hand, the Riemann tensor \( \text{in vacuum} \) becomes the Weyl tensor, which analogously is completely decomposed (c.f. Eq. (30) of [10]) in its electric part \( \mathbb{E}_{\alpha\beta} = R_{\alpha\mu\beta\nu} u^\mu u^\nu \) and magnetic part \( \mathbb{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu} u^\mu u^\nu \) wrt \( u^\alpha \). Moreover, \( \mathbb{E}_{\alpha\beta} \) and \( \mathbb{H}_{\alpha\beta} \) form quadratic invariants \( \text{formally analogous to (1)}: \)

\[ \mathbb{E}^{\alpha\gamma} \mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} = \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{1}{8} \mathbb{R} \cdot \mathbb{R} \quad \mathbb{E}^{\alpha\gamma} \mathbb{H}_{\alpha\gamma} = \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} = \frac{1}{16} \star \mathbb{R} \cdot \mathbb{R}, \quad (2) \]

which are usually combined into \( I \equiv (\mathbb{R} \cdot \mathbb{R} + i \star \mathbb{R} \cdot \mathbb{R})/8. \) However, whereas the invariants (1) are the only two independent scalar invariants of \( F^{\alpha\beta} \), in the case of \( R_{\alpha\beta\gamma\delta} \) there also two independent \( \text{cubic invariants} \), given by \( A \equiv R^{\lambda\beta}_{\text{\lambda}\mu} R_{\text{\lambda}\rho\sigma} R_{\text{\alpha}\beta}/16 \) and \( B \equiv R^{\lambda\rho}_{\text{\beta}\mu} R_{\text{\rho}\sigma} \star R_{\text{\alpha}\beta}/16, \) and combined into \( J \equiv A - i B. \) It turns out [7, 8] that one basically obtains formally equivalents statements (i)-(iii) as above by replacing \( F \) by \( \mathbb{R} \), \text{provided that the condition} \( M \equiv I^2/J^2 - 6 \geq 0 \) \((\text{real or infinite})\) \text{is added to (ii). Further comments and a more detailed treatment hereof shall be given in [6].}

3. Interpretation of the invariant’s structure for relevant setups

For astrophysical applications we will be interested essentially in the gravitational far field of non-spinning and spinning bodies obeying criterion \( M \geq 0 \), which thus have an electromagnetic counterpart. Now the important point is that \( \mathbb{E}_{\alpha\gamma} \) and \( \mathbb{H}_{\alpha\gamma} \) also obey transformation laws in a change of observer/frame which exhibit a degree of similarity with the transformation of the electromagnetic fields that allows for invariant structure of a gravitational setup to be understood comparing with the analogous electromagnetic setup. The magnetic part of the Riemann tensor \( (\mathbb{H})_{\alpha\gamma} \) measured by a given observer \( U^\alpha = U^0(1, \vec{v}), \) can be obtained, in terms of the tensors \( (\mathbb{E})_{\alpha\gamma} \) and \( (\mathbb{H})_{\alpha\gamma} \) measured by \( U^\alpha = u^\alpha 0 \), from decomposition (30) of [10], yielding [13], to first order in \( V \), Eq. (3i) below, which exhibits suggestive similarities with its electromagnetic counterpart (3ii):

\[ \mathbb{H}(U) \simeq \vec{E}(u) \times \vec{v} - \vec{v} \times \vec{E}(u) + \mathbb{H}(u) \quad (i) \quad \vec{B}(U) \simeq -\vec{v} \times \vec{E}(u) + \vec{B}(u) \quad (ii) \quad (3) \]

where \((\vec{E} \times \vec{v})_{kl} \equiv e_{ij} E_{ik} v_j, \quad (\vec{v} \times \vec{E})_{kl} \equiv e_{ij} v_i E_{lj}. \)

The examples of interest will be discussed in detail in [6]. Consider first a single point charge; for a generic observer \( U^\alpha \), we have \( \vec{B}(U) \neq 0 \); but as one can read from (3ii), always \( \vec{B} \perp \vec{E} \), since there are observers (those comoving with the charge) for which \( \vec{B} \) vanishes everywhere; and the invariant \( \star \mathbb{R} \cdot \mathbb{R} = 0 \) reflects that. Analogous arguments explain the vanishing of \( \star \mathbb{R} \cdot \mathbb{R} = 0 \) in Schwarzschild spacetime.
If we consider a system of two charges in arbitrary motion, then in general $\mathbf{F} \cdot \mathbf{F} \neq 0$, there is no frame where $\mathbf{B} = 0$ globally, which we relate with the fact that there is no observer for which both particles are at rest (i.e., the currents involved cannot be made to vanish by a Lorentz boost). However, as explicitly shown in [6], in the special case of planar motion, in the plane of the motion we have $\mathbf{F} \cdot \mathbf{F} = 0$, meaning that therein (and only therein) there are observers, whose 4-velocity $U^\alpha$ varies from point to point, relative to which $\mathbf{B}$ vanishes, which arises from an exact cancellation between the magnetic field produced by the motion of the two charges relative to the observer $U^\alpha$. It should be emphasized that even in the plane of the motion $\mathbf{B}$ does not globally vanish in any rigid frame (as the velocity field $U^\alpha$ for which $B^\alpha = 0$ is a shearing congruence), which is a clearly distinct situation from the case of a single charge, despite both being characterized by $\mathbf{F} \cdot \mathbf{F} \neq 0$. As shall be shown in [6], the analogous gravitational two body systems (which are studied to Post Newtonian order) exhibit an analogous invariant structure: $\star \mathbf{R} \cdot \mathbf{R} \neq 0$ in the most general case, telling us that in general there are no observers for which $\mathbf{H}_{\alpha\beta} = 0$; and for planar motion, which is the case of the Earth-Sun system discussed in [1] we have $\star \mathbf{R} \cdot \mathbf{R} = 0$ in the orbital plane, and $\star \mathbf{R} \cdot \mathbf{R} \neq 0$ elsewhere. This structure can be explained by precisely the same reasoning as in the electromagnetic case, since the tensor $\mathbf{H}_{\alpha\beta}$ is linear to Post Newtonian order and therefore to this accuracy a superposition principle applies.

One of the key points of this work is that the very same arguments are shown to apply to the case of a spinning body. The structure of the electromagnetic invariants of the field of a spinning charge is similar to the one of a system of charges in planar motion: $\star \mathbf{F} \cdot \mathbf{F} = 0$ in the equatorial plane, $\star \mathbf{F} \cdot \mathbf{F} \neq 0$ elsewhere (explicit expressions will be given in [6]). A spinning body can be viewed as an assembly of translating elements; and there is no observer for which every part of it is at rest\(^1\); hence (except in the equatorial plane) it is not possible to make $B^\alpha$ vanish. In the equatorial plane, again by exactly the same principle from the case of $N \geq 2$ sources in planar motion, there are observers for which $B^\alpha = 0$ (not the observers co-rotating with the source, despite rotating in the same sense; these observers have a velocity that decays as $r^{-1}$, which is the same asymptotic dependence as case of the two body system in planar motion mentioned above, c.f. [6]). The analogy with the gravitational case again holds: the curvature invariants for Kerr spacetime have the structure $\star \mathbf{R} \cdot \mathbf{R} \neq 0$ elsewhere, and $\star \mathbf{R} \cdot \mathbf{R} = 0$ (exactly!) in the equatorial plane, where, likewise, there are observers for which $\mathbf{H}_{\alpha\beta} = 0$; their angular velocity (which again depends on $r$) asymptotically matches its electromagnetic counterpart up to a factor of 2 [6, 12]. Hence, from the point of view of the structure of the invariants, whilst clearly distinguishing between the fields produced by a single translating body from the ones of a rotating body (which is in agreement with [5]), we found no fundamental distinction between the latter and the

\(^1\) In a co-rotating frame indeed the whole spinning body would be seen to be at “rest”; but a (rigid) rotating frame consists of a congruence of observers all with different 4-velocities $U^\alpha$, whilst having the same angular velocity. $\mathbf{B}$ does not vanish in the co-rotating frame, even though there are no currents therein; the reason being (taking the perspective of the rotating frame) that the vorticity of the congruence also contributes as a source for $\mathbf{B}$ (see Eqs. (17)-(20) of [10]). It is crucial to realize that unlike the translation of a single body, which can always be made to vanish by a simple boost, rotation (like the motion of $N \geq 2$ translating bodies in general) imprints itself in intrinsic properties (like the intrinsic angular momentum) that cannot be made to vanish by any change of frame.
ones from a system of translating bodies in planar motion. This is one point that (we hope) may shed some light on the ongoing debate.

It should also be remarked that, as the examples above make clear, whereas \( \mathbf{F} \cdot \mathbf{F} = 0 \) (\( \mathbf{R} \cdot \mathbf{R} \neq 0 \)) signals intrinsic magnetic field (curvature), the case \( \mathbf{R} \cdot \mathbf{R} = 0 \) (with \( M \geq 0 \)) does not distinguish static solutions, e.g. Coulomb field of a point charge (Schwarzschild solution), where \( \mathcal{B}^\alpha (\mathbb{H}_{\alpha\beta}) \) vanishes globally in a rigid frame, from setups where \( \mathcal{B}^\alpha (\mathbb{H}_{\alpha\beta}) \) can only be made to vanish for a class of observers specific to each point (i.e., forming shearing congruences), as is the case of the equatorial plane of a spinning charge (equatorial plane of Kerr spacetime).

4. Dynamical implications of the invariants. Gravitomagnetism.

In spite of the insight it provides to interpret the structure of the gravitational invariants, the analogy with the electromagnetic invariants is purely formal (see Sec. 4 of [14]), and should not be used as a physical guiding principle: in one case we are dealing with quantities built on electromagnetic fields \( \mathcal{E}^\alpha, \mathcal{B}^\alpha \); in the other case with gravitational tidal tensors \( \mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta} \), which, as shown in [9], play in gravity analogous dynamical roles not to \( \mathcal{E}^\alpha, \mathcal{B}^\alpha \), but to the electromagnetic tidal tensors (which are quantities one order higher in differentiation). This distinction is crucial, because the effects involved may actually be opposite: \( \mathbf{F} \cdot \mathbf{F} = 0 \), telling us that there there are observers for which \( \mathcal{B} = 0 \), means that magnetic dipoles carried by them do not undergo Larmor precession, but in general they will feel a force (since the magnetic tidal field does not vanish [12, 6]); by contrast what \( \mathbf{R} \cdot \mathbf{R} = 0 \) (with \( M \geq 0 \)) tells us is that there is a class of observers carrying gyroscopes that to not feel gravitational force (see Eq. (28) of [9]), but in general they “precess” (since it is the gravitomagnetic tidal tensor \( \mathbb{H}_{\alpha\beta} \) that vanishes, not the so called [1, 4, 5] “gravitomagnetic field”). In [6] this will be shown explicitly for the physical systems considered in the previous section. To conclude, curvature invariants do not tell us about the “gravitomagnetic field” itself, nor the Lense Thirring effect (gravitomagnetic field and gyroscope “precession” are actually artifacts of the reference frame, thus cannot be manifest in invariant quantities). Curvature invariants tell us about the gravitomagnetic tidal field (magnetic curvature), and the appropriate probe to measure it is the force (not the precession!) on a gyroscope.

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