A Characterization of $T_{2g+1,2}$ among Alternating Knots

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Abstract Let $K$ be a genus $g$ alternating knot with Alexander polynomial $\Delta_K(T) = \sum_{i=-g}^{g} a_i T^i$. We show that if $|a_g| = |a_{g-1}|$, then $K$ is the torus knot $T_{2g+1, \pm 2}$. This is a special case of the Fox Trapezoidal Conjecture. The proof uses Ozsváth and Szabó’s work on alternating knots.

Keywords Alternating knots, Alexander polynomial, strongly quasipositive fibered knots

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1 Introduction

Alternating knots have many good properties. For example, the information from the Alexander polynomial of an alternating knot $K$ determines the genus of $K$ and whether $K$ is fibered [2, 11]. Even so, there are still some open problems about alternating knots. One of these problems is the following conjecture made by Fox [3, Problem 12].

Conjecture 1.1 (Fox Trapezoidal Conjecture) Let $K$ be an alternating knot with normalized Alexander polynomial

$$\Delta_K(T) = \sum_{i=-g}^{g} a_i T^i,$$

(1.1)

where $g$ is the genus of $K$. Then

$$|a_i| \leq |a_{i-1}| \quad \text{when} \quad 0 < i \leq g.$$

Moreover, if $|a_i| = |a_{i-1}|$ for some $i$, then $|a_j| = |a_i|$ whenever $0 \leq j \leq i$.

This conjecture was known for 2–bridge knots [9] and alternating arborescent knots [12]. Using Heegaard Floer homology, Ozsváth and Szabó [14] proved the first part of the conjecture for $i = g$. See (2.2) for the precise inequality. As a result, they proved the conjecture for genus–2 knots.

In this paper, we will prove the second part of Conjecture 1.1 for $i = g$. In this case, we will get a stronger conclusion.

Theorem 1.2 Let $K$ be an alternating knot with normalized Alexander polynomial given by (1.1), where $g$ is the genus of $K$. If $|a_g| = |a_{g-1}|$, then $K$ or its mirror is the torus knot $T_{2g+1, \pm 2}$. 

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Our proof uses Ozsváth and Szabó’s work [14].

This paper is organized as follows. In Section 2, we prove that if a knot $K$ has thin knot Floer homology, and $|a_g| = |a_{g-1}|$, then $K$ is a strongly quasipositive fibered knot. In Section 3, we prove that strongly quasipositive fibered alternating knots are connected sums of torus knots of the form $T_{2n+1,2}$. Hence we get a proof of Theorem 1.2.

2 Thin Knots with $|a_g| = |a_{g-1}|$

Let $K \subset S^3$ be a knot with knot Floer homology $[16, 18]$

$$\widehat{HF}K(S^3, K) = \bigoplus_{i,j \in \mathbb{Z}} \widehat{HF}K_j(S^3, K, i).$$

We say the knot Floer homology is thin, if it is supported in the line

$$j = i - \tau,$$

where $\tau = \tau(K)$ is the concordance invariant defined in [15].

By work of Hedden [10], we will make the following definition of strongly quasipositive fibered knots. We do not need the original definition of strong quasipositivity in [19].

**Definition 2.1** A strongly quasipositive fibered knot is a fibered knot $K \subset S^3$, such that the open book with binding $K$ supports the tight contact structure on $S^3$.

Now we can state the main result we will prove in this section.

**Proposition 2.2** Let $K \subset S^3$ be a knot with thin knot Floer homology. Let the normalized Alexander polynomial be given by (1.1). If $|a_g| = |a_{g-1}|$, then $K$ or its mirror is a strongly quasipositive fibered knot.

Let $S^3_0(K)$ be the manifold obtained by 0-surgery on $K$. Ozsváth and Szabó proved that if $\widehat{HF}K(S^3, K)$ is thin and $\tau(K) \geq 0$, then

$$HF^+(S^3_0(K), s) \cong \mathbb{Z}b_s + (\mathbb{Z}[U]/U^{\delta(-2\tau,s)})$$

for $s > 0$, where

$$\delta(-2\tau, s) = \max\left\{0, \left\lfloor \frac{|\tau| - |s|}{2} \right\rfloor \right\}$$

and

$$(-1)^{s-\tau}b_s = \delta(-2\tau, s) - t_s(K)$$

with

$$t_s(K) = \sum_{j=1}^{\infty} ja_{s+j}.$$

See [14, Theorem 1.4] and the paragraph after it.

Using (2.1), one can deduce the following inequality as in [14]:

$$|a_{g-1}| \geq 2|a_g| + \begin{cases} -1 & \text{if } |\tau| = g, \\ 1 & \text{if } |\tau| = g - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2.2)
Proof of Proposition 2.2 It follows from [17] that $a_g \neq 0$. If $|a_g| = |a_{g-1}|$, then by (2.2) we must have

$$|a_g| = 1, \quad |\tau| = g.$$ 

By [6, 13], $K$ is fibered. Replacing $K$ with its mirror if necessary, we may assume $\tau = g$. It follows from [14, Corollary 1.7] that the open book with binding $K$ supports the tight contact structure. \hfill \Box

3 Strongly Quasipositive Fibered Alternating Knots

Suppose that $K$ is a fibered alternating link. Let $D \subset S^2$ be a reduced connected alternating diagram of $K$. Applying Seifert’s algorithm to $D$, we can get a Seifert surface $F$ which is a union of disks and twisted bands corresponding to the crossings in $D$. We call the disks Seifert disks with boundary Seifert circles, and call the twisted bands Seifert bands. By [5, Theorem 5.1], $F$ is a fiber of the fibration of $S^3 \setminus K$ over $S^1$.

Following [7], we say a Seifert circle is nested, if each of its complementary regions contains another Seifert circle. It is well-known that $F$ decomposes as a Murasugi sum of two surfaces along a nested Seifert circle $C$ [11, 20]. More precisely, let $D_1, D_2$ be the two disks bounded by $C$. Let $B_i$ be the union of Seifert bands connecting $C$ to Seifert circles in $D_i$, $i = 1, 2$. We cut $F$ open along $B_{3-i} \cap C$ to get a disconnected surface. Let $F_i$ be the component such that the projection of $\partial F_i$ is supported in $D_i$. Then $F$ is a Murasugi sum of $F_1$ and $F_2$. Gabai [4] proved that $F$ is a fiber of a fibration of $S^3 \setminus K$ if and only if each $F_i$ is a fiber of a fibration of $S^3 \setminus \partial F_i$, $i = 1, 2$.

Definition 3.1 If a diagram contains no nested Seifert circles, then this diagram is special as defined in [11].

Suppose that $D \subset S^2$ is a reduced connected special alternating diagram for a link $K$. Let $S_1, \ldots, S_k$ be the Seifert circles in $D$. Since $D$ is special, these Seifert circles bound disjoint disks $D_1, \ldots, D_k$. We color the complementary regions of $D$ by two colors black and white, so that two regions sharing an edge have different colors. The coloring convention is that the disks $D_1, \ldots, D_k$ have the black color. Clearly, there are no other black regions. We will construct the black graph $\Gamma_B$ and the white graph $\Gamma_W$ as usual. Namely, the vertices in $\Gamma_B$ (or $\Gamma_W$) are the black (or white) regions, and the edges correspond to the crossings. These two graphs are embedded in $S^2$ as a pair of dual graphs. We also construct the reduced black graph $\Gamma'_B$ by deleting all but one edges connecting two vertices $v_i$ and $v_j$ if there is any edge connecting them.

The following proposition can be found in [1, Propositions 13.24 and 13.25].

Proposition 3.2 Suppose that $D \subset S^2$ is a reduced connected special alternating diagram for a fibered link $K$, then all but one vertices in $\Gamma_W$ have valence 2. As a result, $K$ is a connected sum of torus links

$$K = \#_{i=1}^\ell T_{k_i, 2}.$$ 

From Proposition 3.2, it is not hard to get the following characterization of $D$ in terms of $\Gamma'_B$.

Lemma 3.3 Under the same assumptions as in Proposition 3.2, the graph $\Gamma'_B$ is a tree.
Proof Since \( D \) is connected, \( \Gamma^r_B \) is also connected. If \( \Gamma^r_B \) contains only two vertices, there is exactly one edge by the definition of \( \Gamma^r_B \), so our conclusion holds. From now on, we assume \( \Gamma^r_B \) has at least three vertices. Let \( R \) be a complementary region of \( \Gamma^r_B \), then it is not a bigon since any two vertices in \( \Gamma^r_B \) are connected by at most one edge and \( \Gamma^r_B \) has at least three vertices. Let \( v \) be the vertex corresponding to \( R \) in \( \Gamma_W \), then \( v \) has valence \( > 2 \). By Proposition 3.2, \( \Gamma^r_B \) has at most one complementary region, which means that \( \Gamma^r_B \) is a tree. □

Lemma 3.4 Under the same assumptions as in Proposition 3.2, if two vertices in \( \Gamma_B \) are connected by an edge, then they are connected by at least two edges.

Proof Using Lemma 3.3, if \( D_i \) and \( D_j \) are connected through only one crossing, then \( D \) is not reduced, a contradiction. □

We say two Seifert bands are parallel if they connect the same two Seifert disks. The following lemma is well-known. See, for example, [7, Proposition 5.1].

Lemma 3.5 If two Seifert bands are parallel, then we can deplumb a Hopf band from \( F \). The resulting surface can be obtained by removing one of the bands from \( F \).

Lemma 3.6 Let \( K \) be a strongly quasipositive fibered alternating knot, and let \( D \) be a reduced connected alternating diagram for \( K \). Let \( C \) be a nested Seifert circle. If \( C \) is connected to two pairs of parallel bands, then these two pairs of bands are on the same side of \( C \).

Proof If \( C \) is connected to two pairs of parallel bands on different sides of \( C \), then we can deplumb a negative Hopf band from \( F \). See Figure 1. Hence the open book with page \( F \) supports an overtwisted contact structure [8, Lemma 4.1], a contradiction. □

![Figure 1](image)

Figure 1 If two collections of parallel bands are on different sides of a nested Seifert circle, we can deplumb a positive Hopf band and a negative Hopf band. The two dashed circles are the cores of the Hopf bands.

Proposition 3.7 Let \( K \) be a strongly quasipositive fibered alternating knot. Then \( K \) is a connected sum of torus knots of the form \( T_{2n_i+1,2} \) for \( n_i > 0 \).

Proof If \( D \) is special, by Proposition 3.2, \( K \) is a connected sum of torus knots \( T_{2n_i+1,2} \). Since \( K \) is strongly quasipositive, each \( n_i \) must be positive, so our conclusion holds.

Now we assume that \( D \) contains at least one nested Seifert circle. We say a nested Seifert circle is extremal, if one of its complementary regions contains no other nested Seifert circles.
Let $C_1, \ldots, C_m$ be a maximal collection of extremal nested Seifert circles in $\mathcal{D}$, and let $R_i$ be the complementary region of $C_i$ which contains no other nested Seifert circles. Then $R_1, \ldots, R_m$ are mutually disjoint. Let $\mathcal{D}'$ be the diagram obtained from $\mathcal{D}$ by Murasugi desumming along $C_1 \cup \cdots \cup C_m$. Let $\mathcal{D}_i$ be the part of $\mathcal{D}'$ supported in $R_i$, and let

$$\mathcal{D}^* = \mathcal{D}' \setminus \left( \bigcup_{i=1}^{m} \mathcal{D}_i \right).$$

By [4], $\mathcal{D}^*$ and $\mathcal{D}_i$ are alternating diagrams representing fibered links.

Since $R_i$ contains no other nested Seifert circles, $\mathcal{D}_i$ is special. By Lemma 3.4, $C_i$ is connected to another circle in $R_i$ by at least a pair of parallel bands.

We claim that $\mathcal{D}^*$ is special. Otherwise, let $C$ be an extremal nested Seifert circle, and let $R$ be the complementary region of $C$ which contains no other nested Seifert circles in $\mathcal{D}^*$. Since $C_1, \ldots, C_m$ is a maximal collection of extremal nested Seifert circles, $R$ must contain at least one $C_i$. By Lemma 3.4, $C_i$ is connected to another circle in $R \setminus R_i$ (including $C$) by at least a pair of parallel bands. This is a contradiction to Lemma 3.6.

Now $\mathcal{D}^*$ is special. There are at least two Seifert circles in $\mathcal{D}^*$, since $C_1$ is nested in $\mathcal{D}$. By Lemma 3.4, $C_1$ is connected to another Seifert circle in $\mathcal{D}^*$ by at least a pair of parallel bands. We again get a contradiction to Lemma 3.6. Hence $\mathcal{D}$ does not contain any nested Seifert circle. This finishes our proof.

**Proof of Theorem 1.2** By [14], $\widehat{HF}(S^3, K)$ is thin. It follows from Proposition 2.2 that $K$ is strongly quasipositive and fibered. Using Proposition 3.7, $K$ is a connected sum of $T_{2g+1,2}$. The condition on the Alexander polynomial forces $K$ to be $T_{2g+1,2}$. \qed

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