Tripartite Entanglement in a Bose Condensate by Stimulated Bragg Scattering

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We show that it is possible to entangle three different many-particle states by Bragg spectroscopy with nonclassical light in a Bose condensate of weakly interacting atomic gases. Among these three states, two are of atoms corresponding to two opposite momentum side-modes of the condensate; and the other is of single-mode photons of the output probe beam. We demonstrate strong dependence of the multiparticle entanglement on the quantum statistics of the probe light. We present detailed results on entanglement keeping in view of the possible experimental situation.

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Entanglement is the most intriguing feature of quantum mechanics. Of late, it has been recognized as a wonderful resource for quantum information processing. Creation of various entangled states is the first step towards development of quantum communication. In recent times, many schemes for entanglement production have been proposed and demonstrated. Entanglement in a Bose condensate arises quite naturally. For instance, Bogoliubov’s theory of Bose-Einstein condensation (BEC) in weakly interacting gases predicts that, in the condensate ground state, a pair of particles moving with opposite momentum are absolutely correlated (entangled). This nonclassical feature makes Bose condensates of weakly interacting atomic gases an excellent source for entanglement in massive particles. Many authors have proposed the production of nonclassical states in a two-component Bose condensate. Bragg scattering has also been considered for producing entanglement and squeezing.

Entanglement between two particles is quite common, for example, EPR states, polarization states of twin-photons, down converted two-photon states in optical parametric oscillator and so on. In contrast, three particle entanglement is not so common, though recently three-photon GHZ entangled states have been experimentally realized. Because of two dominant momentum side-modes $q$ and $-q$ involved in the Bragg scattering, a condensate seems to be a suitable candidate for exploring tripartite entanglement among these two condensate side-modes and the probe photon mode. We demonstrate that Bragg scattering with two input light beams - one pump and the other probe, generates entanglement between the output probe and the two side-modes excited due to the scattering of light. We analyze in detail the relevant parameter regimes where such tripartite entanglement will show up. We consider three different input probes: coherent, vacuum and one-photon Fock state. Although vacuum field as an input has been considered earlier and the entanglement between the scattered field and one condensate side-mode ($q$) is discussed, here we report another interesting regime where strong entanglement in long time limit persists. Our results show that although a vacuum or a coherent probe field can induce entanglement between the scattered field and one excited side-mode ($q$) of the condensate, the other condensate side-mode ($-q$) remains immune to entanglement with the field. The most interesting result we obtain is that a nonclassical one-photon field state as an input probe can cause this other mode to become entangled with the field and thus generates the desired tripartite entanglement. We also find that nonclassical field is useful for detecting the entanglement between the two condensate side-modes.

There are mainly four types of processes in pump-probe or Bragg scattering of light by a condensate. First, one pump photon is transformed into a probe photon causing conversion of a zero-momentum atom into an atom of momentum $q$. In the second process, an atom moving with a momentum $-q$ is scattered back into a zero-momentum atom by the transformation of a photon from pump to probe mode. The other two are the reverse processes of these two. Both the pump and the probe laser beams are detuned far off resonance from an electronic excited state of the atoms in order to avoid heating of the condensate. Because of interplay and simultaneity of such processes, Bragg spectroscopy holds the key for generating many-particle entanglement in the motional states of the atoms and photons.

The Hamiltonian of the system $H = H_A + H_F + H_{AF}$, with $H_F = \hbar \omega_1 c_{k_1} \hat{c}_{k_1} + \hbar \omega_2 c_{k_2} \hat{c}_{k_2}$ and

$$H_A = \sum_k \hbar \omega_k \hat{a}^\dagger_k \hat{a}_k$$

$$+ \frac{4\pi\hbar^2 a_s}{2mV} \sum_{k_3,k_4,k_5,k_6} \hat{a}_{k_3}^\dagger \hat{a}_{k_4} \hat{a}_{k_5} \hat{a}_{k_6} \hat{a}_{k_3+k_4+k_5+k_6}$$

(1)

$$H_{AF} = \hbar \Omega c_{k_2}^\dagger \hat{c}_{k_1} \sum_k \left( \hat{a}_{q+k}^\dagger \hat{a}_k + \hat{a}_{-q+k} \hat{a}_k^\dagger \right) + H.c.$$  (2)

where $\hat{c}_k (\hat{a}_k^\dagger)$ represents the annihilation (creation) operator for a laser photon with momentum $k$, $\hat{a}_{q+k} (\hat{a}_{-q+k}^\dagger)$ is the annihilation (creation) operator for atoms with momentum $k$ and frequency $\omega_k = \frac{\hbar k^2}{2m}$; $\Omega = (\hat{E}_1 \hat{d}_{13}^\dagger)(\hat{E}_2 \hat{d}_{32}^\dagger)/\Delta$ is the two-photon Rabi frequency, where $E_{1(2)}$ is the the
pump(probe) field amplitude with frequency \(\omega_{1(2)}\) and the \(\vec{d}_{ij}\) is the electronic transition dipole moment between the states \(|i\rangle\) and \(|j\rangle\) of an atom. Here \(a_s\) is the s-wave scattering length of the atoms and \(V\) is the volume of the condensate.

We assume that zero-momentum \((k = 0)\) condensate state is macroscopically occupied and therefore the atom-atom interaction characterized by s-wave scattering length for a weakly interacting atomic gas is mainly due to the collision between zero- and and non-zero momentum atoms. Applying Bogoliubov's prescription \(\hat{a}_0, \hat{\alpha}_0 \rightarrow \sqrt{N_0}\), while the condensate fraction \(N_0/N\), with \(N\) being the total number of atoms, and the number density \(n_0 = N_0/V\) remaining fixed in the thermodynamic limit, we convert the Hamiltonian \(H\) into a quadratic form \[\hat{a}_k = u_k \hat{\alpha}_k - v_k \hat{\alpha}^+_k \] where \(v_k = (u_k^2 - 1)^{1/2} = \frac{1}{2} \left(\frac{\omega_k + \mu/\hbar}{\omega_k^2 - 1}\right)^{1/2}\) and

\[
\omega_k^B = \left(\omega_k + \frac{\mu}{\hbar}\right)^2 - \left(\frac{\mu}{\hbar}\right)^2 \right)^{1/2},
\] (3)

is energy of Bogoliubov's quasi-particle. Here \(\mu = \frac{\hbar^2 \xi^2}{2m}\) is the chemical potential with \(\xi = (8\pi n_0 a_s)^{-1/2}\) being a characteristic length scale known as the healing length. The condensate ground state energy is \(E_0 = \frac{1}{2} N_0 \mu - \frac{1}{2} \sum_{k \neq 0} v_k^2 E_k\). We thus diagonalize \(H_A\) and rewrite the entire Hamiltonian in terms of Bogoliubov's quasi-particle operators \(\hat{\alpha}_k\). We consider the condensate ground state energy as the zero of the energy scale. By treating the pump light beam classically, the effective Hamiltonian can be written in the standard form

\[
H_{\text{eff}} = \hbar \omega_q^B \left(\hat{\alpha}_q^\dagger \hat{\alpha}_q + \hat{\alpha}_q^\dagger \hat{\alpha}_q - 1\right) - \hbar \delta \hat{\alpha}_k^\dagger \hat{\alpha}_k + \h. c.
\] (4)

where \(\delta = \omega_1 - \omega_2\), \(q = k_1 - k_2\) and \(\eta = \sqrt{N} f_q \Omega\); where \(f_q = u_q - v_q\). In writing Eq.(4), we have retained only two dominant momentum side-modes of the Bragg resonance condition \((\delta \simeq \omega_q)\). Equation (4) involves three coupled operators. The evolution operator exp\((-iH_{\text{eff}} t/\hbar)\) can not be disentangled into those of individual operators, since there exists no such disentanglement theorem for such Hamiltonian. We therefore, emphasize that the Hamiltonian (4) can be solved exactly and more easily in Heisenberg picture. The Heisenberg equations of motion for a triad of operators \(X = (\hat{\alpha}_q, \hat{\alpha}_q^\dagger, \hat{\alpha}_k^\dagger)\) can be written in a matrix form \(\dot{X} = i \omega_q^B \mathbf{M} \dot{X}\). The dynamics of these three coupled modes (two momentum side modes \(\pm q\) plus one probe field mode) is controlled by the eigenvalues of \(\mathbf{M}\) matrix. The long time behavior is oscillatory or growing depending on the momentum transfer \(q\) and the effective coupling constant \(\eta\). For a given value of \(q\), if \(\eta\) exceeds a threshold value \(\eta_h\), the long-time dynamics becomes hyperbolic dominated by the complex eigenvalues. In contrast, if \(\eta\) is less than \(\eta_h\), the long time behavior is oscillatory. Fig.1 shows the dependence of \(\eta_h\) on the momentum \(q\) of the recoiling atoms. In the phonon or quasi-particle regime \((\xi q < 1)\), \(\eta_h\) is much larger compared to that in particle regime \((\xi q >> 1)\). For experimental situation of Ref. [13] with Na condensate, where quasi-particle regime \((\xi q \approx 0.47)\) is probed by Bragg scattering of light, \(\eta_h = 0.314 \omega_q^B\) with \(\omega_q^B = (2\pi) \times 4.7\ kHz\). The value of \(\eta\) used in that experiment is about 150 \(\omega_q^B\).

![FIG. 1. The threshold value of \(\eta (\eta_h/\omega_q^B)\) as a function of \(\xi q\).](image)

To demonstrate many-particle entanglement among two motional (momentum) states of the condensate and one probe field state, following Ref. [10,13], we define two-mode entanglement parameter

\[
\xi_{i,j} = \frac{\langle (\hat{n}_i - \hat{n}_j)^2 \rangle}{\langle \hat{n}_i \rangle + \langle \hat{n}_j \rangle}, \quad i, j = q, -q, k_2
\] (5)

where \(\langle (\hat{n}^2) \rangle = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2\) and \(\hat{n}_{i,j}\) are the two particle number operators \(\hat{\alpha}_q^\dagger \hat{\alpha}_q \hat{\alpha}_q^\dagger \hat{\alpha}_q \hat{\alpha}_k^\dagger \hat{\alpha}_k \hat{\alpha}_k^\dagger \hat{\alpha}_k\) and one photon number operator \(\hat{\alpha}_q^\dagger \hat{\alpha}_q\). The particle operators \(\hat{\alpha}\) are related to the quasi-particle operators \(\hat{\alpha}\) by Bogoliubov’s transformation. If \(\xi_{i,j}\) is below unity, the corresponding two-modes ‘i’, and ‘j’ are entangled. We specifically consider a condensate of \(5 \times 10^6\) Na atoms. First, we examine entanglement by light scattering in phonon regime \((\xi q < 1)\). While choosing the relevant parameters, we followed the experiment of Ref. [13], i.e., chemical potential \(\mu = 6.7\ kHz\), the pump and the probe fields intersecting at an angle \(14^\circ\) transfer a momentum \(q = 0.47\xi^{-1}\) to the atoms. Figure 2 illustrates the different two-mode entanglement parameters as a function of time with probe field being initially in a coherent state \(|\beta\rangle\) with average photon number \(|\beta|^2 = 1.0\). Compared to long-time hyperbolic limit (Fig.2(a)), the oscillatory limit (Fig.2(b))
seems to be more interesting from the two-mode or multimode entanglement viewpoint. As shown by Fig.2(b), entanglement between the \((q, -q)\) and \((q, k_2)\) revives and disappears with time in the long-time limit, while in the hyperbolic limit, entanglement completely vanishes after a certain time (Fig.2(a)). Before vanishing, \(\xi_{q, k_2}\) makes small oscillations eventually reaching a minimum (maximum entanglement). As the coupling \(\eta\) increases, these oscillations become less prominent (inset to Fig.2(a)), while for higher momentum (\(\xi q > 1\)), these oscillations completely die down (inset to Fig.2(b)). The oscillations are caused by the interference of the three closely lying eigenvalues of \(M\). From Fig.2, we infer that although the two condensate side-modes \(q\) and \(-q\) and one side-mode \(q\) and the scattered field mode \(k_2\) are entangled in different time and parameter zones, modes \(-q\) and \(k_2\) does not exhibit any mutual entanglement in any zone. However, by changing the character of the applied probe field, as we show below, it is possible to obtain mutual entanglement among all the three modes. By comparing Fig.2(b) with Fig.3(a) which exhibits \(\xi_{ij}\) when the probe field is in vacuum, we further infer that Bragg scattering with either coherent or vacuum probe field generates almost similar entanglement characteristics in the system. Although there arises mutual entanglement between the modes \((q, -q)\) and \((q, k_2)\) due to either of these two fields, entanglement between \((-q, k_2)\) is not developed in either case. The results depicted in Fig.3(a) may be contrasted with those of Ref. [11] which studied entanglement between \(q\) and \(k_2\) modes in the hyperbolic limit for a vacuum probe. In contrast, the oscillatory limit we describe in the present paper is a new result not discussed before thus far. We emphasize that entanglement in the oscillatory limit is more significant; since it sustains at long time, while in the hyperbolic limit it does not exist at long time. We also point out that these two limits depend on the two parameters \(q\) and \(\eta\), and not due the dynamical processes of the three coupled modes.

Next, we consider the probe field as a one-photon Fock state [4]. In this case, all the three modes exhibit mutual entanglement at different times and duration as is evident from the Fig.3(b). Thus three-mode or tripartite entanglement can be generated in a Bose condensate by Bragg spectroscopy with nonclassical light fields (Fock states) which constitutes another central result of this present paper. It should be pointed out that neither vacuum nor coherent probes are able to generate such tripartite entanglement in any parameter space. Why one-photon field state and neither vacuum nor the coherent state can generate tripartite entanglement may be explained by examining the respective phase-space density distribution functions. Since, coherent state is a displaced vacuum state, both of them have almost similar phase-space structure, except a displacement of the equilibrium position. We have developed a formalism based on time-dependent Wigner distribution functions which provides significant insight into the entanglement properties in various cases discussed in this paper.

![Fig. 2. (a) Two-mode entanglement parameter \(\xi_{q,-q}\) (solid line), \(\xi_{q,k_2}\) (dashed lines) and \(\xi_{-q,k_2}\) (dotted line) as a function of time \(t\) in microsecond (\(\mu s\)). Initially, the condensate is assumed to be in the ground state and the probe field is in a coherent state with average photon number \(\langle n_p\rangle = 1\). The other parameters: \(q = 0.47\xi^{-1}\), \(\omega_q^B = (2\pi) \times 4.7\) kHz, the two-photon Rabi frequency \(\Omega = 8\) Hz, \(\eta/\omega_q^B = 0.34\) corresponding to hyperbolic regime, since \(\eta/k_h/\omega_q^B = 0.313\). The eigenvalues of \(M\) matrix are 1.07975, -0.69788 + 0.14153i and -0.69788 - 0.14153i. (b) Same as in (a), but \(\Omega = 7\) Hz corresponding to oscillatory regime \((\eta/k_h/\omega_q^B = 0.29)\) with eigenvalues 1.06235, -0.79248, and -0.69788. The inset to Fig.4(a) shows the variation of \(\xi_{q,k_2}\) with \(t\) in \(\mu s\) for \(\Omega = 16\) Hz and \(q = 0.47\xi^{-1}\). The inset to (b): \(\xi_{q,k_2}\) Vs \(t\) in \(\mu s\) for \(\Omega = 16\) Hz, \(q = 2\xi^{-1}\). Now, to know the entanglement property in the particle regime \((\xi q > 1)\), we show the \(\xi_{ij}\) as a function of time for \(q = 8.329\xi^{-1}\) in Fig.4. While considering phonon \((\omega_q^B \propto q)\) or particle \((\omega_q^B \propto q^2)\) regime, exact value of \(\omega_q^B\) should be taken, otherwise an approximation in \(\omega_q^B\) may lead to wrong results if \(q\) is close to the threshold value \(\eta_{th}\). Here also we consider a Na condensate of \(5 \times 10^6\) atoms, but the other parameters \((\mu, q)\) are chosen from the experimental paper of Ref. [14]. From Fig.4(a), we observe that before reaching the hyperbolic regime, all the three modes remain entangled for a considerable duration. The inset to Fig.4(a) shows the variation of \(\xi_{ij}\) as a function of the two-photon Rabi frequency \(\Omega\) at time \(t = 10\mu s\). From this figure we conclude that the tripartite entanglement is significant at weak coupling (low \(\Omega\) or low \(\eta\)) near the threshold \((\eta_{th})\). Figure 4(b) exhibits entanglement property in the same particle regime but
in the long-time oscillatory limit. In this case, unlike the other cases, entanglement among the three modes persists for almost all the time, although in the long-time limit the modes \( q \) and \(-q\) (solid curve) are marginally entangled. But, in contrast, the modes \( q \) and \( k_2 \) remain largely entangled for all the time.

FIG. 3. (a) Same as in Fig.2(b), but for the probe field being initially in vacuum (cavity). The inset to Fig.3 (a) shows the \( Q \) parameter \( Q_p \) of the output probe field as a function of time in \( \mu s \). This \( Q \) parameter always remains greater than unity. (b) Same as in (a), but for probe field being initially in one photon Fock state. The inset to Fig.3 (b) shows the corresponding \( Q_p \) as a function of time in \( \mu s \). In this case, the \( Q \) parameter becomes less than unity implying nonclassicality of the output light field.

To see whether the photon number fluctuation of the output (scattered) light field has any connection with the entanglement parameters discussed so far, we calculate the Mandel \( Q \) parameter \( Q_p = \langle (\Delta \hat{n}_{k_2})^2 \rangle / \langle \hat{n}_{k_2} \rangle \). We find that neither a coherent nor a vacuum field as an input probe can generate nonclassical light output, while a nonclassical one-photon Fock state can do so. The time-evolution of the \( Q \) parameter has a strong link with the time-evolution of the entanglement parameter \( \xi_{q,-q} \), i.e., entanglement between the atomic modes \( q \) and \(-q\). With a nonclassical input probe, both the parameters \( Q_p \) and \( \xi_{q,-q} \) become less than unity at the same time (Fig.3 (b) inset) indicating that the entanglement between the two atomic side-modes may be inferred by measuring \( Q_p \) using beam splitter at the output of the probe beam and detecting the photon number fluctuations with two detectors.

In conclusion, we have presented an exact treatment of the dynamics of excitations in a condensate by Bragg scattering of light under Bragg resonance condition. We have shown that by using a nonclassical light (Fock states) as input probe, it is possible to generate tripartite entanglement in the many-particle states of three motional modes which include two momentum side-modes of the condensate and one probe field mode. The relevant physical parameter regimes where such tripartite entanglement can be observed are analyzed. The important question which needs detailed study is how to detect the generated entanglement. Our results suggest that by measuring the second order correlation function of the scattered field, it is possible to measure the entanglement between the two modes of the condensate. The number variance of the \( q \) and \(-q\) momentum atoms may also be measured by outcoupling them with a change in the hyperfine spin state by applying Doppler-sensitive two radio frequency pulses and then by measuring the intensity variation at the output of the two pulses.

FIG. 4. (a) Same as in Fig.3(b), but \( \mu = (2\pi) \times 1.23 \text{ kHz} \), \( q = 8.329 \xi^{-1} \), \( \omega_B^q = (2\pi) \times 86.65 \text{ kHz} \), \( \Omega = 7 \text{ Hz} \), \( \eta = 0.0285 \omega_B^q \) and \( \eta_h = 0.0071 \omega_B^q \). The eigenvalues are 1.00041, -0.99315 + 0.02771i, -0.99315 - 0.02771i. (b) Same as in (a), but \( \Omega = 1 \text{ Hz} \) corresponding to oscillatory regime (\( \eta = 0.0041 \omega_B^q \)) with eigenvalues 1.00001, -0.99970 - 0.9872i. The inset to Fig.(a) shows the variation of \( \xi_{ij} \) with two-photon Rabi frequency \( \Omega \) in Hz at time \( t = 10 \mu s \) with other parameters remaining the same as in the main figure of (a).
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