Constitutive Relations for the Thermoviscoelasticity of Diphase Medium with a Kelvin-Voigt Rheological Skeleton

Monika Bartlewska-Urban 1, Tomasz Strzelecki 2

1 Wroclaw University of Science and Technology, Faculty of Geoengineering, Mining and Geology, 15 Na Grobli st. 50-421 Wroclaw, Poland
2 Wroclaw University of Science and Technology, Faculty of Technology and Natural Sciences, 9 S. Batory st. 59-200 Legnica, Poland

monika.bartlewska@pwr.wroc.pl

Abstract. Constitutive relations for a diphase medium consisting of a viscoelastic Kelvin-Voigt skeleton and a compressible Newtonian liquid filling the pores of the medium were derived from the equations of irreversible thermodynamics, assuming the thermoviscoelasticity process to be non-isothermal. The rate of change in internal energy in the two phases of the diphase medium was determined from the first law of irreversible thermodynamics. Physical relations for the solid phase and the liquid filling the pores of the medium, and an equation of entropy were determined using Helmholtz free energy and the second law of thermodynamics for open systems. The temperature of the medium’s two phases in the heat conduction process was assumed to be identical for the representative elementary volume (RVE). Stresses, related to the total surface of RVE, for the diphase medium solid phase and liquid phase, were defined according to the Darcy-Biot consolidation theory. The derived constitutive relations and the heat conduction equation are linear since the expansion of the function of Helmholtz free energy into a Taylor series was limited to the second order. In reality, however, as the temperature increases, the model becomes nonlinear. Therefore, the applicability of the proposed model is limited to certain intervals of change in the temperature of the solid body and the liquid. The constitutive relations for the elastic skeleton come down to relations for the thermal consolidation of the Biot-Darcy body.

1. Introduction

The aim of this study was to develop constitutive relations for a diphase medium consisting of a solid phase, represented by the viscoelastic medium of the Kelvin-Voight body, and a compressible Newtonian liquid flowing through the pores of the solid phase. Constitutive relations for a diphase medium consisting of an elastic skeleton and a compressible liquid, for the isothermal process were first given by Biot [1], who then published them in a modified form in [2], indicating that they satisfied Onsager’s postulate. The physical relations given by Biot were:

\[ \sigma_{ij} = 2N\varepsilon_{ij} + (A\varepsilon + Q\theta)\delta_{ij} \]
\[ \sigma = Q\varepsilon + R\theta \]
where: $\sigma_{ij}$ stands for the fuzzy stress tensor components in the solid phase and $\sigma$ represents the fuzzy stress in the liquid, expressed through pressure $p$ in the liquid by the formula:

$$\sigma = -f_\Omega p$$

(2)

where $f_\Omega$ stands for the volume porosity of the solid phase, $\varepsilon_{ij}$ represents the strain tensor components of the diphase medium solid phase, $\varepsilon$ is the dilatation of the skeleton and $\Theta$ – the dilatation of the liquid, N, A stand for the Biot body elasticity constants related to the fuzzy stresses, corresponding to Lamé constants $\lambda$ and $\mu$ in the theory of elasticity, constant Q represents the influence of volumetric strain $\Theta$ of the liquid on the stress tensor in the solid phase of the medium or the influence of skeleton dilatation $\varepsilon$ on stress $\sigma$ in the liquid (thus constant Q is a parameter representing the coupling between the two phases of the medium) and R defines the compressibility of the liquid phase.

In 1972 in Derski’s work [3] constitutive relations for the Biot body were determined on the basis of the irreversible thermodynamics theorems, assuming a constant temperature. Many authors have addressed the problem of building a rheological diphase medium model, assuming the presence of a scale of heterogeneity, on the basis of the homogenization theory theorems. Using the asymptotic homogenization method, constitutive relations were derived in the form of relations (1) in work [4] by Auriault and Sanches Palenzia and in Auriault’s work [5], which also included a methodology for determining effective model parameters when the geometry of the medium at a pore scale is known.

A model of the consolidation of the Biot body with the Kelvin-Voight rheological skeleton was the subject of M. Bartlewska’s PhD thesis [6] in which the author on the basis of the irreversible thermodynamics theorems derived constitutive relations for a diphase rheological medium. In [6] she determined the effective parameters of a mathematical model proposed for isothermal processes in the soil medium. The proposed model of a two-phase medium has a significant advantage over Biot’s model. The solutions based on classic Biot’s model are characterized by the occurrence of instantaneous strains, which, depending on the effective parameters of the soil skeleton, often amount to over 50% of the total strains in the medium. Experiments conducted on diphase media (e.g. soils) show that from the instant of load application creep takes place and no instantaneous strains occur. The model proposed by Bartlewska [6] eliminates instantaneous strains and creep, whereby the creep process, which can be related to the creep of soil, better represents the reality.

In this paper constitutive relations are derived for the Biot body with a rheological skeleton, for non-isothermal processes, using irreversible thermodynamics laws for open thermodynamic systems.

2. Assumptions of thermoconsolidation theory

According to M.A. Biot’s initial assumptions, the medium consisted of a porous solid forming a continuous medium. The theory assumed that it was a diphase medium consisting of a porous elastic skeleton and a compressible Newtonian liquid filling the pores of the skeleton and that the medium was homogenous and isotropic. It was also assumed that the deformations of the skeleton were small whereby the nonlinear members of Green strain tensor $\varepsilon_{ij}$ could be omitted. Thus, it follows that:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

(3)

Let us assume that $\Omega$ is a REV filled with a diphase medium consisting of a porous skeleton described by the Kelvin-Voight rheological model, and a Newtonian liquid filling its pores, $S$
represents the surface bounding spatial element $\Omega$, vector $\vec{n}$ is a unit vector normal to surface $S$, directed outwards of element $\Omega$, $\vec{v}_2$ and $\vec{v}_1$ stand for the filtration rate vectors of respectively the liquid and the skeleton, and $v'_i = v_{2i} - v_{1i}$ represents the relative rate of the filtration flow of the liquid through the porous medium. If $\rho_s$ and $\rho_l$ stand for the specific density of respectively the skeleton and the liquid, then one can calculate skeleton density $\rho_1$ and liquid density $\rho_2$, each of them related to the total volume of area $\Omega$: $\rho_1 = (1 - f_A)\rho_s$ and $\rho_2 = f_A\rho_l$. The diphase medium density equal to the sum $\rho = \rho_1 + \rho_2$ is denoted as $\rho$ while $\overline{\rho}$ represents the density of the liquid flowing through wall $S$: $\overline{\rho} = f_A\rho_1$, where $f_A$ stands for surface porosity.

It is also assumed that the permeability coefficient and the porosity coefficient are constant in the course of consolidation and do not depend on the state of strain.

In Biot’s work [2] one can find the following definition of the kinetic energy of the Biot diphase body:

$$2K = \int_{\Omega} [\rho_1v_{1i}v_{1i} + 2\rho_{12}v_{1i}v_{2i} + \rho_{22}v_{2i}v_{2i}]d\Omega$$  (4)

where $\rho_1 + \rho_{12} = \rho_1 > 0; \rho_{22} + \rho_{12} = \rho_2 > 0; \rho_{12} < 0$.

The momenta of the medium’s two phases are expressed by the formulas:

$$P_{1i} = \frac{\partial K}{\partial v_{1i}} = \int_{\Omega} (\rho_{11}v_{1i} + \rho_{12}v_{2i})d\Omega$$  (5)

$$P_{2i} = \frac{\partial K}{\partial v_{2i}} = \int_{\Omega} (\rho_{12}v_{1i} + \rho_{22}v_{2i})d\Omega$$

According to Strzelecki’s work [7], the function of energy dissipation during the flow of a liquid through a porous medium is expressed by the quadratic form:

$$2W_d = \int_{\Omega} b v'_i v'_i d\Omega$$  (6)

where $b$ stands for the viscosity resistance of the liquid flowing through the soil.

Using the energy dissipation function, one can determine the volume forces of the interaction between the two phases of the medium, resulting from the viscosity resistance of the liquid. The forces are expressed by the formulas:

$$M_{1i} = \frac{\partial W_d}{\partial v_{1i}} = -\int_{\Omega} b v'_i d\Omega$$  (7)

$$M_{2i} = \frac{\partial W_d}{\partial v_{2i}} = \int_{\Omega} b v'_i d\Omega$$

Let us assume that the consolidation process satisfies the Kelvin-Voight model assumptions, then:

$$\sigma_{ij} = \sigma_{ij}^s + \sigma_{ij}^f$$

$$\varepsilon_{ij} = \varepsilon_{ij}^s = \varepsilon_{ij}^f$$  (8)
which, according to Bartlewska’s work [6], leads to the equation:

$$\sigma_{ij} = (1 + \tilde{C} \Psi) \sigma_{ij}^s$$  \hfill (9)

where $\tilde{C}$ is a factor of proportionality, $\Psi$ is differential operator $\frac{\partial}{\partial t}$, $\sigma_{ij}^s$ stand for an elastic stress tensor and $\sigma_{ij}^l$ is a viscous stress tensor equal to:

$$\sigma_{ij}^l = \tilde{C} \Psi \sigma_{ij}^s$$  \hfill (10)

3. Equations of flow continuity

According to work [8] by Strzelecki et al., the equation of continuity of a porous medium (skeleton + liquid) has the following form:

$$\rho \frac{D^l \rho}{D t} + \rho \dot{\varepsilon} = -(\bar{\rho} \dot{v}_{li})_{ri}$$  \hfill (11)

where $\frac{D^l \rho}{D t}$ is a material derivative equal in value to $\frac{\partial \rho}{\partial t} + v^l \frac{\partial \rho}{\partial x_i}$. The liquid continuity equation can be presented in the form:

$$\rho \frac{D^l \rho}{D t} + \bar{\rho} \dot{\varepsilon} = -(\bar{\rho} \dot{v}_{li})_{ri}$$  \hfill (12)

where $\frac{D^l \rho}{D t}$ is a material derivative equal in value to $\frac{\partial \bar{\rho}}{\partial t} + v_{li} \frac{\partial \bar{\rho}}{\partial x_i}$.

4. Equation of momentum conservation

According to [5], the momentum conservation equation for the solid phase of a diphase medium leads to the following soil skeleton motion equation:

$$\sigma_{ij,ij} + b \dot{v}_{ij} + (\rho - \bar{\rho}) X_i = \rho_{11} \frac{\partial v_{li}}{\partial t} + \rho_{12} \frac{\partial v_{2i}}{\partial t}$$  \hfill (13)

For quasistatic processes the forces of inertia are omitted and the motion equation can be written in the following simplified form:

$$\sigma_{ij,ij} + (\rho - \bar{\rho}) X_i = -b \dot{v}_{ij}$$  \hfill (14)

Having the momentum conservation equation for the liquid phase one can determine the following motion equation for this phase:

$$\sigma_{li} - b \dot{v}_{li} + \bar{\rho} X_i = \rho_{12} \frac{\partial v_{li}}{\partial t} + \rho_{22} \frac{\partial v_{2i}}{\partial t}$$  \hfill (15)

For quasistatic processes the terms representing the forces of inertia are omitted and the motion equation for the liquid phase is written as:

$$\sigma_{li} + \bar{\rho} X_i = -b \dot{v}_{li}$$  \hfill (16)
Equation (16) can be presented in the form of Darcy’s equation:

\[ v'_i = -kH_i \]  

where \( k \) stands for Darcy’s coefficient of permeability and \( H \) is the hydraulic head expressed by the formula:

\[ H = -\frac{\sigma}{f_{\Omega}\rho g} + x_i\delta_{i\lambda} \]  

In the case of quasistatic processes, the system of motion equations for both phases of the porous medium has the form:

\[
\begin{cases}
\sigma_{ij,j} + (\rho - \bar{\rho})X_i = -bv'_i \\
\sigma_{i\lambda} + \bar{\rho}X_i = bv'_i
\end{cases}
\]  

In the literature on Biot’s theory [1] the system of motion equations for a diphase medium is written differently, i.e.

\[
\begin{cases}
\sigma_{ij,j} + \sigma_{i\lambda} + \rho X_i = 0 \\
\sigma_{i\lambda} + \rho_2X_i = bv'_i
\end{cases}
\]  

In the above equation it was assumed, that \( f_{\Omega} \approx f_{s} \) and so one can assume that \( \bar{\rho} = \rho_2 \).

5. First law of thermodynamics
The first law of thermodynamics can be written in the form:

\[
\dot{L} + \dot{Q} = \frac{\partial}{\partial t}(W + K)
\]  

where: \( \dot{L} \) is a sum of the work done by the internal forces, the gravity forces and the forces generated by the viscosity resistance of the flowing liquid, \( \dot{Q} \) stands for the change of heat over time, \( W \) and \( K \) express respectively internal energy and kinetic energy. The first law of thermodynamics was written for each phase separately, using index 1 for the solid phase and index 2 for the liquid phase.

The power of the surface forces, the gravity forces and the viscous resistance carried by the skeleton of a porous medium was written as:

\[
\dot{L}_s = \int_S (\sigma_{ij} + \sigma\delta_{ij})v_i n_j dS + \int_\Omega \rho_i X_i v_i d\Omega - \int_\Omega bv'_i v_i d\Omega
\]  

The rate of change of heat in the skeleton of the medium is expressed by the formula:

\[
\dot{Q}_s = -\int_\Omega q_{hi,i} d\Omega
\]  

where \( q_{hi} \) represents the components of the heat flux flowing through the skeleton.

The partial derivative of the internal energy of the medium’s solid phase with respect to time is defined as:
where $\hat{W}_1$ stands for the rate of change of the local internal energy of the skeleton. The partial derivative of the kinetic energy is:

$$\frac{\partial K_1}{\partial t} = \int \left[ \rho_1 \frac{\partial v_{1i}}{\partial t} v_{1i} + \rho_{12} \frac{\partial v_{2i}}{\partial t} v_{1i} \right] d\Omega$$

(25)

Taking into account solid phase motion equation (13), the first law of the thermodynamics can be presented in the form of a local relation expressing the quantity of free energy with respect to the solid phase of the medium:

$$\hat{w}_1 = \left( \sigma_{ij} + \delta_{ij}\sigma \right) \hat{e}_{ij} + \sigma_{vi} v_{1i} - q_{1i,1i}$$

(26)

For the liquid phase the power of the surface forces, the gravity forces and the viscosity of the porous medium can be written as:

$$L_\omega = \int (\sigma_{ij} v'_{ij} n_j) dS + \int \rho_i X v_{2i} d\Omega + \int b v'_{ij} v_{2i} d\Omega$$

(27)

The rate of change of heat in the liquid phase of the medium is expressed by the formula:

$$\dot{Q}_2 = -\int q_{2i,1i} d\Omega$$

(28)

where $q_{2i,1}$ represents the components of the heat flux flowing through the liquid.

The partial derivative of the internal energy of the liquid with respect to time in area $\Omega$ of the medium is defined as:

$$\frac{\partial W_2}{\partial t} = \int \hat{w}_2 d\Omega$$

(29)

where $\hat{W}_2$ stands for the rate of change of the local internal energy of the liquid. The derivative of the kinetic energy with respect to time is:

$$\frac{\partial K_2}{\partial t} = \int \left[ \rho_{12} \frac{\partial v_{1i}}{\partial t} v_{2i} + \rho_{22} \frac{\partial v_{2i}}{\partial t} v_{2i} \right] d\Omega$$

(30)

Taking into account liquid phase motion equation (15), the first law of thermodynamics can be presented in the form of a local relation expressing the quantity of free energy with respect to the liquid phase of the medium:

$$\hat{w}_2 = -\sigma_{vi} v_{1i} + \sigma \left( \dot{e} - \hat{e} \right) - q_{2i,1i}$$

(31)

For a diphase medium the rate of change of internal energy ($\hat{\omega}$) is equal to the sum of the rates of change of each of the phases ($\hat{W}_1$ and $\hat{W}_2$) of the medium, and the total heat flux expressed through components $q_i$ is equal to the sum of component heat fluxes $q_{1i}, q_{2i}$ for each of the phases. Thus

$$\hat{\omega} = \sigma_{ij} \hat{e}_{ij} + \sigma \dot{\rho} - q_{1i,1i}$$

(32)
Taking relations (9) and (10) into account in formula (32) one can write:

\[ w = \sigma_j^i \dot{e}_{ij} + \sigma_j^i \ddot{e}_{ij} + \sigma^i \dot{q}, \]

(33)

The first law of thermodynamics defines the relationship between mechanical work and heat and it does not impose any constraints on the direction of the change in a body’s state. The irreversibility of the thermodynamic process means no possibility of return of the system and its environment to the initial state. In thermodynamics in order to describe irreversible phenomena the following function of the rate of change in the state called entropy is introduced:

\[ \dot{S} = \int_{\Omega} \frac{q}{T} d\Omega = \int_{\Omega} \dot{S} d\Omega \]

(34)

Since in the general case, the body can exchange heat with the environment, the rate of change in entropy is the sum of the changes in external entropy \( S_z \), resulting from the exchange of heat with the environment, and the rate of change in internal entropy \( S_w \), which can be written as follows:

\[ \dot{S} = \dot{S}_z + \dot{S}_w \]

(35)

The entropy connected with the exchange with the environment is expressed by the relation:

\[ \dot{S}_z = \int_{\Omega} \left( \frac{q}{T} \right)_z d\Omega \]

(36)

which comes down to the following local equation:

\[ \dot{S}_z = \left( \frac{q}{T} \right)_z \]

(37)

It follows from relation (37) that entropy \( S_z \) can be positive, negative or equal to zero, depending on the direction of flow of the heat and its divergence. The situation is different in the case of internal entropy \( S_w \). Its change in a system unit volume must satisfy the Clausius-Duhem inequality, which in the local notation can be presented as follows:

\[ \dot{S}_w = \int_{\Omega} \frac{q}{T^2} d\Omega \geq 0 \]

(38)

Let us assume that the temperature of the skeleton of the diphasic medium is different from that of the liquid flowing through the pores. Denoting the absolute temperature of the diphasic medium as \( T \), the rate of change of entropy can be expressed by the formula:

\[ \dot{S} = -\frac{q}{T} \]

(39)

Using the Clausius-Duhem relation for the considered medium:

\[- \frac{q}{T^2} \geq 0 \]

(40)

one gets the heat conduction law in the form:
\[ q_i = -\lambda_{ij} T_j \]  

Helmholtz free energy

Let us introduce Helmholtz free energy for the diphase medium, in the form:

\[ F = W - ST \]  

and assume that Helmholtz energy is expressed through local free energy by the formula:

\[ F = \int_{\Omega} \chi d\Omega \]  

The rate of change in local free energy can be expressed by the formula:

\[ \dot{\chi} = \dot{w} - s \dot{T} - \dot{s} T \]  

Substituting formula (33) into formula (44) one gets:

\[ \sigma_{ij} \dot{\varepsilon}_{ij} + \sigma \dot{\theta} - s \dot{T} - \dot{\chi} = 0 \]  

Assuming then that locally free energy is a function of solid phase strain tensor \( \varepsilon_{ij} \), liquid dilatation \( \theta \) and the absolute temperatures of the solid phase and the liquid phase one can write:

\[ \chi = \chi(\varepsilon_{ij}, \dot{\varepsilon}_{ij}, \theta, T) \]  

Having determined the total differential of function \( \chi \), one gets:

\[ \left( \sigma_{ij}' - \frac{\partial \chi}{\partial \varepsilon_{ij}} \right) \dot{\varepsilon}_{ij} + \left( \sigma_{ij}' - \frac{\partial \chi}{\partial \dot{\varepsilon}_{ij}} \right) \dot{\varepsilon}_{ij} + \left( \sigma - \frac{\partial \chi}{\partial \theta} \right) \dot{\theta} + \left( -s - \frac{\partial \chi}{\partial T} \right) \dot{T} \geq 0 \]  

The above relation leads to:

\[ \sigma_{ij}' = \frac{\partial \chi}{\partial \varepsilon_{ij}}, \quad \sigma_{ij}' = \frac{\partial \chi}{\partial \dot{\varepsilon}_{ij}}, \quad \sigma = \frac{\partial \chi}{\partial \theta}, \quad -s = \frac{\partial \chi}{\partial T} \]  

Let us now expand the function of local free energy in the neighbourhood of a natural state, i.e. a state when \( \varepsilon_{ij} = 0, \theta = 0, T = T_0 \). Temperature \( T_0 \) is the natural state absolute temperature for the skeleton and the fluid, which can be arbitrarily assumed to be equal to, e.g., \( 0^\circ K \). The temperature of the two phases of the medium can be presented as the sum \( T = T_0 + \theta \) where \( T_0 \) is a temperature arbitrarily assumed as the reference temperature, e.g. \( 273,15^\circ K \). Expanding free energy function \( \chi \) in the neighbourhood of the natural state one gets:
\( \chi(e_{ij}, \dot{e}_{ij}, \theta, T) = \chi(0, 0, 0, T_0) + \frac{\partial \chi(0, 0, 0, T_0)}{\partial e_{ij}} e_{ij} + \frac{\partial \chi(0, 0, 0, T_0)}{\partial \dot{e}_{ij}} \dot{e}_{ij} + \frac{\partial \chi(0, 0, 0, T_0)}{\partial \theta} \theta + \frac{\partial \chi(0, 0, 0, T_0)}{\partial T} T + \frac{1}{2} \left( \frac{\partial^2 \chi(0, 0, 0, T_0)}{\partial e_{ij} \partial e_{kl}} e_{ij} e_{kl} + \frac{\partial^2 \chi(0, 0, 0, T_0)}{\partial \dot{e}_{ij} \partial \dot{e}_{kl}} \dot{e}_{ij} \dot{e}_{kl} \right) \right.

\left. + \frac{\partial^2 \chi(0, 0, 0, T_0)}{\partial \theta \partial \dot{e}_{ij}} \dot{e}_{ij} \theta + \frac{\partial^2 \chi(0, 0, 0, T_0)}{\partial T \partial \dot{e}_{ij}} \dot{e}_{ij} T \right] + \ldots \tag{48}

Assuming that in the natural state

\[ \chi(0, 0, 0, T_0) = 0, \quad \frac{\partial \chi(0, 0, 0, T_0)}{\partial e_{ij}} = 0, \quad \frac{\partial \chi(0, 0, 0, T_0)}{\partial \theta} = \sigma_a, \quad \frac{\partial \chi(0, 0, 0, T_0)}{\partial T} = 0 \tag{49} \]

the free energy function can be expressed by the formula:

\[ 2 \chi(e_{ij}, \dot{e}_{ij}, \theta, T) = \sigma_a^* \theta + c_{ijkl} e_{ij} e_{kl} + d_{ijkl} \dot{e}_{ij} \dot{e}_{kl} + a_2 \theta \theta + \alpha_s T T + 2 e_{ijkl} \dot{e}_{ij} \dot{e}_{kl} + 2 \alpha_y \theta \dot{e}_{ij} + 2 \gamma_y T \dot{e}_{ij} + 2 \alpha_s T \theta \]

\[ + 2 \alpha_y \theta \dot{e}_{ij} + 2 \beta_y \theta \dot{e}_{ij} + 2 \alpha_s T \dot{e}_{ij} + 2 \gamma_y T \dot{e}_{ij} + 2 \alpha_y T \theta \]

\[ + 2 \alpha_y \theta \dot{e}_{ij} + 2 \beta_y \theta \dot{e}_{ij} + 2 \alpha_s T \dot{e}_{ij} + 2 \gamma_y T \dot{e}_{ij} + 2 \alpha_y T \theta \]

\[ + \frac{\partial \chi(0, 0, 0, T_0)}{\partial \theta} \theta + \frac{\partial \chi(0, 0, 0, T_0)}{\partial T} T \theta \tag{50} \]

Using the above formula and relations (47) one finally gets:

\[ \sigma_{ij} = \frac{\partial \chi}{\partial e_{ij}} = c_{ijkl} e_{ij} e_{kl} + d_{ijkl} \dot{e}_{ij} \dot{e}_{kl} + \beta_{ij} \theta + \gamma_{ij} T \]

\[ \sigma_{ij} = \frac{\partial \chi}{\partial \dot{e}_{ij}} = d_{ijkl} e_{ij} e_{kl} + e_{ijkl} \dot{e}_{ij} \dot{e}_{kl} + \alpha_{ij} \theta + \alpha_{ij} T \]

\[ \sigma - \sigma_a = \frac{\partial \chi}{\partial \theta} = \beta_{ij} e_{ij} + \alpha_{ij} \dot{e}_{ij} + \alpha_{ij} \theta + \alpha_{ij} T \]

\[ -s = \frac{\partial \chi}{\partial T} = \gamma_{ij} e_{ij} + \beta_{ij} \dot{e}_{ij} + \alpha_{ij} \theta + \alpha_{ij} T \tag{51} \]

It follows from the properties of the linear elastic skeleton that:

\[ e_{ijkl} = 0 \tag{52} \]

Let us assume that for a Newtonian liquid viscous stresses do not depend on the velocity of the strain tensor. Therefore the fourth-order tensor:

\[ \alpha_{ij} = \theta_{ij} = 0 \tag{53} \]
Considering that only the volumetric strains of the diphase medium skeleton have an influence on the magnitude of the pressure in the pores and on the absolute temperature of the liquid, it can be assumed that \( \beta_{ij} = \beta \delta_{ij} \) and \( \gamma_{ij} = \gamma \delta_{ij} \), whereby the following simplified physical relations are obtained:

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl} + d_{ijkl} \dot{\varepsilon}_{kl} + (\beta \theta + \gamma T) \delta_{ij}
\]

\[
\sigma = \beta \varepsilon + a_2 \theta + a_1 T
\]

\[-s = \gamma \varepsilon + a_1 \theta + a_3 T\]  

\( (54) \)

For an isotropic homogenous solid phase, both the fourth-order elasticity tensor and the viscosity tensor of the skeleton are expressed by two constants. In the case of the elasticity tensor, they are Lamé constants, which using Biot’s denotations are expressed by the formula:

\[
c_{ijkl} = A \delta_{ij} \delta_{kl} + N (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]  

\( (55) \)

In the case of the viscosity tensor of the solid phase, they are non-dilatational viscosity constant \( \eta_s \) and volume viscosity constant \( \lambda_s \), which is expressed by the formula:

\[
d_{ijkl} = \lambda_s \delta_{ij} \delta_{kl} + \eta_s (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]  

\( (56) \)

Introducing Biot’s denotations \( \beta = Q \) and \( a_2 = R \) into relations \( (54) \) one gets the following constitutive relations together with the equation of entropy:

\[
\sigma_{ij} = 2N \varepsilon_{ij} + A \varepsilon_{ij} + 2 \eta_s \delta_{ij} + \lambda_s \dot{\varepsilon}_{ij} + (Q \theta + \gamma T) \delta_{ij}
\]

\[
\sigma = Q \varepsilon + R \theta + a_1 T
\]

\[-s = \gamma \varepsilon + a_1 \theta + a_3 T\]  

\( (57) \)

Let us then consider the physical sense of material constants \( \gamma, a_1, a_3 \). By contracting the first formula in relations \( (51) \) one gets the formula:

\[
3 \sigma^*_{m} = (2N + 3A) \varepsilon + 3Q \theta + 3\gamma T
\]  

\( (58) \)

Thus, one gets the mean stress prevailing in the skeleton, in the form:

\[
\sigma^*_{m} = K \varepsilon + Q \theta + \gamma T
\]  

\( (59) \)

where \( K = A + 2N / 3 \) is the volume elasticity modulus of the porous medium skeleton. If it is assumed that the only cause of the deformations of the liquid and the solid is temperature \( T \), then \( \sigma^*_{m} = 0 \) and one can write:

\[
\varepsilon = 3 \alpha_1 T
\]

\[
\theta = \alpha_{21} T
\]  

\( (60) \)

Hence one gets:

\[
\gamma = -(3K \alpha_1 + Q \alpha_{21})
\]  

\( (61) \)

where \( \alpha_1 \) is the linear expansion coefficient of the solid phase of the medium and \( \alpha_{21} \) is a coefficient representing the influence of the thermal expansion of the skeleton on the dilatation of the liquid. The
above formula differs from a similar formula derived by O. Coussy in [9]. According to the latter work, $\gamma$ depends solely on the linear thermal expansion of the skeleton, which means that the Biot model’s classic phase coupling coefficient $\alpha_{21}$ does not occur here. Let us consider now the second physical relation for a case when temperature is the only cause of deformations. Then $\sigma = 0$ and it is found that:

$$\varepsilon = \alpha_{21} T$$  
$$\theta = \alpha_{21} T$$  \hspace{1cm} (62)

and one gets the value of constant $a_1$ in the form:

$$a_1 = -\left( Q\alpha_{21} + R\alpha_{21} \right)$$  \hspace{1cm} (63)

where $\alpha_{21}$ is the thermal expansion coefficient of the liquid. Both $\gamma$ and $a_1$ are negative.

6. Equation of heat conduction

Let us consider now the third of equations (57), which can be written as:

$$-s = \gamma \varepsilon + a_1 \theta + a_3 T$$  \hspace{1cm} (64)

Let us perform the operation of differentiation over time on the above equation and multiply both its both sides by $T$:

$$T\dot{s} = -\gamma \dot{\varepsilon} T - a_1 \dot{\theta} T - a_3 T \dot{T}$$  \hspace{1cm} (65)

Quantity $T\dot{s}$ is a measure of the heat generated in a unit area in the course of the temperature change process when the strain of the consolidating medium is constant and so skeleton dilatation rate $\dot{\varepsilon}$ and liquid dilatation rate $\dot{\theta}$ are equal to zero. Thus, at a unit temperature change rate, $a_3$ is:

$$a_3 = -\frac{\rho c_v}{T}$$  \hspace{1cm} (66)

Therefore the equation of entropy can be written as:

$$\dot{T}s = -T \gamma \dot{\varepsilon} - Ta_1 \dot{\theta} + \rho c_v \dot{T}$$  \hspace{1cm} (67)

Since

$$\dot{T}s = -q_{i\gamma}$$  \hspace{1cm} (68)

and using the Fourier heat conduction law:

$$q_{i\gamma} = -\theta T_j$$  \hspace{1cm} (69)

where $\theta$ is the coefficient of heat conduction through the diphasic medium, the equation of entropy can be written in the form of the following heat conduction equation:

$$\text{div}(\text{grad}(T)) = P_1 \dot{\varepsilon} + P_2 \dot{\theta} + \kappa \dot{T}$$  \hspace{1cm} (70)

where:

$$P_1 = \frac{(3K\alpha_{11} + Q\alpha_{21}) T_0}{\gamma}$$  
$$P_2 = \frac{(Q\alpha_{21} + R\alpha_{21}) T_0}{\gamma}$$  
$$\kappa = \frac{\rho c_v}{\gamma}$$
and $T_0 = 273^\circ K$.

The constitutive relations for a non-isothermal process have the form:

\[
\sigma_{ij} = 2N(1+C\Psi)e_{ij} + A(1+C_1\Psi)e\delta_{ij} + (Q\theta + \gamma T)\delta_{ij}
\]
\[
\sigma = Q\varepsilon + R\theta + a_iT
\]

where $C = \frac{n}{N}$ and $C_1 = \frac{\lambda}{A}$, while the equation of heat conduction through the diphase medium reduces itself to:

\[
T_{ij} = P^i_{j} + P^j_{i} + \kappa\dot{T}
\]

7. Conclusion

Starting from the laws of irreversible thermodynamics constitutive relations and a heat conduction equation have been derived for a diphase medium consisting of a viscoelastic skeleton and a Newtonian liquid filling the pores of the skeleton. The derived relations and the equation have a linear character and are applicable to a limited range of temperatures.

For quasistatic processes the proposed model of the thermal consolidation of the diphase medium can be presented in the form of a system of five differential equations, including three displacement equations of consolidation, an equation of the flow of the liquid through a porous medium and a heat conduction equation. The model has the form:

\[
\begin{align*}
\sigma_{ij}^{\alpha} + \left( \sigma \delta_{ij} \right)_{ij} + \rho \left( X \dot{\delta}_{ij} \right)_{ij} &= 0 \\
\text{div}(\text{grad}(k,\sigma)) &= (\dot{\theta} - \dot{\varepsilon}) \\
\text{div}(\text{grad}(T)) &= P^i_{j} + P^j_{i} + \kappa\dot{T}
\end{align*}
\]

When isothermal processes are considered, and zero skeleton viscosity values are assumed, the above model comes down to the classic Biot-Darcy model. In the case of higher temperatures and pressures one should take into account the phase transition of the liquid into a gas, whereby the above system of equations with the phase transition taken into account becomes nonlinear.

Acknowledgment

Authors wishing to acknowledge financial support from statutory research: 0401/0127/17.

References

[1] Biot M. A., “General Theory of Three-Dimensional Consolidation,” J. Appl. Physics, 12, p. 155-164, 1941
[2] Biot M.A., “General Solutions of the Equations of Elasticity and Consolidation of a Porous Material”, Journal of Applied Mechanics, vol. 78, pp. 91-96, 1956
[3] Derski W., “Law of behavior in the theory of consolidation,” Studia Geotechnica, Vol. III, no.1, 1972 (in French)
[4] Auriault J.L., Sanchez Palencia E. “Study of the macroscopic behavior of a deformable saturated porous medium”, J. de Mecanique,16, 4, p. 575-603, 1977 (in French)
[5] Auriault J.L., “Behaviour of Porous Saturated Deformable Media in Geomaterials: Constitutive Equations and Modelling”, Ed. Darv. F, Elsevier Applied Science, London, Chap. 14, p. 311-328, 1990
[6] Bartlewska M., PhD thesis, “Determination of the effective parameters of rheological models of cohesive soils” (in Polish), Wroclaw University of Technology, Faculty of Geoengineering,
Mining and Geology, Wrocław, 2009
[7] Strzelecki T., “Law of behavior in the theory of electrohydrodynamics consolidation,” Studia Geotechnica et Mechanica, vol. 1, No. 3-4, p. 39-49, 1979 (in French)
[8] Strzelecki T., Kostecki S., Żak S., “Modelling flows through porous media” (in Polish), DWE, 2008.
[9] Coussy O., “Mechanics and Physics of Porous Solids”, John Wiley&Sons, 2011