Large-eddy simulations of adverse pressure gradient turbulent boundary layers

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Abstract. Adverse pressure-gradient (APG) turbulent boundary layers (TBL) are studied by performing well-resolved large-eddy simulations. The pressure gradient is imposed by defining the free-stream velocity distribution with the description of a power law. Different inflow conditions, box sizes and upper boundary conditions are tested in order to determine the final set-up. The statistics of turbulent boundary layers with two different power-law coefficients and thus magnitudes of adverse pressure gradients are then compared to zero pressure-gradient (ZPG) data. The effect of the APG on TBLs is manifested in the mean flow through a much more prominent wake region and in the Reynolds stresses through the existence of an outer peak. The pre-multiplied energy budgets show, that more energy is transported from the near-wall region to farther away from the wall.

1. Introduction
Due to the difficulty of finding analytical solutions to the Navier–Stokes equations in turbulent flows, numerical simulations are performed from which relevant physics can be extracted and understood. Complex flows can then be reduced to canonical cases, which give the possibility to general statements and laws. A number of different flow types has been widely investigated, of which some are closer to industrial applications such as of pipe flow, while others are more idealized configurations which provide deeper understanding of some particular flow features. This is the case, for instance, of the zero pressure-gradient (ZPG) turbulent boundary layer (TBL). A less investigated case is the TBL with pressure gradient (PG). This type of TBL is observed in both external (such as in the flow around an airfoil) and internal (as is the case of a diffuser) configurations. With respect to such flows, many investigations are of experimental nature, as in Refs. [1, 2]. These studies have investigated the one-point statistics of PG flows in different ranges of Reynolds numbers ($Re$) and different strengths of PG, even comparatively strong positive PGs leading to separation [3]. In more recent works the change of one-point statistics with increasing PG strengths (e.g. Ref. [4]) has been studied, in which a second peak was noticed in the wall-normal profiles of the root-mean-square (rms) of streamwise velocity fluctuations with strong adverse pressure-gradient (APG). Harun et al. [5] investigated the modification of the large-scale features of TBLs for different types of PGs. On the other hand, there are several relevant numerical studies of PG TBLs; cases close to separation were investigated [6, 7], while also direct numerical simulations (DNS) were performed with a wide range of PGs [8, 9, 10]. The Reynolds number was, however, low in all studies, except the more recent ones by Gungor et al. [10] and Kitsios et al. [11].
A number of parameters have been used in the literature to characterize PG TBLs. This complicates comparisons between the various studies, even if the particular PG parameters are matched between cases. History effects play an important role in the development of TBLs as well, and are not taken into account in these parameters. Therefore, one of our initial goals is to consider the impact of history effects on the state of the boundary-layer flows, in order to properly assess the effect of the pressure gradient and facilitate the determination of adequate scaling laws. The present paper does however not directly deal with this goal, but can be seen as a foundation for such a follow-up study. The set-up for PG flows in equilibrium (to be defined in section 2) is discussed, analyzed, and the influence of PGs in turbulent flows is shown.

The paper is organized as follows: Section 2 gives a description of the flow cases and introduces the equilibrium boundary layer. Section 3 explains the numerical method and boundary conditions. The preliminary simulations performed to assess the best set-up are presented in Section 4, together with a discussion of the results of the main simulations. Finally, Section 5 summarizes the paper and gives an outlook to forthcoming studies. The description of accompanying RANS computations together with the determination of the virtual origin can be found in the Appendix.

2. Definitions

Earlier studies of TBLs subjected to pressure gradients date back to the 1930s [12, 13]. Laminar boundary layers with pressure gradients are well known, and under some specific assumptions can theoretically be described by the Falkner–Skan similarity solution, which was developed in 1931 [14]. These solutions are valid when the free-stream velocity $U_\infty$ follows a power law of the form $U_\infty = ax^m$ (where $x$ is the streamwise coordinate and $m$ the power-law parameter), so that the boundary-layer approximation is valid [15]. The dimensionless velocity is $U = U_\infty (df/d\eta)$, with an unknown dimensionless function $f$, and the dimensionless similarity parameter $\eta = y/\delta(x)$, with $\delta(x) = \sqrt{\nu x/U_\infty}$. The Hartree parameter $\beta_H$ from the Falkner–Skan equation [16] can be written in terms of $m$ as $m = \beta_H/(2 - \beta_H)$. Note that, for instance, the range $0 < \beta_H < 1$ corresponds to the flow around a wedge. Therefore, the relation between $m$ and $\beta_H$ can be determined from the potential solution of such a flow. Clauser [17] found that near-equilibrium conditions are obtained in laminar case when the term $\delta^*(dP/dx)/\tau_w$ (where $\delta^*$ is the displacement thickness, $\tau_w$ the wall shear stress and $dP/dx$ the streamwise pressure gradient) is constant. This was then later referred to as the Clauser parameter $\beta$ and has been widely used. Several other studies [9, 18, 19] claim that a near-equilibrium state is reached in turbulent boundary layers when the Clauser parameter $\beta$ remains constant, as in the laminar case.

An additional condition for near-equilibrium, according to Ref. [9], is that the form parameter $G = (H_{12} - 1)/(H_{12}\sqrt{c_f/2})$ (a measure of the velocity defect) has to be constant, where $H_{12} = \delta^*/\theta$ denotes the shape factor. Mellor and Gibson [19], as well as Townsend [18], found that a power-law distribution for the free-stream velocity leads to a near-equilibrium state and assessed the downstream evolution of the boundary-layer thickness using the Rotta–Clauser length scale $\Delta = (U_\infty/u_r)\delta^*$, where $u_r$ denotes the friction velocity. Skåre and Krogstad [4] produced a region of near-equilibrium conditions in their experimental study, where the Clauser parameter $\beta$ was constant over a significant part of the test section. Although they obtained their experimental free-stream velocity distribution by adjusting the wind tunnel ceiling, they could fit it to a power law defined as $U_\infty \approx (x - x_{0,e})^m$, where $x_{0,e}$ is the virtual origin.

A relation between the exponent $m$ and the Clauser parameter $\beta$ in turbulent flows was obtained in Ref. [20] by assuming $u_r/U_\infty \to 0$: $m = -\beta/(1 + 3\beta)$. To account for the empirical fact that $u_r/U_\infty$ approaches a constant rather than zero, Skote et al. [8] performed a non-linear analysis instead and obtained $m = -\beta/(H(1 + \beta) + 2\beta)$. 

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The non-dimensional acceleration parameter

\[ K = \frac{\nu}{U^2} \frac{dU_\infty}{dx} \]  

(1)
is also sometimes used \[21\] to describe the distribution of the free-stream velocity. However, it was shown in the study by Monty \textit{et al.} \[21\] that characterizing the effect of the pressure gradient on the TBL was difficult with this parameter. The authors showed that the turbulence intensity was further increasing when \( K \) was matched but \( \beta \) increased, while for constant \( \beta \) the profiles of the turbulence intensity showed a good collapse within the logarithmic region and a slight offset in the outer region. The parameter \( K \) is also used to set the limit for relaminarization in favorable pressure-gradient TBLs as in Ref. \[22\].

An alternative definition of equilibrium was introduced in Refs. \[23, 24\] where instead of assuming the validity of the scaling laws, scaling parameters were derived from the conditions for similarity. The pressure gradient was defined as

\[ \Lambda = \frac{\delta}{\rho U_\infty^2 d\delta/dx} \approx \text{const.}, \]  

(2)

with the boundary-layer thickness \( \delta \). According to these authors the velocity scale for the velocity deficit law is \( U_\infty \) and for the inner region it is \( u_\tau \). The Clauser parameter \( \beta \) is an asymptotic limit of \( \Lambda \) for infinitely high \( Re \), since \( Re \sim U_\infty^2 d\delta/dx \sim u_\tau^2 \) such as for ZPG TBLs. A more detailed description can be found in Ref. \[24\]. In the present work we will consider the region where \( \beta \) remains constant to be in near-equilibrium conditions, as in Townsend \[18\] and Mellor and Gibson \[19\]. As mentioned in the Introduction, a wide range of studies are focused on APG TBLs, and in many of them the exponent \( m \) was used to describe the pressure gradient. However, the results from the various studies cannot be easily compared, even if the value of \( \beta \) is provided, due to history effects and the different conditions for equilibrium state. In the present study the pressure gradient will be imposed by considering a free-stream velocity following the power law with constant exponent \( m \)

\[ U_\infty(x) = U_{\infty,0} \left(1 - \frac{x}{x_{0,l}}\right)^m \]  

(3)

with \( x_{0,l} \) being the virtual leading edge, where the displacement thickness is \( \delta^* = 0 \). The derivation of the free-stream velocity equation can be found in Ref. \[8\]. This case is referred to as the constant \( m \) case. The corresponding free-stream distribution is shown in Figure 1(a).

In a future follow-up study the history effects will be taken into account and the set up will be modified in order to impose a constant value of \( \beta \). There, the PG will be described by a free-stream velocity distribution resulting in a constant pressure gradient coefficient \( \beta \) over a long streamwise distance (as shown in Figure 1(c)) in order to eliminate upstream effects and to reach a near-equilibrium. During the entire study the integrated parameters are evaluated up to the boundary-layer thickness \( \delta_{99} \). As shown by Vinuesa \textit{et al.} \[25\], the diagnostic plot concept \[26\] is a robust method to determine the 99\% boundary-layer thickness \( \delta_{99} \) in pressure gradient TBLs. The boundary layer edge is found as the wall-normal location where \( u_{rms}/(U^{\sqrt{H_{12}}}) = 0.02 \) corresponds to \( U/U_\infty = 0.99 \).

3. Numerical method

Numerical simulations were performed to investigate turbulent boundary layers with APGs. The spectral code SIMSON \[27\] developed at KTH was used to conduct large-eddy simulations (LES) in order to reduce the computational cost of the numerical campaign. The approximate
deconvolution relaxation–term model (ADM-RT [27]) was used as the sub-grid scale (SGS) model to describe the small unresolved scales. In contrast to other SGS models such as the Smagorinsky model, here the SGS force acts directly on the resolved velocity components $u_i$,

$$\frac{\partial \tau_{ij}}{\partial x_j} = \chi H_N \ast \bar{\pi}_i,$$

The symbol $\ast$ denotes the convolution and the overbar the implicit grid filter due to the lower resolution in the LES. The model coefficient $\chi = 0.2$ is chosen as constant similar to the time step of integration, $1/\Delta t$. The high-order filter $H_N$ with a cut-off frequency of $\omega_c \approx 0.86\pi$ affects only the small scales, whereas the energetic large scales are not taken into account by the model. As shown in earlier studies [28, 29], the flow predicted by this SGS model is in good agreement with DNS results in terms of mean flow, Reynolds-stress tensor and turbulence budgets. Even the budget terms are described well by LES data based on the current SGS model. The numerical method is based on Fourier discretization with dealiasing in the streamwise and spanwise directions as well as an expansion in Chebychev polynomials along the wall-normal
Table 1. Summary of the simulations of turbulent boundary layers with APG. Reynolds number \(Re_{\delta_s}\), different domain sizes (in terms of the initial laminar displacement thickness \(\delta_s\)), grid points (before dealiasing), boundary conditions (BC) at the top boundary: 110 for asymptotic BC and 101 for Neumann BC, power-law exponent \(m\) (equation (3)), and range of pressure-gradient parameter \(\beta\).

| Case            | \(Re_{\delta_s}\) | Domain size \(L_x \times L_y \times L_z\) | Grid points in \(x \times y \times z\) | Top BC | \(m\)     | \(\beta\)          |
|-----------------|-------------------|------------------------------------------|--------------------------------------|--------|-----------|-------------------|
| Skote [32]      | 450               | 450×24×24                                | 480×161×96                           | 101    | -0.15     | [0.42; 0.68]     |
| Fringe          | 400               | 800×48×48                                | 480×161×96                           | 101    | -0.15     | [0.52; 1.40]     |
| Blasius inlet   | 450               | 800×48×48                                | 480×161×96                           | 101    | -0.15     | [0.58; 1.15]     |
| Boxxsize 1      | 450               | 3000×80×160                              | 1024×121×128                         | 101    | -0.1      | –                 |
| Boxxsize 2      | 450               | 3000×120×160                             | 1024×161×128                         | 101    | -0.1      | –                 |
| Boxxsize 3      | 450               | 3000×160×160                             | 1024×201×128                         | 101    | -0.1      | –                 |
| BC              | 450               | 3000×120×160                             | 1024×161×128                         | 110    | -0.13     | [0.86; 1.49]     |
| m1              | 450               | 3000×120×160                             | 2048×301×256                         | 101    | -0.16     | [1.55; 2.55]     |
| m2              | 450               | 3000×180×220                             | 2048×361×384                         | 101    | -0.16     | [1.55; 2.55]     |

A fringe region extends from \(x = 2700\) up to the end of the box, where the flow is forced back to the Blasius inflow profile. The exact forcing is described in Ref. [27]. In this way, periodic boundary conditions can be imposed in the streamwise direction while simulating a spatially developing flow. At the wall the no-slip condition \((u, v, w)_{wall} = (0, 0, 0)\) was imposed, while at the upper boundary different boundary conditions were tested as described in Section 4.1. The inflow at \(x = 0\) is a laminar Blasius profile with the Reynolds number based on the displacement thickness \(Re_{\delta_s} = 450\). For the remainder of the paper, all length scales are given in terms of this displacement thickness \(\delta_s\) and all velocities in terms of \(U_{\infty,0}\) at \(t = 0\) and \(x = 0\), unless stated otherwise. At \(x = 10\) the flow is tripped [30] and transitions thereafter into a turbulent flow. The length of the box is chosen as \(L_x = 3000\) for all LES. During the study different box sizes and resolutions were tested before the final cases were defined. The chosen parameters of the auxiliary simulations as well as of the final cases are shown in Table 1. A coarse resolution was considered in the auxiliary cases in order to obtain general trends for the design of the final cases. For the main simulations the grid spacing was chosen to be of the same order as the zero pressure-gradient case by Schlatter et al. [31]: i.e. \(\Delta x^+ = 21.5\), \(\Delta y_{max}^+ = 13.9\), \(\Delta z^+ = 9.2\), and a total of 12 points below \(y^+ = 10\).

4. Results

4.1. Auxiliary simulations

The final simulation parameters were determined by performing several auxiliary simulations, in which different set-ups and boundary conditions together with different box sizes were tested. The investigation started with the set-up as considered by Skote [32]. The box size and resolution (shown in Table 1) were chosen as in Case 2 from the DNS by Skote. At the inflow a Falkner–Skan profile with \(m = -0.077\) was imposed, which was tripped at \(x = 10\). The pressure gradient is imposed by setting the free-stream velocity distribution at the top of the box as in equation (3) over the entire box, where the exponent is chosen as \(m = -0.15\) and \(x_{0,t} = -60\). The exact condition at the upper boundary is described below. These results showed a strong dependence on the inflow conditions, which motivated further tests to investigate the influence of those. Hereby the test case Fringe was simulated once with the same fringe forcing as in Skote and another time with a stronger fringe forcing. The box was extended in all three directions to investigate the effect on the boundary-layer development at higher \(Re\), while the number of grid points was kept constant. In Figure 2 the two cases (moderate and strong forcing) are compared with each other. In addition, the data of the case Blasius inlet (see Table 1) is shown, which differs from the moderate fringe forcing case by using a Blasius inflow instead of a Falkner–
Figure 2. Comparison between the Fringe data set of moderate fringe forcing (black), the one with strong fringe forcing (red) and the data set with Blasius inlet profile (blue) of (a) the pressure gradient parameter $\beta$, (b) the skin friction coefficient $c_f$ and (c) the shape factor $H_{12}$.

Skan profile. The case with the strong fringe forcing shows an unphysical behavior in all three parameters $\beta$, $c_f$ and $H_{12}$ for $x < 100$. The effect of the fringe diminishes at higher Reynolds numbers. The resulting $\beta$ range is higher than in the Skote case, since higher $Re$ are observed as well as resolution effects have to be considered. The test cases are not as finely resolved as the Skote case, but as mentioned an SGS model was employed instead. In all three cases $H_{12}$ converges towards a constant value of about 1.6, but shows different developments right after the inlet. The three parameters show a strong dependency on the inflow conditions. One should note that the statistics are not fully converged as it can be seen in the figures.

To ensure that the PG TBL is independent of the inflow conditions, the set-up had to be modified. Instead of a Falkner–Skan profile with pressure gradient at the inlet, the inflow was set to a Blasius profile. This approach emulates the procedure carried out in wind tunnel experiments. The fringe and tripping parameters remained unchanged. A ZPG region extends up to $x \approx 350$, henceforward the pressure gradient will be applied following the power–law, equation (3), with $x = [350, L_x]$. The exponent $m$ is initially chosen as zero at $t = 0$. Once the flow is fully turbulent over the whole box, the exponent is slowly adapted towards the final value. In comparison to the case of Skote, the computational box was extended in the streamwise...
direction to investigate higher Reynolds numbers. The free-stream velocity of the new set-up is shown in Figure 1(b). This modified set-up enables the study of turbulent boundary layers with various strengths of pressure gradient independent of the choice of the inflow profile and the fringe parameters. All the following simulations were performed with this new set-up.

Upon deciding the method to impose the PG, we investigate the effect of the box size on the statistics of the simulations. The streamwise box length $L_x$ was chosen in order to achieve the required Reynolds number before the fringe region. The spanwise box width $L_z$ was chosen according to earlier investigations, see e.g. Ref. [29], where $L_z > 2\delta_{99}$ (taken at the end of the domain) to ensure correct development of large-scale structures in the outer region.

The height of the box $L_y$ was tested in three different simulations (see Table 1), where the streamwise and spanwise extend was kept constant while the height of the box was varied. In Figure 3 the contour lines of the rms for all velocity components are shown for the three different box sizes. This figure shows that there is not a strong effect of the box height on the inner layer for all the components. The outer layer, on the other hand, is influenced strongly by the height of the box. In all three rms-components in Boxsize 1 it can be seen that the large structures

Figure 3. Contour lines (equidistantly spaced between 0 and the maximum rms values on 10 contour lines) of rms-profiles for cases Boxsize 1 (red), Boxsize 2 (green), Boxsize 3 (blue) Table 1: (a) $u_{rms}$, (b) $v_{rms}$, (c) $w_{rms}$ scaled by $U_{\infty,0}$. The boundary-layer thickness $\delta_{99}$ (black) and the respective box heights are indicated through dashed lines.
are restricted by the height of the box, while no influence of the upper boundary is observed in Boxsize 2 and Boxsize 3. The height of Boxsize 2 is consequently a good choice and we conclude that a box height of $L_y = 2 \delta_{99}$ will be deemed sufficient for further investigations.

As already mentioned, two different boundary conditions were tested for the upper boundary. One condition is the Neumann condition (denoted as 101 in Table 1), where the wall-normal derivatives of the streamwise and spanwise components are fixed to

$$\frac{\partial u}{\partial y} \bigg|_{y=L_y} = \frac{\partial w}{\partial y} \bigg|_{y=L_y} = 0.$$  \hfill (5)

at $y = L_y$. The wall-normal boundary condition is obtained from the continuity equation and expressed as:

$$\frac{\partial v}{\partial y} \bigg|_{y=L_y} = -\frac{\partial U_\infty}{\partial x}. \hfill (6)$$

The second investigated boundary condition is the so called asymptotic condition (denoted as 110 in Table 1) based on Fasel [33] and Malik et al. [34]. The condition is set by the asymptotic decay of the velocity perturbation in wall-normal direction. The base flow $\hat{U}$ is calculated from the Blasius solution, which was chosen as the initial condition and is scaled for each streamwise position according to the outer velocity as imposed at the upper boundary. The boundary condition is written as

$$\left( \frac{\partial \hat{u}}{\partial y} + |k| \hat{u} \right) \bigg|_{y=L_y} = \left( \frac{\partial \hat{U}}{\partial y} + |k| \hat{U} \right) \bigg|_{y=L_y}, \hfill (7)$$

where $(\cdot)$ denotes the horizontal Fourier transform with respect to the horizontal coordinates and $k^2 = \alpha^2 + \beta^2$ with the horizontal wave-numbers $\alpha$ and $\beta$ [27].

In Figures 4 and 5, first-order statistics are compared between simulations using the two boundary conditions. The free-stream distribution is defined exactly when using the Neumann condition. Conversely, in the case of the asymptotic condition $U_\infty$ does not follow the exact power law as specified. Overshoots in the velocity are observed in the transition from ZPG

![Figure 4](image_url)

**Figure 4.** Comparison between the Neumann (black) and asymptotic (red) as in Ref. [33, 34] boundary condition. (a) shows the free stream velocity distribution $U_\infty$ at the top of the box and (b) the displacement thickness $\delta^*$ as a function of the streamwise position $x$. 


to APG and at the entrance of the fringe region. While it is possible to define a power law describing the distribution without the overshoots, it will not exactly match the prescribed one, and therefore the direct connection between the predetermined \( m \) and the produced \( \beta \) is not possible. The \( U_\infty \) distribution also leads to differences in the displacement thickness \( \delta^* \) as depicted in Figure 4(b). Since \( U_\infty \) decreases less for the asymptotic condition downstream, \( \delta^* \) increases slower and reaches lower values. The position of the maxima is at the same location but less pronounced when using the asymptotic condition. In Figure 5 the rms of the pressure fluctuations are compared in the \( xy \)-plane. An unexpected behavior can be noticed for the asymptotic condition in the free-stream, which cannot be further explained. This occurrence and the inexact definition of the free-stream velocity excludes the asymptotic condition as a boundary condition. While the asymptotic condition can be placed relatively close to the wall, the use of the Neumann condition is restricted by the size of the disturbance velocity: sufficiently high boxes have to be chosen, where the disturbance velocity is small. An alternative procedure to impose the pressure gradient on the TBL was developed by Spalart and Watmuff [35], who basically decomposed the velocity field \( U \) into the sum of several fields as \( U_0 + U_1 + U_2 + U_3 \), \( U_1 \) being the unknown. Whereas \( U_2 \) is related to the fringe region and \( U_3 \) is a correction that depends on \( U_1 \), the pressure gradient is imposed indirectly through the velocity \( U_0 \). The idea is that \( U_0 \) sets a slip velocity throughout the boundary layer that basically is an extrapolation of the velocity in the rotational region across the boundary layer and all the way down to the wall. Nevertheless, in the present study the Neumann condition will be considered as the upper boundary condition.

4.2. Main simulations

The statistics of the final cases with the constant power–law exponent \( m = −0.13 \) and \( m = −0.16 \) (cf. Table 1) are compared with each other and the influence of the strength of the PG on the flow statistics is investigated. The statistics were obtained after averaging over 11.5 eddy turnover times (ETT) in the case of \( m = −0.13 \) and over 2.3 ETT in case of \( m = −0.16 \). Here ETT is defined as \( ETT = \delta_9/u_\tau \) at \( x = 2300 \), i.e. close to the end of the physical domain. According to Lozano-Durán and Jiménez [37], turbulent channel-flow simulations at similar \( Re \) require an averaging time of around 11–12 ETT in order to provide fully converged statistics.

**Figure 5.** Comparison of Neumann (bottom) and asymptotic (top) as in Ref. [33, 34] boundary condition: rms of the pressure in terms of the reference quantities \( U_\infty \) and \( \delta_0^* \).
Figure 6. (a) Clauser parameter $\beta$, (b) skin-friction coefficient $c_f$, (c) shape factor $H_{12}$, (d) displacement thickness $\delta^*$ (solid) and momentum thickness $\theta$ (dashed), (e) boundary-layer thickness $\delta_{99}$, (f) ratio between $\delta_{99}$ and $\delta^*$ (solid) and $\delta_{99}$ and $\theta$ (dashed) for the two $m$-cases $m = -0.13$ (black), $m = -0.16$ (red) and ZPG [36] (grey) over the streamwise position $x$. 
channel-flow simulations the streamwise and spanwise direction are homogeneous, which allows to increase the effective number of statistical samples by averaging in those directions. In the case of the spatially developing TBL only the spanwise direction is homogeneous, which means that converged statistics will require longer averaging times than those for a channel flow.

Therefore, the averaging time is not sufficient in the present simulation with $m = -0.16$, consequently some of the statistics presented here are not fully converged. Streamwise one-point statistics are shown in Figure 6 comparing the PG cases to a ZPG case [36]. In Figure 6(a) the streamwise distribution of the Clauser parameter $\beta$ is shown. A constant $m$ leads to a downstream decrease in $\beta$, while a larger absolute value of $m$ results in a larger value of $\beta$, since the velocity gradient over the streamwise direction is stronger. The skin-friction coefficient $c_f$, shown in Figure 6(b), first decreases downstream, but converges to a seemingly constant value, while for the ZPG $c_f$ continues to decrease over the whole box length as expected. Since the APG thickens the boundary layer, the velocity gradient at the wall is reduced and therefore APG TBLs exhibit a lower $c_f$ value than the corresponding ZPG case. The shape factor $H_{12}$, depicted in Figure 6(c), is higher for flows with APG as the ratio between the displacement thickness $\delta^*$ and momentum thickness $\theta$ increases as seen in Figure 6(d). In $\delta^*$ and $\theta$ upstream influences of the APG and effects of the different tripping [30] can be noticed at the streamwise positions $x > 110$. The boundary-layer thicknesses $\delta^*$ and $\theta$, as well as $\delta_{99}$ (Figure 6(e)), increase, since the APG decelerates the flow and it basically lifts up the boundary layer. The APG TBL is twice as thick as the ZPG case. For instance, comparisons with ZPG datasets (Ref. [30]) show that the boundary-layer thickness $\delta_{99}$ is five to six times $\delta^*$ and eight times $\theta$. In the present APG TBL the ratios are four to five times $\delta^*$ and seven times $\theta$. This clearly shows the effect of the APG in the development of the TBL.

In Figure 7 the mean velocity is shown and compared to a ZPG case at matched friction Reynolds number $Re_\tau = u_\tau \delta_{99} / \nu$, which fixes the inner-scaled thickness of the TBL, allowing the characterization of PG effects on the profiles. The profiles of two different APG strengths are compared to the ZPG case at $Re_\tau = 380$ and $Re_\tau = 600$. In Figure 7(a) the profiles of the mean velocity $U^+$ scale in the viscous sublayer as expected. In the log layer the APG profiles are consistently below the ZPG ones. In the outer region the influence of the APG is manifested in a stronger wake parameter $\Pi$, which is the consequence of more energetic

![Figure 7](image-url)
structures in the outer layer. Note that the velocity gradient is not necessarily zero for \( y > \delta \), i.e. \( U(y > \delta) \neq \text{const.} \) as it can be observed in Figure 7(a). Further insight in the outer region can be gained by examining the defect mean velocity profile \( (U_\infty^+ - U^+) \) as in Figure 7(b). This figure shows clearly the large influence of the APG in the outer region.

The inner-scaled root mean square values \( u_{rms}^+, v_{rms}^+ \) and \( \langle uv \rangle^+ \) are depicted in Figure 8. The near-wall peak is higher for stronger PG when compared to ZPG TBLs, while the position of the \( u_{rms}^+ \) peak is invariant around \( y^+ = 15 \). In the outer region a second peak can be noticed as in Ref. [21]. It is interesting to observe that an outer peak emerges in all the components of the Reynolds-stress tensor, the magnitude of which increases with PG, as well as with \( Re \).

The strength of the outer peak in case \( m = -0.16 \) is of a comparable magnitude as the inner peak. In the case of Skote [32] with \( m = -0.15 \), a second hump can be noticed with values lower than 2. Even though the exponents \( m \) are close to each other, it is difficult to compare these cases since both history and Reynolds number are different. In the study of Vinuesa et al. [38] the TBL around a wing was analyzed. Large PG with \( \beta = 4.54 \) lead to a strong outer peak in \( u_{rms}^+ \), however this peak amplitude is smaller than the inner peak. Low PG with \( \beta = 0.53 \) (comparable to Skote [32]), leads to a less pronounced outer peak and the shape of the \( u_{rms}^+ \) can be compared to the shape as presented in Skote. The possible connection between the outer peak in the turbulent intensity found in high \( \beta \) and high \( Re \) ZPG boundary layers will be further explored in future studies.

The decomposition of total shear stress \( \tau \) into viscous stress and Reynolds shear stress is shown in Figure 8(b). In ZPG flows the maximum total shear stress is unity. In APG cases the decomposition of \( \tau \) is the same, while the value of unity can be exceeded due to a much larger Reynolds shear–stress term in the outer region. The Reynolds shear stress \( \langle uv \rangle^+ \) is influenced strongly as indicated in Figure 8(b) with a strong outer peak, while the viscous stress term appears unaffected by PG effects. Since the free-stream velocity is defined with a power law, the TBL exhibits near-equilibrium features (Ref. [19]). This could be connected to the aforementioned fact that the viscous shear stress is independent of the PG.

The skewness and flatness factors for the streamwise velocity component are compared in

![Figure 8](image-url)

**Figure 8.** (a) Inner-scaled root mean square values \( (u_{rms}^+, v_{rms}^+) \) for the two \( m \)-cases: \( m = -0.13 \) (black), \( m = -0.16 \) (red) at \( Re_\tau = 380 \) (dashed) and \( Re_\tau = 600 \) (solid) and ZPG [36] (grey). (b) Decomposition of total shear stress \( \tau^+ \) into viscous (dotted) and Reynolds shear stress (dashed) for the two \( m \)-cases: \( m = -0.13 \) (black), \( m = -0.16 \) (red) at \( Re_\tau = 600 \) (solid) and ZPG [36] (grey).
Figure 9. (a) Skewness and (b) flatness factor for the two $m$-cases: $m = -0.13$ (black), $m = -0.16$ (red) and ZPG [36] (grey) at $Re_\tau = 600$.

In general the values at the wall of skewness and flatness are higher in PG flows and are increasing with the strength of the APG. This is also seen in ZPGs with increasing $Re$. The increase of the values with the PG is also apparent throughout the buffer layer. As described in Refs. [28, 39] for large $Re$ in ZPGs, the large-scale motions become progressively stronger at higher $Re$ in ZPG TBLs. It can therefore be assumed that large-scale motions also become stronger for increasing APGs, and consequently their influence on the small scales. In the flatness factor of the APG cases a second peak can be noticed farther away from the wall and the skewness does not become negative within the buffer layer. As stated in Ref. [40] the zero-crossing of the skewness factor and the minimum value of the flatness factor are at the same wall-normal position in ZPG TBLs (see also Eitel-Amor et al. [28]). In the present APG cases the first minimum of the flatness is at the same position as in the ZPG, while the second minimum coincides with the zero-crossing of the skewness. It is interesting to note that the

Figure 10. Inner-scaled pre-multiplied turbulent kinetic energy budget: production (solid) and dissipation (dashed) for the two $m$-cases: $m = -0.13$ (black), $m = -0.16$ (red) and ZPG [36] (grey) at $Re_\tau = 600$. 

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latter position is close to the one of the second peak occurring in the turbulence intensities. This behavior is in agreement with large Re ZPGs as reported in Vallikivi et al. [41].

Finally, the production and dissipation terms of the pre-multiplied kinetic energy budget are shown in Figure 10. The production exhibits a low inner peak and production and dissipation show a pronounced peak farther away from the wall at a position \( y/\delta_{99} \approx 0.45 \) in the APG cases, while the peak is stronger for stronger APG. The outer peak of the production was also noticed in Skare and Krogstad [4]. They found a diffusion mechanism from the outer region of the boundary layer towards the wall. This is another manifestation of the impact of the APG on the large-scale structures of the flow. The dominant outer peak in the production, as seen in Figure 10, is a second energy source and dissipation is not restricted to the wall, but energy is transferred from large to small scales throughout the whole layer.

5. Conclusion and Outlook

In the present work we developed a suitable set-up to numerically study adverse pressure gradient turbulent boundary layers, with the future aim of assessing history and development effects on the flow. The set-up suggested by Skote et al. [8], where a power law was used to describe the velocity distribution at the free-stream, was modified in order to ensure the proper tripping and development of the TBL. The new approach mimics the procedure carried out in wind tunnel experiments. Here the APG is applied once the flow has reached a fully turbulent state. The computational box size has been assessed by testing different heights of the box and it could be concluded that the flow develops independently from the box height when \( L_y > 2\delta_{99} \) as in ZPG TBLs. The spanwise length of the box was decided based on earlier investigations on ZPG flows as in Ref. [30], where a value of \( L_z = 2 - 3\delta_{99} \) was appropriate. The correct definition of the boundary condition on the top boundary was systematically tested. Although the asymptotic condition suggested by Fasel [33] would in principle allow to use smaller box heights, we found that this BC does not lead to correct prescribed pressure distributions. The Neumann condition was found to describe the required velocity distribution exactly and was chosen as the upper boundary condition. After deciding on the final set-up, we performed the main simulations with following power-law exponents: \( m = -0.13 \) and \( m = -0.16 \), in the \( U_{\infty} \) distribution given by equation (3). The effect of the APG on TBLs was manifested in the mean flow through a much more prominent wake region and in the Reynolds stresses through the existence of an outer peak. The pre-multiplied energy budget shows in the production term a pronounced second peak in the outer region, which underlines influence on the distribution of the turbulent kinetic energy transfer mechanism across the boundary layer. This might also indicate that the large-scale structures are stronger in APG TBLs than in ZPG TBLs. Note that the structures are not necessarily stronger when scaling with the local mean shear is considered. Another observation is that the displacement thickness and momentum thickness are around three times larger than the respective thicknesses in the ZPG case. The boundary layer decelerates by the influence of the APG, the skin friction is reduced and the boundary layer thickness increases. In this study we defined the set-up to investigate APG TBLs with constant \( m \) and we also performed first tests to set-up a case with constant \( \beta \) (see Appendix). In future work we will compare the two cases of constant \( m \) and constant \( \beta \) with each other to identify history effects and to further explain the characteristics of APG flows.

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Appendix: Design of a case with constant $\beta$

As discussed above, the pressure gradient is imposed on the TBL through the streamwise evolution of $U_\infty$. The $U_\infty(x)$ curve can be determined by the geometry, as it is the case in the flow around wings, by modifying the ceiling configuration in wind-tunnel experiments, or by setting its evolution as a power law with exponent $m$. All those flows have different histories, which are not taken into account in the analysis of scaling laws. With $\beta$ being constant over a long streamwise distance the same pressure gradient can be observed from low to high $Re$. It is therefore necessary to determine the $U_\infty(x)$ distribution leading to a constant $\beta$ over a long streamwise distance in the domain. Our initial idea was to mimic a wind-tunnel experiment by performing Reynolds-Averaged Navier–Stokes (RANS) computations, and to define the shape of the ceiling (initially chosen as in Ref. [4]) through an iterative process until a constant $\beta$ distribution was obtained in the $Re$ range of interest. Once a constant $\beta$ distribution was reached, the corresponding $U_\infty(x)$ could be extracted and then imposed in the LES.

The RANS computations were carried out by considering the two equation SST (shear stress transport) model [43], implemented in the commercially available CFD code Fluent (v.6.3). The inflow condition in the RANS is a uniform profile, which together with 0.5% turbulence intensity, leads to a turbulent flow few grid points downstream the inflow. However, in the LES the inflow is set as a laminar boundary layer defined by a Blasius profile with $Re_{\delta_0} = 450$, which is then tripped to turbulence. The tripping parameters were chosen as follows: time-dependent forcing amplitude $tampt = 0.4$, streamwise length scale of tripping $txsc = 4.0$, streamwise origin of tripping $tx0 = 10$, wall-normal length scale of tripping $tysc = 1.0$, time scale of tripping $tdt = 4.0$. It is important to ensure that the development of the boundary layer computed in the RANS is equivalent to the one in the LES. Since the RANS essentially starts from a turbulent profile, it is necessary to determine the value of $Re_{\delta_0}^{\text{turb}}$ that a turbulent boundary layer would have at the location where in the LES the laminar $Re_{\delta_0}^{\text{lam}} = 450$, i.e. at $x/\delta_0$. To this end, we considered the ZPG DNS data [36], which also has $Re_{\delta_0}^{\text{lam}} = 450$, and computed the virtual origin $x_0$ based on the data with $x/\delta_0 > 400$. Note that although the flow is tripped at $x/\delta_0 = 10$, we did not consider the part of the domain downstream the trip to ensure that a fully-developed turbulent region is used in the analysis. We initially considered the following

![Figure 11. Displacement thickness $\delta^*$ of the ZPG DNS data (black solid), equation from Ref. [42] (blue dashed), fitted data (red solid).](image-url)
empirical correlation by Spurk and Aksel [42] for the displacement thickness $\delta^*$,

$$\delta^* = 0.046 Re_x^{-1/5} (x - x_0) \quad \text{with} \quad Re_x = \frac{U_\infty}{\nu} (x - x_0),$$

and calculated, for each $x$ position, the corresponding virtual origin $x_0$. Interestingly, this method did not lead to a constant $x_0$ over $x$, which is an indication of the fact that equation (8) does not properly represent the DNS results in this low Re range. Additional confirmation of this fact is given in Figure 11, where the displacement thickness from the DNS is shown as a function of $x/\delta^*_0$ together with the result of equation (8) with an average value of $x_0$. The insufficient agreement between both curves motivated the use of another method to determine $x_0$, and therefore we considered the following more general form of the $\delta^*$ curve,

$$\delta^* = A Re_x^B (x - x_0),$$

which has three fitting parameters: $A$, $B$ and $x_0$. Fitting equation (9) to the region of the DNS with $x/\delta^*_0 > 400$, we obtained the values $A = 0.06527$, $B = -0.2209$ and the virtual origin of a TBL with $Re_{\delta^*_0}^\text{lam} = 450$ as inflow condition, i.e. $x_0 = -170.1$. In this case, equation (9) represents the DNS data very well, and interestingly, small deviations are observed between the fit and the DNS close to $x/\delta^*_0 \simeq 0$. This is of course due to the fact that close to the inflow the simulated boundary layer is still laminar, and the fit proposed here is aimed at representing the turbulent region. We then used equation (9) to determine the value of $\delta^*_\text{turb}$ at the origin of the LES domain ($\delta^*_\text{turb} = 0.9259$), and consequently the value that will be considered at the inflow in the RANS simulation: $Re_{\delta^*_0}^\text{turb} = 417$. It is interesting to note that the displacement thickness of the turbulent boundary layer is in fact lower than the laminar one, which may come as a surprise, but this is a result of the different developments of both boundary layers: farther downstream where the turbulent boundary layer is the one with larger $\delta^*$ values.

After ensuring that both numerical boundary layers would theoretically exhibit the same streamwise development, we performed a RANS simulation of a ZPG TBL with $Re_{\delta^*_0}^\text{turb} = 417$. Since the length scale of the simulations is $\delta^*_0$, and they have different Reynolds numbers, the results from the RANS have to be scaled (taking into account that the ratio $U_\infty/\nu$ is still the same in both simulations) with the length $l = (Re_{\delta^*_0}^\text{lam}/Re_{\delta^*_0}^\text{turb}) \delta^*_0 = (450/417) \delta^*_0$. Figure 12 shows the correct scaling between RANS data (black) and DNS data [36] for ZPG (red): a) skin friction coefficient $c_f$ and b) displacement thickness $\delta^*$.

![Figure 12](image-url)
shows a comparison of the downstream evolution of skin friction coefficient $c_f$ and $\delta^*$ from the DNS and from the RANS, which are appropriately scaled as discussed above. Unfortunately, there are significant deviations between both boundary layers, and interestingly the RANS provides better predictions of skin friction at higher Reynolds numbers, whereas the boundary-layer growth deviates more from the DNS as one moves downstream. The first observation is explained by the fact that RANS models are calibrated with high-$Re$ datasets, and are focused on skin friction predictions, therefore the deviations decrease with $Re$. In the case of the displacement thickness, the two boundary layers start with similar initial conditions (note that $\delta^*$ is 0.9259 in the RANS), but the poor modeling of the boundary-layer development in the RANS leads to progressively larger deviations between the two simulations. As a consequence, the route of using the RANS to define the $U_\infty(x)$ distribution of the LES was not further pursued, and an alternative methodology where the free-stream velocity distribution is adapted dynamically during the simulation is currently being developed for future extensions of the present study.