ON THE CONTINUED FRACTION EXPANSION
OF A CLASS OF NUMBERS

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Au Professeur Wolfgang Schmidt,
avec mes meilleurs vœux et toute mon admiration.

1. Introduction

A classical result of Dirichlet asserts that, for each real number \( \xi \) and each real \( X \geq 1 \), there exists a pair of integers \((x_0, x_1)\) satisfying

\[
1 \leq x_0 \leq X \quad \text{and} \quad |x_0 \xi - x_1| \leq X^{-1}
\]

(a general reference is Chapter I of [10]). If \( \xi \) is irrational, then, by letting \( X \) tend to infinity, this provides infinitely many rational numbers \( x_1/x_0 \) with \( |\xi - x_1/x_0| \leq x_0^{-2} \). By contrast, an irrational real number \( \xi \) is said to be badly approximable if there exists a constant \( c_1 > 0 \) such that \( |\xi - p/q| > c_1 q^{-2} \) for each \( p/q \in \mathbb{Q} \) or, equivalently, if \( \xi \) has bounded partial quotients in its continued fraction expansion. Thanks to H. Davenport and W. M. Schmidt, the badly approximable real numbers can also be described as those \( \xi \in \mathbb{R} \setminus \mathbb{Q} \) for which the result of Dirichlet can be improved in the sense that there exists a constant \( c_2 < 1 \) such that the inequalities \( 1 \leq x_0 \leq X \) and \( |x_0 \xi - x_1| \leq c_2 X^{-1} \) admit a solution \((x_0, x_1) \in \mathbb{Z}^2\) for each sufficiently large \( X \) (see Theorem 1 of [2]).

If \( \xi \) is rational or quadratic real, then, upon writing \( \xi^2 = (q \xi + r)/p \) for integers \( p, q \) and \( r \) with \( p \neq 0 \) and putting \( c_3 = |p| \max\{|p|, |q|\} \), one deduces from the result of Dirichlet that, for each \( X \geq 1 \), there exists a point \((x_0, x_1, x_2) \in \mathbb{Z}^3\) satisfying

\[
1 \leq x_0 \leq X, \quad |x_0 \xi - x_1| \leq c_3 X^{-1} \quad \text{and} \quad |x_0 \xi^2 - x_2| \leq c_3 X^{-1}.
\]

Conversely, Davenport and Schmidt proved that, for each real number \( \xi \) which is neither rational nor quadratic over \( \mathbb{Q} \), there is a constant \( c_4 > 0 \) such that, upon writing \( \gamma = (1 + \sqrt{5})/2 \), the system of inequations

\[
|x_0| \leq X, \quad |x_0 \xi - x_1| \leq c_4 X^{-1/\gamma}, \quad |x_0 \xi^2 - x_2| \leq c_4 X^{-1/\gamma},
\]

admits no non-zero integer solution \((x_0, x_1, x_2) \in \mathbb{Z}^3\) for arbitrarily large values of \( X \) (Theorem 1a of [3]). Since \( 1/\gamma \simeq 0.618 < 1 \), this establishes a clear gap between the set of rational or quadratic real numbers and the remaining real numbers. Moreover, this result of

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Davenport and Schmidt is best possible in the following sense. There exist real numbers $\xi$ which are neither rational nor quadratic and for which there is a constant $c_5 > 0$ such that the system (1), with $c_4$ replaced by $c_5$, admits a non-zero integer solution for each $X \geq 1$ (Theorem 1.1 of [7]). These real numbers, which we call extremal, present from this point of view a closest behavior to quadratic real numbers. An application of Schmidt’s subspace theorem proves them to be transcendental over $\mathbb{Q}$ (see Theorem 1B in Chapter VI of [10]). Still they possess several properties that make them resemble to quadratic real numbers. In the present paper, we are interested in their approximation by rational numbers.

It is well known that each quadratic real number has an ultimately periodic continued fraction expansion and so is badly approximable. Since there exist extremal real numbers which are badly approximable [6], this raises the question as to whether or not each extremal real number is such. At present, we simply know that an extremal real number $\xi$ satisfies a measure of approximation by rational numbers $p/q$ of the form
\[
\left| \xi - \frac{p}{q} \right| \geq c_6 q^{-2} (1 + \log |q|)^{-t},
\]
with constants $c_6 > 0$ and $t \geq 0$ depending only on $\xi$ (Theorem 1.3 of [7]). In this paper, we establish a sufficient condition for an extremal real number to have bounded partial quotients and construct new examples of such numbers.

2. Notation and statements of the main results

A Fibonacci sequence in a monoid is a sequence $(w_i)_{i \geq 1}$ of elements of this monoid which satisfies the recurrence relation $w_{i+2} = w_{i+1} w_i$ for each $i \geq 1$. Here, we shall work with two types of monoids.

One is the monoid of words $E^*$ on an alphabet $E$, with the product given by concatenation of words. A Fibonacci sequence $(w_i)_{i \geq 1}$ in $E^*$ has the property that $w_i$ is a prefix (left-factor) of $w_{i+1}$ for each $i \geq 2$ and so, it admits a limit $w_\infty = \lim w_i$ in the completion of $E^*$ for pointwise convergence. This limit is an infinite word unless $w_1$ and $w_2$ are empty. For example, if $E = \{a,b\}$ consists of two distinct elements $a$ and $b$, then the Fibonacci sequence of words starting with $w_1 = b$ and $w_2 = a$ converges to the infinite word $f_{a,b} = abaababa \ldots$. In general, the limit of any Fibonacci sequence of words $(w_i)_{i \geq 1}$ derives from this generic infinite word $f_{a,b}$ by substituting into it the words $w_1$ and $w_2$ for the letters $b$ and $a$ respectively. It will come out indirectly of our analysis that such a limit is an infinite non-ultimately periodic word if and only if $w_1$ and $w_2$ do not commute (see the remark after Theorem 2.2 below). A direct proof of this fact has been recently provided by B. Lucier [5].

The other monoid is in fact a group. It is constructed as follows. Define the content $c(A)$ of a non-zero matrix $A$ in $\text{Mat}_{2 \times 2}(\mathbb{Z})$ to be the greatest positive common divisor of its coefficients and say that such a matrix $A$ is primitive if $c(A) = 1$. For each non-zero
A ∈ Mat_{2×2}(\mathbb{Z})$, denote by \( A^{\text{red}} \) the unique primitive integer matrix such that \( A = c(A)A^{\text{red}} \). Then, the set \( \mathcal{P} \) of all primitive matrices with non-zero determinant in Mat_{2×2}(\mathbb{Z}) is a group for the operation \( * \) given by \( A*B = (AB)^{\text{red}} \). Its quotient \( \mathcal{P}/\{±I\} \) is isomorphic to PGL_{2}(\mathbb{Q})

**Definition 1.** We say that a Fibonacci sequence \((W_i)_{i≥1}\) in \( \mathcal{P} \) is **admissible** if there exists a non-symmetric and non-skew-symmetric matrix \( N \in \mathcal{P} \) such that, upon putting \( N_i = t_iN \) for \( i \) odd and \( N_i = N \) for \( i \) even, the product \( W_iN_i \) is a symmetric matrix for each \( i ≥ 1 \).

This definition differs slightly from that in §3 of [9]. However, the same argument as in the proof of Proposition 3.1 of [9] shows that most Fibonacci sequences in \( \mathcal{P} \) are admissible in the sense that there exists a non-empty Zariski open subset \( \mathcal{U} \) of GL_{2}(\mathbb{C})^2 such that any pair \((W_1, W_2) ∈ \mathcal{U} ∩ \mathcal{P} \) generates an admissible Fibonacci sequence in \( \mathcal{P} \).

We define also the **norm** \( \|A\| \) of a matrix \( A \) with real coefficients to be the largest absolute value of its coefficients. With this notation, we will prove in §3 the following characterization of extremal real numbers which translates in the present setting several results from [7] and [8].

**Theorem 2.1.** For each extremal real number \( ξ \) there exists an unbounded admissible Fibonacci sequence \((W_i)_{i≥1}\) in \( \mathcal{P} \) which satisfies

\[
\|W_{i+1}\| ≫ ≪ \|W_i\|^γ, \quad \|(ξ, -1)W_i\| ≫ ≪ \|W_i\|^{-1} \quad \text{and} \quad |\det W_i| ≫ ≪ 1,
\]

with implied constants that are independent of \( i \). Such a sequence is uniquely determined by \( ξ \) up to its first terms, and up to term-by-term multiplication by a Fibonacci sequence in \( \{±1\} \). Conversely, any unbounded admissible Fibonacci sequence \((W_i)_{i≥1}\) in \( \mathcal{P} \) which satisfies

\[
\|W_{i+2}\| ≫ \|W_{i+1}\|\|W_i\| \quad \text{and} \quad |\det W_i| ≪ 1,
\]

also satisfies the conditions (2) for some extremal real number \( ξ \).

Thus, any unbounded admissible Fibonacci sequence in \( \mathcal{P} \) satisfying (3) is **associated** to some extremal real number \( ξ \) in the sense that it satisfies (2). Note also that, since \( γ^2 = γ + 1 \), the first condition in (2) is stronger than the first condition in (3).

It is shown in §2 of [6] and in §6 of [7] that, for any choice of distinct positive integers \( a \) and \( b \), the real number \( ξ_{a,b} \) whose continued fraction expansion \( ξ_{a,b} = [0, a, b, a, a, \ldots] \) is given by 0 followed by the elements of \( f_{a,b} \) is an extremal real number. More generally, we will prove the following result (see [11]).

**Theorem 2.2.** A real number \( ξ \) is extremal with an associated Fibonacci sequence in GL_{2}(\mathbb{Z}) if and only if the sequence of its partial quotients in its continued fraction expansion coincides, up to its first terms, with the limit of a Fibonacci sequence of words \((w_i)_{i≥1}\) in \((\mathbb{N} \setminus \{0\})^*\) starting with two non-commuting words \( w_1 \) and \( w_2 \).
Let \( w_\infty = a_1 a_2 a_3 \ldots \) be the limit of a Fibonacci sequence of words \( (w_i)_{i \geq 1} \) in \((\mathbb{N} \setminus \{0\})^*\) starting with non-empty words \( w_1 \) and \( w_2 \). If \( w_1 \) and \( w_2 \) commute, then \( w_\infty = \lim_{i \to \infty} (w_i)^i \) is a periodic word and so \( \xi = [0, a_1, a_2, \ldots] \) is a quadratic real number. Conversely, if \( w_1 \) and \( w_2 \) do not commute, the above theorem shows that this real number \( \xi \) is extremal. Since an extremal real number is not quadratic, the infinite word \( w_\infty \) cannot in this case be ultimately periodic.

The next result provides a sufficient condition for an extremal real number to be badly approximable.

**Theorem 2.3.** Let \( E = \{a, b\} \) be an alphabet of two letters, let \( (w_k)_{k \geq 1} \) be the Fibonacci sequence in \( E^* \) generated by \( w_1 = b \) and \( w_2 = a \), and let \( f_{a,b} = \lim_{k \to \infty} w_k \). Let \( \xi \) be an extremal real number and let \( (W_k)_{k \geq 1} \) be a Fibonacci sequence in \( \mathcal{P} \) which is associated to \( \xi \). Consider the morphism of monoids \( \Phi: E^* \to \mathcal{P} \) mapping \( w_k \) to \( W_k \) for each \( k \geq 1 \). For each \( i \geq 1 \), denote by \( u_i \) the prefix of \( f_{a,b} \) with length \( i \), and put \( U_i = \Phi(u_i) \). Then, we have

\[
\| (\xi, -1) U_i \| \gg \ll \left| \frac{\det U_i}{\| U_i \|} \right|
\]

with implied constants that do not depend on \( i \). Moreover, if the sequence \( (\det U_i)_{i \geq 1} \) is bounded, then \( \xi \) is badly approximable.

It would be interesting to know if, conversely, the sequence \( (\det U_i)_{i \geq 1} \) is bounded when \( \xi \) is badly approximable. The proof of the above result is given in §5.

Going back to the definitions, we note that, if \( \xi \) is badly approximable (resp. extremal) and if \( a, b \in \mathbb{Q} \) with \( a \neq 0 \), then \( a\xi + b \) and \( 1/\xi \) are as well badly approximable (resp. extremal). This implies that the set of badly approximable real numbers is stable under the action of \( \text{GL}_2(\mathbb{Q}) \) on \( \mathbb{R} \setminus \mathbb{Q} \) by linear fractional transformations. Our last main result, proved in §6, is that there exist orbits which do not contain any of the numbers produced by Theorem 2.2.

**Theorem 2.4.** There exist badly approximable extremal real numbers which are not conjugate under the action of \( \text{GL}_2(\mathbb{Q}) \) to any extremal real number having an associated Fibonacci sequence in \( \text{GL}_2(\mathbb{Z}) \).

3. **Proof of Theorem 2.2**

The following lemma gathers essentially all facts that we will need from [7] and [8].

**Lemma 3.1.** Let \( \xi \) be an extremal real number. Then, there exists an unbounded sequence of symmetric matrices \( (y_i)_{i \geq 1} \) in \( \mathcal{P} \) such that, for each \( i \geq 1 \), we have

\[
\| y_{i+1} \| \gg \ll \| y_i \|^{\gamma}, \quad \| (\xi, -1) y_i \| \gg \ll \| y_i \|^{-1} \quad \text{and} \quad |\det y_i| \gg \ll 1,
\]

with implied constants that are independent of \( i \). Such a sequence \( (y_i)_{i \geq 1} \) is uniquely determined by \( \xi \) up to its first terms and up to multiplication of each of its terms by \( \pm 1 \). Moreover,
for any such sequence, there exists a non-symmetric and non-skew-symmetric matrix \( M \in \mathcal{P} \) such that
\[
 y_{i+2} = \pm \begin{cases} 
 y_{i+1} \ast M \ast y_i & \text{if } i \text{ is odd,} \\
 y_{i+1} \ast tM \ast y_i & \text{if } i \text{ is even,}
\end{cases}
\]
for any sufficiently large index \( i \). Conversely, if \((y_i)_{i \geq 1}\) is an unbounded sequence of symmetric matrices in \( \mathcal{P} \) which satisfies a recurrence relation of the type (6) for some non-symmetric matrix \( M \in \mathcal{P} \), and if
\[
 \|y_{i+2}\| \gg \|y_{i+1}\| \|y_i\| \quad \text{and} \quad |\det y_i| \ll 1,
\]
then \((y_i)_{i \geq 1}\) also satisfies the estimates (5) for some extremal real number \( \xi \).

**Proof.** The first assertion in this proposition comes from Theorem 5.1 of [7] upon noting that, for an arbitrary symmetric matrix \( y = \begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} \), we have
\[
 \|\begin{pmatrix} \xi, -1 \end{pmatrix} y\| = \max\{|y_0 \xi - y_1|, |y_1 \xi - y_2|\} \gg \max\{|y_0 \xi - y_1|, |y_0 \xi^2 - y_2|\},
\]
with implied constants depending only on \( \xi \). The second assertion follows from Proposition 4.1 of [8], the third one from Corollary 4.3 of [8], and the last one from Proposition 5.1 of [8]. □

In the proof of Theorem 2.1 below, we use repeatedly the following observation (the proof of which is omitted).

**Lemma 3.2.** Let \((W_i)_{i \geq 1}\), \((y_i)_{i \geq 1}\) and \((N_i)_{i \geq 1}\) be sequences in \( \mathcal{P} \), and let \( \xi \in \mathbb{R} \). Assume that the sequence \((N_i)_{i \geq 1}\) is bounded and that \( y_i = W_i \ast N_i \) for each \( i \geq 1 \). Then, we have
\[
 \|W_i\| \ll \|y_i\|, \quad \|\begin{pmatrix} \xi, -1 \end{pmatrix} W_i\| \ll \|\begin{pmatrix} \xi, -1 \end{pmatrix} y_i\| \quad \text{and} \quad |\det(W_i)| \ll |\det(y_i)|,
\]
with implied constants that do not depend on \( i \).

**Proof of Theorem 2.1.** Let \( \xi \in \mathbb{R} \) be extremal. Then, lemma 3.1 provides an unbounded sequence of symmetric matrices \((y_i)_{i \geq 1}\) in \( \mathcal{P} \) and a non-symmetric and non-skew-symmetric matrix \( M \in \mathcal{P} \) satisfying both the estimates (5) and the recurrence relation (6) for each sufficiently large \( i \). Omitting if necessary a finite even number of initial terms in the sequence \((y_i)_{i \geq 1}\), we may assume, without loss of generality, that (6) holds for each \( i \geq 1 \). Then, for a suitable choice of signs, the formula \( W_i = \pm (y_i \ast M_i) \) with \( M_i = tM \) if \( i \) is odd and \( M_i = M \) if \( i \) is even, defines an admissible Fibonacci sequence in \( \mathcal{P} \) (the corresponding matrix \( N \) is the inverse of \( M \) in the group \( \mathcal{P} \)). Moreover, the estimates (5) together with Lemma 3.2 show that this sequence satisfies the conditions (2) of Theorem 2.1. This proves the first assertion of the theorem.
Now, let \((W_i')_{i \geq 1}\) be any unbounded admissible Fibonacci sequence satisfying, like \((W_i)_{i \geq 1}\), the conditions \((2)\), and let \(N' \in \mathcal{P}\) such that, upon putting \(N_i' = \lceil N' \rceil\) for \(i\) odd and \(N_i' = N'\) for \(i\) even, the matrix \(y_i' = W_i' * N_i'\) is symmetric for each \(i \geq 1\). Then, lemma 3.2 shows that \((y_i')_{i \geq 1}\) satisfies, like \((y_i)_{i \geq 1}\), the estimates \((5)\). Consequently, by Lemma 3.1 there exist integers \(k, \ell \geq 0\) such that \(y_{i+k} = \pm y_{i+\ell}\) for each \(i \geq 1\). Since we have \(y_{i+2} = \pm (W_{i+1} * y_i)\) and \(y_{i+2} = \pm (W_{i+1} * y_i')\) for \(i \geq 1\), this implies that \(W'_{i+k} = \pm W_{i+\ell}\) for each \(i \geq 2\). Moreover, the signs \(\pm\) in the last formula must come from a Fibonacci sequence in \(\{\pm 1\}\). This proves the second assertion of the theorem.

Finally, let \((W_i)_{i \geq 1}\) be any unbounded admissible Fibonacci sequence satisfying the conditions \((3)\) in Theorem 2.1 without reference to a given extremal real number \(\xi\), and let \(N \in \mathcal{P}\) such that, upon putting \(N_i = \lceil N \rceil\) for \(i\) odd and \(N_i = N\) for \(i\) even, the matrix \(y_i = W_i * N_i\) is symmetric for each \(i \geq 1\). Then, lemma 3.2 shows that \((y_i)_{i \geq 1}\) satisfies the conditions \((7)\) in Lemma 3.1. Consequently it satisfies the stronger conditions \((5)\) for some extremal real number \(\xi\), and thus, by Lemma 3.2 satisfies the estimates \((2)\) of Theorem 2.1 for the same \(\xi\).

4. Proof of Theorem 2.2

Serret’s theorem asserts that two real numbers have continued fraction expansions which coincide up to their first terms if and only if these numbers belong to the same orbit under the action of \(GL_2(\mathbb{Z})\) by linear fractional transformations (Theorem 6B of [10]). Our proof for Theorem 2.2 is inspired from the proof of this result given by Cassels in §3, Chap. I of [1]. We break it into two propositions. To establish the first one, we need the following auxiliary result which provides the link with continued fractions.

**Lemma 4.1.** Let \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})\) with \(d \geq 1\). Then, there is one and only one choice of integers \(s \geq 1\) and \(a_0, a_1, \ldots, a_s\) with \(a_1, \ldots, a_{s-1} \geq 1\) such that

\[
A = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_s & 1 \\ 1 & 0 \end{pmatrix}.
\]

These integers are also characterized by the properties

\[
\frac{b}{d} = [a_0, \ldots, a_{s-1}], \quad \frac{c}{d} = [a_s, \ldots, a_1], \quad \det(A) = (-1)^{s+1}.
\]

**Proof.** Induction on \(s\) shows that, if \(A\) can be written in the form \((8)\) for a choice of integers \(s \geq 1\) and \(a_0, a_1, \ldots, a_s\) with \(a_1, \ldots, a_{s-1} \geq 1\), then we have \(b/d = [a_0, \ldots, a_{s-1}]\). Taking the transpose of both sides of \((8)\), this observation also provides \(c/d = [a_s, \ldots, a_1]\). Moreover the last equality in \((9)\) follows from the multiplicativity of the determinant. Since each rational number has exactly two continued fraction expansions with lengths differing by one, this proves the uniqueness of the factorization \((8)\), when it exists.
Now, without making assumptions on \( A \), define an integer \( s \geq 1 \) and a sequence of integers \( a_1, \ldots, a_s \) with \( a_1, \ldots, a_{s-1} \geq 1 \) by the conditions \( c/d = [a_s, \ldots, a_1] \) and \( \det(A) = (-1)^{s+1} \). Define also \( a_0 \) to be the integer for which the distance between \( b/d \) and \( [a_0, \ldots, a_{s-1}] \) is at most 1/2. Then, by the above observations, the right hand side of (8) is a matrix \( A' = \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \) with the same determinant as \( A \), satisfying \( d' \geq 1 \), \( c'/d' = c/d \) and \( |b'/d' - b/d| \leq 1/2 \). Since \((c, d)\) and \((c', d')\) are rows of matrices in \( \text{GL}_2(\mathbb{Z}) \), they are primitive points of \( \mathbb{Z}^2 \) and the relation \( c'/d' = c/d \) implies \( (c', d') = \pm(c, d) \). Since \( d \) and \( d' \) are positive, we deduce that \( A \) and \( A' \) have the same second row \((c', d') = (c, d) \). Since these matrices also have the same determinant, this forces \((a', b') = (a, b) + k(c, d)\) for some integer \( k \). Then, we find \( |b'/d' - b/d| = |k| \), thus \( k = 0 \) and therefore \( A' = A \). \( \square \)

**Corollary 4.2.** Let \( S_1 \) denote the set of matrices \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Mat}_{2 \times 2}(\mathbb{R}) \) with \( a \geq \max\{b, c\} \) and \( \min\{b, c\} \geq d \geq 0 \), and define \( S = S_1 \cap \text{GL}_2(\mathbb{Z}) \). Then, \( S_1 \) and \( S \) are closed under multiplication and transposition. Moreover, the map from \( \mathbb{N} \setminus \{0\} \) to \( S \) sending \( a \) to \( \left( \begin{array}{cc} a & 1 \\ 0 & 1 \end{array} \right) \) for each \( a \in \mathbb{N} \setminus \{0\} \) extends to an isomorphism of monoids \( \sigma: (\mathbb{N} \setminus \{0\})^* \rightarrow S \cup \{I\} \).

**Proof.** The only delicate point here is the surjectivity of the map \( \sigma \). Clearly, any matrix \( A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in S \) is in the image of \( \sigma \) if \( d = 0 \) because then we have \( a \geq b = c = 1 \). If \( d \geq 1 \), we note that the integers \( a_0 \) and \( a_s \) given by Lemma 4.1 are respectively the integral parts of \( b/d \) and \( c/d \) except in the case where \( d = 1 \) and \( \det A = -1 \). In the latter case, we have \( s = 2 \), \( a_0 = b - 1, a_1 = 1, a_2 = c - 1 \), and also \( a = bc - 1 \). Then, the condition \( a \geq \max\{b, c\} \geq 1 \) implies \( b, c \geq 2 \) and so \( a_0, a_s \geq 1 \). Otherwise, the condition \( \min\{b, c\} \geq d \geq 1 \) ensures that \( a_0, a_s \geq 1 \). So, in both cases, the integers \( a_0, \ldots, a_s \) are positive, and \( A \) is the image of \( (a_0, \ldots, a_s) \) under \( \sigma \). \( \square \)

The following proposition presents a first step towards the proof of Theorem 2.2.

**Proposition 4.3.** The set of extremal real numbers with an associated (admissible) Fibonacci sequence in \( \text{GL}_2(\mathbb{Z}) \) is stable under the action of \( \text{GL}_2(\mathbb{Z}) \) by linear fractional transformations. Any orbit contains an extremal real number with an associated Fibonacci sequence in \( S \).

**Proof.** Let \( \xi \) be an extremal real number with an associated Fibonacci sequence \( (W_i)_{i \geq 1} \) in \( \text{GL}_2(\mathbb{Z}) \). This sequence being admissible, there exists \( N \in \mathcal{P} \) with \( N \neq \pm \mathbf{i}N \) such that, upon putting \( N_i = N \) if \( i \) is even and \( N_i = \mathbf{i}N \) if \( i \) is odd, the product \( y_i = W_i N_i \) is symmetric for each \( i \geq 1 \).

For each \( U \in \text{GL}_2(\mathbb{Z}) \), the sequence \( (W'_i)_{i \geq 1} = (U^{-1}W_iU)_{i \geq 1} \) is a Fibonacci sequence in \( \text{GL}_2(\mathbb{Z}) \). It is admissible with corresponding matrix \( N' = U^{-1}N \mathbf{i}U^{-1} \), and it satisfies the conditions (2) of Theorem 2.1 with \( W_i \) replaced by \( W'_i \) and \( \xi \) replaced by the real number
\(\eta\) such that \((\eta, -1)\) is proportional to \((\xi, -1)U\). By varying \(U\), we get in this way all real numbers \(\eta\) which are conjugate to \(\xi\) under \(GL_2(\mathbb{Z})\). So, these numbers are extremal with an associated Fibonacci sequence in \(GL_2(\mathbb{Z})\). This proves the first assertion of the lemma.

For the second assertion, let \(1/\xi = [a_0, a_1, a_2, \ldots]\) be the continued fraction expansion of \(1/\xi\). Put \(d = \text{det}(N)\) and \(M = dN^{-1}\), so that we have \(M \in \mathcal{P}\) and \(W_i = d^{-1}y_iM_i\) where \(M_i = M\) if \(i\) is even and \(M_i = ^tM\) if \(i\) is odd. Since \(M\) is not skew-symmetric and since \([\mathbb{Q}(\xi) : \mathbb{Q}] > 2\), the product

\[
\theta = \begin{pmatrix} 1/\xi & 1 \\ 1 & 1 \end{pmatrix} M \begin{pmatrix} 1/\xi \\ 1 \end{pmatrix}
\]

is non-zero. Replacing \(N\) by \(-N\) if necessary, so that \(M\) is replaced by \(-M\), we may assume without loss of generality that this number \(\theta\) is positive. For each \(k \geq 1\), define

\[
U_k = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.
\]

Then, the standard recurrence relations in the theory of continued fractions show that we have \(U_k = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}\) where \(p_k/q_k = [a_0, \ldots, a_k]\) denotes the \(k\)-th convergent of \(1/\xi\) written in reduced form. Since \(|q_k(1/\xi) - p_k| \leq 1/q_{k+1}\) for each \(k \geq 0\), this gives

\[
U_k = \begin{pmatrix} 1/\xi & 1 \\ 1 & 1 \end{pmatrix} (q_k q_{k-1}) + \mathcal{O}(1/q_k)
\]

and thus

\[
^tU_k MU_k = \theta \begin{pmatrix} q_k^2 & q_{k-1}q_k \\ q_k q_{k-1} & q_{k-1}^2 \end{pmatrix} + \mathcal{O}(1).
\]

The latter matrix belongs to \(\mathcal{S}_1\) if \(k\) sufficiently large, because we have \(q_k > q_{k-1}\) for each \(k \geq 2\), and \(q_{k-1}\) tends to infinity with \(k\). Fix such a value of \(k\). Since \(\mathcal{S}_1\) is closed under transposition, we get \(^tU_k M U_k \in \mathcal{S}_1\) for each \(i \geq 1\). We claim that \(\epsilon_i U_k^{-1} W_i U_k\) also belongs to \(\mathcal{S}_1\) for an appropriate choice of \(\epsilon_i \in \{-1, 1\}\) and each sufficiently large \(i\). To prove this, we note that the product \(U_k^{-1} \begin{pmatrix} 1/\xi & 1 \\ 1 & 1 \end{pmatrix}\) is proportional to \(\begin{pmatrix} r \xi & 1 \\ 1 & 1 \end{pmatrix}\) where \(r = [a_{k+1}, a_{k+2}, \ldots]\) is a real number with \(r > 1\). Since, for each \(i \geq 1\), we have

\[
y_i = y_{i,2} \begin{pmatrix} \xi^{-2} & \xi^{-1} \\ \xi^{-1} & 1 \end{pmatrix} + \mathcal{O}(\|W_i\|^{-1})
\]

with \(y_{i,2} \in \mathbb{Z}\), we find

\[
U_k^{-1} y_i U_k^{-1} = c_i \begin{pmatrix} r^2 & r \\ r & 1 \end{pmatrix} + \mathcal{O}(\|W_i\|^{-1}),
\]

for some \(c_i \in \mathbb{R}\). Thus, if \(i\) is sufficiently large, say \(i \geq i_0\), the matrix \(\pm U_k^{-1} y_i U_k^{-1}\) belongs to \(\mathcal{S}_1\) for an appropriate choice of sign \(\pm\). Multiplying this matrix on the right by \(^tU_k M_i U_k\) which also belongs to \(\mathcal{S}_1\), we deduce that \(\pm U_k^{-1} y_i M_i U_k \in \mathcal{S}_1\) for the same choice of sign and thus that \(\epsilon_i U_k^{-1} W_i U_k \in \mathcal{S}_1\) for some \(\epsilon_i \in \{-1, 1\}\). Since \(U_k \in GL_2(\mathbb{Z})\) and since \(W_i \in GL_2(\mathbb{Z})\) for each \(i \geq 1\), we conclude that \((\epsilon_i U_k^{-1} W_i U_k)_{i \geq i_0}\) is an admissible Fibonacci sequence in \(\mathcal{S}\).
By the first part of the proof, it is associated to an extremal real number $\eta$ in the same \(GL_2(\mathbb{Z})\)-orbit as $\xi$. \hfill \Box

We also need the following technical result.

**Lemma 4.4.** Let \((W_i)_{i \geq 1}\) be a Fibonacci sequence in \(\mathcal{P}\). If \(W_1\) and \(W_2\) do not have a common eigenvector in \(\mathbb{Q}^2\), and satisfy \(W_1W_2 \neq \pm W_2W_1\), then \((W_i)_{i \geq 1}\) is an admissible Fibonacci sequence.

**Proof.** We first note that there exists a non-zero primitive matrix \(N \in \text{Mat}_{2 \times 2}(\mathbb{Z})\) such that \(W_1^tN\), \(W_2N\) and \(W_3^tN\) are symmetric because these three conditions translate into a system of three homogeneous linear equations in the four unknown coefficients of \(N\). Fix such a choice of \(N\) and define accordingly \(N_i = t^i N\) for \(i\) odd and \(N_i = N\) for \(i\) even. Then, the product \(W_iN_i\) is symmetric for \(i = 1, 2, 3\) and using the relation of proportionality

\[
W_{i+3}N_{i+3} \propto (W_{i+1}N_{i+1})N_{i+1}^{-1}(W_iN_i)N_i^{-1}(W_{i+1}N_{i+1}),
\]

we deduce by induction on \(i\) that \(W_iN_i\) is symmetric for each \(i \geq 1\).

If \(\det N = 0\), then we can write \(N = A^tB\) with non-zero column vectors \(A\) and \(B\) in \(\mathbb{Q}^2\). Since \(W_1^tN = (W_1B)^tA\) is symmetric, we deduce that \(W_1B \propto A\). Similarly, since \(W_2N = (W_2A)^tB\) and \(W_3^tN = (W_3B)^tA\) are symmetric, we find that \(W_2A \propto B\) and \(W_3B \propto A\). Using the first two relations of proportionality, we also get \(W_3B \propto W_2(W_1B) \propto W_2A \propto B\). As \(W_3B \neq 0\), this shows that \(A \propto B\), and thus that \(B\) is a common eigenvector of \(W_1\) and \(W_2\), against the hypothesis. Thus we have \(N \in \mathcal{P}\). We also note that

\[
W_2W_1^tN = t(W_2W_1^tN) = t(W_1^tN)^tW_2 = (W_1^tN)^tW_2 = W_1^t(W_2N) = W_1W_2N.
\]

Since \(W_1W_2 \neq \pm W_2W_1\), this implies that \(tN \neq \pm N\). Thus, the sequence \((W_i)_{i \geq 1}\) is admissible. \hfill \Box

The hypotheses of Lemma 4.4 are satisfied for example when the matrices \(W_1\), \(W_2\), \(W_1W_2\) and \(W_2W_1\) are linearly independent over \(\mathbb{Q}\). The corollary below provides another instance where this lemma applies.

**Corollary 4.5.** Any Fibonacci sequence \((W_i)_{i \geq 1}\) in \(\mathcal{S}\) generated by two non-commuting matrices \(W_1, W_2 \in \mathcal{S}\) is admissible.

**Proof.** Since \(\mathcal{S} \subset GL_2(\mathbb{Z})\), the eigenvalues of a matrix \(W \in \mathcal{S}\) are algebraic units. So, if one of them is rational, both of them belong to \(\{-1, 1\}\). Since the only matrices of \(\mathcal{S}\) with trace at most 2 are \(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\) which have no rational eigenvalue, we deduce that no matrix of \(\mathcal{S}\) has a rational eigenvalue. In particular, any \(W_1, W_2 \in \mathcal{S}\) do not share a common eigenvector in \(\mathbb{Q}^2\). Since such matrices have non-negative coefficients and non-zero product,
they also satisfy $W_1 W_2 \neq -W_2 W_1$. Thus, if they do not commute, lemma 4.4 shows that they generate an admissible Fibonacci sequence. \qed

Serret’s theorem combined with Proposition 4.3 reduces the proof of Theorem 2.2 to the following statement.

**Proposition 4.6.** A real number $\xi$ is extremal with an associated Fibonacci sequence in $S$ if and only if its continued fraction expansion is of the form $[0, a_1, a_2, \ldots]$ where $(a_1, a_2, \ldots)$ is the limit of a Fibonacci sequence of words $(w_i)_{i \geq 1}$ in $(\mathbb{N} \setminus \{0\})^*$ starting with two non-commuting words $w_1$ and $w_2$.

Proof. Let $\xi = [0, a_1, a_2, \ldots]$ where $(a_1, a_2, \ldots)$ is the limit of a sequence of words $(w_i)_{i \geq 1}$ in $(\mathbb{N} \setminus \{0\})^*$ starting with two non-commuting words $w_1$ and $w_2$. Denote by $(W_i)_{i \geq 1}$ the image of the sequence $(w_i)_{i \geq 1}$ under the isomorphism of monoids $\sigma: (\mathbb{N} \setminus \{0\})^* \rightarrow S \cup \{I\}$ defined in Corollary 4.2. Since $w_1$ and $w_2$ do not commute, the same is true of $W_1$ and $W_2$ and so, by Corollary 4.3, $(W_i)_{i \geq 1}$ is an admissible Fibonacci sequence in $S$. We also note that, for each pair of matrices $A, B \in S$, we have $\|AB\| > \|A\|\|B\|$. Then, the relation $W_{i+2} = W_{i+1} W_i$ implies $\|W_{i+2}\| > \|W_{i+1}\|\|W_i\|$ for each $i \geq 1$. In particular, the sequence $(W_i)_{i \geq 1}$ is unbounded. As $|\det W_i| = 1$ for each $i$, it also satisfies the conditions (3) of Theorem 2.1. Thus, the sequence $(W_i)_{i \geq 1}$ is associated to some extremal real number $\eta$. On the other hand, the theory of continued fractions shows that $\|(\xi, -1) W_i\| \ll \|W_i\|^{-1}$ since the ratios of the elements in the columns of $W_i$ are successive convergents of $1/\xi$. Thus, $\xi = \eta$ is extremal.

Conversely, let $\xi$ be an extremal real number with an associated Fibonacci sequence $(W_i)_{i \geq 1}$ in $S$. The inverse image of this sequence under $\sigma$ is a Fibonacci sequence $(w_i)_{i \geq 1}$ in $(\mathbb{N} \setminus \{0\})^*$ and, as above, we deduce that $\xi = [0, a_1, a_2, \ldots]$ where $(a_1, a_2, \ldots) = \lim_{i \to \infty} w_i$. Since $\xi$ is neither rational nor quadratic, this sequence is infinite and ultimately not periodic. In particular, $w_1$ and $w_2$ are not both powers of the same word, and so they do not commute (Proposition 1.3.2 of Chapter 1 of [4]). \qed

**Remark.** Let $w_\infty = (1, 2, 3, 1, 2, 1, 2, 3, \ldots)$ be the limit of the Fibonacci sequence $(w_i)_{i \geq 1}$ generated by $w_1 = (3)$ and $w_2 = (1, 2)$. Since $w_1 w_2 \neq w_2 w_1$, the corresponding real number $\xi = [0, 1, 2, 3, 1, 2, 1, 2, 3, \ldots]$ is extremal. However, contrary to the generic Fibonacci word $f_{a,b}$ which contains palindromes of arbitrary length as prefixes, the infinite word $w_\infty$ contains no factor of length greater than 3 which is a palindrome.

5. **Proof of Theorem 2.3**

Throughout this section, the notation is the same as in Theorem 2.3. Namely, we fix an alphabet $E = \{a, b\}$ of two letters and denote by $(w_k)_{k \geq 1}$ the Fibonacci sequence in $E^*$ generated by $w_1 = b$ and $w_2 = a$, with limit $f_{a,b}$. We also fix an extremal real number $\xi$ with
an associated Fibonacci sequence \( (W_k)_{k \geq 1} \) in \( \mathcal{P} \), and denote by \( \Phi: E^* \to \mathcal{P} \) the morphism of monoids mapping \( w_k \) to \( W_k \) for each \( k \geq 1 \). We start with the following observation.

**Lemma 5.1.** Let \( k \) and \( \ell \) be integers with \( k \geq \ell \geq 2 \), and let \( w_k = uv \) be a factorization of \( w_k \) in \( E^* \). Then, there exist a prefix \( u_0 \) of \( w_\ell \) and strictly decreasing sequences of integers \( i_1 > i_2 > \cdots > i_s \) and \( j_1 > j_2 > \cdots > j_t \) bounded below by \( \ell \) such that

\[
u = w_{i_1}w_{i_2} \cdots w_{i_s}u_0 \quad \text{and} \quad u_0v = w_{j_1} \cdots w_{j_t}w_{j_1}.
\]

If \( u \) is not a prefix of \( w_\ell \), we can ask that \( i_1 \leq k - 1 \) and \( j_1 \leq k - 2 \).

**Proof.** If \( u \) is a prefix of \( w_\ell \), we take \( u_0 = u \) so that \( u_0v = w_k \). Otherwise, we have \( k > \ell \), thus \( k \geq 3 \) and the factorization \( w_k = w_{k-1}w_{k-2} \) implies that either there is a word \( u' \) such that \( u = w_{k-1}u' \) and \( u'v = w_{k-2} \), or we have \( k \geq \ell + 2 \) and there is a word \( v' \) such that \( v = v'w_{k-2} \) and \( uv' = w_{k-1} \). The result then follows by induction on \( k \). \( \square \)

Since the sequence \( (W_i)_{i \geq 1} \) is admissible, there exists a non-symmetric and non-skew-symmetric matrix \( N \) such that, upon putting \( N_i = N \) if \( i \) is even and \( N_i = \acute{N} \) if \( i \) is odd, the product \( y_i = W_iN_i \) is symmetric for each \( i \geq 1 \). This matrix \( y_i \) may not be primitive but, for the next result, it is convenient not to normalize it.

**Lemma 5.2.** Define \( L = \max\{1, |\xi|\}^{-1}(1, \xi) \) and \( \theta = LN^{-1}(\acute{L}) \). Then, there exist an index \( \ell \geq 1 \) and a constant \( c \geq 1 \) such that, for any sequence of integers \( (i_1, \ldots, i_s) \) with entries bounded below by \( \ell \) and repeated at most twice, we have

\[
\frac{1}{c} \leq \frac{\|W_{i_1}W_{i_2} \cdots W_{i_s}\|}{|\theta|^s\|y_{i_1}\|\|y_{i_2}\| \cdots \|y_{i_s}\|} \leq c.
\]

Note that we have \( \theta \neq 0 \) since \( \xi \) is transcendental and \( N \) is not skew-symmetric.

**Proof.** Write \( y_i = \begin{pmatrix} y_{i,0} & y_{i,1} \\ y_{i,1} & y_{i,2} \end{pmatrix} \) for each \( i \geq 1 \). As \( |(\xi,-1)y_i| \ll |(\xi,-1)W_i| \ll \|W_i\|^{-1} \), we have

\[y_i = y_{i,0} \begin{pmatrix} 1 \\ \xi \end{pmatrix} (1, \xi) + O(\|W_i\|^{-1}),\]

and so \( \|y_i\| = |y_{i,0}| \max\{1, |\xi|\}^2 + O(\|W_i\|^{-1}) \). In particular, this shows that \( y_{i,0} \neq 0 \) for each sufficiently large \( i \), say for \( i \geq \ell \). Then, for those values of \( i \), we find

\[
\frac{W_i}{\theta\|y_i\|} = A_i + R_i
\]

where \( R_i = O(\|W_i\|^{-2}) \) and where \( A_i = \pm \theta^{-1}(\acute{L})LN_i^{-1} \) belongs to the set

\[\mathcal{A} = \left\{ \pm I, \pm \frac{1}{\theta}LLN^{-1}, \pm LLL'N^{-1}\right\} .\]

Since \( \theta = LN^{-1}(\acute{L}) = L(\acute{N}^{-1})(\acute{L}) \), the set \( \mathcal{A} \) is stable under multiplication.
Now, let \((i_1, \ldots, i_s)\) be any sequence of integers bounded below by \(\ell\), with no entry repeated more than twice. Using (10), we find

\[
\frac{W_{i_1} \cdots W_{i_s}}{\theta^s \|y_{i_1}\| \cdots \|y_{i_s}\|} = A + R
\]

where \(A = A_{i_1} \cdots A_{i_s}\) belongs to \(A\) and where \(R\) is a sum, indexed by all non-empty subsequences \((j_1, \ldots, j_t)\) of \((i_1, \ldots, i_s)\), of products of the form \(B_1R_{j_1} \cdots B_tR_{j_t}B_{t+1}\) with \(B_1, \ldots, B_{t+1} \in A\). Thus, for an appropriate constant \(\kappa > 0\), we have

\[
\|R\| \leq (1 + \kappa\|W_{i_1}\|^{-2}) \cdots (1 + \kappa\|W_{i_s}\|^{-2}) - 1 \leq \exp\left(2\kappa \sum_{i=\ell}^{\infty} \|W_i\|^{-2}\right) - 1.
\]

If \(\ell\) is sufficiently large, this gives \(\|R\| \leq \|A\|/2\), and so \(\|A + R\| \ll \|A\| \gg 1\), as requested.

\[\square\]

**Lemma 5.3.** Let \(k\) be a positive integer and let \(w_k = uv\) be a factorization of \(w_k\) in \(E^*\). Put \(U = \Phi(u)\) and \(V = \Phi(v)\). Then, we have \(\|UV\| \ll \|U\|\|V\|\) with implied constants that are independent of \(k, u\) and \(v\).

**Proof.** Let \(\ell\) be as in Lemma 5.2. Without loss of generality, we may assume that \(k \geq \ell\). Then, according to Lemma 5.1, we can write \(u = w_{i_1} \cdots w_{i_s}u_0\) and \(u_0v = w_{j_1} \cdots w_{j_t}\), where \(u_0\) is a prefix of \(w_\ell\) and where \((i_1, \ldots, i_s)\) and \((j_1, \ldots, j_t)\) are strictly decreasing sequences of integers bounded below by \(\ell\). Put

\[P = W_{i_1} \cdots W_{i_s}, \quad Q = W_{j_1} \cdots W_{j_t} \quad \text{and} \quad U_0 = \Phi(u_0).
\]

Then, we have \(U = aPU_0\) and \(U_0V = bQ\) with non-zero rational numbers \(a\) and \(b\). Since \(U_0\) belongs to a finite set of matrices in \(P\), we deduce that

\[\|U\| \gg \|a\|\|P\| \quad \text{and} \quad \|V\| \gg \|b\|\|Q\|.
\]

Moreover, since the sequence \((i_1, \ldots, i_s, j_1, \ldots, j_t)\) has its entries repeated at most twice and bounded below by \(\ell\), Lemma 5.2 gives

\[\|PQ\| \ll |\theta|^{s+t}\|y_{i_1}\| \cdots \|y_{i_s}\|\|y_{j_1}\| \cdots \|y_{j_t}\| \gg \|P\|\|Q\|.
\]

The conclusion follows because \(UV = abPQ\).

\[\square\]

**Proof of Theorem 2.3.** We first note that, for any \((x, y) \in \mathbb{R}^2\) and \(U \in \text{GL}_2(\mathbb{R})\), we have

\[
\|(x, y)U\| \geq \frac{\|(x, y)UU^{-1}\|}{2\|U^{-1}\|} = \frac{\|(x, y)\|\det U}{2\|U\|}.
\]

Applying this to the point \((x, y) = (\xi, -1)\) and the matrix \(U_i = \Phi(u_i)\) where \(u_i\) denotes the prefix of \(f_{a,b}\) of length \(i\), we get

\[
\|((\xi, -1)U_i\| \geq \frac{\det U_i}{2\|U_i\|}.
\]
for each $i \geq 1$. To prove an upper bound of the same type for $\| (\xi, -1) U_i \|$, we denote by $k = k(i)$ the smallest positive integer such that $u_i$ is a prefix of $w_k$, and write $w_k = u_i v_i$ with $v_i \in E^\ast$. Putting $V_i = \Phi(v_i)$, we then have

\[(12)\quad U_i V_i = m_i W_k\]

for some integer $m_i \geq 1$. Applying \((11)\) to the point $(x, y) = (\xi, -1) U_i$ and the matrix $U = V_i$, we find

\[\| (\xi, -1) W_k \| = \frac{1}{m_i} \| (\xi, -1) U_i V_i \| \geq \frac{\| (\xi, -1) U_i \| \| \det V_i \|}{2m_i \| V_i \|} .\]

Since $\| (\xi, -1) W_k \| \ll \| W_k \|^{-1}$, this gives

\[(13)\quad \| (\xi, -1) U_i \| \ll \frac{m_i \| V_i \|}{\| W_k \| \| \det V_i \|} .\]

Applying Lemma 5.3 to the factorization \((12)\) on one hand, and taking determinants of both sides of \((12)\) on the other hand, we also find

\[\| V_i \| \gg \frac{m_i \| W_k \|}{\| U_i \|} \quad \text{and} \quad | \det V_i | = \frac{m_i^2 \| W_k \|}{| \det U_i |} \geq \frac{m_i^2}{| \det U_i |} .\]

These estimates combined with \((13)\) lead to

\[(14)\quad \| (\xi, -1) U_i \| \ll \frac{| \det U_i |}{\| U_i \|} ,\]

which completes the proof of \((4)\) in Theorem 2.3.

Now, assume that the integers $\det U_i$ are bounded independently of $i$ and, for each $i \geq 1$, choose a column $\left( \begin{array}{c} q_i \\ p_i \end{array} \right)$ of $U_i$ with the largest norm. Then, \((14)\) leads to

\[| q_i \xi - p_i | \ll \| U_i \|^{-1} .\]

Since $U_{i+1}$ is either equal to $U_i \ast W_1$ or to $U_i \ast W_2$, we also have $\| U_{i+1} \| \ll \| U_i \|$, and thus $| q_{i+1} | \ll \| U_i \| \ll | q_i |$. Combining these estimates and noting that $\gcd(p_i, q_i)$ is a divisor of $\det U_i$, we deduce the existence of a constant $c \geq 1$ such that

\[(15)\quad | q_{i+1}(q_i \xi - p_i) | \leq c \quad \text{and} \quad | q_{i+1} | \leq c \left| \frac{q_i}{\gcd(p_i, q_i)} \right| ,\]

for each $i \geq 1$. Moreover, we have $\lim \sup_{i \to \infty} | q_i | = \infty$ since $(U_i)_{i \geq 1}$ contains the unbounded sequence $(W_k)_{k \geq 1}$ as a subsequence. These facts imply that $\xi$ is badly approximable. Indeed, if $p/q$ is an arbitrary rational number, then, at the expense of replacing $c$ by a larger constant if necessary, we may assume that there exists an index $i \geq 1$ such that $0 < | q_i | \leq 2c | q | \leq | q_{i+1} |$. Using \((13)\), this gives $2 | q | \leq | q_i / \gcd(p_i, q_i) |$, thus $p/q \neq p_i/q_i$ and so we find

\[| \xi - \frac{p}{q} | \geq \left| \frac{p_i}{q_i} - \frac{p}{q} \right| - \left| \frac{p_i}{q_i} - \xi \right| \geq \frac{1}{| q q_i |} - \frac{c}{| q_i q_{i+1} |} \geq \frac{1}{2 | q q_i |} \geq \frac{1}{4c q^2} .\]

\[\square\]
6. Proof of Theorem 2.4

Again, let $E$ be a set of two elements $a$ and $b$, and let $(w_i)_{i \geq 1}$ be the Fibonacci sequence in $E^*$ determined by the conditions $w_1 = b$ and $w_2 = a$, with limit $f_{a,b}$. The following lemma is our main-tool for constructing more extremal real numbers.

**Lemma 6.1.** Let $m$ be a non-zero integer and let $W \in \text{Mat}_{2 \times 2}(\mathbb{Z})$ with $W^2 \equiv 0 \mod m$. Assume that there exist primitive matrices $W_1, W_2 \in \text{Mat}_{2 \times 2}(\mathbb{Z})$ of determinant $m$ with $W_1 \equiv W_2 \equiv W \mod m$, and consider the morphism of monoids $\Phi: E^* \rightarrow \mathcal{P}$ mapping $a$ to $W_2$ and $b$ to $W_1$. Then, for each word $u \in E^*$, the determinant of $\Phi(u)$ is 1 if $u$ has even length and it is $m$ if $u$ has odd length.

**Proof.** We proceed by recurrence on the length $\ell$ of $u$. If $\ell \leq 1$, the result is clear (for the empty word 1, the matrix $\Phi(1)$ is the identity). If $\ell = 2$, we have $\Phi(u) = (W_iW_j)_{\text{red}}$ for some choice of indices $i, j \in \{1, 2\}$. Then, since $W_iW_j \equiv W^2 \equiv 0 \mod m$ and since $\det(W_iW_j) = m^2$, the matrix $W_iW_j$ has content $|m|$, and so $\Phi(u) = |m|^{-1}W_iW_j$ has determinant 1. Now, assume that $\ell > 2$ and that the result is true for words of smaller length. Write $u = u'u''$ where $u'$ has even length and $u''$ has length 1 or 2. By induction hypothesis, $\Phi(u')$ has determinant 1 while $\Phi(u'')$ is primitive with det $\Phi(u'') = 1$ if $\ell$ is even and det $\Phi(u'') = m$ if $\ell$ is odd. Then the product $\Phi(u')\Phi(u'')$ is primitive and so $\Phi(u) = \Phi(u')\Phi(u'')$ has the same determinant as $\Phi(u'')$. \hfill $\Box$

We also need the following technical result.

**Lemma 6.2.** Let $r$ be a real number with $0 < r \leq 1$ and let $\mathcal{S}_r$ denote the set of matrices $A \in \text{Mat}_{2 \times 2}(\mathbb{R})$ with positive coefficients whose elements of the first row are bounded below by $r$ times those of the second row, and whose elements of the first column are bounded below by $r$ times those of the second column. Then, $\mathcal{S}_r$ is closed under multiplication and, for each $A, A' \in \mathcal{S}_r$, we have $\|AA'\| > r\|A\|\|A'\|$.\n
**Proof.** The set $\mathcal{S}_r$ consists of all $2 \times 2$ matrices $A$ with positive coefficients such that the products $(1, -r)A$ and $(1, -r)'A$ have non-negative coefficients. The fact that this set is closed under multiplication then follows from the associativity of the matrix product. To prove the second assertion, take $A, A' \in \mathcal{S}_r$. Let $(a, b)$ and $(a', b')$ denote respectively rows of $A$ and $'A'$ with largest norm. Since $a \geq rb$, we have

$$\|AA'\| \geq aa' + bb' \geq rb(a' + b') > rb\|A'\|.$$ 

Similarly, since $a' \geq rb'$, we find $\|AA'\| > rb'\|A\|$. If $b = \|A\|$ or $b' = \|A'\|$, this gives $\|AA'\| > r\|A\|\|A'\|$ as requested. Otherwise, we have $a = \|A\|$ and $a' = \|A'\|$ and we get the stronger inequality $\|AA'\| > \|A\|\|A'\|$. \hfill $\Box$

The next proposition is more specific than Theorem 2.4 and thereby proves it.
Proposition 6.3. Put \( W_1 = \begin{pmatrix} m & m \\ m-1 & m \end{pmatrix} \) and \( W_2 = \begin{pmatrix} 2m & m \\ 2m-1 & m \end{pmatrix} \) for a non-zero integer \( m \). Then the Fibonacci sequence \((W_i)_{i \geq 1}\) of \( P \) generated by these two matrices is associated to a badly approximable real number \( \xi \), and it satisfies \( \det W_i = m \) for each index \( i \) which is not divisible by 3. If \( |m| \) is not the square of an integer, then \( \xi \) is not conjugate under the action of \( \text{GL}_2(\mathbb{Q}) \) to an extremal real number having an associated Fibonacci sequence in \( \text{GL}_2(\mathbb{Z}) \).

Proof. A short computation shows that \( W_1, W_2, W_1W_2 \) and \( W_2W_1 \) are linearly independent over \( \mathbb{Q} \). Then, \( W_1 \) and \( W_2 \) fulfill the hypotheses of lemma 6.4 and so the sequence \((W_i)_{i \geq 1}\) is admissible. One can check that a corresponding matrix \( N \) is \( \begin{pmatrix} m & -m \\ -2m & 2m-1 \end{pmatrix} \). Moreover, for the given choice of \( m \), the matrices \( W_1 \) and \( W_2 \) satisfy the hypotheses of Lemma 6.1 with \( W = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \). Thus, defining the map \( \Phi: E^* \to P \) as in this lemma, we have \( \det \Phi(u) = 1 \) for each word \( u \in E^* \) of even length and \( \det \Phi(u) = m \) for each \( u \in E^* \) of odd length. Since the length of \( w_i \) is even if and only if \( i \) is divisible by 3, we deduce that \( W_i = \Phi(w_i) \) has determinant 1 when \( i \) is divisible by 3 and determinant \( m \) otherwise. In particular, we have \( |\det W_i| \leq |m| \) for each \( i \geq 1 \).

A short computation also gives \( W_3 = \pm \begin{pmatrix} 3m-1 & 3m \\ 3m-2 & 3m-1 \end{pmatrix} \) and shows, in the notation of Lemma 6.2, that \( \pm W_2 \) and \( \pm W_3 \) both belong to \( S_{1/2} \) for some appropriate choice of signs. Thus, for each \( i \geq 2 \), one of the matrices \( \pm W_i \) belongs to \( S_{1/2} \) and we have

\[
\|W_{i+1}W_i\| > \frac{1}{2}\|W_{i+1}\|\|W_i\|.
\]

Since the determinant of \( W_{i+1}W_i \) is a divisor of \( m^2 \), the content of this product is a divisor of \( m \) and so the matrix \( W_{i+2} = (W_{i+1}W_i)^{\text{red}} \) satisfies \( \|W_{i+2}\| \geq |m|^{-1}\|W_{i+1}W_i]\). Combining this inequality with the previous one, we deduce that

\[
\|W_{i+2}\| > \frac{1}{2|m|}\|W_{i+1}\|\|W_i\|,
\]

for each \( i \geq 2 \). By induction, this implies \( \|W_{i+1}\| > \|W_i\| \geq 2|m| \) for each \( i \geq 2 \), and so the sequence \((W_i)_{i \geq 1}\) is unbounded. Applying Theorem 2.1 we deduce that the sequence \((W_i)_{i \geq 1}\) is associated to some extremal real number \( \xi \). Moreover, since we have \( |\det \Phi(u)| \leq |m| \) for each \( u \in E^* \), Theorem 2.3 shows that \( \xi \) is badly approximable.

Finally, suppose that \( \xi \) is \( \text{GL}_2(\mathbb{Q}) \)-conjugate to an extremal real number \( \eta \) with an associated Fibonacci sequence \((W'_i)_{i \geq 1}\) in \( \text{GL}_2(\mathbb{Z}) \). Then, there exists a matrix \( A \in P \) such that \((\eta, -1)\) is proportional to \((\xi, -1)A\) and, upon denoting by \( B \) the inverse of \( A \) in \( P \), we find that \((A * W'_i * B)_{i \geq 1}\) is a Fibonacci sequence in \( P \) which is associated to \( \xi \). So, by Theorem 2.1 the sequences \((W_i)_{i \geq 1}\) and \((A * W'_i * B)_{i \geq 1}\) differ only up to their first terms and up to
multiplication by a Fibonacci sequence in $\{−1, 1\}$. Comparing determinants, this implies that $|m|$ is the square of an integer. □

Remark. The Fibonacci sequence $(W_i)_{i \geq 1}$ in $\mathcal{P}$ starting with

$$W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$W_4 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_5 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \quad W_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is periodic of period 6 as one finds that $W_7 = W_1$ and $W_8 = W_2$. Therefore, $(W_i)_{i \geq 1}$ is a Fibonacci sequence of matrices with bounded determinant. It does not correspond to an extremal real number as the sequence itself is bounded. However, if $\Phi : E^{*} \to \mathcal{P}$ denotes the morphism of monoids sending $w_i$ to $W_i$ for each $i \geq 1$, then, for each $i \geq 1$, the word $v_i = w_{6i+1} \cdots w_7w_1$ is a prefix of $f_{a,b}$ whose image under $\Phi$ is the matrix $W_i^{i+1}$ which has determinant $2^{i+1}$ tending to infinity with $i$.

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