Hyperbolic structures on link complements, octahedral decompositions,
and quantum $\mathfrak{sl}_2$

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Abstract
Hyperbolic structures (equivalently, principal $\text{PSL}_2(\mathbb{C})$-bundles with connection) on link complements can be described algebraically by using the \textit{octahedral decomposition}, which assigns an ideal triangulation to any diagram of the link. The decomposition (like any ideal triangulation) gives a set of \textit{gluing equations} in \textit{shape parameters} whose solutions are hyperbolic structures. We show that these equations are closely related to a certain presentation of the \textit{Kac-de Concini quantum group} $\mathcal{U}_q(\mathfrak{sl}_2)$ in terms of cluster algebras at $q = \xi$ a root of unity. Specifically, we identify ratios of the shape parameters of the octahedral decomposition with central characters of $\mathcal{U}_\xi(\mathfrak{sl}_2)$. The quantum braiding on these characters is known to be closely related to $\text{SL}_2(\mathbb{C})$-bundles on link complements, and our work provides a geometric perspective on this construction.

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1. Introduction

Let \( L \) be a link in \( S^3 \). In low-dimensional topology, it is frequently useful to understand representations \( \rho : \pi_1(S^3 \setminus L) \to \text{SL}_2(\mathbb{C}) \), or in our language \( \text{SL}_2(\mathbb{C}) \)-structures. A major motivation is hyperbolic geometry, because the isometry group \( \text{Isom}(\mathbb{H}^3) = \text{PSL}_2(\mathbb{C}) \) of hyperbolic 3-space is double-covered by \( \text{SL}_2(\mathbb{C}) \). In particular, the complete finite-volume hyperbolic metric on a hyperbolic knot corresponds to a distinguished \( \rho : \pi_1(S^3 \setminus L) \to \text{SL}_2(\mathbb{C}) \), so computing hyperbolic structures is a special case of our problem.

In this article, we discuss algebraic methods for computing \( \text{SL}_2(\mathbb{C}) \)-structures on link complements. A shape is an ordered triple \( \chi = (a, b, m) \in (\mathbb{C} \setminus \{0\})^3 \) of nonzero complex numbers. A link diagram is shaped when each segment is colored with a shape and the shapes satisfy certain algebraic relations at the crossings. We show that:

1. A shaping of a diagram \( D \) of a link \( L \) corresponds to an \( \text{SL}_2(\mathbb{C}) \)-structure (a representation \( \rho : \pi_1(S^3 \setminus L) \to \text{SL}_2(\mathbb{C}) \)) and up to conjugacy every \( \text{SL}_2(\mathbb{C}) \)-structure arises in this way.
2. The shapes of a diagram correspond in a simple and explicit way to the shape parameters of the ideal tetrahedra of the octahedral decomposition associated to the diagram.
3. The relations between the shapes at a crossing are determined by the braiding on the quantum group \( U_\xi(sl_2) \) for \( \xi \) a root of unity.

Previously Blanchet et al. [Bla+20] have shown versions of 1 and 3, and Kim, Kim, and Yoon [KKY18] have shown versions of 1 and 2. Closely related to their work is a description of boundary-parabolic \( SL_2(C) \)-structures for braid closures in terms of cluster variables due to Hikami and Inoue [HI15].

This paper improves these results and places them in a unified context. Our key idea is to consider a presentation of \( U_\xi(sl_2) \) in terms of a quantum cluster algebra. This presentation leads to cluster-type coordinates on the \( SL_2(C) \)-representation variety of a tangle complement, which are in turn naturally related to the octahedral decomposition.

1.1. Plan of the paper

- In the remainder of the introduction we give more background on the hyperbolic geometry of 3-manifolds and connections to quantum topology.

- In Section 2 we define shaped tangle diagrams and explain how they relate to \( SL_2(C) \)-structures. We give an algebraic proof that, for any diagram \( D \) of a link \( L \), up to conjugacy every \( SL_2(C) \)-structure on \( L \) is detected by a shaping of \( D \). We also give some examples of shaped link diagrams.

- In Section 3 we relate shaped tangle diagrams to geometry using the octahedral decomposition. We use this perspective to strengthen our existence result: we can always find geometrically nondegenerate shapings for any nontrivial \( SL_2(C) \)-structure.

- In Section 4 we show how to compute the eigenvalues of the restriction of \( \rho \) to the peripheral subgroups of \( \pi_1(S^3 \setminus L) \). This is closely related to a decoration of the representation.

- In Section 5 we explain the connection to quantum groups and cluster algebras. This perspective allows us to interpret our results as a stronger version of the Hikami-Inoue conjecture [HI15] and its solution by Cho, Yoon, and Zickert [CYZ20].

- In Section 6 we explain how to compute \( SL_2(C) \)-structures in practice: if we restrict to geometrically non-degenerate structures, then we can eliminate half the variables. As an example, we compute all shapings of \((2, 2n + 1)\)-torus knots corresponding to irreducible holonomy representations.

1.2. Hyperbolic 3-manifolds and \( SL_2(C) \)-structures

A manifold is hyperbolic if it admits a complete, finite-volume Riemannian metric of curvature \(-1\). Following Thurston [Thu02], it is more helpful it describe the hyperbolic structure on a 3-manifold \( M \) as a \((\text{Isom}(\mathbb{H}^3), \mathbb{H}^3)\) structure. Informally, this means that instead of just building \( M \) out of open sets glued together with continuous maps, we build it out of subsets of hyperbolic 3-space \( \mathbb{H}^3 \) glued together with hyperbolic isometries. We can further abstract these gluing maps into a representation

\[
\rho : \pi_1(M) \to \text{Isom}(\mathbb{H}^3) = PSL_2(C)
\]
called the holonomy of the hyperbolic structure. It is well-defined up to conjugation, and this gives us an algebraic way to study hyperbolic structures. We consider a slightly more general idea:
1. Introduction

Definition 1.1. An $\text{SL}_2(\mathbb{C})$-structure on a 3-manifold $M$ is a representation
\[ \rho : \pi_1(M) \to \text{SL}_2(\mathbb{C}). \]

We call the pair $(M, \rho)$ a $\text{SL}_2(\mathbb{C})$-manifold. Two $\text{SL}_2(\mathbb{C})$-structures $\rho, \rho'$ are conjugate or gauge-equivalent if if $\rho' = g \rho g^{-1}$ for some $g \in \text{SL}_2(\mathbb{C})$. A $\text{SL}_2(\mathbb{C})$-structure is hyperbolic (gives a complete finite-volume metric) when $\rho$ is discrete and faithful.

To study the hyperbolic geometry of $M$ it suffices to find only the hyperbolic $\rho$. Remarkably, such a $\rho$ is unique (up to conjugacy and multiplication by $\pm 1$) and when it exists is determined by $\pi_1(M)$; this fact is known as Mostow-Prasad rigidity. However, for a few reasons we are interested instead in computing all $\text{SL}_2(\mathbb{C})$-structures, not just the hyperbolic ones.

Motivating Problem 1. Given a combinatorial description of a 3-manifold $M$, compute all $\text{SL}_2(\mathbb{C})$-structures $\rho$ on $M$.

The space $\mathcal{R}_M$ of all such $\rho$ is usually called the $\text{SL}_2(\mathbb{C})$-representation variety of $M$. Taking the quotient\footnote{Sometimes a general representation $\rho : \pi_1(M) \to \text{PSL}_2(\mathbb{C})$ is called a pseudo-hyperbolic structure. It is convenient for us to also specify a lift to the double cover $\text{SL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C})$.} by the conjugation action of $\text{SL}_2(\mathbb{C})$ gives the $\text{SL}_2(\mathbb{C})$-character variety $\mathcal{X}_M$ of $M$. It determines important information about the topology of $M$, such as essential surfaces [CS83]. One reason to try to find all $\text{SL}_2(\mathbb{C})$-structures is so that we can understand character varieties.

In this paper we are specifically interested in link complements $M_L := S^3 \setminus L$. From a geometric viewpoint, link complements are slightly different than the closed case: when $L$ has $n$ components $M_L$ is the interior of a compact manifold $\overline{M_L}$ with $n$ torus boundary components. $M_L$ itself is not compact, but we say that it has $n$ cusps (corresponding to the boundary components) which we think of as lying at infinity.\footnote{There are some technicalities here: taking a naive quotient by the conjugation action of $\text{SL}_2(\mathbb{C})$ gives a badly-behaved space. Instead the character variety is defined to be $\text{Spec} \left( \mathcal{R}_M^{\text{PSL}_2(\mathbb{C})} \right)$, where $\mathcal{R}_M^{\text{PSL}_2(\mathbb{C})}$ is the ring of conjugation-invariant functions on the representation variety. Since these issues are not important here, we mostly ignore them. [CS83] M. Culler and P. B. Shalen, “Varieties of group representations and splittings of 3-manifolds”. doi} Despite these differences we can still describe the hyperbolic geometry of $M_L$ in terms of $\text{SL}_2(\mathbb{C})$-structures $\rho : \pi_1(M_L) \to \text{SL}_2(\mathbb{C})$, which is one reason for introducing them. We can now rephrase our problem as:

Motivating Problem 2. Given a combinatorial description of a link $L$ in $S^3$, effectively compute all $\text{SL}_2(\mathbb{C})$-structures on $M_L = S^3 \setminus L$.

Even if one is only interested in hyperbolic 3-manifolds it is useful to consider more general $\text{SL}_2(\mathbb{C})$-structures on links, because every hyperbolic 3-manifold $M$ can be obtained as Dehn surgery on an $\text{SL}_2(\mathbb{C})$-link $(L, \rho)$. In general, the $\rho$ giving the complete hyperbolic structure on $M$ will give an incomplete structure on the original link complement $M_L$, so to understand manifolds obtained by surgery we want to compute all of $\mathcal{R}_{M_L}$.

1.3. Ideal triangulations

In principle we can compute $\text{SL}_2(\mathbb{C})$-structures $\rho$ from a presentation $\langle x_i | r_j \rangle$ of $\pi_1(M_L)$; the most obvious method is to introduce a variable for each entry of each matrix $\rho(x_i)$, then consider the algebraic equations in these variables given by the relations $r_j$. In practice this is usually too difficult unless $L$ is very simple. In addition, these coordinates are not very geometrically enlightening.

Thurston [Thu02] introduced the idea of computing $\text{SL}_2(\mathbb{C})$-structures using ideal triangulations, which give more tractable equations and a more geometric coordinate system.

Definition 1.2. An ideal tetrahedron is a tetrahedron $\Delta \subseteq \mathbb{H}^3$ whose ideal vertices lie on the boundary at infinity of $\mathbb{H}^3$. An ideal triangulation of $M_L$ is a triangulation of $M_L$ by ideal tetrahedra such that the ideal vertices all lie on $L$.

More formally, let $M_L$ be the space obtained by collapsing each component $L_j$ of $L$ to a point $P_j$. An ideal triangulation of $M_L$ is a triangulation of $M_L$ in which the 0-skeleton lies
1.4. Quantum holonomy invariants

One motivation for computing SL₂(ℂ)-structures on ML in terms of a diagram of L is to construct quantum holonomy invariants. These are enhanced versions of ordinary quantum invariants (like the Jones polynomial) that depend on a choice of SL₂(ℂ)-structure. A representative example is a construction of Blanchet et al. [Bla+20, Corollary 6.11], as extended in the author’s thesis [McP21]:

**Theorem 1.3.** Let L be an oriented link in S³ with components L₁, . . . , Lₙ, and let ρ be an SL₂(ℂ)-structure on ML. Then for each integer N ≥ 2 there is an invariant

Fₙ(L, ρ, s) ∈ ℂ

defined up to multiplication by 2Nth roots of unity. It depends only on the isotopy class of L and the conjugacy class of ρ. Furthermore, when ρ = 1 is the trivial representation,

Fₙ(L, 1, s) = Jₙ(L)

where zₖᵢ is one of z, 1/(1 − z), or 1 − 1/z depending on the combinatorics of the triangulation.

We can compute SL₂(ℂ)-structures on ML by specifying an ideal triangulation T and solving the associated gluing equations. For example, this technique is used by SnapPy [Cul+] to compute hyperbolic structures. However, there are a few drawbacks to this approach:

1. Not all triangulations T detect all SL₂(ℂ)-structures: we might need to subdivide T first.
2. Solving the gluing equations is computationally expensive, and the difficulty grows quickly with the number of tetrahedra.
3. It is not straightforward to recover the matrix coefficients of ρ from the shape parameters; these are needed to compute the quantum holonomy invariants discussed below.
4. Finding an efficient ideal triangulation of a link from its diagram is quite difficult.

The last one is particularly relevant for quantum topology: If we want to work with tangle diagrams and surgery presentations, it is difficult to translate to the language of ideal triangulations and back.

We can address this problem by systematically computing ideal triangulations from link diagrams. The standard way to do this is the octahedral decomposition, which was introduced by Thurston [Thu99] and implicit in the work of Kashaev [Kas95, Section 4]. We follow the extensive description by Kim, Kim, and Yoon [KKY18]. The idea is to place an ideal octahedron at each crossing of the link diagram. While the resulting triangulation uses many tetrahedra (4 or 5 per crossing, depending on the version) we can eliminate many of the shape parameters to obtain an efficient description of the gluing equations. In addition, it is possible to reconstruct the map ρ from the shapes of the ideal octahedra.

4 Usually we also consider some extra equations that ensure the restriction of ρ the boundary tori of ML has the right eigenvalues; we will discuss these later.

5 More accurately, the shape parameters compute PSL₂(ℂ)-structures.

[Thu99] D. Thurston, Hyperbolic volume and the Jones polynomial

[Kas95] R. Kashaev, "A link invariant from quantum dilogarithm". arXiv

[Bla+20] J. Blanchet, G. Wagner, A. Virelizier, "Quantum invariants of 3-manifolds and links".

[McP21] C. McPhail-Snyder, “SL₂(ℂ)-holonomy invariants of links". arXiv
is (up to an overall normalization) the colored Jones polynomial of $L$ evaluated at $q = \exp(\pi i/N)$, also known as the Kashaev invariant \cite{MM01}.

**Remark 1.4.** Here $s$ is a type of generalized spin structure on $M_L$. Specifically, if $x_i$ is a meridian of the $i$th component of $L$, then $s$ is a choice of complex numbers $\mu_i$ such that $\mu_i^N$ is an eigenvalue of $\rho(x_i)$ for each $i$. In addition, there are some restrictions when $\text{tr} \rho(x_i) = \pm 2$ \cite[Introduction §3]{McP21}.

Computing holonomy invariants like $F_N$ requires efficiently computing $\text{SL}_2(\mathbb{C})$-structures on links in a particular coordinate system related to the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. In the language of \cite{Bla+20}, we must use the *factorized biquandle* associated to $\text{SL}_2(\mathbb{C})^*$ instead of the *conjugation quandle* of $\text{SL}_2(\mathbb{C})$. Remarkably, this coordinate system is naturally connected to the octahedral decomposition, and describing this connection is a major motivation for this paper.

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I would like to thank Ian Agol for several helpful conversations about hyperbolic knot theory and Seokbeom Yoon for informing me about the recurrences in Remark 6.18.

This paper is an elaboration of ideas first published in my my thesis \cite{McP21}. The presentation of $\mathcal{U}_q(\mathfrak{sl}_2)$ leading to the shape coordinates on tangles was originally discovered by Nicolai Reshetikhin; future joint work \cite{MR22} will apply the ideas in this paper to quantum holonomy invariants.

## 2. The shape biquandle

### 2.1. Shaped link diagrams

**Definition 2.1.** Let $L$ be a link in $S^3$ with $n$ components with a link diagram $D$. We assume all link diagrams are oriented. Thinking of $D$ as a decorated 4-valent graph $G$ embedded in $S^2$, the *segments* of $D$ are the edges of $G$. A *region* of a diagram is a connected component of the complement of $G$, equivalently a vertex of the dual graph of $G$.

For example, Figure 1 shows an (oriented) diagram with the segments labeled. In an oriented diagram all crossings are positive or negative, as shown in Figure 2. Our preference is to read crossings left-to-right. As shown there, we usually refer to the segments at a given crossing by $1, 2, 1',$ and $2'$. We similarly refer to the regions touching the crossing as $N, S, E,$ and $W$. The labeling conventions are summarized in Figure 3.

**Definition 2.2.** A *shape* is a triple of nonzero complex numbers. We usually denote a shape by $\chi = (a, b, m) \in (\mathbb{C} \setminus \{0\})^3$, and when it is assigned to a segment $i$ of a tangle diagram we denote it $\chi_i = (a_i, b_i, m_i)$.

\[ \text{Figure 2: Positive (left) and negative (right) crossings.} \]

\[ \text{Figure 1: A diagram of the figure-eight knot, with the 8 segments indexed by 1, \ldots, 8.} \]

\[ \text{Figure 3: Segments and regions near a crossing.} \]
Remark 2.3. We can think of a shape as:

- An element of the group \(SL_2(\mathbb{C})^*\):
  \[
  (a, b, m) = \left( \begin{bmatrix} a & 0 \\ (a - 1/m)/b & 1 \end{bmatrix} \right) \cdot \left( \begin{bmatrix} 1 & (a - m)b \\ 0 & a \end{bmatrix} \right)
  \in \left\{ \left( \begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix} \right) \cdot \left( \begin{bmatrix} 1 & 0 \\ \epsilon & \kappa \end{bmatrix} \right) \mid \kappa \neq 0 \right\} = \text{SL}_2(\mathbb{C})^* \subseteq \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})
  \]

Here \(\text{SL}_2(\mathbb{C})^*\) is the Poisson dual group [McP21, Section 0.1] of \(\text{SL}_2(\mathbb{C})\).

- A central character of \(U_\xi(\mathfrak{sl}_2)\) for \(\xi\) a root of unity.

- Data determining the shapes (complex dihedral angles) of the octahedral decomposition of a link diagram. In the language of Kim, Kim, and Yoon [KKY18] the \(b_i\) and \(m_i\) correspond to segment variables and \(a_i\) and \(m_i\) to (ratios of) region variables.

- Data determining the value of an \(\text{SL}_2(\mathbb{C})\)-structure \(\rho\) on the meridian around a segment of a knot diagram.

Definition 2.4. The braiding \(B\) is the partially-defined map given by \(B(\chi_1, \chi_2) = (\chi_2', \chi_1')\), where

\[
\begin{align*}
a_1' &= a_1 A^{-1} \\
a_2' &= a_2 A \\
a &= 1 - \frac{m_1 b_1}{b_2} \left(1 - \frac{a_1}{m_1}\right) \left(1 - \frac{1}{m_2 a_2}\right) \\
b_1' &= m_2 b_2 \left(1 - m_2 a_2 \left(1 - \frac{b_2}{m_1 b_1}\right)\right)^{-1} \\
b_2' &= b_1 \left(1 - \frac{m_1}{a_1} \left(1 - \frac{b_2}{m_1 b_1}\right)\right) \\
m_1' &= m_1 \quad m_2' = m_2
\end{align*}
\]

We think of \(B\) as being associated to a positive crossing with incoming strands 1 and 2 and outgoing strands 2’ and 1’, as in Figure 3.

\(B\) is a partially-defined map on the space of pairs of shapes, and it satisfies braid relations; in Section 2.3 will formalize this by saying that \(B\) gives a generically defined biquandle [Bla+20, Section 5].

Lemma 2.5. The map \(B\) is generically invertible, and if \((\chi_2', \chi_1') = B^{-1}(\chi_1, \chi_2)\), then

\[
\begin{align*}
a_1' &= a_1 \tilde{A}^{-1} \\
a_2' &= a_2 \tilde{A} \\
\tilde{A} &= 1 - \frac{b_2}{m_1 b_1} \left(1 - m_1 a_1\right) \left(1 - \frac{m_2}{a_2}\right) \\
b_1' &= m_2 b_2 \left(1 - \frac{a_2}{m_2} \left(1 - \frac{m_1 b_1}{b_2}\right)\right) \\
b_2' &= b_1 \left(1 - \frac{1}{m_1 a_1} \left(1 - \frac{m_1 b_1}{b_2}\right)\right)^{-1} \\
m_1' &= m_1 \quad m_2' = m_2
\end{align*}
\]
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Definition 2.6. We say that a tangle diagram $D$ is shaped if its segments are assigned shapes \{\chi_i\} so that at each positive crossing (labeled as in Figure 3) we have $B(\chi_1, \chi_2) = (\chi_2', \chi_1')$, and similarly for negative crossings and $B^{-1}$. We also require that all of the components of $\chi_1'$ and $\chi_2'$ lie in $\mathbb{C}^\times$. For example, this means that at a positive crossing we must assign $\chi_1$ and $\chi_2$ so that

$$A = 1 - \frac{b_2}{m_1b_1} (1 - m_1a_1) \left(1 - \frac{m_2}{a_2}\right)$$

is not 0 or $\infty$.

Example 2.7. Consider the diagram of the trefoil in Figure 4 with the segments labeled by 1, 4, 6. An assignment of shapes $\chi_i = (a_i, b_i, m_i), i = 1, \ldots, 6$ to the diagram is valid if

$$B(\chi_1, \chi_2) = (\chi_3, \chi_4), B(\chi_3, \chi_4) = (\chi_5, \chi_6), \text{ and } B(\chi_5, \chi_6) = (\chi_1, \chi_2).$$

This immediately implies that $m_1 = m_2 = \cdots = m_6 = m$; in general there is only one variable $m$ for each component of the link.

A family of solutions is given by

$$\chi_1 = \frac{(b_1m - b_2)}{(b_1 - b_3)} \cdot b_1, m$$

$$\chi_2 = \frac{b_2m^2 - b_2b_3m + b_1b_3}{(b_1m - b_2)(b_2m - b_3)} \cdot b_2, m$$

$$\chi_3 = \frac{b_2b_3m^3 + b_1b_3m^2 - b_2^2m^2 - b_1b_2m + b_1b_3}{(b_2m - b_3)(b_1 - b_3)m} \cdot b_3, m$$

$$\chi_4 = \frac{(b_2m - b_2b_3m + b_1b_3)m}{b_2b_3m^3 + b_1b_3m^2 - b_2^2m^2 - b_1b_2m + b_1b_3} \cdot (b_2m + b_1 - b_3)m, m$$

$$\chi_5 = \frac{-b_2^2m^4 - b_2b_3m^3 + b_2b_3m + b_1b_3 - b_2^2}{(b_2m - b_3)(b_1 - b_3)m} \cdot b_1b_2m \cdot b_3m + b_1 - b_3, m$$

$$\chi_6 = \frac{(b_2^2m^4 - b_2b_3m^3 + b_2b_3m + b_1b_3 - b_2^2)}{(b_2^2m^4 - b_2b_3m^3 + b_2b_3m + b_1b_3 - b_2^2)} \cdot \frac{b_1b_3}{(b_2m - b_3)m}, m$$

where $b_1, b_2, b_3$ can be freely chosen as long as none of the $a_i$ or $b_i$ are 0 or $\infty$. It turns out that the choice of $b_1, b_2, b_3$ does not affect the conjugacy class of the $\text{SL}_2(\mathbb{C})$-structure determined by this shaping. We show how to compute these solutions in Section 6.3.

In practice, the equations for all the $a_i$ and $b_i$ are usually difficult to solve. We can simplify them by either eliminating the $b_i$ and solving them in terms of the $a_i$, or vice-versa. For example, the solutions in the previous example were determined by first solving for the $b_i$, then using them to determine the $a_i$. We discuss this in detail in Section 6. For now, we give two relevant lemmas, which also have a geometric interpretation (see Section 3).

Definition 2.8. A crossing (labeled as in Figure 3) is pinched if any of the equations

$$b_2 = m_1b_1, \quad m_2b_2 = m_1b_1', \quad b_2' = b_1, \quad m_2b_2' = b_1'$$

hold, in which case all of them do. A crossing is degenerate if either of the equations

$$a_1 = a_1', \quad a_2 = a_2'$$

hold, in which case both do. A degenerate crossing is necessarily pinched but a pinched crossing can be non-degenerate.
Remark 2.9. We will see in Section 3 that a crossing is pinched when the tetrahedra of the four-term decomposition are geometrically degenerate, while it is degenerate if the tetrahedra of the five-term decomposition are geometrically degenerate.

Lemma 2.10. At a non-pinched positive crossing,

\[
\begin{align*}
ad_1 &= \frac{b_2 - m_1 b_1}{b_2' - b_1} & ad_1' &= \frac{m_2 b_2 - m_1 b_1'}{m_2 b_2' - b_1'} \\
d_2 &= \frac{b_1 m_2 b_2 - m_1 b_1'}{m_2 b_2' - b_2 - m_1 b_1} & d_2' &= \frac{b_1 m_2 b_2' - b_1'}{m_2 b_2' - b_2 - b_1}
\end{align*}
\]  

(7)

while at a non-pinched negative crossing

\[
\begin{align*}
ad_1 &= \frac{b_2' - m_1 b_1}{m_1 b_2' - b_1} & ad_1' &= \frac{b_2' m_2 b_2 - m_1 b_1'}{m_2 b_2' - m_1 b_1'} \\
d_2 &= \frac{m_2 b_2 - m_1 b_1'}{b_2 - m_1 b_1} & d_2' &= \frac{m_2 b_2' - b_1'}{b_2' - b_1}
\end{align*}
\]  

(8)

Proof. Once we know (7) and (8) it is easy to check them against (1–6).  

Lemma 2.11. At any non-degenerate positive crossing,

\[
\begin{align*}
b_2 &= \frac{(1 - a_1/m_1)(1 - 1/m_2 a_2)}{1 - a_1/a_1'} \\
m_2 b_2' &= \frac{(1 - m_1/a_1')(1 - m_2 a_2')}{1 - a_1'/a_1} \\
b_1 &= \frac{1 - a_1'/a_1}{1 - m_1/a_1} \\
m_1 b_1' &= \frac{1 - a_1'/a_1}{1 - m_1/a_1}
\end{align*}
\]  

(9)

while at a non-degenerate negative crossing,

\[
\begin{align*}
b_2 &= \frac{1 - a_1/a_1'}{1 - a_1/m_1(1 - m_2 a_2)} \\
m_2 b_2' &= \frac{1 - a_1/a_1'}{1 - a_1/m_1(1 - m_2 a_2')} \\
b_1' &= \frac{(1 - 1/m_1 a_1')(1 - a_2'/m_2)}{(1 - 1/m_1 a_1)(1 - m_2 a_2')} \\
b_2' &= \frac{1 - a_1'/a_1}{1 - m_1/a_1} \\
m_1 b_1' &= \frac{(1 - m_1 a_1')(1 - a_2/m_2)}{1 - a_1'/a_1}
\end{align*}
\]  

(10)

Proof. As before, once we know the right equations to check this is a straightforward verification.
2. The shape biquandle

2.2. The holonomy of a shaped diagram

We can now explain how shaped diagrams relate to $\text{SL}_2(\mathbb{C})$-structures.

**Definition 2.12.** Let $D$ be a shaped diagram of $L$. The fundamental groupoid $\Pi_1(D)$ of $D$ has one object for each region of $D$ and two generators $x_j^\pm$ for each segment $j$. These represent paths above and below the segment, as in Figure 5. There are three relations for each crossing, given by

$$x_1^\pm x_2^\pm = x_2^\pm x_1^\pm \text{ and } \begin{cases} x_1^- x_2^+ = x_2^+ x_1^-, \\ x_1^+ x_2^- = x_2^- x_1^+ \end{cases} \text{ for a positive crossing, or}$$

$$\begin{cases} x_1^- x_2^- = x_2^- x_1^- \\ x_1^+ x_2^+ = x_2^+ x_1^+ \end{cases} \text{ for a negative crossing.} \quad (11)$$

**Proposition 2.13.** For any diagram $D$ of a link $L$, the groupoid $\Pi_1(D)$ is equivalent to the fundamental group $\pi_1(M_L)$ of the link complement. ♦

To use more abstract language, the claim is that the group $\pi_1(M_L)$ is a skeleton of the groupoid $\Pi_1(D)$.

**Proof.** A detailed proof is given in [Bla+20, Section 3]. It is instructive to consider an example: In Figure 7, we have expressed a path $w_3 \in \pi_1(M_L)$ representing a generator of the Wirtinger presentation of $\pi_1(M_L)$ in terms of elements of $\Pi_1(D)$.

**Remark 2.14.** The standard way to study $\pi_1(M_L)$ using a diagram of $L$ is the Wirtinger presentation, which has one generator for each arc (arcs don’t break at overcrossings, unlike segments) and one relation for each crossing. We can think of $\Pi_1(D)$ as using a greater number of more local generators. This turns out to be more convenient when we discuss face pairings in Section 3.

**Definition 2.15.** Let $D$ be a shaped link diagram. The holonomy representation of $D$ is the representation

$$\rho : \Pi_1(D) \to \text{SL}_2(\mathbb{C})$$

Figure 7: The path $w_3$ in $\pi_1(M_L)$ and the path $x_1^+ x_2^+ x_3^- (x_3^-)^{-1} (x_2^+)^{-1} (x_1^+)^{-1}$ in $\Pi_1(D)$ are equivalent.

Figure 5: Generators of the fundamental groupoid $\Pi_1(D)$ of a tangle diagram.

Figure 6: Deriving the middle relation at a crossing.
given by

$$\rho \left( x^+ \right) = \begin{bmatrix} a & 0 \\ (a - 1/m)b & 1 \end{bmatrix}, \quad \rho \left( x^- \right) = \begin{bmatrix} 1 & (a - m)b \\ 0 & a \end{bmatrix},$$

(12)

where the generators $x^\pm$ are associated to a strand of $D$ colored with the shape $\chi = (a, b, m)$.

By Theorem 2.17 below, $\rho$ also gives a representation $\pi_1(M_L) \to \text{SL}_2(\mathbb{C})$, and we do not usually distinguish between these.

**Remark 2.16.** A link group $\pi_1(M_L)$ has certain distinguished elements called meridians, which correspond to paths around a single strand of $L$. The generators of the Wirtinger presentation are meridians, and all meridians of the same component of $L$ are conjugate. In terms of $\Pi_1(D)$, the meridian around a strand with shape $\chi$ is conjugate to the matrix

$$\rho \left( x^+(x^-)^{-1} \right) = \begin{bmatrix} a & -(a-m)b \\ (a - 1/m)b & m + m^{-1} - a \end{bmatrix}$$

(13)

which has trace $m + m^{-1}$. In general, the meridian is *not* equal to $\rho \left( x^+(x^-)^{-1} \right)$, as shown in Figure 7.

**Theorem 2.17.** The holonomy representation of $\Pi_1(D)$ is well-defined and gives a representation $\rho : \pi_1(M_L) \to \text{SL}_2(\mathbb{C})$.

**Proof.** To make sure $\rho$ is well-defined, we need to check that the braiding rules on the $\chi_i$ are compatible with the relations (11), which is straightforward. Then both claims follow from Proposition 2.13. $\square$

For this theorem to be useful, we need to make sure that shapings of link diagrams actually give all $\text{SL}_2(\mathbb{C})$-structures. This is false, but a slightly weaker statement is true. Let $L$ be a link in $S^3$ and $D$ any oriented diagram of $L$. In practice we are mostly interested in $\text{SL}_2(\mathbb{C})$-structures up to conjugation, and we have the following existence theorem:

**Theorem 2.18.** We say a decorated $\text{SL}_2(\mathbb{C})$-structure $\rho$ is *detected* by $D$ if there is a shaping of $D$ with holonomy representation $\rho$. Every decorated $\text{SL}_2(\mathbb{C})$-structure on $L$ is conjugate to one detected by $D$. $\diamond$

This theorem is one of the major results of Blanchet et al. [Bla+20]. We sketch the proof in the next section. In Theorem 3.10 we give a stronger result: when $\rho \neq \pm 1$ is nontrivial, then up to conjugacy $\rho$ is detected by a *geometrically nondegenerate* shaping.

### 2.3. Biquandle factorizations and the existence of shapings

Blanchet et al. [Bla+20] have developed a general theory for dealing with coordinate systems like ours by considering two related algebraic structures. We briefly explain how it relates to our example.

**Definition 2.19.** A *quandle* is a set $Q$ with a binary operation $\triangleright$ such that:

1. For all $a, b, c \in Q$, $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$,
2. for all $a, b \in Q$ there is a unique $c \in Q$ such that $a = b \triangleright c$, and
3. for any $a \in Q$, $a \triangleright a = a$.

We say a link diagram is *colored* by a quandle $Q$ if its segments are assigned elements of $Q$ according to the rule in Figure 8 (with a similar one for negative crossings).

![Figure 8: Labeling segments at a crossing using a quandle. Notice that both ends of the over-arc have the same label.](image)
The prototypical example is the conjugation structure of a group:

**Example 2.20.** Let $G$ be a group. The *conjugation quandle* associated to $G$ has $Q = G$ and

$$g 	riangleright h = hgh^{-1}.$$  

Using the Wirtinger presentation of the fundamental group of a link complement, it is straightforward to see that colorings of diagrams of $L$ by the conjugation quandle of $G$ are exactly the same as representations $\pi_1(M_L) \to G$. In particular, $SL_2(\mathbb{C})$-structures on a link diagram are exactly colorings by the conjugation quandle of $SL_2(\mathbb{C})$.

However, to construct quantum invariants of links with $SL_2(\mathbb{C})$ structures a different description is needed. Consider the following factorization of elements\(^8\) of $SL_2(\mathbb{C})$: $g = \begin{bmatrix} \kappa & -\epsilon \\ \phi & (1 - \epsilon\phi)/\kappa \end{bmatrix} = \begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} = g^+(g^-)^{-1}$

We can think of the pair $(g^+, g^-)$ as an element of the group

$SL_2(\mathbb{C})^* = \left\{ \left( \begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} \right) | \kappa \neq 0 \right\} \subseteq GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$.

Instead of giving the holonomy $g$ around a strand, we split $g$ into the holonomy $g^+$ from passing above the strand and $g^-$ from passing below. In general this works for any *factorizable algebraic group* [KR05, Section 2]. The motivation for working with group factorizations has to do with the representation theory of quantum groups at roots of unity. If $G$ is a simple Lie group with algebra $\mathfrak{g}$ and $\xi$ is a root of unity, the center of (the Kac-de Concini form of) $U_{\xi}(\mathfrak{g})$ is an algebraic group $G^*$ distinct from the Lie group $G$. In general, $G^*$ is related to $G$ in the same way $SL_2(\mathbb{C})^*$ is to $SL_2(\mathbb{C})$. We discuss this in more detail in Section 5.

When passing from $G$ to $G^*$ we pass from the Wirtinger-like relations $g_1' = g_1$ and $g_2' = g_1g_2g_1^{-1}$ to the factorized relations (11). In particular, we have to allow the case $g_1^+ \neq g_1^-$.

This is captured by a more general structure called a biquandle.

**Definition 2.21.** A *biquandle* is a set $X$ together with an invertible map $B = (B_1, B_2) : X \times X \to X \times X$ such that:

1. The map $B$ satisfies the braid relation:
   $$ (id \times B)(B \times id)(id \times B) = (B \times id)(id \times B)(B \times id). $$

2. The map $B$ is sideways invertible: there is a unique bijection $S : X \times X \to X \times X$ such that
   $$ S(B_1(x, y), x) = (B_2(x, y), y) $$
   for all $x, y \in X$.

3. The map $S$ induces a bijection $\alpha : X \to X$ on the diagonal:
   $$ S(x, x) = (\alpha(x), \alpha(x)). $$

We say a link diagram is *colored* by $X$ if its segments are assigned elements of $X$ such that at every positive crossing labeled as in Figure 2,

$$ B(x_1, x_2) = (x_2', x_1') $$

and similarly for negative crossings and $B^{-1}$.

\[\text{[KR05]}\] R. Kashaev and N. Reshetikhin, "Invariants of tangles with flat connections in their complements". arXiv not
A biquandle is a generalization of a quandle, and it is the right language for dealing with factorizations like (13). We think of axiom (2) as saying that specifying any two adjacent elements $x_i$ at a crossing determines the other two, and axiom (3) as regulating the behavior of the biquandle at kinks.

It turns out that in order to handle our coordinate system we need a further generalization than biquandles. The problem is that the map $B$ of Definition 2.4 is only partially defined: there are certain singular pairs $(\chi_1, \chi_2)$ where, say, $a_1'$ becomes 0. The main example [Bla+20, Example 5.2] of a biquandle factorization in the literature is very closely related to our shape coordinates, and suffers from the same problem. To fix it, we can introduce generically defined biquandles [Bla+20, Section 5]. The idea is that if such singular pairs are rare (say, the complement of a Zariski open dense set) then we can always avoid them, possibly by global conjugation (also known as gauge transformation).

By using these techniques, proving Theorem 2.18 reduces to showing that the braiding of Definition 2.4 defines a generic biquandle factorization of the conjugation quandle of $\text{SL}_2(\mathbb{C})$. The idea is that a biquandle coloring naturally gives a representation of the groupoid $\Pi_1(D)$, while a quandle coloring is more naturally related to $\pi_1(M_L)$, and these need to be compatible as in Theorem 2.17.

Once the general theory is constructed, it is not too hard to show [Bla+20, Theorem 5.5] that a closely related example is a generic factorization of the conjugation quandle of $\text{SL}_2(\mathbb{C})$. We can repeat essentially the same proof to prove Theorem 2.18, with some modifications to one step: In the original proof [Bla+20, Appendix B] we consider the Zariski open subset

$$U = \left\{ \begin{bmatrix} \kappa & -\epsilon \\ \phi & (1-\epsilon \phi)/\kappa \end{bmatrix} \bigg| \kappa, \epsilon, \phi \in \mathbb{C}, \kappa \neq 0 \right\} \subset \text{SL}_2(\mathbb{C})$$

of matrices representable in terms of the factorization $g = g^+(g^-)^{-1}$. In our example, we consider the smaller set

$$U' = \left\{ \begin{bmatrix} a & -(a-m)b \\ (a-1/m)/b & m + m^{-1} - a \end{bmatrix} \bigg| a, b, m \in \mathbb{C} \setminus \{0\} \right\} \subset \text{SL}_2(\mathbb{C})$$

which is still Zariski open, so the same argument goes through.

We sketch the argument briefly. Fixing a diagram $D$ we can consider the set of representations $\rho : \pi_1(D) \to \text{SL}_2(\mathbb{C})$ such that $\rho(x_i) \in U$ for every Wirtinger generator $x_i$. This is a finite intersection of open sets, so it is again open. By the general theory developed in [Bla+20, Section 5, Appendix A] this is enough to guarantee that every $\rho$ is conjugate to some $\rho'$ with $\rho'(x_i) \in U$ for all $i$, i.e. to one representable in terms of the factorized matrices $g^\pm$.

### 3. The octahedral decomposition and hyperbolic geometry

We can now describe a geometric interpretation of our coordinates $\chi_i = (a_i, b_i, m_i)$.

#### 3.1. Ideal octahedra and their shapes

Thurston [Thu02] introduced a way to combinatorially describe hyperbolic structures on link complements (more generally, on cusped 3-manifolds) by using ideal triangulations. The idea is to triangulate $S^3 \setminus L$ with all the 0-vertices “at infinity”, that is lying on $L$; we can think of this in terms of ideal polyhedra, which are polyhedra with their vertices removed.

We can describe hyperbolic structures on ideal polyhedra by splitting them into ideal tetrahedra. The hyperbolic geometry of an ideal tetrahedron is summarized by a shape parameter...
\( z \in \mathbb{C} \setminus \{0,1\} \) whose argument is the dihedral angle at a particular edge of the tetrahedron. (The shape parameters at the other edges are \(1/(1-z)\) and \(1 - 1/z\).) When the ideal simplices are glued together we can check certain gluing equations on the parameters at the involved edges; if they are satisfied, we get a hyperbolic structure on the glued manifold. We refer to Purcell [Pur20] for details.

One method to construct ideal triangulations of link complements systematically from diagrams is the octahedral decomposition [Thu99; Kas95], which decomposes the link complement into ideal octahedra. We briefly summarize the treatment of Kim, Kim, and Yoon [KKY18]. Fix a diagram \( D \) of \( L \). We put an ideal octahedron at each crossing of \( D \) with its top and bottom ideal vertices on the strands of the link, labeled as \( P_1 \) and \( P_2 \) in Figure 9. There are four extra ideal vertices \( P_+, P_-, P_+', \) and \( P'_- \), which we pull above and below the diagram, in the process identifying \( P_+ \) with \( P'_+ \) and \( P_- \) with \( P'_- \). The resulting simplicial complex is called a twisted octahedron.

The twisted octahedra have two types of edges to glue, which we call vertical and horizontal edges as in Figure 10. We can determine the gluing patterns of the horizontal edges by looking at the regions of the link diagram \( D \), while the gluing patterns of the vertical edges come from the arcs of \( D \); see [KKY18, Section 4] for details. The result is an ideal triangulation of \( S^3 \setminus (L \cup \{P^+, P^-\}) \), where \( P_\pm \) are the two extra ideal points above and below the diagram. These extra points are not a problem in practice; because their neighborhoods are balls (not tori) we can cap them off canonically.

Given a shaped diagram \( D \) of our link \( L \) we can assign shape parameters to the edges of the ideal octahedra. The idea is that the \( a \)-variables are associated to the vertical edges and the \( b \)-variables are associated to the horizontal edges.

**Definition 3.1.** Let \( D \) be a shaped diagram of a link \( L \). Consider a twisted octahedron \( O \) at a positive crossing of \( D \). In terms of the characters \( \chi \) of the segments of the crossing, we assign the following shape parameters to the vertical and horizontal edges of \( O \):

\[
\begin{align*}
o_1 &= \frac{a_1}{m_1} & o_2 &= \frac{1}{m_2b_2} & o_1' &= \frac{m_1}{a_1'} & o_2' &= \frac{m_2a_2'}{m_2b_2} \\
o_N &= \frac{b_2'}{b_1'} & o_W &= \frac{m_1b_1}{b_2} & o_S &= \frac{m_2b_2}{m_1b_1'} & o_E &= \frac{b_1'}{m_2b_2'}
\end{align*}
\]

(14) Here by \( o_j \) we mean the shape of the vertical edge immediately below or above segment \( j \), and by \( o_k \) we mean the shape of the horizontal edge near region \( k \). It may be more convenient to consult Figure 11.

At a negative crossing, we instead assign

\[
\begin{align*}
o_1 &= \frac{1}{m_1a_1} & o_2 &= \frac{a_2}{m_2} & o_1' &= \frac{m_1a_1'}{a_2'} & o_2' &= \frac{m_2}{a_2'} \\
o_N &= \frac{b_2'}{b_1'} & o_W &= \frac{m_1b_1}{b_2} & o_S &= \frac{m_2b_2}{m_1b_1'} & o_E &= \frac{b_1'}{m_2b_2'}
\end{align*}
\]

(15) Note that the horizontal shapes (17) are the same as (15).

**Theorem 3.2.** For any shaped link diagram \( D \), the shape assignments of Definition 3.1 satisfy the gluing relations of the octahedral decomposition.

**Proof.** We refer to [KKY18, Section 3.2] for the derivation of these equations. There are two\(^8\) types to check: region equations and segment equations. We also need to check the behavior around the cusps (that is, the strands of the link) to make sure it matches the eigenvalues.
Figure 11: Shapes of edges at a positive crossing. There are four horizontal edges at the four corners and four vertical edges below and above the four segments.

Figure 13: Two examples of the region gluing equations. The product of all the parameters is 1 regardless of the orientation of the boundary segments.

$m_j$; doing this carefully requires considering the triangulation of the boundary induced by truncating our tetrahedra.

The region equations [KKY18, eq. 7] say that the product of horizontal edge shapes around any region of the diagram must be 1. It is straightforward to see that this always holds, because the horizontal edge shapes are ratios of parameters assigned to the segments of the diagram, so checking this becomes a combinatorial fact about oriented planar graphs. We give two examples in Figure 13.

The segment equations are more complicated because the vertical edges of the octahedra are glued together along the over-arcs and under-arcs of the diagram. This involves multiple segments at different crossings, so there is something nonlocal to check. A key observation of Kim, Kim, and Yoon is that the vertical edge gluing equations follow from a stronger local condition at each segment, which they call the $m$-hyperbolicity equations [KKY18, eq. 10]. We give them in our conventions in Definition 4.9. It is easy to check that the vertical edges at each segment satisfy the corresponding $m$-hyperbolicity equation and this implies that the gluing equations for the vertical edges of all the octahedra hold.

As suggested by the name the $m$-hyperbolicity equation asserts that $m$ really is an eigenvalue of the holonomy of a meridian with shape $\chi = (a,b,m)$; we discuss this further in Section 4.
We have described the geometry of ideal octahedra, but to connect with the standard language of hyperbolic geometry we need to subdivide these into ideal 3-simplices, that is into ideal tetrahedra. There are two ways (Figure 14) to do this, which we call the four-term and five-term decompositions. The latter is sometimes more convenient because we can always (see Theorem 3.10) find geometrically nondegenerate shapes for its octahedra, at least when the representation \( \rho \) is nontrivial. However, the four-term decomposition is more closely connected to the quantum algebra of Section 5, and in Section 3.3 we use it to prove that the holonomy representation associated to the octahedral decomposition agrees with the algebraically defined holonomy representation of Section 2.

3.2. Conventions on ideal tetrahedra

Before discussing the decomposition of the ideal octahedra we need to establish some conventions. We think of the vertices of an ideal tetrahedron \( \tau \) as lying on the Riemann sphere, which is the boundary at infinity of hyperbolic 3-space \( \mathbb{H}^3 \). By using the upper half-plane model we can identify the boundary of \( \mathbb{H}^3 \) at infinity with the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). (This is one way to compute \( \text{Isom}(\mathbb{H}^3) = \text{PSL}_2(\mathbb{C}) \): isometries preserve the boundary, whose isometries are given by the group \( \text{PSL}_2(\mathbb{C}) \) of fractional linear transformations.) The geometry of an ideal tetrahedron \( \Delta \) is determined by the locations \( p_i \in \hat{\mathbb{C}}, i = 0, 1, 2, 3 \) of the points, which is summarized by their cross-ratio, in this context called a shape parameter \( z \in \hat{\mathbb{C}} \). One pair of edges of \( \Delta \) has shape parameter \( z \), and the others have shapes \( 1/(1-z) \) and \( 1-1/z \).

Specifying where they are determines the geometry of \( \tau \).

**Definition 3.3.** For \( p_0, p_1, p_2, p_3 \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), the cross-ratio is

\[
[p_0 : p_1 : p_2 : p_3] = \frac{(p_0 - p_3)(p_1 - p_2)}{(p_0 - p_2)(p_1 - p_3)}.
\]

It is well known that \([p_0 : p_1 : p_2 : p_3]\) is invariant under the action of \( \text{PSL}_2(\mathbb{C}) \) by fractional linear transformations.

We follow [Cho18, Definition 2.6] and use a slightly nonstandard convention on edges and shape parameters. Our tetrahedra have signs, and the relationship between the shape parameters and the cross-ratio depends on the sign. It turns out that this is much more convenient for our purposes.

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**Figure 14:** The four-term and five-term decompositions of an octahedron.

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[Cho18] J. Cho, "Quandle theory and the optimistic limits of the representations of link groups", arXiv doi
3. The octahedral decomposition and hyperbolic geometry

Definition 3.4. An ideal tetrahedron is labeled if its vertices are totally ordered by labeling them with the set \( \{0, 1, 2, 3\} \) and it is assigned a sign \( \epsilon \in \{1, -1\} \). If the vertices of a labeled tetrahedron \( \tau \) are at points \( p_0, p_1, p_2, p_3 \in \mathbb{C} \), then we assign the edges 01 and 23 the shape parameter \( z^0 \) given by the cross-ratio

\[
z^0 := [p_0 : p_1 : p_2 : p_3]^\epsilon.
\]

We assign the edges 12 and 03 the shape \( z^1 \) and the edges 02 and 13 the shape \( z^2 \) given by

\[
(z^1)^\epsilon = \frac{1}{1 - (z^0)^\epsilon} \quad \text{and} \quad (z^2)^\epsilon = 1 - \frac{1}{(z^0)^\epsilon}.
\]

A tetrahedron is degenerate if one (hence all of) its shape parameters is 0, 1, or \( \infty \).

This means that

\[
\begin{align*}
z^1 &= 1 - \frac{1}{z^0} & z^2 &= 1 - \frac{1}{z^0} & \text{for } \epsilon = 1 \\
\frac{z^1}{z^0} &= 1 & \frac{z^2}{z^0} &= 1 & \text{for } \epsilon = -1
\end{align*}
\]

In general, if we index tetrahedra by a symbol \( j \), we write \( z_j^k \) for the \( k \)th shape parameter of tetrahedron \( j \) and \( \epsilon_j \) for its sign. It is frequently useful to use the identity

\[
z_{j+1}^k = \begin{cases} 
\frac{1}{1 - z_j^k} & \epsilon_j = 1 \\
\frac{1}{1 - \frac{1}{z_j^k}} & \epsilon_j = -1
\end{cases}
\]

The index \( k \) is modulo 3, so \( z_j^3 = z_j^0 \) regardless of \( \epsilon_j \).

3.3. The four-term decomposition

We can now describe the four-term decomposition, which we will use to show that the holonomy representation of a diagram given in Definition 2.15 agrees with the shapes of the ideal octahedra from Section 3.1. The idea is to draw a single vertical edge from \( P_1 \) to \( P_2 \), as in Figure 16. This divides each octahedron into four tetrahedra, each of which lies between two segments of the diagram, and we label them \( N, S, E, W \) as with the regions near a crossing (see Figure 3).

To describe shapes for the tetrahedra we label them and identify their vertices with points of \( \mathbb{C} \). At a positive crossing, our convention is that the vertices are always ordered \( P_2, P_1, P_-, P_+ \), that \( \tau_N \) and \( \tau_S \) are positive, and that \( \tau_E \) and \( \tau_W \) are negative. (Recall that \( P_+, P'_+ \) and \( P_-, P'_- \) are identified.) Geometrically \( P_- \) is located at 0, \( P_+ \) is located at \( \infty \), and we vary the locations of \( P_1 \) and \( P_2 \) to give the correct shapes. At a negative crossing we flip the signs of the tetrahedra. Both sets of conventions are summarized in Tables 1 and 2.

Theorem 3.5. At any non-pinched crossing the shaped tetrahedra of Tables 1 and 2 are geometrically non-degenerate and glue together to give an octahedron matching Definition 3.1.

Proof. The non-degeneracy claim is obvious from Definition 2.8, so consider the claim about gluing. The horizontal edges are automatic. For example, at a positive crossing the edge \( P_+ P'_+ \) is the 12 edge of \( \tau_S \), so it is assigned the shape

\[
z_{0S}^0 = \frac{m_2 b_2}{m_1 b_1'} = o_S
\]
Figure 16: The four-term decomposition of an ideal octahedron at a positive crossing.

### Table 1: Geometric data associated to the four-term decomposition at a positive crossing.

|    | vertices       | sign $\epsilon$ | $P_1$       | $P_2$       | shape $z^0$       |
|----|----------------|------------------|-------------|-------------|-------------------|
| $\tau_N$ | $P_1 P_2 P'_+ P'_-$ | 1                | $-1/b_1$    | $-1/b_2'$   | $b_2'/b_1$       |
| $\tau_W$ | $P_1 P_2 P'_- P'_+$ | $-1$          | $-1/m_1 b_1$ | $-1/b_2$    | $m_1 b_1/b_2$    |
| $\tau_S$ | $P_1 P_2 P'_- P'_+$ | 1                | $-1/m_1 b_1'$ | $-1/m_2 b_2$ | $m_2 b_2/m_1 b_1'$ |
| $\tau_E$ | $P_1 P_2 P'_- P'_+$ | $-1$          | $-1/m_1 b_1'$ | $-1/m_2 b_2'$ | $b_1'/m_2 b_2'$  |

### Table 2: Geometric data associated to the four-term decomposition at a negative crossing. The only difference from the positive case is that all the tetrahedra have the opposite sign.

|    | vertices       | sign $\epsilon$ | $P_1$       | $P_2$       | shape $z^0$       |
|----|----------------|------------------|-------------|-------------|-------------------|
| $\tau_N$ | $P_1 P_2 P'_+ P'_-$ | 1                | $-1/b_1$    | $-1/b_2'$   | $b_2'/b_1$       |
| $\tau_W$ | $P_1 P_2 P'_- P'_+$ | $-1$          | $-1/m_1 b_1$ | $-1/b_2$    | $m_1 b_1/b_2$    |
| $\tau_S$ | $P_1 P_2 P'_- P'_+$ | 1                | $-1/m_1 b_1'$ | $-1/m_2 b_2$ | $m_2 b_2/m_1 b_1'$ |
| $\tau_E$ | $P_1 P_2 P'_- P'_+$ | $-1$          | $-1/m_1 b_1'$ | $-1/m_2 b_2'$ | $b_1'/m_2 b_2'$  |
The significant part is checking the vertical edges. Again, we compute a representative case. Consider the edge $P_1P_2$ at a positive crossing. Its shape has contributions from $\tau_W$ and $\tau_S$; from Table 1 they are

$$z_W^1z_S^1 = \left(1 - \frac{b_2}{m_1b_1}\right)\left(1 - \frac{m_2b_2}{m_1b_1}\right)^{-1} = \frac{b_1 - b_2 - m_1b_1}{m_1b_2 - m_1b_1},$$

and by Lemma 2.10 this is equal to $1/m_2a_2 = a_2$.

Unfortunately, the four-term decomposition has a technical problem: at any pinched crossing (in the sense of Definition 2.8) all four tetrahedra have shape parameters $\{0, 1, \infty\}$, so they are geometrically degenerate. As shown in Example 6.4 and Section 6.3 there are plenty of interesting examples without pinched crossings, but to prove general existence theorems like Theorem 3.10 we need to use the five-term decomposition.

However, the four-term decomposition is still quite useful. We have discussed two notions of holonomy representation for a shaped diagram $D$ of a link $L$:

1. The holonomy representation $\rho : \Pi_1(D) \to \mathrm{SL}_2(\mathbb{C})$ given by Definition 2.15.
2. The holonomy representation $\rho : \pi_1(M_L) \to \mathrm{PSL}_2(\mathbb{C})$ induced by the geometric data of Tables 1 and 2.

We can use the four-term decomposition to make sure that these definitions agree. To be precise about the second one, we use face pairings.

The idea is as follows: We determine the geometry of the tetrahedra in an ideal triangulation by choosing where on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$ their ideal vertices lie. When we glue two tetrahedra together along a face, they will in general disagree about where the vertices of that face are. To fix this, we include a face map $g \in \mathrm{PSL}_2(\mathbb{C}) = \text{Isom}(\hat{\mathbb{C}})$ sending the vertices of one face to the other by a fractional linear transformation. Together all the face maps give a representation $\pi_1(M_L) \to \mathrm{PSL}_2(\mathbb{C})$. By carefully choosing the locations of the points $P_1$ and $P_2$, we see that the face-pairing maps at a crossing exactly correspond to the matrices in Definition 2.15.

**Theorem 3.6.** The holonomy representation of a shaped diagram agrees with the holonomy representation generated by the face maps of the associated four-term decomposition.

To prove the theorem it is helpful to consider a slightly different description of an ideal octahedron at a positive crossing related to ideal triangulations of discs. Every link $L$ can be represented as the closure of a braid $\beta$. If we view $\beta$ as an element of the mapping class group of the $n$-punctured disc $D_n$, then the complement $M_L$ of $L$ is the mapping torus $\Sigma$ of $\beta$. If we ideally triangulate $D_n$, and interpret the action of $\beta$ in terms of this triangulation, we can get an ideal triangulation of the mapping torus of $\beta$, that is of $M_L$. We describe this process in Figure 17. (To visualize it, it may help to examine Figure 18.)

We start with the triangulation in Figure 17a. For simplicity we consider a single crossing at a time, so we only need to consider two punctures $P_1$ and $P_2$ (plus two auxiliary punctures $P_+$ at the top and $P_-$ at the bottom). We think of these punctures as corresponding to strands oriented out of the page.

We can modify ideal triangulations by flipping the diagonal of a quadrilateral. From a 3-dimensional perspective, we are attaching the final edge of a tetrahedron above its base. In Figure 17b we add a red edge to build an ideal tetrahedron $P'_2 P_2 P_- P_1$. We then add two green edges, building two more tetrahedra. Finally, we add the blue edge to finish.

10 This has something to do with cluster algebras, as discussed in Section 5.
11 If $f : \Sigma \to \Sigma$ is a homeomorphism, then mapping torus of $f$ is the space $\Sigma \times [0, 1]$ modulo the relation $(x, 0) \sim (f(x), 1)$.
3. The octahedral decomposition and hyperbolic geometry

(a) The initial triangulation.
(b) Building a tetrahedron on top of a quadrilateral.
(c) Adding two more tetrahedra.
(d) The final result.

Figure 17: Building an ideal octahedron.
Ignoring the interior dashed edges, which are now below the tetrahedra we have added, we have a new, twisted copy of the triangulation in Figure 17a. By rotating $P_1$ above $P_2$, we pull the green edges taut and obtain our original picture, but with the points $P_1$ and $P_2$ swapped. In the process, we have braided the point $P_1$ over the point $P_2$. This corresponds to a positive braiding in our conventions, assuming that the strands are oriented out of the page in Figure 17. At the same time we have built a twisted octahedron at the crossing as required.

We are almost ready to compute the face maps, but first we need to pick a slightly unusual convention on fractional linear transformations, in order to match our convention that words in $\Pi_1(D)$ are read left-to-right.

**Definition 3.7.** Elements of $\text{PSL}_2(\mathbb{C})$ act on $\hat{\mathbb{C}}$ on the right by

$$z \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{az + c}{bz + d}.$$

**Lemma 3.8.** The face maps (as elements of $\text{PSL}_2(\mathbb{C})$) of the octahedron at a positive crossing agree with the holonomies assigned to the diagram complement by the shape parameters. \(\diamondsuit\)

**Proof.** If we think of the face map in Figure 19 as going from $\tau_W$ to $\tau_N$, then it represents the holonomy from travelling above strand 2, which should be mapped to $\chi_2^+$. Observe that for any $z \in \hat{\mathbb{C}}$,

$$z \cdot \chi_2^+ = z \cdot \begin{bmatrix} a_2 \\ (a_2 - 1/m_2)/b_2 \\ 0 \\ 1 \end{bmatrix} = a_2z + \frac{a_2}{b_2} - \frac{1}{m_2b_2}.$$

In particular, we see that $\chi_2^+$ fixes $\infty$, maps $-1/b_2$ to $-1/m_2b_2$, and maps $-1/m_1b_1$ to

$$(-1/m_1b_1) \cdot \chi_2^+ = a_2 \left( \frac{1}{b_2} - \frac{1}{m_1b_1} \right) - \frac{1}{m_2b_2} = -\frac{1}{m_1b_1^2}.$$

Because fractional linear transformations are totally determined by their action on three points of $\hat{\mathbb{C}}$, we conclude that the face map agrees with $\chi_2^+$. The negative holonomy of strand 2 does not correspond directly to a face map, but the face map going from $\tau_W$ to $\tau_N$ similarly corresponds to the inverse negative holonomy of $\chi_1$. We see that the transformation

$$z \cdot (\chi_1^-)^{-1} = z \cdot \begin{bmatrix} 1 \\ 0 \\ -(1 + m_1/a_1)b_1 \\ 1/a_1 \end{bmatrix} \left( -b_1 - \frac{m_1b_1}{a_1} + \frac{1}{za_1} \right)^{-1}$$

preserves $0$, maps $1/m_1b_1$ to $-1/b_1$, and maps $-1/m_2b_2$ to

$$(-1/m_2b_2) \cdot g^-(\chi_1)^{-1} = \left( -b_1 - \frac{m_1b_1}{a_1} - \frac{b_2}{a_1} \right)^{-1} = -\frac{1}{b_2}.$$

There is a parallel characterization of the holonomies on the other side of the crossing. For example, $\chi_2^+$ corresponds to the glu ng map between $\tau_N$ and $\tau_E$, and correspondingly acts on the vertices of $\tau_N$ by

$$\infty \cdot \chi_2^+ = \infty$$

$$(-1/b_2) \cdot \chi_2^+ = -1/m_2b_2$$

$$(-1/b_1) \cdot \chi_2^+ = -\frac{a_2}{b_1} + \frac{a_2}{b_2} - \frac{1}{m_2b_2} = -1/b_1,$$

and similarly the face map gluing $\tau_S$ to $\tau_E$ is $(\chi_1^-)^{-1}$. \(\square\)
Figure 20: Decomposition of the octahedron at a positive crossing into five tetrahedra.

Table 3: Geometric data for the five-term decomposition at a positive crossing.

|   | vertices   | sign $\epsilon$ | shape $z^0$ |
|---|------------|------------------|-------------|
| $\tau_1$ | $P_1 P_- P_+ P'^+$ | 1 | $a_1/m_1$ |
| $\tau_2$ | $P_2 P_+ P_- P'^+$ | 1 | $1/m_2 a_2$ |
| $\tau_{1'}$ | $P_1 P'^+ P_+ P'^+$ | $-1$ | $m_1/a_1'$ |
| $\tau_{2'}$ | $P_2 P'^+ P_- P'^+$ | $-1$ | $m_2 a_2'$ |
| $\tau_m$ | $P_- P'^+ P_+ P'^+$ | 1 | $a_1'/a_1$ |

**Proof of Theorem 3.6.** The theorem is a corollary of the previous lemma. At a positive crossing, we have shown the matrices $\chi_2^+, \chi_1, \chi_2^+$, and $\chi_1^-$ agree with the corresponding face maps. $\chi_2$ is now the unique matrix of the form

$$\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$$

such that

$$\text{tr} \chi_2^+ (\chi_2^-)^{-1} = m_2 + m_2^{-1}$$

and similarly for the other strands. This shows that we have agreement at any positive crossing. By repeating the computation in Lemma 3.8 for negative crossings we obtain the theorem. 

3.4. The five-term decomposition

We can divide the octahedron at a crossing into five tetrahedra, as in Figure 20. We think of this decomposition as being associated to the $a$-variables. We record the geometric data of the tetrahedra in Table 3.

Proposition 3.9. At any non-degenerate crossing the shaped tetrahedra of Tables 3 and 4 are non-degenerate and glue together to give an octahedron matching Definition 3.1.

$\diamondsuit$
Table 4: Geometric data for the five-term decomposition at a negative crossing.

| $\tau_1$ | $P_1 P'_+ P'_-$ | $\text{sign } \epsilon$ | $z^0$ |
|-------|-----------------|----------------|------|
| $\tau_2$ | $P_2 P_-' P'_+$ | $-1$ | $1/a_1 m_1$ |
| $\tau_1'$ | $P_1 P'_- P'_+$ | $-1$ | $a_2/m_2$ |
| $\tau_2'$ | $P_2 P'_+ P'_+$ | $1$ | $m_1 a_1'$ |
| $\tau_m$ | $P_-' P'_+ P'_+$ | $-1$ | $a_1/a_1'$ |

Proof. We first check the non-degeneracy claim. A crossing is non-degenerate if $a_1/a_1' = a_2/a_2'$ is not equal to 1, which is the same as saying that $\tau_m$ is geometrically non-degenerate. Suppose the crossing is positive. Then since

$$\frac{a_1}{a_1'} = 1 - \frac{m_1 b_1}{b_2} \left( 1 - \frac{a_1}{m_1} \right) \left( 1 - \frac{1}{m_2 a_2} \right) \neq 1$$

we cannot have $a_1 = m_1$ or $a_2 = 1/m_2$, which says that $\tau_1$ and $\tau_2$ are geometrically non-degenerate. There is a similar expression for $a_1/a_1'$ in terms of $a_1'$, $a_2'$, $b_1'$, and $b_2'$ which comes from inverting the map $B$, and it shows that $a_1/a_1' \neq 1$ implies $a_1' \neq m_1$ and $a_2' \neq 1/m_2$, so $\tau_1$ and $\tau_2$ are geometrically non-degenerate. If the crossing is negative, a similar argument shows that $a_1/a_1' \neq 1$ implies $a_1, a_1' \neq 1/m_1$ and $a_2, a_2' \neq m_2$.

Now we check the gluing relations. In contrast to before, the vertical edges are automatic. For example, at a positive crossing the total shape of $P_- P_1$ should be $o_1 = a_1/m_1$, and the only contributing tetrahedron is $\tau_1$:

$$z_1^0 = \frac{a_1}{m_1} = o_1.$$  

The horizontal edges require using some identities on $a_1/a_1' = a_2'/a_2$. As always, we compute some representative examples. Consider the edge $P_- P_+$ at a positive crossing, which should have shape $o_{12} = b_2/m_1 b_1$. It has contributions from $\tau_1$, $\tau_2$, and $\tau_m$, which give a shape

$$z_1^1 z_2^1 z_m^2 = \left( 1 - \frac{a_1}{m_1} \right)^{-1} \left( 1 - \frac{1}{m_2 a_2} \right)^{-1} \left( 1 - \frac{a_1}{a_1'} \right).$$

By Definition 2.4,

$$1 - \frac{a_1}{a_1'} = \frac{m_1 b_1}{b_2} \left( 1 - \frac{a_1}{m_1} \right) \left( 1 - \frac{1}{m_2 a_2} \right)$$

so we see that

$$z_1^1 z_2^1 z_m^2 = \frac{m_1 b_1}{b_2} = o_W.$$  

Similarly, at a positive crossing the edge $P'_+ P'_+$ has contributions from $\tau_1'$, $\tau_2'$, and $\tau_m$, which give a shape

$$z_1^1 z_2^1 z_m^2 = \left( 1 - \frac{m_1}{a_1'} \right)^{-1} \left( 1 - m_2 a_2' \right)^{-1} \left( 1 - \frac{a_1}{a_1'} \right).$$

By using the inverse $B^{-1}$ of the braiding we can compute that

$$1 - \frac{a_1}{a_1'} = \frac{b_1'}{m_2 b_2} \left( 1 - \frac{m_1}{a_1'} \right) \left( 1 - m_2 a_2' \right).$$
4. Decorated representations

and we conclude that
\[ z_1^2 z_2 z_m^2 = \frac{b_1}{m_2 b_2} = o_E. \]

While the five-term decomposition uses more tetrahedra, it has a nice nondegeneracy property:

**Theorem 3.10.** Let \( \rho \) be a \( \mathrm{SL}_2(\mathbb{C}) \)-structure on a link \( L \). We say \( \rho \) is **meridian-trivial** if \( \rho(m) = \pm 1 \), that is if sends any meridian of \( L \) to plus or minus the identity matrix. If \( \rho \) is meridian-nontrivial, then for any diagram \( D \) of \( L \) there is a shaping of \( D \) such that:

1. the holonomy representation of \( D \) is conjugate to \( \rho \), and
2. all the tetrahedra in the five-term decomposition associated to \( D \) are geometrically nondegenerate.

**Proof.** This is due to Yoon, who adapted a result of Cho [Cho16]. As discussed in Section 6.2 we can solve for non-degenerate shapings in terms of only the variables \( a_i \) and \( m_i \). Using the region variables instead of the \( a_i \), such solutions are exactly the “non-degenerate points” of [Yoo21], so the claim is Theorem 1.2 of [Yoo21].

4. Decorated representations

It turns out that our coordinates do not just describe a representation \( \rho : \pi_1(S^3 \setminus L) \rightarrow \mathrm{SL}_2(\mathbb{C}) \), but actually give slightly more information called a **decoration.** These show up naturally in a few contexts: some indeterminacies in the \( A \)-polynomial come from a choice of decoration,\(^\text{13}\) and decorations occur naturally in Ptolemy coordinates and the computation of complex volume.

In this section we discuss decorations, then explain how to compute them from shaping of a link diagram. In particular we show how to compute the longitude eigenvalues of the holonomy representation.

4.1. Peripheral subgroups and decorations

**Definition 4.1.** For a link \( L \) in \( S^3 \) the **exterior** \( E(L) := S^3 \setminus \nu(L) \) is the complement of an open regular neighborhood of \( L \). It is a compact manifold with boundary \( \partial E(L) = T_1 \cup \cdots \cup T_n \) a disjoint union of tori, one for each component of \( L \). We call the image \( H_j \subset \pi_1(S^3 \setminus L) \) of \( \pi_1(T_j) \) the **peripheral subgroup** associated to the component \( L_j \).

Each \( H_j \) is isomorphic to \( \mathbb{Z}^2 \). The usual choice of generators are a **meridian** \( m_j \) and **longitude** \( l_j \). (An example is given in Figure 21.) Unless noted we always use the zero-framed longitude.

Because \( H_j \) is abelian we can always conjugate \( \rho \) so that \( \rho(m_j) \) and \( \rho(l_j) \) are both lower-triangular. Another way to say this is that we can always identify \( \rho(H_j) \) with a subgroup of \( B \), where \( B \subset \mathrm{SL}_2(\mathbb{C}) \) is the subgroup of lower-triangular matrices.

**Definition 4.2.** A **decoration** [GTZ15, Section 4] of \( \rho : \pi_1(S^3 \setminus L) \rightarrow \mathrm{SL}_2(\mathbb{C}) \) is a choice of identification of each peripheral \( \rho(H_j) \) with a subgroup of \( B \).

Strictly speaking we are using the characterization of decorations up to equivalence given in [GTZ15, Proposition 4.6]. The choice to use lower-triangular matrices is nonstandard but it matches our conventions in Definition 3.7.

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\[ \text{[Cho16]} \] J. Cho, “Optimistic limit of the colored Jones polynomial and the existence of a solution”. arXiv doi

\[ \text{[Yoo21]} \] S. Yoon, "On the potential functions for a link diagram". arXiv doi

\[ ^\text{13} \] The \( A \)-polynomial is usually described as a Laurent polynomial in two variables \( m \) and \( \ell \) that is well-defined up to simultaneous inversion \( m \mapsto m^{-1}, \ell \mapsto \ell^{-1} \). These variables represent the meridian and longitude eigenvalues determined by the decorated representation and the inversion comes from changing the choice of decoration.

---

Figure 21: A meridian \( m \) (in red) and longitude \( l \) (in blue) for the figure-eight knot.

[GTZ15] S. Garoufalidis, D. P. Thurston, and C. K. Zickert, ”The complex volume of \( \mathrm{SL}(n, \mathbb{C}) \)-representations of 3-manifolds”. arXiv doi
Remark 4.3. A decoration determines preferred eigenvalues $m_j$ of each $\rho(m_j)$ and $l_j$ of each $\rho(l_j)$, because we identify them with matrices

$$\rho(m_j) \sim \begin{pmatrix} m_j & 0 \\ * & m_j^{-1} \end{pmatrix}, \quad \rho(l_j) \sim \begin{pmatrix} l_j & 0 \\ * & l_j^{-1} \end{pmatrix}$$

Conversely, when $m_j \neq \pm 1$ the choice of $m_j$ versus $m_j^{-1}$ determines a decoration. In the boundary-parabolic case $m_j = \pm 1$ the correspondence is more complicated: we have

$$\rho(m_j) \sim \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho(l_j) \sim \begin{pmatrix} \epsilon & 0 \\ \epsilon & \epsilon \end{pmatrix}$$

where $\epsilon \in \{1, -1\}$ and $c \in \mathbb{C}$ is the cusp shape.

More generally a decoration gives a choice $\delta(x)$ of eigenvalue of $\rho(x)$ for every $x \in \pi_1(T_j)$, since $\rho(x)$ is conjugate to

$$\begin{pmatrix} \delta(x) & 0 \\ * & \delta(x)^{-1} \end{pmatrix}$$

Another way to phrase this is to say that $\rho$ induces a homomorphism

$$\rho : H_1(\partial M; \mathbb{Z}) = \bigoplus_{j=1}^n H_1(T_j; \mathbb{Z}) \to \text{SL}_2(\mathbb{C}).$$

and a decoration gives lift

$$\delta : H_1(\partial M; \mathbb{Z}) \to \mathbb{C}^\times.$$

We sometimes call $\delta$ an eigenvalue decoration because it is almost but not quite a decoration: when our representation is boundary-parabolic we also need to know the cusp shape to determine one. Cusp shapes can be computed from the octahedral decomposition [KKY19, Section 5.1] with a similar method but we are most interested in $\delta$. Our main result in this section is an explicit formula for it in terms of a diagram shaping.

**Theorem 4.4.** Let $D$ be a diagram of a link $L$. A shaping $\chi$ of $D$ determines an eigenvalue decoration $\delta = \delta_{\rho}$ of its holonomy representation $\rho$. Explicitly, the homomorphism $\delta$ is given by

$$\delta(m_j) = m_j, \quad \delta(l_j) = m_j^{-w_j} \prod_{k} b_{k}^{\eta_k}$$

where $w_j$ is the writhe of component $j$, the product is over all segments in component $j$, and

$$\eta_k := \begin{cases} 1 & \text{if segment } k \text{ is over-under,} \\ -1 & \text{if it is under-over, and} \\ 0 & \text{otherwise.} \end{cases}$$

We typically do not work directly with the matrices of the holonomy to determine the decoration. Instead, we use a more geometric characterization. Thinking of the boundary torus $T_j$ of $E(L)$ as the boundary of a cusp it has an affine structure locally modeled on the Euclidean plane $\mathbb{C}$. The holonomy acts by affine transformations, and the eigenvalues $m_j, \ell_j$ are related to the scaling factors of these transformations. We refer to [Pur20, Section 4.3] for a
This perspective lets us compute the holonomies directly from an ideal triangulation of $S^3 \setminus L$. By truncating our tetrahedra we get a triangulation of the cusps and we can read off the eigenvalues in terms of the shapes.

**Definition 4.5.** Let $\gamma$ be an oriented simple curve in the boundary $T$ of a cusp of $E(L)$, which we triangulate by truncating an ideal triangulation of $S^3 \setminus L$. Isotope $\gamma$ so it intersects only edges of the triangulation transversely and cuts a single corner off of each triangle. This corner is associated to the edge of an ideal tetrahedron, and we assign it the shape parameter $z_k$ of that edge. The *holonomy* of $\gamma$ is

$$\text{Hol}(\gamma) := \prod_{k=1}^{n} z_k^{\epsilon_k}$$

where the product is over all the triangles $\gamma$ passes through,

$$\epsilon_k = \begin{cases} +1 & \text{if the corner is right of } \gamma, \\ -1 & \text{if the corner is left of } \gamma, \end{cases}$$

and we view the boundary triangles from outside $S^3 \setminus L$.\(^{14}\) We give an example in Figure 22.

For the octahedral decomposition this triangulation comes from squares, which are subdivided in different ways depending on whether we use the four-term or five-term-decomposition. In the rest of this section we compute the holonomy of the meridian and longitude using this construction, which along with the next proposition gives a proof of Theorem 4.4.

**Proposition 4.6.** Consider an ideal triangulation of $S^3 \setminus L$ with a shaping corresponding to the representation $\rho : \pi_1(S^3 \setminus L) \to \text{SL}_2(\mathbb{C})$. The holonomy map $\text{Hol}$ of the triangulation gives a homomorphism

$$\text{Hol} : H_1(\partial E(L); \mathbb{Z}) \to \mathbb{C}^\times$$

and any decoration of $S^3 \setminus L$ satisfies

$$\delta(x)^2 = \text{Hol}(x) \text{ for all } x \in H_1(\partial E(L); \mathbb{Z}).$$

**Proof.** The square root comes from the difference between the action of $\mathbb{C}^\times$ on $\mathbb{C}$ by multiplication and the action of $\text{SL}_2(\mathbb{C})$ by fractional linear transformations. Geometrically, the holonomy $\text{Hol}(x)$ represents a scaling/rotation of $\mathbb{C}$ by multiplication by an element of $\mathbb{C}^\times$. On the other hand we can also compute this action directly from the matrix $\rho(x) \in \text{SL}_2(\mathbb{C})$. Using the decoration $\delta$, $\rho(x)$ is conjugate to

$$\tilde{\rho}(x) = \begin{pmatrix} \delta(x) & 0 \\ b & \delta(x)^{-1} \end{pmatrix}$$

for some $b \in \mathbb{C}$ and (following Definition 3.7)

$$\tilde{\rho}(x) \cdot z = \frac{\delta(x)z + b}{\delta(x)^{-1}} = \delta(x)^2 z + b\delta(x)^{-1}$$

We are interested only in the scaling action $\delta(x)^2$, which is the *square* of the eigenvalue $\delta(x)$ as claimed. \(\square\)
For a $\text{PSL}_2(\mathbb{C})$ representation the eigenvalues $\delta(x)$ are only determined up to sign, so knowing $\text{Hol}$ determines $\delta$. For $\text{SL}_2(\mathbb{C})$ representations it is not obvious how to choose the sign of $\sqrt{\text{Hol}}$; we discuss this further in Remark 4.12.

This can be somewhat subtle: for a boundary-parabolic representation we can always choose a lift where the meridians have trace 2, but the longitudes might still have trace $-2$. This is related to the obstruction class of the representation [CYZ20, Section 2.1].

4.2. The meridians

We first compute the holonomy of the meridian $m$ of a segment of a shaped diagram. We already know that the answer should be $m^2$ when the segment is assigned the shape $(a, b, m)$, so our goal is to check this against the geometry. We can express $m$ as a composition of the curves $\sigma^\pm, \tau^\pm$ shown in Figure 23. The exact form depends on the type of segment:

**Proposition 4.7.** Consider a segment between crossings labeled $k$ and $k+1$. Using the segment

\[
\begin{array}{c|c|c}
& k & k+1 \\
\hline
k & & k+1
\end{array}
\]

Figure 25: An over-under (left) and under-over (right) segment between crossings $k$ and $k+1$. 

\[
\begin{aligned}
P_1 & \quad \sigma^+ \quad \tau^+ \quad \beta^-
\end{aligned}
\]

\[
\begin{aligned}
P_2 & \quad \sigma^- \quad \tau^- \quad \beta^+
\end{aligned}
\]
types in Figure 25,
\[ \tau_k^+ \sigma_{k+1}^- = m \quad \text{at an over-under segment,} \]
\[ \tau_k^- \sigma_{k+1}^+ = m \quad \text{at an under-over segment, and} \]
\[ \tau_k^\pm \sigma_{k+1}^\pm = m \quad \text{at an over-over or under-under segment.} \]

where \( m \) is the meridian of the segment. \( \Diamond \)

**Proof.** We can see this directly by composing the curves in Figures 23 and 24 during the gluing. Alternately, it follows from the discussion in [KKY18, Section 4.1] and in particular [KKY18, eq. 10]. Notice that their meridians are the inverse of ours. \( \square \)

Now we can read off the holonomy of the curves directly from the cusp triangulation:

**Lemma 4.8.** At a positive crossing,
\[ \text{Hol}(\sigma^+) = o_1^{-1} = \frac{m_1}{a_1} \]
\[ \text{Hol}(\tau^+) = o_1' = \frac{m_1}{a_1'} \]
\[ \text{Hol}(\sigma^-) = o_2^{-1} = m_2a_2 \]
\[ \text{Hol}(\tau^-) = o_2' = m_2a_2' \]

while at a negative crossing
\[ \text{Hol}(\sigma^+) = o_2^{-1} = m_2a_2 \]
\[ \text{Hol}(\tau^+) = o_2' = m_2a_2' \]
\[ \text{Hol}(\sigma^-) = o_1^{-1} = m_1a_1 \]
\[ \text{Hol}(\tau^-) = o_1' = m_1a_1' \]

where the \( o_j \) are the shapes of the vertical edges given in equations (14) and (16). \( \Diamond \)

**Proof.** To apply Definition 4.5 we divide the squares of Figure 26 into triangles by dividing our octahedra into tetrahedra. Using the five-term decomposition this looks like Figure 26, and then at a positive crossing we have
\[ \text{Hol}(\sigma^+) = \left( z_1^{-1} \right) m_1 \]
\[ \text{Hol}(\sigma^-) = \left( z_2^{-1} \right) m_2a_2 \]
\[ \text{Hol}(\tau^+) = \left( z_1^{-1} \right) \]
\[ \text{Hol}(\tau^-) = \left( z_2^{-1} \right) \]

because we view the boundary from outside the link exterior. The other cases follow from similar computations. \( \square \)

**Proof of equation (20).** Consider a segment assigned the shape \( \chi = (a, b, m) \) between crossings \( k \) and \( k+1 \). If it is an over-under segment, then
\[ \text{Hol}(m) = \text{Hol}(\tau_k^+ \sigma_{k+1}^-) = \frac{m}{a} m a = m^2 \]
as claimed. Notice that this computation does not rely on the signs of the crossings. Similarly at an under-over segment we have
\[ \text{Hol}(m) = \text{Hol}(\tau_k^- \sigma_{k+1}^+) = \frac{m}{a} m a = m^2. \]

Taking the square root gives equation (20). We know that \( m \) (and not \( -m \)) is the right sign because we can explicitly check that it is an eigenvalue of (13).\(^\text{15}\) We only need to check equation (20) for one segment of each link component, and at least one segment of any component of any link diagram is either over-under or under-over. (Actually, this is only true if the component has at least one crossing. By adding kinks we can always assume this.) \( \square \)

\(^{15}\) Actually, we didn’t really need to go through this whole derivation using boundary holonomies: we could have just used this argument. However, it’s a good warm-up for the longitude computation.
The equations for $\text{Hol}(m)$ are examples of $m$-hyperbolicity equations:

**Definition 4.9.** The $m$-**hyperbolicity equation** for a segment is

$$
\frac{o'}{o} = \begin{cases} 
m^2 & \text{if the segment is over-under or under-over, and} \\
1 & \text{otherwise.}
\end{cases}
$$

where $o$ is the shape of the vertical edge at the start of the segment and $o'$ is the shape of the vertical edge at the end.

We just showed that when using shape coordinates the $m$-hyperbolicity equations automatically hold. As discussed in the proof of Theorem 3.2 they imply the gluing equations for the vertical edges.

### 4.3. The longitudes

Our convention is to obtain the blackboard-framed longitude $\tilde{l}$ by pushing off to the right, so it is given by

$$
\tilde{l} = \prod_k \beta_k^{\eta_k}
$$

where $\eta_k$ is $+$ at an overcrossing and $-$ at an undercrossing and the product is over all intersections of our component with the rest of the diagram. In general the zero-framed longitude is

$$
l = \tilde{l} - w m
$$

where $w$ is the writhe of the link component we are considering.

**Example 4.10.** In Figure 27 the blue curve is $\tilde{l}$, so it is given by

$$
\tilde{l} = \beta_1^{-} \beta_2^{+} \beta_3^{-} \beta_1^{+} \beta_2^{-} \beta_3^{+}.
$$

Notice that a crossing can appear twice in the product in equation (22), and for knots they always do. The zero-framed longitude is $l = \tilde{l} - 3m$ because this diagram has writhe 3.

**Lemma 4.11.** At a positive crossing

$$
\text{Hol}(\beta^+) = \frac{a_2}{a_1} \frac{b_1'}{b_2}
$$

and at a negative crossing

$$
\text{Hol}(\beta^-) = \frac{a_2}{b_1} \frac{b_1'}{b_2'}
$$

**Proof.** For $\beta^+$ at a positive crossing, we can use the five-term decomposition as before to compute

$$
\text{Hol}(\beta^+) = z_1^1 z_1^1 = \left(1 - \frac{a_1}{m_1}\right)^{-1} \left(1 - \frac{a_1'}{m_1'}\right) = \frac{b_2'}{b_1'}
$$

using equation (7). The other computations follow similarly.

---

![Figure 27: Here the blue curve is the blackboard-framed longitude $\tilde{l}$.](image)

![Figure 28: Following the gold strand, this crossing contributes a factor of $(r'/r)(b'/b)^o$ to the longitude holonomy $\text{Hol}(l)$.](image)
4. Decorated representations

Proof of equation (21). We can now compute \( \text{Hol}(\tilde{l}) \) and thus \( \text{Hol}(l) \) as a product over the crossings of our diagram. To prove our claim it suffices to show that this product is the square of the right-hand-side of equation (21).

The expressions in Lemma 4.11 follow a simple pattern in terms of the region variables of Section 6.2. The idea is to assign variables \( r_k \) to the regions of the diagram so ratios of adjacent region variables give the \( a \)-variable of the strand between them, as in Figure 30. It is easy to see (Lemma 6.7) we can always assign region variables to any shaped diagram.

Once we do this we can summarize Lemma 4.11 by saying that the contribution of the crossing in Figure 28 is

\[
\frac{r'}{r} \left( \frac{b'}{b} \right)^\eta
\]  

(24)

where \( \eta \) is 1 if the gold strand passes over the black strand and -1 if it passes under. The variables \( r, r' \) and \( b, b' \) correspond to the regions and segments adjacent to the crossing. We can prove equation (24) by a trivial case-by-case check, as usual.

Now we need to translate our product over crossings into a product over segments. First consider an over-under segment, like in Figure 29. The crossings at each end contribute a factor

\[
a \left( \frac{b_1}{b_0} \right)^{+1} \cdot \frac{1}{a'} \left( \frac{b_2}{b_1} \right)^{-1} = \frac{r_1 b_1 r_2 b_1}{r_0 b_0 r_1 b_1} = \frac{r_2}{r_0 b_0} \frac{1}{b_1^2}
\]

to \( \text{Hol}(\tilde{l}) \). In particular, we see that \( r_1 \) does not contribute, and the exponent of \( b_1 \) is +2. If instead we had an over-over segment, the contribution would be

\[
\frac{r_1 b_1 r_2 b_2}{r_0 b_0 r_1 b_1} = \frac{r_2}{r_0} \frac{b_2}{b_0}
\]

and \( b_1 \) does not appear at all.

More generally, when following a component of a link diagram, the region variables appear as a telescoping product

\[
\frac{r_0}{r_1} \cdot \frac{r_1}{r_2} \cdots \frac{r_n}{r_0} = 1
\]

and the \( b \)-variables only show up with even exponents: +2 if their segment is over-under, -2 if it is under-over, and 0 otherwise. We have shown that when we write \( \text{Hol}(\tilde{l}) \) as a product over segments,

\[
\text{Hol}(\tilde{l}) = \prod_k t_k^{2\sigma_k}
\]

so

\[
\text{Hol}(l) = m^{-2w_j} \prod_k t_k^{2\sigma_k} = \delta(l)^2.
\]

It remains only to show that we have taken the correct sign of \( \sqrt{\text{Hol}(l)} \). (We have already showed that we picked the right sign of \( \sqrt{\text{Hol}(m)} \) because \( m \) is an eigenvalue of (13), not \( -m \).)

Remark 4.12. When \( \rho \) is a boundary-parabolic \( \text{PSL}_2(\mathbb{C}) \)-representation \( \text{Hol}(m) = (\pm 1)^2 = 1 \), so \( \delta(m) = \pm 1 \). Because the meridians generate \( \pi_1(\text{E}(L)) \) we can always lift \( \rho \) to \( \text{SL}_2(\mathbb{C}) \) by picking a sign of the meridians, say \( \delta(m) = 1 \). However, in general the sign of \( \delta(l) \) can be \(-1 \), regardless of the choice of lift. In fact, for a hyperbolic knot complement we have \( \delta(l) = -1 \) for any lift of the geometric representation \[\text{Cal06, Corollary 2.4}\].
We can think of this as an obstruction to lifting \( \rho \) to a boundary-unipotent \( SL_2(\mathbb{C}) \)-representation, and in this context \( \delta(l) \in \{1, -1\} \) is called the obstruction class \([\text{GTZ15; KKY19}]\) of the representation. Determining the sign of \( \delta(l) = \sqrt{\text{Hol}(l)} \) is closely related to these obstruction classes.

**Proof of Theorem 4.4.** The trick is again to work with a triangulation of the boundary, just as for \( \text{Hol} \), but this time with matrices instead of elements of \( \mathbb{C}^\times \). This is done in detail in the boundary-parabolic case in \([\text{KKY19, Section 4}]\); the general argument follows by extending their work to deformed Ptolemy varieties, as in \([\text{Yoo18, Section 2}]\). We sketch the idea below.

We want to encode the restriction \( \rho|_{H_j} \) of \( \rho \) to the peripheral subgroup \( H_j \) of our boundary component in terms of a cocycle \( N^1 \to B \), where by \( N^1 \) we mean the 1-skeleton of the triangulation of the boundary \( \partial E(L) \). It turns out this gives a particularly nice description of \( \rho \) on all of \( M \) in terms of a natural cocycle \( B : M^1 \to SL_2(\mathbb{C}) \) on the 1-skeleton of our triangulation of \( E(L) \), and we can then read off \( \delta(l) \) directly.

It is easy to express \( B \) using the \( b \)-variables, and we get an expression for the longitude just like the ones for \( \text{Hol}(\beta^\pm) \). (Compare the formula \( \sigma(e_i) = z_a/z_c \) in \([\text{KKY19, Section 4}]\) to equation \((24)\).) Once we do this it is clear that equation \((21)\) has the correct sign. \( \square \)

5. A connection to quantum groups

The shape coordinates (Section 2) are deeply related to the hyperbolic geometry of the knot complement (Section 3). However, they were originally discovered in the context of quantum groups, specifically the representation theory of quantum \( sl_2 \). When \( q \) is a root of unity the braiding of the quantum group has some unusual properties, and understanding them leads directly to the braiding given in Definition 2.4. To obtain the geometrically interesting shape coordinates we use a presentation of \( \mathcal{U}_\xi(sl_2) \) in terms of a quantum Weyl algebra. This presentation is derived from cluster algebras associated to the ideal triangulation in Figure 17a.

While most of this story has either previously appeared in the literature or in the author’s thesis \([\text{McP21}]\), in this section we give a brief overview for non-specialists.

5.1. Quantum groups at roots of unity

For any simple Lie algebra \( g \), the quantum group \( \mathcal{U}_q(g) \) is a \( q \)-analogue of the universal enveloping algebra of \( g \). Quantum groups have a number of interesting algebraic properties; the most important for topology is the existence of an element \( R \in \mathcal{U}_q(g) \otimes \mathcal{U}_q(g) \) called the universal R-matrix. It satisfies braid relations that are the key ingredient for constructing quantum invariants of knots and links \([\text{RT90; Oht01}]\). This construction is universal in the sense that any choice of \( \mathcal{U}_q(g) \)-module \( V \) determines a link invariant. For example, if we choose \( g = sl_2 \) and \( V \) the irreducible \( N \)-dimensional representation of \( \mathcal{U}_q(sl_2) \) we get the \( N \)th colored Jones polynomial.

When \( q \) is not a root of unity, the representation theory of \( \mathcal{U}_q(g) \) is quite similar to the classical representation theory of \( g \). However, at a root of unity things become much more complicated and depend on exactly which form of the quantum group we use. For the Kac-de Concini form \([\text{DKP91}]\) of the quantum group, this is because the center gets much larger. We can make this precise as follows:

**Theorem 5.1.** When \( q = \xi \) is a primitive \( 2N \)th root of unity, \( \mathcal{U}_\xi(sl_2) \) contains a large central subalgebra \( \mathcal{Z}_0 \subset \mathcal{U}_\xi(sl_2) \). Because \( \mathcal{Z}_0 \) is a commutative Hopf algebra, we can interpret it as

\[ \text{Yoo18} \] S. Yoon, The volume and Chern-Simons invariant of a Dehn-filled manifold. arXiv DOI

\[ \text{RT90} \] N. Y. Reshetikhin and V. G. Turaev, “Ribbon graphs and their invariants derived from quantum groups”. DOI

\[ \text{Oht01} \] T. Ohtsuki, Quantum Invariants. DOI

\[ \text{DKP91} \] C. De Concini, V. G. Kac, and C. Procesi, “Representations of quantum groups at roots of 1”
the algebra of functions on an algebraic group, in this case the group
\[ \text{SL}_2(\mathbb{C})^* = \left\{ \begin{pmatrix} \kappa & 0 \\ \phi & 1 \end{pmatrix}, \begin{pmatrix} 1 & \epsilon \\ 0 & \kappa \end{pmatrix} \mid \kappa \neq 0 \right\}. \]

Concretely, characters (i.e. algebra homomorphisms) \( \chi : \mathcal{Z}_0 \to \mathbb{C} \) give points of \( \text{SL}_2(\mathbb{C})^* \), and the product is given by
\[ (\chi_1 \cdot \chi_2)(x) := (\chi_1 \otimes \chi_2)(\Delta(x)) \]
where \( \Delta \) is the coproduct of \( \mathcal{U}_q(\mathfrak{sl}_2) \). The algebra \( \mathcal{Z}_0 \) is large in the sense that \( \mathcal{U}_q(\mathfrak{sl}_2)/\ker \chi \) has dimension \( N^2 \) for any \( \chi \), and the whole center
\[ Z(\mathcal{U}_q(\mathfrak{sl}_2)) = \mathcal{Z}_0[\mathcal{O}] / \text{(polynomial relation)} \]
is generated by \( \mathcal{Z}_0 \) and the Casimir element \( \Omega \), modulo a degree \( N \) polynomial relation given by a Chebyshev polynomial.

\[ \square \]

Proof. See [McP21, Chapter 0] or [Bla+20, Section 6]. These results are due to work of De Concini, Kac, and Procesi [DKP91; DKP92].

We can be more concrete by giving a presentation of \( \mathcal{U}_q(\mathfrak{sl}_2) \): it is the algebra over \( \mathbb{C}[q, q^{-1}] \) with generators \( K^\pm, E, F \) and relations
\[ KK^{-1} = 1, \quad KE = q^2 KE, \quadKF = q^{-2} FK, \quad EF - FE = (q - q^{-1})(K - K^{-1}). \]

It is a Hopf algebra with coproduct
\[ \Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \]
counit
\[ \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0, \]
and antipode
\[ S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \]

For \( q \) not a root of unity, the center \( Z(\mathcal{U}_q(\mathfrak{sl}_2)) \) is generated by the Casimir
\[ \Omega = EF + q^{-1}K + qK^{-1} = FE + qK + q^{-1}K^{-1}. \]

**Proposition 5.2.** At \( q = \xi \) a \( 2N \)th root of unity the subalgebra \( \mathcal{Z}_0 \) is generated by \( K^{\pm N}, E^N, \) and \( F^N \). Characters \( \chi : \mathcal{Z}_0 \to \mathbb{C} \) correspond to points of \( \text{SL}_2(\mathbb{C})^* \) via
\[ \chi \mapsto \begin{pmatrix} \chi(K^N) & 0 \\ \chi(K^N F^N) & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{C})^* \]
and this correspondence gives an isomorphism \( \text{Spec}(\mathcal{Z}_0) \to \text{SL}_2(\mathbb{C})^* \) of algebraic groups.

As a consequence, to construct quantum invariants from \( \mathcal{U}_q(\mathfrak{sl}_2) \) we must understand the group \( \text{SL}_2(\mathbb{C})^* \). To see why, use Schur’s Lemma: if \( V \) is any simple \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module, the action of the central subalgebra \( \mathcal{Z}_0 \) factors through some character \( \chi : \mathcal{Z}_0 \to \mathbb{C} \). Furthermore, if \( V_1, V_2 \) are two simple modules with characters \( \chi_1, \chi_2 \), then their tensor product \( V_1 \otimes V_2 \) will have central character \( \chi_1 \chi_2 \). One way to say this is that \( \mathcal{U}_q(\mathfrak{sl}_2)-\text{Mod} \) is a \( \text{SL}_2(\mathbb{C})^* \)-graded category.\[16\]

\[ \square \]

[DKP92] C. De Concini, V. G. Kac, and C. Procesi, "Quantum coadjoint action". DOI

[RT91] N. Reshetikhin and V. G. Turaev, "Invariants of 3-manifolds via link polynomials and quantum groups". DOI

[Tur16] V. G. Turaev, Quantum invariants of knots and 3-manifolds. DOI
One approach to dealing with this grading is to mostly eliminate it: if we take the quotient of \( \mathcal{U}_q(\mathfrak{sl}_2) \) by the relations \( K^{2N} = 1, E^N = F^N = 0 \) we obtain the \textit{small quantum group} \( \overline{\mathcal{U}}_q \). This corresponds to considering only representations whose \( Z_0 \)-character is plus or minus the identity element of \( \text{SL}_2(\mathbb{C})^* \). The category \( \overline{\mathcal{U}}_q \text{-Mod} \) is not semisimple, but by killing the so-called negligible morphisms (those with quantum trace 0) we can obtain a \textit{modular} category, which has the necessary algebraic properties to construct a surgery TQFT [RT91; Tur16]. The corresponding link invariants are colored Jones polynomials evaluated at roots of unity.

However, we want to go in a different direction and take full advantage of the \( \text{SL}_2(\mathbb{C})^* \)-grading. This idea leads to \textit{quantum holonomy invariants} [KR05; Bla+20; McP22; McP21]. In the usual construction, picking a single \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module \( V \) gives a link invariant; for example, choosing \( V \) to be the simple \( N \)-dimensional representation of \( \mathcal{U}_q(\mathfrak{sl}_2) \) gives the colored Jones polynomial at a root of unity \([MM01]\). For quantum holonomy invariants, we instead pick a \textit{family} \( V_\chi \) of \( \mathcal{U}_q(\mathfrak{sl}_2) \)-modules indexed by points of \( \text{SL}_2(\mathbb{C})^* \). We think of \( V_\chi \) as a deformation of \( V \) by the character \( \chi \in \text{SL}_2(\mathbb{C})^* \), and \( V \) is the case where \( \chi \) is the identity element. The input to our construction is no longer a link diagram, but a link diagram with segments colored by elements of \( \text{SL}_2(\mathbb{C})^* \). As shown in Section 2, such a coloring is a choice of hyperbolic structure on the link complement, so we think of holonomy invariants as geometrically twisted versions of ordinary quantum invariants.

### 5.2. The braiding at a root of unity

In order for the construction described in the last paragraph to work the coloring by elements of \( \text{SL}_2(\mathbb{C})^* \) has to be compatible with the structure of \( \mathcal{U}_q(\mathfrak{sl}_2) \), in particular with the braiding. For \( q = \xi \) a root of unity there are some technical issues, which we briefly explain.

It is frequently said that \( \mathcal{U}_q(\mathfrak{sl}_2) \) is a \textit{quasitriangular} Hopf algebra, but strictly speaking this is false. Instead this is true for a version \( \mathcal{U}_h(\mathfrak{sl}_2) \) defined over formal power series in \( \hbar \), where \( q = e^{\hbar} \). Saying that the Hopf algebra \( \mathcal{U}_h(\mathfrak{sl}_2) \) is quasitriangular means in particular that it has a \textit{universal} \( \hbar \)-matrix

\[
R = q^{H \otimes H/2} \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{\{n\}!}{\{n\}!} (E \otimes F)^n \in \mathcal{U}_h(\mathfrak{sl}_2) \otimes \mathcal{U}_h(\mathfrak{sl}_2)
\]

(25)

where \( \{n\} := q^n - q^{-n} \) is a quantum integer and \( \{n\}! := \{n\}\{n-1\} \cdots \{1\} \) is a quantum factorial. The key properties of \( R \) are that it intertwines the coproduct and opposite coproduct

\[
R \Delta = \Delta^{op} R
\]

and satisfies the \textit{Yang-Baxter relation}

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
\]

which is a version of the braid relation. (Here \( R_{12} = R \otimes 1 \) and so on.) Specifically, for any \( \mathcal{U}_h(\mathfrak{sl}_2) \)-module \( V \), write \( R \) for the action of \( R \) on \( V \otimes V \), let \( \tau(x \otimes y) = y \otimes x \). Then \( c = \tau R \) is a map \( V \otimes V \to V \otimes V \) of \( \mathcal{U}_h(\mathfrak{sl}_2) \)-modules satisfying the braid relation

\[
(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).
\]

The map \( c \) is the braiding used to define quantum link invariants.

Usually we work with \( \mathcal{U}_q(\mathfrak{sl}_2) \), not \( \mathcal{U}_h(\mathfrak{sl}_2) \), even though the element \( R \) involves power series in \( \hbar \). It turns out that for any finite-dimensional \( \mathcal{U}_q(\mathfrak{sl}_2) \)-module \( V \) the action of \( R \)

[McP22] C. McPhail-Snyder, "Holonomy invariants of links and nonabelian Reidemeister torsion". arXiv doi

\[\text{17 This is what Blanchet et al. [Bla+20] call a representation of a biquandle in a pivotal category.}\]
converges and can be written in terms of \( q \) only, which gives the \( R \)-matrices defining the colored Jones polynomials. The reason this works is that when \( q \) is not a root of unity the elements \( E \) and \( F \) act nilpotently on any finite-dimensional representation.

Even when \( q = \xi \) is a root of unity one can choose modules for which \( E \) and \( F \) act nilpotently. This leads to Kashaev’s invariant [MM01] and to the ADO invariants [ADO92, Bla+16]. However, these modules correspond to \( \text{SL}_2(\mathbb{C}) \)-structures with reducible or abelian image; to capture geometrically interesting \( \text{SL}_2(\mathbb{C}) \)-structures we need to allow \( E \) and \( F \) to act invertibly, that is to consider cyclic \( U_{\xi}(\mathfrak{sl}_2) \)-modules. Unfortunately the action of the \( R \)-matrix (25) on a tensor product of two cyclic modules diverges. We can work around this by instead considering its conjugation action.

**Proposition 5.3.** Consider the automorphism \( \mathcal{R} \) of \( \mathcal{U}_\xi(\mathfrak{sl}_2)^{\otimes 2} \) given by
\[
\mathcal{R}(x) := R_x R^{-1}.
\]
Write \( W = 1 - K^{-N} E^N \otimes F^N K^N \in \mathcal{U}_\xi(\mathfrak{sl}_2)^{\otimes 2} \). Then \( \mathcal{R} \) defines an algebra homomorphism
\[
\mathcal{R} : \mathcal{U}_\xi(\mathfrak{sl}_2)^{\otimes 2} \to \mathcal{U}_\xi(\mathfrak{sl}_2)^{\otimes 2}[W^{-1}]
\]
characterized uniquely by
\[
\begin{align*}
\mathcal{R}(1 \otimes K) &= (1 \otimes K)(1 - \xi^{-1} K^{-1} E \otimes FK) \\
\mathcal{R}(E \otimes 1) &= E \otimes K \\
\mathcal{R}(1 \otimes F) &= K^{-1} \otimes F
\end{align*}
\]
and
\[
\mathcal{R}(\Delta(u)) = \Delta^{op}(u), \quad u \in \mathcal{U}_\xi(\mathfrak{sl}_2).
\]

The requirement that \( W \) be invertible is related to the fact that the shape biquandle is only partially defined.

**Proof.** See [KR04]. \( \Box \)

For a tensor product \( V_1 \otimes V_2 \) of \( U_{\xi}(\mathfrak{sl}_2) \)-modules the \( R \)-matrix defining the braiding is no longer given by the action of \( \mathcal{R} \). Instead we say a linear map
\[
R : V_1 \otimes V_2 \to V_1' \otimes V_2
\]
is a **holonomy \( R \)-matrix** if it intertwines \( \mathcal{R} \) in the sense that
\[
R(x \cdot v) = \mathcal{R}(x) \cdot R(v) \text{ for every } v \in V_1 \otimes V_2, x \in \mathcal{U}_\xi(\mathfrak{sl}_2)^{\otimes 2}.
\]

For irreducible \( V_1 \) and \( V_2 \) this characterizes \( R \) up to an overall scalar, but it is not obvious how to compute the matrix elements of \( R \) or to choose the normalization. This problem was the original motivation to consider the Weyl presentation of \( U_{\xi}(\mathfrak{sl}_2) \) discussed in the next section; a partial solution is given in the author’s thesis [McP21, Chapter 3], and a full solution is forthcoming [MR22].

More relevant to the us is the fact that the module \( V_1' \otimes V_2' \) on the right-hand side of (26) will in general not be isomorphic to \( V_1 \otimes V_2 \). One way to see this is to consider their
At a To obtain the shape biquandle, we need to use a specific presentation of to better understand \( R \) is helps to use a particular presentation of \( \mathcal{U}_q(\mathfrak{sl}_2) \).

**Definition 5.4.** The extended Weyl algebra is the algebra \( \mathcal{W}_q \) generated over \( \mathbb{C}[q, q^{-1}] \) by a central invertible element \( z \) and invertible \( x, y \) subject to the relation

\[
xy = q^2yx.
\]

**Proposition 5.5.** The map \( \phi : \mathcal{W}_q \to \mathcal{U}_q(\mathfrak{sl}_2) \) given by

\[
K \mapsto x \quad E \mapsto qy(z - x) \quad F \mapsto y^{-1}(1 - z^{-1}x^{-1})
\]

is an algebra homomorphism. It acts on the Casimir by

\[
\Omega \mapsto qz + (qz)^{-1}.
\]

At a \( 2N \)th root of unity \( q = \xi \) the center of \( \mathcal{W}_\xi \) is generated by \( x^N, y^N, \) and \( z \). The automorphism \( \phi \) takes the center of \( \mathcal{U}_\xi(\mathfrak{sl}_2) \) to the center of \( \mathcal{W}_\xi \). Explicitly,

\[
\phi(K^N) = x^N \quad \phi(E^N) = y^N(x^N - z^N) \quad \phi(F^N) = y^{-N}(1 - z^{-N}x^{-N}).
\]

**Remark 5.6.** This presentation was obtained from one given by Faddeev [Fad00] in terms of a quantum cluster algebra with generators \( w_1, w_2, w_3, w_4 \). These generators \( q^2 \)-commute according to a certain quiver [SS19, Figure 4] associated to the triangulation of a punctured disc given in Figure 17a. It is known [Fad00; SS19] that this presentation explains the factorization of the \( R \)-matrix of \( \mathcal{U}_q(\mathfrak{sl}_2) \) into four terms; a version of this works at a root of unity [McP21, Chapter 3], and we should think of this factorization as being associated to the four-term decomposition of the octahedron at a crossing of a link diagram. Kashaev’s construction [Kas95] of link invariants from quantum dilogarithms is closely related.
The map $\phi$ sends central characters of $\mathcal{W}_\xi$ to central characters of $\mathcal{U}_c(\mathfrak{sl}_2)$ via $\chi \mapsto \chi \phi$. If $\chi$ is a central character with $\chi(x^N) = a$, $\chi(y^N) = b$, and $\chi(z^N) = m$, the map $\phi$ identifies $\chi$ with
\[
\begin{pmatrix}
a & 0 \\
(a - 1/m)b & 1
\end{pmatrix}, \begin{pmatrix}1 & (a - m)b \end{pmatrix}
\]
which is exactly the holonomy of the shape $(a, b, m)$ associated to $\chi$. We can similarly pull back the map $R$ along $\phi$ to give an automorphism $R^w$ of $W_{q^{\otimes 2}}$ characterized by
\[
R^w(x_1) = x_1 g, \\
R^w(x_2) = g^{-1} x_2, \\
R^w(y_1^{-1}) = y_2^{-1} + (y_1^{-1} - y_2^{-1}) x_2^{-1}, \\
R^w(y_2) = \frac{z_1}{z_2} y_1 + (y_2 - y_1^{-1}) x_1, \\
R^w(z_1) = z_1 \\
R^w(z_2) = z_2
\]
where
\[
g = 1 - x_1^{-1} y_1 (z_1 - x_1) y_2^{-1} (x_2 - z_2^{-1}).
\]
We can use $R^w$ to define a braiding on the central characters of $\mathcal{W}_\xi$, which will turn out to be exactly the braiding of Definition 2.4.

**Lemma 5.7.** The action of $R^w$ on the center of $W_{q^{\otimes 2}}$ is given by
\[
R^w(x_1^N) = x_1^N G, \\
R^w(x_2^N) = x_2^N G^{-1}, \\
R^w(y_1^{-N}) = y_2^{-N} + \left( y_1^{-N} - \frac{y_2^{-N}}{z_2} \right) x_2^{-N}, \\
R^w(y_2^N) = \frac{z_1}{z_2} y_1^N + \left( y_2^N - \frac{y_1^N}{z_2} \right) x_1^N,
\]
where
\[
G = 1 + x_1^{-N} y_1^N (x_1^N - z_1^N) (x_2^N - z_2^{-N})
\]
with inverse action
\[
(R^w)^{-1}(x_1^N) = x_1^N \bar{G}^{-1}, \\
(R^w)^{-1}(x_2^N) = x_2^N \bar{G}, \\
(R^w)^{-1}(y_1^{-N}) = \frac{z_1}{z_2} y_2^{-N} + (y_1^{-N} - z_1^N y_2^{-N}) x_2^N, \\
(R^w)^{-1}(y_2^N) = y_1^N + (y_2^N - z_1^N y_1^N) x_1^{-N},
\]
where
\[
\bar{G} = 1 + x_2^{-N} y_1^N (x_1^N - z_1^N) (x_2^N - z_2^{-N}).
\]
Theorem 5.8. Consider the map $B^W$ defined on pairs of central characters of $W_\xi$ by
\[ B^W(\chi_1, \chi_2) = (\chi_2', \chi_1'), \text{ where } (\chi_1' \otimes \chi_2') = (\chi_1 \otimes \chi_2)R^{-1}. \]
Identifying central characters of $W_\xi$ with shapes via $\chi \mapsto (\chi(x^N), \chi(y^N), \chi(z^N))$, the map $B^W$ is exactly the map $B$ defined by (1–3).

Proof. We have
\[
\begin{align*}
\frac{b_{1'}}{b_2m_2} &= \chi_1(y^{-N}) \\
&= (\chi_1 \otimes \chi_2) \left( \frac{z_1^N}{z_2^N} y_2^{-N} + (y_1^{-N} - z_1^N y_2^{-N}) x_2^N \right) \nonumber \\
&= \frac{m_1}{b_2m_2} + \left( \frac{1}{b_2} - \frac{m_1}{b_2} \right) a_2
\end{align*}
\]
which after some algebraic manipulation gives the expression for $b_{1'}$ in (2). The other variables follow similarly. \(\square\)

Remark 5.9. While it is possible to derive (4–6) by inverting the map $B$, it is much easier to compute them by repeating the above proof with $R$.

6. Gluing equations

The conditions (1–3) on the shapes at each crossing seem rather complicated. In practice, we can usually make some simplifications. For example, we have already seen that we only need to make one choice of $m$ per link component. It is possible to go further and either eliminate the $a$-variables or the $b$-variables, as long as we avoid certain geometrically degenerate solutions. To make this precise, we introduce some terminology.

Definition 6.1. Let $L$ be a link in $S^3$ with $c$ components. The representation variety of $L$ is the set $R_L$ of representations $\rho : \pi_1(S^3 \setminus L) \to \text{SL}_2(\mathbb{C})$ of the link complement into $\text{SL}_2(\mathbb{C})$.

Now suppose $D$ is a diagram of $L$ with $s$ segments. We associate variables
\[
a_1, \ldots, a_s, b_1, \ldots, b_s \in \mathbb{C} \setminus \{0\}
\]
to the segments of $D$ and variables
\[
m_1, \ldots, m_c \in \mathbb{C} \setminus \{0\}
\]
to the components. Writing $C(i)$ for the component of segment $i$, we assign each segment the shape $\chi_i = (a_i, b_i, m_{C(i)})$. The shape variety of $D$ is the set $\mathcal{S}_L \subset (\mathbb{C} \setminus \{0\})^{2s+c}$ of shapings satisfying the relations of Definition 2.6.

In this language Theorem 2.17 says that there is an inclusion $\mathcal{S}_D \hookrightarrow R_L$. The inclusion is not surjective, but it is effectively surjective: we usually only care about representations up to the conjugation action of $\text{SL}_2(\mathbb{C})$, and Theorem 2.18 says that the $\text{SL}_2(\mathbb{C})$-orbit of every $\rho \in R_L$ intersects the image of $\mathcal{S}_D$.

In this section we will define two smaller sets $\mathcal{A}_D$ and $\mathcal{B}_D$ and show that they map injectively into $\mathcal{S}_D$. $\mathcal{B}_D$ comes from eliminating the $a$-variables in terms of the $b$-variables, but it only detects non-pinched solutions in the sense of Definition 2.8. To check membership in $\mathcal{B}_D$ we only need to check one equation in the $b_i$ and $m_i$ for each segment of $D$. In parallel, $\mathcal{A}_D$ comes from eliminating the $b$-variables in terms of the $a$-variables, and it only detects
non-degenerate solutions. The equations defining \( \mathfrak{A}_D \) are instead associated to the regions of the diagram \( D \): for each region we check that a certain product involving \( a \)-variables and \( m \)-variables is 1. Because every pinched solution is degenerate, but not vice-versa, we have injective maps

\[
\mathcal{B}_D \hookrightarrow \mathfrak{A}_D \hookrightarrow \mathfrak{S}_D \hookrightarrow \mathcal{R}_L
\]

for any diagram \( D \) of \( L \). There are examples of geometrically interesting points of \( \mathfrak{S}_D \) that do not lie in the image of \( \mathcal{B}_D \), but Theorem 3.10 says that every point of \( \mathfrak{S}_D \) with nontrivial (not \( \pm 1 \)) holonomy lies in the image of \( \mathfrak{A}_D \).

### 6.1. The b-shape variety \( \mathcal{B}_D \) and the segment equations

Let \( D \) be a diagram of \( L \) with \( c \) components and \( s \) segments. Associate variables \( b_1, \ldots, b_s \) to the segments and \( m_1, \ldots, m_c \) to the components of \( D \); as before this gives a tuple \( (b_i, m_{C(i)}) \) for each segment \( i \) of \( D \). Now consider a crossing of \( D \). If the crossing is not pinched (which can be checked in terms of the \( b_i \) and \( m_i \) alone), Lemma 2.10 determines the \( a \)-variables for each segment at the crossing.

**Definition 6.2.** Because a segment is adjacent to two crossings, this procedure assigns two different \( a \)-variables to each segment. The **segment equation** of a segment says that the two \( a \)-variables agree. The **b-shape variety** \( \mathcal{B}_D \) of the diagram \( D \) is the set of all \( b_1, \ldots, b_s, m_1, \ldots, m_c \) satisfying the segment equations.

**Theorem 6.3.** There is an inclusion \( \mathcal{B}_D \hookrightarrow \mathfrak{S}_D \) whose image consists of all non-pinched shapings of \( D \).

**Proof.** The segment equations are written in terms of two of the three variables of the shapes \( \chi_i = (a_i, b_i, m_i) \). When the segment equations are satisfied each segment is also assigned a well-defined \( a \)-variable, hence a shape. These assignments determine a shaping because the equations (7) imply the braiding relations \( B(\chi_1, \chi_2) = (\chi_2', \chi_1') \) at every positive crossing, and similarly for negative crossings.

We can only compute non-pinched shapings in this manner because at a pinched crossing the expressions appearing in (7) and (8) are indeterminate. Conversely, any non-pinched shaping determines a solution of the \( b \)-gluing equations by forgetting the \( a \)-variables.

**Example 6.4.** Consider the diagram \( D \) of the figure-eight knot given in Figure 1. For simplicity, we restrict to the boundary-parabolic case where the meridian eigenvalue \( m \) is 1. We assign \( b \)-variables \( b_1, \ldots, b_8 \) to the segments of \( D \). The equation for segment 1 is

\[
\frac{b_4 b_1 - b_5}{b_5 b_1 - b_4} = \frac{b_5 - b_1}{b_6 - b_1}
\]

Following [KKY18, Example 4.6] we see that there is a 3-parameter family of solutions given in terms of \( p, q, r \) by

\[
(b_1, \ldots, b_8) = \left( pr, pq(1 + q\Lambda), -\frac{pr\Lambda(1 + q\Lambda)}{1 - p}, \frac{pqr}{1 - p}, -qr, r - qr, -\frac{pr(1 - q)\Lambda^2}{1 + p\Lambda}, \frac{pr}{1 + p\Lambda} \right)
\]

where \( \Lambda \) satisfies \( \Lambda^2 + \Lambda + 1 \). The solution space is parametrized by one discrete parameter \( \Lambda \) and three continuous parameters \( p, q, r \), which can be freely chosen as long as we avoid pinched crossings and all the \( b_i \) are nonzero.
Solving for all of $\mathcal{B}_D$ and not just the part with $m = 1$ is significantly more difficult. We discuss this further and compute more examples in Section 6.3.

Remark 6.5. In general our solutions have three extra parameters. Kim, Kim, and Yoon [KKY18, Example 4.6] explain this as follows: one degree of freedom comes from the homogeneity of the segment equations in the $b_i$, while the other two come from the arbitrary locations of the extra ideal points $P_z$ of the octahedral decomposition (Section 3).

6.2. The $a$-shape variety $\mathcal{A}_D$ and the region equations

Instead of using Lemma 2.10 to eliminate the $a$-variables, we can use Lemma 2.11 to eliminate the $b$-variables. However, it is most convenient to do this using a slightly different set of variables.

Definition 6.6. Let $D$ be a diagram of a link $L$ with $s$ segments; because $D$ is a 4-valent planar graph there are $s - 1$ regions, one of which is unbounded. To any shaping of $D$ we associate $s - 1$ region variables $r_0, \ldots, r_{s-2}$ by the rule given in Figure 30: when passing from $r_i$ to $r_i'$ across a strand $\chi = (a, b, m)$ we should have $r_i' = ar_i$.

Lemma 6.7. Any shaping of $D$ gives well-defined region variables. \[\diamond\]

Proof. The rules of Definition 6.6 determine the $r_i$ up to an overall constant from any shaping: we can walk from the unbounded region of $D$ to any other region, picking up factors of $a_i$ in the process. We need to make sure this assignment is well-defined.

It’s enough to check it’s well-defined near each crossing. Given a choice of $r_N$ at a crossing (labeled as in Figure 3), we have both

$$r_S = a_1a_2r_N \quad \text{and} \quad r_S = a_1'a_2'r_N.$$ 

However, these give the same value for $r_S$ because by (1) and (4) we have $a_1a_2 = a_1'a_2'$ at any crossing. \[\square\]

We want to work the other way and use the region variables (and meridian eigenvalues) to determine a shaping. Suppose $D$ has $c$ components and we have chosen meridian eigenvalues $m_1, \ldots, m_c$ and region variables $r_0, \ldots, r_{s-2}$. Lemma 2.11 gives the ratios of the $b$-variables near any crossing; given an arbitrary choice of one $b_i$, this determines the rest of them, and should give a shaping. However, there is a consistency condition: for these ratios to come from an assignment of $b$-variables to each segment bounding a region, the product of the ratios (one from each corner) must be 1.

Definition 6.8. Let $D$ be a diagram with region variables $\{r_i\}$ and meridian eigenvalues $\{m_i\}$. At non-degenerate positive crossings, the corner terms are

$$f_N = \left(\frac{r_Wr_E - r_Nr_S}{r_W - m_1r_N}(r_E - r_N/m_2)\right),$$

$$f_W = \left(\frac{r_Nr_S - r_Wr_E}{r_N - r_W/m_1}(r_S - r_W/m_2)\right),$$

$$f_S = \left(\frac{r_Wr_E - r_NR_S}{r_N - m_1r_N}(r_W - m_2r_S)\right),$$

$$f_E = \left(\frac{r_Nr_S - r_Wr_E}{r_S - m_1r_E}(r_N - m_2r_E)\right).$$

Figure 30: The correspondence between region variables and $a$-variables.
and at non-degenerate negative crossings they are

\[
\begin{align*}
    f_N &= \frac{(r_W - r_N/m_1)(r_E - m_2r_N)}{r_Wr_E - r_Nr_S} \\
    f_W &= \frac{r_Nr_S - r_Wr_E}{(r_N - m_1r_W)(r_S - m_2r_W)} \\
    f_S &= \frac{r_Wr_E - r_Nr_S}{(r_E - m_1r_S)(r_W - r_S/m_2)} \\
    f_E &= \frac{r_Nr_S - r_Wr_E}{(r_S - m_1r_E/m_1)(r_N - r_E/m_2)}
\end{align*}
\]

(28)

Here by \( r_N \) we mean the region variable north of the crossing (viewed left-to-right) as in Figure 3 and similarly for \( r_E, r_S, r_W \). The region equation associated to any region of \( D \) says that the product of all the corner terms near a region is 1. We call the set \( \mathcal{R}_D \) consisting of solutions \( (r_0, \ldots, r_{s-2}, m_1, \ldots, m_2) \) to the region equations the shape variety of \( D \). We require that solutions in \( D \) satisfy the non-degeneracy conditions, which in terms of the region variables are

\[ r_W \neq m_1r_N \]

(29)

Example 6.9. In Figure 31 the central region labeled 7 has three segments and three corners. The corner terms are

\[
\begin{align*}
    f_{13} &= \frac{r_1r_3 - r_2r_7}{(r_3 - r_7/m_2)(r_1 - m_1r_7)} \\
    f_{35} &= \frac{r_3r_5 - r_4r_7}{(r_5 - r_7/m_3)(r_3 - m_2r_7)} \\
    f_{51} &= \frac{r_1r_5 - r_6r_7}{(r_1 - r_7/m_1)(r_5 - m_3r_7)}
\end{align*}
\]

and the region equation is

\[ f_{13}f_{35}f_{51} = 1. \]

The non-degeneracy relations at each crossing are

\[ r_1r_3 \neq r_2r_5, \quad r_3r_5 \neq r_4r_7, \quad \text{and} \quad r_1r_5 \neq r_6r_7. \]

Remark 6.10. Because the region equations are homogeneous the region equations have an extra degree of freedom. We can remove this by fixing the variable for some region; an obvious choice is to fix the value for the unbounded region as \( r_0 = 1 \).

Theorem 6.11. There is an inclusion \( \mathcal{R}_D \hookrightarrow \mathcal{S}_D \) whose image consists of all non-degenerate shapings of \( D \). \( \Box \)

Proof. Lemma 6.7 says that any non-degenerate shaping determines a solution of the region equations. Conversely, a point of \( \mathcal{R}_D \) is a choice of meridian eigenvalue \( m_i \) for each component and a choice of region variables, which uniquely determines the \( a \)-variables. The relations (9) and (10) determine the ratios between the \( b \)-variables of every segment, so if we pick the \( b \)-variable \( b_1 \) of one segment arbitrarily we determine all of them. \( \Box \)

Example 6.12. Consider the diagram of the trefoil knot in Figure 32 with labeled regions. For simplicity, assume that the meridian eigenvalue \( m \) is 1. Then the region equation for region 2 is

\[
\frac{(r_4 - r_2)(r_1 - r_2)(r_1 - r_3)(r_4 - r_2)}{r_1r_4 - r_0r_2} = 1
\]

Figure 31: A diagram region with three edges.
6. The segment equations of a twist region

A twist region of a knot diagram with \( N \) positive twists is shown in Figure 33. We call it a parallel twist region because both strands are oriented in the same direction. Kim, Kim, and Yoon [KKY18, Section 6] showed how to solve the segment equations in a twist region in when the holonomy representation is boundary-parabolic (that is, when \( m = \pm 1 \)). By doing so, they can compute boundary-parabolic \( \text{SL}_2(\mathbb{C}) \)-structures on some infinite families of knots like \((2, N)\) torus knots and twist knots. In this section we translate their computation to our conventions and explain how to find all the \( \text{SL}_2(\mathbb{C}) \)-structures on the \((2, N)\) torus knots, not just the boundary parabolic ones.

**Definition 6.13.** A sequence \( \{F_i\}_{i \in \mathbb{Z}} \) is \( W \)-Fibonacci for \( W \in \mathbb{C} \) if it satisfies

\[
F_{i+1} = W \cdot F_i + F_{i-1}.
\]

The sequence \( \{B_i\} \) with \( B_0 = 0 \) and \( B_1 = 1 \) is called the base \( W \)-Fibonacci sequence.

**Lemma 6.14.** Let \( \{F_i\} \) and \( \{G_i\} \) be \( W \)-Fibonacci sequences and \( \{B_i\} \) the base \( W \)-Fibonacci sequence. Then for all \( i \),

(a) \( F_i = F_0 B_{i-1} + F_1 B_i \),

(b) \( B_i^2 - B_{i-1} B_{i+1} = (-1)^{i+1} \),

(c) and if \( i \geq 0 \)

\[
B_{i+1} = \sum_{0 \leq j \leq i/2} \binom{i-j}{j} W^{i-2j}.
\]

\( \Box \)
6. Gluing equations

Proof. [KKY18, Lemma 6.2].

Lemma 6.15. Consider a parallel twist region with \( N \) positive twists, labeled as in Figure 33, in which both strands have meridian eigenvalue \( m \). If the equations

\[
x_i = \frac{F_i}{G_i} \quad y_i = \frac{F_{i-1}}{mG_{i+1}}
\]

hold for \( i = 1, 2 \) (i.e. for segments 1, 2, 3, 4) then they hold for all \( 1 \leq i \leq n + 1 \). In a region with \( n \) negative twists, the same holds with (32) replaced with

\[
x_i = \frac{F_{i-1}}{G_{i+1}} \quad y_i = \frac{F_i}{mG_i}
\]

Proof. Consider the gluing equation for the segment labeled \( x_i \). Because we are using \( b \)-variables, we think of the four \( b \)-variables associated to the segments at each crossing as determining the \( a \)-variables via (7). The gluing equation of a segment is then checking that the \( a \)-variables for each side agree. In this case, it is

\[
x_{i-1} \quad mx_i - y_i = \frac{y_i - mx_i}{my_i} \quad x_i - x_{i-1} = \frac{x_{i+1} - x_i}{x_i y_i + 1}
\]

Solving for \( x_{i+1} \) and repeating this argument for the segment \( y_i \) gives the recurrence relations

\[
x_{i+1} = my_i - \frac{y_i - mx_i}{x_{i-1}} + x_i
\]

\[
\frac{1}{y_{i+1}} = \frac{m}{x_i} \quad \frac{my_{i-1}}{x_i y_i + 1} + \frac{1}{y_i}
\]

(These are the relations of [KKY18, Lemma 6.3] up to some factors of \( m \).) The result follows by induction, and the negative case works similarly.

To give some details, substitute (32) into the recurrence for \( x_{i+1} \) to obtain

\[
\frac{F_{i+1}}{G_{i+1}} = \frac{F_{i-1}}{G_{i+1}} - \frac{F_i}{G_{i+1}} + \frac{F_i}{G_i}
\]

equivalently

\[
F_{i+1} = G_{i+1}^{-1} [G_i F_{i-1} + F_i (G_{i+1} - G_{i-1})]
\]

After applying the recurrence for \( G_{i+1} \) the right-hand side becomes

\[
G_{i+1}^{-1} [G_i F_{i-1} + F_i (WG_i + G_{i-1} - G_{i-1})] = F_{i-1} + WF_i
\]

\[
= F_{i+1}
\]

as required. Something similar works for the recurrence for \( y_i \).

Proposition 6.16. In a parallel twist region with positive twists in which both strands have meridian eigenvalue \( m \), the adjusted segment variables are given in terms of \( x_1, x_2, y_1, y_2 \) by

\[
x_i = \frac{mx_1 y_1 W \Fib_{i-1}(W) + x_1 (x_2 - my_1) \Fib_i(W)}{(x_2 - my_1) \Fib_{i-2}(W) + x_1 W \Fib_{i-1}(W)}
\]

\[
y_i = \frac{m^{-1} mx_2 y_1 W \Fib_{i-2}(W) + x_1 (x_2 - my_1) \Fib_{i-1}(W)}{(x_2 - my_1) \Fib_{i-1}(W) + x_1 W \Fib_i(W)}
\]
where
\[ W^2 = (m y_1 - x_2) \left( \frac{1}{x_1} - \frac{1}{m y_2} \right), \]

Fib_i is the polynomial Fib_i(W) = B_i, and B_i is the base W-Fibonacci sequence determined by
\[ B_{i+1} = W B_i + B_{i-1}, B_1 = 1, B_0 = 0. \]

\[ \text{Remark 6.17.} \text{ The Fib}_i \text{ are sometimes called the Fibonacci polynomials. When } i \text{ is odd Fib}_i(W) \text{ is a polynomial in } W^2, \text{ and when } i \text{ is even Fib}_i(W) \text{ is } W \text{ times a polynomial in } W^2. \text{ In particular, the solutions in (34–35) depend only on } W^2, \text{ not } W. \]

\[ \text{Proof.} \text{ We need to pick the right initial conditions for } F_i \text{ and } G_i. \text{ A convenient way to do this is to set} \]
\[ F_0 = m W x_1 y_1, \quad F_1 = x_1 (x_2 - m y_1) \]
\[ G_1 = x_2 - m y_1, \quad G_2 = W x_1. \]

Then we can check that
\[ x_i = \frac{F_i}{G_i}, \text{ and } y_i = \frac{F_{i-1}}{m G_{i+1}} \]
holds for \( i = 1, 2 \), so by Lemma 6.15 they hold for all \( i \). We can now apply (a) of Lemma 6.14

\[ \text{Remark 6.18.} \text{ To compute more examples, we would need to extend this computation to:} \]

1. parallel twist regions where the meridian eigenvalues \( m_1, m_2 \) on the two strands differ, and

2. to antiparallel twist regions in the boundary non-parabolic (\( m \neq \pm 1 \)) case.

In fact, the segment equations in these two cases are closely related, because there is a simple formula [McP21, Definition 4.2] for reversing the orientation of a strand. Following [HL18, Section 7], the right generalization is to consider sequences of the form
\[ A_{i+1} = W A_i + \frac{m_i}{m_{i+1}} A_{i-1} \]
where the index on \( m_i \) is understood mod 2.

We can at use Proposition 6.16 to compute an infinite family of boundary non-parabolic examples. A \((2, N)\)-torus knot is obtained by attaching segments \( x_1 \) and \( x_{N+1} \) and segments \( y_1 \) and \( y_{N+1} \) in Figure 33. We assume \( N = 2n + 1 \) is odd, so that we obtain a knot and not a link. We can now solve for its segment variables. We think of of \( x_1, x_2, \) and \( y_1 \) as parameters and use the ansatz
\[ x_i = \frac{F_i}{G_i} \text{ and } y_i = \frac{F_{i-1}}{m G_{i+1}} \]
of Proposition 6.16. We need to choose \( \Lambda \) (equivalently, choose \( y_2 \)) so that the gluing equations of the edges \( x_1 = x_{N+1} \) and \( y_1 = y_{N+1} \) are satisfied. The former is
\[ \frac{m x_1 - y_1}{m y_1} \frac{x_1 - x_N}{x_1 - x_N} = \frac{y_1 - m x_1}{x_2 - x_1} \]
or
\[ 1 = \frac{x_N}{m y_1} \frac{x_1 - x_2}{x_1 - x_N} \]

[HL18] J.-Y. Ham and J. Lee, On the volume and the Chern-Simons invariant for the hyperbolic alternating knot orbifolds. arXiv
which is equivalent to
\[
\frac{G_N}{F_N} = \frac{G_0}{F_0}. \tag{36}
\]
Now, the sequence \( H_i = F_0 G_i - F_i G_0 \) is a \( \sqrt{\Lambda} \)-Fibonacci sequence satisfying \( H_0 = 0 \) and \( H_1 \neq 0 \), and (36) holds if and only if \( H_N = 0 \). By part (a) of Lemma 6.14 we have \( H_i = H_i B_i \) for all \( i \), and we conclude that (36) holds if and only if \( B_N = 0 \). Using the fact that \( N = 2n + 1 \) is odd, the condition is
\[
B_{2n+1} = \sum_{0 \leq j \leq (2n+1)/2} \left( \begin{array}{c} 2n-j \\ j \end{array} \right) \Lambda^{2n-2j}
\]
\[
= \sum_{0 \leq j \leq n} \left( \begin{array}{c} 2n-j \\ j \end{array} \right) \Lambda^{n-j}
\]
It turns out that this also implies the gluing relation for \( y_1 = y_{N+1} \). We have shown:

**Theorem 6.19.** Taking the braid closure of Figure 33 for \( N = 2n + 1 \) gives a diagram \( D \) of a \((2, 2n + 1)\)-torus knot. For any meridian eigenvalue \( m \) the \( b \)-variables of a shaping of \( D \) are given by
\[
x_i = \frac{q \sqrt{\Lambda} \text{Fib}_{i-1}(\sqrt{\Lambda}) + pr \text{Fib}_i(\sqrt{\Lambda})}{r \text{Fib}_{i-2}(\sqrt{\Lambda}) + \sqrt{\Lambda} \text{Fib}_{i-1}(\sqrt{\Lambda})} \tag{37}
\]
\[
y_i = \frac{1}{m} \frac{q \sqrt{\Lambda} \text{Fib}_{i-2}(\sqrt{\Lambda}) + pr \text{Fib}_{i-1}(\sqrt{\Lambda})}{r \text{Fib}_{i-1}(\sqrt{\Lambda}) + \sqrt{\Lambda} \text{Fib}_i(\sqrt{\Lambda})} \tag{38}
\]
where \( \Lambda \) satisfies
\[
\sum_{0 \leq j \leq n} \left( \begin{array}{c} 2n-j \\ j \end{array} \right) \Lambda^{n-j} = 0 \tag{39}
\]
and
\[
p = x_1, \quad q = my_1, \quad r = (x_2 - my_1)/x_1
\]
are arbitrary nonzero parameters chosen so that the first crossing is not pinched.\(^{19}\) As discussed in Remark 6.17 the expressions for \( x_i \) and \( y_i \) depend only on \( \Lambda \), not \( \sqrt{\Lambda} \). \( \diamond \)

**Remark 6.20.** The polynomial in (39) is sometimes called the Riley polynomial of the knot, and this specific case is discussed in [Ril72, Section 5]. In particular, (39) is a monic polynomial with \( n \) distinct roots, which corresponds to the general fact [Muñ09, Theorem 3.1] that the character variety of a \((p, q)\) torus knot has \((p-1)(q-1)/2\) components coming from irreducible representations.\(^{20}\) In our family of examples \( p = 2 \) and \( q = 2n + 1 \).

The roots of (39) can be given explicitly [Ril72, Theorem 5]. In particular, when \( n = 1 \) the only root is \( \Lambda = -1 \). This gives us the solutions (40) for the trefoil knot.

**Remark 6.21.** The geometry of these solutions is not affected by the choice of \( p, q, r \), but is instead determined by \( m \) and the choice of root \( \Lambda \) of the Riley polynomial. In our conventions this is less obvious, but there is a different presentation [KKY18, Section 5] of the holonomy of a shaped diagram that is manifestly independent of the choice of \( p, q, r \).

We can still give an informal explanation of this independence. The components of the character variety of a torus knot corresponding to irreducible holonomy representations are all 1-dimensional [Muñ09, Theorem 3.1]. Picking the discrete parameter \( \Lambda \) determines the component, which is then parametrized by a rational function of \( m \). (The components are isomorphic to \( \mathbb{C} \) via the trace \( \mu + \mu^{-1} \) of a generator of the knot group, but this generator is not a meridian, so \( \mu \) is some rational function of \( m \).)

\(^{19}\) \( p, q, r \) are closely related to but not exactly the variables \( p, q, r \) in [KKY18, Theorem 6.5].

\(^{20}\) There is also a component describing the representations with abelian image. It is not detected by the \( b \)-gluing equations because they only detect non-pinched solutions, which always correspond to a nonabelian holonomy representation.
Example 6.22. Consider the diagram of the trefoil in Figure 4. For any meridian eigenvalue \( m \neq 0 \), we can freely choose the variables \( b_1, b_2, b_3 \), as long as
\[
\frac{b_2}{mb_1}, \frac{b_3}{b_1} \neq 1
\]
so that the solution is not pinched. The remaining segment variables are given by
\[
(b_4, b_5, b_6) = \left( \frac{b_1(mb_2 - b_3)}{m(b_1 + mb_2 - b_3)}, \frac{mb_2}{b_1 + mb_2 - b_3}, -\frac{b_1b_3}{m(mb_2 - b_3)} \right). \tag{40}
\]
This solution is the case \( n = 1 \) of Theorem 6.19 after the substitutions \( p = b_1, q = mb_2, r = (b_1 - mb_2)/b_1 \).

We can use Theorem 4.4 to compute the boundary eigenvalues of this representation. We have
\[
\delta^-(m) = m \quad \text{and} \quad \ell = \delta^-(l) = m^{-3}\frac{b_2b_4b_6}{b_1b_3b_5} = -1/m^6
\]
which matches the \( A \)-polynomial \( m^6\ell + 1 \) of the trefoil knot.

As before, the choice of \( b_1, b_2, b_3 \) does not affect the conjugacy class of the \( SL_2(\mathbb{C}) \)-structure, which is uniquely determined by \( m \). In this case, we can check this against the character variety of the trefoil knot. Non-pinched solutions correspond to \( SL_2(\mathbb{C}) \)-structures with nonabelian image, and the \( SL_2(\mathbb{C}) \)-character variety of the trefoil has a single nonabelian component [Muñoz, Theorem 3.1] cut out by the equation
\[
m^6\ell + 1 = 0.
\]
This shows explicitly that (40) yields at least one \( SL_2(\mathbb{C}) \)-structure in every nonabelian conjugacy class. We can do a similar computation for general \((2, 2n + 1)\)-torus knots:

Example 6.23. Consider the shaping of the \((2, N)\)-torus knot given in Theorem 6.19. The longitude is
\[
\delta(l) = m^N \prod_{k=1}^{N} \frac{x_i}{y_i} = \prod_{k=1}^{N} \frac{F_{k-1}}{mG_{k+1}} \frac{G_k}{F_k}
\]
where in the last step we used (36). Because \( B_N = 0 \), by Lemma 6.14 we have
\[
\frac{G_0G_1}{G_NG_{N+1}} = \frac{G_0G_1}{(G_0B_{N-1} + G_1B_N)(G_0B_N + G_1B_{N+1})} = \frac{1}{B_{N-1}B_N} = (-1)^N = -1
\]
because \( N = 2n + 1 \) is odd. We conclude that
\[
\delta(l) = -m^{-2N} \tag{41}
\]
which recovers the \( A \)-polynomial of the right-handed \((2, N)\)-torus knot.

The discrete parameter \( \Lambda \) does not appear in this computation; this corresponds to the fact [Muñoz, Theorem 3.1] that the \( n \) nonabelian components of the character variety are all isomorphic. Choosing \( \Lambda \) picks out a component which is then continuously parametrized by \( m + m^{-1} \).
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