Reduction of divisors and Kowalevski top

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Abstract

In the modern theory of the Kowalevski top there are two elliptic curves introduced by Kowalevski and by Reyman and Semenov-Tian-Shansky. The Kowalevski variables of separation and poles of the Baker-Akhiezer function define two classes of linearly equivalent divisors on these elliptic curves. According to the Riemann-Roch theorem each class has a unique reduced representative and we construct such reduced divisors for the Kowalevski top.

1 Introduction

In the realm of algebraic geometry usually associated with many Liouville integrable systems, the Hamiltonian evolution equations are written as a Lax equation

\begin{equation}
\frac{d}{dt}L(x) = [L(x), A(x)],
\end{equation}

for two \(N \times N\) matrix functions \(L(x)\) and \(A(x)\) on the phase space depending on the auxiliary spectral parameter \(x\). The time-independent spectral equation

\begin{equation}
L(x) \psi(x, y) = y \psi(x, y)
\end{equation}

allows us to represent the vector Baker-Akhiezer function \(\psi\) in terms of the Riemann theta function on a nonsingular compactification of the spectral curve defined by the equation

\begin{equation}
\Gamma : \quad f(x, y) = \det(L(x) - y) = 0.
\end{equation}

There is a large freedom of similarity transformations of the Lax matrix,

\(L(x) \rightarrow VL(x)V^{-1}\),

which do not change the spectrum of \(L(x)\), but change poles of \(\psi\) which form a \(D\) on \(\Gamma\). This freedom can be characterized, and therefore fixed, by introducing a normalization of the Baker-Akhiezer function

\(\vec{\alpha} \cdot \psi = \sum_{i=1}^{N} \alpha_i \psi_i = 1\),

which is given by a normalization (row-) vector \(\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)\) [32].

Any divisor \(D\) determines a class of linear equivalent divisors

\(D \rightarrow \{D, D_1, D_2, \ldots, D_k, \ldots\}\), \quad \(D - D_k = \text{div}(f_k)\),

where \(\text{div}(f_k)\) is divisor of the rational function \(f_k\) on \(\Gamma\) [24, 25]. It follows from the Riemann-Roch theorem that each class has a unique reduced representative \(\rho(D)\). Similarly, any normalization \(\vec{\alpha}\) determines a family of equivalent normalizations

\(\vec{\alpha} \rightarrow \{\vec{\alpha}, \vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_k, \ldots\}\)

associated and unique normalization \(\rho(\vec{\alpha})\) associated with \(\rho(D)\), see examples in [37].

In this note, we construct two reduced divisors using Kowalevski’s variables of separation and poles of the Baker-Akhiezer function which is an eigenfunction of the \(4 \times 4\) Lax matrix proposed by Reyman and Semenov-Tian-Shansky [30]. According to Kuznetsov [19] these poles are also variables of separation in a particular case of the Kowalevski top which is contrary to the Dubrovin-Skrypnyk theory [7] that does not take into account reducibility of abelian varieties.
1.1 Description of the model

Let two vectors \( \ell \) and \( g \) are coordinates on the phase space \( M \). As a Poisson manifold \( M \) is identified with Euclidean algebra \( e(3)^* \) with the Lie-Poisson brackets

\[
\{ \ell_i, \ell_j \} = \varepsilon_{ijk} \ell_k, \quad \{ \ell_i, g_j \} = \varepsilon_{ijk} g_k, \quad \{ g_i, g_j \} = 0, \quad (1.3)
\]

having two Casimir functions

\[
c_1 = g_1^2 + g_2^2 + g_3^2, \quad c_2 = g_1\ell_1 + g_2\ell_2 + g_3\ell_3. \quad (1.4)
\]

Here \( \varepsilon_{ijk} \) is the skew-symmetric tensor.

The Euler-Poisson equations (1.5) were integrated by Kowalevski by using change of variables which

\[
\text{To remove cross-terms } \dot{\ell}_i = \ell \times \frac{\partial H}{\partial \ell} + g \times \frac{\partial H}{\partial g}, \quad \dot{g} = g \times \frac{\partial H}{\partial \ell}, \quad (1.5)
\]

where \( x \times y \) means a cross product of two vectors.

The Kowalevski top is defined by the Hamiltonian \( H \),

\[
H = \ell_1^2 + \ell_2^2 + 2\ell_3^2 - 2b\ell_1, \quad b \in \mathbb{R} \quad (1.6)
\]

and the second integral \( K \),

\[
K = (\ell_1^2 + \ell_2^2)^2 + 4b(g_1\ell_1 - \ell_2) + 4b^2(g_1^2 + g_2^2), \quad (1.7)
\]

which are in involution \( \{ H, K \} = 0 \) with respect to the Poisson brackets (1.3).

2 Reduced divisor on the Kowalevski elliptic curve

The Euler-Poisson equations (1.5) were integrated by Kowalevski by using change of variables which reduced the problem to hyperelliptic quadratures [16]. Let us briefly discuss her calculations.

At the first step, Kowalevski introduced two pairs of Lagrangian variables \( z_{1,2} \) and \( \dot{z}_{1,2} \) such that

\[
H = -\frac{\dot{z}_1 \dot{z}_2 + R(z_1, z_2)}{(z_1 - z_2)^2}, \quad K = \frac{\left(\dot{z}_1^2 - R(z_1, z_1)\right)\left(\dot{z}_2^2 - R(z_2, z_2)\right)}{(z_1 - z_2)^4}. \quad (2.1)
\]

Here

\[
z_1 = \ell_1 + i\ell_2, \quad z_2 = \ell_1 - i\ell_2 \quad (2.2)
\]

and

\[
R(z_1, z_2) = z_1^2z_2 - H(z_1^2 + z_2^2) - 4b c_2(z_1 + z_2) - 4b^2 c_1 + K.
\]

To remove cross-terms \( \dot{z}_1 \dot{z}_2 \) in Hamiltonian (2.1), Kowalevski used arithmetic of divisors

\[
P_1' = P_1 + P_2, \quad P_2' = P_1 - P_2, \quad (2.3)
\]

on the elliptic curve \( E \) defined by equation

\[
E : \quad Z^2 = R(z, z), \quad R(z, z) = z^4 - 2Hz^2 - 8b c_2z - 4b^2 c_1 + K \equiv \sum_{k=0}^{4} a_k z^k. \quad (2.4)
\]

Here \( P_k = (z_k, Z_k) \) and \( P'_k = (z'_k, Z'_k) \) are two pairs of points defining two semi-reduced divisors \( D = P_1 + P_2 \) and \( D' = P'_1 + P'_2 \) on the abelian variety \( E \). Transformation (2.3) can be rewritten in the matrix form

\[
\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \to \begin{pmatrix} P'_1 \\ P'_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}
\]

similar to the standard rotation which reduces quadratic form to a diagonal form.
According to Abel [1], affine coordinates of points \( P_1' = (z_1', Z_1') \) and \( P_2' = (z_2', Z_2') \) on \( E \) are equal to

\[
\begin{align*}
z_{1,2}' &= -z_1 - z_2 - \frac{2b_0b_2 + b_1^2 - a_2}{2b_1b_2 - a_3} \quad \text{and} \quad Z_{1,2}' = -\mathcal{V}_{1,2}(z_{1,2}') .
\end{align*}
\]  

(2.5)

Here \( a_j \) are given by (2.4) and \( b_j \) are coefficients of the interpolation polynomials

\[
\mathcal{V}_{1,2}(z) = b_2 z^2 + b_1 z + b_0 = \sqrt{a_4}(z - z_1)(z - z_2) + \frac{(z - z_2)Z_1}{z_1 - z_2} + \frac{(z - z_1)Z_2}{z_2 - z_1} ,
\]

which now are called second Mumford’s coordinates of the semi-reduced divisors \( E_1 \) on hyperelliptic curve by a birational transformation. For these reasons Kowalevski reduced the imaginary hyperelliptic curve by a birational transformation. From this birational transformation maps \( E \) to variables \( w_{1,2} \) which satisfy Abel’s differential equations, see (2.14) below.

A formal description of the group structure and arithmetic (2.5) in a real elliptic curve

\[
y^2 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 , \quad a_4 \neq 0
\]

is a little more difficult than arithmetic in an imaginary elliptic curve

\[
y^2 = a_3x^3 + a_2x^2 + a_1x + a_0 , \quad a_3 \neq 0 ,
\]

because these curves differ by the number of points at infinity, see [26] and references within.

According to Weierstrass, any real hyperelliptic curve with a ramified prime divisor can be reduced to the imaginary hyperelliptic curve by a birational transformation. For these reasons Kowalevski made an additional transformation from the unpublished Weierstrass lectures [16].

This birational transformation maps \( z_{1,2}' \) to variables \( w_{1,2} \) which satisfy Abel’s differential equations

\[
\begin{align*}
\frac{\dot{w}_1}{\sqrt{\mathcal{R}_5(w_1)}} + \frac{\dot{w}_2}{\sqrt{\mathcal{R}_5(w_2)}} &= 0, \\
\frac{w_1\dot{w}_1}{\sqrt{\mathcal{R}_5(w_1)}} + \frac{w_2\dot{w}_2}{\sqrt{\mathcal{R}_5(w_2)}} &= 1 
\end{align*}
\]

(2.7)

on hyperelliptic curve

\[
C : \quad y^2 = \mathcal{R}_5(w) ,
\]

where \( \mathcal{R}_5(w) \) is the reducible polynomial

\[
\mathcal{R}_5(w) = \left( \frac{(6w + H)^2}{9} - K \right) W^2 = \left( \frac{(6w + H)^2}{9} - K \right) (4w^3 + aw + b) .
\]

Birational transformation destroys commutativity of \( z_{1,2}' \) and we have

\[
\{w_1, w_2\} \neq 0 .
\]

Because Abel’s equations (2.7) involve the reducible polynomial \( \mathcal{R}_5(w) \), Kowalevski made an additional transformation

\[
w \to s + \frac{H}{3}
\]

(2.8)

which reduce the standard Weierstrass equation (2.6) for an elliptic curve to the following equation

\[
E : \quad S^2 = 4s^3 + 4Hs^2 + (4b^2c_1 + H^2 - K)s + 4b^2c_2^2 ,
\]

(2.9)

which appears in the theory of degenerate Abel’s integrals at \( c_2 \neq 0 \).
The Kowalevski variables of separation

\[ s_{1,2} = w_{1,2} - \frac{H}{3} = \frac{R(z_1, z_2) \pm \sqrt{R(z_1, z_1) R(z_2, z_2)}}{2(z_1 - z_2)^2}, \tag{2.10} \]

satisfy Abel’s equations

\[ \frac{\dot{s}_1}{\sqrt{\mathcal{P}_5(s_1)}} + \frac{\dot{s}_2}{\sqrt{\mathcal{P}_5(s_2)}} = 0, \quad \frac{s_1 \dot{s}_1}{\sqrt{\mathcal{P}_5(s_1)}} + \frac{s_2 \dot{s}_2}{\sqrt{\mathcal{P}_5(s_2)}} = 1 \tag{2.11} \]

on the hyperelliptic curve defining by the reducible polynomial \( \mathcal{P}_5(s) \):

\[ C : \quad S^2 = \mathcal{P}_5(s), \quad \mathcal{P}_5(s) = \left( (H + 2s)^2 - K \right) S^2. \tag{2.12} \]

Variables \( s_{1,2} \) are in the involution

\[ \{ s_1, s_2 \} = 0 \]

with respect to the Lie-Poisson brackets on the Euclidean algebra \( e^\ast(3) \). The corresponding canonically conjugated momenta, separation relations and \( 2 \times 2 \) are discussed in \([18]\).

Of course, Kowalevski never computed the Poisson brackets between variables \( s_{1,2} \) and her reason for the transformation of variables \((2.5)\) was probably related to the investigations of degenerate Abel’s integrals in \([19]\), where she formulated so-called Weierstrass theorem on the periods of reducible integrals from her unpublished lectures \([6]\).

The meaning of the Kowalevski calculations has been discussed by many authors, see a shortlist \([2,10,12,13,14,18,20,22,40]\) and references within. Now any computer algebra system performs all these calculations for a few seconds.

### 2.1 Abel’s equations in \( z’ \)-variables

Birational transformation between coordinates of the point \((z’, Z’)(2.5)\) on \( E (2.4) \) and coordinates \((s, S)\) of the same point on \( E (2.9) \) reads as

\[
s = \frac{4b^2 c_1 + 4bc_2 z’ + Hz’^2 - \sqrt{K - 4b^2 c_1} Z’ - K}{2z’^2},\]

\[
S = \frac{(2bc_2 z’ + 4b^2 c_1 - K)Z’ + \sqrt{K - 4b^2 c_1} (Hz’^2 + 6bc_2 z’ + 4b^2 c_1 - K)}{z’^3}. \tag{2.13} \]

Because

\[ \{ s_1, s_2 \} = \{ z_1’, z_2’ \} = 0 \]

we have a canonical transformation of variables so that \( ds/S = dz’/Z’ \) or

\[
\frac{ds}{\sqrt{4s^4 + 4Hz^2 + (4b^2 c_1 + H^2 - K)s + 4b^2 c_1^2}} = \frac{dz’}{\sqrt{z’^4 - 2Hz’^2 - 8bc_2 z’ + K - 4b^2 c_1}}.
\]

Thus, Abel’s equations in \( z’ \)-variables

\[ \frac{\dot{z}_1’}{\phi(z_1’, Z_1’)} + \frac{\dot{z}_2’}{\phi(z_2’, Z_2’)} = 0, \quad \frac{\dot{z}_1’}{\phi(z_1’, Z_1’)} + \frac{\dot{z}_2’}{\phi(z_2’, Z_2’)} = 1 \tag{2.14} \]

also have two factors in the denominators:

\[ \phi(z’, Z’) = Z’ \sqrt{4s^2 + 4Hz^2 + H^2 - K}, \quad \phi(z’, Z’) = Z’ s^{-1} \phi(z’, Z’), \]

where \( s \) is given by \((2.13)\).

We present here these simple calculations because variables of separation \( z_1’ \) and \( z_2’ \) do not satisfy the so-called Kowalevski conditions \([21]\). It allows us to say that these Kowalevski conditions are noninvariant conditions with respect to canonical transformations of variables preserving the additive separation of variables in the Hamilton-Jacobi equation.
2.2 Degenerate Abel’s integrals

As early as 1832 Legendre had shown that two hyperelliptic integrals
\[ \int \frac{dx}{\sqrt{x(1-x^2)(1-\kappa^2x^2)}} \quad \text{and} \quad \int \frac{x \, dx}{\sqrt{x(1-x^2)(1-\kappa^2x^2)}} \]
are each expressible in terms of two elliptic integrals of the first kind through a quadratic transformation. Immediately after, Jacobi pointed out that this property holds for integrals on the hyperelliptic curve
\[ \int \frac{dx}{\sqrt{R(x)}} \quad \text{and} \quad \int \frac{x \, dx}{\sqrt{R(x)}} \]
defined by equation
\[ C : \quad y^2 = R(x), \quad R(x) = x(1-x)(1-\kappa \lambda x)(1+\kappa x)(1+\lambda x). \] 
(2.15)

Invariant conditions for reducibility of abelian integrals involve the so-called Weierstrass-Picard theorem on the periods of reducible integrals. This theorem was formulated by Picard [28] in 1883 and Kowalevski [15] in 1884, see discussion in [6]. Thus, it is clear that Kowalevski was familiar with the Weierstrass theory of reducible Abel’s integrals on hyperelliptic curves, which generalize the Jacobi calculations.

The Jacobi hyperelliptic curve \( C \) \((2.15)\) is isomorphic to a curve with an affine equation
\[ C : \quad y^2 = x^6 - c_1 x^4 + c_2 x^2 - 1, \]
having standard elliptic involutions \( \sigma_{1,2} \), see detailed calculations in [3]. The quotients \( E_i = C/\sigma_i \) are two elliptic curves
\[ E_1 : \quad y^2 = x^3 - a_1 x^2 + a_2 x - 1 \quad \text{and} \quad E_2 : \quad y^2 = x(x^3 - a_1 x^2 + a_2 x - 1) \] 
(2.16)
so that Jacobian of \( C \) decomposes as \( Jac(C) = E_1 \times E_2 \). Now such elliptic fibrations of the reducible abelian varieties are studied extensively due to the promising post-quantum cryptography applications [3].

Let us come back to the Kowalevski top:
- First curve \( E_1 \) in \((2.16)\) coincides with the Kowalevski elliptic curve \( E \) \((2.9)\) at the special values of parameters \( b \) and \( c_2 \neq 0 \);
- Second curve \( E_2 \) in \((2.16)\) is related to a spectral curve \( \Gamma \) of the 4 \( \times \) 4 Lax matrix proposed by Reyman and Semenov-Tian-Shansky [30] at \( c_2 \neq 0 \).

According to Theorem 7.8 in [4] if \( c_2 \neq 0 \) the common level surface of the integrals of motion \( H, K, c_1, c_2 \) consists of two components (Liouville tori). If \( c_2 = 0 \) the level surface is irreducible.

To continue Kowalevski calculations in [15, 16] we can consider two-dimensional abelian variety \( E_1 \times E_2 \) and the corresponding group law on this reducible algebraic group. For instance, when an elliptic curve is realized as a nonsingular cubic curve, its group structure can be described in terms of the sets of three points in which lines intersect the curve. In our case two points \( P_1 = (s_1, S_1) \) and \( P_2 = (s_2, S_2) \) determine a line
\[ \Upsilon : \quad S = \frac{s - s_2}{s_1 - s_2} S_1 + \frac{s - s_1}{s_2 - s_1} S_2 , \] 
(2.17)
which has an intersection with the elliptic curve \( E \) \((2.9)\) at the third point \( P_3 = (s_3, S_3) \) so that
\[ \det \begin{pmatrix} s_1 & S_1 & 1 \\ s_2 & S_2 & 1 \\ s_3 & S_3 & 1 \end{pmatrix} = 0 \]
and
\[ \varphi_1(s_1, \dot{s}_1, \dot{s}_3, c_1, c_2, H, K) = \frac{ds_1}{S_1} + \frac{ds_2}{S_2} + \frac{ds_3}{S_3} = 0 . \] 
(2.18)
Affine coordinates of the third point $P_3$ are equal to

$$s_3 = -(s_1 + s_2) - H + \frac{(S_1 - S_2)^2}{4(s_1 - s_2)^2} = \frac{\eta}{\nu}$$

where

$$\eta = b^2 \ell_2^2 (\gamma_1 - i\gamma_2)^2 - 2i(\ell_3 z_2 + b \gamma_3)(b(\gamma_1 - \gamma_2) + z_2^2)\ell_2 \ell_3 - (\ell_3^2 + z_2^2)(\ell_3 z_2 + b \gamma_3)^2,$$

$$\nu = (2b(\gamma_1 - i\gamma_2) + z_2^2)\ell_2^2 + (\ell_3 z_2 + b \gamma_3)^2,$$

and

$$S_3 = \frac{s_3 - s_2}{s_1 - s_2} S_1 + \frac{s_3 - s_1}{s_2 - s_1} S_2.$$

Point $P_3 = (s_3, S_3)$ is the desired reduced divisor $\rho(D')$ on the Kowalevski elliptic curve $E$ [23]. Affine coordinates of this reduced divisor are dynamical variables which satisfy equations

$$\frac{s_3}{2S_3} = \frac{\ell_3 z_2 - b \gamma_3}{\sqrt{\eta}} \quad \text{and} \quad \frac{s_3}{4S_3} = \frac{(2\ell_2 \ell_3 - \ell_3 z_2 - b \gamma_3)(2b(\gamma_1 - i\gamma_2) + z_2^2)}{\sqrt{\eta}}.$$ 

Using these equation we can easy check equation (2.18) with elliptic differentials and obtain two equations

$$\varphi_2(s_1, s_2, s_3, c_1, c_2, H, K) = 0 \quad \text{and} \quad \varphi_3(s_1, s_2, s_3, c_1, c_2, H, K) = 1$$

similar to three equations for the three poles of the Backer-Akhiezer function associated with $4 \times 4$ Lax matrix [4].

Only new thing is the construction of the $2 \times 2$ Lax matrix on the Kowalevski elliptic curve $E = E_1$, which is a factor of reducible abelian variety $E_1 \times E_2$. Let us compute Mumford’s coordinates of the intersection divisor $D_3 = P_1 + P_2 + P_3 = (U, V)$ [11, 25].

$$U(s) = (s - s_1)(s - s_2)(s - s_3) \quad \text{and} \quad V(s) = \frac{s - s_2}{s_1 - s_2} S_1 + \frac{s - s_1}{s_2 - s_1} S_2 + f(s)U(s) \quad (2.19)$$

where $f(s)$ is a rational function on $E$ without poles in $s_k$. Sometimes Mumford’s coordinates of the divisor $D$ are called the Jacobi-Mumford polynomials themselves $(U, V)$, and sometimes Mumford’s coordinates are called the coefficients of these polynomials $U$ and $V$. Both these polynomials $U$ and $V$ appeared in Abel’s memories [1], when he also used rational functions instead of polynomials.

Using these polynomials we can construct a new $2 \times 2$ Lax matrix for the Kowalevski top

$$L(s) = \left( \begin{array}{cc} V & U \\ W & -V \end{array} \right), \quad W = \frac{S^2 - V^2}{U},$$

where $S$ is given by [23]. Thus, using reduced divisor we can get a new formal Lax matrix on the first factor $E_1 = E$ of the reduced abelian variety $E_1 \times E_2$ [39]. The corresponding Backer-Akhiezer function $\psi$ [11] with the standard normalization $\alpha$ has three poles $P_1, P_2$ and $P_3$, which lie on the line $Y$ [2.17]. Spectral curves of the similar $2 \times 2$ Lax matrices from [27] and [18] coincide with the Kowalevski hyperelliptic curve $C$ [2.12].

Of course, variables $s_{1,2}$ and the corresponding Abel’s equations [2.11] are well studied. Nevertheless, in the literature, we do not find a discussion of the group operations on a reducible abelian variety $E_1 \times E_2$ generating the following transformations of the Kowalevski variables

$$\tau: \quad (s_1, S_1) + (s_2, S_2) \rightarrow (s'_1, S'_1) + (s'_2, S'_2)$$

so that

1. $(s'_1, S'_1)$ and $(s'_2, S'_2)$ are points on the first factor $E_1$;
2. $(s'_1, S'_1)$ and $(s'_2, S'_2)$ are points on the second factor $E_2$;
3. $(s'_1, S'_1)$ and $(s'_2, S'_2)$ are points on $E_1$ and $E_2$, respectively.

The group law on the reducible abelian variety $E_1 \times E_2$ is beyond the scope of this note dedicated to the reduced divisors on elliptic curves $E_1$ and $E_2$. 

6
3 Reduced divisor on the spectral curve of Lax matrix

In [30] Reyman and Semenov-Tian-Shansky found Lax matrices for the Kowalevski top. In [4] these Lax matrices were used to integrate the problem in terms of theta-functions, see also textbook [31].

Let us take Lax matrix $L$ (6.3) from [4] and multiply it’s first term on $b$ that corresponds to scaling $g_i \to bg_i$. As a result, we obtain the Lax matrix

$$L(\lambda) = \frac{ib}{\lambda} \begin{pmatrix} 0 & g_1 - ig_2 & 0 & -g_3 \\ -g_1 - ig_2 & 0 & g_3 & 0 \\ 0 & -g_3 & 0 & -g_2 - ig_2 \\ g_3 & 0 & g_1 - ig_2 & 0 \end{pmatrix}$$

(3.1)

with the spectral curve defined by equation

$$\Gamma : \quad \det (L(\lambda) - \mu) = \mu^4 - 2\left(2\lambda^2 - H + \frac{b^2 c_1}{\lambda^2}\right)\mu^2 + K - \frac{b^2}{\lambda^2}\frac{(c_1 H - c_2^2)}{\lambda^2} + \frac{b^4 c_2^2}{\lambda^4} = 0,$$

(3.2)

where $c_1, c_2$ are the Casimir functions \((1.4)\) and integrals of motion $H$ and $K$ are given by \((1.6,1.7)\).

Symmetries of Lax matrices give rise to the two commuting involutions $\tau_1$ and $\tau_2$ on $\Gamma$, that allows us to consider quotient elliptic curve $E = \Gamma/(\tau_1, \tau_2)$ [4]. Indeed, substituting

$$\mu^2 = v, \quad \text{and} \quad \lambda^2 = u$$

into (3.2) we obtain the following equation

$$E : \quad \Phi(u,v) = (uv)^2 - 2\left(2u^2 + Hu + b^2 c_1\right)uv + Ku^2 - 2b^2 c_2^2 u + b^4 c_1^2 = 0,$$

(3.3)

which after birational transformation

$$v \to \frac{y - 2u^2 - Hu - b^2 c_1}{u} \quad \text{and} \quad u = -x$$

(3.4)

looks like

$$E : \quad y^2 = xS^2, \quad S^2 = 4x^3 + 4Hx^2 + (4b^2 c_1 + H^2 - K)x + 4b^2 c_2^2.$$

(3.5)

Here $S^2$ is the cubic polynomial from the definition of the Kowalevski elliptic curve $E = E_1 (2.9)$.

At $c_2 \neq 0$ we can identify curve $E$ with the second factor $E_2$ (2.16) in the reduced abelian variety $E_1 \times E_2$.

3.1 Standard normalization and $r$-matrix

Let us impose a linear constraint

$$\vec{\alpha} \cdot \psi = \sum_{i=1}^{N} \alpha_i \psi_i = 1,$$

i.e. fix normalization $\vec{\alpha}$ of the Backer-Akhiezer function $\psi$ (1.2). Poles of $\psi$ form a divisor $D$ on the spectral curve $\Gamma$ (1.2). Following the Sklyanin scheme, we substitute matrices

$$L^{(p)} = L \left(\text{tr} L^{(p-1)}\right) - (p - 1) L^{(p-1)}L, \quad \text{with} \quad L^{(1)} = L,$$

labelled by number $p = 1, \cdots, N$ into the $N \times N$ matrix

$$B(x) = \begin{pmatrix} \vec{\alpha} \cdot L^{(1)}(x) & L^{-1}(x) \\ \vec{\alpha} \cdot L^{(2)}(x) & L^{-1}(x) \\ \frac{1}{2} \vec{\alpha} \cdot L^{(3)}(x) & L^{-1}(x) \\ \cdots \\ \frac{1}{(N-1)!} \vec{\alpha} \cdot L^{(N)}(x) & L^{-1}(x) \end{pmatrix},$$

(3.6)
This matrix determine first Mumford’s coordinate of the divisor \( D = (U(x), V(x)) \)

\[
U(x) = \text{MakeMonic} \det B(x),
\]

(3.7)

and a finite set of second Mumford’s coordinates

\[
y^m - V_m(x) = 0, \quad y_i^m = V_m(x_i), \quad i = 1, \ldots, N, \quad m = 1, \ldots, N - 1,
\]

see [17, 19, 32] and references within. As usual, second Mumford’s coordinates

\[
(-y)^{j-i} = \frac{(B^\wedge(x))_{k,i}}{(B^\wedge(x))_{k,j}},
\]

(3.8)

are equivalent up to mod \( U(x) \), see [2, 19], and the MakeMonic means that we take the numerator of a rational function and divide the corresponding polynomial by its leading coefficient.

For the \( 4 \times 4 \) Lax matrix \( L \) (3.1) and the standard normalization vector

\[
\vec{\alpha} = (1, 0, 0, 0),
\]

(3.9)

matrix \( B \) (3.6) is equal to

\[
B(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{i b_2}{\lambda} & -i \ell & \frac{i b_2}{\lambda} \\
-4\lambda^2 + \ell_1^2 + \ell_2^2 + 4\ell_1^3 - 4b_2\lambda - \frac{b_2^2}{\lambda} & 0 & 2\ell_1\ell_2 + 2b_3 \\
-2i(\ell_1^2 + 2b_3\ell_1 + \ell_2^2) - \frac{2ib^2g_3g_\perp}{\lambda} & b_{24} & b_{34} & \frac{2ib^2g_3g_\perp + ib\ell_+ (g_\perp - 2g_\parallel)}{\lambda} - \frac{ib^3g_\perp g_\parallel}{\lambda^3}
\end{pmatrix}
\]

where

\[
b_{24} = 2i\left(\ell_2^2 + 2bg_\perp\right) + \frac{b\left(4i\ell_3(g_3\ell_\perp - g_\perp \ell_3) + 2ib(g_3^2 + 2g_1g_\perp) + ig_\perp \ell_+\lambda^2\right)}{\lambda^2} + \frac{ib^3c_1g_\perp}{\lambda^3},
\]

\[
b_{34} = -i\ell_+ \left(\ell_2^2 + 2bg_\perp\right) + \frac{ib^2\left(g_3^2\ell_\perp - 2g_2g_\perp \ell_3 - g_\perp^2\ell_+\right)}{\lambda^2},
\]

and

\[
g_\pm = g_1 \pm ig_2, \quad \ell_\pm = \ell_1 \pm i\ell_2.
\]

The corresponding first Mumford’s coordinate \( U(\lambda) \) (3.7) has the following form

\[
U(\lambda) = \lambda^6 + u_4\lambda^4 + u_2\lambda^2 + c_2u_0,
\]

(3.10)

where coefficients \( u_k \) may be recovered from the definition (3.7) or found in [19].

In 2002 Kuznetsov proved that roots \( \lambda_{1,2} \) of the polynomial \( U(\lambda) \) (3.10) at \( c_2 = (\ell, \gamma) = 0 \)

\[
U(\lambda) = \lambda^2(\lambda^2 - \lambda_{1}^2)(\lambda^2 - \lambda_{2}^2)
\]

are variables of separation in the corresponding Hamilton-Jacobi equation [19]. Thus, poles of the Backer-Akhiezer function with the standard normalization \( \vec{\alpha} \) can not generate variables of separation in the framework of the Sklyanin method. Nevertheless, these variables exist and we can say that the Dubrovin-Skrypnyk theory is wrong at least in the Kowalevski case.

In 2018 Dubrovin and Skrypnyk introduced the necessary and sufficient conditions for satisfiability of the Sklyanin scheme in the terms of the classical \( r \)-matrix [8]. It is easy to show that \( r \)-matrix for the Kowalevski top [30] does not satisfy these conditions at any value of \( c_2 \). Thus, according to [8] poles of the Backer-Akhiezer function with numerical normalization \( \vec{\alpha} \) can not generate variables of separation in the framework of the Sklyanin method. Nevertheless, these variables exist and we can say that the Dubrovin-Skrypnyk theory is wrong at least in the Kowalevski case.

At \( c_2 = 0 \) we have irreducible abelian variety and it is a principal cause for applicability of the Sklyanin scheme which is independent on the \( r \)-matrix describing the Poisson brackets between entries of the Lax matrix. The mapping of the Liouville torus to JacC becomes an unramified two-sheeted covering, see Sect. 7.9 in [4].

Another variables of separation in the partial case at \( c_2 = 0 \) were proposed in [33, 34]. Relations between various algebraic curves for the Kowalevski top are discussed in [2, 10, 9, 23].
3.2 Semi-reduced divisors

Below we study the only case
\[ c_2 = \ell_1g_2 + \ell_2g_2 + \ell_3g_3 \neq 0 \]
when we can construct reducible abelian variety \( E_1 \times E_2 \). In this case evolution of the three poles \( P_1, P_2 \) and \( P_3 \) of the Backer-Akhiezer function is defined by three equations involving elliptic and Prym differentials, see equations (7.69) in [4]. An interesting problem is how to compute the action-angle variables of the Kowalewski top [7] directly from these three poles of the Backer-Akhiezer function [4].

At \( c_2 \neq 0 \) and \( \lambda^2 = u \) we have a cubic polynomial \( U(u) \) or first Mumford’s coordinate of the divisor of poles on the elliptic curve \( \mathcal{E} \)
\[ D = P_1 + P_2 + P_3. \]

According to the Riemann-Roch theorem dimension of the linear system \(|D|\), which is the set of all nonnegative divisors which are linearly equivalent to \( D \)
\[ |D| = \{ D' \in \text{Div}(C) \mid D' \sim D \text{ and } D' > 0 \}, \]
is equal to
\[ \dim |D| = \deg D - g, \quad \text{at } \deg D = n > g, \]
where \( g \) is a genus of \( C \) and \( \deg D \) is a degree of the divisor, see definitions and other details in textbook [24]. In our case
\[ \dim |D| = \deg D - g(\mathcal{E}) = 3 - 1 = 2, \]
and we have nontrivial space of equivalent divisors involving a chain of divisors
\[ D \to D' \to D'', \quad \deg D' = 2, \quad \deg D'' = 1. \]

We aim to construct these divisors and to study the evolution of these divisors.

Let us calculate six possible second Mumford’s coordinates (3.8) for \( D = (U(x), V(x)) \)
\[ \mu^2 - V_2(\lambda), \quad V_2(\lambda) = \frac{(B^\wedge(\lambda))_{k,j}}{(B^\wedge(\lambda))_{k,j+2}}, \quad j = 1, 2, \quad k = 2, 3, 4. \]

Substituting \( \mu^2 = V_2(\lambda) \) into the equation for spectral curve \( \Gamma \) (3.2) we obtain rational function on \( \lambda \)
\[ V_2^4(\lambda) - 2d_1(\lambda)V_2(\lambda) + d_2(\lambda) = 0. \]

Its numerator is so-called Abel’s polynomial [1, 11]
\[ \Psi(\lambda) = \theta U(\lambda)U'(\lambda) = 0, \]
generating coordinates of divisors \( D \) and \( D' \) so that
\[ D + D' + D_\infty = 0, \]
where \( D_\infty \) is a linear combination of the points at infinity. It is easy to prove that

- at \( k = 3, 4 \) and \( j = 1 \) divisor \( D' \) has degree more then degree of divisor \( D \);
- at \( k = 2, i = 1 \) and \( k = 3, j = 2 \) or \( k = 4, j = 2 \) divisor \( D' \) is a constant divisor of degree two with coordinate
\[ U'(\lambda) = \lambda^2; \]
- at \( k = j = 2 \) divisor \( D' = (U'(x), V'(x)) \) has degree four and its first Mumford’s coordinates is equal to
\[ U'(\lambda) = \lambda^4 - \left( \frac{2b_1g_1\ell_1 + g_2\ell_3}{\ell_1^2 + \ell_2^2} \right) \lambda^2 + \frac{b_2c_2^2}{\ell_1^2 + \ell_2^2}. \]
Entries of the corresponding normalization $\tilde{a''}$ are bulky functions on $e^*(3)$ that we will omit for brevity.

Summing up, using numerical normalization vector $\tilde{a}$ and Abel’s reduction of divisors we obtain divisor of degree two $D' = (U'(x), V'(x))$ on the elliptic curve $E$ with following Mumford’s coordinates

$$U'(u) = (u - u_1)(u - u_2) \equiv u^2 - \left( \ell_3^2 + \frac{2b_1 \ell_1 (g_1 \ell_1 + g_2 \ell_2) + b_2 g_3^2}{\ell_1^2 + \ell_2^2} \right) u + \frac{b_2 c_2^4}{\ell_1^2 + \ell_2^2} . \quad (3.11)$$

and

$$v - V'(u), \quad V'(u) = \frac{(B^\wedge (\lambda))_{2,2}}{(B^\wedge (\lambda))_{2,4}} \bigg|_{\lambda^2 = u} . \quad (3.12)$$

Because

$$\dim |D'| = \deg D' - g(E) = 2 - 1 = 1,$$

divisor $D'$ is a semi-reduced divisor on the elliptic curve, which can be reduced to the one-degree equivalent divisor $D''$.

### 3.3 Reduced divisor

Birational transformation $(v, u) \rightarrow (x, y)$ transforms elliptic curve $E$ to canonical form that allows as to directly apply Abel’s calculations. Indeed, two points $P'_1 = (x_1, y_1)$ and $P'_2 = (x_2, y_2)$ of divisor $D'$ on an elliptic curve

$$E : \quad y^2 = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

determine parabola

$$Y' : \quad y = \sqrt{a_4} x^2 + b_1 x + b_0 = \sqrt{a_4} (x - x_1)(x - x_2) + \frac{(x - x_2)y_1}{x_1 - x_2} + \frac{(x - x_1)y_2}{x_2 - x_1},$$

which has a finite third point of intersection $P'_3 = (x_3, y_3)$ with affine coordinates

$$x_3 = -x_1 - x_2 - \frac{2\sqrt{a_4} b_0 + b_1^2 - a_2}{2\sqrt{a_4} b_1 - a_3}, \quad y_3 = \sqrt{a_4} x_3^2 + b_1 x_3 + b_0 . \quad (3.13)$$

These equations describe the reduction of divisors

$$D' = P'_1 + P'_2 \rightarrow D'' = P'_3 .$$

In our case, a unique reduced divisor $D''$ in a class of equivalent divisors associated with standard normalization $\tilde{a''}$ consists of the point

$$\rho(D) = D'' = P''(x_3, y_3), \quad \dim |D''| = \deg D'' - g(E) = 1 - 1 = 0,$$

with affine coordinates

$$x_3 = \frac{4b_2 c_2^2}{4b^2 c_1 - K}, \quad y_3 = \frac{4b^2 c_1 (4b^2 c_2^2 - 8b^2 c_3^2 - HK)}{(4b^2 c_1 - K)^2} .$$

Entries of the corresponding normalization $\tilde{a''}$ are bulky functions on $e^*(3)$ that we will omit for brevity.

Affine coordinates of the reduced divisor $\rho(D)$ can be considered as action coordinates on the phase space $e^*(3)$ which were obtained from the three poles $P_1, P_2$ and $P_3$ of the Backer-Akhiezer function by a standard reduction procedure.

### 3.4 Evolution of semi-reduced divisor

Two variable points $P'_1, P'_2$ and one fixed point $P'_3$ lie on the parabola $Y'$ and, therefore, so-called geometric Abel’s integral

$$\frac{dx_1}{y_1} + \frac{dx_2}{y_2} + \frac{dx_3}{y_3} = 0 ,$$

10
determines evolution of the points $P_1'(x_1, y_1)$ and $P_2'(x_2, y_2)$ on elliptic curve
\[ \frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 0. \]  \hspace{1cm} (3.14)

So, similar to the Kepler problem and harmonic oscillator [35 36], we have an evolution of the parabola $Y'$ around a fixed point on an elliptic curve.

Abscissas $u_{1,2}$ (3.11) and ordinates (3.12)
\[ v_{1,2} = V'(u = u_{1,2}), \]
of the points in support of the semi-reduced divisor $D'$ satisfy to equation (3.3) for the elliptic curve $E \cong E_2$
\[ \Phi(u_i, v_i) = 0. \]

Variables $u_{1,2}$ do not commute to each other
\[ \{u_1, u_2\} \neq 0, \]
similar to $w_{1,2}$ variables in the first component $E = L_1$ of the reducible abelian variety $E_1 \times E_2$.

Affine coordinates of poles $u_{1,2}$ and $v_{1,2}$ satisfy to the following differential equations
\[ \Omega_1(u_1, v_1) \dot{u}_1 + \Omega_1(u_2, v_2) \dot{u}_2 = 0, \]
\[ \Omega_2(u_1, v_1) \dot{u}_1 + \Omega_2(u_2, v_2) \dot{u}_2 = 0, \]  \hspace{1cm} (3.15)
on the elliptic curve $E$ (3.3). Here
\[ \dot{u}_k = \{H, u_k\} \]
and
\[ \begin{align*}
\Omega_1(u, v) &= \frac{1}{u} \frac{\partial_H \Phi(u, v)}{\partial u} = \frac{b^2 c_1 - u v}{u(b^2 c_1 - H u - u v + 2u^2)}, \\
\Omega_2(u, v) &= \frac{1}{u} \frac{\partial_K \Phi(u, v)}{\partial u} = \frac{1}{2(b^2 c_1 - H u - u v + 2u^2)}. 
\end{align*} \hspace{1cm} (3.16)
Birational transformation $(v, u) \to (x, y)$ (3.4) transforms elliptic curve $E$ to canonical form (3.5)
\[ y^2 = xP_3(x), \quad P_3(x) = 4x^3 + 4Hx^2 + (4b^2 c_1 + H^2 - K)x + 4b^2c_2 \]
and also reduces equations (3.15) to Abel’s equations
\[ \frac{dx_1}{y_1} + \frac{dx_2}{y_2} = 0, \]
\[ \left( \frac{dx_1}{x_1} + \frac{2x_1 dx_1}{y_1} \right) + \left( \frac{dx_2}{x_2} + \frac{2x_2 dx_2}{y_2} \right) = 0, \]  \hspace{1cm} (3.17)
involving holomorphic 1-form $\omega_1$ and logarithmic 1-form $\omega_2$
\[ \omega_1 = \frac{dx}{y}, \quad \omega_2 = \frac{dx}{x} - \frac{2xdx}{y}. \]
A differential of the third kind $\omega_3 = 1/xy$ appears in the following equation
\[ \frac{\dot{x}_1}{x_1y_1} + \frac{\dot{x}_2}{x_2y_2} = \frac{1}{2} \frac{d}{dt} \frac{1}{x_1x_2}. \]
(3.18)
Equation involving time involves Prym differentials on the corresponding double covering of elliptic curve $E = E_2$.

It is easy to see that independent on the phase space variables $x_{1,2}$ become dependent on the common level of integrals of motion because
\[ (x + x_1)(x + x_2) = x^2 + \frac{X(X - 2H)}{4X} + K - 4c_1 b^2 \frac{4^2 c_2}{X}, \quad X = z_1 z_2 \equiv \ell_1^2 + \ell_2^2. \]
Thus, equations (3.17) and (3.18) on a second factor $E = E_2$ of a reducible abelian variety $E_1 \times E_2$ describe the evolution of the one variable $X$ similar to the Clebsch case [38].
3.5 Kowalevski gyrostat

Our calculations can be directly generalized to the Kowalevski gyrostat with the Lax matrix

\[ \tilde{L} = L + i\gamma \text{diag}(-1, 1, -1, 1), \quad \gamma \in \mathbb{R}, \]

and the following integrals of motion

\[ \tilde{H} = \ell_1^2 + \ell_2^2 + \ell_3^2 + (\ell_3 + \gamma)^2 - 2bg_1, \]
\[ \tilde{K} = (4g_1^2 + 4g_2^2)b^2 + (4\ell_1^2g_1 + 8\ell_1\ell_2g_2 - 8\gamma\ell_1g_3 - 4\ell_2^2g_1 - 4\gamma^2g_1)b + (\ell_1^2 + \ell_2^2 - 2\ell_3^2\gamma - \gamma^2)^2. \]

The corresponding spectral curve \( \tilde{\Gamma} \) is also twofold coverings of the elliptic curve \( \tilde{E} \) defined by the equation

\[ \tilde{\Phi}(u, v) = \Phi(u, v) - 4\gamma^2u^3 = 0, \]

whereas first Mumford’s coordinate for semi-reduced divisor \( \tilde{D}' \) on \( \tilde{E} \) is equal to

\[ \tilde{U}'(u) = (u - \tilde{u}_1)(u - \tilde{u}_2) = U'(u) - \gamma u \left( 2\ell_3 + \gamma + \frac{2b\ell_3g_3}{\ell_1^2 + \ell_2^2} \right). \]

As above coordinates of poles \( \tilde{u}_{1,2} \) and \( \tilde{v}_{1,2} \) satisfy equations of the form \([3.15]\) and \([3.16]\) on the elliptic curve \( \tilde{E} \). Similar equations appear also for other generalizations of the Kowalevski curve associated with elliptic curves \([4, 12, 13]\).

4 Conclusion

We construct and study reduced divisors associated with the Kowalevski variables of separation and the poles of the Baker-Akhiezer function with a standard normalization.

Affine coordinates of the first reduced divisor are dynamical variables which can be useful to a description of the group law on the reducible abelian variety \( E_1 \times E_2 \), where elliptic curves \( E_1 \) and \( E_2 \) were introduced by Kowalevski [16] and by Reyman and Semenov-Tian-Shansky up to isogenies [20], respectively. For instance, we can use this reduced divisor to construct \( 2 \times 2 \) Lax matrix on the first factor in the reducible abelian variety \( E_1 \times E_2 \), i.e. spectral curve of this Lax matrix is the Kowalevski elliptic curve.

Classical \( r \)-matrix for the \( 4 \times 4 \) Lax matrix introduced by Reyman and Semenov-Tian-Shansky does not satisfy to the Dubrovin-Skrypnyk necessary and sufficient conditions for satisfiability of the Sklyanin construction of the variables of separation in the Hamilton-Jacobi equation. Nevertheless, Kuznetsov proved that poles of the Baker-Akhiezer function with a standard normalization are variables of separation when one of the Casimir functions on \( e^+(3) \) is equal to zero. Thus, the Dubrovin-Skrypnyk theory is wrong, at least in the Kowalevski case.

It was one of the reasons to study reduced representative of a class of equivalent divisors on an elliptic curve determined by poles of the Baker-Akhiezer function with a standard normalization. If one of the Casimir functions is equal to zero, the spectral curve of the \( 4 \times 4 \) Lax matrix determines irreducible abelian variety and the Sklyanin scheme works. If this Casimir function is nonzero we can consider the reducible abelian variety and the corresponding reduced divisor on the second elliptic curve in a product \( E_1 \times E_2 \). Affine coordinates of this reduced divisor can be considered as action variables, which are variables of separation with trivial dynamics.

As a by-product, we obtain equations describing the evolution of the semi-reduced divisor of degree 2 on the elliptic curve \( E_2 \) defined by a spectral curve of the \( 4 \times 4 \) Lax matrix. We plan to discuss these equations on the elliptic curve \( E_2 \) and similar equations on the Kowalevski elliptic curve \( E_1 \) in the forthcoming publication.

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