Limit theorems of Chatterjee’s rank correlation

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Abstract

Establishing the limiting distribution of Chatterjee’s rank correlation for a general, possibly non-independent, pair of random variables has been eagerly awaited by many. This paper shows that (a) Chatterjee’s rank correlation is asymptotically normal as long as one variable is not a measurable function of the other, (b) the corresponding asymptotic variance is uniformly bounded by 36, and (c) a consistent variance estimator exists. Similar results also hold for Azadkia-Chatterjee’s graph-based correlation coefficient, a multivariate analogue of Chatterjee’s original proposal. The proof is given by appealing to Hájek representation and Chatterjee’s nearest-neighbor CLT.

Keywords: dependence measure, rank-based statistics, graph-based statistics, Hájek representation, nearest-neighbor CLT.

1 Introduction

Let $Y$ be a random variable in $\mathbb{R}$ and $X$ be a random vector in $\mathbb{R}^d$ that are defined on the same probability space and of joint and marginal distribution functions $F_{X,Y}$ and $F_X, F_Y$, respectively. Throughout the paper, we consider $F_{X,Y}$ to be fixed and continuous.

To measure the dependence strength between $X$ and $Y$, Dette et al. (2013) introduced the following population quantity,

$$\xi = \xi(X,Y) := \frac{\int \text{Var}\{\mathbb{E}[\mathbb{1}(Y \geq y) | X]\}dF_Y(y)}{\int \text{Var}\{\mathbb{1}(Y \geq y)\}dF_Y(y)},$$

(1.1)

with $\mathbb{1}(\cdot)$ representing the indicator function. This quantity, termed the Dette-Siburg-Stoimenov’s dependence measure in literature, enjoys desirable properties of being between 0 and 1 and being (a) 0 if and only if $Y$ is independent of $X$; and (b) 1 if and only if $Y$ is a measurable function of $X$.

Consider $(X_1, Y_1), \ldots, (X_n, Y_n)$ to be $n$ independent copies of $(X,Y)$. For any $i \in \{1, \ldots, n\}$, let $R_i := \sum_{j=1}^n \mathbb{1}(Y_j \leq Y_i)$ denote the rank of $Y_i$, and let $N(i)$ and $\mathbf{N}(i)$ index the nearest neighbor (NN) of $X_i$ among $\{X_j\}_{j=1}^n$ (under the Euclidean metric $\| \cdot \|$) and the right NN of $X_i$ among $\{X_j\}_{j=1}^n$ (when $d = 1$, with $\mathbf{N}(i) := i$ if $X_i = \max\{X_1, \ldots, X_n\}$), respectively. To estimate $\xi$ based

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only on \((X_i, Y_i)\)'s, Azadkia and Chatterjee (2021) and Chatterjee (2021) introduced the following two correlation coefficients:

\[
\text{(Azadkia-Chatterjee)} \quad \xi_n := \frac{6}{n^2 - 1} \sum_{i=1}^{n} \min \{R_i, R_{N(i)}\} - \frac{2n + 1}{n - 1}, \quad \text{for } d \geq 1; \quad (1.2)
\]

\[
\text{(Chatterjee)} \quad \overline{\xi}_n := 1 - \frac{3}{n^2 - 1} \sum_{i=1}^{n} \left| R_{\overline{N}(i)} - R_i \right|, \quad \text{when } d = 1. \quad (1.3)
\]

Azadkia and Chatterjee (2021, Theorem 2.2) and Chatterjee (2021, Theorem 1.1) showed that, under some very mild conditions, both \(\xi_n\) and \(\overline{\xi}_n\) constitute strongly consistent estimators of \(\xi\). Unfortunately, deriving the limiting distributions of \(\xi_n\) and \(\overline{\xi}_n\) is also of interest to statisticians. Unfortunately, unless \(X\) and \(Y\) are independent — implying that \(N(i)\) and \(\overline{N}(i)\)'s are independent of \(Y_1, \ldots, Y_n\) — this is apparently still an open problem.

The following two theorems answer this call, and are the main results of this paper.

**Theorem 1.1** (Asymptotic normality). For any fixed and continuous \(F_{X,Y}\) such that \(Y\) is not a measurable function of \(X\) almost surely, we have

\[
\frac{(\xi_n - E[\xi_n])}{\sqrt{\text{Var}[\xi_n]}} \rightarrow N(0,1) \quad \text{in distribution,} \quad (1.4)
\]

and

\[
\frac{(\overline{\xi}_n - E[\overline{\xi}_n])}{\sqrt{\text{Var}[\overline{\xi}_n]}} \rightarrow N(0,1) \quad \text{in distribution (if } d = 1). \quad (1.5)
\]

Let \(N_k(i)\) and \(\overline{N}_k(i)\) index the \(k\)-th NN of \(X_i\) among \(\{X_j\}_{j=1}^{n}\) and the right \(k\)-th NN of \(X_i\) among \(\{X_j\}_{j=1}^{n}\) (when \(d = 1\), with \(\overline{N}_k(i) := i\) if \(X_i\) is among the \(k\) largest). For any \(a, b \in \mathbb{R}\), write \(a \lor b = \max\{a, b\}\) and \(a \land b = \min\{a, b\}\). Define

\[
\sigma^2 :=
\]

\[
36 \left\{ \frac{1}{n^3} \sum_{i=1}^{n} \left( R_i \land R_{N(i)} \right)^2 \left( 1 + 1 \left( i = N(N(i)) \right) \right) + \frac{1}{n^3} \sum_{i=1}^{n} \left( R_i \land R_{N(i)} \right) \left( R_i \land R_{N_2(i)} \right) \left( 1 + 1 \left( i \neq N(N(i)) \right) \left( 1 \left( j \neq N_2(i) \right) \right) \right) \right. \]

\[
- \frac{1}{n^3} \sum_{i=1}^{n} \left( R_i \land R_{N(i)} \right) \left( R_{N_2(i)} \land R_{N_3(i)} \right) \left( 1 + 1 \left( i \neq N(N(i)) \right) \left( 1 \left( j \neq N_2(i) \right) \right) \right) \left( 1 \left( j \neq N_3(i) \right) \right) \right\}
\]

\[
+ 4 \sum_{i,j=1 \atop i \neq j}^{n} \mathbb{1} \left( R_i \leq R_j \land R_{N(j)} \right) \left( R_i \land R_{N(i)} \right)
\]

\[
- \frac{2}{n^2(n - 1)} \sum_{i,j=1 \atop i \neq j}^{n} \mathbb{1} \left( R_i \leq R_j \land R_{N(j)} \right) \left( R_{N(i)} \land R_{N_2(i)} \right)
\]

\[
+ \frac{1}{n^2(n - 1)} \sum_{i,j=1 \atop i \neq j}^{n} \mathbb{1} \left( R_i \land R_{N(i)} \land R_j \land R_{N(j)} \right) - 4 \left[ \frac{1}{n^2} \sum_{i=1}^{n} \left( R_i \land R_{N(i)} \right) \right]^2 \},
\]
and
\[ \hat{\sigma}^2 := \]
\[ 36 \left( \frac{1}{n^3} \sum_{i=1}^{n} \left( R_i \wedge R_{N(i)}^\circ \right)^2 + \frac{2}{n^3} \sum_{i=1}^{n} \left( R_i \wedge R_{N(i)}^\circ \right) \left( R_i \wedge R_{N_d(i)}^\circ \right) \right) - \frac{2}{n^2} \sum_{i=1}^{n} \left( R_i \wedge R_{N(i)}^\circ \right) \left( R_{N_{2d}(i)}^\circ \wedge R_{N_2(i)}^\circ \right) + \frac{4}{n^2(n-1)} \sum_{i,j=1}^{n} 1 \left( R_i \leq R_j \wedge R_{N(j)}^\circ \right) \left( R_i \wedge R_{N(i)}^\circ \right) \]
\[ - \frac{2}{n^2(n-1)} \sum_{i,j=1}^{n} 1 \left( R_i \leq R_j \wedge R_{N(j)}^\circ \right) \left( R_{N(i)}^\circ \wedge R_{N_2(i)}^\circ \right) \]
\[ + \frac{1}{n^2(n-1)} \sum_{i,j=1}^{n} \left( R_i \wedge R_{N(i)}^\circ \wedge R_j \wedge R_{N(j)}^\circ \right) - 4 \left[ \frac{1}{n^2} \sum_{i=1}^{n} \left( R_i \wedge R_{N(i)}^\circ \right) \right]^2 \} \]

**Theorem 1.2** (Variance estimation). For any fixed continuous \( F_{X,Y} \), it holds true that
\[ \hat{\sigma}^2 - n \text{Var}[\xi_n] \rightarrow 0 \quad \text{in probability}, \quad (1.5) \]
and
\[ \tilde{\hat{\sigma}}^2 - n \text{Var}[\tilde{\xi}_n] \rightarrow 0 \quad \text{in probability}. \quad (1.6) \]

The following two propositions further complement Theorems 1.1 and 1.2.

**Proposition 1.1** (Asymptotic bias, Azadkia and Chatterjee (2021)). Assume \( F_{X,Y} \) to be fixed and continuous.

(i) If \( X \) and \( Y \) are independent, then
\[ E[\xi_n] = -\frac{1}{n-1} \quad \text{and} \quad E[\tilde{\xi}_n] = 0 \quad \text{(if } d = 1). \]

(ii) If there exist fixed constants \( \beta, C, C_1, C_2 > 0 \) such that for any \( t \in \mathbb{R} \) and \( x, x' \in \mathbb{R}^d \),
\[ P(Y \geq t \mid X = x) - P(Y \geq t \mid X = x') \leq C(1 + \|x\|^\beta + \|x'\|^\beta)\|x - x'\| \]
and
\[ P(\|X\| \geq t) \leq C_1 e^{-C_2 t}, \]
we then have
\[ |E[\xi_n] - \xi| = O\left( \frac{(\log n)^{d+\beta+1}(d=1)}{n^{1/d}} \right) \quad \text{and} \quad |E[\tilde{\xi}_n] - \xi| = O\left( \frac{(\log n)^{\beta+3}}{n^{d}} \right) \quad \text{(if } d = 1). \]

**Proposition 1.2** (Asymptotic variance). Assume \( F_{X,Y} \) to be fixed and continuous.

(i) If \( Y \) is not a measurable function of \( X \) almost surely,
\[ \liminf_{n \to \infty} \left\{ n \text{Var}[\xi_n] \right\} > 0 \quad \text{and} \quad \liminf_{n \to \infty} \left\{ n \text{Var}[\tilde{\xi}_n] \right\} > 0 \quad \text{(if } d = 1). \]

On the other hand, if \( Y \) is a measurable function of \( X \) almost surely, then
\[ \lim_{n \to \infty} \left\{ n \text{Var}[\xi_n] \right\} = 0 \quad \text{and} \quad \lim_{n \to \infty} \left\{ n \text{Var}[\tilde{\xi}_n] \right\} = 0 \quad \text{(if } d = 1). \]
(ii) It holds true that
\[ \limsup_{n \to \infty} \left\{ n \operatorname{Var}[^{\xi_n}] \right\} < \infty \quad \text{and} \quad \limsup_{n \to \infty} \left\{ n \operatorname{Var}[\bar{\xi}_n] \right\} \leq 36 \quad \text{(if } d = 1) . \] (1.7)

If in addition \( F_X \) is absolutely continuous, then
\[ \limsup_{n \to \infty} \left\{ n \operatorname{Var}[\xi_n] \right\} \leq 36 - 9q_d + 9\sigma_d , \] (1.8)

where \( q_d \) and \( \sigma_d \) are two positive constants depending only on \( d \), with explicit values:
\[ q_d := \left\{ 2 - I_{3/4} \left( \frac{d + 1}{2}, \frac{1}{2} \right) \right\} ^{-1} , \quad I_x(a, b) := \int_0^x t^{a-1}(1 - t)^{b-1} dt , \] (1.9)
\[ \sigma_d := \int_{\Gamma_{d,2}} \exp \left\{ - \lambda \left\{ B(w_1, \| w_1 \|) \cup B(w_2, \| w_2 \|) \right\} \right\} d(w_1, w_2) , \] (1.10)
\[ \Gamma_{d,2} := \left\{ (w_1, w_2) \in (\mathbb{R}^d)^2 : \max(\| w_1 \|, \| w_2 \|) < \| w_1 - w_2 \| \right\} , \]

\( B(w_1, r) \) denotes the ball of radius \( r \) centered at \( w_1 \), and \( \lambda(\cdot) \) denotes the Lebesgue measure.

**Remark 1.1.** It is worth noting that (1.4) and (1.5) hold without requiring \( F_X \) to be absolutely continuous (with regard to the Lebesgue measure). In particular, \( \xi_n \) is still asymptotically normal even when \( X \) is supported on a low-dimensional manifold in \( \mathbb{R}^d \), e.g., the \( (d - 1) \)-dimensional unit sphere.

**Remark 1.2.** For establishing asymptotic normality, Theorem 1.1 requires \( Y \) to be not a measurable function of \( X \). When \( Y \) is perfectly dependent on \( X \), Proposition 1.2 suggests that \( \xi_n \) and \( \bar{\xi}_n \) are degenerate; indeed, Chatterjee (2021, Remark 9 after Theorem 1.1) showed that when \( Y \) is an increasing transformation of \( X \), \( \bar{\xi}_n = (n - 2)/(n + 1) \), which reduces to a deterministic constant.

**Remark 1.3.** The assumptions in Proposition 1.1(ii) correspond to Assumptions A1 and A2 in Azadkia and Chatterjee (2021). Its proof is a minor twist to that of Azadkia and Chatterjee (2021, Theorem 4.1), which we credit this proposition to. On the other hand, Proposition 1.2 is genuinely new, although the constants in (1.9) and (1.10) can be traced to Devroye (1988), Henze (1987), and in particular, Shi et al. (2021a, Theorem 3.1).

**Remark 1.4.** Combining Theorems 1.1, 1.2 with Propositions 1.1 and 1.2, when \( d = 1 \), one could immediately establish confidence intervals for \( \xi \) using either \( \xi_n \) or \( \bar{\xi}_n \) since the asymptotic bias in this case is root-\( n \) ignorable. For instance, as \( d = 1 \) and \( n \) large enough, an \( 1 - \alpha \) confidence interval of \( \xi \) can be constructed as
\[ (\bar{\xi}_n - z_{1-\alpha/2} \cdot \hat{\sigma} / \sqrt{n}, \quad \bar{\xi}_n + z_{1-\alpha/2} \cdot \hat{\sigma} / \sqrt{n}) , \]
where for any \( \beta \in (0, 1) \), \( z_\beta \) represents the \( \beta \)-quantile of a standard normal distribution. One could similarly construct large-sample tests for the following null hypothesis
\[ H_0 : \xi \leq \kappa , \quad \text{(for a given and fixed } \kappa < 1) \]
using, e.g., the test
\[ T := \mathbb{1}(\bar{\xi}_n > \kappa + z_{1-\alpha} \hat{\sigma} / \sqrt{n}) . \]
Remark 1.5. Checking Proposition 1.1, when $d > 1$, an asymptotically non-ignorable bias term may appear in the central limit theorem (CLT) and thus confidence intervals can only be established for $E\xi_n$ instead of $\xi$. To further debias $\xi_n$, enforcing more assumptions on $F_{X,Y}$ seems inevitable to us. A possible approach is to follow the similar derivations made in Berrett et al. (2019), who studied the problem of multivariate entropy estimation using NN methods.

Remark 1.6. The codes for computing $\xi_n$ and $\hat{\sigma}^2$ are available at https://github.com/zhexiaolin/Limit-theorems-of-Chatterjee-s-rank-correlation. Simulations support the theoretical results made in this paper.

1.1 Related literature

The study of Dette-Siburg-Stoimenov’s dependence measure (Dette et al., 2013) is receiving considerably increasing attention, partly due to the introduction of Chatterjee’s rank correlation (Chatterjee, 2021) as an elegant approach to estimating it. Nowadays, this growing literature has included Azadkia and Chatterjee (2021), Cao and Bickel (2020), Gamboa et al. (2022), Deb et al. (2020), Huang et al. (2020), Auddy et al. (2021), Shi et al. (2021a), Lin and Han (2022), Fuchs (2021), Azadkia et al. (2021), Griessenberger et al. (2022), Strothmann et al. (2022), Zhang (2022), Bickel (2022), and Chatterjee and Vidyasagar (2022), among many others. We also refer the readers to Han (2021) for a short survey on some most recent progress.

Below we outline the results in literature that are most relevant to Theorem 1.1.

(1) In his original paper, Chatterjee established the asymptotic normality of $\xi_n$ under an important additional assumption that $X$ is independent of $Y$. In particular, he showed

$$\sqrt{n}\xi_n \to N(0, 2/5) \text{ in distribution,}$$

if $Y$ is continuous and independent of $X$ (Chatterjee, 2021, Theorem 2.1).

(2) Although Azadkia and Chatterjee introduced $\xi_n$ as an extension of $\xi_n$ to multivariate $X$, their results did not include a CLT for $\xi_n$, which was listed as an open problem in Azadkia and Chatterjee (2021). Notable progress was later made by Deb et al. (2020) and Shi et al. (2021a), which we shall detail below.

(3) In Deb et al. (2020), the authors generalized Azadkia and Chatterjee’s original proposal to arbitrary metric space via combining the graph- and kernel-based methods. In particular, under independence between $X$ and $Y$ and some additional assumptions on $F_{X,Y}$, Deb et al. (2020, Theorem 4.1) established the following CLT for $\xi_n$,

$$\xi_n/S_n \to N(0, 1) \text{ in distribution,}$$

where $S_n$ is a data-dependent normalizing statistic.

(4) In Shi et al. (2021a), the authors re-investigated the proof of Deb et al. (2020) and, in particular, derived the closed form of the limit of $\text{Var}[\xi_n]$. More specifically, Shi et al. (2021a, Theorem 3.1(ii)) showed that, under independence between $X$ and $Y$ and some additional assumptions on $F_{X,Y}$,

$$\sqrt{n}\xi_n \to N\left(0, \frac{2}{5} + \frac{2}{5}q_d + \frac{4}{5}o_d\right) \text{ in distribution,}$$

(1.12)
where \( q_d \) and \( \alpha_d \) are two positive constants that only depend on \( d \) and were explicitly defined in Proposition 1.2(ii).

(5) In a related study, in order to boost the power of independence testing, Lin and Han (2022) revised \( \xi_n \) via incorporating more than one right nearest neighbor to its construction. Assuming independence between \( X \) and \( Y \) and some assumptions on \( F_{X,Y} \), Lin and Han (2022, Theorem 3.2) established the following CLT for their correlation coefficient \( \xi_{n,M} \) (with \( M \) representing the number of right NNs to be included):

\[
\sqrt{nM}\xi_{n,M} \rightarrow N(0, 2/5) \quad \text{in distribution},
\]

as long as \( M \) is increasing at a certain rate.

All the above CLTs only hold when \( Y \) is independent of \( X \). The following papers, on the other hand, studied the statistics’ behavior when \( Y \) is possibly dependent on \( X \). They, however, can only handle local alternatives, i.e., such distributions where the dependence between \( X \) and \( Y \) is so weak that \( F_{X,Y} \) is very close to \( F_X F_Y \).

(7) Assuming \( \xi = \xi^{(n)} \rightarrow 0 \) as \( n \rightarrow \infty \) at a certain rate, Auddy et al. (2021, Theorem 2.3) showed

\[
\sqrt{n}(\xi_n - \xi^{(n)}) \rightarrow N(0, 2/5) \quad \text{in distribution}.
\]

(8) For quadratic mean differentiable (QMD) classes of alternatives to the null independence one, Cao and Bickel (2020), Shi et al. (2021b), Auddy et al. (2021), and Lin and Han (2022), although reducing to it under independence; see Remark 1.8 ahead for more discussions about this point.

1.2 Proof sketch

To establish Theorem 1.1, the first and most important step is to find the correct forms of Hájek representations (Hájek et al., 1999) for \( \xi_n \) and \( \xi_n^* \) with regard to a general distribution function \( F_{X,Y} \) that is not necessary equal to \( F_X F_Y \). This step is technically highly challenging as we have to carefully monitor the dependence between \( X \) and \( Y \); it shall occupy most of the rest paper. Interestingly, the newly found Hájek representation is distinct from that used in Deb et al. (2020), Cao and Bickel (2020), Shi et al. (2021b), Auddy et al. (2021), and Lin and Han (2022), although reducing to it under independence; see Remark 1.8 ahead for more discussions about this point.

For sketching the proof of Theorem 1.1, let us first introduce some necessary notation. For any \( t \in \mathbb{R} \), define

\[
G_X(t) := \mathbb{P}(Y \geq t \mid X) \quad \text{and} \quad h(t) := \mathbb{E}[G_X^2(t)].
\]

Ahead we will show that the Hájek representations of \( \xi_n \) and \( \xi_n^* \) take the forms

\[
\xi_n^* := \frac{6n}{2 - 1^n} \left( \sum_{i=1}^n \min \{F_Y(Y_i), F_Y(Y_{N(i)})\} + \sum_{i=1}^n h(Y_i) \right) \quad \text{(1.14)}
\]
and
\[ \tilde{\xi}_n^* := \frac{6n}{n^2 - 1} \left( \sum_{i=1}^{n} \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \} + \sum_{i=1}^{n} h(Y_i) \right). \quad (1.15) \]

Why so? Below we give some intuition. Let us use \( \wedge \) to represent the minimum of two numbers and focus on \( \xi_n \) as the analysis for \( \tilde{\xi}_n \) is identical. From (1.2), \( \xi_n \) takes the form
\[
\sum_{i=1}^{n} [R_i \wedge R_{N(i)}] \quad (1.16)
\]
and a natural component of its Hájek representation shall be
\[
\sum_{i=1}^{n} [F_Y(Y_i) \wedge F_Y(Y_{N(i)})], \quad (1.17)
\]
which is via replacing the empirical distribution by the population one. We use Hájek projection (van der Vaart, 1998, Lemma 11.10) to find the remaining component via checking the difference between (1.16) and (1.17).

Fix an integer \( k \in [1, n] \) and consider the projection of (1.16) on \( (X_k, Y_k) \). From the definition of ranks, we have
\[
R_i \wedge R_{N(i)} = \sum_{j=1}^{n} 1(Y_j \leq Y_i \wedge Y_{N(i)})
= 1(Y_k \leq Y_i \wedge Y_{N(i)}) + \sum_{j=1,j \neq k}^{n} 1(Y_j \leq Y_i \wedge Y_{N(i)}).
\]

Then \( \xi_n \), of the form \( n^{-2} \sum_{i=1}^{n} [R_i \wedge R_{N(i)}] \), can be decomposed as the summation of the following two terms:
\[
\sum_{i=1}^{n} \sum_{j=1,j \neq k}^{n} 1(Y_j \leq Y_i \wedge Y_{N(i)}) \quad \text{and} \quad \sum_{i=1}^{n} 1(Y_k \leq Y_i \wedge Y_{N(i)}).
\]

For the first term, since \( j \neq k \), \( (X_j, Y_j) \) is independent of \( (X_k, Y_k) \) and hence
\[
\mathbb{E} \left[ n^{-2} \sum_{i=1, j=1,j \neq k}^{n} 1(Y_j \leq Y_i \wedge Y_{N(i)}) \mid X_k, Y_k \right] \approx \mathbb{E} \left[ n^{-1} \sum_{i=1}^{n} F_{Y}(Y_i \wedge Y_{N(i)}) \mid X_k, Y_k \right],
\]
which corresponds exactly to the “natural component of the Hájek representation” (1.17) when projected to \( (X_k, Y_k) \).

What about the second term in (1.18)? Notice that when the sample size is sufficiently large, the NN distance is small, and hence for any \( k \neq 1 \),
\[
\mathbb{E} \left[ n^{-2} \sum_{i=1}^{n} 1(Y_k \leq Y_i \wedge Y_{N(i)}) \mid X_k, Y_k \right] \approx n^{-1} \mathbb{E} \left[ 1(Y_k \leq Y_1 \wedge Y_{N(1)}) \mid X_k, Y_k \right]
\approx n^{-1} \mathbb{E} \left[ 1(Y_k \leq Y_1 \wedge \tilde{Y_1}) \mid X_k, Y_k \right],
\]
where \( \tilde{Y_1} \) is sampled independently from the conditional distribution of \( Y \) given \( X_1 \). By the definition
of the function $h(\cdot)$ in (1.13),

$$E[\mathbb{1}(Y_k \leq Y_1 \wedge \tilde{Y}_1) | X_k, Y_k] = h(Y_k).$$

Then using the Hájek projection, the difference between

$$n^{-2} \sum_{i=1}^{n} [R_i \wedge R_{N(i)}] \text{ and } n^{-1} \sum_{i=1}^{n} [F_{Y}(Y_i) \wedge F_{Y}(Y_{N(i)})]$$

after projection into sums is $n^{-1} \sum_{k=1}^{n} h(Y_k)$ up to a constant. This gives rise to (1.14).

In detail, we have the following theorem.

**Theorem 1.3** (Hájek representation). It holds true (for any fixed continuous $F_{X,Y}$) that

$$\lim_{n \to \infty} \left\{ n \text{Var}[\xi_n - \xi_n^\star] \right\} = 0 \quad \text{and} \quad \lim_{n \to \infty} \left\{ n \text{Var}[\bar{\xi}_n - \bar{\xi}_n^\star] \right\} = 0 \quad (\text{if } d = 1).$$

Using Theorem 1.3, as long as $n \liminf_{n \to \infty} \text{Var}[\xi_n] > 0$, normalized $\xi_n$ ($\bar{\xi}_n^\star$) and $\xi_n^\star$ ($\bar{\xi}_n^\star$) share the same asymptotic distribution and it suffices to establish the CLT for $\xi_n^\star$ ($\bar{\xi}_n^\star$). In the second step, we establish the CLT of $\xi_n^\star$ and $\bar{\xi}_n^\star$ by noticing that it merely consists of a linear sum of nearest neighbor statistics. Leveraging the normal approximation theorem under local dependence (Chatterjee, 2008), one can then reach the following two CLTs.

**Theorem 1.4.** As long as $Y$ is not a measurable function of $X$ almost surely, it holds true (for any fixed continuous $F_{X,Y}$) that

$$\frac{(\xi_n^\star - E[\xi_n^\star])}{\sqrt{\text{Var}[\xi_n^\star]}} \to N(0,1) \quad \text{in distribution,} \quad (1.19)$$

and

$$\frac{(\bar{\xi}_n^\star - E[\bar{\xi}_n^\star])}{\sqrt{\text{Var}[\bar{\xi}_n^\star]}} \to N(0,1) \quad \text{in distribution.}$$

**Remark 1.7.** Of note, in conducting global sensitivity analysis via the first-order Sobol indices, Gamboa et al. (2022, Theorem 4.1) obtained a CLT similar to (1.19) above. Both results, however, do not have to handle the randomness from ranking $Y_i$’s that we addressed in Theorem 1.3 and is to us the most difficult part.

Finally, Theorem 1.1 is proved by combining Theorems 1.3 and 1.4.

**Remark 1.8.** The Hájek representation of $\xi_n$ under independence between $X$ and $Y$ was established in, e.g., Deb et al. (2020, Lemma D.1), Cao and Bickel (2020, Equ. (4.9)), Shi et al. (2021a, Lemma 7.1), and Lin and Han (2022, Remark 3.2). See also Auddy et al. (2021, Theorem 2.1). The remaining component there is a U-statistic of the form

$$-\frac{1}{n(n-1)} \sum_{i \neq j} F_{Y}(Y_i \wedge Y_j). \quad (1.20)$$

Using standard U-statistic theory (van der Vaart, 1998, Theorem 12.3), the main term of (1.20) is

$$-n^{-1} \sum_{i=1}^{n} \left( 2F_{Y}(Y_i) - F_{Y}^{2}(Y_i) - \frac{1}{3} \right). \quad (1.21)$$
Noticing that \( E[G_X(\cdot)] = 1 - F_Y(\cdot) \), we have
\[
h(\cdot) = \text{Var}[G_X^2(\cdot)] + (E[G_X(\cdot)])^2 = \text{Var}[G_X^2(\cdot)] - (2F_Y(\cdot) - F_Y^2(\cdot)) + 1.
\]
Under the null, one is then ready to check \( \text{Var}[G_X^2(\cdot)] = 0 \), and thus \( h(\cdot) \) reduces to (1.21) (up to some constants).

## 2 Proof of the main results

**Notation.** For any integers \( n, d \geq 1 \), let \( [n] := \{1, 2, \ldots, n\} \), and \( \mathbb{R}^d \) be the \( d \)-dimensional real space. A set consisting of distinct elements \( x_1, \ldots, x_n \) is written as either \( \{x_1, \ldots, x_n\} \) or \( \{x_i\}_{i=1}^n \), and its cardinality is written by \( |\{x_i\}_{i=1}^n| \). The corresponding sequence is denoted by \( [x_1, \ldots, x_n] \) or \( [x_i]_{i=1}^n \). For any two real sequences \( \{a_n\} \) and \( \{b_n\} \), write \( a_n \lesssim b_n \) (or equivalently, \( b_n \gtrsim a_n \)) if there exists a universal constant \( C > 0 \) such that \( a_n/b_n \leq C \) for all sufficiently large \( n \), and write \( a_n \prec b_n \) (or equivalently, \( b_n \succ a_n \)) if \( a_n/b_n \to 0 \) as \( n \) goes to infinity. Write \( a_n = O(b_n) \) if \( |a_n| \lesssim b_n \) and \( a_n = o(b_n) \) if \( |a_n| \prec b_n \). We shorthand \( (X_1, \ldots, X_n) \) by \( X \). We use \( \overset{d}{\to} \) and \( \overset{p}{\to} \) to denote convergences in distribution and in probability, respectively.

**Proof of Theorem 1.1.** From Proposition 1.2 and Theorem 1.3,
\[
\limsup_{n \to \infty} E \left[ \frac{\xi_n^* - E[\xi_n^*]}{\sqrt{\text{Var}[\xi_n^*]}} - \frac{\xi_n - E[\xi_n]}{\sqrt{\text{Var}[\xi_n]}} \right]^2 = \limsup_{n \to \infty} \frac{\text{Var}[\xi_n - \xi_n^*]}{\text{Var}[\xi_n]} \leq \frac{\limsup_{n \to \infty} n \text{Var}[\xi_n - \xi_n^*]}{\liminf_{n \to \infty} n \text{Var}[\xi_n]} = 0,
\]
and
\[
\limsup_{n \to \infty} \left| \frac{\text{Cov}[\xi_n, \xi_n - \xi_n^*]}{\text{Var}[\xi_n]} \right| \leq \limsup_{n \to \infty} \left( \frac{\text{Var}[\xi_n - \xi_n^*]}{\text{Var}[\xi_n]} \right)^{\frac{1}{2}} \leq \left( \frac{\limsup_{n \to \infty} n \text{Var}[\xi_n - \xi_n^*]}{\liminf_{n \to \infty} n \text{Var}[\xi_n]} \right)^{\frac{1}{2}} = 0.
\]
One can then deduce
\[
\frac{\xi_n^* - E[\xi_n^*]}{\sqrt{\text{Var}[\xi_n^*]}} - \frac{\xi_n - E[\xi_n]}{\sqrt{\text{Var}[\xi_n]}} \overset{p}{\to} 0 \quad \text{and} \quad \frac{\text{Var}[\xi_n^*]}{\text{Var}[\xi_n]} \to 1.
\]
We then complete the proof for \( \xi_n \) by using Theorem 1.4. The proof for \( \bar{\xi}_n \) can be established in the same way.

For better readability, we defer the proof of Theorem 1.2 to the end of this section.

**Proof of Theorem 1.3.** We first introduce some necessary notation for the proof.

For any \( t \in \mathbb{R} \), recall \( G_X(t) = P(Y \geq t \mid X) \) and define
\[
G(t) := P(Y \geq t) = 1 - F_Y(t), \quad g(t) := \text{Var} \left[ G_X(t) \right] = E \left[ G_X^2(t) \right] - G^2(t).
\]
For any \( x \in \mathbb{R}^d \), define
\[
h_0(x) := E[h(Y) \mid X = x] = \int E[G_X^2(t)]dF_Y \mid X=x(t),
\]
where $F_{Y|X=x}$ is the conditional distribution of $Y$ conditional on $X = x$.

We then introduce an intermediate statistic $\tilde{\xi}_n$ as follows,

$$\tilde{\xi}_n := \frac{6n}{n^2 - 1} \left( \sum_{i=1}^{n} \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \} - \frac{1}{n-1} \sum_{i,j=1 \atop i \neq j}^{n} \min \{ F_Y(Y_i), F_Y(Y_j) \} \right) \quad (2.3)$$

$$+ \sum_{i=1}^{n} g(Y_i) + \frac{1}{n-1} \sum_{i,j=1 \atop i \neq j}^{n} E \left[ \min \{ F_Y(Y_i), F_Y(Y_j) \} \mid X_i, X_j \right]$$

$$- \sum_{i=1}^{n} E \left[ g(Y_i) \mid X_i \right] + \sum_{i=1}^{n} h_0(X_i).$$

Notice that

$$\text{Var}[\xi_n - \xi_n^*] = \text{Var}[\xi_n - \tilde{\xi}_n] + \text{Var}[\tilde{\xi}_n - \xi_n^*] + 2 \text{Cov}[\xi_n - \tilde{\xi}_n, \tilde{\xi}_n - \xi_n^*]$$

$$\leq \text{Var}[\xi_n - \tilde{\xi}_n] + \text{Var}[\tilde{\xi}_n - \xi_n^*] + 2(\text{Var}[\xi_n - \tilde{\xi}_n] \text{Var}[\tilde{\xi}_n - \xi_n^*])^{1/2}.\quad (2.4)$$

As long as

$$\lim_{n \to \infty} n \text{Var}[\xi_n - \tilde{\xi}_n] = 0 \quad \text{and} \quad \lim_{n \to \infty} n \text{Var}[\tilde{\xi}_n - \xi_n^*] = 0,$$

the proof for $\xi_n$ is complete. The proof for $\tilde{\xi}_n$ is similar and accordingly omitted.

For the first equation in (2.4), by the law of total variance, one can decompose $\text{Var}[\xi_n - \tilde{\xi}_n]$ as follows,

$$n \text{Var}[\xi_n - \tilde{\xi}_n] = nE[\text{Var}[\xi_n - \tilde{\xi}_n \mid X]] + n \text{Var}[E[\xi_n - \tilde{\xi}_n \mid X]].$$

**Step I.** $\lim_{n \to \infty} nE[\text{Var}[\xi_n - \tilde{\xi}_n \mid X]] = 0$.

We decompose $nE[\text{Var}[\xi_n - \tilde{\xi}_n \mid X]]$ as:

$$nE[\text{Var}[\xi_n - \tilde{\xi}_n \mid X]] = nE[\text{Var}[\xi_n \mid X]] + nE[\text{Var}[\tilde{\xi}_n \mid X]] - 2nE[\text{Cov}[\xi_n, \tilde{\xi}_n \mid X]].\quad (2.5)$$

For the first term in (2.5), using (1.2), we have

$$n \text{Var}[\xi_n \mid X]$$

$$= \frac{36n}{(n^2 - 1)^2} \text{Var} \left[ \sum_{i=1}^{n} \min \{ R_i, R_{N(i)} \} \mid X \right]$$

$$= \frac{36n^4}{(n^2 - 1)^2} \left\{ \frac{1}{n^3} \sum_{i=1}^{n} \text{Var} \left[ \min \{ R_i, R_{N(i)} \} \mid X \right] \right\}$$

$$+ \frac{1}{n^2} \sum_{j=N(i), i \neq N(i) \atop \text{or } i=N(j), j \neq N(i)} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \mid X \right]$$

$$+ \frac{1}{n^3} \sum_{i \neq j \atop N(i)=N(j)} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \mid X \right].$$
\[ + \frac{1}{n^3} \sum_{j=N(i), j=N(j)} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \bigg| X \right] \]

\[ + \frac{1}{n^3} \sum_{i, j, N(i), N(j) \text{ distinct}} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \bigg| X \right] \]

\[ =: \frac{36n^4}{(n^2 - 1)^2} \left( T_1 + T_2 + T_3 + T_4 + T_5 \right). \]

For the second term in (2.5), noticing that the last three terms in (2.3) are constants conditional on \( X \), we have

\[ n \text{Var}[\xi_n \mid X] \]

\[ = \frac{36n^3}{(n^2 - 1)^2} \text{Var} \left[ \sum_{i=1}^n \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \} - \frac{1}{n-1} \sum_{i,j=1}^n \min \{ F_Y(Y_i), F_Y(Y_j) \} \right] \]

\[ + \frac{1}{n} \sum_{j=N(i), i \neq N(j) \text{ or } i=N(j), j \neq N(i)} \text{Cov} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \min \{ F_Y(Y_j), F_Y(Y_{N(j)}) \} \bigg| X \right] \]

\[ + \frac{1}{n} \sum_{i \neq j} \text{Cov} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \min \{ F_Y(Y_j), F_Y(Y_{N(j)}) \} \bigg| X \right] \]

\[ + \frac{1}{n} \sum_{j=N(i), i=N(j)} \text{Cov} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \min \{ F_Y(Y_j), F_Y(Y_{N(j)}) \} \bigg| X \right] \]

\[ + \frac{1}{n} \sum_{i,j,N(i),N(j) \text{ distinct}} \text{Cov} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \min \{ F_Y(Y_j), F_Y(Y_{N(j)}) \} \bigg| X \right] \]

\[ - \frac{2}{n(n-1)} \sum_{i=1}^n \text{Cov} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \sum_{i,j=1}^n \min \{ F_Y(Y_i), F_Y(Y_j) \} \bigg| X \right] \]

\[ + \frac{1}{n(n-1)^2} \text{Var} \left[ \sum_{i,j=1}^n \min \{ F_Y(Y_i), F_Y(Y_j) \} \bigg| X \right] \]

\[ + \frac{2}{n} \sum_{i=1}^n \text{Cov} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \sum_{i=1}^n g(Y_i) \bigg| X \right] \]

\[ - \frac{2}{n(n-1)} \text{Cov} \left[ \sum_{i,j=1}^n \min \{ F_Y(Y_i), F_Y(Y_j) \}, \sum_{i=1}^n g(Y_i) \bigg| X \right] \]
\[
+ \frac{1}{n} \operatorname{Var}\left[ \sum_{i=1}^{n} g(Y_i) \middle| X \right] \\
= \frac{36n^4}{(n^2-1)^2} \left( T_1^* + T_2^* + T_3^* + T_4^* + T_5^* - 2T_6^* + T_7^* + 2T_8^* - 2T_9^* + T_{10}^* \right) 
\]

For the third term in (2.5), from (1.2) and (2.3), we have
\[
\begin{align*}
&\quad n \operatorname{Cov}[\xi_n, \bar{\xi}_n \mid X] \\
&= \frac{36n^2}{(n^2-1)^2} \operatorname{Cov}\left[ \sum_{i=1}^{n} \min \{ R_i, R_{N(i)} \}, \sum_{i=1}^{n} \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \} \mid X \right] \\
&\quad - \frac{1}{n-1} \sum_{i,j=1 \atop i \neq j}^{n} \min \{ F_Y(Y_i), F_Y(Y_j) \} + \sum_{i=1}^{n} g(Y_i) \mid X \\
&= \frac{36n^4}{(n^2-1)^2} \left\{ \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{Cov}\left[ \min \{ R_i, R_{N(i)} \}, \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \} \mid X \right] \\
&\quad + \frac{1}{n^2} \sum_{j=N(i), i \neq N(j) \text{ or } i=N(j), j \neq N(i)} \operatorname{Cov}\left[ \min \{ R_i, R_{N(i)} \}, \min \{ F_Y(Y_j), F_Y(Y_{N(j)}) \} \mid X \right] \\
&\quad + \frac{1}{n^2} \sum_{j=N(i), i = N(j)} \operatorname{Cov}\left[ \min \{ R_i, R_{N(i)} \}, \min \{ F_Y(Y_j), F_Y(Y_{N(j)}) \} \mid X \right] \\
&\quad + \frac{1}{n^2} \sum_{i,j,N(i),N(j) \text{ distinct}} \operatorname{Cov}\left[ \min \{ R_i, R_{N(i)} \}, \min \{ F_Y(Y_i), F_Y(Y_{j}) \} \mid X \right] \\
&\quad - \frac{1}{n^2(n-1)} \sum_{i=1}^{n} \operatorname{Cov}\left[ \min \{ R_i, R_{N(i)} \}, \sum_{i,j=1 \atop i \neq j}^{n} \min \{ F_Y(Y_i), F_Y(Y_j) \} \mid X \right] \\
&\quad + \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{Cov}\left[ \min \{ R_i, R_{N(i)} \}, \sum_{i=1}^{n} g(Y_i) \mid X \right] \\
&= \frac{36n^4}{(n^2-1)^2} \left( T_1^* + T_2^* + T_3^* + T_4^* + T_5^* - T_6^* + T_7^* \right).
\end{align*}
\]

Let \( Y, \tilde{Y} \sim F_Y, \tilde{Y}_1, \tilde{Y}_1' \sim F_{Y_1 \mid X=x_1}, \tilde{Y}_2 \sim F_{Y_2 \mid X=x_2} \) be mutually independently drawn. We then establish the following five lemmas that control the terms of (2.6)-(2.8).

**Lemma 2.1.** For \( i = 1, 2, 3, 4, \)
\[
\lim_{n \to \infty} \left| \operatorname{E}[T_i] - \operatorname{E}[T_i^*] \right| = 0, \quad \lim_{n \to \infty} \left| \operatorname{E}[T_i'] - \operatorname{E}[T_i'^*] \right| = 0,
\]
and
\[
\lim_{n \to \infty} \left| \operatorname{E}[T_1] - \operatorname{E}[\operatorname{Var}\left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right]] \right| = 0,
\]

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\[ \lim_{n \to \infty} E[T_2] - 2E\left[ \text{Cov}\left(F_Y(Y_1 \wedge \bar{Y}_1), F_Y(\bar{Y}_1 \wedge \bar{Y}_1') \mid X_1\right) \mathbb{I}\left(1 \neq N(N(1))\right) \right] = 0, \]
\[ \lim_{n \to \infty} E[T_3] - E\left[ \text{Cov}\left(F_Y(Y_1 \wedge \bar{Y}_1), F_Y(\bar{Y}_1 \wedge \bar{Y}_1') \mid X_1\right) \mathbb{I}\left\{j : j \neq 1, N(j) = N(1)\right\} \right] = 0, \]
\[ \lim_{n \to \infty} E[T_4] - E\left[ \text{Var}\left(F_Y(Y_1 \wedge \bar{Y}_1) \mid X_1\right) \mathbb{I}\left(1 = N(N(1))\right) \right] = 0. \]

**Lemma 2.2.**
\[ \lim_{n \to \infty} \left[ E[T_5] - 2E[T_5'] \right] = E\left[ \text{Cov}\left(\mathbb{I}\left(Y_3 \leq Y_1 \wedge \bar{Y}_1\right), \mathbb{I}\left(Y_3 \leq Y_2 \wedge \bar{Y}_2\right) \mid X_1, X_2, X_3\right) \right] =: a_1, \]
\[ E[T_5] = 0, \quad \text{and} \quad \lim_{n \to \infty} E\left[T_5'\right] - 2E\left[ \text{Cov}\left(\mathbb{I}\left(Y_2 \leq Y_1 \wedge \bar{Y}_1\right), F_Y(Y_2 \wedge \bar{Y}_2) \mid X_1, X_2\right) \right] = 0. \]

**Lemma 2.3.**
\[ \lim_{n \to \infty} \left[ E[T_6] - E[T_6'] \right] = 2E\left[ \text{Cov}\left(\mathbb{I}\left(Y_2 \leq Y_1 \wedge \bar{Y}_1\right), F_Y(Y_2 \wedge Y) \mid X_1, X_2\right) \right] =: 2a_2. \]

**Lemma 2.4.**
\[ \lim_{n \to \infty} E[T_7'] = 4E\left[ \text{Cov}\left(F_Y(Y_1 \wedge Y), F_Y(Y_1 \wedge \bar{Y}) \mid X_1\right) \right] =: 4a_3. \]

**Lemma 2.5.**
\[ \lim_{n \to \infty} E[T_8] = 2E\left[ \text{Cov}\left(F_Y(Y_1 \wedge \bar{Y}_1), g(Y_1) \mid X_1\right) \right] =: 2b_1, \]
\[ \lim_{n \to \infty} E[T_9] = 2E\left[ \text{Cov}\left(F_Y(Y_1 \wedge Y), g(Y_1) \mid X_1\right) \right] =: 2b_2, \]
\[ \lim_{n \to \infty} E[T_7] = E\left[ \text{Cov}\left(\mathbb{I}\left(Y_2 \leq Y_1 \wedge \bar{Y}_1\right), g(Y_2) \mid X_1, X_2\right) + 2E\left[ \text{Cov}\left(F_Y(Y_1 \wedge \bar{Y}_1), g(Y_1) \mid X_1\right) \right] \right] =: b_3, \]
\[ \lim_{n \to \infty} E[T_{10}'] = E\left[ \text{Var}\left(g(Y_1) \mid X_1\right) \right]. \]

Plugging (2.6)-(2.8) to (2.5) and using Lemmas 2.1-2.5, one obtains
\[ \lim_{n \to \infty} nE[\text{Var}[\xi_n - \tilde{\xi}_n \mid X]] = 36 \lim_{n \to \infty} E\left[ \sum_{i=1}^{4} \left( T_i + T_i^* - 2T_i' \right) + \left( T_5 + T_5^* - 2T_5' \right) - 2\left( T_6^* - T_6' \right) \right. \]
\[ \left. + T_7^* + 2T_8^* - 2T_9^* - 2T_7' + T_{10}^* \right] \]
\[ = 36\left( a_1 + 4a_2 + 4a_3 - 2\left( b_3 - 2b_1 + 2b_2 \right) + E\left[ \text{Var}\left[g(Y_1) \mid X_1\right] \right] \right). \]

For the relationship of \(a_1, a_2, a_3\) and \(b_1, b_2, b_3\), we establish the following identity.

**Lemma 2.6 (A key identity).** We have
\[ a_1 + 4a_2 + 4a_3 = b_3 - 2b_1 + 2b_2 = E\left[ \text{Var}\left[g(Y_1) \mid X_1\right] \right]. \]

Combining Lemma 2.6 with (2.9) proves
\[ \lim_{n \to \infty} nE[\text{Var}[\xi_n - \tilde{\xi}_n \mid X]] = 0. \]
Step II. lim_{n \to \infty} n \text{Var}[E[\xi_n | X]] = 0.

Checking (1.2), one has

\[
E[\xi_n | X] = E\left[\frac{6}{n^2 - 1} \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) - \frac{2n+1}{n-1} \middle| X\right]
\]

\[
= \frac{6}{n^2 - 1} \sum_{i=1}^{n} \sum_{k=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] - \frac{2n+1}{n-1}.
\]

Checking (2.3), one has

\[
E[\xi_n | X] = \frac{6n}{n^2 - 1} \left( \sum_{i=1}^{n} E\left[F_Y(Y_i \cap Y_{N(i)}) \middle| X\right] + \sum_{i=1}^{n} h_0(X_i) \right).
\]

Consequently, we obtain

\[
\text{Var}[E[\xi_n - \tilde{\xi}_n | X]] = \frac{36n^2}{(n^2 - 1)^2} \cdot \text{Var}\left[\sum_{i=1}^{n} \sum_{k=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] - \sum_{i=1}^{n} E\left[F_Y(Y_i \cap Y_{N(i)}) \middle| X\right] - \sum_{i=1}^{n} h_0(X_i) \right].
\]

To apply the Efron-Stein inequality (Theorem 3.1 in Boucheron et al. (2013)), recall \( X = (X_1, \ldots, X_n) \) and define, for any \( \ell \in [n] \),

\[
X_\ell := (X_1, \ldots, X_{\ell-1}, \tilde{X}_\ell, X_{\ell+1}, \ldots, X_n),
\]

where \([\tilde{X}_\ell]_\ell=1^n\) are independent copies of \([X_\ell]_\ell=1^n\).

We fix one \( \ell \in [n] \). For any \( i \in [n] \), let \( \tilde{N}(i) \) be the index of the NN of \( i \) in \( X_\ell \).

For the first term in (2.11), we first decompose it as

\[
\sum_{i=1}^{n} \sum_{k=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] = \sum_{i=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] + \sum_{i=1}^{n} \sum_{k=1, k \neq \ell} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right].
\]

Notice that \( E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] \) only depends on \( X_k, X_i, X_{N(i)} \). Then for any \( i \in [n] \) such that \( i \neq \ell, N(i) \neq \ell, \tilde{N}(i) \neq \ell \), we have \( N(i) = \tilde{N}(i) \), and then

\[
\sum_{k=1, k \neq \ell} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] - \sum_{k=1, k \neq \ell} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{\tilde{N}(i)}) \middle| X_\ell\right] = 0.
\]

One then has

\[
\sum_{i=1}^{n} \sum_{k=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] - \sum_{i=1}^{n} \sum_{k=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{\tilde{N}(i)}) \middle| X_\ell\right]
\]

\[
= \sum_{i=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{N(i)}) \middle| X\right] - \sum_{i=1}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \cap Y_{\tilde{N}(i)}) \middle| X_\ell\right]
\]
For the third term in (2.11), noticing that \( E[F_Y(Y_i \wedge Y_{N(i)}) \mid X] \) only depends on \( X_i, X_{N(i)} \), we have

\[
\sum_{i=1}^{n} E\left[F_Y(Y_i \wedge Y_{N(i)}) \mid X\right] - \sum_{i=1}^{n} E\left[F_Y(Y_i \wedge Y_{N(i)}) \mid X_\ell\right] = E\left[F_Y(Y_\ell \wedge Y_{N(\ell)}) \mid X\right] - E\left[F_Y(Y_\ell \wedge Y_{N(\ell)}) \mid X_\ell\right] + \sum_{i=1}^{n} \left[E\left[F_Y(Y_i \wedge Y_{N(i)}) \mid X\right] - E\left[F_Y(Y_i \wedge Y_{N(i)}) \mid X_\ell\right]\right].
\]

For the second term in (2.11), we have

\[
\sum_{i=1}^{n} h_0(X_i) - \sum_{i=1, i \neq \ell}^{n} h_0(X_i) - h_0(\tilde{X}_\ell) = h_0(X_\ell) - h_0(\tilde{X}_\ell).
\]

Plugging (2.12)-(2.14) to (2.11) and using the Efron-Stein inequality then yields

\[
n \text{Var}[\mathcal{E}[\xi_n - \xi_n \mid X]] \\
\leq \frac{18n^3}{(n^2 - 1)^2} \sum_{\ell=1}^{n} E\left\{ \frac{1}{n} \sum_{i=1}^{n} E\left[\mathbb{I}(Y_\ell \leq Y_i \wedge Y_{N(i)}) \mid X\right] - h_0(X_\ell) \right\}
\]

\[
+ \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E\left[\mathbb{I}(Y_k \leq Y_\ell \wedge Y_{N(\ell)}) \mid X\right] - E\left[F_Y(Y_\ell \wedge Y_{N(\ell)}) \mid X\right]
\]

\[
+ \frac{1}{n} \sum_{\ell=1}^{n} \sum_{k=1, k \neq \ell}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \wedge Y_{N(i)}) \mid X\right] - \sum_{i=1}^{n} E\left[F_Y(Y_i \wedge Y_{N(i)}) \mid X\right]
\]

\[
- \frac{1}{n} \sum_{i=1}^{n} E\left[\mathbb{I}(Y_i \leq Y_\ell \wedge Y_{N(i)}) \mid X_\ell\right] + h_0(\tilde{X}_\ell)
\]

\[
- \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E\left[\mathbb{I}(Y_k \leq Y_\ell \wedge Y_{N(\ell)}) \mid X_\ell\right] + E\left[F_Y(Y_\ell \wedge Y_{N(\ell)}) \mid X_\ell\right]
\]

\[
- \frac{1}{n} \sum_{\ell=1}^{n} \sum_{k=1, k \neq \ell}^{n} E\left[\mathbb{I}(Y_k \leq Y_i \wedge Y_{N(i)}) \mid X_\ell\right] + \sum_{i=1}^{n} E\left[F_Y(Y_i \wedge Y_{N(i)}) \mid X_\ell\right]
\]

\[
\leq \frac{72n^4}{(n^2 - 1)^2} E\left\{ \frac{1}{n} \sum_{i=1}^{n} E\left[\mathbb{I}(Y_\ell \leq Y_i \wedge Y_{N(i)}) \mid X\right] - h_0(X_\ell) \right\}
\]

\[15\]
\[ + \frac{1}{n} \sum_{k=1 \atop k \neq \ell}^{n} \mathbb{E} \left[ \mathbb{I} (Y_k \leq Y_\ell \land Y_{N(\ell)}) \mid \mathbf{X} \right] - \mathbb{E} \left[ F_Y (Y_\ell \land Y_{N(\ell)}) \mid \mathbf{X} \right] \]

\[ + \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1 \atop k \neq \ell}^{n} \mathbb{E} \left[ \mathbb{I} (Y_k \leq Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] - \sum_{i=1}^{n} \mathbb{E} \left[ F_Y (Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] \]

\[ \leq \frac{216n^4}{(n^2 - 1)^2} \left\{ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{I} (Y_\ell \leq Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] - h_0 (X_\ell) \right]^2 \right\} + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \mathbb{I} (Y_\ell \leq Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] - \mathbb{E} \left[ F_Y (Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] \right]^2 \]

\[ + \mathbb{E} \left[ \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{k=1 \atop k \neq i}^{n} \mathbb{E} \left[ \mathbb{I} (Y_k \leq Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] - \mathbb{E} \left[ F_Y (Y_i \land Y_{N(i)}) \mid \mathbf{X} \right] \right)^2 \right] \]

\[ =: \frac{216n^4}{(n^2 - 1)^2} (\overline{T}_1 + \overline{T}_2 + \overline{T}_3); \]

recall that \([X_i]_{i=1}^{n}\) are independent and identically distributed (i.i.d.), and \([\overline{X}_i]_{i=1}^{n}\) are independent copies of \([X_\ell]_{\ell=1}^{n}\).

We then establish the following three lemmas.

**Lemma 2.7.** \( \lim_{n \to \infty} \overline{T}_1 = 0. \)

**Lemma 2.8.** \( \lim_{n \to \infty} \overline{T}_2 = 0. \)

**Lemma 2.9.** \( \lim_{n \to \infty} \overline{T}_3 = 0. \)

Applying Lemmas 2.7-2.9 to (2.15) yields

\[ \lim_{n \to \infty} n \operatorname{Var} [\xi_n - \tilde{\xi}_n \mid \mathbf{X}] = 0. \quad (2.16) \]

**Step III.** \( \lim_{n \to \infty} n \operatorname{Var} [\tilde{\xi}_n - \xi^*_n] = 0. \)

By the definition of \( \tilde{\xi}_n \) in (2.3), one has

\[ \tilde{\xi}_n = \frac{6n^2}{n^2 - 1} \left( \frac{1}{n} \sum_{i=1}^{n} \min \left\{ F_Y (Y_i), F_Y (Y_{N(i)}) \right\} - \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \min \left\{ F_Y (Y_i), F_Y (Y_j) \right\} \right) \]

\[ + \frac{1}{n} \sum_{i=1}^{n} g(Y_i) + \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \mathbb{E} \left[ \min \left\{ F_Y (Y_i), F_Y (Y_j) \right\} \mid X_i, X_j \right] \]

\[ - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ g(Y_i) \mid X_i \right] + \frac{1}{n} \sum_{i=1}^{n} h_0 (X_i). \]

Notice that \( \tilde{\xi}_n \) consists of U-statistic terms. For any \( x \in \mathbb{R}^d \) and \( t \in \mathbb{R} \), define

\[ \tilde{h}(t) := 2 \mathbb{E} \left[ \min \left\{ F_Y (Y), F_Y (t) \right\} \right] - \frac{1}{3} \quad \text{and} \quad \tilde{h}_0 (x) := 2 \mathbb{E} \left[ \min \left\{ F_Y (Y), F_Y (Y_x) \right\} \right] - \frac{1}{3}. \]
where \( Y \sim F_Y, X \sim F_Y|X=x \) and are independent. Using the probability integral transform and the boundedness of \( F_Y \),

\[
\begin{align*}
\mathbb{E}
\left[
\min \{ F_Y(Y_1), F_Y(Y_2) \}
\right] &= 1/3, \\
\mathbb{E}
\left[
\min \{ F_Y(Y_1), F_Y(Y_2) \} \mid X_1, X_2
\right] &= 1/3, \\
\mathbb{E}
\left[
\min \{ F_Y(Y_1), F_Y(Y_2) \}^2
\right] &\leq 1, \\
\mathbb{E}
\left[
\mathbb{E}
\left[
\min \{ F_Y(Y_1), F_Y(Y_2) \} \mid X_1, X_2
\right]^2
\right] &\leq 1.
\end{align*}
\]

Then the standard U-statistic Hájek projection (van der Vaart, 1998, Theorem 12.3) gives

\[
\sqrt{n} \xi_n = \frac{6n^2}{n^2 - 1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} F_Y(Y_i \wedge Y_{N(i)}) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{h}(Y_i) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(Y_i) \right) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{h}_0(X_i) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}[g(Y_i) \mid X_i] + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h_0(X_i) + Q,
\]

(2.17)

with \( \mathbb{E}[Q^2] \lesssim n^{-1} \).

Notice that for \( \tilde{h} \) and \( \tilde{h}_0, F_Y(Y) \) follows a uniform distribution on \([0,1]\) with \( Y \sim F_Y \). Then it is ready to check

\[
\tilde{h}(t) = 2F_Y(t) - F_Y^2(t) - \frac{1}{3} \quad \text{and} \quad \tilde{h}_0(x) = 2\mathbb{E}[F_Y(Y) \mid X = x] - \mathbb{E}[F_Y^2(Y) \mid X = x] - \frac{1}{3}.
\]

Recall that for any \( t \in \mathbb{R}, h(t) = \mathbb{E}[G_X^2(t)] \) and \( g(t) = \mathbb{E}[G_X(t)] - G^2(t) = h(t) - G^2(t) \). Then

\[
g(t) - \tilde{h}(t) = h(t) - G^2(t) - \left[ 2F_Y(t) - F_Y^2(t) - \frac{1}{3} \right] = h(t) - \left( 1 - F_Y(t) \right)^2 - \left[ 2F_Y(t) - F_Y^2(t) - \frac{1}{3} \right] = h(t) - \frac{2}{3}.
\]

Similarly, recall that \( h_0(x) = \mathbb{E}[h(Y) \mid X = x] \) and \( g(t) = h(t) - G^2(t) \). Then for any \( x \in \mathbb{R}^d \),

\[
\tilde{h}_0(x) - \mathbb{E}[g(Y) \mid X = x] + h_0(x) = \tilde{h}_0(x) - \mathbb{E}[g(Y) \mid X = x] + \mathbb{E}[h(Y) \mid X = x]
\]

\[
= \mathbb{E}[G^2(Y) \mid X = x] + 2\mathbb{E}[F_Y(Y) \mid X = x] - \mathbb{E}[F_Y^2(Y) \mid X = x] - \frac{1}{3} = 2/3.
\]

(2.19)

Plugging (2.18) and (2.19) to (2.17) yields

\[
\sqrt{n} \xi_n = \frac{6n^2}{n^2 - 1} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} F_Y(Y_i \wedge Y_{N(i)}) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(Y_i) \right) + Q = \sqrt{n} \xi_n^* + Q.
\]

Since \( \mathbb{E}[Q^2] \lesssim n^{-1} \), we obtain

\[
\lim_{n \to \infty} n \text{Var}\left( \xi_n - \xi_n^* \right) = 0.
\]

(2.20)

Lastly, combining (2.10), (2.16), and (2.20) completes the proof. \( \square \)
Proof of Theorem 1.4. Let

$$W_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} F_Y(Y_i \wedge Y_{N(i)}) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(Y_i).$$

Then $\sqrt{n} \xi_n^* = \frac{6n \sigma^2}{n^2 - 1} W_n$, and

$$\left( \xi_n^* - E[\xi_n^*] \right) / \sqrt{\text{Var}[\xi_n^*]} = (W_n - E[W_n]) / \sqrt{\text{Var}[W_n]}. \quad (2.21)$$

It suffices to establish the self-normalization central limit theorem for $W_n$.

Let $\delta_n$ be the Kantorovich–Wasserstein distance between the laws of

$$(W_n - E[W_n]) / \sqrt{\text{Var}[W_n]}$$

and the standard Gaussian. Notice that

(i) for any $i \in [n]$, $F_Y(Y_i \wedge Y_{N(i)}) + h(Y_i)$ is the function of $(X_i, Y_i)$ and its NN $(X_{N(i)}, Y_{N(i)})$, with NN graph constructed by $\{X_i\}_{i=1}^n$;

(ii) both $F_Y$ and $h$ are bounded;

(iii) by Proposition 1.2 and Theorem 1.3, $\liminf_{n \to \infty} \text{Var}[W_n] = \liminf_{n \to \infty} n \text{Var}[\xi_n^*]/36$, which is further equal to $\liminf_{n \to \infty} n \text{Var}[\xi_n^*]/36 > 0$.

Then using Theorem 3.4 in Chatterjee (2008) with some minor modification since we now consider $[(X_i, Y_i)]_{i=1}^n$ instead of $[X_i]_{i=1}^n$, one can show $\lim_{n \to \infty} \delta_n = 0$. Since Kantorovich–Wasserstein distance is stronger than weak convergence, we obtain

$$(W_n - E[W_n]) / \sqrt{\text{Var}[W_n]} \overset{d}{\to} N(0,1). \quad (2.22)$$

Combining (2.21) and (2.22) completes the proof for $\xi_n^*$.

For $\xi_n^*$, the only difference is that this time we consider the right NN instead of NN. While Theorem 3.4 in Chatterjee (2008) cannot be directly applied, we can identify an interaction rule as Step III of the proof of Theorem 3.2 in Lin and Han (2022) with the number of right NN to be 1. Then the self-normalization central limit theorem for $\xi_n^*$ is followed. \(\square\)

Proof of Theorem 1.2. Invoking (2.6) and Lemmas 2.1 and 2.2, one has

$$n \text{E}[\text{Var}[\xi_n | X]] = 36(1 + O(n^{-2})) \left\{ \text{E} \left[ \text{Var} \left[ F_Y(Y_1 \wedge \bar{Y}_1) \right] | X_1 \right] \right.$$

$$+ 2 \text{E} \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \bar{Y}_1), F_Y(Y_1 \wedge \bar{Y}_1') \right] | X_1 \right] \mathbb{I} \left( 1 \neq N(N(1)) \right) \right.$$

$$+ \text{E} \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \bar{Y}_1), F_Y(Y_1 \wedge \bar{Y}_1') \right] | X_1 \right] \left\{ \left\{ j : j \neq 1, N(j) = N(1) \right\} \right\} \right.$$

$$+ \text{E} \left[ \text{Var} \left[ F_Y(Y_1 \wedge \bar{Y}_1) \right] | X_1 \right] \mathbb{I} \left( 1 = N(N(1)) \right) \right.$$

$$+ 4 \text{E} \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \bar{Y}_1), F_Y(Y_2 \wedge \bar{Y}_2) \right] | X_1, X_2 \right] \right.$$

$$+ \text{E} \left[ \text{Cov} \left[ \mathbb{I}(Y_3 \leq Y_1 \wedge \bar{Y}_1), \mathbb{I}(Y_3 \leq Y_2 \wedge \bar{Y}_2) \right] | X_1, X_2, X_3 \right] \right\} + o(1).$$

The following lemma establishes approximation for each term above.
Lemma 2.10.

\[
\frac{1}{n^3} \sum_{i=1}^{n} \left[ \left( R_i \wedge R_N(i) \right) \left( R_i \wedge R_N(i) - R_{N2}(i) \wedge R_{N3}(i) \right) \right] - \mathbb{E} \left[ \text{Var} \left( F_Y (Y_1 \wedge \bar{Y}_1) \mid X_1 \right) \right] \xrightarrow{p} 0,
\]

\[
\frac{1}{n^3} \sum_{i=1}^{n} \left[ \left( R_i \wedge R_N(i) \right) \left( R_i \wedge R_{N2}(i) - R_{N2}(i) \wedge R_{N3}(i) \right) \mathbb{1} \left( i \neq N(N(i)) \right) \right] - \mathbb{E} \left[ \text{Cov} \left( F_Y (Y_1 \wedge \bar{Y}_1), F_Y (\bar{Y}_1 \wedge \bar{Y}_1) \mid X_1 \right) \mathbb{1} \left( 1 \neq N(N(1)) \right) \right] \xrightarrow{p} 0,
\]

\[
\frac{1}{n^3} \sum_{i=1}^{n} \left[ \left( R_i \wedge R_N(i) \right) \left( R_i \wedge R_{N2}(i) - R_{N2}(i) \wedge R_{N3}(i) \right) \left\{ j : j \neq i, N(j) = N(i) \right\} \right] - \mathbb{E} \left[ \text{Cov} \left( F_Y (Y_1 \wedge \bar{Y}_1), F_Y (\bar{Y}_1 \wedge \bar{Y}_1) \mid X_1 \right) \left\{ j : j \neq 1, N(j) = N(1) \right\} \right] \xrightarrow{p} 0,
\]

\[
\frac{1}{n^3} \sum_{i=1}^{n} \left[ \left( R_i \wedge R_N(i) \right) \left( R_i \wedge R_N(i) - R_{N2}(i) \wedge R_{N3}(i) \right) \mathbb{1} \left( i = N(N(i)) \right) \right] - \mathbb{E} \left[ \text{Var} \left( F_Y (Y_1 \wedge \bar{Y}_1) \mid X_1 \right) \mathbb{1} \left( 1 = N(N(1)) \right) \right] \xrightarrow{p} 0,
\]

\[
\frac{1}{n^2(n-1)} \sum_{i,j=1, i \neq j}^{n} \left[ \mathbb{1} \left( R_i \leq R_j \wedge R_N(j) \right) \left( R_i \wedge R_N(i) - R_{N2}(i) \wedge R_{N3}(i) \right) \right] - \mathbb{E} \left[ \text{Cov} \left( F_Y (Y_1 \wedge \bar{Y}_1), F_Y (\bar{Y}_2 \wedge \bar{Y}_2) \mid X_1, X_2 \right) \right] \xrightarrow{p} 0,
\]

\[
\frac{1}{n(n-1)(n-2)} \sum_{i,j,k=1 \atop i \neq j \neq k}^{n} \left[ \mathbb{1} \left( R_i \leq R_j \wedge R_N(j) \right) \left( R_i \wedge R_N(i) - R_{N2}(i) \wedge R_{N3}(i) \right) \mathbb{1} \left( R_N(i) \leq R_k \wedge R_N(k) \right) \right] - \mathbb{E} \left[ \text{Cov} \left( F_Y (Y_1 \wedge \bar{Y}_1), F_Y (Y_3 \wedge \bar{Y}_3) \mid X_1, X_2, X_3 \right) \right] \xrightarrow{p} 0.
\]

On the other hand, Lemma 3.1 ahead yields

\[ n \text{Var} [\mathbb{E} [X_n \mid X]] = 36(1 + O(n^{-2})) \text{Var} \left[ h_1(X_1) + h_0(X_1) \right] + o(1), \]

where we define \( h_0(x) = \mathbb{E}[h(Y) \mid X = x] \) and \( h_1(x) = \mathbb{E}[F_Y(Y \wedge \bar{Y}) \mid X = x] \) with \( Y, \bar{Y} \) independently drawn from \( Y \mid X = x \).

The following lemma establishes approximation for each term above.

Lemma 2.11.

\[
\frac{1}{n(n-1)(n-2)} \sum_{i,j,k=1 \atop i \neq j \neq k}^{n} \mathbb{1} \left( R_i \leq R_j \wedge R_N(j) \right) \mathbb{1} \left( R_N(i) \leq R_k \wedge R_N(k) \right) - \left[ \frac{1}{n^2} \sum_{i=1}^{n} \left( R_i \wedge R_N(i) \right) \right]^2 \xrightarrow{p} 0,
\]

\[
\frac{1}{n^2(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \mathbb{1} \left( R_i \leq R_j \wedge R_N(j) \right) \left( R_N(i) \wedge R_{N2}(i) \right) - \left[ \frac{1}{n^2} \sum_{i=1}^{n} \left( R_i \wedge R_N(i) \right) \right]^2 \xrightarrow{p} 0.
\]
In the sequel, denote the law of \( Y \) by \( \mu \), and the conditional law of \( Y \) given \( X = x \) by \( \mu_x \).

### 3 Proofs of the rest results

In the sequel, denote the law of \( Y \) by \( \mu \), and the conditional law of \( Y \) given \( X = x \) by \( \mu_x \).

#### 3.1 Proof of Proposition 1.1

**Proof of Proposition 1.1(i).** Lemma 6.1 in Lin and Han (2022) showed \( \mathbb{E} \left[ \min \left\{ R_1, R_2 \right\} \right] = (n + 1)/3 \). Then if \( X \) and \( Y \) are independent,

\[
\mathbb{E}[\xi_n] = \frac{6n}{n^2 - 1} \mathbb{E} \left[ \min \left\{ R_1, R_{N(1)} \right\} \right] - \frac{2n + 1}{n - 1} = \frac{6n}{n^2 - 1} \mathbb{E} \left[ \min \left\{ R_1, R_2 \right\} \right] - \frac{2n + 1}{n - 1} = - \frac{1}{n - 1}.
\]

When \( d = 1 \), there exists only one index \( i \in [n] \) such that \( \overline{N}(i) = i \). Then

\[
\mathbb{E}[\xi_n] = 1 - \frac{3(n - 1)}{n^2 - 1} \mathbb{E} \left[ \sum_{i=1}^{n} \left| R_{\overline{N}(i)} - R_i \right| \right] = 1 - \frac{3(n - 1)}{n^2 - 1} \mathbb{E} \left[ \left| R_2 - R_1 \right| \right]
\]

\[= 1 - \frac{3(n - 1)}{n^2 - 1} \left( \mathbb{E}[R_1] + \mathbb{E}[R_2] - 2 \mathbb{E} \left[ \min \left\{ R_1, R_2 \right\} \right] \right) = 1 - \frac{3(n - 1)}{n^2 - 1} \cdot \frac{3}{3} = 0.
\]

This completes the proof. \( \square \)

**Proof of Proposition 1.1(ii).** Notice that for any \( i \in [n] \), \( \min \left\{ R_i, R_{N(i)} \right\} = \sum_{k=1}^{n} \mathbb{1}(Y_k \leq Y_i \wedge Y_{N(i)}) \). From (1.2) and since \( [(X_i, Y_i)]_{i=1}^{n} \) are i.i.d., we have

\[
\mathbb{E}[\xi_n] = \frac{6}{n^2 - 1} \mathbb{E} \left[ \sum_{i=1}^{n} \min \left\{ R_i, R_{N(i)} \right\} \right] - \frac{2n + 1}{n - 1}
\]

\[= \frac{6n}{n^2 - 1} \mathbb{E} \left[ \min \left\{ R_1, R_{N(1)} \right\} \right] - \frac{2n + 1}{n - 1}
\]

\[= \frac{6n(n - 1)}{n^2 - 1} \mathbb{E} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}) \right] + \frac{6n}{n^2 - 1} \mathbb{E} \left[ \mathbb{1}(Y_1 \leq Y_1 \wedge Y_{N(1)}) \right] - \frac{2n + 1}{n - 1}
\]

\[= 6 \mathbb{E} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \tilde{Y}_1) \right] - 2 + 6 \left( \mathbb{E} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}) \right] - \mathbb{E} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \tilde{Y}_1) \right] \right)
\]

\[= \frac{6}{n + 1} \mathbb{E} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}) \right] + \frac{6n}{n^2 - 1} \mathbb{E} \left[ \mathbb{1}(Y_1 \leq Y_1 \wedge Y_{N(1)}) \right] - \frac{3}{n - 1}
\]

\[= 6 \mathbb{E} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \tilde{Y}_1) \right] - 2 + Q. \tag{3.1}
\]
For the first term in (3.1),
\[ E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1) \right] = E \left[ \int G_{X_1}(t) d\mu_{X_2}(t) \right] = \int E[G_{X_1}(t)] d\mu(t). \]

Noticing that \( \int G^2(t) d\mu(t) = 1/3 \), one has
\[ 6E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1) \right] - 2 = 6 \int (E[G_{X_1}^2(t)] - G^2(t)) d\mu(t). \]

On the other hand, it is ready to check
\[ \int \text{Var} \{ E[\mathbb{1}(Y \geq t) | X] \} d\mu(t) = \int (E[G_{X_1}^2(t)] - G^2(t)) d\mu(t), \]

and
\[ \int \text{Var} \{ \mathbb{1}(Y \geq t) \} d\mu(t) = \frac{1}{6}. \]

Accordingly, combining (1.1) and (3.1), we obtain
\[ E[\xi_n] - \xi = E[\xi_n] - 6E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1) \right] + 2 = Q. \]

Let \( N^{-2}(1) \) index the NN of \( X_1 \) among \( \{X_i\}_{i=1}^n \setminus \{X_2\} \). Using the definition of \( Q \) and noticing that the indicator function is bounded by 1, we have
\[
|Q| \leq \left| E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}) \right] - E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1) \right] \right| + \frac{1}{n} \\
\leq \left| E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N-2(1)}) \right] - E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1) \right] \right| + 2P(N(1) = 2) + \frac{1}{n} \\
= \left| E \left[ F_Y(Y_1 \wedge Y_{N-2(1)}) \right] - E \left[ F_Y(Y_1 \wedge \bar{Y}_1) \right] \right| + 2P(N(1) = 2) + \frac{1}{n} \\
\leq \left| E \left[ F_Y(Y_1 \wedge Y_{N(1)}) \right] - E \left[ F_Y(Y_1 \wedge \bar{Y}_1) \right] \right| + 4P(N(1) = 2) + \frac{1}{n}.
\]

For the second term above, \( P(N(1) = 2) = 1/(n-1) \). For the first term above, recall that \( G_X(t) = P(Y \geq t | X) \). Then since \( 0 \leq G_X(t) \leq 1 \) holds for any \( t \in \mathbb{R} \), one has
\[
\left| E \left[ F_Y(Y_1 \wedge Y_{N(1)}) \right] - E \left[ F_Y(Y_1 \wedge \bar{Y}_1) \right] \right| = \left| \int \left( E \left[ G_{X_1}(t) G_{X_{N(1)}}(t) \right] - E \left[ G_{X_1}^2(t) \right] \right) d\mu(t) \right| \\
\leq \int \left| E \left[ G_{X_{N(1)}}(t) - G_{X_1}(t) \right] \right| d\mu(t).
\]

In the same way as the proof of Theorem 4.1 in Azadkia and Chatterjee (2021), essentially Lemma 14.1 and the proof of Lemma 14.2 therein, and from the assumptions, one could deduce
\[ \int E \left\| G_{X_{N(1)}}(t) - G_{X_1}(t) \right\| d\mu(t) \leq \frac{(\log n)^{d+\beta+1+I(d=1)}}{n^{1/d}}, \]

and the proof for \( \xi_n \) is thus complete.

Similar analyses can be performed for \( \xi_n \) as well and details are accordingly omitted. \qed
3.2 Proof of Proposition 1.2

Recall that $h_0(x) = E[h(Y) \mid X = x] = \int E[G_X^2(t)]d\mu_x(t)$ and let us further define

$$h_1(x) := E[F_Y(Y \wedge \tilde{Y}) \mid X = x] = \int F_Y(t \wedge t')d\mu_x(t)d\mu_x(t').$$

The following lemma about $E[\xi^*_n \mid X]$ will be used.

**Lemma 3.1.** We have

$$\lim_{n \to \infty} \left\{ n \operatorname{Var} \left[ \frac{6n}{n^2 - 1} \sum_{i=1}^{n} \left( h_1(X_i) + h_0(X_i) \right) - E[\xi^*_n \mid X] \right] \right\} = 0.$$

**Proof of Proposition 1.2** (i). From (2.6),

$$n \operatorname{Var}[\xi_n] \geq nE[\operatorname{Var}[\xi_n \mid X]] = \frac{36n^4}{(n^2 - 1)^2} \left( E[T_1] + E[T_2] + E[T_3] + E[T_4] + E[T_5] \right).$$

Using Lemmas 2.1 and 2.2, and then noticing that for any $X_1 \in \mathbb{R}^d$, we have

$$\operatorname{Cov}[F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1] \geq 0,$$

one can deduce

$$n \operatorname{Var}[\xi_n] \geq 36(1 + O(n^{-2})) \left\{ E \left[ \operatorname{Var} \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] + 2E \left[ \operatorname{Cov} \left[ F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1 \right] \mathbf{1}(1 \neq N(N(1))) \right] + E \left[ \operatorname{Var} \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \mathbf{1}(1 = N(N(1))) \right] + 4E \left[ \operatorname{Cov} \left[ \mathbf{1}(Y_2 \leq Y_1 \wedge \tilde{Y}_1'), F_Y(Y_2 \wedge \tilde{Y}_2) \mid X_1, X_2 \right] \right] + E \left[ \operatorname{Cov} \left[ \mathbf{1}(Y_3 \leq Y_1 \wedge \tilde{Y}_1), \mathbf{1}(Y_3 \leq Y_2 \wedge \tilde{Y}_2) \mid X_1, X_2, X_3 \right] \right] \right\} + o(1)$$

For the last term above, recalling that $h(t) = E[G_X^2(t)]$ from (1.13), one has

$$E \left[ \operatorname{Cov} \left[ \mathbf{1}(Y_3 \leq Y_1 \wedge \tilde{Y}_1), \mathbf{1}(Y_3 \leq Y_2 \wedge \tilde{Y}_2) \mid X_1, X_2, X_3 \right] \right] = E \left[ \int G_{X_1}(t)G_{X_2}(t)d\mu_{X_3}(t) - \left( \int G_{X_1}(t)d\mu_{X_3}(t) \right) \left( \int G_{X_2}(t)d\mu_{X_3}(t) \right) \right] = E \left[ \int h^2(t)d\mu_{X_3}(t) - \left( \int h(t)d\mu_{X_3}(t) \right) \left( \int h(t)d\mu_{X_3}(t) \right) \right] = E \left[ \operatorname{Var} [h(Y_1) \mid X_1] \right].$$

For the second last term,

$$E \left[ \operatorname{Cov} \left[ \mathbf{1}(Y_2 \leq Y_1 \wedge \tilde{Y}_1), F_Y(Y_2 \wedge \tilde{Y}_2) \mid X_1, X_2 \right] \right] = E \left[ \int G_{X_1}(t)F_Y(t \wedge t')d\mu_{X_2}(t)d\mu_{X_2}(t') - \left( \int G_{X_1}(t)d\mu_{X_2}(t) \right) \left( \int F_Y(t \wedge t')d\mu_{X_2}(t)d\mu_{X_2}(t') \right) \right] = E \left[ \int h(t)F_Y(t \wedge t')d\mu_{X_2}(t)d\mu_{X_2}(t') - \left( \int h(t)d\mu_{X_2}(t) \right) \left( \int F_Y(t \wedge t')d\mu_{X_2}(t)d\mu_{X_2}(t') \right) \right] = E \left[ \operatorname{Cov} [h(Y_1), F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1] \right].$$
We then have
\[
\begin{align*}
n \Var[\xi_n] \geq & 36(1 + O(n^{-2})) \left\{ \E \left[ \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] \\
& + \E \left[ \left( 2 \Cov \left[ F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1 \right] \right) \bigvee \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right) \\
& + 4\E \left[ \Cov \left[ h(Y_1), F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] + \E \left[ \Var \left[ h(Y_1) \mid X_1 \right] \right] \} + o(1). \tag{3.2}
\end{align*}
\]

Notice that
\[
\begin{align*}
2 \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] & + 4 \Cov \left[ h(Y_1), F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] + \Var \left[ h(Y_1) \mid X_1 \right] \\
& = 2 \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \right] + \frac{1}{2} h(Y_1) + \frac{1}{2} h(\tilde{Y}_1) \mid X_1, \tag{3.3}
\end{align*}
\]
and
\[
\begin{align*}
\Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] & + 2 \Cov \left[ F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1 \right] \\
& + 4 \Cov \left[ h(Y_1), F_Y(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] + \Var \left[ h(Y_1) \mid X_1 \right] \\
& = \frac{1}{3} \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) + F_Y(Y_1 \wedge \tilde{Y}_1') + F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') + h(Y_1) + h(\tilde{Y}_1) + h(\tilde{Y}_1') \mid X_1 \right]. \tag{3.4}
\end{align*}
\]

**Case I.** If \( Y \) is not a measurable function of \( X \) almost surely, then
\[
\E \left[ \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) + \frac{1}{2} h(Y_1) + \frac{1}{2} h(\tilde{Y}_1) \mid X_1 \right] \right] > 0,
\]
and
\[
\E \left[ \Var \left[ F_Y(Y_1 \wedge \tilde{Y}_1) + F_Y(Y_1 \wedge \tilde{Y}_1') + F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') + h(Y_1) + h(\tilde{Y}_1) + h(\tilde{Y}_1') \mid X_1 \right] \right] > 0.
\]
Combining the above two bounds with (3.2), (3.3), and (3.4) then yields
\[
\liminf_{n \to \infty} \left\{ n \Var[\xi_n] \right\} > 0.
\]

**Case II.** If \( Y \) is a measurable function of \( X \) almost surely, it is ready to check that
\[
\lim_{n \to \infty} \E[T_1] = \lim_{n \to \infty} \E[T_2] = \lim_{n \to \infty} \E[T_3] = \lim_{n \to \infty} \E[T_4] = \lim_{n \to \infty} \E[T_5] = 0
\]
using Lemmas 2.1 and 2.2 since the variance and the covariance terms there are zero conditional on \( X \). Accordingly, one has
\[
\lim_{n \to \infty} n\E[\Var[\xi_n \mid X]] = 0
\]
invoking (2.6).

It remains to establish \( \lim_{n \to \infty} n \Var[\E[\xi_n \mid X]] = 0 \). From Theorem 1.3, we have
\[
\limsup_{n \to \infty} n \Var[\E[\xi_n - \xi_n^* \mid X]] \leq \limsup_{n \to \infty} n \Var[\xi_n - \xi_n^*] = 0.
\]
Then it suffices to establish \( \lim_{n \to \infty} n \Var[\E[\xi_n^* \mid X]] = 0 \).

From Lemma 3.1, we consider \( \Var[\sum_{i=1}^n (h_1(X_i) + h_0(X_i))] \). Let \( Y = \phi(X) \) almost surely with \( \phi \) to be a measurable function. Then
\[
h_1(X_i) = \E[F_Y(Y \wedge \tilde{Y}) \mid X = X_i] = F_Y(\phi(X_i))
\]

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and
\[ h_0(X_i) = E[h(Y) \mid X = X_i] = h(\phi(X_i)). \]

Notice that for any \( t \in \mathbb{R} \),
\[ h(t) = E[G_X^2(t)] = E[P(Y \geq t \mid X)]^2 = E[E(\phi(X) \geq t)] = P(\phi(X) \geq t), \]
and
\[ F_Y(t) = P(Y \leq t) = P(\phi(X) \leq t). \]

We then have
\[ h_1(X_i) + h_0(X_i) = F_Y(\phi(X_i)) + h(\phi(X_i)) = P(\phi(X) \leq \phi(X_i)) + P(\phi(X) \geq \phi(X_i)) \]
\[ = 1 + P(\phi(X) = \phi(X_i)) = 1 + P(Y = \phi(X_i)) = 1 \]
from the continuity of \( F_Y \). Then \( \text{Var} \left[ \sum_{i=1}^n (h_1(X_i) + h_0(X_i)) \right] = 0 \) and then \( \lim_{n \to \infty} n \text{Var}[E[\xi_n \mid \mathbf{X}]] = 0 \) from Lemma 3.1.

The two claims for \( \overline{\xi}_n \) can be established in the same way by simply replacing \( N(\cdot) \) by \( \overline{N}(\cdot) \). □

**Proof of Proposition 1.2 (ii).** Invoking (2.6) and Lemmas 2.1 and 2.2,
\[ nE[\text{Var}[\xi_n \mid \mathbf{X}]] = 36(1 + O(n^{-2})) \left\{ E \left[ \text{Var} \left[ F_Y (Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] \right. \]
\[ + 2E \left[ \text{Cov} \left[ F_Y (Y_1 \wedge \tilde{Y}_1), F_Y (\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1 \right] \mathbb{I} \left( 1 \neq N(N(1)) \right) \right] \]
\[ + E \left[ \text{Cov} \left[ F_Y (Y_1 \wedge \tilde{Y}_1), F_Y (\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1 \right] \mathbb{I} \left( 1 = N(N(1)) \right) \right] \]
\[ + 4E \left[ \text{Cov} \left[ \mathbb{I} (Y_2 \leq Y_1 \wedge \tilde{Y}_1), F_Y (Y_2 \wedge \tilde{Y}_2) \mid X_1, X_2 \right] \right] \]
\[ + E \left[ \text{Cov} \left[ \mathbb{I} (Y_3 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{I} (Y_3 \leq Y_2 \wedge \tilde{Y}_2) \mid X_1, X_2, X_3 \right] \right] \} + o(1). \]

From (3.3) and (3.4), one deduces
\[ nE[\text{Var}[\xi_n \mid \mathbf{X}]] = 36(1 + O(n^{-2})) \left\{ 2E \left[ \text{Var} \left[ F_Y (Y_1 \wedge \tilde{Y}_1) + \frac{1}{2} h(Y_1) + \frac{1}{2} h(\tilde{Y}_1) \mid X_1 \right] \mathbb{I} \left( 1 = N(N(1)) \right) \right] \right. \]
\[ + 3E \left[ \text{Var} \left[ \frac{1}{3} F_Y (Y_1 \wedge \tilde{Y}_1) + \frac{1}{3} F_Y (Y_1 \wedge \tilde{Y}_1') + \frac{1}{3} h(Y_1) + \frac{1}{3} h(\tilde{Y}_1) + \frac{1}{3} h(\tilde{Y}_1') \mid X_1 \right] \right. \]
\[ \mathbb{I} \left( 1 \neq N(N(1)) \right) \right] \left. + E \left[ \text{Cov} \left[ F_Y (Y_1 \wedge \tilde{Y}_1), F_Y (\tilde{Y}_1 \wedge \tilde{Y}_1') \mid X_1 \right] \right. \right. \left. \right. \left. \mathbb{I} \left( 1 = N(N(1)) \right) \right] \left. \right. \left. \right. \left. \right. \left. \right. \left. \mathbb{I} \left( j \neq 1, N(j) = N(1) \right) \right. \right. \}
\[ + o(1). \]

Notice that for any \( t, t' \in \mathbb{R} \), \( F_Y (t \wedge t') \leq (F_Y (t) + F_Y (t'))/2 \). In addition, we have
\[ h(t) = E[G_X^2(t)] \leq E[G_X(t)] = 1 - F_Y (t). \]

Then for any \( Y_1, \tilde{Y}_1, \tilde{Y}_1' \in \mathbb{R} \),
\[ 0 \leq F_Y (Y_1 \wedge \tilde{Y}_1) + \frac{1}{2} h(Y_1) + \frac{1}{2} h(\tilde{Y}_1) \leq 1, \]

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and

\[ 0 \leq \frac{1}{3} F_Y(Y_1 \wedge \tilde{Y}_1) + \frac{1}{3} F_Y(Y_1 \wedge \tilde{Y}_1') + \frac{1}{3} F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') + \frac{1}{3} h(Y_1) + \frac{1}{3} h(\tilde{Y}_1) + \frac{1}{3} h(\tilde{Y}_1') \leq 1. \]

Leveraging Popoviciu’s inequality, for any \( X_1 \in \mathbb{R} \), we deduce

\[
\begin{align*}
\Var\left[F_Y(Y_1 \wedge \tilde{Y}_1) \big| X_1\right] &\leq \frac{1}{4}, \\
\Var\left[F_Y(Y_1 \wedge \tilde{Y}_1') \big| X_1\right] &\leq \frac{1}{4}, \\
\Cov\left[F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \big| X_1\right] &\leq \Var\left[F_Y(Y_1 \wedge \tilde{Y}_1) \big| X_1\right] \leq \frac{1}{4}.
\end{align*}
\]

Then we have

\[
\begin{align*}
nE[\Var[\xi_n \mid X]] &\leq 36(1 + O(n^{-2})) \left[ \frac{1}{2} P\left(1 = N(N(1))\right) + \frac{3}{4} P\left(1 \neq N(N(1))\right) \right] + \frac{1}{4} E\left[ \left\{ j : j \neq 1, N(j) = N(1) \right\} \right] + o(1).
\end{align*}
\]

From Lemma 20.6 together with Theorem 20.16 in Biau and Devroye (2015), the size of the set

\[
\left\{ j : j \neq 1, N(j) = N(1) \right\}
\]

is always bounded by a constant that only depends on \( d \). Accordingly, we have

\[
\limsup_{n \to \infty} nE[\Var[\xi_n \mid X]] < \infty. \tag{3.5}
\]

If we further assume \( F_X \) to be absolutely continuous, then Lemmas 3.2 and 3.3 in Shi et al. (2021a) show

\[
\lim_{n \to \infty} P\left(1 = N(N(1))\right) = q_d, \quad \lim_{n \to \infty} E\left[ \left\{ j : j \neq 1, N(j) = N(1) \right\} \right] = o_d.
\]

It then holds true that

\[
\limsup_{n \to \infty} nE[\Var[\xi_n \mid X]] \leq 27 - 9q_d + 9o_d. \tag{3.6}
\]

On the other hand, Lemma 3.1 yields

\[
n \Var[E[\xi_n \mid X]] = 36(1 + O(n^{-2})) \Var[ h_1(X_1) + h_0(X_1) ] + o(1).
\]

Using the definition of \( h_0 \) and \( h_1 \),

\[ 0 \leq h_1(X_1) + h_0(X_1) \leq E\left[ F_Y(Y_1 \wedge \tilde{Y}_1) + \frac{1}{2} h(Y_1) + \frac{1}{2} h(\tilde{Y}_1) \big| X_1\right] \leq 1. \]

Then Popoviciu’s inequality implies

\[
\limsup_{n \to \infty} n \Var[E[\xi_n \mid X]] \leq 9. \tag{3.7}
\]

Combining (3.5), (3.6), (3.7) completes the proof for \( \xi_n \).

For \( \tilde{\xi}_n \), the only difference is that we have

\[
\lim_{n \to \infty} P\left(1 = \tilde{N}(N(1))\right) = \lim_{n \to \infty} E\left[ \left\{ j : j \neq 1, \tilde{N}(j) = \tilde{N}(1) \right\} \right] = 0,
\]

\[
(3.5)
\]

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and thusly one can replace the bound (3.6) by
\[
\limsup_{n \to \infty} n \mathbb{E}[\text{Var}[\xi_n | \mathbf{X}]] \leq 27.
\]
We thus complete the proof. \hfill \square

### 3.3 Proof of Lemma 2.1

**Proof of Lemma 2.1.** We establish the two claims for \(i = 1, 2, 3, 4\) separately.

**Part I.** \(i = 1\).
Since \([(X_i, Y_i)]_{i=1}^n\) are i.i.d., we have

\[
\mathbb{E}[T_1] = \mathbb{E}\left[\frac{1}{n^3} \sum_{i=1}^n \mathbb{V} \left[ \min \{R_i, R_{N(i)}\} \mid \mathbf{X} \right] \right]
\]

\[
= \frac{1}{n^2} \mathbb{E} \left[ \mathbb{V} \left[ \min \{R_1, R_{N(1)}\} \mid \mathbf{X} \right] \right]
\]

\[
= \frac{1}{n^2} \mathbb{E} \left[ \mathbb{V} \left[ \sum_{k=1}^n \mathbb{I} (Y_k \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right]
\]

\[
= \frac{(n-1)(n-2)}{n^2} \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_2 \leq Y_1 \land Y_{N(1)}), \mathbb{I} (Y_3 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right]
\]

\[
+ \frac{1}{n^2} \mathbb{E} \left[ \mathbb{V} \left[ \mathbb{I} (Y_1 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right] + \frac{n-1}{n^2} \mathbb{E} \left[ \mathbb{V} \left[ \mathbb{I} (Y_2 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right]
\]

\[
+ \frac{2(n-1)}{n^2} \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_1 \leq Y_1 \land Y_{N(1)}), \mathbb{I} (Y_2 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right]
\]

\[
= \frac{(n-1)(n-2)}{n^2} \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_2 \leq Y_1 \land Y_{N(1)}), \mathbb{I} (Y_3 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right] + S_1
\]

\[
= (1 + O(n^{-1})) \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_2 \leq Y_1 \land \bar{Y}_1), \mathbb{I} (Y_3 \leq Y_1 \land \bar{Y}_1) \mid \mathbf{X} \right] \right] + S_1
\]

\[
+ (1 + O(n^{-1})) \left\{ \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_2 \leq Y_1 \land Y_{N(1)}), \mathbb{I} (Y_3 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right] \right] - \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_2 \leq Y_1 \land \bar{Y}_1), \mathbb{I} (Y_3 \leq Y_1 \land \bar{Y}_1) \mid \mathbf{X} \right] \right] \right\}
\]

\[
= (1 + O(n^{-1})) \mathbb{E} \left[ \mathbb{C} \left[ \mathbb{I} (Y_2 \leq Y_1 \land \bar{Y}_1), \mathbb{I} (Y_3 \leq Y_1 \land \bar{Y}_1) \mid \mathbf{X} \right] \right] + S_1 + (1 + O(n^{-1})) S_2,
\]

where \(\bar{Y}_1\) is sampled from \(F_{Y_{1 \mid X=X_1}}\) independent of the data.

For \(S_1\) in (3.8), noticing that the variance of the indicator function is bounded by 1 and then invoking the Cauchy–Schwarz inequality yields

\[
|S_1| \leq \frac{3n - 2}{n^2} = O(n^{-1}).
\]

For \(S_2\) in (3.8), we first have

\[
\mathbb{E} \left[ \mathbb{I} (Y_2 \leq Y_1 \land Y_{N(1)}), \mathbb{I} (Y_3 \leq Y_1 \land Y_{N(1)}) \mid \mathbf{X} \right]
\]

\[
= \int \mathbb{I} (y_2 \leq y_1 \land y_4) \mathbb{I} (y_3 \leq y_1 \land y_4) d\mu_X(y_1) d\mu_X(y_2) d\mu_X(y_3) d\mu_X(y_4) \mathbb{I} (N(1) \neq 2, 3)
\]
\[
+ \int \mathbb{1}(y_2 \leq y_1 \land y_2) \mathbb{1}(y_3 \leq y_1 \land y_2) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)\mathbb{1}(N(1) = 2)
\]
\[
+ \int \mathbb{1}(y_2 \leq y_1 \land y_3) \mathbb{1}(y_3 \leq y_1 \land y_3) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)\mathbb{1}(N(1) = 3)
\]
\[
= \int \mathbb{1}(y_2 \leq y_1 \land y_4) \mathbb{1}(y_3 \leq y_1 \land y_4) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)d\mu_{X_N(1)}(y_4) + Q_1
\]
\[
= \int G_{X_1}(y_2 \lor y_3)G_{X_N(1)}(y_2 \lor y_3)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + Q_1.
\]
From the boundedness of the indicator function and \(P(N(1) = 2) = P(N(1) = 3) = 1/(n - 1)\), we then have \(E[|Q_1|] = O(n^{-1})\).
We can establish in the same way that
\[
E\left[ \mathbb{1}(Y_2 \leq Y_1 \land Y_{N(1)}) \mathbb{1}(Y_3 \leq Y_1 \land Y_{N(1)}) \right] = \int \mathbb{1}(y_2 \leq y_1 \land y_4) \mathbb{1}(y_3 \leq y_1 \land y_4) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)d\mu_{X_N(1)}(y_4)d\mu_{X_1}(y_5)d\mu_{X_N(1)}(y_6) + Q_2
\]
\[
= \int G_{X_1}(y_2)G_{X_N(1)}(y_2)G_{X_1}(y_3)G_{X_N(1)}(y_3)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + Q_2,
\]
with \(E[|Q_2|] = O(n^{-1})\).
On the other hand,
\[
E\left[ \mathbb{1}(Y_2 \leq Y_1 \land \tilde{Y}_1) \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1) \right] = \int \mathbb{1}(y_2 \leq y_1 \land y_4) \mathbb{1}(y_3 \leq y_1 \land y_4) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)d\mu_{X_1}(y_4)
\]
\[
= \int G_{X_1}^2(y_2 \lor y_3)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3),
\]
and
\[
E\left[ \mathbb{1}(Y_2 \leq Y_1 \land \tilde{Y}_1) \right] E\left[ \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1) \right] = \int \mathbb{1}(y_2 \leq y_1 \land y_4) \mathbb{1}(y_3 \leq y_5 \land y_6) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)d\mu_{X_1}(y_4)d\mu_{X_1}(y_5)d\mu_{X_1}(y_6)
\]
\[
= \int G_{X_1}^2(y_2)G_{X_1}(y_3)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3).
\]
Then, since \(G_x\) is uniformly bounded by 1 for any \(x \in \mathbb{R}^d\),
\[
\left| E\left[ \mathbb{1}(Y_2 \leq Y_1 \land Y_{N(1)}) \mathbb{1}(Y_3 \leq Y_1 \land Y_{N(1)}) \right] - E\left[ \mathbb{1}(Y_2 \leq Y_1 \land \tilde{Y}_1) \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1) \right] \right|
\]
\[
= \left| \int G_{X_1}(y_2 \lor y_3)(G_{X_N(1)}(y_2 \lor y_3) - G_{X_1}(y_2 \lor y_3))d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + Q_1 \right|
\]
\[
\leq \int \left| G_{X_N(1)}(y_2 \lor y_3) - G_{X_1}(y_2 \lor y_3) \right| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + |Q_1|,
\]
and
\[
\left| E\left[ \mathbb{1}(Y_2 \leq Y_1 \land Y_{N(1)}) \right] E\left[ \mathbb{1}(Y_3 \leq Y_1 \land Y_{N(1)}) \right] - E\left[ \mathbb{1}(Y_2 \leq Y_1 \land \tilde{Y}_1) \right] E\left[ \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1) \right] \right|
\]
\[
= \left| \int G_{X_1}(y_2)G_{X_1}(y_3)(G_{X_N(1)}(y_2)G_{X_N(1)}(y_3) - G_{X_1}(y_2)G_{X_1}(y_3))d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + Q_2 \right|
\]

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\[ \leq \int |G_{X_N(1)}(y_2) - G_{X_Y(1)}(y_2)| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + \int |G_{X_N(1)}(y_3) - G_{X_Y(1)}(y_3)| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) + |Q_2|. \]

We then have
\[
|S_2| = \mathbb{E}\left[ \text{Cov}\left[ \mathbb{I}(Y_2 \leq Y_1 \wedge Y_{N(1)}), \mathbb{I}(Y_3 \leq Y_1 \wedge Y_{N(1)}) \right] \right] - \mathbb{E}\left[ \text{Cov}\left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{I}(Y_3 \leq Y_1 \wedge \tilde{Y}_1) \right] \right] \\
\leq \mathbb{E}\left[ \int |G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3)| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) \right] \\
+ 2\mathbb{E}\left[ \int |G_{X_N(1)}(y_2) - G_{X_Y(1)}(y_2)| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) \right] + \mathbb{E}[|Q_1|] + \mathbb{E}[|Q_2|].
\]

For the first term above, since \( G_x \) is uniformly bounded by 1 for \( x \in \mathbb{R}^d \), we have
\[ \int |G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3)| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) \leq 2 \int \mu_{X_2}(y_2)d\mu_{X_3}(y_3) = 2. \]

Invoking Fatou’s lemma then yields
\[
\limsup_{n \to \infty} \mathbb{E}\left[ \int \left| G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3) \right| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) \right] \\
= \limsup_{n \to \infty} \mathbb{E}\left[ \int \left( G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3) \right) \mid X_2, X_3 \right] \] \\
= \limsup_{n \to \infty} \mathbb{E}\left[ \int \left( G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3) \right) \mid X_2, X_3 \right] d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) \] \\
\leq \mathbb{E}\left[ \int \limsup_{n \to \infty} \left( G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3) \right) \mid X_2, X_3 \right] d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) .
\]

Notice that for any \( t \in \mathbb{R} \), the map \( x \to G_x(t) \) is a measurable function. Then from Lemma 11.7 in Azadka and Chatterjee (2021), \( G_{X_N(1)}(t) - G_{X_Y(1)}(t) \xrightarrow{p} 0 \). Then for all \( t \in \mathbb{R} \) and almost all \( X_2, X_3 \in \mathbb{R}^d \),
\[ \limsup_{n \to \infty} \mathbb{E}\left[ \left| G_{X_N(1)}(t) - G_{X_Y(1)}(t) \right| \mid X_2, X_3 \right] = 0, \]
and accordingly
\[ \lim_{n \to \infty} \mathbb{E}\left[ \int \left| G_{X_N(1)}(y_2 \vee y_3) - G_{X_Y(1)}(y_2 \vee y_3) \right| d\mu_{X_2}(y_2)d\mu_{X_3}(y_3) \right] = 0. \]

We can handle the second term in the upper bound of \( |S_2| \) in the same way. Recall that \( \mathbb{E}[|Q_1|], \mathbb{E}[|Q_2|] = O(n^{-1}) \). We then obtain
\[ |S_2| = o(1). \] (3.10)

In the end, let’s study the first term in (3.8). Notice that
\[
\mathbb{E}\left[ \mathbb{E}\left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \tilde{Y}_1) \mathbb{I}(Y_3 \leq Y_1 \wedge \tilde{Y}_1) \mid \mathbb{X} \right] \right] \\
= \mathbb{E}\left[ \int \mathbb{I}(y_2 \leq y_1 \wedge y_4) \mathbb{I}(y_3 \leq y_1 \wedge y_4) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)d\mu_{X_4}(y_4) \right] \\
= \mathbb{E}\left[ \int \mathbb{I}(y_2 \leq y_1 \wedge y_4) \mathbb{I}(y_3 \leq y_1 \wedge y_4) d\mu_{X_1}(y_1)d\mu_{X_2}(y_2)d\mu_{X_3}(y_3)d\mu_{X_4}(y_4) \mid X_1 \right] .
\]

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\[ E \left[ \int \mathbb{1}(y_2 \leq y_1 \wedge y_4) \mathbb{1}(y_3 \leq y_1 \wedge y_4) d\mu_X(y_1) d\mu(y_2) d\mu(y_3) d\mu_X(y_4) \right] \]
\[ = E \left[ \int F_Y^2(y_1 \wedge y_4) d\mu_X(y_1) d\mu_X(y_4) \right] = E \left[ E \left[ F_Y^2(Y_1 \wedge Y_1) \mid X_1 \right] \right]. \]

We can establish
\[ E \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_1) \mid X \right] E \left[ \mathbb{1}(Y_3 \leq Y_1 \wedge Y_1) \mid X \right] = E \left[ \left( E \left[ F_Y(Y_1 \wedge Y_1) \mid X_1 \right] \right)^2 \right]. \]

Then
\[ E \left[ \text{Cov} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_1), \mathbb{1}(Y_3 \leq Y_1 \wedge Y_1) \mid X \right] \right] = E \left[ \text{Var} \left[ F_Y(Y_1 \wedge Y_1) \mid X_1 \right] \right]. \quad (3.11) \]

Plugging (3.9)-(3.11) to (3.8) yields
\[ E[T_1] = (1 + O(n^{-1})) E \left[ \text{Var} \left[ F_Y(Y_1 \wedge Y_1) \mid X_1 \right] \right] + o(1). \]

Similar to (3.10), we also have
\[ E[T'_1] = E \left[ \frac{1}{n} \sum_{i=1}^n \text{Var} \left[ \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \} \mid X \right] \right] \]
\[ = E \left[ \text{Var} \left[ F_Y(Y_1) \wedge F_Y(Y_{N(1)}) \mid X \right] \right] = E \left[ \text{Var} \left[ F_Y(Y_1 \wedge Y_{N(1)}) \mid X \right] \right] \]
\[ = E \left[ \text{Var} \left[ F_Y(Y_1 \wedge Y_1) \mid X \right] \right] + o(1) \]
\[ = E \left[ \text{Var} \left[ F_Y(Y_1 \wedge Y_1) \mid X_1 \right] \right] + o(1). \]

Using the fact that \( F_Y \leq 1 \), we complete the proof of the first claim, and the second claim can be established in the same way.

**Part II.** \( i = 2 \).

Since \( (X_i, Y_i)_{i=1}^n \) are i.i.d. and the indicator function is bounded, we have
\[ E[T_2] = \frac{1}{n^3} E \left[ \sum_{j=N(i); i \neq N(j) \text{ or } i = N(j), j \neq N(i)} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \mid X \right] \right] \]
\[ = \frac{2}{n^3} E \left[ \sum_{j=N(i), i \neq N(j)} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \mid X \right] \right] \]
\[ = \frac{2}{n^3} E \left[ \sum_{i=1}^n \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_{N(i)}, R_{N(N(i))} \} \mid X \right] \mathbb{1}(i \neq N(N(i))) \right] \]
\[ = \frac{2}{n^2} E \left[ \text{Cov} \left[ \min \{ R_1, R_{N(1)} \}, \min \{ R_{N(1)}, R_{N(N(1))} \} \mid X \right] \mathbb{1}(1 \neq N(N(1))) \right] \]
\[ = \frac{2}{n^2} E \left[ \text{Cov} \left[ \sum_{k=1}^n \mathbb{1}(Y_k \leq Y_1 \wedge Y_N(1)), \sum_{\ell=1}^n \mathbb{1}(Y_\ell \leq Y_{N(1)} \wedge Y_{N(N(1)))} \mid X \right] \mathbb{1}(1 \neq N(N(1))) \right] \]
\[ = \frac{2(n-1)(n-2)}{n^2} E \left[ \text{Cov} \left[ \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1),}), \mathbb{1}(Y_3 \leq Y_{N(1)} \wedge Y_{N(N(1)))} \mid X \right] \mathbb{1}(1 \neq N(N(1))) \right] + O(n^{-1}). \]

Lemma 11.3 in Azadkia and Chatterjee (2021) shows \( X_{N(1)} \to X_1 \) almost surely. Notice that
\[ \|X_{N(N(1))} - X_1\| \leq \|X_{N(1)} - X_1\| + \|X_{N(N(1))} - X_{N(1)}\| \leq 2\|X_{N(1)} - X_1\|. \]
Then $X_{N(N(1))} \rightarrow X_1$ almost surely. Similar to the proof of Lemma 11.7 in Azadkia and Chatterjee (2021), for any $t \in \mathbb{R}$, one can prove

$$G_{X_{N(N(1))}}(t) - G_{X_1}(t) \overset{p}{\rightarrow} 0.$$  

Notice that $P(N(1) = 2, 3) = 2/(n - 1)$ and $P(N(N(1)) = 2, 3) \leq 2/(n - 1)$. Then, similar to the proof of (3.10),

$$E \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge Y_{N(1)}), \mathbb{I}(Y_3 \leq Y_{N(1)} \wedge Y_{N(N(1)))}) \bigg| X \right] \mathbb{I}(1 \neq N(N(1))) \right]$$

$$= E \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{I}(Y_3 \leq \tilde{Y}_1 \wedge \tilde{Y}_1') \bigg| X \right] \mathbb{I}(1 \neq N(N(1))) + o(1) \right].$$

Let $X^{-2,3} := X \setminus \{X_2, X_3\}$, and let $N^{-2,3}(j)$ index the NN of $X_j$ in $X^{-2,3}$ for $j \in \llbracket n \rrbracket$ and $j \neq 2, 3$. If $N(1) \neq 2, 3$ and $N(N(1)) \neq 2, 3$, then $N(1) = N^{-2,3}(1)$ and $N(N(1)) = N^{-2,3}(N(1))$. Then $N^{-2,3}(N^{-2,3}(1)) = N(N(1))$. Notice that $P(N(1) = 2, 3), P(N(N(1)) = 2, 3) = O(n^{-1})$ and the event $\{1 \neq N^{-2,3}(N^{-2,3}(1))\}$ is a function of $X^{-2,3}$. From the boundedness of the indicator function and $F_Y$,

$$E \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{I}(Y_3 \leq \tilde{Y}_1 \wedge \tilde{Y}_1') \bigg| X \right] \mathbb{I}(1 \neq N(N(1))) \right]$$

$$= E \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{I}(Y_3 \leq \tilde{Y}_1 \wedge \tilde{Y}_1') \bigg| X \right] \mathbb{I}(1 \neq N^{-2,3}(N^{-2,3}(1))) + O(n^{-1}) \right].$$

$$= E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \bigg| X \right] \mathbb{I}(1 \neq N^{-2,3}(N^{-2,3}(1))) + O(n^{-1}) \right].$$

We then obtain

$$E[T_2] = 2(1 + O(n^{-1}))E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge \tilde{Y}_1') \bigg| X \right] \mathbb{I}(1 \neq N(N(1))) \right] + o(1).$$

For $T_2^*$, we have

$$E[T_2^*] = \frac{1}{n} E \left[ \sum_{j=N(1), i \neq N(j)} \text{Cov} \left[ \min \{F_Y(Y_i), F_Y(Y_{N(i)})\}, \min \{F_Y(Y_j), F_Y(Y_{N(j)})\} \bigg| X \right] \right]$$

$$= \frac{2}{n} E \left[ \sum_{j=N(1), i \neq N(j)} \text{Cov} \left[ \min \{F_Y(Y_i), F_Y(Y_{N(i)})\}, \min \{F_Y(Y_j), F_Y(Y_{N(j)})\} \bigg| X \right] \right]$$

$$= 2E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge Y_{N(1)}), F_Y(Y_{N(1)} \wedge Y_{N(N(1)))}) \bigg| X \right] \mathbb{I}(1 \neq N(N(1))) \right]$$

From the boundedness of $F_Y$, we complete the proof of the first claim.

The second claim can be established in the same way. Both claims for $i = 4$ can be established in the same way by replacing the event $\{1 \neq N(N(1))\}$ by $\{1 = N(N(1))\}$. We can obtain

$$E[T_4] = (1 + O(n^{-1}))E \left[ \text{Var} \left[ F_Y(Y_1 \wedge \tilde{Y}_1) \bigg| X \right] \mathbb{I}(1 = N(N(1))) \right] + o(1),$$

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and

\[ E[T_4^2] = E \left[ \text{Var} \left[ F_Y (Y_1 \wedge \tilde{Y}_1) \big| X_1 \right] \mathbb{1} \left( 1 = N(N(1)) \right) \right] + o(1). \]

**Part III. i = 3.**

Conditional on \( X \), let \( A_1 = A_1(X) := \{j : j \neq 1, N(j) = N(1)\} \), i.e., the set of all indices \( j \) such that \( X_j \) and \( X_1 \) share the same NN. Let \( \pi(1) \) be the random variable that assigns the same probability mass on the elements of \( A_1 \), and are independent of \( Y \) conditional on \( X \), i.e., for any \( j \in A_1 \), \( P(\pi(1) = j) = 1/|A_1| \).

Then

\[
E[T_3] = \frac{1}{n^3} E \left[ \sum_{i \neq j} \sum_{N(i) = N(j)} \text{Cov} \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \big| X \right] \right]
\]

\[
= \frac{1}{n^2} E \left[ \sum_{j \in A_1} \text{Cov} \left[ \min \{ R_1, R_{N(1)} \}, \min \{ R_j, R_{N(j)} \} \big| X \right] \right]
\]

\[
= \frac{1}{n^2} E \left[ |A_1| \text{Cov} \left[ \min \{ R_1, R_{N(1)} \}, \min \{ R_{\pi(1)}, R_{N(1)} \} \big| X \right] \right]
\]

\[
= \frac{(n-1)(n-2)}{n^2} E \left[ |A_1| \text{Cov} \left[ \mathbb{1} (Y_2 \leq Y_1 \wedge Y_{N(1)}) , \mathbb{1} (Y_3 \leq Y_{\pi(1)} \wedge Y_{N(1)}) \big| X \right] \right] + O \left( \frac{E[|A_1|]}{n} \right).
\]

From Lemma 20.6 together with Theorem 20.16 in Biau and Devroye (2015), \(|A_1|\) is always bounded by a constant only depending on \( d \). Then

\[
E[T_3] = (1 + O(n^{-1})) E \left[ |A_1| \text{Cov} \left[ \mathbb{1} (Y_2 \leq Y_1 \wedge \tilde{Y}_1) , \mathbb{1} (Y_3 \leq \tilde{Y}_1' \wedge \tilde{Y}_1) \big| X \right] \right] + o(1).
\]

Recall the definition of \( X^{-2,3} \) and \( N^{-2,3}(-) \) in the second part. Let

\[
A_1^{-2,3} = A_1^{-2,3}(X^{-2,3}) := \{ j : j \neq 1, N^{-2,3}(j) = N^{-2,3}(1) \}.
\]

We consider the event \( N(1) \neq 2, 3 \). For any \( j \in A_1 \), we have \( j \neq 1, N(j) = N(1) \). If \( j \neq 2, 3 \), then \( N^{-2,3}(j) = N^{-2,3}(1) \) from \( N(1) \neq 2, 3 \), and then \( j \in A_1^{-2,3} \). Then

\[
|A_1 \setminus A_1^{-2,3}| \leq \mathbb{1} (N(2) = N(1)) + \mathbb{1} (N(3) = N(1)).
\]

On the other hand, for any \( j \in A_1^{-2,3} \), we have \( N^{-2,3}(j) = N^{-2,3}(1) = N(1) \). If \( N(j) \neq N(1) \), then the possible case is \( N(j) = 2, 3, N(N(j)) = 2, 3, N(1) = N(N(j)) \), or \( N(j) = 2, 3, N(N(j)) \neq 2, 3, N(1) = N(N(j)) \). Then

\[
|A_1^{-2,3} \setminus A_1| \leq \sum_{j : N(j) = 2, 3} \left( \mathbb{1} (N(1) = N(N(j))) + \mathbb{1} (N(1) = N(N(j))) \right)
\]

\[
\leq \sum_{j : N(j) = 2} \left( \mathbb{1} (N(1) = N(2)) + \mathbb{1} (N(1) = N(2)) \right)
\]

\[
+ \sum_{j : N(j) = 3} \left( \mathbb{1} (N(1) = N(3)) + \mathbb{1} (N(1) = N(3)) \right).
\]

Notice that for any \( i \in [n] \), the number of \( j \in [n] \) such that \( N(j) = i \) is always bounded
by a constant depending only on $d$. Then $E[|A_1 \setminus A_i|^{-2.3}]$, $E[|A_i^{-2.3} \setminus A_1|] = O(n^{-1})$. Notice that $P(N(1) = 2, 3) = O(n^{-1})$. Then

$$E[|A_1| \text{ Cov } \left[ \mathbb{1}(Y_2 \leq Y_1 \land \bar{Y}_1), \mathbb{1}(Y_3 \leq \bar{Y}_1' \land \bar{Y}_1) \right| \mathbf{X}] + O(n^{-1})$$

$$= E[|A_i^{-2.3}| \text{ Cov } \left[ \mathbb{1}(Y_2 \leq Y_1 \land \bar{Y}_1), \mathbb{1}(Y_3 \leq \bar{Y}_1' \land \bar{Y}_1) \right| \mathbf{X}] + O(n^{-1})$$

$$= E[|A_i^{-2.3}| \text{ Cov } \left[ F_Y(Y_1 \land \bar{Y}_1), F_Y(\bar{Y}_1' \land \bar{Y}_1') \right| \mathbf{X}] + O(n^{-1})$$

$$= E[|A_i| \text{ Cov } \left[ F_Y(Y_1 \land \bar{Y}_1), F_Y(\bar{Y}_1' \land \bar{Y}_1') \right| \mathbf{X}] + O(n^{-1}).$$

We then obtain

$$E[T_3] = (1 + O(n^{-1})) E[|A_1| \text{ Cov } \left[ F_Y(Y_1 \land \bar{Y}_1), F_Y(\bar{Y}_1' \land \bar{Y}_1') \right| \mathbf{X}] + o(1).$$

For $T_3^*$,

$$E[T_3^*] = \frac{1}{n} E \left[ \sum_{i \neq j, N(i) = N(j)} \text{ Cov } \left[ \min \{ F_Y(Y_i), F_Y(Y_N(i)) \}, \min \{ F_Y(Y_j), F_Y(Y_N(j)) \} \right| \mathbf{X} \right]$$

$$= E[|A_1| \text{ Cov } \left[ F_Y(Y_1 \land Y_N(1)), F_Y(Y_N(1) \land Y_N(1)) \right| \mathbf{X}]$$

$$= E[|A_1| \text{ Cov } \left[ F_Y(Y_1 \land \bar{Y}_1), F_Y(\bar{Y}_1' \land \bar{Y}_1') \right| X_1] + o(1).$$

Then we complete the proof of the first claim and the second claim can be similarly derived. \[ \Box \]

### 3.4 Proof of Lemma 2.2

**Proof of Lemma 2.2.** Since $\{(X_i, Y_i)\}_{i=1}^n$ are i.i.d. and $\min \{ R_i, R_{N(i)} \} = \sum_{k=1}^n \mathbb{1}(Y_k \leq Y_i \land Y_{N(i)})$ for any $i \in [n]$, we have

$$E[T_3] = E \left[ \frac{1}{n^3} \sum_{i,j,N(i),N(j) \text{ distinct}} \text{ Cov } \left[ \min \{ R_i, R_{N(i)} \}, \min \{ R_j, R_{N(j)} \} \right| \mathbf{X} \right]$$

$$= \frac{n(n-1)}{n^3} E \left[ \text{ Cov } \left[ \min \{ R_1, R_{N(1)} \}, \min \{ R_2, R_{N(2)} \} \right| \mathbf{X} \right] \mathbb{1}(1, 2, N(1), N(2) \text{ distinct})$$

$$= \frac{n-1}{n^2} E \left[ \text{ Cov } \left[ \sum_{k=1}^n \mathbb{1}(Y_k \leq Y_1 \land Y_{N(1)}), \sum_{\ell=1}^n \mathbb{1}(Y_\ell \leq Y_2 \land Y_{N(2)}) \right| \mathbf{X} \right] \mathbb{1}(1, 2, N(1), N(2) \text{ distinct}) \right].$$

Notice that for $k, \ell \neq 1, 2, N(1), N(2)$ and $k \neq \ell$, under the event $\{1, 2, N(1), N(2) \text{ distinct}\}$, we have

$$\text{ Cov } \left[ \mathbb{1}(Y_k \leq Y_1 \land Y_{N(1)}), \mathbb{1}(Y_\ell \leq Y_2 \land Y_{N(2)}) \right| \mathbf{X} \right] = 0.$$

Then by the symmetry,

$$E \left[ \text{ Cov } \left[ \sum_{k=1}^n \mathbb{1}(Y_k \leq Y_1 \land Y_{N(1)}), \sum_{\ell=1}^n \mathbb{1}(Y_\ell \leq Y_2 \land Y_{N(2)}) \right| \mathbf{X} \right] \mathbb{1}(1, 2, N(1), N(2) \text{ distinct}) \right]$$

$$= (n-2) E \left[ \text{ Cov } \left[ \mathbb{1}(Y_3 \leq Y_1 \land Y_{N(1)}), \mathbb{1}(Y_3 \leq Y_2 \land Y_{N(2)}) \right| \mathbf{X} \right] \mathbb{1}(1, 2, N(1), N(2) \text{ distinct}) \right]$$

$$= 0.$$
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_1 \leq Y_1 \wedge Y_{N(1)}), \mathbb{1} (Y_3 \leq Y_2 \wedge Y_{N(2)}) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) \\
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_{N(1)} \leq Y_1 \wedge Y_{N(1)}), \mathbb{1} (Y_3 \leq Y_2 \wedge Y_{N(2)}) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) \\
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_2 \leq Y_1 \wedge Y_{N(1)}), \mathbb{1} (Y_3 \leq Y_2 \wedge Y_{N(2)}) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) \\
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_{N(2)} \leq Y_1 \wedge Y_{N(1)}), \mathbb{1} (Y_3 \leq Y_2 \wedge Y_{N(2)}) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) \} + O(1),

where \( O(1) \) is from the boundedness of the indicator function and the number of the remaining terms and the overlap terms is \( O(1) \).

Noticing \( P(1, 2, N(1), N(2) \text{ distinct}) = 1 - O(n^{-1}) \), we have

\[
E \left[ \text{Cov} \left\{ \mathbb{1} (Y_3 \leq Y_1 \wedge Y_{N(1)}), \mathbb{1} (Y_3 \leq Y_2 \wedge Y_{N(2)}) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) \\
= E \left[ \text{Cov} \left\{ \mathbb{1} (Y_3 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{1} (Y_3 \leq Y_2 \wedge \tilde{Y}_2) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) + o(1)
\]

Similarly,

\[
E \left[ \text{Cov} \left\{ \mathbb{1} (Y_1 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{1} (Y_3 \leq Y_2 \wedge \tilde{Y}_2) \right\} \left| X \right. \right] + o(1)
\]

where the last step is by expanding the covariance in the same way as (3.11).

Then it holds true that

\[
E[T_5] = (1 + O(n^{-1})) \left\{ E \left[ \text{Cov} \left\{ \mathbb{1} (Y_3 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{1} (Y_3 \leq Y_2 \wedge \tilde{Y}_2) \right\} \left| X_1, X_2, X_3 \right. \right] \\
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_1 \leq Y_1 \wedge \tilde{Y}_1), F_Y (Y_2 \wedge \tilde{Y}_2) \right\} \left| X_1, X_2 \right. \right] \\
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_1 \leq Y_1 \wedge \tilde{Y}_1), F_Y (Y_2 \wedge \tilde{Y}_2) \right\} \left| X_1, X_2 \right. \right] \\
+ 2E \left[ \text{Cov} \left\{ \mathbb{1} (Y_2 \leq Y_1 \wedge \tilde{Y}_1), F_Y (Y_2 \wedge \tilde{Y}_2) \right\} \left| X_1, X_2 \right. \right] \} + o(1)
\]

\[
= (1 + O(n^{-1})) \left\{ E \left[ \text{Cov} \left\{ \mathbb{1} (Y_3 \leq Y_1 \wedge \tilde{Y}_1), \mathbb{1} (Y_3 \leq Y_2 \wedge \tilde{Y}_2) \right\} \left| X_1, X_2, X_3 \right. \right] \\
+ 4E \left[ \text{Cov} \left\{ \mathbb{1} (Y_2 \leq Y_1 \wedge \tilde{Y}_1), F_Y (Y_2 \wedge \tilde{Y}_2) \right\} \left| X_1, X_2 \right. \right] \} + o(1).
\] (3.12)

On the other hand,

\[
E[T_5^*] = E \left[ \frac{1}{n^2} \sum_{i,j,N(1),N(j) \text{ distinct}} \text{Cov} \left\{ \min \{ R_i, R_{N(i)} \}, \min \{ F_Y (Y_j), F_Y (Y_{N(j)}) \} \right\} \left| X \right. \right]
\]

\[
= \frac{n(n-1)}{n^2} E \left[ \text{Cov} \left\{ \min \{ R_i, R_{N(i)} \}, \min \{ F_Y (Y_{N(1)}), F_Y (Y_{N(2)}) \} \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right)
\]

\[
= \frac{n-1}{n} \left\{ E \left[ \text{Cov} \left\{ \mathbb{1} (Y_1 \leq Y_1 \wedge Y_{N(1)}), F_Y (Y_2 \wedge Y_{N(2)}) \right\} \left| X \right. \right] \mathbb{1} \left( 1, 2, N(1), N(2) \text{ distinct} \right) \right\}
\]
Proof of Lemma 3.5

Proof of Lemma 2.3. Since \( (X_i, Y_i)_{i=1}^n \) are i.i.d. and \( \min \{ R_i, R_N(i) \} = \sum_{k=1}^n \mathbb{1}(Y_k \leq Y_i \land Y_N(i)) \) for any \( i \in \{n\} \), we have

\[
E[T_6] = E \left[ \frac{1}{n^3} \sum_{i=1}^n \text{Cov} \left[ \min \{ R_i, R_N(i) \}, \sum_{i,j=1 \atop i \neq j}^n \min \{ F_Y(Y_i), F_Y(Y_j) \} \bigg\vert X \right] \right]
\]

\[
= \frac{1}{n^2} E \left[ \text{Cov} \left[ \min \{ R_1, R_N(1) \}, \sum_{i,j=1 \atop i \neq j}^n \min \{ F_Y(Y_i), F_Y(Y_j) \} \bigg\vert X \right] \right]
\]

\[
= \frac{1}{n^2} E \left[ \text{Cov} \left[ \sum_{k=1}^n \mathbb{1}(Y_k \leq Y_1 \land Y_N(1)), \sum_{i,j=1 \atop i \neq j}^n F_Y(Y_i \land Y_j) \bigg\vert X \right] \right]
\]

\[
= \frac{1}{n^2} \sum_{k=1}^n E \left[ \text{Cov} \left[ \mathbb{1}(Y_k \leq Y_1 \land Y_N(1)), \sum_{i,j=1 \atop i \neq j}^n F_Y(Y_i \land Y_j) \bigg\vert X \right] \right] + O(n^{-1})
\]

\[
= \frac{(n-1)(n-2)}{n^2} \left\{ 2E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 \leq Y_1 \land Y_N(1)), F_Y(Y_1 \land Y_2) \bigg\vert X \right] \right] \right. \\
+ 2E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 \leq Y_1 \land Y_N(1)), F_Y(Y_N(1) \land Y_2) \bigg\vert X \right] \right] \\
+ 2E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 \leq Y_1 \land Y_N(1)), F_Y(Y_3 \land Y_2) \bigg\vert X \right] \right] \right\} + O(n^{-1})
\]

\[
= (1 + O(n^{-1})) \left\{ 2E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1), F_Y(Y_1 \land Y_2) \bigg\vert X \right] \right] \right. \\
+ 2E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1), F_Y(Y_3 \land Y_2) \bigg\vert X \right] \right] \\
+ 2E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 \leq Y_1 \land \tilde{Y}_1), F_Y(Y_3 \land Y_2) \bigg\vert X \right] \right] \right\} + o(1)
\]

Combining (3.12) and (3.13) completes the proof of the first claim. The second claim is direct from the definition of \( T_5^* \). \( \square \)
\begin{align*}
&= (1 + O(n^{-1})) \left\{ 4E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \bar{Y}_1), F_Y(Y_1 \wedge Y_2) \ \big| \ X, X_2 \right] \right] \\
&\quad + 2E \left[ \text{Cov} \left[ \mathbb{I}(Y_3 \leq Y_1 \wedge \bar{Y}_1), F_Y(Y_3 \wedge Y_2) \ \big| \ X \right] \right] \right\} + o(1).
\end{align*}

On the other hand,
\begin{align*}
E[T_n^*] &= E \left[ \frac{1}{n^3} \sum_{i=1}^n \text{Cov} \left[ \min \left\{ F_Y(Y_i), F_Y(Y_{N(i)}) \right\}, \sum_{i,j=1}^n \min_{i \neq j} \left\{ F_Y(Y_i), F_Y(Y_j) \right\} \ \big| \ X \right] \right] \\
&= \frac{1}{n} \left[ \text{Cov} \left[ \min \left\{ F_Y(Y_1), F_Y(Y_{N(1)}) \right\}, \sum_{i,j=1}^n \min_{i \neq j} \left\{ F_Y(Y_i), F_Y(Y_j) \right\} \ \big| \ X \right] \right] \\
&= \frac{n-1}{n} \left\{ 2E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge Y_{N(1)}), F_Y(Y_1 \wedge Y_2) \ \big| \ X \right] \right] \\
&\quad + 2E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge Y_{N(1)}), F_Y(Y_{N(1)} \wedge Y_2) \ \big| \ X \right] \right] \right\} + O(n^{-1}) \\
&= (1 + O(n^{-1})) \left\{ 2E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \bar{Y}_1), F_Y(Y_1 \wedge Y_2) \ \big| \ X \right] \right] \\
&\quad + 2E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \bar{Y}_1), F_Y(\bar{Y}_1 \wedge Y_2) \ \big| \ X \right] \right] \right\} + o(1) \\
&= (1 + O(n^{-1})) 4E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge \bar{Y}_1), F_Y(Y_1 \wedge Y_2) \ \big| \ X, X_2 \right] \right] + o(1).
\end{align*}

Combining (3.14) and (3.15) and expanding the covariance, we obtain
\begin{align*}
\lim_{n \to \infty} \left[ E[T_n^*] - E[T_n] \right] &= 2E \left[ \text{Cov} \left[ \mathbb{I}(Y_3 \leq Y_1 \wedge \bar{Y}_1), F_Y(Y_3 \wedge Y_2) \ \big| \ X \right] \right] \\
&= 2E \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \bar{Y}_1), F_Y(Y_2 \wedge Y) \ \big| \ X, X_2 \right] \right],
\end{align*}
and thus complete the proof. \hfill \Box

### 3.6 Proof of Lemma 2.4

**Proof of Lemma 2.4.** Since \([(X_i, Y_i)]_{i=1}^n\] are i.i.d., we have
\begin{align*}
E[T_n^*] &= E \left[ \frac{1}{n^3} \text{Var} \left[ \sum_{i,j=1}^n \min_{i \neq j} \left\{ F_Y(Y_i), F_Y(Y_j) \right\} \ \big| \ X \right] \right] \\
&= \frac{1}{n^3} E \left[ \text{Cov} \left[ \sum_{i,j=1}^n F_Y(Y_i \wedge Y_j), \sum_{k,\ell=1}^n F_Y(Y_k \wedge Y_\ell) \ \big| \ X \right] \right] \\
&= \frac{1}{n^3} E \left[ \sum_{i,j=1}^n \sum_{k,\ell=1}^n \text{Cov} \left[ F_Y(Y_i \wedge Y_j), F_Y(Y_k \wedge Y_\ell) \ \big| \ X \right] \right].
\end{align*}

Notice that when \(i, j, k, \ell\) are distinct, the covariance is zero. Then
\begin{align*}
E[T_n^*] &= \frac{4n(n-1)(n-2)}{n^3} E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge Y_2), F_Y(Y_1 \wedge Y_3) \ \big| \ X \right] \right] + \frac{2n(n-1)}{n^3} E \left[ \text{Var} \left[ F_Y(Y_1 \wedge Y_2) \ \big| \ X \right] \right] \\
&= (1 + O(n^{-1})) 4E \left[ \text{Cov} \left[ F_Y(Y_1 \wedge Y_2), F_Y(Y_1 \wedge Y_3) \ \big| \ X \right] \right] + O(n^{-1}).
\end{align*}
Expanding the covariance, we obtain
\[
\lim_{n \to \infty} E[T_r^*] = 4E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y_2), F_Y(Y_1 \wedge Y_3) \mid X \right) \right] = 4E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y), F_Y(Y_1 \wedge \bar{Y}) \mid X_1 \right) \right]
\]
and thus complete the proof. \(\square\)

3.7 Proof of Lemma 2.5

Proof of Lemma 2.5. For \(T^*_8\),
\[
E[T_8^*] = E \left[ \frac{1}{n} \sum_{i=1}^{n} \text{Cov} \left( \min \{ F_Y(Y_i), F_Y(Y_{N(i)}) \}, \sum_{i=1}^{n} g(Y_i) \mid X \right) \right]
\]
\[
= E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y_{N(1)}), \sum_{i=1}^{n} g(Y_i) \mid X \right) \right]
\]
\[
= E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y_{N(1)}), g(Y_1) \mid X \right) \right] + E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y_{N(1)}), g(Y_{N(1)}) \mid X \right) \right]
\]
\[= 2E \left[ \text{Cov} \left( F_Y(Y_1 \wedge \bar{Y}_1), g(Y_1) \mid X_1 \right) \right] + o(1).
\]

For \(T^*_9\),
\[
E[T_9^*] = E \left[ \frac{1}{n^2} \sum_{i,j=1 \atop i \neq j}^{n} g(Y_i, Y_j) \mid X \right]
\]
\[
= \frac{n(n-1)}{n^2} E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y_2), \sum_{i=1}^{n} g(Y_i) \mid X \right) \right]
\]
\[
= 2(1 + O(n^{-1})) E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y_2), g(Y_1) \mid X_1, X_2 \right) \right]
\]
\[= 2(1 + O(n^{-1})) E \left[ \text{Cov} \left( F_Y(Y_1 \wedge Y), g(Y_1) \mid X_1 \right) \right].
\]

For \(T^*_7\), we have
\[
E[T_7^*] = E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \text{Cov} \left( \min \{ R_i, R_{N(i)} \}, \sum_{i=1}^{n} g(Y_i) \mid X \right) \right]
\]
\[
= \frac{1}{n} E \left[ \text{Cov} \left( R_1 \wedge R_{N(1)}, \sum_{i=1}^{n} g(Y_i) \mid X \right) \right] = \frac{1}{n} E \left[ \text{Cov} \left( \sum_{k=1}^{n} (Y_k \leq Y_1 \wedge Y_{N(1)}), \sum_{i=1}^{n} g(Y_i) \mid X \right) \right]
\]
\[
= \frac{n-1}{n} E \left[ \text{Cov} \left( \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}), \sum_{i=1}^{n} g(Y_i) \mid X \right) \right] + O(n^{-1})
\]
\[= (1 + O(n^{-1})) E \left[ \text{Cov} \left( \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}), g(Y_1) \mid X \right) \right] + E \left[ \text{Cov} \left( \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}), g(Y_2) \mid X \right) \right]
\]
\[+ E \left[ \text{Cov} \left( \mathbb{1}(Y_2 \leq Y_1 \wedge Y_{N(1)}), g(Y_{N(1)}) \mid X \right) \right] + O(n^{-1})
\]
\[= (1 + O(n^{-1})) \left[ 2E \left[ \text{Cov} \left( \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1), g(Y_1) \mid X_1 \right) \right] + E \left[ \text{Cov} \left( \mathbb{1}(Y_2 \leq Y_1 \wedge \bar{Y}_1), g(Y_2) \mid X_1, X_2 \right) \right] \right] + o(1)
\]
\[
= (1 + O(n^{-1})) \left[ 2E \left[ \text{Cov} \left[ F_Y(Y_1 ∧ \tilde{Y}_1), g(Y_1) \mid X_1 \right] \right] + E \left[ \text{Cov} \left[ \mathbb{1}(Y_2 ≤ Y_1 ∧ \tilde{Y}_1), g(Y_2) \mid X_1, X_2 \right] \right] \right] + o(1).
\]

For \( T_{10}^\prime \), the result is direct from the variance of the sample mean. \( \Box \)

### 3.8 Proof of Lemma 2.6

**Proof of Lemma 2.6.**

**Part I.** \( a_1 + 4a_2 + 4a_3 \).

Recall that \( G_X(t) = P(Y ≥ t \mid X) \) and \( h(t) = E[G_Y^2(t)] \). Then

\[
a_1 = E \left[ \text{Cov} \left[ \mathbb{1}(Y_3 ≤ Y_1 ∧ \tilde{Y}_1), \mathbb{1}(Y_3 ≤ Y_2 ∧ \tilde{Y}_2) \mid X_1, X_2, X_3 \right] \right]
\]

\[
= E \left[ \int G_{X_1}(t)G_{X_2}(t) d\mu_X(t) \right] - \left( \int G_{X_1}(t) d\mu_X(t) \right) \left( \int G_{X_2}(t) d\mu_X(t) \right) \]

\[
= E \left[ \int h^2(t) d\mu_X(t) - \left( \int h(t) d\mu_X(t) \right) \left( \int h(t) d\mu_X(t) \right) \right]
\]

\[
= E \left[ \text{Var} \left[ h(Y_1) \mid X_1 \right] \right]. \quad (3.16)
\]

Let \( h'(t) := F_Y(t) - F_Y^2(t)/2 \). Notice that for \( Y \sim F_Y \), we have \( F_Y(Y) \sim U(0, 1) \) from the probability integral transform. Then

\[
a_3 = E \left[ \text{Cov} \left[ F_Y(Y_1 ∧ Y), F_Y(Y_1 ∧ \tilde{Y}) \mid X_1 \right] \right]
\]

\[
= E \left[ \int E \left[ \left( F_Y(t) ∧ F_Y(Y) \right) \left( F_Y(t) ∧ F_Y(\tilde{Y}) \right) \right] d\mu_X(t) \right]
\]

\[
- \left( \int E \left[ F_Y(t) ∧ F_Y(Y) \right] d\mu_X(t) \right) \left( \int E \left[ F_Y(t) ∧ F_Y(\tilde{Y}) \right] d\mu_X(t) \right) \]

\[
= E \left[ \int (F_Y(t) - F_Y^2(t)/2)^2 d\mu_X(t) \right]
\]

\[
- \left( \int (F_Y(t) - F_Y^2(t)/2) d\mu_X(t) \right) \left( \int (F_Y(t) - F_Y^2(t)/2) d\mu_X(t) \right) \]

\[
= E \left[ \text{Var} \left[ h'(Y_1) \mid X_1 \right] \right]. \quad (3.17)
\]

In the same way as \( a_1 \) and \( a_3 \),

\[
a_2 = E \left[ \text{Cov} \left[ \mathbb{1}(Y_2 ≤ Y_1 ∧ \tilde{Y}_1), F_Y(Y_2 ∧ Y) \mid X_1, X_2 \right] \right]
\]

\[
= E \left[ \int G_{X_1}(t)(F_Y(t) - F_Y^2(t)/2) d\mu_X(t) \right]
\]

\[
- \left( \int G_{X_1}(t) d\mu_X(t) \right) \left( \int (F_Y(t) - F_Y^2(t)/2) d\mu_X(t) \right) \]

\[
= E \left[ \int h(t)h'(t) d\mu_X(t) - \left( \int h(t) d\mu_X(t) \right) \left( \int h'(t) d\mu_X(t) \right) \right]
\]

\[
= E \left[ \text{Cov} \left[ h(Y_1), h'(Y_1) \mid X_1 \right] \right]. \quad (3.18)
\]
Noticing that
\[ h(t) = \mathbb{E}[G_X^2(t)] = g(t) + G^2(t) = g(t) + (1 - F_Y(t))^2 = 1 - 2h'(t) + g(t). \] (3.19)
and combining (3.16)-(3.19) yields
\[ a_1 + 4a_2 + 4a_3 = \mathbb{E} \left[ \text{Var} \left[ h(Y_1) \mid X_1 \right] \right] + 4\mathbb{E} \left[ \text{Cov} \left[ h(Y_1), h'(Y_1) \mid X_1 \right] \right] + 4\mathbb{E} \left[ \text{Var} \left[ h'(Y_1) \mid X_1 \right] \right]
= \mathbb{E} \left[ \text{Var} \left[ h(Y_1) + 2h'(Y_1) \mid X_1 \right] \right]
= \mathbb{E} \left[ \text{Var} \left[ g(Y_1) \mid X_1 \right] \right].
\]
The first part’s proof is then complete.

**Part II.** \( b_3 - 2b_1 + 2b_2. \)

In the same way as the first part,
\[ b_3 - 2b_1 = \mathbb{E} \left[ \text{Cov} \left[ \mathbb{I}(Y_2 \leq Y_1 \wedge \bar{Y}_1), g(Y_2) \mid X_1, X_2 \right] \right]
= \mathbb{E} \left[ \int G_{X_1}^2(t)g(t)d\mu_{X_2}(t) - \left( \int G_{X_1}^2(t)d\mu_{X_2}(t) \right) \left( \int g(t)d\mu_{X_2}(t) \right) \right]
= \mathbb{E} \left[ \int h(t)g(t)d\mu_{X_2}(t) - \left( \int h'(t)d\mu_{X_2}(t) \right) \left( \int g(t)d\mu_{X_2}(t) \right) \right]
= \mathbb{E} \left[ \text{Cov} \left[ h(Y_1), g(Y_1) \mid X_1 \right] \right]. \] (3.20)

and
\[ b_2 = \mathbb{E} \left[ \text{Cov} \left[ F_Y(Y_1 \wedge Y), g(Y_1) \mid X_1 \right] \right]
= \mathbb{E} \left[ \int (F_Y(t) - F_Y^2(t)/2)g(t)d\mu_{X_1}(t) - \left( \int (F_Y(t) - F_Y^2(t)/2)d\mu_{X_1}(t) \right) \left( \int g(t)d\mu_{X_1}(t) \right) \right]
= \mathbb{E} \left[ \int h'(t)g(t)d\mu_{X_1}(t) - \left( \int h'(t)d\mu_{X_1}(t) \right) \left( \int g(t)d\mu_{X_1}(t) \right) \right]
= \mathbb{E} \left[ \text{Cov} \left[ h'(Y_1), g(Y_1) \mid X_1 \right] \right]. \] (3.21)

Combining (3.19)-(3.21) yields
\[ b_3 - 2b_1 + 2b_2 = \mathbb{E} \left[ \text{Cov} \left[ h(Y_1), g(Y_1) \mid X_1 \right] \right] + 2\mathbb{E} \left[ \text{Cov} \left[ h'(Y_1), g(Y_1) \mid X_1 \right] \right]
= \mathbb{E} \left[ \text{Var} \left[ h(Y_1) + 2h'(Y_1), g(Y_1) \mid X_1 \right] \right]
= \mathbb{E} \left[ \text{Var} \left[ g(Y_1) \mid X_1 \right] \right]. \]
The second part’s proof is then complete. \( \square \)

### 3.9 Proof of Lemma 2.7

**Proof of Lemma 2.7.** From the boundedness of the indicator function and \( h, \) we have
\[ \bar{T}_1 \]
\[ = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \mathbb{E} \left[ \mathbb{I}(Y_{\ell} \leq Y_i \wedge Y_{N(i)}) \mid X \right] - h(X_{\ell}) \right)^2 \right] \]

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We then complete the proof.

\((1 + O(n^{-1})) \mathbb{E} \left[ \left( \mathbb{E} \left[ \mathbf{1} (Y_3 \leq Y_1 \wedge Y_{N(1)}) \mid X_1, X_3 \right] - h(X_3) \right)^2 \right] = 0.

We then complete the proof. \hfill \square

3.10 Proof of Lemma 2.8

Proof of Lemma 2.8. Since the indicator function and \(F_Y\) are both bounded and \([X_i]_{i=1}^{n}\) are i.i.d.,

\[
\tilde{T}_2 = \mathbb{E} \left[ \frac{1}{n} \sum_{k=1, k \neq \ell} \left( \mathbb{E} \left[ \mathbf{1} (Y_k \leq Y_\ell \wedge Y_{N(\ell)}) \mid X_1 \right] - \mathbb{E} \left[ F_Y (Y_\ell \wedge Y_{N(\ell)}) \mid X_1 \right] \right) \right] = 1 + O(n^{-1}) \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1} (Y_2 \leq Y_1 \wedge Y_{N(1)}) \mid X_1, X_2 \right] - \mathbb{E} \left[ F_Y (Y_1 \wedge Y_{N(1)}) \mid X_1 \right] \right] + O(n^{-1})
\]

\[
= 1 + O(n^{-1}) \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1} (Y_2 \leq Y_1 \wedge \tilde{Y}_1) \mid X_1, X_2 \right] - \mathbb{E} \left[ F_Y (Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] + O(n^{-1})
\]

The proof is then complete. \hfill \square

3.11 Proof of Lemma 2.9

Proof of Lemma 2.9. Lemma 20.6 together with Theorem 20.16 in Biou and Devroye (2015) show that \(|\{i : N(i) = \ell\}|, |\{i : \tilde{N}(i) = \ell\}|\) are both bounded by a constant that only depend on \(d\). Notice that \(P(N(1) = 4), P(\tilde{N}(1) = 4) = O(n^{-1})\). We assume \(\ell = 4\) without loss of generality. Then from
the Cauchy–Schwarz inequality,

\[ \tilde{T}_3 = E \left[ \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E \left[ \mathbb{1} (Y_k \leq Y_i \wedge Y_{N(i)}) \mid X \right] - E \left[ F_Y (Y_i \wedge Y_{N(i)}) \mid X \right] \right)^2 \right] \]

\[ \leq E \left[ \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E \left[ \mathbb{1} (Y_k \leq Y_i \wedge Y_{N(i)}) \mid X \right] - E \left[ F_Y (Y_i \wedge Y_{N(i)}) \mid X \right] \right)^2 \right] \]

\[ \leq (n - 1) E \left[ \left( \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E \left[ \mathbb{1} (Y_k \leq Y_1 \wedge Y_{N(1)}) \mid X \right] - E \left[ F_Y (Y_1 \wedge Y_{N(1)}) \mid X \right] \right)^2 \right] \]

\[ + \mathbb{1} \left( N(1) = \ell \text{ or } \tilde{N}(1) = \ell \right) \]

\[ = n(1 + O(n^{-1})) E \left[ \left( \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E \left[ \mathbb{1} (Y_k \leq Y_1 \wedge Y_{N(1)}) \mid X \right] - E \left[ F_Y (Y_1 \wedge Y_{N(1)}) \mid X \right] \right)^2 \right] \]

\[ + O \left( P \left( N(1) = \ell \right) + P \left( \tilde{N}(1) = \ell \right) \right) \]

\[ = 2(1 + O(n^{-1})) E \left[ \left( \frac{1}{n} \sum_{k=1, k \neq \ell}^{n} E \left[ \mathbb{1} (Y_k \leq Y_1 \wedge Y_{N(1)}) \mid X \right] - E \left[ F_Y (Y_1 \wedge Y_{N(1)}) \mid X \right] \right)^2 \right] + O(n^{-1}). \]

The last step is true since \( \sum_{k=1}^{n} \mathbb{1}(N(1) = k) = 1 - \mathbb{1}(N(1) = 2, 3), P(N(1) = 2, 3) = O(n^{-1}), \) and \([X_i]_{i=1}^{n}\) are i.i.d..

Invoking the same idea as used in the proof of Lemma 2.8 then completes the proof.

\[ \square \]

### 3.12 Proof of Lemma 2.10

**Proof of Lemma 2.10.** For the first statement, notice that

\[ E \left[ \text{Var} \left[ F_Y (Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] = E \left[ E \left[ F_Y^2 (Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] - E \left[ \left( E \left[ F_Y (Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right)^2 \right] \]

\[ = E \left[ E \left[ F_Y^2 (Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] - E \left[ E \left[ F_Y (Y_1 \wedge \tilde{Y}_1) F_Y (\tilde{Y}_1' \wedge \tilde{Y}_1'') \mid X_1 \right] \right], \]

where \( \tilde{Y}_1', \tilde{Y}_1'' \) are independently drawn from \( Y \mid X_1 \) and are further independent of \( Y_1, \tilde{Y}_1 \) conditional on \( X_1 \).
For the first term above, letting $F_Y^{(n)}$ be the empirical distribution of $\{Y_i\}_{i=1}^n$, one then has

$$\frac{1}{n^3} \sum_{i=1}^n \left( R_i \wedge R_{N(i)} \right)^2 = \frac{1}{n} \sum_{i=1}^n \left( F_Y^{(n)}(Y_i \wedge Y_{N(i)}) \right)^2$$

$$= \left[ \frac{1}{n} \sum_{i=1}^n \left( F_Y^{(n)}(Y_i \wedge Y_{N(i)}) \right)^2 - \frac{1}{n} \sum_{i=1}^n \left( F_Y(Y_i \wedge Y_{N(i)}) \right)^2 \right] + \frac{1}{n} \sum_{i=1}^n F_Y^2(Y_i \wedge Y_{N(i)}).$$

Using the Glivenko-Cantelli theorem (Theorem 19.1 in van der Vaart (1998)) and that fact that $F_Y, F_Y^{(n)}$ are bounded by 1, one has

$$\left| \frac{1}{n} \sum_{i=1}^n \left( F_Y^{(n)}(Y_i \wedge Y_{N(i)}) \right)^2 - \frac{1}{n} \sum_{i=1}^n \left( F_Y(Y_i \wedge Y_{N(i)}) \right)^2 \right| \leq 2\|F_Y^{(n)} - F_Y\|_{\infty} \xrightarrow{a.s.} 0,$$

with "\xrightarrow{a.s.}" representing strong convergence.

Then it suffices to consider the second term. We use bias-variance decomposition. Notice that

$$E \left[ \frac{1}{n} \sum_{i=1}^n F_Y^2(Y_i \wedge Y_{N(i)}) \right] = E \left[ E \left[ F_Y^2(Y_1 \wedge Y_{N(1)}) \mid X \right] \right]$$

$$= E \left[ \int 1(Y_1 \wedge Y_{N(1)} \geq t_1) 1(Y_1 \wedge Y_{N(1)} \geq t_2) d\mu_Y(t_1) d\mu_Y(t_2) \mid X \right]$$

$$= E \left[ \int 1(Y_1 \geq t_1 \vee t_2) 1(Y_{N(1)} \geq t_1 \vee t_2) d\mu_Y(t_1) d\mu_Y(t_2) \mid X \right]$$

$$= E \left[ \int G_{X_1}(t_1 \vee t_2) G_{X_{N(1)}}(t_1 \vee t_2) d\mu_Y(t_1) d\mu_Y(t_2) \right].$$

On the other hand, one can check that

$$E \left[ E \left[ F_Y^2(Y_1 \wedge \tilde{Y}_1) \mid X \right] \right] = E \left[ \int G_{X_1}(t_1 \vee t_2) d\mu_Y(t_1) d\mu_Y(t_2) \right].$$

Lemma 11.7 in Azadkia and Chatterjee (2021) then implies that the bias is

$$\limsup_{n \to \infty} E \left[ \frac{1}{n} \sum_{i=1}^n F_Y^2(Y_i \wedge Y_{N(i)}) \right] - E \left[ E \left[ F_Y^2(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right]$$

$$= \limsup_{n \to \infty} \left| E \left[ \int G_{X_1}(t_1 \vee t_2) \left( G_{X_{N(i)}}(t_1 \vee t_2) - G_{X_1}(t_1 \vee t_2) \right) d\mu_Y(t_1) d\mu_Y(t_2) \right] \right| = 0.$$

From the Efron-Stein inequality and the fact that $|\{j : N(j) = i\}|$ is always bounded for any $i \in [n]$, the variance is

$$\text{Var} \left[ \frac{1}{n} \sum_{i=1}^n F_Y^2(Y_i \wedge Y_{N(i)}) \right] = O\left( \frac{1}{n} \right).$$

Combining the bias and the variance yields

$$\frac{1}{n^2} \sum_{i=1}^n \left( R_i \wedge R_{N(i)} \right)^2 - E \left[ E \left[ F_Y^2(Y_1 \wedge \tilde{Y}_1) \mid X_1 \right] \right] \xrightarrow{p} 0.$$
In the same way and noticing that \( i, N(i), N_2(i), N_3(i) \) are all different for any \( i \in \{n\} \),

\[
\frac{1}{n^3} \sum_{i=1}^{n} \left( R_i \wedge R_N(i) \left( R_{N_2(i)} \wedge R_{N_3(i)} \right) \right) = \frac{1}{n} \sum_{i=1}^{n} F_Y(n) \left( Y_i \wedge Y_N(i) \right) F_Y(n) \left( Y_{N_2(i)} \wedge Y_{N_3(i)} \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} F_Y(Y_i \wedge Y_N(i)) F_Y(Y_{N_2(i)} \wedge Y_{N_3(i)}) + o_{p}(1)
\]

\[
= E \left[ E \left( F_Y(Y_1 \wedge \tilde{Y}_1) F_Y(\tilde{Y}_1' \wedge \tilde{Y}_1'') \left| X_1 \right. \right) \right] + o_{p}(1).
\]

We then complete the proof of the first statement, and the fourth statement holds in the same way. The second and the third statements can be established similarly by noticing that

\[
\text{Cov} \left[ F_Y(Y_1 \wedge \tilde{Y}_1), F_Y(\tilde{Y}_1 \wedge Y_1') \left| X_1 \right. \right] = E \left[ F_Y(Y_1 \wedge \tilde{Y}_1) F_Y(\tilde{Y}_1 \wedge Y_1') \left| X_1 \right. \right] - E \left[ F_Y(Y_1 \wedge \tilde{Y}_1) F_Y(\tilde{Y}_1' \wedge \tilde{Y}_1'') \left| X_1 \right. \right].
\]

For the fifth statement,

\[
\frac{1}{n^2(n-1)} \sum_{i,j=1}^{n} 1 \left( R_i \leq R_j \wedge R_{N(j)} \right) R_i \wedge R_{N(i)} = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} 1 \left( Y_i \leq Y_j \wedge Y_{N(j)} \right) F_Y^2(n) \left( Y_i \wedge Y_{N(i)} \right)
\]

\[
= \frac{1}{n(n-1)} \sum_{i,j=1}^{n} 1 \left( Y_i \leq Y_j \wedge Y_{N(j)} \right) F_Y(Y_i \wedge Y_{N(i)}) + o_{p}(1).
\]

Notice that \( P(N(1) = 2) \) and \( P(N(1) = N(2)) \) are both \( O(n^{-1}) \). Then the expectation is

\[
E \left[ \frac{1}{n(n-1)} \sum_{i,j=1}^{n} 1 \left( Y_i \leq Y_j \wedge Y_{N(j)} \right) F_Y(Y_i \wedge Y_{N(i)}) \right] = E \left[ 1 \left( Y_2 \leq Y_1 \wedge Y_{N(1)} \right) F_Y(Y_2 \wedge Y_{N(2)}) \right]
\]

\[
= E \left[ \int 1 \left( Y_2 \leq Y_1 \wedge Y_{N(1)} \right) 1 \left( Y_2 \geq t \right) 1 \left( Y_{N(2)} \geq t \right) d\mu_Y(t) \right]
\]

\[
= E \left[ \int G_X(1)(Y_2)G_{N(1)}(Y_2) 1 \left( Y_2 \geq t \right) 1 \left( Y_{N(2)} \geq t \right) d\mu_Y(t) \right] + O \left( \frac{1}{n} \right)
\]

\[
= E \left[ \int G_X^2(1)(Y_2) 1 \left( Y_2 \geq t \right) 1 \left( Y_{N(2)} \geq t \right) d\mu_Y(t) \right] + o(1)
\]

\[
= E \left[ \int h(Y_2) 1 \left( Y_2 \geq t \right) 1 \left( Y_{N(2)} \geq t \right) d\mu_Y(t) \right] + o(1)
\]

\[
= E \left[ \int G_{N(2)}(t) d\mu_Y(t) \right] + o(1) + E \left[ \int G_X^2(t) G_{N(2)}(t) d\mu_Y(t) \right] + o(1)
\]

\[
= \int E \left[ G_X^2(t) G_{N(2)}(t) d\mu_Y(t) \right] + o(1),
\]

where \( G_x^*(t) := E[h(Y) 1(Y \geq t) \mid X = x] \) for \( x \in \mathbb{R}^d \).

On the other hand, we can check

\[
E \left[ 1 \left( Y_2 \leq Y_1 \wedge \tilde{Y}_1 \right) F_Y(Y_2 \wedge \tilde{Y}_2) \left| X_1, X_2 \right. \right] = \int E \left[ G_X^*(t) G_X(t) d\mu_Y(t) \right].
\]

Then the fifth statement is established by using the same argument as before. The sixth statement can also be established in the same way.
3.13 Proof of Lemma 2.11

Proof of Lemma 2.11. The proof is similar to that of Lemma 2.10. The key is to notice that from the definitions of \( h_0 \) and \( h_1 \),

\[
\begin{align*}
\text{Var} \left[ h_0(X_1) \right] &= E \left[ h_0^2(X_1) \right] - \left( E \left[ h_0(X_1) \right] \right)^2, \\
&= E \left[ E \left[ 1(Y_3 \leq Y_1 \wedge \bar{Y}_1) 1(\bar{Y}_3 \leq Y_2 \wedge \bar{Y}_2) \mid X_1, X_2, X_3 \right] \right] - \left( E \left[ E \left[ F_Y(Y_1 \wedge \bar{Y}_1) \mid X_1 \right] \right] \right)^2,
\end{align*}
\]

\[
\begin{align*}
\text{Cov} \left[ h_0(X_1), h_1(X_1) \right] &= E \left[ h_0(X_1)h_1(X_1) \right] - E \left[ h_0(X_1) \right]E \left[ h_1(X_1) \right]
= E \left[ E \left[ 1(Y_2 \leq Y_1 \wedge \bar{Y}_1) F_Y(\bar{Y}_2 \wedge \bar{Y}_2') \mid X_1, X_2 \right] \right] - \left( E \left[ E \left[ F_Y(Y_1 \wedge \bar{Y}_1) \mid X_1 \right] \right] \right)^2,
\end{align*}
\]

\[
\begin{align*}
\text{Var} \left[ h_1(X_1) \right] &= E \left[ h_1^2(X_1) \right] - \left( E \left[ h_1(X_1) \right] \right)^2, \\
&= E \left[ E \left[ F_Y(Y_1 \wedge \bar{Y}_1) F_Y(\bar{Y}_1' \wedge \bar{Y}_1'') \mid X_1 \right] \right] - \left( E \left[ E \left[ F_Y(Y_1 \wedge \bar{Y}_1) \mid X_1 \right] \right] \right)^2.
\end{align*}
\]

All the rest is the same. \( \square \)

3.14 Proof of Lemma 3.1

Proof of Lemma 3.1. For any \( x_1, x_2 \in \mathbb{R}^d \), define \( \Phi(x_1, x_2) := E[F_Y(Y_1 \wedge Y_2) \mid X_1 = x_1, X_2 = x_2] \). Then by the definition of \( \xi_n \) in (1.14),

\[
E[\xi_n^* \mid X] = \frac{6n}{n^2 - 1} E \left[ \sum_{i=1}^n \min \left\{ F_Y(Y_i), F_Y(Y_{N(i)}) \right\} + \sum_{i=1}^n h(Y_i) \mid X \right]
= \frac{6n}{n^2 - 1} \left( \sum_{i=1}^n \Phi(X_i, X_{N(i)}) + \sum_{i=1}^n h_0(X_i) \right).
\]

To apply the Efron-Stein inequality, we implement the same notation as used in the Step II in the proof of Theorem 1.3. It is then true that

\[
\begin{align*}
n \text{Var} \left[ \frac{6n}{n^2 - 1} \sum_{i=1}^n \left( h_1(X_i) + h_0(X_i) \right) - E[\xi_n^* \mid X] \right]
&= n \text{Var} \left[ \frac{6n}{n^2 - 1} \sum_{i=1}^n \left( \Phi(X_i, X_{N(i)}) - h_1(X_i) \right) \right]
= \frac{36n^3}{(n^2 - 1)^2} \text{Var} \left[ \sum_{i=1}^n \left( \Phi(X_i, X_{N(i)}) - h_1(X_i) \right) \right]
\leq \frac{18n^3}{(n^2 - 1)^2} \sum_{\ell=1}^n E \left[ \Phi(X_\ell, X_{N(\ell)}) - h_1(X_\ell) - \Phi(\bar{X}_\ell, X_{\bar{N}(\ell)}) + h_1(\bar{X}_\ell) \right]
+ \sum_{i=1}^n \left( \Phi(X_i, X_{N(i)}) - \Phi(X_i, X_{\bar{N}(i)}) \right)^2
\leq \frac{18n^4}{(n^2 - 1)^2} E \left[ \Phi(X_\ell, X_{N(\ell)}) - h_1(X_\ell) - \Phi(\bar{X}_\ell, X_{\bar{N}(\ell)}) + h_1(\bar{X}_\ell) \right]
= \frac{18n^4}{(n^2 - 1)^2} \text{Var} \left[ \Phi(X_\ell, X_{N(\ell)}) - h_1(X_\ell) - \Phi(\bar{X}_\ell, X_{\bar{N}(\ell)}) + h_1(\bar{X}_\ell) \right].
\end{align*}
\]
\[
+ \sum_{i=1}^{n} \left( \Phi(X_i, X_N(i)) - \Phi(X_i, X'_N(i)) \right)^2,
\]

where \( X'_N(i) = X_N(i) \) if \( \tilde{N}(i) \neq \ell \) and \( X'_N(i) = \tilde{X}_N(i) \) if \( \tilde{N}(i) = \ell \).

From Lemma 11.3 in Azadkia and Chatterjee (2021), \( X_{N(i)} \to X_1 \) almost surely. Then similar to the proof of Lemma 11.7 in Azadkia and Chatterjee (2021), one can establish \( \Phi(X_\ell, X_{N(\ell)}) - \Phi(X_\ell, X_\ell) \) converges to zero in probability. Noticing that \( \Phi(X_\ell, X_\ell) = h_1(X_\ell) \) from the definition of \( h_1 \), one deduces

\[
\lim_{n \to \infty} E \left[ \Phi(X_\ell, X_{N(\ell)}) - h_1(X_\ell) \right] = 0, \quad \lim_{n \to \infty} E \left[ \Phi(\tilde{X}_\ell, X_{\tilde{N}(\ell)}) - h_1(\tilde{X}_\ell) \right] = 0.
\]

Similar to the proof of Lemma 2.9, we then have

\[
\lim_{n \to \infty} E \left[ \sum_{i=1}^{n} \left( \Phi(X_i, X_N(i)) - \Phi(X_i, X'_N(i)) \right)^2 \right] = 0.
\]

Leveraging the Cauchy–Schwarz inequality then completes the proof.

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