Test of the nonrelativistic $c\bar{c}$ potential

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We analyze the charmonium states by testing a phenomenological nonrelativistic potential and propose a new set of parameters. This new set of parameters is fixed using only the lowest lying $S$-wave states of charmonia where the spin-orbit and tensor interactions will not contribute. After fitting the parameters we analyze the whole fine structure of charmonium states taking into account the spin-orbit and tensor interactions too. Calculations showed that the nonrelativistic potential model with the phenomenologically defined parameters is indeed well approximation for describing the charmonium states.

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I. INTRODUCTION

An applicability of any nonrelativistic potential model during the studies of heavy hadrons spectra can be well checked by reproducing the heavy quarkonium states. The best candidate for this role is the charmonium consisting one heavy quark $Q$ and one heavy antiquark $\bar{Q}$ [1,2]. Analyzing the charmonium spectra one can establish some interesting features. For that purpose one keeps in mind some facts in formulating the better approach. First of all, one can note the large value of the current quark mass in the characteristic energy scale $m_Q \gg \Lambda_{QCD}$. The second, the smallness of heavy-quark velocity $v_Q \ll c$ inside the charmonia which can be crudely estimated from the radial excitation energy differences corresponding to the given quantum numbers. These two facts indicate that the relativistic effects could be taken into account as the systematic corrections (for example, see review [3]).

From other side, one can also note the smallness of nonperturbative effects in the spectra of charmonia. The authors of Ref. [4] estimated contribution to the spin-independent heavy quark potential due to the nonperturbative dynamics in the framework of instanton vacuum model of quantum chromodynamics (QCD) [5,6]. They also found the relatively small value of difference between the current and constituent quark masses in comparison with the constituent or current quark mass [6]. The recent calculations [7,8], based on the estimations above, showed that the contributions due to the nonperturbative dynamics also can be considered as the small corrections. In particular, the authors of Ref. [8] showed that the instanton effects are the first order perturbative corrections. Nevertheless, in the Ref. [6] it was discussed that the instantons may shed some light on the origin of parameters of the potentials used in the phenomenological approaches [7,9]. The studies performed in Refs. [4,7] may lead to the conclusion that, although it is very nontrivial at low energies, at the high energies the nonperturbative dynamics seems to be tightly hidden behind the confinement mechanism which is not yet fully understood.

All said above more or less explains the success of phenomenological potential model [11] where the complicated and unknown dynamics is expressed in terms of the effective values of phenomenological parameters. Therefore, one has readily a nonrelativistic Schrödinger approach for describing the energy spectra of heavy quarkonium. Technically, the quarkonium is very similar to the Hydrogen atom, i.e. one can solve the corresponding problem in the given external potential field instead of considering the two body relativistic system. The difference from the Hydrogen atom problem is only due to the nature of interactions and its corresponding range. Consequently, due to the strong nature of interactions the excitation energies of charmonia will be much large than the electron excitation energies in the Hydrogen atom. The size of a charmonium also will be much less in comparison with the Hydrogen atom size due to the short range nature of interactions. One can use these obvious facts in applying a numerical method to the charmonium problem and easily find the appropriate variational parameters of the model.

Starting the discussions of the heavy quark potential one can note that the basic spin-independent central interaction between the quark and antiquark can be well separated into two parts. The first scalar exchange part is fully phenomenological because of an unknown confinement mechanism. The most popular choice for this interaction is expressed as a linearly increasing potential due to the area law of Wilson loop [12] for the heavy-quark potential. The second, the vector exchange part is due to the perturbative one gluon exchange mechanism at short distances and in the lowest order has the Coulomb interaction like form with the corresponding running coupling constant. The spin-dependent parts of interactions can be reproduced from the central potential in the framework of nonrelativistic expansions [13]. The correspond-
ing model is called a nonrelativistic constituent quark model.

In the present work, we discuss the interesting features of the nonrelativistic constituent quark model and propose the new set of parameters for describing the charmonium states on a basis of updated experimental data. While we are doing that, as an input we concentrate only to the minimal part of spectrum instead of considering the whole spectrum. After fitting the parameters in a most compact way we concentrate to the whole spectrum and analyze the applicability of nonrelativistic potential model approach.

The paper is organized in the following way. In the next section we briefly repeat the main features of the model and describe very shortly a variational approach to the problem. In the section the results from calculations will be presented and discussed. In the last section we summarize our results and make the corresponding conclusions.

II. QQ POTENTIAL AND VARIATIONAL METHOD

In the simple constituent quark model, the total \( Q\bar{Q} \) potential has the following standard form

\[
V_{Q\bar{Q}}(r) = V_C(r) + V_{SS}(r)(\mathbf{S}_Q \cdot \mathbf{S}_{\bar{Q}}) + V_{LS}(r)(\mathbf{L} \cdot \mathbf{S}) + V_T(r) [3(\mathbf{S}_Q \cdot \mathbf{n})(\mathbf{S}_{\bar{Q}} \cdot \mathbf{n}) - \mathbf{S}_Q \cdot \mathbf{S}_{\bar{Q}}],
\]

where \( \mathbf{S}_Q \) (\( \mathbf{S}_{\bar{Q}} \)) spin of the quark (antiquark), \( \mathbf{L} \) is relative orbital momentum, \( \mathbf{S} = \mathbf{S}_Q + \mathbf{S}_{\bar{Q}} \) is total spin of the quarkonium system. We work in the center of mass frame and, therefore, a radius vector \( r \) is given in terms of the relative coordinates \( r = r_Q - r_{\bar{Q}} \) and \( \mathbf{n} = r/r \) defines the unit vector in direction of the radius vector. In Eq. (1) \( V_C(r) \), \( V_{SS}(r) \), \( V_{LS}(r) \) and \( V_T(r) \) are central, spin-spin, spin-orbit and tensor potentials depending on the relative distance between the quark and antiquarks.

The central part of the potential in a nonrelativistic reduction employs the following “Coulomb+linear” form

\[
V_C(r) = \kappa r - \frac{4\alpha_s}{3r},
\]

where \( \kappa \) is parameter of string tension and

\[
\alpha_s(\mu) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{QCD}^2)}
\]

is the strong running coupling constant at the one-loop level. Its value is determined from the characteristic energy scale \( \mu \) corresponding to the problem. Further, \( \beta_0 = (33 - 2N_f)/(12\pi) \) is the beta function at the one-loop level and \( \Lambda_{QCD} \) is the dimensional transmutation parameter. The nonrelativistic expansion of \( Q\bar{Q} \) interactions allows to relate the spin-dependent parts of the potential to the central part. So, the spin-dependent interactions corresponding to the “vector one-gluon-exchange+scalar confinement” are given as

\[
V_{SS}^{(P)}(r) = \frac{32\pi\alpha_s}{9m_Q^2} \delta_s(r),
\]

\[
V_{LS}^{(P)}(r) = \frac{1}{2m_Q^2} \left( \frac{4\alpha_s}{r^2} - \frac{\kappa_r}{r} \right),
\]

\[
V_T^{(P)}(r) = \frac{4\alpha_s}{m_Q^2},
\]

where \( m_Q \) is heavy quark mass. In practical calculations, the pointlike spin-spin interaction in Eq. (3) is “smeared” by using an exponential function of the form

\[
\delta_s(r) = \left( \frac{\sigma}{\sqrt{\pi}} \right)^3 e^{-\sigma^2 r^2},
\]

where \( \sigma \) is smearing parameter. In such a way \( Q\bar{Q} \) potential is described in terms of only four parameters, \( \kappa, \alpha_s, m_Q \) and \( \sigma \). Usually, these parameters are found by fitting the whole charmonium spectrum. In the present work we will follow the phenomenological approach but find those parameters by fitting only some minimal part of S-wave spectrum instead of considering the whole spectrum.

After fitting the form of potential, in order to evaluate the energy states of quarkonia in a nonrelativistic potential approach, one needs to solve the Schrödinger equation

\[
\langle \hat{H} - E | \Psi_{J, J_s} \rangle = 0.
\]

Here \( \hat{H} \) is Hamilton operator and \( | \Psi_{J, J_s} \rangle \) represents the state vector with the total angular momentum \( J \) and its third component \( J_s \). The coordinate space projection of the state vector \( \langle r | \Psi_{J, J_s} \rangle \) will reproduce the coordinate space representation of the Hamiltonian

\[
\hat{H}(r) = -\frac{\hbar^2}{m_Q} \nabla^2 + V_{Q\bar{Q}}(r),
\]

where \( m_Q \) arises from the doubled reduced mass of the quarkonium system. The matrix elements of \( Q\bar{Q} \) potential in the standard basis \( |2S+1L_j\rangle \), which is given in terms of the total spin \( S \), the orbital angular momentum \( L \), and the total angular momentum \( J \) satisfying the relation \( J = L + S \), has the following form

\[
V_{Q\bar{Q}}(r) = \langle 2S+1L_j | V_{Q\bar{Q}}(r) | 2S+1L_j \rangle
\]

\[
= V(r) + \left[ \frac{1}{2} S(S + 1) - \frac{3}{4} \right] V_{SS}(r)
\]

\[
+ (\mathbf{L} \cdot \mathbf{S}) V_{LS}(r) + \left[ \frac{\langle \mathbf{L} \cdot \mathbf{S} \rangle (\mathbf{L} \cdot \mathbf{S}) - 2}{(2L - 1)(2L + 3)} \right] V_T(r),
\]

where \( \langle \mathbf{L} \cdot \mathbf{S} \rangle \) is defined as

\[
\langle \mathbf{L} \cdot \mathbf{S} \rangle = \frac{1}{2} [J(J + 1) - L(L + 1) - S(S + 1)].
\]
The corresponding radial part of the wave function for a given orbital momentum \( L \) is a solution of the Schrödinger equation
\[
\left( -\frac{\hbar^2}{m_Q} \nabla^2 + V_{QQ}(r) - E \right) \psi_{LL_3}(r) = 0, \tag{11}
\]
where an angular part of the wave function \( \psi_{LL_3}(r) \) is represented in terms of the standard spherical harmonics \( Y_{LL_3}(\hat{r}) \). In order to solve Eq. (11) numerically, we will follow the gaussian expansion method (for details, see review [14]), where the state vector \( |\psi_{LL_3}\rangle \) is expanded in terms of a set of basis vectors \( \{|\phi_{nLL_3}\rangle; n = 1, 2, \ldots, n_{\text{max}}\} \) as
\[
|\psi_{LL_3}\rangle = \sum_{n=1}^{n_{\text{max}}} C_n^{(L)} |\phi_{nLL_3}\rangle. \tag{12}
\]
Here \( n \) is a radial quantum number. So, the radial excitations corresponding to the given angular momentum value will be reproduced naturally. In the gaussian expansion method, the radial part \( \phi_{nLL_3}^{(r)}(r) \) of the total eigenfunction in the spherical coordinate basis
\[
\phi_{nLL_3}(r) = \phi_{nL}^{(r)}(r) Y_{LL_3}(\hat{r}) \tag{13}
\]
is expressed in terms of gaussian trial functions
\[
\phi_{nL}^{(r)}(r) = \left( \frac{2^{2L+7/2}r^L e^{-(r/r_n)^2}}{\sqrt{\pi}(2L+1)!!} \right)^{1/2} n^{2L+1/2} r^L e^{-(r/r_n)^2}. \tag{14}
\]
For the given set \( n \) runs the values \( n = 1, 2, \ldots, n_{\text{max}} \) and the corresponding \( r_n \)'s are playing the role of variational parameters. The variational parameters could be optimized using a geometric progression [14]
\[
r_n = r_1 n^{-1}, \quad n = 1, 2, \ldots, n_{\text{max}} \tag{15}
\]
and, therefore, the actual number of parameters is reduced to the three (e.g. \( r_1, r_{n_{\text{max}}} \) and \( n_{\text{max}} \)) for the given values of the orbital quantum number \( L \), the spin \( S \) and the total angular momentum \( J \).

The expansion coefficients \( C_n^{(L)} \) in Eq. (12) and the eigenenergies \( E_n^{(L)} \) are determined by employing Rayleigh-Ritz variational principle. This leads to a generalized matrix eigenvalue problem
\[
\sum_{n=1}^{n_{\text{max}}} \left( K_{mn}^{(L)} + V_{mn}^{(L)} - E_n^{(L)} N_{mn}^{(L)} \right) C_n^{(L)} = 0, \tag{16}
\]
where \( m = 1, 2, \ldots, n_{\text{max}} \),

where the corresponding matrix elements are defined in the following way
\[
K_{mn}^{(L)} = \langle \phi_{mLL_3} | \frac{-\hbar^2}{m_Q} | \phi_{nLL_3} \rangle, \tag{17}
\]
\[
V_{mn}^{(L)} = \langle \phi_{mLL_3} | V_{QQ} | \phi_{nLL_3} \rangle, \tag{18}
\]
\[
N_{mn}^{(L)} = \langle \phi_{mLL_3} | N_{mn}^{(L)} | \phi_{nLL_3} \rangle. \tag{19}
\]

### III. RESULTS AND DISCUSSIONS

As we mentioned above in the phenomenological approaches the parameters of the model, \( \kappa, \alpha_s, \sigma \) and \( m_c \), are fitted to the spectra of experimentally known charmonium states. For example, the authors of Ref. [11] proposed the set of potential parameters given in Table I (see the model referred as NR). Using that set of parameters they calculated all allowed E1 radiative partial width and some important M1 width. As an input for the fitting of parameters they used 11 meson states: 6 states corresponding to the S-wave, 3 states corresponding to the P-wave and 2 states corresponding to the D-wave, respectively. The input values of these energy states are given in Table I (see the 2nd column). The results of their calculations showed that the nonrelativistic potential model with the certain set of parameters describes the charmonium spectrum very well.

However, nowadays the experimental data is improved and some new states were fixed in the particle data [13]. The values of charmonium states extracted from the current experimental data are given in the last column of the Table I. The natural question arises — “How the parameters of the nonrelativistic potential model will change if one concentrates to the updated experimental data?” Partially, our aim in the present work is to answer this question. However, our main aim in the present work is not only fitting the updated experimental data by means of the new set of parameters. In addition to the fitting process we want to check “How well does a nonrelativistic expansion work in the potential approaches?”

As we said above the authors of Ref. [11] fitted the parameters of the potential to the all eleven, experimentally known at that time, states of the charmonium spectrum. Therefore, a beauty of nonrelativistic expansion seems remained to be hidden behind. We want to emphasize that, in principle, one can concentrate to the part of spectrum in order to fit the parameters of model. For example, one can concentrate to the S-wave part of spectrum for fitting the parameters of model. During this process the spin-orbit and tensor interactions are not contributing to the total interaction. One can also act in opposite form by concentrating to the part of spectra where the spin-orbit and tensor interactions are important. The reason for the possibility of such choices is due to the fact that the central and spin dependent parts of the potential are

| The model | \( m_c \) | \( \alpha_s \) | \( \kappa \) | \( \sigma \) |
|----------|-------|-------|-------|-------|
| NR       | 1.4794 | 0.5461 | 0.1425 | 1.0946 |
| NR4      | 1.4796 | 0.5426 | 0.1444 | 1.1510 |
related to each-other in the nonrelativistic expansion and described by the same set of parameters. In an ideal case, only four input states are enough to fit four “arbitrary” parameters of the potential model.

Consequently, as a possible test of nonrelativistic expansion, in the present work we consider the “ideal case” and fit the parameters of model according to some part of S-wave charmonia. In such a way we ignore the spin-orbit and tensor interactions during the fitting process. More specifically, we propose a potential model where the parameters are fitted using the four lowest S-wave spin 0 and spin 1 states. On top of that we will fit the lowest spin zero $^1S_0$ and $^2S_0$ states exactly. The parameters of the corresponding interaction potential are also given in Table II (see the model referred as NR4) and the values of corresponding input states are given in Table II respectively (see the 4th column).

In order to keep a good accuracy of numerical calculations, during the fitting process we used a basis set with 30 to 50 gaussian functions corresponding to the given values of $L$, $S$ and $J$. So, the value of the first parameter $n_{\text{max}}$ from the three variational parameters is free input and equals to the definite integer number belonging to the interval $n_{\text{max}} \in [30, 50]$. The values of other two corresponding variational parameters, $r_1$ and $r_{\text{max}}$, are found by minimizing not only the ground state energy $E_1$ but also minimizing simultaneously the lowest 10 to 20 radial excitation energies $\sum_{n=1}^{n_{\text{max}}} E_n$ (i.e. $n_{\text{min}} \in [5, 20]$) from the possible 30 to 50 energy states. The convergence of the results are checked by increasing the number of total radial excitations starting from $n_{\text{max}} \sim 10$ with $n_{\text{min}} \sim 5$ to the above mentioned final values, $n_{\text{max}} \in [30, 50]$ and $n_{\text{min}} \in [10, 20]$.

Now let us discuss our results. Analyzing the S-wave results of NR4 model one can note that the values of the first two spin 0 and the first two spin 1 S-wave states are obviously reproduced very well. Naturally, these energy states are input and reproduced better in comparison with the results of NR model. Quick look for the calculated $^3S_1$-state energy value for NR4 and comparing it with the corresponding experimental value shows the large difference, around 45 MeV. Nevertheless, this problem seems to be unavoidable if one fits the parameters to the whole spectrum, e.g. compare the corresponding NR result where the difference from the experimental value is around 33 MeV. However, the next excited $^4S_1$-state energy for NR4 is reproduced at almost its averaged experimental value while NR model gives the relatively different result. One can conclude that, in general, S-wave states are reproduced better in NR4 model in comparison with NR model.

The power of nonrelativistic expansion becomes more obvious when we include the spin-orbit and tensor interactions for the analysis of whole spectrum. While the input parameters are already fixed we do not need to play with them anymore. Therefore, the calculations of $L \geq 1$ states are straightforward and does not require any fitting process.

The beauty of the nonrelativistic expansion is realized when we analyze the P-wave states. One can see that, although we are not fitting them, among the six experimentally known states three of them $^3P_0, ^1P_1$ and $^3P_2$ are reproduced at their experimental value for NR4 model. Two states, $^1P_2$ and $^3P_1$, among the remaining three P-wave states in the table are also reproduced quite well with the differences 3 MeV and 6 MeV from the experiment, respectively. Only one state $^2P_3$ is far from its experimental value, the difference is 51 MeV. For com-

| State | Ref. [11] | This work | Exp. [15] |
|-------|-----------|-----------|-----------|
| $J/\psi(1^3S_1)$ | 3097 | 3097 | 3098 | 3096.900 ± 0.006 |
| $\eta(1^1S_0)$ | 2797 | 2982 | 2984 | 2983.9 ± 0.5 |
| $\psi(2^3S_1)$ | 3686 | 3672 | 3682 | 3686.907 ± 0.025 |
| $\eta(2^1S_0)$ | 3638 | 3630 | 3638 | 3637.6 ± 1.2 |
| $\psi(3^3S_1)$ | 4040 | 4072 | 4084 | 4039 ± 1 |
| $\eta(3^1S_0)$ | 4043 | 4045 | 4055 | 4055 |
| $\psi(4^3S_1)$ | 4415 | 4406 | 4422 | 4421 ± 4 |
| $\eta(4^1S_0)$ | 4384 | 4397 | 4397 | 4397 |
| $\chi(2^1P_2)$ | 3556 | 3556 | 3559 | 3556.17 ± 0.07 |
| $\chi(1^1P_1)$ | 3511 | 3505 | 3505 | 3510.67 ± 0.05 |
| $\chi(0^1P_0)$ | 3415 | 3424 | 3415 | 3414.71 ± 0.30 |
| $h(0^1P_1)$ | 3516 | 3524 | 3525.38 ± 0.11 |
| $\chi(2^3P_2)$ | 3972 | 3978 | 3978 | 3972.2 ± 2.6 |
| $\chi(1^1P_1)$ | 3925 | 3925 | 3925 | 3925.35 ± 0.11 |
| $\chi(2^3P_2)$ | 3852 | 3864 | 3864 | 3861.94 ± 0.04 |
| $h(1^1P_1)$ | 3934 | 3945 | 3945 | 3945 |
| $\chi(3^3P_2)$ | 4317 | 4325 | 4325 | 4325.17 ± 0.04 |
| $\chi(2^3P_1)$ | 4271 | 4293 | 4293 | 4293.48 ± 0.11 |
| $\chi(2^3P_0)$ | 4202 | 4227 | 4227 | 4227.14 ± 0.04 |
| $h(2^3P_1)$ | 4279 | 4293 | 4293 | 4293.14 ± 0.04 |
| $\psi(1^1D_3)$ | 3806 | 3816 | 3816 | 3816.33 ± 0.04 |
| $\psi(1^3D_2)$ | 3800 | 3807 | 3807 | 3822.2 ± 1.2 |
| $\psi(1^1D_2)$ | 3870 | 3785 | 3785 | 3778.1 ± 1.2 |
| $\eta(2^3D_3)$ | 3799 | 3809 | 3809 | 3809.1 ± 1.2 |
| $\psi(2^3D_2)$ | 4167 | 4179 | 4179 | 4179.17 ± 0.04 |
| $\psi(2^3D_2)$ | 4158 | 4167 | 4167 | 4167.14 ± 0.04 |
| $\chi(2^1D_2)$ | 4159 | 4142 | 4153 | 4191 ± 5 |
| $\eta(2^1D_2)$ | 4158 | 4170 | 4170 | 4170.17 ± 0.04 |
| $\chi(3^1F_4)$ | 4021 | 4033 | 4033 | 4033.17 ± 0.04 |
| $\chi(3^1F_4)$ | 4029 | 4039 | 4039 | 4039.17 ± 0.04 |
| $\chi(3^1F_2)$ | 4029 | 4041 | 4041 | 4041.17 ± 0.04 |
| $h(3^1F_3)$ | 4026 | 4037 | 4037 | 4037.17 ± 0.04 |
| $\chi(2^3F_4)$ | 4348 | 4362 | 4362 | 4362.17 ± 0.04 |
| $\chi(2^3F_3)$ | 4352 | 4365 | 4365 | 4365.17 ± 0.04 |
| $\chi(2^3F_3)$ | 4351 | 4365 | 4365 | 4365.17 ± 0.04 |
| $h(3^1F_3)$ | 4350 | 4364 | 4364 | 4364.17 ± 0.04 |
parison, the general fit using NR model reproduces only one state $1^3P_2$ at its experimental value. Another state $1^3P_1$ is same as in the case of NR4 and three $(1^3P_0, 1^1P_1$ and $2^3P_0$) from the remaining four states are reproduced approximately with 10 MeV differences from the experimental values, respectively. The last state $2^3P_2$ is very far from the experimental value and difference 47 MeV is almost same as NR4 case. One could also conclude, that the concentration, respectively, to S-wave (e.g. $1^1S_0$ and $2^1S_0$) and P-wave (e.g. $1^1P_0$ and $1^3P_1$) states as an input will lead to more or less similar values of the potential parameters in comparison with the values in Table III. Summarizing analysis of P-wave states one can conclude that NR4 model much better reproduces the experimental data in comparison with NR model.

It is also interesting to analyze more higher energy states corresponding to NR and NR4 models and compare them with the available experimental data. Coming to the D-wave states, one can note that $1^3D_1$-state energy value is reproduced relatively better in NR model. However, the situation becomes opposite if we analyze $1^3D_2$-state energy. Here NR4 model gives relatively better result in comparison with NR model. Finally, an experimentally available the highest energy state $2^3D_1$ is again reproduced relatively better in NR4 model. Again and in general, D-wave states are also better reproduced in NR4 model.

Consequently, one can conclude that, although the number of input parameters in NR4 model are chosen in a maximum compact form it gives better result than the NR model. From the Table III one can also make a general conclusion that the fitting to the S-wave part of the spectrum is completely satisfactory. In a such way we see that the nonrelativistic expansion in the potential models for describing the fine structure of charmonium states indeed works pretty well.

IV. SUMMARY

In the present work we aimed at testing the nonrelativistic potential model and reparametrization of the nonrelativistic potential based on the currently available experimental data. In particular, we investigated the applicability of nonrelativistic expansion in the potential approaches to the charmonium spectrum. For that purpose we concentrated only on the four lowest S-wave states of charmonium spectrum. By doing that we demonstrated, that the concentration to the minimal part of charmonium spectrum is enough during the fitting process of the values of potential parameters. The model quite satisfactorily described the whole spectrum of charmonia. From our studies, one can make a general conclusion that the nonrelativistic potential approach is indeed good approximation in describing the spectrum including the fine-structure of charmonium states.

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