A fractional parabolic inverse problem involving a
time-dependent magnetic potential

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ABSTRACT. We study a class of fractional parabolic equations involving a time-dependent magnetic potential and formulate the corresponding inverse problem. We determine both the magnetic potential and the electric potential from the exterior partial measurements of the Dirichlet-to-Neumann map.

1 Introduction

The study of fractional operators has been an active research field in past decades. Differential equations involving fractional derivatives have been introduced to describe anomalous diffusion and random processes with jumps in physics and probability theory. See for instance, [19, 20, 25].

Correspondingly, various kinds of inverse problems associated with fractional operators have been extensively studied so far. The study of the inverse problem for space-fractional operators (very different from the one for time-fractional operators studied in [4, 13]), namely the fractional Calderón problem, was initiated in [12] where the authors considered the exterior Dirichlet problem

$$\left((-\Delta)^s + q\right)u = 0 \quad \text{in } \Omega, \quad u|_{\Omega_e} = f$$

and they showed that the electric potential $q$ in $\Omega$ can be determined from the exterior partial measurements of the Dirichlet-to-Neumann map

$$\Lambda_q : f \rightarrow (-\Delta)^s u_f|_{\Omega_e}.$$ 

See [1, 3, 11, 10, 24] for further studies based on [12].

As variants of the inverse problem introduced in [12], parabolic analogues of the fractional Calderón problem have been studied in recent years (see [2, 5] for results in the local parabolic inverse problem). One related work can be found in [15] where the authors studied the inverse problem for the fractional operator

$$(\partial_t - \Delta)^s + q.$$ 

Another related work can be found in [23] where the authors established the well-posedness of the initial exterior problem associated with the fractional operator

$$\partial_t + (-\Delta)^s$$

1
and its Runge approximation property.

In this paper, we study an inverse problem for a fractional operator generalizing \( \partial_t + (-\Delta)^s \). Our operator contains a time-dependent space-fractional derivative and our inverse problem can be viewed as a parabolic analogue of the the fractional magnetic Calderón problem introduced in \cite{16,17}. See \cite{9,14,21,26} for results in the local magnetic Calderón problem. Also see \cite{6} for the study of a different fractional magnetic Calderón problem.

More precisely, we consider the time-dependent operator \( \mathcal{R}^s_{A(t)} \), which is formally defined by

\[
\langle \mathcal{R}^s_{A(t)} u, v \rangle := 2 \int \int (u(x) - R_{A(t)}(x,y)u(y))v(x)K(x,y)\,dxdy
\]

for each \( t \). Here \( K \) is a function associated with a heat kernel (see Subsection 2.2 in \cite{16} or Section 2 in \cite{10}) satisfying

\[
K(x,y) = K(y,x), \quad K(x,y) \sim \frac{1}{|x-y|^{n+2s}},
\]

\( A(\cdot, t) \) is a time-dependent real vector-valued magnetic potential and

\[
R_{A(t)}(x,y) := \cos((x-y) \cdot A(x+y/2, t)).
\]

Clearly, the operator \( \mathcal{R}^s_{A(t)} \) coincides with the fractional Laplacian \( (-\Delta)^s \) when \( A \equiv 0 \) and \( K(x,y) = c_{n,s}/|x-y|^{n+2s} \).

Under appropriate assumptions on \( A \) and the time-dependent electric potential \( q(\cdot, t) \), the initial exterior problem

\[
\begin{cases}
\partial_t u + \mathcal{R}^s_{A(t)} u + q(t)u = 0 & \text{in } \Omega \times (-T,T) \\
u = g & \text{in } \Omega_e \times (-T,T) \\
u = 0 & \text{in } \mathbb{R}^n \times \{-T\},
\end{cases}
\]

is well-posed so we can define the solution operator \( P_{A,q} : g \to u_g \) and the Dirichlet-to-Neumann map \( \Lambda_{A,q} \), which is formally given by

\[
\Lambda_{A,q} g := \mathcal{R}^s_{A}(u_g)|_{\Omega_e \times (-T,T)}.
\]

Our goal here is to determine both \( A \) and \( q \) from the exterior partial measurements of \( \Lambda_{A,q} \).

The following theorem is the main result in this paper.

**Theorem 1.1.** Suppose \( \Omega \subset B_r(0) \) for some constant \( r > 0 \), \( A_j, q_j \in C^2([-T,T]; L^\infty(\Omega)) \) and \( q_j \geq c \) for some constant \( c > 0 \), \( W_j \) are open sets s.t. \( W_j \cap B_{3r}(0) = \emptyset \) \((j = 1,2)\). Let

\[
W^{(1,2)} = \left\{ \frac{x+y}{2} : x \in W_1, y \in W_2 \right\}.
\]

Also assume \( W^{(1,2)} \setminus \Omega \neq \emptyset \). If

\[
\Lambda_{A_1,q_1} g|_{W_2 \times (-T,T)} = \Lambda_{A_2,q_2} g|_{W_2 \times (-T,T)}
\]

for any \( g \in C^\infty_c(W_1 \times (-T,T)) \), then \( A_1(t) = \pm A_2(t) \) and \( q_1 = q_2 \) in \( \Omega \times (-T,T) \).
The rest of this paper is organized in the following way. In Section 2, we summarize the background knowledge. We show the well-posedness of the initial exterior problem in Section 3. We introduce the associated Dirichlet-to-Neumann map, prove the Runge approximation property of our fractional operator and the main theorem in Section 4.

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2 Preliminaries

Throughout this paper

- Fix the space dimension \( n \geq 2 \) and the fractional power \( 0 < s < 1 \)
- Fix the constant \( T > 0 \) and \( t \) denotes the time variable
- \( \Omega \) denotes a bounded Lipschitz domain and \( \Omega_e := \mathbb{R}^n \setminus \overline{\Omega} \)
- \( B_r(0) \) denotes the open ball centered at the origin with radius \( r > 0 \) in \( \mathbb{R}^n \)
- If \( u(\cdot, t) \) is an \((n+1)\)-variable function, then \( u(t) \) denotes the corresponding \( n \)-variable function for each \( t \)
- \( A(\cdot, t) \) denotes a time-dependent \( \mathbb{R}^n \)-valued magnetic potential and \( q(\cdot, t) \) denotes a time-dependent electric potential
- If \( A(t) \in L^\infty(\Omega) \), then identify \( A(t) \) with its zero extension in \( L^\infty(\mathbb{R}^n) \)
- \( c, C, C', C_1, \cdots \) denote positive constants (which may depend on some parameters)
- \( \int \cdots \int = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \)
- \( X^* \) denotes the continuous dual space of \( X \) and write \( \langle f, u \rangle = f(u) \) for \( u \in X, f \in X^* \) when \( X \) is an \( n \)-variable function space.

2.1 Function spaces

Throughout this paper we refer all function spaces to real-valued function spaces.

For \( \alpha \in \mathbb{R} \), \( H^\alpha(\mathbb{R}^n) \) denotes the Sobolev space \( W^{\alpha, 2}(\mathbb{R}^n) \).

We have the natural identification

\[
H^{-\alpha}(\mathbb{R}^n) = H^\alpha(\mathbb{R}^n)^*. 
\]

Let \( U \) be an open set and \( F \) be a closed set in \( \mathbb{R}^n \),

\[
H^\alpha(U) := \{ u|_U : u \in H^\alpha(\mathbb{R}^n) \}, \quad H^\alpha_F(\mathbb{R}^n) := \{ u \in H^\alpha(\mathbb{R}^n) : \text{supp } u \subset F \},
\]

\[
\hat{H}^\alpha(U) := \text{the closure of } C^\infty_c(U) \text{ in } H^\alpha(\mathbb{R}^n).
\]

\( \Omega \) is Lipschitz bounded implies

\[
\hat{H}^\alpha(\Omega) = H^\alpha_{\Omega}(\mathbb{R}^n).
\]
Let $X$ be a Banach space. We use $C^m([-T,T];X)$ (resp. $AC([-T,T];X)$) to denote the space consisting of the corresponding Banach space-valued continuously differentiable (resp. absolutely continuous) functions on $[-T,T]$.

$L^p(-T,T;X)$ denotes the space consisting of the corresponding Banach space-valued $L^p$ functions, equipped with the standard norm

$$||u||_{L^p(-T,T;X)} := \left( \int_{-T}^{T} ||u(t)||_X^p \, dt \right)^{1/p}.$$

### 2.2 The operator $R_{A(t)}^s$

Recall that we gave the formal definition of $R_{A(t)}^s$ in (1). Note that

$$\iint (u(x) - R_{A(t)}(x,y)u(y))v(x)K(x,y) \, dx \, dy = \iint (u(y) - R_{A(t)}(x,y)u(x))v(y)K(x,y) \, dy \, dx$$

so it is easy to verify that

$$\langle R_{A(t)}^s u, v \rangle = \Re \iint (u(x) - e^{i(x-y)\cdot A(\cdot \cdot \cdot ,t)}u(y))(v(x) - e^{-i(x-y)\cdot A(\cdot \cdot \cdot ,t)}v(y))K(x,y) \, dx \, dy$$

and

$$\langle R_{A(t)}^s u, v \rangle = \langle R_{A(t)}^s v, u \rangle. \quad (6)$$

The following lemma is a time-dependent version of Lemma 3.3 in (16).

**Lemma 2.1.** Suppose $A \in C([-T,T], L^\infty(\mathbb{R}^n))$, then for $u \in H^s(\mathbb{R}^n)$, we have

$$c||u||_{H^s} \leq ||u||_{H_A^s(t)} \leq C||u||_{H^s}$$

where the magnetic Sobolev norm $|| \cdot ||_{H_A^s(t)}$ is defined by

$$||u||_{H_A^s(t)} := (||u||^2_{L^2} + \langle R_{A(t)}^s u, u \rangle)^{1/2}$$

and $c, C$ depend on $\sup_{t \in [-T,T]} ||A(t)||_{L^\infty}$ but do not depend on $t$.

**Definition 2.2.** We define the time-dependent bilinear form associated with $A, q$ by

$$B_t[u, v] := \langle R_{A(t)}^s u, v \rangle + \int_\Omega q(t)uv, \quad t \in [-T,T] \quad (7)$$

The symmetry of $B_t$ follows immediately from the symmetry of $R_{A(t)}^s$.

The following estimates will be useful when we show the well-posedness of the initial exterior problem later.

**Lemma 2.3.** Suppose $A, q \in C^2([-T,T]; L^\infty(\Omega))$ and $q \geq c'$ in $\Omega \times [-T,T]$ for some $c'$. Then

$$|B_t[u, v]| \leq C_0||u||_{H^s}||v||_{H^s}, \quad u, v \in H^s(\mathbb{R}^n) \quad (8)$$

and for $u, v \in \dot{H}^s(\Omega)$, we have

$$|B_t[u, u]| \geq c_0||u||^2_{H^s}, \quad (9)$$

$$|B_{t+h}[u, v] - B_t[u, v]| \leq C_1 h||u||_{H^s}||v||_{H^s}, \quad (10)$$

$$|B_{t+h}[u, v] + B_{t-h}[u, v] - 2B_t[u, v]| \leq C_2 h^2||u||_{H^s}||v||_{H^s}, \quad (11)$$

where the constants $c_0, C_0, C_1, C_2$ do not depend on $t, h, u, v$. 

4
Proof. \((\ref{eq:1})\) and \((\ref{eq:2})\) follow from Lemma 2.1 immediately.

Note that for \(u, v \in \dot{H}^s(\Omega)\), we have
\[
\frac{d}{dt} \langle R_{A(t)}^s u, v \rangle = -2 \int_\Omega \frac{d}{dt} R_{A(t)}(x, y) u(y) v(x) K(x, y) \, dx dy
\]
so we have
\[
= -2 \int_\Omega \int_\Omega \frac{d}{dt} R_{A(t)}(x, y) u(y) v(x) K(x, y) \, dx dy
\]
so we have
\[
\frac{d}{dt} \langle R_{A(t)}^s u, v \rangle \leq C \int_\Omega \int_\Omega \frac{|u(y) v(x)|}{|x-y|^{n+2s-2}} \, dx dy
\]
\[
\leq C \int_\Omega \int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s-2}} \, dx dy \int_\Omega \int_\Omega \frac{|v(x)|^2}{|x-y|^{n+2s-2}} \, dx dy \frac{1}{2}.
\]
Note that
\[
\int_\Omega \int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s-2}} \, dx dy = \int_\Omega \left( \int_\Omega \frac{1}{|x-y|^{n+2s-2}} \, dx \right) |u(y)|^2 \, dy \leq C' \|u\|^2_{L^2}
\]
and similarly
\[
\int_\Omega \int_\Omega \frac{|v(x)|^2}{|x-y|^{n+2s-2}} \, dx dy \leq C' \|v\|^2_{L^2},
\]
so we have
\[
\frac{d}{dt} \langle R_{A(t)}^s u, v \rangle \leq C'' \|u\|_{L^2} \|v\|_{L^2} \leq C'' \|u\|_{H^s} \|v\|_{H^s},
\]
which implies \((\ref{eq:3})\) holds.

Also note that
\[
\frac{d^2}{dt^2} R_{A(t)}(x, y) = -\sin((x - y) \cdot A(x-y) \cdot \partial_t A(x+y, t))
\]
so we have
\[
\frac{d^2}{dt^2} \langle R_{A(t)}^s u, v \rangle \leq C''' |x-y|^2
\]
and we can similarly show that
\[
\frac{d^2}{dt^2} \langle R_{A(t)}^s u, v \rangle \leq C'''' \|u\|_{H^s} \|v\|_{H^s}, \quad u, v \in \dot{H}^s(\Omega),
\]
which implies \((\ref{eq:4})\) holds.

We will use the following proposition to prove the Runge approximation property later.

Proposition 2.4. (Proposition 2.4 in [18]) Suppose \(\Omega \cup \text{supp } A(t) \subset B_r(0)\) for some \(r > 0\), \(W\) is an open set s.t. \(W \cap B_{3r}(0) \neq \emptyset\). If \(u \in \dot{H}^s(\Omega), \quad R_{A(t)}^s u|_W = 0\) then \(u = 0\) in \(\mathbb{R}^n\).
3 Initial Exterior Problem

From now on we always assume \( A, q \in C^2([-T, T]; L^\infty(\Omega)) \) and \( q \geq c' \) in \( \Omega \times [-T, T] \) for some \( c' \).

3.1 Discretization in time

First we study the initial value problem

\[
\begin{aligned}
\partial_t u + \mathcal{R}_A^s u + q(t)u &= f & \text{in } \Omega \times (-T, T) \\
u &= 0 & \text{in } \Omega \times \{-T\}.
\end{aligned}
\]  

(12)

**Proposition 3.1.** Suppose \( f \in L^2(\Omega \times (-T, T)) \) and

\[
||f(t + h) - f(t)||_{L^2(\Omega)} \leq Ch
\]

for some \( C \) independent of \( t, h \), then (12) has a unique (weak) solution satisfying

\[
 u \in L^2(-T, T; \tilde{H}^s(\Omega)) \cap AC([-T, T]; L^2(\Omega)), \quad \partial_t u \in L^2(\Omega \times (-T, T)).
\]

**Remark.** The initial value problem associated with \( \partial_t + (-\Delta)^s \) has been studied in [23] where the authors used a Galerkin approximation to show the existence of solutions. Here the time-dependent fractional operator \( \mathcal{R}_A^s \) makes the problem much more complicated. We will use the method of the Rothe’s method for local parabolic problems (see Chapter 15 in [23]).

**Proof.** **Existence:** Divide \([-T, T]\) into \( p \) subintervals of length \( h = 2T/p < 1/2 \) and let \( t_j = -T + jh \).

Consider the discretization in \( t \)

\[
\frac{z_j - z_{j-1}}{h} + \mathcal{R}_A^s(t_j)z_j + q(t_j)z_j = f(t_j) \quad j = 1, \ldots, p
\]

\[z_0 = 0.\]  

(14)

We can iteratively determine \( z_j \in \tilde{H}^s(\Omega) \), which solves the elliptic equation

\[
\mathcal{R}_A^s(t_j)z_j + (q(t_j) + \frac{1}{h})z_j = f(t_j) + \frac{z_{j-1}}{h}.
\]

(8) and (9) ensure the existence and uniqueness of \( z_j \) by Lax-Milgram Theorem.) Define \( Z_j := (z_j - z_{j-1})/h \) and \( u^{(1)} : [-T, T] \to \tilde{H}^s(\Omega) \) given by

\[
u^{(1)}(t) := z_{j-1} + Z_j(t - t_{j-1}), \quad t \in [t_{j-1}, t_j].
\]

Now divide \([-T, T]\) into \( 2^{m-1}p \) subintervals of length \( h = 2T/(2^{m-1}p) \) and let \( t_j^{(m)} = -T + jh_m \). Similarly, we consider the discretization, obtain a sequence

\[
\{z_0^{(m)}, \ldots, z_{2^{m-1}p}^{(m)}\}
\]
in \( \tilde{H}^s(\Omega) \), define \( Z_j^{(m)} := (z_j^{(m)} - z_{j-1}^{(m)})/h_m \) and the poly-line

\[
u^{(m)}(t) := z_{j-1}^{(m)} + Z_j^{(m)}(t - t_{j-1}^{(m)}), \quad t \in [t_{j-1}^{(m)}, t_j^{(m)}].
\]
Also define the step functions
\[ \tilde{q}^{(m)}(t) := z_j^{(m)}, \quad \tilde{U}^{(m)}(t) := Z_j^{(m)}, \quad t \in (t_{j-1}^{(m)}, t_j^{(m)}). \]

Note that the constants in Lemma 3.2 do not depend on \( h \) so for general \( m \),
\[ \|z_j^{(m)}\|_{H^s} \leq c_2, \quad \|Z_j^{(m)}\|_{L^2} \leq c_3 \]
hold. This implies the boundedness of \( \{u^{(m)}\} \) in \( L^2(-T,T;\dot{H}^s(\Omega)) \) and the boundedness of \( \{\tilde{U}^{(m)}\} \) in \( L^2(\Omega \times (-T,T)) \). Hence we can choose weakly convergent sequences s.t.
\[ u^{(m_k)} \rightharpoonup u \quad \text{in} \quad L^2(-T,T;\dot{H}^s(\Omega)), \quad \tilde{U}^{(m_k)} \rightharpoonup \tilde{U} \quad \text{in} \quad L^2(\Omega \times (-T,T)). \]

Note that \( \tilde{U}^{(m)} \) is the weak derivative as well as the pointwise derivative of \( u^{(m)} \). Let \( k \to \infty \) in
\[ u^{(m_k)}(t) = \int_{-T}^t \tilde{U}^{(m_k)}(\tau) d\tau \]
to obtain
\[ u(t) = \int_{-T}^t \tilde{U}(\tau) d\tau. \]

Hence \( u \) is absolutely continuous in \( t \), \( u(-T) = 0 \) and \( \partial_t u(t) = \tilde{U}(t) \).

Now we show that this \( u \) satisfies the equation in (12).

Define the step function
\[ \tilde{f}^{(m)}(t) := f(t_j^{(m)}), \quad t \in (t_{j-1}^{(m)}, t_j^{(m)}) \]
and the step bilinear form
\[ B_{t_j^{(m)}}[\cdot, \cdot] := B_{t_j^{(m)}}[\cdot, \cdot], \quad t \in (t_{j-1}^{(m)}, t_j^{(m)}). \]

Fix \( v \in L^2(-T,T;\dot{H}^s(\Omega)) \) and let both sides of the discretized equation
\[ Z_j^{(m)} + \mathcal{R}^s_{s_A(t_j^{(m)})} z_j^{(m)} + q(t_j^{(m)}) z_j^{(m)} = f(t_j^{(m)}) \]
act on \( v(t) \) for \( t \in (t_{j-1}^{(m)}, t_j^{(m)}) \) and integrate from \(-T\) to \( T \), then we have
\[ \int_{-T}^T \langle \tilde{U}^{(m)}(t), v(t) \rangle dt + \int_{-T}^T B_{t_j^{(m)}}[\tilde{u}^{(m)}(t), v(t)] dt = \int_{-T}^T \langle \tilde{f}^{(m)}(t), v(t) \rangle dt. \] (15)

[13] ensures that
\[ \int_{-T}^T \langle \tilde{f}^{(m)}(t), v(t) \rangle dt \to \int_{-T}^T \langle f(t), v(t) \rangle dt. \] (16)

The weak convergence of \( \tilde{U}^{(m_k)} \) implies
\[ \int_{-T}^T \langle \tilde{U}^{(m_k)}(t), v(t) \rangle dt \to \int_{-T}^T \langle \partial_t u(t), v(t) \rangle dt. \] (17)
Note that (8) ensures that
\[
\int_{-T}^{T} B_t[\cdot, v(t)] \, dt
\]
is a bounded linear functional on \(L^2(-T, T; \hat{H}^s(\Omega))\) and by Lemma 3.3 we have the weak convergence of \(\{\hat{u}^{(m_k)}\}\) so
\[
\int_{-T}^{T} B_t[\hat{u}^{(m_k)}(t), v(t)] \, dt \to \int_{-T}^{T} B_t[u(t), v(t)] \, dt \tag{18}
\]
Also we can show
\[
\int_{-T}^{T} B_t^{(m_k)}[\hat{u}^{(m_k)}(t), v(t)] \, dt - \int_{-T}^{T} B_t[\hat{u}^{(m_k)}(t), v(t)] \, dt \to 0 \tag{19}
\]
In fact, we first assume \(v(t) = 1_{[\alpha, \beta]}(t)v_0\) where \(v_0 \in \hat{H}^s(\Omega)\) and \(\alpha, \beta\) are endpoints of subintervals in the \(m_k\)-division for some \(k\). For each large \(m_k\), we write \(\alpha = \iota_{j_1}^{(m_k)}, \beta = \iota_{j_2}^{(m_k)}\) for some \(j_1, j_2\), then
\[
\left| \int_{-T}^{T} B_t^{(m_k)}[\hat{u}^{(m_k)}(t), v(t)] \, dt - \int_{-T}^{T} B_t[\hat{u}^{(m_k)}(t), v(t)] \, dt \right|
\]
\[
= \left| \sum_{j=j_1+1}^{j_2} \int_{t_{j-1}}^{t_j} B_t^{(m_k)}[z_{j}^{(m_k)}, v_0(t)] - B_t[z_{j}^{(m_k)}, v_0(t)] \right|
\]
\[
\leq \sum_{j=j_1+1}^{j_2} \int_{t_{j-1}}^{t_j} C_{1, 2} \|z_{j}^{(m_k)}\|_{H^s} \|v_0\|_{H^s} \, dt \leq 2TC_{1, 2} h_{m_k} \|v_0\|_{H^s}
\]
by using (10) and the boundedness of \(\{z_{j}^{(m_k)}\}\).

Since the set consisting of such \(v\) spans a space dense in \(L^2(-T, T; \hat{H}^s(\Omega))\), we know (19) holds for all \(v \in L^2(-T, T; \hat{H}^s(\Omega))\).

Combine (10), (17), (18), (19) with (15). We conclude that
\[
\int_{-T}^{T} \langle \partial_t u(t), v(t) \rangle \, dt + \int_{-T}^{T} B_t[u(t), v(t)] \, dt = \int_{-T}^{T} \langle f(t), v(t) \rangle \, dt. \tag{20}
\]

**Uniqueness:** Let \(v = u\) be the solution constructed in the existence part, then (20) becomes
\[
\frac{1}{2} \|u(T)\|_{L^2(\Omega)}^2 + \int_{-T}^{T} B_t[u(t), u(t)] \, dt = \int_{-T}^{T} \langle f(t), u(t) \rangle \, dt,
\]
which implies
\[
\|u\|_{L^2(-T, T; \hat{H}^s(\Omega))} \leq C' \|f\|_{L^2(-T, T; H^{-s}(\Omega)).
\]

By (12) and (4),
\[
\|\partial_t u\|_{L^2(-T, T; H^{-s}(\Omega))} \leq \|f\|_{L^2(-T, T; H^{-s}(\Omega))} + \|\mathbf{R} u + qu\|_{L^2(-T, T; H^{-s}(\Omega))}
\]
\[
\leq \|f\|_{L^2(-T, T; H^{-s}(\Omega))} + C'' \|u\|_{L^2(-T, T; H^s(\Omega))},
\]
Hence we have
\[
\|u\|_{L^2(-T, T; \hat{H}^s(\Omega))} + \|\partial_t u\|_{L^2(-T, T; H^{-s}(\Omega))} \leq \|f\|_{L^2(-T, T; H^{-s}(\Omega)). \tag{21}
\]
The uniqueness is clear when we let \(f = 0\).
We need the following two lemmas to complete the proof of Proposition 3.1. They are essentially the same as their counterparts in the local parabolic problem (see Page 286-294 in [22] for details). We include the proofs here for completeness.

**Lemma 3.2.** \( \{z_j\} \) and \( \{Z_j\} \) defined in the proof of Proposition 3.1 satisfy

\[
\|z_j\|_{H^s} \leq c_2, \quad \|Z_j\|_{L^2} \leq c_3
\]

where \( c_2, c_3 \) do not depend on \( h \).

**Proof.** We first show that \( \{z_j\} \) is bounded in \( L^2(\Omega) \).

In fact, let both sides of the equation in (14) act on \( z_j \), then we have

\[
B_{t_j}[z_j, z_j] + \frac{1}{h} \langle z_j - z_{j-1}, z_j \rangle = \langle f(t_j), z_j \rangle,
\]

which implies

\[
\|z_j\|_{L^2} \leq \|z_{j-1}\|_{L^2} + h\|f(t_j)\|_{L^2},
\]

then iteratively we can show

\[
\|z_j\|_{L^2} \leq h \sum_{l=1}^{j} \|f(t_l)\|_{L^2} \leq jhC_f \leq 2TC_f =: c_1.
\]

Next we show that \( \{z_j\} \) is bounded in \( \tilde{H}^s(\Omega) \).

In fact, let both sides of the equation in (14) act on \( z_j - z_{j-1} \), then we have

\[
B_{t_j}[z_j, z_j - z_{j-1}] + \frac{1}{h} \|z_j - z_{j-1}\|_{L^2} = \langle f(t_j), z_j - z_{j-1} \rangle. \tag{22}
\]

Note that

\[
B_{t_l}[z_l, z_l - z_{l-1}] = \frac{1}{2} \left( B_{t_l}[z_l, z_l] + B_{t_l}[z_l - z_{l-1}, z_l - z_{l-1}] - B_{t_l}[z_{l-1}, z_{l-1}] \right),
\]

then sum up all the identities in the form (22) for \( 1 \leq l \leq j \) and omit all the non-negative terms

\[
\|z_l - z_{l-1}\|_{L^2}, \quad B_{t_l}[z_l - z_{l-1}, z_l - z_{l-1}]
\]

to obtain the inequality

\[
B_{t_j}[z_j, z_j] - \sum_{l=1}^{j-1} (B_{t_{l+1}}[z_l, z_l] - B_{t_l}[z_l, z_l]) \leq 2\langle f(t_j), z_j \rangle + \sum_{l=1}^{j-1} \langle f(t_l) - f(t_{l+1}), z_l \rangle.
\]

By (9), (10) and (13), this inequality implies

\[
c_0\|z_j\|_{H^s}^2 \leq C_1 h \sum_{l=1}^{j-1} \|z_l\|_{H^s}^2 + 2\|f(t_j)\|_{L^2} \|z_j\|_{L^2} + \sum_{l=1}^{j-1} \|f(t_l) - f(t_{l+1})\|_{L^2} \|z_l\|_{L^2}
\]
\[
\leq C_1 h \sum_{l=1}^{j-1} \|z_l\|_{H^s}^2 + 2c_1C_f + 2c_1C(j-1)h.
\]
Since \((j-1)h \leq 2T\), we have
\[
||z_j||_{H^s}^2 \leq C_2' h \sum_{i=1}^{j-1} ||z_i||_{H^s}^2
\]
where \(C_1', C_2'\) do not depend on \(h\). This implies
\[
||z_j||_{H^s}^2 \leq C_2' e^{C_1'(j-1)h}, \quad ||z_j||_{H^s} \leq (C_2' e^{2TC_1'})^{\frac{1}{2}} =: c_2.
\]
Now we show \(\{Z_j\}\) is bounded in \(L^2(\Omega)\).
In fact, combine the consecutive equations in (14) to get
\[
Z_j - Z_{j-1} + (R^A_{\lambda(t_j)} z_j + q(t_j) z_j) - (R^A_{\lambda(t_{j-1})} z_{j-1} + q(t_{j-1}) z_{j-1}) = f(t_j) - f(t_{j-1}).
\]
Let both sides act on \(Z_j\), then we have
\[
\langle Z_j - Z_{j-1}, Z_j \rangle - B_{t_j}[z_j, Z_j] - B_{t_{j-1}}[z_{j-1}, Z_j] = \langle f(t_j) - f(t_{j-1}), Z_j \rangle. \tag{23}
\]
Note that
\[
B_{t_j}[z_j, Z_j] = \frac{1}{2h}(B_{t_j}[z_j, z_j] - B_{t_j}[z_{j-1}, z_{j-1}] + h^2 B_{t_j}[Z_j, Z_j]),
\]
\[
B_{t_{j-1}}[z_{j-1}, Z_j] = \frac{1}{2h}(B_{t_{j-1}}[z_{j-1}, z_{j-1}] - B_{t_{j-1}}[z_{j-1}, z_{j-1}] - h^2 B_{t_{j-1}}[Z_j, Z_j])
\]
so we have
\[
B_{t_j}[z_j, Z_j] - B_{t_{j-1}}[z_{j-1}, Z_j] = \frac{h}{2}(B_{t_j}[Z_j, Z_j] + B_{t_{j-1}}[Z_j, Z_j]) + \frac{1}{2h}(B_{t_j}[z_j, z_j] - B_{t_j}[z_{j-1}, z_{j-1}] + B_{t_{j-1}}[z_{j-1}, z_{j-1}] - B_{t_{j-1}}[z_{j-1}, z_{j-1}] - B_{t_{j-1}}[z_{j-1}, z_{j-1}] - B_{t_{j-1}}[z_{j-1}, z_{j-1}]).
\]
Also note that
\[
\langle Z_j - Z_{j-1}, Z_j \rangle = \frac{1}{2}||Z_j||_{L^2}^2 + ||Z_j - Z_{j-1}||_{L^2}^2 - ||Z_{j-1}||_{L^2}^2.
\]
\[
\left|\frac{\langle f(t_j) - f(t_{j-1}), Z_j \rangle}{h}\right| \leq \frac{1}{2}||Z_j||_{L^2}^2 + C^2.
\]
Now sum up all the identities in the form (23) for \(2 \leq l \leq j\) and omit all the non-negative terms
\[
B_{t_l}[Z_l, Z_l], \quad B_{t_{l-1}}[Z_l, Z_l], \quad ||Z_l - Z_{l-1}||_{L^2}^2
\]
to obtain the inequality
\[
||Z_j||_{L^2}^2 - ||Z_1||_{L^2}^2 + \frac{1}{h}(B_{t_j}[z_j, z_j] - B_{t_{j-1}}[z_j, z_j]) + \frac{1}{h} \sum_{i=2}^{j-1} (-B_{t_{i+1}}[z_i, z_i] + 2B_{t_i}[z_i, z_i] - B_{t_{i-1}}[z_i, z_i]) + \frac{1}{h}(B_{t_{l}}[z_l, z_1] - B_{t_{l-1}}[z_l, z_1]) + \frac{1}{h}(B_{t_{l}}[z_1, z_l] - B_{t_{l-1}}[z_1, z_l]) \leq h \sum_{l=2}^{j} (||Z_l||_{L^2}^2 + C^2). \tag{24}
\]
By (10) and (11), we have
\[ |B_{t_i}[z_i, z_i] - B_{t_{i-1}}[z_i, z_i]| \leq C_1 h \|z_i\|_{H^*}^2 \leq c_2^2 C_1 h, \]
\[ | - B_{t_{i+1}}[z_i, z_i] + 2B_{t_i}[z_i, z_i] - B_{t_{i-1}}[z_i, z_i]| \leq C_2 h^2 \|z_i\|_{H^*}^2 \leq c_2^2 C_2 h^2 \]
so (24) implies
\[ ||Z_j||_{L^2}^2 \leq ||Z_1||_{L^2}^2 + (j-1) h C^2 + 2c_2^2 C_1 + (j-2) c_2^2 C_2 h + h \sum_{i=2}^j ||Z_i||_{L^2}^2. \]
Since \( ||Z_1||_{L^2} \leq C_f \) and \((j-1) h \leq 2 T\), we can write
\[ ||Z_j||_{L^2}^2 \leq C''_2 + h \sum_{i=2}^j ||Z_i||_{L^2}^2 \]
where \(C''_2\) does not depend on \(h\). Since \(h < 1/2\), we have the estimate
\[ ||Z_j||_{L^2}^2 \leq 2C''_2 e^{2(j-1)h}, \quad ||Z_j||_{L^2} \leq (2C''_2 e^{4T})^{1/2} =: c_3. \]

Lemma 3.3. \(u\) and \(\{\tilde{u}^{(m_k)}\}\) defined in the proof of Proposition 3.1 satisfy
\[ \tilde{u}^{(m_k)} \to u \quad \text{in} \quad L^2(-T, T; \tilde{H}^s(\Omega)). \]

Proof. Since \(u^{(m_k)} \to u\) in \(L^2(-T, T; \tilde{H}^s(\Omega))\), we only need to show that
\[ u^{(m_k)} - \tilde{u}^{(m_k)} \to 0 \quad \text{in} \quad L^2(-T, T; \tilde{H}^s(\Omega)). \]

Consider \(v(t) = 1_{[\alpha, \beta]}(t)v_0\) where \(v_0 \in H^{-s}(\Omega)\) and \(\alpha, \beta\) are endpoints of subintervals in the \(m_k\)-division for some \(k\). For each large \(m_k\), we write \(\alpha = t^{(m_k)}_{j_1}, \beta = t^{(m_k)}_{j_2}\) for some \(j_1, j_2\), then
\[ \int_{-T}^T \langle v(t), u^{(m_k)}(t) - \tilde{u}^{(m_k)}(t) \rangle dt = \sum_{j=j_1+1}^{j_2} \int_{t^{(m_k)}_{j-1}}^{t^{(m_k)}_{j}} \langle v_0, u^{(m_k)}(t) - \tilde{u}^{(m_k)}(t) \rangle dt \]
\[ = \sum_{j=j_1+1}^{j_2} \int_{t^{(m_k)}_{j-1}}^{t^{(m_k)}_{j}} \langle v_0, (z^{(m_k)}_j - z^{(m_k)}_{j-1}) \frac{t - t^{(m_k)}_{j-1}}{h_{m_k}} \rangle dt \]
\[ = \sum_{j=j_1+1}^{j_2} \frac{h_{m_k}}{2} \langle v_0, (z^{(m_k)}_{j-1} - z^{(m_k)}_j) \rangle = \frac{h_{m_k}}{2} \langle v_0, (z^{(m_k)}_{j_1} - z^{(m_k)}_{j_2}) \rangle. \]
By the boundedness of \(\{z^{(m)}_j\}\), it converges to zero.

By using a density argument, we can conclude that
\[ \int_{-T}^T \langle v(t), u^{(m_k)}(t) - \tilde{u}^{(m_k)}(t) \rangle dt \to 0 \]
for general \(v \in L^2(-T, T; H^{-s}(\Omega)). \)
3.2 Well-posedness

Now we consider \( f \in L^2(\Omega \times (-T, T)) \). In fact, we can choose \( f_m \) satisfying \( f_m \to f \) in \( L^2(\Omega \times (-T, T)) \).

Let \( u_m \) be the solution corresponding to \( f_m \). Then we have

\[
||u_m - u||_{L^2(-T,T;H^s(\Omega))} + ||\partial_t(u_m - u)||_{L^2(-T,T;H^{-s}(\Omega))} \leq ||f_m - f||_{L^2(-T,T;H^{-s}(\Omega))}
\]

so

\[
u_m \to u \text{ in } L^2(-T,T;\dot{H}^s(\Omega)), \quad \partial_t u_m \to v \text{ in } L^2(-T,T;H^{-s}(\Omega))
\]

for some \( u, v \) and \( \partial_t u = v \).

This implies the convergence of \( \{u_m\} \) in \( C([-T,T];L^2(\Omega)) \) (see, for instance, Theorem 1 in Section 1.2 in Chapter 18 in [7]) and this \( u \) satisfies the estimate \( (21) \).

Hence we reach the following conclusion.

**Corollary 3.4.** Suppose \( f \in L^2(\Omega \times (-T, T)) \), then \((12)\) has a unique (weak) solution satisfying

\[
u \in L^2(-T,T;\dot{H}^s(\Omega)) \cap C([-T,T];L^2(\Omega)), \quad \partial_t u \in L^2(-T,T;H^{-s}(\Omega)).
\]

From now on we always assume \( \Omega \subset B_r(0) \) for some constant \( r > 0 \) and \( W \) is an open set in \( \mathbb{R}^n \) s.t. \( W \cap B_{3r}(0) = \emptyset \).

**Proposition 3.5.** Suppose \( g \in C^\infty_c(W \times (-T, T)) \), then \((15)\) has a unique (weak) solution \( u \) satisfying

\[
u \in L^2(-T,T;\dot{H}^s(\Omega)) \cap C([-T,T];L^2(\Omega)), \quad \partial_t w \in L^2(-T,T;H^{-s}(\Omega))
\]

where \( w := u - g \).

**Proof.** By (11), we have

\[
\mathcal{R}_A g|_{\Omega \times (-T,T)} = (-\Delta)^s g|_{\Omega \times (-T,T)}
\]

where \((-\Delta)^s\) acts on space variables. This is because \((x, y) \in \Omega \times W \) implies \(|x + y|/2 \geq r \) and thus \( A(t) = 0, R_A(t) = 1 \). Consider the problem

\[
\begin{cases}
\partial_t w + \mathcal{R}_A^s w + q(t)w = f & \text{in } \Omega \times (-T, T) \\
u = 0 & \text{in } \Omega \times \{-T\}
\end{cases}
\]

(25)

where \( f := -\mathcal{R}_A^s g|_{\Omega \times (-T,T)} \). Since

\((-\Delta)^s : H^\alpha(\mathbb{R}^n) \to H^{\alpha-2s}(\mathbb{R}^n), \quad \alpha \in \mathbb{R}\)

(see Lemma 2.1 in [12]), it is clear that \( f \in L^2(\Omega \times (-T, T)) \). Now apply the Corollary above. \( \Box \)

Consider the substitutions \( \tilde{A}(x,t) := A(x,-t), \tilde{q}(x,t) := q(x,-t), \tilde{g}(x,t) := g(x,-t) \) and \( \tilde{u}(x,t) := u(x,-t) \). Then we know the proposition above holds for the dual problem

\[
\begin{cases}
-\partial_t u + \mathcal{R}_A^s u + q(t)u = 0 & \text{in } \Omega \times (-T, T) \\
u = g & \text{in } \Omega \times (-T, T) \\
u = 0 & \text{in } \mathbb{R}^n \times \{T\}.
\end{cases}
\]

(26)

We denote the solution operator \( g \to u_g \) associated with (3) (resp. (26)) by \( P_{A,q} \) (resp. \( P^{*}_{A,q} \)).
4 Inverse Problem

4.1 Dirichlet-to-Neumann map

Proposition 3.5 ensures that the Dirichlet-to-Neumann map \( \Lambda_{A,q} \) given by (4) is well-defined at least for \( g \in C^\infty_c(W \times (-T, T)) \).

Now let \( g \in C^\infty_c(W_1 \times (-T, T)) \) and \( h \in C^\infty_c(W_2 \times (-T, T)) \).

By the definition of the solution operator \( P_{A,q} \) we have

\[
\int_{-T}^{T} \langle \Lambda_{A,q} g(t), h(t) \rangle dt = \int_{-T}^{T} B_1[u(t), \tilde{h}(t)] dt + \int_{-T}^{T} \langle \partial_t u(t), \tilde{h}(t) \rangle \Omega dt
\]

for any \( \tilde{h} \) satisfying \( \tilde{h} - h \in L^2(-T, T; \tilde{H}^s(\Omega)) \). Here \( u := P_{A,q} g, \ w := u - g \) and

\[
\langle \partial_t u(t), \tilde{h}(t) \rangle \Omega := \langle \partial_t w(t), \tilde{h}(t) - h(t) \rangle.
\]

Similarly we can define

\[
\Lambda_{A,q}^* h := \mathcal{R}_A u^* |_{\Omega_e \times (-T, T)}
\]

where \( u^* := P_{A,q}^* h \) and we have

\[
\int_{-T}^{T} \langle \Lambda_{A,q}^* h(t), g(t) \rangle dt = \int_{-T}^{T} B_1[u^*(t), \tilde{g}(t)] dt + \int_{-T}^{T} \langle -\partial_t u^*(t), \tilde{g}(t) \rangle \Omega dt
\]

for any \( \tilde{g} \) satisfying \( \tilde{g} - g \in L^2(-T, T; \tilde{H}^s(\Omega)) \).

**Proposition 4.1.** For \( g \in C^\infty_c(W_1 \times (-T, T)) \) and \( h \in C^\infty_c(W_2 \times (-T, T)) \), we have

\[
\int_{-T}^{T} \langle \Lambda_{A,q} g(t), h(t) \rangle dt = \int_{-T}^{T} \langle \Lambda_{A,q}^* h(t), g(t) \rangle dt.
\]

**Proof.** Let \( \tilde{h} = u^* \) in (24) and let \( \tilde{g} = u \) in (28). Since \( u(-T) = u^*(T) = 0 \), we have

\[
\int_{-T}^{T} \langle \Lambda_{A,q} g(t), h(t) \rangle dt - \int_{-T}^{T} \langle \Lambda_{A,q}^* h(t), g(t) \rangle dt
\]

\[
= \int_{-T}^{T} \langle \partial_t u(t), u^*(t) \rangle \Omega + \langle \partial_t u^*(t), u(t) \rangle \Omega dt = \langle u(t), u^*(t) \rangle \Omega |_{t=-T}^{+T} = 0.
\]

Now we build the integral identity, which will be used in the proof of the main theorem.

For \( g_j \in C^\infty_c(W_j \times (-T, T)) \) \((j = 1, 2)\), let \( u_1 = P_{A_1,q_1}(g_1) \) and \( u_2^* = P_{A_2,q_2}^*(g_2) \), i.e. \( u_1 \) is the unique weak solution of

\[
\begin{cases}
\partial_t u + \mathcal{R}_{A_1}(u) + q_1(t) u = 0 & \text{in } \Omega \times (-T, T) \\
u = g_1 & \text{in } \Omega_e \times (-T, T) \\
u = 0 & \text{in } \mathbb{R}^n \times \{-T\}.
\end{cases}
\]

(30)
and \( u_2^* \) is the unique weak solution of

\[
\begin{cases}
-\partial_t u + R_{A_2(t)}^s u + q_2(t) u = 0 & \text{in } \Omega \times (-T, T) \\
u = g_2 & \text{in } \Omega_c \times (-T, T) \\
u = 0 & \text{in } \mathbb{R}^n \setminus \{T\}.
\end{cases}
\] (31)

then we have

\[
\int_{-T}^{T} (\Lambda_{A_1, q_1} g_1(t), g_2(t)) - (\Lambda_{A_2, q_2} g_1(t), g_2(t)) dt
= \int_{-T}^{T} (\Lambda_{A_1, q_1} g_1(t), g_2(t)) dt - \int_{-T}^{T} (\Lambda_{A_2, q_2} g_1(t), g_2(t)) dt
= \int_{-T}^{T} B_t^{(1)}[u_1(t), u_2^*(t)] dt - \int_{-T}^{T} B_t^{(2)}[u_2^*(t), u_1(t)] dt
= \int_{-T}^{T} B_t^{(1)}[u_1(t), u_2^*(t)] dt - \int_{-T}^{T} B_t^{(2)}[u_1(t), u_2^*(t)] dt
= \int_{-T}^{T} \int \int \int G(x, y, t) u_1(y, t) u_2^*(x, t) - \int_{-T}^{T} \int (q_2 - q_1) u_1 u_2^* \tag{32}
\]

where

\[G(x, y, t) := 2(R_{A_2(t)}(x, y) - R_{A_1(t)}(x, y))K(x, y).\]

### 4.2 Runge approximation

**Proposition 4.2.** Suppose \( \Omega \subset B_r(0) \) for some constant \( r > 0 \) and \( W \) is an open set in \( \mathbb{R}^n \) s.t. \( W \cap B_{3r}(0) = \emptyset \), then

\[S := \{ P_{A,q} g |_{\Omega \times (-T, T)} : g \in C_c^{\infty} (W \times (-T, T)) \},
\]

\[S^* := \{ P_{A,q}^* g |_{\Omega \times (-T, T)} : g \in C_c^{\infty} (W \times (-T, T)) \}
\]

are dense in \( L^2(\Omega \times (-T, T)) \).

**Proof.** By the Hahn-Banach Theorem, it suffices to show that:

If \( v \in L^2(\Omega \times (-T, T)) \) and \( \int_{-T}^{T} \int_{\Omega} vw = 0 \) for all \( w \in S \), then \( v = 0 \) in \( \Omega \times (-T, T) \).

In fact, for a given \( v \in L^2(\Omega \times (-T, T)) \), let \( \phi \in L^2(-T, T; \hat{H}^s(\Omega)) \) be the solution of

\[
\begin{cases}
-\partial_t \phi + R_{A(t)}^s \phi + q(t) \phi = v & \text{in } \Omega \times (-T, T) \\
\phi = 0 & \text{in } \Omega \times \{T\}.
\end{cases}
\] (33)

For \( g \in C_c^{\infty}(W \times (-T, T)) \), write \( u_g := P_{A,q} g \), then we have

\[
\int_{-T}^{T} \int_{\Omega} v u_g = \int_{-T}^{T} (\int_{-T}^{T} (v(t) + R_{A(t)}^s \phi(t) + q(t) \phi, u_g(t) - g(t)) dt
= \int_{-T}^{T} (\int_{-T}^{T} (R_{A(t)}^s \phi(t), \phi(t)) dt - \int_{-T}^{T} (R_{A(t)}^s \phi(t), g(t)) dt.
\]
The first equality holds since $u_g - g \in L^2(-T,T;\tilde{H}^s(\Omega))$, the second equality holds since $u_g(T) = \phi(T) = 0$ and the last equality holds since $\phi \in L^2(-T,T;\tilde{H}^s(\Omega))$ and $u_g$ is the solution of (33).

Hence, if $\int_{-T}^{T} \int_{\Omega} u w = 0$ for all $w \in S$, then for each $t$ we have

$$\phi(t) \in \tilde{H}^s(\Omega), \quad R_{A(t)}^s \phi(t)|_W = 0,$$

which implies $\phi(t) = 0$ in $\mathbb{R}^n$ for each $t$ by Proposition 2.4 and thus $v = 0$ in $\Omega \times (-T,T)$.

Similarly we can show $S^*$ is dense in $L^2(\Omega \times (-T,T))$.

\textbf{Remark.} Proposition 4.2 can be viewed as a generalization of Theorem 2 in [23]. We refer readers to [3] [22] for more approximation properties of solutions of nonlocal evolution problems.

### 4.3 Proof of the main theorem

Now we are ready to prove Theorem 1.1. As in the proof of Theorem 1.1 in [17], we exploit the integral identity and the Runge approximation property associated with our operator.

\textbf{Proof.} Write $u_1 = P_{A_1,q_1}(g_1)$ and $u_2^* = P_{A_2,q_2}(g_2)$ for $g_j \in C^\infty_c(W_j \times (-T,T))$.

As in the proof of Theorem 1.1 in [17], the assumptions on $W_1, W_2, W^{(1,2)}$ ensure that

$$\int_{-T}^{T} \int_{\Omega} G(x,y,t) u_1(y,t) u_2^*(x,t) \, dx \, dy = \int_{-T}^{T} \int_{\Omega} G(x,y,t) u_1(y,t) u_2^*(x,t) \, dx \, dy$$

for each $t$ (if we shrink $W_1, W_2$ when necessary).

By the integral identity [22], [5] implies

$$\int_{-T}^{T} \int_{\Omega} G(x,y,t) u_1(y,t) u_2^*(x,t) = \int_{-T}^{T} \int_{\Omega} (q_2 - q_1) u_1 u_2^*. \quad (34)$$

\textbf{Determine $A$:} Fix open sets $\Omega_j \subset \Omega$ s.t. $\Omega_1 \cap \Omega_2 = \emptyset$. Also fix $\phi_j \in C^\infty_c(\Omega_j)$ and the constants $a, b \in (-T,T)$ and $\epsilon > 0$. Write

$$\tilde{\phi}_j(x,t) := 1_{[a,b]}(t)\phi_j(x).$$

By Proposition 4.2, we can choose $g_1 \in C^\infty_c(W_1 \times (-T,T))$ s.t.

$$\|u_1 - \tilde{\phi}_1\|_{L^2(\Omega \times (-T,T))} \leq \epsilon$$

and for this chosen $g_1$, we can choose $g_2 \in C^\infty_c(W_2 \times (-T,T))$ s.t.

$$\|u_1\|_{L^2(\Omega \times (-T,T))} \|u_2^* - \tilde{\phi}_2\|_{L^2(\Omega \times (-T,T))} \leq \epsilon.$$

Note that $\phi_1(x) \tilde{\phi}_2(x) = 0$ for $x \in \Omega$ so

$$|\int_{-T}^{T} \int_{\Omega} (q_2 - q_1) u_1 u_2^*| = |\int_{-T}^{T} \int_{\Omega} (q_2 - q_1)(u_1 - \tilde{\phi}_1) \tilde{\phi}_2 + \int_{-T}^{T} \int_{\Omega} (q_2 - q_1) u_1 (u_2^* - \tilde{\phi}_2)|$$

$$\leq \|q_2 - q_1\|_{L^\infty} \|\tilde{\phi}_2\|_{L^2} \|u_1 - \tilde{\phi}_1\|_{L^2} + \|q_2 - q_1\|_{L^\infty} \|u_1\|_{L^2} \|u_2^* - \tilde{\phi}_2\|_{L^2} \leq C \epsilon. \quad (35)$$

Also note that

$$|G(x,y,t)| \leq \frac{C'}{|x-y|^{n+2s-2}},$$
By Cauchy-Schwarz inequality, we have the estimate
\[ a, b \Rightarrow C' \leq C'', \quad y \in \Omega \]
where \( C', C'' \) do not depend on \( t \). By the generalized Young’s Inequality,
\[ ||T_t f||_{L^2(\Omega)} \leq C''||f||_{L^2(\Omega)}, \quad (T_t f)(x) := \int_{\Omega} |G(x, y, t)f(y)| \, dy. \]

Now note that
\[
\int_{-T}^{T} \int_{\Omega} G(x, y, t)u_1(y, t)u_2^*(x, t) \, dxdydt = \int_{a}^{b} \int_{\Omega_1} \int_{\Omega_2} G(x, y, t)\phi_1(y)\phi_2(x) \, dxdydt
\]

By Cauchy-Schwarz inequality, we have the estimate
\[
\int_{-T}^{T} \int_{\Omega} |G(x, y, t)u_1(y, t)||u_2^*(x, t) - \tilde{\phi}_2(x, t)| \, dxdydt
\leq \left( \int_{-T}^{T} \int_{\Omega} ||(T_t u_1(t))(x)||^2 \, dxdydt \right)^{\frac{1}{2}} ||u_2^* - \tilde{\phi}_2||_{L^2(\Omega\times(-T,T))}
\]

Similarly, we have
\[
\int_{-T}^{T} \int_{\Omega} |G(x, y, t)\tilde{\phi}_2(x, t)||u_1(y, t) - \tilde{\phi}_1(y, t)| \, dxdydt
\leq C''||\tilde{\phi}_2||_{L^2(\Omega\times(-T,T))}||u_1 - \tilde{\phi}_1||_{L^2(\Omega\times(-T,T))}.
\]

Hence
\[
\left| \int_{-T}^{T} \int_{\Omega} G(x, y, t)u_1(y, t)u_2^*(x, t) - \int_{a}^{b} \int_{\Omega_1} \int_{\Omega_2} G(x, y, t)\phi_1(y)\phi_2(x) \, dxdydt \right| \leq C''' \epsilon. \quad (36)
\]

Combine (34), (35) with (31). \( \epsilon \) is arbitrary implies
\[
\int_{a}^{b} \int_{\Omega_1} \int_{\Omega_2} G(x, y, t)\phi_1(y)\phi_2(x) \, dxdydt = 0.
\]

Then \([a, b] \) is arbitrary implies
\[
\int_{\Omega_1} \int_{\Omega_2} G(x, y, t)\phi_1(y)\phi_2(x) \, dxdy = 0
\]

16
for each $t$ and thus $G(x, y, t) = 0$ in $\Omega_1 \times \Omega_2$ for each $t$ since $\phi_1, \phi_2$ are arbitrary. Now we can conclude that $G(x, y, t) = 0$ for $x, y \in \Omega$ whenever $x \neq y$ since $\Omega_1, \Omega_2$ are arbitrary. Hence

$$R_{A_1(t)}(x, y) = R_{A_2(t)}(x, y), \quad x, y \in \Omega$$

(37)

for each $t$, which implies $A_1(t) = \pm A_2(t)$ as in the proof of Theorem 1.1 in [17].

**Determine $q$:** Now (34) becomes

$$\int_{-T}^{T} \int_{\Omega} (q_2 - q_1)u_1u_2^* = 0.$$

Fix $\epsilon > 0$ and $f \in L^2(\Omega \times (-T, T))$. Choose $g_1 \in C_0^\infty(W_1 \times (-T, T))$ s.t.

$$||u_1 - f||_{L^2(\Omega \times (-T,T))} \leq \epsilon.$$

For this chosen $u_1$, choose $g_2 \in C_0^\infty(W_2)$ s.t.

$$||u_1||_{L^2(\Omega \times (-T,T))}||u_2^* - 1||_{L^2(\Omega \times (-T,T))} \leq \epsilon.$$

Now we have

$$|\int_{-T}^{T} \int_{\Omega} (q_1 - q_2)f| = |\int_{-T}^{T} \int_{\Omega} (q_1 - q_2)(f - u_1) + \int_{-T}^{T} \int_{\Omega} (q_1 - q_2)u_1(1 - u_2^*)| \leq C\epsilon.$$

We conclude that $q_1 = q_2$ since $\epsilon, f$ are arbitrary. \qed

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