Einstein solvmanifolds and nilsolitons

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Dedicated to Isabel Dotti and Roberto Miasek on the occasion of their 60th birthday.

Abstract. The purpose of the present expository paper is to give an account of the recent progress and present status of the classification of solvable Lie groups admitting an Einstein left invariant Riemannian metric, the only known examples so far of noncompact Einstein homogeneous manifolds. The problem turns to be equivalent to the classification of Ricci soliton left invariant metrics on nilpotent Lie groups.

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1. Introduction

Let $M$ be a differentiable manifold. The question of whether there is a ‘best’ Riemannian metric on $M$ is intriguing. A great deal of deep results in Riemannian geometry have been motivated, and even inspired, by this single natural question. For several good reasons, an Einstein metric is a good candidate, if not the best, at

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least a very distinguished one (see [Besse 87, Chapter 0]). A Riemannian metric $g$ on $M$ is called Einstein if its Ricci tensor $\text{ric}_g$ satisfies

\begin{equation}
\text{ric}_g = cg, \quad \text{for some } c \in \mathbb{R}.
\end{equation}

This notion can be traced back to [Hilbert 15], where Einstein metrics emerged as critical points of the total scalar curvature functional on the space of all metrics on $M$ of a given volume. Equation (1.1) is a non-linear second order PDE (recall that the number of parameters is $\frac{n(n+1)}{2}$ on both sides, $n = \dim M$), which also gives rise to some hope, but a good understanding of the solutions in the general case seems far from being attained. A classical reference for Einstein manifolds is the book [Besse 87], and some updated expository articles are [Anderson 94], [Lebrun-Wang 99], [Berger 00] III,C., and [Berger 03] 11.4.

The Einstein condition (1.1) is very subtle, even when restricted to almost any subclass of metrics on $M$ one may like. It is too strong to allow general existence results, and sometimes even just to find a single example, and at the same time, it is too weak to get obstructions or classification results.

But maybe the difficulty comes from PDEs, so let us ‘algebrize’ the problem (algebra is always easier for a geometer ...). Let us consider homogeneous Riemannian manifolds. Indeed, the Einstein equation for a homogeneous metric is just a system of $\frac{n(n+1)}{2}$ algebraic equations, but unfortunately, a quite involved one, and the following main general question is still open in both compact and noncompact cases:

Which homogeneous spaces $G/K$ admit a $G$-invariant Einstein Riemannian metric?

We refer to [Böhm-Wang-Ziller 04] and the references therein for an update in the compact case. In the noncompact case, the only known examples until now are all of a very particular kind; namely, simply connected solvable Lie groups endowed with a left invariant metric (so called solvmanifolds). According to the following long standing conjecture, these might exhaust all the possibilities for noncompact homogeneous Einstein manifolds.

Alekseevskii’s conjecture [Besse 87, 7.57]. If $G/K$ is a homogeneous Einstein manifold of negative scalar curvature then $K$ is a maximal compact subgroup of $G$ (which implies that $G/K$ is a solvmanifold when $G$ is a linear group).

The conjecture is wide open, and it is known to be true only for $\dim \leq 5$, a result which follows from the complete classification in these dimensions given in [Nikonorov 05]. One of the most intriguing facts related to this conjecture, and maybe the only reason so far to consider Alekseevskii’s conjecture as too optimistic, is that the Lie groups $\text{SL}_n(\mathbb{R})$, $n \geq 3$, do admit left invariant metrics of negative Ricci curvature, as well as does any complex simple Lie group (see [Dotti-Leite 82], [Dotti-Leite-Miatello 84]). However, an inspection of the eigenvalues of the Ricci tensors in these examples shows that they are far from being close to each other, giving back some hope.

Let us now consider the case of left invariant metrics on Lie groups. Let $\mathfrak{g}$ be a real Lie algebra. Each basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$ determines structural constants
\{c^k_{ij}\} \subset \mathbb{R} \text{ given by } 

[X_i, X_j] = \sum_{k=1}^{n} c^k_{ij} X_k, \quad 1 \leq i, j \leq n.

The left invariant metric on any Lie group with Lie algebra \( \mathfrak{g} \) defined by the inner product given by \( \langle X_i, X_j \rangle = \delta_{ij} \) is Einstein if and only if the \( n^2(n+1)/2 \) numbers \( c^k_{ij} \)'s satisfy the following \( n(n+1)/2 \) algebraic equations for some \( c \in \mathbb{R} \):

\[
(1.2) \quad \sum_{kl} -\frac{1}{2} c^d_{ik} c^d_{jk} + \frac{1}{4} c^d_{kl} c^d_{kl} - \frac{1}{2} c^d_{jk} c^d_{ik} + \frac{1}{2} c^d_{kl} c^d_{kl} + \frac{1}{2} c^d_{kl} c^d_{kl} = c\delta_{ij}, \quad 1 \leq i \leq j \leq n.
\]

In view of this, one may naively think that the classification of Einstein left invariant metrics on Lie groups is at hand. However, the following natural questions remain open:

(i) Is any Lie group admitting an Einstein left invariant metric either solvable or compact?

(ii) Does every compact Lie group admit only finitely many Einstein left invariant metrics up to isometry and scaling?

(iii) Which solvable Lie groups admit an Einstein left invariant metric?

We note that question (i) is just Alekseevskii Conjecture restricted to Lie groups, and question (ii) is contained in [Besse 87, 7.55]. The only group for which the answer to (ii) is known is SU(2), where there is only one (see [Milnor 76]). For most of the other compact simple Lie groups many Einstein left invariant metrics other than minus the Killing form are explicitly known (see [D’Atri-Ziller 79]).

Even if one is very optimistic and believes that Alekseevskii Conjecture is true, a classification of Einstein metrics in the noncompact homogeneous case will depend on some kind of answer to question (iii). The aim of this expository paper is indeed to give a report on the present status of the study of Einstein solvmanifolds.

Perhaps the main difficulty in trying to decide if a given Lie algebra \( \mathfrak{g} \) admits an Einstein inner product is that one must check condition (1.2) for any basis of \( \mathfrak{g} \), and there are really too many of them. In other words, there are too many left invariant metrics on a given Lie group, any inner product on the vector space \( \mathfrak{g} \) is playing. This is quite in contrast to what happens in homogeneous spaces \( G/K \) with not many different \( \text{Ad}(K) \)-irreducible components in the decomposition of the tangent space \( T_{eK}(G/K) \). Another obstacle is how to recognize your Lie algebra by just looking at the structural constants \( c^k_{ij} \)'s. Though even in the case when we have two solutions to (1.2), and we know they define the same Lie algebra, to be able to guarantee that they are not isometric, i.e. that we really have two Einstein metrics, is usually involved.

If we fix a basis \( \{X_1, \ldots, X_n\} \) of \( \mathfrak{g} \), then instead of varying all possible sets of structural constants \( \{c^k_{ij}\} \)'s by running over all bases, one may act on the Lie bracket \([\cdot, \cdot]\) by \( g \cdot [\cdot, \cdot] = g[g^{-1}, \cdot, g^{-1}] \), for any \( g \in \text{GL}(\mathfrak{g}) \), and look at the structural constants of \( g \cdot [\cdot, \cdot] \) with respect to the fixed basis \( \{X_1, \ldots, X_n\} \). This give rises to an orbit \( \text{GL}(\mathfrak{g}), [\cdot, \cdot] \) in the vector space \( V := \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} \) of all skew-symmetric
bilinear maps from \( \mathfrak{g} \times \mathfrak{g} \) to \( \mathfrak{g} \), which parameterizes, from a different point of view, the set of all inner products on \( \mathfrak{g} \). Indeed, if \( \langle \cdot, \cdot \rangle \) is the inner product defined by \( \langle X_i, X_j \rangle = \delta_{ij} \) then

\[
\langle g, [\cdot, \cdot], \langle \cdot, \cdot \rangle \rangle \text{ is isometric to } \langle \mathfrak{g}, [\cdot, \cdot], \langle g, \cdot \rangle \rangle \text{ for any } g \in \text{GL}(\mathfrak{g}).
\]

The subset \( \mathcal{L} \subset V \) of those elements satisfying the Jacobi condition is algebraic, \( \text{GL}(\mathfrak{g}) \)-invariant and the \( \text{GL}(\mathfrak{g}) \)-orbits in \( \mathcal{L} \) are precisely the isomorphism classes of Lie algebras. \( \mathcal{L} \) is called the variety of Lie algebras. Furthermore, if \( \text{O}(\mathfrak{g}) \subset \text{GL}(\mathfrak{g}) \) denotes the subgroup of \( \langle \cdot, \cdot \rangle \)-orthogonal maps, then two points in \( \text{GL}(\mathfrak{g}) \cdot [\cdot, \cdot] \) which lie in the same \( \text{O}(\mathfrak{g}) \)-orbit determine isometric left invariant metrics, and the converse holds if \( \mathfrak{g} \) is completely solvable (see [Alekseevskii 71]).

This point of view is certainly a rather tempting invitation to try to use geometric invariant theory in any problem which needs a running over all left invariant metrics on a given Lie group, or even on all Lie groups of a given dimension. We shall see throughout this article that indeed, starting in [Heber 98], the approach ‘by varying Lie brackets’ has been very fruitful in the study of Einstein solvmanifolds during the last decade.

The latest fashion generalization of Einstein metrics, although they were introduced by R. Hamilton more than twenty years ago, is the notion of Ricci soliton:

\[
\text{ric}_g = cg + L_X g, \quad \text{for some } c \in \mathbb{R}, \; X \in \chi(M),
\]

where \( L_X g \) is the usual Lie derivative of \( g \) in the direction of the field \( X \). A more intuitive equivalent condition to (1.3) is that \( \text{ric}_g \) is tangent at \( g \) to the space of all metrics which are homothetic to \( g \) (i.e. isometric up to a constant scalar multiple). Recall that Einstein means \( \text{ric}_g \) tangent to \( \mathbb{R}_{>0}g \). Ricci solitons correspond to solutions of the Ricci flow

\[
\frac{d}{dt}g(t) = -2 \text{ric}_g(t),
\]

that evolves self similarly, that is, only by scaling and the action by diffeomorphisms, and often arise as limits of dilations of singularities of the Ricci flow. We refer to [L. 01a], [Guenther-Isenberg-Knopf 06], [Chow et al. 07] and the references therein for further information on the Hamilton-Perelman theory of Ricci flow and Ricci solitons and the role played by nilpotent Lie groups in the story.

A remarkable fact is that if \( S \) is an Einstein solvmanifold, then the metric restricted to the submanifold \( N := [S, S] \) is a Ricci soliton, and conversely, any Ricci soliton left invariant metric on a nilpotent Lie group \( N \) (called nilsolitons) can be uniquely ‘extended’ to an Einstein solvmanifold. This one-to-one correspondence is complemented with the uniqueness up to isometry of nilsolitons, which finally turns the classification of Einstein solvmanifolds into a classification problem on nilpotent Lie algebras. These are not precisely good news. Historically, as the literature and experience shows us, any classification problem involving nilpotent Lie algebras is simply a headache.

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2. Structure and uniqueness results on Einstein solvmanifolds

A solvmanifold is a simply connected solvable Lie group $S$ endowed with a left invariant Riemannian metric. A left invariant metric on a Lie group $G$ will be always identified with the inner product $\langle \cdot, \cdot \rangle$ determined on the Lie algebra $\mathfrak{g}$ of $G$, and the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ will be referred to as a metric Lie algebra. If $S$ is a solvmanifold and $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$ is its metric solvable Lie algebra, then we consider the $\langle \cdot, \cdot \rangle$-orthogonal decomposition

$$ \mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, $$

where $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is the derived algebra (recall that $\mathfrak{n}$ is nilpotent).

**Definition 2.1.** A solvmanifold $S$ is said to be standard if $[\mathfrak{a}, \mathfrak{a}] = 0$.

This is a very simple algebraic condition, which may appear as kind of technical, but it has nevertheless played an important role in many questions in homogeneous Riemannian geometry:

- [Gindikin-Piatetskii Shapirov-Vinberg 67] Kähler-Einstein noncompact homogeneous manifolds are all standard solvmanifolds.
- [Alekseevskii 75, Cortés 96] Every quaternionic Kähler solvmanifold (completely real) is standard.
- [Azencott-Wilson 76] Any homogeneous manifold of nonpositive sectional curvature is a standard solvmanifold.
- [Heber 06] All harmonic noncompact homogeneous manifolds are standard solvmanifolds (with $\dim \mathfrak{a} = 1$).

Partial results on the question of whether Einstein solvmanifolds are all standard were obtained for instance in [Heber 98] and [Schueth 04], who gave several sufficient conditions. The answer was known to be yes in dimension $\leq 6$ (see [Nikitenko-Nikonorov 06]) and followed from a complete classification of Einstein solvmanifolds in these dimensions. On the other hand, it is proved in [Nikolayevsky 06b] that many classes of nilpotent Lie algebras can not be the nilradical of a non-standard Einstein solvmanifold.

**Theorem 2.2.** [L. 07] Any Einstein solvmanifold is standard.

An idea of the proof of this theorem will be given in Section 8. Standard Einstein solvmanifolds were extensively investigated in [Heber 98], where the remarkable structural and uniqueness results we next describe are derived. Recall that combined with Theorem 2.2 all of these results are now valid for any Einstein solvmanifold.

**Theorem 2.3.** [Heber 98, Section 5] (Uniqueness) A simply connected solvable Lie group admits at most one standard Einstein left invariant metric up to isometry and scaling.

A more general result is actually valid: if a noncompact homogeneous manifold $G/K$ with $K$ maximal compact in $G$ admits a $G$-invariant metric $g$ isometric to an Einstein solvmanifold, then $g$ is the unique $G$-invariant Einstein metric on $G/K$ up to isometry and scaling. This is in contrast to the compact homogeneous case, where many pairwise non isometric $G$-invariant Einstein metrics might exist (see [Böhm-Wang-Ziller 04] and the references therein), although it is open if only finitely many (see [Besse 87, 7.55]).
In the study of Einstein homogeneous manifolds, the compact case is characterized by the positivity of the scalar curvature and Ricci flat implies flat (see [Alekseevskii-Kimel’fel’d 75]). The following conditions on an Einstein solvmanifold $S$ are equivalent:

(i) $s$ is unimodular (i.e. $\text{tr ad } X = 0$ for all $X \in s$).
(ii) $S$ is Ricci flat (i.e. $\text{sc}(S) = 0$).
(iii) $S$ is flat.

We can therefore consider from now on only nonunimodular solvable Lie algebras.

Theorem 2.4. [Heber 98] (Rank-one reduction) Let $s = a \oplus n$ be a nonunimodular solvable Lie algebra endowed with a standard Einstein inner product $(\cdot, \cdot)$, say with $\text{ric}(\cdot, \cdot) = c(\cdot, \cdot)$. Then $c < 0$ and, up to isometry, it can be assumed that $\text{ad} A$ is symmetric for any $A \in a$. In that case, the following conditions hold.

(i) There exists $H \in a$ such that the eigenvalues of $\text{ad} H|_n$ are all positive integers without a common divisor.
(ii) The restriction of $(\cdot, \cdot)$ to the solvable Lie algebra $\mathbb{R}H \oplus n$ is also Einstein.
(iii) $a$ is an abelian algebra of symmetric derivations of $n$ and the inner product on $a$ must be given by $(A, A') = -\frac{1}{c} \text{tr ad} A \text{ad} A'$ for all $A, A' \in a$.

The Ricci tensor for these solvmanifolds has the following simple formula.

Lemma 2.5. Let $S$ be a standard solvmanifold such that $\text{ad} A$ is symmetric and nonzero for any $A \in a$. Then the Ricci tensor of $S$ is given by

(i) $\text{ric}(A, A') = -\text{tr ad} A \text{ad} A'$ for all $A, A' \in a$.
(ii) $\text{ric}(a, n) = 0$.
(iii) $\text{ric}(X, Y) = \text{ric}_n(X, Y) - (\text{ad} H(X), Y)$, for all $X, Y \in n$, where $\text{ric}_n$ is the Ricci tensor of $(n, (\cdot, \cdot)|_n \times n)$ and $H \in a$ is defined by $(H, A) = \text{tr ad} A$ for any $A \in a$.

The natural numbers which have appeared as the eigenvalues of $\text{ad} H$ when $(s, (\cdot, \cdot))$ is Einstein play a very important role.

Definition 2.6. If $d_1, ..., d_r$ denote the corresponding multiplicities of the positive integers without a common divisor $k_1 < ... < k_r$ given by Theorem 2.4 (i), then the tuple

$$(k; d) = (k_1 < ... < k_r; d_1, ..., d_r)$$

is called the eigenvalue type of the Einstein solvmanifold $(s, (\cdot, \cdot))$.

We find here the first obstruction: if a solvable Lie algebra admits an Einstein inner product then the nilpotent Lie algebra $n = [s, s]$ is $\mathbb{N}$-graded, that is, there is a decomposition $n = n_1 \oplus ... \oplus n_r$ such that $[n_i, n_j] \subset n_{i+j}$ for all $i, j$ (recall that some of the $n_i$’s might be trivial). This is precisely the decomposition into eigenspaces of the derivation with positive integer eigenvalues $\text{ad} H$. Another important consequence of Theorem 2.4 is that to study Einstein solvmanifolds, it will be enough to consider rank-one (i.e. $\dim a = 1$) metric solvable Lie algebras, since every higher rank Einstein solvmanifold will correspond to a unique rank-one Einstein solvmanifold and certain abelian subalgebra $a$ of derivations of $n$ containing $\text{ad} H$. Recall that how to extend the inner product is determined by Theorem 2.4 (iii).
Let \( \mathcal{M} \) be the moduli space of all the isometry classes of Einstein solvmanifolds of a given dimension with scalar curvature equal to \(-1\), endowed with the \( C^\infty \)-topology. Notice that any \( n \)-dimensional solvmanifold \( S \) is diffeomorphic to the euclidean space \( \mathbb{R}^n \), and so any \( S \) can be viewed as a Riemannian metric on \( \mathbb{R}^n \) (which is in addition invariant by some transitive solvable Lie group of diffeomorphisms of \( \mathbb{R}^n \)).

**Theorem 2.7.** [Heber 98, Section 6] (Moduli space) In every dimension, only finitely many eigenvalue types occur, and each eigenvalue type \((k; d)\) determines a compact path connected component \( \mathcal{M}_{(k; d)} \) of \( \mathcal{M} \), homeomorphic to a real semialgebraic set.

Results on the topology and 'dimension' of the moduli spaces \( \mathcal{M}_{(k; d)} \) near a rank-one symmetric space are obtained in [Heber 98, Section 6.5]. This has also been done for many other symmetric spaces \((\text{rank} \geq 2)\) in [Gordon-Kerr 01], where even explicit examples are exhibited to describe a neighborhood. The moduli spaces \( \mathcal{M}_{1 < 2^{q,p}} \) are studied in detail in [Eberlein 07] and [Nikolayevsky 08a].

In the light of Theorem 2.4 and Lemma 2.5, it is reasonable to expect that Einstein solvmanifolds are actually completely determined by their nilpotent parts. Let us describe this more precisely.

**Definition 2.8.** Given a metric nilpotent Lie algebra \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\), a metric solvable Lie algebra \((\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle \cdot, \cdot \rangle')\) is called a metric solvable extension of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) if \( \mathfrak{n} \) is an ideal of \( \mathfrak{s} \), \([\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{n} \) and \( \langle \cdot, \cdot \rangle'_{|\mathfrak{n} \times \mathfrak{n}} = \langle \cdot, \cdot \rangle \).

It turns out that for each \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) there exists a unique rank-one metric solvable extension \((\mathfrak{s}, \langle \cdot, \cdot \rangle')\) which stands a chance of being an Einstein manifold. Indeed, since the inner product on \( \mathfrak{a} \) is determined by Theorem 2.4 (iii), the only datum we need to recover \((\mathfrak{s}, \langle \cdot, \cdot \rangle')\) from \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is the way \( \mathfrak{a} \) is acting on \( \mathfrak{n} \) by derivations. Recall that \( \dim \mathfrak{a} = 1 \). It follows from Lemma 2.5 that if \( A \in \mathfrak{a} \) satisfies \( ||A||^2 = \text{tr} A \) and we set \( D := \text{ad} A|_{\mathfrak{n}} \), then the Ricci operator \( R_{\langle \cdot, \cdot \rangle} \) of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) equals

\[
R_{\langle \cdot, \cdot \rangle} = cI + D.
\]

But \( R_{\langle \cdot, \cdot \rangle} \) is orthogonal to \( D \) and actually to any symmetric derivation of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) (see [L. 01b, (2)]), thus

\[
(2.1) \quad \text{tr } D A = -c \text{ tr } A, \quad c = \frac{\text{tr } R^{2}_{\langle \cdot, \cdot \rangle}}{\text{tr } R_{\langle \cdot, \cdot \rangle}},
\]

for any symmetric derivation \( A \) of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\). This determines \( D \) in terms of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) (recall that if \( \mathfrak{n} \) is nonabelian then \( \text{tr } R_{\langle \cdot, \cdot \rangle} < 0 \), and in the abelian case both \( R_{\langle \cdot, \cdot \rangle} \) and \( D \) equal zero). We have therefore seen that a rank-one Einstein solvmanifold is completely determined by its (metric) nilpotent part. This fact turns the study of rank-one Einstein solvmanifolds into a problem on nilpotent Lie algebras.

**Definition 2.9.** A nilpotent Lie algebra \( \mathfrak{n} \) is said to be an *Einstein nilradical* if it admits an inner product \( \langle \cdot, \cdot \rangle \) such that there is a metric solvable extension \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) which is Einstein.

Such a solvable extension must satisfy \([\mathfrak{a}, \mathfrak{a}] = 0\) by Theorem 2.2 and it can be assumed that \( \dim \mathfrak{a} = 1 \) by Theorem 2.4 (ii). In other words, Einstein nilradicals are precisely the *nilradicals* (i.e. the maximal nilpotent ideal) of Lie algebras of Einstein solvmanifolds.
Given a nilpotent Lie algebra $\mathfrak{n}$, an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ can be extended to construct an Einstein solvmanifold if and only if any of the following equivalent conditions hold, which shows us that these left invariant metrics on nilpotent Lie groups are very special from many other points of view:

(i) $R_{\langle \cdot, \cdot \rangle} = cI + D$ for some $c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{n})$, where $R_{\langle \cdot, \cdot \rangle}$ is the Ricci operator of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ and $\text{Der}(\mathfrak{n})$ is the space of all derivations of $\mathfrak{n}$.

(ii) $\langle \cdot, \cdot \rangle$ is a Ricci soliton metric: the solution $\langle \cdot, \cdot \rangle_t$ with initial point $\langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle$ to the Ricci flow

$$\frac{d}{dt} \langle \cdot, \cdot \rangle_t = -2 \text{ric}(\cdot, \cdot)_t,$$

remains isometric up to scaling to $\langle \cdot, \cdot \rangle$, that is, $\langle \cdot, \cdot \rangle_t = c_t \varphi_t^* \langle \cdot, \cdot \rangle$ for some one parameter group of diffeomorphisms $\{\varphi_t\}$ of $N$ and $c_t \in \mathbb{R}$ (see for instance [Chow et al. 07]), where $N$ denotes the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$.

(iii) $\langle \cdot, \cdot \rangle$ is a quasi-Einstein metric:

$$\text{ric}(\cdot, \cdot) = c \langle \cdot, \cdot \rangle + L_X \langle \cdot, \cdot \rangle$$

for some $C^\infty$ vector field $X$ in $N$ and $c \in \mathbb{R}$, where $L_X \langle \cdot, \cdot \rangle$ denotes the usual Lie derivative. This class of metrics were actually first considered in theoretical physics (see [Friedan 85] and [Chave-Valent 96]).

(iv) $\langle \cdot, \cdot \rangle$ is a minimal metric:

$$\|\text{ric}(\cdot, \cdot)\| = \min \{\|\text{ric}(\cdot, \cdot)'\| : \text{sc}(\cdot, \cdot)' = \text{sc}(\cdot, \cdot)\},$$

where $\langle \cdot, \cdot \rangle'$ runs over all left invariant metrics on $N$ and $\text{sc}(\cdot, \cdot)$ denotes scalar curvature of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ (see [L. 04]). A nilpotent Lie group $N$ can never admit an Einstein left invariant metric, unless it is abelian, and a way of getting as close as possible to satisfy the Einstein condition is to have a minimal metric. Indeed,

$$\|\text{ric}(\cdot, \cdot) - \frac{\text{sc}(\cdot, \cdot)}{n} \langle \cdot, \cdot \rangle\|^2 = \|\text{ric}(\cdot, \cdot)\|^2 - \frac{\text{sc}(\cdot, \cdot)^2}{n}.$$

**Definition 2.10.** A left invariant metric on a nilpotent Lie group (or equivalently an inner product on a nilpotent Lie algebra) is called a nilsoliton if it satisfies any of the above conditions (i)-(iv).

**Theorem 2.11.** A nilpotent Lie algebra $\mathfrak{n}$ is an Einstein nilradical if and only if $\mathfrak{n}$ admits a nilsoliton metric.

It is then reasonable to expect that uniqueness for nilsolitons should hold, as it does for standard Einstein solvmanifolds (see Theorem 2.3). This is actually true, and even a very similar proof worked out.

**Theorem 2.12.** [L. 01a] There is at most one nilsoliton metric on a nilpotent Lie group up to isometry and scaling.
We therefore obtain the following picture of one-to-one correspondences on the classification problem:

\[
\begin{align*}
\{\text{Rank-one Einstein solvmanifolds}\} & \xrightarrow{\text{isometry and scaling}} \\
\{\text{Ricci soliton (simply connected) nilmanifolds}\} & \xrightarrow{\text{isometry and scaling}} \\
\{\text{Einstein nilradicals}\} & \xrightarrow{\text{isomorphism}}
\end{align*}
\]

Thus the classification of Einstein solvmanifolds reduces to a completely ‘algebraic’ problem; namely, the classification of nilpotent Lie algebras which are Einstein nilradicals. This problem will be treated in Section 5.

3. Technical background

In this section, we fix the notation and give all the definitions and elementary results we need to use throughout the paper. All this mainly concerns the vector space where the Lie algebras of a given dimension live and the action determining the isomorphism relation between them.

Let us consider the space of all skew-symmetric algebras of dimension \(n\), which is parameterized by the vector space

\[
V = \Lambda^2(R^n)^* \otimes R^n = \{\mu : R^n \times R^n \rightarrow R^n : \mu \text{ bilinear and skew-symmetric}\}.
\]

Then

\[
\mathcal{N} = \{\mu \in V : \mu \text{ satisfies Jacobi and is nilpotent}\}
\]

is an algebraic subset of \(V\) as the Jacobi identity and the nilpotency condition can both be written as zeroes of polynomial functions. \(\mathcal{N}\) is often called the \textit{variety of nilpotent Lie algebras} (of dimension \(n\)). There is a natural action of \(GL_n(R)\) on \(V\) given by

\[
g.\mu(X,Y) = g\mu(g^{-1}X, g^{-1}Y), \quad X,Y \in R^n, \quad g \in GL_n(R), \quad \mu \in V.
\]

Recall that \(\mathcal{N}\) is \(GL_n(R)\)-invariant and the Lie algebra isomorphism classes are precisely the \(GL_n(R)\)-orbits. The action of \(\mathfrak{gl}_n(R)\) on \(V\) obtained by differentiation of (3.1) is given by

\[
\pi(\alpha)\mu = \alpha\mu(\cdot, \cdot) - \mu(\cdot, \alpha \cdot) - \mu(\alpha \cdot, \cdot), \quad \alpha \in \mathfrak{gl}_n(R), \quad \mu \in V.
\]

We note that \(\pi(\alpha)\mu = 0\) if and only if \(\alpha \in \text{Der}(\mu)\), the Lie algebra of derivations of the algebra \(\mu\), which is actually the Lie algebra of \(\text{Aut}(\mu)\), the group of automorphisms of the algebra \(\mu\). Recall that \(\text{Aut}(\mu)\) is the isotropy subgroup at \(\mu\) for the action (3.1), and so \(\dim GL_n(R),\mu = n^2 - \dim \text{Der}(\mu)\).
The canonical inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^n \) determines an \( O(n) \)-invariant inner product on \( V \), also denoted by \( \langle \cdot, \cdot \rangle \), as follows:
\[
(\alpha, \beta) = \sum_{i,j} (\alpha(e_i), \beta(e_j)) = \sum_{i,j} (\alpha(e_i), \beta(e_j)), \quad \alpha, \beta \in \mathfrak{gl}_n(\mathbb{R}).
\]
and also the standard \( \text{Ad}(O(n)) \)-invariant inner product on \( \mathfrak{gl}_n(\mathbb{R}) \) given by
\[
(\alpha, \beta) = \text{tr } \alpha \beta^t = \sum_i (\alpha e_i, \beta e_i) = \sum_i (\alpha e_i, \beta e_i), \quad \alpha, \beta \in \mathfrak{gl}_n(\mathbb{R}).
\]

**Remark 3.1.** We have made several abuses of notation concerning inner products. Recall that \( \langle \cdot, \cdot \rangle \) has been used to denote an inner product on \( s, n, \mathbb{R}^n, V \) and \( \mathfrak{gl}_n(\mathbb{R}) \).

We note that \( \pi(\alpha)^t = \pi(\alpha^t) \) and \( (\text{ad } \alpha)^t = \text{ad } \alpha^t \) for any \( \alpha \in \mathfrak{gl}_n(\mathbb{R}) \), due to the choice of canonical inner products everywhere.

Let \( t \) denote the set of all diagonal \( n \times n \) matrices. If \( \{e'_1, \ldots, e'_n\} \) is the basis of \( (\mathbb{R}^n)^* \) dual to the canonical basis \( \{e_1, \ldots, e_n\} \) then
\[
\{v_{ijk} = (e'_i \wedge e'_j) \otimes e_k : 1 \leq i < j \leq n, 1 \leq k \leq n\}
\]
is a basis of weight vectors of \( V \) for the action (3.1), where \( v_{ijk} \) is actually the bilinear form on \( \mathbb{R}^n \) defined by \( v_{ijk}(e_i, e_j) = -v_{ijk}(e_j, e_i) = e_k \) and zero otherwise. The corresponding weights \( \alpha^k_{ij} \) are given by
\[
(\alpha, v_{ijk}) = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha^k_{ij} \rangle v_{ijk}, \quad \forall \alpha = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in t,
\]
where \( \alpha^k_{ij} = E_{kk} - E_{ii} - E_{jj} \) and \( \langle \cdot, \cdot \rangle \) is the inner product defined in (3.4). As usual \( E_{rs} \) denotes the matrix whose only nonzero coefficient is 1 at entry \( rs \). The structural constants \( \mu^k_{ij} \) of an algebra \( \mu \in V \) are then given by
\[
\mu(e_i, e_j) = \sum_{k=1}^n \mu^k_{ij} e_k, \quad \text{or } \mu = \sum_{i,j} \mu^k_{ij} v_{ijk}, \quad i < j.
\]
Let \( t^+ \) denote the Weyl chamber of \( \mathfrak{gl}_n(\mathbb{R}) \) given by
\[
t^+ = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \in t : a_1 \leq \ldots \leq a_n \right\}.
\]

For \( \alpha \in t^+ \) we define the parabolic subgroup
\[
P_\alpha = BGL_n(\mathbb{R})_\alpha,
\]
where \( B \) is the subgroup of \( GL_n(\mathbb{R}) \) of all lower triangular invertible matrices and
\[
GL_n(\mathbb{R})_\alpha = \{ g \in GL_n(\mathbb{R}) : g \alpha g^{-1} = \alpha \}.
\]
In general, for any \( \alpha' \in \mathfrak{gl}_n(\mathbb{R}) \) which is diagonalizable over \( \mathbb{R} \), we let \( P_{\alpha'} := gP_{\alpha}g^{-1} \) if \( \alpha' = g \alpha g^{-1}, \alpha \in t^+ \). This is well defined since \( h \alpha h^{-1} = g \alpha g^{-1} \) implies that \( h^{-1}g \in GL_n(\mathbb{R})_\alpha \subset P_\alpha \) and so \( h^{-1}gP_{\alpha}g^{-1}h = P_\alpha \).

There is an ordered basis of \( V \) with respect to which the action of \( g \) on \( V \) is lower triangular for any \( g \in B \), and furthermore the eigenvalues of \( \pi(\alpha) \) are increasing for any \( \alpha \in t^+ \).

Given a finite subset \( X \) of \( t \), we denote by \text{CH}(X) the convex hull of \( X \) and by \text{mcc}(X) the minimal convex combination of \( X \), that is, the (unique) vector
of minimal norm (or closest to the origin) in \( \text{CH}(X) \). If \( X = \{\alpha_1, ..., \alpha_r\} \subset \mathfrak{t} \) and \( \beta := \text{mcc}(X) \), then there exist \( c_i \geq 0, i = 1, ..., r \), such that \( \sum_{i=1}^{r} c_i = 1 \) and

\[
\beta = \sum_{i=1}^{r} c_i \alpha_i.
\]

Since \( \langle \beta, \alpha_i \rangle \geq ||\beta||^2 \) for all \( i \) (why?), we have that

\[
||\beta||^2 = \sum_{i=1}^{r} c_i \langle \beta, \alpha_i \rangle \geq ||\beta||^2,
\]

from which follows that \( \langle \beta, \alpha_i \rangle = ||\beta||^2 \) for all \( i \) such that \( c_i > 0 \). We can therefore assume that \( \langle \beta, \alpha_i \rangle = ||\beta||^2 \) for all \( i \), and also that \( \beta = \text{mcc}(\{\alpha_1, ..., \alpha_s\}) \), where \( \{\alpha_1, ..., \alpha_s\} \) is a linearly independent subset of \( \mathfrak{t} \). Thus the \( s \times s \) matrix \( U := \begin{bmatrix} \langle \alpha_1, \alpha_1 \rangle & \cdots & \langle \alpha_1, \alpha_s \rangle \\ \vdots & \ddots & \vdots \\ \langle \alpha_s, \alpha_1 \rangle & \cdots & \langle \alpha_s, \alpha_s \rangle \end{bmatrix} \) is invertible and satisfies

\[
U \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix} = ||\beta||^2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.
\]

(3.8)

In particular, if all the entries of \( \alpha_i \) are in \( \mathbb{Q} \) for any \( i = 1, ..., r \), then also the entries of \( \beta := \text{mcc}(X) \) are all in \( \mathbb{Q} \). Indeed, \( \frac{c_i}{||\beta||^2} \in \mathbb{Q} \) for all \( i \) and so their sum \( \frac{c_i}{||\beta||^2} \in \mathbb{Q} \), which implies that \( c_i \in \mathbb{Q} \) for all \( i \) and consequently \( \beta \) has all its coefficients in \( \mathbb{Q} \).

4. Variational approach to Einstein solvmanifolds

Einstein metrics are often considered as the nicest, or most privileged ones on a given differentiable manifold (see for instance [Besse 87, Introduction]). One of the justifications is the following result due to Hilbert (see [Hilbert 15]): the Einstein condition for a compact Riemannian manifold \((M, g)\) of volume one is equivalent to the fact that the total scalar curvature functional

\[
\text{sc} : g \mapsto \int_M \text{sc}(g) \mu_g
\]

admits \( g_0 \) as a critical point on the space of all metrics of volume one (see also [Besse 87, 4.21]). This variational approach still works for \( G \)-invariant metrics on \( M \), where \( G \) is any compact Lie group acting transitively on \( M \) (see [Besse 87, 4.23]).

On the other hand, it is proved in [Jensen 71] that in a unimodular \( n \)-dimensional Lie group, the Einstein left invariant metrics are precisely the critical points of the scalar curvature functional on the set of all left invariant metrics having a fixed volume element. However, this fails in the non-unimodular case. For instance, if \( \mathfrak{s} \) is a solvable non-unimodular Lie algebra, then the scalar curvature functional restricted to any leaf \( F = \{t\} \times \text{SL}(\mathfrak{s})/\text{SO}(\mathfrak{s}) \subset \mathcal{P} \) of inner products, has no critical points (see [Heber 98, 3.5]). Thus, the approach to study Einstein solvmanifolds by a variational method should be different.

In this section, we shall describe the approach proposed in the introduction: to vary Lie brackets rather than inner products. Recall that when \( \mathfrak{n} \) is an \( n \)-dimensional nilpotent Lie algebra, then the set of all inner products on \( \mathfrak{n} \) is very nice, it is parameterized by the symmetric space \( \text{GL}_n(\mathbb{R})/\text{O}(n) \). However, isometry classes are precisely the orbits of the action on \( \text{GL}_n(\mathbb{R})/\text{O}(n) \) of the group of automorphisms \( \text{Aut}(\mathfrak{n}) \), a group mostly unknown, hard to compute, and far from being reductive, that is, ugly from the point of view of invariant theory. If we instead vary
Lie brackets, isometry classes will be given by $O(n)$-orbits, a beautiful group. But since nothing is for free in mathematics, the set of left invariant metrics will now be parameterized by a $GL_n(\mathbb{R})$-orbit in the variety $\mathcal{N}$ of $n$-dimensional nilpotent Lie algebras, a terrible space.

We fix an inner product vector space $$(s = \mathbb{R}H \oplus \mathbb{R}^n, \langle \cdot, \cdot \rangle), \quad \langle H, \mathbb{R}^n \rangle = 0, \quad \langle H, H \rangle = 1,$$
such that the restriction $\langle \cdot, \cdot \rangle|_{\mathbb{R}^n \times \mathbb{R}^n}$ is the canonical inner product on $\mathbb{R}^n$, which will also be denoted by $\langle \cdot, \cdot \rangle$. A linear operator on $\mathbb{R}^n$ will be sometimes identified with its matrix in the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. The metric Lie algebra corresponding to any $(n + 1)$-dimensional rank-one solvmanifold, can be modeled on $(s = \mathbb{R}H \oplus \mathbb{R}^n, \langle \cdot, \cdot \rangle)$ for some nilpotent Lie bracket $\mu$ on $\mathbb{R}^n$ and some $D \in \text{Der}(\mu)$, the space of derivations of $(\mathbb{R}^n, \mu)$. Indeed, these data define a solvable Lie bracket $[\cdot, \cdot]$ on $s$ by

$$\langle [H, X], Y \rangle = DX, \quad \langle [X, Y], Z \rangle = \mu(X, Y), \quad X, Y, Z \in \mathbb{R}^n,$$

and the solvmanifold is then the simply connected Lie group $S$ with Lie algebra $(s, [\cdot, \cdot])$ endowed with the left invariant Riemannian metric determined by $\langle \cdot, \cdot \rangle$. We shall assume from now on that $\mu \neq 0$ since the case $\mu = 0$ (i.e. abelian nilradical) is well understood (see [Heber 98 Proposition 6.12]). We have seen in the paragraph above Definition 2.9 that for a given $\mu$ there exists a unique symmetric derivation $D_\mu$ to consider if we want to get Einstein solvmanifolds. We can therefore associate to each nilpotent Lie bracket $\mu$ on $\mathbb{R}^n$ a distinguished rank-one solvmanifold $S_\mu$, defined by the data $\mu, D_\mu$ as in (4.1), which is the only one with a chance of being Einstein among all those metric solvable extensions of $(\mu, \langle \cdot, \cdot \rangle)$.

We note that conversely, any $(n + 1)$-dimensional rank-one Einstein solvmanifold is isometric to $S_\mu$ for some nilpotent $\mu$. Thus the set $\mathcal{N}$ of all nilpotent Lie brackets on $\mathbb{R}^n$ parameterizes a space of $(n + 1)$-dimensional rank-one solvmanifolds

$$\{S_\mu : \mu \in \mathcal{N}\},$$

containing all those which are Einstein in that dimension.

Concerning the identification

$$\mu \mapsto (N_\mu, \langle \cdot, \cdot \rangle),$$

where $N_\mu$ is the simply connected nilpotent Lie group with Lie algebra $(\mathbb{R}^n, \mu)$, the $GL_n(\mathbb{R})$-action on $\mathcal{N}$ defined in (4.1) has the following geometric interpretation: each $g \in GL_n(\mathbb{R})$ determines a Riemannian isometry

$$(N_{g, \mu}, \langle \cdot, \cdot \rangle) \longrightarrow (N_{\mu}, \langle g \cdot \cdot, g \cdot \rangle)$$

by exponentiating the Lie algebra isomorphism $g^{-1} : (\mathbb{R}^n, g, \mu) \longrightarrow (\mathbb{R}^n, \mu)$. Thus the orbit $GL_n(\mathbb{R}).\mu$ may be viewed as a parametrization of the set of all left invariant metrics on $N_\mu$. By a result of E. Wilson, two pairs $(N_\mu, \langle \cdot, \cdot \rangle), (N_\lambda, \langle \cdot, \cdot \rangle)$ are isometric if and only if $\mu$ and $\lambda$ are in the same $O(n)$-orbit (see [L. 06 Appendix]), where $O(n)$ denotes the subgroup of $GL_n(\mathbb{R})$ of orthogonal matrices. Also, two solvmanifolds $S_\mu$ and $S_\lambda$ with $\mu, \lambda \in \mathcal{N}$ are isometric if and only if there exists $g \in O(n)$ such that $g \cdot \mu = \lambda$ (see [L. 011b Proposition 4]). From (4.2) and the definition of $S_\mu$ we obtain the following result.

**Lemma 4.1.** If $\mu \in \mathcal{N}$ then the nilpotent Lie algebra $(\mathbb{R}^n, \mu)$ is an Einstein nilradical if and only if $S_{g, \mu}$ is Einstein for some $g \in GL_n(\mathbb{R})$. 

Recall that being an Einstein nilradical is a property of a whole \( \text{GL}_n(\mathbb{R}) \)-orbit in \( \mathcal{N} \), that is, of the isomorphism class of a given \( \mu \).

For any \( \mu \in \mathcal{N} \) we have that the scalar curvature of \( (N_\mu, \langle \cdot, \cdot \rangle) \) is given by \( \text{sc}(\mu) = -\frac{1}{4}||\mu||^2 \), which says that normalizing by scalar curvature and by the spheres of \( V \) is actually equivalent. The critical points of any scaling invariant curvature functional on \( \mathcal{N} \) appear then as very natural candidates to be distinguished left invariant metrics on nilpotent Lie groups.

**Theorem 4.2.** \cite{L.01a, L.01b, L.-Will 06} For a nonzero \( \mu \in \mathcal{N} \), the following conditions are equivalent:

1. \( S_\mu \) is Einstein.
2. \( (N_\mu, \langle \cdot, \cdot \rangle) \) is a nilsoliton.
3. \( \mu \) is a critical point of the functional \( F : V \longrightarrow \mathbb{R} \) defined by
   \[
   F(\mu) = \frac{16}{||\mu||^2} \text{tr} R_\mu^2,
   \]
   where \( R_\mu \) denotes the Ricci operator of \( (N_\mu, \langle \cdot, \cdot \rangle) \).
4. \( \mu \) is a minimum of \( F|_{\text{GL}_n(\mathbb{R}), \mu} \) (i.e. \( (N_\mu, \langle \cdot, \cdot \rangle) \) is minimal).
5. \( R_\mu \in \mathbb{R}I \oplus \text{Der} \mu \).

Under these conditions, the set of critical points of \( F \) lying in \( \text{GL}_n(\mathbb{R}), \mu \) equals \( O(n), \mu \) (up to scaling).

Thus another natural approach to find rank-one Einstein solvmanifolds would be to use the negative gradient flow of the functional \( F \). It follows from \cite{L.01b} Lemma 6] that if \( \pi \) is the representation defined in \( (3.2) \) then
\[
\text{grad}(F)_\mu = \frac{16}{||\mu||^2} \left( ||\mu||^2 \pi(R_\mu)\mu - 4 \text{tr} R_\mu^2 \mu \right).
\]
Since \( F \) is invariant under scaling we know that \( ||\mu|| \) will remain constant in time along the flow. We may therefore restrict ourselves to the sphere of radius 2, where the negative gradient flow \( \mu = \mu(t) \) of \( F \) becomes
\[
\frac{d}{dt} \mu = -\pi(R_\mu)\mu + \text{tr} R_\mu^2 \mu.
\]
Notice that \( \mu(t) \) is a solution to this differential equation if and only if \( g \mu(t) \) is so for any \( g \in O(n) \), according to the \( O(n) \)-invariance of \( F \). The existence of \( \lim_{t \to \infty} \mu(t) \) is guaranteed by the compactness of the sphere and the fact that \( F \) is a polynomial (see for instance \cite{Sjamaar 98} Section 2.5)).

**Lemma 4.3.** \cite{L.-Will 06} For \( \mu_0 \in V \), \( ||\mu_0|| = 2 \), let \( \mu(t) \) be the flow defined in \( (4.3) \) with \( \mu(0) = \mu_0 \) and put \( \lambda = \lim_{t \to \infty} \mu(t) \). Then

1. \( \mu(t) \in \text{GL}_n(\mathbb{R}), \mu_0 \) for all \( t \).
2. \( \lambda \in \text{GL}_n(\mathbb{R}), \mu_0 \).
3. \( S_\lambda \) is Einstein.

Part (i) follows from the fact that \( \frac{d}{dt} \mu \in T_\mu \text{GL}_n(\mathbb{R}), \mu \) for all \( t \) (see \( (4.3) \)), and part (ii) is just a consequence of (i). Condition (ii) is often referred in the literature as the Lie algebra \( \mu_0 \) degenerates to the Lie algebra \( \lambda \). Some interplays between degenerations and Riemannian geometry of Lie groups have been explored in \cite{L.03b}, by using the fact that for us, the orbit \( \text{GL}_n(\mathbb{R}), \mu_0 \) is the set of all left invariant metrics on \( N_{\mu_0} \). We note that if the limit \( \lambda \in \text{GL}_n(\mathbb{R}), \mu_0 \), then \( \mu_0 \) is an Einstein nilradical. We do not know if the converse holds. Since \( \lambda \) is a critical point
of $F$ and $\lambda \in \mathcal{N}$ by (ii) and the fact that $\mathcal{N}$ is closed, we have that part (iii) follows from Theorem 4.2.

In geometric invariant theory, a moment map for linear reductive Lie group actions over $\mathbb{C}$ has been defined in [Ness 84] and [Kirwan 84] (see Appendix). In our situation, it is an $O(n)$-equivariant map

$$m : V \setminus \{0\} \rightarrow \text{sym}(n),$$

defined implicitly by

$$(4.4) \quad \langle m(\mu), \alpha \rangle \equiv \frac{1}{||\mu||^2} \langle \pi(\alpha)\mu, \mu \rangle, \quad \mu \in V \setminus \{0\}, \alpha \in \text{sym}(n).$$

We are using $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \text{sym}(n)$ as the Cartan decomposition for the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $\text{GL}_n(\mathbb{R})$, where $\mathfrak{so}(n)$ and $\text{sym}(n)$ denote the subspaces of skew-symmetric and symmetric matrices, respectively.

Recall that $\mathcal{N} \subset V$ and each $\mu \in \mathcal{N}$ determines two Riemannian manifolds $S_\mu$ and $(N_\mu, \langle \cdot, \cdot \rangle)$. A remarkable fact is that this moment map encodes geometric information on $S_\mu$ and $(N_\mu, \langle \cdot, \cdot \rangle)$; indeed, it was proved in [L. 06] that

$$(4.5) \quad m(\mu) = \frac{4}{||\mu||^2} R_\mu.$$

This allows us to use strong and well-known results on the moment given in [Kirwan 84] and [Ness 84], and proved in [Marian 01] for the real case (see the Appendix for an overview on such results). We note that the functional $F$ defined in Theorem 4.2 (iii) is precisely $F(\mu) = ||m(\mu)||^2$, and so the equivalence between (iii) and (iv) in Theorem 4.2 follows from Theorem 11.3 (i). It should be pointed out that actually most of the results in Theorem 4.2 follow from general results on the moment map proved in [Marian 01]. For instance, the last sentence about uniqueness of critical points of $F$ (see Theorem 11.3 (ii)), is easily seen to be equivalent to the uniqueness of standard Einstein solvmanifolds (see Theorem 2.3) and nilsolitons (see Theorem 2.12).

In Section 7, we shall see that one can go further in the application of geometric invariant theory to the study of Einstein solvmanifolds, by considering a stratification for $\mathcal{N}$ intimately related to the moment map.

5. On the classification of Einstein solvmanifolds

As we have seen in Section 2, the classification of Einstein solvmanifolds is essentially reduced to the rank-one case. There is a bijection between the set of all isometry classes of rank-one Einstein solvmanifolds and the set of isometry classes of certain distinguished left invariant metrics on nilpotent Lie groups called nilsolitons, and the uniqueness up to isometry of nilsolitons finally determines a new bijection with the set of all isomorphism classes of Einstein nilradicals. For better or worse, what we get in the end is then a classification problem on nilpotent Lie algebras.

Recall that a nilpotent Lie algebra $n$ is an Einstein nilradical if and only if $n$ admits a nilsoliton, that is, an inner product $\langle \cdot, \cdot \rangle$ such that the corresponding Ricci operator $R_{\langle \cdot, \cdot \rangle}$ satisfies

$$R_{\langle \cdot, \cdot \rangle} = cI + D, \quad \text{for some} \ c \in \mathbb{R}, \ D \in \text{Der}(n).$$

Therefore, in order to understand or classify Einstein nilradicals, a main problem would be how to translate this condition based on the existence of an inner product on $n$ having a certain property into purely Lie theoretic conditions on $n$. The following questions also arise:
(A) Besides the existence of an N-gradation, is there any other neat structural obstruction for a nilpotent Lie algebra to be an Einstein nilradical?

(B) Is there any algebraic condition on a nilpotent Lie algebra which is sufficient to be an Einstein nilradical?

(C) An N-graded nilpotent Lie algebra can or can not be an Einstein nilradical, what is more likely?

Let us now review what we do know on the classification of Einstein nilradicals. Any nilpotent Lie algebra of dimension \( \leq 6 \) is an Einstein nilradical (see [Will 03]). There are 34 of them in dimension 6, giving rise to 29 different eigenvalue-types (there are 5 eigenvalue-types with exactly two algebras). In dimension 7, the first nilpotent Lie algebras without any N-gradation appear, but also do the first examples of N-graded Lie algebras which are not Einstein nilradicals. The family of 7-dimensional nilpotent Lie algebras defined for any \( t \in \mathbb{R} \) by

\[
\begin{align*}
[X_1, X_2]_t &= X_3, & [X_1, X_5]_t &= X_6, & [X_2, X_4]_t &= X_6, \\
[X_1, X_3]_t &= X_4, & [X_1, X_6]_t &= X_7, & [X_2, X_5]_t &= tX_7, \\
[X_1, X_4]_t &= X_5, & [X_2, X_3]_t &= X_5, & [X_3, X_4]_t &= (1-t)X_5,
\end{align*}
\]

is really a curve in the set of isomorphism classes of algebras (i.e. \([\cdot, \cdot]_t \simeq [\cdot, \cdot]_s\) if and only if \( t = s \)) and \([\cdot, \cdot]_t\) turns to be an Einstein nilradical if and only if \( t \neq 0, 1 \) (see [L.-Will 06]). Recall that all of them admit the gradation \( n = n_1 \oplus n_2 \oplus \cdots \oplus n_7 \), \( n_i = \mathbb{R}X_i \) for all \( i \). This example in particular shows that to be an Einstein nilradical is not a property which depends continuously on the structural constants of the Lie algebra.

Perhaps the nicest source of examples of Einstein nilradicals is the following.

**Theorem 5.1.** [Tamaru 07] Let \( \mathfrak{g} \) be a real semisimple Lie algebra. Then the nilradical of any parabolic subalgebra of \( \mathfrak{g} \) is an Einstein nilradical.

If we add to this that H-type Lie algebras and any nilpotent Lie algebra admitting a naturally reductive left invariant metric are Einstein nilradicals, one may get the impression that any nilpotent Lie algebra which is special or distinguished in some way, or just has a ‘name’, will be an Einstein nilradical. This is contradicted by the following surprising result, which asserts that free nilpotent Lie algebras are rarely Einstein nilradicals.

**Theorem 5.2.** [Nikolayevsky 06a] A free \( p \)-step nilpotent Lie algebra on \( m \) generators is an Einstein nilradical if and only if

- \( p = 1, 2 \);
- \( p = 3 \) and \( m = 2, 3, 4, 5 \);
- \( p = 4 \) and \( m = 2 \);
- \( p = 5 \) and \( m = 2 \).

A nilpotent Lie algebra \( \mathfrak{n} \) is said to be filiform if \( \dim \mathfrak{n} = n \) and \( \mathfrak{n} \) is \((n-1)\)-step nilpotent. These algebras may be seen as those which are as far as possible from being abelian along the class of nilpotent Lie algebras, and in fact most of them admit at most one N-gradation. Several families of filiform algebras which are not Einstein nilradicals have been found in [Nikolayevsky 07], as well as...
many isolated examples of non-Einstein nilradicals belonging to a curve of Einstein nilradicals as in example (5.1). In [Arroyo 08], a weaker version of Theorem 5.10 given in [Nikolayevsky 07] is used to get a classification of 8-dimensional filiform Einstein nilradicals.

The lack of $\mathbb{N}$-gradations is not however the only obstacle one can find for Einstein nilradicals. Several examples of non-Einstein nilradicals are already known in the class of 2-step nilpotent Lie algebras (i.e. $[n, [n, n]] = 0$), the closest ones to being abelian and so algebras which usually admit plenty of different $\mathbb{N}$-gradations.

**Definition 5.3.** A 2-step nilpotent Lie algebra $n$ is said to be of type $(p, q)$ if $\dim n = p + q$ and $\dim [n, n] = p$.

In [L.-Will 06], certain 2-step nilpotent Lie algebras attached to graphs are considered (of type $(p, q)$ if the graph has $q$ vertices and $p$ edges) and it is proved that they are Einstein nilradicals if and only if the graph is positive (i.e. when certain uniquely defined weighting on the set of edges is positive). For instance, any regular graph and also any tree such that any of its edges is adjacent to at most three other edges is positive. On the other hand, a graph is not positive under the following condition: there are two joined vertices $v$ and $w$ such that $v$ is joined to $r$ vertices of valency 1, $w$ is joined to $s$ vertices of valency 1, both are joined to $t$ vertices of valency 2 and $(r, s, t)$ is not in a set of only a few exceptional small triples. This provides a great deal of 2-step non-Einstein nilradicals, starting from types $(5, 6)$ and $(7, 5)$, and any dimension $\geq 11$ is attained.

Many other 2-step algebras of type $(6, 5)$ and $(7, 5)$ which are not Einstein nilradicals have appeared from the complete classification for types $(p, q)$ with $q \leq 5$ and $(p, q) \neq (5, 5)$ carried out in [Nikolayevsky 08a].

Curiously enough, at this point of the story, with so many examples of non-Einstein nilradicals available, a curve was still missing. In each fixed dimension, only finitely many nilpotent Lie algebras which are not Einstein nilradicals have showed up. But this potential candidate to a conjecture has recently been dismissed by the following result.

**Theorem 5.4.** [Will 08] Let $n_t$ be the 9-dimensional Lie algebra with Lie bracket defined by

- $[X_5, X_4]_t = X_7, \quad [X_1, X_6]_t = X_8, \quad [X_3, X_2]_t = X_9,$
- $[X_3, X_6]_t = tX_7, \quad [X_5, X_2]_t = tX_8, \quad [X_1, X_4]_t = tX_9,$
- $[X_1, X_2]_t = X_7.$

Then $n_t, \, t \in (1, \infty)$, is a curve of pairwise non-isomorphic 2-step nilpotent Lie algebras of type $(3, 6)$, none of which is an Einstein nilradical.

The following definition is motivated by (2.1), a condition a rank-one solvable extension of a nilpotent Lie algebra must satisfy in order to have a chance of being Einstein.

**Definition 5.5.** A derivation $\phi$ of a real Lie algebra $g$ is called pre-Einstein if it is diagonalizable over $\mathbb{R}$ and

- $\text{tr } \phi \psi = \text{tr } \psi, \quad \forall \psi \in \text{Der}(g).$

The following result is based on the fact that Aut($g$) is an algebraic group.
Theorem 5.6. [Nikolayevsky 08a] Any Lie algebra $\mathfrak{g}$ admits a pre-Einstein derivation, which is unique up to $\text{Aut}(\mathfrak{g})$-conjugation and has eigenvalues in $\mathbb{Q}$.

Let $\mathfrak{n}$ be a nilpotent Lie algebra with pre-Einstein derivation $\phi$. We note that if $\mathfrak{n}$ admits a nilsoliton metric, say with $R(\cdot,\cdot) = eI + D$, then $D$ necessarily equals $\phi$ up to scaling and conjugation (see Section 2.1), and thus the eigenvalue-type of the corresponding Einstein solvmanifold is the set of eigenvalues of $\phi$ up to scaling.

In particular, $\phi > 0$. It is proved in [Nikolayevsky 08a] that also $\text{ad}\phi \geq 0$ as long as $\mathfrak{n}$ is an Einstein nilradical. These conditions are not however sufficient to guarantee that $\mathfrak{n}$ is an Einstein nilradical (see [Nikolayevsky 06b]). In order to get a necessary and sufficient condition in terms of $\phi$ we have to work harder.

Let us first consider

$$
\mathfrak{g}_{\phi} := \{ \alpha \in \mathfrak{gl}(\mathfrak{n}) : [\alpha, \phi] = 0, \quad \text{tr} \, \alpha \phi = 0, \quad \text{tr} \, \alpha = 0 \}
$$

and let $G_{\phi}$ be the connected Lie subgroup of $\text{GL}(\mathfrak{n})$ with Lie algebra $\mathfrak{g}_{\phi}$. Recall that the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{n}$ belongs to the vector space $\Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n}$ of skew-symmetric bilinear maps from $\mathfrak{n} \times \mathfrak{n}$ to $\mathfrak{n}$, on which $\text{GL}(\mathfrak{n})$ is acting naturally by $g.[\cdot, \cdot] = g[g^{-1}\cdot, g^{-1}\cdot]$.

Theorem 5.7. [Nikolayevsky 08a] Let $\mathfrak{n}$ be a nilpotent Lie algebra with pre-Einstein derivation $\phi$. Then $\mathfrak{n}$ is an Einstein nilradical if and only if the orbit $G_{\phi}.[\cdot, \cdot]$ is closed in $\Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n}$.

This is certainly the strongest general result we know so far concerning questions (A) and (B) above, and of course it has many useful applications, some of which we will now describe (see also Theorem 5.4 for a turned to be equivalent result).

Definition 5.8. Let $\{X_1, ..., X_n\}$ be a basis for a nilpotent Lie algebra $\mathfrak{n}$, with structural constants $c_{ij}^k$'s given by $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$. Then the basis $\{X_i\}$ is said to be nice if the following conditions hold:

- for all $i < j$ there is at most one $k$ such that $c_{ij}^k \neq 0$,
- if $c_{ij}^k$ and $c_{ij}^{k'}$ are nonzero then either $\{i, j\} = \{i', j'\}$ or $\{i, j\} \cap \{i', j'\} = \emptyset$.

A nice property a nice basis $\{X_i\}$ has is that the Ricci operator $R(\cdot, \cdot)$ of any inner product $\langle \cdot, \cdot \rangle$ for which $\{X_i\}$ is orthogonal diagonalizes with respect to $\{X_i\}$ (see [L.-Will 06 Lemma 3.9]). Uniform bases considered in [Deloff 79] and [Wolter 91] are nice. The existence of a nice basis for a nilpotent Lie algebra looks like a strong condition, although we do not know of any example for which we can prove the non-existence of a nice basis. Not even an existence result for such example is available.

Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric nilpotent Lie algebra with orthogonal basis $\{X_i\}$ and structural constants $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$. If we fix an enumeration of the set $\{\alpha_{ij}^k : c_{ij}^k \neq 0\}$ (see Section 3), we can define the symmetric matrix

$$
U = \left[ \langle \alpha_{ij}^k, \alpha_{i'j'}^{k'} \rangle \right],
$$

and state the following useful result.
THEOREM 5.9. [Payne 05] Assume that \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) satisfies \(R_{\langle \cdot, \cdot \rangle} \in \mathfrak{t}\). Then \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is a nilsoliton if and only if
\[
U \left( (c^k_{ij})^2 \right) = c[1], \quad c \in \mathbb{R},
\]
where \( (c^k_{ij})^2 \) is meant as a column vector in the same order used in \((5.3)\) for defining \(U\) and \([1]\) is the column vector with all entries equal to 1.

It turns out that equations \(U \left( (c^k_{ij})^2 \right) = c[1]\) are precisely those given by the Lagrange method applied to find critical points of the functional \(F\) in Theorem 4.2.

In [Payne 05], a Cartan matrix is associated to \(U\) and the theory of Kac-Moody algebras is applied to analyze the solutions space of such a linear system.

Recall that \(\text{mcc}(X)\) denotes the unique vector of minimal norm in the convex hull \(\text{CH}(X)\) of a finite subset \(X\) of \(t\).

THEOREM 5.10. [Nikolayevsky 08a] A nonabelian nilpotent Lie algebra \(\mathfrak{n}\) with a nice basis \(\{X_i\}\) and structural constants \([X_i, X_j] = \sum_{k=1}^{n} c^k_{ij} X_k\) is an Einstein nilradical if and only if any of the following equivalent conditions hold:
(i) \(\text{mcc}\{\alpha^k_{ij} : c^k_{ij} \neq 0\}\) lies in the interior of \(\text{CH}\{\{\alpha^k_{ij} : c^k_{ij} \neq 0\}\}\).
(ii) Equation \(U[x^k_{ij}] = [1]\) has a positive solution \([x^k_{ij}]\).

This is a non-constructive result, in the sense that it is in general very difficult to explicitly find the nilsoliton metric. The absence of an inner product in its statement (compare with Theorem 5.9), however, makes of Theorem 5.10 quite a useful result.

THEOREM 5.11. [Nikolayevsky 08a] Let \(\mathfrak{n}_1, \mathfrak{n}_2\) be real nilpotent Lie algebras which are isomorphic as complex Lie algebras (i.e. they have isomorphic complexifications \(\mathfrak{n}_i \otimes \mathbb{C}\)). Then \(\mathfrak{n}_1\) is an Einstein nilradical if and only if \(\mathfrak{n}_2\) is so, and in that case, they have the same eigenvalue-type.

This turns our classification of Einstein nilradicals into a problem on complex nilpotent Lie algebras, with all the advantages an algebraically closed field has if we want to use known classifications in the literature or results from algebraic geometry and geometric invariant theory.

The following result reduces the classification of Einstein nilradicals to those which are indecomposable (i.e. non-isomorphic to a direct sum of Lie algebras).

THEOREM 5.12. [Nikolayevsky 08a] Let \(\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2\) be a nilpotent Lie algebra which is the direct sum of two ideals \(\mathfrak{n}_1\) and \(\mathfrak{n}_2\). Then \(\mathfrak{n}\) is an Einstein nilradical if and only if both \(\mathfrak{n}_1\) and \(\mathfrak{n}_2\) are Einstein nilradicals.

Any 2-step nilpotent Lie algebra of type \((p, q)\) can be identified with an element in the vector space \(V_{q,p} := \Lambda^2(\mathbb{R}^p)^* \otimes \mathbb{R}^p\), and it is easy to see that two of them are isomorphic if and only if they lie in the same \(\text{GL}_q(\mathbb{R}) \times \text{GL}_p(\mathbb{R})\)-orbit.

THEOREM 5.13. [Eberlein 07, Nikolayevsky 08a] If \((p, q) \neq (2, 2k + 1)\), then the vector space \(V_{q,p}\) contains an open and dense subset of Einstein nilradicals of eigenvalue-type \((1 < 2; q, p)\). This is no doubt an important indicator related to question (C)
above, but we must go carefully. What Theorem 5.13 is actually asserting is that if one throws a dart on $V_{q,p}$, then, with probability one, the dart will hit at a Lie bracket $[\cdot,\cdot] \in V_{q,p}$ which is an Einstein nilradical of eigenvalue-type $(1 < 2; q, p)$. Recall that each algebra of type $(p, q)$ is identified with a whole $\text{GL}_q(\mathbb{R}) \times \text{GL}_p(\mathbb{R})$-orbit in $V_{q,p}$, not with a single point, and some of these orbits can be much thicker than others.

Let us consider a simple example to illustrate this phenomenon. There are exactly 7 algebras up to isomorphism in the vector space $V_{4,2}$, including the abelian one. Only two of them are Einstein nilradicals of type $(1 < 2; 4, 2)$; namely, the H-type Lie algebra $\mathfrak{h}_3 \otimes \mathbb{C}$ (i.e. the complexification of $\mathfrak{h}_3$ viewed as real) and $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, where $\mathfrak{h}_3$ denotes the 3-dimensional Heisenberg algebra (see [Will 03, Table 4]). If we fix basis $\{X_1, \ldots, X_4\}$ and $\{Z_1, Z_2\}$ of $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively, then each $[\cdot, \cdot] \in V_{4,2}$ is determined by 12 structural constants as follows:

\[
[X_i, X_j] = c_{ij}^k Z_1 + c_{ij}^2 Z_2, \quad c_{ij}^k \in \mathbb{R}, \quad 1 \leq i < j \leq 4, \quad k = 1, 2.
\]

If we take variables $x, y$ and define the skew-symmetric matrix $J$ with $ij$ entry, $i < j$, given by $c_{ij}^1 x + c_{ij}^2 y$, then det $J$ is a 4-degree homogeneous polynomial on $(x, y)$ with a ‘square root’ $f(x, y)$, a 2-degree homogeneous polynomial called the Pfaffian form of $[\cdot, \cdot]$ (see [L. 08a, Section 2]). Thus the Hessian of $f$ is a real number $h([\cdot, \cdot])$ which depends polynomially on the $c_{ij}^k$’s. This defines a polynomial function $h: V_{4,2} \rightarrow \mathbb{R}$, which turns to be $\text{SL}_4(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$-invariant.

It is not hard to see that $h([\cdot, \cdot]) \neq 0$ if and only if $[\cdot, \cdot]$ is isomorphic to either $\mathfrak{h}_3 \otimes \mathbb{C}$ ($h > 0$) or $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ ($h < 0$). This implies that the union of the two $\text{GL}_4(\mathbb{R}) \times \text{GL}_2(\mathbb{R})$-orbits corresponding to $\mathfrak{h}_3 \otimes \mathbb{C}$ and $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, which coincides with the set of all Einstein nilradicals of eigenvalue type $(1 < 2; 4, 2)$ in $V_{4,2}$, is open and dense in $V_{4,2}$. However, recall that the net probability of being an Einstein nilradical of eigenvalue type $(1 < 2; 4, 2)$ in $V_{4,2}$ is $\mathbb{R}$.

One may try to avoid this by working on the quotient space $V_{q,p}/\text{GL}_q(\mathbb{R}) \times \text{GL}_p(\mathbb{R})$, where Theorem 5.13 is by the way also true, but the topology here is so ugly that an open and dense subset can never be taken as a probability one subset. In fact, there could be a single point set which is open and dense. On the other hand, the coset of 0 is always in the closure of any other subset, which shows that this quotient space is far from being $T_1$.

It has very recently appeared in [Nikolayevsky 08b] a complete classification for 2-step Einstein nilradicals of type $(2, q)$ for any $q$. In [Jablonsky 08], a construction called concatenation of 2-step nilpotent Lie algebras is used to obtain Einstein nilradicals of type $(1 < 2; q, p)$ from smaller ones, as well as many new examples of 2-step non-Einstein nilradicals.

6. Known examples and non examples

As far as we know, the following is a complete chronological list of nilpotent Lie algebras which are known to be Einstein nilradicals, or equivalently, of known examples of rank-one Einstein solvmanifolds:

- [Cartan 27] The Lie algebra of an Iwasawa $N$-group: $G/K$ irreducible symmetric space of noncompact type and $G = K \text{AN}$ the Iwasawa decomposition.
• **Gindikin-Piatetskii Shapiro-Vinberg 67** Nilradicals of normal \(j\)-algebras (i.e. of noncompact homogeneous Kähler Einstein spaces).

• **Alekseevskii 75, Cortés 96** Nilradicals of homogeneous quaternionic Kähler spaces.

• **Deloff 79** Certain 2-step nilpotent Lie algebras for which there is a basis with very uniform properties (see also [Wolter 91 1.9]).

• **Boggino 85** \(H\)-type Lie algebras (see also [Lanzerof 97]).

• **Eberlein-Heber 96, L. 99** Nilpotent Lie algebras admitting a naturally reductive left invariant metric.

• **Heber 98** Families of deformations of Lie algebras of Iwasawa \(N\)-groups in the rank-one case.

• **Fanai 00, Fanai 02** Certain 2-step nilpotent Lie algebras constructed via Clifford modules.

• **Gordon-Kerr 01** A 2-parameter family of 2-step nilpotent Lie algebras of type \(3,6\) and certain modifications of the Lie algebras of Iwasawa \(N\)-groups (rank \(\geq 2\)).

• **L. 02** Any nilpotent Lie algebra with a codimension one abelian ideal.

• **L. 02** A curve of 6-step nilpotent Lie algebras of dimension 7, which is the lowest possible dimension for a continuous family.

• **Mori 02** (and Yamada), Certain 2-step nilpotent Lie algebras defined from subsets of fundamental roots of complex simple Lie algebras.

• **L. 02** Any nilpotent Lie algebra of dimension \(\leq 5\).

• **Will 03** Any nilpotent Lie algebra of dimension 6.

• **L. 03b** A curve of 2-step nilpotent Lie algebras of type \((5,5)\).

• **Kerr 06** A 2-parameter family of deformations of the nilradical of the 12-dimensional quaternionic hyperbolic space.

• **Payne 05** Any filiform (i.e. \(n\)-dimensional and \((n - 1)\)-step nilpotent) Lie algebra with at least two linearly independent semisimple derivations.

• **L.-Will 06** Certain 2-step nilpotent Lie algebras attached to graphs as soon as a uniquely defined weighting on the graph is positive. Regular graphs and trees without any edge adjacent to four or more edges are positive.
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- **Nikolayevsky 06a** The free $p$-step nilpotent Lie algebras $f(m, p)$ on $m$ generators for $p = 1, 2$; $p = 3$ and $m = 2, 3, 4, 5$; $p = 4$ and $m = 2$; $p = 5$ and $m = 2$.

- **Nikolayevsky 07** Several families of filiform Lie algebras.

- **Tamaru 07** The nilradical of any parabolic subalgebra of a semisimple Lie algebra.

- **Nikolayevsky 08a** Any 2-step nilpotent Lie algebra of type $(p, q)$ (i.e. $p + q$-dimensional and $p$-dimensional derived algebra) with $q \leq 5$ and $(p, q) \neq (5, 5)$, with the only exceptions of the real forms of six complex algebras of type $(6, 5)$ and three of type $(7, 5)$.

We now give an up to date list of $\mathbb{N}$-graded nilpotent Lie algebras which are not Einstein nilradicals, that is, they do not admit any nilsoliton metric.

- **L.-Will 06** Three 6-step nilpotent Lie algebras of dimension 7, and certain 2-step nilpotent Lie algebras attached to graphs in any dimension $\geq 11$ (only finitely many in each dimension).

- **Nikolayevsky 06a** The free $p$-step nilpotent Lie algebras $f(m, p)$ on $m$ generators for $p = 3$ and $m \geq 6$; $p = 4$ and $m \geq 3$; $p = 5$ and $m \geq 3$; $p \geq 6$.

- **Nikolayevsky 07** Many filiform Lie algebras starting from dimension 8 (see also [Arroyo 08]).

- **Nikolayevsky 08a** Real forms of six complex 2-step nilpotent Lie algebras of type $(6, 5)$ and three of type $(7, 5)$.

- **Will 08** Two curves of 2-step nilpotent Lie algebras of type $(3, 6)$.

7. A stratification for the variety of nilpotent Lie algebras

In this section, we define a $\text{GL}_n(\mathbb{R})$-invariant stratification for the representation $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ of $\text{GL}_n(\mathbb{R})$ by adapting to this context the construction given in [Kirwan 84, Section 12] for reductive group representations over an algebraically closed field. This construction, in turn, is based on some instability results proved in [Kempf 78] and [Hesselink 78]. We decided to give in [L. 07, Section 2] a self-contained proof of all these results, bearing in mind that a direct application of them does not seem feasible (see also [L. 03a]).

We shall use the notation given in Section 3. For any $\mu \in V$ we have that

$$\lim_{t \to -\infty} e^{tI} \cdot \mu = \lim_{t \to -\infty} e^{-t} \cdot \mu = 0,$$

and hence $0 \in \text{GL}_n(\mathbb{R}) \cdot \mu$, that is, any element of $V$ is unstable for our $\text{GL}_n(\mathbb{R})$-action (see Appendix). Therefore, in order to distinguish two elements of $V$ from
the point of view of geometric invariant theory, we would need to measure in some sense ‘how’ unstable each element of \( V \) is. Maybe the above is not the optimal way to go to 0 along the orbit starting from \( \mu \).

Let us consider \( \mu \in V \) and \( \alpha \in D \), where \( D \) denotes the set of all \( n \times n \) matrices which are diagonalizable, that is,

\[
D = \bigcup_{g \in \text{GL}_n(\mathbb{R})} g \cdot \text{Diag} \cdot g^{-1}.
\]

Thus \( \pi(\alpha) \) is also diagonalizable (see (3.2)), say with eigenvalues \( a_1, \ldots, a_r \) and eigenspace decomposition \( V = V_1 \oplus \cdots \oplus V_r \). This implies that if \( \mu \neq 0 \) and \( \mu = \mu_1 + \cdots + \mu_r, \mu_i \in V_i \), then

\[
e^{-\alpha \cdot \mu} = \sum_{i=1}^{r} e^{-a_i \cdot \mu_i},
\]

and so \( e^{-\alpha \cdot \mu} \) goes to 0 when \( t \to \infty \) if and only if \( \mu_i = 0 \) as soon as \( a_i \leq 0 \).

Moreover, in that case, the positive number

\[
m(\mu, \alpha) := \min \{a_i : \mu_i \neq 0\},
\]

measures the degree of instability of \( \mu \) relative to \( \alpha \), in the sense that the train has not arrived until the last wagon has. Indeed, the larger \( m(\mu, \alpha) \) is, the faster \( e^{-\alpha \cdot \mu} \) will converge to 0 when \( t \to \infty \). Recall that for an action in general the existence of such \( \alpha \) for any unstable element is guaranteed by Theorem 11.1 (iv).

Notice that \( m(\mu, c \alpha) = cm(\mu, \alpha) \) for any \( c > 0 \). We can therefore consider the most efficient directions (up to the natural normalization) for a given \( \mu \in V \), given by

\[
\Lambda(\mu) := \left\{ \beta \in D : m(\mu, \beta) = 1 = \sup_{\alpha \in D} \{m(\mu, \alpha) : \text{tr}\alpha^2 = \text{tr}\beta^2\} \right\}.
\]

A remarkable fact is that \( \Lambda(\mu) \) lie in a single conjugacy class, that is, there exists an essentially unique direction which is ‘most responsible’ for the instability of \( \mu \). All the parabolic subgroups \( P_\beta \) of \( \text{GL}_n(\mathbb{R}) \) naturally associated to any \( \beta \in \Lambda(\mu) \) defined in (3.7) coincide, and hence they define a unique parabolic subgroup \( P_\mu \) which acts transitively on \( \Lambda(\mu) \) by conjugation. A very nice property \( P_\mu \) has is that

\[
\text{Aut}(\mu) \subset P_\mu.
\]

Since

\[
\Lambda(g, \mu) = g \Lambda(\mu) g^{-1}, \quad \forall \mu \in V, \quad g \in \text{GL}_n(\mathbb{R}),
\]

we obtain that \( \Lambda(g, \mu) \) will meet the Weyl chamber \( t^+ \) for some \( g \in \text{GL}_n(\mathbb{R}) \), and the intersection set will consist of a single element \( \beta \in t^+ \) (see (3.0)).

Summarizing, we have been able to attach to each nonzero \( \mu \in V \), and actually to each nonzero \( \text{GL}_n(\mathbb{R}) \)-orbit in \( V \), a uniquely defined \( \beta \in t^+ \) which comes from instability considerations.

**Definition 7.1.** Under the above conditions, we say that \( \mu \in S_\beta \) and call the subset \( S_\beta \subset V \) a *stratum*.

We note that \( S_\beta \) is \( \text{GL}_n(\mathbb{R}) \)-invariant for any \( \beta \in t^+ \) and

\[
V \setminus \{0\} = \bigcup_{\beta \in t^+} S_\beta.
\]
a disjoint union. An alternative way to define $S_\beta$ is

$$S_\beta = \text{GL}_n(\mathbb{R}) \cdot \left\{ \mu \in V : \frac{\beta}{||\mu||^2} \in \Lambda(\mu) \right\},$$

which actually works for any $\beta \in t$. From now on, we will always denote by $\mu_{ij}^k$ the structure constants of a vector $\mu \in V$ with respect to the basis $\{v_{ijk}\}$:

$$\mu = \sum \mu_{ij}^k v_{ijk}, \quad \mu_{ij}^k \in \mathbb{R}, \quad \text{i.e.} \quad \mu(e_i, e_j) = \sum_{k=1}^n \mu_{ij}^k e_k, \quad i < j.$$  

Each nonzero $\mu \in V$ uniquely determines an element $\beta_{\mu} \in t$ given by

$$\beta_{\mu} = \text{mcc} \left\{ \alpha_{ij}^k : \mu_{ij}^k \neq 0 \right\}.$$

Recall that $\text{mcc}(X)$ denotes the unique element of minimal norm in the convex hull $\text{CH}(X)$ of a subset $X \subset t$, and thus $\beta_{\mu}$ has rational coefficients (see (3.8)). We also note that $\beta_{\mu}$ is always nonzero since $\text{tr} \alpha_{ij}^k = -1$ for all $i < j$ and consequently $\text{tr} \beta_{\mu} = -1$. If for $\mu \in V$ we define $\Lambda_T(\mu)$ as above but by replacing $D$ with the set of diagonal matrices $t$, then one can prove that

$$\Lambda_T(\mu) = \left\{ \frac{\beta_{\mu}}{||\mu||^2} \right\},$$

that is, $\beta_{\mu}$ is the (unique) ‘most responsible’ direction for the instability of $\mu$ with respect to the action of the torus $T$ with Lie algebra $t$ on $V$. Another equivalent definition for the stratum $S_\beta$, $\beta \in t$, is given by

$$S_\beta = \left\{ \mu \in V \setminus \{0\} : \beta \text{ is an element of maximal norm in } \{\beta_{g,\mu} : g \in \text{GL}_n(\mathbb{R})\} \right\}.$$

If $\mu$ runs through $V$, there are only finitely many possible vectors $\beta_{\mu}$, and consequently the set $\{\beta \in t : S_\beta \neq \emptyset\}$ is finite. We furthermore get from this new description that if $\beta \in t$ satisfies $S_\beta \neq \emptyset$ then $\beta$ has rational coefficients and

$$(7.2) \quad \text{tr} \beta = -1.$$

**Remark 7.2.** A very illustrative exercise is to consider the action given in Example 11.5, draw the nice picture of its weights, detect all possible $\beta_{\mu}$’s and try to figure out which of them actually determine a nonempty stratum (i.e. $S_\beta_\mu \neq \emptyset$).

Recall from Section 4 that the moment map $m$ for the $\text{GL}_n(\mathbb{R})$-representation $V$ plays a fundamental role in the study of Einstein solvmanifolds and nilsolitons, as $m(\mu) = -\frac{1}{\text{sc}(\mu)} R_\mu$, where $R_\mu$ and $\text{sc}(\mu)$ denote the Ricci operator and the scalar curvature of $(N_\mu, (\cdot, \cdot))$, respectively. The square norm functional $F(\mu) = ||m(\mu)||^2$ therefore provides a natural curvature functional on the space $N$ of all left invariant metrics on $n$-dimensional nilpotent Lie groups whose critical points are precisely nilsoliton metrics (see Theorem 1.2).

We have collected in the following theorem some relationships between $m$, $F$ and the strata. Let $p_t(\alpha)$ denote the orthogonal projection on $t$ of an $\alpha \in \text{sym}(n)$ (i.e. the diagonal part of $\alpha$).

**Theorem 7.3.** [L.-Will 06] Let $\mu = \sum \mu_{ij}^k v_{ijk}$ be a nonzero element of $V$.

(i) $p_{t}(m(\mu)) = \frac{1}{||\mu||^2} \sum_{i<j} (\mu_{ij}^k)^2 \alpha_{ij}^k \in \text{CH} \left\{ \alpha_{ij}^k : \mu_{ij}^k \neq 0 \right\}$.

(ii) $F(\mu) \geq ||\beta||^2$ for any $\mu \in S_\beta$.

(iii) If $\inf F(\text{GL}_n(\mathbb{R}), \mu) = ||\beta_{\mu}||^2$ then $\mu \in S_{\beta_{\mu}}$.

(iv) If $\mu \in N$, $m(\mu) \in t$ and $S_\mu$ is Einstein then $\mu \in S_{m(\mu)}$. 
(v) For \( \mu \in S_\beta \cap N \), the following conditions are equivalent:

(a) \( S_\beta \) is Einstein.

(b) \( S_\beta \) is Einstein of eigenvalue-type \( \beta+||\beta||^2I \) (up to a positive multiple).

(c) \( m(\mu) \) is conjugate to \( \beta \).

(d) \( F(\mu) = ||\beta||^2 \).

It follows from part (v) in the above theorem that the stratum \( S_\beta \) to which \( \mu \) belongs determines the eigenvalue type of a potential Einstein solvmanifold \( S_{g,\mu} \), \( g \in \text{GL}_n(\mathbb{R}) \) (if any), and so the stratification provides a convenient tool to produce existence results as well as obstructions for nilpotent Lie algebras to be an Einstein nilradical. Thus \( \beta \) plays a role similar to the one played by the pre-Einstein derivation \( \phi \) (see Definition 5.5). The subtle relationship between \( \beta \) and \( \phi \) will be explained in Section 9.

We will now give a description of the strata in terms of semistable vectors (see Appendix). For each \( \beta \in t \) consider the sets

\[
Z_\beta = \{ \mu \in V : \langle \beta, \alpha^k_{ij} \rangle = ||\beta||^2, \ \forall \mu^k_{ij} \neq 0 \},
\]

\[
W_\beta = \{ \mu \in V : \langle \beta, \alpha^k_{ij} \rangle \geq ||\beta||^2, \ \forall \mu^k_{ij} \neq 0 \},
\]

\[
Y_\beta = \{ \mu \in W_\beta : \langle \beta, \alpha^k_{ij} \rangle = ||\beta||^2, \ \text{for at least one} \ \mu^k_{ij} \neq 0 \}.
\]

Notice that \( Z_\beta \) is actually the eigenspace of \( \pi(\beta) \) with eigenvalue \( ||\beta||^2 \), and so \( \mu \in Z_\beta \) if and only if \( \beta+||\beta||^2I \in \text{Der}(\mu) \). We also note that \( W_\beta \) is the direct sum of all the eigenspaces of \( \pi(\beta) \) with eigenvalues \( \geq ||\beta||^2 \), and since \( Z_\beta \subset Y_\beta \subset W_\beta \), they are all \( \text{GL}_n(\mathbb{R})_\beta \)-invariant, where \( \text{GL}_n(\mathbb{R})_\beta \) is the centralizer of \( \beta \) in \( \text{GL}_n(\mathbb{R}) \).

Let \( \mathfrak{g}_n(\mathbb{R})_\beta \) denote the Lie algebra of \( \text{GL}_n(\mathbb{R})_\beta \), that is,

\[
\mathfrak{g}_n(\mathbb{R})_\beta = \{ \alpha \in \mathfrak{g}_n(\mathbb{R}) : [\alpha, \beta] = 0 \},
\]

and let \( G_\beta \) be any reductive subgroup of \( \text{GL}_n(\mathbb{R}) \) with Lie algebra \( \mathfrak{g}_\beta \), the orthogonal complement of \( \beta \) in \( \mathfrak{g}_n(\mathbb{R})_\beta \). Thus

\[
(7.3) \quad \mathfrak{g}_n(\mathbb{R})_\beta = \mathfrak{g}_\beta \oplus \mathbb{R}\beta,
\]

is an orthogonal decomposition,

\[
\mathfrak{g}_\beta = (\mathfrak{so}(n) \cap \mathfrak{g}_\beta) \oplus (\mathfrak{sym}(n) \cap \mathfrak{g}_\beta)
\]

is a Cartan decomposition and \( t \cap \mathfrak{g}_\beta = \{ \alpha \in t : \langle \alpha, \beta \rangle = 0 \} \) is a maximal abelian subalgebra of \( \mathfrak{sym}(n) \cap \mathfrak{g}_\beta \). Recall that if \( S_\beta \neq \emptyset \) then \( \beta \) is rational and so such a reductive group \( G_\beta \) does exist.

**Definition 7.4.** A vector \( \mu \in V \) is called \( G_\beta \)-semistable if \( 0 \notin \overline{G_\beta.\mu} \).

**Theorem 7.5.** For any \( \beta \in t \), the \( \text{GL}_n(\mathbb{R})_\beta \)-invariant subsets \( Z_\beta^{ss} := Z_\beta \cap S_\beta \) and \( Y_\beta^{ss} := Y_\beta \cap S_\beta \) satisfy:

(i) \( S_\beta = O(n).Y_\beta^{ss} \).

(ii) \( Y_\beta^{ss} = \{ \mu \in S_\beta : \beta_\mu = \beta \} \).

(iii) \( Z_\beta^{ss} \) is the set of \( G_\beta \)-semistable vectors in \( Z_\beta \).

(iv) \( Y_\beta^{ss} \) is the set of \( G_\beta \)-semistable vectors in \( W_\beta \).

We summarize in the following theorem the main properties of the \( \text{GL}_n(\mathbb{R}) \)-invariant stratification of the vector space \( V \) given above.
Theorem 7.6. \[ \text{There exists a finite subset } \mathcal{B} \subset t^+, \text{ and for each } \beta \in \mathcal{B} \text{ a } \text{GL}_n(\mathbb{R})-\text{invariant subset } \mathcal{S}_\beta \subset V \text{ (a stratum) such that} \\
V \setminus \{0\} = \bigcup_{\beta \in \mathcal{B}} \mathcal{S}_\beta \text{ (disjoint union).} \]

If \( \mu \in \mathcal{S}_\beta \) then
\[ \langle [\beta, D], D \rangle \geq 0 \quad \forall \ D \in \text{Der}(\mu) \quad (\text{equality holds } \iff [\beta, D] = 0) \]

and
\[ \beta + ||\beta||^2 I \text{ is positive definite for all } \beta \in \mathcal{B} \text{ such that } \mathcal{S}_\beta \cap N \neq \emptyset. \]

If in addition \( \mu \in Y^{ss}_\beta \), i.e.
\[ \min \left\{ \langle \beta, \alpha^k_{ij} \rangle : \mu^k_{ij} \neq 0 \right\} = ||\beta||^2, \quad \text{or equivalently} \ 0 \notin G_\beta \mu, \]
then
\[ \text{tr } \beta D = 0 \quad \forall \ D \in \text{Der}(\mu), \]

and
\[ \langle \pi (\beta + ||\beta||^2 I), \mu \rangle \geq 0 \quad (\text{equality holds } \iff \beta + ||\beta||^2 I \in \text{Der}(\mu)). \]

Moreover, condition (7.6) is always satisfied by some \( g.\mu \) with \( g \in O(n) \), for any \( \mu \in \mathcal{S}_\beta \).

Remark 7.7. We note that (7.5) is actually the only result stated in this section where we really need \( \mu \) to be a nilpotent Lie algebra, and not just any vector in \( V \). It is known for instance that semisimple Lie algebras lie in the stratum \( \mathcal{S}_\beta \) for \( \beta = -\frac{1}{n} I \), and consequently \( \beta + ||\beta||^2 I = 0 \) (see [L. 03a]).

8. The stratification and the standard condition

We now apply the stratification described in Section 7 to prove that Einstein solvmanifolds are all standard.

Let \( S \) be a solvmanifold, that is, a simply connected solvable Lie group endowed with a left invariant Riemannian metric. Let \( \mathfrak{s} \) be the Lie algebra of \( S \) and let \( \langle \cdot, \cdot \rangle \) denote the inner product on \( \mathfrak{s} \) determined by the metric. We consider the orthogonal decomposition \( \mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n} \), where \( \mathfrak{n} = [\mathfrak{s}, \mathfrak{s}] \). Recall that \( S \) is called standard if \([\mathfrak{a}, \mathfrak{a}] = 0\). The mean curvature vector of \( S \) is the only element \( H \in \mathfrak{a} \) which satisfies \( \langle H, A \rangle = \text{tr} \text{ad} A \) for any \( A \in \mathfrak{a} \). If \( B \) denotes the symmetric map defined by the Killing form of \( \mathfrak{s} \) relative to \( \langle \cdot, \cdot \rangle \) then \( B(\mathfrak{a}) \subset \mathfrak{a} \) and \( B|_n = 0 \) as \( \mathfrak{n} \) is contained in the nilradical of \( \mathfrak{s} \). The Ricci operator \( \text{Ric} \) of \( S \) is given by (see for instance [Besse 87, 7.38]):
\[ \text{Ric} = R - \frac{1}{2} B - S(\text{ad} H), \]

where \( S(\text{ad} H) = \frac{1}{2}(\text{ad} H + (\text{ad} H)^t) \) is the symmetric part of \( \text{ad} H \) and \( R \) is the symmetric operator defined by
\[ \langle Rx, y \rangle = -\frac{1}{2} \sum_{ij} \langle [x, x_i], [x_j, x], x \rangle \langle y, x_i, x_j \rangle + \frac{1}{4} \sum_{ij} \langle [x_i, x], x \rangle \langle [x, x_j], y \rangle, \]

for all \( x, y \in \mathfrak{s} \), where \( \{x_i\} \) is any orthonormal basis of \( \langle \mathfrak{s}, \langle \cdot, \cdot \rangle \rangle \).
It is proved in [L, 06] Propositions 3.5, 4.2 that $R$ is the only symmetric operator on $\mathfrak{s}$ such that

\[(8.3) \quad \text{tr} RE = \frac{1}{4} \langle \pi(E)[\cdot, \cdot], [\cdot, \cdot] \rangle, \quad \forall E \in \text{End}(\mathfrak{s}),\]

where we are considering $[\cdot, \cdot]$ as a vector in $\Lambda^2 \mathfrak{s}^* \otimes \mathfrak{s}$, $\langle \cdot, \cdot \rangle$ is the inner product defined in (3.3) and $\pi$ is the representation given in (3.2) (see the notation in Section 3 and replace $\mathbb{R}^n$ by $\mathfrak{s}$). This is equivalent to saying that

\[m([\cdot, \cdot]) = \frac{4}{||\cdot||^2} R,\]

where $m : \Lambda^2 \mathfrak{s}^* \otimes \mathfrak{s} \rightarrow \text{sym}(\mathfrak{s})$ is the moment map for the action of $\text{GL}(\mathfrak{s})$ on $\Lambda^2 \mathfrak{s}^* \otimes \mathfrak{s}$ (see (4.4)). Thus the ‘anonymous’ tensor $R$ in formula (8.1) for the Ricci operator is precisely the value of the moment map at the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{s}$ (up to scaling).

We therefore obtain from (8.1) and (8.3) that $S$ is an Einstein solvmanifold with $\text{Ric} = cI$, if and only if, for any $E \in \text{End}(\mathfrak{s})$,

\[(8.4) \quad \text{tr} \left( cI + \frac{1}{2} B + S(\text{ad} H) \right) E = \frac{1}{4} \langle \pi(E)[\cdot, \cdot], [\cdot, \cdot] \rangle.\]

Let $S$ be an Einstein solvmanifold with $\text{Ric} = cI$. We can assume that $S$ is not unimodular by using [Dotti 82], thus $H \neq 0$ and $\text{tr} \text{ad} H = ||H||^2 > 0$. By letting $E = \text{ad} H$ in (8.4) we get

\[(8.5) \quad c = -\frac{\text{tr} S(\text{ad} H)^2}{\text{tr} S(\text{ad} H)} < 0.\]

In order to apply the results in Section 7 we identify $n$ with $\mathbb{R}^n$ via an orthonormal basis $\{e_1, \ldots, e_n\}$ of $n$ and we set $\mu := [\cdot, \cdot]|_{n \times n}$. In this way, $\mu$ can be viewed as an element of $\mathcal{N} \subset V$. If $\mu \neq 0$ then $\mu$ lies in a unique stratum $S_\beta$, $\beta \in B$, by Theorem (7.6) and it is easy to see that we can assume (up to isometry) that $\mu$ satisfies (7.6), so that one can use all the additional properties stated in the theorem. In particular, the following crucial technical result follows. Consider $E_\beta \in \text{End}(\mathfrak{s})$ defined by

\[E_\beta = \begin{bmatrix} 0 & 0 \\ 0 & \beta + ||\beta||^2 I \end{bmatrix},\]

that is, $E|_n = 0$ and $E|_n = \beta + ||\beta||^2 I$.

**Lemma 8.1.** If $\mu \in S_\beta$ satisfies (7.6) then $\langle \pi(E_\beta)[\cdot, \cdot], [\cdot, \cdot] \rangle \geq 0$.

We then apply (8.4) to $E_\beta \in \text{End}(\mathfrak{s})$ and obtain from Lemma (8.1) and (7.2) that

\[\text{tr} S(\text{ad} H)^2 \text{tr} E_\beta^2 \leq (\text{tr} S(\text{ad} H) E_\beta)^2,\]

a ‘backwards’ Cauchy-Schwartz inequality. This turns all inequalities which appeared in the proof of Lemma (8.1) into equalities, in particular:

\[\frac{1}{4} \sum_{rs} \langle (\beta + ||\beta||^2 I)[A_r, A_s], [A_r, A_s] \rangle = 0,\]

where $\{A_i\}$ is an orthonormal basis of $\mathfrak{a}$. We finally get that $\mathfrak{a}$ is abelian since $\beta + ||\beta||^2 I$ is positive definite by (7.5).
9. The stratification and Einstein solvmanifolds via closed orbits

We shall describe in this section some other applications of the strata defined in Section 7 to the study of Einstein solvmanifolds.

Let $\mathfrak{n}$ be a nonabelian nilpotent Lie algebra of dimension $n$. We fix any basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{n}$ and consider the corresponding structural constants:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \quad 1 \leq i < j \leq n.$$ 

Let $\beta$ denote the unique element of minimal norm in the convex hull of the set $\{\alpha_{ij}^k : c_{ij}^k \neq 0\}$, where $\alpha_{ij}^k$ is the diagonal $n \times n$ matrix $-E_{ii} - E_{jj} + E_{kk}$. Notice that $tr \beta = -1$, and so $\beta$ is always nonzero. We define the Lie algebra

$$\mathfrak{g}_\beta = \{\alpha \in \text{End}(\mathfrak{n}) : [\alpha, \beta] = 0, \quad tr \alpha \beta = 0\},$$

and take any reductive subgroup $G_\beta$ of $\text{GL}(\mathfrak{n})$ with Lie algebra $\mathfrak{g}_\beta$ (existence is guaranteed by rationality of $\beta$).

Recall that the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{n}$ belongs to the vector space $\Lambda^2\mathfrak{n}^* \otimes \mathfrak{n}$ of skew-symmetric bilinear maps from $\mathfrak{n} \times \mathfrak{n}$ to $\mathfrak{n}$, on which $\text{GL}(\mathfrak{n})$ is acting naturally by $g[\cdot, \cdot] = g[\cdot, \cdot] g^{-1} \cdot g^{-1}$.

**Theorem 9.1.** (L. 08b) Let $\mathfrak{n}$ be a nonabelian nilpotent Lie algebra and for any basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{n}$ consider $\beta$ and $G_\beta \subset \text{GL}(\mathfrak{n})$ as defined above.

(i) If the orbit $G_\beta[\cdot, \cdot]$ is closed in $\Lambda^2\mathfrak{n}^* \otimes \mathfrak{n}$ then $\mathfrak{n}$ is an Einstein nilradical and $\beta + ||\beta||^2 I \in \text{Der}(\mathfrak{n})$.

(ii) If $\mathfrak{n}$ is an Einstein nilradical, $\beta + ||\beta||^2 I \in \text{Der}(\mathfrak{n})$ and $0 \notin G_\beta[\cdot, \cdot]$, then the orbit $G_\beta[\cdot, \cdot]$ is closed in $\Lambda^2\mathfrak{n}^* \otimes \mathfrak{n}$.

**Remark 9.2.** The following example shows that condition $\beta + ||\beta||^2 I \in \text{Der}(\mathfrak{n})$ is necessary in part (ii) of Theorem 9.1. Let $\mathfrak{n}$ be the 4-dimensional 3-step nilpotent Lie algebra with Lie bracket given by

$$[X_1, X_2] = X_3 + X_4, \quad [X_1, X_3] = X_4.$$

It is easy to see that $\beta = (-1, -\frac{1}{2}, 0, \frac{1}{2})$ and $0 \notin G_{\beta}[\cdot, \cdot]$. If $\lambda$ is defined by

$$\lambda(X_1, X_2) = X_3, \quad \lambda(X_1, X_3) = X_4,$$

then $\lambda \in \text{GL}_4(\mathbb{R})[\cdot, \cdot], m(\lambda) = \beta$ and $\beta + ||\beta||^2 I \in \text{Der}(\lambda)$, from which follows that $\lambda$ is a nilsoliton and so $\mathfrak{n}$ is an Einstein nilradical. However,

$$\lambda = \lim_{t \to \infty} e^{-t\alpha}[\cdot, \cdot] \in G_{\beta}[\cdot, \cdot], \quad \text{for} \quad \alpha = (1, 0, 1, 2),$$

and thus $G_{\beta}[\cdot, \cdot]$ is not closed. Indeed, $\lambda \notin G_{\beta}[\cdot, \cdot]$ since $\beta + ||\beta||^2 I \in \text{Der}(\lambda)$ and $\beta + ||\beta||^2 I \notin \text{Der}(\mathfrak{n})$.

Recall that $\beta$ has entries in $\mathbb{Q}$ and so if $\beta \in t^+$ and has eigenvalues $b_1 < \ldots < b_r$ with multiplicities $n_1, \ldots, n_r$, respectively, then one can for instance take the reductive group $G_{\beta}$ given by

$$G_{\beta} = \left\{ \begin{bmatrix} g_1 & \cdots & g_r \\ \vdots & \ddots & \vdots \\ g_1 & \cdots & g_r \end{bmatrix} : \det g_1^{n_1} \cdots \det g_r^{n_r} = 1, \quad g_i \in \text{GL}_{n_i}(\mathbb{R}) \right\},$$

where $g_1, \ldots, g_r$ are matrices acting on $\mathfrak{n}$ as defined in the previous section.
where \( m \) is the least common multiple of the denominators of the \( b_i \)'s. If \( \sigma \) is a permutation of \( \{1, \ldots, n\} \) then the new basis \( \{X_{\sigma(1)}, \ldots, X_{\sigma(n)}\} \) of \( n \) has structural constants

\[
[X_{\sigma(i)}, X_{\sigma(j)}] = \sum_{k=1}^{n} c_{\sigma(i)\sigma(j)}^{(k)} X_{\sigma(k)}, \quad 1 \leq i < j \leq n,
\]

and so the new \( \beta \) has eigenvalues \( b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(n)} \) with respective eigenvectors \( X_{\sigma(1)}, \ldots, X_{\sigma(n)} \) (see the beginning of the proof of [L. 07, Theorem 2.10]). Therefore, we can always assume that \( \beta \in \mathfrak{t}^+ \), up to just a permutation of the basis \( \{X_i\} \).

Otherwise, if one insists on keeping the original basis, one may take as \( G_\beta \) the group \( h^{-1}G_{h\beta h^{-1}}h \), where \( h \in \text{GL}(n) \) is a permutation matrix such that \( h\beta h^{-1} \in \mathfrak{t}^+ \).

The following two results show the interplay between the stratum \( \beta \) and the pre-Einstein derivation \( \phi \) (see Definition 5.5), providing in particular a new method to compute \( \phi \).

**Lemma 9.3.** [L. 08b] Let \( n \) be a nonabelian nilpotent Lie algebra and for any basis \( \{X_1, \ldots, X_n\} \) of \( n \) consider \( \beta \) and \( G_\beta \subset \text{GL}(n) \) as defined above.

(i) \( \{\cdot, \cdot\} \in S_\beta \) if and only if \( 0 \notin G_\beta \cdot \{\cdot, \cdot\} \).

(ii) If \( \beta + \frac{1}{2}I \in \text{Der}(n) \) and \( 0 \notin G_\beta \cdot \{\cdot, \cdot\} \) then

\[
\phi := \frac{1}{\|\beta\|^2}(\beta + \|\beta\|^2I)
\]

is a pre-Einstein derivation of \( n \) and \( G_\phi \cdot \{\cdot, \cdot\} \) is closed if and only if \( G_\phi \cdot \{\cdot, \cdot\} \) is closed.

**Remark 9.4.** We conclude from Lemma 9.3 that Theorem 5.7 and Theorem 9.1 are equivalent.

**Remark 9.5.** The stratum a given nilpotent Lie algebra belongs to provides useful information on its automorphism group. Indeed, let \( n \) be a nilpotent Lie algebra and for any basis \( \{X_1, \ldots, X_n\} \) of \( n \) consider \( \beta \) as defined above. If \( \{\cdot, \cdot\} \in S_\beta \), then \( \text{Aut}(n) \subset P_\beta \) by 7.3.

**Lemma 9.6.** [L. 08b] Let \( n \) be a nonabelian nilpotent Lie algebra and let \( \phi \) be a pre-Einstein derivation of \( n \) with basis of eigenvectors \( \{X_1, \ldots, X_n\} \) and define

\[
\beta := \frac{1}{\|\phi\|^2}(\phi - I).
\]

Then \( \{\cdot, \cdot\} \in S_\beta \) if and only if \( 0 \notin G_\beta \cdot \{\cdot, \cdot\} \), and in that case, \( \beta = \text{mcc}(\{c_{ij}^k : c_{ij}^k \neq 0\}) \).

It follows from Lemma 9.6 that if \( \phi \) is a pre-Einstein derivation of \( n \) then \( \phi > 0 \) (see 5.3) and \( \text{ad} \phi \geq 0 \) (see 7.1) are necessary conditions in order to have \( 0 \notin G_\beta \cdot \{\cdot, \cdot\} \) (i.e. \( \{\cdot, \cdot\} \in S_\beta \)). These conditions are not however sufficient (compare with the paragraph below Theorem 5.6). For instance, any free nilpotent Lie algebra which is not an Einstein nilradical provides a counterexample (see Nikolayevsky 08a, Remark 2).

10. Open problems

Let \( n \) be an \( \mathbb{N} \)-graded nilpotent Lie algebra.

1. **Obstructions.** To find algebraic necessary conditions on \( n \) to be an Einstein nilradical.

2. **Existence.** Are there algebraic conditions on \( n \) which are sufficient to be an Einstein nilradical?
(3) Does the assertion ‘\( n \) is an Einstein nilradical’ have probability 1 in some sense?

(4) Does the assertion ‘\( n \) is not an Einstein nilradical’ have probability 1 in some sense?

(5) Assume that \( n \) is an Einstein nilradical with Lie bracket \( \mu_0 \in \mathcal{N} \), and consider the flow \( \mu(t) \) defined in (4.3) with \( \mu(0) = \mu_0 \). Does \( \lambda = \lim_{t \to \infty} \mu(t) \) necessarily belong to \( \text{GL}_n(\mathbb{R}) \)? (this would provide a nice obstruction).

(6) To exhibit an explicit example or prove the existence of a nilpotent Lie algebra which does not admit a nice basis (see Definition 5.8).

(7) Are there only finitely many \( \mathbb{N} \)-graded filiform Lie algebras which are not Einstein nilradicals in each dimension?

11. Appendix: Real geometric invariant theory

Let \( G \) be a real reductive group acting linearly on a finite dimensional real vector space \( V \) via \( (g,v) \mapsto g.v, g \in G, v \in V \). The precise definition of our setting is the one considered in [Richardson-Slodowy 90]. We also refer to [Eberlein-Jablonsky 07], where many results from geometric invariant theory are adapted and proved over \( \mathbb{R} \).

The Lie algebra \( \mathfrak{g} \) of \( G \) also acts linearly on \( V \) by the derivative of the above action, which will be denoted by \( (\alpha,v) \mapsto \pi(\alpha)v, \alpha \in \mathfrak{g}, v \in V \). We consider a Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \), where \( \mathfrak{k} \) is the Lie algebra of a maximal compact subgroup \( K \) of \( G \). Endow \( V \) with a fixed from now on \( K \)-invariant inner product \( \langle \cdot, \cdot \rangle \) such that \( \mathfrak{p} \) acts by symmetric operators, and endow \( \mathfrak{p} \) with an \( \text{Ad}(K) \)-invariant inner product \( \langle \cdot, \cdot \rangle \).

The function \( m : V \setminus \{0\} \longrightarrow \mathfrak{p} \) implicitly defined by

\[
(m(v),\alpha) = \frac{1}{||v||}\langle \pi(\alpha)v,v \rangle, \quad \forall \alpha \in \mathfrak{p}, \ v \in V,
\]

is called the moment map for the representation \( V \) of \( G \). Since \( m(cv) = m(v) \) for any nonzero \( c \in \mathbb{R} \), we also may consider the moment map on the projective space of \( V \), \( m : \mathbb{P}V \longrightarrow \mathfrak{p} \), with the same notation and definition as above for \( m([v]) \), \( [v] \) the class of \( v \) in \( \mathbb{P}V \). It is easy to see that \( m \) is \( K \)-equivariant: \( m(k.v) = \text{Ad}(k)m(v) \) for all \( k \in K \).

In the complex case (i.e. for a complex representation of a complex reductive algebraic group), under the natural identifications \( \mathfrak{p} = \mathfrak{p}^* = (\mathfrak{k}^*)^* = \mathfrak{t}^* \), the function \( m \) is precisely the moment map from symplectic geometry, corresponding to the Hamiltonian action of \( K \) on the symplectic manifold \( \mathbb{P}V \) (see for instance the survey [Kirwan 98] or [Mumford-Fogarty-Kirwan 94] Chapter 8] for further information). For real actions, this nice interplay with symplectic geometry is lost (\( \mathbb{P}V \) could even be odd dimensional), but the moment map is nevertheless a very natural object attached to a real representation encoding a lot of information on the geometry of \( G \)-orbits and the orbit space \( V/G \).

Let \( \mathcal{M} = \mathcal{M}(G,V) \) denote the set of minimal vectors, that is,

\[
\mathcal{M} = \{v \in V : ||v|| \leq ||g.v|| \text{ } \forall g \in G\}.
\]

For each \( v \in V \) define

\[
\rho_v : G \mapsto \mathbb{R}, \quad \rho_v(g) = ||g.v||^2.
\]
In [Richardson-Slodowy 90], it is shown that the nice interplay between closed orbits and minimal vectors discovered in [Kempf-Ness 79] for actions of complex reductive algebraic groups, is still valid in the real situation.

**Theorem 11.1.** [Richardson-Slodowy 90] Let $V$ be a real representation of a real reductive group $G$, and let $v \in V$.

(i) The orbit $G.v$ is closed if and only if $G.v$ meets $\mathcal{M}$.

(ii) $v \in \mathcal{M}$ if and only if $\rho_v$ has a critical point at $e \in G$.

(iii) If $v \in \mathcal{M}$ then $G.v \cap \mathcal{M} = K.v$.

(iv) The closure $\overline{G.v}$ of any orbit $G.v$ always meets $\mathcal{M}$. Moreover, there always exists $\alpha \in \mathfrak{p}$ such that $\lim_{t \to \infty} \exp(-t\alpha).v = w$ exists and $G.w$ is closed.

(v) $\overline{G.v} \cap \mathcal{M}$ is a single $K$-orbit, or in other words, $\overline{G.v}$ contains a unique closed $G$-orbit.

As usual in the real case, classical topology of $V$ is always considered rather than Zariski topology, unless explicitly indicated.

Let $(d \rho_v)_e : \mathfrak{g} \to \mathbb{R}$ denote the differential of $\rho_v$ at the identity $e$ of $G$. It follows from the $K$-invariance of $\langle \cdot, \cdot \rangle$ that $(d \rho_v)_e$ vanishes on $\mathfrak{k}$, and so we can assume that $(d \rho_v)_e \in \mathfrak{p}^*$, the vector space of real-valued functionals on $\mathfrak{p}$. If we identify $\mathfrak{p}$ and $\mathfrak{p}^*$ by using $\langle \cdot, \cdot \rangle$, then it is easy to see that

$$m(v) = \frac{1}{\|v\|^2} (d \rho_v)_e.$$  

The moment map at $v$ is therefore an indicator of the behavior of the norm along the orbit $G.v$ in a neighborhood of $v$. It follows from Theorem [11.1] (ii) that

$$\mathcal{M} \setminus \{0\} = \{v \in V \setminus \{0\} : m(v) = 0\}.$$  

Thus if we consider the functional square norm of the moment map

$$(11.1) \quad F : V \setminus \{0\} \to \mathbb{R}, \quad F(v) = \|m(v)\|^2,$$

which is a 4-degree homogeneous polynomial times $\|v\|^{-4}$, $\mathcal{M} \setminus \{0\}$ coincides with the set of zeros of $F$. It then follows from Theorem [11.1] parts (i) and (iii), that a nonzero orbit $G.v$ is closed if and only if $F(w) = 0$ for some $w \in G.v$, and in that case, the set of zeros of $F|_{G.v}$ coincides with $K.v$. Recall that $F$ is scaling invariant and so it is actually a function on any sphere of $V$ or on $\mathbb{P}V$.

A natural question arises: what is the role played by the remaining critical points of $F$ (i.e. those for which $F(v) > 0$) in the study of the $G$-orbit space of the action of $G$ on $V$? This was independently shown in [Kirwan 84] and [Ness 84] in the complex case, who have shown that non-minimal critical points still enjoy most of the nice properties of minimal vectors stated in Theorem [11.1]. In the real case, the analogues of some of these results have been proved in [Marian 01].

We endow $\mathbb{P}V$ with the Fubini-Study metric defined by $\langle \cdot, \cdot \rangle$ and denote by $x \mapsto \alpha_x$ the vector field on $\mathbb{P}V$ defined by $\alpha \in \mathfrak{g}$ via the action of $G$ on $\mathbb{P}V$, that is, $\alpha_x = \frac{dt}{dt}|_0 \exp(t\alpha)x$. We will also denote by $F$ the functional $F : \mathbb{P}V \to \mathbb{R}$, $F([v]) = \|m([v])\|^2$.

**Lemma 11.2.** [Marian 01] The gradient of the functional $F : V \setminus \{0\} \to \mathbb{R}$ is given by

$$\text{grad}(F)_v = \frac{1}{\|v\|^4} \left( \pi(m(v))v - \|m(v)\|^2 v \right), \quad v \in V \setminus \{0\},$$
and for $F : \mathbb{P}V \to \mathbb{R}$ we have that
\[
\text{grad}(F)_{[v]} = 4m([v])_{[v]}, \quad [v] \in \mathbb{P}V.
\]

Therefore, $v$ is a critical point of $F$ (or equivalently, of $F|_{G.x}$) if and only if $v$ is an eigenvector of $\pi(m(v))$, and $[v]$ is a critical point of $F$ (or equivalently, of $F|_{G,[v]}$) if and only if $\exp tm([v])$ fixes $[v]$.

**Theorem 11.3.** [Marian 01] Let $V$ be a real representation of a real semisimple Lie group $G$.

(i) If $x \in \mathbb{P}V$ is a critical point of $F$ then the functional $F|_{G.x}$ attains its minimum value at $x$.

(ii) If nonempty, the critical set of $F|_{G.x}$ consists of a single $K$-orbit.

**Definition 11.4.** A nonzero vector $v \in V$ is called unstable if $0 \in \overline{G.v}$, and semistable otherwise. If a semistable vector has in addition compact isotropy subgroup then it is called stable.

If the orbit of a nonzero $v \in V$ is closed then $v$ is clearly semistable. More generally, $v \in V$ is semistable if and only if the unique (up to $K$-action) zero of $F$ which belongs to $\overline{G.v}$ is a nonzero vector. On the contrary, any critical point of $F$ which is not a zero of $F$ is unstable. Indeed, if $\pi(m(v))v = cv$, $c = |m(v)|^2 > 0$ (see Lemma 11.2), then
\[
\lim_{t \to -\infty} \exp(-tm(v))v = \lim_{t \to -\infty} e^{-tc}v = 0,
\]
and so $0 \in \overline{G.v}$. Thus the study of critical points of $F$ other than zeroes gives useful information on the orbit space structure of the subset of all unstable vectors, often called the nullcone of $V$.

**Example 11.5.** Let us consider the example of $G = \text{SL}_3(\mathbb{R})$ and $V = P_{3,3}(\mathbb{R})$, the vector space of all homogeneous polynomials of degree 3 on 3 variables. The action is given by a linear change of variables on the left
\[
(g.p)(x_1, x_2, x_3) = p \left( g^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right), \quad \forall g \in \text{SL}_3(\mathbb{R}), \quad p \in P_{3,3}(\mathbb{R}).
\]
It follows that $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$, $K = \text{SO}(3)$, $\mathfrak{k} = \mathfrak{so}(3)$ and $\mathfrak{p} = \text{sym}_0(3)$ is the space of traceless symmetric $3 \times 3$ matrices. As an $\text{Ad}(K)$-invariant inner product on $\mathfrak{p}$ we take $(\alpha, \beta) = \text{tr} \alpha \beta$, and it is easy to see that the inner product $\langle \cdot, \cdot \rangle$ on $V$ for which the basis of monomials
\[
\{ x^D := x_1^{d_1} x_2^{d_2} x_3^{d_3} : d_1 + d_2 + d_3 = 3, \ D = (d_1, d_2, d_3) \}
\]
is orthogonal and
\[
||x^D||^2 = d_1!d_2!d_3!, \quad \forall D = (d_1, d_2, d_3),
\]
satisfies the required conditions. Let $E_{ij}$ denote as usual the $n \times n$ matrix whose only nonzero coefficient is a 1 in the entries $ij$. Since
\[
\pi(E_{ij})p = \frac{\partial}{\partial t} p(e^{-tE_{ij}}) = -x_j \frac{\partial p}{\partial x_i},
\]
we obtain that the moment map $m : P_{3,3}(\mathbb{R}) \to \text{sym}_0(3)$ is given by
\[
m(p) = I - \frac{1}{||p||^2} \left( x_j \frac{\partial p}{\partial x_i}, p \right).
\]
We are using here that $\langle x_j \frac{\partial p}{\partial x_i}, p \rangle = \langle x, \frac{\partial p}{\partial x_j}, p \rangle$ for all $i, j$. 

It is also easy to see that the action of a diagonal matrix $\alpha \in A_{0} \mathfrak{s} l_{4}(\mathbb{R})$ with entries $a_{1}, a_{2}, a_{3}$ is given by

$$\pi(\alpha)x^{D} = - \left( \sum_{i=1}^{3} a_{i}d_{i} \right)x^{D}, \quad \forall D = (d_{1}, d_{2}, d_{3}).$$

A first general observation is that any monomial is a critical point of $F$. Indeed,

$$m(x^{D}) = \begin{bmatrix} 1-d_{1} & 1-d_{2} & 1-d_{3} \end{bmatrix},$$

and so $x^{D}$ is an eigenvector of $m(x^{D})$ with eigenvalue $F(x^{D}) = \sum d_{i}^{2} - 1$ (see Lemma 11.2). It follows that $m(p) = 0$ for $p = x_{1}x_{2}x_{3}$, that is, $p$ is a minimal vector and its $\mathbb{S}L_{3}(\mathbb{R})$-orbit is therefore closed. We also have in such case that $p_{1} = p + x_{1}^{3}$ is a semistable vector whose orbit is not closed. Indeed, by acting by diagonal elements with entries $t, \frac{1}{t}, 1$ we get that $p + t^{3}x_{1}^{3} \in \mathbb{S}L_{3}(\mathbb{R}).p$ for all $t \neq 0$ and so $p \in \mathbb{S}L_{3}(\mathbb{R}).p_{1}$ (recall that $p$ and $p_{1}$ can never lie in the same orbit since they have non-isomorphic isotropy subgroups).

For the vector $q = x_{1}^{2}x_{3} + x_{1}x_{2}^{2}$ we have that

$$m(q) = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

It follows from (11.2) that $\pi(m(q))q = \frac{1}{2}q$ proving that $q$ is a critical point of $F$ with critical value $F(q) = \frac{1}{2} > 0$. On the other hand, the family $p_{a,b} = ax_{1}^{2}x_{3} + bx_{2}^{2}$, $a, b \neq 0$, lie in a single orbit and

$$m(p_{a,b}) = I - \frac{1}{2a^{2}+6b^{2}} \begin{bmatrix} 4a^{2} & 18b^{2} \\ 18b^{2} & 2a^{2} \end{bmatrix}.$$

It is then easy to see by using (11.2) that $p_{a,b}$ is a critical point if and only if $5a^{2} = 27b^{2}$ and the critical value equals $\frac{155}{19} - 3$, a number smaller than $\frac{1}{2}$. In particular, $p_{a,b}$ can not be in the closure of the orbit of $q$ by Theorem 11.3 (i).

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