In this note we discuss the frame property in $L^2(\mathbb{R})$ of the irregular Gabor system generated by a single Cauchy kernel, i.e., the system

$$G(\Lambda, M) = \{ \phi_{\lambda, \mu}(t) \}_{\lambda \in \Lambda, \mu \in M} = \left\{ \frac{e^{-2i\pi \mu t}}{t - \lambda - iw} \right\}_{\lambda \in \Lambda, \mu \in M},$$

here $w \in \mathbb{C}$, $\Re w \neq 0$, and $\Lambda = \{ \lambda \} \subset \mathbb{R}$, $M = \{ \mu \} \subset \mathbb{R}$ are, generally speaking, irregular sets, their properties will be described further in this note. In other words, we look for the frame inequality

$$A \| f \|^2 \leq \sum_{\lambda \in \Lambda, \mu \in M} |\langle f, \phi_{\lambda, \mu} \rangle|^2 \leq B \| f \|^2, \; f \in L^2(\mathbb{R}).$$

The reason we wrote this note is twofold. First, in contrast to the (now) classical rectangular lattices $\alpha \mathbb{Z} \times \beta \mathbb{Z}$, not much is known about irregular ones $\Lambda \times M$. The recent breakthrough related to semiregular lattices of the form $\Lambda \times \beta \mathbb{Z}$ has been achieved in [1], where the authors considered the Gabor frames, generated by Gaussian totally positive functions of finite type. We also refer the reader to [1] for the history of the problem. Unfortunately, we are not able to apply the techniques from [1] to $G(\Lambda, M)$ even in the case, when $M = \beta \mathbb{Z}$ for some $\beta > 0$. Second, the proof we suggest is very simple, much simpler than the known ones for rectangular lattices, see [2, 3]. We expect this proof can serve as a model in more general settings.

We use the following notation and definition.

**Definition 1.** Given $\beta > 0$, by the Paley-Wiener space $PW_{[0, \beta]}$ we mean

$$PW_{[0, \beta]} = \{ f : f(z) = \int_0^\beta e^{2i\pi \xi z} \hat{f}(\xi) d\xi, \; \hat{f} \in L^2(0, \beta) \}.$$

This space consists of entire functions of exponential type, which belong to $L^2(\mathbb{R})$ and have the indicator diagram included in $[-2i\beta, 0]$. We refer the reader to [4] for the detailed description of the Paley-Wiener spaces as well as for other facts on entire functions.

**Definition 2.** We say that the set $\Lambda \subset \mathbb{R}$ is *sampling* for $PW_{[0, \beta]}$ if, for some constants $0 < A, B < \infty$, (sampling constants)

$$A \| f \|^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B \| f \|^2, \; f \in PW_{[0, \beta]}.$$
Such sets have been completely described in [5]. Typically, they have density bigger than $1/\beta$, yet their structure may be rather complicated, in particular they need not have density strictly bigger than $1/\beta$, also they need not contain subsets which are complete interpolating sequences for $PW_{[0,\beta]}$.

**Definition 3.** The set $M = \{\mu_n\} \subset \mathbb{R}$, $\mu_n < \mu_{n+1}$ is *locally finite* if

\begin{equation}
\beta(M) := \sup \{\beta_n\} < \infty, \quad \text{here } \beta_n = \mu_{n+1} - \mu_n,
\end{equation}

and

\begin{equation}
\sup_{x \in \mathbb{R}} \# \{M \cap [x, x+1]\} < \infty.
\end{equation}

**Theorem** Let $M \subset \mathbb{R}$ and $\Lambda \subset \mathbb{R}$. The following statements are equivalent:

1) $M$ is a locally finite set and $\Lambda$ is a sampling set for $PW_{[0,\beta(M)]}$;
2) The system $G(\Lambda, M)$ is a frame in $L^2(\mathbb{R})$.

**Proof** 1) $\Rightarrow$ 2). Let for definiteness $\Re w > 0$ so $iw$ lies in the upper half-plane. In what follows it is convenient to simplify notation, by writing $\phi_{\lambda,n}$ instead of $\phi_{\lambda,\mu_n}$. Given $f \in L^2(\mathbb{R})$, denote

\begin{equation}
c_{\lambda,n} = \langle \phi_{\lambda,n}, \tilde{f} \rangle = \int_{-\infty}^{\infty} f(t) \frac{e^{-2i\pi \mu_n t}}{t - \lambda - iw} dt.
\end{equation}

Let

\[f_k(t) = \int_{\mu_k}^{\mu_{k+1}} \hat{f}(\xi) e^{2i\pi \xi t} d\xi,
\]

as always, $\hat{f}$ stays for the Fourier transform of $f$. We have

\begin{equation}f(t) = \sum_k f_k(t), \quad \|f\|^2 = \sum_k \|f_k\|^2.
\end{equation}

Denote also

\[h_k(t) := e^{-2i\pi \mu_k t} f_k(t) \in PW_{[0,\beta(M)]}.
\]

A straightforward calculation yields

\[\langle \phi_{\lambda,n}, \tilde{f}_k \rangle = \int_{-\infty}^{\infty} h_k(t) e^{-2i\pi (\mu_n - \mu_k) t} \frac{e^{-2i\pi \mu_n t}}{t - (\lambda + iw)} dt,
\]

\[= \begin{cases} e^{-2i\pi \mu_n \lambda} h_k(\lambda + iw) e^{2i\pi \mu_k \lambda} e^{2\pi w(\mu_n - \mu_k)}, & k \geq n; \\
0, & k < n,
\end{cases}
\]

and

\[c_{\lambda,n} = e^{-2i\pi \mu_n \lambda} \sum_{k \geq n} h_k(\lambda + iw) e^{2i\pi \mu_k \lambda} e^{2\pi w(\mu_n - \mu_k)}.
\]

Let $d_{\lambda,n} := c_{\lambda,n} e^{2i\pi \mu_n \lambda}$ and

\[c_n := \{c_{\lambda,n}\}_{\lambda \in \Lambda} \in l^2(\Lambda), \quad d_n := \{d_{\lambda,n}\}_{\lambda \in \Lambda} \in l^2(\Lambda),
\]

\[\omega_{\lambda} = \{\omega_{\lambda,k}\}_{k \in \mathbb{Z}} := \{h_k(\lambda + iw) e^{2i\pi \mu_k \lambda} \}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}).
\]
We have
\[ \| \{ c_{\lambda,n} \} \|^2_{l^2(\Lambda \times \mathbb{Z})} = \sum_n \| c_n \|^2 = \sum_n \| d_n \|^2, \]
\[ \sum_{\lambda} \| \omega_{\lambda} \|^2 = \sum_{\lambda, k} |h_k(\lambda + i)|^2 \asymp \| f \|^2, \]
the later follows from (5) and the fact that \( \Lambda \) is a sampling set for \( PW_{[0,\beta]} \).

We also have
\[ (6) \quad d_n = A \omega_{\lambda}, \]
where the matrix \( A \) is defined as
\[ (7) \quad A = (a_{k,n})_{k,n \in \mathbb{Z}}, \quad a_{k,n} = \begin{cases} e^{-2\pi(\mu_k - \mu_n)w}, & k \geq n; \\ 0, & k < n. \end{cases} \]

Denote \( \gamma_n = 2\pi w(\mu_{n+1} - \mu_n) \) and consider the operator \( B = (b_{p,q}) : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) with the matrix
\[ b_{p,q} = \begin{cases} e^{-\gamma_p}, & q = p + 1; \\ 0, & \text{otherwise}. \end{cases} \]

We have
\[ A = I + \sum_{j \geq 1} B^j. \]

For sufficiently large \( N \) such that \( \mu_{n+N} - \mu_n \geq 1 \) (i.e. \( \gamma_n + \ldots + \gamma_{n+N} \geq 1 \)) for all \( n \) we have \( \| B^N \| \leq e^{-\Re w} \). Therefore, this series converges to \( (I - B)^{-1} \), in particular \( A \) is invertible.

Therefore, \( \| d_n \| \asymp \| \omega_{\lambda} \| \) and, finally,
\[ \sum_n \| c_n \|^2 \asymp \sum_{\lambda} \| \omega_{\lambda} \|^2 \asymp \| f \|^2. \]

This completes the proof.

The statement 2) \( \Rightarrow \) 1) follows easily from the frame inequality. We restrict ourselves just to brief outline of its proof.

Indeed, relation (3) follows from the right-hand side of (1). In case (3) fails the lattice \( \Lambda \times M \) contains arbitrary large clusters of points. Taking a function whose time-frequency support is located (mainly) in such cluster we come to contradiction with (1).

Relation (2) is straightforward. If the set \( M \) contains arbitrary large gaps, then the time-frequency shifts of the Gaussian, centred to the middle of these gaps bring contradiction to (1).

As in the proof of the previous part it now follows that the operator \( A \) defined in (7), is invertible.

In order to verify that \( \Lambda \) is a sampling set in \( PW_{[0,\beta(M)]} \) it suffices to check that \( \Lambda \) is a sampling set in each \( PW_{[0,\beta_n]} \) with the sampling constants independent of \( n \). The latter is also straightforward. Indeed, take \( n \in \mathbb{Z} \), \( h \in PW_{\beta_n} \), and set \( f(t) = h(t)e^{2\pi i \mu_n t} \) (we keep the same notation as in
the first part of the Theorem). The sampling property follows now from representation (6), invertibility of the operator $A$, and also from the frame inequality.

**Remark** It is worth to mention that, in contrast to the case of the Gaussian window, the condition for $\mathcal{G}(\Lambda, M)$ to be a frame is essentially non-symmetric with respect to $\Lambda$ and $M$. This reflects the asymmetry of the Cauchy kernel under the Fourier transform.

**References**

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