Phenomenology of cosmological particle creation, Dirac sea and all that

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Abstract. We constructed the conformally invariant model for scalar particle creation induced by strong gravitational fields. Starting from the “usual” hydrodynamical description of the particle motion written in the Eulerian coordinates we substituted the particle number conservation law (which enters the formalism) by “the particle creation law”, proportional to the square of the Weyl tensor (following the famous result by Ya.B. Zel’dovich and A.A. Starobinsky). Then, demanding the conformal invariance of the whole dynamical system, we have got both the (Weyl)-conformal gravity and the Einstein-Hilbert gravity action integral with dilaton field. Thus, we obtained something like the induced gravity suggested first by A.D. Sakharov. It is shown that the resulting system is self-consistent. We considered also the vacuum equations. It is shown that, beside the “empty vacuum”, there may exist the “dynamical vacuum”, which is nothing more but the Dirac sea. The latter is described by the unexpectedly elegant equation which includes both the Bach and Einstein tensors and the cosmological terms. All the vacuum solutions violate the local conformal invariance.

1. Introduction

The discovery of the relic (CMB) radiation proved that the early universe was hot and filled with the soup of elementary particles. Tracing this situation back in time (and replacing the elementary particles by more and more elementary ones) we come inevitably to the Big Bang scenario. It seems more plausible that the universe was creating quantum mechanically from “nothing” [1] or reemerged after some previous circle [2-3] in a pure quantum state. And only then it was filled with some matter constituents. This reminds the Starobinsky inflationary model [4], in which the present Friedman stage of the universe is originated from decay of the initial de Sitter space-time (which is unstable due to the presence of terms in the Lagrangian quadratic in the curvature). The cosmological particle creation in the framework of General Relativity were studied extensively in 70-s of the last century by many authors [5-16]. Due to results of their works we know much about the structure of the counter-terms, the importance of the trace anomaly in the particle creation processes, the rate of particle production and so on.

All the above-mentioned investigations were confined to considering the quantum scalar field on the given background metrics, namely, cosmological homogeneous, but slightly anisotropic, space-times. What about the back reaction? The main obstacle in accounting for the back reaction is that the rigorous solution of the quantum problem requires the knowledge of the boundary conditions, while the latter can be imposed only after solving the (classical) Einstein equations. Thus, we have got the “vicious circle”.

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The full self-consistent treatment of the pure quantum phenomenon of particle creation is impossible when the space-time is considered as the classical entity. This is because we need to impose the boundary conditions in order to define the very notion of the quantum particle, but to do so we have to know the global space-time geometry, which, in turn, can be constructed only after solving the gravitational field equations with these very particles (among the other constituents) as the source. To avoid this difficulty we use phenomenological description of the particle creation process, which should be quite applicable when the fields responsible for it are strong enough, as well as the gravitational field serving as a trigger. All these requirements are met in the very early universe.

We describe the already created particles by some hydrodynamics, the “creation law” being incorporated into the hydrodynamical action integral (written using Eulerian variables). For this “creation law” we adopted the well known result for cosmological particle creation [8] that the particle production rate is proportional to the square of the Weyl tensor. As the quantum field responsible for the particle creation we considered the specific scalar field and wrote the corresponding Lagrangian, as simplest as possible, i.e., it consists of the kinetic and “mass” terms only with no ad hoc self-interaction. But the scalar field in the Lagrangian is not the fundamental one, since one part of the fundamental field is already exists in the form of the created particles, while another one, namely, the square of the Weyl tensor is some part of the quantum trace anomaly.

We proceeded by demanding the local conformal invariance of the total action. This was inspired by the appearance of the square of the Weyl tensor in the total action and some speculations by R. Penrose and G.’tHooft [2,17] on this subject. The immediate result is the emergence of the scalar curvature in the scalar field Lagrangian. Together with the square of the Weyl tensor this is in spirit of the A.D. Sakharov’s idea about the induced gravity [18]. The other result appeared completely unexpected. Namely, the “mass” term becomes converted into the self-interaction term which exponent (=4) is due to the space-time dimensionality. There are other consequences of the local conformal invariance in our model, which will be discusses in due course. Throughout the paper we use the units $\hbar = c = 1$, the signature of the metric tensor $g_{\mu\nu}$ is $(+,−,−,−)$, the Riemann curvature tensor is defined as

$$R^\mu_{\nu\sigma\tau} = \frac{\partial \tilde{A}^\mu_{\sigma\tau}}{\partial x^\nu} - \frac{\partial \tilde{A}^\mu_{\nu\sigma}}{\partial x^\tau} + \tilde{A}^\mu_{\nu\xi} \tilde{A}^\xi_{\sigma\tau} - \tilde{A}^\mu_{\nu\xi} \tilde{A}^\xi_{\sigma\tau},$$

while the Ricci tensor is the convolution $R_{\nu\sigma} = R^\mu_{\nu\sigma \mu}$. The scalar curvature $R = g^{\nu\sigma} R_{\nu\sigma}$, and $\tilde{A}^\mu_{\nu\sigma}$ are the metric connections, i.e., the covariant derivatives of the metric tensor are zero.

2. Phenomenology of particle creation

We start with construction of the hydrodynamical part of our model. In the “classical” hydrodynamics there exist two different sets of dynamical variables, the so called Lagrangian and Eulerian coordinates. The first of them are comoving, i.e., the observer is sitting on some world-line. So, using the least action principle, one has to vary the trajectory of the (quasi)-particles. Since in such a case we cannot take into account the very processes of both creation and annihilation of particles (i.e., trajectories), it is not appropriate for our purposes. Therefore, we need to use the Eulerian description, when the dynamical variables are fields, namely, the particle number density $n(x)$ and the four-velocities $u^\mu$. The action integral in this case is [19] (for details see also [20]):

$$S_{\text{hydro}} = -\int c(X,n) \sqrt{-g} dx + \int \lambda_1(u^\mu u_{\mu} - 1) \sqrt{-g} dx + \int \lambda_2 (m u^\mu) \sqrt{-g} dx + \int \lambda_3 X_\mu u^\mu \sqrt{-g} dx,$$

where $c(X,n)$ is the invariant energy density, $n(x)$ - invariant particle number density, $u^\mu(x)$ - four-velocity of the particle flow, $X(x)$ is the auxiliary dynamical variable introduced in order to avoid the identically zero vorticity of particle flow. It enters the action integral with the Lagrange multiplier $\lambda_3$, indicating the constraint $X_\mu u^\mu = 0$, i.e., $X(x) = \text{const}$ on the trajectories, thus enumerating them. The other two Lagrange multipliers, $\lambda_1(x)$ and $\lambda_2(x)$ are responsible, respectively, for the constraints
$u^\mu u_\mu = 1$ (natural normalization of the four-velocities) and $(nu^\mu)_\mu = 0$ - particle number conservation law. The semicolon "$;" denotes a covariant derivative with respect to the metric $g_{\mu\nu}$.

Our aim is to incorporate into the formalism the particle “creation law” $(nu^\mu)_\mu = \dot{O}(\mathrm{inv}) \neq 0$. Evidently, the function $\dot{O}$ should depend on some invariants of the fields causing this particle creation. Here we would like to explore the fundamental result by Ya.B. Zel’dovich and A.A. Starobinsky [15] obtained for the cosmological particle production

$$(nu^\mu)_\mu = \beta \tilde{N}^2,$$

where $\tilde{N}^2$ is the square of the Weyl tensor $C^a_{\alpha\beta\gamma\delta}$ (its definition as well as some most important properties see e.g., in [21-23]) and the coefficient $\beta$ depends on the type of particles under consideration. We will consider this “creation law” as our first postulate. So, the hydrodynamical part of the action integral now becomes

$$S_{\text{hydro}} = -\int e(X,n)\sqrt{-g} dx + \int \left[ \lambda_0 (u^\mu u_\mu - 1) \sqrt{-g} dx + \int \lambda_1 \left( (nu^\mu)_\mu - \beta C^2 \right) \sqrt{-g} dx + \int \lambda_2 X_\mu u^\mu \sqrt{-g} dx. \right]$$

Very important note. The Lagrange multiplier $\lambda_1$ is, actually, defined up to the additive constant. Indeed, let us replace $\lambda_1 \rightarrow \lambda_1 + \gamma_0 \beta_0 = \text{const}$, then

$$\gamma_0 \int \left( (nu^\mu)_\mu - \beta C^2 \right) \sqrt{-g} dx = \gamma_0 \left( n(\sqrt{-g}) u^\mu \right)_\mu - \beta C^2 \sqrt{-g}. \right) \sqrt{-g} dx.$$ (4)

Due to the identity $(nu^\mu)_\mu \sqrt{-g} = \left( n(\sqrt{-g}) u^\mu \right)_\mu$, the corresponding volume integral transforms into the surface integral with no effect on the dynamical equations. In result, we are left with the same “creation law” as before plus the Weyl gravitational action

$$S_{\text{Wey}}^{\text{grav}} = -\gamma_0 \beta \int C^2 \sqrt{-g} dx.$$ (5)

This means that we can start with the action that includes the Weyl gravity and then incorporate it into the hydrodynamical action simply by redefinition of the Lagrange multiplier $\lambda_1$. Such a procedure, of course, will not reduce the number of dynamical degrees of freedom.

3. Scalar field and conformal invariance

By the conformal transformation we will understand the space-time dependent scaling of the metric tensor $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \Omega^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu = \Omega^2 (x) \hat{dx}^2.$$ (6)

The conformal invariance means

$$\frac{\delta S_{\text{tot}}}{\delta \Omega} = 0.$$ (7)

Therefore, we can (and will) consider the conformal factor $\Omega$ as a dynamical variable and make variations independently in $\Omega$ and in $\hat{g}_{\mu\nu}$ [17]. Let us $S_{\text{tot}} = S_{\text{grav}} + S_{\text{matter}}$. By definition

$$\delta S_{\text{matter}} = \frac{1}{2} \int T_{\mu\nu} \sqrt{-\hat{g}} \delta \hat{g}^{\mu\nu} dx$$ or $\delta S_{\text{matter}} = \frac{1}{2} \int \hat{T}_{\mu\nu} \sqrt{-\hat{g}} \delta \hat{g}^{\mu\nu} dx,$ (8)

where $T_{\mu\nu}$ ($\hat{T}_{\mu\nu}$) is the matter energy-momentum tensor. Consider, first, the following transformation of the metric tensor

$$\delta \hat{g}^{\mu\nu} = -\frac{1}{2\Omega^2} \hat{g}^{\mu\nu} \delta \Omega = -\frac{1}{2\Omega} g^{\mu\nu} \delta \Omega.$$ (9)
Suppose \( \frac{\delta S_{\text{tot}}}{\delta \Omega} = 0 \), then
\[
0 = \delta S_{\text{matter}} = \int T_{\mu\nu} g^{\mu\nu} \frac{\delta \Omega}{\Omega} \sqrt{-g} \, dx ,
\]
that is, the trace of the energy-momentum tensor should be zero: \( \text{Tr} [ T_{\mu\nu} ] = \text{Tr} [ \hat{T}_{\mu\nu} ] = 0 \). If one considers the metric tensor transformation of the kind \( \delta g^{\mu\nu} = \Omega^2 \delta \hat{g}^{\mu\nu} \), then, as can be easily seen,
\[
\hat{T}_{\mu\nu} = \Omega^2 T_{\mu\nu}, \quad \hat{T}^\nu = \Omega^4 T^\nu, \quad \hat{\rho}^{\mu\nu} = \Omega^6 g^{\mu\nu}.
\]
Let us go further on. The question arises: quanta of what kind a field is creating? The simplest choice is the scalar field. And the simplest action integral is
\[
\delta S_{\text{scalar}} = \int \left( \frac{1}{2} \chi'' \chi + \frac{R}{12} \chi^2 - \frac{1}{2} m^2 \chi^2 \right) \sqrt{-g} \, dx .
\]
Here \( \chi_{,\mu} = \chi_{,\mu} \) (comma denotes the partial derivative), \( \chi'' = g^{\alpha\beta} \chi_{,\alpha} \chi_{,\beta} \), \( m \) is some constant with the dimension of mass. The term \( \frac{R}{12} \chi^2 \), where \( R \) is the scalar curvature, constructing from the metric \( g_{\mu\nu} \), is added to make this action conformally covariant. After the “standard” conformal transformations, namely, \( g^{\mu\nu} = \Omega^2 \hat{g}^{\mu\nu} \) and \( \chi = \frac{\hat{\chi}}{\Omega} \), one gets
\[
\delta S_{\text{scalar}} = \int \left( \frac{1}{2} \hat{\chi}'' \hat{\chi} + \frac{\hat{R}}{12} \hat{\chi}^2 - \frac{1}{2} m^2 \Omega^2 \hat{\chi}^2 \right) \sqrt{-\hat{g}} \, dx - \frac{1}{2} \left( \frac{\hat{R}}{2} \Omega^4 \right) \sqrt{-\hat{g}} \, dx .
\]
Now indices are raising and lowering with the metric \( \hat{g}_{\mu\nu} \) and vertical line “\( | \)” denotes the covariant derivative also with respect to the metric \( \hat{g}_{\mu\nu} \). The last term can be transformed to the surface integral, it does not effect the dynamics. Remarkably enough, that started with no gravitational action at all, we have got now both the conformal gravity (as a part of the conformally covariant scalar field Lagrangian).

It seems that if one puts \( m = 0 \), everything else will be all right. But, it is not so easy; there exists a problem [17]. This problem concerns the signs. With the “correct” sign for the kinetic term \( \frac{1}{2} \chi'' \chi \) we have the “wrong” sign for the Einstein-Hilbert-dilaton part, \( \frac{\hat{R}}{12} \chi^2 \) (with our sign convention there should be “-” instead of “+”), and vice-versa. Our choice is the “correct” sign for \( \hat{R} \), i.e., \( -\frac{\hat{R}}{12} \chi^2 \), and the “wrong” sign for the kinetic term, i.e., \( -\frac{1}{2} \chi'' \chi \). This requires some explanation. First of all, we do not care about the “correct” sign for the kinetic term, because our scalar \( \chi \) is not the “genuine” (i.e., fundamental) one. Some part of it we have already “used” as the created particles. The residual part can be viewed as the vacuum fluctuations that consist of virtual particles, including the conformal anomaly, which is responsible for the creation process. Moreover, the “wrong” sign in the kinetic term means the absence of the lower bound for the energy and allows even infinite number of the created particles (let us remember the C-field in the “steady state” cosmological model by [24]). Besides, we are not going to consider our field \( \chi \) as an independent dynamical variable. One more thing. If the scalar field \( \chi \) is an independent dynamical variable, then, why it “knows” about the conformal transformation \( g^{\mu\nu} = \Omega^2 \hat{g}^{\mu\nu} \) and adjusts itself properly, i.e., \( \chi = \frac{\hat{\chi}}{\Omega} \)? Only, when this field is a part of it! Fortunately, in our case it is not so, and one can always choose the conformal factor,
Ω = φ, in such a way that \( x = \frac{1}{l} \), where \( l \) is some factor having dimension of length (it is introduced in order to keep the action integral dimensionless). Then, the action integral for the scalar field takes the form

\[
\delta S_{\text{scalar}} = -\frac{1}{2} \int \left( \frac{1}{2} \phi'' \phi + \frac{\hat{R}}{12} \phi^2 + \frac{1}{2} m^2 \phi^4 \right) \sqrt{-\hat{g}} \, dx .
\] (13)

There appears the self-interaction term, \( \phi^4 \). It must be noted that the power 4 in this term is only in the case of the four-dimensional space-time (it depends on the space-time dimensions). Here, two comments are in order. First, the above action is covariant under the conformal transformation

\[
\varphi(\text{new}) = \hat{\Omega} \varphi(\text{old}) , \quad g^\mu (\text{old}) = \hat{\Omega}^2 g^\mu (\text{new}) , \quad \sqrt{-g(\text{old})} = \hat{\Omega} \sqrt{-g(\text{new})} ,
\]

what can be easily checked. Second, it is now evident, that \( 3m^2 = A \) plays the role of the (bare) cosmological term.

To finish this Section we write down the energy-momentum tensor \( T_{\mu\nu} \) for our (new) scalar field \( \varphi \) obtained by varying \( S_{\text{scalar}} \) in \( \hat{g}_{\mu\nu} \):

\[
\hat{T}_{\mu\nu} = -\frac{1}{2} \phi \phi_{,\mu} + \frac{1}{2l^2} \phi'^2 \phi_{,\mu} \hat{g}_{\mu\nu} + \frac{1}{2l^2} m^2 \phi^2 \hat{g}_{\mu\nu} - \frac{1}{6l^2} \left( \phi'' - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} - 2 \left( \phi' \hat{g}_{\mu\nu} \right) \right) .
\] (14)

Note the appearance of the second derivatives. The trace of this tensor equals

\[
Tr[\hat{T}_{\mu\nu}] = -\frac{1}{l^2} \left( \phi'^2 - \frac{\hat{R}}{6} \hat{g}'\hat{g} - 2m^2 \phi^4 \right) .
\] (15)

4. Hydrodynamics and conformal covariance

Since we consider now the conformal factor \( \phi \) and transformed metric tensor \( \hat{g}_{\mu\nu} \), as the independent dynamical variables, the above-written hydrodynamical action integral should be properly “updated”. Let us start with analyzing the “creation law”,

\[
\left( nu^\mu \right)_\mu - \beta \hat{N}^2 = \left( nu^\mu \right)_\mu - \beta \hat{N}^2 \sqrt{-\hat{g}} .
\] (16)

It is well known that in the four-dimensional space-time the combination \( C^2 \sqrt{-g} \) is invariant under conformal transformation, i.e., \( C^2 \sqrt{-g} = \hat{C}^2 \sqrt{-\hat{g}} \). So should be the full derivative \( \left( nu^\mu \right)_\mu \). The square of the interval \( ds^2 \) transforms as

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \phi^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu = \phi^2 dx^2 .
\] (17)

Therefore, the four-velocity behaves as \( u^\mu = \frac{dx^\mu}{ds} = \frac{u^\mu}{\phi} \), and, respectively, \( u_\mu = g_{\mu\nu} u^\nu = \phi \hat{u}_\mu \). Thus \( n\sqrt{-g} u^\mu = n\phi \sqrt{-\hat{g}} \hat{u}^\mu = \hat{n} \hat{u}^\mu \), where we introduced the new notation \( \hat{n} = n\phi \). \( \sqrt{-\hat{g}} \).

It is clear that in the comoving coordinate system \( \hat{n} \) is nothing but the particle number per unit spatial coordinate volume, and, thus, the conformally invariant quantity. So, the “creation law” does not contain the conformal factor \( \phi \) explicitly. Therefore, the hydrodynamical part of the total action integral becomes now
\[ S_{\text{hydro}} = -\int E \left( X, \frac{\hat{n}}{\sqrt{-g}} \right) \phi^4 \sqrt{-g} dx + \int \lambda_0 \left( \hat{u}^\mu \hat{u}_\mu - 1 \right) \phi^4 \sqrt{-g} dx + \int \lambda \left( \hat{n} \hat{u}^\mu \right)_\mu - \beta \mathcal{C}^2 \sqrt{-g} dx + \int \lambda_2 X, \hat{u}^\mu \phi^3 \sqrt{-g} dx. \]  

(18)

and now the hydrodynamical variables are \( \hat{n}, \hat{u}^\mu \) and. Let us write down the corresponding equations of motion

\[ \frac{\delta S_{\text{hydro}}}{\delta \hat{n}} = -\frac{\partial E}{\partial \hat{n}} \frac{\phi^4}{\sqrt{-g}} - \lambda_\alpha \hat{u}^\alpha = 0, \]

(19)

\[ \frac{\delta S_{\text{hydro}}}{\delta \hat{u}^\alpha} = 2 \lambda_0 \hat{u}_\mu \phi^3 + \lambda_2 \phi^3 X, \hat{u}_\mu - \lambda, \phi^3 \phi^3 \frac{\phi^3}{\sqrt{-g}} = 0, \]

(20)

\[ \frac{\delta S_{\text{hydro}}}{\delta X} = -\frac{\partial E}{\partial X} \frac{\phi^3}{\sqrt{-g}} - \left( \lambda_0 \phi^3 \sqrt{-g} \phi^3 \right)_\alpha = 0. \]

(21)

To these we should add, of course, the constraints that follow from variation of the action integral in Lagrange multipliers \( \lambda_0, \lambda_1 \) and \( \lambda_2 : \hat{u}^\mu \hat{u}_\mu = u^\alpha u_\alpha = 1, X, \hat{u}^\alpha = X, u^\alpha = 0, \phi^3 \phi^3 = \beta \mathcal{C}^2 \sqrt{-g} \). The last of them being equivalent to \( \left( nu^\mu \right)_\mu = \beta \mathcal{C}^2 \sqrt{-g} \). The above equations of motion can be written in terms of the quantities without “hats”. Namely,

\[ -\frac{\partial E}{\partial \hat{n}} - \lambda_\alpha u^\alpha, 2 \lambda_0 u_\mu + \lambda_2 X, \hat{u}_\mu - \lambda, u_\mu = 0, \frac{\partial E}{\partial X} + \left( \lambda_0 u^\alpha \right)_\alpha = 0. \]  

(22)

It is not difficult to extract the Lagrange multiplier \( \lambda_0 \) from these equations. Indeed, by making the convolution of the second of the equations with the four-velocity vector \( u^\mu \) and using the constraints, we get, after comparing the results with the first of the equations, that \( 2 \lambda_0 = -n \frac{\partial E}{\partial \hat{n}} \). Then, introducing the pressure \( p \) in the usual way, \( p = -\epsilon + \frac{n \partial E}{\partial \hat{n}} \), one obtains \( 2 \lambda_0 = -(\epsilon + p) \). The next step is to compute the hydrodynamical part of the total energy-momentum tensor. Omitting the details, we present here the result:

\[ \hat{T}_{\mu \nu}^{\text{hydro}} = (\epsilon + p) \phi^4 \hat{u}^\mu \hat{u}^\nu - p \phi^4 \hat{g}_{\mu \nu} - 4 \beta \left( \lambda_0 \hat{C}_{\mu \nu \rho} \right)^{\nu \rho} + \frac{1}{2} \lambda_2 \hat{C}_{\mu \nu \rho} \hat{R}^{\rho \sigma}, \]  

(23)

with the trace, equal to

\[ \text{Tr} \left[ \hat{T}_{\mu \nu}^{\text{hydro}} \right] = (\epsilon - 3p) \phi^4. \]  

(24)

In result, the total trace equals

\[ \text{Tr} \left[ \hat{T}_{\mu \nu}^{\text{hydro}} \right] = -\frac{1}{l^2} \left( \phi^{\nu \nu} - \frac{\hat{R}}{6} \phi^2 - 2m^2 \phi^4 \right) + (\epsilon - 3p) \phi^4. \]  

(25)

Finally, let us write the result of the variation of the total action integral in \( \phi \), which can be considered as one of the equations of motion as well as the consequence of the postulated conformal invariance. One gets:

\[ \frac{1}{l^2} \left( \phi^{\nu \nu} - \frac{1}{6} \hat{R} \phi - 2m^2 \phi^4 \right) + (\epsilon - 3p) \phi^4 = 0, \]  

(26)

as it should be: \( \text{Tr} \left[ \hat{T}_{\mu \nu}^{\text{hydro}} \right] = 0 \). The relation \( \hat{T}_{\mu \nu} = \phi^\nu \hat{T}_{\mu \nu} \) can be also easily verified. This proves the self-consistency of our model. Note that in no way \( \phi \) can be zero value, since this would lead to to the degeneracy of the whole space-time, i.e., actually, to its disappearance.
Introducing new notations, $6\ell^2 = 8\pi G$ and $3m^2 = \Lambda$, we are able to write our equations in a more familiar form (without “hats”, corresponding to $\phi = 1$).

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - A g_{\mu\nu} = 8\pi G T_{\mu\nu}^{\text{hydro}}.$$  \hspace{1cm} (27)

These equations look like the ordinary Einstein equations with a cosmological constant, but now the hydrodynamical energy-momentum tensor is modified by the presence of terms originated from the “creation law”, namely

$$T_{\mu\nu}^{\text{hydro}} = (\varepsilon + p) u\mu u\nu - pg_{\mu\nu} - 4\beta B_{\mu\nu} \left[ \lambda_1 \right],$$  \hspace{1cm} (28)

where

$$B_{\mu\nu} \left[ \lambda_1 \right] = \left( C_{\mu\nu\alpha\beta} \right)^{\lambda_1} + 2 \lambda_1 \mathcal{C}_{\mu\nu\alpha\beta} R^{\lambda_1},$$  \hspace{1cm} (29)

for $\lambda_1 = 1$ it is just the Bach tensor $B_{\mu\nu}$. Note, that $B_{\mu\nu} = \phi^2 B_{\mu\nu}$ and $\tilde{T}_{\mu\nu} = \phi^2 T_{\mu\nu}$. It can be checked that derived equations are conformally covariant, i.e., if one makes the conformal transformation $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = g_{\mu\nu} = \left( \phi^2 \Omega^2 \right) \hat{g}_{\mu\nu}$, then the equations written in terms of $= \left\{ \phi \Omega, \hat{O} \right\}$ the same as for $= \left\{ \phi \Omega, \hat{O} \right\}$.

5. Vacuum equations and empty vacuum

By the empty vacuum we mean the space-time both without particles (i.e., $\varepsilon = p = 0$) and without particle creation (i.e., $C^2 = 0$). Then, in the case of spherical symmetry the Weyl tensor itself is also zero, $C_{\mu\nu\alpha\beta} = 0$ [22]. The vacuum equations become simply the Einstein equations with the cosmological term. The only solution is the (anti-) de Sitter ones. It can be shown that there are no other spherically symmetric vacuum metrics (i.e., conformal to (anti-) de Sitter). The detailed calculations will be published elsewhere. The gravitational equations now look as follows

$$32\pi G \beta B_{\mu\nu} \left[ \lambda_1 \right] = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - A g_{\mu\nu}.$$  \hspace{1cm} (30)

The equation for scalar field (=conformal factor) becomes

$$\psi^\mu = \frac{1}{6} \phi \left( \hat{R} + 4A \phi^2 \right) = - \frac{\partial V_{\text{eff}}}{\partial \phi},$$  \hspace{1cm} (31)

where $V_{\text{eff}}$ – the effective potential and $\hat{R}$ is the scalar curvature constructed of the conformally transformed metric $\left( ds^2 = \phi^2 ds^2, d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu \right)$, we will call it the “reference metric”. For this reference metric it is more convenient to choose the solution of the vacuum equation, corresponding to $\phi = 1$ (i.e., without “hats”). The equation (24) for the scale factor becomes now

$$\psi^\mu + \frac{2}{3} \phi \left( 1 - \phi^2 \right).$$  \hspace{1cm} (32)

Then, due to the tracelessness of the “Bach tensor” $B_{\mu\nu} \left[ \lambda_1 \right]$ we have immediately $\hat{R} = -4A$, and

$$V_{\text{eff}} = V_0 - \frac{2}{3} A \left( \frac{\phi^2}{2} - \frac{\phi^4}{4} \right),$$  \hspace{1cm} (33)

where $V_0$ is an arbitrary constant. We see that for the negative cosmological constant ($\Lambda < 0$) it is just the famous Ginzburg-Landau potential (Higgs potential), while for $\Lambda > 0$ it is turned upside down.
It is evident that for \( \Lambda < 0 \) we have the (stable) minima at \( \phi = \pm 1 \) and the (unstable) maximum at \( \phi = 0 \) (forbidden in our model). Respectively, we for \( \Lambda > 0 \) we have the (unstable) maxima at \( \phi = \pm 1 \) and the (stable) minimum at \( \phi = 0 \). The instability in the case of the cosmological term means that such a vacuum can exist only one single moment, the moment of the creation “from nothing”. And then the universe starts filling with some matter (particles). In the case of the negative cosmological term, the vacuum is stable, but the infinite volume prevents it from the creation “from nothing”.

6. Vacuum equations and Dirac sea (dynamical vacuum)

We found that there may exist much more general vacuum solutions when the requirement \( C^2 = 0 \) is removed. Namely, the requirement \( C^2 = 0 \) may appear too restrictive (e.g., in the case of the baby universe creation in the dilatonic inflation [25-27]). Instead, one can imagine the particle creation of both positive and negative energies. This is something like the Dirac sea. For the vacuum solutions they must compensate each other. One should not be afraid of fluctuations having negative energies above the vacuum state. Due to the self-antigravitation they will be gone away, while those with positive energies will undergo the usual gravitational instability and form the structures. Thus we need two (instead of one) parts of hydrodynamical action with, correspondingly, two sets of dynamical variables (labeled by “\( \pm \)”). Let us write down the corresponding equations of motion

\[
\left( E_{(\pm)} + P_{(\pm)} \right) u^{(\pm)}_{\mu} + \lambda^{(\pm)}_{2} X^{(\pm)}_{\mu} - n_{(\pm)} A^{(\pm)}_{\mu} = 0, \quad (34)
\]

\[
\frac{\partial E_{(\pm)}}{\partial X^{(\pm)}} \left( \lambda^{(\pm)}_{2}, u^{(\pm)}, n^{(\pm)} \right)_{\sigma} = 0. \quad (35)
\]

In the vacuum, exactly as in the Dirac sea, from \( e_+ = -e_- \) and \( n_+ = n_- \), it follows that \( p_+ = -p_- \). Since there must be no energy or particle number flows in the vacuum, we get

\[
u^{(\pm)} = u^{(\pm)}, \quad (36)
\]

i.e., the trajectories of these two types of “matter” are the same. For this reason, the auxiliary variables \( X^{(\pm)} \) are also the same. Therefore, the second (scalar) equation of motion (35) gives us

\[
X^{(\pm)} = X^{(\pm)} \Rightarrow \lambda^{(\pm)}_{2} = -\lambda^{(\pm)}_{2}, \quad (37)
\]

and from the first (vectorial) equation of motion (34) it follows that

\[
\lambda^{(\pm)}_{\mu} = -\lambda^{(\pm)}_{\mu}. \quad (38)
\]

The Lagrange multiplier \( \lambda^{(\pm)}_{\sigma} \) will enter as a sum in our vacuum equation, so

\[
\lambda^{(\pm)}_{\sigma} - \lambda^{(-)}_{\sigma} = \text{const}. \quad (39)
\]

Finally, we obtain the following equation for what can be called “the dynamical vacuum”
\begin{align}
4\alpha_0 B_{\mu\nu} + \frac{1}{16\pi G} G_{\mu\nu} - \frac{A}{16\pi G} g_{\mu\nu} = 0, \quad (40)
\end{align}

where

\begin{align}
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,
\end{align}

is the Einstein tensor and

\begin{align}
B_{\mu\nu} = C_{\mu\nu\alpha\beta} \tilde{\alpha} \tilde{\beta} + \frac{1}{2} C_{\mu\nu\alpha\beta} R_{\alpha\beta} \quad (42)
\end{align}

is the Bach tensor. The equation (40) appeared unexpectedly elegant.

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