Extended Structures in Topological Quantum Field Theory

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An n dimensional quantum field theory typically deals with partition functions and correlation functions of n dimensional manifolds and quantum Hilbert spaces of n − 1 dimensional manifolds. One of the novel ideas in topological field theories is to extend these notions to manifolds of dimension n − 2 and lower. Such extensions inevitably lead to the introduction of categories. These ideas are very much “in the air”. Some of the people involved are Kazhdan, Segal, Lawrence, Kapranov, Voevodsky, Crane, and Yetter. Mostly this has been considered for 3 dimensional theories, but recently such ideas have also appeared in relation to the 4 dimensional Donaldson invariants (see [Fu], for example). Our motivation comes from a detailed understanding of classical topological field theories, which we also extend to manifolds of codimension two and higher. In the particular case of gauge theory with finite gauge group we define extensions of the usual “path integral” for the extended classical theory [F1]. For an n-manifold this is the usual path integral, and for an (n − 1)-manifold we recover the quantum Hilbert space. The result of this integration for an (n − 2)-manifold is a 2-Hilbert space.

In this note we briefly explain the consequences of this extended notion in an arbitrary 3 dimensional topological theory. The 2-Hilbert space $\mathcal{E}$ of a circle has additional structure: it is a “commutative, associative algebra with identity and involution”. This must be understood in the categorical sense, since $\mathcal{E}$ is a category. In good cases $\mathcal{E}$ can be realized as the category of representations of a quantum group. Hence we explain the appearance of quantum groups in 3 dimensional field theories in terms of our extended path integral. We refer to [F1] for more details as well as for an example: gauge theory with finite gauge group.

Reshetikhin/Turaev [RT] start with a Hopf algebra (of a certain type) and from it construct a 3 dimensional field theory. Recent work of Kazhdan/Reshetikhin starts instead with a special type of category and construct the field theory from it. That category is then the 2-Hilbert space of the circle in the resulting theory. We find these algebraic data unnatural in isolation; our purpose is to explain their introduction in terms of general properties of field theory. Our point of view is that

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they are the “solution” to a field theory, rather than a natural starting point. In the last section we discuss framed tangles, and so make more direct contact with the starting point in their work. However, except in the case of finite theories we cannot offer an alternative to their constructions of invariants.

The 3 dimensional Chern-Simons theory with positive dimensional gauge group \([W]\) is only defined projectively for oriented manifolds; there are central extensions of diffeomorphism groups which appear. Witten realized these central extensions by certain framings. We briefly discuss a different topological structure—a rigging—which realizes these central extensions. Our discussion is loosely based on some remarks in Segal [S].

### 2-Hilbert Spaces.

A finite dimensional Hilbert space over \(\mathbb{C}\) is a set \(W\) with operations of addition, scalar multiplication, and hermitian inner product:

\[
+ : W \times W \rightarrow W \\
\cdot : \mathbb{C} \times W \rightarrow W \\
(\cdot, \cdot) : W \times W \rightarrow \mathbb{C}
\]

These operations satisfy the usual axioms, which we do not list here. A 2-Hilbert space has an analogous definition, except we replace the set \(W\) by a category \(W\) and the field \(\mathbb{C}\) by the category \(\mathcal{V}\) of all finite dimensional Hilbert spaces.

A category differs from a set in that its “elements” (usually called “objects”) may have automorphisms.\(^1\) More generally, there are “morphisms” \(V \rightarrow V'\) between objects in a category. For example, the morphisms between any two inner product spaces \(V, V' \in \mathcal{V}\) are linear maps \(V \rightarrow V'\). But \(\mathcal{V}\) is much more than a category—it has a structure analogous to a (semi)ring structure on a set. Namely, there is an addition (direct sum), a multiplication (tensor product), an additive identity (the zero vector space), and a multiplicative identity (the ground field \(\mathbb{C}\)).

So a 2-Hilbert space \(W\) is a module over \(\mathcal{V}\) with an inner product. In other words, it is a category endowed with addition, scalar multiplication, and an inner product:

\[
+ : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W} \\
\cdot : \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{W} \\
(\cdot, \cdot) : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{V}
\]

These maps are functors. Of course, \(\mathcal{V}\) itself is a one dimensional 2-Hilbert space; the inner product is

\[
(V_1, V_2) = V_1 \otimes V_2.
\]

Analogous to the Hilbert space \(\mathbb{C}^n\) is the \(n\) dimensional 2-Hilbert space \(\mathcal{V}^n\) whose objects are \(n\)-tuples \((V^{(1)}, \ldots, V^{(n)})\) of inner product spaces. Note that the multiplicative identity is \((\mathbb{C}, \ldots, \mathbb{C})\); the inner product is

\[
((V_1^{(1)}, \ldots, V_1^{(n)}), (V_2^{(1)}, \ldots, V_2^{(n)})) = \bigoplus_{i=1}^{n} V_1^{(i)} \otimes V_2^{(i)}.
\]

Since a 2-Hilbert space is a category, there is an extra layer of structure beyond spaces and maps. Namely, there are maps (morphisms) between elements (objects)

\(^{1}\)We abuse notation and write ‘\(A \in \mathcal{C}\)’ for an object \(A\) in a category \(\mathcal{C}\).
of a 2-Hilbert space, and so also maps (natural transformations) between maps (functors). Thus a familiar axiom for the hermitian inner product now asserts the existence of a preferred isometry

\[(W_1, W_2) \rightarrow (W_2, W_1)\]

for any elements \(W_1, W_2\) in a 2-Hilbert space \(W\). There are new properties as well. We assume the existence of preferred maps

\[C \rightarrow (W, W) \rightarrow \mathbb{C}\]
\[(W_2, W_1) \cdot W_1 \rightarrow W_2\]

for \(W, W_1, W_2 \in W\). Note from (1) that \((W, W)\) has a preferred real structure, and we assume that (2) is compatible. The composition is a real number attached to each \(W\), which we denote \(\dim W\). The ‘·’ in (3) is scalar multiplication.

Here is a less trivial example of a 2-Hilbert space. Let \(G\) be a finite group and \(W = \text{Rep}(G)\) the category of finite dimensional unitary representations of \(G\). Addition is direct sum, scalar multiplication is tensor product (with \(G\) acting trivially on the “scalar” vector space), and the inner product is

\[(W_1, W_2) = (W_1 \otimes \overline{W_2})^G,\]

the vector space of \(G\)-invariants in \(W_1 \otimes \overline{W_2}\). (There is an additional operation—tensor products of representations—but we ignore it for now.) An orthonormal basis of \(W\) consists of a collection of representations \(W_1, \ldots, W_n\), one from each equivalence class of irreducible representations. Then for any representation \(W\) the maps in (3) give a preferred isometry

\[\bigoplus_{i=1}^n \frac{1}{\dim W_i} (W, W_i) \cdot W_i \rightarrow W.\]

Here \(\frac{1}{\dim W_i}(W, W_i)\) is the vector space \((W, W_i)\) with its inner product multiplied by \(\frac{1}{\dim W_i}\). This equation asserts that \(\{W_i/\sqrt{\dim W_i}\}\) is an “orthonormal basis” of \(\text{Rep}(G)\). If we replace the finite group \(G\) by a compact Lie group of positive dimension, then we obtain an infinite dimensional 2-Hilbert space.

Many constructions in linear algebra have analogues for 2-Hilbert spaces. These include the dual space, direct sum, and tensor product. The trace of an endomorphism of a 2-Hilbert space is a Hilbert space. For example, the “dimension” of \(\text{Rep}(G)\) is the Hilbert space \(R(G)\) of equivalence classes of representations of \(G\). (There is also a ring structure on \(R(G)\) from the tensor product of representations in \(\text{Rep}(G)\).)

The reader may object that a 2-Hilbert space resembles an integral lattice in a Hilbert space more than it does a Hilbert space. Note, however, that if \(V \in \mathcal{V}\) is a Hilbert space, and \(\mu \in \mathbb{R}^+\) a positive real number, then \(\mu V \in \mathcal{V}\) makes sense—it is the same underlying vector space \(V\) with inner product multiplied by \(\mu\). Note also that \(\mu V\) is isometric to \(V\). We can extend this multiplication formally to complex numbers with nonzero phase, and then allow these scalars in 2-Hilbert spaces. From this point of view there is no natural lattice in \(\text{Rep}(G)\), since we have nothing to fix the scale of the inner product in an irreducible representation. In this regard there is an isometry (4) with \(W_i\) replaced by \(\mu_i W_i\) for any positive scalars \(\mu_i\). The map in (4) then depends on the \(\mu_i\).
Topological Field Theories.

An $n$ dimensional topological field theory typically consists of assignments

\begin{align}
Y^{n-1} & \mapsto E(Y) \\
X^n & \mapsto Z_X
\end{align}

of a finite dimensional Hilbert space $E(Y)$ to a closed $(n-1)$-manifold, and a “path integral” $Z_X \in E(\partial X)$ to a compact $n$-manifold. In most theories the manifolds carry additional topological structure, such as an orientation. Symmetries of the manifolds which preserve the extra structure are implemented as symmetries on the corresponding objects in the field theory. Thus symmetries of $Y$ act as unitary transformations on $E(Y)$ and symmetries of $X$ leave $Z_X$ invariant. Oppositely oriented manifolds map to the conjugate object. Most importantly, there is a gluing law for pasting together components of the boundary of an $n$-manifold. The axioms are spelled out in [A1] (cf. [FQ], for example). They capture the gross structure common to all topological theories; specific theories have more detailed structure, of course.

The consequences of these axioms in a 2 dimensional topological field theory are standard, and provide a good warmup to the 3 dimensional case. Let $E = E(S^1)$ denote the quantum Hilbert space of the circle. Up to isotopy there is only one orientation-reversing diffeomorphism of $S^1$, and it determines an antilinear map

\[ c: E \rightarrow E. \]

Note that $c^2 = \text{id}$ since any orientation-preserving diffeomorphism of $S^1$ is isotopic to the identity. So $c$ determines a real structure on $E$. The path integral $Z_X$ over any surface $X$ is real. The path integral $Z_P$ over the pair of pants $P$ determines a multiplication

\[ \circ: E \otimes E \rightarrow E, \]

and $Z_{D^2}$ acts as the identity, where $D^2$ is the disk. One easily shows that the trilinear form

\[ x \otimes y \otimes z \mapsto (x \circ y, c(z))_E, \quad x, y, z \in E, \]

is symmetric, and that the multiplication is associative. It follows that $E$ is semisimple. Conversely, let $E$ be a commutative, associative algebra $E$ with identity, compatible real structure, and compatible inner product. Then we can construct a 2 dimensional field theory whose associated algebra is $E$.

We now extend the assignments (5) to codimension two manifolds in an $n$ dimensional theory. As mentioned in the introduction, we understand this extension in terms of a generalized “path integral” based on an extended classical theory. The result is that to a closed $(n-2)$-manifold $S$ we assign a 2-Hilbert space $E(S)$:

\[ S^{n-2} \mapsto E(S). \]

There are “higher” notions for lower dimensional manifolds, but they will not concern us here. The theory also assigns to an $(n-1)$-manifold $Y$ with boundary an element $E(Y) \in E(S)$. We postulate a gluing law in terms of the inner product on $E(S)$ for $(n-1)$-manifolds pasted together along $S$. We also assume that symmetries of $S$ act on $E(S)$ by unitary transformations, and further that homotopies...
of symmetries act as natural transformations (which respect the 2-Hilbert space structure.) In particular, this implies that $\pi_1 \text{Diff}(S)$ acts as unitary automorphisms of the identity$^2$ on $\mathcal{E}(S)$. Compare with the standard assertion that for a closed $(n-1)$-manifold $Y$ there is an action of $\pi_0 \text{Diff}(Y)$ by unitary transformations of $E(Y)$.

Our main interest is in 3 dimensional theories of oriented manifolds. We examine the consequences of the axioms for the structure of the 2-Hilbert space $\mathcal{E} = \mathcal{E}(S^1)$, where $S^1$ is the standard oriented circle. (In the next section we discuss some modifications necessary in a projective theory, for example the Chern-Simons theory with positive dimensional compact gauge group.) First, $\pi_1 \text{Diff}^+(S^1) \cong \mathbb{Z}$ and the positive generator, represented by a loop of rotations, acts as an automorphism of the identity on $\mathcal{E}$. That is, for each $W \in \mathcal{E}$ there is a morphism

$$\theta_W : W \rightarrow W$$

which commutes with all morphisms in $\mathcal{E}$. Next, reflection induces a conjugation $c : \mathcal{E} \rightarrow \mathcal{E}$ as in (6), and we denote

$$c(W) = W^*.$$

Since $c^2 = \text{id}$ we have $(W^*)^* = W$. Then $\theta_{W^*} = \theta_W^*$ follows from an equation in $\text{Diff}(S^1)$ relating rotations and reflections.

For any connected compact oriented 1-manifold $S$ there is a unique isotopy class of orientation-preserving diffeomorphisms $S \rightarrow S^1$. In a 2 dimensional theory that suffices to identify the Hilbert spaces assigned to $S$ and $S^1$ uniquely. In a 3 dimensional theory, however, different diffeomorphisms $S \rightarrow S^1$ yield different isometries $\mathcal{E}(S) \cong \mathcal{E}(S^1)$, although there is an isometry (natural transformation) between any two such isometries. The latter depends nontrivially on a choice of isotopy, due to the nontrivial $\pi_1 \text{Diff}^+(S^1)$. Hence we must make a rigid convention to identify different circles. Namely, we require any circle $S$ to appear in the complex line $\mathbb{C}$ and have its center in $\mathbb{R} \subset \mathbb{C}$. Then there is a unique composition of a translation and a dilation which identifies $S$ with the standard circle $S^1 \subset \mathbb{C}$. The reflection which induces duality is reflection in the real axis. We also standardize surfaces diffeomorphic to a disk with a finite number of smaller disjoint disks removed. Such surfaces embed in $\mathbb{C}$ with standardized boundaries.

Now $\mathcal{E} = \mathcal{E}(S^1)$ has a structure analogous to the real commutative associative algebra structure (with compatible inner product) on $E(S^1)$ in a 2 dimensional theory. The path integral $Z_P$ on the standardized pair of pants $P$ determines a multiplication

$$\odot : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}.$$  

The “reality” statement is a preferred isometry

$$(W_1 \odot W_2)^* \cong W_1^* \odot W_2^*$$

$^2$If $\mathcal{E} = \text{Rep}(G)$ then a unitary automorphism of the identity is multiplication by a phase on each irreducible representation.
for all $W_1, W_2 \in \mathcal{E}$. The path integral $Z_{D^2} = 1 \in \mathcal{E}$ over the standard disk $D^2 \subset \mathbb{C}$ acts as an identity for the multiplication in the sense that there are preferred isometries

$$1 \circ W \cong W \circ 1 \cong W$$

for all $W \in \mathcal{E}$. The associativity is a natural isometry

$$(10) \quad \varphi_{W_1, W_2, W_3}: (W_1 \circ W_2) \circ W_3 \rightarrow W_1 \circ (W_2 \circ W_3)$$

which satisfies the “pentagon relation”. The isometry (10) is constructed from the gluing law, and the pentagon follows from cuttings and pastings of the surface $Q$ which is a disk with 3 interior disks removed. Also, the braiding diffeomorphism $\beta: P \rightarrow P$ of the pair of pants induces a natural isometry

$$R_{W_1, W_2}: W_1 \circ W_2 \rightarrow W_2 \circ W_1.$$

Equations in $\text{Diff}^+(Q)$ imply two “hexagon relations”, while the relation

$$(\theta_{W_2} \circ \theta_{W_1}) \circ R_{W_1, W_2} = R_{W_2, W_1}^{-1} \circ \theta_{W_1} \circ \theta_{W_2}$$

follows from an equation in $\text{Diff}^+(P)$.

This is a taste of what may be extracted from the functoriality and the gluing laws. See [F1] for more details. I imagine there is an appropriate semisimplicity statement which can be made as well.

Category enthusiasts may prefer to regard $\mathcal{E}$ as an abelian category endowed with extra structure—monoidal structure (9), duality or rigidity (8), balancing (7), and braiding (10). If we are also given a fiber functor, that is a functor $\mathcal{E} \rightarrow \mathcal{V}$ which preserves the monoidal structure, then $\mathcal{E}$ can be recognized as the category of representations $\text{Rep}(A)$ of a quasitriangular quasi-Hopf algebra $A$, also known as a quantum group. In [F1] we construct an obvious fiber functor for the Chern-Simons theory with finite gauge group, and so recover Hopf algebras which appeared previously in this context. In general, a field theory does not construct a fiber functor, and they do not exist for all theories.\(^3\)

We remark that if $\mathcal{E} = \text{Rep}(A)$ and $W \in \mathcal{E}$ is an irreducible representation, then $\theta_W$ is multiplication by $e^{2\pi i h_W}$, where $h_W$ is the conformal weight corresponding to $W$.

Just as one can construct a 2 dimensional theory starting from an algebra $E$ of the appropriate sort, so too one can construct a 3 dimensional theory starting from a 2-Hilbert space $\mathcal{E}$ with multiplication, braiding, etc. A precise version of this statement is contained in recent work of Kazhdan/Reshetikhin.

\(^3\)For a finite $\sigma$-model into a space with $n$ points, the 2-Hilbert space $\mathcal{E}$ is $\mathcal{V}^n$, which does not admit a functor $\mathcal{V}^n \rightarrow \mathcal{V}$ which preserves direct sums and tensor products. (Indeed, the image of the unit object $1 = \langle \mathbb{C}, \ldots, \mathbb{C} \rangle$ must be one dimensional, but $1$ is the sum of $n$ “basis” elements.) This is related to the fact that the quantum Hilbert space of $S^2$ is $n$ dimensional. In Chern-Simons theories, on the other hand, the Hilbert space of $S^2$ is one dimensional. A related observation: The endomorphisms of the unit object $1 = \langle \mathbb{C}, \ldots, \mathbb{C} \rangle$ form a ring which is not a field. I believe that an arbitrary unitary theory decomposes into a direct sum of such “irreducible” theories—the idempotents in this ring give such a decomposition—and it is possible that fiber functors always exist for irreducible theories.
Central Extensions.

The Chern-Simons theory introduced by Witten [W] involves certain central extensions, which he realizes in terms of “2-framings”. We follow Segal [S] and instead propose that manifolds in the theory be endowed with an extra topological structure termed a “rigging”. A rigging is a trivialization of a topological invariant which for a closed oriented 4-manifold \( W \) is the signature \( \text{Sign}(W) \). There are corresponding topological invariants of 3-, 2-, and 1-manifolds which we briefly explain below. Observe that three times the signature is the Pontrjagin number \( p_1(W) \in \mathbb{Z} \). Topological invariants in lower dimensions stemming from \( p_1(W) \) are more easily constructed than those stemming from \( \text{Sign}(W) \). The difference here is one between \( K \)-theory and cohomology. Also, there are invariants of spin manifolds which stem from the \( \hat{A} \)-genus \( \hat{A}(W) \). In all three cases we can define riggings, though we focus here on the signature and associated invariants. Our constructions in this section are similar to constructions of Segal [S]. This material is preliminary as we cannot yet check all of the details.

A word about the abstractions which follow. *Torsors* and *gerbes* are concrete realizations of integral cohomology, somewhat analogous to the way that elements of \( K \)-theory are realized by vector bundles. For example, a family of \( \mathbb{Z} \)-torsors is a principal \( \mathbb{Z} \) bundle, and it has a characteristic class in the first cohomology of the parameter space. Families of higher \( \mathbb{Z} \)-gerbes have characteristic classes in higher integral cohomology.

Our starting point is the observation in [F2] that a Dirac operator on a 4-manifold \( W \) with boundary has a topological index which lives in a \( \mathbb{Z} \)-torsor which is a topological invariant of the operator on the boundary. ‘\( \mathbb{Z} \)-torsor’ is by definition ‘principal homogeneous space for the integers’. In the case of the signature operator that \( \mathbb{Z} \)-torsor \( T_{\partial W} \) has a canonical trivialization, and the signature is, of course, an integer. One can describe the \( \mathbb{Z} \)-torsor \( T_X(g) \) of a closed oriented 3-manifold \( X \) with metric \( g \) in terms of the \( \xi \)-invariant (\( \frac{1}{2} \eta \)-invariant) of Atiyah/Patodi/Singer. The \( \xi \)-invariant of the metric determines the \( \mathbb{Z} \)-torsor \( T_X(g) \) of real numbers \( x \) which satisfy \( e^{2\pi ix} = e^{2\pi i\xi} \). The \( \mathbb{Z} \)-torsor \( T_X \) is the space of sections of this bundle of \( \mathbb{Z} \)-torsors over the space of metrics; it is a topological invariant of \( X \).

Next, the exponentiated \( \xi \)-invariant of a Riemannian 3-manifold \( X \) with boundary is meant to live in the determinant circle \( C_Y \) of the boundary \( Y = \partial X \). (We now omit the metric from the notation.) This is mentioned in [S] and is currently under investigation with Dai. The determinant circle is the set of elements of unit norm in the determinant line with respect to the Quillen metric. It is a torsor for the group \( T \) of unit size complex numbers. Consider the collection of all \( \mathbb{R} \)-torsors \( \widetilde{C}_Y \) which cover the \( T \)-torsor \( C_Y \), i.e., the collection of all covering maps \( \widetilde{C}_Y \to C_Y \) compatible with the \( \mathbb{R} \) and \( T \) actions. The set of morphisms between any two such is a \( \mathbb{Z} \)-torsor, and the collection \( \mathcal{G}_Y \) of all these covering maps is an example of a \( \mathbb{Z} \)-*gerbe*. As before, we eliminate the choice of metric by working with smooth families over the space of metrics. This construction works for any closed oriented 2-manifold \( Y \). The exponentiated \( \xi \)-invariant of a compact oriented 3-manifold is a point in \( C_{\partial X} \), and using this point we can construct a particular cover \( T_X \in \mathcal{G}_{\partial X} \).

The corresponding topological invariant of a closed oriented 1-manifold \( S \) is more complicated to describe. Briefly, for each metric on \( S \) there is a self-adjoint signature operator, which is essentially two copies of the operator \( i \frac{d}{dx} \) on functions. It has discrete real spectrum extending to both \( \infty \) and \( -\infty \). Let \( A \subset \mathbb{R} \) be the
complement of the spectrum. Then for \( a, b \in A \) there is a finite dimensional Hilbert space of eigenvectors with eigenvalue between \( a \) and \( b \). Let \( C_{a,b} \) be the determinant circle of this space. These circles fit together to form a flat circle bundle over \( A \times A \). Now consider a flat circle bundle \( C \to A \) together with consistent isomorphisms \( C_a \otimes C_{a,b} \to C_b \). The collection of all such is a \( T \)-gerbe. Then the collection of liftings of this \( T \)-gerbe to \( \mathbb{R} \)-gerbes is a “2-gerbe” over \( Z \). Finally, we factor out the metric to obtain a topological invariant \( G_S \).

| dim | closed manifold | compact manifold with boundary |
|-----|-----------------|-------------------------------|
| 4   | \( \text{Sign}(W) \in \mathbb{Z} \) | \( \text{Sign}(W) \in T_{\partial X} \) |
| 3   | \( \mathbb{Z} \)-torsor \( T_X \) (lifts of \( e^{2\pi i \xi} \)) | \( T_X \in \mathcal{G}_{\partial X} \) |
| 2   | \( \mathbb{Z} \)-gerbe \( \mathcal{G}_Y \) (covers of determinant circle) | \( \mathcal{G}_Y \in \mathcal{G}_{\partial Y} \) |
| 1   | 2-gerbe \( \mathfrak{G}_S \) (covers of “determinant \( T \)-gerbe”) | |

Table 1: Topological invariants of oriented manifolds

We summarize this discussion in Table 1. Each entry is a topological invariant, which means it is functorial under orientation-preserving diffeomorphisms. The invariants also obey gluing laws, and they “change sign” when the orientation is reversed. As mentioned, the \( \mathbb{Z} \)-torsor \( T_X \) has a natural trivialization \( T_X \cong \mathbb{Z} \) when \( X \) is closed.

A rigging of an oriented manifold is a trivialization of the topological invariant in Table 1. For 4-manifolds this is meaningless, or it demands that we only consider 4-manifolds with vanishing signature. For a closed 3-manifold \( X \) a rigging is a choice of an element in \( T_X \). Recalling that \( T_X \) has a natural trivialization, this amounts to choosing an integer. The existence of a canonical element in \( T_X \) corresponds to Atiyah’s canonical 2-framing [A2]. For a closed 2-manifold \( Y \) a rigging is a choice of cover of the determinant circle bundle over the space of metrics. I believe that our definition of a rigging of a 1-manifold differs from Segal’s. Diffeomorphisms of rigged manifolds are required to preserve the rigging. Thus the group of diffeomorphisms \( \text{Diff}^+_{\text{rig}}(Y) \) of a closed oriented rigged 2-manifold \( Y \) is a central extension by \( \mathbb{Z} \) of the group of orientation-preserving diffeomorphisms \( \text{Diff}^+(Y) \). The fundamental group of rigged diffeomorphisms \( \pi_1 \text{Diff}^+_{\text{rig}}(S) \) of a rigged oriented closed 1-manifold \( S \) is a central extension by \( \mathbb{Z} \) of \( \pi_1 \text{Diff}^+(S) \). Note that the boundary of any oriented manifold is rigged. Also, riggings glue together.

A 3 dimensional (unitary) field theory of rigged oriented manifolds has the structure described in the previous section, but modified to account for the riggings. The
example we have in mind here is Chern-Simons theory with positive dimensional compact gauge group. There are homomorphisms $Z \to T$ induced by: (i) the change in the path integral over a closed 3-manifold $X$ under change of rigging; (ii) the action of the kernel of $\pi_0 \Diff^{+}_{\text{rig}}(Y) / \pi_0 \Diff^{+}(Y)$ on $E(Y)$ for a closed rigged 2-manifold $Y$; and (iii) the action of the kernel of $\pi_1 \Diff^{+}_{\text{rig}}(S) / \pi_1 \Diff^{+}(S)$ on $\mathcal{E}(S)$ (by automorphisms of the identity) for a closed rigged 1-manifold $S$. I believe that these homomorphisms are universal in a theory—that is, independent of $X, Y, S$—and that the generator maps to $e^{2\pi i c/24}$ where $c$ is the central charge of the theory. Other computations must now take into account the riggings as well. For example, I believe that it is no longer necessarily true that the square of the conjugation (8) is the identity, but I cannot yet see this in terms of riggings.

**Invariants of Framed Tangles.**

We briefly explain how a 3 dimensional topological field theory, extended as previously discussed, yields invariants of framed tangles. This is meant to make contact with the Kazhdan/Reshetikhin work. See [RT] for a precise definition of framed tangles. For simplicity we ignore riggings in this section.

A framed tangle $D$ is represented by a diagram like Figure 1, where $b$ strands intersect the line labeled 0 and $t$ strands intersect the line labeled 1. View the lines as copies of $\mathbb{C}$ and view the whole picture as embedded in $\mathbb{R}^3$. Extend the picture to $[0,1] \times S^2$ by adding a point at $\infty$ to each plane and an extra strand $[0,1] \times \{\infty\}$. Now cut out tubular neighborhoods of each strand to obtain a 3-manifold $X$. The boundary of $X$ is

$$\partial X = \{(1) \times S^2 - (t+1)D^2\} \cup \{0\} \times S^2 - (b+1)D^2\} \cup \text{(annulus at } \infty) \cup \bigcup_{\pi_0(D)} \text{(annulus or torus)}.$$

Using the framings we can identify, up to isotopy, the annuli and tori with the standard annulus $[0,1] \times S^1$ and the standard torus $S^1 \times S^1$. Now the quantum invariant $Z_X$ is an element of the Hilbert space $E(\partial X)$. We reinterpret it according to the decomposition induced by (4).

Applying the gluing law to (9) we find that $E([1] \times (S^2 - (t+1)D^2))$ is the $(t-1)$-fold tensor product

$$\odot : \mathcal{E} \otimes \cdots \otimes \mathcal{E} \longrightarrow \mathcal{E}.$$

We must choose a specific order for the multiplications since multiplication is not associative (cf. (10)). For the annuli and tori we have

$$E([0,1] \times S^1) \cong \text{(id: } \mathcal{E} \to \mathcal{E})$$

$$E(S^1 \times S^1) = V \cong \text{Trace(id: } \mathcal{E} \to \mathcal{E}).$$

This last equation defines $V$. Now we glue the annuli to the ends $\{0\} \times S^2$ and $\{1\} \times S^2$. When we glue along a circle we obtain an inner product in $\mathcal{E}$. We only have to

\[\text{In Hopf algebra terms this corresponds to asserting that the square of the antipode is not necessarily the identity.}\]

\[\text{as explained to me by Reshetikhin, who I warmly thank.}\]
be careful about orientations. We geometrically pass between the two orientations of $S^1$ via reflection, and so in the field theory via duality, according to (8).

This abstract description can be made somewhat more concrete. Notice that by (3) we can convert the inner products from the gluing into maps. Write

$$\pi_0(D) = \pi_0(D)_{\text{annuli}} \cup \pi_0(D)_{\text{tori}}.$$ 

Define maps (functors)

$$F_i: \underbrace{\mathcal{E} \otimes \cdots \otimes \mathcal{E}}_{|\pi_0(D)_{\text{annuli}}|} \to \mathcal{E}, \quad i = 0, 1,$$

by

$$F_i(A_1 \otimes \cdots \otimes A_n) = A_{i_1}^\pm \otimes \cdots \otimes A_{i_n}^\pm,$$

where $i_j$ is the position of the boundary of the $j$th annulus among the circles in $\{1\} \times S^2$, the sign is chosen according to the orientation (a minus sign is the dual), and the product on the right hand side is (11). Then the vector space obtained by gluing the annuli to the ends $\{0\} \times S^2$ and $\{1\} \times S^2$ maps to the vector space of natural transformations from $F_0$ to $F_1$. To include the tori we simply take the tensor product with $V \otimes |\pi_0(D)_{\text{tori}}|$. So, finally, the partition function $Z_X$ can be seen as an element of

$$\text{NatTrans}(F_0, F_1) \otimes V \otimes |\pi_0(D)_{\text{tori}}|.$$ 

This element is an invariant of the framed link.

This is almost the description of Kazhdan/Reshetikhin, except that they realize $V$ as the space of natural automorphisms of the trivial functor $W \mapsto 1$. This is correct if $\text{End}(1) \cong \mathbb{C}$.

As a simple example we see that the partition function of the tangle pictured in Figure 2 is a natural transformation $W \otimes W^* \to 1$. Similarly, we find a natural transformation $1 \to W \otimes W^*$. This fills a gap in [F1,§5].

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