Tweaking Ramanujan’s Approximation of $n!$

by

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Abstract: About 1730 James Stirling, building on the work of Abraham de Moivre, published what is known as Stirling’s approximation of $n!$. He gave a good formula which is asymptotic to $n!$. Since then hundreds of papers have given alternative proofs of his result and improved upon it, including notably by Burside, Gosper, and Mortici. However Srinivasa Ramanujan gave a remarkably better asymptotic formula. Hirschhorn and Villarino gave a nice proof of Ramanujan’s result and an error estimate for the approximation. In recent years there have been several improvements of Stirling’s formula including by Nemes, Windschitl, and Chen. Here it is shown (i) how all these asymptotic results can be easily verified; (ii) how Hirschhorn and Villarino’s argument allows a tweaking of Ramanujan’s result to give a better approximation; (iii) that a new asymptotic formula can be obtained by further tweaking of Ramanujan’s result; (iv) that Chen’s asymptotic formula is better than the others mentioned here, and the new asymptotic formula is comparable with Chen’s.
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1. Introduction

About 1730 James Stirling, building on the work of Abraham de Moivre, published what is known as Stirling’s approximation of \( n! \). In fact, Stirling proved that

\[
\sqrt{2\pi n} \left( \frac{n}{e} \right)^n \sim n!
\]

De Moivre had been considering a gambling problem and needed to approximate \( \binom{2n}{n} \) for large \( n \). The Stirling approximation gave a very satisfactory solution to this problem.

The problem of extending the factorial from the positive integers to a wider class of numbers was first investigated by Daniell Bernoulli and Christian Goldbach in the 1720s. In 1729 Leonhard Euler succeeded and in 1730 he proved that for \( z \) any complex number with positive real part,

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^t \, dt,
\]

where \( \Gamma(n) = (n-1)! \), for any positive integer \( n \). The name gamma function is due to Adrien-Marie Legendre.

In 1774 Pierre-Simon Laplace noticed that Stirling’s formula for \( n! \) has a generalization to the gamma function, namely that for \( x \) a positive real number,

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x.
\]

One of the most elementary
proofs of Stirling’s formula for the gamma function is by Reinhard Michel [5].

Most of the proofs in the literature of Stirling’s formula and its extensions prove that they are asymptotic by establishing an error estimate such as

\[ \Gamma(x + 1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x (1 + O(x^{-1})) . \]

In fact most of the effort goes into proving such error estimates.

In this paper we observe that once one knows that Stirling’s formula is asymptotic to \( \Gamma(x + 1) \), all of the other known asymptotic formulae can be verified trivially without the need to establish any error estimates.

In 1917 William Burnside [11] published a modest improvement on Stirling’s formula, namely \( \Gamma(x + 1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x + \frac{1}{3}} \). How modest an improvement it is can be ascertained from Table 1 below. In 1978 Ralph William (Bill) Gosper Jr. [3], published a significant improvement on Stirling and Burnside’s formulae. It was that

\[ \Gamma(x + 1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(\frac{x + 1}{2}\right)^{x+1/2} . \]

In a web post in 2002, Robert H. Windschitl [11], gave an elegant and good asymptotic approximation of \( n! \), namely that \( \Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \left(\frac{1}{x}\right)\right)^{\frac{x}{2}} \). In 2010 Gergő Nemes gave an asymptotic approximation which is almost as good as Windschitl’s but better than all the others at that time. It was that \( \Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right) \). An asymptotic formula of a different style, which is much better than Gosper’s, was published in 2011 by Cristinel Mortici [6]. It was \( \Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e} + \frac{1}{12e x}\right)^x \).

Pierre-Simon Laplace discovered what is now known as the Stirling series for the gamma function.

\[
\Gamma(x + 1) \sim e^{-x} x^{x+\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \sum_{n=5}^{\infty} \frac{a_n}{b_n x^n}\right),
\]

where the real numbers \( a_n \) and \( b_n \) are explicitly calculated in [9]. As stated in [7], “the performance deteriorates as the number of terms is increased beyond a certain value”. In Table 2 below we show how
using up to the term \(x^{-4}\) in this divergent series compares with the other approximations.

A major advance in producing an asymptotic formula for \(n!\) was made by the extraordinary Indian mathematician Srinivasa Ramanujan (1887–1920) in the last year of his life. Ramanujan’s claim, recorded in [10, p. 339], was that

\[
\Gamma(x + 1) = \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{\theta_x}{30} \right)^{\frac{1}{2}},
\]

where \(\theta_x \to 1\) as \(x \to \infty\) and \(\frac{3}{10} < \theta_x < 1\) and he gave numerical evidence for his claim.

Ramanujan’s approximation is substantially better than all those which were published in the subsequent 80 years. For example, when \(n = 1\) million, the percentage error of Ramanujan’s approximation is one million million times better than Gosper’s.

In 2013 Michael Hirschhorn and Mark B. Villarino [4] proved the correctness of Ramanujan’s claim above for positive integers. They showed that Ramanujan’s \(\theta_n\) satisfies for each positive integer \(n\):

\[
1 - \frac{11}{8n} + \frac{79}{112n^2} < \theta_n < 1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3}.
\]

Although they did not explicitly say it, it is clear from their work that

\[
\Gamma(x + 1) \sim \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2}}{30} \right)^{\frac{1}{2}},
\]

at least for positive integers. This approximation, as can be seen in Table 3, is better than all that preceded it. Indeed for \(n = 1\) million, it has a percentage error at least one million times better than each one.

In 2016 Chao-Ping Chen [2] produced an asymptotic approximation which for \(n = 1\) million has a percentage error one million times better than that of Hirschhorn and Villarino. His asymptotic approximation is

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{20}{7}} \right)^{x^2 + \frac{53}{30}}.
\]

A more detailed analysis of Hirschhorn and Villarino’s improvement on that of Ramanujan, suggests a tweaking of their approximation. That tweaking produces an approximation which is stated in Corollary 2.3 and is comparable to Chen’s for \(n = 1\) to \(n = 10,000\) and much better than Chen’s for \(n = 1\) million, as is evidenced in Table 3.

Let me make, with some hesitation, a controversial remark. Chen points out that Burnside’s approximation involves an error of order \(O(n^{-1})\), Ramanujan’s approximation involves an error of \(O(n^{-4})\), Nemes
and Windschitl’s approximations involves an error of $O(n^{-5})$, and his own approximation involves an error of order $O(n^{-7})$. But in my opinion, these statements are not very informative not only because all the approximations are asymptotic to $n!$, but also because of the following extreme example:

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{72}} \left(1 + \frac{10^{100}}{n^8}\right) \sim n!$$

and has an error of the order of $O(n^{-7})$ but is an absurdly bad approximation even for $n = 1$ million. The order estimate can be used to compare approximations for “very large” $n$, but does not tell us how large is “very large”.

### 2. The Approximations of $\Gamma(x + 1)$

As suggested in §1, once we know Stirling’s asymptotic formula for $\Gamma(x + 1)$, all of the others follow trivially. This fact is captured in Theorem 2.1.

**Theorem 2.1.** Let $f$ be a function from a subset $(a, \infty)$ to $\mathbb{R}$, where $a \in \mathbb{R}, a > 0$. If $\lim_{x \to \infty} f(x) = 1$, then $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \cdot f(x)$.

**Proof.** This follows immediately from the Stirling asymptotic approximation, namely that $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x$. □

As an immediate corollary of Theorem 2.1, we obtain that all of the other mentioned approximations are asymptotic to $\Gamma(x + 1)$. Some of these were proved by the authors only for $x$ a positive integer.

**Corollary 2.2.** For $x$ a positive real number:

1. **Burnside** [1]: $\Gamma(x + 1) \sim \sqrt{2\pi} \left(\frac{x + 1/2}{e}\right)^{x+1/2}$;
2. **Gosper** [3]: $\Gamma(x + 1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x + \frac{1}{3}}$;
3. **Mortici** [6]: $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e} + \frac{1}{12e^2}\right)^x$;
4. **Ramanujan** [10]: $\Gamma(x + 1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{\frac{1}{6}}$;
5. **Laplace** ($n$): Fix $n \in \mathbb{N}$. For $a_i, b_i \in \mathbb{N}$,

$$\Gamma(x + 1) \sim e^{-x}x^{x+\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \sum_{i=3}^{n} \frac{a_i}{b_i x^i}\right);$$

\[\text{and Windschitl’s approximations involves an error of } O(n^{-5}), \text{ and his own approximation involves an error of order } O(n^{-7}). \text{ But in my opinion, these statements are not very informative not only because all the approximations are asymptotic to } n!, \text{ but also because of the following extreme example:}

\[
\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{72}} \left(1 + \frac{10^{100}}{n^8}\right) \sim n!
\]

\[\text{and has an error of the order of } O(n^{-7}) \text{ but is an absurdly bad approximation even for } n = 1 \text{ million. The order estimate can be used to compare approximations for "very large" } n, \text{ but does not tell us how large is "very large".} \]
(vi) Nemes: $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$.

(vii) Windschitl [11]: $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \left(\frac{1}{x}\right)\right)^\frac{x}{2}$.

(viii) Hirschhorn & Villarino [4]:
$$\Gamma(x + 1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(x \frac{e}{8x^3 + 4x^2 + x + \frac{1}{30} \frac{11}{8x} + \frac{79}{112x^2}}\right)^\frac{1}{2}.$$  

(ix) Chen [2]: $\Gamma(x + 1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7} x - \frac{1}{2}}\right)^{x^2 + \frac{53}{710}}$.

Proof. In each case it is sufficient to determine the function $f$ in Theorem 2.1 and observe that $\lim_{x \to \infty} f(x) = 1$.

(i) Use $f(x) = \left(1 + \frac{1}{2x}\right)^x \left(1 + \frac{1}{2x}\right)^{\frac{1}{2}}$.

(ii) Use $f(x) = \sqrt{1 + \frac{1}{6x}}$.

(iii) Use $f(x) = \left(1 + \frac{1}{12x^2}\right)^x$.

(iv) Use $f(x) = \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}\right)^\frac{1}{2}$.

(v) Use $f(x) = \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \sum_{i=3}^{\infty} \frac{a_i}{b_i x^i}\right)$.

(vi) Use $f(x) = \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$.

(vii) Use $f(x) = (x \sinh \left(\frac{1}{x}\right))^{\frac{x}{2}}$.

(viii) Use $f(x) = \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2}}{240x^3}\right)^\frac{1}{2}$.

(ix) Use $f(x) = \left(1 + \frac{1}{12x^3 + \frac{24}{7} x - \frac{1}{2}}\right)^{x^2 + \frac{53}{710}}$.

□

In fact, at the expense of a little more complication, we can tweak Ramanujan’s approximation again to get an even better approximation for large values of $x$, which we refer to in the table below as the SAM approximation. The proof of the corollary uses an obvious modification of the proof of (ix) above.
Corollary 2.3. For $x$ a positive real number,

$$\Gamma(x+1) \sim \sqrt{\pi} \left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{11}{8x} + \frac{79}{112x^2} + \frac{A}{x^3}\right)^{\frac{1}{6}},$$

where $A = \frac{380279456577}{722091376690}$. \hfill \square

3. Numerical Analysis of the Approximations

The tables in this section were calculated using the WolframAlpha software package. (See [https://www.wolframalpha.com](https://www.wolframalpha.com/)) They demonstrate the performance of the asymptotic approximations.

Each of the approximations gets further and further from $n!$ as $n$ tends to infinity. So the quality of the approximations is best judged by considering the percentage error, that is $100 \times \frac{\text{approximation} - n!}{n!}$.

In the tables $S =$ Stirling, $B =$ Burnside, $G =$ Gosper, $L4 =$ (Laplace) Stirling series up to $x^{-4}$, $M =$ Mortici, $R =$ Ramanujan, $HV =$ Hirschhorn and Villarino, $C =$ Chen, and $SAM =$ the author of this paper.

From the tables it is abundantly clear that Gosper’s approximation is a much better approximation than Stirling’s, and Mortici’s elegant approximation is closer in accuracy to Ramanujan’s. Ramanujan’s approximation is amazingly good. The tweaking of Ramanujan’s approximation using the Hirschhorn-Villarino results significantly improves the approximation. Chen’s approximation is better than all that precede it. The SAM approximation obtained by extra tweaking of Ramanujan’s approximation produces an approximation similar to Chen’s up to $n = 10,000$ and much better for $n = 1,000,000$. 
Table 1.

| n   | n!   | S %error | B %error | G %error |
|-----|------|----------|----------|----------|
| 2   | 2    | 1.2×10^-2 | 3.3×10^-3 | 1.4×10^-2 | 1.7×10^-2 |
| 5   | 1.7×10^-3 | 7.6×10^-1 | 2.5×10^-2 |
| 10  | 8.3×10^-1 | 4.0×10^-1 | 6.6×10^-3 |
| 20  | 8.3×10^-3 | 2.0×10^-1 | 1.7×10^-3 |
| 50  | 8.3×10^-2 | 8.3×10^-2 | 2.7×10^-4 |
| 100 | 8.3×10^-1 | 4.1×10^-2 | 6.9×10^-5 |
| 10^4| 8.3×10^-3 | 4.2×10^-3 | 6.9×10^-7 |
| 10^4| 8.3×10^-4 | 4.2×10^-4 | 6.9×10^-9 |
| 10^6| 8.3×10^-6 | 4.2×10^-6 | 6.9×10^-13|

Table 2.

| n   | n!   | M %error | R %error | L4 %error | N %error |
|-----|------|----------|----------|-----------|----------|
| 2   | 2    | 1.0×10^-2 | 3.3×10^-3 | 1.4×10^-2 | 1.7×10^-3 |
| 5   | 1.2×10^-3 | 7.6×10^-1 | 2.5×10^-2 |
| 10  | 8.3×10^-1 | 4.0×10^-1 | 6.6×10^-3 |
| 20  | 8.3×10^-3 | 2.0×10^-1 | 1.7×10^-3 |
| 50  | 8.3×10^-2 | 8.3×10^-2 | 2.7×10^-4 |
| 100 | 8.3×10^-1 | 4.1×10^-2 | 6.9×10^-5 |
| 10^4| 8.3×10^-3 | 4.2×10^-3 | 6.9×10^-7 |
| 10^4| 8.3×10^-4 | 4.2×10^-4 | 6.9×10^-9 |
| 10^6| 8.3×10^-6 | 4.2×10^-6 | 6.9×10^-13|

Table 3.

| n   | n!   | W % error | HV % error | C % error | SAM % error |
|-----|------|-----------|------------|-----------|-------------|
| 2   | 2    | 1.6×10^-3 | 1.6×10^-4 | 2.2×10^-3 | 2.9×10^-4 |
| 5   | 1.9×10^-3 | 5.0×10^-7 | 6.0×10^-7 |
| 10  | 1.5×10^-6 | 4.1×10^-9 | 4.9×10^-9 |
| 20  | 5.2×10^-10 | 3.2×10^-11 | 3.8×10^-11 |
| 50  | 2.3×10^-12 | 5.3×10^-14 | 6.3×10^-14 |
| 100 | 3.6×10^-14 | 4.2×10^-16 | 4.9×10^-16 |
| 10^4| 3.7×10^-20 | 4.2×10^-23 | 4.9×10^-23 |
| 10^4| 6.2×10^-22 | 4.2×10^-30 | 4.9×10^-30 |
| 10^6| 6.2×10^-32 | 4.2×10^-44 | 1.3×10^-50 |
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