On The $b$-Chromatic Number of Regular Bounded Graphs

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Abstract

A $b$-coloring of a graph is a proper coloring such that every color class contains a vertex adjacent to at least one vertex in each of the other color classes. The $b$-chromatic number of a graph $G$, denoted by $b(G)$, is the maximum integer $k$ such that $G$ admits a $b$-coloring with $k$ colors. El Sahili and Kouider conjectured that $b(G) = d + 1$ for $d$-regular graph with girth 5, $d \geq 4$. In this paper, we prove that this conjecture holds for $d$-regular graph with at least $d^3 + d$ vertices. More precisely we show that $b(G) = d + 1$ for $d$-regular graph with at least $d^3 + d$ vertices and containing no cycle of order 4. We also prove that $b(G) = d + 1$ for $d$-regular graphs with at least $2d^3 + 2d - 2d^2$ vertices improving Cabello and Jakovac bound.

Keywords: proper coloring, $b$-coloring, $b$-chromatic number.

1 Introduction

A proper coloring of a graph $G = (V; E)$ is an assignment of colors to the vertices of $G$, such that any two adjacent vertices have different colors. The chromatic number of $G$, denoted by $\chi(G)$ is the smallest integer $k$ such that $G$ has a proper coloring with $k$ colors. A color class in a proper coloring of a graph $G$ is the subset of $V$ containing all the vertices of same color. A proper coloring of a graph is called a $b$-coloring, if each color class contains a vertex adjacent to at least one vertex of each of the other color classes. Such a vertex is called a dominant vertex. The $b$-chromatic number of a graph $G$, denoted by $b(G)$, is the largest integer $k$ such that $G$ has a $b$-coloring with $k$ colors. For a given graph $G$, it may be easily remarked that $\chi(G) \leq b(G) \leq \Delta(G) + 1$.

The $b$-chromatic number of a graph was introduced by Irving and Manlove [10] when considering minimal proper colorings with respect to a partial order defined on the set of all partitions of the vertices of a graph. They proved
that determining \( b(G) \) is NP-hard for general graphs, but polynomial-time solvable for trees.

Recently, Kratochvıl et al.\cite{11} have shown that determining \( b(G) \) is NP-hard even for bipartite graphs while Corteel, Valencia-Pabon, and Vera \cite{5} proved that there is no constant \( \epsilon > 0 \) for which the \( b \)-chromatic number can be approximated within a factor of \( 120/133-\epsilon \) in polynomial time (unless \( P = NP \)).

Finally, Balakrishnan and Francis Raj \cite{1,2} investigated the \( b \)-chromatic number of the Mycielskians and vertex deleted subgraphs. Hoang and Kouider \cite{9} characterize all bipartite graphs \( G \) and all \( P_4 \)-sparse graphs \( G \) such that each induced subgraph \( H \) of \( G \) satisfies \( b(H) = \chi(H) \). In \cite{8}, Effantin and Kheddouci gave the exact value for the \( b \)-chromatic number of power graphs of a path and determined bounds for the \( b \)-chromatic number of power graphs of a cycle.

In \cite{6}, Kouider and El-Sahili formulated the following conjecture: For a \( d \)-regular graph with girth 5, \( b(G) = d + 1 \) for \( d \geq 4 \). They proved that for every graph \( G \) with girth at least 6, \( b(G) \) is at least the minimum degree of the graph, and if this graph is \( d \)-regular then \( b(G) = d + 1 \). This conjecture have been proved in \cite{6} for the particular case when \( G \) contains no cycles of order 6. In \cite{3}, Maffray et al. proved that the conjecture holds for \( d \)-regular graphs different from the Petersen graph and \( d \leq 6 \). The \( f \)-chromatic vertex number of a \( d \)-regular graph \( G \), denoted by \( f(G) \), is the maximum number of dominant vertices with distinct color classes in a proper \( d + 1 \)-coloring of \( G \). In \cite{7}, El Sahili et al. proved that \( f(G) \leq b(G) \) and then reformulated El Sahili and Kouider conjecture as follows: \( f(G) = d + 1 \) for \( d \)-regular graph with no cycle of order 4. They proved that (i) \( f(G) \geq \left\lfloor \frac{d+1}{2} \right\rfloor + 2 \) for a \( d \)-regular graph containing no cycle of order 4; (ii) \( f(G) \geq \left\lceil \frac{d+1}{2} \right\rceil + 4 \) for \( d \)-regular graph containing no cycle of order 4 and of diameter 5; (iii) \( b(G) = d + 1 \) for a \( d \)-regular graph containing no cycle of order 4 nor of order 6; (v) \( b(G) = d + 1 \) for a \( d \)-regular graph with no cycle of order 4 and of diameter at least 6. In this paper, we prove in two different methods that El Sahili and Kouider conjecture holds for a \( d \)-regular graph containing at least \( d^3 + d \) vertices and more precisely we prove that \( b(G) = d + 1 \) for a \( d \)-regular graph containing no cycle of order 4 and at least \( d^3 + d \) vertices.
In [11], Kratochvíl et al. proved that for a $d$-regular graph $G$ with at least $d^4$ vertices, $b(G) = d + 1$. It follows from their result that for any $d$, there is only a finite number of $d$-regular graphs $G$ with $b(G) \leq d$. In [4], using matchings, Cabello and Jakovac reduced the bound of $d^4$ vertices to $2d^3 - d^2 + d$ vertices. In this paper, we show that, without using matching, $b(G) = d + 1$ for a $d$-regular graphs with $v(G) \geq 2d^3 + 2d - 2d^2$ improving Cabello and Jakovac bound.

2 Lower Bounded Graphs

Consider a $d$-regular graph $G$ and let $K$ and $F$ be 2 disjoint and fixed induced subgraphs of $G$. Suppose that the vertices of $K$ are colored by a proper $d + 1$-coloring. Also, suppose that the vertices of $F$ are colored by a proper $d + 1$-coloring $c$. We define a digraph $\Delta_c$ where $V(\Delta_c) = \{1, 2, ..., d + 1\}$ and $E(\Delta_c) = \{(i, j) : \text{a vertex of color } i \text{ in } F \text{ is not adjacent to a vertex of color } j \text{ in } K\}$. $\Delta_c$ is called a coloring digraph. Note that the coloring digraph $\Delta_c$ may contain loops and circuits of length 2. The number of loops in $\Delta_c$ is denoted by $\ell(\Delta_c)$. We introduce the following lemma:

Lemma 2.1. If $(i, i) \notin E(\Delta_c)$ and if there exists a circuit $C$ in $\Delta_c$ containing $i$, then we can recolor $V(F)$ by a proper $d + 1$-coloring $c'$ such that $\ell(\Delta_{c'}) > \ell(\Delta_c)$.

Proof. Suppose that $(i, i) \notin E(\Delta_c)$ and there exists a circuit $C$ in $\Delta_c$ containing $i$. Without loss of generality, suppose that $C = 1 \ 2 \ ... \ i$. We define a new proper coloring $c'$, where for $v \in F$

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \notin C \\ c(v) + 1 & \text{if } c(v) \in C\setminus\{i\} \\ 1 & \text{if } c(v) = i \end{cases}$$

A loop $(s, s)$ in $\Delta_c$ is clearly a loop in $\Delta_{c'}$ whenever $s \geq i + 1$. Since $(s, s + 1)$ and $(i, 1) \in E(\Delta_c)$, $1 \leq s \leq i - 1$, then $(l, l)$ is a loop in $\Delta_{c'}$, $\forall l$ $1 \leq l \leq i$. Thus, $\ell(\Delta_{c'}) > \ell(\Delta_c)$. \hfill $\Box$

Theorem 2.1. Let $G$ be a $d$-regular graph with no cycle of order 4. If $v(G) \geq d^3 + d$, then $b(G) = d + 1$
Proof. Suppose that $k$ vertices and their neighbors are colored by a proper $d + 1$-coloring in such a way that these $k$ vertices are dominant of color 1, 2, ..., $k$, $k \leq d$. Let $C$ be the set of colored vertices, then $|C| = k + kd \leq d(d + 1)$.

Let $R_i = \{v \in R : v$ has exactly $i$ neighbors in $C\}$, $0 \leq i \leq d$, and set $R = N(C) = \bigcup_{i=1}^{d} R_i$.

Let $R_a = R_2 \cup \ldots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, $R_b = R_1 \cup \ldots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, and $R_c = R_{\lfloor \frac{d+1}{2} \rfloor+1} \cup \ldots \cup R_d$.

Since dominant vertices has no neighbors in $V(G) \setminus C$ and a neighbor of a dominant vertex has at most $d - 1$ neighbors in $R$, then by double counting the edges between $C$ and $(R_1 \cup R_2 \cup \ldots \cup R_d)$ we can say that:

$$|R| + |R_a| + 2|R_b| + \left\lfloor \frac{d+1}{2} \right\rfloor |R_c| \leq d^2(d - 1) \quad (a)$$

Let

$$S_1 = \{v \notin C \cup R : |N(v) \cap R_a| \geq d - 2\}$$

$$S_2 = \{v \notin C \cup R : |N(v) \cap R_b| \geq \left\lceil \frac{d-1}{2} \right\rceil\}$$

$$S_3 = \{v \notin C \cup R : |N(v) \cap R_c| \geq 1\}$$

We have

$$(d - 2)|S_1| \leq (d - 2)|R_a|$$

$$\left\lceil \frac{d-1}{2} \right\rceil |S_2| \leq (d - 4)|R_b|$$

and

$$|S_3| \leq \left(\left\lceil \frac{d-1}{2} \right\rceil - 1\right)|R_c|$$

So,

$$|C| + |R| + |S_1| + |S_2| + |S_3| < |C| + |R| + |R_a| + 2|R_b| + \left(\left\lceil \frac{d-1}{2} \right\rceil - 1\right)|R_c|$$

But, by $(a)$, we have $\left\lceil \frac{d+1}{2} \right\rceil |R_c| \leq d^2(d - 1) - |R| - |R_a| - 2|R_b|$, thus

$$|C| + |R| + |S_1| + |S_2| + |S_3| < |C| + |R| + |R_a| + 2|R_b| + d^2(d - 1) - |R| - |R_a| - 2|R_b| \leq d(d + 1) + d^2(d - 1) \leq d^3 + d$$

Since $v(G) \geq d^3 + d$, then there exists a vertex $y$ such that $y \notin C \cup R \cup S_1 \cup S_2 \cup S_3$. We note that

$$|N(y) \cap R_a| \leq d - 3, \quad |N(y) \cap R_b| \leq \left\lceil \frac{d-1}{2} \right\rceil - 1, \quad |N(y) \cap R_c| = 0.$$  

Thus, we have

$$|N(y) \cap (R_0 \cup R_1)| \geq 3 \quad (b).$$

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Let $K$ and $F$ be two induced subgraphs of $G$ where $V(K) = C$ and $V(F) = N(y) \cup \{y\}$. Color $y$ and its neighbors by a proper $d+1$-coloring $c$ in such a way that $y$ is a dominant vertex of color $k+1$ and $\ell(\Delta_c)$ is maximal. If $\ell(\Delta_c) = d + 1$, then $y$ is a dominant vertex in a proper $d+1$-coloring for $V(K) \cup V(F)$. Else, there exists $i \neq k + 1$ such that $(i, i) \notin E(\Delta_c)$. Let $x$ be the vertex of color $i$ in $N(y)$.

There exists at least a neighbor of $y$, say $w_1$, such that $w_1$ has no neighbor of color $i$ in $K$ since $G$ has no $C_4$ and $k \leq d$. Also, by (b), there exist at least three neighbors of $y$, say $w_2, w_3$ and $w_4$, such that each of them has at most one neighbor in $K$. Since $x$ has at most $\left\lfloor \frac{d+1}{2} \right\rfloor$ neighbors in $K$ and $y \notin C \cup R \cup S_1 \cup S_2 \cup S_3$, then there exists a neighbor of $y$, say $w_5$, such that $x$ has no neighbor of color $c(w_5)$ ($c(w_5) \neq k + 1$) in $K$ and $w_5$ has at most 3 neighbors in $K$. If $x$ has no neighbor of color $c(w_1)$ in $K$, then $C_1 = i \ c(w_1)$ is a circuit in $\Delta_c$. Else, $i$ has a neighbor in $K$ of color $c(w_1)$. If there exists $j$, $2 \leq j \leq 4$, such that $x$ has no neighbor of color $c(w_j)$ in $K$, then $C_2$ or $C_3$ is a circuit in $\Delta_c$, where $C_2 = i \ c(w_j)$ and $C_3 = i \ c(w_j) \ c(w_1)$, since $w_j$ has at most one neighbor in $K$. Otherwise, neither $x$ nor $w_5$ belongs to the set $\{w_1, w_2, w_3, w_4\}$ since $x$ has no neighbor in $K$ of color $c(w_5)$ and it has more than one neighbor. If $w_5$ is not adjacent to a vertex of color $i$ in $K$, then $C_3 = i \ c(w_5)$ is a circuit in $\Delta_c$. Otherwise, there exists $j$, $1 \leq j \leq 4$, such that $w_5$ has no neighbor of color $c(w_j)$ in $K$ since $w_5$ has at most 3 neighbors in $K$. If $j=1$, then $C_4 = c(w_5) \ c(w_1) \ i$ is a circuit in $\Delta_c$. Else, $C_5$ or $C_6$ is a circuit in $\Delta_c$, where $C_5 = c(w_5) \ c(w_j) \ i$ and $C_6 = c(w_5) \ c(w_j) \ c(w_1) \ i$, since $w_j$ has at most one neighbor in $K$. In all cases, there exists a circuit containing $i$. Then, by Lemma 2.1 we can find a proper $d+1$-coloring $c'$ of $V(F)$ such that $\ell(\Delta_{c'}) > \ell(\Delta_c)$, a contradiction. Thus, $\ell(\Delta_c) = d + 1$ and so $c$ is a proper $d+1$-coloring for $V(F) \cup V(K)$ and $y$ is a dominant vertex of color $k + 1$. This proves that we can find a proper $d+1$-coloring of $G$ that contains $d + 1$ dominant vertices of distinct colors.

The coloring digraph, which is used to prove Theorem 2.1, can be used also to establish the following result improving Cabello and Jakovac bound:

**Theorem 2.2.** Let $G$ be a $d$-regular graph such that $v(G) \geq 2d^3 + 2d - 2d^2$, then $b(G) = d + 1$.

**Proof.** Suppose that $k$ vertices and their neighbors are colored by a proper $d + 1$-coloring in such a way that these $k$ vertices are dominant of color $1, 2, \ldots, k$, $k \leq d$. Define $C$, $R$ and $R_i$, $0 \leq i \leq d$, as in the proof of Theorem...
2.1. Let $C_i = \{ v \in R : v \text{ has a neighbor of color } i \text{ in } C \}$, $1 \leq i \leq d + 1$. Let $R_a = R_3 \cup \ldots \cup R_{\left\lceil \frac{d+1}{2} \right\rceil}$ and $R_b = R_{\left\lceil \frac{d+1}{2} \right\rceil+1} \cup \ldots \cup R_d$.

Dominant vertices have no neighbors in $V(G) \setminus C$ and a neighbor of a dominant vertex has at most $d - 1$ neighbors in $R$, so we can say that:

$$|R| + 2|R_a| + \left\lceil \frac{d+1}{2} \right\rceil |R_b| \leq d^2(d - 1)$$

For $k < d$, we have

$$|C_i| \leq k(d - 1) \leq (d - 1)^2 \quad 1 \leq i \leq d + 1$$

For $k = d$, since only $d - 1$ vertices of color $i$, $i \neq d + 1$, can have neighbors outside $C$ while $d$ vertices of color $d + 1$ can have neighbors outside $C$, then

$$|C_i| \leq (d - 1)^2 \text{ for } i \neq d + 1, \text{ and } |C_{d+1}| \leq d(d - 1)$$

Let

$$S_1 = \{ v \notin C \cup R : |N(v) \cap R_a| \geq \left\lceil \frac{d+1}{2} \right\rceil \}$$

$$S_2 = \{ v \notin C \cup R : |N(v) \cap R_b| \geq 1 \}$$

$$S'_i = \{ v \notin C \cup R : |N(v) \cap C_i| \geq d - 1 \}, \quad 1 \leq i \leq d + 1 \text{ such that } i \neq k + 1.$$ The union of the sets $S'_i$, $1 \leq i \leq d + 1$ and $i \neq k + 1$, is denoted by $S'$. We have

$$\left\lceil \frac{d+1}{2} \right\rceil |S_1| \leq (d - 3)|R_a|, \text{ thus } |S_1| < 2|R_a|$$

$$|S_2| = (\left\lceil \frac{d+1}{2} \right\rceil - 1)|R_b|$$

$$(d - 1)|S'_i| \leq (d - 1)|C_i|, \text{ thus } |S'_i| \leq (d - 1)^2$$

and $|S'| \leq d(d - 1)^2$.

So,

$$|C| + |R| + |S_1| + |S_2| + |S'| < |C| + |R| + 2|R_a| + (\left\lceil \frac{d+1}{2} \right\rceil - 1)|R_b| + d(d - 1)^2$$

But $\left\lceil \frac{d+1}{2} \right\rceil |R_b| \leq d^2(d - 1) - |R| - 2|R_a|$, thus

$$|C| + |R| + |S_1| + |S_2| + |S'| < d(d + 1) + d^2(d - 1) + d(d - 1)^2 \leq 2d^3 + 2d - 2d^2$$

Since $v(G) \geq 2d^3 + 2d - 2d^2$, then there exists a vertex $y$ such that $y \notin C \cup R \cup S_1 \cup S_2 \cup S'$. We note that

$$|N(y) \cap R_a| \leq \left\lceil \frac{d+1}{2} \right\rceil - 1, |N(y) \cap R_b| = 0, |N(y) \cap C_i| \leq d - 2, \quad \forall i \neq k + 1$$

(*)
Let $K$ and $F$ be two induced subgraphs where $V(K) = C$ and $V(F) = N(y) \cup \{y\}$. Color $y$ and its neighbors by a proper $d+1$-coloring $c$ in such a way that $y$ is a dominant vertex of color $k+1$ and $\ell(\Delta_c)$ is maximal. If $\ell(\Delta_c) = d + 1$, then $y$ is a dominant vertex in a proper $d+1$-coloring for $V(K) \cup V(F)$. Else, there exists $i \neq k + 1$, such that $(i, i) \notin E(\Delta_c)$. Let $x$ be the vertex of color $i$ in $N(y)$.

By (*), we can find at least 2 neighbors of $y$, say $w_1$ and $w_2$, such that $w_1$ and $w_2$ has no neighbor of color $i$ in $K$. If $x$ has no neighbor of color $c(w_j)$, $j \in \{1, 2\}$, then $C_1 = i \in c(w_j)$ is a circuit in $\Delta_c$. Otherwise, since $x$ has at most $\lfloor \frac{d+1}{2} \rfloor$ neighbors in $K$, then there exists a neighbor of $y$, say $w_3$, such that $x$ has no neighbor of color $c(w_3)$ in $K$ where $c(w_3) \neq k+1$, $w_3 \notin \{w_1, w_2\}$ and $w_3$ has at most 2 neighbors in $K$. If $w_3$ has no neighbor of color $i$ in $K$, then $C_2 = c(w_3) i$ is a circuit in $\Delta_c$. Else, $w_3$ has no neighbor in $K$ of color $c(w_k)$ where $k = 1$ or 2. So, $C_3 = c(w_3) c(w_k) i$ is a circuit in $\Delta_c$. In all cases, there exists a circuit containing $i$, then by Lemma 2.1 we can find a proper $d+1$-coloring $c'$ of $V(F)$ such that $\ell(\Delta_{c'}) > \ell(\Delta_c)$, a contradiction. Thus, $\ell(\Delta_c) = d + 1$ and so $c$ is a proper $d+1$-coloring for $V(F) \cup V(K)$ and $y$ is a dominant vertex of color $k + 1$. Consequently, we can find a $d + 1$ dominant vertices of distinct colors.

\section{Matching and $b$-coloring}

Using matching Cabello and Jacovac proved that $b(G) = d + 1$ for any $d$-regular graph with at least $2d^3 + d - d^2$ vertices. Matching also yields another proof for Theorem 2.1. This proof is based on the following Lemma:

**Lemma 3.1.** Let $t$ be a fixed integer. Let $L$ and $V$ be two sets of cardinality $t$. Let $H$ be a bipartite graph with partition $V$ and $L$ such that for every $v \in V$ and every $u \in L$, $d_H(v) + d_H(u) \geq t$. Then $H$ has a perfect matching.

*Proof.* The proof is by contradiction. Let $M$ be the maximum matching such that $M$ is not perfect. Then, there exist at least two vertices, say $u \in L$ and $v \in V$, outside $M$. Since $d_H(v) + d_H(u) \geq t$, then there exists an edge in $M$, say $ab$, such that $a \in N_H(u)$ and $b \in N_H(v)$. Then let $M'$ be the set of edges such that $M' = (M \setminus \{ab\}) \cup \{au, bv\}$. It is clear that $M'$ is a matching with $|M'| > |M|$, a contradiction. \qed
Another Proof of Theorem 2.1.

We have a partial $b$ coloring of the graph $G$, the set of $k$ dominant vertices of the colors $1, 2, \ldots, k$ and their neighbors. $C$, $R$ and $R_i$, $0 \leq i \leq d$, are defined as in the previous proof. Let $C_i = \{v \in C : v \text{ is of color } i \}$, $1 \leq i \leq d + 1$. Let $R_a = R_1 \cup R_2 \cup \ldots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, $R_b = R_3 \cup R_4 \cup \ldots \cup R_{\lfloor \frac{d+1}{2} \rfloor}$, and $R_c = R_{\lfloor \frac{d+1}{2} \rfloor + 1} \cup \ldots \cup R_d$.

Dominant vertices has no neighbors in $\overline{V(G) \setminus C}$ and a neighbor of a dominant vertex has at most $d - 1$ neighbors in $R$, so we can say that:

$$|R| + |R_2| + 2|R_b| + \lfloor \frac{d-1}{2} \rfloor |R_c| \leq d^2(d - 1) \quad (a)$$

For each $y \in G - (C \cup R)$, let us set:

$R_c(v) = N(v) \cap R_c$,

$R_b(v) = N(v) \cap R_b$,

$R_a(v) = N(v) \cap (R_0 \cup R_1 \cup R_2)$,

$S_1 = \{v \notin (C \cup R), |R_b(v)| \geq \lfloor \frac{d-1}{2} \rfloor \}$.

$S_2 = \{v \notin (C \cup R), |R_c(v)| \geq 1 \}$.

Then, we have

$$\lfloor \frac{d-1}{2} \rfloor |S_1| \leq (d - 3)|R_b|, \text{ then } |S_1| \leq 2|R_b|$$

and

$$|S_2| \leq (\lfloor \frac{d-1}{2} \rfloor - 1)|R_c|$$

Let $S_0 = \{y \in G \setminus (C \cup R \cup S_1 \cup S_2) : N(v) \cap (R_2 \cup \ldots \cup R_{\lfloor \frac{d+1}{2} \rfloor}) \geq d - 2 \}$ and $S_{0,i} = \{v \in S_0 : R_b(v) = i \}$, $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor - 1$. So, $S_0 = \cup_{0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor - 1} S_{0,i}$.

We have:

$$\sum_{0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor - 1} i |S_{0,i}| + \lfloor \frac{d-1}{2} \rfloor |S_1| \leq (d - 3).|R_b| \quad (1)$$

$$\sum_{0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor - 1} (d - 2 - i).|S_{0,i}| \leq (d - 2).|R_2| \quad (2)$$

From these 2 last inequalities, we deduce that

$$(d - 2).|S_0| + 2 \lfloor \frac{d-1}{2} \rfloor |S_1| \leq 2(d - 3)|R_b| + (d - 2)|R_2|$$

Thus, we get

$$|S_0| + |S_1| < 2|R_b| + |R_2| \quad (b)$$

Thus, by (a) and (b), we get:
\[ |C| + |R| + |S_0| + |S_1| + |S_2| < |C| + |R| + 2|R_b| + |R_2| + (\lceil \frac{d-1}{2} \rceil - 1)|R_c| \leq d^2(d-1) + d(d+1) \leq d^3 + d. \]

Since, \(|V(G)| \geq (d^3 + d)\), then we can find a vertex \(y\) such that \(y \in G - (C \cup R \cup S_0 \cup S_1 \cup S_2)\). Color \(y\) by \(k + 1\). Now, we color separately \(R_b(y)\) and \(R'_b(y)\). Let \(B\) be a subset of \(N(y)\). For any color \(j\), let \(e(C_j, B)\) be the number of edges with one extremity in \(C_j\) and the other one in \(B\). We remark that for any color \(j\)

\[ e(C_j, N(y)) \leq d - 1 \quad (*) \]

By definition of \(y\), \(|R_c(y)| = 0\), \(|R_b(y)| \leq \lceil \frac{d-1}{2} \rceil - 1\) and \(|N(y) \cap (R - R_1)| \leq (d - 3)\), so

\[ |N(y) \cap (R_0 \cup R_1)| \geq 3 \quad (**) \]

Let us note that

\[ e(C, R'_b(y)) = |N(y) \cap R_1| + 2 |N(y) \cap R_2| \]

Let 1 be a color such that \(e(C_1, R'_b(y))\) is maximum.

(1) If \(e(C_1, R'_b(y)) = |R'_b(y) \setminus R_0|\), by (*) we can choose one vertex \(w_1 \notin R'_b(y)\) not neighbor of \(c_1\) and we color it by \(c_1\). We have: \(e(C \setminus C_1, R'_b(y)) = |R'_b(y) \cap R_2| \leq |R'_b(y)| - 3\) by inequality (**).

(2) If \(e(C_1, R'_b(y)) = |R'_b(y) \setminus R_0| - 1\), we choose one vertex \(w_1 \in R'_b(y)\) not neighbor of \(c_1\) and we color it by \(c_1\). Now, by (**), \(e(C - C_1, R'_b(y) - w_1) = |R'_b(y) \cap R_2| \leq |R'_b(y) \setminus \{w_1\}| - 2\)

(3) If (1) and (2) are excluded, for any color \(j\), \(e(C_j, R'_b(y)) \leq |R'_b(y) \setminus R_0| - 2\).

Now, we are going to color \(R_b(y) \cup R'_b(y) \setminus \{w_1\}\) using colors in \(L\), where \(L = \{1, 2, ..., d + 1\} \setminus \{1, (k + 1)\}\). By definition of \(R_b(y)\), each vertex \(u\) of \(R_b(y)\) is adjacent to at most \(\lceil \frac{d-1}{2} \rceil - 1\) colored vertices in \(C\), so \(u\) is colorable by at least \(\lfloor \frac{d+1}{2} \rfloor\) colors of \(L\) in a proper coloring. Thus, we can color easily the vertices of \(R_b(y)\) by colors of \(L\) such that 2 by 2 they get different colors.

There remains a set \(L' \subset \{1, ..., d\}\) (or \(\{2, ..., d\}\)) of \(|R'_b(y) \setminus w_1|\) colors not used yet.
For each remaining color $j$, we have $e(C_j, R'_b(y)\{w_1\}) \leq |R'_b(y)\{w_1\}|-2$, and for each $u \in R'_b(y)$, $u$ has at most two colored neighbors in $C$. Let $H$ be the bipartite graph with bipartition $L'$ and $V = R'_b(y)\{w_1\}$ such that $uj$ is an edge in $H$ whenever $u$ has no neighbor of color $j$ in $G$, where $u \in V$ and $j \in L'$. Let $t = |R'_b(y)\{w_1\}|$. For each $u \in V$, $d_H(u) \geq t-2$, for each $j \in L'$, $d_H(j) \geq 2$. Thus, by lemma 3.1, there exists a perfect matching in $H$. Now, if $uj$ is an edge in the matching then color $u$ by $j$. Finally, we get a dominant vertex $y$ for the color $k + 1$. $\square$
References

[1] R. Balakrishnan, S. Francis Raj, Bounds for the b-chromatic number of the Mycielskian of some families of graphs, manuscript.

[2] R. Balakrishnan, S. Francis Raj, Bounds for the b-chromatic number of \( G - v \), manuscript.

[3] M. Blidia, F. Maffray, Z. Zemir, On b-colorings in regular graphs, Discrete Appl. Math. 157 (2009) 1787-1793.

[4] Sergio Cabello, Marko Jakovac, On the b-chromatic number of regular graphs, Discrete Applied Mathematics 159(13). (2011) 1303-1310.

[5] S. Corteel, M. Valencia-Pabon, J-C. Vera, On approximating the b-chromatic number, Discrete Appl. Math. 146 (2005) 106-110.

[6] A. El Sahili and M. Kouider, b-chromatic of regular graphs, Utilitas Math Vol.80. (2009) 211-216.

[7] A. El Sahili, H. Kheddouci , M. Kouider , M. Mortada , The b-chromatic number and f-chromatic vertex number of regular graphs, submitted to the Journal of Graph Theory.

[8] B. Effantin, H.Kheddouci , The b-chromatic number of power graphs, Discrete Mathematics and Theoretical Computer Science 6(1), 45-54 (2003)

[9] C.T. Hoang, Kouider, M., On the b-dominating coloring of graphs, Discrete Applied Maths, 152 (2005) no.1-3, 176-186.

[10] R.W. Irving, D.F. Manlove, The b-chromatic number of a graph, Discrete Appl. Math. 91 (1999) 127-141.

[11] J. Kratochvil, Zs Tuza, and M. Voigt, On the b-chromatic number of graphs, Lectures Notes in Computer Science, Springer, Berlin, 2573 (2002), 310-320.