On condition numbers of symmetric and nonsymmetric domain decomposition methods

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Abstract

Using oblique projections and angles between subspaces we write condition number estimates for abstract nonsymmetric domain decomposition methods. In particular, we consider a restricted additive method for the Poisson equation and write a bound for the condition number of the preconditioned operator. We also obtain the non-negativity of the preconditioned operator. Condition number estimates are not enough for the convergence of iterative methods such as GMRES but these bounds

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may lead to further understanding of nonsymmetric domain decomposi-
tion methods.

Keywords: Restricted Additive Schwarz, Domain Decomposition Meth-
ods, Oblique Projections.

1 Introduction

The restricted additive Schwarz (RAS) was originally introduced by Cai and
Sarkis in [3] in 1999. RAS outperforms the classical additive Schwarz (AS)
preconditioner in the sense that it requires fewer iterations, as well as lower
communication and CPU time costs when implemented on distributed memory
computers [3]. Unfortunately, RAS in its original form is nonsymmetric, and
therefore the conjugate gradient (CG) method cannot be used. Pursuing the
analysis of RAS, several interesting methods have been developed. Some of
these versions have been completely or partially analyzed and some of them
outperform the classical AS. Despite of many contributions, the analysis of this
method remains incomplete.

We mention some of the developments related to the RAS method. The
methods was introduced in [3]. The authors introduced the RAS as a cheaper
and faster variants of the classical AS preconditioner for general sparse linear
systems. The new method was shown to perform better that the AS according
to the numerical studies presented there (see also [2]). The authors of [3] quoted
that

...RAS was found accidentally. While working on a AS/GMRES
algorithm in a Euler simulation, we removed part of the communica-
tion routine and surprisingly the “then AS” method converged
faster both in terms of iteration counts and CPU time. We note that
RAS is the default parallel preconditioner for nonsymmetric sparse
linear systems in PETSc ...

Many works have been devoted to RAS and therefore it would be difficult to
present a complete review of them. Here we mention that in [6, 5] an algebraic
convergence analysis is presented. In [13, 2] the authors provide and extension
of RAS using the so-called harmonic overlaps (RASHO). Both RAS and RASHO
outperform their counterparts of the classical additive Schwarz variants. An
almost optimal convergence theory is presented for the RASHO. In [4], it is
shown that a matrix interpretation of RAS iteration can be related to the the
continuous level of the underlying problem. The authors explain how this inter-
pretation reveals why RAS converges faster than classical AS. Still, bounds
for of the condition number of the RAS preconditioned operator remains to be
satisfactory. In [12], a by now classical book introducing domain decomposi-
tion methods, the authors comment

To our knowledge, a comprehensive theory of this algorithm is still
missing. We note however that the restricted additive Schwarz pre-
conditioner is the default parallel preconditioner for nonsymmetric systems in the PETSc library ... and has been used for the solution of very large problems...

In this paper we re-visit the classical one-level additive method and the restricted additive method proposed by Cai and Sarkis. Inspired by these methods, we develop an abstract setting that may be useful for further understanding of nonsymmetric methods. We write a Hilbert space framework for the analysis of the classical additive method. Then we generalize this Hilbert space framework and apply this extension to write bounds for the condition numbers of several preconditioned operators where the construction of the preconditioner uses restrictions onto original subdomains (instead of restrictions to the overlapping subdomains). We present abstract results that may be useful to analyze non-symmetric domain decomposition method in general. We illustrate in particular how to use the results for a one-level restricted additive method with similar local problems as the local problems of the OBDD (Overlapping Balancing Domain Decomposition) introduced in [8]. Several other models and similar methods can be considered as well. For instance, restricted method for the elasticity equation, two-level domain decomposition method with classical or modern coarse spaces design, etc.

The rest of the paper is organized as follows. In Section 2 we review the classical domain decomposition methods results in a simple Hilbert space framework. In Section 3 we recall the classical AS one level method. In Section 4 we present the abstract analysis of symmetric methods. We first revisit the analysis for symmetric methods using projections and angles between sub-spaces. We generalize this analysis to nonsymmetric methods. In particular we apply this analysis to a special family of nonsymmetric methods. In Section 6 we define the restricted method that we analyze. In Section 7 we write a condition number estimate of the restricted method of Section 6.

2 A Hilbert space framework

Let $H$ and $\hat{G}$ be real Hilbert spaces with inner products $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, respectively. The case of complex Hilbert spaces is similar. Consider $R : H \to \hat{G}$ to be a bounded operator and denote by $\|R\|_{a,b}$ its operator norm. In domain decomposition methods literature $R$ is referred to as a restriction operator. Introduce the transpose operator $R^{T,b} : \hat{G} \to H$ defined by

$$a(R^{T,b}x, v) = b(x, Rv) \quad \text{for all } v \in H, x \in \hat{G}. \quad (1)$$

Despite of the fact that $R^{T,b}$ depends on inner products $a$ and $b$, our notation makes explicit only the dependence on $b$.

Assume there is a (closed) subspace $G \subset \hat{G}$ such that

$$E = R^{T,b}|_G \quad (2)$$
is easy to compute. The operator $E$ is known as an *extension operator*. Note also that $E : G \to H$ and that we have $E^{T,b} : H \to G \subset \hat{G}$ with

$$E^{T,b} = \Pi_{G,b}R$$

where $\Pi_{G,b} : \hat{G} \to G$ is the orthogonal projection on $G$ using the inner product $b$. We want to study the operator

$$E E^{T,b} = R^{T,b} \Pi_{G,b}R.$$  

See Section 3 for a particular example in the case of a one-level domain decomposition method. This operator is clearly symmetric and non-negative definite in the $b$ inner product. If we want $EE^{T,b}$ to be non-singular and since $\mathcal{N}(E) = \mathcal{N}(R^{T,b}) \cap G$ and $\mathcal{R}(E^{T,b}) = \Pi_{G,b}R(R)$ are $b$-orthogonal in $G$, we need to be sure $E^{T,b} = \Pi_{G,b}R$ is 1-1 or, equivalently, $E$ is onto. A sufficient condition for the symmetric operator $EE^{T,b} = R^{T,b} \Pi_{G,b}R$ to be invertible is given by the following lemma known as *stable decomposition lemma* or *Lion’s lemma* in the domain decomposition community. For the sake of completeness we show a detailed proof as it is usually presented in the domain decomposition literature; see for instance [12, Chapter 2] or [9] and references therein. We note that we do not need to refer to the space $\hat{G}$ at this moment. Later we revise some of these inequalities in a more natural way to obtain a sharper estimate.

**Lemma 1 (Lions Lemma)** Assume that there exists a bounded right inverse of $E$. That is, there exists a bounded operator $\hat{E} : H \to G$ such that $E \hat{E} v = v$ for all $v \in H$. Then, the mapping $EE^{T,b} : H \to H$ is non-singular. Moreover, we have

$$\| \hat{E} \|_{a,b}^{-2} \| v \|_a^2 \leq a(v, EE^{T,b} v) = \| E^{T,b} v \|_b^2 \leq \| E \|_{b,a}^2 \| v \|_a^2$$

for all $v \in H$.

**Proof.** Note that for $v \in H$ we have,

$$\| v \|_a^2 = a(v, v) = a(E \hat{E} v, v) = b(E v, E^{T,b} v) \leq b(\hat{E} v, \hat{E} v)^{1/2} b(E^{T,b} v, E^{T,b} v)^{1/2} = \| \hat{E} v \|_b a(v, EE^{T,b} v)^{1/2} \leq \| \hat{E} \|_{a,b} \| v \|_a a(v, EE^{T,b} v)^{1/2}.$$ 

Using this last inequality we obtain

$$\| \hat{E} \|_{a,b}^{-2} \| v \|_a^2 \leq a(v, EE^{T,b} v).$$
To obtain the upper bound we proceed as follows using properties of subordinated norm of operators,

\[ \| EE^{T,b}v \|_a^2 \leq \| E \|_{b,a}^2 \| EE^{T,b}v \|_b^2 \]
\[ = \| E \|_{b,a}^2 b(E^{T,b}v, E^{T,b}v) \]
\[ \leq \| E \|_{b,a}^2 a(v, EE^{T,b}v) \]
\[ \leq \| E \|_{b,a}^2 \| v \|_a \| EE^{T,b}v \|_a \]

and therefore \( \| EE^{T,b}v \|_a \leq \| E \|_{b,a}^2 \| v \|_a \). We also have,

\[ a(v, EE^{T,b}v) \leq \| v \|_a \| EE^{T,b}v \|_a \leq \| E \|_{b,a}^2 \| v \|_a^2. \]

This finishes the proof.

\[ \blacksquare \]

\textbf{Remark 2} Note that what it is needed is the existence of operator \( \widehat{E} : H \to G \subset \hat{G} \) such that \( T = \widehat{E} : H \to H \) is invertible. In this case we have that \( \hat{E} = ET^{-1} \) is an stable right inverse of \( E \).

If, in addition, the extension operator \( E \) comes from a restriction operator \( R \), as in \([2]\), we can state the following corollaries.

\textbf{Corollary 3} Let \( R \) be a restriction operator such that \( E = RT^{T,b}\big|_G \). Assume that there exits a bounded operator \( J_b : \hat{G} \to G \subset \hat{G} \) such that \( R^{T,b}J_bRv = v \) for all \( v \in H \). Then, the mapping \( EE^{T,b} = \Pi_G^{b}R : H \to H \) is non-singular with

\[ \| J_bR \|_{a,b} \| v \|_a^2 \leq a(v, EE^{T,b}v) = \| EE^{T,b}v \|_b^2 \leq \| R \|_{b,a}^2 \| v \|_a^2 \]

for all \( v \in H \).

\textbf{Corollary 4} Let \( R \) be a restriction operator such that \( E = RT^{T,b}\big|_G \). Assume that there exits a bounded operator \( \widehat{E} : H \to G \subset \hat{G} \) such that \( R^{T,b}\widehat{E}v = v \) for all \( v \in H \). Then, the mapping \( R^{T,b}R : H \to H \) is non-singular with

\[ \| \widehat{E} \|_{a,b} \| v \|_a^2 \leq a(v, R^{T,b}Rv) = \| Rv \|_b^2 \leq \| R \|_{b,a}^2 \| v \|_a^2 \]

for all \( v \in H \).

Let \( L : H \to \mathbb{R} \) be a bounded linear functional on \( H \). Denote by \( u \in H \) the solution of the following variational equation,

\[ a(u, v) = L(v) \quad \text{for all } v \in H. \quad (5) \]

Assuming that \( E \) is easy to compute. We see that, for the solution \( u, E^{T,b}u \) is possible to compute using this variational equation (without explicitly knowing or computing the function \( u \)). In fact, we have

\[ b(E^{T,b}u, \phi) = a(u, E\phi) = L(E\phi) = LE\phi \quad \text{for all } \phi \in G. \quad (6) \]
This equation might be easier to solve numerically than the original problem. Therefore, we can alternatively compute the solution of (5) by iteratively solving the equation,

$$EE^T b u = \tilde{L}$$

where the right hand side $\tilde{L}$ can be computed by solving (6) and applying the extension operator $E$. When implementing an iterative method to solve (7), in each iteration we have to apply the operator $EE^T b$ to a residual vector, say $r$. More precisely, we have to

1. Compute $x = E^T b r$, this can be done by solving the equation

$$b(x, \phi) = a(r, E\phi) \quad \text{for all } \phi \in H.$$

In terms of the restriction operator $R$ we have $x = \Pi_{G,b} R r$.

2. Compute $s = E x = E E^T b r$ by applying the extension operator $E$, that is $s = R^T b \Pi_{G,b} R r = E \Pi_{G,b} R r$ which is assumed possible and numerically efficient to compute.

The practicality of using the iteration depends on the possibility to inexpensively compute the right hand side $\tilde{L}$ and the number of iterations needed until convergence. The condition of $EE^T b$ gives us some information about the difficulty in solving the corresponding equation. In particular, since $EE^T b$ is symmetric (and positive-definite as we will see later), PCG could be applied. In this case the performance of the iterative procedure depends on the condition number of the associated operator equation. If we use the spectral condition number of the operator $EE^T b$, we see from Lemma [4] that

$$\kappa_{\text{spectral}}(EE^T b) \leq \|\hat{E}\|_{b,a}^2 \|E\|_{b,a}^2.$$

Then, for some iterative methods such as CG, the number of iterations for solving the equation (7) (up to a desired tolerance) will depend on $\|\hat{E}\|_{b,a}^2 \|E\|_{b,a}^2$.

In general, the condition number of an operator $T : H \rightarrow H$ is defined by

$$\kappa(T) = \|T^{-1}\|_{a,a} \|T\|_{a,a}.$$

For general iterative methods such as GMRES that could be applied to non-symmetric problems, bounds for the condition number alone are not enough for the convergence of the method. In this case we need information about the distribution of the eigenvalues. Nevertheless, condition number bounds may lead to further understanding of iterative methods applied to non-symmetric methods.

3 Classical additive method for Laplace equation

In this section we use the Hilbert space framework above to review the analysis of the classical additive method. As usual, we consider a subdomain $D$ with a
non-overlapping partition of the domain into subdomains \( \{ D_\ell \}_{\ell=1}^{N_S} \). By enlarging these subdomains an specific width \( \delta \) we obtain and overlapping decomposition \( \{ O_\ell \}_{\ell=1}^{N_S} \). For more details see [12].

Let \( H = H_0^1(D), \tilde{G} = \times_{\ell=1}^{N_S} H^1(O_\ell) \) and \( G = \times_{\ell=1}^{N_S} H_0^1(O_\ell) \subset \tilde{G} \). In this case consider
\[
 a(u, v) = \int_D \nabla u \nabla v \quad \text{for all } u, v \in H.
\]

Denoting by \( \{ v_\ell \} \) the elements of \( \tilde{G} \), we define for \( \{ v_\ell \}, \{ w_\ell \} \in \tilde{G} \)
\[
b(\{ v_\ell \}, \{ w_\ell \}) = \sum_{\ell=1}^{N_S} \int_{O_\ell} \nabla v_\ell \nabla w_\ell + \int_{\partial O_\ell} v_\ell w_\ell = \sum_{\ell=1}^{N_S} b_\ell(v_\ell, w_\ell) + b^0_\ell(v_\ell, w_\ell)
\]
where we have put \( b_\ell(v_\ell, w_\ell) = \int_{O_\ell} \nabla v_\ell \nabla w_\ell \) and \( b^0_\ell(v_\ell, w_\ell) = \int_{\partial O_\ell} v_\ell w_\ell \).

**Remark 5 (Norm boundary term)** The role of \( b^0_\ell \) is not essential and can be replaced by any other bilinear form that vanish for functions on \( G \) which makes \( b \) a positive definite bilinear form on \( G \).

Introduce also \( R : H \to \tilde{G} \) defined by \( Ru = \{ u_{\mid O_\ell} \} \). Equation [1] defining \( R^{T,b} \) corresponds to
\[
\int_D \nabla R^{T,b} \{ v_\ell \} \nabla z = b(\{ v_\ell \}, \{ z_{\mid O_\ell} \}) \quad \text{for all } z \in H.
\]
This definition implies that \( E : G \to H \) where \( E = R^{T,b}_{\mid G} \), is given by
\[
 E\{ v_\ell \} = \sum_{\ell=1}^{N_S} E_\ell v_\ell
\]
where \( E_\ell \) is the extension by zero outside \( O_\ell \) operator. To see this note that
\[
\int_D \nabla E\{ v_\ell \} \nabla z = \sum_{\ell=1}^{N_S} \int_{O_\ell} \nabla v_\ell \nabla z \quad \text{for all } z \in H.
\]
We have that \( E^{T,b} : H \to G \) is given by
\[
b(E^{T,b} u, \{ v_\ell \}) = a(u, E\{ v_\ell \}).
\]
Note that \( E^{T,b} u = \{ E_\ell^{T,b} u \} \) where \( E_\ell^{T,b} u \) solves the local equation
\[
\int_{O_\ell} \nabla E_\ell^{T,b} u \nabla v_\ell = \int_D \nabla u \nabla E_\ell v_\ell \quad \text{for all } v_\ell \in H_0^1(O_\ell).
\]
Observe that \( E_\ell^{T,b} \) can be obtained by solving a local problem. We define the additive method by \( P_{add} E^{T,b} \). Therefore,
\[
P_{add} u = E E^{T,b} u = \sum_{\ell=1}^{N_S} E_\ell E_\ell^{T,b} u = \left( \sum_{\ell=1}^{N_S} P_\ell \right) u.
\]
Here we denote $P_\ell = E_\ell E_\ell^T b$. The existence of a right inverse can be stated as follows as it is common in domain decomposition literature. In fact, to obtain the stable inverse assume:

- **Stable decomposition**: There exists a constant $C_E$ such that for all $v \in V$ there exist $v_\ell \in H_0^1(\Omega_\ell)$, $\ell = 1, \ldots, N_S$ such that $v = \sum_{\ell=1}^{N_S} E_\ell v_\ell$ and
  \[
  \sum_{\ell=1}^{N_S} b_\ell(v_\ell, v_\ell) \leq C_E^2 a(v, v). \tag{9}
  \]

- **Strengthened Cauchy inequalities.** There exits a matrix $\mu = (\mu_{\ell k})_{\ell, k}$ with $\mu_{\ell k} \leq 1$ and such that
  \[
  a(E_\ell v_\ell, E_k u_k) \leq \mu_{\ell k} b_\ell(v_\ell, v_\ell)^{1/2} b_k(v_k, v_k)^{1/2}.
  \]

The stable decomposition assumption clearly implies the existence of $\hat{E}$ and $\|\hat{E}\|_{b,a} \leq C_E$. In fact, $\hat{E}v = \{v_\ell\}$ where the functions $v_\ell$ are the ones given by the stable decomposition assumption.

By using bilinearity and vector Cauchy inequalities, this clearly implies that
\[
  a(E\{v_\ell\}, E\{v_k\}) = \sum_{\ell,k} a(E_\ell u, E_k u) \\
  \leq \sum_{\ell,k} \mu_{\ell k} b_\ell(v_\ell, v_\ell)^{1/2} b_k(v_k, v_k)^{1/2} \\
  \leq \rho(\mu) \left( \sum_{\ell=1}^{N_S} b_\ell(v_\ell, v_\ell) \right) \leq \rho(\mu) b(\{v_\ell\}, \{v_\ell\}),
\]
where $\rho(\mu)$ is the spectral radius of the matrix $\mu$ above. Then $\|\hat{E}\|_{b,a} \leq \sqrt{\rho(\mu)}$. Using this and the fact that $\|\hat{E}\|_{b,a} \leq C_E$ in Lemma 1 we have the following result.

**Corollary 6** For all $u \in V$ we have
\[
  C_E^{-2} a(u, u) \leq a(P_{add} u, u) \leq \rho(\mu) a(u, u) \tag{10}
\]
where $C_E$ is the stable decomposition constant and $\rho(\mu)$ is the spectral radius of the matrix $\mu$ above.

**Remark 7** Let us consider the case of the one level additive method setting. More levels can be analyzed in a similar way. In the one level setting, with original domains of diameter $\tau$ and overlap of size $\delta$ a usual bound for $C_E$ is given as follows by constructing a stable decomposition as follows; see [12]. Start by constructing cut functions $\eta_\ell$ such that
\[
  \eta_\ell(x) = 1 \text{ for all } x \in D_\ell, \quad \eta_\ell(x) = 0 \text{ for all } x \notin \Omega_\ell, \quad |\nabla \eta_\ell(x)| \leq C_{cut} \frac{1}{\delta}. \tag{11}
\]
Define the partition of unity function
\[ \chi_\ell = \frac{\eta_\ell}{\eta} \quad \text{where} \quad \eta = \sum_{\ell=1}^{N_3} \eta_\ell. \]

We see that
\[ \chi_\ell(x) = 1 \quad \text{for all} \quad x \in D_\ell \setminus \cup_{\ell' \neq \ell} \mathcal{O}_{\ell'}, \quad \chi_\ell(x) = 0 \quad \text{for all} \quad x \notin \mathcal{O}_\ell, \]
and therefore
\[ |\nabla \chi_\ell(x)| \leq \frac{C_{\text{pu}}}{\delta}. \tag{12} \]

We should have \( C_{\text{cut}} \leq C_{\text{pu}} \). Then define the stable right inverse by \( \tilde{E} u = \{ \chi_\ell u \} \).

Denoting \( \text{Neigh}(j) = \{ j : \mathcal{O}_j \cap \mathcal{O}_i \neq \emptyset \} \) and \( \nu = \max_j \# \text{Neigh}(j) \) \( \tag{13} \)

we have (see [12])
\[ C_E^2 \leq \nu C_{\text{pu}} \left( 1 + \frac{1}{\tau_0} \right). \]

The norm of \( E \) and \( \rho(\mu) \) are bounded by
\[ \| E \|_{b,a}^2 \leq \rho(\mu) \leq \nu. \]

In case we want to approximate the solution of problem \( (5) \), we see that \( u \) also solves the operator equation,
\[ E E^T b \ u = \left( \sum_{\ell=1}^{N_3} P_\ell \right) u = \tilde{L}. \tag{14} \]

Here \( \tilde{L} \) is obtaining by \( E \)-assembling the solutions of the local problems,
\[ b_\ell(E_T^T u, v_\ell) = L(E_T v_\ell) \quad \text{for all} \quad v_\ell \in H^1_0(\mathcal{O}_\ell). \]

The linear system \((14)\) is usually better conditioned that the original linear system \( a(u, v) = L(v) \).

4 Nonsymmetric methods obtained by changing restrictions and the inner product

We use the Hilbert space framework introduced in Section 2. Recall that we have \( H \) and \( G \subset \tilde{G} \) Hilbert spaces with inner products \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \), respectively. We also used the bounded restriction operator \( R : H \rightarrow \tilde{G} \) for the definition of the extension operator \( E \).
To develop a framework for non-symmetric methods we additionally introduce a second bi-linear form $c(\cdot, \cdot)$ defined on $\hat{G}$. Let us introduce a possibly different and bounded restriction operator $S : H \to \hat{G}$ and the transpose $S^{T,c}$ defined analogously to (1) by

$$a(S^{T,c} \phi, v) = c(\phi, Sv) \quad \text{for all } v \in H. \quad (15)$$

Define $F = S^{T,c}|_G$ as a second extension operator. As before, assume that there is an stable left-inverse for $F$, say $\hat{F}$ such that $F\hat{F}v = v$ for all $v \in H$ (that is, $\hat{F}$ is bounded in the $c$ and $b$ inner product norms). We can then conclude about $F$ and $F^{T,c}$ similar inequalities than the given before in the case $c$ is symmetric and positive definite. In particular $F^{T,c}$ is a bijective application from $H$ onto $R(F^{T,c})$.

**Corollary 8** Let $S$ be a restriction operator and $F = S^{T,c}|_G$. Assume that there exits a bounded right inverse of $F = S^{T,c}|_G$, say $\hat{F}$. Then, the mapping $FF^{T,c} = R^{T,c}\Pi_G R$ is non-singular with

$$\|\hat{F}\|^{-2}_{a,c} \|v\|_a^2 \leq a(v, FF^{T,c}v) \leq \|F\|^2_{c,a} \|v\|_a^2$$

and

$$\|\hat{F}\|^{-2}_{a,c} \|v\|_a^2 \leq a(v, S^{T,c}Sv) \leq \|S\|^2_{c,a} \|v\|_a^2$$

for all $v \in H$.

We want to study the nonsingularity of the operator $FE^{T,b} : H \to H$. Note that

$$FE^{T,b} = S^{T,c}\Pi_{G,b} R. \quad (16)$$

See (4). This operator is nonsymmetric for general bi-linear forms $b$ and $c$. This is due to the fact that $\Pi_{G,b}$ might not be symmetric in the $c$ bilinear form.

**Example 9** As a particular case of our general construction we can put $S = R$. In this case, $F = R^{T,c}|_G$ and $F^{T,c} = \Pi_G R$. We can then obtain the operators

$$FE^{T,b} = R^{T,c}\Pi_{G,b} R \quad \text{and} \quad EF^{T,c} = R^{T,b}\Pi_{G,c} R. \quad (17)$$

In Section 5.3 we obtain condition number bounds for operator $FE^{T,b}$ in this example. This will allows us to write condition number estimates for a non-symmetric method that uses local problems similar to the local problems of the OBDD in [8]. See Section 6.

**Example 10** Another particular case is when $c = b$ and $S = M^{T,b}R$ where $M : G \to \hat{G}$ is a bounded operator. Then $F = S^{T,b}|_G = (R^{T,b}M)|_G$ and $F^{T,b} = \Pi_{G,b}S = \Pi_{G,b}M^{T,b}R$. We can write

$$FE^{T,b} = R^{T,b}M\Pi_{G,b} R \quad (18)$$

and also

$$EF^{T,b} = R^{T,b}\Pi_{G,b}M^{T,b}R. \quad (19)$$
If $M(G) \subset G$ we are left with $F = EM$ and $F E^{T,b} = EM E^{T,b}$. In addition if $M(G) : G \rightarrow G$ has a bounded inverse, $EF^{T,b} = FM^{-1}F^{T,b}$. Therefore, the spectral properties of the resulting operators (18) and (19) will depend on the spectral properties of $M$. A case we explicitly mention is the case where $M$ is defined as point-wise multiplication operator in the context of Section 3.

Let us consider the example of Section 3 and select a cut-off functions $\eta^\text{cut} = \{\eta^\text{cut}_\ell\} \in \hat{G}$. Define the operator $M : \hat{G} \rightarrow \hat{G}$ by

$$M\{v_\ell\} = \{\eta^\text{cut}_\ell v_\ell\}.$$

This operator is not symmetric with respect to the $b$ bilinear form. In Section 3, $E$ is the extension by zero operator and then the extension operator $F = EM$ corresponds to extending by zero after a pointwise multiplication by $\eta^\text{cut}_\ell$ in each overlapping subdomain $O_\ell$. According to (18) and recalling the definition of $E^{T,b}$ in Section 3 we see that $FE^{T,b} = EME^{T,b} = \sum_{\ell=1}^{N_S} E_\ell \eta^\text{cut}_\ell E^{T,b}_\ell$. It needs the solution of local problems, these local solutions are then point-wise multiplied by the cut functions and after that an extension by zero and addition follows. This is exactly the RAS preconditioned operator as introduced by Cai and Sarkis in [3]. On the other hand, according to (19), $EF^{T,b}u = R^{T,b}\Pi_{G,b}M^{T,b}Ru = R^{T,b}\Pi_{G,b}M^{T,b}\{u|_{O_\ell}\}$. To compute $M^{T,b}\{u|_{O_\ell}\}$ we need to solve problem (61) and then extend the local solutions by zero. See (61) and Remark 27 later in Section 6. The local problem (61) is the local problem used in the OBDD method in [8].

A successful application of the abstract analysis developed in this paper (see Section 5) to the analysis of the operators in this example was not obtained here. This is a topic of ongoing research. The main issue is that we need results on the angle between subspaces $R(F^{T,b}) = MR(E^{T,b})$ and $R(E^{T,b})$ that are not available (see Section 5.7). In Section 6 we are able to bound the condition number of a one level method that uses local problems similar to the local problems of the OBDD. See Remark 27.

Remark 11 (Perturbation theory) Note that we can write

$$EF^{T,b} = EE^{T,b} + E(F - E)^{T,b} = EE^{T,b} + J$$

where $J = E(F - E)^{T,b}$ is a perturbation of $EE^{T,b}$ of size $\|J\|_{a,a} = \|E(F - E)^{T,b}\|_{a,a}$. Several results can be pursued of the type: If $\|J\|$ is small enough, then the operator $EF^{T,b}$ will be invertible and it is possible to estimate its condition number. We think that this approach is not practical for analyzing domain decomposition methods.
5 Condition number estimates using norms of projections

In this section we present a different analysis that may turn useful when estimating condition number of preconditioned operators (not-necessarily constructed by a domain decomposition design). We present a series of projection arguments to study nonsymmetric methods. As presented earlier, the idea is to estimate the condition number of an operator of the form $EF^T,b$ where $E,F : G \to H$ are different extension operators. In particular we are able to bound the condition numbers for the family of nonsymmetric methods presented in Section 4 where the extension operators are defined from restriction operator from $H$ to a bigger space $\hat{G} \supset G$. Before going to nonsymmetric methods we review norms of projections.

5.1 Norms of projections

We need the following definitions and results; see [11, 1, 7]. Let $X$ and $Y$ be subspaces of $G$ (or $\hat{G}$). Introduce the minimal angle between subspaces $X$ and $Y$ with respect to the inner product $b$, $\theta_b(X,Y)$ as

$$\sin(\theta_b(X,Y)) = \inf_{\|x\| = 1, x \in X} \text{dist}_b(x,Y) = \inf_{\|x\| = 1, x \in X} \sqrt{1 - \|\Pi_Y x\|^2_b}.$$  

(20)

Equivalently, we have $\sin(\theta_b(X,Y))^2 = 1 - \cos^2(\theta_b(X,Y))$ where

$$\cos(\theta_b(X,Y)) = \sup_{x \in X, y \in Y} \frac{b(x,y)}{\|x\|_b \|y\|_b} = \|\Pi_X \Pi_Y\|_{b,b} = \|\Pi_Y \Pi_X\|_{b,b}.$$  

(21)

Still equivalent, we have,

$$\sin(\theta_b(X,Y)) = \frac{1}{\|Q(X,Y)\|_{b,b}}$$  

(22)

where $Q(X,Y)$ is the (oblique) projection on $X$ along $Y$. Note that if $Y = X^\bot,b$ then $Q(X,Y) = \Pi_{X,b}$. Introduce the maximal angle between subspaces $X$ and $Y$, $\Theta_b(X,Y)$ as

$$\sin(\Theta_b(X,Y)) = \sup_{\|x\| = 1, x \in X} \text{dist}_b(x,Y)$$

$$= \sup_{\|x\| = 1, x \in X} \sqrt{1 - \|\Pi_Y b x\|^2_b} = \|\Pi_X b\|_{b,b} - \Pi_Y b\|_{b,b}.$$  

(23)

Equivalently we have $\sin(\Theta_b(X,Y))^2 = 1 - \cos^2(\Theta_b(X,Y))$ where

$$\cos(\Theta_b(X,Y)) = \inf_{x \in X, y \in Y} \frac{b(x,y)}{\|x\|_b \|y\|_b}.$$  

(21)

We also have,

$$\theta_b(X,Y) + \Theta_b(X,Y^\bot,b) = \frac{\pi}{2},$$

We also have,
Figure 1: Illustration of subspaces of $G$. In order to illustrate angles we picture $\mathcal{R}(\hat{F})$ as a cone. We also illustrate the procedure presented in the proof of Theorem 12 and the oblique projection $Q = Q(\mathcal{R}(E^T,b), \mathcal{N}(F))$.

and

$$\sin(\theta_b(X, Y)) = \cos(\Theta_b(X, Y^\perp,b)).$$  \hspace{1cm} (24)

5.2 General nonsymmetric method analysis using projections

Let $F : G \to H$ be a second extension operator. We want to study the operator $F E^T,b$. See Figure 1 for an illustration.

Theorem 12 Consider extensions operators $E$ and $F$ with stable right inverse $\hat{E}$ and $\hat{F}$, respectively. Assume the boundedness of $Q = Q(\mathcal{R}(E^T,b), \mathcal{N}(F))$, the oblique projection onto $\mathcal{R}(E^T,b)$ along $\mathcal{N}(F)$. Then, the operator $F E^T,b : H \to H$ is invertible. Moreover,

$$\| (FE^T,b)^{-1} \|_{a,a} \leq \| Q \|_{\mathcal{R}(\hat{E})} \| b,b \| \hat{E} \|_{a,b} \| \hat{F} \|_{b,a}. $$

Proof. We solve the equation

$$FE^T,b w = u.$$

Let $u \in H$ be given.
1. Define \( y = \hat{F}u \in G \). Then we readily see that \( Fy = u \). By assumption we then have
\[
\|y\|_b \leq \|\hat{F}\|_{a,b}\|u\|_a. \tag{25}
\]

2. Construct \( x \in \mathcal{R}(E^{T,b}) \) such that \( Fx = Fy = u \). Here we use the oblique projection \( Q = Q(\mathcal{R}(E^{T,b}), \mathcal{N}(F)) \). See Figure 1. In fact, \( x = Qy \). By definition of the projection \( Q \) we have \( F(y - x) = F(y - Qy) = 0 \) so that \( Fx = Fy \). We have,
\[
\|x\|_b \leq \|Q|_{\mathcal{R}(\hat{E})}\|b,b\|y\|b. \tag{26}
\]

3. Take \( w \in H \) such that \( E^{T,b}w = x \). In fact, \( w = \hat{E}^{T,b}x \). This \( w \) is the solution of the equation above since we have \( EE^{T,b}w = Fx = Fy = u \). We can bound
\[
\|w\|_a \leq \|\hat{E}\|_{a,b}\|x\|b. \tag{27}
\]
By combining the estimates in (25), (26) and (27) above we finish the proof. \( \blacksquare \)

We can now give a bound for the condition number of the operator \( FE^{T,b} \).

**Corollary 13** We have
\[
\kappa(FE^{T,b}) = \|FE^{T,b}\|_{a,a}\|(FE^{T,b})^{-1}\|_{a,a} \leq \|\hat{F}\|_{a,b}\|F\|_{b,a}\|Q|_{\mathcal{R}(\hat{E})}\|b,b\|E\|_{b,a}\|\hat{E}\|_{b,a}. \tag{32}
\]

We also have the following corollary.

**Corollary 14** If \( \mathcal{N}(F) \) is orthogonal to \( \mathcal{R}(E^{T,b}) \) (or \( \mathcal{N}(F) \subset \mathcal{N}(E) \)) then
\[
\kappa(FE^{T,b}) = \|\Pi_{\mathcal{R}(E^{T,b})}|_{\mathcal{R}(\hat{E})}\|_{b,b}\|FE^{T,b}\|_{a,a}\|(FE^{T,b})^{-1}\|_{a,a} \leq \|F\|_{b,a}\|E\|_{b,a}\|\hat{F}\|_{a,b}\|\hat{E}\|_{a,b}. \tag{33}
\]

Finally, our result generalizes the analysis of the symmetric method in the sense that we have the following corollary.

**Corollary 15** If \( F = E \) we have,
\[
\kappa(E^{T,b}) = \|E^{T,b}\|_{a,a}\|(E^{T,b})^{-1}\|_{a,a} \leq \cos(\alpha_E)\|E\|_{b,a}^2\|\hat{E}\|_{b,a}^2 \leq \|E\|_{b,a}^2\|\hat{E}\|_{b,a}^2
\]
where \( \alpha_E \) is the minimal angle between subspaces \( \mathcal{R}(\hat{E}) \) and \( \mathcal{R}(E^{T,b}) \), that is,
\[
\alpha_E = \theta_b\left(\mathcal{R}(\hat{E}), \mathcal{R}(E^{T,b})\right). \tag{34}
\]

**Proof.** Denote by \( \Pi_{\mathcal{R}(E^{T})}|_{\mathcal{R}(\hat{E})} \) the restriction of \( \Pi_{\mathcal{R}(E^{T,b})} \) to \( \mathcal{R}(\hat{E}) \). Observe that (see [1][2][3])
\[
\|\Pi_{\mathcal{R}(E^{T,b})}|_{\mathcal{R}(\hat{E})}\|_{b,b} = \sup_{\|x\|_b = 1, x \in \mathcal{R}(\hat{E})} \|\Pi_{\mathcal{R}(E^{T,b})}|_{b,x}\|_b \tag{29}
\]
\[
= \sup_{\|x\|_b = 1, x \in \mathcal{R}(\hat{E})} \sqrt{1 - \|\Pi_{\mathcal{N}(E),b,x}\|_b^2} \tag{30}
\]
\[
= \sin\left(\theta_b\left(\mathcal{R}(\hat{E}), \mathcal{N}(E)\right)\right) \tag{31}
\]
\[
= \cos\left(\theta_b\left(\mathcal{R}(\hat{E}), \mathcal{R}(E^{T,b})\right)\right). \tag{32}
\]

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Figure 2: Illustration of subspaces of $G$. In order to illustrate angles we picture $\mathcal{R}(\hat{E})$ as a cone. We also illustrate the projection $x = \Pi_{\mathcal{R}(E^{T,b})} y$.

See Figure 2 for an illustration of this case. See [11, 1, 7] for more details and related results on oblique projections.

There is another interesting observation that is useful for the analysis and it is worth saying as a result before we move on.

**Lemma 16** The operator $Q_{E} = \hat{E}E$ is a projection on $\mathcal{R}(\hat{E})$ along $\mathcal{N}(E)$. Analogously, the operator $Q_{E}^{T,b} = E^{T,b}\hat{E}^{T,b}$ is a projection on $\mathcal{R}(E^{T,b})$ and along $\mathcal{N}(\hat{E}^{c})$.

Using this lemma we can study the relative position of subspaces of interest. For instance we have,

$$\cos(\Theta_{b}(\mathcal{R}(\hat{E}), \mathcal{R}(E^{T,b}))) = \sin(\theta_{b}(\mathcal{R}(\hat{E}), \mathcal{N}(E))) = \frac{1}{\|Q_{E}\|_{a,a}} \geq \frac{1}{\|\hat{E}\|_{a,b} \|E\|_{b,a}}.$$  \hfill (33)

See Figure 2

**Remark 17** Note that $\Pi_{\mathcal{R}(E^{T,b})} Q_{E} x = x$ for all $x \in \mathcal{R}(E^{T,b})$ and $Q_{E} \Pi_{\mathcal{R}(E^{T,b})} y = y$ for all $y \in \mathcal{R}(\hat{E})$. 


We now come back to the general case where \( F \neq E \). In practice we have to estimate the norms \( \| F \|_{b,a}, \| E \|_{b,a}, \| \hat{E} \|_{a,b}, \| Q \|_{b,b} \) and \( \| \hat{F} \|_{a,b} \). The norm \( \| \hat{E} \|_{b,a} \) it is usually required in symmetric methods. The norm \( \| \hat{F} \|_{a,b} \) corresponds to the new extension operator used to obtain the nonsymmetric method. The norm \( \| Q \|_{\mathcal{R}(\hat{E})} \) corresponds to a compatibility of them both extension operators.

There are ways to try to estimate \( \| Q \|_{\mathcal{R}(\hat{E})} \) that may lead to different analysis for nonsymmetric methods. See [11, 1, 17]. In case it is technically difficult to get a bound for \( \| Q \|_{\mathcal{R}(\hat{E})} \), we can use the fact that

\[
Q y = Q \Pi_{\mathcal{R}(F^T),b} y
\]

for all \( y \in \tilde{G} \) and therefore we can use the bound

\[
\| Q \|_{\mathcal{R}(\tilde{F})} \|_{b,b} \leq \| Q \|_{\mathcal{R}(F^T)} \|_{b,b} \| \Pi_{\mathcal{R}(F^T),b} \|_{\mathcal{R}(\tilde{F})} \|_{b,b} \leq \| Q \|_{b,b} \cos(\alpha_F) = \frac{\cos(\alpha_F)}{\cos(\beta_{E,F})}
\]

where \( \beta_{E,F} \) is the maximal angle between subspaces \( \mathcal{R}(F^T,b) \) and \( \mathcal{R}(E^T,b) \), that is,

\[
\beta_{E,F} = \Theta_b(\mathcal{R}(F^T,b), \mathcal{R}(E^T,b)) = \Theta_b(\mathcal{N}(F), \mathcal{N}(E)).
\]

See Figure 1. Here we used (22) to obtain,

\[
\| Q \|_{b,b}^{-1} = \sin(\Theta_b(\mathcal{R}(F^T,b), \mathcal{N}(E))) = \cos(\Theta_b(\mathcal{R}(F^T,b), \mathcal{R}(E^T,b))) = \cos(\beta_{E,F}).
\]

In this case we have the following result. See Figure 1 for an illustration.

**Corollary 18** Under the assumptions of Theorem 12 we have

\[
\kappa(F^{E^T,b}) = \| F E^T,b \|_{a,a}, Q(F E^T,b)^{-1} \|_{a,a} \leq \| \hat{F} \|_{a,b} \| F \|_{b,a} \frac{\cos(\alpha_F)}{\cos(\beta_{E,F})} \| E \|_{b,a} \| \hat{E} \|_{a,b}
\]

where \( \alpha_F \) is defined in (28) and \( \beta_{E,F} \) is defined in (35).

Then we can try to study the angle related to subspaces \( \mathcal{R}(F^{T,b}) \) and \( \mathcal{R}(\tilde{F}) \), \( \mathcal{R}(\tilde{F}) \) and \( \mathcal{R}(\hat{E}) \) and the angles between subspaces \( \mathcal{R}(\hat{E}) \) and \( \mathcal{R}(E^T,b) \). Recall that we have (33) and the analogous expression for \( F \), that is

\[
\cos(\Theta_b(\mathcal{R}(\tilde{F}), \mathcal{R}(F^{T,b}))) = \sin(\Theta_b(\mathcal{R}(\tilde{F}), \mathcal{N}(F))) = \frac{1}{\| Q_F \|_{b,b}} \geq \frac{1}{\| F \|_{b,a} \| F \|_{b,a}}.
\]

Here \( Q_F = \hat{F} F \).

In order to make the presentation simpler we only present the case where we can chose \( \tilde{E} \) and \( \hat{F} \) such that \( \mathcal{R}(\tilde{F}) = \mathcal{R}(\hat{E}) \). Recall that \( Q_E = \tilde{E} E \) and \( Q_F = \hat{F} F \).

**Theorem 19** Consider the assumptions of Theorem 12. Assume additionally \( E \neq F \) and that the following two conditions hold,
1. We can choose \( \hat{E} \) and \( \hat{F} \) such that \( \hat{H} = \mathcal{R}(\hat{F}) = \mathcal{R}(\hat{E}) \subset G \).

2. It holds

\[ \beta_E + \beta_F < \frac{\pi}{2}, \]

where

\[ \beta_E = \Theta_b(\hat{H}, \mathcal{R}(E^T)) = \cos^{-1}(\|Q_E\|_{b,b}^{-1}), \]

and

\[ \beta_F = \Theta_b(\hat{H}, \mathcal{R}(F^T)) = \cos^{-1}(\|Q_F\|_{b,b}^{-1}). \]

Then we have,

\[ \|(EF^T)^{-1}\|_{a,a} \leq \frac{\cos(\beta_E)}{\cos(\beta_E + \beta_F)} \|\hat{E}\|_{a,b} \|\hat{F}\|_{b,a}. \]

For other characterizations of \( \|Q\|_{a,b} \) and \( \|Q|_{\mathcal{R}(\hat{E})}\|_{a,b} \) that may lead to possible analysis of nonsymmetric method see [11, 1, 7].

5.3 Special nonsymmetric methods

We consider the case of the family of nonsymmetric methods of Section 4, in particular we focus on Example 9. For simplicity of the presentation we consider only the case where \( S = R \). The general case can be also consider from the results presented next. In the case \( S = R \) we can estimate the norm \( \|Q|_{\mathcal{R}(\hat{E})}\|_{b,b} \) in a simple way.

![Figure 3: Illustration of relative position of subspaces of \( \hat{G} \). In particular, we illustrate the angle between subspaces \( \mathcal{N}(E) \) and \( \mathcal{N}(F) \).](image-url)
Theorem 20 Assume there is a bounded restriction operator $R : H \to \hat{G}$ and bilinear forms $b$ and $c$ such that $E = R T,b|_G$ and $F = R T,c|_G$ where $G \subset \hat{G}$. Suppose that the extensions operators $E$ and $F$ have stable right inverse $\hat{E}$ and $\hat{F}$, respectively. Assume also that
\[ c(\phi, \phi) \leq r_0^2 b(\phi, \phi) \text{ for all } \phi \in \hat{G} \]
and
\[ b(\psi, \psi) \leq r_1^2 c(\psi, \psi) \text{ for all } \psi \in \mathcal{R}(R). \]
We have
\[ \|\Pi_{\mathcal{R}(R),c}\|_{b,b} \leq r_0 r_1 \]
and therefore,
\[ \cos \left( \Theta_b \left( \mathcal{N}(R T,b), \mathcal{N}(R T,c) \right) \right) = \frac{1}{\|\Pi_{\mathcal{R}(R),c}\|_{b,b}} \geq \frac{1}{r_0 r_1}. \tag{38} \]
Finally we can bound
\[ \kappa(F E T,b) = \kappa(R T,c \Pi_{G,b} R) \leq \|\hat{E}\|_{a,b} \|F\|_{b,a} \cos(\alpha_E) r_0 r_1 \|E\|_{b,a} \|\hat{E}\|_{a,b}. \tag{39} \]

**Proof.** Note that in this case there are bilinear forms $b$ and $c$ such that $E = R T,b|_G$ and $F = R T,c|_G$. We then have,
\[ \mathcal{N}(E) = \mathcal{N}(R T,b) \cap G \text{ and } \mathcal{N}(F) = \mathcal{N}(R T,c) \cap G. \tag{40} \]
Therefore,
\[ \beta_{E,F} = \Theta_b(\mathcal{N}(E),\mathcal{N}(F)) \leq \Theta_b(\mathcal{N}(R T,b),\mathcal{N}(R T,c)). \tag{41} \]
It is clear that no element in $\mathcal{N}(R T,b)$ is $c$–orthogonal to the space $\mathcal{N}(R T,c)$. Note that (by using (24)),
\[ \cos(\Theta_b(\mathcal{N}(R T,c),\mathcal{N}(R T,b))) = \sin(\theta_b(\mathcal{N}(R T,c),\mathcal{R}(R))). \tag{42} \]
The $c$-orthogonal projection $\Pi_{\mathcal{R}(R),c}$ is the $b$–oblique projection on $\mathcal{R}(R)$ and in the direction of $\mathcal{N}(R T,c)$. That is, $\Pi_{\mathcal{R}(R),c} = Q(\mathcal{R}(R),\mathcal{N}(R T,c))$. We only need to estimate the $c$–norm of this projection. See an illustration in Figure 3.

Due to Corollary 18 we need only to bound $\cos(\beta_{E,F})$ defined in (35). By (40), (41), (42) and (22) we need only to bound the norm $\|\Pi_{\mathcal{R}(R),c}\|_{a,b}$.

We have,
\[ \|\Pi_{\mathcal{R}(R),c}\|_{b,b} = \sup_{\phi \neq 0} \frac{\|\Pi_{\mathcal{R}(R),c} \phi\|_b}{\|\phi\|_b} \leq r_0 r_1 \sup_{\phi \neq 0} \frac{\|\Pi_{\mathcal{R}(R),c} \phi\|_c}{\|\phi\|_c} \leq r_0 r_1. \]
If no more information is available about the restriction operator $R$ we can use the following result.

**Lemma 21** Under the assumption of Theorem 20 we can bound $r_1 \leq \|R\|_{a,b}\hat{F}\|c\|$. 

**Proof.** Note that for any $\psi = Rw \in \mathcal{R}(R)$ we can combine Corollaries 4 and 8 (with $S = R$) to obtain

$$\|\psi\|_b \leq \|R\|_{a,b}\|w\|_a \leq \|R\|_{a,b}\hat{F}\|_{a,c}\|\psi\|_c. \quad (43)$$

In the case of the operator in (17), we note that if the image of $R$ is in the appropriate relative position (in sense of angles measured in the $c$–inner product) with respect to the subspace $G$ and $G^\perp,c$, then the operator (17) is positive definite in the sense that $c(FE^Tu,u) \geq 0$ for all $u \in H$. See Remark 25.

Note that $\Pi_{G,b}$ is the ($c$-oblique) projection onto $G$ and in the direction of $G^\perp,c$. Let us consider $y \in G$, $z \in G^\perp,c$ and put $x = y + z$. Any $x \in \hat{G}$ can be obtained in this manner. Note that $\Pi_{G,b}x = y + \Pi_{G,b}z$

$$c(x, \Pi_{G,b}x) = c(y,y) + c(y, \Pi_{G,b}z) \geq \frac{1}{2} (\|y\|_c^2 - \|y\|_c\|\Pi_{G,b}z\|_c) .$$

We conclude that if $x$ is such that

$$\|\Pi_{G,b}z\|_c^2 \leq \|y\|_c^2$$

then $c(x, \Pi_{G,b}x) \geq 0$. This happens in particular if

$$\|z\|_c^2\|\Pi_{G,b}\|_{G^\perp,c}^2 \leq \|y\|_c^2. \quad (45)$$

We have the following result.

**Theorem 22** Assume that $\hat{G}$ is finite dimensional. If $y \in G$, $z \in G^\perp,c$ and

$$x = y + z \text{ with } \|z\|_c \tan(\Theta_c(G^\perp,b,G^\perp,c)) \leq \alpha \|y\|_c \text{ with } \alpha \leq 1,$$

then $c(x, \Pi_{G,b}x) \geq (1 - \alpha^2)\|y\|_c^2$.

**Proof.** According to (45) we need only to bound $\|\Pi_{G,b}\|_{G^\perp,c}$. Recall that $\Pi_{G,b}$ is the $c$–oblique projection onto $G$ and in the direction of $G^\perp,b$. Using Lemma 2.80 (p.76) we can bound the norm $\|\Pi_{G,b}\|_{G^\perp,c}$ as follows,

$$\|\Pi_{G,b}\|_{G^\perp,c} \leq \|\Pi_{G,b}\|_{c,c} \sin \left( \Theta_c \left( G^\perp,b, G^\perp,c \right) \right) \quad (46)$$

$$= \frac{\sin \left( \Theta_c \left( G^\perp,b, G^\perp,c \right) \right)}{\sin(\theta_c(G,G^\perp,b))} \quad (47)$$

$$= \frac{\sin \left( \Theta_c \left( G^\perp,b, G^\perp,c \right) \right)}{\cos(\Theta_c(G^\perp,b,G^\perp,c))} \quad (48)$$

$$= \tan(\Theta_c(G^\perp,b,G^\perp,c)). \quad (49)$$
Here we have used \cite{22}.

Introduce the operator $C : \hat{G} \to \hat{G}$ defined by

$$b(Cu, v) = c(u, v).$$

(50)

This operator is symmetric and positive definite. We recall the Wielandt inequality. See for instance \cite{10, 7}.

**Lemma 23** Assume that $mb(x, x) \leq c(x, x) \leq Mb(x, x)$ for all $x \in \hat{G}$. For any pair of vectors $z, w$ with $b(z, w) = 0$ we have

$$\frac{b(z, Cw)^2}{b(z, Cz)b(w, Cw)} \leq \left( \frac{M - m}{M + m} \right)^2$$

(51)

Taking $x = C^{-1/2}z$ and $y = C^{1/2}w$ we obtain,

**Corollary 24** For any pair of vectors $x, y$ with $b(x, y) = 0$ we have

$$\frac{b(x, Cy)^2}{b(x, x)b(Cy, Cy)} \leq \left( \frac{M - m}{M + m} \right)^2$$

(52)

We see that $\cos(\theta_c(G, G^\perp, b)) \leq \left( \frac{M - m}{M + m} \right)$. Then

$\sin(\theta_c(G, G^\perp, b)) \geq \sqrt{1 - \left( \frac{M - m}{M + m} \right)^2} = \frac{\sqrt{mM}}{2(M + m)}$ and therefore

$\tan(\theta_c(G^\perp, b)) = 1 / \tan(\theta_c(G, G^\perp, b)) \leq \frac{\sqrt{mM(M - m)}}{2(M + m)}$. We have the following corollary of the previous result and Theorem 22.

**Theorem 25 (Positivity of special non-symmetric methods)** Assume that there is a constant $\alpha_R$ such that

$$\| (I - \Pi_{G,c})Ru \|_{c,c} \frac{\sqrt{mM(M - m)^2}}{2(M + m)} \leq \alpha_R \| \Pi_{G,c}Ru \|_{c,c} \quad \text{for all } u \in H.$$ (53)

Then we have that $c(R^T, \Pi_{G,b}Ru, u) \geq (1 - \alpha^2_R)c(\Pi_{G,c}Ru, \Pi_{G,c}Ru)$.

**Remark 26** Similar results hold when $S \neq R$ and $c = b$. In this case $E = R^T, b|_G$ and $F = S^T, b|_G$,

$$N(E) = N(R^T, b) \cap G \quad \text{and} \quad N(F) = N(S^T, b) \cap G$$

(54)

Therefore,

$$\beta_{E,F} = \Theta_b(N(E), N(F)) \leq \Theta_b \left( N(R^T, b), N(S^T, b) \right).$$ (55)

This last angle can be bound in terms of $\Theta_b(R(S), R(R))$, that would require and assumption on $S$ when compared to $R$. 

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6 Restricted methods

In this section we consider a particular case of Example 10. We now use the Hilbert space framework previously introduced to obtain a bound for the condition number of a restricted additive method. For simplicity and readability we consider the one level method. Similar results can be obtained using a multilevel setting. We use the notation and setup introduced in Section 3, in particular, we consider the Hilbert spaces $H$ and $G = \times_{\ell=1}^{N_S} H_0^1(\mathcal{O}_\ell)$ as before (with the same inner products $a$ and $b$).

Start by defining the (harmonic-like) extension operator $F_\ell : H_0^1(\mathcal{O}_\ell) \to H$ as follows. Given $v_\ell \in H_0^1(\mathcal{O}_\ell)$ define $F_\ell v_\ell \in H$ as the unique solution of

$$\int_D \nabla F_\ell v_\ell \nabla z = \int_{D_\ell} \nabla v_\ell \nabla z \quad \text{for all } z \in H. \quad (56)$$

Note that the integration on the right is on the domain $D_\ell$. In the case of an interior subdomain, this is the weak form of the strong form given by,

$$\begin{aligned}
-\Delta F_\ell v_\ell &= -\Delta v_\ell \quad \text{in } D_\ell, \\
-\Delta F_\ell v_\ell &= 0 \quad \text{in } D \setminus D_\ell, \\
\frac{\partial F_\ell v_\ell}{\partial \eta} + \frac{\partial F_\ell v_\ell}{\partial \eta} &= \frac{\partial v_\ell}{\partial \eta} \quad \text{on } \partial D_\ell, \\
F_\ell v_\ell &= 0 \quad \text{on } \partial D.
\end{aligned} \quad (57)$$

We introduce the bilinear form $\tilde{b}_\ell$ defined by

$$\tilde{b}_\ell(v, z) = \int_{D_\ell} \nabla v \nabla z$$

for any $v$ and $z$ that can be restricted to $D_\ell$. Using our bilinear forms notation we have

$$a(F_\ell v_\ell, z) = \tilde{b}_\ell(v_\ell, z) \quad \text{for all } z \in H.$$ 

Define, in analogy with the previous discussions, the extension operator $F : G \to H$ by

$$F\{v_\ell\} = \sum_{\ell=1}^{N_S} F_\ell v_\ell. \quad (58)$$

Consider also the operator $F_T^{T,b}$ which are given by the problem, $F_T^{T,b}u \in H_0^1(\mathcal{O}_\ell)$ with,

$$\int_{\mathcal{O}_\ell} \nabla F_T^{T,b} u \nabla v_\ell = \int_D \nabla u \nabla F_\ell v_\ell = \int_{D_\ell} \nabla u \nabla v_\ell \quad \text{for all } v_\ell \in H_0^1(\mathcal{O}_\ell). \quad (59)$$

Here in the last step we used the definition of $F_\ell$. Note that the weak from
above correspond to the strong form
\[
\begin{align*}
-\Delta F_{\ell}^{T,b} v_\ell &= -\Delta u \quad \text{in } D_\ell, \\
-\Delta F_{\ell}^{T,b} u_\ell &= 0 \quad \text{in } O_\ell \setminus D_\ell, \\
\frac{\partial F_{\ell}^{T,b} v_\ell}{\partial \eta^+} + \frac{\partial F_{\ell}^{T,b} v_\ell}{\partial \eta^-} &= \frac{\partial u}{\partial \eta} \quad \text{on } \partial D_\ell, \\
F_{\ell}^{T,b} u_\ell &= 0 \quad \text{on } \partial O_\ell.
\end{align*}
\]

This equation corresponds to a local problem with the same computational cost of the local problem used to obtain $E_{\ell}^{T,b}$ in the additive method. We then have
\[F^{T,b} = \{ F_{\ell}^{T,b} \}.
\]

6.1 The operator $E F^{T,b}$

Let $u$ be the solution of (5) and introduce $w$ such that $E F^{T,b} w = u$. Then we consider the equation,

\[a(E F^{T,b} w, v) = L(v) \text{ for all } v \in H.
\]

Note that, given $w$ the computation of $E F^{T,b} w = \sum_{\ell=1}^{N_S} E_{\ell} F_{\ell}^{T,b} w$ requires the solution of local problems posed on the overlapping subdomains. See (59) and (60). Then we can iteratively solve this equation. After computing $w$ we can compute $u = E F^{T,b} w$ by solving one more round of local problems.

Remark 27 For comparison the local problem of the OBDD in [8] can be written as
\[ -\Delta w_\ell = \eta_{\ell}^{\text{cut}} \Delta v_\ell \quad \text{in } O_\ell
\]
where $\eta_{\ell}^{\text{cut}}$ is a cut function that is 1 in $D_\ell \subset O_\ell$ and decays to zero. See Example 10. See the comments in Example 10. The method analyzed in this section use local problems in (60). A main difference is that (when applied to e.g. finite element implementations) the local problem in (60) needs the Neumann stiffness matrix associated to subdomains $\{ D_\ell \}^{N_S}_{\ell=1}$ which is not the case for the local problem in (61). See the weak form of (60) in (59).

6.2 The operator $F E^{T,b}$

If now we consider the method $F E^{T,b}$ which correspondes to the RAS preconditioner. The solution of problem (5) satisfies,

\[a(F E^{T,b} u, v) = a(F^{T,b} u, F^{T,b} v)
\]

\[= \sum_{\ell=1}^{N_S} \int_{O_\ell} \nabla E_{\ell}^{T,b} u \nabla F_{\ell}^{T,b} v
\]

\[= \sum_{\ell=1}^{N_S} \int_{D_\ell} \nabla E_{\ell}^{T,b} u \nabla v
\]

\[= L(F E^{T,b} v) = \tilde{L}(v).
\]
Here we see that each term $\int_{D_\ell} \nabla E_{T_\ell} u \nabla v$ can be computed to assemble $\tilde{L}$.

After $\tilde{L}$ is assembled, we solve iteratively

$$\sum_{\ell=1}^{N_\Omega} \int_{D_\ell} \nabla E_{T_\ell} u \nabla v = \tilde{L}(v).$$

Recall that the right hand above is equivalent to $a(F E_{T,b}^T u, v)$. Note that the computation of the residual needs to update the solution only on the subdomains $D_\ell$. Note also that in the case of an implementation in finite element spaces we need access to the Neumann stiffness matrix associated to subdomains $\{D_\ell\}_{\ell=1}^{N_\Omega}$.

### 7 Condition number estimates

We can consider the operator $F$ introduced in Section 6 and use the results of our Hilbert space framework to obtain the non-singularity of $E F_{T,b}^T$ or $F_{T,b}^T$ as before. For $0 < \epsilon$ define

$$c_\epsilon(\{u_\ell\}, \{v_\ell\}) = \sum_{\ell=1}^{N_\Omega} \int_{D_\ell} \nabla u_\ell \nabla v_\ell + \epsilon \sum_{\ell=1}^{N_\Omega} \int_{O_\ell \setminus D_\ell} \nabla u_\ell \nabla v_\ell + \sum_{\ell=1}^{N_\Omega} b_{\ell}^0(u_\ell, v_\ell).$$

Recall the restriction operator $R$ introduced in Section 3. Define $F_\epsilon = R^{T,c_\epsilon}|_G$ so that for $\{v_\ell\} \in G$ we have

$$\int_D \nabla F_{T,c_\epsilon}^T v_\ell \nabla z = \sum_{\ell=1}^{N_\Omega} \int_{D_\ell} \nabla v_\ell \nabla z + \epsilon \sum_{\ell=1}^{N_\Omega} \int_{O_\ell \setminus D_\ell} \nabla v_\ell \nabla z \quad \text{for all } z \in H.$$  

Denote by $F = F_0$ is the extension operator used in Section 6. Define the operator $F_{ov}$ by

$$\int_D \nabla F_{ov} v_\ell \nabla z = \sum_{\ell=1}^{N_\Omega} \int_{O_\ell \setminus D_\ell} \nabla v_\ell \nabla z \quad \text{for all } z \in H. \quad (62)$$

The operator $F_{ov}$ is clearly bounded with $\|F_{ov}\|_{a,b} \leq \|E\|_{a,b}$ and

$$F_\epsilon = F + \epsilon F_{ov}$$

where $F$ was defined in (58).

Note that $F_\epsilon E_{T,b}^T = R^{T,c_\epsilon} \Pi_{\ell \ell,b} R$ and therefore we are in the case of special non-symmetric methods of Section 4 that were analyzed in Section 5.3.

We can find a stable right inverse of $F_\epsilon$ as follows.

**Lemma 28 (Stable right inverse of $F$)** There exists $C_0F$ such that for every $v \in H$ there exists $v_\ell \in H_0^1(O_\ell)$ such that

$$\sum_{\ell=1}^{N_\Omega} F_\ell v_\ell = v$$
and
\[ \sum_{\ell=1}^{N_S} b_\ell(v_\ell, v_\ell) \leq C_F^2 a(v, v). \]

If we put \( \hat{F}v = \{v_\ell\} \) we then have \( \|\hat{F}\|_{a,b} \leq C_F \). We can estimate \( C_F^2 \geq \nu C_{cut}(1 + \frac{1}{\tau\delta}) \). We also have \( \|\hat{F}\|_{a,c} \leq C_F \).

**Proof.** This proof is similar to the stable decomposition for the operator \( E \); see [12]. Let us consider cut of functions \( \eta_\ell \) introduced in (11) Define \( v_\ell = \eta_\ell v \).

We have that
\[
\sum_{\ell=1}^{N_S} F_\ell v_\ell, z = \sum_{\ell=1}^{N_S} a(F_\ell v_\ell, z) = \sum_{\ell=1}^{N_S} \int_{D_\ell} \nabla v_\ell \nabla z = \sum_{\ell=1}^{N_S} \int_{D_\ell} \nabla v \nabla z = a(v, z).
\]

We conclude that \( \sum_{\ell=1}^{N_S} F_\ell v_\ell = v \). As in the case of classical additive method stable decomposition -which uses the gradient of the product rule plus a Friedrichs inequality, it is easy to see that,
\[
\sum_{\ell=1}^{N_S} b_\ell(v_\ell, v_\ell) = \sum_{\ell=1}^{N_S} \int_{\partial \Omega_\ell} |\nabla v_\ell|^2 = \sum_{\ell=1}^{N_S} \frac{1}{\sigma_\ell} \int_{\partial \Omega_\ell} |\nabla \eta_\ell v|^2 \leq \nu C_{cut}(1 + \frac{1}{\tau\delta}) \int_D |\nabla v|^2.
\]

A stable right inverse of \( F_\epsilon \) can be also obtained.

**Corollary 29** For \( \epsilon \) small enough, \( F_\epsilon \hat{F} \) is non-singular. Moreover, \( \hat{F}_\epsilon = \hat{F}(F_\epsilon \hat{F})^{-1} \) is an stable right inverse of \( F_\epsilon \) with \( \|\hat{F}_\epsilon\|_{b,b} \leq \|\hat{F}\|_{b,b}/(1 - \epsilon \|F_\epsilon \hat{F}\|_{a,a}) \).

**Proof.** Note that \( F_\epsilon \hat{F} = \hat{F} + \epsilon F_\epsilon \hat{F} = I + \epsilon F_\epsilon \hat{F} \). This is invertible for small enough \( \epsilon \) and \( \|F_\epsilon \hat{F}\|_{b,b} \leq 1/(1 - \epsilon \|F_\epsilon \hat{F}\|_{b,b}) \).

We now estimate the norm of \( F_\epsilon \).

**Lemma 30 (Norm \( F \))** We have \( \|F\|_{b,b} \leq 1 \) and \( \|F_\epsilon\|_{b,b} \leq 1 + \epsilon \|F_{\epsilon\hat{F}}\|_{b,b} \).

**Proof.** From the definition of \( F_\ell \) we have for every \( z \in H \).
\[ a(F\{v_\ell\}, z) = a(\sum_{\ell=1}^{N_S} F_\ell v_\ell, z) = \sum_{\ell=1}^{N_S} \int_{D_\ell} \nabla v_\ell \nabla z \]
\[ = \sum_{\ell=1}^{N_S} \int_{D_\ell} \nabla v \nabla z \]
\[ \leq \left( \sum_{\ell=1}^{N_S} \int_{D_\ell} |\nabla v_\ell|^2 \right)^{1/2} \left( \sum_{\ell=1}^{N_S} \int_{D_\ell} |\nabla z|^2 \right)^{1/2} \]
\[ \leq \left( \sum_{\ell=1}^{N_S} \int_{O_\ell} |\nabla v_\ell|^2 \right)^{1/2} \left( \int_D |\nabla z|^2 \right)^{1/2} \]
\[ \leq \|\{v_\ell\}\|_b \|z\|_a. \]

Taking \( z = F\{v_\ell\} = \sum_{\ell=1}^{N_S} F_\ell v_\ell \) we see that \( \|F\{v_\ell\}\|_a \leq \|\{v_\ell\}\|_b \) and the result follows.

As a corollary we have the following result.

**Corollary 31 (Angle \( \alpha_F \))** We have \( \alpha_F = \theta_b \left( \mathcal{R}(\hat{F}), \mathcal{R}(F^{T,b}) \right) = 0. \)

We do not need the following results but we stated for completeness. The range of \( \hat{E} \) and \( \hat{F} \) coincide.

**Lemma 32** We can choose \( \hat{E} \) and \( \hat{F} \) such that \( \mathcal{R}(\hat{E}) = \mathcal{R}(\hat{F}) \).

**Proof.** As in the proof of Lemma 28 chose \( \hat{F}u = \{\eta_\ell u\} \). Define
\[ \eta = \sum_{\ell=1}^{N_S} \eta_\ell. \]

We recall that an classical construction of the operator \( \hat{E} \) is given by
\[ \hat{E}u = \{\chi_\ell u\} \text{ with } \chi_\ell = \frac{\eta_\ell}{\eta}. \]

We readily see that \( \hat{F}u = \eta \hat{E}u \) and since \( \eta \geq 1 \) is bounded with bounded gradient we have the result.

We can estimate the parameter \( r_1 \) in Theorem 20 as follows.

**Lemma 33** We have that \( b(\mathcal{R}v, \mathcal{R}v) \leq \nu_c(\mathcal{R}v, \mathcal{R}v) \) for all \( v \in H \). Here \( \nu \) is defined in (13).

**Proof.** Observe that,
\[ \sum_{\ell=1}^{N_S} b_e(v|_{O_\ell}, v|_{O_\ell}) = \sum_{\ell=1}^{N_S} \int_{O_\ell} |\nabla v|^2 \leq \nu \int_D |\nabla v|^2 = \nu \sum_{\ell=1}^{N_S} \int_{D_\ell} |\nabla v_\ell|^2 \leq \nu \left( \sum_{\ell=1}^{N_S} \int_{D_\ell} |\nabla v_\ell|^2 + \epsilon \sum_{\ell=1}^{N_S} \int_{O_\ell\setminus D_\ell} |\nabla v_\ell|^2 \right). \]

Therefore, if we include the boundary terms we have \( b(Rv, Rv) \leq \nu c_e(Rv, Rv). \)

Putting together the previous bounds and Theorem 20 we can write condition number bounds. Recall that:

- For \( E \) define in Section 3 we have \( \|E\|_{b,a} \leq \sqrt{\rho(\mu)} \leq \nu. \) See Remark 7.
- For \( E \) define in Section 3 we have \( \|\hat{E}\|_{a,b} \leq \nu C_{pu}(1 + 1/(\tau \delta)). \)
- For \( \hat{F} \) defined in Lemma 28 we have \( \|\hat{F}\|_{a,b} \leq \nu C_{cut}(1 + 1/(\tau \delta)). \)
- From Lemma 30 we have \( \|F\|_{b,a} \leq 1, \|\hat{F}\|_{b,a} \leq 1 + \epsilon \|F_{ov}\|_{b,a}. \)
- From Corollary 29 we have \( \|\hat{F}\|_{b,a} \leq \|\hat{F}\|_{b,a}/(1 - \epsilon \|F_{ov}\|_{a,a}). \)
- From Corollary 31 we have \( \cos(\alpha_E) = 1. \)
- From \( c_e, 0 < \epsilon < 1, \) defined in and \( b \) defined in Section 3 we have \( r_0 = 1. \)
- From Lemma 33 we have \( r_1 = \sqrt{\nu}. \)

Replacing in (39) we get the following result.

**Theorem 34** Let \( E \) and \( F_\epsilon \) be defined as before. Then we have that \( F_\epsilon E^T,b \) is invertible and

\[ \kappa(F_\epsilon E^T,b) \leq \frac{\sqrt{\nu} \sqrt{\rho(\mu)} C_F C_E (1 + \epsilon \|F_{ov}\|_{b,a}) \cos(\alpha_E)}{(1 - \epsilon \|F_{ov}\|_{a,a})} \leq \frac{\sqrt{\nu} \sqrt{\rho(\mu)} C_F C_E (1 + \epsilon \|F_{ov}\|_{b,a})}{(1 - \epsilon \|F_{ov}\|_{a,a})}. \]

Finally, taking \( \epsilon \to 0 \) we obtain the condition number of the RAS,

\[ \kappa(FE^T,b) \leq \sqrt{\nu} \sqrt{\rho(\mu)} C_F C_E. \]

Note that we use \( \sqrt{\rho(\mu)} \leq \sqrt{\nu} \) and \( C_F \leq C_E \) we have that the bound of \( \kappa(FE^T,b) \) is smaller than the bound for \( \kappa(EE^T,b) \). That is, the bound for the condition number of this restricted method is smaller than the bound obtained for AS.
We now investigate the positivity of the operator $F_E^{T,b}$. Recalling (50) we have $c_e(B,x,y) = b(x,y)$. We also have that $\epsilon b(x,x) \leq c_e(x,x) \leq b(x,x)$. In order to use Theorem $25$ we need

$$\| (I - \Pi_{G,c_e} R u \|_{c_e,c_e} \leq \alpha_R \| \Pi_{G,c_e} R u \|_{c_e,c_e} \quad \text{for all } u \in H. \quad (63)$$

The left hand side multiplier of the norm vanishes when $\epsilon \to 0$ and therefore we conclude (using Theorem $25$) that, for $\epsilon$ small enough, $c_e(FE^{T,b} u, u) \geq 0$ for all $u \in H$.

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**References**

[1] A. Böttcher and I. M. Spitkovsky. A gentle guide to the basics of two projections theory. *Linear Algebra Appl.*, 432(6):1412–1459, 2010.

[2] Xiao-Chuan Cai, Charbel Farhat, and Marcus Sarkis. A minimum overlap restricted additive Schwarz preconditioner and applications in 3D flow simulations. In *Domain decomposition methods, 10 (Boulder, CO, 1997)*, volume 218 of *Contemp. Math.*, pages 479–485. Amer. Math. Soc., Providence, RI, 1998.

[3] Xiao-Chuan Cai and Marcus Sarkis. A restricted additive Schwarz preconditioner for general sparse linear systems. *SIAM J. Sci. Comput.*, 21(2):792–797 (electronic), 1999.

[4] Evridiki Efthathiou and Martin J. Gander. Why restricted additive Schwarz converges faster than additive Schwarz. *BIT*, 43(suppl.):945–959, 2003.

[5] A. Frommer, R. Nabben, and D. B. Szyld. An algebraic convergence theory for restricted additive and multiplicative Schwarz methods. In *Domain decomposition methods in science and engineering (Lyon, 2000)*, Theory Eng. Appl. Comput. Methods, pages 371–377. Internat. Center Numer. Methods Eng. (CIMNE), Barcelona, 2002.

[6] Andreas Frommer and Daniel B. Szyld. An algebraic convergence theory for restricted additive Schwarz methods using weighted max norms. *SIAM J. Numer. Anal.*, 39(2):463–479 (electronic), 2001.
[7] Aurél Galántai. *Projectors and projection methods*, volume 6. Springer Science & Business Media, 2013.

[8] Jung-Han Kimn and Marcus Sarkis. Obdd: Overlapping balancing domain decomposition methods and generalizations to the helmholtz equation. In *Domain Decomposition Methods in Science and Engineering XVI*, pages 317–324. Springer, 2007.

[9] Tarek P. A. Mathew. *Domain decomposition methods for the numerical solution of partial differential equations*, volume 61 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2008.

[10] Mohammad Sal Moslehian. Recent developments of the operator Kantorovich inequality. *Expo. Math.*, 30(4):376–388, 2012.

[11] Daniel B. Szyld. The many proofs of an identity on the norm of oblique projections. *Numer. Algorithms*, 42(3-4):309–323, 2006.

[12] Andrea Toselli and Olof Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[13] M. Dryja X.-C. Cai and M. Sarkis. RASHO: A restricted additive schwarz preconditioner with harmonic overlap. In *Domain Decomposition Methods in Science and Engineering*, N. Debit, M. Garbey, R. Hoppe, D. Keyes, Y. Kuznetsov, J. Periaux, eds., CIMNE, Contemp. Math., pages 337–344. 2002.