MOTIVIC DE RHAM-WITT COMPLEX

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Abstract. We show that additive higher Chow groups in the Milnor range on smooth varieties over a perfect field of characteristic $p \neq 2$ induce a Zariski sheaf of pro-differential graded algebras, whose Milnor range is isomorphic to the Zariski sheaf of the big de Rham-Witt complexes of Hesselholt and Madsen. When $p > 2$, the Zariski hypercohomology of the $p$-typical part of the sheaf arising from additive higher Chow groups computes the crystalline cohomology of smooth proper varieties. This revisits the 1970s results of S. Bloch and L. Illusie on crystalline cohomology.

1. Introduction

One of the aims of this paper is to build a bridge between two different worlds, where S. Bloch had made fundamental contributions decades ago: at one end is the world of crystalline cohomology of P. Berthelot [3] of smooth proper varieties over $k$ over a perfect field of characteristic $p > 0$. This is computable via parts of Quillen relative higher algebraic $K$-groups of certain nilpotent schemes as shown in [4]. At the other end is the world of algebraic cycles, in particular higher Chow groups [5], which is the motivic cohomology theory for smooth $k$-schemes. Sure, the cycle class maps from the motivic cohomology to crystalline cohomology do provide a bridge of certain kind, but what we are after here is not this sort of Hodge-Tate type question. We ask whether there is a way to construct the crystalline cohomology itself from groups arising from algebraic cycles.

This might sound odd to some people, but several works indicate that this is plausible. In the 1970s, L. Illusie [18] proved that the crystalline cohomology is computable as hypercohomology of the $p$-typical de Rham-Witt complex. This object is isomorphic to the $p$-typical curves of S. Bloch in [4]. The $p$-typical de Rham-Witt complex was generalized to the “multi-prime” big de Rham-Witt complex $\mathbb{W}_m^\infty X$ by L. Hesselholt and I. Madsen in [16] in the 1990s. More recently, in the 2000s, S. Bloch and H. Esnault [6] and K. Rülling [38] proved that for $X = \text{Spec} \ (k)$, the big de Rham-Witt complex comes from algebraic cycles, via the (cubical) additive higher Chow group of 0-cycles.

Back then, we did not know much about the additive higher Chow groups, but during the past several years they were developed and studied from various perspectives (see [19], [23], [24], [25], [26], [34], [35]). Based on these recent developments, this paper attempts to go back the question on describing the big de Rham-Witt complexes from additive higher Chow groups.

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Our main result below is a generalization of the theorem of K. Rülling [38] as well as the additive analogue of results of P. Elbaz-Vincent and S. Müller-Stach [10], M. Kerz and S. Müller-Stach [20], and M. Kerz [21]. A good part of it is stated as Theorem 3.1.

**Theorem 1.1.** Let $k$ be a perfect field of characteristic $p \neq 2$. Let $R$ be a regular semi-local $k$-algebra essentially of finite type. Then, for all $n, m \geq 1$, we have isomorphisms between the big de Rham-Witt forms and the additive higher Chow groups

$$\tau_{n,m}^R : W_m \Omega^{-1}_R \cong \text{TCH}^n(R; n; m).$$

In particular, additive higher Chow groups of $\text{Spec}(R)$ in the Milnor range form a restricted Witt-complex over $R$, which is isomorphic to the big de Rham-Witt complex of $R$.

Using this theorem, one can identify the Zariski sheaf of the big de Rham-Witt complexes with a Zariski sheafification of a presheaf constructed from additive higher Chow groups in the Milnor range. All sections of the paper, other than Section 11 which discusses some applications, are devoted to proving this theorem. In Section 2, we recall the definitions of various objects appearing in the paper as well as some results we will use. In Section 3, we show that this map $\tau_{n,m}^R$ is injective using several results, including one of M. Gros [13] regarding the Cousin resolution of the $p$-typical de Rham-Witt complexes of a smooth scheme.

The remaining surjectivity part of $\tau_{n,m}^R$ is more challenging, and Sections 4 ∼ 10 are all about it. Several of them develop a semi-local version of moving lemma in the Milnor range. We first show in Sections 4 and 5 that every class in $\text{TCH}^n(R; n; m)$ can be represented by a cycle whose supports are all finite and surjective over $R$. If we denote the subgroup of such cycle classes as $\text{TCH}^n_{fs}(R; n; m)$, then we have $\text{TCH}^n_{fs}(R; n; m) = \text{TCH}^n(R; n; m)$. Sections 6 ∼ 9 go further to show that the cycle classes in $\text{TCH}^n_{fs}(R; n; m)$ can be represented by cycles that have additional smoothness. The subgroup of such cycle classes is denoted by $\text{TCH}^n_{sfs}(R; n; m)$. The following summarizes this:

**Theorem 1.2.** Let $k$ be an arbitrary field and let $R$ be a smooth semi-local $k$-algebra essentially of finite type. Then, $\text{TCH}^n_{sfs}(R; n; m) = \text{TCH}^n_{fs}(R; n; m) = \text{TCH}^n(R; n; m)$.

Under this result and the main theorems of [10] and [20], we can reduce every cycle class of $\text{TCH}^n(R; n; m)$ to a sum of special type of cycles, that we call *Witt-Milnor graph cycles* or *symbolic cycles* over an extension ring $R'$ over $R$. Unlike the case of [10] and [20], this does not yet solve our surjectivity problem, especially due to lack of the construction of the trace maps on the big de Rham-Witt forms for suitable finite extensions of rings $R$. Our approach is to use the map $\tau_{n,m}^R$ and the groups $\text{TCH}^n(R; n; m)$ to define the notion of “traceability” for forms in $W_m \Omega^{-1}_S$, where $S$ is a finite simple ring extension of $R$, and to use the Witt-complex structure of the additive higher Chow groups as proven in [29] to show that all forms in $W_m \Omega^{-1}_S$ are traceable. From this, we can deduce the desired surjectivity of $\tau_{n,m}^R$.

In Section 11 we discuss applications of Theorem 1.1. We list some of them:
Theorem 1.3. Let \( k \) be a perfect field of characteristic \( p \neq 2 \). Let \( R \) be a regular semi-local \( k \)-algebra essentially of finite type. Then, additive higher Chow groups of \( R \) in the Milnor range satisfy the Gersten conjecture in the sense that the Cousin complex is a resolution. In particular, when \( K = \text{Frac}(R) \), the flat pull-back map \( \text{TCH}^n(R,n;m) \to \text{TCH}^n(K,n;m) \) is injective.

Theorem 1.1 also allows us to prove the existence of the trace maps on the big de Rham-Witt forms for finite ring extensions of smooth \( k \)-algebras, by way of push-forwards of additive higher Chow cycles:

Theorem 1.4. Let \( k \) be a perfect field of characteristic \( \neq 2 \). Then, for any finite ring extension \( R \subset R' \) of smooth \( k \)-algebras essentially of finite type, there exists a trace map \( \text{Tr}_{R'/R} : \mathbb{W}_m\Omega^n_{R'} \to \mathbb{W}_m\Omega^n_R \), which is transitive, and compatible with the push-forward of additive higher Chow cycles, i.e. the diagram commutes:

\[
\begin{array}{ccc}
\mathbb{W}_m\Omega^{n-1}_{R'} & \xrightarrow{\tau_{R',m}} & \text{TCH}^n(R',n;m) \\
\downarrow \text{Tr}_{R'/R} & & \downarrow f_* \\
\mathbb{W}_m\Omega^{n-1}_R & \xrightarrow{\tau_{R,m}} & \text{TCH}^n(R,n;m).
\end{array}
\]

Combining Theorem 1.1 with the main theorems of [4] and [18], we obtain:

Theorem 1.5. Let \( k \) be a perfect field of characteristic \( p > 2 \) and let \( X \) be a smooth proper \( k \)-scheme. Then, we have an isomorphism

\[
\text{H}^n_{\text{crys}}(X/W) \simeq \lim_{\leftarrow i} \text{H}^n_{\text{Zar}}(X, \mathcal{TCH}^M_{(p)}(-;p^i)_{\text{Zar}}),
\]

where \( \mathcal{TCH}^M_{(p)}(-;p^i)_{\text{Zar}} \) is the Zariski sheafification of the \( i \)-th level of the \( p \)-typical part of the additive higher Chow presheaf of \( X \) in the Milnor range.

Since the group on the right hand side originates from objects defined in terms of algebraic cycles, in a sense this isomorphism gives an algebraic-cycle description of the crystalline cohomology groups. In [4], the crystalline cohomology group was described by the hypercohomology of some sheaves given in terms of some relative algebraic \( K \)-groups, while one of the insights in the development of additive higher Chow groups comes from the desire to describe such relative \( K \)-groups in terms of algebraic cycles. So, the authors are excited about this description of the crystalline cohomology in terms of algebraic cycles.

**Conventions** In this paper, a \( k \)-scheme is a separated scheme of finite type over \( k \), unless we say otherwise. A \( k \)-variety is a reduced \( k \)-scheme. The product \( X \times Y \) means usually \( X \times_k Y \), unless we specify otherwise. We let \( \text{Sch}_k \) be the category of \( k \)-schemes, \( \text{Sm}_k \) of smooth \( k \)-schemes, and \( \text{SmAff}_k \) of smooth affine \( k \)-schemes. A scheme essentially of finite type is a scheme obtained by localizing at a finite subset (including \( \emptyset \)) of a finite type \( k \)-scheme. For \( C = \text{Sch}_k, \text{Sm}_k, \text{SmAff}_k \), we let \( C^\text{ess} \) be the extension of category \( C \) obtained by localizing at a finite subset (including \( \emptyset \)) of objects in \( C \). We let \( \text{SmLoc}_k \) be the category of smooth semi-local \( k \)-schemes essentially of finite type over \( k \). So, \( \text{SmAff}_k^\text{ess} = \text{SmAff}_k \cup \text{SmLoc}_k \) for the objects. When we say a semi-local \( k \)-scheme, we always mean one that is essentially of finite type over \( k \). Let \( \text{SmProj}_k \) be the category of smooth projective \( k \)-schemes.
2. Recollection of some basic definitions

Let $k$ be any field. In this section, we recall some definitions used throughout the paper, including the cubical version of higher Chow groups, additive higher Chow groups, the Milnor $K$-groups, the ring of big Witt vectors, and the big de Rham-Witt complexes.

2.1. Higher Chow groups. We recall (cf. [5, 39]) the definition of higher Chow groups as follows. Let $X \in \text{Sch}^\text{ess}_k$ be equidimensional. Let $\mathbb{P}^1 = \text{Proj } k[Y_0, Y_1]$, and $\square^n = (\mathbb{P}^1 \setminus \{1\})^n$. Let $(y_1, \ldots, y_n)$ be the coordinates on $\square^n$. A face of $\square^n$ is a closed subscheme defined by a set of equations of the form $y_{i_1} = \epsilon_1, \ldots, y_{i_s} = \epsilon_s$, where $\epsilon_j \in \{0, \infty\}$. For $1 \leq i \leq n$ and $\epsilon = 0, \infty$, we let $\iota^i : \square^{n-1} \rightarrow \square^n$ be the closed immersion given by $(y_1, \ldots, y_{n-1}) \mapsto (y_1, \ldots, y_{i-1}, \epsilon, y_i, \ldots, y_{n-1})$. Its image gives a codimension 1 face.

Let $q, n \geq 0$. Let $z^q(X, n)$ be the free abelian group on the set of integral closed subschemes of $X \times \square^n$ of codimension $q$, that intersects properly with $X \times F$ for each face $F$ of $\square^n$. We define the boundary map $\partial^q_i (Z) := [(\text{Id}_X \times \iota^i)^* (Z)]$. This collection of data gives a cubical abelian group $(n \mapsto \bigoplus z^q(X, n))$ in the sense of [23 §1.1], and the groups $z^q(X, n) := \bigoplus z^q(X, n) \text{deg}_n$ (in the notations of loc. cit.) give a complex of abelian groups, whose boundary map at level $n$ is given by $\partial := \sum_{i=1}^n (-1)^i (\partial_i^\infty - \partial_i^0)$. The homology $\text{CH}^q(X, n) := H_n(z^q(X, \bullet), \partial)$ is called the higher Chow group of $X$.

2.2. Additive higher Chow groups. We recall the definition of additive higher Chow groups from [20 Section 2].

Let $X \in \text{Sch}^\text{ess}_k$ be equidimensional. Let $\mathbb{A}^1 = \text{Spec } k[t]$, $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$, and $\square = \mathbb{P}^1$. For $n \geq 1$, let $B_n = \mathbb{G}_m \times \square^{n-1}$, $\widehat{B}_n = \mathbb{A}^1 \times \square^{n-1}$ and $\hat{B}_n = \mathbb{P}^1 \times \square^{n-1} \supset \widehat{B}_n$. Let $(t, y_1, \ldots, y_{n-1})$ be the coordinates on $\hat{B}_n$.

On $\hat{B}_n$, define the Cartier divisors $F_{n, i}^1 := \{y_i = 1\}$ for $1 \leq i \leq n - 1$, $F_{n, 0} := \{t = 0\}$ (which is contained in $\widehat{B}_n$), and let $F_n^1 := \sum_{i=1}^{n-1} F_{n, i}^1$. A face of $B_n$ is a closed subscheme defined by a set of equations of the form $y_{i_1} = \epsilon_1, \ldots, y_{i_s} = \epsilon_s$ where $\epsilon_j \in \{0, \infty\}$. For $1 \leq i \leq n - 1$ and $\epsilon = 0, \infty$, let $\iota_{n, i, \epsilon} : B_{n-1} \rightarrow B_n$ be the inclusion $(t, y_1, \ldots, y_{n-2}) \mapsto (t, y_1, \ldots, y_{i-1}, \epsilon, y_i, \ldots, y_{n-2})$. Its image is identified with a codimension 1 face.

The additive higher Chow complex is defined similarly using the spaces $B_n$ instead of $\square^n$, but other than proper intersection with all faces, we impose an additional condition called the modulus conditions, that control how the cycles should behave at “infinity”:

2.2.1. Modulus conditions. We recall the modulus condition ($M_{\text{sum}}$) from [20 Definition 2.1].

**Definition 2.1.** Let $X$ be a $k$-scheme, and let $V$ be an integral closed subscheme of $X \times B_n$. Let $\nabla$ denote the Zariski closure of $V$ in $X \times \hat{B}_n$ and let $\nu : \nabla^N \rightarrow V \subset X \times \hat{B}_n$ be the normalization of $V$. Let $m, n \geq 1$ be integers. When $n = 1$, the notations $F_1^1$ and $F_1^1$ below are regarded as the zero divisors. We say that $V$ satisfies the modulus $m$ condition on $X \times B_n$, if as Weil divisors on $\nabla^N$ we have $(m + 1)[\nu^* (F_{n, 0})] \leq [\nu^* (F_{n, 1}^1)]$. If $V$ is a cycle on $X \times B_n$, we say that $V$ satisfies
the modulus condition if all of its irreducible components satisfy the modulus condition.

When $m$ is understood, often we just say that $V$ satisfies the modulus condition.

**Remark 2.2.** For $n \geq 1$, let $V \subseteq X \times B_n$ be an irreducible closed subscheme satisfying the modulus condition in Definition 2.1. Then, the closure of $V$ in $X \times \hat{B}_n$ can intersect $\{t = 0\}$ only along the divisor $F_n^1$. Hence, the closure of $V$ in $X \times \mathbb{A}^1 \times \square^{n-1}$ does not intersect $\{t = 0\}$. For $n = 1$, that the closure of $V$ in $X \times \mathbb{A}^1$ does not intersect $\{t = 0\}$ is equivalent to that $V$ satisfies the modulus condition for all $m \geq 1$. So, we may often use $\mathbb{A}^1 \times \square^{n-1}$ instead of $B_n = \mathbb{G}_m \times \square^{n-1}$.

### 2.2.2. Additive cycle complexes

We recall the following from [26, Definition 2.5]:

**Definition 2.3.** Let $X \in \text{Sch}_k^{\text{ess}}$, and let $r, m, n$ be integers with $m, n \geq 1$.

(0) $T^u Z_n(X, 1; m)$ is the free abelian group on integral closed subschemes $Z$ of $X \times \mathbb{G}_m$ of dimension $r$, satisfying the modulus condition.

For $n > 1$, $T^u Z_n(X, n; m)$ is the free abelian group on integral closed subschemes $Z$ of $X \times B_n$ of dimension $r + n - 1$ such that:

1. For each face $F$ of $B_n$, $Z$ intersects $X \times F$ properly on $X \times B_n$.
2. $Z$ satisfies the modulus condition on $X \times B_n$.

For each $1 \leq i \leq n$ and $\epsilon = 0, \infty$, let $\partial_i^\epsilon(Z) := [(\text{Id}_X \times \iota_{n,i,\epsilon})^\ast(Z)]$.

The proper intersection with faces ensures that $\partial_i^\epsilon(Z)$ are well-defined.

Algebraic cycles on $X \times B_n$ that belong to $T^u Z_n(X, n; m)$ are called the admissible cycles (additive higher Chow cycles, or additive cycles). When the scheme $X$ is equidimensional of dimension $d$ over $k$, we write for $q \geq 0$, $T^q(Z_n, n; m) := T^q Z_{d+1-q}(X, n; m)$.

This gives the cubical abelian group $(n \mapsto T^q Z_n(X, n+1; m))$ in the sense of [23, §1.1]. Using the containment lemma [24, Proposition 2.4], that each face $\partial_i^\epsilon(Z)$ lies in $T^q Z_n(X, n-1; m)$ is implied from (1) and (2).

**Definition 2.4.** Let $X \in \text{Sch}_k^{\text{ess}}$ be equidimensional. The additive higher Chow complex, or just the additive cycle complex, $T^q(Z_n; m)$ of $X$ in codimension $q$ and with the modulus $m$ condition is the nondegenerate complex associated to the cubical abelian group $(n \mapsto T^q Z_n(X, n+1; m))$, i.e., $T^q(Z_n; m) := T^q Z_n(X, n; m) / T^q Z_n(X, n; m)_{\text{deg}}$.

The boundary map of this complex at level $n$ is given by $\partial := \sum_{i=1}^{n-1} (-1)^i (\partial_i^\infty - \partial_i^0)$, and it satisfies $\partial^2 = 0$. The homology $\text{CH}^q(X, n; m) := H_n(T^q(Z_n; m))$ for $n \geq 1$ is the additive higher Chow group of $X$ with modulus $m$ condition.

### 2.3. Subcomplexes associated to some algebraic subsets

Here are subgroups of $T^q(Z_n, n; m)$ with a finer intersection property with a given finite set $W$ of locally closed algebraic subsets of an equidimensional $X \in \text{Sch}_k^{\text{ess}}$:

**Definition 2.5** (cf. [24, Definition 4.2]). Define $T^{q}_W(X, n; m)$ to be the subgroup of $T^q(Z_n, n; m)$ generated by integral closed subschemes $Z \subset X \times B_n$ such that $Z \in T^q(Z_n, n; m)$ and

$$\text{codim}_{W \times F}(Z \cap (W \times F)) \geq q$$

for all $W \in W$ and all faces $F \subset B_n$.
The groups $T^q_{qW}(X, \bullet; m)$ form a cubical subgroup of $T^q_{W}(X, \bullet; m)$ and they give the subcomplex $T^q_{qW}(X, \bullet; m) \subset T^q(X, \bullet; m)$ by modding out by the degenerate cycles. The homology groups are denoted by $TCH^q_{qW}(X, n; m)$.

Recall that (cf. [12, §2.2]) we say a scheme $X$ is an FA-scheme if given any finite subset $\Sigma \subset X$, there exists an affine open subset $U \subset X$ such that $\Sigma \subset U$. We have the following (see [12, §2.2]):

**Lemma 2.6.** Any quasi-projective $k$-scheme is FA. Any open subset of an FA-scheme is FA. Given any finite subset $\Sigma$ of a quasi-projective $k$-scheme, and an open subset $U \subset X$ containing $\Sigma$, there exists an affine open subset $W \subset U$ containing $\Sigma$.

**Definition 2.7.** Recall a semi-local $k$-algebra $R$ is essentially of finite type if there is a connected quasi-projective $k$-scheme $X = \text{Spec}(A)$ of finite type over $k$ and a finite set of (not necessarily closed) points $\Sigma$ of $X$ such that $R = O_{X, \Sigma}$. We say that it is of geometric type if $\Sigma$ consists of only closed points. In this case, the pair $(X, \Sigma)$ will be called an atlas for $V = \text{Spec}(R)$. An affine open subatlas $(Y, \Sigma)$ of $(X, \Sigma)$ for $V$ is an atlas for $V$ such that $Y \subset X$ is an affine open subset.

**Remark 2.8.** By FA-scheme property, we may assume that the schemes $X$ in Definition 2.7 are affine $k$-schemes. Hence, for any semi-local $k$-algebra $R$ essentially of finite type, we always have an atlas $(U, \Sigma)$, where $U$ is affine. So, usually we take only affine atlases.

We recall the following results from [29] that are frequently used in this paper:

**Lemma 2.9.** Let $V = \text{Spec}(R)$ be a regular semi-local $k$-scheme essentially of finite type with a finite set of points $\Sigma$. Let $m, n, q \geq 1$ and let $\alpha \in T^q(V, n; m)$ be a cycle. Then, there exists an atlas $(X, \Sigma)$ for $V$ and a cycle $\overline{\alpha} \in T^q(X, n; m)$ such that $\overline{\alpha}_V = \alpha$. If $\partial(\alpha) = 0$, we can assume that $\partial(\overline{\alpha}) = 0$. If $\alpha \in T^q_2(V, n; m)$, then $\overline{\alpha}$ is necessarily in $T^q_2(X, n; m)$.

2.4. Milnor $K$-groups. Let $R$ be a commutative ring with unity. Recall that the Milnor $K$-ring $K^M_n(R)$ of $R$ is defined as follows: let $T^q_Z(R^\times)$ be the tensor algebra on the multiplicative group $R^\times$ of units of $R$. Its degree zero part is $Z$. Let $I(R)$ be the two-sided ideal of $T^q_Z(R^\times)$ generated by elements of the form $a \otimes (1 - a)$, where $a \in R^\times$ is such that $1 - a \in R^\times$. The Milnor $K$-ring of $R$ is defined to be the graded ring $T^q_Z(R^\times)/I(R)$. Its degree $n \geq 0$ part is denoted by $K^M_n(R)$, and it is called the $n$-th Milnor $K$-group of $R$. The image of $a_1 \otimes \cdots \otimes a_n$ in $K^M_n(R)$ is denoted by $\{a_1, \cdots, a_n\}$.  

2.5. De Rham-Witt complexes.

2.5.1. Rings of Witt-vectors. Let $R$ be a commutative ring with unity. We briefly recall and sketch the definition of the big Witt rings of $R$ from [38, Appendix A]. A truncation set $S \subset \mathbb{N}$ is a nonempty set such that if $s \in S$ and $t|s$, then $t \in S$. As a set, let $\mathbb{W}_S(R) := R^S$ and define the map $w : \mathbb{W}_S(R) \to R^S$ by sending $a = (a_s)_{s \in S}$ to $w(a) = (w(a)_s)_{s \in S}$, where $w(a)_s := \sum_{t|s} t a_t^s$. When $R^S$ on the target of $w$ is given the component-wise ring structure, it is known (see e.g. [17, Proposition 1.2]) that there is a unique functorial ring structure on
\(\mathbb{W}_S(R)\) such that \(w\) is a ring homomorphism. For two truncation sets \(S \subseteq S'\), there is a restriction \(\mathcal{R} : \mathbb{W}_{S'}^m \to \mathbb{W}_S^m\). When \(S = \{1, \cdots, m\}\), we write \(\mathbb{W}_S^m = \mathbb{W}_S(R)\). For a fixed prime number \(p\), when \(S = \{1, p, p^2, \cdots\}\), and \(S_i = \{1, p, \cdots, p^{i-1}\}\), we write \(W = \mathbb{W}_S(R)\), \(W_i = \mathbb{W}_{S_i}(R)\), that are \(p\)-typical rings of Witt vectors.

There is an alternative description of the ring \(\mathbb{W}_S(R)\) for a truncation set \(S \subseteq \mathbb{N}\); let \(\mathbb{W}(R) := \mathbb{W}_{\mathbb{N}}(R)\). Consider the multiplicative group \((1 + TR[[T]])^\infty\), where \(T\) is an indeterminate. Then, there is a natural bijection \(\mathbb{W}(R) \simeq (1 + TR[[T]])^\infty\), where the addition of the ring \(\mathbb{W}(R)\) corresponds to the multiplication of the formal power series. For a truncation set \(S\), we can describe \(\mathbb{W}_S(R)\) as the quotient of \((1 + TR[[T]])^\infty\) by a suitable subgroup \(I_S\). See [38, A.7] for details. In case \(S = \{1, \cdots, m\}\), we have an isomorphism

\[
(2.2) \gamma : \mathbb{W}_m(R) \simeq (1 + TR[[T]])^\infty / (1 + T^{m+1} R[[T]])^\infty \quad ; \quad \prod_{i=1}^m (1 - a_i T^i) \mapsto (a_i)_{1 \leq i \leq m}
\]

There is the Teichmüller lift map \([-] : R \to \mathbb{W}_m(R)\) given by \(a \mapsto 1 - a T\), and for each \(i \geq 1\), we have the Verschiebung given by \(V_i([a]_{m/i}) = (1 - a T^i)\). Using this idea, one deduces (see [38, Properties A.4(i)]) that for \(x = (x_i) \in \mathbb{W}_m(R)\), we have

\[
(2.3) x = \sum_{i=1}^m V_i([x_i]_{m/i})
\]

2.5.2. De Rham-Witt complexes. Let \(R\) be a \(\mathbb{Z}_p\)-algebra for a prime \(p\). (Note that if \(R\) is a \(K\)-algebra for a field \(K\), whether its characteristic is 0 or positive, this is always a \(\mathbb{Z}_p\)-algebra for some \(p\).) For each truncation set \(S\), there is a differential graded algebra \(\mathbb{W}_S^m \Omega_R^\bullet\), which defines a functor on the category of truncation sets, and it is called the big de Rham-Witt complex over \(R\). This is an initial object in the category of \(V\)-complex and in the category of Witt-complexes over \(R\). Its rigorous definition is lengthy so that we direct the reader to [16] and [38]. In case \(S\) is a finite truncation set, we have \(\mathbb{W}_S^m \Omega_R^\bullet = \mathbb{W}_S \Omega_{S/R}^\bullet / N_S^\bullet\), where \(N_S^\bullet\) is the differential graded ideal given by some generators. See [38, Proposition 1.2] for details. In case \(S = \{1, 2, \cdots, m\}\), we write \(\mathbb{W}_m^m \Omega_R^\bullet\) for this object. For a prime number \(p\), when \(S = \{1, p, p^2, \cdots\}\), and \(S_i = \{1, p, \cdots, p^{i-1}\}\), we write \(\mathbb{W}_R^\bullet = \mathbb{W}_R^\bullet\), \(W_i \Omega_R^\bullet = \mathbb{W}_i \Omega_R^\bullet\), that are \(p\)-typical de Rham-Witt complexes.

The more relevant objects for this paper are restricted Witt-complexes over \(R\), for which we recall the full definition from [16, Definition 1.1.1]; a restricted Witt-complex over \(R\) is a projective system of differential graded \(\mathbb{Z}\)-algebras \((E_m)_{m \in \mathbb{N}}, \mathcal{R} : E_{m+1} \to E_m\), together with families of homomorphisms of graded rings \((F_r : E_{m+r-1} \to E_m)_{m, r \in \mathbb{N}}\) called Frobenius maps, and homomorphisms of graded groups \((V_r : E_m \to E_{m+r-1})_{m, r \in \mathbb{N}}\) called Verschiebung maps, satisfying the following relations for all \(n, r \in \mathbb{N}\):

(i) \(\mathcal{R}F_r = F_r \mathcal{R}, \mathcal{R}V_r = V_r \mathcal{R}, F_1 = V_1 = \text{Id}, F_r F_s = F_{rs}, V_r V_s = V_{rs}\);

(ii) \(F_r V_r = r\). When \((r, s) = 1\), then \(F_r V_s = V_s F_r\) on \(E_{m+r-1}\);

(iii) \(V_r(F_r(x)y) = x V_r(y)\) for all \(x \in E_{m+r-1}\) and \(y \in E_m\); (projection formula)

(iv) \(F_r dV_r = d\) (where \(d\) is the differential of the DGAs).
Furthermore, there is a homomorphism of projective system of rings \((\lambda : \mathbb{W}_m(R) \to E_m^0)_{m \in \mathbb{N}}\) that commutes with \(F_r\) and \(V_r\), and we have
\[ F_r d\lambda([a]) = \lambda([a]^{r-1}) d\lambda([a]) \] for all \(a \in R\) and \(r \in \mathbb{N}\),
where \([a]\) is the Teichmüller lift in \(\mathbb{W}_m(R)\) of \(a \in R\).

The system \(\{\mathbb{W}_m \Omega^R\}_{m \geq 1}\) is an initial object in the category of restricted Witt-complexes over \(R\). See \[38\], Proposition 1.15.

### 2.6. The presheaf \(\mathcal{TCH}\).

Recall from \[29\] that additive higher Chow groups\( T^q_{n,m}(X) = \mathcal{TCH}^q(X,n;m)\) give a presheaf on \(\text{SmAff}_k\), by using \[19\]. Furthermore, we further have a “presheafification” \(\mathcal{TCH}^q(-,n;m)\) on \(\text{Sch}_k\) defined as follows:

**Definition 2.10** ([29, §4.4]). Let \(X \in \text{Sch}_k\). The functor \(T^q_{n,m} : \text{SmAff}_k^{\text{op}} \to (Ab)\) induces the functor \(T^q_{n,m} : (X \downarrow \text{SmAff}_k)^{\text{op}} \to (Ab)\). Here, \((X \downarrow \text{SmAff}_k)\) is the category whose objects are the \(k\)-morphisms \(X \to A\), with \(A \in \text{SmAff}_k\), and a morphism from \(h_1 : X \to A\) to \(h_2 : X \to B\), with \(A,B \in \text{SmAff}_k\) is given by a \(k\)-morphism \(g : A \to B\) such that \(g \circ h_1 = h_2\). It is cofiltered. Define
\[
\mathcal{TCH}^q(X,n;m) := \colim_{(X,\text{SmAff}_k)^{\text{op}}} T^q_{n,m}.
\]

This definition occurred to the authors while working on the paper \[27\]. By \[29\] Proposition 4.8, we know that \(\mathcal{TCH}^q(-,n;m)\) is a presheaf on \(\text{Sm}_k\) and \(\text{Sch}_k\). There is a natural homomorphism \(\alpha_X : \mathcal{TCH}^q(X,n;m) \to \mathcal{TCH}^q(X,n;m)\), that becomes an isomorphism if \(X \in \text{SmAff}_k\).

Recall from \[29\] Remark 6.12] that using the above idea in \(\mathcal{TCH}^q\), we may redefine \(\mathbb{W}_m \Omega^{n-1}_R\) as follows: for \(X \in \text{Sm}_k\), define \(\mathbb{W}_m \Omega^n_{X}\) to be the presheaf \(\colim_{(X,\text{SmAff}_k)^{\text{op}}} \mathbb{W}_m \Omega^n_{(-)}\). By loc. cit., for each \(X = \text{Spec}(R) \in \text{SmAff}_k\), we have a natural homomorphism \(\tau_{n,m} : \mathbb{W}_m \Omega^{n-1}_R \to \mathcal{TCH}^q(R,n;m)\). So, by taking the colimits and the Zariski sheafifications, we obtain the morphism of Zariski sheaves \(\tau_{n,m} : \mathbb{W}_m \Omega^{n-1}_{\text{zar}} \to \mathcal{TCH}^q(-,n,m)_{\text{zar}}\). Theorem \[11\] then shows that this is an isomorphism of Zariski sheaves on \(\text{Sm}_k\).

### 3. The de Rham-Witt-Chow homomorphism and injectivity

In Section 3, we let \(k\) be a perfect field of characteristic \(\neq 2\).

#### 3.1. The de Rham-Witt-Chow homomorphism.

In this paper, we will mostly consider \(X \in \text{SmAff}_k^{\text{ess}}\). We let \(\mathcal{TCH}^M(X;m) := \mathcal{TCH}^*(X,\bullet;m)\) in the Milnor range. Let \(\mathcal{TCH}^M(X) := \{\mathcal{TCH}^M(X;m)\}_{m \geq 1}\). We similarly define \(\mathcal{TCH}^M(X;m)\) and \(\mathcal{TCH}^M(X)\).

Recall from \[29\] Theorem 1.3 that, for any ring \(R\) such that \(\text{Spec}(R) \in \text{SmAff}_k^{\text{ess}}\), the pro-system \(\mathcal{TCH}^M(R) = \{\mathcal{TCH}^*(R,\bullet;m)\}_{m \in \mathbb{N}}\) is a restricted Witt-complex over \(R\), in the sense of Section 2.5.2 with respect to the restriction \(\mathfrak{R}\), the differential \(\delta\), the Frobenius \(F_r\) and the Verschiebung \(V_r\) for \(r \geq 1\), and the ring homomorphisms \(\tau_R : \mathbb{W}_m(R) \to \mathcal{TCH}^1(R,1;m)\). Hence, the universal property
of the big de Rham-Witt complex \( \{ \mathbb{W}_m \Omega_R^* \}_{m \in \mathbb{N}} \) induces the system of homomorphisms for \( n, m \geq 1 \)

\[
\tau_{n,m}^R : \mathbb{W}_m \Omega_R^{n-1} \to \text{TCH}^n(R, n; m), \quad \tau_{1,m}^R = \tau_R,
\]

that we would like to call the de Rham-Witt-Chow homomorphism. These maps \( \{ \tau_{n,m}^R \}_{n,m \in \mathbb{N}} \) give a morphism of restricted Witt-complexes over \( R \). From this, we deduce the following identities that we use often in this paper.

\[
\tau_{n,m}^R d = \delta \tau_{n-1,m}^R; \quad \tau_{n,rm+r-1}^R V_r = V_r \tau_{n,m}^R; \quad \tau_{n,m}^R F_r = F_r \tau_{n,rm+r-1}^R.
\]

The second identity has the following variation that we use, up to applying \( \mathfrak{R} \):

\[
\tau_{n,m}^R V_r = V_r \tau_{n,\lfloor m/r \rfloor}^R,
\]

where for a non-negative real number \( x \), one denotes by \( \lfloor x \rfloor \) the greatest integer not bigger than \( x \). Since we use only some of the properties of the operators \( \mathfrak{R}, \delta, F_r, V_r, \tau_R \) in this paper, we won’t recall their definitions unless we need.

The main theorem of this paper is the following:

**Theorem 3.1.** Let \( R \) be a regular semi-local \( k \)-algebra essentially of finite type over a perfect field \( k \) of characteristic \( \neq 2 \). Then, for \( m, n \geq 1 \), the map \( \tau_{n,m}^R : \mathbb{W}_m \Omega_R^{n-1} \to \text{TCH}^n(R, n; m) \) is an isomorphism.

Here, the de Rham-Witt-Chow map \( \tau_{n,m}^R \) is from (3.1). This theorem generalizes the isomorphism of [38, Theorem 1], and it is an additive analogue of [10, Theorem 3.4], [20], and [21].

For such \( R \), by [29, Theorem 1.3(3)], we already have an isomorphism \( \tau_{1,m}^R = \tau_R : \mathbb{W}_m(R) \xrightarrow{\sim} \text{TCH}^1(R, 1; m) \), but extending it to \( n \geq 2 \) is very nontrivial. In this section, we first prove that \( \tau_{n,m}^R \) is injective in Corollary 3.6. For surjectivity, we will need a somewhat technical moving lemma for additive cycles in the Milnor range in Theorem 4.13 and this is proved over next several sections. This moving lemma will be used to complete the proof of the desired surjectivity in Section 10.

### 3.2. Functoriality and 2-functoriality

Recall the following functoriality results for pull-backs and push-forwards for additive higher Chow groups:

**Lemma 3.2** ([29, Theorems 6.1, 6.16]). Let \( R \to S \) be a finite extension of regular semi-local rings essentially of finite type over \( k \) and let \( f : \text{Spec}(S) \to \text{Spec}(R) \) denote the induced morphism of schemes. Let \( n, m, r \geq 1 \) be integers. Then, the pull-back \( f^* : \text{TCH}^n(R, n; m) \to \text{TCH}^n(S, n; m) \) and the push-forward \( f_* : \text{TCH}^n(S, n; m) \to \text{TCH}^n(R, n; m) \) induce morphisms of Witt-complexes over \( k \). In particular, we have identities:

\[
\mathfrak{R} f^* = f^* \mathfrak{R}; \quad \delta f^* = f^* \delta; \quad F_r f^* = f^* F_r; \quad V_r f^* = f^* V_r;
\]

\[
\mathfrak{R} f_* = f_* \mathfrak{R}; \quad \delta f_* = f_* \delta; \quad F_r f_* = f_* F_r; \quad V_r f_* = f_* V_r.
\]

We now discuss the 2-functoriality of \( \tau_{n,m} \).

**Lemma 3.3.** Let \( R \to S \) be a flat extension of regular semi-local rings essentially of finite type over \( k \). Let \( f : \text{Spec}(S) \to \text{Spec}(R) \) be the induced map of semi-local schemes. Then \( f^* \circ \tau_{1,m}^R([a]) = \tau_{1,m}^S \circ f^*([a]) \) for all \( a \in R \).
Proof. Let $\Gamma_{(1-at)}$ denote the cycle in $\text{TCH}^1(R, 1; m)$ corresponding to the ideal $(1-at) \subset R[t]$. By definition of $\tau_{1,m}^R$, we have $f^* \circ \tau_{1,m}^R([a]) = f^*([\Gamma_{(1-at)}])$. Since $R \to S$ is a flat extension, $f^*([\Gamma_{(1-at)}])$ is the cycle in $\text{TCH}^1(S, 1; m)$ corresponding to the ideal $(1-at) \subset S[t]$. We conclude that $f^*([\Gamma_{(1-at)}]) = \tau_{1,m}^S([a])$, which proves the lemma. \qed

**Proposition 3.4.** Let $R \to S$ be a flat extension of regular semi-local rings which are essentially of finite type over $k$. Let $f : \text{Spec}(S) \to \text{Spec}(R)$ be the induced map of semi-local schemes. Then, the diagram

\[
\begin{array}{ccc}
\mathbb{W}_m \Omega_R^{n-1} -\tau_{m,n}^R & \xrightarrow{\tau_{n,m}^R} & \text{TCH}^n(R, n; m) \\
f^* \downarrow & & \downarrow f^* \\
\mathbb{W}_m \Omega_S^{n-1} -\tau_{m,n}^S & \xrightarrow{\tau_{n,m}^S} & \text{TCH}^n(S, n; m)
\end{array}
\]

commutes for all integers $m, n \geq 1$.

**Proof.** Using the surjection $\Omega_{\mathbb{W}_m(R)}^{n-1} \to \mathbb{W}_m \Omega_R^{n-1}$ and the presentation (2.3), it suffices to show the commutativity for the de Rham-Witt forms of the kind $\omega = V_{i_0}([a]) \cdot dV_{i_1}([b_1]) \wedge \cdots \wedge dV_{i_{n-1}}([b_{n-1}])$, where $a, b_1, \ldots, b_{n-1} \in R$. We will use that $\tau_{n,m}^R$ and $\tau_{n,m}^S$ are morphisms of Witt-complexes over $R$ and $S$ to get

\[
f^* \circ \tau_{n,m}^R(w) = f^*(V_{i_0}([a]) \cdot \bigwedge_{j=1}^{n-1} dV_{i_j}([b_j])) = f^*(V_{i_0}(\tau_{1,m}^R([a])) \cdot \bigwedge_{j=1}^{n-1} \delta V_{i_j}(\tau_{1,m}^R([b_j])))
\]

\[
= V_{i_0} f^*(\tau_{1,m}^S([a])) \cdot \bigwedge_{j=1}^{n-1} \delta V_{i_j} f^*(\tau_{1,m}^S([b_j]))
\]

\[
= \tau_{n,m}^S \left( V_{i_0} f^*(\tau_{1,m}^S([a])) \cdot \bigwedge_{j=1}^{n-1} \delta V_{i_j} f^*(\tau_{1,m}^S([b_j])) \right)
\]

\[
= \tau_{n,m}^S \left( f^*(V_{i_0}([a]) \cdot \bigwedge_{j=1}^{n-1} dV_{i_j}([b_j])) \right)
\]

where the equalities $\dagger$ hold by Lemma 3.2 and $\ddagger$ holds by Lemma 3.3. All other equalities follow from that $\tau_{n,m}^R$ and $\tau_{n,m}^S$ are the morphisms of Witt-complexes. This proves the proposition. \qed

### 3.3. Injectivity

Injectivity of $\tau_{n,m}^R$ requires the following:

**Proposition 3.5.** Let $R$ be a regular semi-local $k$-algebra and let $K = \text{Frac}(R)$. Then, for $m \geq 1$ and $n \geq 0$, the natural map $\mathbb{W}_m \Omega^n_R \to \mathbb{W}_m \Omega^n_K$ is injective.

**Proof.** We prove it in two steps.

**Step 1.** Consider the case $m = 1$, i.e. we show that $\Omega^n_R \to \Omega^n_K$ is injective. **Claim:** $\Omega^n_R$ is a free $R$-module, possibly of an infinite rank.

This is obvious for $i = 0$. Suppose $i \geq 1$. Consider the Jacobi-Zariski exact sequence of the maps $\mathbb{Z} \to k \to R$ from [32, 3.5.5.1] (which generalizes [15, Proposition 8.3A]):

\[
\cdots \to D_1(R/k) \to \Omega^1_{k/\mathbb{Z}} \otimes_k R \to \Omega^1_{R/\mathbb{Z}} \to \Omega^1_{R/k} \to 0,
\]

where $D_1(R/k)$ is the first André-Quillen homology of M. André [2] and D. Quillen [36] (see [32, 3.5.4]). Since $R$ is smooth over $k$, we have $D_1(R/k) = 0$ by [32.
Theorem 3.5.6]. On the other hand, since \( R \) is a regular local \( k \)-algebra, \( \Omega^1_{R/k} \) is a free \( R \)-module so that we have an isomorphism \( \Omega^1_{R/k} \simeq \Omega^1_{R/k} \oplus (\Omega^1_{k/Z} \otimes_k R) \). Since \( \Omega^1_{k/Z} \) is a free \( k \)-module (a \( k \)-vector space), the space \( \Omega^1_{k/Z} \otimes_k R \) is a free \( R \)-module. Hence, \( \Omega^1_{R/Z} \) is a free \( R \)-module. Taking wedge products, we deduce that \( \Omega^n_{R/Z} \) is a free \( R \)-module for all \( i \geq 1 \), proving the Claim.

Going back to the proof of Step 1, apply the functor \( - \otimes_R \Omega^1_{R/Z} \) to the inclusion \( R \hookrightarrow K \). By Claim, the module \( \Omega^1_{R/Z} \) is free so that we get the injection \( \Omega^1_{R/Z} \hookrightarrow K \otimes_R \Omega^1_{R/Z} \), where the latter group is isomorphic to \( \Omega^1_{K/Z} \) by [15] Proposition 8.2A. Hence, Step 1 is proven.

**Step 2.** We now suppose \( m \geq 1 \) is any integer. When \( \text{char}(k) = 0 \), by [38] Remark 1.12, \( W_m \Omega^n_R \to W_m \Omega^n_K \) decompose into a direct product of \( \Omega^n_{R/Z} \to \Omega^n_{K/Z} \), each of which is injective by Step 1. Hence, their direct product is also injective.

When \( \text{char}(k) = p > 0 \), recall that by [17] Example 1.11 (also [38] Theorem 1.11]), \( W_m \Omega^n_R \to W_m \Omega^n_K \) decomposes into a direct product of some copies of maps of \( p \)-typical de Rham-Witt forms \( W_s \Omega^n_R \to W_s \Omega^n_K \) of finite lengths, where the product is over some values of \( s \) (see loc. cit for the precise values). But, it is proven in by M. Gros [13] Proposition 5.1.2 that for any smooth \( k \)-scheme \( X \), the Cousin complex of \( W_s \Omega^n_X \) is a resolution of \( W_s \Omega^n_X \). In particular, each \( W_s \Omega^n_R \to W_s \Omega^n_K \) is injective. Hence, \( W_m \Omega^n_R \to W_m \Omega^n_K \) is injective. This completes the proof of the proposition.

**Corollary 3.6.** Let \( R \) be a regular semi-local \( k \)-algebra. Then, \( \tau^n_{R,m} \) is injective.

**Proof.** Let \( K = \text{Frac}(R) \). We have a commutative diagram

\[
\begin{array}{ccc}
W_m \Omega^{n-1}_R & \longrightarrow & W_m \Omega^{n-1}_K \\
\tau^n_{R,m} \downarrow & & \tau^n_K \downarrow \\
\text{TCH}^n(R, n; m) & \longrightarrow & \text{TCH}^n(K, n; m),
\end{array}
\]

where the bottom map is the flat pull-back via \( \text{Spec}(K) \to \text{Spec}(R) \), the map \( \tau^n_K \) is the isomorphism of [38] Theorem 1, and the top horizontal map is injective by Proposition 3.5. In particular, the map \( \tau^n_{R,m} \) is injective. \( \square \)

Thus, it remains to prove the surjectivity part of Theorem 3.1.

### 3.4. A reduction.

Here is one reduction that simplifies our proof of surjectivity:

**Lemma 3.7.** If the statement of Theorem 3.1 holds for all regular semi-local \( k \)-algebras obtained by localizing at a finite set of closed points of a regular affine variety over \( k \), then it holds for any regular semi-local \( k \)-algebra essentially of finite type over \( k \).

**Proof.** Let \( R \) be any regular semi-local \( k \)-algebra essentially of finite type, i.e. \( R = O_{V, \Sigma} \), where \( V \) is a smooth \( k \)-variety and \( \Sigma \subset V \) is a finite subset. Since the map \( \tau^n_{R,m} \) is injective by Corollary 3.6 it is enough to prove that \( \tau^n_{R,m} \) is surjective.

Given any cycle class \( \alpha \in \text{TCH}^n(R, n; m) \), choose a representative, also denoted by \( \alpha \). Then, by Lemma 2.4 there exists an affine open subset \( U \subset V \) containing \( \Sigma \) such that the Zariski closure \( \alpha_U \) of \( \alpha \) on \( U \times B_n \) is still an admissible cycle.
For each $p \in \Sigma$, choose a closed point $m_p \in U$ that is a specialization of $p$ (which exists by the basic fact in commutative algebra that any proper ideal of a ring is contained in a maximal ideal). We let $\Sigma' = \{m_p \mid p \in \Sigma\}$, and let $R' := \mathcal{O}_{U, \Sigma'}$. Here $\alpha_U \in \TCH^n(U, n; m)$, which gives $\alpha' \in \TCH^n(R', n; m)$ by pulling back via the flat map $\Spec (R') \to U$. We also have the localization map $\phi : R' \to R$, which gives the diagram by Proposition 3.3.

\[ \begin{array}{c}
\mathbb{W}_m \Omega_R^{n-1} \\
\phi_1
\end{array} \xrightarrow{\tau_{n,m}^{R'}} \TCH^{n}(R', n; m) \xrightarrow{\phi_2} \mathbb{W}_m \Omega_R^{n-1} \xrightarrow{\tau_{n,m}^{R}} \TCH^{n}(R, n; m). \]

Here the vertical maps are the flat pull-back maps induced by the localization $\phi : R' \to R$, and by construction $\phi_2(\alpha') = \alpha$. By the given assumption, Theorem 3.1 holds for $R'$ so that the map $\tau_{n,m}^{R'}(\beta') = \alpha'$. Then, by the commutativity of the diagram (3.6), we have $\tau_{n,m}^R(\phi_1(\beta')) = \alpha$, which proves the surjectivity of $\tau_{n,m}^R$ for $R$. \hfill \Box

3.5. Product with higher Chow groups. We use the following later:

**Lemma 3.8.** Let $X \in \Sch^{ess}_k$ be equidimensional and $m, n, q \geq 1$. Then, there is a homomorphism $\zeta_n : \CH^q(X, n - 1) \to \TCH^{q+1}(X, n; m)$ given by sending an irreducible cycle $Z \in z^q(X, n - 1)$ to $\zeta_n(Z) = \tau(\{1\} \times Z) \subset X \times B_n$, where $\tau : G_m \times X \times \square^{n-1} \to X \times G_m \times \square^{n-1}$ is the obvious transposition map. This map $\zeta_n$ commutes with the pull-back and push-forward whenever they exist.

**Proof.** To show that $\zeta_n(Z)$ is indeed admissible, one can note that $\overline{\zeta_n(Z)} = \tau(\{1\} \times Z) \subset X \times \overline{B_n}$, and it is admissible by definition. Since $\zeta_n$ clearly commutes with the boundary operators, it defines a map of the cycle complexes and on the homology. The functoriality part is obvious. \hfill \Box

**Proposition 3.9.** Let $R$ be a smooth $k$-algebra. Then, there is a natural cap product map

$$\cap_R : \TCH^p(R, n; m) \otimes \CH^p(R', n') \to \TCH^{p+q}(R, n + n'; m),$$

that commutes with the pull-back and push-forward whenever they exist. This is given by $b \cap_R a = \Delta^*_R(a \times \zeta(b))$, where $\Delta^*_R$ is the pull-back via the diagonal map $\Delta_R : \Spec (R) \to \Spec (R) \times \Spec (R)$.

**Proof.** This is a refinement of [23, Theorem 4.10] to the case when $X$ is affine. Here we write it as a right action of the Chow group on the additive Chow group. Here, the moving lemma on additive higher Chow groups of smooth affine $k$-scheme by [19] shows that $\Delta^*_R$ is defined on the external products of higher Chow cycles and the additive higher Chow cycles. \hfill \Box

We remark that we may use [28, Theorem 3.12] together with the homotopy invariance for $\CH^p(R, n')$ to give a different proof. We do not attempt it here.
4. The fs and sfs-cycles

In Section 4, we suppose $k$ is any field, unless said otherwise. The objective of Sections 4 ~ 9 is to prove Theorem 4.13, which shows that any additive higher Chow cycle in the Milnor range with trivial boundary is equivalent to a sum of very special types of cycles, called sfs-cycles. All these sections are devoted to prove this result, and unfortunately the arguments involved are rather lengthy and complicated.

In this section, we first introduce two special classes of cycles, the fs-cycles and the sfs-cycles, and address some basic properties.

Let $V$ be a regular semi-local $k$-algebra essentially of finite type. Given any atlas $(X, \Sigma)$ for $V$ and a map $Z \to X$, the scheme $Z \times_X V$ will be denoted by $Z_V$, and we will call it the restriction of $Z$ on $V$. More generally, for any $k$-scheme $B$ and a cycle $Z = \sum_{i=1}^s a_i Z_i$ on $X \times B$, where $a_i \in \mathbb{Z}$ and $Z_i$ irreducible, we will define $Z_V$ to be the cycle $\sum_{i=1}^s a_i(Z_i)_V$.

4.1. fs and sfs-cycles. Let $k$ be any field.

Definition 4.1. Let $X \in \text{Sch}_k^{\text{ess}}$. We fix integers $m, n \geq 1$. Recall that for $n \geq 1$, $\widehat{B}_n$ denotes the scheme $\mathbb{P}_k^n \times (\mathbb{P}_k^1)^{n-1}$ with coordinates $(t, y_1, \ldots, y_{n-1})$ and we have the canonical open inclusion $B_n \subseteq \widehat{B}_n$ with complement $F_n$. For $1 \leq j \leq n$, let $\pi_j : B_n \to B_j$ and $\widehat{\pi}_j : \widehat{B}_n \to \widehat{B}_j$ be the projection maps given by $(t, y_1, \ldots, y_{n-1}) \mapsto (t, y_1, \ldots, y_{j-1})$. For any irreducible closed subscheme $Z \subseteq X \times B_n$, let $Z^{(j)} = (\text{Id}_X \times \pi_j)(Z)$. We extend it $\mathbb{Z}$-linearly to cycles $Z$ on $X \times B_n$. This $Z^{(j)}$ is not necessarily a closed subscheme of $X \times B_j$. However, if $Z$ is finite over $X$, then the morphisms in the sequence $Z = Z^{(n)} \to Z^{(n-1)} \to \cdots \to Z^{(1)} \to X$ are all finite, so each $Z^{(j)}$ is closed in $X \times B_j$.

Definition 4.2. Let $X = \text{Spec} (A)$ be a smooth affine $k$-scheme of finite type over $k$, and let $\Sigma$ be a finite set of closed points on $X$. Let $V = \text{Spec} (\mathcal{O}_{X, \Sigma})$.

For a smooth affine geometrically integral $k$-variety $B$ of dimension $n$, an algebraic cycle $Z \in z^n(X \times B)$ is called an fs-cycle along $\Sigma$, if each irreducible component of $Z$ is finite and surjective over an affine open neighborhood of $\Sigma$ under the projection map $X \times B \to X$. For simplicity, we call them just fs-cycles when $(X, \Sigma)$ is understood.

In case $B = B_n$, the subgroup of admissible fs-cycles in $Tz^n(X, n; m)$ will be denoted by $Tz^n_{\Sigma, \text{fs}}(X, n; m)$.

We shall say that a cycle $Z \in Tz^n(X, n, m)$ is an sfs-cycle along $\Sigma$ if for every irreducible component $Z_i$ of $Z$, the following hold:

1. $[Z_i] \in Tz^n_{\Sigma, \text{fs}}(X, n; m)$.
2. $Z^{(j)}_i$ is smooth over $k$ for each $1 \leq j \leq n$.

The subgroup of sfs-cycles will be denoted by $Tz^n_{\Sigma, \text{sfs}}(X, n; m)$. We can replace $X$ by $V$ and define the above groups. In this case, an sfs-cycle along $\Sigma$ will be simply called an sfs-cycle. Moreover, we often suppress $\Sigma$ in the notations of $Tz^n_{\Sigma, \text{fs}}(V, n; m)$ and $Tz^n_{\Sigma, \text{sfs}}(V, n; m)$.

We will use the following notion:
**Definition 4.3.** Let $V = \text{Spec} (R)$ be a smooth semi-local $k$-scheme of geometric type with the set of closed points $\Sigma$. Let $m, n, p \geq 1$ and let $\alpha \in Tz^n(V, n; m)$ be a cycle. We say that $V$ is $\alpha$-linear if there is an atlas $(X, \Sigma)$ as in Lemma 2.9 such that $X$ is an affine space over $k$.

Here is one convenient result on finiteness often used in the paper:

**Lemma 4.4.** Let $X$ be an irreducible smooth affine $k$-scheme. Let $B$ be a smooth affine geometrically integral scheme of finite type over $k$ of dimension $n > 0$, and let $\hat{B}$ be a smooth projective geometrically integral scheme over $k$, with an open immersion $B \subset \hat{B}$.

Let $Z \in z^n(X \times B)$ be an irreducible cycle. Then, $Z \rightarrow X$ is finite and surjective over $X$ if and only if $Z$ is closed in $X \times \hat{B}$.

**Proof.** Let $f : Z \rightarrow X \times \hat{B} \rightarrow X$ be the composite map. Suppose $f$ is finite and surjective. Since the last map is projective, by [15] Corollary II-4.8-(e), Theorem II-4.9 it follows that the first map is a closed immersion. This proves $(\Rightarrow)$. Conversely, suppose that $Z$ is closed in $X \times \hat{B}$, i.e. the first map is a closed immersion (thus projective). Since the second map is projective, the composite $f$ is projective. Hence, $f$ is a projective morphism of affine schemes, so that it must be finite by [15] Exercise II-4.6. Moreover, being a finite map of irreducible affine schemes of the same dimension, it must also be surjective. This proves $(\Leftarrow)$. □

**Lemma 4.5.** Let $V = \text{Spec} (R)$ be a smooth semi-local $k$-scheme of geometric type with the set of closed points $\Sigma$. Let $B, \hat{B}$ be as in Lemma 4.4. Let $F := \hat{B} \setminus B$. Let $Z \in z^n(V \times B)$ be an irreducible cycle and let $\hat{Z}$ be the closure of $Z$ in $V \times \hat{B}$.

Suppose that $\hat{Z} \cap (\Sigma \times F) = \emptyset$. Then, given any atlas $(X, \Sigma)$ for $V$, there exists an affine open subatlas $(U, \Sigma)$ for $V$ such that for the Zariski closure $\overline{Z}$ of $Z$ in $X \times B$, the projection map $\overline{Z}_U \rightarrow U$ finite and surjective.

**Proof.** Let $(X, \Sigma)$ be a given atlas. Let $\hat{Z}$ be the Zariski closure of $Z$ in $X \times \hat{B}$ and let $\hat{f} : \hat{Z} \rightarrow X$ be the projection. Let $Y := \hat{f}(\hat{Z} \cap (X \times F))$. Since $\hat{f}$ is projective and since $\hat{Z} \cap (\Sigma \times F) = \emptyset$, we see that $Y \subset X$ is a closed subset disjoint from $\Sigma$. Hence, $X \setminus Y$ is an open neighborhood of $\Sigma$ such that $\hat{Z} \cap ((X \setminus Y) \times F) = \emptyset$. By Lemma 2.6, we can find an affine open neighborhood $U$ of $\Sigma$ in $X \setminus Y$, so we have $\hat{Z} \cap (U \times F) = \emptyset$, and in particular, $\hat{Z} \cap (U \times \hat{B}) = \overline{Z} \cap (U \times \hat{B})$. This means $\overline{Z}_U$ is closed in $U \times \hat{B}$. Hence, by Lemma 4.4 the map $\overline{Z}_U \rightarrow U$ is finite and surjective. □

We now get the following characterization of $\text{fs}$-cycles in $Tz^n(V, n; m)$:

**Proposition 4.6.** Let $V = \text{Spec} (R)$ be a smooth semi-local $k$-scheme of geometric type with the set of closed points $\Sigma$. Let $m, n \geq 1$ and let $[Z] \in Tz^n(V, n; m)$ be an irreducible cycle. Then, $[Z]$ is an $\text{fs}$-cycle if and only if there is an atlas $(X, \Sigma)$ for $V$ such that for the closure $\overline{Z}$ in $X \times B_n$ and $\hat{Z}$ in $V \times \hat{B}_n$, we have $[Z] \in Tz^n_\Sigma(X, n; m)$ and $\hat{Z} \cap (\Sigma \times F_n) = \emptyset$. 
First suppose that $Z \in T_z^n(V,n;m)$, by Lemma 2.9 one can find an affine atlas $(X,\Sigma)$ for $V$ such that $\overline{Z} \in T_z^n(X,n;m)$. Since $Z \to V$ is finite surjective, by Lemma 4.4, $\overline{Z} \cap (\Sigma \times F_n) = \emptyset$. Then, the assertion follows by Lemma 4.3.

Conversely, suppose that for an atlas $(X,\Sigma)$ and the closure $\overline{Z}$ in $X \times B_n$, we have $\overline{Z} \in T_z^n(X,n;m)$ and $\overline{Z} \cap (\Sigma \times F_n) = \emptyset$. Then, by Lemma 4.3 we may shrink $(X,\Sigma)$ to an affine open atlas $(U,\Sigma)$ such that $\overline{Z}_U \to U$ is finite and surjective. We still have $\overline{Z}_U \in T_z^n(U,n;m)$ and it shows $Z \in T_z^n(V,n;m)$. Since finite surjectivity is stable under base change to $V$, we deduce that $Z \to V$ is finite and surjective. Hence $Z$ is an $fs$-cycle.

Lemma 4.7. Let $V = \text{Spec}(R)$ be a smooth semi-local $k$-scheme of geometric type with the set of closed points $\Sigma$. Let $m, n \geq 1$ and let $\alpha \in T_z^n(V,n;m)$ be a cycle each of whose component is finite over $V$. Then $\alpha$ does not intersect any face $F \subset \Box^{n-1}$ at all.

Proof. We can assume that $\alpha = [Z]$ is an irreducible cycle. We may further assume that $\Sigma$ is a singleton set so that $V$ is the spectrum of a regular local ring $R$. By taking successive quotients of $R$ by the subsets of a chosen system of parameters for $R$ and using an induction on the dimension of $R$, we can reduce to the case when $R$ is a DVR. This induction step requires finiteness of $\alpha \to V$. Let $i : \{x\} \to V$ and $j : \eta \to V$ be the inclusions of the close and the generic points of $V$. If $Z$ intersects any face $F$, then either $i^*(Z)$ or $j^*(Z)$ must intersect $F$. On the other hand, proper intersection of $Z$ with $\{x\} \times F$ and $X \times F$ implies that neither of $i^*(Z)$ or $j^*(Z)$ intersect with $F$. Hence $Z$ does not intersect $F$. these can happen. 

Proposition 4.8. Let $V = \text{Spec}(R)$ be a smooth semi-local scheme of geometric type over a perfect field $k$ with the set of closed points $\Sigma$. Then, an irreducible cycle $[Z] \in T_z^n(V,n;m)$ is an $fs$-cycle if and only if there is an atlas $(X,\Sigma)$ for $V$ and an irreducible cycle $[\overline{Z}] \in T_z^n(X,n,m)$ for which the following hold.

(1) $Z = \overline{Z} \times_X V$.

(2) $\overline{Z}$ is closed in $X \times \hat{B}_n$, contained in $A_k^1 \times (A_k^1)^{n-1}$, and does not intersect any face $F \subset \Box^{n-1}$. (In particular, by Lemma 4.4, $\overline{Z}$ is finite and surjective over $X$.)

(3) For each $1 \leq j \leq n$, each $\overline{Z}^{(j)} = (\text{Id}_X \times \pi_j)(\overline{Z}) \subset X \times B_j$ is an irreducible closed subscheme, where $\pi_j$ is given by $(t,y_1,\cdots,y_{n-1}) \mapsto (t,y_1,\cdots,y_{j-1})$, and its coordinate ring is given by

$$k[\overline{Z}^{(j)}] = A[t,y_1,\cdots,y_{j-1}]/I(\overline{Z}^{(j)}) = A[a,b_1,\cdots,b_{j-1}],$$

where $A = k[X]$, and they form a sequence of finite extensions of integral domains

$$A \subset A[a] \subset A[a,b_1] \subset \cdots \subset A[a,b_1,\cdots,b_{n-1}]$$

such that each ring in the sequence is smooth over $k$.

(4) There are irreducible monic polynomials $P(t) \in A[t]$ in $t$ and $Q_j(y_j) \in A[a,b_1,\cdots,b_{j-1}][y_j]$ in $y_j$ for $1 \leq j \leq n-1$ such that $A[a] = A[t]/(P(t))$ and $A[a,b_1,\cdots,b_j] = A[a,b_1,\cdots,b_{j-1}][y_j]/(Q_j(y_j))$. 


Proof. For the ($\iff$) direction, it is clear that the existence of an atlas $(X, \Sigma)$ for $V$ and a cycle $[Z] \in \text{Tz}_{\Sigma}^n(X, n, m)$ satisfying (1)$\sim$(4) implies that $[Z] \in \text{Tz}_{\Sigma}^n(V, n, m)$ is an sfs-cycle over $V$. So we need to prove the converse ($\Rightarrow$).

Suppose that $[Z] \in \text{Tz}_{\Sigma}^n(V, n, m)$ is an irreducible sfs-cycle. Since it is an fs-cycle, by definition, by Lemma 4.4, Proposition 4.6 and Lemma 4.7 there is an atlas $(X, \Sigma)$ for $V$ such that the closure $\overline{Z}$ of $Z$ in $X \times B_n$ is an irreducible admissible cycle in $\text{Tz}_{\Sigma}^n(X, n, m)$ which satisfies (1) and (2).

Since $\overline{Z} \to X$ is finite surjective, there is a sequence of finite and surjective maps $\overline{Z} = \overline{Z}^{(n)} \to \cdots \to \overline{Z}^{(1)} \to X = \text{Spec}(A)$ such each $\overline{Z}^{(j)}$ is finite and surjective over $X$. Furthermore, each $\overline{Z}^{(j)}_V$ is smooth because $Z$ is an sfs-cycle.

We now want to show that there is an affine open subatlas $(U, \Sigma)$ of $(X, \Sigma)$ for $V$ such that the restrictions $\overline{Z}^{(j)}_U$ to $U$ are all smooth. To prove it, set $A^j = k[\overline{Z}^{(j)}]$. By the perfectness of $k$, each $\Omega^1_{A^j/k}$ is a finite $A$-module, such that $\Omega^1_{S_{\Sigma}^{-1}A^j/k}$ is a free $R$-module, where $R = S_{\Sigma}^{-1}A$ for the multiplicative subset $S_{\Sigma} \subset A$ corresponding to the finite set of closed points $\Sigma$. Hence, there is an affine open neighborhood $U$ of $\Sigma$ in $X$ such that each $\Omega^1_{A^j/k|U}$ is free $k[U]$-module. Replacing the given atlas $(X, \Sigma)$ by the new $(U, \Sigma)$, we may thus assume that each $\overline{Z}^{(j)}$ is smooth. Hence, we proved (3).

To prove (4), we observe that if we replace $A$ in the sequence (1.1) by its semi-local ring $R$ by localization, then we get a sequence of finite extensions of smooth semi-local rings. Note that such rings are UFDs by Auslander-Buchsbaum and $R[a] = R[t]/I_1$ for the prime ideal $I_1 = I(Z^{(1)})$. Since $\dim(R) = \dim(R[a]) = \dim(R[t]) - 1$, we have $ht(I_1) = 1$. But, $R$ is a UFD so that $I_1$ must be principal by Proposition I.1.12A]. Thus, if $P(t) \in R[t]$ is a monic irreducible polynomial of $a$, then we have $I_1 = P(t)$. Similarly, we have $R[a,b_1] = R[a][y_1]/I_2$ for the prime ideal $I_2 = I(Z^{(2)})$, and since $R[a]$ is a UFD, $R[a][y_1]$ is also a UFD. Hence, we obtain $I_2 = \langle Q_1(y_1) \rangle$ in the same way. Continuing this way, we get the irreducible monic polynomials $P(t) \in R[t]$ and $Q_j(y_j) \in R[a,b_1,\cdots,b_{j-1}]$ for which the property (4) holds over $R$. Choose lifts of these polynomials over $A$, and then there is a localization $A' = A[f^{-1}]$ with the inclusions $A \hookrightarrow A' \hookrightarrow R$ such that the property (4) holds over $A'$. Replacing $(X, \Sigma)$ again by $(\text{Spec}(A'), \Sigma)$, we obtain a new atlas for $V$ for which all of (1)$\sim$(4) hold. \hfill $\square$

Remark 4.9. The reader should observe in the above proof that without the perfectness assumption on $k$, Proposition 4.8 is still valid if we replace the atlas $(X, \Sigma)$ by $V$ (and the ring $A$ by $R$) and each $\overline{Z}^j$ by $Z^j$.

We should also mention a consequence of Proposition 4.8 which is that if $[Z] \in \text{Tz}_{\Sigma}^n(V, n, m)$ is an sfs-cycle, then $[Z^j] \in \text{Tz}_{\Sigma}^{j+1}(V, j + 1, m)$ for all $0 \leq j \leq n - 1$.

4.2. Additive higher Chow groups of fs and sfs-cycles. We now define certain subgroups of additive higher Chow groups of smooth $k$-schemes in the Milnor range. As part of proving our main results, the goal is to show that these subgroups actually coincide with the additive higher Chow groups in the Milnor range for a smooth semi-local $k$-scheme of geometric type. This goal is achieved by means...
of the fs-moving lemma and sfs-moving lemma, that we prove in the following sections.

**Definition 4.10.** Given a smooth affine k-scheme X and a finite set of closed points Σ, we define

\[
\widetilde{TCH}_\Sigma^n(X, n; m) = \frac{\ker(\partial : Tz^n_\Sigma(X, n; m) \to Tz^n(X, n-1; m))}{\text{im}(\partial : Tz^n(X, n+1; m) \to Tz^n(X, n; m)) \cap Tz^n_\Sigma(X, n; m)}.
\]

\[
TCH^n_{\Sigma,fs}(X, n; m) = \frac{\ker(\partial : Tz^n_{\Sigma,fs}(X, n; m) \to Tz^n(X, n-1; m))}{\text{im}(\partial : Tz^n(X, n+1; m) \to Tz^n(X, n; m)) \cap Tz^n_{\Sigma,fs}(X, n; m)}.
\]

\[
TCH^n_{\Sigma,sfs}(X, n; m) = \frac{\ker(\partial : Tz^n_{\Sigma,sfs}(X, n; m) \to Tz^n(X, n-1; m))}{\text{im}(\partial : Tz^n(X, n+1; m) \to Tz^n(X, n; m)) \cap Tz^n_{\Sigma,sfs}(X, n; m)}.
\]

Here, the definition of the group \(\widetilde{TCH}_\Sigma^n(X, n; m)\) is slightly different from that of \(TCH^n_\Sigma(X, n; m)\) in Definition 2.5. However, we have:

**Lemma 4.11.** The natural surjection \(TCH^n_\Sigma(X, n; m) \to \widetilde{TCH}_\Sigma^n(X; n; m)\) is an isomorphism.

**Proof.** By the moving lemma for additive higher Chow groups of smooth affine schemes in [19], the composition \(TCH^n_\Sigma(X, n; m) \to \widetilde{TCH}_\Sigma^n(X; n; m) \to TCH^n(X, n; m)\) is an isomorphism. Hence, the first arrow is injective. \(\square\)

If \(V = \text{Spec}(R)\) is a smooth semi-local k-scheme of geometric type with the set of closed points Σ, we shall write \(TCH^n_{\Sigma,fs}(V, n; m)\) and \(TCH^n_{\Sigma,sfs}(V, n; m)\) for \(TCH^n_{\Sigma,fs}(X, n; m)\) and \(TCH^n_{\Sigma,sfs}(X, n; m)\), respectively.

Note that there natural maps

\[
(4.2)
TCH^n_{\Sigma,fs}(X, n; m) \to TCH^n_{\Sigma,fs}(X, n; m) \to \widetilde{TCH}_\Sigma^n(X; n; m) \to TCH^n(X, n; m),
\]

and the third group can be replaced by \(TCH^n_\Sigma(X, n; m)\) by Lemma 4.11. Our goal is to show that all arrows in this sequence are isomorphisms for regular semi-local k-schemes of geometric type. The next result is meant to show how we can assume the base field to be an infinite perfect field in order to establish these isomorphisms.

**Lemma 4.12.** Let \(V = \text{Spec}(R)\) be a smooth semi-local k-scheme of geometric type with the set of closed points Σ. Suppose that the map

\[
sfs_V : TCH^n_{sfs}(V, n; m) \to TCH^n(V, n; m)
\]

is an isomorphism when \(k\) is an infinite perfect field. Then, this map is an isomorphism also for an arbitrary field \(k\). The same holds for the map \(fs_V : TCH^n_{fs}(V, n; m) \to TCH^n(V, n; m)\).

**Proof.** We prove the lemma for the sfs-cycles only for the case of fs-cycles is similar. Suppose that the lemma holds when \(k\) is an infinite perfect field. First assume that \(k\) is a finite field (hence perfect). We use the pro-\(\ell\)-extension trick to reduce it to the case of infinite perfect field. (cf. [30] Proof of Theorem 3.5.14]) Namely, choose two distinct primes \(\ell_1\) and \(\ell_2\) different from \(\text{char}(k)\) and let \(k_i\) denote the pro-\(\ell_i\) extension of \(k\) for \(i = 1, 2\). Notice that these fields are infinite and perfect.
Suppose that \( \alpha \in TCH_{\text{sfs}}^n(V, n; m) \) is such that \( \text{sfs}_V(\alpha) = 0 \). From the case of infinite perfect field, we see that \( \alpha_{k_i} = 0 \) for \( i = 1, 2 \). It follows from Proposition[4.8] that \( TCH_{\text{sfs}}^n(V, n; m) \) commutes with the base-change of \( V \) by any direct limit of separable field extensions of \( k \). In particular, we get two finite extensions \( k'_1 \) and \( k'_2 \) of \( k \) of relatively prime degrees such that \( \alpha_{k'_i} = 0 \) for \( i = 1, 2 \). We conclude from [24] Lemma 4.6] that \( \alpha = 0 \).

Next suppose that \( \beta \in TCH^n(V, n; m) \). Using the case of infinite perfect field and the above property of commutativity with direct limits of separable extensions of \( k \), we can again get two finite extensions \( k'_1 \) and \( k'_2 \) of \( k \) of relatively prime degrees such that \( \beta_{k'_i} = \text{sfs}_V(\alpha_i) \) for some \( \alpha_i \in TCH_{\text{sfs}}^n(V_{k'_i}, n; m) \) for \( i = 1, 2 \). We again conclude from [24] Lemma 4.6] that \( \beta = \text{sfs}_V(\alpha) \) for some \( \alpha \in TCH_{\text{sfs}}^n(V, n; m) \). This completes the case of finite fields.

Suppose now that \( k \) is an infinite imperfect field. Using Quillen’s trick (see [37] Proof of Theorem 5.11, p. 133]), we can find a subfield \( k' \) of \( k \) finitely generated over the prime field \( \mathbb{F}_p \), a regular semi-local scheme \( V' \) of geometric type over \( k' \) with the set of closed points \( \Sigma' \) such that \( (V, \Sigma) = (V'_k, \Sigma'_k) \). Letting \( k_i \) run through the subfields of \( k' \) that are finitely generated over the prime field, we get \( V = \lim_{\rightarrow i} V'_k \).

Since each \( k_i \) is separable (see [33] Theorem 26.3]) and hence smooth over \( \mathbb{F}_p \), it follows from Proposition[4.8] that \( TCH_{\text{sfs}}^n(V, n; m) = \lim_{\rightarrow i} TCH_{\text{sfs}}^n(V'_k, n; m) \) and \( TCH^n(V, n; m) = \lim_{\rightarrow i} TCH^n(V'_k, n; m) \). Since each \( V'_k \) is now a regular semi-local scheme of geometric type over \( \mathbb{F}_p \), it follows from the case of finite fields that \( \text{sfs}_V(V'_k) \) is an isomorphism. We conclude that \( \text{sfs}_V = \lim_{\rightarrow i} \text{sfs}_{V'_k} \) is an isomorphism, too. \( \square \)

Here is the main theorem for the next a few sections:

**Theorem 4.13** (The sfs-moving lemma). Let \( k \) be any field. Let \( V = \text{Spec}(R) \) be an \( r \)-dimensional smooth semi-local \( k \)-scheme of geometric type with the set of closed points \( \Sigma \). Assume that \( r \geq 1 \) and that \( m, n \geq 1 \) are two integers. Then the map \( \text{sfs}_V : TCH_{\text{sfs}}^n(V, n; m) \rightarrow TCH^n(V, n; m) \) is an isomorphism.

The proof is long enough so we prove it in various steps. Roughly speaking, in Theorem[4.16] we first show that, if there is an admissible cycle \( Z \in Tz^n(\mathbb{A}^r, n; m) \), then, we can move it into a cycle in \( Tz_{\Sigma,\text{sfs}}^n(U, n; m) \) for some affine open subset \( U \subset \mathbb{A}^r \) containing \( \Sigma \). Then, using a somewhat complicated generic projection machine, we prove in Theorem[5.11] that the map \( TCH_{\text{sfs}}^n(V, n; m) \rightarrow TCH^n(V, n; m) \) is an isomorphism. The most delicate step is to move cycles in \( TCH_{\text{sfs}}^n(V, n; m) \) to cycles in \( TCH_{\text{sfs}}^n(V, n; m) \), using further local generic projections with various loci.

We consider some results that will be needed. First consider the following local analogue of the spreading lemma:

**Lemma 4.14** (Local spreading lemma). Let \( k \) be an infinite perfect field and let \( X = \text{Spec}(R) \), where \( R \) is a smooth \( k \)-algebra essentially of finite type. Let \( K \) be a purely transcendental extension of \( k \). Then, the base change map

\[
p_{K/k}^* : \frac{Tz^n(X, n; m)}{Tz_{\text{sfs}}^n(X, n; m)} \rightarrow \frac{Tz^n(X_K, n; m)}{Tz_{\text{sfs}}^n(X_K, n; m)}
\]

is injective in homology in the following sense: let \( \alpha \in Tz^n(X, n; m) \) be a cycle such that \( \partial(\alpha) = 0 \) and its base change \( \alpha_K \in Tz^n(X_K, n; m) \) is equivalent to a
cycle in $T_{sz}^n(X_K, n; m)$ modulo the boundary of a cycle in $T^m(X_K, n + 1; m)$. Then, $\alpha$ is equivalent to a cycle in $T_{sz}^n(X, n; m)$ modulo the boundary of a cycle in $T^m(X, n + 1; m)$. The same result holds if $X$ is replaced by its atlas $(X, \Sigma)$.

**Proof.** Its proof is almost identical to usual spreading argument (cf. [24 Proposition 4.7]), except that we should check that the required smoothness, and finiteness and surjectivity over the base are preserved under the arguments of loc. cit.

By induction, we may reduce to the case when $K$ is a purely transcendental extension of $k$, with tr. deg $k = 1$, i.e. $K = k(\mathbb{A}_k^1)$. By the given assumptions, write $\alpha_K = \beta_K + \partial \gamma_K$, where $\beta_K \in T_{sz}^n(X_K, n; m)$ and $\gamma_K \in T^m(X_K, n + 1; m)$. Note that each irreducible component of $\beta_K$ is finite and surjective over $X_K$ and smooth over $K$, and its projections to $X_K \times B_i$ for $1 \leq i \leq n - 1$ are all smooth over $K$. Express $\beta_K = \sum_{j=1}^N r_j V^j_K$, where $r_j \in \mathbb{Z}$ and each $V^j_K$ is an irreducible cycle in $T_{sz}^n(X_K, n; m)$.

Then, for each $V^j_K$, there exists a nonempty open subset $U_j \subset \mathbb{A}_k^1$ and a cycle $V^j_{U_j} \in T^m(X \times U, n; m)$ that restricts to $V^j_K$, such that $V^j_{U_j}$ is finite and surjective over $X \times U_j$ and smooth over $U_j$, and its projections to $X \times U_j \times B_i$ for $1 \leq i \leq n - 1$ are all smooth over $U_j$. Take $\mathcal{U} \coloneqq \cap_{j=1}^N U_j$, which is yet a dense open subset. Let $V^j_{U_j}$ be the restriction of $V^j_{U_j}$ on $X \times \mathcal{U}$. This satisfies the analogous property. Let $\beta_\mathcal{U} \coloneqq \sum_{j=1}^N m_j V^j_{U_j}$.

Then, after shrinking $\mathcal{U}$ if necessary, there exist cycles $\alpha_\mathcal{U} \in T^m(X \times \mathcal{U}, n; m)$ and $\gamma_\mathcal{U} \in T^m(X \times \mathcal{U}, n + 1; m)$, that restrict to $\alpha_K$ and $\gamma_K$ over $X_K$ respectively, such that $\alpha_\mathcal{U} = \beta_\mathcal{U} + \partial \gamma_\mathcal{U}$.

Now, since $k$ is an infinite field, there is a $k$-rational point $u \in \mathcal{U}(k)$. Since $X$ is smooth over $k$, the inclusion $\iota_u : X \to X \times U$, $x \mapsto x \times u$ induces the pull-back map $\iota_u^*$ of the admissible cycles by the existence of pull-back. Applying it to the above equation of cycles over $X \times \mathcal{U}$, we obtain $\alpha = \iota_u^* \alpha_\mathcal{U} = \iota_u^* \beta_\mathcal{U} + \partial \iota_u^* \gamma_\mathcal{U}$, where we have $\iota_u^* \gamma_\mathcal{U} \in T^m(X, n + 1; m)$ and $\iota_u^* \beta_\mathcal{U} \in T^m(X, n; m)$. But, in fact, $\iota_u^* \beta_\mathcal{U} \in T_{sz}^n(X, n; m)$ because smoothness, finiteness and surjectivity of morphisms are stable under base change via $\iota_u$. This finishes the proof of the lemma.

Here is a lemma that we use often:

**Lemma 4.15 ([24 Lemma 1.2]).** Let $X$ be an algebraic $k$-scheme and $G$ a connected algebraic $k$-group acting on $X$. Let $A, B \subset X$ be closed subsets, and assume the fibers of the map $G \times A \to X$, $(g, a) \mapsto g \cdot a$ all have the same dimension, and that this map is dominant.

Moreover, suppose that for an overfield $K \supset k$ and a $K$-morphism $\psi : X_K \to G_K$, there is a nonempty open subset $U \subset X$ such that for every $x \in U_K$, a scheme point, we have tr. deg $k(\psi \circ \phi(x), \pi(x)) \geq \dim G$, where $\pi : X_K \to X_k$ and $\phi : G_K \to G_k$. Define $\phi : X_K \to X_K$ by $\phi(x) = \psi(x) \cdot x$ and suppose $\phi$ is an isomorphism. Then, the intersection $\phi(A \cap U) \cap B$ is proper.

### 4.3 Affine space case

Now, let $k$ be an infinite perfect field. The goal of Section 4.3 is to prove the following, which is weaker than a special case of Theorem 4.13, to which we reduce the general case eventually.

**Theorem 4.16.** Let $m \geq 1$. Let $\alpha \in T^m(\mathbb{A}_k^n, n; m)$. Let $V$ denote the spectrum of the semi-local ring of a finite set $\Sigma$ of closed points on $\mathbb{A}_k^n$ with the localization
map \( j : V \to \mathbb{A}_k^n \). Then, there is an sfs-cycle \( \beta \in Tz^n(V,n;m) \) and a cycle \( \gamma \in Tz^n(V,n+1;m) \) such that \( \partial(\gamma) = j^*(\alpha) - \beta \).

We discuss first some lemmas before we prove Theorem \ref{main_theorem}. Let \( x = (x_1, \ldots, x_r) \), \( x' = (x'_1, \ldots, x'_r) \), and \( t \) be variables. For any \( s > 0 \), consider the homomorphism \( \phi^s : k[x,t] \to k[x,t,x'] \) of polynomial rings given by \( x \mapsto x + ts^{(m+1)}x' \), \( t \mapsto t \). The induced map of schemes is \( \phi_s : \mathbb{A}^r \times \mathbb{A}^1 \times \mathbb{A}^r \to \mathbb{A}^r \times \mathbb{A}^1 \), \((x,t,x') \mapsto (x+ts^{(m+1)}x', t) \). This morphism is flat, hence open. In particular, for each scheme point \( g \in \mathbb{A}^r \), it induces a morphism \( \phi_{g,s} = \phi_s(-,-,g) : \mathbb{A}^r \times \mathbb{A}_{k(g)}^1 \to \mathbb{A}^r \times \mathbb{A}_{k(g)}^1 \), which is an isomorphism. When \( g = \eta \in \mathbb{A}^r \) is the generic point, we let \( K = k(\eta) \). Let \( \phi^s_{h,s} : K[x,t] \to K[x,t] \) be the projection morphism to the first factor \( \mathbb{A}^r \). We first prove:

**Lemma 4.17.** Let \( f(x,t) \in k[x,t] \) be a nonzero polynomial. Then, there is a nonempty open subset \( U \subset \mathbb{A}^r \), such that for each \( g \in U(F) \) for some overfield \( F \supset k \) and sufficiently large \( s > 0 \), the polynomial \( \phi_{g,s}(f) \) is monic in \( t \), i.e. integral over \( F[x] \).

**Proof.** Write \( f(x,t) = \sum_{i=0}^{M} f_i(x)t^M-i \) for some \( f_i \in k[x] \) and \( M \geq 0 \). Let \( d_i = \deg_x(f_i) \), which is the total degree in \( x \). We first consider the case \( r = 1 \). Then, \( f(x+ts^{(m+1)}g,t) = \sum_{i=0}^{M} f_i(x+ts^{(m+1)}g)t^{M-i} = \sum_{i=0}^{M} (gt^{d_i}+s^{(m+1)}M-i) + (i\text{-th lower order terms in } t) \). Here, we suppose \( g \neq 0 \), i.e. \( U_Z = \mathbb{A}^1 \setminus \{0\} \). Let \( d_{i_0} = \max\{d_0, d_1, \ldots, d_M\} \), and suppose \( i_0 \) is the smallest such integer. If \( d_{i_0} = 0 \), then each \( f_i(x) \) is a constant, so \( f(x+ts^{(m+1)}g,t) \) gives an integral relation of \( t \) as desired. Suppose \( d_{i_0} > 0 \). If \( i_0 = 0 \), then for each \( i > 0 \) and each \( s > 0 \), we have \( d_is(m+1)+M \geq d_is(m+1)+M > d_is(m+1)+M - i \). Hence, the leading coefficient of the highest degree term in \( t \) is \( g \in k^\times \), so, it is integral. If \( i_0 > 0 \), then for each \( i > i_0 \) and each \( s > 0 \), we have \( d_is(m+1)+M - i_0 \geq d_is(m+1)+M - i_0 > d_is(m+1)+M - i \), while for \( 0 \leq i < i_0 \), we have \( d_i < d_{i_0} \) so that for every sufficiently large \( s > 0 \), having \( d_is(m+1)+M - i < d_{i_0}s(m+1)+M - i_0 \). Hence, for every sufficiently large \( s > 0 \), again the leading coefficient of highest degree in \( t \) is \( g \in F^\times \), and it gives the desired integral relation. In case \( r \geq 2 \), the backbone of the proof is the same, but one problem is a possible cancellation of the highest degree terms in \( t \), namely, if \( d_i \) is the total degree of \( f_i(x_1, \ldots, x_r) \), then possibly a multiple number of monomials could have the same total degree \( d_i \). However, such \( g \)'s form a closed subscheme of \( \mathbb{A}^r \) (depends on \( f(x,t) \)), so for a general \( g \in U \) for some nonempty open subset \( U \subset \mathbb{A}^r \), we can avoid it. \( \square \)

We prove Theorem \ref{main_theorem} in the special case of \( n = 1 \), up to a purely transcendental base change:

**Lemma 4.18.** Let \( m \geq 0 \) be an integer. Let \( \alpha \in Tz^1(\mathbb{A}^1,1;m) \), and let \( V \) be the spectrum of the semi-local ring of a finite subset \( \Sigma \) of closed points in \( \mathbb{A}^1_k \), with the localization map \( j : V \to \mathbb{A}^1_k \). Then, there exists an sfs-cycle \( \beta \in Tz^1(V_K,1;m) \) and a cycle \( \gamma \in Tz^1(V_K,2;m) \) such that \( \partial(\gamma) = j^*(\alpha) - \beta \).

**Proof.** We may assume \( \alpha = Z \subset \mathbb{A}^r \times \mathbb{A}^1 \) is an integral closed subscheme. Since \( \mathbb{A}^r \times \mathbb{A}^1 \) is factorial, there exists an irreducible polynomial \( f(x,t) \in k[x,t] \) such
that $Z = \text{Spec} (k[\underline{z}, t]/(f(\underline{z}, t)))$. The modulus condition mandates that the cycle does not intersect the divisor $\{ t = 0 \}$ in $\mathbb{A}^r \times \mathbb{A}^1$, so that we must have $f = th - 1$ for some $h(\underline{z}, t) \in k[\underline{z}, t]$. By Lemma 4.17, we can choose some $g \in \mathbb{A}^r(k)$ and a sufficiently large $s > 0$ such that $\phi_{g,s}(th - 1)$ is monic in $t$. This is equivalent to saying that $\phi_{g,s}(Z) \to \mathbb{A}^r_s$ is finite. However, they are both integral and of the same dimension, this is automatically finite. In particular, by base change to $\text{Spec}(K) = \text{Spec}(k(\eta))$, the map $\phi_{\eta,s}(Z) \to \mathbb{A}^r_s$ is finite surjective.

Let $Z_{\text{sm}} \subset Z$ be the set of smooth points over $k$. Since $k$ is perfect, this is a dense open subset. Let $Z_{\text{sing}} := Z \setminus Z_{\text{sm}}$. Here, $\dim(Z_{\text{sing}}) \leq r - 1$. Since $Z_{\text{sm}} \to \text{Spec}(k)$ is smooth, by base change $(Z_{\text{sm}})_K \to \text{Spec}(K)$ is also smooth.

We now apply Lemma 4.15 with $X = \mathbb{A}^r_K \times \mathbb{A}^1_k$, $G = \mathbb{A}^r_k$, $\psi(\underline{z}, t) = (-\eta)t^{s(m+1)}$, $A = \Sigma \times \mathbb{A}^1_k$ and $B = Z_{\text{sing}}$, where $\eta \in \mathbb{A}^1_k$ is the generic point, $G$ acts on $\mathbb{A}^r_K \times \mathbb{A}^1_k$ by $g \cdot (\underline{z}, t) := (g + \underline{z}, t)$. Now, one sees immediately that the conditions of Lemma 4.15 are satisfied. It follows that the intersection $\phi_{\eta,s}^{-1}(A_K) \cap B_K$ is proper. Since $\phi_{\eta,s}$ is an isomorphism, this is equivalent to saying that the intersection $A_K \cap \phi_{s,K}(Z_{\text{sing}})_K$ is proper. But, by dimension counting, this means $\phi_{\eta,s}(Z_{\text{sing}})_K \cap A_K = \emptyset$.

We saw $(Z_{\text{sm}})_K$ is smooth over $K$, while its complement in $Z_K$ is $(Z_{\text{sing}})_K$. (N.B. In general, $(Z_{\text{sm}})_K$ is a subset of the set of points in $Z_K$ smooth over $K$.) Hence, $\phi_{\eta,s}((Z_{\text{sm}})_K)$ is smooth over $K$, while its complement is $\phi_{\eta,s}((Z_{\text{sing}})_K)$, because $\phi_{\eta,s}$ is an isomorphism. Since $\phi_{\eta,s}(Z_K) \to \mathbb{A}^r_s$ is finite as shown above, the image of $\phi_{\eta,s}(Z_{\text{sing}})_K \to \mathbb{A}^r_s$ is a proper closed subset of $\mathbb{A}^r_s$, disjoint from $\Sigma$. Hence, there is an open neighborhood $U \subset \mathbb{A}^r_s$ of $\Sigma$, such that $\phi_{\eta,s}(Z_K) \cap q^{-1}(U)$ is smooth over $K$. Hence $j^*(\phi_{\eta,s}(Z_K))$ is an sfs-cycle.

On the other hand, it follows from [19] Lemma 2.3, Proposition 2.5 that there is a cycle $\bigcirc \in Tz^n(\mathbb{A}^r_s, n + 1; m)$ such that $\partial(\bigcirc) = [Z_K] - [\phi_{\eta,s}(Z_K)]$. Setting $\beta = j^*(\phi_{s,K}(Z_K))$ and $\gamma = j^*(\bigcirc)$, we conclude $\partial(\gamma) = j^*(\alpha_K) - \beta$.

Before we consider the general case, we make the following observations. We consider three types of additive cycles.

**Lemma 4.19.** Let $Z \subset \mathbb{A}^r \times \mathbb{A}^1 \times \mathbb{A}^{n-1}$ be a closed irreducible admissible subscheme of dimension $r$. Suppose the projection to the first factor $Z \to \mathbb{A}^r$ is dominant. Then, there is a dense open subset $U \subset \mathbb{A}^r$ such that for each $g \in U$ and $s > 0$, the projection to the first factor $\phi_{g,s}(Z) \to \mathbb{A}^r$ is still dominant. In particular, $\dim(\phi_{g,s}(Z)) = \dim Z$.

**Proof.** This is immediate from the definition of $\phi_{g,s}$. 

**Lemma 4.20.** Let $Z \subset \mathbb{A}^r_K \times \mathbb{A}^1 \times \mathbb{A}^{n-1}$ be a closed irreducible admissible subscheme of dimension $r$ such that (a) the projection $q_n : Z \to \mathbb{A}^r$ is not dominant, while (b) the projection $pr_2 : Z \to \mathbb{A}^1_k$ is dominant. Then, there is a dense open subset $U \subset \mathbb{A}^r_k$ such that for each $g \in U$ and $s > 0$, we have

1. $\dim(q_n(\phi_{g,s}(Z))) = \dim(q_n(Z)) + 1$ and
2. the projection $pr_2 : \phi_{g,s}(Z) \to \mathbb{A}^1$ is dominant.

**Proof.** By (b), $pr_2(Z) \subset \mathbb{A}^1$ is a dimension 1 locally closed subset. Then, for each $g \in \mathbb{A}^r$ and $s > 0$, we have a surjection $\Theta : q_n(Z) \times pr_2(Z) \to q_n(\phi_{g,s}(Z))$, given by sending $(x, t)$ to $x + t^{s(m+1)}g$. Thus, $\dim q_n(\phi_{g,s}(Z)) \leq \dim q_n(Z) + 1$. 

On the other hand, for each fixed $t_0 \in \text{pr}_2(Z)$, $\Phi(q_n(Z), t_0)$ is isomorphic to $q_n(Z)$, while it is a proper closed irreducible subset of $q_n(\phi_{s,g}(Z))$ when $g$ is a general member, i.e., in an open subset of $A_k^r$. Since $\text{pr}_2(Z)$ is dense open in $A^1$ and hence of positive dimension, we must have $\dim(q_n(\phi_{s,g}(Z))) > \dim(q_n(Z))$. The second property is obvious because $\phi_{g,s}$ does not modify the $A_k^1$-coordinate.

**Lemma 4.21.** Let $Z \subset A^r \times A^1 \times \mathbb{P}^{n-1}$ be a closed irreducible admissible subscheme of dimension $r$ such that (a) the projection to the first factor $q_n : Z \to A^r$ is not dominant, and (b) the projection to the second factor $\text{pr}_2 : Z \to A^1$ is not dominant, either. Let $\Sigma \subset A^r$ be a finite subset of closed points.

Then, there is a dense open subset $U \subset A^r$ such that for each $g \in U$, there is an open neighborhood $W_g \subset A^r$ of $\Sigma$ such that $\phi_{g,s}(Z)$ restricted over $W_g$ is empty.

**Proof.** Let $W = \overline{q_n(Z)}$, which is a closed subscheme of $A^r$ of dimension $< r$. Let $U_0 := A^r \setminus W$, which is dense open in $A^r$. Since $\text{pr}_2 : Z \to A^1$ is not dominant, $\text{pr}_2(Z) \subset A^1$ is irreducible and closed. Hence, it must be a singleton $\{t_0\}$. By the modulus condition $Z$ satisfies, we must have $t_0 \neq 0$.

Now, for each $p \in \Sigma$, take $U_p := p - U_0$ and let $U = \bigcap_{p \in \Sigma} U_p$. This is a finite intersection, so it is a dense open subset of $A^r$. Let $U := U_0^{-s(m+1)}$, which is still dense open in $A^r$. Notice that for each $g \in U$, we have $p_i \in U_0 + t_0^{s(m+1)} g$, so that this open set contains $\Sigma$. Let $W_g := U_0 + t_0^{s(m+1)} g$, which is dense open in $A^r$. Because $Z$ restricted over $U_0$ is empty by construction, by applying $\phi_{g,s}$ for $g \in U$, the variety $\phi_{g,s}(Z)$ restricted over $\phi_{g,s}(U_0) = W_g$ is empty. This proves the lemma.

Now, we prove Theorem 4.19 for $n \geq 1$, up to a purely transcendental base change:

**Lemma 4.22.** Let $m \geq 0$, $n \geq 1$ be integers. Let $\alpha \in \text{Tz}^n(A^r, n; m)$ and let $V$ be the spectrum of the semi-local ring of a finite subset $\Sigma$ of closed points in $A^r$, with the localization map $j : V \to A^r$. Then, there exists an sfs-cycle $\beta \in \text{Tz}^n(V_K, n; m)$ and a cycle $\gamma \in \text{Tz}^n(V_K, n+1; m)$ such that $\partial(\gamma) = j^*(\alpha_K) - \beta$.

**Proof.** We may assume $\alpha$ is irreducible, and let $Z \hookrightarrow A^r_k \times A^1_k \times \mathbb{P}^{n-1}$ be the integral closed subscheme giving $\alpha$. Consider the commutative diagram

\[
\begin{array}{ccc}
A^r_k \times A^1_k \times \mathbb{P}^{n-1} & \xrightarrow{q_n} & \overline{Z} \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
$B \in \textbf{Sch}_K$. Since $p$ is projective, it follows that $\phi_{\eta,s}(\overline{Z}_K)$ is finite and surjective over $A^r_K$ for $s \gg 1$.

Set $Y = \overline{Z} \setminus Z = \overline{Z} \cap (A^r_K \times A^1_k \times F^1_n)$ so that $\dim_k(Y) \leq r - 1$. We now apply Lemma 4.15 with (4.4) $X = A^r_k \times A^1 \times \square^{n-1}; G = A^r_k; \psi(x, t, y) = (-\eta)t^{s(m+1)}; A = \Sigma \times A^1 \times \square^{n-1}; B = Y,$

where $G$ acts on $A^r_k \times A^1 \times \square^{n-1}$ by $g \cdot (x, t, y) = (g + x, t, y)$. One checks immediately that the conditions of Lemma 4.15 are satisfied. It follows that the intersection $\phi^{-1}_\eta(A_K) \cap B_K$ is proper. But this means, by a dimension counting, that $\phi^{-1}_\eta(A_K) \cap B_K = \emptyset$. Equivalently, we have $A_K \cap \phi_{\eta,s}(Y_K) = \emptyset$. Since $\phi_{\eta,s}(\overline{Z}_K) \to A^r_K$ is finite, we see that the image of $\phi_{\eta,s}(Y_K) \to A^r_K$ is a closed subset disjoint from $\Sigma$. Taking $U$ to be its complement, we conclude that $U$ is an open neighborhood of $\Sigma$ such that $\phi_{\eta,s}(\overline{Z}_K) \cap q^{-1}_n(U) = \phi_{\eta,s}(Z_K) \cap q^{-1}_n(U)$. Since $\phi_{\eta,s}(\overline{Z}_K) \cap q^{-1}_n(U) \to U$ is finite and surjective, we conclude that $\phi_{\eta,s}(Z_K) \cap q^{-1}_n(U) \to U$ is finite and surjective.

To show that $\phi_{\eta,s}(Z_K)$ is an sfs-cycle over some open neighborhood of $\Sigma$ for all sufficiently large $s \gg 1$, we first take $A = \Sigma \times A^1 \times \square^{n-1}$ and $B = \overline{Z}_\text{sing}$ and apply Lemma 4.15 with the same situation as in (4.4). We repeat the argument in the proof of Lemma 4.18 ($n = 1$ case) in verbatim to find an open neighborhood $\Sigma_K \subseteq U_n \subseteq A^r_K$ such that $\phi_{\eta,s}(\overline{Z}_K) \cap q^{-1}_n(U_n)$ is smooth over $K$ for all large $s \gg 1$. In particular, $\phi_{\eta,s}(Z_K) \cap q^{-1}_n(U_n)$ is smooth over $K$ for all large $s \gg 1$.

Since $\phi_{\eta,s}(\overline{Z}_K)$ finite over $A^r_K$, we see that each projection of $\overline{Z}_K$, via $\pi_j : A^r_K \times A^1 \times \square^{n-1} \to A^r_K \times A^1 \times \square^{j-1}$, is finite over $A^r_K$ for $1 \leq j \leq n$. We can now apply the above argument to each of these projections to successively get open neighborhoods $\Sigma_K \subseteq U_j \subseteq A^r_K$ such that

1. $\phi_{\eta,s}(\overline{Z}_K) \cap q^{-1}_n(U_j) = \phi_{\eta,s}(Z_K) \cap q^{-1}_n(U_j)$,
2. $\phi_{\eta,s}(\overline{Z}_K) \cap q^{-1}_n(U_j)$ is finite and surjective over $U_j$, and
3. $\pi_i(\phi_{\eta,s}(\overline{Z}_K) \cap q^{-1}_n(U_j))$ is smooth over $K$ for all $1 \leq i \leq n$.

Setting $U = \bigcap_{j=1}^n U_j$ and noting that $\phi_{\eta,s}$ commutes with each projection $A^r_K \times A^1 \times \square^{r-j} \to A^r_K \times A^1 \times \square^{j-1}$, we conclude that $U \subseteq A^r_K$ is an open neighborhood of $D_K$ such that $\phi_{\eta,s}(Z_K) \cap q^{-1}_n(U)$ is an sfs-cycle over $U$. Since each $\phi_{g,s}(Z)$ is congruent to $Z$ as admissible additive cycle by [19] Lemma 2.3, Proposition 2.5, we are done in this case.

Case 2. We next consider the case when the map $Z \to A^r_K$ is not dominant. Here, we have two further cases. Suppose first that the projection $pr_2 : Z \to A^1_k$ is dominant. In this case, we can apply Lemma 4.19 repeatedly to conclude that the composite $\phi_{g_1,s} \circ \cdots \circ \phi_{g_r,s}(Z) \to A^r_K$ is dominant for a finite sequence of general members of $A^r$. So, we are reduced to the case when $Z \to A^r_K$ is dominant, which we already treated in Case 1.

Case 3. The only case remaining is when neither of the maps $Z \to A^r_K$ and $Z \to A^1_k$ is dominant. In this case, we see by Lemma 4.20 that there is an open neighborhood $U \subset A^r$ containing $\Sigma$ such that for a general $g \in U$ and $s > 0$, the map $\phi_{g,s}(Z) \cap q^{-1}_n(U) = \emptyset$. In particular, $[j^* (\phi_{g,s}(Z))] = 0$, where $j : V \to A^r$ is the localization map. So, by applying [19] Lemma 2.3, Proposition 2.5, we conclude that such $Z$ is equivalent to 0. The proof of the lemma is now complete.  \[\square\]
So, we deduce:

Proof of Theorem 4.10. This follows immediately from Lemmas 4.14 and 4.22. □

5. The fs-moving lemma

Let $k$ be an infinite perfect field until the end of the proof of Theorem 4.13, unless stated otherwise. As a step forward, the goal of Section 5 is to prove Theorem 5.11, which says that the map $\text{fs}_V : \text{TC}_k^n(V; n; m) \to \text{TC}_k^n(V; n; m)$ is an isomorphism for a regular semi-local $k$-scheme $V$ of geometric type. In order to attack this question, we need various techniques of linear projections inside projective or affine spaces.

5.1. Some algebraic results. We collect some algebraic results before we delve into linear projections. The following results are repeatedly used in this text.

Lemma 5.1. Let $k$ be a field and let $A \xrightarrow{g} B \xrightarrow{f} C$ be local and flat morphisms of local rings which are essentially of finite type over $k$ and whose residue fields are finite over $k$. Assume that $fg$ and $f$ are étale. Then $g$ is also étale.

Proof. We only need to show that $\Omega^1_{B/A} = 0$. However, as $f$ is étale, the relative differential and the André-Quillen homology of $C$ over $B$ vanishes. This implies in particular that $0 = \Omega^1_{C/A} \simeq \Omega^1_{B/A} \otimes_B C$. Since $f$ is faithfully flat, we conclude that $\Omega^1_{B/A} = 0$. □

Lemma 5.2. Let $f : A \to B$ be an injective local morphism of noetherian local rings such that $f$ is finite, unramified and induces an isomorphism on the level of residue fields. Then $f$ is an isomorphism.

Proof. Let $m_A$ and $m_B$ denote the maximal ideals of $A$ and $B$, respectively. We only need to show that $B/A = 0$. Using Nakayama’s lemma, suffices to show that $A/m_A \to B/m_B$ is surjective. But this follows because the map $A/m_A \to B/m_B$ is an isomorphism and so is the map $B/m_A \to B/m_B$ as $f$ is unramified. □

Lemma 5.3. Let $f : X \to Y$ be a finite and flat map of connected smooth schemes over $k$. Let $W \subset Y$ be an irreducible closed subset and let $y \in W$ be a closed point. Set $S = f^{-1}(y)$ and $Z' = X \times_Y W$. Let $x \in f^{-1}(y)$ and let $Z \subset Z'$ be an irreducible component with $x \in Z$. Assume that $f$ is étale at $x$ and $k(y) \cong k(x)$. Then $Z \cap S = \{x\}$ if and only if $Z$ is the only component of $Z'$ passing through $x$.

Proof. We first observe that under the given assumptions, we must have $|S| > 1$ unless $f$ is an isomorphism (in which case, there is nothing to prove). To see this, suppose $S = \{x\}$. It follows from Lemma 5.2 that the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,y}$ is an isomorphism. This in turn implies that $f$ is a finite and flat map with $[k(X) : k(Y)] = 1$ (see [31, Ex. 5.1.25]) and hence must be an isomorphism.
We now consider the commutative diagram of semi-local rings:

\[
\begin{array}{ccc}
\mathcal{O}_{Y,y} & \xrightarrow{\alpha_1} & \mathcal{O}_{X,y} \\
\beta_1 \downarrow & & \beta_2 \downarrow \\
\mathcal{O}_{W,y} & \xrightarrow{\alpha_3} & \mathcal{O}_{Z',y}
\end{array}
\quad \begin{array}{ccc}
\mathcal{O}_{X,S} & \xrightarrow{\alpha_2} & \mathcal{O}_{X,x} \\
\beta_3 \downarrow & & \beta_4 \downarrow \\
\mathcal{O}_{Z,S} & \xrightarrow{\alpha_4} & \mathcal{O}_{Z',x}
\end{array}
\quad \begin{array}{ccc}
\mathcal{O}_{Z,S,x} & \xrightarrow{\alpha_5} & \mathcal{O}_{Z,x}.
\end{array}
\]

In this commutative diagram, \( \alpha_1 \) and \( \alpha_3 \) are finite and flat maps. Using this observation, we see that the lemma is equivalent to showing that \( \alpha_5 \) is an isomorphism if and only if \( \beta_5 \) is so given that \( \alpha_2 \circ \alpha_1 \) is étale.

Assume first that \( \alpha_5 \) is an isomorphism. Since \( \beta_4 \) is surjective and \( \alpha_3 \) is finite, we conclude that \( \gamma \) is finite. In particular, \( \gamma' := \alpha_5 \circ \gamma \) is a finite map of local rings.

Next, since \( \alpha_2 \circ \alpha_1 \) is étale, we see that \( \alpha_4 \circ \alpha_3 \) is also étale. Since \( \beta_5 \) is surjective, we see that \( \gamma' \) is unramified. In particular, \( \gamma' \) is a finite and unramified map of local rings. Since \( Z \to W \) is surjective and \( k(y) \simeq k(x) \), we conclude from Lemma 5.2 that \( \gamma' \) is an isomorphism. In particular, \( \alpha_4 \circ \alpha_3 \) is an étale map of local rings such that \( \beta_5 \circ \alpha_4 \circ \alpha_3 \) is an isomorphism. In particular, it is étale. It follows easily that \( \beta_5 \) is étale. We conclude that \( \beta_5 \) is a surjective map of local rings which is étale. This can happen only if it is an isomorphism.

To prove the converse, suppose that \( \beta_5 \) is an isomorphism. Let \( p \) denote the minimal prime of \( \mathcal{O}_{Z',S} \) such that \( \mathcal{O}_{Z',S}/p = \mathcal{O}_{Z,S} \) and let \( \{p_1, \ldots, p_m\} \) denote the set of minimal primes of \( \mathcal{O}_{Z',S} \) different from \( p \). To show that \( \alpha_5 \) is an isomorphism, we need to show that \( p + p_i = \mathcal{O}_{Z',S} \) for all \( 1 \leq i \leq m \).

We first make the following

**Claim 1:** \( p_i \mathcal{O}_{Z',x} = \mathcal{O}_{Z',x} \) for all \( 1 \leq i \leq m \).

(\( \vdots \)) Since \( \mathcal{O}_{Z,x} \) is an integral domain and \( \beta_5 \) is an isomorphism, it follows that \( \mathcal{O}_{Z',x} \) is an integral domain. In particular, we must either have \( p_i \mathcal{O}_{Z',x} = 0 \) or \( p_i \mathcal{O}_{Z',x} = \mathcal{O}_{Z',x} \). In the first case, we must have \( p_i \mathcal{O}_{Z,x} = 0 \) for each \( i \) as \( \beta_5 \) is an isomorphism. Equivalently, \( \alpha_5 \circ \beta_4(p_i) = 0 \) for each \( i \).

On the other hand, as \( p_i \neq p \), there is \( a_i \in p_i \setminus p \) such that \( \beta_4(a_i) \neq 0 \) and hence \( \alpha_5 \circ \beta_4(a_i) \neq 0 \). We thus arrive at a contradiction and this shows that we must have \( p_i \mathcal{O}_{Z',x} = \mathcal{O}_{Z',x} \) for each \( i \), proving the claim.

Let \( m \) denote the maximal ideal of \( \mathcal{O}_{Z',S} \) defining the point \( x \). Using Claim 1, we see that for any given \( 1 \leq i \leq m \), there exists \( a_i \in p_i \setminus m \) in \( \mathcal{O}_{Z',S} \) such that \( \alpha_4(a_i) \) is invertible. Setting \( a = \prod_{i=1}^m a_i \), we see that there are non-zero elements \( b, c \in \mathcal{O}_{Z',S} \) with \( c \notin m \) such that \( c(1-ab) = 0 \).

**Claim 2:** \( 1-ab \in p \).

(\( \vdots \)) Setting \( v = 1-ab \), we see that as \( cv = 0 \) and \( c \notin m \), we must have \( v \in m \) and \( \alpha_4(v) = 0 \). If \( v \notin p \), then \( v \in m \setminus p \) and \( \beta_4(v) \neq 0 \). This in turn means that \( \beta_5 \circ \alpha_4(v) = \alpha_5 \circ \beta_4(v) \neq 0 \). But this contradicts our above conclusion that \( \alpha_4(v) = 0 \). Hence, we must have \( v \in p \), which proves the claim.

Using Claim 2, we see that \( v \in p \), \( ab \in p \), for all \( i \) and \( v - ab = 1 \). This shows that \( p + p_i = \mathcal{O}_{Z',S} \) for all \( 1 \leq i \leq m \) and \( \alpha_5 \) is an isomorphism. \( \square \)
5.2. Linear projections. We recall some known facts and set up our terminology which will be used throughout our proofs of the fs-moving and the sfs-moving lemmas in the next several sections.

**Notations:** Given integers \( N, n \geq 0 \) with \( 0 \leq n < N \), let \( Gr(n, \mathbb{P}^N_k) \) denote the Grassmannian scheme of \( n \)-dimensional linear subspaces of \( \mathbb{P}^N_k \). A point \([x_0, \ldots, x_n] \in \mathbb{P}^n_k \) will be often denoted in short by \([x]\).

Given two closed subschemes \( Y, Y' \subseteq \mathbb{P}^N_k \), we let \( \text{Sec}(Y, Y') = \text{Sec}(Y', Y) \) denote the union of all lines \( \ell_{yy'} \) joining distinct points \( y \in Y, y' \in Y' \). This is called the *join* of \( Y \) and \( Y' \). One checks that \( \dim(\text{Sec}(Y, Y')) = \dim(Y) + \dim(Y') - \dim(Y \cap Y') \) with the convention that \( \dim(\emptyset) = -1 \). If \( Y = Y' \), the scheme \( \text{Sec}(Y, Y') = \text{Sec}(Y) \) is the secant variety of \( Y \). If \( Y' = L \) is a linear subspace, then \( \text{Sec}(Y, L) = C_L(Y) \) is the cone over \( Y \) with vertices in \( L \).

Given a locally closed subset \( S \subseteq \mathbb{P}^N_k \), we shall denote the set of \( n \)-dimensional linear subspaces of \( \mathbb{P}^N_k \) which *do not* intersect \( S \) by \( Gr(S, n, \mathbb{P}^N_k) \). If \( S = \{x\} \), we shall write \( Gr(S, n, \mathbb{P}^N_k) \) as \( Gr(x, n, \mathbb{P}^N_k) \).

We shall denote the set of \( n \)-dimensional linear subspaces of \( \mathbb{P}^N_k \) containing a locally closed subscheme \( S \subseteq \mathbb{P}^N_k \) by \( Gr_S(n, \mathbb{P}^N_k) \). We shall write \( Gr_S(n, \mathbb{P}^N_k) \) as \( Gr_x(n, \mathbb{P}^N_k) \) if \( S = \{x\} \) is a closed point. It is easy to check that \( Gr(n, \mathbb{P}^N_k) \) is a homogeneous space of dimension \( (N - n)(n + 1) \). If \( M \subsetneq \mathbb{P}^N_k \) is a linear subspace of dimension \( 0 \leq m \leq n \), then \( Gr_M(n, \mathbb{P}^N_k) \) is a homogeneous space which is an irreducible closed subscheme of \( Gr(n, \mathbb{P}^N_k) \) of dimension \( (N - n)(n - m) \). In particular, \( Gr(S, n, \mathbb{P}^N_k) \) is a dense open subset of \( Gr(n, \mathbb{P}^N_k) \) if \( S \) is finite. Given two distinct locally closed subsets \( S, S' \subseteq \mathbb{P}^N \), we set \( Gr_S(S', n, \mathbb{P}^N_k) := Gr_S(n, \mathbb{P}^N_k) \cap Gr(S', n, \mathbb{P}^N_k) \).

For a scheme \( X \), let \( X_{\text{sing}} \subset X \) be the singular locus of \( X \) and let \( X_{\text{sm}} \) be its complement. For a closed subscheme \( X \subseteq \mathbb{P}^N_k \), let \( Gr^\text{tr}(X, n, \mathbb{P}^N_k) \) denote the set of \( n \)-dimensional linear subspaces which *do not* intersect \( X_{\text{sing}} \), and possibly intersect \( X_{\text{sm}} \) transversely. Recall here our assumption is that \( n < N \). In the above, for any linear subscheme \( L \subset \mathbb{P}^N \), we can define \( Gr(S, n, L), Gr_S(n, L), \) or \( Gr^\text{tr}(X, n, L) \) similarly. The following result is elementary.

**Lemma 5.4.** For any finite set \( S \subseteq \mathbb{P}^N_k \), \( Gr(S, n, \mathbb{P}^N_k) \subseteq Gr(n, \mathbb{P}^N_k) \) is a dense open subset such that \( Gr(S, n, \mathbb{P}^N_k) \subseteq Gr(S', n, \mathbb{P}^N_k) \) if \( S' \subseteq S \). Moreover, if \( S' \subseteq S \), then \( Gr(S, n, \mathbb{P}^N_k) \subseteq Gr(S', n, \mathbb{P}^N_k) \).

**Proof.** The lemma is easily reduced to showing that \( Gr(x', n, \mathbb{P}^N_k) \cap Gr_x(n, \mathbb{P}^N_k) \neq \emptyset \) whenever \( x \neq x' \in \mathbb{P}^N_k \). But this is immediate since \( N > n \). \( \square \)

5.2.1. **Affine Veronese embedding.** Recall that for positive integers \( m, d \geq 1 \), the Veronese embedding \( v_{m,d} : \mathbb{P}^m_k \rightarrow \mathbb{P}^N_k \) is a closed embedding given by \( v_{m,d}([x]) = [M_0(x), \ldots, M_N(x)] = [M(x)] \), where \( N = \binom{m+d}{m} - 1 \) and \( \{M_0, \ldots, M_N\} \) are all monomials in \( \{x_0, \ldots, x_m\} \) of degree \( d \), arranged in the lexicographic order of their powers \( \alpha = (\alpha_0, \ldots, \alpha_m) \).

If we write the coordinate of a point \([y] \in \mathbb{P}^N_k\) by \([y] = [y_0, \ldots, y_N]\), it is clear that \( v_{m,d}^{-1}(\{y_0 = 0\}) = \{x_0^d = 0\} \). In particular, the Veronese embedding yields a Cartesian squares of schemes:
5.2.2. Linear projections. We shall say that the two linear subspaces $L, L' \subseteq \mathbb{P}^N_k$ are complementary if $L \cap L' = \emptyset$ and $\text{Sec}(L, L') = \mathbb{P}^N_k$. Given two complementary linear subspaces $L$ and $L'$ of dimensions $d$ and $N - d - 1$, respectively, there is a linear projection morphism $\phi_{L'} : \mathbb{P}^N_k \setminus L' \to L$. After a linear change of coordinates in $\mathbb{P}^N_k$, this map is defined by the sections $\{s_0, \ldots, s_d\}$ of $\mathcal{O}_{\mathbb{P}^N_k}(1)$ such that $\{s_0, \ldots, s_d, s_{d+1}, \ldots, s_N\}$ is a basis of $H^0(\mathbb{P}^N_k, \mathcal{O}(1))$. Notice that $\phi_{L'}$ defines a vector bundle morphism over $L$ of rank $N - d$ whose fiber over a point $x \in L$ is the affine space $C_x(L') \setminus L'$, where $C_x(L') = \text{Sec}(x, L')$.

We should also observe that if there is a hyperplane $H \subseteq \mathbb{P}^N_k$ containing $L'$ and if $X \subseteq \mathbb{P}^N_k$ is a closed subscheme not intersecting $L'$, then $\phi_{L'}$ defines the Cartesian squares of morphisms

\[
\begin{array}{c}
X \setminus H \to X \leftarrow X \cap H \\
\downarrow \quad \downarrow \quad \downarrow \\
L \setminus H \to L \leftarrow L \cap H.
\end{array}
\]

Since the morphism $X \to L$ is projective with affine fibers, it must be finite. In particular, the map $X \setminus H \to L \setminus H$ of affine schemes is also finite. As a consequence of this and (5.2), we get the following fact, which we shall use often in this text.

**Lemma 5.5.** Let $X \hookrightarrow \mathbb{A}^m_k$ be an affine scheme of dimension $r \geq 1$ and let $\overline{X} \hookrightarrow \mathbb{P}^m_k$ be its projective closure. Then for $N \gg m$, the Veronese embedding $v_{m,d} : \mathbb{P}^m_k \hookrightarrow \mathbb{P}^r_k$ and the linear projection away from $L \in Gr(N - r - 1, \mathbb{P}^N_k)$ yield a Cartesian square of finite maps

\[
\begin{array}{c}
X \leftarrow \overline{X} \\
\downarrow \quad \downarrow \\
\mathbb{A}^r_k \leftarrow \mathbb{P}^r_k
\end{array}
\]

if $L \in Gr(X, N - r - 1, H_{N,0}) \subseteq Gr(N - r - 1, \mathbb{P}^N_k)$, where $H_{N,0} = \{x_0 = 0\} \subset \mathbb{P}^N$ as in (5.2).

In order to ensure that such linear subspaces always exist, we use the following result often in this text.
Lemma 5.6. Assume that $k$ is algebraically closed. Let $X \subseteq \mathbb{P}^N_k$ be a projective scheme of dimension $r \geq 1$ with $N \gg r$ and let $H \hookrightarrow \mathbb{P}^N_k$ be a hyperplane. Then $\text{Gr}(X, N - r - 1, H)$ is a dense open subset of $\text{Gr}(N - r - 1, H)$.

Proof. Consider the incidence variety $S = \{(x, L) \in X \times \text{Gr}(N - r - 1, H) \mid x \in L\}$ and set $Y = X \cap H$. We have the projection maps

\[ X \leftarrow \pi_1 S \rightarrow \pi_2 \text{Gr}(N - r - 1, H). \]

The fiber of $\pi_1$ over $X \setminus Y$ is empty and it is a smooth fibration over $Y$ with each fiber isomorphic to $\text{Gr}(N - r - 2, \mathbb{P}^N_k)$. It follows that $\dim(S) = \dim(Y) + r(N - r - 1) = r + (N - r - 1) - 1 = r(N - r) - 1$. We conclude from this that $\pi_2(S)$ is a closed subscheme of $\text{Gr}(N - r - 1, H)$ of dimension bounded by $r(N - r) - 1$. On the other hand, $\dim(\text{Gr}(N - r - 1, H)) = r(N - r)$. It follows that $\text{Gr}(X, N - r - 1, H) = \text{Gr}(N - r - 1, H) \setminus \pi_2(S)$ is a dense open subset. 

Lemma 5.7. Assume that $k$ is algebraically closed. Let $r \geq 2$ be an integer and assume $N \gg r$. Let $H \hookrightarrow \mathbb{P}^N_k$ be a hyperplane. Let $L \subsetneq \mathbb{P}^N_k$ be a linear subspace of dimension $N - r + 1$ intersecting $H$ transversely and let $X \subsetneq L$ be a curve (not necessarily connected). Then the set of linear subspaces in $\text{Gr}^r(L, N - 2, H)$ which do not intersect $X$, is a dense open subset of $\text{Gr}(N - 2, H)$.

Proof. It is easy to check that $\text{Gr}^r(L, N - 2, H)$ is a dense open subset of $\text{Gr}(N - 2, H)$. Let us now consider the map $\nu_L : \text{Gr}^r(L, N - 2, H) \rightarrow \text{Gr}(N - r - 1, L \cap H)$ given by $\nu_L(M) = L \cap M$. Then $\nu_L$ is a smooth and surjective morphism of relative dimension $2(r - 1)$. It follows from Lemma 5.6 that $\text{Gr}(X, N - r - 1, L \cap H)$ is a dense open subset of $\text{Gr}(N - r - 1, L \cap H)$. We conclude that $\nu_L^{-1}(\text{Gr}(X, N - r - 1, L \cap H))$ is a dense open subset of $\text{Gr}(N - 2, H)$ and hence a dense open subset of $\text{Gr}(N - 2, H)$.

5.3. The fs-moving lemma. Let $X$ be a smooth affine $k$-scheme of dimension $r \geq 1$. Let $B$ be a geometrically integral smooth affine $k$-scheme of positive dimension with a geometrically integral smooth projective compactification $\hat{B}$ such that $F := \hat{B} \setminus B$ is an effective divisor. Let $Z \subsetneq X \times B$ be a nonempty closed subscheme of dimension at most $r$ and let $\hat{Z} \subset X \times \hat{B}$ be its Zariski closure. (Later, we will consider $B = \mathbb{A}^1 \times \square^{n-1}$ and $\hat{B} = \hat{B}_n = \mathbb{P}^1_k \times (\mathbb{P}^1_k)^{n-1}$.)

Let $f : Z \rightarrow X$ and $\hat{f} : \hat{Z} \rightarrow X$ denote the projection maps. Let $\{Z_1, \ldots, Z_s\}$ be the set of irreducible components of $Z$. For given $1 \leq i \leq s$, we fix closed points $x_i \in X$, $b_i \in B$ such that $\alpha_i = (x_i, b_i) \in Z_i$, with $\alpha_i \notin \hat{Z}_j$ if $j \neq i$. Since each $Z_i \neq \emptyset$, such closed points always exist. Let $D_0 \subsetneq X$ be a finite set of closed points containing $\{x_1, \ldots, x_s\}$ and let $E^0 \subsetneq B$ be a finite set of closed points containing $\{b_1, \ldots, b_s\}$. Set $F^0 = F \cup E^0$.

Notations: Let $\overline{X}$ be a compactification of $X$ and let $W \hookrightarrow \overline{X} \times \hat{B}$ be a closed subscheme. Let $\overline{f} : W \rightarrow \overline{X}$ and $\overline{g} : W \rightarrow \hat{B}$ be the composites with the projections. Then, for any morphisms $X' \rightarrow \overline{X}$ and $B' \rightarrow \hat{B}$, let $W_{X'}$ and $W_{B'}$ be
Proof. The closure of the general Lemma 5.9.
Assume that (5.7) additive cycles exist (see [23, X
For any (5.2) the reader should observe that an irreducible cycle [X
The reader should observe that an irreducible cycle [X
For every (6) For any (5) (5) (4) k
φ(2) L
For an (2) φ
We often write it just as φ
φ
For any (1)
Since (1.1) Let h : X → X' and a cycle Z on X × B, the cycle
h∗ ◦ h∗([Z]) = [Z] is called the residual cycle of [Z], and denoted by h+(|Z|). In
Let φLA : X × A → Pk × A be the map φL × IdA for any
k-scheme A. We often write it just as φL.
Given a finite and flat map h : X → X' and a cycle Z on X × B, the cycle
h∗ ◦ h∗([Z]) = [Z] is called the residual cycle of [Z], and denoted by h+(|Z|). In
case Z ∈ Tz′(W, n; m), the flat pull-back h∗ and the projective push-forward h∗ of
additive cycles exist (see [23, § 3.3, 3.4]).
The reader should observe that an irreducible cycle [Z] may be a component of
h+(|Z|) in general. By extending linearly, h+(−) is defined on the full cycle
complex Tz(W, n; m). For a map φL : X → A as above, (φL)+([Z]) will be
written as L+(|Z|). We shall denote the support of the cycle L+(|Z|) by L+(Z).
For every x ∈ X, let L+(x) = (φL)−1(φL(x)) \ {x} and let L+(D0) = φL−1(φL(D0)) \ D0. Under these conditions, we wish to prove the following.
Lemma 5.9. Assume that X is not isomorphic to an affine space over k and
that no component of Z is contained in X × E0. After replacing the embedding
X ⇐ Pk by its composition with a suitable Veronese embedding Pm Pk, a
general L ∈ Gr(X, N − r − 1, HN,0) satisfies the following, where HN,0 is as in
[5.2]:
(1) φL is étale at D0.
(2) φL(x) ≠ φL(x′) for x ≠ x′ ∈ D0.
(3) k(φL(x)) ≃ k(x) for all x ∈ D0.
(4) L+(x) ≠ ∅ for all x ∈ D0.
(5) L+(D0) ∩ ̃f(ZF0) = ∅.
For any 1 ≤ i ≤ s such that f : Zi → X is not dominant, we also have
(6) L+(D0) ∩ ̃f(Zi) = ∅.
Proof. By replacing the given embedding X ⇐ Pm by its composition with a
Veronese embedding, we can assume that there is a closed embedding X ⇐ Pn
such that N >> r and the degree of X in Pn is bigger than one. Let Xsing denote
the singular locus of X. Since k is perfect, we see that dim(Xsing) ≤ r − 1.
Step 1. We first assume that k is algebraically closed. Let W ⊆ X × B denote
the closure of ̃B in X × B. Since Z is an r-dimensional scheme none of whose
component is contained in X × E0, and F ⊆ B is of codimension 1, we see that
dim(WF0) ≤ r − 1. Since ̃f is projective, we see that ̃f(WF0) is a closed subscheme
of X of dimension ≤ r − 1. Let Znd be the union of irreducible components of
Z which are not dominant over X. Then ̃f(Znd) is a closed subscheme of X.
of dimension $\leq r - 1$. Setting $\mathcal{X}^F_0 = \mathcal{X}_{\text{sing}} \cup \widehat{f}(W^F_0) \cup \widehat{f}(Z_{\mathrm{sing}})$, we conclude that $\mathcal{X}^F_0$ is a closed subscheme of $\overline{X}$ of dimension $\leq r - 1$. We conclude that $\dim(\Sec(D_1, \mathcal{X}^F_0 \cup D_2)) \leq r$ for any finite closed subsets $D_1, D_2 \subseteq \mathcal{X}$.

Let $T_{D_0,\overline{X}} \subseteq \mathbb{P}^N_k$ denote the union of tangent spaces to $\overline{X}$ at all points of $D_0 \subseteq X$. Then $T_{D_0,\overline{X}} \subseteq \mathbb{P}^N_k$ is a finite union of linear subspaces of dimension $r$. We conclude that for every $x \in D_0$, $Z^F_x := \overline{X} \cup T_{D_0,\overline{X}} \cup \Sec(\{x\}, \mathcal{X}^F_0 \cup (D_0 \setminus \{x\}))$ is a closed subset of $\mathbb{P}^N_k$ of dimension $r$.

It follows from Lemmas 5.5 and 5.6 that $U^F := \bigcap_{x \in D_0} \text{Gr}(Z^F_x, N - r - 1, H_{N,0})$ is dense open in $\text{Gr}(N - r - 1, H_{N,0})$. In particular, for every $L \subseteq U^F$, the map $\phi_L : \mathbb{P}^N_k \setminus L \rightarrow \mathbb{P}^r_k$ induces a finite map $\phi_L : X \rightarrow \mathbb{A}^r_k$ which satisfies properties (2), (3), (5) and (6) and is unramified at $D_0$. Since $\phi_L$ is a finite map of smooth schemes of same dimension, it is also flat by [15, Exercise III-10.9, p.26]. We conclude that $\phi_L$ is étale at $D_0$ and hence satisfies (1) as well. (4) follows because $\deg(\phi_L) > 1$ by the assumption on $X$.

**Step 2.** Now suppose that $k$ is an infinite perfect field. Let $\pi_X : X_{\mathbb{T}} \rightarrow X$ denote the base change projection map. Set $D_0 = \{x_1, \ldots, x_s, x_{s+1}, \ldots, x_p\}$. For $1 \leq i \leq p$, let $S_i = (\pi_X)^{-1}(x_i) = \{x_{i,1}, \ldots, x_{i,r}\}$ and set $S = \bigcup_{i=1}^p S_i$. Since $D_0$ is a set of smooth points of $\overline{X}$, we see that $S$ is a set of smooth points of $X_{\mathbb{T}}$. We set $\overline{\mathbb{T}} = (\pi_{X \times \mathbb{T}})^{-1}(\overline{Z})$ and $\overline{F_0} = F_{\mathbb{T}} \cup (\pi_{B \times \mathbb{T}})^{-1}(F_0)$. Notice that no component of $\overline{\mathbb{T}}$ is contained in $X_{\mathbb{T}} \times \overline{F_0}$.

We choose a Veronese embedding $\eta : \overline{X} \hookrightarrow \mathbb{P}^m \rightarrow \mathbb{P}^N_k$ and the open subset $U \subseteq \text{Gr}(N - r - 1, H_{N,0,k})$ such that the inclusion $X_{\mathbb{T}} \hookrightarrow \mathbb{P}^N_k$ satisfies the assertion of the lemma with $S \subseteq X_{\mathbb{T}}$, $\overline{F_0} \subseteq B_{\mathbb{T}}$ and $\overline{\mathbb{T}} \subseteq X_{\mathbb{T}} \times \mathbb{T}$ chosen as above.

Since $\text{Gr}(N - r - 1, H_{N,0,k})$ contains an affine space $\mathbb{A}^M_k$ as an open subset, we can assume that $U \subseteq \mathbb{A}^M_k$. Since $k$ is infinite, the set of points in $\mathbb{A}^M_k$ with coordinates in $k$ is dense in $\mathbb{A}^M_k$. We conclude that there is a dense subset of $U$ each of whose point $L$ is defined over $k$. In other words, $L \in \text{Gr}(N - r - 1, H_{N,0})$. Since $\overline{X} \cap L = \emptyset$ by the choice of $L$, we can apply Lemma 5.5 to get a finite map $\phi_L : X \rightarrow \mathbb{A}^r_k$ over $k$. We now show that $\phi_L$ satisfies the properties (1)~(6) of the lemma.

Since $\phi_L$, being a finite map of smooth schemes, is already flat, we only need to check that it is unramified at each point of $D_0$ to prove (1). So we fix a point $x \in D_0$ and set $y = \phi_L(x)$. Given a point $x' \in (\pi_X)^{-1}(x)$, set $y' = \phi_L(x')$ and $l = k(x')$. This yields a commutative diagram of (regular) local rings

$$
\begin{array}{ccc}
O_{\mathbb{A}^r_k,y} & \rightarrow & O_{\mathbb{A}^r_k,y'} \\
\downarrow & & \downarrow \\
O_{X,x} & \rightarrow & O_{X_1,x'}.
\end{array}
$$

Since $k$ is perfect, the two horizontal maps are étale. We have shown above that the map $O_{\mathbb{A}^r_k,y} \rightarrow O_{X_1,x'}$ is étale. Equivalently, the map $O_{\mathbb{A}^r_k,y'} \rightarrow O_{X_1,x'}$ is étale. We conclude that the composite map $O_{\mathbb{A}^r_k,y} \rightarrow O_{X,x} \rightarrow O_{X_1,x'}$ is étale. It follows from Lemma 5.1 that the map $O_{\mathbb{A}^r_k,y} \rightarrow O_{X,x}$ is étale. This proves (1).
Before proving the other properties, we verify the following claim. Consider the Cartesian square

\[
\begin{array}{ccc}
X \times \mathbb{A}^r_X & \overset{\phi_{L_X}}{\longrightarrow} & \mathbb{A}^r_X \\
\pi_X & \downarrow & \pi_{L_X} \\
X & \overset{\phi_L}{\longrightarrow} & \mathbb{A}^r_k.
\end{array}
\]

**Claim:** Given a closed point \( x \in X \) and \( y = \phi_L(x) \), one has \( |(\pi_{A^r})^{-1}(y)| \leq |(\pi_X)^{-1}(x)| \) and the equality occurs if and only if \([k(x) : k(y)] = 1\). Furthermore, this equality occurs if the map \( \phi_{L_X} : (\pi_X)^{-1}(x) \to (\pi_{A^r})^{-1}(y) \) is injective.

(\cdot) Since \( k \) is perfect, one checks that \( |(\pi_X)^{-1}(x)| = [k(x) : k] \) and \( |(\pi_{A^r})^{-1}(y)| = [k(y) : k] \). The inclusions \( k \hookrightarrow k(y) \hookrightarrow k(x) \) now prove the first assertion. Next, the injectivity of the map \( \phi_{L_X} : (\pi_X)^{-1}(x) \to (\pi_{A^r})^{-1}(y) \) implies that \( |(\pi_{A^r})^{-1}(y)| \geq |(\pi_X)^{-1}(x)| \). The second part of the Claim is now easily deduced.

We now prove the remaining properties of \( \phi_L \). Since the map \( \phi_{L_X} \) is injective on \( S \), the properties (2) and (3) follow directly from the above Claim and our choice of \( S \).

Since \( L^+(S) \cap \hat{\mathbb{Z}}_{\mathbb{F}}(\hat{\mathbb{Z}}_{\mathbb{F}}) = L^+(S) \cap \hat{\mathbb{Z}}_{\mathbb{F}}(\hat{\mathbb{Z}}_{\mathbb{F}}) = \emptyset \), properties (5) and (6) follow, if the following holds:

(\ast) \( (\pi_X)^{-1}(L^+(D_0)) = L^+(S) \) and \( (\pi_X)^{-1}(\hat{f}(\hat{\mathbb{Z}}_{\mathbb{F}})) = \hat{\mathbb{Z}}_{\mathbb{F}}(\hat{\mathbb{Z}}_{\mathbb{F}}) \).

To prove (\ast), it is clear from the choice of \( S \) that \( L^+(S) \subseteq (\pi_X)^{-1}(L^+(D_0)) \). Suppose now that there is an \( x' \in X_X \setminus S \) such that \( \pi_{A^r} \circ \phi_{L_X}(x') = \phi_L(x) = y \) for some \( x \in D_0 \). This implies that \( x' \in (\phi_{L_X})^{-1}((\pi_{A^r})^{-1}(y)) \).

On the other hand, it follows from the Claim and our choice of \( S \) that \( (\pi_{A^r})^{-1}(y) = \phi_{L_X}((\pi_X)^{-1}(x)) \). We deduce that \( x' \in (\phi_{L_X})^{-1}(\phi_{L_X}((\pi_X)^{-1}(x))) \setminus S \) and hence \( x' \in L^+(S) \).

To prove the second equality, it is again clear that \( \hat{\mathbb{Z}}_{\mathbb{F}}(\hat{\mathbb{Z}}_{\mathbb{F}}) \subseteq (\pi_X)^{-1}(\hat{f}(\hat{\mathbb{Z}}_{\mathbb{F}})) \).

For the reverse inclusion, let \( x' \in X_X \) be such that \( x = \pi_X(x') \in \hat{f}(\hat{\mathbb{Z}}_{\mathbb{F}}) \). This implies that there is \( x, b \in \hat{\mathbb{Z}}_{\mathbb{F}} \) with \( \hat{f}(x, b) \in F_0 \). This implies that \( \alpha' = (x', b) \in (\pi_X \times \hat{\mathbb{Z}}_{\mathbb{F}})^{-1}(\hat{\mathbb{Z}}_{\mathbb{F}}) = \hat{\mathbb{Z}}_{\mathbb{F}}(\hat{\mathbb{Z}}_{\mathbb{F}}) \) and \( \hat{f}(\alpha') = x' \). This proves (\ast).

To prove (4), we observe that \( \phi_L : X \to \mathbb{A}^r_k \) is a finite map of smooth schemes of same dimension and hence it is flat. Our assumption on \( X \) that it is not isomorphic to an affine space implies that \( \phi_L \) can not be an isomorphism. Since \( \phi_L \) is étale at \( x \in D_0 \) with \( k(\phi_L(x)) \simeq k(x) \), the set \( L^+(x) \) must be nonempty, as shown in the proof of Lemma 5.3. This proves property (4) and completes the proof of the lemma. \( \square \)

5.4. The fs-moving for additive cycles. We now restrict the general situation of Lemma 5.3 to additive cycles, where we take \( B = \mathbb{A}^1 \times \square^{n-1} \), \( \hat{B} = \hat{B}_n = \mathbb{P}^1 \times \square^{n-1} \) and \( F = F_n = \hat{B}_n \setminus B_n \). This yields the following fs-moving result for additive cycles. We remark that one can check that exactly the same proof also yields an fs-moving lemma for higher Chow cycles, with \( B = \square^n \) and \( \hat{B} = \square^n \).
Proposition 5.10. Let $V = \text{Spec}(R)$ be an $r$-dimensional regular semi-local $k$-scheme of geometric type with the set of closed points $\Sigma$. Let $m \geq 1$, $n \geq 1$ be two integers and let $\alpha \in Tz^n(X; n; m)$ be a cycle.

Assume that no component of $\alpha$ is an $fs$-cycle. Assume further that $r \geq 1$ and that $V$ is not $\alpha$-linear (see Definition 4.3). Then, we can find

1. an atlas $(X, \Sigma)$ for $V$,
2. a cycle $\alpha \in Tz^n(X, n; m)$ with $\alpha = \alpha_V$,
3. a finite and flat map $\phi : X \to \mathbb{A}^n_k$, and
4. an affine open neighborhood $U \subseteq X$ of $\Sigma$

such that the following hold:

(A) If $Z_i$ is a component of $\alpha$ which is dominant over $V$, then for every component $Z'_i$ of $L^+([Z_i])$, the map $Z'_i \to U$ is finite and surjective.

(B) If $Z_i$ is a component of $\alpha$ which is not dominant over $V$, then $L^+([Z_i])_U = 0$.

Proof. Let $(X, \Sigma)$ be an atlas for $V$ and let $\alpha \in Tz^n(X, n; m)$ be such that $\alpha = \alpha_V$ (Lemma 2.9). Here, every component $\alpha$ is the closure of one and only one component of $\alpha$ in $X \times B_n$. Since $V$ is not $\alpha$-linear, $X$ is not isomorphic to an affine space over $k$. Let $\{Z_1, \ldots, Z_T\}$ be the set of irreducible components of $\alpha$ and let $Z := \text{Supp}(\alpha)$. So, $\text{Supp}(\alpha) = \overline{Z} = \bigcup_{i=1}^T \overline{Z}_i$.

Let $f : \overline{Z} \to X$ denote the projection map. Let $\tilde{Z}$ denote the closure of $\overline{Z}$ in $X \times \hat{B}_n$. Then the map $\tilde{f} : \tilde{Z} \to X$ is projective and the map $\tilde{g} : \tilde{Z} \to \hat{B}_n$ preserves closed points. For given $1 \leq i \leq s$, we fix closed points $x_i \in X$, $b_i \in B_n$ such that $\alpha_i = (x_i, b_i) \in \overline{Z}_i$ and $\alpha_i \notin \overline{Z}_j$ for $i \neq j$. Since each $\overline{Z}_i \neq \emptyset$, such closed points always exist. Notice that $\overline{Z}$ is an $r$-dimensional cycle in $X \times B_n$, none of whose component is an $fs$-cycle over $V$, so in particular, no component of $\overline{Z}$ is contained in $X \times \{b\}$ for any closed point $b \in B_n$. Set $D_0 = \{x_1, \ldots, x_s\} \cup \Sigma$, $E_0 = \{b_1, \ldots, b_s\}$ and $F_0 = F_n \cup E_0$.

Let $\phi_L : \overline{X} \to \mathbb{P}^r_k$ and $\phi_f : \overline{X} \to \mathbb{A}^n_k$ be the finite maps which satisfy properties (1)−(6) stated in Lemma 5.9 with $\overline{Z}$, $D_0$ and $F_0$ as chosen above. Notice that the map $\phi_L : \overline{X} \to \mathbb{A}^r_k$ is flat as $X$ is smooth of dimension $r$. We set $\phi = \phi_L$ and write $\phi_{B_n} = \phi_L \times \text{Id}_{B_n}$, $\phi_{\overline{Z}_i} = \phi_L \times \text{Id}_{\overline{Z}_i}$, for simplicity.

For an irreducible component $\overline{Z}_i$ of $\overline{Z}$, observe that the irreducible components of $L^+(\overline{Z}_i)$ are exactly the restrictions of the irreducible components of $L^+(\tilde{Z}_i)$ to $X \times B_n$. We first show that no irreducible component of $L^+(\tilde{Z}_i)$ coincides with $\tilde{Z}_i$. Let $y_i = \phi(x_i)$ and $\beta_i = \phi_{B_n}(\alpha_i) = (y_i, b_i)$. Set

$$Y_i = \tilde{f}(\overline{Z}_i), \quad Y^0_i = \tilde{f}(\overline{Z}_i^0), \quad W_i = \phi_{B_n}(\overline{Z}_i) \text{ and } \widehat{W}_i = \phi_{B_n}(\tilde{Z}_i).$$

One checks from the definitions of the flat pull-back and proper push-forward of cycles (see [23], §3.3, 3.4) that $\tilde{Z}_i$ is not a component of $L^+(\tilde{Z}_i)$ if and only if the map of semi-local rings $\mathcal{O}_{W_i, \beta_i} \to \mathcal{O}_{\overline{Z}_i, \alpha_i}$ is an isomorphism. We prove the latter.

It follows from the property (5) of Lemma 5.9 that the map $\mathcal{O}_{Z_i, \alpha_i} \to \mathcal{O}_{Z_i, \beta_i}$ is an isomorphism. By the property (1) of Lemma 5.9 of the map $\phi$, it is étale in an affine neighborhood $U'$ of $D_0$, and hence $\phi_{B_n}$ is étale in $U' \times \hat{B}_n$. In particular, it is étale at $\alpha_i$. This in turn implies that the map $\mathcal{O}_{W_i, \beta_i} \to \mathcal{O}_{\overline{Z}_i, \alpha_i}$ is unramified.
By the property (3) of Lemma 5.9 of the map \( \phi \), we have \( k(\beta_i) \xrightarrow{\sim} k(\alpha_i) \). We conclude that \( \mathcal{O}_{V_i, \beta_i} \to \mathcal{O}_{\hat{Z}_i, \beta_i} \) is an injective, finite and unramified map of local rings which induces isomorphism of the residue fields. But, by Lemma 5.2, it is an isomorphism. We have thus shown that no irreducible component of \( L^+ (\hat{Z}_i) \) coincides with \( \hat{Z}_i \).

We now prove (A). Let \( \overline{Z}_i \) be an irreducible component of \( Z \) which is dominant over \( V \). Let \( Z' \) be an irreducible component of \( L^+ (\hat{Z}_i) \). By Lemma 4.3, it is enough to show that \( Z' \cap (\Sigma \times F_n) = \emptyset \). Let \( D_1 \) be the set of points \( x \in D_0 \) such that \( \hat{Z}_i \cap (\{x\} \times F_n) \neq \emptyset \) and set \( D_2 = D_0 \setminus D_1 \).

Toward contradiction, suppose there are closed points \( x \in \Sigma \) and \( b \in F_n \) such that \( \lambda = (x, b) \in Z' \). Note \( x \in \Sigma \subset D_0 = D_1 \cup D_2 \). Let us first assume that \( x \in D_2 \). This means that there is \( x' \in L^+(x) = \phi^{-1}(\phi(x)) \setminus \{x\} \) such that \( \lambda' = (x', b) \in \hat{Z}_i \). In particular, \( x' \in Y_0 \) and this can not happen by the property (5) of Lemma 5.9 of \( \phi \). So, \( x \in D_1 \).

So, we have points \( x \in D_1 \) and \( b \in F_n \) such that \( \lambda = (x, b) \in Z' \). This means that there is a point \( x' \in \phi^{-1}(\phi(x)) \) such that \( \lambda' = (x', b) \in \hat{Z}_i \). If \( x' \in L^+(x) \), then we must have \( x' \in L^+(x) \cap Y_i \), which is empty. So, it cannot happen. If \( x' = x \), then we must have \( \lambda = (x, b) \neq \hat{Z}_i \). We show this cannot happen either.

Let \( \xi = \phi_{\hat{B}_n}(\lambda) = (\phi(x), b) = (y, b) \), say. Let \( S := (\phi_{\hat{B}_n})^{-1}(\xi) = \phi^{-1}(y) \times \{b\} \). Since \( \phi_{\hat{B}_n} \) is étale in \( U' \times \hat{B}_n \), it is étale at \( \lambda \). By the property (3) of Lemma 5.9 of the map \( \phi \), we have \( k(\xi) \xrightarrow{\sim} k(\lambda) \). If there is a point \( \lambda' = (x', b) \in \hat{Z}_i \) with \( \lambda' \in S \setminus \{\lambda\} \), then we must have \( \lambda' \in \hat{Z}_i \) and \( x' \in L^+(x) \). In particular, \( x' \in Y_0 \cap L^+(x) \). But this set is empty by the property (5) of Lemma 5.9 of the map \( \phi \). We conclude that \( \hat{Z}_i \cap S = \{\lambda\} \). It follows from Lemma 5.3 that \( Z' \) can not pass through \( \lambda \), so it contradicts \( \lambda \in Z' \). This proves (A).

We prove (B) now. Suppose next that \( \overline{Z}_i \) is an irreducible component of \( Z \) which is not dominant over \( V \). Suppose that \( Z' \) is some component of \( L^+ (\hat{Z}_i) \) such that \( Z' \cap (\Sigma \times \hat{B}_n) \neq \emptyset \). Then, we can find \( x \in \Sigma \) and \( b \in \hat{B}_n \) such that \( \lambda = (x, b) \in Z' \). This means that there is a point \( x' \in \phi^{-1}(\phi(x)) \) such that \( \lambda' = (x', b) \in \hat{Z}_i \). If \( x' \in L^+(x) \), then we must have \( x' \in L^+(x) \cap Y_i \), which is empty. If \( x' = x \), then we must have \( \lambda = (x, b) \neq \hat{Z}_i \). But the same proof as above shows that this cannot occur. We conclude that \( \hat{f}(L^+(\hat{Z}_i)) \) is a closed subset of \( X \) disjoint from \( D \). Hence we can choose an affine open neighborhood \( U \) of \( \Sigma \) in \( X \) such that \( L^+ ((\hat{Z}_i)_U) = \emptyset \). This proves (B), and proof of the proposition is now complete. \( \square \)

**Theorem 5.11 (The fs-moving lemma).** Let \( V = \text{Spec} \ (R) \) be an \( r \)-dimensional regular semi-local \( k \)-scheme of geometric type with the set of closed points \( \Sigma \). Assume that \( r \geq 1 \) and that \( m \geq 0, n \geq 1 \) are two integers. Then the map \( \text{fs}_V : \text{TCH}^r_{\text{fs}}(V, n; m) \to \text{TCH}^r(V, n; m) \) is an isomorphism.

*Proof.* It is clear from the definition of \( \text{TCH}^r_{\text{fs}}(V, n; m) \) that the map \( \text{fs}_V \) is injective. So the main point is to prove its surjectivity. Let \( \gamma \in \text{TCH}^r_{\text{fs}}(V, n; m) \) be a cycle with \( \partial(\gamma) = 0 \). We can write \( \gamma = \alpha + \beta \), where no component of \( \alpha \) is an fs-cycle and every component of \( \beta \) is an fs-cycle.
First suppose that \( V \) is \( \alpha \)-linear, so that by there is an atlas \((\mathbb{A}^r, \Sigma)\) for \( \alpha \). In this case, by Theorem \( 4.16 \) we can write \( \alpha = \alpha_1 + \partial (\alpha_2) \), where \( \alpha_1 \in Tz^n_{fs}(V, n; m) \subseteq Tz^n_{fs}(V, n; m) \) and \( \alpha_2 \in Tz^n(V, n + 1; m) \). In particular, \( \gamma = \alpha_1 + \beta + \partial (\alpha_2) \). So, for \( \gamma' := \alpha_1 + \beta \in Tz^n_{fs}(V, n; m) \), \( \partial (\gamma') = 0 \), we have \( \gamma - \gamma' = \partial (\alpha_2) \), proving the theorem in this case.

Now suppose \( V \) is not \( \alpha \)-linear. We apply Proposition \( 5.10 \) to \( \alpha \) and Lemma \( 2.9 \) to \( \beta \). This yields an atlas \((X, \Sigma)\) for \( V \) and cycles \( \overline{\alpha}, \overline{\beta} \in Tz^n(X, n; m) \) with \( \overline{\gamma} = \overline{\alpha} + \overline{\beta} \) and \( \partial (\overline{\gamma}) = 0 \) and \( X \) is not isomorphic to \( \mathbb{A}^r \). Moreover, \( \overline{\alpha} \) satisfies all the properties stated in Proposition \( 5.10 \) Let \( \phi : X \to \mathbb{A}^r_k \) be a finite and flat map as in Proposition \( 5.10 \) and let \( \Sigma' = \phi (\Sigma) \), which consists of finitely many closed points of \( \mathbb{A}^r_k \). Let \( V' \) be the regular semi-local \( k \)-scheme of geometric type with the set of closed points \( \Sigma' \). Let \( W := X \times_{\mathbb{A}^r} V' \). Here there are inclusions \( \Sigma \subseteq V \subseteq W \subseteq X \), and a finite flat morphism \( \phi : W \to V' \). Furthermore \( V' \) is \( \phi_* (\gamma) \)-linear by definition.

Write \( \overline{\alpha} = \overline{\alpha}_1 + \overline{\alpha}_2 \), where each component of \( \overline{\alpha}_1 \) is dominant over \( X \) and no component of \( \overline{\alpha}_2 \) is dominant over \( X \). So, we can write

\[
\overline{\gamma} = \overline{\alpha}_1 + \overline{\alpha}_2 + \overline{\beta} = (\overline{\alpha}_1 - \phi^* \phi_* (\overline{\alpha}_1)) + (\overline{\alpha}_2 - \phi^* \phi_* (\overline{\alpha}_2)) + (\overline{\beta} - \phi^* \phi_* (\overline{\beta})) + \phi^* \phi_* (\overline{\gamma}).
\]

Set \( \overline{\alpha}' := \overline{\alpha}_1 - \phi^* \phi_* (\overline{\alpha}_1) \) and \( \overline{\beta}' := \overline{\beta} - \phi^* \phi_* (\overline{\beta}) \). Note that \( \phi_* \) and \( \phi^* \) preserve the fs-cycles so that \( \overline{\beta}' \) is an fs-cycle. On the other hand, by Proposition \( 5.10 \)B, we have \( (\overline{\alpha}'_2)_V = 0 \), while by Proposition \( 5.10 \)A we have \( (\overline{\alpha}'_1)_V \in Tz^n_{fs}(V, n; m) \).

Finally, since \( \overline{\gamma} \in Tz^n(X, n; m) \) with \( \partial (\overline{\gamma}) = 0 \), we have \( \phi_* (\overline{\gamma}) \in Tz^n(\mathbb{A}^r, n; m) \) with \( \partial (\phi_* (\overline{\gamma})) = 0 \). Furthermore \( V' \) is \( \phi_* (\gamma) \)-linear, so, by Theorem \( 4.16 \) there are cycles \( \eta_1 \in Tz^n(V', n; m) \) \( \eta_2 \in Tz^n(V', n + 1; m) \) such that \( j^* (\phi_* (\overline{\gamma})) = \eta_1 + \partial \eta_2 \).

This is equivalent to \( \phi_* (\overline{\gamma}_W) = \eta_1 + \partial \eta_2 \). Hence, \( \phi^* \phi_* (\overline{\gamma}_W) = \phi^* (\eta_1) + \phi^* (\partial \eta_2) = \phi^* (\eta_1) + \partial (\phi^* \eta_2) \). Note \( \phi^* \phi_* (\overline{\gamma}_W)_V = \phi^* \phi_* (\overline{\gamma})_V \).

Hence, combining these, we have \( \gamma = (\overline{\gamma})_V = (\overline{\alpha}'_1)_V + \overline{\beta}'_V + (\phi^* (\eta_1))_V + \partial ((\phi^* \eta_2)_V) \), where \( \gamma_1 := (\overline{\alpha}'_1)_V + \overline{\beta}'_V + (\phi^* (\eta_1))_V \in Tz^n_{fs}(V, n; m) \).

Since \( \partial \gamma = 0 \), we also deduce \( \partial \gamma_1 = 0 \). This complete the proof.

6. The fs-s-moving lemma I: Admissible linear subspaces

Let \( k \) be any infinite perfect field. From Section \( 6.2 \) we suppose \( k \) is algebraically closed. The goal of the next few sections is to show that the map \( \text{sfs}_V : TCH^n_{sfs}(V, n; m) \to TCH^n_{fs}(V, n; m) \) is an isomorphism for a regular semi-local \( k \)-scheme \( V \) of geometric type. Like in the case of fs-moving lemma discussed in Section \( 5 \) we use techniques of linear projections inside projective and affine space to reduce to the case when \( V \) is linear with respect to a given admissible cycle. In this section, we define certain admissible linear subspaces and loci in a Grassmannian space and prove some results on them. These results contribute to the final proof of the sfs-moving lemma. For the definition and some elementary properties of sfs-cycles, see Section \( 3 \) We use the notations and terminologies of Section \( 5.2 \) on linear projections.

6.1. Rectifiable cycles. Let \( X \) be an irreducible quasi-projective scheme over \( k \) of dimension \( r \geq 1 \). Let \( X_{fs} \subseteq X \) be a fixed dense open subset which is smooth over \( k \) (possibly strictly smaller than \( X_{sm} \)) and let \( X_{sfs} \) be its complement. Notice
that $X_{\text{sing}} \subseteq X_{\text{fss}}$. Let $B$ be a geometrically integral smooth affine scheme over $k$ of positive dimension, and let $\hat{B}$ be a geometrically integral smooth projective compactification of $B$. Let $Z$ be a cycle on $X \times \hat{B}$ and let $\{Z_1, \cdots, Z_s\}$ be its irreducible components. Let $f : X \to X$ and $g : Z \to \hat{B}$ denote the projection maps.

**Definition 6.1.** We say that $Z$ is a rectifiable cycle, if the following hold for each irreducible component $Z_i$ of $Z$:

1. the map $Z_i \to X$ is finite and surjective over $X_{\text{fss}}$, and
2. the map $Z_i \to B$ is not constant.

Since we would eventually localize $X_{\text{fss}}$ at a finite subset of closed subsets, when no confusion arises, we will say that $Z$ is an fs-cycle, if each $Z_i$ satisfies only (1). Certainly this notion depends on the choice of $X_{\text{fss}}$.

**6.2. Admissible linear subspaces.** We assume in the rest of Section 6 that the ground field $k$ is algebraically closed. The general case will be considered in the following sections.

Suppose $X$ is projective and fix a closed embedding $\eta : X \hookrightarrow \mathbb{P}^N$ of degree $d + 1 \gg 0$ such that $N \gg r$. We fix $X_{\text{fss}}$ and fix a closed point $x \in X_{\text{fss}}$. Let $Z$ be a rectifiable cycle on $X \times \hat{B}$, with irreducible components $\{Z_1, \cdots, Z_s\}$. Let $S \subset X$ be a given finite set of closed points.

**Definition 6.2.** We say that a linear subspace $L \in \text{Gr}(N - r, \mathbb{P}^N)$ is $(Z, x)$-admissible, if the following hold.

1. $L \cap (X_{\text{fss}} \cup S) = \emptyset$.
2. $L$ intersects $X_{\text{fss}}$ transversely.
3. $L \cap X = \{x = x_0, x_1, \cdots, x_d\}$ with $x_i \neq x_j$ for $i \neq j$.
4. Each $Z_i$ is regular at all points lying over $\{x_1, \cdots, x_d\}$.
5. $g(Z_{x_i}) \cap g(Z_{x_j}) = \emptyset$ for $0 \leq i \neq j \leq d$.

Note that $L \in \text{Gr}(N - r, \mathbb{P}^N)(k)$ is $(Z, x)$-admissible if and only if it is $(Z_i, x)$-admissible for every irreducible component $Z_i$ of $Z$. The following is an “intermediate” version of Definition 6.2.

**Definition 6.3.** For $1 \leq n \leq d - 1$ and $0 \leq m \leq n$, we say that a member $\underline{L} = (L, x_1, \cdots, x_d) \in \text{Gr}(N - r, \mathbb{P}^N) \times X^d$ is $(Z, x, m, n)$-admissible, if the conditions (1)~(4) of Definition 6.2 as well as the following hold:

5. $g(Z_{x_i}) \cap g(Z_{x_j}) = \emptyset$ for $0 \leq i \neq j \leq n$.
6. $g(Z_{x_i}) \cap g(Z_{x_{n+1}}) = \emptyset$ for $0 \leq i \leq m$.

Note that $\underline{L}$ is $(Z, x, m, n)$-admissible if and only if it is $(Z_i, x, m, n)$-admissible for every irreducible component $Z_i$ of $Z$. Note also that $L \in \text{Gr}(N - r, \mathbb{P}^N)$ is $(Z, x)$-admissible if and only if $\underline{L}$ is $(Z, x, d - 1, d - 1)$-admissible. We now define the admissible locus of the simplest kind:

**Definition 6.4.** Let $\tilde{U}^{x, S}_{m, n} \subset \text{Gr}_x(N - r, \mathbb{P}^N) \times X^d$ be the subset parametrizing all $(Z, x, m, n)$-admissible points and let $U^{x, S}_{m, n} \subset \text{Gr}_x(N - r, \mathbb{P}^N)$ be the image of $\tilde{U}^{x, S}_{m, n}$ under the projection $\text{Gr}_x(N - r, \mathbb{P}^N) \times X^d \to \text{Gr}_x(N - r, \mathbb{P}^N)$. Let $U^{x, S}_{\text{adm}} \subset \text{Gr}_x(N - r, \mathbb{P}^N)$ be the subset parametrizing all $(Z, x)$-admissible points.
Before we proceed to the relative notion of admissibility, we prove the following:

**Lemma 6.5.** Assume $N \gg r = 1$ and let $x \neq y$ be two closed points on $X_{sm}$. Let $Gr_{x+2y}(N-1, \mathbb{P}^N) \subsetneq Gr(N-1, \mathbb{P}^N)$ be the set of hyperplanes containing $\{x, y\}$ that do not intersect $X$ transversely at $y$. Then $Gr_{x+2y}(N-1, \mathbb{P}^N) \simeq \mathbb{P}^{N-3}$.

**Proof.** Since $N \gg r = 1$, we can find a linear form $s_1 \in W = H^0(\mathbb{P}^N, \mathcal{O}(1))$ which does not vanish at $\{x, y\}$. This yields a $k$-linear map $\alpha : W \to \mathcal{O}_{X, (x, y)}/m_xm_y$ given by $\alpha(s) = s/s_1$. Since $k$ is algebraically closed and so $m_y$ is generated by linear forms vanishing at $y$, we see that the composite map $W \to \mathcal{O}_{X, (x, y)}/m_xm_y \to \mathcal{O}_{X, y}/m_y^2$ is surjective and $\alpha^{-1}(m_y^2)$ is precisely the set of linear forms in $W$ which are not transverse to $X$ at $y$.

Since $x, y$ are two distinct smooth closed points of $X$, the set $Gr_y(x, N-1, \mathbb{P}^N)$ is non-empty and hence $m_y/m_xm_y \to \mathcal{O}_{(x)}$ and there is a commutative diagram of short exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \alpha^{-1}(m_xm_y) & \longrightarrow & \alpha^{-1}(m_y) & \longrightarrow \mathcal{O}_{\{x\}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & \\
0 & \longrightarrow & m_xm_y/m_xm_y & \longrightarrow & m_y/m_y^2 & \longrightarrow \mathcal{O}_{\{x\}} \longrightarrow 0.
\end{array}
\]

In particular, the first vertical map is surjective. Since $Gr_x(y, N-1, \mathbb{P}^N) \neq \emptyset$, we conclude that $\alpha$ is surjective. We now have a commutative diagram of short exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & W(-x - 2y) & \longrightarrow & W & \longrightarrow \mathcal{O}_{\{x+2y\}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & \\
0 & \longrightarrow & \alpha^{-1}(m_y^2) & \longrightarrow & W & \longrightarrow \mathcal{O}_{\{2y\}} \longrightarrow 0.
\end{array}
\]

Since the last vertical arrow is surjective with one-dimensional kernel, it follows that the first vertical arrow is injective with one-dimensional cokernel. Since $|\alpha^{-1}(m_y^2)| \simeq \mathbb{P}^{N-2}$, we conclude that $Gr_{x+2y}(N-1, \mathbb{P}^N) \simeq |W(-x - 2y)| \simeq \mathbb{P}^{N-3}$. \qed

**Lemma 6.6.** Assume $r = 1$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points in $X$. Then the set $\mathcal{U}_S^{x, 1-4} \subseteq Gr_x(N - 1, \mathbb{P}^N)$ consisting of hyperplanes $H$ such that $H \in Gr_x(N - 1, \mathbb{P}^N) \times X^d$ satisfies the conditions (1) – (4) of Definition 6.3 is a dense open subset of $Gr_x(N - 1, \mathbb{P}^N)$.

**Proof.** Since $f : Z \to X$ is finite and since $\dim(Z_{\text{sing}}) = 0$, we see that $f(Z_{\text{sing}})$ is a finite closed subset of $X$ and so is $X_{\text{sing}}$. In particular, $T = (f(Z_{\text{sing}}) \cup X_{\text{sing}} \cup S) \setminus \{x\}$ is a finite closed subset of $X$. Hence, the hyperplanes not intersecting $T$ form an open subset of $Gr(N - 1, \mathbb{P}^N)$. It is well known and easy to check that the set of hyperplanes in $Gr(N - 1, \mathbb{P}^N)$ which do not intersect $T$ and intersect $X$ transversely along $X_{\text{sm}}$ also form an open subset of $Gr(N - 1, \mathbb{P}^N)$. We conclude that the hyperplanes $H$ satisfying the conditions (1) – (4) of Definition 6.3 form an open subset of $Gr_x(N - 1, \mathbb{P}^N)$. We only need to show that this set is nonempty.
For any \( y \in X_{\text{sm}} \), let \( m_y \) denote the maximal ideal of \( O_{X,y} \). Let \( V \) denote the set of linear subspaces of \( W = H^0(\mathbb{P}^N, O(1)) \) which vanish at \( x \). Since \( x \in X_{\text{sm}} \), we see that \( m_x \) is generated by the elements of \( V \). In particular, if we choose \( s_1 \) not vanishing at \( x \), then the map \( \alpha : V \to m_x/m_x^2 \) given by \( \alpha(s) = s/s_1 \) is surjective and \( D_x = \mathbb{P}(\ker(\alpha)) \) is exactly the set of hyperplanes passing through \( x \) and not transverse to \( X \) at \( x \). It is clear that \( \dim(D_x) = N - 2 < \dim(Gr_x(N-1, \mathbb{P}^N)) \).

Next suppose that \( y \neq x \) is a smooth closed point of \( X \). Let us choose \( s_1 \in W \) such that \( s_1(x) = 0 \) and \( s_1(y) \neq 0 \). Consider the map \( \beta : V \to O_{X,y}/m_y^2 \) given by \( \beta(s) = s/s_1 \). Since \( Gr_{(x,y)}(N-1, \mathbb{P}^N) \cong \mathbb{P}^{N-2} \), it follows from Lemma 6.3 that \( m_y \) is generated by linear forms vanishing at \( x \). We conclude that \( \beta \) is surjective and \( \ker(\beta) \) is precisely the set of linear forms in \( V \) which are not transverse to \( X \) at \( y \).

Let \( B' \subset X \times [V] \) denote the set of points \( (y,H) \) such that \( H \) is not transverse to \( X \) at \( y \). Since \( \dim(O_{X,y}/m_y^2) = 2 \) for any \( y \in X_{\text{sm}} \), we see that the general fiber of the map \( B' \to X \) has dimension not more than \( N - 3 \). In particular, \( \dim(B') \leq N - 2 \). Taking the image of \( B' \) in \([V]\) under the projection, we see that the set of hyperplanes in \( Gr_x(N-1, \mathbb{P}^N) \) which are not transverse to \( X \) at its smooth points form a proper closed subset. Since \( N \gg 0 \) and \( T \) is finite, we can always find hyperplanes which pass through \( x \) but not through \( T \). We have thus shown that \( U_{S}^{1-4,1} \subseteq Gr_x(N-1, \mathbb{P}^N) \) is a dense open subset.

### 6.3. Admissibility relative to a linear subspace

Let us assume that \( r \geq 2 \).

Let \( L_0 \in Gr(N - r + 1, \mathbb{P}^N) \) be a fixed \((N - r + 1)\)-dimensional linear subspace. A hyperplane in \( \mathbb{P}^N \) intersects \( L_0 \) in a linear subspace of dimension at least \( N - r \). The subset \( Gr^r(L_0, N - 1, \mathbb{P}^N) \subseteq Gr(N - 1, \mathbb{P}^N) \) of hyperplanes intersecting \( L_0 \) transversely, is open whose complement is isomorphic to \( \mathbb{P}^{r-2} \). In particular, \( Gr^r(L_0, N - 1, \mathbb{P}^N) \) is a dense open subset of \( Gr(N - 1, \mathbb{P}^N) \).

**Definition 6.7.** Under the above notations, define the regular map of schemes

\[
\theta_{L_0} : Gr^r(L_0, N - 1, \mathbb{P}^N) \to Gr(N - r, L_0)
\]

given by \( \theta_{L_0}(H) = H \cap L_0 \).

One checks that \( \theta_{L_0} \) is a surjective smooth morphism of relative dimension \( r - 1 \).

Here is a relative version of Definition 6.3.

**Definition 6.8.** Let \( S \subset X \setminus \{x\} \) be a finite set of closed points. For \( 1 \leq n \leq d - 1 \) and \( 0 \leq m \leq n \), we say that \( H = (H, x_1, \ldots, x_d) \in Gr^r(L_0, N - 1, \mathbb{P}^N) \times X^d \) is \((Z, x, m, n)\)-admissible relative to \( L_0 \) if \( \theta_{L_0}(H), x_1, \ldots, x_d \in Gr(N - r, L_0) \times X^d \) is \((Z, x, m, n)\)-admissible.

We say that \( H \in Gr^r(L_0, N - 1, \mathbb{P}^N) \) is \((Z, x)\)-admissible relative to \( L_0 \) if there exists \( H = (H, x_1, \ldots, x_d) \in Gr^r(L_0, N - 1, \mathbb{P}^N) \times X^d \) which is \((Z, x, d - 1, d - 1)\)-admissible relative to \( L_0 \).

We see that \( H \) is \((Z, x, m, n)\)-admissible relative to \( L_0 \) if and only if it is \((Z_i, x, m, n)\)-admissible relative to \( L_0 \) for every irreducible component \( Z_i \) of \( Z \). Here is a relative version of the admissible loci of Definition 6.4.

**Definition 6.9.** Recall \( Gr^r_x(L_0, N - 1, \mathbb{P}^N) : = Gr^r(L_0, N - 1, \mathbb{P}^N) \cap Gr_x(N - 1, \mathbb{P}^N) \).

Let \( \widetilde{U}_{m,n} \subset Gr_x(L_0^r, N - 1, \mathbb{P}^N) \times X^d \) be the set of all points that are \((Z, x, m, n)\)-admissible relative to \( L_0 \) and let \( U_{m,n} \subset Gr_x(L_0^r, N - 1, \mathbb{P}^N) \) be the image of
$U_{x,m,n}^{L}$ under the projection $Gr_x(L_0^N, N-1, \mathbb{P}^N) \times X^d \to Gr_x(L_0^N, N-1, \mathbb{P}^N)$. Let $U_{adm}^{x,S,L_0} \subseteq Gr_x(L_0^N, N-1, \mathbb{P}^N)$ be the set of all points that are $(Z, x)$-admissible relative to $L_0$.

Before we move on to the proof of openness of admissible loci $U_{x,m,n}^{S,L_0}$ and $U_{x,m,n}^{x,S,L_0}$, we prove the following higher dimensional analogue of Lemma 6.6, which requires the relative version we considered just now:

**Lemma 6.10.** Assume $r \geq 2$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points. After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}^N$ by its composition with a Veronese embedding, one has the following.

Given a hyperplane $H_0 \subset \mathbb{P}^N$ disjoint from $S \cup \{x\}$, and a general $L_0 \in Gr(H_0, N-r+1, \mathbb{P}^N)$, the set $U_{x,L_0}^{S,1-4} \subseteq Gr_x^{tr}(L_0, N-1, \mathbb{P}^N)$ consisting of hypersurfaces $H$ such that $H \in Gr_x^{tr}(L_0, N-1, \mathbb{P}^N) \times X^d$ satisfy the conditions (1) $\sim$ (4) of Definition 6.8, is a dense open subset of $Gr_x^{tr}(L_0, N-1, \mathbb{P}^N)$.

**Proof.** By the Bertini theorems of Altman and Kleiman [1], a general intersection of $(r-1)$ hypersurfaces containing $S \cup \{x\}$, all of a fixed degree in $\mathbb{P}^N$ (depending only on $X$ and $S$), is an irreducible curve $C$. This curve contains $S \cup \{x\}$, is not contained in $f(Z_{\text{sing}}) \cup X_{\text{ufs}}$, is smooth away from $X_{\text{sing}}$, and the map $f^{-1}(C) \to B$ is not constant.

Hence, after replacing the embedding $\eta : X \hookrightarrow \mathbb{P}^N$ by its composition with the Veronese embedding of $\mathbb{P}^N$ given by the above degree of hypersurfaces, we can find an $(r-1)$-tuple of hyperplanes $(H_1, \cdots, H_{r-1})$, each of which is from $Gr_{S \cup \{x\}}(N-1, \mathbb{P}^N)$, such that the linear subspace $L_0 = H_1 \cap \cdots \cap H_{r-1}$ has the following property: $L_0$ is transverse to $H_0$, $C = L_0 \cap X$ is an irreducible curve such that $C \notin f(Z_{\text{sing}}) \cup X_{\text{ufs}}$, is smooth away from $X_{\text{sing}}$ and the map $f^{-1}(C) \to B$ is not constant. Moreover, any general $(r-1)$-tuple of hyperplanes $(H_1, \cdots, H_{r-1})$, each from $Gr_{S \cup \{x\}}(N-1, \mathbb{P}^N)$, has this property. Set $S' = (C \setminus \{x\}) \cap (f(Z_{\text{sing}}) \cup X_{\text{ufs}} \cup S)$. The choice of $C$ implies that $S'$ is a finite closed subset of $C \setminus \{x\}$.

It follows from the definition of the degree of the embedding $\eta : X \hookrightarrow \mathbb{P}^N$ that a general hyperplane in $L_0$ will intersect $C$ at $d+1$ points. It follows from Lemma 6.6 that the set $U_{x}^{S,1-4} \subseteq Gr_x(N-r, L_0)$ consisting of hyperplanes $L \subseteq L_0$ such that $L \cap S' = \emptyset$ and $L \in Gr_x(N-r, L_0) \times C^d$ satisfy the conditions (1) $\sim$ (4) of Definition 6.3, is a dense open subset of $Gr_x(N-r, L_0)$. Since $\theta_{L_0}$ is a smooth and surjective morphism such that $(\theta_{L_0})^{-1}(Gr_x(N-r, L_0)) = Gr_x^{tr}(L_0, N-1, \mathbb{P}^N)$, we see that $(\theta_{L_0})^{-1}(U_{x}^{S,1-4})$ is a dense open subset of $Gr_x^{tr}(L_0, N-1, \mathbb{P}^N)$. We are thus only left with showing that $(\theta_{L_0})^{-1}(U_{x}^{S,1-4}) = U_{S}^{x,L_0,1-4}$ to finish the proof of the lemma. It follows immediately from the definition of $U_{S}^{x,L_0,1-4}$ and the choice of $U_{x}^{S,1-4}$ that $U_{S}^{x,L_0,1-4} \subseteq (\theta_{L_0})^{-1}(U_{x}^{S,1-4})$. So we need to show the opposite inclusion.

Suppose $H \in (\theta_{L_0})^{-1}(U_{x}^{S,1-4})$. This means that $\theta_{L_0}(H) \cap S' = \emptyset$ and $\theta_{L_0}(H) = H \cap L_0$ satisfies (1)-(4) of Definition 6.3 with $Z$ replaced by $f^{-1}(C)$. Since $\theta_{L_0}(H) \cap (X_{\text{ufs}} \cup S) = H \cap (L_0 \cap X) \cap (X_{\text{ufs}} \cup S) = \theta_{L_0}(H) \cap C \cap (X_{\text{ufs}} \cup S) \subseteq \theta_{L_0}(H) \cap S'$, and since $x \in X_{\text{uf}}$, we see that $\theta_{L_0}(H) \cap (X_{\text{ufs}} \cup S) = \emptyset$.

Since $H$ intersects $L_0$ transversely, which in turn intersects $X$ transversely along $X_{\text{sm}}$, we see that $\theta_{L_0}(H)$ intersects $X$ transversely along $X_{\text{fs}}$. Also, $\theta_{L_0}(H) \cap X = \theta_{L_0}(H) \cap C = \{x = x_0, x_1, \ldots, x_d\}$ with $x_i \neq x_j$ for $i \neq j$. Finally, since
\[(C \cap f(Z_{\text{sing}})) \setminus \{x\} \subseteq S'\] and since \(\theta_{L_0}(H) \cap S' = \emptyset\), we see that \(Z\) is smooth along all points lying over \(x_i\) for \(1 \leq i \leq d\). This shows that \((\theta_{L_0})^{-1}(U_{S_i}^{x,-1}) \subseteq U_{S_i}^{x,L_0,-1}\) and completes the proof of the lemma. \(\square\)

7. The sf-\(s\)-moving lemma II: Openness of admissible loci

We continue to assume in this section that \(k\) is algebraically closed. Our goal is to prove a set of results about the sets \(U_{m,n}^{x,S}\) and \(U_{m,n,L_0}^{x,S}\). The main interest in Section 7.1 is to show that they form dense open subsets of various parameter spaces. We shall later consider a more general situation where \(\{x\}\) is replaced by a finite set of smooth closed points of \(X\) in Section 7.2.

7.1. Openness of admissible loci I. Let \(\eta : X \hookrightarrow \mathbb{P}^N\) be a closed embedding of degree \(d + 1 \gg 0\) such that \(N \gg r = \dim(X)\) as in Section 6.2 and let \(x \in X\) be a fixed smooth point. Let \(S \subseteq X\) \(\setminus \{x\}\) be a finite set of closed points.

In the following, Lemmas 7.1, 7.2, 7.3 and Proposition 7.4 are about \(U_{m,n}^{x,S}\) when \(r = 1\), and Proposition 7.5 is about \(U_{m,n,L_0}^{x,S}\) when \(r \geq 2\).

**Lemma 7.1.** Suppose \(r = 1\). For any \(1 \leq n \leq d - 1\) and \(0 \leq m \leq n\), the set \(U_{m,n}^{x,S} \subseteq Gr_2(N - 1, \mathbb{P}^N)\) is open.

**Proof.** We have shown in Lemma 6.6 that the set \(U_{S_i}^{x,-1} \subseteq Gr_2(N - 1, \mathbb{P}^N)\) is open. Let \(X \xleftarrow{f} Z \xrightarrow{g} \hat{B}\) be the projection maps. Set \(A = g(Z_x)\) and \(A_Z = g^{-1}(A)\). Since \(Z\) is a rectifiable curve, the maps \(f\) and \(g\) are finite, so we see that \(A\) and \(A_Z\) are finite closed subsets of \(\hat{B}\) and \(Z\), respectively.

Let \(V_d \subset X^{d}\) be the open subset defined by

\[(7.1) \quad V_d = \{(y_1, \cdots, y_d) | y_i \neq y_j \text{ for } 1 \leq i \neq j \leq d \text{ and } y_i \neq x \text{ for } 1 \leq i \leq d\}.

Let \(D_1, D_2 \subset V_d\) be given by

\[(7.2) \quad \begin{cases} D_1 = \{(y_1, \cdots, y_d) \in V_d | g(Z_{y_i}) \cap g(Z_{y_j}) \neq \emptyset \text{ for some } 1 \leq i \neq j \leq n\}, \\ D_2 = \{(y_1, \cdots, y_d) \in V_d | g(Z_{y_i}) \cap g(Z_x) \neq \emptyset \text{ for some } 1 \leq i \leq n\}. \end{cases}

For \(0 \leq i \leq m\), let \(G_i \subset V_d\) be given by

\[(7.3) \quad \begin{cases} G_0 = \{(y_1, \cdots, y_d) \in V_d | g(Z_{y_i}) \cap g(Z_{y_{n+1}}) \neq \emptyset\}, \\ G_i = \{(y_1, \cdots, y_d) \in V_d | g(Z_{y_i}) \cap g(Z_{y_{n+1}}) \neq \emptyset\} \text{ for } 1 \leq i \leq m. \end{cases}

**Claim:** \(D_1, D_2\) and \(G_i, 0 \leq i \leq m\) are all closed subsets of \(V_d\).

\(\vdash\) Let \(E_1, E_2 \subset (\hat{B})^d\) be the closed subsets given by

\[(7.4) \quad \begin{cases} E_1 = \{(b_1, \cdots, b_d) \in (\hat{B})^d | b_i = b_j \text{ for some } 1 \leq i \neq j \leq n\}, \\ E_2 = \{(b_1, \cdots, b_d) \in (\hat{B})^d | b_i \in A \text{ for some } 1 \leq i \leq n\}. \end{cases}

For \(0 \leq i \leq m\), let \(F_i \subset (\hat{B})^d\) be given by

\[(7.5) \quad \begin{cases} F_0 = \{(b_1, \cdots, b_d) \in (\hat{B})^d | b_{n+1} \in A\}, \\ F_i = \{(b_1, \cdots, b_d) \in (\hat{B})^d | b_i = b_{n+1}\} \text{ for } 1 \leq i \leq m. \end{cases}

One easily checks that

\[(7.6) \quad \begin{cases} D_1 = V_d \cap (f^d((g^d)^{-1}(E_1))), \\ D_2 = V_d \cap (f^d((g^d)^{-1}(E_2))), \\ G_i = V_d \cap (f^d((g^d)^{-1}(F_i))) \text{ for } 0 \leq i \leq m. \end{cases}\]
Since $f : Z \to X$ is finite, we conclude that $D_1, D_2, G_i$ are all closed in $V_d$. This proves the claim.

Let $\pi : X^d \to \text{Sym}^d(X)$ be the quotient map under the action of the symmetric group $\mathfrak{S}_d$ which permutes the coordinates of $X^d$. Notice also that the open subset $V_d \subset X^d$ is $\mathfrak{S}_d$-invariant and $\mathfrak{S}_d$ acts freely on $V_d$. In particular, the map $V_d \to \pi(V_d)$ is a finite étale map of degree $d!$. Let $\mathcal{U}^{x,1-4}_S \to \text{Sym}^d(X)$ be the map $H \mapsto \sum_{i=1}^d [y_i]$, where $H \cap X = \{x = x_0, y_1, \ldots, y_d\}$. The property (3) in Definition 6.3 implies that the image of $\mathcal{U}^{x,1-4}_S$ under this map lies in $V_d$. Consider the Cartesian square

$$
\begin{array}{ccc}
\mathcal{V}^{x,1-4} & \xrightarrow{e} & V_d \\
\downarrow \psi & & \downarrow \pi \\
\mathcal{U}^{x,1-4}_S & \xrightarrow{\pi} & \pi(V_d)
\end{array}
$$

so that $\psi$ is a finite étale map.

From what we have shown above, it follows that $e^{-1} (D_1 \cup D_2 \cup G_0 \cup \cdots \cup G_m)$ is closed in $\mathcal{V}^{x,1-4}_S$, whose complement is $\mathcal{U}^{x,1-4}_{m,n}$. Since $\psi$ is an open map, we conclude that $\mathcal{U}^{x,1-4}_{m,n} = \psi(\mathcal{U}^{x,1-4}_{m,n})$ is open in $\mathcal{U}^{x,1-4}_S$ and hence in $Gr_x(N - 1, \mathbb{P}^N)$. \hfill \Box

**Lemma 7.2.** Suppose $r = 1$ and fix $1 \leq n \leq d - 2$. If $\mathcal{U}^{x,1-4}_{m,n} \neq \emptyset$, then $\mathcal{U}^{x,1-4}_{m,n+1} \neq \emptyset$.

**Proof.** Set $T = S \cup (f(A_Z) \setminus \{x\})$, where $A_Z$ is as in the proof of Lemma 7.1. Then, apply Lemma 6.6 with $S$ replaced by $T$. We conclude that $\mathcal{U}^{x,1-4}_T$ is a dense open subset of $Gr_x(N - 1, \mathbb{P}^N)$. In particular, that $\mathcal{U}^{x,1-4}_{0,1}$ is a dense open subset of $Gr_x(N - 1, \mathbb{P}^N)$.

If $\mathcal{U}^{x,1-4}_{m,n} \neq \emptyset$, then it follows from Lemma 7.1 that it is a dense open subset of $Gr_x(N - 1, \mathbb{P}^N)$. In particular, $\mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T$ is open and dense in $Gr_x(N - 1, \mathbb{P}^N)$. But, since $\mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T \subset \mathcal{U}^{x,1-4}_{m,n+1}$, we are done. \hfill \Box

**Lemma 7.3.** Suppose $r = 1$, $1 \leq n \leq d - 1$, and $0 \leq m \leq n - 1$. If $\mathcal{U}^{x,1-4}_{m,n} \neq \emptyset$, then $\mathcal{U}^{x,1-4}_{m,n+1} \neq \emptyset$.

**Proof.** Let $T \subset X \setminus \{x\}$ be as in Lemma 7.2. If $\mathcal{U}^{x,1-4}_{m,n} \neq \emptyset$, then it follows from Lemma 7.1 that it is a dense open subset of $Gr_x(N - 1, \mathbb{P}^N)$. In particular, $\mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T$ is open dense in $Gr_x(N - 1, \mathbb{P}^N)$. Let us fix an element $H_0$ of $\mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T$ with $H_0 \cap X = \{x = x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_d\}$.

Since $N \gg 0$, there is a one-parameter family (isomorphic to $\mathbb{P}^1$) $B$ in $Gr_x(N - 1, \mathbb{P}^N)$ containing $H_0$ such that every member of this family passes through $\{x_0, x_{m+1}\}$ and a general member does not pass through $x_{n+1}$. Since $H_0 \in \mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T$, a general member of $B$ is in $\mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T$. Let $W \subset B \cap \mathcal{U}^{x,1-4}_{m,n} \cap \mathcal{U}^{x,1-4}_T$ be a smooth affine irreducible (rational) curve containing $H_0$. 
Let $W \to \pi(V_d)$ be the map $H \mapsto \sum_{i=1}^{d} [y_i]$, where $H \cap X = \{ x = x_0, y_1, \ldots, y_d \}$ and $V_d$ is as in (7.1). This yields a Cartesian square

\[
\begin{array}{ccc}
W' & \xrightarrow{e} & V_d \\
\psi \downarrow & & \downarrow \pi \\
W & \xrightarrow{\pi} & \pi(V_d)
\end{array}
\]  

(7.8)

as in (7.7) so that $\psi$ is finite and étale. Observe that $W'$ is irreducible.

Let $D_1, D_2 \subset V_d$ be as given by (7.2), and for $1 \leq i \leq m+1$, let $G_i \subset V_d$ be given by the same formula (7.3). We saw in the proof of Lemma 7.1 that $D_1, D_2, G_i$ are closed in $V_d$, by (7.6).

**Claim:** $Y := e^{-1}(D_1 \cup D_2 \cup G_0 \cup \cdots \cup G_{m+1})$ is finite.

(·) By the definition of $W$, we have $e^{-1}(D_2) = \emptyset$. Since $H_0 \notin e^{-1}(D_1)$ and since $W'$ is an irreducible curve, we see that $e^{-1}(D_1)$ must be finite. By our choice of $W$, no member of $W$ passes through $f(A_Z) \setminus \{x\}$ and hence $e^{-1}(G_0) = \emptyset$. For $1 \leq i \leq m$, we see that $H_0 \notin e^{-1}(G_i)$, and hence $e^{-1}(G_i)$ must be a proper closed subset of $W'$, thus finite.

We now show that $e^{-1}(G_{m+1})$ is finite. To show this part, we consider the composite map $q : W' \twoheadrightarrow V_d \to X^2$ which takes $H = (H, y_1, \ldots, y_d)$ to $(y_{m+1}, y_{n+1}) \in X^2$. Since all $H \in W$ contain $x_{m+1}$, the composition of $q$ with the first projection is the constant map which takes all $H \in W'$ to $x_{m+1}$. On the other hand, since a general member of $W$ does not contain $x_{n+1}$, we see that the composition of $q$ with the second projection of $X^2$ is not a constant map. In other words, the map $q$ is not constant. In particular, the image $q(W') \subset X^2$ is an irreducible curve such that $q(W') \subset x_{m+1} \times X$. Let $W' \twoheadrightarrow q(W') \twoheadrightarrow X$ denote the map $(H, y_1, \ldots, y_d) \mapsto y_{n+1}$. We see that $u$ and $v$ are both non-constant morphisms of irreducible curves, hence dominant and quasi-finite.

On the other hand, one checks that $e^{-1}(G_{m+1})$ is a subset of $\{(H, y_1, \ldots, y_d) \in W'| y_{n+1} \in f(g^{-1}(g(Z_{x,m+1})))\}$. Since $f$ and $g$ are finite, the set $S_1 = f(g^{-1}(g(Z_{x,m+1})))$ must be finite. We conclude that $e^{-1}(G_{m+1}) \subset (v \circ u)^{-1}(S_1)$, thus it is finite. This proves the **Claim**.

Now, choose any $H = (H, y_1, \ldots, y_d) \in W' \setminus Y$. Then, it is clear from the choice of $W$ and $Y$ that $H \in U_{m,n}^{x,S}$. This completes the proof. \hfill $\Box$

**Proposition 7.4.** Suppose $r = 1$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points in $X$. Then, for every $1 \leq n \leq d-1$ and $0 \leq m \leq n$, the set $U_{m,n}^{x,S} \subset \text{Gr}_{x}(N-1, \mathbb{P}^N)$ is a dense open subset. In particular, the set $U_{\text{adm}}^{x,S} \subset \text{Gr}_{x}(N-1, \mathbb{P}^N)$, consisting of $(Z, x)$-admissible hyperplanes, is a dense open subset.

**Proof.** By Lemma 7.1 each $U_{m,n}^{x,S}$ is open. We saw in the proof of Lemma 7.2 that $U_{0,1}^{x,S}$ is open and dense. Applying Lemmas 7.2 and 7.3 repeatedly, we conclude that each $U_{m,n}^{x,S}$ is nonempty and open, thus dense open. The last assertion follows because $U_{\text{adm}}^{x,S} = U_{d-1,d-1}^{x,S}$. \hfill $\Box$

**Proposition 7.5.** Suppose $r \geq 2$ and let $S \subset X \setminus \{x\}$ be a finite set of closed points in $X$. After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}^N$ by its composition with a Veronese embedding, one has the following:
For a general \( L_0 \in Gr(N - r + 1, \mathbb{P}^N) \) and for \( 1 \leq n \leq d - 1, \ 0 \leq m \leq n \), the set \( U_{m,n}^{x,L_0} \subset Gr_x^{tr}(L_0, N - 1, \mathbb{P}^N) \) is a dense open subset. In particular, the set \( U_{\text{adim}}^{x,L_0} \subset Gr_x^{tr}(L_0, N - 1, \mathbb{P}^N) \), consisting of hyperplanes which are \((Z, x)\)-admissible relative to \( L_0 \), is a dense open subset.

**Proof.** The last assertion follows from the first by taking \( m = n = d - 1 \), so we prove the first one. Choose a reembedding \( \eta : X \hookrightarrow \mathbb{P}^N \), a general \( L_0 \in Gr(N - r + 1, \mathbb{P}^N) \) and \( C = L_0 \cap X \) as in Lemma 6.10. Set \( S' = (C \setminus \{x\}) \cap (f(Z_{\text{sing}}) \cup X_{\text{nfs}} \cup S) \) and \( W = f^{-1}(C) \). We use the notations from the proof of Lemma 6.10. Recall the smooth surjective morphism \( \theta_{L_0} : Gr_x^{tr}(L_0, N - 1, \mathbb{P}^N) \to Gr_x(N - r, L_0) \) from (6.3). By applying Proposition 7.4 to \( C \) and \( S' \subset C \setminus \{x\} \), we know that the subset \( U_{m,n}^{x,S'} \subset Gr_x(N - r, L_0) \) is dense open. Hence, its inverse image via \( \theta_{L_0} \) is a dense open subset of \( Gr_x^{tr}(L_0, N - 1, \mathbb{P}^N) \). Thus, it only remains to show that \( U_{m,n}^{x,S,L_0} = (\theta_{L_0})^{-1}(U_{m,n}^{x,S'}) \). We have already shown in the proof of Lemma 6.10 that 
\[
(\theta_{L_0})^{-1}(U_{m,n}^{x,S'}) = U_{m,n}^{x,S,L_0 - 1}.
\]
Since \( W = f^{-1}(C) \), we see that \( Z_y = W_y \) and hence \( g(Z_y) = g(W_y) \) for any \( y \in C \). It follows that for any \( H \in Gr_x^{tr}(L_0, N - 1, \mathbb{P}^N) \) with \( \theta_{L_0}(H) \cap X = (H \cap L_0) \cap C = \{x = x_0, x_1, \ldots, x_d\} \), the conditions (5)\( \sim \)(6) of Definition 6.8 are satisfied if and only if the conditions (5)\( \sim \)(6) of Definition 6.3 are satisfied for \( \theta_{L_0}(H) \) with \( X \) replaced by \( C \). In other words, \( (\theta_{L_0})^{-1}(U_{m,n}^{x,S} - 6) = U_{m,n}^{x,S,L_0 - 6} \). We conclude that \( U_{m,n}^{x,S,L_0} = (\theta_{L_0})^{-1}(U_{m,n}^{x,S'}) \). This proves the proposition. \( \square \)

### 7.2. Openness of the admissible loci II.

In Section 7.2, we strengthen the openness assertion on the admissible loci, where we replace a fixed smooth closed point \( x \in X \) by any finite set of smooth closed points of \( X \).

Let \( \eta : X \hookrightarrow \mathbb{P}^N \) be a closed embedding of degree \( d + 1 \gg 0 \) such that \( N \gg r = \dim(X) \) as in Section 6.2, and let \( S \subset X \) be a fixed set smooth closed points. Recall from Section 5.2 that for closed subsets \( Y, Y' \subseteq X \) with \( Y \cap Y' = \emptyset \), the set \( Gr_Y(Y', n, \mathbb{P}^N) \) denotes the set of \( n \)-dimensional linear subspaces of \( \mathbb{P}^N \) which contain \( Y \) but do not intersect \( Y' \). Given a closed point \( x \in X \) and \( L \in Gr(x, n, \mathbb{P}^N) \), let \( C_x(L) \) denote the linear span of \( x \) and \( L \) in \( \mathbb{P}^N \). Then \( C_x(L) \) is the unique element of \( Gr(n + 1, \mathbb{P}^N) \) containing \( L \) and \( x \). The association \( L \mapsto C_x(L) \) defines a smooth, surjective and affine morphism of schemes

\[
(7.9) \quad \vartheta_x : Gr(x, n, \mathbb{P}^N) \to Gr_x(n + 1, \mathbb{P}^N); \quad \vartheta_x(L) = C_x(L)
\]

of relative dimension \( n + 1 \) whose fiber over a point \( M \) is the smooth affine scheme \( Gr(x, n, M) \). In fact, it is a vector bundle morphism of rank \( n + 1 \). If \( L_0 \) is any proper linear subspace of \( \mathbb{P}^N \) containing \( x \), then \( \vartheta_x \) induces a smooth surjective map

\[
(7.10) \quad \vartheta_x^{L_0} : Gr_x^{tr}(x, L_0, n, \mathbb{P}^N) \to Gr_x^{tr}(L_0, n + 1, \mathbb{P}^N),
\]

where \( Gr_x^{tr}(S, L_0, n, \mathbb{P}^N) = Gr_x^{tr}(L_0, n, \mathbb{P}^N) \cap Gr(S, n, \mathbb{P}^N) \).

**Lemma 7.6.** Given any proper linear subspace \( L_0 \subsetneq \mathbb{P}^N \) and any element \( L \in Gr_x^{tr}(L_0, N - r, \mathbb{P}^N) \) intersecting \( X \) properly, the set \( Gr_x^{tr}(X, L_0, N - r - 1, L) =: \{M \in Gr_x^{tr}(L_0, N - r - 1, L)|M \cap X = \emptyset\} \) is a dense open of \( Gr(N - r - 1, L) \).
Proof. Since \( L \) intersects \( X \) properly in \( \mathbb{P}^N \) and \( \text{codim}_{\mathbb{P}^N}(L) = \dim X = r \), we see that \( D := X \cap L \) is either empty or a 0-dimensional closed subscheme of \( L \). In particular, \( \lvert D \rvert < \infty \). Since \( N > r \), the subscheme \( G(\lvert D \rvert) = \{ M \in Gr(N - r - 1, L) : M \cap \lvert D \rvert \neq \emptyset \} \) is a proper closed subset of \( Gr(N - r - 1, L) \). Hence, \( Gr(X, N - r - 1, L) = Gr(N - r - 1, L) \setminus G(\lvert D \rvert) \) is dense open in \( Gr(N - r - 1, L) \).

Since the intersection \( L \cap L_0 \) is transversal, \( Gr^{\text{tr}}(L_0, N - r - 1, L) \) is also dense open in \( Gr(N - r - 1, L) \), so, \( Gr^{\text{tr}}(X, L_0, N - r - 1, L) := Gr(X, N - r - 1, L) \cap Gr^{\text{tr}}(L_0, N - r - 1, L) \) is also dense open in \( Gr(N - r - 1, L) \), as desired. \( \square \)

**Definition 7.7.** Let \( x \in X_{\text{fs}} \) be a closed point and let \( S \subset X \setminus \{ x \} \) be a finite set of closed points. For \( r = 1 \), let \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \) be the set of linear subspaces \( M \) such that

1. \( M \cap X = \emptyset \)
2. \( C_x(M) \text{ is } (Z, x)-\text{admissible (see Definition 6.3).} \)

For \( r \geq 2 \) and \( L \in Gr(N - r + 1, \mathbb{P}^N) \), let \( U^r_{\text{adm}}(S, L, N - 2, \mathbb{P}^N) \) be the set of all \( (N - 2) \)-dimensional linear subspaces \( M \) such that

1. \( M \text{ intersects } L \text{ transversely,} \)
2. \( M \cap L \cap X = \emptyset \), and
3. \( C_x(M) \text{ is } (Z, x)-\text{admissible relative to } L \text{ (see Definition 6.8).} \)

**Definition 7.8.** Let \( H_0 \hookrightarrow \mathbb{P}^N \) be a hyperplane disjoint from \( S \setminus \{ x \} \). For \( r = 1 \), let \( U^r_{\text{adm}}(S, L, N - 2, \mathbb{P}^N) \) denote the intersection \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \cap Gr(N - 2, H_0) \). For \( r \geq 2 \), let \( U^r_{\text{adm}}(S, L, N - 2, \mathbb{P}^N) \) denote the intersection \( U^r_{\text{adm}}(S, L, N - 2, \mathbb{P}^N) \cap Gr(N - 2, H_0) \).

**Lemma 7.9.** Let \( x \in X_{\text{fs}} \) be a closed point and let \( S \subset X \setminus \{ x \} \) be a finite set of closed points.

(A) Suppose \( r = 1 \). Then, given a hyperplane \( H_0 \hookrightarrow \mathbb{P}^N \) disjoint from \( S \setminus \{ x \} \), the set \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \) is dense open in \( Gr(N - 2, H_0) \).

(B) Suppose \( r \geq 2 \). After replacing the given embedding \( \eta : X \hookrightarrow \mathbb{P}^N \) by its composition with a Veronese embedding, one has the following: given a hyperplane \( H_0 \hookrightarrow \mathbb{P}^N \) disjoint from \( S \setminus \{ x \} \) and a general \( L_0 \in Gr^{\text{tr}}(H_0, N - r + 1, \mathbb{P}^N) \), the set \( U^r_{\text{adm}}(S, L_0, N - 2, \mathbb{P}^N) \) is dense open in \( Gr(N - 2, H_0) \).

Proof. We often drop \( Z \) from the notations when we write \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \), \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \), \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \), and \( U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \).

Suppose \( r = 1 \). It follows from Proposition 7.4 that \( U^r_{\text{adm}}(S) \) is a dense open subset of \( Gr_x(N - 1, \mathbb{P}^N) \). We have seen above that the map \( \vartheta_x : Gr(x, N - 2, \mathbb{P}^N) \to Gr_x(N - 1, \mathbb{P}^N) \) is a vector bundle of rank \( N - 1 \). It is easily checked that \( Gr(N - 2, H_0) \hookrightarrow Gr(x, N - 2, \mathbb{P}^N) \) is a closed immersion and the restriction \( \vartheta_{x,H_0} : Gr(N - 2, H_0) \to Gr_x(N - 1, \mathbb{P}^N) \) is an isomorphism. It follows that \( \vartheta_{x,H_0}^{-1} U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \) is a dense open subset of \( Gr(N - 2, H_0) \). Since \( Gr(N - 2, H_0) \) is irreducible, we conclude from Lemma 7.6 that \( U^r_{\text{adm}}(H_0) = Gr(X, N - 2, H_0) \cap \vartheta_{x,H_0}^{-1} U^r_{\text{adm}}(S, N - 2, \mathbb{P}^N) \) is a dense open subset of \( Gr(N - 2, H_0) \). This proves (A).

Suppose now that \( r \geq 2 \). Let us choose a reimbedding \( \eta : X \hookrightarrow \mathbb{P}^N \), a general \( L_0 \in Gr^{\text{tr}}(H_0, N - r + 1, \mathbb{P}^N) \) and \( C = L \cap X \) as in Lemma 6.10.
It follows from Proposition 7.9 that $U_{\text{adm}}^{x,S,L_0}$ is a dense open subset of $Gr^{tr}_x(L_0, N-1, \mathbb{P}^N)$. We know the map $\vartheta_x : Gr^{tr}(x, L_0, N-2, \mathbb{P}^N) \to Gr^{tr}_x(L_0, N-1, \mathbb{P}^N)$ is smooth and surjective. Consider its restriction

$$\vartheta_{x,H_0} : Gr^{tr}(L_0, N-2, H_0) \to Gr^{tr}_x(L_0, N-1, \mathbb{P}^N).$$

One checks that this map is an inclusion whose image is the dense open subset $Gr^{tr}_x(L_0 \cap H_0, N-1, \mathbb{P}^N)$. On the other hand, $H_0 \cap \{x\} = \emptyset$ implies that $Gr^{tr}_x(L_0 \cap H_0, N-1, \mathbb{P}^N) = Gr^{tr}(L_0, N-1, \mathbb{P}^N)$. In particular, (7.11) is an isomorphism and we conclude that $\vartheta_{x,H_0}^{-1}(U_{\text{adm}}^{x,S,L_0})$ is dense open in $Gr^{tr}(L_0, N-2, H_0)$ and hence dense open in $Gr(N-2, H_0)$. Combining this with Lemma 5.6 we conclude that $U_{\text{adm}}^{x,S,L_0,N-2}(H_0) = Gr(X, N-2, H_0) \cap \vartheta_{x,H_0}^{-1}(U_{\text{adm}}^{x,S,L_0})$ is dense open in $Gr(N-2, H_0)$. This proves (B).

**Definition 7.10.** Let $S = \{x_1, \ldots, x_n\}$ be a set of distinct closed points of $X_{fs}$ and let $H_0 \hookrightarrow \mathbb{P}^N$ be a hyperplane disjoint from $S$. For $r = 1$, let $U_{\text{adm}}^{S,N-2}(Z, H_0)$ be the subset of $Gr(N-2, H_0)$ consisting of linear subspaces $L \in Gr(N-2, H_0)$ such that $L \in U_{\text{adm}}^{x, \{x\},N-2}(Z, H_0)$ for all $x \in S$.

For $r \geq 2$ and $L_0 \in Gr^{tr}(H_0, N-r+1, \mathbb{P}^N)$, let $U_{\text{adm}}^{S,L_0,N-2}(Z, H_0)$ be the subset of $Gr(N-2, H_0)$ consisting of linear subspaces $L \in Gr(N-2, H_0)$ such that $L \in U_{\text{adm}}^{x, \{x\},N-2}(Z, H_0)$ for all $x \in S$.

**Theorem 7.11.** Let $S = \{x_1, \ldots, x_n\}$ be a set of distinct closed points of $X_{fs}$.

(A) Suppose $r = 1$. Then, given a hyperplane $H_0 \hookrightarrow \mathbb{P}^N$ disjoint from $S$, the set $U_{\text{adm}}^{S,N-2}(Z, H_0)$ is dense open in $Gr(N-2, H_0)$.

(B) Suppose $r \geq 2$. After replacing the given embedding $\eta : X \hookrightarrow \mathbb{P}^N$ by its composition with a Veronese embedding, one has the following: given a hyperplane $H_0 \hookrightarrow \mathbb{P}^N$ disjoint from $S$ and a general $L_0 \in Gr^{tr}(H_0, N-r+1, \mathbb{P}^N)$, the set $U_{\text{adm}}^{S,L_0,N-2}(Z, H_0)$ is dense open in $Gr(N-2, H_0)$.

**Proof.** When $r = 1$, it follows directly from Lemma 7.9 that the set $U_{\text{adm}}^{S,N-2}(Z, H_0) = \bigcap_{i=1}^n U_{\text{adm}}^{x_i, \{x_i\},N-2}(Z, H_0)$ is a dense open subset of $Gr(N-2, H_0)$.

When $r \geq 2$, it follows from Lemma 7.9 that after replacing the embedding $\eta : X \hookrightarrow \mathbb{P}^N$ by its composition with a Veronese embedding, the following holds: for a general $L_0 \in Gr^{tr}(H_0, N-r+1, \mathbb{P}^N)$, $U_{\text{adm}}^{x_i, \{x_i\},N-2}(Z, H_0)$ is a dense open subset of $Gr(N-2, H_0)$ for each $1 \leq i \leq n$. In particular, $U_{\text{adm}}^{S,L_0,N-2}(Z, H_0) = \bigcap_{i=1}^n U_{\text{adm}}^{x_i, \{x_i\},L_0,N-2}(Z, H_0)$ is also dense open subset of $Gr(N-2, H_0)$.

8. The sfs-moving lemma III: Admissible linear projections

Now, we assume $k$ is any infinite perfect field and let $\overline{k}$ denote its algebraic closure. For any $X \in \text{Sch}_{k}^{\text{ss}}$, let $X_{\overline{k}} = X \times \text{Spec}(k) \text{ Spec}(\overline{k})$ and let $\pi_X : X_{\overline{k}} \to X$ denote the projection map.

Let $X$ be an irreducible quasi-projective scheme of dimension $r \geq 1$ with a smooth dense open subset $X_{fs}$. Let $x \in X_{fs}$ be a closed point. Let $B$ be a geometrically integral smooth affine $k$-scheme of positive dimension, and let $\widehat{B}$ be a geometrically integral smooth compactification of $B$. Given a closed subscheme $Z \subset X \times \widehat{B}$, let $X \overset{\varphi}{\leftarrow} Z \overset{\vartheta}{\rightarrow} \widehat{B}$ be the projection maps.
Definition 8.1. Let $Z \subset X \times \hat{B}$ be an irreducible fs-cycle in the sense of Definition 6.1. A finite set $D_x = \{x = x_0, x_1, \cdots, x_d\}$ of distinct closed points on $X$ is called $(Z, x)$-admissible, if the following hold:

1. $D_x \subsetneq X_{fs}$.  
2. $D_x$ contains the chosen closed point $x$.  
3. $Z$ is regular on the points of $(D_x \setminus \{x\}) \times \hat{B}$.  
4. Either each $Z = X \times \{b\}$ for some closed point $b \in \hat{B}$, or $g(Z_x) \cap g(Z_y) = \emptyset$ for $0 \leq i \neq j \leq d$.

For an fs-cycle $Z$ on $X \times \hat{B}$ with irreducible components $\{Z_1, \cdots, Z_s\}$, we say that $D_x$ is $(Z, x)$-admissible if it is $(Z_i, x)$-admissible for each $1 \leq i \leq s$.

The following result is an immediate consequence of this definition and smoothness of $X_{fs}$.

Lemma 8.2. Let $Z$ be an fs-cycle in the sense of Definition 6.1. Write $Z = Z_1 + Z_2$, where each component of $Z_1$ is rectifiable, and each component of $Z_2$ is not. Then, a finite set $D_x = \{x = x_0, x_1, \cdots, x_d\}$ of distinct closed points on $X$ is $(Z, x)$-admissible if and only if it is $(Z_1, x)$-admissible.

8.1. Global nature of admissible linear projections. Let $X$ be a projective $k$-scheme of dimension $r$. In this section, for an embedding into a big enough projective space $\eta : X \hookrightarrow \mathbb{P}^N$, we prove some results about the maps $\phi : X \to \mathbb{P}_k^r$ obtained from the linear projections away from the admissible $(N - r - 1)$-dimensional linear subspaces in $\mathbb{P}_k^N$.

Proposition 8.3. Let $X$ be a projective $k$-scheme and let $Z$ be an fs-cycle as in Definition 6.1 on $X \times \hat{B}$, with irreducible components $\{Z_1, \cdots, Z_s\}$.

Let $\Sigma = \{x_1, \cdots, x_n\}$ be a finite set of distinct closed points of $X_{fs}$ and let $Y \subset X$ be a closed subscheme of dimension at most $r - 1$. Then, there is a closed embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ such that for a given hyperplane $H_0 \hookrightarrow \mathbb{P}_k^N$ disjoint from $\Sigma$ and a general linear subspace $L \in Gr(N - r - 1, H_0)$, the linear projection $\phi : \Sigma \to \mathbb{P}_k^r \setminus L \to \mathbb{P}_k^r$ restricts to a finite map $\phi : X \to \mathbb{P}_k^r$ with the following additional properties:

1. $\phi$ is étale at $\phi^{-1}(\phi(\Sigma))$.  
2. $\phi(x_i) \neq \phi(x_j)$ for $1 \leq i \neq j \leq n$.  
3. $k(\phi(x)) \cong k(x)$ for all $x \in \phi^{-1}(\phi(\Sigma))$.  
4. $\phi^{-1}(\phi(\Sigma))$ is $(Z, x_i)$-admissible for all $1 \leq i \leq n$.  
5. $L^+(\Sigma) \cap Y = \emptyset$.

Proof. Using Lemma 8.2, we can assume that $Z$ is rectifiable. For $1 \leq i \leq n$, let $S_i = (\pi_X)^{-1}(x_i) = \{x_{i}^1, \cdots, x_{i}^t\}$ and set $S = \bigcup_{i=1}^n S_i \subsetneq (X_{fs})^\infty$. In particular, $S$ is a set of smooth closed points of $X^\infty$.

We choose a closed embedding $\eta : X \hookrightarrow \mathbb{P}_k^N$ such that the inclusion $X^\infty \hookrightarrow \mathbb{P}_k^N$ satisfies the assertions of Theorem 7.11 with $S \subsetneq X^\infty$ as above so that $S \cap (H_0)^\infty = \emptyset$. Notice that $\eta$ is obtained by replacing $\mathbb{P}_k^N$ by one of its Veronese embeddings of large degrees. For a finite set $S \subsetneq \mathbb{P}_k^N$ and a closed subscheme $W \subsetneq \mathbb{P}_k^N$, let $C_S(W)$ denote the set of secant lines joining the points of $S$ and $W$. One knows that $\dim(C_S(W)) = \dim(W) + 1$. 
If $r = 1$, we take any $L \in \mathcal{U}_{\text{adm}}^{S,N-2}(Z_{\mathcal{T}}, (H_0)^{\mathcal{T}}) \cap Gr(C_S(Y_{\mathcal{T}}), N - 2, (H_0)^{\mathcal{T}})$. It follows from Lemma 5.9 that $Gr(C_S(Y_{\mathcal{T}}), N - 2, (H_0)^{\mathcal{T}})$ is a dense open subset of $Gr(N - 2, (H_0)^{\mathcal{T}})$.

If $r \geq 2$, we choose a general $L_0 \in Gr^{tr}((H_0)^{\mathcal{T}}, N - r + 1, \mathbb{P}_{\mathcal{T}}^N)$ such that $L_0 \cap X_{\mathcal{T}}$ is a curve $C$, none of whose component is contained in $Y_{\mathcal{T}}$. Let $Gr^{tr}(C_S(Y_{\mathcal{T}} \cap C), L_0, N - 2, (H_0)^{\mathcal{T}})$ be the set of linear subspaces in $Gr(N - 2, (H_0)^{\mathcal{T}})$ which intersect $L_0$ transversely and do not intersect $C_S(Y_{\mathcal{T}} \cap C)$. It follows from Lemma 5.9 that $Gr^{tr}(C_S(Y_{\mathcal{T}} \cap C), L_0, N - 2, (H_0)^{\mathcal{T}})$ is a dense open subset of $Gr(N - 2, (H_0)^{\mathcal{T}})$. We take $L = M \cap L_0$ for any $M \in \mathcal{U}_{\text{adm}}^{S,Lo,N-2}(Z_{\mathcal{T}}, (H_0)^{\mathcal{T}}) \cap Gr^{tr}(C_S(Y_{\mathcal{T}} \cap C), L, N - 2, (H_0)^{\mathcal{T}})$.

As shown in the proof of Lemma 5.9 we can find dense subsets of $\mathcal{U}_{\text{adm}}^{S,N-2}(Z_{\mathcal{T}}, (H_0)^{\mathcal{T}}) \cap Gr(C_S(Y_{\mathcal{T}}), N - 2, (H_0)^{\mathcal{T}})$, $Gr(N - r + 1, \mathbb{P}_{\mathcal{T}}^N)$ and $\mathcal{U}_{\text{adm}}^{S,Lo,N-2}(Z_{\mathcal{T}}, (H_0)^{\mathcal{T}}) \cap Gr^{tr}(C_S(Y_{\mathcal{T}} \cap C), L_0, N - 2, (H_0)^{\mathcal{T}})$ each of whose points $L$, $L_0$ are $M$, respectively, are all defined over $k$. Since $X \cap L = \emptyset$, we get a finite map $\phi = \phi_L : X \to \mathbb{P}_k$ over $k$. We now show that $\phi$ satisfies the properties (1)~(5) of the proposition.

It follows from the admissibility condition for $C_x(L)$ for all $x \in S$ (see Definition 7.7) that $\phi_{\mathcal{T}}^{-1}(\phi_{\mathcal{T}}(S)) \subset (X_S)_{\mathcal{T}}$ and equivalently, $\phi^{-1}(\phi(D)) \subset X_k$. It follows from the admissibility of $C_x(L)$ and Lemma 8.3 below that $\phi_{\mathcal{T}}$ is étale at all points of $\phi_{\mathcal{T}}^{-1}(\phi_{\mathcal{T}}(S))$. But this is equivalent (by descent) to saying that $\phi$ is étale at all points of $\phi^{-1}(\phi(S))$. This proves (1).

Since the map $\phi_{\mathcal{T}}$ is injective on $S$, the properties (2) and (3) follow directly from the ‘Claim’ in the proof of Lemma 5.9.

For $1 \leq i \leq n$, set $\Sigma_i = \phi^{-1}(\phi(x_i))$. We have already shown that $\Sigma_i \subset X_k$. It follows from the admissibility condition that $Z_{\mathcal{T}}$ is smooth at all points of $(\pi_{\mathcal{T}})^{-1}(\Sigma_i \setminus \{x_i\}) \times \hat{B}_{\mathcal{T}}$. We conclude from the faithfully flat descent of smoothness that $Z$ is smooth at all points of $(\Sigma_i \setminus \{x_i\}) \times \hat{B}$. The third and the fourth conditions of 8.1 follow directly from the Definition 7.10 of $\mathcal{U}_{\text{adm}}^{S,N-2}$ and $\mathcal{U}_{\text{adm}}^{S,Lo,N-2}$. This shows that each $\Sigma_i$ is $(Z,x_i)$-admissible, proving (4). The property (5) follows at once from our choice of $L$, as shown in the proof of the last property of Lemma 5.9. \hfill \Box

In the middle of the proof of above Proposition 8.3, we used the following:

**Lemma 8.4.** Let $X \hookrightarrow \mathbb{P}^N_k$ be a closed embedding of a projective scheme of dimension $r \geq 1$. Let $L \subset \mathbb{P}^N_k$ be a linear subspace of dimension $N - r - 1$ such that $X \cap L = \emptyset$. Let $P \in \mathbb{P}^r_k$ be a closed point such that $C_P(L) \cap X_{\text{sing}} = \emptyset$. Then, the map $\phi_L : X \to \mathbb{P}^r_k$ obtained by the linear projection away from $L$, is finite and étale over an affine neighborhood of $P$ in $\mathbb{P}^r_k$ if and only if $C_P(L)$ intersects $X$ transversely in $\mathbb{P}^N_k$.

**Proof.** Suppose that $C_P(L)$ intersects $X$ transversely in $\mathbb{P}^N_k$ and let $E$ be this scheme-theoretic intersection, with $\text{Supp}(E) = \{x_0, \ldots, x_e\}$. Since $k$ is perfect, the transversal intersection is equivalent to saying that $E$ is smooth (but disconnected). However, as $L \subset C_P(L)$ and $X \cap L = \emptyset$, we see that $C_P(L) \cap X = (C_P(L) \setminus L) \cap X$ as schemes. But it is easy to observe that the latter is same as the scheme-theoretic
fiber $\phi^*_L(P)$. In other words, the diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\phi^*_L} & \phi^{-1}_L(U) \\
\downarrow_{\phi^*_L} & & \downarrow_{\phi_L} \\
\text{Spec } (k(P)) & \longrightarrow & U
\end{array}
\]
is Cartesian such that $\phi^*_L$ is smooth, where $U \subseteq \mathbb{P}^r_k$ is an affine neighborhood of $P$. Since $\phi_L$ is finite map of affine schemes over $k$, it follows from \cite[Theorem 24.3]{33} that it is flat over an affine neighborhood of $P$. We can now apply \cite[Ex. III.10.2]{15} to conclude that there is an affine neighborhood of $P$ in $U$ over which the map $\phi_L$ is smooth, and hence finite and étale.

The converse is easy to see, because smoothness of $\phi_L$ over a neighborhood of $P$ implies that the map $\phi^*_L$ in (8.1) is étale. Since $k \hookrightarrow k(P)$ is smooth, this precisely means that $E = C_P(L) \cap X$ is smooth, which in turn is equivalent to saying that this intersection is transverse. \hfill $\square$

8.2. Local nature of admissible linear projections. Let $k$ be an infinite perfect field. Let $X$ be a (connected) smooth affine $k$-scheme of dimension $r \geq 1$ and $\Sigma \subset X$ be a finite set of closed points. Let $X \hookrightarrow \mathbb{A}^m_k$ be a closed embedding and let $\overline{X}$ denote the closure of $X$ under the inclusion $\mathbb{A}^m_k \hookrightarrow \mathbb{P}^m_k$. Let $H_{m,0}$ denote the hyperplane $\mathbb{P}^m_k \setminus \mathbb{A}^m_k$. Let $B$ be a geometrically integral smooth affine $k$-scheme of positive dimension, and let $\mathring{B}$ be a geometrically integral smooth compactification of $B$. Let $Z$ be an fs-cycle on $X \times \mathring{B}$ in the sense of Definition 6.1 with irreducible components $\{Z_1, \ldots, Z_s\}$. Let $Z_i$ and $Z$ be the closures of $Z_i$ and $Z$ in $\overline{X} \times \mathring{B}$. Notice that $Z$ is an fs-cycle, too. It is a rectifiable cycle if $Z$ is so. Let $Y \subset \overline{X}$ be a closed subscheme of dimension at most $r - 1$.

Lemma 8.5. Given a cycle $Z \subset X \times \mathring{B}$ as above, there exists a closed embedding $\overline{X} \hookrightarrow \mathbb{P}^N_k$ and $L \in Gr(\overline{X}, N - r - 1, \mathbb{P}^N_k)$ such that the following hold:

1. The linear projection away from $L$ defines a Cartesian square
\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \mathbb{A}^r_k \\
\downarrow & & \downarrow \\
\mathbb{P}^r_k & \xrightarrow{\phi} & \mathbb{P}^r_k
\end{array}
\]
such that the vertical maps are finite and the horizontal maps are open immersions with $\phi(\Sigma) \subset \mathbb{A}^r_k$.
2. The map $\phi : X \to \mathbb{A}^r_k$ is étale over an affine neighborhood $U$ of $\phi(\Sigma)$.
3. $\phi(x) \neq \phi(x')$ for $x \neq x' \in \Sigma$.
4. $k(\phi(x)) \xrightarrow{\mathring{L}} k(x)$ for all $x \in \phi^{-1}(\phi(\Sigma))$.
5. $L^+(\Sigma) \cap Y = \emptyset$.
6. $\phi^{-1}(\phi(\Sigma))$ is $(Z, x)$-admissible as in Definition 8.1 for all $x \in \Sigma$. In particular, $\phi^{-1}(\phi(\Sigma)) \subset X$.

Proof. We apply Proposition 8.3 with $\overline{X}_{fs} = X$ and $H_0 = H_{N,0}$ (see (5.2)) to get a closed embedding $\eta : X \hookrightarrow \mathbb{P}^N_k$ and a linear subspace $L \in Gr(\overline{X}, N - r - 1, H_{N,0})$ such that the linear projection $\phi_L : \mathbb{P}^N_k \setminus L \to \mathbb{P}^r_k$ restricts to a finite map $\phi : \overline{X} \to \mathbb{P}^r_k$.
satisfying properties (3)-(6) of the lemma. It follows from Lemma 5.5 that the property (1) holds and \( \phi(\Sigma) \not\subseteq A_k^r \).

Since \( \phi \) is flat along \( \phi(\Sigma) \), it follows from [33, Theorem 24.3] that there is an affine open neighborhood \( U' \subseteq A_k^r \) of \( \phi(\Sigma) \) such that the map \( \phi^{-1}(U') \to U' \) is finite and flat. We can now apply [15, Ex. III.10.2] to conclude that there is an affine open neighborhood \( U \subset A_k^r \) of \( \phi(\Sigma) \) such that the map \( \phi^{-1}(U) \to U \) is finite and étale. This proves property (2) of the lemma. \( \square \)

We now prove the following improvement of Lemma 8.5 when \( \widehat{B} \) has a more specific form. Let \( n \geq 1 \) be an integer. Let \( A_0, A_1, \ldots, A_{n-1} \) be smooth affine geometrically integral schemes of positive dimension, and let \( \widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_{n-1} \) be smooth projective geometrically integral schemes over \( k \) such that \( A_j \not\subseteq \widehat{A}_j \) is open for \( 0 \leq j \leq n - 1 \). For \( 1 \leq j \leq n \), we set

\[
C_j := \prod_{i=0}^{j-1} A_i, \quad \widehat{C}_j := \prod_{i=0}^{j-1} \widehat{A}_i, \quad B := C_n, \quad \widehat{B} = \widehat{C}_n, \quad \pi_j : \widehat{B} \to \widehat{C}_j,
\]

where \( \pi_j \) is the projection.

Let \( X \) be a smooth affine \( k \)-scheme of dimension \( r \geq 1 \) and let \( \Sigma \subseteq X \) be a finite subset of closed points. Let \( Z \) be an fs-cycle on \( X \times \widehat{B} \) as in Definition 6.1 with irreducible components \( \{Z_1, \ldots, Z_s\} \). For \( 1 \leq j \leq n \), set \( Z^{(j)} = \pi_j(Z) \). Then, each \( Z^{(j)} \) is an fs-cycle on \( X \times \widehat{C}_j \). Suppose for each \( 1 \leq i \leq s \), \( Z_i^{(j)} \) is singular for some \( 1 \leq j \leq n \). Let \( Y \subseteq X \) be a closed subscheme of dimension at most \( r - 1 \).

**Proposition 8.6.** Under the above notations, there exists a closed embedding \( X \hookrightarrow \mathbb{P}_k^n \) and \( L \in Gr(X, N - r - 1, \mathbb{P}_k^N) \) such that the following hold:

1. The linear projection away from \( L \) defines a Cartesian square

\[
\begin{array}{ccc}
X & \hookrightarrow & \overline{X} \\
\phi \downarrow & & \phi \\
A_k^r & \hookrightarrow & \mathbb{P}_k^r
\end{array}
\]

such that the vertical maps are finite and the horizontal maps are open immersions with \( \phi(\Sigma) \not\subseteq A_k^r \).

2. The map \( \phi : X \to A_k^r \) is étale over an affine neighborhood \( U \) of \( \phi(\Sigma) \).
3. \( \phi(x) \neq \phi(x') \) for \( x \neq x' \in \Sigma \).
4. \( k(\phi(x)) \cong k(x) \) for all \( x \in \phi^{-1}(\phi(\Sigma)) \).
5. \( L^+(\Sigma) \cap Y = \emptyset \).
6. \( \phi^{-1}(\phi(x)) \) is \( (Z^{(j)}, x) \)-admissible in the sense of Definition 8.1 for all \( x \in \Sigma \) and all \( 0 \leq j \leq n - 1 \). In particular, \( \phi^{-1}(\phi(\Sigma)) \not\subseteq X \).

**Proof.** It follows from Proposition 8.3 that there is an embedding \( \eta : X \hookrightarrow \overline{X} \hookrightarrow \mathbb{P}_k^n \) such that for a general linear subspace \( L \in Gr(\overline{X}, N - r - 1, H_{N,0}) \) (with \( H_{N,0} \) as in Lemma 8.5), the linear projection \( \phi_L : \mathbb{P}_k^n \setminus L \to \mathbb{P}_k^r \) restricts to a finite map \( \phi : \overline{X} \to \mathbb{P}_k^r \) satisfying properties (3)-(6) of the Proposition for each \( \overline{Z}^{(j)} \). If we let \( U^{(j)} \) denote the open subset of \( Gr(N - r - 1, H_{N,0}) \) satisfying this property and set \( U = \bigcap_{j=1}^n U^{(j)} \), we see that for a general member \( L \) of \( Gr(N - r - 1, H_{N,0}) \), the
associated linear projection $\phi_L$ satisfies properties (3)-(6) of the Proposition. Now one can repeat the argument of the proof of Lemma 8.5 to complete the proof. □

9. The sfs-moving lemma IV: the main results

Now, we again suppose $k$ is any infinite perfect field. In this section, we complete the proof of Theorem 4.13, i.e., the proof of the sfs-moving lemma for additive higher Chow cycles in the Milnor range on a smooth semi-local scheme of geometric type. The interested reader can apply the machines of the paper to the Milnor range of higher Chow cycles, as well.

9.1. Smoothness for rectifiable cycles. Let $X$ be a connected smooth affine $k$-scheme. Let $\Sigma$ be a finite set of closed points of $X$. Let $X \hookrightarrow A^m_k$ be a closed embedding. Let $X$ denote the closure of $X$ under the inclusion $A^m_k \hookrightarrow \mathbb{P}^m_k$ and let $H_{n,0} = \mathbb{P}^m_k \setminus A^m_k$. We now apply Proposition 8.6 with $C_j = A^1_k \times \Box^{j-1}$ and $\tilde{C}_j = \tilde{B}_j = \mathbb{P}^1_k \times \Box^{j-1}$ with $1 \leq j \leq n - 1$ and $n \geq 1$.

Let $Z$ be an fs-cycle on $X \times \tilde{B}_n$ with irreducible components $\{Z_1, \ldots, Z_s\}$. Let $\overline{Z}_i$ and $\overline{Z}$ denote the closures of $Z_i$ and $Z$ in $\overline{X} \times \tilde{B}_n$. Note that $\overline{Z}$ is an fs-cycle on $\overline{X} \times \tilde{B}_n$. Furthermore, it is a rectifiable cycle if $Z$ is so.

**Proposition 9.1.** Under the above notations, suppose that $Z$ is rectifiable. Then, there exists a closed embedding $\overline{X} \hookrightarrow \mathbb{P}^N_k$ and $L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}^N_k)$ such that the following hold:

1. The linear projection away from $L$ defines a Cartesian square

\[
\begin{array}{ccc}
X & \overset{\phi}{\hookrightarrow} & \overline{X} \\
\downarrow & & \downarrow \phi \\
A^r_k & \overset{\phi}{\hookrightarrow} & \mathbb{P}^r_k
\end{array}
\]

such that the vertical maps are finite and the horizontal maps are open immersions with $\phi(\Sigma) \subsetneq A^r_k$.

2. The map $\phi : X \rightarrow A^r_k$ is étale over an affine neighborhood $U$ of $\phi(\Sigma)$.
3. $\phi^{-1}(\phi(\Sigma)) \subsetneq X$.
4. Given any $1 \leq i \leq s$ and irreducible component $Z'$ of $L^+(Z_i)$, the scheme $\pi_j(Z')$ is an fs-cycle which is also smooth at all points over $\Sigma \times \tilde{B}_j$ for all $1 \leq j \leq n$.

**Proof.** For given $1 \leq i \leq s$, we fix closed points $x_i \in X$, $b_i \in \tilde{B}_n$ such that $\alpha_i = (x_i, b_i) \in Z_i$. Since each $Z_i \neq \emptyset$, such closed points always exist. Set $D_0 = \Sigma \cup \{x_1, \ldots, x_s\}$ and $E^0 = \{b_1, \ldots, b_s\}$. Let $Y$ denote the closure of $f(Z \cap (X \times E^0))$ in $\overline{X}$. Since $Z$ is rectifiable, we see that $Z \subsetneq (X \times E^0)$. Since $f$ is projective, $f(Z \cap (X \times E^0))$ is a closed subset of $X$ dimension at most $r - 1$. We conclude that $Y \subsetneq \overline{X}$ is a closed subset of dimension at most $r - 1$.

Let $\phi : \overline{X} \rightarrow \mathbb{P}^r_k$ be the finite map satisfying properties (1)-(6) of Proposition 8.6 with $D_0 \subsetneq X$ and $Y \subsetneq \overline{X}$. The only thing we need to show is (4). We set $\hat{\phi} = \phi_{\tilde{B}_n}$ and $\hat{\phi}_j = \phi_{\tilde{B}_j}$ for $1 \leq j \leq n$.  


For each \(1 \leq i \leq s\), recall that \(L^+([Z_i])\) is the cycle \(\hat{\phi}^* \circ \hat{\phi}_*([Z]) - [Z]\) and \(L^+(Z_i)\) is the support of \(L^+([Z_i])\). Under the assumptions of the proposition, we have already shown in the proof of Proposition \(5.10\) that \(Z_i\) is not a component of \(L^+(Z_i)\). This uses properties (2), (4) and (5) of Proposition \(5.6\) and our choice of \(\alpha_i \in Z_i\) and \(Y\).

To prove the remaining part of (4), it is enough to prove it for \(\overline{Z}_i\). Since we need to prove (4) for \(L^+(\overline{Z}_i)\) for all \(i\) and at all points of \(\Sigma \times \hat{B}_n\), we concentrate on \(L^+(\overline{Z}_i)\) for a fixed \(1 \leq i \leq s\) and a fixed point \(x \in \Sigma\). Set \(y = \phi(x)\) and \(\overline{Z} = \overline{Z}_i\). We first prove the following.

**Claim 1:** \(\pi_j(L^+(\overline{Z})) = L^+(\overline{Z}^{(j)})\) for each \(1 \leq j \leq n\).

\((.:)\) It is clear that \(\pi_j(L^+(\overline{Z})) \subseteq L^+(\overline{Z}^{(j)})\). To prove the reverse inclusion, let \((x, b_0, \ldots, b_{j-1}) \in L^+(\overline{Z}^{(j)})\). This means that there exists \(z' = (x', b'_0, \ldots, b'_{n-1}) \in \overline{Z}\) such that

\[
(\phi(x), b_0, \ldots, b_{j-1}) = \phi \circ \pi_j((x', b'_0, \ldots, b'_{n-1})) = (\phi(x'), b'_0, \ldots, b'_{j-1}).
\]

But this implies that \(\phi(x) = \phi(x')\) and \(b_l = b'_l\) for \(0 \leq l \leq j - 1\). Setting \(z = (x, b_0, \ldots, b_{n-1})\), we see that \(\hat{\phi}(z) = \hat{\phi}(z')\) and \(\pi_j(z) = (x, b_0, \ldots, b_{j-1})\). This proves the claim.

For \(1 \leq j \leq n\), let \(T^{(j)} = \hat{\phi}_j(\overline{Z}(j))\) and let \(\overline{Z}(j)\) denote the scheme-theoretic inverse image of \(T^{(j)}\). Set \(T = T_1^{(n)}\) and \(\overline{Z} = \overline{Z}^{(n)}\). It follows from Claim 1 that \(\overline{Z}(j) = \pi_j((\overline{\phi}(n)^{-1}(T)) = \pi_j(\overline{Z})\).

We first show that \(L^+(\overline{Z})\) is an fs-cycle. Let \(Z'\) be a component of \(L^+(\overline{Z})\). Since \(Z' \to \overline{X}\) is a projective morphism of schemes of the same dimension, it is enough to show that this map is quasi-finite. Suppose on the contrary that there is a closed point \(x \in \overline{X}\) such that \(\dim(Z'_x) \geq 1\) and let \(E = p_{\overline{B}_n}(Z'_x)\). Since \(Z'\) is a part of the residual cycle of \(\overline{Z}\), we must have \(\dim(\overline{Z} \cap (D_x \times E)) \geq 1\), where \(D_x = \phi^{-1}(\phi(x))\). But this contradicts the fact that \(\overline{Z}\) is an fs-cycle and \(\phi\) is finite. We have thus shown that \(L^+(\overline{Z})\) is an fs-cycle.

We now show the smoothness of the components of \(L^+(\overline{Z})\) along \(\Sigma \times \hat{B}_n\). We have seen before that \(\overline{Z} \not\subseteq \overline{X} \times \{b\}\) for any \(b \in \hat{B}_n\). Set \(\hat{B}_0 = \text{Spec}(k), \overline{Z}' = \overline{X}\) and let \(m \in \{0, 1, \ldots, n - 1\}\) be the largest integer such that \(\overline{Z}(m) = \overline{X} \times \{b\}\) for some \(b \in \hat{B}_m\). It follows from Claim 1 that \((L^+(\overline{Z}))^{(m)} = \overline{X} \times \{b\}\). In particular, for every component \(Z'\) of \(L^+(\overline{Z})\), the scheme \(\pi_j(Z')\) is an fs-cycle which is smooth along \(X \times \hat{B}_j\) for \(1 \leq j \leq m\).

To show the smoothness of \(\pi_j(Z')\) along \(\{x\} \times \hat{B}_j\) for \(m + 1 \leq j \leq n\), it is equivalent to show using Claim 1, that the components of \(L^+(\overline{Z}^{(j)})\) are smooth along \(\{x\} \times \hat{B}_j\) for \(m + 1 \leq j \leq n\). So we fix \(j \in \{m + 1, \ldots, n\}\), a component \(Z'\) of \(L^+(\overline{Z}^{(j)})\). It is enough to show the smoothness of \(Z'\) along \(\{x\} \times \hat{B}_j\).

We now fix \(m + 1 \leq j \leq n\). Set \(D_x = \phi^{-1}(y)\), fix a point \(a = (a, b) \in D_x \times \hat{B}_j\) and set \(\beta = \hat{\phi}_j(a) = (y, b)\). We write \(D^\beta_x = D_x \times \{b\} = (\hat{\phi}_j)^{-1}(\beta)\). Set \(W^j_1 = U \times \hat{B}_j\) and \(W^j_2 = \phi^{-1}(U) \times \hat{B}_j\). Notice that \(W^j_2\) is smooth by property (2). We next prove the following.
Claim 2: If \( \alpha \in \tilde{Z}^{(j)} \), then \( \tilde{Z}^{(j)} \) is the only irreducible component of \( \tilde{Z}^{(j)} \) which passes through \( \alpha \).

\( \vdash \) Since \( \tilde{\phi}_j \) is finite and étale over \( U \times \tilde{B}_j \), we see that the map \( \mathcal{O}_{W_1, \beta} \to \mathcal{O}_{W_2, \beta} \) is finite and étale. In particular, the map \( \mathcal{O}_{T^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \beta} \) is finite and étale. This in turn implies that the map \( \mathcal{O}_{T^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \beta} \) is finite and unramified.

Since \( j > m \), we see that \( \tilde{Z}^{(j)} \not\subseteq X \times \{ b' \} \) for any \( b' \in \tilde{B}_j \). It follows from the second part of the condition (4) in Definition (8.1) and property (6) in Proposition 8.6 that the map \( \mathcal{O}_{\tilde{Z}^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) is an isomorphism. Combining this with property (4) of Proposition 8.6, we conclude that the map \( \mathcal{O}_{T^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) is an injective (since \( \tilde{Z}^{(j)} \to T^{(j)} \)), finite and unramified map of local rings of closed points of affine integral domains over \( k \) which induces isomorphism of the residue fields. It follows from Lemma 5.2 that the map \( \mathcal{O}_{T^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) is an isomorphism.

On the other hand, the map \( \mathcal{O}_{T^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \beta} \) being finite and étale, shows that the map \( \mathcal{O}_{T^{(j)}, \beta} \to \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) is étale. In particular, the map of completions \( \tilde{\mathcal{O}}_{T^{(j)}, \beta} \to \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, \alpha} \) is finite and étale. Since it induces isomorphism at the level of residue fields, it follows again from Lemma 5.2 that the map \( \tilde{\mathcal{O}}_{T^{(j)}, \beta} \to \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, \alpha} \) is an isomorphism.

We conclude that there are local homomorphisms of complete local rings

\[
\tilde{\mathcal{O}}_{T^{(j)}, \beta} \to \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, \alpha} \to \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, \alpha},
\]

where the first map and the composite map are both isomorphisms. Thus, the second map is an isomorphism too. Since \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \to \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \), the Krull intersection theorem (333 Theorem 8.10) says that the second map in (9.2) is an isomorphism if and only if the map \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \to \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) is an isomorphism. This is in turn equivalent to saying that \( \tilde{Z}^{(j)} \) is the only irreducible component of \( \tilde{Z}^{(j)} \) which passes through \( \alpha \). The Claim 2 is now proven.

We now complete the proof of smoothness of \( Z' \) along \( \{ x \} \times \tilde{B}_j \). Let \( \alpha' = (x, b) \in Z' \) for some \( b \in \tilde{B}_j \). Set \( \beta = (y, b) \in U \times \tilde{B}_j \).

We have shown in the course of proving Claim 2 (see (9.2)) that the map \( \tilde{\mathcal{O}}_{T^{(j)}, (y, b)} \to \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, (a, b)} \) of completions is an isomorphism for every \( a \in D_x \). Since \( \alpha' \in Z' \not\subseteq \tilde{Z}^{(j)} \), there must exist a point \( \alpha = (a, b) \in \tilde{Z}^{(j)} \) for some \( a \in D_x \). Since \( Z' \) is a residual component, Claim 2 implies that we must have \( a \neq x \). In this case, Claim 2 again shows that \( \tilde{Z}^{(j)} \) is the only irreducible component of \( \tilde{Z}^{(j)} \) which passes through \( \alpha \). In particular, \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \cong \mathcal{O}_{\tilde{Z}^{(j)}, \alpha'} \).

Since \( a \neq x \), it follows from the condition (3) in Definition (8.1) and property (6) in Proposition 8.6 that \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) is smooth and hence so is \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha'} \). Using the isomorphisms \( \tilde{\mathcal{O}}_{T^{(j)}, \beta} \cong \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, \alpha} \) and \( \tilde{\mathcal{O}}_{T^{(j)}, \beta} \cong \tilde{\mathcal{O}}_{\tilde{Z}^{(j)}, \alpha'} \), we conclude that \( \mathcal{O}_{T^{(j)}, \beta} \) and \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha} \) are both smooth. Since \( \alpha' \in Z' \), the local ring \( \mathcal{O}_{\tilde{Z}^{(j)}, \alpha'} \) will be smooth only if \( Z' \) is the only component of \( \tilde{Z}^{(j)} \) passing through \( \alpha' = (x, b) \) and is smooth at this point. The proof of the proposition is now complete. \( \square \)
Remark 9.2. We have in fact shown that if a point \( \alpha = (x, b) \) lies in any irreducible component \( Z' \) of \( L^+(\overline{Z}) \), then \( Z' \) is the only component passing through \( \alpha \). Moreover, \( Z' \) is smooth at \( \alpha \) if \( Z' \neq \overline{Z}_i \) for all \( 1 \leq i \leq s \). The same holds when \( Z \) is replaced by \( Z^{(j)} \).

Remark 9.3. The reader can observe that Claim 2 in the proof of Proposition 9.1 can be deduced from Lemma 5.3. But we had to give another proof because we needed a stronger assertion than Lemma 5.3 in order to prove smoothness of residual components.

Corollary 9.4. Let \( Z \) be an fs-cycle on \( X \times \hat{B}_n \) as in Proposition 9.1. Then, there exists a closed embedding \( \overline{X} \hookrightarrow \mathbb{P}_k^N \) and \( L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N) \) such that the following hold:

1. The linear projection away from \( L \) defines a Cartesian square

\[
\begin{array}{ccc}
X & \longrightarrow & \overline{X} \\
\phi \downarrow & & \phi \\
\mathbb{A}_k^r & \hookrightarrow & \mathbb{P}_k^r
\end{array}
\]

such that the vertical maps are finite and the horizontal maps are open immersions with \( \phi(\Sigma) \subsetneq \mathbb{A}_k^r \).

2. The map \( \phi : X \to \mathbb{A}_k^r \) is étale over an affine neighborhood \( U \) of \( \phi(\Sigma) \).

3. \( \phi^{-1}(\phi(\Sigma)) \subset X \).

4. For every \( 1 \leq i \leq s \), one of the following holds:
   (A) \( Z_i \) is smooth and \( Z_i = \phi^{-1}(\phi(Z_i))_{\text{red}} \).
   (B) Given any irreducible component \( Z' \) of \( L^+(Z_i) \), and any \( 1 \leq j \leq n \), the scheme \( \pi_j(Z') \) is an fs-cycle which is also smooth at all points over \( \Sigma \times \hat{B}_j \).

Proof. We write \( Z = W_1 + W_2 \), where \( W_1 \) is rectifiable and \( W_2 \) is not rectifiable. We now apply Proposition 9.1 to get a closed embedding \( \overline{X} \hookrightarrow \mathbb{P}_k^N \) and \( L \in \text{Gr}(\overline{X}, N - r - 1, \mathbb{P}_k^N) \) such that (1)–(3) and (4)(B) are satisfied for \( W_1 \).

Suppose now that \( Z_i \) is a component of \( Z \) which is not rectifiable. This means that \( Z_i \subseteq X \times \{b_i\} \) for some closed point \( b_i \in \hat{B}_n \). Since \( Z_i \) is an irreducible scheme of dimension \( r \) which is finite and surjective over \( X \), we must have \( Z_i = X \times \{b_i\} \). In particular, \( Z_i \) is smooth and \( \overline{Z}_i = \overline{X} \times \{b_i\} \). But this means that \( \phi(\overline{Z}_i) = \mathbb{P}_k^r \times \{b_i\} \) and hence \( \phi^{-1}(\phi(\overline{Z}_i))_{\text{red}} = \overline{Z}_i \). This proves the corollary.

9.2. The proof.

Theorem 9.5. Let \( V = \text{Spec}(R) \) be an \( r \)-dimensional smooth semi-local \( k \)-scheme of geometric type with the set of closed points \( \Sigma \). Assume that \( r \geq 1 \) and that \( m \geq 0, n \geq 1 \) are integers. Let \( \alpha \in \text{Tz}_n^m(V; n; m) \) be a cycle. Then, we can find:

1. an atlas \( (X, \Sigma) \) for \( V \) and a finite surjective map \( \phi : X \to \mathbb{A}_k^r \),
2. a cycle \( \overline{\alpha} \in \text{Tz}_n^m(X, n; m) \) with \( \alpha = \overline{\alpha}_V \), and
3. an open subset \( U \subseteq \mathbb{A}_k^r \) with a finite and étale map \( \phi : \phi^{-1}(U) \to U \)

such that for every component \( Z \) of \( \overline{\alpha} \), one of the following holds:

(A) \( Z \) is smooth and \( Z = \phi^{-1}(\phi(Z))_{\text{red}} \).
(B) \( \phi^+([Z]) \) is an sfs-cycle over \( V \).

**Proof.** By Lemma 2.9, we can find an atlas \((X, \Sigma)\) for \( V \) and a cycle \( \overline{\alpha} \in Tz^n_D(X, n; m) \) satisfying (1) and (2). Our goal now is to modify this atlas to get the desired result.

We choose a closed embedding \( X \hookrightarrow \mathbb{A}^k \) satisfying (1) and (2). Our goal now is to modify this atlas to get the desired result.

We choose a closed embedding \( X \hookrightarrow \mathbb{P}^N_k \) with \( N \gg r \) such that for a general linear subspace \( L \in \text{Gr}(X, N - r - 1, \mathbb{P}^N_k) \), the assertion of Corollary 9.4 holds. Since \( L \) is a general member of \( \text{Gr}(X, N - r - 1, \mathbb{P}^N_k) \), we can apply the moving lemma for additive higher Chow groups of smooth affine \( k \)-schemes (see [19]) to choose \( L \) so that every component of \( \phi^* \phi_*(\overline{\alpha}) \) lies in \( Tz^n_{\text{sfs}}(X, n; m) \).

Let \( \phi : \phi^{-1}(U) \to U \) be the map of smooth affine schemes as in Corollary 9.4. Let \( V' \) denote the spectrum of the semi-local ring of \( U \) at \( \phi(\Sigma) \). Since \( \phi^{-1}(\phi(\Sigma)) \subset X \), we see that the restriction of \( \phi \) at \( V' \) is finite and étale. It follows that \( \phi \) is finite (and hence flat) in an affine neighborhood of \( \phi(\Sigma) \) in \( U \). It follows now from [15, Exercise III.10.2] that \( \phi \) is finite and étale over an affine neighborhood of \( \phi(\Sigma) \) in \( U \). By shrinking \( U \) suitably and replacing \( X \) by the corresponding inverse image under \( \phi \), we get an atlas \((X, \Sigma)\) for \( V \) satisfying (1)~(3). The last assertion now follows directly by applying property (4) in Corollary 9.4 to \( \overline{\alpha} \).

Finally, we get to the conclusion of Sections 4 ~ 9.

**Proof of Theorem 4.13.** It is clear from the definition of \( TCH^n_{\text{sfs}}(V, n; m) \) that the map \( \text{sfs}_V \) is injective. So, the main point is to prove its surjectivity. By Theorem 5.11 we may replace \( TCH^n(\cdot, n; m) \) by \( TCH^n_{\text{sfs}}(\cdot, n; m) \).

Let \( \alpha \in Tz^n_{\text{sfs}}(V, n; m) \) be a cycle with \( \partial(\alpha) = 0 \). We can apply Lemma 2.9 to choose an atlas \((X, \Sigma)\) for \( V \) and a cycle \( \overline{\alpha} \in Tz^n(X, n; m) \) such that \( \partial(\alpha) = 0 \). If \( X \simeq \mathbb{A}^k \), we can apply Theorem 4.10 to write \( \alpha = \beta + \partial(\gamma) \), where \( \beta \in Tz^n_{\text{sfs}}(V, n; m) \subset Tz^n_{\text{sfs}}(V, n; m) \) and \( \gamma \in Tz^n(V, n + 1; m) \).

If \( X \) is not an affine space, we let \( \phi : X \to \mathbb{A}^k \) be the finite and flat map as in Theorem 9.5 and let \( \Sigma' = \phi(\Sigma) \). Let \( V' \) denote the localization of \( \mathbb{A}^k \) at \( \Sigma' \) and let \( W = X \times_{\mathbb{A}^k} V' \), so there are inclusions \( \Sigma \subset V \subset W \hookrightarrow X \) and \( \mathbb{P}^n_k : W \to V' \) is a finite and flat morphism of regular semi-local schemes, where \( V' \) is \( \phi_*(\overline{\alpha}_W) \)-linear.

Let \( j : V \to X \) be the localization map.

We can write \( \overline{\alpha}_W = (\overline{\alpha}_W - \phi^*\phi_*(\overline{\alpha}_W)) + \phi^*\phi_*(\overline{\alpha}_W) \). We also have \( \partial(\phi_*(\overline{\alpha}_W)) = \phi_*\partial(\overline{\alpha}_W) = 0 \). Since \( V' \) is \( \phi_*\overline{\alpha}_W \)-linear, we can write \( \phi_*(\overline{\alpha}_W) = \eta_1 + \partial(\eta_2) \), where \( \eta_1 \in Tz^n_{\text{sfs}}(V', n; m) \) and \( \eta_2 \in Tz^n(V', n + 1; m) \). This yields \( \phi^*\phi_*(\overline{\alpha}_W) = \phi^*(\eta_1) + \partial(\phi^*(\eta_2)) \). Since \( \phi \) is finite and étale, \( \phi^* \) preserves the sfs-cycles. In particular, \( \phi^*(\eta_1) \in Tz^n_{\text{sfs}}(W, n; m) \).

It follows from Theorem 9.5 that \( j^*(\overline{\alpha}_W - \phi^*\phi_*(\overline{\alpha}_W)) \in Tz^n_{\text{sfs}}(V, n; m) \). Setting \( \beta = j^*(\overline{\alpha}_W - \phi^*\phi_*(\overline{\alpha}_W)) + j^*(\phi^*(\eta_1)) \) and \( \gamma = j^*(\phi^*(\eta_2)) \), we get

\[
\begin{align*}
\alpha &= j^*(\overline{\alpha}_W) \\
&= j^*(\overline{\alpha}_W - \phi^*\phi_*(\overline{\alpha}_W)) + j^*(\phi^*(\eta_1)) + j^*(\partial(\phi^*(\eta_2))) \\
&= j^*(\overline{\alpha}_W - \phi^*\phi_*(\overline{\alpha}_W)) + j^*(\phi^*(\eta_1)) + \partial(j^*(\phi^*(\eta_2))) \\
&= \beta + \partial(\gamma)
\end{align*}
\]

with \( \beta \in Tz^n_{\text{sfs}}(V, n; m) \) and \( \gamma \in Tz^n(V, n + 1; m) \). Since \( \partial(\alpha) = 0 \), we must have \( \partial(\beta) = 0 \) as well. This proves the theorem. \( \square \)
10. Surjectivity of the de Rham-Witt-Chow homomorphism

Let $k$ be any infinite perfect field of characteristic $\neq 2$, unless otherwise stated. We now prove the surjectivity of our de Rham-Witt-Chow homomorphism for regular semi-local rings and complete the proof of Theorem 3.1. In view of Theorem 4.13, we need to show that every sfs-cycle is generated by cycles that are Witt-Milnor cycles over $R$, or symbolic over $R$, that is, in the image of the map $\tau_{n,m}^R$. This will be achieved by a delicate usage of the Witt-complex structure on the additive higher Chow groups of regular semi-local rings of Section 3.1 and [29].

10. Traceability of de Rham-Witt forms via cycles.

10.1. Notion of traceability. Let $R \rightarrow S$ be a finite extension of regular semi-local rings essentially of finite type over $k$ and let $f : \text{Spec}(S) \rightarrow \text{Spec}(R)$ denote the induced morphism of schemes. Let $m, n \geq 1$ be two integers. Given this, we obtain a diagram:

\[ \begin{array}{ccc}
\mathbb{W}_m\Omega_{S}^{n-1} & \tau_{n,m}^{S} & \text{TCH}^n(S, n; m) \\
\downarrow & \downarrow f_* & \\
\mathbb{W}_m\Omega_{R}^{n-1} & \tau_{n,m}^{R} & \text{TCH}^n(R, n; m).
\end{array} \]

A problem one faces is that it is unknown if there is a trace map $\mathbb{W}_m\Omega_{S}^{n-1} \rightarrow \mathbb{W}_m\Omega_{R}^{n-1}$ so that the above square can be completed. The main idea here is to use the push-forward $f_*$ as a trace-like operation on the de Rham-Witt forms via algebraic cycles.

Definition 10.1. Let $R \rightarrow S$ be as above. We say that a de Rham-Witt form $\omega \in \mathbb{W}_m\Omega_{S}^{n-1}$ is traceable to $R$ (via cycles) if $f_* \circ \tau_{n,m}^{S}(\omega) \in \text{Image}(\tau_{n,m}^{R})$.

10.1.2. Traceability under simple ring extensions. We want to show the traceability of the de Rham-Witt forms under “simple extensions” of regular semi-local rings in Proposition 10.5.

Definition 10.2. A ring extension $R \subset S$ is said to be a simple extension if there exists a monic irreducible polynomial $p(t) \in R[t]$ such that $S \simeq R[t]/(p(t))$. Let $e = \deg p(t)$. Let $a := t \mod (p(t))$ in $S$. Then $S$ is a finite free extension of $R$ with an $R$-basis $\{1, a, a^2, \cdots, a^{e-1}\}$. We need the following basic fact about the ring of Witt vectors.

Lemma 10.3. Let $S$ be a free $R$-algebra with $x_1, \cdots, x_e$ as an $R$-basis. Let $T$ be a finite truncation set. Then, every $\omega \in \mathbb{W}_T(S)$ is uniquely written as

\[ \omega = \sum_{n \in T} \sum_{i=1}^{e} V_n([-c_{n,i}]_{T/n} \cdot [x_i]_{T/n}), \]

for some $c_{n,i} \in R$, where $[-]_{T/n}$ denotes the Teichmüller lift in $\mathbb{W}_T(R)$.

Proof. Its proof is similar to that of [38] Lemma 2.20, for instance. Let $\omega = (\omega_n)_{n \in T} \in \mathbb{W}_T(S)$. Suppose $\omega \neq 0$ for otherwise there is nothing to prove. Define an operator $\varphi$ as follows: first choose $s_0 = \min\{s \in T \mid \omega_s \neq 0\}$. This minimum
exists because $\omega \neq 0$. Here, $\omega_{s_0} \in S$ so that there exists a unique expression $\omega_{s_0} = \sum_{i=1}^e c_{s_0,i} \cdot x_i$ in $S$ for some $c_{s_0,i} \in R$. Now, define $\varphi(\omega) := \omega - V_{s_0}(\sum_{i=1}^e c_{s_0,i})T/s_0 \cdot [x_i]T/s_0$. For $\varphi(\omega)$, we have either $\varphi(\omega) = 0$, or $\varphi(\omega) \neq 0$ and so there exists $s_1 := \min\{s \in T| \varphi(\omega)_s \neq 0\}$. In the former, the argument stops, while in the latter case, by construction $s_1 > s_0$. We repeat this process. Since $|T| < \infty$, there exists $N \geq 1$ such that eventually $\varphi^N(\omega) = 0$. \hfill $\square$

When $S$ is a simple extension, and $T = \{1, \ldots, m\}$, we immediately deduce $\omega = \sum_{i=1}^m \sum_{j=0}^{e-1} V_i([c_{ij}][m/i] \cdot [a^j_{m/i}]).$

Recall from [38, Proposition A.9] that for a finite free extension of rings $R \to S$, and $m \geq 1$, there is a trace map $Tr_{S/R} : \mathbb{W}_m(S) \to \mathbb{W}_m(R)$ which commutes with the Frobenius and the Verschiebung operators and satisfies other usual properties of the trace maps. This $Tr_{S/R}$ is given as follows: for the finite free extension $R[[t]] \to S[[t]]$, we have the norm map $N_{S/R} : (S[[t]])^x \to (R[[t]])^x$ given by the determinant of the left multiplication maps, which induces $N_{S/R} : (1 + tS[[t]])^x / (1 + t^{m+1}S[[t]])^x \to (1 + tR[[t]])^x / (1 + t^{m+1}R[[t]])^x$. This $N_{S/R}$ is the definition of $Tr_{S/R}$ via the identification (2.2).

**Lemma 10.4.** Let $R \subset S$ be a simple extension of regular semi-local rings essentially of finite type over $k$ and let $m \geq 1$ be an integer. Then, the diagram

$$
\begin{array}{ccc}
\mathbb{W}_m(S) & \xrightarrow{\tau^S_{1,m}} & \text{TCH}^1(S, 1; m) \\
\downarrow{Tr_{S/R}} & & \downarrow{f_*} \\
\mathbb{W}_m(R) & \xrightarrow{\tau^R_{1,m}} & \text{TCH}^1(R, 1; m)
\end{array}
$$

commutes, where $f : \text{Spec}(S) \to \text{Spec}(R)$ is the induced map.

**Proof.** Using Lemmas [3.2 and 10.3] we only need to show that $\tau^R_{1,m}(Tr_{S/R}([x])) = f_*([\Gamma(1-xt)])$ for all $x \in S$, where $\Gamma(1-xt)$ is the cycle in $\text{TCH}^1(S, 1; m)$ corresponding to the ideal $(1-xt) \subset S[t]$. Since $[x] \in \mathbb{W}_m(S)$ corresponds to $1-xt \in (1 + tS[[t]])^x / (1 + t^{m+1}S[[t]])^x$, by the definition of $Tr_{S/R}$ we have $Tr_{S/R}([x]) = N_{S/R}(1-xt)$. On the other hand, for a polynomial representative $g(t) \in (1 + tR[[t]])^x / (1 + t^{m+1}R[[t]])^x$, we have by definition $\tau^R_{1,m}(g(t)) = [\Gamma(g(0))]$. Hence, $\tau^R_{1,m}(Tr_{S/R}([x])) = \tau^R_{1,m}(N_{R/S}(1-xt)) = [\Gamma(N_{S/R}(1-xt))]$. On the other hand, by [11, Proposition 1.4(2)], we have $f_*([\text{div}(1-xt)] = [\text{div}(N_{S/R}(1-xt))]$. Since $1-xt$ and $N_{S/R}(1-xt)$ are regular functions, we have $\text{div}(1-xt) = \Gamma(1-xt)$ and $\text{div}(N_{S/R}(1-xt)) = \Gamma(N_{S/R}(1-xt))$. Hence, we have $[\Gamma(N_{S/R}(1-xt))] = f_*([\Gamma(1-xt)])$. Hence, we have $\tau^R_{1,m}(Tr_{S/R}([x])) = [\Gamma(N_{S/R}(1-xt))] = f_*([\Gamma(1-xt)])$, as desired. \hfill $\square$

**Proposition 10.5.** Let $R \to S$ be a simple extension of regular semi-local rings which are essentially of finite type over $k$ and let $m, n \geq 1$ be two integers. Then all elements in $\mathbb{W}_m \Omega_S^{n-1}$ are traceable to $R$.

**Proof.** Let $p(t) \in R[t]$ be a monic polynomial of degree $e$ such that $S \simeq R[t]/(p(t))$. For $m, n \geq 1$, let $P_{n,m}$ be the statement

$P_{n,m}$: all elements in $\mathbb{W}_m \Omega_S^{n-1}$ are traceable to $R$. 


We prove the proposition by a double induction on the variables \((n, m) \in \mathbb{N} \times \mathbb{N}\).

We begin with the boundary cases:

**Case 1:** \(P_{1,m}\) and \(P_{n,1}\) are true.

Notice that the statement \(P_{1,m}\) is an immediate consequence of Lemma 10.4. In particular, \(P_{1,1}\) is also true.

**Subcase 1-1:** To show \(P_{2,1}\), note that every element of \(\mathbb{W}_1 \Omega^1_S \cong \Omega^1_{S/Z}\) is a finite sum of the type \(ca'd(c^a) = ca'de' + jke'a^{i+j}da\) for some \(c, c' \in R\).

If \(c, c' \in R\) and \(i \geq 0\).

For \(ca'dc'\), we have

\[
\begin{align*}
f_* \circ \tau^S_{2,1}(ca'dc') = & f_* \left( \tau^S_{1,1}(a') \cdot \tau^S_{2,1}(cdc') \right) = f_* \left( \tau^S_{1,1}(a') \cdot \tau^S_{2,1}(f^*(cdc')) \right) \\
= & f_* \left( \tau^S_{1,1}(a') \cdot f^* \left( \tau^R_{1}(cdc') \right) \right) = f_* \left( \tau^S_{1,1}(a') \cdot \tau^R_{2}(cdc') \right) \\
= & 2 \tau^R_{1,1}(\text{Tr}_S/R(a')) \cdot \tau^R_{2,1}(cdc') = \tau^R_{2,1}(\text{Tr}_S/R(a') \cdot (cdc')).
\end{align*}
\]

Here, the equalities \(\dagger\) hold because \(\tau^R_{m,n}, \tau^S_{m,n}\) are morphisms of DGAs. The equality \(\dagger\) holds by Proposition 3.4 \(= \) by the projection formula for the additive higher Chow groups and \(\dagger\) holds by Lemma 10.4. We conclude that 1-forms of the type \(ca'dc'\) are traceable to \(R\).

Next, for \(ca'da\), note that by the part of definition of a Witt-complex in Section 2.5.2(\(v\)), we can write \(ca'da = cF_{i+1}d[a]\). Hence,

\[
\begin{align*}
f_* \circ \tau^S_{2,1}(ca'da) = & f_* \circ \tau^S_{2,1}(cF_{i+1}d[a]) = f_* \circ \tau^S_{2,1}(f^*(c)F_{i+1}d[a]) \\
= & f_* \left( \tau^S_{1,1}(f^*(c)) \cdot F_{i+1} + \delta \tau^S_{1,2i+1}([], [a]) \right) = f_* \left( \tau^R_{1,1}(c) \cdot F_{i+1} + \delta \tau^S_{1,2i+1}([], [a]) \right) \\
= & 2 \tau^R_{1,1}(c) \cdot F_{i+1} + \delta \tau^S_{1,2i+1}([], [a]) = 2 \tau^R_{1,1}(c) \cdot F_{i+1} \cdot \text{Tr}_S/R([a]) \\
= & 2 \tau^R_{1,1}(c) \cdot \tau^S_{2,1}F_{i+1}d(\text{Tr}_S/R([a])) = 2 \tau^R_{2,1} \left( c \cdot F_{i+1}d(\text{Tr}_S/R(a)) \right),
\end{align*}
\]

where the equalities \(\checkmark\) hold by (3.2), the equality \(\dagger\) holds by Proposition 3.4 \(= \) by the projection formula for \(f_*\) and \(f^*\), \(\dagger\) holds by Lemma 3.2 and \(\dagger\) holds by \(\tau^S_{m,n}\) being a morphism of DGAs. We conclude that 1-forms of the type \(ca'da\) are traceable to \(R\). Hence \(P_{2,1}\) is true.

**Subcase 1-2:** Suppose now that \(n > 2\) and that the statement \(P_{i,1}\) is true for any \(2 \leq i < n\). It suffices to show that the forms of the type \(\omega = c_0a^{i_0}d(c_1a^{i_1}) \wedge \ldots \wedge d(c_{n-1}a^{i_{n-1}})\) is traceable to \(R\), where \(c_0, \ldots, c_{n-1} \in R\) and \(i_0, \ldots, i_{n-1} \geq 0\) are integers. Each \(d(c_ja^{i_j})\) is equal to \(a^{i_j}dc_j + id_ja^{i_j-1}1\) \(da\) by the Leibniz rule, so that expanding the terms of \(\omega\), we are reduced to showing that every element of the form

\[
\omega_0 := c_0a^{i_0}dc_1 \wedge \ldots \wedge dc_s \wedge da \wedge \ldots \wedge da
\]

is traceable to \(R\), where \(0 \leq s \leq n - 1\) and \(c_0, \ldots, c_s \in R\).

- If \(n - s - 1 = 0\), then the traceability of \(\omega\) follows by repeating the steps in (10.3) verbatim.
- If \(n - s - 1 = 1\), let \(\omega'_0 := a^{i_0}da\) so that \(\omega_0 = c_0dc_1 \wedge \ldots \wedge dc_s \wedge \omega'_0\). Since \(P_{2,1}\) is true, we can write \(f_* \tau^S_{2,1}(\omega'_0) = \tau^R_{2,1}(\omega''_0)\) for some \(\omega''_0 \in \Omega^1_{R/Z}\). Set \(\eta :=
\]
$c_0d_1 \wedge \cdots \wedge d_c \in \Omega^*_R/Z$. Then, we have

$$
\begin{align*}
    f_* \circ \tau^S_n(\omega) &= f_* \circ \tau^S_n(\eta \wedge \omega') = f_* \circ \tau^S_n(f^*(\eta) \wedge \omega') \\
    &= \tau^S_n(f^* \circ \tau^S_n(\eta) \wedge \omega') = \tau^S_n(f^* \tau^S_n(\eta) \wedge \omega') \\
    &= \tau^R_n(\eta) \wedge f^* \tau^S_n(\omega') = \tau^S_n(\omega') \wedge \tau^R_n(\eta) = \tau^R_n(\eta \wedge \omega'),
\end{align*}
$$

where the equalities $\dagger$ hold because $\tau^S_{n,m}, \tau^R_{n,m}$ are morphisms of DGAs, $\dagger\dagger$ holds by Proposition \[3.4\] and $\dagger$ holds by the projection formula for $f_*$ and $f^*$. We conclude that $\omega_0$ is traceable to $R$.

- If $n - 1 - s > 1$, then $\omega_0 = 0$ since $2 \in S^\infty$, so traceable to $R$.

We have thus shown that $P_n, P, P_{1,m}$ are true for all $n, m \geq 1$.

**Case 2:** The general case. We now show that $P_{n,m}$ is true in general by using double induction on $(n, m)$. Fix $n, m \geq 2$ and assume that $P_{i,j}$ is true for all $1 \leq i \leq n, 1 \leq j \leq m$, except $(i, j) = (n, m)$.

Through the surjection $\Omega^{\leq}_{s}(\omega) \to \Omega^{\leq}_{s}(\omega)$ we know that every element in $\Omega^{\leq}_{s}$ is a sum of de Rham-Witt forms of the type $\omega = V_r([c_0][a]^{i_0}) \wedge \cdots \wedge dV_{r-1}([c_{n-1}][a]^{i_{n-1}})$, where $c_0, \cdots, c_{n-1} \in R$, $r_0, \cdots, r_{n-1} \in \{1, \cdots, m\}$, and $0 \leq i_0, \cdots, i_{n-1} \leq e - 1$.

**Subcase 2-1:** First, consider the case $r_0 > 1$. Let $\omega_0 := dV_r([c_1][a]^{i_1}) \wedge \cdots \wedge dV_{r-1}([c_{n-1}][a]^{i_{n-1}})$. In this case, we can write

$$
\omega = V_r([c_0][a]^{i_0}) \cdot \omega_0 = V_r([c_0][a]^{i_0} \cdot F_r(\omega_0))
$$

by the projection formula for $V_r$ and $F_r$ (see Section \[2.5.2\] iii). Since $\omega'_0 := [c_0][a]^{i_0} \cdot F_r(\omega_0) \in \Omega^{\leq}_{s}$, it is traceable to $R$ by the induction hypothesis $P_{n,m/r_0}$. In particular, there exists $\eta \in \Omega^{\leq}_{s}$ such that $f_* \tau^S_{n,m/r_0}(\omega'_0) = \tau^S_{n,m/r_0}(\eta)$. This in turn yields

$$
\begin{align*}
    f_* \tau^S_{n,m}(\omega) &= f_* \tau^S_{n,m}V_r(\omega'_0) = f_* V_r \tau^S_{n,m/r_0}(\omega'_0) \\
    &= f_* \tau^S_{n,m}(\omega'_0) = \tau^S_{n,m}(\eta) = \tau^S_{n,m}(V_r(\eta)),
\end{align*}
$$

where the equalities $\dagger$ hold by \[3.3\], $\dagger\dagger$ holds by Lemma \[3.2\]. This shows that $\omega$ is traceable to $R$.

**Subcase 2-2:** Suppose now that $r_0 = 1$, but for some $j > 0$, we have $r_j > 1$. We may assume that $r_1 > 1$, without loss of generality. We let $\omega_0 := dV_{r_2}([c_2][a]^{i_2}) \wedge \cdots \wedge dV_{r_{n-1}}([c_{n-1}][a]^{i_{n-1}})$. By the Leibniz rule, we have

$$
V_r([c_0][a]^{i_0}) \cdot dV_{r_1}([c_1][a]^{i_1}) = V_r([c_0][a]^{i_0} \cdot dV_{r_1}([c_1][a]^{i_1}) = d([c_0][a]^{i_0} \cdot V_r([c_1][a]^{i_1})),
$$

Hence, $\omega = \omega_1 - \omega_2$, where $\omega_1 := d([c_0][a]^{i_0} \cdot V_r([c_1][a]^{i_1})) \wedge \omega_0$ and $\omega_2 := V_r([c_1][a]^{i_1}) \cdot [c_0][a]^{i_0} \wedge \omega_0$. Let $\omega'_1 := [c_0][a]^{i_0} \cdot V_r([c_1][a]^{i_1})$ so that $\omega_1 = d\omega'_1 \wedge \omega_0 = d(\omega'_1 \wedge \omega_0)$.

Since $\omega'_1 \cdot \omega_0 \in \Omega^{\leq}_{s}$, it follows by the induction hypothesis $P_{n-1,m}$ that there is an element $\eta \in \Omega^{\leq}_{s}$ such that $f_* \tau^S_{n-1,m}(\omega'_1 \wedge \omega_0) = \tau^S_{n-1,m}(\eta)$. Thus,

$$
\begin{align*}
    f_* \tau^S_{n-1,m}(\omega_1) &= f_* \tau^S_{n-1,m}(d(\omega'_1 \wedge \omega_0)) = f_* \delta(\tau^S_{n-1,m}(\omega'_1 \wedge \omega_0)) \\
    &= \tau^S_{n-1,m}(\eta) = \tau^S_{n-1,m}(\omega'_1 \wedge \omega_0)
\end{align*}
$$
Symbolicity of sfs-cycles and Proof of Theorem 3.1.

10.2. Subcase 2-1: Now, the remaining case is when all $r_0 = r_1 = \cdots = r_{n-1} = 1$, i.e., $\omega = [c_0][a]^i d(([c_1][a])^i) \wedge \cdots \wedge d([c_{n-1}][a])$. Its proof is almost identical to that of Subcase 1-2, which we argue now. Each $d([c_j][a]^i)$ is equal to $[a]^i d[c_j] + ij[c_j][a]^i d[a]$ by the Leibniz rule, so that expanding the terms of $\omega$, we are reduced to show that elements of the form

$$\omega_0 := [c_0][a]^i d[c_1] \wedge \cdots \wedge d[c_s] \wedge \underbrace{d[a] \wedge \cdots \wedge d[a]}_{n-s-1}$$

are traceable, where $0 \leq s \leq n-1$ and $c_0, \ldots, c_s \in R$.

- If $n-s-1 = 0$, then we can use Proposition 3.4, Lemma 10.3 and repeat the steps of (10.3) verbatim to conclude that $\omega_0$ is traceable to $R$.
- If $n-s-1 = 1$, let $\omega'_0 := [a]^i d[a]$ so that $\omega_0 = [c_0][a]^i d[c_1] \wedge \cdots \wedge d[c_{n-2}] \wedge \omega'_0$. By the part of definition of a Witt-complex in Section 2.5.2(v), we can write $\omega'_0 = [a]^i d[a] = F_{i+1} d[a]$. Set $\eta = [c_0][a]^i \wedge \cdots \wedge d[c_{n-2}] \in \mathbb{W}_m \Omega^{-2}_R$, so that $\omega_0 = \eta \wedge F_{i+1} d[a]$. (Remember, here $n \geq 2$.) This yields

$$f_* \tau^S_{n,m}(\omega_0) = f_* \tau^S_{n,m}(\eta \wedge F_{i+1} d[a]) = f_* \tau^S_{n,m}(f^*(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a]) = f_* (f^* \tau^S_{n-2,m}(\eta) \wedge F_{i+1} d[a])$$

where the equalities = hold because $\tau^R_{n,m}$ and $\tau^S_{n,m}$ are morphisms of DGAs, the equality = is the projection formula for $f_*$ and $f^*$, the equalities = hold by (3.2), the equality $=^2$ holds by Lemma (3.2) and $=^3$ follows from Lemma 10.3. This shows that $\omega_0$ is traceable to $R$.

- If $n-s-1 > 1$, we set $\omega'_0 = [a]^i d[a] \wedge \underbrace{d[a] \wedge \cdots \wedge d[a]}_{n-s-1}$. Since $2 \leq S$ and since the Teichmüller lift is multiplicative, we see that $2 \in (\mathbb{W}_m(S))^\times$. In particular, $d[a] \wedge d[a] = 0$ in $\Omega^2_{\mathbb{W}_m(S)/\mathbb{Z}}$ and hence it is zero in $\mathbb{W}_m \Omega^2_S$. In particular, $\omega'_0 = 0$ in $\mathbb{W}_m \Omega^{n-s-1}_S$ so that $\omega_0 = 0$, which is traceable to $R$. We have thus shown that $P_{n,m}$ is true. The proof of the proposition is now complete. 

10.2. Symbolicity of sfs-cycles and Proof of Theorem 3.1. In Section 10.2, we complete the proof of Theorem 3.1 using Theorem 4.13 and Proposition 10.5 as two key ingredients. We begin with the following result that relates Milnor $K$-theory, de Rham-Witt complex, higher Chow groups and additive higher Chow groups.

Consider the composition of natural homomorphisms $\theta^R_R : \mathbb{W}_m(R) \otimes_{\mathbb{Z}} K^M_{n-1}(R) \to \mathbb{W}_m(R) \otimes_{\mathbb{Z}} \Omega_{n-1}^{R/\mathbb{Z}} \to \Omega_{n-1}^{R/\mathbb{Z}}$. It is determined by $\theta^R_R(V_i([a]) \otimes \{b_1, \cdots, b_{n-1}\}) = V_i([a]) \cdot \text{dlog}[b_1] \wedge \cdots \wedge \text{dlog}[b_{n-1}]$ for $a \in R$ and $b_1, \cdots, b_{n-1} \in R^S$. We further have the composite $\theta_R : \mathbb{W}_m(R) \otimes_{\mathbb{Z}} K^M_{n-1}(R) \xrightarrow{\theta^R_R} \Omega^{n-1}_{\mathbb{W}_m(R)/\mathbb{Z}} \to \mathbb{W}_m \Omega^{n-1}_R$. 

On the other hand, consider the composite \( \xi_R : \mathbb{W}_m(R) \otimes \mathbf{Z} H^{*-1}(R, \bullet - 1) \xrightarrow{\tau_{1,m}^R \otimes \text{Id}} \text{TCH}^1(R, 1; m) \otimes \mathbf{Z} H^{*-1}(R, \bullet - 1) \rightarrow \text{TCH}^*(R, \bullet; m) \), where the second arrow is given by the cap product in Proposition 3.9.

**Proposition 10.6.** Let \( R \) be a regular semi-local ring essentially of finite type over \( k \). Then the following diagram commutes

\[
\begin{array}{ccc}
\mathbb{W}_m(R) \otimes \mathbf{Z} K^M_{n-1}(R) & \xrightarrow{\psi_R} & \mathbb{W}_m(R) \otimes \mathbf{Z} H^{n-1}(R, n - 1) \\
\downarrow \phi_R & & \downarrow \xi_R \\
\mathbb{W}_m \Omega^{n-1}_R & \xrightarrow{\tau_{n,m}^R} & \text{TCH}^n(R, n; m),
\end{array}
\]

where \( \psi_R = \text{Id} \otimes \phi_R \) and \( \phi_R(\{b_1, \cdots, b_{n-1}\}) \) is the cycle on \( \text{Spec}(R) \times \square^{n-1} \) given by \( W_{b_1, \cdots, b_{n-1}} := \text{Spec} \left( R[g_1, \cdots, g_{n-1}] \right) \).

**Proof.** That \( \phi_R \) is well-defined is proven in [39] §2. (The proof in the reference is stated for fields, but since Milnor \( K \)-groups are constructed with \( R^* \), the identical argument stands. cf. [10] Lemma 2.1). Thus \( \psi_R \) is also well-defined. So, all maps in the above are well-defined. By (2.3), it is enough to prove the commutativity for \( x := V_i([a]) \otimes \{b_1, \cdots, b_{n-1}\} \) for some \( 1 \leq i \leq m \) and \( a \in R, b_1, \cdots, b_{n-1} \in R^* \).

Note that \( \theta_R(x) = V_i([a]) \cdot \log[b_1] \land \cdots \land \log[b_{n-1}] \), and \( \tau_{n,m}(\theta_R(x)) \) is given by

\[
Z_x = Z_{a,b_1,\cdots,b_{n-1}} := \text{Spec} \left( \frac{R[t, y_1, \cdots, y_{n-1}]}{1 - at^i, y_1 - b_1, \cdots, y_{n-1} - b_{n-1}} \right)
\]

by [29] Example 6.11. On the other hand, \( \psi_R(x) = V_i([a]) \otimes [W_{b_1,\cdots,b_{n-1}}] \), and (\( \tau_{1,m}^R \otimes \text{Id} \))\( \psi_R(x) \) = \( \text{Spec} \left( \frac{R[t]}{1 - at^i} \right) \otimes \text{Spec} \left( \frac{R[y_1, \cdots, y_{n-1}]}{(y_1 - b_1, \cdots, y_{n-1} - b_{n-1})} \right) \)

in \( \text{TCH}^1(R, 1; m) \otimes \mathbf{Z} H^{n-1}(R, n - 1) \) so that their cap product is given by

\[
\Delta_R \left( \text{Spec} \left( \frac{(R \otimes_k R)[t, y_1, \cdots, y_{n-1}]}{(1 - (a \otimes 1)t^i, y_1 - 1 \otimes b_1, \cdots, y_{n-1} - 1 \otimes b_{n-1})} \right) \right),
\]

which is equal to \( Z_x \) in (10.8) because \( \Delta_R : \text{Spec}(R) \rightarrow \text{Spec}(R) \times \text{Spec}(R) \simeq \text{Spec}(R \otimes_k R) \) is given by the product map \( R \otimes_k R \rightarrow R \). This proves that \( \tau_{n,m}^R \circ \theta_R(x) = [Z_x] = \xi_R \circ \psi_R(x) \) as desired. This finishes the proof. \( \square \)

**Proposition 10.7.** Let \( R \) be a regular semi-local ring essentially of finite type over \( k \), and let \( m, n \geq 1 \) be integers. Then every sfs-cycle in \( \text{TCH}^n(R, n; m) \) is symbolic over \( R \), that is, it is in the image of \( \tau_{n,m}^R \).

**Proof.** Let \( [Z] \in Tz^n(R, n; m) \) be an irreducible sfs-cycle. By Proposition 4.8 we know that \( Z \) is a closed subscheme of \( \text{Spec}(R) \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \), which is in fact contained in \( \text{Spec}(R) \times \mathbb{A}^1 \times (\mathbb{A}^1)^{n-1} \). Moreover, the ideal \( I(Z) \) of \( Z \) inside \( R[t, y_1, \cdots, y_{n-1}] \) is given by the equations:

\[
\begin{align*}
p(t) &= 0, \\
q_1(t, y_1) &= 0, \\
&\vdots \\
q_{n-1}(t, y_1, \cdots, y_{n-1}) &= 0,
\end{align*}
\]
with the following additional properties: let \( R_0 = R, R_1 = R[t]/(p(t)) \) and \( R_i = R_{i-1}[t_i]/(q_{i-1}) \) for \( 2 \leq i \leq n \). Here, the rings \( \{R_i\}_{1 \leq i \leq n} \) are all regular semi-local \( k \)-algebra such that each \( R_i \) is a simple extension of \( R_{i-1} \). Let \( f_i : \text{Spec}(R_i) \to \text{Spec}(R_{i-1}) \) be the induced finite surjective map of semi-local schemes for \( 1 \leq i \leq n \). Set \( f = f_1 \circ \cdots \circ f_n \).

Set \( c^{-1} := t \mod I(Z) \) and \( b_i := y_i \mod I(Z) \) for \( 1 \leq i \leq n - 1 \). Notice that another consequence of the sfs-property of \( Z \) is that \( c^{-1}, b_i \in (R_n)^\times \) for all \( 1 \leq i \leq n - 1 \). Set \( Z_n = \text{Spec}\left( \frac{R_n[y_1, \ldots, y_{n-1}]}{(1-ct, y_1, \ldots, y_{n-1} - b_{n-1})} \right) \) and \( W_n = \text{Spec}\left( \frac{R_n[y_1, \ldots, y_{n-1}]}{(y_1 - b_1, \ldots, y_{n-1} - b_{n-1})} \right) \). Let \( \eta_n := [c]\text{dlog}[b_1] \land \cdots \land \text{dlog}[b_{n-1}] \).

It follows then that \( [Z_n] \in \text{Te}_n(R_n, n; m) \) such that \( [Z] = f_*([Z_n]) \) and \( [W_n] \in C_n\text{H}^{-1}(R_n, n-1) \). Moreover, we check that \( [Z_n] = \xi_{R_n}([c] \otimes [W_n]) \) is \( \xi_{R_n} \circ \psi_{R_n}([c] \otimes \{b_1, \ldots, b_{n-1}\}) = \tau_{R_n}([c]\text{dlog}[b_1] \land \cdots \land \text{dlog}[b_{n-1}]) = \tau_{n,m}(\eta_n) \), where \( \tau_{n,m} \) holds by Proposition 10.6. Since \( f_n \) is a simple extension, by Proposition 10.5, we have \( f_n^* \tau_{n,m}(\eta_n) = \tau_{n,m-1}(\eta_{n-1}) \) for some \( \eta_{n-1} \in \mathbb{W}_m \Omega_{R_{n-1}}^{-1} \). Since \( f = f_1 \circ \cdots \circ f_n \) and each \( f_i \) is a simple extension, by successive applications of \( f_* \) and Proposition 10.5, we obtain \( [Z] = f_*[Z_n] = \tau_{n,m}(\eta_0) \) for some \( \eta_0 \in \mathbb{W}_m \Omega_{R}^{-1} \). This means, \( [Z] \) is symbolic over \( R \) as desired.

Thus, we finally get to:

**Proof of Theorem 3.1.** We have already shown in Corollary 3.6 that \( \tau_{n,m} \) is injective. The surjectivity of \( \tau_{n,m} \) follows by Theorem 4.13 and Proposition 10.7.

**Corollary 10.8.** Let \( k \) be a perfect field of characteristic \( \neq 2 \). Then, the morphism \( \mathbb{W}_m \Omega_{(-)\text{zar}}^{n-1} \to \mathcal{TCH}^n(-, n; m)_{\text{zar}} \) of Section 2.6 is an isomorphism of Zariski sheaves on \( \text{Sm}_k \).

### 11. Applications

In this section, we discuss some applications of Theorem 3.1.

#### 11.1. Gersten conjecture for additive higher Chow groups.

Let \( X \) be a smooth scheme over \( k \). For a reasonable functor \( F \) defined on \( k \)-schemes, the Gersten conjecture for the functor \( F \) is whether the Cousin complex of \( F \) is exact. Such results were proven for the higher algebraic \( K \)-theory on \( \text{Sm}_k \) by Quillen [37] using a sort of presentation lemma, and since then it is known that various functors satisfy the Gersten conjecture. For instance, it was proven for Milnor \( K \)-theory by Kerz [21] and for the de Rham-Witt complex by a result of Gros [13]. We have the following answer for additive higher Chow groups:

**Theorem 11.1.** When \( X \) is a smooth scheme of dimension \( r \) over a perfect field \( k \) of characteristic \( p \neq 2 \), the Gersten conjecture holds for additive higher Chow presheaves on \( X \) in the Milnor range. More precisely, the following complex is exact after Zariski sheafifications:

\[
0 \to \mathcal{TCH}^n(X, n; m) \to \mathcal{TCH}^n(K, n; m) \to \bigoplus_{x \in X^{(1)}} (i_x)_* \mathcal{H}^1_x(\mathcal{TCH}^n(-, n; m)) \to \bigoplus_{x \in X^{(2)}} (i_x)_* \mathcal{H}^2_x(\mathcal{TCH}^n(-, n; m)) \to \cdots
\]
where $K = k(X)$, and $X^{(i)}$ is the set of codimension $i$ points. In particular, for any point $p \in X$, the natural map $\text{TCH}^n(\mathcal{O}_{X,p}, n; m) \to \text{TCH}^n(K, n; m)$ is injective.

Proof. This follows from Theorem 3.1, because it implies that we have an isomorphism of Zariski sheaves $\mathcal{TCH}^n(-; n; m)_{\text{Zar}} \cong \mathbb{W}_m \Omega^{n-1}_{(-)_{\text{Zar}}}$ on $\text{Sm}_k$, and the Cousin complex of the big de Rham-Witt complex on a smooth scheme is a flasque resolution. In characteristic 0, this is just a direct product of couplets of the Cousin flasque resolution of the absolute Kähler differential forms $\Omega^{n-1}_X$, and in characteristic $p > 2$, it is just a direct product of the Cousin flasque resolutions of $p$-typical de Rham-Witt forms $W_i \Omega^{n-1}_X$ ([13 Proposition 5.1.2]).

Remark 11.2. Using a Gabber-style presentation lemma, in [7, Corollary 6.2.4] it was proven that any reasonable cohomology functors satisfying the Nisnevich descent property (COH1) and the projective bundle formula (COH5) must satisfy the Gersten conjecture. By [24, Theorem 3.2] (see also [23, Theorem 5.6]), we have the latter for additive higher Chow groups, but the authors yet do not know if the Nisnevich descent property holds.

11.2. Trace maps for big de Rham-Witt forms. In Proposition 10.5 we saw that for a simple extension $R \subset S$ of regular semi-local algebras essentially of finite type over a perfect field $k$ of characteristic $\neq 2$, all forms in $\mathbb{W}_m \Omega^{n-1}_S$ are traceable to $R$ for $m, n \geq 1$. We used it to prove Theorem 3.1 but this in turn allows us to construct the trace map on de Rham-Witt forms on all finite extensions of smooth algebras essentially of finite type over $k$. This answers a question of L. Hesselholt:

**Theorem 11.3.** Let $k$ be a perfect field of characteristic $p \neq 2$. Let $R \subset R'$ be any finite extension of regular $k$-algebras essentially of finite type. Then, there exists a trace map $\text{Tr}_{R'/R} : \mathbb{W}_m \Omega^n_R \to \mathbb{W}_m \Omega^n_R$.

If $R \subset R' \subset R''$ are finite extensions of regular $k$-algebras essentially of finite type, then, we have $\text{Tr}_{R''/R} = \text{Tr}_{R'/R} \circ \text{Tr}_{R'/R''}$.

Moreover, there is a commutative diagram

$$
\begin{array}{ccc}
\mathbb{W}_m \Omega^{n-1}_{R'} & \xrightarrow{\text{Tr}_{R'/R}} & \mathcal{TCH}^n(R', n; m) \\
\text{Tr}_{R'/R} \downarrow & & \downarrow f_* \\
\mathbb{W}_m \Omega^{n-1}_R & \xrightarrow{\text{Tr}_{R'/R}} & \mathcal{TCH}^n(R, n; m).
\end{array}
$$

Proof. Let $X = \text{Spec}(R)$, $Y = \text{Spec}(R')$, and $f : Y \to X$ be the associated finite map for the extension $R \subset R'$. We have canonical maps $\mathbb{W}_m \Omega^{n-1}_R \overset{\text{Tr}_{R'/R}}{\to} \mathcal{TCH}^n(R, n; m) \to H^0_{\text{Zar}}(X, \mathcal{TCH}^n(-; n; m)_{\text{Zar}})$, and similarly for $R'$. The push-forward map on the additive higher Chow groups yields the corresponding map of presheaves $f_*([\mathcal{TCH}^n(-; n; m)_{Y}]) \to \mathcal{TCH}^n(-; n; m)_{X}$, and taking the Zariski sheafifications and the global sections, we get $f_* : H^0_{\text{Zar}}(Y, \mathcal{TCH}^n(Y, n; m)_{\text{Zar}}) \to H^0_{\text{Zar}}(X, \mathcal{TCH}^n(-; n; m)_{\text{Zar}})$. On the other hand, by Section 2.6 we have a morphism of Zariski sheaves $\mathbb{W}_m \Omega^{n-1}_{(-)_{\text{Zar}}} \to \mathcal{TCH}^n(-; n; m)_{\text{Zar}}$ on $\text{Sm}_{\text{Aff}}^{\text{ess}}_k$, which is an isomorphism by Theorem 3.1. Hence, if we let $\mathbb{W}_m \Omega^{n-1}_X$ denote the Zariski sheaf $\mathbb{W}_m \Omega^{n-1}_{(-)_{\text{Zar}}} |_X$ (similarly, for $Y$), then we have a push-forward $f_* : H^0_{\text{Zar}}(Y, \mathbb{W}_m \Omega^{n-1}_Y) \to H^0_{\text{Zar}}(X, \mathbb{W}_m \Omega^{n-1}_X)$. But, the correspondence $X \mapsto \mathbb{W}_m \Omega^{n-1}_X$ is a quasi-coherent
sheaf of \( \mathbb{W}_m \mathcal{O}_X \)-modules (see [17] §5, for instance), so that the map \( \mathbb{W}_m \Omega^n_{R,1} \to H^0_{Zar}(X, \mathbb{W}_m \Omega^n_{X,1}) \) is an isomorphism. Similarly, \( \mathbb{W}_m \Omega^n_{R,1} \to H^0_{Zar}(Y, \mathbb{W}_m \Omega^n_{Y,1}) \) is an isomorphism. Hence, \( f_* \) uniquely defines a map, denoted \( \text{Tr}_{R'/R} : \mathbb{W}_m \Omega^n_{R,1} \to \mathbb{W}_m \Omega^n_{R,1} \). By construction, commutativity of the diagram \((11.1)\) holds.

For a tower of finite extensions \( R \subset R' \subset R'' \), we repeat the above procedure to \( g : \text{Spec}(R'') \to \text{Spec}(R') \) and \( f \circ g : \text{Spec}(R'') \to \text{Spec}(R) \), and combining with the transitivity of proper push-forwards of additive higher Chow cycles, we immediately deduce that \( (f \circ g)_* = f_* \circ g_* \). \( \square \)

**Remark 11.4.** Sometime around 2013, the second named author JP was told by K. Rülling that, using [8], [9] and the duality machine in [14], one can prove the existence of the trace map abstractly. On the other hand, S. Kelly contacted JP that using his thesis [22], one can also deduce the existence of the trace map. We believe these constructions should coincide with the one in Theorem \((11.3)\). In fact, by modifying the double induction argument of Proposition \((10.5)\) a bit, one may show the uniqueness of the trace map subject to the compatibility of \((11.1)\).

### 11.3. Comparison with crystalline cohomology.

#### 11.3.1. \( p \)-typicalization.

Let \( R \) be a \( \mathbb{Z}(p) \)-algebra for a prime number \( p \). Recall that there is a projection \( \mathbb{W}(R) \to W(R) \), where \( W(R) \) is the \( p \)-typical Witt-ring of \( R \). This projection is given by

\[
\pi := \sum_{n \in \mathbb{Z}(p)} \frac{\mu(n)}{n} V_n F_n,
\]

where \( I(p) \) is the set of integers \( \geq 1 \) not divisible by \( p \), and \( \mu(n) \) is the Möbius function. In fact, for any restricted Witt-complex, we can also define it using the same formula:

**Lemma 11.5.** Let \( R \) be a \( \mathbb{Z}(p) \)-algebra and let \( E = \{E_m\} \) be a restricted Witt-complex over \( R \). Let \( \mathbb{E} := \lim_{\leftarrow m} E_m \). Let \( \pi := \sum_{n \in I(p)} \frac{\mu(n)}{n} V_n F_n \). Then, \( \pi \) is a projection on \( \mathbb{E} \), that is, \( \pi^2 = \pi \). Furthermore, \( \pi(\mathbb{E}) = \bigcap_{n \in I(p), n > 1} \ker F_n \).

**Proof.** This is proven formally using the identities satisfied by \( F_r \) and \( V_r \), following exactly the same argument of [4, Proposition (3.1)]]. We omit the details. \( \square \)

**Definition 11.6.** Let \( R \) be a \( \mathbb{Z}(p) \)-algebra, and let \( E \) be a restricted Witt-complex over \( R \). We let \( E(p) := \pi(\mathbb{E}) \) using \( \pi \) of \((11.2)\). We will call it the \( p \)-typical part or \( p \)-typicalization of \( E \). If we are given a presheaf of restricted Witt-complexes, we define its \( p \)-typicalization similarly.

When \( k \) is a perfect field of characteristic \( p > 2 \), we saw in Section 2.4 that \( \{TCH^M(\_; m)\} \) is a presheaf of restricted Witt-complexes over \( k \). By applying the projector \( \pi \), we define

**Definition 11.7.** The \( p \)-typical additive higher Chow presheaf in Milnor range is

\[
TCH^M(p)(\_; p^\infty) := \pi(\lim_{\leftarrow m} TCH^M(\_; m)),
\]
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\[ \mathcal{TCH}_\mathbf{M}(−; p^i) := \mathcal{TCH}_\mathbf{M}(−; p^\infty)/p^i. \]

For a Grothendieck topology \( \tau \) on \( \mathbf{Sm}_k \) or \( \mathbf{Sch}_k \), we also define a \( \tau \)-sheaf \( \mathcal{TCH}_\mathbf{M}(-; p^i)_\tau \), where the subscript \( \tau \) denotes the \( \tau \)-sheafification.

Apply the projector \( \pi \) to \( \lim ←_m \) of the isomorphism of Zariski sheaves \( \mathcal{TCH}_\mathbf{M}(−; m)_{\text{Zar}} ≃ \mathbb{W}_mΩ^\bullet_{−;} \) on \( \mathbf{Sm}_k \) of Corollary 10.8. Then, we have:

**Corollary 11.8.** Let \( k \) be a perfect field of characteristic \( p > 2 \). Then, we have a quasi-isomorphism of Zariski sheaves \( \mathcal{TCH}_\mathbf{M}(−; p^i)_{\text{Zar}} ≃ \text{q.iso } W_iΩ^\bullet_{−;} \) on \( \mathbf{Sm}_k \).

**Proof.** By definition, \( \mathcal{TCH}_\mathbf{M}(−; p^\infty)_{\text{Zar}} ≃ WΩ^\bullet_− \). By [18, Proposition 3.2, p. 568], we have \( W_iΩ^\bullet = WΩ^\bullet_−/\text{Fil}^iWΩ^\bullet_− \), where the filtration \( \text{Fil}^iWΩ^\bullet_− \) is \( \text{ker}(WΩ^\bullet_− \to W_iΩ^\bullet_−) \) as given in [18, (3.1.2), p. 568], and by [18, Corollaire 3.17, p. 577], the map \( WΩ^\bullet_−/p^i \to W_iΩ^\bullet_− \) is a quasi-isomorphism. Hence, we obtain the result. \( \square \)

Recall that the results of Bloch [4, Theorem (0.1)] and Illusie [18, II, Th. 1.4] (via the comparison between these given in [18, §5]), when \( X \) is a smooth proper scheme over a perfect field of characteristic \( p > 0 \), we have an isomorphism \( H^n_{\text{crys}}(X/W) \simeq \lim ←_i H^n_{\text{Zar}}(X, (W_iΩ^\bullet)_{\text{Zar}}) \), which computes the \( n \)-th crystalline cohomology \( H^n_{\text{crys}}(X/W) \) of Berthelot [3], in terms of Zariski hypercohomology groups of Zariski sheaves of DGAs. Combining it with Corollary 11.8, we deduce the following “computation” of crystalline cohomology in terms of groups originating from algebraic cycles:

**Theorem 11.9.** Let \( X \) be a smooth proper scheme over a perfect field \( k \) of characteristic \( p > 2 \). Then, for all \( n \geq 0 \), we have an isomorphism

\[ H^n_{\text{crys}}(X/W) \simeq \lim ←_i H^n_{\text{Zar}}(X, \mathcal{TCH}_\mathbf{M}(−; p^i)_{\text{Zar}}). \]

We won’t recall the definition of the crystalline cohomology \( H^n_{\text{crys}} \), because we do not really need to know it for the proof of the above; we are accessing to this group only via the theorems of Bloch and Illusie. The interested reader should see [3] as well as [4] and [18] for details of the definitions. We wonder what new information could be obtained from this description of crystalline cohomology by groups originating from cycles.

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