A PDE approach for open-loop equilibriums in time-inconsistent stochastic optimal control problems

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Abstract

This paper studies open-loop equilibriums for a general class of time-inconsistent stochastic control problems under jump-diffusion SDEs with deterministic coefficients. Inspired by the idea of Four-Step-Scheme for forward-backward stochastic differential equations with jumps (FBSDEJs, for short), we derive two systems of integro-partial differential equations (IPDEs, for short). Then, we rigorously prove a verification theorem which provides a sufficient condition for open-loop equilibrium strategies. As an illustration of the general theory, we discuss a mean-variance portfolio selection problem under a jump-diffusion model.

Keywords: Time-inconsistency, stochastic control, open-loop equilibrium control, Poisson point process, parabolic integro-partial differential equations, mean-variance problem.

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1 Introduction

Traditional optimal control problems are time-consistent in the sense that an optimal control constructed for a given initial pair of time and state will remain optimal thereafter. Time-consistency provides a powerful approach to solving optimal control problems, namely, the method of dynamic programming. The basic idea of this method is to consider a family of (time-consistent) optimal control problems with different initial times and states, to establish relationships among these problems via the so-called Hamilton-Jacobi-Bellman equation (HJB, for short). If the HJB equation is solvable, then one can obtain an optimal control in closed-loop form by taking the maximizer/minimizer of the Hamiltonian involved in the HJB equation; see e.g. the textbook [49] for a detailed discussion.

However, there are growing evidences which tend to support that the time-consistency could be lost in real world situations (i.e., an optimal control selected at a given moment might not remain optimal at later time moments). Among many possible reasons causing the time-inconsistency, there are two playing some essential roles: (i) People are impatient about choices in the short term but are patient when choosing between long-term alternatives (see e.g. [27]); (ii) and people’s attitude regarding risks is subjective rather than objective, meaning that, different (group of) people will have different opinions on the risks that contained in some coming event (see e.g. [44]). Mathematically, the first situation can be described by the general discounting, and the second situation can lead to a certain nonlinear appearance of conditional expectations for the state process and/or control process in the cost functional. Two well-known examples are Merton’s portfolio problem with non-exponential discounting [13] and continuous-time mean-variance (MV, for short) portfolio selection model [52].

The theory of time-inconsistent optimal control problems was pioneered by R.H. Strotz in his work [34] on a general discounting Ramsay problem. It became now very popular, and is an important field of research due to its application in mathematical finance. To deal with the time-inconsistency, Strotz [34] suggested the so-called consistent planning approach. The basic idea of this method is that an action the controller selects at every instant of time is considered as a non-cooperative game against all the actions the controller is going to make in the future. A Nash equilibrium strategy is therefore a decision such that any deviation from it at any time instant will be worse off. Further work along this line in continuous and discrete time can be found in [15], [32], [25], [33] and [26]. However, at that time, the results in continuous-time setting were

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essentially obtained from intuitive descriptions of the Nash equilibrium concept. The first precise definition of the equilibrium strategy in continuous-time setting was introduced in the papers of Ekeland and Lazrak [12] and Ekeland and Pirvu [13], where the source of inconsistency is general discounting. In the series of works carried out by Björk and Murgoci [4] and Björk et al. [6], the authors investigated equilibrium strategies within the class of closed-loop controls for a general class of time inconsistent Markovian problems. Inspired by the discrete case, they first derived in an heuristic way the so-called extended HJB equations (that is a system of three-coupled fully nonlinear parabolic PDEs), and then rigorously proved a verification theorem. Recently, Lindensjö [24] derived rigourously the extended HJB equations without using arguments from the discrete-time case. He and Jiang [19] and Hernández and Possamaï [20] generalized [6] by refining the definition of the closed-loop equilibrium concept. For equilibrium stopping times in time-inconsistent Markovian Problems, see e.g. [7], [8] and references therein.

In the series of works carried out by Yong [[45], [46], [47], [48]], the author studied a general class of time inconsistent optimal control problems. He constructed closed-loop equilibrium strategies in a multi-person differential game framework with a hierarchical structure and introduced the so-called equilibrium HJB equation; a new type of forward-backward Riccati-Volterra equations were also introduced. Yong’s approach has inspired many studies over the last five years. Let us just mention a few: see [36] for controlled systems with random coefficients, see [42], [30] for stochastic systems with Markov regime switching and see [43] for time-inconsistent recursive utilities. More recently, Wang and Yong [41] considered the case where the cost functional is determined by a backward stochastic Volterra integral equation which covers most of the existence literature on the general discounting situation. Dou and Lü [11] dealt with closed-loop equilibrium controls in the framework of linear-quadratic optimal control problems with time-inconsistent cost functionals for stochastic evolution equations in a Hilbert space.

The extended HJB-method and Yong’s differential game approach are extensions of the classical dynamic programming approach for the determination of closed-loop equilibrium strategies. In contrast to the above mentioned literature, Hu et al. ([21], [22]) introduced the concept of open-loop equilibrium control by using a spike variation formulation, which is different from the closed-loop equilibrium concepts. The authors considered a stochastic linear-quadratic model (SLQ, for short) in a non-Markovian framework, where the time-inconsistency arises from the presence of a quadratic term of the expected state as well as a state-dependent term in the objective functional. They used a variational method in the spirit of Peng’s stochastic maximum principle to characterize the existence and uniqueness of the equilibrium control in terms of the solvability of a ”flow” of FBSDEs; we refer the readers to the work of Hamaguchi [16] for a detailed discussion about this new type of FBSDEs. Some recent studies devoted to the open-loop equilibrium concept can be found in [9], [1], [3], [17], [37], [39], [40], [23], [35] and [50]. Specially, Wang [37] discussed open-loop equilibrium controls and their particular closed-loop representations for a class of SLQ problems under mean-field SDEs. Alia [1] investigated open-loop equilibrium controls in general discounting situations. Sun and Guo [35] generalized [22] to the case of SLQ models with random jumps. Wang et al. [39] found the open-loop equilibrium strategy to a mean-variance problem under a non-Markovian regime-switching model. Hamaguchi [17] performed a sophisticated FBSDEs-approach and characterized the unique open-loop equilibrium solution to a general discounting Merton portfolio problem in an incomplete market with random parameters. Finally, in the work of Hamaguchi [18], the author investigated open-loop equilibrium controls in the framework of time-inconsistent stochastic recursive control problems, where the cost functional is defined by the solution to a backward stochastic Volterra integral equation. He managed, by a number of very clever ideas, to derive a necessary and sufficient condition for an open-loop equilibrium control via variational methods.

In this paper, we suggest a new PDEs-approach to characterize open-loop equilibrium controls for a general class of time-inconsistent stochastic control problems with deterministic coefficients. We combine the ideas from the generalized HJB-approaches [[6], [47]] and forward-backward stochastic differential equations with jumps [28]. In the following, we provide a brief outline of our approach:

1. Due to the Markovian structure of our problem\(^1\), we introduce the concept of open-loop equilibrium strategy (see Definition 3.4 below) which can be seen as a closed-loop representation of an open-loop equilibrium control; see e.g. [37] and [40].

2. Combining the spike perturbation of the equilibrium strategy with the idea of Four-Step-Scheme for FBSDEs with jumps introduced in [28], we derive two systems of parabolic integro-partial differential equations; we point out that this type of IPDEs appears for the first time in the literature.

\(^1\)All the coefficients in our model are deterministic functions.
3. We then rigorously prove a verification theorem by which one can construct an open-loop equilibrium strategy by solving the above-mentioned IPDEs.

Recently, Alia [2] investigated open-loop equilibriums for a similar class of time-inconsistent stochastic control problems by using a backward stochastic partial differential equations (BSPDEs, for short) approach. He derived a verification theorem that provides a general sufficient condition for equilibriums via a coupled stochastic system of a single forward SDE and a flow of BSPDEs. Note that the author introduced BSPDEs in his approach while all the involved coefficients are deterministic. In contrast to the above mentioned paper, here we derive a verification theorem by using deterministic IPDEs instead of BSPDEs. Moreover, it is worth mentioning that compared with the existing literature, the previous proposed approach demonstrates several new advantages on the treatment of open-loop equilibrium controls; see Remark 4.5 below for more details.

The plan of the paper is as follows, in the second section, we give necessary notations and some preliminaries on FBSDEs with jumps. In Section 3, we formulate our time-inconsistent stochastic optimal control problem. Section 4 is devoted to the presentation of the verification theorem. In Section 5, we discuss the connections between the PDEs-approach of the present article and the variational approach of Hu et al. ([21], [22]). Finally, in Section 6, we discuss a mean-variance portfolio selection model.

2 Preliminaries

2.1 Notations

Throughout this paper \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) is a filtered probability space such that \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets, \(\mathcal{F}_T = \mathcal{F}\) for an arbitrarily fixed finite time horizon \(T > 0\), and \((\mathcal{F}_t)_{t \in [0,T]}\) satisfies the usual conditions. We assume that \((\mathcal{F}_t)_{t \in [0,T]}\) is generated by a \(d\)-dimensional standard Brownian motion \((W(t))_{t \in [0,T]}\) and an independent Poisson measure \(N\) on \([0,T] \times E\) where \(E \subseteq \mathbb{R}\) \(\setminus \{0\}\). We assume that the compensator of \(N\) has the form \(\mu(ds, de) = \theta(de) ds\) for some positive and finite Lévy measure on \(E\), endowed with its Borel \(\sigma\)-field \(\mathcal{B}(E)\). We write \(\hat{N}(ds, de) = N(ds, de) - \theta(de) ds\) for the compensated jump martingale random measure of \(N\). Obviously, we have \(\mathcal{F}_t = \sigma\left[\int_{(0,s]} A N(dr, de); s \leq t, A \in \mathcal{B}(E)\right] \vee \sigma[W(s); s \leq t] \vee \mathcal{N}\), where \(\mathcal{N}\) denotes the totality of \(\theta\)-null sets, and \(\sigma_1 \vee \sigma_2\) denotes the \(\sigma\)-field generated by \(\sigma_1 \cup \sigma_2\).

We use \(H^t\) to denote the transpose of any vector or matrix \(H\) and \(\chi_A\) to denote the indicator function of the set \(A\). For a function \(f\) we denote by \(\nabla f(x, z)\) (resp. \(\nabla^2 f(x, y)\)) the gradient (resp. the Hessian) of \(f\) with respect to the variables \((x, z) \in \mathbb{R}^n \times \mathbb{R}^n\). Particularly, \(f_x(x, z)\) (resp. \(f_{yy}(x, z)\)) denotes the first derivative (resp. the second derivative) of \(f\) with respect to the variable \(y = x, z\). When \((X(s))_{s \in [t,T]}\) is a càdlàg processes, we define the process \((X_-(s))_{s \in [t,T]}\) by

\[
X_-(t) = X(t) \quad \text{and} \quad X_-(s) = \lim_{r \uparrow s} X(r), \quad \text{for } s \in (t,T].
\]

In addition, we use the following notations for several sets and spaces of processes on the filtered probability space, which will be used later:

- \(D[0,T] = \{(t, s) \in [0, T] \times [0, T] \mid s \geq t\}\).
- \(\mathbb{S}^n\) : the set of \((n \times n)\) symmetric matrices.
- \(L^q(\Omega, \mathcal{F}_t, \mathbb{P}^n)\) : the set of \(\mathbb{R}^n\)-valued, \(\mathcal{F}_t\)-measurable random variables \(\zeta\), with \(\|\zeta\|_{L^q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)} = \mathbb{E}[|\zeta|^q] < \infty\).
- \(L^q(E, \mathcal{B}(E), \mathbb{P}; \mathbb{R}^n)\) : the space of functions \(r : E \rightarrow \mathbb{R}^n\) such that \(r(\cdot)\) is \(\mathcal{B}(E)\) -measurable, with \(\|r(\cdot)\|_{L^q(E, \mathcal{B}(E), \mathbb{P}; \mathbb{R}^n)} = \int_E |r(e)|^q \vartheta(de) < \infty\).
- \(\mathbb{S}^2_{\mathcal{F}}(t, T; \mathbb{R}^n)\) : the space of \(\mathbb{R}^n\)-valued, \((\mathcal{F}_s)_{s \in [t,T]}\)-adapted càdlàg processes \(X(\cdot)\), with \(\|X(\cdot)\|_{\mathbb{S}^2_{\mathcal{F}}(t, T; \mathbb{R}^n)} = \mathbb{E}\left[\sup_{s \in [t,T]} |X(s)|^2\right] < \infty\).
• $\mathcal{L}^p (t, T; \mathbb{R}^{n \times d})$ : the space of $\mathbb{R}^{n \times d}$-valued, $(\mathcal{F}_s)_{s \in [t, T]}$-adapted processes $Q (\cdot)$, with

$$\|Q (\cdot)\|^2_{\mathcal{L}^p (t, T; \mathbb{R}^{n \times d})} = \mathbb{E} \left[ \int_t^T \left( Q (s) \right)^T Q (s) \right] ds < \infty.$$

• $\mathcal{L}^p (t, T; \mathbb{R}^l)$ : the space of $\mathbb{R}^l$-valued, $(\mathcal{F}_s)_{s \in [t, T]}$-predictable processes $u (\cdot)$, with

$$\|u (\cdot)\|^p_{\mathcal{L}^p (t, T; \mathbb{R}^l)} = \mathbb{E} \left[ \int_t^T |u (s)|^p ds \right] < \infty.$$

• $\mathcal{L}^{p, q} (\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ : the space of $\mathbb{R}^n$-valued, $(\mathcal{F}_s)_{s \in [t, T]}$-predictable processes $R (\cdot, \cdot)$, with

$$\|R (\cdot, \cdot)\|^p_{\mathcal{L}^{p, q} (\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})} = \mathbb{E} \left[ \int_t^T \int_E |R (s, e)|^q \vartheta (de) ds \right] < \infty.$$

• $C^{1,2} ([t, T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ : the space of functions $V : [t, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, such that $V$, $V_s$, $\nabla V$ and $\nabla^2 V$ are continuous.

### 2.2 FBSDEs with jumps and IPDEs

As preparations, in this paragraph we investigate the connection between a specific class of FBSDEs with jumps and IPDEs. Let $\mu : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$, $c : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times E \to \mathbb{R}^n$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\bar{F} : \mathbb{R}^n \to \mathbb{R}$ be deterministic measurable functions.

For some fixed $(t_0, \xi, \eta) \in [0, T] \times L^4 (\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^n) \times L^4 (\Omega, \mathcal{F}_{T}, \mathbb{P}; \mathbb{R}^n)$, consider in time interval $[t_0, T]$ the following coupled system of FBSDEs,

\[
\begin{align*}
&dZ (s) = \mu (s, Z (s), Z (s)) ds + \sigma (s, Z (s), Z (s)) dW (s) \\
&\quad + \int_E \bar{c} (s, Z_+ (s), Z_+ (s), e) \tilde{N} (ds, de), \\
&dX (s) = \mu (s, X (s), Z (s)) ds + \sigma (s, X (s), Z (s)) dW (s) \\
&\quad + \int_E \bar{c} (s, X_+ (s), Z_+ (s), e) \tilde{N} (ds, de), \\
&dY (s) = -f (s, X (s), Z (s)) ds + Q (s)^T dW (s) \\
&\quad + \int_{E} R (s, e) \tilde{N} (ds, de), \text{ for } s \in [t_0, T], \\
Z (t_0) = \xi, X (t_0) = \eta, Y (T) = \bar{F} (X (T)).
\end{align*}
\]

We introduce the following assumptions.

**(A1)** The maps $\mu$, $\sigma$ and $c$ are continuous and there exists a constant $C > 0$ such that for all $s \in [0, T]$, $e \in E$, $(x, x', z, z') \in (\mathbb{R}^n)^4$, $q \geq 2$

\[
|\mu (s, x, z) - \mu (s, x', z')| + |\sigma (s, x, z) - \sigma (s, x', z')| \\
+ |\bar{c} (s, x, z, e) - \bar{c} (s, x', z', e)| \leq C (|x - x'| + |z - z'|)
\]

and

\[
|\mu (s, x, z)| + |\sigma (s, x, z)| + \left( \int_E |\bar{c} (s, x, z, e)|^q \vartheta (de) \right)^{\frac{1}{q}} \leq C (1 + |x| + |z|).
\]

**(A2)** The maps $f$ and $\bar{F}$ are continuous and quadratic growth on $(x, z)$ uniformly in time, i.e. there exists a constant $C > 0$ such that for all $(s, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

\[
|f (s, x, z)| \leq C \left( 1 + |x|^2 + |z|^2 \right)
\]

and

\[
|\bar{F} (x)| \leq C \left( 1 + |x|^2 \right).
\]
Under Assumptions \((A1)-(A2)\), it is not difficult to verify that the forward SDEs in \((2.1)\) are uniquely solvable in \((Z(\cdot), X(\cdot)) \in S^2_t (t_0, T; \mathbb{R}^n)^2\), and the BSDEs in \((2.1)\) admits a unique adapted solution of FBSDEs \((2.1)\) are closely linked to the following linear integro-partial differential equation:

\[
0 = \Theta_z(s, x, z) + \langle \Theta_x(s, x, z), \mu(s, x, z) \rangle + \langle \Theta_x(s, x, z), \hat{\mu}(s, x, z) \rangle + \frac{1}{2} \text{tr} \left[ \sigma(s, x, z) \Theta_{xx}(s, x, z) \right] + \frac{1}{2} \text{tr} \left[ \hat{\sigma}(s, x, z) \Theta_{zz}(s, x, z) \right] + \int_E \left\{ \Theta(s, x + \tilde{c}(s, x, z, e), z + \tilde{\sigma}(s, z, e)) - \Theta(s, x, z) \right. \\
- \langle \Theta_x(s, x, z), \tilde{c}(s, x, z, e) \rangle - \langle \Theta_z(s, x, z), \tilde{c}(s, x, z, e) \rangle \left\} \theta(\text{d}e), \right.
\]

for \((s, x, z) \in [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n\),

\[\Theta(T, x, z) = \tilde{F}(x), \text{ for } (x, z) \in \mathbb{R}^n \times \mathbb{R}^n.\]

The following result will be used later.

**Theorem 2.1.** Assume that \((A1)-(A2)\) are satisfied. Let \((X(\cdot), Z(\cdot), Y(\cdot), Q(\cdot), R(\cdot, \cdot))\) be the unique adapted solution of FBSDEs \((2.1)\) and suppose that \(\Theta(\cdot, \cdot, \cdot) \in \mathcal{C}^{1,2,2}([t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})\) is a classical solution to the IPDE \((2.2)\) such that the following hold

\[
\int_{t_0}^T \mathbb{E} \left[ \left| \Theta_z(s, X(s), Z(s)) \right| \right]^2 ds < \infty, \tag{2.3}
\]

\[
\int_{t_0}^T \mathbb{E} \left[ \left| \Theta_x(s, X(s), Z(s)) \right| \right]^2 ds < \infty, \tag{2.4}
\]

and

\[
\mathbb{E} \int_{t_0}^T \int_E \left| \Theta(s, X(s) + \tilde{c}_1(s, e), Z(s) + \tilde{c}_2(s, e)) - \Theta(s, X(s), Z(s)) \right|^2 \theta(\text{d}e) ds < \infty. \tag{2.5}
\]

where

\[
\tilde{c}_1(s, e) := \tilde{c}(s, X_-(s), Z_-(s), e), \tag{2.6}
\]

\[
\tilde{c}_2(s, e) := \tilde{c}(s, Z_-(s), Z_-(s), e). \tag{2.7}
\]

Then for a.e. \((s, e) \in [t_0, T] \times E, \text{ a.s.},\)

\[
Y(s) = \Theta(s, X(s), Z(s)),
\]

\[Q(s) = \tilde{\sigma}(s, X(s), Z(s)) \Theta_x(s, X(s), Z(s)) \]

\[
+ \tilde{\sigma}(s, Z(s), Z(s)) \Theta_z(s, X(s), Z(s)),
\]

\[
R(s, e) = \Theta(s, X-(s) + \tilde{c}_1(s, e), Z-(s) + \tilde{c}_2(s, e)) - \Theta(s, X(s), Z(s)).
\]

Before presenting a proof of the above theorem we first recall a version of Itô’s formula to jump-diffusion processes; see e.g. [31], Theorem 1.16.

**Lemma 2.2.** Let \(X_1(\cdot), X_2(\cdot) \in S^2_\mathcal{F} (t_0, T; \mathbb{R}^n)\) be two processes of the form

\[
dX_i(s) = b_i(s) ds + \sigma_i(s) dW(s) + \int_E c_i(s, e) \tilde{N}(ds, de), \text{ for } i = 1, 2,
\]

where \(b_i(\cdot) \in \mathcal{L}^1_\mathcal{F} (t_0, T; \mathbb{R}^n), \sigma_i(\cdot) \in \mathcal{L}^2_\mathcal{F} (t_0, T; \mathbb{R}^{n \times d})\) and \(c_i(\cdot, \cdot) \in \mathcal{L}^{d \times q}_{\mathcal{F}, P} ([t_0, T] \times E; \mathbb{R}^n).\) Suppose that \(V(\cdot, \cdot, \cdot) \in \mathcal{C}^{1,2,2}([t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}).\) Then for all \(s \in [t_0, T], \text{ a.s.,}\)

\[
dV(s, X_1(s), X_2(s)) = \{V_x(s, X_1(s), X_2(s)) + V_z(s, X_1(s), X_2(s)), b_1(s) \}
\]

\[
+ \langle V_x(s, X_1(s), X_2(s)), b_2(s) \rangle + \frac{1}{2} \text{tr} \left[ \sigma_1(s) \sigma_1(s) \top V_{xx}(s, X_1(s), X_2(s)) \right] \]

\[
+ \frac{1}{2} \text{tr} \left[ \sigma_2(s) \sigma_2(s) \top V_{zz}(s, X_1(s), X_2(s)) \right] + \text{tr} \left[ \sigma_1(s) \top V_{xz}(s, X_1(s), X_2(s)) \right] \sigma_2(s) \]
where $X$ and $Y$.

Proof of Theorem 2.1. First define for $s \in [t, T]$, 
\[
\begin{align*}
\tilde{Y} (s) & := \Theta (s, X (s) , Z (s)) , \\
\tilde{Q} (s) & := \sigma (s, X (s), Z (s))^{\top} \Theta_{z} (s, X (s), Z (s)) \\
& \quad + \sigma (s, Z (s))^{\top} \Theta_{z} (s, X (s), Z (s)),
\end{align*}
\]
and 
\[
\tilde{R} (s, e) := \Theta (s, X_{-} (s) + \bar{c}_{1} (s, e), Z_{-} (s) + \bar{c}_{2} (s, e)) - \Theta (s, X (s), Z (s)),
\]
where $X (\cdot), Z (\cdot)$ are the corresponding solutions to the forward equations in (2.1) and $c_{1} (s, e), c_{2} (s, e)$ are as introduced in (2.6)-(2.7). Applying Itô’s formula to $\Theta (\cdot, X (\cdot), Z (\cdot))$, we get 
\[
d\tilde{Y} (s) = \Theta (s, X (s) , Z (s)) + \langle \Theta_{z} (s, X (s), Z (s)) , \mu (s, X (s), Z (s)) \rangle \\
+ \langle \Theta_{x} (s, X (s), Z (s)) , \mu (s, Z (s), Z (s)) \rangle + \frac{1}{2} \text{tr} \left[ \tilde{\sigma} \tilde{\sigma}^{\top} (s, X (s), Z (s)) \Theta_{xx} (s, X (s), Z (s)) \right] \\
+ \frac{1}{2} \text{tr} \left[ \tilde{\sigma} (s, X (s), Z (s))^{\top} \Theta_{zz} (s, X (s), Z (s)) \right] \\
+ \text{tr} \left[ \Theta (s, X (s) + \bar{c}_{1} (s, e), Z (s) + \bar{c}_{2} (s, e)) - \Theta (s, X (s), Z (s)) \right] \\
- \langle \Theta_{x} (s, X (s), Z (s)), \bar{c}_{1} (s, e) \rangle - \langle \Theta_{z} (s, X (s), Z (s)), \bar{c}_{2} (s, e) \rangle \rangle \text{d} (de) \\
+ \left\{ \Theta_{x} (s, X (s), Z (s)), \bar{c}_{1} (s, e), \Theta_{z} (s, X (s), Z (s)), \bar{c}_{2} (s, e) \right\} \text{d} (de) \\
+ \int_{E} \left\{ \Theta (s, X (s) + \bar{c}_{1} (s, e), Z (s) + \bar{c}_{2} (s, e)) - \Theta (s, X (s), Z (s)) \right\} \tilde{N} (ds, dc) .
\]

On the other hand, it follows from the IPDE (2.2) that 
\[
\begin{align*}
\Theta_{x} (s, X (s), Z (s)) &= - \bar{f} (s, X (s), Z (s)) - \langle \Theta_{z} (s, X (s), Z (s)) , \mu (s, X (s), Z (s)) \rangle \\
- \langle \Theta_{z} (s, X (s), Z (s)) , \mu (s, Z (s), Z (s)) \rangle - \frac{1}{2} \text{tr} \left[ \tilde{\sigma} \tilde{\sigma}^{\top} (s, X (s), Z (s)) \Theta_{xx} (s, X (s), Z (s)) \right] \\
&\quad - \langle \Theta_{z} (s, X (s), Z (s)), \bar{c}_{1} (s, e), \Theta_{z} (s, X (s), Z (s)), \bar{c}_{2} (s, e) \rangle \rangle \text{d} (de) \\
- \langle \Theta_{x} (s, X (s), Z (s)), \bar{c}_{1} (s, e) \rangle - \langle \Theta_{z} (s, X (s), Z (s)), \bar{c}_{2} (s, e) \rangle \rangle \text{d} (de) .
\end{align*}
\]

Invoking this into (2.8), we obtain that $\left( \tilde{Y} (\cdot), \tilde{Q} (\cdot), \tilde{R} (\cdot, \cdot) \right)$ satisfies the following BSDE 
\[
\begin{align*}
\left\{ \\
&d\tilde{Y} (s) = - \bar{f} (s, X (s), Z (s))\text{d}s + \tilde{Q} (s)^{\top} \text{d}W (s) + \int_{E} \tilde{R} (s, e) \tilde{N} (ds, dc) , \\
&\tilde{Y} (T) = \bar{F} (X (T)) .
\end{align*}
\]
Hence, by the uniqueness of the solutions to the BSDE in (2.1), we obtain that
\[ (Y(s),Q(s),R(s,e)) = (\tilde{Y}(s),\tilde{Q}(s),\tilde{R}(s,e)) \text{, a.s., a.e. } (s,e) \in [t_0,T] \times E. \]

3 Formulation of the problem

Given a subset \( U \subset \mathbb{R}^l \), let \( \mu : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n \), \( \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d} \) and \( c : [0,T] \times \mathbb{R}^n \times U \times E \to \mathbb{R}^n \) be three deterministic measurable functions. Consider on the time interval \([0,T]\)

the following controlled stochastic differential equation with jumps

\[
\begin{aligned}
&\begin{cases}
\frac{d}{dt} X_t^{x,u}(\cdot) = \mu(s, X_t^{x,u}(\cdot), u(s)) \, ds + \sigma(s, X_t^{x,u}(\cdot), u(s)) \, dW(s) \\
+ \int_{E} c(s, X_t^{x,u}(\cdot), u(s), e) \, \tilde{N}(ds,de), \quad s \in [0,T],
\end{cases}
\end{aligned}
\]

where \( u : [0,T] \times \Omega \to U \) represents the control process, \( X_t^{x,u}(\cdot) \) is the controlled state process and \( x_0 \in \mathbb{R}^n \) is regarded as the initial state.

As time evolves, we need to consider the following controlled stochastic differential equation starting from the situation \((t,y) \in [0,T] \times \mathbb{R}^n\),

\[
\begin{aligned}
&\begin{cases}
\frac{d}{dt} X_t = \mu(s, X_t, u(s)) \, ds + \sigma(s, X_t, u(s)) \, dW(s) \\
+ \int_{E} c(s, X_t, u(s), e) \, \tilde{N}(ds,de), \quad s \in [t,T],
\end{cases}
\end{aligned}
\]

where \( X(\cdot) = X^{t,y,u(\cdot)}(\cdot) \) denotes its solution. For any initial state \((t,y) \in [0,T] \times \mathbb{R}^n\), in order to evaluate the performance of a control process \( u(\cdot) \), we introduce the cost functional

\[
J(t,y;u(\cdot)) := \mathbb{E}_t \left[ \int_t^T f(t, s, X(s), u(s)) \, ds + F(t, X(T)) \right] + G(t, y, \mathbb{E}_t[\Psi(X(T))]),
\]

where \( \mathbb{E}_t[\cdot] = \mathbb{E}_t[\cdot|\mathcal{F}_t] ; f : [0,T] \times [0,T] \times \mathbb{R}^n \times U \to \mathbb{R} ; F : [0,T] \times \mathbb{R}^n \to \mathbb{R} ; G : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) and \( \Psi = (\Psi_1, \ldots, \Psi_m)^T : \mathbb{R}^n \to \mathbb{R}^m \) are four deterministic measurable functions.

Of course the appearance of \( t, y \) and \( \mathbb{E}_t[\Psi(X(T))] \) in the coefficients \( f, F \) and \( G \) is not just means an extension from mathematical point of view, but also be of great importance in applications. More specifically:

(i) The dependence of \( f(t, s, X(s), u(s)) \) and \( F(t, X(T)) \) on the initial time \( t \) is motivated by general discounting situations; see e.g. \([34], [13], [47], [41] \) and \([43] \).

(iii) In the term \( G(t, y, \mathbb{E}_t[\Psi(X(T))]) \) we have a non linear function \( G \) acting on \((y, \mathbb{E}_t[\Psi(X(T))])\), which can be motivated by the mean-variance criterion with state dependent risk aversion \([5] \).

**Remark 3.1.** (i) Let \( Q : [0,T] \to \mathbb{S}^n, R : [0,T] \to \mathbb{S}^l \) be two deterministic measurable maps and \( h, \bar{h}, h \in \mathbb{S}^n \), \( \mu_1 \in \mathbb{R}^{n \times n}, \mu_2 \in \mathbb{R}^n \) be four matrices. If \( f, F, \Psi \) and \( G \) are of the following form

\[
\begin{aligned}
f(t, s, x, u) := & \frac{1}{2} \left( (Q(s)x,x) + (R(s)u,u) \right), \\
F(t, x) := & \frac{1}{2} \langle h x, x \rangle, \quad \Psi(x) := x \text{ and}
\end{aligned}
\]

\[
G(t, y, \bar{x}) := - \left\{ \frac{1}{2} \bar{h} \bar{x} + \mu_1 y + \mu_2, \bar{x} \right\},
\]

then the cost functional \((3.3)\) becomes the same as \((2.3)\) in \([21] \) (in the case when the coefficients are deterministic).

(ii) In the case when \( G(t, y, \bar{x}) \equiv 0 \), the cost functional \((3.3)\) reduces to the same as \((3.2)\) in \([47] \).

We introduce the following assumptions.
(H1) The maps $\mu$, $\sigma$ and $c$ are continuous and there exists a constant $C > 0$ such that for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $(t, u, e) \in [0, T] \times U \times E$ and $q \geq 2$

$$|\mu(s, x, u) - \mu(s, y, u)| + |\sigma(s, x, u) - \sigma(s, y, u)| + |c(s, x, u, e) - c(s, y, u, e)| \leq C|x - y|$$

and

$$|\mu(s, 0, u)| + |\sigma(s, 0, u)| + \left(\frac{1}{E} \int_{E} |c(s, 0, u, e)^{q} \vartheta(de)\right)^{1/q} \leq C(1 + |u|).$$

(H2) (i) The maps $f$, $h$ are continuous and quadratic growth on $(x, u)$ uniformly in time, i.e. there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^n$ and $(t, u) \in [0, T] \times [0, T] \times U$,

$$|f(t, s, x, u)| \leq C\left(1 + |x|^2 + |u|^2\right),$$

$$|F(t, x)| \leq C\left(1 + |x|^2\right).$$

(ii) The map $G$ is continuously differentiable with respect to $\bar{x}$ and there exists a constant $C > 0$ such that for all $(t, y, \bar{x}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$,

$$|G(t, y, \bar{x})| \leq C\left(1 + |y|^2 + |\bar{x}|^2\right),$$

$$|G_s(t, y, \bar{x})| \leq C\left(1 + |y| + |\bar{x}|\right).$$

(iii) The map $\Psi$ is continuous and linear growth on $x$, i.e.
there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^n$,

$$|\Psi(x)| \leq C\left(1 + |x|\right).$$

Under Assumptions (H1)-(H2), for any initial pair $(t, y) \in [0, T] \times \mathbb{R}^n$ and a control $u(\cdot) \in L_{\mathcal{F}}^{2, p}(t, T; \mathbb{R}^l)$, with $q \geq 2$, the state equation (3.1) admits a unique solution $X(\cdot) = X^{t, y, u(\cdot)}(\cdot) \in \mathcal{S}_X^2(t, T; \mathbb{R}^n)$ (see e.g. [14], pp. 125-126) and the cost functional (3.3) is well-defined. Moreover, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X(s)|^q \right] \leq C(1 + |y|^q).$$

We now introduce the class of admissible controls.

**Definition 3.2** (Admissible control). An admissible control $u(\cdot)$ over $[t, T]$ is a $U$-valued $(\mathcal{F}_s)_{s \in [t, T]}$-predictable process such that

$$\mathbb{E}\left[\int_{t}^{T} |u(s)|^4 \, ds\right] < \infty.$$

The class of admissible controls over $[t, T]$ is denoted by $\mathcal{U}[t, T]$.

We also introduce the class of admissible closed-loop controls.

**Definition 3.3** (Admissible Strategy). A map $\varphi : [0, T] \times \mathbb{R}^n \to U$ is called an admissible strategy if for every $(t, y) \in [0, T] \times \mathbb{R}^n$ the following SDE

$$\begin{cases} dX(s) = \mu(s, X(s), \varphi(s, X(s))) \, ds + \sigma(s, X(s), \varphi(s, X(s))) \, dW(s) \\ + \int_{E} c(s, X_s(s), \varphi(s, X_(s)), e) \, N(ds, de), s \in [t, T], \end{cases}$$

(3.4)

admits a unique strong solution $X(\cdot) = X^{t, y, \varphi(\cdot)}(\cdot) \in \mathcal{S}_X^2(t, T; \mathbb{R}^n)$ and the control process $u^{t, y, \varphi}(\cdot) = \varphi(\cdot, X^{t, y, \varphi}_t(\cdot)) \in \mathcal{U}[t, T]$. The class of admissible strategies is denoted by $\mathcal{S}$.

Subsequently, we will use the notation $\varphi^*(z)$ instead of $\varphi(s, z)$, whenever no confusion arises.

Our stochastic optimal control problem can be stated as follows.

**Problem (N).** For any given initial pair $(t, y) \in [0, T] \times \mathbb{R}^n$, find a $\bar{u}^{t, y}(\cdot) \in \mathcal{U}[t, T]$ such that

$$J(t, y; u^{t, y}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, y; u(\cdot)). \quad (3.5)$$
For a given \((t, y) \in [0, T] \times \mathbb{R}^n\), any \(\hat{u}^{t,y}(\cdot) \in \mathcal{U}[t, T]\) satisfying the above is called a pre-commitment optimal control for Problem (N) at \((t, y)\).

As well documented in the review paper [44], the appearance of the terms \(t\), \(y\) and \(E_t[\Psi(X(T))]\) in the performance functional (3.3) destroys the time-consistency of pre-committed optimal controls of Problem (N): If we find some \(\hat{u}^{t,y}(\cdot)\) satisfying (3.5) for some initial pair \((t, y)\), we do not have

\[
\text{J} \left(s, \hat{X}^{t,y} (s); \hat{u}^{t,y}_{[s,T]}(\cdot) \right) = \inf_{u(\cdot)\in \mathcal{U}[s, T]} \text{J} \left(s, \hat{X}^{t,y} (s); u(\cdot) \right),
\]
in general, for any \(s \in (t, T)\), where \(\hat{X}^{t,y}(\cdot)\) is the corresponding solution of (3.2) with \(u(\cdot) = \hat{u}^{t,y}(\cdot)\) and \(\hat{u}^{t,y}_{[s,T]}(\cdot)\) denotes the restriction of \(\hat{u}^{t,y}(\cdot)\) on time interval \([s, T]\).

Since time-consistency is important for a rational controller, the concept “optimality” needs to be reconsidered in a more sophisticated way. Here we adopt the concept of equilibrium strategy which is, at any \(t \in [0, T]\), optimal “infinitesimally” via spike variation.

Let \(\hat{\varphi}(\cdot, \cdot) \in \mathcal{S}\) be a given admissible strategy and \(\hat{X}^{x_0}(\cdot) = X^{x_0, \hat{\varphi}(\cdot, \cdot)}(\cdot)\) be the corresponding solution of (3.4) with \(\varphi(\cdot, \cdot) = \hat{\varphi}(\cdot, \cdot)\) and \((t, y) = (0, x_0)\). For any \(t \in [0, T]\), \(v \in L^4(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{U})\) and for any \(\varepsilon \in [0, T-t)\) define

\[
u^\varepsilon (s) = \begin{cases} v, & \text{for } s \in [t, t + \varepsilon], \\ \hat{\varphi}^s \left(\hat{X}^{x_0}(s)\right), & \text{for } s \in (t + \varepsilon, T) \end{cases}
\]  (3.6)

It is easy to verify that \(u^\varepsilon (\cdot) \in \mathcal{U}[t, T]\). The following definition is slightly different from the original definition of open-loop equilibrium controls defined by Hu et al. ([21], [22]) in a time-inconsistent LQ control problem.

**Definition 3.4 (Open-Loop Equilibrium Strategy).** A strategy \(\hat{\varphi}(\cdot, \cdot) \in \mathcal{S}\) is called an open-loop equilibrium strategy for Problem (N) if

\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ \text{J} \left(t, \hat{X}^{x_0}(t); v^\varepsilon (\cdot) \right) - \text{J} \left(t, \hat{X}^{x_0}(t); \hat{\varphi}(\cdot, \cdot) \right) \right\} \geq 0,
\]  (3.7)

where \(u^\varepsilon (\cdot)\) is defined by (3.6), for any \(t \in [0, T]\), \(v \in L^4(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{U})\) and

\[
\text{J} \left(t, \hat{X}^{x_0}(t); \hat{\varphi}(\cdot, \cdot) \right) := E_t \left[ \int_t^T f \left(t, s, \hat{X}^{x_0}(s), \hat{\varphi}^s \left(\hat{X}^{x_0}(s)\right) \right) ds + F \left(t, \hat{X}^{x_0}(T)\right) \right] + G \left(t, \hat{X}^{x_0}(t)\right) E_t \left[ \Psi \left(\hat{X}^{x_0}(T)\right) \right]
\]

**Remark 3.5.** The intuition behind this definition is similar to that of [6]: For any \(t \in [0, T]\), on the premise that for every \(s > t\), the optimal decision for the controller at \(s\) is \(\hat{\varphi}^s \left(\hat{X}^{x_0}(s)\right)\), then the optimal choice for the controller at \(t\) is that he/she also uses the decision \(\hat{\varphi}^t \left(\hat{X}^{x_0}(t)\right)\). However, even thought the above-described equilibrium concept concerns closed-loop controls only, there is a fundamental difference between the definition here and the ones in [6] and [47]: indeed, in the above definition, the perturbation of \(\hat{\varphi}(\cdot, \cdot)\) in \([t, t + \varepsilon]\) will not affect the control \(\hat{\varphi}^s \left(\hat{X}^{x_0}(s)\right)\) in \((t + \varepsilon, T)\), which is not the case with closed-loop equilibrium concepts considered in [6] and [47].

**Remark 3.6.** In the work of Yan and Yong [44] the definition of an open-loop equilibrium control \(\hat{u}(\cdot) \in \mathcal{U}[0, T]\) is given by

\[
\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ \text{J} \left(t, \hat{X}(t); u^{t,\varepsilon,u}(\cdot) \right) - \text{J} \left(t, \hat{X}(t); \hat{u}(\cdot) \right) \right\} \geq 0,
\]  (3.8)

where \(u^{t,\varepsilon,u}(\cdot) := u \chi_{[t, t + \varepsilon)}(\cdot) + \hat{u}(\cdot) \chi_{(t + \varepsilon, T)}(\cdot)\) for any \(t \in [0, T]\), \(u \in \mathcal{U}\); \(\hat{X}(\cdot)\) is the corresponding solution of (3.1) with \(u(\cdot) = \hat{u}(\cdot)\). It is easy to see that if \(\hat{\varphi}(\cdot, \cdot) \in \mathcal{S}\) is an open-loop equilibrium strategy then the control process \(\hat{u}^{x_0}(\cdot)\) defined by

\[
\hat{u}^{x_0}(s) := \hat{\varphi}^s \left(\hat{X}^{x_0}(s)\right), \text{ for } s \in [0, T],
\]
is an open-loop equilibrium control. Note that the open-loop equilibrium strategy \(\hat{\varphi}(\cdot, \cdot)\) does not depend on the initial state \(x_0\), while the open-loop equilibrium control \(\hat{u}^{x_0}(\cdot)\) does. It is also worth mentioning that \(\hat{\varphi}(\cdot, \cdot)\) is a generalization of the notion ”closed-loop representation to an open-loop equilibrium control” introduced by Wang [[37], [40]] in the framework of time-inconsistent LQ models.
We close this section with the following definition of a function space that will be used below.

Definition 3.7. Let $\hat{\psi}(\cdot, \cdot) \in \mathcal{S}$ be a given strategy. A function $h(\cdot, \cdot) \in C^{1,2,2}(t, T) \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}$ is said to belong to the space $C^{1,2,2}_{4,4}(t, T) \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}$ if it satisfies the condition: There exists a constant $C > 0$ such that for all $(s, u, z) \in [t, T] \times U \times \mathbb{R}^n \times \mathbb{R}^n$ we have

$$C \left( 1 + |x|^2 + |u|^2 + |\hat{\psi}(z)|^2 + |z|^2 \right)$$

$$\geq [h(s, x, z)] + [h_x(s, x, z)] |\mu(s, x, u)| + |h_x(s, x, z)| |\sigma(s, x, u)|$$

$$+ [h_{xx}(s, x, z)] |\sigma(s, z, \hat{\psi}(z))| + [h_{xx}(s, x, z)] |\sigma(s, x, u)|$$

$$+ \int_E \left| h_x(s, x, z) \right| c(\tau, x, u, e) \vartheta(de) .$$

4 PDEs and Verification Theorem

A large number of existing studies in the literature have examined open-loop equilibrium controls by adopting FBSDEs-approaches. For instance, in the series of papers [21], [9], [17], [37], [39], [40], [23], [35] and [50] the open-loop equilibrium control process $\hat{u}^{x_0}(\cdot)$ is constructed by solving a flow of FBSDEs. Recently, the work [2] investigated open-loop equilibrium controls by using a BSPDEs-approach. Therein, an open-loop equilibrium control can be constructed by solving a system of forward-backward stochastic partial differential equations (FBSPDEs, for short).

Unlike to the above mentioned papers, our aim in the present article is to characterize open-loop equilibrium controls by using a PDEs-approach. The main idea is to introduce two deterministic functions $\hat{\theta}(s, x, z)$ and $\hat{\varphi}(s, x, z)$ as solutions to two systems of parabolic integro-partial differential equations. A verification theorem for open-loop equilibrium strategies can be derived in a natural way via these functions.

As in [2], before providing the precise statement of the main results in this paper, let us begin by establishing some heuristic derivations. Let $\hat{\psi}(\cdot, \cdot) \in \mathcal{S}$ be a given admissible strategy and $X^{x_0}(\cdot)$ be the unique strong solution of the SDE

$$\begin{cases} dX^{x_0}(s) = \mu(s, X^{x_0}(s), \hat{\varphi}(X^{x_0}(s))) ds + \sigma(s, X^{x_0}(s), \hat{\varphi}(X^{x_0}(s))) dW(s) \\
\quad + \int_E c(s, X^{x_0}(s), \hat{\varphi}(X^{x_0}(s)), e) \tilde{N}(ds, de), \quad s \in [0, T], \\
X^{x_0}(0) = x_0. \end{cases} \quad (4.1)$$

Consider the perturbed strategy $u^\varepsilon(\cdot)$ defined by the spike variation (3.6) for some fixed arbitrarily $t \in [0, T], v \in L^4(\Omega, \mathcal{F}_t, \mathbb{P}; U)$ and $\varepsilon \in [0, T - t)$. We would like to determine a suitable expression to the difference

$$\Delta \tilde{J}^\varepsilon(t) := J\left( t, X^{x_0}(t); u^\varepsilon(\cdot) \right) - J\left( t, X^{x_0}(t); \hat{\psi}(\cdot, \cdot) \right),$$

in order to be able to evaluate the limit in (3.7). To this end, let $X^\varepsilon(\cdot)$ be the corresponding solution of $u^\varepsilon(\cdot)$, i.e.

$$\begin{cases} dX^\varepsilon(s) = \mu(s, X^\varepsilon(s), v) ds + \sigma(s, X^\varepsilon(s), v) dW(s) \\
\quad + \int_E c(s, X^\varepsilon(s), v, e) \tilde{N}(ds, de), \quad s \in [t, t + \varepsilon], \\
X^\varepsilon(t) = \hat{X}^{x_0}(t) \end{cases} \quad (4.2)$$

and for each fixed $(s, x, z) \in [t, T] \times \mathbb{R}^4(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n) \times \mathbb{R}^4(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n)$, denote by $\hat{X}^{s,x}(\cdot)$ and $X^{s,x}(\cdot)$
the corresponding solutions of the following coupled system of SDEs:
\[
\begin{align*}
\text{(4.3)}
\begin{cases}
\begin{align*}
\text{d}X_\cdot^s\cdot &= b\left(\tau, \dot{X}^s\cdot (\tau), \dot{\varphi}^s\left(\dot{X}^s\cdot (\tau)\right)\right) \text{d}\tau + \sigma\left(\tau, \dot{X}^s\cdot (\tau), \dot{\varphi}^s\left(\dot{X}^s\cdot (\tau)\right)\right) \text{d}W(\tau) \\
+ \int_E c\left(\tau, \dot{X}^s\cdot (\tau), \dot{\varphi}^s\left(\dot{X}^s\cdot (\tau)\right), \epsilon\right) \tilde{N}(d\tau, d\epsilon),
\end{align*}
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\text{d}X_\cdot^s\cdot (\tau) &= b\left(\tau, X^s\cdot, \varphi^s\left(\dot{X}^s\cdot (\tau)\right)\right) \text{d}\tau + \sigma\left(\tau, X^s\cdot, \varphi^s\left(\dot{X}^s\cdot (\tau)\right)\right) \text{d}W(\tau)
\end{align*}
\]
\[
\dot{X}^s\cdot (s) = z, \quad X^s\cdot (s) = x.
\]
Observe that for each fixed \((s, z) \in [t, T] \times L^4(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n)\), we have
\[
X^s\cdot (\cdot) = \dot{X}^s\cdot (\cdot).
\] (4.4)

By the definition of the cost functional, we have
\[
\begin{align*}
\text{J} \left(t, \dot{X}^{x_0}(t) ; u^\varepsilon(\cdot) \right) &= \mathbb{E}_t \left[\int_t^T f \left(t, \tau, \dot{X}^{x_0}(\tau), u^\varepsilon(\tau)\right) \text{d}\tau + F \left(t, X^\varepsilon(T)\right) \right] \\
&\quad + G \left(t, \dot{X}^{x_0}(t) \right), \mathbb{E}_t \left[\Psi \left(X^\varepsilon(T)\right)\right] \right).
\end{align*}
\] (4.5)
and since \(u^\varepsilon(\tau) \equiv v_{\chi(t, t+\varepsilon)}(\tau) + \dot{\varphi}^t \left(X^{x_0}(\tau)\right)(t+\varepsilon)\), the objective functional of \(u^\varepsilon(\cdot)\) can be written as
\[
\begin{align*}
\text{J} \left(t, \dot{X}^{x_0}(t) ; u^\varepsilon(\cdot) \right) &= \mathbb{E}_t \left[\int_t^T f \left(t, \tau, X^\varepsilon(\tau), u^\varepsilon(\tau)\right) \text{d}\tau + F \left(t, X^\varepsilon(T)\right) \right] \\
&\quad + G \left(t, \dot{X}^{x_0}(t) \right), \mathbb{E}_t \left[\Psi \left(X^\varepsilon(T)\right)\right] \right).
\end{align*}
\]
On the other hand, it follows from the uniqueness of solutions to SDEs (4.1)-(4.2) that for all \(\tau \in [t + \varepsilon, T]\),
\[
X^\varepsilon(\tau) = X^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(\tau), \ a.s.
\]
and
\[
\dot{X}^{x_0}(\tau) = \dot{X}^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(\tau), \ a.s.
\]
where \(\dot{X}^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(\cdot)\) and \(X^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(\cdot)\) are the corresponding solutions of (4.3) with \(s = t + \varepsilon, \ x = X^\varepsilon(t+\varepsilon)\) and \(z = X^{x_0}(t+\varepsilon)\).

Accordingly, the objective functional of \(u^\varepsilon(\cdot)\) can be rewritten as
\[
\begin{align*}
\text{J} \left(t, \dot{X}^{x_0}(t) ; u^\varepsilon(\cdot) \right) &= \mathbb{E}_t \left[\int_t^{t+\varepsilon} f \left(t, \tau, X^\varepsilon(\tau), v\right) \text{d}\tau \right. \\
&\quad + \int_t^T f \left(t, \tau, X^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(\tau), \dot{\varphi}^t \left(X^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(\tau)\right)\right) \text{d}\tau \\
&\quad + F \left(t, \dot{X}^{x_0}(t) \right), \mathbb{E}_t \left[\Psi \left(X^{t+\varepsilon, X^{x_0}(t+\varepsilon)}(T)\right)\right] \right] \right). \quad (4.6)
\end{align*}
\]
Now, if we define for all \((t, s, x, z) \in D[0, T] \times L^4(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n) \times L^4(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n),\)
\[
h^t(s, x, z) := \mathbb{E}_s \left[\int_s^T f \left(t, \tau, X^{s\cdot, x\cdot, z\cdot}(\tau), \dot{\varphi}^s \left(X^{s\cdot, x\cdot, z\cdot}(\tau)\right)\right) \text{d}\tau + F \left(t, X^{s\cdot, x\cdot, z\cdot}(T)\right) \right]. \quad (4.7)
\] \footnote{Observe that the control \(\dot{\varphi}^t \left(X^{x_0}(\tau)\right)\) in \((t+\varepsilon, T)\) is not influenced by the initial pair \((t+\varepsilon, X^\varepsilon(t+\varepsilon))\); this is quite different from the closed-loop control concept adopted by Björk et al. [6].}
Under Assumptions (H1)-(H2) with the boundary conditions we introduce the following system of FBSDEs:

\[ \Delta \hat{J} = h^S \left( t, \hat{X}^x(t), \hat{X}^x(t) \right) + G \left( t, \hat{X}^x(t), \eta \left( t, \hat{X}^x(t), \hat{X}^x(t) \right) \right) \]

and \[ J \left( t, \hat{X}^x(t); \hat{\phi} (\cdot, \cdot) \right) = h^S \left( t, \hat{X}^x(t), \hat{X}^x(t) \right) + G \left( t, \hat{X}^x(t), \eta \left( t, \hat{X}^x(t), \hat{X}^x(t) \right) \right) \]

respectively. Consequently, \[ \Delta \hat{J}^* \left( t \right) \] takes the form

\[ \Delta \hat{J}^* \left( t \right) = E_t \left[ \int_t^{t+\varepsilon} f \left( t, \tau, X^\varepsilon (\tau), v \right) d\tau + \hat{\Delta}_1^\varepsilon + \hat{\Delta}_2^\varepsilon \right] \]

where \[ \Delta_1^\varepsilon := h^S \left( t+\varepsilon, X^\varepsilon (t+\varepsilon), \hat{X}^x(t+\varepsilon) \right) - h^S \left( t, X^\varepsilon (t), \hat{X}^x(t) \right) \]

and \[ \Delta_2^\varepsilon := G \left( t, \hat{X}^x(t), E_t \left[ \eta \left( t+\varepsilon, X^\varepsilon (t+\varepsilon), \hat{X}^x(t+\varepsilon) \right) \right] \right) - G \left( t, \hat{X}^x(t), E_t \left[ \eta \left( t, X^\varepsilon (t), \hat{X}^x(t) \right) \right] \right) \]

However, the above expression of \[ \Delta \hat{J}^* \left( t \right) \] is still very difficult to handle, since the terms \[ \hat{\Delta}_1^\varepsilon \] and \[ \hat{\Delta}_2^\varepsilon \] involved on the right-hand side of (4.9) are too complicated. Thus, we need to explore (4.9) further, in order to be able to evaluate the limit in (3.7). Inspired by the idea of Four Step Scheme introduced in [28] for FBSDEs with jumps, we proceed as follows. For each fixed \( (t, s, x, z) \in D [0, T] \times \mathbb{L}^1 (\Omega, F_s, \mathbb{P}; \mathbb{R}^n) \times \mathbb{L}^1 (\Omega, F_x, \mathbb{P}; \mathbb{R}^n) \), we introduce the following system of FBSDEs:

\[
\begin{align*}
\mathrm{d}\hat{X}^{s,x}(\tau) & = b \left( \tau, \hat{X}^{s,x}(\tau), \hat{\phi}^\tau \left( \hat{X}^{s,x}(\tau) \right) \right) d\tau + \sigma \left( \tau, \hat{X}^{s,x}(\tau), \hat{\phi}^\tau \left( \hat{X}^{s,x}(\tau) \right) \right) dW(\tau) \\
& \quad + \int_{E} c \left( \tau, \hat{X}^{s,x}(\tau), \hat{\phi}^\tau \left( \hat{X}^{s,x}(\tau) \right), e \right) \tilde{N}(ds, de), \\
\mathrm{d}X^{s,x}(\tau) & = b \left( \tau, X^{s,x}(\tau), \phi^\tau \left( X^{s,x}(\tau) \right) \right) d\tau + \sigma \left( \tau, X^{s,x}(\tau), \phi^\tau \left( X^{s,x}(\tau) \right) \right) dW(\tau) \\
& \quad + \int_{E} c \left( \tau, X^{s,x}(\tau), \phi^\tau \left( X^{s,x}(\tau) \right), e \right) \tilde{N}(ds, de), \\
\mathrm{d}Y^{s,x}(\tau; t) & = -f \left( \tau, X^{s,x}(\tau), \phi^\tau \left( X^{s,x}(\tau) \right) \right) d\tau + Q^{s,x}(\tau; t)^\top dW(\tau) \\
& \quad + \int_{E} R^{s,x}(\tau, c, e) \tilde{N}(d\tau, de), \\
\mathrm{d}K^{s,x}(\tau) & = H^{s,x}(\tau) dW(\tau) + \int_{E} \Gamma^{s,x}(\tau, e) \tilde{N}(d\tau, de), \quad \text{for } \tau \in [s, T],
\end{align*}
\]
Note that by virtue of Definition 3.7, it is not difficult to verify that all needed integrability conditions to the above representations are satisfied. Thus, setting \( r = s \) in (4.13)-(4.14), we get

\[
\hat{\theta}^t (s, x, z) = Y (s; t) = h^t (s, x, z)
\]

and

\[
\hat{g} (s, x, z) = K (s) = \eta (s, x, z).
\]

Hence we can rewrite \( \Delta^e_1 \) and \( \Delta^e_2 \) as follows:

\[
\Delta^e_1 := \hat{\theta}^t \left( t + \varepsilon, X^\varepsilon (t + \varepsilon), X^{\varepsilon_0 (t + \varepsilon)} \right) - \hat{\theta}^t \left( t, X^\varepsilon (t), X^{\varepsilon_0 (t)} \right)
\]
and
\[
\Delta_t^\varepsilon := G \left( t, X_{t^0}^\varepsilon (t), E_t \left[ \hat{g} \left( t + \varepsilon, X^\varepsilon (t + \varepsilon), \hat{X}_{t^0}^\varepsilon (t + \varepsilon) \right) \right] \right) - G \left( t, X_{t^0}^\varepsilon (t), \hat{g} \left( t, X^\varepsilon (t), \hat{X}_{t^0}^\varepsilon (t) \right) \right).
\]

Invoking this into (4.9), we obtain that
\[
\Delta_t^\varepsilon (t) = E_t \left[ \int_t^{t+\varepsilon} f \left( t, s, X^\varepsilon (s), v \right) ds \right] + E_t \left[ \frac{1}{2} \text{tr} \left( \begin{pmatrix} \hat{g}_t \left( s, X, Z \right) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z) \end{pmatrix} \sigma \sigma^T \right) \right] + \frac{1}{2} \text{tr} \left( \begin{pmatrix} \hat{g}_s \left( s, X, Z \right) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z) \end{pmatrix} \sigma \sigma^T \right) + \frac{1}{2} \text{tr} \left( \begin{pmatrix} \hat{g}_\tau \left( s, X, Z \right) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z) \end{pmatrix} \sigma \sigma^T \right)
\]

Before going further, let us introduce an \( \mathcal{H} \)-function associated to the 3-tuple \( \left( \hat{\varphi} (\cdot, \cdot), \hat{\theta} (\cdot, \cdot, \cdot), \hat{g} (\cdot, \cdot, \cdot) \right) \) as follows: For each \( (t, s) \in D [0, T], (u, y) \in U \times \mathbb{R}^n \) and for each \( X, Z \in L^4 (\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n) \)
\[
\mathcal{H} (t, y, s, X, Z, u)
:= \left\{ \hat{\theta}^t (s, X, Z) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z), \mu (s, X, u) \right\}
+ \frac{1}{2} \text{tr} \left( \begin{pmatrix} \hat{g}_t \left( s, X, Z \right) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z) \end{pmatrix} \sigma \sigma^T \right) (s, X, u)
+ \text{tr} \left( \begin{pmatrix} \hat{g}_s \left( s, X, Z \right) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z) \end{pmatrix} \sigma \sigma^T \right) (s, X, Z)
+ \int_E \left( \hat{\theta}^t (s, X, c (s, X, u, e), Z + c (s, Z, \hat{\varphi} (Z), e)) - \left( \hat{\theta}^t (s, X, Z), c (s, X, u) \right) \right) d (de)
+ \frac{1}{2} \text{tr} \left( \begin{pmatrix} \hat{g}_t \left( s, X, Z \right) + \sum_{i=1}^m G_x \left( t, y, E_t [\hat{g} (s, X, Z)] \right) \hat{g}_x^i (s, X, Z) \end{pmatrix} \sigma \sigma^T \right) (s, X, Z)
\]
\times \int_E \left( \hat{g}^i (s, X, c (s, X, u, e), Z + c (s, Z, \hat{\varphi} (Z), e)) - \left( \hat{g}_x^i (s, X, Z), c (s, X, u) \right) \right) d (de)
+ \int \left( \hat{g}^i (s, X, c (s, X, u, e), Z + c (s, Z, \hat{\varphi} (Z), e)) - \left( \hat{g}_x^i (s, X, Z), c (s, X, u) \right) \right) d (de)
+ f (t, s, X, u),
\]
where for all \( (t, y, x) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \), \( G_x (t, y, \bar{x}) = \frac{\partial}{\partial y} G (t, y, \bar{x}_1, \bar{x}_2, ..., \bar{x}_m) \) and \( \hat{g}^i (\cdot, \cdot, \cdot) \) the i-th coordinate of \( \hat{g} (\cdot, \cdot, \cdot) \) = \( \left( \hat{g}^1 (\cdot, \cdot, \cdot), ..., \hat{g}^m (\cdot, \cdot, \cdot) \right)^T \).

The above derivation can be summarized as follows.

**Proposition 4.1.** Let \( (H_1)-(H_2) \) hold. Given an admissible strategy \( \hat{\varphi} (\cdot, \cdot) \in \mathcal{S} \), suppose that for each \( t \in [0, T] \), the PDEs (4.11) and (4.12) admit the classical solutions \( \hat{\theta}^t (\cdot, \cdot, \cdot) \) and \( \hat{g} (\cdot, \cdot, \cdot) \), respectively, such that
\[
\hat{\theta}^t (\cdot, \cdot, \cdot) \in C^{1,2,2}_\mathbb{R} ([t, T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})
\]
and
\[
\hat{g} (\cdot, \cdot, \cdot) \in C^{1,2,2}_\mathbb{R} ([0, T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}).
\]

Then for each \( (t, s, x, z) \in D [0, T] \times L^4 (\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n) \times L^4 (\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n) \), \( \hat{\theta}^t (s, x, z) \) and \( \hat{g} (s, x, z) \) have the following probabilistic representations:
\[
\hat{\theta}^t (s, x, z) = E_x \left[ \int_s^T f \left( t, \tau, X^{s,x,z} (\tau), \hat{\varphi} \left( \hat{X}^{s,x,z} (\tau) \right) \right) d \tau + F \left( t, X^{s,x,z} (T) \right) \right]
\]
and
\[
\hat{g} (s, x, z) = E_x \left[ \Psi \left( X^{s,x,z} (T) \right) \right],
\]
respectively, where $\hat{X}^{s,x}(\cdot)$ and $X^{s,X}(\cdot)$ are the corresponding solutions of (4.3). Furthermore, for any $t \in [0,T)$, $v \in L^1(\Omega,\mathcal{F}_t,\mathbb{P};U)$ and for any $\varepsilon \in [0,T-t)$, the following equality holds

$$\begin{align*}
\mathbf{J}\left(t, \hat{X}^{x_0}(t); u^\varepsilon (\cdot)\right) - \mathbf{J}\left(t, \hat{X}^{x_0}(t); \hat{\varphi} (\cdot, \cdot)\right) &= \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \mathcal{H}\left(t, \hat{X}^{x_0}(t), s, X^\varepsilon (s), \hat{X}^{x_0}(s), v\right) - \mathcal{H}\left(t, \hat{X}^{x_0}(t), s, X^\varepsilon (s), \hat{X}^{x_0}(s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \right] ds, \\
\text{where } \mathcal{H} \text{ is as introduced in (4.17), } u^\varepsilon (\cdot) \text{ is defined by (3.6) and } X^\varepsilon (\cdot) \text{ is unique strong solution of (4.2).}
\end{align*}$$

**Proof.** The probabilistic representations in (4.18)-(4.19) follow from Equalities (4.15)-(4.16). So we only need to show (4.20). Consider the difference

$$\begin{align*}
\mathbf{J}\left(t, \hat{X}^{x_0}(t); u^\varepsilon (\cdot)\right) - \mathbf{J}\left(t, \hat{X}^{x_0}(t); \hat{\varphi} (\cdot, \cdot)\right) &= \mathbb{E}_t \left[ \int_t^{t+\varepsilon} f\left(t, s, X^\varepsilon (s), v\right) ds \right] \\
&+ \mathbb{E}_t \left[ \hat{\varphi}^t \left(t + \varepsilon, X^\varepsilon (t + \varepsilon), \hat{X}^{x_0}(t)\right) - \hat{\varphi}^t \left(t, X^\varepsilon (t), \hat{X}^{x_0}(t)\right) \right] \\
&+ G\left(t, \hat{X}^{x_0}(t), \mathbb{E}_t \left[ g\left(t + \varepsilon, X^\varepsilon (t + \varepsilon), \hat{X}^{x_0}(t)\right) \right] \right) \\
&- G\left(t, \hat{X}^{x_0}(t), \hat{g}\left(t, X^\varepsilon (t), \hat{X}^{x_0}(t)\right) \right).
\end{align*}$$

Applying Itô’s formula to $\hat{\varphi}^t \left(t, X^\varepsilon (\cdot), \hat{X}^{x_0}(\cdot)\right)$ on time interval $[t, t+\varepsilon)$ and taking conditional expectations, we obtain that

$$\begin{align*}
\mathbb{E}_t &\left[ \hat{\varphi}^t \left(t + \varepsilon, X^\varepsilon (t + \varepsilon), \hat{X}^{x_0}(t+\varepsilon)\right) - \hat{\varphi}^t \left(t, X^\varepsilon (t), \hat{X}^{x_0}(t)\right) \right] \\
&= \int_t^{t+\varepsilon} \mathbb{E}_t \left[ \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) + \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right), \mu \left(s, X^\varepsilon (s), v\right) \right] ds \\
&+ \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top \left(s, X^\varepsilon (s), v\right) \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) \right] \\
&+ \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top \left(s, \hat{X}^{x_0}(s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) \right] \\
&+ \text{tr} \left[ \sigma \left(s, X^\varepsilon (s), v\right) \hat{\varphi} \left(\hat{X}^{x_0}(s)\right) \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) \right] \\
&+ \int_E \left[ \hat{\varphi}^t \left(s, X^\varepsilon (s) + c(s, X^\varepsilon (s), v, e), \hat{X}^{x_0}(s) + c(s, \hat{X}^{x_0}(s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right), e)\right) \\
&- \hat{\varphi}^t \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) - \hat{\varphi}^t \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right), \mu \left(s, X^\varepsilon (s), v\right) \right] \vartheta (de) ds.
\end{align*}$$

On the other hand, it follows from the PDE (4.11) that

$$\begin{align*}
\hat{\varphi}^t \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) &= - f \left(t, s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) + \hat{\varphi}^t \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right), \mu \left(s, X^\varepsilon (s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \\
&- \hat{\varphi}^t \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right), \mu \left(s, X^\varepsilon (s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \\
&- \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top \left(s, X^\varepsilon (s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) \right] \\
&- \frac{1}{2} \text{tr} \left[ \sigma \sigma^\top \left(s, \hat{X}^{x_0}(s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \hat{\varphi}^t_s \left(s, X^\varepsilon (s), \hat{X}^{x_0}(s)\right) \right] \\
&- \text{tr} \left[ \sigma \left(s, X^\varepsilon (s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \hat{\varphi} \left(\hat{X}^{x_0}(s)\right) \right] \\
&- \text{tr} \left[ \sigma \left(s, X^\varepsilon (s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \hat{\varphi} \left(\hat{X}^{x_0}(s)\right) \right] \\
&- \text{tr} \left[ \sigma \left(s, X^\varepsilon (s), \hat{\varphi} \left(\hat{X}^{x_0}(s)\right)\right) \hat{\varphi} \left(\hat{X}^{x_0}(s)\right) \right].
\end{align*}$$
taking the conditional expectations, we get

Thus, using the system of PDEs (4.22), we get

Invoking this into (4.22), we get

Next, for each $1 \leq i \leq m$, applying Itô’s formula to $\hat{g}^i(\cdot, X^\varepsilon(\cdot), \hat{X}^{x_0}(\cdot))$ on time interval $[t, t + \varepsilon]$, and taking the conditional expectations, we get

Thus, using the system of PDEs (4.12) yields
This completes the proof.
Remark 4.2. For each $t \in [0,T]$ fixed, setting $(s,x,z) = \left(t, \hat{X}^x_0(t), \hat{X}^z_0(t)\right)$ in (4.18)-(4.19), we obtain that
\[
\begin{align*}
\hat{\theta}^t \left(t, \hat{X}^x_0(t), \hat{X}^z_0(t)\right) &= \mathbb{E}_t \left[ \int_t^T f \left(t,s, X^t,X^z(t), \hat{X}^x_0(t)(s), \hat{\varphi}^s \left(\hat{X}^x_0(t)(s)\right)\right) ds + F \left(t,X^t,X^z(t)\right) \right] \\
&= \mathbb{E}_t \left[ \int_t^T f \left(t,s, \hat{X}^z_0(s), \hat{\varphi}^s \left(\hat{X}^z_0(s)\right)\right) ds + F \left(t,\hat{X}^z_0(T)\right) \right]
\end{align*}
\]
and
\[
\hat{g} \left(t, \hat{X}^z_0(t), \hat{X}^z_0(t)\right) = \mathbb{E}_t \left[ \Psi \left(X^t,\hat{X}^z_0(t)\right) \right] = \mathbb{E}_t \left[ \Psi \left(\hat{X}^z_0(T)\right) \right],
\]
where we have used Equality (4.4).

4.1 Verification Theorem

Now, we are ready to state the main result of this work.

Theorem 4.3 (Verification Theorem). Let (H1)-(H2) hold. Given an admissible strategy $\hat{\varphi}(\cdot,\cdot) \in \mathcal{S}$, suppose that for each $t \in [0,T]$, the IPDEs (4.11) and (4.12) admit the classical solutions $\hat{\theta}^t (\cdot,\cdot,\cdot) \in C^{\nu,\mu,\nu\mu}_2([t,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ and $\hat{g} (\cdot,\cdot,\cdot) \in C^{\nu,\mu,\nu\mu}_2([0,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^m)$, respectively, such that the following hold:

(i) $\hat{\varphi}(\cdot,\cdot)$ is a continuous function and there exists a constant $C > 0$ such that for all $(s,z) \in [0,T] \times \mathbb{R}^n$
we have
\[
|\hat{\varphi}^s(z)| \leq C \left(1 + |z|\right).
\]

(ii) $\hat{\theta}^t (\cdot,\cdot,\cdot) \in C^{\nu,\mu,\nu\mu}_2([t,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ and $\hat{g} (\cdot,\cdot,\cdot) \in C^{\nu,\mu,\nu\mu}_2([0,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^m)$.

(iii) For all $(t,z,u) \in [0,T] \times \mathbb{R}^n \times U$,
\[
0 \leq \mathcal{H} (t,z,t,z,z,u) - \mathcal{H} (t,z,t,z,z,\hat{\varphi}^t(z)).
\]

Then $\hat{\varphi}(\cdot,\cdot)$ is an open-loop equilibrium strategy. Furthermore, the objective value of $\hat{\varphi}(\cdot,\cdot)$ at time $t \in [0,T]$ is given by
\[
\mathcal{J} \left(t, \hat{X}^z_0(t); \hat{\varphi}(\cdot,\cdot)\right) = \hat{\theta}^t \left(t, \hat{X}^z_0(t), \hat{X}^z_0(t)\right) + \mathcal{G} \left(t, \hat{X}^z_0(t), \hat{g} \left(t, \hat{X}^z_0(t), \hat{X}^z_0(t)\right)\right).
\]

Proof. In this proof, $C > 0$ denotes a universal constant which may vary from line to line. Let $\hat{\varphi}(\cdot,\cdot) \in \mathcal{S}$ be an admissible strategy for which Conditions (i)-(iii) in Theorem 4.3 hold. We are going to show that $\hat{\varphi}(\cdot,\cdot)$ is an open-loop equilibrium strategy. Consider the perturbed strategy $u^v(\cdot)$ defined by the spike variation (3.6) for some fixed arbitrarily $t \in [0,T]$, $v \in L^4 (\Omega, \mathcal{F}_t, \mathbb{P}; U)$ and $\varepsilon \in [0, T-t]$. Let $X^\varepsilon(\cdot)$ be the unique strong solution of (4.2) and $X^{t,v}(\cdot)$ be the solution of the following SDE
\[
\begin{cases}
\begin{align*}
dX^{t,v}(s) &= b(s, X^{t,v}(s), v) ds + \sigma (s, X^{t,v}(s), v) dW(s) \\
&+ \int_E c \left(s, X^{t,v}(s), v, \varepsilon\right) \tilde{N} (ds, de), \text{ for } s \in [t,T], \\
X^{t,v}(t) &= 0.
\end{align*}
\end{cases}
\]
Under Assumption (H1) the above SDE admits a unique solution $X^{t,v}(\cdot)$ and there exists a constant $C > 0$ such that
\[
\mathbb{E} \left[ \sup_{s \in [t,T]} |X^{t,v}(s)|^4 \right] \leq C \left(1 + \mathbb{E} \left[ |\hat{X}^z_0(t)|^4 \right] \right).
\]
Moreover, since $u^v(s) = v$ on $[t,t+\varepsilon]$, we have
\[
X^\varepsilon(s) = X^{t,v}(s), \text{ a.s., for } s \in [t,t+\varepsilon].
\]

Moreover...
First, we begin by showing that
\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J \left( t, \dot{X}^{x_0}(t); u^{\varepsilon}(\cdot) \right) - J \left( t, \dot{X}^{x_0}(t); \dot{\varphi}(\cdot, \cdot) \right) \right\} = \mathcal{H} \left( t, \dot{X}^{x_0}(t), t, \dot{X}^{x_0}(t), \dot{X}^{x_0}(t), v \right) - \mathcal{H} \left( t, \dot{X}^{x_0}(t), t, \dot{X}^{x_0}(t), \dot{X}^{x_0}(t), \dot{\varphi}(\cdot, \cdot) \right).
\]

To this end, let us introduce in the time interval \([t, T]\) the process \(\Lambda(\cdot; t)\) defined by
\[
\Lambda(s; t) := \mathcal{H} \left( t, \dot{X}^{x_0}(t), s, X^{t,u}(s), \dot{X}^{x_0}(s), \dot{\varphi}(s, \cdot) \right) - \mathcal{H} \left( t, \dot{X}^{x_0}(t), s, X^{t,u}(s), \dot{X}^{x_0}(s), \dot{\varphi}(\cdot, \cdot) \right).
\]

Under Assumptions \((H1)-(H2)\) together with Condition (i), it is easy to see that \(\Lambda(\cdot; t)\) is a right-continuous \((\mathcal{F}_s)_{s \in [t,T]}\)-progressively measurable process. Moreover, it follows from Assumptions \((H1)-(H2)\) together with Condition (ii) and Definition 3.7 that there exists a constant \(C > 0\), such that
\[
|\mathcal{H} (t, y, s, Z, u)| \leq \left| \hat{g}^s_x(s, X, Z) \right| \left| \mu(s, X, u) \right| + \frac{1}{2} \left| \hat{g}^s_{xx}(s, X, Z) \right| \left| \sigma(s, X, u) \right|^2 + \int_E \left| \hat{g}^s_x(s, X, Z) \right| \left| \sigma(s, X, u) \right| \left| \sigma(s, Z, \dot{\varphi}(Z)) \right| + |f(t, s, X, u)|
\]
\[
+ \int_E \left| \hat{g}^s_x(s, X, Z) \right| \left| c(s, X, u, e) \right| \vartheta(de)
\]
\[
+ \frac{1}{2} \left| \hat{g}^s_{xx}(s, X, Z) \right| \left| \sigma(s, X, u) \right|^2 + \left| \hat{g}^s_{xx}(s, X, Z) \right| \left| \sigma(s, X, u) \right| \left| \sigma(s, Z, \dot{\varphi}(Z)) \right|
\]
\[
+ \int_E \left| \hat{g}^s_x(s, X, Z) \right| \left| c(s, X, u, e) \right| \vartheta(de)
\]
\[
\leq C \left( 1 + |X|^2 + |u|^2 + |\varphi(Z)|^2 + |Z|^2 \right)
\]
\[
+ \sum_{i=1}^m C \left[ 1 + |y| + E_t \left[ |X|^2 \right] + |u|^2 + E_t \left[ |\varphi(Z)|^2 \right] + E_t \left[ |Z|^2 \right] \right] \left( 1 + |X|^2 + |u|^2 + |\varphi(Z)|^2 + |Z|^2 \right)
\]
\[
\leq C \left( 1 + |y|^4 + |Z|^4 + E_t \left[ |Z|^4 \right] + |X|^4 + E_t \left[ |X|^4 \right] + E_t \left[ |\varphi(Z)|^4 \right] + |\varphi(Z)|^4 + |u|^4 \right),
\]
for all \((t, s) \in D [0, T]\), \((u, y) \in U \times \mathbb{R}^n\) and for each \(X, Z \in L^4(\Omega, \mathcal{F}_s, \mathbb{P}; \mathbb{R}^n)\), where we have used Jensen inequality together with the inequalities \(a_i^2 \leq \frac{1}{2} (a_i^4 + 1)\), \(a_1 a_2 \leq \frac{1}{2} (a_1^2 + b_2^2)\) and \(\left( \sum_{i=1}^n a_i^2 \right)^2 \leq n \sum_{i=1}^n a_i^4\) for any \(n \geq 2, a_1, \ldots, a_n \in \mathbb{R}\).

Accordingly, we obtain that
\[
|\Lambda(s; t)| \leq C \left( 1 + \left| \dot{X}^{x_0}(t) \right|^4 + |v|^4 + |X^{t,u}(s)|^4 + \left| \dot{X}^{x_0}(s) \right|^4 \right)
\]
\[
+ E_t \left[ \left| \dot{X}^{x_0}(s) \right|^4 \right] + \left| \dot{\varphi}(\dot{X}^{x_0}(s)) \right|^4 + E_t \left[ \left| \dot{\varphi}(\dot{X}^{x_0}(s)) \right|^4 \right]
\]
\[
\leq C \left( 1 + \left| \dot{X}^{x_0}(t) \right|^4 + |v|^4 + |X^{t,u}(s)|^4 + \left| \dot{X}^{x_0}(s) \right|^4 \right)
\]
\[
+ E_t \left[ \left| \dot{X}^{x_0}(s) \right|^4 \right],
\]
for any \(n \geq 2, a_1, \ldots, a_n \in \mathbb{R}\).
where we have used (4.27). Thus noting that \( \sup_{s \in [t; T]} E_t \left[ |X(s)|^4 \right] \leq E_t \left[ \sup_{s \in [t; T]} |X(s)|^4 \right] \), for any \( X(\cdot) \in \mathcal{S}^4_{\epsilon} (t; T; \mathbb{R}^n) \), we get

\[
\mathbb{E} \left[ \sup_{s \in [t; T]} |\Lambda(s; t)| \right] \\
\leq C \left( 1 + \mathbb{E} \left[ |\hat{X}^{x_0} (t)|^4 \right] + \mathbb{E} \left[ |v|^4 \right] + \mathbb{E} \left[ \sup_{s \in [t; T]} |\hat{X}^{t,v} (s)|^4 \right] \right) \\
+ \mathbb{E} \left[ \sup_{s \in [t; T]} |\hat{X}^{x_0} (s)|^4 \right] \leq \infty.
\]

Applying Dominated Convergence Theorem together with (4.20) and (4.30), we get

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J \left( t, \hat{X}^{x_0} (t) ; u^\varepsilon (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t) ; \hat{\varphi} (\cdot, \cdot) \right) \right\} \\
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} \mathbb{E}_t \left[ |\Lambda (s; t)| \right] \, ds \\
= \lim_{s \downarrow t} \mathbb{E}_t \left[ |\Lambda (s; t)| \right] \\
= \Lambda (t; t) \\
= H \left( t, \hat{X}^{x_0} (t) , t, \hat{X}^{x_0} (t) , \hat{X}^{x_0} (t) , v \right) \\
- H \left( t, \hat{X}^{x_0} (t) , t, \hat{X}^{x_0} (t) , \hat{X}^{x_0} (t) , \hat{\varphi} \left( \hat{X}^{x_0} (t) \right) \right).
\]

Hence, in view of (4.29), we obtain that

\[
\lim_{\varepsilon \downarrow 0} \inf \frac{1}{\varepsilon} \left\{ J \left( t, \hat{X}^{x_0} (t) ; u^\varepsilon (\cdot) \right) - J \left( t, \hat{X}^{x_0} (t) ; \hat{\varphi} (\cdot, \cdot) \right) \right\} \\
= H \left( t, \hat{X}^{x_0} (t) , t, \hat{X}^{x_0} (t) , \hat{X}^{x_0} (t) , v \right) - H \left( t, \hat{X}^{x_0} (t) , t, \hat{X}^{x_0} (t) , \hat{X}^{x_0} (t) , \hat{\varphi} \left( \hat{X}^{x_0} (t) \right) \right) \\
\geq 0, \text{ a.s.}
\]

Since \( t \in [0, T] \) and \( v \in L^4 (\Omega, \mathcal{F}_t, \mathbb{P}; U) \) are arbitrary, we deduce that \( \hat{\varphi} (\cdot, \cdot) \) is an equilibrium strategy. The equality in (4.29) is an immediate consequence of (4.25)-(4.26). This completes the proof. \( \blacksquare \)

Let us make some remarks on Theorem 4.3.

**Remark 4.4.** Note that, the verification argument in Theorem 4.3 permits us to construct an open-loop equilibrium strategy by solving the following coupled system of IPDEs

\[
\begin{align*}
0 &= \partial_s \hat{\sigma}_s (s, x, z) + \partial_x \hat{\sigma}_s (s, x, z), \mu (s, x, \hat{\varphi}^s (z)) \\
+ &\frac{1}{2} \text{tr} \left[ \sigma \sigma^T (s, x, \hat{\varphi}^s (z)) \theta^T_{s x} (s, x, z) \right] + \frac{1}{2} \text{tr} \left[ \sigma \sigma^T (s, z, \hat{\varphi}^s (z)) \theta^T_{s z} (s, x, z) \right] \\
+ &\text{tr} \left[ \sigma (s, x, \hat{\varphi}^s (z))^T \hat{\theta}^T_t (s, x, z) \sigma (s, z, \hat{\varphi}^s (z)) \right] + \frac{1}{2} \text{tr} \left[ \sigma \sigma^T (s, z, \hat{\varphi}^s (z)) \theta^T_{z z} (s, x, z) \right] \\
+ &\left\{ \hat{\theta}^T_t (s, x, z) + c(s, x, \hat{\varphi}^s (z), e), z + c(s, z, \hat{\varphi}^s (z), e) \right\} \partial (de), \\
0 &= \hat{g}_1^t (s, x, z) + \hat{g}_1^t (s, x, z), \mu (s, x, \hat{\varphi}^s (z)) \\
+ &\frac{1}{2} \text{tr} \left[ \sigma \sigma^T (s, x, \hat{\varphi}^s (z)) \theta^T_{s x} (s, x, z) \right] + \frac{1}{2} \text{tr} \left[ \sigma \sigma^T (s, z, \hat{\varphi}^s (z)) \theta^T_{z x} (s, x, z) \right] \\
+ &\text{tr} \left[ \sigma (s, x, \hat{\varphi}^s (z))^T \hat{g}^T_{z x} (s, x, z) \sigma (s, z, \hat{\varphi}^s (z)) \right] \\
+ &\left\{ \hat{g}^t (s, x, z) + c(s, x, \hat{\varphi}^s (z), e), z + c(s, z, \hat{\varphi}^s (z), e) \right\} \partial (de), \\
\hat{\varphi}^s (z) &\in \arg \min \mathcal{H} (s, z, s, z, z, \cdot), \text{ for } (s, z) \in [0, T] \times \mathbb{R}^n.
\end{align*}
\]
with the boundary conditions

\[
\hat{\theta}^i (T, x, z) = F (t, x), \quad \hat{g}^i (T, x, z) = \Psi_i (x), \quad \text{for} \quad (t, x, z) \in [0, T] \times (\mathbb{R}^n)^2, \quad \text{for} \quad 1 \leq i \leq m.
\]

To the best of our knowledge, coupled IPDEs of the above form appear for the first time in the literature. However, proving the general existence for these IPDEs remains an outstanding open problem even for the simplest case when \( n = m = d = 1 \).

**Remark 4.5.** Compared with some existing studies in the literature, our PDEs-method demonstrates several new advantages on the treatment of open-loop equilibrium controls. Some of them are listed as follows:

(i) Unlike \([21, 22, 9, 35]\), our approach avoids the complex variation arguments of the second order expansion in the spike variation. Moreover, our results do not require any differentiability assumptions on the involved coefficients, except the function \( G (\cdot, \cdot, \cdot) \).

(ii) By solving the system of IPDEs (4.11)-(4.12), one can derive simultaneously the equilibrium strategy \( \hat{\varphi} (\cdot, \cdot) \), as well as, its corresponding equilibrium objective value \( J \left( t, \bar{X}_x (t) ; \hat{\varphi} (\cdot, \cdot) \right) \), at each \( t \in [0, T] \); this is different from most of the existing methodologies in the literature, since they do not provide the objective value \( J \left( t, \bar{X}_x (t) ; \bar{u} (\cdot) \right) \).

(iii) Our methodology in constructing an open-loop equilibrium strategy is purely deterministic; indeed an equilibrium strategy \( \hat{\varphi} (\cdot, \cdot) \) can be constructed by solving (4.31), which is actually a deterministic coupled system of IPDEs.

(iv) Even thought we are concerned with the open-loop equilibrium framework, our approach looks more like a dynamical programming than a maximum principle.

**Remark 4.6.** Note that some of the above-listed "advantages" are also satisfied by the BSPDEs-approach introduced by Alia \([2]\). However, since Problem \((N)\) is formulated in the Markovian framework, it seems to us that it is more natural to use PDEs, instead of BSPDEs, to characterize open-loop equilibriums.

**Remark 4.7.** The PDEs-approach is closely linked to the strong Markovian property of the controlled process. Thus when the problem is not Markovian, the PDEs-approach does not apply while the variational method of Hu et al. \([21, 22]\) does.

### 5 Connection between the PDEs-approach and the variational approach

As previously mentioned in Introduction section, the idea of defining the equilibrium control within the whole class of open-loop controls goes back to Hu et al. \([21, 22]\), where the authors undertook a deep study of a time-inconsistent stochastic LQ model. They performed a variational method in the spirit of Peng's stochastic maximum principle and constructed an equilibrium solution by solving a "flow" of coupled FBSDEs. In this section, we briefly discuss the connection between the PDEs approach of the present paper and the variational approach of Hu et al. \([21, 22]\). For sake of simplicity and clarity of the presentation, we suppose that \( \Psi (x) \equiv x \) and all the coefficients are assumed to be one dimensional (i.e. \( n = d = l = 1 \)).

Specially, the PDEs (4.11)-(4.12) reduce, respectively, to

\[
\begin{cases}
0 = \hat{\beta}^x_1 (s, x, z) + \hat{\theta}^x_1 (s, x, z) \mu (s, x, \hat{\varphi}^\circ (z)) + \hat{\theta}^x_2 (s, x, z) \mu (s, z, \hat{\varphi}^\circ (z)) + \frac{1}{2} \sigma (s, x, \hat{\varphi}^\circ (z))^2 \hat{\theta}^x_3 (s, x, z) + \frac{1}{2} \sigma (s, z, \hat{\varphi}^\circ (z))^2 \hat{\theta}^x_3 (s, x, z) \\
+ \sigma (s, x, \hat{\varphi}^\circ (z)) \sigma (s, z, \hat{\varphi}^\circ (z)) \hat{\theta}^x_3 (s, x, z) + f (t, s, x, \hat{\varphi}^\circ (z)) \\
+ \int_E \left\{ \hat{\theta}^x_3 (s, x, \hat{\varphi}^\circ (z), e) + c (s, x, \hat{\varphi}^\circ (z) + e) + c (s, z, \hat{\varphi}^\circ (z), e) \right\} \hat{\beta} (de), \\
\hat{\theta}^T (T, x, z) = F (t, x), \quad \text{for} \quad (s, x, z) \in [t, T] \times \mathbb{R} \times \mathbb{R},
\end{cases}
\]

(5.1)
and

\[
\begin{aligned}
0 &= \dot{g}(s, x, z) + \dot{g}_x(s, x, z) \mu(s, x, \varphi^z(z)) + \dot{g}_z(s, x, z) \mu(s, z, \varphi^x(z)) \\
&+ \frac{1}{2} \sigma(s, x, \varphi^z(z))^2 \dot{g}^2_x(s, x, z) + \frac{1}{2} \sigma(s, z, \varphi^x(z))^2 \dot{g}^2_z(s, x, z) \\
&+ \sigma(s, x, \varphi^z(z)) \dot{g}^s_x(s, x, z) \sigma(s, z, \varphi^x(z)) \\
&+ \int_F \left\{ \dot{g}(s, x + c(s, x, \varphi^z(z), e), z + c(s, z, \varphi^x(z), e)) - \dot{g}(s, x, z) \right\} \vartheta(de),
\end{aligned}
\]  
for \((s, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \)

\(\dot{g}(T, x, z) = x, \) for \((x, z) \in \mathbb{R} \times \mathbb{R}. \)

In order to establish the link between our results and the variational approach of Hu et al. ([21], [22]), we first need to derive a version of Peng’s type stochastic maximum principle that characterizes open-loop equilibrium controls. To this end, let \(\varphi^z(\cdot, \cdot) \in \mathcal{S} \) be a fixed admissible strategy and \(X^0(\cdot) \) be the unique strong solution of the SDE (4.1). For some fixed arbitrary \(U \), we put for \(\rho = \mu, \sigma:\)

\[
\begin{aligned}
\rho(t) &= \rho(t, X_0(t), \varphi^t)\left(X^0(t)\right), \ c(t, e) = c\left(t, X_0(t), \varphi^t\right)\left(X^0(t)\right), \\
\rho_x(t) &= \rho_x(t, X_0(t), \varphi^t)\left(X^0(t)\right), \ \rho_{xx}(t) = \rho_{xx}(t, X_0(t), \varphi^t)\left(X^0(t)\right), \\
\delta f(t, s; u) &= f(t, X_0(t), \varphi^t)\left(X^0(t)\right) - f(t, s, X_0(t), \varphi^t)\left(X^0(t)\right), \\
c_x(t, e) &= c_x\left(t, X_0(t), \varphi^t\right)\left(X^0(t)\right), \ c_{xx}(t, e) = c_{xx}(t, X_0(t), \varphi^t)\left(X^0(t)\right), \\
\delta c(t, e; u) &= c\left(t, X_0(t), \varphi^t\right)\left(X^0(t)\right) - c\left(t, X_0(t), \varphi^t\right)\left(X^0(t)\right),
\end{aligned}
\]

We impose the following assumptions.

\((H1^*)\) The maps \(\mu(s, x, u), \sigma(s, x, u) \) and \(c(s, x, u, e) \) are twice continuously differentiable with respect to \(x\).
They and their derivatives in \(x\) are continuous in \((s, x, u)\), and bounded.

\((H2^*)\) (i) The functions \(F(t, x) \) and \(f(t, s, x, u) \) are twice continuously differentiable with respect to \(x\). They and their derivatives in \(x\) are continuous in \((s, x, u)\), and bounded.

(ii) The function \(G(t, y, \bar{x}) \) is twice continuously differentiable with respect to \(\bar{x}\). \(G\) and its derivatives in \(\bar{x}\) are continuous in \(\bar{x}\), and bounded.

We point out that Assumptions \((H1^*)-(H2^*)\) can be substantially relaxed, but we do not focus on this here. For any \(t \in [0, T]\), define in the time interval \([t, T]\) the processes \((p^t(\cdot), q^t(\cdot), r^t(\cdot, \cdot)) \in \mathcal{S}^2\)\((t, T; \mathbb{R}) \times \mathcal{L}^2_\mathbb{F}(t, T; \mathbb{R}) \times \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}([t, T] \times \mathbb{E}) ; \mathbb{R})\) and \((P^t(\cdot), \Phi^t(\cdot), \Upsilon^t(\cdot, \cdot)) \in \mathcal{S}^2\)\((t, T; \mathbb{R}) \times \mathcal{L}^2_\mathbb{F}(t, T; \mathbb{R}) \times \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}([t, T] \times \mathbb{E}) ; \mathbb{R})\) as the solution of the following equations:

\[
\begin{aligned}
dp(s) &= - \left\{ \mu_x(s) p(s) + \sigma_x(s) q^t(s) + \int_{E} c_x(s, e) r(s, e) \vartheta(de) \\
&+ f_x(t, s) \right\} ds + q^t(s) dW(s) + \int_{E} r(s, e) \bar{N}(ds, de), \ s \in [t, T], \\
P^t(T) &= F_x\left(t, X^0(T)\right) + G_x\left(t, X^0(t), \mathbb{E}_t\left[X^0(T)\right]\right),
\end{aligned}
\]

\[
\begin{aligned}
dP(s) &= - \left\{ \left(2 \mu_x(s) + \sigma_x(s)\right)^2 P^t(s) + 2 \sigma_x(s) \Phi^t(s) \\
&+ \int_{E} \left( \Upsilon^t(s, e) + P^t(s) c_x(s, e) \right) ds + c_x(s, e)^2 + 2 c_x(s, e) \Upsilon^t(s, e) \right\} \vartheta(de) \\
&+ \mathbb{E}_{xx}(s, X^0(s), \varphi^s(s)) \left[ 2 \mu_x(s) + \sigma_x(s) \Phi^t(s) \right] ds \\
&+ \Phi^t(s) dW(s) + \int_{E} \Upsilon^t(s, e) \bar{N}(ds, de), \ s \in [t, T], \\
P^t(T) &= F_{xx}\left(t, X^0(T)\right),
\end{aligned}
\]

where the Hamiltonian \(H\) is defined by

\[
H(t, s, x, u, p, q, r(\cdot)) := \mu(s, x, u) p + \sigma(s, x, u) q \\
+ \int_{E} c(s, x, u, e) r(e) \vartheta(de) + f(t, s, x, u),
\]

for any \((t, s, x, u, p, q, r(\cdot)) \in [0, T] \times [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{L}^2(\mathbb{E}, \mathcal{B}(\mathbb{E}), \vartheta; \mathbb{R}).\)
We also need to introduce an $H$-function associated with the family of processes $(\hat{\varphi}(\cdot,\hat{X}^{x_0}(\cdot)),\hat{X}^{x_0}(\cdot))$
\{$(p^t(\cdot),q^t(\cdot),r^t(\cdot,\cdot),P^t(\cdot),\Gamma^t(\cdot,\cdot))_{t\in[0,T]}$\} as follows:
\[
\mathcal{H}(t,s,x,u) := H(t,s,x,u,p^t(s),q^t(s),r^t(s,\cdot)) \\
+ \frac{1}{2} P^t(s) \left( \sigma(s,x,u) - \sigma(s,\hat{X}^{x_0}(s),\hat{\varphi}^s\left(\hat{X}^{x_0}(s)\right)) \right)^2 \\
+ \frac{1}{2} \int_{E} \left( \Upsilon^t(s,e) + P^t(s) \right) \left( e(s,x,u,z) - e(s,\hat{X}^{x_0}(s),\hat{\varphi}^s\left(\hat{X}^{x_0}(s)\right),e) \right)^2 \vartheta(de),
\]
for any $(t,s,x,u) \in D[0,T] \times U \times \mathbb{R}$.

The following theorem is comparable with Theorem 4.3.

**Theorem 5.1** (Stochastic Maximum Principle). Let $(H1^*)-(H2^*)$ hold. Given an admissible strategy $\hat{\varphi}(\cdot,\cdot) \in \mathcal{S}$, let for each $t \in [0,T]$, $(p^t(\cdot),q^t(\cdot),r^t(\cdot,\cdot),\Upsilon^t(\cdot,\cdot)) \in S^2(t,T;\mathbb{R}) \times \mathcal{L}^2_p(t,T;\mathbb{R}) \times \mathcal{L}^0_{p,p}(t,T;\mathcal{E};\mathbb{R})$ and $(P^t(\cdot),\Psi^t(\cdot,\cdot)) \in S^2(t,T;\mathbb{R}) \times \mathcal{L}^2_p(t,T;\mathbb{R}) \times \mathcal{L}^0_{p,p}(t,T;\mathcal{E};\mathbb{R})$ be the unique solutions to the BSDEs (5.3) and (5.4), respectively. Suppose that the following hold:

(i) $\hat{\varphi}(\cdot,\cdot)$ is a continuous function.
(ii) For each $(t,e) \in [0,T] \times E$, $s \to (q^t(s),r^t(s,e),\Upsilon^t(s,e))$ is right-continuous at $s = t$ a.s..
(iii) For each $t \in [0,T]$, there exists a constant $C > 0$ such that
\[
C \geq \mathbb{E} \left[ \sup_{s \in [t,T]} |q^t(s)|^2 \right] + \mathbb{E} \left[ \int_{E} \sup_{s \in [t,T]} |r^t(s,e)|^2 \vartheta(de) \right] \\
+ \mathbb{E} \left[ \int_{E} \sup_{s \in [t,T]} |\Upsilon^t(s,e)|^2 \vartheta(de) \right], \text{ a.s.}
\]
(iv) For any $t \in [0,T]$, 
\[
\hat{H}(t,t,\hat{X}^{x_0}(t),\hat{\varphi}^t(\hat{X}^{x_0}(t))) = \min_{u \in \mathcal{U}} \hat{H}(t,t,\hat{X}^{x_0}(t),u).
\]

Then $\hat{\varphi}(\cdot,\cdot)$ is an open-loop equilibrium strategy.

**Proof.** The proof follows in a similar way as in [2], Theorem 4.1. We omit the details here. \[\Box\]

We are now ready to state the main result of this section. For brevity, we put for $g = \theta^t,g$ 

\[
\begin{align*}
\varrho(s) &= \varrho\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), & \varrho_x(s) &= \varrho_x\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), \\
\varrho_{xx}(s) &= \varrho_{xx}\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), & \varrho_{x}(s) &= \varrho_{x}\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), \\
\varrho_{xxx}(s) &= \varrho_{xxx}\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), & \varrho_{xx}(s) &= \varrho_{xx}\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), \\
\Delta \varrho(s,e) &= \varrho\left(s,\hat{X}^{x_0}(s) + \hat{\varphi}(s,e),\hat{X}^{x_0}(s) + \hat{\varphi}(s,e)\right) - \varrho\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), \\
\Delta \varrho_{x}(s,e) &= \varrho_{x}\left(s,\hat{X}^{x_0}(s) + \hat{\varphi}(s,e),\hat{X}^{x_0}(s) + \hat{\varphi}(s,e)\right) - \varrho_{x}\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), \\
\Delta \varrho_{xx}(s,e) &= \varrho_{xx}\left(s,\hat{X}^{x_0}(s) + \hat{\varphi}(s,e),\hat{X}^{x_0}(s) + \hat{\varphi}(s,e)\right) - \varrho_{xx}\left(s,\hat{X}^{x_0}(s),\hat{X}^{x_0}(s)\right), \\
\end{align*}
\]

where 
\[
\hat{c}(s,e) := c\left(s,\hat{X}^{x_0}(s),\hat{\varphi}^s\left(\hat{X}^{x_0}(s)\right),e\right).
\]

**Theorem 5.2.** Suppose that $(H1^*)-(H2^*)$ hold. Let $\hat{\varphi}(\cdot,\cdot) \in \mathcal{S}$ be a fixed admissible control and $\hat{X}^{x_0}(\cdot)$ be the unique strong solution of the SDE (4.1). Suppose that, for each $t \in [0,T]$, the BSPDEs (5.1)-(5.2) admit two unique classical solutions $\hat{\theta}^t(\cdot,\cdot) \in C^{1,4,4}(t,T;\mathbb{R} \times \mathbb{R};\mathbb{R})$ and $\hat{g}(\cdot,\cdot) \in C^{1,4,4}(t,T;\mathbb{R} \times \mathbb{R};\mathbb{R})$, respectively, such that 
\[
\left(\hat{\theta}^{x_0}(\cdot,\cdot),\hat{\theta}^{xx}(\cdot,\cdot,\cdot)\right) \in C^{2,2,2}(t,T;\mathbb{R} \times \mathbb{R};\mathbb{R})^2
\]
and 
\[
\left(\hat{g}(\cdot,\cdot),\hat{g}_{xx}(\cdot,\cdot,\cdot)\right) \in C^{1,2,2}(0,T;\mathbb{R} \times \mathbb{R};\mathbb{R})^2.
\]
For each $0 \leq t \leq s \leq T$, $e \in E$, define

$$
\begin{align*}
 p^t (s) &:= \hat{\theta}^t_x (s) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \hat{g}_x (s), \\
 q^t (s) &:= \sigma (s) \left( \hat{\theta}^t_{xx} (s) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \hat{g}_{xx} (s) \right) \\
 + \sigma (s) \left( \hat{\theta}^t_{x} (s) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \hat{g}_{x} (s) \right), \\
 r^t (s, e) &:= \Delta \hat{\theta}_x (s, e) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \Delta \hat{g}_x (s, e), \\
 P^t (s) &:= \hat{\theta}^t_{xx} (s) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \hat{g}_{xx} (s), \\
 \Phi^t (s) &:= \sigma (s) \left( \hat{\theta}^t_{x} (s) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \hat{g}_{x} (s) \right) \\
 + \sigma (s) \left( \hat{\theta}^t_{xx} (s) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \hat{g}_{xx} (s) \right), \\
 Y^t (s, e) &:= \Delta \hat{\theta}_x (s, e) + G_x \left( t, X^{x_0} (t), E_t \left[ X^{x_0} (T) \right] \right) \Delta \hat{g}_x (s, e).
\end{align*}
$$

Then $(p^t (\cdot), q^t (\cdot), r^t (\cdot, \cdot))$ satisfies the BSDE (5.3) and $(P^t (\cdot), \Phi^t (\cdot), Y^t (\cdot, \cdot))$ satisfies the BSDE (5.4).


**Proof.** Let $t \in [0, T]$ be fixed. Differentiate equations (5.1)-(5.2) in $x$, we obtain that $\hat{\theta}^t_x (\cdot, \cdot, \cdot)$ and $\hat{g}_x (\cdot, \cdot, \cdot)$ satisfy the following IPDEs:

$$
\begin{align*}
 0 &= \hat{\theta}^t_{xx} (s, x, z) + \hat{\theta}^t_{x} (s, x, z) \mu (s, x, \phi^* (z)) + \hat{\theta}^t_x (s, x, z) \mu_x (s, x, \phi^* (z)) \\
 &+ \hat{\theta}^t_{xx} (s, x, z) \mu (s, x, \phi^* (z)) + \frac{1}{2} \sigma (s, x, \phi^* (z)) \hat{\theta}^t_{xx} (s, x, z) \\
 &+ \sigma (s, x, \phi^* (z)) \sigma_x (s, x, \phi^* (z)) \hat{\theta}^t_{x} (s, x, z) + \frac{1}{2} \sigma (s, x, \phi^* (z))^2 \hat{\theta}^t_{x} (s, x, z) \\
 &+ \sigma (s, x, \phi^* (z)) \sigma (s, z, \phi^* (z)) \hat{\theta}^t_{xx} (s, x, z) + \sigma (s, x, \phi^* (z)) \sigma (s, z, \phi^* (z)) \hat{\theta}^t_{xx} (s, x, z) \\
 &+ \hat{\theta}^t_{x} (s, x, c (s, x, \phi^* (z), e), z + c (s, z, \phi^* (z), e)) + \hat{\theta}^t_{xx} (s, x, c (s, z, \phi^* (z), e)) + \theta (de) + f_x (t, s, x, \phi^* (z)), \\
 &\quad \text{for } (s, x, z) \in [t, T] \times \mathbb{R} \times \mathbb{R}, \\
 \hat{\theta}^t_x (T, x, z) &= F_x (t, x), \text{ for } (x, z) \in \mathbb{R} \times \mathbb{R},
\end{align*}
$$

and

$$
\begin{align*}
 0 &= \hat{g}_x (s, x, z) + \hat{g}_x (s, x, z) \mu (s, x, \phi^* (z)) + \hat{g}_x (s, x, z) \mu_x (s, x, \phi^* (z)) \\
 &+ \hat{g}_x (s, x, z) \mu (s, x, \phi^* (z)) + \frac{1}{2} \sigma (s, x, \phi^* (z)) \hat{g}_x (s, x, z) \\
 &+ \sigma (s, x, \phi^* (z)) \sigma_x (s, x, \phi^* (z)) \hat{g}_x (s, x, z) + \frac{1}{2} \sigma (s, x, \phi^* (z))^2 \hat{g}_x (s, x, z) \\
 &+ \sigma (s, x, \phi^* (z)) \sigma (s, z, \phi^* (z)) \hat{g}_x (s, x, z) + \sigma (s, x, \phi^* (z)) \sigma (s, z, \phi^* (z)) \hat{g}_x (s, x, z) \\
 &+ \hat{g}_x (s, x, c (s, x, \phi^* (z), e), z + c (s, z, \phi^* (z), e)) + \hat{g}_x (s, x, c (s, z, \phi^* (z), e)) + \theta (de), \\
 &\quad \text{for } (s, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \\
 \hat{g}_x (T, x, z) &= 1, \text{ for } (x, z) \in \mathbb{R} \times \mathbb{R}.
\end{align*}
$$

Then applying Itô formula to $\theta^t_x (\cdot, X^{x_0} (\cdot), \hat{X}^{x_0} (\cdot))$, $\hat{g}_x (\cdot, X^{x_0} (\cdot), \hat{X}^{x_0} (\cdot))$ and using the above IPDEs, we get

$$
\begin{align*}
 d\hat{\theta}^t_x (s) &= - \left\{ \mu_x (s) \hat{\theta}^t_x (s) + \sigma_x (s) \sigma (s) \left( \hat{\theta}^t_{xx} (s) + \hat{\theta}^t_{x} (s) \right) \\
 &+ \int_E c_x (s, e) \Delta \hat{\theta}_x (s, e) \theta (de) + f_x (t, s) \right\} ds \\
 &+ \sigma (s) \left( \hat{\theta}^t_{xx} (s) + \hat{\theta}^t_{x} (s) \right) dW (s) + \int_E \Delta \hat{\theta}_x (s, e) \tilde{N} (ds, de), \ s \in [t, T], \\
 \hat{\theta}^t_x (T) &= F_x \left( t, \hat{X}^{x_0} (T) \right)
\end{align*}
$$

and

$$
\begin{align*}
 d\hat{g}_x (s) &= - \left\{ \mu_x (s) \hat{g}_x (s) + \sigma_x (s) \sigma (s) \left( \hat{g}_x (s) + \hat{g}_x (s) \right) \\
 &+ \int_E c_x (s, e) \Delta \hat{g}_x (s, e) \theta (de) \right\} ds \\
 &+ \sigma (s) \left( \hat{g}_x (s) + \hat{g}_x (s) \right) dW (s) + \int_E \Delta \hat{g}_x (s, e) \tilde{N} (ds, de), \ s \in [t, T], \\
 \hat{g}_x (T) &= 1.
\end{align*}
$$
Now define \((p^t (\cdot), q^t (\cdot), r^t (\cdot))\) as in (5.6). Using (5.8)-(5.9), we can easily verify that the 3-tuple of processes \((p^t (\cdot), q^t (\cdot), r^t (\cdot))\) satisfies the following BSDE
\[
\begin{cases}
dp^t (s) = - \{μ_x (s) p^t (s) + σ_x (s) q^t (s) + \int_E c_x (s, e) r^t (s, e) \vartheta (de) + f_x (t, s) \} \, ds \\
+ q^t (s) \, dW (s) + \int_E \bar{r} (s, e) \, \bar{N} (ds, de), \ s \in [t, T], \\
p^t (T) = F_x (t, X^{x_0} (T)) + G_x (t, X^{x_0} (t), E_x \left[ X^{x_0} (T) \right]).
\end{cases}
\]

Hence, by the uniqueness of the solution to (5.3), we obtain that
\[
(p^t (\cdot), q^t (\cdot), r^t (\cdot)) = (p^t (\cdot), q^t (\cdot), r^t (\cdot)), \ \text{a.s., a.e.} \ [s, e] \in [t, T] \times E.
\]

Similarly, we can easily verify that the 3-tuple of processes defined by (5.7) coincides with the solution of BSDE (5.4). This completes the proof. 

\section{A Mean-variance portfolio problem}

As an application of the general theory, we consider a Markowitz mean-variance portfolio selection problem associated to a jump-diffusion model with deterministic coefficients. We apply the verification argument in Theorem 4.3 to derive the equilibrium investment strategy in an explicit form. Note that, this problem was discussed in [35], where the authors derived the open-loop equilibrium solution by solving a flow of FBSDEs with jumps. We emphasize that, we solve this problem for illustrative purposes only. Our goal is to show that the solution we obtain via our verification theorem coincides with the one derived in [21]; which indirectly indicates that our PDEs-approach is correct.

In this section, we assume that \(d = 1\) (i.e., the Brownian motion is one-dimensional). Suppose that there is a financial market, in which two securities are traded continuously. One of them is a bond, with price \(S_0 (s)\) at time \(s \in [0, T]\) governed by
\[
\frac{dS_0 (s)}{S_0 (s)} = r_0 (s) \, ds, \ S_0 (0) = s_0 > 0,
\]
where \(r_0 : [0, T] \to (0, +\infty)\) is a deterministic function which represents the risk-free rate. The other asset is called the risky stock, whose price process \(S_1 (\cdot)\) satisfies the following stochastic differential equation
\[
\frac{dS_1 (s)}{S_1 (s)} = r (s) \, ds + σ (s) \, dW (s) + \int_E φ (s, e) \, \bar{N} (ds, de), \ S_1 (0) = s_1 > 0,
\]
where \(r : [0, T] \to (0, +\infty), \ σ : [0, T] \to (0, +\infty)\) and \(φ : [0, T] \times E \to \mathbb{R}\) are deterministic measurable functions; \(r (\cdot), \ σ (\cdot)\) and \(φ (\cdot, \cdot)\) represent the appreciation rate, the volatility and the jump coefficient of the risky stock, respectively.

We assume that \(r_0 (\cdot), \ r (\cdot), \ σ (\cdot)\) and \(φ (\cdot, \cdot)\) are continuous and uniformly bounded functions, such that \(r (s) > r_0 (s)\) and \(φ (s, e) \geq -1, \ \forall s \in [0, T]\) and \(φ - \text{a.e.} \ e \in E\). We also require a uniform ellipticity condition as follows:
\[
σ (s)^2 + \int_E φ (s, e)^2 \, \vartheta (de) \geq \epsilon, \ \text{for all} \ s \in [0, T],
\]
for some \(\epsilon > 0\).

Starting from an initial capital \(x_0 > 0\), the investor is allowed to invest in both the financial market. A trading strategy is a one-dimensional stochastic process \(u (\cdot)\), where \(u (s)\) represents the amount invested in the risky stock at time \(s \in [0, T]\). The dollar amount invested in the bond at time \(s\) is \(X_{x_0, u(\cdot)} (s) - u (s)\), where \(X_{x_0,u(\cdot)} (\cdot)\) is the wealth process associated with the strategy \(u (\cdot)\) and the initial capital \(x_0\). Then the evolution of \(X_{x_0,u(\cdot)} (\cdot)\) can be described as
\[
\begin{cases}
dX_{x_0,u(\cdot)} (s) = \left( r_0 (s) X_{x_0,u(\cdot)} (s) + u (s) \rho (s) \right) \, ds + u (s) σ (s) \, dW (s) \\
+ u (s) \int_E φ (s, e) \, \bar{N} (ds, de), \ \text{for} \ s \in [0, T],
\end{cases}
\]
where \(\rho (\cdot) := r (\cdot) - r_0 (\cdot)\).
when time evolves, we consider the controlled stochastic differential equation parameterized by \((t,y) \in [0,T] \times \mathbb{R}\),

\[
\begin{aligned}
dX(s) &= \{r_0(s)X(s) + u(s)\rho(s)\} \, ds + u(s)\sigma(s) \, dW(s) \\
&\quad + \int_E u(s)\phi(s,z)\, N(\, ds,\, dz), \text{ for } s \in [t,T], \\
X(t) &= y.
\end{aligned}
\]

(6.1)

A trading strategy \(u(\cdot)\) is said to be admissible over \([t,T]\), if it is a \(\mathbb{R}\)-valued \((\mathcal{F}_s)_{s \in [t,T]}\)-predictable process such that:

\[
\mathbb{E}\left[\int_t^T |u(s)|^4 \, ds\right] < \infty.
\]

For any fixed initial capital \((t,y) \in [0,T] \times \mathbb{R}\), the investor’s aim is to choose an investment strategy \(u(\cdot)\) in order to maximize the conditional expectation of the terminal wealth over the period \([t,T]\), while trying at the same time to minimize the conditional variance of the terminal wealth (i.e. \(\text{Var}_t[X(T)] = \mathbb{E}_t[X(T)^2] - \mathbb{E}_t[X(T)]^2\)). The optimization problem is therefore to minimize

\[
J(t,y,u(\cdot)) := \frac{1}{2}\text{Var}_t[X(T)] - \mu y \mathbb{E}_t[X(T)]
\]

\[
\quad := \frac{1}{2}\mathbb{E}_t[X(T)^2] - \left(\frac{1}{2}\mathbb{E}_t[X(T)]^2 + \mu y \mathbb{E}_t[X(T)]\right)
\]

(6.2)

over \(\mathcal{L}^4_{\mathbb{F}}(t,T;\mathbb{R})\), where \(\mu y\), with \(\mu \geq 0\), denotes the weight between the conditional variance and the conditional expectation.

It is easy to see that, the above model is mathematically a special case of Problem (N) formulated earlier in this paper, with

\[
n = d = m = 1, \quad U = \mathbb{R},
\]

\[
\sigma(s,x,u) \equiv u\sigma(s), \quad c(s,x,u,e) \equiv u\phi(s,e),
\]

\[
\mu(s,x,u) \equiv r_0(s)x + u\rho(s),
\]

\[
f(t,y,s,x,u) \equiv 0, \quad F(t,y,x) \equiv \frac{1}{2}x^2,
\]

\[
\Psi(x) = x \quad \text{and} \quad G(t,y,x) \equiv -\frac{1}{2}x^2 - \mu y x.
\]

Accordingly, the IPDEs associated to an admissible strategy \(\hat{\phi}(\cdot,\cdot)\) are defined as follows:

\[
0 = \hat{\theta}_s(s,x,z) + \hat{\varphi}_x(s,x,z) (r_0(s)x + \hat{\varphi}_z(z)\rho(s))
\]

\[
\quad + \hat{\varphi}_s(z)\sigma(s)
\]

\[
\quad + \frac{1}{2} |\hat{\varphi}_z(z)(\sigma(s))|^2 \left(\hat{\theta}_{xx} + \hat{\theta}_{xz} z + 2\hat{\theta}_{xz}(s,x,z)\right)
\]

\[
\quad + \int_E \left\{ \hat{\theta}(s,x,\hat{\varphi}_z(z)\phi(s,e), z + \hat{\varphi}_z(z)\phi(s,e)) - \hat{\varphi}_z(s,x,z) \right\} \, \vartheta(\, de),
\]

\[
\text{for } (s,x,z) \in [0,T] \times \mathbb{R} \times \mathbb{R},
\]

\[
\hat{\varphi}(T,x,z) = \frac{1}{2}x^2, \text{ for } (x,z) \in \mathbb{R} \times \mathbb{R},
\]

\[
0 = \hat{g}_s(s,x,z) + \hat{g}_x(s,x,z) (r_0(s)x + \hat{\varphi}_z(z)\rho(s))
\]

\[
\quad + \hat{g}_z(z)\sigma(s)
\]

\[
\quad + \frac{1}{2} |\hat{\varphi}_z(z)(\sigma(s))|^2 \left(\hat{g}_{xx} + \hat{g}_{xz} z + 2\hat{g}_{xz}(s,x,z)\right)
\]

\[
\quad + \int_E \left\{ \hat{g}(s,x,\hat{\varphi}_z(z)\phi(s,e), z + \hat{\varphi}_z(z)\phi(s,e)) - \hat{\varphi}_z(s,x,z) \right\} \, \vartheta(\, de),
\]

\[
\text{for } (s,x,z) \in [0,T] \times \mathbb{R} \times \mathbb{R},
\]

\[
\hat{g}(T,x,z) = x, \text{ for } (x,z) \in \mathbb{R} \times \mathbb{R},
\]
and the $\mathcal{H}$-function associated to $\left(\hat{\varphi}(\cdot, \cdot), \hat{\theta}(\cdot, \cdot), \hat{g}(\cdot, \cdot)\right)$ defined in (4.17) takes the form,

$$
\mathcal{H}(t, y, s, X, Z, u) := \frac{1}{2} \left( \hat{\theta}_{xx}(s, X, Z) - (\mu y + E_t [\hat{g}(s, X, Z)]) \right) \hat{\varphi}_{xx}(s, X, Z) \left( \sigma(s) u \right)^2 \\
+ \left( \hat{\theta}_{x}(s, X, Z) - (\mu y + E_t [\hat{g}(s, X, Z)]) \right) \hat{\varphi}_{x}(s, X, Z) \left( r_0(s) x + u \rho(s) \right) \\
+ \left( \hat{\theta}_{zz}(s, X, Z) - (\mu y + E_t [\hat{g}(s, X, Z)]) \right) \hat{\varphi}_{zz}(s, X, Z) \left( \sigma(s) \right)^2 \\
+ \left\{ \hat{\theta}(s, X + u \phi(s, e), z + \varphi(s)(Z)) - \hat{\theta}(s, X, Z) u \phi(s, e) \right\} \vartheta(de) \\
- (\mu y + E_t [\hat{g}(s, X, Z)]) \left\{ \hat{g}(s, X + u \phi(s, e), z + \varphi(s)(Z)) \phi(s, e) \right\} \vartheta(de).
$$

### 6.1 Equilibrium solution

In the next, we derive the equilibrium investment strategy in an explicit form. First, letting for $s \in [0, T]$, 

$$
\kappa(s) := \frac{\rho(s)}{\left( \sigma(s)^2 + \phi(s, e)^2 \vartheta(de) \right)}.
$$

**Theorem 6.1.** The mean-variance portfolio problem in (6.1)-(6.2) has an open-loop equilibrium strategy that can be represented by

$$
\hat{\varphi}^s(z) = \mu e^{\int_s^T r_o(t) dt} \left( 1 + \mu \int_s^T e^{\int_s^T r_o(t) dt} \kappa(t) \rho(t) dt \right)^{-1} \kappa(s) z,
$$

for all $(s, z) \in [0, T] \times \mathbb{R}$.

**Proof.** Assume for the time being that the conditions of Theorem 4.3 hold. The verification argument in Theorem 4.3 leads to the following system of IPDEs,

$$
0 = \hat{\theta}_s(s, x, z) + \hat{\theta}_z(s, x, z) (r_0(s) x + \varphi(s)(z) \rho(s)) + \hat{\theta}_x(s, x, z) (r_0(s) z + \varphi(s)(z) \rho(s)) \\
+ \frac{1}{2} \hat{\varphi}^s(z) \sigma(s)^2 \left( \hat{\theta}_{xx}(s, x, z) + \hat{\theta}_{zz}(s, x, z) + 2 \hat{\theta}_{xz}(s, x, z) \right) \\
+ \left\{ \hat{\theta}(s, x + \varphi(s)(z)) \phi(s, e), z + \varphi(s)(z) \phi(s, e) \right\} \vartheta(de), \\
0 = \hat{\varphi}_s(s, x, z) + \hat{\varphi}_z(s, x, z) (r_0(s) x + \varphi(s)(z) \rho(s)) + \hat{\varphi}_x(s, x, z) (r_0(s) z + \varphi(s)(z) \rho(s)) \\
+ \frac{1}{2} \hat{\varphi}^s(z) \sigma(s)^2 \left( \hat{\varphi}_{xx}(s, x, z) + \hat{\varphi}_{zz}(s, x, z) + 2 \hat{\varphi}_{xz}(s, x, z) \right) \\
+ \left\{ \hat{\varphi}(s, x + \varphi(s)(z)) \phi(s, e), z + \varphi(s)(z) \phi(s, e) \right\} \vartheta(de), \\
$$

with the condition: for all $(s, z) \in [0, T] \times \mathbb{R}$,

$$
\hat{\theta}(T, x, z) = \frac{1}{2} x^2, \hat{g}(T, x, z) = x, \text{ for } (x, z) \in \mathbb{R} \times \mathbb{R},
$$

(6.5)

To solve the above system of IPDEs, we suggest the following ansatz: For all $(s, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$
\hat{\theta}(s, x, z) = M_4(s) \frac{x^2}{2} + M_2(s) \frac{z^2}{2} + M_3(s) x z
$$

To solve the above system of IPDEs, we suggest the following ansatz: For all $(s, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,
Thus, by substituting \( \hat{\theta} (s, x, z) \) and \( \hat{g} (s, x, z) \) together with the above derivatives into (6.5), this leads to

\[
0 = \frac{dM_1 (s)}{ds} x^2 + \frac{dM_2 (s)}{ds} z^2 + \frac{dM_3 (s)}{ds} xx
+ (M_1 (s) x + M_3 (s) z) (r_0 (s) x + \hat{\varphi}^* (z) \rho (s))
+ (M_2 (s) z + M_3 (s) x) (r_0 (s) z + \hat{\varphi}^* (z) \rho (s))
+ \frac{1}{2} (\hat{\varphi}^* (z) \sigma (s))^2 (M_1 (s) + M_2 (s) + 2M_3 (s))
+ \int_{E} \left\{ M_1 (s) \left( x + \hat{\varphi}^* (z) \phi (s, e) \right)^2 + M_2 (s) \left( z + \hat{\varphi}^* (z) \phi (s, e) \right)^2 \right\}
\]

\[
- \phi (s, e) \hat{\varphi}^* (z) \left( \left( M_1 (s) + M_3 (s) \right) x + \left( M_3 (s) + M_2 (s) \right) z \right) \theta (de)
\]

and

\[
0 = \frac{dN_1 (s)}{ds} x + \frac{dN_2 (s)}{ds} z
+ N_1 (s) \left( r_0 (s) x + \hat{\varphi}^* (z) \rho (s) \right)
+ N_2 (s) \left( r_0 (s) z + \hat{\varphi}^* (z) \rho (s) \right)
+ \int_{E} \left\{ N_1 (s) \left( x + \hat{\varphi}^* (z) \phi (s, e) \right) + N_2 (s) \left( z + \hat{\varphi}^* (z) \phi (s, e) \right) \right\}
\]

\[
- N_1 (s) x - N_2 (s) z - \left( N_1 (s) + N_2 (s) \right) \hat{\varphi}^* (z) \phi (s, e) \theta (de).
\]

On the other hand, the minimum condition in (6.6) suggests that: for all \((s, z) \in [0, T] \times \mathbb{R}\),

\[
0 = \mathcal{H}_u (s, z, s, z, z, \hat{\varphi}^* (z))
\]

\[
= \left( \hat{\theta}_{xx} (s, z, z) - (\mu z + \hat{g} (s, z, z)) \hat{g}_{xx} (s, z, z) \right) \sigma (s) \hat{\varphi}^* (z)
+ \left( \hat{\theta}_x (s, z, z) - (\mu z + \hat{g} (s, z, z)) \hat{g}_x (s, z, z) \right) \rho (s)
+ \left( \hat{\theta}_{zz} (s, z, z) - (\mu z + \hat{g} (s, z, z)) \hat{g}_{zz} (s, z, z) \right) \hat{\varphi}^* (z) \sigma (s)^2
+ \int_{E} \left\{ \hat{\theta}_x (s, z + \hat{\varphi}^* (z) \phi (s, e), z + \hat{\varphi}^* (z) \phi (s, e)) \phi (s, e) - \hat{\theta}_x (s, z, z) \phi (s, e) \right\} \theta (de)
\]

\[
- (\mu z + \hat{g} (s, z, z)) \int_{E} \left\{ \hat{g}_x (s, z + \hat{\varphi}^* (z) \phi (s, e), z + \hat{\varphi}^* (z) \phi (s, e)) \phi (s, e) - \hat{g}_x (s, z, z) \phi (s, e) \right\} \theta (de)
\]
and coming back to (6.7)-(6.8), we get: For all \( (s, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \)

\[
0 = \frac{x^2}{2} \left( \frac{dM_1}{ds} (s) + M_1 (s) 2r_0 (s) \right) \\
+ \frac{z^2}{2} \left( \frac{dM_2}{ds} (s) + 2r_0 (s) M_2 (s) + 2 (M_2 (s) + M_3 (s)) \alpha (s) \rho (s) \right) \\
+ (M_1 (s) + M_2 (s) + 2M_3 (s)) \alpha (s) (s) \frac{\sigma^2 (s)}{\sigma^2 (s) + \int_E \phi (s, e)^2 \vartheta (de)} \\
+ zx \left( \frac{dM_3}{ds} + 2r_0 (s) M_3 (s) + (M_1 (s) + M_3 (s)) \alpha (s) \rho (s) \right)
\]

and

\[
0 = \left( \frac{dN_1}{ds} (s) + N_1 (s) r_0 (s) \right) x \\
+ \left( \frac{dN_2}{ds} (s) + r_0 (s) N_2 (s) + (N_1 (s) + N_2 (s)) \alpha (s) \rho (s) \right) z,
\]

which leads to the following systems of ODEs (suppressing \( (s) \)):

\[
\begin{align*}
\frac{dM_1}{ds} &= -2r_0 M_1, \\
\frac{dM_2}{ds} &= -2r_0 M_2 - 2 (M_2 + M_3) \alpha \rho \\
&\quad - (M_1 + M_2 + 2M_3) \alpha^2 \left( \frac{\sigma^2 (s)}{\sigma^2 (s) + \int_E \phi (s, e)^2 \vartheta (de)} \right), \\
\frac{dM_3}{ds} &= -2r_0 M_3 - (M_3 + M_1) \alpha \rho, \\
\frac{dN_1}{ds} &= -r_0 N_1, \\
\frac{dN_2}{ds} &= -r_0 N_2 - (N_1 + N_2) \alpha \rho, \\
M_1 (T) &= N_1 (T) = 1, \\
N_2 (T) &= M_2 (T) = M_3 (T) = 0.
\end{align*}
\] (6.11)

Using the above system of ODEs, it is not difficult to verify that: For all \( s \in [0, T] \),

\[ M_1 (s) - N_1 (s)^2 = 0 \]

and

\[ M_3 (s) - N_1 (s) N_2 (s) = 0. \]
Consequently, it follows from (6.10) that for all \( s \in [0, T] \),
\[
\alpha(s) = \frac{\mu (N_1(s) + N_2(s))^{-1} \rho(s)}{\sigma(s)^2 + \int_E \phi(s,e)^2 \vartheta(de)} = \mu (N_1(s) + N_2(s))^{-1} \kappa(s),
\]
where \( \kappa(s) \) is as introduced in (6.3).

Using the above expressions for \( \alpha(\cdot) \), we can easily solve the system of ODEs (6.11), whose solutions are
\[
N_1(s) = e^{\int_s^T r_0(\tau)d\tau}, \quad N_2(s) = e^{\int_s^T \int_0^\tau r_0(\nu)d\nu \kappa(\tau) \rho(\tau)d\tau}, \quad M_1(s) = e^{2\int_s^T r_0(\tau)d\tau}, \quad M_3(s) = e^{2\int_s^T \int_0^\tau r_0(\nu)d\nu \kappa(\tau) \rho(\tau)d\tau}
\]
and
\[
M_2(s) = e^{\int_s^T (r_0(\tau)d\tau + e^{-\int_s^\tau r_0(\nu)d\nu} \kappa(\tau) \rho(\tau))d\tau}
\]
where, \( \chi(\tau) \equiv 2r_0(\tau) + \left( \frac{N_1(\tau)+N_0(\tau)+\mu}{N_1(\tau)+N_2(\tau)} \right)^2 \kappa(\tau) \rho(\tau) - \kappa(\tau) \rho(\tau). \)

By (6.9) we get
\[
\hat{\varphi}(z) = \mu (N_1(s) + N_2(s))^{-1} \kappa(s) z, \text{ for all } (s, z) \in [0, T] \times \mathbb{R}. \quad (6.17)
\]

Substituting this into the wealth equation results
\[
\begin{aligned}
\dot{X}^{t,y}(s) &= \left\{ r_0(s) + \mu (N_1(s) + N_2(s))^{-1} \kappa(s) \rho(s) \right\} X^{t,y}(s) ds \\
&\quad + \int_E \kappa(s) \rho(s) X^{t,y}(s) \sigma(s) dW(s) \\
&\quad + \int_E \mu (N_1(s) + N_2(s))^{-1} \kappa(s) \sigma(s,e) \tilde{N}(ds,de),
\end{aligned}
\]
for \( s \in [t, T] \).

The above SDE has a unique solution \( X^{t,y}(\cdot) \in \mathcal{S}^\mathbb{R}_+(t, T; \mathbb{R}) \), for any \( (t, y) \in [0, T] \times \mathbb{R} \). So the closed-loop strategy \( \hat{\varphi}(\cdot, \cdot) \) defined by (6.17) is admissible. Moreover, it is not difficult to check that Assumptions (i), (ii) and (iii) in Theorem 4.3 are satisfied. Hence, \( \hat{\varphi}(\cdot, \cdot) \) is an open-loop equilibrium strategy. ■

Remark 6.2. Let \( N_1(\cdot), N_2(\cdot), M_1(\cdot), M_2(\cdot), M_3(\cdot) \) be the deterministic functions given by (6.12)-(6.16), respectively. Then the open-loop equilibrium control can be represented by
\[
\hat{\varphi}^o(s) = \mu e^{-\int_s^T r_0(\tau)d\tau} \left( 1 + \mu \int_s^T e^{-\int_\tau^T r_0(\nu)d\nu} \kappa(\tau) \rho(\tau)d\tau \right)^{-1} \kappa(s) \hat{X}^{x_0}(s),
\]
for \( s \in [0, T] \), where the equilibrium wealth process is explicitly given by
\[
\hat{X}^{x_0}(s) = x_0 \exp \left\{ \int_0^s \left( r_0(\tau) + \mu L(\tau) \kappa(\tau) \rho(\tau) - \frac{1}{2} (\mu L(\tau) \kappa(\tau) \sigma(\tau))^2 \right) d\tau \\
\quad + \int_0^s \int_E \ln \left( 1 + \mu L(\tau) \kappa(\tau) \phi(\tau,e) \right) - \mu L(\tau) \kappa(\tau) \phi(\tau,e) \vartheta(de) d\tau \\
\quad + \int_0^s \mu L(\tau) \kappa(\tau) \sigma(\tau) dW(\tau) + \int_0^s \int_E \ln \left( 1 + \mu L(\tau) \kappa(\tau) \phi(\tau,e) \right) \tilde{N}(d\tau,de) \right\},
\]
for \( s \in [0, T] \),
with \( L(s) \equiv e^{-\int_s^T r_0(\tau)d\tau} \left( 1 + \mu \int_s^T e^{-\int_s^\tau r_0(\nu)d\nu} \kappa(\tau) \rho(\tau) d\tau \right)^{-1} \). Moreover, simple calculations show that

\[
J \left( t, X^{x_0}(t) ; \hat{\varphi}(\cdot, \cdot) \right) = \left( M_2(t) - N_2(t)^2 - 2\mu_1 (N_2(t) + N_1(t)) \right) \frac{\dot{X}^{x_0}(t)^2}{2} ,
\]

for any \( t \in [0, T] \).

**Remark 6.3.** If the modeling framework is without jumps, then the function \( \kappa(s) \) introduced in (6.3) reduces to

\[
\kappa(s) := \frac{\rho(s)}{\sigma(s)^2} .
\]

Accordingly, the open-loop equilibrium control \( \hat{u}^{x_0}(\cdot) \) takes the form

\[
\hat{u}^{x_0}(s) = \mu e^{-\int_s^T r_0(\tau)d\tau} \left( 1 + \mu \int_s^T e^{-\int_s^\tau r_0(\nu)d\nu} \frac{\rho(\tau)^2}{\sigma(\tau)^2} d\tau \right)^{-1} \frac{\rho(s)}{\sigma(s)^2} \dot{X}^{x_0}(s) ,
\]

and the equilibrium wealth process becomes

\[
\dot{X}^{x_0}(s) = x_0 \exp \left\{ \int_0^s \left[ r_0(\tau) + \mu L^0(\tau) \kappa(\tau) \rho(\tau) - \frac{1}{2} \mu L^0(\tau) \kappa(\tau) \sigma(s)^2 \right] d\tau + \int_0^s \mu L^0(\tau) \kappa(\tau) \sigma(s) dW(\tau) \right\} ,
\]

for \( s \in [0, T] \),

with \( L^0(s) \equiv e^{-\int_s^T r_0(\tau)d\tau} \left( 1 + \mu \int_s^T e^{-\int_s^\tau r_0(\nu)d\nu} \frac{\rho(\tau)^2}{\sigma(\tau)^2} d\tau \right)^{-1} \). This essentially coincide with the result derived by Hu et al. [21], Subsection 5.4.1] by solving a flow of FBSDEs.

### 7 Conclusion and future work

In this paper we have investigated open-loop equilibrium controls to a general class of time-inconsistent stochastic optimal control problems. Compared with the existing literature, the novelty of this paper is that we propose a new method that enables us to construct open-loop equilibrium controls by solving a deterministic coupled system of IPDEs. In Remark 4.5, we have listed several advantages of this new approach. Moreover, we illustrated our main result by solving a mean-variance portfolio selection problem and the solution we obtained coincides with that obtained in [35] and [21] by solving a flow of FBSDEs. Some obvious open research problems are the following.

(i) The task of proving existence and/or uniqueness of solutions to the system of IPDEs (4.31) to be technically extremely difficult. We have no idea about how to proceed so we leave it for future research.

(ii) We characterized open-loop equilibriums via a sufficient condition only, it remains to derive the necessary condition and investigate the uniqueness of the equilibrium solution.

(iii) Assumptions (i) and (ii) in Theorem 4.3 may seem restrictive conditions. We hope that in our future publications, these conditions can be made weaker.

(iv) Finally, the present work can be extended in several ways. For example it is interesting to generalize our results to the case where the running cost also depend on conditional expectations for the state process and control process.

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