AN ANALYTIC VERSION OF THE MELVIN-MORTON-ROZANSKY CONJECTURE

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Abstract. To a knot in 3-space, one can associate a sequence of Laurent polynomials, whose nth term is the nth colored Jones polynomial. The Volume Conjecture for small angles states that the value of the n-th colored Jones polynomial at $e^{\alpha/n}$ is a sequence of complex numbers that grows subexponentially, for a fixed small complex angle $\alpha$. In an earlier publication, the authors proved the Volume Conjecture for small purely imaginary angles, using estimates of the cyclotomic expansion of a knot. The goal of the present paper is to identify the polynomial growth rate of the above sequence to all orders with the loop expansion of the colored Jones function. Among other things, this provides a strong analytic form of the Melvin-Morton-Rozansky conjecture.

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1. Introduction

1.1. The volume conjecture for small angles. In an earlier publication, the authors stated and proved the Volume Conjecture for small purely imaginary angles; see [GL2]. More precisely, the authors proved that
for every knot $K$ in $S^3$ there exists a positive angle $\alpha(K) > 0$ such that

$$\lim_{n \to \infty} \frac{\log |J_{K,n}(e^{\alpha/n})|}{n} = 0$$

for all $\alpha \in i[0, \alpha(K))$, where

- $f(e^{\alpha/n})$ denotes the evaluation of a rational function $f(q)$ at $q = e^{\alpha/n}$,
- $J_{K,n}(q) \in \mathbb{Z}[q^\pm]$ is the Jones polynomial of a knot colored with the $n$-dimensional irreducible representation of $\mathfrak{sl}_2$, normalized so that it equals to 1 for the unknot (see [4, 11]).

In the following, we will refer to the complex parameter $\alpha$ as the angle, making contact with standard terminology from hyperbolic geometry. As was explained in [GL2], the above result agrees with the fact that $\text{vol}(\rho_\alpha) = 0$

in the following, we will refer to the complex parameter $\alpha$ as the angle, making contact with standard terminology from hyperbolic geometry. As was explained in [GL2], the above result agrees with the fact that $\text{vol}(\rho_\alpha) = 0$

where

$$\rho_\alpha : \pi_1(S^3 - K) \to \text{SL}_2(\mathbb{C}), \quad \rho_\alpha(m) = \begin{pmatrix} e^{\alpha/n} & 0 \\ 0 & e^{-\alpha/n} \end{pmatrix}. $$

is a reducible representation of the knot group in $\text{SL}_2(\mathbb{C})$ with prescribed behavior on a meridian $m$ of the knot $K$.

For further reading concerning the history of the volume conjecture, we refer the reader to [GM, K, MM], as well as [GL2].

Notice that $\rho_\alpha$ is a 1-parameter deformation of the trivial representation $\rho_0 = I$.

Moreover, Equation (1) implies that the sequence $J_{K,n}(e^{\alpha/n})$ grows at a subexponential rate, as $n$ approaches infinity, and $\alpha$ is small and purely imaginary.

The purpose of the present paper is to identify the polynomial growth rate of $J_{K,n}(e^{\alpha/n})$ in terms of the inverse Alexander polynomial $\Delta_K$ of $K$, symmetrized by $\Delta_K(t^{-1}) = \Delta_K(t)$, and normalized by $\Delta_K(1) = 1$, and $\Delta_{\text{unknot}}(t) = 1$. More precisely, we have the following theorem.

**Theorem 1.** For every knot $K$ there exists an open neighborhood $U_K$ of $0 \in \mathbb{C}$ such that for all complex angles $\alpha \in U_K$, we have:

$$\lim_{n \to \infty} \frac{J_{K,n}(e^{\alpha/n})}{\Delta_K(e^{\alpha})} = \frac{1}{\Delta_K(e^{\alpha})} \in \mathbb{C}. $$

Moreover, the convergence with respect to $\alpha$ is uniform on compact subsets of $U_K$.

In particular since $\Delta_K(1) = 1$, (3) implies (1).

The reader may compare the above theorem with the famous Melvin-Morton-Rozansky (MMR, in short) Conjecture, which was settled by Bar-Natan and the first author in [10]. Let $\mathbb{Q}[[h]]$ denote the ring of formal power series in a variable $h$ with rational coefficients.

**Theorem 2.** [10] For every knot $K$ we have the following equality in the ring $\mathbb{Q}[[h]]$:

$$\lim_{n \to \infty} J_{K,n}(e^{h/n}) = \frac{1}{\Delta_K(e^h)} \in \mathbb{Q}[[h]], $$

To avoid confusion, let us point out that Equation (4) is a statement about coefficients of formal power series. In other words, (4) can be phrased as follows: for every $m \geq 0$, we have:

$$\lim_{n \to \infty} \text{coeff} \left( J_{K,n}(e^{h/n}), h^m \right) = \text{coeff} \left( \frac{1}{\Delta_K(e^h)}, h^m \right), $$

where for an analytic function $f(x)$ we define:

$$\text{coeff}(f(h), h^m) = \frac{1}{m!} \frac{d^m}{dh^m}|_{h=0} f(h). $$
Actually, for every \( m \geq 0 \), \( \text{coeff} \left( J_{K,n}(e^{h/n}), h^m \right) \) is a polynomial in \( 1/n \) of degree \( m \) (see also Section 2.1 below). Thus, the limit with respect to \( n \to \infty \) in (5) exists and is simply the constant term of the above-mentioned polynomial. Identifying that constant term with the right hand side of (5) is the non-trivial part of the MMR Conjecture.

Let us compare Theorems 1 and 2. Since convergence with respect to \( \alpha \) is uniform on compact subsets, it is easy to see that Theorem 1 implies Theorem 2. In that sense, we may say that Theorem 1 is an analytic form of the MMR Conjecture.

Thus, Theorem 1 can be viewed as a statement about the volume conjecture for small angles, as well as an analytic form of the MMR Conjecture.

Armed with Theorem 1, one may ask for a full asymptotic expansion of the left hand side of (5) in terms of powers of \( 1/n \). Before we answer this question, let us recall what is known on the level of formal power series, that is, about the \( 1/n \) terms of (5).

Rozansky discovered that after resummation, for every fixed \( m \geq 0 \), the \( 1/n^m \) terms of (5) are rational functions in a variable \( e^h \). Let us state Rozansky’s discovery concretely.

**Theorem 3. [Ro]** For every knot \( K \) there exists a sequence \( P_{K,k}(q) \in \mathbb{Q}[q^\pm] \) of Laurent polynomials with \( P_{K,0}(q) = 1 \) such that

\[
J_{K,n}(e^{h/n}) \sim_{n \to \infty} \sum_{k=0}^{\infty} \frac{P_{K,k}(e^h)}{\Delta_K(e^h)^{2k+1}} \left( \frac{h}{n} \right)^k \in \mathbb{Q}[[h]]
\]

in the ring \( \mathbb{Q}[[h]] \) of formal power series in \( h \).

A different proof, valid for all simple Lie groups, was given in [Ga1], using work of [GK].

Let us point out that (6) means the following: for every \( N \geq 0 \) we have:

\[
\lim_{n \to \infty} \left( \frac{n}{h} \right)^N \left( J_{K,n}(e^{h/n}) - \sum_{k=0}^{N-1} \frac{P_{K,k}(e^h)}{\Delta_K(e^h)^{2k+1}} \left( \frac{h}{n} \right)^k \right) = \frac{P_{K,N}(e^h)}{\Delta_K^{N+1}(e^h)} \in \mathbb{Q}[[h]].
\]

### 1.2. Asymptotics to all orders.

Our results are the following:

**Theorem 4.** For every knot \( K \) there exists an open neighborhood \( U_K \) of \( 0 \in \mathbb{C} \) such that for all complex angles \( \alpha \in U_K \), we have an asymptotic expansion (uniform on compact subsets of \( U_K \) with respect to \( \alpha \)):

\[
J_{K,n}(e^{\alpha/n}) \sim_{n \to \infty} \sum_{k=0}^{\infty} \frac{P_{K,k}(e^\alpha)}{\Delta_K(e^\alpha)^{2k+1}} \left( \frac{\alpha}{n} \right)^k.
\]

In other words, for \( \alpha \in U_K \) and every \( N \geq 0 \),

\[
\lim_{n \to \infty} \left( \frac{n}{\alpha} \right)^N \left( J_{K,n}(e^{\alpha/n}) - \sum_{k=0}^{N-1} \frac{P_{K,k}(e^\alpha)}{\Delta_K(e^\alpha)^{2k+1}} \left( \frac{\alpha}{n} \right)^k \right) = \frac{P_{K,N}(e^\alpha)}{\Delta_K^{N+1}(e^\alpha)} \in \mathbb{C}.
\]

Moreover, convergence with respect to \( \alpha \) is uniform on compact subsets of \( U_K \).

Thus, the above theorem determines to all orders the asymptotic expansion of the volume conjecture for small angles.

### 1.3. A small dose of physics.

One does not need to know the relation of the colored Jones function and quantum field theory in order to understand the statement and proof of Theorem 1. Nevertheless, we want to add some philosophical comments, for the benefit of the willing reader. According to Witten (see [AW]), the Jones polynomial \( J_{K,n} \) can be expressed by a partition function of a topological quantum field theory in 3 dimensions—a gauge theory with Chern-Simons Lagrangian. The stationary points of the Lagrangian correspond to \( SU(2) \)-flat connections on an ambient manifold, and the observables are knots, colored by the \( n \)-dimensional irreducible representation of \( SU(2) \). In case of a knot in \( S^3 \), there is only one ambient flat connection, and the corresponding perturbation theory is a formal power series in \( h = \log q \).

Rozansky exploited a cut-and-paste property of the Chern-Simons path integral and considered perturbation theory of the knot complement, along an abelian flat connection with monodromy given by (2). In
fact, Rozansky calls such an expansion the $U(1)$-$RCC$ \textit{connection contribution} to the Chern-Simons path integral, where RCC stands for \textit{reducible connection contribution}, and $U(1)$ stands for the fact that the flat $SU(2)$ connections are actually $U(1)$-valued abelian connections. Formal properties of such a perturbative expansion, enabled Rozansky to deduce (in physics terms) the loop expansion of the colored Jones function. In a later publication, Rozansky proved the existence of the loop expansion using an explicit state-sum description of the colored Jones function.

Of course, perturbation theory means studying formal power series that rarely converge. Perturbation theory at the trivial flat connection in a knot complement converges, as it resums to a Laurent polynomial in $e^h$: namely the colored Jones polynomial. The volume conjecture for small complex angles is precisely the statement that perturbation theory for abelian flat connections (near the trivial one) does converge.

At the moment, there is no physics (or otherwise) formulation of perturbation theory of the Chern-Simons path integral along a discrete and faithful $SL_2(\mathbb{C})$ representation. Nor is there an adequate explanation of the relation between $SU(2)$ gauge theory (valid near $\alpha = 0$) and a complexified $SL_2(\mathbb{C})$ gauge theory, valid near $\alpha = 2\pi i$. These are important and tantalizing questions, with no answers at present.

### 1.4. WKB

Since we are discussing physics interpretations of Theorem 4, let us make some more comments. Obviously, when the angle $\alpha$ is sufficiently big, the asymptotic expansion of Equation (8) may break down. For example, when $e^\alpha$ is a complex root of the Alexander polynomial, then the right hand side of (8) does not make sense, even to leading order. In fact, when $\alpha$ is near $2\pi i$, then the solutions are expected to grow exponentially, and not polynomially, according to the Volume Conjecture.

The breakdown and change of rate of asymptotics is a well-documented phenomenon well-known in physics, associated with WKB analysis, after Wentzel-Krammer-Brillouin; see for example [O]. In fact, one may obtain an independent proof of Theorem 4 using \textit{WKB analysis}, that is, the study of asymptotics of solutions of difference equations with a small parameter. The key idea is that the sequence of colored Jones functions is a solution of a linear $q$-difference equation, as was established in [GL]. A discussion on WKB analysis of $q$-difference equations was given by Geronimo and the first author in [GG].

The WKB analysis can, in particular, determine \textit{small exponential corrections} of the form $e^{-c_\alpha n}$ to the asymptotic expansion of Theorem 4, where $c_\alpha$ depends on $\alpha$, with $\text{Re}(c_\alpha) < 0$ for $\alpha$ sufficiently small. These exciting small exponential corrections cannot be captured by classical asymptotic analysis (since they vanish to all orders in $n$), but they are important and dominant (i.e., $\text{Re}(c_\alpha) > 0$) when $\alpha$ is near $2\pi i$, according to the volume conjecture. Understanding the change of sign of $\text{Re}(c_\alpha)$ past certain so-called Stokes directions is an important question that WKB addresses.

We will not elaborate or use the WKB analysis in the present paper. Let us only mention that the loop expansion of the colored Jones function can be interpreted as WKB asymptotics on a $q$-difference equation satisfied by the colored Jones function.

### 1.5. The main ideas

The main ideas of Theorem 4 is to compare three different views of the Jones polynomial: one coming from perturbative quantum field theory, one from a resummation of quantum field theory (known as the loop expansion), and a third non-perturbative view, in terms of the cyclotomic function.

The main advantage of the cyclotomic function of a knot is a key integrality property, due to Habiro, and a priori exponential estimates for the $L^1$-norm and quadratic bounds for the degrees of the relevant polynomials. The latter were established in [GL2]. Using these bounds, we can prove that for small enough complex angles, a sequence of holomorphic functions is uniformly bounded, and the limit of derivatives of any order (at zero) exists; see Theorem 5. A key lemma from complex analysis on normal families guarantees under the above hypothesis that the sequence of holomorphic functions converges, uniformly on compact sets, to a holomorphic function whose derivatives (at zero) are the limits of the derivatives of the original sequence of holomorphic functions.

### 1.6. Acknowledgement

Soon after the completion of the authors’ work [GL2], H. Murakami posted an interesting paper, in which he identified the polynomial growth of the volume conjecture for small angles, for the case of the 4$_1$ knot; see [M]. Upon reading Murakami’s paper, it became clear that the methods of [GL2] can be adapted to all knots, and to all orders, for small complex angles. We wish to thank Murakami who motivated our present work.
2. Three expansions of the Jones polynomial

2.1. Finite type invariants and the Jones polynomial. The colored Jones function of a knot is a 2-parameter invariant, that depends on the color $n$ and the formal parameter $h = \log q$.

**Perturbative quantum field theory** (formalized mathematically by the Kontsevich integral of a knot, and its image under the $\mathfrak{sl}_2$ weight system, described for example in [B-N]) gives the following expansion of the colored Jones function:

$$J_{K,n}(e^h) = \sum_{0 \leq i, 0 \leq j \leq i} c_{K,i,j} n^j h^i$$

(10)

$$= \sum_{0 \leq i, 0 \leq j \leq i} c_{K,i,j} (nh)^j h^{i-j}$$

$$= \sum_{0 \leq j, k} c_{K,j+k,j} (nh)^j h^k.$$  

Here, $K \rightarrow c_{K,i,j}$ are **finite type knot invariants** of type $i$; see [B-N]. The important property is that $c_{K,i,j} = 0$ in the $(i, j)$ plane and above the diagonal $i = j$. Thus, one can resum the formal power series as follows:

$$J_{K,n}(e^h) = \sum_{k=0}^{\infty} R_{K,k}(nh) h^k,$$

(11)

where

$$R_{K,k}(x) = \sum_{0 \leq j} c_{K,j+k,j} x^j \in \mathbb{Q}[[x]].$$

2.2. The loop expansion of the Jones polynomial. The Melvin-Morton-Rozansky Conjecture states that

$$R_{K,0}(x) = \frac{1}{\Delta_K(e^x)}.$$  

More generally, in [Ro], Rozansky proves that

$$R_{K,k}(x) = \frac{P_{K,k}(e^x)}{\Delta_K(e^x)^{2k+1}}$$

for Laurent polynomials $P_{K,k}(q) \in \mathbb{Q}[q^\pm]$.

Although the polynomials $P_{K,k}(q)$ are not finite type invariants (with respect to the usual crossing change of knots), they are indeed finite type invariants with respect to a loop move described in [GR]. We will not use this fact in our paper.

Rozansky conjectured that the resummation given by the above equations could be performed on the level of a universal perturbative invariant (the Kontsevich integral of a knot; see [B-N]), and this was proven to be the case in [YK]. As a result, one obtains a proof of this resummation property valid for all simple Lie algebras, see [Ga1].

2.3. The cyclotomic expansion of the Jones polynomial. In [Ha], Habiro introduced an alternative packaging of the colored Jones function $J_{K,n}$; using the so-called **cyclotomic function** $C_{K,n}$. The latter is related to the former by the following

$$J_{K,n}(q) = \sum_{k=0}^{n} C_{n,k}(q) C_{K,k}(q),$$

(12)
where
\[
C_{n,k}(q) := \frac{1}{q^{n/2} - q^{-n/2}} \prod_{j=n-k}^{n+k} (q^{j/2} - q^{-j/2}) \\
= \prod_{j=1}^{k} ((q^{n/2} - q^{-n/2})^2 - (q^{j/2} - q^{-j/2})^2) \\
= \prod_{j=1}^{k} ((q^{n/2} + q^{-n/2})^2 - (q^{j/2} + q^{-j/2})^2).
\]

Thus, in a sense $J_{K,n}$ and $C_{K,n}$ are related by a lower-diagonal invertible matrix. For an explicit inversion of the above equation (which we will not use in the present paper), we refer the reader to [GL1, Sec.4].

2.4. Comparing the cyclotomic and the loop expansion. So far, we have three expansions: the finite type expansion, the loop expansion and the cyclotomic expansion. Now, we’ll compare the last two. In other words, we’ll compare Equations (11) and (12).

Let \( q = e^h, \quad x = nh. \)

For a function \( f(q) \), let us denote by \( \langle f \rangle_k \) the \( k \)-th coefficient in the Taylor expansion of \( f(e^h) \) around \( h = 0 \). Of course,

\[
\langle f \rangle_k = \frac{d^k}{dh^k} \bigg|_{h=0} f(e^h).
\]

In other words, we have:

\[
f(e^h) = \sum_{k=0}^{\infty} \langle f \rangle_k h^k \in \mathbb{Q}[[h]].
\]

**Lemma 2.1.** (a) For every knot \( K \), we have the following equality in \( \mathbb{Q}[[x, h]] \):

\[
\sum_{k=0}^{\infty} R_{K,k}(x)h^k = \sum_{k=0}^{\infty} C_{K,k}(e^h) \prod_{l=0}^{k} (e^{x/2} - e^{-x/2})^2 - (e^{jh/2} - e^{-jh/2})^2 \in \mathbb{Q}[[x, h]].
\]

(b) It follows that for every \( k \),

\[
R_{K,k}(x) = \sum_{l=0}^{\infty} \sum_{j=0}^{k} (C_{K,l})_j z^{2l-[j/2]} p_{l,j,k}(z)
\]

where \( z = e^{x/2} - e^{-x/2} \), and \( p_{l,j,k}(z) \) is an even polynomial of \( z \) of degree \( |j/2| \), with coefficients polynomials of \( l \) of degree \( k + 1 \).

(c) In particular, we have:

\[
R_{K,0}(x) = \sum_{l=0}^{\infty} (C_{K,l})_0 z^{2l} \\
R_{K,1}(x) = \sum_{l=0}^{\infty} (C_{K,l})_1 z^{2l} \\
R_{K,2}(x) = \sum_{l=0}^{\infty} (C_{K,l})_2 z^{2l} - \sum_{l=0}^{\infty} (C_{K,l})_0 \frac{l(l+1)(2l+1)}{6} z^{2l-2} \\
R_{K,3}(x) = \sum_{l=0}^{\infty} (C_{K,l})_3 z^{2l} - \sum_{l=0}^{\infty} (C_{K,l})_1 \frac{l(l+1)(2l+1)}{6} z^{2l-2}
\]

in \( \mathbb{Q}[[x]] \).
Proof. It follows easily, working in the ring $\mathbb{Q}[[x, h]]$, and using the fact that the map:

$$Q(e^x)[[h]] \rightarrow Q[[x, h]]$$

given by $e^x = \sum_{k=0}^{\infty} x^k/k!$ is 1-1. □

3. Proof of Theorem 1

Let us assume for the moment the following theorem, whose proof will be given in the next section.

**Theorem 5.** (a) For every knot $K$ there exist an open neighborhood $U_K$ of $0 \in \mathbb{C}$ and a positive number $M$ such that for $\alpha \in U_K$, and all $n \geq 0$, we have:

$$|J_{K,n}(e^{\alpha/n})| < M.$$  

(b) Moreover, for every $m \geq 0$, the following limit exists and given by:

$$\lim_{n \to \infty} \frac{d^m}{d\alpha^m}|_{\alpha=0} J_{K,n}(e^{\alpha/n}) = m! \text{coeff} \left( \frac{1}{\Delta_K(e^\alpha)}, \alpha^m \right).$$

3.1. A lemma from complex analysis. The proof of Theorem 1 will use the following lemma on normal families that is sometimes referred to by the name of Vitali and Montel’s theorem. For a reference, see [Hi, Sch]. The lemma exhibits the power of holomorphy, coupled with uniform boundedness.

Let $\Delta_r = \{ z \in \mathbb{C} : |z| < r \}$ denote the open complex disk around $0$ of radius $r > 0$.

**Lemma 3.1.** If $f_n : \Delta_r \to \bar{\Delta}_M$ is a sequence of holomorphic functions such that for every $m \geq 0$, we have:

$$\lim_{n \to \infty} f_n^{(m)}(0) = a_m.$$ 

Then,

- The limit $f(z) = \lim_n f_n(z)$ exists pointwise for $z \in D_r$.
- $f : D_r \to \bar{\Delta}_M$ is holomorphic,
- The convergence is uniform on compact subsets, and
- For every $m$, $f^{(m)}(0) = a_m$.

Proof. $\{f_n\}_n$ is uniformly bounded, so it is a normal family, and contains a convergent subsequence $f_j \to f$. Convergence is uniform on compact sets, and $f$ is holomorphic, and for every $m \geq 0$, $\lim_{j} f_j^{(m)}(0) = f^{(m)}(0) = a_m$.

If $\{f_n\}_n$ is not convergent, since it is a normal family, then there exist two subsequences that converge to $f$ and $g$ respectively, with $f \neq g$. Applying the above discussion, it follows that $f$ and $g$ are holomorphic functions with equal derivatives of all orders at $0$. Thus, $f = g$, giving a contradiction. Thus, $\{f_n\}_n$ is convergent and the result follows from the above discussion. □

**Remark 3.2.** We have seen that the hypotheses in Lemma 3.1 are sufficient to ensure existence of the limit and uniform convergence on compact sets. It is easy to see that these hypotheses are also necessary.

3.2. Proof of Theorem 1. Fix a knot $K$ and an open neighborhood $U_K$ of $0 \in \mathbb{C}$ as in Theorem 5. Theorem 4 and Lemma 3.1 imply that for $\alpha \in U_K$,

$$\lim_{n \to \infty} J_{K,n}(e^{\alpha/n}) = \frac{1}{\Delta_K(e^\alpha)}.$$ 

Moreover, convergence with respect to $\alpha$ is uniform on compact subsets of $U_K$. This proves Theorem 1. □
4. Estimates of the cyclotomic function

This section is devoted to the proof of Theorem 5. Our main tool will be estimates in the cyclotomic expansion of a knot, similar to the ones used in [GL2].

A key result of Habiro is an integrality property of the cyclotomic function $n \rightarrow C_{K,n}$ of a knot. Namely,

$$C_{K,n}(q) \in \mathbb{Z}[q^\pm]$$

for all knots $K$ and all $n$; see [Ha].

We will use two further results from [GL2]: an exponential bound on the size of the coefficients of $C_{K,n}$, and a quadratic bound on the min and max degrees of $C_{K,n}$. Recall that for a Laurent polynomial $f(q) = \sum_{k} a_k q^k$, we define its $l^1$ norm by

$$||f||_1 = \sum_k |a_k|.$$

**Theorem 6.** (a) For every knot $K$ we have:

$$||C_{K,n}||_1 \leq e^{C n + C' \log n}$$

(b) Moreover,

$$\maxdeg_q(C_{K,n}) = O(n^2), \quad \mindeg_q(C_{K,n}) = O(n^2).$$

Here, and below, the $O(f(n))$ notation means that a quantity bounded by a constant times $f(n)$.

**Theorem 7.** For every knot $K$, there exist constants $C, C', C''$ and $C'''$ (that depend on $K$) such that for all $n \geq 0$ and $k \geq 0$ we have:

$$|C_{K,n}^{(k)}(e^{\alpha})| \leq e^{C n + C' (k+1) \log n + \Re(\alpha) C'' n^2 + C'''},$$

where $C_{K,n}^{(k)}$ denotes the $k$-th derivative of $C_{K,n}(e^h)$ with respect to $h$.

**Proof.** Let us write

$$C_{K,n}(q) = \sum_{j=-C_1 n^2}^{C_1 n^2} a_{j,n} q^j.$$

Then,

$$C_{K,n}^{(k)}(e^h) = \sum_{j=-C_1 n^2}^{C_1 n^2} a_{j,n} j^k e^{j h}.$$

We will estimate each coefficient and each monomial by:

$$|a_{j,n}| \leq ||C_{K,n}||_1 \leq e^{C n + C' \log n}$$

$$|j|^k \leq (C_1 n^2)^k$$

$$|e^{j \alpha}| \leq e^{\Re(\alpha) C_1 n^2}.$$
Proof. (of Theorem 5) Combining Corollary 4.1 and Lemma 4.2, it follows that for all \( 0 \leq k \leq n \), we have:

\[
|C_{n,k}(e^{\alpha/n})C_{K,k}(e^{\alpha/n})| \leq e^{ck+c'\log k+|\text{Re}(\alpha)|C''k+C'''+C_1k \log |\alpha|+C_2k \log k+C_3}.
\]

Let us choose \( \alpha \in U_K \), where

\[
U_K = \{ \alpha \in \mathbb{C} \mid C + C'|\text{Re}(\alpha)| + C_1 \log |\alpha| < 0 \}.
\]

Then, equation (12) and the above estimate conclude the first part of Theorem 5.

The second part follows from Equation (11) and the MMR Conjecture. Indeed, consider the sequence

\[
f_n : U_K \to \{ z : |z| < N \}, \quad \alpha \to f_n(\alpha) = J_{K,n}(e^{\alpha/n}).
\]

Since \( J_{K,n}(q) \) is a Laurent polynomial in \( q \), it follows that \( f_n \) is an entire function. Equation (11) implies that

\[
f_n(\alpha) = \sum_{k=0}^{\infty} R_{K,k}(\alpha) \left( \frac{\alpha}{n} \right)^k.
\]

Thus, for every \( m \geq 0 \),

\[
f_n^{(m)}(0) = m! \left( \text{coeff}(R_{K,0}(\alpha), \alpha^m) + \frac{1}{n} \text{coeff}(R_{K,1}(\alpha), \alpha^{m-1}) + \ldots + \frac{1}{n^m} \text{coeff}(R_{K,m}(\alpha), \alpha^0) \right).
\]

Thus, using the MMR Conjecture, we obtain:

\[
\lim_{m \to \infty} f_n^{(m)}(0) = m! \text{coeff}(R_{K,0}(\alpha), \alpha^m) = m! \text{coeff} \left( \frac{1}{\Delta_K(e^{\alpha})}, \alpha^m \right).
\]

The result follows. \( \square \)

5. Proof of Theorem 1

To leading order (i.e., \( N = 0 \) in (7)) Theorem 4 is Theorem 1. By now, it should be clear the strategy for proving Theorem 1 to all orders. To simplify notation, let us define:

\[
J_{K,n}^{(N)}(e^{\alpha/n}) = J_{K,n}(e^{\alpha/n}) - \sum_{k=0}^{N-1} \frac{P_{K,k}(e^{\alpha})}{\Delta_K(e^{\alpha})^{2k+1}} \left( \frac{\alpha}{n} \right)^k.
\]

Theorem 4 follows from the following result and the argument of Section 3.2.

**Theorem 8.** (a) For every knot \( K \) there exists an open neighborhood \( U_K \) of 0 in \( \mathbb{C} \) such that for every \( N \geq 0 \) there exists a positive number \( M_N \) such that for \( \alpha \in U_K \), and all \( n \geq 0 \), we have:

\[
\left| \left( \frac{n}{\alpha} \right)^N J_{K,n}^{(N)}(e^{\alpha/n}) \right| < M_N.
\]

(b) Moreover, for every \( m \geq 0 \), the following limit exists and given by:

\[
\lim_{n \to \infty} \frac{\partial^m}{\partial \alpha^m} \bigg|_{\alpha=0} \left( \left( \frac{n}{\alpha} \right)^N J_{K,n}^{(N)}(e^{\alpha/n}) \right) = m! \text{coeff} \left( \frac{P_{K,N}(e^{\alpha})}{\Delta_K(e^{\alpha})^{2N+1}}, \alpha^m \right).
\]

**Proof.** We will prove the theorem by induction on \( N \). For \( N = 0 \), this is Theorem 1 proven in Section 3. Let us assume that it is true for \( N - 1 \).

Let us define for every \( k \geq 0 \), two auxiliary biholomorphic functions

\[
c_k(x, \epsilon) = \prod_{j=1}^{k} \left( e^{x/2} - e^{-x/2} \right)^2 - \left( e^{j\epsilon/2} - e^{-j\epsilon/2} \right)^2,
\]

\[
g_{K,k}(x, \epsilon) = c_k(x, \epsilon)C_{K,k}(e^{\epsilon}).
\]
Thus, using the definition of $C_{n,k}$ and Equation (12), it follows that:

\begin{equation}
C_{n,k}(e^{\alpha/n}) = c_k(\alpha, \alpha/n), \quad J_{K,n}(e^{\alpha/n}) = \sum_{k=0}^{n} g_{K,k}(\alpha, \alpha/n).
\end{equation}

For a function $h = h(x)$, let us define the $N$-th Taylor approximation by:

\[
\text{Taylor}^N(h, x) = \sum_{j=0}^{N} \frac{h^{(j)}(0)}{j!} x^j.
\]

Applying Lemma 2.1 to the function $\epsilon \rightarrow g_{K,k}(\alpha, \epsilon)$, and evaluating at $\epsilon = \alpha/n$, it follows that:

\begin{equation}
\begin{aligned}
\sum_{k=0}^{N-1} P_{K, k}(e^{\alpha}) \frac{1}{\Delta_k(e^{\alpha})^{2k+1}} \left( \frac{\alpha}{n} \right)^k &= \sum_{k=0}^{\infty} \text{Taylor}^{N-1}(g_{K,k}(\alpha, \cdot), \frac{\alpha}{n}) \\
&= \sum_{k=0}^{n} \text{Taylor}^{N-1}(g_{K,k}(\alpha, \cdot), \frac{\alpha}{n}) + \text{err}_n(\alpha).
\end{aligned}
\end{equation}

Equations (16), (17) and (19) and Taylor’s theorem imply that:

\[
J_{K,n}(e^{\alpha/n}) = J_{K,n}(e^{\alpha/n}) - \sum_{k=0}^{N-1} P_{K, k}(e^{\alpha}) \frac{1}{\Delta_k(e^{\alpha})^{2k+1}} \left( \frac{\alpha}{n} \right)^k
\]

\[
= \sum_{k=0}^{n} g_{K,k}(\alpha, \alpha/n) - \sum_{k=0}^{n} \text{Taylor}^{N-1}(g_{K,k}(\alpha, \cdot), \frac{\alpha}{n}) - \text{err}_n(\alpha)
\]

\[
\approx \left( \frac{\alpha}{n} \right)^N \sum_{k=0}^{n} \frac{1}{N!} \frac{\partial^N}{\partial \epsilon^N} |_{\epsilon = \alpha/n} g_{K,k}(\alpha, \epsilon) - \text{err}_n.
\]

The analyticity of $g_{K,k}$ and Theorem 7 implies that there exists a positive $M_N'$ such that for all $\alpha \in U_K$ (defined in 15), we have:

\[
|\text{err}_n(\alpha)| < M_N'.
\]

Corollary 4.1 and Equation (17) imply that there exists a positive $M_N$ such that

\[
\left| \left( \frac{\alpha}{n} \right)^N J_{K,n}(e^{\alpha/n}) \right| < M_N
\]

for all $n \geq 0$ and for all $\alpha \in U_K$. This proves part (a) of Theorem 8.

For part (b), we will use Equation (11), which implies that:

\[
J_{K,n}(e^{\alpha/n}) = \sum_{k=N}^{\infty} R_{K,k}(\alpha) \left( \frac{\alpha}{n} \right)^k.
\]

Thus, for every $m \geq 0$,

\[
\frac{d^m}{d\alpha^m} |_{\alpha=0} \left( \frac{n}{\alpha} \right)^N J_{K,n}(e^{\alpha/n}) = m! \left( \text{coeff}(R_{K,N}(\alpha), \alpha^m) + \frac{1}{n} \text{coeff}(R_{K,N+1}(\alpha), \alpha^{m-1}) + \cdots + \frac{1}{n^m} \text{coeff}(R_{K,N+m}(\alpha), \alpha^0) \right).
\]

Using Rozansky’s theorem 3 and Equation (7), we obtain:

\[
\lim_{m \to \infty} \frac{d^m}{d\alpha^m} |_{\alpha=0} \left( \frac{n}{\alpha} \right)^N J_{K,n}(e^{\alpha/n}) = m! \text{coeff}(R_{K,N}(\alpha), \alpha^m)
\]

\[
= m! \text{coeff} \left( \frac{P_{K,N}(e^{\alpha})}{\Delta_k(e^{\alpha})^{2N+1}}, \alpha^m \right).
\]

The result follows. \qed
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