Ruled surfaces right normalized

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Abstract

This paper deals with skew ruled surfaces $\Phi$ in the Euclidean space $\mathbb{E}^3$ which are right normalized, that is they are equipped with relative normalizations, whose support function is of the form $q(u,v) = f(u) + g(u)v$, where $w^2(u,v)$ is the discriminant of the first fundamental form of $\Phi$. This class of relatively normalized ruled surfaces contains surfaces such that their relative image $\Phi^*$ is either a curve or it is as well as $\Phi$ a ruled surface whose generators are, additionally, parallel to those of $\Phi$. Moreover we investigate various properties concerning the Tchebychev vector field and the support vector field of such ruled surfaces.

Key Words: Ruled surfaces, Relative normalizations, Tchebychev vector field, Pick invariant

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1 Preliminaries

In this section we present briefly some definitions, results and formulae of relative Differential Geometry of surfaces and Differential Geometry of ruled surfaces in the Euclidean space $\mathbb{E}^3$. The reader can use [3] and [5] as general references.

In the three-dimensional Euclidean space $\mathbb{E}^3$ we denote by $\Phi = (U, \mathbf{x})$ a ruled $C^r$-surface of nonvanishing Gaussian curvature, $r \geq 3$, defined by an injective $C^r$-immersion $\mathbf{x} = \mathbf{x}(u,v)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$. We denote by $(\cdot, \cdot)$ the standard scalar product in $\mathbb{E}^3$. We introduce the so-called standard parameters $u \in I, v \in \mathbb{R}$ of $\Phi$, such that

$$\mathbf{x}(u,v) = s(u) + ve(u),$$

where the differentiation with respect to $u$ is denoted by a prime. Here $\Gamma : \mathbf{s} = \mathbf{s}(u)$ is the striction curve of $\Phi$ and the parameter $u$ is the arc length along the spherical curve $\mathbf{e} = \mathbf{e}(u)$.

The distribution parameter

$$\delta(u) := (\mathbf{s}', \mathbf{e}, \mathbf{e}')$$

the conical curvature

$$\kappa(u) := (\mathbf{e}, \mathbf{e}', \mathbf{e}'')$$

and
and the function
\[ \lambda(u) := \cot \sigma, \]
where
\[ \sigma(u) := \angle(\overrightarrow{\tau}, \overrightarrow{\tau}') \]
is the striction of \( \Phi \),
\[ -\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}, \quad \text{sign } \sigma = \text{sign } \delta, \]
are the fundamental invariants of \( \Phi \) and determine uniquely the ruled surface \( \Phi \) up to Euclidean rigid motions.

We also consider the moving frame \( D := \{ \overrightarrow{\tau}, \overrightarrow{\pi}, \overrightarrow{\kappa} \} \) of \( \Phi \), where
\[ \overrightarrow{\kappa}(u) := e \times n \]
is the central normal vector and
\[ \overrightarrow{\pi}(u) := e \times e' \]
is the central tangent vector. It is well known that the following equations
\[ e' = n, \quad n' = -e + \kappa \overrightarrow{\pi}, \quad \overrightarrow{\pi}' = -\kappa n \]
are valid (see [3, p. 280]). Then we have
\[ s' = \delta \lambda e + \delta \overrightarrow{\pi}. \]

We denote partial derivatives of a function (or a vector-valued function) \( f \) in the coordinates \( u^1 := u, u^2 := v \) by \( f_{i}, f_{ij} \) etc. Then from (1.1) and (1.4) we obtain
\[ \frac{x}{1} = \delta \lambda e + v n + \delta \overrightarrow{\pi}, \quad \frac{x}{2} = e, \]
and thus the unit normal vector \( \overrightarrow{\xi}(u, v) \) to \( \Phi \) is expressed by
\[ \overrightarrow{\xi} = \frac{\delta \overrightarrow{\pi} - v \overrightarrow{\tau}}{w}, \]
where
\[ w^2 := \delta^2 + v^2 \]
is the discriminant of the first fundamental form of \( \Phi \). Let \( II = h_{ij} du^i du^j \) be the second fundamental form of \( \Phi \), where
\[ h_{11} = -\frac{\kappa w^2 + \delta' v - \delta^2 \lambda}{w}, \quad h_{12} = \frac{\delta}{w}, \quad h_{22} = 0. \]
The Gaussian curvature \( \widetilde{K}(u, v) \) and the mean curvature \( \widetilde{H}(u, v) \) of \( \Phi \) are given by (see [3])
\[ \widetilde{K} = -\frac{\delta^2}{w^2}, \quad \widetilde{H} = \frac{\kappa w^2 + \delta' v + \delta^2 \lambda}{2w^3}. \]

A \( C^s \)-relative normalization of \( \Phi \) is a \( C^s \)-mapping \( \Phi = \Phi(u, v) \), \( 1 \leq s < r \), defined on \( U \), such that
\[ \text{rank}(\{ \overrightarrow{\tau}/1, \overrightarrow{\tau}/2, \overrightarrow{\gamma} \}) = 3, \quad \text{rank}(\{ \overrightarrow{\tau}/1, \overrightarrow{\tau}/2, \overrightarrow{\gamma}/i \}) = 2, \quad i = 1, 2, \quad \forall (u, v) \in U. \]
The pair \( (\Phi, \overrightarrow{\gamma}) \) is called a relatively normalized ruled surface and the line issuing from a point \( P \in \Phi \) in the direction of \( \overrightarrow{\gamma} \) is called the relative normal of \( \Phi \) at \( P \). The pair \( \Phi^* = (U, \overrightarrow{\gamma}) \) is called the relative image of \( (\Phi, \overrightarrow{\gamma}) \).
The support function of the relative normalization $\bar{y}$ is defined by

$$q(u, v) := \langle \bar{\xi}, \bar{y} \rangle$$

(see [2]). Because of (1.9), $q$ never vanishes on $U$. Conversely, when a support function $q$ is given, the relative normalization $\bar{y}$ of the ruled surface $\Phi$ is uniquely determined and can be expressed in terms of the moving frame $D$ as follows [6, p. 179]:

$$\bar{y} = y_1 \bar{v} + y_2 \bar{\pi} + y_3 \bar{z},$$

(1.10)

where

$$y_1 = -w \frac{\delta q_1 + q_2 (\kappa w^2 + \delta' v)}{\delta^2}, \quad y_2 = \frac{\delta^2 q - w^2 v q_2}{\delta w}, \quad y_3 = -v q + w^2 q_2 \frac{\lambda}{w}.$$  

(1.11)

The coefficients $G_{ij}(u, v)$ of the relative metric $G(u, v)$ of $(\Phi, \bar{y})$, which is indefinite, are given by

$$G_{ij} = q^{-1} h_{ij}.$$  

Then, by taking (1.7) into consideration, the coefficients of the inverse relative metric tensor are computed by

$$G^{(11)} = 0, \quad G^{(12)} = \frac{w q}{\delta}, \quad G^{(22)} = w \frac{\kappa w^2 + \delta' v - \delta^2 \lambda}{\delta^2}.$$  

For a function (or a vector-valued function) $f$ we denote by $\nabla^G f$ the first Beltrami differential operator and by $\nabla^G_i f$ the covariant derivative in the direction $u^i$, both with respect to the relative metric. The coefficients $A_{ijk}(u, v)$ of the Darboux tensor are defined by

$$A_{ijk} := q^{-1} \langle \bar{\xi}, \nabla^G_i \nabla^G_j \bar{y}_k \rangle.$$  

Then, by using the relative metric tensor $G_{ij}$ for “raising and lowering the indices”, the Pick invariant $J(u, v)$ of $(\Phi, \bar{y})$ is given by

$$J := \frac{1}{2} A_{ijk} A^{ijk}.$$  

As we showed in [8] (see equation (2.2)) the Pick invariant is calculated by

$$J = \frac{3}{2} \left( \frac{w^2 q_2 + v q}{2 \delta^2 w^3 q} \right) \left\{ w^2 [\kappa q v + 2 \delta q_1 + q_2 (\kappa w^2 + \delta' v - \delta^2 \lambda)] - \delta^2 q (\lambda v - \delta') \right\}. \quad (1.12)$$

The relative shape operator has the coefficients $B^j_i(u, v)$ defined by

$$\bar{y}_i = -B^j_i \bar{y}_j.$$  

(1.13)

Then, for the relative curvature $K(u, v)$ and the relative mean curvature $H(u, v)$ of $(\Phi, \bar{y})$ we have

$$K := \det (B^j_i), \quad H := \frac{B^1_1 + B^2_2}{2}. \quad (1.14)$$

We mention finally, that among the surfaces $\Phi \subset \mathbb{E}^3$ with negative Gaussian curvature the ruled surfaces are characterized by the relation

$$3H - J - 3S = 0 \quad (1.15)$$

(see [7]), where $S(u, v)$ is the scalar curvature of the relative metric $G$, which is defined formally as the curvature of the pseudo-Riemannian manifold $(\Phi, G)$.  

3
2 Right normalizations

We focus now our investigation on the main subject of this paper, namely the right normalizations of a skew ruled surface $\Phi$, that is, relative normalizations which are given by (1.10) and (1.11) by means of the support function

$$q = \frac{f + g v}{w},$$

where $f$ and $g$ are arbitrary $C^{s+1}$-functions of $u$, such that $q \neq 0$. These normalizations are introduced in [8] by the authors.

When the function $g$ vanishes in $I$, the relative normal at each point $P \in \Phi$ lies on the corresponding asymptotic plane $\{P; \overline{\tau}, \overline{\pi}\}$ of $\Phi$. Normalizations of this type are called asymptotic and they have been studied by I. Kaffas and S. Stamatakis [6].

Another special case arises when the function $f$ vanishes in $I$. Then the relative normal at each point $P \in \Phi$ lies on the corresponding central plane $\{P; e, \overline{z}\}$ of $\Phi$. Normalizations of this type are called central and they have been studied in [8].

Since both asymptotic and central normalizations belong to the right ones and they have been studied thoroughly in the above mentioned papers, we assume that in what follows none of the functions $f$ and $g$ is vanishing.

From (1.10), (1.11) and (2.1) it follows that a right normalization of the given ruled surface $\Phi$ is

$$\overline{y} = \frac{(\kappa f - \delta g')v + \delta f' - \delta^2 \kappa g}{\delta^2} \overline{\tau} + \frac{f}{\delta} \overline{\pi} - g \overline{\tau}.$$  

(2.2)

Then, by using (1.3), (1.5), (1.13) and (2.2), we obtain the coefficients $B^i_j$ of the relative shape operator of a right normalization:

$$B^1_1 = \frac{\delta g' - \kappa f}{\delta^2},$$

$$B^1_2 = 0,$$

$$B^2_1 = \frac{\delta f - \delta^2 \kappa g}{\delta^2} - \frac{\kappa f'}{\delta^2},$$

$$B^2_2 = \frac{\delta f' - \kappa f}{\delta^2}.$$  

(2.3)

Hence, via (1.14), the relative mean curvature $H$ and the relative curvature $K$ are

$$H = \frac{\delta g' - \kappa f}{\delta^2}, \quad K = H^2.$$  

(2.4)

Firstly, we observe that all points of $\Phi$ are relative umbilics ($H^2 - K \equiv 0$). Thus, for the relative principal curvatures $k_1$ and $k_2$, which by definition are the eigenvalues of the relative shape operator (see [5, p. 215]),

$$k_1 = k_2 = H$$

holds.

Then, from (1.12) we find for the Pick invariant

$$J = 3g \frac{\kappa g v^2 + 2 \delta g' v + \delta^2 g (\kappa - \lambda) - \delta f' + 2 \delta f'}{2 \delta^2 (f + g v)}.$$
Consequently $J$ vanishes identically iff 
\[
\kappa g v^2 + 2\delta g'v + \delta^2 g (\kappa - \lambda) - \delta' f + 2\delta f' = 0,
\]
or, equivalently, after successive differentiations of this last equation relative to $v$, iff 
\[
\kappa = g' = \delta^2 g (\kappa - \lambda) - \delta' f + 2\delta f' = 0,
\]
from which we have 
\[
\kappa = 0,
\]
i.e., $\Phi$ is conoidal, 
\[
g = c_1 \in \mathbb{R}^*
\]
and 
\[
f = |\delta|^{1/2} \left( \frac{c_1}{2} \int |\delta|^{1/2} \lambda \, du + c_2 \right), \quad c_2 \in \mathbb{R}.
\]
Thus, the following has been shown

**Proposition 2.1.** The Pick invariant of a right normalized skew ruled surface $\Phi \subset \mathbb{E}^3$ vanishes identically iff $\Phi$ is conoidal, the function $g$ is a nonvanishing constant $c_1$ and the function $f$ is given by 
\[
f = |\delta|^{1/2} \left( \frac{c_1}{2} \int |\delta|^{1/2} \lambda \, du + c_2 \right), \quad c_2 \in \mathbb{R}.
\]

Additionally, in view of (2.3a) and (2.5), a right normalized ruled surface with vanishing Pick invariant is relatively minimal.

By using (1.15), (2.3a) and (2.4) we obtain the scalar curvature of the relative metric 
\[
S = -\frac{\kappa g^2 v^2 + 2\kappa f g v + \delta^2 g^2 (\kappa - \lambda) + 2\kappa f^2 - \delta f g + 2\delta (f'g - f g')}{2\delta^2 (f + g v)}.
\]
The scalar curvature of the relative metric $G$ vanishes identically iff 
\[
\kappa = \delta^2 g^2 (\kappa - \lambda) + 2\kappa f^2 - \delta f g + 2\delta (f'g - f g') = 0,
\]
that is, iff 
\[
\kappa = 0
\]
and 
\[
f = \frac{1}{2} |\delta|^{1/2} g \left( \int |\delta|^{1/2} \lambda \, du + e \right), \quad e \in \mathbb{R}.
\]
So, we have:

**Proposition 2.2.** The scalar curvature $S$ of a right normalized skew ruled surface $\Phi \subset \mathbb{E}^3$ vanishes identically iff $\Phi$ is conoidal and the function $f$ is given by 
\[
f = \frac{1}{2} |\delta|^{1/2} g \left( \int |\delta|^{1/2} \lambda \, du + e \right), \quad e \in \mathbb{R}.
\]

We distinguish the right normalizations in two types.
2.1 Right normalizations of the first type

We say that a right relative normalization \( \overline{y} \) is of the first type if the relative image \( \Phi^* \) of \((\Phi, \overline{y})\) degenerates into a curve. Obviously this occurs iff
\[
\delta g' - \kappa f = 0
\]
(cf. (2.2)). Thus, on account of (2.2) and (2.3), we conclude:

**Proposition 2.3.** Let \((\Phi, \overline{y})\) be a right normalized ruled surface. Then the following properties are equivalent:

(a) \( \overline{y} \) is a right normalization of the first type.
(b) \((\Phi, \overline{y})\) is relatively minimal.
(c) The function \( g \) is given by
\[
g = \int \frac{\kappa f}{\delta} \, du + c, \quad c \in \mathbb{R}.
\]

The right normalized ruled surfaces with vanishing Pick invariant belong obviously to this subclass.

The relative image \( \Phi^* \) is the curve parametrized by
\[
\overline{y} = \frac{\delta f - \delta f' - \delta^2 \kappa g}{\delta^2} \tau + \frac{f}{\delta} \pi - g \pi.
\]

2.2 Right normalizations of the second type

A right relative normalization \( \overline{y} \) is said to be of the second type if the relative image \( \Phi^* \) of \((\Phi, \overline{y})\) does not degenerate into a curve of \( \mathbb{E}^3 \). Then \( \Phi^* \) is a ruled surface whose generators are parallel to those of \( \Phi \). From (2.2) we find the following parametrization of the striction curve of \( \Phi^* \):
\[
\Gamma^*: \tau = \frac{\delta f - \delta f' - \delta^2 \kappa g}{\delta^2} \tau + \frac{f}{\delta} \pi - g \pi.
\]

Consequently \( \Phi^* \) can be parametrized like (1.1) and (1.2):
\[
\Phi^*: \overline{y} = \tau + v^* \pi,
\]
where
\[
v^* := \frac{(\kappa f - \delta g') v}{\delta^2}.
\]

Considering \( D \) as moving frame of \( \Phi^* \) we compute its fundamental invariants:
\[
\kappa^* = \kappa, \quad \delta^* = \frac{\kappa f - \delta g'}{\delta}, \quad \lambda^* = -\frac{\delta^3 (\kappa g' + \kappa' g) + \delta^2 (f + f'') - \delta (\delta'' f + 2 \delta' f') + 2 \delta^2 f}{\delta^2 (\kappa f - \delta g')}.
\]

By using (1.6b) we infer that
\[
w^* = |H| w
\]
and, thus, by means of (1.8a), the Gaussian curvature \( \tilde{K}^* \) of \( \Phi^* \) is
\[
\tilde{K}^* = -\frac{\delta^6}{w^4 (\kappa f - \delta g')^2}.
\]
The focal surfaces, which are the loci of the edges of regression of the developable surfaces consisting of the relative normals along the relative lines of curvature, coincide. The parametrization of the unique relative focal surface of $\Phi$, which initially reads

$$\vec{x}^* = \vec{s} + v \vec{e} + \frac{1}{H} \vec{g},$$

in view of (2.2) and (2.3a) becomes

$$\vec{x}^* = \vec{s} + \frac{(\delta f - \delta f' - \delta^2 \kappa g) \vec{e} + \delta^2 g \vec{v}}{\delta g' - \kappa f},$$

i.e., the focal surface degenerates into a curve $\Lambda^*$ and all relative normals along each generator form a pencil of straight lines.

3 The Tchebychev vector field of a right normalization

In [6] it was shown that the coordinate functions of the Tchebychev vector $\vec{T}(u,v)$ of $(\Phi, \vec{g})$, which is defined by

$$\vec{T} := T^m \vec{x}/m,$$

where

$$T^m := \frac{1}{2} A^{im}_1,$$

are given by

$$T^1 = \frac{w^2 q_2/2 + v q}{\delta w}, \quad T^2 = \frac{2 \delta w^2 q_1/2 + \delta' q (\delta^2 - v^2)}{2 \delta^2 w} + \frac{T^1 (\kappa w^2 + \delta' v - \delta^2 \lambda)}{\delta}.$$

By means of (1.5) and (2.1) the Tchebychev vector of a right normalization can be expressed in terms of the moving frame $\mathcal{D}$ as follows:

$$\vec{T} = \frac{2 \kappa g v^2 + (\delta' g + 2 \delta g') v + 2 \delta^2 \kappa g - \delta' f + 2 \delta f'}{2 \delta^2} \vec{e} + \frac{g}{\delta} (v \vec{w} + \delta \vec{v}).$$

The vectors $\vec{T}$ are orthogonal to the generators iff $\langle \vec{e}, \vec{T} \rangle = 0$. Taking (3.2) into consideration we find

$$2 \kappa g v^2 + (\delta' g + 2 \delta g') v + 2 \delta^2 \kappa g - \delta' f + 2 \delta f' = 0,$$

or, after successive differentiations of this last equation relative to $v$, iff

$$2 \kappa g = \delta' g + 2 \delta g' = 2 \delta^2 \kappa g - \delta' f + 2 \delta f' = 0.$$

After standard treatment of this system we deduce that

$$\kappa = 0,$$

$$g = c_1 |\delta|^{-1/2}, \quad c_1 \in \mathbb{R}^*$$

and

$$f = c_2 |\delta|^{1/2}, \quad c_2 \in \mathbb{R}^*.$$

So, we have the following
Proposition 3.1. The Tchebychev vector field $\mathbf{T}$ of a right normalized skew ruled surface $\Phi \subset \mathbb{E}^3$ is orthogonal to the generators of $\Phi$ iff $\Phi$ is conoidal and the functions $g$ and $f$ are given by

$$g = c_1|\delta|^{-1/2}, \ c_1 \in \mathbb{R}^* \text{ and } f = c_2|\delta|^{1/2}, \ c_2 \in \mathbb{R}^*.$$ 

We turn now to the right normalized ruled surfaces $(\Phi, \mathbf{T})$, whose Tchebychev vectors are tangent or orthogonal to one of the following geometrically distinguished families of curves of $\Phi$:

a. the curves of constant striction distance ($u$-curves),

b. the curved asymptotic lines and
c. the $\tilde{K}$-curves, i.e., the curves along which the Gaussian curvature is constant [4].

The corresponding differential equations of these families of curves are

$$v' = 0, \tag{3.3}$$

$$\kappa v^2 + \delta' v + \delta^2(\kappa - \lambda) - 2\delta v' = 0, \tag{3.4}$$

$$2\delta vv' + \delta' (\delta^2 - v^2) = 0. \tag{3.5}$$

We will investigate necessary and sufficient conditions for the Tchebychev vector field $\mathbf{T}$ to be tangential or orthogonal to each one of these families of curves.

We consider a directrix $\Lambda : v = v(u)$ of $\Phi$. Then we have

$$\mathbf{T}' = (\delta \lambda + v')\mathbf{T} + v \mathbf{N} + \delta \mathbf{N}. \tag{3.6}$$

From (3.2) and (3.6) it follows: $\mathbf{T}'$ and $\mathbf{T}$ are parallel or orthogonal iff

$$2\kappa g v^2 + (\delta' g + 2\delta g') v + 2\delta^2 \kappa g - \delta' f + 2\delta f' - 2\delta g (\delta \lambda + v') = 0 \tag{3.7}$$

or

$$[2\kappa g v^2 + (\delta' g + 2\delta g') v + 2\delta^2 \kappa g - \delta' f + 2\delta f'] (\delta \lambda + v') + 2\delta g w^2 = 0, \tag{3.8}$$

respectively.

From (3.3) and (3.7), resp. (3.8), we have: $\mathbf{T}$ is tangential or orthogonal to the $u$-curves iff

$$2\kappa g v^2 + (\delta' g + 2\delta g') v + 2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = 0 \tag{3.9}$$

or

$$2g(\kappa \lambda + 1)v^2 + \lambda(\delta' g + 2\delta g') v + 2\delta^2 g(\kappa \lambda + 1) + \lambda(2\delta f' - \delta f) = 0, \tag{3.10}$$

respectively. From (3.9) we find that $\mathbf{T}$ is tangential to the $u$-curves iff

$$\kappa = \delta' g + 2\delta g' = 2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = 0,$$

that is, iff

$$\kappa = 0,$$

$$g = c_1|\delta|^{-1/2}, \ c_1 \in \mathbb{R}^*$$

and

$$f = |\delta|^{1/2} \left( c_1 \int \lambda du + c_2 \right), \ c_2 \in \mathbb{R}.$$
From (3.10) we derive that \( \overline{T} \) is orthogonal to the \( u \)-curves iff
\[
\kappa \lambda + 1 = \lambda (\delta' g + 2\delta g') = 2\delta^2 g (\kappa \lambda + 1) + \lambda (2\delta f' - \delta' f) = 0.
\]
By direct computation we deduce that
\[
\kappa \lambda + 1 = 0,
\]
i.e., the striction curve of \( \Phi \) is an Euclidean line of curvature,
\[
g = c_1 |\delta|^{-1/2}, \quad c_1 \in \mathbb{R}^*
\]
and
\[
f = c_2 |\delta|^{1/2}, \quad c_2 \in \mathbb{R}^*.
\]
Therefore, we obtain

**Proposition 3.2.** The Tchebychev vector field \( \overline{T} \) of a right normalized skew ruled surface \( \Phi \subset \mathbb{E}^3 \) is
(a) tangential to the \( u \)-curves of \( \Phi \) iff \( \Phi \) is conoidal and the functions \( g \) and \( f \) are given by
\[
g = c_1 |\delta|^{-1/2}, \quad c_1 \in \mathbb{R}^* \quad \text{and} \quad f = |\delta|^{1/2} \left( c_1 \int \lambda \, du + c_2 \right), \quad c_2 \in \mathbb{R}.
\]
(b) orthogonal to the \( u \)-curves of \( \Phi \) iff the striction curve of \( \Phi \) is an Euclidean line of curvature and the functions \( g \) and \( f \) are given by
\[
g = c_1 |\delta|^{-1/2}, \quad c_1 \in \mathbb{R}^* \quad \text{and} \quad f = c_2 |\delta|^{1/2}, \quad c_2 \in \mathbb{R}^*.
\]
From (3.3) and (3.7) we infer, that \( \overline{T} \) is tangential to the curved asymptotic lines iff
\[
\kappa g v^2 + 2\delta g' v + \delta^2 g (\kappa - \lambda) - \delta' f + 2\delta f' = 0,
\]
that is, iff
\[
\kappa = g' = \delta^2 g (\kappa - \lambda) - \delta' f + 2\delta f' = 0,
\]
from which we have
\[
\kappa = 0,
\]
\[
g = c_1 \in \mathbb{R}^*
\]
and
\[
f = |\delta|^{1/2} \left( \frac{c_1}{2} \int |\delta|^{1/2} \lambda \, du + c_2 \right), \quad c_2 \in \mathbb{R}.
\]
So, we arrive at

**Proposition 3.3.** The Tchebychev vector field \( \overline{T} \) of a right normalized skew ruled surface \( \Phi \subset \mathbb{E}^3 \) is tangential to the curved asymptotic lines of \( \Phi \) iff \( \Phi \) is conoidal, the function \( g \) is a nonvanishing constant \( c_1 \) and the function \( f \) is given by
\[
f = |\delta|^{1/2} \left( \frac{c_1}{2} \int |\delta|^{1/2} \lambda \, du + c_2 \right), \quad c_2 \in \mathbb{R}.
\]
From (3.5) and (3.7), resp. (3.8), we infer: $\bar{T}$ is tangential or orthogonal to the $\tilde{K}$-curves iff

$$2\kappa gv^3 + 2\delta g' v^2 + \left[2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' \right] v + \delta^2 \delta' g = 0 \quad (3.11)$$

or

$$2\kappa \delta' gv^4 + \left[4\delta^2 g(\kappa\lambda + 1) + \delta'(\delta' g + 2\delta g') \right] v^3 
+ (2\delta^2 \delta' \lambda g - \delta^2 f + 2\delta' f' + 4\delta^3 \lambda g') v^2 
+ \delta^2 \left[4\delta^2 g(\kappa\lambda + 1) - 2\delta' \lambda f - \delta^2 g + 4\delta\lambda f' - 2\delta' g' \right] v 
- \delta^2 \delta' (2\delta^2 \kappa g - \delta' f + 2\delta f') = 0, \quad (3.12)$$

respectively. From (3.11) we find that $\bar{T}$ is tangential to the $\tilde{K}$-curves iff

$$\kappa = g' = 2\delta^2 g(\kappa - \lambda) - \delta' f + 2\delta f' = \delta' = 0,$$

i.e., iff

$$\kappa = 0,$$

$$\delta = c_1 \in \mathbb{R}^*,$$

$$g = c_2 \in \mathbb{R}^*$$

and

$$f = c_1 c_2 \int \lambda \, du + c_3, \quad c_3 \in \mathbb{R}.$$  

From (3.12) we deduce that $\bar{T}$ is orthogonal to the $\tilde{K}$-curves iff

$$\kappa \delta' = 4\delta^2 g(\kappa\lambda + 1) + \delta'(\delta' g + 2\delta g') = 2\delta^2 \delta' \lambda g - \delta^2 f + 2\delta' f' + 4\delta^3 \lambda g' = 0,$$

$$4\delta^2 g(\kappa\lambda + 1) - 2\delta' \lambda f - \delta^2 g + 4\delta\lambda f' - 2\delta' g' = \delta' (2\delta^2 \kappa g - \delta' f + 2\delta f') = 0,$$

that is, iff

$$\delta = c \in \mathbb{R}^*$$

or

$$\kappa = 0.$$

If

$$\delta = c \in \mathbb{R}^*,$$

we deduce that

$$\kappa \lambda + 1 = 0,$$

i.e., $\Phi$ is an Edlinger surface \footnote{i.e., a ruled surface whose osculating quadrics are rotational hyperboloids. The Edlinger surfaces are characterized by the conditions $\delta = \kappa \lambda + 1 = 0$ (see \cite{1}, p. 36), \cite{1}.},

$$g = c_1 \in \mathbb{R}^*$$

and

$$f = c_2 \in \mathbb{R}^*.$$  

If

$$\kappa = 0$$

and

$$\delta \neq c \in \mathbb{R}^*$$

we arrive at a contradiction. Thus, the following has been shown
Proposition 3.4. The Tchebychev vector field $\mathbf{T}$ of a right normalized skew ruled surface $\Phi \subset \mathbb{E}^3$ is

(a) tangential to the $\tilde{K}$-curves of $\Phi$ iff $\Phi$ is conoidal of constant distribution parameter $c_1$, the function $g$ is a nonvanishing constant $c_2$ and the function $f$ is given by

$$f = c_1 c_2 \int \lambda \, du + c_3, \; c_3 \in \mathbb{R}.$$ 

(b) orthogonal to the $\tilde{K}$-curves of $\Phi$ iff $\Phi$ is an Edlinger surface and the functions $g$ and $f$ are nonvanishing constants $c_1$ and $c_2$, respectively.

The following table summarizes the results:

| $\mathbf{T}$ is . . . | Type of the ruled surface $\Phi$ | $g$ | $f$ |
|-----------------------|---------------------------------|-----|-----|
| orthogonal to the generators | conoidal                        | $g = c_1 |\delta|^{-1/2}$, $c_1 \in \mathbb{R}^*$ | $f = c_2 |\delta|^{1/2}$, $c_2 \in \mathbb{R}^*$ |
| tangential to the $u$-curves | conoidal                        | $g = c_1 |\delta|^{-1/2}$, $c_1 \in \mathbb{R}^*$ | $f = |\delta|^{1/2}$ ($c_1 f \lambda \, du + c_2$), $c_2 \in \mathbb{R}$ |
| orthogonal to the $u$-curves | the striction curve is an Euclidean line of curvature | $g = c_1 |\delta|^{-1/2}$, $c_1 \in \mathbb{R}^*$ | $f = c_2 |\delta|^{1/2}$, $c_2 \in \mathbb{R}^*$ |
| tangential to the curved asympt. lines | conoidal                        | $g = c_1 \in \mathbb{R}^*$ | $f = |\delta|^{1/2}$ ($\frac{1}{2} f |\delta|^{1/2} \lambda \, du + c_2$), $c_2 \in \mathbb{R}$ |
| tangential to the $\tilde{K}$-curves | conoidal                        | $g = c_2 \in \mathbb{R}^*$ | $f = c_1 c_2 \lambda \, du + c_3, \; c_3 \in \mathbb{R}$ |
| orthogonal to the $\tilde{K}$-curves | Edlinger surface                | $g = c_1 \in \mathbb{R}^*$ | $f = c_2 \in \mathbb{R}^*$ |

The divergence $\text{div}^\mathbf{I} \mathbf{T}$ of $\mathbf{T}$ with respect to the first fundamental form $\mathbf{I}$ of $\Phi$, which initially reads (see [6])

$$\text{div}^\mathbf{I} \mathbf{T} = \frac{(w T^i) / \partial_i}{w}$$

becomes, on account of (3.1) and (2.1),

$$\text{div}^\mathbf{I} \mathbf{T} = \frac{6 \kappa g v^3 + 6 \delta g' v^2 + (6 \delta^2 \kappa g - 2 \delta^2 \lambda g - \delta' f + 2 \delta f') v + \delta^2 (\delta' g + 4 \delta g')}{2 \delta^2 w^2},$$

from which we have that the Tchebychev vector field $\mathbf{T}$ is incompressible with respect to the first fundamental form of $\Phi$ ($\text{div}^\mathbf{I} \mathbf{T} = 0$) iff

$$\kappa = g' = 6 \delta^2 \kappa g - 2 \delta^2 \lambda g - \delta' f + 2 \delta f' = \delta' g + 4 \delta g' = 0,$$

or iff

$$\kappa = 0,$$

$$g = c_1 \in \mathbb{R}^*, \quad \delta = c_2 \in \mathbb{R}^*$$

and

$$f = c_1 c_2 \int \lambda \, du + c_3, \; c_3 \in \mathbb{R}.$$
Proposition 3.5. The Tchebychev vector field $\mathbf{T}$ of a right normalized skew ruled surface $\Phi \subset \mathbb{E}^3$ is incompressible with respect to the first fundamental form of $\Phi$ iff $\Phi$ is conoidal of constant distribution parameter $c_2$, the function $g$ is a nonvanishing constant $c_1$ and the function $f$ is given by

$$f = c_1 c_2 \int \lambda du + c_3, \; c_3 \in \mathbb{R}.$$ 

Let, now, $\text{div}^G \mathbf{T}$ be the divergence of $\mathbf{T}$ with respect to the relative metric of $(\Phi, \mathbf{y})$. Analogously to the above computation, by using (1.7), we get

$$\text{div}^G \mathbf{T} = \frac{\kappa g^2 v^2 + 2\kappa f g v - \delta^2 g^2 (\kappa - \lambda) + \delta' f g - 2\delta g f' + 2\delta f g'}{\delta^2 (g v + f)}.$$ 

The Tchebychev vector field $\mathbf{T}$ is incompressible with respect to the relative metric ($\text{div}^G \mathbf{T} = 0$) iff

$$\kappa = -\delta^2 g^2 (\kappa - \lambda) + \delta' f g - 2\delta g f' + 2\delta f g' = 0,$$

i.e., iff

$$\kappa = 0$$

and

$$f = \frac{1}{2} |\delta|^{1/2} g \left( \int |\delta|^{1/2} \lambda du + c \right), \; c \in \mathbb{R}.$$ 

So, by taking into consideration Proposition 2.2 we deduce:

Proposition 3.6. Let $\Phi \subset \mathbb{E}^3$ be a right normalized skew ruled surface. The following properties are equivalent:

(a) The Tchebychev vector field $\mathbf{T}$ is incompressible with respect to the relative metric.

(b) The scalar curvature $S$ of the relative metric vanishes identically.

(c) $\Phi$ is conoidal and the function $f$ is given by

$$f = \frac{1}{2} |\delta|^{1/2} g \left( \int |\delta|^{1/2} \lambda du + c \right), \; c \in \mathbb{R}.$$ 

4 The support vector field of a right normalization

Let

$$\overline{\mathbf{Q}} := \frac{1}{4} \nabla^G \left( \frac{1}{\kappa} \mathbf{T} \right)$$

be the support vector $\overline{\mathbf{Q}}(u, v)$ of $(\Phi, \mathbf{y})$, which is introduced in [6]. On account of (1.5), (1.7) and (2.1) we express the support vector in terms of the moving frame $\mathcal{D}$ as follows:

$$\overline{\mathbf{Q}} = -w \left( \delta g' - \kappa f \right) v + \delta^2 \kappa g - \delta' f + \delta f' \overline{\mathbf{r}} + \frac{f v - \delta^2 g}{4\delta w (g v + f)} \left( v \overline{\mathbf{r}} + \delta \overline{\mathbf{z}} \right). \tag{4.1}$$

The vectors $\overline{\mathbf{Q}}$ are orthogonal to the generators iff $\langle \overline{\mathbf{r}}, \overline{\mathbf{Q}} \rangle = 0$. Taking (4.1) into consideration we have

$$\left( \delta g' - \kappa f \right) v + \delta^2 \kappa g - \delta' f + \delta f' = 0,$$

that is, iff

$$\delta g' - \kappa f = \delta^2 \kappa g - \delta' f + \delta f' = 0,$$
from which we find that $\Phi$ is relative minimal and
\[ f = \pm \delta |c - g^2|^{1/2}, \ c \in \mathbb{R}, \ g^2 \neq c. \]

Thus, we arrive at:

**Proposition 4.1.** The support vector field $\overrightarrow{Q}$ of a right normalized skew ruled surface $\Phi \subset \mathbb{E}^3$ is orthogonal to the generators of $\Phi$ iff $\Phi$ is relative minimal and the function $f$ is given by
\[ f = \pm \delta |c - g^2|^{1/2}, \ c \in \mathbb{R}, \ g^2 \neq c. \]

We will investigate, now, the right normalized ruled surfaces $\Phi$, whose support vectors are tangent or orthogonal to the above mentioned geometrically distinguished families of curves of $\Phi$. From (3.6) and (4.1) we have:

\[ x' \text{ and } Q \text{ are parallel or orthogonal iff } w^2 [(\delta g' - \kappa f) v + \delta^2 \kappa g - \delta' f + \delta f'] + \delta (fv - \delta^2 g) (\delta \lambda + v') = 0 \]  \hspace{1cm} (4.2)

or
\[ - (\delta \lambda + v') [(\delta g' - \kappa f) v + \delta^2 \kappa g - \delta' f + \delta f'] + \delta (fv - \delta^2 g) = 0. \]  \hspace{1cm} (4.3)

From (4.1) and (4.2), resp. (4.3), we find: $\overrightarrow{Q}$ is tangential or orthogonal to the $u$-curves iff
\[ (\kappa f - \delta g') v^3 + (-\delta^2 \kappa g + \delta' f - \delta f') v^2 + \delta^2 [f (\kappa - \lambda) - \delta g'] v - \delta^2 \delta g (\kappa - \lambda) - \delta' f + \delta f' = 0 \]  \hspace{1cm} (4.4)

or
\[ [f (\kappa \lambda + 1) - \delta \lambda g'] v - \delta^2 g (\kappa \lambda + 1) + \lambda (\delta' f - \delta f') = 0, \]  \hspace{1cm} (4.5)

respectively. From (4.4) we infer that $\overrightarrow{Q}$ is tangential to the $u$-curves iff
\[ \kappa f - \delta g' = -\delta^2 \kappa g + \delta' f - \delta f' = f (\kappa - \lambda) - \delta g' = \delta^2 g (\kappa - \lambda) - \delta' f + \delta f' = 0, \]

that is, iff $\Phi$ is relative minimal,
\[ \lambda = 0, \]

i.e., $\Phi$ is orthoid\(^2\) and
\[ f = \pm \delta |c - g^2|^{1/2}, \ c \in \mathbb{R}, \ g^2 \neq c. \]

From (4.2) we take that $\overrightarrow{Q}$ is orthogonal to the $u$-curves iff
\[ f (\kappa \lambda + 1) - \delta \lambda g' = -\delta^2 g (\kappa \lambda + 1) + \lambda (\delta' f - \delta f') = 0, \]

i.e., iff
\[ \kappa \lambda + 1 = \frac{\delta \lambda g'}{f} \]

and
\[ f = \pm \delta |c - g^2|^{1/2}, \ c \in \mathbb{R}, \ g^2 \neq c, \]

hence
\[ \kappa = \pm g' |c - g^2|^{-1/2} - \lambda^{-1}, \ \lambda \neq 0. \]

Therefore, we obtain
\(^2\)that is, a ruled surface whose striction curve is an orthogonal trajectory of the generators. The ortoid ruled surfaces are characterized by the condition $\lambda = 0$. 

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Proposition 4.2. The support vector field $Q$ of a right normalized skew ruled surface $\Phi \subset E^3$ is

(a) tangential to the $u$-curves of $\Phi$ iff $\Phi$ is an orthoid, relative minimal surface and the function $f$ is given by

$$ f = \pm \delta |c - g^2|^{1/2}, \ c \in \mathbb{R}, \ g^2 \neq c. $$

(b) orthogonal to the $u$-curves of $\Phi$ iff the conical curvature and the function $f$ are given by

$$ \kappa = \pm g' |c - g^2|^{-1/2} - \lambda^{-1}, \ c \in \mathbb{R}, \ \lambda \neq 0, \ g^2 \neq c \quad \text{and} \quad f = \pm \delta |c - g^2|^{1/2}. $$

From (3.3) and (4.2) we have, that $Q$ is tangential to the curved asymptotic lines iff

$$ (\kappa f - 2\delta g') v^3 + (-\delta^2 \kappa g + \delta' f - 2\delta f') v^2 + \delta^2 \left[ f (\kappa - \lambda) + \delta' g - 2\delta g' \right] v $$

$$ - \delta^2 \left[ \delta^2 g (\kappa - \lambda) - 2\delta' f + 2\delta f' \right] = 0, $$

i.e., iff

$$ \kappa f - 2\delta g' = -\delta^2 \kappa g + \delta' f - 2\delta f' = 0, $$

$$ f (\kappa - \lambda) + \delta' g - 2\delta g' = \delta^2 g (\kappa - \lambda) - 2\delta' f + 2\delta f' = 0. $$

Treating the above system in the standard way we find that

$$ \lambda = \delta' = 0. $$

If

$$ \kappa = 0, $$

$\Phi$ is right helicoid \(^3\)

$$ f = c_1 \in \mathbb{R}^* $$

and

$$ g = c_2 \in \mathbb{R}^*. $$

If

$$ \kappa \neq 0, $$

$\Phi$ is orthoid of constant distribution parameter $c_3$,

$$ \kappa = \pm 2\delta' |c_4 - g^2|^{-1/2}, \ c_4 \in \mathbb{R}^*, \ g' \neq 0, \ g^2 \neq c_4 $$

and

$$ f = \pm c_3 |c_4 - g^2|^{1/2}. $$

So, we can state

Proposition 4.3. The support vector field $Q$ of a right normalized skew ruled surface $\Phi \subset E^3$ is tangential to the curved asymptotic lines of $\Phi$ iff

(a) $\Phi$ is right helicoid, the function $f$ is a nonvanishing constant $c_1$ and the function $g$ is a nonvanishing constant $c_2$, or

(b) $\Phi$ is orthoid of constant distribution parameter $c_3$ and the conical curvature and the function $f$ are given by

$$ \kappa = \pm 2\delta' |c_4 - g^2|^{-1/2}, \ c_4 \in \mathbb{R}^*, \ g' \neq 0, \ g^2 \neq c_4 \quad \text{and} \quad f = \pm c_3 |c_4 - g^2|^{1/2}. $$

\(^3\)The right helicoids are characterized by the conditions $\delta = c \in \mathbb{R}^*$ and $\kappa = \lambda = 0.$
From (4.3) and (4.2), resp. (4.4), we deduce: \( \overline{Q} \) is tangential or orthogonal to the \( \tilde{K} \)-curves iff

\[
2 (\kappa f - \delta g') v^4 - (2\delta^2 \kappa g + \delta' f + 2\delta f') v^3 + \delta^2 [2f (\kappa - \lambda) + \delta' g - 2\delta g'] v^2
- \delta^2 [2\delta^2 g (\kappa - \lambda) - 3\delta' f + 2\delta f'] v - \delta^4 \delta' g = 0
\]  

(4.6)

or

\[
\delta' (\kappa f - \delta g') v^4 + [2\delta^2 f (\kappa \lambda + 1) - \delta^2 \delta' \kappa g + \delta^2 f - \delta \delta' f' - 2\delta^3 \lambda g'] v^2
- \delta^2 [2\delta^2 g (\kappa \lambda + 1) + \delta' \kappa f + 2\lambda (\delta f' - \delta f) - \delta \delta' g'] v
+ \delta^2 \delta' (\delta^2 \kappa g - \delta' f + \delta f') = 0,
\]  

(4.7)

respectively. From (4.6) we have that \( \overline{Q} \) is tangential to the \( \tilde{K} \)-curves iff

\[
\begin{align*}
\kappa f - \delta g' &= 2\delta^2 \kappa g - \delta' f + 2\delta f' = 2f (\kappa - \lambda) + \delta' g - 2\delta g' = 0, \\
2\delta^2 g (\kappa - \lambda) - 3\delta' f + 2\delta f' &= \delta' = 0,
\end{align*}
\]

from which we obtain that \( \Phi \) is relative minimal,

\[
\delta = c_1 \in \mathbb{R}^*, \quad \lambda = 0
\]

and

\[
f = \pm |c_2 - c_1^2 g^2|^{1/2}, \quad c_2 \in \mathbb{R}, \quad c_1^2 g^2 \neq c_2.
\]

From (4.7) we infer that \( \overline{Q} \) is orthogonal to the \( \tilde{K} \)-curves iff

\[
\begin{align*}
\delta' (\kappa f - \delta g') &= 2\delta^2 f (\kappa \lambda + 1) - \delta^2 \delta' \kappa g + \delta^2 f - \delta \delta' f' - 2\delta^3 \lambda g' = 0, \\
2\delta^2 g (\kappa \lambda + 1) + \delta' \kappa f + 2\lambda (\delta f' - \delta f) - \delta \delta' g' &= \delta' (\delta^2 \kappa g - \delta' f + \delta f') = 0,
\end{align*}
\]

that is, iff \( \Phi \) is relative minimal or

\[
\delta = c_1 \in \mathbb{R}^*.
\]

If \( \Phi \) is relative minimal we arrive at a contradiction.

If

\[
\delta = c_1 \in \mathbb{R}^*,
\]

we have

\[
\kappa \lambda + 1 = \frac{c_1 \lambda g'}{f}
\]

and

\[
f = \pm |c_2 - c_1^2 g^2|^{1/2}, \quad c_2 \in \mathbb{R}, \quad c_1^2 g^2 \neq c_2,
\]

hence

\[
\kappa = \pm c_1 g' |c_2 - c_1^2 g^2|^{-1/2} - \lambda^{-1}, \quad \lambda \neq 0.
\]

Thus, we deduce

**Proposition 4.4.** The support vector field \( \overline{Q} \) of a right normalized skew ruled surface \( \Phi \subset E^3 \) is

(a) tangential to the \( \tilde{K} \)-curves of \( \Phi \) iff \( \Phi \) is an orthoid, relative minimal surface of constant distribution parameter \( c_1 \) and the function \( f \) is given by

\[
f = \pm |c_2 - c_1^2 g^2|^{1/2}, \quad c_2 \in \mathbb{R}, \quad c_1^2 g^2 \neq c_2.
\]
The following table summarizes the results:

| $\mathbb{Q}$ is . . . | Type of the ruled surface $\Phi$ | $f, g$ |
|----------------------|---------------------------------|-------|
| orthogonal to the generators | relative minimal | $f = \pm \delta |c - g^2|^{1/2}, c \in \mathbb{R}, g^2 \neq c$. |
| tangential to the $u$-curves | orthoid, relative minimal | $f = \pm \delta |c - g^2|^{1/2}, c \in \mathbb{R}, g^2 \neq c$. |
| orthogonal to the $u$-curves | $\kappa = \pm g |c - g^2|^{1/2} - \lambda^{-1}, c \in \mathbb{R}, \lambda \neq 0, g^2 \neq c$ | $f = \pm \delta |c - g^2|^{1/2}$ |
| tangential to the curved asympt. lines | right helicoid | $f = c_1 \in \mathbb{R}^*, g = c_2 \in \mathbb{R}^*$ |
| tangential to the $K$-curves | orthoid, relative minimal, $\delta = c_1 \in \mathbb{R}^*$ | $f = \pm |c_3 - c_2 g^2|^{1/2}, c_2 \in \mathbb{R}, c_1^2 g^2 \neq c_2$ |
| orthogonal to the $K$-curves | $\delta = c_1 \in \mathbb{R}^*$, $\kappa = \pm g |c - c_2 g^2|^{1/2} - \lambda^{-1}, c_2 \in \mathbb{R}, c_2 \neq 0, c_1^2 g^2 \neq c_2$ | $f = \pm |c_2 - c_1 g^2|^{1/2}$ |

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