More on logarithmic sums of convex bodies

Christos Saroglou
August, 2014

Abstract

We prove that the log-Brunn-Minkowski inequality (log-BMI) for the Lebesgue measure in dimension \( n \) would imply the log-BMI and, therefore, the B-conjecture for any log-concave density in dimension \( n \). As a consequence, we prove the log-BMI and the B-conjecture for any log-concave density, in the plane. Moreover, we prove that the log-BMI reduces to the following: For each dimension \( n \), there is a density \( f_n \), which satisfies an integrability assumption, so that the log-BMI holds for parallelepipeds with parallel facets, for the density \( f_n \). As byproduct of our methods, we study possible log-concavity of the function \( t \mapsto \| (K +_p e^t L)^{\alpha} \| \), where \( p \geq 1 \) and \( K, L \) are symmetric convex bodies, which we are able to prove in some instances and as a further application, we confirm the variance conjecture in a special class of convex bodies. Finally, we establish a non-trivial dual form of the log-BMI.

1 Introduction

Let \( K, L \) be convex bodies in \( \mathbb{R}^n \) (i.e. compact, convex sets, with non-empty interiors), that contains the origin in their interiors. For \( p \geq 1 \), the \( L^p \)-Minkowski-Firey sum \( a \cdot K +_p b \cdot L \) of \( K \) and \( L \) with respect to some positive numbers \( a, b \) is defined by its support function

\[
h_{a \cdot K +_p b \cdot L} = (a h^p_K + b h^p_L)^{1/p}.
\]

(1)

The case \( p = 1 \) corresponds to the classical Minkowski sum \( aK + bL = \{ ax + by \mid x \in K, y \in L \} \). In the pioneer work of Böröczky, Lutwak, Yang and Zhang \[4\], the \( L^p \)-convex combination of \( K \) and \( L \), with respect to some \( \lambda \in (0, 1) \), for all \( p \geq 0 \) is defined:

\[
\lambda \cdot K +_p (1 - \lambda) \cdot L = \{ x \in \mathbb{R}^n \mid x \cdot u \leq [\lambda h^p_K(u) + (1 - \lambda) h^p_L(u)]^{1/p}, \text{ for all } u \in S^{n-1} \}
\]

and

\[
\lambda \cdot K +_0 (1 - \lambda) \cdot L = \{ x \in \mathbb{R}^n \mid x \cdot u \leq h^\lambda_K(u) h^{1-\lambda}_L(u), \text{ for all } u \in S^{n-1} \}.
\]

Note that if \( 0 \leq p < 1 \), \( \lambda \cdot K +_p (1 - \lambda) \cdot L \) cannot be defined by (1), simply because the resulting function is not always convex. Nevertheless, the two definitions coincide for \( p \geq 1 \). Let us state the fundamental \( L^p \)-Brunn-Minkowski inequality (see e.g \[17\], \[41\], \[18\]), for \( p \geq 1 \) in its dimension-free form:

\[
|\lambda \cdot K +_p (1 - \lambda) \cdot L| \geq |K|^\lambda |L|^{1-\lambda},
\]

where \( | \cdot | = | \cdot |_n \) is the \( n \)-dimensional Lebesgue measure.

Although for \( p \geq 1 \), the \( L^p \)-Brunn-Minkowski theory has been considerably developed in the previous years (see e.g. \[31\], \[32\], \[33\], \[34\], \[35\]), much less seem to be known for \( 0 \leq p < 1 \). The following is conjectured in \[4\] (without the equality cases):
Conjecture 1.1. (The logarithmic-Brunn-Minkowski inequality) Let $K$, $L$ be symmetric convex bodies in $\mathbb{R}^n$ and $\lambda \in (0, 1)$. Then,

$$|\lambda \cdot K + o(1 - \lambda) \cdot L| \geq |K|^\lambda |L|^{1 - \lambda},$$

with equality in the following case: Whenever $K = K_1 \times \cdots \times K_m$, for some convex sets $K_1, \ldots, K_m$, that cannot be written as cartesian products of lower dimensional sets, then there exist positive numbers $c_1, \ldots, c_m$, such that $L = c_1 K_1 \times \cdots \times c_m K_m$.

The conjecture can easily be seen to be wrong for general convex bodies, even for $n = 1$. Note, also, that for $0 \leq p \leq q$, $\lambda \cdot K + p(1 - \lambda)L \subseteq \lambda \cdot K + q(1 - \lambda)L$, thus the log-Brunn-Minkowski inequality (if true) implies the $L^p$-Brunn-Minkowski inequality for all $p > 0$.

The cone-volume measure of $K$ is defined as: $S_0(K, \cdot) = h_K S(K, \cdot)$, where $S(K, \cdot)$ is the surface area measure of $K$, viewed as a measure on the sphere (see e.g. [41]). In [5], a necessary and sufficient condition was discovered (the planar case was treated by Stancu [42] [43]; see also [6] for other applications of the cone-volume measure and [8] for a possible functional generalization of the classical Minkowski problem). A confirmation of Conjecture 1.1 would answer the following open problem: When do two symmetric convex bodies $K, L$ have proportional cone-volume measures? If Conjecture 1.1 was proven to be true, the pairs $(K, L)$ would be exactly the ones for which equality holds in the log-Brunn-Minkowski inequality. The planar case was settled in [4]:

**Theorem A.** [4] Conjecture 1.1 is true in dimension two.

It was shown in [40] that Conjecture 1.1 holds true for pairs of unconditional bodies with respect to the same orthonormal basis. Actually, the proof (based on a result from [9]) shows that this result (as for the inequality) remains true if we replace the Lebesgue measure with any unconditional log-concave measure in $\mathbb{R}^n$. Recall that a measure $\mu$ is called log-concave if for all convex bodies $K$, $L$, it satisfies the Brunn-Minkowski inequality:

$$\mu(\lambda K + (1 - \lambda)L) \geq \mu(K)^\lambda \mu(L)^{1 - \lambda}, \quad \lambda \in (0, 1).$$

By a result of C. Borell [3], the absolutely continuous log-concave measures in $\mathbb{R}^n$ are exactly the ones having log-concave densities, i.e. their logarithms are concave functions. It is reasonable to conjecture the following:

**Conjecture 1.2.** Let $\mu$ be an even log-concave measure in $\mathbb{R}^n$, $K$, $L$ be symmetric convex bodies and $\lambda \in (0, 1)$. Then, $\mu(\lambda \cdot K + o(1 - \lambda) \cdot L) \geq \mu(K)^\lambda \mu(L)^{1 - \lambda}$.

Conjecture 1.2 is closely connected (actually implies; see Corollary 3.2) with the so called B-conjecture.

**Conjecture 1.3.** (B-conjecture) Let $\mu$ be an even log-concave measure in $\mathbb{R}^n$ and $K$ be a symmetric convex body. Then, the function $\mathbb{R} \ni t \mapsto \mu(e^t K)$ is log-concave.

This was conjectured in [9] [27]. This was previously conjectured by Banaszczyk in [29] for the standard Gaussian measure $\gamma_n$ (i.e. the measure that has density $e^{-\|x\|_2^2/2}$) and it was confirmed by Cordero-Erasquin, Fradelizi and Maurey in [9], a fact known as the B-theorem (see also [28] for an application). More generally it was shown that
Theorem B. (The B-Theorem for the Gaussian measure [9]) Let $A$ be a diagonal $n \times n$ matrix and $K$ be a symmetric convex body. Then, the function $\mathbb{R} \ni t \mapsto \gamma_n(e^{At}K)$ is log-concave. In particular, the standard Gaussian measure satisfies the B-conjecture.

Moreover, the following fact, also from [9], will be used:

Theorem C. [9] Let $A$ be a diagonal $n \times n$ matrix, $\mu$ be an unconditional log-concave measure and $K$ be an unconditional convex body. Then, the function $\mathbb{R} \ni t \mapsto \mu(e^{At}K)$ is log-concave.

A connection between the log-Brunn-Minkowski inequality for the Lebesgue measure (Conjecture 1.1) and the B-conjecture for uniform measures of symmetric convex bodies (i.e. measures of the form $|K \cap \cdot|$, where $K$ is a symmetric convex body) was established in [40]. Namely, it was proven that (i) the log-Brunn-Minkowski inequality for the Lebesgue measure in dimension $n$ implies the B-conjecture for uniform measures in dimension $n$. Thus, by Theorem A, the B-conjecture for uniform measures in the plane follows (this fact was proven independently in [30]). (ii) The log-Brunn-Minkowski inequality for the Lebesgue measure holds in any dimension if and only if, in any dimension, the function $|(e^{At}C_n) \cap K|$ is log-concave in $t$ for any symmetric convex body $K$ and for any diagonal matrix $A$. Here $C_n$ denotes the cube $[-1,1]^n$.

Our first goal is to continue the ideas from [40] and extend the formentioned results even further. Let us briefly describe our main results towards this direction. In Section 3 we prove (see Theorem 3.1) that actually the log-Brunn-Minkowski inequality for the Lebesgue density implies the log-Brunn-Minkowski inequality for any log-concave density and, therefore, the B-conjecture in full generality. Thus, again by Theorem A we establish (see Corollary 3.3) Conjectures 1.2 and 1.3 in the plane.

On the other hand, in Section 5 we modify the proof of fact (ii) mentioned earlier to prove that actually in order to confirm Conjecture 1.2, one needs for any dimension $n$ to find a density $f_n$ which satisfies a mild integrability assumption and the function

$$t \mapsto \int_{e^{At}C_n} f_n(Tx)dx$$

is log-concave for any choice of the diagonal matrix $A$ and for any invertible linear map $T$. The reader should focus in the case of the Gaussian density; see Remark 5.2.

Our second goal is to study log-concavity and log-convexity properties for dual bodies. In Section 4, as byproduct of our method from Section 3, we show (see Proposition 4.3) that the B-conjecture for uniform measures or for measures with densities of the form $e^{-\|x\|^p}_K$, $p \geq 1$ would imply the log-concavity of the function

$$t \mapsto |(K + p \cdot e^tL)^\circ|,$$

where $M^\circ$ stands for the dual body of $M$. Using the cases where the B-Conjecture is known to hold, we establish this log-concavity property in some special cases (see Corollary 4.6). As a further application, in Theorem 4.8 we confirm the variance conjecture (see Section 4 for more information) in a special class of convex bodies.

Finally, in Section 6, we establish the $L^0$-analogue of Firey’s dual Brunn-Minkowski inequality [13] (and its extension to other quermassintegrals):

$$|((\lambda K + (1 - \lambda)L)^\circ| \leq |K^\circ|^\lambda|L^\circ|^{1-\lambda}.$$
Note that the $L^0$-version is clearly a stronger inequality. Also, since no explicit formula is valid for the support function of the logarithmic sum, no classical arithmetic inequalities (such as Hölder) can be used directly towards the proof. Therefore, our inequality is a non-trivial extension of Firey’s result.

2 Preliminaries

Let us state some results that will be needed subsequently. We refer to [41] [17] for more information.

Let $K$ be a convex body that contains 0 in its interior. The polar body of $K$ is defined as:

$$K^c = \{ x \in \mathbb{R}^n \mid x \cdot y \leq 1, \forall y \in K \}.$$  

Then, $K^c$ is also a convex body that contains 0 in its interior and $(K^c)^c = K$.

The $i$-th quermassintegral $W_i(K)$ of $K$ is defined by the Steiner formula

$$|K + tB^n_2| = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i , \quad t > 0 , \quad i = 0, \ldots, n ,$$

where $B^n_2$ is the Euclidean unit ball. Note that $W_0(K)$ is the volume of $K$ and $W_1(K), W_{n-1}(K),$ $W_n(K)$ are proportional to the surface area, the mean width and the Euler characteristic respectively. Moreover, the functional $W_i$ is $(n-i)$-homogeneous, that is $W_i(tK) = t^{n-i}W_i(K)$. A useful formula for the quermassintegrals of $K$ is the Kubota recursion formula:

$$W_i(K) = \int_{G_{n,n-i}} |K|H|_{n-i} dH , \quad i = 1, \ldots, n - 1 .$$

Here, $\int_{G_{n,n-i}} dH$ denotes the integral of a function defined on the Grassmannian $G_{n,n-i}$, with respect to the Haar-measure on $G_{n,n-i}$ and $K|H$ is the orthogonal projection of $K$ onto the subspace $H$.

For the rest of this section, $K$ will be denoting a symmetric convex body (i.e. $K = -K$). The norm $\| \cdot \|_K$ of $K$ is the unique norm in $\mathbb{R}^n$, such that $K = \{ x \mid \| x \|_K \leq 1 \}$. Recall that every norm in $\mathbb{R}^n$ is the norm of a unique symmetric convex body.

The support function of $K$ is defined by $h_K(x) = \max_{y \in K} (x \cdot y) , \quad x \in \mathbb{R}^n$. There is a duality relation between the norm and the support function of $K$: $h_K = \| \cdot \|_{K^c}$.

The inradius and the outradius of $K$ are defined as:

$$\text{inradius}(K) = \min_{x \in \partial K} \| x \|_2 , \quad \text{outradius}(K) = \max_{x \in K} \| x \|_2 .$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a homogeneous of degree $p$ function. Then, by integration in polar coordinates, we have:

$$\int_K f(x)dx = \frac{1}{n+p} \int_{S^{n-1}} f(x)\| x \|_{K^c}^{n-p}dx ,$$

where $S^{n-1} = \{ x \in \mathbb{R}^n \mid \| x \|_2 = 1 \}$, the unit sphere in $\mathbb{R}^n$.  

The Prékopa-Leindler inequality is probably the most famous functional generalization of the Brunn-Minkowski inequality which states that whenever \( f, g, h \) are non-negative measurable functions with the property that for some \( \lambda \in (0, 1) \),
\[
\lambda(x + (1-\lambda)y) \geq \lambda f(x)g^{1-\lambda}(y), \text{ for all } x, y \in \mathbb{R}^n,
\]
then
\[
\int_{\mathbb{R}^n} h(x)dx \geq \left[ \int_{\mathbb{R}^n} f(x)dx \right]^{\lambda} \left[ \int_{\mathbb{R}^n} g(x)dx \right]^{1-\lambda}. \tag{2}
\]
We will need the 1-dimensional Prékopa-Leindler inequality in the following form, proven in [14]:

**Theorem D.** (Multiplicative version of the Prékopa-Leindler inequality)
Let \( \lambda \in (0, 1) \), \( f, g, h : \mathbb{R}_+ \to \mathbb{R}_+ \) be non-negative measurable functions, such that \( h(x^\lambda y^{1-\lambda}) \geq f^\lambda(x)g^{1-\lambda}(y) \), for all \( x, y > 0 \). Then, (2) holds.

The proof follows by applying the Prékopa-Leindler inequality to the functions \( \overline{f}(x) = e^x f(e^x) \), \( \overline{g}(x) = e^x g(e^x) \), \( \overline{h}(x) = e^x h(e^x) \) and the change of variables \( y = e^x \).

It is well-known that there exists a unique-up to isometry-volume preserving linear map \( T \) such that the quantity
\[
L^2_{TK} := \frac{1}{|K|^{\frac{n+2}{n}}} \int_{TK} (x \cdot y)^2 dx
\]
is constant as a function of \( y \in S^{n-1} \). Then, \( TK \) is said to be isotropic and the number \( L_{TK} \) is called the isotropic constant of \( K \) (see [38] for basic results on this concept). It is true that
\[
L^2_{TK} = \frac{1}{n|K|^\frac{n+2}{n}} \min_{T \in SL_n} \int_{TK} \|x\|_2^2 dx.
\]
It has been conjectured that the isotropic constants of symmetric convex bodies are bounded from above by an absolute constant; this problem is known as the slicing problem. The isotropic constant is known to be bounded form below by an absolute constant (see again [38]). The best estimate up to date for the upper bound is of the order \( n^{1/4} \), due to Klartag [25] after improving the previous estimate \( Cn^{1/4} \log n \) by Bourgain [7].

Let \( H \) be a \( k \)-dimensional subspace of \( \mathbb{R}^n \). Define the Schwartz-symmetrization \( S_H(K) \) of \( K \) with respect to \( H \) as the set that is constructed by replacing every cross-section, orthogonal to \( H \), of \( K \) with a Euclidean ball of the same \((n-k)\)-dimensional volume. It is an easy application of the Brunn-Minkowski inequality that \( S_H(K) \) is also a convex body. If \( H = \mathbb{R}u \), for some unit vector \( u \), we abbreviate \( S_u(K) = S_H(K) \). Notice, furthermore, that \( h_K(u) = h_{S_u(K)}(u) \).

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function. The epigraph of \( f \) is defined as
\[
\text{Epi}(f) := \{(x, t) \mid x \in \mathbb{R}^n, t \geq f(x)\} \subseteq \mathbb{R}^{n+1}.
\]
It is true that \( f \) is convex if and only if its epigraph is a convex set. Moreover, \( \text{Epi}(f) \) characterizes \( f \). For \( u \in S^{n-1} \), define the Schwartz-symmetrization \( S_u(f) \) with respect to \( u \), as the function with
\[
\text{Epi}(S_u(f)) = S_H(\text{Epi}(f)) ,
\]
where \( H \) is the subspace spanned by \( u \) and an orthogonal to \( \mathbb{R}^n \equiv \text{Domain of } f \), vector of \( \mathbb{R}^{n+1} \). By the previous discussion, if \( f \) is convex, then \( S_u(f) \) is convex as well.
3 On the log-Brunn-Minkowski inequality for general log-concave measures

The main result of this section is the following:

**Theorem 3.1.** Assume that the log-Brunn-Minkowski inequality holds in dimension \( n \) for the Lebesgue measure. Then, the log-Brunn-Minkowski inequality holds in dimension \( n \) for any even log-concave density.

**Corollary 3.2.** Assume that the log-Brunn-Minkowski inequality holds in dimension \( n \) for the Lebesgue measure. Then, the B-conjecture holds in dimension \( n \), for any even log-concave density.

Proof. Let \( \mu \) be an even log-concave measure, \( K \) be a symmetric convex body, \( \lambda \in (0, 1) \) and \( s, t \in \mathbb{R} \). Then, by Theorem 3.1, the log-Brunn-Minkowski inequality holds for the measure \( \mu \), therefore
\[
\mu(e^{\lambda s+(1-\lambda)t}K) = \mu(\lambda e^{sK} +_0 (1-\lambda)(e^tK)) \geq \mu(e^{sK})^\lambda \mu(e^{tK})^{1-\lambda}
\]
and the assertion follows. \( \square \)

Combining Theorem 3.1, Corollary 3.2 and Theorem A, we immediately obtain:

**Corollary 3.3.** The log-Brunn-Minkowski inequality and the B-conjecture hold in the plane, for any even log-concave density.

Corollary 3.3 (in particular, the B-Theorem in the plane) together with [37, Proposition 3.1] (see also [19]) immediately imply the following:

**Corollary 3.4.** Let \( \mu \) be an even log-concave measure in the plane, \( M \) be a symmetric convex body in the plane and \( \lambda \in (0, 1) \). Then for every \( K, L \in \{\alpha M; \alpha \geq 0\} \), one has
\[
\mu(\lambda K + (1-\lambda)L)^{\frac{1}{2}} \geq \lambda \mu(K)^{\frac{1}{2}} + (1-\lambda)\mu(L)^{\frac{1}{2}}.
\]

For the proof of Theorem 3.1, some geometric lemmas are required.

**Lemma 3.5.** Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) be an even convex function and \( t \in \mathbb{R} \), so that the sets \( \{\varphi \leq t\} \) and \( \{\varphi = \varphi(0)\} \) are convex bodies. Then, there exists \( b > 0 \), depending only on \( \varphi, t \), such that for all \( u \in S^{n-1}, r, s \in \mathbb{R} \) with \( \varphi(0) < r < s \leq t \), the following inequality holds:
\[
s - r \leq b(h_{\{\varphi \leq s\}}(u) - h_{\{\varphi \leq r\}}(u))\ .
\]

Proof. Note that the restriction of \( \varphi \) into the set \( \{\varphi \leq t\} \) is Lipschitz with some constant \( A > 0 \). Let \( r < s, u \in S^{n-1} \) and \( x \in \mathbb{R}^n \), such that \( \varphi(x) = r \) and \( x \cdot u = h_{\{\varphi \leq r\}}(u) \). Then, there exists a \( \xi > 1 \), such that for \( x' := \xi x, \varphi(x') = s \). Then, \( x' \cdot u \leq h_{\{\varphi \leq s\}}(u) \). Thus,
\[
h_{\{\varphi \leq s\}}(u) - h_{\{\varphi \leq r\}}(u) \geq x' \cdot u - x \cdot u = \|x' - x\|_2 \frac{x' - x}{\|x' - x\|_2} \cdot u
\]
\[
= \|x' - x\|_2 \frac{x}{\|x\|_2} \cdot u
\]
\[
= \|x' - x\|_2 \frac{h_{\{\varphi \leq r\}}(u)}{\|x\|_2} .
\]
We have \( \varphi(x) = r \leq t \), therefore \( \|x\|_2 \leq \text{outradius}(\{\varphi \leq t\}) \). Also, \( h_{\{\varphi \leq r\}}(u) \geq h_{\{\varphi = \varphi(0)\}}(u) \geq \text{inradius}(\{\varphi = \varphi(0)\}) \). Hence,

\[
\frac{\varphi(x') - \varphi(x) \text{ inradius}(\{\varphi = \varphi(0)\})}{A \text{ outradius}(\{\varphi \leq t\})} = \frac{s - r \text{ inradius}(\{\varphi = \varphi(0)\})}{A \text{ outradius}(\{\varphi \leq t\})} .
\]

**Lemma 3.6.** Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) be a function and \( t \) be a real number, both satisfying the conditions of Lemma 3.5. There exists \( c > |\varphi(0)| \), so that if we set

\[
\overline{\varphi}(x) = \begin{cases} 
\varphi(x) + c & , \quad \varphi(x) \leq t \\
\infty & , \quad \varphi(x) > t
\end{cases}
\]

then for any \( r, s \geq 0 \),

\[
\{\overline{\varphi} \leq r s^{1-\lambda}\} \supseteq \lambda \cdot \{\overline{\varphi} \leq r\} +_0 (1 - \lambda) \cdot \{\overline{\varphi} \leq s\} .
\]

**Proof.** Let \( b \) be the constant from Lemma 3.5. Set

\[
c := b \cdot \text{outradius}(\{\varphi \leq t\}) + |\varphi(0)|
\]

and define \( \overline{\varphi} \) by (3). Clearly, \( \overline{\varphi} \geq 0 \). It suffices to prove that for every \( r, s > 0 \),

\[
h_{\{\overline{\varphi} \leq r s^{1-\lambda}\}}(u) \geq h_{\{\overline{\varphi} \leq r\}}(u) h_{\{\overline{\varphi} \leq s\}}(u) .
\]

First assume that \( \overline{\varphi}(0) \leq r, s \leq t + c \). Fix \( u \in S^{n-1} \) and consider the Schwartz symmetrization \( S_u(\overline{\varphi}) \) of \( \overline{\varphi} \). Note that

\[
h_{\{\overline{\varphi} \leq p\}}(u) = h_{\{S_u(\overline{\varphi}) \leq p\}}(u) ,
\]

for every \( p \geq \overline{\varphi}(0) \). Moreover, since the body \( \{S_u(\overline{\varphi}) \leq p\} \) is unconditional with respect to an orthonormal basis that contains \( u \), one can easily see that

\[
S_u(\overline{\varphi}) \left( h_{\{S_u(\overline{\varphi}) \leq p\}}(u) u \right) = p = S_u(\overline{\varphi}) \left( h_{\{\overline{\varphi} \leq p\}}(u) u \right) .
\]

For \( t + c \geq p \geq \overline{\varphi}(0) \), set

\[
f(p) := S_u(\overline{\varphi})(pu) .
\]

Then, \( f \) is a convex and strictly increasing function and also,

\[
h_{\{\overline{\varphi} \leq p\}}(u) = f^{-1}(p) .
\]

We will show that for \( t + c \geq s > r \geq \overline{\varphi}(0) \),

\[
f^{-1}(r s^{1-\lambda}) \geq f^{-1}(r)^\lambda f^{-1}(s)^{1-\lambda} .
\]

Consider the line through the points \( (f^{-1}(r), r) \) and \( (f^{-1}(s), s) \) and suppose that this is defined by the equation \( x_2 = c_1 x_1 + d_1 \), for points \((x_1, x_2)\) of the plane. Since \( f \) is strictly increasing, it is
clear that $c_1 > 0$. We claim that $d_1 \geq 0$. Indeed, by (4), we have:

$$d_1 = s - \frac{s - r}{f^{-1}(s) - f^{-1}(r)} f^{-1}(s) = s - \frac{s - r}{h_{\{\varphi < s\}}(u) - h_{\{\varphi \leq s\}}(u)} h_{\{\varphi \leq s\}}(u) \geq s - bh_{\{\varphi \leq s\}}(u) \geq s - b \cdot \text{outradius}(\{\varphi \leq t\}) \geq \varphi(0) - b \cdot \text{outradius}(\{\varphi \leq t\}) = \varphi(0) + |\varphi(0)| \geq 0.$$ 

Now, the convexity of $f$ implies that

$$r^\lambda s^{1-\lambda} \leq c_1 f^{-1}(r^\lambda s^{1-\lambda}) + d_1.$$

It follows that

$$f^{-1}(r^\lambda s^{1-\lambda}) \geq (r/c_1)^\lambda (s/c_1)^{1-\lambda} - d_1/c_1 \geq \left(\frac{r}{c_1} - \frac{d_1}{c_1}\right)^\lambda \left(\frac{s}{c_1} - \frac{d_1}{c_1}\right)^{1-\lambda} = f^{-1}(r)^\lambda f^{-1}(s)^{1-\lambda}.$$ 

This proves (4) in the case where $t + c \geq s$, $r \geq \varphi(0)$. If $s < \varphi(0)$ (or $r < \varphi(0)$), then $\{\varphi \leq s\} = \emptyset$ and (4) holds trivially. On the other hand, if $s > t + c$, then $\{\varphi \leq s\} = \{\varphi \leq t + c\}$ and

$$\{\varphi \leq r^\lambda s^{1-\lambda}\} \supseteq \{\varphi \leq r^\lambda (t + c)^{1-\lambda}\}.$$ 

Thus, if $r \leq t + c$, we fall in the previous cases, otherwise $\{\varphi \leq r\} \supseteq \{\varphi \leq t + c\}$ and (4) is again trivial. \(\Box\)

**Lemma 3.7.** Let $\lambda \in (0, 1)$, $a_1, a_2 > 0$, $\mu$ be a measure, $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a non-negative even convex function and $K$, $L$ be symmetric convex bodies. Assume that for all $r_1, r_2 > 0$,

$$\mu\left(\left[\lambda \cdot K +_0 (1 - \lambda) \cdot L\right] \cap \{\varphi \leq r_1^\lambda r_2^{1-\lambda}\}\right) \geq \mu\left(\left[\lambda \cdot K \cap \{\varphi \leq r_1\}\right] \cap \{\varphi \leq r_2\}\right)^{1-\lambda}.$$ 

Then,

$$\int_{\lambda \cdot K +_0 (1 - \lambda) \cdot L} e^{-a_1^\lambda a_2^{1-\lambda} \varphi(x)} d\mu(x) \geq \left[\int_K e^{-a_1 \varphi(x)} d\mu(x)\right]^\lambda \left[\int_L e^{-a_2 \varphi(x)} d\mu(x)\right]^{1-\lambda}.$$ 

Proof. By the Fubini Theorem we have:

$$\int_{\lambda \cdot K +_0 (1 - \lambda) \cdot L} e^{-a_1^\lambda a_2^{1-\lambda} \varphi(x)} d\mu(x) = \int_0^\infty \mu\left(\left[\lambda \cdot K +_0 (1 - \lambda) \cdot L\right] \cap \{e^{-a_1^\lambda a_2^{1-\lambda} \varphi} \geq s\}\right) ds$$

$$= \int_0^1 \mu\left(\left[\lambda \cdot K +_0 (1 - \lambda) \cdot L\right] \cap \{a_1^\lambda a_2^{1-\lambda} \varphi \leq -\log s\}\right) ds$$

$$= \int_0^\infty \mu\left(\left[\lambda \cdot K +_0 (1 - \lambda) \cdot L\right] \cap \{a_1^\lambda a_2^{1-\lambda} \varphi \leq r\}\right) e^{-r} dr.$$
Set \( h(r) = \mu(\{\lambda \cdot K + (1 - \lambda) \cdot L \} \cap \{a_1 r_1^{1 - \lambda} \leq r\}) e^{-r}, \) \( f(r) = \mu(K \cap \{a_1 \varphi \leq r\}) e^{-r} \) and \( g(r) = \mu(L \cap \{a_2 \varphi \leq r\}) e^{-r}. \) We will make use of the multiplicative form of the Prékopa-Leindler inequality. If \( r_1, r_2 > 0, \) using our assumption, we have:

\[
h(r_1 r_2^{1 - \lambda}) = \mu(\{\lambda \cdot K + (1 - \lambda) \cdot L \} \cap \{\varphi \leq (a_1^{-1} r_1)^{\lambda} (a_2^{-1} r_2)^{1 - \lambda}\}) e^{-r_1 r_2^{1 - \lambda}} \\
\geq \mu(K \cap \{\varphi \leq a_1^{-1} r_1\})^{\lambda} \mu(L \cap \{\varphi \leq a_2^{-1} r_2\})^{1 - \lambda} e^{-r_1 r_2^{1 - \lambda}} \\
\geq \mu(K \cap \{\varphi \leq a_1^{-1} r_1\})^{\lambda} \mu(L \cap \{\varphi \leq a_2^{-1} r_2\})^{1 - \lambda} e^{-|\lambda r_1 + (1 - \lambda) r_2|} \\
= \left[\mu(K \cap \{a_1 \varphi \leq r_1\}) e^{-r_1}\right]^{\lambda} \left[\mu(L \cap \{a_2 \varphi \leq r_2\}) e^{-r_2}\right]^{1 - \lambda} \\
= f(r_1)^{\lambda} g(r_2)^{1 - \lambda}.
\]

Thus, by Theorem \[D\], we have

\[
\int_{\lambda \cdot K + (1 - \lambda) \cdot L} e^{-a_1^{-1} a_2^{-1} \varphi(x)} d\mu(x) = \int_0^{\infty} h(r) dr \\
\geq \left[\int_0^{\infty} f(r) dr\right]^{\lambda} \left[\int_0^{\infty} g(r) dr\right]^{1 - \lambda} \\
\geq \left[\int_K e^{-a_1 \varphi(x)} d\mu(x)\right]^{\lambda} \left[\int_L e^{-a_2 \varphi(x)} d\mu(x)\right]^{1 - \lambda}.
\]

**Proof of Theorem 3.1**

It is clearly sufficient (by approximation) to prove the log-Brunn-Minkowski inequality for densities of the form \( e^{-\varphi_{s,t}}, \) where \( \varphi \) is any even convex function defined in \( \mathbb{R}^n, \) \( t > s > \varphi(0) \) and \( \varphi_{s,t} \) is given by

\[
\varphi_{s,t}(x) = \begin{cases} 
\varphi(x), & s \leq \varphi(x) \leq t \\
 s, & \varphi(x) < s \\
\infty, & \varphi(x) > t
\end{cases}
\]

Note that \( \varphi_{s,t} \) satisfies the conditions of Lemma 3.5, for all choices of \( s \) and \( t. \) Fix \( s, t \) and let \( c > 0 \) and \( \overline{\varphi} \) be as in Lemma 3.6, i.e. \( \overline{\varphi}(x) = \varphi_{s,t}(x) + c. \) Then,

\[
\int_{\lambda \cdot K + (1 - \lambda) \cdot L} e^{-\varphi_{s,t}} dx = e^c \int_{\lambda \cdot K + (1 - \lambda) \cdot L} e^{-\overline{\varphi}} dx.
\]

Therefore, we need to prove the log-Brunn-Minkowski inequality for the even log-concave density \( e^{-\overline{\varphi}}. \) Note that \( \overline{\varphi} \) is non-negative. We need to show that the assumption of Lemma 3.7 is satisfied. Let \( r_1, r_2 > 0. \) By Lemma 3.6, we have:

\[
\left[\lambda \cdot K + (1 - \lambda) \cdot L \right] \cap \{\overline{\varphi} \leq r_1 r_2^{1 - \lambda}\} \supseteq \left[\lambda \cdot K + (1 - \lambda) \cdot L \right] \cap \left[\lambda \cdot \left\{\overline{\varphi} \leq r_1\right\} + (1 - \lambda) \cdot \left\{\overline{\varphi} \leq r_2\right\}\right] \\
\supseteq \lambda \cdot \left(K \cap \{\overline{\varphi} \leq r_1\}\right) + (1 - \lambda) \cdot \left(L \cap \{\overline{\varphi} \leq r_2\}\right).
\]

Since we assumed that the log-Brunn-Minkowski inequality holds for the Lebesgue measure, the assertion follows by taking volumes in the previous inclusion and by Lemma 3.7 (used with \( \mu = |\cdot| \) and \( a_1 = a_2 = 1. \) \( \square \)
4 Log-concavity properties for dual bodies

**Lemma 4.1.** Let $M$ be a symmetric convex body and $\mu$ be a measure, such that the function

$$\mathbb{R} \ni t \mapsto \mu(e^t M)$$

is log-concave. Then, for $p \geq 1$, $a \in \mathbb{R}$, the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} e^{-at \|x\|^p_M} d\mu(x)$$

is also log-concave.

**Proof.** This is a special case of Lemma 3.7. Indeed, set $K = \mathbb{R}^n = L$ and $\varphi(x) = \|x\|^p_M$. For $\lambda \in (0,1)$, $r_1, r_2 > 0$, we have:

$$\mu \left( \left[ \lambda \cdot K + (1-\lambda) \cdot L \right] \cap \{ \varphi \leq r_1^{\lambda} r_2^{1-\lambda} \} \right) = \mu \left( \{ \| \cdot \|^p_M \leq r_1^{\lambda} r_2^{1-\lambda} \} \right)$$

$$= \mu (r_1^{\lambda/p} r_2^{1-\lambda/p} M)$$

$$\geq \mu (r_1^{\lambda/p} M)^{\lambda} \mu (r_2^{1/p} M)^{1-\lambda}$$

$$= \mu (\{ \| \cdot \|_M \leq r_1^{1/p} \})^{\lambda} \mu (\{ \| \cdot \|_M \leq r_2^{1/p} \})^{1-\lambda}$$

$$= \mu (K \cap \{ \varphi \leq r_1 \})^{\lambda} \mu (L \cap \{ \varphi \leq r_2 \})^{1-\lambda}.$$ 

Therefore, by Lemma 3.7 if $a_1, a_2 > 0$,

$$\int_{\mathbb{R}^n} e^{-a_1^{\lambda} a_2^{1-\lambda} \|x\|^p_M} d\mu(x) \geq \left[ \int_{\mathbb{R}^n} e^{-a_1 \|x\|^p_M} d\mu(x) \right]^{\lambda} \left[ \int_{\mathbb{R}^n} e^{-a_2 \|x\|^p_M} d\mu(x) \right]^{1-\lambda},$$

proving our claim. □

The following is well known. We include its simple proof for the sake of completeness.

**Lemma 4.2.** Let $M$ be a convex body that contains the origin in its interior. For $p \geq 0$, there exists a constant $c_{n,p} > 0$, that depends only on $n$ and $p$, such that

$$|M| = c_{n,p} \int_{\mathbb{R}^n} e^{-\|x\|^p_M} dx.$$

**Proof.** Write

$$\int_{\mathbb{R}^n} e^{-\|x\|^p_M} dx = \int_0^\infty \left| \{ e^{-\|\cdot\|^p_M} > s \} \right| ds$$

$$= \int_0^1 \left| \{ \| \cdot \|_M \leq (-\log s)^{1/p} \} \right| ds$$

$$= \int_0^1 (-\log s)^{1/p} M ds$$

$$= \int_0^1 (-\log s)^{n/p} |M| ds = c_{n,p}^{-1} |M|.$$ □
Proposition 4.3. Let $p \geq 1$ and $L$ be a symmetric convex body which has one of the following two properties:

i) The measure with density $e^{-\|\cdot\|^p_L}$ satisfies the B-Theorem.

ii) The uniform measure of $L$ satisfies the B-Theorem.

Then, for any symmetric convex body $K$, the function

$$\mathbb{R} \ni t \mapsto |(K^o + p e^t \cdot L^o)^o|$$

is log-concave.

Proof. Suppose that (i) holds. Using Lemma 4.1 with $\mu = e^{-\|\cdot\|^p_L} dx$, we obtain that the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} e^{-e^{-t} \|x\|^p_K} d\mu(x) = \int_{\mathbb{R}^n} e^{-\|x\|^p_L - e^{-t} \|x\|^p_K} dx =: \phi(t) .$$

is log-concave. Note that $\left(\|\cdot\|^p_L + e^{-t} \|\cdot\|^p_K\right)^{1/p}$ is the support function (= dual norm) of the convex body $L^o + p e^{-t} \cdot K^o$. Therefore, by Lemma 4.2, $\phi(t) = c_{n,p}^{-1} |(K^o + p e^t L^o)^o|$ and the function

$$|(K^o + p e^t L^o)^o| = e^{nt/p} |(L^o + p e^{-t} \cdot K^o)^o|$$

is log-concave.

Assume now that (ii) holds. Use again Lemma 4.1 with $\mu = 1_K(x) dx$ to get that the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} e^{-e^t \|x\|^p_L} dx$$

is log-concave. Write

$$\int_{K} e^{-e^t \|x\|^p_L} dx = \int_{K} e^{-\|e^{t/p}x\|^p_L} dx = e^{-nt/p} \int_{e^{-t/p}K} e^{\|x\|^p_L} dx = : e^{-nt/p} \mu'(e^{-t/p} K) ,$$

where $\mu'$ is the measure with density $e^{-\|x\|^p_L}$. Since the function $e^{-nt/p}$ is log-affine, it follows that the assumption of Lemma 4.1 holds with $\mu'$ instead of $\mu$, thus the function

$$\mathbb{R} \ni t \mapsto \int_{\mathbb{R}^n} e^{-e^t \|x\|^p_K} d\mu'(x) = \phi(t)$$

is log-concave, where $\phi(t)$ was defined previously and was proven to be proportional to $|(K^o + p e^t \cdot L^o)^o|$. This proves our claim. $\square$

Remark 4.4. Proposition 4.3 asserts that the B-conjecture for uniform measures implies the log-concavity of the function $t \mapsto |(K^o + p e^t \cdot L^o)^o|$. The opposite is also true, since the limiting case $p = \infty$ is just the B-conjecture for uniform measures.
Next, let us confirm the B-conjecture for uniform measures in its most simple case: The case of the symmetric strips.

**Theorem 4.5.** Let $u \in S^{n-1}$, $a > 0$. Set $E = \{x \in \mathbb{R}^n \mid |x \cdot u| \leq a\}$. Then, for every symmetric convex body $K$, the function $\mathbb{R} \ni t \mapsto |E \cap e^t K|$ is log-concave.

**Proof.** We need to prove that for every $\lambda \in (0,1)$, $t_1, t_2 \in \mathbb{R}$, then:

$$|E \cap e^{\lambda t_1 + (1-\lambda)t_2}| \geq |E \cap e^{t_1} K|^\lambda |E \cap e^{t_2} K|^{1-\lambda}.$$  \hfill (6)

One can easily verify that for each $b > 0$,

$$S_u(E \cap bK) = E \cap bS_u K,$$

hence nothing changes in (6) in terms of volumes if we replace $K$ with the Schwartz symmetrization $S_u K$. But then, $E, S_u K$ are unconditional with respect to some (any) orthonormal basis that contains $u$. Now, Theorem C proves our claim. \(\Box\)

It follows immediately by Proposition 4.3, Theorem 4.5, Theorem A and Theorems B, C that:

**Corollary 4.6.** Let $K, L$ be symmetric convex bodies, $p \geq 1$ and $u$ be a unit vector. The function $\mathbb{R} \ni t \mapsto |(K^{\circ} + p e^t \cdot L)^{\circ}|$ is log-concave (at least) in the following cases:

i) $p = 2$ and $L = B_2^n$.

ii) $L$ is an origin symmetric line segment.

iii) $K$ and $L$ are unconditional, with respect to the same orthonormal basis.

iv) $K$ and $L$ are planar.

The variance conjecture \cite{1} \cite{2} states that if $X$ is a random vector with log concave probability density $f$, whose barycenter is at the origin and its covariance matrix is the identity (i.e. $X$ is isotropic), then the variance of $\|X\|_2^2$ satisfies

$$\text{Var}(\|X\|_2^2) \leq Cn,$$

where $C > 0$ is an absolute constant. The variance conjecture plays a central role in modern convex geometry. Surprisingly, it implies other major conjectures (see \cite{11} \cite{12}), such as the slicing problem and the KLS conjecture \cite{24} up to a logarithmic factor. The best general known estimate up to date is of order $n^{5/3}$, due to O. Guedon and E. Milman \cite{21} (see also \cite{16}). It has been confirmed for random vectors with unconditional log-concave densities \cite{26} (see also \cite{15}, \cite{10}). We refer to \cite{22} for more information and references.

We would like to restrict our attention in the class of symmetric convex bodies, i.e. the density $f$ is the indicator function of a symmetric convex body. In this case the variance conjecture becomes: Let $K$ be a symmetric isotropic convex body. Then,

$$\sigma^2(K) := \frac{|K| \int_K \|x\|_2^4 dx - \left[ \int_K \|x\|_2^2 dx \right]^2}{\frac{1}{n} \left[ \int_K \|x\|_2^2 dx \right]^2} \leq C.$$
Lemma 4.7. Let $K$, $L$ be symmetric convex bodies, $p \geq 1$, $a > 0$. Set $T := (K^o + p \cdot a \cdot L^o)^o$. If the function
$$
\mathbb{R} \ni t \mapsto \left| (K^o + p \cdot a \cdot L^o)^o \right|
$$
is log-concave, then

$$
|T| \int_T \|x\|^p_L dx - \left[ \int_T \|x\|^p_L dx \right]^2 \leq \frac{p}{a(n+p)} |T| \int_T \|x\|^p_L dx .
$$

Proof. Set $f(t) = \left| (K^o + p \cdot a \cdot L^o)^o \right|$. Then, $f(0) = |T|$ and $f$ is log-concave. Integrating in polar coordinates we obtain:

$$
f(t) = \frac{1}{n} \int_{S^{n-1}} \left( \|x\|^p_K + e^t a \|x\|^p_L \right)^{-n/p} dx ,
$$

thus

$$
f'(t) = \frac{1}{n} \int_{S^{n-1}} -\frac{n}{p} e^t a \|x\|^2_L \left( \|x\|^p_K + e^t a \|x\|^p_L \right)^{-(n+p)/p} dx .
$$

So,

$$
f'(0) = -\frac{1}{p} \int_{S^{n-1}} a \|x\|^p_L \|x\|^n_T dx = -a \frac{n+p}{p} \int_T \|x\|^p_L dx .
$$

Also,

$$
f''(t) = f'(t) + \frac{1}{n} \int_{S^{n-1}} \frac{n(n+p)}{p^2} e^{2t} a^2 \|x\|^{2p}_L \left( \|x\|^p_K + e^t a \|x\|^p_L \right)^{-(n+2p)/p} dx .
$$

Therefore,

$$
f''(0) = -a \frac{n+p}{p} \int_T \|x\|^p_L dx + \frac{a^2(n+p)(n+2p)}{p^2} \int_{S^{n-1}} \frac{1}{n+2p} \|x\|^2_L \|x\|^n_T \|x\|^{(n+2p)} dx
$$

$$
= -a \frac{n+p}{p} \int_T \|x\|^p_L dx + \frac{a^2(n+p)(n+2p)}{p^2} \int_T \|x\|^2_L dx
$$

$$
\geq -a \frac{n+p}{p} \int_T \|x\|^p_L dx + \left[ \frac{a(n+p)}{p} \right]^2 \int_T \|x\|^2_L dx .
$$

Now, the log-concavity of $f$ implies $f''(0) f(0) \leq [f'(0)]^2$ and the assertion follows. \qed

For $a > 0$, define the class of convex bodies $C_a$ as follows:

$$
C_a = \left\{ (K^o + a \cdot B_2^n)^o \mid K \text{ is a symmetric convex body, } (K^o + a \cdot B_2^n)^o \text{ is isotropic } \right\} .
$$

Combining Lemma 4.7 with Corollary 4.6 we immediately obtain:

**Theorem 4.8.** Let $T \in C_a$, with $|T| = 1$, for some $a > 0$. Then,

$$
\sigma^2(T) \leq \frac{2}{a(n+2)L_T^2} .
$$

In particular, if $a > c/n$, for some absolute constant $c > 0$, then $T$ satisfies the variance conjecture.

Before ending this section, we would like to give an alternative description of the class $C_a$. 

13
Lemma 4.9. Let $K$ be a symmetric convex body and $a$ be a positive number. Then, $(K + 2a \cdot B_2^n)^\circ$ is isotropic if and only if

$$|(K + 2a \cdot B_2^n)^\circ| = \max_{T \in SL_n} |(TK + 2a \cdot B_2^n)^\circ|.$$  \hfill (7)

Proof. It is easy to check that the quantity $|(TK + 2a \cdot B_2^n)^\circ|$ indeed attains a maximum, among $T \in SL(n)$. Let $v \in S^{n-1}$, $t \in \mathbb{R}$, $|t| < 1$. Define the linear map

$$T_t(x) = \left(\frac{1}{1+t}\right)^n (x + t(x \cdot v)v).$$

Then, $T_t \in SL_n$. Using polar coordinates, one may compute:

$$\frac{\partial}{\partial t} \bigg|_{t=0} \left|(T_t^{-1}K + 2a \cdot B_2^n)^\circ\right| = \frac{\partial}{\partial t} \bigg|_{t=0} \left|(K + 2a \cdot T_tB_2^n)^\circ\right| = \frac{\partial}{\partial t} \bigg|_{t=0} \left(1 \int_{S^{n-1}} (h_K(x)^2 + a\|T_t x\|_2)^{-n/2} dx\right)$$

$$= \frac{\partial}{\partial t} \bigg|_{t=0} \frac{1}{n} \int_{S^{n-1}} (h_K(x)^2 + a(1+t)^{-2/n}\|x + t(x \cdot v)\|_2^2)^{-n/2} dx$$

$$= \frac{1}{n} \int_{S^{n-1}} \frac{n}{2} \left([2a/n]\|x\|_2^2 - 2a(x \cdot v)^2\right) \left(h_K(x)^2 + a\|x\|_2^2\right)^{-(n+2)/2} dx$$

$$= (n+2)a \left[-\int_{(K+2aB_2^n)^\circ} (x \cdot v)^2 dx + \frac{1}{n} \int_{(K+2aB_2^n)^\circ} \|x\|_2^2 dx \right].$$  \hfill (8)

Therefore, if (7) holds, then the derivative at $t = 0$ of the volume of $(T_t^{-1}K + 2a \cdot B_2^n)^\circ$ equals zero, so by (8),

$$\int_{(K+2aB_2^n)^\circ} (x \cdot v)^2 dx = \frac{1}{n} \int_{(K+2aB_2^n)^\circ} \|x\|_2^2 dx.$$  \hfill (9)

Since this is true for all $v \in S^{n-1}$, it follows that $(K + 2a \cdot B_2^n)^\circ$ is isotropic. On the other hand, if $T_0$ is a critical point of the function $SL_n \ni T \mapsto |(TK + 2a \cdot B_2^n)^\circ|$, we have proved that $(T_0K + 2a \cdot B_2^n)^\circ$ is isotropic. By the uniqueness up to isometry-of the isotropic position, it follows that this critical point is unique, thus if $(K + 2a \cdot B_2^n)^\circ$ is isotropic, then (7) holds. \hfill \Box

5 Reduction to the log-BM inequality for coordinate parallelepipeds

Theorem 5.1.

i) Assume that for all $n \in \mathbb{N}$, there exists an even function $f_n : \mathbb{R}^n \to \mathbb{R}$, whose restriction in any subspace of $\mathbb{R}^n$ is integrable, with the following property: For all $T \in GL_n$ and for all diagonal $n \times n$-matrices $A$, the function

$$\mathbb{R} \ni t \mapsto \int_{e^{\lambda A}C_n} f_n(Tx) dx$$

is log-concave. Then, the log-Brunn-Minkowski inequality holds for all even log-concave densities $g : \mathbb{R}^n \to \mathbb{R}$, for all $n \in \mathbb{N}$.
ii) Assume that for all \( n \in \mathbb{N} \), there exists an even function \( f_n : \mathbb{R}^n \to \mathbb{R} \), whose restriction in any subspace of \( \mathbb{R}^n \) is integrable, with the following property: The log-Brunn-Minkowski inequality holds for the density \( f_n \), for any two parallelepipeds with parallel facets. Then, the log-Brunn-Minkowski inequality holds for all even log-concave densities \( g : \mathbb{R}^n \to \mathbb{R} \), for all \( n \in \mathbb{N} \).

iii) Fix \( n \in \mathbb{N} \). If the log-Brunn-Minkowski inequality holds for some density \( f : \mathbb{R}^n \to \mathbb{R} \), then for all \( T \in \text{GL}_n \) and for all diagonal \( n \times n \)-matrices \( A \), the function

\[
\mathbb{R} \ni t \mapsto \int_{e^{tA}C_n} f(Tx)dx
\]

is log-concave.

Proof. Let us first prove (iii). It is easily verified (see [40] [4]) that if \( s, t \in \mathbb{R} \) and \( A \) is a diagonal \( n \times n \)-matrix, then for \( \lambda \in (0, 1) \),

\[
e^{[\lambda s+(1-\lambda)t]A}C_n = \lambda \cdot (e^{sA}C_n) +_0 (1-\lambda) \cdot (e^{tA}C_n) .
\]

Therefore, for \( T \in \text{GL}_n \),

\[
\lambda \cdot (Te^{sA}C_n) +_0 (1-\lambda) \cdot (Te^{tA}C_n) = T[\lambda \cdot (e^{sA}C_n) +_0 (1-\lambda) \cdot (e^{tA}C_n)] = Te^{[\lambda s+(1-\lambda)t]A}C_n . \tag{9}
\]

Thus,

\[
\int_{e^{[\lambda s+(1-\lambda)t]A}C_n} f(Tx)dx = |\det T| \int_{Te^{[\lambda s+(1-\lambda)t]A}C_n} f(x)dx
\]

\[
= |\det T| \int_{\lambda . (Te^{sA}C_n) +_0 (1-\lambda) \cdot (Te^{tA}C_n)} f(x)dx
\]

\[
\geq |\det T| \left[ \int_{Te^{sA}C_n} f(x)dx \right]^{\lambda} \left[ \int_{Te^{tA}C_n} f(x)dx \right]^{1-\lambda}
\]

\[
= \left[ \int_{e^{sA}C_n} f(Tx)dx \right]^{\lambda} \left[ \int_{e^{tA}C_n} f(Tx)dx \right]^{1-\lambda} .
\]

Assertion (ii) is just a reformulation of (i). Indeed, one can check that if \( P_1, P_2 \) are two parallelepipeds with parallel facets, then there exist \( s_1, s_2 \in \mathbb{R} \), a diagonal matrix \( A \) and a \( \text{GL}(n) \)-map \( T \), such that \( P_i = Te^{s_iA}C_n, i = 1, 2 \). Thus, by (9),

\[
\int_{\lambda . P_1 +_0 (1-\lambda) \cdot P_2} f_n(x)dx = |\det T| \int_{e^{[\lambda s+(1-\lambda)t]A}C_n} f_n(Tx)dx .
\]

It remains to prove (i). Let \( K, L \) be symmetric convex bodies in \( \mathbb{R}^n \) and \( \lambda \in (0, 1) \). As in [40], Theorem 1.5], consider the following discretized version of the logarithmic sum of \( K \) and \( L \): Let \( v_1, \ldots, v_m \) be unit vectors in \( \mathbb{R}^n \), \( m \geq n \). Set

\[
R_\lambda := \{ x \in \mathbb{R}^n \mid |x \cdot v_i| \leq r_i^\lambda s_i^{1-\lambda}, i = 1, \ldots, m \} ,
\]

where \( r_i = h_K(v_i), s_i = h_L(v_i), i = 1, \ldots, m \). We will prove that under the assumption of (i),

\[
|R_\lambda| \geq |R_0|^{1-\lambda} |R_1|^\lambda . \tag{10}
\]
Since $R_\lambda$ can be chosen to be arbitrarily close to $\lambda \cdot K + 0 \cdot (1 - \lambda) \cdot L$, as $m \to \infty$, if (10) is proved for any choice of the $v_i$’s, $r_i$’s, $s_i$’s, then the log-Brunn-Minkowski inequality will be established for the Lebesgue measure. But then, by Theorem 3.1, the log-Brunn-Minkowski inequality for any log-concave measure will follow. Therefore, it suffices to prove (10) for any choice of $m, r_i > 0, s_i > 0, v_i \in \mathbb{S}^{n-1}, i = 1, \ldots, m$. As in [10], write

$$|R_\lambda| = \int_{\mathbb{R}^n} \prod_{i=1}^m 1_{[-r_i s_i^{-1}, r_i s_i^{-1}, -1, -1]}(x \cdot v_i)dx .$$

Set also,

$$G_\lambda(\varepsilon) := \int_{x \in \mathbb{R}^n} \int_{u \in \mathbb{R}^m} \prod_{i=1}^m 1_{[-r_i s_i^{-1}, r_i s_i^{-1}, -1, -1]}(x \cdot v_i + u_i)\varepsilon^{-n} f_{m+n}(u/\varepsilon)dudx ,$$

where $\varepsilon > 0, u_i = u \cdot e_i, i = 1, \ldots, m$ and $\{e_1, \ldots, e_m\}$ is an orthonormal basis in $\mathbb{R}^m$. It follows by the change of variables $U := u/\varepsilon$ that

$$G_\lambda(\varepsilon) := \int_{x \in \mathbb{R}^n} \int_{U \in \mathbb{R}^m} \prod_{i=1}^m 1_{[-r_i s_i^{-1}, r_i s_i^{-1}, -1, -1]}(x \cdot v_i + \varepsilon U_i) f_{m+n}(U)dU dU dx$$

$$\xrightarrow{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \prod_{i=1}^m 1_{[-r_i s_i^{-1}, r_i s_i^{-1}, -1, -1]}(x \cdot v_i)dx \int_{U \in \mathbb{R}^m} f_{m+n}(U)dU = \|f_{m+n}|_{\mathbb{R}^m} \|_1 \cdot |R_\lambda| .$$

Thus, it suffices to prove that, for $\varepsilon > 0$,

$$G_\lambda(\varepsilon) \geq G_1(\varepsilon)^\lambda G_0(\varepsilon)^{1-\lambda} .$$

Using the change of variables $w_i := \frac{u_i + x \cdot v_i}{s_i}, i = 1, \ldots, m$, we get:

$$G_\lambda(\varepsilon) = A \int_{w \in \mathbb{R}^m} \int_{x \in \mathbb{R}^n} \prod_{i=1}^m 1_{[-r_i s_i^{-1}, r_i s_i^{-1}, -1, -1]}(s_i w_i) f_{m+n}\left(\varepsilon^{-1} \sum_{i=1}^m (s_i w_i - x \cdot v_i) e_i\right)dw dx$$

$$= \lim_{a \to \infty} A \int_{w \in \mathbb{R}^m} \int_{x \in \mathbb{R}^n} \prod_{i=1}^m 1_{[-r_i s_i^{-1}, r_i s_i^{-1}, -1, -1]}(s_i w_i) 1_{a C_n}(x) f_{m+n}\left(\varepsilon^{-1} \sum_{i=1}^m (s_i w_i - x \cdot v_i) e_i\right) dw dx ,$$

where $A = \varepsilon^{-n} s_1 \ldots s_n$. Define the (singular) linear map $T : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, with $T(w, x) = \varepsilon^{-1} \sum_{i=1}^m (s_i w_i - x \cdot v_i) e_i$. Note that the linear map $T_\delta := T + \delta I_{\mathbb{R}^{m+n}}$ becomes invertible, for $\delta > 0$, small enough. Therefore, if

$$A_a := \text{diag}\left(\log(r_1 s_1^{-1}), \ldots, \log(r_m s_m^{-1})\right), \log a, \ldots, \log a\right), a > 1 ,$$

then

$$G_\lambda(\varepsilon) = A \lim_{a \to \infty} \lim_{\delta \to 0^+} \int_{\varepsilon^{|x|} \cdot (1 - \lambda) \cdot (\delta - 0^+) \cdot A_a C_{m+n}} f_{m+n}(T_\delta z)dz$$

$$= A \lim_{a \to \infty} \lim_{\delta \to 0^+} F(a, \delta, \lambda) .$$

Using our assumption,

$$F(a, \delta, \lambda) \geq F(a, \delta, 1)^\lambda F(a, \delta, 0)^{1-\lambda} ,$$

for all $a > 1, \delta > 0$ (\delta small enough). This proves our claim. \square
Remark 5.2. The case of $f_n$ being the Gaussian density seems to be the most promising in the attempt of proving the log-Brunn-Minkowski inequality. It follows by the previous Theorem and Theorem 3.1 that the log-Brunn-Minkowski inequality is true in any dimension and for every log-concave density if and only if it holds true for the (standard) Gaussian density and for parallelepipeds with parallel facets, in all dimensions.

6 The dual log-Brunn-Minkowski inequality

The main goal of this section is to establish the following dual logarithmic Brunn-Minkowski inequality (see Corollary 6.5 below).

Theorem 6.1. Let $K$, $L$ be two convex bodies in $\mathbb{R}^n$ and $\lambda \in (0,1)$. Then,
\[
|\langle \lambda \cdot K +_0 (1-\lambda) \cdot L \rangle^0| \leq |K^0|^{\lambda} |L^0|^{1-\lambda}.
\] (11)

Once Theorem 6.1 is established, one can follow Firey’s argument [14] to prove Corollary 6.5 (see below), where the volume is replaced by the other quermassintegrals. Since the dual $L^0$-sum contains the dual $L^p$-sum, for $p \geq 0$, Theorem 6.1 extends immediately to the $L^p$-setting, for all $p \geq 0$. Therefore, it is stronger than Firey’s [13] dual Brunn-Minkowski inequality. It is also stronger than the dual Brunn-Minkowski inequality with respect to $L^0$-radial sums, established in [20]. It seems plausible that the equality cases in (11) are exactly the equality cases in Conjecture 1.1 (here of course non-symmetric bodies are allowed); we do not address this here.

Without loss of generality, we may assume that $K$ and $L$ contain 0 in their interiors. Otherwise, the assertion would be trivial. As in the previous section, we will prove our claim for the (asymmetric) discrete approximations of the logarithmic sum $K$ and $L$. The rest of the proof will follow by compactness. Set
\[
AR_\lambda = \{ x \in \mathbb{R}^n \mid x \cdot v_i \leq r_i^{\lambda} s_i^{1-\lambda}, \ i = 1, \ldots, m \},
\]
where $v_1, \ldots, v_m \in S^{n-1}$, $r_i = h_K(v_i)$, $s_i = h_L(v_i)$, $i = 1, \ldots, m$. Since $K$ and $L$ contain 0 in their interiors, it is true that $r_i, s_i > 0$, $i = 1, \ldots, m$, thus $AR_\lambda$ is well defined. One, then, needs to prove that the function $(0,1) \ni \lambda \mapsto |(AR_\lambda)^0|$ is log-convex (i.e. its logarithm is convex). On the other hand,
\[
(AR_\lambda)^0 = \text{conv}\{(r_i^{-1})^{\lambda}(s_i^{-1})^{1-\lambda} u_i \mid i = 1, \ldots, m\},
\]
therefore the proof of Theorem 6.1 reduces to the proof of the following:

Theorem 6.2. Let $x_1, \ldots, x_m \in \mathbb{R}^n$, $a_1, \ldots, a_m \in \mathbb{R}$ and consider the family of polytopes
\[
P_t = \text{conv}\{e^{at} x_i \mid i = 1, \ldots, m\}, \ t \in (t_1, t_2),
\]
for some $t_1 < t_2$. If $P_t$ contains the origin in its interior for all $t \in (t_1, t_2)$, then the function
\[
(t_1, t_2) \ni t \mapsto |P_t|
\]
is log-convex.
We remark here that Theorem 6.2 may be viewed as the dual version of the B-conjecture for uniform measures in the following sense: If \( K, L \) are convex bodies that contain the origin in their interiors, then the function \( t \mapsto \left| (e^tK) \cap L \right|^{\delta} \) is log-convex. The idea for the proof of Theorem 6.2 is taken from Saroglou [39, Theorem 3.1]. First we will need two easy lemmas.

**Lemma 6.3.** Let \( f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}_+ \) be log-convex functions. Then, their sum is log-convex.

**Proof.** For \( \lambda \in (0, 1), t_1, t_2 \in \mathbb{R}^n \), we have
\[
\sum_{i=1}^{m} f_i(\lambda t_1 + (1 - \lambda)t_2) \leq \sum_{i=1}^{m} f_i^\lambda(t_1)f_i^{1-\lambda}(t_2) \leq \left[ \sum_{i=1}^{m} f_i(t_1) \right]^\lambda \left[ \sum_{i=1}^{m} f_i(t_2) \right]^{1-\lambda}.
\]
This proves our assertion. \( \Box \)

**Lemma 6.4.** Let \( x_1, \ldots, x_n \in \mathbb{R}^n, a_1, \ldots, a_n \in \mathbb{R} \). Then, the function
\[
t \mapsto \left| \text{conv}\{0, e^{a_1t}x_1, \ldots, e^{a_nt}x_n\} \right|
\]
is log-affine and therefore log-convex.

**Proof.** We have:
\[
\left| \text{conv}\{0, e^{a_1t}x_1, \ldots, e^{a_nt}x_n\} \right| = \frac{1}{n!} \left| \det(e^{a_1t}x_1, \ldots, e^{a_nt}x_n) \right| = \frac{e^{a_1t} \cdots e^{a_nt}}{n!},
\]
proving our claim. \( \Box \)

**Proof of Theorem 6.2**

We need to prove that the function \( (t_1, t_2) \mapsto t \mapsto |P_t| \) is log-convex. Actually, we need to prove that for any \( s_1, s_2 \in (t_1, t_2), s_1 < s_2, \)
\[
|P_{s_1+s_2}| \leq |P_{s_1}|^{1/2} |P_{s_2}|^{1/2}.
\]
Set \( s = (s_1 + s_2)/2, p = (s_2 - s_1)/2 \). Let \( \{T_1, \ldots, T_k\} \) be a triangulation of the boundary of \( P_s \); that is a subdivision of the boundary of \( P_s \) into non-overlapping simplices, whose vertices are vertices of \( P_s \). Set
\[
\Delta_i := \text{conv}\{\{0\} \cup T_i\}, i = 1, \ldots, k.
\]
Then, the family \( \{\Delta_1, \ldots, \Delta_k\} \) is a triangulation of \( P_s \). For \( i = 1, \ldots, k \), consider the following transformation of \( \Delta_i \): If \( \Delta_i = \text{conv}\{0, e^{a_{j_1}t}x_{j_1}, \ldots, e^{a_{j_n}t}x_{j_n}\} \), for some \( 1 \leq j_1 < \cdots < j_n \leq m \), set
\[
\Delta_{i,r} = \text{conv}\{0, e^{a_{j_1}(r+s)}x_{j_1}, \ldots, e^{a_{j_n}(r+s)}x_{j_n}\}, \quad r \in [-p, p].
\]
It is clear that the \( \Delta_{i,r} \)'s are non-overlapping, for \( r \in [-p, p] \). This is because, for \( i = 1, \ldots, k \), \( r \in [-p, p] \), \( \Delta_{i,r} \) is contained in the positive cone spanned by \( \Delta_i \) and every two such cones are, by construction, non-overlapping. Now, it is clear that
\[
|P_{s+r}| = \left| \text{conv}\left( \bigcup_{i=1}^{k} \Delta_{i,r} \right) \right| \geq \left( \sum_{i=1}^{k} |\Delta_{i,r}| \right), \quad r \in [-p, p]. \quad (12)
\]
By Lemmas 6.3 and 6.4 the function \([-p, p] \ni r \mapsto \sum_{i=1}^{k} |\Delta_{i,r}|\) is log-convex. Thus,

\[|P_s| = \sum_{i=1}^{k} |\Delta_i| = \sum_{i=1}^{k} |\Delta_{i,0}| \leq \left( \sum_{i=1}^{k} |\Delta_{i,-p}| \right)^{1/2} \left( \sum_{i=1}^{k} |\Delta_{i,p}| \right)^{1/2} \leq |P_{s-p}|^{1/2} |P_{s+p}|^{1/2} = |P_{s_1}|^{1/2} |P_{s_2}|^{1/2},\]
as required. \(\square\)

**Corollary 6.5.** Let \(K, L\) be two convex bodies that contain 0 in their interior. For \(i = 1, \ldots, n-1, p \geq 0\) and \(\lambda \in (0, 1)\), the following is true:

\[W_i \left( [\lambda \cdot K +_p (1 - \lambda) \cdot L]^{\circ} \right) \leq W_i (K^{\circ})^{\lambda} W_i (L^{\circ})^{1-\lambda} .\]

Corollary 6.5 also generalizes a result of Firey [14], who proved this for the \(L^1\)-sum. This, was recently extended in the \(L^p\)-case, for \(p \geq 1\) in [23], where it was explained that by the homogeneity of the quermassintegrals, dual Brunn-Minkowski inequalities have dimension-dependent equivalent forms (in the same manner as the original Brunn-Minkowski inequality does; see e.g. [18]). Adopting the same argument we obtain:

**Corollary 6.6.** Let \(K, L\) be two convex bodies that contain the origin in their interior. If \(i \in \{1, \ldots, n-1\}, p \geq 0\) and \(\lambda \in (0, 1)\), then

\[W_i \left( [\lambda \cdot K +_p (1 - \lambda) \cdot L]^{\circ} \right)^{-\frac{p}{i}} \leq \lambda W_i (K^{\circ})^{-\frac{p}{i}} + (1 - \lambda) W_i (L^{\circ})^{-\frac{p}{i}} .\]

**Proof.** Use Corollary 6.5 with \(K = W_i (K^{\circ})^{1/n-i}, L = W_i (L^{\circ})^{1/n-i}, \lambda = \lambda W_i (L^{\circ})^{-p/n-i} \cdot \mu^{-1}\), where \(\mu = \lambda W_i (K^{\circ})^{-\frac{p}{i}} + (1 - \lambda) W_i (L^{\circ})^{-\frac{p}{i}}\), in the place of \(K, L, \lambda\) respectively. \(\square\)

For the proof of Corollary 6.5 the following (contained in an earlier version of [40]) is required.

**Lemma 6.7.** Let \(\lambda \in [0, 1], K, L\) be convex bodies in \(\mathbb{R}^n\) and \(H\) be a subspace of \(\mathbb{R}^n\). Then,

\[\lambda \cdot (K \cap H) +_0 (1 - \lambda) \cdot (L \cap H) \subseteq [\lambda \cdot K +_0 (1 - \lambda) \cdot L] \cap H,\]

where the logarithmic sum in the first part of the previous inclusion is considered with respect to the subspace \(H\).

**Proof.** Note that if \(x, u \in H\) and \(y \in H^\perp\), then \(x \cdot (u + y) = x \cdot u,\)

\[h_{K \cap H}(u + y) = \max_{z \in K \cap H} z \cdot (u + y) = \max_{z \in K \cap H} z \cdot u = h_{K \cap H}(u)\]

and, similarly, \(h_{L \cap H}(u + y) = h_{L \cap H}(u).\) Thus, \(\lambda \cdot (K \cap H) +_0 (1 - \lambda) \cdot (L \cap H)\)

\[= \{ x \in H \mid x \cdot (u + y) \leq h_{K \cap H}(u + y) h_{L \cap H}^{1-\lambda}(u + y), \text{ for all } u \in H, y \in H^\perp \} \]
\[= \{ x \in H \mid x \cdot w \leq h_{K \cap H}(w) h_{L \cap H}^{1-\lambda}(w), \text{ for all } w \in \mathbb{R}^n \} \]
\[\subseteq \{ x \in H \mid x \cdot w \leq h_{K}^{\lambda}(w) h_{L}^{1-\lambda}(w), \text{ for all } w \in \mathbb{R}^n \} \]
\[= [\lambda \cdot K +_0 (1 - \lambda) \cdot L] \cap H. \quad \square\]
Proof of Corollary 6.5
We will make use of Firey’s argument for passing from the volume to other quermassintegrals (see [14]) and the fact that the \( p \)-convex combination of convex bodies contains the logarithmic convex combination, \( p > 0 \). It follows immediately by Lemma 6.7, that

\[
\left[ \lambda \cdot K +_0 (1 - \lambda) \cdot L \right]^o |H| \subseteq \left[ \lambda \cdot (K \cap H) +_0 (1 - \lambda) \cdot (L \cap H) \right]^o .
\]

Therefore, by the Kubota formula and Theorem 6.1, we obtain:

\[
W_i \left( \left[ \lambda \cdot K +_p (1 - \lambda) \cdot L \right]^o \right) \leq \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} \left[ \left[ \lambda \cdot K +_0 (1 - \lambda) \cdot L \right]^o |H| \right]_{n-i} dH \\
\leq \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} \left[ \lambda \cdot (K \cap H) +_0 (1 - \lambda) \cdot (L \cap H) \right]^o |H|_{n-i} dH \\
\leq \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} \left[ (K \cap H)^o |H|_{n-i}^\lambda (L \cap H)^o |H|_{n-i}^{1-\lambda} \right] dH \\
= \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} \left[ K^o |H|_{n-i}^\lambda L^o |H|_{n-i}^{1-\lambda} \right] dH \\
\leq \left[ \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} K^o |H|_{n-i} \right] \lambda \left[ \frac{\omega_n}{\omega_{n-i}} \int_{G_{n,n-i}} L^o |H|_{n-i} \right]^{1-\lambda} \\
= \left[ W_i(K^o)^\lambda W_i(L^o)^{1-\lambda} \right] , \quad i = 1, \ldots, n - 1 . \quad \Box
\]

Before ending this note, we would like to state a consequence of Theorem 6.1 that concerns the logarithmic sum itself, rather than its dual.

**Corollary 6.8.** Let \( \Delta_1, \Delta_2 \subseteq \mathbb{R}^2 \) be two triangles whose centroids are at the origin. Then,

\[
|\lambda \cdot \Delta_1 +_0 (1 - \lambda) \cdot \Delta_2| \geq |\Delta_1|^\lambda |\Delta_2|^{1-\lambda} . \tag{13}
\]

Proof. It is well known (see [36]) that if \( K \) is any planar convex body, then

\[
|K| \cdot |K^o| \geq |\Delta_1| \cdot |\Delta_2^o| , \tag{14}
\]

Now, if (13) is not true, then by Theorem 6.1 and (14) we get:

\[
|\lambda \cdot \Delta_1 +_0 (1 - \lambda) \cdot \Delta_2| \cdot |(\lambda \cdot \Delta_1 +_0 (1 - \lambda) \cdot \Delta_2)^o| < |\Delta_1|^\lambda |\Delta_2|^{1-\lambda} |\Delta_1^o|^\lambda |\Delta_2^o|^{1-\lambda} = |\Delta_1| \cdot |\Delta_2^o| ,
\]

which contradicts (14). \( \Box \)

**Acknowledgement.** I would like to thank A. Marsiglietti for reading carefully this manuscript and for many useful comments, especially for pointing me Corollary 3.4. I would also like to thank A. Zvavitch and O. Guedon for explaining me some very nice things.
References

[1] M. Anttila, K. Ball, I. Perissinaki, The central limit problem for convex bodies, Trans. Amer. Math. Soc., 355 (2003), 4723–4735.

[2] S. Bobkov, A. Koldobsky, On the central limit properties of convex bodies, Lect. Notes in Math. 1807 (2003), 44-52

[3] C. Borell, Convex measures on locally convex spaces, Ark. Mat. 12 (1974), 239-252.

[4] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, The log-Brunn-Minkowski inequality, Adv. Math. 231 (2012), 1974-1997.

[5] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. 26 (2013), 831–852.

[6] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Affine images of isotropic measures, J. Diff. Geom., accepted.

[7] J. Bourgain, On the distribution of polynomials on high-dimensional convex sets, Geometric aspects of functional analysis (1989-90), Lecture Notes in Math., 1469, Springer, Berlin, (1991), 127-137.

[8] A. Colesanti, I. Fragalà, The first variation of the total mass of log-concave functions and related inequalities, Advances in Mathematics 244 (2013), pp. 708-749.

[9] D. Cordero-Erausquin, M. Fradelizi and B. Maurey, The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems, J. Funct. Anal. 214 (2004), 410–427.

[10] D. Cordero-Erausquin, N. Gozlan, Transport proofs of weighted Poincaré inequalities for log-concave distributions, preprint available at arXiv:1407.3217

[11] R. Eldan, Thin shell implies spectral gap via a stochastic localization scheme, Geometric and Functional Analysis, 23 (2013), Issue 2, 532-569.

[12] R. Eldan, B. Klartag, Approximately gaussian marginals and the hyperplane conjecture, Proc. of a workshop on “Concentration, Functional Inequalities and Isoperimetry”, Contemporary Math., 545 (2011), Amer.Math. Soc., 55–68.

[13] W. J. Firey, Polar means of convex bodies and a dual to the Brunn-Minkowski theorem, Canad. J. Math. 13 (1961), 444-453.

[14] W. J. Firey, Mean cross-section measures of harmonic means of convex bodies, Pacific J. Math. 11 (1961), 1263-1266.

[15] B. Fleury, Between Paouris concentration inequality and variance conjecture, Ann. Inst. Henri Poincaré Probab. Stat. 46, 2 (2010), 299–312.

[16] M. Fradelizi, O. Guédon, A. Pajor, Spherical thin-shell concentration for convex measures, preprint.

[17] R. J. Gardner, Geometric Tomography, Second ed., Cambridge University Press, Cambridge, 2006.

[18] R. J. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002), 355–405.

[19] R. J. Gardner, A. Zvavitch, Gaussian Brunn-Minkowski inequalities, Trans. Amer. Math. Soc. 362 (2010), no. 10, 5333–5353.

[20] R.J. Gardner, D. Hug, W. Weil, D. Ye, The dual Orlicz-Brunn-Minkowski theory, 2014, preprint.
[21] O. Guédon, E. Milman, Interpolating Thin-Shell and Sharp Large-Deviation Estimates For Isotropic Log-Concave Measures, Geom. Funct. Anal. 21 (5) (2011), 1043-1068.

[22] O. Guédon, Concentration phenomena in high dimensional geometry. In ESAIM Proceedings., vol. 44. SMAI, 2014, 47–60.

[23] M.A. Hernández Cifre, J.Y. Nicolás, On Brunn-Minkowski type inequalities for polar bodies, to appear in J. Geom. Anal.

[24] R. Kannan, R., L. Lovász, M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, Discrete Comput. Geom. 13, 3-4 (1995), 541-559.

[25] B. Klartag On convex perturbations with a bounded isotropic constant, Geom. and Funct. Anal., 16, (6) (2006), 1274–1290.

[26] B. Klartag, A Berry-Esseen type inequality for convex bodies with an unconditional basis, Probab. Theory Related Fields, 45, no. 1, (2009), 1 – 33.

[27] B. Klartag, High-dimensional distributions with convexity properties, Proc. of the Fifth Euro. Congress of Math., Amsterdam, July 2008. Eur. Math. Soc. publishing house, (2010), 401–417.

[28] R. Latala, K. Oleszkiewicz, Small ball probability estimate in terms of width, Studia Math. 169 (2005), 305-314.

[29] R. Latala, On some inequalities for Gaussian measures, Proceedings of the International Congress of Mathematicians, Beijing, Vol. II, Higher Ed. Press, Beijing, 2002, pp. 813-822.

[30] A. Livne Bar-on, The (B) conjecture for uniform measures in the plane, 2013, preprint, arXiv:1311.6584.

[31] E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, J. Differential Geom. 38 (1993), 131–150.

[32] E. Lutwak, The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas, Adv. Math. 118 (1996), 224–294.

[33] E. Lutwak, D. Yang, G. Zhang, Lp affine isoperimetric inequalities, J. Differential Geom. 56 (2000), 111–132.

[34] E. Lutwak, D. Yang, G. Zhang, On the Lp-Minkowski problem, Trans. Amer. Math. Soc. 356 (2004), 4359-4370.

[35] E. Lutwak, D. Yang, G. Zhang Lp John Ellipsoids, Proceedings of the London Mathematical Society, 90 (2005), 497–520.

[36] K. Mahler, Ein Minimalproblem für konvexe Polygone, Mathematica (Zupphen) B 7 (1939), 118-127.

[37] A. Marsiglietti, On the improvement of concavity of convex measures, preprint, arXiv:1403.7643.

[38] V. D. Milman and A. Pajor, Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, Geometric aspects of functional analysis (1987-88), Lecture Notes in Math., 1376, Springer, Berlin, 1989, 64-104.

[39] C. Saroglou, Shadow systems: remarks and extensions, Arch. Math., 100 (2013), 389–399.

[40] C. Saroglou, Remarks on the conjectured log-Brunn-Minkowski inequality, 2013, Geom. Dedicata (to appear).
[41] R. Schneider, Convex Bodies: The Brunn-Minkowski theory, second edition, Cambridge University Press, Cambridge, 2014.

[42] A. Stancu, The discrete planar $L_0$-Minkowski problem, Adv. Math. 167 (2002), 160–174.

[43] A. Stancu, On the number of solutions to the discrete two-dimensional $L_0$-Minkowski problem, Adv. Math. 180 (2003), 290–323.

[44] B. Uhrin, Curvilinear Extensions of the Brunn-Minkowski-Lusternik Inequality, Adv. Math., 109 (1994), no. 2, 288-312.

Ch. Saroglou: Department of Mathematics, Texas A&M University, 77840 College Station, TX, USA.
E-mail: saroglou@math.tamu.edu & christos.saroglou@gmail.com