Anomalous Hall effect sensitive to magnetic monopoles and skyrmion helicity in spin–orbit coupled systems

Jun Mochida and Hiroaki Ishizuka

1 Department of Physics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
2 Department of Physics, Tokyo Institute of Technology, Meguro, Tokyo 152-8551, Japan

* Author to whom any correspondence should be addressed.
E-mail: ishizuka@phys.titech.ac.jp

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Abstract

Magnetic textures, such as skyrmions and domain walls, engender rich transport phenomena, including anomalous Hall effect and nonlinear response. In this work, we discuss an anomalous Hall effect proportional to the net magnetic monopole charge and dependent on the skyrmion helicity that occurs by a skew scattering in a noncentrosymmetric two-dimensional magnet. This mechanism, which arises from the spin–orbit interaction (SOI), gives rise to a finite anomalous Hall effect in a ferromagnetic domain wall whose spins rotate in the $xy$ plane despite no out-of-plane magnetic moment. We show that the presence and absence of the monopole contribution is related to crystal symmetry, which gives a guideline for finding candidate materials beyond the Rashba model. The results demonstrate the rich features arising from the interplay of SOI and magnetic textures, and their potential for detecting various magnetic textures in micrometer devices.

1. Introduction

Noncollinear magnetic textures give rise to novel phenomena, such as anomalous [1–3] and spin [4, 5] Hall effects, multiferroics [6, 7], and electrical magnetochiral effect [8]. These phenomena are often related to the scalar and vector spin chiralities defined by $S_i \cdot S_j \times S_k$ and $S_i \times S_j$, respectively. These phenomena were also discussed experimentally: For instance, the anomalous Hall effect (AHE) was studied in transition metal magnets, such as in materials with non-coplanar magnetic order [9] and magnetic skyrmions [10, 11], and the electrical magnetochiral effect in helical magnets [12, 13].

Among various examples, a particularly interesting example is the continuum limit, in which the scalar spin chirality is related to skyrmion number [1]. In the limit, the Hall conductivity is related to skyrmion density, as in perturbation theory [14] and the skew scattering argument [15]. The relation to a topological quantity implies the robustness of the AHE against fluctuations in the spin texture. Such robustness is desirable for applications as it can be used as a robust probe for detecting a skyrmion in small devices, such as in a racetrack memory [16–18]. However, such a relation between the transport coefficient and a quantity characterizing the magnetic texture is rare.

To explore the possibility of novel AHE by magnetic textures, we focus on the skew scattering mechanism by multiple magnetic moments (figure 1(a)). Theoretically, the mechanism of AHE is broadly classified into two groups: the intrinsic mechanism related to Berry curvature [19] and the extrinsic mechanism involving impurity scattering [20, 21]. Both non-magnetic [20–23] and magnetic impurity scattering [24–26] were discussed for the latter. Besides the AHE in ferromagnets, non-coplanar magnetic states induce an AHE [1, 2, 14] as in skyrmion materials, an effect known as topological Hall effect. The AHE by non-coplanar magnetic order raised interesting questions on the interplay of non-collinear magnetic states and the spin–orbit interaction (SOI) [3, 27], especially in the continuous limit relevant to AHE by skyrmions [28–30]. However, in the cases considered so far, the correction remains relatively small [28] or vanishing [29]. In contrast, the...
effect of SOI on AHE is much less known for skew scattering by multiple magnetic moments [5, 15], which potentially gives rise to a larger Hall effect as recently observed in experiment [31–33].

In this work, we study the AHE by the interplay of magnetic multiple scattering and SOI in two-dimensional magnetic semiconductors. We study the AHE in Rashba and Dresselhaus models coupled to local moments by the Kondo coupling, combining scattering theory and semiclassical Boltzmann theory. Our calculation for the Rashba model finds that, when the magnetic moments lie in the $xy$ plane, the Hall conductivity is proportional to the magnetic monopole charge characterized by the divergence of magnetic moments. We predict that the monopole Hall effect induces an observable AHE, especially in micrometer-size devices. In addition, we find that the Heisenberg spin contribution induces an AHE that depends on the skyrmion helicity. This contribution has new possibilities for the detection and control of skyrmions with different helicity. We argue that these terms are strongly restricted by crystal symmetry, as we show from a phenomenological formula and comparison to the Dresselhaus model. Lastly, we discuss the effect of lattice distortion.

2. Rashba and Dresselhaus models

To study the effect of SOI on AHE by skew scattering, we focus on Rashba and Dresselhaus models in which electrons are coupled to localized classical $XY$ spins by Kondo coupling. The Hamiltonian for the Rashba model reads

$$H = H_0 + H_K,$$

$$H_0 = \sum_{\alpha,\beta} \sum_{k,\alpha,\beta} c_{\alpha}^\dagger (r_k) \left[ \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \delta_{\alpha,\beta} + \hbar \lambda k \times \sigma_{\alpha,\beta} \right] c_{\beta},$$

$$H_K = -J a^2 \sum_{i,\alpha,\beta} c_{\alpha}^\dagger (r_i) [S (r_i) \cdot \sigma_{\alpha,\beta}] c_{\beta} (r_i),$$

where $c_{\alpha}$ ($c_{\alpha}^\dagger$) is the electron annihilation (creation) operator with momentum $k = (k_x, k_y)$ and spin $\alpha$, $c_{\alpha} (r_k) = \frac{1}{\sqrt{N}} \sum_k e^{i k \cdot r_k} c_{\alpha}^\dagger (r_k)$ is the electron annihilation (creation) operator at position $r$ and spin $\alpha$, $S (r_i)$ is the classical $XY$ spin at position $r_i$, and $\sigma = (\sigma^x, \sigma^y, \sigma^z)$ is the vector of Pauli matrices $\sigma^{\lambda, \gamma}$. The constant $m$ is the effective mass of the electron, $\mu$ is the chemical potential, $\lambda$ is the spin–orbit coupling, $J$ is the Kondo coupling, $a$ is the lattice constant, and $\hbar$ is the Planck constant. The triple and inner products are $\hat{z} \cdot k = k_z$ and $S (r) \cdot \sigma = (S^x (r) \sigma^x + S^y (r) \sigma^y + S^z (r) \sigma^z)$. The eigenergy of $H_0$ reads $\varepsilon_{k_0} = \frac{\hbar^2 k_0^2}{2m} + \eta \hbar \lambda k_0$ with $\eta = \pm 1$.

The Hamiltonian for the Dresselhaus model reads

$$H = H_0 + H_K,$$

$$H_0 = \sum_{k,\alpha} c_{\alpha}^\dagger \left[ \left( \frac{\hbar^2 k^2}{2m} - \mu \right) \delta_{\alpha,\beta} + \frac{\beta}{\hbar} \sum_{k,\alpha,\beta} c_{\alpha}^\dagger (k_\beta \sigma_{\alpha,\beta}^{\alpha', \beta'} - k_\gamma \sigma_{\alpha,\beta}^{\alpha', \gamma}) c_{\alpha',\beta'},$$

and $H_K$ is the same as in equation (1). This model lacks the mirror symmetries about $x$ and $y$ axes, $M_x$ and $M_y$, respectively. As we argue in the later sections, the difference in symmetry gives rise to distinct features in AHE.
this asymmetric scattering reads

\[ W_1 = \frac{(k_{\ell}' + k_{\ell})}{2h^2} \frac{\alpha}{2m\lambda^2 + 2m\mu} - \frac{m^2\lambda}{2h^2} \]

\[ W_2 = \frac{\sigma}{2h^2} \frac{m^2\lambda}{2m\lambda^2 + 2m\mu} - \frac{m^2\lambda}{2h^2} \]

\[ W_3 = \frac{\sigma}{2h^2} \frac{m^2(\mu + m\lambda^2)}{2m\lambda^2 + 2m\mu} \]

\[ W_4 = \frac{(k_{\ell} - k_{\ell})}{2h^2} \frac{m^2\lambda}{2m\lambda^2 + 2m\mu} - \frac{m^2\lambda}{2h^2} \]

Table 1. Coefficients of the skew scattering in the Rashba electron in equations (3) and (12). The left column is for chemical potential \( \mu < 0 \) and the right column is for \( \mu \geq 0 \).

| \( \mu < 0 \) | \( 0 \leq \mu \) |
|----------------|----------------|
| \( W_1 \) \( \frac{(k_{\ell} + k_{\ell})}{2h^2} \frac{\alpha}{2m\lambda^2 + 2m\mu} - \frac{m^2\lambda}{2h^2} \) | \( W_1 \) \( \frac{m^2\lambda}{2h^2} \) |
| \( W_2 \) \( - \frac{m^2\lambda}{2h^2} \) | \( W_2 \) \( \frac{m^2\lambda}{2h^2} \) |
| \( W_3 \) \( \frac{m^2(\mu + m\lambda^2)}{2m\lambda^2 + 2m\mu} \) | \( W_3 \) \( \frac{m^2\lambda}{2h^2} \) |
| \( W_4 \) \( \frac{(k_{\ell} - k_{\ell})}{2h^2} \frac{m^2\lambda}{2m\lambda^2 + 2m\mu} - \frac{m^2\lambda}{2h^2} \) | \( W_4 \) \( \frac{m^2\lambda}{2h^2} \) |

Table 2. Coefficients of the Hall conductivities in equation (6). The center column is for chemical potential \( \mu < 0 \) and the right column is for \( \mu \geq 0 \).

| \( \mu < 0 \) | \( 0 \leq \mu \) |
|----------------|----------------|
| \( \sigma_1 \) \( \frac{e^2}{2\pi^2} \frac{\alpha}{h^2} \frac{m^2\lambda a(\mu + m\lambda^2)}{(\lambda m)^2 + 2m\mu} \) | \( \sigma_1 \) \( \frac{e^2}{2\pi^2} \frac{\alpha}{h^2} \frac{m^2\lambda a(\mu + m\lambda^2)}{(\lambda m)^2 + 2m\mu} \) |
| \( \sigma_2 \) \( \frac{e^2}{4\pi^2} \frac{\alpha}{h^2} \frac{m^2\lambda^2}{(\lambda m)^2 + 2m\mu} \) | \( \sigma_2 \) \( \frac{e^2}{4\pi^2} \frac{\alpha}{h^2} \frac{m^2\lambda^2}{(\lambda m)^2 + 2m\mu} \) |
| \( \sigma_3 \) \( \frac{e^2}{4\pi^2} \frac{\alpha}{h^2} \frac{m^2\lambda a(\mu + m\lambda^2)}{(\lambda m)^2 + 2m\mu} \) | \( \sigma_3 \) \( \frac{e^2}{4\pi^2} \frac{\alpha}{h^2} \frac{m^2\lambda a(\mu + m\lambda^2)}{(\lambda m)^2 + 2m\mu} \) |

3. Results

3.1. Rashba model

We investigate the Hall effect by evaluating the skew scattering probability using the Born approximation and calculating the Hall conductivity using the Boltzmann equation [see the Appendices for details]. The scattering probability from \( k \) state to \( k' \) \( \eta' \) is calculated within second-Born approximation, in which the skew scattering probability \( W_{k\eta \rightarrow k'\eta'} = (W_{k\eta \rightarrow k'\eta'} - W_{k'\eta' \rightarrow k\eta})/2 \) reads

\[ W_{k\eta \rightarrow k'\eta'} = \frac{\varepsilon_{k\eta} - \varepsilon_{k'\eta'}}{(aL)^2} \frac{\delta(\varepsilon_{k\eta} - \varepsilon_{k'\eta'})}{\sum_{i,j,l} (S_i \times r_{ij}) \cdot (S_i \times S_j) \sin \delta \phi, \]  

(5)

in the leading order in \( ka \), where \( L \) is the length of the system, \( \delta \phi = \sin(\phi_{k\eta} - \phi_{k'\eta'} / (kF)) \) is the scattering angle, and \( S_i \) = \( S(r_i) \) is the magnetic moment at \( r_i \). The coefficient \( W_1 \) is shown in table 1. We have used the perturbation condition by neglecting higher-order terms of \( O((kF)^3) \) with the Fermi wave vector \( k_F \), and keeping the Kondo coupling small as \( \varepsilon_F \gg \lambda a^2 \) with the Fermi energy \( \varepsilon_F \). The scattering term in equation (5) induces skew scattering without a finite scalar spin chirality or magnetic moment perpendicular to the plane, both of which are zero when the spins lie in the \( xy \) plane. The Hall conductivity stemming from this asymmetric scattering reads

\[ \sigma_{xy} = -\frac{\sigma_1}{aL^2} \sum_{i,j,l} (S_i \times r_{ij}) \cdot (S_i \times S_j), \]

(6)

Here, the coefficient \( \sigma_1 \) depends on the chemical potential \( \mu \), as given in table 2; in \( \sigma_1 \), we introduce a phenomenological relaxation rate \( \tau \). The coefficients show different \( \mu \) dependence reflecting the distinct nature of electron bands above and below \( \mu = 0 \).

The outer product in the sum of equation (6) shows that the scattering by two spins is necessary for a finite \( \sigma_{xy} \) in this system as shown in figure 1(a). Note that, due to the zero out-of-plane magnetization, the AHE contributions known in ferromagnets vanish. However, a finite \( \sigma_{xy} \) appears due to the scattering by multiple spins.
Figure 2. Schematics of domain walls. (a) A domain wall whose spin rotates in the xy plane. In view of magnetic monopoles, the domain wall corresponds to magnetic monopoles (yellow circles) aligned along the domain wall. (b) A similar argument holds for domain walls with spins rotating out-of-plane. (c) A schematic of a system containing multiple magnetic walls and monopoles, and its Hall effect. The plus and minus circles on the figure are the sign of the spin monopoles.

Focusing on the scattering processes involving two nearest-neighbor spins, \(i\) and \(j\), the Hall conductivity reads

\[
\sigma_{xy} = \frac{2\sigma_1}{aL^2} \sum_{\langle i,j \rangle} \left( 1 + S_i \cdot S_j \right) \left[ r_{ij} \cdot (S_i - S_j) \right].
\]  

Here, we assume multiple scattering by the nearest-neighbor spins dominates. In the continuum limit assuming \(S_i \sim S_j\), this formula becomes

\[
\sigma_{xy} = \frac{4\sigma_1}{aL^2} \int_V d\mathbf{x} \nabla \cdot \mathbf{S(x)},
\]  

where \(\nabla = (\partial_x, \partial_y)\) and \(\nabla \cdot \mathbf{S} = \partial_x S_x + \partial_y S_y\). Here, we use the gradient expansion assuming \(|S_i| = 1\). This expansion is valid when the spin texture changes gradually compared to the lattice constant so that the nearest-neighbor spins are almost parallel to each other. The divergence \(\nabla \cdot \mathbf{S(x)}\) defines the monopole charge of spins, analogous to the definition of an electric charge in electromagnetism. Equation (8) suggests that the monopole charge of the magnetic moments induces the AHE through the skew scattering.

Using the divergence theorem, the above formula reads

\[
\sigma_{xy} = \frac{4\sigma_1}{aL^2} \int_{\partial V} n(x) \cdot \mathbf{S(x)} \, dl,
\]  

as in the Gauss’ theorem in electromagnetism. Here, the integral is taken over the boundary of the system \(\partial V\), where \(n(x)\) is the unit vector perpendicular to the boundary (figure 1(b)). We note that, as the integral is over the surface, the internal spin texture does not affect the overall conductivity. In other words, the anomalous Hall conductivity is related to the net magnetic charge, not to the fine details of the spin structure.

The simple expression in equation (7) can be straightforwardly extended to more complex systems of real materials. For example, this AHE should naturally occur in a two-dimensional magnet with multiple magnetic domains (figure 2(c)). In the presence of ferromagnetic domains, each domain wall and its intersection is a monopole. Therefore, a finite net charge of the domain wall is expected to provide a finite Hall effect. In such a case, the anomalous Hall conductivity is given as the simple summation of the contribution from each domain wall.

To provide a quantitative estimate, we compute the contribution to the anomalous Hall conductivity from a monopole (figure 1(b)) and a domain wall (figure 2(a)). For the monopole that lies in the xy plane, the spin configuration reads \(\mathbf{S(r)} = (\cos(\phi), \sin(\phi), 0)\), where \(\phi\) is the azimuth angle of the polar coordinate, i.e. \(\mathbf{r} = (r\cos(\phi), r\sin(\phi))\). This spin configuration describes a magnetic monopole located at the origin as shown in figure 1(b). By applying equation (5), the Hall conductivity reads

\[
\sigma_{xy} = \frac{8\pi\sigma_1}{aL}.
\]  

A finite Hall effect by the magnetic monopole implies that the monopoles are electrically detectable using the Hall effect, similar to the detection of skyrmions using the AHE. For the domain wall whose magnetization points along opposite directions (figure 2(a)), the contribution to the Hall conductivity from a domain wall is given by

\[
\sigma_{xy} = \frac{8\sigma_1}{aL}.
\]
Here, we assumed that a domain wall in the form \( S(r) = (\tanh(x), \frac{1}{\cosh(x)}, 0) \). Note that, in both cases, the conductivity scales \( L^{-1} \) to the system size \( L \). However, it gives an observable consequence if the coefficient \( \sigma_1 \) is sufficiently large.

To examine the typical magnitude of the response equation (11) gives for a system with multiple domain walls, we estimate the Hall conductivity using a set of parameters for Rashba electron known in experiment [34]: \( m = 10^{-30} \) kg, \( \mu = 0.19 \) eV, \( a = 4.3 \) Å, and \( h\lambda = 3.8 \) eVÅ. Assuming the Kondo coupling \( J = 0.1 \) eV, the Hall conductivity becomes \( \sigma_{xy} \sim 10^4 \Omega^{-1} \text{cm}^{-1} \) for a domain wall in \( L = 1 \) \( \mu \text{m} \) in size device, which is sufficiently large to be considered a bulk material in most cases. We reemphasize that the Hall conductivity is determined solely by the direction of magnetic moments in the two adjacent domains or the magnetization at the boundary of the device (the boundary of the shaded region in figure 2(a)), as discussed above. In other words, the detail of the domain wall structure does not affect the result. The observable magnitude of the Hall conductivity and its robustness against the fine structure of the domain wall implies that this AHE is a good probe of domain walls.

3.2. Heisenberg spins

For the case of Heisenberg spins \( S_i = (S^x_i, S^y_i, S^z_i) \), whose magnetic moment can point perpendicular to the plane, a similar result of the skew scattering is given as

\[
W_{k\eta \to k'\eta'} = \pi \delta (\varepsilon_{k\eta} - \varepsilon_{k'\eta'}) \sum_{i,j} \left[ \frac{W_1}{(aL^2)^2} (S_i \times r_{ij}) \cdot (S_i \times S_j) - \frac{W_2}{(aL^2)^2} S^z_i (S_i \cdot S_j) \right.
\]

\[
+ \left. \frac{W_3}{(aL^2)^2} S^z_i (\hat{z} \times r_{ij}) \cdot (S_i \times S_j) \right] \sin \delta \phi + \delta (\varepsilon_{k\eta} - \varepsilon_{k'\eta'}) \cdot \frac{W_4}{(aL^2)^2} \sum_{i,j,l} (S_i \cdot S_j) (\hat{z} \cdot r_{ij} \times S_l) \cos \delta \phi.
\]

Subsequently, the Hall conductivity of the Rashba model with Heisenberg spins reads

\[
\sigma_{xy} = -\frac{\sigma_1}{aL^2} \sum_{i,j,l} (S_i \times r_{ij}) \cdot (S_i \times S_j) + \frac{\sigma_3}{aL^2} \sum_{i,j,l} S^z_i (S_i \cdot S_j) - \frac{\sigma_3}{aL^2} \sum_{i,j,l} S^z_i (\hat{z} \times r_{ij}) \cdot (S_i \times S_j).
\]

In addition to the \( \sigma_1 \) term in equation (3), two additional terms \( \sigma_2 \) and \( \sigma_3 \); the explicit form given in table 1 appear reflecting the uniaxial anisotropy of Rashba model. This formula reduces to equation (3) when \( S^z_i = 0 \). The \( \sigma_2 \) term in equation (13) is the generalization of a scattering mechanism pointed out by Kondo [24], in which they considered the scattering by single spin. On the other hand, \( \sigma_3 \) is another term related to the vector spin chirality of \( S_i \) and \( S_j \) that does not exist in a system without SOI. For a smooth magnetic structure on the square lattice, equation (13) reads

\[
\sigma_{xy} = \frac{16\sigma_2}{aL^2} \int \text{d}x^2 S^z \cdot \text{d}x^2 \nabla \cdot S + \frac{4\sigma_2}{aL^2} \int \text{d}x^2 \left[ \partial_x S^x + \partial_y S^y + S^z \{ S \cdot \partial_x S + S \cdot \partial_y S \} \right]
\]

\[
+ \frac{4\sigma_3}{aL^2} \int \text{d}x^2 \left[ (S \times \partial_x S)_y - (S \times \partial_y S)_x \right].
\]

The last term, with one spatial gradient, resembles that recently discussed as a correction to the topological Hall effect in skyrmion materials [28]. Note that all these terms arise from two-spin scattering, which is different from the three-spin contribution that gives rise to the AHE related to scalar spin chirality [1, 9, 15].

To see how the novel term arising in the Heisenberg spins affects the AHE, we consider the AHE by magnetic skyrmion crystal. To this end, we consider the scattering by magnetic skyrmions whose spin configuration is given by

\[
S(r) = \begin{pmatrix}
\cos (\phi + \gamma) \\
\sin (\phi + \gamma) \\
\tanh \left( \frac{r - \ell}{\xi} \right)
\end{pmatrix},
\]

where \( \ell, \xi \) are the radius and domain wall length of the skyrmion, respectively. Using equation (14), the Hall conductivity after taking the sum over all skyrmions reads

\[
\sigma_{xy} = \frac{16\sigma_2}{aL^2} \sum_{i,k} n_{ik} \cos \left( \frac{k}{\xi} \right) \Omega \left( \frac{l}{\xi} \right),
\]

\[
\Omega(x) = \frac{\pi}{2} + \tan^{-1} \left( \sinh(x) \right) + \frac{\sinh(x)}{\cosh^2(x)},
\]

where

\[
\Omega(x) = \left[ \frac{\pi}{2} + \tan^{-1} \left( \sinh(x) \right) + \frac{\sinh(x)}{\cosh^2(x)} \right].
\]
where \( m_z = \frac{1}{N} \int \mathrm{d}^2 S^r \) is the magnetization of the skyrmions and \( n_{sk} \) is the number density of the skyrmions. The phase term \( \gamma \) represents the helicity of the skyrmion. The skyrmions are of Néel type if \( \gamma = 0, \pi \), and are of Bloch type if \( \gamma = \pm \pi/2 \). Now we consider the contribution of the AHE from the averaged skyrmion in a skyrmion crystal where the skyrmions are aligned in the bulk, therefore the second term of the conductivity is multiplied by \( n_{sk} \).

The first term in equation (16) is a contribution similar to the ordinary AHE studied by Kondo [24], and the second term is the contribution from the last term in equation (14); note that the monopole term proportional to \( \sigma_1 \) vanishes because the spins at the boundary all points upward. The second term in equation (16) resembles that of the scalar-chirality AHE in the sense that it is proportional to \( n_{sk} \). However, it shows \( \cos(\gamma) \) helicity dependence, which is distinct from the scalar chirality term. The helicity dependence indicates that the second term in equation (16) remains finite for the Néel-type skyrmion whereas it vanishes for the Bloch-type skyrmion.

For the case of domain walls whose spins point out of plane (figure 2(b)), the Hall conductivity reads

\[
\sigma_{xy} = 16 \sigma_2 m_z + \frac{8 \sigma_1}{aL} + \frac{8 \sigma_3}{aL}.
\]

(18)

In this case, both \( \sigma_1 \) and \( \sigma_3 \) contribute to the AHE by domain walls, in addition to the ferromagnetic contribution proportional to \( \sigma_2 \). However, since \( \sigma_1 \) and \( \sigma_3 \) scales \( L^{-1} \), the contribution from \( \sigma_2 \) becomes larger than \( \sigma_1 \) and \( \sigma_3 \) in a large device.

As discussed in the previous section, the anomalous Hall conductivity equation (16) can be applied to systems with complex magnetic structures by simply summing the individual contributions. This is justified by the separability of each local scattering term in the Boltzmann equation. As demonstrated through Gauss’s theorem in equation (3), the contribution arises from the surface of the system. Utilizing this property, we can calculate the skyrmion crystal equation (18) from calculating a single skyrmion and then taking the sum over the entire bulk system.

### 3.3. Dresselhaus model

To gain some idea of the model dependence of the result, we next look into the Dresselhaus model. The Dresselhaus model has exactly the same dispersion with differences in the symmetry and eigenstate wavefunction. Hence, the Dresselhaus model is expected to be a good model for demonstrating how the difference in the symmetry of electronic state affects the transport property.

Using the same method as in the Rashba model, the general formula for anomalous Hall conductivity reads

\[
\sigma_{xy} = -\frac{\sigma_1}{aL^2} \sum_{i,j,l} (S_i \times S_j)_z (S_i^r S_{ij}^r - S_j^r S_{ij}^r) + \frac{\sigma_1}{aL^2} \sum_{i,j,l} (S_i \cdot S_j) (S_i^r S_{ij}^r + S_j^r S_{ij}^r) - \frac{\sigma_3}{aL} \sum_{i,j,l} S_i^2 (S_i \cdot S_j)
\]

\[+ \frac{\sigma_1}{aL^2} \sum_{i,j,l} S_i^2 \left( (S_i \times S_j)_x r_{ij}^y - (S_i \times S_j)_y r_{ij}^x \right) + \frac{\sigma_3}{aL^2} \sum_{i,j,l} S_i^2 \left( (S_i \times S_j)_x r_{ij}^y - (S_i \times S_j)_y r_{ij}^x \right),
\]

(19)

where the coefficients \( \sigma_i \) are given in table 3. The last two terms have a different dependence to the vector chirality \( (S_i \times S_j)_\alpha \) for \( \alpha = x, y \) and \( S_i^2 \) compared to the third term in the Rashba model case, equation (13). Most importantly, the fourth term depends on the \( r_{ij} \) and not on \( {}_n r_{ij} \). On the other hand, the third term in the equation (19) represents conventional AHE as same as the second term in the equation (13).

The first two terms gives rise to a finite AHE without the out-of-plane magnetic moment. For the XY spins, i.e. \( S_i^z = 0 \), the Hall conductivity reduces to

\[
\sigma_{xy} = -\frac{\sigma_1}{aL^2} \sum_{i,j,l} (S_i \times S_j)_z (S_i^r S_{ij}^r - S_j^r S_{ij}^r) + \sum_{i,j,l} (S_i \cdot S_j) (S_i^r S_{ij}^r + S_j^r S_{ij}^r).
\]

(20)

These two terms are different from the conductivity in the Rashba model. For instance, the continuous limit of equation (20) reads

\[
\sigma_{xy} = -\frac{2 \sigma_1}{aL^2} \int \mathrm{d}^2 S^r [S^r (S \times \partial_z S)_z - S^r (S \times \partial_z S)_z] - \frac{2 \sigma_1}{aL^2} \int \mathrm{d}^2 \left[ \partial_z S^r + \partial_y S^r \right].
\]

(21)

Unlike the Rashba model, the \( \nabla \cdot S \) term does not appear in the case of the Dresselhaus model.
Table 3. Coefficients of the Hall conductivities in equations (19). The center column is for a chemical potential $\mu < 0$, and the right column is for $\mu \geq 0$.

| $\sigma$ | $\mu < 0$ | $0 \leq \mu$ |
|---------|------------|-------------|
| $\sigma_1$ | $\frac{e^2 r^2 p^d}{4\pi^2} \frac{m^d a^2 (2^2 m^b + m^d b^2)}{h^2 \mu + 2 m^d b^2}$ | $\frac{e^2 r^2 p^d}{4\pi^2} \frac{m^d a^2 (2^2 m^b + m^d b^2)}{h^2 \mu + 2 m^d b^2}$ |
| $\sigma_2$ | $\frac{e^2 r^2 p^d}{4\pi^2} \frac{m^d b^3}{h^2 \mu + 2 m^d b^2}$ | $\frac{e^2 r^2 p^d}{4\pi^2} \frac{m^d b^3}{h^2 \mu + 2 m^d b^2}$ |
| $\sigma_3$ | $\frac{e^2 r^2 p^d}{4\pi^2} \frac{m^d a^2 (2^2 m^b + m^d b^2)}{h^2 \mu + 2 m^d b^2}$ | $\frac{e^2 r^2 p^d}{4\pi^2} \frac{m^d a^2 (2^2 m^b + m^d b^2)}{h^2 \mu + 2 m^d b^2}$ |

Table 4. Symmetries of AHE and SOI. In the table, it is written as "0" if each conductivity in the phenomenological AHE formula vanishes under each symmetry. $\nabla \cdot S_{\perp}$ and $(\nabla \times S_{\perp})^2$ are the Hall current proportional to the divergence and rotation of the magnetic texture, respectively. $\sigma^\dagger_{\text{Dressel}}$ and $\sigma^\dagger_{\text{Dressel}}$ are respectively the first and second terms in equation (21).

| $\nabla \cdot S_{\perp}$ | $\nabla \times S_{\perp}^2$ | $\nabla \cdot S_{\perp}$ | $\nabla \times S_{\perp}^2$ | $\nabla \cdot S_{\perp}$ | $\nabla \times S_{\perp}^2$ | $\nabla \cdot S_{\perp}$ | $\nabla \times S_{\perp}^2$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\pi$ | $C_2$ | $C_4$ | $C_5$ | $M_2$ | $M_4$ | $M_6$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
| $\sigma_{\text{Dressel}}$ | $\sigma_{\text{Dressel}}$ |
| Rashba | $\times$ | $\circ$ | $\times$ | $\times$ | $\times$ | $\circ$ | $\circ$ | $\circ$ | $\circ$ |
| Dresselhaus | $\times$ | $\circ$ | $\circ$ | $\circ$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |

3.4. The role of symmetry in AHE

The strong sensitivity of $\sigma_{xy}$ to the model is understandable from the symmetry viewpoint. The AHE by magnetic monopole is phenomenologically described by a formula $J_r = \sigma \int \nabla \cdot S_{\perp}^2 (r) dr^2 E_r$. For example, when the inversion symmetry exists in the system, the symmetry operation transforms $\pi : S_{\perp} (r) = S_{\perp} (-r)$, $J_r = -J_r$, and $E_r = -E_r$. Hence, the phenomenological formula becomes $J_r = -\sigma \int \nabla \cdot S_{\perp}^2 (r) dr^2 E_r$. Thus, $\sigma = -\sigma$ for the system with inversion symmetry, implying that the AHE proportional to $\nabla \cdot S_{\perp}^2$ vanishes. Similarly, the mirror operation about the $z$ axis, $M_z : x, y \rightarrow x, y$ and $z \rightarrow -z$, gives $\sigma = -\sigma$, again implying the vanishing Hall conductivity. The lower half of the table shows the symmetry of the Rashba and Dresselhaus models, where the $\times$ in the table denotes that the symmetry does not exist in the model.

As a result, reflecting the symmetry property, the $\nabla \cdot S$ term is allowed in the Rashba model as seen in equation (21) whereas the Hall effect related to other forms of spin structure is possible in the Dresselhaus model. As demonstrated by two models, the form of spin textures reflected in AHE is controllable using crystal symmetry.

3.5. Effect of lattice distortion in Rashba model

In the last, we discuss how the lattice distortion affects the AHE by focusing on the formation of clusters. Up to now, we considered a uniform lattice without lattice distortion. In some materials, however, the lattice distortion forms a cluster of spins, such as in trimerized triangular lattice [33] and in breathing kagomé [35, 36] and pyrochlore [37, 38] magnets. As a concrete example, we consider a breathing kagomé lattice with a canted 120° spin configuration as in figure 3(a). By extending equation (6) to the spins with nonzero $S^z$, the Hall conductivity reads

$$\sigma_{xy} = 12\sigma_1 \delta \cos \theta (1 + 3 \sin^2 \theta) + 24\sqrt{3}\sigma_2 \sin^3 \theta - 48\sigma_3 \delta \cos \theta \sin^2 \theta,$$

where $\theta$ is the out-of-plane canting angle, and the bond length of the upward (downward) triangle is $(\frac{1}{2} + \delta) a$ [$\frac{1}{2} - \delta] a$ (figure 3(b)). When $\theta = 0$, i.e. for the coplanar 120° order, the conductivity becomes $\sigma_{xy} = 12\sigma_1 \delta$ Unlike the uniform case discussed in the previous sections, the Hall conductivity in the distorted kagomé lattice remains finite in the bulk limit.

In view of the magnetic monopole, the 120° order is a pair of monopole and anti-monomopole (the monopole with the opposite charge). Hence, the total monopole charge is zero which is consistent with the zero Hall conductivity at $\delta = 0$. In the presence of breathing, however, the breathing breaks the cancellation between monopole and anti-monopole contributions, giving a finite Hall effect.
4. Discussion

In this work, we systematically study the skew scattering by multiple spins in systems with strong SOI. A particularly interesting term is that proportional to $(1 + S_i \cdot S_j)[r_{ij} \cdot (S_i - S_j)]$, which is related to the magnetic monopoles in the continuous limit $\nabla \cdot S(r)$. We showed that this is the only term that contributes to AHE when the spins lie in the $xy$ plane, e.g. for magnetic metals with an easy-plane anisotropy. In a material of $L = 1\,\mu m$ size, the $\nabla \cdot S(r)$ term gives rise to observable AHE in the presence of vortex-like defects or domain walls. We also find that, in the presence of a breathing-type lattice distortion, a similar AHE also occurs. Lastly, our estimate for the Rashba model gives a contribution of $\sigma_{xy} \sim 10^{3} \Omega^{-1} cm^{-1}$ from each domain wall of $100 \, nm$ length, which should be observable in an experiment. In the case of skyrmions, we find that the AHE contains a term proportional to $\cos(\gamma)$, where $\gamma$ is the skyrmion helicity.

The Hall effect proportional to $\nabla \cdot S(r)$ depends only on the spin configuration of the bulk surrounding the vortex or the domain wall, and not on the fine details of the spin structure. Such robustness to the detail is also interesting from an application viewpoint, especially as a local probe for detecting magnetic structures, which is a key technology in racetrack memory [16–18].

On the other hand, the $\cos(\gamma)$ helicity dependence of the AHE enables the delineation of two Néel-type skyrmions $\gamma = 0$ or $\pi$, as well as the Néel skyrmions from Bloch skyrmions ($\gamma = \pi/2, 3\pi/2$). The helicity dependence opens a route to utilize the skyrmion helicity as a memory storage. Note that a similar study on a centrosymmetric model finds that such a term does not exist in the centrosymmetric model [39]. Hence, it is presumably another feature of the AHE unique to noncentrosymmetric materials.

We note that while our main focus is on a system with multiple monopoles and domain walls, the theory should be applicable to a small device with one monopole or domain wall. For the small device, a difference from a larger device is the possible existence of spatial inhomogeneity. In our formalism, such effects can be described as the modification of the electron distribution function in the Boltzmann equation, which is proportional to the Kondo coupling. Hence, the inhomogeneity gives a higher-order correction to our result in the perturbative limit. However, it does not affect the leading order contribution which we discussed in this work. Therefore, the result here is also expected to be valid for a small device with only one monopole or domain wall, as long as the system is in the perturbative limit.

In the last, we note that a recent work studying the effect of SOI on the AHE by non-collinear magnetic states finds a vanishing contribution from the SOI [29]. This work focuses on the intrinsic AHE from the Berry phase viewpoint which is valid in the strong exchange-interaction limit. They find only two relevant contributions: the conventional AHE by the ferromagnetic moment and the chirality-related AHE. In contrast, our study finds that the skew scattering by magnetic moments is strongly affected by the SOI, which gives rise to rich features not seen in systems without SOI.

Data availability statement

No new data were created or analysed in this study.
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Appendix A. Second Born approximation

In this appendix, we provide the details of the methods used in the main text. First, we investigate the scattering rate of the magnetic scattering processes using a second Born approximation. The scattering rate an electron with momentum $k$ and spin $\alpha$ to that of $k'$ and $\beta$ is given as

$$W_{k\alpha \to k'\beta} = \frac{2\pi}{\hbar} \left| N_{k\alpha,k'\beta}^{(1)} + N_{k\alpha,k'\beta}^{(2)} \right|^2 \delta (\varepsilon_{k\alpha} - \varepsilon_{k'\beta}) ,$$  \hspace{1cm} (A.1)

where $N_{k\alpha,k'\beta}^{(1)}$ and $N_{k\alpha,k'\beta}^{(2)}$ are the amplitudes of first- and second-order scattering, respectively. In our work, we treat the Kondo Hamiltonian $H_K$ as a perturbation to the non-perturbative Hamiltonian $H_0$ in equation (1). Then, scattering amplitudes are given by

$$N_{k\alpha,k'\beta}^{(1)} = \langle k\alpha | H_K | k'\beta \rangle ,$$ \hspace{1cm} (A.2)

$$N_{k\alpha,k'\beta}^{(2)} = \sum_{\eta \eta'} \langle k\alpha | H_K | p\eta \rangle \langle p\eta | H_K | k' \beta \rangle ,$$ \hspace{1cm} (A.3)

where $| k\alpha \rangle$ denotes an eigenstate of $H_0$ and $\varepsilon_{k\alpha}$ is an eigenenergy of that. Since the band structure of the system with SOI is qualitatively different between $\varepsilon_{k\alpha} > 0$ and $\varepsilon_{k\alpha} < 0$, physical properties such as skew scattering and transport coefficients change qualitatively through the sign of the chemical potential. As the scattering process is elastic, $\varepsilon_{k\alpha} = \varepsilon_{k'\beta}$, we can express the magnitude of momentum $k$ as a function of the chemical potential $\mu$ and the band index $\eta$,

$$k_{\xi} (\mu, \eta) = -\frac{\eta \lambda m + \xi \sqrt{\lambda^2 m^2 + 2 m \mu}}{\hbar} ,$$ \hspace{1cm} (A.4)

where $\xi, \eta = \pm 1$. For $\mu \geq 0$, only $\xi = +1$ appears while only $\eta = -1$ appears in $\mu < 0$. To avoid redundancy, the notations $k_{\xi} = k_{\xi} (\mu < 0, \eta = -1), k = k_{\xi = +1} (\mu \geq 0, \eta)$ and $k' = k_{\xi = +1} (\mu > 0, \eta')$ are used in this section.

To examine the effect of magnetic scattering on the AHE, we mainly focus on the antisymmetric part of the scattering rate defined as

$$W_{k\alpha \to k'\beta} = \frac{W_{k\alpha \to k'\beta} - W_{k'\beta \to k'\alpha}}{2} = \frac{4\pi^2}{\hbar} \sum_{\eta \eta'} \mathcal{Z} \left[ | k'\beta | H_K | k\alpha \rangle \langle k\alpha | H_K | p\eta \rangle \langle p\eta | H_K | k' \beta \rangle \right] \times \delta (\varepsilon_{k\alpha} - \varepsilon_{k'\beta}) \delta (\varepsilon_{p\eta} - \varepsilon_{k\alpha}) \delta (\varepsilon_{p\eta} - \varepsilon_{k'\beta}) ,$$ \hspace{1cm} (A.5)

where $\delta (x)$ is the Dirac delta function. We derived the second line of equation (A.5) from equation (A.1). For instance, equation (A.5) for the Rashba system reads

$$W_{k\eta \to k'\eta'} = \pi \delta (\varepsilon_{k\eta} - \varepsilon_{k'\eta'}) \cdot \sum_{i,j} \left[ \frac{W_1}{(aL)^2} \left( S_i \times r_j \right) \cdot \left( S_i \times S_j \right) - \frac{W_2}{(L)^2} S_i^z \left( S_i \cdot S_j \right) \right] + \frac{W_3}{(aL)^2} S^z \left( \hat{z} \times r_j \right) \cdot \left( S_i \times S_j \right) \sin \delta \phi + \delta (\varepsilon_{k\eta} - \varepsilon_{k'\eta'}) \cdot \frac{W_4}{(aL)^2} \sum_{i,j} \left( S_i \cdot S_j \right) \left( \hat{z} \cdot r_j \times S_j \right) \cos \delta \phi ,$$ \hspace{1cm} (A.6)

for Heisenberg spins $S_i$; the $S^z_i = 0$ case corresponds to the XY spin case. Here, we neglect higher order terms of $O \left( (k_f a)^2 \right)$, $O (L)$, and keep the Kondo coupling small as $\varepsilon_f \ll J a^2$. The skew terms are averaged over the mean angle $\phi = \phi_k + \phi_f / 2$, where $\phi_k$ is the direction of the electron momentum $k$. Then, equation (A.6) depends only on the scattering angle $\delta \phi = \phi_k - \phi_f$. Each coefficient $W_i$ is summarized in table 1. As mentioned in the main text, all terms in equation (A.6) are different from the scalar spin chirality term that appears without SOI, which is in the order of $O ((k_f a)^3)$. 

\hspace{1cm} (A.6)
Appendix B. Boltzmann theory

Next, we evaluate the Hall conductivity which arises from the skew scattering in equation (12) within the semi-classical Boltzmann theory. In the presence of the uniform static electric field \( E \), the Boltzmann equation reads

\[
e E \cdot v_{k0} f_0(\mu) = \frac{g k_0}{\tau} - \frac{V}{(2\pi)^2} \sum_\eta \int d^2k W_{k'\eta' \rightarrow k_0\eta} \delta k',
\]

where \( v_{k0} = \nabla k \varepsilon_{k0}/h \) is the velocity of the electron with momentum \( k \), and \( f_0(\varepsilon) \) is the Fermi–Dirac distribution with its energy derivative \( f'_0(\varepsilon) \). We assume that the electron distribution is expanded as \( f_{k0} = f_0(\varepsilon_{k0}) + \delta f_{k0} \) and that the displacement from the equilibrium distribution \( \delta f_{k0} \) is of the order \( E \). For the scattering terms on the right-hand side of equation (B.1), the symmetric part of the scattering rate \( W_{k\eta \rightarrow k'\eta'} = (W_{k\eta \rightarrow k'\eta'} + W_{k'\eta' \rightarrow k\eta})/2 \) is replaced by the relaxation time \( \tau \).

Equation (B.1) is analytically solvable for \( W^- \) [15]. In order to evaluate the displacement \( \delta f_{k0} \), we define a parameter \( P_\eta(\mu) \) related to the integral of \( \delta f_{k0} \) with respect to the angle as

\[
P_\eta(\mu) = \int_0^{2\pi} d\phi \hat{k} \delta f_{k0},
\]

where \( \hat{k} \) is the unit vector along \( k \). Using \( P_\eta \), we rewrite the Boltzmann equation as

\[
\delta f_{k0} = e v E \cdot v_{k0} f_0(\varepsilon) + \pi \bar{W} v \sum_\eta \bar{V}^{\eta\eta'}(\mu) \hat{k} \times P_\eta(\mu),
\]

where \( \bar{W} \) and \( \bar{V}^{\eta\eta'} \) are the coefficients related to the \( \sin \delta \phi \) term in equation (12) as follows,

\[
\bar{W} = \frac{1}{4\pi v(\mu)} \int d^6k \frac{m}{E^2 h^4},
\]

\[
\bar{V}^{\eta\eta'}(\mu) = \frac{\lambda k_0}{v(\mu)} \left[ -\frac{1}{2} (k_\eta + k_{\eta'}) \sum_{ij} (S_i \times r_j) \cdot (S_i \times S_j) \right.
\]

\[
- \frac{\mu + m \lambda^2}{h \lambda} \sum_{ij} S_i^j (\hat{x} \times r_j) \cdot (S_i \times S_j)
\]

\[
\left. - \eta \eta' \frac{m \lambda}{h} \sum_{ij} S_i^j (\hat{x} \times r_j) \cdot (S_i \times S_j) \right] \quad (\mu \leq 0)
\]

\[
\bar{V}^{\eta\eta'}(\mu) = k' \left[ \frac{1}{2} (\eta k' + \eta' k) \sum_{ij} (S_i \times r_j) \cdot (S_i \times S_j) + \eta \eta' \sum_{ij} S_i^j (S_i \times S_j) \right.
\]

\[
- \frac{\eta \eta' m \lambda}{h} \sum_{ij} S_i^j (\hat{x} \times r_j) \cdot (S_i \times S_j) \right] \quad (\mu \geq 0)
\]

where \( v(\mu) = |v_k| \). By multiplying \( \hat{k} \) to both sides of equation (B.3) and integrating over \( \phi_k \), we obtain the following self-consistent equation for \( P_\eta \):

\[
P_\eta = e \pi v E f_0(\mu) + \pi \bar{W} \sum_\eta \bar{V}^{\eta\eta'} P_\eta(\mu) \times \hat{x}.
\]

Equation (B.7) is a linear equation of \( P_\eta \) and can be solved analytically. To the leading order of \( O(\lambda^3) \), \( P_\eta \) is expressed as

\[
P_\eta(\mu) = e \pi v f_0(\mu) \left( E - \pi \bar{W} \hat{x} \times E \sum_\eta \bar{V}^{\eta\eta'} \right).
\]

Finally, the Hall conductivity is calculated using the current formula

\[
j = \frac{-e}{(2\pi)^2} \sum_\eta \int d^2k v_{k0} \delta f_{k0},
\]

by substituting equation (B.8) for equation (B.3). The results for the XY spins are given in equation (6), and for the Heisenberg spins in equation (13).
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