Twistor theory of hyper-Kähler metrics with hidden symmetries

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Abstract

We review the hierarchy for the hyper-Kähler equations and define a notion of symmetry for solutions of this hierarchy. A four-dimensional hyper-Kähler metric admits a hidden symmetry if it embeds into a hierarchy with a symmetry. It is shown that a hyper-Kähler metric admits a hidden symmetry if it admits a certain Killing spinor. We show that if the hidden symmetry is tri-holomorphic, then this is equivalent to requiring symmetry along a higher time and the hidden symmetry determines a ‘twistor group’ action as introduced by Bielawski [3]. This leads to a construction for the solution to the hierarchy in terms of linear equations and variants of the generalised Legendre transform for the hyper-Kähler metric itself given by Ivanov & Rocek [17]. We show that the ALE spaces are examples of hyper-Kähler metrics admitting three tri-holomorphic Killing spinors. These metrics are in this sense analogous to the ‘finite gap’ solutions in soliton theory. Finally we extend the concept of a hierarchy from that of [8] for the four-dimensional hyper-Kähler equations to a generalisation of the conformal anti-self-duality equations and briefly discuss hidden symmetries for these equations.
1 Introduction

It is well known that finding exact solutions to a nonlinear partial differential equation (PDE) is greatly simplified by the existence of symmetries. In differential geometric language the symmetries of a hyper-Kähler structure or, more generally, anti-self-dual conformal structures in four dimensions, correspond to (conformal) Killing vectors. Equations for hyper-Kähler four-manifolds with conformal Killing vectors have been studied and, in many cases, solved \[11, 31, 9, 10\].

Apart from natural Lie-point symmetries, integrable soliton equations possess infinitely many hidden symmetries, which can also be effectively used to construct solutions. It is less well know how to find such solutions in hyper-Kähler geometry (although we will see that such solutions have indeed been found in another guise, \[3, 20, 17\]). In this paper we shall show that hidden symmetries correspond to Killing tensors and spinors (which, classically, occur in Riemannian geometry as additional integrals of the geodesic flow) and propose two methods of finding hyper-Kähler metrics with such symmetries.

We start by briefly reviewing a beautiful construction of Novikov [24] which we shall posit as a motivation and a guiding principle. Consider the Korteweg–de Vries (KdV) equation

\[
\frac{\partial u}{\partial t_1} = 6uu_x - u_{xxx}, \quad \text{where} \quad u = u(x, t_1), \tag{1.1}
\]

together with the associated hierarchy of equations for \(u(x, t_1, t_2, \ldots)\)

\[
\frac{\partial u}{\partial t_i} = \frac{\partial}{\partial x} \frac{\delta H_i}{\delta u}. \tag{1.2}
\]

Here

\[
H_0 = \int \frac{1}{2} u^2 \, dx, \quad H_1 = \int \left( \frac{1}{2} u^2 + u^3 \right) \, dx, \quad H_2 = \int \left( \frac{1}{2} u_x^2 - \frac{5}{2} u^2 u_{xx} + \frac{5}{2} u^4 \right) \, dx, \ldots
\]

are constants of motions which can be found recursively by solving the Riccati equation. Imposing a constraint

\[
\frac{\partial u}{\partial t_k} + c_1 \frac{\partial u}{\partial t_{k-1}} + \ldots + c_k \frac{\partial u}{\partial t_0} = c_0 \tag{1.3}
\]

reduces (1.1) to an ODE. This ODE is a completely integrable Hamiltonian system with \(k\) first integrals in involution. In the simplest non-trivial case the solution is

\[
x = \int \frac{du}{\sqrt{2u^3 + c_1 u^2 + 2c_0 u + E}}
\]

In the case of the KdV equation, it is then possible to proceed to obtain explicit formulae for such solutions in terms of theta functions.

For a general integrable system, hidden symmetries are constructed systematically by studying a hierarchy of commuting flows associated to the original equations. A hidden symmetry is then an explicit point symmetry of the hierarchy which, in particular, include the higher flows themselves.

In this paper we shall propose an analogous construction of hidden symmetries for the hyper-Kähler equations in four dimensions and its integrable generalisations, which include quaternionic structures in \(4k\) dimensions. Recall that a four-dimensional Riemannian manifold \((\mathcal{M}, g)\) is hyper-Kähler if it admits three Kähler structures \(\Sigma_I, \Sigma_J, \Sigma_K\) compatible with \(g\) and such that the endomorphisms \(I, J, K\) given by \(g(I X, Y) = \Sigma_I(X, Y)\), etc., satisfy \(IJ = K = -JI\). To impose higher symmetries on this system one needs to:

1. reformulate the hyper-Kähler condition on a metric as an integrable PDE (the heavenly equation),
2. construct the associated hierarchy,
3. look for solutions invariant under the hidden symmetries, and characterise twistor spaces corresponding to these solutions.
Steps (1) and (2) were taken in [27], and [30] [29] [8], respectively. We shall review the approach taken in [8] in [2] which focuses on hierarchies associated to 4-dimensional hyper-Kähler spaces. This is generalised in [4] to give a hierarchy associated to general conformally ASD spaces both in four dimensions and their higher dimensional generalisations such as quaternion Kähler spaces. The other sections deal with (3).

In §3 and §4 we discuss symmetry and hidden symmetry reduction in the context of the 4-dimensional hyper-Kähler equations. In §3 we first discuss and classify symmetries of solutions to the hyper-Kähler hierarchy. We use the well known twistor description [31] of the Gibbons-Hawking solution as a guiding example in our analysis of the case of the hierarchy; we show that when the symmetry is triholomorphic, the solutions of the reduced equations are linear and we briefly discuss the corresponding twistor theory. The hyper-Kähler hierarchy is in particular foliated by four-dimensional Hyper-Kähler manifolds that admit a hidden symmetry in a variant of the Ivanov and Rocek construction [17].

In §4 we discuss hyper-Kähler spaces with a hidden symmetry, defined to be a space that embeds into a hierarchy that has a symmetry. In the general case, we show that a hidden symmetry corresponds to the existence of a spinor $K^{B}_{B_0'...B_k'}$ satisfying

$$\nabla^{(A}(A',K^{B)}_{B_0'...B_k')} = 0. \tag{1.4}$$

When the hidden symmetry is ‘triholomorphic’ in an appropriate sense, we find that $K^{B}_{B_0'...B_k'} = \nabla^{B}_{(B_0'} L_{B_1'...B_k')}^{(1)}$ for some spinor field $L_{B_1'...B_k'}$ satisfying

$$\nabla_{A'(A,L_{B_1'...B_k')} = 0. \tag{1.5}$$

A non-constant solution $L_{A_1'...A_k'}$ to the killing-spinor equation (1.5) is said to be a Killing spinor of type $(0,k)$ and is also sometimes known as a solution to the valence–$(0,k)$ twistor equation. If $k = 2$, then $K_{A'A'} = \nabla^{B}_{A'} L_{A'B'}$ is a tri-holomorphic Killing vector of the given hyper-Kähler space, and the corresponding metric is of the Gibbons-Hawking form [12].

If the metric admits a hyper-Kähler hidden symmetry and hence Killing spinor, then the corresponding twistor space admits a globally defined twistor function $Q$ homogeneous of degree $k$. This is because the existence of a Killing spinor implies that the twistor space of the hyper-Kähler hierarchy admits an action of a Hamiltonian vector field with Hamiltonian $Q$ and factor space $O(k)$. The hyper-Kähler twistor space therefore arises as an affine bundle over $O(k)$, described by a cohomology class $f \in H^1(\mathbb{CP}^1,O(2-k))$. The corresponding space-time can be determined directly and the construction followed through to give explicit formulae for a basis of the self-dual two–forms (and therefore for the metric).

In Section 5 we demonstrate that the asymptotically locally Euclidean (ALE) spaces constructed by Hitchin [15] and Kronheimer [18, 19] admit three triholomorphic hidden symmetries. We show that the corresponding twistor spaces are elliptic fibrations over $O(k)$ for some $k$, and the transition functions defining these bundles can be found in terms of elliptic integrals.

Finally in §6 we extend the concept of a hierarchy from that of §4 for the four-dimensional hyper-Kähler equations to a generalisation of the conformal anti-self-duality equations and give a brief discussion of hidden symmetries in this context.

The two-component spinor notation used in the paper is summarised in the appendix.

2 The hyper-Kähler hierarchy.

Let $\mathcal{M}$ be a complex 4-manifold equipped with a holomorphic metric $g$ and compatible volume form $\nu$; we shall refer to this triple as a space-time. For a four-manifold with metric, we have that $T\mathcal{M} = \mathcal{S} \otimes \tilde{\mathcal{S}}$ where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are the bundles of self-dual and anti-self-dual spinors respectively each being rank two complex vector bundles on $\mathcal{M}$. The metric connection necessarily preserves this factorisation, and the hyper-Kähler condition is equivalent to the condition that the induced connection on $\tilde{\mathcal{S}}$ be flat. (The $I$, $J$ and $K$ act trivially on the $\tilde{\mathcal{S}}$ factor and on the $\mathcal{S}$ factor by Pauli
This implies that, on translation of the 1-form indices to indices $A, B, \ldots$ denoting membership of $S$ and $A', B', \ldots$ denoting membership of $\tilde{S}$, the curvature has the form

$$R_{A'B'C'D'} = \varepsilon_{A'B'} C_{ABCD} \varepsilon^{C'D'}, \quad C_{ABCD} = C(ABCD),$$

where $\varepsilon_{AB}$ and $\varepsilon_{A'B'}$ are skew, and $\varepsilon_{AB} \varepsilon_{A'B'}$ induces the metric under $TM = S \otimes \tilde{S}$. In four-dimensions this amounts to the ASD vacuum equations

$$\Phi_{ABA'B'} = 0, \quad R = 0, \quad C_{A'B'C'D'} = 0, \quad (2.6)$$

Here $R$ is the Ricci scalar, $\Phi_{ABA'B'}$ is the trace-free part of the Ricci tensor, and $C_{A'B'C'D'}$ is the SD part of the Weyl tensor (we use the conventions of Penrose and Rindler [26]).

We now show how the geometrical characterisation of the hyper-Kähler equations and its hierarchy can be reduced to a differential equations. We use a potential formulation, due to Plebański [27], based on the fact that the equations locally imply the existence of a complex-valued function $\Theta$ and coordinate system $(w, z, x, y)$ such that the metric is given by

$$g = 2dwdx + 2dzdy - 2\Theta_{xz}dz^2 - 2\Theta_{yy}dw^2 + 4\Theta_{xy}dwz, \quad (2.7)$$

and $\Theta$ satisfies so called second heavenly equation

$$\Theta_{xx} + \Theta_{yz} + \Theta_{xx} \Theta_{yy} - \Theta_{xy}^2 = 0. \quad (2.8)$$

The associated hierarchy is a differential equation with higher times generalising one of the formulations of the hyper-Kähler equations in terms of potentials. We introduce the coordinates $x^{Ai}, A = 0, 1, i = 0, \ldots n$ on a $(2n + 2)$ dimensional manifold $\mathcal{N}$. The dependent variable $\Theta(x^{Ai})$ satisfies the equations

$$\partial_{AI} \partial_{BJ - 1} \Theta - \partial_{Bj} \partial_{AI - 1} \Theta + \{\partial_{AI - 1} \Theta, \partial_{Bj - 1} \Theta\}_{yx} = 0, \quad i, j = 1, \ldots n. \quad (2.9)$$

Here $\{\ldots\}_{yx}$ is the Poisson bracket with respect to the Poisson structure $\partial/\partial x^{A0} \wedge \partial/\partial x^{A0}$. (In order to make contact with the above for $n = 1$, put $x^{A0} = (y, -x)$ and $x^{A1} = (w, z)$ and note that $\mathcal{N}$ is $\mathbb{C}P^1$ with $i = j = 1$.)

This hierarchy has a Lax representation

$$L_{AI} \Phi = \partial_{AI - 1} \Phi + \lambda(\partial_{AI} \Phi + \{\partial_{AI - 1} \Theta, \Phi\}) = 0, \quad (2.10)$$

where $A = 0, 1, i = 1, \ldots n$, $\lambda$ is an affine coordinate on $\mathbb{C}P^1$ and $\Phi(x^{Ai}, \lambda)$ is a function on $\mathcal{N} \times \mathbb{C}P^1$. It is clear that this provides the point of contact with the abstract definition.

It is clear from the form of the equations [28] that the space $\mathcal{N}$ is foliated by 4-dimensional hyper-Kähler manifolds parametrized by $x^{Ai} = \text{constant}$ for $i \geq 2$.

In [29] the hierarchies were obtained both via a recursion operator construction and via a twistor construction. We shall summarize these constructions in the remaining part of this section.

Let

$$\Box_{\Theta} = \partial_x \partial_w + \partial_y \partial_z + \Theta_{yy} \partial_x^2 + \Theta_{xx} \partial_y^2 - 2\Theta_{xy} \partial_x \partial_y$$

denote the wave operator on the ASD background determined by $\Theta$, and let $W_\Theta = \text{Ker} \Box_{\Theta}$.

**Proposition 2.1** [27]

(i) Elements of $W_\Theta$ can be identified with linearised solutions $\delta \Theta$ (i.e. $\Theta + \delta \Theta$) satisfies up to the linear terms in $\delta \Theta$ of the heavenly equation [28].

(ii) Let $(\delta \Theta_1, \delta \Theta_2) \in W_\Theta \times W_\Theta$. The ‘recursion operator’ $\mathcal{R}$ is defined to be the subspace $\mathcal{R} \subset W_\Theta \times W_\Theta$ on which

$$\partial_y (\delta \Theta_2) = (\partial_w - \Theta_{xy} \partial_y + \Theta_{yy} \partial_x) \delta \Theta_1, \quad -\partial_x (\delta \Theta_2) = (\partial_z + \Theta_{xx} \partial_y - \Theta_{xy} \partial_x) \delta \Theta_1. \quad (2.11)$$
Note that the recursion operator is only an operator in the usual sense when the subspace \( \mathcal{R} \) can be realized as a graph of a genuine operator \( R : \mathcal{W}_0 \longrightarrow \mathcal{W}_0 \) given by the recursion relations

\[
R\delta\Theta = \partial_y^{-1}(\partial_w - \Theta_{xy}\partial_y + \Theta_{yx}\partial_x)\delta\Theta), \quad R\delta\Theta = -\partial_x^{-1}(\partial_z + \Theta_{xz}\partial_y - \Theta_{xy}\partial_x)\delta\Theta). \quad (2.12)
\]

This identification with a genuine operator will only be possible when we impose appropriate boundary conditions. However, for the definition of the hierarchy as a local system of equations, we will need only the definition of \( \mathcal{R} \) above.

The first few iterations can be explicitly integrated to give

\[
w \longrightarrow y \longrightarrow -\Theta_x \longrightarrow \Theta_z \longrightarrow \ldots, \quad z \longrightarrow -x \longrightarrow -\Theta_y \longrightarrow -\Theta_w \longrightarrow \ldots.
\]

We introduce the new coordinates \( x^{A_i}, A = 0, 1, i = 0 \ldots n \). For \( i = 0, 1, x^{A_i} = x^{A_i'} = (w, z, x, y) \) are coordinates on \( \mathcal{M} \), and for \( 1 < i \leq n, x^{A_i} \) are the parameters for the new flows (with \( 2n - 2 \) dimensional parameter space \( \mathbb{X} \)). The propagation of \( \Theta \) along these parameters is determined by the recursion operator \( \Theta_{A} \rightarrow R\Theta_{A} \). However the consistency conditions imply that in addition \( \Theta \) satisfies the equations (2.9) with Lax system generated by the operators (2.10).

The twistor theory is summarised in the following:

**Theorem 2.2** There is a 1-1 correspondence between solutions to (2.9) on \( \mathcal{M} \times \mathbb{X} \) and twistor spaces \( \mathcal{PT}_n \) defined as follows.

The twistor space \( \mathcal{PT}_n \) is a three-dimensional complex manifold with the following structures

1. a projection \( \mu : \mathcal{PT} \longrightarrow \mathbb{CP}^1 \),
2. a section \( s : \mathbb{CP}^1 \rightarrow \mathcal{PT} \) of \( \mu \) with normal bundle \( \mathcal{O}(n) \oplus \mathcal{O}(n) \),
3. a non-degenerate 2-form \( \Sigma \) on the fibres of \( \mu \), with values in the pullback from \( \mathbb{CP}^1 \) of \( \mathcal{O}(2n) \).
4. The choice of coordinate systems and potential \( \Theta \) in the second Plebanski form for the hierarchy corresponds on the twistor space to a choice of point \( [\omega^A] \in \mathbb{CP}^1 \) and canonical homogeneous degree \( n \) coordinates \( \omega^A \) (i.e., \( \Sigma = d\omega^0 \wedge d\omega^1 \)) on a neighbourhood of the fibre of \( \mathcal{PT}_n \) over \( [\omega^A] \) defined up to 2\( n \)th order away from this fibre.

Briefly, the space \( \mathcal{M} \times \mathbb{X} \) is reconstructed as the moduli space \( \mathcal{N} \) of deformations of the section \( s \) given in condition (2) above. Then \( \mathcal{N} \) is 2\( n + 2 \) dimensional and we can introduce coordinates and the function \( \Theta \) as follows:

We use homogeneous coordinates \( \pi_{A'} = (\pi_{A'}, \pi_{A'}) \) and affine coordinate \( \lambda = \pi_{A'}/\pi_{A'} \) on \( \mathbb{CP}^1 \) so that the point \( o \) is represented by \( o_{A'} = (0, 1) \), or \( \lambda = 0 \). The homogeneous coordinates \( \omega^A \) (i.e., \( \omega^A, \pi_{A'} \) \( \sim (c^A \omega^A, c \pi_{A'}) \) for \( c \in (\mathbb{C} - 0) \)) can be pulled back to \( \mathcal{N} \times \mathbb{CP}^1 \) and the expansion

\[
\omega^A = \sum_{i=0}^{n} x^{A_i'} \lambda^i + O(\lambda^{n+1}) \quad \text{and this defines coordinates } x^{A_i'} \text{ on } \mathcal{N}.
\]

Expanding \( \omega^A \) further, we discover, as shown in [8], that the twistor coordinates \( \omega^A \) pulled back to the correspondence space \( \mathbb{CP}^1 \times \mathcal{N} \) can be expanded further as

\[
\omega^0_n = (\pi_{A'})^n [x_0^n + \lambda x_0^{n-1} + \ldots + \lambda^n x_0^0 + \lambda^{n+1} \frac{\partial\Theta}{\partial x_{10}} + \ldots + \lambda^{2n+1} \frac{\partial\Theta}{\partial x_{1n}} + \ldots],
\]

\[
\omega^1_n = (\pi_{A'})^n [x_1^n + \lambda x_1^{n-1} + \ldots + \lambda^n x_1^0 + \lambda^{n+1} \frac{\partial\Theta}{\partial x_{01}} + \ldots + \lambda^{2n+1} \frac{\partial\Theta}{\partial x_{0n}} + \ldots], \quad (2.13)
\]

and this determines \( \Theta(x^{A_i}) \) (up to a constant) satisfying (2.9). The form of the Lax system determined by the fact that \( \omega^B \) is solutions to \( L_{A} \omega^B = 0 \).

The form of the expansions (2.13) and equations (2.14) can be obtained from the fact that the expansion of \( \Sigma = d\omega^0 \wedge d\omega^1 \) on \( \mathcal{N} \times \mathbb{CP}^1 \) in powers of \( \lambda \) must truncate after \( \lambda^{2n} \).

There is a 2\( n \)-dimensional distribution on the ‘spin bundle’ \( D \subset T(\mathcal{N} \times \mathbb{CP}^1) \) that is tangent to the fibres of the projection \( \mathcal{N} \times \mathbb{CP}^1 \longrightarrow \mathcal{PT}_n \). The distribution \( D \) has an identification with \( \mathcal{O}(-1) \otimes \mathbb{C}^{2n} \) and is generated by the Lax system (2.10).

This correspondence is stable under small perturbations of the complex structure on \( \mathcal{PT}_n \) preserving (1-3).
One can find the twistor spaces for the four-dimensional hyper-Kähler slices given by $x^{A_i} = \text{const.}$, $i \geq 2$ by taking a sequence of $n - 1$ blowups of points in the fibre over $a_{A'} \in \mathbb{CP}^1$, the choice of point in the fibre to blow up at the $n - i + 1$'th blowup corresponding precisely to the choice of the values of $x^{A_i}$. If one wishes to respect Euclidean reality conditions, one can instead blow up complex conjugate points in the fibres over $a_{A'}$ and $\bar{a}_{A'}$.

## 3 Symmetries of Hyper-Kähler hierarchies

For a hyper-Kähler space, we can characterise conformal symmetries as follows: let $\Sigma^{A'B'} = (\Sigma^{0'0'}, \Sigma^{0'1'}, \Sigma^{1'1'})$ be a basis of SD two-forms, and let $\Sigma = \Sigma^{A'B'} \pi_{A'} \pi_{B'}$.

**Definition 3.1** A solution to the hyper-Kähler equations admits a symmetry if there exists a vector field $K$ on $\mathcal{N}$ together with a lift $\tilde{K}$ to $S^A$ over $\mathcal{N}$ such that $\mathcal{L}_{\tilde{K}} \Sigma = 0$.

Here the lift must be $\tilde{K} = K + \phi^{A'}_{B'} \pi_{A'} \partial / \partial \pi_{B'}$ where $\phi^{A'}_{B'} = -\frac{1}{2} \nabla_{BB'} K^{BA'} + \frac{1}{2} \nabla_0 K^{c_{A'}} \epsilon_{A'}^{A} \epsilon_{B'}^{B}$ according to the standard theory of Lie derivatives of spinors, see [26].

We will use this also as a definition for symmetries of the hyper-Kähler hierarchy where now $\Sigma$ will be the pull-back from twistor space to the spin bundle of the corresponding 2-form that is homogeneous of degree $2n$ in $\pi_{A'}$.

**Definition 3.2** A solution to the hyper-Kähler hierarchy admits a symmetry if there exists a vector field $K$ on $\mathcal{N}$ together with a lift $\tilde{K}$ to $S^A$ over $\mathcal{N}$ such that $\mathcal{L}_{\tilde{K}} \Sigma = 0$.

Again, by homogeneity, we must have that the lift must be $\tilde{K} = K + \phi^{A'}_{B'} \pi_{A'} \partial / \partial \pi_{B'}$ where $\phi^{A'}_{B'}$ will be determined by $K$.

In particular, $\tilde{K}$ is therefore in involution with the Lax distribution $D$ so that it projects down to a global holomorphic vector field $K$ on the twistor space $\mathcal{PT}$. We can classify symmetries according to the extent to which they preserve the various structures on $S^A$ or on the twistor space. This is most easily seen by examining the vertical part $\phi = \phi^{B'}_{A'} \pi_{B'} \partial / \partial \pi_{A'} = \tilde{K} - K$ (where here $\tilde{K}$ denotes the horizontal lift of $K$). The matrix $\phi^{B'}_{A'}$ generically has two constant eigenvalues (the space-time must be of Petrov type $N$ for non-constant eigenvalues to be admissible).

1. $K$ will be said to be **tri-holomorphic** if the eigenvalues of $\phi^{B'}_{A'}$ are equal. The projection of $\tilde{K}$ to the projective spin bundle, $PS^A$, is then horizontal, and the vertical part $\phi$ of $\tilde{K}$ on $S^A$ is a multiple of the Euler homogeneity operator $\pi^{A'} \partial / \partial \pi^{A'}$, i.e., $\phi^{B'}_{A'} = \mu \delta^{B'}_{A'}$. Equivalently, it is tri-holomorphic if $\pi$ is tangent to the fibres of the projection $p : \mathcal{PT} \rightarrow \mathbb{CP}^1$.

2. $K$ is said to be Killing if the trace of $\phi$ vanishes, $\phi^{A'}_{A'} = 0$. Then $\phi$ preserves the form $\pi_A \partial \pi^{A'}$ on the spin bundle.

3. $K$ is said to be a homothety if the trace of $\phi$, $\phi^{A'}_{A'} = 0$, is constant, i.e., $\mathcal{L}_K \epsilon^{A'B'} \pi_{A'} \partial \pi_{B'} = \mu \epsilon^{A'B'} \pi_{A'} \partial \pi_{B'}$, for some constant $\mu \neq 0$.

In the case that the symmetry is not tri-holomorphic, we can further distinguish the case where $\phi$, on projection to the projective spin bundle, has one or two zeroes. In the single zero case in particular, $\phi^{B'}_{A'}$ will not be diagonalizable. We will not pursue this distinction here but see [9, 7] for a study of such symmetries on $(+ + - -)$ hyper-Kähler space.

### 3.1 Examples with triholomorphic Killing symmetry

We first consider well known reductions of the hyper-Kähler equations, and then analogous reductions of the hierarchy.

#### 3.1.1 Gibbons-Hawking metrics revisited.

The heavenly equation [3, 4] with $R(\Theta_x) = \Theta_z = 0$ can be expressed as

$$d\Theta_x \wedge dx \wedge dy + dw \wedge d\Theta_x \wedge d\Theta_y = 0.$$  

(3.14)
Introduce \( p := \Theta_x \) and perform a Legendre transform
\[
F(p, y, w) := px(w, y, p) - \Theta(w, z, y, x(w, y, p)).
\]
Then \( x = F_p, \Theta_y = -F_y \) and \((3.14)\) yields the wave equation \[11\]
\[
F_{pw} + F_{yy} = 0. \tag{3.15}
\]
Implicit differentiation gives
\[
\Theta_{yy} = -F_{yy} + \frac{F_{pp}}{F_{pp}} \quad \Theta_{xy} = -\frac{F_{py}}{F_{pp}} \quad \Theta_{xx} = \frac{1}{F_{pp}},
\]
and so (with the help of \((3.14)\) and \((3.16)\))
\[
g = F_{pp} \left( \frac{1}{4} \text{d}y^2 + \text{d}wdp \right) - \frac{1}{F_{pp}} \left( \text{d}z - \frac{F_{pp}}{2} \text{d}y + F_{py} \text{d}w \right)^2
\]
\[
= \psi \left( \frac{1}{4} \text{d}y^2 + \text{d}wdp \right) - \psi^{-1} (\text{d}z + \Omega)^2, \tag{3.16}
\]
where \( \psi = F_{pp} \) and \( \Omega = F_{py} \text{d}w - (F_{pp}/2) \text{d}y \) satisfy the monopole equation \( *d\psi = d\Omega \) from \((3.15)\).

Thus \((3.10)\) is of the Gibbons-Hawking form \[12\].

The twistor description is as follows: the twistor coordinates pull back to the spin bundle as
\[
\omega^0 = \pi_1 \left[ w + \lambda y - \lambda^2 \Theta_x + \lambda^3 \Theta_x + \ldots \right],
\]
\[
\omega^1 = \pi_1 \left[ z - \lambda x - \lambda^2 \Theta_y - \lambda^3 \Theta_w + \ldots \right]. \tag{3.17}
\]
The vanishing of \( \Theta_z \) implies that the whole series for \( \omega^0 \) truncates at 2nd order. Thus the twistor space admits a global section of \( O(2) \), and this is the Hamiltonian with respect to \( \Sigma \), for the holomorphic vector field corresponding to the Killing field \( \partial_z = K^{AA'} \partial_{AA'} \). Conversely, given a tri-holomorphic symmetry, the tri-holomorphicity condition means that its lift to the spin bundle \( \mathcal{M} \) is horizontal and so on twistor space, the corresponding holomorphic vector field is tangent to the fibres of \( \mu \). It also preserves \( \Sigma \) and so is Hamiltonian with Hamiltonian given by a homogeneity degree-2 global function. We can choose \( \omega^0 \) to be this preferred section divided by \( \pi_1 \), so that the series for \( \omega^0 \) terminates after \( \lambda^2 \).

Substituting the Legendre transform into \((3.14)\) yields
\[
\omega^0 = \pi_1 \left[ w + \lambda y - \lambda^2 p \right], \tag{3.18}
\]
\[
\omega^1 = \pi_1 \left[ z - \lambda F_p + \lambda^3 F_y + \lambda^3 F_w + \ldots \right].
\]
With the definition \( \Sigma = d\omega^0 \wedge d\omega^1 \rvert_{\lambda=\text{const}} \) the equation \((3.14)\) can be rewritten as \( \Sigma \wedge \Sigma = 0 \). The basis of SD two forms can be read off from \( \Sigma = \Sigma^{A'B'} \pi_A \pi_{B'} \cdot \Sigma^{00} = -dz \wedge dp + dy \wedge dF_p - dw \wedge dF_y, \Sigma^{01} = dz \wedge dw, \Sigma^{1'1'} = dz \wedge dw \).

and these determine the metric above.

### 3.1.2 Triholomorphic symmetry reductions of the hierarchy

In this sub-section we shall generalize the construction of Gibbons-Hawking metrics described in the last subsection, and generate solutions to the hyper-Kähler hierarchy such that \( R^n \Theta_x = \partial_1 \Theta = 0 \). These are the cases of a tri-holomorphic Killing symmetry.

**Proposition 3.3** The Hyper-Kähler hierarchy \((2.9)\) with symmetry \( \partial \Theta / \partial x^{1n} = 0 \) reduces (in appropriate coordinates) to an overdetermined system of \( n(2n-1) \) linear equations for \( F(i^0, \ldots, i^{2n}) \):
\[
\frac{\partial^2 F}{\partial i^{i+1} \partial \bar{u}^i} = \frac{\partial^2 F}{\partial i^i \partial \bar{u}^{i+1}}, \quad i, j = 0, \ldots, 2n-1. \tag{3.20}
\]
Proof. Let $PT_n$ be the twistor space from Proposition 2.18, and let $(\omega^A, \pi_A)$ be homogeneous coordinates$^1$ on the neighborhood of $\pi_A = o_A$.

Now impose the symmetry condition, i.e., assume that $R^x \Theta_x = \partial_1 \Theta = 0$. Again, the vanishing of $\partial_1 \Theta$ implies that the series in $\lambda$ for $\omega^0$ truncates at degree $2n$. Thus, $\pi_1^n \omega^0$ is a global holomorphic function of homogeneity degree-2 on $PT_n$. Conversely, again, this symmetry corresponds to a global holomorphic vector field on $PT_n$ that is vertical up the fibres of $\mu$ and preserves $\Sigma$. It therefore is generated by a global Hamiltonian $Q$ homogeneous of degree $2n$, and we can take as before $\omega^0 = Q/\pi_1^n$.

We can now perform the Legendre transform

$$p^i = \partial_1 \Theta \quad i = 0,\ldots,n-1, \quad F(p', x^{0j}) = \sum_{i=0}^{n-1} p^i x^{1i}(p', x^{0j}) - \Theta(x^{0j}, x^{1i}(p', x^{0j})), \quad x^{1n} = T.$$ (3.21)

Therefore $\partial_0 F = -\partial_0 \Theta, \partial_0 F = x^{1i}$. Define $2n+1$ functions $(t^0, \ldots, t^{2n})$ by

$$t^{n-i} = p^i, i = 0,\ldots,n-1 \quad t^{n+i} = x^{0i}, i = 0,\ldots,n.$$ This implies

$$\omega_n^0 = (\pi_1)^n [t^{2n} + \lambda t^{2n-1} + \ldots + \lambda^n t^0]$$

$$\omega_n^1 = (\pi_1)^n [T + \lambda \partial F / \partial t^0 + \lambda^2 \partial F / \partial t^1 + \ldots + \lambda^{2n+1} \partial F / \partial t^{2n} + \ldots].$$ (3.22)

The equations (3.20) arise from the vanishing of coefficient $\lambda^{2n+2}$ in $d\omega_n^0 \wedge d\omega_n^1$:

$$\sum_{i=0}^{2n-1} dt^i \wedge d \frac{\partial F}{\partial t^{i+1}} = 0.$$ It can also be verified by cross-differentiating that all integrability conditions for the system (3.20) are satisfied.

The geometry on twistor space can be understood as follows. The section $Q$ of $O(2n)$ generates a Hamiltonian flow

$$\mathcal{K} = \varepsilon^{AB} \frac{\partial Q}{\partial \omega^A} \frac{\partial}{\partial \omega^B}, \quad \varepsilon^{AB} = -\varepsilon^{BA}, \quad \varepsilon^{01} = 1$$
on the extended twistor space $PT_n$. This flow corresponds to $K^{A_1\ldots A_n} \partial_{A_1\ldots A_n} = \partial / \partial x^{1n}$ on $N$.

Since $Q$ is constant along $\mathcal{K}$, the quotient space $PT_n / \mathcal{K}$ is the total space of $O(2n) \rightarrow \mathbb{CP}^1$ where the map to $O(2n)$ is furnished by $(\omega^A, \pi_A)$.

The full twistor space $PT_n$ is an affine line bundle with trivial underlying translation bundle and so it corresponds to an element $G(Q, \pi_A)$ of $H^1(O(2n), \mathcal{O})$.

Sections of $O(2n) \rightarrow \mathbb{CP}^1$ are parametrised by $\mathbb{C}^{2n+1}$ with coordinates $t = (t^0, \ldots, t^{2n})$. The $2n + 2$ dimensional space of sections of $PT_n \rightarrow \mathbb{CP}^1$ maps onto this with fibre $\mathbb{C}$ parametrized by $T$. Choosing linear coordinates up the line bundle $\eta$ and $\eta$ over open sets $U$ and $\bar{U}$, the problem of lifting a curve $L_t$ in $O(2n)$ to one in $PT_n$ is one of finding a trivialization of the line bundle over $L_t$, i.e. of finding functions $g(t, \pi_A)$ and $\bar{g}(t, \pi_A)$ such that, on $U \cap \bar{U}$, $g - \bar{g} = G$ on restriction to $L_t$ where $G$ is the log of the patching function for the line bundle, and therefore has homogeneity degree zero. We can then take $\eta = T + g$. We can give a formula for $g$ as

$$g(t, \lambda) = \int_\eta G(t^{2n} + \zeta t^{2n-1} + \ldots + \zeta^{2n} t^0, \zeta) d\zeta / (\zeta - \lambda).$$

$^1$In the previous section $PT := PT_1$ and $\omega^A := \omega_1^A$ correspond to the standard situtation of the nonlinear graviton construction.
where the contour $\gamma$ is taken in $L_\iota \cap U \cap \breve{U}$ surrounding $\zeta$ in $U$. With an identical expression for $\breve{g}$ but with contour $\breve{\gamma}$ such that $\gamma - \breve{\gamma}$ surrounds $\zeta$, we see that $g - \breve{g} = G$ follows from Cauchy’s integral formula. Then the expression for the expansion of the coordinate $\eta = T + g$ about $\lambda = 0$ is

$$\eta = T + \sum_{i=0}^{\infty} \lambda^i \oint G(t, \zeta, \zeta) d\zeta / \zeta_{i+1}.$$ 

It can then be seen that if we define

$$F(t) = \oint G\left(t^{2n} + \lambda t^{2n-1} + \cdots + \lambda^{2n+1}, \lambda\right) \frac{1}{\lambda^2} d\lambda,$$

then $F$ clearly satisfies the equations of (3.20) and we obtain the expansions (3.22) for $(\omega^0, \omega^1) = (Q/\pi_n^1, \pi_n^1, \eta)$. These can then be used to obtain concrete expressions for $\Sigma = d\omega^0 \wedge \omega^1$ to determine the geometric structures of the hyper-Kähler hierarchy.

Clearly we have:

Lemma 3.4 The full space of the hierarchy is foliated by hyper-Kähler four-manifolds with $x^{A_i} = \text{constant for } i > 1$ and metric

$$2dx^{10} dx^{01} + 2dx^{11} dx^{00} - 2\frac{\partial^2 \Theta}{\partial(x^{10})^2} (dx^{11})^2 - 2\frac{\partial^2 \Theta}{\partial(x^{00})^2} (dx^{01})^2 - 4\frac{\partial^2 \Theta}{\partial x^{10} \partial x^{11}} dx^{01} dx^{11}. \quad (3.24)$$

This gives a variant of the Legendre transform of Ivanov and Rocek [17] (see also [8]).

3.2 Example for $n = 2$

We saw above that for $n = 1$ the construction is equivalent to the Gibbons-Hawking anzatz. The $n = 2$ case goes as follows:

Let $F_i := \partial_i F$. Implicit differentiation of $\partial_0 F = -\partial_0 \Theta, \partial_i F = x^{1i}$ with respect to $p_0, p_1, y$ yields

$$\Theta_{xx} = -\frac{F_{00}}{M}, \quad \Theta_{xy} = \frac{-F_{01} F_{02} + F_{00} F_{12}}{M}, \quad \Theta_{yy} = -F_{22} - \frac{F_{00} (F_{12})^2 - 2F_{01} F_{12} F_{02} + F_{11} (F_{02})^2}{M},$$

where $M := (F_{01})^2 - F_{00} F_{11}$. The metric (2.7) with $x = x(t), z = z(t)$ is defined on the surface $F_4 = 0$. The formula for the metric in terms of $F$ is not very illuminating, but we shall give it for the sake of completeness:

$$g = M^{-1} (F_{10} N dt^1 dt^2 + F_{00} N dt^0 dt^2 + F_{11} (F_{01})^2 (dt^2)^2 + F_{11} (dt^3)^2 + 2F_{01} (F_{11})^2 dt^2 dt^3 + 2F_{01} (F_{02})^2 dt^0 dt^1 + F_{00} (dt^0)^2 + F_{00} (F_{02})^2 (dt^1)^2 + [F_{11} N + F_{01} F_{00} F_{03}] dt^1 dt^3 + [3F_{01} F_{00} F_{11} - (F_{01})^3 + (F_{00})^2 F_{03}] dt^0 dt^3, \quad (3.25)$$

where $N := (F_{01})^2 + F_{00} F_{11}$.

4 Hyper-Kähler spaces with hidden symmetries

If we have a hyper-Kähler space that embeds into a hyper-Kähler hierarchy that admits a symmetry we will say that that the original hyper-Kähler space admits a hidden symmetry.

The first question we wish to address is of how to recognise when a hyper-Kähler space admits such a hidden symmetry.

Proposition 4.1 If a hyper-Kähler space $\mathcal{M}$ admits a hidden symmetry then it admits a solution to the equation

$$\nabla^{(A_i} K^{B_j)}_{(A'_i} A'_{j} \cdots A'_i) = 0 \quad (4.26)$$
Proof: This result is most easily seen from the twistor theory. The symmetry vector $K$ gives rise to a global holomorphic vector field $\mathcal{K}$ on the twistor space for $\mathcal{N}$, $\mathcal{PT}_n$. The twistor space $\mathcal{PT}_1$ for $\mathcal{M}$ is a region in the blowup of $\mathcal{PT}_n$ at a number of points. Thus we have a map $p: \mathcal{PT}_1 \to \mathcal{PT}_n$. So rather than consider $\mathcal{K}$ itself, we consider the 2-form $\mathcal{K} \cdot \nu_n$ weight $2n + 2$ where $\nu_n \in \Gamma(\mathcal{PT}_n, \Omega^1(2n + 2))$ is given by $\Sigma_n \wedge \pi_{A^I} d\pi_A$.

This 2-form can be pulled back to give $p^* \mathcal{K} \cdot \nu_n$ a global 2-form of weight $2n + 2$ on the twistor space $\mathcal{PT}_1$ for $\mathcal{M}$. This 2-form can then be pulled back to give a 2-form on the spin-bundle $S^{A^I}$ which must take the form

$$\mathcal{K} \cdot \nu_n = K^{A^I_1 \cdots A^I_{2n-1}} e^{A_{2n}} \pi_{A^I_1} \cdots \pi_{A^I_{2n}} \wedge \pi_B d\pi_B^r + \chi_{A^I_1 \cdots A^I_{2n}} \pi_{A^I_1} \cdots \pi_{A^I_{2n}} \Sigma_1$$

for some $K^{A^I_1 \cdots A^I_{2n-1}}$ and $\chi_{A^I_1 \cdots A^I_{2n}}$.

The condition that this 2-form descends to twistor space is the condition that

$$\pi^{A^I} \nabla_{A^I} (\mathcal{K} \cdot \nu_n) = 0.$$ 

This leads to equation (1.26) and

$$\nabla^A (K A^I_1 \cdots A^I_{2n}) A = \chi_{A^I_1 \cdots A^I_{2n}} , \quad \nabla A (\mathcal{K} \cdot \nu_n) = 0.$$

However, it can be checked that these two equations are a consequence of (1.26), if the first equation is taken to be the definition of $\chi_{A^I_1 \cdots A^I_{2n}}$.

4.1 The case of a hidden tri-holomorphic Killing symmetry

The case of a hidden tri-holomorphic Killing symmetry reduces to linear equations and can be worked through completely modulo some intergations and solving for implicit functions. This is effectively the case studied by Ivanov and Rocek [17] and generalised by Bielawski [3].

In the tri-holomorphic Killing case, we have

**Lemma 4.2** Suppose $\mathcal{M}$ admits a hidden tri-holomorphic Killing symmetry, then $\chi_{A^I_1 \cdots A^I_{2n}} = 0$ and there exists a spinor $\phi_{A^I_2 \cdots A^I_{2n}}$ such that

$$\nabla A A^I \phi_{A^I_2 \cdots A^I_{2n}} = K_{A^I_1 \cdots A^I_{2n-1}} e^{A_{2n}} A^I.$$

Proof: The vanishing of $\chi_{A^I_1 \cdots A^I_{2n}} = 0$ follows from the fact that $\mathcal{K}$ is tangent to the fibres of twistor space over $\mathbb{C}P^1$. The existence of $\phi_{A^I_2 \cdots A^I_{2n}}$ follows from the fact that $\mathcal{K}$ is Hamiltonian with respect to the symplectic forms $\Sigma_n$ up the fibres of $\mu$ and so is generated by a Hamiltonian $Q \in \Gamma(\mathcal{O}(2n))$. On pullback to the spin bundle $Q = \phi_{A^I_2 \cdots A^I_{2n}} \pi_{A^I_1} \cdots \pi_{A^I_{2n}}$ and the condition that $Q$ descends to twistor space is $\pi^{A^I} \nabla_{A^I} Q = 0$ and this gives the equation above. 

The above lemma shows that a Killing spinor on an ASD vacuum determines a function $Q$ homogeneous of degree $k$ on its twistor space $\mathcal{PT}$. This in turn implies

**Lemma 4.3** If an ASD vacuum space-time admits a Killing spinor, its twistor space $\mathcal{PT}$ is an affine line bundle over $\mathcal{O}(k)$ with underlying translation bundle $\mathcal{O}(2-k)$.

Proof: The existence of a global twistor function homogeneous of degree $k$ on $\mathcal{PT}$ gives a projection onto $p: \mathcal{PT} \to \mathcal{O}(k)$. Furthermore the fibre is spanned by the Hamiltonian vector field of $Q$ with respect to $\Sigma$, in local coordinates,

$$\mathcal{K} = \varepsilon^{AB} \frac{\partial Q}{\partial a^A} \frac{\partial}{\partial a^B}.$$ 

This is a vector field with values in $\mathcal{O}(k-2)$. This gives each fibre an affine linear structure which is twisted globally by $\mathcal{O}(2-k)$, since, if $a$ is a local section of $\mathcal{O}(2-k)$ over $\mathbb{C}P^1$, then $a \mathcal{K}$ is a vector whose flows determine an action of $\mathbb{C}$. Thus $\mathcal{PT} \to \mathcal{O}(k)$ is an affine line bundle over $\mathcal{O}(k)$ with underlying translation bundle $\mathcal{O}(2-k).$
Such affine line bundles are classified by elements \([f]\) of \(H^1(\mathcal{O}(k), \mathcal{O}(2 - k))\). In a Čech description, cover \(\mathcal{O}(k)\) by open sets, \(U_i\), and represent \([f]\) by its Čech representative \(f_{ijk} \in \Gamma(\mathcal{O}(2 - k), U_i \cap U_j)\). Then \(\mathcal{P}T\) is constructed by patching together the total space of \(\mathcal{O}(2 - k)\) to \(U_i\) to \(\mathcal{O}(2 - k)\) to \(U_j\) by translating the zero section by \(f_{ij}\). The data \([f]\) therefore determines the twistor space. This proves the first part of:

**Theorem 4.4** There is a one-to-one correspondence between ASD vacuum space-times \((\mathcal{M}, g)\) admitting a valence \((0, k)\) Killing spinor and elements \([f]\) of \(H^1(\mathcal{O}(k), \mathcal{O}(2 - k))\).

In this case, for \(k \geq 3\), \(\mathcal{M}\) admits a natural map into \(\mathbb{C}^{k+1} = i^{k+1}S^A\) which we coordinatise with \(t^A_1 \cdots A_k\). The hyper-kähler space \(\mathcal{M}\) is determined as a subset of \(\mathbb{C}^{k+1}\) by the \(k - 3\) constraints

\[
f_{A_1' \cdots A_k'-4} = \oint f(Q, \pi_A)\pi \cdot d\tau = 0.
\]

The basis of SD two forms for \(g\) is then given by the restriction of the forms

\[
\Sigma^{A_1'B_1} = \psi_{A_2' \cdots A_k'A_{k+1} \cdots A_{2k-4}} dt^{A_1' \cdots A_k'} \wedge dt^{B_1' \cdots B_k'} ,
\]

(4.27)

to \(\mathcal{M}\), where

\[
\psi_{A_1' \cdots A_{2k-4}} = \frac{1}{2\pi i} \oint_{\gamma} \frac{dt^{A_1' \cdots A_{2k-4}}}{\partial f} \cdot d\pi = 0
\]

(4.28)

is a field determined by an arbitrary element of \(H^1(\mathcal{O}(k), \mathcal{O}(2 - k))\).

**Proof.** If one wishes to obtain the space-time \((\mathcal{M}, g, \mathbf{v})\) determined by a given twistor space, the first task is to locate the 4-dimensional family of sections of the fibration \(\mathcal{P}T \to \mathbb{C}P^1\).

Let \(t^A_1 \cdots A_k\) be coordinates on the \(\mathbb{C}^{k+1} = i^{k+1}S^A\) parameter space of sections \(\sigma_t: \pi_A \to \mathcal{Q} = t^{A_1+1} \cdots A_k' \cdots \pi_{A_k} \in \Gamma(\mathcal{O}(k))\). Sections of \(\mathcal{P}T \to \mathbb{C}P^1\) determine sections of \(\mathcal{O}(k)\) by projection onto \(\mathcal{O}(k)\). However, the affine bundle \(\mathcal{P}T\) only admits a section over some \(\sigma_t\) if the cohomology class \([f]\) vanishes on restriction to \(\sigma_t\). If \([f]\) vanishes on restriction to \(\sigma_t\), \(\mathcal{P}T\) restricts to become the line bundle \(\mathcal{O}(2 - k)\) so that there is a \(3 - k\)-dimensional family of sections over \(\sigma_t\) for \(3 - k > 0\) or just the 0-section otherwise.

To obtain explicit formulae, we first note that \([f]\) determines a field

\[
f_{A_1' \cdots A_k'-4}(t^{B_1' \cdots B_k'}) = \oint_{\gamma \subset \sigma_t} \pi_{A_1'} \cdots \pi_{A_{k'-4}} f\pi_{B'} d\pi^{B'}
\]

on \(\mathbb{C}^{k+1}\) (here we express the natural pairing as a contour integral over some contour \(\gamma_t\) in \(\sigma_t\) where here \(\chi\) is a Čech representative). This vanishes at some \(t^{A_1' \cdots A_k'}\) iff \([f]\) vanishes on the corresponding \(\sigma_t\). Thus, for \(3 - k > 0\), \(\mathcal{M}\) is fibred over the zero set of \(\chi_{A_1'} \cdots A_{k-4}'\) in \(\mathbb{C}^{k+1}\) with \(3 - k\)-dimensional fibres, and is simply identified with this zero set for \(k \geq 3\).

In order to calculate the SD 2-forms associated to the space-time, we use the method of Gindikin [13], and pullback \(\Sigma\) to the spin-bundle. To this end, introduce local homogeneous coordinates \((\pi_A', Q, \zeta_i) = (c\pi_A', cQ, c^{2-k}\zeta_i)\) on each set \(U_i\) of some Stein cover twistor space: here \(\zeta_i\) is a fibre coordinate up the fibres of the affine line bundle \(\mathcal{P}T \to \mathcal{O}(k)\) on \(U_i\) with patching relations \(\zeta_i = \zeta_j + f_{ij}\) on \(U_i \cap U_j\). In these coordinates

\[
\Sigma = d_h Q \wedge d_h \zeta_i ,
\]

where \(d_h\) denotes the exterior derivative in which \(\pi_A'\) is held constant, i.e., horizontal on the spin bundle over space-time (although slightly confusingly, vertical with respect to the fibration \(\mathcal{P}T \to \mathbb{C}P^1\)). This form \(\Sigma\) is globally defined on vector fields tangent to the fibres of \(\mathcal{P}T \to \mathbb{C}P^1\) as \(f_{ij}\) does not depend on \(\zeta_i\).

In order to evaluate this, we need to find the values of \(\zeta_i\) on the sections of \(\mathcal{P}T \to \mathbb{C}P^1\). These are obtained by a splitting formula due to Sparling.

On a \(\sigma_t\) for which \(f_{A_1' \cdots A_{k'-4}}(t^{B_1' \cdots B_k'}) = 0\), we can find \(\zeta_i(t, \pi_A')\) such that

\[
\zeta_i(t, \pi_{A'}) = \zeta_j(t, \pi_{A'}) + f_{ij}(t^{A_1' \cdots A_k'} \pi_{A_1'} \cdots \pi_{A_{k'-4}} , \pi_{B'}) .
\]

(4.29)
For $k \geq 3$ this solution will be unique, but for $k < 3$ we will be free to add $x^{A_1' \ldots A_{k-1}' \pi A_1' \ldots \pi A_{k-2}}$ to the solution.

In the formula for $\Sigma$ we can rearrange so that we have

$$\Sigma = d_h Q \wedge d_h \zeta = dt^{A_1' \ldots A_k'} \wedge dh(\pi A_1' \ldots \pi A_{k-2} \zeta).$$

Applying $d_h$ to equation (1.29), and multiplying by $k - 3$ of the $\pi$s, we obtain

$$d_h(\pi A_1' \ldots \pi A_{k-3} \zeta) = d_h(\pi A_1' \ldots \pi A_{k-3} \zeta) + \frac{\partial f_{ij}}{\partial Q} \pi A_1' \ldots \pi A_{k-3} \pi B_{1'} \ldots \pi B_{k'} dt^{B_{1'} \ldots B_{k'}}.$$

The cocycle $\partial f_{ij}/\partial Q$ defines a class in $H^1(O(k),O(2-2k))$ so that the expression above takes values in $O(-1)$ on $\mathbb{CP}^1$ for each fixed $t$. Thus the splitting as stated exists and is unique since $H^0(\mathbb{CP}^1,O(-1)) = H^1(\mathbb{CP}^1,O(-1)) = 0$. This gives

$$d_h(\pi A_1' \ldots \pi A_{k-3} \zeta) = k_{i,A_1' \ldots A_{k-2} - k_{j,A_1' \ldots A_{k-2}} = \pi A_1' \ldots \pi A_{k-2} \frac{\partial f_{ij}}{\partial Q} \rho, \quad (4.30)$$

where for simplicity we assume a two set cover and the contour $\gamma_i$ is chosen so that $\gamma_i - \gamma_j$ surrounds $\pi = \rho$. It follows that

$$\psi A_1' \ldots A_{k-4} = \pi A_{k-3} k_{A_1' \ldots A_{k-3}} A_{k-4} = \int_{\gamma_i} \rho A_1' \ldots \rho A_{k-4} \frac{\partial f_{ij}}{\partial Q} \rho \cdot d\rho, \quad (4.31)$$

is the field on $\mathbb{CP}^{k+1}$ naturally associated to $\partial f/\partial Q$.

We therefore obtain the formula

$$\Sigma = dt^{A_1' \ldots A_k'} \wedge \pi A_1' \ldots \pi A_{k-1}' A_{k-1}' \ldots A_{k-2} dt^{A_{k+1} \ldots A_{2k}}.$$

Define the indexed 2-forms $\Sigma(B_{1'} B_{2'})/(A_{1} \ldots A_{2k})$ by

$$d(B_{1'} B_{2'})/(A_{1} \ldots A_{2k}) = \epsilon(A_1' \Sigma B_{1'} B_{2'})/(A_{1} \ldots A_{2k}) C.$$  

With this, we find that a $\pi A_1'$ is contracted onto $k_{i,A_1' \ldots A_{k-2}}$ so that (4.31) gives

$$\Sigma = \pi A_1' \pi A_2' \psi A_1' \ldots A_{2k-2} \Sigma A_1' \ldots A_{2k-2}$$

Thus, the result follows.  

Remarks

- If $k > 3$ then there exists a potential for $\psi A_1' \ldots A_{2k-4}'$;

$$\psi A_1' \ldots A_{2k-4} = \partial A_{k-3} \ldots A_{k-4} f A_1' \ldots A_{k-4};$$

where

$$f A_1' \ldots A_{k-4} = \int_{\Gamma} \rho A_1' \ldots \rho A_{k-4} f \rho \cdot d\rho.$$

The space-time is a four-dimensional surface $\chi A_1' \ldots A_{k-4} = 0$ in $k + 1$ dimensional moduli space of $O(k)$ sections coordinatized by $x^{A_1' \ldots A_k'}$.

---

This indexed 2-form can be represented as

$$\Sigma(B_{1'} B_{2'})/(A_{1} \ldots A_{2k}) = \frac{3k}{2} d \epsilon(B_{1'} B_{2'})/(A_{1} \ldots A_{k+1} \wedge dt^{A_{k+2} \ldots A_{2k}}) C' + a_k \epsilon(A_1' (B_{1'} B_{2'})) A_1' dt^{A_{1} \ldots A_{k+1}} \wedge dt^{A_{k+2} \ldots A_{2k}} C' d't' E', $$

where $a_k$ is a combinatorial constant depending on $k$. 

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• \( k = 3 \) the field \( \psi_{A_1' A_2'} \) doesn’t have a potential, and no conditions have to be imposed to on the moduli space of \( \mathcal{O}(3) \) to find the space time. This is because \( H^0(\mathbb{CP}^1, \mathcal{O}(3)) = H^0(\mathbb{CP}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \), and \( x^{A_1' A_2'} \) has as many components as \( x^{A A'} \). Ward [32] regards \( \psi_{A_1' A_2'} \) as a self-dual Maxwell field on \( \mathbb{C}^4 \).

• The case \( k = 2 \) implies the existence of a tri-holomorphic Killing vector and was consider by Tod and Ward in [31]. Now
  \[
  \psi = \oint \frac{\partial f}{\partial Q} Q \cdot d\rho
  \]
  is a solution to the three-dimensional wave equation. The relation between our construction and the description of the Gibbons-Hawking metric form Section 3.1.1 given by
  \[
  F(x^{A' B'}) = \oint \frac{G(\pi_{A'}, Q)}{(\pi \cdot o)^2} \pi \cdot d\pi, \quad \psi = \oint (\pi \cdot o)^2 \frac{\partial^2 G}{\partial Q^2} Q \cdot d\pi,
  \]
  where
  \[
  x^{A' B'} := \left( \begin{array}{c}
  -p \\
  y/2 \\
  w
  \end{array} \right).
  \]

4.2 Relation between the two constructions

In this section we relate the hyper-Kähler slices (4.1) of the symmetric hierarchy introduced in 4.2 to the method described above.

**Proposition 4.5** Let \( f \in H^1(\mathcal{O}(2n), \mathcal{O}(2 - 2n)) \) give rise to the ZRM field (4.25) with \( k = 2n \). Then hyper-Kähler metrics arising from Proposition 3.2 form a subclass of metrics from Proposition 4.4 if
  \[
  F = \oint (\pi \cdot o)^{-2} G(Q, \pi_{A'}) \pi \cdot d\pi, \quad \text{where} \quad (\pi \cdot o)^2 \frac{\partial G}{\partial Q} = f \in H^1(\mathcal{O}(2n), \mathcal{O}(2 - 2n)),
  \]
  where \( o_{A'} \) is a constant spinor.

**Proof.** Let \( (Q, \pi_{A'}) \) be homogeneous coordinates on the total space of \( \mathcal{O}(2n) \) bundle. Let us choose a constant spinor \( o_{A'} \) and parameterize a section of the \( \mathcal{O}(2n) \rightarrow \mathbb{CP}^1 \) by \( 2n + 1 \) complex numbers
  \[
  x^{A_1' \cdots A_{2n}} = \frac{\partial^{2n} Q}{\partial \pi A_1' \cdots \partial \pi A_{2n}} |_{\pi_{A'} = o_{A'}}.
  \]
  The coordinates \( x^{A_1' \cdots A_{2n}} \) on \( \mathbb{C}^{2n+1} \) correspond to \( t^0, \ldots, t^{2n} \) by
  \[
  t^i = \left( \begin{array}{c}
  2n \\
  i
  \end{array} \right) x^{A_1' \cdots A_{2n}} \partial_{A_1'} \cdots \partial_{A_{2n}} (-1)^{n-i}, \quad i = 0, \ldots, 2n.
  \]
  Define
  \[
  \frac{\partial}{\partial t^i} = t^{A_1'} \cdots t^{A_{i-1}'} o_{A_{i+1}} \cdots o_{A_{2n}} \frac{\partial}{\partial x^{A_1' \cdots A_{2n}}}.
  \]
  Let
  \[
  F = \oint (\pi \cdot o)^{-2} G(Q, \pi_{A'}) \pi \cdot d\pi, \quad \text{where} \quad (\pi \cdot o)^2 \frac{\partial G}{\partial Q} = f \in H^1(\mathcal{O}(2n), \mathcal{O}(2 - 2n)).
  \]
  We have
  \[
  f_{A_1' \cdots A_{2n-4}} = \oint \pi_{A_1'} \cdots \pi_{A_{2n-4}} f \pi \cdot d\pi = \oint \pi_{A_1'} \cdots \pi_{A_{2n-4}} (\pi \cdot o)^2 \frac{\partial G}{\partial Q} \pi \cdot d\pi
  \]
  \[
  = o^{A_{2n-3} \cdots A_{2n}} \frac{\partial}{\partial x^{A_1' \cdots A_{2n}}} \oint (\pi \cdot o)^{-2} G \pi \cdot d\pi = \frac{\partial F}{\partial x^{A_1' \cdots A_{2n-4} o' o' o' o'}}.
  \]
Therefore fixing $f_{A_1 \ldots A_{2n-4}}$ is equivalent to fixing $\partial F/\partial t^i$ for $i < 2n - 3$. Moreover the global twistor function is given by

$$Q = (\omega_n \cdot \iota)(\pi \cdot \iota)^n = (\pi \cdot \iota)^{2n} \sum_{i=0}^{2n} \lambda^i I^{2n-i}.$$ 

\[\square\]

5 ALE spaces revisited; Finite-Gap solutions of ASD vacuum equations

One way to generalise the Novikov construction of finite gap solutions of the Korteweg de Vries equation to hyper-Kähler equations would be to study solutions to (2.8) which are invariant under three commuting hidden symmetries that we shall take to be tri-holomorphic:

$$\partial_{T_1} := \sum_{i=1}^{k} a_i \frac{\partial}{\partial t_i}, \quad \partial_{T_2} := \sum_{i=1}^{l} b_i \frac{\partial}{\partial t_i}, \quad \partial_{T_3} := \sum_{i=1}^{m} c_i \frac{\partial}{\partial t_i},$$

where $a_i, b_i, c_i$ are constant, and the propagation of $\Theta$ along the parameters $t_i$ is determined by the recursion relations (2.9). This would reduce (2.8) down to an ODE.

Rather than performing the explicit reduction to an ODE, we see from the twistor picture that the twistor space must have three projections onto the total space of the line bundle $\mathcal{O}(n)$ for three values of $n$. Thus we have a map of the twistor space $\mathcal{PT}$ into $\mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r)$ and so $\mathcal{PT}$ can be realized as a hypersurface in this space (although there may need to some blowup or resolution of singularities where the map fails to be an embedding). If we realize $\mathcal{PT}$ as the zero set of a function $F$ taking values in a line bundle of degree $s$, then we must have, for rational curves to have the appropriate normal bundle, that $p + q + r = 2 + s$.

We will now see that the ALE hyper-Kähler spaces falls precisely into this above class.

It is well know that hyper-Kähler manifolds $(\mathcal{M}, g)$ which have the topology of $\mathbb{R}^4$ at infinity, and approach the flat Euclidean metric $\eta = dx_1^2 + \ldots + dx_4^2$ sufficiently fast, in the sense that

$$g_{ab} = \eta_{ab} + O(r^{-4}), \quad (\partial_a)^p (g_{bc}) = O(r^{-4-p}), \quad r^2 = x_1^2 + \ldots + x_4^2$$

(5.32)

have to be flat. A weaker asymptotic condition one can impose on $g$ is asymptotically locally Euclidean (ALE).

The ALE spaces are non-compact, complete hyper-Kähler manifolds which satisfy (5.32) only locally for $r \to \infty$. Globally the neighbourhood of infinity must look like $S^3/\Gamma \times \mathbb{R}$, where $\Gamma$ is a finite group of isometries acting freely on $S^3$ (a Kleinian group). These manifolds belong to the class of gravitational instantons because their curvature is localised in a ‘finite region’ of a space-time.

Finite subgroups of $\Gamma \subset SU(2)$ correspond Platonic solids in $\mathbb{R}^3$. They are the cyclic groups, and the binary dihedral, tetrahedral, octahedral and icosahedral groups (one can think about the last three as Möbius transformations of $S^2 = \mathbb{CP}^1$ which leave the points corresponding to vertices of a given Platonic solid fixed). Each of them can be related to a Dynkin diagram of a simple Lie algebra. All Kleinian groups act on $\mathbb{C}^2$, and the ‘infinity’ $S^3 \subset \mathbb{C}^2$. Let $(z_1, z_2) \in \mathbb{C}^2$. For each $\Gamma$ there exist three invariants $x, y, z$ which are polynomials in $(z_1, z_2)$ invariant under $\Gamma$. These invariants satisfy some algebraic relations which we list below:

| Group       | Dynkin diagram | Relation $F_\Gamma(x, y, z) = 0$                                      |
|-------------|----------------|---------------------------------------------------------------------|
| cyclic      | $A_k$          | $xy - z^k = 0$                                                      |
| dihedral    | $D_{k-1}$      | $x^2 + y^2z + z^k = 0$                                              |
| tetrahedral | $E_6$          | $x^2 + y^3 + z^4 = 0$                                               |
| octahedral  | $E_7$          | $x^2 + y^3 + z^3 = 0$                                               |
| icosahedral | $E_8$          | $x^2 + y^3 + z^5 = 0$                                               |

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In each case
\[ C^2/\Gamma \subset C^3 = \{ (x, y, z) \in C^3, F_T(x, y, z) = 0 \} \]
The manifold \( M \) on which an ALE metric is defined is obtained by minimally resolving the singularity at the origin of \( C^2/\Gamma \). This desingularisation is achieved by taking \( M \) to be the zero set of
\[ \tilde{F}_T(x, y, z) = F_T(x, y, z) + \sum_{i=1}^{r} a_i f_i(x, y, z), \]
where \( f_i \) span the ring of polynomials in \( (x, y, z) \) which do not vanish when \( \partial_x F_T = \partial_y F_T = \partial_z F_T = 0 \). The dimension \( r \) of this ring is equal to the number of non-trivial conjugacy classes of \( \Gamma \) which is \( k - 1, k + 1, 6, 7 \) and \( 8 \) respectively [5]. Kronheimer [18, 19] proved that for each \( \Gamma \) a unique hyper-Kähler metric exists on a minimal resolution \( M \) and that this metric is precisely the ALE metric with \( \mathbb{R}^4/\Gamma \) as its infinity. His construction was a combination of the hyper-Kähler quotient [16] with twistor theory.

In each case the twistor space is the three dimensional hyper-surface \( \tilde{F}_T(x, y, z, \lambda) = 0 \) in the rank-three bundle \( \mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \rightarrow \mathbb{C}P^1 \). Now \( x(\lambda) \in \mathcal{O}(p), y(\lambda) \in \mathcal{O}(q), z(\lambda) \in \mathcal{O}(r) \) are polynomials in \( \lambda, f_i = f_i(x, y, z) \), and \( a_i = a_i(\lambda) \). Therefore
\[ \mathcal{PT} \rightarrow \mathcal{O}(p), \quad \mathcal{PT} \rightarrow \mathcal{O}(q), \quad \mathcal{PT} \rightarrow \mathcal{O}(r), \]
and Lemma [12] implies that the corresponding hyper-Kähler metrics admits three commuting hidden symmetries, and the heavenly equation [28] reduces to an ODE.

The degrees \( p, q \) and \( r \) are such that \( \tilde{F}_T(x, y, z, \lambda) \) is a function homogeneous of some degree \( s \). Therefore
\[ \tilde{F}_T : \mathcal{O}(p) \oplus \mathcal{O}(q) \oplus \mathcal{O}(r) \rightarrow \mathcal{O}(s). \]
To determine the integers \( p, q, r, s \) take the determinants of the above, and notice that the normal bundle to an \( \mathcal{O}(1) \oplus \mathcal{O}(1) \) section of \( \mathcal{PT} \rightarrow \mathbb{C}P^1 \) will have the Chern class \( p + q + r - s = 2 \). This gives us the following
\[
\begin{align*}
A_k & \quad \mathcal{PT} = \{ (x, y, z, \lambda) \in \mathcal{O}(k) \oplus \mathcal{O}(k) \oplus \mathcal{O}(2) \rightarrow \mathbb{C}P^1, \\
& \quad \quad xy - z^k - a_1 z^{k-2} - \ldots a_{k-1} = 0 \}, \\
D_{k-1} & \quad \mathcal{PT} = \{ (x, y, z, \lambda) \in \mathcal{O}(2k) \oplus \mathcal{O}(2k-2) \oplus \mathcal{O}(4) \rightarrow \mathbb{C}P^1, \\
& \quad \quad x^2 + y^2 z + z^k + a_1 y^2 + a_2 y + a_3 z^{k-2} + \ldots + a_{k-1} z + a_{k-1} = 0 \}, \\
E_6 & \quad \mathcal{PT} = \{ (x, y, z, \lambda) \in \mathcal{O}(12) \oplus \mathcal{O}(8) \oplus \mathcal{O}(6) \rightarrow \mathbb{C}P^1, \\
& \quad \quad x^2 + y^3 + z^4 + y(a_1 z^2 + a_2 z + a_3) + a_4 z^2 + a_5 z + a_6 = 0 \}, \\
E_7 & \quad \mathcal{PT} = \{ (x, y, z, \lambda) \in \mathcal{O}(18) \oplus \mathcal{O}(12) \oplus \mathcal{O}(8) \rightarrow \mathbb{C}P^1, \\
& \quad \quad x^2 + y^3 + z^3 + y^2 (a_1 z + a_2) + y(a_3 z + a_4) + a_5 z^2 + a_6 z + a_7 = 0 \}, \\
E_8 & \quad \mathcal{PT} = \{ (x, y, z, \lambda) \in \mathcal{O}(30) \oplus \mathcal{O}(20) \oplus \mathcal{O}(12) \rightarrow \mathbb{C}P^1, \\
& \quad \quad x^2 + y^3 + z^5 + y(a_1 z^3 + a_2 z^2 + a_3 z + a_4) + a_5 z^3 + a_6 z^2 + a_7 z + a_8 = 0 \}
\end{align*}
\]
Note that these spaces are not quite the full non-singular twistor space as there will be singular points where \( \tilde{F}_T \) vanishes together with its first derivatives. These singularities can, however, be resolved, see [19].

We observe that these twistor spaces have projections onto \( \mathcal{O}(2n) \) for \( 2n = p, q, r \) and this corresponds to the existence of three independent commuting tri-holomorphic hidden symmetries. The simplest description along the lines of [16] arises for the lowest value of \( n \), i.e., when we project onto the \( z \) coordinates in the above construction. It is clear from the above formulæ that the fibres of this projection are affine conics in the \( A_k \) and \( D_k \) cases, and affine elliptic curves in the \( E_k \) cases.

From now on we shall drop the subscript \( \Gamma \), because the construction we shall describe applies to all cases. These twistor spaces can all be described as affine line bundles over \( \mathcal{O}(r) \) in effect by uniformizing the affine conics or elliptic curves that make up the fibres of \( \mathcal{PT} \rightarrow \mathcal{O}(r) \). The affine line bundle is determined by a cohomology class in \( H^1(\mathcal{O}(r), \mathcal{O}(2 - r)) \) which can be evaluated as a
linear field on $\mathbb{C}^{r+1}$. The ALE space can then be realised as (a branched cover of) the zero set of this linear field in the real slice $\mathbb{R}^{r+1}$ as in theorem [13].

To make the description more concrete, we now find a patching description of the relevant cohomology class in $H^1(O(r),O(2-r))$. We exclude the curve(s) on which both $F_x = 0$ and $F_y = 0$ so that we can cover the twistor space by the two open sets $U, \bar{U}$ such that $F_x \neq 0$ in $U$ and $\bar{F}_y \neq 0$ in $\bar{U}$. We use $(y, z, \lambda)$ and $(x, z, \lambda)$ as local coordinates in $U$ and $\bar{U}$ respectively. The symplectic form $\Sigma$ on each fibre of $\mathcal{PT} \rightarrow \mathbb{CP}^3$ is given by

$$\frac{dy \wedge dz}{F_x} \text{ in } U,$$

or

$$\frac{-dx \wedge dz}{\bar{F}_y} \text{ in } \bar{U}.$$ These arise from the formula $\Sigma = \oint dz \wedge dy \wedge d\bar{z} / \bar{F}$ with the contour being a small circle surrounding $\bar{F} = 0$. The global homogeneous function $F$ gives rise to a homogeneity $r - 2$ Hamiltonian vector field $X_z$ tangent to the fibres of $\mathcal{PT} \rightarrow O(r)$. From the formula $X_z \cdot \Sigma = dz$ we deduce that

$$X_z = \frac{F_x}{\bar{F}_x} \frac{\partial}{\partial y} \text{ in } U,$$

$$X_z = -\frac{\bar{F}_y}{\bar{F}_x} \frac{\partial}{\partial x} \text{ in } \bar{U}.$$ We now introduce new coordinates $(u, z, \lambda)$ and $(\bar{u}, z, \lambda)$ on $U$ and $\bar{U}$ respectively, where the fibre coordinates in $u$ in $U \rightarrow O(r)$ and $\bar{u}$ in $\bar{U} \rightarrow O(r)$ satisfy $X_z(u) = X_z(\bar{u}) = 1$. Therefore

$$u(y, z) = \int_{\sigma}^{F_x=1} \frac{dy}{F_x}, \quad \bar{u}(x, z) = -\int_{\sigma}^{\bar{F}_y=1} \frac{dx}{\bar{F}_y}$$

for some $\sigma$, and the patching function is given on the overlap by

$$f(z, \lambda) = u - \bar{u} = \int_{F_x=1}^{\bar{F}_y=1} \frac{dy}{F_x}.$$ In the above formula $x$ should be determined in terms of $(y, z, \lambda)$ using $\bar{F} = 0$ before the integral is evaluated. The upper and lower limits will then involve $y = y(z, \lambda)$.

In the case of $A_k$ ALE space we can assume that

$$z^k + a_1 z^{k-2} + \ldots + a_{k-1} = \prod_{j=1}^{k} (z - p_j(\lambda)), \quad \text{where } p_j \in \Gamma(O(2)).$$

A simple integration yields $f = \ln \prod_{j=1}^{k} (z - p_j(\lambda))$, and from Proposition 15 we find

$$G = \sum_{j=1}^{k} (z - p_j)(\ln (z - p_j) - 1).$$

For $D_{k-1}$ we can redefine $z$ and $a_j$ to get rid of terms linear and quadratic in $y$, and write

$$\bar{F} = x^2 - y^2 z - \prod_{j=1}^{k} (z - q_j(\lambda)) \quad \text{where } q_j \in \Gamma(O(4)).$$

Now 13 yields

$$f = \frac{1}{\sqrt{z}} \ln \left(\frac{z - 4z \prod_{j=1}^{k} (z - q_j(\lambda))^{1/2}}{1 + 4z \prod_{j=1}^{k} (z - q_j(\lambda))^{1/2}} - 1 \right)^{1/2}$$

In the remaining cases $E_6$, $E_7$ and $E_8$ the fibres of $\mathcal{PT} \rightarrow O(r)$ are elliptic curves $x^2 = 4y^2 + g_1 y + g_2$ (in case of $E_7$ one needs to redefine $y, z, a_j$ to obtain this canonical form). The periods $g_1, g_2$ are

\[\text{See } [15] \text{ and } [13] \text{ for further discussion of this point.}\]
polynomials in $z$ of order less or equal to 5 which can be determined form (5.33). The fibres can therefore be parametrised by the Weierstrass elliptic function. The cohomology class is represented by an elliptic integral

$$f = \frac{1}{2} \int_{y_0}^{y_1} \frac{dy}{\sqrt{4y^3 + g_1 y + g_2}},$$

where $y_1$ and $y_0$ are roots of $4y^3 + g_1 y + g_2 - 1/4 = 0$ and $12y^2 + g_1 - 1 = 0$ respectively.

One can now, in principle, take these cohomology classes and integrate them to obtain a linear field with $r - 3$ components on $\mathbb{C}^{r+1}$ the vanishing of which will determine the complexified ALE space as a submanifold. This above description is not completely satisfactory for two related reasons. Firstly the description of the ALE space will not be global; the projection from the true ALE space to $\mathbb{C}^{r+1}$ can be many to one, and can have irregular values. Secondly, the limits of integration above defining the cohomology classes actually branch and are not completely well defined.

Further work is required to make this a useful description of ALE spaces. It seems likely that these are the only complete hyper-Kähler metrics with three tri-holomorphic hidden symmetries.

### 6 Hierarchies for the generalised conformal anti-self-duality equations

In this section we extend the concept of a hierarchy from that of $\mathbb{S}$ for the 4-dimensional hyper-Kähler equations to a generalisation of the conformal anti-self-duality equations (and in the process give new and more geometric formulations for the hyper-Kähler hierarchy than in $\mathbb{S}$). The guiding motivation for these definitions come from the twistor theory. However, we first define the various concepts in space-time terms, and then discuss the twistor theory subsequently. We shall, for convenience, work in the holomorphic category. Real versions of the various structures and equations can then be obtained subsequently by demanding the existence of an anti-holomorphic involution $\sigma$ fixing a real slice and with specified action on the various geometric structures.

We will abbreviate the term conformal anti-self-duality to CASD and generalised CASD to GCASD. Unfortunately this terminology is non standard but is designed to be consistent with the corresponding discussion for the anti-self-dual Yang-Mills equations given in [21]. The generalisation of the CASD case is a mild generalisation of quaternionic structures discussed in [28] and have been termed paraconformal structures, see [2] and Grassman structures, [14] where many properties, including the twistor theory, of these spaces are studied. Here we shall refer to them as generalised CASD, GCASD, spaces. The hierarchies defined here are a special case of the $\mathcal{P}$-structures of Gindikin, [14] and references therein.

**Definition 6.1** A solution to the GCASD hierarchy consists of the data $(\mathcal{M}, \mathbb{S}, \mathbb{S}, e^{AA'}_{A''} \cdots \mathbb{A}_n')$ defined as follows: $\mathcal{M}$ is a manifold of dimension $r(n+1)$, $\mathbb{S}$ and $\mathbb{S}$ are vector bundles of rank $r$ and 2 respectively, we use abstract indices $A$ and $A'$ to denote membership of $\mathbb{S}$ and $\mathbb{S}$ respectively; when realised concretely, $A = 0, 1, \cdots, r - 1$ and $A' = 0', 1'$. The indexed 1-form $e^{AA'}_{A''} \cdots \mathbb{A}_n'$, symmetric over its primed indices, determines an isomorphism $T\mathcal{M} = \mathbb{S} \otimes \mathbb{S}^\ast$ at every point.

An element $\pi_{A'}$ of $\mathbb{S}^\ast$ at $m \in \mathcal{M}$ determines an $rn$-plane element

$$z(m)_\pi = \{V \in T_m \mathcal{M}, V \cdot e^{AA'}_{A''} \cdots \mathbb{A}_n' \pi_{A'} \cdots \pi_{A_n'} = 0\}.$$  

Such an $rn$-plane element will be said to be an $\alpha$-plane element at $m$. An $\alpha$-surface is an $rn$-dimensional surface whose tangent space defines an $\alpha$-plane element at each of its points.

The GCASD hierarchy equations are the requirement that there exists a full family of $\alpha$-surfaces, with a unique $\alpha$-surface through each $z(m)_\pi$.

The notation derives from the identification of these bundles with the spin bundles of a conformal structure in 4-dimensions, $r = 2$, $n = 1$. It will also be convenient to introduce a `clumped’ index $i$ for the $n + 1$-dimensional vector space $\mathbb{S}$. When the indices are realised concretely by a choice of a frame for $\mathbb{S}$ with components labelled by 0 and 1, there is a standard correspondence between
the $i$th component for the clumped index, and the component with $i$ 1s and $n-i$ 0s, so that $i$ naturally goes from 0 to $n$.

We now assume that we have a solution to the CASD hierarchy so that we have a full complement of $\alpha$-surfaces and that, shrinking $\mathcal{M}$ to a convex neighbourhood of a point if necessary, the space of these $\alpha$-surfaces is a manifold. We can then define

**Definition 6.2** The space of such $\alpha$-surfaces will be called the twistor space and is denoted $\mathcal{PT}$.

Twistor space is an $r+1$ dimensional complex manifold.

**Theorem 6.3** The twistor space determines and is determined by the GCASD hierarchy. The correspondence is stable under small deformations of the complex structure of $\mathcal{PT}$ or of the GCASD hierarchy.

**Proof:** This is a straightforward extension of Penrose’s nonlinear graviton construction. The correspondence can be studied by means of the double fibration

$$
\begin{array}{ccc}
\mathcal{PT} & \leftarrow & Q \\
\mathcal{M} & \downarrow & \pi \\
\mathcal{PT} & \rightarrow & \mathcal{M}
\end{array}
$$

Points $m \in \mathcal{M}$ correspond to rational curves $L_m := q(p^{-1}(m)) \equiv \mathbb{CP}^1$ in $\mathcal{PT}$. The normal bundle of these rational curves is $N = S \otimes \mathcal{O}(n)$ where $\mathcal{O}(n)$ is the line bundle of Chern class $n$ on $\mathbb{CP}^1$; this follows from the fact that, since the tangent space of $\mathcal{Z}(m)_{\mathbb{R}}$ is the kernel of $e^{A_{A_1^{\prime}}^{\prime} \cdots A_n^\prime} \pi_{A_1^\prime} \cdots \pi_{A_n^\prime}$, the section of the normal bundle corresponding to a vector $V$ can be identified with $V \cdot e^{A_{A_1^{\prime}}^{\prime} \cdots A_n^\prime} \pi_{A_1^\prime} \cdots \pi_{A_n^\prime}$, a function with values in $\mathbb{R}$ with homogeneity $n$. However, sections of $\mathcal{O}(n)$ can be identified with functions homogeneous degree $n$, and so the normal bundle is $S \otimes \mathcal{O}(n)$ as claimed.

With knowledge of the normal bundle, Kodaira theory can now be applied and shows that, since $H^1(\mathbb{CP}^1, N) = H^1(\mathbb{CP}^1, \mathcal{End}(N)) = 0$, the moduli space of curves has $\dim (H^0(\mathbb{CP}^1, S \otimes \mathcal{O}(n))) = r(n+1)$ dimensions, contains $\mathcal{M}$ and $T_m \mathcal{M} \equiv \Gamma(S \otimes \mathcal{O}(n)) \equiv S \otimes \mathcal{O}(n)$. Points of $\mathcal{PT}$ clearly then correspond to integrable $\alpha$-surfaces in $\mathcal{M}$. Kodaira theory also provides the stability of the correspondence under small deformations. See [22, 23] for general constructions that apply to these situations.

**Remark:** The $\mathcal{P}$-structures of Gindikin are more general but can be understood easily in this context as arising naturally on moduli spaces of rational curves in some complex manifold whose normal bundles are $\mathcal{O}(k_1) \oplus \cdots \mathcal{O}(k_r)$ with the $k_i$ not being required to be equal. Such prescriptions for the normal bundle are unstable under deformations of the underlying complex manifold unless no two of the $k_i$ differ by more than one. Under deformations of the complex structure, in the moduli space of such rational curves, the normal bundle will jump so that a dense open set will be the stable case where the normal bundle will be $E \otimes \mathcal{O}(k) \oplus F \otimes \mathcal{O}(k+1)$ with $E$ and $F$ trivial bundles.

This additional generality can be important. For example, if we wish to discuss generalisations of the Ward construction, the twistor space has the structure of a holomorphic vector bundle over a lower dimensional space. In this case, the normal bundle of the rational curves along the fibres is usually taken to have degree 0, whereas the normal bundle of its projection into the bases will usually be required to have a higher degree normal bundle.

These geometric structures fall into the category of involutive $G$-structures studied by Merkulov [22, 23]. In particular, one can exploit his theorems to deduce the existence of connections compatible with the geometric structure. However, for the most part, they must have torsion, although they fall into Merkulov’s category of ‘$G$-structures with very little torsion’, [23].

**Lemma 6.4** For $r \geq 2$, $n \geq 1$, there exists connections on $\mathcal{S}$ and $\tilde{\mathcal{S}}$ such that the induced connection on $TM$ has torsion with non-vanishing irreducible parts only in $\mathcal{S} \otimes \mathcal{O}^{2} \mathcal{S} \otimes \mathcal{O}^{n-2} \mathcal{S}$ and $\mathcal{S} \otimes \mathcal{O}^{n-4} \mathcal{S}$.
where we take $\mathcal{O}^n\tilde{S} = \mathbb{C}$ for $n = 0$ or zero for $n < 0$. There exists a unique choice for such a connections when $n > 1$ and unique up to a 1-form for $n = 1$ (which can be taken to be exact with appropriate choices).

**Proof:** Merkulov reformulates the moduli spaces considered above as Legendrian moduli spaces of holomorphically embedded $\mathbb{CP}^1 \times \mathbb{CP}^{r-1}$'s in the projective cotangent bundle $PT^{r}\mathcal{P}T$ of twistor space. A rational curve, $\mathbb{CP}^1$ in $\mathcal{P}T$ determines its projective conormal bundle in $PT^{r}\mathcal{P}T$, i.e., the 1-forms up to scale that annihilate the tangent space of $\mathbb{CP}^1$. This correspondence is studied by means of the following double fibration

$$P\tilde{S} \times P\tilde{S} \subset P(T^*M)$$

$$\mu \not\propto \nu$$

$$M \not\propto P(T^*\mathcal{P}T).$$

Merkulov’s method uses the contact line bundle $L$ which is the dual to the tautological line bundle $T^*\mathcal{P}T \to P(T^*\mathcal{P}T)$. On restriction to a $\mathbb{CP}^1 \times \mathbb{CP}^{r-1}$, it gives $O(n, 1)$ where $O(p, q)$ is the product of the pullback of $O(p)$ from $\mathbb{CP}^1$ with the pullback of $O(q)$ from $\mathbb{CP}^{r-1}$.

Merkulov shows that the minimal torsion of an affine connection preserving the $G$-structure (or obstruction to obtaining a torsion free connection preserving the tensor decomposition of the tangent space) is then measured by a geometrically obtained class in $H^1(\mathbb{CP}^1 \times \mathbb{CP}^{r-1}, L \otimes \tilde{S}(J^1L)^*)$ and the freedom in the resulting connection is given by $H^0(\mathbb{CP}^1 \times \mathbb{CP}^{r-1}, L \otimes \tilde{S}(J^1L)^*)$.

These groups can be computed as follows. The first jet of a section of $O(n, 1)$ can be encoded into the derivative of a homogeneous degree $n$ function with respect to the $r + 1$ homogeneous coordinates. The value of the function is then retrieved from this by Euler’s homogeneity equations, $\pi_A \partial f / \partial \pi_A = nf$. Thus, the sheaf $J^1L$ on $\mathbb{CP}^1 \times \mathbb{CP}^r$ can be understood as the kernel

$$0 \to J^1L \to S^*(n - 1, 1) \oplus \tilde{S}^*(n, 0) \oplus \tilde{S}(n, 1) \to 0$$

since in the third map, we are imposing the requirement that the Euler homogeneity relation for each factor leads to the same value for $f$.

The cohomology groups $H^i(\mathbb{CP}^1 \times \mathbb{CP}^{r-1}, L \otimes \tilde{S}(J^1L)^*)$ can therefore be calculated by consideration of the long exact cohomology sequence arising from the short exact sequence

$$0 \to S^*(-n, 0) \oplus \tilde{S}^*(1 - n, -1) \oplus \tilde{S}(n, 0) \oplus \tilde{S}(1 - n, 0) \oplus \tilde{S}(2 - n, -1) \to L \otimes \tilde{S}(J^1L^*) \to 0$$

where $\pi_A$ and $\pi_A'$ are the homogenous coordinates on $\mathbb{CP}^{r-1}$ and $\mathbb{CP}^1$ respectively. 

Note that the Merkulov framework is not quite equivalent to ours in the sense that it only requires knowledge of the total space $P(T^*\mathcal{P}T)$ but does not require that it be realised as the projective cotangent bundle of some $\mathcal{P}T$. The results are only inequivalent for $r = 1, n < 3$ and $r = 2, n = 1$. In these cases the results are well known, for example, the full theory of the latter case goes back to Penrose 1976, [25]. The Merkulov framework does not see the curvature conditions that arise from existence of $\mathcal{P}T$, but gives the correct result for the existence of and freedom in choosing compatible torsion-free affine connections.

The cases $r = 1$ are also well known, but for $n = 1, 2$ do not fall satisfactorily into the Merkulov framework. For $n = 1$, there is a projective structure, i.e., an equivalence class of torsion-free connections that share the same unparametrised geodesics, with freedom given by a 1-form. For $n = 2$ there exists a unique torsion-free connection compatible with a conformal structure. For $n = 3$ the connection is still torsion-free, but not subsequently for higher $n$.

The general $n = 1$ case was studied in [25] [2]. It also follows from the calculations of [2] that the torsion must be non zero for a non flat structure in the $n > 1, r > 2$ cases as a consequence of the fact that in these cases the decomposition of the tangent space as a tensor product of $\mathbb{S}$ with $\mathcal{O}^n\mathbb{S}$ determines a paraconformal structure in which both factors have dimension greater than two, and in that case the torsion-free condition implies flatness.
Lemma 6.5 For $r > 1$, the requirement of uniqueness for the $\alpha$ surface through $z(m)_\alpha$ is redundant.

Proof: The integrability equations in particular give a propagation equation for $\pi$ across the $\alpha$-surface. \hfill \square

There are a number of specializations of the GCASD equations: hypercomplex, scalar-flat Kähler, Einstein, hyper-Kähler. The hypercomplex and hyper-Kähler cases have straightforward extensions to the hierarchy.

- The hyper-complex case for $r$ even, $n = 1$, where there exists a flat connection on $\hat{S}$ such that the distribution $D$ on $\hat{P}\hat{S}$ is horizontal. This is equivalent to the existence of a fibration $\mathcal{PT} \rightarrow \mathbb{C}P^1$. This condition (flatness of the induced connection of $\mathcal{PT}$ or a fibration of the associated twistor space over $\mathbb{C}P^1$) can clearly be imposed consistently on any $\mathcal{M}_{r,n}$ to give a hypercomplex hierarchy.

- In case of the hyper-Kähler hierarchy, we require that there exists a connection compatible with $\mathcal{M}_r$ that induces a flat connection on $\hat{S}$ and preserves skew forms $\varepsilon_{AB}$ on $\hat{S}$ such that the forms $\varepsilon_{AB}e^{A(A_1\ldots A_n)}_B\wedge e^{B(B_1\ldots B_n)}_A$ are closed. This then implies that $\eta = \varepsilon^{A'B'}\pi_{A'}D\pi_{B'}$ and $\eta\wedge\varepsilon_{AB}\pi_A\pi_1\ldots\pi_{A_n}e^{A(A_1\ldots A_n)}_B\wedge\pi_{B_1}\ldots\pi_{B_n}e^{B(B_1\ldots B_n)}_A$ descend to $\mathcal{PT}$, in such a way that $\eta$ is the annihilator of an integrable distribution determining a fibration over $\mathbb{C}P^1$.

6.1 Reality structures

The imposition of reality conditions is standard; it is imposed by requiring the existence of an anti-holomorphic involution $\sigma$ on $\mathcal{M}$ that fixes a real slice and preserves the geometric structures (i.e., sends $\alpha$-surfaces to $\alpha$-surfaces). In the hyper-Kähler case, we can talk in terms of the signature of the associated metric (although the following conditions can be applied more generally). For Euclidean signature, we require that it induces a quaternionic involution on $\hat{S}$ given by $\sigma^2 = -1$. In particular there are no non-zero fixed points. It will then also induce a quaternionic involution on $\mathcal{S}$ which will have to be even dimensional and we must also require that the Hermitian form $\pi^A\pi^B\varepsilon_{AB}$ be definite (it is trivially definite for $r = 2$). For non-Euclidean signature we can have different signatures for $\pi^A\pi^B\varepsilon_{AB}$, or impose a conjugation whose action on $\mathcal{S}$ sends $\hat{S}$ is an ordinary complex conjugation.

The conjugation will also lead to an anti-holomorphic involution on the twistor space, without fixed points in the quaternionic case, and with a fixed real slice otherwise. Points of the real slice of $\mathcal{M}$ will then correspond to $\sigma$ invariant rational curves.

6.2 Embedding into hierarchies

In the usual definition of a hierarchy, the hierarchy is an overdetermined, but compatible system of equations which contains the original system. A given solution to the original system may not actually extend to a solution of the hierarchy in general (there can be obstructions, see p249 and p253 footnote 2 of [1] for some discussion of this behaviour for the Drinfeld Sokolov hierarchies). Furthermore, if such an extension does exist, it will not in general be unique without the imposition of boundary conditions.

Our definition of a hidden symmetry in Section 4 require the existence of an extension of a solution to the original equation to the hierarchy that happens to admit a symmetry, but only when thought of as a solution to the hierarchy.

We state the embedding definition in slightly greater generality as for one hierarchy into another:

Definition 6.6 A solution $\mathcal{M}_{r_1,n_1}$ to the GCASD hierarchy embeds into another solution $\mathcal{M}_{r_2,n_2}$, $n_1 < n_2$, $r_1 \leq r_2$ if $\mathcal{M}_{r,n_1}$ embeds into $\mathcal{M}_{r_2,n_2}$ as a manifold in such a way that the $\alpha$-surfaces of $\mathcal{M}_{r_2,n_2}$ intersect $\mathcal{M}_{r_1,n_1}$ in the $\alpha$-surfaces of $\mathcal{M}_{r_1,n_1}$ and all $\alpha$-surfaces of $\mathcal{M}_{r,n_1}$ arise in this way.

Due to a remarkable theorem of Bernstein & Gindikin, [14], the twistor characterisation of such an embedding in the most interesting case, $r_1 = r_2$, is remarkably simple:
Theorem 6.7 (Bernstein & Gindikin) A solution $\mathcal{M}_{r,n}$ to the GCASD hierarchy embeds into $\mathcal{M}_{r,n_2}$ iff the twistor space $\mathcal{PT}_{n_1}$ for $\mathcal{M}_{r,n_1}$ is obtained from that, $\mathcal{PT}_{n_2}$ for $\mathcal{M}_{r,n_2}$ by choosing submanifolds $\Gamma_1, \Gamma_2, \cdots$ of codimension greater than one and $S_1, S_2, \cdots$ of codimension= 1 and blowing up along each $\Gamma_1, \Gamma_2, \cdots$ and taking a branched covers branching with some multiplicity over each $S_i$.

In $\mathcal{M}$ (see also Section 3), we embedded $\mathcal{M}_{2,n_1}$ into $\mathcal{M}_{2,n_2}$ by blowing up the twistor space $\mathcal{PT}_{n_2}$ at one point $n_2 - n_1$ times.

Note that it is natural in the quaternionic cases, or in the hyper-CASD cases to require that the additional structures be compatible. This is straightforward in the hypercomplex case in which one wishes the embedded twistor space to inherit a fibration over $\mathbb{CP}^1$, but when line bundle valued forms need to be pulled back also, there is the problem that the forms that have been pulled back will in general take values in an inappropriate line bundle unless particular care has been taken.

6.3 Symmetries and hidden symmetries

We can define a symmetry for a GCASD structure to be the requirement that the twistor space admits a global holomorphic vector field $K$. This will in turn determine global holomorphic vector fields $\tilde{K}$ on the correspondence space and $K$ on $M$, such that $\tilde{K}$ projects to $K$. The essential requirement on $\tilde{K}$ will be that it preserves the twistor distribution and this will lead to a generalisation of the conformal Killing vector equations on $K$ whose precise form will depend on $r$ and $n$.

Clearly the concept of hidden symmetry can be applied as before but with greater generality; a solution $\mathcal{M}_{r,n}$ admits a hidden symmetry if it can be embedded as above into an $\mathcal{M}_{r,m}$ that admits an explicit symmetry. This will, as in the proof of proposition \ref{prop4.1}, lead to a global vector field on $\mathcal{PT}_{r,n}$ with values in a line bundle $\mathcal{L}$ of degree $m - n$ (the restriction of the canonical bundle of $\mathcal{PT}_{r,m}$ tensored with the inverse of that of $\mathcal{PT}_{r,n}$). There will also be a generalisation of theorem \ref{thm4.1} this global vector field with values in $\mathcal{L}$ will lead to the realisation of $\mathcal{PT}_{r,n}$ as the total space of an affine line bundle, with underlying translation bundle $\mathcal{L}^*$, over some reduced twistor space which will generically be $\mathcal{PT}_{r-1,l}$ where $l = n + m/(r - 1)$, if $l$ is an integer, although if $l$ is fractional, or in non-generic situations, the normal bundle of lines in the reduced twistor space must be $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_{r - 1})$ with $\sum k_i = (r - 1)n + m$. Thus the original $\mathcal{PT}_{r,n}$ with a hidden symmetry can be determined in terms of a lower dimensional twistor space together with a linear cohomology class on that space.

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Appendix

In four complex dimensions orthogonal transformations decompose into products of ASD and SD rotations $SO(4,\mathbb{C}) = (SL(2,\mathbb{C}) \times \tilde{SL}(2,\mathbb{C}))/\mathbb{Z}_2$. The spinor calculus in four dimensions is based on this isomorphism. We use the conventions of Penrose and Rindler \cite{Penrose-Rindler}. The tangent space at each point of $M$ is isomorphic to a tensor product of the two spin spaces $T^aM = S^A \otimes S^A$. Spin dyads $(\theta^A, \iota^A)$ and $(\theta^{A'}, \iota^{A'})$ span $S^A$ and $S^{A'}$ respectively. The spin spaces $S^A$ and $S^{A'}$ are equipped with symplectic forms $\varepsilon_{AB}$ and $\varepsilon_{A'B'}$ such that $\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These anti-symmetric objects are used to raise and lower the spinor indices via $\iota_A = \iota^B \varepsilon_{BA}, \iota^B = \varepsilon^{AB} \iota_B$ Let $\Gamma_{AB}$ and $\Gamma_{A'B'}$
are the $SL(2, \mathbb{C})$ and $\widetilde{SL}(2, \mathbb{C})$ spin connections. The curvature of the unprimed spin connection $R^A_{\ B} = d\Gamma^A_{\ BC} + \Gamma^A_{\ CD} \wedge \Gamma^C_{\ B}$ decomposes as

$$R^A_{\ B} = C^A_{\ BCD} \Sigma^{CD} + (1/12) R \Sigma^A_{\ B} + \Phi^A_{\ BCD} \Sigma^{CD},$$

and similarly for $R^{A'}_{\ B'}$. Here $R$ is the Ricci scalar, $\Phi_{\ ABA'B'}$ is the trace-free part of the Ricci tensor $R_{\ ab}$, and $C_{ABCD}$ is the ASD part of the Weyl tensor

$$C_{abcd} = \varepsilon_{AB'C'D'} C_{ABCD} + \varepsilon_{AB} \varepsilon_{CD} C_{A'B'C'D'},$$

and the two forms $\Sigma^{A'B'}$ span the three dimensional space of SD two forms.

Given a complex four-dimensional manifold $\mathcal{M}$ with curved metric $g$, a twistor in $\mathcal{M}$ is an $\alpha$-surface, i.e. a null two-dimensional surface whose tangent space at each point is an $\alpha$ plane (a null two-dimensional plane with a SD bi-vector). There are Frobenius integrability conditions for the existence of such $\alpha$-surfaces through each $\alpha$-plane element at each point and these are equivalent, after some calculation, to the vanishing of the self-dual part of the Weyl curvature, $C^A_{\ BCD}$. Thus, given $C_{A'B'C'D'} = 0$, we can define a twistor space $\mathcal{PT}$ to be the three complex dimensional manifold of $\alpha$-surfaces in $\mathcal{M}$. If $g$ is also Ricci flat then $\mathcal{PT}$ has further structures which are listed in the Nonlinear Graviton Theorem:

**Theorem 6.8 (Penrose [25])** There is a 1-1 correspondence between complex ASD vacuum metrics on complex four-manifolds and three dimensional complex manifolds $\mathcal{PT}$ such that

- There exists a holomorphic projection $\mu : \mathcal{PT} \rightarrow \mathbb{CP}^1$
- $\mathcal{PT}$ is equipped with a four complex parameter family of sections of $\mu$ each with a normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$, (this will follow from the existence of one such curve by Kodaira theory),
- Each fibre of $\mu$ has a symplectic structure $\Sigma_{\lambda} \in \Gamma(\Lambda^2(\mu^{-1}(\lambda)) \otimes \mathcal{O}(2))$, where $\lambda \in \mathbb{CP}^1$.

To obtain real metrics on a real four-manifold, we can require further that the twistor space admit an anti-holomorphic involution.

The correspondence space $\mathcal{F} = \mathcal{M} \times \mathbb{CP}^1$ is coordinatized by $(x, \lambda)$, where $x$ denotes the coordinates on $\mathcal{M}$ and $\lambda$ is the coordinate on $\mathbb{CP}^1$ that parametrises the $\alpha$-surfaces through $x$ in $\mathcal{M}$. We represent $\mathcal{F}$ as the quotient of the primed-spin bundle $S^{A'}$ with fibre coordinates $\pi_{A'}$ by the Euler vector field $\Upsilon = \pi^{A'} \partial / \partial \pi_{A'}$. We relate the fibre coordinates to $\lambda$ by $\lambda = \pi_0 / \pi_1$. A form with values in the line bundle $\mathcal{O}(n)$ on $\mathcal{F}$ can be represented by a homogeneous form $\alpha$ on the non-projective spin bundle satisfying $\Upsilon \cdot \alpha = 0$, $L_\Upsilon \alpha = n\alpha$.

The correspondence space has the alternate definition

$$\mathcal{F} = \mathcal{PT} \times \mathcal{M}|_{Z \in l_x} = \mathcal{M} \times \mathbb{CP}^1$$

where $l_x$ is the line in $\mathcal{PT}$ that corresponds to $x \in \mathcal{M}$ and $Z \in \mathcal{PT}$ lies on $l_x$. This leads to a double fibration

$$\mathcal{M} \to \mathcal{PT} \to \mathcal{F} \to l_x \to \mathcal{PT}. \quad (6.35)$$

Points in $\mathcal{M}$ correspond to rational curves in $\mathcal{PT}$ with normal bundle $\mathcal{O}^A(1) := \mathcal{O}(1) \oplus \mathcal{O}(1)$. The normal bundle to $l_x$ consists of vectors tangent to $x$ (horizontally lifted to $T(x, \lambda)\mathcal{F}$) modulo the twistor distribution. We have a sequence of sheaves over $\mathbb{CP}^1$

$$0 \to D \to \mathcal{C} \to \mathcal{O}^{A}(1) \to 0.$$  

The map $\mathcal{C} \to \mathcal{O}^{A}(1)$ is given by $V^{AA'} \mapsto V^{AA'} \pi_{A'}$. Its kernel consists of vectors of the form $\pi^A \lambda^A$ with $\lambda^A$ varying. The twistor distribution is therefore $D = \mathcal{O}(-1) \otimes S^{A}$ and so there is a canonical $L_A \in \Gamma(D \otimes \mathcal{O}(1) \otimes S_A)$ given by $L_A = \pi^{A'} \nabla_{AA'}$. The projective twistor space $\mathcal{PT}$ arises as a quotient of $\mathcal{F}$ by the twistor distribution. Functions on $\mathcal{F}$ which are constant along $L_{A'}$ are called twistor functions.
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