The Toulouse Limit of the Multi-Channel Kondo Model.

A. M. Tsvelik

March 23, 2022

Department of Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, UK

Abstract

We study the Toulouse limit of the multi-channel Kondo model defined as the limit of maximal anisotropy which can be achieved without changing the infrared behaviour of the model. It is shown that when the number of channels $k$ exceeds two, the interactions do not vanish and the Toulouse limit is described by a non-trivial field theory. Considerable simplifications take place only at $k = 2$ where the Bethe ansatz reproduces the results obtained by Emery and Kivelson [Phys. Rev. B 47, 10812 (1992)].

PACS numbers: 72.10.Fk, 72.15.Qm, 73.20. Dx

1 Introduction

In his original paper Toulouse suggested that the anisotropic spin-$1/2$ Kondo model greatly simplifies at some special value of anisotropy, where it becomes a model of free fermions\(^1\). Later it was discovered that a similar point exists for the two-channel Kondo model, but the limiting theory is a theory of free Majorana fermions\(^2,3\). As we shall demonstrate below, if the number of channels $k$ exceeds two, the Toulouse limit does not correspond to a theory of free particles. For example, it has been established by Fabrizio and Gogolin\(^4\) that for $k = 4$ the Toulouse limit describes the boundary sine-Gordon model with the scaling
dimension of the boundary term $\Delta = 3/4$. The latter is a non-trivial field theory whose infrared behaviour is governed by the same scaling laws as for the isotropic 4-channel Kondo model\textsuperscript{5}. Under these circumstances one may wonder how to define the Toulouse limit. We think that the logical choice is to define it as the limit of maximal anisotropy for which the infrared behaviour still remains the same as for the isotropic model. Since the Toulouse limit plays a considerable role in many studies of the Kondo models (see Refs. 2,3 and 6), we undertake to study it in detail using the Bethe ansatz.

2 Derivation of the Thermodynamic Equations

The Bethe ansatz equations for an anisotropic k-channel Kondo model are similar to those for the Heisenberg chain with spin $k/2$ and are given by (see, for example, Refs. 6-8):

\[ [e_k(\eta; u_a)]^N e_{2S}(\eta; u_a - 1/g)^M \prod_{b=1} e_2(\eta; u_a - u_b) \]  \hspace{1cm} (1)

\[ E = \sum_{a=1}^M \frac{1}{2i} \ln e_k(\eta; u_a) \]  \hspace{1cm} (2)

\[ e_n(\eta; u) = \frac{\sinh[\eta(u - i)]}{\sinh[\eta(u + i)]} \]  \hspace{1cm} (3)

where $S$ is the impurity spin, $g$ is the Kondo coupling constant, $\eta$ is the anisotropy and $N$ is the length of the system and $M = kN/2 + S - S^z$ is the number of up spins. The universal relationship between the quantities $g$ and $\eta$ and the parameters of the Hamiltonian exists only in the limit of weak anisotropy $\eta \ll 1$.

As we have mentioned above, the difficulty in determining the Toulouse limit is resolved if we define it as corresponding to the maximal value of $\eta$ at which the IR fixed point of the Kondo model still belongs to the same universality class as at $\eta \to 0$. The periodicity of the trigonometric functions in Eqs.(1) suggests that for $k > 1$ the Toulouse limit is realized at $\eta = \pi/2(k + 1/\nu)$ ($\nu \to \infty$). Here we perceive a profound difference between the single-channel ($k = 1$) and multi-channel Kondo models. In the single-channel case the Toulouse limit is defined as the point where the scattering vanishes. It occurs at $\eta = \pi/4$ where the right hand side of Eqs.(1) becomes 1. At $k > 2$ the point $\eta = \pi/4$ cannot be continuously connected with the isotropic point $\eta \to 0$, since at $\eta = \pi/2k$ the left hand side becomes equal to 1, which marks a discontinuity. Therefore the Toulouse limit
defined as the limit of maximal anisotropy is no longer equivalent to free particles and corresponds to a non-trivial field theory. Thus, as we have already mentioned, according to the arguments given in Ref. 4, at $k = 4$ the Toulouse limit describes the boundary sine-Gordon model with the scaling dimension of the boundary term $\Delta = 3/4$.

The Toulouse limit has been considered in Refs. 7-9 in the context of the model of the Anisotropic Principal Chiral Field. The most detailed derivation of the thermodynamic Bethe ansatz (TBA) equations is given in Ref. 9. The equations for the Kondo model are essentially the same as for the Principal Chiral Field, but, nevertheless, some details need explanation which is presented in this article. We will show that the TBA equations in the Toulouse limit have the following form:

$$
\epsilon_n = s \ln (1 + e^{\epsilon_{n-1}}) (1 + e^{\epsilon_{n+1}}) + \delta_{n,k-2} s \ln (1 + e^{\epsilon_k}) + \delta_{n,k-1} s f(v), \ n = 1, \ldots, k - 1
$$

(4)

$$
\epsilon_k = s \ln (1 + e^{\epsilon_{k-2}}) - 2 \exp(-\pi v/2) - s f(v)
$$

(5)

$$
F_{imp} = -T \int_{-\infty}^{\infty} dv s(v + 2/\pi \ln T_K/T) \ln \left[1 + e^{\epsilon_{2s(v)}}\right]
$$

(6)

where

$$
\epsilon_n = \int_{-\infty}^{\infty} du g(u) \frac{4 \cosh[\pi(v - u)/2]}{\pi(v - u)/2}
$$

(7)

the function $f(v)$ is defined later in the text (see Eq.(29) and the Appendix) and the Kondo temperature $T_K$ is defined as:

$$
T_K = \lim_{\nu, \Lambda \to \infty} \frac{\pi \Lambda}{2\nu} \exp(-\pi/2g)
$$

At zero magnetic field $f(v) = 0$. When taking the Toulouse limit one has to be careful to keep the Kondo energy scale, $T_K$, finite. This is because at $\nu \to \infty$ the left hand side of Eqs.(4) formally becomes equal to one. In order to get non-trivial solutions one has to renormalize the cut-off, $\Lambda \sim \nu$.

As it was shown by Takahashi and Suzuki for $k = 1$, and later for general $k$ by Kirillov and Reshetikhin, at $\eta = \pi/(k + 1/\nu)$ Eq.(4) has the following string solutions: (i) there are $k - 1$ strings of lengths $n = 1, 2, \ldots, k - 1$ whose energies we denote as $\epsilon_n$, (ii) there are kink and antikink excitations $E_\sigma, \sigma = \pm$, (iii) there are $\nu - 1$ breathers with energies $\kappa_j, j = 1, 2, \ldots, \nu - 1$. The total number of excitation branches is equal to $k + \nu$. Kinks and antikinks have spin 1/2 with a Landee factor which is a function of the interactions, whilst the breathers are spinless.
The TBA equations for the strings are:

\[
\epsilon_n = s * \ln (1 + e^{\epsilon_{n-1}}) (1 + e^{\epsilon_{n+1}}) \\
+ \delta_{n,k-1} \left[ s * \sum_{\sigma} \ln \left(1 + e^{-E_{\sigma}}\right) + \sum_j \xi_j * \ln \left(1 + e^{\kappa_j}\right) \right], \ n = 1, \ldots, k-1
\]

where

\[
\xi_j(\omega) = \frac{\cosh[(1 - j/\nu)\omega]}{\cosh \omega}
\]

In the limit \(\nu \to \infty\) this kernel becomes a delta-function.

The TBA equations for the breathers are:

\[
\ln (1 + e^{\kappa_j}) - B_{jm} * \ln \left(1 + e^{-\kappa_m}\right) = \\
\frac{2\nu}{\pi} \sin(\pi j/2\nu)e^{-\pi v/2} + b_j * \sum_{\sigma} \ln \left(1 + e^{-E_{\sigma}}\right) - \xi_j * \ln \left(1 + e^{\kappa_{k-1}}\right)
\]

where the Fourier transforms of the kernels are given by:

\[
B_{jm}(\omega) = 2 \coth(\omega/\nu) \cosh[(1 - j/\nu)\omega] \sinh(m \omega/\nu), \ (j > m)
\]

\[
b_j(\omega) = \frac{\coth(\omega/\nu) \sinh(j \omega/\nu)}{\cosh \omega}
\]

In the limit \(\nu \to \infty\) we have

\[
B_{jm}(v) = 2 \min(j, m) \delta(v), \ b_j(v) = 2js(v)
\]

In this limit Eqs.(9) can be solved and the functions \(\xi_j\) expressed in terms of other excitation energies.

The TBA equations for kink and anti-kink excitations are

\[
E_\sigma(v) - \sum_{\sigma'} K * \left(1 + e^{-E_{\sigma'}}\right) = \\
\frac{\nu}{\pi} e^{-\pi v/2} - g_L \sigma H/T - \sum_j s * \ln \left(1 + e^{\epsilon_j}\right) + \sum_j b_j * \ln \left(1 + e^{-\kappa_j}\right)
\]

where

\[
K(\omega) = \frac{\sinh[(1 - \nu^{-1})\omega]}{2 \cosh \omega \sinh(\omega/\nu)}
\]

The free energy has the following form:

\[
\frac{1}{L} F_{bulk} = \\
- \frac{\nu}{\Lambda} T^2 \int_{-\infty}^{\infty} dv e^{-\pi v/2} \left[ \frac{1}{\pi} \sum_{\sigma} \left(1 + e^{-E_{\sigma}}\right) + \nu^{-1} \sum_j j \ln \left(1 + e^{-\kappa_j}\right) \right]
\]

\[
F_{imp} = -T \int_{-\infty}^{\infty} dv s[v + \frac{2}{\pi} \ln(T \kappa/T)] \ln \left(1 + e^{2s(v)}\right)
\]
where $\Lambda = N/L$ and $L$ is the length of the system. The Kondo temperature $T_K$ is defined by Eq. (7).

Now we shall consider the limit $\nu \to \infty$. As we have said above, we shall keep $T_K$ finite when taking this limit, which implies keeping the combination $\Lambda/\nu$ finite. Let us consider the equations for $E_\sigma$ first. At $\nu \to \infty$ the kernel $K(\omega) \to \nu \tilde{K}(\omega)$ where

$$
\tilde{K}(\omega) = \frac{\tanh \omega}{2\omega}
$$

Since the right hand side of Eq. (12) is proportional to $\nu$, $E_-(\nu)$ goes to infinity and only $E_+(\nu)$ remains finite and only in a finite magnetic field. The corresponding equation is

$$
\tilde{K} \ast \ln \left(1 + e^{-E_+(\nu)}\right) = \tilde{g}_L H/2T - \frac{1}{\pi} e^{-\pi v/2}
$$

(15)

where

$$
\tilde{g}_L = \lim_{\nu \to \infty} \frac{g_L}{\nu}
$$

(16)

The latter quantity is the effective impurity Landee factor which is difficult to determine in the Bethe ansatz framework. Similar difficulties are common for all models with linear spectrum and originate from the fact that in the Bethe ansatz one chooses as the reference state a ferromagnetic state with a very high energy (see the discussions in Ref. 11). According to Refs. 3 and 6 the impurity magnetic susceptibility vanishes in the Toulouse limit, which indicates that $\tilde{g}_L = 0$. However, this result was obtained under the condition that the impurity and the conduction electrons have the same Landee factors. For the general case $g_{imp} \neq g_c$, where $g_{imp}$ and $g_c$ are the bare Landee factors of the impurity and the conduction electrons, it was demonstrated by Fabrizio et al.\textsuperscript{12} that

$$
\tilde{g}_L = g_{imp} - g_c
$$

(17)

It makes sense to consider $g_{imp} \neq g_c$ and keep $\tilde{g}_L$ finite. In what follows we shall denote $f(\nu) = \ln \left(1 + e^{-E_+(\nu)}\right)$. The solution of Eq. (15) is given in the Appendix.

The $\nu \to \infty$ limit of the TBA equations for breathers are given by

$$
\ln \left(1 + e^{\kappa_j(\nu)}\right) - 2 \sum_m \min(j, m) \ln \left(1 + e^{-\kappa_m(\nu)}\right) =
$$

$$
\frac{1}{2} \sum_j \left(e^{-\pi v/2} + 2s \ast f(\nu)\right) - \ln \left(1 + e^{\kappa_{j-1}(\nu)}\right)
$$

(18)

Inverting the kernel we transform these equations to the more familiar form:

$$
\kappa_j(\nu) = \frac{1}{2} \ln \left(1 + e^{\kappa_{j-1}(\nu)}\right) \left(1 + e^{\kappa_{j+1}(\nu)}\right) - \frac{1}{2} \delta_{j,1} \ln \left(1 + e^{\kappa_{j-1}(\nu)}\right)
$$

$$
\lim_{\nu \to \infty} \frac{\kappa_j(\nu)}{j} = \left(e^{-\pi v/2} + 2s \ast f(\nu)\right) \equiv \epsilon_0(\nu)
$$

(19)
The general solution of this system was obtained by Takahashi\textsuperscript{13}:

\[ 1 + e^{\kappa_j} = \left\{ \frac{\sinh\left(\frac{1}{2}\epsilon_0(j + 1) - u(v)\right)}{\sinh\left(\frac{1}{2}\epsilon_0\right)} \right\}^2 \]  
(20)

In our case the function \( u(v) \) is determined by the boundary condition at \( j = 1 \):

\[ 1 + e^{\epsilon_k-1} = \left[ \frac{\sinh\left(\frac{1}{2}\epsilon_0\right)}{\sinh\left(\frac{1}{2}\epsilon_0 - u\right)} \right]^2 \]  
(21)

Using this result we can calculate the infinite sums appearing in other TBA equations:

\[ \sum_j \ln (1 + e^{\kappa_j}) = \ln \left[ \frac{e^{-\epsilon_0/2\sinh(\epsilon_0 - u)}}{\sinh(\frac{1}{2}\epsilon_0 - u)} \right] \]  
(22)

\[ \sum_j j \ln (1 + e^{\kappa_j}) = - \ln \left( 1 - e^{2u-\epsilon_0} \right) \equiv \ln \left( 1 + e^{\epsilon_k} \right) / \left( 1 - e^{\epsilon_0} \right) \]  
(23)

Here we introduced the new energy \( \epsilon_k \) in order to get rid of \( u \). We shall also redefine \( \epsilon_{k-1} \) introducing a new function \( \epsilon'_{k-1} \):

\[ 1 + e^{\epsilon_{k-1}} = (1 + e^{\epsilon_k}) \left( 1 + e^{\epsilon'_{k-1}} \right) \]  
(24)

Substituting Eqs.(22),(23) and Eq.(24) into Eqs.(9) we obtain Eqs.(5), and (6) with \( f(v) \) given by Eq.(15).

3 Thermodynamic Properties in Zero Magnetic Field

The system of TBA equations (5) and (6) really have different topologies at \( k = 2 \) and \( k > 2 \). In the former case the equations decouple and we have

\[ \epsilon_2(v) = -2 \exp(-\pi v/2) - s * f(v) \]
\[ \epsilon_1 = s * f(v) \]  
(25)

The function \( f \) is independent of \( \epsilon_{1,2} \), the explicit expression for it is derived in the Appendix (see Eq.(38)). Substituting Eq.(38) into Eq.(6) we obtain the following expression for the impurity free energy:

\[ F_{imp} = -T \int dv s[v + \frac{2}{\pi} \ln T_K] \ln \left\{ 1 + \exp\left[ s * f(v + \frac{2}{\pi} \ln T) \right] \right\} \]
\[ = -\frac{\pi}{T} \int_0^{\infty} \frac{dx}{1 + x^2} \ln \left\{ 1 + \exp\left[ \frac{T_K}{2T} \left( \sqrt{g_L H/T_K} \right)^2 + x^2 - x \right] \right\} \]  
(26)
This formula describes the free energy of a Majorana fermion with dispersion \( E(v) \). Since \( f \sim H \), \( E(v) \) vanishes in zero magnetic field together with the impurity part of the specific heat. Thus we reproduce the results obtained in Refs. 2,3 and 6.

At \( k > 2 \) no simplifications take place. Neither the impurity contribution to the specific heat disappears at \( H = 0 \), and the Toulouse limit is described by a non-trivial field theory. We can solve the TBA equations only approximately for \( T >> T_K \) and \( T << T_K \). At \( T << T_K \) we can expand Eqs. (5) and (6) in powers of \( g_k \equiv \ln[1 + \exp(\epsilon_k)] \) and \( f \), as was done in Ref. 14. The answer is:

\[
\Phi(n) = \frac{\sin[(\pi n) + 1]/(k + 2)]}{\sin[\pi/(k + 2)]}
\]

The first non-vanishing pole of the kernel is at \( \omega = -2\pi i/(k + 2) \) which gives the correct low temperature asymptotics for the impurity free energy:

\[
F_{\text{imp}} = -AT(T/T_K)^{4/(k+2)}
\]

\[
A = \int dv e^{-\pi v/(k+2)}[g_k(v) + \frac{\Phi(k-2)}{4\cos^2(\pi/k+2)}s \ast f(v)]
\]

### 4 Thermodynamic Properties at \( T = 0 \)

\[
f^{(-)}(v) + \int_Q^{\infty} K(v-u)f^{(+)}(u)du = \frac{\hat{g}_L H}{2T} - \frac{1}{\pi}e^{-\pi v/2}
\]
For $k = 2$, $S = 1/2$ this integral can be calculated:

$$M_{\text{imp}} = \frac{h}{\pi \sqrt{|1 - h^2|}} \left[ \ln \left( h^{-1} + \sqrt{h^{-2} - 1} \right) \theta(1 - h) + \sin^{-1} \sqrt{1 - h^{-2}} \theta(h - 1) \right]$$

(31)

For $k = 4$, $S = 1$ the integral for the magnetization also looks rather simple, but, nevertheless, cannot be expressed in elementary functions:

$$M_{\text{imp}} = 4\sqrt{h} \int_0^\infty \frac{dy}{y(1 + y^2)(1 + hy)}$$

(32)

In both cases $h = \tilde{g}_L H/T_K$.

5 Conclusion

Here we just give a brief summary of the picture described in the main text. It turns out that the Toulouse limit is very different for $k = 2$ and $k > 2$. In the former case one can solve the TBA equations explicitly for any temperatures and magnetic fields. It is also true that at $H = 0$ the impurity contribution to the free energy becomes trivial $F_{\text{imp}} = -T \ln \sqrt{2}$. This result holds for any $H$ when $g_{\text{imp}} = g_c$ (see Refs. 3, 6). Nothing of that sort happens at $k > 2$. The TBA equations remain non-trivial and no explicit analytic solution is available.

6 Acknowledgements

The author expresses his gratitude to A. Gogolin for very helpful critical remarks and interest to the work and to P. de Sa for his help in preparation of the manuscript.

7 Appendix

Eq. (13) is quite special in that respect that its solution can be expressed in elementary functions. Namely, one can treat this equation as a limiting case of the equation

$$\int_{-Q}^Q du \ln | \coth[\frac{\pi}{4}(v - u)]| \tilde{f}(u) = -\pi \tilde{g}_L H + 4m \cosh(\pi v/2)$$

(33)
The latter equation becomes Eq. (13) at $m \to 0$, $m \exp(\pi Q/2) = \text{const}$ with
\[ \tilde{f}(v) = T f(v + \frac{2}{\pi} \ln T) \]
Differentiating this equation with respect to $v$ and using the substitution
\[ \tilde{f}(u) = [\cosh(\pi u/2)]^{-1} F(u), \ x = \tanh(\pi u/2) \]
we reduce it to the canonical form
\[ P \int_{-B}^{B} \frac{dy F(y)}{x-y} = g(x); \ g(x) = -\pi m \frac{x}{1-x^2} \] (34)
This equation has the following general solution\textsuperscript{15}:
\[ F(x) = -\frac{1}{\pi^2} \sqrt{B^2-x^2} P \int_{-B}^{B} \frac{g(y)dy}{(y-x)^2} \] (35)
In our case we get the following expression for $\tilde{f}(u)$:
\[ \tilde{f}(u) = -m \left[ \frac{B^2 \cosh^2(\pi u/2) - \sinh^2(\pi u/2)}{(1-B^2)^{1/2}} \right]^{1/2} \] (36)
The limit $B$ can be determined from the original equation. In particular, substituting there the solution (36) and putting $v = 0$ we obtain the following condition:
\[ \int_{-Q}^{Q} dv \ln |\coth(\pi v/4)| \left[ \sinh^2(\pi Q/2) \cosh^2(\pi v/2) - \cosh^2(\pi Q/2) \sinh^2(\pi v/2) \right]^{1/2} \]
\[ = -4 + \pi H \tilde{g}_L/m \] (37)
where $B = \tanh(\pi Q/2)$. In the limit $H/m \to \infty$ when $Q \to \infty$ we obtain from Eq. (37):
\[ m e^{Q\pi/2} = \tilde{g}_L H \]
and
\[ s * \tilde{f} = \frac{\tilde{g}_L H}{2} \left( -e^{-\pi v/2} + \sqrt{1+e^{-\pi v}} \right) \] (38)

References

[1] G. Toulouse, Phys. Rev. B2, 270 (1970).
[2] V. Emery and S. Kivelson, Phys. Rev. B 47, 10812 (1992).
[3] D. G. Clarke, T. Giamarchi and B. Shraiman, Phys. Rev. B 48, 7070 (1993).
[4] M. Fabrizio and A. O. Gogolin, Phys. Rev. B50, 17732 (1994).

[5] A. M. Tsvelik, cond-mat/9502022; P. de Sa and A. M. Tsvelik, unpublished.

[6] A. M. Sengupta and A. Georges, Phys. Rev. B49, 10020 (1994), Phys. Rev. Lett. to be published.

[7] P. B. Wiegmann, Pis'ma Zh. Eksp. Teor. Fiz. 41, 79 (1984) [JETP Lett. 41, 95 (1984)]; Phys. Lett. B141, 217 (1984).

[8] A. N. Kirillov and N. Yu. Reshetikhin, Zapiski Nauch. Seminara LOMI 145, 109; 146, 47 (1985).

[9] A. M. Tsvelik, Zh. Eksp. Teor. Fiz. 93, 385 (1987) [JETP 66, 221 (1987)].

[10] M. Takahashi and K. Suzuki, Prog. Theor. Phys.48, 2187 (1972).

[11] A. M. Tsvelik and P. B. Wiegmann, Adv. Phys. 32, 331 (1983); see the discussion on p. 596.

[12] M. Fabrizio and A. O. Gogolin and Ph. Nozières, submitted to Phys. Rev. B.

[13] M. Takahashi, Prog. Theor. Phys.46, 401 (1971).

[14] A. M. Tsvelik, J. Phys. C18, 159 (1985).

[15] S. G. Mikhlin, Integral Equations, Pergamon, Oxford (1964); see also, for example, C. W. J. Beenakker, Nucl. Phys. B422, 515 (1994).