Cosmological evolution of vacuum and cosmic acceleration

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Abstract

It is known that the unregularized expressions for the stress–energy tensor components corresponding to subhorizon and superhorizon vacuum fluctuations of a massless scalar field in a Friedmann–Robertson–Walker background are characterized by the equation of state parameters $\omega = 1/3$ and $\omega = -1/3$, which are not sufficient to produce cosmological acceleration. However, the form of the adiabatically regularized finite stress–energy tensor turns out to be completely different. By using the fact that vacuum subhorizon modes evolve nearly adiabatically and superhorizon modes have $\omega = -1/3$, we approximately determine the regularized stress–energy tensor, whose conservation is utilized to fix the time dependence of the vacuum energy density. We then show that vacuum energy density grows from zero up to $H^4$ in about one Hubble time, vacuum fluctuations give positive acceleration of the order of $H^4/M_p^2$ and they can completely alter the cosmic evolution of the universe dominated otherwise by the cosmological constant, radiation or pressureless dust. Although the magnitude of the acceleration is tiny to explain the observed value today, our findings indicate that the cosmological backreaction of vacuum fluctuations must be taken into account in early stages of cosmic evolution.

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1. Introduction

Explaining the observed cosmological acceleration of the universe is one of the most important and difficult problems of modern cosmology awaiting a solution. Whether the acceleration is due to a bare cosmological constant or mysterious dark energy, understanding its cause in terms of fundamental physics is challenging. This leads many researchers to seek for unorthodox solutions and there are a lot of new ideas and directions in the literature claiming...
different answers to the problem. In the absence of any other experimental or observational input to test these ideas, it is difficult to judge them based solely on theoretical arguments and decide whether a natural solution is really obtained.

On the other hand, one may wonder if at least a partial explanation in terms of standard physical notions can be given to the acceleration phenomena. In this paper, we show by a careful treatment that quantum vacuum fluctuations of a massless scalar field in a Friedmann–Robertson–Walker (FRW) spacetime can give cosmological acceleration. Usually, vacuum fluctuations of quantized fields are thought to contribute to the bare cosmological constant. In fact, the famous statement that there is a $10^{122}$ order of magnitude discrepancy between the expected and the observed values of the cosmological constant depends on this view (see e.g. [1]). However, the cosmological constant is characterized by the equation of state $P = \omega \rho$ with $\omega = -1$, which cannot easily be fulfilled by vacuum fluctuations. For example, the (unregularized) stress–energy tensor of a quantized massless scalar field in Minkowski spacetime has $\omega = 1/3$, which is naively equivalent to the radiation. Similarly, for a massless scalar in an expanding FRW universe while the subhorizon modes are described by the parameter $\omega = 1/3$, the superhorizon modes have $\omega = -1/3$, which are not close to the equation of state of the cosmological constant. As discussed in [2, 3] it may not even be possible to mimic the cosmological constant by quantum fluctuations with trans-Planckian modifications of the dispersion relation.

One may then think that the quantum vacuum fluctuations cannot give rise to cosmic acceleration, which requires $\omega < -1/3$. However, the above-mentioned naive expressions for the stress–energy tensor components contain ultraviolet (UV) divergences and they must be regularized in a suitable way, which may alter the equation of state (this will be the case as we will see below). Moreover, vacuum fluctuations and the Hubble parameter influence each other in a very non-trivial way: while the expansion speed separates the subhorizon and superhorizon excitations, the vacuum energy density directly affects the magnitude of the Hubble parameter. In other words, simple general arguments based solely on the equation of state might be misleading since the dynamical impact of the vacuum fluctuations is different from that of a single component perfect fluid.

In this paper, we study the cosmological evolution of a quantized massless scalar field in a FRW spacetime placed initially in its ground state. The problem is truly very hard to tackle: the adiabatically regularized finite stress–energy tensor, which drives the expansion, can be determined in terms of the integrals of the mode functions, which in turn obey the massless Klein–Gordon equation on the background. Therefore, one encounters a complicated but consistent system of coupled integro-differential equations involving the scale factor of the universe and the mode functions. The assumption that the scalar is in its ground state at some time specifies the initial conditions for the mode functions. Supplying also the expansion speed of the universe at the same time as an initial condition, the system of equations can in principle be uniquely solved to determine the subsequent evolution.

Of course it is very difficult, if not impossible, to exactly solve this complicated system. In order to figure out the cosmological evolution, we try to simplify these equations by using the fact that the subhorizon modes must evolve adiabatically and the superhorizon modes can be characterized by the equation of state parameter $\omega = -1/3$. In this way, the stress–energy tensor can approximately be found without knowing the time dependence of the mode functions explicitly. Instead, the vacuum energy density can be shown to obey a modified conservation equation determining its time evolution. The final, approximate but consistent differential equation system is simple enough to analyze analytically.

Partially due to the terms coming from adiabatic regularization, the stress–energy tensor of vacuum fluctuations turns out to depend on the Hubble parameter and its time derivative.
Consequently, the cosmological evolution is significantly altered and one can see from the field equations that the expansion is always accelerated whose order of magnitude is given by $H^4 / M_p^2$. As a matter of fact, it is possible to obtain an analytical solution describing the cosmological evolution of vacuum. Depending on the initial expansion speed, the solution gives an increasing Hubble parameter blowing up in a finite proper time (i.e. a big-rip singularity) or a constant Hubble parameter (i.e. de Sitter space) or a decreasing but always positive acceleration parameter which asymptotically vanishes. Although the magnitude of the acceleration $H^4 / M_p^2$ is very small to explain the accelerated expansion today, it is encouraging to see that a seemingly simple and standard physical system yields cosmic acceleration. Besides, we observe that vacuum fluctuations can modify the standard cosmic evolution of matter in a very non-trivial way. For example, in the presence of a cosmological constant we show that the Hubble parameter increases in time and it either blows up in a finite proper time or asymptotes to a larger fixed value. Similarly, when the initial expansion speed is about the Planck scale, the radiation or dust-dominated expansions are also significantly modified. While the validity of this last statement is questionable due to the possible modifications of the standard equations near the Planck scale, it still indicates that a proper understanding of vacuum dynamics in an expanding universe is very crucial in early time cosmology.

Without doubt, the problem of determining the stress–energy tensor of a quantized field in a cosmological background is a well-studied problem. For example, it was shown in an earlier work [4] that a massive quantized field in a closed FRW geometry can stop a cosmological collapse and convert it into an expansion, hence avoiding a cosmic singularity. Similarly, it was pointed out in [5] that due to the trace anomaly induced by the broken conformal invariance, the strong energy condition can be violated, which indeed gives acceleration. Although our approach in this paper is akin to that of [4, 5], the final set of equations we obtain are much simpler and easy to interpret physically. Consequently, we are able to construct analytical solutions describing the cosmological impact of vacuum. On the other hand, more recently, the backreaction of a quantum scalar field in a curved spacetime has also been analyzed in relation to the cosmic acceleration phenomena (see e.g. [6–12]). In the next section, we will compare our findings with these results that already exist in the literature.

2. Cosmological evolution of vacuum

We consider a massless scalar field $\phi$ in a FRW spacetime which has the metric

$$\begin{align*}
\text{d}s^2 &= -\text{d}t^2 + a(t)^2 (\text{d}x^2 + \text{d}y^2 + \text{d}z^2) \\
&= a(\eta)^2 (-\text{d}\eta^2 + \text{d}x^2 + \text{d}y^2 + \text{d}z^2).
\end{align*}$$

(1)

The corresponding Hubble parameters in the conformal and the proper time coordinates are defined as

$$\begin{align*}
h &= \frac{a'}{a}, \\
H &= \frac{\dot{a}}{a},
\end{align*}$$

(2)

where the prime and the dot represent derivatives with respect to $\eta$ and $t$, respectively. In this paper we only consider expanding geometries and thus assume $H > 0$. A real, massless scalar field $\phi$ propagating in this background obeys

$$\nabla^2 \phi = 0.$$

(3)

For quantization it is convenient to define a new field $\mu$ by

$$\mu = a\phi.$$

(4)
After applying the standard canonical quantization procedure, one can see that the field operator $\mu$ can be decomposed in terms of the \textit{time-independent} ladder operators as

$$
\mu = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ \mu_k(\eta) e^{i\vec{k} \cdot \vec{a}_k} + \mu_k(\eta)^* e^{-i\vec{k} \cdot \vec{a}_k^+} \right].
$$

where $\vec{k}$ is the comoving momentum variable, $[a_k, a_k^+] = \delta(\vec{k} - \vec{k}')$ and the mode functions satisfy

$$
\mu''_k + \left[ k^2 - \frac{a''}{a} \right] \mu_k = 0.
$$

The ground state $|0\rangle$ of the system at time $\eta_0$ can be defined by imposing

$$
a_k|0\rangle = 0
$$

and

$$
\mu_k(\eta_0) = \frac{1}{\sqrt{2k}}, \quad \mu_k'(\eta_0) = -i\sqrt{\frac{k}{2}}.
$$

The subhorizon and superhorizon modes are given by $k > h$ and $k < h$, respectively. The approximate solutions of (6) in these two regimes can be determined as

$$
\mu_k \approx e^{\pm ik\eta} \quad \text{(subhorizon)}
$$

and

$$
\left(\frac{\mu_k}{a}\right)^\prime \approx 0 \quad \text{(superhorizon)}
$$

Using the definition of the stress–energy–momentum tensor $T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu}(\nabla \phi)^2$, one can easily calculate the following vacuum-expectation values:

$$
\langle 0 | \rho | 0 \rangle = \frac{1}{4\pi^2 a^4} \int_0^{\infty} k^2 \left[ a^2 \left| \left( \frac{\mu_k}{a} \right)^\prime \right|^2 + k^2 |\mu_k|^2 \right] dk,
$$

$$
\langle 0 | P | 0 \rangle = \frac{1}{4\pi^2 a^4} \int_0^{\infty} k^2 \left[ a^2 \left| \left( \frac{\mu_k}{a} \right)^\prime \right|^2 - \frac{k^2}{3} |\mu_k|^2 \right] dk.
$$

From these expressions we see that for subhorizon modes obeying $\mu_k \approx e^{ik\eta}$ (and also for $k \gg h$) the stress–energy tensor is characterized by the equation of state parameter $\omega = 1/3$. Likewise, the stress–energy tensor of superhorizon modes satisfying $\left(\mu_k/a\right)^\prime \approx 0$ has $\omega = -1/3$. One can verify that the conservation equation $\rho + 3h (\rho + P) = 0$ is identically satisfied provided that the mode functions obey (6).

The expectation values given in (9) contain UV divergences and must be regularized. The most straightforward way of regularization is to place a UV cutoff for momentum integrals. However, to preserve stress–energy conservation the \textit{comoving} cutoff scale must be chosen to be time independent. Thus, the physical cutoff becomes time-dependent which is puzzling.

An alternative way is to apply adiabatic (or WKB) regularization [13, 14], which gives finite results without the need of introducing a new scale to the problem. Let us briefly review the adiabatic regularization procedure. One first writes the mode function $\mu_k$ in terms of a new variable $\Omega_k$ in the following WKB form:

$$
\mu_k = \frac{1}{\sqrt{2\Omega_k}} e^{-i f(\Omega_k, \eta)}.
$$

Using (6), $\Omega_k$ can be seen to obey

$$
\Omega_k'' = k^2 - \frac{a''}{a} + \frac{3}{4} \frac{\Omega_k'^2}{\Omega_k^2} - \frac{1}{2} \frac{\Omega_k''}{\Omega_k}.
$$

It is now possible to solve this equation iteratively by including terms with more and more time derivatives (the zeroth-order solution is $\Omega_k = k$). To regularize (9), one simply subtracts

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the corresponding stress–energy tensor expressions obtained from the adiabatic mode function (10). To cancel the quartic and the quadratic divergences it is enough to proceed up to the adiabatic order 2, i.e. to keep the terms containing up to two time derivatives. To remove the remaining logarithmic divergence, one should also subtract the fourth-order adiabatic terms.

In this paper, we only utilize adiabatic regularization up to second order. As we will see below, after applying a further approximation, one can still obtain a finite and consistent set of field equations even at this order. Furthermore, these equations turn out to be simple enough to be analyzed analytically. When the fourth-order adiabatic subtractions are included, the system becomes much more difficult to be examined analytically. Moreover, in that case the scale factor obeys a fourth-order differential equation, which is problematic when one tries to interpret the dynamical evolution in the Hamiltonian formulation (in that case to determine the evolution uniquely one should know not only the initial expansion speed but also the initial acceleration and the third derivative of the scale factor). To avoid the logarithmic divergence, the momentum integrals may still be imagined to be restricted by the Planck scale. As we will see in a moment, the dependence of the stress–energy tensor on the cut-off scale will disappear in the final set of equations.

Up to adiabatic order 2, (11) can be solved as

$$\Omega_k = k \left[ 1 - \frac{a''}{2ak^2} \right].$$

(12)

After a straightforward calculation one can then obtain the following regularized expressions for the stress–energy tensor components:

$$\rho_V = \frac{1}{4\pi^2a^4} \int_0^\infty k^2 \left[ a^2 \left( \frac{\mu_k}{a} \right)' \right]^2 + k^2 |\mu_k|^2 - k - \frac{h^2}{2k} \right] dk,$$

(13)

$$P_V = \frac{1}{4\pi^2a^4} \int_0^\infty k^2 \left[ a^2 \left( \frac{\mu_k}{a} \right)' \right]^2 - \frac{k^2}{3} |\mu_k|^2 - k - \frac{h^2}{2k} + \frac{a''}{3ak} \right] dk.$$  

(14)

As shown in [13], \(\rho_V\) and \(P_V\) are guaranteed to be free of quartic and quadratic infinities and obey the conservation equation \(\dot{\rho}_V + 3h(\rho_V + P_V) = 0\). Using them as sources in the field equations, one finds

$$h^2 = \frac{8\pi a^2}{3M_p^2} \rho_V,$$

(15)

$$h' = -\frac{4\pi a^2}{3M_p^2} (\rho_V + 3P_V).$$

(16)

Giving \(h(\eta_0)\), setting \(a(\eta_0) = 1\) and imposing further the initial conditions\(^1\) (8), equations (6) and (13)–(16) give a complicated but (up to a possibly logarithmic divergence) well-defined integro-differential equation system for \(\mu_k\) and the scale factor of the universe \(a(\eta)\), which uniquely determine the cosmic evolution.

This complicated system is very difficult to analyze analytically. To simplify these equations we point out that there should not be too much difference between the physical and adiabatic modes for subhorizon excitations with \(k > h\). In other words, subhorizon modes

\(^1\) The initial conditions defining the vacuum state must be modified due to the counter-terms added to the action. Moreover, imposing (8) for superhorizon modes can be criticized based on the principle of locality. Since we will not use (8) in our calculations, we do not discuss this issue further.
can approximately be viewed to evolve adiabatically because ‘the particle creation effects’ can be ignored for them. Therefore, in (13) and (14) the momentum integrals can be neglected in the range \((h, \infty)\), since the terms coming from the mode functions \(\mu_k\) will be canceled by adiabatic subtractions. On the other hand, by defining

\[
\rho_S = \frac{1}{4\pi^2a^2} \int_0^h k^2 \left[ a^2 \left( \frac{\mu_k}{a} \right)' \right]^2 dk, \tag{17}
\]

\[
P_S = \frac{1}{4\pi^2a^2} \int_0^h k^2 \left[ a^2 \left( \frac{\mu_k}{a} \right)' \right]^2 - \frac{k^2}{3} |\mu_k|^2 dk, \tag{18}
\]

which encode the contributions coming from the non-adiabatically evolving superhorizon modes, one can write

\[
\rho_V = \rho_S - \frac{h^4}{8\pi^2a^2}, \tag{19}
\]

\[
P_V = P_S - \frac{h^4}{12\pi^2a^2} + \frac{h^2}{24\pi^2a^2} a'' a. \tag{20}
\]

One must now fix the time evolution of the superhorizon modes or correspondingly \(\rho_V\) and \(P_S\). To achieve this we recall that superhorizon modes obey \((\mu_k/a)' \simeq 0\) and \(P_S = -\rho_S/3\). As a result, the stress–energy conservation can now be used to determine \(\rho_S\). Namely, the conservation equation

\[
\rho_V' + 2H\rho_V = \frac{1}{4\pi^2} H^5 - \frac{1}{8\pi^2} H^3 \dot{H}. \tag{23}
\]

Note that (23) follows from the conservation equation, which dictates how vacuum energy density is forced by the expansion of the background. Likewise, the vacuum stress–energy tensor alters the standard equation for the scale factor in a very non-trivial way and yields (22). It is straightforward to check that these last three equations for two unknown functions \(H\) and \(\rho_V\) are consistent, i.e. (22) follows from (21) and (23), as it should. Remarkably, one sees from (22) that vacuum fluctuations give rise to acceleration of the order of \(H^4/M_p^2\).

In fact, there is an important *loophole* in the above derivation: in neglecting the momentum integrals in (13) and (14) in the interval \((h, \infty)\), one assumes that all subhorizon modes with \(k > h\) at a given time \(t > t_0\) have evolved adiabatically in their entire history. This is clearly wrong for the modes which were born as superhorizon and then later became subhorizon, since in that case the difference between the real mode \(\mu_k\) and the corresponding adiabatic mode cannot be ignored. Therefore, the decomposition of the stress–energy tensor components
given in (19) and (20), and consequently the field equations (21)–(23), can only be used when the acceleration is always positive in the whole cosmic history, which forbids the superhorizon modes to turn into the subhorizon regime. From (22), we see that a cosmology dominated by vacuum fluctuations has always positive acceleration, which is an important self-consistency check of these equations.

The above comments also explain why it is not correct to evaluate the integrals in (13) and (14) by using the approximate solutions $\mu_k \simeq e^{ik\eta}$ for subhorizon modes and $(\mu_k/a)' \simeq 0$ for superhorizon modes. In other words, a mode may evolve from one regime to the other, and indeed the main reason for the emergence of acceleration in this setup, despite the fact that neither superhorizon nor subhorizon modes give $\omega < -1/3$, is precisely these cross overs. We should note that the terms coming from adiabatic regularization also play a role in obtaining the acceleration. We thus observe that as regularization changes the naive spectrum of cosmological perturbations produced during inflation [15, 16], it also significantly changes the cosmic evolution due to vacuum fluctuations.

Naturally one should impose $\rho_V(t_0) = 0$ as an initial condition. However from the Friedmann equation (21), this implies $H(t_0) = 0$. The unique solution with these initial conditions is $H = 0$ and $\rho_V = 0$, which in some sense shows the stability of the flat space. To obtain non-trivial cosmological solutions in this picture we assume that $\rho_V(t_0) \neq 0$. Physically, this corresponds to a situation where the vacuum fluctuations have been created previously and started to dominate the expansion after some time. As we will see in the next section, by adding matter in the form of a cosmological constant, radiation or dust, this scenario can actually be realized.

It is possible to solve (22) exactly. There is a special solution with the constant Hubble parameter where

$$H = \sqrt{3} M_p. \quad (24)$$

Other than $H = 0$, this is the unique solution with the constant $H$ which mimics the cosmological constant. For $H \neq \sqrt{3} M_p$, the following implicit solution can be found:

$$\frac{1}{H} + \frac{\sqrt{3}}{4M_p} \ln \left| \frac{H - \sqrt{3} M_p}{H + \sqrt{3} M_p} \right| = t. \quad (25)$$

If $H > \sqrt{3} M_p$, the Hubble parameter can be seen to increase indefinitely and one encounters a big-rip singularity at finite proper time where $H$ blows up. If $H < \sqrt{3} M_p$, the Hubble parameter continuously decreases in time. Although in that case $H < 0$, the acceleration is always positive $\ddot{a} > 0$ and one can see that $a \to t$ asymptotically as $t \to \infty$. The graphs of $H$ for these cases are given in figure 1.

The non-homogenous equation of motion (23) shows that in one Hubble time the vacuum energy density $\rho_V$ is driven by the expansion to increase up to $H^2$. However, the homogeneous piece of $\rho_V$ decreases as $\rho_V \sim 1/a^2$. Therefore, when $H^2$ decreases faster than $1/a^2$ during cosmological evolution, $\rho_V$ is expected to increase from zero to a maximum value and then decrease like $1/a^2$.

One may wonder if quantum vacuum fluctuations can be used to realize an inflationary scenario. We found that for $H = \sqrt{3} M_p$, the solution is exactly the de Sitter space, which would give eternal inflation. Alternatively, by tuning $H$ close to this value, it is possible to get

\[ \text{This actually follows from (13) if one uses the initial conditions (8) in the integral. However, to determine the initial value of the pressure $P_V(t_0)$ from (14) is more subtle. In our approach, $P_V(t_0)$ is fixed by the conservation equation.} \]

\[ \text{In flat space the adiabatic regularization is equivalent to normal ordering and thus both vacuum energy density and pressure vanish.} \]
a nearly exponential expansion with a suitable number of e-foldings. Note that the exit from inflation is naturally achieved since $H$ decreases in time.

We close this section by comparing our results with other approaches in the literature. In [6–8], the cosmological impact of a very light massive scalar field is studied using the one-loop effective action, which is determined by applying the zeta-function regularization and heat kernel methods. As shown in [6], the effective action gives the standard trace anomaly in the massless conformally coupled limit. Using the modified field equations, and approximately treating the scalar curvature as a constant, the authors of [6–8] manage to obtain an approximate accelerating solution, which can be glued with continuous first and second derivatives to a matter-dominated FRW spacetime. Moreover the free parameters of this solution, which are the mass of the scalar and its curvature coupling, can be fit to agree with the type Ia supernova data. It is clear that the stress–energy–momentum tensor obtained from the one loop effective action is expected to agree (or at least imply the same physics) with the adiabatically regularized stress–energy tensor. The fact that the trace anomaly of the massless conformally coupled scalar can be obtained from the one loop effective action supports this claim. However, the stress–energy–momentum tensor, either adiabatically regularized or obtained from the one-loop effective action, is very complicated and highly nonlinear, involving terms up to fourth derivatives of the metric functions. Therefore, it is difficult to trust any approximate background since the nonlinear differential equations containing higher derivative terms can easily give runaway solutions and stability can be an issue. Moreover, quantum effects are expected to be much stronger near the big bang and therefore one should also make sure that the one loop effective action does not alter the standard cosmological evolution (this is a much more difficult problem to deal with since the curvatures are large). By employing the adiabatic regularization up to second order and canceling the remaining logarithmic divergence by a physical argument, we manage to obtain a very simple set of self-consistent field equations, whose solutions physically make sense. For example, the vacuum energy density obtained from the modified conservation equation (23) turns out to be positive and grows up to $H^2$, which is consistent with the order of magnitude estimate given in [5]. An interesting possibility, which is worth elaborating, is to see whether the one loop effective action of [6–8] truncated in a certain limit agrees with the simplified equations obtained in our paper.

In another interesting work [9], the dynamical evolution of a massive quantum scalar field in a FRW background is studied using the Born–Oppenheimer reduction of the mini-superspace...
Wheeler–De Witt equation. In this way, the authors of [9] succeed in obtaining coupled equations which describe the backreaction of quantum matter on the semiclassical evolution of the scale factor of the universe obeying a Hamilton–Jacobi-type equation. They show that an inflationary period is possible for a large set of initial quantum states. The approach of [9] mainly differs from ours in that the scale factor is treated semiclassically using the mini-superspace approximation (in our case the background metric is assumed to be purely classical). Therefore, in some way they accomplish taking into account the ‘quantum gravitational effects’. On the other hand, the matter sector is analyzed not in the field theory context, but in its quantum mechanical truncation, which can be seen as a drawback of this method.

To realize an inflationary scenario based on quantum backreaction effects, rather than using classical scalar fields, in [10] a class of purely gravitational but non-local models are studied. Quantum gravitational corrections are encoded in an effective energy–momentum tensor, which is assumed to have the perfect fluid form. These models are characterized by a distinctive phase of oscillations, which occur both during and after inflation. The oscillations terminate at the onset of matter domination and can induce a positive cosmological constant having the right size. The approach of [10] is completely different from the method of the present paper. Whereas in [10] the contribution of quantum matter is ignored, here we simply neglect the quantum gravitational effects. We can justify our approximation by referring to the mainstream view that the quantum gravitational effects must be negligible on scales smaller than the Planck scale (indeed all inflationary models, which use classical scalar fields, rely on this common belief), but the findings of [10] show that in some models this may not be the case.

Finally, we should mention that, as discussed in [11, 12], determining the strength of the quantum backreaction effects in nonlinear theories can be a delicate issue. In [11], the backreaction of the quadratic scalar fluctuations on the leading order linearized metric perturbations is studied in the exact de Sitter spacetime, where the cosmological constant is generated by the background scalar field placed in the minimum of its potential. In general, the linearized equations are shown to have instabilities, which can be avoided by taking strictly de Sitter invariant quantum states. It is difficult to apply the results of [11] directly to our case, since other than the special finely tuned solution with the constant Hubble parameter, the symmetry group is not the de Sitter one. However, since this problem does not arise for the ground state in the de Sitter spacetime, one may hope that it also does not emerge in our setup, since the quantum scalar is assumed to be placed in its ground state. Similarly in [12], certain integrals of the second-order metric and matter fluctuations are shown to be much larger than the similar integrals involving first-order terms in FRW spacetimes close to the de Sitter background. Although the main argument of [12] is based on purely classical manipulations, it is also proved in the same paper that when fluctuations are taken to be quantum distributions, the result is not spoiled by quantum anomalies. The analysis of [12] shows that the strength of the perturbations does not necessarily agree with the order of the perturbative expansion. In our method, we simply ignore the metric perturbations and consider a free quantum scalar field, i.e the backreaction is not determined in a perturbative expansion, so the complication discussed in [12] is expected not to arise in our analysis.

### 3. Adding matter

In the previous section we have seen that the quantized massless scalar field in a FRW spacetime gives positive acceleration. In this section, we include contributions coming from other viable cosmological sources and try to see whether this result continues to hold. As
usual we assume that these sources, which we call matter, can be modeled by a perfect fluid obeying the equation of state

$$P_M = w \rho_M.$$  \hfill (26)

We further assume that matter does not directly couple to the scalar field and thus its stress–energy tensor obeys the standard conservation equation, which gives $\rho_M = 1/a^{3(1+w)}$. On the other hand, the vacuum energy density continues to obey (23) and the field equations become

$$H^2 = \frac{8\pi}{3M_p^2} (\rho_V + \rho_M),$$ \hfill (27)

$$H + \dot{H}^2 = \left(1 + \frac{H^2}{6M_p^2}\right)^{-1}\left[\frac{H^4}{2M_p^2} - \frac{4\pi(1 + 3w)}{3M_p^2} \rho_M\right].$$ \hfill (28)

Initially the scalar is placed in its ground state at time $t_0$ which means

$$\rho_V(t_0) = 0.$$ \hfill (29)

Typically the time $t_0$ can be thought to correspond to the big bang, after which the largest possible particle creation or vacuum fluctuation effects are supposed to occur.

From the decomposition given in (19) and (20), and recalling that $P_S = -\rho_S/3$, one finds $P_V(t_0) \neq 0$. This is actually necessary to satisfy the stress–energy conservation, since ‘the particle creation effects’ give $\dot{\rho}_V(t_0) > 0$ and the conservation equation implies $P_V(t_0) = -\dot{\rho}_V(t_0)/(3H_0) < 0$. Therefore, at least in the beginning of its evolution the vacuum has negative pressure in an expanding universe.

Using $\rho_V(t_0) = 0$, one can see from (27) and (28) that to obtain acceleration at $t_0$ the initial expansion speed must obey

$$H(t_0)^2 > (1 + 3w)M_p^2.$$ \hfill (30)

Similarly for

$$H(t_0)^2 > \frac{9(1 + w)}{2} M_p^2,$$ \hfill (31)

one has $H(t_0) > 0$. Recall that having acceleration is necessary for the validity of the stress–energy tensor expressions (19) and (20), since otherwise the modes which were born as superhorizon and later became subhorizon demand a special treatment. When the initial expansion speed obeys (30), the corresponding initial negative pressure of vacuum fluctuations turns out to be large enough to produce acceleration even when matter has $w > -1/3$. Note that for $w > -1/3$, (30) in general gives $H(t_0) \sim M_p$. One must be aware of the fact that near such Planck scale expansion speeds, the quantum gravitational effects must be taken into account and it is difficult to trust purely classical equations. Keeping this remark in mind, let us proceed to analyze the field equations in different cases.

### 3.1. Cosmological constant

Assume now that in addition to the quantum massless scalar field, there also exists a cosmological constant obeying $P_M = -\rho_M$ and $\dot{\rho}_M = 0$. When the backreaction of quantum fluctuations is ignored, the scale factor becomes $a = 1/(H_0\eta)$, where $H_0$ is the constant
Hubble parameter of the de Sitter space, and the corresponding mode functions (obeying (6) and suitable initial conditions) can be fixed as

$$\mu_k = \frac{1}{\sqrt{2k}} \left[ 1 - \frac{i}{k\eta} \right] e^{-i k \eta}. \quad (32)$$

Using $\mu_k$ in (13) and (14) one finds that $\rho_V = P_V = 0$. Therefore, in the exact de Sitter background the stress–energy tensor of vacuum fluctuations vanishes as in the case of the flat space\(^4\). However, taking an exact de Sitter space in reality is very questionable since there inevitably exist other cosmological sources. Moreover, during inflation, which is supposedly the era closest to the de Sitter phase, there exists an 'effective' varying cosmological constant generated by classical scalar fields. In that case, the particle creation effects are not negligible and the vacuum energy density obeying (23) necessarily grows.

To analyze the evolution let us rewrite (28) as

$$\dot{H} = \left(1 + \frac{H^2}{6M_p^2}\right)^{-1} \left[ \frac{H^4}{3M_p^2} - H^2 + \frac{8\pi}{3M_p^2} \rho_M \right]. \quad (33)$$

For $\rho_M > 9M_p^4/(32\pi)$ the right-hand side is strictly positive for any value of $H$, which implies $\dot{H} > 0$. In that case the Hubble parameter indefinitely increases and from (33) it is possible to see that $H$ blows up in a finite proper time.

For $\rho_M = 9M_p^4/(32\pi)$, the polynomial in the square brackets in (33) can be written as $\left[H^2 - 3M_p^2/2\right]^2$. On the other hand, the initial expansion speed can be determined from (27) as $H_0^2 = 3M_p^2/4$ (recall that we set $\rho_V(t_0) = 0$), which is smaller than the double root of the polynomial $3M_p^2/2$. Therefore, from (33), one sees that starting from its initial value $H_0^2 = 3M_p^2/4$, the Hubble parameter increases in time to reach asymptotically the value $H^2 = 3M_p^2/2$.

Finally for $0 < \rho_M < 9M_p^4/(32\pi)$ the polynomial on the right-hand side of (33) can be written as

$$\frac{1}{3M_p^2} \left[(H^2 - H_0^2)(H^2 - H_\pm^2)\right]. \quad (34)$$

where

$$H^2_\pm = \frac{3M_p^2}{2} \pm \frac{3M_p^2}{2} \sqrt{1 - \frac{32\pi \rho_M}{9M_p^2}}. \quad (35)$$

This time the initial expansion speed, which is determined by $\rho_M$ as $H_0^2 = 8\pi \rho_M/(3M_p^2)$, turns out to be smaller than $H_\pm$. For $H < H_\pm$, one sees from (33) that $\dot{H} > 0$. Therefore, in that case $H$ will again increase and become asymptotically $H_\pm$.

In summary, we find that for $0 < \rho_M < 9M_p^4/(32\pi)$ the vacuum fluctuations of the massless scalar field renormalizes the bare cosmological constant by increasing its value from $H_0$ to $H_\pm$ given in (35). For larger values of the bare cosmological constant $\rho_M > 9M_p^4/(32\pi)$, the scalar field yields a big-rip singularity in a finite proper time.

3.2. Radiation

As an important case, let us now consider the quantum scalar field in the presence of radiation $P_M = \rho_M/3$. From (30) we see that to get initial accelerated expansion the Hubble parameter

\(^4\) This fact has been used in [17] to criticize the inflationary mechanism of generating cosmological perturbations. This criticism was answered in [18] by pointing out that inflation does not correspond to an exact de Sitter phase.
must be very large $H_0 \geq \sqrt{2} M_p$. Thus, as noted above, the results presented in this subsection are likely to be modified by quantum gravitational effects. In any case, let us proceed to see how vacuum fluctuations of the scalar field naively change the evolution of radiation.

For $H_0 = \sqrt{2} M_p$, equations (27) and (28) can be solved exactly which gives

$$a = \frac{t}{t_0},$$

where $t_0 = 1/\left(\sqrt{2} M_p\right)$. Here the initial expansion speed is tuned to yield $\ddot{a} = 0$, which then implies that both $H^4$ and $\rho_M$ in (28) decrease like $1/a^4$ and cancel each other for all times. Integrating (23) the evolution of the vacuum energy density can be found as

$$\rho_V = \frac{3 M_p^2}{8 \pi} \left[ 1 - \frac{t_0^2}{t^2} \right] \frac{1}{t^2}, \quad t \geq t_0.$$  

As pointed out in the previous section, nearly in one Hubble time $\rho_V$ increases to a maximum, of the order of $H^4$, and then decreases like $1/a^2$.

For $\sqrt{6} M_p > H_0 > \sqrt{2} M_p$, one has $\dot{a}(t_0) > 0$ and $\dot{H}(t_0) < 0$. Since $H^4 = \dot{a}^4/a^4$, $\rho_M = C/a^4$ and initially $\dot{a}$ increases (since $\ddot{a}(t_0) > 0$), $\rho_M$ decreases faster than $H^2$, which shows by (28) that $\dot{a}$ continues to be positive. On the other hand, by a similar comparison, it is possible to see that $H < 0$. Therefore, for this given initial expansion rate the universe continues to expand with positive acceleration since radiation always decays faster than the vacuum energy density. As radiation becomes negligible the evolution can be approximately described by the vacuum-only solution (25) (see figure 1(b)). Note that for $H_0^2 < 2 M_p^2$ our equations cannot be used to analyze the expansion of the universe since evolution starts with deceleration.

### 3.3. Pressureless dust

Finally let us determine the cosmological evolution in the presence of pressureless dust with $P_M = 0$. From (30), for $H_0 > M_p$, the acceleration is initially positive. On the other hand, (31) implies that for $H_0 \geq 3 M_p/\sqrt{2}$ the initial rate of change of $H$ is also non-negative. Again, these expansion speeds are very large and the results presented in this subsection must be taken with care.

Rewriting (28) as

$$\dot{H} = \left( 1 + \frac{H^2}{6 M_p^2} \right)^{-1} \left[ \frac{H^4}{3 M_p^2} - H^2 - \frac{4\pi}{3 M_p^2} \rho_M \right].$$

one sees that for $H_0 \geq 3 M_p/\sqrt{2}$, which gives $\dot{H}(t_0) \geq 0$, the first two terms in the square brackets are at least non-decreasing while $\rho_M$ decreases. In that case $H$ will always be positive and the dust will become more and more insignificant at later times. Thus, the metric must approach the purely vacuum solution (25) with increasing $H$ and one encounters a big-rip singularity in a finite proper time (see figure 1(a)).

For $3 M_p/\sqrt{2} > H_0 > M_p$, the expansion starts with acceleration but by comparing the positive and negative terms in (38) one can see that the Hubble parameter always decreases. To determine whether the acceleration survives, one must compare the rate of change of the two terms in (28), namely $H^2$ and $\rho_M$, which give positive and negative contributions, respectively. If $H^2$ decreases faster than $\rho_M$, the acceleration will eventually stop. Since $\rho_M = C/a^4$, the ratio $H^4/\rho_M$ becomes proportional to $f(t) = \dot{a}^4/a$. By using (28) one can calculate

$$f = \dot{a}^4 \left[ \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right] \sim \left[ \frac{11 H^4}{6 M_p^2} - H^2 - \frac{16\pi \rho_M}{3 M_p^2} \right].$$
which shows that $f < 0$ for $H^2 < 6M_p^2/11$. Since $H$ continuously decreases $f$ will eventually become negative and $f$ will start decreasing. This proves that in the end the $H^4$ term decays faster than $\rho_M$ and thus acceleration must stop at some time and deceleration should start over. Unlike the case with radiation, the acceleration supported by vacuum fluctuations will be defeated by the deceleration of the pressureless dust.

4. Conclusions

In this paper we try to determine the cosmological evolution of a FRW spacetime driven by the vacuum fluctuations of a quantized massless scalar field. It turns out that to solve this problem one must deal with a coupled integro-differential equation system involving the mode functions of the scalar field and the scale factor of the FRW metric. Namely, the mode functions obey the massless Klein–Gordon equation on the background and thus they are affected by the expansion; at the same time their integrals in momentum space determine the stress–energy tensor of the scalar field, which then fixes the evolution of the scale factor. To simplify these equations we try to pin down the stress–energy tensor by noting that subhorizon modes evolve nearly adiabatically and thus their contribution to any adiabatically regularized expression can be neglected. On the other hand, since superhorizon modes have equation of state parameter $\omega = -1/3$, their contribution to the stress–energy tensor can be encoded by one unknown function. One can then use stress–energy conservation to determine the time evolution of the total vacuum energy density and pressure. In this way (when the background is accelerating) a simple and consistent set of equations for two unknown functions, which are the scale factor of the universe and the vacuum energy density, can be obtained.

The above-mentioned approximate stress–energy tensor of vacuum fluctuations has some peculiar properties. Maybe the most important new feature is that it appears to depend on the Hubble parameter $H$ and its time derivative $\dot{H}$. The stress–energy conservation yields a non-homogeneous first-order linear equation for the vacuum energy density. The non-homogeneous source terms are given by the Hubble parameter and they dictate how vacuum energy density is forced to increase by the expansion. Similarly, the homogeneous part of this equation shows how vacuum energy density decreases with expansion.

We are able to solve the system exactly and obtain analytical expressions describing the cosmological evolution of vacuum. From the field equations one can see that there is always positive acceleration, of the order of $H^4/M_p^2$. Since the acceleration depends on the speed of the expansion, the exact time evolution becomes completely different from that of a perfect fluid with $\omega < -1/3$. Namely, for different values of the initial Hubble parameter, one encounters different behavior. Since we are employing adiabatic regularization, there does not appear any extra renormalization scale and not surprisingly the critical expansion speed, which separates different regimes, turns out to be of the order of the Planck scale.

Since the magnitude of the acceleration $H^4/M_p^2$ is tiny today one may think that vacuum fluctuations cannot have any cosmological impact at the present time. However, once the vacuum energy density starts to increase in the early universe, in about one Hubble time it reaches the value $H^4$, which is not necessarily very very small compared to the total energy density $H^2M_p^2$. Furthermore, as pointed out above, the vacuum energy density then decreases like $1/a^2$, which is much slower than radiation. Therefore, even though the acceleration is wiped out by other sources giving deceleration, the vacuum energy density can still be significant compared to the total energy density of the universe at an earlier epoch or even at present.
In this paper we also try to answer how the quantized massless scalar field possibly changes the standard cosmic evolution of matter in the form of a cosmological constant, radiation or pressureless dust. We show rigorously that vacuum fluctuations can renormalize the magnitude of the cosmological constant by increasing its bare value. On the other hand, to obtain acceleration in the presence of radiation or dust the initial expansion speed must be very large, which is of the order of the Planck scale, and thus these calculations must be taken with care. In any case, we show that once vacuum is allowed to produce acceleration in the beginning, it will dominate over the radiation and always give an accelerated expansion, but it will eventually be defeated by the pressureless dust and the acceleration will be taken over by deceleration. As a result we show that vacuum fluctuations can alter the well-known standard cosmological evolutions in a very non-trivial way.

It might be interesting to extend these results in different directions. First, one can work out the cosmological impact of a quantized massive field. This is a harder problem since the mass of the scalar field introduces a new scale to the problem and the behavior of the mode functions will be more complicated to determine. Obviously, when the Hubble parameter is much larger than the mass of the scalar, the analysis will be similar to that of the massless case. However, when the mass becomes larger than the Hubble parameter all modes start to evolve adiabatically. Second, it would be interesting to study the impact of the quantized massless scalar field when the universe decelerates. In that case one must determine the effects of the modes which were born as superhorizon and later became subhorizon. The contributions of these modes are expected to be close to radiation with $\omega \sim 1/3$. Of course this prediction must be checked by an explicit calculation. Third, one wonders if a viable scenario of inflation can be realized in this setup. We noted earlier that by fine tuning it is possible to obtain a suitable number of e-foldings by vacuum fluctuations and exit from inflation is naturally achieved. As shown in [19], there inevitably exist (as a matter of fact large) deviations about the quantum vacuum expectation values. One may be tempted to interpret these deviations as the source of cosmological perturbations and it would be interesting to determine the corresponding spectrum.

Let us note that in some solutions presented in this paper, the Hubble parameter increases and this corresponds to an effective equation of state parameter $\omega < -1$. It is known that when the self-interactions are taken into account in determining the stress–energy tensor of the $\lambda \phi^4$ scalar in the de Sitter background, one also finds $\omega < -1$ [20, 21] (it is also possible to obtain $\omega < -1$ in the so-called classical quintom models, for a review see [22]). In our case, super-acceleration does not arise from interactions but it appears due to the vacuum pressure becoming very negative at very large expansion speeds. This happens since at such speeds the vacuum energy density tends to increase faster (due to large particle creation effects) and conservation of the stress–energy tensor necessarily gives correspondingly large negative pressure. We see here that when super-acceleration occurs due to vacuum fluctuations there always appears a big-rip singularity in a finite proper time. This can be interpreted as a semiclassical instability of general relativity in the presence of quantized fields and it would be interesting to sharpen this observation by other semiclassical methods.

Finally, in this paper we consider adiabatic subtractions up to second order. In principle, one can repeat the above calculations with the fully regularized stress–energy tensor by including the fourth-order adiabatic subtraction terms. As noted above, in that case the field equations contain fourth-order time derivatives of the scale factor and to specify the evolution uniquely one must impose not only the initial expansion speed but also the initial acceleration and the third derivative of the scale factor. Although this is puzzling since the physical fields are expected to obey second-order dynamical equations, it would be interesting to study the cosmic evolution with these terms included.
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