DONALDSON AND SEIBERG-WITTEN INVARIANTS OF ALGEBRAIC SURFACES

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1. Introduction

Donaldson theory and more recently Seiberg-Witten theory have led to dramatic breakthroughs in the study of smooth 4-manifolds, and in particular of algebraic surfaces and their generalizations, symplectic 4-manifolds. In this paper, we shall survey some of the main results. Many expositions of Seiberg-Witten theory have appeared, for example the survey article of Donaldson [10] and the book of Morgan [28], and the only claim to originality made here is in the focus on algebraic surfaces. A general reference for Donaldson theory is the book of Donaldson and Kronheimer [11]. A discussion of the invariants from the point of view of physics has been given by Witten [41]. A more detailed exposition of the results described here for Kähler manifolds is given in the papers of the author and Morgan [16] and Brussee [5], as well as in Okonek-Teleman [33], [34].

From the viewpoint of algebraic geometry, the major numerical invariants of an algebraic surface \(X\) are the plurigenera \(P_n(X) = H^0(X; nK_X)\), \(n \geq 1\). Indeed, the second plurigenus \(P_2(X)\) already appears in Castelnuovo’s criterion for rationality, and invariants \(P_4(X)\) and \(P_6(X)\) appear in Enriques’ criterion for when a surface is rational or ruled. Unlike the more natural invariants \(p_g(X)\) and \(q(X)\), however, which are linked to the oriented homotopy theory of \(X\) via Hodge theory, there is no natural topological interpretation (as yet) for the higher invariants \(P_n(X)\) for \(n \geq 2\). A deep geometric fact about surfaces is the existence of strong minimal models for algebraic surfaces of Kodaira dimension at least zero (in other words, those surfaces which are not rational or ruled, or equivalently for which \(P_n(X) \neq 0\) for some \(n\)). Assuming for simplicity that \(X\) is the blowup of a minimal surface \(X_0\) at distinct points, the curve fibers of the blowup morphism are then distinguished holomorphically embedded copies of \(\mathbb{P}^1\) in \(X\) with self-intersection \(-1\), and their cohomology classes are special cohomology classes as far as the algebraic geometry of \(X\) is concerned. A further marked class is that of \(K_X\). However, from the point of view of diffeomorphism, it is better to work with the pullback of \(K_{X_0}\), since if \(X\) is not minimal then \(c_1(K_X)\) is almost never invariant up to sign under diffeomorphisms of \(X\).

The main questions in 1935 concerning the topology of algebraic surfaces described in, say, Zariski’s book [42] are mainly concerned with what would now be viewed as the consequences of Hodge theory, and as such give invariants which depend only on the oriented homotopy type of \(X\). Questions more directly concerned

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with the smooth topology of \( X \) date back to Severi’s problem [38] of giving a topological criterion for rationality (1949). Actually, given the examples of the Enriques and Godeaux surfaces, Severi asked if a surface with \( p_g = 0 \) and \( H_1(X; \mathbb{Z}) = 0 \) was necessarily rational. Later Dolgachev [6] constructed examples of simply connected elliptic surfaces which were not rational (1967), and Barlow [1] gave the only known example of a simply connected surface of general type with \( p_g = 0 \) (1980). By Freedman’s work (1981), it is known that these surfaces are all homeomorphic to rational surfaces. In a related vein, Kodaira [21] classified all of the complex surfaces homotopy equivalent to \( K3 \) surfaces (1967) and asked if they were homeomorphic to \( K3 \) surfaces; again this is a consequence of Freedman’s theorem. Thanks to the examples of Dolgachev and Kodaira, elliptic surfaces were intensively investigated during the 1970’s by Moishezon and Mandelbaum (see for example [27]), as well as by Harer-Kas-Kirby [19], using Kirby’s handlebody calculus. Using these techniques one can show for example that all homotopy equivalent simply connected elliptic surfaces become diffeomorphic once we connect sum one copy of \( \mathbb{C}P^2 \). However, these techniques proved unable to give much information about the actual diffeomorphism classification of elliptic surfaces, and matters remained at an impasse until Donaldson’s famous counterexample [8] to the \( h \)-cobordism theorem in dimension 4, which showed via gauge theory that a certain Dolgachev surface was not diffeomorphic to a rational surface. At this point, the power of gauge theory as a tool in attacking the smooth classification of complex surfaces became apparent, and it was natural to conjecture that the smooth topology of an algebraic surface would in many ways closely reflect its algebro-geometric structure. A related question is the Thom conjecture and its generalizations. The following is a list of theorems along these lines which can be proved by Donaldson theory or by Seiberg-Witten theory (which has also simplified the previous proofs in case the result was already known by Donaldson theory).

**Theorem.** For a fixed diffeomorphism type of a 4-manifold \( M \), there are only finitely many complex structures on \( M \) up to deformation of complex structure, at least so that the resulting complex surface is Kähler.

A proof is given in the book [15]. (In the non-Kähler case, the difficulty lies in the classification of surfaces of type VII in Kodaira’s notation, but presumably the result is still true.) The main part of the argument involves studying elliptic surfaces, and these surfaces have now been classified up to diffeomorphism [13], [15], [31], [2], [3], [12], [30]:

**Theorem.** Two elliptic surfaces with positive holomorphic Euler characteristic are diffeomorphic if and only if they are deformation equivalent.

As a consequence, there is an answer to the question raised by Kodaira:

**Theorem.** A complex surface diffeomorphic to a \( K3 \) surface is a \( K3 \) surface.

One can also classify elliptic surfaces with holomorphic Euler characteristic equal to zero, and show that diffeomorphism implies deformation equivalence up to a two-to-one ambiguity caused by orientation questions [15]. Thus, for elliptic surfaces \( X \) and \( X' \), if \( X \) is diffeomorphic to \( X' \) then \( X \) is deformation equivalent either to \( X' \) or to its complex conjugate.

**Theorem (the van de Ven conjecture).** If \( X \) and \( X' \) are diffeomorphic complex surfaces, then \( X \) and \( X' \) have the same Kodaira dimension.
A proof is given in [18] as well as [36] and [35]. A main step in the proof is the answer to the question raised by Severi:

**Theorem.** A complex surface diffeomorphic to a rational surface is rational.

Using Seiberg-Witten theory, one can prove a stronger statement:

**Theorem.** If $X$ and $X'$ are diffeomorphic complex surfaces, then $P_n(X) = P_n(X')$ for all $n \geq 1$.

For an elliptic surface, $P_n(X)$ can be calculated from the homotopy type once we know the multiple fibers, so the theorem for elliptic surfaces follows from the diffeomorphism classification of elliptic surfaces. For a surface $X$ of general type, the plurigenera are determined by $K_{X_0}^2$, where $X_0$ is the minimal model of $X$. Thus the proof of this theorem follows once we know that the $C^\infty$ topology of $X$ determines the number of times $X$ is blown up from its minimal model.

**Theorem.** Suppose that $X$ and $X'$ are complex surfaces with $\kappa(X) \geq 0$, and that $f : X \to X'$ is an orientation-preserving diffeomorphism. (Thus necessarily $\kappa(X') \geq 0$ as well.) Let $X_0$ be the minimal model of $X$ and $X_0'$ the minimal model of $X'$. Likewise let $[E_i], 1 \leq i \leq r$, be the classes of the exceptional curves on $X$ and let $[E'_j], 1 \leq j \leq s$, be the classes of the exceptional curves on $X$. Then $r = s$ and for all $j, f^*[E'_j] = \pm [E_i]$ for some $i$. Moreover, after identifying $[K_{X_0}]$ with a cohomology class in $H^2(X; \mathbb{Z})$ and likewise for $[K_{X'_0}]$, we have $f^*[K_{X_0}] = \pm [K_{X'_0}]$.

This theorem, whose proof seemed inaccessible to Donaldson theory, follows quite easily from the form of the Seiberg-Witten invariants for a Kähler surface as worked out by Witten, Taubes, and Kronheimer-Mrowka, and seems to have been first explicitly noted for Kähler surfaces with $b_2^+ \geq 3$ by Kronheimer. The case $b_2^+ = 1$, or equivalently $p_g = 0$, needs some further analysis of chamber structures which is rather easy to handle in the Seiberg-Witten case and is worked out in [16] as well as [5].

**Theorem.** Suppose that $X$ is a complex surface and that $X$ is diffeomorphic to a connected sum $M_1 \# M_2$. Then one of $M_1, M_2$ is negative definite (Donaldson [9]). If $\kappa(X) \geq 0$ and, say, $M_2$ is negative definite, then $H_2(M_2)$ is contained in the subgroup of $H_2(X)$ spanned by the classes of the exceptional curves on $X$. In particular, if $\kappa(X) \geq 0$, then every smoothly embedded 2-sphere in $X$ with self-intersection $−1$ is homologous up to sign to the class of an exceptional curve.

Again this follows in a straightforward way in case $b_2^+ \geq 3$, and for $b_2^+ = 1$ by analyzing the chamber structure.

Finally there is the Thom conjecture and its generalizations:

**Theorem.** Let $X$ be a Kähler surface and let $\alpha \in H_2(X; \mathbb{Z})$ satisfy $\alpha^2 \geq 0$. If $\alpha$ is represented by a closed oriented smooth 2-manifold of genus $g$, then $2g - 2 \geq \alpha^2 + \alpha \cdot K_X$.

Of course, by adjunction, if $\alpha = [C]$ where $C$ is a holomorphically embedded algebraic curve, then $2g(C) - 2 = \alpha^2 + \alpha \cdot K_X$, and so the genus of $C$ is minimal among all closed oriented smooth 2-manifolds whose fundamental class is $\alpha$. The Thom conjecture is the special case $X = \mathbb{P}^2$. Following a program outlined by Kronheimer [22], Kronheimer and Mrowka attacked this conjecture in a series of deep papers [23] and proved many special cases (for example in case $X$ is a $K3$
surface). In particular they were led to the discovery of the structure of Donaldson polynomials for manifolds of simple type (to be explained below). The general conjecture was proved via Seiberg-Witten theory by Kronheimer and Mrowka [26] and independently by Morgan-Szabó-Taubes [32].

2. A brief review of Donaldson theory

As we shall see, one fundamental reason that Donaldson theory and Seiberg-Witten theory exist to give such powerful information about dimension 4 is that 4 is the unique solution to the equation $n = 2 - 2$.

Let $M$ be a closed oriented Riemannian 4-manifold and let $P \to M$ be a principal $SU(2)$-bundle over $M$ (the theory also studies the case of an $SO(3)$-bundle, but we shall not discuss this). Such bundles are classified by the integer $c = c_2(P) \in H^4(M; \mathbb{Z}) \cong \mathbb{Z}$. If $A$ is a connection on $P$, then its curvature $F_A$ is a 2-form with values in $adP$. It is natural to consider the energy

$$\int_M |F_A|^2, \quad \text{the Yang-Mills functional},$$

and to try to minimize this energy. In this sense, Yang-Mills theory is a nonabelian analogue of Hodge theory, which attempts to minimize the norm of a form representing a fixed cohomology class. The critical points of the Yang-Mills functional are given by the Euler-Lagrange equations $D_A F_A = 0, D_A * F_A = 0$. Here $D_A$ is the differential operator on sections of $\Omega^2(M; adP)$ associated to $A$, the equation $D_A F_A = 0$ is the Bianchi identity which is always satisfied, and $*$ is the Hodge $*$-operator from $\Omega^n(M; adP)$ to $\Omega^{n-2}(M; adP)$. In case $\dim M = 4$, $*F_A$ is also a section of $\Omega^2(M; adP)$. Thus if $*F_A$ is a scalar multiple of $F_A$, then the Bianchi identity also implies that $D_A * F_A = 0$. Since $** = Id$, the only possibilities are $*F_A = F_A$, in which case we say that $A$ is a self-dual connection, or $*F_A = -F_A$, in which case we say that $A$ is an anti-self-dual (ASD) connection. These connections, if they exist, are absolute minima for the Yang-Mills functional.

For a 4-manifold $M$, we can break up the space of sections $\Omega^2(M; adP)$ into its $+1$ and $-1$ eigenspaces for the Hodge $*$-operator:

$$\Omega^2(M; adP) = \Omega^2_+(M; adP) \oplus \Omega^2_-(M; adP).$$

Write $F^+_A$ for the component of $F_A$ lying in $\Omega^2_+(M; adP)$, so that $A$ is ASD if and only if $F^+_A = 0$ if and only if $F_A \in \Omega^2_-(M; adP)$. Given the bundle $P$ and the metric $g$, the set of all anti-self-dual connections, modulo the group of symmetries of $P$, forms a finite-dimensional oriented smooth manifold $M(P, g)$, at least for a generic metric $g$. While $M(P, g)$ is not in general compact, it can be compactified to a stratified space $X(P, g)$ which carries a fundamental class $[X(P, g)]$. Now morally speaking there is a tautological $SU(2)$-bundle $P$ over $M \times M(P, g)$. Taking the slant product with $c_2(P)$ induces a homomorphism $\mu : H_*(M; \mathbb{Z}) \to H^{4-*}(M(P, g); \mathbb{Z})$. Again morally speaking the classes $\mu(\alpha)$ extend to classes in $H^{4-*}(X(P, g); \mathbb{Z})$, and appropriate combinations can then be evaluated on the fundamental class $[X(P, g)]$ to produce polynomial invariants for $M$, which will be independent of $g$ as long as $b^2_+(M) > 1$.

Why should Donaldson theory give more information about the smooth topology than the standard homotopy or homeomorphism invariants? This is a deep
mystery, but one could attempt to give a partial answer, or perhaps an ideologi-
ical underpinning to Donaldson theory, by making the following points about the
structure of the ASD equations and the corresponding moduli spaces:

(1) The groups SU(2) and SO(3) are nonabelian, and so the associated non-
abelian Hodge theory might be able to reveal more structure than the usual
abelian Hodge theory, which only gives homotopy-theoretic information
about $M$.

(2) The ASD equations are conformally invariant.

(3) As a consequence of the conformal invariance, which allows for bubbling off,
the ASD moduli spaces are almost never compact, and the compactification
involves the underlying manifold $M$ in an interesting and nontrivial way.
(Sometimes, however, even compact moduli spaces can contain the essential
information in Donaldson theory.)

(4) The theory works best for simply connected 4-manifolds. One can define
the polynomial invariants for arbitrary 4-manifolds, but the definiton is
much more difficult\cite{29}, and there are ideological grounds for concen-
trating on the class of simply connected 4-manifolds. (Arbitrary 4-manifolds are
technically unknowable, because every finitely presented group is the funda-
mental group of a smooth 4-manifold, and on the other hand, modulo the
three-dimensional Poincaré conjecture, dimension 4 is the first dimension
where we encounter really interesting simply connected manifolds.)

Now suppose that $M$ is a Kähler surface $X$, with Kähler form $\omega \in \Omega^{1,1}(X)$. Then it is an easy calculation using the Hodge identities that

\[
\Omega^2_{\mathbb{C}}(X; \text{ad } P) \otimes \mathbb{C} = \Omega^{0,2}(X; \text{ad } P) \oplus \Omega^{2,0}(X; \text{ad } P) \oplus \Omega^2(X; \mathbb{C}) \cdot \omega;
\]
\[
\Omega^2_{\mathbb{C}}(X; \text{ad } P) \otimes \mathbb{C} = (\omega^1 \cap \Omega^{1,1}(X; \text{ad } P)),
\]

where this last equation means that the elements in $\Omega^2_{\mathbb{C}}(X; \text{ad } P) \otimes \mathbb{C}$ are $(1,1)$-
forms pointwise orthogonal to $\omega$. Let $F_A^{0,2}$ be the projection of the 2-form $F_A$ onto
the subspace $\Omega^{0,2}(X; \text{ad } P)$, and similarly for $F_A^{1,1}$. Thus the equation $F_A^{+} = 0$ is
equivalent to the two equations

\[
F_A^{0,2} = 0;
\]
\[
F_A^{1,1} \text{ is pointwise orthogonal to } \omega.
\]

We can also write this last equation as $\Lambda F_A^{1,1} = 0$, where the operator $\Lambda$ is con-
traction against $\omega$. The first equation says that the projection $A^{0,1}$ of $A$ onto the
(0,1)-forms defines the $\bar{\partial}$-operator for a holomorphic structure, with holomorphi-
cally trivial determinant, on the complex 2-plane bundle associated to $P$. This is an
easy integrability result, which follows for example from the Newlander-Nirenber-
gh theorem. The second equation, according to the Kobayashi-Hitchin conjecture,
proved by Donaldson\cite{7} in the case of Kähler surfaces and by Uhlenbeck-Yau\cite{40}
for general Kähler manifolds, is equivalent to saying that the holomorphic vector
bundle structure on $V$ is $\omega$-stable. In the case of a rank two holomorphic vector bun-
dle $V$ on a surface $X$ with $c_1(V) = 0$, this condition amounts to the following: if $L$
is a line bundle and there is a nonzero holomorphic map $L \to V$ (not necessarily an
inclusion on each fiber), then $c_1(L) \cdot \omega < 0$. In case $\omega$ is the Kähler form of a Hodge
metric corresponding to an ample divisor $H$, then $\omega$-stability exactly corresponds to stability with respect to $H$, a notion introduced by Mumford and later Takemoto in order to construct moduli spaces on curves and higher-dimensional projective varieties. While not completely successful in constructing compact moduli spaces (one needs instead to consider all Gieseker semistable torsion free sheaves), stability is a fundamental non-degeneracy condition for vector bundles on a projective variety. As such, it has numerical consequences for $V$, in the form of Bogomolov’s inequality. In the case of $V$ of rank two and trivial determinant, this inequality simply reads $c_2(V) \geq 0$, and is an easy consequence of the existence of an ASD metric on $V$. Moreover, again by using the existence of an ASD metric, the case $c_2(V) = 0$ corresponds to the case where $V$ is flat, in other words associated to an irreducible representation of $\pi_1(X)$ in $SU(2)$. But not every bundle which satisfies the numerical condition $c_2 \geq 0$ is automatically stable, so that there are also more subtle, not strictly numerical consequences of stability which are necessary in order to construct separated moduli spaces.

In any case, the connection between ASD connections and stable holomorphic vector bundles leads to some calculation of polynomial invariants by algebro-geometric methods, and suggests that there is a link between the algebraic geometry of a Kähler surface and its 4-manifold topology. Of course, a little experience shows that the moduli spaces of vector bundles on algebraic surfaces are very subtle invariants to calculate. Already for $\mathbb{P}^2$, they become quite complicated when $c_2$ grows, and as the complexity of the surface increases, the complexity of the corresponding moduli spaces also seems to increase.

To return to the general story of Donaldson theory, one can ask if the Donaldson polynomial invariants can detect special cohomology classes on $M$ (where $b_2^+(M) \geq 3$). One way to do this is the following: under certain circumstances, the Donaldson polynomials lie in the subring $\mathbb{C}[q_M, k]$ of the symmetric algebra, where $q_M$ is the symmetric polynomial corresponding to the intersection form on $M$ and $k$ is a class in $H^2(M; \mathbb{Z})$. If there exists a polynomial of this type not lying in $\mathbb{C}[q_M]$, then it is an easy algebraic argument that the class $k$ is preserved up to $\pm 1$ by orientation-preserving self-diffeomorphisms of $M$. Another circumstance where Donaldson theory can find special classes is the following: suppose that there is a smoothly embedded 2-sphere in $M$ whose associated cohomology class $\alpha$ satisfies $\alpha^2 = -1$; for example this is the case for the classes of exceptional curves. Then the Donaldson polynomials have a rather restricted form. If there exists a nonzero Donaldson invariant for $M$, then the possible such classes $\alpha$ represented by smoothly embedded 2-spheres of self-intersection $-1$ are either equal up to $\pm 1$ or orthogonal, and in particular there are only finitely many such classes. For a description of these and related results, see [15]. After a period involving a considerable amount of difficult calculation of Donaldson polynomials, by algebro-geometric and other means, Kronheimer and Mrowka proved a deep structure theorem for a large class of 4-manifolds, those of simple type. While we shall not give a precise definition, a 4-manifold $M$ with $b_2^+(M) \geq 3$ is of simple type if evaluating the polynomial invariant on the 4-dimensional class $\mu(\text{pt})$ essentially just gives back a polynomial invariant of a smaller-dimensional moduli space. Large classes of 4-manifolds have simple type, and there is no known example of a 4-manifold with $b_2^+ \geq 3$ which does not have simple type. Kronheimer and Mrowka [24, 25] showed that, if $M$ is a simply connected 4-manifold of simple type, then the Donaldson polynomials can be expressed via certain recurrence relations as polynomials involving certain
basic classes $\kappa_i \in H^2(M; \mathbb{Z})$ and rational numbers $a_i$. Here the $\kappa_i$ are characteristic elements of $H^2(M; \mathbb{Z})$, i.e. for all $\alpha \in H^3(M; \mathbb{Z})$, $\alpha^2 \equiv \alpha \cdot \kappa \mod 2$.

3. Definition of the Seiberg-Witten invariants

One basic motivation which led to the definition of the Seiberg-Witten invariants was the attempt to find an a priori description of the Kronheimer-Mrowka basic classes. To define the Seiberg-Witten invariants, we need to recall the definition of a Spin$^c$ structure on $M$. Let $M$ be an oriented Riemannian 4-manifold (actually we can define a Spin$^c$ structure in every dimension). Then the tangent bundle of $M$ corresponds to a principal $SO(4)$-bundle. Now $SO(4)$ has a unique double cover $\text{Spin}(4)$, which by one of the coincidences of Lie group theory in low dimensions is isomorphic to $SU(2) \times SU(2)$, and one can ask if the tangent bundle of $M$ lifts to a Spin$^c(4)$ bundle. The answer is that such a lift exists if and only if the second Stiefel-Whitney class $w_2(M)$ is zero, and in case $M$ is simply connected (or more generally if $H^2(M; \mathbb{Z})$ has no 2-torsion) this condition is equivalent to assuming that $H^2(M; \mathbb{Z})$ has an even intersection form. In this case a lift of $TM$ to $\text{Spin}(4)$ is called a Spin structure on $M$. Of course, most 4-manifolds do not have a Spin structure. However, if one does exist, then from the isomorphism $\text{Spin}(4) \cong SU(2) \times SU(2)$, there are two associated rank two complex vector bundles $\mathbb{S}^+, \mathbb{S}^-$ with trivial determinant. A fundamental fact is that the Levi-Civita connection on $TM$ induces a differential operator, the Dirac operator $\bar{\theta}: \mathbb{S}^+ \to \mathbb{S}^-$. Here $\bar{\theta}$ is a first order elliptic formally self-adjoint operator.

If there is no Spin structure, there is still a related construction. Define

$$\text{Spin}^c(4) = \text{Spin}(4) \times U(1)/\{\pm 1\},$$

where the group $\{\pm 1\}$ acts diagonally on both factors. While these groups can be defined in general, in dimension 4 we also have the isomorphism

$$\text{Spin}^c(4) \cong \{(A, B) \in U(2) \times U(2) : \det A = \det B\}.$$

By definition a Spin$^c$ structure on $M$ is a lift of the $SO(4)$-bundle $TM$ to a bundle with structure group Spin$^c(4)$. There is an exact sequence

$$\{1\} \to U(1) \to \text{Spin}^c(4) \to SO(4) \to \{1\},$$

and from this exact sequence it is straightforward to see that $M$ has a Spin$^c$ structure if and only if the image of $w_2(M)$ under the Bockstein homomorphism $H^2(M; \mathbb{Z}/2\mathbb{Z}) \to H^3(M; \mathbb{Z})$ is zero, or equivalently if $w_2(M)$ is the mod 2 reduction of an integral class. In fact, this condition is satisfied by every 4-manifold $M$. In case $M$ is an almost complex surface $X$, a natural lift of $w_2(X)$ is $c_1(X)$. The set of all Spin$^c$ structures on $M$ is a principal homogeneous space over $H^2(M; \mathbb{Z})$.

Let $\xi$ be a Spin$^c$ structure on $M$. Since Spin$^c(4) \subset U(2) \times U(2)$, there are two associated complex 2-plane bundles $\mathbb{S}^+(\xi), \mathbb{S}^-(\xi)$, and det $\mathbb{S}^+(\xi) = \det \mathbb{S}^-(\xi)$. We call the complex line bundle $L = \det \mathbb{S}^+(\xi) = \det \mathbb{S}^-(\xi)$ the determinant of $\xi$ and will write it as det $\xi$. Note that if $\xi$ and $\xi'$ differ by $\alpha \in H^2(M; \mathbb{Z})$, then $\mathbb{S}^+(\xi') = \mathbb{S}^+(\xi) \otimes L_\alpha$, where $L_\alpha$ is a complex line bundle such that $c_1(L_\alpha) = \alpha$, and $c_1(\det \xi') = c_1(\det \xi) + 2\alpha$. In particular, the mod two reduction of $c_1(\det \xi)$ depends only on $M$, and it is easy to see that this reduction is exactly $w_2(M)$. Choosing a
connection \( A \) on \( \det \xi \) gives a Dirac operator \( \bar{\partial}_A : S^+(\xi) \to S^-(\xi) \), which is again a first order elliptic formally self-adjoint operator. The index of \( \bar{\partial}_A \) is given by the Atiyah-Singer index theorem:

\[
\text{index} \bar{\partial}_A = \frac{1}{8} (c_1(\det \xi)^2 - \sigma(M)),
\]

where \( \sigma(M) = b_2^+(M) - b_2^-(M) \) is the signature of \( M \).

Now suppose that \( M = X \) is an almost complex 4-manifold. Then the structure group of \( M \) reduces to \( U(2) \), and we seek a lift of the inclusion \( U(2) \subset SO(4) \) to the group \( \text{Spin}^c(4) \subset U(2) \times U(2) \). One checks that a natural lift is given by

\[
T \in U(2) \mapsto \left( \begin{pmatrix} \text{Id} & 0 \\ 0 & \det T \end{pmatrix}, T \right) \in U(2) \times U(2).
\]

For this lift, if \( \xi_0 \) denotes the corresponding \( \text{Spin}^c \) structure, we have \( S^+(\xi_0) = \Omega^0(X) \oplus K_X^{-1} = \Omega^0(X) \oplus \Omega^2(X) \) and \( S^-(\xi_0) = TX = \Omega^{1,1}(X) \), where \( K_X \) is the canonical bundle of the almost complex structure. Thus every \( \text{Spin}^c \) structure \( \xi \) on \( X \) is given by choosing a complex line bundle \( L_0 \) on \( M \), corresponding to a class in \( H^2(X; \mathbb{Z}) \), and twisting \( \xi_0 \) and \( S^+(\xi_0) \) by \( L_0 \). In this case

\[
S^+(\xi) = \Omega^0(L_0) \oplus \Omega^{0,2}(L_0), \quad S^-(\xi) = \Omega^{0,1}(L_0),
\]

and \( \det \xi = L_0^2 \otimes K_X^{-1} \), or equivalently \( L_0 \) is a square root of \( K_X \otimes L \), where \( L = \det \xi \) is a characteristic line bundle (the mod two reduction of \( c_1(L) \) is \( w_2(X) \)). There are two obvious choices (not necessarily distinct) for a \( \text{Spin}^c \) structure on \( X \): we can choose \( L_0 \) to be trivial and \( L = K_X^{-1} \) or \( L_0 = K_X \) and \( L = K_X \). We will refer to these two choices as the trivial \( \text{Spin}^c \) structures on \( X \). Finally, in case \( X \) is Kähler, the operator \( \bar{\partial}_A \) corresponds under this isomorphism to \( \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^0) \). Here \( \bar{\partial}_A \) is the \( \partial \)-operator on \( X \) coupled with the connection induced on \( L_0 \) by \( A \), which can be defined for every \( C^\infty \) complex line bundle \( L \) (not necessarily holomorphic) and connection \( A \) on \( L \). This is essentially a calculation in Euclidean space, based on the fact that a Hermitian metric is Kähler if and only if it looks like the standard metric to second order.

We can now give the Seiberg-Witten equations for a smooth Riemannian 4-manifold \( M \): let \( \xi \) be a \( \text{Spin}^c \) structure on \( M \), with \( \det \xi = L \). Then the equations, for a \( C^\infty \) section \( \psi \) of \( S^+(\xi) \) and a connection \( A \) on \( L \) are as follows:

\[
\bar{\partial}_A \psi = 0;
\]

\[
F^+_A = \psi \otimes \psi^* - \frac{|\psi|^2}{2} \text{Id}.
\]

Here the first equation can be paraphrased by saying that \( \psi \) is a harmonic spinor. The second equation means the following: using the metric to identify \( S^+(\xi) \) with its dual, both \( \psi \otimes \psi^* \) and \( \frac{|\psi|^2}{2} \text{Id} \) are sections of \( S^+(\xi) \otimes (S^+(\xi))^* = \text{Hom}(S^+(\xi), S^+(\xi)) \). The difference has trace zero. Now using Clifford multiplication on \( S^+(\xi) \), one can identify the traceless endomorphisms from \( S^+(\xi) \) to itself with \( \Omega^2_+(M) \otimes \mathbb{C} \), in such a way that \( \psi \otimes \psi^* - \frac{|\psi|^2}{2} \text{Id} \) becomes a purely imaginary self-dual 2-form, and we
can compare it with $F_A^+$. There is also a symmetry group, the gauge group of automorphisms of $\xi$ (which are understood to induce the identity on the tangent bundle). Up to a factor of 2, this group is just the group of automorphisms of the line bundle $L$ and hence it is abelian. The quotient space of the set of solutions to the Seiberg-Witten equations by the gauge group is the Seiberg-Witten moduli space. It is a compact space locally modeled on a real analytic variety. A solution $(\psi, A)$ of the Seiberg-Witten equations is reducible if $\psi = 0$, in which case $A$ is necessarily a flat connection. For simplicity we shall assume henceforth that there are no reducible solutions to the Seiberg-Witten equations. Provided that $b_2^+ (M) > 0$, for generic perturbations of the equations, the Seiberg-Witten moduli space becomes a smooth manifold $\mathcal{M}(\xi)$ of dimension

$$\frac{1}{4} \left( L^2 - (2\chi(M) + 3\sigma(M)) \right) ,$$

and in particular $\mathcal{M}(\xi)$ is empty whenever this integer is negative. An orientation on $\mathcal{M}(\xi)$ can be given by choosing orientations on $H^0 (M; \mathbb{R}), H^1 (M; \mathbb{R})$, and $H^2_+ (M; \mathbb{R})$. Thus for example once we have chosen such an orientation, in case $\mathcal{M}(\xi)$ has dimension zero it is a signed collection of points, and we can add up these signs to obtain an integer. Changing the orientation simply changes the sign of this integer. In case $\mathcal{M}(\xi)$ has dimension greater than zero, there is a procedure for obtaining an integer invariant as well: there is a tautological complex line bundle $\mathcal{L}$ on $M \times \mathcal{M}(\xi)$, and one can use slant product with $c_1(\mathcal{L})$ to obtain a class $\mu \in H^2 (\mathcal{M}(\xi); \mathbb{Z})$ whose top power can be integrated against the fundamental class of $\mathcal{M}(\xi)$ to give an integer. As long as $b_2^+ (M) > 1$, the invariants obtained are independent of the choice of the Riemannian metric $g$, and in case $b_2^+ (M) = 1$, there is a formula for the dependence on the metric.

Thus, assuming for simplicity that $b_2^+ (M) > 1$ and that we have oriented $H^0 (M; \mathbb{R}), H^1 (M; \mathbb{R})$, and $H^2_+ (M; \mathbb{R})$, we have assigned an integer $SW(\xi)$ to every Spin$^c$ structure $\xi$ on $M$. A compactness result shows that there are only finitely many $\xi$ such that $SW(\xi) \neq 0$. If there exists a Riemannian metric $g$ on $M$ with positive scalar curvature, then in fact $SW(\xi) = 0$ for every Spin$^c$ structure $\xi$ on $M$. A Spin$^c$ structure $\xi$ such that $SW(\xi) \neq 0$ will be called a basic Spin$^c$ structure, and if $L = \det \xi$ where $\xi$ is a basic Spin$^c$ structure, then we will call $L$ a basic class. Witten has conjectured that, in case $M$ is of simple type, then the basic classes in this sense are exactly the Kronheimer-Mrowka basic classes $\kappa_i$, and has also conjectured the precise form of the rational numbers $\alpha_i$ in the Kronheimer-Mrowka formula: up to a universal factor depending only on the homotopy type of $M$, they are just $\sum_{\det \xi = \kappa_i} SW(\xi)$. Moreover, there is a corresponding notion of simple type in Seiberg-Witten theory, that the expected dimension of the moduli space is always 0, and it is natural to expect that a 4-manifold is of simple type in the sense of Kronheimer-Mrowka if and only if it is of simple type in the sense of Seiberg-Witten theory.

Let us make the following points about Seiberg-Witten theory:

1. Seiberg-Witten theory is essentially an abelian theory. Although the group Spin$^c(4)$ is not abelian, the requirement that we only consider Spin$^c(4)$-bundles lifting the tangent bundle puts all of the extra degrees of freedom in the choice of a complex line bundle on $M$. In particular the gauge group associated to Seiberg-Witten theory is abelian and the main calculations involve the curvature of complex line bundles, as opposed to rank two complex
vector bundles. Thus, in Yang-Mills theory, curvature calculations do not give much useful information, whereas in Seiberg-Witten theory they give the crucial compactness results as well as vanishing in the case of positive scalar curvature.

(2) The Seiberg-Witten equations are not conformally invariant.

(3) In connection with (1) and (2), no bubbling phenomena occur and the moduli spaces are compact.

(4) The fundamental group seems to play no major role in the definition or the computation of the invariants.

Thus the ideological underpinnings of Donaldson theory do not seem to address the fundamental mystery of dimension 4. Indeed, the Seiberg-Witten equations do not have anything like the natural interpretation of the Yang-Mills equations, and there does not as yet exist a mathematical understanding of why the very complicated information of Donaldson theory should condense into the much more manageable information of Seiberg-Witten theory. (A strategy for proving Witten’s conjecture has been proposed by Pidstrigach and Tyurin. This strategy gives a reason why there should be a link between the two theories, but it does not, at least to me, suggest why the Seiberg-Witten equations should be the right way to attack 4-manifold topology.)

4. The case of a Kähler surface

In case $X$ is a Kähler surface, then $S^+(\xi) = \Omega^0(L_0) \oplus \Omega^{0,2}(L_0)$, we can write the Dirac operator in terms of the $\bar{\partial}$-operator, and the Clifford multiplication used to identify the traceless endomorphisms from $S^+(\xi)$ to itself with $\Omega_2^+(M) \otimes \mathbb{C}$ can be identified with wedge product or its adjoint on $\Omega^0(L_0) \oplus \Omega^{0,2}(L_0)$. A spinor field $\psi$ can be written in terms of its components $(\alpha, \beta) \in \Omega^0(L_0) \oplus \Omega^{0,2}(L_0)$. Working out the Seiberg-Witten equations in this case gives the equations:

$$\bar{\partial}_A \alpha + \bar{\partial}^*_A \beta = 0;$$
$$F^0_2 = \bar{\partial}A^{0,1} = \bar{\alpha} \beta;$$
$$(F^+_A)^{1,1} = \frac{i}{2} (|\alpha|^2 - |\beta|^2) \omega.$$  

(Here the metric $g$ defines a Hermitian metric on $L_0$ and thus a conjugate linear isomorphism $\Omega^0(L_0) \rightarrow \Omega^0(L_0^*)$, and $\bar{\alpha}$ denotes the image of $\alpha$ under this isomorphism.) We assume that $(\psi, A)$ is an irreducible solution to the Seiberg-Witten equations. To analyze the solutions, we argue as follows: applying $\bar{\partial}^*_A$ to the first equation and plugging in the second, we get

$$\bar{\partial}^2_A \alpha + \bar{\partial}_A \bar{\partial}^*_A \beta = 0 = F^0,2 \alpha + \bar{\partial}_A \bar{\partial}^*_A \beta = |\alpha|^2 \beta + \bar{\partial}_A \bar{\partial}^*_A \beta.$$  

Taking the inner product of this expression (namely 0) with $\beta$ and integrating over $X$ gives

$$0 = \int_X (\langle |\alpha|^2 \beta, \beta \rangle + \langle \bar{\partial}_A \bar{\partial}^*_A \beta, \beta \rangle)$$
$$= \int_X |\alpha|^2 |\beta|^2 + \|\bar{\partial}^*_A \beta\|^2.$$
Since both terms are positive, we must have \(|\alpha|^2|\beta|^2 = 0\) (pointwise) and \(\|\bar{\partial}_A \beta\|^2 = 0\), so that \(\bar{\partial}_A \beta = 0\). Now since \(|\alpha|^2|\beta|^2 = 0\), the product \(\bar{\alpha} \beta = 0\) as well, so that \(F_A^{0,2} = 0\). Thus as in the Yang-Mills case \(A^{0,1}\) defines a holomorphic structure on \(L\) and so on \(L_0\). Since \(\bar{\partial}_A \beta = 0\), the first equation implies that \(\bar{\partial}_A \alpha = 0\), and so \(\alpha\) is a holomorphic section of \(L_0\) and \(* \beta\) is a holomorphic section of \(K_X \otimes L_0^{-1}\). On the other hand, since \(\bar{\alpha} \beta = 0\), \(\beta\) must vanish on the open set \(\{\alpha(x) \neq 0\}\). If this open set is nonempty, i.e. if \(\alpha \neq 0\), then \(\beta\) is identically zero, and conversely if \(\beta \neq 0\) then \(\alpha\) is identically zero. For simplicity let us assume that \(\alpha \neq 0\). In this case, since \(L_0\) has the nonzero holomorphic section \(L_0\), we can write \(L_0 = O_X(D_0)\) for an effective curve \(D_0\) on \(X\). The first two equations for \((\alpha, \beta, A)\) just say that \(\beta = 0\), that \(A\) defines a holomorphic structure on \(L_0\), and that \(\alpha\) is a nonzero holomorphic section in the structure defined by \(A\). In other words, the \(A\) and \(\alpha\) give us effective curves \(D_0\) on \(X\) whose associated cohomology class is equal to \(c_1(L_0)\). Now the set of all curves on \(X\) with a fixed cohomology class is parametrized by a projective variety, the Hilbert scheme of \(X\) (with an appropriate choice of Hilbert polynomial). The Hilbert scheme is an interesting and often highly nontrivial moduli space associated to \(X\), and so as algebraic geometers we can get to work on this problem.

However, there is a third equation for \(A\) and \(\alpha\) which is part of the Seiberg-Witten equations, namely the equation \((F_A^+)^{1,1} = \frac{i}{2} (|\alpha|^2 - |\beta|^2) \omega\). This equation for the \((1, 1)\) part of the curvature is analogous to the equation \(AF_A^{1,1} = 0\) in Yang-Mills theory, and should be thought of as a stability type condition for the divisor \(D_0\). As we shall see, however, the consequence of stability in this case is strictly a numerical one. In case \(\beta = 0\), the equation becomes \((F_A^+)^{1,1} = \frac{i}{2} |\alpha|^2 \omega\). In particular since \(c_1(L)\) is represented by the form \(\frac{i}{2\pi} F_A\), it follows that \(c_1(L) \cdot \omega < 0\) in real cohomology. Thus we are led to the necessary conditions for \((\alpha, 0, A)\) to be a solution to the Seiberg-Witten equations:

1. \(A\) defines a holomorphic structure on \(L\);
2. \(\alpha\) defines a holomorphic section of \(L_0 = (K_X \otimes L)^{1/2}\);
3. \(c_1(L) \cdot \omega < 0\).

(In case \(\alpha = 0, \beta \neq 0\), the appropriate change is: \(\beta\) defines a holomorphic section of \(K_X \otimes L_0^{-1} = (K_X \otimes L^{-1})^{1/2}\) and \(c_1(L) \cdot \omega > 0\), and these conditions are Serre dual to the previous ones.) There is also the condition that the expected dimension of the moduli space is nonnegative, which for a complex surface is simply \(L^2 \geq K_X^2\), because \(2\chi(X) + 3\sigma(X) = K_X^2\) by the Hodge index theorem and Noether’s formula.

In fact, these necessary conditions are also sufficient. An easy calculation with the Kähler identities shows that the existence of a solution involves solving a PDE of the form

\[\Delta h - fe^{h/2} - g = 0\]

for the unknown function \(h\), where \(f\) and \(g\) are given \(C^\infty\) functions on \(X\) such that \(f \geq 0\) pointwise, where \(f \neq 0\), and \(f \neq f\), and \(\Delta\) is the negative definite Laplacian on \(X\). This vortex equation has a unique \(C^\infty\) solution, by a result of Kazdan-Warner [20] (first exploited in gauge theory by Bradlow [4]). The analysis in the proof of this result is nontrivial, but is much easier than that used in the proof of the theorem of Donaldson and Uhlenbeck-Yau linking ASD connections and stable bundles on a Kähler surface. In this way, the part of the Seiberg-Witten
moduli space corresponding to $\alpha \neq 0, \beta = 0$ (for the unperturbed equations) can be identified as a set with the Hilbert scheme of all effective curves $D_0$ on $X$ such that, if we set $L = \mathcal{O}_X(2D_0) \otimes K_X^{-1}$, then $L$ satisfies the numerical condition $c_1(L) \cdot \omega < 0$. There is a similar description in case $\alpha = 0, \beta \neq 0$, which is Serre dual to the first one. Finally we should require that $L^2 \geq K_X^2$, since otherwise the moduli space will perturb away to the empty set and the invariants we define will all be zero.

In this way we have identified the (unperturbed) Seiberg-Witten moduli space with the union of finitely many components of the Hilbert scheme of curves on $X$. In fact, both spaces have additional structure: the Seiberg-Witten moduli space is locally modeled on a real analytic space, and the Hilbert scheme has a scheme structure and so is a complex analytic space (possibly nonreduced). One can show that the two spaces are isomorphic as real analytic spaces [17]. However, as we shall see in a minute, aside from a few very special cases this result is more of a virtual theorem than an actual method of computation.

We can now try to identify the Seiberg-Witten basic classes and then the moduli spaces for various algebraic surfaces. For an algebraic geometer, the end result is disappointing: the geometric interest of the basic classes and the moduli space is essentially inversely proportional to the interest of the surface in question. Thus, if $X$ is a minimal surface of general type, the basic classes are exactly $K_X^{\pm 1}$, corresponding to the trivial Spin$^c$ structures on $X$. Of course, it is exactly this fact which shows that the classes $\pm [K_X]$ are preserved under diffeomorphisms. In fact, the following is an easy consequence of the Hodge index theorem:

**Theorem.** Let $X$ be a minimal algebraic surface of general type, let $\omega$ be a Kähler form on $X$ and let $L$ be a holomorphic line bundle on $X$ such that:

1. $L^2 \geq K_X^2$;
2. $\omega \cdot L < 0$;
3. There is an effective divisor $D_0$ such that $\mathcal{O}_X(2D_0) = K_X \otimes L$.

Then $L = K_X^{-1}$ and $D_0$ is the trivial divisor.

**Corollary.** If $X$ is a minimal algebraic surface of general type, then the only basic classes on $X$ are $\pm [K_X]$, with the trivial Spin$^c$ structures.

**Proof of the theorem.** We write $K_X$ and $L$ in the additive notation of divisors, and will just consider the case corresponding to $\alpha \neq 0$. Since $\frac{1}{2}(K_X + L)$ is effective, $\omega \cdot (K_X + L) \geq 0$, and likewise $K_X \cdot (K_X + L) \geq 0$ since $K_X$ is nef. Thus $K_X \cdot L \geq -K_X^2$. On the other hand, since $\omega \cdot L < 0$, there exists a $t \geq 1$ such that $\omega \cdot (K_X + tL) = 0$. By the Hodge index theorem, $(K_X + tL)^2 \leq 0$. Thus, since $t \geq 1$ and $L^2 \geq K_X^2$,

$$
(K_X + tL)^2 = K_X^2 + 2t(K_X \cdot L) + t^2 L^2 \\
\geq K_X^2 - 2tK_X^2 + t^2 K_X^2 = (1 - 2t + t^2)K_X^2 = (1 - t)^2 K_X^2 \geq 0.
$$

So $(K_X + tL)^2 = 0$, which can only happen if $t = 1$ and $K_X + L$ is numerically trivial. Thus $D_0 = \frac{1}{2}(K_X + L)$ is numerically trivial, and it is also effective, so that it is zero. It follows that the corresponding line bundle $L_0 = \mathcal{O}_X(D_0)$ is the trivial line bundle, and so $L = K_X^{-1}$ with the trivial Spin$^c$ structure. \qed

We note that the value of the function $SW$ on $\pm [K_X]$ is $\pm 1$, which follows formally since the trivial divisor is the unique smooth point of its moduli space. In particular, $\pm [K_X]$ really are basic classes.

Similar arguments show the following:
**Theorem.** Let $X$ be a minimal algebraic surface with $\kappa(X) \geq 0$ and $K_X^2 = 0$. Then the basic classes on $X$, as elements of rational cohomology, are of the form $rK_X$ with $r \in \mathbb{Q}$ and $|r| \leq 1$. Moreover the case $r = \pm 1$ does occur, and in this case the only corresponding $\text{Spin}^c$ structures are the trivial ones on $\pm[K_X]$.

In the elliptic case, the invariant need not take on the value $\pm 1$. The calculation of the value of the invariant is given in [41] in case $b_2^+ \geq 3$ and in [5] and [17] in general. In case $X$ has at most two multiple fibers, the divisibility of $[K_X]$ and the largest $r \in \mathbb{Q}$ with $r \neq 1$ such that $r[K_X]$ is again a basic class will then determine the possible multiplicities. In this way we can classify elliptic surfaces up to diffeomorphism.

For a nonminimal surface $X$, there is the following result, which is easy to check directly from the holomorphic criteria for Seiberg-Witten classes:

**Theorem.** Let $X$ be a surface of general type with $b_2^+(X) \geq 3$, let $\rho : X \to X_0$ be the blowdown to the minimal model, and let $K_0$ be the preimage in $H^2(X;\mathbb{Z})$ of the canonical class of $X_0$ and $E_1, \ldots, E_r$ be the exceptional curves of $\rho$. Then the basic classes for $X$ are exactly the classes $\pm K_0 \pm E_1 \pm \cdots \pm E_r$, and the invariant takes the value $\pm 1$ on each of these.

Similar results hold for elliptic surfaces or in case $b_2^+ = 1$, in which case the statement means that the classes above are the basic classes for a particular chamber. It follows that, if $b_2^+(X) \geq 3$, then $X$ is always of simple type in the Seiberg-Witten sense. Using the above theorem, one can show that the classes $K_0$ and the $E_i$ are $C^\infty$ invariants up to sign and permutation of the $E_i$. In fact this is clear up to 2-torsion for the case $b_2^+ \geq 3$, and follows by an analysis of the chamber structure in case $b_2^+ = 1$ [5], [16]. In case $b_2^+(X) \geq 3$ and $X$ is minimal, we see that the Seiberg-Witten basic classes are equal to $\pm[K_X]$ if $X$ is of general type, and are of the form $r[K_X]$ where $r \in \mathbb{Q}$ in general. Thus, if Witten’s conjecture on the form of the Donaldson polynomials is true, then the Donaldson polynomials of a minimal algebraic surface $X$ with $b_2^+(X) \geq 3$ always lie in $\mathbb{C}[q_X, k]$, where $k = [K_X]$, and actually involve $k$, as long as $k$ is nonzero in rational cohomology.

There remains the case where $X$ is rational or ruled. In this case $b_2^+(X) = 1$, and the invariants depend on a chamber structure. It is easy to see from the holomorphic criterion for Seiberg-Witten invariants or from the existence of metrics with positive scalar curvature on $X$ that there is always a chamber on $X$ for which all of the invariants vanish, i.e. for which there are no basic classes. On the other hand, there are chambers where the invariants are nonzero. Some of these values correspond to moduli spaces where the expected dimension of the Seiberg-Witten moduli space is nonzero, i.e. $X$ does not necessarily have simple type in case $b_2^+(X) = 1$. For the case of rational surfaces the nonzero values are always $\pm 1$. However, for irrational ruled surfaces the values are more complicated, and the associated moduli spaces involve interesting algebraic geometry. For example, in case $X = C \times \mathbb{P}^1$, the moduli spaces are connected with correspondences on $X$ and tie in with the theory of special divisors on $C$. In case $X$ is a general ruled surface over $C$, the structure of the Seiberg-Witten moduli space involves the study of the Hilbert scheme on $X$ and is related to questions concerning general rank two stable vector bundles over $C$. For a discussion of these results, see [17]. However, while the algebraic geometry involved is nontrivial, Seiberg-Witten theory in this case does not seem to have any consequences for the $C^\infty$ topology of $X$, which can be analyzed by elementary methods.
5. CONCLUDING REMARKS

The ability of the Seiberg-Witten invariants to solve seemingly intractable questions on the smooth topology of algebraic surfaces is a stunning achievement. Of course, like all such achievements, there is also a certain amount of disappointment: the Seiberg-Witten invariants, and therefore presumably also Donaldson invariants, are only able to tell us about the pullback of the canonical class of the minimal model and the exceptional curves (at least for $\kappa(X) \geq 0$), and these classes are the obvious ones. Is this the end of the story as far as gauge theory invariants are concerned? There are surfaces of general type, for example some of the Horikawa surfaces, which are homeomorphic but not deformation equivalent, and which cannot be distinguished by Seiberg-Witten or Donaldson invariants. Perhaps some new multi-monopole invariants will be able to show that such surfaces are not diffeomorphic, perhaps completely new invariants are needed, or perhaps the surfaces are indeed diffeomorphic but not deformation equivalent. Still other questions, such as the problem of understanding the mapping class groups of algebraic surfaces or more general 4-manifolds, or in other words understanding the difference between pseudo-isotopy and isotopy in dimension 4, remain and do not seem accessible either. Beyond these questions, the unruly world of surfaces of general type, for which no reasonable classification is known to exist, is now known to account for only a small part of the even more complicated world of smooth 4-manifolds.

On the positive side, even as the story on gauge theory and algebraic surfaces seems to be coming to a conclusion, or at least a natural pause, the study of symplectic 4-manifolds has opened up dramatically thanks mainly to deep new work of Taubes [39]. Whereas for algebraic surfaces the classification theory leads to a simple description of the Seiberg-Witten invariants, in the case of symplectic 4-manifolds it is the invariants themselves which have led to a deep series of results paralleling the classification theory of algebraic surfaces. This story is still in progress.

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