Tunneling between Luttinger liquids

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We consider the problem of tunneling between spinless 1D Luttinger liquid chains. We show how to map the problem onto the 4-state chiral clock model together with a free boson for the total charge mode. We use a variety of results, some of them exact, from the integrable chiral Potts model and the study of commensurate/incommensurate transitions to deduce the physics of coupled Luttinger liquids. For those intrachain interaction strengths for which interchain tunneling is relevant, we find that it can lead to the formation of symmetric and antisymmetric bands with split Fermi surfaces, depending on the relative strengths of interchain tunneling and interchain interactions. With split Fermi surfaces, interchain transport is coherent. It is not possible to have two gapless Fermi surfaces with the same Fermi momentum when interchain tunneling is relevant. However, interchain interactions can drive the formation of a gap, in which case interchain transport will be incoherent. We comment on the possible relevance of our results to c-axis transport in high-\(T_c\) superconductors.

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I. INTRODUCTION

The problem of tunneling between 1D Luttinger liquids has been a focus of interest and controversy. Much of the interest stems from the unusual phenomenology of the cuprate superconductors. The normal state of these materials exhibits insulating \((d\rho_c/dT < 0)\) or, at best, very poor \(c\)-axis transport coexisting with metallic \(ab\)-plane transport [8]. Moreover, angle-resolved photoemission [9] reveals a Fermi surface with no dispersion along the \(c\)-axis, despite the fact that the hopping matrix elements are quite substantial between the two planes of a Bi2212 bi-layer. Apparently, single-electron motion along the \(c\)-axis is very strongly suppressed while motion in the \(ab\)-plane is not.

It has been suggested that this peculiar situation is characteristic of a putative non-Fermi liquid normal state. 1D Luttinger liquids are the best-understood non-Fermi liquid metals; as a result, they are a natural testing ground for such ideas. Indeed, it has been suggested [10] that the type of behavior seen in the cuprates occurs when two 1D Luttinger liquids are coupled by weak single-electron hopping between the 1D chains.

When intrachain interactions are extremely strong, interchain hopping between 1D Luttinger liquids is irrelevant in the renormalization group sense. Extremely strong means much stronger than can be reached in the infinite-\(U\) Hubbard model, and so strong that the fermion spectral function has no singularity whatsoever at the Fermi surface. In such a case, it is possible to have metallic behavior within a chain but insulating transport between chains. For weak and even moderately strong intrachain interaction strengths, however, interchain hopping is a relevant perturbation. The natural assumption is that this leads to metallic interchain conduction.

Many analyses of [4] have claimed that there is a similar range of intrachain interaction strengths for which interchain tunneling is relevant but does not lead to metallic interchain transport. These claims have been supported by appealing to the spin-boson problem – in which a two-level system is coupled to a bath of bosons – which has a parameter regime in which tunneling between the two levels is relevant, but there are no coherent oscillations [11]. The authors of [11] have claimed that there is a similar range of intrachain interaction strengths for which interchain hopping between 1D Luttinger liquids is relevant but does not lead to metallic interchain transport.

To resolve this issue, we must address whether there
is $c$-axis dispersion and metallic interchain transport. If the minimum energy for adding an electron occurs at two different Fermi surfaces, $\pm k_F^a$ and $\pm k_F^s$, corresponding to the symmetric and antisymmetric bands (i.e. if there are two different wavevectors for the minimum energy excitations which are even and odd under the $Z_2$ symmetry which exchanges the two Luttinger liquids), then there is $c$-axis dispersion. If, in addition, this minimum energy is zero, then interchain transport will be metallic. In this case, various long-wavelength power-law correlation functions will oscillate at wavevector $\pm (k_F^a - k_F^s)$. According to [4], there is no gap, but this splitting does not occur: $k_F^c = k_F^a$.

However, the problem is complicated somewhat by the fact that interchain interactions can drive the formation of a gap. When this occurs, it is possible to have $c$-axis dispersion without metallic interchain transport. In this case, the minimum energy for adding an electron occurs at two different ‘Fermi surfaces’, $\pm k_F^c$ and $\pm k_F^s$, but this energy is non-zero. The opening of such a gap would not explain the absence of a bi-layer splitting in photoemission studies of the cuprates [22], since it occurs ‘on top’ of the bi-layer splitting. On the other hand, it is also possible for the gapped phase to show no $c$-axis dispersion, so that the minimum energy (which is non-zero) for adding an electron occurs at a single ‘Fermi surface’. Clearly, there is no phase transition between these two limits of the gapped phase, but they embody a qualitative distinction which is important for photoemission in the cuprates. The purpose of this paper is to determine which of these possibilities is realized. We do this by diagonalizing intrachain interactions first and then treating interchain tunneling and interchain interactions by non-perturbative methods, some of which are exact.

The main technical difficulty which has prevented the resolution of the problem of tunneling between Luttinger liquids is that tunneling is a chiral perturbation – i.e. the interchain tunneling term transforms non-trivially under the $O(2)$ rotation group of 1 + 1-dimensional Euclidean (in imaginary-time formalism) spacetime. Physically, the tunneling operator is chiral because it mixes right-moving fermions in one chain with right-moving fermions in the other, and similarly for left-moving fermions; it does not mix right-movers with left-movers. This stands in contrast to the case of a non-chiral perturbation – which mixes right-movers with left-movers – for which the 1 + 1-dimensional sine-Gordon model is the archetype. As we know from this exactly soluble models, non-chiral perturbations, if relevant, usually lead to the formation of a gap. Chiral perturbations are poorly understood, however. At the free fermion point, interchain tunneling is a chiral perturbation which is soluble because it is quadratic in fermion operators; it leads to the formation of symmetric and antisymmetric bands with Fermi surfaces which are shifted from the unperturbed Fermi surface. However, the behavior at this trivial point may not be generic. Until now, it has not been possible to appeal to a non-trivial exactly soluble model in which the fate of a relevant chiral perturbation is understood.

In this paper, we consider spinless 1D Luttinger liquids coupled by tunneling. We believe that spin is an inessential complication for the issue at hand; the claims of [4] apply to spinful and spinless Luttinger liquids, as do possible analogies to the spin-boson problem. We map this model to a statistical mechanical model known as the generalized 4-state chiral clock model. The phase structure of this model is fairly well understood [4,5,23]. This chiral clock model has a disordered phase and both commensurate and incommensurate ordered phases. As we discuss below, the incommensurate phase corresponds to an electronic state with $c$-axis dispersion and metallic interchain transport. This picture receives further support from the consideration of a particular limit in which the model becomes the integrable chiral Potts model which, according to exact results [16,17], is in an incommensurate phase.

In the interesting regime in which interchain tunneling is relevant, there are two possible phases. Depending on the relative strength of interchain interactions and interchain hopping, the system can either be in a gapless incommensurate phase or a gapped phase. In the incommensurate gapless phase, the coupled Luttinger chains exhibit $c$-axis dispersion and metallic interchain transport despite strong intrachain non-Fermi liquid properties. It is not possible for the system to be gapless and commensurate; in other words, there is no gapless phase with two identical Fermi surfaces if the tunneling matrix element is non-vanishing. Hence, some of the claims of [4] are incorrect. The gapped phase, on the other hand, may or may not have displaced Fermi surfaces. Thus, it is possible that the mysterious $c$-axis physics of the high-$T_c$ cuprates can be explained by a simple analogy with the 1D Luttinger liquid, but only in the somewhat extreme parameter regimes in which $c$-axis tunneling is irrelevant or a gap forms which is strong enough to preclude $c$-axis dispersion.

In section 2, we introduce the model of coupled Luttinger liquids and give its bosonized formulation. In section 3, we review the results for tunneling between Fermi liquids, and see how incommensurability arises there. In section 4, we map the Luttinger problem to a generalized 4-state chiral clock model. In section 5, we use results from the chiral clock model to understand the physics of coupled Luttinger liquids.

**II. THE MODEL**

The Luttinger liquid is a model of interacting gapless fermions in one spatial dimension. At the microscopic level, it might be described by the lattice Hamiltonian
\[ H_0 = -t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + \text{h.c.}) + \sum_{i,j} V(|i-j|) n_i n_j \]  
(1)

where \(i, j\) label sites along the 1D chains, \(c_i\) annihilates a fermion at site \(i\), and \(n_i = c_i^\dagger c_i\). In the long-wavelength limit, the corresponding Luttinger liquid action is:

\[ S_0 = \int dx \int dt \left( c_{R,L}^\dagger (\partial_t \pm v \partial_x) c_{R,L} - u c_{R,L}^\dagger c_{R,L} \right) \]  
(2)

where the fermion creation operator is

\[ c(x) = e^{-ikF x} c_R(x) + e^{ikF x} c_L(x). \]  
(3)

As is well known (see [18] and references therein), the Luttinger model may be rewritten in terms of a single free boson with action

\[ S_0 = \int dx \int dt \frac{1}{8\pi} \left( (\partial_t \varphi)^2 - v^2 (\partial_x \varphi)^2 \right) \]  
(4)

The field, \(\varphi\), is taken to be an angular variable satisfying the periodicity condition \(\varphi \equiv \varphi + 2\pi r\). Here \(r = \sqrt{g}\), where \(g\) is determined by the interaction strength, \(g = \sqrt{(v-u)/(v+u)}\). The coupling \(g < 1\) for repulsive interactions, \(g > 1\) for attractive interactions. Our results apply to both cases because the Hamiltonian is invariant under the duality symmetry \(g \rightarrow 1/g\). In the long-wavelength limit, the theory is ‘Lorentz invariant’, with \(v\) playing the role of the speed of light.

The operators of the theory are defined in terms of chiral fields obeying \((\partial_t \pm v \partial_x) \varphi_{R,L} = 0\). Then \(\varphi = \varphi_L + \varphi_R\), while the dual field \(\tilde{\varphi} = \varphi_L - \varphi_R\). It is convenient to label the operators by their “electric” charge \(m\) and “magnetic” charge \(n\), defined as

\[ \{m,n\} = \exp \left( i \frac{m}{2r} \varphi + inr \tilde{\varphi} \right) \]  
(5)

The left and right scaling dimensions of this operator are then

\[ (h_L, h_R) = \left( \frac{1}{2} \left( \frac{m}{2r} + nr \right)^2, \frac{1}{2} \left( \frac{m}{2r} - nr \right)^2 \right). \]  
(6)

This means that when \(r = \sqrt{g}\), the overall scaling dimension \(x = h_L + h_R\) and Lorentz spin \(s = h_L - h_R\) of the operator \(\{m,n\}\) are

\[ x = \frac{m^2}{4g} + n^2 g, \quad s = mn. \]

The electron operator in particular is the \((1, 1/2)\) operator, namely

\[ c_L = e^{\frac{\varphi}{2}((\sqrt{3}+1)/\sqrt{3})\varphi_L - (\sqrt{3}-1)/\sqrt{3})\varphi_R} \]  
(7)

Then \(c_L^\dagger = \{-1, -1/2\}\), while \(c_R = \{1, -1/2\}\) and \(c_R^\dagger = \{-1, 1/2\}\). The electron operators have Lorentz spin \(\pm 1/2\), meaning that they have non-trivial properties under Lorentz transformations (i.e. rotations in the in 1+1 dimensional spacetime). Operators with non-zero Lorentz spin are called chiral. Adding a chiral operator to the action breaks Lorentz invariance.

In this paper, we consider the effects of coupling two identical spinless Luttinger liquids. There are two types of couplings which will concern us here, interchain hopping and interchain interactions. Interchain hopping takes the form:

\[ H_c = -t_\perp \sum_{i,j} \left( c_{i+1}^\dagger c_{i,j} + \text{h.c.} \right) \]  
(8)

while interchain interactions take the form:

\[ H_{V_\perp} = \sum_{i,j} V_\perp (|i-j|) n_{i+1} n_{j,2} \]  
(9)

Here, \(c_{i,1}\) annihilates an electron at site \(i\) of chain 1 while \(c_{i,2}\) annihilates an electron at site \(i\) of chain 2; \(n_{i,1}\) and \(n_{j,2}\) have corresponding definitions.

In the long-wavelength limit, we can rewrite the model in terms of two free bosons \(\varphi_1\) and \(\varphi_2\). However, the interactions depend only on the difference of the two, so we define the linear combinations

\[ \varphi_\pm = \frac{1}{\sqrt{2}} (\varphi_1 \pm \varphi_2) \]  
(10)

Then interchain hopping takes the form:

\[ \mathcal{L}_c = -t_\perp e^{\frac{\varphi_\perp}{2}[(\sqrt{3}+1)/\sqrt{3}]\varphi \pm (\sqrt{3}-1)/\sqrt{3} \varphi_\perp] + \text{c.c.} + \frac{1}{2} t_\perp e^{\frac{\varphi_\perp}{2}[(\sqrt{3}+1)/\sqrt{3}]\varphi \pm (\sqrt{3}-1)/\sqrt{3} \varphi_\perp] + \text{c.c.} \]  
(11)

In the decoupled theory, the dimensions just add, so this operator has dimension and spin twice as large as the fermion, namely \(x = (g + 1/g)/2\) and Lorentz spin \(\pm 1\). Thus this theory has chiral operators in the action and breaks Lorentz invariance. In a model of coupled chains which is parity-invariant, \(t_\perp^R = t_\perp^L\). However, it will be convenient to allow these two hopping parameters to be independent as so as to separate various effects. In particular, the case \(t_\perp^L = 0, t_\perp^R \neq 0\) will be of special interest. Interchain interactions take the form:

\[ \mathcal{L}_{V_\perp} = V_{2k_F} \cos(\sqrt{2g}\varphi_\perp) + J \cos(\sqrt{2/g}\varphi_\perp). \]  
(12)

The first is a pure interaction term of dimension \(2g\): no charge is transferred between the two liquids. The second is a Josephson tunneling or pair hopping term: it transfers a pair of electrons from one chain to the other. Even if we set these terms to zero by hand (and there is no reason to do so), second-order perturbation theory in the interchain hopping induces them. This model is invariant under the electric-magnetic duality \(g \rightarrow 1/g\): under this symmetry the pair hopping and interchain interactions are swapped. We could also add zero-momentum
density-density and current-current interactions of the form $\partial_{\perp} \varphi_1 \partial_{\perp} \varphi_2$, but they will not change the results described here: they only renormalize the coupling $g$ and the Fermi velocity $v$, so we absorb them in redefinitions of the couplings.

Thus the interactions of two coupled Luttinger liquids can be described by a single boson $\varphi_\perp$; the boson $\varphi_\perp$ is free and describes the total charge, which is unaffected by the interchain coupling. It is useful to rework the interaction terms in the action in terms of electric and magnetic charges with a radius $r = \sqrt{g/2}$. The interchain hopping terms are of the form (1), namely $\{ \pm 1, \pm 1 \}_-$, where the subscript indicates the extra $\sqrt{2}$ in the relation of the radius to $g$. The $V_{2kF}$ term is $\{ 0, \pm 2 \}_-$, while the $J$ term is $\{ \pm 2, 0 \}_-$.

The lowest-order RG equations for these perturbations are:

\[
\begin{align*}
\frac{dt_{\perp}^{R,L}}{dt} & = \left( 2 - \frac{1}{2} \left( g + \frac{1}{g} \right) \right) t_{\perp}^{R,L} + O(t_{\perp}^{R,L} t_{\perp}^{R,L}) \\
\frac{dJ}{dt} & = (2 - 2/g) J + O(t_{\perp}^{R,L} t_{\perp}^{R,L}) \\
\frac{dV_{2kF}}{dt} & = (2 - 2g) V_{2kF} + O(t_{\perp}^{R,L} t_{\perp}^{R,L})
\end{align*}
\]

(13)

Several important features of the model are apparent in these equations.

(1) For $g \neq 1$, either the pair-hopping or the $2k_F$ interaction is relevant, and the other one is irrelevant. Thus when the electrons interact, there is always a relevant non-chiral term in the action.

(2) For $1 > g > 2 - \sqrt{3} = 0.768 \ldots$, $t_{\perp}^{R,L}$, $t_{\perp}^{L}$ and $V_{2kF}$ are relevant. For $g < 2 - \sqrt{3}$ tunneling between Luttinger liquids is irrelevant. For $g < 2 - \sqrt{3}$, the fermion operator has dimension 1, so its spectral function has no singularity at the Fermi surface (for $2 - \sqrt{3} < g < 1$, there is at least a power-law divergence; for $g = 1$, i.e. free fermions, a $\delta$-function singularity); hence, this is the limit of extremely strong interactions.

(3) If $t_{\perp}^{R,L} = 0$ and $V_{2kF} = J = 0$ but $t_{\perp}^{R} \neq 0$, then all the higher-order terms in the RG equation for $t_{\perp}^{R,L}$ vanish: higher powers of $t_{\perp}^{R,L}$ are of higher Lorentz spin and thus cannot renormalize $t_{\perp}^{R,L}$. This is in stark contrast to the pair-hopping and $2k_F$ interaction terms which do renormalize themselves.

Our aim in this paper is to discuss the fate of the system under the flow of these relevant perturbations.

### III. WEAK-COUPLING PHYSICS

We first consider the free fermion limit, $g = 1$. If we set $V_{2kF} = J = 0$, then we simply have two Fermi liquid chains coupled by single-fermion hopping. This special case was discussed in detail in \[13\]. The hopping operators are of dimension $(1,0)$ and $(0,1)$. As a result of the interchain hopping, symmetric and antisymmetric bands are formed with Fermi surfaces displaced from $k_F$. For the purpose of comparison with $g \neq 1$, we describe this in a little more detail.

The action is

\[
S_0 = \sum_{I = 1,2} \int dx \int dt \left( \frac{1}{2} \left( \partial_t \varphi_\perp \right)^2 - v^2 \left( \partial_x \varphi_\perp \right)^2 \right) + \int dx \int dt \left( t_{\perp}^{R,L} \partial_t (\varphi_\perp \pm v \partial_x) c_{I,R,L} + \frac{1}{2} \varphi_\perp^{c.c.} \right)
\]

(14)

where $I = 1,2$ specifies the chain. We diagonalize this by forming symmetric and antisymmetric bands in terms of

\[
c_{s,a,R,L}(x) = \left( c_{1,R,L}(x) \pm c_{2,R,L}(x) \right) e^{\pi i t_{\perp}^{R,L} (vt \mp x)}
\]

(15)

The action is simply:

\[
S = \int dx \int dt \left( \frac{1}{2} \left( \partial_t \varphi_\perp \right)^2 - v^2 \left( \partial_x \varphi_\perp \right)^2 \right) + \frac{1}{2} \varphi_\perp^{c.c.}
\]

(17)

If the fermion spectra are truly strictly linear for $t_{\perp}^{R,L} = 0$, then $v_s = v_a = v$; otherwise, these velocities will be shifted.

It is instructive to re-cast these steps in the bosonic representation, which is the most useful representation for $g \neq 1$. Then,

\[
S = \int dx \int dt \frac{1}{8\pi} \left( \partial_t \varphi_\perp \right)^2 - v^2 \left( \partial_x \varphi_\perp \right)^2 - \int dx \int dt \left( t_{\perp}^{R,L} e^{i\varphi_\perp \sqrt{2}} + t_{\perp}^{L} e^{i\varphi_- \sqrt{2}} + c.c. \right)
\]

The $SU(2)$ symmetry which interchanges the two uncoupled chains can be used to transform this into a more tractable form. Performing a rotation generated by the unitary operator

\[
U = \exp \left( \frac{\pi}{2} \left( e^{i\varphi_- \sqrt{2}} - e^{-i\varphi_\perp \sqrt{2}} + L \leftrightarrow R \right) \right)
\]

(18)

we transform the action into:

\[
S = \int dx \int dt \frac{1}{8\pi} \left( \partial_t \varphi' \right)^2 - v^2 \left( \partial_x \varphi' \right)^2 - \int dx \int dt \left( t_{\perp}^{R,L} \partial_- \varphi'_{-R} + t_{\perp}^{L} \partial_+ \varphi'_{-L} \right)
\]

(19)

where $\varphi'$ is the rotated field, $\partial_\perp = \partial_t \pm v \partial_x$ and, for simplicity, we have assumed that $t_{\perp}^{R,L}$ are real. The second line of (19) can be eliminated by making the change of variables: $\varphi'_{-R,L} \rightarrow \varphi'_{-R,L} + \frac{1}{2} t_{\perp}^{R,L} (vt \mp x)$. As a result, various correlations functions of the unshifted variables will be spatially modulated:

\[
\langle e^{i\varphi'_{-R}(x)}/\sqrt{2} e^{-i\varphi'_{-R}(0)}/\sqrt{2} \rangle = \frac{1}{x^2} e^{\pi t_{\perp}^{R,L} x}
\]

(20)
The same is true of the unprimed fields. This spatial modulation of a correlation function is characteristic of an incommensurate phase, and can be taken as the definition of such a phase\textsuperscript{[13]}. This behavior still occurs even if $t_{11}^{R} \neq t_{11}^{L}$. In particular, if $t_{11}^{R} \neq 0$, but $t_{11}^{L} = 0$, the modulation occurs in the right-moving sector, but not the left-moving one. The relation to the fermionic representation is:

$$c_{s,a,R,L}(x) = e^{i \sqrt{\pi \Delta} R \cdot x} e^{ \pm i \sqrt{\pi \Delta} R \cdot x} e^{i \pi \frac{\eta}{\sqrt{\pi \Delta} R \cdot x}}$$

(21)

so comparing with \textsuperscript{[8]} we see how in an incommensurate phase the fermi surfaces are shifted.

To summarize, the hallmark of $c$-axis dispersion and metallic $c$-axis transport is the oscillation of the power-law correlation functions of $\varphi$, as exemplified by eq. \textsuperscript{[21]}. This is equivalent to the statement that gapless fermionic excitations occur at two different Fermi surfaces.

Let us now turn on weak interactions and move away from the free fermion point, but remaining in the weak-coupling regime. In this regime, it makes sense to consider both intra- and interchain interactions as a perturbation of the system in which interchain hopping has already been solved \textsuperscript{[10]}. These interactions take the form:

$$S_{int} = \int dx \int dt \left[ u_s c_{s,R}^\dagger c_{s,L} + u_a c_{a,R}^\dagger c_{a,L} + u_x \left( c_{s,R}^\dagger c_{a,L} + c_{a,R}^\dagger c_{s,L} \right) + u_t \left( c_{s,R}^\dagger c_{s,L} + c_{a,R}^\dagger c_{a,L} \right) \right]$$

(22)

The renormalization group equations for these interactions read:

$$\frac{du_s,a}{d\ell} = \frac{u_s + u_a}{2u_{s,a}} u_t^2$$

$$\frac{du_x}{d\ell} = -u_t^2$$

$$\frac{du_t}{d\ell} = - \left[ \frac{u_s + u_a}{2u_s} u_s + \frac{u_s + u_a}{2u_a} u_a - 2u_x \right] u_t$$

(23)

For $u_s = u_a$, these are the Kosterlitz-Thouless equations for $u_t$ and $u_s + u_a - 2u_x$. In the limit of small $u_t$, the system is described by two gapless decoupled Luttinger liquids (corresponding to the symmetric and antisymmetric bands) if $u_s + u_a - 2u_x > 0$. If, in the same limit, we instead have $u_s + u_a - 2u_x < 0$, then a gap is opened. The action \textsuperscript{[22]} is Lorentz invariant in this limit, so this gap cannot affect the splitting between the Fermi momenta of the symmetric and antisymmetric bands. The lowest energy single-fermion excitations will be at $\approx k_F + \sqrt{\pi \Delta}$. When we move away from $v_x = v_a$ (which will certainly happen if the interaction strengths are different in the two bands), we can no longer appeal to Lorentz invariance, and it is possible that the symmetric and antisymmetric bands are not split in momentum. In the limit of very large $u_s + u_a + 2u_x$, the entire approach breaks down because interchain tunneling between the two chains is irrelevant. In this limit, symmetric and antisymmetric bands do not form, and $c$-axis transport is insulating. In subsequent sections, we address the question of whether this can happen in the strong-coupling limit even when interchain tunneling is still relevant.

IV. THE CHIRAL CLOCK MODEL

In this section we introduce a lattice model which, in the continuum limit, is described by the above coupled Luttinger liquids, including the chiral operators. In the next section we will utilize known results on the incommensurate phases of this lattice model to extract physics for our problem.

The lattice model is a generalized chiral clock model, with nearest-neighbor interactions on a square lattice. It is written in terms of discrete “clock” variables $2\pi x/N$, where $a$ is an integer ranging from 0 to $N - 1$. The interaction energy between neighbors of values $a$ and $b$ is required to have a $\mathbb{Z}_n$ symmetry and, thus, is of the form $E(a, b) = E(a - b) = E(a - b + N)$. The general action is

$$S = -\sum_{x,y}^{N-1} (K_j \cos \left[ \frac{2\pi}{N} (jDx - \Delta_j) \right] + \hat{K}_j \cos \left[ \frac{2\pi}{N} (jDy - \hat{\Delta}_j) \right])$$

(24)

where $D$ is the lattice derivative, namely $D_x a \equiv a(x + 1, y) - a(x, y)$ and $D_y a \equiv a(x, y + 1) - a(x, y)$. We can obtain the physics we want by taking

$$K_j = \hat{K}_j \quad \Delta_j = \hat{\Delta}_j \quad \Delta_{j-1} = \Delta \quad \hat{\Delta}_{j-1} = \hat{\Delta} \quad \Delta = 0 \quad \text{otherwise}$$

We will also take $K_j$ to be real, but $\Delta$ and $\hat{\Delta}$ need not be real. Usually, clock models have $K_j = 0$ for $j > 1$, but it will prove very useful to relax this condition. This model was originally introduced in \textsuperscript{[14]}, which discussed the special case $K_1 = 1/T$ and $K_j = 0$ for $j > 1$, and $\hat{\Delta} = 0$. When $\Delta \neq 0$, the model is chiral. For $0 < \Delta < 1/2$, it is energetically more favorable for $D_x a$ to be 1 than for it to be $-1$: $\alpha$ tends to rise (cyclically) as $x$ moves to the right. The fact that at low temperature there are two possibilities (no rise or a rise of $+1$) leads to the existence of an incommensurate phase, to which we will return in the next section.

For couplings close to a critical point, one expects the continuum physics of the chiral clock model to be described by a field theory. For $\Delta = \hat{\Delta} = 0$, a number of
critical points are well understood. For $N=3$, there is a single critical point at $K_1 = [\ln(\sqrt{3} + 1)]/3$, the $S_3$-symmetric 3-state Potts model critical point. The corresponding field theory is a conformal field theory with central charge $c = 4/5$. For $N \geq 5$, there is a critical “Villain” line. This line appears by arguments familiar from the Coulomb gas approach to the Kosterlitz-Thouless transition in the XY model. Namely, one assumes that the discrete clock variable $a(x, y)$ can be replaced by a bosonic field $\phi(x, y)$ taking continuous values: the effect of the original discreteness is a potential $\cos(N\phi)$ (the operator $\{N, 0\}$ in the notation of section 2). When $N \geq 5$ there is a regime in which both vortices (operators $\{0, n\}$ for $n$ integer) and the $\cos(N\phi)$ potential are irrelevant. In this regime the free boson fixed-line is therefore stable. In the full parameter space of the generalized clock model with no chirality, there is a region which flows to this stable fixed line. However, there is an additional fixed point in this parameter space, known as the Fateev-Zamolodchikov or parafermion fixed point [21]. This unstable fixed point is at the edge of the region of stability of the Villain line, and is at a self-dual value of the couplings. The parafermion conformal field theory describing this fixed point has central charge $c = 2(N - 1)/(N + 2)$ while the Villain line has central charge $c = 1$. As $N$ gets higher, the structure gets more elaborate, but the parafermion and Villain points are the only ones we need to discuss here; see [23] for a thorough discussion of the phase structure of the $Z_5$ and $Z_6$ models.

The case of interest for the coupled Luttinger liquids turns out to be $N = 4$. The existence of the Villain line follows directly from the fact that the (non-chiral) $Z_4$ clock model is equivalent to the Ashkin-Teller model. The Ashkin-Teller model is two coupled Ising models, with $s, \sigma = \pm 1$ on every site related to $a$ via

$$e^{i\pi a/2} = \frac{s + i\sigma}{1 + i}$$

so at $\Delta = \tilde{\Delta} = 0$ the action becomes

$$S = -\sum_{<ij>} (K_1 (\sigma_i \sigma_j + s_i s_j) + K_2 \sigma_i \sigma_j s_i s_j). \quad (25)$$

The original chiral clock model is at $K_2 = 0$, where the two Ising models decouple. On the self-dual line $\sinh(2K_1) = \exp(-2K_2)$, the Ashkin-Teller model is, in turn, equivalent to a special case of the lattice 8-vertex model which is not only integrable but critical for $K_2 \leq \ln(3)/4$ [24]. The continuum field theory description of this critical line is precisely a free boson (or to be precise, its orbifold [25]). That the Ashkin-Teller model is equivalent to a free boson should not be surprising. The two-dimensional Ising model is equivalent to a free Majorana (real) fermion, so two Ising models are equivalent to a Dirac fermion. In the continuum theory, the four-fermion coupling is the only way to add interactions without moving off the critical line. The $N = 4$ parafermion critical point is a special point on this line, namely $K_2 = (\ln(\sqrt{2} + 1))/4$. The critical point of the four-state Potts model (which has a full $S_4$ symmetry) is on the self-dual line at $K_2 = K_1 = (\ln 3)/4$.

Now that we have found a critical line of the lattice model and the corresponding field theory, we need to find the operators in the field theory which are relevant perturbations. The thermal perturbation is not chiral: it corresponds to moving off the self-dual line while keeping $\Delta = \tilde{\Delta} = 0$. This model is not integrable on the lattice, but the dimension of the thermal operator is known by combining the exact solution at $T = T_c$ [24] with standard scaling assumptions and a Coulomb-gas analysis [26]. We can put this result in our notation for free bosons by defining the radius $r_{AT}$ via

$$\cos \left( \frac{\pi}{S_4 \Delta_{AT}} \right) \equiv \frac{1}{2} (1 - e^{4K_2})$$

The thermal operator then is $\{0, \pm 2\}_{AT}$, with left and right dimensions $(2r_{AT}^2, 2r_{AT}^2)$. At the $K_2 = 0$ decoupled point, $r_{AT} = 1/2$, so the thermal perturbation has dimensions $(1/2, 1/2)$: it corresponds to giving a mass to the Ising fermions. At the parafermion fixed point, the thermal operator is of dimension $(1/3, 1/3)$.

We find the operators in the long-wavelength field theory which couple to $\Delta$ and $\tilde{\Delta}$ by following the argument in [24]. When $\Delta$ and $\tilde{\Delta}$ are small, the linear term in [24] can be rewritten as

$$K_1 \left( \Delta \sin \left[ \frac{2\pi j}{N} (D_x a) \right] + \tilde{\Delta} \sin \left[ \frac{2\pi j}{N} (D_y a) \right] \right).$$

As one would expect from difference operators, $\sin D_x f$ and $\sin D_y f$ transform into each other under $90^\circ$ lattice rotations. In other words, these operators are not invariant under Lorentz transformations: they transform like vectors. In the continuum theory, they should therefore have Lorentz spin $\pm 1 + 4n$, for some integer $n$. The chiral perturbation must correspond to adding to the action operators with Lorentz spin $\pm 1$ (operators with higher spin are irrelevant). We can identify these operators unambiguously by using one other fact about this perturbation: second-order contributions of the chiral terms renormalize the original thermal perturbation, because $2\sin^2 x = 1 + \cos 2x$. Thus the operator product of the chiral operator (Lorentz spin 1) with the antichiral operator (Lorentz spin $-1$) must contain the thermal operator. Under operator products, the electric and magnetic charges merely add. This means that the chiral operators must be $\{1, 1\}_{AT}$ and $\{-1, -1\}_{AT}$, while the antichiral operators are $\{1, -1\}_{AT}$ and $\{-1, 1\}_{AT}$.

For example, at the point $K_2 = 0$ where the Ising models decouple, the interaction between Ising spins connected by a bond in the $x$ direction is

$$K_1 \cos \Delta (\sigma_i \sigma_{i+1} + s_is_{i+1}) + K_1 \sin \Delta (\sigma_i s_{i+1} - s_i \sigma_{i+1}).$$
The second term is thus proportional to $\sigma D_x s - s D_x \sigma$. The continuum field theory therefore has action

$$S = S_{I2} + \Delta_L \int (\sigma(x,t)\partial_L s(x,t) - s(x,t)\partial_L \sigma(x,t)) + \Delta_R \int (\sigma(x,t)\partial_R s(x,t) - s(x,t)\partial_R \sigma(x,t))$$

where $S_{I2}$ is the action of two decoupled Ising models, and $\Delta_{L,R} = (\Delta \pm i\Delta)$. We have continued to real time $t$ via $y = it$. The operator product of two magnetization operators in the Ising model indeed contains the thermal operator, so the second-order contribution of this term indeed obeys our above condition. Since the magnetization fields $\sigma, s$ of the Ising models have dimensions $(1/16, 1/16)$, the operator coupling to $\Delta_L$ has dimensions $(9/8, 1/8)$, while the operator coupling to $\Delta_R$ has dimensions $(1/8, 9/8)$. These operators are not total derivatives, so in the bosonic language they do indeed correspond to the operators $\{\pm 1, \pm 1\}_{AT}$ and $\{\pm 1, \mp 1\}_{AT}$ at the decoupling radius $r_{AT} = 1/2$.

The action of the continuum field theory describing the 4-state chiral clock model near its critical line is therefore found by adding all the different perturbations to the free boson action $S_0$:

$$S = S_0 + (T - T_c) \int (\{0, 2\} + \{0, -2\}) + \Delta_L \int (\{1, 1\} + \{-1, -1\}) + \Delta_R \int (\{1, -1\} + \{-1, 1\})$$

All the operators are defined with radius $r_{AT}$. This is the action of our coupled Luttinger liquid written in terms of $\varphi$. ! One needs to make the identification $t^{L,R}_{1,2} = \Delta_{L,R}$, $T - T_c = V_{2k_F}$, and $r_{AT} = \sqrt{g/2}$. The critical line and hence the description is valid for $e^{4K_2} < 3$, which corresponds to $1/8 < r^2_{AT} < 3/8$. Thus the coupled Luttinger liquid is equivalent to the continuum limit of the generalized chiral clock model if $1/4 < g < 3/4$. For the dual regime $4 > g > 3/4$, one instead identifies $r_{AT} = \sqrt{1/2g}$ and $T - T_c = F$.

**V. COUPLED LUTTINGER LIQUID PHYSICS FROM THE CHIRAL CLOCK MODEL**

The chiral clock models were the subject of quite a bit of study a few decades ago; a review can be found in [13]. The four-state model with $K_2 \neq 0$ was discussed in [7]. When $\Delta = 0$, the four-state model is believed to have a phase diagram which looks like Fig. 1.

This picture is for some fixed value of $K_2/K_1$ (fixed $g$ in the Luttinger problem). The critical point at $\Delta = 0$ is the corresponding critical point on the Ashkin-Teller self-dual line; changing the temperature corresponds to changing $K_2$ and $K_1$ while keeping $K_2/K_1$ constant. This phase diagram should be valid for $2 - \sqrt{3} < g < 3/4$. Since everything is invariant under the transformations $g \rightarrow 1/g$, the models are also equivalent for $4/3 < g < 2 + \sqrt{3}$. For $g < 2 - \sqrt{3}$, the chiral perturbation is irrelevant, so there is no incommensurate phase. The other bound of the map, $g = 3/4$, is the value $K_2 = -\infty$. As shown in section 3, there is definitely an incommensurate phase at the free fermion point $g = 1$. As we will argue below, we expect three types of behavior. For $g < 2 - \sqrt{3}$ and its dual, the model is always commensurate. For $2 - \sqrt{3} < g < 3/4$ and its dual, both phases are possible, depending on the relative strength of the interchain interactions and the interchain hopping. Because as one brings $g$ closer to $1$ the hopping operator becomes more relevant and the non-chiral operator less relevant, our guess is that the incommensurate phase takes a larger portion of the phase diagram as $g \rightarrow 3/4$. For $4/3 > g > 3/4$, we argue that the commensurate phase vanishes. These possibilities are summarized in figure 2. Explaining these phase diagrams is the purpose of this section.
The evidence for the existence of an incommensurate phase in the chiral clock model derives from several arguments. In the parity-invariant model ($\Delta = 0$), the main evidence comes from looking at the lattice model close to zero temperature ($K_1 \to \infty$, $K_2 \to \infty$ with $K_2/K_1$ fixed). At zero temperature, the system is completely ordered, except at special points at which there are two lowest-energy states. In the four-state model, this happens at $\Delta = \Delta_f \leq 1/2$, where

$$
\frac{K_2}{K_1} = \sqrt{2} \sin \left[ \frac{\pi}{2} \left( \Delta_f - \frac{1}{2} \right) \right].
$$

(26)

At zero temperature, if $\Delta < \Delta_f$, $a$ takes the same value at every site; but if $\Delta_f < \Delta < 2 - \Delta_f$, $D_x a = 1$, i.e. $a$ increases by one (cyclically) every site as one moves to the right. These are incommensurate phases. At $\Delta = \Delta_f$, it is equally favorable energetically for $D_x a$ to be 0 or 1. Thus there are a huge number of ground states, characteristic of an incommensurate phase. At $\Delta = \Delta_f$ (while still $\Delta = 0$ and $T = 0$), the system forms stripes with constant $a$ in the $y$-direction. The values of $a$ increases (cyclically) from stripe to stripe as $x$ increases.

We can extend the argument of [14] to $K_2 \neq 0$ to show that there is an incommensurate phase at small but nonzero values of temperature. At non-zero temperature, the stripe boundaries are no longer necessarily strictly parallel to the $y$-axis: they can develop kinks. The partition function can be computed at low temperatures if one neglects stripe creation (i.e. stripes which do not exist for all values of $y$). Defining $\rho$ to be the number of stripes per unit length, one finds

$$
\cos(\pi \rho) = e^{2K_1 + 2K_2} \left( K_1 \sqrt{2} \sin \left[ \frac{\pi}{2} \left( \frac{1}{2} - \Delta \right) \right] + K_2 \right)
$$

(27)

When the magnitude of the right-hand side is greater than one, the phase is ordered and commensurate. In the range of $K_1$ and $\Delta$ where this magnitude is less than 1, the density of stripes varies continuously with the parameters. This is the incommensurate phase.

The low-temperature approximation thus accounts for the bottom part of the phase diagram. We now consider the evidence supporting the upper part of the phase diagram of figure 1. First of all, it is useful to consider what happens for clock models for $N > 4$. As noted before, there is a region of the phase diagram in which vortices (which correspond in the low-temperature picture to stripe-creation operators) and the $\cos N\phi$ clock operator are irrelevant. The chiral operators coupling to $\Delta$ still remain relevant. Thus even at $\Delta = 0$ there is a range of temperatures with a phase intermediate between commensurate and disordered, in the manner of [28]. It is natural to assume that as one turns on $\Delta$, this intermediate phase turns into the incommensurate phase. Thus the phase diagram for $N > 4$ looks like figure 1, except that the incommensurate/disordered transition line hits the $y$-axis above the commensurate/incommensurate line. For $N = 4$, there is a particular point on the $y$-axis at which the vortex and the clock operators become (naively) marginal, while, as we have seen, the chiral operators remain relevant. This results in the picture of figure 1, in which the disordered/incommensurate and commensurate/incommensurate lines hit the $y$-axis at the same point. The system can be tuned across the disordered/incommensurate boundary by varying the chiral and non-chiral perturbations, i.e. by changing the interchain tunneling and interaction strengths in the coupled Luttinger liquid model.

As $g$ increases to 3/4, the size of the incommensurate region increases because the exponent in (26) decreases. In fact, as $g \to 3/4$, $K_2 \to -\infty$ while $K_2/K_1 \to -1$. This means effectively that the critical point at $T = T_c$ is moving towards the $T = 0$ axis. In addition, the incommensurate value $\Delta_f$ also is moving to 0. Moreover, at $g = 3/4$, $2(K_1 + K_2) = \ln 2$ on the self-dual line so the exponential factor in (27) goes away. Thus the commensurate phase in the field theory is being pushed out. Therefore, we believe that in the low-temperature phase at $g = 3/4$ the model is gapless even at $\Delta = 0$ (and that the low-energy fixed point turns out to be the three-state Potts critical point). There is no longer a commensurate phase in the field theory. We believe that for $4/3 > g > 3/4$, only the incommensurate and disordered phases are possible, which dovetails with our analysis of the weak-coupling RG equations [28]. These equations are of Kosterlitz-Thouless type, and there is no evidence for any commensurate phase.

---

**FIG. 2. The phases of the coupled Luttinger liquids as a function of the intrachain interaction strength $g$.**

| $g$ | incoherent |
|-----|------------|
| 3.73 | coherent/incoherent (fig.1 applicable) |
| 4/3 | weak coupling applicable |
| 3/4 | coherent/incoherent (fig.1 applicable) |
| .268 | incoherent |
Thus we see that the $g \approx 1$ weak-coupling picture should persist until $g$ is lowered to $3/4$. For $2 - \sqrt{3} < g < 3/4$, the phase diagram in Fig. 1 is applicable. In the continuum picture, this phase diagram results from the interplay between the chiral and non-chiral operators. At $T = T_c$, and $\Delta = 0$, the coefficient of the thermal (interchain interaction) operator is tuned to be zero, and we have a critical point. Taking $\Delta > 0$ while keeping $|V_{2kF}|$ small then results in the incommensurate phase. Thus we believe that the phase structure is determined by the relative strengths of the tunneling and the interactions. (We will present additional evidence for this below.) We expect the size of the incommensurate phase to get bigger as $g$ increases to $3/4$, because of the low-temperature argument above, and because the chiral tunneling operator is getting more relevant while the non-chiral interactions are getting less relevant.

The incommensurate phase here is believed to be gapless. The main evidence for this comes from a closely-related model: the sine-Gordon field theory in a background field coupling to the charge. This model clearly has a phase diagram similar to that of figure 1. The commensurate phase comes when the $\cos(N\phi)$ term is relevant and the background field is small. For large enough background field, it is energetically favorable to have kinks in the ground state, even though they are gapped. This results in the incommensurate phase, which clearly is gapless, because increasing the background field continuously increases the number of kinks in the ground state.

Note that the disordered phase can have minimum energy excitations which are commensurate or incommensurate. The former case occurs in the limit in which the non-chiral perturbation is much larger than the chiral perturbation. The latter was discussed in the context of the weak-coupling analysis at the end of section III. Both of these possibilities lie within the disordered phase of the chiral clock model and are, hence, adiabatically connectable.

Note also that our results predict the existence of a Lifshitz point (i.e. a point at which the commensurate, incommensurate, and disordered phases all coexist) in the chiral clock model for $1/4 < K_2/K_1 < 2 - \sqrt{3} = .268 \ldots$. In this regime, the chiral operator is irrelevant near the $\Delta = 0$ critical point, so the disordered/commensurate transition line should extend out for finite $\Delta$. However, for $g > 1/4$ there is still an incommensurate phase at low-enough temperature. The phase diagram should look almost like figure 1, except there is a direct phase transition between the disordered and commensurate phases for $\Delta$ small. The phase boundary between the disordered and commensurate phases ends at a Lifshitz point, beyond which these two phases are separated by the incommensurate phase.

Strong support for the above picture comes from an integrable model, usually called the chiral Potts model. This model comprises a special subspace of the couplings in the general clock Hamiltonian. The name is somewhat misleading: the integrable model is not $S_N$ symmetric but only $Z_N$ symmetric. When there is no chiral interaction, the integrable model reduces to the $Z_N$ Fateev-Zamolodchikov parafermion critical point (there is no Potts critical point for $N > 4$).

Unfortunately, except at the critical point the couplings in the integrable model do not include the value $\Delta = 0$ in which we are most interested. However, it does include a very interesting special case, corresponding to $\Delta = \pm i\sqrt{3}$. Because of the $i$, the Boltzmann weights are not real, but the Hamiltonian in real time is Hermitian. In the language of the coupled Luttinger model, this corresponds to setting the tunneling at one of the Fermi points, say $\Delta_R = t^{\perp}_R = 0$. With this condition, the model remains self-dual even away from the critical line.

The thermal operator is not self-dual, so it cannot be in the continuum action either. Thus the field theory of the integrable model on this line is equivalent to that of Luttinger liquids at $g = 1/3$ (the parafermion value) coupled by a single chiral tunneling operator. This allows us to isolate the physics of the chiral operator (with coefficient $\Delta_L = t^{\perp}_L$) from that of the thermal ($V_{2kF}$) perturbation.

At a special value of $\Delta_L$ known as the superintegrable point, the exact excitation energies can be computed. One finds that level crossing has occurred: there are states with $Z_N$ charge (not to be confused with the electric charge of our Luttinger liquids) which have energy below that of the original ground state, which is chargeless. Thus the ground state at non-zero temperature contains a mixture of these $Z_N$ charged states, just as in the sine-Gordon model at large enough background fields. Thus, at the superintegrable point on the self-dual line, the model is gapless and incommensurate. Moreover, this phase persists even when a small non-zero $\Delta_R$ is allowed.

One very interesting exact result is that the exponents are the same at the superintegrable point and the parafermion critical point, once the anisotropy is properly accounted for. This is a strong indication that the model is gapless and incommensurate as one tunes $\Delta$ from $0$ to the superintegrable point, and in fact all along the self-dual line. All other exact results support this assertion as well. The continuum language also strongly supports this picture. Because the critical point is perturbed only by a single chiral operator, the thermal operator is not induced in perturbation theory. The only effect of renormalization is to scale the coefficient of the chiral operator, and not create a gap. This is also implied by the analysis of the sine-Gordon model in a background field, which indicates that there is no Lifshitz point in the three-state model.

Thus the integrable model provides strong evidence that adding a single chiral operator to the action results
in a gapless incommensurate phase. This is in perfect harmony with the picture that adding chiral operators to the action tends to cause an incommensurate phase, while adding non-chiral ones causes commensurability. A transition occurs as the relative couplings are varied. Unfortunately, we do not know how to determine the exact location of the transition line (except at low temperatures in the lattice model). Nevertheless, we believe that all available evidence agrees with the phase diagram of figure 1.

VI. CONCLUSIONS

Our main conclusions are

(1) For \( g = 1/3 \), interchain tunneling alone (which can be disentangled from interactions by taking \( t^L_{\perp} = V_{2k_F} = 0 \) but \( t^R_{\perp} \neq 0 \)) leads to the formation of symmetric and antisymmetric bands with displaced Fermi surfaces – i.e. \( c \)-axis dispersion and metallic interchain transport.

(2) Interchain interactions which couple the \( 2k_F \) density oscillations of the two chains suppress \( c \)-axis dispersion and transport. As a result, for \( 3/4 > g > 2 - \sqrt{3} \) (and the mathematically equivalent dual case \( 4/3 < g < 2 + \sqrt{3} \)), the relative strengths of interchain tunneling and interchain interactions determine the phase boundary between a state with \( c \)-axis dispersion and metallic \( c \)-axis transport and a state with a partially gapped spectrum (the total charge mode remains gapless) which exhibits insulating \( c \)-axis transport. As interchain interactions are increased within the gapped phase, the \( c \)-axis dispersion smoothly disappears; no phase boundary is crossed as the minimum energy excitations move to the same point in momentum space.

(3) There is no fully gapless phase without \( c \)-axis dispersion and metallic interchain transport, contrary to the claim of [4].

These results were drawn by mapping the problem of coupled Luttinger liquids onto the 4-state chiral clock model. The phase diagram of the latter model can be obtained from an analysis of the low- and high-temperature expansions of this model, together with an analysis of the \( Z_4 \) parafermion critical point of the non-chiral limiting case of the model. Result (2) follows from this phase diagram. Note, in particular, that this phase diagram has a single gapless phase: the incommensurate phase. Hence, it is not possible to have a completely gapless phase with two identical Fermi surfaces unless \( g < 2 - \sqrt{3} \), in which case interchain tunneling is irrelevant (this case is beyond the regime of validity of the mapping to the 4-state chiral clock model). Result (1) was obtained from exact solutions of the 4-state chiral Potts model (a special case of the general 4-state chiral clock model). These conclusions are consistent with earlier analytical [5-10] and numerical [33] studies. They contradict some of the claims of [4], although we do find that it is possible for \( c \)-axis transport to be suppressed in a regime in which the tunneling is still relevant.

These results suggest the following lessons for the interlayer physics of the high-\( T_c \) superconductors. In-plane non-Fermi liquid behavior alone is not likely to be enough to frustrate single-electron hopping between copper-oxide planes in high-\( T_c \) superconductors. However, the formation of a gap (possibly with interlayer correlations) could have this effect, thereby leading to a state which is metallic within the planes but does not show \( c \)-axis dispersion.

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