Determining a Points Configuration on the Line from a Subset of the Pairwise Distances

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Abstract

We investigate rigidity-type problems on the real line and the circle in the non-generic setting. Specifically, we consider the problem of uniquely determining the positions of \( n \) distinct points \( V = v_1, \ldots, v_n \) given a set of mutual distances \( P \subseteq \binom{V}{2} \).

We establish an extremal result: if \(|P| = \Omega(n^{3/2})\), then the positions of a large subset \( V' \subseteq V \), where large means \(|V'| = \Omega(|P|n)\), can be uniquely determined up to isometry.

As a main ingredient in the proof, which may be of independent interest, we show that dense graphs \( G = (V, E) \) for which every two non-adjacent vertices have only a few common neighbours must have large cliques.

Furthermore, we examine the problem of reconstructing \( V \) from a random distance set \( P \). We establish that if the distance between each pair of points is known independently with probability \( p = \frac{C\ln(n)}{n} \) for some universal constant \( C > 0 \), then \( V \) can be reconstructed from the distances with high probability. We provide a randomized algorithm with linear expected running time that returns the correct embedding of \( V \) to the line with high probability.

Since we posted a preliminary version of the paper on arxiv, follow-up works have improved upon our results in the random setting. Girão, Illingworth, Michel, Powierski, and Scott proved a hitting time result for the first moment at which one can reconstruct \( V \) when \( P \) is revealed using the Erdős–Rényi evolution, our extremal result lies in the heart of their argument. Montgomery, Nenadov and Szabó resolved a conjecture we posed in a previous version and proved that w.h.p a graph sampled from the Erdős–Rényi evolution becomes globally rigid in \( \mathbb{R} \) at the moment it’s minimum degree is 2.

1 Introduction

Let \( V \) be a set of \( n \) distinct points in a metric space. Given a subset of pairwise distances \( P \) of \( V \), we inquire whether all other distances in \( V \) can be uniquely determined. In \( \mathbb{R}^d \) with \( d > 1 \), it is easy to construct examples where all distances are known except one distance, yet this distance cannot be deduced from the others. In order to say something meaningful on rigidity a popular line of work is to assume general position, e.g. that the points are in generic position, meaning that all the coordinates of elements of \( V \) are algebraically independent over \( \mathbb{Q} \). In that case, it is known to be a property of the graph \((V, P)\) that does not depend on the distances of elements in \( P \).

In \( \mathbb{R} \), it is well known that if the points are in generic position, then all distances are determined by \( P \) if and only if \((V, P)\) is 2-connected, as shown in [7]. However, when the points are not in generic position, it is easy to build an example with 4 points and 4 given distances in which 2-connectivity is no longer sufficient.

We study rigidity \textit{without} assuming general position but only assuming the weaker assumption that \( V \) are different points, and show that when the points are in \( \mathbb{R} \) the set \( P \) of pairwise distances known does carry meaningful information.

A graph \( G = (V, P) \) is said to be \textit{globally rigid} in \( \mathbb{R} \) if, for any \textit{injective} map \( f: V \rightarrow \mathbb{R} \), the distance between \((f(i), f(j))\) for \( i, j \in P \) determines uniquely the distances of all \( \binom{V}{2} \) pairs. A characterization of globally rigid graphs in \( \mathbb{R} \) in the injective setting was obtained by [4] (Theorem 2.4). They established that certain edge colorings of the graph serve as witnesses for the graph being \textit{not} globally rigid in \( \mathbb{R} \). Furthermore, they showed that recognizing whether a graph is globally rigid in \( \mathbb{R} \) is an NP-complete problem.

To conclude the preceding discussion, we emphasize that although there exists a significant amount of research on globally rigid graphs, it focuses to the generic setting, where the embedding function \( f \) satisfies a strong condition of being generic. In contrast, the results presented in this paper apply under a much weaker assumption, namely that \( f \) is injective.
We consider $n$ points in either $\mathbb{R}$ or $S^1$, which may not necessarily be in generic position. We first examine an arbitrary distance sets $P$. We prove that every dense enough graph $(V, P)$ contains a dense globally rigid subgraph.

**Theorem 1**

Let $V$ be a subset of either $\mathbb{R}$ or $S^1$ of size $n$ and let $P$ be the set of given distances. If $|P| > 40\sqrt{n}$, then there exists $V' \subseteq V$ s.t. all distances between elements of $V'$ can be deduced from $P$ and $|V'| = \Omega\left(\frac{|P|}{n}\right)$.

We prove the following stronger claim. Consider $V = \{v_1, \ldots, v_n\}$ a set of objects whose explicit representation is unknown. Assume we have knowledge of the result of a certain function $f(v_i, v_j)$ for some pairs of elements in $V$ and the objective is to reconstruct some subset of the elements of $V$ from these measurements. That is, we would like to determine the $f$ valuations of all the pairs in some subset $V'$ of $V$, with $|V'|$ large.

We say that a function $f$ is $k$-locally determined if $f(v_i, v_j)$ can be determined from the values of $f(v_i, x), f(v_j, x)$ for $k$ different elements $x$ in $V$. Let $P$ be the set of pairs for which the $f$ valuations are known, then:

**Theorem 2**

Let $V$ be a set of $n$ elements, and let $f$ be a $k$-locally determined function with a measurement set $P \subseteq \binom{V}{2}$. If $|P| \geq 8n\sqrt{n}$, there exists a subset $V' \subseteq V$ of size $\Omega\left(\frac{|P|}{n}\right)$ for which all the measurements on $\binom{V'}{2}$ are determined from $P$.

To prove the bound given in Theorem 1 we demonstrate that distances in $\mathbb{R}$ and $S^1$ are 3 and 5 locally-determined, respectively, and then apply Theorem 2.

We also study what can be deduced from a random $P$, and show that the situation differs significantly. In this case, we show that if the distance between each pair of elements in $\binom{V}{2}$ is known with probability $p = \frac{C \ln(n)}{n}$, independently, and for a large enough constant $C$, then there is a linear expected running time algorithm that returns a correct embedding to the line with high probability.

**Theorem 3**

Let $V = \{v_1, \ldots, v_n\}$ be $n$ distinct points in $\mathbb{R}$. There exists a constant $C$ such that when each pairwise distance is known with probability $p = \frac{C \ln(n)}{n}$, independently, then all other distances can be deduced from $P$ with high probability.

The main ingredient in the proof of the claim above is proving a sharp threshold for having a monotone path from 1 to $n$ in an Erdős–Rényi random graph with vertex set $[n]$. Specifically, we prove that there is a sharp threshold for the existence of such a path at $p = \frac{\ln(n)}{n}$.

**Remarks on advancements after first arxiv version**

Since the first publication Girão, Illingworth, Michel, Powierski, and Scott proved a hitting time result for the first moment at which an time at which one can reconstruct $V$ when $P$ is revealed using the Erdős–Rényi evolution. They proved that one can reconstruct $V$ at the moment where the minimum degree of the graph is 2. In particular they proved Theorem 3 holds with any constant $C > 1$. Their proofs relies heavily on Theorem 1.

The work of Barnes, Petr, Portier, Randall Shaw and Sergeev made progress on a question of Girão, Illingworth, Michel, Powierski, and Scott and gave the first non trivial bounds on the required density of a random set $P$ of distances, such that given $n$ points in $\mathbb{R}^d$, one can reconstruct all distances between $(1 - o(1))n$ of the points from the distances in $P$.

In the work of Montgomery, Nenadov and Szabó they resolved a conjecture from previous version, proved that w.h.p a graph sampled from the Erdős–Rényi evolution becomes globally rigid in $\mathbb{R}$ at the moment it’s minimum degree is 2. Their result also improve upon Theorem 2 and does not use Theorem 4.

### 2 Reconstructing a subset of $\mathbb{R}$ or $S^1$

We start by showing that the distance functions in $\mathbb{R}$ and $S^1$ are 3, 5 locally determined respectively.
Claim 4
Given $v_i, v_j \in \mathbb{R}$, and 3 common neighbours in $G = (V, E)$ one can determine the distance $d(v_i, v_j)$ from their distances to the common neighbours.

Proof. Consider the functions $f(z) = |d(v_i, z) - d(z, v_j)|$. As $f$ attains maximum outside the segment $v_iv_j$, and attains any other value at most twice inside $v_iv_j$. Let $z_1, z_2, z_3$ be three common neighbours of $v_i, v_j$. If all $f(z_i)$ are equal then $f(z_i) = d(v_i, v_j)$. Otherwise, pick the element with minimal $f$ value among $z_1, z_2, z_3$. w.l.o.g it is $z_1$, this implies $z_1$ lies inside $v_iv_j$ and hence $d(v_i, v_j) = (v_i, z_1) + d(z_1, v_j)$ can be determined. \hfill \Box

In $S^1$ five common neighbors are enough.

Claim 5
Given $v_i, v_j \in S^1$, and 5 common neighbours in $G = (V, P)$ one can determine the distance $d(v_i, v_j)$ from their distances to the common neighbours.

Proof. Let $\{z_k\}_{k=1}^5$ be common neighbors of $v_i, v_j$. Consider the 3 regions in Figure 1 and again consider the function $f(z) = |d(v_i, z) - d(z, v_j)|$. If $f(z_k)$ has the same value for all $k = 1, ..., 5$ then necessarily this is the distance from $x_1$ to $x_1$ and all points $z_1, ..., z_5$ are in region $B$.

Otherwise w.l.o.g we may assume that $z_1$ attains the minimum over all $|d(v_i, z_k) - d(z_k, v_j)|$. This means that $z_1$ is not in the region $B$. Hence if $d(x_1, z_1) + d(z_1, x_j) \geq \frac{2}{3}$ then $z_1$ is at region $C$ and $d(v_i, v_j) = 1 - (d(v_i, z_1) + d(z_1, v_j))$, and if $d(v_i, z_1) + d(v_i, x_j) \leq \frac{2}{3}$ then $z_1$ is at region $A$ and $d(v_i, v_j) = d(v_i, z_1) + d(v_1, x_j)$.

By the previous claims, it suffices to prove Theorem 2 to get Theorem 2.

![Figure 1: Three regions on the circle](image)

2.1 Proof of Theorem 2

Let $(V, P)$ be the graph of given measurements. Define $G = (V, E)$ the graph of determined $f$ valuations, which contains as edges all possible pairs that can be determined from $P$ by the locality of $f$. Let $\Gamma(v)$ denote the set of neighbors of $v$ in $G$. Since $E$ contains all possible pairs that can be determined it means that $|\Gamma(v) \cap \Gamma(v_j)| \geq k \implies \{v_i, v_j\} \in E$.

Denote by $\alpha(G)$ the size of a maximum independent set of $G$ and by $\omega(G)$ the size of the maximum clique in $G$.

Therefore the following implies Theorem 2:

Theorem 6

Let $G = (V, E)$ be a graph on $n$ vertices, such that $\{i, j\} \not\in E \implies |\Gamma(i) \cap \Gamma(j)| \leq k$.

If $|E| \geq 8n\sqrt{Kn}$ then $\omega(G) \geq \frac{1}{4}|E|$.

We remind the following standard lemma:

Lemma 7

Let $G$ be a graph of average degree $d = \frac{2|E|}{n}$. There exists an induced subgraph $G'$ with minimal degree $\delta(G')$ satisfying $\delta(G') \geq \frac{d}{2}$.

Proof. Iteratively delete from $G$ vertices of degree smaller than $\frac{d}{2}$. Each deletion only increases the average degree, so this process must terminate and the final set of vertices $V'$ induce the desired subgraph. \hfill \Box

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Consequently, from now we assume that $G$ has minimal degree $\delta(G) \geq \frac{|P|}{n}$ with $V = V'$.

To ease the notation we refer to $\delta(G)$ as $\delta$.

To bound the size of the maximal independent set of $G$. We need the following result due to Corrádi, whose proof appears in [8] (Lemma 2.1):

**Lemma 8**

Let $S_1, \ldots, S_N$ be subsets of a set $X$, each of size $\geq r$ with $|S_i \cap S_j| \leq k$ then:

$$|X| \geq \frac{r^2N}{r(N-1)k}$$

We now bound the size of the maximal independent set in $G$ by applying Corrádi's Lemma:

**Claim 9**

Let $G$ be a graph on $n$ vertices such that all non-neighbours $i, j$ satisfy $|\Gamma(s_i) \cap \Gamma(s_j)| \leq k$.

If the minimal degree of $G$ satisfies $\delta \geq \sqrt{2nk}$ then the size of a maximum independent set in $G$ satisfies $\alpha(G) \leq 2n\delta$.  

**Proof.** Denote the elements of the maximum independent set by $s_1, \ldots, s_\alpha$. By Lemma 8 applied to $S_i = \Gamma(s_i)$:

$$n \geq \frac{\delta^2 \alpha(G)}{\delta + k(\alpha(G) - 1)}$$

Rearranging this inequality yields:

$$\alpha(G) \leq \frac{n(\delta - k)}{\delta^2 - kn} \leq \frac{n\delta}{1/2 \cdot \delta^2} \leq 2\frac{n}{\delta}$$

**Proof of Theorem 6.** We assume that the graph $G$ has minimal degree $\delta \geq \frac{|E|}{2n}$. Let $I = \{s_1, \ldots, s_\alpha\}$ be any independent set of maximum size. Since by assumption $|E| \geq 8n\sqrt{kn}$ one can apply Claim 9 to obtain that $\alpha \leq 2\frac{n}{\delta}$. Define $B_i$ to be the set of elements that has only $s_i$ as a neighbour in $s_1, \ldots, s_\alpha$. Formally $B_i = \{u \in G \mid \Gamma(u) \cap \{s_1, \ldots, s_\alpha\} = s_i\}$, i.e. it is the set of elements in $\Gamma(s_i)$ for which $s_i$ is their unique neighbor from the set $s_1, \ldots, s_\alpha$.

The following observation is the heart of the argument:

**Observation 10**

Each $B_i$ form a clique in $G$.

The observation follows by noticing that if $x, y \in B_i$ are not adjacent then $I \cup x, y \setminus s_i$ is an independent set. (see Figure 2).

To estimate $|\bigcup_{i=1}^{\alpha(G)} B_i|$, first estimate $|\bigcup_{i=1}^{\alpha(G)} \Gamma(s_i)|$ by using Bonferroni’s inequality and get:

$$\omega(G) \geq \frac{|\bigcup_{i=1}^{\alpha(G)} B_i|}{\alpha(G)}$$

Figure 2: Left: non-adjacent vertices have at most $k$ common neighbors.

Right: $B_i$ is a clique since replacing $i$ with $x, y$ non-adjacent increases the size of a maximum independent set.

Thus it follows that:

$$\omega(G) \geq \frac{|\bigcup_{i=1}^{\alpha(G)} B_i|}{\alpha(G)}$$

To estimate $|\bigcup_{i=1}^{\alpha(G)} B_i|$, first estimate $|\bigcup_{i=1}^{\alpha(G)} \Gamma(s_i)|$ by using Bonferroni’s inequality and get:
\[
\left| \bigcup_{i=1}^{\alpha(G)} \Gamma(s_i) \right| \geq \sum_{i=1}^{\alpha(G)} |\Gamma(s_i)| - \sum_{i<j} |\Gamma(s_i) \cap \Gamma(s_j)| \geq \delta \alpha(G) - k\left(\frac{\alpha(G)}{2}\right)
\]

Observe that the set \( \bigcup_i B_i \) consists of exactly the elements in \( \bigcup_i \Gamma(s_i) \) that do not appear in any \( \Gamma(s_i) \cap \Gamma(s_j) \) for all \( i < j \), therefore:

\[
\left| \bigcup_{i=1}^{\alpha(G)} B_i \right| \geq \left| \bigcup_{i=1}^{\alpha(G)} \Gamma(s_i) \right| - \sum_{i<j} |\Gamma(s_i) \cap \Gamma(s_j)| \geq \delta \alpha(G) - k\left(\frac{\alpha(G)}{2}\right) - k\left(\frac{\alpha(G)}{2}\right)
\]

Plugging this into Equation (1) and using Claim [9] we get since \( \delta \geq 4\sqrt{k}n \):

\[
\omega(G) \geq \delta - 2k\alpha(G) \geq \delta - 4k\frac{n}{\delta} \geq \frac{\delta}{2} \geq \frac{|E|}{4n}
\]

Notice that the clique size achieved by Theorem [6] is asymptotically tight by considering disjoint union of cliques of size roughly \( \Theta\left(\frac{|P|}{n}\right) \). We note that the requirement \( |E| = \Omega\left(\sqrt{k} \cdot n \sqrt{n}\right) \) cannot be improved as well.

**Claim 11**

For every \( k \), there are graphs \( G \) satisfying the assumptions of Theorem [8] with \( |E| = \Theta\left(\sqrt{k} \cdot n \sqrt{n}\right) \) and \( \omega(G) = O(k) \).

**Proof.** Let \( G \) be a \( C_4 \)-free graph with \( |E(G)| = \Theta(n \sqrt{n}) \) edges, e.g. one may take the well-known "lines vs planes" graph of the incidences of lines and planes of \( \mathbb{F}_p \) for a prime \( p \approx \sqrt{n} \). Denote by \( V \) it’s vertex set, and let \( n \) denote the size of \( V \). Observe that by being \( C_4 \)-free \( |\Gamma(v) \cap \Gamma(u)| \leq 1 \) for any \( v, u \in V \).

If we consider the graph \( G' \), obtained by replacing each vertex of \( V \) by a \( k \)-clique and each edge of \( G \) by all possible edges between the new cliques, then we obtain \( G' = (V', E') \) with \( |E(G')| = \Omega\left(k^2 |E(G)|\right) \) and \( |V(G')| = nk \), moreover every two vertices in this graph has \( \leq 2k \) common neighbors, hence this graph satisfies the conditions of this claim as:

\[
|E(G')| = \Omega(k^2 n \sqrt{n}) = \Omega(\sqrt{k} n k \sqrt{n}) = \Omega(\sqrt{k} |V(G')|^{3/2})
\]

And \( G' \) has maximum clique of size \( O(k) \) as needed.

**Reconstructing 3 points on the 3-Regular tree requires \( \Omega(n^2) \) distances**

It is natural to ask whether we can find analogs of Theorem [6] for subsets \( S \) of metric spaces that are not \( \mathbb{R} \) or \( S^1 \). We now consider the 3-regular tree \( T_3 \) with the graph metric. We give an example of a set \( \mathcal{P} \) of size \( \Omega(n^2) \) for which we cannot fully reconstruct a subset \( S' \) of size 3.

Consider the infinite binary tree, and associate each vertex of distance \( k \) from some root \( r \) by a length \( k \) string over \( \{0,1\} \). Consider the two subsets of vertices of level 3n, \( S_1 = \{x \in \{0,1\}^{3n} \mid x \text{ starts in } 2n \text{ zeros}\} \) and \( S_2 = \{y \in \{0,1\}^{3n} \mid y \text{ ends in } 2n \text{ zeros}\} \). Notice that every two vertices of \( S_1 \) are at distance at most 2n, and every two vertices of \( S_2 \) are at distance at least 4n.

For the 3-regular tree, notice that for an edge \( e \) the graph \( T_3 \setminus e \) is isomorphic to two copies of the infinite binary tree. Define the set \( R_1 \) to be taking two copies of \( S_1 \) in each infinite binary tree in \( T_3 \setminus e \). If the set of distances known, \( \mathcal{P} \), is the set of all distances between vertices in the different binary trees, then all distances in \( \mathcal{P} \) is 6n and we cannot distinguish \( R_1, R_2 \) which have all other distances different, see Figure [3]. We also note this construction is tight since we constructed \( 2t \) points and \( t^2 \) known distances for which no 3 points can be reconstructed and by Mantel’s theorem adding one more pair to \( \mathcal{P} \) yields a triangle in the distance graph.
Sparse globally rigid graphs in $\mathbb{R}$ Claim 4 has also the following application to injective global rigidity. A graph $G = (V, E)$ is globally rigid in $\mathbb{R}$ if given an injective map $\varphi : V \to \mathbb{R}$ all the distances of $e \in E(G)$ determine all other distances uniquely (or equivalently, determine $\varphi$ up to isometry). The proof of Theorem 6 gives that dense enough graphs contain large globally rigid subgraphs. We note that Claim 4 implies that there are infinitely many globally rigid graphs with average degree at most $\frac{12}{5}$.

To see this, first notice that if $G$ is globally rigid in $\mathbb{R}$ then adding a new vertex $v$ along with two edges to vertices of $G$ yields a new graph that is also rigid. Given a graph $G = (V, E)$, define $T(G)$ to be a graph with $V(T(G)) = V \cup (E \times \{i\})_{i=1}^{|E|}$ and $E(T(G)) = \{(v, (e,i)) | v \in e\}$, i.e. each edge is replace by 3 paths of length two. Note that for every $\{u,v\} \in E(G)$, $|\Gamma_{T(G)}(u) \cap \Gamma_{T(G)}(v)| = 3$ thus by Claim 4 all the distances of vertices that correspond to the original edges of $G$ can be deduced from $T(G)$. This means that $T(G)$ is rigid since all distances of $G$ can be recovered from it, and the vertices of $V(T(G)) \setminus V(G)$ all contain two edges that connect to $G$. Note that if $\frac{|E|}{n} = \alpha$ then $\frac{|E(T(G))|}{|V(T(G))|} = \frac{6\alpha}{1+3\alpha}$. Hence one can construct an infinite family of graphs by $G_1 = K_3$ a clique of size 3 and take $G_i = T(G_{i-1})$. This infinite family has average degree $< \frac{10}{3}$ since the sequence $a_n = \frac{|E(G_n)|}{|V(G_n)|}$ is the sequence $a_1 = 1, a_{n+1} = \frac{6a_n}{1+3a_n}$ which converges to $\frac{5}{3}$.

We note that the average degree of a globally rigid graph must be substantially larger than what’s possible in the generic setting:

**Proposition 12** Any globally rigid graph with at least 4 vertices has average degree of at least $\frac{12}{5}$.

**Proof.** Let $G = (V, E)$ be a connected graph and let $V_2$ be the set of vertices of degree 2. Observe that if $G$ is globally rigid in $\mathbb{R}$ then $V_2$ must be an independent set. Indeed, for $x, y \in V_2$ assuming $x \sim y$ they must lie on a path $(x, y, z)$ for $w \neq z$ since $G$ is connected and $|V| \geq 4$. Let $\varphi : V \setminus \{x, y\} \to \mathbb{R}$ be an injective map and consider:

$$\varphi^{(c)}(a) = \begin{cases} \varphi(a), & \text{if } a \in V \setminus \{x, y\} \\ \varphi(c) + c, & \text{if } a = x \\ \varphi(z) + c, & \text{if } a = z \end{cases}$$

Notice that $\varphi^{(c)}$ and $\varphi^{(-c)}$ induce the same distances on $e \in E$. As $c \in \mathbb{R}$ is arbitrary, then for some $c$ both $\varphi^{(c)}$ and $\varphi^{(-c)}$ are injective and therefore $G$ is not globally rigid.

Therefore $V_2$ is an independent set. Furthermore, it is clear that a globally rigid graph must have minimum degree $> 1$. Now consider the size of $V_2$. If $|V_2| \geq \frac{3}{8}n$ then $|E| \geq 2|V_2| \geq \frac{3}{8}n$ and therefore $\frac{2|E|}{n} \geq \frac{12}{5}$. If $|V_2| \leq \frac{3}{8}n$ then $\sum_{v \in V} \deg(v) = \sum_{v \in V_2} \deg(v) + \sum_{v \in V \setminus V_2} \deg(v) \geq 2|V_2| + 3(n - |V_2|) \geq \frac{12}{5}n$ as needed. \hfill $\square$

### 3 Reconstruction from Random $\mathcal{P}$

We now consider the case of a fixed set of points $V$ and a set of pairs $\mathcal{P}$ sampled randomly, i.e. for each pair $\{i, j\}$ we know the distance between point $i$ and point $j$ with probability $p$. We show that for $p = \frac{C}{\ln(n)}$ with $C$ being some universal constant it is possible to reconstruct $V \subset \mathbb{R}$ w.h.p. We also show that at this $p$, there is a randomized algorithm with expected running time linear in $|\mathcal{P}|$ that returns the correct location of the points of $V$ up to isometry w.h.p.
Before giving the full proof we sketch the strategy of the proof. Let $P$ be the random pairs samples, and $(V, P)$ is a $G(n, p)$ Erdős–Rényi random graph.

At the probability $p$ specified above we do not expect to vertices to have common neighbors as this event occurs at $p \approx \frac{1}{\sqrt{n}}$. Since the $G(n, p)$ graph at our range of $p$ looks locally like a tree and the local structure is not rigid. Denote by $W(x, y)$ the set of all walks between $x, y$ in the graph $(V, P)$. We can define an estimate of the distance between two nodes $x, y$ to be $Est(x, y) := \min_{\omega \in W(x, y)} \sum_i d(v_i, v_{i+1})$ where $\gamma = (v_0, \ldots, v_k)$ is a walk from $v_0 = x$ to $v_k = y$, i.e. it is the distance in the graph metric between $x, y$ in the graph $(V, P)$. It is clear that $Est(x, y) \geq d(x, y)$. We will show that for $p = \frac{\gamma n^{k(n)}}{n}$ far enough points $x, y$ (with $\Omega(n)$ points between them in the order induced by $R$) have $Est(x, y) = d(x, y)$ w.h.p.

The estimate $Est(x, y)$, even though correctly estimates far enough points, is usually incorrect for close points, see Figure 4. To bypass this, we observe that if $Est(x, y)$ is incorrect, then with many $z$, the three numbers $Est(x, y), Est(x, z), Est(y, z)$ do not satisfy the triangle inequality (w.h.p). We also show the other direction, that is: if $Est(x, y)$ is correct, then with many $z$, the terms $Est(x, y), Est(x, z), Est(y, z)$ satisfy the triangle inequality. Combining these we get that we can find (w.h.p) the correct $Est(x, y)$ efficiently, by checking triangle equalities.

\[ Est(x, y) \approx 2Diam(S) \]

Figure 4: Close points with $Est \approx 2Diam(S)$

Therefore, let $V = \{v_1, ..., v_n\}$ be $n$ points in $R$ and assume $v_1 < v_2 < ... < v_n$ and let $G \sim G(n, p)$ be an Erdős–Rényi random graph on vertex set $[n]$ where we associate the vertex $i$ with $v_i$, then by the preceding discussion we’d like to understand when is $Est(v_i, v_j) = d(v_i, v_j)$ which is exactly when there is a monotone path from $i$ to $j$ in $G$.

We show that there is a sharp threshold at $\frac{\ln(n)}{n}$ for the existence of a monotone path from 1 to $n$ in a random graph, that is:

**Theorem 13**

Let $G \sim G(n, p)$ be a random graph with vertex set $\{1, ..., n\}$, then for any $\varepsilon > 0$: 

1. (Subcritical) If $p = (1 - \varepsilon)\frac{\ln(n)}{n}$ then w.h.p there is no monotone path from 1 to $n$.
2. (Supercritical) If $p = (1 + \varepsilon)\frac{\ln(n)}{n}$ then w.h.p there is a monotone path from 1 to $n$.

**Remark 14**

In fact, we show in the supercritical regime that for each $\varepsilon$ the corresponding events happen with probability $1 - n^{1(\varepsilon)}$ where $f(\varepsilon)$ is some constant depending on $\varepsilon$ and $\lim_{\varepsilon \to \infty} f(\varepsilon) = \infty$.

By the remark above we have:

**Corollary 15**

There exist a universal constant $C$ such that a $G \sim G(n, p)$ random graph, with vertex set $[n]$ and $p = \frac{\ln(n)}{n}$ there is a monotone path between all vertices $i, j \in [n]$ with $|i - j| \geq \frac{\varepsilon}{C}$. 

**Proof.** Pick $\varepsilon > 0$ such that $f(\varepsilon) \geq 3$ for $x \geq \varepsilon$ and $f$ as in Remark 14. Take $C = \frac{\varepsilon}{C}$, by a union bound one gets that the probability some $i, j, |i - j| \geq \frac{\varepsilon}{C}$ do not have a monotone path is at most $n^2 \cdot \frac{1}{n} = O(\frac{1}{n})$.

Before we prove Theorem 13 we show it implies that we can reconstruct all distances w.h.p.

The following lemma states that if one can reconstruct the $d(u, v)$ from $d(P)$ for many pairs $(u, v)$ then one can reconstruct $d(u, v)$ for all pairs $(u, v)$. Let $d : [n]_2 \to \mathbb{R}_{\geq 0}$ be the distances of $V$, and let $d' : [n]_2 \to \mathbb{R}_{\geq 0}$ be a corrupted distance function, whereby corrupted we mean that $d, d'$ agree for many pairs (to be defined precisely in the following lemma) then we claim that if they are sufficiently close then $d'$ uniquely defines $d$.

---

1. where by monotone we mean the sequence of vertex indices along the path is monotone
Lemma 16

Let \( d : \{0, 1\}^n \rightarrow \mathbb{R}_{\geq 0} \) be the distances between elements of \( V \) and \( d' \) another function with the same domain, range. Assume for \( u \in V : \) we have \( \{|v \in V : d(u, v) \neq d'(u, v)\}| \geq cn \) then:

1. If \( c < \frac{1}{8} \) then it is possible to uniquely reconstruct \( d \) from \( d' \).
2. If \( c > \frac{1}{8} \) then it is not possible to uniquely reconstruct \( d \) from \( d' \).

Proof. For the first part, we can find all distances that are not corrupted by observing that:

\[ d'(i, j) \text{ satisfies triangle equality with more than } \frac{c}{2} \text{ other points if and only if } d'(i, j) = d(i, j). \]

It is clear that for any \( i, j \) the number of \( k \) such that \( d'(i, k) = d(i, k), d'(j, k) = d(j, k) \) is at least \( n - 2cn - 2 = (1 - 2c)n - 2 \) since this is the total number of vertices minus corrupted neighbors of \( i \) or \( j \). Therefore if \( d'(i, j) = d(i, j) \) it will satisfy triangle equality with at least \((1 - 2c)n - 2 > \frac{n}{2} \) vertices.

Assume that \( d'(i, j) \neq d(i, j) \), we know by the argument above that there is a set \( S \) consisting of \((1 - 2c)n - 2 \) points for which their distance to \( i \) and \( j \) is correct. For any point \( k \in S \) that is contained in \( I \) for which \( i, j, k \) satisfy triangle inequality we must have that \( d'(i, j) \) is not the largest edge in the triangle \( \{i, j, k\} \) as this implies \( d'(i, j) = d(i, j) \) since \( k \in S \). Therefore \( d'(i, j) \) satisfies with \( k \in S \cap I \):

\[ d'(i, j) = \max(d'(i, k), d'(j, k)) - \min(d'(i, k), d'(j, k)) = \max(d(i, k), d(j, k)) - \min(d(i, k), d(j, k)) \]

Where the right inequality follows by assuming \( k \in S \). The RHS above doesn’t take any value except the correct one more than twice by Claim 4 therefore there are at most two points in \( S \cap I \) that satisfy triangle inequality with \( i, j \). A similar argument for points outside the segment between \( i, j \) implies that at most two points of \( S \) that lie outside the segment \( I \) satisfy triangle equality with \( i, j \) therefore if \( d'(i, j) \neq d(i, j) \) then we have that \( d'(i, j) \) satisfies triangle equality with at most \((2cn + 2) + 2 + 2 = 2cn + 6 < \frac{n}{2} \) points for large enough \( n \) while in the case \( d'(i, j) = d(i, j) \) we have that \( d'(i, j) \) satisfies triangle equality with at least \( \frac{n}{2} \) points by the assumption on \( c \), hence one may observe \( d' \) and distinguish the \( d' \) values corresponding to \( d \), and those which does not. This proves that the graph \((V, E')\) with \( E' = \{\epsilon \mid d'(\epsilon) = d(\epsilon)\} \) satisfies that all pairs of vertices share at least \( \frac{n}{2} \) neighbors we get by Claim 4 that \( d' \) uniquely determines \( d \).

For the second part, take any \( n = 4k \) points labeled by \( [n] \) and let \( s \) be their pairwise distances. Notice that if we partition the \( n \) vertices to two sets of size \( 2k \) we can find a \( k \)-regular subgraph of \( K_{2k, 2k} \) (since \( K_{2k, 2k} \) is a union of \( 2k \) perfect matchings), denote the edges of this subgraph by \( E' \). For the \( n \) points in the line, if we shift the points associated with \( 1, ..., 2k \) (while the other vertices stay in place) then the distances that change are only those that correspond to edges in the cut \([1, ..., 2k] \cup \{2k + 1, ..., 4k\}\). If the edges of \( E' \) are corrupted according to the shifted configuration to produce a corrupted set of distances \( d'(i, j) \) then we cannot distinguish between the original set and the shifted set after the corruption.

Proof of Theorem 4. By Corollary 15 w.h.p each pair of nodes that have at least \( \frac{1}{4}n \) points between them there is a monotone path with edges in \( \mathcal{P} \).

Therefore for all \( i \) : \( |\{j \mid Est(i, j) \neq d(i, j)\}| \leq 2\delta n < \frac{1}{4}n \) and thus by Lemma 16 one can correct \( Est(x, y) \) to \( d(x, y) \) and reconstruct the distances of all the pairs of vertices w.h.p.

We note that from an algorithmic standpoint Lemma 16 is inefficient since one needs to run over all pairs to find which \( Est \) are corrupted and which are not. We now explain how to obtain a simple linear time algorithm, i.e. \( O(n^2 \delta) \) expected running time that finds an embedding of \( V \) to the line w.h.p for \( p \) in the range above.

3.1 Linear Time Algorithm to Find an Embedding

We note that it is possible to obtain a linear time algorithm that returns the correct embedding w.h.p for a random \( \mathcal{P} \) when \( p = \frac{C}{\log n} \) for large enough \( C \). Specifically, sample two vertices \( u, v \) and compute shortest paths (value of \( Est \)) from them to all other vertices. Define:

\[dist(u, v) = \max\{d(u, x) + d(v, y) : x, y \in V\} \]
Int(u,v) = \{ x \ | \ Est(u,v), Est(u,x), Est(x,v) satisfy triangle equality, \\
Est(u,v) = \max(\{Est(u,v), Est(u,x), Est(x,v)\}\}

Conditioning on the event that every two points u,v with \(\frac{1}{n}n\) between them are connected by a monotone path we have that \(|Int(u,v)| \geq \frac{n}{2}\) happens only for points for which \(Est(u,v) = d(u,v)\) and in this case all the points of \(Int(u,v)\) are indeed interior points of the segment u,v (this follows since \(Est\) is always at least d). Even though \(Int(u,v)\) consists of point in the segment between u,v it would be easier to work with points that are contagious. This can be done by removing from \(Int(u,v)\) the \(\frac{1}{n}\) points closest to u and removing the \(\frac{1}{n}\) points closest to v, denote this smaller set by \(Int'(u,v)\), which is a set of contiguous points by assuming the event of Corollary 14 holds. To construct the embedding, we first embed u at 0 and all vertices of \(Int'(u,v)\) are embedded by \(x \mapsto Est(u,x)\). Using \(Int'(u,v)\) we can embed all other points as follows. Let \(x,y\) be the middle two elements of \(Int'(u,v)\) and denote the one of them which is closer to u by \(x_u\) and the other by \(x_v\). All points in \(V \setminus Int'(u,v)\) have between them and each of \(x_u, x_v\) at least \(\frac{1}{n}\) points (since \(Int'(u,v)\) is a contagious set, of at least \(\frac{1}{n}\) elements with \(x_u, x_v\) being the middle elements). Therefore it is possible to embed correctly \(V \setminus (Int'(u,v) \cup u)\) by computing shortest path from \(x_u, x_v\) to all other points since the \(Est\) is correct for pairs in \(\{x_u, x_v\} \times V \setminus (Int'(u,v) \cup u)\). We describe below the algorithm more formally which returns an embedding \(Emb : V \mapsto \mathbb{R}\).

Algorithm 1 Reconstruction(P)
1. for \(i = 1 \ldots \ln(n)\) do
2. Sample \(u,v\) from \(V\) uniformly.
3. Compute \(Est\) on \((u,v) \times V\).
4. Compute \(Int(u,v)\).
5. if \(|Int(u,v)| \geq \frac{n}{2}\) then
6. Compute \(Int'(u,v)\).
7. for \(w \in Int'(u,v)\) do
8. \(Emb(w) = Est(u,w)\).
9. \(x_u, x_v \leftarrow \) middle vertices of \(Int'(u,v)\).
10. for \(w \notin Int'(u,v)\) do
11. if \(Est(x_u, w) > Est(x_v, w)\) then
12. \(Emb(w) = Emb(x_u) + Est(x_u, w)\).
13. if \(Est(x_u, w) < Est(x_v, w)\) then
14. \(Emb(w) = Emb(x_v) - Est(x_u, w)\)
return
15. Return \(Emb\)
16. return FALSE

Correctness (w.h.p) of Algorithm 1. The algorithm may be incorrect if the event of Corollary 15 does not hold, though this is \(o(1)\) in our range of \(p\). Assuming the event of Corollary 15 we get that \(Est\) is correct for all \(u,v\) with \(\frac{1}{n}\) points between them. We note that if the sample used by the algorithm is of two vertices with \(\frac{1}{n}\) points between them then necessarily Line 5 of Algorithm 1 will be executed and will produce an isometric embedding. Hence, the only event remaining for which the algorithm is incorrect is the case where all the samples didn’t satisfy Line 5 which is \(o(1)\) since each sample has constant probability to satisfy Line 5 and \(\ln(n)\) samples are used.

Running time of Algorithm 1. Computing \(Est\) between the samples and \(V\) can be done in time \(O(|V| \ln(|V|) + |E|) = O(n^2p)\) by applying Dijkstra’s algorithm from \(u,v\). Since each sample has constant probability to invoke Line 5 of Algorithm 1 the expected number of executions of the loop is constant. Therefore, the expected running time is \(O(n^2p)\) (which is linear in the size of the random graph).

3.2 Proof of Theorem 13
Before we analyze the threshold for monotone path at \(p = \frac{\ln(n)}{n}\) we sketch the main ideas of the proof. The case of \(p\) below the threshold follow from a standard first moment method. For \(p\) above the threshold the analysis is more involved. Set \(p = \frac{\ln(n)}{n}\) and let \(V_1\) be the
set of vertices in 1, ..., \( \frac{n}{2} \) that connects to 1 by a monotone path and define \( V_2 \) similarly for vertices in \( \frac{n}{2} + 1, ..., n \) that connect to \( n \). We note that each monotone path from 1 to \( n \) passes only once from \( V_1 \) to \( V_2 \). Therefore if any element of the set \( V_1 \times V_2 \) is in our random graph we will find a monotone 1 to \( n \) path. We show that with high probability \( |V_1| \cdot |V_2| = \Omega(n) \) in the super critical regime and therefore we will be done as w.h.p an edge in \( V_1 \times V_2 \) will be in the random graph in our range of \( p \).

To estimate the growth of \( V_1 \) divide the \( n \) vertices to contiguous intervals of size \( \frac{n}{\ln(n)} \). Consider the first interval \( I_1 \) before giving the vertices labels (meaning do not reveal yet the order of \( I_1 \) on \( \mathbb{R} \)). The random graph induced on \( I_1 \) is distributed as \( G(\frac{n}{\ln(n)}, c_{\ln(n)} \cdot \frac{\ln(n)}{n}) \), and it is well known that the neighbourhood of a vertex in \( G(\frac{n}{\ln(n)}, c_{\ln(n)} \cdot \frac{\ln(n)}{n}) \) converges Benjamini-Schramm (see [3]) to a Galton-Watson process with off-spring distribution Poisson(\( c \)), this in particular means that we expect the vertex 1 to have \( c^i \) neighbours of distance \( i \) in the interval \( I_1 \). Recall that we do not care just about connectivity to vertices but we want to be connected by a monotone path.

To understand the number of vertices in \( I_1 \) that \( v \) connects to via a monotone path it will be useful to think first of the neighbourhood structure being picked and only then revealing for each vertex a random “label” from \([n] \) not yet used (again, by label we mean it’s index in \( \mathbb{R} \) with respect to the order induced from \( \mathbb{R} \)). When giving those vertices a random label we get that vertices with distance \( i \) from 1 are connected to 1 by a monotone path w.p. \( \frac{1}{i^2} \). Therefore, we expect that 1 connects by a monotone path to \( 1 + c + \frac{c^2}{2} + ... = e^c \) vertices from the first interval.

In general we expect the number of vertices that connect to 1 by a monotone path to grow by \( e^c \) with each new interval since each vertex that connects to 1 by a monotone path connects by a monotone path to \( e^c \) vertices in the next interval therefore \( |V_1| \approx (e^c)^{\frac{n}{\ln(n)}} = n^c \) and \( |V_1| \cdot |V_2| \approx n^c \). Therefore if \( c > 1 \) then w.h.p an edge between \( V_1 \) to \( V_2 \) will be found.

**Lemma 17**

Let \( T \) be a tree with \( n \) vertices rooted at \( v \), and denote by \( r_i \) the number of vertices of distance \( i \), from \( v \). Let \( \pi \) be a uniformly random bijection \( \mathbb{V}(T) \rightarrow \mathbb{V} \) that satisfies \( \pi(v) = 1 \). Denote by \( M_\pi \) the set of all vertices \( u \) for which the \( \pi \) values on the path from \( v \) to \( u \) form an increasing sequence. Then \( E[|M_\pi|] = \sum \frac{r_i}{i^2} \).

**Proof.** Let \( u \) be a vertex of distance \( k \) and let \( I_u \) be an indicator to the event that the path from \( v \) to \( u \) is increasing. Fixing the \( k + 1 \) labels of the path there are \( k! \) ways to label the path (since \( \pi(v) \) is fixed to be 1) and exactly one of them is increasing, thus \( E[I_u] = P[I_k = 1] = \frac{1}{k!} \) and the proof follows by linearity of expectation.

**Corollary 18**

Let \( T \) be a Branching Process with a root \( r \), offspring distribution \( D \) with mean \( \mu \), and let the process be stopped at time \( M \). If the vertices of each tree in the distribution are randomly labeled with \( \pi \) such that \( \pi(v) = 1 \) then \( E[|M_\pi|] = \sum_{i=0}^{M} \frac{\mu^i}{i!} \).

**Proof.** Let \( r_i \) be the size of the \( i \)th generation, then \( E[r_i] = \mu^i \). The proof follows from Lemma 17 and the law of total expectation.

Finally, we remind the following inequality by Bernstein which may be found in [10]:

**Theorem 19**

Let \( X_1, ... X_n \) be i.i.d R.Vs with \( 0 \leq X_i \leq K \) a.s. , \( E[X_i] = \mu \) then:

\[
\mathbb{P}[\left| \sum_{i=1}^{n} X_i - n\mu \right| \geq t] \leq 2 \exp \left( -\frac{t^2/2}{nK^2 + K^2} \right)
\]

**Proof of Theorem 19** In the subcritical regime a simple computation yields that the expected number of monotone paths from 1 to \( n \) is \( \sum \binom{n-1}{i} p^{i+1} = p \cdot (1 + p)^{n-2} \leq e\mu^n \) and this is \( o(1) \) in the subcritical regime.

For the range above the threshold, let \( G \) be a random graph on vertices 1, ..., \( n \) at \( p = \frac{\ln(n)}{t} \) for \( c > 1 \), denote \( I_j = \{(i-1)\binom{n}{i}, ..., i\binom{n}{i}\} \) and let \( S_{ij} \) be the number of vertices in \( \cup_{j=1}^{l} I_j \) that are connected to 1 by a monotone path and define a R.V. \( X_i = |S_{ij}| \). By the discussion above, it is enough to show that the probability \( X_i \binom{n}{i} < n^{1/2+\delta} \) is
By the probabilistic Bonferroni's inequality:

Thus by Markov's inequality this means that since \( c > c^* > 1 \) such that for large enough \( n \) the expected number of neighbours of \( v \) in \( I_{i+1} \) is at least \( c^* \). We can also truncate the distribution, meaning that there is \( T \) such that if \( v \) has more then \( T \) neighbours we take only \( T \) of them (and do not use the other for constructing monotone paths). Also, for any \( c > c^* > 1 \) there is an \( M \) such that \( \sum_{i=0}^{M} (\frac{c^*}{c})^i > \epsilon \). To ease the notation we denote by \( c \) this \( c^* \) and \( T, M \) as above and we use this truncated distribution on the neighbours. Hence from now on we assume each element from \( S_i \) has at most \( T \) neighbours, and consider only monotone paths from it for \( I_j \) for \( j > i \), analyzed as follows.

We first claim that if \( X_i \leq n^{\frac{3}{2}} \) then for large enough \( n \) with probability \( \geq \frac{1}{2} \) we have \( X_{i+1} \geq (\frac{\alpha}{2} + \frac{\epsilon}{3})X_i \). Fix \( v \in I_{i+1} \) and \( u \in S_i \). By considering the event that \( v \) adjacent to \( u \), by linearity of expectation we have:

\[
\mathbb{E}[X_{i+1}|X_i] = X_i + \frac{n}{\ln(n)}\mathbb{P}[v \in I_{i+1} \text{ connects to some } u \in S_i]
\]  

(2)

By the probabilistic Bonferroni’s inequality:

\[
\mathbb{P}[v \in I_{i+1} \text{ connects to some } u \in S_i] \geq \frac{c\ln(n)}{n}X_i - \left( \frac{X_i}{2} \right)^2 \left( \frac{c\ln(n)}{n} \right)^2 \geq \frac{c\ln(n)}{n}X_i \left( 1 - \frac{c\ln(n)}{n} \cdot X_i \right)
\]

(3)

(4)

If \( X_i \leq n^{\frac{3}{2}} \) and \( n \) is large enough, plugging Equation (3) into Equation (2) we get:

\[
\mathbb{E}[X_{i+1}|X_i] \geq (1 + c - o(1))X_i
\]

Therefore by Markov’s inequality this means that since \( c > 1 \):

\[
\mathbb{P}[X_{i+1} \geq \left( \frac{3}{4} + \frac{c}{3} \right) X_i] \geq \frac{1}{2}
\]

(5)

And since \( c > 1 \) this implies for large enough \( n \):

\[
\mathbb{P}[X_{i+1} > \left( \frac{3}{4} + \frac{c}{3} \right) X_i] \geq \frac{1}{2}
\]

(6)

To show this implies that for suitable \( \alpha, \beta \) one has: \( X_{\beta\ln(n)} \geq n^\alpha \), define:

\[
Y_i = \begin{cases} 
    n^\alpha, & \text{if } i = 1 \\
    \left( \frac{\alpha}{2} + \frac{\epsilon}{3} \right)^{-1} Y_{i-1}, & \text{if } X_i > \left( \frac{\alpha}{2} + \frac{\epsilon}{3} \right) X_{i-1} \\
    Y_{i-1}, & \text{otherwise}
\end{cases}
\]

Notice that \( \mathbb{P}[X_i \leq n^\alpha] \leq \mathbb{P}[Y_i \geq 1] \) since each multiplicative decrease of \( Y \) is complemented by a multiplicative increase in \( X \) by at least the same constant. Also, note that \( \mathbb{E}[Y_1] = n^\alpha \). By combining Equation (6) with the law of total expectation this implies that:

\[
\mathbb{E}[Y_k] \leq \left( \frac{1}{2} + \frac{1}{2} \left( \frac{3}{4} + \frac{c}{3} \right) \right)^{-1} \mathbb{E}[Y_{k-1}]
\]

Therefore by setting \( k = \beta \cdot \ln(n) \) and \( \gamma = \alpha - \beta \ln \left( \frac{\alpha}{2} + \frac{\epsilon}{3} \right) \) we get that:

\[^2\text{We need polynomially small probability for Corollary} 15, \]
\[ E[Y_{\beta \ln(n)}] \leq \left( \frac{7}{8} + \frac{c}{8^k} \right)^{-\beta \ln(n)} n^{\alpha} \leq \exp(\gamma \ln(n)) = n^\gamma \quad (7) \]

Notice that for any fixed \( c, \beta \) for small enough \( \alpha \) we have \( \gamma < 0 \) thus the expectation is polynomially small and by Markov’s inequality:

\[ P[Y_{\beta \ln(n)} \geq 1] \leq \frac{E[Y_{\beta \ln(n)}]}{1} \leq n^\gamma \quad (8) \]

Since \( P[X_i \leq n^\alpha] \leq P[Y_i \geq 1] \) we get \( P[X_i \leq n^\alpha] \leq n^\gamma \) which is what we aimed to achieve in the first phase.

For the second phase, consider the neighbourhood of some \( v \in V \) restricted to \( I_{i+1} \), this neighbourhood is distributed according to \( D \). The ball around \( v \) of size \( M \) is a random rooted tree with offspring distribution \( D \) of height \( M \). Let \( Z_v \) denote the random variable of the number of monotonically connected vertices to \( v \) among these neighbourhoods. By Corollary \[8\] we know that \( E[Z_v] = \sum_{i=0}^{M} \frac{\gamma}{\beta} > e \) for our choice of \( M \). Denote \( E[Z_v] \) by \( \mu > e \), and pick \( \mu > \mu' > 1 \). Denote \( X_i = t \) and the elements of \( S_i \) by \( x_1, \ldots, x_t \), notice that \( X_{i+1} \geq \sum_{i=1}^{M} Z_{x_i} \), each \( Z_{x_i} \) is bounded by \( \sum_{i=0}^{M} T_i \leq T^{M+1} \) since we use the truncated distribution, and the \( Z_{x_i} \) are independent therefore by Bernstein inequality we get that

\[ P[i \sum_{j=1}^{t} Z_{x_i} - \mu t \leq (\mu - \mu') t] \leq 2 \exp \left( -\frac{(\mu - \mu')^2 t^2/2}{(T^{M+1})^2 + T^{M+1} \cdot \frac{\mu - \mu'}{e}} \right) \quad (9) \]

Since \( \mu - \mu' < 1 \) we know that \( T^{M+1} \cdot \frac{\mu - \mu'}{e} \leq t \left( T^{M+1} \right)^2 \) and thus:

\[ 2 \exp \left( -\frac{(\mu - \mu')^2 t^2/2}{(T^{M+1})^2 + T^{M+1} \cdot \frac{\mu - \mu'}{e}} \right) \leq 2 \exp \left( -\frac{(\mu - \mu')^2}{4T^{2M+2}} t \right) = e^{-\Omega(\tau)} \quad (10) \]

If we condition on the event that \( X_{\beta \ln(n)} > n^\alpha \) then this probability is \( e^{-\Omega(\alpha^\gamma)} \) which implies that \( X_{\beta \ln(n)} > \left( \frac{\mu'}{\beta} \right)^{\frac{1}{1-\beta}} \ln(n) \) \( X_{\beta \ln(n)} = n^{\ln(\mu')(1 - \beta)} \) w.h.p. We now note that we may first choose \( T, M, \mu' \), then find \( \beta \) such that \( \ln(\mu')(\frac{1}{\beta} - \beta) > \frac{1}{2} \), denote this quantity by \( \tau > \frac{1}{2} \). Then pick \( \alpha \) such that \( \gamma := \alpha - \beta \ln \left( \frac{7}{8} + \frac{c}{8^k} \right) < 0 \). This implies that \( P[X_{\beta \ln(n)} \geq n^\gamma] > 1 - n^{-\gamma - \ln(n)}e^{-\Omega(n^\gamma)} \geq (1 - \frac{1}{n^\gamma}) \).

Therefore the probability that \( 1, n \) are not connected by a monotone path is at most \( \frac{1}{n^\gamma} + \frac{2}{n^\gamma + (1 - p)^{n^\gamma}} = O \left( \frac{1}{n^\gamma} + (1 - p)^{n^\gamma} \right) \) where the left term corresponds to the event that \( 1 \) connects to less than \( n^\gamma \) vertices in \( \{1, \ldots, \frac{n}{2} \} \), the middle term corresponds to the event that \( n \) connects to less than \( n^\gamma \) vertices in \( \{ \frac{n}{2} + 1, \ldots, n \} \), and the last term corresponds to the probability that both \( 1 \) and \( n \) connect by a monotone path to \( n^\gamma \) vertices in their half, and each edge between those sets is not present in the random graph. Since \( \tau > \frac{1}{2} \) we have \( (1 - p)^{n^\gamma} = O \left( \frac{1}{n^\gamma} \right) \) hence the probability of finding a monotone 1 to \( n \) path is at least \( 1 - O \left( \frac{1}{n^\gamma} \right) \) as needed.

4 Concluding Remarks and Open Problems

We remark that the proof of our extremal result is related to a result of \[3\] about large cliques in induced \( C_4 \) free graphs.

Another direction is to improve the sufficient condition \( |P| = \Omega(n\sqrt{n}) \) in Theorem \[1\] to a weaker one. We conjecture that Theorem \[1\] holds for any graph without density assumptions. Explicitly we conjecture that any graph \( G = (V, P) \) has a globally rigid subgraph in \( \mathbb{R} \) and \( \Omega \left( \frac{|P|}{\gamma} \right) \) vertices.

In a previous version we conjectured that for large enough constant \( k \), every \( k \) connected graph is globally rigid. This was proved to be false in the work of Girão, Illingworth, Michel, Powierski, and Scott \[3\] and separates rigidity in \( \mathbb{R} \) and generic rigidity in higher dimensions by the recent breakthrough of Villányi \[11\] which proved that \( d(d+1) \) connected graphs are globally rigid in \( \mathbb{R}^d \) assuming the embedding is generic.
In regard to the threshold of monotone path we believe it is interesting to understand what happens at $c = 1$? Another problem is to understand the distribution of the number of monotonically connected vertices to a given vertex. In particular consider a Galton-Watson process with off spring distribution $D$ and for which each node $v$ gets an auxiliary R.V $X_v \sim U[0,1]$, and the root $r$ has $X_r = 0$. What is the distribution of the number of monotonically connected vertices to the root?

We conjectured that random graph $G$ sampled via the Erdős-Rényi evolution becomes globally rigid in $\mathbb{R}$ exactly at the time its minimum degree is 2. This was proved by a clever argument of Montgomery, Nenadov and Szabó \[9\] and extends to other models of random graphs, e.g. $d$-regular graphs with $d \geq 6$.

We also believe it is interesting to construct sparse globally rigid graphs with average degree as small as possible.

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