Relativistic Quantum Information of Anyons

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In this paper, a method is developed to investigate the relativistic quantum information of anyons. Anyons are particles with intermediate statistics ranging between Bose-Einstein and Fermi-Dirac statistics, with a parameter $\alpha$ ($0 < \alpha < 1$) characteristic of this intermediate statistics. A density matrix is also introduced as a combination of the density matrices of bosons and fermions with a continuous parameter, $\alpha$, that represents the behavior of anyons. This density matrix reduces to bosonic and fermionic density matrices in the limits $\alpha \to 0$ and $\alpha \to 1$, respectively. We compute entanglement entropy, negativity, and coherency for anyons in non-inertial frames as a function of $\alpha$. For small values of acceleration, the anyons with $\alpha = 0.5$ were observed to have the least entanglement. At high acceleration limits, however, the negativity for anyons with $\alpha < 0.5$ was found to be less than that for anyons with $\alpha > 0.5$. Semions, which are particles with $\alpha = 0.5$, were found to be less coherent than those with other values of fractional parameter.

I. INTRODUCTION

Particles in the three-dimensional or higher space are classified based on their statistical behavior, as bosons and fermions. The multi-particle wave function of identical bosons (fermions) is symmetric (antisymmetric) under interchange of any pair of particles. It has been shown that quasiparticles in the two-dimensional space may have intermediate statistics between bosons and fermions with a continuous parameter. This can be written in the two particles’ case as follows:

$$|\psi_1\psi_2\rangle = e^{\pi i \alpha} |\psi_2\psi_1\rangle,$$

(1)

where, $\alpha$ is the fractional statistical exchange quantum number, also called the statistical parameter, ranging over $0 \leq \alpha \leq 1$, with $\alpha = 0$ standing for bosons and $\alpha = 1$ for fermions. The theoretical possibility of these particles was first propounded by J. M. Leinaas and J. Myrheim [1]. They later came to be called anyon by F. Wilczek [2].

The Pauli exclusion principle was generalized to yield another sort of generalized statistics introduced by F. D. M. Haldane [3]. This generalization is independent of the dimension of the system. A fractional parameter, $g$, was defined for the fractional exclusion statistics. In this exclusion statistics two limits are defined for $g$, where $g = 0$ ($g = 1$) corresponds to bosons (fermions). Despite the radical differences in the basic definitions of the fractional exchange and fractional exclusion statistics, the relationship between these fractional statistics has been investigated in the two-dimensional space. Particles with a fractional statistics in the two-dimensional space are sometimes called anyons. For our purposes in this study, we will consider the Haldane fractional exclusion statistics in two dimensions and use anyon to call a particle.

Using Haldane’s fraction exclusion statistics, Wu derived the statistical distribution function of anyons as follows [4]:

$$n_i = \frac{1}{\omega(e^{(\epsilon_i - \mu)/kT} + \alpha)}.$$

(2)

The functional equation of $\omega$ is $\omega(x) = \omega(1 + \omega(x))^{1-\alpha} = x$. This is an active area of research for its important role in such different fields as quantum computations [5] and fractional quantum Hall effect [6].

Recently, quantum information in a relativistic limit and in non-inertial frames has attracted the attention of many researchers to the new field of relativistic quantum information [7–14]. The entanglement of bosons and fermions have been studied with interesting results [7, 8]. It has been found that entanglement degrades at high limit accelerations in both bosonic and fermionic cases as a result of appearance of a horizon in accelerated frames. Entanglement in bosonic modes has been found to vanish but that of fermionic modes to survive at the infinite limit of acceleration. Researchers have also investigated entanglement generation for boson and fermion modes in an expanding universe [11, 12, 15]. Moreover, investigations have shown behavioral differences between boson and fermion modes in relativistic frames. The present study considers anyon modes in a non-inertial frame and investigates variations in entanglement with respect to the acceleration as an attempt to shed more light on the differences between boson and fermion modes. To achieve this goal, a model is introduced for the study of entanglement of anyons in non-inertial frames. The results thus obtained will be compared with those obtained for bosons and fermions in non-inertial frames.

The paper comprises the following four sections. In Section II, a brief review is presented of previous studies of entanglement of fermion and boson modes in non-inertial frames. In Section III, a density matrix is proposed for anyon modes and entanglement variation in
response to varying accelerations is studied for different values of the fractional parameter. Finally, a summary of the results is presented in Section IV.

II. ENTANGLEMENT ENTROPY AND NEGATIVITY OF BOSONS AND FERMIONS

We consider an inertial observer, named Alice, who has a detector sensitive to modes $k_A$. Another observer, named Rob who moves with a uniform acceleration ($a$), has a detector sensitive to modes $k_R$. Then, we consider an entangled Bell state for the two maximally fermionic modes, $k_A$ and $k_R$ as follows:

$$|\psi_{k_A,k_R}\rangle = \frac{1}{\sqrt{2}}(|0_{k_A}\rangle + |0_{k_R}\rangle + |1_{k_A}\rangle + |1_{k_R}\rangle),$$  \hspace{1cm} (3)

where, $+$ is used to show the positive answers of Dirac fields (particles). We write the expansion of $|0_{k_A}\rangle$ and $|1_{k_A}\rangle$ in Rob’s and antiRob’s states in Rindler coordinates, which are in two different causally disconnected regions, $I$ and $II$, respectively. Finally, the reduced density matrix for the entangled state observed by Alice and Rob $|\rho(A,I)\rangle = Tr_{II}(|\psi\rangle\langle\psi|)$ for fermions is found as follows [8]:

$$\rho_f(A,I) = \frac{1}{2}((\frac{1}{1 + e^{(\frac{2\pi\omega_f}{a})}}) |0,1\rangle\langle 0,1| + |1,1\rangle\langle 1,1|$$
$$+ (\frac{1}{\sqrt{1 + e^{(\frac{2\pi\omega_f}{a})}}} |0,0\rangle\langle 0,1| + h.c.)$$
$$+ (\frac{1}{1 + e^{(\frac{2\pi\omega_f}{a})}} |0,0\rangle\langle 0,0|),$$  \hspace{1cm} (4)

where, $a$ is the relative acceleration, $\omega_f$ is the frequency of the modes detected by Alice and Rob, and $|a,b\rangle = |a\rangle_A |b\rangle_I$. One can construct the matrix form on the basis of $|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$, which has the eigenvalues

$$0, 0, \frac{1}{2} + \frac{1}{2 + e^{(2\pi\omega_f/a)}} |1,0\rangle, \frac{1}{2 + e^{(2\pi\omega_f/a)}} |0,0\rangle, \frac{1}{2 + e^{(2\pi\omega_f/a)}} |0,1\rangle, \frac{1}{2 + e^{(2\pi\omega_f/a)}} |1,1\rangle.$$

The density matrix for the bosonic case ($\rho_b$) is obtained as follows [7]:

$$\rho_b(A,I) = \frac{1}{2}(1 - exp(-\frac{2\pi\omega_b}{a})) \sum_{n=0}^{\infty} (exp(-\frac{\pi\omega_b}{a}))^{2n} \rho_n^{(n)}$$  \hspace{1cm} (5)

where, $\rho_n$ is

$$\rho_n^{(n)} = |0_k, n_k\rangle\langle 0_k, n_k|$$
$$+ (n + 1)(1 - e^{(-\frac{2\pi\omega_b}{a})}) |1_k, (n + 1)\rangle\langle 1_k, (n + 1)|$$
$$+ (n + 1) \sqrt{1 - e^{(-\frac{2\pi\omega_b}{a})}} |0_k, n_k\rangle\langle 1_k, (n + 1)| + h.c.)$$

Although $\rho_b$ is infinite-dimensional, it is block diagonal ($\rho_n^{(n)}$), allowing it to be diagonalized block by block, and its eigenvalues to be found. The entanglement entropies for the above density matrices, Eqs. (4) and (5) are plotted in Fig. (1) (for $\omega_f = \omega_b = 1$).

III. ENTANGLEMENTS OF ANYON MODES

As mentioned in the introduction, anyons are particles with intermediate statistics. It has been shown that a system of anyons in the two-dimensional space can be considered as the ensemble average of bosons and fermions with the fractions of $\alpha$ and $(1 - \alpha)$, respectively. This property stems from the fact that the density of states is constant in a two-dimensional space. Thus, the ensemble average of any thermodynamic quantity, like the internal energy or particle number, can be factorized as follows:

$$Q(\alpha) = \alpha Q_b + (1 - \alpha)Q_f$$  \hspace{1cm} (8)

where, $Q(\alpha)$ denotes the thermodynamic quantity of anyons and $Q_f$ and $Q_b$ refer to those of fermions and bosons, respectively [17–20]. We may note that the thermodynamics of a system can be obtained from the partition function or, equivalently, from the density matrix.
Exploiting this idea, we introduce a new density matrix as a combination of boson’s and fermion’s density matrices that mimics the behavior of anyons in non-inertial frames:

\[ \rho_a(A, I) = (1 - \alpha)\rho_b(A, I) \otimes I_f + \alpha I_b \otimes \rho_f(A, I), \quad (9) \]

where, \( \alpha \) is a parameter called the statistical parameter. \( I_f \) and \( I_b \) are identity matrices of dimensions \( \rho_f \) and \( \rho_b \), respectively. Since \( \rho_0 \) is a block diagonal matrix, we may diagonalize \( \rho_a \) block by block to find its eigenvalues. The block diagonal section \( (n, n + 1) \) of \( \rho_a \) is given by

\[ \rho_a^{(n)}(A, I) = (1 - \alpha)\rho_b^{(n)} \otimes I_{4 \times 4} + \alpha I_{4 \times 4} \otimes \rho_f, \quad (10) \]

and its eigenvalues \( \lambda_i^{(n)}(\rho_a) \) are the sum of those of \( \rho_f \) and \( \rho_b \)

\[ \lambda_i^{(n)}(\rho_a) = \{ (1 - \alpha)\lambda_i(\rho_b^{(n)}) + \alpha \lambda_j(\rho_f) \}. \quad (11) \]

We may now use the following identity for entanglement entropy, which is valid for both arbitrary density matrices \( \rho_1 \) and \( \rho_2 \) [22]:

\[ s(\rho_1 \otimes \rho_2) = Tr(\rho_1)s(\rho_2) + Tr(\rho_2)s(\rho_1). \quad (12) \]

The entanglement entropy of \( \rho_b^{(n)} \) may be defined as follows:

\[ s(\rho_b^{(n)}) := \frac{1}{4} \sum_i \lambda_i^{(n)}(\rho_b^{(n)}) \log_2(\lambda_i^{(n)}(\rho_b^{(n)})), \quad (13) \]

where, \( \frac{1}{4} \) is added to remove the additional counting of eigenvalues related to \( Tr(I_{4 \times 4}) \). The entanglement entropy of anyons will, then, be of the following form:

\[ s(\rho_a) = \sum_n s(\rho_a^{(n)}). \quad (14) \]

Entanglement entropy of anyons is plotted in Fig.(3). It is easy to check that \( \alpha = 0 \) and \( \alpha = 1 \) in Fig.(4) are the bosonic and fermionic cases as plotted in Fig. (1).

It is seen that the entanglement entropy increases with increasing acceleration. This is expected as entanglement entropy is not a suitable measure for mixed states [21]. Therefore other measures, like negativity, may be employed to explore the entanglement of the system. To compute the logarithmic negativity, we need to find the partial transpose of \( \rho_a \). For a density matrix of the following form

\[ \rho = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

the partial transpose is defined as follows:

\[ \rho_a^{PT} = \begin{pmatrix} A^{PT} & B^{PT} \\ C^{PT} & D^{PT} \end{pmatrix}. \quad (15) \]

This leads to (17) below:

\[ \rho_a^{PT}(A, I) = (1 - \alpha)\rho_b^{PT} \otimes I_{4 \times 4} + \alpha I_{4 \times 4} \otimes \rho_f^{PT}. \quad (17) \]

And, the eigenvalues of \( \rho_a^{PT} \) are given by those of \( \rho_b^{PT} \) and \( \rho_f^{PT} \) as follows:

\[ \lambda_i^{(\rho_a^{PT})} = \{ \alpha \lambda_i(\rho_f^{PT}) + (1 - \alpha)\lambda_j(\rho_b^{PT}) \}. \quad (18) \]

Finally, we evaluate the negativity of the anyon mode as follows:

\[ N(\rho) := \frac{1}{4} \sum_k |\lambda_k(\rho_a^{PT})| - \lambda_k(\rho_a^{PT})^2. \quad (19) \]
Again, $\frac{1}{4}$ is added to remove the additional counting of eigenvalues related to $\rho \otimes I_{4 \times 4}$ in Eq. (10). In this case, the logarithmic negativity of anyon modes is given by:

$$E_N = Log_2(1 + 2N(\rho)).$$

(20)

This is plotted for different values of $\alpha$ in Fig. (6).

![Logarithmic negativity of anyons as a function of $\alpha$.](image)

**FIG. 6.** Logarithmic negativity of anyons as a function of acceleration for different values of statistic parameter. Solid red curve: $\alpha = 1$, Solid yellow curve: $\alpha = 0.9$. Solid black curve: $\alpha = 0.8$. Solid green curve: $\alpha = 0.5$. Dashed orange curve: $\alpha = 0.4$. DotDashed gray curve: $\alpha = 0.2$. Solid blue curve: $\alpha = 0$.

As we see, $\alpha = 0$ and $\alpha = 1$ are the bosonic and fermionic cases, respectively. Whenever $\alpha$ nears 0, negativity declines with increasing acceleration. Ultimately, in the bosonic case of $\alpha = 0$, negativity becomes 0 in the limit $\alpha \rightarrow \infty$. Anyons with $\alpha = 0.5$ exhibit the least negativity at lower acceleration limits.

IV. COHERENCY

Quantum coherence forms another useful subject in quantifying quantum correlations. Unlike entanglement which is used for an interacting system, quantum coherence is of interest in systems with no interaction. Coherence measures more quantum correlations than entanglement does. From among the coherence quantifiers available, we use the relative entropy of coherence defined as follows: [23]

$$C_r = S(\rho_{\text{diagonal}}) - S(\rho),$$

(21)

where, $\rho_{\text{diagonal}}$ is obtained by deleting the off-diagonal elements of $\rho$. Using Eqs. (10) and (13), we can compute the relative entropy of coherence (Eq. (21)).

In Fig. (7), the relative entropy of coherence of this system is plotted for different values of the statistic parameter. As shown in Fig. (7), $C_r$ is the minimum value for $\alpha = 0.5$. As $\alpha$ distances more away from 0.5, $C_r$ rises to reach its maximum at $\alpha = 0$ (bosons).

![Relative entropy of coherence of anyons as a function of acceleration for different values $\alpha$.](image)

**FIG. 7.** Relative entropy of coherence of anyons as a function of acceleration for different values $\alpha$.

We proposed a density matrix for anyon modes as a linear combination of the density matrices of boson and fermion modes (Eq. (9)). We found the entanglement entropy of the system and compared it with those of the bosonic and fermionic cases. The entanglement between anyon modes was investigated by computing the logarithmic negativity of the system. In the bosonic limit, $\alpha = 0$, negativity was observed to vanish for $a \rightarrow \infty$, as expected. To consider all the possible correlations, we also computed the relative entropy of coherence to find that anyons are always coherent for all values of both acceleration and statistical parameter. Semions, with $\alpha = 0.5$, were found to be less coherent than those with other values of $\alpha$. It is interesting to further explore relativistic quantum information of anyons in other curved spacetimes [24].

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