REPRESENTATIONS OF THE MIRACULOUS KLEIN GROUP

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Abstract. The Klein group contains only four elements. Nevertheless this little group contains a number of remarkable entry points to current highways of modern representation theory of groups. In this paper, we shall describe all possible ways in which the Klein group can act on vector spaces over a field of two elements. These are called representations of the Klein group. This description involves some powerful visual methods of representation theory which builds on the work of generations of mathematicians starting roughly with the work of K. Weiestrass. We also discuss some applications to properties of duality and Heller shifts of the representations of the Klein group.

1. Introduction

Consider the familiar complex plane $\mathbb{C} = \{x + iy \mid x, y \text{ are real numbers}\}$ with two reflections $\sigma$ and $\tau$ in the standard axes $X$ and $Y$ respectively. Precisely, we have

\[
\sigma(x + iy) = x - iy, \quad \text{and} \quad \tau(x + iy) = -x + iy.
\]

Thus, $\sigma$ is the complex conjugation and $\tau$ is like a real brother of $\sigma$. Note that if we apply $\sigma$ or $\tau$ twice, we get the identity map: $\sigma^2 = 1 = \tau^2$. Also, we see that $\sigma \tau = \tau \sigma = -1$. Geometrically, the maps $\sigma \tau$ and $\tau \sigma$ are rotations by 180 degrees in the complex plane. The set of maps $\{1, \sigma, \tau, \sigma \tau\}$ forms a group under composition and is called the Klein four group or just Klein group, often denoted by $V_4$. One would guess that the letter $V$ here is a sign of victory but the reason is that “Vier” in German means “four.” “Klein” in German also means “small” and indeed Klein group $V_4$ having only four elements is quite small. It is an absolutely amazing fact that this small and ostensibly innocent group contains remarkable richness and that important mathematics can be developed by just studying this one group. The world’s smallest field is $\mathbb{F}_2 = \{0, 1\}$, and one can think of this as a toy model of complex numbers. The problem to be investigated in this paper is the following: What are all the finite dimensional representations of $V_4$ over $\mathbb{F}_2$? That is, can one describe all possible actions of the group $V_4$ on finite dimensional vector spaces $W$ over $\mathbb{F}_2$. Although, we work over an arbitrary

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field of characteristic two, not much is lost if the reader assumes throughout that
the ground field \( k \) is \( \mathbb{F}_2 \). The reason for restricting to fields of characteristic 2 is
due to the fact that when the characteristic of the ground field \( k \) is either zero
or odd, the finite dimensional representations \( W \) of \( V_4 \) have a very simple nature.
Namely, \( W \) is a sum of one dimensional representations. One each of these one-
dimensional subspaces, the generators \( \sigma \) and \( \tau \) act as multiplication by 1 or \(-1\). We
therefore stick with fields of characteristic 2. This bring us to the world of modular
representations. (That is, the characteristic of the field is a positive divisor of the
order of the group.)

Note that \( V_4 \) is a product of two cyclic groups of order two. In terms of generators
and relations, \( V_4 \) has the following presentation.

\[
V_4 = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma \rangle.
\]

The group algebra \( kV_4 \) is then isomorphic to

\[
k[a, b]/(a^2, b^2),
\]

where \( a \) corresponds to \( 1 + \sigma \) and \( b \) to \( 1 + \tau \). We define an ideal \( U \) of the \( kV_4 \) as
the ideal generated by \( a \) and \( b \). This is an extremely important ideal called the
augmentation ideal of our group ring. Sometime it will be convenient to divide
\( kV_4 \) by ideal generated by \( ab \). This simply amounts adding further relation \( ab = 0 \).
The reason for this is that often \( ab \) acts on our vector spaces as 0 and therefore
why not simplify our ring even further and add the relation \( ab = 0 \)? We still call
the image of \( U \) in this new ring \( U \) as we do not want to make our notation too
complicated. In this paper we present a rather accessible proof of the well-known
classification of all the finite dimensional representations of \( V_4 \), or equivalently, of
the finitely generated indecomposable \( kV_4 \)-modules. These are also known as the
modular representations of \( V_4 \). Note that a \( V_4 \) representation where \( ab \) acts as zero
can be viewed as a finite dimensional \( k \)-linear space equipped with a pair commuting
linear maps \( a \) and \( b \) both of which square to zero.

Having explained what a representation of \( V_4 \) is, the following two questions have
to be answered.

1. Why do we care about the representations of \( V_4 \)?
2. What is unique about our approach?

In answer to the first question, first note that groups act naturally on various
algebraic objects including vector spaces, rings, algebraic varieties and topological
spaces. These actions tend to be quite complex in general. Therefore it is important
to find simple pieces of this action and find ways to glue these pieces together
to reconstruct the original action. Often this is related to other invariants of the
group or our given representation like cohomology groups and support varieties.
Amazingly, this goal in modular representation theory turns out to be exceedingly
difficult. It turns out that besides the cyclic groups whose representations are very
easily understood, Klein four group is one of the very few (other groups are the dihe-
dral groups) interesting yet non-trivial examples for which representation theorists
are able to completely classify all the finite dimensional modular representations.
There is a lot to be learned by studying the representation theory of this one group and it goes to tell how complex the study of modular representations can be for an arbitrary group.

Now we turn to the second question. Although the classification of the finite dimensional representations of $V_4$ is well-known and many proofs can be found in the literature, we could not find a proof to our heart’s content. This is what motivated us to write up one – one that is transparent and which takes a minimal background. Furthermore, our approach is diagrammatic, so the reader can see what is happening through pictures. These methods, besides making the statements of theorems and proofs elegant and conceptual, give a better insight into the subject.

We mostly follow Benson’s approach [?] but we approach some parts of his proof from a different point of view and simplify them and in particular we make our proof accessible for a general reader. One ingredient that is new in our approach is Auslander-Reiten sequences which will be introduced later in the paper.

The subject of classifying the indecomposable representations of the Klein group has a long and rich history that can be traced all the way back to V. A. Bašev [?], a student of I.R. Šafarevič who observed that an old result of L. Kronecker on pairs of matrices can be used effectively in the classification, but over algebraically closed fields. This result of L. Kronecker on pairs of matrices was actually a completion of the work of K. Weierstrass. Then later on I. M. Gelfand and V. Ponomarev [?] observed in their analysis of the representations of the Lorentz group that quiver techniques were quite useful and they both knew that G.Szekeres had a result in this direction. However, they did not know enough details about Szekeres’s techniques and therefore they invented their new innovative and influential quiver method which is influenced by Maclane’s notion of relations – a generalization of a linear map. In [?] A. Heller and I. Reiner provided another nice approach to the classification where they also worked over fields that are not necessarily algebraically closed. Finally D. Benson [?] wrote a modern treatment of the classification of the indecomposable representations of the Klein group in which he combined some of the crucial ideas in the works of the aforementioned people. The diagrammatic methods in our paper are inspired by S. B. Conlon who introduced these in [?]. It is quite remarkable that a complete understanding of an innocent looking group on four elements would take the works of some of the great minds of the 19th, 20th, and 21st centuries.

Before going further, we remind the reader some basis facts and terminology. We refer the reader to Carlson lecture notes for basic representation theory [?]. In the category of modules over a the Klein group (or more generally, over a p-group), the three terms “injective”, “projective” and “free” are synonymous. Given a $V_4$-module $M$, its Heller shift $\Omega(M)$ is defined to be the kernel of a minimal projective cover of $M$. It can be shown that minimal projective covers are unique up to isomorphism and from that it follows that $\Omega(M)$ is well-defined. Inductively one defines $\Omega^n(M)$ to be $\Omega(\Omega^{n-1}M)$. Similarly, $\Omega^{-1}M$ is defined to be the cokernel of an injective envelope of $M$, and $\Omega^{-n}(M)$ to be $\Omega^{-1}(\Omega^{-n+1}M)$. Again one can shown that these are well-defined modules. The modules $\Omega^nM$ are also known as
the syzygies of $M$. By the classical Krull-Remak-Schmidt theorem, one knows that every representation of a finite group decomposes as a direct sum of indecomposable ones. Thus it suffices to classify the indecomposable representations.

Advice for the novice: some arguments in our paper are only sketched and some notions maybe still unfamiliar for a novice. If that is the case, we advice readers to skip these parts on the first reading as they may became more clear later on and they most likely will not influence the basic understanding of the key ideas. The main point of this article is to provide overview of the remarkable proof of classification of representations of Klein group $V_4$ with appreciation of the works of number of people and to show that this proof open doors to study modern group representation where Auslander-Reiten sequences play increasingly important role. We hope that after reading our article a reader will read more texts in the references and possibly go on to further exciting heights in group representation theory.

2. INDECOMPOSABLE REPRESENTATIONS OF KLEIN’S FOUR GROUP

We list all the indecomposable representations of $V_4$ below. Note that these are just the finitely generated modules over the group algebra $kV_4 \cong k[a, b]/(a^2, b^2)$ which cannot be written as a sum of strictly smaller modules (much the same way prime numbers cannot be written as product of smaller numbers). Since we take a diagrammatic approach, we first explain the diagrams that follow. Each bullet represents a one dimensional $k$ vector space, a southwest arrow "\downarrow\bullet" connecting two bullets corresponds to the action of $a$ and maps one bullet to the other in the indicated direction, and similarly the south east arrows "\downarrow\bullet" correspond to the action of $b$. If no arrow emanates from a bullet in given direction, then the corresponding linear action is understood to be zero.

**Theorem 2.1.** (Kronecker, Weierstrass, Basev, Gelfand, Ponomarev, Conlon, Heller, Reiner, Benson) [?, ?, ?, ?, ?] Let $k$ be a field of characteristic 2. Every isomorphism class of an indecomposable $V_4$ representation over $k$ is precisely one of the following.

1. The projective indecomposable module $kV_4$ of dimension 4.

\[ a \quad \rightarrow \quad b \]

2. The (non-projective) indecomposable even dimensional modules:
   (a) For each even dimension $2n$ and an indecomposable rational canonical from corresponding to the power of an irreducible monic polynomial $f(x)^l = \sum_{i=0}^{n} \theta_i x^i$, $(\theta_n = 1)$ there is an indecomposable representation

\[ a \quad \rightarrow \quad b \]

\[ \bullet \quad \downarrow \bullet \quad \downarrow \bullet \quad \downarrow \bullet \quad \downarrow \bullet \]
given by
\[ g_{n-1} \rightarrow g_{n-2} \rightarrow g_{n-3} \rightarrow \cdots \rightarrow g_0 \]
where \( a(g_{n-1}) = \sum_{i=0}^{n-1} \theta_i f_i \), as represented by the vertical dotted arrow emanating from \( g_{n-1} \) above.

(b) For each even dimension \( 2n \) there is an indecomposable representation given by
\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]
(3) The (non-projective) indecomposable odd dimensional modules:
(a) The trivial representation \( k \).

(b) For each odd dimension \( 2n + 1 \) greater than one, there is an indecomposable representation given by
\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]

(c) For each odd dimension \( 2n + 1 \) greater than one, there is an indecomposable representation given by
\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]

The reader may decide to make a pleasant check that the above diagrams are indeed representations of \( V_4 \). For each \( V_4 \)-module \( M \), one can define the dual \( V_4 \)-module \( M^* \), where \( M^* \) is the dual \( k \)-vector space of \( M \), and a group element \( \sigma \) of \( V_4 \) acts on \( f \) in \( M \) via the rule \( \sigma f(m) = f(\sigma^{-1}m) \). Then, as a fun exercise we ask the reader to verify that the diagrams in 3(a) and 3(b) are dual to each other. This will help the reader to get acquainted with some of the diagrammatic methods that will appear later on. We now begin by proving the easy part of the theorem.

**Lemma 2.2.** All representations of \( V_4 \) that appear in the above theorem are indecomposable and pair-wise non-isomorphic.

**Proof.** Item (1) is not isomorphic to the rest because it is the only module that contains a non-zero element \( x \) such that \((ab)x \neq 0\). Modules in item (2) are even dimensional and those in (3) are odd dimensional and hence there is no overlap between the two. To see that all the \( 2n \) dimensional representations of item 2(a)
are distinct, it is enough to observe that the rational canonical forms of the linear transformations on the co-invariant submodules,

\[ b^{-1}a : M/UM \rightarrow M/UM \]

are distinct, where \( U \) is the ideal generated by \( a \) and \( b \). To see that the \( 2n \)-dimensional representation of item 2\((b)\) does not occur in item 2\((a)\) observe that kernel of the \( b \)-action in both cases have different dimensions: \( n \) for the module in 2\((a)\) and \( n + 1 \) for that in 2\((b)\). The two \( 2n + 1 \)-dimensional modules in items 3\((b)\) and 3\((c)\) are non-isomorphic because it is clear from the diagrams that the dimensions of the invariant submodules in both cases are different: \( n \) for those in item 3\((b)\), and \( n + 1 \) for those in item 3\((c)\).

\[ \square \]

Of course the hard thing is to show that every indecomposable representation of \( V_4 \) is isomorphic to one in the above list. Since projective modules over \( p \)-groups are free, there is only one indecomposable projective \( V_4 \)-module, namely \( kV_4 \) which occurs as item (1) in the list. Therefore we only consider indecomposable projective-free (modules which do not have projective summands) \( V_4 \)-modules.

One can get a better handle on the projective-free representations of \( V_4 \) by studying the representations of the so called Kronecker Quiver, which is a directed graph \( Q \) on two vertices as shown below.

\[ u_1 \bullet \overset{f}{\longrightarrow} \overset{g}{\longrightarrow} \bullet u_2 \]

A representations of the above quiver is just a pair of finite dimensional \( k \)-vector spaces \( V \) and \( W \) and a pair of \( k \)-linear maps \( \psi_1 \) and \( \psi_2 \) from \( V \) to \( W \). Such a representation will be denoted by the four tuple \([V,W;\psi_1,\psi_2]\), and given two such representations, the notion of direct sum, and morphisms between them are defined in the obvious way. Thus it makes sense to talk about the isomorphism class of an indecomposable representation of \( Q \). Let us call a representation of \( Q \) special if the following conditions hold:

\[ \text{Ker}(\psi_1) \cap \text{Ker}(\psi_2) = 0 \]

\[ \text{Image}(\psi_1) + \text{Image}(\psi_2) = V_2. \]

**Proposition 2.3.** [?] There is a one-one correspondence between the isomorphism classes of (indecomposable) projective-free representations of \( V_4 \) and those of the special (indecomposable) representations of the Kronecker quiver. Under this correspondence, an (indecomposable) projective-free representation \( M \) of \( G \) corresponds to the (indecomposable) representation of \( Q \) that is given by \([M/UM,UM;a,b]\). Conversely, given an (indecomposable) special representation \([V,W;\psi_1,\psi_2]\) of \( Q \), the corresponding (indecomposable) \( G \)-module \( M \) is given by \( M = V \oplus W \) where \( a(\alpha,\beta) := (0,\psi_1(\alpha)) \) and \( b(\alpha,\beta) := (0,\psi_2(\alpha)) \).

We will use this translation between the representations of the Klein group and the Kronecker Quiver freely throughout the paper.
If \( M = [V_1, V_2; a, b] \) is an indecomposable projective-free representation, then we have
\[
V_1 = 0 \iff M = 0 \\
V_2 = 0 \iff M = k.
\]
So henceforth it will be assumed that the spaces \( V_1 \) and \( V_2 \) are non-zero, i.e., \( M \) is an indecomposable projective-free and a non-trivial representation of \( V_4 \).

We begin with some lemmas that will help streamline the proof of the classification theorem. The proofs of these lemmas will be deferred to the last section. It should be noted that these lemmas are also of independent interest.

**Lemma 2.4.** Let \( M \) be a projective-free \( V_4 \)-module given by \( [V_1, V_2; a, b] \). Then we have the following.

1. If \( l \) is the smallest positive integer such that \( \Omega^l(k) \) is isomorphic to a submodule of \( M \), then \( \Omega^l(k) \) is a summand of \( M \).

**Lemma 2.5.** Let \( M \) be a projective-free \( V_4 \)-module. Then we have the following.

1. If \( l \) is the smallest positive integer such that \( \Omega^l(k) \) is isomorphic to a submodule of \( M \), then \( \Omega^l(k) \) is a summand of \( M \).

**Lemma 2.6.** For all integers \( n \), \( \Omega^n(k) \) is isomorphic to the dual representation \( \Omega^{-n}(k)^* \). Furthermore,

1. If \( n \) is positive, then \( \Omega^n(k) \) is a \( 2n + 1 \) dimensional indecomposable representation given by

\[
\begin{array}{cccccccc}
\bullet & \downarrow b & & \bullet & & \bullet & & \bullet & & \bullet \\
& & & \bullet & & \bullet & & \bullet & & \bullet \\
& & & & & \bullet & & \bullet & & \bullet \\
& & & & & & \bullet & & \bullet & & \bullet \\
& & & & & & & & \bullet & & \bullet \\
\end{array}
\]
(2) If $n$ is a negative integer, then $\Omega^n(k)$ is a $2n+1$ dimensional indecomposable representation given by

We now give the proof of the classification theorem assuming these lemmas. The lemmas will be proved in the last section. Let $M = (V_1, V_2; a, b)$ be an indecomposable projective-free representation of $V_4$. We will show that $M$ is isomorphic to one of the representation that appear in items (2) it is even dimensional, and to those in item (3) if it odd dimensional.

2.1. Even dimensional representations. Let $M$ be an even dimensional (2$n$ say) indecomposable representation. We break the argument into cases for clarity.

Case 1: $\det(a + \lambda b)$ is non-zero. We have two subcases. First assume that $\det b \neq 0$. Then consider the map $b^{-1}a : V_1 \to V_1$.

We claim that this map is indecomposable. Suppose we have a decomposition $f \oplus g$ of $b^{-1}a$ as follows

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Set $C := b(A)$ and $D := b(B)$. Then it is straightforward to verify that $M = (V_1, V_2; a, b)$ decomposes as

Since $M$ is indecomposable it follows that the map $b^{-1}a$ is indecomposable. Now since $b^{-1}a$ is indecomposable we can choose a basis $\{g_0, g_1, \ldots, g_{n-1}\}$ of $V_1$ such that the rational canonical form of $b^{-1}a$ has only one block which corresponds to some power of an irreducible polynomial $f(x) = \sum_{i=0}^{n-1} \theta_i x^i$. This means we have

Now the vectors $f_i := b(g_i)$ for $0 \leq i \leq n - 1$ define a basis for $V_2$ because $b$ is non-singular. With respect to the bases $(g_i)$ of $V_1$ and $(f_i)$ of $V_2$, it is now clear
that $M$ has the description

\[
\begin{array}{cccc}
  g_{n-1} & g_{n-2} & g_{n-2} & \ldots \\
  b & a & f_{n-1} & f_{n-2} \\
  f_{n} & f_{0} & \ldots & \\
\end{array}
\]

The action of $a$ on $g_{n-1}$ can be seen by applying $b$ on both sides of the equation (*) above: $a(g_{n-1}) = \sum_{i=0}^{n-2} b(g_i) = \sum_{i=0}^{n-1} f_i$. These representations are the exactly ones in item 3(a).

Now if $\det(b) = 0$, we do a change of coordinate trick. We assume that $k$ is an infinite field. If $k$ is finite, we can pass to an extension field and do a descent argument; see [2] for details. Then there exists some $\lambda_0$ in $k$ such that $\det(a + \lambda_0 b) \neq 0$.

Consider the tuple $(V_1, V_2; b, a + \lambda_0 b)$. By case (i), we know that there exist bases for $V_1$ and $V_2$ such that $a + \lambda_0 b = I$ and $b = J_0$ (the rational canonical form [2, 4] corresponding to any indecomposable singular transformation). This gives the representation in item 2(b).

**Case 2:** $\det(a + \lambda b) = 0$. We will show that this case cannot arise. First suppose that there is a copy of $\Omega^l(k)$ in $M$ for some positive integer $l$. Now pick $l$ to be the smallest such integer, then by lemma 2.5 we know that $\Omega^l(k)$ is a direct summand of $M$. Since $M$ is indecomposable, this means $M$ has to be isomorphic to $\Omega^l(k)$, which is impossible since the latter is odd dimensional while $M$ was assumed to be even dimensional. So the upshot is that $M$ does not contain $\Omega^l(k)$ for any positive $l$. By lemma 2.4 this is equivalent to the fact $\det(a + \lambda b) \neq 0$ in the ring $k[\lambda]$ which contradicts our hypothesis.

2.2. **Odd dimensional representations.** If $M$ is odd dimensional, then clearly $\dim V_1 \neq \dim V_2$. We consider the two cases.

**Case 1:** $\dim V_1 > \dim V_2$. Then there is a non-zero vector $\omega(\lambda)$ in $V_1 \otimes_k K[\lambda]$ such that $(a + \lambda b)(\omega(\lambda)) = 0$ which then implies, by lemma 2.4 the existence of a copy of $\Omega^l(k)$ inside $M$ for some $l > 0$. Picking $l$ to be minimal, we can conclude from lemma 2.5 that $\Omega^l(k)$ is a direct summand of $M$. Since $M$ is indecomposable, we have $M \cong \Omega^l(k)$. This gives the modules in item 3(b).

**Case 2:** $\dim V_1 < \dim V_2$. Dualising $M = (V_1, V_2; a, b)$, we get the dual representation $M^* = (V_2^*, V_1^*; a^*, b^*)$ which is also indecomposable. Now $\dim V_2^* > \dim V_1^*$, so by Case(1) we know that $M^* \cong \Omega^l(k)$ for some $l$ positive. Taking duals on both sides and invoking lemma 2.7 we get $M \cong \Omega^{-l}(k)$. This recovers the modules in item 3(c).

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1Most readers are familiar with the Jordan canonical form of an operator acting on a vector space over $\mathbb{C}$ or other algebraically closed fields. These forms use critically the fact that non-constant polynomials have roots. However, a parallel and beautiful theory also exists when the field is not algebraically closed, and this is not so well-known. One often thinks about the base field as the field of rational numbers and the name “The rational canonical form” stick also to completely different fields including $\mathbb{F}_2$. 

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This completes the proof of the classification of the indecomposable representations of \( V_4 \).

3. SOME APPLICATIONS

Having a good classification of the indecomposable representations of a finite group helps a great deal in answering general module theoretic questions. In this section, we illustrate this by proving some facts about module over the Klein group. Note that we don’t know of any direct proofs of the statements below that do not depend on the classification of the indecomposable representations.

3.1. Heller Shifts of the \( V_4 \)-representations. We will show how our knowledge of the representations of \( V_4 \) can be used to give a homological characterisation of the parity of the dimensions of the representations. Proofs of the propositions are given in the last section.

**Proposition 3.1.** [?] If \( M \) is an even dimensional indecomposable projective-free representation of \( V_4 \), then \( \Omega(M) \cong M \).

**Corollary 3.2.** A finite dimensional projective-free representation \( M \) of \( V_4 \) is even dimensional if and only if \( \Omega(M) \cong M \).

**Proof.** We only have to show that if \( M \) is an odd dimensional indecomposable then \( \Omega(M) \not\cong M \). By the classification theorem, we know that \( M \) is isomorphic to \( \Omega^l(k) \) for some integer \( l \). Then \( \Omega(M) \cong \Omega(\Omega^l(k)) \cong \Omega^{l+1}(k) \), which is clearly not isomorphic to \( M \) just for dimensional reasons: just note that dimension of \( \Omega^n(k) \) is \( 2n+1 \).

3.2. Dual representations of \( V_4 \). We will use our knowledge of the representations of \( V_4 \) to characterise the parity of the dimension of a representation using the concept of self-duality.

Recall that if \( M \) is a finite dimensional representation of a group \( G \), then one can talk about the dual representation \( M^* := \text{Hom}(M,k) \), where a group element \( g \) acts on a linear functional \( \phi \) by \( (g \cdot \phi)(x) := \phi(xg^{-1}) \). A representation of \( G \) is self-dual if it is isomorphic to its dual.

When \( G = V_4 \), it is not hard to see that if \( M = (V_1,V_2; a,b) \) is a projective-free representation of \( V_4 \), then \( M^* = (V_2^*,V_1^*; a^*,b^*) \).

**Proposition 3.3.** Even dimensional indecomposable representations of \( V_4 \) are self-dual.

**Corollary 3.4.** A non-trivial indecomposable representation of \( V_4 \) is even dimensional if and only if it is self-dual.

**Proof.** If \( M \) is a non-trivial odd dimensional representation of \( V_4 \), then we know that \( M \cong \Omega^l(k) \) for some \( l \neq 0 \). Then \( M^* \cong (\Omega^l(k))^* \cong \Omega^{-l}(k) \). In particular, \( M^* \not\cong M \).
4. Proofs

In this section we give the proofs of the lemmas and propositions that were used in the classification theorem and applications.

4.1. **Proof of proposition 2.3.** Let $M$ be a projective-free $V_4$ module. Then we have we have $ab(M) = 0$, it follows that $UM$ is included in $U_4'$. Remarkably one can show that if $M$ is additionally not trivial representation and $M$ is indecomposable then $UM$ is actually equal $M_{1'}$, the $V_4$ invariant submodule of $M$. Consider short exact sequence of $V_4$ modules

$$0 \to UM \to M \to M/UM \to 0.$$ 

Let $\pi : M \to UM$ be a vector space retraction of the inclusion $UM \hookrightarrow M$. Define a $V_4$ action on the vector space $M/UM \oplus M$ as follows:

$$a(x, y) := (0, ax)$$

$$b(x, y) := (0, by).$$

Then it is easy to verify that the map $x \mapsto (x, \pi(x))$ establishes an isomorphism of $V_4$ modules between $M$ and $M/UM \oplus UM$. Thus $M$ is determined by the vector spaces $M/UM$ and $UM$ and the linear maps $a, b : M/UM \to M$. This data amounts to giving a special representation of $Q$.

In the other direction, suppose $[V_1, V_2; \psi_1, \psi_2]$ is a special representation of $Q$. Define a $V_4$ action on the vector space $V_1 \oplus V_2$ by setting $a(x, y) := (0, \psi_1(x))$ and $b(x, y) := (0, \psi_2(x))$. This is easily shown to be a projective free $V_4$ module.

We leave it as an exercise to the reader to verify that the recipes are inverses to each other.

It is also clear that these recipes respect direct sum of representations. Thus the indecomposables are also in 1-1 correspondence.

4.2. **Proof of lemma 2.4.** Suppose $M$ contains a copy of $\Omega_l(k)$, for some $l \geq 1$.

Define a vector $V(\lambda) := g_0 + g_1\lambda + g_2\lambda^2 + \cdots + g_l\lambda^l$. A trivial verification shows that $(a + \lambda b)(V(\lambda)) = 0$ and therefore $a + \lambda b$ is a singular transformation as desired.

Conversely, suppose $a + \lambda b$ is singular. Then there is a non-zero vector $V(\lambda) = g_0 + g_1\lambda + g_2\lambda^2 \cdots g_l\lambda^l$ of smallest degree $l$ in $V_1 \otimes \mathbb{k}[\lambda]$ (so $g_l \neq 0$) such that $(a + b\lambda)(V(\lambda)) = 0$. This means: $a(g_0) = 0$, $b(g_i) = a(g_{i+1})$ for $0 \leq i \leq l-1$, and $b(g_l) = 0$. We now argue that these equations give a copy of $\Omega_l(k)$ inside $M$. To this end, it suffices to show that the vectors $\{g_0, g_1, g_2, \cdots, g_l\}$ are linearly independent. As a further reduction, we claim that it suffices to show that $\{a(g_1), a(g_2), \cdots, a(g_l)\}$ are linearly independent. For, then it will be clear that $\{g_1, g_2, \cdots, g_l\}$ is linearly independent, and moreover if $g_0 = \sum_{i=1}^l c_i g_i$, applying $a$ on both sides we get
\[ a(g_0) = 0 = \sum_{i=1}^t c_i a(g_i). \] Linear independence of \( a(g_i) \) forces all the \( c_i = 0 \). Thus we will have shown that \( \{g_0, g_1, g_2, \cdots, g_t\} \) is linearly independent. So it remains to establish our claim that \( \{a(g_1), a(g_2), \cdots, a(g_t)\} \) is a linearly independent set.

Suppose to the contrary that there is a non-trivial linear combination of \( a(g_i) \)'s which is zero: say \( \sum_{i=1}^t \gamma_i a(g_i) = 0 \) (*). We will get a contradiction by showing that there is a vector of smaller degree \( \langle \ell \rangle \) in \( \text{Ker}(a + \lambda b) \). It suffices to produce elements \( (\tilde{g}_i)_{0 \leq i \leq t - 1} \) such that \( a(\tilde{g}_0) = 0, b(g_{t-1}) = 0 \), and for \( 0 \leq i \leq t - 2 \), \( b(\tilde{g}_i) = a(g_{i+1}) \) (*). For then the vector \( \sum_{i=0}^{t-1} \tilde{g}_i \lambda^i \) will be of degree less than \( \ell \) belonging to the kernel of \( a + \lambda b \).

To start, we set \( \tilde{g}_0 = \sum_{i=0}^t \gamma_i g_i \). The condition \( a(\tilde{g}_0) = 0 \) is satisfied by assumption (*). Now define \( \tilde{f}_0 := b(\tilde{g}_0) = \sum_{i=1}^t \gamma_i b(g_i) = \sum_{i=0}^{t-1} \gamma_i b(g_i) \) (since \( b(g_t) = 0 \)). Then we define \( \tilde{g}_1 = \sum_{i=0}^{t-1} \gamma_i g_{i+1} \) so that we have the required condition \( a(\tilde{g}_1) = b(\tilde{g}_0) \).

Now we simply repeat this process: Inductively we define, for \( 0 \leq t \leq l - 1 \),

\[
\begin{align*}
\tilde{g}_t &= \sum_{i=1}^{t-1} \gamma_i g_{i+t}, \\
\tilde{f}_t &= \sum_{i=1}^{t-1} \gamma_i b(g_{i+t}).
\end{align*}
\]

When \( t = l - 1 \), we have \( \tilde{g}_{l-1} = \gamma_1 g_1 \) and \( \tilde{f}_{l-1} = 0 \). So this inductive process terminates at \( t = l - 1 \langle \ell \rangle \) and the requirements (*) are satisfied by construction. Thus we have shown that the vector \( \sum_{i=0}^{l-1} \tilde{g}_i \lambda^i \) is of smaller degree in the kernel of \( a + \lambda b \) contradicting the minimality of \( \ell \). Therefore the vectors \( \{a(g_1), a(g_2), \cdots, a(g_t)\} \) should be linearly independent. This completes the proof of the first statement in the lemma. The second statement follows by a straightforward duality argument.

### 4.3. Proof of lemma 2.5

First note that the second part of this lemma follows by dualising the first part; here we also use the fact that \( (\Omega^l k)^* \cong \Omega^{-l} k \) which will be proved in the next lemma. So it is enough to prove the first part. Although this lemma is secretly hidden in Benson’s treatment [2, Theorem 4.3.2], it is hard very to extract it. So we give a clean proof of this lemma using almost split sequences, a.k.a Auslander-Reiten sequences. Recall that a short exact sequence

\[ 0 \to A \xrightarrow{f} B \to C \to 0 \]

of finitely generated modules over a group \( G \) is an almost split sequence if it is a non-split sequence with the property that every map out of \( A \) which is not split injective factors through \( f \). It has been shown in [2] that given an finitely generated indecomposable non-projective \( kG \)-module \( C \), there exists a unique (up to isomorphism of short exact sequences) almost split sequence terminating in \( C \). In particular, if \( G = V_4 \) and \( C = \Omega^l k \), these sequences are of the form; see [2, Appendix, p 180].

\[
\begin{align*}
0 &\to \Omega^{l+2} k \to \Omega^{l+1} k \oplus \Omega^{l+1} k \to \Omega^l k \to 0 & l \neq -1 \\
0 &\to \Omega^l k \to kV_4 \oplus k \to \Omega^{-1} k \to 0
\end{align*}
\]
To start the proof, let \( l \) be the smallest positive integer such that \( \Omega^l k \) embeds in a projective-free \( V_4 \)-module \( M \). If this embedding does not split, then by the property of an almost split sequence, it should factor through \( \Omega^{l-1} k \oplus \Omega^{l-1} k \) as shown in the diagram below.

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^l k & \rightarrow & \Omega^{l-1} k \oplus \Omega^{l-1} k & \rightarrow & \Omega^{l-2} k & \rightarrow & 0 \\
& & \downarrow f \oplus g & & \downarrow & & \downarrow & & \\
& & M & & & & & & \\
\end{array}
\]

Now if either \( f \) or \( g \) is injective, that would contradict the minimality of \( l \), so they cannot be injective. So both \( f \) and \( g \) should factor through \( \Omega^{l-2} k \oplus \Omega^{l-2} k \) as shown in the diagrams below.

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^{l-1} k & \rightarrow & \Omega^{l-2} k \oplus \Omega^{l-2} k & \rightarrow & \Omega^{l-3} k & \rightarrow & 0 \\
& & \downarrow f \rightarrow (f_1 \oplus f_2) & & \downarrow & & \downarrow & & \\
& & M & & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^{l-1} k & \rightarrow & \Omega^{l-2} k \oplus \Omega^{l-2} k & \rightarrow & \Omega^{l-3} k & \rightarrow & 0 \\
& & \downarrow g \rightarrow (g_1 \oplus g_2) & & \downarrow & & \downarrow & & \\
& & M & & & & & & \\
\end{array}
\]

Proceeding in this way we can assemble all the lifts obtained using the almost split sequences into one diagram as shown below.
So it suffices to show that for a projective-free $M$ there cannot exist a factorisation of the form

\[
\begin{array}{c}
\Omega^l k^c \\
\downarrow \phi \\
(kV_4)^* \oplus k^t
\end{array} \rightarrow M
\]

where $l$ is a positive integer. It is not hard to see that the invariance $(\Omega^l k)^G$ of $\Omega^l k$ maps into $((kV_4)^*)^G$. We will arrive at a contradiction by showing $((kV_4)^*)^G$ maps to zero under the map $\phi$. Since $((kV_4)^*)^G \cong ((kV_4)^G)^*$ it is enough to show that $\phi$ maps each $(kV_4)^G$ to zero. $(kV_4)^G$ is a one-dimensional subspace, generated by say $v$. It $v$ maps to a non-zero element, then it is easy to see that the restriction of $\phi$ on the corresponding copy of $kV_4$ is injective, but $M$ is projective-free, so this is impossible. In other words $\phi(v) = 0$ and that completes the proof of the lemma.

4.4. Proof of lemma 2.6. Recall that $\Omega^1(k)$ is defined to be the kernel of the augmentation map $kV_4 \rightarrow k$. Dualising the short exact sequence

\[
0 \rightarrow \Omega^1(k) \rightarrow kV_4 \rightarrow k \rightarrow 0,
\]

we get

\[
0 \leftarrow \Omega^1(k)^* \leftarrow kV_4 \leftarrow k \leftarrow 0
\]

because $kV_4$ and $k$ are self-dual. This shows that $\Omega^{-1}(k) \cong \Omega^1(k)^*$. Now a straightforward induction gives $\Omega^{-l}(k) \cong \Omega^l(k)^*$ for all $l \geq 1$.

So it is enough to prove the part (1) of the lemma because it is not hard to see that the representations in part (2) are precisely the duals of those in part (1). We leave this as an easy exercise to the reader.

As for (1) we will prove the cases $n = 1$ and $n = 2$. The general case will then be abundantly clear. For $n = 1$, we have to identify the kernel of the augmentation map $kV_4 \rightarrow k$ which is defined by mapping the generator $e_0$ of $kV_4$ to the basis element $g_0$ of $k$, so the kernel $\Omega^1(k)$ is a three dimensional representation as shown in the diagram below

\[
0 \rightarrow e_0 \rightarrow a_0 \rightarrow b_0 \rightarrow g_0 \rightarrow 0
\]

Now consider the case $n = 2$. Note the $\Omega^1(k)$ is generated by two elements $g_0$ and $g_1$. So a minimal projective cover will be $kV_4 \oplus kV_4$ generated by $e_0$ and $e_1$. The projective covering maps $e_i$ to $g_i$, $i = 0, 1$. The kernel $\Omega^2(k)$ of this projective
covering will be 5-dimensional and can be easily seen in the diagram below.

\[
\begin{array}{c}
0 \to a_1 \downarrow b_0 \oplus a_0 \to c_1 \oplus a_0 \to b_0 \oplus a_0 \to 0
\end{array}
\]

Now it is clear that in general \( \Omega^l(k) \) for \( l \geq 1 \) will be a \( 2l + 1 \) dimensional representation and has the shape of the zig-zag diagram as shown in the statement of the lemma.

4.5. **Proof of proposition 3.1.** We begin by showing the modules in item 2(b) are fixed by the Heller shift operator. Recall that these have the form

\[
\begin{array}{c}
g_{n-1} \to g_{n-2} \to \cdots \to g_1 \to g_0
\end{array}
\]

It is clear that the \( \{ g_0, g_1, g_2, \cdots, g_{n-1} \} \) is a minimal generating set for the above module, \( M \) say. So a minimal projective cover of this module will be a free \( V_4 \)-module of rank \( n \) generated by basis elements \( \{ e_0, e_1, \cdots, e_{n-1} \} \), and the covering map sends \( e_i \) to \( g_i \), for all \( i \). Counting dimensions, it is then clear that the dimension of the kernel \( (\Omega(M)) \) of this projective cover is of dimension \( 2n \). We only have to show that the \( V_4 \)-module structure on the kernel is isomorphic to the one on \( M \). This will be clear from the following diagrams. We consider the cases \( n = 2 \) and \( 3 \), the general case will then be clear.

\[
\begin{array}{c}
0 \to b_0 \to a_0 \to b_0 \to f_0 \to 0
\end{array}
\]

\[
\begin{array}{c}
0 \to b_1 + a_0 \to b_0 \oplus a_0 \to b_0 \oplus a_0 \to 0
\end{array}
\]
We now show that the modules in item 2(a) are fixed under the Heller. Recall that in each even dimension $2n$, these modules correspond to indecomposable rational canonical forms given by powers of an irreducible polynomials $f(x)^l = \sum_{i=0}^{n} \theta_i x^i$, schematically:

$\begin{align*}
&g_{n-1} \quad g_{n-2} \quad g_{n-3} \quad \cdots \quad g_0 \\
&\downarrow b \quad \downarrow a \quad \downarrow \quad \downarrow \\
&f_{n-1} \quad f_{n-2} \quad f_1 \quad \cdots \quad f_0
\end{align*}$

where $a(g_{n-1}) = \sum_{i=0}^{n-1} \theta_i f_i$. It is again clear that $\{g_0, g_1, g_2, \cdots, g_{n-1}\}$ is a minimal generating set, and hence a projective cover can be taken to be a free $V_4$-module of rank $n$ with basis elements $\{e_0, e_1, e_2, \cdots, e_n\}$, and the mapping sends the elements $e_i$ to the generators $g_i$. We will again convince the reader that these modules are fixed under the Heller by examining the cases $n = 1$ and $n = 2$. We begin with the case $n = 1$. Here the rational canonical form is determined by constant $\theta_0$, and $a(g_0) = \theta_0 f_0$. The following diagram shows that the Heller fixes these two dimensional modules.

$\begin{align*}
0 &\rightarrow a_0 + \theta_0 b_0 \\
&\downarrow c_0 \\
&\rightarrow a_0 \\
&\downarrow b_0 \\
&\rightarrow b_0 \\
&\downarrow c_0 \\
&\rightarrow g_0 \\
&\downarrow f_0 \\
&\rightarrow 0
\end{align*}$

Now consider the four dimensional modules: $n = 2$ and the rational conical form corresponds to a polynomial $x^2 + \theta_1 x + \theta_0$. In the diagram below $a(g_1) = \theta_0 f_0 + \theta_1 f_1$.

$\begin{align*}
0 &\rightarrow \gamma \\
&\downarrow c_1 \\
&\rightarrow a_1 \\
&\downarrow b_1 \\
&\rightarrow b_0 \\
&\downarrow a_0 \\
&\rightarrow a_0 \\
&\downarrow b_0 \\
&\rightarrow b_0 \\
&\downarrow c_0 \\
&\rightarrow c_0 \\
&\rightarrow c_0 \\
&\rightarrow c_0 \\
&\rightarrow c_0 \\
&\rightarrow c_1 \\
&\rightarrow c_1 \\
&\rightarrow c_1 \\
&\rightarrow c_1 \\
&\rightarrow g_1 \\
&\downarrow f_0 \\
&\rightarrow 0
\end{align*}$

where $\gamma = a_1 + \theta_0 b_0 + \theta_1 b_1$. Note that $a(\gamma) = \theta_0 c_0 + \theta_1 c_1$, as desired.

4.6. **Proof of proposition 3.3**  Note that it suffices to show that the indecomposable representations in item 2(a) are self-dual; for that forces the representations in item 2(b) to be self-dual, and it is well known that $kV_4$ is self-dual.
A $2n$ dimensional representation $M$ of item 2(a) can be chosen to be of the form (after a suitable choice of bases)

$$M = (V, V; I, J)$$

where $V$ is an $n$-dimensional vector space, $I$ denotes the identity transformation, and $J$ an indecomposable rational canonical form. It is then clear that the dual of $M$ is given by

$$M^* = (V^*, V^*; I, J^T)$$

It is an interesting exercise\(^2\) to show that a square matrix is similar to its transpose, so there exists an invertible matrix $D$ such that $J^T = DJD^{-1}$. The following commutative diagram then tells us that $M$ is isomorphic to $M^*$.

\[
\begin{array}{ccc}
V & \xrightarrow{J} & V \\
\downarrow{D} & \cong & \downarrow{D} \\
V^* & \xrightarrow{J^T} & V^*
\end{array}
\]

5. **The quest continues**

In our paper we concentrated on Klein group but what about $C_3 \oplus C_3$? What are all the representation of this group? Interestingly enough this is an extremely difficult question. Yet, some progress has been made very recently which involves more sophisticated machinery of representation theory. For the curious reader we refer to a recent paper [?].

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\(^2\)Hint: Use Jordan decomposition