Transport in driven systems has received widespread interest for several years because of its potential applicability to nonequilibrium processes [1]. One canonical model reduces to the motion of a particle in a one-dimensional space-periodic potential in the presence of friction and a time-dependent stochastic force $\chi(t)$. If $\chi(t)$ contains correlations, a nonzero current may be realized even in the case of zero average $\langle \chi(t) \rangle = 0$ [1]. Despite an enormous accumulation of results in this area [2] we are still lacking a full understanding of the microscopic mechanisms of current occurrence. If such an understanding is realizable, it should make use of the true dynamical evolution of the system rather than of properties of equations for probabilities. A first step in such a direction requires to separate the essential time correlations from the pure Gaussian white noise in $\chi(t)$. The simplest way is to assume that $\chi(t) = E(t) + \xi(t)$ where $E(t) = E(t + T)$ is a time-periodic function with zero mean $\langle E(t) \rangle = 0$ and $\xi(t)$ is a Gaussian white noise term.

The next step is to skip the $\xi(t)$ term which leaves us with a regular dynamical problem. In [3] such a case was considered and the relevant space-time symmetries of the dynamical problem have been obtained. It was shown that a breaking of those symmetries leads to a nonzero dc current. The mechanism of current occurrence for the dissipative case was identified with a desymmetrization of attractor basins. In contrast the nondissipative (Hamiltonian) case is much less understood [4]. A time-dependent Hamiltonian system is usually nonintegrable [5]. A strong dc current component was found in the corresponding stochastic layer of such a system [6] when mentioned absence or presence of symmetries, the dynamical nature of directed transport in the stochastic layer lacks full understanding. The importance of this understanding can be seen from e.g. results in [7] where kinetic equations for probability functions have been studied. In particular it was found that the approaching of the Hamiltonian (dissipationless) limit leads to an increase of the dc current value by 2-3 orders of magnitude. Thus the description of the dynamical mechanisms of directed current generation in the stochastic layer of a driven Hamiltonian system will provide with very useful information for dissipative systems as well.

Let us consider the canonical example of a particle moving in a spatially periodic nonlinear potential $U(x) = -\cos x$ under the influence of time-periodic zero-mean force $E(t)$. The Hamiltonian and the equation of motion are given by:

$$H = \frac{p^2}{2} - \cos x - xE(t) , \quad \ddot{x} = -\sin x + E(t) .$$  \hspace{1cm} (1)

Here $p$ and $x$ are canonically conjugated momentum and coordinate and $\ddot{x} \equiv d^2x/dt^2$.

We restrict our consideration to the choice

$$E(t) = E_1 \cos(t) + E_2 \cos(2t + \phi) .$$  \hspace{1cm} (2)

According to [3] for $E_2 \neq 0$ and $\phi \neq 0, \pi$ all possible symmetries which yield zero dc current are broken. Note that the phase space dimension $d$ of (2) is $d = 3$.

In the case of a nonzero field $E(t)$ the phase space of (2) is characterized by the presence of a stochastic layer which originates from the destroyed separatrix of the undriven system [8]. For $\phi = 0, \pi$ this layer is invariant under the transformation $(p \rightarrow -p, t \rightarrow -t, x \rightarrow x)$. At the same time the average velocity for any trajectory in this layer vanishes, so we find zero dc current. The symmetry will be broken when tuning $\phi$ away from the values $(0, \pi)$. The stochastic layer will deform. Most importantly any trajectory in the layer will then be characterized by a nonzero value of the average velocity. Due to ergodicity inside the layer this value will be unique for all trajectories from the layer. While the fact that it may become nonzero is understandable using the symmetry analysis, its appearance and magnitude is due to dynamical mechanisms of motion inside the stochastic layer.

In this paper, we show that the dc current is induced by
the presence and desymmetrization of ballistic channels inside the stochastic layer. The existence of these channels is due to resonances. The characterization of the realization of flights inside ballistic channels is described by distribution functions. We obtain these distribution functions numerically and find very good agreement with simulation data.

System (3) has a mixed phase space, which contains of chaotic areas and regular resonance islands (4). These islands are impermeable for chaotic trajectories and, at a first glance, may be excluded from the consideration of the phase space flow inside the stochastic layer. In reality the phase space topology inside the stochastic layer is very complex precisely at the boundary between chaotic and regular regions (6). Close to resonances the stochastic layer shows up with a hierarchical set of cantori, which form partial barriers for a trajectory from the layer. Due to the presence of these barriers a chaotic trajectory can be trapped for a long time near a regular island resonance. This trapping or sticking effect leads to the appearance of strongly nonergodic episodes during the overall chaotic motion. Regular islands are characterized by a corresponding rational winding number \( \omega = \Delta x/T \) which defines the distance \( \Delta x \) travelled during one period \( T = 2\pi \) of the drive. If the winding number \( \omega \) is nonzero, the corresponding sticking episode of the chaotic trajectory is a ballistic-like unidirectional flight. For \( \omega = 0 \) the sticking episode corresponds to trapped oscillations.

Thus the complicated evolution of a trajectory in the stochastic layer can be subdivided into several parts (6). The first one is a fast diffusion in the bulk of the layer, while the other ones are stickings to the above mentioned regular islands and correspond to propagation in ballistic channels. The switching from the diffusion process into a ballistic flight will be described by some probability distribution. The same will be true for the actual residence or sticking time inside a given channel. We will show for the cases studied that the fast diffusion alone is not capable of explaining the observed dc current. The main point is, that the leading mechanism of current generation in the stochastic layer is related to the desymmetrization of the strongly non-stochastic part of the overall stochastic dynamics inside the layer. Note that our kinetic energy choice \( p^2/2 \) in (5) implies that the stochastic layer is bounded in \( p \), so we will always expect ballistic channels to appear.

Let us study the case of weak driving \( E_1 = 0.252 \) and \( E_2 = 0.052 \). A Poincare map of the phase space flow for \( \phi = 0 \) is shown in Fig.1(a) (6). The main stochastic layer (central location) shows up with zero average velocity due to symmetry arguments (Fig.2). The large hole in the middle of this layer corresponds to regular trapped motion in the well of \( U(x) \). Additional resonances are seen above and below the central layer. These thin ballistic-like but yet stochastic channels have no overlap with the central layer.

![Fig. 1. Poincare map for (a) \( \phi = 0 \) and (b) \( \phi = \pi/2 \).](image)

A weak asymmetry \( \phi = \pi/5 \) leads to a slight deformation of the main stochastic layer and to a desymmetrization of the overlap of the chaotic layer with higher-order resonances and to the appearance of a positive current in the system. Note that it still does not overlap with the thin ballistic channels seen in Fig.1(a). Most importantly we observe a nonzero average velocity \( \langle \dot{x} \rangle \approx 0.05 \) (Fig.2).

Further increase of the asymmetry, \( \phi = \pi/2 \), results in an overlapping of the main stochastic layer with the upper ballistic resonance Fig.1(b). Note that at the same time the lower ballistic resonance is not overlapping. The average velocity increases to \( \langle \dot{x} \rangle \approx 0.2 \), which is four times larger than the result for \( \phi = \pi/5 \). Standard harmonic mixing theories (see (6), (7)) would predict a dependence \( \langle \dot{x} \rangle \sim \sin \phi \) and thus only an increase by a factor of 1.7.

In the \( x(t) \) curves in Fig.2 we observe many ballistic flights. For one of them an inset shows the corresponding Poincare map result, which verifies that these flights correspond to stickings of the chaotic trajectory to the upper ballistic resonances. A zooming of the \( x(t) \) curves shows self-similarity, i.e. the seemingly random dynamics between observable long flights is actually again composed of shorter flights and seemingly random dynamics etc (cf. insets in Fig.2).

In order to quantify our analysis of the symmetry broken dynamics we compute the distribution of travelling times of ‘uniform’ flights to the left \( P_-(t_f) \) and to the right \( P_+(t_f) \) separately. Here ‘uniform’ means no change
of direction of motion [10]. For each separate flight we note both the time \( t_f \) spent in this motion and the distance \( x_f \) travelled. The dependence of \( x_f \) on \( t_f \) is shown in the inset of Fig.3(a) for \( \phi = \pi/2 \). Similar to the other cases \( \phi = 0, \pi/5 \) we observe a simple fork-like structure. This is due to the fact that any considerable distance covering in the stochastic layer is realized through flights while sticking to the boundary of the stochastic layer. The slopes in the inset of Fig.3(a) are given by the corresponding winding numbers of the layer boundaries. Note that the two fork parts merge at values of \( t_f \approx 10T \). In principle other ballistic channels with different winding numbers might be present. Here they were too weak to be detected.

In Fig.3(a) we show the corresponding distribution functions \( P_{\pm}(t_f) \) (again for \( \phi = \pi/2 \)). They are obtained by counting the number of flights with \( t_f \) falling into a time window of size \( T \). For \( t_f < 10T \) we observe exponential dependence of \( P_{\pm} \) on \( t_f \). These short flight distributions are in fact independent of the direction of flight. Skipping all longer flights would lead to the prediction of nearly zero average velocity (restricting to flights of length \( t_f < 10T \) yields about one percent of the numerically observed current). Thus the desymmetrization will manifest itself for longer flights. For \( t_f > 10T \) a crossover to a power law \( P_{\pm} \sim t_f^{\phi/2} \) takes place. Here we find a significant desymmetrization for \( \phi = \pi/5, \pi/2 \). Estimating the exponents [11] we find for \( \phi = \pi/5: \alpha_- \approx 2.5, \alpha_+ \approx 2.4 \) and for \( \phi = \pi/2: \alpha_- \approx 3.7, \alpha_+ \approx 2.3 \). It is worthwhile noting that \( \alpha < 3 \) implies unidirectional anomalous diffusion with diverging second moments of \( \phi \). The flights are coined \textit{Levy flights} in such a case [12].

Following the continuous-time random walk (CTRW) formalism [13] we propose a generalized asymmetrical flight model capable of reproducing the above results. The applicability of the CTRW model follows from the assumption that the presence of a random phase with fast decaying correlations leads to the absence of correlations between consecutive flights, since they are almost always separated by dispersive chaotic motion.

Assume that there exist \( N \) different resonances with winding numbers \( w_i, i = 1, \ldots, N \). Every resonance is characterized by a probability distribution function (PDF) of sticking time \( S_i(t) \). After finishing a random phase event, the probability of sticking to the \( i \)th resonance is \( \rho_i, \sum_{i=1}^{N} \rho_i = 1 \). The random phase residing time is characterized by a PDF \( S_r(t) \). All functions \( S_i(t) \) and \( S_r(t) \) must have finite first moments, due to the Kac theorem about finiteness of recurrence times in Hamiltonian systems [8]. With these definitions we obtain the following expression for the current:

\[
J = \frac{\sum_{i=1}^{N} \omega_i \rho_i(t_i)}{\sum_{i=1}^{N} \rho_i(t_i) + \langle t_r \rangle}
\]

where \( \langle t_r \rangle = \int t P_r(t) dt \).

For the above discussed cases of \( \phi = \pi/5, \pi/2 \) we find only two relevant ballistic channels - one with positive winding number and a second one with negative winding number. In order to properly obtain \( S(t) \), we note that our numerically obtained function \( P(t) \) consists of a lot of short ‘flights’ as defined through the numerics [10]. These may be either stickings to islands with zero winding number or chaotic motion. We observe that for flight times \( t_f > 10T \) only ballistic flights with nonzero winding number are obtained. So the functions \( S_{\pm}(t) \) may be easily obtained from \( P_{\pm}(t_f) \) by cutting the central part \( t < 10T \) out and properly normalizing. In this case the

FIG. 2. \( x(t) \) for \( \phi = 0, \pi/5, \pi/2 \) (lower, middle and upper curves respectively). Left upper inset: Poincare map for ballistic flight with \( \phi = \pi/2 \) as indicated by arrow. Right inset: zoom of \( x(t) \) for \( \phi = \pi/2 \).

FIG. 3. \( P_+(t_f) \) (solid line) and \( P_-(t_f) \) (thick dashed line). Insets: \( x_f \) versus \( t_f \). For parameters see text.
expression for the average current simplifies to

\[ J = \frac{1}{\kappa(1 + f)} (\omega_+ \langle t_+ \rangle + f \omega_- \langle t_- \rangle) \]  

(4)

where the two constants \( \kappa \) and \( f \) can be obtained from the total time of a simulation \( T_{\text{tot}} \) and the numbers \( N_{\pm} \) of ballistic flights, \( \kappa = T_{\text{tot}}/(N_+ + N_-) \) and \( f = N_-/N_+ \).

For \( \phi = \pi/5 \) we obtain from the numerical runs \( f \approx 0.57 \), \( \kappa \approx 1900 \), \( \langle t_+ \rangle \approx \langle t_- \rangle \approx 220 \) and \( \omega_+ = 10/6 \approx 1.67 \), \( \omega_- = -1.5 \). In this case of \textit{weak desymmetrization} the main source of a nonzero current is the different probability to enter a right or left going flight because \( f \neq 1 \). At the same time the average flight times in both ballistic channels are nearly identical. With the help of (4) we find \( J \approx 0.056 \) which is close to the numerically observed value 0.05.

For the case \( \phi = \pi/2 \) we find \( f \approx 0.16 \), \( \kappa \approx 2600 \), \( \langle t_+ \rangle \approx 400 \), \( \langle t_- \rangle \approx 150 \) and \( \omega_+ = 2 \), \( \omega_- \approx -1.4 \). Note that the above discussed overlap with the upper resonance yields a further \textit{strong desymmetrization} in the probability to realize a left or right going flight, and in addition the average flight times in both channels significantly differ. Expression (4) yields \( J \approx 0.22 \) which is in good agreement with the numerically observed value 0.2.

For stronger driving amplitude \( E_1 = 3.26 \), \( E_2 = 1.2 \) and \( \phi = \pi/2 \) we obtain an average velocity \( \langle \dot{x} \rangle \approx 0.85 \). The corresponding \( x_f(t_f) \) dependence and the PDFs \( P_{\pm}(t_f) \) are shown in Fig. 3(b). The \( x_f(t_f) \) dependence shows that more than two ballistic channels are involved. The asymmetry of the PDFs at short flight times indicates that a considerable number of left-going flights becomes dominating at short times, in agreement with the tendency of the previous results. This makes the application of the simplified sum rule (4) impossible, instead the original definition (3) should be used. Careful analysis of the structure of the stochastic layer shows that relevant resonances become embedded in the bulk of the stochastic layer. While these structures are of rather small size, they are frequently visited. A restriction to short flights \( t_f < 10T \) now yields a considerable nonzero current which is however \textit{negative}, i.e. opposite to the total current value. Again the long ballistic flights are necessary in order to properly obtain the observed current value.

In summary, we have explained the dynamical mechanisms of current appearance in driven Hamiltonian systems inside the stochastic layer with broken time-reversal symmetry. The key source of such a directed transport is the desymmetrization of flight probabilities in ballistic channels inside the layer. As it follows from our sum rule (3) the resulting current depends among other parameters on the average time spent in a ballistic channel. These times will sensitively depend on control parameters of the system if the exponent \( \alpha \) becomes less than 3. In such a case small changes may significantly alter the current value as shown above.

A recently proposed geometric approach of counting areas and winding numbers is in principle also capable of obtaining the observed mean value for the current (4). This approach may also require sophisticated studies of the fractal structure of the chaotic layer. It represents a nontrivial complementary result, since it, although not explaining the dynamical mechanisms of current generation, is capable of obtaining the average current value (provided the sums in Eq.(3) of (14) converge fast enough).

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