SOME RESULTS ON $\eta$-RICCI SOLITON AND GRADIENT $\rho$-EINSTEIN SOLITON IN A COMPLETE RIEMANNIAN MANIFOLD

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Abstract. The main purpose of the paper is to prove that if a compact Riemannian manifold admits a gradient $\rho$-Einstein soliton such that the gradient Einstein potential is a non-trivial conformal vector field, then the manifold is isometric to the Euclidean sphere. We have showed that a Riemannian manifold satisfying gradient $\rho$-Einstein soliton with convex Einstein potential possesses non-negative scalar curvature. We have also deduced a sufficient condition for a Riemannian manifold to be compact which satisfies almost $\eta$-Ricci soliton (see Theorem 2).

1. Introduction

In 1982, Hamilton [10] introduced the notion of Ricci flow in a Riemannian manifold $(M, g_0)$ to find the various geometric and topological structures of Riemannian manifolds. The Ricci flow is defined by an evolution equation for metrics on $(M, g_0)$:

$$\frac{\partial}{\partial t} g(t) = -2Ric, \quad g(0) = g_0.$$ 

A Ricci soliton on a Riemannian manifold $(M, g)$ is a generalization of Einstein metric and is defined as

$$Ric + \frac{1}{2} \mathcal{L}_X g = \lambda g,$$

where $X$ is a smooth vector field on $M$, $\mathcal{L}$ denotes the Lie-derivative operator and $\lambda \in \mathbb{R}$. It is observed that Ricci solitons are self-similar solutions to the Ricci flow. Ricci soliton is called shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. The vector field $X$ is called the potential vector field of the Ricci soliton. If $X$ is either Killing or vanishing vector field, then Ricci soliton is called trivial Ricci soliton and (1) reduces to an Einstein metric. If $X$ becomes the gradient of a smooth function $f \in C^\infty(M)$, the ring of smooth functions on $M$, then the Ricci soliton is called gradient Ricci soliton and (1) reduces...
to the form

\[ \text{Ric} + \nabla^2 f = \lambda g, \]

where \( \nabla^2 f \) is the Hessian of \( f \). Perelman [12] showed that Ricci soliton on any complete manifold is always a gradient Ricci soliton. If we replace the constant \( \lambda \) in (1) with a smooth function \( \lambda \in C^\infty(M) \), called soliton function, then we say that \((M, g)\) is an almost Ricci soliton, see ( [3], [4], [14]).

Almost gradient Ricci soliton motivated Catino [7] to introduce a new class of Riemannian metrics which are natural generalization of Einstein metrics. In particular, a Riemannian manifold \((M^n, g), n \geq 2\), is called a generalized quasi-Einstein manifold if there are smooth functions \( f, \lambda \) and \( \mu \) on \( M \) such that

\[ \text{Ric} + \nabla^2 f = \lambda g + \mu \eta \otimes \eta, \]

Cho and Kimura [9] further generalized the notion of Ricci soliton and developed the concept of \( \eta \)-Ricci soliton. If a Riemannian manifold \( M \) satisfies

\[ \text{Ric} + \frac{1}{2} L_X g = \lambda g + \mu \eta \otimes \eta, \]

for some constant \( \lambda \) and \( \mu \), then \( M \) is said to admit \( \eta \)-Ricci soliton with soliton vector field \( X \). A further generalization is the notion of almost \( \eta \)-Ricci soliton defined by Blaga [5].

**Definition 1.1.** [5] A complete Riemannian manifold \((M, g)\) is said to satisfy almost \( \eta \)-Ricci soliton if there exists a smooth vector field \( X \in \chi(M) \), the algebra of smooth vector fields on \( M \), such that

\[ \text{Ric} + \frac{1}{2} L_X g = \lambda g + \mu \eta \otimes \eta, \]

where \( \lambda \) and \( \mu \) are smooth functions on \( M \) and \( \eta \) is an 1-form on \( M \).

If \( X \) is the gradient of \( f \in C^\infty(M) \), then \((M, g)\) is called a gradient almost \( \eta \)-Ricci soliton. Hence (3) reduces to the form

\[ \text{Ric} + \nabla^2 f = \lambda g + \mu \eta \otimes \eta. \]

Instead of Ricci flow, Catino and Mazzieri [6] considered the following gradient flow

\[ \frac{\partial}{\partial t} g(t) = -2(\text{Ric} - \frac{1}{2} Rg), \]

and introduced the concept of gradient Einstein soliton in a Riemannian manifold, where \( R \) is the scalar curvature of the manifold.
Definition 1.2. [6] A Riemannian manifold $(M, g)$ of dimension $n$ is said to be the gradient Einstein Ricci soliton if

$$Ric - \frac{1}{2}Rg + \nabla^2 f = \lambda g,$$

for some function $f \in C^\infty(M)$ and some constant $\lambda \in \mathbb{R}$.

A more general type gradient Einstein soliton has been deduced by considering the following Ricci-Bourguignon flows [8]:

$$\frac{\partial}{\partial t}g(t) = -2(Ric - \rho Rg),$$

where $\rho$ is a real non-zero constant.

Definition 1.3. [6] A Riemannian manifold $(M, g)$ of dimension $n \geq 3$ is said to be the gradient $\rho$-Einstein Ricci soliton if

$$Ric + \nabla^2 f = \lambda g + \rho Rg, \quad \rho \in \mathbb{R}, \quad \rho \neq 0,$$

for some function $f \in C^\infty(M)$ and some constant $\lambda \in \mathbb{R}$. The function $f$ is called Einstein potential. The gradient $\rho$-Einstein soliton is called expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$.

The paper is arranged as follows: Section 2 discusses some basic concepts of Riemannian manifold and some definitions, which are needed for the rest of the paper. Section 3 deals with the study of almost $\eta$-Ricci soliton in a complete Riemannian manifold and it is shown that in a compact manifold the potential of such soliton turns into the Hodge-de Rham potential, up to a constant. In this section we have also deduced a sufficient condition for a Riemannian manifold admitting almost $\eta$-Ricci soliton to be compact. In the last section as the main result of the paper we have proved that a compact Riemannian manifold satisfying a gradient $\rho$-Einstein soliton with gradient of Einstein potential as a conformal vector field, is isometric to the Euclidean sphere. We have also studied some properties of gradient $\rho$-Einstein soliton in a complete Riemannian manifold. Among others it is proved that if $(M, g)$ is a compact gradient $\rho$-Einstein soliton with $\rho$ as non-positive real number and gradient of the Einstein potential is a conformal vector field, then such soliton can never be expanding.

2. Preliminaries

Throughout this paper by $M$ we mean a complete Riemannian manifold of dimension $n$ endowed with some positive definite metric $g$ unless otherwise stated. In this section we have discussed some rudimentary facts of $M$ (for reference see [13]). The tangent space at the point
\(p \in M\) is denoted by \(T_p M\). The geodesic with initial point \(p\) and final point \(q\) is denoted by \(\gamma_{pq}\). A smooth section of the tangent bundle \(TM\) is called smooth vector field. The gradient of a smooth function \(u : M \to \mathbb{R}\) at the point \(p \in M\) is defined by \(\nabla u(p) = g^{ij} \frac{\partial u}{\partial x^j} \big|_p\). It is the unique vector field such that any smooth vector field \(X\) in \(M\) satisfies \(g(\nabla u, X) = X(u)\). The Hessian \(\text{Hess}(u)\) is the symmetric \((0, 2)\)-tensor field and is defined by \(\nabla^2 u(X, Y) = \text{Hess}(u)(X, Y) = g(\nabla_X \nabla u, Y)\) for all smooth vector fields \(X, Y\) of \(M\). In local coordinates this can be written as

\[
(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma^k_{ij} \partial_k u,
\]

where \(\Gamma^k_{ij}\) is the Christoffel symbol of \(g\). For any vector field \(X \in \chi(M)\) and a covariant tensor field \(\omega\) of order \(r\) on \(M\), the Lie derivative of \(\omega\) with respect to \(X\) is defined by

\[
(L_X \omega)(X_1, \ldots, X_r) = X(\omega(X_1, \ldots, X_r)) - \sum_{i=1}^r \omega(X_1, \ldots, [X, X_i], \ldots, X_n),
\]

where \(X_i \in \chi(M)\) for \(i = 1, \ldots, r\). In particular, when \(\omega = g\), then

\[
(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)\]

for \(Y, Z \in \chi(M)\).

Given a vector field \(X\), the divergence of \(X\) is defined by

\[
div(X) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \sqrt{|g|} X^j,
\]

where \(g = \det(g_{ij})\) and \(X^j \frac{\partial}{\partial x^j}\). The Laplacian of \(u\) is defined by \(\Delta u = div(\nabla u)\).

**Definition 2.1.** [17] A \(C^2\)-function \(u : M \to \mathbb{R}\) is said to be harmonic if \(\Delta u = 0\). The function \(u\) is called subharmonic (resp. superharmonic) if \(\Delta \geq 0\) (resp. \(\Delta u \leq 0\)), where \(\Delta\) is the Laplacian operator in \(M\).

**Definition 2.2.** [15] A function \(u : M \to \mathbb{R}\) is called convex if the following inequality holds

\[
u \circ \gamma(t) \leq (1 - t) \circ \gamma(0) + t \circ \gamma(1) \quad \forall t \in [0, 1],
\]

and for any geodesic \(\gamma : [0, 1] \to M\). And in case of \(u\) is differentiable, then \(u\) is convex if and only if \(u\) satisfies

\[
g(\nabla u, X)_x \leq u(exp_x \nabla u) - u(x), \quad \forall X \in T_x M.
\]
3. Some results of almost $\eta$-Ricci soliton in a compact Riemannian manifold

We consider $M$ as a compact orientable Riemannian manifold and $X \in \chi(M)$. Then Hodge-de Rham decomposition theorem [2] implies that $X$ can be expressed as

$$X = \nabla h + Y,$$

where $h \in C^\infty(M)$ and $\text{div}(Y) = 0$. The function $h$ is called the Hodge-de Rham potential [3].

**Theorem 1.** Let $(M, g, X, \lambda)$ be a compact gradient almost $\eta$-Ricci soliton. If $M$ is also a gradient almost $\eta$-Ricci soliton with potential function $f$ then, up to a constant, $f$ equals to the Hodge-de Rham potential.

**Proof.** Since $(M, g, X, \lambda)$ is a compact almost $\eta$-Ricci soliton, so taking trace of (3), we get

$$R + \text{div}(X) = \lambda n + \text{tr}(\mu \eta \otimes \eta).$$

Now Hodge-de Rham decomposition implies that $\text{div}(X) = \Delta h$, hence from the above equation, we obtain

$$R = \lambda n - \Delta h + \text{tr}(\mu \eta \otimes \eta).$$

Again since $M$ is gradient almost $\eta$-Ricci soliton with Perelman potential $f$, hence taking trace of (4), we have

$$R = \lambda n - \Delta f + \text{tr}(\mu \eta \otimes \eta).$$

Equating the last two equations, we get $\Delta (f - h) = 0$. Hence $f - h$ is a harmonic function in $M$, but $M$ is compact. Hence $f = h + c$, for some constant $c$. $\square$

**Theorem 2.** Let $(M, g)$ be a complete Riemannian manifold satisfying

(6) $$\text{Ric} + \frac{1}{2} \mathcal{L}_g \geq \lambda g + \mu \eta \otimes \eta,$$

where $X$ is a smooth vector field, $\mu$ and $\lambda$ are smooth functions and $\eta$ is an 1-form. Then $M$ is compact if $\|X\|$ is bounded and one of the following conditions holds:

(i) $\lambda \geq 0$ and $\mu > c > 0$,
(ii) $\lambda > c > 0$ and $\mu \geq 0$,

for some constant $c > 0$.

**Proof.** Let $p \in M$ be a fixed point and $\gamma : (0, \infty] \to M$ be a geodesic ray such that $\gamma(0) = p$. Then along $\gamma$ we calculate

$$\mathcal{L}_X g(\gamma', \gamma') = 2g(\nabla_{\gamma'} X, \gamma') = 2 \frac{d}{dt} [g(X, \gamma')] .$$
Now from (6) and above equation, we have
\[ \int_0^T \text{Ric}(\gamma', \gamma') dt \geq \int_0^T \lambda(\gamma(t)) g(\gamma', \gamma') dt - \int_0^T \frac{d}{dt} [g(X, \gamma')] dt + \int_0^T \mu(\gamma(t)) (\eta \otimes \eta)(\gamma', \gamma') dt \]
\[ = \int_0^T \lambda(\gamma(t)) dt + g(X_p, \gamma'(0)) - g(X_{\gamma(T)}, \gamma'(T)) + \int_0^T \mu(\gamma(t)) \eta^2(\gamma') dt \]
\[ \geq \int_0^T \lambda(\gamma(t)) dt + g(X_p, \gamma'(0)) - \|X_{\gamma(T)}\| + \int_0^T \mu(\gamma(t)) \eta^2(\gamma') dt. \]

The last inequality follows from Cauchy-Schwarz inequality. If any one of the conditions (i) and (ii) holds, then above inequality implies that
\[ \int_0^\infty \text{Ric}(\gamma', \gamma') dt = \infty. \]

Hence Ambrose’s compactness theorem [1] implies that \( M \) is compact.

\[ \square \]

4. Gradient \( \rho \)-Einstein soliton in a compact Riemannian manifold

Throughout this section \( M \) is a complete Riemannian manifold with dimension \( n \geq 2 \).

**Theorem 3.** [16] Suppose \((M, g)\) is a compact Riemannian manifold with constant scalar curvature and \( M \) admits a non-trivial conformal vector field \( X \). If \( \mathcal{L}_X \text{Ric} = \alpha g \) for some \( \alpha \in C^\infty(M) \), then \( M \) is isometric to the Euclidean sphere \( \mathbb{S}^n \).

Let \((M, g)\) be a gradient \( \rho \)-Einstein soliton. Then
\[ \text{Ric} + \nabla^2 f = \rho R g + \lambda g. \]

If \( \nabla f \) is conformal vector field, then \( \nabla^2 f = \psi g \), for some \( \psi \in C^\infty(M) \). Hence above equation reduces to the form
\[ (7) \quad \text{Ric} = (\rho R + \lambda - \psi) g. \]

Hence Ricci curvature depends only on the points of \( M \). Then it follows from Schur’s lemma that \( R \) is constant. Again by taking \( X = \nabla f \), we have
\[ \mathcal{L}_X \text{Ric} = (\rho R + \lambda - \psi) \mathcal{L}_X g = (\rho R + \lambda - \psi) \psi g. \]

Hence from Theorem 3 we can state the main theorem of the paper:

**Theorem 4.** Let \((M, g)\) be a compact gradient \( \rho \)-Einstein soliton with Einstein potential \( f \). If \( \nabla f \) is a non-trivial conformal vector field, then \( M \) is isometric to the Euclidean sphere \( \mathbb{S}^n \).
Theorem 5. [16] If $M$ is compact with constant scalar curvature and admits a non-trivial conformal vector field $X: \mathcal{L}_X g = 2\psi g$, $\psi \neq 0$, then
\[
\int_M \psi dV = 0.
\]
Taking the trace in (7), we get
\[
R = n(\rho R + \lambda - \psi),
\]
which implies that
\[
\int_M (1 - n\rho)R = n \int_M (\lambda - \psi).
\]
If $X$ is conformal vector field and $M$ is of constant scalar curvature, then applying Theorem 5 we get
\[
R \int_M (1 - n\rho) = n \int_M \lambda.
\]
Now if $\lambda < 0$, then the above equation becomes
\[
R \int_M (1 - n\rho) < 0.
\]
If $M$ is compact, then Theorem 4 implies that $M$ is isometric to $\mathbb{S}^n$. Since isometry preserves scalar curvature so $R > 0$. Hence the above equation implies that
\[
Vol(M) < n \int_M \rho.
\]
Hence we can state the following:

Theorem 6. Let $(M, g)$ be a compact gradient $\rho$-Einstein soliton with Einstein potential $f$ and $\rho \leq 0$. If $\nabla f$ is conformal vector field then $M$ is shrinking or steady gradient $\rho$-Einstein soliton.

Lemma 7. [6] Let $(M, g)$ be gradient $\rho$-Einstein Ricci soliton with Einstein potential $f$. Then we have
\[
\Delta f = -(1 - n\rho)R + n\lambda.
\]

Proposition 8. Suppose $(M, g)$ is an expanding or steady gradient $\rho$-Einstein Ricci soliton with Einstein potential $f$ and $n\rho > 1$. If $f$ is a convex function, then $M$ has non-negative scalar curvature.

Proof. The convexity of $f$ implies that $f$ is subharmonic [11], i.e., $\Delta f \geq 0$. Hence (10) implies that
\[
(1 - n\rho)R - n\lambda \leq 0.
\]
Now take $1 - n\rho = -h$, where $h > 0$ is a real constant. Then we get

$$R \geq -\frac{n\lambda}{h}.$$  \hspace{1cm} (11)

Since $M$ is expanding or steady, so $\lambda \leq 0$. Hence we can conclude from (11) that $R \geq 0$. \hfill \square

The following can be easily derived from (10):

**Proposition 9.** Suppose $(M, g)$ is a steady gradient $\rho$-Einstein Ricci soliton with Einstein potential $f$ and $n\rho > 1$. If $f$ is a harmonic function, then the scalar curvature of $M$ vanishes.

Integrating (8) on $M$, we get

$$R(1 - n\rho)Vol(M) = n\lambda Vol(M),$$

which yields

$$R = \frac{n\lambda}{1 - n\rho}.$$  \hspace{1cm} (9)

If $R > 0$, then $n\lambda > 1 - n\rho$, i.e., $\rho > \frac{1}{n}(1 - n\lambda)$. Hence Theorem 4 implies that

**Proposition 10.** Let $(M, g)$ be a compact gradient $\rho$-Einstein soliton with Einstein potential $f$. If $\nabla f$ is a non-trivial conformal vector field, then $\rho$ satisfies

$$\rho > \frac{1}{n}(1 - n\lambda).$$

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