TORUS ACTIONS ON ORIENTED MANIFOLDS OF 
GENERALIZED ODD TYPE

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Abstract. In [LS], Landweber and Stong prove that if a closed spin 
manifold $M$ admits a smooth $S^1$-action of odd type, then its signature 
$\text{sign}(M)$ vanishes. In this paper, we extend the result to a torus action 
on a closed oriented manifold with generalized odd type.

1. Introduction

The existence of a non-trivial group action on a manifold gives certain 
restrictions on the manifold and one of them is a characteristic class. Atiyah 
and Hirzebruch prove that if a closed spin manifold admits a non-trivial 
circle action, then its $\hat{A}$-genus vanishes [AH]. Hattori generalizes the result 
to spin$^c$-manifolds [Ha]. It is shown later that $\hat{A}$-genus vanishes if an oriented 
manifold with finite second and fourth homotopy groups admits an $S^1$-action 
[HH], [HH3].

In this paper, we discuss the vanishing of $L$-genus of an oriented manifold, 
that is, the signature of the manifold. The $L$-genus is the characteristic class 
belonging to the power series $f(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}$. The signature of an oriented 
manifold $M$ is the index of the signature operator on $M$. The Atiyah-Singer index theorem states that the $L$-genus of an oriented manifold $M$ is 
equal to the signature of $M$ [AS]. Kawakubo and Uchida prove that if a 
closed oriented manifold $M$ admits a semi-free $S^1$-action with $\dim(M^{S^1}) < \frac{1}{2} \dim M$, then the signature of $M$ vanishes [KU]. Li and Liu generalize 
the result to a so-called prime action [LL]. For a vanishing result on the 
signature of a manifold with a finite group action, see [E] for instance.

Let $M$ be an orientable manifold. Introduce a Riemannian metric on $M$. 
A spin structure on $M$ is an equivariant lift $P$ (called a principal $Spin(n)$- 
bundle) of the oriented orthonormal frame bundle $Q$ (called the principal 
$SO(n)$-bundle) over $M$ with respect to the double covering $\pi : Spin(n) \to SO(n)$.

Let the circle act on a spin manifold $M$. Then the action lifts to an action 
on the principal $SO(n)$-bundle $Q$. The action is called of even type, if it
further lifts to an action on the principal $\text{Spin}(n)$-bundle $P$. The action is
called of odd type, if it fails to lift to an action on $P$.

Given an action of a Lie group $G$ on a manifold $M$, denote $M^G$ by the set of points fixed by the $G$-action on $M$, i.e.,
$M^G = \{ p \in M | g \cdot p = p, \forall g \in G \}$. If $H$ is a subgroup of $G$, then define the set $M^H$ of points fixed by the
$H$-action in the same way.

Given a circle action on a spin manifold $M$, as a subgroup of $S^1$, $\mathbb{Z}_2$ also acts on $M$. The $S^1$-action on $M$ is of even type if and only if each
connected component of the set $M^{\mathbb{Z}_2}$ has codimension congruent to 0 modulo 4. Similarly, the $S^1$-action on $M$ is of odd type if and only if each connected
component of $M^{\mathbb{Z}_2}$ has codimension congruent to 2 modulo 4. For this, see [AH].

Now consider a circle action on a manifold $M$. Since $M$ need not allow a
spin structure, we use the latter equivalent definition to define an action of
even type and an action of odd type. The $S^1$-action is called of even type,
if each connected component of $M^{\mathbb{Z}_2}$ has codimension congruent to 0 modulo 4 and of odd type, if each connected component of $M^{\mathbb{Z}_2}$ has codimension congruent to 2 modulo 4. As before, $\mathbb{Z}_2$ acts on $M$ as a subgroup of $S^1$. In [HH2], H. Herrera and R. Herrera adapt these alternative definitions for
circle actions on oriented manifolds.

Landweber and Stong prove that a closed spin manifold admitting a circle
action of odd type must have vanishing $L$-genus [LS].

Theorem 1.1. [LS] If a closed spin manifold $M$ admits a smooth $S^1$-action
of odd type, then its signature $\text{sign}(M)$ vanishes.

In this paper, we generalize Theorem 1.1 in three directions:
(1) from spin manifolds to oriented manifolds.
(2) from circle actions to torus actions.
(3) from odd type to generalized odd type.

In this paper, for a torus action on a manifold, we introduce the notion
of an action of generalized odd type.

Definition 1.2. Let a $k$-torus $T^k$ act on a manifold $M$. Let $S$ be a closed
subgroup of $T^k$.

(1) The $T^k$-action is called of even type with respect to $S$, if for any
connected component $Z$ of $M^S$, we have $\dim Z \equiv \dim M \mod 4$. An
action of a torus $T^k$ on a manifold is called of generalized even type,
if it is of even type with respect to some closed subgroup of $T^k$.

(2) The $T^k$-action is called of odd type with respect to $S$, if for any
connected component $Z$ of $M^S$, we have $\dim Z \equiv \dim M - 2 \mod 4$. An
action of a torus $T^k$ on a manifold is called of generalized odd
type, if it is of odd type with respect to some closed subgroup of $T^k$.

Therefore, a circle action on a manifold being of odd type is a special of
a torus action of generalized odd type where the torus is the circle, and the
closed subgroup is $\mathbb{Z}_2$. The main result of this paper is the following:
Theorem 1.3. Let a torus act on a closed oriented manifold \( M \). If the action is of generalized odd type, then the signature of \( M \) vanishes.

In [HH2], H. Herrera and R. Herrera prove vanishing results on characteristic classes of manifolds admitting circle actions. One of them is that if a \( 4n \)-dimensional oriented manifold with finite second homotopy group admits a circle action of odd type, then its signature vanishes. As a special case of Theorem 1.3 where a torus group is the circle group and the closed subgroup is \( \mathbb{Z}_2 \), we recover the result without the assumption on the second homotopy group of the manifold.

2. Preliminaries and the proof of the main result

Let the circle act on a closed oriented manifold \( M \). Then the equivariant index of the signature operator is defined for each element of \( S^1 \). In [AS], it is proved that the equivariant index is rigid under the circle action, i.e., is independent of the choice of an element of \( S^1 \), and is equal to the signature of \( M \). Consequently, the signature of \( M \) is equal to the sum of the signatures of the connected components of \( M^{S^1} \).

Now, let a \( k \)-torus \( \mathbb{T}^k \) act on a closed oriented manifold \( M \). Then there exists a circle \( S^1 \) inside \( \mathbb{T}^k \) that has the same fixed point set as \( \mathbb{T}^k \), i.e., \( M^{S^1} = M^{\mathbb{T}^k} \). As for \( S^1 \)-actions, for an action of a torus the signature of \( M \) is equal to the sum of the signatures of the connected components of \( M^{\mathbb{T}^k} \).

It follows from Theorem 6.12 in [AS] and is stated explicitly in [KR].

Theorem 2.1. [AS], [KR] Let a \( k \)-torus \( \mathbb{T}^k \) act on a closed oriented manifold \( M \). Then \( \text{sign}(M) = \text{sign}(M^{\mathbb{T}^k}) \).

By \( \text{sign}(M^{\mathbb{T}^k}) \), it means the sum of the signatures of all connected components of \( M^{\mathbb{T}^k} \), i.e., \( \text{sign}(M^{\mathbb{T}^k}) = \sum_{N \subset M^{\mathbb{T}^k}} \text{sign}(N) \).

In [Ko], Kobayashi proves that the fixed point set of a torus action on an orientable manifold is orientable.

Lemma 2.2. [Ko] Let a torus act on an orientable manifold \( M \). Then the fixed point set is a union of closed orientable manifolds.

Given a circle action on an oriented manifold, H. Herrera and R. Herrera prove the orientability of the set of points fixed by a subgroup of \( S^1 \).

Lemma 2.3. [HH] Let \( M \) be an oriented, \( 2n \)-dimensional, smooth manifold endowed with a smooth \( S^1 \)-action. Consider \( \mathbb{Z}_k \subset S^1 \) and its corresponding action on \( M \). If \( k \) is odd then the fixed point set \( M^{\mathbb{Z}_k} \) of the \( \mathbb{Z}_k \)-action is orientable. If \( k \) is even and a connected component \( Z \) of \( M^{\mathbb{Z}_k} \) contains a fixed point of the \( S^1 \)-action, then \( Z \) is orientable.

Let \( S \) be a closed subgroup of \( \mathbb{T}^k \). Then \( S \) is Lie isomorphic to a product of \( S^1 \)'s and \( \mathbb{Z}_a \)'s, i.e., \( S \approx S^1 \times \cdots \times S^1 \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_m} \), where \( a_i \)'s are positive integers bigger than 1. Note that the \( a_i \)'s may have repeated
elements. By using Lemma 2.2 and Lemma 2.3, we extend Lemma 2.3 to torus actions.

Lemma 2.4. Let a $k$-torus $\mathbb{T}^k$ act on a $2n$-dimensional orientable manifold $M$ and $S$ a closed subgroup of $\mathbb{T}^k$. Let $Z$ be a connected component of $M^S$. If $Z$ contains a $\mathbb{T}^k$-fixed point (i.e., if $Z \cap M^{\mathbb{T}^k} \neq \emptyset$), then $Z$ is orientable.

Proof. Without loss of generality, by choosing an orientation, assume that $M$ is oriented. Since $S$ is a closed subgroup of $\mathbb{T}^k$, $S$ is isomorphic to

$$S = (S^1)^l \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_m}$$

for some $l \geq 0$ and positive integers $a_i > 1$ for $i = 1, \ldots, m$.

Denote $S_i = (S^1)^l \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_i}$ for $i = 0, 1, \ldots, m$. Also, denote $M_i$ by the set of points fixed by $S_i$-action, i.e., $M_i = M^{S_i}$. We prove that for any $i$ if $Z_i$ is a connected component of $M_i$ that contains $Z$, then $Z_i$ is orientable.

Consider the case that $i = 0$. Then $S_0 = (S^1)^l$. By Lemma 2.2, the set $M^{S_0} = M^{(S^1)^l}$ of points fixed by the $(S^1)^l$-action is a union of smaller dimensional closed orientable manifolds.

Suppose that a connected component $Z_{i-1}$ of $M_{i-1}$ is orientable and contains $Z$. On $Z_{i-1}$, we have an induced action of $G_i = \mathbb{T}^k/S_i = (S^1)^k/(S^1)^l \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_i} = (S^1)^{k-l} = \mathbb{T}^{k-l}$. Moreover, as a subgroup of $G_i$, $\mathbb{Z}_{a_i}$ acts on $Z_{i-1}$. Given a generator $b_i$ of $\mathbb{Z}_{a_i}$, there exists $X_i$ in the Lie algebra of $\mathbb{T}^{k-l}$ such that $\exp(\frac{1}{a_i}X_i) = b_i$. In other words, $X_i$ generates a circle $S^1$ such that $\mathbb{Z}_{a_i} \subset S^1$. Denote the circle by $H_i$.

Therefore, on the orientable manifold $Z_{i-1}$, we have the action of the circle $H_i$ and the action of $\mathbb{Z}_{a_i}$, as a subgroup of $H_i$. A connected component $Z_i$, the set of points in $Z_{i-1}$ that are fixed by the $\mathbb{Z}_{a_i}$-action and contains $Z$, contains a $H_i$-fixed point, since it contains $Z$ which is fixed by the $\mathbb{T}^k$-action and $H_i$ is a subgroup of $\mathbb{T}^k$.

Apply Lemma 2.3 for the action of the circle $H_i$ on $Z_{i-1}$ with its subgroup $\mathbb{Z}_{a_i}$. Since the connected component $Z_i$ of $Z_{i-1}$ contains a $H_i$-fixed point, it follows that $Z_i$ is orientable.

Note that $Z_m = Z$. The lemma then follows by inductive argument. \qed

Note that in Lemma 2.4, we can remove the condition that $Z$ contains a $\mathbb{T}^k$-fixed point, if all $a_i$ are odd. With Lemma 2.4, we are ready to prove our main result.

Proof of Theorem 1.3. If the dimension of $M$ is not divisible by 4, then the signature of $M$ is defined to be 0 and hence the theorem follows. Therefore, from now on, suppose that the dimension of $M$ is divisible by 4.

Let $\mathbb{T}^k$ be the torus which acts on $M$. Let $S$ be the subgroup of the torus $\mathbb{T}^k$ such that the torus action is of odd type with respect to $S$. Let $Z$ be a connected component of the set $M^S$ of points fixed by the $S$-action on
$M$ that contains a $\mathbb{T}^k$-fixed point, i.e., $Z \cap M^{\mathbb{T}^k} \neq \emptyset$. Then by Lemma 2.4, $Z$ is orientable. Choose an orientation of $Z$. Since the $\mathbb{T}^k$-action is of odd type with respect to the closed subgroup $S$, $\dim Z \equiv \dim M - 2 \mod 4$. Since $\dim M \equiv 0 \mod 4$, we have that $\dim Z \equiv 2 \mod 4$. Therefore, we have that $\text{sign}(Z) = 0$. On $Z$, there is an induced action of $\mathbb{T}^k/S = \mathbb{T}^{k'}$. Moreover, the set of points in $Z$ that are fixed by the induced $\mathbb{T}^{k'}$-action is precisely the set of points in $Z$ that are fixed by the entire $\mathbb{T}^k$-action on $M$, i.e., $Z^{\mathbb{T}^{k'}} = Z \cap M^{\mathbb{T}^k}$. Applying Theorem 2.1 to the induced action of the torus $\mathbb{T}^{k'} = \mathbb{T}^k/S$ on $Z$, we have that

$$\sum_{N \subset Z^{\mathbb{T}^{k'}}} \text{sign}(N) = \text{sign}(Z) = 0.$$ 

On the other hand, by directly applying Theorem 2.1 to the action of the $k$-torus $\mathbb{T}^k$ on $M$, we have that

$$\text{sign}(M) = \sum_{N \subset M^{\mathbb{T}^k}} \text{sign}(N).$$

Since $M^{\mathbb{T}^k} \subset M^S \subset M$, each connected component $N$ of $M^{\mathbb{T}^k}$ is contained in a unique connected component $Z$ of $M^S$ which contains a $\mathbb{T}^k$-fixed point, as $Z$ contains the $\mathbb{T}^k$-fixed component $N$. Therefore, we have that

$$\text{sign}(M) = \sum_{N \subset M^{\mathbb{T}^k}} \text{sign}(N) = \sum_{Z \subset M^S, Z \cap M^{\mathbb{T}^k} \neq \emptyset} \sum_{N \subset Z \cap M^{\mathbb{T}^k}} \text{sign}(N) = \sum_{Z \subset M^S, Z \cap M^{\mathbb{T}^k} \neq \emptyset} \text{sign}(Z) = 0.$$

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