The role of monopoles in a Gluon Plasma

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Abstract

We study the role of magnetic monopoles at high enough temperature $T > 2T_c$, when they can be considered heavy, rare objects embedded into matter consisting mostly of the usual “electric” quasiparticles, quarks and gluons. We review available lattice results on monopoles at finite temperatures. Then we proceed to classical and quantum charge-monopole scattering, solving the problem of gluon-monopole scattering for the first time. We find that, while this process hardly influences thermodynamic quantities, it does produce a large transport cross section, significantly exceeding that for pQCD gluon-gluon scattering up to quite high $T$. Thus, in spite of their relatively small density at high $T$, monopoles are extremely important for QGP transport properties, keeping viscosity small enough for hydrodynamics to work at LHC.

1 Introduction

1.1 Overview

As it is known from 1970’s, QCD at high temperature $T$ is weakly coupled [1] and provides perturbative screening of the charge [2], thus being called Quark-Gluon Plasma (QGP). Creating and studying this phase of matter in the laboratory has been the goal of experiments at CERN SPS and recently at the Relativistic Heavy Ion Collider (RHIC) facility in Brookhaven National Laboratory, soon to be continued by the ALICE collaboration at the Large Hadron Collider (LHC). RHIC experiments have revealed robust collective phenomena in the form of radial and elliptic flows, which turned out to be quite accurately described by near-ideal hydrodynamics. QGP thus seems to be the most perfect liquid known, with the smallest viscosity-to-entropy ratio $\eta/s$. 

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The theory of QGP has shifted from the perturbative-based one, appropriate for a weakly coupled (gas) regime, to the non-perturbative methods needed to address the strongly coupled QGP (sQGP for short) regime. This “paradigm shift”, documented in Refs. [3, 4], is still profoundly affecting the developments. The methods addressing strongly coupled gauge theories include in particular the so-called AdS/CFT correspondence, relating strongly coupled gauge theory to weakly coupled string theory in a particular setting. We will not discuss it in this paper, for a recent review see e.g. [5]. On pure phenomenological grounds, it has been argued that, since many substances exhibit a minimum of the viscosity at some phase transitions, perhaps QGP is the “best liquid” at the QCD phase transition as well, namely at $T = T_c$ [6].

Another duality which has been used to explain unusual properties of the sQGP is the electric – magnetic duality. Liao and one of us have proposed the so-called “magnetic scenario” [7], according to which the near-$T_c$ region is dominated by magnetic monopoles. This is not surprising, if the deconfinement phase transition is basically interpreted as their Bose condensation. Based on molecular dynamics of classical plasmas with both electric and magnetic quasiparticles, it has been further argued in that work that the minimal viscosity/entropy ratio (the “best liquid”) does not correspond to the phase transition point $T = T_c$, but rather to the “electric-magnetic equilibrium”, at $T \approx 1.4T_c$, where both components of QGP contribute about equally to transport coefficients. We will review more recent works on the subject below.

One of the central questions is how sQGP with “perfect fluidity” will change into a weakly coupled wQGP with increasing $T$. In view of the next round of heavy ion experiments at LHC, a quite urgent question is what transport properties are expected to be observed there, at temperatures reaching about twice those reached at RHIC. In order to answer this question, one of course has to understand where the “perfect fluidity” property of QGP comes from. As an important example of perturbative point of view, we mention the work by Xu, C. Greiner and Stöcker [8] who argued that the QGP is only moderately coupled, with $\alpha_s = 0.3..0.6$, explaining the small viscosity by inclusion of the next-order radiative processes, $gg \leftrightarrow ggg$. We will discuss this issue partially in the next section, dealing with parametric dependences of densities and scattering rates, and also at the end of the paper in the discussion section. Here we only notice that, if this should be the explanation, one would expect a very slow transition to weakly coupled QGP, induced by the logarithmic running of the coupling.

In this paper we address the issue of QGP transport properties in the “magnetic scenario” framework, moving away from the phase transition region to higher temperatures, where QGP is still dominated by the usual electric quasiparticles – quarks and gluons – and the coupling is moderately small. Our goal is to study the interaction between electric and magnetic sectors. The main difference, with respect
to the Liao-Shuryak paper [7], is that their classical (manybody) treatment of this interaction (via Lorentz force) is replaced by our (two-body) quantum-mechanical calculation. Our main result is the explicit solution of the rather difficult problem of quantum gluon-monopole scattering, from which we calculate the corresponding transport cross sections.

Two types of “enhancement mechanisms” are found: one is related to the couplings and the other one (of much more nontrivial origin) is due to enhanced large-angle scattering. The former is rather easy to explain: while “electric” $qq, gg, gg$ scattering has cross sections proportional to small $\alpha_{\text{electric}}^2 \ll 1$, the scattering of electric-magnetic type contains the product $\alpha_{\text{electric}} \alpha_{\text{magnetic}}$ which, according to the celebrated Dirac condition, cannot run and is equal to an integer (just one for the ‘t Hooft-Polyakov monopole we will consider). Therefore, in this problem there is no small parameter at any temperature. For many other details the reader should read the paper further; nevertheless, we emphasize here our main result, namely that the gluon-monopole scattering rates are very large. Parametrically they are $\dot{w}_{gm}/T \sim \log(T)^{-3}$, just one power of the log down compared to the gluon-gluon pQCD scattering rates $\dot{w}_{gg}/T \sim \log(T)^{-2}$. Thus, if the gluon-monopole scattering is the dominant one, one should expect stronger decrease of scattering ($\eta/s$ growth) at LHC, compared to the perturbative scenario [8].

Below we include a rather extensive introduction to the electric-magnetic duality in gauge theories in general, and to the magnetic sector of QCD at high $T$ in particular. This choice, which we hope will be useful to many readers, can be justified by the somewhat intermittent nature of this field, with many important results obtained and then semi-forgotten outside a relatively small group of experts. In particular, the scattering of an electric particle on a magnetic monopole is a century-old problem. It is hard to explain the quantum scattering without introducing classical notations and results first. Thus, although this part of the story is rather old, we include some of this material for completeness as an extended introduction. We will then proceed to the extension of this theory to gauge particles.

### 1.2 Scattering on monopole in classical approximation

Let us start with a very old (19th century) problem in classical electrodynamics: an electrically charged particle with charge $e$ is moving in the magnetic field of a static monopole with magnetic coupling $g$. Classically, one does not need the vector potential, thus many subtleties are absent. The interaction is simply given by the Lorentz force

$$m\ddot{\vec{r}} = -e\frac{\vec{r} \times \hat{r}}{r^2},$$

(1)

where $\times$ indicates the vector product and $\hat{r} = \vec{r}/r$ is the unit vector along the line connecting both charges. It is worth noticing that only the product of the couplings
Furthermore, the cross product of the magnetic field of the monopole and electric field of the charge leads to a nonzero Poynting vector, which rotates around $\vec{r}$: thus, the field itself has a nonzero angular momentum. The total conserved angular momentum for this problem has two parts

$$\vec{J} = m\vec{r} \times \dot{\vec{r}} + e g \hat{r}. \quad (2)$$

The traditional potential scattering only has the first part: therefore, in that case, the motion entirely takes place in the so-called “reaction plane” normal to $\vec{J}$. In the charge-monompole problem, the second term restricts the motion to the so-called Poincaré cone [9]: its half-opening angle $\frac{\pi}{2} - \xi$ being

$$\sin(\xi) = \frac{eg}{J}. \quad (3)$$

Only at large $J$ (large impact parameter scattering) the angle $\xi$ is small and thus the cone opens up, approaching the scattering plane.

Following Ref. [10], one can project the motion on the cone to a planar motion, by introducing

$$\vec{R} = \frac{1}{\cos(\xi)} [\vec{r} - \hat{J}(\vec{r} \cdot \hat{J})] \quad (4)$$

where the first scale factor is introduced to keep the same length for both vectors $\vec{R}^2 = \vec{r}^2$. Now, two integrals of motion are

$$\vec{J} = m \vec{R} \times \dot{\vec{R}} \quad (5)$$

$$E = \frac{m\vec{R}^2}{2} - \frac{(eg)^2}{2mR^2} \quad (6)$$

and the problem seems to be reduced to the motion of a particle of mass $m$ in an inverse-square potential. The scattering angle $\Delta \psi$ for this planar problem can be readily found: it is the variation of $\psi$ as $R$ goes from $\infty$ to its minimum $b$ and back to $\infty$

$$\Delta \psi = \pi \left( \frac{1}{\cos \xi} - 1 \right) = \pi \left( \sqrt{1 + \left(\frac{eg}{mvb}\right)^2} - 1 \right). \quad (7)$$

Note that at large $b$ (small $\xi$) we have $\Delta \psi \sim 1/b^2$, as expected for the inverse-square potential. Yet this is not the scattering angle of the original problem, because one has to project the motion back to the Poincaré cone. The true scattering angle – namely the angle between the initial and final velocities – is $\cos \theta = - (\hat{v}_i \cdot \hat{v}_f)$. By relating velocities on the plane and on the cone one can find it to be

$$\left( \frac{\cos \theta}{2} \right)^2 = (\cos \xi)^2 \left( \sin \frac{\pi}{2 \cos \xi} \right)^2. \quad (8)$$
Thus for distant scattering – small $\xi$ – one gets $\theta \approx 2\xi = 2eg/(mvb)$, which is much larger than $\Delta\psi \sim 1/b^2$. The important lesson that we learn from these formulae is that the small scattering angle is given by the opening angle of the cone, rather than the scattering angle in the planar, inverse-square effective potential. Calculating the cross section by $d\sigma = 2\pi bdb$ one finds that, at small angles, it is

$$\frac{d\sigma}{d\Omega} = \left(\frac{2eg}{mv}\right)^2 \frac{1}{\theta^4}, \quad (9)$$

similar to the Rutherford scattering of two charges. The difference (apart from different charges) is also the additional second power of velocity, originating from the Lorentz force.

### 1.3 Quantum charge-monopole scattering problem

Jumping from 1900’s to 1930’s and from classical to quantum mechanics, one first has to deal with the necessity of introducing the vector potential $A_\mu(x)$. If this is done, the divergence of the magnetic field is zero. Dirac [11] brilliantly solved this difficulty by introducing the celebrated Dirac string: he then proved it to be an unphysical gauge artefact, provided the charge quantization condition is fulfilled

$$\frac{eg}{4\pi} = \text{integer}. \quad (10)$$

The same condition appears, if we recognize the necessity of quantizing the field angular momentum to (semi-integer) multiples of $\hbar$.

Now we jump to mid-1970’s, when the quantum scattering problem was solved in Refs. [10, 12] for the scattering of a scalar particle on a monopole, and in Ref. [13] for a monopole-spin 1/2 particle scattering.

The wave function in spherical coordinates is as usual a sum of products of certain $r$-dependent radial functions, times the angular functions. The former basically follow from the inverse-square law potential and thus are easily solved in Bessel functions: they correspond to the auxiliary planar projection of the classical problem of the previous subsection. The nontrivial part happens to be in the unusual angular functions. Before introducing those, let us hint why the usual set of angular harmonics $Y_{lm}(\theta, \varphi)$ should not to be used. The reason is that their classical limit – for large values of the indices $m \approx l \gg 1$ – corresponds to flat planar motion near the $z = 0$ plane. Nevertheless, we have already learned from the classical limit to expect the motion to be concentrated around the Poincaré cone instead!

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1 One can introduce the so-called dual vector potential related to $\vec{B}, \vec{E}$ in the opposite way: then it is the electric field which would be forced to have zero divergence and Dirac strings.
The functions that we need (for example in the scalar sector) must satisfy the following set of conditions

\[
\begin{cases}
\vec{T}^2 \\
T_3 \vec{I}^2 \\
(\hat{r} \cdot \vec{I})
\end{cases}
\]

\[
\phi_{mn}^{ti}(\theta, \varphi) = \begin{cases}
t(t + 1) \\
m \\
i(i + 1) \\
n
\end{cases}
\phi_{ni}^{mn}(\theta, \varphi).
\]

(11)

where \(\vec{T}\) is the total angular momentum \(\vec{T} = \vec{L} + \vec{I}\), \(\vec{L}\) is the orbital angular momentum and \(\vec{I}\) the isospin. The unusual condition in the above set is the last one, since the vector \(\vec{I}\) must be projected to the (space-dependent) radial unit vector. The functions satisfying this requirement \[10, 12\] will be introduced in Sec. 3 in the case of a scalar particle, and in Sec. 4 in the case of a vector one. Here we can anticipate that, in order to check if the hint we provided above is satisfied, it is enough to explore the large \(l, n\) limit of the \(D\)-functions involved. The result

\[
D^l_{nl} \sim e^{i(l-n)\phi} \exp[-l(\theta - \theta^*)^2/2]
\]

(12)

where \(\cos(\theta^*) = n/l\), shows that they indeed correspond to the Poincaré cone.

1.4 Brief review on QCD monopoles

The discussion in the previous two subsections was restricted to electrodynamics, in which there is neither an explicit explanation of the monopole structure nor a real necessity to have them. However, non-Abelian gauge theories do have monopole-like solitons, as demonstrated by 't Hooft and Polyakov \[14, 15\] in the Georgi-Glashow model framework. Three decades later, Seiberg and Witten \[16\] have discovered beautiful ways in which monopoles and dyons can get alive and even replace the usual electric particles (gluons/gluinos) in \(\mathcal{N}=2\) Supersymmetric-Yang-Mills theory.

Since those developments are very well documented in books and reviews, we directly jump to finite-\(T\) QCD. Explicit calculations show that the static magnetic field is not screened perturbatively in QGP \[2\]; yet Polyakov conjectured \[17\] that magnetic screening should appear non-perturbatively, at the so-called “magnetic scale” \(E_M \sim e^2T\). Linde \[18\] related this magnetic scale and screening to monopole solutions, which appear at finite \(T\) when the Higgs fundamental scalar of the Georgi-Glashow model is replaced by \(A_0\), the 0 component of the gauge field. Shortly afterwards, the monopole Bose-condensation was the central idea of the so-called

\[2\] The famous Dirac’s quote is that he would be surprised that Nature would not make use of them.

\[3\] In order to keep with the notations used in monopole problems, in this work we use \(e\) to indicate the usual gauge coupling, reserving \(g\) for magnetic coupling. We hope \(e\) will not be confused with QED charge, which is not used in this work.

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“color superconductor” picture of confinement, proposed by Mandelstam and ‘t Hooft [19]. The abelian Higgs model (an incarnation of Landau-Ginzburg action) was worked out in detail and shown to be in excellent agreement with lattice results on the structure of confining flux tubes at $T = 0$ by a large number of people. The profiles of the flux tubes nicely follow the celebrated Abrikosov solution, and lattice monopoles do rotate around them, providing a “coil” needed to contain the electric flux.

Lattice monopoles are defined by the procedure [20] which basically locates the ends of singular Dirac strings by calculating the total magnetic flux through the boundary of elementary 3-d boxes. Since the strings and thus the procedure depend on a certain gauge, for decades sceptics kept the viewpoint that those objects are just unphysical UV gauge noise. Yet, many specific questions – e.g. monopole density and correlations [21] – produced very reasonable and consistent answers, which are apparently independent of the particular lattice parameters. It is hard to imagine that lattice gauge artefact have consistent correlations at relatively large distances, accurately reproducing those in a Coulomb plasma. We thus assume instead, that these monopoles are indeed meaningful physical degrees of freedom, present in QGP as quasiparticles and being the source of a Coulomb-like magnetic field.

The picture of the (uncondensed) monopole gas at $T > T_c$ was discussed in particular in the important paper by Smit et al. [22], in which the relation between the monopole density and such observables as the spatial string tension were considered. One question (especially important for our work) is whether monopoles form a gas or a liquid. To that purpose, the authors of Ref. [22] have pointed out that the combination $n_m/M_m^3 \sim .03 \ll 1$ is small while in a weakly coupled (Debye) plasma it should be large. (Indeed, in high-$T$ electric plasma $n \sim T^3, E_F \sim eT$, and thus $n/E_F^3 \sim 1/e^3 \gg 1$). Yet such ratio can be small in a liquid; in fact the ensemble of instantons in the QCD vacuum – the “instanton liquid” extensively studied by one of us [23, 24] – has the same property. The $T = 0$ instanton density $n_{\text{inst}} \sim 1$ fm$^{-4}$ while the screening mass of the topological charge is $m_{\eta'} \approx .94$ GeV, thus $n_{\text{inst}}/M_{\eta'}^4 \sim .002 \ll 1$ as well. Taking the corresponding roots for comparison, one even finds the small parameters in both cases to be similar: $n_m^{1/3}/M_m \approx .3$, $n_{\text{inst}}^{1/4}/M_{\eta'} \approx .2$.

Recent interest in QGP monopoles (at $T > T_c$) started with the “magnetic scenario” [7] which basically suggested the magnetic sector to be a strongly coupled (magnetic) Coulomb plasma of monopoles, in its liquid form. Another line of work based on lattice monopoles has led Chernodub and Zakharov [25] at the same time to a very similar conclusion. An important feature of this scenario [7] is the opposite running of the electric coupling $e$ and the magnetic one $g$, induced by the Dirac condition $eg = \text{const}$. As recently shown in [26], this feature has been dramatically

\footnote{For more recent discussion of this feature see Korthals Altes [27].}
confirmed by the behavior of the lattice correlation functions [21], which indeed display monopole-monopole and antimonopole-monopole correlations increasing with \( T \). As shown in [26], those correlations are well described by a picture of classical Coulomb gas. The main input in those calculations is the magnetic Coulomb coupling \( g \) growing with \( T \). Furthermore, it was shown to be simply the inverse of the gauge coupling \( e \), as Dirac predicted. We consider the correlations observed in [21] to be a decisive confirmation of the existence of the long-distance magnetic Coulomb field of the monopoles. If so, the scattering of the dominant “electric” particles on this magnetic field is an important component of transport, and this is what we are going to study below.

In order to complete this brief overview, let us comment on the issue of “Higgsing” in hot gauge theories. It is well known that, at high \( T \), one can consider a dimensionally reduced effective 3d theory, which is basically a Georgi-Glashow model with \( A_0 \) being the adjoint scalar. Its lattice-derived phase diagram, as well as the effective lines of the high-\( T \) 4-d gauge theories derived to 1 and 2 loops can be found in Ref. [28]. If taken literally, the line for QCD crosses into the broken phase, except at extremely high \( T \); but the authors themselves argued that this line should in fact remain in the symmetric phase. Thus, even the asymptotically high-\( T \) behavior is under debate. In the temperature range of few-\( T_c \) we know that the expectation value of the Polyakov line changes from zero to unit value, and that its effect on quarks [29] and gluons [30] is large, producing significant suppression. This approach in particular gives a very good description of the quark number susceptibilities [31].

One more practical aspect of the issue comes from heavy-ion phenomenology. Dumitru and collaborators [32] have used this form of the effective Lagrangian to study the real-time evolution of \( A_0 \). The main conclusion from their work is that \( \langle A_0 \rangle \) belongs to the class of so-called slow variables, and its evolution in heavy ion collisions has to be treated separately from the overall equilibration. They have numerically solved the EOM for \( A_0 \), starting from the “suddenly quenched” value corresponding to its vacuum form, moving toward its minimum at the deconfined phase at \( T = 2T_c \). The main finding of this work is that the relaxation of this variable is very slow, taking approximately 40 fm/c. This time significantly exceeds the QGP lifetime at RHIC, which is only about 5 fm/c, which suggests that in real collisions we should treat \( A_0 \) essentially as a random variable frozen at some value and color direction during hydro evolution. This means that there is a chaotic out-of-equilibrium Higgsing, slowly rolling down, like in cosmological inflationary models: thus one would like to know as much as possible about phase transitions and EoS for all values of the Higgs VEV.
Figure 1: (a) The monopole density $n_m$ in units of $[\text{fm}^{-3}]$ versus $T/T_c$ for the confined phase $T < T_c$. (b) The normalized density $n/T^3$ versus $T/T_c$ for the deconfined phase $T > T_c$. 
2 Monopoles in gauge theories at high $T$

2.1 The monopole density and properties

In section 1.4 we have already briefly reviewed the main issues and ideas related with magnetic monopoles in hot QCD. Now we begin a much more quantitative discussion of the main parameters and lattice results involved, which will be needed for our discussion of transport cross sections.

In Fig. 1 we compile the lattice-based data on the monopole density. Since we compare various theories, SU(2) and SU(3) pure gauge theories as well as those with quarks, we need to explain the units. Following the lattice tradition, physical units are defined by insisting that the string tension is the same in all of them, $\sqrt{\sigma} = 426$ MeV. We will always show the temperatures in units of the corresponding deconfinement transition temperature $T_c$.

In Fig. 1(a) the vacuum value of the density is taken from Bornyakov et al. [33], $(n_m)_{T=0}/\sigma^{3/2} = 0.5$. The dots and crosses are from Liao and Shuryak [34]: they correspond to “condensed” and “decondensed” monopoles, respectively. Note that these two add together to an approximate constant total monopole density, for all $T < T_c$.

In Fig. 1(b) we summarize what is known about monopoles in the deconfined phase. The diamonds of different colors show the direct lattice observation of monopole density by D’Alessandro and D’Elia [21], scaled up by the factor 2 which accounts for the transition from SU(2) to SU(3) gauge group. The best fit to the data of D’Alessandro and D’Elia (not shown) is

$$n_m/T^3 = A/\log(T/\Lambda_{\text{eff}})^\alpha$$

with $A = 0.48, \alpha = 1.89$ and $T_c/\Lambda_{\text{eff}} = 2.48$: we discuss the expected parametric dependence at high $T$ below.

The crosses in Fig. 1(b) are from Ref. [34], as those in Fig. 1(a), but for temperatures larger than $T_c$. They have been obtained from (metastable) flux tubes in the plasma phase. The latter should be compared to the solid line, which represents gluons, affected by the nonzero VEV of the Polyakov line effects [30]. (At high $T$ the normalized gluon density becomes a constant corresponding to Stefan-Boltzmann ideal gas of massless SU(3) gluons.) Note that the densities of gluons and monopoles cross each other at $T \simeq 1.3T_c$.

The available lattice results on the internal monopole structure are not too detailed. Fig. 2 is taken from the talk given by Ilgenfritz at Lattice 2007 [35]. Panel

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5 The origin of those points resides in the lattice results on string tensions corresponding to free energy and potential energy, respectively.

6 The reader is reminded that it reflects the nonzero VEV of $A_0$, which acts on electrically charged particles as an imaginary chemical potential.
(a) and panel (b) show the structure, transversal to a monopole world line, of the “non-Abelianicity” (the gauge potential not in the Abelian direction) and of the modulus of the topological density, respectively. These two quantities are definitely correlated with the monopole current, however they display different sizes. The non-Abelianicity shows a narrower peak, with a small width of only about

$$r_m \approx 1.5a \approx 0.15 \text{ fm}$$  (14)

(The lattice spacing was $a = 0.105 \text{ fm}$ in this simulation). It is hardly surprising that we don’t have any real insights about the monopole structure, as their size is so close to $a$ (the UV cutoff) of the best lattice calculations.

From Fig.(b) it is evident that the topological charge shows a larger radius: the interpretation is that this is actually the size of dyons having both $\vec{E}$, $\vec{B}$ along the radius. Although we will not go into this subject, let us note that selfdual dyons, possessing the topological charge and zero modes, have been studied via fermionic methods without any gauge fixing: nice correspondence to Kraan-vanBaal solution was found.

The issue of monopole mass is currently under intense study. We have been informed by D’Elia and D’Alessandro about their derivation of the mass from

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7Provided by the unfiltered overlap-defined topological density.
8The reader is reminded that the lattice definition of monopoles, based on the end of the Dirac strings, includes all magnetically charged objects. The density that we showed in Fig. also includes all monopoles and dyons of all kinds.
monopole paths. We will only mention that the mass grows with $T$ substantially, justifying the static monopole approximation adopted in this work. Only close to $T_c$ the monopole mass becomes lighter than the gluonic one, reaching about 300 MeV at $T_c$, as anticipated from Bose condensation arguments [5]. This region is excluded from discussion in the present work.

2.2 Parametric dependence at high $T$

Let us now switch to the domain of asymptotically high $T$ and discuss what should happen when the electric sector gets really weakly coupled, namely when $e^2(T)/4\pi \ll 1$. At high $T \gg T_c$ we are in weakly coupled QGP, with “electric” degrees of freedom – quarks and gluons$^9$ – dominating. Perturbative scales include (i) the hard scale $T$; (ii) the electric scale, e.g. that of the electric mass found perturbatively [2], $E_E \sim eT$. In the magnetic sector, perturbatively there is no screening but power infrared divergencies of the diagrams, on which basis Polyakov [17] conjectured that magnetic screening will be developed nonperturbatively at the (iii) magnetic scale

$$E_M \sim e^2T \sim T/\log(T).$$

This conjecture is supported by the lattice data at high $T$ (e.g. by Nakamura et al. [37]). To give an idea of the magnitude, let us mention that at $T/T_c \approx 10$ the numerical values of the two screening masses are $M_{M}^{\text{screening}}/T \approx 1$, $M_{E}^{\text{screening}}/T \approx 2.3$. The total contribution of the magnetic sector to thermodynamics (pressure) is of the order of

$$p_m \sim e^6 N_c^5 T^4,$$

similar to what one obtains from 4 (and more) gluonic loops. This value is much smaller than the total pressure $p \sim N_c^2 T^4$. 

Perturbative arguments for the magnetic sector suggest for its entropy, monopole density, magnetic screening mass and spatial string tension to scale with Polyakov’s “magnetic scale” $e^2T$. In fact, lattice results for the monopole density [21] that we discussed above can be fitted as $n_m \sim \log(T)^{-3}$ for $T/T_c > 4$ (omitting the near-$T_c$ points which are not expected to obey the asymptotic behavior).

One general property of the magnetic scenario [7] is the expectation of a strong magnetic coupling at high $T$. More specifically, it is expected [26] to be a monopole plasma in liquid form, with the ratio of potential to kinetic energies (known as plasma parameter $\Gamma$) to be constant $\Gamma(T) \to \Gamma^*(\infty) \approx 5$. Yet, there are many more specific

$^9$One may ask why we call gluons “electric” excitations, as they have $\vec{E}$ and $\vec{B}$ of equal strength. A more accurate formulation of what we mean is: they are the excitations which can be inferred directly from the Lagrangian of the “electric” formulation of the gauge theory, in which the magnetic objects are solitonic objects. The “magnetic” formulation would have a Lagrangian with monopoles as sources, containing of course the gauge fields but no quarks.
questions about the magnetic sector which remain open. In general, the monopole properties are formed under the influence of the following three important effects: (i) interaction with $A_0$, the “Higgs field”; (ii) interaction with other monopoles; (iii) interaction with “electric” quasiparticles, quarks and gluons.

Let us comment on them, subsequently.

(i) At nonzero $T$ the role of scalar adjoint Higgs is taken by $A_0$, whose nonzero VEV breaks the color group into massive non-Abelian and massless Abelian gauge bosons. This phenomenon, related to the expectation of the Polyakov loop, is widely studied, see e.g. [31]: it is most important at $T \sim (1 - 4)T_c$. The Euclidean-real $eA_0/T$ produces a $O(1)$ phase factor – also known as imaginary chemical potential – for electric particles, quarks and gluons. The corresponding extra $i$ in Minkovski VEV does not allow to use standard ’t Hooft-Polyakov monopole solution of the Georgi-Glashow model directly: yet one would still expect that the monopole mass is heavier than that for quarks/gluons by the factor $1/e^2(T) \sim \log(T)$, thus with $M_m/T$ growing at high $T$.

(ii) Monopole-monopole interaction is Coulombic, and (as shown in [26]) the available lattice data on same and opposite sign monopole correlations are in good correspondence with MD simulations for such plasma. One important point is that, at high $T$, the stronger magnetic coupling cancels the relevant diluteness of monopoles, thus the corrections to their mass are finite

$$\Delta M_m/T \sim g^2 n_m^{-1/3}/T \sim g^2 e^2 \sim O(1)$$

(17)

similar implications hold for the magnetic screening mass

$$\left(M^\text{screening}_M\right)^2 \sim g^2 n_m/T \sim g^2 e^6 T^2 \sim (e^2 T)^2$$

(18)

which is thus at the expected “magnetic scale”. Below we will consider more detailed models of the monopole correlations and the collective (magnetic or dual) potential resulting from them.

(iii) Electric-magnetic interaction is what this paper is mostly about. Let us only note here that, since such interaction is based on the Lorentz force, it contains the coupling combination $ge \sim O(1)$ locked by the Dirac condition, thus here there is no small parameter involved.

### 2.3 Parametric estimates of the scattering rates

We define the dimensionless scattering rate of a particle of type $i$ on all particles of type $j$ as

$$\frac{\dot{w}_{ij}}{T} = \frac{\langle v_{ij} n_j \sigma_{ij} \rangle}{T}$$

(19)

where $n_j$ is the density of “scatterers” and $v_{ij} \sigma_{ij}$ is the relative velocity and corresponding (transport) cross section.
We start by counting the powers of $\log(T)$ in the well-known results for $gg \rightarrow gg$, $ggg$ processes, to be compared with the monopole-related ones.

We have

$$\frac{\dot{w}_{gg}}{T} = \frac{\langle n_g \sigma_{gg} \rangle}{T} \sim \alpha_s^2 \sim \left( \frac{1}{\log(T)} \right)^2$$

(20)

Although the radiative process $gg \rightarrow ggg$ is a higher-order process and has extra coupling, it also has one “soft” propagator carrying momentum at the “electric scale” $eT$, which puts it at the same level as the elastic process.

As we argued above, the monopole density scales with the “magnetic” scale: $n_m \sim (e^2 T)^3$ and thus it is small; nevertheless, there is no coupling constant in the cross section. Thus

$$\frac{\dot{w}_{gm}}{T} = \frac{\langle n_m \sigma_{gm} \rangle}{T} \sim \frac{\log(\log(T))}{\log^3(T)}$$

(21)

where the factor $\log(\log(T))$ comes from divergent transport cross section (to be derived below). Thus, the process that we consider is parametrically subleading at high $T$, but not by much.

There are many other details to be included below – such as numerical constants in all densities and cross sections – but at this point we only address the dependence on one more general parameter, the number of colors $N_c$. The number of gluons scales as $N_c^2$, while that of monopoles scales as $N_c^1$. The perturbative $gg$ cross section contains the 't Hooft coupling squared, $\lambda^2 = (e^2 N_c)^2$ with extra powers of $N_c$, but the result only makes sense as an expansion in $\lambda \ll 1$, so the limit of large $N_c$, irrespective of $e$, cannot be taken. The non-perturbative gluon-monopole amplitude has partial scattering phases which are pure numbers $O(1)$, thus one can rather write that the ratio of interest is $\dot{w}_{gm}/\dot{w}_{gg} \sim 1/\lambda^2 N_c$, with small $\lambda$ but large $N_c$.

### 2.4 The monopole mass

In the original context of the Georgi-Glashow model, the monopole mass can be written as a perturbative series

$$M_m = \frac{4\pi M_W}{g^2} \left[ f_0 \left( \frac{M_H}{M_W} \right) + \frac{e^2}{4\pi} f_1 \left( \frac{M_H}{M_W} \right) + O(e^4) \right].$$

(22)

The first classical term contains the known functions $f_0(0) = 1, f_0(1) = 1.238$, obtained from the numerical solution of the classical EOM. As for the second term, only its limit at small argument has been calculated \[38\]

$$f_1(z) = \frac{1}{2\pi} \log(z^2) \quad \text{for} \quad z \rightarrow 0.$$  

(23)

There was one lattice calculation of the monopole mass, by Rajantie \[39\], which was performed at $z = M_H/M_W = 1$, with very weak coupling $e^2 = 1/5$, at which the
deviations from the classical limit were small. From these results one can estimate that \( f_1(1) \approx -(1.5 - 2) \). If so, extrapolating it to \( \alpha_e = \varepsilon^2 / 4\pi \sim 0.7 \) (which we have at \( T \sim a \) few times \( T_c \) in QGP), one finds that quantum corrections would be as large as the classical mass, cancelling it to zero. While one cannot trust this extrapolation quantitatively, it certainly shows the trend.

What is the mass which is needed to reproduce the observed monopole density? We write the expression for the density as

\[
\rho_m = \frac{(\text{dof})^2}{2\pi^2} \int \frac{p^2 dp}{\exp[(\varepsilon_p + V(0))/T] - 1}
\]

with \( \varepsilon_p^2 = M_m^2 + \vec{p}^2 \). We included in the above expression the mass as well as the correlation energy, which is due to a (dual Coulomb) potential induced by all other monopoles at the position of one of them. We will write it as

\[
\frac{V(0)}{T} = -\left( \frac{g^2}{4\pi aT} \right) M
\]

with \( a = (n_m)^{-1/3} \) and \( M \) the so called “Madelung” dimensionless constant: its value is the subject of the next subsection. The effective degrees of freedom are \( \text{dof} = 2 \) for the SU(2) gauge group and 4 for the SU(3) one, which corresponds to (spinless) monopoles and antimonopoles.

If one wants the monopole ensemble to mimic the scaling relations mentioned above, the mass (minus \( V(0) \)) should slowly grow with \( T \)

\[
\frac{M_m + V(0)}{T} \approx 3\log(\log(T))
\]

in contrast to the perturbative masses of quarks and gluons which slowly decrease: \( m_q/T \sim e \sim [\log(T)]^{-1/2} \). This leads to the picture that we consider here, namely heavy, rare monopoles embedded into a dense plasma of light gluons and quarks. If so, the main interaction to be included is a strong-coupling mutual magnetic Coulomb one.

### 2.5 Monopole interactions

Before we get into the details of the possible models describing the monopole ensemble, we provide the numerical values of some key variables at two temperatures (we choose them near the ends of the interval but, prudently, we do not take the endpoints):

| \( T/T_c \) | \( n_m (\text{fm}^{-3}) \) | \( aT \) | \( n_m/T^3 \) | \( g^2/(4\pi aT) \) |
|---|---|---|---|---|
| 1.424 | 2.834 | 1.5 | 0.29 | 1.3 |
| 9.865 | 169.2 | 2.6 | 0.056 | 2.4 |
Let us start by comparing the dimensionless densities $n_m/T^3 = 1/(aT)^3$ in the table, to that of a massless Bose gas with two degrees of freedom, $n/T^3 = .244$ (like that for the blackbody photons). While at the higher $T/T_c = 9.865$ the monopole density is several times larger than this number (ascribed to a mass-induced suppression above), at the lower $T/T_c = 1.424$ it is somewhat above it, which would require “overcompensation” $M_m + V(0) < 0$. Since in this work we will not discuss the near-$T_c$ region, with all its complications, we conclude for now that $T/T_c = 1.424$ is a bit too low for our basic picture to apply in full. The reason why we will still include its discussion below is pragmatic: monopole correlations are not yet known at really high $T$.

The value of the effective plasma parameter $(g^2/(4\pi aT))$ is given in the above table, the coupling is from correlations [21]. Note that, if the Madelung constant is simply $\sim 1$, the interaction-induced factor is about $\exp(g^2/(4\pi aT)) = 3.611$ at the two considered temperature values. This cannot be taken literally, but shows that the interaction effect can be quite substantial. The values of $\Gamma$ itself grow from 1 at $T = T_c$, to about 4.5 at $T/T_c = 4$, as calculated in ref. [25]. Its $T$-dependence shows that it approaches a constant value $\Gamma^*(\infty) \approx 5$ for large $T$.

In the expression for the density, $V(0)$ implicitly depends on how the other monopoles are distributed in space, so we have to solve a manybody plasma problem. The distribution of those monopoles in space can be described by three models, corresponding to (i) “weak”, (ii) “medium” and (iii) “strong” correlations. Physically, they correspond to a “gas”, “liquid” and “molten salt” examples, respectively, all well known and studied for classical Coulomb plasmas.

In the first case (i) we simply ignore the correlations (perturbative Debye theory includes a cloud of correlated charges with total charge -1, but with large size and small Debye correlation energy $E = -(g^2/(4\pi R_D)) \sim g^3T$).

The second model [40] (ii) is based on two assumptions/inputs: that the static two-body correlation between particles is given by the lattice results for monopoles [21]; and that manybody correlations are absent. If so, the correlated charge density is spherically symmetric and thus

$$\Delta Q(r) = (n_m/2) \int_0^r (g_{++} - g_{+-}) 4\pi r^2 dr. \quad (27)$$

The interaction field and the potential induced by the correlated charge can be calculated via Gauss’ law. For one typical case discussed in [40] in detail, namely for $T/T_c = 1.42$ when $\alpha_m = g^2/4\pi \approx 2$ and $a \approx .24$ fm, the potential induced by the correlated charge is harmonic at small $r$. Its value at the origin is $V(0) \approx 35$ MeV, thus the correction induced by it is $\exp(-V(0)/T) \approx 1.086$. Such corrections $O(10\%)$ are typical for this model (ii).

---

\[^{10}We will discuss its realistic values below.\]
The third model (iii) has two different assumptions: (a) that local multi-body correlations are significant; (b) that those can be described by a cubic lattice of alternating charges, since this is known to be the ground state for ion lattices such as NaCl (thus the name “molten salt”). Collective potentials in solid plasmas are easily calculable from obvious triple sums over the atoms. Although it is not difficult to calculate them numerically, as the sum over expanding cubes converges well, quite a lot of mathematical efforts have been made over nearly a century to get the best analytic representation of it. The latest one that we found is ref. [41], which tells that the so called Madelung constant is

\[ M = -\frac{1}{8} - \frac{\log 2}{4\pi} - \frac{4\pi}{3} + \frac{\Gamma(1/8)\Gamma(3/8)}{\pi^{3/2}2^{1/2}} + S \approx -1.74764594 \]  

(28)

where \( S \) is explicitly exponentially small \( O(10^{-10}) \).

Physicwise, by elevating the Coulomb energy of a particle at the exact crystal node to a Coulomb potential, induced by all charges but one, one can see the directions in which it decreases. This reflects the obvious fact that classically the ion lattice is simply unstable against opposite charges falling at each other. That does not happen quantum-mechanically, of course, because of the “localization potential” \( \sim \hbar^2/4mr^2 \). Furthermore, both atoms in salts and monopoles in QGP are extended objects, which only interact via the Coulomb law at large distances \( r \), while there exist some repulsive cores at small \( r \). For molecular dynamics simulations [7] we used a potential of the form

\[ V(r) = \frac{g^2}{4\pi a} \left[ \frac{1}{(r/a)} - \frac{1}{p(r/a)^p} \right] \]  

(29)

where \( p \) is some power, chosen to be 9. (The factor \( 1/p \) in the second term is needed to make the force zero at distance \( r = a \).) It leads to a nice stable in-lattice potential with a near-isotropic harmonic well for small deviations from the node. The energy per particle for this potential is

\[ V(0) \approx -1.1 \frac{g^2}{4\pi a} \]  

(30)

So far we don’t have lattice results on manybody correlations of monopoles and thus we are not sure which of those is more realistic. We have presented those results for future comparison, and also to emphasize that the collective magnetic (dual) potential is large and non-negligible compared to the monopole mass. Both should be included in the monopole density calculations, which are thus more involved compared to a weakly coupled gas considered before.
3 Scalar fluctuations

We set up the problem of quantum scattering on monopoles in the framework of the Georgi-Glashow model \[42\] (see Appendix A for the details on the model and the notations we use). The generalization to QCD will be discussed in the following. The Georgi-Glashow model describes the interaction between gauge fields (two massive, spin-1 \(W\)s and a massless photon) and a scalar Higgs field. We start with the scattering of the scalar fluctuations on monopoles. We introduce a total angular momentum operator \(\vec{T}\), which is the sum of the orbital angular momentum \(\vec{L}\) and the isotopic spin \(\vec{I}\):

\[
\vec{T} = -i\vec{r} \times \nabla + \vec{I}
\]  

(31)

with \((I^a)_{bc} = -i\epsilon_{abc}\). In terms of these operators, the wave equation for the scalar fluctuations can be written in the form (for a derivation of this equation see Appendix A):

\[
\begin{pmatrix}
\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \left(\frac{\vec{T}^2 - (\hat{\vec{r}} \cdot \vec{I})^2}{r^2}\right) - \partial_6^2
\end{pmatrix} \chi + \frac{2K(\xi)}{r^2} \left(\vec{I} \cdot \vec{T} - (\hat{\vec{r}} \cdot \vec{I})^2\right) \chi

- \frac{K(\xi)^2}{r^2} \left[\vec{I}^2 - (\hat{\vec{r}} \cdot \vec{I})^2\right] \chi - \lambda \left[\frac{2\vec{r} \cdot \chi}{e^2 r^4} H(\xi)^2 \vec{r} + \left(\frac{H(\xi)^2}{e^2 r^2} - v^2\right) \chi\right] = 0.
\]  

(32)

The term which is proportional to \(K[\xi]\) induces charge-exchange reactions.

We can define a simultaneous eigenfunction \(\phi_{nm}(\hat{\vec{r}})_{a}\) of the commuting operators \(\vec{T}^2, T_3, \vec{I}\) and \(\hat{\vec{r}} \cdot \vec{I}\) (see eq. (11)). This function depends only on the angular variables specified by \(\hat{\vec{r}}\). A solution to the equation (a specific partial wave) can be written as the product of the angular function \(\phi_{nm}(\hat{\vec{r}})_{a}\) and a radial function \(S^n_t(r)\):

\[
\chi(\vec{r})_{a} = \phi_{nm}^{(\hat{\vec{r}})_{a}} S^n_t(r).
\]  

(33)

The angular function \(\phi_{nm}^{(\hat{\vec{r}})_{a}}\) is peculiar because the operator \(\vec{I}\) is projected along \(\hat{\vec{r}}\). Therefore, the angular function must be rotated, from the standard cartesian frame, to a “radial” frame. This construction can be achieved by making use of a spatially dependent unitary matrix which rotates \(\hat{\vec{r}} \cdot \vec{I}\) into \(I_3\):

\[
U(-\varphi, -\theta, \varphi) = e^{-i\varphi l_3} e^{-i\theta l_2} e^{i\varphi l_3}.
\]  

(34)

We therefore have

\[
\hat{\vec{r}} \cdot \vec{I} U(-\varphi, -\theta, \varphi) = U(-\varphi, -\theta, \varphi) l_3
\]

\[
\vec{T} U(-\varphi, -\theta, \varphi) = U(-\varphi, -\theta, \varphi) \vec{T}
\]  

(35)

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where
\[
\vec{T} = -\vec{r} \times \left( i\vec{\nabla} + e\vec{A} I_3 \right) + \hat{r} I_3.
\] (36)

and
\[
e\vec{A} = \frac{\hat{r} \times \hat{z}}{r + z}.
\] (37)

Eqs. (36) are satisfied by the following function
\[
\phi_{\alpha\beta}(\vec{r}) = (U(-\varphi, -\theta, \varphi))_{\text{a}} \mathcal{D}(\hat{r})
\] (38)

where \((\chi^{\alpha}_{i})_{\text{a}}\) is an eigenvector of \(I_3\) in the cartesian basis
\[
I_3(\chi^{\alpha}_{i})_{\text{a}} = n(\chi^{\alpha}_{i})_{\text{a}}
\] (39)

and the function \(\mathcal{D}(\hat{r})\) obeys
\[
\left\{ \begin{array}{c}
\vec{T}^2 \\
T_3
\end{array} \right\} \mathcal{D}(\hat{r}) = \left\{ \begin{array}{c}
t(t + 1) \\
m
\end{array} \right\} \mathcal{D}(\hat{r})
\] (40)

where \(I_3\) in eq. (36) is now replaced by its eigenvalue, \(n\). We have
\[
\mathcal{D}(\hat{r}) = \mathcal{D}_{\text{nm}}^{(l)}(-\varphi, \theta, \varphi) = \langle t, n|e^{-i\varphi T_3}e^{i\theta T_2}e^{i\varphi T_3}|t, m\rangle.
\] (41)

We can write the function \(\phi_{\alpha\beta}(\vec{r})_{\text{a}}\) by making use of the following expansion
\[
\phi_{\alpha\beta}(\vec{r})_{\text{a}} = \sum_{n'\alpha}\langle \chi^{n'}_{i}\rangle_{\text{a}}(-1)^{n-n'} \times
\]
\[
\times \sum_{l=|t-i|}^{t+i} \langle i, -n, t, n|l, 0\rangle \mathcal{D}_{0, m-n'}^{(l)}(-\varphi, \theta, \varphi) \langle l, m-n'|i, -n', t, m\rangle
\] (42)

where
\[
\mathcal{D}_{0, m-n'}^{(l)}(\alpha, \beta, \gamma) = \sqrt{\frac{4\pi}{2l+1}} Y_l^m(\beta, \gamma).
\] (43)

These functions are normalized in the following way
\[
\int_{-1}^{1} d\cos\theta \phi_{\alpha\beta}^{m_1 n_1}(-\varphi, \theta, \varphi)^{\dagger} \phi_{\alpha\beta}^{m_2 n_2}(\theta, \varphi) = \frac{4\pi}{2l_1 + 1} \delta_{l_1, l_2} \delta_{m_1, m_2} \delta_{n_1, n_2}.
\] (44)

Since the above function is a polynomial in \(x/r, y/r, z/r\), the function in eq. (42) is analytic everywhere. This clearly shows that there are compensating singularities in \(U(-\varphi, -\theta, \varphi)\) and \(\mathcal{D}(\hat{r})\). Another way of building the spherical harmonics [42] would be to act with the vector operators \(\vec{r}, (r\vec{\nabla} - i\vec{\Lambda}), (r\vec{\nabla} + i\vec{\Lambda})\) on the standard angular functions \(Y_{l_m}(\theta, \varphi)\) (with \(\vec{\Lambda} = \vec{r} \times \vec{\nabla}\)). The two methods are totally equivalent, but
the one we choose here, following ref. [10], provides an easy way to understand the
mechanism of rotating from the standard, cartesian frame to a “radial” one. Since
equation (32) in its most general form admits mixing between particles of different
charge, in order to get the equation for the radial functions \( S^n_t \) we have to write the
order 1 fluctuations of the field around the classical solutions as a superposition of
the functions describing a particle with definite charge \( n \):
\[
\chi(\vec{r}, x_0)_a = \sum_{n=-1,0,1} e^{i\omega x_0} \frac{S^n_t(r)}{r} \phi_{tn}^m(\theta, \varphi)_a
\]  
(45)

where the three functions \( S^n_t(r) \) correspond to the three physical fluctuations with
charge \( n = 0, \pm 1 \). We plug the above expansion for \( \chi(\vec{r}) \) into Eq. (32); through
this procedure we obtain the following system of equations for the radial functions

\[
S^{0''}_t(\xi) - \left( \frac{t(t+1)}{\xi^2} + 2 \frac{K(\xi)^2}{\xi^2} - \omega^2 \right) S^0_t(\xi) - \lambda \left( 3 \frac{H(\xi)^2}{\xi^2} - 1 \right) S^0_t(\xi)
+ \frac{\sqrt{2t(t+1)}}{\xi^2} K(\xi) (S^1_t(\xi) + S^{-1}_t(\xi)) = 0 
\]  
(46a)

\[
S^{1''}_t(\xi) - \left( \frac{t(t+1) - 1}{\xi^2} + \frac{K(\xi)^2}{\xi^2} - \omega^2 \right) S^1_t(\xi) - \lambda \left( \frac{H(\xi)^2}{\xi^2} - 1 \right) S^1_t(\xi)
+ \frac{\sqrt{2t(t+1)}}{\xi^2} K(\xi) S^0_t(\xi) = 0 
\]  
(46b)

\[
S^{-1''}_t(\xi) - \left( \frac{t(t+1) - 1}{\xi^2} + \frac{K(\xi)^2}{\xi^2} - \omega^2 \right) S^{-1}_t(\xi) - \lambda \left( \frac{H(\xi)^2}{\xi^2} - 1 \right) S^{-1}_t(\xi)
+ \frac{\sqrt{2t(t+1)}}{\xi^2} K(\xi) S^0_t(\xi) = 0. 
\]  
(46c)

where we have introduced the dimensionless variable \( \xi = evr \). At the end of this
section we will discuss how to fix the scale and go to physical units. As it is evident
from the above system, a mixing occurs in the monopole core between different
charges: the term \( \propto K(\xi) \) involves a mixing between charges that differ by one
unit. The above system of equations has been obtained in the most general case,
for generic angular momentum \( t \). Nevertheless, we have to keep in mind that there
are some restrictions due to the following requirement:
\[
\hat{r} \cdot \vec{T} = \hat{r} \cdot \vec{L} + \hat{r} \cdot \vec{I} = \hat{r} \cdot \vec{I} = n. 
\]  
(47)
For this reason, in the case $t = 0$ only the $n = 0$ scalar fluctuation is allowed. The equation for this special case and its solution will be discussed in the following.

For $t > 0$, the system of equations (46) is difficult to solve, due to the mixing between the different radial functions. This mixing is due to the charge-exchange reactions that can occur inside the monopole core. If the monopole core is small (we have seen in Section 2.1 that lattice-based estimates for the monopole size give $r_m \simeq 0.15 \text{ fm}$) we can neglect the charge-exchange reactions. This corresponds to considering the above system of equations (46) in the limit

\[ K(\xi) \to 0 \quad H(\xi) \to \xi. \]  \hspace{1cm} (48)

In this approximation, it reduces to

\[ S^{0''}_t(\xi) - \left( \frac{t(t+1)}{\xi^2} - \omega^2 + 2\lambda \right) S^{0'}_t(\xi) = 0 \] \hspace{1cm} (49a)

\[ S^{1''}_t(\xi) - \left( \frac{t(t+1)-1}{\xi^2} - \omega^2 \right) S^1_t(\xi) = 0 \] \hspace{1cm} (49b)

\[ S^{-1''}_t(\xi) - \left( \frac{t(t+1)-1}{\xi^2} - \omega^2 \right) S^{-1}_t(\xi) = 0. \] \hspace{1cm} (49c)

From the above system it is clear that we can identify the radial functions with spherical Bessel functions having index $t'$, which is the positive root of

\[ t'(t' + 1) = t(t + 1) - n^2. \] \hspace{1cm} (50)

Namely, in the limit of small monopole core we have $S^n_t(r) \to j_n(kr)$. In general, $t'$ is not an integer number. The corresponding scattering phase will be $\delta_{t'} = t'\pi/2$, independent of the energy of the incoming particle.

For $t = 0$ we have only one fluctuation allowed, namely the one having zero-charge: $S^0_0(\xi)$. It obeys the following equation

\[ S^{0''}_0(\xi) - \left( \frac{2K(\xi)^2}{\xi^2} \right) S^0_0(\xi) - \lambda \left( 3\frac{H(\xi)^2}{\xi^2} - 1 \right) S^0_0(\xi) = -\omega^2 S^0_0(\xi). \] \hspace{1cm} (51)

In this case, there is no Coulomb potential of the form $1/\xi^2$, which is obvious since a charge-neutral particle does not feel the Lorentz force. The scattering in this case is entirely due to the monopole core. We can solve the above equation numerically, thus obtaining the scattering phase as a function of the energy of the incoming particle, by imposing the following boundary conditions

\[ S^0_0(\xi) = \sin \left[ \xi \sqrt{\omega^2 - 2\lambda + \delta_0} \right] \quad \xi \to \infty \]

\[ S^{0'}_0(\xi) = \sqrt{\omega^2 - 2\lambda} \cos \left[ \xi \sqrt{\omega^2 - 2\lambda + \delta_0} \right] \quad \xi \to \infty. \] \hspace{1cm} (52)
Figure 3: Scattering phase $\delta_0$ as a function of $|\vec{k}| = \sqrt{\omega^2 - 2\lambda}$. $\delta_0$ is obtained by solving Eq. (51) with the boundary conditions (52). The continuous line corresponds to the BPS limit ($\lambda = 0$), while the dashed line corresponds to $\lambda = 1$.

The scattering phase as a function of $|\vec{k}| = \sqrt{\omega^2 - 2\lambda}$ is plotted in Fig. 3 in the BPS (non-interacting) case, and also for a finite value of the coupling $\lambda$ in the Higgs potential ($\lambda = 1$). Fig. 4 shows the classical solutions $H(\xi)$ and $K'(\xi)$, both in the BPS limit and for $\lambda = 1$. At this point we need to fix the scale in our problem, and to estimate the scattering length in physical units. In order to do it, we have to connect the physical size of the core of the monopole, $r_m$ (14), to the dimensionless units that we used in the Georgi-Glashow model. We recall that we obtained $r_m$ through the width at half height of the “non-Abelianicity”, which is defined as the square of the non-abelian components of the gauge potential. Therefore, it is natural to fix the scale by imposing that, in the Georgi-Glashow model, $r_m$ coincides with $\xi_m$, the width at half height of the function $K'(\xi)^2$. By looking at Fig. 4 we can see that, in the BPS limit, we have $\xi_m = 1.49$, while for $\lambda = 1$ we have $\xi_m = 0.87$. Thus, $ev = 9.94$ fm$^{-1}$ in the BPS limit, while $ev = 5.8$ fm$^{-1}$ for $\lambda = 1$. The scattering length is defined as $a_{sl} \simeq -d\delta_0/dk$. From Fig. 3 which shows the behavior of $\delta_0$ as a function of $|\vec{k}|$, we get $a_{sl} = 1$ in the BPS limit and $a_{sl} = 0.6$ for $\lambda = 1$. Converting to physical units, we obtain $a_{sl} \simeq 0.1$ fm for both values of $\lambda$. 

22
4 Vector fluctuations

In the case of vector particles, the generalized angular momentum $\vec{J}$ is made up of three components: the orbital angular momentum $\vec{L}$, the isotopic spin $\vec{I}$ and the spin $\vec{S}$:

$$\vec{J} = \vec{L} + \vec{I} + \vec{S} = \vec{T} + \vec{S}. \quad (53)$$

There are generally two different ways of composing three vectors, depending on the two vectors to be added first. The monopole vector spherical harmonics are eigenfunctions of $\vec{J}^2$ and $J_3$. Due to the following relation

$$\hat{r} \cdot \vec{J} = \hat{r} \cdot \vec{L} + \hat{r} \cdot \vec{I} + \hat{r} \cdot \vec{S} = \hat{r} \cdot \vec{I} + \hat{r} \cdot \vec{S} = n + \sigma, \quad (54)$$

the allowed values of the total angular momentum quantum number $j$ are $|n| - 1, |n|, ...$ except in the case of $n = 0$, where $|n| - 1$ is absent. There are different ways of building the monopole vector spherical harmonics; for example, they can be constructed by making use of the standard Clebsch-Gordan technique of addition of momenta.

Another possibility, which we will adopt here, is to build the vector harmonics with $j \geq n$ by applying vector operators to the scalar harmonics, as we will see in the
following [44]. By definition, these harmonics can be introduced as eigenfunctions of the operator of the radial component of the spin, $\vec{S} \cdot \hat{r}$. We will show that this is a very useful choice for the "hedgehog" configuration we are working in: in fact, there is a natural separation between radial and transverse vectors, making this choice particularly useful for studying spherically symmetric problems. The vector harmonics with the minimum allowed angular momentum $j = |n| - 1$ cannot be constructed in this way and must be treated specially. As already mentioned, there is more than one way to obtain a given value of $j$, and thus several multiplets of harmonics with the same total angular momentum. In the following, we will classify the multiplets by the eigenvalue of $\hat{r} \cdot \vec{S} = \sigma$. In general, $\sigma = 0, \pm 1$, but it is further restricted by the requirement [54], which implies that $n + \sigma$ lies in the range $-j$ to $j$. This gives

- for $j = 0$:
  - $n = 0$ and $\sigma = 0$
  - $n = 1$ and $\sigma = -1$
  - $n = -1$ and $\sigma = 1$

- for $j = 1$ all combinations are allowed, except:
  - $n = -1$ and $\sigma = -1$
  - $n = 1$ and $\sigma = 1$

We denote the vector harmonics by $\Phi_{j,n}^{m,\sigma}(\theta, \varphi)_{ai}$. They obey the following eigenvalue equations

$$
\begin{align*}
\left\{ \begin{array}{c}
\vec{J}^2 \\
J_3 \\
(\hat{r} \cdot \vec{I}) \\
(\hat{r} \cdot \vec{S})
\end{array} \right\} \Phi_{j,n}^{m,\sigma}(\theta, \varphi)_{ai} &= \begin{pmatrix}
j(j+1) \\
m \\
n \\
\sigma
\end{pmatrix} \Phi_{j,n}^{m,\sigma}(\theta, \varphi)_{ai}.
\end{align*}
$$

(55)

4.1  $j \geq 2$

In this case, all the possible combinations of $n$ and $\sigma$ are allowed. We can therefore construct a set of vector spherical harmonics $\Phi_{j,n}^{m,\sigma}(\theta, \varphi)_{ia}$ that will obey the
eigenvalue equations \(55\) (in the following, \(\vec{\Lambda} = \vec{r} \times \vec{\nabla}\)):

\[
\Phi_{j,0}^{m,1}(\theta, \varphi)_{ia} = \frac{1}{\sqrt{j(j+1)(2j+1)}} \left[ \left( r\vec{\nabla} + i\vec{\Lambda} \right)_i \hat{r}_a Y_j^m(\theta, \varphi) \right.
+ \frac{1}{j(j+1)} \left( r\vec{\nabla} + i\vec{\Lambda} \right)_i \left( r\vec{\nabla} - i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \left. \right]
\]

\[
\Phi_{j,0}^{m,-1}(\theta, \varphi)_{ia} = \frac{1}{\sqrt{j(j+1)(2j+1)}} \left[ \left( r\vec{\nabla} - i\vec{\Lambda} \right)_i \hat{r}_a Y_j^m(\theta, \varphi) \right.
+ \frac{1}{j(j+1)} \left( r\vec{\nabla} - i\vec{\Lambda} \right)_i \left( r\vec{\nabla} + i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \left. \right]
\]

\[
\Phi_{j,1}^{m,-1}(\theta, \varphi)_{ia} = \frac{1}{j(j+1)\sqrt{2(2j+1)}} \left[ \left( r\vec{\nabla} - i\vec{\Lambda} \right)_i \left( r\vec{\nabla} + i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \right]
\]

\[
\Phi_{j,-1}^{m,1}(\theta, \varphi)_{ia} = \frac{1}{j(j+1)\sqrt{2(2j+1)}} \left[ \left( r\vec{\nabla} + i\vec{\Lambda} \right)_i \left( r\vec{\nabla} - i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \right]
\]

\[
\Phi_{j,1}^{m,1}(\theta, \varphi)_{ia} = \sqrt{\frac{2}{(2j+1)(j-1)j(j+1)(j+2)}} \left[ \left( r\vec{\nabla} + i\vec{\Lambda} \right)_i \hat{r}_a Y_j^m(\theta, \varphi) \right.
+ \frac{1}{2} \left( r\vec{\nabla} + i\vec{\Lambda} \right)_i \left( r\vec{\nabla} + i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \left. \right]
+ \frac{1}{j(j+1)} \left( r\vec{\nabla} + i\vec{\Lambda} \right)_i \left( r\vec{\nabla} - i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \left. \right]
\]

\[
\Phi_{j,-1}^{m,-1}(\theta, \varphi)_{ia} = \sqrt{\frac{2}{(2j+1)(j-1)j(j+1)(j+2)}} \left[ \left( r\vec{\nabla} - i\vec{\Lambda} \right)_i \hat{r}_a Y_j^m(\theta, \varphi) \right.
+ \frac{1}{2} \left( r\vec{\nabla} - i\vec{\Lambda} \right)_i \left( r\vec{\nabla} - i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \left. \right]
+ \frac{1}{j(j+1)} \left( r\vec{\nabla} - i\vec{\Lambda} \right)_i \left( r\vec{\nabla} + i\vec{\Lambda} \right)_a Y_j^m(\theta, \varphi) \left. \right]
\]
The above functions are normalized in the following way:

\[ \Phi_{j,0}^{m,0}(\theta, \varphi)_{ia} = \sqrt{\frac{2}{2j+1}} \hat{r}_i \hat{r}_a Y_j^m(\theta, \varphi) \]

\[ \Phi_{j,1}^{m,0}(\theta, \varphi)_{ia} = \frac{1}{\sqrt{j(j+1)(2j+1)}} \hat{r}_i \left(r \tilde{\nabla} + i \tilde{\Lambda} \right) Y_j^m(\theta, \varphi) \]

\[ \Phi_{j,-1}^{m,0}(\theta, \varphi)_{ia} = \frac{1}{\sqrt{j(j+1)(2j+1)}} \hat{r}_i \left(r \tilde{\nabla} - i \tilde{\Lambda} \right) Y_j^m(\theta, \varphi). \]

The above functions are normalized in the following way:

\[ \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta) (\Phi_{j,n}^{m,\sigma}(\theta, \varphi))^\dagger \Phi_{j,n'}^{m',\sigma'}(\theta, \varphi) = \frac{2}{2j+1} \delta_{jj'} \delta_{mm'} \delta_{\sigma\sigma'}. \]

We can therefore expand the vector fluctuations in terms of the above spherical harmonics basis:

\[ a(\vec{r}, x_0)_{ia} = \frac{e^{i\omega x_0}}{r} \left[ T_{j1}(r) \Phi_{j,0}^{m,1}(\theta, \varphi) + T_{j2}(r) \Phi_{j,1}^{m,1}(\theta, \varphi) + T_{j3}(r) \Phi_{j,-1}^{m,1}(\theta, \varphi) + \right. \]

\[ + T_{j4}(r) \Phi_{j,0}^{m,-1}(\theta, \varphi) + T_{j5}(r) \Phi_{j,1}^{m,-1}(\theta, \varphi) + T_{j6}(r) \Phi_{j,-1}^{m,-1}(\theta, \varphi) + \]

\[ T_{j7}(r) \Phi_{j,0}^{m,0}(\theta, \varphi) + T_{j8}(r) \Phi_{j,1}^{m,0}(\theta, \varphi) + T_{j9}(r) \Phi_{j,-1}^{m,0}(\theta, \varphi) \right]_{ia}. \]

The coefficients of the expansion are radial functions; by plugging eq. (58) into the corresponding field equation for the vector fluctuations (\ref{A-15}), we obtain a system of differential equations for these coefficients. We find that the equations for \( T_{j7}(r), T_{j8}(r) \) and \( T_{j9}(r) \) do not contain a second derivative with respect to \( \xi \). These amplitudes are closely related to infinitesimal gauge transformations (see for example the discussion in Ref. \cite{45}) and can in fact be removed by such transformations. Explicitly we have

\[ T_{j1}''(\xi) - \sqrt{\frac{j(j+1)}{2}} \xi T_{j7}(\xi) + \frac{K(\xi)}{\xi} T_{j8}(\xi) - \left[ -\omega^2 + \frac{j(j+1) + 2K(\xi)^2}{2\xi^2} \right] T_{j1}(\xi) \]

\[ + \sqrt{\frac{(j-1)(j+2)}{2}} \xi^2 T_{j2}(\xi) + \sqrt{\frac{j(j+1)}{2}} \xi^2 T_{j3}(\xi) + \frac{j(j+1)}{2\xi^2} T_{j4}(\xi) \]

\[ - \sqrt{2j(j+1)} \xi^2 T_{j5}(\xi) + \sqrt{\frac{j(j+1)}{2}} \xi^2 T_{j7}(\xi) + \frac{2K'(\xi)\xi - K(\xi)}{\xi^2} T_{j8}(\xi) = 0 \]

(59a)
\[
T''_{j_2}(\xi) - \sqrt{\frac{(j - 1)(j + 2)}{2}} \frac{1}{\xi} T'_{j_8}(\xi) - \left[ -\omega^2 + \frac{j (j + 1) - 2 K (\xi)^2 + 2 H (\xi)^2}{2\xi^2} \right] T_{j_2}(\xi)
+ \sqrt{\frac{(j - 1)(j + 2)}{2}} K (\xi) \xi T_{j(1)} + \sqrt{\frac{(j - 1)j(j + 1)(j + 2)}{2\xi^2}} T_{j_5}(\xi)
+ \sqrt{\frac{(j - 1)(j + 2)}{2}} \frac{1}{\xi^2} T_{j_8}(\xi) = 0
\quad (59b)
\]

\[
T''_{j_3}(\xi) + \frac{K(\xi)}{\xi} T'_{j_7}(\xi) - \sqrt{\frac{j(j + 1)}{2}} \frac{1}{\xi} T'_{j_9}(\xi)
- \left[ -\omega^2 + \frac{j (j + 1) - 2 + 4 K (\xi)^2 + 2 H (\xi)^2}{2\xi^2} \right] T_{j_3}(\xi)
+ \sqrt{\frac{j(j + 1) K (\xi)}{2\xi^2}} T_{j_1}(\xi)
- \sqrt{2j(j + 1)} \frac{K(\xi)}{\xi^2} T_{j_5}(\xi) + \sqrt{\frac{(j - 1)j(j + 1)(j + 2)}{2\xi^2}} T_{j_6}(\xi)
+ \frac{2 K'(\xi) \xi - K(\xi)}{\xi^2} T_{j_7}(\xi) + \sqrt{\frac{j(j + 1)}{2}} \frac{1}{\xi^2} T_{j_9}(\xi) = 0
\quad (59c)
\]

\[
T''_{j_4}(\xi) - \sqrt{\frac{j(j + 1)}{2}} \frac{1}{\xi} T'_{j_7}(\xi) + \frac{K(\xi)}{\xi} T'_{j_9}(\xi) - \left[ -\omega^2 + \frac{j (j + 1) + 2 K (\xi)^2}{2\xi^2} \right] T_{j_4}(\xi)
+ \frac{j(j + 1)}{2\xi^2} T_{j_1}(\xi) - \sqrt{2j(j + 1)} \frac{K(\xi)}{\xi^2} T_{j_3}(\xi) + \sqrt{\frac{j(j + 1) K(\xi)}{2\xi^2}} T_{j_5}(\xi)
+ \sqrt{\frac{(j - 1)(j + 2)}{2}} \frac{K(\xi)}{\xi^2} T_{j_6}(\xi) + \sqrt{\frac{j(j + 1)}{2}} \frac{1}{\xi^2} T_{j_7}(\xi) + \frac{2 K'(\xi) \xi - K(\xi)}{\xi^2} T_{j_9}(\xi) = 0
\quad (59d)
\]

\[
T''_{j_5}(\xi) + \frac{K(\xi)}{\xi} T'_{j_7}(\xi) - \sqrt{\frac{j(j + 1)}{2}} \frac{1}{\xi} T'_{j_8}(\xi)
- \left[ -\omega^2 + \frac{j (j + 1) - 2 + 4 K (\xi)^2 + 2 H (\xi)^2}{2\xi^2} \right] T_{j_5}(\xi) - \sqrt{2j(j + 1)} \frac{K(\xi)}{\xi^2} T_{j_1}(\xi)
+ \sqrt{\frac{(j - 1)j(j + 1)(j + 2)}{2\xi^2}} T_{j_2}(\xi) + \frac{K(\xi)^2}{\xi^2} T_{j_3}(\xi) + \sqrt{\frac{j(j + 1) K(\xi)}{2\xi^2}} T_{j_4}(\xi)
+ \frac{2 K'(\xi) \xi - K(\xi)}{\xi^2} T_{j_7}(\xi) + \sqrt{\frac{j(j + 1)}{2}} \frac{1}{\xi^2} T_{j_8}(\xi) = 0
\quad (59e)
\]
\[ T''_j(\xi) - \sqrt{\frac{(j-1)(j+2)}{2}} \frac{1}{\xi} T'_j(\xi) - \left[-\omega^2 + \frac{j(j+1) - 2K(\xi)^2 + 2H(\xi)^2}{2\xi^2}\right] T_j(\xi) \\
+ \frac{\sqrt{(j-1)(j+1)(j+2)}}{2\xi^2} T_{j3}(\xi) + \sqrt{\frac{(j-1)(j+2)}{2}} \frac{K(\xi)}{\xi^2} T_{j4}(\xi) \\
+ \sqrt{\frac{(j-1)(j+2)}{2}} \frac{1}{\xi^2} T_{j9}(\xi) = 0 \quad (59f) \]

\[ \frac{T'_{j1}(\xi)}{\xi} + \left(1 - \frac{2K(\xi)^2}{j(j+1)}\right) \frac{T'_{j4}(\xi)}{\xi} - \frac{\sqrt{2}K(\xi)}{\xi} \frac{T'_{j5}(\xi)}{\xi} \\
+ \frac{\sqrt{2}(j-1)(j+2)K(\xi)}{j(j+1)} \frac{T_{j6}(\xi)}{\xi} + \frac{\sqrt{2}K'(\xi)}{\xi} \frac{T_{j3}(\xi)}{\xi} + \frac{2K(\xi)K'(\xi)}{j(j+1)} \frac{T_{j4}(\xi)}{\xi} \\
+ \frac{\sqrt{2}K'(\xi)}{\sqrt{j(j+1)}} \frac{T_{j5}(\xi)}{\xi} + \frac{\sqrt{2}}{\sqrt{j(j+1)}} \left(\omega^2 - \frac{j(j+1)}{\xi^2}\right) T_{j7}(\xi) + \frac{2K(\xi)}{\xi^2} T_{j8}(\xi) \\
- \frac{2K(\xi)(-1 + H(\xi)^2 + K(\xi)^2 - \omega^2\xi^2)}{j(j+1)\xi^2} T_{j9}(\xi) = 0 \quad (59g) \]

\[ \frac{T'_{j3}(\xi)}{\xi} - \frac{\sqrt{2}K(\xi)}{\sqrt{j(j+1)}} \frac{T'_{j4}(\xi)}{\xi} + \sqrt{\frac{(j-1)(2+j)}{j(j+1)}} \frac{T'_{j6}(\xi)}{\xi} + \frac{\sqrt{2}K'(\xi)}{\sqrt{j(j+1)}} \frac{T_{j4}(\xi)}{\xi} \\
+ \frac{2K(\xi)}{\xi^2} T_{j7}(\xi) - \frac{\sqrt{2}(H(\xi)^2 + j(j+1) - 1 + K(\xi)^2 - \omega^2\xi^2)}{\sqrt{j(j+1)\xi^2}} T_{j9}(\xi) = 0 \quad (59h) \]

\[ \frac{T'_{j2}(\xi)}{\xi} + \frac{\sqrt{2}K(\xi)(j(j+1) - 2K(\xi)^2)}{j(j+1)\sqrt{(j-1)(j+2)}} \frac{T'_{j4}(\xi)}{\xi} + \frac{(j(j+1) - 2K(\xi)^2)}{\sqrt{(j-1)j(j+1)(j+2)}} \frac{T'_{j5}(\xi)}{\xi} \\
+ \frac{2K(\xi)^2}{j(j+1)} \frac{T_{j6}(\xi)}{\xi} + \frac{\sqrt{2}K'(\xi)}{\sqrt{(j-1)(j+1)}} \frac{T_{j1}(\xi)}{\xi} + \frac{2K(\xi)K'(\xi)}{\sqrt{(j-1)j(j+1)(j+2)}} \frac{T_{j3}(\xi)}{\xi} \\
+ \frac{2\sqrt{2}K(\xi)^2K'(\xi)}{j(j+1)\sqrt{(j-1)(j+2)}} \frac{T_{j4}(\xi)}{\xi} + \frac{2K(\xi)K'(\xi)}{\sqrt{(j-1)j(j+1)(j+2)}} \frac{T_{j5}(\xi)}{\xi} \\
- \frac{2\omega^2K(\xi)}{\sqrt{(j-1)j(j+1)(j+2)}} T_{j7}(\xi) - \frac{\sqrt{5}(H(\xi)^2 + j(j+1) - 1 + K(\xi)^2 - \omega^2\xi^2)}{\sqrt{(j-1)(j+2)\xi^2}} T_{j8}(\xi) \\
- \frac{2\sqrt{2}K(\xi)^2(H(\xi)^2 - 1 + K(\xi)^2 - \omega^2\xi^2)}{j(j+1)\sqrt{(j-1)(j+2)}} T_{j9}(\xi) = 0 \quad (59i) \]
As we already did for the scalar channel, we consider the above system in the limit of small monopole core. This corresponds to the limit

$$K(\xi) \to 0 \quad H(\xi) \to \xi.$$  \hspace{1cm} (60)

Besides, we gauge away the unphysical longitudinal components of our fields ($T_{j7}(\xi)$, $T_{j8}(\xi)$, $T_{j9}(\xi)$), after which the above system reduces to:

$$T''_{j1}(\xi) = \left[-\omega^2 + \frac{j(j+1)}{2\xi^2} \right] T_{j1}(\xi) + \frac{j(j+1)}{2\xi^2} T_{j4}(\xi) = 0$$  \hspace{1cm} (61a)

$$T''_{j4}(\xi) = \left[-\omega^2 + \frac{j(j+1)}{2\xi^2} \right] T_{j4}(\xi) + \frac{j(j+1)}{2\xi^2} T_{j1}(\xi) = 0$$  \hspace{1cm} (61b)

$$T''_{j2}(\xi) = \left[-\omega^2 + \frac{j(j+1)}{2\xi^2} \right] T_{j2}(\xi) + \frac{\sqrt{(j-1)(j+1)(j+2)}}{2\xi^2} T_{j5}(\xi) = 0$$  \hspace{1cm} (61c)

$$T''_{j5}(\xi) = \left[-\omega^2 + \frac{j(j+1)}{2\xi^2} \right] T_{j5}(\xi) + \frac{\sqrt{(j-1)(j+1)(j+2)}}{2\xi^2} T_{j2}(\xi) = 0$$  \hspace{1cm} (61d)

$$T''_{j3}(\xi) = \left[-\omega^2 + \frac{j(j+1)}{2\xi^2} \right] T_{j3}(\xi) + \frac{\sqrt{(j-1)(j+1)(j+2)}}{2\xi^2} T_{j6}(\xi) = 0$$  \hspace{1cm} (61e)

$$T''_{j6}(\xi) = \left[-\omega^2 + \frac{j(j+1)}{2\xi^2} \right] T_{j6}(\xi) + \frac{\sqrt{(j-1)(j+1)(j+2)}}{2\xi^2} T_{j3}(\xi) = 0$$  \hspace{1cm} (61f)

$$T'_{j1}(\xi) = -T'_{j4}(\xi)$$  \hspace{1cm} (61g)

$$T'_{j2}(\xi) = -\sqrt{\frac{j(j+1)}{(j-1)(j+2)}} T'_{j5}(\xi)$$  \hspace{1cm} (61h)

$$T'_{j3}(\xi) = -\sqrt{\frac{(j-1)(j+2)}{j(j+1)}} T'_{j6}(\xi)$$  \hspace{1cm} (61i)
The last three equations in the above set are what remains of the equations for the longitudinal components of the fields, namely for $T_{j7}(\xi)$, $T_{j8}(\xi)$, $T_{j9}(\xi)$, after we gauge these components away. They can be used as constraints for the other six $T_{ji}$s. Indeed, after taking into account these constraints, we end up with the following set of (diagonal) equations for our radial functions:

\[
T_{j1}''(\xi) - \left[ -\omega^2 + \frac{j(j+1)}{\xi^2} \right] T_{j1}(\xi) = 0 \quad (62a)
\]

\[
T_{j4}''(\xi) - \left[ -\omega^2 + \frac{j(j+1)}{\xi^2} \right] T_{j4}(\xi) = 0 \quad (62b)
\]

\[
T_{j2}''(\xi) - \left[ -\omega^2 + 1 + \frac{j(j+1)-1}{\xi^2} \right] T_{j2}(\xi) = 0 \quad (62c)
\]

\[
T_{j5}''(\xi) - \left[ -\omega^2 + 1 + \frac{j(j+1)-1}{\xi^2} \right] T_{j5}(\xi) = 0 \quad (62d)
\]

\[
T_{j3}''(\xi) - \left[ -\omega^2 + 1 + \frac{j(j+1)-1}{\xi^2} \right] T_{j3}(\xi) = 0 \quad (62e)
\]

\[
T_{j6}''(\xi) - \left[ -\omega^2 + 1 + \frac{j(j+1)-1}{\xi^2} \right] T_{j6}(\xi) = 0. \quad (62f)
\]

We observe that the above equations are those corresponding to Bessel functions with index $j' = \frac{1}{2} (-1 + \sqrt{(2j+1)^2 - 4n^2})$. In analogy with the scalar fluctuations, also in this case we have a constant scattering phase $\delta_{j'} = j'\pi/2$. The same set of equations can be obtained, as a check, if we consider the general field equations for the fluctuations, in the limit of pointlike monopole core for the three fields $W_n$ ($n = 0, \pm 1$):

\[
\left[ \frac{1^2}{\xi} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial}{\partial \xi} \right) - \frac{\vec{T}^2 - \left( \vec{\bar{r}} \cdot \bar{r} \right)^2 + 2n \left( \vec{\bar{S}} \cdot \bar{r} \right)}{\xi^2} \right] W_n = 0. \quad (63)
\]

We remember that the operator $\vec{T}$ is given by

\[
\vec{T} = \vec{L} + \vec{I} = \vec{J} - \vec{S}
\]

and therefore we have

\[
\vec{T}^2 = \vec{J}^2 + \vec{S}^2 - 2\vec{J} \cdot \vec{S}. \quad (65)
\]
In the above equation, the only nondiagonal operator is $2\vec{J} \cdot \vec{S}$ and we therefore have to understand how it acts on our angular functions. We obtain:

$$2\vec{J} \cdot \vec{S}\Phi_{j,n}^{0,1}(\theta, \varphi) = \sqrt{2(j-n)(j+1+n)}\Phi_{j,n}^{0,0}(\theta, \varphi) + 2(1+n)\Phi_{j,n}^{0,1}(\theta, \varphi)$$

$$2\vec{J} \cdot \vec{S}\Phi_{j,n}^{0,-1}(\theta, \varphi) = \sqrt{2(j+n)(j+1-n)}\Phi_{j,n}^{0,0}(\theta, \varphi) + 2(1-n)\Phi_{j,n}^{0,-1}(\theta, \varphi).$$

(66)

Since the above equations involve mixing with the longitudinal components only (for example $\Phi_{j,n}^{0,1}(\theta, \varphi)$ only gets mixed with $\Phi_{j,n}^{0,0}(\theta, \varphi)$ and not with $\Phi_{j,n}^{0,-1}(\theta, \varphi)$, and since we have seen that the longitudinal components are spurious and can be gauged away, we conclude that (63) will produce diagonal equations for the remaining radial functions. In particular we get

$$\begin{bmatrix}
\vec{J}^2 + \vec{S}^2 - 2\vec{J} \cdot \vec{S} - \left(\vec{I} \cdot \hat{r}\right)^2 \pm 2\left(\vec{S} \cdot \hat{r}\right) \\
- j(j+1) + s(s+1) - 2(1+n) - n^2 + 2n
\end{bmatrix} \xi^2 \Phi_{j,n}^{0,1}(\theta, \varphi) =$$

$$\begin{bmatrix}
\vec{J}^2 + \vec{S}^2 - 2\vec{J} \cdot \vec{S} - \left(\vec{I} \cdot \hat{r}\right)^2 \pm 2\left(\vec{S} \cdot \hat{r}\right) \\
- j(j+1) + s(s+1) - 2(1-n) - n^2 + 2n
\end{bmatrix} \xi^2 \Phi_{j,n}^{0,-1}(\theta, \varphi).$$

which results in the set of equations (62) for the radial functions.

### 4.2 $j=0$

As we have seen, for $j = 0$, only three combinations of $n$ and $\sigma$ are allowed

- $n = 0$ and $\sigma = 0$
- $n = 1$ and $\sigma = -1$
- $n = -1$ and $\sigma = 1$. 


We construct the corresponding set of vector spherical harmonics in the following way

\[
(\Phi_{0,0}^{0,0}(\theta, \varphi))_{ia} = \sqrt{2} \hat{r}_i \hat{r}_a Y_0^0(\theta, \varphi) \\
(\Phi_{0,1}^{0,-1}(\theta, \varphi))_{ia} = \frac{1}{\sqrt{2}} \left[ r \vec{\nabla} - i \vec{\Lambda} \right] \hat{r}_a Y_0^0(\theta, \varphi) \\
(\Phi_{0,-1}^{0,1}(\theta, \varphi))_{ia} = \frac{1}{\sqrt{2}} \left[ r \vec{\nabla} + i \vec{\Lambda} \right] \hat{r}_a Y_0^0(\theta, \varphi).
\] (68)

The above functions are normalized in the following way

\[
\int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta) (\Phi_{0,n}^0(\theta, \varphi))^{*} \Phi_{0,n'}^0(\theta, \varphi) = 2 \delta_{nn'} \delta_{\sigma\sigma'}.
\] (69)

We can write the vector fluctuations for \( j = 0 \) as a superposition of the above vector spherical harmonics:

\[
a(r, x_0)_{ia} = \frac{e^{i \omega x_0}}{r} (T_{07}(r) \Phi_{0,0}^0(\theta, \varphi) + T_{03}(r) \Phi_{0,1}^{0,-1}(\theta, \varphi) + T_{05}(r) \Phi_{0,-1}^{0,1}(\theta, \varphi))_{ia}.
\] (70)

The coefficients \( T_{0i}(r) \) in the above expansion are radial functions which obey a set of coupled differential equations. The equation for the function \( T_{07}(r) \) does not contain a second derivative with respect to \( \xi \). This amplitude is closely related to an infinitesimal gauge transformation (see for example the discussion in Ref. [45]) and can in fact be removed by such transformation. The remaining two radial functions obey the following set of equations

\[
T_{03}''(\xi) - \left( \frac{H(\xi)^2 + 2K(\xi)^2 - 1}{\xi^2} - \omega^2 \right) T_{03}(\xi) + \frac{K(\xi)^2}{\xi^2} T_{05}(\xi) = 0
\] (71a)

\[
T_{05}''(\xi) - \left( \frac{H(\xi)^2 + 2K(\xi)^2 - 1}{\xi^2} - \omega^2 \right) T_{05}(\xi) + \frac{K(\xi)^2}{\xi^2} T_{03}(\xi) = 0
\] (71b)

\[
T_{03}'(\xi) + T_{05}'(\xi) - \frac{K'(\xi)}{K(\xi)} T_{03}(\xi) - \frac{K'(\xi)}{K(\xi)} T_{05}(\xi) = 0
\] (71c)

The last equation above can be used as a constraint on \( T_{03}(\xi) \) and \( T_{05}(\xi) \), in order to obtain two decoupled equations for these functions:

\[
T_{03}''(\xi) - \left( \frac{H(\xi)^2 + 3K(\xi)^2 - 1}{\xi^2} - \omega^2 \right) T_{03}(\xi) = 0
\] (72a)

\[
T_{05}''(\xi) - \left( \frac{H(\xi)^2 + 3K(\xi)^2 - 1}{\xi^2} - \omega^2 \right) T_{05}(\xi) = 0
\] (72b)
We can solve the above equations both in the BPS limit and for a finite value of $\lambda$ (for example we choose $\lambda = 1$ in analogy with what we did in the scalar channel), by imposing the following boundary conditions for $\xi \to \infty$:

\begin{align*}
T_{03}(\xi) &= \sin \left[ \xi \sqrt{\omega^2 - 1} + \delta_0 \right] \\
T'_{03}(\xi) &= \sqrt{\omega^2 - 1} \cos \left[ \xi \sqrt{\omega^2 - 1} + \delta_0 \right] \\
T_{05}(\xi) &= \sin \left[ \xi \sqrt{\omega^2 - 1} + \delta_0 \right] \\
T'_{05}(\xi) &= \sqrt{\omega^2 - 1} \cos \left[ \xi \sqrt{\omega^2 - 1} + \delta_0 \right].
\end{align*}

(73)

We recall that the classical solutions for $K(\xi)$ and $H(\xi)$ that we plug in the equations are the ones that we show in Fig. 4 both in the BPS limit and for $\lambda = 1$. We plot the potential $V(\xi) = (H(\xi)^2 + 3K(\xi)^2 - 1)/\xi^2 - 1$ in Fig. 5 for these two values of $\lambda$ (we have subtracted the mass of the particle, 1 in dimensionless units); we see that it can in principle admit bound states. In particular, in the BPS limit $V(\xi) \sim 1/\xi$ at large $\xi$, namely we have a Coulomb potential which admits infinitely many bound states. For $\lambda = 1$ the number of bound states is reduced: we still find two of them, as it can be seen from Fig. 7, which shows the wave function $T_{03}(\xi)$ in the case of negative (quantized) energy.

By solving the above equations, we obtain the resulting scattering phase $\delta_0$ as a function of the momentum $|\vec{k}|$ of the incoming particle: it is shown in Fig. 6 for the two values of $\lambda$ that we consider. As we already did for the scalar case, by calculating $-d\delta_0/dk$ at $|\vec{k}| = 0$ we can obtain the value of the scattering length $a_{sl}$. In the BPS limit, since we have infinitely many bound states, we also have an undetermined scattering length. For $\lambda = 1$ we get $d\delta_0/dk \simeq -10$ in dimensionless units, which translated to physical units gives a scattering length $a_{sl} \simeq 1.7 \text{ fm}$. This rather large value can be explained by the presence of the two bound states: in particular, the second one is very weakly bound, therefore it shows a rather large effective radius.

4.3 $j = 1$

In this case, the set of vector spherical harmonics coincides with \ref{50}, but two of them are missing, namely $\Phi_{1,1}^{m,1}(\theta, \varphi)$ and $\Phi_{1,-1}^{m,-1}(\theta, \varphi)$. The corresponding equations

\begin{align*}
43 \quad a_{sl} \simeq 1.7 \text{ fm}
\end{align*}
Figure 5: Behavior of $V(\xi)$, the potential in the radial equations at $j = 0$, in the BPS limit (solid line) and for $\lambda = 1$ (dashed line).

for the radial functions are in this case:

$$T''_{11}(\xi) - \left[ -\omega^2 + \frac{1 + K(\xi)^2}{\xi^2} \right] T_{11}(\xi) + \frac{K(\xi)}{\xi^2} T_{13}(\xi)$$

$$+ \frac{1}{\xi^2} T_{14}(\xi) - 2 \frac{K(\xi)}{\xi^2} T_{15}(\xi) = 0$$

(75a)

$$T''_{13}(\xi) - \left[ -\omega^2 + \frac{2K(\xi)^2 + H(\xi)^2}{\xi^2} \right] T_{13}(\xi)$$

$$+ \frac{K(\xi)}{\xi^2} T_{11}(\xi) - 2 \frac{K(\xi)}{\xi^2} T_{14}(\xi) + \frac{K(\xi)^2}{\xi^2} T_{15}(\xi) = 0$$

(75b)
Figure 6: Scattering phase $\delta_0$ as a function of $|\vec{k}| = \sqrt{\omega^2 - 1}$. The continuous line corresponds to the BPS limit ($\lambda = 0$), while the dashed line corresponds to $\lambda = 1$.

Figure 7: Some radial wave functions in the BPS limit (left panel) and for $\lambda = 1$ (right panel) corresponding to bound states in the $j=0$ channel.
\[ T''_{14}(\xi) - \left[ -\omega^2 + \frac{1 + K(\xi)^2}{\xi^2} \right] T_{14}(\xi) + \frac{1}{\xi^2} T_{11}(\xi) \]
\[ - 2 \frac{K(\xi)}{\xi^2} T_{13}(\xi) + \frac{K(\xi)}{\xi^2} T_{15}(\xi) = 0 \] (75c)

\[ T''_{15}(\xi) - \left[ -\omega^2 + \frac{2K(\xi)^2 + H(\xi)^2}{\xi^2} \right] T_{15}(\xi) - 2 \frac{K(\xi)}{\xi^2} T_{11}(\xi) \]
\[ + \frac{K(\xi)^2}{\xi^2} T_{13}(\xi) + \frac{K(\xi)}{\xi^2} T_{14}(\xi) = 0. \] (75d)

\[ \frac{T'_{11}(\xi)}{\xi} + (1 - K(\xi)^2) \frac{T'_{14}(\xi)}{\xi} - K(\xi) \frac{T'_{15}(\xi)}{\xi} \]
\[ + K'(\xi) \frac{T_{13}(\xi)}{\xi} + K(\xi) K'(\xi) \frac{T_{14}(\xi)}{\xi} + K'(\xi) \frac{T_{15}(\xi)}{\xi} = 0 \] (75e)

\[ \frac{T'_{13}(\xi)}{\xi} - K(\xi) \frac{T'_{14}(\xi)}{\xi} + K'(\xi) \frac{T_{14}(\xi)}{\xi} = 0 \] (75f)

\[ K(\xi) (1 - K(\xi)^2) \frac{T'_{14}(\xi)}{\xi} + (1 - K(\xi)^2) \frac{T'_{15}(\xi)}{\xi} + K'(\xi) \frac{T_{11}(\xi)}{\xi} \]
\[ + K(\xi) K'(\xi) \frac{T_{13}(\xi)}{\xi} + K(\xi)^2 K'(\xi) \frac{T_{14}(\xi)}{\xi} + K(\xi) K'(\xi) \frac{T_{15}(\xi)}{\xi} = 0 \] (75g)

The above system, in the limit of large distances reduces to

\[ T''_{11}(\xi) - \left[ -\omega^2 + \frac{1}{\xi^2} \right] T_{11}(\xi) + \frac{1}{\xi^2} T_{14}(\xi) = 0 \] (76a)

\[ T''_{13}(\xi) - \left[ -\omega^2 + 1 \right] T_{13}(\xi) = 0 \] (76b)

\[ T''_{14}(\xi) - \left[ -\omega^2 + \frac{1}{\xi^2} \right] T_{14}(\xi) + \frac{1}{\xi^2} T_{11}(\xi) = 0 \] (76c)
\( T''_{15}(\xi) - \left[ -\omega^2 + 1 \right] T_{15}(\xi) = 0 \) \hfill (76d)

\( \frac{T'_{11}(\xi)}{\xi} + \frac{T'_{14}(\xi)}{\xi} = 0 \) \hfill (76e)

\( T'_{13}(\xi) = 0 \) \hfill (76f)

\( T'_{15}(\xi) = 0 \) \hfill (76g)

Due to the constraint given by the last two equations, we find that, in the case of \( j = 1 \), the two radial functions \( T_{13}(\xi) \) and \( T_{15}(\xi) \) are actually constant. This means that, for this value of the angular momentum, only \( T_{11}(\xi) \) and \( T_{14}(\xi) \) will appear in the expansion of the general solution in terms of partial waves.

5 Applications

5.1 Density of states and thermodynamics

Among the applications of the results obtained above, we start with thermodynamic quantities. Indeed, with a small admixture of monopoles in the electric plasma, one can use the standard virial expansion for these quantities. The Beth-Uhlenbeck-like formula from the theory of nonideal gases gives well known corrections to the partition function, induced by the scattering on monopoles

\[
\delta M_m = -\frac{T}{\pi} \sum_j (2j + 1) \int dk \frac{d\delta_j}{dk} f(k,T)
\]  \hfill (78)

where \( \delta M_m \) is the modification of the monopole mass, \( \delta_j(k) \) is the scattering phase and \( f(k,T) \) is the Bose (Fermi) distribution for gluons (quarks). These partial waves imply the decomposition of scattering amplitudes into the usual basis of angular polynomials

\[
2ik f(\theta,k) = \sum_j (2j + 1)(e^{2i\delta_j(k)} - 1)P_j(\cos \theta)
\]  \hfill (79)

rather than the Wigner d-functions we have to use in the current problem. One may think that a complicated decomposition of the amplitude should be needed. This however can be avoided in general because of the following identity

\[
\sum_j (2j + 1) \frac{d\delta_j}{dk} = \frac{d}{dk} Re(f(\theta = 0)) + \frac{i k^2}{4\pi} \int d\Omega \left( \frac{\partial f^*}{\partial k} - f^* \frac{\partial f}{\partial k} \right)
\]  \hfill (80)
expressing the relevant sum in terms of the amplitude itself.

It has been shown above, that the scattering problem involves (a) many partial waves with \( j > 0 \) and (b) the exceptional \( j = 0 \) channels. In the former, the scattering is dominated by the Coulomb magnetic field of the monopole, while in the latter the Abelian magnetic field has no effect and the scattering happens on the monopole core. Let us discuss the \( j > 0 \) case first, returning to the exceptional \( j = 0 \) channel at the end of the section.

Since the Coulombic magnetic field possesses no dimensional scale, one finds that the dimensionless combination \( 2ikf(\theta, k) \) has no dependence on momentum, being a function of the angle only. The scattering phases are just numbers: they don’t have any logarithmic dependence on \( k \) familiar from electric Coulomb problem. Therefore, in this approximation the whole Beth-Uhlenbeck correction to the statistical sum vanishes identically!

The origin of this somewhat unexpected result can be traced to the fact that the Beth-Uhlenbeck expression was derived from a semiclassical counting of the density of states in a large (spherical) box containing the monopole. However the semiclassical density of states, related to classical phase space, is insensitive to magnetic fields because the corresponding integral

\[
\Omega_{st}(E) = \int \frac{d^3p d^3x}{(2\pi)^3} \delta(E - H(p, x))
\]

for an electric particle in any magnetic field \( H = (\vec{p} - e\vec{A}(x))^2/2m \) does not depend on the field at all (in order to see that this explanation is correct, consider for example a scattering on a dyon, which was also solved in the same papers [10, 12]. In this case there is a Coulomb phase \( \delta_j \sim (e_1 g_1 + e_2 g_2) \log(k) \) and thus a nonzero change in the density of states).

Let us now turn to the contribution of the exceptional \( j = 0 \) channels. The corresponding scattering phase has a non-trivial \( k \)-dependence, and thus a nonzero contribution to thermodynamics appears.

Before we estimate the effect of this scattering on the thermal monopole mass, let us briefly clarify the issue of “mean gluon momentum”. When one calculates the typical momentum of black-body radiation, related to integrals like \( \int d^3k \delta(k/k) \), one finds that those are \( k \sim 3T \). However such estimate is not adequate for the integrals which appear in both the thermal mass (considered here) and the scattering rates (considered in the next section). Indeed, if in the expression (78) one can replace \( d\delta_j/dk \approx a_{st} \) where \( a_{st} \) is the scattering length, assuming that \( ka_{st} \) is a small parameter (to be justified in the following), then the integral is of the 1d type \( \int d\delta j(k/k) \) and thus the dominant \( k \sim 1T \). The same argument applies to the calculation of the scattering rates, since the integral is similar

\[
\dot{w} \sim \int d\sigma \sim \int d^3k \delta(k/k/k^2)
\]
with the cross section being $\sigma \sim 1/k^2$ (for non-exceptional channels). These differences between bulk and single-particle properties are basically enforced by the dimension.

Let us now look at the absolute magnitude of the scattering lengths. Our results for $\lambda = 0, 1$ for the exceptional channels show that the gluon-monopole scattering length is large $a_{gm}^s \approx 10a$ where $a \approx .15$ fm = .76 GeV$^{-1}$ is the monopole core size. This happens due to shallow gluon-monopole bound states: we remind that we found two of them (one near-zero) for $\lambda = 1$ and infinitely many in the BPS limit.

However we don’t expect to have these states in physical QCD because the effective Higgs selfcoupling (and thus mass) are expected to be large, as seen from near-Coulomb monopole correlations [21]. We already noted that currently the monopole structure is too closed to UV lattice cutoff to get a realistic value of the effective Higgs selfcoupling $\lambda$. As soon as it is determined, one may solve the classical equations for $K(\xi)$ and $H(\xi)$ at the corresponding value of $\lambda$ and determine the scattering phase. It seems highly probable that this coupling is large enough to prevent formation of the bound states. One expects that, without bound states the scattering length is small $a_{gm}^s \approx 0.1 - 0.2$ fm, analogous to that in the scalar channel.

Thus indeed $ka < 1$ for temperatures of the order of $T_c$ (.19 GeV in QCD and .27 GeV in pure gauge $SU(3)$ theory) and the contribution to the thermal monopole mass is small

$$\delta M_m \approx \frac{a_{si} T^2}{\pi} \int \frac{dk}{\exp(\sqrt{k^2 + m^2/T}) - 1} \approx \frac{a_{si} T^2}{\pi} \log\left(\frac{m}{T}\right)$$

where the latter integral has only weak (logarithmic) dependence on the gluon mass $m$, provided it is small. Since this was obtained under the assumption that monopoles have a small size, $a_{si} T \ll 1$, this contribution to the monopole mass is relatively small.

At asymptotically high $T$ this assumption gets violated and monopoles size would be at the magnetic scale, so that this parameter gets large $ka \sim 1/e^2(T) \sim \log(T) \gg 1$. So to say, hard thermal gluons now resolve the monopole structure. The answer in this regime then is the perturbative one-loop correction to the classical monopole mass, already mentioned in the introduction.

5.2 Transport coefficients: the benchmarks

In the gas approximation, all transport coefficients are inversely proportional to the so-called transport cross section, normally defined as

$$\sigma_t = \int (1 - \cos \theta)d\sigma$$

(84)
where \( \theta \) is the scattering angle. While the factor in brackets vanishes at small angles, the Rutherford singularity in the cross section, for any charged particle, leads to its logarithmic divergence. Since we will be comparing the gluon scattering on monopoles with that on gluons, let us first introduce those benchmarks, namely the well-known (lowest order) QCD processes, the \( gg \) and \( \bar{q}q \) scatterings:

\[
\frac{d\sigma_{\bar{q}q}}{dt} = \frac{e^4}{36\pi} \left( \frac{s^4 + t^4 + u^4}{s^2 t^2 u^2} - \frac{8}{3tu} \right) \\
\frac{d\sigma_{gg}}{dt} = \frac{9e^4}{128\pi} \frac{(s^4 + t^4 + u^4)(s^2 + t^2 + u^2)}{s^4 t^2 u^2}
\]

where (we remind) the electric coupling is related to \( \alpha_s \) as usual: \( e^2/4\pi = \alpha_s \).

While for non-identical particles the transport cross section is simply given by the cross section weighted by momentum transport \( t \sim (1 - z) \), for identical ones such as \( gg \) one needs to introduce the additional factor \((1 + z)/2\) in order to suppress backward scattering as well. The integrands of the transport cross section for these two processes are compared in Fig. 8. The integrated transport cross sections themselves are given by

\[
\sigma_{gg}^t = \frac{3e^4}{320\pi s} \left( 105 \log(3) - 16 + 30 \log \left( \frac{4}{\theta_{\min}^2} \right) \right) \\
\sigma_{\bar{q}q}^t = \frac{e^4}{54\pi s} \left( 4 + 7 \log(3) + 3 \log \left( \frac{4}{\theta_{\min}^2} \right) \right)
\]

where the smallest scattering angle can be related to the (electric) screening mass by \( \theta_{\min}^2 = 2 \ast M_D^2/s \). Note that the forward scattering log in the \( gg \) case has a coefficient which is roughly four times larger, as a consequence of the gluon color being roughly twice that of a fundamental quark. Note also that the \( gg \) scattering is significantly larger at large angles, as compared to the \( \bar{q}q \) scattering.

### 5.3 Transport cross section on monopoles

We explained (already in the introduction) that the charge-monopole scattering is Rutherford-like at small angles: this comes from harmonics with large angular momenta (large impact parameters). However, in matter there is a finite density of monopoles, so the issue of the scattering should be reconsidered. A sketch of the setting, assuming strong correlation of monopoles into a crystal-like structure, is shown in Fig. 9. A “sphere of influence of one monopole” (the dotted circle) gives the maximal impact parameter to be used.

As a result, the impact parameter is limited from above by some \( b_{max} \), which implies that only a finite number of partial waves should be included. The range of
Figure 8: Angular dependence of the integrand of the transport cross sections vs \( z = \cos \theta \), where \( \theta \) is the scattering angle. The (red) solid line shows the integrand for the \( gg \) scattering, \( \frac{d\sigma_{gg}}{dt} \ast s \ast t \ast \left(1 + z\right)/2/e^4 \), while the (black) dashed line shows that for the \( \bar{q}q \) scattering scaled up by a factor 4: \( 4 \ast \frac{d\sigma_{\bar{q}q}}{dt} \ast s \ast t/e^4 \).

Partial waves to be included in the scattering amplitude can be estimated as follows:

\[
J_{\text{max}} = \langle p_z \rangle n_m^{-1/3}/2 \sim aT \sim 1/e^2(T) \sim \log(T).
\]  

(89)

Since at asymptotically high \( T \) the monopole density \( n_m \sim (e^2T)^3 \) is small compared to the density of quarks and gluons \( \sim T^3 \), \( J_{\text{max}} \) asymptotically grows logarithmically with \( T \). So, only in the academic limit \( T \to \infty \) one gets \( J_{\text{max}} \to \infty \) and the usual free-space scattering amplitudes calculated in [10] where all partial waves are recovered. However, in reality we have to recalculate the scattering, retaining only several lowest partial waves from the sum. As we will see, this dramatically changes the angular distribution, by strongly depleting scattering at small angles and enhancing scattering backwards.

5.3.1 Scalar particle

By making use of the partial-wave solutions to the wave equations that we obtained in Sec. 3 we can proceed to construct the scattering solutions. At large distance from the origin, they should behave as a plane wave plus an outgoing spherical wave.

We consider a particle entering from \( z = -\infty \); the \( z \) component of its total angular momentum is well defined: \( J_3 = T_3 = -(\bar{T} \cdot \hat{r}) = -n \). Therefore, the appropriate
Figure 9: A charge scattering on a 2-dimensional array of correlated monopoles (open points) and antimonopoles (closed points). The dotted circle indicates a region of impact parameters for which scattering on a single monopole is a reasonable approximation.

Angular functions that we must use to expand our solution are $\phi^{\pm mn}_{ji}(\theta, \varphi)$. In the following, we consider only $n \geq 0$; a negative $n$ would just produce an overall phase factor. We consider the following partial-wave sum

$$
\phi^{(+)\mathbf{r}} = e^{-i\pi n} \sum_{j=|n|}^{j_{\text{max}}} (2j + 1) e^{i\pi j} e^{-i\pi j'/2} j_{j'}(kr) [U(-\varphi, \theta, \varphi) \chi^m_{ji}] D^{(j)}_{n,-n}(-\varphi, \theta, \varphi) \psi^{(+)}(\mathbf{r}).
$$

We recall that the index $j'$ in the above formula is the positive root of

$$
j'(j' + 1) = j(j + 1) - n^2.
$$

The square brackets in the above formula contain a vector which is the charge eigenstate. Since in our approximation we don’t have any charge-flip reactions (they would be possible inside the monopole core, which we consider pointlike), this factor will remain the same for the incoming and outgoing waves. The physically significant
wave function is therefore given by

\[ \psi^{(+)}(\vec{r}) = e^{-i\pi n} \sum_{j=|n|}^{j_{\text{max}}} (2j + 1)e^{i\pi j}e^{-i\pi j'/2}j'_{\text{max}}(kr)D_{n,-n}^{(j)}(-\varphi, \theta, \varphi) \]

\[ = e^{-i\pi n} \sum_{j=|n|}^{j_{\text{max}}} (2j + 1)e^{i\pi j}e^{-i\pi j'/2}j'_{\text{max}}(kr)e^{-2im\varphi}d_{n,-n}^{(j)}(\theta). \quad (92) \]

The above expression contains the functions \(d_{n,-n}^{(j)}(\theta)\) which can be written via the generalized Rodriguez formula as \((z = \cos \theta)\)

\[ d_{n,-n}^{(j)}(z) = \frac{(-1)^{j-n}}{2i(j-n)!}(1-z)^{-n}\left(\frac{d}{dz}\right)^{j-n}\left[(1-z)^{j+n}(1+z)^{-n}\right] \quad (93) \]

So, the first term with \(j = n\) is simply \(\sim (1 - \cos \theta)^n\), a function strongly peaked backwards at large \(n\).

Obviously, a tendency to scatter backwards is very important for transport properties and equilibration of electric plasma.

The asymptotic value of the wave function \(\psi^{(+)}(\vec{r})\) is then

\[ \psi^{(+)}(\vec{r}) \sim e^{-2im\varphi}\left[ e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \right] \quad (94) \]

with the scattering amplitude \(f(\theta)\) given by

\[ 2ikf(\theta) = \sum_{j=|n|}^{j_{\text{max}}} (2j + 1)e^{i\pi(j-j')}d_{n,-n}^{(j)}(\theta). \quad (95) \]

The above sum would be badly divergent if we would sum over \(j\) up to \(\infty\). It is possible to improve its convergence by separating the most singular behavior (see for example the discussion in \([10]\)). We do not have such convergence problems, since we have an ultraviolet cutoff on \(j\) due to the finite density of monopoles. Therefore we calculate the scattering amplitude by directly using Eq. \((95)\).

The integrands of the transport cross section \((1 - \cos \theta)|f(\theta)|^2\) are shown in Fig. 10 for \(n = 0, j_{\text{max}} = 2, 4, 6\) (left panel), \(n = \pm 1, j_{\text{max}} = 2, 4, 6\) (right panel). One can see how much their angular distribution is distorted. Strong oscillations of this function occur because we use a sharp cutoff for the higher harmonics, which represents diffraction of a sharp edge. This edge in reality does not exist and can be removed by any smooth edge prescription (for example a gaussian weight). However, we further found that the transport cross section itself is rather insensitive to these oscillations, and thus there is no need in smoothening the scattering amplitude. The transport cross section as a function of \(j_{\text{max}}\) is shown in Fig. 11: it is large and smoothly rising with the cutoff.
Figure 10: Scaled Integrand of the trasport cross section \( g(\theta) = k^2(1 - \cos \theta)|f(\theta)|^2 \) with only 2, 4 and 6 lowest partial waves included, for a scalar particle with \( n = 0 \) (left) and \( n = \pm 1 \) (right). The curves can be easily recognized by higher \( j_{\max} \) having more oscillations.

Figure 11: Normalized transport cross section as a function of cutoff in maximal harmonics retained, for \( n = 0, 1 \).
5.3.2 Vector particle

We consider a vector particle entering from \( z = -\infty \). As we already saw in the case of the scalar particle, the \( z \) component of its total angular momentum is well-defined: we have seen that, if we consider angular functions that are simultaneous eigenfunctions of \((\vec{T} \cdot \hat{r})\) and \((\vec{S} \cdot \hat{r})\), the equations for the corresponding radial functions are diagonal. This means that, once we neglect the monopole core, we can consider an incoming particle with definite charge and radial polarization, and these quantities will be conserved in the final state, after the scattering. The third component of the angular momentum will then be

\[
J_3 = - \left[ (\vec{T} \cdot \hat{r}) + (\vec{S} \cdot \hat{r}) \right] = - [n + \sigma] = -\nu. \tag{96}
\]

We have seen that, after gauge fixing, for each particle of given charge we have two possible spin polarizations \( \sigma = \pm 1 \). This means that, for a vector particle with charge \( n = 1(-1) \), we can have \( \nu = 0, 2(0, -2) \) respectively, while for the vector particle with \( n = 0 \) (the photon) we have \( \nu = \pm 1 \). We consider the following partial-wave sum

\[
\Phi^{(+)}(\vec{r}) = e^{-i\pi\nu} \sum_{j=|\nu|}^{j_{\text{max}}} (2j + 1) e^{i\pi j} e^{-i\pi j'/2} j_j (kr) \left[ \Xi (\theta, \varphi)^\sigma_{n} \right] D_{\nu,-\nu}^{(j)} (-\varphi, \theta, \varphi)
\]

\[
= \left[ \Xi (\theta, \varphi)^\sigma_{n} \right] \Psi^{(+)}(\vec{r}) \tag{97}
\]

where the factor \( \left[ \Xi (\theta, \varphi)^\sigma_{n} \right] \) is a two-index tensor representing the charge and \((\vec{S} \cdot \hat{r})\) eigenstate, which is not altered in the peripheral collision that we are considering. As we already did for the scalar channel, we consider the physically significant wave function

\[
\Psi^{(+)}(\vec{r}) = e^{-i\pi\nu} \sum_{j=|\nu|}^{j_{\text{max}}} (2j + 1) e^{i\pi j} e^{-i\pi j'/2} j_j (kr) D_{\nu,-\nu}^{(j)} (-\varphi, \theta, \varphi)
\]

\[
= e^{-i\pi\nu} \sum_{j=|\nu|}^{j_{\text{max}}} (2j + 1) e^{i\pi j} e^{-i\pi j'/2} j_j (kr) e^{-2i\nu\varphi} d_{\nu,-\nu}^{(j)}(\theta). \tag{98}
\]

for which we can write the asymptotic behavior

\[
\Psi^{(+)}(\vec{r}) \sim e^{-2i\nu\varphi} \left[ e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right] \tag{99}
\]

with the scattering amplitude \( f(\theta) \) given by

\[
2ikf(\theta)_{n,\nu} = \sum_{j=|\nu|}^{j_{\text{max}}} (2j + 1) e^{i\pi(j'-j)} d_{\nu,-\nu}^{(j)}(\theta). \tag{100}
\]
Actually, there are some exceptions to the above expression for $f(\theta)$: in Sec. 4 we have seen that the cases $j = 0$ and $j = 1$ are somehow exceptional. For $j = 0$ we have only two modes, with charge $\pm 1$. The corresponding radial function obeys a differential equation which is not Bessel-like. Yet, this solution still has a sinusoidal behavior at large distances, with a momentum-dependent scattering phase. We have also seen that, for $j = 1$, the only surviving modes are the two transverse polarizations of the particle with charge $n = 0$. Therefore, the above general formula for the scattering amplitude (100) must be separately written, for particles with different charge. For the neutral particle (with $\nu = \pm 1$) we have

$$2ikf(\theta)_{0,\pm 1} = \sum_{j=1}^{j_{\text{max}}} (2j + 1)d_{1,-1}^{(j)}(\theta), \quad (101)$$

where the factor $e^{i\pi(j'-j)}$ is absent in this case, since for a neutral particle we have $j' = j$. We have seen that, for the positive (negative) charge particle, we have two possible values of $\nu = 0, 2 (-2)$. Therefore we have:

$$2ikf(\theta)_{\pm 1, \pm 2} = \sum_{j=2}^{j_{\text{max}}} (2j + 1)e^{i\pi(j'-j)}d_{2,-2}^{(j)}(\theta), \quad (102)$$

where $\delta_0(k)$ is the scattering phase plotted in Fig. 6. The integrands of the transport cross section $(1 - \cos(\theta))|f(\theta)_{n,\nu}|^2$ are shown in Fig. 12 for $j_{\text{max}} = 6$, $n = 0, \pm 1$ and the corresponding values of $\nu$. One can see how much their angular distribution is distorted. The resulting transport cross section,

$$\langle \sigma_t \rangle_{n,\nu} = \int_{-1}^{1} d\cos \theta (1 - \cos \theta)|f(\theta)_{n,\nu}|^2 \quad (103)$$

is shown in Fig. 13 as a function of $j_{\text{max}}$ for the different possible combinations of charge and polarization. As we can notice, it is large and smoothly rising with the cutoff, in spite of the large oscillations in the integrand, due to our choice of a sharp cutoff. In the following we will use $j_{\text{max}} = 6$. Now we proceed to evaluate the scattering rate of gluons on monopoles. We recall that it is defined as

$$\frac{\dot{w}_{gm}}{T} = \frac{\langle n_m(\sigma_t)_{gm} \rangle}{T} \quad (104)$$

where the $\langle \ldots \rangle$ indicates an average over the incoming gluon. Following Ref. [30], we take into account the interaction of transverse gluons with a temporal background.
Figure 12: Integrand of the transport cross section \( g(\theta) = (1 - \cos(\theta))|f(\theta)|^2 \) with only 6 lowest partial waves included, for a vector particle with \( n = 0, \nu = \pm 1 \), \( n = \pm 1, \nu = 0 \) and \( n = \pm 1, \nu = \pm 2 \).

Figure 13: Normalized transport cross section as a function of cutoff in maximal harmonics retained, for \( n = 0, \pm 1 \) and the corresponding polarizations.
gauge field $A_0$ which is related to the Polyakov loop through

$$\Phi = \frac{1}{N_c} \text{tr} \left[ \mathcal{P} \exp \left( i \int_0^\beta A_0 d\tau \right) \right] = \frac{1}{3} \text{tr} \exp \left[ \frac{i A^a_0 \lambda_a}{T} \right]$$

(105)

(the above definition of the gauge field actually includes the electric coupling $e$). If we select the abelian gauge for $A_0$, for example along $\lambda_3$, we get:

$$\Phi = \frac{1}{3} \left[ 2 \cos \frac{A_3}{T} + 1 \right].$$

(106)

The presence of this background gauge field breaks color symmetry. The eight gluons interact with $A^3_0$ in different ways, depending on their color charge. For example, since we chose $A_0$ along the abelian direction, the two abelian gluons don’t interact with $A_0$ at all. In particular, the gluon density has the following form

$$n_g(T) = \frac{8\pi}{(2\pi)^3} \int k^2 dk \left[ \frac{2}{\exp(\beta \epsilon_k) - 1} + \frac{2}{\exp(\beta \epsilon_k) \exp(i\beta A^3_0) - 1} \right.
\left. + \frac{1}{\exp(\beta \epsilon_k) \exp(-i\beta A^3_0) - 1} + \frac{1}{\exp(\beta \epsilon_k) \exp(2i\beta A^3_0) - 1} \right] = \frac{4\pi}{(2\pi)^3} \int k^2 dk \rho_g(k, T)$$

(107)

where we use for $A^3_0$ the expectation value which fits the available lattice results for the Polyakov loop in pure-gauge $SU(3)$ gauge theory (see Fig. 15) and we define $\epsilon_k = \sqrt{k^2 + m_g(T)^2}$, with an effective gluon mass $m_g(T) = e(T)T/\sqrt{2}$. For $\alpha_s(T)$ we use the estimate given in Ref. [26]. Notice that $\rho_g$ contains a factor 2 to account for gluon spin polarization. The coupling of gluons to the Polyakov loop gives a suppression of these degrees of freedom approaching $T_c$ from above. This is evident from Fig. 1, where we plot $n_g(T)/T^3$ as defined in eq. (107) as a function of the temperature (continuous line).

In order to obtain the formula for the (scaled) scattering rate, we need to average $\sigma_t(k)$ over the momentum of the incoming gluon: we use the distribution taken from eq. (107); in taking the integral over momenta, we have to recall that the transport cross section, and the gluon screening due to $A_0$, both depend on the charge of gluons: for this reason, we have to split $\rho_g(k, T)$ into a “charge-neutral” part, which will be used as a weight for $(\sigma_t(k))_{0,1}$, and a “charged part”, which will be used to
weight \((\sigma_t(k))_{1,0}\) and \((\sigma_t(k))_{1,2}\):

\[
\frac{\dot{w}_{gm}}{T} = \frac{n_m(T)}{n_g(T)T} \frac{4\pi}{(2\pi)^3} \int k^2 dk \left[ \frac{4(\sigma_t(k))_{0,1}}{\exp(\beta\epsilon_k) - 1} \right.
\]
\[
+ \left( (\sigma_t(k))_{1,0} + (\sigma_t(k))_{1,2} \right) \left( \frac{2}{\exp(\beta\epsilon_k) \exp(i\beta A_0^3) - 1} \right)
\]
\[
+ \frac{2}{\exp(\beta\epsilon_k) \exp(-i\beta A_0^3) - 1} + \frac{1}{\exp(\beta\epsilon_k) \exp(2i\beta A_0^3) - 1}
\]
\[
+ \frac{1}{\exp(\beta\epsilon_k) \exp(-2i\beta A_0^3) - 1} \right].
\]  

We recall that the momentum dependence of the transport cross section is trivially \(\sim 1/k^2\), except for the case \((\sigma_t(k))_{1,0}\) for which we have the exceptional case \(j = 0\), with a momentum-dependent scattering phase \(\delta_0(k)\). In the above equation, \(n_m(T)\) is the monopole density as a function of the temperature. We take this information from the available lattice results for this quantity, plotted in Fig. 1. We show \(\dot{w}_{gm}/T\) in Fig. 14 (the red, continuous line in the left panel). Also shown is the same quantity for the \(gg\) scattering process (black, dotted line), obtained through the following equation

\[
\frac{\dot{w}_{gg}}{T} = \frac{1}{n_{g1}} \int \frac{4\pi k_1^2 dk_1}{(2\pi)^3} \int \frac{2\pi k_2^2 dk_2}{(2\pi)^3} \int_{-1}^{1} d\cos\theta \sigma_{gg}^s(k_1, k_2, \cos\theta) \rho_g(k_1, T) \rho_g(k_2, T)
\]  

where \(\sigma_{gg}^s(k_1, k_2, \cos\theta)\) was defined in eq. (87), with

\[
s = 2k_1k_2(1 - \cos\theta) + 2m_g(T)^2.
\]

### 6 Conclusions and discussion

In this paper we studied the role of magnetic monopoles in the “electric” QGP phase. In distinction with papers by Liao and Shuryak, we have not focused on the near-\(T_c\) region, in which monopoles seem to be dominant over gluons in number, and may even expel electric fields into flux tubes as they do in the confined phase. Instead, we considered the more traditional QGP away from \(T_c\), dominated by the usual “electric” constituents – quarks and gluons, the magnetic monopoles being subleading in number.

More specifically, we focused on scalar-monopole and gluon-monopole scattering. Classical molecular dynamics, studied in refs. \[7\], suggested a very interesting mechanism of “mutual trapping” by electric/magnetic quasiparticles. Based on results of quantum scattering on monopoles \[10\] \[12\] \[13\] for scalar-monopole and spinor-monopole scatterings, in this paper we studied this effect further. Indeed, we found
significant large angle and even backward scattering for small impact parameters, complemented with a Rutherford-like forward scattering peak.

Our main result is a detailed derivation of the gluon-monopole scattering amplitude, especially in the lowest partial waves. For gauge particles, the existence of spin and isospin leads to complications which so far precluded this problem from being solved.

Our first important finding is that the gluon-monopole scattering has little effect on thermodynamics, because scattering phases – albeit large – are mostly energy independent, except the “exceptional channel” with \( j = 0 \). This explains why lattice studies which focused on thermodynamic observables were unable to see such effects.

Our second (and the main) finding is that the contribution of gluon-monopole scattering is very important for transport properties. While the monopole density may be small, the \( gm \) scattering amplitudes have \( e^2 g^2 \sim O(1) \) coupling instead of small \( e^4 \ll 1 \). Furthermore, in our setting (with a limited number of partial waves \( j < j_{\text{max}} \) included) there is an additional enhancement for large angle (or even backward) scattering.

Convoluting the cross sections found with the monopole density and gluon momentum distribution, we plot the scattering rates \( n\sigma_t \) per gluon vs \( T \) in Fig. 14.

It follows from this comparison of the gluon-monopole curve with the gluon-gluon one that the former remains the leading effect till very high \( T \), although asymptotically it is expected to get subleading. This maximal \( T \) expected at LHC does not exceed \( 4T_c \), where \( \eta/s \sim 0.2 \). This value is well in the region which would ensure hydrodynamical radial and elliptic flows, although deviations from ideal hydro would be larger than at RHIC (and measurable!).

The approximate relation of these rates to viscosity/entropy ratio is

\[
\frac{\eta}{s} \approx \frac{T}{5\bar{w}}; \tag{111}
\]

We plot \( \eta/s \) in the right panel of Fig. 14.

Let us now comment on some details of the calculations, which were not yet taken into account and should be included later. One important fact is that the lattice data for the monopole density we used actually include all magnetically charged particles, including dyons. Their electric charge would lead to additional Rutherford-like scattering which we have ignored.

In the calculation above we have not included Bose enhancement for scattered gluons. This effect adds the factor \( (1 + f(p')) \) for \( gm \) and \( (1 + f(p'_1))(1 + f(p'_2)) \) for \( gg \), with the prime marking the secondary gluons. If the gluon mass is small (high \( T \)) those corrections are small: their magnitude is \( (f) \sim (T/m) \sim 1/e(T) \gg 1 \). In the experimentally relevant region, when \( m/T = O(1) \), the effect is not enhanced and is additionally suppressed by the expectation values of the Polyakov lines\(^{11}\) \( \sim \langle L \rangle^2 \).

\(^{11}\) We do not agree with Hidaka and Pisarski \(^{47}\) in their conclusion that \( \langle L \rangle < 1 \) makes viscosity
As we have already mentioned in the introduction, Xu, C. Greiner and Stöcker \cite{8} have suggested an alternative explanation for small QGP viscosity, namely the next-order radiative processes, \( gg \leftrightarrow ggg \). Using perturbative matrix elements and \( \alpha_s = 0.3..0.6 \), they found \( \eta/s \) several times smaller than for the \( gg \leftrightarrow gg \) process, close to what we get from the \( gm \) scattering. Obviously, both mechanisms, albeit having such different origin, would thus be sufficient to explain the well-known hydrodynamic results for radial and elliptic flow at RHIC.

It will require much more work to see how both results will change, when further refinements are performed. We have discussed those for monopoles above: let us now mention a few questions for \( gg \leftrightarrow ggg \) :

(i) Xu et al used near-massless perturbative gluons: while in RHIC-LHC range the lattice quasiparticle masses are instead much larger than \( T \), about \( 3T \) or so. This would suppress emission of extra gluons.

(ii) in RHIC-LHC range one should include the suppression by the Polyakov VEV \( \langle L \rangle \) for any gluon effects (see Fig. 1).

(iii) Inclusion of higher order corrections in badly divergent perturbative series needs further studies. As shown years ago in [46], similarly treated processes \( gg \rightarrow ng \) with larger \( n = 4, \ldots \) lead to even larger rates! The development of convergent series for \( \eta/s \) itself still remains to be an open challenging problem.

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\( \eta \) smaller by \( \langle L \rangle^2 \).
Appendix

A The Georgi-Glashow model

The Lagrangian of the Georgi-Glashow model describes coupled gauge and Higgs fields and reads

\[ L = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} (D^{\mu} \phi) (D_{\mu} \phi) - V(\phi) \]

\[ = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} (D^{\mu} \phi^a) (D_{\mu} \phi^a) - V(\phi), \quad (A-1) \]

where \( F_{\mu\nu} = F^a_{\mu\nu} I^a \), \( \phi = \phi^a I^a \) and \( \text{Tr}(I^a I^b) = \frac{1}{2} \delta_{ab} \), \( a, b = 1, 2, 3 \). For the gauge group \( SU(2) \), the matrices \( I_a \) satisfy the algebra

\[ [I^a, I^b] = i \epsilon_{abc} I^c. \quad (A-2) \]

The covariant derivative is defined as

\[ D_\mu = \partial_\mu + ieA_\mu \quad (A-3) \]

which gives, for the Higgs field,

\[ D_\mu \phi = \partial_\mu \phi + ie [A_\mu, \phi] \quad \text{or} \quad D_\mu \phi^a = \partial_\mu \phi^a - e \epsilon_{abc} A^b_\mu \phi^c \quad (A-4) \]

and the potential for the scalar field reads

\[ V(\phi) = \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2. \quad (A-5) \]

In the above formulas, \( e \) and \( \lambda \) are the gauge and scalar coupling constants, respectively. The field strength tensor is

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - e \epsilon_{abc} A^b_\mu A^c_\nu \quad (A-6) \]

or, in matrix form,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie [A_\mu, A_\nu] = \frac{1}{ie} [D_\mu, D_\nu]. \quad (A-7) \]

The field equations corresponding to the Lagrangian \((A-1)\) can be written as

\[ D_\nu F^a_{\mu\nu} = -e \epsilon_{abc} \phi^b D^\mu \phi^c, \quad D_\mu D^\mu \phi^a = -\lambda \phi^a (\phi^b \phi^b - v^2). \quad (A-8) \]

One looks for a classical solution that is invariant under a combined space and isospin rotation generated by the \( \vec{J} \) spin operator \( \vec{J} = \vec{T} + \vec{S} \), with \( \vec{T} = \vec{L} + \vec{I} \), and
$S$ is the spin of the particle. An ansatz with this generalized “spherical symmetry” is [14, 15]

$$
\Phi^a = \frac{r^a}{er^2} H (\xi), \quad A_n^a = \epsilon_{amn} \frac{r^m}{er^2} [1 - K (\xi)], \quad A_0^a = 0, \quad (A-9)
$$

were $H(\xi)$ and $K(\xi)$ are functions of the dimensionless variable $\xi = ver$. These functions are solutions of the following system of coupled equations

$$
\xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K (K^2 - 1), \quad \xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H (H^2 - \xi^2) \quad (A-10)
$$

with boundary conditions

$$
K (\xi) \to 1, \quad H (\xi) \to 0 \quad \text{as} \quad \xi \to 0
$$

$$
K (\xi) \to 0, \quad H (\xi) \to \xi \quad \text{as} \quad \xi \to \infty. \quad (A-11)
$$

In the Bogomol’nyi-Prasad-Sommerfield (BPS) limit, namely for $\lambda = 0$, the equations (A-10) have an analytic solution in terms of elementary functions

$$
K (\xi) = \frac{\xi}{\sinh \xi}, \quad H (\xi) = \xi \coth \xi - 1. \quad (A-12)
$$

We are interested in fluctuations around the classical solutions (A-9) in the general case in which $\lambda \neq 0$. We therefore write the fields in the Lagrangian (A-1) as the sum of the classical solution, considered to be of order 0, and a fluctuation field which is considered to be of order 1:

$$
A_\mu^a = A_\mu^a + a_\mu^a, \quad \phi^a = \Phi^a + \chi^a \quad (A-13)
$$

The Lagrangian is then expanded up to order 2 and is therefore quadratic in the fluctuations $a_\mu^a$ and $\chi^a$:

$$
\mathcal{L}^{(2)} = -\frac{1}{4} \left\{ \partial_\mu a_\nu^a \partial_\mu a_\nu^a - \partial_\mu a_\nu^a \partial_\nu a_\mu^a - \partial_\nu a_\mu^a \partial_\mu a_\nu^a + \partial_\nu a_\mu^a \partial_\nu a_\mu^a \right\}
- 2e \epsilon_{abc} \left[ (\partial_\mu a_\nu^b) (A_\mu^b a_\nu^c + a_\mu^b A_\nu^c) + (\partial_\nu a_\mu^b) - (\partial_\mu A_\nu^b) a_\nu^b \right]
+ e^2 \left[ 4 A_\mu^b A_\mu^c a_\nu^b a_\nu^c - 2 A_\mu^b A_\mu^c a_\nu^c a_\nu^b + 2 A_\mu^b A_\mu^c a_\nu^c a_\nu^b - 2 A_\mu^b A_\mu^c a_\nu^c a_\nu^b \right]
+ \frac{1}{2} \partial_\mu \chi^a \partial_\mu \chi^a - e \epsilon_{abc} \left[ (\partial_\mu \Phi^a) a_\mu^b \chi^c + (\partial_\mu \chi^a) \Phi^a + (\partial_\mu \chi^a) A_\mu^b \chi^c \right]
+ e^2 \left[ 2 A_\mu^b \Phi^a \Phi^b a_\mu^c - A_\mu^b \Phi^a \Phi^b a_\mu^c - A_\mu^b \Phi^a \Phi^b a_\mu^c + \frac{1}{2} a_\mu^b a_\mu^b \Phi^c \Phi^c - \frac{1}{2} a_\mu^b a_\mu^b \Phi^c \Phi^c \right]
+ \frac{1}{2} A_\mu^b A_\mu^c \chi^a \chi^c - \frac{1}{2} A_\mu^b A_\mu^c \chi^a \chi^c \right\] - \frac{\lambda}{4} \left[ 2 \chi^a \chi^a (\Phi^b \Phi^b - v^2) + 4 (\Phi^a \chi^a)^2 \right] \quad (A-14)
$$
so that their equations of motion are linear.

If we only allow fluctuations in the scalar and vector channel separately, we get the following equations

$$\partial_{\mu} \partial^{\mu} \chi^a - 2\varepsilon_{abc} \mathcal{A}_{\mu}^b (\partial_{\mu} \chi^c) + e^2 [\mathcal{A}_{\mu}^a \mathcal{A}_{\mu}^b - \mathcal{A}_{\mu}^b \mathcal{A}_{\mu}^a] - \lambda [2\Phi^a \Phi^b \chi^b + \chi^a (\Phi^b \Phi^b - v^2)] = 0$$

$$\partial_{\nu} \partial_{\mu} a^a_{\nu} - \partial_{\nu} \partial_{\nu} a^a_{\mu} + e\varepsilon_{abc} [2 (\partial_{\nu} a_{\mu}^b) \mathcal{A}_{\nu}^b + (\partial_{\mu} a_{\nu}^b) \mathcal{A}_{\mu}^c - \mathcal{A}_{\nu}^b (\partial_{\mu} a_{\nu}^c) - a_{\nu}^b \partial_{\mu} \mathcal{A}_{\nu}^a] + e^2 [2\mathcal{A}_{\mu}^a \mathcal{A}_{\nu}^b - \mathcal{A}_{\mu}^b \mathcal{A}_{\nu}^a + \mathcal{A}_{\nu}^b \mathcal{A}_{\mu}^a - \mathcal{A}_{\nu}^a \mathcal{A}_{\mu}^b - a_{\mu}^a \Phi^b + a_{\mu}^a \Phi^a] = 0.$$  \hfill (A-15)

The equation for the scalar particles can be rewritten in polar coordinates by making use of the following definition

$$\vec{L} = -i\hat{r} \times \vec{\nabla}; \quad L_k = -i\varepsilon_{kmn} r_m \partial_n$$  \hfill (A-16)

and by remembering that

$$\partial_{\mu} \partial^{\mu} = \frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \frac{L^2}{r^2} - \partial_0^2.$$  \hfill (A-17)

Therefore, we can write

$$\left[ \frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \frac{L^2}{r^2} - \partial_0^2 \right] \chi^a - 2\varepsilon_{abc} \mathcal{A}_{\mu}^b (\partial_{\mu} \chi^c) + e^2 [\mathcal{A}_{\mu}^a \mathcal{A}_{\mu}^b - \mathcal{A}_{\mu}^b \mathcal{A}_{\mu}^a] - \lambda [2\Phi^a \Phi^b \chi^b + \chi^a (\Phi^b \Phi^b - v^2)] = 0.$$  \hfill (A-18)

Now we remember the classical solution for the monopole \hfill \boxed{(A-9)} so that we can write

$$-2e\varepsilon_{abc} \mathcal{A}_{\mu}^b (\partial_{\mu} \chi^c) = -2i\frac{\varepsilon_{ack} L_k}{r^2} \chi^c [1 - K(\xi)],$$

$$e^2 [\mathcal{A}_{\mu}^a \mathcal{A}_{\mu}^b - \mathcal{A}_{\mu}^b \mathcal{A}_{\mu}^a] = -\left( \frac{\delta_{ab}}{r^2} + \frac{r a r b}{r^4} \right) [1 - K(\xi)]^2$$  \hfill (A-19)

and by recalling that the isospin operator is \hfill \boxed{(I^a)_{bc} = -i\varepsilon_{abc}} we get

$$\left[ \frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} \left( \vec{L} + \vec{T} [1 - K(\xi)] \right)^2 - \partial_0^2 \right] \chi + \left( \frac{\hat{r} \cdot \vec{T} [1 - K(\xi)]}{r^2} \right)^2 \chi$$

$$-\lambda \left[ 2 \frac{r a r b}{e^2 r^4} H(\xi)^2 \chi^b + \chi^a \left( \frac{H(\xi)^2}{e^2 r^2} - v^2 \right) \right] = 0.$$  \hfill (A-20)
The above equation can be rewritten as
\[
\begin{align*}
\left[ \frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \left( \nabla \cdot \vec{I} \right)^2 \right] \chi + 2K(\xi) \left( \vec{I} \cdot \vec{T} - (\hat{r} \cdot \vec{I})^2 \right) \\
- \frac{K(\xi)^2 \left( \vec{I}^2 - (\hat{r} \cdot \vec{I})^2 \right)}{r^2} - \lambda \left[ 2 \frac{r^{a}r^{b}}{c^2r^4} H(\xi)^2 \chi^b + \chi^a \left( \frac{H(\xi)^2}{c^2r^2} - v^2 \right) \right] = 0.
\end{align*}
\]

(A-21)

The term \( \propto K(\xi) \) in the above equation induces charge-exchange reactions. The term proportional to \( K(\xi)^2 \), modifies the deep-scattering amplitude for the charge-preserving processes.

\section{SU(3) 't Hooft-Polyakov monopoles}

Although in the text above we deal with monopoles in a simplified way, we briefly review the elements of the \( SU(3) \) Lie algebra and monopoles. It is given by a set of traceless Hermitian \( 3 \times 3 \) matrices
\[
T^a = \lambda^a / 2 \quad a = 1, 2...8
\]

(B-1)

where \( \lambda^a \) are the standard Gell-Mann matrices. They are normalized as \( TrT^aT^b = \delta^{ab}/2 \). The structure constants of the Lie algebra are \( f^{abc} = \frac{1}{4} Tr[\lambda^a, \lambda^b] \lambda^c \) and in the adjoint representation we have \( (T^a)_{bc} = f^{abc} \). We are interested in the diagonal, or Cartan, subalgebra of \( SU(3) \). It is given by two generators:
\[
H_1 = T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad H_2 = T^8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

(B-2)

which are composed into the vector \( \vec{H} = (H_1, H_2) \). The number of positive roots is 3, since the dimension of the group is 8. We can take the basis of simple roots as
\[
\vec{\beta}_1 = (1,0), \quad \vec{\beta}_2 = (1/2, -\sqrt{3}/2).
\]

(B-3)

The third positive root is given by the composition of the first two roots \( \vec{\beta}_3 = \vec{\beta}_1 - \vec{\beta}_2 = (1/2, \sqrt{3}/2) \). Since all these roots have unit length, our choice corresponds to the self-dual basis \( \vec{\beta}^*_i = \vec{\beta}_i \).
Let us now remind the usual construction of a ’t Hooft-Polyakov monopole in QCD, once we assume that the role of the Higgs is played by the expectation value of the field $A_0$, here considered to be real in Minkowski time. The QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4} \left[ F_{\mu
u}^a [A] + D_\mu^{ac} Q_{ve} - D_\nu^{ab} Q_{\rho b} - g f^{abc} Q^b_{\mu} Q^c_{\nu} \right]$$

$$\times \left[ F_{a\mu}^\rho [A] + D_{\alpha}^{\mu c} Q_{\nu e} - D_{\nu d}^{\alpha} Q_{\mu d} - g f_{\alpha}^{\mu c} Q^d_{\nu e} \right] - V[A_0^a],$$

where we have written the gauge field as the sum of a background field and fluctuations

$$A^a_\mu = A^a_\mu + Q^a_\mu$$

$$A^a_\mu \rightarrow \text{background field}$$

$$Q^a_\mu \rightarrow \text{fluctuations} \quad (B-5)$$

and the potential $V[A_0^a]$ will be defined later. In our case, the background field will have the following form:

$$A^a_\mu = \delta_{a0} \delta^{\beta3} A_0^3.$$  \hspace{1cm} (B-6)

The field $A_0^a$ will play the role of a Higgs field. Since we suppose that our background field is diagonal in color space, we can rewrite it as follows:

$$A_0 = A_0^3 \vec{h} \cdot \vec{H}$$  \hspace{1cm} (B-7)

where $\vec{H}$ was defined after Eq. (B-2) and clearly $\vec{h} = (1, 0)$. If the monopole solution obeys the Bogomol’nyi equations, in the direction chosen to define $A_0$, the asymptotic magnetic field of a BPS monopole is of the form

$$B_n = \vec{g} \cdot \vec{H} r_n.$$  \hspace{1cm} (B-8)

Here, the magnetic charge $g = \vec{g} \cdot \vec{H}$ is defined as a vector in the root space. The charge quantization condition may be obtained from the requirement of topological stability, namely the phase factor must be restricted by

$$\exp \left\{ i e \vec{g} \cdot \vec{H} \right\} = 1.$$

A general solution of this equation is given by

$$g = \vec{g} \cdot \vec{H} = \frac{4\pi}{e} \left[ \left( n_1 + \frac{n_2}{2} \right) H_1 - \frac{\sqrt{3}}{2} n_2 H_2 \right].$$

(B-11)
The magnetic charge is not so trivially quantized as in the $SU(2)$ model.

Since our Higgs vector $\vec{h}$ is not orthogonal to any of the simple roots $\vec{\beta}_i$, there is a unique set of simple roots with positive inner product with $\vec{h}$, Thus, the symmetry is maximally broken on the maximal abelian torus $U(1) \times U(1)$. In this case, both the numbers $n_1$ and $n_2$ have the meaning of topological charges. The topological charge of a non-Abelian monopole is given by the integral over the surface of sphere $S^2$

$$G = \frac{1}{\mathcal{A}_0} \int dS_n Tr (B_n A_0) = \vec{g} \cdot \vec{h} = g_1 + \frac{1}{2} g_2.$$  \hspace{1cm} (B-12)

Therefore, if $\vec{h}$ is orthogonal to one of the roots $\vec{\beta}_i$, only one component of $\vec{g}$ may be associated with the topological charge. In our case, there are two topological integers that are associated with a monopole.

The above definition of topological charge can be used in order to generalize the Bogomol’nyi bound for the $SU(N)$ monopoles. If we do not consider degrees of freedom that are related with electric charges of the configuration, it becomes

$$M = \mathcal{A}_0^3 |G| = \frac{4\pi A_0^3}{e} \sum_{i=1}^{r} n_i \left( \vec{h} \cdot \vec{\beta}_i \right) = \sum_{i=1}^{r} n_i M_i = \frac{4\pi A_0^3}{e} \left( n_1 + \frac{n_2}{2} \right)$$ \hspace{1cm} (B-13)

where $M_i = \frac{4\pi A_0^3}{e} \vec{h} \cdot \vec{\beta}_i$ and we suppose that the orientation of the Higgs field uniquely determines a set of simple roots that satisfies the condition $\vec{h} \cdot \vec{\beta}_i \geq 0$ for all $i$. Thus, it looks like there are $r$ individual monopoles of masses $M_i$. In Figure 15 we show the $A_0$ that we obtain through a fit of the lattice results for the Polyakov loop, as a function of the temperature.

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Figure 15: $A_0$ as a function of $T/T_c$.

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