Area Product and Mass Formula for Kerr-Newman-Taub-NUT Space-time

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Abstract We derive some important physical black hole thermodynamic products (e.g. “area product”, “entropy product”, irreducible mass product etc.) of the event horizon and Cauchy horizons of Kerr-Newman-Taub-NUT(Newman-Unti-Tamburino) space-time. We observe that these thermodynamic products are not universal (mass-independence) for Kerr-Newman-Taub-NUT(KNTN) black hole(BH), whereas for Kerr-Newman(KN) BH such products are universal (mass-independence). We also examine the “entropy sum” and “area sum” for this BH. It is shown that they are all depends on mass, charge and NUT parameter of the background space-time. Thus we can conclude that the universal(mass-independence) behaviour of “area product”, “entropy product”, “area sum” and “entropy sum” for KNTN BH is broken down and which is also quite different from KN BH. We further show that the KNTN BH do not possess first law of BH thermodynamics and Smarr-Gibbs-Duhem realations and that such relations are unlikely in the KN case. The BH mass formula and Christodoulou-Ruffini mass formula for KNTN black holes are derived.

1 Introduction

In an earlier work [1], we have shown that the area product, entropy product and irreducible mass product of the inner and outer horizons are universal (independent of ADM(Arnovitt-Deser-Misner) mass and depends only on the quantized charges), whereas the surface gravity product, surface temperature product and Komar energy product of the said horizons are not universal for...
KN BH. We have also verified that for KN BH, the first law of BH thermodynamics holds on the Cauchy horizon ($H^-$) as well as event horizon ($H^+$):

$$dM = \frac{\kappa_\pm}{8\pi} dA_\pm + \Omega_\pm dJ + \Phi_\pm dQ .$$ (1)

where, $\kappa_\pm$ is the surface gravity of $H^\pm$, $\Omega_\pm$ is the angular velocity of $H^\pm$ and $\Phi_\pm$ is the electromagnetic potentials of $H^\pm$.

The second law

$$dA_\pm = \frac{4A_\pm}{r_\pm} (dM - \Omega_\pm dJ - \Phi_\pm dQ) \geq 0$$ (2)

states that the area of both the horizons ($H^\pm$) will always increases with out violating the dominant energy condition. It has been shown that Smarr’s mass formula is valid on the inner horizon in agreement with the outer horizon.

In [2,3,4,5,6,7,8,9,10] the authors have been pointed out that the area product of the event horizons and the Cauchy horizons for various stationary axially symmetric space-time are often (but not always) independent of ADM(Arnovitt-Deser-Misner) mass of the space-time. For example, in case of KN space-time it is easy to check that the product of the area $A_\pm$ of the horizons $H^\pm$ can be expressed by the relation:

$$A_+ A_- = (8\pi J)^2 + (4\pi Q^2)^2 .$$ (3)

which is remarkably independent of the mass ($M$). But they depends on the angular momentum $J$ and charge $Q$ of the black hole, respectively.

Curir [12] in 1979 first calculated the “area sum” and “entropy sum” of Kerr BH for the interpretation of the spin entropy of the area of the inner horizon. In a recent work [11], the authors have examined a new universal property of “entropy sum” relation for various BHs in the asymptotically AdS space-time background.

It is now well known by fact that Cauchy horizon is an infinite blue-shift surface whereas event horizon is an infinite red-shift surface [13]. Thus when an observer cross the surface $r = r_+$, following a future directed time-like trajectory, is forever ‘lost’ to an external observer and any radiation transmitted by such an observer at the moment of crossing will be infinitely red shifted. Whereas, the same observer when cross the the surface $r = r_-$, following a future directed time-like trajectory and at the moment of crossing he/she will observe a panorama of the entire history of the external universe and any radiation transmitted by such an observer at the moment of crossing will be infinitely blue shifted. This is the fundamental differences between the $H^+$ and $H^-$. It is also true that the Cauchy horizon is classically unstable due to the external perturbation. Thus when an observer attempting to cross the Cauchy horizon, by following a time-like geodesics, and to emancipate herself or himself from the past, she or he would experience the impact of an infinite flux of radiation at the moment of crossing. Thus crossing the Cauchy horizon is fraught with danger [13].
Despite the above characteristics of the Cauchy horizon, yet Cauchy horizon takes now an important place in BH physics where, we have studied the following features of the Kerr-Newman-Taub-NUT space-time \[14, 16\]. Due to the presence of gravito-magnetic mass or NUT parameter what would be the changes are manifested in the “area product”, “entropy product”, “entropy sum”, “area sum”, Smarr’s mass formula \[17\] and Christodoulou-Ruffini mass formula. This is the principal objective of this paper.

We also prove that due to the presence of the NUT parameter Smarr-Gibbs-Duhem relation does not hold for KNTN spacetime. We also verify explicitly that the first law of BH thermodynamics does not hold for KNTN spacetime. It is also shown to be unlikely that such relations exist for KN BHs.

The structure of the paper is as follows. In section (2), we prove that like the “area product”, “entropy product”, “area sum”, “entropy sum”, the surface gravity product and surface temperature or BH temperature product of both the inner horizon and outer horizons do not shows any universal properties due to the mass dependence. Such products are not universal in nature. In section (3), We explicitly show that the BH mass or ADM mass can be expressed as in terms of the area of both horizons $H^\pm$. Also we derive in section (4) the Christodoulou-Ruffini \[19,20\] mass formula for KNTN space-time. We also point out that the product of Christodoulou’s irreducible mass of inner horizon (Cauchy horizon) and outer horizon (event horizon) are not independent of mass. In section (5), we discuss the extremal limit of the KNTN space-time and finally we conclude in section (6).

2 Kerr-Newman-Taub-NUT Geometry:

In Boyer-Lindquist like spherical coordinates ($t, r, \theta, \phi$) the KNTN is completely determined by the four parameters i.e., the mass ($M$), charge ($Q$), angular momentum ($J = aM$) and gravito-magnetic monopole or NUT parameter ($n$) or magnetic mass. Thus the corresponding metric \[14,16\] is described by

$$ds^2 = -\frac{\Delta}{\rho^2} [dt - P d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2 + n^2) d\phi - adt]^2 + \rho^2 \left[\frac{dr^2}{\Delta} + d\theta^2\right].$$

where

$$a \equiv \frac{J}{M}, \quad \rho^2 \equiv r^2 + (n + a \cos \theta)^2,$$

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 - n^2,$$

$$P \equiv a \sin^2 \theta - 2a \cos \theta.$$

The electromagnetic field 2-form would be given by

$$F = \frac{Q}{\rho^2} [r^2 - (n + a \cos \theta)^2] dr \wedge (dt - P d\phi) +$$
\[
\frac{2aQr \sin \theta \cos \theta}{\rho^4} \, d\theta \wedge [(r^2 + a^2 + n^2) \, d\phi - adt].
\] (8)

Note that when \( Q = 0 \), the electromagnetic field tensor vanishes and the metric satisfies the vacuum Einstein equations. When \( n = 0 \), the specific geometry reduces to Kerr-Newman geometry and when \( Q = n = 0 \), the geometry reduces to Kerr geometry.

The remarkable feature of this space-time is that: it has duality between the mass and the NUT parameter. Therefore the solution is invariant under the duality transformation, \( M \leftrightarrow n, r \leftrightarrow ia\chi \), where \( \chi \) is an angle coordinate and \( a \) is the Kerr parameter. This is in fact duality between gravito-electric mass (\( M \)) and gravito-magnetic charges (\( n \)) [15].

The radius of the horizon is determined by the solution of the function \( \Delta = 0 \). i.e.,
\[
r = r_{\pm} \equiv M \pm \sqrt{M^2 - a^2 - Q^2 + n^2} \quad \text{and} \quad r_+ > r_-. \quad (9)
\]

The static limit surface (outer region of the ergo-sphere) is at \( g_{tt} = 0 \) i.e.
\[
r = r_{\text{ergo}} \equiv M + \sqrt{M^2 - a^2 \cos^2 \theta - Q^2 + n^2}. \quad (10)
\]

As long as \( M^2 + n^2 \geq Q^2 + a^2 \) the KNTN metric describes a BH, otherwise it has a naked ringlike singularity. When \( M^2 + n^2 = Q^2 + a^2 \), the situation is called extremal situation in gravitational physics.

Now the area of both the horizons (\( H^\pm \)) for KNTN space-time is
\[
A_{\pm} = \int \int \sqrt{g} d\theta d\phi = 4\pi (r_{\pm}^2 + a^2 + n^2) = 4\pi \left[ 2(M^2 + n^2) - Q^2 \pm 2M\sqrt{M^2 + n^2 - a^2 - Q^2} \right]. \quad (11)
\]

The angular velocity of \( H^\pm \) is
\[
\Omega_{\pm} = \frac{a}{r_{\pm}^2 + a^2 + n^2} = \frac{a}{2Mr_{\pm} - Q^2 + 2n^2}. \quad (13)
\]

The semi classical Bekenstein-Hawking [21][22] entropy of \( H^\pm \) reads (in units in which \( G = \hbar = c = 1 \))
\[
S_{\pm} = \frac{A_{\pm}}{4} = \pi (r_{\pm}^2 + a^2 + n^2) = \pi (2Mr_{\pm} - Q^2 + 2n^2). \quad (14)
\]

The surface gravity of \( H^\pm \) is
\[
\kappa_{\pm} = \frac{r_{\pm} - r_{\mp}}{2(r_{\pm}^2 + a^2 + n^2)} = \frac{r_{\pm} - r_{\mp}}{2(2Mr_{\pm} - Q^2 + 2n^2)} \quad \text{and} \quad \kappa_+ > \kappa_. \quad (15)
\]

and the BH temperature or Hawking temperature of \( H^\pm \) reads as
\[
T_{\pm} = \frac{\kappa_{\pm}}{2\pi} = \frac{r_{\pm} - r_{\mp}}{4\pi (r_{\pm}^2 + a^2 + n^2)}. \quad (16)
\]
It should be noted that $T_+ > T_-$. Finally, the horizon Killing vector field may be defined for $\mathcal{H}^\pm$ is

$$\chi^\pm_a = (\partial_t)^a + \Omega^a (\partial_\phi)^a.$$  (17)

Now we compute the different horizons product, for instance the proper radii product of $\mathcal{H}^\pm$ for KNTN space-time:

$$r_+ r_- = a^2 + Q^2 - n^2.$$  (18)

It implies that, it does not depend on mass but depends on charge, Kerr parameter and NUT parameter.

Remarkably, if the inner Cauchy horizon exists (i.e. if $J$, $Q$ and $n$ do not vanish simultaneously), then the product of the area of the Cauchy horizon and event horizon of KNTN space-time holds the following relation:

$$A_+ A_- = (8\pi)^2 \left( J^2 + \frac{Q^4}{4} + n^2 (M^2 + n^2 - Q^2) \right).$$  (19)

Unfortunately, the area product strictly depends upon the mass of the space-time. Thus this product is not universal for stationary, axially symmetric KNTN space-time.

Now the sum of the area of both the horizons are

$$A_+ + A_- = 8\pi \left[ 2(M^2 + n^2) - Q^2 \right].$$  (20)

It shows that the area sum of both the horizons also depend on the mass, charge and the NUT parameter. It does not manifested of any universal character on the “area sum”.

It is well known that non-extremal BH filled up with the trapped surface, whereas extremal BH don’t have any trapped surface. Thus for non-extremal BH, the following relation holds:

$$A_+ > \sqrt{(8\pi)^2 \left( J^2 + \frac{Q^4}{4} + n^2 (M^2 + n^2 - Q^2) \right)} > A_-.$$  (21)

Also the product of entropy is

$$S_+ S_- = (2\pi)^2 \left( J^2 + \frac{Q^4}{4} + n^2 (M^2 + n^2 - Q^2) \right).$$  (22)

It is also depends on the mass ($M$) parameter. Similarly, for our completeness, we also calculate the “entropy sum” of both the horizons and found to be

$$S_+ + S_- = 2\pi \left[ 2(M^2 + n^2) - Q^2 \right].$$  (23)

It seems that it is also mass dependent. The entropy of the non-extremal case satisfied the following inequality:

$$S_+ > \sqrt{(2\pi)^2 \left( J^2 + \frac{Q^4}{4} + n^2 (M^2 + n^2 - Q^2) \right)} > S_-.$$  (24)
Analogously, we can compute the product of surface gravity of $\mathcal{H}^\pm$:

$$\kappa_+\kappa_- = -\frac{(r_+ - r_-)^2}{4(r_+^2 + a^2 + n^2)(r_-^2 + a^2 + n^2)} = -\frac{M^2 - a^2 - Q^2 + n^2}{(r_+^2 + a^2 + n^2)(r_-^2 + a^2 + n^2)}$$  \hspace{1cm} (25)$$

and the product of surface temperature of $\mathcal{H}^\pm$ reads

$$T_+T_- = -\frac{(r_+ - r_-)^2}{(4\pi)^2(r_+^2 + a^2 + n^2)(r_-^2 + a^2 + n^2)} = -\frac{M^2 - a^2 - Q^2 + n^2}{(2\pi)^2(r_+^2 + a^2 + n^2)(r_-^2 + a^2 + n^2)}$$  \hspace{1cm} (26)$$

Thus the universal behavior of “area product”, “entropy product”, “area sum” and “entropy sum” do not hold for KNTN space-time.

There are some useful relations for KNTN BH as derived by Curir\cite{12} for Kerr BH:

$$T_+S_+ + T_-S_- = 0.$$  \hspace{1cm} (27)$$

and

$$\frac{\Omega_+}{T_+} + \frac{\Omega_-}{T_-} = 0.$$  \hspace{1cm} (28)$$

Now, we shall compute various physical thermodynamic quantities for KN BH, Kerr BH in comparison with KNTN BH.
### 3 The Mass Formula for KNTN Spacetime:

To evaluate the mass formula for KNTN BH, first we need to compute the surface area of this BH. The surface area of the BH is the two dimensional surface formed by the intersection of the outer horizon or event horizon with a space-like hyper-surface. According to Hawking’s area theorem \( [22] \), the surface area of a BH always increases i.e.,

\[
dA \geq 0
\]  \( (29) \)

Now for KNTN BH the surface area of both event horizon and Cauchy horizon is indeed constant. can be defined as

\[
A_\pm = 4\pi \left( 2M^2 - Q^2 + 2n^2 \pm 2\sqrt{M^4 - J^2 - M^2Q^2 + n^2M^2} \right).
\]  \( (30) \)

It may be noted that \( A_+ > A_- \).
Inverting the above relation one can obtain the BH mass or ADM mass can be expressed as in terms of area of both the horizons $H^\pm$,

$$\mathcal{M}^2 = \frac{1}{(A_\pm - 4\pi n^2)} \left[ \frac{A_\pm^2}{16\pi} + 4\pi J^2 + \frac{A_\pm Q^2}{2} + \pi Q^4 - n^2(A_\pm - 4\pi n^2 - 4\pi Q^2) \right] \tag{31}$$

It is remarkable that the mass can be expressed as in terms of both the area of $H^+$ and $H^-$.

Now we can deduce the mass differential for KNTN space-time. It is indeed expressed as four physical invariants of both $H^+$ and $H^-$,

$$d\mathcal{M} = \Gamma_\pm dA_\pm + \Omega_\pm dJ + \Phi_\pm dQ + \Phi_n^\pm dn \tag{32}$$

where

$$\Gamma_\pm = \frac{1}{2M(A_\pm - 4\pi n^2)^2} \left( \frac{A_\pm^2}{16\pi} - 4\pi J^2 - \pi Q^4 - n^2A_\pm - 2\pi n^2Q^2 \right) \tag{33}$$

$$\Omega_\pm = \frac{4\pi J}{M(A_\pm - 4\pi n^2)} \tag{34}$$

$$\Phi_\pm = \frac{1}{2M(A_\pm - 4\pi n^2)} (QA_\pm + 4\pi Q^3 - 8\pi n^2Q) \tag{35}$$

$$\Phi_n^\pm = \frac{(16\pi n^3A_\pm - 4\pi \pi Q^2A_\pm - \frac{3}{2}nA_\pm^2 - 32n^2J^2 + 32\pi J^2 + 8n^2Q^4)}{2M(A_\pm - 4\pi n^2)^2} \tag{36}$$

where

- $\Gamma_\pm$ = Effective surface tension of $H^+$ and $H^-$
- $\Omega_\pm$ = Angular velocity of $H^\pm$
- $\Phi_\pm$ = Electromagnetic potentials of $H^\pm$ for electric charge
- $\Phi_n^\pm$ = Electromagnetic potentials of $H^\pm$ for NUT charge

Using the Euler’s theorem on homogenous functions to $\mathcal{M}$ of degree $\frac{1}{2}$ in $(A_\pm, J, Q, n)$. Thus we can deduce the mass can be expressed in terms of these quantities both for $H^\pm$ as a bilinear form

$$\mathcal{M} = 2\Gamma_\pm A_\pm + 2J\Omega_\pm + \Phi_\pm Q + \Phi_n^\pm n \tag{38}$$

This has been derived from the homogenous function of degree $\frac{1}{2}$ in $(A_\pm, J, Q^2, n^2)$. Remarkably, $\Gamma_\pm$, $\Omega_\pm$, $\Phi_\pm$ and $\Phi_n^\pm$ can be defined and are constant on the $H^+$ and $H^-$ for any stationary, axially symmetric spacetime.

Since the $d\mathcal{M}$ is perfect differential, one may choose freely any path of integration in $(A_\pm, J, Q, n)$ space. Thus the surface energy $\mathcal{E}_{\pm}$ for $H^+$ and $H^-$ can be defined as

$$\mathcal{E}_{\pm} = \int_0^{A_\pm} \Gamma(\tilde{A}_\pm, 0, 0, 0) d\tilde{A}_\pm \tag{39}$$
The rotational energy for $\mathcal{H}^+$ and $\mathcal{H}^-$ can be defined by

$$E_{r\pm} = \int_0^J \Omega_{\pm}(A_{\pm}, \tilde{J}, 0, 0)d\tilde{J}, \ A_{\pm} \text{ fixed}; \quad (40)$$

The electromagnetic energy for $\mathcal{H}^+$ and $\mathcal{H}^-$ due to the charge $Q$ is given by

$$E_{em\pm} = \int_0^Q \Phi_{\pm}(A_{\pm}, J, \tilde{Q}, 0)d\tilde{Q}, \ A_{\pm}, J \text{ fixed}; \quad (41)$$

and finally the electromagnetic energy for $\mathcal{H}^+$ and $\mathcal{H}^-$ due to the NUT charge $n$ is given by

$$E_{em\pm} = \int_0^n \Phi_{\pm}(A_{\pm}, J, \tilde{Q}, \tilde{n})d\tilde{n}, \ A_{\pm}, J, \tilde{Q} \text{ fixed}; \quad (42)$$

Generally, combining the mass differential Eq. (32) with the first law leads to the Smarr-Gibbs-Duhem relation. We observe that from Eqs. (32) and (38), such an equation does not hold for KNTN spacetime because

$$\Gamma_{\pm} = \frac{\partial M}{\partial A_{\pm}} \neq \frac{\kappa_{\pm}}{8\pi} = \frac{T_{\pm}}{4} \quad (43)$$

An important point should be noted here that the first law of BH thermodynamics do not satisfied for the KNTN spacetime.

### 4 Christodoulou’s Irreducible Mass for KNTN Space-time:

Floyd and Penrose were first noticed that in an example of the extraction of the energy from a Kerr BH, the surface area of the horizon increases when a BH undergoing any transformations. What is now called the Penrose process [18]. Independently, Christodoulou [19] has shown that a quantity which he named the “irreducible mass” of the BH, $M_{irr}$ unchanged by means of any process. In fact, most processes result in an increase in $M_{irr}$ and during reversible process this quantity also does not change. This result indicates that there exists a relation between area and irreducible mass. Now it is well known that $M_{irr}$ is proportional to the square root of the BH’s area. Since the KNTN spacetime has regular event horizon and Cauchy horizon. Thus the irreducible mass can be defined for both the horizons are

$$M_{irr,\pm} = \sqrt{\frac{A_{\pm}}{16\pi}} = \sqrt{\frac{r_{\pm}^2 + a^2 + n^2}{2}}. \quad (44)$$

where $+$ indicates for $\mathcal{H}^+$ and $-$ indicates for $\mathcal{H}^-$. The area and angular velocity can be expressed as in terms of $M_{irr,\pm}$:

$$A_{\pm} = 16\pi (M_{irr,\pm})^2. \quad (45)$$
\[ \Omega_{\pm} = \frac{a}{r_{\pm}^2 + a^2 + n^2} = \frac{a}{4(M_{\text{irr}, \pm})^2}. \] 

Interestingly, the product of the irreducible mass of \(H^\pm\) for KNTN space-time:

\[ M_{\text{irr}, +}M_{\text{irr}, -} = \frac{\sqrt{A_+A_-}}{(16\pi)^2} = \sqrt{J^2 + \frac{Q^4}{4} + n^2(M^2 + n^2 - Q^2)}. \]

It shows that this product is not universal because it depends on the mass parameter.

In fact the Christodoulou-Ruffini mass formula for KNTN spacetime in term of irreducible mass \(M_{\text{irr}, \pm}\), angular momentum \(J\), charge \(Q\) and NUT parameter \(n\) is given by

\[ M^2 = \left[ (M_{\text{irr}, \pm} + \frac{Q^2}{4M_{\text{irr}, \pm}})^2 + \frac{J^2}{4(M_{\text{irr}, \pm})^2} - n^2 \left( 1 + \frac{Q^2}{4(M_{\text{irr}, \pm})^2} - \frac{n^2}{4(M_{\text{irr}, \pm})^2} \right) \right] \times \left( 1 - \frac{n^2}{4(M_{\text{irr}, \pm})^2} \right)^{-1}. \]

When the NUT parameter and charge parameter goes to zero we get the mass formula for Kerr-Newman space-time [20]. When the charge parameter vanishes, we obtain the mass formula for Kerr-Taub-NUT space-time:

\[ M^2 = \left[ (M_{\text{irr}, \pm})^2 + \frac{J^2}{4(M_{\text{irr}, \pm})^2} - n^2 \left( 1 - \frac{n^2}{4(M_{\text{irr}, \pm})^2} \right) \right] \left( 1 - \frac{n^2}{4(M_{\text{irr}, \pm})^2} \right)^{-1}. \]

Again when \(n = 0\), we recover the mass formula for Kerr BH [20].

5 Extremal KNTN BH:

In the case of extremal KNTN black hole, where the radii of both the horizons are coincide. That means \(r_+ = r_-\) or \(A_+ = A_-\) or \(\kappa_+ = \kappa_-\) or \(T_+ = T_-\) or \(M_{\text{irr}, +} = M_{\text{irr}, -}\). Thus, we get the equality in Eq. (19):

\[ A_+^2 = A_-^2 = (8\pi)^2 \left( J^2 + \frac{Q^4}{4} + n^2(M^2 + n^2 - Q^2) \right). \]

when the areas of the both horizons are coincide i.e. \(A_+ = A_-\). As a result of (51), we obtain the extremal condition for KNTN space-time:

\[ a^2 + Q^2 - n^2 = M_{\text{CR}}^2. \]

where \(M_{\text{CR}}\) denotes the Christodoulou-Ruffini mass of the BH which agrees with the ADM mass of KNTN space-time.
6 Discussions:

In this work, we have derived the Smarr formula for KNTN space-time. We have also demonstrated that the “area product”, “entropy product”, horizon radii product, irreducible mass product, the surface gravity product and surface temperature product of the event horizons and Cauchy horizons are not universal for KNTN BH. We also examined the “entropy sum” or “area sum” for this BH, it is shown that these properties do not showing any universal character in the back ground geometry. We have also defined the Christodoulou and Ruffini mass formula for KNTN space-time. Thus in conclusion “the area product is independent of mass ” and “ entropy sum is independent of mass ” do not hold for axially symmetric stationary KNTN space-time. We also expect that this argument is also true for the most general axially symmetric stationary space-time like Plebański and Demiański, which contains seven parameters - acceleration, mass \( M \), Kerr parameter \( a \), electric charge \( Q_e \), magnetic charge \( Q_m \), NUT parameter \( n \) and the Cosmological constant \( \Lambda \).

The failure of the first law of BH thermodynamics and Smarr-Gibbs-Duhem relations for the KNTN spacetime due to the presence of the NUT parameter. Such results are unlikely in the KN BH.

It is known that in the presence of NUT charges, the entropy of a spacetime acquires a contribution from the Dirac-Misner strings that the NUT charge introduces- this is in addition to the usual contribution from horizon area. Thus it seems quite natural to expect that the formula for the “area product” or “entropy product” and “entropy sum” or “area sum” get modified due to the Dirac-Misner strings. This investigation might also be interesting and it could be found elsewhere.

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