On a conjecture by Blocki-Zwonek
by
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Abstract. This paper gives a counterexample to a conjecture by Blocki and Zwonek on the area of sublevel sets of the Green’s function.

1. Introduction

In [1] Blocki and Zwonek make the following conjecture:

Conjecture 1.1. If $\Omega$ is a pseudoconvex domain in $\mathbb{C}^n$, then the function

$$t \to \log \lambda(\{G_{\Omega,w} < t\})$$

is convex.

Here $w$ is any given point in $\Omega$ and $G$ denotes the pluricomplex Green function with pole at $w$. Also $\lambda$ denotes Lebesgue measure.

In this note we provide a counterexample in $\mathbb{C}$. The example is constructed in the next section. It is a triply connected domain where the Green function has a higher order critical point.

2. The Example

Lemma 2.1. There exists a bounded domain $\Omega \subset \mathbb{C}$ and a point $q \in \Omega$ such that the Green function $G$ with pole at $q$ has a critical point $p \in \Omega$ so that $G(z) - G(p) = O(|z - p|^3)$ in a neighborhood of $p$. The domain is triply connected with boundary consisting of three simply closed real analytic smooth curves.

Proof. We construct the domain $\Omega$ by modifying domains in steps:

$$\Omega_1, \Omega_2, \Omega_3, \Omega_4 = \Omega.$$ 

First, we define $\Omega_1 \subset \mathbb{P}^1$. The point $\infty$ will be an interior point. The constant $r = e^{-1/3}$, $0 < r < 1$.

$$\Omega_1 := \{\eta \in \mathbb{P}^1; |\eta + 1| > r\}$$

So $\Omega_1$ is a topological disc. $\eta = 0 \in \Omega_1$. The complement of $\Omega_1$ is the closed disc centered at $\eta = -1$ with radius $r$.

Next we let $\Omega_2 := \{\tau \in \mathbb{P}^1; |\tau^3 + 1| > r\}$. Again $\infty$ is in the domain. The points $-1, e^{i\pm 2\pi/3} = c_1, c_2, c_3$ belong to three simply connected domains $D_j$ with real analytic boundary $\gamma_1, \gamma_2, \gamma_3$. The union of their (pairwise disjoint) closures constitute the complement of $\Omega_2$ in $\mathbb{P}^1$. The origin and infinity are in the interior of the domain.
We translate 1 unit to the right:

$$\Omega_3 := \{ \sigma \in \mathbb{P}^1; |(\sigma - 1)^3 + 1| > r \}$$

The points $0, 1 + e^{\pm i \pi/3} = c'_1, c'_2, c'_3$ belong to three simply connected domains $D'_j$ with real analytic boundary $\gamma'_1, \gamma'_2, \gamma'_3$. The union of their (pairwise disjoint) closures constitute the complement of $\Omega_3$. The point 1 and infinity belong to the interior of $\Omega_3$. Also $c' = 0$ is contained in a neighborhood $D'_1$ which is in the complement of $\Omega_3$.

Next we make an inversion so that the domain becomes bounded.

$$\Omega_4 = \Omega_3 = \{ w \in \mathbb{P}^1; |(1/w - 1)^3 + 1| > r \}$$

The points $\infty, 1/e^{1+e^{-1/3}} = c''_1, c''_2, c''_3$ belong to three simply connected domains $D''_j$ with real analytic boundary $\gamma''_1, \gamma''_2, \gamma''_3$. The union of their closures constitute the complement of $\Omega$. The point 1 and 0 belong to the interior of $\Omega$. $\Omega$ lies inside the curve $\gamma''_j$ and there are two holes, bounded by $\gamma''_2, \gamma''_3$ respectively.

We rewrite the description of $\Omega$:

$$\Omega = \{ w \in \mathbb{P}^1; \log |(1/w - 1)^3 + 1| - \log r > 0 \}$$

$$\Omega = \{ w \in \mathbb{P}^1; \log |(1-w)^3 + w^3|/w^3 - \log r > 0 \}$$

$$\Omega = \{ w \in \mathbb{P}^1; \log |w^3/(1-w)^3 + w^3| + \log r < 0 \}$$

$$\Omega = \{ w \in \mathbb{P}^1; \log |w^3/(1-3w + 3w^2)| + \log e^{-1/3} < 0 \}$$

$$\Omega = \{ w \in \mathbb{P}^1; 1/3 \log |w^3/(1-3w + 3w^2)| + 1/3(-1/3) < 0 \}$$

Let $G(w) = 1/3 \log |w^3/(1-3w + 3w^2)| - 1/3$. Then $\Omega$ is the set where this function is negative. In a neighborhood of the origin, we have $G(w) = \log |w| + O(1)$. In the rest of $\Omega$ the function is harmonic. Hence $G(w)$ is the Green function for $\Omega$ with a pole at the origin.

Next we consider the function $G$ in a neighborhood of the interior point $w = 1$: We get $G(1) = -1/9$. We estimate $G(w) + 1/9$:

\[ \text{...} \]
\[ G(w) + \frac{1}{9} = \frac{1}{3} \log \left| \frac{w^3}{w^3 + (1 - w)^3} \right| \]
\[ = \frac{1}{3} \log \left| \frac{1}{1 + (\frac{1-w}{w})^3} \right| \]
\[ = \frac{1}{3} \log \left| 1 - \left( \frac{1-w}{w} \right)^3 + \mathcal{O} \left( \frac{1-w}{w} \right)^6 \right| \]
\[ = \frac{1}{3} (Re) \log \left( 1 - \left( \frac{1-w}{w} \right)^3 + \mathcal{O} \left( \frac{1-w}{w} \right)^6 \right) \]
\[ = \frac{1}{3} (Re) \left( - \left( \frac{1-w}{w} \right)^3 + \mathcal{O} \left( \frac{1-w}{w} \right)^6 \right) \]
\[ = 3(Re) (- (1-w)^3 + \mathcal{O}(1-w)^4) \]
\[ = \mathcal{O}(\left| w - 1 \right|^3) \]

\[ \square \]

Lemma 2.2. Let \( S_\epsilon \) denote the sector \( S_\epsilon = \{ w = 1 + re^{i\theta}; 0 < r < \epsilon^{1/3}/2, \frac{11\pi}{12} < \theta < \frac{13\pi}{12} \} \). There is an \( \epsilon_0 > 0 \) so that if \( 0 < \epsilon < \epsilon_0 \) then

\[ S_\epsilon \subset \{ w \in \Omega; -\epsilon < G(w) - G(1) < 0 \}. \]

Proof. Suppose that \( w \in S_\epsilon \). Then
\[
G(w) - G(1) = G(w) + \frac{1}{9}
\]
\[
= \frac{1}{3}(Re)(w - 1)^3 + O(w - 1)^4
\]
\[
\Rightarrow \quad \frac{1}{3}(Re)((w - 1)^3) - Cr^4 \leq G(w) - G(1)
\]
\[
\leq \frac{1}{3}(Re)((w - 1)^3) + Cr^4
\]
\[
\Rightarrow \quad 3r^3 \cos(3\theta) - Cr^4 \leq G(w) + 1
\]
\[
\leq 3r^3 \cos(3\theta) + Cr^4
\]
\[
2\pi + \frac{3}{4}\pi \leq 3\theta
\]
\[
\leq 2\pi + \frac{5}{4}\pi
\]
\[
\Rightarrow \quad -1 \leq \cos(3\theta)
\]
\[
\leq -\sqrt{2}/2
\]
\[
-\frac{1}{3}r^3 - Cr^4 \leq G(w) - G(1)
\]
\[
\leq -\frac{1}{3}\sqrt{2}/2r^3 + Cr^4
\]
\[
-\frac{\epsilon}{24} - \frac{C\epsilon^{4/3}}{16} \leq G(w) - G(1)
\]
\[
\leq -\frac{\sqrt{2}\epsilon}{48}
\]
\[
\Rightarrow \quad -\epsilon < G(w) - G(1) < 0
\]

\[
□
\]

Define the function \( s(t) \) for \( -\infty < t < 0 \).

\[
s(t) = \lambda(\{G(w) < t\})
\]

i.e. the area of the sublevel set where \( G < t \).

**Corollary 2.3.** For all small enough \( \epsilon > 0 \) we have that

\[
s(-\frac{1}{9}) - s(-\frac{1}{9} - \epsilon) \geq \pi \frac{\epsilon^{2/3}}{48}.
\]
Proof. We have that
\[
s(-\frac{1}{9}) - s(-\frac{1}{9} - \epsilon) = \lambda\{-\frac{1}{9} - \epsilon < G < -\frac{1}{9}\}
\geq \lambda(S_\epsilon)
= \pi(\frac{\epsilon^{1/3}}{2})^2 \frac{1}{12}
\]

This implies that
\[
s(-\frac{1}{9} - \epsilon) \leq s(-\frac{1}{9}) - \pi \frac{\epsilon^{2/3}}{48}.
\]

The following is immediate.

**Theorem 2.1.** The function is not convex.

**Corollary 2.4.** The function \( t \to \log \lambda\{G < t\} \) is not convex.

**References**

1. Blocki, Z., Zwonek, W.; Estimates for the Bergman kernel and the multidimensional Suita conjecture. *New York J. of Math.* 21 (2015), 151–161