BOUNDARY EXPANSIONS AND CONVERGENCE FOR COMPLEX MONGE-AMPÈRE EQUATIONS

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Abstract. We study boundary expansions of solutions of complex Monge-Ampère equations and discuss the convergence of such expansions. We prove a global convergence result under the assumption that the entire boundary is analytic. If a portion of the boundary is assumed to be analytic, the expansions may not converge locally.

1. Introduction

Kähler-Einstein metrics play an important role in complex geometry. Let \( \Omega \subseteq \mathbb{C}^n \) be a smooth, bounded, and strictly pseudoconvex domain, \( n \geq 2 \). Consider

\[
\det(w_{ij}) = e^{(n+1)w} \quad \text{in } \Omega,
\]

\[
w_{ij} > 0 \quad \text{in } \Omega,
\]

\[
w = \infty \quad \text{on } \partial \Omega,
\]

(1.1)

Then, \( w_{ij}dz^id\bar{z}^j \) defines a complete Kähler-Einstein metric in \( \Omega \).

Cheng and Yau [9] studied the problem (1.1) and proved the existence and regularity of its solution. Lee and Melrose [26] proved the optimal regularity and constructed an expansion of the solution \( w \) near \( \partial \Omega \). Specifically, let \( \rho \) be a (negative) strictly plurisubharmonic defining function of \( \Omega \) and consider a reference metric given by

\[-(\log(-\rho))_{ij}dz^id\bar{z}^j.\]

Cheng and Yau [9] proved that the problem (1.1) admits a solution \( w \) of the form

\[w = -\log(-\rho) + u,\]

for some \( u \in C^{n,\frac{1}{2}-\delta}(\Omega) \), for any \( \delta > 0 \). Lee and Melrose [26] proved \( u \in C^{n,\alpha}(\Omega) \), for any \( \alpha \in (0,1) \), and that \( u \) has an expansion near \( \partial \Omega \) given by

\[u = c_0 + c_1\rho + \cdots + c_n\rho^n + c_{n+1,1}\rho^{n+1}\log\rho + c_{n+1}\rho^{n+1} + \cdots.\]

In general, solutions are not better than \( C^{n+1}(\Omega) \), due to the presence of the logarithmic factors.

In this paper, we will study the convergence of the boundary expansions associated with (1.1). First, we prove a local boundary expansion using techniques developed in [18].

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Theorem 1.1. Let $\Omega \subseteq \mathbb{C}^n$ be a $C^7$ bounded strictly pseudoconvex domain and $w$ be a solution of (1.1). Assume $\Gamma$ is a smooth open portion of $\partial \Omega$ and $U$ is an open subset of $\Gamma$, with $\overline{U} \subseteq \Gamma$. Then, in $U \times \{-r < \rho \leq 0\}$ for some $r > 0$ and for any $k \geq n + 1$,

$$w = -\log(-\rho) + c_0 + c_1 \rho \cdots + c_n \rho^n + \sum_{i=n+1}^{k} \sum_{j=0}^{N_i} c_{i,j} \rho^i (\log(-\rho))^j + R_k,$$

where $N_i$ is a positive integer with $N_{n+1} = 1$ and the $R_k$ is a $C^{k,\alpha}$-function in $U \times \{-r < \rho \leq 0\}$ and satisfies, for any $\alpha \in (0, 1)$,

$$|R_k| \leq \rho^{k+\alpha} \text{ in } U \times \{-r < \rho \leq 0\}.$$

As a consequence, we conclude that $w + \log \rho$ is at most $C^{n,\alpha}$ near $\Gamma$ in general, due to the presence of the logarithmic term, and that $w + \log \rho$ is smooth up to $\Gamma$ if $c_{n+1,1} = 0$. Note that $c_{n+1,1}$ is the coefficient of the first logarithmic term. In fact, if $c_{n+1,1} = 0$, then all coefficients of the logarithmic terms vanish.

Next, we discuss the convergence of the boundary expansions associated with (1.1). We prove the convergence under the assumption that the entire boundary $\partial \Omega$ is analytic.

Theorem 1.2. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and $w$ be a solution of (1.1). Assume $\partial \Omega$ is analytic and parallel. Then, the expansion in Theorem 1.1

$$w = -\log(-\rho) + \sum_{i=0}^{n} c_i \rho^i + \sum_{i=n+1}^{\infty} \sum_{j=0}^{N_i} c_{i,j} \rho^i (\log(-\rho))^j \tag{1.2}$$

converges uniformly in $\partial \Omega \times \{-r < \rho \leq 0\}$ for some small $r$.

By the Stiefel’s Theorem, all oriented three manifolds are parallel. Hence, the assumption that $\partial \Omega$ is parallel is redundant for $\Omega \subseteq \mathbb{C}^2$. However, such an assumption is necessary for general $n$.

Theorem 1.2 concerns the global convergence near the entire boundary. If only a portion of the boundary is assumed to be analytic, the boundary expansions may not necessarily be convergent. We will prove that solutions belongs to the Gevrey space in tangential directions locally if a portion of the boundary is analytic.

Theorem 1.3. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and $w$ be a solution of (1.1). Assume $\Gamma$ is an analytic open portion of $\partial \Omega$ and $U$ is an open subset of $\Gamma$, with $\overline{U} \subseteq \Gamma$. Then, for some $r > 0$ and any integer $p$,

$$|D^p_T u| \leq DB^p(p!)^2 \text{ in } U \times \{-r < \rho \leq 0\},$$

where $B$ and $D$ are positive constants independent of $p$, and $D^p_T u$ denotes a $p$-th order tangential derivative of $u$.

The possibility of the divergence mainly originates from the complex structure, in which one tangential direction, conjugate to the normal direction, differs from other tangential directions. This is sharply different from problems in the real space, where all tangential directions are the same. We usually have the local convergence in the real
space, such as the local convergence for the minimal surface equation in the hyperbolic space. (Refer to [19].)

This paper is organized as follows. In Section 2 we introduce the basic set up for our main equation. In Section 3 we provide a formal computation of expansions near boundary. In Section 4 we derive some basic estimates of solutions. In Section 5 we prove the tangential regularity and establish estimates involving tangential derivatives. In Section 6 we discuss the regularity along the normal direction and expansions near boundary. In Section 7 we prove the convergence of boundary expansions under the assumption that the entire boundary is analytic. In Section 8 we prove that solutions are in the Gevrey class tangentially. In Section 9 we construct a counterexample for the local convergence for a class of linear equations.

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2. Preliminary

In this paper, for an index \( \alpha \) (or \( \beta, \gamma, \) etc.), “\( \alpha \neq n, 2n \)” means \( 1 \leq \alpha \leq 2n - 1 \) and \( \alpha \neq n \), and “\( \alpha \neq 2n \)” means \( 1 \leq \alpha \leq 2n - 1 \). In addition, we always denote by \( d \) the Euclidean distance function to the boundary.

Assume \( \rho \) is a strictly plurisubharmonic defining function of \( \Omega \), i.e., \( \rho = 0 \) on \( \partial \Omega \), \( d\rho \neq 0 \) on \( \partial \Omega \), \( \rho_{ij} > 0 \) in \( \bar{\Omega} \) and \( \Omega = \{ \rho < 0 \} \). If \( d \) is the distance function to \( \partial \Omega \), we can choose
\[
\rho = e^{-\lambda d} - 1 \quad \text{near} \ \partial \Omega,
\]
for some \( \lambda > 0 \) large. Set
\[
g = -\log(-\rho).
\]
Denote by \( g_{ij} = (-\log(-\rho))_{ij} \) a reference metric. We define \( |u|_{C^0_g} = |u| \) and, for each \( k \geq 1 \),
\[
|u|_{C^k_g} = (\nabla^k u, \nabla^k u)_g + |u|_{C^{k-1}_g}.
\]
We point out that this is pointwisely defined. We also define the \( C^k \)-norm by
\[
\|u\|_{C^k_g(U)} = \sup_U |u|_{C^k_g}.
\]
To define the \( C^{k,\alpha}_g \)-norm, we fix a boundary point, say the origin \( O \in \partial \Omega \). Assume \( P \in \Omega \), with \( d(P, O) = d(P) \). Cheng and Yau [9] proved \( g_{ij} \) satisfies the condition of bounded geometry in the coordinates chart
\[
(\nu_1, \cdots, \nu_n) = \frac{(2d(P) - d(P)^2)^{\frac{1}{2}}}{z_n + d(P) - z_n d(P)} (z_1, z_2, \cdots, - \frac{z_n - d(P)}{(2d(P) - d(P)^2)^{\frac{1}{2}}}).
\]
When \( d(P) \) is small, we can define a \( C^{k,\alpha}_g \)-norm in the Euclidean ball \( B_{d(P)/2}(P) \) with respect to \( \{\nu_1, \cdots, \nu_n\} \) and then apply a covering to define the global norm.

By [9], we have the following result on the existence and interior estimates of solutions.
Theorem 2.1. Suppose $\Omega$ is a $C^{k+2}$ strictly pseudoconvex domain and $\rho$ is a $C^{k+2}$ defining function of $\Omega$, for some $k \geq 5$. Then for any $F \in C^{k-2,\alpha}_g(\Omega)$, $\alpha \in (0, 1)$, there exists a unique $u \in C^{k,\alpha}_g(\Omega)$, such that

$$
\det(g_{i\bar{j}} + u_{i\bar{j}}) = e^{(n+1)u} e^F \det(g_{i\bar{j}}) \quad \text{in } \Omega,
$$

$$
\frac{1}{c} g_{i\bar{j}} \leq u_{i\bar{j}} \leq c g_{i\bar{j}}.
$$

If $F = -\log(\det(\rho_{i\bar{j}})(-\rho + g_{i\bar{j}}\rho_{i\bar{j}}))$, then $w = -\log(-\rho) + u$ is a solution of (1.1).

In the following, we set $k = 5$. Hence, a $C^7$ domain $\Omega$ implies $u \in C^{5,\alpha}_g(\Omega)$.

Assume a portion $\Gamma$ of $\partial\Omega$ is smooth. We will discuss behaviors of $u$ near $\Gamma$.

3. Formal Computations

Fefferman [11] discussed formal boundary expansions for (1.1). In this section, we present such expansions slightly differently.

We write the equation (2.3) as

$$
\log \det(I + (D^2 g)^{-1} \cdot D^2 u) = (n + 1)u + F.
$$

Set

$$
Q(u) = \log \det(I + (D^2 g)^{-1} \cdot D^2 u) - (n + 1)u + F.
$$

We can find functions $c_0, \cdots, c_k$ on boundary such that the function defined by

$$
u_k = c_0(y') + c_1(y')d + \cdots + c_k(y')d^k
$$

satisfies

$$
Q(u_k) = O(d^{k+1}).
$$

Note the following identity

$$
\log \det(I + M^{-1}N) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{Tr}(M^{-1}N)^k,
$$

if all eigenvalues of $M^{-1}N$ are bounded by 1. Set

$$
M_{i\bar{j}} = g_{i\bar{j}}, \quad N_{i\bar{j}} = u_{i\bar{j}}.
$$

Then,

$$
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{Tr}(M^{-1}N)^k - (n + 1)u = F.
$$

First, we consider $\text{Tr}(M^{-1}N)$, which will be proven to be the dominating term.

In the next result, we will adopt the local frame discussed in Appendix [C].
Lemma 3.1. Let \( w \) be a \( C^2 \) function in \( \Omega \) and \( Q \) be an arbitrary point on boundary. Then, in the local frame as in Appendix C at any \( P \) close to boundary with \( d(P) = \text{dist}(P, Q) \),

\[
M_{ij}^{-1} w_{ji} = d^2 (1 + dC_1)(w_{dd} + Y_n^2 w) + (-n - 1)d + d^2 C_d)w_d \\
+ d^2 \sum_{\beta \neq n, 2n} T_{\beta d} Y_{\beta} w_d + d^2 \sum_{\beta \neq n, 2n} T_{n\beta} Y_n Y_{\beta} w + d \sum_{\beta, \gamma \neq n, 2n} T_{\beta \gamma} Y_{\beta} Y_{\gamma} w \\
+ d \sum_{\beta \neq 2n} T_{\beta} Y_{\beta} w.
\]

(3.3)

Proof. We point out that the equation is invariant under a change of complex coordinates \( \{z^i_Q\} \) with the same complex structure. Recall that \( \rho = -\lambda d + O(d^2) \). Since \( d_i = \delta m/2 \), we have

\[
M_{ij}^{-1} = -\rho (\rho^{ij} + \rho^{i\bar{j}} \rho^{\bar{m}m} \rho^{\bar{n}n}) \\
-\rho \frac{1}{\lambda^2} \rho^{i\bar{j}} \rho^{\bar{n}n} + \frac{1}{\lambda^2} \rho^{\bar{n}n} \rho^{\bar{m}m}.
\]

(3.4)

Hence, \( M_{ij}^{-1} \) is \( O(\rho^2) \) if \( i \) or \( j \) = \( n \), and is \( O(\rho) \) otherwise.

In the geodesic coordinates around \( Q \), we have, at point \( P \),

\[
\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = B_{ij} \frac{\partial}{\partial y^i} = B_{pi} \frac{\partial}{\partial y^p} B_{ij} \frac{\partial}{\partial y^j} + B_{mij} \frac{\partial}{\partial y^m} \frac{\partial}{\partial y^j}.
\]

Recall, at \( P \), by (C.1),

\[
[A]_{2n \times 2n} = \begin{pmatrix}
1 - \frac{\partial^2 \varphi}{\partial y^1 \partial y^1} d & -\frac{\partial^2 \varphi}{\partial y^1 \partial y^2} d & \cdots & -\frac{\partial^2 \varphi}{\partial y^1 \partial y^{2n}} d & 0 \\
-\frac{\partial^2 \varphi}{\partial y^2 \partial y^1} d & 1 - \frac{\partial^2 \varphi}{\partial y^2 \partial y^2} d & \cdots & -\frac{\partial^2 \varphi}{\partial y^2 \partial y^{2n}} d & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{\partial^2 \varphi}{\partial y^{2n+1} \partial y^1} d & -\frac{\partial^2 \varphi}{\partial y^{2n+1} \partial y^2} d & \cdots & 1 - \frac{\partial^2 \varphi}{\partial y^{2n+1} \partial y^{2n}} d & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

and by the proof of Lemma 14.17 in [15],

\[
\frac{\partial^2 d}{\partial x^i \partial x^j} = B_{kij} \frac{\partial N_i}{\partial y^k} = -\frac{\partial^2 \varphi}{\partial y^i \partial y^j} + dC_{ij},
\]

for some bounded matrix \( C_{ij} \). We have, for \( 1 \leq i, j < 2n \),

\[
\frac{\partial^2 \rho}{\partial x^i \partial x^j} = e^{-\lambda d} (-\lambda \frac{\partial^2 d}{\partial x^i \partial x^j} + \lambda^2 \frac{\partial d}{\partial x^i} \frac{\partial d}{\partial x^j}) = \lambda \frac{\partial^2 \varphi}{\partial y^i \partial y^j} + dC_{ij},
\]
for some new bounded smooth $C_{ij}$. Thus, we obtain $\rho_{ij}$. Moreover,

$$
\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial^2 \varphi}{\partial y^i \partial y^j} \frac{\partial}{\partial d} + \frac{\partial^2}{\partial y^i \partial y^j} + d \sum_{\alpha, \beta \neq n, 2n} C_{ij}^{\alpha \beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} + d \sum_{\alpha \neq 2n} C_{ij}^\alpha \frac{\partial}{\partial y^\alpha},
$$

where we used, at $P$ and for $i \neq 2n$,

$$
B_{pi} \left( \frac{\partial}{\partial y^p} B_{2n,j} \right) \frac{\partial}{\partial d} = \frac{\partial}{\partial y^i} \left( \frac{A_{j,2n}^*}{\det A} \right) \frac{\partial}{\partial d} + O(d) = \frac{\partial}{\partial y^i} \left( \frac{\partial \varphi}{\partial y^j} \right) \frac{1}{1 + |D\varphi|^2} \frac{\partial}{\partial d} + O(d) = \frac{\partial^2 \varphi}{\partial y^i \partial y^j} \frac{\partial}{\partial d} + O(d).
$$

In summary, we obtain

$$
M_{ij}^{-1} w_{ji} = d^2(1 + dC_1)(w_{dd} + \frac{\partial^2 w}{\partial y^n \partial y^n}) + (-n - 1)d + d^2 C_d w_d
$$

$$
(3.6)
+ d^2 \sum_{\beta \neq n, 2n} T_{\beta d} \frac{\partial w_{dd}}{\partial y^\beta} + d^2 \sum_{\beta \neq n, 2n} T_{\beta d} \frac{\partial^2 w}{\partial y^\beta \partial y^n} + d \sum_{\beta, \gamma \neq n, 2n} T_{\beta \gamma} w_{\beta \gamma}
$$

$$
+ d \sum_{\beta \neq 2n} T_{\beta} \frac{\partial w}{\partial y^\beta}.
$$

Note $Y_i = \partial / \partial y^i$ at $P$. The above form does not change under the local frame. □

Now, we discuss $\sum_{k=2}^{\infty} \text{Tr}(M^{-1} N)^k$. By Lemma [C.3], the norm $C_g^2$ can be defined under the local frame system $\{Y_i\}$.

**Lemma 3.2.** For $k \geq 2$, in the local frame as in Appendix C, Tr$(M^{-1} N)^k$ is a polynomial in

$$
(3.7)
d^2 u_{dd}, d^2 Y^n u, d^2 Y^n Y^n, d^2 Y^n Y^n Y^n, u, d Y^n Y^n, u, d Y^n Y^n Y^n, u,
$$

for $\beta, \gamma \neq n, 2n, 1 \leq i \leq 2n$, with smooth coefficients. Moreover,

$$
(3.8)
\text{Tr}(M^{-1} N)^k \leq C^k |u|_{C_g^2}^k,
$$

for some positive constant $C$.

By Lemma [3.2] $\sum_{k=2}^{\infty} \text{Tr}(M^{-1} N)^k$ is analytic in $d^2 Y^n u, d^2 Y^n Y^n$ and terms in (3.7) in a domain $U$ if $C\|u\|_{C_g^2(U)} < 1$.

**Proof.** In the local frame and same setting as in Lemma [3.1] for fixed point $P$, we have (3.3). Hence, $M_{ij}^{-1}$ is $O(\rho^2)$ if $i$ or $j = n$ and is $O(\rho)$ otherwise. Using this fact, we can
count the power of \( \rho \) (or \( d \)) in each term of the summation,

\[
Tr((M^{-1}N)^k) = \sum_{i_1, \ldots, i_k} M^{-1}_{i_1} u_{i_1} \cdots M^{-1}_{i_k} u_{i_k},
\]

depending on how many lower indices in each term equal \( n \).

First, we have a factor \( d^k \) in the term

\[
M^{-1}_{i_1} u_{i_1} \cdots M^{-1}_{i_k} u_{i_k},
\]

(There is no summation above.) If only one of these lower indices is \( n \), we have an extra \( d \) factor. After that, each time we insert two \( n \)'s in lower indices, we add at least one extra \( d \) factor. So for \( k \geq 2 \), \( Tr((M^{-1}N)^k) \) is a linear combination of

\[
(d^2 u_{n\beta})^{k_1} (d^3 u_{n\alpha})^{k_2} (d^3 u_{\beta\gamma})^{k_3} (d u_{\alpha\beta})^{k_4},
\]

for \( k_1 + \cdots + k_4 > 2 \). Fefferman pointed out that a logarithmic term \( d^{n+1} \log d \) is needed to find

\[
u_{n+1} = c_0 + c_1 d + \cdots + c_n d^n + c_{n+1,1} d^{n+1} \log d
\]

such that

\[
Q(v_{n+1}) = O(d^{n+2} \log d).
\]

Note that \( c_{n+1} \) is nonlocal and relies on the entire boundary \( \partial \Omega \). If \( c_{n+1} \) is known by other methods, then for any \( k > n+1 \), we can find

\[
u_k = \sum_{i=0}^n c_i d^i + \sum_{i=n+1}^k N_i \sum_{j=0}^i c_{i,j} d^i (\log d)^j
\]

such that

\[
Q(u_k) = O(d^{k+1}).
\]

Such a formal calculation is local, so we do it near a smooth portion of boundary \( \Gamma \).
4. Basic Estimates

In this section, we derive some basic estimates.

Let \( \psi = u_1 \) be the polynomial of degree 1 from the formal computation, which is smooth up to boundary. Set

\[ u = \psi + v, \]

and rewrite the equation (2.3) as

\[
\log \det(g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}} + v_{\bar{i} \bar{j}}) = (n + 1)(\psi + v) + F - \log \frac{\det(g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}})}{\det(g_{\bar{i} \bar{j}})},
\]

near \( \partial \Omega \) where \( g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}} > \frac{1}{2} g_{\bar{i} \bar{j}} \). Set

\[ G_{\bar{i} \bar{j}} = g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}}, \]

which is equivalent to \( g_{\bar{i} \bar{j}} \). The right-hand side in (4.1) is given by \( (n + 1)v + F \) for a \( C^3 \)-function \( F_1 \) near \( \partial \Omega \). We now derive some estimates of \( v \).

**Lemma 4.1.** For every \( 0 < \epsilon < 1 \), there holds

\[ |v| \leq C|\rho|^{1+\epsilon}. \]

**Proof.** We first prove \( |v| \leq C|\rho| \). By Theorem 2.1, \( v \) is uniformly bounded. For some small \( r \) to be fixed such that the frame system \( \{Y_i\} \) is well-defined in \( \{0 \leq d < r\} \), we take a constant \( b \) such that

\[ b > \frac{\|v\|_{L^\infty}}{1 - e^{-\lambda r}}. \]

Then, \( v + bp < 0 \) for \( d \geq r \). Consider the function

\[ \sigma = v + bp. \]

If \( \sigma \geq 0 \) somewhere, it must occur near boundary. Take a sequence of points \( \{p_i\} \) such that \( \sigma(p_i) \) approaching the maximum of \( \sigma \). By the maximum principal on noncompact manifolds by Cheng-Yau [9], we have

\[ \lim \sup \nabla^2 \sigma(p_i) \leq 0, \]

and hence, by \( \rho_{\bar{i} \bar{j}} > 0 \),

\[ \lim \sup \frac{\det(g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}} + v_{\bar{i} \bar{j}})}{\det(g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}})}(p_i) \leq \lim \sup \frac{\det(g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}} + \sigma_{\bar{i} \bar{j}})}{\det(g_{\bar{i} \bar{j}} + \psi_{\bar{i} \bar{j}})}(p_i) \leq 1. \]

By (4.1), we get

\[ \lim \sup e^{(n+1)v + F_1 d^2}(p_i) \leq 1, \]

and, by the definition of \( \sigma \),

\[ \lim \sup ((n + 1)\sigma - (n + 1)bp + F_1 d^2)(p_i) \leq 0. \]

Take \( b \) large such that \(- (n + 1)bp + F_1 \rho^2 > 0 \). Therefore,

\[ \lim \sigma(p_i) \leq 0. \]
Thus, \( v \leq -bp \). For the other direction, we set \( \sigma_1 = v - bp \) and apply the same method for a sequence of points \( \{ q_i \} \) approaching the minimum point of \( \sigma_1 \) to get \( \sigma_1 \geq 0 \).

After getting \( |v| \leq C|\rho| \), the rest is a standard maximum principle argument. In fact, we use the test functions

\[
M_\pm = \pm b(-\rho)^{1+\epsilon},
\]

and apply the maximum principle to equation (4.1) on \( \Omega \), using Lemmas 3.1 and 3.2 which still hold for \( M_{i\bar{j}} = G_{i\bar{j}} \) and \( N_{i\bar{j}} = v_{i\bar{j}} \) as (4.2). The only difference is that we need to replace \( (\rho_{i\bar{j}}) \) by \( (H_{i\bar{j}}) \), which is the inverse of the matrix,

\[
H_{i\bar{j}} = \rho_{i\bar{j}} - \rho v_{i\bar{j}}.
\]

Then,

\[
M_{i\bar{j}}^{-1} = -\rho(H_{i\bar{j}}^{-1}) + \frac{H_{i\bar{j}}^{-1} H_{m\bar{j}} \rho_{m\bar{m}}}{\rho - \rho \rho_{m\bar{m}}}
= -\rho\left(\frac{H_{i\bar{j}}^{-1} \rho - \frac{1}{4} \lambda^2 H_{i\bar{j}}^{-1} H_{n\bar{n}} + \frac{1}{4} \lambda^2 H_{i\bar{n}}^{-1} H_{n\bar{j}}^{-1}}{\rho - \frac{1}{4} \lambda^2 H_{n\bar{n}}^{-1}}\right).
\]

This ends the proof. \( \square \)

Next, we consider expansions near \( \Gamma \subset \partial \Omega \). Around a point \( P \in \Gamma \times \{ 0 \leq d \leq r \} \), we set

(4.2) \[ M_{i\bar{j}} = G_{i\bar{j}}, \quad N_{i\bar{j}} = v_{i\bar{j}}, \]

and consider the equation of \( v \) given by

(4.3) \[ \log \det(I + G^{-1} \cdot D^2v) - (n + 1)v = F_1 d^2. \]

We use coordinates \( \{ \nu_1, \cdots, \nu_n \} \) around point \( P \in \Gamma \times \{ 0 \leq d \leq r \} \). Then,

(4.4) \[ \|v\|_C^{2\alpha,(B(x,1/2))} \leq C(\|v\|_L^\infty + C_1 d^2) \leq C_2 d^{1+\epsilon}. \]

By Lemma [C.3], it implies the weighted estimate

\[
|v| + d|Y_n v| + d|v_d| + \sqrt{d}|Y_1 v|
+ d^2|Y_n^2 v| + d^2|v_{dd}| + d|Y_1 Y_2 v| + d^2|Y_n Y_1 v| + d^2|Y_1 v_d| \leq C_3 d^{1+\epsilon},
\]

for \( i, j \neq n, 2n \).

Using the fact that Lemma 3.1 and Lemma 3.2 still hold for \( M_{i\bar{j}} \) and \( N_{i\bar{j}} \) defined as (4.2), locally the main equation can be expressed as

\[
d^2(1 + C_1 d)(v_{dd} + Y_n^2 v) + (- (n - 1)d + C_2 d^2)v_d
+ d^2 \sum_{\beta \neq n, 2n} T_{\beta\alpha} Y_\alpha v_d + d^2 \sum_{\beta \neq n, 2n} T_{\alpha\beta} Y_\alpha Y_\beta v
+ d^2 \sum_{\beta, \gamma \neq n, 2n} T_{\beta\gamma} Y_\beta Y_\gamma v
\]

\[
+ d \sum_{\beta \neq 2n} T_{\beta\gamma} Y_\beta v - (n + 1)v = F_1 d^2 + F_2,
\]

(4.5)
where \( C_1, C_d, T_{\beta d}, T_{\beta} \) are smooth functions, and \( F_2 \) can be written as a convergent series in

\[
(4.6) \quad d^2 v_{d d}, d^2 Y_n^2 v, d^2 Y_{\beta d} v, d^2 Y_{\alpha d} v, d Y_{\beta d} Y_{\gamma} v, d Y_{\beta} Y_{\gamma} v, d Y_i v, \]

for \( 0 \leq l < p, \beta, \gamma \neq n, 2n, 1 \leq i \leq 2n \), with smooth coefficients in \( x \). (We may need to replace \( d^2 \) in \( (4.6) \) by \( d^2 \) if necessary.) In fact, every monomial in \( F_2 \) contains at least two terms in \( (4.6) \). We may have \( d^2 Y_n Y_{\beta} v \cdot d^2 Y_{\beta} v_d \), or \( d^2 Y_n Y_{\beta} v \cdot d^2 v_{d d} \). By the equation, the latter case should really be \( d^2 Y_n Y_{\beta} v \cdot d^2 v_d \).

5. Tangential Estimates

In this section, we derive estimates along tangent directions. Set \( w = Y_\alpha v \), for \( 1 \leq \alpha \leq 2n - 1 \). Applying \( Y_\alpha \) to \( (3.2) \) yields

\[
\sum_{k=1}^{\infty} (-1)^{k-1} Tr((M^{-1} N)^{k-1}(M^{-1} Y_\alpha N)) - (n + 1)w
\]

(5.1)

\[
= Y_\alpha F_1 d^2 - \sum_{k=1}^{\infty} (-1)^{k-1} Tr((M^{-1} N)^{k-1}(Y_\alpha M^{-1} \cdot N)).
\]

In the left-hand side, the coefficients from \( k = 1 \) in form of Lemma 3.1 are dominating coefficients. Then, the main equation is linearized.

For some local tangential vector \( Y_{\alpha l}, l = 1, \cdots, p \), set

\[
w^p = Y_\alpha \cdots Y_{\alpha_p} v.
\]

Lemma 5.1. Let \( k \) be a nonnegative integer. Then, for any \( 0 \leq p \leq k \),

\[
\|w^p\|_{C^\alpha_x \left( B(x,1/2) \right)} \leq C d(x)^{1+\epsilon},
\]

and, for \( 1 \leq p \leq k \), \( w^p \) satisfies an equation of the form

\[
\begin{align*}
&d^2 (1 + dC_1)(w^p_{d d} + Y_n^2 w^p) + (-n - 1)d + d^2 C_d)w^p_d \\
&\quad + \sum_{\beta \neq n, 2n} (d^2 T_{\beta d} + d^2 C_{\beta d})Y_{\alpha} w^p_d + \sum_{\beta \neq n, 2n} (d^2 T_{n \beta} + d^2 C_{n \beta})Y_{\gamma} w^p_d \\
&\quad + d \sum_{\beta, \gamma \neq n, 2n} (T_{\beta \gamma} + C_{\beta \gamma})Y_{\beta} Y_{\gamma} w^p_d + d \sum_{\beta \neq 2n} (T_{ij} + C_{ij})Y_{\beta} w^p_d - (n + 1)w \\
&\quad = F_1 d^2 + F_2,
\end{align*}
\]

where \( C_1, C_d, C_{ad}, C_{\alpha}, C_{\alpha \beta} \), and \( F_1 \) are smooth functions in \( x \), and \( F_2 \) is a convergent series in

\[
(5.4) \quad d^2 w_{d d}^l, d^2 Y_n^2 w^l, d^2 Y_{\beta d} w^l, d^2 Y_{\gamma} w^l, d Y_{\beta} Y_{\gamma} w^l, d Y_i w^l,
\]

for \( 0 \leq l < p, \beta, \gamma \neq n, 2n, 1 \leq i \leq 2n \), with smooth coefficients (with \( d^2 \) in \( (5.4) \) replaced by \( d^2 \) if necessary). Every monomial in \( F_2 \) contains at least one term in \( (5.4) \).
Proof. We prove by an induction on $k$. For $k = 0$, the desired results follow by (4.4). We assume they hold for $k - 1$ and proceed to prove for $k$.

If $k = 1$, $w^k$ satisfies an equation of form (5.3) by applying Lemma 3.1 and Lemma 3.2 to (5.1). If $k > 1$, note that $w^{k-1}$ satisfies an equation of form (5.3) by induction. Taking $Y_\alpha$ of (5.3), for $1 \leq \alpha \leq 2n - 1$, we derive an equation of $w^k = Y_\alpha w^{k-1}$ of the form of (5.3). In both cases, we only need to consider the terms generated by the exchange of orders in $dY_\alpha \beta \gamma w^{k-1}$, which may not be bounded by $\|w^{k-1}\|_{C^2_G}$.

Case 1: $\alpha = n$ and $w^k = Y_n w^{k-1}$. By an exchange of the orders of vector fields, we have

$$dY_n \beta Y_\gamma w^{k-1} = dY_\beta Y_n Y_\gamma w^{k-1} + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1},$$

where, if $\theta = n$, we have a term not bounded by $\|w^{k-1}\|_{C^2_G}$. We can continue and get

$$dY_n \beta Y_\gamma w^{k-1} = dY_\beta Y_n w^k + d \sum_{\theta \neq 2n} Y_\beta (g_\theta Y_\theta w^{k-1}) + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$$

$$= dY_\beta Y_\gamma w^k + d \sum_{\theta \neq 2n} Y_\beta (g_\theta Y_\theta w^{k-1}) + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$$

$$+ dY_\beta (g_n w^k) + df_n Y_\gamma w^{k-1}$$

$$= dY_\beta Y_\gamma w^k + d \sum_{\theta \neq 2n} Y_\beta (g_\theta Y_\theta w^{k-1}) + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$$

$$+ dY_\beta (g_n w^k) + df_n Y_\gamma w^k + d \sum_{\theta \neq 2n} (f_\theta h_\theta) Y_\theta w^{k-1},$$

where $dY_\beta Y_\gamma w^k, dg_\theta Y_\beta w^k, df_\theta Y_\gamma w^k$ are regarded as main terms and the rest are bounded by $\|w^{k-1}\|_{C^2_G}$. We conclude that $w^k$ satisfies the equation (5.3), with $F_2$ as series in (5.4), for $1 \leq l < k$.

By induction, we have

$$\|w^{k-1}\|_{C^2_G(B(x,1/2))} \leq Cd(x)^{1+\epsilon},$$

and hence $w^k = O(d^k)$. Consider a function

$$M = a|y|^2 d^\epsilon + bd^{1+\epsilon}.$$

As in [18], we get

$$|w^k| \leq Cd^{1+\epsilon}.$$

Then, by the Schauder estimate, we obtain

$$\|w^k\|_{C^2_G(B(x,1/2))} \leq |w^k|_{L^\infty} + C(\|w^1\|_{C^2_G(B(x,1))} + \cdots + \|w^{k-1}\|_{C^2_G(B(x,1))} + d^2)$$

$$\leq Cd(x)^{1+\epsilon}.$$
Case 2: $Y_\alpha \neq Y_n$ and there is a $Y_n$ in the expression of $w^{k-1}$, for example, $w^{k-1} = Y_n w^{k-2}$ for some $w^{k-2}$. Then,

$$w_k = Y_\alpha w^{k-1} = Y_n (Y_\alpha w^{k-2}) + \sum_{\theta \neq 2n} f_\theta Y_\theta w^{k-2},$$

where the first term is discussed in the first case and the second term is done by induction.

Case 3: $Y_\alpha \neq Y_n$ and $w^{k-1}$ does not contain $Y^n$ in its expression. Assume $w^{k-1} = Y_\lambda w^{k-2}$ for $\lambda \neq n, 2n$. By an exchange of orders of differentiations, we get

$$d Y_\alpha Y_\beta Y_\gamma w^{k-1} = d Y_\beta Y_\alpha Y_\gamma w^{k-1} + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$$

$$= d Y_\beta Y_\alpha w^{k} + d \sum_{\theta \neq 2n} Y_\beta (g_\theta Y_\theta w^{k-1}) + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}.$$  

We now consider the last two terms. Taking $d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$ for example, we have

$$d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1} = d f_\theta Y_\theta Y_\gamma w^{k-1} + d \sum_{\theta \neq 2n, 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$$

$$= d f_n Y_n Y_\gamma w^{k-1} + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1}$$

$$= d f_n Y_n Y_\gamma w^{k-1} + d \sum_{\theta \neq 2n} f_n g_\theta Y_\theta w^{k-1} + d \sum_{\theta \neq 2n} f_\theta Y_\theta Y_\gamma w^{k-1},$$

where $d f_n Y_n Y_\gamma w^{k-1} = d f_n Y_n (Y_n w^{k-1})$ is considered in the first case, where we proved $Y_n w^{k-1}$ satisfies (5.2). The rest terms are estimated by induction for $p = k - 1$. As earlier, we can apply the maximum principle.

We now finish the proof of tangential regularity.  

\[\square\]

6. Regularity and Expansions along the Normal Direction

In this section, we derive estimates along the normal direction and study expansions of solutions.

The equation (4.5), expressed in a frame system, is fully nonlinear. The term $v_{dd}$ appears in the right-hand side of (4.5), which in not present in quasilinear equations in (18) and (19).

We introduce a variable new variable $t$ such that

$$d = \frac{1}{2} t^2.$$ 

Then,

$$\frac{\partial}{\partial d} = t^{-1} \frac{\partial}{\partial t}, \quad \frac{\partial^2}{\partial d^2} = t^{-2} \frac{\partial^2}{\partial t^2} - t^{-3} \frac{\partial}{\partial t}.$$  

In the Euclidean metric, we get from (5.2), for any $p \geq 0$,

$$\|D^p_{Y} v_{tt}\|_{C^0(G(B_G(x,1/2)))}, \|D^p_{Y} v_{tt}\|_{C^0(G(B_G(x,1/2)))}, \|D^p_{Y} v_{tt}\|_{C^0(G(B_G(x,1/2)))} \leq C_p.$$
where $B_G(x, 1/2)$ is a ball with radius $1/2$ in the metric $G$. The advantage here is that all these estimates are under Euclidean coordinates. Here, $C^*_G$ is the weighted Hölder space with respect to the metric $G$.

We rewrite equation (3.2) as

$$v_{tt} - (2n - 1)\frac{v_t}{t} - 4(n + 1)\frac{v}{t^2} = \overline{F},$$

where $\overline{F}$ is a function of $y', t, v/t, v_t, tv_t, D_Yv, D^2_Yv$, and in fact, given by

$$\overline{F} = -C_1tv_{tt} - (n - 1)tC_d v_t$$

$$+ t \sum_{\beta \neq n, 2n} T_{\beta}Y_\beta v_t + t^2 \sum_{\beta \neq n, 2n} T_{n\beta}Y_nY_\beta v + \sum_{\beta, \gamma \neq n, 2n} T_{\beta\gamma}Y_\beta Y_\gamma v$$

$$+ \sum_{\beta \neq 2n} T_{Y_\beta v} + F_1 t^2 - t^{-2} F_2,$$

where $F_1$ and $F_2$ are as in Lemma 5.1. Now we discuss $v_{tt}$ in $\overline{F}$. We point out that $F_2$ contains $v_{tt}$. Set

$$v_1 = v' - 2\frac{v}{t},$$

and hence

$$tv''' = tv'' - 2t(\frac{v}{t})''.$$\hspace{1cm}(6.3)

Then we differentiate (6.2) with respect to $t$, substitute $tv'''$ by (6.3), and then move the generated terms that contains $tv''_t$ (also from $\frac{d}{dt} F_2$) to the left-hand side. A further simplification yields

$$v''_1 - (2n - 3)\frac{v'_1}{t} - (6n + 3)\frac{v_1}{t^2} = F,$$

where

$$F = F(y', t, \frac{v}{t}, v_t, \frac{v_1}{t}, tv_t, D^2_Y v', D^2_Y v'_1).$$

Then, we can follow the quasilinear case as in [18] to develop the boundary expansion.

7. Global Convergence for Analytic Boundary

In this section, we discuss the convergence of the boundary expansions. Consider the following boundary expansions we derived in earlier sections:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{N_i} c_{i,j} t^i (\log t)^j.$$\hspace{1cm}(6.4)

Under the assumption that the entire boundary $\partial \Omega$ is analytic, we prove the convergence of this series.

For technical reasons, we need to assume that $\partial \Omega$ is parallel; namely, the tangent bundle of $\partial \Omega$ is trivial. By Stiefel’s theorem, all oriented three manifolds are parallel. Hence, the assumption that $\partial \Omega$ is parallel is redundant in $C^2$. 


By Remark C.3, the choice of the frame system is flexible. When a manifold is parallel, its local forms \( \{ Y_i \} \) can be defined globally. Note that \( Y_n \) is globally defined. We can pull back the metric on \( T(\partial\Omega) \) to \( \partial\Omega \times \mathbb{R}^{2n-1} \), and introduce the orthogonal group actions \( O(n) \) on \( T(\partial\Omega) \). Then, for all \( \alpha \neq n, 2n \), we can define \( Y_\alpha = T_\alpha Y_n \), for some \( T_\alpha \in O(n) \) that rotates at an angle \( \frac{\pi}{2} \). Meanwhile, we make all these vector fields orthogonal to each other.

Set
\[
\theta = \frac{1}{2\sqrt{-1}}(\partial \rho - \bar{\partial} \rho),
\]
and
\[
(7.1) \quad H(\partial\Omega) = \text{span}\{ Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_{2n-1} \}.
\]

Then, \( (\partial\Omega, H(\partial\Omega), \theta) \) is a CR-manifold as in Appendix B.

**Lemma 7.1.** Let \( T \) be defined in Appendix B. Then under the new frame system,
\[
\tilde{Y}_n = T, \quad \tilde{Y}_\alpha = Y_\alpha \text{ for } \alpha \neq n, 2n,
\]
Lemma 3.1 and Lemma C.4 still hold.

**Proof.** For any interior point \( P \) near boundary, we fix \( Q \in \partial\Omega \) such that \( d(P, Q) = d(P, \partial\Omega) \). Under the coordinates \( \{ z^Q_i \} \), we have
\[
\theta = \text{Im}(\partial \rho) = \frac{\partial \rho}{\partial x_{i+n}} dx_i - \frac{\partial \rho}{\partial x_i} dx_{i+n},
\]
which is \( -\lambda dx_n \) at \( P \), for \( \lambda \) as in (2.1). Thus on \( M = \partial\Omega \), \( \theta(Y_n) = -\lambda \) and \( \theta(Y_\alpha) = 0 \) for \( \alpha \neq n, 2n \). Hence,
\[
H(M) = \text{span}\{ Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_{2n-1} \},
\]
and \( Y_n + \lambda T \in H(M) \). Therefore,
\[
Y_\alpha = \tilde{Y}_\alpha \quad \text{if } \alpha \neq n, 2n,
\]
\[
Y_n = -\lambda \tilde{Y}_n + \sum_{\alpha \neq n, 2n} f_\alpha \tilde{Y}_\alpha,
\]
for some functions \( f_\alpha \). Then, the form of (3.3) still holds, since
\[
d^2 Y_n w = d^2 \lambda^2 \tilde{Y}_n^2 + d^2 \{ \ldots \},
\]
where omitted terms do not affect the form of (3.3) with the presence of the \( d^2 \) factor. Lemma C.4 follows similarly. \( \square \)

In the rest of this paper, we replace the frame system \( \{ Y_i \} \) by \( \{ \tilde{Y}_i \} \), but still denote it by \( \{ Y_i \} \). We proceed to prove the convergence for \( 0 \leq t \leq r \), with some \( r > 0 \), if the entire boundary \( \partial\Omega \) is analytic.

First, we prove that \( v \) is tangentially analytic. We assume \( \partial\Omega \) is analytic and
\[
[Y_i, Y_j] = S^m_{ij} Y_m,
\]
where $S_{ij}^m$ are analytic functions on $\partial \Omega$. By Lemma 3.1 we have for $\alpha \neq n, 2n$,
\begin{equation}
S_{n\alpha}^\alpha = 0.
\end{equation}
This implies the Lie bracket of $Y_\alpha, Y_\alpha$ does not include a $Y_\alpha$ term. There are positive constants $D$ and $B > 0$ such that, for all $k \geq 0$ and $1 \leq i, j, m \leq 2n - 1$,
\begin{equation}
\|D^k v, S_{ij}^m\|_{C^\alpha} \leq DB^{k-1}(k-3)!,
\end{equation}
where $! = 1$ if $l$ is an negative integer, and the $C^\alpha$-norm is defined by the metric on $\partial \Omega$.
We may also use
\begin{equation}
\|D^k v, S_{ij}^m\|_{C^\alpha} \leq DB^k(k-2)!.\end{equation}

**Theorem 7.2.** Assume $\partial \Omega$ is analytic and a global frame system $\{Y_i\}$ is defined as above. Then,
\begin{equation}
|D_{ij}^l v|_{G}(x) \leq DB^{l-1}(l-1)!d(x)^{1+\epsilon},
\end{equation}
where $B_g(x, 1/2)$ is the metric ball centered at $x$ with radius 1/2 under $g$, and $D_{ij}^l v$ denotes the maximum of the norms of all tangential derivatives of $v$ of order $l$.

**Proof.** We prove (7.4) by a similar method for the tangential regularity and aim to derive estimates of analyticity type for higher order derivatives. No shrinking technique as in [19] is needed. However, we have to deal with the exchange of orders of vector fields.

We prove inductively (7.4) and the following two estimates:
\begin{equation}
|d(x)D_{ij}^l v| \leq DB^{l-1}(l-1)!d(x)^{1+\epsilon},
\end{equation}
where $i = l + 1$ if exactly one of the $Y_i$'s in (7.5) is not $Y_\alpha$ and $i = l + 2$ if at least two of the $Y_i$'s in (7.5) are not $Y_\alpha$, and
\begin{equation}
|d(x)^2 D_{ij}^{l+2} v| \leq DB^{l-1}(l-1)!d(x)^{1+\epsilon},
\end{equation}
if exactly one of the $Y_i$'s in (7.5) is not $Y_\alpha$. Note that all $Y_\alpha$ case, which is not included in (7.5), is covered by (7.4).

Set $\Omega_r = \{0 < d < r\}$. As $v$ is analytic in $\Omega$, we assume (7.4) holds in $\Omega_r \setminus \Omega_{r/2}$ for all $l \in \mathbb{Z}$, for a fixed small $r$.

Assume (7.4), (7.5), and (7.6) hold for all $l$ with $l < p$.

First, we prove
\begin{equation}
|D^p v| \leq DB^{p-1}(p-1)!d^{1+\epsilon}.
\end{equation}
By applying $D^p v$ to (5.1), we have
\begin{equation}
\tilde{G}^{ij} D^p Y_i Y_j v + P_i dD^p v, D^p v - (n+1)D^p v = H_p,
\end{equation}
where $\tilde{G}^{ij}$ is the real metric obtained from the coefficient matrix of $Y_\alpha N$ in the linearized equation (5.1), and $H_p$ denotes the rest terms, which involve at most $(p+1)$th derivative of $v$. In the left-hand side of (7.7), we need to exchange the order of derivatives, from $D^p Y_i Y_j v$ to $Y_i Y_j D^p v$. In the right-hand side, we can discuss $H_p$ in a similar way as in [19], using (7.5) and (7.6) for the exchange of orders of derivatives. We will concentrate on the left-hand side and skip the discussion of the right-hand side.
We need to analyze \((p + 1)\)th derivatives. Taking \(dd^p Y_1 Y_2 v\) for example, we have to change it to \(dY_1 Y_2 d^p v\), which results in \(2p\) terms of \((p+1)\)th order tangential derivatives. We will prove that they are bounded by the inductive estimates \((7.4)\) and \((7.5)\) in several cases.

Case 1: \(d_Y^p = d_Y^{p-1}\). We first prove a lemma.

**Lemma 7.3.** Assume the estimates \((7.3)\), \((7.4)\), and \((7.5)\) hold for \(1 \leq l \leq p - 1\). Then,
\[
\| d_Y^p \alpha_Y Y_\beta^p v - d_Y^l \alpha_Y Y_\beta^l v \|_{L^\infty(B_y(x, 1/2))} \leq CB^{p-2}(p - 1)!d(x)^{1+\epsilon}.
\]

**Proof.** From \(d_Y^p \alpha_Y Y_\beta^p v\) to \(d_Y^l \alpha_Y Y_\beta^l v\), we generate terms of the form, with \(p_1 + p_2 = p\),
\[
(7.8) \quad d_Y^{p_1-1}(S_{\alpha x}^m \cdot Y_m d_Y^{p_2} Y_\beta^p v),
\]
which is actually generated from \(d_Y^{p_1} \alpha_Y Y_\beta^p v\) to \(d_Y^{p_1-1} \alpha_Y Y_\beta^{p+1} v\). Similarly, from \(d_Y^p \alpha_Y Y_\beta^p v\) to \(d_Y^l \alpha_Y Y_\beta^l v\), we generate terms of the form, with \(p_1 + p_2 = p\),
\[
(7.9) \quad d_Y^l \alpha_Y Y_\beta^{p+1} v.
\]

By \((7.2)\), \(Y_m\) cannot be \(Y_n\) in \((7.8)\) and \((7.9)\).

Since both \(m\) and \(\beta\) are not \(n\) in \((7.8)\), by \((7.5)\), we get
\[
d\| d_Y^{p_1-1}(S_{\alpha x}^m \cdot Y_m d_Y^{p_2} Y_\beta^p v) \|_{L^\infty(B_y(x, 1/2))}
\leq d \sum_{k=0}^{p_1-1} C_{p_1-1}^k \| (D_Y^k S_{\alpha x}^m)(d_Y^{p_1-1-k}Y_m d_Y^{p_2} Y_\beta^p v) \|_{L^\infty(B_y(x, 1/2))}.
\]
\[
\leq \sum_{k=0}^{p_1-1} C_{p_1-1}^k (DB^k(k - 2)!)(DB^{p_1-2-k}(p_1 + p_2 - 2 - k)!)(p_1 + p_2 - 2 - k)!d(x)^{1+\epsilon}
\]
\[
= \sum_{k=0}^{p_1-1} C_{p_1-1}^k (k - 2)!(p - 2 - k)!D^2 B^{p_1-2} d(x)^{1+\epsilon}.
\]
\[
(7.10) \quad \leq \sum_{k=0}^{p_1-1} C_{p_1-1}^k (k - 2)!(p_1 - 2 - k)! \frac{(p - 2 - k)!}{(p_1 - 2 - k)!} D^2 B^{p_1-2} d(x)^{1+\epsilon}.
\]

Note, for any \(0 \leq k \leq p_1 - 1\),
\[
\frac{(p - 2 - k)!}{(p_1 - 2 - k)!} \leq \frac{(p - 2)!}{(p_1 - 2)!},
\]
and
\[
\sum_{k=0}^{p_1-1} C_{p_1-1}^k (k - 2)!(p_1 - 2 - k)! \leq C(p_1 - 2)!,
\]
as in proof of Lemma A.2. Combining all these, we obtain
\[
\sum_{p_1=1}^{p} \| d_Y^{p_1-1}(S_{\alpha x}^m \cdot Y_m d_Y^{p_2} Y_2 v) \|_{L^\infty(B_y(x, 1/2))} \leq CB^{p-2}(p - 1)!d(x)^{1+\epsilon}.
\]
This finishes the estimate of terms in the form of (7.8).

To estimate (7.9), we have to consider two types of terms: applying $Y_\alpha$ to $S_{ij\beta}^m$, or to $Y_mD_{Y_n}^{p_2}v$. Then,

$$
\begin{align*}
&d\|Y_\alpha D_{Y_n}^{p_1-1}(S_{ij\beta}^m, Y_m D_{Y_n}^{p_2}v)\|_{L^\infty(B_{\theta}(x,1/2))} \\
&\leq d \sum_{k=0}^{p_1-1} C_{p_1-1}^k \|Y_\alpha D_{Y_n}^k S_{ij\beta}^m(D_{Y_n}^{p_1-1-k}Y_mD_{Y_n}^{p_2}v)\|_{L^\infty(B_{\theta}(x,1/2))} \\
&+ d \sum_{k=0}^{p_1-1} C_{p_1-1}^k \|D_{Y_n}^k S_{ij\beta}^m(Y_\alpha D_{Y_n}^{p_1-1-k}Y_mD_{Y_n}^{p_2}v)\|_{L^\infty(B_{\theta}(x,1/2))}.
\end{align*}
$$

We only need to estimate the first term. We have

$$
\begin{align*}
&d \sum_{k=0}^{p_1-1} C_{p_1-1}^k \|Y_\alpha D_{Y_n}^k S_{ij\beta}^m(D_{Y_n}^{p_1-1-k}Y_mD_{Y_n}^{p_2}v)\|_{L^\infty(B_{\theta}(x,1/2))} \\
&\leq \sum_{k=0}^{p_1-1} C_{p_1-1}^k (DB^k(k-2)!(DB^{p_2-2-k}(p_1 + p_2 - 2 - k)!)d(x)^{1+\epsilon} \\
&= \sum_{k=0}^{p_1-1} C_{p_1-1}^k (k-2)!(p-2-k)!D^2 B^{p_2-2}d(x)^{1+\epsilon},
\end{align*}
$$

which is the same as (7.10), and already estimated. □

To continue, we consider the test function

$$M(x) = bd^{1+\epsilon} \quad \text{in } \Omega_\epsilon,$$

and obtain

$$|D_{Y_n}^p v| \leq DB^{p-1}(p-1)!d^{1+\epsilon}. $$

We apply the interior $C^{1,\alpha}$- and $C^{2,\alpha}$-estimates in $B_{\theta}(x,1/2)$. Note that Lemma 7.3 still holds if we replace the $L^\infty$-norm by the $C^\alpha$-norm. Hence, (7.11) follows for case $l = p$ and $D_{Y_n}^p = D_{Y_n}^p$.

**Case 2:** $D_{Y_n}^p v$ has exactly $(p-1)$ $Y_n$’s. We first consider $D_{Y_n}^p = Y_\theta D_{Y_n}^{p-1}$, for some $\theta \neq n, 2n$. We need to consider terms in (7.8) and (7.9) with $D_{Y_n}^{p_1-1}$ replaced by $Y_\theta D_{Y_n}^{p_1-2}$, and also terms of the following form

$$
\begin{align*}
&dS_{\theta,\alpha}^p D_{Y_n}^p Y_\beta v, \quad dY_\alpha(S_{\theta,\beta}^p D_{Y_n}^p v). 
\end{align*}
$$

The terms in (7.11) come from the exchange of $Y_\theta$ with $Y_\alpha$ and $Y_\beta$, and cannot be estimated as in Lemma 7.3. We can change them to $dY_\alpha D_{Y_n}^p v$ and $dY_\beta D_{Y_n}^p v$, plus a proper number of $p$th order derivatives. Note that $dY_\alpha D_{Y_n}^p v$, $dY_\beta D_{Y_n}^p v$ are already estimated in Case 1. If $D_{Y_n}^p = Y_n Y_\theta D_{Y_n}^{p-2}, \cdots, D_{Y_n}^{p_1-1} Y_\theta$, the discussion is similar. Then, we can apply the maximum principle and Schauder estimates to derive (7.11).

**Case 3:** $D_{Y_n}^p v$ has at most $(p-2)$ $Y_n$’s. Note that all $(p+1)$th derivatives generated by the change from $D_{Y_n}^p, D_{Y_\alpha} D_{Y_\beta} v$ to $dD_{Y_n}^p D_{Y_\alpha} D_{Y_\beta} v$ have at least 2 non-$Y_n$ vector fields.
in its form. We can estimate these terms as estimating (7.3) in the proof of Lemma 7.3. Then, we can apply the maximum principle and Schauder estimates to derive (7.4).

Finally, we can verify (7.5) and (7.6), using (7.4) and estimates as in Lemma 7.3.

With bigger constants $D$ and $B$, it follows from Theorem 7.2 that, in local coordinates \{\mathbf{y}^i\}, the tangential derivatives satisfy

\[
\|D^l_{\mathbf{y}^i}v\|_{C^2(B_G(x,1/2))} \leq DB^{l-1}(l-1)!d(x)^{1+\epsilon}.
\]

In view of $d = t^2/2$, we have, similar to (6.1),

\[
\|D^p_{\mathbf{y}^i}v_t\|_{L^\infty(B_G(x,1/2))}, \|D^p_{\mathbf{y}^i}v_t\|_{L^\infty(B_G(x,1/2))} \leq DB^{p-1}(p-1)!
\]

The rest follows as in [19].

8. The Local Case and the Gevrey Space

In this section, we discuss behaviors of solutions near a portion of the analytic boundary. We will prove that solutions belong to the Gevrey space of order 2 along tangential directions, which consists of functions $v$ such that

\[
\|D^p_{\mathbf{y}^i}v\|_{L^\infty} \leq DB^p(p!)^2.
\]

In the next section, we will demonstrate that solutions of linear equations with similar structures may not have convergent expansions.

Define

\[
G_R = \{y = (y',d) : |y'| < R, 0 \leq d < R\}.
\]

Assume $v$ is a solution defined in $G_R$. Fix any $r < R$.

We first prove the following result.

**Theorem 8.1.** Assume the boundary portion $\Gamma = G_R \cap \{d = 0\}$ is analytic. Then, for any $y \in G_R$,

\[
\delta_l^{-1}|D^l_{\mathbf{y}^i}v|_{C^2_G}(y) \leq DB^{l-1}(l-1)!d^{1+\epsilon}(y)\tilde{d}^{-(l-1)}(y), \tag{8.1}
\]

where $\delta_l = \tilde{d}/l$, $\tilde{d}$ is the distance function to the cylinder $|y'| = R$, and $D^l_{\mathbf{y}^i}v$ denotes any $l$-th order derivative of $v$ along $Y_1, \cdots, Y_{2n-1}$.

**Proof.** By the interior analyticity, we assume (8.1) holds in $G_R \cap \{r/2 \leq d < R\}$.

Define

\[
T_l = G_R \cap \{0 \leq d < (\frac{\tilde{d}}{l})^2\},
\]

where $\tilde{d}$ is the distance from $y$ to the cylinder $|y'| = R$. So, $T_l$ is a circular cone and shrinks while $l$ increases. In this way, we divide $G_R$ into two parts $G_R = T_l \cup T_l^C$.

We prove (8.1) and the following two estimates by induction:

\[
\delta_l^{-1}|d(y)D^l_{\mathbf{y}^i}v| \leq DB^{l-1}(l-1)!d(y)^{1+\epsilon}\tilde{d}^{-(l-1)}(y), \tag{8.2}
\]
where \( i = l + 1 \) if exactly one of the \( Y' \)'s in (8.2) is not \( Y_n \) and \( i = l + 2 \) if at least two of the \( Y' \)'s in (8.2) are not \( Y_n \), and

\[
\delta l^{-1} |d(y)^3 D l^2 v| \leq DB^{l-1}(l-1)!d(y)^{1+\epsilon}d^{-(l-1)}(y),
\]

if exactly one of the \( Y' \)'s in (8.3) is not \( Y_n \). Note that all other cases, which are not included in (8.2), are covered by (8.1).

Assume

\[
\delta l^{-1} \|D_l Y, v\|_{C^2_{G_l}} \leq DB^{l-1}(l-1)!d^{1+\epsilon}d^{-(l-1)}
\]

holds in \( T_i \) and (8.1), (8.2), (8.3) hold for all \( 0 \leq l \leq p - 1 \). We prove case \( p \) through several steps.

**Step 1.** We prove

\[
|D_p Y, v| \leq \delta_{p+1} DB^{p-1}(p-2)!d^{1+\epsilon}d^{-(p-1)} \quad \text{in } G_R.
\]

Consider any \( y_0 \in T_p \). Without loss of generality, we assume \( y_0 \) is the origin, and find a column \( G_{\delta_{\delta^2}} \) of \( \{y : |y'| < \delta, 0 \leq d < \delta^2\} \subseteq T_{p-1} \) containing \( y_0 \), for \( \delta = d(y_0)/p \).

Note \( d(y_0) < \delta \). Set

\[
M(y) = a|y'|^2d^\epsilon + bd^{1+\epsilon}.
\]

Then,

\[
M(y', \delta^2) = a|y'|^2\delta^{2\epsilon} + b\delta^{1+\epsilon},
\]

\[
M(y', d) = a\delta^2d^\epsilon + bd^{1+\epsilon} \quad \text{if } |y'| = \delta.
\]

By (8.1) for \( l = p - 1 \), we have

\[
|D_p Y, v| \leq \delta^{p+2} DB^{p-2}(p-2)!d^\epsilon d^{-(p-2)}.
\]

If

\[
a \geq \delta^{-(p-1)} DB^{p-2}(p-2)!(\tilde{d}(y_0) - \delta)^{-(p-2)},
\]

\[
b \geq \delta^{-(p-1)} DB^{p-2}(p-2)!(\tilde{d}(y_0) - \delta)^{-(p-2)},
\]

then, \( v \leq M \) on \( \partial G_\delta \). If a point \( y \) close to \( y_0 \) satisfies \( \tilde{d}(y) \geq \tilde{d}(y_0) - \delta \), then

\[
\tilde{d}(y)^{-(p-2)} \leq (\tilde{d}(y_0) - \delta)^{-(p-2)} \leq (\tilde{d}(y_0) - \frac{\tilde{d}(y_0)}{p})^{-(p-2)} \leq C\tilde{d}^{-(p-2)}(y_0),
\]

where \( C \) is some constant independent of \( p \). We require

\[
a \geq \delta^{-(p-1)} CDB^{p-2}(p-1)!\tilde{d}^{p+1}(y_0),
\]

\[
b \geq \delta^{-(p-1)} CDB^{p-2}(p-1)!\tilde{d}^{p+1}(y_0).
\]

To apply the maximum principle as in the proof of Theorem 7.2 in \( G_{\delta_{\delta^2}} \), we set

\[
a = \delta^{-(p-1)} CDB^{p-2}(p-1)!\tilde{d}^{p+1}(y_0),
\]

\[
b = \delta^{-(p-1)} DB^{p-1-\frac{1}{4}}(p-1)!\tilde{d}^{p+1}(y_0).
\]

Different from [19], we need to track the power of \( \delta^{-1} \).
To bound the lower order terms $H_p$ in (7.7), we need a lemma similar to Lemma 7.3.

**Lemma 8.2.** Assume (7.3) holds, and (8.1), (8.2) hold for $1 \leq l \leq p - 1$. Then, at any $y \in G_R$,

$$\delta_p^{-1}|dD_{Y'}^p, Y_\alpha Y_\beta v - dY_\alpha Y_\beta D_{Y'}^p, v| \leq CB^{p-2}(p-1)!d(y)^{1+\epsilon} \tilde{d}^{-(p-1)}(y),$$

where $C$ is a positive constant depending on $D, R$.

**Proof.** Switching from $dD_{Y'}^p, Y_\alpha Y_\beta v$ to $dY_\alpha Y_\beta D_{Y'}^p, v$, we have terms of the form, with $p_1 + p_2 = p$,

$$dD_{Y'}^{p_1-1}(S_{*,\alpha}^m \cdot Y_m D_{Y'}^{p_2} Y_\beta v),$$

and

$$dY_\alpha D_{Y'}^{p_1-1}(S_{*,\beta}^m \cdot Y_m D_{Y'}^{p_2} v),$$

where * is some index from 1 to $2n - 1$. We point out that $m$ can assume the value $n$. This is different from Lemma 7.3.

To estimate (8.5), we first have

$$d|D_{Y'}^{p_1-1}(S_{*,\alpha}^m \cdot Y_m D_{Y'}^{p_2} Y_\beta v)| \leq d \sum_{k=0}^{p_1-1} C_{p_1-1}^k ((D_{Y'}^{k}, S_{*,\alpha}^m) (D_{Y'}^{p_1-k} Y_m D_{Y'}^{p_2} Y_\beta v)).$$

For $k = 0$, we have at least two subindices in $D_{Y'}^{p_1-k} Y_m D_{Y'}^{p_2} Y_\beta v$ which are not $n$. This is because $\beta \neq n$ and, if all $Y_\alpha$'s in $D_{Y'}^{p_1-k} Y_m D_{Y'}^{p_2} Y_\beta v$ equal $Y_n$, then $m \neq n$ by (7.2). For $k > 0$, there may be only one non-$Y_n$ operator in $D_{Y'}^{p_1-k} Y_m D_{Y'}^{p_2} Y_\beta v$. Now, for $k = 0$, we apply the $C^2$ part of the estimate (8.1) with $p - 1$ and, for $k > 0$, we apply the $C^1$ part of the estimate (8.1) with $p - k$. Then,

$$d|D_{Y'}^{p_1-1}(S_{*,\alpha}^m \cdot Y_m D_{Y'}^{p_2} Y_\beta v)|$$

$$\leq 2 \sum_{k=1}^{p_1-1} C_{p_1-1}^k (DB^{k-1}(k-3)!) \cdot \delta_2^{k+1-p} D(B\tilde{d}(y)^{-1})^{p_1-k}(p_1 - k)!d(y)^{1+\epsilon}$$

$$= 2 \sum_{k=1}^{p_1-1} C_{p_1-1}^k (k-3)!(p_1 - k)!D^2 B^{p_2-2} \delta_2^{2-p} d(x)^{1+\epsilon} \tilde{d}(y)^{-(p-2)}$$

$$\leq C(p_1 - 1)! D^2 B^{p_2-2} \delta_2^{2-p} d(y)^{1+\epsilon} \tilde{d}(y)^{-(p-2)},$$

since

$$\sum_{k=1}^{p_1-1} C_{p_1-1}^k (k-3)!(p_1 - k)! \leq \sum_{k=1}^{p_1-1} C_{p_1-1}^k (p_1 - 3)!(p_1 - 1)! \cdot \frac{(p_1 - 1)!}{(p_1 - 1)!}$$

$$\leq \sum_{k=1}^{p_1-1} (p_1 - 1)! \cdot \frac{(k-3)!}{k!} \cdot \frac{(p_1 - 1)!}{(p_1 - 1)!}$$

$$\leq C(p_1 - 1)!. $$
Thus,
\[ \sum_{p_1=1}^{p} |dD_{Y_n}^{p_1-1}(S_{\alpha\alpha} \cdot Y_m D_{Y_n}^{p_2} Y_2 v)| \leq C p! D^2 B^{p-2} \delta_{p}^{2-p} d(y)^{1+\epsilon} \tilde{d}(y)^{-(p-2)} \]
\[ \leq C(p-1)! D^2 B^{p-2} R \cdot \delta_{p}^{1-p} d(y)^{1+\epsilon} \tilde{d}(y)^{-(p-2)}. \]
This finishes the estimate of terms in the form of (8.5).

For (8.6), the discussion is similar. \hfill \Box

Taking B sufficiently larger than C, we have, by maximum principle,
\[ |D^p_{Y^0} v| \leq \delta^{-(p-1)} DB^{p-1} \frac{4}{3} (p-1)! d^{1+\epsilon} \tilde{d}^{-p+1}(y_0), \]
where the extra factor \( B^{-1/3} \) is for a later purpose.

For \( y \in T^C_p = (T_{p-1} \setminus T_p) \cup T^C_{p-1} \), by induction for case \( p-2 \), if we can write \( D^p_{Y^0} v = D^2_{Y_n} D^{p-2}_{Y^0} v \), then
\[ |D^p_{Y^0} v| \leq d^{-2} \delta^{-3} DB^{p-3} (p-3)! d^{1+\epsilon} \tilde{d}^{-p+3} \]
\[ \leq \delta^{-p+1} DB^{p-1} (p-1)! d^{1+\epsilon} \tilde{d}^{-p+1}(y_0), \]
since \( d \geq \delta^2 \). We can discuss the cases \( D^p_{Y^0} v = D_{Y_n} D_{Y^0} D^{p-2}_{Y^0} v \) or \( D^p_{Y^0} v = D_{Y_n} D_{Y^0} D^{p-2}_{Y^0} v \) similarly.

**Step 2.** By gradient estimates for elliptic equations, we have, for fixed \( 0 < \rho' < \rho < 1 \),
\[ \|D^p_{Y^0} v\|_{C^{1,\alpha}_G(B_G(y_0,\rho'))} \leq C(\|D^p_{Y^0} v\|_{L^\infty_G(B_G(y_0,\rho))} + \|H_p\|_{L^\infty(B_G(y_0,\rho))}), \]
and hence
\[ (8.7) \quad \|D^p_{Y^0} v\|_{C^{1,\alpha}_G(B_G(y_0,\rho'))} \leq \delta^{-p+1} DB^{p-1} \frac{4}{3} (p-1)! d^{1+\epsilon} \tilde{d}^{-p+1}(y_0). \]
This also holds if we replace \( p \) by any \( l \leq p \).

Inductively, we can also prove, for \( l \leq p-1 \),
\[ \|D^l_{Y^0} v\|_{C^{2,\alpha}_G(B_G(y_0,\rho'))} \leq \delta^{-l+1} DB^{l-1} \frac{4}{3} (l-1)! d^{1+\epsilon} \tilde{d}^{-l+1}(y_0). \]

We consider \( l = p-1 \) for an illustration. For derivatives other than \( D^2_{ad} D^p_{Y^0} v \), it is implied by (8.7). For \( D^2_{ad} D^p_{Y^0} v \), we consider the equation (5.3) for \( p-1 \) and apply Lemma A.2 and Remark A.3. Detailed discussion can be found in [19].

Now we have
\[ \|H_p\|_{C^{2,\alpha}_G(B_G(y_0,\rho'))} \leq \delta^{-p+1} DB^{p-1} \frac{4}{3} (p-1)! d^{1+\epsilon} \tilde{d}^{-p+1}(y_0). \]
Then, by Schauder estimates,
\[ \|D^p_{Y^0} v\|_{C^{2,\alpha}_G(B_G(y_0,\rho'))} \leq \delta^{-p+1} DB^{p-1} \frac{4}{3} (p-1)! d^{1+\epsilon} \tilde{d}^{-p+1}(y_0). \]
Therefore, we have (8.4) in \( T_p \) for the case \( p \).

**Step 3.** We now prove (8.1) for \( l = p \) in \( T^C_p \). Define
\[ \|v\|_\theta = \sup \{v(y) : y \in T^C_p, \tilde{d}(y) \geq \theta\}. \]
Set
\[ w_{p,k} = D_{Y'}^{-k}D_{Y_n}^{-1}v. \]
Here and hereafter, none of the \( Y' \)'s equals \( Y_n \).

We claim
\[
\| \delta_{p-1} \max_{1 \leq k \leq p} [w_{p,k}]_{C_G^2} \|_\theta \leq \frac{1}{10} \| \delta_{p-1} \max_{1 \leq k \leq p} [w_{p,k}]_{C_G^2} \|_{(1 - \frac{1}{p+2})\theta} + DB^{p-\frac{5}{4}}(p - 1)!\theta^{-p+1}.
\]
We need the following result from [14].

**Lemma 8.3.** Let \( g(\theta) \) be a positive monotone decreasing function, defined in the interval \( 0 \leq \theta \leq 1 \) and satisfying
\[
g(\theta) \leq \frac{1}{10}g(\theta(1 - \frac{1}{n})) + C \frac{\theta}{\theta^{n-3}},
\]
for some \( n \geq 4 \) and \( C > 0 \). Then,
\[
g(\theta) \leq CA\theta^{-n+3},
\]
for some constant \( A \).

By applying Lemma 8.3 to (8.8), we obtain
\[
\| \delta_{p-1} \max_{1 \leq k \leq p} [w_{p,k}]_{C_G^2} \|_\theta \leq DB^{p-\frac{5}{4}}(p - 1)!\theta^{-p+1}.
\]
This finishes the induction. Note that Lemma 8.3 requires \( g(0) \) to be defined. We can shrink \( R \) to achieve this.

Fix a \( \theta \). For any \( y_0 \in T_p^C \) with \( \tilde{d}(y_0) \geq \theta \), set
\[
(8.10) \quad \delta_1 = \min\left\{ \frac{\tilde{d}}{(p + 2)\sqrt{d}}, \frac{\delta^2}{d} \right\}.
\]
Since \( \delta \leq \sqrt{d} \), then
\[
\frac{p}{p + 2} \cdot \frac{\delta^2}{d} \leq \delta_1 \leq \frac{\delta^2}{d}.
\]
Note that if we perturb the variables by a distance \( \delta_1 \) under the metric \( G \), the analyticity estimates still hold. In fact, \( \tilde{d}^p \) will vary by a bound independent of \( p \) and \( \tilde{d} \).

We now estimate derivatives of \( w_{p,k} \).

For \( D_{Y_n}D_{Y'}w_{p,k} \), we consider \( w_{p+1,k} = D_{Y'}w_{p,k} \). We multiply (5.3) for the case \( p + 1 \) by \( \delta_1^2 \) and consider it under \( G/\delta_1^2 \), the metric \( G \) scaled by a factor \( 1/\delta_1^2 \). By (8.1) for the case \( l = p - 1 \) and \( \delta \leq \sqrt{d} \), we have
\[
(8.11) \quad \| w_{p+1,k} \| \leq \delta^{-p+2}DB^{p-2}(p - 2)!d^{-\frac{1}{2}} + \epsilon d^{-\frac{1}{2}} \cdot \delta^{-p+2}.
\]
By gradient estimates for elliptic equations, we obtain, for metric balls centered at \( y_0 \),
\[
\| w_{p+1,k} \|_{C^{1,\alpha}_{G/\delta_1^2}(B_{G/\delta_1^2}(y_0,\rho))} \leq C_1 \left( \| w_{p+1,k} \|_{L^\infty_{G/\delta_1^2}(B_{G/\delta_1^2}(y_0,\rho))} + \delta_1^2 \| H_p+1 \|_{L^\infty_{G/\delta_1^2}(B_{G/\delta_1^2}(y_0,\rho))} \right).
\]
Hence,

\[ \|w_{p+1,k}\|_{C^{1,\alpha}_{G/s_1}^1(B_{G/s_1}^1(y_0,\rho))} \leq \delta^{-p+2} C_1 DB^{p-\frac{3}{2}}(p-2)!d^{-\frac{1}{2}} + \tilde{D}^{-(p+2)}(y_0) + C_1 \delta^2 \|H_{p+1}\|_{L^\infty_{G/s_1}^1(B_{G/s_1}^1(y_0,\rho))}. \]

We note that the power of \( B \) is increased by \( 1/2 \). The extra power was used to control the value change caused by a variation of \( \tilde{d} \).

We now apply \( D_{Y_\gamma}^{p+1-k}D_{Y_n}^k \) to (8.11) and write the resulting equation in a similar form as (7.1), with \( H_{p+1} \) in the right-hand side. There are at most \( C_1 p \) terms of form \( D^2w_{p,k}, D^2w_{p,k-1} \) in \( H_{p+1} \), which are all of the \( (p+2) \)-th derivatives in \( H_{p+1} \). We have estimates of the rest terms in \( H_{p+1} \), just as before. Hence,

\[ \|w_{p+1,k}\|_{C^{1,\alpha}_{G/s_1}^1(B_{G/s_1}^1(y_0,\rho))} \leq \delta^{-p+2} C_1 DB^{p-\frac{3}{2}}(p-2)!d^{-\frac{1}{2}} + \tilde{D}^{-(p+2)}(y_0) + C_1 p \delta^2 \|w_{p,k}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))} + C_1 p \delta^2 \|w_{p,k-1}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))}, \]

where \( [\cdot]_{C^2_G} \) denotes the \( C^2 \) semi-norm. Then,

\[ \delta_1 d|D_{Y_n}D_{Y_\gamma}w_{p,k}| \leq \delta^{-p+2} DB^{p-\frac{3}{2}}(p-2)!d^{-\frac{1}{2}} + \tilde{D}^{-(p+2)}(y_0) + C_1 p \delta^2 \|w_{p,k}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))} + C_1 p \delta^2 \|w_{p,k-1}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))}, \]

or equivalently,

\[ d^\frac{2}{3}|D_{Y_n}D_{Y_\gamma}w_{p,k}| \leq \delta^{-p+1} DB^{p-\frac{3}{4}}(p-1)!d^{1+\epsilon} + \tilde{d}^{-(p+1)}(y_0) + p C_1 \delta^2 \|w_{p,k}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))} + \frac{p C_1 \delta^2}{\sqrt{d}} \|w_{p,k-1}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))}, \]

\[ \leq \delta^{-p+1} DB^{p-\frac{3}{4}}(p-1)!d^{1+\epsilon} + \tilde{d}^{-(p+1)}(y_0) + C_1 \delta \tilde{d}^{k}\|w_{p,k}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))} + C_1 \delta \tilde{d}^{k}\|w_{p,k-1}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))}, \]

\[ (8.12) \]

If \( k = 0 \), the last term is absent.

For \( D^2_{Y_\gamma}w_{p,k} \), we have similarly, by taking \( \delta_1 = \frac{\delta}{\sqrt{n}} \),

\[ d|D^2_{Y_\gamma}w_{p,k}| \leq \delta^{-p+1} C_1 DB^{p-\frac{3}{4}}(p-1)!d^{1+\epsilon} + \tilde{d}^{-(p-1)}(y_0) + C_1 \delta \tilde{d}^{k}\|w_{p,k}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))} + C_1 \delta \tilde{d}^{k}\|w_{p,k-1}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))}, \]

\[ (8.13) \]

For \( D^2_{Y_n}w_{p,k} \) with \( k < p \), we have

\[ d^2|D_{Y_n}D_{Y_n}w_{p,k}| = d^2|D_{Y_n}D_{Y_\gamma}w_{p,k+1}| + \text{ lower order terms} \]

\[ \leq \delta^{-p+1} DB^{p-\frac{3}{4}}(p-1)!d^{1+\epsilon} + \tilde{d}^{-(p-1)}(y_0) + C_1 \delta \tilde{d}^{k}\|w_{p,k+1}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))} + C_1 \delta \tilde{d}^{k}\|w_{p,k}\|_{C^2_G(B_{G}(y_0,\delta_1\rho))}, \]
A similar estimate holds for all second derivatives of \( w_{p,k} \) weighted by coefficients of the metric \( G \), except \( D^2_{dd}w_{p,k} \), which can be estimated by using the equation (5.3). Thus, for \( k < p \),

\[
[w_{p,k}]_{C^2_G} \leq \delta^{-p+1}DB^{p-\frac{3}{2}}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)}(y_0)
\]

(8.14)

\[
+ C_1\delta d[w_{p,k+1}]_{C^2_G(B_G(y_0,\delta_1\rho))} + C_1\tilde{d}[w_{p,k}]_{C^2_G(B_G(y_0,\delta_1\rho))}
\]

\[
+ C_1\tilde{d}[w_{p,k-1}]_{C^2_G(B_G(y_0,\delta_1\rho))}.
\]

For \( D^2_{Y_n}w_{p,k} \) with \( k = p \), we consider \( w_{p+1,p+1} = D_{Y^n}w_{p,p} \). By the gradient estimates, we have

\[
\|w_{p+1,p+1}\|_{C^1_G(B_{Y^n})} \leq \delta^{-p+2}C_1DB^{p-\frac{3}{2}}(p-2)!d^{1+\epsilon}\tilde{d}^{-(p-2)}(y_0)
\]

(8.15)

\[
+ C_1(p+1)\delta^2[w_{p,p}]_{C^2_G(B_G(y_0,\delta_1\rho))},
\]

and hence

\[
d^2|D^2_{Y_n}w_{p,p}| \leq \delta^{-p+1}DB^{p-\frac{3}{2}}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)}(y_0)
\]

\[
+ C_1\delta d[w_{p,p}]_{C^2_G(B_G(y_0,\delta_1\rho))}.
\]

Similarly,

\[
[w_{p,p}]_{C^2_G} \leq \delta^{-p+1}DB^{p-\frac{3}{2}}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)}(y_0)
\]

\[
+ C_1\delta d[w_{p,p}]_{C^2_G(B_G(y_0,\delta_1\rho))} + C_1\tilde{d}[w_{p-1,p}]_{C^2_G(B_G(y_0,\delta_1\rho))}.
\]

Finally, we multiply (8.14) and (8.15) by \( \delta^{p-1} \), and take a maximum of all these \( C^2_G \) semi-norms over \( k = 1, \ldots, p \). We obtain

\[
\max_{1 \leq k \leq p} \delta^{p-1}[w_{p,k}]_{C^2_G} \leq DB^{p-\frac{3}{2}}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)}(y_0)
\]

\[
+ C_1C_2\delta \max_{1 \leq k \leq p} \delta^{p-1}[w_{p,k}]_{C^2_G(B_G(y_0,\delta_1\rho))}
\]

\[
\leq DB^{p-\frac{3}{2}}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)}(y_0)
\]

\[
+ C_1C_2\delta \max_{1 \leq k \leq p} \delta^{p-1}[w_{p,k}]_{C^2_G(B_G(y_0,\delta_1\rho))},
\]

since \( \tilde{d}(y_0) \geq \theta \). At points in \( B_G(y_0,\delta_1\rho) \) but not in \( T^C_p \), \([w_{p,k}]_{C^2_G} \) is already estimated in Step 2. We have

\[
\| \max_{1 \leq k \leq p} \delta^{p-1}[w_{p,k}]_{C^2_G} \|_\theta \leq \frac{1}{10} \| \max_{1 \leq k \leq p} \delta^{p-1}[w_{p,k}]_{C^2_G} \|_{(1-\frac{1}{10})\theta}
\]

\[
+ DB^{p-\frac{3}{2}}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)},
\]

provided \( C_1C_2\delta \leq 1/10 \). We point out that we can replace the \( C^2_G \) semi-norm in (8.16) by the \( C^2_G \) norm. This is because the \( C^1_G \) semi-norm of \( w_{p,k} \) is the summation norm of
$\hat{d}^2 |Y' w_{p,k}|, \, d|Y_n w_{p,k}|, \, \text{and } d|D_d w_{p,k}|$, and satisfy
\[\| \max_{1 \leq k \leq p} \delta^{p-1} |w_{p,k}|_{C^2_{\tilde{G}}} \| \leq \hat{d}^{-1} \| \max_{1 \leq k \leq p-1} \delta^{p-1} |w_{p-1,k}|_{C^2_{\tilde{G}}} \| \leq \hat{d}^{-1} \| \max_{1 \leq k \leq p-1} \delta^{p-2} |w_{p-1,k}|_{C^2_{\tilde{G}}} \| \leq \hat{d}^{-1} \| \max_{1 \leq k \leq p-1} \delta^{p-3} |w_{p-1,k}|_{C^2_{\tilde{G}}} \| \leq DB^{p-2}(p-1)!d^{1+\epsilon} \theta^{-p+1}.
\]

We can assume $\hat{d}$ small since we can prove the theorem first for $R$ small and then extend to the general case by the interior analyticity. This finishes the proof of (8.8). \qed

In (8.1), $Y'$ could be $Y_n$. In the domain $G_R \cap \{d(y) \geq \eta\}$, (8.1) implies
\[|D_Y^l v|_{C^2_{\tilde{G}}}(y) \leq \tilde{q}^{-l}(l-1)!DB^{l-1}(l-1)!d^{1+\epsilon}(y)\tilde{d}^{-l}(y) \leq l^{-l}DB^{l-1}(l-1)!d^{1+\epsilon}(y)\tilde{d}^{-2(l-1)}(y) \leq D(\eta^{-2}B)^{-l}d^{1+\epsilon}(y) \leq D(\eta^{-2}B)^{-l}(2l)!d^{1+\epsilon}(y).
\]
Note $(2l)! = (1 \cdot 3 \cdot 5 \cdots (2l-1)) \cdot (2 \cdot 4 \cdot 6 \cdots (2l)) = (2 \cdot 4 \cdot 6 \cdots (2l))^2 = 2^{2l}(l!)^2$. Hence,
\[|D_Y^l v|_{C^2_{\tilde{G}}}(y) < 4D(\eta^{-2}B)^{-l}(l!)^2d^{1+\epsilon}(y).
\]
Therefore, $v$ is in the Gevrey space of order 2 along the tangential directions.

A natural question is whether we can get an estimate with $l!$ replacing $(l!)^2$ in (8.17) and hence obtain the tangential analyticity. It turns out that we cannot do this in general for the local setting. In the next section, we construct an example demonstrating solutions are not necessary analytic locally up to boundary.

In the following, we introduce an extra assumption in order to derive analyticity-type estimates up to boundary for the $2n-2$ tangential directions $Y_1, \cdots, Y_{n-1}, Y_{n+1}, \cdots, Y_{2n-1}$. Let $\Gamma = G_R \cap \{d = 0\}$ be a portion of the boundary. We assume that its CR structure $H(\Gamma)$ is integrable, i.e., for any $Y_\alpha, Y_\beta \in H(\Gamma)$,
\[(8.18) \quad [Y_\alpha, Y_\beta] \in H(\Gamma),
\]or equivalently, $H(\Gamma)$ is the tangent field of a $2n-2$ dimensional submanifold of $\Gamma$. Refer to (7.1) for the definition of $H(\Gamma)$.

Before stating the next result, we discuss briefly the role of the assumption (8.18). By (7.2), there is no $Y_n$ term in $[Y_n, Y_\alpha]$, and by (8.18), there is no $Y_n$ term in $[Y_\alpha, Y_\beta]$. In the absence of the assumption (8.18), when we switch orders, we have, for $Y' \neq Y_n$,
\[dD_Y^p Y_\alpha Y_\beta v = dY_\alpha Y_\beta D_Y^p v + pS_{*,\alpha}^m dY_\beta Y_n D_Y^{p-1} v + pS_{*,\beta}^m dY_\alpha Y_n D_Y^{p-1} v + \cdots,
\]where $*$ denotes some index which is not $n, 2n$. In general, the terms $pS_{*,\alpha}^m dY_\beta Y_n D_Y^{p-1} v, \, pS_{*,\beta}^m dY_\alpha Y_n D_Y^{p-1} v$ should not have worse estimates than the inductive estimates of
$dD^p_Y, Y_\alpha Y_\beta v$. However, by induction (8.19),
\[ |pS^n_{*,a}dY_\beta Y_n D_Y^{-1}v| \leq p\delta_{p-1}^{D} DB^{p-2}(p-2)!d^{1+\epsilon}y\tilde{d}^{-(p-2)}(y) \]
\[ \leq DB^{p-2}p!d^{1+\epsilon}y\tilde{d}^{-(p-3)}(y). \]
This is worse than the inductive estimate of $dY_\alpha Y_\beta D^p_Y, v$, which is, by (8.19),
\[ |dY_\alpha Y_\beta D^p_Y, v| \leq DB^{p-1}(p-1)!d^{1+\epsilon}y\tilde{d}^{-(p-1)}(y). \]

We note that this is not an issue if we allow adding two $p$ factors in each induction step. Refer to Lemma 8.2 for details. Under the assumption (8.18), there are no terms such as $pS^n_{*,a}dY_\beta Y_n D_Y^{p-1}v, pS^n_{*,a}dY_\alpha Y_v D_Y^{p-1}v$ while switching orders from $dD^p_Y, Y_\alpha Y_\beta v$ to $dD^p_Y, Y_\alpha Y_\beta v$.

**Theorem 8.4.** Assume a boundary portion $\Gamma = G_R \cap \{d = 0\}$ of $G_R$ is analytic and, in addition, (8.18) holds. Then, for any $l \geq 0$, $0 \leq k \leq l$, and any $y \in G_R$,
\[ \delta_l^k |D_Y^{l-k}Y_n v|_{C^2_G}(y) \leq DB^{l-1}(l-1)!d^{1+\epsilon}y\tilde{d}^{-(l-1)}(y), \]
where $\delta_l = \tilde{d}/l$, $\tilde{d}$ is the distance function to the cylinder $|y'| = R$, and $Y'$ is any one of $Y_1, \ldots, Y_{n-1}, Y_{n+1}, \ldots, Y_{2n-1}$, and $B$ and $D$ are positive constants.

**Proof of Theorem 8.4** We now sketch the proof. Set
\[ w_{p,k} = D^{p-k}_Y, Y_n v. \]
Here and hereafter, none of the $Y'$s equals $Y_n$.

By the interior analyticity, we assume (8.19) holds in $G_R \cap \{r/2 \leq d < R\}$. Define $T_l$ as in the proof of Theorem 8.1. We prove (8.19) by induction. The proof is parallel to that of (8.1). We record in this proof only those estimates different from those in the proof of Theorem 8.1.

Take a positive integer $p$. Assume
\[ \delta_l^k \|w_{l,k}\|_{C^2_G} \leq DB^{l-1}(l-1)!d^{1+\epsilon}\tilde{d}^{-(l-1)} \]
holds in $T_l$ and (8.19) holds for any $0 \leq l \leq p-1$ and $0 \leq k \leq l$.

Step 1. For $w_{p,k} = D_Y, Y_{p-1,k-1}$, for some $1 \leq k \leq p$, we proceed similarly as in Step 1 in the proof of Theorem 8.1 and obtain
\[ |w_{p,k}| \leq \delta_p^k DB^{p-1}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)} \quad \text{in } T_l. \]
For $w_{p,k} = D_Y, Y_{p-1,k}$, we consider the test function
\[ M(y) = a|y'|^2d^{1+\epsilon} + bd^{1+\epsilon}, \]
and obtain
\[ |w_{p,k}| \leq \delta_p^k DB^{p-1}(p-1)!d^{1+\epsilon}\tilde{d}^{-(p-1)} \quad \text{in } T_l. \]
For $y \in T_p^C$, by the induction for the case $p - 2$, if we can write $w_{p,k} = D_{Y_n}^2 w_{p-2,k-2}$, then
\[
|w_{p,k}| \leq d^{-2} \delta_p^{-(k-2)} DB^{p-3}(p-3)!d^{1+\epsilon} \tilde{d}^{p+3} \\
\leq \delta_p^{-k} DB^{p-1}(p-1)!d^{1+\epsilon} \tilde{d}^{p+1}(y_0),
\]
since $d \geq \delta_p^2$. For cases $w_{p,k} = D_{Y_n} D_{Y'} w_{p-2,k-1}$, $w_{p,k} = D_{Y'}^2 w_{p-2,k}$, we can discuss similarly.

**Step 2.** By gradient estimates for elliptic equations, for universal constants $\rho''$, $\rho'$, $\rho$ such that $0 < \rho'' < \rho' < \rho < 1$, we have,
\[
\|w_{p,k}\|_{C^{1,\alpha}_{G}(B_G(y_0,\rho'))} \leq C(\|w_{p,k}\|_{L^\infty_G(B_G(y_0,\rho'))} + \|H_p\|_{L^\infty_B(B_G(y_0,\rho'))}),
\]
and hence
\[
(8.21) \quad \|w_{p,k}\|_{C^{1,\alpha}_{G}(B_G(y_0,\rho'))} \leq \delta_p^{-k} DB^{p-1-\frac{1}{2}}(p-1)!d^{1+\epsilon} \tilde{d}^{p+1}(y_0).
\]
This also holds if we replace $p$ by any $l \leq p$ and $0 \leq k \leq l$. Inductively, we can also prove, for $l \leq p - 1$,
\[
\|w_{l,j}\|_{C^{2,\alpha}_{G}(B_G(y_0,\rho'))} \leq \delta_l^{-j} DB^{l-\frac{1}{2}}(l-1)!d^{1+\epsilon} \tilde{d}^{l+1}(y_0).
\]
Thus,
\[
\|H_p\|_{C^{2}_{G}(B_G(y_0,\rho'))} \leq \delta_p^{-k} DB^{p-1-\frac{1}{2}}(p-1)!d^{1+\epsilon} \tilde{d}^{p+1}(y_0),
\]
and by Schauder estimates, for $0 \leq k \leq p$,
\[
\|w_{p,k}\|_{C^2_{G}(B_G(y_0,\rho'))} \leq \delta_p^{-k} DB^{p-1-\frac{1}{2}}(p-1)!d^{1+\epsilon} \tilde{d}^{p+1}(y_0).
\]
Therefore, we have \textbf{(8.20)} in $T_p$ for the case $p$.

**Step 3.** We claim
\[
(8.22) \quad \|\max_{1 \leq k \leq p} \delta_p^k \|w_{p,k}\|_{C^{2}_{G}}\|_{\theta} \leq \frac{1}{10} \|\max_{1 \leq k \leq p} \delta_p^k \|w_{p,k}\|_{C^{2}_{G}}\|_{(1 - \frac{1}{p^{1/2}})\theta} + DB^{p-\frac{7}{4}}(p-1)!\theta^{-p+1}.
\]
Then, by Lemma \textbf{[8.3]}

\[
(8.23) \quad \|\max_{1 \leq k \leq p} \delta_p^k \|w_{p,k}\|_{C^{2}_{G}}\|_{\theta} \leq ADB^{p-\frac{7}{4}}(p-1)!\theta^{-p+1} \leq DB^{p-1}(p-1)!\theta^{-p+1}.
\]
We hence finish the proof by induction.

Fix a $\theta$. For any $y_0 \in T_p^C$ with $d(y_0) \geq \theta$, set $\delta_1$ by \textbf{[8.10]}

We consider $w_{p+1,k} = D_{Y'}^2 w_{p,k}$, for $Y' \neq Y_n$. By \textbf{(8.19)} for case $l = p - 1$ and $\delta \leq \sqrt{d}$, we have
\[
|w_{p+1,k}| \leq \delta^{-k+1} DB^{p-2}(p-2)!d^{-\frac{1}{2}+\epsilon} \tilde{d}^{-(p-2)}.
\]
By gradient estimates for elliptic equations, we obtain, for balls centered at $y_0$,
\[
\|w_{p+1,k}\|_{C^{1,\alpha}_{G/\delta^2}(B_{\rho'})} \leq C_1(\|w_{p+1,k}\|_{L^\infty_{G/\delta^2}(B_{\rho'})} + \delta^2 \|H_p\|_{L^\infty_{G/\delta^2}(B_{\rho'})}),
\]
which implies

\[
d_{Y}^{\frac{3}{2}}|D_{Y_{0}}D_{Y_{0}}w_{p,k}| \leq \delta^{-k}C_{1}DB^{p-\frac{4}{5}}(p-1)!d^{1+\epsilon}\bar{d}^{-(p-1)}(y_{0})
\]

(8.24)

\[
+ \frac{C_{1}\delta\bar{d}}{\sqrt{d}}[w_{p,k}]C_{G}^{2}(B_{\delta_{1}\rho}) + \frac{C_{1}\delta\bar{d}}{\sqrt{d}}[w_{p,k-1}]C_{G}^{2}(B_{\delta_{1}\rho}).
\]

Comparing (8.24) with (8.12), we note that \(\delta^{-k}\) appears in the first term in the right-hand side of (8.24) instead of \(\delta^{-p+1}\) in (8.12). We have similar improvements for the rest of estimates.

For \(D_{Y}^{2}, w_{p,k}\) we derive similarly, with \(\delta_{1} = \frac{\delta}{\sqrt{d}}\),

\[
d|D_{Y}^{2}w_{p,k}| \leq \delta^{-k}C_{1}DB^{p-\frac{4}{5}}(p-1)!d^{1+\epsilon}\bar{d}^{-(p-1)}(y_{0})
\]

\[
+ \frac{C_{1}\delta\bar{d}}{\sqrt{d}}[w_{p,k+1}]C_{G}^{2}(B_{\delta_{1}\rho}) + \frac{C_{1}\delta\bar{d}}{\sqrt{d}}[w_{p,k}]C_{G}^{2}(B_{\delta_{1}\rho})
\]

\[
+ \frac{C_{1}\delta\bar{d}}{\sqrt{d}}[w_{p,k-1}]C_{G}^{2}(B_{\delta_{1}\rho}).
\]

For \(D_{Y_{0}}^{2}, w_{p,k}\) with \(k < p\), we have

\[
[w_{p,k}]C_{G}^{2} \leq \delta^{-k}DB^{p-\frac{4}{5}}(p-1)!d^{1+\epsilon}\bar{d}^{-(p-1)}(y_{0})
\]

(8.25)

\[
+ C_{1}\delta\bar{d}[w_{p,k+1}]C_{G}^{2}(B_{\delta_{1}\rho}) + C_{1}\delta\bar{d}[w_{p,k}]C_{G}^{2}(B_{\delta_{1}\rho})
\]

\[
+ C_{1}\delta\bar{d}[w_{p,k-1}]C_{G}^{2}(B_{\delta_{1}\rho}).
\]

For the case \(k = p\), we have

\[
[w_{p,p}]C_{G}^{2} \leq \delta^{-p}DB^{p-\frac{4}{5}}(p-1)!d^{1+\epsilon}\bar{d}^{-(p-1)}(y_{0})
\]

(8.26)

\[
+ C_{1}\delta\bar{d}[w_{p,p}]C_{G}^{2}(B_{\delta_{1}\rho}) + C_{1}\delta\bar{d}[w_{p,p-1}]C_{G}^{2}(B_{\delta_{1}\rho}).
\]

Finally, we multiply (8.25) by \(\delta^{k}\), multiply (8.26) by \(\delta^{p}\), and take a maximum of all these \(C_{G}^{2}\) semi-norms over \(k = 1, \cdots, p\) to obtain

\[
\max_{1 \leq k \leq p} (\delta^{k}[w_{p,k}]C_{G}^{2}) \leq DB^{p-\frac{4}{5}}(p-1)!d^{1+\epsilon}\bar{d}^{-(p-1)}(y_{0}) + C_{1}C_{2}\bar{d} \max_{1 \leq k \leq p} \delta^{k}[w_{p,k}]C_{G}^{2}(B_{\delta_{1}\rho}).
\]

As at points in \(B_{\delta_{1}\rho}(y_{0})\) under the metric \(G\) but not in \(T_{p}^{C}\), \(\|w_{p,k}\|_{C_{G}^{2}}\) is already estimated in Step 2, we can ignore these points and actually have

\[
\| \max_{1 \leq k \leq p} \delta^{k}[w_{p,k}]C_{G}^{2} \|_{\theta} \leq \frac{1}{10}\| \max_{1 \leq k \leq p} \delta^{k}[w_{p,k}]C_{G}^{2} \|_{(1-\frac{\epsilon}{p+1})\theta}
\]

\[
+ DB^{p-\frac{4}{5}}(p-1)!d^{1+\epsilon}\theta^{-p+1},
\]

provided \(C_{1}C_{2}\bar{d} \leq 1/10\), where the \(C_{G}^{2}\) semi-norm can be replaced by the \(C_{G}^{1}\) norm. In fact, the \(C_{G}^{1}\) norm of \(w_{p,k}\) is the summation norm of \(d^{\frac{3}{2}}|Y'w_{p,k}|, d|Y_{n}w_{p,k}|, d|D_{d}w_{p,k}|,\) so
we check

\[ \| \max_{1 \leq k \leq p} \delta^k |w_{p,k}|_{C^0_G} \| \theta \leq \| \max_{1 \leq k \leq p-1} \delta^{\frac{1}{2}} d \partial^k |w_{p-1,k}|_{C^0_G} \| \theta \]

\[ \leq \delta^{-1} \| \max_{1 \leq k \leq p-1} \delta^k |w_{p-1,k}|_{C^0_G} \| \theta \]

\[ \leq DB^{p-2}(p-1)!d^{1+\epsilon} \theta^{-p+1}, \]

where we applied the inductive estimate (8.20) for case \( p-1 \). This finishes the proof of (8.22).

We point out that the power of \( \delta \) in the left-hand side of (8.19) is \( k \), the order of the differentiation with respect to \( Y_n \). This is a significant improvement over (8.1). Hence, \( \delta \) is absent from the left-hand side of (8.19) if \( k = 0 \). As a consequence, we have, for any \( Y' \neq Y_n, Y_2n \),

\[ |D_{Y'} v| \leq DB^{p-1}(p-1)! \]

In other words, \( u \) is analytic in the directions in \( H(\Gamma) \).

We note that (8.18) usually does not hold for the complex Monge-Ampère equation in the setting of this paper. As we discussed, this is mainly due to the presence of complex structure.

In fact, Theorem 8.4 asserts (8.19) holds for solutions of a large class of equations which has linearization

\[ d^2 v_{dd} + P(y)dv_d + Q(y)v + d^2 v_{y_1y_1} + d \sum_{i=2}^{n-1} v_{y_1y_i} + R(y) \sum_{j=1}^{n-1} v_{y_j} = F, \]

where \( \frac{\partial}{\partial y_1} \) plays the role of \( Y_n \), and \( \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_{n-1}} \) are other tangential directions.

9. A Counterexample to the Local Convergence

In [19], we discussed degenerate elliptic equations of form

\[ d^2 v_{dd} + d^2 T v - (n-1)dv_d - (n+1)v = F, \]

where \( T \) is a uniformly elliptic operator in tangential coordinates. All second derivatives have the same rate of degeneracy. We proved the local convergence of the series solutions.

In this section, we study a class of equations where second derivatives have different rates of degeneracy. Specifically, we consider

\[ d^2 v_{dd} + d^2 v_{tt} + dv_{ss} - (n-1)dv_d - (n+1)v = 0, \]

in domain \( G_r = \{ 0 < d < r \} \times \{ t^2 + s^2 < r^2 \} \). We construct a counterexample to the local convergence.

Consider the operator

\[ A = d^2 \partial_{dd} + d^2 \partial_{tt} - (n-1)d\partial_d - (n+1) = d^2 \Delta_{d,t} - (n-1)d\partial_d - (n+1). \]
By Proposition 1 of [7], there exist a nonanalytic function \( w \) in \( d \) and \( t \), defined in a neighborhood of the origin in \( \mathbb{R}^+ \times \mathbb{R} \), such that, for each \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^2 \),

\[
\| \partial^\alpha \Delta^k d,t w \|_{L^2} \leq C|\alpha|+k+1(2k)!(2\alpha)!.
\]

The same estimates hold for \( \| \partial^\alpha \Delta^k d,t w \|_{C^2} \) if we enlarge \( C \).

Let \( v \) be a bounded solution of (9.1). By a formal computation, all local terms in the expansion of \( v \) vanish. Assume the first global term of \( v \) is \( c_{n+1} \). Then, \( v \) has a formal expansion given by

\[
v = c_{n+1}d^{n+1} + c_{n+2}d^{n+2} + \cdots + c_k d^k + \cdots.
\]

A convergent series solution of \( Av + dv_{ss} = 0 \) (or (9.1)) can be constructed as

\[
\overline{v} = \sum_{k=0}^{\infty} \left( -\frac{A}{d} \right)^k (d^{n+1}w) \frac{1}{(2k)!} s^{2k},
\]

where \( w \) is a nonanalytic function satisfying (9.2). Note that the coefficient of \( s^{2k} \) has an explicit factor \( d^{n+1} \) for each \( k \). In fact, formally,

\[
(-A/d)(d^{n+1}h) = -d^{n+1}(dh_{dd} + dh_{tt} + 2h_d),
\]

for any function \( h \). So inductively, we can prove the existence of \( d^{n+1} \) factor in the coefficients. Since \( \overline{v} = d^{n+1}w \) is not analytic in \( d \) for \( s = 0 \), we conclude \( \overline{v} \) is not analytic in \( d \). Further calculations show \( \overline{v} \) belongs to Gevrey space \( G_2 \).

The example constructed above is for a linear equation with the same structure as the linear part of the nonlinear equation discussed in this paper. It strongly suggests that it is impossible to prove the convergence in the local setting simply by writing the nonlinear equation in the form of a perturbation of the linear equation. It is believed that a similar example can be constructed for the nonlinear equation. We will not pursue this in this paper.

**Appendix A. Analyticity Type Estimates**

In this section, we present some lemmas on analyticity type estimates for composition functions by following Friedman [14].

Our main concern is the validity of the analyticity type estimates of the form, for any \( l \geq 0 \),

\[
\| D^l u \| \leq DB^{(l-3)^+} (l-3)!,
\]

where \( D^l u \) is an arbitrary derivative of \( u \) of order \( l \), \( D, B \) are positive constants independent of \( l \), and \( \| \cdot \| \) is a norm compatible with algebra of smooth functions, such as \( L^\infty \)- and \( C^{0,\alpha} \)-norms.

We introduce the following notation. For any \( C^1 \)-function \( u = u(x) \), set

\[
G_u(x) = (x, u(x), \nabla u(x)).
\]

We simply write \( G \) instead of \( G_u \) if clear from the context.
By Lemma 1 in [14] and its proof and assuming $M_l = l!$ in [14], we have the following result.

**Lemma A.1.** Assume $S$ is analytic in its arguments $G$ and satisfies, for any $l \geq 0$,

$$|D_G^lS| \leq CA^l(l - 2)!,$$

for some positive constants $A$ and $C$. Then, for an analytic function $u = u(x)$, the function $(S \circ G_u)(x) \equiv S(x, u, Du(x))$ satisfies, for any $p \geq 0$,

$$\|D^p(S \circ G_u)\| \leq DB^{p-2}(p - 2)!,$$

where $B, D$ are constants independent of $p$, with $B >> A$.

An immediate consequence is the following result.

**Lemma A.2.** Assume $S$ and $T_{ij}$ are analytic in its arguments $G$ and satisfies, for any $l \geq 0$,

$$|D_G^lS| + |D_G^lT_{ij}| \leq CA^l(l - 2)!,$$

for some positive constants $A$ and $C$. Then, for an analytic function $u = u(x)$, the function $R = S \circ G_u + T_{ij} \circ G_u \cdot u_{ij}$ satisfies, for any $p \geq 0$,

$$\|D^pR\| \leq MDB^{p-2}(p - 1)! + \|T_{ij} \circ G_u D^{p+2}u\|,$$

where $B, D, M$ are constants independent of $p$.

**Proof.** Note

$$\|D^p(T_{ij} \circ G_u \cdot u_{ij})\| \leq \sum_{l=1}^{p} \|C^l_pD^l(T_{ij} \circ G_u)D^{p-l}u_{ij}\| + \|T_{ij} \circ G_u \cdot D^{p+2}u\|.$$

Hence,

$$\sum_{l=1}^{p} \|C^l_pD^l(T_{ij} \cdot G_u)D^{p-l}u_{ij}\|$$

$$\leq \sum_{l=1}^{p} \|C^l_pED^lB^{l-2}(l - 2)! \cdot DB^{p-l-1}(p - l - 1)!\|$$

$$\leq pED^2B^{p-2}(p - 2)! + ED^2B^{p-2}(p - 2)!$$

$$+ \sum_{l=2}^{p-1} \frac{p}{l(l-1)(p-l)}ED^2B^{p-3}(p - 1)!$$

$$\leq MDB^{p-2}(p - 1)!,$$
for some $M$ independent of $B, p$, if we set $B$ much larger than other constants. Here, we used the following estimates:

\[
\sum_{l=2}^{p-1} \frac{1}{l(l-1)(p-l)} = \sum_{l=2}^{p-1} \frac{1}{l-1} \left( \frac{1}{l} + \frac{1}{p-l} \right) = \sum_{l=2}^{p-1} \frac{1}{(l-1)l} + \frac{1}{p-1} \sum_{l=2}^{p-1} \left( \frac{1}{l-1} + \frac{1}{p-l} \right) < 3.
\]

We have the desired result. □

**Remark A.3.** Note that all above estimates still hold if we increase the power of $B$ by less than or equal to 1.

**Appendix B. A Lie Bracket Lemma by CR-Geometry**

As in [38], let $(M, H(M), J)$ be an oriented CR manifold, and $\theta$ be the one-form on $M$ that annihilates exactly $H(M)$. We can define a pseudo-Riemannian metric $g_\theta$ on $M$, which is a Riemannian metric if we assume $(M, \theta)$ is strictly pseudoconvex. In addition, there is a unique vector field $T$ on $M$ such that $\theta(T) = 1, d\theta(T, \cdot) = 0$. For $X, Y \in \Gamma(H(M))$, we have

\[
g_\theta(X, Y) = d\theta(X, JY), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1.
\]

Then, we have the Tanaka-Webster connection $\nabla$ on $M$. Let $\tau$ be the torsion. Then, $A(X) = \tau(T, X)$ satisfies $AH(M) \subseteq H(M)$, i.e.,

\[
\nabla_T X - \nabla_X T - [T, X] \in \Gamma(H(M))
\]

if $X \in \Gamma(H(M))$. Note that $\nabla_T X \in \Gamma(H(M))$ since $H(M)$ is parallel, and $\nabla_X T \in H(M)$ since $g_\theta(\nabla_X T, T) = \frac{1}{2} X(g_\theta(T, T)) = 0$. Hence, we have the following result.

**Lemma B.1.** $[T, X] \in \Gamma(H(M))$ if $X \in \Gamma(H(M))$.

An easier proof follows from the Cartan’s formula as follows:

\[
0 = d\theta(T, X) = T\theta(X) - X\theta(T) - \theta([T, X]) = -\theta([T, X]).
\]

**Appendix C. A Frame System near $\partial \Omega$**

First, we build a frame system $\{Y_i\}_{i=1, \ldots, 2n}$ on near $\partial \Omega$. Fix any boundary point in $\partial \Omega$, say the origin $O$. After a unitary transform of coordinates, we can assume the tangent plane of $\partial \Omega$ at the origin is given by $x^{2n} = 0$. Denote by $\varphi$ the function on the tangent plane, whose graph is $\partial \Omega$ near the origin. We will use the Cartesian coordinates $x^1, \ldots, x^{2n}$, the complex coordinates $z_1 = \sqrt{-1} x_1 + x_{n+1}, \ldots, z_n = \sqrt{-1} x_n + x_{2n}$, and the geodesic coordinates $y^1, \ldots, y^{2n}$, where $y^{2n} = d = dist(x, \partial \Omega)$. For $\tilde{y} = (y^1, \ldots, y^{2n-1})$, \ldots
we have $x = (\tilde{y}, \varphi(\tilde{y})) + N(\tilde{y})d$. Then,

$$
(C.1) \quad \left[ \frac{Dx}{Dy} \right]_{2n \times 2n} = \begin{pmatrix}
1 + \frac{\partial N_1}{\partial y^1}d & 1 + \frac{\partial N_2}{\partial y^1}d & \cdots & 1 + \frac{\partial N_{2n-1}}{\partial y^1}d & N_1 \\
\frac{\partial N_1}{\partial y^2}d & 1 + \frac{\partial N_2}{\partial y^2}d & \cdots & 1 + \frac{\partial N_{2n-1}}{\partial y^2}d & N_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial N_1}{\partial y^{2n}}d & \frac{\partial N_2}{\partial y^{2n}}d & \cdots & \frac{\partial N_{2n-1}}{\partial y^{2n}}d & N_{2n-1} \\
\frac{\partial y^1}{\partial y^1} + \frac{\partial y^2}{\partial y^2} + \frac{\partial y^3}{\partial y^2}d & \frac{\partial y^2}{\partial y^2} + \frac{\partial y^3}{\partial y^3}d & \cdots & \frac{\partial y^2}{\partial y^{2n}} + \frac{\partial y^3}{\partial y^{2n}}d & N_{2n}
\end{pmatrix},
$$

where $N$ is the unit inner normal vector on $\partial \Omega$. Denote by $A$ this matrix, and by $B$ its inverse matrix.

For any point $Q$ on the boundary, denote by $n(Q) \in \mathbb{C}^n$ its unit inner normal direction. When $Q$ is at $O$, $n(Q) = (0, \cdots, 0, 1)^T$. When $Q$ is near $O$, we can find a unitary transform $T_1$ depending on $Q$, such that,

$$
T_1(Q) : n(O) \rightarrow n(Q),
$$

where $T_1(Q)$ can be computed as a unitary matrix from the Gram-Schmidt orthonormalization of the matrix

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 & n(Q)_1 \\
0 & 1 & \cdots & 0 & n(Q)_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & n(Q)_{n-1} \\
0 & 0 & \cdots & 0 & n(Q)_n
\end{pmatrix}.
$$

In procedure, we keep the $n$-th column. Note that $n(Q)_n$ is close to 1 and $T_1$ is $C^{k-1,\alpha}$ if $Q$ is given under a $C^{k,\alpha}$-coordinates chart of $\partial \Omega$ near $O$. Define $T_2$ as the translation from $O$ to $Q$.

At point $P$ such that $d(P) = dist(P, Q)$, we use complex coordinates

$$(z_1, \cdots, z_n)^T_Q = T_1^{-1}(Q)T_2^{-1}(Q)(z_1, \cdots, z_n)^T_O.$$ 

Here, $Q$ is the origin, and on the ray $QP$, $x^\alpha_P = 0$ if $\alpha \neq 2n$. Then we can use geodesic coordinates $(y^1, \cdots, y^{2n})_Q$, and denote a frame system, for $1 \leq i \leq 2n$,

$$Y_i(P) = \frac{\partial}{\partial y^i_Q}(P).$$

This is a $C^{k-1,\alpha}$ frame system under the coordinates chart $(z_1, \cdots, z_n)_O$. It is not orthonormal but $(Y_i, Y_j) = \delta_{ij} + O(d)$, since $Y_i(P) = \frac{\partial}{\partial x^i_Q}(P) + O(d)$. We note that $Y_{2n} = \frac{\partial}{\partial \varphi}Q$ is globally defined.

We have following properties.

**Lemma C.1.** $(Y_\alpha, \frac{\partial}{\partial \varphi}) = 0$ for $\alpha \neq 2n$.

**Lemma C.2.** $[Y_\alpha, \frac{\partial}{\partial \varphi}] = 0$ for $\alpha \neq 2n$. 
Lemma C.4. \( \{ Y_i \}_{i=1,...,2n} \) is defined on the product topology of \( \partial \Omega \times \{ 0 \leq d \leq r \} \), and \( Y_\alpha \) does not depend on \( d \), for \( \alpha \neq 2n \).

Assume \( Y_\alpha = \delta_{\alpha,i} \frac{\partial}{\partial q_i} \) near \( P \), where \( \delta_{\alpha,i} \)'s are coefficients, and \( \delta_{\alpha,2n} = 0 \) since \( (Y_\alpha, Y_{2n}) = 0 \). Then,

\[
[Y_\alpha, \frac{\partial}{\partial d}] = - \frac{\partial \delta_{\alpha,i}}{\partial d} \frac{\partial}{\partial q_i} = 0,
\]

since \( \delta_{\alpha,i} \) does not change on the integral curve of \( \frac{\partial}{\partial \alpha} \). Too see this, notice at point \( P' \) near \( P \), if \( Q' \in \partial \Omega \) satisfies \( d(P') = d(P', Q') \), then \( Y_\alpha(P') = \frac{\partial}{\partial q_i} \). We can find a transformation between the two coordinates system \( \{ y_Q^i \}_{i=1,...,2n} \) and \( \{ y_{Q'}^i \}_{i=1,...,2n} \). For any point \( A \in \Omega \) under these two geodesic coordinates system, \( y_Q^{2n} = y_{Q'}^{2n} \). Assume \( A' \in \partial \Omega \) is the closest boundary point to \( A \). The orthonormal projection of \( A' \) to the tangent plane at \( Q \) (resp., \( Q' \)) defines the tangential coordinates \( \{ y_Q^{\alpha} \}_{\alpha=1,...,2n-1} \) (resp., \( \{ y_{Q'}^{\alpha} \}_{\alpha=1,...,2n-1} \)) of \( A \). This correspondence defines the transformation between tangential coordinates, and only depends on where \( A' \) is, independent of \( d \). Then, we can find a transformation between \( \{ \frac{\partial}{\partial q_i} \}_{i=1,...,2n} \) and \( \{ \frac{\partial}{\partial q_{i'}} \}_{i=1,...,2n} \), which does not depend on \( d \). \( \square \)

Remark C.3. Only \( Y_n, Y_{2n} \) are defined globally near \( \partial \Omega \). With the validity of Lemma 3.1 we do not need to restrict \( Y_\alpha \) as defined above, for \( 1 \leq \alpha \leq 2n-1, \alpha \neq n \). In fact, we can define new frames \( \{ \tilde{Y}_i \} \) such that, for \( 1 \leq \alpha \leq 2n-1, \alpha \neq n \),

\[
(C.2) \quad \tilde{Y}_\alpha = \sum_{\beta=1}^{2n-1} \theta_{\alpha,\beta} Y_\beta,
\]

where \( \theta_{\alpha,n} = 0 \). Note that \( \theta_{\alpha,\beta} \)'s are independent of \( d \). Then, Lemma 3.1 also holds under the frames \( \{ \tilde{Y}_i \} \).

Consider the metric \( g_{ij} = (-\log(-\rho))_{ij} \). Then, we have the following result.

Lemma C.4. (1) The \( C^1 \)-norm \( \|w\|_{C^1_g} \) is equivalent to the norm which is the summation of \( d|Y_n u|, d|Y_{2n} u|, \sqrt{d}|Y_i u|, |u| \), with \( i, j \neq n, 2n \).

(2) The \( C^2 \)-norm \( \|w\|_{C^2_g} \) is equivalent to the norm which is the summation of \( d^2|Y_n^2 w|, d^2|Y_{2n}^2 w|, d^2|Y_i Y_j w|, d^2|Y_n Y_i w|, d^2|Y_{2n} Y_i w|, \|w\|_{C^1_g} \), with \( i, j \neq n, 2n \).

Proof. Similarly as in Lemma 3.1 we can calculate at \( Q \), using \( d_i = \frac{1}{2} \delta_{in} \),

\[
G(Y_i, Y_j) = \frac{1}{d} T_{ij} + \frac{1}{d^2} T_{nn} (\delta_{in} \delta_{jn} + \delta_{i,2n} \delta_{j,2n}),
\]

for some smooth positive matrix \( T_{ij} \) with respect to Euclidean coordinates near \( \Gamma \). This verifies that we can express \( \|u\|_{C^1_g} \) under frame system \( \{ Y_i \} \) with suitable weight; namely, that \( |u|_{C^1_g} \) is bounded is equivalent to that \( d|Y_n u|, d|Y_{2n} u|, \sqrt{d}|Y_i u|, |u| \) are bounded for \( i \neq n, 2n \).
Since $G$ is Kähler, we have
\[
\|u\|_{C^2_g}^2 = g^{i\bar{j}} g^{p\bar{q}} (\nabla_g)_{i\bar{j}}^2 u (\nabla_g)_{p\bar{q}}^2 u + g^{i\bar{j}} g^{p\bar{q}} (\nabla_g)_{i\bar{j}}^2 u (\nabla_g)_{p\bar{q}}^2 u + \|u\|_{C^1_g}^2 = g^{i\bar{j}} g^{p\bar{q}} u_{i\bar{j}} u_{p\bar{q}} u + g^{i\bar{j}} g^{p\bar{q}} (\nabla_g)_{i\bar{j}}^2 u (\nabla_g)_{p\bar{q}}^2 u + \|u\|_{C^1_g}^2.
\]

Note $g^{i\bar{j}}$ is $O(\rho^2)$ if $i$ or $j = n$. So $\|u\|_{C^2_g} \leq C$ is equivalent to $\|u\|_{C^1_g} \leq C_1$ and
\[
d^{2+\delta_n+\delta} \|u_{i\bar{j}}\| \leq C_1,
\]
\[
d^{2+\delta_n+\delta} \| (\nabla_g)_{i\bar{j}}^2 u \| \leq C_1.
\]

We have
\[
(\nabla_g)_{i\bar{j}}^2 u = u_{i\bar{j}} + (\nabla_g)_i \frac{\partial}{\partial z^{\bar{j}}} u = u_{i\bar{j}} + (\Gamma^G)_{i\bar{j}}^k u_k = u_{i\bar{j}} + g^{k\bar{l}} \partial_k g_{i\bar{l}} \cdot u_k,
\]
where
\[
\partial_k g_{i\bar{l}} = -\frac{\rho_{i\bar{j}}}{\rho} + \frac{\rho_{i\bar{j}} \rho_{\bar{k}i} + \rho_{i\bar{j}} \rho_{\bar{i}k} + \rho_{j\bar{i}} \rho_{\bar{i}k}}{\rho^2} - \frac{2\rho_{i\bar{j}} \rho_{j\bar{i}}}{\rho^3},
\]
whose order of $d$ depends on whether $i, j,$ or $l = n$. Comparing the order of $d,$ we conclude that $\|u\|_{C^2_g} \leq C$ is equivalent to the summation norm defined as in the lemma. \qed

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