Thermality of the Rindler horizon: A simple derivation from the structure of the inertial propagator

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Abstract

The Feynman propagator \( G(x_1, x_2) \) encodes all the physics contained in a free field and transforms as a covariant biscal. Therefore, we should be able to discover the thermality of Rindler horizon, just by probing the structure of the propagator, expressed in the Rindler coordinates. I show that the thermal nature of the Rindler horizon is indeed contained — though hidden — in the standard, inertial, Feynman propagator. The probability \( P(E) \) for a particle to propagate between two events, with energy \( E \), can be related to the temporal Fourier transform of the propagator. A strikingly simple computation reveals that: (i) \( P(E) \) is equal to \( P(-E) \) if the propagation is between two events in the same Rindler wedge while (ii) they are related by a Boltzmann factor with temperature \( T = g/2\pi \), if the two events are separated by a horizon. A more detailed computation reveals that the propagator itself can be expressed as a sum of two terms, governing absorption and emission, weighted correctly by the factors \( (1 + n_\nu) \) and \( n_\nu \) where \( n_\nu \) is a Planck distribution at the temperature \( T = g/2\pi \). In fact, one can discover the Rindler vacuum and the alternative (Rindler) quantization, just by probing the structure of the inertial propagator. These results can be extended to local Rindler horizons around any event in a curved spacetime. The implications are discussed.

1 The main result: Inertial propagator knows all!

The path integral representation of the (Feynman) propagator is given by the sum over paths prescription using the (square-root) action for a relativistic particle:

\[
\sum_{\text{paths}} \exp \left[ -im\ell(x_1, x_2) \right] = G(x_1, x_2) \tag{1}
\]

where \( \ell(x_1, x_2) \) is the length of the path. This suggests that one can interpret \( G(x_1, x_2) \) as an amplitude for a particle/antiparticle to propagate between two events in the spacetime.\(^1\). This interpretation acquires an operational meaning in the presence of a source \( J(x) \) capable of emitting/absorbing the particles [1]. Then the vacuum persistence amplitude

\[
\langle \text{out}|\text{in} \rangle_J = \langle \text{out}|\text{in} \rangle_{J=0} \exp \left\{ -\frac{1}{2} \int d^Dx_1 \sqrt{-g_1} \int d^Dx_2 \sqrt{-g_2} J(x_1) G(x_1, x_2) J(x_2) \right\} \tag{2}
\]

\(^1\)I use mostly negative signature — except when specified otherwise — and natural units. The propagator in momentum space \( G(p) = i(p^2 - m^2 - i\epsilon)^{-1} \) is defined with an \( i \) factor, so that \( G(x_1, x_2) = \langle 0|T[\phi(x_1)\phi(x_2)]|0 \rangle \).
can be thought of describing the emission/absorption at the two events (controlled by \( J(x_1), J(x_2) \)) and the propagation between the events governed by \( G(x_1, x_2) \).

I am interested in the stationary situations in which the propagator depends on the time coordinates only through the time difference, so that \( G(x_1, x_2) = G(\tau; x_1, x_2) \) with \( \tau \equiv (x_1^0 - x_2^0) \equiv (\tau_1 - \tau_2) \). Such stationarity is assured if there exists a Killing vector field \( \xi_a \), which, in suitable coordinate system, can be represented as \( \xi = \partial/\partial \tau \). One can then interpret the temporal Fourier transform

\[
A(\Omega; x_1, x_2) = \int_{-\infty}^{\infty} d\tau \, G(\tau; x_1, x_2) \, e^{i\Omega \tau}; \quad \tau = (\tau_1 - \tau_2)
\]  

as the amplitude for the particle to propagate between \( x_1 \) and \( x_2 \) with energy \( \Omega \), introduced as the Fourier conjugate to the time coordinate \( \tau \). In what follows I will simplify the notation and write \( G(\tau) \) for \( G(\tau; x_1, x_2) \) and \( A(\Omega) \) for \( A(\Omega; x_1, x_2) \), suppressing the spatial coordinates. While evaluating the amplitude \( A(\Omega) \) in Eq. (3) it is convenient to assume that \( \Omega > 0 \) and interpret \( A(-\Omega) \) as the expression obtained by replacing \( \Omega \) by \(-\Omega \) in the result of the integral in Eq. (3). My interest lies in comparing \( A(-\Omega) \) with \( A(\Omega) \). If they are equal then the amplitudes for the particle to propagate with an energy \( \Omega \) or \(-\Omega \) are the same; but when they are unequal it indicates some interesting physics.

To probe this issue, let us consider the explicit form of \( G(\tau) \) in a \( D \)-dimensional flat spacetime given by (with \( m^2 \) treated as \( m^2 - i\varepsilon \)):

\[
G(\tau) = i \left( \frac{1}{4\pi i} \right)^{D/2} \int_0^\infty \frac{ds}{s^{D/2}} \exp \left[ -ism^2 - i \frac{s}{4} \sigma^2(\tau) \right]
\]  

where \( \sigma^2(\tau) \equiv \sigma^2(x_1, x_2) \) is the squared line interval between the two events. The fact that \( \sigma^2 \) depends only on \( \tau = \tau_1 - \tau_2 \) again arises from the stationarity of the background and the existence of the Killing vector \( \partial/\partial \tau \). From the structure of the integral in Eq. (3) it is obvious that, if \( G(\tau) = G(-\tau) \), then \( A(\Omega) = A(-\Omega) \) so that nothing very interesting happens. This is, of course, trivially true if we take \( \tau \) to be the standard inertial time coordinate \( t \) so that \( \sigma^2(t) = t^2 - |x_1 - x_2|^2 \). This makes \( \sigma^2 \) and \( G \) even functions of the time difference, leading to \( A(\Omega) = A(-\Omega) \).

Interestingly enough, the same result holds even when both events \( x_1 \) and \( x_2 \) are on the right Rindler wedge (R) with \( \tau \) being the Rindler time coordinate. In R the Rindler coordinates \((\tau, \rho, x_\perp)\) can be defined\(^2\) in the usual manner as \( t = \rho \sinh \tau, \ x = \rho \cosh \tau \). The line interval \( \sigma^2_{RR}(\tau) \) for two events in the right wedge has the form

\[
\sigma^2_{RR}(\tau) = -L_1^2 + 2\rho_1 \rho_2 \cosh \tau
\]  

where \( L_1^2 = (\Delta x_1^2 + 2\xi_1 + 2\xi_2) \), with the \( \xi \) coordinate defined through the relation \( x^2 - t^2 = 2\xi \). (In R, \( 2\xi = \rho^2 \)). The \( \sigma^2_{RR}(\tau) \) is clearly an even function of \( \tau \) and hence we reach the following conclusion: When a particle propagates between any two events within the right Rindler wedge R, we have \( A(\Omega) = A(-\Omega) \) and nothing fancy happens.\(^3\)

Let us next consider what happens when we take one event to be in R and the second event to be in F where the Rindler-like coordinate system is introduced through \( t = \rho \cosh \tau \) and \( x = \rho \sinh \tau \). (If

\(^2\)We will work with units such that the acceleration \( g \) of the Rindler frame is unity. In the coordinate transformation from \((t, x, x_\perp)\) to \((\tau, \rho, x_\perp)\), the transverse coordinates \( x_\perp \) go for a ride and I will not display them unless necessary.

\(^3\)The Unruh-Dewitt detector response [2], for a uniformly accelerated trajectory in R, is computed by a Fourier transform similar to the one in Eq. (3), for events with \( \rho_2 = \rho_1, \Delta x_\perp = 0 \), with the Wightman function replacing the propagator. This, of course, leads to \( A(-\Omega) \neq A(\Omega) \). The difference arises due to the difference in the structure of Wightman function and the Feynman propagator. Algebraically, \(|\sinh^2(\tau/2) - i\varepsilon|\) — which occurs in the propagator — is an even function of \( \tau \) while \( \sinh^2[(\tau/2) - i\varepsilon] \) — which occurs in the Wightman function — is not.
one uses the $\xi$ coordinate, then the relation $x^2 - t^2 = 2\xi$ allows the region $F$ to be covered by the range $-\infty < \xi < 0$ and the region $R$ to be covered by the range $0 < \xi < \infty$.) The line interval $\sigma_{FR}^2(\tau)$ between an event $(\tau_F, \rho_F)$ in $F$ and an event $(\tau_R, \rho_R)$ in $R$ is given by

$$
\sigma_{FR}^2(\tau) \equiv (t_F - t_R)^2 - (x_F - x_R)^2 - \Delta x_{\pm}^2 \tag{6}
$$

$$
= (\rho_F \cosh \tau_F - \rho_R \sinh \tau_R)^2 - (\rho_F \sinh \tau_F - \rho_R \cosh \tau_R)^2 - \Delta x_{\pm}^2 \tag{7}
$$

$$
= \rho_F^2 - \rho_R^2 - 2\rho_F\rho_R \sinh(\tau_R - \tau_F) - \Delta x_{\pm}^2 \tag{8}
$$

$$
= -L_2^2 - 2\rho_F\rho_R \sinh \tau; \quad \tau \equiv (\tau_R - \tau_F) \tag{9}
$$

with $L_2^2 \equiv (\Delta x_{\pm}^2 + 2\xi_F + 2|\xi_F|)$. I displayed this calculation in gory detail because there is a bit of algebraic sorcery involved in it. (This is the only non-trivial calculation in this paper!) The line interval $\sigma^2(\mathcal{P}_1, \mathcal{P}_2)$ between any two events in the spacetime, of course, is symmetric with respect to the interchange of events, $\sigma^2(\mathcal{P}_1, \mathcal{P}_2) = \sigma^2(\mathcal{P}_2, \mathcal{P}_1)$. In our case, the two events have the coordinates

$$
\mathcal{P}_1 = \mathcal{P}_F = (t_F, x_F, x_F^\perp) = (\rho_F \cosh \tau_F, \rho_F \sinh \tau_F, x_F^\perp) \tag{10}
$$

and

$$
\mathcal{P}_2 = \mathcal{P}_R = (t_R, x_R, x_R^\perp) = (\rho_R \cosh \tau_R, \rho_R \cosh \tau_R, x_R^\perp). \tag{11}
$$

The symmetry of the line interval is manifest in the inertial coordinates and we have $\sigma^2(t_F, x_F; t_R, x_R) = \sigma^2(t_R, x_R; t_F, x_F)$. But you cannot display the same symmetry by interchanging the relevant Rindler coordinates! From the Eq. (8) we see that

$$
\sigma^2(\tau_F, \rho_F, x_F^\perp; \tau_R, \rho_R, x_R^\perp) \neq \sigma^2(\tau_R, \rho_R, x_R^\perp; \tau_F, \rho_F, x_F^\perp) \tag{12}
$$

Of course, if you introduce arbitrary coordinate labels to events in spacetime, there is no assurance that the interchange of coordinate labels will correspond to the interchange of events, when two different coordinate charts are involved. This is precisely what happens here: It is obvious from Eq. (10) and Eq. (11) that the interchange $(\tau_F, \rho_F) \leftrightarrow (\tau_R, \rho_R)$, of coordinate labels we are using, does not lead to the interchange of the events $\mathcal{P}_1 \leftrightarrow \mathcal{P}_2$ because two different coordinate charts are used in $R$ and $F$.

We will now compute the Fourier transform in Eq. (3) with respect to $\tau \equiv (\tau_R - \tau_F)$. The sign convention in Eq. (3) implies that $G$ picks up a contribution $A(\Omega) \exp -i\Omega(\tau_R - \tau_F)$ which will correspond to a positive energy with respect to $\tau_R$ when $\Omega > 0$ (and negative energy when $\Omega < 0$). These are defined with respect to $\tau_R$ which is a valid time coordinate in $R$. (So I do not have to worry about the fact that $\tau_F$ has no clear meaning as a time coordinate in $F$; it is an ignorable constant which goes away when I do the integral over the range $-\infty < \tau < \infty$.) The Fourier transform in Eq. (3) requires us to compute the integral:

$$
I = \int_{-\infty}^{\infty} d\tau \ e^{i\Omega \tau - \frac{i}{2} \sigma_{FR}^2(\tau)} = 2 \ e^{\frac{ic_i^2}{2}} \ e^{-\pi\Omega/2} K_{\Omega}(2\alpha) \tag{13}
$$

where $\alpha \equiv (\rho_1\rho_2/2s)$. This was done using the standard integral representation for the McDonald function, leading to:

$$
\int_0^{\infty} dq \ q \ e^{\alpha(q - t)} = 2 \ e^{-\pi\omega/2} \ K_{\omega}(2\alpha); \quad (\alpha > 0) \tag{14}
$$

\footnote{How come $\sigma^2$ between the events in $R$ and $F$ only depends on the difference in ‘time’ labels, especially since $\tau$ is not even a time variable in $F$? This has to do with the fact that one can indeed introduce, a (Schwarzschild-like) coordinate system covering both $R$ and $F$ in which the 2-D metric takes the form $ds^2 = (\xi^2)\rho^2 - (2\xi)^{-2}d\xi^2$. We see that $\tau$ retains its Killing character both in $R$ and $F$, though $\partial/\partial \tau$ is timelike only in $R$. It is the Killing character which ensures that $\sigma_{FR}^2$ only depends on the difference in the ‘time’ labels.}
Substituting Eq. (13) into Eq. (3), we find that the relevant amplitude is given by

\[ A(\Omega) = e^{-\pi\Omega/2} \int_0^\infty ds \, F(s)K_{i\Omega}(2\alpha) \]  

where

\[ F(s) = 2i \left( \frac{1}{4\pi is} \right)^{D/2} e^{-im^2s+i/2} \]  

Since \( K_{i\Omega} = K_{-i\Omega} \) is an even function of \( \Omega \), it follows that

\[ A(-\Omega) = e^{\pi\Omega/2} \int_0^\infty ds \, F(s)K_{i\Omega}(2\alpha) = e^{\pi\Omega} A(\Omega) \]  

leading to the familiar Boltzmann factor

\[ \frac{|A(\Omega)|^2}{|A(-\Omega)|^2} = e^{-2\pi\Omega} \]  

corresponding to the Davis-Unruh \[3\] temperature \( T = g/2\pi = 1/2\pi \) in our units. Interpreting \( |A(\Omega)|^2 \) etc. as probabilities and considering two energy “levels” \( \Omega \) and \(-\Omega\), we find that the probability for the particle to propagate with the higher energy is suppressed by the Boltzmann factor.

This result tells us that something non-trivial happens when a particle propagates between any pair of events located on two sides of the horizon,\(^5\) such as between events in the region F and region R. Clearly the situation now is different compared to what happens when the particles are propagating between events within the region R, when nothing interesting happens. It is nice to see that the standard thermal behaviour of the Rindler horizon has left a trace in the inertial propagator.

I find it particularly gratifying that the propagator can distinguish so nicely between the propagation across the horizon from the propagation within one side of the horizon. Let me stress how this fact prevents you from interpreting (‘understanding’) the Eq. (18) in a trivial manner: You might think, at first sight, that if I am Fourier transforming \( G \) with respect to the Rindler time \( \tau \) (and define positive/negative energies through \( \exp \pm i\Omega \tau \)) then it is a foregone conclusion that I will get the thermal factor. \textit{This is simply not true.} Recall that, when I do the Fourier transform with respect to Rindler time etc. but for two events within the right wedge R, I get \textit{nothing} interesting. So the usual suspect, viz., \( \exp -i\Omega \tau \) being a superposition of \( \exp \pm i\Omega \tau \), is \textit{not} responsible for this result. There are two other crucial ingredients which go into it. First, you need horizon crossing to break the symmetry between \( G(\tau) \) and \( G(-\tau) \); this is obtained, as I said, by the only non-trivial calculation in this paper, leading to Eq. (9). Second, it is crucial that the result in Eq. (9) depends only on the difference \( \tau \equiv (\tau_R - \tau_F) \). So when I integrate over all \( \tau \), I don’t have to worry what \( \tau_F \) means, since it is not a time coordinate in F. I can stay in R and interpret everything using \( \tau_R \). Therefore, it is not just using the Rindler time coordinate which leads to the result. The structure of the propagator is more nontrivial than one would first imagine.

In obtaining this result, I worked entirely in the Lorentzian sector with a well-defined causal structure and the horizons at \( x^2 - t^2 = 0 \). I have also emphasized the key role played by the horizon in obtaining this result. You may wonder what happens to this analysis if it is done with the inertial propagator in the Euclidean sector. In the conventional approach, the right wedge (with \( t = \rho \sinh \tau, x = \rho \cosh \tau \)) itself will fill the \textit{entire} Euclidean plane \((t_E, x_E)\) if we take \( it = t_E, i\tau = \tau_E \) leading to \( t_E = \rho \sin \tau_E, x = \rho \cos \tau_E \).

\(^5\)The analysis leads to similar conclusions for other situations when the events are separated by a horizon, like for e.g., between region P and region L. I will concentrate on F and R.
The horizons \((x^2 - t^2 = 0)\) map to the origin \((x^2 + t_E^2 = 0)\) and the F,P,L wedges seem to disappear! At first sight, it is not clear how to recover the information contained in the F,P,L wedges if we start with the Euclidean, inertial, propagator. However, it can be done but one needs to use four different types of analytic continuations to proceed from the Euclidean plane to the four Lorentzian sectors (R, F, L, P). I have described this briefly in Appendix A for the sake of completeness.

2 The horizon thermality hiding in the inertial propagator

Given these facts, let me probe the structure of the inertial propagator a little more closely. While obtaining the above result I did not compute the final integral in Eq. (15) because it was unnecessary. However, this can be done both for events in R and for two events separated by a horizon. The relevant integrals are simpler to exhibit if we first get rid of the transverse coordinates, by Fourier transforming both sides of Eq. (3) with respect to the transverse coordinate difference \((x_1^\perp - x_2^\perp)\), thereby introducing the conjugate variable \(k^\perp\). (As usual, I will simply write \(\perp\).)

On the other hand, because \(K_{i\Omega} = K_{-i\Omega}\) we trivially get \(A_{RR}(\Omega) = A_{RR}(-\Omega)\). So the explicit computation verifies the previous result but — as I will argue later — the original approach offers greater generality.

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As before, I have already Fourier transformed with respect to the transverse coordinate difference \((x_1^\perp - x_2^\perp)\) thereby introducing the conjugate variable \(k^\perp\). Further \(\mu^2 = k_\perp^2 + m^2\) (This expression, with a \(|\theta - \theta'|\) is well-known in literature and is very easy to derive. In Appendix A, I have given the derivation as well.
as its relation with the form in Eq. (19), which is based on another variant with \((\theta - \theta')\); this one is a bit nontrivial to derive. Using just a series of Bessel function identities and no physics input, this result can be re-expressed in the following form:

\[
G_{\text{Eu}}^{\text{inertial}}(\theta - \theta') = \sum_{n=-\infty}^{\infty} G_{\text{Eu}}^{\text{Rindler}}[\theta - \theta' + 2\pi n]
\]

(23)

where the function \(G_{\text{Eu}}^{\text{Rindler}}\) is given by:

\[
G_{\text{Eu}}^{\text{Rindler}} = \frac{1}{\pi^2} \int_0^\infty d\omega \ (\sinh \pi\omega) K_{i\omega}(\mu \rho) K_{i\omega}(\mu' \rho') e^{-\omega|\theta-\theta'|}
\]

(24)

This results tells us two things: (a) First, the Euclidean version of our standard inertial propagator can be expressed as an infinite, periodic sum in the (Euclideanised) Rindler time. The fact that inertial propagator is periodic in (Euclideanised) Rindler time is a trivial result; you only need to note that the \(\sigma_{RR}^2\) in Eq. (5) is periodic in \(i\tau\). But Eq. (23) and Eq. (24) give us a lot more information. They explicitly express \(G_{\text{Eu}}^{\text{inertial}}\) as this an infinite periodic sum of another specific function \(G_{\text{Eu}}^{\text{Rindler}}\). (b) From the product structure of \(G_{\text{Eu}}^{\text{Rindler}}\), we learn that, when analytically continued back to Lorentzian sector, it can be thought of as a propagator built from another set of mode functions:

\[
\phi_{\nu}(\tau, \rho) = \frac{1}{\pi} (\sinh \pi\nu)^{1/2} K_{i\nu}(\mu \rho) e^{-i\nu\tau}
\]

(25)

in the standard fashion with time ordering with respect to \(\tau\). This allows us to discover the Rindler mode functions, Rindler vacuum and the Rindler propagator, just from analyzing the inertial propagator and rewriting it as in Eq. (23) and Eq. (24). (Of course, the modes in Eq. (25) satisfy the Klein-Gordon equation and are properly normalized.) So just staring at the inertial propagator, you can discover the Rindler modes and the Rindler vacuum.

There is another, closely related, feature. To bring this out, I will introduce a reflected wave function \(\phi^{(r)}_{\nu}\) by the definition

\[
\phi^{(r)}_{\nu}(\rho, \tau) = \phi_{\nu}(-\rho, \tau - i\pi) = \phi_{\nu}(\rho', \tau')
\]

(26)

The adjective “reflected” is justified by the facts that: (i) The coordinates \(\rho\) and \(-\rho\) are obtained by a reflection through the origin and (ii) the replacement of \(\tau\) by \(\tau - i\pi\) in the Rindler coordinate transformation takes you from \(R\) to \(L\). (If you replace \(\rho\) by \(-\rho\) and also replace \(\tau\) by \(\tau - i\pi\) in the coordinate relations \(x = \rho \cosh \tau, t = \rho \sinh \tau\), you will get back to the same event in \(R\). But \(\phi^{(r)}_{\nu}(\rho, \tau) \neq \phi_{\nu}(\rho, \tau)\), making the reflected wave function different from the original one.) It turns out that propagator for two events within the right wedge can be expressed in a very suggestive form as:

\[
G^{(RR)} = \int_0^\infty d\nu \left[ (n_{\nu} + 1) \phi_{\nu} \phi^{(r)}_{\nu} + n_{\nu} \phi^{*}_{\nu} \phi^{(r)*}_{\nu} \right]
\]

(27)

where \(n_{\nu}\) is the thermal population:

\[
n_{\nu} = \frac{1}{e^{2\pi \nu} - 1}
\]

(28)

Obviously, the second term in Eq. (27) suggests an absorption process weighted by \(n_{\nu}\) while the first term could represent emission with the factor \(n_{\nu} + 1\) coming from a combination of stimulated emission and

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6The proofs for all these claims, like e.g., Eq. (22), Eq. (23), Eq. (24) are sketched in Appendix A
spontaneous emission. If we think of $\phi_\nu$ and $\phi^*_\nu$ as the wave functions for a fictitious particle, then this structure again encodes the usual thermality.  

Since the Rindler frame is just a coordinate transformation of the inertial frame and the propagator $G(x_1, x_2)$ transforms as a bi-scalar under coordinate transformation, we can trivially represent it in Rindler coordinates. Further because $G(x_1, x_2)$ encodes all the physics contained in a free field we should be able to discover the thermality just by staring at $G(x_1, x_2)$. In other words, it should not be necessary for me to quantize the field in Rindler coordinates, identify positive frequency modes, construct Rindler vacuum and particles etc etc. Everything should flow out of $G(x_1, x_2)$ expressed in Rindler coordinates including the alternative, Rindler, quantization. This is what I have achieved in the above discussion.

3 Discussion

3.1 Comparison with other approaches

There are three other main approaches which follows similar philosophy — viz., to obtain the Davies-Unruh temperature without using explicit Rindler quantization — as far as thermality of the horizon is concerned. (None of them, however, takes you beyond that, to the results I have obtained in Section 2.) The first one is through the response of Unruh-Dewitt detector in which one merely calculates a Fourier transform of the Wightman function. The second is the path integral approach used in [5]. Finally, the horizon tunneling approach (see, for example, [6]) has some superficial similarity with the ideas presented above.

The approach in Section 1 of this paper is quite different from all the three approaches mentioned above. To begin with, it makes use of the Feynman propagator, the central quantity in QFT, and obtains the thermality from it. I stress that the Feynman propagator has a hidden structure which ensures that: (i) nothing non-trivial happens when the two events are in R but (ii) the notion of thermality arises when one events are separated by a horizon. So the ‘horizon crossing’ plays a crucial but hidden role. This is not the case with the calculation of the detector response. It is not obvious (in a calculation confined within R) what exactly is the role played by the horizon, if any. In fact a detector in any non-trivial trajectory will click — albeit in a complicated and time dependent manner — even if there is no horizon. So the superficial similarity — of evaluating a Fourier transform of a two-point function — should not mislead you in this matter.

The path integral approach in [5], again, has a superficial similarity with what I have done here. However, there are some significant differences. First, the derivation in [5] suggests that the probability, for the absorption of a particle by a region beyond the horizon, is related by a thermal factor to the probability for the emission from that region. This is very different from the interpretation I am trying to advocate. I just look at the propagation amplitude $A(\Omega)$ in energy domain and ask how $A(\Omega)$ and $A(-\Omega)$ are related, for propagation between the same pair of events. I have to again stress that the non-trivial structure of the Feynman propagator ensures that when the events are separated by a horizon a thermal relationship arises. Second, the analysis in [5] crucially uses the white hole region (P) to arrive at the conclusion. My approach just uses F and R and hence is conceptually clearer.

7In the usual approach, the Bogoluibov transformation between inertial and Rindler modes involves $|\beta|^2 \sim n_\nu, |\alpha|^2 \sim (1+n_\nu)$ and one can transform $G(x_1, x_2) = \langle 0 | T [\phi(x_1) \phi(x_2)] | 0 \rangle$, expressed in inertial modes to one involving Rindler modes. This is a way of connecting up Eq. (27) to something more familiar. The factors multiplying $(1+n)$ and $n$ can be related to the Bremsstrahlung by an accelerating source. In fact, both terms will correspond to emission when viewed in the inertial frame.
While on the topic of path integrals, it is important to stress the following fact. When you calculate the Feynman propagator between two events in R, you do sum over paths which criss-cross the horizon and meander into F. All this criss-crossing does not lead to any non-trivial feature when both the events are in R. (Roughly speaking, such paths have to cross the horizon an even number of times and the net effect between a pair of such crossings is zero.) So, even though $G(x_1, x_2)$ for two events in R does have contribution from paths which visit region F, it does not lead to any non-trivial result. On the other hand, when the two events are in F and R, we do get something non-trivial.

Finally, my approach is quite distinct from the standard lore of deriving thermality from horizon tunneling. First, the tunneling approach — like the path integral approach — tries to relate the amplitude for absorption by F to the emission from F and claims that these two are different because of the pole structure in complex plane. I did not have to resort procedures like analytic continuation in the main derivation. Further, it is not very clear how structure of quantum field theory — encoded in the propagator — is incorporated in the tunneling approach. In contrast, it is very clear in what I have done.

### 3.2 Generalizations

The approach, and the result, have obvious generalizations to more complicated situations and I concentrated on the Rindler thermality only for keeping things simple. To begin with, the result can be extended to de Sitter spacetime in a straightforward manner because the dependence of the propagator on the geodesic distance (see, for e.g., [7]) allows the same derivation to go through. More generally, one can use this approach to attribute thermality to any local Rindler horizon along the following lines.

In an arbitrary spacetime, pick an event $P$ and introduce the Riemann normal coordinates around $P$. These coordinates will be valid in a region, $V$, of size $L$ where the typical background curvature is of the order of $L^{-2}$. Introduce now a local Rindler coordinate system by boosting with an acceleration $g$ with respect to the local inertial frame, defined in $V$. If we now concentrate on events $(x_1, x_2)$ within $V$, then the standard Schwinger-DeWitt expansion of the propagator tells us that the form in Eq. (4) will be (approximately) valid. The Fourier integral in Eq. (3) can be defined formally, though the range of $\tau$ outside the domain $V$ is not meaningful. To circumvent this, we have to arrange matters such that most of the contribution to the integral in Eq. (3) comes from the range $\tau \lesssim L$. This, in turn, requires us to concentrate on the high frequencies with $\Omega \gg L^{-1}$. In this high frequency limit everything will go through as before and one will obtain the local Rindler temperature to be $T = g/2\pi$. For consistency, we also need to ensure that $gL \gg 1$ which, of course, can be done around any event with finite $L$. (In fact, this approach suggests a procedure for obtaining the curvature corrections to the temperature systematically, using the Schwinger-DeWitt expansion.) I stress that — in this very general context of a bifurcate Killing horizon, introduced into a local inertial frame — my approach gets you whatever you could reasonably expect. After all, in a curved spacetime, one can expect thermality (with approximately constant temperature) only when the modes do not probe the curvature scale; this is what is achieved by concentrating on the Feynman propagator at two events which are localized within $V$.

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8In the case curved spacetimes with horizons, like in Schwarzschild, Reissner-Nordstrom etc. we get the same result by explicit computation in $D = 2$. In $D > 2$, we do not have closed expressions for $G(x, x')$, but one can compute it close to the horizon. This is because, close to the horizon, you again get a 2D CFT and one can compute approximate form of the modes — and through them — the propagator $G(x, x')$. This will lead to the same result.
3.3 Future directions

There are three avenues of further work which seem interesting. First is to probe the uniqueness of the result in Eq. (23) and Eq. (24). I have shown that, starting from just the Euclidean version of the inertial propagator and the coordinate transformation in the right wedge, one can obtain Eq. (23) and Eq. (24). This is just Bessel function gymnastics with no physics input. But the resulting structure in Eq. (24) — involving product of mode functions and time ordering with respect to $\tau$, when analytically continued back into Lorentzian sector — immediately suggests an alternative set of mode functions (with positive/negative frequency decomposition with respect to $\tau$), corresponding Rindler vacuum and the Rindler propagator. Then Eq. (23) tells us that inertial vacuum will appear as a thermal state in the new representation. Only thing missing is a proof that the form of the infinite periodic sum in Eq. (23) and Eq. (24) is unique. I think this is true but might require some analyticity assumptions.

Second, one might like to probe the details of emission/absorption by localized sources (e.g, on two sides of a horizon) using the expression in Eq. (2) and connecting up with the structure in Eq. (27). This will throw more light on how such processes appear in inertial coordinates versus Rindler coordinates. In fact, I expect both processes to appear as emission in the inertial frame.

Third, it will be interesting to see whether the path integral in Eq. (1) can be computed from first principles in the Rindler coordinates. It can be done (even with a non-quadratic action) in inertial coordinates by a lattice regularization [8]. But it is not clear how to introduce a suitable lattice, either in polar coordinates in the Euclidean sector, or in the Rindler frame in the Lorentzian sector. These and related issues are under investigation.

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Appendix A: The unreasonable effectiveness of the Euclidean continuation

I will briefly outline the steps involved in obtaining Eq. (19), Eq. (20), Eq. (23), Eq. (24), Eq. (27) and some related results, postponing their detailed discussion to another publication. I will now use mostly positive signature so that the analytic continuation of the time coordinate leads to a positive definite metric.

One can obtain Eq. (19) and Eq. (20) by doing the remaining integral in Eq. (15) (and the analogous one for RR case) but this requires fairly complicated manipulation of known integrals over Bessel functions. But, since I also want to describe how to do the analytic continuation from the Euclidean sector to get all the four wedges (R, F, L, P), I will follow an alternative route. I will start from the Euclidean propagator and obtain all the relevant results we need by careful analytic continuation.

The Euclidean (inertial) propagator can be expressed in polar coordinates (with $x = \rho \cos \theta$, $t_E = \rho \sin \theta$) in the following form

$$G_{\text{Eu}}(k_\perp; \rho_1, \rho_2, \theta) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\nu \, e^{\pi \nu} K_{i\nu}(\mu \rho_2) K_{i\nu}(\mu \rho_1) \, e^{-\nu|\theta|}$$

In obtaining this propagator, I have already Fourier transformed with respect to the transverse coordinate difference $(x_1^\perp - x_2^\perp)$ thereby introducing the conjugate variable $k_\perp$. Further $\mu^2 = k_{\perp}^2 + m^2$. This result
is well known in literature and is trivial to obtain. One begins by noting that if you Fourier transform the transverse coordinates in the Euclidean version of the propagator in Eq. (4) you just get the reduced (two-dimensional) propagator, viz. $K_0(\mu \ell)/2\pi$ where $\ell = |\rho_1 - \rho_2|$. One can then use a standard identity

$$
\frac{1}{2\pi} K_0(\mu \ell) = \frac{1}{\pi^2} \int_0^\infty d\nu \; K_{i\nu}(\mu \rho_1) \; K_{i\nu}(\mu \rho_2) \; \cosh[\nu(\pi - |\theta|)]
$$

(30)
to express it as an integral over the range $0 < \nu < \infty$. Extending the integration range to $(-\infty < \nu < \infty)$ we obtain Eq. (29).

To proceed from Eq. (30) (which has $|\theta_1 - \theta_2|$) to Eq. (19) or Eq. (20) (which have $(\theta_1 - \theta_2)$), one needs to do the analytic continuation of the variables in a specific way. Let me start with the approach to obtain Eq. (19). Usually one does the analytic continuation by $\theta_1 \rightarrow i\tau_1, \theta_2 \rightarrow i\tau_2$ and interpret $|\theta_1 - \theta_2|$ as $i|\tau_1 - \tau_2|$, transferring the ordering to $\tau$ coordinate. This, of course, will give the correct Lorentzian propagator but with an exp($-i\nu|\tau_1 - \tau_2|$) factor. To get $(\tau_1 - \tau_2)$ without the modulus, we need to employ the following analytic continuation: $(\rho_>, \theta) \rightarrow (\rho_>, i\tau)$ and $(\rho_>, \theta') \rightarrow (-\rho_>, \pi + i\tau)$ with the ordering $\rho_2 > \rho_<$. For complex numbers, we will interpret the relative ordering in $|z - z'|$ based on the real parts. This leads to the nice result that we now end up replacing

$$
e^{\pi \nu - \nu|\theta - \theta'|} \Rightarrow e^{-i\nu(\tau - \tau')}
$$

(31)

Substituting this into Eq. (29), one immediately obtains

$$
G_{Min} = \frac{i}{\pi} \int_{-\infty}^\infty \frac{d\nu}{2\pi} K_{i\nu}(\mu \rho_>) \; K_{i\nu}(\mu \rho_<) \; e^{-i\nu(\tau - \tau')}
$$

(32)
from which Eq. (19) follows. This is a simple way to get the result.

But if you don’t like simplicity and feel this is a bit too slick, let me show you how to get this result from published tables of integrals. You again begin by recalling that, when you Fourier transform with respect to transverse coordinates in the Lorentzian propagator, you get the two-dimensional result $G_{Min} = iK_0(\mu \ell)/2\pi$ with $\ell^2 = \rho_2^2 - 2\rho_<\rho_> \cosh(\tau_2 - \tau_1)$ where we have ordered the $\rho$-s as $\rho_2 > \rho_<$ for future convenience. (The $\tau$ ordering is irrelevant; note that, in Eq. (32), interchanging $\tau$ and $\tau'$ corresponds to reversing the sign of $\nu$ which makes no difference because $K_{i\nu}$ is an even function of $\nu$.) Next, you look up the integral 6.792 (2) of [11] which gives, as a special case, the result:

$$
\int_{-\infty}^\infty \frac{d\omega}{\pi} e^{-i\omega \tau} K_{i\omega}(a)K_{i\omega}(b) = K_0(\sqrt{a^2 + b^2 + 2ab \cosh \tau}); \quad (|\arg[a]| + |\arg[b]| + |\text{Im}[\tau]| < \pi)
$$

(33)
The left hand side almost looks like what we want but in the right hand side, the argument of $K_0$ has a term with $(+ \cosh \tau)$ while our $\ell^2$ has $(- \cosh \tau)$. We need to take care of this and also ensure that $\sigma^2$ comes up as the limit of $\sigma^2 + i\epsilon$ in the Lorentzian sector (i.e, $\text{Im}(\sigma^2) > 0$). To this end, make the following identification in Eq. (33):

$$
a = \mu \rho_< e^{i(\pi - \epsilon)}; \quad b = \mu \rho_>
$$

(34)
with real $\tau$. Then we have $|\arg[a]| + |\arg[b]| + |\text{Im}[\tau]| = \pi - \epsilon < \pi$ taking care of the condition in Eq. (33). Further, you can verify that the ordering $\rho_2 > \rho_<$ also ensures that $\text{Im}(\ell^2) > 0$ leading to

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"It is straightforward to verify that the coordinates transforms correctly from the Euclidean Rindler to Lorentzian Rindler under this transformation. To get the correct $i\epsilon$ prescription in the Lorentzian sector, it is important to interpret $(-\rho_<)$ as the limit of $\rho_< \exp[i(\pi - \epsilon)]$. This aspect has been noticed previously, in a different context, in Ref. [10]."
the correct $i\epsilon$ prescription in the Lorentzian sector. (The sign of imaginary part is decided by the sign of $(\rho_+ \cosh(\tau) - \rho_-)$ which remains positive due to our ordering of $\rho$-s.) We thus get our advertised result:

$$
\frac{i}{2\pi^2} \int_{-\infty}^{\infty} d\omega \ e^{-i\omega \tau} K_{i\omega}(-\mu \rho_-) K_{i\omega}(\rho_+) = \frac{i}{2\pi} K_0(\mu \ell) = G_{\text{Min}}
$$

which is the same as Eq. (32). I prefer the simpler derivation, though.

To obtain the structure in Eq. (20) we need to know how to proceed from the Euclidean sector to the wedge $F$. This is nontrivial because, in the usual procedure of analytic continuation ($\theta \rightarrow i \tau$) you go from $(\rho \sin \theta, \rho \cos \theta)$ to $(i \rho \sin \tau, \rho \cos \tau)$ which only covers the right wedge! But one can actually get all the four wedges from the Euclidean sector by using the following four sets of analytic continuations. (This is discussed in greater detail in [9].):

$$
R: \quad \rho \rightarrow \rho, \ \theta \rightarrow i \tau; \quad x = \rho \cosh \tau, \ t = \rho \sinh \tau
$$

$$
F: \quad \rho \rightarrow i \rho, \ \theta \rightarrow i \tau + \frac{\pi}{2}; \quad x = \rho \sinh \tau, \ t = \rho \cosh \tau
$$

$$
L: \quad \rho \rightarrow \rho, \ \theta \rightarrow i \tau - \pi; \quad x = -\rho \cosh \tau, \ t = -\rho \sinh \tau
$$

$$
P: \quad \rho \rightarrow i \rho, \ \theta = i \tau - \frac{\pi}{2}; \quad x = -\rho \sinh \tau, \ t = -\rho \cosh \tau
$$

Now using in $R$, $(\rho, \theta) \rightarrow (\rho, i \tau)$ and using in $F$, $(\rho, \theta) \rightarrow (i \rho, i \tau + \pi/2)$ along with the identity

$$
K_{i\nu}(iz) = -\frac{i\pi}{2} e^{-\pi \nu/2} H^{(2)}_{i\nu}(z) = -\frac{i\pi}{2} e^{\pi \nu/2} H^{(2)}_{i\nu}(z)
$$

one obtains a result similar to Eq. (32) with a Hankel function replacing one McDonald function. This gives you Eq. (20).

In fact the analytic continuations in Eq. (36) to Eq. (39) allow us to obtain the propagator for any pair of points located in any two wedges directly — and rather easily — from the Euclidean propagator. You get a $K_{i\nu} K_{i\nu}$ structure in RR, LL, RL and LR. (The notation AB corresponds to first event being in wedge A and second in wedge B.) In FF, PP, FP and PF the McDonald functions are replaced by the Hankel functions. In PR, FL, RF, LP, RP and LF you get a product of a Hankel and McDonald function. The interchange of F with P or R with L reverses the sign of $\nu$; so does the interchange of the two events. The similarity in structure with Minkowski-Bessel modes [4] is obvious. (These results agree with the ones in [13], obtained by more complicated procedure, except for some inadvertent typos in [13]). We will discuss this procedure and results in detail in another publication [9].

You can now obtain Eq. (27), working in the Lorentzian sector, by some further straightforward manipulations. One starts with Eq. (32) and converts it to an integral the range $(0 < \nu < \infty)$. Then using the results

$$
n_\nu = \frac{e^{-\pi \nu}}{2 \sinh \pi \nu}; \quad 1 + n_\nu = \frac{e^{\pi \nu}}{2 \sinh \pi \nu}
$$

we can rewrite the propagator as

$$
G^{(RR)} = \frac{i}{\pi^2} \int_0^{\infty} d\nu \ K_{i\nu}(\mu \rho_+) K_{i\nu}(-\mu \rho_-) \sinh \pi \nu \left[ e^{-\pi \nu} (n_\nu + 1) e^{-i\nu \tau} + n_\nu e^{i\nu \tau} \right]
$$

$$
= \frac{i}{\pi^2} \int_0^{\infty} d\nu \ K_{i\nu}(\mu \rho_+) K_{i\nu}(-\mu \rho_-) \sinh \pi \nu \left[ (n_\nu + 1)e^{-i\nu(\tau - i \pi)} + n_\nu e^{i\nu(\tau - i \pi)} \right]
$$

(42)
The pre-factors (outside the square bracket) lead to the product of wave functions in Eq. (27) and the shift \((\tau - i\pi)\) leads to the reflected coordinate.

However, the thermal factor in Eq. (27) finds a more natural home in the Euclidean sector. Let me show you how this comes about — using again a set of identities related to Bessel functions — when we work in the Euclidean sector. First, the Euclidean propagator \(K_0(\mu\ell)/2\pi\) (obtained after transverse coordinates are removed by a Fourier transform) satisfies a Bessel function addition theorem (see page 351 (8) of [12]) given by:

\[
G_E = \frac{1}{2\pi} K_0(\mu\ell) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} K_m(\mu\rho_+) I_m(\mu\rho_-) \cos m(\theta - \theta')
\]

The \(K_mI_m\) part of the above result can be rewritten in terms of another identity you can look up (see 6.794(10) of [11]):

\[
2\pi \int_0^\infty d\omega \, \omega \sinh \pi \omega \frac{K_{i\omega}(\mu\rho) K_{i\omega}(\mu\rho')}{\omega^2 + m^2} = K_m(\mu\rho_+) I_m(\mu\rho_-)
\]

reaching:

\[
G_E = \frac{1}{\pi^3} \sum_{m=-\infty}^{\infty} \int_0^\infty d\omega \, \omega \sinh \pi \omega \frac{K_{i\omega}(\mu\rho) K_{i\omega}(\mu\rho')}{\omega^2 + m^2} \cos m(\theta - \theta')
\]

The sum in the above expression can again be looked up (see 1.445 (2) of [11]); it is precisely the thermal factor in Eq. (27) written in Euclidean sector:

\[
T_\omega(\theta - \theta') = \sum_{n=-\infty}^{\infty} e^{-\omega |\theta - \theta'| + 2\pi n|}
\]

reaching:

\[
G_E = \frac{1}{\pi^3} \int_0^\infty d\omega \, (\sinh \pi \omega) K_{i\omega}(\mu\rho) K_{i\omega}(\mu\rho') T_\omega(\theta - \theta')
\]

What is nice is that the thermal factor in the Euclidean sector can also be expressed as a periodic sum in the Euclidean angle; that is, we can easily show that:

\[
T_\omega(\theta - \theta') = \sum_{n=-\infty}^{\infty} e^{-\omega |\theta - \theta'| + 2\pi n|}
\]

thereby making the periodicity in the Euclidean, Rindler time obvious. This is yet another hidden thermal feature of the inertial propagator! This allows us to write the Euclidean, inertial, propagator as a thermal sum:

\[
G_E = \sum_{n=-\infty}^{\infty} \frac{1}{\pi^2} \int_0^\infty d\omega \, (\sinh \pi \omega) K_{i\omega}(\mu\rho) K_{i\omega}(\mu\rho') e^{-\omega |\theta - \theta'| + 2\pi n|}
\]

This equation has a simple interpretation (which will be explored extensively in [9]): In the right hand side the \(n = 0\) terms is just the Euclidean propagator in the Rindler vacuum. The periodic, infinite, sum ‘thermalises’ it thereby producing the inertial propagator.
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