Commutators of the Fractional Hardy Operator on Weighted Variable Herz-Morrey Spaces

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1. Introduction

Hardy operators and related commutators play an indispensable role in the theory of partial differential equations [1, 2] and the characterization of function spaces [3–5]. Without going into much details, let us first define the fractional Hardy operators [3]

\[ H g(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|<|z|} g(t) dt, \quad H^{\ast} g(z) = \int_{|t|>|z|} \frac{g(t)}{|t|^{n-\beta}} dt, \quad z \in \mathbb{R}^n \setminus \{0\} \]  

(1)

and related commutators:

\[ [b, H_{\beta}] g = b H g - H(bg), \quad [b, H^{\ast}_{\beta}] g = b H^{\ast} g - H^{\ast}(bg). \]  

(2)

It is important to note that taking \( \beta = 0 \) in (1), we get multidimensional Hardy operator defined and studied in [6, 7]. Also, (1) reduces to the one dimensional Hardy operator [8] if we choose \( \beta = 0 \) and \( n = 1 \). Here, we cite some important literature with regards to the study of Hardy-type operators on different function spaces which include [9–15].

The new development of variable exponent commenced with the work of Kov’akračik and R’akosn’ik in [16], where a class of function spaces having variable exponent was defined, and basic properties of variable exponent Lebesgue space were explored. Recently, the theory of variable exponent analysis is modeled in terms of the boundedness of the Hardy Littlewood maximal operator \( M \) [17–21]:

\[ M g(z) = \sup_{B:\text{ball},z \in B} \frac{1}{|B|} \int_{|t|<|z|} |g(t)| dt. \]  

(3)

Besides, Muckenhoupt \( A_p \) theory [22] is generalized in the recent span of time with regard to variable exponent spaces ([23–28]). By taking into account the generalization of function spaces with variable exponents and the same with weights, many results like duality, boundedness of sublinear operators, the wavelet characterization, and commutators of fractional and singular integrals have been studied [29–38].

Recently, authors have studied generalized Herz space in terms of both Muckenhoupt weights and variable exponent [39–41]. Moreover, an idea of combining two function spaces to develop a new one is also an interesting problem in Harmonic analysis. One such problem is considered in [42] in which Herz-Morrey space was defined. Although,
the weighted versions of Herz-Morrey spaces were introduced recently in [43, 44].

In this piece of work, our main focus is on establishing the boundedness of commutators of fractional Hardy operators on a class of function spaces called the weighted Herz-Morrey space with variable exponents. We seek to find the boundedness of these commutators with symbol functions in BMO (bounded mean oscillation) spaces. In establishing such a boundedness, we make use of the boundedness of the fractional integral operator

$$I_β(g)(z) = \int_{\mathbb{R}^n} \frac{g(t)}{|z - t|^{n-β}} dt$$

on weighted Lebesgue space which was done in [39].

In the rest of this paper, the symbol $C$ expresses a constant whose value may differ at all of its occurrences. The Greek letter $χ_5$ denotes the characteristic function of a sphere $S$ where $S$ is a measurable subset of $\mathbb{R}^n$ and $|S|$ represents its Lebesgue measure. Before turning to our key results, let us first define the relevant variable exponent function spaces.

2. Preliminaries

Let us consider a measurable function $p(\cdot)$ on $\mathbb{R}^n$ having range $[1, ∞)$. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions $f$ such that

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{C} \mid \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\sigma} \right)^{p(x)} dx < \infty, \text{for some } \sigma > 0 \right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ turns out to be Banach function space under the norm:

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \sigma > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\sigma} \right)^{p(x)} dx \leq 1 \right\}.$$  

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot): \mathbb{R}^n \rightarrow (1, ∞)$ such that

$$1 < p_- ≤ p(x) ≤ p_+ < ∞,$$

where

$$p_- = \text{essinf}_{x \in \mathbb{R}^n} p(x), p_+ = \text{esssup}_{x \in \mathbb{R}^n} p(x).$$

**Definition 1.** Suppose $p(\cdot)$ is a real valued function on $\mathbb{R}^n$. We say that

(i) $\mathcal{C}^∗_{p(\cdot)}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying

$$|p(x) - p(y)| ≤ \frac{C}{\log ((x - y)/|x|)}, \quad |x - y| < \frac{1}{2}, \quad x, y \in \mathbb{R}^n.$$

(ii) $\mathcal{C}^\log_{p(\cdot)}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous function $p(\cdot)$ satisfying at the origin

$$|p(x) - p(0)| ≤ \frac{C}{\log ((x + 1/|x|)/|x|)}, \quad |x| < \frac{1}{2}, \quad x \in \mathbb{R}^n.$$  

(iii) $\mathcal{C}_C^\log(\mathbb{R}^n)$ is the set of all log-Hölder continuous functions satisfying at infinity

$$|p(x) - p_∞| ≤ \frac{C}{\log (e + |x|)}, \quad x \in \mathbb{R}^n.$$  

(iv) $\mathcal{C}_∞^\log(\mathbb{R}^n) = \mathcal{C}^\log_{p(\cdot)} \cap \mathcal{C}^\log_{loc}$ denotes the set of all global log-Hölder continuous functions $p(\cdot)$.

It was proved in [21] that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \mathcal{C}^\log(\mathbb{R}^n)$, then Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Suppose $w(\cdot)$ is a weight function on $\mathbb{R}^n$, which is non-negative and locally integrable on $\mathbb{R}^n$. Let $L^{p(\cdot)}(w)$ be the space of all complex-valued functions $f$ on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f(x)|^p w(x) dx < \infty$. The space $L^{p(\cdot)}(w)$ is a Banach function space equipped with the norm:

$$\|f\|_{L^{p(\cdot)}(w)} = \left\| f w^{1/p(\cdot)} \right\|_{L^{p(\cdot)}(w)}.$$  

Benjamin Muckenhoupt introduced the theory of $A_p(1 < p < ∞)$ weights on $\mathbb{R}^n$ in [22]. Recently, in [39, 40], Izuki and Noi generalized the Muckenhoupt $A_p$ class by taking $p$ as a variable.

**Definition 2.** Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight $w$ is an $A_{p(\cdot)}$ weight if

$$\sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \left\| w^{1/p(\cdot)} \chi_B \right\|_{L^{p(\cdot)}} \left\| w^{-1/p(\cdot)} \chi_B \right\|_{L^{p(\cdot)}} < ∞.$$  

In [25], the authors proved that $w \in A_{p(\cdot)}$ if and only if $M$ is bounded on the space $L^{p(\cdot)}$.

**Remark 3** (see [39]). Suppose $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \mathcal{C}^\log(\mathbb{R}^n)$ and $p(\cdot) ≤ q(\cdot)$, then we have

$$A_{1} \subset A_{p(\cdot)} \subset A_{q(\cdot)}.$$  

**Definition 4.** Suppose $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $β \in (0, n)$ such that $1/p_2(x) = 1/p_1(x) - β/n$. A weight $w$ is said to be $A(p_1(\cdot), p_2(\cdot))$ weight if

$$\|X_B\|_{\mathcal{L}(p_1(\cdot)/w^{p_1(\cdot)})} \|X_B\|_{\mathcal{L}(p_2(\cdot)/w^{p_2(\cdot)})} ≤ C|B|^{1−\frac{β}{n}}.$$  


Definition 5 (see [39]). Suppose \( p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( \beta \in (0, n) \) such that \( 1/p_2(x) = 1/p_1(x) - \beta/n. \) Then, \( w \in A_{(p_1(\cdot), p_2(\cdot))} \) if and only if \( w_{p_2(\cdot)} \in A_{1+p_1(\cdot)} \).

Now, we define the variable exponent weighted Morrey-Herz space \( MK_{\mathbb{R}^n}^{p(\cdot), L}(w) \). Let \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \}, A_k = B_k/B_{k-1}, \) and \( \chi_k = \chi_n \) for \( k \in \mathbb{Z}. \)

Definition 6. Let \( w \) be a weight on \( \mathbb{R}^n, \lambda \in [0, \infty), q \in (0, \infty), \rho(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( \alpha(\cdot) \colon \mathbb{R}^n \to \mathbb{R} \) with \( \alpha(\cdot) \in L^{\infty}(\mathbb{R}^n). \) The space \( MK_{\mathbb{R}^n}^{p(\cdot), L}(w) \) is the set of all measurable functions which is given by

\[
MK_{\mathbb{R}^n}^{p(\cdot), L}(w) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, w) : \|f\|_{MK_{\mathbb{R}^n}^{p(\cdot), L}(w)} < \infty \right\},
\]

where

\[
\|f\|_{MK_{\mathbb{R}^n}^{p(\cdot), L}(w)} = \sup_{k \in \mathbb{Z}} 2^{-k\lambda} \left( \sum_{k=\infty}^{\infty} 2^{k\lambda} q \|f\|_{L^{p(\cdot)}(w)} \right)^{1/q}. \quad (16)
\]

Obviously, \( MK_{\mathbb{R}^n}^{\rho(\cdot), L}(w) = MK_{\mathbb{R}^n}^{\rho(\cdot), L}(w) \) is the weighted Herz space with variable exponent (see [30]). Here, it is important to refer to some of the pioneering studies of the Herz space with constant exponents made in [45, 46].

3. Some Useful Lemmas

We start this section with some useful lemmas that will be helpful in proving our main results.

Lemma 7 (see [47]). If \( X \) is Banach function space, then

(i) The associated space \( X' \) is also Banach function space

(ii) \( \|\cdot\|_X \) and \( \|\cdot\|_X' \) are equivalent

(iii) If \( g \in X \) and \( f \in X' \), then

\[
\int_{\mathbb{R}^n} |f(x)g(x)| \leq \|g\|_X \|f\|_{X'},
\]

is the generalized Hölder inequality.

Lemma 8 (see [39]). Suppose \( X \) is a Banach function space. Then, we have that for all balls \( B, \)

\[
I \leq \frac{1}{|B|} \|X_B\|_{X'} \|X_B\|_X. \quad (19)
\]

Lemma 9 (see [28, 39]). Let \( X \) be a Banach function space. Suppose that the Hardy Littlewood maximal operator \( M \) is weakly bounded on \( X; \) that is,

\[
\|X_{\{M \geq \sigma\}}\|_X \leq \sigma^{-1} \|f\|_X
\]
is true for \( \sigma > 0 \) and for all \( f \in X. \) Then, we have

\[
\sup_{B \ni x} \frac{1}{|B|} \|X_B\|_{X'} \|X_B\|_X' < \infty. \quad (21)
\]

Lemma 10 (see [39, 48]).

(a) \( X(\mathbb{R}^n, W) \) is Banach function space equipped with the norm

\[
\|f\|_{X(\mathbb{R}^n, W)} = \|f\|_{X'}, \quad (22)
\]

where

\[
X(\mathbb{R}^n, W) = \{ f \in M : fW \in X \}. \quad (23)
\]

(b) The associate space \( X'(\mathbb{R}^n, W^{-1}) \) is also a Banach function space

Lemma 11 (see [39]). Let \( X \) be a Banach function space. Assume that \( M \) is bounded on \( X' \), then there exists a constant \( \delta \in (0, 1) \) for all \( B \subset \mathbb{R}^n \) and \( E \subset B, \)

\[
\|X_E\|_X \|X_B\|_{X'} \leq \left( \frac{|E|}{|B|} \right)^\delta. \quad (24)
\]

The paper [16] shows that \( L^{p(\cdot)}(\mathbb{R}^n) \) is a Banach function space and the associated space \( L^{p(\cdot)}(\mathbb{R}^n) \) with equivalent norm.

Remark 12. Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), and by comparing the Lebesgue space \( L^{p(\cdot)}(w^{p(\cdot)}) \) and \( L^{p(\cdot)}(w^{-p(\cdot)}) \) with the definition of \( X(\mathbb{R}^n, W) \), we have

(1) If we take \( W = w \) and \( X = L^{p(\cdot)}(\mathbb{R}^n) \), then we get \( L^{p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(w^{p(\cdot)}) \)

(2) If we consider \( W = w^{-1} \) and \( X = L^{p(\cdot)}(\mathbb{R}^n) \), then we have \( L^{p(\cdot)}(w^{-p(\cdot)}) = L^{p(\cdot)}(\mathbb{R}^n, w^{-1}) \)

By virtue of Lemma 10, we get \( (L^{p(\cdot)}(\mathbb{R}^n, w))' = (L^{p(\cdot)}(w^{p(\cdot)}))' = L^{p(\cdot)}(w^{-p(\cdot)}) = L^{p(\cdot)}(\mathbb{R}^n, w^{-1}). \) Next, in view of Lemma 11 and Remark 12, we have the following Lemma.

Lemma 13 (see [41]). Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \mathcal{C} \) and \( \log \mathbb{R}^n \) be a Log Hölder continuous function both at infinity and at origin, if \( u^{p(\cdot)} \in A_{p(\cdot)} \) implies \( u^{-p(\cdot)} \in A_{p(\cdot)}. \) Thus, the Hardy Littlewood operator is bounded on \( L^{p(\cdot)}(w^{p(\cdot)}) \), and there exist constants \( \delta_1, \delta_2 \in (0, 1) \) such that

\[
\frac{\|X_E\|_{L^{p(\cdot)}}(w^{p(\cdot)})}{\|X_B\|_{L^{p(\cdot)}}(w^{p(\cdot)})} \leq \frac{\|X_E\|_{L^{p(\cdot)}}(w^{-p(\cdot)})}{\|X_B\|_{L^{p(\cdot)}}(w^{-p(\cdot)})} \leq \left( \frac{|E|}{|B|} \right)^\delta_1, \quad (25)
\]

\[
\frac{\|X_E\|_{L^{p(\cdot)}}(w^{p(\cdot)})}{\|X_B\|_{L^{p(\cdot)}}(w^{p(\cdot)})} \leq \frac{\|X_E\|_{L^{p(\cdot)}}(w^{-p(\cdot)})}{\|X_B\|_{L^{p(\cdot)}}(w^{-p(\cdot)})} \leq \left( \frac{|E|}{|B|} \right)^\delta_2,
\]

for all balls \( B \) and all measurable sets \( E \subset B.\)
Lemma 14 (see [39]). Let $p_1 \in \mathcal P(\mathbb R^n) \cap \mathcal C^{\log}(\mathbb R^n)$ and $0 < \beta < n/p_1$, and let $\rho_1 \in \mathcal P(\mathbb R^n) \cap \mathcal C^{\log}(\mathbb R^n)$, and $\rho_2 \in \mathcal P(\mathbb R^n) \cap \mathcal C^{\log}(\mathbb R^n)$. If $\rho_1 \ll \rho_2$, then $\rho_1^\beta$ is bounded from $L^p_\mu(\mathbb R^n)$ to $L^q_\mu(\mathbb R^n)$.

4. Main Results and their Proofs

Definition 15. Let $f \in L^1_\text{loc}(\mathbb R^n)$ and set

$$\|b\|_{\text{BMO}} = \sup_B \int_B |b(x) - b_B|dx,$$

where the supremum is taken all over the balls $B \in \mathbb R^n$ and $b_B = [B]^{-1} \int_B b(y)dy$. The function $b$ is a bounded mean oscillation if $\|b\|_{\text{BMO}} < \infty$ and $\text{BMO}^\prime(\mathbb R^n)$ consist of all $f \in L^1_\text{loc}(\mathbb R^n)$ with $\text{BMO}^\prime(\mathbb R^n) \subset \text{BMO}^\prime(\mathbb R^n)$. For a comprehensive review of the $\text{BMO}$ space, we suggest the reader to follow the books [49, 50].

Lemma 16. Let $q(\cdot) \in \mathcal P(\mathbb R^n)$ and $w$ be an $A_q(\cdot)$ weight. Then, for all $b \in \text{BMO}$ and all $l, i \in \mathbb Z$ with $l > i$, we have

$$\|b\|_{\text{BMO}} \sim \sup_{B \text{Ball}} \|b\|_{L^{q_i(\cdot)}(\mathbb R^n)} \|B\|_{L^{q_i(\cdot)}(\mathbb R^n)}.$$  

Proof. First part of this lemma is a consequence of [41, Theorem 18]. Next, we will prove (28), for all $l, i \in \mathbb Z$ with $l > i$

$$\left\| (b - b_{Bi}) X_{Bi} \right\|_{L^{q_i(\cdot)}(\mathbb R^n)} \leq C \left\| (b - b_{Bi}) X_{Bi} \right\|_{L^{q_i(\cdot)}(\mathbb R^n)} + \left\| (b_{Bi} - b_{Bi}) X_{Bi} \right\|_{L^{q_i(\cdot)}(\mathbb R^n)}.$$  

In the view of (27), we have

$$\left\| (b - b_{Bi}) X_{Bi} \right\|_{L^{q_i(\cdot)}(\mathbb R^n)} \leq C \|b\|_{\text{BMO}} \|X_{Bi}\|_{L^{q_i(\cdot)}(\mathbb R^n)}.$$

Also, it is easy to see that

$$\left| b_{Bi} - b_{Bi} \right| \leq \sum_{n=1}^{l-i} |b_{n+1} - b_n| \leq \sum_{n=1}^{l-i} \frac{1}{|B_{n+1}|} \int_{B_{n+1}} |b_{n+1} - b(x)|dx$$

$$\leq C \sum_{n=1}^{l-i} \frac{1}{|B_{n+1}|} \int_{B_{n+1}} |b_{n+1} - b(x)|dx = C(l - i) \|b\|_{\text{BMO}(\mathbb R^n)}.$$

Combining (29), (30), and (31), we get (28). □

Proposition 17. Let $q(\cdot) \in \mathcal P(\mathbb R^n)$, $0 < p < \infty$, and $0 \leq \lambda \leq \infty$.

Proof. The proof is similar to the proof of Proposition 17 in [44]. So, we omit the details. □

Theorem 18. Let $0 < p_1 \leq p_2 < \infty$, $q_1(\cdot) \in \mathcal P(\mathbb R^n) \cap \mathcal C^{\log}(\mathbb R^n)$, and $q_1(\cdot)$ be such that $1/q_1(\cdot) = 1/q_2(\cdot) - \beta/n$.

Also, let $w^{\beta}(\cdot) \in A_1$, $b \in \text{BMO}(\mathbb R^n)$, $\lambda > 0$, and $\alpha(\cdot) \in L^\infty(\mathbb R^n) \cap \mathcal C^{\log}(\mathbb R^n)$ be log Hölder continuous at the origin, with $\alpha(0) \leq a(\infty) = \lambda + \delta/2 - \beta$, where $0 < \delta_2 < 1$, then

$$\|b \cdot H_{\alpha} f\|_{\text{MK}^{\alpha(\cdot)}_{p_2,q_1(\cdot)}(w^{\beta}(\cdot))} \leq C \|b\|_{\text{BMO}} \|f\|_{\text{MK}^{\alpha(\cdot)}_{p_2,q_1(\cdot)}(w^{\beta}(\cdot))}.$$  

Proof. For any $f \in \text{MK}^{\alpha(\cdot)}_{p_2,q_1(\cdot)}(w^{\beta}(\cdot))$, if we denote $f_i = f \cdot X_i$, and for each $l \in \mathbb Z$,

$$f(x) = \sum_{1 = -\infty}^{\infty} f_i(x),$$

then it is not difficult to see that

$$\|b \cdot H_{\alpha} f_i(x)\|_{\text{MK}^{\alpha(\cdot)}_{p_2,q_1(\cdot)}(w^{\beta}(\cdot))} \leq \frac{1}{|x|^{\beta - \beta}} \int_{B_j} |(b(y) - b(y))f(y)|dy \cdot X_j(x)$$

$$\leq 2^{j(n - \beta)} \int_{B_j} |(b(y) - b(y))f(y)|dy \cdot X_j(x)$$

$$\leq 2^{j(n - \beta)} \int_{B_j} |(b(y) - b_{Bi})f(y)|dy \cdot X_j(x)$$

$$+ 2^{j(n - \beta)} \int_{B_j} |(b(y) - b_{Bi})f(y)|dy \cdot X_j(x)$$

$$= E_1 + E_2.$$  

(35)
The generalized Hölder inequality (Lemma 7) yields the following inequality for $E_1$:

$$E_1 = 2^{j(n-\beta)} \sum_{l=\infty}^{j} \int B_l \left[ \left( b(x) - b_{B_l} \right) f(y) |dy \cdot \chi_j(x) \right] \leq 2^{j(n-\beta)} \sum_{l=\infty}^{j} \left[ \left( b(x) - b_{B_l} \right) f(y) \right] \| f \|_{L^q \left( \omega^1 \right)} \sup_{l=\infty}^{j} \left\| \chi_j(x) \right\| \sum_{l=\infty}^{j} \left( \| f \|_{L^q \left( \omega^1 \right)} \right),$$

which by virtue of Lemma 9 reduces to

$$\| [b, H_{\beta}] f \chi_j \|_{L^{q, \omega^1}} \leq 2^{j\beta} \| b \|_{BMO} \sum_{l=\infty}^{j} (j-1) \| f \|_{L^{q, \omega^1}} \sup_{l=\infty}^{j} \left\| \chi_j(x) \right\|^1 \| X_{B_l} \|_{L^{q, \omega^1}} \| X_{B_l} \|_{L^{q, \omega^1}},$$

Now using Lemma 13, we learn

$$\| [b, H_{\beta}] f \chi_j \|_{L^{q, \omega^1}} \leq 2^{j\beta} \| b \|_{BMO} \sum_{l=\infty}^{j} (j-1) \| f \|_{L^{q, \omega^1}} \sup_{l=\infty}^{j} \left\| \chi_j(x) \right\|^1 \| X_{B_l} \|_{L^{q, \omega^1}} \| X_{B_l} \|_{L^{q, \omega^1}},$$

In the definition of the fraction integral $I_{\beta}$, we replace $f$ by $X_{B_l}$ to obtain

$$I_{\beta} \left( X_{B_l} \right) (x) \geq C 2^{j\beta} X_{B_l} (x),$$

from which we infer that

$$X_{B_l} (x) \leq C 2^{-j\beta} I_{\beta} \left( X_{B_l} \right) (x).$$

Taking the norm on both sides and using Lemmas 14 and 9, respectively, we get

$$\left\| X_{B_l} \right\|_{L^{q, \omega^1}} \leq C 2^{j\beta} \left\| I_{\beta} \left( X_{B_l} \right) \right\|_{L^{q, \omega^1}} \leq C 2^{j\beta} \left\| X_{B_l} \right\|_{L^{q, \omega^1}} \leq C 2^{j(n-\beta)} \left\| X_{B_l} \right\|^1_{L^{q, \omega^1}}.$$
In view of Lemmas 8 and 9, the use of (44) into (41) results in the following inequality:

\[
\begin{align*}
\| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})} & \leq C \| b \|_{BMO} \sum_{l = -\infty}^{j} 2^{l(n-p_0)} 2^j (j - l) 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^j \times \left( \| X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^l \| X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^j \right)^{-1} \\
& \leq C \| b \|_{BMO} \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} (j - l) \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^j \times \left( 2^{-\lambda} \| X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^l \| X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^j \right)^{-1} \\
& \leq C \| b \|_{BMO} \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} (j - l) \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^j .
\end{align*}
\]

(45)

Now, by virtue of the condition \( p_1 \leq p_2 \) and Proposition 17, we have

\[
\| [b, H_\beta] f X_i \|_{MK^{\alpha(\cdot),\lambda}_{1+\delta_2+\lambda - \beta}(w_{\alpha(\cdot)})} \leq \max \left\{ \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right), \sup_{k_0 \in \mathbb{Z}} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right) \right\} \\
= \max \{ X_1, X_2, X_3 \},
\]

(46)

where

\[
\begin{align*}
X_1 &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right), \\
X_2 &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right), \\
X_3 &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right).
\end{align*}
\]

(47)

To estimate \( X_1, X_2, \) and \( X_3, \) we make use of the conditions on \( \alpha(\cdot), \) such that for \( l < 0, \) we have

\[
\begin{align*}
\| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})} = 2^{-l \alpha(0)} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right)^{\frac{1}{p_i}} \\
\leq 2^{-l \alpha(0)} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right)^{\frac{1}{p_i}} \\
\leq 2^{(l - \alpha(0))} 2^{-l \lambda} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right)^{\frac{1}{p_i}} \\
\leq C 2^{(l - \alpha(0))} \| f \|_{MK^{\alpha(\cdot),\lambda}_{1+\delta_2+\lambda - \beta}(w_{\alpha(\cdot)})}^{p_i} .
\end{align*}
\]

(48)

and for \( l \geq 0, \) we obtain

\[
\begin{align*}
\| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})} = 2^{-l \alpha(\infty)} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right)^{\frac{1}{p_i}} \\
\leq 2^{-l \alpha(\infty)} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right)^{\frac{1}{p_i}} \\
\leq 2^{(l - \alpha(\infty))} 2^{-l \lambda} \left( \sum_{l = -\infty}^{j} 2^{l(\beta_2 - \beta_1)} \| f \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} \right)^{\frac{1}{p_i}} \\
\leq C 2^{(l - \alpha(\infty))} \| f \|_{MK^{\alpha(\cdot),\lambda}_{1+\delta_2+\lambda - \beta}(w_{\alpha(\cdot)})}^{p_i} .
\end{align*}
\]

(49)

In order to estimate \( X_1, \) we need to use \( \alpha(0) \leq \alpha(\infty) < n \delta_2 + \lambda - \beta. \)

\[
X_1 \leq \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda p_1} \sum_{j = -\infty}^{j} 2^{j(\beta_2 - \beta_1)} \| [b, H_\beta] f X_i \|_{L^{p_i/(\alpha(\cdot))}(w_{\alpha(\cdot)})}^{p_i} .
\]

(46)
\[ \|b\|_{\text{BMO}} \leq C \sum_{k=0}^{C} 2^{k \cdot \lambda} \sum_{j=0}^{k} 2^{j \cdot \lambda} \left( \sum_{j=0}^{k} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \right)^{p_i} \]

The result of \(X_3\) is similar to that of \(X_1\). Next, we will estimate \(X_3\) below.

\[ X_3 \leq \sum_{k=0}^{C} 2^{k \cdot \lambda} \sum_{j=0}^{k} 2^{j \cdot \lambda} \left( \sum_{j=0}^{k} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \right)^{p_i} \]

We estimate \(F_1\) and \(F_2\) separately. A use of generalized inequality results in the following:

\[ F_1 \leq C \sum_{j=1}^{C} 2^{-j(\beta-\delta_2-w(0)\cdot j)} \left( \sum_{j=0}^{k} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \right)^{p_i} \]

Finally, we combine the estimates for \(X_i (i = 1, 2, 3)\), to have the desired result. \(\square\)

**Theorem 19.** Let \(p_1, p_2, q_1, q_2, \beta, \alpha\) and \(w\) be as in Theorem 18. In addition, if \(\lambda - n\delta_2 < \alpha(0) \leq \alpha(\infty)\), where \(1 < \delta_2 < 0\), then

\[ \left\| \left[ b, H^\beta \right] f \right\|_{\text{MK}^p_{\text{F}^+}} \leq C \|b\|_{\text{BMO}} \left\| f \right\|_{\text{MK}^p_{\text{F}^+} (u^{q_1}(\cdot))} \]

We write

\[ \left[ b, H^\beta \right] f(x) = \sum_{j=0}^{C} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \left( \sum_{j=0}^{k} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \right)^{p_i} \]

Applying the weighted Lebesgue space norm on both sides and using Lemma 16, we obtain

\[ \left\| F_1 \right\|_{\text{MK}^p_{\text{F}^+} (u^{q_1}(\cdot))} \leq C \sum_{j=1}^{C} 2^{-j(\beta-\delta_2-w(0)\cdot j)} \left( \sum_{j=0}^{k} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \right)^{p_i} \]

Similarly,

\[ F_2 \leq C \sum_{j=1}^{C} 2^{-j(\beta-\delta_2-w(0)\cdot j)} \left( \sum_{j=0}^{k} (j-1)2^{(j-1)\cdot(\beta-\delta_2-w(0)\cdot j)} \right)^{p_i} \]
In view of the weighted Lebesgue norm and Lemma 16, we get
\[
\|F_2\|_{L^{q_i}(\omega_0^{1/i})} \leq \sum_{k=1}^{\infty} 2^{-\beta(n-p)} \left( \| (y) - b_{R_k} \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| f \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right)
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| f \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| f \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]
\[
\leq \sum_{k=1}^{\infty} 2^{n\beta(j-1)} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| f \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]

Hence, from (53), (55), and (57), we obtain
\[
\left\| \left[ b, H^*_\beta \right] f X_j \right\|_{L^{q_i}(\omega_0^{1/i})} \leq \sum_{k=1}^{\infty} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| f \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right)
\]
\[
\leq \sum_{k=1}^{\infty} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| f \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]
\[
\leq \sum_{k=1}^{\infty} 2^{n\beta(j-1)} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| f \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]

Using the condition of $A(q_1(\cdot), q_2(\cdot))$ weights given in the Definition 4, the above inequality reduces to
\[
\left\| \left[ b, H^*_\beta \right] f X_j \right\|_{L^{q_i}(\omega_0^{1/i})} \leq \sum_{k=1}^{\infty} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| f \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]
\[
\leq \sum_{k=1}^{\infty} 2^{n\beta(j-1)} 2^{-\beta(n-p)} (I - j) \| b \|_{\text{BMO}} \| f \|_{L^{q_i}(\omega_0^{1/i})} \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right) \times \left( \| X_j \|_{L^{q_i}(\omega_0^{1/i})} \right).
\]

Lastly, in view of the condition $-n\delta_j + \lambda < \alpha(0) \leq \omega(\infty)$, we estimate $Y_i, i = 1, 2, 3$, as we estimated $X_i, i = 1, 2, 3$, in Theorem 18. Hence, we finish the proof. \qed

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

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