FORMULAE FOR A NUMERICAL COMPUTATION OF ONE-LOOP TENSOR INTEGRALS

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A numerical and iterative approach for computing one-loop tensor integrals is presented.

1 Introduction

In Reference \(^1\) a new approach has been introduced for computing, recursively and numerically, one-loop tensor integrals. Here we describe a few modifications of the original method that allow a more efficient numerical implementation of the algorithm. We keep all of our notations as in Ref. \(^1\) and, in particular, put a bar over \(n\)-dimensional quantities and a tilde over \(\epsilon\)-dimensional objects \((n = 4 + \epsilon)\). The formulae we want to modify are Eqs. (9), (35), (48), (52) and (54) of Ref. \(^1\), that, all together, allow to reduce any \((m+1)\)-point tensor integral with \(m \geq 2\) to the standard set of scalar one-loop functions \(^2\).

Such formulae are not symmetric when interchanging any pair of loop denominators \(\bar{D}_k\), because are derived under the assumption that at least one of them (identified with \(\bar{D}_0\)) carries a vanishing external momentum, namely

\[
\bar{D}_k = (\bar{q} + p_k)^2 - m_k^2, \quad k = 0, \cdots, m, \quad p_0^\mu = 0. \tag{1}
\]

Already after the first iteration, terms appear in which the denominator \(\bar{D}_0\) is canceled by a \(\bar{D}_0\) reconstructed in the numerator, so that the resulting integrals do not fulfill any longer the assumption of Eq. (1). A shift of the integration variable \(\bar{q}\) is then needed to bring them back to a form suitable to apply the algorithm again. However, shifting \(\bar{q}\) may generate a large amount of terms, especially when dealing with high rank tensors, so that deriving more symmetric formulae, in which \(p_0^\mu \neq 0\), is clearly preferable.

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A second useful modification is related to the problem outlined in Sec. 6 of Ref. 1, that occur when $p_1^2 = 0$ and $p_2^2 \neq 0$ ($p_1^2 \neq 0$ and $p_2^2 = 0$) and $(p_1 \cdot p_2) \sim 0$. For those kinematical configurations a new linear combination of the momenta $p_1$ and $p_2$ is needed as a basis of the reduction to ensure the numerical stability of the algorithm. Once again, it is better to include such cases right from the beginning.

2 The General Recursion Formula

When $p_0^\mu \neq 0$, the $n$-dimensional version of Eq. (9) of Ref. 1 should be modified as follows

\[
I^{(n)}_{\mu \nu \rho \tau} = \frac{\beta}{2\gamma} T_{\mu \nu \lambda \sigma} \left\{ J^{(n)}_{\mu \nu \rho \tau} \right\} 
- \frac{1}{4\gamma} T_{\mu \nu} \left\{ (m_0^2 - p_0^2) I^{(n)}_{\mu \nu \rho \tau} + I^{(n)}_{\mu \nu \rho \tau} (0) - 2 p_0 \alpha I^{(n)}_{\mu \nu \rho \tau} - I^{(n:2)}_{\mu \nu \rho \tau} \right\} 
- \frac{1}{4\gamma} T_{\mu \nu \lambda \sigma} \left\{ \left( h_1 I^{(n)}_{\mu \nu \rho \tau} + I^{(n)}_{\mu \nu \rho \tau} (3) - I^{(n)}_{\mu \nu \rho \tau} (0) \right) \right. 
\left. - \frac{2\beta}{\gamma} k_{\lambda \sigma} J^{(n)}_{\mu \nu \rho \tau} \right\},
\]

where

\[
J^{(n)}_{\mu \nu \rho \tau} = (h_1 r_2 + h_2 r_3) I^{(n)}_{\mu \nu \rho \tau} + (r_1^2 + \xi r_1^3) I^{(n)}_{\mu \nu \rho \tau} (1) 
+ (r_1^3 + \xi_1 r_1^3) I^{(n)}_{\mu \nu \rho \tau} (2) - \left( r_1^2 + \xi_2 r_1^3 \right) I^{(n)}_{\mu \nu \rho \tau} (0).
\]

and where the extra integrals $I^{(n:2)}_{\mu \nu \rho \tau}$ are defined in Eq. (77) of Ref. 1.

In the previous Equations, $k_i = p_i - p_0$ and the massless 4-momenta $\ell_{1,2}$ to be used as a basis of the reduction algorithm, as in Eq. (13) of Ref. 1, are such that

\[
s_1 = \ell_1 + \alpha_1 \ell_2, \quad s_2 = \ell_2 + \alpha_2 \ell_1,
\]

where $s_{1,2}$ are suitable linear combinations of $k_{1,2}$

\[
s_1 = k_1 + \xi_1 k_2, \quad s_2 = k_2 + \xi_2 k_1.
\]

By choosing

\[
\xi_1 = \frac{1}{2} \text{sign}(k_2^2) \text{sign}(k_1 \cdot k_2) \quad \text{and} \quad \xi_2 = \frac{1}{2} \text{sign}(k_1^2) \text{sign}(k_1 \cdot k_2),
\]

the quantity

\[
\gamma = \frac{s_1^2 s_2}{(s_1 \cdot s_2) \mp \sqrt{\Delta}} \equiv (s_1 \cdot s_2) \pm \sqrt{\Delta}, \quad \Delta = (s_1 \cdot s_2)^2 - s_1^2 s_2^2,
\]

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defined in Eq. (62) of Ref. \(^1\) only vanishes when \(k_1^2 = k_2^2 = (k_1 \cdot k_2) = 0\), that always corresponds to collinear configurations cut away in physical observables, therefore solving the second problem outlined in the Introduction. The tensors \(T_{\mu\nu\lambda\sigma}, T_{\mu\nu\lambda}, T_{\mu\nu}\) and the 4-vectors \(r_{12}\) are defined as in Ref. \(^1\), but in terms of \(\ell_{1,2}\) given in Eq. (4) and with the replacement \(p_3 \rightarrow k_3\). Finally

\[
\begin{align*}
    h_1 &= (m_1^2 - p_1^2) + \xi_1 (m_2^2 - p_2^2) - (1 + \xi_1) (m_0^2 - p_0^2), \\
    h_2 &= (m_2^2 - p_2^2) + \xi_2 (m_1^2 - p_1^2) - (1 + \xi_2) (m_0^2 - p_0^2), \\
    h_3 &= (m_3^2 - p_3^2) - (m_0^2 - p_0^2), \\
    \frac{\beta}{\gamma} &= \pm \frac{1}{2\sqrt{\Delta}}.
\end{align*}
\] (8)

The derivation of Eq. (2) closely follows the derivation of Eq. (9) of Ref. \(^1\). For example, choosing \(\ell_{1,2}\) as in Eq. (4), the quantity

\[
D_{\mu} = \frac{1}{\beta} [2(q \cdot \ell_1)\ell_{2\mu} + 2(q \cdot \ell_2)\ell_{1\mu}]
\] (9)

defined in Eq. (18) of Ref. \(^1\) can be rewritten as

\[
D_{\mu} = [\bar{D}_1 + \xi_1 \bar{D}_2 - (1 + \xi_1)\bar{D}_0 + h_1] r_{2\mu} + [\bar{D}_2 + \xi_2 \bar{D}_1 - (1 + \xi_2)\bar{D}_0 + h_2] r_{1\mu},
\] (10)

and generates the term \(J^{(n)}\) of Eq. (3). Analogously, choosing \(b = k_3\) in Eq. (22) of Ref. \(^1\) generates the first part of the last term of Eq. (2), because, when \(p_0^\mu \neq 0\)

\[
2(q \cdot k_3) = \bar{D}_3 - \bar{D}_0 + h_3.
\] (11)

Finally, the equality

\[
q^2 = \bar{D}_0 + (m_0^2 - p_0^2) - 2(q \cdot p_0) - \tilde{q}^2,
\] (12)

is the origin of the second row of Eq. (2).

3 Three-point Tensors

When \(p_0^\mu \neq 0\), rank 2 and rank 3 three-point tensor integrals need a separate treatment. The relevant formulae follow by adapting the theorems in Eqs. (37) and (40) of Ref. \(^1\) to the case \(p_0^\mu \neq 0\):

\[
\int d^n q \frac{1}{D_0 D_1 D_2} [(q + p_0) \cdot \ell_3]^i = 0,
\]
\[ \int d^nq \frac{1}{D_0 D_1 D_2} [(q + p_0) \cdot \ell_4]^i = 0, \quad \forall i = 1, 2, 3 \ldots \quad \text{and} \]

\[ \int d^nq \frac{1}{D_0 D_1 D_2} [(q + p_0) \cdot \ell_{3,4}] q_\rho = 0. \quad (13) \]

The final results read as follows

\[ I^{(n)}(n)_{\mu \nu} = \frac{\beta}{2\gamma} T'_{\mu \lambda \sigma} J^{(n)}_{\lambda \sigma} - \frac{1}{4\gamma} t_{\mu \nu} \left\{ (m_0^2 - p_0^2) I^{(n)}_{2; \mu} + I^{(n)}_{1}(0) \right\} - 2p_{0\alpha} I^{(n)}_{2; \alpha} - I^{(n; 2)}_{2; \mu} \right\} \frac{1}{4\gamma} t_{\mu \nu} \left\{ -p_0^2 I^{(n)}_{2; \mu} - 2 I^{(n)}_{2; \mu} \right\}, \quad (14) \]

\[ I^{(n)}(n)_{\mu \nu} = \frac{\beta}{2\gamma} T'_{\mu \lambda \sigma} J^{(n)}_{\lambda \sigma} - \frac{1}{4\gamma} t_{\mu \nu} \left\{ (m_0^2 - p_0^2) I^{(n)}_{2; \mu} + I^{(n)}_{1}(0) \right\} - 2p_{0\alpha} I^{(n)}_{2; \alpha} - I^{(n; 2)}_{2; \mu} \right\} \frac{1}{4\gamma} t_{\mu \nu} \left\{ -p_0^2 I^{(n)}_{2; \mu} - 2 I^{(n)}_{2; \mu} \right\}, \quad (14) \]

\[ J^{(n)} \] is given in Eq. (3) and

\[ t_{\mu \nu} = \ell_{3\mu} \ell_{4\nu} + \ell_{4\mu} \ell_{3\nu}, \]

\[ T'_{\mu \nu \lambda} = -\frac{\ell_{3\mu} \ell_{3\nu} \ell_{3\lambda} (p_0 \cdot \ell_4) + \ell_{4\mu} \ell_{4\nu} \ell_{3\lambda} (p_0 \cdot \ell_3)}{\gamma}. \quad (15) \]

### 4 Rank One Tensors

In this section we adapt Eqs. (48), (52) and (54) of Ref. 1 to the case \( p_0^\nu \neq 0. \)

#### 4.1 The \( m = 2 \) case

With \( t_{\alpha \mu} \) defined in Eq. (15) we get

\[ I^{(n)}_{2; \mu} = \frac{\beta}{\gamma} J^{(n)}_{2; \mu} + \frac{p_0^\alpha}{2\gamma} t_{\alpha \mu} I^{(n)}_{2; \mu}. \quad (16) \]

#### 4.2 The \( m = 3 \) case

With \( T_{\mu \nu \lambda} \) defined as in Eq. (23) of Ref. 1 we get

\[ I^{(n)}_{3; \mu} = \frac{\beta}{\gamma} J^{(n)}_{3; \mu} + \frac{1}{4} \left[ \frac{\ell_{3\mu}}{(k_3 \cdot \ell_3)} + \frac{\ell_{4\mu}}{(k_3 \cdot \ell_4)} \right] \times \left\{ h_3 I^{(n)}_{3; \mu} + I^{(n)}_{2; \mu}(3) - I^{(n)}_{2; \mu}(0) - \frac{2\beta}{\gamma} k_3^\lambda J^{(n)}_{3; \lambda} \right\} \]

\[ - \frac{1}{4\gamma} T_{\mu \nu \lambda} (p_0^\mu k_3^\lambda - p_0^\lambda k_3^\mu) I^{(n)}_{3; \mu}. \quad (17) \]
4.3 The $m > 3$ case

The generalization of Eq. (54) of Ref. 1 reads

$$I^{(n)}_{m,\mu} = \frac{\beta}{\gamma} f^{(n)}_{m,\mu} + \frac{\ell_3 \ell_4 \alpha - \ell_3 \ell_4 \mu}{2 \delta} \times \left\{ k_3^\alpha \left[ h_4 I^{(n)}_m + I^{(n)}_{m-1}(4) - I^{(n)}_{m-1}(0) - \frac{2\beta}{\gamma} \ell_4 \lambda J^{(n)}_m \right] \right. $$

$$ - k_4^\alpha \left[ h_3 I^{(n)}_m + I^{(n)}_{m-1}(3) - I^{(n)}_{m-1}(0) - \frac{2\beta}{\gamma} k_3 \lambda J^{(n)}_m \right] \right\} \quad (18)$$

where $\delta = (\ell_3 \cdot k_4)(\ell_4 \cdot k_3) - (\ell_3 \cdot k_3)(\ell_4 \cdot k_4)$, and $h_4 = (m_1^2 - p_4^2) - (m_0^2 - p_0^2)$.

5 The Extra Integrals

In most practical cases, the extra integrals, such as $I^{(n;2)}_{m;\rho_\tau}$ in Eq. (2), are either zero or scaleless, so that, even when $p_{\rho_\tau}^0 \neq 0$, one can directly use the results given in Appendix B of Ref. 1. In all other cases modifications are needed. For example, Eqs. (78) and (83) of Ref. 1 must be replaced by

$$I^{(n;2)}_{2;\mu} = \frac{i \pi^2}{6} (p_{0\mu} + p_{1\mu} + p_{2\mu}) + \mathcal{O}(\epsilon),$$

$$I^{(n;2)}_1 = -i \frac{\pi^2}{2} \left[ m_1^2 + m_0^2 - \frac{(p_1 - p_0)^2}{3} \right] + \mathcal{O}(\epsilon). \quad (19)$$

6 Conclusion

We have derived a set of formulae to efficiently implement the $n$-dimensional reduction algorithm presented in Ref. 1. As for the three-point tensors, we limited our analysis to ranks $\leq 3$. For higher ranks, a general formula can be easily derived, with the help of Appendix C of Ref. 1, using the theorems of Eq. (13).

References

1. F. del Aguila and R. Pittau, “Recursive numerical calculus of one-loop tensor integrals,” arXiv:hep-ph/0404120.
2. G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B 153 (1979) 365.