Some aspects of the three dimensional gravity theories with temporal scalar field.

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Abstract

The present paper proposes a new explanation for the 3-dimensional Einstein general theory of relativity which is free of contradictions and consistent with usual 4-dimensional physics. We discuss the property of the new gravity theory with temporal scalar field arise in lower-dimensional theories as the reduction of timelike extra dimension. These ideas we continued by using the 3-dimensional analog of Jordan, Brans-Dicke theory with temporal scalar field where space and time are treated in different ways.

1 Introduction

It is well known since the ancient times that two concepts of space as absolute or relative are possible; for example Democritos believed in the existence of both atoms and empty space, while Aristotle with his opinion that space only epitomizes the place of the objects comes close to a relational concept of space. Consequently, although Mach was not the first who formulated mechanics in terms of purely relational quantities his profound critique of foundations of Newtonian mechanics played a key role in Einsteins development of general relativity. It has been difficult to identify spacetime geometries uniquely from statement of Machs principle since it is difficult to translate it into some precise mathematical language. Moreover, each statement of Machs principle should be accompanied by a declaration of the theoretical framework in which it is applied.
Mach's principle has also guided us to developments of multidimensional gravitation theories. The idea of the ordinary spacetime is viewed as a hypersurface embedded in a higher dimensional space have been considered in several different contexts, such as strings [1], D-branes [2], Randall-Sundrum models [3], [4] and non-compactified versions of Kaluza-Klein theories [5]. A Campbell-Magaards theorem [6], [7] and its extension [8], [9] has acquired fundamental relevance for granting the mathematical consistency of multidimensional embedding theories and also has been applied to investigate how lower-dimensional theories could be related to (3+1)-dimensional Einstein gravity.

The dimensional reduction in order to explain the property of nature is now widely used in theoretical physics [10], [11], [12]. From the pioneering paper by Jordan [13] physicists have explored the odd, however apparently fruitful thought, that Kaluza-Klein theories with special concern for the idea that the new 5-dimensional metric component, a spacetime scalar, might play the role of a varying gravitational constant. However, Jordan and his colleagues took the next step of separating the scalar field from the original 5-dimensional metric context unified gravitational-electromagnetic framework.

On the other hand, many physicists have expressed reservations about the 4-dimensional formalism. Though Lorentz recognized Einstein's hypothesis of geometrization of gravity as one of the basic principles, his enthusiasm for it gradually waned soon afterwards. In 1910 Lorentz said "Die Vorstellung (die auch Redner nur ungen aufgeben würde), dass Raum und Zeit etwas völlig Verschiedenes seien und dass es eine wahre Zeit gebe (die Gleichzeitigkeit würde denn unabhängig vom Orte bestehen)" [14].

An investigation into how lower and higher-dimensional theories of gravity are related to 3-dimensional Universe is therefore well motivated from a physical point of view. A potential bridge between gravitational theories of different dimensionality may be found at least two distinct ways by employing the embedding or foliation approaches [16]. In the embedding approach, the geometry is determined in terms of quantities which are defined exclusively in lower-dimensional. In the foliation approach, the lower-dimensional space-time geometry is determined by inducing the metric on the leaves, so that in this case the metric tensor depends on the extra coordinate. According to this world-view, our perception of normal 3-dimensional space arises because we leave on a domain wall (time-brane) in a bulk space. This kind of infinitely thin branes models can, however, be only treated as approxima-
tion. One can describe a thick brane in framework of 4-dimensional general relativity as domain wall separating two different states past and future and to an observer in ordinary space feels the thickness of brane as present.

This paper details the new approach to the problem of separate the temporal scalar field from the 4-dimensional metric context of general relativity which where outlined in previous letter [17]. Section 2 develops the formalism related to the correspondence between the ordinary 4-dimensional gravity with 3-dimensional theories with temporal scalar field. In Section 3 the exact solutions for the static sphere has been obtained. Finally, Section 4 briefly discussed some aspects of these theories. We choose units such that $G_N = c = 1$, and let Latin indices run 1-4 and Greek indices run 1-3.

2 Dimensional reduction approach

Embedding theorems of differential geometry provide a natural framework for relating higher and lower-dimensional theories of gravity. Procedure introduced by Campbell- Magaards theorems to embed a Riemannian manifold into Einstein space has been applied to investigate how low-dimensional theories could be related to (3+1) dimensional Einstein gravity. Such an embedding has a number of applications. For example, employing classical dimensional reduction techniques [15], [11] a 3-dimensional theory containing temporal scalar field [17], [18] can be obtained. Such a possibility naturally appears within the frame of multidimensional models type of Kaluza-Klein theory.

The claim that, in the spirit of Newtonian theory, the space and time are treated in completely different ways can be obtained by separate the temporal scalar field from the original 4-dimensional metric context of general relativity.

A second tenet of this theory is that all classical macroscopic physical quantities, such as matter density and pressure, could be given a geometrical interpretation. In this way, it is proposed that the classical energy-momentum tensor, which enters the right-hand side of the 3-dimensional Einstein equations, can be generated by pure geometrical means. In other words, it is claimed that geometrical curvature induces matter in three dimensions, and to an observer in the ordinary space the extra dimensions (time) appear as the matter source for gravity. However, induced stress-
energy tensor does not in general uniquely determine the matter content and the interpretation can lead to quite different kinematic quantities.

One can use the local an isotropic embedding of the 3-dimensional, Riemannian manifold with line element

\[ ds^2 = g_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta, \] (1)

in (3+1)-dimensional manifold defined by the metric

\[ ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + \phi^2 dt^2, \] (2)

where \( h_{\alpha\beta} = h_{\alpha\beta}(x^\mu,t) \) and \( \phi = \phi(x^\mu,t) \) are functions of the (3+1) variables \( \{x^\mu,t\} \) [8]. The claim that any energy-momentum can be generated by an embedding mechanism may be translated in geometrical language as saying that Riemannian 3-dimensional manifold is embeddable into a 4-dimensional Ricci manifold. In the coordinates (2), the field equations take the form

\[ R_{\alpha\beta} = (3) R_{\alpha\beta} - \left. \frac{\phi_{,\alpha\beta}}{\phi} \right. - \frac{1}{2\phi^2} \left( \ddot{\phi} h_{\alpha\beta} - \dot{h}_{\alpha\beta} - \frac{1}{2} h_{\gamma\delta} \dot{h}_{\gamma\delta} h_{\alpha\beta} + h_{\gamma\delta} \dot{h}_{\alpha\gamma} \dot{h}_{\beta\delta} \right), \] (3)

\[ R_{\alpha\gamma} = \frac{\phi}{2} \nabla_\beta \left( \frac{1}{\phi} h_{\beta\gamma} h_{\alpha\delta} - \frac{1}{\phi} \delta_{\beta} \dot{h}_{\gamma\delta} h_{\alpha\delta} \right), \] (4)

\[ R_{\gamma\delta} = \phi \phi_{,\alpha\beta} - \frac{1}{2} h^\gamma_{\delta} \dot{h}_{\gamma\delta} - \frac{1}{2} \dot{h} h_{\gamma\delta} + \frac{1}{2} h_{\gamma\delta} \dot{h}_{\gamma\delta} \frac{\dot{\phi}}{\phi} - \frac{1}{4} h h_{\gamma\delta} \dot{h}_{\delta\alpha} \dot{h}_{\gamma\alpha}. \] (5)

where overdots denote normal time derivatives. As in classical general relativity the metric is determined by Einstein’s equations.

\[ (3) R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (3) R = -\kappa (3) T_{\alpha\beta}, \] (6)

where \( \kappa \) is the Einstein constant and \( (3) T_{\alpha\beta} \) energy-momentum tensor, which is a function of all matter fields and metric. These give the components of the induced energy-momentum tensor since Einsteins equations (6) hold.
\[ T_{\alpha\beta} = \frac{\phi_{,\alpha\beta}}{\phi} - \frac{1}{2\phi} \left[ \frac{\phi}{\phi} \, h_{\alpha\beta} - \ddot{h}_{\alpha\beta} + h^{\lambda\mu} \, \ddot{h}_{\alpha\lambda} \, \ddot{h}_{\beta\mu} - \frac{1}{2} h^{\mu\nu} \, \dddot{h}_{\mu\nu} \, \dddot{h}_{\alpha\beta} \right] + \]
\[ + \frac{h_{\alpha\beta}}{8\phi^2} \left[ \phi_{,\mu\nu} \, \phi_{,\mu\nu} + \left( h^{\mu\nu} \, \phi_{,\mu\nu} \right)^2 \right]. \tag{7} \]

It can be shown that the field equations of this theory can be obtained from the reduced action that are formally similar to the Jordan, Brans-Dicke action with \( \omega = 0 \) [11]. The Jordan, Brans-Dicke theory of gravity [13], [19] represents a natural extension of general relativity, where a nonminimally-coupled scalar field parametrizes the space-time dependence of Newton's constant. It is expected on the general ground that dimensional reduction of multi-dimensional theory yields a (generalized) Jordan, Brans-Dicke theory. It is well known that the Jordan, Brans-Dicke theory accommodates both Mach's principle and Dirac's large number hypothesis [13], [19]. Note that this conclusion depends crucially on the particular formulation of Mach's principle used [23]. Newton's gravitational constant \( G_N \) replaced by dynamical scalar field acts as the source of the (local) gravitational coupling with \( G_N \sim \phi^{-1} \) and consequently is determined by the total matter in the universe through auxiliary scalar field equations.

By the way the presented 3-dimensional effective theory is derived by the integration of the action with respect to the extra dimension as it customary in Kaluza-Klein compactification. In contrast with standard Jordan, Brans-Dicke theory this temporal scalar field don’t play the role of a varying gravitational ”constant”. Take it into account one can show that such 3-dimensional theory introduce a temporal scalar field, which will (locally and approximately) play the role of Newtonian gravitational potential. The next step is to suppose that it is possible to find a special coordinate system, such that we can make the following split of 4-dimensional metric

\[ g_{ab} = \begin{pmatrix} g_{\alpha\beta} + \phi^2 A_\alpha A_\beta & \phi^2 A_\alpha \\ \phi^2 A_\beta & \phi^2 \end{pmatrix} \tag{8} \]

where \( \phi \) may be regarded as a temporal scalar field, and \( A_\alpha \) is 3-dimensional vector, whose role is to be determined later.
3 Three dimensional gravity theories with temporal scalar field.

Alternative ways to construct the temporal scalar field theory is to consider the more general 3-dimensional theory governed by the action where gravity is coupled to temporary scalar field. From the formal viewpoint, scalar-tensor models are purely metric theories including a nonminimal coupling between a temporal scalar field and the curvature scalar $R$ of the metric $g$. The most general action can be submitted by allowing $\omega(\phi)$ to depend on the scalar field $\phi$ and introducing a cosmological function $\lambda(\phi)$. The action for these theories in the Jordan frame is:

$$S = \int \left( f(\phi) (3)R + \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi,_{\mu} \phi,_{\nu} - 2\phi \lambda(\phi) + L_{\text{matt}} \right) \sqrt{|g|} d^3x \, dt,$$  \hspace{1cm} (9)

The dynamics of the scalar field $\phi$ depends on the functions $f(\phi)$, $\omega(\phi)$ and $\lambda(\phi)$. The matter part of the action $L_{\text{matt}}$ depends on the material fields and the metric but does not involve the scalar field $\phi$. This ensures a satisfaction of the "weak equivalence principle", that is, the statement that the paths of test particles in gravitational field are independent of their masses. We may demand that, in a spacetime consistent with Mach's principle, the metric tensor should be completely determined by the mass distribution in the universe. Different choices of function $\omega(\phi)$ and $\lambda(\phi)$ give different temporal scalar tensor theories. A simple example elaborates on further possibilities. One can neglect the cosmological function assuming $\lambda(\omega) = 0$ and use $f(\phi) = \phi$, $\omega(\phi) = \text{const}$. Then the variation of (9) with respect to $g^{\mu\nu}$ and $\phi$ gives, respectively, the field equations

$$^{(3)}R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, ^{(3)}R = -\frac{8\pi}{\phi} \, ^{(3)}T_{\mu\nu} - \frac{\omega}{\phi^2} \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\rho} \phi_{,\rho} \right) - \frac{1}{\phi} \left( \phi_{;\mu;\nu} - g_{\mu\nu} \phi_{,\rho} \phi^{,\rho} \right),$$  \hspace{1cm} (10)

$$\phi_{;\mu} = -\frac{8\pi}{3 + 2\omega} T,$$  \hspace{1cm} (11)

where $^{(3)}T_{\mu\nu}$ is the energy-momentum tensor of matter fields.
The system of equations (10), (11) have the vacuum solution $g^{\mu\nu} = \eta^{\mu\nu}$ and $\phi = \text{const}$, where $\eta^{\mu\nu}$ is the metric tensor of a flat space. Obviously, this solution identical to Minkowski’s space of general relativity and this solution fixes the arbitrariness in the coordinate system. Notice that this explanation viable in a more general theories when $f(\phi) \neq \phi$ and $\omega(\phi) \neq \text{const}$. It is necessary to remark that, according to Mach, the inertial forces observed locally in accelerated laboratory may be interpreted as gravitational effects having their origin in distant matter accelerated relative to laboratory. Consequently, there must be energy density of matter everywhere in machian universe. In completely empty space there can not be no gravitational field at all; in this case it would not be possible neither distribution of light, nor existence of scales and hours \cite{20}. In this case we must choose the solution $g^{\mu\nu} = 0$ and $\phi = 0$. On the other hand if we choose the case $g^{\mu\nu} = \eta^{\mu\nu}$ and $\phi = 0$ then we came to ”Newtonian” world.

The most appropriate coordinates in which to study the static spherically symmetric field are isotropic. Hence, the metric will be assumed to have the form

$$ds^2 = e^\beta \left( dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right),$$

(12)

where, $\beta$ and $\phi$ will be assumed to be function of $r$ only. Expressing the line element in isotropic form equation (10), (11) gives:

$$\frac{\beta'^2}{r} - \frac{2\omega'}{r} - 2\omega \left( \frac{\phi'}{r} \right)^2 - 2\beta'' - \frac{2\phi''}{\phi} = 0$$

$$\frac{3\beta'}{r} + \frac{\beta'^2}{2} + \beta'' + \left( \frac{2}{r} + \beta' \right) \frac{2\phi'}{\phi} = 0$$

(13)

$$\frac{4}{r} + \beta' + \frac{2\phi''}{\phi} = 0$$

The solution of equations (13) for the static spherically symmetric configuration can be easily obtained. This gives
\[ \beta = -\ln \left( \frac{r^{1+\mu}}{r^{2}+\mu^{2}} \sqrt{\frac{2}{2+\omega} (2+\omega)(r^{2}-\mu^{2})^{2}} \right) + c_{1}, \]  

\[ \phi = c_{2} \frac{r^{+\mu}}{r^{2}-\mu} \sqrt{\frac{2}{2+\omega}}, \]  

where \( \mu, c_{1} \) and \( c_{2} \) arbitrary constant.

A point to be noted is that under choices the case \( \omega=0 \) the solution (14) becomes identical to Schwarzschild solution of general relativity. It is well known that the exterior Schwarzschild solution of general relativity does not incorporate Mach's principle. Moreover, the anti-Machian character of the general relativity field equations also shows up in solutions which allow for a curved spacetime even in the absence of any matter [21]. The point is that the field equations admit curved vacuum solutions and also solutions that are asymptotically flat. However, a single mass point have inertia but in a consistent relativity theory there cannot be inertia relative to "space" but in a Mach's philosophy the inertia of a body gets determined by the presence of all other bodies in the universe.

However, that despite the mathematical equivalence of the Schwarzschild and (14) with \( \omega=0 \) solutions, the fact that they come from different physical theories makes them conceptually distinct. It is surprising that in 3-dimensional temporal scalar tensor theories there is not single matter particle solution. Thus in this sense the fourth dimension (time) generates effective matter sources and in turn act to curve 3-dimensional space. In other words, existence of even a sole particle in the space fills the entire universe with matter. Consequently, according to the Machian hypothesis [24] there are the matter "everywhere" in this universe.

4 Conclusion

There has been a lot of activity on describes our world as a 4-dimensional surface (brane) world embedded in higher dimensional space-time (bulk) [1], [2], [3], [4], [5]. This concept leads to a great variety of specific models both in cosmological context and in the description of local self-gravitating objects. Modifying the usual picture, where the our Universe represent as
4-dimensional differentiable manifolds, these our developments are based on the idea that matter fields could be confined to 3-dimensional world, while time could live in a higher dimensional space. So, like many authors, we try to describe N-dimensional gravitational models with extra fields were obtained from some multidimensional model by dimensional reduction based on decomposition of the manifold [13]. Most of theories built on this idea represent the 4-dimensional space-time and its associated extra space by differentiable manifolds hence such a gravity theory is equivalent to the Jordan, Brans-Dicke theory with vanishing parameter $\omega$ and a potential term [15], [22]. On the other hand after the conceptual foundations of non-Euclidian geometry had been laid by Lobachevsky, Bolyai, Gauss and the examinations on mechanics in non-Euclidean spaces had been clarified (at the end of XIX centuries [25], [26]), it was natural to consider how Newton’s law should fit into the new framework. In this scheme, in the spirit of Newtonian theory, space and time are treated in completely different ways.

We have presented here a new point of view to the General relativity theory. In present article the possibility of description of 3-dimensional gravity theories in terms of dimensional reduction is analyzed. By the way the idea that the matter in three dimensions can be explained from a 4-dimensional Riemannian manifold is a consequence of this point of view. This theory represents some modification of general relativity can be interpreted as a dynamical theory of evolution of 3-dimensional Riemannian geometry. A related question in whether plausible brane physics can generically be recovered and whether there are testable correction to Newtonian gravity. It would be interesting to consider these question further.

If the brane world is real, one may find some evidences of higher-dimensions. The present can be viewed as confined to a time-brane, a 3-dimensional hypersurface embedded in space-time with time extra dimensions. It is necessary to remark that, the man feels the present (the thickness of brane) as 0.5 seconds. The field equations of this theory formally have a structure similar to the Jordan, Brans-Dicke theory in three dimension with the free parameter $\omega = 0$. It is expected on the general ground that dimensional reduction of a multi-dimensional theory in the 3-dimensional scheme yields a temporal scalar field theory. Consequently, these ideas may be continued by suggested considering the more general case leading to the new theory with an additional temporal scalar field. These models based on temporal scalars have the theoretically appealing property that the same scalar field that participates
in the gravity sector is enhanced by potential. Introducing the temporal scalar field a different actions (9) with different choices of functions $\lambda(\omega)$, $f(\phi)$ and $\omega(\phi)$ has been proposed.

Finally an important question with regards to three dimensional gravity theories with temporal scalar field theories is how to choose reasonable forms of the field equations and the energy-momentum tensor. An examination based upon case of static spherically symmetric configuration with restriction to the functions $\lambda(\omega) = 0$, $f(\phi) = \phi$ and $\omega(\phi) = \text{const.}$, show that there is solution of fields equations (13) identical to the Schwarzschild solution in general relativity. However, the geometrical curvature induces matter in three dimensions, and the extra dimensions (time) appear as the additional effective matter source for gravity.

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