WEIGHTED WEAK TYPE ENDPOINT ESTIMATES FOR THE
COMPOSITIONS OF CALDERÓN–ZYGMDUND OPERATORS

GUOEN HU

(Received 4 August 2018; accepted 20 February 2019; first published online 8 April 2019)

Communicated by C. Meaney

Abstract

Let $T_1$, $T_2$ be two Calderón–Zygmund operators and $T_{1,b}$ be the commutator of $T_1$ with symbol $b \in \text{BMO}(\mathbb{R}^n)$. In this paper, by establishing new bilinear sparse dominations and a new weighted estimate for bilinear sparse operators, we prove that the composite operator $T_1 T_2$ satisfies the following estimate: for $\lambda > 0$ and weight $w \in A_1(\mathbb{R}^n)$,

$$w(\{x \in \mathbb{R}^n : |T_1 T_2 f(x)| > \lambda\}) \lesssim [w]_{A_1} [w]_{A_1} \log(e + [w]_{A_1}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(e + \frac{|f(x)|}{\lambda}\right) w(x) \, dx,$$

while the composite operator $T_{1,b} T_2$ satisfies

$$w(\{x \in \mathbb{R}^n : |T_{1,b} T_2 f(x)| > \lambda\}) \lesssim [w]_{A_1} [w]_{A_1}^2 \log(e + [w]_{A_1}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^2\left(e + \frac{|f(x)|}{\lambda}\right) w(x) \, dx.$$

2010 Mathematics subject classification: primary 42B20; secondary 42B25, 47B33.

Keywords and phrases: weighted bound, Calderón–Zygmund operator, bilinear sparse operator, grand maximal operator.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let $p \in [1, \infty)$ and $w$ be a nonnegative, locally integrable function on $\mathbb{R}^n$. We say that $w \in A_p(\mathbb{R}^n)$ if the $A_p$ constant $[w]_{A_p}$ is finite, where

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(x) \, dx \right)^{p-1}, \quad p \in (1, \infty),$$

the supremum is taken over all cubes in $\mathbb{R}^n$ and

$$[w]_{A_1} := \sup_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)}.$$
For properties of $A_p(\mathbb{R}^n)$, we refer the reader to the monograph [6]. In the last two decades, considerable attention has been paid to the sharp weighted bounds with $A_p$ weights for the classical operators in harmonic analysis. A prototypical work in this area is Buckley’s paper [2], in which it was proved that if $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, then the Hardy–Littlewood maximal operator $M$ satisfies

$$
||Mf||_{L^p(\mathbb{R}^n, w)} \leq n, p \quad [w]_{A_p}^{1/(p-1)} ||f||_{L^p(\mathbb{R}^n, w)}. \tag{1.1}
$$

Moreover, the estimate (1.1) is sharp in the sense that the exponent $1/(p-1)$ cannot be replaced by a smaller one. Hytönen and Pérez [15] improved the estimate (1.1) and showed that

$$
||Mf||_{L^p(\mathbb{R}^n, w)} \leq n, p \quad ([w] A_{p'} [w^{-1/(p-1)}]_{A_{\infty}})^{1/p} ||f||_{L^p(\mathbb{R}^n, w)},
$$

where, and in the following, for a weight $u \in A_{\infty}(\mathbb{R}^n) = \bigcup_{p \geq 1} A_p(\mathbb{R}^n)$, $[u]_{A_{\infty}}$ is the $A_{\infty}$ constant of $u$, defined by

$$
[u]_{A_{\infty}} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{u(Q)} \int_Q M(u \chi_Q(x)) \, dx;
$$

see [28].

Let $K$ be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y): x \neq y\}$. We say that $K$ is a Calderón–Zygmund kernel if $K$ satisfies the size condition that for $x, y \in \mathbb{R}^n$, $x \neq y$,

$$
|K(x, y)| \leq |x - y|^{-n}
$$

and the regularity condition that for any $x, y, y' \in \mathbb{R}^n$ with $|x - y| \geq 2|y - y'|$,

$$
|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \leq \frac{|y - y'|^\varepsilon}{|x - y|^{n+\varepsilon}}, \tag{1.2}
$$

where $\varepsilon \in (0, 1)$ is a constant. A linear operator $T$ is said to be a Calderón–Zygmund operator with kernel $K$ if it is bounded on $L^2(\mathbb{R}^n)$ and satisfies

$$
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy \tag{1.3}
$$

for all $f \in L^2(\mathbb{R}^n)$ with compact support and almost every $x \in \mathbb{R}^n \setminus \text{supp } f$. Hytönen [12] proved that for a Calderón–Zygmund operator $T$ and $w \in A_2(\mathbb{R}^n)$,

$$
||Tf||_{L^2(\mathbb{R}^n, w)} \leq n \quad [w]_{A_2} ||f||_{L^2(\mathbb{R}^n, w)}. \tag{1.4}
$$

This solved the so-called $A_2$ conjecture. Hytönen and Lacey [13] improved the estimate (1.4) and proved that for a Calderón–Zygmund operator $T$, $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$
||Tf||_{L^p(\mathbb{R}^n, w)} \leq n, p \quad [w]_{A_p}^{1/p} ([w]_{A_{\infty}}^{1/p'} + [\sigma]_{A_{\infty}}^{1/p}) ||f||_{L^p(\mathbb{R}^n, w)},
$$

where, and in the following, for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$, $p' = p/(p-1)$ and $\sigma = w^{1/(p-1)}$. Hytönen and Pérez [16] proved that if $T$ is a Calderón–Zygmund operator and $w \in A_1(\mathbb{R}^n)$, then

$$
||Tf||_{L^{1,\infty}(\mathbb{R}^n, w)} \leq [w]_{A_1} ||f||_{L^1(\mathbb{R}^n, w)}.
$$
For other works about quantitative weighted bounds of Calderón–Zygmund operators, see [14, 15, 17, 19, 20, 23] and the related references therein.

Let $T_1, T_2$ be two Calderón–Zygmund operators and $T_2^*$ be the adjoint operator of $T_2$. It was pointed out in [5, Section 9] that if $T_1(1) = T_2^*(1) = 0$, then the composite operator $T_1 T_2$ is also a Calderón–Zygmund operator; thus, for $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$||T_1 T_2 f||_{L^p(\mathbb{R}^n, w)} \lesssim [w]_{A_p}^{1/p} [w]_{A_\infty}^{1/p'} + [\sigma]_{A_\infty}^{1/p} ||f||_{L^p(\mathbb{R}^n, w)}.$$ 

Benea and Bernicot [1] considered the weighted bounds for $T_1 T_2$ when $T_1(1) = 0$ or $T_2^*(1) = 0$. In fact, the results in [1] imply the following conclusion (see Remark 3.3 in Section 3).

**Theorem 1.1.** Let $T_1, T_2$ be two Calderón–Zygmund operators, $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$.

(i) If $T_1(1) = 0$, then

$$||T_1 T_2 f||_{L^p(\mathbb{R}^n, w)} \lesssim [w]_{A_p}^{1/p} [w]_{A_\infty}^{1/p'} + [\sigma]_{A_\infty}^{1/p} ||f||_{L^p(\mathbb{R}^n, w)}.$$ 

(ii) If $T_2^*(1) = 0$, then

$$||T_1 T_2 f||_{L^p(\mathbb{R}^n, w)} \lesssim [w]_{A_p}^{1/p} [w]_{A_\infty}^{1/p'} + [\sigma]_{A_\infty}^{1/p} ||f||_{L^p(\mathbb{R}^n, w)}.$$ 

Fairly recently, Hu [9] considered the quantitative bounds on $L^p(\mathbb{R}^n, w)$ for the composite operator $T_{1,b} T_2$, with $T_1, T_2$ two Calderón–Zygmund operators, $b \in \text{BMO}(\mathbb{R}^n)$ and $T_{1,b}$ the commutator of $T_1$ defined by

$$T_{1,b} f(x) = b(x) T_1 f(x) - T_1 (b f)(x);$$ 

see [4, 15, 23] for the quantitative weighted bounds of the commutator of Calderón–Zygmund operators. Employing the ideas of Lerner [21], Hu [9] proved that if $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$||T_{1,b} T_2 f||_{L^p(\mathbb{R}^n, w)} \lesssim ||b||_{\text{BMO}(\mathbb{R}^n)} [w]_{A_p}^{1/p} [w]_{A_\infty}^{1/p'} + [\sigma]_{A_\infty}^{1/p}$$

$$\times ([w]_{A_\infty} + [\sigma]_{A_\infty}^2) ||f||_{L^p(\mathbb{R}^n, w)}. \quad (1.5)$$

We remark that the operator $T_{1,b} T_2$ was introduced by Krantz and Li [18] in the study of the Toeplitz-type operator of singular integral operators.

The main purpose of this paper is to establish the weighted weak type endpoint estimates for the composite operators $T_1 T_2$ and $T_{1,b} T_2$. Our main results can be stated as follows.

**Theorem 1.2.** Let $T_1$ and $T_2$ be Calderón–Zygmund operators. Then, for $w \in A_1(\mathbb{R}^n)$ and $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : |T_1 T_2 f(x)| > \lambda\}) \lesssim [w]_{A_1} [w]_{A_\infty} \log(e + [w]_{A_\infty})$$

$$\times \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(e + \frac{|f(x)|}{\lambda}\right) w(x) \, dx. \quad (1.6)$$
Moreover, if \( T_1(1) = 0 \), then
\[
w(\{ x \in \mathbb{R}^n : |T_1 T_2 f(x)| > \lambda \}) \leq [w] A_1 \log^2 (e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left( e + \frac{|f(x)|}{\lambda} \right) w(x) \, dx.
\]

Theorem 1.3. Let \( T_1 \) and \( T_2 \) be two Calderón–Zygmund operators and \( b \in \text{BMO}(\mathbb{R}^n) \). Then, for \( w \in A_1(\mathbb{R}^n) \) and \( \lambda > 0 \),
\[
w(\{ x \in \mathbb{R}^n : |T_{1,b} T_2 f(x)| > \lambda \}) \lesssim [w] A_1 [w]_{A_\infty}^2 \log (e + [w]_{A_\infty}) \times \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^2 \left( e + \frac{|f(x)|}{\lambda} \right) w(x) \, dx.
\]

Remark 1.4. It was proved in [9] that for each \( q \in (1, 2) \) and bounded functions \( f \) and \( g \) with compact support, there exists a sparse family of cubes \( S \) such that
\[
\int_{\mathbb{R}^n} |g(x)T_{1,b} T_2 f(x)| \, dx \lesssim A_{S,L_1 \log L^2, L^1}(f, g) + A_{S,L_1 \log L^1, L_1 \log L^2}(f, g) + q^2 A_{S, L^1, L^2}(f, g),
\]
where \( A_{S,L_1 \log L^2, L^1} \), \( A_{S,L_1 \log L^1, L_1 \log L^2} \) and \( A_{S, L^1, L^2} \) are bilinear sparse operators (for the definition of sparse family and bilinear sparse operator, see Section 3 below). Although the estimate (1.9) is adequate for establishing the weighted estimate (1.5), we do not know if it implies the weighted weak type endpoint estimates for \( T_{1,b} T_2 \). In this paper, we will establish a more refined bilinear sparse domination (Theorem 3.2), which, along with a new weighted estimate relating to the sparse operator \( A_{S, L_1 \log L^2, L_1 \log L^2} \) (see Lemma 3.1 in Section 3), leads to (1.8). It should be pointed out that Lemma 3.1 is a weighted version of [1, Proposition 6] and its proof is different from and simpler than what was used in the proof of Proposition 6 of [1].

Throughout the article, \( C \) always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol \( A \lesssim B \) to denote that there exists a universal constant \( C \) such that \( A \leq CB \). Especially, we use \( A \lesssim_{n, p} B \) to denote that there exists a positive constant \( C \) depending only on \( n, p \) such that \( A \leq CB \). A constant with a subscript such as \( c_1 \) does not change in different occurrences. For any set \( E \subset \mathbb{R}^n \), \( \chi_E \) denotes its characteristic function. For a cube \( Q \subset \mathbb{R}^n \) and \( \lambda \in (0, \infty) \), we use \( \lambda Q \) to denote the cube with the same center as \( Q \) and whose side length is \( \lambda \) times that of \( Q \). For \( \beta \in [0, \infty) \), a cube \( Q \subset \mathbb{R}^n \) and a suitable function \( g \), \( \|g\|_{L(\log L)^\beta, Q} \) is defined by
\[
\|g\|_{L(\log L)^\beta, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|g(y)|}{\lambda} \log^\beta \left( e + \frac{|g(y)|}{\lambda} \right) dy \leq 1 \right\}.
\]
We denote \( \|g\|_{L(\log L)^\beta, Q} \) by \( \langle |g| \rangle_Q \). For \( r \in (0, \infty) \), we set \( \langle |g| \rangle_{Q, r} = (\|g\|_{L(\log L)^r, Q})^{1/r} \) and define \( M_r \) by
\[
M_r g(x) = \sup_{Q \ni x} \langle |g| \rangle_{Q, r}.
\]
For $\beta \in [0, \infty)$, the maximal operator $M_{L(\log L)^\beta}$ is defined by

$$M_{L(\log L)^\beta}f(x) = \sup_{Q \ni x} \|f\|_{L(\log L)^\beta, Q}.$$  

It is well known that

$$|\{x \in \mathbb{R}^n : M_{L(\log L)^\beta}f(x) > \lambda\}| \lesssim \int_{\mathbb{R}^n} |f(x)| \lambda \log^\beta \left( e + \frac{|f(x)|}{\lambda} \right) dx. \quad (1.10)$$

### 2. Two grand maximal operators

For a linear operator $T$, we define the corresponding grand maximal operator $M_T$ by

$$M_T f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$. This operator was introduced by Lerner [20]. Moreover, Lerner [20] proved that a Calderón–Zygmund operator $T$ with kernel $K$ in the sense of (1.3) satisfies

$$M_T f(x) \lesssim T^* f(x) + M f(x), \quad (2.1)$$

where $T^*$ denotes the maximal operator associated with $T$, defined by

$$T^* f(x) = \sup_{\epsilon > 0} \left| \int_{|x-y| > \epsilon} K(x, y) f(y) dy \right|.$$  

Let $T_1, T_2$ be Calderón–Zygmund operators and $b \in \text{BMO}(\mathbb{R}^n)$. We define the grand maximal operators $M^*_{T_1T_2}$ and $M^*_{T_1T_2,b}$ by

$$M^*_{T_1T_2} f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T_1(f\chi_{\mathbb{R}^n \setminus 3Q}T_2(f\chi_{\mathbb{R}^n \setminus 9Q}))(\xi)|$$

and

$$M^*_{T_1T_2,b} f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T_1(f\chi_{\mathbb{R}^n \setminus 3Q}T_{2,b}(f\chi_{\mathbb{R}^n \setminus 9Q}))(\xi)|,$$

respectively. As we will see in Section 3, these two operators play important roles in the proof of Theorems 1.2 and 1.3. This section is devoted to the endpoint estimates for the operators $M^*_{T_1T_2}$ and $M^*_{T_1T_2,b}$. We begin with some preliminary lemmas.

**Lemma 2.1.** Let $p_0 \in (1, \infty)$, $\varphi \in [0, \infty)$ and $S$ be a sublinear operator. Suppose that

$$\|S f\|_{L^{p_0}(\mathbb{R}^n)} \leq A_1 \|f\|_{L^{p_0}(\mathbb{R}^n)}$$

and, for all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : |S f(x)| > \lambda\}| \leq A_2 \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^\varphi \left( e + \frac{|f(x)|}{\lambda} \right) dx.$$
Then, for $\beta \in [0, \infty)$ and two cubes $Q_2, Q_1 \subset \mathbb{R}^n$,
\[
\int_{Q_1} |S(f\chi_{Q_2})(x)| \log^\beta (e + |S(f\chi_{Q_2})(x)|) \, dx \leq |Q_1| + (A_1^{P_0} + A_2) \int_{Q_2} |f(x)| \log^{\beta + 1} (e + |f(x)|) \, dx.
\] (2.2)
Moreover, if $Q_0 \subset \mathbb{R}^n$ is a cube and $\{Q_j\}$ is a family of cubes contained in $Q_0$ with pairwise disjoint interiors, then
\[
\sum_j \int_{Q_j} |S(f\chi_{Q_j})(x)| \, dx \leq (1 + A_1^{P_0} + A_2)|Q_0||\|f\|_{L(\log L)^{\beta + 1}, Q_0}.
\] (2.3)

For the case of $\beta = 0$, (2.2) was proved in [11]. For the case of $\beta \in (0, \infty)$, the proof is similar. And (2.3) follows from (2.2) by homogeneity.

**Lemma 2.2.** Let $s \in [0, \infty)$ and $S$ be a sublinear operator which satisfies that for any $\lambda > 0$,
\[
|\{x \in \mathbb{R}^n : |Sf(x)| > \lambda\}| \leq \int_{\mathbb{R}^n} \left|\frac{|f(x)|}{\lambda}\right| \log^s \left(e + \frac{|f(x)|}{\lambda}\right) \, dx.
\]
Then, for any $\varrho \in (0, 1)$ and cube $Q \subset \mathbb{R}^n$,
\[
\left(\frac{1}{|Q|} \int_Q |S(f\chi_{Q})(x)|^\varrho \, dx\right)^{1/\varrho} \leq \|f\|_{L(\log L)^s, Q}.
\]

For the proof of Lemma 2.2, see [10, page 643].

**Lemma 2.3.** Let $R > 1$ and $\Omega \subset \mathbb{R}^n$ be an open set. Then $\Omega$ can be decomposed as $\Omega = \bigcup_j Q_j$, where $\{Q_j\}$ is a sequence of cubes with disjoint interiors, and:
(i) $5R \leq \frac{\text{dist}(Q_j, \mathbb{R}^n\setminus \Omega)}{\text{diam}Q_j} \leq 15R$;
(ii) $\sum_j \chi_{RQ_j}(x) \leq nR \chi_\Omega(x)$.

For the proof of Lemma 2.3, see [27, page 256].

**Lemma 2.4.** Let $\beta \in [0, \infty)$ and $U$ be a sublinear operator which is bounded on $L^2(\mathbb{R}^n)$ and satisfies that for any $t > 0$,
\[
|\{x \in \mathbb{R}^n : |Uf(x)| > t\}| \leq \int_{\mathbb{R}^n} \frac{|f(x)|}{t} \log^\beta \left(e + \frac{|f(x)|}{t}\right) \, dx.
\]
Let $T$ be a Calderón–Zygmund operator and $b \in \text{BMO}$ with $\|b\|_{\text{BMO}(\mathbb{R}^n)} = 1$. Then, for any $\lambda > 0$,
\[
|\{x \in \mathbb{R}^n : |UTf(x)| > \lambda\}| \leq \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\beta + 1} \left(e + \frac{|f(x)|}{\lambda}\right) \, dx
\] (2.4)
and
\[
|\{x \in \mathbb{R}^n : |UT_bf(x)| > \lambda\}| \leq \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\beta + 2} \left(e + \frac{|f(x)|}{\lambda}\right) \, dx.
\] (2.5)
Proof. We only prove (2.5). The proof of (2.4) is similar and will be omitted. By homogeneity, it suffices to prove the inequality (2.5) for the case of $\lambda = 1$. Applying Lemma 2.3 to the set $\{ x \in \mathbb{R}^n : Mf(x) > 1 \}$, we obtain a sequence of cubes $\{ Q_l \}$ with disjoint interiors, such that $\{ x \in \mathbb{R}^n : Mf(x) > 1 \} = \bigcup_l Q_l$, and

$$\sum_l \chi_{5Q_l}(x) \lesssim 1, \int_{Q_l} |f(y)| dy \leq |Q_l|.$$

Let

$$g(x) = f(x)\chi_{\mathbb{R}^n \cup \bigcup_l Q_l}(x) + \sum_l \langle f \rangle_{Q_l} \chi_{Q_l}(x)$$

and

$$h(x) = \sum l (f(x) - \langle f \rangle_{Q_l}) \chi_{Q_l}(x) := \sum_l h_l(x).$$

Recall that $UTb$ is bounded on $L^2(\mathbb{R}^n)$. Thus, by the fact that $\|g\|_{L^\infty(\mathbb{R}^n)} \lesssim 1$,

$$|\{ x \in \mathbb{R}^n : |UTb g(x)| > 1/2 \}| \lesssim \int_{\mathbb{R}^n} |g(x)|^2 dx \lesssim \int_{\mathbb{R}^n} |f(x)| dx.$$

To estimate $UTb h$, write

$$|UTb h(x)| \leq \left| UT\left( \sum_l (b - \langle b \rangle_{Q_l}) h_l \right)(x) \right| + \left| U\left( \sum_l (b - \langle b \rangle_{Q_l}) \chi_{5Q_l} Th_l \right)(x) \right|$$

$$+ \left| U\left( \sum_{l' \not= l} \chi_{\mathbb{R}^n \setminus 5Q_l} (b - \langle b \rangle_{Q_l}) Th_l \right)(x) \right|$$

$$= V_1(x) + V_2(x) + V_3(x).$$

We first consider the term $V_1$. Employing Jensen’s inequality, we have that for $\gamma > 0$,

$$\langle |f| \rangle_Q \log^\gamma(e + \langle |f| \rangle_Q) \leq \frac{1}{|Q|} \int_Q |f(y)| \log^\gamma(e + |f(y)|) dy.$$

Observe that, for $t_1, t_2 \in (0, \infty)$,

$$(t_1 + t_2) \log^\gamma(e + t_1 + t_2) \lesssim \gamma t_1 \log^\gamma(e + t_1) + t_2 \log^\gamma(e + t_2).$$

It then follows that

$$\int_{Q_l} |h_l(y)| \log^\gamma(e + |h_l(y)|) dy \lesssim \int_{Q_l} |f(y)| \log^\gamma(e + |f(y)|) dy. \quad (2.6)$$

On the other hand, the generalization of Hölder’s inequality (see [26]) tells us that

$$t_1 t_2 \log^\beta(e + t_1 t_2) \leq \exp t_1 + \log^\beta+1(e + t_2). \quad (2.7)$$
We deduce from inequalities (2.4), (2.6) and (2.7) that
\[
|x \in \mathbb{R}^n : |V_1(x)| > 1/6| \\
\leq \sum_l \int_{Q_l} |b(x) - \langle b \rangle_{Q_l}| |h_l(x)| \log^{\beta+1}(e + |b(x) - \langle b \rangle_{Q_l}| |h_l(x)|) dx \\
\leq \sum_l \int_{Q_l} \exp\left(\frac{|b(x) - \langle b \rangle_{Q_l}|}{C\|b\|_{\text{BMO}({\mathbb{R}^n})}}\right) dx + \sum_l \int_{Q_l} |h_l(x)| \log^{\beta+2}(e + |h_l(x)|) dx \\
\leq \int_{\mathbb{R}^n} |f(x)| \log^{\beta+2}(e + |f(x)|) dx. \tag{2.8}
\]
Recall that $\chi_{Q \cup 5Q_l} \leq 1$. It follows from inequality (2.7), Lemma 2.1 and inequality (2.6) that
\[
|x \in \mathbb{R}^n : |V_2(x)| > 1/6| \\
\leq \sum_l \int_{5Q_l} |b(x) - \langle b \rangle_{Q_l}| |Th_l(x)| \log^\beta(e + |b(x) - \langle b \rangle_{Q_l}| |Th_l(x)|) dx \\
\leq \sum_l \left( |Q_l| + \int_{Q_l} |Th_l(y)| \log^{\beta+1}(e + |Th_l(y)|) dy \right) \\
\leq \int_{\mathbb{R}^n} |f(y)| \log^{\beta+2}(e + |f(y)|) dy. \tag{2.9}
\]
To estimate the term $V_3$, we first observe that for each $l$ and $y \in \mathbb{R}^n \setminus 5Q_l$,
\[
|Th_l(y)| \leq \frac{\ell(Q_l)^\epsilon}{|y - z_l|^{n+\epsilon}} \|h_l\|_{L^1(\mathbb{R}^n)};
\]
here $z_l$ is the center of $Q_l$ and $\epsilon \in (0,1]$ is the constant in (1.2). Thus, for each $v \in L^2(\mathbb{R}^n)$ with $\|v\|_{L^2(\mathbb{R}^n)} = 1$, we have by the John–Nirenberg inequality that
\[
\sum_l \left\| \int_{\mathbb{R}^n \setminus 5Q_l} (b(y) - \langle b \rangle_{Q_l}) Th_l(y)v(y) dy \right\| \\
\leq \sum_l \int_{\mathbb{R}^n} |h_l(y)| \int_{\mathbb{R}^n \setminus 5Q_l} \frac{|b(y) - \langle b \rangle_{Q_l}| |Q_l|^{\epsilon/n}}{|y - z_l|^{n+\epsilon}} |v(y)| dy dz \\
\leq \sum_l \int_{Q_l} M_{L,\log} L v(y) dy \lesssim \left( \sum_l |Q_l| \right)^{1/2}.
\]
This, via a standard duality argument, gives that
\[
\left| \left\{ x \in \mathbb{R}^n : |V_3(x)| > \frac{1}{6} \right\} \right| \lesssim \left\| \sum_l \chi_{\mathbb{R}^n \setminus 5Q_l}(b(\cdot) - \langle b \rangle_{Q_l}) Th_l \right\|_{L^2(\mathbb{R}^n)}^2 \\
\lesssim \int_{\mathbb{R}^n} |f(y)| dy. \tag{2.10}
\]
Combining the estimates (2.8)–(2.9) and (2.10) leads to our desired conclusion. \qed
COROLLARY 2.5. Let $T_1$ and $T_2$ be two Calderón–Zygmund operators. Then, for each $\lambda > 0$,

$$
|x \in \mathbb{R}^n : MT_2 f(x) + T_1^* T_2 f(x) > \lambda| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) dx,
$$

(2.11)

and

$$
|x \in \mathbb{R}^n : MT_{2,b} f(x) > \lambda| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^2 \left( e + \frac{|f(x)|}{\lambda} \right) dx.
$$

(2.12)

and, for $r \in (0, 1)$,

$$
|x \in \mathbb{R}^n : M_r T_{1}^* T_{2,b} f(x) > \lambda| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^2 \left( e + \frac{|f(x)|}{\lambda} \right) dx.
$$

(2.13)

**Proof.** Inequalities (2.11) and (2.12) follow from Lemma 2.4 directly. Also, we know from Lemma 2.4 that

$$
|x \in \mathbb{R}^n : T_1^* T_{2,b} f(x) > \lambda| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) dx.
$$

Recall that for $r \in (0, 1)$,

$$
|x \in \mathbb{R}^n : M_r f(x) > \lambda| \lesssim \lambda^{-1} \sup_{s \in 2^{-1/\lambda}} |x \in \mathbb{R}^n : |f(x)| > s|;
$$

see [10, page 651]. Combining the last two inequalities establishes (2.13). \hfill \Box

We are now ready to establish our main conclusion in this section.

**Theorem 2.6.** Let $T_1$, $T_2$ be two Calderón–Zygmund operators and $b \in BMO(\mathbb{R}^n)$ with $\|b\|_{BMO(\mathbb{R}^n)} = 1$. Then, for each bounded function $f$ with compact support and $\lambda > 0$,

$$
|x \in \mathbb{R}^n : M_{T_1, T_2}^* f(x) > \lambda| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) dx
$$

(2.14)

and

$$
|x \in \mathbb{R}^n : M_{T_1, T_2,b}^* f(x) > \lambda| \lesssim \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^2 \left( e + \frac{|f(x)|}{\lambda} \right) dx.
$$

(2.15)

**Proof.** We first prove the estimate (2.14). By Corollary 2.5 and estimate (1.10), it suffices to prove that

$$
M_{T_1, T_2}^* f(x) \lesssim M_{1/2}^* T_1^* T_2 f(x) + M_{L \log L} f(x) + MT_2 f(x).
$$

(2.16)

Let $x \in \mathbb{R}^n$ and $Q$ be a cube containing $x$. A trivial computation involving inequality (2.1) leads to that for each $\xi \in Q$,

$$
|T_1(\chi_{\mathbb{R}^n \setminus 3Q} T_2(f\chi_{\mathbb{R}^n \setminus 9Q})(\xi))| \lesssim \inf_{z \in Q} M_{T_1}(T_2 f \chi_{\mathbb{R}^n \setminus 9Q})(z)
$$

$$
\lesssim \left( \frac{1}{|Q|} \int_{Q} (M_{T_1}(T_2(f\chi_{\mathbb{R}^n \setminus 9Q}))(z))^{1/2} dz \right)^2
$$

$$
\lesssim \left( \frac{1}{|Q|} \int_{Q} |T_1^* T_2 f(z)|^{1/2} dz \right)^2 + \left( \frac{1}{|Q|} \int_{Q} [MT_2 f(z)]^{1/2} dz \right)^2
$$

$$
+ \left( \frac{1}{|Q|} \int_{Q} [T_1^* T_2 (f\chi_{\mathbb{R}^n \setminus 9Q})(\xi)]^{1/2} d\xi \right)^2 + \left( \frac{1}{|Q|} \int_{Q} [MT_2 (f\chi_{\mathbb{R}^n \setminus 9Q})(\xi)]^{1/2} d\xi \right)^2.
$$
Recalling that $M_{1/2}Mh(x) \leq Mh(x)$, we know that
\[
\left( \frac{1}{|Q|} \int_Q [MT_2f(\xi)]^{1/2} \, d\xi \right)^2 \leq MT_2f(x).
\]
On the other hand, it follows from estimate (2.11) and Lemma 2.2 that
\[
\left( \frac{1}{|Q|} \int_Q [T_1^*T_2(f\chi_{[0,1]}Q)(\xi)]^{1/2} \, d\xi \right)^2 + \left( \frac{1}{|Q|} \int_Q [MT_2(f\chi_{[0,1]}Q)(\xi)]^{1/2} \, d\xi \right)^2
\]
\[
\leq \|f\|_{L\log L,9Q} \leq M_{L\log L}f(x).
\]
This establishes (2.16).

We turn our attention to the inequality (2.15). Again by Corollary 2.5 and estimate (1.10), it suffices to prove that
\[
M_{T_1^*T_2,b}f(x) \leq M_{T_1^*T_2,b}f(x) + MT_2,bf(x) + M_{L\log L}f(x). \tag{2.17}
\]
Let $x \in \mathbb{R}^n$ and $Q$ be a cube containing $x$. As in the proof of (2.16), we have that for each $\xi \in Q$,
\[
|T_1(\chi_{[0,1]}Q T_2.b(f\chi_{[0,1]}Q))| \xi \rangle
\]
\[
\leq \left( \frac{1}{|Q|} \int_Q [MT_i(T_2,b(f\chi_{[0,1]}Q))(z)]^{1/2} \, dz \right)^2
\]
\[
\leq \left( \frac{1}{|Q|} \int_Q [T_1^*(T_2,b(f\chi_{[0,1]}Q))(z)]^{1/2} \, dz \right)^2
\]
\[
+ \left( \frac{1}{|Q|} \int_Q [M(T_2,b(f\chi_{[0,1]}Q))(z)]^{1/2} \, dz \right)^2 = I + II.
\]
A trivial computation involving estimate (2.13) and Lemma 2.2 shows that
\[
I \leq \left( \frac{1}{|Q|} \int_Q [T_1^*T_2,bf(z)]^{1/2} \, dz \right)^2 + \left( \frac{1}{|Q|} \int_Q [T_1^*T_2,bf(\chi_{[0,1]}Q)]^{1/2} \, dz \right)^2
\]
\[
\leq M_{1/2}T_1^*T_2,bf(x) + M_{L\log L}f(x).
\]
Similarly, we can obtain that
\[
II \leq MT_2,bf(x) + M_{L\log L}f(x).
\]
Combining the estimates above yields (2.17). \qed

3. Proof of Theorems 1.2 and 1.3

Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of all cubes of the form
\[
2^{-k}([0,1]^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n.
\]
Denote the standard grid by $D$. For a fixed cube $Q$, denote by $D(Q)$ the set of dyadic cubes with respect to $Q$, that is, the cubes from $D(Q)$ are formed by repeating the subdivision of $Q$ and each of its descendants into $2^n$ congruent subcubes.
As usual, by a general dyadic grid \( \mathcal{D} \) we mean a collection of cubes with the following properties: (i) for any cube \( Q \in \mathcal{D} \), its side length \( \ell(Q) \) is of the form \( 2^k \) for some \( k \in \mathbb{Z} \); (ii) for any cubes \( Q_1, Q_2 \in \mathcal{D}, \) \( Q_1 \cap Q_2 \in \{Q_1, Q_2, \emptyset\} \); (iii) for each \( k \in \mathbb{Z} \), the cubes of side length \( 2^k \) form a partition of \( \mathbb{R}^n \).

Let \( \eta \in (0, 1) \) and \( S = \{Q_j\} \) be a family of cubes. We say that \( S \) is \( \eta \)-sparse if, for each fixed \( Q \in S \), there exists a measurable subset \( E_Q \subset Q \) such that \( |E_Q| \geq \eta |Q| \) and the \( E_Q \) are pairwise disjoint. Associated with the sparse family \( S \) and \( \beta \in [0, \infty) \), define the sparse operator \( \mathcal{A}_{S, L(\log L)^\beta} \) by

\[
\mathcal{A}_{S, L(\log L)^\beta} f(x) = \sum_{Q \in S} |\langle f, \chi_Q \rangle|_{L(\log L)^\beta, Q}(x).
\]

For a sparse family \( S \) and constants \( \beta_1, \beta_2 \in [0, \infty) \), we define the bilinear sparse operator \( \mathcal{A}_{S, L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} \) by

\[
\mathcal{A}_{S, L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} (f, g) = \sum_{Q \in S} |\langle f, \chi_Q \rangle|_{L(\log L)^{\beta_1}, Q} |\langle g, \chi_Q \rangle|_{L(\log L)^{\beta_2}, Q}.
\]

We denote \( \mathcal{A}_{S, L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} \) by \( \mathcal{A}_{S, L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} \) for simplicity, and \( \mathcal{A}_{S, L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} \) by \( \mathcal{A}_{S; L(\log L)^{\beta_1}, L} \).

**Lemma 3.1.** Let \( \beta_1, \beta_2 \in \mathbb{N} \cup \{0\} \) and \( U \) be an operator. Suppose that for a bounded function \( f \) with compact support, there exists a sparse family of cubes \( S \) such that for any function \( g \in L^1(\mathbb{R}^n) \),

\[
\left| \int_{\mathbb{R}^n} U f(x) g(x) \, dx \right| \leq \mathcal{A}_{S; L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} (f, g) \tag{3.1}
\]

Then, for any \( \epsilon \in (0, 1) \) and weight \( u \),

\[
u\left(\left\{ x \in \mathbb{R}^n : |U f(x)| > \lambda \right\}\right) \leq \frac{1}{\epsilon^{1+\beta_1}} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^{\beta_1} \left( 1 + \frac{|f(x)|}{\lambda} \right) M_{L(\log L)^{\beta_2}, u}(x) \, dx.
\]

**Proof.** Let \( f \) be a bounded function with compact support and \( S \) be the sparse family such that (3.1) holds true. By the one-third trick (see [14, Lemma 2.5] or [22, Theorem 3.1]), there exist dyadic grids \( \mathcal{D}_1, \ldots, \mathcal{D}_{3^n} \) and sparse families \( S_1, \ldots, S_{3^n} \) such that for \( j = 1, \ldots, 3^n \), \( S_j \subset \mathcal{D}_j \), and

\[
\mathcal{A}_{S; L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} (f, g) \leq \sum_{j=1}^{3^n} \mathcal{A}_{S_j; L(\log L)^{\beta_1}, L(\log L)^{\beta_2}} (f, g).
\]

It was proved in [16, pages 618–619] that

\[
\|Mg\|_{L^{p}(\mathbb{R}^n, (M_{L(\log L)^{\beta_1}, u})^{-1/p})} \leq p^2 \left( \frac{1}{\epsilon} \right)^{1/p'} \|g\|_{L^{p}(\mathbb{R}^n, u^{-1/p'})}.
\]
Repeating the argument in the proof of Theorem 1.8 in [25], we know that if $p \in (1, \infty)$ and $\|h\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} = 1$, then

$$
\mathcal{A}_{S_j; L\log L}^0, L\log L \|f\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \leq p^{1+\beta_1} \|M_{L\log L}^2 g\|_{L^p} \|f\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \leq p^{1+\beta_1} \left[p^2 \left(\frac{1}{\varepsilon}\right)^{1/p'}\right]^{\beta_2+1} \|g\|_{L^p} \|f\|_{L^p(\mathbb{R}^n, e^{L^p} u^1)},
$$

since $M_{L\log L}^2 g(x) \approx M_{L\log L}^{2+1} g(x)$; see [3]. This, via homogeneity, yields that

$$
\mathcal{A}_{S_j; L\log L}^0, L\log L \|f\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \leq p^{1+\beta_1} \left[p^2 \left(\frac{1}{\varepsilon}\right)^{1/p'}\right]^{\beta_2+1} \|h\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \|f\|_{L^p(\mathbb{R}^n, e^{L^p} u^1)}. 
$$

(3.2)

Now let $M_{\mathcal{D}_j, L\log L}^{\beta_1}$ be the maximal operator defined by

$$
M_{\mathcal{D}_j, L\log L}^{\beta_1} f(x) = \sup_{Qx \in \mathcal{D}_j} \|f\|_{L^p(\mathbb{R}^n, Qx)}.
$$

For each fixed $j = 1, \ldots, 3^n$, we decompose the set \{ $x \in \mathbb{R}^n : M_{\mathcal{D}_j, L\log L}^{\beta_1} f(x) > 1$ \} as

\{ $x \in \mathbb{R}^n : M_{\mathcal{D}_j, L\log L}^{\beta_1} f(x) > 1$ \} $= \bigcup_k Q_{j,k}$,

with $Q_{j,k}$ the maximal cubes in $\mathcal{D}_j$ such that $\|f\|_{L^p(\mathbb{R}^n, Q_{j,k})} > 1$. We have that

$$
1 < \|f\|_{L^p(\mathbb{R}^n, Q_{j,k})} \leq 2^n.
$$

Let

$$
\begin{align*}
f_1^j(y) &= f(y) \chi_{\mathbb{R}^n \setminus \bigcup_k Q_{j,k}}(y), \quad f_2^j(y) = \sum_k f(y) \chi_{Q_{j,k}}(y) \\
f_3^j(y) &= \sum_k \|f\|_{L^p(\mathbb{R}^n, Q_{j,k})} \chi_{Q_{j,k}}(y).
\end{align*}
$$

It is obvious that $\|f_1^j\|_{L^{\infty}(\mathbb{R}^n)} \leq 1$ and $\|f_3^j\|_{L^{\infty}(\mathbb{R}^n)} \leq 1$. Take $p_1 = 1 + \varepsilon/(2\beta_2 + 1)$. It then follows from the inequality (3.2) that

$$
\mathcal{A}_{S_j; L\log L}^{\beta_1} \|f_1^j\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \leq p_1^{1+\beta_1} \left[p_1^2 \left(\frac{1}{\varepsilon}\right)^{1/p_1'}\right]^{\beta_2+1} \|f_1^j\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \leq \frac{1}{\varepsilon^{1+\beta_1}} \|f_1^j\|_{L^p(\mathbb{R}^n, M_{L\log L}^{2(1+\eta)p-1+\eta})} \|f_1^j\|_{L^p(\mathbb{R}^n, e^{L^p} u^1)}. 
$$

(3.3)

Let $E = \bigcup_j \bigcup_k 4n Q_{j,k}$ and $\bar{u}(y) = u(y) \chi_{\mathbb{R}^n \setminus E}(y)$. It is obvious that

$$
\begin{align*}
u(E) &\leq \sum_{j=1}^{3^n} \sum_k \inf_{z \in Q_{j,k}} \|Mz\|_{Q_{j,k}} \\
&\leq \int_{\mathbb{R}^n} |f(y)| \log^h (e + |f(y)|) M u(y) dy.
\end{align*}
$$

(3.4)
Moreover, by the fact that
\[
\inf_{y \in Q_{jk}} M_{L(\log L)^p} \overline{u}(y) \approx \sup_{z \in Q_{jk}} M_{L(\log L)^p} \overline{u}(z),
\]
we obtain that for \( \gamma \in [0, \infty) \),
\[
\| f_j^2 \|_{L^1(\mathbb{R}^n, M_{L(\log L)^p} \overline{u})} \leq \sum_{k} \inf_{y \in Q_{jk}} M_{L(\log L)^p} \overline{u}(z) \| f_j^2 \|_{L^1(\mathbb{R}^n, M_{L(\log L)^p} \overline{u})}
\leq \int_{\mathbb{R}^n} |f(y)| \log^{\beta_1}(e + |f(y)|) M_{L(\log L)^p} u(y) \, dy. \tag{3.5}
\]
Let
\[
S_j^* = \{ I \in S_j : I \cap (\mathbb{R}^n \setminus E) \neq \emptyset \}.
\]
Note that if \( \text{supp } g \subset \mathbb{R}^n \setminus E \), then
\[
\mathcal{A}_{S_j^*, L(\log L)^p_1, L(\log L)^p_2} (f_j^1, g) = \mathcal{A}_{S_j^*, L(\log L)^p_1, L(\log L)^p_2} (f_j^1, g).
\]
As in the argument in [8, pages 160–161], we can verify that for each fixed \( I \in S_j^* \),
\[
\| f_j^2 \|_{L^1(\mathbb{R}^n, \tilde{u}^\gamma)} \leq \| f_j^2 \|_{L^1(\mathbb{R}^n, M_{L(\log L)^p} \overline{u})}.
\]
Therefore, for \( g \in L^1(\mathbb{R}^n) \) with \( \text{supp } g \subset \mathbb{R}^n \setminus E \),
\[
\mathcal{A}_{S_j, L(\log L)^p_1, L(\log L)^p_2} (f_j^1, g) \leq \mathcal{A}_{S_j, L(\log L)^p_1, L(\log L)^p_2} (f_j^1, g)
\leq \frac{1}{\epsilon^{1 + \beta_1}} \| f_j^1 \|_{L^1(\mathbb{R}^n, M_{L(\log L)^p} \overline{u})} \| g \|_{L^2(\mathbb{R}^n, \tilde{u}^1-\gamma)} \tag{3.6}
\]
Inequalities (3.3), (3.5) and (3.6) tell us that for a function \( g \in L^p(\mathbb{R}^n \setminus E, \tilde{u}^{1-p'}) \) with
\[
\| g \|_{L^p(\mathbb{R}^n \setminus E)} \leq 1,
\]
\[
\left| \int_{\mathbb{R}^n} U f(x) g(x) \, dx \right| \leq \sum_{j=1}^{3^n} \mathcal{A}_{S_j, L(\log L)^p_1, L(\log L)^p_2} (f_j^1, g)
+ \sum_{j=1}^{3^n} \mathcal{A}_{S_j, L(\log L)^p_1, L(\log L)^p_2} (f_j^2, g)
\leq \frac{1}{\epsilon^{1 + \beta_1}} \| f \|_{L^1(\mathbb{R}^n, M_{L(\log L)^p} \overline{u})}
+ \frac{1}{\epsilon^{1 + \beta_1}} \| f \|_{L^1(\mathbb{R}^n, M_{L(\log L)^p} \overline{u})}.
\]
Thus, by a duality argument and (3.4),
\[
\frac{1}{\epsilon^{1 + \beta_1}} \int_{\mathbb{R}^n} |f(y)| \log^{\beta_1}(e + |f(y)|) M_{L(\log L)^p} u(y) \, dy.
\]
This, via homogeneity, leads to our desired conclusion. \( \square \)
**Theorem 3.2.** Let $T_1$ and $T_2$ be two Calderón–Zygmund operators and $b \in \text{BMO}(\mathbb{R}^n)$ with $\|b\|_{\text{BMO}(\mathbb{R}^n)} = 1$.

(i) For a bounded function $f$ with compact support, there exist a $\frac{1}{2}(1/9^n)$-sparse family of cubes $S = \{Q\}$ and functions $J_0, J_1$ such that for each $j = 0, 1$ and function $g$,

$$\left| \int_{\mathbb{R}^n} J_j(x)g(x) \, dx \right| \leq \mathcal{A}_{S; L(\log L)^{1/2}, L(\log L)^{1/2}}(f, g);$$

and, for almost every $x \in \mathbb{R}^n$,

$$T_1 T_2 f(x) = J_0(x) + J_1(x).$$

(ii) For a bounded function $f$ with compact support, there exist a $\frac{1}{2}(1/9^n)$-sparse family of cubes $S = \{Q\}$ and functions $H_0, H_1$ and $H_2$ such that for each $j = 0, 1, 2$ and function $g$,

$$\left| \int_{\mathbb{R}^n} H_j(x)g(x) \, dx \right| \leq \mathcal{A}_{S; L(\log L)^{3/2}, L(\log L)^{3/2}}(f, g);$$

and, for almost every $x \in \mathbb{R}^n$,

$$T_{1,b} T_2 f(x) = H_0(x) + H_1(x) + H_2(x).$$

**Proof.** We only prove the conclusion (ii). The proof of the conclusion (i) is similar and will be omitted. We will employ the argument in [20]. For a fixed cube $Q_0$, define the local analogy of $\mathcal{M}_{2}, \mathcal{M}^*_{2,T_2}$ and $\mathcal{M}^*_{T_1 T_2,b}$ by

$$\mathcal{M}_{T_2,Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \text{ess sup}_{\xi \in Q} |T_2(f(x)\chi_{3Q}(\xi))|,$$

$$\mathcal{M}^*_{T_1 T_2; Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \text{ess sup}_{\xi \in Q} |T_1(\chi_{3Q} T_2(f(x)\chi_{2Q}(\xi)))|,$$

and

$$\mathcal{M}^*_{T_1 T_2,b; Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \text{ess sup}_{\xi \in Q} |T_1(\chi_{3Q} T_2,b(f(x)\chi_{2Q}(\xi)))|,$$

respectively. Let $E = \bigcup_{j=1}^{5} E_j$ with

$$E_1 = \{x \in Q_0 : |T_{1,b} T_2(f(x)\chi_{Q_0}(\xi))| > D\|f\|_{L(\log L)^{1/2}, 9Q_0}\},$$

$$E_2 = \{x \in Q_0 : \mathcal{M}_{T_2,Q_0} f(x) > D\|f\|_{9Q_0}\},$$

$$E_3 = \{x \in Q_0 : \mathcal{M}^*_{T_1 T_2; Q_0} f(x) > D\|f\|_{L(\log L)^{1/2}, 9Q_0}\}$$

and

$$E_4 = \{x \in Q_0 : \mathcal{M}^*_{T_1 T_2,b; Q_0} f(x) > D\|f\|_{L(\log L)^{3/2}, 9Q_0}\},$$

$$E_5 = \{x \in Q_0 : \mathcal{M}^*_{T_1 T_2; Q_0} ((b - (b)_{Q_0}) f(x) > D\|f\|_{L(\log L)^{1/2}, 9Q_0}\},$$

with $D$ a positive constant. It then follows from Theorem 2.6 and Corollary 2.5 that

$$|E| \leq \frac{1}{2^{n+2}} |Q_0|,$$
if we choose $D$ large enough. Now, on the cube $Q_0$, we apply the Calderón–Zygmund decomposition to $\chi_E$ at level $1/2^{n+1}$ and obtain pairwise disjoint cubes $\{P_j\} \subset D(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and $|E \setminus \cup_j P_j| = 0$. Observe that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$. Let

$$G_0(x) = T_{1,b}T_2(f\chi_{Q_0})(x)\chi_{Q_0 \setminus \cup P_j}(x) - \sum_{l} T_1(\chi_{\mathbb{R}^n \setminus 3P_j}T_{2,b}(f\chi_{Q_0 \setminus 9P_j}))(x)\chi_{P_l}(x)$$

and

$$G_0(x) = T_{1,b}T_2((b - \langle b \rangle_{Q_0})f\chi_{Q_0 \setminus 9P_j}))(x)\chi_{P_l}(x).$$

The facts that $P_j \cap E^c \neq \emptyset$ and $|E \setminus \cup_j P_j| = 0$ imply that

$$|G_0(x)| \leq |T_{1,b}T_2(f\chi_{Q_0})(x)|_{\mathcal{Q}_0 \setminus \cup P_j} + \sum_{l} \inf_{\xi \in P_j} M^*_{T_{1,b}T_2}f(\xi)\chi_{P_l}(x)$$

$$\leq \|f\|_{L(\log L)^{2}, Q_0}.$$ (3.7)

Also, we define functions $G_1$ and $G_2$ by

$$G_1(x) = (b(x) - \langle b \rangle_{Q_0}) \sum_{l} T_1(\chi_{\mathbb{R}^n \setminus 3P_j}T_{2}(f\chi_{Q_0 \setminus 9P_j}))(x)\chi_{P_l}(x)$$

and

$$G_2(x) = \sum_{l} T_{1,b}T_2(f\chi_{Q_0 \setminus 9P_j}))(x)\chi_{P_l}(x).$$

Then

$$|G_1(x)| \leq |b(x) - \langle b \rangle_{Q_0}| \sum_{l} \inf_{\xi \in P_j} M^*_{T_{1,b}T_2}f(\xi)\chi_{P_l}(x)$$

$$\leq |b(x) - \langle b \rangle_{Q_0}|\|f\|_{L(\log L)^{2}, Q_0}(x).$$ (3.8)

Let $\overline{T}_{1}$ be the adjoint operator of $T_1$ and $\overline{T}_{1,b}$ the commutator of $\overline{T}_{1}$. For each function $g$, we have by estimate (2.3) in Lemma 2.1 that

$$\left| \int_{\mathbb{R}^n} G_2(x)g(x) \, dx \right| \leq \sum_{l} \int_{3P_j} |T_2(f\chi_{Q_0 \setminus 9P_j}))(x)| \overline{T}_{1,b}(g\chi_{P_l})(x) \, dx$$

$$\leq \sum_{l} \inf_{\xi \in P_j} M_{T_{2,Q_0}}(\xi) \int_{3P_j} |\overline{T}_{1,b}(g\chi_{P_l})(x)| \, dx$$

$$\leq \|f\|_{9Q_0} \|g\|_{L(\log L)^{2}, Q_0} |Q_0|.$$ (3.9)

Moreover,

$$T_{1,b}T_2(f\chi_{Q_0})(x)\chi_{Q_0}(x) = G_0(x) + G_1(x) + G_2(x) + \sum_{l} T_{1,b}T_2(f\chi_{P_l})(x)\chi_{P_l}(x).$$
We now repeat the argument above with \( T_{1,b} T_2(f \chi_{Q_0})(x) x_{Q_0} \) replaced by each
\( T_{1,b} T_2(f \chi_{Q_j})(x) x_{Q_j} \) and so on. Let \( Q_{0,j}^0 = P_j \) and, for fixed \( j_1, \ldots, j_{m-1}, \{ Q_{0,j_1,\ldots,j_{m-1}} \}_{j_m} \) be the cubes obtained at the \( m \)th stage of the decomposition process to the cube
\( Q_{0,j_1,\ldots,j_{m-1}} \). For each fixed \( j_1, \ldots, j_m \), define the functions \( H_{Q_0,1}^{j_1,\ldots,j_m} \) and \( H_{Q_0,2}^{j_1,\ldots,j_m} \) by

\[
H_{Q_0,1}^{j_1,\ldots,j_m} f(x) = T_1(x \chi_{Q_0^{j_1,\ldots,j_m}}) x_{Q_0^{j_1,\ldots,j_m}}(x)
\]

and

\[
H_{Q_0,2}^{j_1,\ldots,j_m} f(x) = T_1(x \chi_{Q_0^{j_1,\ldots,j_m}}) x_{Q_0^{j_1,\ldots,j_m}}(x),
\]

respectively. Set \( \mathcal{F} = \{ Q_0 \} \cup \bigcup_{m=1}^{\infty} \bigcup_{j_1,\ldots,j_m} \{ Q_{0,j_1,\ldots,j_m} \} \). Then \( \mathcal{F} \subset D(Q_0) \) is a \( \frac{1}{2} \)-sparse family. Let

\[
H_{0,Q_0}(x) = T_{1,b} T_2(f \chi_{Q_0})(x) x_{Q_0^{j_1,\ldots,j_m}}(x)
\]

Also, we define the functions \( H_{1,Q_0} \) and \( H_{2,Q_0} \) by

\[
H_{1,Q_0}(x) = \sum_{m=1}^{\infty} \sum_{j_1,\ldots,j_m} H_{Q_0,1}^{j_1,\ldots,j_m} f(x) \chi_{Q_0^{j_1,\ldots,j_m}}(x)
\]

and

\[
H_{2,Q_0}(x) = \sum_{m=1}^{\infty} \sum_{j_1,\ldots,j_m} H_{Q_0,2}^{j_1,\ldots,j_m} f(x) \chi_{Q_0^{j_1,\ldots,j_m}}(x).
\]

Then, for almost every \( x \in Q_0 \),

\[
T_{1,b} T_2(f \chi_{Q_0})(x) = H_{0,Q_0}(x) + H_{1,Q_0}(x) + H_{2,Q_0}(x).
\]

Moreover, as in inequalities (3.7)–(3.9), the process of producing \( \{ Q_{0,j_1,\ldots,j_{m-1}} \} \) leads to

\[
|H_{0,Q_0}(x) \chi_{Q_0}| \leq \sum_{Q \in \mathcal{F}} \|f\|_{L^1} \|\log L\|^2 \|g\|_{L^1} \chi_{Q}(x).
\]

For any function \( g \), we can verify that

\[
\left| \int_{Q_0} g(x) H_{1,Q_0}(x) \, dx \right| \leq \sum_{Q \in \mathcal{F}} |Q| \|f\|_{L^1} \|\log L\|^2 \|g\|_{L^1} \chi_{Q}(x).
\]
and
\[ \left| \int_{Q_l} g(x)H_{2,Q_l}(x) \, dx \right| \lesssim \sum_{Q \in F} |Q| \|f\|_g \|g\|_{L(\log L)^{\frac{1}{2}}},Q. \]

We can now conclude the proof of Theorem 3.2. In fact, as in [20], we decompose \( \mathbb{R}^n \) by cubes \( \{R_l\} \) such that \( \text{supp} f \subset 3R_l \) for each \( l \) and the \( R_l \) have disjoint interiors. Then, for almost every \( x \in \mathbb{R}^n \),

\[ T_{1,b} T_2 f(x) = \sum_l H_{0,R_l} f(x) + \sum_l H_{1,R_l} f(x) + \sum_l H_{2,R_l} f(x) \]

:= \( H_0 f(x) + H_1 f(x) + H_2 f(x) \).

Obviously, for \( j = 0, 1, 2 \) and any function \( g \),

\[ \left| \int_{\mathbb{R}^n} H_j(x)g(x) \, dx \right| \lesssim A_{S,L(\log L)^{\frac{3}{2}}},L(\log L)^j \langle f, g \rangle. \]

Our desired conclusion then follows directly. \( \square \)

**Proof of Theorem 1.2.** By Theorem 3.2 and Lemma 3.1, we know that for each \( \epsilon \in (0, 1) \), weight \( u \) and \( \lambda > 0 \),

\[ u(\{x \in \mathbb{R}^n : |T_1 T_2 f(x)| > \lambda \}) \]

\[ \leq \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) M_{L(\log L)^j} u(x) \, dx \]

\[ + \frac{1}{\lambda \epsilon} \int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^{j+1}} u(x) \, dx \]

\[ \leq \frac{1}{\epsilon} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) M_{L(\log L)^{j+1}} u(x) \, dx. \]

Applying the ideas used in [16, page 608] (see also the proof of Corollary 1.3 in [21]), we know that the last inequality implies (1.6).

The inequality (1.7) is essentially an application of [1, Proposition 9] and Lemma 3.1. Recall that \( T_1(1) = 0 \). It then follows from [1, Proposition 9] that for \( f \in L^2(\mathbb{R}^n) \), there exists a sparse family of cubes \( S \) such that

\[ |T_1 T_2 f(x)| \lesssim \sum_{Q \in S} \text{osc}_Q(T_2 f) \chi_Q(x); \]

here \( \text{osc}_Q(T_2 f) \) is defined by

\[ \text{osc}_Q(T_2 f) = \frac{1}{|Q|} \int_Q |T_2 f(x) - \langle T_2 f \rangle_Q| \, dx. \]

A trivial computation leads to

\[ \text{osc}_Q(T_2 f) \lesssim \|f\|_{L,8n} + \sum_{k=1}^{\infty} 2^{-k\epsilon} \|f\|_{2^k},Q. \]
with $\varepsilon$ the constant in (1.2). Let $G$ be the operator defined by

$$
G f(x) = \sum_{k=1}^{\infty} 2^{-ke} \sum_{Q \in S} \langle f \rangle_{2^k Q} \chi_Q(x).
$$

We then have that

$$
|T_1 T_2 f(x)| \leq \mathcal{A}_{S, L \log L} f(x) + G f(x). \tag{3.10}
$$

On the other hand, it was proved in [19] that there exist sparse families of cubes $S_1, \ldots, S_{2^n+1}$ such that for any function $g$,

$$
\int_{\mathbb{R}^n} |G f(x)g(x)| \, dx \leq \sum_{j=1}^{2^n+1} \mathcal{A}_{S_j, \log L} (f, g). \tag{3.11}
$$

Thus, by inequalities (3.10), (3.11) and Lemma 3.1, we know that for each fixed $\lambda > 0$, $\varepsilon \in (0, 1)$ and weight $u$,

$$
u(x \in \mathbb{R}^n : |T_1 T_2 f(x)| > \lambda) \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^n} \left| f(x) \right| \frac{\log \left( 1 + \frac{|f(x)|}{\lambda} \right)}{\lambda} M_{L(\log L)^2} u(x) \, dx.
$$

This implies (1.7).

\[\square\]

**Remark 3.3.** By the estimate of the bilinear sparse operator (see [24] or [7]), we proved in [9] that for $b_1, b_2 \in [0, \infty)$, $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$
\mathcal{A}_{S, L \log L} f, L \log L \mathcal{A}_2 (f, g) \leq |\sigma|_{A_p}^{b_1} [w]_{A_p}^{b_2} [w]^{1/p} [w]^{1/p'} + |\sigma|_{A_p} [f]_{L^p(\mathbb{R}^n)} [g]_{L^{p'}(\mathbb{R}^n, \sigma)}. \tag{3.12}
$$

The conclusions in Theorem 1.1 now follow from inequalities (3.10)–(3.12).

**Proof of Theorem 1.3.** As was shown in the proof of Theorem 1.2, by Theorem 3.2 and Lemma 3.1, we know that for each $\varepsilon \in (0, 1)$, weight $u$ and $\lambda > 0$,

$$
u(x \in \mathbb{R}^n : |T_1 b T_2 f(x)| > \lambda) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \left| f(x) \right| \frac{\log \left( 1 + \frac{|f(x)|}{\lambda} \right)}{\lambda} M_{L(\log L)^{2+u}} u(x) \, dx.
$$

The inequality (1.8) then follows immediately. \[\square\]

**Remark 3.4.** Let $m \in \mathbb{N}$ and $0 \leq k \leq m$, $T_1, \ldots, T_m$ be Calderón–Zygmund operators, $b_1, \ldots, b_m \in \text{BMO}(\mathbb{R}^n)$ and $T_{j,b_j}$ ($j = 1, \ldots, m$) be the commutator of $T_j$. Repeating the proofs of Theorem 3.2, we can verify that for each bounded function $f$, there exist a $\frac{1}{2}(1/3^n)$-sparse family of cubes $S$ and functions $J_0, \ldots, J_{m+k-1}$ such that for each $j = 0, \ldots, m + k - 1$ and function $g \in L^1(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} J_j(x) g(x) \, dx \leq \mathcal{A}_{S, \text{BMO}(L^{2+1})} (f, g).
$$
and, for almost every \( x \in \mathbb{R}^n \),

\[
T_{1,b_1} \ldots T_{k,b_k} T_{k+1} \ldots T_m f(x) = \sum_{j=0}^{m+k-1} J_j(x).
\]

This, via Lemma 3.1 and estimate (3.12), shows that:

(i) for \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}^n) \),

\[
\|T_{1,b} \ldots T_{k,b_k} T_{k+1} \ldots T_m f\|_{L^p(\mathbb{R}^n, w)} \leq [w]_{A_p}^{1/p} ([w]_{A_\infty}^{1/p'} + [\sigma]_{A_\infty}^{1/p}) ([w]_{A_\infty} + [\sigma]_{A_\infty})^{m+k-1} \|f\|_{L^p(\mathbb{R}^n, w)};
\]

(ii) for each \( \epsilon \in (0, 1) \), weight \( u \) and \( \lambda > 0 \),

\[
u(\{x \in \mathbb{R}^n : |T_{1,b_1} \ldots T_{k,b_k} T_{k+1} \ldots T_m f(x)| > \lambda\}) \leq \frac{1}{\epsilon} \int_{\mathbb{R}^n} \frac{|f(x)| \log^{m+k-1} (e + |f(x)|)}{\lambda} M_L(\log L)^{m+k-1} u(x) \, dx.
\]

References

[1] C. Benea and F. Bernicot, ‘Conservation de certaines propriétés à travers un contrôle épars d’un opérateur et applications au projecteur de Leray–Hopf’, Preprint, 2017, arXiv:1703:00228.

[2] S. M. Buckley, ‘Estimates for operator norms on weighted spaces and reverse Jensen inequalities’, Trans. Amer. Math. Soc. 340 (1993), 253–272.

[3] N. Carozza and A. Passarelli di Napoli, ‘Composition of maximal operators’, Publ. Mat. 40 (1996), 397–409.

[4] D. Chung, M. C. Pereyra and C. Pérez, ‘Sharp bounds for general commutators on weighted Lebesgue spaces’, Trans. Amer. Math. Soc. 364 (2012), 1163–1177.

[5] R. R. Coifman and Y. Meyer, Wavelets: Calderón–Zygmund Operators and Multilinear Operators (Cambridge University Press, Cambridge, 1997).

[6] L. Grafakos, Modern Fourier Analysis, 2nd edn, Graduate Texts in Mathematics, 250 (Springer, New York, 2008).

[7] T. Hänninen, T. Hytönen and K. Li, ‘Two-weight \( L^p - L^q \) bounds for positive dyadic operators: unified approach to \( p \leq q \) and \( p > q \)’, Potential Anal. 45 (2016), 579–608.

[8] G. Hu, ‘Weighted vector-valued estimates for a non-standard Calderón–Zygmund operator’, Nonlinear Anal. 165 (2017), 143–162.

[9] G. Hu, ‘Quantitative weighted bounds for the composition of Calderón–Zygmund operators’, Banach J. Math. Anal. 13 (2019), 133–150.

[10] G. Hu and D. Li, ‘A Cotlar type inequality for the multilinear singular integral operators and its applications’, J. Math. Anal. Appl. 290 (2004), 639–653.

[11] G. Hu and D. Yang, ‘Weighted estimates for singular integral operators with nonsmooth kernels’, J. Aust. Math. Soc. 85 (2008), 377–417.

[12] T. Hytönen, ‘The sharp weighted bound for general Calderón–Zygmund operators’, Ann. of Math. (2) 175 (2012), 1473–1506.

[13] T. Hytönen and M. T. Lacey, ‘The \( A_p - A_\infty \) inequality for general Calderón–Zygmund operators’, Indiana Univ. Math. J. 61 (2012), 2041–2052.

[14] T. Hytönen, M. T. Lacey and C. Pérez, ‘Sharp weighted bounds for the \( q \)-variation of singular integrals’, Bull. Lond. Math. Soc. 45 (2013), 529–540.

[15] T. Hytönen and C. Pérez, ‘Sharp weighted bounds involving \( A_\infty \)’, Anal. PDE 6 (2013), 777–818.

[16] T. Hytönen and C. Pérez, ‘The \( L(\log L)^p \) endpoint estimate for maximal singular integral operators’, J. Math. Anal. Appl. 428 (2015), 605–626.
[17] T. Hytönen, C. Pérez and E. Rela, ‘Sharp reverse Hölder property for $A_1$ weights on spaces of homogeneous type’, *J. Funct. Anal.* **263** (2012), 3883–3899.

[18] S. G. Krantz and S. Li, ‘Boundedness and compactness of integral operators on spaces of homogeneous type and applications’, *J. Math. Anal. Appl.* **258** (2001), 629–641.

[19] A. K. Lerner, ‘A simple proof of the $A_2$ conjecture’, *Int. Math. Res. Not.* **14** (2013), 3159–3170.

[20] A. K. Lerner, ‘On pointwise estimate involving sparse operator’, *New York J. Math.* **22** (2016), 341–349.

[21] A. K. Lerner, ‘A weak type estimate for rough singular integrals’, Preprint, 2017, arXiv:1705.07397.

[22] A. K. Lerner and F. Nazarov, ‘Intuitive dyadic calculus: the basics’, *Expo. Math.* doi:10.1016/j.exmath.2018.01.001.

[23] A. K. Lerner, S. Ombrosi and I. Rivera-Rios, ‘On pointwise and weighted estimates for commutators of Calderón–Zygmund operators’, *Adv. Math.* **319** (2017), 153–181.

[24] K. Li, ‘Two weight inequalities for bilinear forms’, *Collect. Math.* **68** (2017), 129–144.

[25] K. Li, C. Pérez, I. P. Rivera-Rios and L. Roncal, ‘Weighted norm inequalities for rough singular integral operators’, *J. Geom. Anal.* doi:10.1007/s12220-018-0085-4.

[26] M. Rao and Z. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 146 (Marcel Dekker, New York, 1991).

[27] E. Sawyer, ‘Norm inequalities relating singular integrals and the maximal function’, *Studia Math.* **75** (1983), 253–263.

[28] M. J. Wilson, ‘Weighted inequalities for the dyadic square function without dyadic $A_\infty$’, *Duke Math. J.* **55** (1987), 19–50.

GUOEN HU, School of Applied Mathematics, Beijing Normal University, Zhuhai 519087, PR China
e-mail: guoenxx@163.com