STABLE SPIN MAPS, GROMOV-WITTEN INVARIANTS, AND QUANTUM COHOMOLOGY

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Abstract. We introduce the stack $\mathcal{M}_{g,n}^{1/r}(V)$ of $r$-spin maps. These are stable maps into a variety $V$ from $n$-pointed algebraic curves of genus $g$, with the additional data of an $r$-spin structure on the curve.

We prove that $\mathcal{M}_{g,n}^{1/r}(V)$ is a Deligne-Mumford stack, and we define analogs of the Gromov-Witten classes associated to these spaces. We show that these classes yield a cohomological field theory (CohFT) that is the tensor product of the CohFT associated to the usual Gromov-Witten invariants of $V$ and the $r$-spin CohFT. When $r = 2$, our construction gives the usual Gromov-Witten invariants of $V$.

Restricting to genus zero, we obtain the notion of an $r$-spin quantum cohomology of $V$, whose Frobenius structure is isomorphic to the tensor product of the Frobenius manifolds corresponding to the quantum cohomology of $V$ and the $r$-th Gelfand-Dickey hierarchy (or, equivalently, the $A_{r-1}$ singularity). We also prove a generalization of the descent property which, in particular, explains the appearance of the $\psi$ classes in the definition of gravitational descendents. Finally, we compute the small phase space potential function when $r = 3$ and $V = \mathbb{P}^1$.

0. Introduction

In this paper, we present a generalization of the theory of quantum cohomology and Gromov-Witten invariants arising from algebraic curves with higher spin structures. Recall that ordinary Gromov-Witten invariants of a projective variety $V$ are constructed by means of the moduli spaces $\mathcal{M}_{g,n}(V)$ of stable maps to $V$. The space $\mathcal{M}_{g,n}(V)$ is a Deligne-Mumford stack compactifying the space of holomorphic maps to $V$ from Riemann surfaces of genus $g$ with $n$ marked points. In particular, the moduli space of stable maps to a point coincides with the moduli of stable curves $\mathcal{M}_{g,n}$.

Although the space $\mathcal{M}_{g,n}(V)$ is not generally smooth, it does possess a homology class $[\mathcal{M}_{g,n}(V)]^\text{virt}$ (called the virtual fundamental class) which plays the role of the fundamental class in intersection theory. Furthermore, the evaluation maps $\mathcal{M}_{g,n}(V) \to V$ obtained by evaluating the map on each marked point on the curve allow one to pull back cohomology classes from $V$. Together, these classes give rise to the collection $A^V = \{ A^V_{g,n} \}$ of Gromov-Witten classes

$$A^V_{g,n} \in H^*(\mathcal{M}_{g,n}) \otimes \mathcal{T}^n H^*(V)^*,$$

which behave nicely when restricted to the boundary strata of $\mathcal{M}_{g,n}$. This allows one to define a collection of multilinear operations on the space $H^*(V)$.

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parametrized by elements of $H_\bullet (\Mbar_{g,n})$. These operations satisfy the axioms of a cohomological field theory (CohFT) in the sense of Kontsevich-Manin [18]. In particular, their restriction to stable maps of genus zero endows $H^\bullet (V)$ with the structure of a (formal) Frobenius manifold [17, 21], called the quantum cohomology of $V$, whose multiplication is a deformation of the usual cup product in $H^\bullet (V)$.

The diagonal map $\Mbar_{g,n} \rightarrow \Mbar_{g,n} \times \Mbar_{g,n}$ induces a natural tensor product operation on the set of CohFTs. Behrend [4] proved that the tensor product of the CohFTs defined by the Gromov-Witten classes of two smooth projective varieties $V$ and $V'$ is isomorphic to the CohFT defined by the classes $\Lambda^{V \times V'}$. This is nontrivial because $\Mbar_{g,n}(V \times V')$ is not isomorphic to the fiber product $\Mbar_{g,n}(V) \times_{\Mbar_{g,n}} \Mbar_{g,n}(V')$. Restricting to genus zero, we can regard this result as a Künneth formula for quantum cohomology.

In [15], we introduced a new class of CohFTs, one for each integer $r \geq 2$, based on the moduli space $\Mbar_{g,n}^{1/r} = \coprod_{m} \Mbar_{g,n}^{1/r,m}$ of higher spin curves, constructed in [10]. Recall that for $m = (m_1, \ldots, m_n)$, with $m_i \in \mathbb{Z}$, the moduli space $\Mbar_{g,n}^{1/r,m}$ is a compactification of the space of Riemann surfaces of genus $g$ with $n$ marked points $p_1, \ldots, p_n$ and an $r$-th root of the twisted canonical line bundle $\omega \otimes \mathcal{O}(-\sum_i m_i p_i)$.

We proved in [15] that the existence of a special cohomology class $c^{1/r}$ (called the spin virtual class) in $H^\bullet (\Mbar_{g,n}^{1/r})$, satisfying certain axioms similar to the Behrend-Manin axioms [4] for the virtual fundamental class, gives rise to a CohFT of rank $r - 1$.

This $r$-spin CohFT is related to the work of Witten [20], who conjectured that its large phase space potential (the generating function of certain intersection numbers on $\Mbar_{g,n}^{1/r,m}$) is a $\tau$ function of the $r$-th Gelfand-Dickey (or KdV$_r$) hierarchy. When $r = 2$, this conjecture reduces to an earlier conjecture of Witten’s on the intersection numbers of $\Mbar_{g,n}$, which was proved by Kontsevich [17]. In [15], following Witten’s ideas, we constructed the spin virtual class and proved the conjecture in the cases $g = 0$ (for all $r$) and $r = 2$ (for all $g$).

In this paper, we introduce and describe moduli spaces that give an intersection-theoretic realization of the tensor product of the Gromov-Witten CohFT and the $r$-spin CohFT. We construct $\Mbar_{g,n}^{1/r}(V)$, the stack of stable $r$-spin maps into a projective variety $V$—objects which combine both the data of a stable map and an $r$-spin structure. We prove that $\Mbar_{g,n}^{1/r}(V)$ is a Deligne-Mumford stack and a ramified cover of $\Mbar_{g,n}(V)$. Similar to the case of ordinary stable maps, stable $r$-spin maps to a point are just stable $r$-spin curves.

We introduce the class $\bar{c}^{1/r} \in H^\bullet (\Mbar_{g,n}^{1/r}(V))$, which is an analog of a virtual class $\bar{c}^{1/r}$ on $\Mbar_{g,n}^{1/r}$. The class $\bar{c}^{1/r}$ is defined by pulling back $c^{1/r}$ from $\Mbar_{g,n}^{1/r}$ when $2g - 2 + n > 0$ and is defined by a direct construction for other values of $g$ and $n$. Using the class $\bar{c}^{1/r}$ and the virtual fundamental class of $\Mbar_{g,n}(V)$, we define the $r$-spin analogs of the Gromov-Witten classes

$$\Lambda^{(V,r)}_{g,n} \in H^\bullet (\Mbar_{g,n}^{1/r}(V)) \otimes T^n \mathcal{H}^{(V,r)},$$

where $\mathcal{H}^{(V,r)} = H^\bullet (V) \otimes \mathcal{H}^{(r)}$ and $\mathcal{H}^{(r)}$ is the state space of the $r$-spin CohFT. The collection of classes $\Lambda^{(V,r)}_{g,n} = \{\Lambda^{(V,r)}_{g,n}\}$ gives rise to a CohFT with the state space...
\[ \mathcal{H}^{(V,r)} \], which is isomorphic to the tensor product of the Gromov-Witten CohFT and the \( r \)-spin CohFT. This fact is not trivial because the space \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \) is not isomorphic to the fiber product \( \overline{\mathcal{M}}_{g,n}(V) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}^{1/r} \).

Our result should be viewed as an analog of the relationship between stable maps to \( V \times V' \) and the tensor product of the CohFTs corresponding to each factor (see [18, 19, 4]). Restricting to genus zero, we obtain the tensor product of the quantum cohomology of \( V \) with the Frobenius manifold associated to KdV, (or equivalently, to the \( A_{r-1} \) singularity). Tensor products of several spin CohFTs were studied in our earlier paper [16]. They correspond to the moduli spaces of curves with multiple spin structures. It would be very interesting to find an enumerative interpretation of the corresponding invariants similar to the interpretation of the ordinary Gromov-Witten invariants.

Finally, it is worth observing that these results have a physical interpretation as well. Our spin Gromov-Witten invariants may be regarded as the correlators in a theory of topological gravity coupled to topological matter, where the matter sector of the theory is the topological sigma model with target space \( V \) coupled with a certain type of gauged \( SU(2)_{r-2}/U(1) \) Wess-Zumino-Witten model. Our moduli spaces \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \) provide an intersection-theoretic realization of this theory.

The structure of the paper is as follows. In the first section, we recall some standard definitions and properties of stable maps and Gromov-Witten invariants. In the second section, we construct the moduli space \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \), establish some of its properties, and prove that it is a Deligne-Mumford stack. In the third section, we introduce decorated graphs associated with the various moduli spaces and the corresponding cohomology classes. In the fourth section, we define the spin virtual class \( \tilde{c}^{1/r} \) on \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \) and study its intersection-theoretic properties. In the fifth section, we define the analogs of the Gromov-Witten classes associated to \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \) and show that the resulting CohFT is the tensor product of CohFTs associated to \( \overline{\mathcal{M}}_{g,n}(V) \) and \( \overline{\mathcal{M}}_{g,n}^{1/r} \). We then define the potential functions of the theory and prove that when \( r = 2 \), this theory reduces to the usual Gromov-Witten invariants of \( V \). Furthermore, we prove that \( \tilde{c}^{1/r} \) satisfies the descent property in genus zero. In the final section, we compute the genus zero small phase space potential function associated to \( \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1) \).

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1. Preliminaries

In this section we recall the basic definitions related to stable maps and Gromov-Witten invariants. For details see [18, 21].

1.1. Prestable curves.

By a curve, in this paper, we mean a reduced, complete, connected, one-dimensional scheme over \( \mathbb{C} \). By genus of a curve \( X \), we mean its arithmetic genus \( g = \dim H^1(X, \mathcal{O}_X) \). An \( n \)-pointed, prestable curve is a curve \( X \) with \( n \) distinct marked points \( p_1, \ldots, p_n \in X \), such that \( X \) has at most ordinary double points (nodes) as singularities and \( p_1, \ldots, p_n \) are nonsingular. A prestable curve \( X \) is called stable.
if, in addition, each irreducible component of $X$ of genus 0 has at least three distinguished points, and each component of genus 1 has at least one distinguished point, where a distinguished point is either a node or a marked point.

A family of prestable, $n$-pointed curves is a flat, proper morphism $X \to T$ with $n$ sections $p_1, \ldots, p_n : T \to X$, such that each geometric fiber $(X_t, p_1(t), \ldots, p_n(t))$ is an $n$-pointed prestable curve.

By a line bundle we mean an invertible (locally free of rank one) coherent sheaf. The canonical sheaf of a family of curves $f : X \to T$ is the relative dualizing sheaf of $f$. This sheaf will be denoted by $\omega_{X/T}$, or $\omega_f$. When $T = \text{Spec } \mathbb{C}$, we will write $\omega_X$ or $\omega$ instead of $\omega_{X/T}$. Note that for a family of prestable curves, the canonical sheaf is a line bundle.

A relatively torsion-free sheaf (or just torsion-free sheaf) on a family of prestable curves $f : X \to T$ is a coherent $\mathcal{O}_X$-module $\mathcal{E}$ that is flat over $T$, such that on each fiber $X_t = X \times_T \text{Spec } k(t)$ the restriction $\mathcal{E}_t$ has no associated primes of height one. We will only be concerned with rank-one torsion-free sheaves. Such sheaves are sometimes called admissible or sheaves of pure dimension 1. On the open set where $f$ is smooth, a torsion-free sheaf is locally free.

1.2. Moduli of stable maps and virtual fundamental class.

Let $V$ be an algebraic variety over $\mathbb{C}$. A stable map to $V$ consists of an $n$-pointed, prestable curve $(X, p_1, \ldots, p_n)$ and a morphism $f : X \to V$, such that each irreducible component $C$ of $X$ mapped by $f$ to a point is stable (i.e. $C$ has at least three distinguished points if it is of genus zero and at least one distinguished point if it is of genus one). A family of stable maps to $V$ is a family of prestable, $n$-pointed curves $\pi : X \to T$ with a morphism $f : X \to V$, such that the restriction of $f$ to each geometric fiber of $\pi$ is stable.

For $\beta \in H_2(V, \mathbb{Z})$, we say that the stable map $f : X \to V$ has class $\beta$ if the pushforward $f_*([X])$ of the fundamental class of $X$ is equal to $\beta$. We denote by $\overline{\mathcal{M}}_{g,n}(V, \beta)$ the stack of $n$-pointed stable maps of genus $g$ to $V$ of class $\beta$, and by $\overline{\mathcal{M}}_{g,n}(V)$ the disjoint union $\displaystyle \bigsqcup_{\beta \in H_2(V, \mathbb{Z})} \overline{\mathcal{M}}_{g,n}(V, \beta)$. If $V$ is a projective variety, $\overline{\mathcal{M}}_{g,n}(V, \beta)$ is a Deligne-Mumford stack. When $V$ is a point, $\overline{\mathcal{M}}_{g,n}(V)$ coincides with the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$.

There are two important canonical morphisms on $\overline{\mathcal{M}}_{g,n}(V)$. The stabilization morphism

\begin{equation}
\text{st} : \overline{\mathcal{M}}_{g,n}(V) \to \overline{\mathcal{M}}_{g,n}
\end{equation}

forgets the map $f$ and collapses the unstable components of the curve $X$. The evaluation morphism

\begin{equation}
ev_i : \overline{\mathcal{M}}_{g,n}(V) \to V
\end{equation}

is obtained by evaluating the stable map $f$ at the $i$-th marked point $p_i$.

The space $\overline{\mathcal{M}}_{g,n}(V)$ has a special Chow homology class

\begin{equation}
[\overline{\mathcal{M}}_{g,n}(V)]^\text{virt} \in H_*(\overline{\mathcal{M}}_{g,n}(V), \mathbb{Q}),
\end{equation}

called the virtual fundamental class, that satisfies various properties (axioms) formulated in [3]. The class $[\overline{\mathcal{M}}_{g,n}(V)]^\text{virt}$ gives rise to the collection $\Lambda^V = \{ \Lambda^V_{g,n} \}$ of Gromov-Witten classes

\begin{equation}
\Lambda^V_{g,n} \in H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes T^n H^\bullet(V)^*.
\end{equation}
We define
\[ \Lambda^{V,\beta}_{g,n}(\gamma_1, \ldots, \gamma_n) := \text{st} \star (\text{ev}_1^* \gamma_1 \ldots \text{ev}_n^* \gamma_n \cap [\overline{\mathcal{M}}_{g,n}(V,\beta)]^\text{virt}), \]
where \( \gamma_j \in H^\bullet(V) \).

2. Stable Spin Maps and Their Moduli

For the remainder of the paper we fix an integer \( r \geq 2 \). In this section we introduce the moduli space of stable \( r \)-spin maps and prove that it is a Deligne-Mumford stack.

2.1. Overview of \( r \)-spin structures.

Although the concept of an \( r \)-spin structure is intuitively simple, its formal definition is somewhat technical. For that reason we first give a brief overview of the ideas involved.

2.1.1. Spin structures on smooth curves.

Definition 2.1. Given a collection of integers \( m = (m_1, \ldots, m_n) \), an \( r \)-spin structure of type \( m \) on a smooth, \( n \)-pointed curve \((X,p_1,\ldots,p_n)\) is a line bundle \( L \) on \( X \) with an isomorphism \( b : L^\otimes r \rightarrow \omega_X(-\sum m_i p_i) \).

In particular, an \( r \)-spin structure of type \( 0 = (0, \ldots, 0) \) on a smooth curve \( X \) is essentially just an \( r \)-th root of the canonical bundle \( \omega_X \).

For degree reasons, an \( r \)-spin structure of type \( m \) exists on a genus \( g \) curve \( X \) only if \( 2g - 2 - \sum m_i \) is divisible by \( r \). When this condition is met, there are \( r^{2g} \) choices of \( L \) on \( X \).

Definition 2.2. A smooth \( r \)-spin curve of type \( m \) is a smooth curve \( X \) with an \( r \)-spin structure of type \( m \). A smooth \( r \)-spin map of type \( m \) into a target \( V \) is a morphism of a smooth curve \( X \) into \( V \) with the additional data of an \( r \)-spin structure of type \( m \) on \( X \).

Example 2.3. A 2-spin structure of type \( 0 \) on a smooth curve \( X \) corresponds to a choice of a theta characteristic \( L \) on \( X \) and an explicit isomorphism \( b : L^\otimes 2 \rightarrow \omega_X \).

Example 2.4. If \( E, p_0 \) is a smooth elliptic curve, then \( \omega_E \) is isomorphic to \( O_E \). An \( r \)-spin structure on \( E \) of type \( 0 \) corresponds to a choice of an \( r \)-torsion point \( q \in E \) and an explicit isomorphism
\[ b : O(q - p_0)^\otimes r \rightarrow O \cong \omega_E \]
from the \( r \)-th tensor power of the invertible sheaf \( O(q - p_0) \) to the canonical sheaf \( \omega_E \).

2.1.2. Spin structures on nodal curves. If we want to compactify the spaces involved by considering stable maps and prestable curves, the above definition of an \( r \)-spin structure is insufficient. In particular, even when the degree condition is satisfied, there may be no line bundle \( L \) on a prestable curve \( X \) such that \( L^\otimes r \) is isomorphic to \( \omega_X(-\sum m_i p_i) \).

The solution involves replacing line bundles by rank-one, torsion-free sheaves, allowing the isomorphism \( b : L^\otimes r \rightarrow \omega_X(-\sum m_i p_i) \) to have non-trivial cokernel at the nodes of the curve, and requiring that \( b \) satisfy some additional technical restrictions (see Definitions 2.6 and 2.7). There are two very different types of
behavior of this torsion-free sheaf $\mathcal{L}$ near a node $q \in X$. When it is still locally free, the sheaf $\mathcal{L}$ is said to be Ramond at the node $q$. If the sheaf $\mathcal{L}$ is not locally free at $q$, it is called Neveu-Schwarz.

In the Ramond case, the homomorphism $b$ is still an isomorphism near the node $q$, but in the Neveu-Schwarz case it cannot be an isomorphism. The local structure of the sheaf $\mathcal{L}$ near a Neveu-Schwarz node can be described as follows.

Near the node $q$, the curve $X$ has two coordinates $x$ and $y$, such that $xy = 0$. The sheaf $\omega_X$ (or $\omega_X(-\sum m_ip_i)$) is locally generated by $\frac{dx}{x} = -\frac{dy}{y}$. Near $q$ the sheaf $\mathcal{L}$ is generated by two elements $\ell_+$ and $\ell_-$, supported on the $x$ and $y$ branches respectively (that is, $x\ell_- = y\ell_+ = 0$). The two generators may be chosen so that the homomorphism $b : \mathcal{L}^{\otimes r} \to \omega_X(-\sum m_ip_i)$ takes $\ell_+^{\otimes r}$ to $x^{m_++1}(\frac{dx}{x}) = x^{m_+}dx$ and $\ell_-^{\otimes r}$ to $y^{m_-+1}(\frac{dy}{y}) = y^{m_-}dy$, where $(m_++1)+(m_-+1) = r$.

**Definition 2.5.** We call $m_+$ (respectively $m_-$) the order of the spin structure along the $x$-branch (respectively $y$-branch).

One more difficulty arises when $r$ is not prime—in this case the moduli of stable curves with $r$-spin structure, as described above, is not smooth. The remedy is to include all $d$-spin structures for every $d$ dividing $r$, satisfying some natural compatibility conditions. This is described in Definition 2.7.

### 2.2. Definition of $r$-spin structures, curves, and maps.

We briefly review here the formal definition of an $r$-spin structure over a fixed prestable curve, given in [10], and we define stable $r$-spin maps.

**Definition 2.6.** Let $(X, p_1, \ldots, p_n)$ be a prestable, $n$-pointed, algebraic curve; let $\mathcal{K}$ be a rank-one, torsion-free sheaf on $X$; and let $\mathbf{m} = (m_1, \ldots, m_n)$ be an $n$-tuple of integers. A $d$-th root of $\mathcal{K}$ of type $\mathbf{m}$ is a pair $(\mathcal{E}, b)$, where $\mathcal{E}$ is a rank-one, torsion-free sheaf, and $b$ is an $\mathcal{O}_X$-module homomorphism

$$b : \mathcal{E}^{\otimes d} \longrightarrow \mathcal{K} \otimes \mathcal{O}_X(-\sum m_ip_i)$$

with the following properties:

- $d \cdot \deg \mathcal{E} = \deg \mathcal{K} - \sum m_i$,
- $b$ is an isomorphism on the locus of $X$ where $\mathcal{E}$ is locally free,
- For every point $p \in X$ where $\mathcal{E}$ is not free, the length of the cokernel of $b$ at $p$ is $d - 1$.

The condition on the cokernel amounts essentially to the condition that at each node the sum of orders of $(\mathcal{E}, b)$ is equal to $d - 2$.

If $\mathbf{m}'$ is congruent to $\mathbf{m}$ modulo $d$, then to any $d$-th root $(\mathcal{E}, b)$ of type $\mathbf{m}$ we can associate a $d$-th root $(\mathcal{E}', b')$ of type $\mathbf{m}'$ simply by taking $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(\frac{1}{d}\sum (m_i - d'm_i)p_i)$. Consequently, the moduli of curves with $d$-th roots of a bundle $\mathcal{K}$ of type $\mathbf{m}$ is canonically isomorphic to the moduli of curves with $d$-th roots of type $\mathbf{m}'$. Therefore, unless otherwise stated, we will always assume the type $\mathbf{m}$ of a $d$-th root satisfies $0 \leq m_i < d$ for all $i$.

Unfortunately, when $d$ is not prime, the moduli space of curves with $d$-th roots of a fixed sheaf $\mathcal{K}$ is not smooth. To fix this problem we must consider not just roots of a bundle, but rather coherent nets of roots [10]. This additional structure suffices to make the moduli space of curves with a coherent net of roots smooth.

**Definition 2.7.** Let $\mathcal{K}$ be a rank-one, torsion-free sheaf on a prestable, $n$-pointed curve $(X, p_1, \ldots, p_n)$. A coherent net of $r$-th roots of $\mathcal{K}$ of type $\mathbf{m} = (m_1, \ldots, m_n)$
is a pair \( (\{\mathcal{E}_d\}, \{c_{d,d'}\}) \) of a set of sheaves and a set of homomorphisms as follows: the set of sheaves consists of a rank-one, torsion-free sheaf \( \mathcal{E}_d \) on \( X \) for every positive divisor \( d \) of \( r \); and the set of homomorphisms consists of an \( \mathcal{O}_X \)-module homomorphism
\[
c_{d,d'} : \mathcal{E}_d^{e^{d/d'}} \rightarrow \mathcal{E}_{d'}
\]
for every pair of divisors \( d', d \) of \( r \), such that \( d' \) divides \( d \). These sheaves and homomorphisms must satisfy the following conditions:

- \( \mathcal{E}_1 = \mathcal{K} \) and \( c_{d,d} = 1_d \), the identity map, for every positive \( d \) dividing \( r \).
- For each divisor \( d \) of \( r \) and each divisor \( d' \) of \( d \), the homomorphism \( c_{d,d'} \) makes \((\mathcal{E}_d, c_{d,d'})\) into a \( d/d' \)-th root of \( \mathcal{E}_{d'} \) of type \( m' \), where \( m' = (m'_1, \ldots, m'_{n}) \) is the reduction of \( m \) modulo \( d/d' \) (i.e. \( 0 \leq m'_i < d/d' \) and \( m_i \equiv m'_i \) (mod \( d/d' \))).
- The homomorphisms \( \{c_{d,d'}\} \) are compatible. That is, the diagram
\[
\begin{array}{ccc}
\mathcal{E}_{d/d''} & \xrightarrow{(c_{d,d'})^{-1}} & \mathcal{E}_{d'/d''} \\
\downarrow & & \downarrow \\
\mathcal{E}_{d'} & \xrightarrow{c_{d,d''}} & \mathcal{E}_{d''}
\end{array}
\]
commutes for every \( d'' | d | r \).

If \( r \) is prime, then a coherent net of \( r \)-th roots is simply an \( r \)-th root of \( \mathcal{K} \). Even when \( d \) is not prime, if the root \( \mathcal{E}_d \) is locally free, then for every divisor \( d' \) of \( d \), the sheaf \( \mathcal{E}_{d'} \) is uniquely determined up to an automorphism of \( \mathcal{E}_{d'} \). In particular, if \( m' \) satisfies the conditions \( m' \equiv m \) (mod \( d' \)) and \( 0 \leq m'_i < d' \), then the sheaf \( \mathcal{E}_{d'} \) is isomorphic to \( \mathcal{E}_{d/d'} \otimes \mathcal{O}(\frac{i}{d'} \mathcal{K}) \).

**Definition 2.8.** Let \( X, p_1, \ldots, p_n \) be an \( n \)-pointed, prestable curve of genus \( g \). Let \( r > 1 \) be an integer and let \( m = (m_1, m_2, \ldots, m_n) \) be an \( n \)-tuple of integers such that \( r \) divides \( 2g-2 - \sum m_i \). An \( r \)-spin structure on \( X \) of type \( m \) is a coherent net of \( r \)-th roots of \( \omega_X \) of type \( m \) on \( X \).

**Definition 2.9.** Let \( r > 1 \) be an integer, and let \( n \) and \( g \) be non-negative integers. Let \( V \) be a Deligne-Mumford stack, and let \( \beta \) be a class in \( H_2(V, \mathbb{Z}) \). Finally, let \( m = (m_1, m_2, \ldots, m_n) \) be an \( n \)-tuple of integers such that \( r \) divides \( 2g-2 - \sum m_i \). A stable, \( n \)-pointed, \( r \)-spin map into \( V \) of genus \( g \), type \( m \), and class \( \beta \) is a pair \((f, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))\) that consists of a stable \( n \)-pointed genus \( g \) map \( f : X \rightarrow V \) of class \( \beta \), and an \( r \)-spin structure \((\{\mathcal{E}_d\}, \{c_{d,d'}\})\) of type \( m \) on \( X \).

**Example 2.10.** If \( V \) is a point, then any stable, \( n \)-pointed, \( r \)-spin map into \( V \) is just a stable \( r \)-spin curve.

**Definition 2.11.** An isomorphism of \( r \)-spin maps \((X \xrightarrow{f} V, p_1, \ldots, p_n, (\{\mathcal{E}_d\}, \{c_{d,d'}\}))\) and \((X' \xrightarrow{f} V, p'_1, \ldots, p'_n, (\{\mathcal{E}'_d\}, \{c'_{d,d'}\}))\) of the same type \( m \)
consists of an isomorphism \( \tau \) of \( n \)-pointed, stable maps

\[
\begin{array}{ccc}
X & \xrightarrow{\tau} & X' \\
f & & f' \\
V & \xrightarrow{=} & V
\end{array}
\]

and a set of \( \mathcal{O}_X \)-module isomorphisms \( \{ \theta_d : \tau^* \mathcal{E}_d' \xrightarrow{\sim} \mathcal{E}_d \} \), with \( \theta_1 \) being the canonical isomorphism \( \tau^* \omega_X'(-\sum_i m_ip_i') \xrightarrow{\sim} \omega_X(-\sum m_ip_i) \), and such that the homomorphisms \( \theta_d \) are compatible with all the maps \( c_{d,d'} \) and \( \tau^* c_{d,d'}' \).

Every \( r \)-spin structure on a smooth curve \( X \) is determined, up to isomorphism, by a choice of a line bundle \( \mathcal{E}_r \), such that \( \mathcal{E}_r^{\otimes r} \cong \omega_X(-\sum m_ip_i) \). Therefore, in the smooth case, the formal Definition 2.8 is equivalent to the intuitive one from the previous subsection (Definition 2.1). In particular, if \( f : X \rightarrow V \) has no automorphisms, then the set of isomorphism classes of \( r \)-spin structures (if non-empty) of type \( \mathbf{m} \) on \( f : X \rightarrow V \) is a principal homogeneous space for the group of \( r \)-torsion points of the Jacobian of \( X \). Thus there are \( r^{2g} \) such isomorphism classes.

2.3. Families of \( r \)-spin structures and stable spin maps.

To define the stack of stable \( r \)-spin maps, we must carefully define how \( r \)-spin structures vary in families. This turns out to be very delicate, since nilpotent elements may arise. In this paper, we use the definition of families of spin curves given in [10]. The main condition is that all of the homomorphisms \( c_{d,d'} \) should be power maps in the sense of [10, §2.3.1]. The definition for families given there reduces to the definition of an \( r \)-spin structure given above, when the base is (the spectrum of) a field; thus they are really only conditions on the fibers, rather than on the families.

The precise definition of families of spin curves is not necessary for understanding the CohFT related to the moduli space of stable \( r \)-spin maps discussed in Sections 3-6. In this paper we only use the formal definition in the proof of Theorem 2.16, which states that the intuitive definition of the stabilization morphism from the stack of stable \( r \)-spin maps to the stack of stable \( r \)-spin curves does, indeed, give a morphism of stacks. The reader willing to accept this result may skip the remainder of this subsection and the proof of Theorem 2.16 in Subsection 2.7.

The following definition is rather technical, but it is precisely what is needed to guarantee that the moduli stacks of spin curves are smooth Deligne-Mumford stacks. Families of spin curves can also be defined (see [1]) in terms of line bundles on the “twisted curves” of Abramovich and Vistoli [2].

**Definition 2.12.** An \( r \)-spin structure of type \( \mathbf{m} \) on a family \( X/T \) of \( n \)-pointed prestable curves is a coherent net of \( r \)-th roots of \( \omega_{X/T} \) of type \( \mathbf{m} \).

Recall that, by Definition 2.3.4 of [10], a coherent net of \( r \)-th roots of \( \omega_{X/T} \) of type \( \mathbf{m} \) is a set of rank-one, torsion-free \( \mathcal{O}_X \)-modules \( \{ \mathcal{E}_d \} \) and a collection of \( \mathcal{O}_X \)-module homomorphisms \( \{ c_{d,d'} : \mathcal{E}_d^{\otimes d}/d \rightarrow \mathcal{E}_{d'} \} \), defined for \( d'|d|r \), such that for each geometric fiber \( X_t \) of \( X/T \), the sheaves \( \{ \mathcal{E}_d \} \) and homomorphisms \( \{ c_{d,d'} \} \) induce a coherent net of \( r \)-th roots of \( \omega_{X_t} \) of type \( \mathbf{m} \), and each homomorphism
$c_{d,r}$ is an isomorphism on the smooth locus of $X/T$. Finally, these sheaves and homomorphisms must have a special type of local structure, which we describe now.

For a node $q \in X_t$ in a fiber of $X/T$ over a geometric point $t \in T$, we denote by $m_{d,+}$ and $m_{d,-}$ the orders of the $d$-th root map

$$c_{d,1} : \mathcal{E}_d^\otimes d \to \omega(-\sum m_ip_i)$$

on the branches of the normalization of $X_t$ at $q$. We define

$$u_d := (m_{d,+} + 1)/\ell_d \quad \text{and} \quad v_d := (m_{d,-} + 1)/\ell_d,$$

where

$$\ell_d := \gcd(m_{d,+} + 1, m_{d,-} + 1).$$

If $c_{d,1}$ is an isomorphism at $q$, we set $u_d = v_d = 0$.

The first requirement on the local structure of a net of coherent roots on a family $X/T$ is the existence of a special local coordinate system near any node $X/t$, with $c_{r,1}$ not an isomorphism (i.e., $\mathcal{E}_r$ is Neveu-Schwarz at $q$). This local coordinate system consists of an étale neighborhood $T'$ of $t$ with an element $\tau \in \mathcal{O}_{T',t}$, and an étale neighborhood $U$ of $q$ in $X \times_T T'$ with sections $x, y \in \mathcal{O}_U$, such that for $s := u_r + v_r$ we have

- $xy = \tau^s$.
- The ideal generated by $x$ and $y$ has the singular locus of $X/T$ as its associated closed subscheme.
- The homomorphism $(\mathcal{O}_{T',t}[[x,y]]/(xy - \tau^s)) \to \mathcal{O}_{U,q}$ induces an isomorphism of the completions

$$\left(\hat{\mathcal{O}}_{T',t}[[x,y]]/(xy - \tau^s)\right) \xrightarrow{\sim} \hat{\mathcal{O}}_{U,q}.$$

Of course, such a local coordinate system would always exist for any $X/T$ if we had $s = 1$, but in general its existence requires that $X/T$ be (locally) a ramified cover of degree $s$ of another family of prestable curves.

The second requirement on the local structure is that the sheaves $\mathcal{E}_d$ must have a special presentation in terms of this special coordinate system. In particular, any rank-one, torsion-free sheaf $\mathcal{F}$ always has a presentation of the form

$$\mathcal{F} \cong \langle \zeta_1, \zeta_2 | e\zeta_1 = x\zeta_2, y\zeta_1 = h\zeta_2 \rangle$$

for some $e$ and $h$ in $\mathcal{O}_{T',t}$, such that $eh = \tau^s$; but for sheaves in the net we require that if $\mathcal{E}_d$ is not locally free at the node $q$, then $\mathcal{E}_d$ must have such a presentation with $e = \tau^{(r/d)(v_d\ell_d)}$ and $h = \tau^{(r/d)(u_d\ell_d)}$.

In other words, $\mathcal{E}_d$ is isomorphic near node $q$ to the sheaf

$$E_d := \langle \zeta_1, \zeta_2 | \tau^{(r/d)(v_d\ell_d)}\zeta_1 = x\zeta_2, y\zeta_1 = \tau^{(r/d)(u_d\ell_d)}\zeta_2 \rangle.$$

If $\mathcal{E}_d$ is locally free at $q$, then for uniformity of notation we will use the unusual presentation $\mathcal{E}_d \cong E_d := \langle \zeta_1, \zeta_2 | \zeta_1 = \zeta_2 \rangle$.

Finally, each homomorphism

$$c_{d,j} : \mathcal{E}_d^\otimes d \to \mathcal{E}_j$$

in the net must be a so-called power map, in the sense of Definition 2.3.1 of [10]. This means that, if we use the local presentations

$$E_{dj} = \langle \xi_1, \xi_2 | \tau^{(r/(dj))}(v_d\ell_a)\xi_1 = x\xi_2, y\xi_1 = \tau^{(r/(dj))(u_d\ell_a)}\xi_2 \rangle,$$
and
\[ E_j = (\zeta_1, \zeta_2) \tau^{(r/j)(v_j \ell_j)} \zeta_1 = x \zeta_2, y \zeta_1 = \tau^{(r/j)(u_j \ell_j)} \zeta_2, \]
of the sheaves \( E_{dj} \) and \( E_j \), then the map
\[ \text{Sym}^d(E_{dj}) \rightarrow E_j, \]
induced by the homomorphism \( c_{dj,j} \), acts on the generators \( \xi^d_{1-i} \xi^i \) of \( \text{Sym}^d(E_{dj}) \) as
\[ \xi^d_{1-i} \xi^i \mapsto \begin{cases} x^{u''-i} \tau^{u} \zeta_1 & \text{if } 0 \leq i \leq u'' \\ y^{v''-d+i} \tau^{(r-j)u} \zeta_2 & \text{if } u'' < i \leq d. \end{cases} \]
Here we require that \( u_j \equiv u_{dj}d \pmod{s} \) and define \( v_j \equiv v_{dj}d \pmod{s}, \ u'' := (u_{dj}d - u_j)/s, \) and \( v'' := (v_{dj}d - v_j)/s. \)

If \( E_{dj} \) is locally free at \( q \), then the existence of a good presentation is automatically satisfied, and we have no additional power map requirement except that the map (5) be an isomorphism.

This completes the definition of \( r \)-spin structures on families of prestable curves, and we can now define families of \( r \)-spin maps.

**Definition 2.13.** Let \( V \) be an algebraic variety and \( \beta \in H_2(V, \mathbb{Z}) \). A family of \( n \)-pointed, stable \( r \)-spin maps to \( V \) of class \( \beta \) and type \( \mathbf{m} \overline{} \) over a base \( T \) is a family of \( n \)-pointed, stable maps \( f : X \rightarrow V \) over \( T \) of class \( \beta \) with an \( r \)-spin structure of type \( \mathbf{m} \) on \( X/T \).

**2.4. Stacks of spin maps.**

Now we are ready to define the main objects of the paper—the stack of stable \( r \)-spin maps.

**Definition 2.14.** Let \( V \) be an algebraic variety over \( \mathbb{C} \), and \( \beta \) an element of \( H_2(V, \mathbb{Z}) \). The stack of stable \( r \)-spin maps to \( V \) \((n\text{-pointed, of genus } g, \text{ and class } \beta)\) is the disjoint union
\[ M_{g,n}^{1/r}(V, \beta) := \bigsqcup_{0 \leq m_i < r} M_{g,n}^{1/r, \mathbf{m}}(V, \beta), \]
of stacks \( M_{g,n}^{1/r, \mathbf{m}}(V, \beta) \) of \((n\text{-pointed, }) \) stable \( r \)-spin maps to \( V \) of genus \( g \), type \( \mathbf{m} = (m_1, \ldots, m_n) \), and class \( \beta \).

We will see in Section 2.6 that \( M_{g,n}^{1/r, \mathbf{m}}(V, \beta) \) (and, therefore, \( M_{g,n}^{1/r}(V, \beta) \)) is a Deligne-Mumford stack whenever \( M_{g,n}(V, \beta) \) is. As the following proposition shows, no information is lost by restricting \( \mathbf{m} \) to the range \( 0 \leq m_i \leq r - 1. \)

**Proposition 2.15.** If \( \mathbf{m} \equiv \mathbf{m}' \pmod{r} \), then \( M_{g,n}^{1/r, \mathbf{m}}(V, \beta) \) is canonically isomorphic to \( M_{g,n}^{1/r, \mathbf{m}'}(V, \beta) \).

**Proof.** When \( \mathbf{m} \equiv \mathbf{m}' \pmod{r} \), every \( r \)-spin structure of type \( \mathbf{m} \) naturally gives an \( r \)-spin structure of type \( \mathbf{m}' \) simply by
\[ E_d \rightarrow E_d \otimes \mathcal{O} \left( \sum \frac{m_i - m'_i}{d} p_i \right). \]
2.5. Canonical morphisms of stacks of stable spin maps.

The stack $\overline{M}^{1/r,m}_{g,n}(V,\beta)$ has a natural projection
\begin{equation}
\tilde{p} : \overline{M}^{1/r,m}_{g,n}(V,\beta) \longrightarrow \overline{M}_{g,n}(V,\beta)
\end{equation}
which forgets the spin structure. The usual evaluation maps
\begin{equation}
evi : \overline{M}_{g,n}(V,\beta) \longrightarrow V,
\end{equation}
which send a point $[X \overset{f}{\rightarrow} V, p_1 \ldots p_n] \in \overline{M}_{g,n}(V,\beta)$ to $f(p_i) \in V$, induce evaluation maps
\begin{equation}
\tilde{ev}_i = ev_i \circ \tilde{p} : \overline{M}^{1/r,m}_{g,n}(V,\beta) \longrightarrow V.
\end{equation}

Less obvious is the fact that for any morphism $s : V \rightarrow V'$ taking $\beta$ to $\beta'$, we have a stabilization morphism
\begin{equation}
\tilde{st} : \overline{M}^{1/r,m}_{g,n}(V,\beta) \longrightarrow \overline{M}^{1/r,m}_{g,n}(V',\beta')
\end{equation}
which takes $f$ to $f' := s \circ f$ and contracts components of the curve that are unstable with respect to $f'$.

**Theorem 2.16.** For any morphism $V \rightarrow V'$, taking $\beta$ to $\beta'$, the stabilization map (8) is a morphism of stacks.

The proof of Theorem 2.16 will be given in Subsection 2.7.

The various canonical maps introduced above are shown in the following commutative diagram.
\begin{equation}
\begin{array}{ccc}
\overline{M}^{1/r,m}_{g,n}(V,\beta) & \xrightarrow{q_1} & \overline{M}_{g,n}(V,\beta) \\
\downarrow \tilde{p} & & \downarrow \beta \\
\overline{M}^{1/r,m}_{g,n} & \xrightarrow{q_2} & \overline{M}_{g,n} \\
\downarrow p & & \downarrow \pi \\
\overline{M}_{g,n} & \xrightarrow{st} & \overline{M}_{g,n}(V,\beta) \xrightarrow{ev_i} V
\end{array}
\end{equation}

We will use the notation of this diagram throughout the remainder of the paper, and we will denote the composition $q_2 \circ q_1$ by $q$.

The universal curves $C_{g,n} \rightarrow \overline{M}_{g,n}$ and $C^{1/r,m}_{g,n} \rightarrow \overline{M}^{1/r,m}_{g,n}(V,\beta)$ will be denoted by $\pi$.

**Remark 2.17.** The stack $\overline{M}^{1/r,m}_{g,n}(V,\beta)$ is not isomorphic to the fibered product $\overline{M}^{1/r,m}_{g,n} \times_{\overline{M}_{g,n}} \overline{M}_{g,n}(V,\beta)$, although on the smooth locus the map
\begin{equation}
q_1 : \overline{M}^{1/r,m}_{g,n}(V,\beta) \longrightarrow \overline{M}^{1/r,m}_{g,n} \times_{\overline{M}_{g,n}} \overline{M}_{g,n}(V,\beta)
\end{equation}
is an isomorphism. This can be seen as follows.

If \( X/T \) is a smooth family of curves, then a stable map \( f: X \to V \) and an \( r \)-spin structure \( (\{ E_d \}, \{ c_{d,d'} \}) \) are precisely the data necessary to construct an \( r \)-spin map, i.e., there is a canonical morphism

\[
j: \mathcal{M}_{g,n}^{1/r,m} \times_{\mathcal{M}_{g,n}} \mathcal{M}_{g,n}(V, \beta) \to \mathcal{M}_{g,n}^{1/r,m}(V, \beta)
\]

which is clearly the inverse of the morphism \( q_1 \).

But if \( X \) is not stable, this morphism \( j \) no longer exists. For example, let \( X \) be a prestable curve that has two irreducible components \( C \) and \( E \), where \( C \) is a smooth curve of genus \( g \), and \( E \) is a smooth, rational curve, without marked points, joined to \( C \) at a single node \( q \). Let \( f: X \to V \) be an embedding of \( X \) in \( V \).

An \( r \)-spin structure \( (\{ E_d \}, \{ c_{d,d'} \}) \) on \( X \) is equivalent to a pair of \( r \)-spin structures \( (\{ E'_d \}, \{ c'_{d,d'} \}) \) on \( C \) and \( (\{ E''_d \}, \{ c''_{d,d'} \}) \) on \( E \) of orders 0 and \( r-2 \), respectively, at \( q \). Thus the automorphism group of the \( r \)-spin map \( (f, (\{ E_d \}, \{ c_{d,d'} \})) \) is \( \mu_r \times \mu_r \), corresponding to multiplication of \( E'_d \) and \( E''_d \) by \( r \)-th roots of unity.

But the stabilization map \( \tilde{\phi} \) takes \( (f, (\{ E_d \}, \{ c_{d,d'} \})) \) to the spin map \( (f|_C, \{ E'_d \}, \{ c'_{d,d'} \}) \) on \( C \), and the automorphism group of

\[
\tilde{\phi}(f, (\{ E_d \}, \{ c_{d,d'} \})) \times \tilde{\phi}(f, (\{ E_d \}, \{ c_{d,d'} \})) = (f|_C, (\{ E'_d \}, \{ c'_{d,d'} \})
\]

is simply \( \mu_r \), since \( C \) is irreducible and \( E'_d \) is invertible on \( C \). Thus the morphism \( q_1 \) is not an isomorphism.

**Proposition 2.18.** The morphism \( q_1 \) is flat and proper.

**Proof.** Flatness follows from the valuative criterion of flatness [11 8.1], which states that it is enough to check flatness of \( q_1 \) over each \( R \)-valued point \( \text{Spec } R \to \mathcal{M}_{g,n}^{1/r,m} \times_{\mathcal{M}_{g,n}} \mathcal{M}_{g,n}(V, \beta) \), where \( R \) is a discrete valuation ring. Since the completion \( \hat{R} \) of \( R \) is faithfully flat over \( R \), it suffices to check this for each complete discrete valuation ring. But in this case, the results of [10] show that the universal deformation (relative to the universal stable map \( f: \mathcal{C} \to V \)) of a spin structure over the central fiber of Spec \( R \) corresponds to the ring homomorphism \( R \to R[[t]]/(t^r - s) \), where \( s \in R \) is a uniformizing parameter for \( R \). In particular, \( R[[t]]/(t^r - s) \) is a free \( R \)-module, and thus is flat over \( R \). Since the universal deformation is faithfully flat (étale) over \( \hat{R} \), this shows that \( q_1 \) is also flat.

Properness also follows by the valuative criterion in exactly the same manner as was proved in [11] for spin structures on stable curves. Nothing in that proof required the underlying curves to be stable—only prestable. \( \square \)

2.6. The algebraic nature of the stack of stable spin maps.

A useful notion in dealing with stacks is the idea of a Deligne-Mumford morphism, or morphism of Deligne-Mumford type. This is analogous to the concept of a representable morphism.

**Definition 2.19.** A morphism of stacks \( f: S \to T \) is called Deligne-Mumford (or of Deligne-Mumford type) if for every \( U \)-valued point \( U \to T \), for a representable \( U \), the fibered product \( S \times_T U \) is a Deligne-Mumford stack.

The most useful fact about these morphisms is that if \( S \to T \) is a Deligne-Mumford morphism, and if \( T \) is a Deligne-Mumford stack, then \( S \) is a Deligne-Mumford stack (see [13 Prop. 3.1.3]).
Theorem 2.20. For all $V$ and $\beta$, the forgetful morphism (see equation 2) is a finite Deligne-Mumford morphism of stacks. In particular, $\overline{M}_{g,n}^{r/m}(V, \beta)$ is a Deligne-Mumford stack whenever $\overline{M}_{g,n}(V, \beta)$ is.

Proof. Given a $T$-valued point $T \to \overline{M}_{g,n}(V, \beta)$ for a representable $T$, we must show that the map $R(X/T) := \overline{M}_{g,n}^{r/m}(V, \beta) \times_T \overline{M}_{g,n}(V, \beta)$ of coherent nets of $r$-th roots of $\omega_X(-\sum m_i p_i)$ on the associated family $X/T$ of prestable curves is a Deligne-Mumford stack, finite over $T$. In particular, we need to construct a smooth cover of $R(X/T)$ and show that the diagonal $\Delta : R(X/T) \times_T R(X/T) \to R(X/T)$ is representable, unramified, and proper.

These facts are all straightforward generalizations of their counterparts over the stack $\overline{M}_{g,n}$ of stable curves as described in [1]. The only real difference is that we are now working with a specific family of prestable curves over $T$, as opposed to working with the universal family of stable curves (over $\overline{M}_{g,n}$), but that changes nothing of substance in the proof.

The proof of properness is also an easy generalization of the case of stable $r$-spin curves, and the morphism is obviously quasi-finite, hence finite. \qed

2.7. Proof of Theorem 2.16.

We now turn to the proof that for any morphism $s : V \to V'$, taking $\beta \in H_2(V, \mathbb{Z})$ to $\beta' \in H_2(V', \mathbb{Z})$, the stabilization map (8) is a morphism of stacks.

It is straightforward to check that the stabilization of the underlying curves preserves $r$-spin structures on each individual fiber, but we must also show that the stabilization morphism on the underlying curves preserves the $r$-spin structure in families.

Theorem 2.16 obviously follows from the following lemma.

Lemma 2.21. Let $st : \tilde{X}/T \to X/T$ be a morphism taking a family of prestable curves $\tilde{X}/T$ to a partial stabilization $X$ of $\tilde{X}$, and let $(\tilde{E}_d, \{\tilde{e}_{d,\beta}\})$ be an $r$-spin structure of type $m = (m_1, \ldots, m_n)$ on $\tilde{X}$, with $0 \leq m_i \leq r - 1$ for every $i$. In this case, the sheaf $R^1 st_* \tilde{E}_d$ is zero for every $d|r$, and the push-forward $((st)_* \tilde{E}_d, \{st_* \tilde{e}_{d,\beta}\})$ is an $r$-spin structure of type $m$ on $X$.

Proof. As mentioned above, it is straightforward to check that the maps $st_* \tilde{e}_{d,\beta}$ and the sheaves $st_* \tilde{E}_d$ are $T$-flat and produce an $r$-spin structure of type $m$ on each fiber of $X/T$ (this will also follow from the computations below). Thus we only need to verify that $R^1 st_* \tilde{E}_d = 0$ (which implies that it commutes with base change), and that the maps and sheaves meet the local conditions outlined in Subsection 2.3 for being a coherent net on the family of curves $X/T$, provided the original sheaves $\{\tilde{E}_d\}$ and maps $\{\tilde{e}_{d,\beta}\}$ form a coherent net on the family $\tilde{X}/T$.

Let us fix a point $p$ of a geometric fiber $X_t$ of $X/T$. There are three cases to consider. First is the case when the point $p$ is not the image of a contracted component (i.e., $st^{-1}(p)$ is a single point). Second is the case when $p$ is a smooth point of the fiber $X_t$, but $p$ is the image of a whole irreducible component of the fiber $\tilde{X}_t$ of $\tilde{X}/T$; that is, $st$ contracts a $-1$-curve to the point $p$. Third is the case that $p$ is a node of the fiber $X_t$ containing it, and it is the image of a contracted component of $\tilde{X}_t$; that is, $p$ is the image of a $-2$-curve $\tilde{E}$. 


Figure 1. A depiction of Case 2 of Lemma 2.21: fibers $\tilde{X}_t$ and $X_t$, the stabilization map $st : \tilde{X}_t \to X_t$, and the normalization of $\tilde{X}_t$. The morphism $st$ contracts the unstable component $C$ to the point $p$ and induces an isomorphism from the rest of the curve $\tilde{Y}$ to $X_t$.

**Case 1:** The first case is easy, since when $st^{-1}(p)$ is a single point, then $st$ is an isomorphism in a neighborhood of $p$ (or of $st^{-1}(p)$). In particular, $st_*$ is an isomorphism, $R^1st_*\mathcal{E}_d = 0$, and $c_{d,d'} = st_*(\tilde{c}_{d,d'})$ is a $d/d'$-th power map near $p$.

The second and third cases are more involved. Before we attack them, we note that the conditions we must verify are local (and analytic) on the base $T$, so it suffices to check the result when $T$ is affine and is the spectrum of a complete local ring $R$. Moreover, the conditions are analytic on $X$; that is, the conditions are all determined by restricting to the completion of the local ring of $X$ near the point $p$. To simplify, we will make the calculations in the case of $d = r$, but all other values of $d$ (dividing $r$) are similar.

**Case 2:** In the second case ($st$ contracts a $-1$-curve of $\tilde{X}$ to the point $p$) we will show that the induced sheaves $st_*\mathcal{E}_d$ are locally free at $p$, and the maps $c_{d,d'}$ are all isomorphisms; thus the local coordinate and power map conditions are automatically fulfilled.

The fiber $\tilde{X}_t$ over $X_t$ has one irreducible component $C$ lying over $p$, and $C$ contains at most one marked point $p'$, labeled with an integer $m$, where $0 \leq m \leq r - 1$. This is indicated in Figure 1. On $C$, the sheaf $(\mathcal{E}_r/torsion)^{\otimes r}$ is isomorphic to $\omega_C(-m^+q^+ - mp)$, where $q^+$ is the point of $C$ which maps to the node $q$ attaching $C$ to the rest of $\tilde{X}_t$. 
Moreover, r must divide \(2gc - 2 - m^+ - m\), so either \(m = r - 1\), which implies that \(\mathcal{E}_r\) is locally free (Ramond) near \(q\), or \(r - 2 = m^+ + m\), which implies that \(\mathcal{E}_r\) is not locally free (it is Neveu-Schwarz) at \(q\). In either case, \(\mathcal{E}_r|_{\overline{C}}\) has degree \(-1\) and thus has no global sections. This also gives \(R^1st_*\mathcal{E}_r = 0\), since this is true on each fiber.

Now, in the Neveu-Schwarz case, the sheaf \(st_*\mathcal{E}_r|_{X_t}\) is simply the sheaf \(\mathcal{E}_r\) restricted (modulo torsion) to the rest of the prestable fiber \(\tilde{Y} = (X_t - C)\). But \(\mathcal{E}_r/torsion\) on \(\tilde{Y}\) is an \(r\)-th root of \(\omega_{C}(-m^-q^- - \sum_{p_i \neq p} m_i p_i)\), where \(q^-\) is the other side of the node defined by \(q^+\). The actual value of \(m^-\) is determined by the relation \(m^+ + m^- = m^+ + m^- = r - 2\), which implies that \(m^- = m\).

In the Ramond case, the vanishing of the global sections of \(\mathcal{E}_r|_{\overline{C}}\) implies that \(st_*\mathcal{E}_r|_{\overline{X}_t}\) is an \((r - 1)\)-th root of \(\omega_{\overline{X}_t}(-q^-)\), so it is an \(r\)-th root of \(\omega_{X_t}(-q^-)\).

In both the Ramond and Neveu-Schwarz cases, the new marked point \(p = st(q^-)\) of \(X_t\) is labeled with \(m\), just as the old marked point \(p'\) was marked with \(m\) on \(\tilde{X}_t\). If no point was marked on \(C\), then the point \(p\) remains unmarked (and \(m^- = 0\)).

Finally, \(st_*\mathcal{E}_d\) is \(T\)-flat and \(R^1st_*\mathcal{E}_r\) vanishes, so we have that \(st_*\mathcal{E}_r\) commutes with base change, and the calculations above on the fibers all hold globally on the family \(X/T\). Thus \(st_*\mathcal{E}_r\) is invertible near \(p\), and \(st_*\mathcal{E}_r|_{\overline{C}}\) is an isomorphism near \(p\). In particular, \(st_*\mathcal{E}_r|_{\overline{C}}\) is an \(r\)-power map. A similar argument holds for each \(\mathcal{E}_d\) and each \(\mathcal{E}_d|_{\overline{C}}\) near \(p\).

**Case 3:** The third case is that of a point \(p \in X\) which is the image of a \(-2\)-curve \(\tilde{C}\) of \(X\). It is easy to see that, just as in case 2, on the unstable (contracted) \(-2\)-curve, the degree of the bundle is \(-1\). Also, we have \(R^1st_*\mathcal{E}_r = 0\); the sheaf \(st_*\mathcal{E}_r\) is \(T\)-flat and commutes with base change; and on the fibers, the induced collection of sheaves and bundles forms an \(r\)-spin structure of type \(m\).

We have still to check that the induced sheaves have the necessary family structure for spin curves (existence of a local coordinate of suitable type, with respect to which the sheaves have the standard presentation—see Definition 2.12), and that the induced maps are power maps, as described in equation (3). For simplicity we will assume that the orders \(m^+, m^-, m'^+, m'^-\) of the \(r\)-spin map \(\mathcal{E}_r|_{\overline{C}}\) along the two nodes \(q\) and \(q'\) where the \(-2\)-curve intersects the rest of the fiber have the property that \(\gcd(m^+ + 1, m^- + 1) = 1 = \gcd(m'^+ + 1, m'^- + 1)\). The case with common divisors larger than 1 is similar.

It is shown in [1], §3.1 that \(\tilde{X}\) is locally isomorphic to

\[
\text{Proj}_A[A[\mu, \nu]/(\nu - \epsilon \mu, h^\nu - \mu y)],
\]

where \(A = \tilde{O}_{X,x} \cong R[[x,y]]/(xy - \pi^r, e, h, \pi)\) are elements of the maximal ideal \(m_R\) of \(R\) with \(eh = \pi\). This shows the existence of the special local coordinate.

Next we show that \(st_*\mathcal{E}_d\) has a presentation of the form

\[
st_*\mathcal{E}_d \cong \langle \zeta_1, \zeta_2 \rangle^{(r/d)(u_1 u_2)} \zeta_1 = x, y = \pi^{(r/d)(u_1 u_2)} \zeta_2.
\]

If we let \(\mu/\nu = s\) and \(\nu/\mu = z\), then near the exceptional \(-2\)-curve \(\tilde{C}\) the curve \(\tilde{X}\) is covered by two open sets,

\[
U = \{\mu \neq 0\} \cong \text{Spec } A[z]/(xz - e^r, y - h^r z)
\]

and

\[
V = \{\nu \neq 0\} \cong \text{Spec } A[s]/(s - e^r s, y s - h^r).
\]
Since \( \{ \tilde{\mathcal{E}}_d \} \) is an \( r \)-spin structure, we can describe \( \tilde{\mathcal{E}}_r \) on \( U \) by \( \tilde{\mathcal{E}}_r \mid U \cong E_U(e^v, e^w) = \langle \zeta_1, \zeta_2 \rangle | \mathbb{C} = e^v \zeta_1, x \zeta_1 = e^v \zeta_2 \rangle \), and on \( V \) by \( \tilde{\mathcal{E}}_r \mid V \cong E_V(h^u, h^v) = \langle \xi_1, \xi_2 \rangle | \mathbb{C} = h^u \xi_1, y \xi_1 = h^v \xi_2 \rangle \), where \( u + v = u' + v' = r \).

On the exceptional curve \( \tilde{C} \cong \mathbb{P}^1 \) the sheaf \( (\tilde{\mathcal{E}}_r/torsion)^{\otimes r} \) is isomorphic to \( \omega_{\mathbb{P}^1}((1 - u) + (1 - v')) \), and degree considerations show that \( u + v' = r \), so \( u = u' \) and \( v = v' \). Moreover, in a neighborhood of \( \tilde{C} \), if \( D_i \) is the image of the \( i \)-th section \( p_i : T \to X \), the invertible sheaf \( \omega_X(- \sum m_i D_i) \) is trivial and is generated by the element \( w = \frac{dx}{z} - \frac{dz}{x} = \frac{dy}{z} - \frac{dz}{y} \). The \( r \)-th power map \( \tilde{c}_{r,1} \) is an isomorphism away from the nodes of \( X \), and since it is a power map (changing the isomorphisms \( \tilde{\mathcal{E}}_r \mid U \cong E_U(e^v, e^w) \) and \( \tilde{\mathcal{E}}_r \mid V \cong E_V(h^u, h^v) \), if necessary), it maps the generators \( \zeta_i \) and \( \xi_i \) as follows:

\[
\zeta_1^r \mapsto z^u w, \quad \zeta_2^r \mapsto x^v w
\]

and

\[
\xi_1^r \mapsto s^u w, \quad \xi_2^r \mapsto y^v w.
\]

Since \( \tilde{c}_{r,1} \) is an isomorphism away from the nodes, we have \( \zeta_1^r = z^r \zeta_1 \), or \( \zeta_1 = z^u \zeta_1 \), for some \( r \)-th root of unity \( \theta \). Changing the isomorphism \( \tilde{\mathcal{E}}_r \mid V \cong E_V(h^u, h^v) \) by \( \theta \), we may assume

\[
\zeta_1 = z \zeta_1.
\]

On \( U \cap V \) we also have

\[
\zeta_2 = se^v \zeta_1 = e^v \zeta_1 \quad \text{and} \quad \xi_2 = z h^u \zeta_1 = h^u \zeta_1.
\]

So global sections of \( \tilde{\mathcal{E}}_r \) are of the form

\[
\Gamma(\tilde{\mathcal{E}}_r) = \{(f_U \zeta_1 + f_V \zeta_2, (f_V \zeta_1 + f_U \zeta_2)) \in E_U \oplus E_V | f_U \zeta_1 + f_U \zeta_2 = f_V \zeta_1 + f_V \zeta_2 \text{ on } U \cap V\}.
\]

We claim that the \( A \)-module

\[
E(\pi^u, \pi^v) := \langle \eta_1, \eta_2 | x \eta_2 = \pi^u \eta_1, y \eta_1 = \pi^v \eta_2 \rangle
\]

is isomorphic to \( \Gamma(\tilde{\mathcal{E}}_r) \) via

\[
\eta_1 \mapsto (\zeta_2, e^v \zeta_1) \quad \text{and} \quad \eta_2 \mapsto (h^u \zeta_1, \zeta_2).
\]

The map is clearly an \( A \)-module homomorphism. Moreover, for any section \( \{(f_U \zeta_1 + f_V \zeta_2, (f_V \zeta_1 + f_U \zeta_2)) \in \Gamma(\tilde{\mathcal{E}}_r) \) we may assume that \( f_U \in R[z] \) and \( f_V \in R[[y]] \). Likewise, we may assume that \( f_V \in R[s] \) and \( f_V \in R[[y]] \).

Consequently, we have

\[
z f_U(z) + e^v f'_U(x) - f_V(s) - z q^u f'_V(y) = 0,
\]

or

\[
z f_U(z) + e^v f'_U(se^v) - f_V(s) - z h^u f'_V(z h^v) = 0.
\]

Thus \( f_U \) and \( f_V \) are completely determined by

\[
f_U = h^u f'_U(y) \quad \text{and} \quad f_V = e^v f'_U(x).
\]

We may, therefore, map \( \Gamma(\tilde{\mathcal{E}}_r) \) to \( E(\pi^u, \pi^v) \) via

\[
(h^u f'_V(y) \zeta_1 + f'_U(x) \zeta_2), (e^v f'_U(x) \xi_1 + f'_V(y) \xi_2) \mapsto f'_U(x) \eta_1 + f'_V(y) \eta_2,
\]

and it is easy to check that this homomorphism is the inverse of the first.

An identical argument shows that \( \Gamma(\mathcal{E}_d) \) is isomorphic to \( E(\pi^u, \pi^v) \), where \( u' \equiv u \pmod{d} \) and \( v' \equiv v \pmod{d} \). This shows the existence of the desired presentation for \( st \cdot \mathcal{E}_d \).
It remains to show that the maps \( s_t(\tilde{c},d,d') \) are power maps (3). Again, since the arguments are essentially identical for each pair \( d \) and \( d' \), it suffices to prove this in the case of \( \tilde{c},r,\sigma \) for some \( \sigma \) dividing \( r \).

As above, we have \( u + v = r \). Let \( \sigma \) be a divisor of \( r \), and let \( d = r/\sigma \). Define integers \( u'' \) and \( v'' \) as

\[
\begin{align*}
    u'' &= \frac{du - u'}{r} \\
v'' &= \frac{dv - v'}{r}.
\end{align*}
\]

The module \( \Gamma(\tilde{E}_r) \equiv E(\pi^u, \pi^v) \) is generated by \( \eta_1 \) and \( \eta_2 \) with \( \eta_1 = (\xi_2, e^v, \xi_1) \) and \( \eta_2 = (h^u\xi_1, \xi_2) \). Further, \( \tilde{E}_r \) may be defined on \( U \) by \( \langle \phi_1, \phi_2 \rangle z\phi_2 = e^u \phi_1 \), \( x\phi_1 = e^v \phi_2 \) and on \( V \) by \( \langle \psi_1, \psi_2 \rangle s\psi_2 = h^u\psi_1, \psi_1 = e^v \psi_2 \), so we may describe \( s^t(\tilde{c},r,\sigma) \) as above: the module \( \Gamma(\tilde{E}_r) \) is isomorphic to \( E(\pi^u, \pi^v) \), and is generated by \( \gamma_1 = (\phi_2, e^v \psi_1) \) and \( \gamma_2 = (h^u \phi_1, \psi_2) \).

We must show that \( \eta_1 \) \& \( \eta_2 \) maps, via \( s_t(\tilde{c},r,\sigma) \), to \( \pi^u x^v - i \gamma_1 \) for \( 0 \leq i \leq u'' \) and to \( \pi^u x^{v''} - i \gamma_2 \) when \( u'' \leq i \leq d \).

We will do the first case—the second case is similar. The element \( \eta_1 \) \& \( \eta_2 \) is of the form

\[
\begin{align*}
    \eta_1 &= (\xi_2, e^v, \xi_1) \in \Gamma(\tilde{E}_r), \\
    \eta_2 &= (h^u\xi_1, \xi_2) \in \Gamma(\tilde{E}_r).
\end{align*}
\]

so on \( U \), this element \( \eta_1 \) \& \( \eta_2 \) maps as

\[
\begin{align*}
    h^u \xi_1 \xi_2 \rightarrow x^{v''} e^u h^u \phi_2 = \pi^u x^{v''} \phi_2.
\end{align*}
\]

On \( V \), the element \( \eta_1 \) \& \( \eta_2 \) maps as

\[
\begin{align*}
    e^{(d-i)v} \xi_1 \xi_2 \rightarrow s^{v''} h^i e^{(d-i)v} \psi_1.
\end{align*}
\]

It is straightforward to check that these are the same on \( U \cap V \). But this is exactly the canonical \( d \)-th power map (1) for \( E(\pi^v, \pi^u)^\otimes d \rightarrow E(\pi^v, \pi^u) \), as desired.

\[\square\]

**Remark 2.22.** It is important to note that if any of the \( m_i \) is greater than \( r - 1 \), Lemma 2.21 is no longer true. In particular, the sheaf \( R^1 s_t E_r \) no longer vanishes in case 2 of the proof, and the subsequent fiber-to-family transitions are not valid.

### 3. Cohomology Classes

#### 3.1. Tautological cohomology classes.

There are many natural cohomology classes in \( H^*(\mathcal{M}_{g,n}^{1/r,m}(V, \beta), \mathbb{Q}) \); these include the classes induced by pullback from \( \mathcal{M}_{g,n}(V, \beta) \) and from \( \mathcal{M}_{g,n}^{1/r,m} \). In particular, we have the \( i \)-th Chern class \( \lambda_i \) of the Hodge bundle \( \pi_* \omega_\pi \), and the components \( \nu_i \) of the Chern character of the Hodge bundle

\[
\frac{c_i \pi_* \omega_\pi}{1 + c_i \pi_* \omega_\pi} = 1 + c_i \pi_* \omega_\pi = g + \nu_1 t + \nu_2 t^2 + \nu_3 t^3 + \ldots.
\]

(The even components of \( c_i \pi_* \omega_\pi \) vanish by Mumford’s theorem (23).) In a similar manner we define the components \( \mu_i \) of the Chern character of the pushforward \( R^i \pi_* E_r \) of the \( r \)-th root bundle \( E_r \) to be

\[
\frac{c_i R^i \pi_* E_r}{1 + c_i R^i \pi_* E_r} = -D + \mu_1 t + \mu_2 t^2 + \ldots.
\]
Here, by $R\pi_*E_r$ we mean the K-theoretic pushforward $\pi_*E_r - R^1\pi_*E_r$, which is generally not the equivalence class of a vector bundle, but only of a coherent sheaf. Here $-D$ is the Euler characteristic $\chi(E_r|_{C_s})$ of $E_r$ on any geometric fiber $C_s$ of $\pi$, and by Riemann-Roch we have

$$D = \frac{1}{r} \left( (r-2)(g-1) + \sum _{i} m_i \right).$$

(12)

In addition to the Hodge-like classes $\lambda_i$, $\nu_i$ and $\mu_i$, there are tautological classes induced by the canonical sections $p_i: \overline{M}_{g,n}^{1/r,m}(V,\beta) \to \mathcal{C}_{g,n}^{1/r,m}$. These are classes

$$\psi_i := c_1(p_i^*(\omega_\pi)) \text{ and } \tilde{\psi}_i := c_1(p_i^*(E_r))$$

(13)

(and also class $\psi_i^{(d)}$ for each divisor $d$ of $r$). When working in $\text{Pic}\overline{M}_{g,n}^{1/r,m}(V,\beta)$, we will abuse notation and use $\psi_i$ to indicate the line bundle $p_i^*(\omega_\pi)$, and $\tilde{\psi}_i$ the line bundle $p_i^*(E_r)$. In [15] it is proved that these classes are closely related:

$$\tilde{\psi}_i = \frac{m_i + 1}{r} \psi_i.$$  

(14)

3.2. Graphs and boundary classes.

Finally, there are the boundary classes. Much of the information about the combinatorial structure of the boundary of $\overline{M}_{g,n}^{1/r,m}(V,\beta)$ can be encoded in terms of decorated graphs.

Recall that the (dual) graph of an $n$-pointed, prestable curve $(X,p_1,\ldots,p_n)$ consists of the following elements:

- **Vertices**, corresponding to the irreducible components of $X$: a vertex $v$ is labeled with a non-negative integer $g(v)$, the (geometric) genus of the component;
- **Edges**, corresponding to the nodes of the curve: an edge connects two vertices (possibly even the same vertex, in which case the edge is called a loop) if and only if the corresponding node lies on the associated irreducible components;
- **Tails**, corresponding to the marked points $p_i \in X$, $i = 1, \ldots, n$: a tail labeled by the integer $i$ is attached at the vertex associated to the component of $X$ that contains $p_i$.

**Definition 3.1.** A half-edge of a graph $\Gamma$ is either a tail or one of the two ends of a “real” edge of $\Gamma$. We denote by $V(\Gamma)$ the set of vertices of $\Gamma$ and by $n(\Gamma)$ the number of half-edges of $\Gamma$ at the vertex $v$.

**Definition 3.2.** Let $\Gamma$ be a graph. The number

$$g(\Gamma) = \dim H^1(\Gamma) + \sum_{v \in V(\Gamma)} g(v)$$

is called the genus of a graph $\Gamma$.

*In [13] the functor $R\pi_*$ was denoted by $\pi_1$. 
Definition 3.3. A pair \((g,n)\) of non-negative integers is called stable if \(2g+n-2 > 0\).

A vertex \(v\) of a graph is called stable if the pair \((g(v),n(v))\) is stable.

A graph \(\Gamma\) is called stable if each vertex \(v\) of \(\Gamma\) is stable.

To describe strata of the moduli space of stable \(r\)-spin maps into \(V\), we decorate the graphs with additional data coming from the \(r\)-spin structure and from \(V\). In particular, the type \(m = (m_1, \ldots, m_n)\) gives a marking to each of the tails, and the homology class \(\beta_v\) of the image in \(V\) of the corresponding irreducible component of the curve gives a marking to each vertex.

Definition 3.4. Fix an integer \(r \geq 2\) and a variety \(V\). A \((V,r)\)-stable graph is a graph \(\Gamma\) with a choice of a homology class \(\beta_v \in H_2(V, \mathbb{Z})\) for each vertex \(v\) of \(\Gamma\), and a marking of each half edge \(h\) by a non-negative integer \(m_h < r\). For each edge \(e\) the marks \(m_+ := m_{h+}\) and \(m_- := m_{h-}\) of the two half-edges \(h_+\) and \(h_-\) of \(e\) must satisfy
\[
m_+ + m_- \equiv r - 2 \pmod{r}.
\] (15)

Finally, the graph should satisfy the stability condition that if \(\beta_v = 0\), then the vertex \(v\) is stable.

If the half-edges of \(\Gamma\) are not marked by integers \(m_h\), but its vertices are marked with classes \(\beta_v\), and all vertices with \(\beta_v = 0\) are stable, then such a graph will be called a \(V\)-stable graph.

Similarly, if the vertices of \(\Gamma\) are stable and not marked, but the half-edges are marked with integers \(m_h\) satisfying (15), then it will be called an \(r\)-stable graph.

Each stable \(r\)-spin map defines a \((V,r)\)-stable graph, called its dual graph.

Definition 3.5. Given a stable \(r\)-spin map \(f : X \rightarrow V, \{E_d\}, \{c_{d,d'}\}\), its \((V,r)\)-decorated dual graph (or just dual graph) is the dual graph \(\Gamma\) of the underlying curve \(X\), with the following additional markings. Each vertex \(v\) is labeled with the class \(\beta_v := [f(X_v)]\) of the image of the corresponding irreducible component \(X_v\). The \(i\)-th tail is marked by \(m_i\), and each half-edge associated to a node of \(X\) is marked by the order (the integer \(m^+\) or \(m^-\)) of the \(r\)-spin structure along the branch of the node associated to that half-edge.

For any \(V\), we let \(G_{g,n}(V)\) denote the set of all \((V,r)\)-stable graphs of genus \(g\) with \(n\) tails.

Definition-Proposition 3.6. For any morphism \(V \xrightarrow{\gamma} V'\) and a stable pair \((g,n)\), there is an associated stabilization map
\[
\gamma_* : G^{1/r}_{g,n}(V) \rightarrow G^{1/r}_{g,n}(V').
\] (16)

defined as follows. The graph \(\Gamma \in G^{1/r}_{g,n}(V)\) is mapped to the graph \(\Gamma' \in G^{1/r}_{g,n}(V')\) obtained from \(\Gamma\) by removing all vertices \(v\) in \(\Gamma\) that fail the stability criterion; that is, \(\gamma_*(\beta_v) = 0\) in \(H_2(V', \mathbb{Z})\) and
\[
2g(v) - 2 + n(v) \leq 0.
\]

Also remove all half-edges attached to each removed vertex \(v\), and join together any other half-edges that were previously connected to half-edges of \(v\). Since there are at most two such half-edges per unstable vertex \(v\), this operation will either produce a well-defined edge, or it will not connect any half-edges at all. Now mark
each remaining vertex \( v \) of the resulting graph with \( \gamma_*(\beta_v) \), and give each half-edge \( h \) the mark \( m_h \) it had in the graph \( \Gamma \). The resulting graph \( \Gamma \) is clearly \((V',r)\)-stable, except in the special case that for every vertex \( v \) of \( \Gamma \) we have \( \gamma_*(\beta_v) = 0 \) and \( 2g(v) - 2 + n(v) \leq 0 \). This only occurs when \( 2g - 2 + n \leq 0 \), so when \( 2g - 2 + n > 0 \) we always have a stabilization map \( \gamma : G_{g,n}^1(V) \to G_{g,n}^1(V') \) associated to \( \gamma \).

If the morphism \( \gamma \) is the constant map taking \( V \) to a point, we will call the image of a \( V \)-stable (or \((V,r)\)-stable) graph \( \Gamma \) under the associated stabilization map \( \gamma \), just the stabilization of \( \Gamma \).

\((V,r)\)-stable graphs of genus \( g \) with \( n \) tails correspond to boundary strata in \( \overline{M}_{g,n}/r(V,\beta) \), although some of these strata may be empty.

**Definition 3.7.** Let \( \Gamma \) be a connected \( V \)-stable graph (or \((V,r)\)-stable, or \( r \)-stable graph) with \( n \) tails and of genus \( g \). We denote by \( \overline{M}_r(V,\beta) \) (or by \( \overline{M}_r(V,\beta) \), or by \( \overline{M}_r(V) \)) the closure in \( \overline{M}_{g,n}(V,\beta) \) (or in \( \overline{M}_{g,n}(V,\beta) \), or in \( \overline{M}_{g,n}(V) \)) of the moduli space of stable maps (or stable \( r \)-spin maps, or stable \( r \)-spin curves) whose dual graph is \( \Gamma \). If \( \Gamma = \sqcup \Gamma_i \) is the disjoint union of connected subgraphs \( \Gamma_i \), then we denote by \( \overline{M}_r(V) \) the product \( \prod \overline{M}_r(\Gamma_i) \), and similarly \( \overline{M}_r(V) = \prod \overline{M}_r(\Gamma_i) \).

These substacks \( \overline{M}_r(V,\beta) \) are generally not irreducible, since different spin structures often cannot be deformed into one another, even for a fixed type \( m \).

For example, when \( r = 2, g > 0, m = 0 \), and the target \( V \) is a point, there are both even and odd spin structures on each stable curve, and these form distinct irreducible components.

Moreover, the classes \( [\overline{M}_r(V,\beta)] \) in the Chow group \( A_*(\overline{M}_{g,n}/r,m(V,\beta)) \) defined by the substacks \( \overline{M}_r(V,\beta) \) are not usually the pullbacks of the corresponding classes in \( A_*(\overline{M}_{g,n}(V,\beta)) \). For example, if \( V \) is a point and the graph \( \Gamma \) is a tree with one edge and two vertices

\[
\Gamma = \xymatrix{ & m_1 \\ m_0 \\ & m_2 \\ & \cdots \\ & m_p \\ k \ar@{-}[ur] & k \ar@{-}[ul] & m_{p+1} },
\]

then there is a unique choice of marking \( m^+ \) and \( m^- \) on the two half-edges of the edge that makes the degree of the twisted canonical bundle divisible by \( r \) on both vertices. In this case it can be shown (see [12]) that in \( \text{Pic} \overline{M}_{g,n}/r,m \) the class \( \tilde{p}^*[\overline{M}_r(\Gamma)] \) is precisely

\[
\tilde{p}^*[\overline{M}_r(\Gamma)] = \frac{r}{\gcd(m^+ + 1, r)}[\overline{M}_r(V,\beta)].
\]

4. Spin Virtual Class

Recall from [12, §4.1] that an \( r \)-spin virtual class on the moduli of stable, \( r \)-spin curves is an assignment of a cohomology class

\[
c_{r}^{1/r} \in H^{2D}(\overline{M}_{r}^{1/r}, \mathbb{Q})
\]

to every genus \( g \), \( r \)-stable graph \( \Gamma \) with \( n \)-tails. Here, if the tails of \( \Gamma \) are marked with the \( n \)-tuple \( m = (m_1, \ldots, m_n) \), then the dimension \( D \) is

\[
D = \frac{1}{r}((r-2)(g-\alpha) + \sum_{i=1}^{n} m_i),
\]

(18)
where $\alpha$ is the number of connected components of $\Gamma$. In the special case where $\Gamma$ has one vertex and no edges, we denote $c^{1/r}_{\Gamma}$ by $c^{1/r}_{\Gamma}(m)$. The classes are required to satisfy the axioms of connected and disconnected graphs, convexity, cutting edges, vanishing, and forgetting tails.

We can use the choice of an $r$-spin virtual class for stable $r$-spin curves to produce a similar $r$-spin class for all stable map spaces.

**Definition 4.1.** Let $\mathcal{G}^{1/r}$ be the set of all $r$-stable graphs. Given an $r$-spin virtual class $\{c^{1/r}_{\Gamma} \in H^{2D}(\overline{\mathcal{M}}^{1/r}_\Gamma, \mathbb{Q})\}_{\Gamma \in \mathcal{G}^{1/r}}$ meeting the axioms of [15, §4.1], then for each $V$ and for each $(V, r)$-stable graph $\Gamma$, define the class

$$c^{1/r}_{\Gamma} = \tilde{s}^* c^{1/r}_{\tilde{\Gamma}},$$

where $\tilde{\Gamma}$ is the stabilization of $\Gamma$.

The only graphs that are $V$-stable but have no stabilization are graphs with $2g - 2 + n \leq 0$, so in these cases we define the $r$-spin virtual class directly.

**Definition 4.2.** If $g = 0$ and $n < 3$ then we define $\tilde{c}^{1/r}_{\Gamma}$ to be the top Chern class of the dual of the first cohomology of the $r$-th root bundle $\mathcal{E}_r$; namely,

$$\tilde{c}^{1/r}_{\Gamma} = c_D(-R^1 \pi_* \mathcal{E}_r),$$

where $\mathcal{E}_r$ is the $r$-th root of the universal spin structure $\{\mathcal{E}_d\}, \{c_{d,d'}\}$ on the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}^{1/r}_r(V, \beta)$.

**Proposition 4.3.** If $g = 0$ and $n < 3$, then for every connected graph $\Gamma \in \mathcal{G}^{1/r}_{0,n}(V)$ with no marking $m_i$ equal to $r - 1$, the $r$-spin virtual class $\tilde{c}^{1/r}_{\Gamma}$ has dimension zero; and thus for any $\Gamma \in \mathcal{G}^{1/r}_{0,n}(V)$ with $n < 3$ we have

$$\tilde{c}^{1/r}_{\Gamma} = \begin{cases} 0 & \text{if any } m_i = r - 1 \\ 1 & \text{otherwise.} \end{cases}$$

**Proof.** The degree of the sheaf $\mathcal{E}_r$ is an integer and is given by

$$\deg \mathcal{E}_r = (2g - 2 - \sum m_i)/r,$$

hence when $g = 0$ we have

$$\sum m_i \equiv -2 \pmod{r}.$$
In the case that \( g = 1 \) and \( n = 0 \), the moduli space \( \overline{M}_{1,0}^{1/r}(V, \beta) \) decomposes into the disjoint union of \( d \) connected components, where \( d \) is the number of positive divisors of \( r \) (including 1 and \( r \)); these components correspond to the fact that (on the smooth locus) \( r \)-spin structures are in one-to-one correspondence with \( r \)-torsion points of the Jacobian of the underlying curve. No deformation of the underlying curve can take a point of order \( i \) to a point of order \( j \) unless \( i = j \), so the moduli space breaks up into disjoint components

\[
\overline{M}_{1,0}^{1/r} = \bigsqcup_{i | r} \overline{M}_{1,0}^{1/r,(i)}(V, \beta).
\]

We call \( i \) the index of the component if the \( r \)-th root is a point of exact order \( i \) in the Jacobian of the underlying curve.

**Definition 4.4.** If \( g = 1 \) and \( n = 0 \), define the \( r \)-spin virtual class \( \tilde{c}_1^{1/r} \) as follows

\[
\tilde{c}_1^{1/r}(V, \beta) = \begin{cases} 
-(r-1) & \text{if the index is 1} \\
1 & \text{otherwise}.
\end{cases}
\]

**Theorem 4.5.** If \( g = 0 \), then \( \tilde{c}_1^{1/r} = \tilde{st}^* c_1^{1/r} \) is the top Chern class \( c_D(-R^1\tilde{\pi}_*\tilde{E}_r) \) of the bundle whose fiber is the dual of the first cohomology of the \( r \)-th root \( \tilde{E}_r \) on the universal curve \( \tilde{\pi} : \tilde{C} \to \overline{M}_{1,r}^{1/r} \).

**Proof.** For \( n < 3 \), this is true by definition.

In the case that \( n \geq 3 \), since \( g = 0 \), the \( r \)-spin virtual class \( c_1^{1/r} \in H^{2D}(\overline{M}_{1,r}^{1/r}, \mathbb{Q}) \) is the top Chern class \( c_D(-R^1\pi_*E_r) \) of the first cohomology of the \( r \)-th root \( E_r \) on the universal curve \( \pi : C \to \overline{M}_{1,r}^{1/r} \), by the convexity axiom of \([13]\) §4.1.

We have the following commutative diagram.

\[
\begin{array}{cc}
\tilde{C} & \xrightarrow{\phi} C \times_{\overline{M}_{1,r}^{1/r}} \overline{M}_{1,r}^{1/r}(V, \beta) \xrightarrow{p_1} C \\
\tilde{\pi} \downarrow & & \downarrow \pi. \\
\overline{M}_{1,r}^{1/r}(V, \beta) & \xrightarrow{\tilde{st}} \overline{M}_{1,r}^{1/r}
\end{array}
\]

Here \( \phi \) is the natural map induced by \( \tilde{\pi} \) and stabilization of \( \tilde{C} \). If \( \tilde{E}_r \) is the \( r \)-th root on \( \tilde{C} \), then by Lemma \( 2.21 \) and the universality of the sheaves involved, \( \phi_*\tilde{E}_r \) is isomorphic to the pullback \( p_1^*E_r \) of the \( r \)-th root \( E_r \) from \( C \), and \( R^1\phi_*\tilde{E}_r = 0 \). By the Leray spectral sequence we have

\[
R^1\tilde{\pi}_*\tilde{E}_r = R^1p_2_*(p_1^*E_r).
\]

But \( \tilde{st} \) is flat (it is the composition of flat morphisms—see the commutative diagram \([\Box]\)), so that

\[
c_D(-R^1p_2_*(p_1^*E_r)) = \tilde{st}^* c_D(-R^1\pi_*E_r) = \tilde{st}^* c_1^{1/r}.
\]

\[\square\]
Remark 4.6. The proof of Theorem 4.5 depends upon the fact that the integer marking $m_h$ of each half edge $h$ lie in the range $0 \leq m_h \leq r - 1$, as required for stable graphs (see Definition 3.4). In particular, when an $m_h$ lies outside that range, Lemma 2.21 fails.

We shall see (in Remark 5.14) that Theorem 4.5 is false in the case that any $m_h$ is larger than $r - 1$.

Definition 4.7. We define $\overline{M}_{g,n}^{1/r}(V, \beta)^{\text{virt}}$ to be the pullback

$$\overline{M}_{g,n}^{1/r}(V, \beta)^{\text{virt}} := \tilde{p}^* \overline{M}_{g,n}(V, \beta)^{\text{virt}}$$

of the usual virtual fundamental class $\overline{M}_{g,n}(V, \beta)^{\text{virt}}$ (see Section 1.2) of $\overline{M}_{g,n}(V, \beta)$ via

$$\tilde{p} : \overline{M}_{g,n}^{1/r}(V, \beta) \rightarrow \overline{M}_{g,n}(V, \beta)$$

Using the notation of the commutative diagram (1), since $\tilde{e}v_i = ev_i \circ \tilde{p}$, for any $\gamma_1, \ldots, \gamma_n \in H^*(V, \mathbb{Q})$ we have the equality

$$\tilde{e}v^*_i \gamma_1 \cup \tilde{e}v^*_2 \gamma_2 \cup \cdots \cup \tilde{e}v^*_n \gamma_n = \tilde{p}^*(ev^*_1 \gamma_1 \cup \cdots \cup ev^*_n \gamma_n).$$

We also have the following important relation on classes, which is the main step in proving that the CohFT defined by stable r-spin maps is the tensor product of the CohFTs of r-spin curves and stable maps (Theorem 5.8).

Theorem 4.8. Given any set $\{\gamma_1, \ldots, \gamma_n\}$ of classes in $A^*(V)$ (or $H^*(V)$), and given an r-spin virtual class $\tilde{c}^{1/r}$ on $\overline{M}_{g,n}^{1/r,m}(V, \beta)$ defined by equations (12), (24) and (27), the relation

$$(22)$$

$$q_*(\tilde{c}^{1/r} \cup \prod_{i=1}^n \tilde{e}v^*_i(\gamma_i) \cap [\overline{M}_{g,n}^{1/r}(V, \beta)]^{\text{virt}}) = p_* c^{1/r} \cup st_*(\prod_{i=1}^n ev^*_i(\gamma) \cap [\overline{M}_{g,n}(V, \beta)]^{\text{virt}})$$

holds.

Proof. We will give the proof on the level of (operational) Chow groups $A^*$ with notation as in [21, V §8]. From [21, VI §2] it will follow then that such results also hold for $H^*(V)$.

In particular, if we denote the identity maps on $\overline{M}_{g,n}$, $\overline{M}_{g,n}^{1/r}$, $\overline{M}_{g,n}(V, \beta)$, $\overline{M}_{g,n}^{1/r} \times_{\overline{M}_{g,n}} \overline{M}_{g,n}(V, \beta)$, and $\overline{M}_{g,n}^{1/r}(V, \beta)$ by $\mathbb{I}$, $\mathbb{I}_r$, $\mathbb{I}_V$, $\mathbb{I}_X$, and $\mathbb{I}_{r,V}$, respectively, then we have $c^{1/r} \in A^*(\overline{M}_{g,n}^{1/r}) := A^*(\mathbb{I} : \overline{M}_{g,n}^{1/r} \rightarrow \overline{M}_{g,n}^{1/r})$, and $\tilde{c}^{1/r} = \tilde{st}^*(c^{1/r}) \in A^*(\overline{M}_{g,n}^{1/r}(V, \beta))$. We take $\gamma_i$ in $A^*(V)$, so that $\tilde{e}v^*_i(\gamma_i)$ is in $A^*(\overline{M}_{g,n}^{1/r}(V, \beta))$. Also, we have $[\overline{M}_{g,n}(V, \beta)]^{\text{virt}} \in A_*(\overline{M}_{g,n}(V, \beta))$.

Finally, by

$$\tilde{c}^{1/r} \cup \prod_{i=1}^n \tilde{e}v^*_i(\gamma_i) \cap [\overline{M}_{g,n}^{1/r}(V, \beta)]^{\text{virt}}$$

we mean

$$\left(\tilde{c}^{1/r} \cup \prod_{i=1}^n \tilde{e}v_i(\gamma_i)\right)_{\mathbb{I}_{r,V}} \cap [\overline{M}_{g,n}^{1/r}(V, \beta)]^{\text{virt}}.$$
As in [21, V §8.9], for any morphism $Y \to X$, we define $f^* : A^*(X) \to A^*(Y)$ to be
\begin{equation}
(f^* h \cap y := \delta_{foh} \cap y, \tag{23}
\end{equation}
where $\delta \in A^*(X)$ and $h : L \to Y$ is an arbitrary morphism, and $y \in A_*(L)$. We also define, for any proper, flat morphism $f : Y \to X$ of Deligne-Mumford stacks $X$ and $Y$, the proper flat pushforward $f_* : A^*(Y) \to A^*(X)$ to be
\begin{equation}
(f_* h \cap c := f_*(\delta_{f'Y} \cap f^*(c)), \tag{24}
\end{equation}
where $g : L \to X$ is an arbitrary morphism, $\alpha$ is an element of $A^*(Y)$, and $c$ is an element of $A_*(X)$.

Remark 4.9. Note that part (ii) of Manin’s definition in [21, V §8.9] of the operational Chow ring $A^*(M)$ for the identity morphism $I : M \to M$ states that elements of $A^*(M)$ only need to commute with pullback along representable, flat morphisms of DM-stacks, despite the fact that standard definitions of general operational Chow rings require that these elements commute with pullback along all flat morphisms of DM-stacks (see Vistoli [25, 5.1.i] and Manin [21, V.8.1.i]).

In what we do below, we will need the definition of $A^*(M)$ that requires commutativity with all flat pullbacks; that is, we require the following.

Let $f : X \to Y$ be a flat morphism of Deligne-Mumford stacks, which is not necessarily representable, and let $h : L \to Y$ be an arbitrary morphism of Deligne-Mumford stacks. For any $\sigma \in A^*(Z)$ and $y \in A_*(Y)$, we have
\begin{equation}
\sigma_{hof} \cap f^*(y) = f^*(\sigma h \cap y). \tag{25}
\end{equation}
This seemingly minor difference in the definition of $A^*$ allows us to prove a projection formula for non-representable morphisms.

Lemma 4.10.
1. Let $f : X \to Y$ be a proper, flat morphism of Deligne-Mumford stacks (which is not necessarily representable), and let $h : L \to Y$ be an arbitrary morphism of Deligne-Mumford stacks. We have
\begin{equation}
h^* f_* = f_* h^*_X. \tag{26}
\end{equation}

2. (Projection formula for $f_*$) Let $f : X \to Y$ be a proper, flat morphism of Deligne-Mumford stacks (which is not necessarily representable). For any $\sigma \in A^*(X)$ and $\beta \in A^*(Y)$ we have
\begin{equation}
f_*(\sigma f^*(\beta)) = f_*(\sigma) \beta. \tag{27}
\end{equation}

Proof. For part 1 of the lemma, the same proof as given by Manin for this equation [21, V.8.30] works exactly for our case, too; nowhere is the representability of $f$ used in Manin’s proof.

For part 2, again Manin’s proof of the projection formula [21, V.8.29] works for non-representable morphisms, the only change needed is that [21, V.8.22] (commutativity with flat, representable pullbacks) must be replaced by our equation (25) for non-representable, flat pullbacks.

One more fact we will need in the proof of Theorem 4.8 is the commutativity with proper pushforwards required by the definition of $A^*$ (cf. [21, V.8.21]); namely,
if \( p : P \rightarrow L \) is proper, and \( h : L \rightarrow M \) is an arbitrary morphism, then by definition of \( A^*(M) \), for any \( \sigma \in A^*(M) \) and for any \( y \in A_*(P) \) we have

\[
\sigma_h \cap p_*(y) = p_*(\sigma_h \cap y).
\]

Now we may proceed with the proof of Theorem 4.8. We will refer throughout the proof to the notation of the commutative diagram (2).

Since \( q_1 \) is a birational map, it is a splitting morphism (i.e., \( q_1q_1^* = \mathbb{1}_X \), as a map on \( A^* \)). Moreover, the morphism \( st \) is flat [3 Prop. 3] and proper, \( p \) is flat [10, Theorem 2.2] and proper [10, Theorem 2.3], and \( q_1 \) is flat and proper by Proposition 2.18. We have the following relations:

\[
q_*\left( \left[ e^{1/r} \cup \prod_{i=1}^n e_{y_i}(\gamma_i) \right] \cap \tilde{p}_*\left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}} \right)
\]

\[
= q_{2*}q_{1*}\left( q_1^*\left( pr_1^* e^{1/r} \cup pr_2^* \prod_{i=1}^n e_{y_i}(\gamma_i) \right) \cap q_1^* pr_2^* \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}} \right)
\]

\[
= q_{2*}\left( q_1 q_1^* \left( pr_1^* e^{1/r} \cup pr_2^* \prod_{i=1}^n e_{y_i}(\gamma_i) \right) \cap pr_2^* \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}} \right)
\]

\[
= q_{2*}\left( pr_1^* e^{1/r} \cup pr_2^* \prod_{i=1}^n e_{y_i}(\gamma_i) \right) \cap pr_2^* \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}}
\]

\[
= st_* pr_2^*\left( pr_1^* e^{1/r} \cup pr_2^* \prod_{i=1}^n e_{y_i}(\gamma_i) \right) \cap \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}}
\]

\[
= st_* \left( pr_2^* \left( pr_2^* \left( pr_1^* e^{1/r} \cup \prod_{i=1}^n e_{y_i}(\gamma_i) \right) \cap \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}} \right) \right)
\]

\[
= st_* \left( (st^* p_* e^{1/r}) \cup \prod_{i=1}^n e_{y_i}(\gamma_i) \right) \cap \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}}
\]

\[
= st_* \left( (st^* p_* e^{1/r}) \cap \left[ \prod_{i=1}^n e_{y_i}(\gamma_i) \right] \cap \left[ \overline{\mathcal{M}}_{g,n}(V,\beta) \right]^{\text{virt}} \right)
\]

This completes the proof of Theorem 4.8.

5. **Gromov-Witten Invariants and Tensor Products of CohFTs**

Let \( V \) be a smooth projective variety. The moduli space of stable \( r \)-spin maps \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \) gives rise to a set of correlators satisfying axioms analogous to those satisfied by Gromov-Witten invariants. This follows from the fact that the CohFT associated to \( \overline{\mathcal{M}}_{g,n}^{1/r}(V) \) is the tensor product of the Gromov-Witten CohFT with the \( r \)-spin CohFT.
5.1. Axioms of Gromov-Witten classes.
For each $\beta \in H_2(V, \mathbb{Z})$ and a stable pair of integers $(g, n)$, define the (cohomological) correlators of Gromov-Witten theory to be linear maps $\Lambda^{(V)}_{g,n,\beta} : H^\bullet(V, \mathbb{C}) \to H^\bullet(M_{g,n}, \mathbb{C})$ such that
\[
\Lambda^{(V)}_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) = \text{st}_* \left[ \prod_{i=1}^n ev_i^* \gamma_i \cap [M_{g,n}(V, \beta)]^\text{virt} \right],
\]
where $\text{st} : M_{g,n}(V) \to M_{g,n}$ is the stabilization morphism, $[M_{g,n}(V, \beta)]^\text{virt}$ is the virtual fundamental class of $M_{g,n}(V)$, and $\gamma_i \in H^\bullet(V, \mathbb{C})$.

Theorem 5.1 ([18]). Let $B(V) \subset H_2(V, \mathbb{Z})$ denote the semigroup of numerical equivalence classes $\beta$ such that $\beta \cdot L \geq 0$ for all ample divisor classes $L$ in $V$. Let $\eta$ be the Poincaré pairing on $H^\bullet(V)$ and let $\eta_{\mu\nu} := \eta(e_\mu, e_\nu)$ be the coefficients of its matrix with respect to a basis $\{e_\mu\}$ for $H^\bullet(V)$. Denote by $(\eta^{\mu\nu})$ the inverse matrix of $(\eta_{\mu\nu})$.

The collection $\{\Lambda^{(V)}_{g,n,\beta}\}$ satisfies the following properties, called the axioms of Gromov-Witten invariants.

1. (Effectivity) $\Lambda^{(V)}_{g,n,\beta} = 0$ if $\beta \notin B(V)$.
2. (S_n-Equivariance) Each map $\Lambda^{(V)}_{g,n,\beta}$ is $S_n$-equivariant, where $S_n$ is the symmetric group on $n$ letters.
3. (Degeneration Axioms)
   (a) Let
   \[
   \rho^{\Gamma_{\text{tree}}} : \overline{\mathcal{M}}_{g-1, n+2} \to \overline{\mathcal{M}}_{g,n}
   \]
   be the gluing map corresponding to the stable graph
   \[
   \Gamma_{\text{tree}} = \begin{array}{c}
   i_1 \\downarrow \quad \vdots \quad k \quad \downarrow \quad i_{j+1} \\
   \end{array}
   \]
   then the forms $\Lambda^{(V)}_{g,n,\beta}$ satisfy the composition property
   \[
   \rho^{\Gamma_{\text{tree}}}_* \Lambda^{(V)}_{g,n,\beta} (\gamma_1, \gamma_2, \ldots, \gamma_n) = \sum_{\beta_1 + \beta_2 = \beta} \Lambda^{(V)}_{k,j+1,\beta_1} (\gamma_{i_1}, \ldots, \gamma_{i_j}, e_\mu) \eta^{\mu\nu} \otimes \Lambda^{(V)}_{g-k,n-j+1,\beta_2} (e_\nu \gamma_{i_{j+1}}, \ldots, \gamma_{i_n})
   \]
   for all $\gamma_i \in \mathcal{H}$.
   (b) Let
   \[
   \rho^{\Gamma_{\text{loop}}} : \overline{\mathcal{M}}_{g-1, n+2} \to \overline{\mathcal{M}}_{g,n}
   \]
   be the gluing map corresponding to the stable graph
   \[
   \Gamma_{\text{loop}} = \begin{array}{c}
   i_1 \quad \downarrow \ldots \downarrow \quad i_n \\
   \end{array}
   \]
   then
   \[
   \rho^{\Gamma_{\text{loop}}}_* \Lambda^{(V)}_{g,n,\beta} (\gamma_1, \gamma_2, \ldots, \gamma_n) = \Lambda^{(V)}_{g-1, n+2,\beta} (\gamma_1, \gamma_2, \ldots, \gamma_n, e_\mu, e_\nu) \eta^{\mu\nu}.
   \]
4. (Identity Axiom) Let $1$ be the unit in $H^\bullet(V)$ then

$$A_{g,n+1,\beta}(\gamma_1, \ldots, \gamma_n, 1) = \pi^* A_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$$

for all $\gamma_i \in H^\bullet(V)$, where $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the forgetful morphism.

5. (Dimension Axiom) Let $K_V$ denote the canonical class of $V$. The linear map $A_{g,n,\beta}$ has the grading

$$\left| A_{g,n,\beta} \right| = 2 \int_\beta K_V + 2(g-2) \dim CV.$$

6. (Divisor Axiom) Let $\alpha$ belong to $H^2(V)$, then

$$\pi_* A_{g,n+1,\beta}(\gamma_1, \ldots, \gamma_n, \alpha) = A_{g,n,\beta}(\gamma_1, \ldots, \gamma_n) \int_\beta \alpha$$

for all $\gamma_i \in H^\bullet(V)$, where $\pi : \overline{M}_{g,n+1}(V) \to \overline{M}_{g,n}$ is the forgetful morphism.

7. (Mapping to a point) For all $\gamma_i \in H^\bullet(V)$ we have

$$A_{g,n}(\gamma_1, \ldots, \gamma_n) = p_2^* \left( \prod_{i=1}^n \gamma_i \cup c_d(TV \otimes L) \right),$$

where $p_1 : V \times \overline{M}_{g,n} \to V$ and $p_2 : V \times \overline{M}_{g,n} \to \overline{M}_{g,n}$ are the canonical projections, $TV$ is the tangent bundle, $L = R^1 \pi_* OC_{g,n}$ where $OC_{g,n}$ is the structure sheaf on the universal curve $\pi : C_{g,n} \to \overline{M}_{g,n}$, and $d = g \dim CV$ (the rank of $TV \otimes L$).

These properties were first presented in [18] and were later proved by various people. The theorem follows from properties of the virtual fundamental class, restriction properties of the Gromov-Witten classes, and the geometry of the moduli space of stable maps into $V$. (See [21] for a summary of the proof.)

**Definition 5.2.** Let $\mathcal{R}$ denote the ring consisting of formal sums of expressions $q^\beta$ with complex coefficients, where $\beta$ belongs to $B(V)$, subject to the relations $q^{\beta_1 + \beta_2} = q^{\beta_1} q^{\beta_2}$. We define $A_{g,n} : H^\bullet(V)^{\otimes n} \to H^\bullet(\overline{M}_{g,n}, \mathcal{R})$ as

$$A_{g,n} := \sum_{\beta} q^\beta A_{g,n,\beta}.$$

Let $A(V)$ denote the collection $\{ A_{g,n}(V) \}$ and $1$ denote the unit in $H^\bullet(V)$.

Because of these properties, the Gromov-Witten invariants form an algebra over the modular operad $H_\bullet(\overline{M})$ or, equivalently, a CohFT. We refer the reader to [18] for the definition of a CohFT.

**Corollary 5.3 ([18]).** The triple $(H^\bullet(V, \mathcal{R}), \eta, A(V))$ forms a CohFT with the flat identity $1$ (over the ground ring $\mathcal{R}$).

5.2. Spin CohFT.

The CohFTs $(\Lambda, H, \eta)$ whose correlators are constructed from classes $A_{g,n,\beta}$ satisfying properties 1 to 7 in Theorem 5.1 form a special class of CohFTs. There is, however, another potential construction of CohFTs.
arising from CohFT associated to the space is associated to Gromov-Witten theory, it is natural to ask if there is a natural closely related to the other two through the operation of tensor product.

\begin{equation}
(15, \text{Theorem 3.8})
\end{equation}

\textbf{Theorem 5.5.} Tensor products of CohFTs.

\textbf{Definition 5.6.} Let \((\mathcal{H'}^{(r)}, \eta^{(r)})\) and \((\mathcal{H''}^{(r)}, \eta''^{(r)}, \Lambda'')\) be CohFTs. Their tensor product is \((\mathcal{H'} \otimes \mathcal{H''}^{(r)}, \eta' \otimes \eta''^{(r)}, \Lambda)\), where

\[\Lambda_{g,n}(v'_1 \otimes v''_1, \ldots, v'_n \otimes v''_n) := (-1)^{\sigma} \Lambda'_{g,n}(v'_1, \ldots, v'_n) \cup \Lambda''_{g,n}(v''_1, \ldots, v''_n)\]

for all \(v'\) in \(\mathcal{H'}\), \(v''\) in \(\mathcal{H''}\), and \((-1)^{\sigma}\) denotes the usual sign associated to the permutation

\[v'_1 \otimes v'_2 \otimes \cdots \otimes v'_n \otimes v''_1 \otimes v''_2 \otimes \cdots \otimes v''_n \mapsto v'_1 \otimes \cdots \otimes v'_n \otimes v''_1 \otimes \cdots \otimes v''_n.\]

This reflects the fact that the diagonal map \(\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}\) is a coproduct with respect to the composition maps of the modular operad \(\{H'_*(\overline{\mathcal{M}}_{g,n})\}\).

In the case of Gromov-Witten invariants, Behrend [14] proved that the CohFT arising from \(\overline{\mathcal{M}}_{g,n}(V' \times V'')\) is the tensor product of that arising from \(\overline{\mathcal{M}}_{g,n}(V')\) and \(\overline{\mathcal{M}}_{g,n}(V'')\). When restricting to genus zero, one can view this result as a deformation of the Künneth theorem. Similarly, it was shown in [15] that the tensor product of an \(r\)-spin CohFT and an \(r'\)-spin CohFT can be geometrically realized by means of the moduli space of \((r, r')\)-spin curves. To complete this picture, what is missing is a description of the tensor product of the Gromov-Witten theory with the \(r\)-spin CohFT.

\textbf{Definition 5.7.} Let \((H^*(V, \mathbb{C}), \eta_P)\) denote the cohomology of \(V\) together with its Poincaré pairing \(\eta_P\). Let \((\mathcal{H}^{(V,r)}, \eta)\) denote the tensor product of \((H^*(V), \eta_P)\) with
(\(H^{(r)}, \eta^{(r)}\)). For each stable pair \((g, n)\) and \(\beta \in H_2(V, \mathbb{Z})\), define the (cohomological) correlators to be linear maps

\[ \Lambda^{(V, r)}_{g, n, \beta} : \mathcal{H}^{(V, r)} \to H^*(\overline{\mathcal{M}}_{g, n}, \mathbb{C}) \]

given by

\[ \Lambda^{(V, r)}_{g, n, \beta}(\gamma_1 \otimes e_{m_1}, \ldots, \gamma_n \otimes e_{m_n}) = Q_s([\hat{\rho}^{1/r}_{g, n} \prod_{i=1}^n e_{\nu_i^*}^* \gamma_i] \cap [\overline{\mathcal{M}}^{1/r}_{g, n}(V, \beta)]^{\text{virt}}), \]

where \(Q : \overline{\mathcal{M}}^{1/r}_{g, n}(V) \to \overline{\mathcal{M}}_{g, n}\) is the morphism that forgets both the stable map and the \(r\)-spin structure, \([\overline{\mathcal{M}}^{1/r}_{g, n}(V, \beta)]^{\text{virt}}\) is the virtual fundamental class of \(\overline{\mathcal{M}}^{1/r}_{g, n}(V)\), and \(\gamma_i \otimes e_{m_i} \in H_* \mathcal{M}_{g, n}\).

The following theorem holds.

**Theorem 5.8.** Let \(\Lambda^{(V, r)}_{g, n} : \mathcal{H}^{(V, r)} \otimes^n \to H^*(\overline{\mathcal{M}}_{g, n}, \mathbb{R})\), where

\[ \Lambda^{(V, r)}_{g, n} := \sum_{\beta} q^\beta \Lambda^{(V, r)}_{g, n, \beta}. \]

Let \(\Lambda^{(V, r)}\) denote the collection \(\{\Lambda^{(V, r)}_{g, n}\}\). The collection \((H^*(V, \mathbb{R}), \eta, \Lambda)\) forms a CohFT (over the ground ring \(\mathbb{R}\)) with flat identity \(1 \otimes e_0\) and is the tensor product of the CohFTs \((\Lambda^{(V)}, H^*(V, \mathbb{R}), \eta)\) and \((\Lambda^{(r)}, H^{(r)}, \eta^{(r)})\).

**Proof.** This is an immediate consequence of Theorem 4.8. 

The \(r\)-spin CohFTs behave as though the elements of \(H^{(r)}\) were cohomology classes of fractional dimension, similar to the orbifold cohomology classes of Chen and Ruan [21]. There is also no analog of the elements \(\beta\) in \(B(V)\) appearing in Gromov-Witten theory. However, the theory associated to \(r\)-spin maps into \(V\) does satisfy analogous axioms. In particular, this theory, like the Gromov-Witten theory, is of qc-type [21, 22].

### 5.4. Spin Gromov-Witten invariants.

The classes \(\Lambda^{(V, r)}_{g, n, \beta}\) have properties analogous to those of Gromov-Witten invariants.

**Theorem 5.9.** Let \((g, n)\) be a stable pair of integers. The collection \(\{\Lambda^{(V, r)}_{g, n, \beta}\}\) satisfies the following properties:

1. **(Effectivity)** \(\Lambda^{(V, r)}_{g, n, \beta} = 0\) if \(\beta \notin B(V)\).
2. **(\(S_n\)-Equivariance)** Each map \(\Lambda^{(V, r)}_{g, n, \beta}\) is \(S_n\)-equivariant.
3. **(Degeneration Axioms)** Given a basis \(\{\epsilon_\mu\}\) for \(H^{(V, r)}\), let \(\eta^{(V, r)}_{\mu \nu} := \eta^{(V, r)}(\epsilon_\mu, \epsilon_\nu)\) and \((\eta^{(V, r)}^{\mu \nu})\) denote the inverse matrix.

   a. Let

   \[ \rho^{\ast}_{\text{tree}} : \overline{\mathcal{M}}_{k, n+1} \times \overline{\mathcal{M}}_{g, n-j+1} \to \overline{\mathcal{M}}_{g, n} \]

   be defined as in Theorem 5.7, then the forms \(\Lambda^{(V, r)}_{g, n, \beta}\) satisfy the composition property:

   \[ \sum_{\beta_1 + \beta_2 = \beta} \Lambda^{(V, r)}_{k, n+1, \beta_1}(\gamma_1, \gamma_2, \ldots, \gamma_j) \cdot \Lambda^{(V, r)}_{g, n-j+1, \beta_2}(\epsilon_\mu, \gamma_{j+1}, \ldots, \gamma_n) \]

   \[ = \Lambda^{(V, r)}_{g, n, \beta}(\gamma_1, \gamma_2, \ldots, \gamma_n) \otimes \Lambda^{(V, r)}_{g, n-j+1, \beta_2}(\epsilon_\mu, \gamma_{j+1}, \ldots, \gamma_n) \]

   \]
Definition 5.10. Consider the

Proof. All axioms follow immediately from Theorems 4.8 and 5.1.

4. (Identity Axiom) Let $1 := 1 \otimes e_0$, where 1 is the unit in $H^\bullet(V)$ and $e_0$ the unit of $H^{(r)}$, then

$$\Lambda_{g,n+1,\beta}(\gamma_1, \ldots, \gamma_n, 1) = \pi^* \Lambda_{g,n,\beta}(\gamma_1, \ldots, \gamma_n)$$

for all $\gamma_i \in H^{(r)}$, where $\pi : \cM_{g,n+1}(V) \to \cM_{g,n}$ is the forgetful morphism.

5. (Dimension Axiom) Let $K_V$ denote the canonical class on $V$. The map $\Lambda_{g,n,\beta}$ of $\mathbb{Z}$-graded modules must be homogeneous of degree

$$\left| \Lambda_{g,n,\beta}(r) \right| = 2 \int_\beta K_V + 2(g - 2) \dim_C V + \frac{2}{r}(r - 2)(g - 1).$$

6. (Divisor Axiom) Let $\alpha \otimes e_0$ belong to $H^2(V) \otimes H^{(r)}$, then

$$\pi^* \Lambda_{g,n+1,\beta}(\gamma_1, \ldots, \gamma_n, \alpha \otimes e_0) = \Lambda_{g,n,\gamma}(\gamma_1, \ldots, \gamma_n) \int_\beta \alpha,$

for all $\gamma_i \in H^{(r)}$, where $\pi : \cM_{g,n+1}(V) \to \cM_{g,n}$ is the forgetful morphism.

7. (Mapping to a Point Axiom)

$$\Lambda_{g,n}(\gamma_1 \otimes e_{m_1}, \ldots, \gamma_n \otimes e_{m_n}) = p_{2*} \left[ p_1^* \left( \prod_{i=1}^n \gamma_i \right) \cup c_d(TV \otimes L) \right] \cup p_* c_{g,r,m}^{vir}$$

for all $\gamma_i \in H^\bullet(V)$, where $p_1 : V \times \cM_{g,n} \to V$ and $p_2 : V \times \cM_{g,n} \to \cM_{g,n}$ are the canonical projections, $TV$ is the tangent bundle, $L = R^1 \pi_* \mathcal{O}_{C_g}$ where $\mathcal{O}_{C_g}$ is the structure sheaf on the universal curve $\pi : C_g \to \cM_{g,n}$, and $d = g \dim_C V$ (the rank of $TV \otimes L$). Finally, $p : \cM_{g,n}^{vir} \to \cM_{g,n}$ is the morphism forgetting the spin structure and $m = (m_1, \ldots, m_n)$.

Proof. All axioms follow immediately from Theorems 4.8 and 5.1. □

5.5. Potential functions and gravitational descendants.

Recall the potential functions associated to $\cM_{g,n}(V)$.

Definition 5.10. Consider the correlation functions

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,\beta} := \int_{[\cM_{g,n}(V,\beta)]^{virt}} \prod_{i=1}^n (\phi_{a_i}^\beta \mu_{a_i}^\gamma \gamma_i)$$

for all integers $a_1, \ldots, a_n \geq 0$ and $\gamma_1, \ldots, \gamma_n$ in $H^\bullet(V)$. Correlation functions such that some of the $a_i$ are nonzero are called gravitational descendants.

The large phase space potential (function) associated to $\cM_{g,n}(V)$ is

$$\Phi(V)(t) := \sum_{g \geq 0} \lambda^{2g-2} \Phi_g(V)(t) \in \lambda^{-2} \mathcal{R}[[\lambda^2]][[t_a]],$$
Proof. Theorem 4.8 shows that the intersection numbers \( \Lambda \) are completely determined by the classes \( \Phi \) determined by the potential functions such that some of the descendants \( x^\alpha \) are nonzero are called gravitational descendants.

The small phase space potential (function), \( \Phi^{(V)}(x) \) where \( x = (x_1, \ldots, x^n) \) are coordinates on \( H^\bullet(V) \) relative to the basis \{ \( \varepsilon_\alpha \) \}, is obtained from \( \Phi^{(V)}(t) \) by setting \( x^\alpha := t^\alpha_0 \) and \( t^\alpha_a := 0 \) for all \( \alpha \geq 1 \) and all \( \alpha, a \).

There are analogous potential functions associated to \( \overline{\mathcal{M}}^{1/r}_{g,n}(V) \).

**Definition 5.11.** Consider the correlation functions

\[
\langle \tau_{a_1}(\gamma_1 \otimes e_{m_1}) \cdots \tau_{a_n}(\gamma_n \otimes e_{m_n}) \rangle_{g,\beta} := \int_{\overline{\mathcal{M}}^{1/r}_{g,n}(V)} \tau^{1-\alpha} \bar{c}^{1/r}(m) \prod_{i=1}^n (\psi_i e_i^* \gamma_i)
\]

for integers \( a_1, \ldots, a_n \geq 0, \gamma_1, \ldots, \gamma_n \in H^\bullet(V) \), and \( e_{m_1}, \ldots, e_{m_n} \in \mathcal{H}^{(r)} \). Correlation functions such that some of the \( a_i \) are nonzero are called gravitational descendants.

The large phase space potential (function) associated to \( \overline{\mathcal{M}}^{1/r}_{g,n}(V) \) is

\[
\Phi^{(V,r)}(u) := \sum_{g \geq 0} \Lambda^{2g-2} \Phi^{(V,r)}_g(u) \in \Lambda^{-2} \mathcal{R}[[\lambda^2]][[\mathcal{H}^{(V,r)}]],
\]

where

\[
\Phi^{(V,r)}_g(u) := \sum_{\beta \in B(V)} \langle \exp(u \cdot \beta) \rangle_{g,\beta} q^\beta
\]

and

\[
u(\cdot) := \sum_{a \geq 0} \sum_{\alpha, m} a^\alpha_m \tau_\alpha(\varepsilon_\alpha \otimes e_m)
\]

relative to the basis \{ \( \varepsilon_\alpha \otimes e_m \) \} for \( \mathcal{H}^{(V,r)} \).

The small phase space potential (function), \( \Phi^{(V,r)}(y) \) where \( y \) consists of coordinates \( \{ y^{a,m} \} \) on \( H^\bullet(V) \) relative to the basis \{ \( \varepsilon_\alpha \otimes e_m \) \}, is obtained from \( \Phi^{(V,r)}(u) \) by setting \( y^\alpha_m := u^\alpha_0 \) and \( u^\alpha_0 := 0 \) for all \( \alpha \geq 1 \) and all \( \alpha, m \).

**Theorem 5.12.** The small phase space potential function \( \Phi^{(V,r)}(y) \) is completely determined by the potential \( \Phi^{(V)}(x) \), the cohomological correlators \{ \( \Lambda^{(V)}_{g,n} \) \}, and \{ \( \Lambda^{(r)}_{g,n} \) \}.

**Proof.** Theorem 4.8 shows that the intersection numbers \( \langle \gamma_1 \otimes e_{m_1} \cdots \gamma_n \otimes e_{m_n} \rangle_{g,n} \) are completely determined by the classes \{ \( \Lambda^{(V)}_{g,n} \) \} and \{ \( \Lambda^{(r)}_{g,n} \) \} if \( (g, n) \) is stable.

We must still address the unstable cases—when \( (g, n) \in \{ (0, 0), (0, 1), (0, 2), (1, 0) \} \). But by Proposition 4.3 and Definition 4.4, these are always of dimension zero.

Let \( \overline{\mathcal{M}}^{1/r,m}_{g,n} := \bigcup_{i} \overline{\mathcal{M}}^{1/r,m,(i)}_{g,n} \) where \( \overline{\mathcal{M}}^{1/r,m,(i)}_{g,n} \) are the connected components of \( \overline{\mathcal{M}}^{1/r,m}_{g,n} \), and let \( \overline{\mathcal{M}}^{1/r,m,(i)}_{g,n}(V, \beta) \to \overline{\mathcal{M}}^{1/r,m,(i)}_{g,n}(V, \beta) \) be the morphisms forgetting the \( r \)-spin structure. Furthermore, let \( \bar{c}^{1/r,(i)} \) be \( \bar{c}^{1/r} \) restricted to \( \overline{\mathcal{M}}^{1/r,m,(i)}_{g,n} \) and
let us assume that $c^{1/r,m,(i)}$ is zero dimensional. For all $\gamma \otimes e := \gamma_1 \otimes e_{m_1} \cdots \gamma_n \otimes e_{m_n}$ in $\mathcal{H}_V^{r,0}$ we have

$$
\langle \gamma \otimes e \rangle_{g,\beta} = r^{-g} \int (\ev^* \gamma \cup \hat{c}^{1/r}) \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

$$
= \sum_i \hat{c}^{1/r,m,(i)}_{g,n}(1-g) \int \ev^* \gamma \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

$$
= \sum_i \hat{c}^{1/r,m,(i)}_{g,n}(1-g) \int \hat{p}_{(i)}^* \ev^* \gamma \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

$$
= \sum_i \hat{c}^{1/r,m,(i)}_{g,n}(1-g) \int (\ev^* \gamma) \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

$$
= \sum_i \hat{c}^{1/r,m,(i)}_{g,n}(1-g) \int \hat{p}_{(i)}^* (\ev^* \gamma) \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

$$
= \sum_i \hat{c}^{1/r,m,(i)}_{g,n}(1-g) \int \ev^* \gamma \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

$$
= \sum_i \hat{c}^{1/r,m,(i)}_{g,n}(1-g) \int \ev^* \gamma \cap [\mathcal{M}_g^{1/r,m}(V,\beta)]_{\text{virt}}
$$

where $\text{deg}$ denotes the (orbifold) degree of $\hat{p}_{(i)}$. This completes the proof. \hfill \Box

5.6. Gravitational descendants. In this subsection, we show that when $g = 0$, our constructions on $\overline{\mathcal{M}}_{g,n}^{1/r}(V,\beta)$ satisfy a generalization of the so-called descent property (introduced in [13]). In particular, this property explicates the geometric origin of the $\psi$ classes (at least in genus zero) in the definition of the Gromov-Witten invariants of $V$.

It may seem curious that $\overline{\mathcal{M}}_{g,n}^{1/r}(V,\beta)$ is defined to be the disjoint union of $\overline{\mathcal{M}}_{g,n}^{1/r,m}(V,\beta)$, where the $n$-tuple of nonnegative integers $m = (m_1, \ldots, m_n)$ is required to satisfy $m_i \leq r - 1$ for all $i = 1, \ldots, n$. The latter restriction, however, is reasonable because of the isomorphism

$$
\overline{\mathcal{M}}_{g,n}^{1/r,m}(V,\beta) \longrightarrow \overline{\mathcal{M}}_{g,n}^{1/r,\tilde{m} + r \delta_i}(V,\beta)
$$

from Proposition 2.13, where $i = 1, \ldots, n$, $\delta_i$ is the $n$-tuple whose $i$-th component is 1 and the rest are zero, and $\tilde{m} := (\tilde{m}_1, \ldots, \tilde{m}_n)$ is any $n$-tuple of nonnegative integers.

On the other hand, in genus zero the classes $c^{1/r} (\tilde{m})$ change under this identification in the following manner.

**Theorem 5.13.** (The descent property) Let $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_n)$ be an $n$-tuple of nonnegative integers and let $m = (m_1, \ldots, m_n)$ be the reduction of $\tilde{m}$ (mod $r$) (i.e., $\tilde{m} \equiv m$ (mod $r$) and $0 \leq m_i \leq r - 1$ for $i = 1, \ldots, n$).

Let $c^{1/r}(\tilde{m})$ be the top Chern class of the vector bundle $R^1 \pi_* \mathcal{E}^{(\tilde{m})}$ on $\overline{\mathcal{M}}_{g,n}^{1/r,m}$. The following equation is satisfied on $\overline{\mathcal{M}}_{g,n}^{1/r,m}$ for all $i = 1, \ldots, n$, where $\delta_i$ is the $n$-tuple whose $i$-th component is 1 and the rest are zero:

$$
c^{1/r} (\tilde{m} + r \delta_i) = -\frac{\tilde{m}_i + 1}{r} \psi_i c^{1/r}(\tilde{m}).
$$
Proof. The proof is identical to the case of $\overline{M}_{g,n}$ in [14]. It follows from the long exact sequence associated to the short exact sequence

$$0 \longrightarrow E_r(\hat{m}) \longrightarrow E_r(\hat{m} + r\delta_i) \longrightarrow \sigma^*_i E_r(\hat{m}) \longrightarrow 0$$

and the fact that

$$\tilde{\psi}_i := c_1(\sigma^*_i E_r(\hat{m})) = \frac{m_i + 1}{r} \psi_i$$

for all $i = 1, \ldots, n$, which follows from an immediate generalization of Proposition 2.2 from [15].

Remark 5.14. The descent property holds on both $\overline{M}_{0,n}$ and $\overline{M}_{0,n}^r(V,\beta)$, but the $\psi$ classes on $\overline{M}_{0,n}^r(V,\beta)$ are not pullbacks of the corresponding $\psi$ classes on $\overline{M}_{g,n}$—just as in the case of stable maps, they differ by divisors that are collapsed under the stabilization map (see [21, 22]). This illustrates the fact, alluded to in Remarks 2.22 and 4.6, that when any $\tilde{m}_i$ is larger than $r - 1$, the class $\tilde{c}^{1/r}(\hat{m})$ is not equal to the class $\tilde{c}^{1/r}(\hat{m})$.

The previous theorem motivates the following generalization of the small phase space potential function in genus zero.

Definition 5.15. Let the $n$-tuples $\hat{m} = (\hat{m}_1, \ldots, \hat{m}_n)$ and $m$ and the class $\tilde{c}^{1/r}(\hat{m})$ on $\overline{M}_{0,n}^r, m$ be the same as in the previous theorem.

Define the correlation functions

$$\langle \tilde{\tau}_0 (\gamma_1 \otimes e_{\hat{m}_1}) \cdots \tilde{\tau}_0 (\gamma_n \otimes e_{\hat{m}_n}) \rangle_{0,\beta} := \int_{\overline{M}_{0,n}^r(V,\beta)^{\text{virt}}} r \tilde{c}^{1/r}(\hat{m}) \prod_{i=1}^n \tilde{\psi}_i^* \gamma_i.$$

Consider the analog of the genus zero small phase space potential

$$\Phi_0^{(V,r)}(\hat{t}) \in \mathcal{R}[[\lambda^2]][[\tilde{t}^{\alpha,\hat{m}}]],$$

where

$$\Phi_0^{(V,r)}(\hat{t}) := \sum_{\beta \in B(V)} (\exp(\hat{t} \cdot \tilde{\tau}))_{0,\beta} q^\beta$$

and

$$\hat{t} \cdot \tilde{\tau} := \sum_{\alpha,\hat{m}} \tilde{t}^{\alpha,\hat{m}} \tilde{\tau}_0 (e_{\alpha} \otimes e_{\hat{m}}),$$

where the sum runs over all $\alpha$ and all nonnegative integers $\hat{m}$.

Corollary 5.16. Let $r \geq 2$ be an integer. The potential functions $\Phi_0^{(V,r)}(\hat{t})$ and $\Phi_0^{(V,r)}(u)$ are equal after making the assignment:

$$\tilde{t}^{\alpha, (ar + m)} := \frac{(-1)^a r^a}{[r(a - 1) + m + 1]_r} \tilde{u}^{\alpha, m},$$

where $a$ and $m$ are nonnegative integers such that $m \leq r - 1$ and

$$[r(a - i) + m + 1]_r := \prod_{i=1}^a ((r(a - i) + m + 1).$$
Remark 5.17. In [24], a candidate class for $c^{1/r}$ was constructed on $\mathcal{M}^{1/r}_{g,n}$, which was shown to obey some of the axioms in [13]. In addition, it satisfies the descent property on $\mathcal{M}^{1/r}_{g,n}$ for nonnegative $n$-tuples $\mathbf{m}$. The construction of [24] can be generalized straightforwardly to $\mathcal{M}_{g,n}(V,\beta)$ to yield a class $\bar{c}^{1/r}(\tilde{\mathbf{m}})$ satisfying equation (4) and, hence, the previous corollary for all $g$ and $n$.

5.7. The case of $r = 2$.

In [13 26], the virtual class $c^{1/r}(\mathbf{m})$ when $r = 2$ was constructed for all genera and $n$-tuples $\mathbf{m} = (m_1, \ldots, m_n)$ with $0 \leq m_i \leq 1$. It was shown that the $r = 2$ case reduced to the Gromov-Witten invariants of a point. Here we show that 2-spin Gromov-Witten invariants are the usual Gromov-Witten invariants.

Theorem 5.18. For a pair of nonnegative integers $(g, n)$ and $\beta \in H_2(V, \mathbb{Z})$ let $\tilde{p} : \bar{\mathcal{M}}^{1/2}_{g,n}(V, \beta) \to \mathcal{M}_{g,n}(V, \beta)$ be the map forgetting the spin structure. For $i = 1, \ldots, n$, let $\gamma_i \otimes e_0$ belong to $\mathcal{H}^{(r)}$, then

$$2^{1-s} \tilde{p}_* \left( \bar{c}^{1/2}(0) \prod_{i=1}^n (e_i^* \gamma_i) \cap [\mathcal{M}^{1/2}_{g,n}(V, \beta)]^{\text{virt}} \right) = \prod_{i=1}^n (e_i^* \gamma_i) \cap [\mathcal{M}_{g,n}(V, \beta)]^{\text{virt}}.$$

Consequently, the large phase space potential functions $\Phi^{(V,2)}(\mathbf{u})$ and $\Phi^{V}(\mathbf{t})$ agree after setting $u_a^{(\alpha,0)} = t_a^\alpha$.

Proof. This was proved in the case where $V$ is a point in [13]. The same proof goes through here using the definition of $c^{1/r}$ (which is now defined in the unstable range) and the fact that $[\mathcal{M}^{1/2}_{g,n}(V, \beta)]^{\text{virt}} = \tilde{p}^* [\mathcal{M}_{g,n}(V, \beta)]^{\text{virt}}$. $\square$

5.8. Genus zero and $\beta = 0$.

Genus zero Gromov-Witten invariants of $V$ give rise to the quantum cohomology of $V$, which is a certain deformation of the cup product on $H^\bullet(V)$. The cup product itself appears as the $\beta = 0$ part of the genus zero potential function. Similarly, the Frobenius structure associated to $\mathcal{M}^{1/r}_{g,n}(V)$ can be regarded as a deformation of the following commutative, associative product on $\mathcal{H}^{(r)}$.

Proposition 5.19. Let $V$ be a smooth projective variety and $n \geq 3$ be an integer. Let $\gamma_1, \ldots, \gamma_n$ belong to $H^\bullet(V)$ and $e_0, \ldots, e_{r-2}$ be the standard basis in $\mathcal{H}^{(r)}$, then

$$\langle \gamma_1 \otimes e_{m_1} \cdots \gamma_n \otimes e_{m_n} \rangle_{g, \beta = 0} = \int_{\mathcal{M}^{1/r}_{0,n}(\mathbb{P}^1)} c^{1/r}(\mathbf{m}) \int_V \gamma_1 \cup \ldots \cup \gamma_n.$$

Proof. This follows from the Mapping to a Point property. $\square$

6. An Example: The Small Phase Space Potential for $\mathcal{M}^{1/3}_{0,n}(\mathbb{P}^1)$

Throughout this section let $r = 3$ and $V = \mathbb{P}^1$. We will now compute its genus zero small phase potential function, denoted by

$$\chi(\mathbf{t}) := \Phi_0^{(\mathbb{P}^1,3)}(\mathbf{t}),$$

where $\mathbf{t}$ is a set of coordinates $t^{\alpha,m}$ associated to the basis $\{ e_\alpha \otimes e_m \}$ (where $\alpha = 0, 1$ and $m = 0, 1$) for $\mathcal{H}^{(2,3)}$. Here $\varepsilon_0$ is the identity element in $H^\bullet(\mathbb{P}^1)$ and $\varepsilon_1$ is the element in $H^2(\mathbb{P}^1)$ Poincaré dual to a point. The metric in this basis is

$\eta(\alpha_1,m_1),\alpha_2,m_2) := \eta(\varepsilon_\alpha \otimes e_{m_1}, \varepsilon_\alpha \otimes e_{m_2}) = \delta_{\alpha_1+\alpha_2,1}\delta_{m_1+m_2,1}.$
The potential function can be broken into two pieces:
\[ \chi(t) = \chi_{\beta=0}(t) + \Psi(t), \]
where \( \chi_{\beta=0}(t) \) consists of only those terms corresponding to the moduli spaces \( \overline{M}_{0,n}(\mathbb{P}^1, 0) \); while \( \Psi(t) \) contains the contributions ("instanton corrections") from \( \overline{M}_{0,n}(\mathbb{P}^1, \beta) \) where \( \beta \neq 0 \). Corollary 5.19 implies that
\[ \chi_{\beta=0}(t) = \left\{ \begin{array}{ll} \frac{1}{2} t^{1,1} (t^{0,0})^2 + t^{0,0} t^{1,0} + \frac{1}{18} t^{1,1} (t^{0,1})^3 & \text{if } n \geq 1, \\
0 & \text{if } n = 0. \end{array} \right. \tag{32} \]

Theorem 5.9 implies that
\[ \Psi(t) = \sum_{\beta \geq 1} \sum_{n_1, n_2 \geq 0} q_0 q_1 \frac{n_1! n_2! (t^{1,1})^{6 \beta + 2 n_1 - 5}}{\eta(\tau_{0,1} \tau_{1,1} \tau_{2,2})}. \tag{33} \]

Furthermore, Theorem 5.4 implies that the potential function must satisfy the WDVV equation
\[ \frac{\partial^3 \chi(t)}{\partial \alpha_1 \partial \alpha_2 \partial \alpha_3} \eta^{(\alpha_+, m_+), (\alpha_-, m_-)} \frac{\partial^3 \chi(t)}{\partial \alpha_+ \partial \alpha_+ \partial \alpha_+} = \frac{\partial^3 \chi(t)}{\partial \alpha_+ \partial \alpha_+ \partial \alpha_+}, \tag{34} \]

for all \( m_1, \alpha_i = 0, 1 \) and \( i = 1, \ldots, 4 \), and where the summation convention has been used.

Setting \( (\alpha_1, m_1) = (1, 0) \), \( (\alpha_2, m_2) = (0, 1) \), and \( (\alpha_3, m_3) = (\alpha_4, m_4) = (1, 1) \) in the WDVV equation and plugging in equation (32), we obtain
\[ \partial_{1,1}^3 \Psi = - \partial_{0,1}^3 \partial_{1,0} \Psi \partial_{1,1} \partial_{1,1}^2 \Psi - \partial_{1,0}^3 \Psi \partial_{0,1} \partial_{1,1}^2 \Psi + \frac{1}{3} \partial_{0,1}^3 \partial_{1,1} \Psi + \partial_{0,1}^2 \partial_{1,1} \Psi \partial_{1,1}^2 + (\partial_{0,1} \partial_{1,0} \partial_{1,1} \Psi)^2, \]

where we have used the shorthand notation
\[ \partial_{\alpha, m}^n = \left( \frac{\partial}{\partial \alpha, m} \right)^n. \]

Together with the Divisor Axiom in Theorem 5.9, we obtain the recursion relations for \( \beta = 1 \) correlators
\[ \langle \tau_{0,1} \tau_{1,1}^3 \rangle_1 = \frac{1}{3} \langle \tau_{1,0}^2 \tau_{1,1} \rangle_1, \]
and, for all \( n_1 \geq 2 \),
\[ \langle \tau_{0,1} \tau_{1,1}^{2 n_1 + 1} \rangle_1 = \frac{n_1}{3} \langle \tau_{0,1}^{n_1 - 1} \tau_{1,1}^{2 n_1 - 1} \rangle_1. \]

These collectively imply that for all \( n_1 \geq 1 \),
\[ \langle \tau_{0,1} \tau_{1,1}^{2 n_1 + 1} \rangle_1 = \frac{n_1}{3 n_1} \langle \tau_{1,0}^2 \tau_{1,1} \rangle_1. \]
Furthermore, the tensor product property implies that \( \langle \tau_{1,0}^2 \rangle_{1,1} = 1 \).

Together with the Divisor Axiom, this determines all of the \( \beta = 1 \) correlators.

If \( \beta \geq 2 \) then we obtain the following recursion relation for all \( n_1 \geq 0 \):

\[
\langle x_0^{n_1} \beta + 2n_1 - 8 \rangle_{1,1} \tau_{1,1} = \frac{n_1 \beta^2}{3} \langle x_0^{n_1+1} \beta + 2n_1 - 7 \rangle_{1,1} \tau_{1,1} + \sum (-\beta' \beta'') \langle n_1' \rangle_{1,1} \tau_{1,1} \tau_{1,1} \langle n_1'' \rangle_{1,1} \tau_{1,1} \langle n_1''' \rangle_{1,1} \tau_{1,1} \tau_{1,1} \langle n_1''' \rangle_{1,1} \tau_{1,1} \tau_{1,1} \langle n_1''' \rangle_{1,1} \tau_{1,1}
\]

where the first summation is over \( \beta', \beta'' \geq 1 \) such that \( \beta = \beta' + \beta'' \), and over \( n_1', n_1'' \geq 0 \) such that \( n_1 = n_1' + n_1'' \). Furthermore, we have defined

\[
\langle x_0^{n_1} \beta + 2n_1 - 7 \rangle_{1,1} = 0.
\]

Together with the Divisor Axiom, these recursion relations completely determine all of the \( n \)-point correlators of the theory where \( n \geq 3 \).

Finally, the 0, 1 and 2 point correlators (those in the unstable range) are determined as a special case of Theorem 5.12. The only nonvanishing correlators of these types are

\[
\langle \tau_{1,1} \rangle_1 = \langle \tau_{1,0} \tau_{1,1} \rangle_1 = 1.
\]

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