GLOBAL ATTRACTOR FOR A NONLINEAR SCHRÖDINGER EQUATION WITH A NONLINEARITY CONCENTRATED IN ONE POINT

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ABSTRACT. We consider the nonlinear Schrödinger equation in dimension one with a nonlinearity concentrated in one point. We prove that this equation provides an infinite dimensional dynamical system. We also study the asymptotic behavior of the dynamics. We prove the existence of a global attractor for the system.

1. Introduction. The focusing nonlinear cubic Schrödinger equation in dimension one with a nonlinearity concentrated in the space origin reads

\[ iu_t + u_{xx} + \delta_0 |u|^2 u = 0, \]

(1)

where the unknown \( u = u(t, x) \) is a complex valued function defined on \( \mathbb{R} \times \mathbb{R} \) and \( \delta_0 \) is the Dirac measure at the origin. This equation was derived as a result of the study of the resonant tunnelling, or in general, the interaction with the impurity in semi-conductors see [4], [9], [14] and [17]. Such an equation exhibits interesting quantitative properties which have been discussed in the physical literature in [9], [14] and [4]. Some rigorous mathematical results are given in [1] and also in [8] where we find that the equation (1) shares many properties with those established for the classical focusing nonlinear Schrödinger equation, without impurity.

The equation (1) is a conservative equation. In some physical contexts, an external forcing term and some damping effects have to be taken into account. This leads to the following weakly damped nonlinear Schrödinger equation with a nonlinearity concentrated in the origin

\[ iu_t + u_{xx} + \delta_0 |u|^2 u + i\gamma u = f, \]

(2)

where \( \gamma > 0 \) denotes the damping parameter and \( f \) is the external forcing term. We assume in the sequel that \( f \) is independent of time and that \( f \) belongs to \( L^2(\mathbb{R}) \). We supplement (2) with the initial condition

\[ u(0, \cdot) = u_0 \in H^1(\mathbb{R}). \]

(3)

In this paper, we plan to study the dynamical system provided by the equation (2) into the infinite dimensional dynamical system framework. We are intended to study the asymptotic behavior, for large enough time, of the equation (2) via the notion of the global attractor. We can refer the reader to the works [21], [18] and

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[15] for details. We begin with recalling some classical results. Consider first the case while the system does not feature any impurity. In this case the equation reads
\[ iu_t + u_{xx} + |u|^2 u + i\gamma u = f. \] (4)
This equation is a weakly damped forced version (focusing case) of the conservative nonlinear Schrödinger equation
\[ iu_t + u_{xx} + |u|^2 u = 0, \] (5)
which has been extensively studied as a fundamental equation in modern mathematical physics [20]. When dissipation and external forcing terms are considered, numerous results appear in the infinite dimensional dynamical system literature. Let us provide an overview of this literature. For unbounded domains, while the space variable \( x \) belongs to the whole line \( \mathbb{R} \), P. Laurençot studied the long time behavior of solutions to subcritical focusing nonlinear Schrödinger equations. The author proved for instance in [12] the existence of a global attractor in the energy space \( H^1(\mathbb{R}) \). This result was improved by N. Akroune in [2] where the author proved that this global attractor is in fact a compact subset of \( H^2(\mathbb{R}) \) following the method introduced by O. Goubet in [7].

In the case while the systems are perturbed by point interactions that can be interpreted as localized defects interacting with the nonlinear Schrödinger equation fields or concentrated nonlinearities, the first feature the nonlinear Schrödinger equation with a point defect in dimension 1 reads
\[ iu_t + u_{xx} + |u|^2 u + \delta_0 u = 0. \] (6)
This equation was studied in [6], where results on local and global existence of the solution are given and conservation laws and conditions on the blow-up solutions are established.

In the presence of dissipation and external forcing term the equation (6) reads
\[ iu_t + u_{xx} + \delta_0 u + |u|^2 u + i\gamma u = f. \] (7)
For the case where the space variable \( x \) belongs to the whole line, W. Kechiche studied in [11] the dynamics of the equation, defined a dynamical system in the energy space \( H^1(\mathbb{R}) \) and also proved the existence of a global attractor in \( H^1(\mathbb{R}) \) which describes the asymptotic behavior of the solutions. Concerning the regularity of such attractor, the author proved in [10] that when the space variable \( x \in \mathbb{R} \) the global attractor is a compact subset of \( H^2(\mathbb{R}) \) following the method introduced by O. Goubet in [7].

In the present article, our concern is that the perturbed system is forced and dissipative. Hence, we plan to study the equation (2) as a dissipative dynamical system. Here we are in the framework of global attractor for dissipative equations. First we study the Cauchy problem for the equation (2). A technical difficulty arises due to the fact that the nonlinearity does not belong to the energy space \( H^1(\mathbb{R}) \) but belongs to \( H^{-1}(\mathbb{R}) \); to overcome it, we use a regularization method described by T. Cazenave in [5]. Then, we define a dynamical system and study the long time
behavior of the solutions. We prove that the asymptotic behavior of the solutions is described by a global attractor in $H^1(\mathbb{R})$.

Our main results state in the following theorems

**Theorem 1.1.** Assume that $\frac{1}{4}\|f\|_{L^2(\mathbb{R})} \leq \frac{1}{2}$. Set

$$E = \left\{ u \in H^1(\mathbb{R}); \|u\|_{L^2(\mathbb{R})} \leq \frac{1}{2} \right\}.$$ 

Then, an initial data $u_0 \in E$ provides a global solution $u(t)$ of the problem (2)-(3) that belongs to $C_b(\mathbb{R}_+; E)$.

Moreover, the mapping $u_0 \mapsto u(t)$ is continuous for $H^1(\mathbb{R})$ strong topology. We define a dynamical system $(E, (S(t))_{t \geq 0})$ by

$$S(t) : u_0 \mapsto S(t)u_0 = u(t) \quad (8)$$

that is dissipative.

To prove these statements, we begin by using an approximation of the nonlinearity $\delta_0|u|^2u$ following the route in [5]: we establish the existence and uniqueness of local solution $u^\epsilon$ to an approximate problem that we will define in the sequel. We obtain uniform estimates with respect to $\epsilon$ to pass to the limit to construct a solution to the initial problem (2)-(3). We then prove that this solution is in fact unique. Then, under an additional condition on the initial data, we prove that the solution is global in time. We then define a dynamical system and we prove the existence of a bounded set to get the dissipativity of the semi-group.

We can state our main result

**Theorem 1.2.** The semi-group $(S(t))_{t \geq 0}$ associated with (2)-(3) possesses a compact global attractor $\mathcal{A}$ in the energy space $H^1(\mathbb{R})$.

We now outline this article. In Section 2 we give mathematical framework and notations. In Section 3 we study the existence and uniqueness of the solution to the problem (2)-(3) and we prove Theorem 1.1. In the last section we prove Theorem 1.2.

In the sequel $C$ denotes a numerical constant that may vary from one line to another. We also denote $K$ as a constant that depends on the data of the equation like $\gamma$, $f$ for instance. We allow $K$ to vary from one line to another in the computations.

2. **Mathematical framework and notation.** The $L^2(\mathbb{R})$ scalar product reads

$$(u, v)_{L^2(\mathbb{R})} = \Re \int_{\mathbb{R}} u(x)\bar{v}(x) \, dx.$$ 

The Sobolev space $H^1(\mathbb{R})$ is defined by

$$H^1(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}); \int_{\mathbb{R}} |\xi|^2|\hat{u}|^2 \, d\xi < +\infty \right\}.$$ 

We now recall that $\delta_0$, the Dirac mass at the origin, is defined by

For $u, v \in H^1(\mathbb{R})$, $(u\delta_0, v) = \Re \int_{\mathbb{R}} u(x)\bar{v}(x) \, d\delta_0 = \Re \left( u(0)\bar{v}(0) \right)$

and its norm in $H^{-1}$ is

$$\|\delta_0\|_{H^{-1}} = \sup_{\|u\|_{H^1}=1} |(\delta_0, u)_{H^{-1}, H^1}| = \sup_{\|u\|_{H^1}=1} |u(0)|.$$
The set $C_c^\infty(\mathbb{R})$ is the space of compactly supported $C^\infty$ functions, while $L^\infty(0, T; H^1(\mathbb{R}))$ denotes the space of essentially bounded functions in the $t$ variable with value in $H^1(\mathbb{R})$. The set $C_0(0, T; H^1(\mathbb{R}))$ denoted the closed subspace of $L^\infty(0, T; H^1(\mathbb{R}))$ consisting of functions that are moreover continuous.

We now state the sharp Gagliardo-Nirenberg inequality. We can guide the reader to [8] for the proof.

**Lemma 2.1.** For any $u \in H^1(\mathbb{R})$

$$|u(0)|^2 \leq \|u\|_{L^2(\mathbb{R})} \|u_x\|_{L^2(\mathbb{R})}. \quad (9)$$

Moreover, the equality is achieved if and only if there exists $\theta \in \mathbb{R}$, $\alpha > 0$ and $\beta > 0$ such that

$$u(x) = e^{\theta \alpha \sqrt{2e}^{-\beta|x|}}.$$

Here we state and prove a compactness result that is an other version of the Ascoli-Theorem.

**Lemma 2.2.** Let $(u_j)_j$ a sequence such that $(u_j)_j$ is bounded in $C([0, T]; H^1(\mathbb{R}))$ and $(\frac{du_j}{dt})_j$ is bounded in $C([0, T]; H^{-1}(\mathbb{R}))$.

Then, $(u_j)_j$ is relatively compact in $C([0, T]; L^2_{loc}(\mathbb{R}))$.

**Proof of Lemma 2.2.** The sequence $u_j$ satisfies

$$u_j \in C([0, T]; H^1(\mathbb{R})) \quad (10)$$

and

$$\frac{du_j}{dt} \text{ is bounded in } C([0, T]; H^{-1}(\mathbb{R})). \quad (11)$$

From (10) we infer that there exists some functions $u$ in $L^\infty(0, T; H^1(\mathbb{R}))$, and some subsequence still denoted by $u_j$ such that

$$u_j \rightharpoonup u \text{ weakly star in } L^\infty(0, T; H^1(\mathbb{R})). \quad (12)$$

On the one hand, from (11) we find that for every test function $w$ in $H^1(\mathbb{R})$ and every $t$ and $t + \tau$ in $[0, T]$,

$$\left(u_j(t + \tau) - u_n(t)\right)_{L^2(\mathbb{R})} = \int_t^{t+\tau} \langle u'_j(s), w \rangle_{H^{-1}(\mathbb{R})} \, ds \leq \tau \|u'_j\|_{L^\infty(0,T;H^{-1}(\mathbb{R}))} \|w\|_{H^1(\mathbb{R})} \quad (13)$$

$$\leq C \tau \|w\|_{H^1(\mathbb{R})},$$

where $C$ is a constant independant of $j$. For fixed $t$ and $\tau$, taking $w = u_j(t + \tau) - u_j(t)$, we obtain

$$\|u_j(t + \tau) - u_j(t)\|_{L^2(\mathbb{R})}^2 \leq 2\tau \|u'_j\|_{L^\infty(0,T;H^{-1}(\mathbb{R}))} \|u_j\|_{L^\infty(0,T;H^1(\mathbb{R}))} \leq C \tau \quad (14)$$

On the other hand we consider a smooth cut off function $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(s) = 1$, if $|s| \leq 1$ and $\phi(s) = 0$, if $|s| \geq 2$. For each $r > 0$, let us set $\phi_r(s) = \phi\left(\frac{s}{r}\right)$. Then, $\phi_r u_j$ belongs to $H^1_0(-r, 2r)$ and from (10) and (14) we see that $(\phi_r u_j)_j$ is equibounded and equicontinuous in $C([0, T]; L^2(-r, r))$ for every $r > 0$, with $(\phi_r u_j)_{(t)_j}$ relatively compact in $C([0, T]; L^2(-2r, 2r))$. By a diagonalization process, and passing to a further subsequence still denoted $u_j$, we find

$$u_j \longrightarrow u \quad \text{strongly in } C([0, T]; L^2(-r, r)), \quad \forall \ r > 0 \quad (15)$$
and the proof is completed.

3. Initial value problem. In this section we focus on the initial value problem related to the problem (2)-(3).

3.1. Existence and uniqueness of local in time solution. Here we will state and prove the following result

**Proposition 1.** For any \( u_0 \in H^1(\mathbb{R}) \), there exists a time \( T > 0 \) and a unique solution \( u \) of (2)-(3) in \( C([0, T]; H^1(\mathbb{R})) \) \( \cap C^1([0, T]; H^{-1}(\mathbb{R})) \) satisfying the following energy equations

\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_{L^2(\mathbb{R})}^2 + \gamma \| u(t) \|_{L^2(\mathbb{R})}^2 = 3m \int_{\mathbb{R}} f \bar{u} \, dx,
\]

\[
\frac{1}{2} \frac{d}{dt} \| u_x(t) \|_{L^2(\mathbb{R})}^2 - \frac{1}{4} \frac{d}{dt} |u(0)|^4 + \frac{d}{dt} \Re \int_{\mathbb{R}} f \bar{u} \, dx + \gamma \| u_x(t) \|_{L^2(\mathbb{R})}^2 - \gamma |u(0)|^4 + \gamma \Re \int_{\mathbb{R}} f \bar{u} \, dx = 0.
\]

Moreover, the mapping \( u_0 \mapsto u(t) \) is continuous in \( H^1(\mathbb{R}) \).

**Proof of Proposition 1.** For the existence result, the difficulty comes from the fact that the nonlinearity term \( \delta_0 |u|^2 u \) is in \( H^{-1}(\mathbb{R}) \). To overcome this difficulty we shall use the regularization method described in Section 3.3 of [5].

Setting \( g(u) = \delta_0 |u|^2 u \) the equation (2) reads in its abstract form

\[ iu_t = -\Delta u - g(u) - i\gamma u + f. \]  

We first approximate the nonlinearity \( g \) by nicer nonlinearities for which we will point arguments to construct approximate solutions by applying a fixed point Theorem. Next, we establish uniform estimates on the approximate solution. Finally, we use these estimates to pass to the limit in the approximate equation.

**Step 1.** Regularizing the nonlinearity.

We introduce for \( \epsilon > 0 \) the operator \( J_\epsilon = (Id - \epsilon \Delta)^{-1} \). We note that the self-adjoint operator \( J_\epsilon \) satisfies the properties

\[ J_\epsilon : H^{-s}(\mathbb{R}) \rightarrow H^{-s+2}(\mathbb{R}), \quad \forall s \geq 0 \]

\[ \| J_\epsilon \|_{L^2} \leq 1, \]  

\[ \| J_\epsilon \|_{H^s} \leq 1, \]  

and

\[ \| J_\epsilon \|_{H^{-1}, H^1} \leq \frac{1}{\epsilon}. \]

We define \( g_\epsilon(u) = J_\epsilon \left( |J_\epsilon u|^2 J_\epsilon u \delta_0 \right) \) for every \( u \in H^1(\mathbb{R}) \).

The abstract regularized form of (2) reads

\[ iu_t = -\Delta u - g_\epsilon(u) - i\gamma u + f. \]  

We state and prove

**Lemma 3.1.** \( g_\epsilon \) is a continuous and a locally Lipschitz map from \( H^1(\mathbb{R}) \) into \( H^1(\mathbb{R}) \).
Proof of Lemma 3.1. Let $u$ and $v$ in $H^1(\mathbb{R})$.
\[
\|g_e(u) - g_e(v)\|_{H^1} \leq \frac{1}{\epsilon} \|g(J_e u) - g(J_e v)\|_{H^{-1}(\mathbb{R})} \\
\leq \frac{1}{\epsilon} \|\delta_0\|_{H^{-1}(\mathbb{R})} \|J_e u\|^2 J_e(u - v) \|J_e v\|^2 \|H^1(\mathbb{R})\| \\
\leq \frac{C}{\epsilon} \|\delta_0\|_{H^{-1}(\mathbb{R})} \left(\|u\|_{H^1(\mathbb{R})}^2 + \|v\|_{H^1(\mathbb{R})}^2\right) \|u - v\|_{H^1(\mathbb{R})}.
\]

\[\square\]

**Step 2.** Construction of an approximate solution

Let $T > 0$. We define
\[
X_{T_e} = C([0, T], H^1(\mathbb{R})) \quad \text{and} \quad \|u\|_{X_{T_e}} = \sup_{t \in [0, T]} \|u(t)\|_{H^1(\mathbb{R})} \quad (25)
\]
a norm on this functional space.

Here, we will perform a solution for the Duhamel’s form of the equation (2) that reads
\[
u'(t) = e^{it \Delta} u_0 + i \int_0^t e^{i(t-s) \Delta} g_e(u'(s)) \, ds - \gamma \int_0^t e^{i(t-s) \Delta} u'(s) \, ds \\
- i \int_0^t e^{i(t-s) \Delta} J_e f \, ds. \quad (26)
\]

We denote by $G_e$ the functional defined by
\[
G_e(u) = e^{it \Delta} u_0 + \int_0^t e^{i(t-s) \Delta} \left(ig_e(u(s)) - \gamma u(s) - iJ_e f\right) \, ds. \quad (27)
\]

**Lemma 3.2.** $G_e$ sends the closed ball $B_{X_{T_e}}(0, R)$ into itself.

**Proof of Lemma 3.2.** Let $R = 2 \left(\|u_0\|_{H^1(\mathbb{R})} + 2C\|f\|_{L^2(\mathbb{R})}\right)$. Let $u \in X_{T_e}$ such that $\|u\|_{X_{T_e}} \leq R$.

\[
\|G_e(u)\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} + i \int_0^t e^{i(t-s) \Delta} \|g_e(u(s))\|_{H^1(\mathbb{R})} \, ds \\
\quad + \gamma \int_0^t e^{i(t-s) \Delta} \|u(s)\|_{H^1(\mathbb{R})} \, ds \\
\quad + i \int_0^t e^{i(t-s) \Delta} \|J_e f\|_{H^1(\mathbb{R})} \, ds. \quad (28)
\]

First, we have
\[
i \int_0^t e^{i(t-s) \Delta} \|g_e(u(s))\|_{H^1(\mathbb{R})} \, ds \leq \frac{1}{\epsilon} \|\delta_0\|_{H^{-1}(\mathbb{R})} \int_0^t \|J_e u\|^2 J_e \, ds \\
\leq \frac{2C}{\epsilon} \|\delta_0\|_{H^{-1}(\mathbb{R})} R^3 \varepsilon. \quad (29)
\]

Second,
\[
\gamma \int_0^t e^{i(t-s) \Delta} \|u(s)\|_{H^1(\mathbb{R})} \, ds \leq \gamma RT_e. \quad (30)
\]
We then get
\[ i \int_0^t e^{i(t-s)\Delta} J_x f \, ds = i(-\Delta)^{-1}(\text{Id} - e^{i\Delta})f(x) \quad (31) \]

We then get
\[ \|i \int_0^t e^{i(t-s)\Delta} J_x f \, ds\|_{L^2(\mathbb{R})} \leq 2\|(-\Delta)^{-1}f\|_{L^2(\mathbb{R})} \leq 2C\|f\|_{L^2(\mathbb{R})}. \quad (32) \]

Hence,
\[ \|G(\epsilon)\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} + 2C\|f\|_{L^2(\mathbb{R})} + \left(\gamma + \frac{\gamma}{2CR^2\|\delta_0\|_{H^{-1}(\mathbb{R})}}\right)RT_\epsilon. \quad (33) \]

We choose \( T_\epsilon \) such that
\[ T_\epsilon = \frac{1}{2\left(\frac{\gamma}{\gamma + 2CR^2\|\delta_0\|_{H^{-1}(\mathbb{R})}}\right)}. \quad (34) \]

Then
\[ \|G(\epsilon)(u)\|_{X_{T_\epsilon}} \leq R. \quad (35) \]

**Lemma 3.3.** \( G(\epsilon) \) is a strictly contraction on the closed ball \( B_{X_{T_\epsilon}}(0, R) \).

**Proof of Lemma 3.3.** Let \( u, v \in B_{X_{T_\epsilon}}(0, R) \) such that \( \|u\|_{H^1(\mathbb{R})} \leq R, \|v\|_{H^1(\mathbb{R})} \leq R. \) Let \( t < T_\epsilon \), where \( T_\epsilon \) is as in (34).

Observing that
\[ \|G(\epsilon)(u) - G(\epsilon)(v)\|_{H^1(\mathbb{R})} \leq \int_0^t \|g(\epsilon)(u) - g(\epsilon)(v)\|_{H^1} \, ds + \gamma \int_0^t e^{i(t-s)\Delta}(u - v) \, ds\|_{H^1} \]
\[ \leq \int_0^t\frac{C\|\delta_0\|_{H^{-1}(\mathbb{R})}}{\epsilon}\left(\|u\|^2_{H^1} + \|v\|^2_{H^1}\right)\|u - v\|_{H^1} \, ds + \gamma T_\epsilon\|u - v\|_{X_{T_\epsilon}}, \]

we then obtain
\[ \|G(\epsilon)(u) - G(\epsilon)(v)\|_{H^1(\mathbb{R})} \leq \frac{2C\|\delta_0\|_{H^{-1}(\mathbb{R})}}{\epsilon} R^2T_\epsilon\|u - v\|_{X_{T_\epsilon}} + \gamma T_\epsilon\|u - v\|_{X_{T_\epsilon}}. \quad (36) \]

Using (34) we then get
\[ \|G(\epsilon)(u) - G(\epsilon)(v)\|_{H^1(\mathbb{R})} < \frac{1}{2}\|u - v\|_{X_{T_\epsilon}}. \quad (37) \]

Now, we can apply the Banach fixed point Theorem to prove the existence of a unique function \( u^\epsilon \) such that \( G(\epsilon)(u^\epsilon) = u^\epsilon \). Hence, we notice the existence of a unique solution \( u^\epsilon \) continuous on \([0, T_\epsilon]\) into \( B_{X_{T_\epsilon}}(0, R) \).

**Step 3.** A priori estimates on the sequence \( u^\epsilon \).

We will first prove that there exists a time \( T > 0 \), independent of \( \epsilon \), such that the unique solution \( u^\epsilon \) is in \( C([0, T]; H^1(\mathbb{R})) \cap C^1([0, T]; H^{-1}(\mathbb{R})) \).

Let \( T_\epsilon \) defined by
\[ T_\epsilon = \inf\left\{ t > 0; \|u^\epsilon(t)\|_{H^1(\mathbb{R})} > 2\|u_0\|_{H^1(\mathbb{R})} + \frac{2}{\gamma}\|f\|_{L^2(\mathbb{R})} \right\}. \quad (38) \]
Let $t < T\epsilon$ we have
\[
\|u'\|_{H^1(\mathbb{R})} \leq 2\|u_0\|_{H^1(\mathbb{R})} + \frac{2}{\gamma}\|f\|_{L^2(\mathbb{R})}.
\] (39)

Using (39) and the equation (24) we see that
\[
\text{for } t \leq T\epsilon, \quad \|u'_t\|_{H^{-1}} \leq k,
\]
where $k$ is a constant that depends on $\|u_0\|_{H^1(\mathbb{R})}$, $\gamma$ and $\|f\|_{L^2(\mathbb{R})}$.

We then infer from (38), (40) and the fact $\|u\|_{L^2(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})}\|u\|_{H^{-1}(\mathbb{R})}$ that
\[
\|u'(t) - u_0\|_{L^2(\mathbb{R})} \leq k t^\frac{1}{2}.
\]
\[\text{Therefore}\]
\[
\|u'(t)\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})} + \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R})} + k t^\frac{1}{2}.
\]
\[\text{Choosing } t = T\text{ such that}\]
\[
k t^\frac{1}{2} = \|u_0\|_{H^1(\mathbb{R})} + \frac{1}{\gamma}\|f\|_{L^2(\mathbb{R})}
\]
we then get $T \leq T\epsilon$.

Hence, there exists $T > 0$, independent of $\epsilon$, and a unique solution $u'$ of the equation (24) that is uniformly bounded in $C\left(\{0, T\}; H^1(\mathbb{R})\right) \cap C^1\left(\{0, T\}; H^{-1}(\mathbb{R})\right)$.

**Step 4.** Passing to the limit

The approximate solution $u'$ of the equation (24) satisfies
\[
u' \in C([0, T], H^1(\mathbb{R})) \cap C^1([0, T], H^{-1}(\mathbb{R})).
\]

Then, we have $u'$ converges to $u$ in $L^\infty([0, T]; H^1(\mathbb{R}))$ weak-star and $u'_t$ converges to $u_t$ in $L^\infty([0, T]; H^{-1}(\mathbb{R}))$ weak-star. By the compactness result given in Lemma 2.2 we find $u'$ converges strongly to $u$ in $L^2_{loc}(\mathbb{R})$ and then $g(u') - g(u)$ converges strongly to 0 in $L^2_{loc}(\mathbb{R})$ and we have $g_u(u)$ converges strongly to $g(u)$ in $H^{-1}(\mathbb{R})$
\[
g_u(w) - g(w)\|_{H^{-1}} = \|g_u(w) - J_\epsilon g(w) + J_\epsilon g(w) - g(w)\|_{H^{-1}} \leq \|
\]
\[
\|J_\epsilon(j w)\|_{L^2_{loc}(\mathbb{R})}^2 J_\epsilon(w\delta_0 - |w|^2w\delta_0)\|_{H^{-1}} + \|J_\epsilon(w)\|_{L^2_{loc}(\mathbb{R})}^2 w\delta_0 - |w|^2w\delta_0\|_{H^{-1}} \leq (44)
\]
\[
\|J_\epsilon(w)\|_{L^2_{loc}(\mathbb{R})}^2 J_\epsilon(w\delta_0 - |w|^2w\delta_0)\|_{H^{-1}} + \|(J_\epsilon - Id)(|w|^2w\delta_0)\|_{H^{-1}}.
\]

We note that
\[
\|J_\epsilon(w)\|_{L^2_{loc}(\mathbb{R})}^2 J_\epsilon(w\delta_0 - |w|^2w\delta_0)\|_{H^{-1}} \leq \|J_\epsilon(w)\|_{L^2_{loc}(\mathbb{R})}^2 (J_\epsilon(w) - w)\delta_0\|_{H^{-1}} + \|w(J_\epsilon(w) - w)\delta_0\|_{H^{-1}} \leq K\|J_\epsilon - Id\|w\|_{H^{-1}},
\]
and
\[
\|(J_\epsilon - Id)(|w|^2w\delta_0)\|_{H^{-1}} \leq K\|(J_\epsilon - Id)\|w\|_{H^{-1}}.
\]
We then deduce the existence of a solution $u$ to the equation (2).

Moreover, this solution $u$ satisfies the energy equations (16) and (17). The proof is standard and is not detailed here for the sake of conciseness.

**Step 5.** Uniqueness and continuous dependence with respect to the initial data

We shall first prove the uniqueness of the solution of the problem (2)-(3).
Let $u$ and $v$ two solutions to the integral form of the equation (2) with respectively initial data $u_0$ and $v_0$.

\[
u(t) - v(t) = e^{-\gamma t} e^{it\Delta} (u_0 - v_0) + i \int_0^t e^{-\gamma(t-s)} e^{i(t-s)\Delta} (|u|^2 u \delta_0 - |v|^2 v \delta_0) \, ds. \tag{45}\]

Using that
\[
\forall \phi \in H^1(\mathbb{R}), \quad e^{it\Delta} \phi = K_t * \phi
\]
where the expression of the Schrödinger kernel $K_t$, (see [5]), is given by
\[
K_t(x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} e^{\frac{-|x|^2}{4it}}, \quad \text{for all } t \neq 0 \text{ and for all } x \in \mathbb{R},
\]
we then get
\[
\|u(t) - v(t)\|_{L^\infty} \leq \|u_0 - v_0\|_{H^1} + \int_0^t \frac{C}{(t-s)^{\frac{d}{2}}} \left( \|u\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 \right) \|u - v\|_{L^\infty} \, ds. \tag{48}\]

Since for every $t \in [0, T]$ the solutions $u$ and $v$ are bounded in the space $X_T = C([0, T], H^1(\mathbb{R}))$ we then infer that
\[
\|u(t) - v(t)\|_{L^\infty} \leq \|u_0 - v_0\|_{H^1} + \sup_{t \in [0, T]} \left( \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \right) \int_0^t \frac{C}{(t-s)^{\frac{d}{2}}} \|u - v\|_{L^\infty} \, ds. \tag{49}\]

Taking $u_0 = v_0$ and using the Gronwall Lemma we get the uniqueness of the solution $u$ to the problem (2)-(3).

Hence, we have obtained a local in time existence and uniqueness for solutions $u(t)$ of (2)-(3) which take values in $H^1(\mathbb{R})$. Moreover, a maximal solution $u(t)$ in $C(0, T_{\text{max}}; H^1(\mathbb{R}))$ satisfies either $T_{\text{max}} = +\infty$ or $\lim_{t \to T_{\text{max}}} \|u(t)\|_{H^1(\mathbb{R})} = \infty$.

We shall give here an overview on the proof of the continuous dependence of the solution with respect to the initial data. We consider $u_0^\alpha$ such that
\[
u_0^\alpha \longrightarrow u_0 \quad \text{strongly in } H^1(\mathbb{R})
\]
and consider $u^\alpha$ and $u$ two solutions of the equation (2) with initial data respectively $u_0^\alpha$ and $u_0$.

On the one hand, using (49) we have for every $T < T_{\text{max}}$
\[
u^\alpha \longrightarrow u_0 \quad \text{strongly in } C(0, T; L^\infty(\mathbb{R})). \tag{50}\]

We then infer from this that for $t \in [0, T]$
\[
u^\alpha(t) \rightharpoonup u(t) \quad \text{weakly in } H^1(\mathbb{R}). \tag{51}\]

On the other hand, using that $u^\alpha$ remains bounded in the space $C(0, T; H^1(\mathbb{R})) \cap C^1(0, T; H^{-1}(\mathbb{R}))$ and the compacity result given in Lemma 2.2 we then obtain the strong convergence
\[
u^\alpha(t) \longrightarrow u(t) \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}). \tag{52}\]

Gathering the weak convergence (51) and the strong convergence (52) and using energy equations (16) and (17) we can prove that
\[
\|u^\alpha(t)\|_{L^2(\mathbb{R})} \longrightarrow \|u(t)\|_{L^2(\mathbb{R})} \tag{53}\]
and
\[ \|u^\alpha(t)\|_{L^2(\mathbb{R})} \rightarrow \|u_\ast(t)\|_{L^2(\mathbb{R})} \] (54)
and then
\[ \|u^\alpha(t)\|_{H^1(\mathbb{R})} \rightarrow \|u(t)\|_{H^1(\mathbb{R})}. \] (55)
Combining the weak convergence (51) and the \( H^1(\mathbb{R}) \) norm convergence (55) we then get the convergence of \( u^\alpha(t) \) to \( u(t) \) strongly in \( H^1(\mathbb{R}) \) and then the continuous dependence of the semigroup is proved. This completes the proof of Proposition 1.

3.2. Global existence of the solution and dissipativity. In this section, we will define a closed subset \( E \) of \( H^1(\mathbb{R}) \) of initial data, which provides global in time solutions for (2)-(3). Moreover, this set is positively invariant by the mapping \( u_0 \mapsto u(t), \ t > 0 \). This allows us to define a semigroup \( S(t) : E \rightarrow E, \ S(t)u_0 = u(t) \) acting on the complete metric space \( E \) endowed with the \( H^1(\mathbb{R}) \) topology. Let us state a result, which moreover describes the dissipativity of \( S(t) \).

**Proposition 2.** Assume that \( \frac{1}{2}\|f\|_{L^2} \leq \frac{1}{2} \). Set
\[ E = \left\{ u \in H^1(\mathbb{R}); \|u\|_{L^2} \leq \frac{1}{2} \right\}. \] (56)
Then for an initial data \( u_0 \in E \) the solution \( u(t) \) of the problem (2)-(3) is global in time and belongs to \( C(\mathbb{R}^+; E) \). Moreover, there exists a ball \( B \) in \( H^1(\mathbb{R}) \) that absorbs any global solution for (2)-(3), the entrance time \( t_1 \) into \( B \) for \( u(t) \) depending only on \( \gamma, f \|u_0\|_{H^1} \). Therefore, \( B \cap E \) is an absorbing set for \( S(t) \).

**Proof of Proposition 2.** We multiply (2) by \( \bar{u} \) and integrate on \( \mathbb{R} \) the imaginary part of the resulting equation to obtain
\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R})}^2 + \gamma \|u(t)\|_{L^2(\mathbb{R})}^2 + \Im \int_{\mathbb{R}} f \bar{u} \, dx = 0. \] (57)
We infer from (57) that for \( t < T_{\text{max}} \),
\[ \|u(t)\|_{L^2(\mathbb{R})}^2 \leq \|u_0\|_{L^2(\mathbb{R})}^2 e^{-\gamma t} + \frac{\|f\|_{L^2(\mathbb{R})}^2}{\gamma^2} (1 - e^{-\gamma t}). \] (58)
Here, we observe that
\[ \|u(t)\|_{L^2(\mathbb{R})}^2 \leq \max \left( \|u_0\|_{L^2(\mathbb{R})}^2, \frac{\|f\|_{L^2(\mathbb{R})}^2}{\gamma^2} \right). \] (59)
Hence, the ball of radius \( \frac{\|f\|_{L^2}^2}{\gamma^2} \) in \( L^2(\mathbb{R}) \) is positively invariant by \( S(t) \) and absorbs global in time solutions of (2)-(3).

Next we multiply (2) by \( \bar{u} + \gamma \bar{u} \) and integrate the real part of the resulting identity on \( \mathbb{R} \) to obtain
\[ \frac{1}{2} \frac{d}{dt} \|u_\ast(t)\|_{L^2(\mathbb{R})}^2 - \frac{1}{4} \frac{d}{dt} |u(0)|^4 + \Re \int_{\mathbb{R}} f \bar{u} \, dx + \gamma \|u_\ast(t)\|_{L^2(\mathbb{R})}^2 - \gamma |u(0)|^4
+ \gamma \Re \int_{\mathbb{R}} f \bar{u} \, dx = 0. \] (60)
Then we deduce from (60) the following energy identity
\[ \frac{1}{2} \frac{d}{dt} \phi(u) + \gamma \phi(u) = \psi(u), \] (61)
where we have set
\[ \phi(u) = \|u_x(t)\|_{L^2(\mathbb{R})}^2 - \frac{1}{2} |u(0)|^4 + 2\Re \int_{\mathbb{R}} f \bar{u} \, dx \] (62)
and
\[ \psi(u) = \frac{\gamma}{2} |u(0)|^4 + \gamma \Re \int_{\mathbb{R}} f \bar{u} \, dx. \] (63)

To begin with, we observe that for initial data \( u_0 \) trapped in \( E \) and for forcing term \( f \) such that \( \|f\|_{L^2}^2 \leq \frac{1}{2} \), we infer from (59)
\[ \|u(t)\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{2}. \] (64)
So that, using the sharp Gagliardo-Nirenberg inequality (9) and the estimates (64), we then obtain the following lower bound for \( \phi(u) \)
\[ \phi(u) \geq \frac{3}{4} \|u_x(t)\|_{L^2(\mathbb{R})}^2 - K. \] (65)
The idea is then to provide an upper bound for the right hand side of (61). For this purpose, using (64), we obtain
\[ \frac{\gamma}{2} |u(0)|^4 \leq \frac{\gamma}{4} \|u_x(t)\|_{L^2(\mathbb{R})}^2, \]
and
\[ \gamma \Re \int_{\mathbb{R}} f \bar{u} \, dx \leq K. \]
Therefore, we have obtained
\[ \frac{1}{2} \frac{d}{dt} \phi(u(t)) + \gamma \phi(u(t)) \leq \frac{\gamma}{3} \phi(u(t)) + K, \] (66)
We now easily infer from (61)-(65) that
\[ \|u_x(t)\|_{L^2}^2 \leq \frac{4}{3} \phi(u) + K \leq \frac{4}{3} \phi(u_0) e^{-\frac{4}{3} t} + K, \] (67)
that leads to the global existence in \( H^1(\mathbb{R}) \) for \( u(t) \) which starts from \( u_0 \) in \( E \).
Moreover, since any global in time solution is absorbed by \( E \) (see (58) for \( t \geq \frac{1}{\gamma} \ln(\frac{1}{\gamma} \|u_0\|_{L^2}^2) = t_0 \)), we infer from the computations (61)-(67) that for \( t \geq t_0 \)
\[ \|u_x(t)\|_{L^2(\mathbb{R})} \leq \frac{4}{3} \phi(u_0) e^{-\frac{4}{3} t} + K \left( f, \frac{2\|f\|_{L^2}^2}{\gamma^2} \right), \] (68)
that leads to the existence of an absorbing set \( B_1 \) in \( H^1(\mathbb{R}) \). Hence, the dissipativity of the semigroup \( \left( S(t) \right)_{t \geq 0} \) is proved.

4. **Existence of the global attractor.** In this section, we prove Theorem 1.2. We proceed in two steps. We first prove the existence of a weak global attractor \( A \) in \( H^1(\mathbb{R}) \) equipped with the weak topology and then we prove that \( A \) is a global attractor for the strong topology in \( H^1(\mathbb{R}) \).
4.1. Global weak attractor. After a transient time, the dynamics reduce to the absorbing set \( \mathcal{B} = B_1 \cap E \). This set is a closed convex set in \( H^1(\mathbb{R}) \) and is then a compact for the weak topology of \( H^1(\mathbb{R}) \). To prove the existence of a global weak attractor defined as the omega limit set of \( \mathcal{B} \) we will prove the weak continuity of the semigroup \( \left( S(t) \right)_{t \geq 0} \) and a compacity result for the trajectory in \( L^2(\mathbb{R}) \).

**Lemma 4.1.** The solution operator \( \left( S(t) \right)_{t \geq 0} \) is weakly continuous in \( H^1(\mathbb{R}) \) in the sens that if \((u_0)_n\) converges weakly in \( H^1(\mathbb{R}) \) to some \( u_0 \), as \( n \to \infty \), then \( \left( S(t)u_0_n \right)_n \) converges to \( S(t)u_0 \) weakly in \( H^1(\mathbb{R}) \) for all \( t \in \mathbb{R}_+ \).

**Proof of Lemma 4.1.** We proceed as in [19]. Let \( u_{0n} \to u_0 \) weakly in \( H^1(\mathbb{R}) \). We fix \( T > 0 \) and consider \( u_n(t) = S(t)u_{0n} \) for \( 0 \leq t \leq T \). Since \((u_0)_n\) is bounded in \( H^1(\mathbb{R}) \), it follows from energy equations that

\[
\text{the sequence } u_n \text{ is bounded in } C(0,T; H^1(\mathbb{R})).
\]  

(69)

From the equation (2), we then see that

\[
\text{the sequence } \frac{du_n}{dt} \text{ is bounded in } C(0,T; H^{-1}(\mathbb{R})).
\]  

(70)

From (69), we infer that there exists some functions \( u \) in \( L^\infty(0,T; H^1(\mathbb{R})) \), and some subsequence still denoted by \( u_n \) such that

\[
u_n \rightharpoonup u \text{ weakly in } L^\infty(0,T; H^1(\mathbb{R})).
\]  

(71)

On the one hand, from (70) we find that for every test function \( w \) in \( H^1(\mathbb{R}) \) and every \( t \) and \( t + \tau \in [0,T] \),

\[
\left( u_n(t + \tau) - u_n(t) \right)_{L^2(\mathbb{R})} = \int_t^{t+\tau} \langle u_n'(s), w \rangle_{H^{-1}(\mathbb{R}) \times H^1(\mathbb{R})} ds
\]

\[
\leq \tau \| u_n' \|_{L^\infty(0,T; H^{-1}(\mathbb{R}))} \| w \|_{H^1(\mathbb{R})}
\]

\[
\leq C\tau \| w \|_{H^1(\mathbb{R})},
\]  

(72)

where \( C \) is a constant independant of \( n \).

For fixed \( t \) and \( \tau \), taking \( w = u_n(t + \tau) - u_n(t) \), we obtain

\[
\|u_n(t + \tau) - u_n(t)\|_{L^2(\mathbb{R})} \leq 2\tau \| u_n' \|_{L^\infty(0,T; H^{-1}(\mathbb{R}))} \| u_n \|_{L^\infty(0,T; H^1(\mathbb{R}))} \leq C\tau \]  

(73)

On the other hand we consider a smooth cut off function \( \phi \in C_0^\infty(\mathbb{R}) \) such that \( \phi(s) = 1 \), if \( |s| \leq 1 \) and \( \phi(s) = 0 \), if \( |s| \geq 2 \). For each \( r > 0 \), let us set \( \phi_r(s) = \phi(s/r) \). Then, \( \phi_r u_n \) belongs to \( H^1_0(-2r,2r) \) and from (69) and (73) we see that \( (\phi_r u_n)_n \) is equibounded and equicontinuous in \( C_0(0,T; L^2(\mathbb{R})) \) for every \( r > 0 \), with \( (\phi_r u_n(t))_n \) relatively compact in \( C_0(0,T; L^2(\mathbb{R})) \). By a diagonalization process, and passing to a further subsequence still denoted \( u_n \), we find

\[
u_n \rightharpoonup u \text{ strongly in } C(0,T; L^2(-r,r)), \quad \forall r > 0.
\]  

(74)

Gathering the weak-star convergence (71) and the strong convergence (74) it is now standard to pass to the limit in the equation (2) to deduce that the limit function \( u \) is a function of (2). It turns out that \( u \) must be the unique solution provided by Theorem 1.1 issued from \( u_0 \). Then, the whole sequence \( u_n \) converges.

It remains to show that \( u_n(t) \) converges weakly in \( H^1(\mathbb{R}) \) to \( u(t) \), for every fixed \( t \in [0,T] \). We know that the convergence is strong in \( L^2(-r,r) \), for every \( r > 0 \).
Thus, taking \( w \) in \( C_c^\infty(\mathbb{R}) \), we find that for every \( r > 0 \) large enough,
\[
\left( u_n(t), w \right)_{H^1(\mathbb{R})} = \left( u_n(t), w - w_{xx} \right)_{L^2(\mathbb{R})} \rightarrow \left( u(t), w - w_{xx} \right)_{L^2(\mathbb{R})} = \left( u(t), w \right)_{H^1(\mathbb{R})},
\]
(75)
Then, from (69) and the density of \( C_c^\infty(\mathbb{R}) \) in \( H^1(\mathbb{R}) \) we find that
\[
\left( u_n(t), w \right)_{H^1(\mathbb{R})} \rightarrow \left( u(t), w \right)_{H^1(\mathbb{R})}
\]
(76)
for every \( w \) in \( H^1(\mathbb{R}) \), and the proof is completed.

**Lemma 4.2.** Consider \( u_n \) that is bounded in \( H^1(\mathbb{R}) \) and \( t_n \rightarrow +\infty \) then the sequence \( S(t_n)u_n \) is relatively compact in \( L^2(\mathbb{R}) \).

**Proof of Lemma 4.2.** We proceed as in [13].
Consider a smooth function \( \theta : \mathbb{R} \rightarrow [0,1] \) such that \( \theta(\xi) = 1 \) if \( |\xi| \leq 1 \) and \( \theta(\xi) = 0 \) if \( |\xi| \geq 2 \). Introduce \( q_R = \int_{\mathbb{R}} |u|^2 (1 - \theta(\frac{\xi}{R})) \, dx \). Then multiplying the equation (2) by \( (1 - \theta(1 - \frac{x}{R})) \) and integrating on \( \mathbb{R} \) and taking the imaginary part of the resulting equation to obtain
\[
\frac{1}{2} \frac{d}{dt} q_R + \gamma q_R = 3m \int_{\mathbb{R}} (1 - \theta(\frac{x}{R})) f \bar{u} \, dx - 3m \int_{\mathbb{R}} (1 - \theta(\frac{x}{R})) \bar{u} u_{xx} \, dx.
\]
(77)
On the one hand
\[
-3m \int_{\mathbb{R}} (1 - \theta(\frac{x}{R})) \bar{u} u_{xx} \, dx = \frac{1}{R} 3m \int_{\mathbb{R}} u_x \theta'(\frac{x}{R}) \bar{u} \, dx.
\]
(78)
Assuming that the trajectory is trapped into the absorbing set \( B \) for \( t \geq 0 \) we have that
\[
| -3m \int_{\mathbb{R}} (1 - \theta(\frac{x}{R})) \bar{u} u_{xx} \, dx | \leq \frac{\| \theta' \|_{L^\infty}}{2R} \int_{R \leq |x| \leq 2R} (|u|^2 + |u_x|^2) \, dx \leq \frac{K}{R},
\]
(79)
On the other hand, by Cauchy-Schwarz inequality
\[
|3m \int_{\mathbb{R}} (1 - \theta(\frac{x}{R})) f \bar{u} \, dx| \leq \frac{\gamma}{2} q_R + \frac{1}{2\gamma} \int_{|x| > R} |f|^2 \, dx,
\]
(80)
and then
\[
\frac{d}{dt} q_R + \gamma q_R \leq \frac{1}{\gamma} \int_{|x| > R} |f|^2 \, dx + \frac{K}{R}.
\]
(81)
Therefore we can choose \( R \) large enough such that \( q_R(t) \leq \epsilon \) for \( t \geq T \).
On the other hand, the sequence \( u \sqrt{\theta(\frac{\xi}{R})} \) remains bounded in \( H_0^1(B(0,2R)) \) that is compactly embedded in \( L^2(\mathbb{R}) \). Hence, \( S(t_n)u_n \) is trapped into a compact set of \( L^2(\mathbb{R}) \) for \( t \geq T \). This completes the proof of the Lemma.

4.2. **Proof of the Theorem 1.2.** We have already proven the existence of a weak global attractor \( A \) in \( H^1(\mathbb{R}) \). We now prove that \( A \) is the global attractor for the strong topology in \( H^1(\mathbb{R}) \). For this purpose, we use the classical argument due to J. Ball (see [3] and [22], [16]) relying essentially on the reversibility in time of the nonlinear Schrödinger equation and an energy equality that reads
\[
\frac{d}{dt} \left[ \| (S(t)u_0)_x \|_{L^2(\mathbb{R})} + G(S(t)u_0) \right] + 2\gamma \left[ \| (S(t)u_0)_x \|_{L^2(\mathbb{R})} + G(S(t)u_0) \right] = H(S(t)u_0),
\]
(82)
where
\[ G(u(t)) = \frac{1}{2}|u(0)|^4 + 2\Re \int_{\mathbb{R}} f \bar{u} \, dx \]
and
\[ H(u(t)) = \gamma|u(0)|^4 + 2\gamma\Re \int_{\mathbb{R}} f \bar{u} \, dx \]

We first prove that \( \mathcal{A} \) attracts trajectories for the \( H^1(\mathbb{R}) \) strong topology. Consider \((u_n)_n\) a sequence in the absorbing set and consider a sequence \((t_n)_n\) that converges to \(+\infty\). We will show that there exists \( u \in H^1(\mathbb{R}) \) and a subsequence \((t_n)_n\) that converges to \( u \) in \( H^1(\mathbb{R}) \).

We know that
\[ S(t_n)u_n \rightarrow u \quad \text{weakly in } H^1(\mathbb{R}), \]  
(83)  
and
\[ S(t_n)u_n \rightarrow u \quad \text{strongly in } L^2(\mathbb{R}), \]  
(84)  
Moreover \( u \) belongs to \( \mathcal{A} \).

Using (83) we have
\[ \|u\|_{H^1(\mathbb{R})} \leq \liminf_{n \to +\infty} \|S(t_n)u_n\|_{H^1(\mathbb{R})} \]  
(85)  
For any \( T > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( \left( S(t_n - T)u_j \right)_{n \geq n_0} \subset \mathcal{B} \).

Hence,
\[ S(t_n - T)u_n \rightarrow y \quad \text{weakly in } H^1(\mathbb{R}). \]

Using the continuity of the semigroup in the bounded set of \( H^1(\mathbb{R}) \) for the \( L^2(\mathbb{R}) \) strong topology, we get
\[ S(t)S(t_n - T)u_n \rightarrow S(t)y \quad \text{strongly in } L^2(\mathbb{R}) \]  
(86)  
Taking \( u_0 = u_m \) in (82) that we integrate between \( t_n - T \) and \( t_n \) we obtain
\begin{align*}
\left( \|(S(t_n)u_n)_x\|_{L^2} + G(S(t_n)u_n) \right) e^{2\gamma t_n} \\
- \left( \|(S(t_n - T)u_n)_x\|_{L^2} + G(S(t_n - T)u_n) \right) e^{2\gamma(t_n - T)} \\
= \int_{t_n - T}^{t_n} e^{2\gamma t} H(S(t)u_n) \, dt
\end{align*}  
(87)  
With a variable change the equality (87) reads
\[ \left\| (S(t_n)u_n)_x \right\|_{L^2} + G(S(t_n)u_n) = \left\| (S(t_n - T)u_n)_x \right\|_{L^2} + \left\| S(t_n - T)u_n \right\|_{L^2} e^{-2\gamma T} + \int_0^T e^{2\gamma(T - s)} H(S(s + t_n - T)u_n) \, ds \]  
(88)  
On the one hand, due to the Gagliardo-Nirenberg inequality and essentially to the strong convergence
\[ S(t_n - T)u_n \rightarrow y \quad \text{strongly in } L^2(\mathbb{R}), \]  
(89)  
\[ \lim_{n \to +\infty} G(S(t_n - T)u_n) = G(y). \]  
(90)  
Moreover, taking \( t = T \) in (86) leads to
\[ S(t_n)u_n \rightarrow S(T)y \quad \text{weakly in } H^1(\mathbb{R}) \]  
(91)  
and
\[ S(t_n)u_n \rightarrow S(T)y \quad \text{strongly in } L^2(\mathbb{R}). \]  
(92)
Therefore,

$$\lim_{n \to +\infty} G(S(t_n)u_n) = G(S(T)y).$$

(93)

On the other hand, due to the dominated convergence theorem of Lebesgue, we know that

$$\int_0^T e^{2\gamma(T-s)} H(S(s + t_n - T)u_n) \, ds \to_{n \to +\infty} \int_0^T e^{2\gamma(T-s)} H(S(s)y) \, ds$$

(94)

Applying once again the energy equality, we know that

$$G(y) + \int_0^T e^{2\gamma(T-s)} H(S(s)y) \, ds = \|(S(T)y)x\|_{L^2(\mathbb{R})}^2 + G(S(T)y) - \|y_x\|_{L^2(\mathbb{R})}^2.$$ (95)

Therefore, using that for $n$ large enough $S(t_n)u_n$ is trapped into the absorbing set,

$$\limsup_{n \to +\infty} \|S(t_n)u_n\|_{L^2(\mathbb{R})} \leq \|y_x\|_{L^2(\mathbb{R})} + Ke^{-2\gamma T}.$$ (96)

Letting $T$ go to infinity leads to the strong $H^1(\mathbb{R})$ convergence of $S(t_n)u_n$ towards $y$.

To prove that $\mathcal{A}$ is a compact subset of $H^1(\mathbb{R})$ is very similar and then omitted. Hence, the Theorem 1.2 is proved.

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