Abstract
A round metric space is the one in which the closure of each open ball is the corresponding closed ball. By a sleek metric space, we mean a metric space in which the interior of each closed ball is the corresponding open ball. In this article, we establish some results on round metric spaces and sleek metric spaces.

Keywords Round metric · Sleek metric · Convexity · Linear metric space

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1 Introduction
All normed linear spaces are known to have the following two properties with the metric induced by the norm:

Property A Closure of every open ball is the corresponding closed ball.

Property B Interior of every closed ball is the corresponding open ball.

Properties A and B may not hold in metric spaces or even in linear metric spaces (see for example, [1, 2]). However, starting with the work of Artémidis [1], there are some necessary and sufficient conditions known in the literature for Property A or B to hold in metric spaces (see [3–5]). For a metrizable space,
Nathanson [3] defined a round metric as the one for which Property A holds. A metrizable space whose topology is induced by a round metric was then defined to be a \textit{round metric space}.

Kiventidis [4] was the first to discuss metric spaces having Property B. Recently, Singh and Narang [5] have discussed metric spaces and linear metric spaces which have Property B under some convexity conditions. Analogously, for a metrizable space, we define a \textit{sleek metric} as the one for which Property B holds. A metrizable space whose topology is induced by a sleek metric will be called a \textit{sleek metric space}.

In [3], Nathanson obtained the following interesting results on round metric spaces.

**Theorem A** (Nathanson [3]).

1. Let \( X = A \cup K \) be a metrizable space, where \( A \) and \( K \) are nonempty, disjoint, closed sets, and \( K \) is compact. Then no metric for \( X \) is round.

2. Let \((X, d_1)\) and \((Y, d_2)\) be metric spaces without isolated points, and let \( f : X \to Y \) be a surjection such that for \( x, y, z, \in X \), if \( d_1(x, z) < d_1(x, y) \), then \( d_2(f(x), f(z)) < d_2(f(x), f(y)) \). If \( d_1 \) is a round metric for \( X \), then \( d_2 \) is a round metric for \( Y \).

3. Let \((X, d)\) be a metric space. Then there exists an equivalent\(^1\) metric on \( X \) that is bounded but not round.

4. Let \( \{(X_k, d_k)\}_{k=1}^{\infty} \) be a countable family of metric spaces such that \( \text{diam}(X_k) < \infty \) for all but finitely many \( k \). The product space \( X = \prod_k X_k \) is metrizable. If \( x = (x_k), y = (y_k) \in X \), define
   \[
   D(x, y) = \sum_{k=1}^{\infty} \frac{d_k(x_k, y_k)}{(\lambda_k 2^k)},
   \]
   where \( \lambda_k = 1 \) if \( \text{diam}(X_k) = \infty \) and \( \lambda_k = \text{diam}(X_k) \) if \( \text{diam}(X_k) < \infty \). Then \( D \) is a metric for \( X \). The metric \( D \) is round for \( X \) if and only if the metric \( d_k \) is round for \( X_k \) for all \( k \).

Interestingly, as we shall see in Sect. 2, the analogues of Theorem A(1)–A(4) hold for sleek metric spaces too.

A metric space \((X, d)\) is said to be

1. \( \lambda \)-convex if for any \( x, y \in X \) and a fixed \( \lambda \in (0, 1) \), there exists \( z \in X \) such that \( d(z, x) = (1 - \lambda)d(x, y) \) and \( d(z, y) = \lambda d(x, y) \).

2. metrical convex or convex if for every pair of distinct points \( x \) and \( y \) in \( X \), there exists \( z \in X \) different from \( x \) and \( y \) such that \( d(x, z) + d(z, y) = d(x, y) \).

3. externally convex if for every pair of distinct points \( x \) and \( y \) of \( X \), there exists \( z \in X \) different from \( x \) and \( y \) such that \( d(x, y) + d(y, z) = d(x, z) \).

\(^1\) Two metrics \( d \) and \( d' \) on a set \( X \) are said to be equivalent if both \( d \) and \( d' \) induce same topology on \( X \).

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Metric spaces satisfying aforementioned convexity conditions turn out to be round or sleek.

**Theorem B** ([3, 5]). Let \((X, d)\) be a metric space.

1. If \((X, d)\) is \(\lambda\)-convex, then the metric \(d\) is round for \(X\) (see [3, Theorem 5]).
2. If \((X, d)\) is a metrically convex and externally convex complete metric space, then the metric \(d\) is sleek for \(X\) (see [5, Theorem 2.4]).
3. If \((X, d)\) is strongly externally convex, then the metric \(d\) is sleek for \(X\) (see [5, Theorem 2.5]).

In [5], the authors obtained the following characterization of sleek metric spaces, which will be used in the sequel.

**Theorem C** (Singh and Narang [5]). A metric \(d\) is sleek for \(X\) if and only if for any \(x \in X\) and \(r > 0\), each \(y \in B_d(x, r)\) satisfying \(d(x, y) = r\) is a limit point of the set \(X \setminus B_d[x, r]\).

In the present paper, in addition to exploring metric spaces, linear metric spaces, and subspaces for which Properties A or B hold, we also investigate the behavior of these properties in spaces by taking unions, intersections, and products. The paper is organized as follows. The main results of this paper are discussed in Sect. 2. An application of the notions of round and sleek metric spaces is discussed in Sect. 3. Finally, in Sect. 4, several examples are discussed to demonstrate the behavior of roundness and sleekness of metrics for subspaces under unions and intersections.

First we fix some notations. Throughout, \((X, d)\) will denote a metrizable topological space having at least two points with the metric \(d\) inducing the topology of \(X\). For any \(x \in X\) and \(r > 0\), the sets \(B_d(x, r) = \{y \in X \mid d(x, y) < r\}\) and \(B_d[x, r] = \{y \in X \mid d(x, y) \leq r\}\), respectively denote the open and the closed balls in \(X\). We also denote by \(B_d(x, r)\), the closure of \(B_d(x, r)\) in \(X\) and \(B^0_d[x, r]\), the interior of \(B_d[x, r]\) in \(X\). For any two subsets \(Y\) and \(Z\) of \(X\), we use \(Y \setminus Z\) to denote the set \(\{y \in Y \mid y \notin Z\}\). Finally, for each natural number \(n\), the symbol \(\rho_n\) will denote the Euclidean metric of \(\mathbb{R}^n\).

**2 Main results**

We begin by proving the following characterization of non-round metrics.

**Theorem 1** Let \((X, d)\) be a metric space. The metric \(d\) is not round for \(X\) if and only if there exists an open set \(U\) and \(x \in X \setminus U\) such that the map \(d(x, \cdot) : U \rightarrow \mathbb{R}\) has a minimum.
Suppose the metric \(d\) is not round for \(X\). Then there exists a pair of distinct points \(u\) and \(v\) in \(X\) for which \(v \not\in \bar{B}_d(u, d(u, v))\). Consequently, there exists an \(\epsilon > 0\), such that \(B_d(v, \epsilon) \cap \bar{B}_d(u, d(u, v)) = \emptyset\), and so, \(B_d(v, \epsilon) \subset X \setminus \bar{B}_d(u, d(u, v))\). Then \(U = X \setminus \bar{B}_d(u, d(u, v))\) is an open set in \(X\) such that \(u \not\in U\), \(v \in U\), and the map \(d(u, \cdot) : U \to \mathbb{R}\) has the minimum value \(d(u, v)\).

Conversely, let there be an open set \(U\) in \(X\) and a point \(x \in X \setminus U\), such that the map \(d(x, \cdot) : U \to \mathbb{R}\) has a minimum value. Suppose \(y \in U\) for which \(d(x, y) = \inf\{d(x, z) \mid z \in U\}\). Since for every \(t \in B_d(x, d(x, y))\), we have \(d(x, t) < d(x, y)\), this in view of the fact that \(d(x, y)\) is the minimum value of \(d(x, \cdot)\) on \(U\) gives \(t \not\in U\). Consequently, \(B_d(x, d(x, y)) \cap U = \emptyset\), and so, \(B_d(x, d(x, y)) \subset X \setminus U\). Then \(\bar{B}_d(x, d(x, y)) \subset X \setminus U = X \setminus U\), and since \(y \not\in X \setminus U\), we find that \(y \not\in \bar{B}_d(x, d(x, y))\). So, \(d\) is not a round metric for \(X\).

**Corollary 2** A metrizable space with at least two points and having an isolated point is not metrically round.

**Proof** Let \(X\) be such a space, and let the metric \(d\) induces the topology of \(X\). If \(y \in X\) is an isolated point, then the set \(\{y\}\) is open in \(X\). So, for any \(x \in X \setminus \{y\}\), the map \(d(x, \cdot) : \{y\} \to \mathbb{R}\) has the minimum value \(d(x, y)\).

A subset \(S\) of a metric space \((X, d)\) is said to be a **metric segment** joining two distinct points \(x\) and \(y\) of \(X\) if there exists a closed interval \([a, b]\) in real line and an isometry \(\gamma : [a, b] \to (X, d)\) which maps \([a, b]\) onto \(S\) with \(\gamma(a) = x\) and \(\gamma(b) = y\). If the metric space \((X, d)\) has the property that every pair of points can be joined by a metric segment, then \((X, d)\) is convex (see [8, p. 35]). The converse known as, the fundamental theorem of metric convexity, is often true, and it states that in a complete convex metric space, any two distinct points can be joined by a metric segment. This result was proved in [7] by Karl Menger, one of the pioneers in the study of metric spaces.

**Theorem 3** If a metric space \((X, d)\) has the property that every pair of distinct points can be joined by a metric segment, then the metric \(d\) is round for \(X\).

**Proof** If possible, suppose that the metric \(d\) is not round for \(X\). By Theorem 1, there exists an open set \(U\) and \(x \in X \setminus U\) such that the map \(d(x, \cdot) : U \to \mathbb{R}\) has the minimum value \(d(x, y)\) for \(y \in U\). Then \(x \neq y\), and by the hypothesis, there exists a closed interval \([a, b]\) in real line and an isometry \(\gamma : [a, b] \to X\) such that \(\gamma(a) = x\), \(\gamma(b) = y\). Since \(\gamma\) is an isometry, we have \(d(x, y) = d(\gamma(a), \gamma(b)) = b - a > 0\). By continuity of \(\gamma\), the inverse image \(\gamma^{-1}(U)\) is an open set containing \(b\), and for all \(t \in \gamma^{-1}(U) \subseteq [a, b]\), we have

\[
\begin{align*}
&b - a = d(x, y) \leq d(x, \gamma(t)) = d(\gamma(a), \gamma(t)) = t - a \leq b - a,
\end{align*}
\]

and so, \(t = b\). Consequently, \(\gamma^{-1}(U) = \{b\}\) which is not open in \([a, b]\). This contradicts the continuity of \(\gamma\). \(\square\)
The fundamental theorem of metric convexity together with Theorem 3 implies that every complete convex metric space is metrically round. This result has been proved earlier in [2, 4], however, our approach is different.

The following result presents an analogue of Theorem 1 for non-sleek metrics.

**Theorem 4** Let \((X, d)\) be a metric space having at least two points. The metric \(d\) is not sleek for \(X\) if and only if there exists \(x \in X\) and an open set \(U\) containing \(x\), such that the map \(d(x, \cdot) : U \to \mathbb{R}\) has a maximum.

**Proof** Suppose there exists \(x \in X\) and an open set \(U\) containing \(x\) such that \(d(x, y) = \max\{d(x, z) \mid z \in U\}\) for some \(y \in U\). Then \(U \subseteq B_d[x, d(x, y)]\), and since \(y \in U\), we have \(y \in B_d^2[x, d(x, y)]\). Consequently, \(y\) is not a limit point of the set \(X \setminus B_d[x, d(x, y)]\). So, by Theorem C, \(d\) is not a sleek metric for \(X\).

Conversely, if \(d\) is not a sleek metric for \(X\), then in view of Theorem C, there exists a pair of distinct points \(u\) and \(v\) in \(X\) for which \(v\) is not a limit point of the set \(X \setminus B_d[u, d(u, v)]\). Also, since \(v \notin (X \setminus B_d[u, d(u, v)])\), we must have \(v \notin X \setminus B_d[u, d(u, v)]\). Consequently, there exists an \(\epsilon > 0\), such that \(B_d(v, \epsilon) \cap (X \setminus B_d[u, d(u, v)]) = \emptyset\). Thus, \(B_d(v, \epsilon) \subseteq B_d[u, d(u, v)]\), and so, \(v \in B_d^2[u, d(u, v)]\). Finally, the map \(d(u, \cdot)\) attains its maximum value \(d(u, v)\) on the open set \(B_d^2[u, d(u, v)]\) containing \(u\), and so, the converse holds. \(\square\)

**Corollary 5**

(a) *A metrizable space with at least two points and having an isolated point is not metrically sleek.*

(b) *A compact metrizable space having at least two points is not metrically sleek.*

**Proof** (a) Let the metric \(d\) induces the topology of the metrizable space \(X\). If \(x \in X\) is an isolated point, then the set \(U = \{x\}\) is open, and so, the map \(d(x, \cdot) : U \to \mathbb{R}\) has the maximum value \(d(x, x) = 0\).

(b) Let the metric \(d\) induces the topology of the compact metrizable space \(X\). Since \(X\) is compact and has at least two points, there exists a pair of distinct points \(x\) and \(y\) in \(X\), such that \(d(x, y) = \sup\{d(u, v) \mid u, v \in X\}\). So, the map \(d(x, \cdot) : X \to \mathbb{R}\) has the maximum value \(d(x, y)\). \(\square\)

A compact metrizable space having no isolated point may or may not be metrically round (see Example 3). An open or dense subspace of a sleek metric space is sleek (see [4]).

The following two Lemmas will be used in proving our next result which discusses round and sleek subspaces of real line with the usual metric.

**Lemma 6** Let \(X\) be a subset of real line in the subspace topology. If \(\rho_1\) is not a round metric for \(X\), then there exist distinct points \(x, y \in X\) and an \(\epsilon > 0\) such that \((y - \epsilon, y) \cap X = \emptyset\) for \(x < y\) and \((y, y + \epsilon) \cap X = \emptyset\) for \(y < x\).

**Proof** By Theorem 1, there exists an open set \(U\) in \(X\), \(x \in X \setminus U\), and \(y \in U\) such that the map \(\rho_1(x, \cdot) : U \to \mathbb{R}\) has the minimum value \(|x - y|\). Since \(U\) is open in \(X\) and \(y \in U\), there exists an \(\epsilon > 0\) for which \((y - \epsilon, y + \epsilon) \cap X \subseteq U\). Observe that
Proof. By Theorem 4, there exists an open set $U$ in $X$ and $x, y \in U$ such that $|x - u| \leq |x - y|$ for all $u \in U$. Since $X$ has no isolated point, we have $x \neq y$. Also, $U \subseteq [2x - y]$ for $x < y$, and $U \subseteq [y, 2x - y]$ for $y < x$. So, $(y, \infty) \cap U = \emptyset$ for $x < y$; and $(-\infty, y) \cap U = \emptyset$ for $y < x$. Since $U$ is open in $X$ and $y \in U$, there exists an $\epsilon > 0$ for which $(y - \epsilon, y + \epsilon) \cap U \subseteq U$. Consequently, $(y, y + \epsilon) \cap X = \emptyset$ for $x < y$ and $(y - \epsilon, y) \cap X = \emptyset$ for $y < x$.

Theorem 8 Let $X$ be a subspace of real line having no isolated point.

(1) If $\rho_1$ is a sleek metric for $X$, then $\rho_1$ is a round metric for $X$.

(2) If $X$ is connected, then $\rho_1$ is round for $X$.

(3) If $X$ is connected and non-singleton, then $\rho_1$ is sleek for $X$ if and only if $X$ is open in real line.

(4) If $X$ is not connected and if no connected component of $X$ is singleton, then $\rho_1$ is sleek for $X$ if and only if $\rho_1$ is sleek for every connected component of $X$.

(5) If $X$ is not connected and if $X$ is not bounded, then $\rho_1$ is round for $X$ if and only if $\rho_1$ is sleek for $X$.

Proof. (1) Assume that $\rho_1$ is not round for $X$. By Lemma 6, there exist $x, y \in X$ with $x \neq y$ and $\epsilon > 0$ such that $(y - \epsilon, y) \cap X = \emptyset$ for $x < y$ and $(y, y + \epsilon) \cap X = \emptyset$ for $y < x$. Letting $U = (y - \epsilon, y + \epsilon) \cap X$, we find that the $U$ is open in the subspace topology of $X$, where $U = [y, y + \epsilon)$ for $x < y$ and $U = (y - \epsilon, y]$ for $y < x$, such that the map $\rho_1(y, \cdot) : U \rightarrow \mathbb{R}$ has the maximum value $\epsilon/2$. By Theorem 4, $\rho_1$ is not a sleek metric for $X$.

(2) If $\rho_1$ is not round for $X$, then using Lemma 6 we have a pair of distinct points $x, y \in X$ and an $\epsilon > 0$ such that $(y - \epsilon, y) \cap X = \emptyset$ for $x < y$ and $(y, y + \epsilon) \cap X = \emptyset$ for $y < x$. In either case, we find that the points $x$ and $y$ do not belong to same connected component of $X$ and so, $X$ is not connected.

(3) Clearly, $\rho_1$ is sleek for any open subset of real line. Conversely, any non-open non-singleton connected subspace of real line is one of the intervals, $[a, b], (a, b], [a, b], (-\infty, b], and [a, \infty]$ where $a$ and $b$ are real numbers with $a < b$, and it is easy to see using Theorem 4 that $\rho_1$ is not sleek for any of these intervals.

(4) Let $\rho_1$ be not sleek for $X$. Then by Lemma 7, there exist $x, y \in X$ with $x \neq y$ and $\epsilon > 0$ for which $(y, y + \epsilon) \cap X = \emptyset$ for $x < y$ and $(y - \epsilon, y) \cap X = \emptyset$ for $y < x$. The set $U = (y - \epsilon, y + \epsilon) \cap X$ is open in $X$, where $U = (y - \epsilon, y]$ for $x < y$ and $U = [y, y + \epsilon)$ for $y < x$. Let $C$ be the connected component of $X$ containing $y$. By the hypothesis, $C$ is not singleton. So, let $c \in C \cap U \setminus \{y\}$. Define $V = [y, c)$ for $y < c$
and \( V = (c, y] \) for \( c < y \). Since \( C \) is connected and \( c, y \in C \), we must have \( C \cap V = V \). So, \( V \) is an open subset of \( C \) such that \( \rho_1(y, \cdot) : V \to \mathbb{R} \) has the maximum value \(|y - c|/2 \). By Theorem 4, \( \rho_1 \) is not sleek for \( C \).

Conversely, suppose there exists a connected component \( Z \) of \( X \) such that \( \rho_1 \) is not sleek for \( Z \). Then \( Z \) is a non-singleton connected subspace of real line. This in view of the statement (3) of the Theorem shows that \( Z \) is a non-open interval in real line. By the hypothesis, \( X \neq \mathbb{R} \), and so, \( \inf Z \notin Z \) or \( \sup Z \notin Z \). Assume that \( \inf Z \in Z \) and let \( z = \inf Z \). Since the collection of all connected components of \( X \) is a partition of \( X \) and each connected component of \( X \) is closed and non-singleton, there exists \( \epsilon > 0 \) for which \( (y - \epsilon, y + \epsilon) \cap X = (z - \epsilon, z + \epsilon) \cap Z = [z, z + \epsilon] \), and so, the set \([z, z + \epsilon]\) is open in \( Z \). Consequently, \( \rho_1(z, \cdot) : [z, z + \epsilon) \to \mathbb{R} \) has the maximum value \( \epsilon/2 \). By Theorem 4, \( \rho_1 \) is not sleek for \( Z \).

(5) If \( \rho_1 \) is sleek for \( Z \), then the statement (1) of the Theorem implies that \( \rho_1 \) is round for \( X \). Suppose then that \( \rho_1 \) is not sleek for \( X \). By Theorem 4, there exists an open set \( V \in X \) and \( x \in V \) such that \( \rho_1(x, \cdot) : V \to \mathbb{R} \) has the maximum value \(|x - y|\) where \( y \in V \). Since \( X \) has no isolated point, we must have \( x \neq y \), and so, \(|x - y| > 0 \). Assume without loss of generality that \( x < y \). Then for all \( z \notin [2x - y, y] \), \(|x - y| < |x - z| \), and so, \( V \subset [2x - y, y] \). Thus, there exists an \( \epsilon > 0 \) for which \( (y - \epsilon, y] \cap X \) is open in \( X \) and \((y - \epsilon, y] \cap X) \subset V \). Since \( X \) is not bounded, there exists \( z \in X \) such that \( y < z \). Now taking \( U = (y - \epsilon, y] \cap X \), we find that \( z \in X \setminus U \) and \( \rho_1(z, \cdot) : U \to \mathbb{R} \) has the minimum value \(|z - x| \). By Theorem 1, \( \rho_1 \) is not round for \( X \).

Theorem 8(1) shows that for real line with the usual metric, the class of metrically sleek subspaces is contained in the class of metrically round subspaces.

On the other hand, there always exists a subspace \( X \) of the Euclidean space \( (\mathbb{R}^n, \rho_n) \) for \( n > 1 \), such that the metric \( \rho_n \) is sleek but not round for \( X \). An explicit example is the subspace \( \mathbb{R} \times \{1, 2, \ldots, n\} \) of \( \mathbb{R}^n \), \( n \geq 2 \).

**Theorem 9** For an index set \( J \), let \( \{A_x\}_{x \in J} \) be a family of subspaces of a metric space \( (X, d) \) such that the metric \( d \) is sleek for the subspace \( A_x \cup A_\beta \) for all \( x, \beta \in J \). Then the metric \( d \) is sleek for the subspace \( \bigcup_{x \in J} A_x \).

**Proof** Let \( A = \bigcup_{x \in J} A_x \). Suppose the contrary that \( d \) is not sleek for \( A \). Then by Theorem 4, there exists \( x \in A \) and an open set \( U \) containing \( x \) such that the map \( d(x, \cdot) : U \to \mathbb{R} \) has the maximum value \( d(x, y) \) for \( y \in U \). As \( U \) is open in \( A \), we have \( U = O \cap A \) for some open subset \( O \) of \( X \). Since \( x, y \in O \cap A = \bigcup_{\alpha \in \mathcal{J}} (O \cap A_\alpha) \), there exist \( x, \beta \in J \) such that \( x \in A_x \) and \( y \in A_\beta \). Then \( x, y \in O \cap (A_x \cup A_\beta) \), such that \( d(x, y) = \sup \{d(x, z) : z \in A_x \cup A_\beta\} \), which in view of Theorem 4 shows that the metric \( d \) is not sleek for the subspace \( A_x \cup A_\beta \). This contradicts the hypothesis.

For sleek metric spaces, Theorems 10, 12, and 13 are analogues of Theorems A (2), A(3), and A (4), respectively.

**Theorem 10** Let \( (X, d_1) \) and \( (Y, d_2) \) be metric spaces. Let \( f : X \to Y \) be a surjection such that for \( x, y, z, \in X \) if \( d_1(x, z) > d_1(x, y) \), then \( d_2(f(x), f(z)) > d_2(f(x), f(y)) \). If \( d_1 \) is a sleek metric for \( X \), then \( d_2 \) is a sleek metric for \( Y \).
Proof The map \( f \) is a homeomorphism of \( X \) onto \( Y \) (see [3]). So any two distinct points of \( Y \) are of the form \( f(x) \) and \( f(x') \) for distinct points \( x \) and \( x' \) in \( X \). Since \( d_1 \) is a sleek metric for \( X \), in view of Theorem C, the point \( x' \) is a limit point of the set \( X \setminus B_{d_1}[x, d_1(x, x')] \). So, there exists a sequence \( \{z_n\} \) of points of \( X \setminus B_{d_1}[x, d_1(x, x')] \) converging to \( x' \). By the continuity of \( f \), the sequence \( \{f(z_n)\} \) in \( Y \) converges to \( f(x') \). Since for each natural number \( n \), \( z_n \notin B_{d_2}[x, d_1(x, x')] \), we must have \( d_1(x, z_n) > d_1(x, x') \). So, by the hypothesis we have \( d_2(f(x), f(z_n)) > d_2(f(x), f(x')) \), which shows that \( f(z_n) \notin B_{d_2}[f(x), d_2(f(x), f(x'))] \). Consequently, \( f(x') \) becomes a limit point of the set \( Y \setminus B_{d_2}[f(x), d_2(f(x), f(x'))] \). By Theorem C, \( d_2 \) is a sleek metric for \( Y \).

The analogues of the first two results in the following corollary have been obtained in [3] for the round metric spaces.

Corollary 11

(a) Let \( d_1 \) be a sleek metric for \( X \). If \( d_2 \) is another metric on \( X \) equivalent to \( d_1 \) such that \( d_2(x, z) < d_2(x, y) \) whenever \( d_1(x, z) < d_1(x, y) \), then \( d_2 \) is also a sleek metric for \( X \).

(b) Let \( d \) be a sleek metric for \( X \). Then there exists an equivalent bounded metric \( d' \) on \( X \) which is sleek for \( X \).

(c) Let \( f : (X, d_1) \to (Y, d_2) \) be a global isometry, that is, \( f \) is a surjective map such that \( d_2(f(x), f(y)) = d_1(x, y) \) for all \( x, y \in X \). Then \( d_1 \) is a sleek metric for \( X \) if and only if \( d_2 \) is a sleek metric for \( Y \).

Proof To prove (a), we take \( X = Y \) and \( f \), the identity map of \( X \) in Theorem 10. To prove (b), let \( d'(x, y) = d(x, y)/(1 + d(x, y)) \) for all \( x, y \in X \). The metric \( d' \) is bounded and is equivalent to \( d \). If \( d(x, z) < d(x, y) \), then we have

\[
d'(x, y) - d'(x, z) = \frac{1}{(1 + d(x, y))(1 + d(x, z))} \{d(x, y) - d(x, z)\} > 0.
\]

By (a), \( d' \) is a sleek metric for \( X \). Finally, (c) follows from the observation that \( d_1(x, z) > d_1(x, y) \) for \( x, y, z \in X \) if and only if \( d_2(f(x), f(z)) = d_1(x, z) > d_1(x, y) = d_2(f(x), f(y)) \).

Theorem 12 Let \((X, d)\) be a metric space. Then there exists an equivalent bounded metric \( d' \) on \( X \) which is not sleek for \( X \).

Proof As in [3], for \( a, b \in X \), \( a \neq b \) and \( 0 < r < d(a, b) \), we let \( d'(x, y) = \min\{d(x, y), r\} \) for all \( x, y \in X \). Then \( d' \) is a bounded and is equivalent to \( d \). We observe that \( B_{d'}^r[a, r] = X \neq B_d^r(a, r) \), since \( b \notin B_d^r(a, r) \). So, \( d' \) is not a sleek metric for \( X \).

Theorem 13 Let \( \{(X_k, d_k)\}_{k=1}^{\infty} \) be a countable family of metric spaces, where \( \text{diam}(X_k) < \infty \) for all but finitely many \( k \). Let \( X = \prod_k X_k \). If the metric \( d_k \) is sleek for \( X_k \) for all \( k \), then the metric \( D \) as defined in Theorem A(4) is sleek for \( X \). The converse is not true.
Proof Assume that the metric \( d_k \) is sleek for \( X_k \) for all \( k \). Let \( x = (x_k) \) and \( y = (y_k) \) be any two distinct points of \( X \). Then there exists an index \( i \) for which \( x_i \neq y_i \). Since the metric \( d_i \) is sleek for \( X_i \), in view of Theorem C, we have a sequence \( \{z^n_i\}_{n=1}^{\infty} \) of points of \( X_i \) such that

\[
\lim_{n \to \infty} d_i(z^n_i, y_i) = 0; \quad d_i(x_i, z^n_i) > d_i(x_i, y_i), \quad \text{for all } n \geq 1.
\]

Taking \( z^n_k = y_k \) if \( k \neq i \), and \( z^n_k = z^n_i \) if \( k = i \), we observe that \( \{z^n\} \) is the sequence of points of \( X \) converging to \( y \), since \( D(z^n, y) = d_i(z^n_i, y_i)/(\lambda_i 2^k) \to 0 \) as \( n \to \infty \). Also,

\[
D(x, z^n) = d_i(x_i, z^n_i)/(\lambda_i 2^i) + \sum_{k=1, k \neq i}^{\infty} d_k(x_k, y_k)/(\lambda_k 2^k)
\]

which shows that \( D(x, z^n) > D(x, y) \); and so, \( z^n \in X \setminus B_D[x, D(x, y)] \) for all \( n \). Thus, \( y \) is a limit point of the set \( X \setminus B_D[x, D(x, y)] \). By Theorem C, \( D \) is a sleek metric for \( X \).

To show that the converse need not be true, we take the product space \( X = \prod_{n=1}^{\infty} (X_n, d_n) \), where

\[
X_1 = \{0, 1\}; \quad d_1 = \rho_1; \quad X_n = \mathbb{R}; \quad d_n(x_n, y_n) = \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad x_n, y_n \in X_n, n \geq 2.
\]

We show that the metric \( D \) as defined in Theorem A(4) is sleek for the product space \( X \). To proceed, we observe that \( \text{diam}(X_n) = 1 \) for all \( n \) so that in view of (1) we have for all \( x = (x_n), y = (y_n) \) in \( X \) that

\[
D(x, y) = \sum_{n=1}^{\infty} d_n(x_n, y_n) 2^{-n} = \begin{cases} \sum_{n=2}^{\infty} \frac{|x_n - y_n|2^{-n}}{1 + |x_n - y_n|}, & \text{if } x_1 = y_1; \\ \frac{1}{2} + \sum_{n=2}^{\infty} \frac{|x_n - y_n|2^{-n}}{1 + |x_n - y_n|}, & \text{if } x_1 \neq y_1. \end{cases}
\]

Consequently, for any point \( a = (a_n) \in X \) and \( r > 0 \), we have the following:

\[
B_D[a, r] = \begin{cases} A_{r, a_1}, & \text{if } 0 < r < (1/2); \\ A_{r, a_1} \cup \{(1 - a_1, a_2, a_3, \ldots)\}, & \text{if } r = 1/2; \\ A_{r, a_1} \cup A_{r - 1/2, 1 - a_1}, & \text{if } (1/2) < r, \end{cases}
\]

where we have

\[
A_{r, a_1} = \{(x_n) \in X \mid x_1 = a_1; \sum_{n=2}^{\infty} \frac{|x_n - y_n|2^{-n}}{1 + |x_n - y_n|} \leq r \}.
\]

We claim that \( B_D^0[a, r] = B_D(a, r) \). To prove our claim, we have the following two cases:

\textbf{Case I: } \( r \geq 1 \). In this case, we observe that \( B_D[x, r] = X = B_D(x, r) \) for all \( x \in X \); and so, in particular, we have \( B_D^0[a, r] = B_D(a, r) \).
Case II: $0 < r < 1$. Let $x = (x_n) \in B_D[a, r]$ be such that $D(a, x) = r$. To prove the claim, it will be enough to show that $x \notin B_D^0[a, r]$. In view of this, we arrive at the following subcases:

Subcase I: $x \in A_{r, a_1}$. In this case for any $\epsilon > 0$, we choose $y = (y_n) \in X$, where for each natural number $n$,  
\[
y_n = \begin{cases} 
x_n, & \text{if } x_n = a_n; \\
x_n + \epsilon \frac{x_n - a_n}{|x_n - a_n|}, & \text{if } x_n \neq a_n.
\end{cases}
\]
Now using (2), we arrive at the following calculations:
\[
D(x, y) = \frac{\epsilon}{1 + \epsilon} \sum_{i \geq 2, x_i \neq a_i} 2^{-n} \leq \frac{\epsilon}{1 + \epsilon} \sum_{n \geq 2} 2^{-n} = \epsilon/2 < \epsilon,
\]
which show that $y \in B_D(x, \epsilon)$. Also, for every nonnegative real number $u$, we have
\[
\frac{u + \epsilon}{1 + u + \epsilon} - \frac{u}{1 + u} = \frac{\epsilon}{(1 + u)(1 + u + \epsilon)} > 0.
\]
Using the inequality (3) for $u = |a_n - x_n|$ for each $n$, we get
\[
D(a, y) = \sum_{(n \geq 2, x_i \neq a_i)} \frac{(|a_n - x_n| + \epsilon)2^{-n}}{1 + |a_n - x_n| + \epsilon} > \sum_{(n \geq 2, x_i \neq a_i)} \frac{|a_n - x_n|2^{-n}}{1 + |a_n - x_n|} = D(a, x).
\]
Consequently, $x \notin B_D^0[a, r]$.

Subcase II: $x \in A_{r-1/2, 1-a_1}$ for $r > 1/2$. Proceeding as in the Subcase I, we find that $x \notin B_D[a, r]$.

Subcase III: $x \in A_{1/2, 1-a_1}$. In this case, $x = (1 - a_1, a_2, a_3, \ldots)$, and for $\epsilon > 0$, $B_D(x, \epsilon)$ contains the point $z = (1 - a_1, a_2 + \epsilon, a_3, a_4, \ldots)$, where
\[
D(a, z) = \frac{|1 - 2a_1|}{2} + \frac{\epsilon}{4(1 + \epsilon)}.
\]
Since $a_1 \in \{0, 1\}$, we must have $|1 - 2a_1| = 1$; and so,  $D(a, z) = (1/2 + \epsilon/(4(1 + \epsilon))) > (1/2)$. Thus, $x \notin B_D^0[a, 1/2]$. In each of the above subcases, we have $x \notin B_D^0[a, r]$, and so, $B_D^0[a, r] = B_D[a, r]$. We conclude that $D$ is a sleek metric for $X$, where we note that the metrizable component $X_1 = \{0, 1\}$ of $X$ is never sleek.

Corollary 14  A countable product of sleek metric spaces is sleek.

Proof Let $\{(X_k, d_k)\}_{k=1}^\infty$ be a countable collection of metric spaces, where $d_k$ is a sleek metric for $X_k$ for all $k$. Let $X = \prod_{k=1}^\infty X_k$ be given the product topology. By Corollary 11(b), there is an equivalent bounded sleek metric $d'_k$ for $X_k$, and by Theorem 13, the product space $\prod_{k=1}^\infty(X_k, d'_k)$ has a sleek metric. The analogue of the above result holds for round metric spaces. This was proved by Nathanson [3].
The counterexample considered in the converse part of Theorem 13 suggests the following generalization.

**Theorem 15** Let \( \{(X_k, d_k)\}_{k=1}^{\infty} \) be a countable family of metric spaces, where \( \text{diam}(X_k) < \infty \) for all but finitely many \( k \). Let \( X = \prod_k X_k \). Let \( D \) be the metric on \( X \) as in Theorem A(4). If there exists at least one positive integer \( k \) for which \( d_k \) is a sleek metric for \( X_k \) with \( \text{diam}(X_k) > 0 \), then \( D \) is a sleek metric for \( X \).

**Proof** Without loss of generality we assume that the metric \( d_1 \) is sleek for \( X_1 \). Let \( x = (x_n) \) and \( y = (y_n) \) be two distinct points of \( X \). We show that \( y \) is a limit point of the set \( X \setminus B_D[x, D(x, y)] \), which in view of Theorem C will prove that the metric \( D \) is sleek for \( X \). So it is enough to prove that there exists a sequence of points of the set \( X \setminus B_D[x, D(x, y)] \) converging to \( y \) in the metric \( D \). We construct such a sequence in each of the following two cases:

First assume that \( x_1 = y_1 \). We define a sequence \( \{\xi_n\} \) of points of \( X \setminus B_D[x, D(x, y)] \) as follows. Since \( X_1 \) is sleek, by Corollary 5 no point of \( X_1 \) is an isolated point. So we can choose a nonconstant sequence \( \{x_1^{(n)}\} \) of points of \( X_1 \) converging to \( x_1 \), that is, \( \lim_{n \to \infty} d_1(x_1^{(n)}, x_1) = 0 \). Without loss of generality, we can assume that \( x_1^{(n)} \neq x_1 \) for all \( n \) so that \( d_1(x_1^{(n)}, x_1) \) is always positive. Let \( \xi_n = (x_1^{(n)}, y_2, y_3, \ldots) \in X \) for all \( n \). Then \( D(\xi_n, y) = \frac{1}{2x_1} d_1(x_1^{(n)}, x_1) \to 0 \) as \( n \to \infty \), which shows that the sequence \( \{\xi_n\} \) converges to \( y \). Also, for every positive integer \( n \), we have

\[
D(\xi_n, x) - D(x, y) = \frac{1}{2x_1} d_1(x_1^{(n)}, y_1) > 0,
\]

which shows that \( \xi_n \in X \setminus B_D[x, D(x, y)] \) for all \( n \).

Now assume that \( x_1 \neq y_1 \). Since the metric \( d_1 \) is sleek for \( X_1 \), by Corollary 5 the point \( y_1 \) is a limit point of the set \( X_1 \setminus B_{d_1}[x_1, d_1(x_1, y_1)] \). Consequently, there exists a sequence \( y_1^{(n)} \) of points of \( X_1 \) such that \( d_1(y_1^{(n)}, y_1) \to 0 \) as \( n \to \infty \), and \( d_1(y_1^{(n)}, x_1) > d_1(y_1, x_1) \) for all \( n \). Here we take \( \eta_n = (y_1^{(n)}, y_2, y_3, \ldots) \in X \) for all \( n \). Then \( D(\eta_n, y) = \frac{1}{2x_1} d_1(y_1^{(n)}, y_1) \to 0 \) as \( n \to \infty \); and for each positive integer \( n \)

\[
D(\eta_n, x) - D(x, y) = \frac{1}{2x_1} \{d_1(y_1^{(n)}, x_1) - d_1(y_1, x_1)\} > 0.
\]

Thus, \( \{\eta_n\} \) is a sequence of points of \( X \setminus B_D[x, D(x, y)] \) converging to \( y \). \( \square \)

**Corollary 16** Let \( \{(X_k, d_k)\}_{k=1}^{\infty} \) be a countable family of metric spaces, such that \( d_k \) is a sleek metric for \( X_k \) with \( \text{diam}(X_k) > 0 \) for at least one value of \( k \). Then the product space \( \prod_k X_k \) is sleek.

**Remark 1** In the dictionary order topology, the topological space \( \mathbb{R}^2 \) can be identified with the (metrizable) product space \( \mathbb{R}_{\text{dis}} \times \mathbb{R} \), where \( \mathbb{R}_{\text{dis}} \) and \( \mathbb{R} \) respectively denote the set of all real numbers with the discrete metric and the usual metric. Let \( \rho \) be the product metric for \( \mathbb{R}_{\text{dis}} \times \mathbb{R} \). Observe that for any two distinct real numbers \( x \) and \( y \), the metric \( \rho \) is sleek for each of the subspaces...
\{x\} \times \mathbb{R}, \{y\} \times \mathbb{R}, and their union \{x,y\} \times \mathbb{R}. By Theorem 9, the metric \(\rho\) is sleek for the subspace \(\bigcup_{x \in \mathbb{R}} \{x\} \times \mathbb{R} = \mathbb{R} \times \mathbb{R} \). 

**Remark 2** Let \((X, d)\) be a metric space. Let \(\phi: [0, \infty) \to [0, \infty)\) be a strictly increasing function such that the composite map \(\phi \circ d\) is a metric on \(X\). If the metric \(d\) is sleek for \(X\), then in view of Corollary 11(a), the metric \(\phi \circ d\) is sleek for \(X\). An explicit example of such a function \(\phi\) is 

\[
\phi(t) = \log(1 + t) \text{ for all } t \in [0, \infty).
\]

**Remark 3** The metric \(\rho_1\) is not sleek for any of the subspaces \([0, 1]\) and \((0, 1]\) of real line but the metric \(\rho_2\) is sleek for each of the product spaces \(\mathbb{R} \times [0, 1]\) and \(\mathbb{R} \times (0, 1]\) of \(\mathbb{R}^2\).

### 3 Roundness and sleekness in linear metric spaces

A linear metric space is a topological vector space with a compatible translation invariant metric. We now investigate strictly convex linear metric spaces for round and sleek translation invariant metrics. A linear metric space \((X, d)\) is said to be strictly convex \([6]\) if for any \(r > 0\) and any two distinct points \(x, y \in X\) such that \(d(x, 0) \leq r\) and \(d(y, 0) \leq r\), we have \(d((x+y)/2, 0) < r\). The closed ball \(B_d[0, r]\) in the linear metric space \((X, d)\) is said to be strictly convex \([9]\) if for any pair of distinct points \(x\) and \(y\) in \(B_d[0, r]\) and \(\lambda \in (0, 1)\), the point \((1-\lambda)x + \lambda y\) belongs to \(B_d[0, r]\).

**Remark 4** If a linear metric space \((X, d)\) is strictly convex, then with the equivalent bounded linear metric \(d'\) on \(X\) defined by \(d'(x,y) = d(x,y)/(1 + d(x,y))\) for all \(x, y \in X\), the linear metric space \((X, d')\) is strictly convex.

The following characterizations of strictly convex linear metric spaces are known in the literature.

**Theorem D** ([5, 9]). In a linear metric space \((X, d)\), the following statements are equivalent:

1. The space \((X, d)\) is strictly convex.
2. All closed balls in \(X\) are strictly convex, and the metric \(d\) is round for \(X\).
3. All closed balls in \(X\) are strictly convex, and the metric \(d\) is sleek for \(X\).

So, in a linear metric space having strict ball convexity, the notions of being round and being sleek are equivalent.

**Remark 5** Let \((X, d)\) be a linear metric space. Then there exists an equivalent bounded translation-invariant metric \(d'\) on \(X\) for which \((X, d')\) is not strictly convex.

**Proof** Let \(0 \neq b \in X\) and \(0 < r < d(0, b)\). Let \(d'(x,y) = \min\{r, d(x,y)\}\) for all \(x, y \in X\). Then \(d'\) is an equivalent bounded metric for \(X\). Since \(d\) is translation-invariant and the function \(\min\) is continuous, the metric \(d'\) is translation invariant.
As in the proof of Theorem 12, we see that the translation-invariant metric \( d' \) is not sleek. So, by Theorem D, the linear metric space \((X, d')\) is not strictly convex. \(\square\)

**Remark 6** Let \((X_i, d_i)_{k=1}^n\) be a collection of \(n\) metric spaces. The product topology of \(\prod_{k=1}^n X_k\) can be induced by the metric \(d\), where

\[
d(x, y) = \sqrt{\sum_{i=1}^n (d_i(x_i, y_i))^2},
\]

for all \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) in \(\prod_{k=1}^n X_k\). Now if \((X_k, d_k)\) is a linear metric space for all \(k\), then the product space \((\prod_{k=1}^n X_k, d)\) is also a linear metric space. If \((X_k, d_k)\) is a strictly convex linear metric space for all \(k\), then \((\prod_{k=1}^n X_k, d)\) need not be strictly convex (see [10, Example 2.2]).

**Remark 7** Let \(\{(X_k, d_k)\}_{k=1}^\infty\) be a countable family of linear metric spaces, such that \(\text{diam}(X_k) < \infty\) for all but finitely many \(k\). Let \(X = \prod_k X_k\), and the metric \(D\) as in Theorem A(4). Let \((X, D)\) and \((X_k, d_k)\) for each \(k\) have strict ball convexity. Then \((X, D)\) is strictly convex if and only if \((X_k, d_k)\) is strictly convex for all \(k\).

**Proof** In view of Theorem D, the metric space \((X, D)\) is strictly convex \(\iff\) the metric \(D\) is round for \(X\) \(\iff\) the metric \(d_k\) is round for \(X_k\) for all \(k\) as follows from Theorem A(4) \(\iff\) the metric space \((X_k, d_k)\) is strictly convex for all \(k\) as follows from Theorem D. This completes the proof. \(\square\)

**Remark 8** Let \(\{(X_k, d_k)\}_{k=1}^\infty\) be a countable collection of strictly convex linear metric spaces. Then there exists a bounded translation-invariant metric for \(X\) under which the product space \(\prod_k X_k\) becomes a strictly convex linear metric space.

### 4 Examples

The following examples show that roundness and sleekness may or may not be preserved by subspaces under unions and intersections.

**Example 1** The metric \(\rho_2\) of \(\mathbb{R}^2\) is round for the subspaces shown in Fig. 1a–c, but \(\rho_2\) is not round for the subspace shown in Fig. 1d. It may be remarked that the

![Fig. 1](image)

Fig. 1 The metric \(\rho_2\) is round but not sleek in (a) and (c), round and sleek in (b), and neither round nor sleek in (d)
shapes corresponding to Fig. 1a, b are round in accordance with the geometric intuition but Fig. 1c is not round as per the geometric perception. One observes that among the various subspaces of $\mathbb{R}^2$ as shown in Fig. 1, the metric $q_2$ is sleek only for the subspace in Fig. 1b. None of the two subspaces given in Fig. 1c or 1d is sleek with respect to the metric $q_2$ due to the presence of corner points. This observation is also in accordance with the intuitive notion of geometric sleekness. On the other hand, the non-sleek case in Fig. 1a is not in agreement with the geometric intuition.

Example 2 The metric $q_1$ is round for $\mathbb{R}$ as well as for each of the subspaces $Q$, $(0, 1)$, and $[0, 1]$.

Example 3 No metric equivalent to $q_1$ is round for the subspace $[0, 1] \cup [2, 3]$ of $\mathbb{R}$.

In the examples below, we follow the notations as in Munkres [11] to denote the point $(x, y)$ in $\mathbb{R}^2$ by $x \times y$ and reserve the notation $(x, y)$ for denoting the open interval $\{t \in \mathbb{R} \mid x < t < y\}$ in real line.

Example 4 The metric $q_2$ of $\mathbb{R}^2$ is round for each of the subspaces

$$S^1 = \{x \times y \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \text{ and } [-1, 1] \times 0$$

but $q_2$ is not round for the subspace $S^1 \cap ([-1, 1] \times 0)$, since this intersection is the two point set as shown in Fig. 2.

Example 5 Consider the following subspaces of $\mathbb{R}^2$ with the metric $q_2$ as shown in Fig. 3a.
We observe that the metric $q_2$ is sleek for $X_1$ as well as $X_2$. Now consider the metric $q_2$ for the subspace $Z = X_1 \cup X_2$. Here,

$$B_{q_2}(1 \times 0, 2) \cap Z \neq B_{q_2}(1 \times 0, 2) \cap Z.$$ 

Consequently, interior of the closed ball $B_{q_2}(1 \times 0, 2) \cap Z$ in $Z$ which is not equal to the corresponding open ball $B_{q_2}(1 \times 0, 2) \cap Z$. Thus, the metric $q_2$ is not sleek for $Z$.

**Example 6** Now consider the following subspaces of $(\mathbb{R}^2, \rho_2)$.

$$Y_1 = \mathbb{R} \times (-\infty, 0]; \quad Y_2 = [0, \infty) \times \mathbb{R}.$$ 

Clearly, the metric $\rho_2$ is sleek for each of the subspaces $Y_1$ and $Y_2$. However, the metric $\rho_2$ is not sleek for the subspace $Y_1 \cap Y_2 = [0, \infty) \times (-\infty, 0]$ as can be visualized from Fig. 3b as the darker-shaded region. We see that

$$0 \times 0 \in B_{\rho_2}^c(1 \times -1, \sqrt{2}] \cap Y_1 \cap Y_2) \quad \text{but} \quad 0 \times 0 \notin B_{\rho_2}(1 \times -1, \sqrt{2}) \cap Y_1 \cap Y_2.$$ 

The following examples show that in general, the notions of round metric space and sleek metric space are different.

**Example 7** Among the subspaces $(0, 1)$, $[0, 1]$, and $[0, 1] \cup [2, 3]$ of real line, the metric $\rho_1$ is round and sleek for $(0, 1)$, round but not sleek for $[0, 1]$, and neither round nor sleek for $[0, 1] \cup [2, 3]$.

**Example 8** Now consider the subspace $X' = \mathbb{R} \times \{0, 1\}$ of $\mathbb{R}^2$ with the metric $\rho_2$. For any point $a \times b$ in $X'$ and $r > 0$, we find that
Fig. 4 The closed ball $B_{p_2}(a \times b, r)$ for different values of $r$

Each of these three type of closed balls are shown in Fig. 4 as the thick parts. In the second case (b), we have $B_{p_2}(a \times (1-b), 1/2) \cap B_{p_2}(a \times b, 1) = \emptyset$, and therefore, $a \times (1-b) \notin B_{p_2}(a \times b, 1)$. Thus, the metric $p_2$ is not round for $X'$. Also, if either $0 < r < 1$, or $1 < r$, then clearly we have $B_{p_2}^o[a \times b, r] = B_{p_2}(a \times b, r)$. Now consider the case $r = 1$. The point $a \times (1-b)$ is not an interior point of $B_{p_2}(a \times b, 1]$, since for every $\epsilon > 0$, $B_{p_2}(a \times (1-b), \epsilon)$ always contains the point $z = (a + \epsilon/2) \times (1-b)$, where $p_2(a \times b, z) = \sqrt{1 + \epsilon^2/4} > 1$. Thus, $B_{p_2}^o[a \times b, 1] = B_{p_2}(a \times b, 1]$. We conclude that the metric $p_2$ is sleek for $X'$.

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Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

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