Standard-model symmetry in complexified spacetime algebra

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March 29, 2022

Complexified spacetime algebra is defined as the geometric (Clifford) algebra of spacetime with complex coefficients, isomorphic $\mathcal{G}_{1,4}$. By resorting to matrix representation by means of Dirac-Pauli gamma matrices, the paper demonstrates isomorphism between subgroups of CSTA and $SU(3)$. It is shown that the symmetry group of those subgroups is indeed $U(1) \otimes SU(2) \otimes SU(3)$ and that there are 4 distinct copies of this group within CSTA.

1. Introduction

The application of geometric algebra $\mathcal{G}_{1,3}$, a.k.a. Clifford algebra $\mathcal{C}l_{1,3}$, to quantum mechanics was initiated by David Hestenes, Hestenes [1, 2, 3], who was also responsible for the designation Spacetime algebra (STA) for the geometric algebra of spacetime; the Cambridge group has also brought important contributions, Lasenby et al. [4], Doran et al. [5]. However none of those authors tackled the problem of extending the application to the Standard Model; this was done under a different approach by another group in Canada, Trayling and Baylis [6], using the 7-dimensional $\mathcal{G}_{4,3}$ mapped to $\mathcal{G}_7$ through the use of scalar imaginary $j$. The choice of 7-dimensional space to accommodate $SU(3)$ symmetry arises quite naturally, Lounesto [7], although it should be clear that this is an oversized dimensionality, since $SU(3)$ is a group of $3 \times 3$ matrices, while 3-dimensional geometric algebra has a $2 \times 2$ matrix representation and 4-dimensional geometric algebra is represented with $4 \times 4$ matrices. One would then expect the space dimensionality corresponding to $SU(3)$ to lie somewhere between 3D and 4D, if that was a possibility.

A suitable basis for $SU(3)$ is provided by the $3 \times 3$ Gell-Mann matrices, Cottingham and
Greenwood [8]; here we use those matrices multiplied by \( j \);

\[
\hat{\lambda}_1 = \begin{pmatrix} 0 & j & 0 \\ j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix} j & 0 & 0 \\ 0 & -j & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & 0 \\ j & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & j & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} j & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & -2j \end{pmatrix};
\]

(1)

a hat (\( \hat{\ } \)) is used to signify that we are dealing with matrices and not with geometric algebra multivectors with corresponding designations. The \( \hat{\lambda} \) matrices satisfy the commutation relations

\[ \hat{\lambda}_a \hat{\lambda}_b - \hat{\lambda}_b \hat{\lambda}_a = -2 \sum_{c=1}^{8} f_{abc} \hat{\lambda}_c, \]

(2)

where the \( f_{abc} \) are the structure constants. The \( f_{abc} \) are odd in the interchange of any pair of indices, and the non-vanishing elements are given by the permutations of \( f_{123} = 1, f_{147} = f_{246} = f_{257} = f_{345} = f_{356} = f_{637} = 1/2, f_{458} = f_{678} = \sqrt{3}/2 \).

Since STA and 4-dimensional geometric algebras in general can be represented by \( 4 \times 4 \) matrices, it is clear that, in terms of dimensionality at least, these algebras are large enough to contain \( SU(3) \); we will show that this is true if complex coefficients are allowed.

The group of gauge symmetries is generally described by \( U(1) \otimes SU(2) \otimes SU(3) \), Trayling and Baylis [6]. Just as \( SU(3) \) is associated to the Gell-Mann matrices, \( SU(2) \) can be represented by the \( 2 \times 2 \) Pauli matrices

\[
\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(3)

which are also associated with the representation of 3-dimensional geometric algebra \( G_3 \). \( U(1) \) is just the algebra of complex numbers. The complexified spacetime algebra (CSTA) contains the gauge symmetry group, avoiding the need to use extra dimensions, as we show below. CSTA is 5-dimensional and can be mapped to other 5-dimensional algebras, although we prefer not to do so in order to facilitate geometrical interpretation.

2. Complexified spacetime algebra

Following Doran et al. [5], with the notation conventions of paper Gull et al. [9], STA is defined as the geometric algebra of Minkowski spacetime and is generated by a frame of orthonormal vectors \( \{ \gamma_\mu \}, \mu = 0 \ldots 3 \), that satisfy the Dirac algebra

\[ \gamma_\mu \cdot \gamma_\nu = \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \text{diag}(+--), \]

(4)

2
but are to be considered as four independent vectors instead of matrices. The full STA is spanned by the basis

\[ 1, \{ \gamma_\mu \}, \{ \sigma_k, i\sigma_k \}, \{ i\gamma_\mu \}, i. \] (5)

Here \( i \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) is the unit pseudoscalar; it anticommutes with vectors and trivectors and \( i^2 = -1 \). The spacetime bivectors \( \{ \sigma_k \}, k = 1 \ldots 3 \) are defined by

\[ \sigma_k \equiv \gamma_k \gamma_0, \] (6)

and represent an orthonormal frame of 3-dimensional space. The \( \{ \sigma_k \} \) generate the Pauli algebra of space, so that relative vectors \( a^k \sigma_k \) are viewed as spacetime bivectors. We will assume that the author is familiar with STA operations and will not go into details in this work; consultation of the cited references should clarify any possible doubts. CSTA doubles STA by intervention of the scalar imaginary so that the coefficients multiplying the basis elements in a multivector can be complex numbers. CSTA is the complex \( G_{1,3} \) algebra, isomorphic to the real 5-dimensional algebra \( G_{1,4} \), but in this work we always use the complex 4-dimensional alternative.

We follow Doran et al. \[5\] for the matrix representation of STA and CSTA defining the \( \hat{\gamma} \)-matrices in the standard Dirac-Pauli representation, Halzen and Martin \[10\],

\[ \hat{\gamma}_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}. \] (7)

When needed we will also make use of \( 4 \times 4 \) \( \hat{\sigma}_\mu \) matrices defined by

\[ \hat{\sigma}_0 = \hat{\gamma}_0, \quad \hat{\sigma}_k = \hat{\gamma}_k \hat{\gamma}_0, \] (8)

which can be seen as the matrix equivalent of relation (6). We denote these \( 4 \times 4 \) matrices in the same way as their \( 2 \times 2 \) counterparts; this should not cause any confusion since the context will always make our intents clear.

In order to discuss \( SU(3) \) symmetry in CSTA we note that \( 4 \times 4 \) counterparts to Gell-Mann’s matrices (1) can be built by inserting an extra column and an extra row of zeroes at position 3; these matrices take the same designation as the original ones, since we should always know the dimension pertinent to each situation and confusion should never arise. The eight \( 4 \times 4 \) matrices are

\[
\hat{\lambda}_1 = \begin{pmatrix} 0 & j & 0 & 0 \\ j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ j & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}, \\
\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2j \end{pmatrix}. \] (9)
These matrices are obviously a basis for $SU(3)$ in the algebra of 4-dimensional matrices, in view of the way they were built. We can now try to find the geometric equations defining the equivalents of the $\lambda$ matrices in CSTA; we do this by defining the four basic elements

$$
\lambda_1 = i(\gamma_1 - \sigma_1)/2, \quad \lambda_2 = i(\gamma_2 - \sigma_2)/2, \\
\lambda_4 = (j\sigma_1 + \gamma_2)/2, \quad \lambda_5 = (j\sigma_2 - \gamma_1)/2; 
$$

(10)

the remaining 4 elements can be defined by recursion

$$
\lambda_3 = -\lambda_1\lambda_2, \quad \lambda_6 = \lambda_1\lambda_5 - \lambda_5\lambda_1 \\
\lambda_7 = \lambda_5\lambda_2 - \lambda_2\lambda_5, \quad \lambda_8 = (-2\lambda_4\lambda_5 - \lambda_3)/\sqrt{3}.
$$

(11)

The $\lambda$-multivectors defined by Eqs. (10) and (11) observe the same commutating relations as their matrix equivalents; it is therefore clear that the CSTA subalgebra provided by the $\lambda$-multivectors is isomorphic to $SU(3)$.

Having demonstrated $SU(3)$ symmetry we need to demonstrate the simultaneous existence of $U(1)$ and $SU(2)$. For the former it is sufficient to note that the dual of Eqs. (10) and (11) is obtained by simply multiplying every $\gamma$-vector by the scalar imaginary $j$. Indeed, by multiplying all $\gamma$-vectors by the same complex number we preserve $SU(3)$ and thus prove the product $U(1) \otimes SU(3)$. Then we note that the definitions allow permutation of the indices 1, 2, 3 in the $\gamma$-vectors, providing the desired $SU(2)$ symmetry. In appendix A we show the matrix equivalents corresponding to these permutations.

Eqs. (10) define two pairs of anti-commuting multivectors, $\lambda_1$–$\lambda_2$ and $\lambda_4$–$\lambda_5$. The first pair produces the elements (1, 2) and (2, 1) of the matrix representation, while the second pair produces elements (1, 4) and (4, 1). Redefining the first pair as

$$
\lambda_1 = -i(\gamma_1 + \sigma_1)/2, \quad \lambda_2 = -i(\gamma_2 + \sigma_2)/2, 
$$

(12)

we get elements (3, 4) and (4, 3), which are compatible with the remaining equations, creating a copy of the gauge group. In a similar fashion we can redefine the second pair as

$$
\lambda_4 = (j\sigma_1 - \gamma_2)/2, \quad \lambda_5 = j(\sigma_2 + \gamma_1)/2, 
$$

(13)

which produces elements (2, 3) and (3, 2) of the matrix representation. Combining the two alternatives for both pairs of multivectors we get a total of 4 copies for the gauge group, present in CSTA; in matrix representation the different copies are characterized by the row/column that gets filled with zeros in the 0th permutation; the matrix representation for the remaining 3 copies is shown in appendix B. Each copy allows multiplication by a complex number, yielding $U(1)$ symmetry, and permutation of the $\gamma$-vector indices, resulting in $SU(2)$ symmetry.

Examination of the matrix representations in appendix B shows that the 4 gauge group copies are not independent; in fact only 4 matrices of copy #1 are independent from those of copy #0, while $\lambda_4$ and $\lambda_5$ are common. Proceeding to copy #2, only two new independent matrices are added, precisely $\lambda_4$ and $\lambda_5$, while all matrices in copy #3 can be obtained by linear combinations from the previous ones; in total there are 16 independent matrices as one could expect from the fact that $G_{1,3}$ is a graded 16-dimensional space. The set of 3 independent copies is possibly connected to another $SU(2)$ symmetry.
3. Conclusion

The geometric algebra of spacetime (STA) is represented by $4 \times 4$ Dirac-Pauli matrices, while $SU(3)$ is the special group of unitary $3 \times 3$ matrices. We introduced complex coefficients in STA, defining the CSTA 5-dimensional algebra, which shares the matrix representation of STA but allows each matrix to be multiplied by a complex number. Within the matrix representation of CSTA we were able to find an isomorphism of $SU(3)$, which we could define in terms of multivectors and geometric products.

$SU(3)$ isomorphism in CSTA is richer than the original $3 \times 3$ matrix one, first because the complex coefficients introduce $U(1)$ symmetry and second because permutation among the frame vectors present in the multivector definitions provide $SU(2)$ symmetry. The result is the standard-model $U(1) \otimes SU(2) \otimes SU(3)$ symmetry group, expressed in Minkowski space-time provided with complex numbers, something familiar to most physicists. We showed also that CSTA contains 4 inter-dependent copies of the standard-model group from which 3 can be considered independent of each other; this is possibly a manifestation another $SU(2)$ symmetry.

We think it will be possible to build on the previous formalism to reformulate the standard-model and gauge theories in terms of geometric algebra, namely CSTA; some work is already under way in that direction.

A. Matrix equivalents for $SU(2)$ permutations

The matrices resulting from permutations of indices 1, 2, 3 in the $\gamma$-vectors can be quickly obtained by performing these permutations in Eqs. (10) and then applying Eqs. (11) unchanged. The first right permutation of $\gamma$-vector produces the equivalent matrices

$$
\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\hat{\lambda}_3 = \begin{pmatrix} 0 & j & 0 & 0 \\ j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix},
$$

$$
\hat{\lambda}_5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & j & -j \\ 0 & 0 & j & -j \\ j & j & 0 & 0 \\ -j & -j & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & j & -j \\ 0 & 0 & -j & j \\ j & -j & 0 & 0 \\ -j & j & 0 & 0 \end{pmatrix},
$$

$$
\hat{\lambda}_7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & -j & j \\ 0 & 0 & j & -j \end{pmatrix}.
$$

(14)
The second right permutation produces the matrices

\[
\hat{\lambda}_1 = \begin{pmatrix}
  j & 0 & 0 & 0 \\
  0 & -j & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix}
  0 & j & 0 & 0 \\
  j & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\lambda}_3 = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
 -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_4 = \frac{1}{2} \begin{pmatrix}
  0 & 0 & j & -1 \\
  0 & 0 & -1 & -j \\
  j & 1 & 0 & 0 \\
  1 & -j & 0 & 0
\end{pmatrix},
\]

\[
\hat{\lambda}_5 = \frac{1}{2} \begin{pmatrix}
  0 & 0 & 1 & j \\
 -1 & j & 0 & 0 \\
  j & 1 & 0 & 0 \\
  j & -j & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_6 = \frac{1}{2} \begin{pmatrix}
  0 & 0 & j & -1 \\
  0 & 0 & 1 & j \\
  j & -1 & 0 & 0 \\
  1 & j & 0 & 0
\end{pmatrix},
\]

\[
\hat{\lambda}_7 = \frac{1}{2} \begin{pmatrix}
  0 & 0 & 1 & j \\
  0 & 0 & -j & 1 \\
 -1 & -j & 0 & 0 \\
  j & -1 & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
  j & 0 & 0 & 0 \\
  0 & j & 0 & 0 \\
  0 & 0 & -j & 1 \\
  0 & 0 & -1 & -j
\end{pmatrix}. \quad (15)
\]

**B. Multiple copies of the gauge group**

Equations (10) define one of the gauge group copies coexisting in CSTA; by convention we label this as copy #0; copy #1 is obtained by replacing the first of those equations with Eq. (12), resulting in the following matrices

\[
\hat{\lambda}_1 = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & j \\
  0 & 0 & j & 0 \\
  0 & j & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & -1 & 0
\end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & j \\
  0 & 0 & j & 0
\end{pmatrix},
\]

\[
\hat{\lambda}_4 = \begin{pmatrix}
  0 & 0 & j & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  j & 0 & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_5 = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & -j & 0 \\
  0 & 0 & 0 & 0 \\
 -j & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{\lambda}_7 = \begin{pmatrix}
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
  2j & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -j & 0 \\
  0 & 0 & 0 & -j
\end{pmatrix}. \quad (16)
\]
Similarly copy #2 is obtained by replacing the second Eq. (10) with Eq. (13)

\[
\hat{\lambda}_1 = \begin{pmatrix} 0 & j & 0 & 0 \\ j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 & -j \\ 0 & 0 & 0 & 0 \\ -j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} -j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 2j & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (17)
\]

Finally copy #3 is obtained by making the two previous replacements simultaneously

\[
\hat{\lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j \\ 0 & 0 & j & 0 \end{pmatrix}, \quad \hat{\lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{\lambda}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & -j \end{pmatrix},
\]

\[
\hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j \\ 0 & j & 0 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2j & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j \end{pmatrix}. \quad (18)
\]

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