THERE IS NO UNIVERSAL PROPER METRIC SPACES FOR ASYMPTOTIC DIMENSION 1

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Abstract. Answering a question of Ma, Siegert, and Dydak we show that there is no universal proper metric space for the asymptotic dimension \( n \geq 1 \).

1. Introduction

The notion of asymptotic dimension is introduced by Gromov \[4\]. A family \( \mathcal{A} \) of subsets of a metric space \( X \) is called uniformly bounded if there is \( M > 0 \) such that \( \text{diam}(A) \leq M \), for every \( A \in \mathcal{A} \). Given \( D > 0 \), we say that a family \( \mathcal{A} \) is \( D \)-discrete if \( d(A, B) = \inf \{ d(a, b) \mid a \in A, b \in B \} \geq D \) for every distinct \( A, B \in \mathcal{A} \).

We say that the asymptotic dimension of \( X \) is \( \leq n \) (written \( \text{asdim} X \leq n \)) if for every \( D > 0 \) there exists a uniformly bounded cover \( \mathcal{U} \) of \( X \) such that \( \mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i \), where every \( \mathcal{U}_i \), \( i = 0, 1, \ldots, n \), is \( D \)-discrete.

Universal spaces for the asymptotic dimensions \( \leq n \) (we do not provide here a precise definition) are constructed in \[2\] and \[1\].

Recall that a metric space \( X \) is proper if every closed ball in it is compact. It is proved in \[5\], Theorem 7.2 that there exists a countable proper ultrametric space \( PU \) such that any proper metric space \( X \) of asymptotic dimension 0 coarsely embeds (see the definition below) in \( PU \). In other words, there is a universal proper metric space of asymptotic dimension 0.

Note that universal spaces for asymptotic dimension are constructed in \[3, 1\].

The following problem is formulated in \[5\]: Given \( n \geq 1 \) is there a universal space in the class of proper metric spaces of asymptotic dimension at most \( n \)?

The aim of this note is to provide a negative answer.

2. Preliminaries

A metric space \( X \) is called geodesic if, for every \( x, y \in X \) there is an isometric embedding \( \alpha : [0, d(x, y)] \rightarrow X \) such that \( \alpha(0) = x \), \( \alpha(d(x, y)) = y \).

A map \( f : X \rightarrow Y \) is called asymptotically Lipschitz if there exist \( \lambda, s > 0 \) such that, for any \( x, y \in X \), \( d(f(x), f(y)) \leq \lambda d(x, y) + s \).

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A map \( f : X \to Y \) is called coarsely uniform if there exists a non-increasing function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{t \to \infty} \phi(t) = \infty \) and
\[
d_Y(f(x), f(y)) \leq \phi(d_X(x, y)), \quad x, y \in X.
\]

In [2] it is proved that any coarse uniform map defined on a geodesic metric space is asymptotically Lipschitz.

A coarse uniform map \( f : X \to Y \) is called a coarse embedding if there exists a non-increasing function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{t \to \infty} \psi(t) = \infty \) and
\[
d_Y(f(x), f(y)) \geq \phi(d_X(x, y)), \quad x, y \in X.
\]

Given \( D > 0 \), we say that a subset \( A \) of a metric space \( X \) is \( D \)-discrete if \( d(x, y) \geq D \) for all \( x, y \in A \) with \( x \neq y \).

Recall that a tree is a connected graph without cycles. We regard any connected graph as a metric space endowed with the geodesic metric. Every edge is assumed to be isometric to the unit line segment.

Given a metric space \((X, d)\) and \( \alpha > 0 \), we define \( \alpha X \) to be the metric space \((X, \alpha d)\).

By \( B_r(x) \) we denote the ball of radius \( r > 0 \) centered at \( x \).

3. Result

**Theorem 3.1.** There is no proper metric space \( X \) with the property that every proper metric space of asymptotic dimension \( \leq 1 \) admits a coarse embedding in \( X \).

**Proof.** Suppose the contrary and let \( X \) be such a space. Let \( x_0 \in X \) be a base point.

For \( r > 0 \), define
\[
\Phi(r) = \max\{|A| \mid A \subset B_r(x_0) \text{ is a 1-discrete subset}\}.
\]

Because of properness of \( X \), \( \Phi(r) \) is well-defined.

For every \( n \in \mathbb{N} \), define \( \Phi_n(r) = \Phi(nr + n) \), for all \( r > 0 \). There exists a nondecreasing function \( \Psi : [0, \infty) \to [0, \infty) \) satisfying the property: \( \lim_{r \to \infty} (\Psi(r)/\Phi_n(r)) = \infty \), for any \( n \in \mathbb{N} \).

Let \( T \) be a rooted tree with the root \( t_0 \). For any \( n \in \mathbb{N} \), define \( \Xi(n) = |\{t \in T \mid d(t, t_0) = n\}| \).

For every \( n \in \mathbb{N} \), attach \( \Psi(n) \) unit segments to \( n - 1 \in \mathbb{R}_+ \) each by one of its endpoints. The obtained rooted tree (the root is \( 0 \in \mathbb{R}_+ \)) is denoted by \( T_1 \). We see that \( \Xi(n) \geq \Psi(n) \).

Let \( S \) be the tree which is obtained by attaching, for any \( n \in \mathbb{N} \), a copy of \( T_n = nT_1 \) to \( n - 1 \in \mathbb{R}_+ \) by its root, which we denote by \( y_n \). Note that \( S \) is a geodesic metric space. For the sake of convenience, any point \( k \in T_n \subset S \) will be denoted by \( k_{T_n} \).

We are going to show that there is no coarse embedding of \( S \) into \( X \). Suppose the contrary, let \( f : S \to X \) be such an embedding. Since \( S \) is geodesic, the map \( f \) is asymptotically Lipschitz. Let \( \lambda, s \) be the corresponding constants from the definition of the asymptotically Lipschitz map.
Since $f$ is a coarse embedding, there exists $k \in \mathbb{N}$ such that $d_X(f(x), f(y)) \geq 1$ whenever $d_S(x, y) \geq k$. We obtain
\[
d_X(x_0, f(nT_k)) \leq d_X(x_0, f(nT_k)) + d_X(f(0T_k), nT_k) \leq d_X(x_0, f(nT_k)) + \lambda kn + s \leq mn + m,
\]
for a constant $m \in \mathbb{N}$ large enough.

We conclude that $\Psi(n) \leq \Xi(n) \leq \Phi(mn + m)$ for all $n \in \mathbb{N}$, which contradicts to the choice of $\Psi$.

\[
\square
\]

4. Remarks

A metric space is said to have bounded geometry if for every $R > 0$ there exists $C < \infty$ such that every 1-discrete set contained in a ball of radius $R$ is of cardinality at most $C$.

Given $n \geq 1$, is there a proper metric into which every proper metric space of bounded geometry and of asymptotic dimension at most $n$ can be coarsely embedded?

References

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