THE CLASS $S$ AS AN ME INVARIANT

HIROKI SAKO

Abstract. We prove that being in Ozawa’s class $S$ is a measure equivalence invariant.

1. Introduction

Measurable group theory is a new field and has been attracting many researchers having various backgrounds. The discipline deals with how much information on countable groups is preserved through measure equivalence. The notion of measure equivalence was given by Gromov [Gr] as a variant of quasi-isometry. Two groups are said to be measure equivalent (ME) if there exists an ME coupling, instead of topological coupling.

In most cases, much information is lost through ME couplings. For example, any two countable amenable groups are ME. This is a consequence of [OrWe], [CoFeWe]. Many people are interested in finding small measure equivalence classes (higher rank lattices [Fu1], mapping class groups with high complexity [Kid]) or in classifying non-amenable groups up to measure equivalence.

We will prove that Ozawa’s class $S$ defined in [Oz3] is an ME invariant class. The class was defined by means of topological amenability [AD] on the largest boundary. Ozawa and Popa proved classification results on group von Neumann algebras of the class $S$ ([Oz2], [OzPo1]).

Definition 1.1 ([Oz3]). A countable group $G$ is said to be in $S$ if the left-times-right translation action of $G \times G$ on $\beta G \cap G^c$ is amenable, where $\beta G \cap G^c$ is the Gelfand spectrum of the commutative $C^*$-algebra $\ell_\infty G/c_0 G$.

The following is the main theorem of this paper.

Theorem 1.2. If $G$ and $\Gamma$ are ME and if $\Gamma \in S$, then $G \in S$.

The class $S$ is an intermediate class between the set of exact groups and that of amenable groups, which are also ME invariant classes. These three classes are also characterized by topological amenability. A countable group is exact if and only if there exists an amenable action on a compact space [Oz1]. A countable group is amenable if and only if any continuous action on any compact space is amenable.

By Hjorth’s theorem [H], a countable group $G$ is treeable in the sense of Pemantle and Peres [PemPer], if and only if $G$ is ME to a free group ($\mathbb{Z}, \mathbb{F}_2$ or $\mathbb{F}_\infty$). As a corollary of Theorem 1.2, the class of treeable groups is an intermediate ME invariant class between $S$ and the set of amenable groups. We get the following fact on group von Neumann algebras:

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Corollary 1.3. If $G$ is ME to a free group, then the group von Neumann algebra $L(G)$ is solid, namely, every diffuse subalgebra has the injective relative commutant.

Since the free groups are in the class $\mathcal{S}$, if $G$ is ME to a free group, then $\Gamma \in \mathcal{S}$. By [Oz2], $L(G)$ is solid.

2. Measure Equivalence and Measure Embedding

The following is a generalization of Gromov’s measure equivalence (0.5.E in [Gr]).

Definition 2.1. Let $G$ and $\Gamma$ be countable groups. We say that the group $G$ measurably embeds into $\Gamma$, if there exist a standard measure space $(\Sigma, \nu)$, a measure preserving action of $G \times \Gamma$ on $\Sigma$ and measurable subsets $X, Y \subset \Sigma$ with the following properties:

$$\Sigma = \bigsqcup_{\gamma \in \Gamma} \gamma(X) = \bigsqcup_{g \in G} g(Y), \quad \nu(X) < \infty.$$  

Then we use the notation $G \preceq_{\text{ME}} \Gamma$. The infinite measure space $\Sigma$ equipped with the $G \times \Gamma$-action is called a measure embedding of $G$ into $\Gamma$. The measure embedding $\Sigma$ is said to be ergodic, if the $G \times \Gamma$-action is ergodic.

If the subset $Y$ also has finite measure, then $\Sigma$ gives an ME coupling between $G$ and $\Gamma$ and these groups are said to be measure equivalent (ME). As in the case of an ME coupling, if there exists a measure embedding of $G$ into $\Gamma$, there exists an ergodic one by using ergodic decomposition. See Lemma 2.2 in Furman [Fu1] for the proof.

Remark 2.2.  
(1) The relation $\preceq_{\text{ME}}$ is transitive; if $H \preceq_{\text{ME}} \Lambda$ and $\Lambda \preceq_{\text{ME}} \Gamma$, then $H \preceq_{\text{ME}} \Gamma$. The proof is same as that of transitivity of measure equivalence.  
(2) If countable groups $G$ and $\Gamma$ satisfy $G \preceq_{\text{ME}} \Gamma$ and if $\Gamma$ is amenable, then $G$ is also amenable. Exactness has the same property (see Remark 3.5).

3. The class $\mathcal{S}$ as an ME Invariant

Theorem 3.1 is stronger than Theorem 1.2 and follows from Proposition 3.2.

Theorem 3.1. If $G \preceq_{\text{ME}} \Gamma$ and $\Gamma \in \mathcal{S}$, then $G \in \mathcal{S}$.

Proposition 3.2. Suppose $\Gamma \in \mathcal{S}$. Let $\beta$ be a free measure preserving (m.p.) action of $\Gamma$ on a standard measure space $(Y, \mu)$ and let $\alpha$ be a free m.p. action of $G$ on a measurable subset $X \subset Y$ with measure 1. If their orbits satisfy $\alpha(G)(x) \subset \beta(\Gamma)(x)$ for a.e. $x \in X$, then $G \in \mathcal{S}$.

In this proposition, $Y$ can be an infinite standard measure space.

Proof of Theorem 3.1 from Proposition 3.2  
Suppose that $(\Sigma, \nu)$ gives an ergodic measure embedding of $G$ into $\Gamma$. Choose $G$ fundamental domain $Y$ and $\Gamma$ fundamental domain $X$. We can replace $\Sigma$ so that the $G$-action on $\Gamma \setminus \Sigma \cong X$ is free, by taking a product of $\Sigma$ and a $G$-probability space on which free m.p. $G$-action is given. Then we can consider $\Sigma$ comes from a stable orbit equivalence with constant $s = \nu(Y)/\nu(X) \in (0, \infty]$. This argument is covered by Lemma 3.2 in [Fu2] and Remark 2.14 in [MoSh] if $s < \infty$. We do the same argument in the case of $s = \infty$. 


We note that \( \Gamma \in \mathcal{S} \) if and only if \( \Gamma \times \mathbb{Z}/n\mathbb{Z} \in \mathcal{S} \). Replacing \( \Gamma \) with \( \Gamma \times \mathbb{Z}/n\mathbb{Z} \), if necessary, we get \( \mathcal{R}(G \rtimes X)^s \cong \mathcal{R}(\Gamma \rtimes Y) \) with constant \( 1 \leq s \), where \( \mathcal{R}(G \rtimes X) \), \( \mathcal{R}(\Gamma \rtimes Y) \) mean the orbit equivalence relations of free m.p. actions of \( G \) and \( \Gamma \). By Proposition 3.2, \( \Gamma \in \mathcal{S} \) implies \( G \in \mathcal{S} \).

To prove Proposition 3.2, we use the notation given as follows. We introduce a measure \( \nu \) on \( \mathcal{R}_\beta \) as the push forward under the map

\[
Y \times \Gamma \ni (y, \gamma) \mapsto (\alpha(\gamma)(y), y) \in \mathcal{R}_\beta,
\]

where the measure on \( Y \times \Gamma \) is given by the product of \( \mu \) and the counting measure. The measure \( \nu \) coincides with the measure defined in Feldman and Moore [FeMo].

The action \( \beta \) (resp. \( \alpha \)) gives a group action of \( \Gamma \) (resp. \( G \)) on \( L^\infty(Y) \) (resp. \( L^\infty(X) \)). We use the same notation \( \beta \) (resp. \( \alpha \)) for this action. Let \( p \in L^\infty(Y) \) be the characteristic function of \( X \). The algebra \( L^\infty(Y) \) and the group \( \Gamma \) are represented on \( L^2(\mathcal{R}_\beta, \nu) \) as

\[
(F\xi)(x,y) = F(x)\xi(x,y), \quad F \in L^\infty(Y),
\]

\[
(u_\gamma\xi)(x,y) = \xi(\beta(\gamma^{-1})(x), y), \quad \gamma \in \Gamma, \xi \in L^2(\mathcal{R}_\beta), (x,y) \in \mathcal{R}_\beta.
\]

Let \( B \) be the \( C^* \)-algebra generated by the images, which is the reduced crossed product algebra \( B = L^\infty(Y) \rtimes_{\text{red}} \Gamma \). Its weak closure is the group measure space construction \( \mathcal{M} = L^\infty(Y) \rtimes_{\text{red}} \Gamma \). We denote by \( \text{tr} \) the canonical faithful normal semi-finite trace on \( \mathcal{M} \) with normalization \( \text{tr}(p) = 1 \). The unitary involution \( J \) of \( (\mathcal{M}, \text{tr}) \) is written as \( (J\xi)(x,y) = \overline{\xi(y,x)}, (\xi \in L^2(\mathcal{R}_\beta), (x,y) \in \mathcal{R}_\beta) \).

The group \( G \) is represented on \( pL^2(\mathcal{R}_\beta) = L^2(\mathcal{R}_\beta \cap X \times Y) \) by

\[
(v_\gamma\xi)(x,y) = \xi(\alpha(g^{-1})(x), y), \quad g \in G, \xi \in pL^2(\mathcal{R}_\beta), (x,y) \in \mathcal{R}_\beta \cap (X \times Y).
\]

We denote by \( C^*_\alpha(G) \) the \( C^* \)-algebra generated by these operators. The algebra is isomorphic to the reduced group \( C^* \)-algebra of \( G \). The Hilbert space \( L^2(\mathcal{R}_\alpha, \nu) \) can be identified with a closed subspace of \( pL^2(\mathcal{R}_\beta) \). The algebra \( C^*_\alpha(G) \) is also represented on \( L^2(\mathcal{R}_\alpha) \) faithfully. We denote by \( P \) the orthogonal projection from \( L^2(\mathcal{R}_\beta) \) onto \( L^2(\mathcal{R}_\alpha) \). We note that the algebra \( pPp \) does not contain \( C^*_\alpha(G) \) in general, although there exists an inclusion between their weak closures.

Let \( e_\Delta \) be the projection from \( L^2(\mathcal{R}_\beta) \) onto the set of \( L^2 \)-functions supported on the diagonal subset of \( \mathcal{R}_\beta \). This is the Jones projection for \( L^\infty(Y) \subset \mathcal{M} \). For \( \gamma \in \Gamma \) and a finite subset \( \Gamma_0 \subset \Gamma \), we define the projections \( e(\gamma), e(\Gamma_0) \) by

\[
e(\gamma) = Ju_\gamma J e_\Delta J u_\gamma^* J, \quad e(\Gamma_0) = \sum_{\gamma \in \Gamma_0} e(\gamma).
\]

For \( g \in G \) and a finite subset \( G_0 \subset G \), we define the projections \( f(g), f(G_0) \) by

\[
f(g) = v_g e_\Delta v^*_g = v_g p e_\Delta v^*_g, \quad f(G_0) = \sum_{g \in G_0} f(g).
\]

Let \( K \subset \mathcal{B}(L^2(\mathcal{R}_\beta)) \) be the hereditary subalgebra of \( \mathcal{B}(L^2(\mathcal{R}_\beta)) \) with approximate units \( \{ e(\Gamma_0) \mid \Gamma_0 \subset \Gamma \text{ finite} \} \), that is,

\[
K = \bigcup_{\Gamma_0} e(\Gamma_0) \mathcal{B}(L^2(\mathcal{R}_\beta)) e(\Gamma_0) \| \cdot \|_w.
\]

It is easy to see that \( B \) and \( JBJ \) are in the multiplier of \( K \), so is \( D = C^*(B, JBJ) \).
The algebra $B$ satisfies the following continuity property, which is similar to Proposition 4.2 in [Oz3].

**Proposition 3.3.** The $*$-homomorphism

$$
\tilde{\Psi} : B \otimes C JBJ \to (D + K)/K
$$

given by $\tilde{\Psi}(b \otimes c) = bc + K$ is continuous with respect to the minimal tensor norm.

For the rest of this paper, $\otimes$ stands for the minimal tensor product between two $C^*$-algebras.

**Proof.** Consider the representation of $\ell_\infty \Gamma$ on $L^2(\mathcal{R}_\beta)$ given by

$$(m_\phi \xi)(\gamma y, y) = \phi(\gamma) \xi(\gamma y, y), \quad \phi \in \ell_\infty \Gamma, \xi \in L^2(\mathcal{R}_\beta), \gamma \in \Gamma, y \in Y.$$

Let $\tilde{D}$ be the $C^*$-algebra generated by $D$ and $m(\ell_\infty \Gamma)$. The algebra $\tilde{D}$ is also in the multiplier of $K$. Since preimage $m^{-1}(K \cap \text{image}(m))$ is $c_0 \Gamma$, $m$ also gives an embedding of $\ell_\infty \Gamma/c_0 \Gamma$ into $(\tilde{D} + K)/K$. The embedding $m$ and the representation of $L^\infty Y$ give a $*$-homomorphism

$$
\tilde{\Psi} : E = \ell_\infty \Gamma/c_0 \Gamma \otimes L^\infty Y \otimes JL^\infty Y J \to (\tilde{D} + K)/K.
$$

Here, we used the fact that abelian $C^*$-algebras are nuclear [Tak]. Consider the action of $\Gamma \times \Gamma$ on $E$ given by

$$
\mathcal{A}(\gamma_1, \gamma_2)((\phi + c_0 \Gamma) \otimes f_1 \otimes J f_2 J) = (l, r_\lambda \phi + c_0 \Gamma) \otimes \beta(\gamma_1)(f_1) \otimes J \beta(\gamma_2)(f_2) J,
$$

where $l.$ and $r.$ stand for the left and the right translations. Since $\ell_\infty \Gamma/c_0 \Gamma$ is in the center of $E$, by [AD], the full crossed product coincides with the reduced crossed product $F = E \rtimes_{\text{red}} (\Gamma \times \Gamma)$ and this is nuclear. The unitary representations $u.$ and $Ju.J$ give a $*$-homomorphism $\tilde{\Psi} : F \to (\tilde{D} + K)/K$. By restricting $\tilde{\Psi}$ on $(L^\infty Y \otimes JL^\infty Y J) \rtimes_{\text{red}} (\Gamma \times \Gamma)$, we get a map satisfying Proposition 3.3. \qed

If the $*$-homomorphism $\Psi : B \otimes C JBJ$ given by $\Psi(b \otimes c) = bc$ is continuous with respect to the minimal tensor norm, then the group $\Gamma$ is amenable. The above proposition can be regarded as a weakened amenability property for the $\Gamma$-action.

We make use of the following characterization of $S$. Proposition 15.2.3 and a variant of Lemma 15.1.4 in [BrOz] imply the following.

**Proposition 3.4.** A countable group $G$ is in $S$ if and only if $G$ is exact and there exists a contractive c.p. map $\Phi : C^*_\alpha(G) \otimes C^*_\beta(G) \to B(\ell_2 G)$, satisfying

$$
\Phi(b \otimes c) - bc \in \mathcal{K}(\ell_2 G), \quad b \in C^*_\alpha(G), c \in C^*_\beta(G),
$$

where $C^*_\alpha(G)$ and $C^*_\beta(G)$ are the $C^*$-algebras generated by the left and right regular representations, respectively.

**Remark 3.5.** By using the notion of weak exactness introduced in Kirchberg [Kir], we get that the exactness on $\Gamma$ implies that on $G$. Indeed, the algebra $\mathcal{M}$ is weakly exact by the exactness of $\Gamma$. The subalgebra $L(G) \subset p\mathcal{M}p$ is also weakly exact. Thus $G$ is exact by Ozawa’s theorem [Oz3].

We have only to show the existence of $\Phi$ in Proposition 3.4. However, lack of inclusion “$C^*_\alpha(G) \subset B$” requires some technical elaboration.

**Lemma 3.6.** There exists a sequence $\{q_n\}_{n=1,2,\ldots}$ of projections in $L^\infty(X)$ satisfying:
The claim reduces to the inequality \( \| q_n f(G_0) \| \leq e(\Gamma_0) \).

(2) For any finite subset \( G_0 \subset G \) and \( n \), there exists a finite subset \( \Gamma_0 \subset \Gamma \) satisfying \( q_n f(G_0) \leq e(\Gamma_0) \).

(3) For any finite subset \( \Gamma_0 \subset \Gamma \) and \( n \), there exists a finite subset \( G_0 \subset G \) satisfying \( q_n e(\Gamma_0) P \leq f(G_0) \).

We note that projections \( P, p, e(\gamma), f(g), (\gamma \in \Gamma, g \in G) \) are in the commutative von Neumann algebra \( L^\infty(\mathcal{R}_\beta) \subset \mathcal{B}(L^2(\mathcal{R}_\beta)) \). Every projection in \( L^\infty(\mathcal{R}_\beta) \) which is less than \( f(g) \) (resp. \( pe(\gamma) \)) is of the form \( qf(g) \) (resp. \( qe(\gamma) \)) for some \( q \in L^\infty(X) \). We also note that \( qf(g) \leq e(\Gamma_0) \) if and only if there exists a partition \( \{ q_\gamma \in L^\infty(X) \mid \gamma \in \Gamma_0 \} \) of \( q \) such that \( qv_\gamma = \sum_{\gamma \in \Gamma_0} q_\gamma u_\gamma \).

**Proof.** We fix an index on \( G \); \( \{ g_1, g_2, \ldots \} = G \). For any \( g_k \), the projection \( e(\Gamma_0) f(g_k) \) can be written as \( Q(g_k, \Gamma_0) f(g_k) \) by some projection \( Q(g_k, \Gamma_0) \in L^\infty(X) \). The net of projections \( \{ e(\Gamma_0) f(g_k) \mid \Gamma_0 \subset \Gamma \) finite \} strongly converges to \( f(g_k) \). For any natural number \( n \), there exists a finite subset \( \Gamma_{k,n} \subset \Gamma \) such that \( \text{tr}(Q(g_k, \Gamma_{k,n})) \geq 1 - 2^{-(n+k)} \).

Then the projections \( Q_n = \bigwedge_{k=1}^\infty Q(g_k, \Gamma_{k,n}) \) satisfy \( \text{tr}(Q_n) \geq 1 - 2^{-2n} \) and

\[
Q_n f(g_k) \leq Q(g_k, \Gamma_{k,n}) f(g_k) = e(\Gamma_{k,n}) f(g_k) \leq e(\Gamma_{k,n}).
\]

Let \( \{ q_n \} \) be the increasing sequence of projections \( \{ \bigvee_{l=1}^n Q_l \} \). Then we have \( \text{tr}(q_n) \geq 1 - 2^{-n} \) and

\[
q_n f(g_k) \leq e \left( \bigcup_{l=1}^n \Gamma_{k,l} \right), \quad g_k \in G.
\]

It turned out that the sequence \( \{ q_n \} \) satisfies (1) and (2).

By a similar technique, we get a sequence \( \{ p_n \} \) with (1) and (3), since

\[
\text{str lim}_{G_0} f(G_0) e(\gamma) = e(\gamma) P, \quad \gamma \in \Gamma.
\]

Taking products \( \{ p_n q_n \} \), we get a sequence which satisfies (1), (2) and (3) at the same time. \( \square \)

The Hilbert space \( \ell_2 G \) embeds into \( L^2(\mathcal{R}_\alpha) \) by the isometry

\[
\ell_2 G \ni \delta_g \mapsto v_g \xi_\Delta = Jv_g^*J\xi_\Delta \in L^2(\mathcal{R}_\alpha),
\]

where the \( L^2 \)-function \( \xi_\Delta \) is the characteristic function of the diagonal subset of \( \mathcal{R}_\alpha \).

We regard \( \ell_2 G \) as a subspace of \( L^2(\mathcal{R}_\alpha) \) by this map. The subspace \( \ell_2 G \) is invariant under the action of \( C^*_\alpha G \) and \( JC^*_\alpha G J \).

**Lemma 3.7.** For a projection \( q \in L^\infty(X) \), the following inequality on operator norm holds true: \( \| (1 - qJqJ) \|_{\ell_2 G} \leq (2 - 2\text{tr}(q))^{1/2} \).

**Proof.** It suffices to show \( \| \eta - qJqJ \eta \| \leq (2 - 2\text{tr}(q)) \| \eta \|^2 \) for any vector \( \eta \in \ell_2 G \). Since \( qJqJ P = PqJqJ \) and \( \eta = P\eta \), we get

\[
\| \eta - qJqJ \eta \|^2 = \sum_{g \in G} \| f(g) \eta - qJqJ \eta \|^2 = \sum_{g \in G} \| f(g) \eta - qJqJ f(g) \eta \|^2,
\]

\[
\| \eta \|^2 = \sum_{g \in G} \| f(g) \eta \|^2.
\]

The claim reduces to the inequality \( \| f(g) \eta - qJqJ f(g) \eta \|^2 \leq (2 - 2\text{tr}(q)) \| f(g) \eta \|^2 \).
We note that \( \eta \) takes a constant value \( \eta(g) \) on the set \( \{ (\alpha(g)(x), x) \in R_\alpha \mid x \in X \} \). By a direct computation, we get
\[
\|f(g)\eta\|^2 = \int_{R_\alpha} |(f(g)\eta)(y, x)|^2d\nu = \int_{x \in X} |\eta(\alpha(g)(x), x)|^2d\mu = |\eta(g)|^2.
\]
Let \( X_0 \subset X \) be a measurable subset such that \( \chi(X_0) = p - q \). The measure of subset \( X_0 = \{ x \in X \mid x \in X_0 \} \) satisfies \( \nu(X_0) \leq 2 - 2\text{tr}(q) \). Then we get
\[
\|f(g)\eta - qJqJf(g)\eta\|^2 = \int_{X_0} |\eta(\alpha(g)(x), x)|^2d\mu = \mu(X_0)|\eta(g)|^2.
\]
Our claim was confirmed.

We finish the proof of Proposition 3.2.

**Proposition 3.2.** We will show the existence of \( \Phi \) in Proposition 3.4. We consider that \( C^*_\lambda(G) \) is a subalgebra of \( B(pL^2(R_\alpha)) \) and \( C^*_\rho(G) \) is \( JC^*_\lambda(G)J \). By \( P_0 \) we denote the orthogonal projection from \( L^2(R_\alpha) \) onto \( \ell_2G \).

Let \( \{ q_n \} \) be the sequence satisfying Lemma 3.6, and let \( B_0 \) be the \( C^* \)-algebra generated by \( p \) and \( \bigcup_n q_nC^*_\rho(G)q_n \). The condition (2) in Lemma 3.6 means \( B_0 \subset B \). We recall that \( B_0 \otimes JB_0J \) is separable and that \( F \) in the proof of Proposition 3.3 is nuclear. By Choi–Effros lifting theorem [Che], there exists a contractive c.p. lifting \( \Psi : B_0 \otimes JB_0J \to D + K \) for \( \tilde{\Psi}|_{B_0 \otimes JB_0J} \). We define contractive c.p. maps \( \Phi_n : C^*_\lambda(G) \otimes C^*_\rho(G) \to B(\ell_2G) \) by
\[
\Phi_n(b \otimes JcJ) = P_0Q_n\Psi(q_nbq_n \otimes Jq_ncq_nJ)Q_nP_0,
\]
where \( Q_n = q_nJq_nJ \). By the condition (3) in Lemma 3.6, we get \( P_0Q_nKQ_nP_0 \subset \mathcal{K}(\ell_2G) \). The element \( \Phi_n(b \otimes JcJ) \) is in
\[
P_0Q_n(bJcJ + K)Q_nP_0 \subset P_0Q_nbJcJQ_nP_0 + \mathcal{K}(\ell_2G).
\]
The sequence \( \{ P_0Q_nbJcJQ_nP_0 + \mathcal{K}(\ell_2G) \} \subset B(\ell_2G)/\mathcal{K}(\ell_2G) \) converges to \( P_0bJcJP_0 + \mathcal{K}(\ell_2G) \), by the inequality
\[
\| P_0bJcJP_0 - P_0Q_nbJcJQ_nP_0 \| \\
\leq \| P_0(1 - Q_n)bJcJP_0 \| + \| P_0Q_nbJcJ(1 - Q_n)P_0 \| \\
\leq 2\| (1 - Q_n)P_0\|\|b\|\|c\| \\
\leq 2(2 - 2\text{tr}(q_n))^{1/2}\|b\|\|c\|.
\]
It follows that the natural \(*\)-homomorphism from the minimal tensor product \( \tilde{\Phi} : C^*_\lambda(G) \otimes C^*_\rho(G) \to B(\ell_2G)/\mathcal{K}(\ell_2G) \) is given and \( \tilde{\Phi} \) is a limit of liftable maps. By Theorem 6 of [Ar], there exists a contractive c.p. lifting \( \Phi \) for \( \tilde{\Phi} \). □

**4. Final remark**

As a consequence of Proposition 3.2, we get the following indecomposability of an equivalence relation given by a class \( S \) group.

**Corollary 4.1.** Let \( \Gamma \) be a countable group and let \( H \subset G \) be an inclusion of countable groups. Suppose that \( \Gamma \in S \) and that the centralizer \( Z_G(H) \) is non-amenable. Let \( \beta \) be a free m.p. action of \( \Gamma \) on a standard measure space \( (Y, \mu) \) and let \( \alpha \) be a
free m.p. action of $G$ on a measurable subset $X \subset Y$ with measure 1. If the orbits satisfy $\alpha(G)(x) \subset \beta(\Gamma)(x)$ for a.e. $x \in X$, then $H$ is finite.

Proof. The class $S$ has the following property: If $G \in S$ and $Z_G(H)$ is non-amenable, then $H$ is finite. □

In particular, the group $G$ is not a direct product group of an infinite group and a non-amenable group. Word hyperbolic groups are typical examples of $S$ groups. Adams [Ad] showed a measurable indecomposability of non-amenable word hyperbolic groups. The above corollary covers some part of Adams’ theorem.

The class $C$ of [MoSh] also contains non-amenable word-hyperbolic groups. A group $G \in C$ satisfies an indecomposability property; $G$ has no infinite normal amenable subgroup. On the other hand, a non-amenable class $S$ group can have an infinite normal amenable subgroup (for example, $\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z}) \in S$ [Oz5]).

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Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo, 153-8914, Japan.

E-mail address: hiroki@ms.u-tokyo.ac.jp