Some exact results of the generalized Turán numbers for paths *

Doudou Hei\textsuperscript{1}, Xinmin Hou\textsuperscript{1,2}, Boyuan Liu\textsuperscript{1}

\textsuperscript{1}School of Mathematics Sciences
University of Science and Technology of China, Hefei, Anhui 230026, China
\textsuperscript{2}CAS Key Laboratory of Wu Wen-Tsun Mathematics
University of Science and Technology of China, Hefei, Anhui 230026, China

Abstract

For graphs $H$ and $F$ with chromatic number $\chi(F) = k$, we call $H$ strictly $F$-Turán-good (or $(H, F)$ strictly Turán-good) if the Turán graph $T_{k-1}(n)$ is the unique $F$-free graph on $n$ vertices containing the largest number of copies of $H$ when $n$ is large enough. Let $F$ be a graph with chromatic number $\chi(F) \geq 3$ and a color-critical edge and let $P_{\ell}$ be a path with $\ell$ vertices. Gerbner and Palmer (2020, arXiv:2006.03756) showed that $(P_{3}, F)$ is strictly Turán-good if $\chi(H) \geq 4$ and they conjectured that (a) this result is true when $\chi(F) = 3$, and, moreover, (b) $(P_{\ell}, K_{k})$ is Turán-good for every pair of integers $\ell$ and $k$. In the present paper, we show that $(H, F)$ is strictly Turán-good when $H$ is a bipartite graph with matching number $\nu(H) = \left\lfloor \frac{|V(H)|}{2} \right\rfloor$ and $\chi(F) = 3$, as a corollary, this result confirms the conjecture (a); we also prove that $(P_{\ell}, F)$ is strictly Turán-good for $2 \leq \ell \leq 6$ and $\chi(F) \geq 4$, this also confirms the conjecture (b) for $2 \leq \ell \leq 6$ and $k \geq 4$.

1 Introduction

Fix a graph $F$, we say that a graph $G$ is $F$-free if it does not contain $F$ as a subgraph. Let $P_{\ell}$, $C_{\ell}$ and $K_{\ell}$ denote a path, cycle and complete graph on $\ell$ vertices, respectively. Fix graphs $H$ and $G$, we denote the number of copies of $H$ in $G$ by $N(H, G)$. We say that an edge $e$ of a graph $F$ is color-critical if deleting $e$ from $F$ results in a graph with smaller chromatic number. Let $e(G)$ denote the number of edges of $G$ and write $\nu(G)$ and $\chi(G)$ for the matching number and chromatic number of $G$, respectively. Throughout the paper, let $[k] = \{1, 2, \ldots, k\}$ for a positive integer $k$.

A fundamental result in extremal graph theory is the Turán Theorem. It states that the Turán graph $T_{k-1}(n)$, which is the complete $(k-1)$-partite graph on $n$ vertices, where each partite class has cardinality $\left\lfloor \frac{n}{k-1} \right\rfloor$ or $\left\lceil \frac{n}{k-1} \right\rceil$, has the largest number of edges among all the $K_{k}$-free graph. Generally, Turán theory deals with the function $ex(n, F)$, which is the largest number of edges in $n$-vertex $F$-free graphs. We call an $n$-vertex $F$-free graph with $ex(n, F)$ edges an extremal graph for $F$. See, for example, [26] for a survey.

*The work was supported by National Natural Science Foundation of China (No. 12071453) and the National Key R and D Program of China(2020YFA0713100).
For two graphs $H$ and $F$, the \textit{generalized Turán number} $	ext{ex}(n,H,F)$ is the largest number of copies of $H$ in an $F$-free graph on $n$ vertices, i.e.,

$$
\text{ex}(n,H,F) = \max\{N(H,G) : G \text{ is an } F\text{-free graph on } n \text{ vertices}\}.
$$

Similarly, we call an $n$-vertex $F$-free graph with $\text{ex}(n,H,F)$ copies of $H$ an \textit{extremal graph}. After several sporadic results, Alon and Shikhelman \cite{2} studied this problem systematically. Since then, this problem has attracted many researchers, see e.g. \cite{7,13,14,15,16,20,21,23}.

However, there are not many pairs $(H,F)$ for which the exact values of $\text{ex}(n,H,F)$ were determined. As pointed by Gerbner and Palmer \cite{15}, there are few $F$-free graphs that are good candidates for being extremal constructions for maximizing copies of $H$, an exception is the Turán graph, they call $H$ to be $F$-Turán-good under this situation. More precisely, given a $k$-chromatic graph $F$ and a graph $H$ that does not contain $F$ as a subgraph, we say that $H$ is $F$-Turán-good if $\text{ex}(n,H,F) = N(H,T_{k-1}(n))$ (and the Turán graph $T_{k-1}(n)$ is the unique extremal graph) for every $n$ large enough, we also call $(H,F)$ to be Turán-good for short. Here is a list of some Turán-good pairs as we have known so far:

(i) (Zykov \cite{27}, Erdős \cite{3}) $(K_r,K_k)$ is strictly Turán-good for $2 \leq r < k$;

(ii) (Simonovits \cite{25}, Ma and Qiu \cite{20}) $(K_r,F)$ is strictly Turán-good, where $F$ is a graph with $\chi(F) > r \geq 2$ and a color-critical edge;

(iii) (Győri, Pach and Simonovits \cite{17}) $(H,K_3)$ is strictly Turán-good, where $H$ is a bipartite graph with matching number $\lfloor \frac{|V(H)|}{2} \rfloor$ (including the path $P_\ell$, the even cycle $C_{2\ell}$ and the Turán graph $T_2(m)$);

(iv) (Győri, Pach and Simonovits \cite{17}) $(K_{2,\ell},K_r)$ are strictly Turán-good for $t = 2,3$.

(v) (Gerbner and Palmer \cite{15}) $(H,K_k)$ is Turán-good for $k \geq k_0$, where $H$ is a complete multipartite graph and $k_0$ is a constant depending on $H$, and they also conjecture that this result is true for any graph $H$.

(vi) (Gerbner and Palmer \cite{15}) $(P_4,C_5)$ and $(C_4,C_5)$ are Turán-good, and in general, they conjectured that $(P_k,C_{2\ell+1})$ and $(C_{2k},C_{2\ell+1})$ are Turán-good. The asymptotic version of this conjecture has been proved by Gerbner et al \cite{13}.

(vii) (Gerbner and Palmer \cite{15}) $(P_4,B_2), (C_4,B_2)$ and $(C_4,F_2)$ are Turán-good, where $B_k$ (resp. $F_k$) is the graph of $k$ triangles all sharing exactly one common edge (resp. one common vertex).

(viii) (Gerbner and Palmer \cite{15}) $(P_3,F)$ is Turán-good, where $F$ is a graph with chromatic number $\chi(F) = k \geq 4$ and a color-critical edge.

(ix) (Murphy and Nir \cite{21}, Qian et al \cite{22}) $(P_4,K_k)$ and $(P_3,K_k)$ are Turán-good for $k \geq 4$.

Gerbner and Palmer \cite{15} conjectured that the result (viii) is still holds when the chromatic number $\chi(F) = 3$ and they also proposed a conjecture that $(P_\ell, K_k)$ is Turán-good for every pair of integers $\ell$ and $k$. 

2
Conjecture 1.1 (Gerbner and Palmer [15]). \((P_3, F)\) is Turán good if \(F\) is a graph with chromatic number \(\chi(F) \geq 3\) and a color-critical edge.

Conjecture 1.2 (Gerbner and Palmer [15]). For every pair of integers \(\ell\) and \(k\), \((P_\ell, K_k)\) is Turán-good.

In this article, we first confirm Conjecture 1.1. In fact, we give a generalized version of (iii) due to Győri, Pach and Simonovits [17], Conjecture 1.1 can be confirmed as a corollary. The following is the first main result.

**Theorem 1.3.** Let \(F\) be a graph with \(\chi(F) = 3\) and a color-critical edge and let \(H\) be a bipartite graph with matching number \(\lfloor \frac{|V(H)|}{2} \rfloor\). Then \(H\) is strictly \(F\)-Turán good, i.e., \(ex(n, H, F) = N(H, T_2(n))\) for every \(n\) large enough. Moreover, the Turán graph \(T_2(n)\) is the unique extremal graph for \((H, F)\).

So Conjecture 1.1 is a straightforward corollary of Theorem 1.3 and (viii).

**Corollary 1.4.** If \(F\) is a graph with chromatic number \(\chi(F) \geq 3\) and a color-critical edge, then \((P_3, F)\) is Turán-good.

**Remarks:** Early this year, Prof. Gerbner told us that he has proved Conjecture 1.1 in [9] by the progressive induction method of Simonovits for generalized Turán problems, in the same paper and several other references provided by Prof. Gerbner, more Turán-good pairs have been proved, we continue to list them here.

(x) (Gerbner [9]) \((M_\ell, F)\) is Turán-good, where \(F\) is a graph with a color-critical edge.

(xi) (Gerbner [10]) For any positive integers \(m\) and \(\ell\), \((P_m, C_{2\ell+1})\) and \((C_{2m}, C_{2\ell+1})\) are Turán-good, these results resolved the conjecture (see (vi)) proposed by Gerbner and Palmer [15]; \((P_m, B_\ell)\) is Turán-good, where \(B_\ell\) is defined as in (vii).

(xii) (Gerbner [11, 12], Győri, Wang, Woolfson [18]) \((K_{a,b}, F)\) and \((S_{a,b}, F)\) are Turán-good, where \(F\) is a 3-chromatic graph with a color-critical edge, \(K_{a,b}\) and \(S_{a,b}\) are a complete bipartite graph and a double star with \(|a - b| \leq 1\), respectively.

For Conjecture 1.2, Murphy and Nir [21], and Qian et al [23] have confirmed this conjecture for \(P_4, P_5\) and \(k \geq 4\), we continue to confirm this conjecture for \(P_6\) and \(k \geq 4\) by showing the following a little more generalized result.

**Theorem 1.5.** Let \(F\) be a graph with chromatic number \(\chi(F) = k \geq 4\) and a color-critical edge. Then the following holds.

(a) If \(2 \leq \ell \leq 6\), then \((P_\ell, F)\) is strictly Turán-good, i.e., \(ex(n, P_\ell, F) = N(P_\ell, T_{k-1}(n))\), and \(T_{k-1}(n)\) is the unique extremal graph for \(n\) large enough.

(b) There is \(k_0\) such that \((P_\ell, F)\) is strictly Turán-good for \(\chi(F) = k \geq k_0\).

The rest of this paper is organized as follows. In Section 2 we give some preliminaries. Next, in Section 3, we prove a technical theorem, which is important in the proofs of Theorems 1.3 and 1.5. We will give the proofs of Theorems 1.3 and 1.5 in Sections 4 and 5, respectively. Finally, we briefly resume this work and give some new lines of research in the Conclusions.
2 Preliminaries

In this section we will present some definitions and results needed in the subsequent sections. Fix a graph $H$ and consider a graph $G$. For each $v \in V(G)$, let $d_G(v, H)$ denote the number of copies of $H$ in $G$ containing the vertex $v$, and let $\delta(G, H) = \min_{x \in V(G)} d_G(x, H)$. Let $\overline{H(G)}$ denote the number of different embeddings $\varphi : V(H) \to V(G)$ such that

1. $v_1 \neq v_2 \Rightarrow \varphi(v_1) \neq \varphi(v_2)$,
2. $v_1, v_2 \in E(H) \Rightarrow \varphi(v_1)\varphi(v_2) \in E(G)$

for every pair $v_1, v_2 \in V(H)$. Evidently, $\overline{H(G)}/N(H, G)$ is equal to the number of automorphisms of $H$. Hence, in any class of graphs $G$, $\overline{H(G)}$ and $N(H, G)$ attain their maximum for the same $G \in G$. Similarly, for each $v \in V(G)$, let $\overline{d_G}(v, H)$ denote the number of embeddings of $H$ in $G$ containing the vertex $v$, and let $\overline{e}(G, H) = \min_{v \in V(G)} \overline{d_G}(v, H)$.

Given a graph $G$, write $\epsilon(G)$ for $|E(G)|$. Let $X, Y$ be disjoint subsets of $V(G)$. By $G[X, Y]$ we denote the bipartite subgraph of $G$ consisting of all edges that have one endpoint in $X$ and another in $Y$. For mutually disjoint subsets $V_1, V_2, \ldots, V_k \subseteq V(G)$, similarly we define $G[V_1, \ldots, V_k]$ to be the $k$-partite subgraph of $G$ consisting of all edges in $\cup_{1 \leq i < j \leq k} E(G[V_i, V_j])$. Write $K(V_1, \ldots, V_k)$ for the complete $k$-partite graph with color classes $V_1, \ldots, V_k$ and write $K_{t_1, \ldots, t_k}$ for a complete $k$-partite graph $K(V_1, \ldots, V_k)$ with $|V_i| = t_i$ for $i \in [k]$.

An $s$ blow-up of a graph $H$ is the graph obtained by replacing each vertex $v$ of $H$ by an independent set $W_v$ of size $s$, and each edge $uv$ of $H$ by a complete bipartite graph between the corresponding two independent sets $W_u$ and $W_v$.

The following result due to Alon and Shikhelman [2] gave an asymptotical value of $\text{ex}(n, H, F)$ for general graphs $H$ and $F$.

**Proposition 2.1** [2]. Let $H$ be a fixed graph with $t$ vertices. Then $\text{ex}(n, H, F) = \Omega(n^t)$ if and only if $F$ is not a subgraph of a blow-up of $H$. Otherwise, $\text{ex}(n, H, F) \leq n^{t-\epsilon}$ for some $\epsilon = \epsilon(H, F) > 0$.

We give a simple observation for the graph with chromatic number $k$ and a color-critical edge.

**Observation 2.2.** If $F$ is a graph with $\chi(F) = k \geq 3$ and a color-critical edge then $F$ is a subgraph of a complete $k$-partite graph with one class of order one.

The following two classical results in extremal graph theory will be used.

**Lemma 2.3** (The stability lemma [1]). Let $F$ be a graph with $\chi(F) = k \geq 3$. Then, for every $\epsilon > 0$, there exist $\xi = \xi(F, \epsilon) > 0$ and $n_0 = n_0(F, \epsilon) \in \mathbb{N}$ such that the following holds. If $G$ is an $F$-free graph on $n \geq n_0$ vertices with $\epsilon(G) \geq \epsilon(T_{k-1}(n)) - \xi n^2$, then there exists a partition of $V(G) = V_1 \cup \ldots \cup V_{k-1}$ such that $\sum_{i=1}^{k-1} \epsilon(G[V_i]) < \epsilon n^2/2$.

**Lemma 2.4** (Erdős-Stone-Simonovits Theorem, [2]). For any graph $H$ with $\chi(H) = r$,

$$\text{ex}(n, H) = \left(1 - \frac{1}{r-1}\right) \binom{n}{2} + o(n^2).$$
The graph removal lemma given by Erdős, Frankl and Rödl [24], initiated from Ruzsa and Szemerédi [23], also plays an important role in our proofs. An improved version has been given by Fox [8]. One well-known application of the graph removal lemma is in property testing (one can see [2] for more details if interested).

**Lemma 2.5** (The graph removal lemma [5]). For each \( \epsilon > 0 \) and graph \( H \) on \( h \) vertices there is \( \delta = \delta(\epsilon,H) > 0 \) such that every graph on \( n \) vertices with at most \( \delta n^h \) copies of \( H \) can be made \( H \)-free by removing at most \( cn^2 \) edges.

Our proof will use the (iii) given by Győri, Pach and Simonovits [17], we restate it here.

**Theorem 2.6** ([17]). Let \( H \) be a bipartite graph with \( m \geq 3 \) vertices and \( \nu(H) = \lfloor \frac{m}{2} \rfloor \). Then, for every \( K_3 \)-free graph \( G \) with \( n > m \) vertices, \( N(H,G) \leq N(H,T_2(n)) \), and equality holds if and only if \( G \cong T_2(n) \).

Given a graph \( G \), write \( \mu(G) \) for the largest eigenvalue of its adjacency matrix, \( \omega(G) \) for its clique number, and \( W_k(G) \) for the number of walks of length \( k \) in \( G \). Nikiforov [22] showed that

**Lemma 2.7** ([22]). For every graph \( G \) and \( r \geq 1 \), \( \mu^r(G) \leq \frac{\omega(G)-1}{\omega(G)} W_{r-1}(G) \).

The following lemma will be used in the proof of Theorem [15](b).

**Lemma 2.8** (Gerber and Palmer [15]). For any graph \( H \) there are integers \( k_0 \) and \( n_0 \) such that if \( k \geq k_0 \) and \( n \geq n_0 \), then for any complete \((k-1)\)-partite \( n \)-vertex \( K \) we have \( N(H,K) \leq N(H,T_{k-1}(n)) \), and the equality holds if and only if \( G \cong T_{k-1}(n) \).

### 3 T-Extremal Case

We say a graph \( H \) has the weak \( k-T \)-property if \( N(H,K) \leq N(H,T_{k-1}(n)) \) for every complete \((k-1)\)-partite graph \( K = K_{t_1,...,t_{k-1}} \) with \( t_1 + \ldots + t_{k-1} = n \) and every \( n \) large enough, and the equality holds if and only if \( K \cong T_{k-1}(n) \). An \( n \)-vertex \( F \)-free graph \( G \) is called \( T \)-extremal if \( |e(G) - e(T_{k-1}(n))| = o(n^2) \).

**Theorem 3.1.** Let \( F \) be a graph with \( \chi(F) = k \geq 3 \) and a color-critical edge and let \( H \) be a connected graph with \( \chi(H) < k \). Suppose every \( F \)-free \( n \)-vertex graph \( G \) with \( N(H,G) = ex(n,H,F) \) is \( T \)-extremal. If \( H \) has the weak \( k-T \)-property, then \( G \cong T_{k-1}(n) \).

**Proof.** Suppose \( G \) is an \( n \)-vertex \( F \)-free graph with \( N(H,G) = ex(n,H,F) \) and \( n \) is large enough. Then \( N(H,G) \geq N(H,T_{k-1}(n)) \). Denote \( m := |V(H)| \). We may assume an additional condition for \( G \) that \( \delta(G,H) \geq \delta(T_{k-1}(n),H) \). Indeed, we can assume \( n \geq n_0 + a(m) \) for some sufficiently large \( n_0 \), where \( a = N(H,K_m) \). If \( G \) does not satisfy the property, then there is a vertex \( v_n \in V(G) \) such that \( d_G(v_n,H) \leq \delta(T_{k-1}(n),H) - 1 \). Set \( G_n = G \) and let \( G_{n-1} = G - v_n \). Then we have

\[
N(H,G_{n-1}) = N(H,G_n) - d_{G_n}(v_n,H) \\
\geq N(H,T_{k-1}(n)) - \delta(H,T_{k-1}(n)) + 1 \\
\geq N(H,T_{k-1}(n-1)) + 1.
\]
Assume that $G_{\ell}$ on $\ell$ vertices with
\[
N(H, G_{\ell}) \geq N(H, T_{k-1}(\ell)) + n - \ell
\]
has been defined for some $\ell \leq n - 1$. If there exists some vertex $v_{\ell} \in V(G)$ such that $d_{G_{\ell}}(v_{\ell}, H) \leq \delta(T_{k-1}(\ell)) - 1$, let $G_{\ell-1} = G_{\ell} - v_{\ell}$. Then we get
\[
N(H, G_{\ell-1}) = N(H, G_{\ell}) - d_{G_{\ell}}(v_{\ell}, H)
\geq N(H, T_{k-1}(\ell)) + n - \ell - \delta(T_{k-1}(\ell), H) + 1
\geq N(H, T_{k-1}(\ell - 1)) + n - \ell + 1;
\]
otherwise, terminate. Let $G_s$ be the graph for which the above iteration terminates. So $G_s$ has exactly $s$ vertices and $\delta(G_s, H) \geq \delta(T_{k-1}(s), H)$. If $s < n_0$, then we have
\[
a \left( \begin{array}{c} n_0 \\ s \end{array} \right) > a \left( \begin{array}{c} m \\ s \end{array} \right) \geq N(H, G_s) \geq N(H, T_{k-1}(s)) + n - s > n - n_0 \geq a \left( \begin{array}{c} n_0 \\ m \end{array} \right),
\]
a contradiction. So we have a subgraph $H$ of sufficiently large order $s(\geq n_0)$ with $N(H, G_s) \geq N(H, T_{k-1}(s)) + n - s$ and $\delta(G_s, H) \geq \delta(T_{k-1}(s), H)$. If we can show $G_s$ has a subgraph of some complete $(k - 1)$-partite graph $K = K_{t_1, \ldots, t_k}$ with $t_1 + \ldots + t_k = s$, then we have $N(H, G_s) \leq N(H, K)$. If $H$ has the weak $k$-T-property, then
\[
N(H, T_{k-1}(s)) + n - s \leq N(H, G_s) \leq N(H, K) \leq N(H, T_{k-1}(s)).
\]
So we have $n = s$ and $G_s = G \cong T_{k-1}(n)$. Therefore, since $s$ is large enough, we can do the same analysis on $G_s$ as $G$. For the sake of writing convenience, in the following proof, we still use $G$ to denote $G_s$ and show that $G \subseteq K_{t_1, \ldots, t_{k-1}}$ with $t_1 + \ldots + t_{k-1} = n$ and $|t_i - \frac{n}{k-1}| = o(n)$.

Let $V_1, \ldots, V_{k-1}$ be a partition of $V(G)$ such that $e(G[V_1]) + \ldots + e(G[V_{k-1}])$ is minimized. Since $G$ is T-extremal, for every $\varepsilon > 0$ (we may choose $\varepsilon$ sufficiently small), choose $\xi = \varepsilon/2$, when $n$ is large enough, we have
\[
e(G[V_1]) + \ldots + e(G[V_{k-1}]) < \varepsilon n^2/2 \tag{1}
\]
and
\[
e(G[V_1, \ldots, V_{k-1}]) > e(T_{k-1}(n)) - \varepsilon n^2. \tag{2}
\]

Claim 3.1. There exists some $\theta = \theta(\varepsilon)$ with $\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$ such that $\left| V_i - \frac{n}{k-1} \right| < \theta n$ for all $i \in [k-1]$.

Proof. Let $p_i := \frac{|V_i|}{n} \in [0, 1]$. So we have
\[
e(K(V_1, \ldots, V_{k-1})) = \sum_{1 \leq i < j \leq k-1} p_i p_j n^2
\geq e(G[V_1, \ldots, V_{k-1}])
\geq e(T_{k-1}(n)) - \varepsilon n^2
\geq \frac{k-2}{2(k-1)} n^2 - 2\varepsilon n^2.
\]
Since $\sum_{1 \leq i < j \leq k-1} p_i p_j$ is maximal if and only if $p_1 = \ldots = p_{k-1} = \frac{1}{k-1}$, and the maximum value is $\frac{k-2}{2(k-1)}$. By the continuity of $\sum_{1 \leq i < j \leq k-1} p_i p_j$, there exists $\theta = \theta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$ such that $\left| p_i - \frac{1}{k-1} \right| < \theta$. This proves the claim. \qed
Let $\beta = 2k\sqrt{\varepsilon}$ and $B_i = \{v \in V_i : |N_G(v) \cap V_i| > \beta n\}$ for $i \in [k-1]$. Let $B = \bigcup_{i=1}^{k-1} B_i$ and $U_i = V_i \setminus B$. Because $\beta > 2\sqrt{\varepsilon}$, we have

$$|B| < \frac{2\varepsilon n^2}{\beta n} < \frac{\beta n}{2}.$$ 

**Claim 3.2.** $B = \emptyset$.

*Proof.* Since $V_1, \ldots, V_{k-1}$ is a partition of $V(G)$ with minimum $e(G[V_1]) + \ldots + e(G[V_{k-1}])$, we get

$$|N_G(v) \cap V_i| \geq |N_G(v) \cap V_i|,$$

for any $v \in V_i$, $i, j \in [k-1]$, and $j \neq i$. This together with the definition of $B$ show that for any $v \in B$ and every $i \in [k-1]$, $|N_G(v) \cap V_i| > \beta n$. Since $U_i = V_i \setminus B$ and $|B| < \frac{\beta n}{2}$, it follows that $|N_G(v) \cap U_i| > \frac{\beta n}{2}$.

Suppose $B \neq \emptyset$. Consider an arbitrary vertex $v \in B$. Choose a subset $S_i \subseteq N_G(v) \cap U_i$ with $|S_i| = \frac{\beta n}{2}$ for $i \in [k-1]$ (this can be done since $|N_G(v) \cap U_i| > \frac{\beta n}{2}$). By the inequality (2)

$$e(G[S_1, \ldots, S_{k-1}]) \geq e(T_{k-1} \setminus ((k-1)\beta n/2)) - \varepsilon n^2 = e(T_{k-1} \setminus ((k-1)\beta n/2)) - o\left((k-1)\beta n/2\right)^2).$$

By Lemma 2.4 for $n$ large enough, $G[S_1, \ldots, S_{k-1}]$ contains a copy of the complete $(k-1)$-partite graph $K_{b_1, \ldots, b_k}$, where $b = |V(F)|$. As $S_i \subseteq N_G(v) \cap U_i$, by Observation 2.2 $G[v, S_1, \ldots, S_{k-1}]$ contains a copy of $F$, which is a contradiction. 

By Claim 3.2 for every $v \in V_i$, $|N_G(v) \cap V_i| \leq \beta n$.

**Claim 3.3.** There exists $\zeta = \zeta(\varepsilon)$ with $\lim_{\varepsilon \to 0} \zeta(\varepsilon) = 0$ such that

$$|N(v) \cap V_j| \geq \left(\frac{1}{k-1} - \zeta\right)n$$

for every $v \in V_i$ and $j \neq i$.

*Proof.* By symmetry, it suffices to prove that the claim is true for every $v \in V_1$. Fix a vertex $v \in V_1$, let $q_j = \frac{|N(v) \cap V_j|}{n}$. We will show this claim by estimating $\overline{d_G}(v, H)$.

First let us estimate the number of $H$-embeddings containing $v$ in $G[V_1, \ldots, V_{k-1}]$. Since $H$ is a connected graph, we can find an order of $V(H)$ starting with $v$ such that, after $v$, each vertex has at least one earlier neighbor (e.g. we can order the vertices of $V(H)$ by the breath-first search). Then we have $\sum_{j=2}^{k-1} q_j n$ ways to pick the second vertex, and always at most $n - |V_1| \leq n - \frac{n}{k-1} + \theta n$ ways to pick a new vertex, where $i$ is some integer in $[k-1]$ and the inequality holds since $|V_i| \geq \frac{1}{k-1} - \theta n$ by Claim 3.1. Thus the number of this kind of embeddings is at most $m \cdot \sum_{j=2}^{k-1} q_j n \cdot ((\frac{k-2}{k-1} + \theta)n)^{m-2}$.

Now, for each embedding of $H$ in $G$ that contains $v$ but is not in $G[V_1, \ldots, V_{k-1}]$, it must contain some edge in $\bigcup_{j=1}^{k-1} E(G[V_j])$. By (1) and $B = \emptyset$, the number of this kind is at most $2e(H) \cdot \beta n \cdot n^{m-2} + 2e(H) \cdot \varepsilon n^2/2 \cdot (m-2)n^{m-3}$. So we have

$$\overline{d_G}(v, H) \leq m \cdot \sum_{j=2}^{k-1} q_j n \cdot \left(\frac{k-2}{k-1} + \theta\right)n^{m-2} + (2\beta + (m-2)\varepsilon)e(H)n^{m-1}.$$
we have

\[ q \geq k \cdot \left( \frac{k-2}{k-1} \right)^{m-1} \cdot n^{m-1} + o(n^{m-1}), \]

we have

\[
\sum_{j=2}^{k-1} q_j \geq \frac{m \cdot \left( \frac{k-2}{k-1} \right)^{m-1} + o(1) - e(H)(2\beta + (m-2)\varepsilon)}{m \cdot \left( \frac{k-2}{k-1} + \theta \right)^{m-2}} = h(\theta, \beta, \varepsilon).
\]

Note that when \( \varepsilon \to 0 \) we have \( \beta \to 0 \) and \( \theta \to 0 \). So, when \( n \) is large enough,

\[
\lim_{\varepsilon \to 0} h(\theta, \beta, \varepsilon) = \frac{k-2}{k-1}.
\]

Since \( q_j < \frac{1}{k-1} + \theta \) and \( \theta = \theta(\varepsilon) \) is small enough, there exists \( \zeta = \zeta(\varepsilon) \) with \( \lim_{\varepsilon \to 0} \zeta(\varepsilon) = 0 \) such that \( q_j \geq \frac{1}{k-1} - \zeta, i \in \{2, \ldots, k-1\} \). This proves the claim.

**Claim 3.4.** Every \( V_i \) is an independent set in \( G \) for \( i \in [k-1] \).

**Proof.** Suppose to the contrary that say, there exists an edge \( e = xy \) in \( G[V_i] \). Choose \( F_1 \subseteq V_1 \) with \( x, y \in F_1 \) and \( |F_1| = m \). If we can find a complete \((k-1)\)-partite subgraph \( K(F_1, \ldots, F_{k-1}) \) in \( G \) such that \( F_i \subseteq V_i \) with \( |F_i| = m \), then, by Observation 2.2 there exists a copy of \( F \) in \( G \), which is a contradiction.

To do this, suppose inductively that for some \( i \in [k-1] \), we have obtained a complete \( i \)-partite subgraph \( K(F_1, \ldots, F_i) \) of \( G \). Then the number of common neighbors of \( H_i = \bigcup_{j=1}^{i} F_j \) in \( V_{i+1} \) is at least

\[
\sum_{v \in H_i} |N_G(v) \cap V_{i+1}| - (|H_i| - 1)|V_{i+1}|
\]

\[
\geq |H_i| \left( \frac{1}{k-1} - \zeta \right) n - (|H_i| - 1) \left( \frac{1}{k-1} + \theta \right) n
\]

\[
\geq \left( \frac{1}{k-1} - |H_i|(\theta + \zeta) \right) n
\]

\[
\geq m,
\]

the last inequality holds since \( \zeta, \theta \) are sufficiently small and \( n \) is sufficiently large. So we can find the desired \( F_{i+1} \) and the proof of the claim is completed.

Therefore, \( G \) must be a subgraph of \( K = K(V_1, \ldots, V_{k-1}) \) with \( ||V_i| - \frac{n}{k-1}| = o(n) \). So \( N(H, G) \leq N(H, K) \). The proof of the theorem is completed.

### 4 Proof of Conjecture [1.1]

It is sufficient to prove Theorem [1.3] A bipartite graph \( H \) is said to have the **strong T-property** with respect to \( F \) if for any \( F \)-free graph \( G \) on \( n \) vertices, \( N(H, G) \leq N(H, T_{\chi(F)-1}(n)) \) for \( n \) large enough, and equality holds if and only if \( G \cong T_{\chi(F)-1}(n) \). Clearly, if \( H \) has the strong T-property with respect to \( F \), then \( H \) has the weak \( \chi(F) \)-T-property.
Proof of Theorem 1.3. Recall that $F$ is a graph with $\chi(F) = 3$ and a color-critical edge and $H$ is a bipartite graph with $m \geq 3$ vertices and $\nu(H) = \lceil \frac{m}{2} \rceil$. It is sufficient to show that $H$ has the strong T-property with respect to $F$. Let $G$ be an $n$-vertex $F$-free graph with the largest number of the copies of $H$. Then $H(G) \geq H(T_2(n))$. We may assume that $H$ is connected. Indeed, let $H_1, \ldots, H_k$ be the connected components of $H$ and let $|V(H_i)| = m_i$ and set $m_0 = 0$. Since $\nu(H) = \lceil \frac{m}{2} \rceil$, each component $H_i$ has $\nu(H_i) = \lceil \frac{m_i}{2} \rceil$. If every $H_i$ has the strong T-property with respect to $F$, then we can embed the components of $H$ into $G$ successively and obtain

$$
H(G) = \prod_{i=1}^{k} H_i(G(n - \sum_{j<i} m_j)) \leq \prod_{i=1}^{k} H_i(T_2(n - \sum_{j<i} m_j)) = H(T_2(n)),
$$

where $G(n - \sum_{j<i} m_j)$ is the subgraph of $G$ on $n - \sum_{j<i} m_j$ vertices. So the equality holds for every $i \in [k]$ and the strong T-property of $H_i$ implies that $G(n - \sum_{j<i} m_j) \cong T_2(n - \sum_{j<i} m_j)$ for $i \in [k]$. Therefore, $G \cong T_2(n)$. By Theorem 2.6 $H$ has the strong T-property with respect to $K_3$. So, by Theorem 3.1 it is sufficient to show that $G$ is T-extremal.

Let $I_k = kK_2$ and $I_k^+ = I_k \cup K_1$. By the color-critical edge theorem of Simonovits [25] (see (ii) in the introduction), $K_2$ has the strong T-property with respect to $F$. So $I_k$ and $I_k^+$ have the strong T-property with respect to $F$ too.

Claim 4.1. $G$ is T-extremal, i.e. $e(G) = e(T_2(n)) - o(n^2)$.

Proof. Since $F$ is a subgraph of a blow-up of $K_3$, by Proposition 2.1

$$N(K_3, G) \leq ex(n, K_3, F) = o(n^3).$$

By Lemma 2.7 we can get a $K_3$-free graph $G^*$ from $G$ by removing $o(n^2)$ edges. The number of copies of $H$ intersecting the $o(n^2)$ removed edges is at most $o(n^2) \cdot O(n^{m-2}) = o(n^m)$. Hence

$$N(H, T_2(n)) \leq N(H, G) \leq N(H, G^*) + o(n^m) \leq N(H, T_2(n)) + o(n^m),$$

the last inequality holds because $G^*$ is $K_3$-free and $H$ has the strong T-property with respect to $K_3$ by Theorem 2.6. So $|H(G^*) - H(T_2(n))| = o(n^m)$.

Let $a_1b_1, \ldots, a_kb_k \in E(H)$ be a maximum matching of $H$ and let $V(H) = A_0 \cup \{a_1, b_1, \ldots, a_k, b_k\}$, where $A_0 = \{a_0\}$ if $m$ is odd and $\emptyset$ otherwise. Assume without loss of generality that $A_0 \cup \{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are the colour classes of $H$. Since $I_k$ (or $I_k^+$) has the strong T-property, there are $I_k(G^*)(\preceq I_k(T_2(n)))$ (or $I_k^+(G^*)(\preceq I_k^+(T_2(n)))$) injections $\varphi : A_0 \cup \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \to V(G)$ such that $\varphi(a_i) \varphi(b_i) \in E(G)$ for every $i \in [k]$. Two such injections $\varphi_1$ and $\varphi_2$ are called equivalent if

1. $\varphi_1(a_1) = \varphi_2(a_1)$ (or $\varphi_1(a_0) = \varphi_2(a_0)$ if $A_0 = \{a_0\}$),
2. $\{\varphi_1(a_i), \varphi_1(b_i)\} = \{\varphi_2(a_i), \varphi_2(b_i)\}$ \ \forall i \in [k].

So there are exactly $2^{k-1}$ (or $2^k$ if $A_0 = \{a_0\}$) elements in every equivalent class. However, due to the fact that $H$ is connected and $G^*$ is triangle-free, each class contains at most one embedding of $H$ into $G^*$. Thus

$$H(G^*) \leq 2^{1-k}I_k(G^*) \leq 2^{1-k}I_k(T_2(n)) = H(T_2(n)).$$

(or $H(G^*) \leq 2^{-k}I_k^+(G^*) \leq 2^{-k}I_k^+(T_2(n)) = H(T_2(n))$ if $A_0 = \{a_0\}$.)
So we have $|\overline{H(G^*)}| = |\overline{H(T_2(n))}| - o(n^m)$. We claim that there exists an $i \in [k]$ such that $H - a_i b_i$ is still connected. Construct a graph $H^*$ as follows. Let $V(H^*) = \{c_1, c_2, \ldots, c_k\}$ and $c_i c_j \in E(H^*)$ if and only if $E(H[\{a_i, b_i, a_j, b_j\}]) \neq \emptyset$. If $H^*$ is a tree, assume $c_i$ is a leaf, then $H - a_i b_i$ is still connected. Otherwise, there exists a cycle $C$ in $H^*$, assume $c_j \in V(C)$, then $H - a_j b_j$ is still connected. Without loss of generality, assume $H' = H - a_1 b_1$ is still connected. Assume $e = xy \in E(G^*)$, let $n(a_1 \to x, b_1 \to y)$ denote the number of different embeddings $\varphi : V(H) \to V(G^*)$ such that

(1) $\varphi(a_1) = x, \varphi(b_1) = y$,
(2) $v_1 \neq v_2 \Rightarrow \varphi(v_1) \neq \varphi(v_2)$,
(3) $v_1 v_2 \in E(H) \Rightarrow \varphi(v_1) \varphi(v_2) \in E(G^*)$

for every pair $v_1, v_2 \in E(H)$. Thus we have

$$\overline{H(G^*)} = \sum_{xy \in E(G^*)} \left( n(a_1 \to x, b_1 \to y) + n(a_1 \to y, b_1 \to x) \right)$$

$$= \sum_{xy \in E(G^*)} \overline{H'(G^* - \{x, y\})}$$

$$\leq \sum_{xy \in E(G^*)} 2^{2-k} \cdot I_{k-1}(G^* - \{x, y\}) \left( \text{ or } \sum_{xy \in E(G^*)} 2^{1-k} \cdot I_{k-1}^+(G^* - \{x, y\}) \right)$$

$$\leq |E(G^*)| \cdot 2^{2-k} \cdot I_{k-1}(T_2(n - 2)) \left( \text{ or } |E(G^*)| \cdot 2^{1-k} \cdot I_{k-1}^+(T_2(n - 2)) \right)$$

$$= |E(G^*)| \cdot 2^{2-k} \prod_{i=1}^{k-1} T_1(T_2(n - 2i)) \left( \text{ or } |E(G^*)| \cdot 2^{1-k} \prod_{i=1}^{k-1} T_1(T_2(n - 2i)) \cdot (n - 2k) \right)$$

$$= |E(G^*)| \cdot 2^{2-k} \prod_{i=1}^{k-1} 2e(T_2(n - 2i)) \left( \text{ or } |E(G^*)| \cdot 2^{1-k} \prod_{i=1}^{k-1} 2e(T_2(n - 2i)) \cdot (n - 2k) \right),$$

where the case of $A_0 = \{a_0\}$ are included in the parentheses. Combining with

$$\overline{H(G^*)} = \overline{H(T_2(n))} - o(n^m) = 2^{1-k} I_k(T_2(n)) - o(n^m)$$

$$= 2^{1-k} \prod_{i=1}^{k} 2e(T_2(n - 2(i - 1))) - o(n^m),$$

or when $A_0 = \{a_0\}$

$$\overline{H(G^*)} = \overline{H(T_2(n))} - o(n^m) = 2^{-k} I_k^+(T_2(n)) - o(n^m)$$

$$= 2^{-k} \prod_{i=1}^{k} 2e(T_2(n - 2(i - 1))) \cdot (n - 2k) - o(n^m),$$

we have

$$e(G) = e(G^*) + o(n^2) = e(T_2(n)) - o(n^2).$$

This completes the proof of Theorem 1.3.\[\square\]
5 Proof of Theorem 1.5

Proof of Theorem 1.5 Recall that $F$ has $\chi(F) = k \geq 4$ and a color-critical edge. Let $G$ be an $n$-vertex $F$-free graph with the largest number of copies of $P_\ell$. So $N(P_\ell, G) \geq N(P_\ell, T_{k-1}(n))$. We first show that $G$ is T-extremal.

Claim 5.1. $G$ is T-extremal, i.e. $e(G) = e(T_{k-1}(n)) - o(n^2)$.

Proof. Since every path corresponds to two walks (one starting from each end-vertex of the path), we have $2N(P_\ell, G) \leq W_{\ell-1}(G)$. It is well known that

$$\frac{W_{\ell-1}(G)}{n} = \frac{1^T A_{\ell-1} 1}{1^T 1} \leq \mu_{\ell-1}(G),$$

where $1$ is the column vector with all entries being 1 and the last inequality holds because the spectral radius of any Hermitian matrix $M$ is the supremum of the quotient $\frac{x^T M x}{x^T x}$, where $x$ ranges over $C^n \setminus \{0\}$. So we have

$$\mu_{\ell-1}(G)n \geq W_{\ell-1}(G) \geq 2N(P_\ell, G) \geq 2N(P_\ell, T_{k-1}(n)) = \left(1 - \frac{1}{k-1} - o(1)\right)^{\ell-1} n^\ell.$$

By Lemma 2.7, $\mu^2(G) \leq \frac{\omega(G)-1}{\omega(G)} W_1(G) = 2 \cdot \frac{\omega(G)-1}{\omega(G)} e(G)$. Hence we have

$$e(G) \geq \frac{1}{2} \frac{\omega(G)}{\omega(G)-1} \mu^2(G) \geq \frac{1}{2} \mu^2(G) \geq \frac{1}{2} \left(1 - \frac{1}{k-1} - o(1)\right)^2 n^2 = e(T_{k-1}(n)) - o(n^2).$$

\qed

Lemma 2.8 tell us that, for the path $P_\ell$, there is a $k_0$ such that $P_\ell$ has the weak T-property for every $k \geq k_0$. So if $\chi(H) = k \geq k_0$ then $G \cong T_{k-1}(n)$ by Theorem 5.1. This completes the proof of Theorem 1.5 (b).

To prove Theorem 1.5 (a), we will apply Theorem 3.1 to $F$ and $P_\ell$ with $2 \leq \ell \leq 6$. Clearly, $P_\ell$ is a connected bipartite graph with $\chi(P_\ell) = \lceil \frac{\ell}{2} \rceil$. The (ii) (Simonovits [25]), (viii) (Gerbner and Palmer [13]) and (ix) (Murphy and Nir [21], Qian et al [23]) imply that $P_\ell$ has the weak T-property when $\ell \leq 5$. To get the result of Theorem 1.5 (a), it is sufficient to show that $P_6$ has the weak T-property.

Claim 5.2. $P_6$ has the weak T-property.

Proof. It suffices to prove among all complete $(k-1)$-partite graphs $K = K(V_1, \ldots, V_{k-1})$ on $n$ vertices, the Turán graph $T_{k-1}(n)$ is the unique one with the largest number of copies of $P_6$. Suppose to the contrary that there is a complete $(k-1)$-partite graph $K = K(V_1, \ldots, V_{k-1})$ on $n$ vertices with $N(H, K) \geq N(H, T_{k-1}(n))$ but $K \not\cong T_{k-1}(n)$. By Claim 5.1 we may assume $\|V_i\| - \frac{n}{k-1} < \theta n$ for some sufficiently small $\theta > 0$. Without loss of generality, we assume $|V_1| > |V_2| + 1$. Let $a = |V_1| - 1$, $b = |V_2|$. Then $a > b$. Let $n_i = |V_i|$ for $3 \leq i \leq k-1$. If moving a vertex $v$ from $V_1$ to $V_2$, the resulting complete $(k-1)$-partite graph $K'$ has more copies of $P_6$ than in $K$, then we have a contradiction and the claim holds.

Now let us move a vertex $v$ from $V_1$ to $V_2$ and count the number of embeddings of $P_6$ destroyed and created after the moving, respectively. Let $P = v_1 v_2 v_3 v_4 v_5 v_6$ be a path of length 5. There are 6
choices of \( v \) to be embedded in the path. First, let us count the number of embeddings \( \phi_1 \) destroyed when \( v_1 \) is embedded to \( v \). Then \( \phi_1(v_2) \) must be in \( V_2 \) and so has \( b \) choices. Thus \( \phi_1(v_3) \in V_1 \) or \( V_i \) for some \( 3 \leq i \leq k - 1 \). We count the total number \( \sharp \phi_1 \) of this kind of destroyed embeddings by dividing them into fifteen cases according to the images of \( v_3, v_4 \) and \( v_5 \):

1. \( \phi_1(v_3) \in V_1, \phi_1(v_4) \in V_2 \) and \( \phi_1(v_5) \in V_1 \) or \( V_i \) for some \( 3 \leq i \leq k - 1 \);
2. \( \phi_1(v_3) \in V_1, \phi_1(v_4) \in V_i \) for some \( 3 \leq i \leq k - 1 \) and \( \phi_1(v_5) \in V_1, V_2 \) or \( V_j \) for some \( j \neq i \), \( 3 \leq j \leq k - 1 \);
3. \( \phi_1(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_1(v_4) \in V_1 \) and \( \phi_1(v_5) \in V_2, V_i \) or \( V_j \) for some \( j \neq i \), \( 3 \leq j \leq k - 1 \);
4. \( \phi_1(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_1(v_4) \in V_2 \) and \( \phi_1(v_5) \in V_1, V_i \) or \( V_j \) for some \( j \neq i \), \( 3 \leq j \leq k - 1 \);
5. \( \phi_1(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_1(v_4) \in V_j \) for some \( j \neq i \), \( 3 \leq j \leq k - 1 \) and \( \phi_1(v_5) \in V_1, V_2, V_i \) or \( V_h \) for some \( h \neq i, j \), \( 3 \leq h \leq k - 1 \). Therefore, we have

\[
\sharp \phi_1 = ba(b - 1) \left[ (a - 1)(n - 3 - a) + \sum_{i=3}^{k-1} n_i(n - 4 - n_i) \right] \\
+ \sum_{i=3}^{k-1} [n_i(b - 1)(n - 3 - b) + (b - 1)(n - 3 - b)] \\
+ \sum_{i=3}^{k-1} bn_i [a(n - 4 - a) + (n_i - 1)(n - 3 - n_i)] \\
+ 2 \sum_{3 \leq i < j \leq k - 1} bn_i n_j [a(n - 4 - a) + (n_i - 1)(n - 3 - n_i)] \\
+ 6 \sum_{3 \leq i < j < h \leq k - 1} bn_i n_j n_h (n - 4 - n_h).
\]

Similarly, we can get the number of destroyed embeddings \( \sharp \phi_2 \) and \( \sharp \phi_3 \) corresponding to the cases \( \phi_2(v_2) = v \) and \( \phi_3(v_3) = v \), respectively (the expressions of them are shown in the Appendix). So the total number of destroyed embeddings

\[
\sharp \phi(a, b, n_3, \ldots, n_{k-1}, n) = 2(\sharp \phi_1 + \sharp \phi_2 + \sharp \phi_3).
\]

Now we count the number of new created embeddings after the moving of \( v \). Similarly, let \( \psi_1 \) denote a created embedding with \( \psi_1(v_1) = v \). Then \( \psi_1(v_2) \) must be in \( V_1 \) and so has \( a \) choices. Thus \( \psi_1(v_3) \in V_2 \) or \( V_i \) for some \( 3 \leq i \leq k - 1 \). So the number of created embeddings \( \psi_1 \) is equal to the function by exchanging the variants \( a \) and \( b \) in \( \sharp \phi_1 \), i.e. \( \sharp \psi_1(a, b, n_3, \ldots, n_{k-1}, n) = \sharp \phi_1(b, a, n_3, \ldots, n_{k-1}) \).

Similarly, we define \( \psi_i \) to be a created embedding with \( \psi_i(v_i) = v \) for \( i = 2, 3 \). We also have \( \sharp \psi_i(a, b, n_3, \ldots, n_{k-1}, n) = \sharp \phi_i(b, a, n_3, \ldots, n_{k-1}) \) for \( i = 2, 3 \). Therefore, the total number of created
embeddings
\[
\hat{\varphi} = \hat{\varphi}(a, b, n_3, \ldots, n_{k-1}, n) = 2 \sum_{i=1}^{3} \hat{\varphi}(a, b, n_3, \ldots, n_{k-1}, n)
\]
\[
= 2 \sum_{i=1}^{3} \hat{\varphi}(b, a, n_3, \ldots, n_{k-1}, n)
\]
\[
= \hat{\varphi}(b, a, p_3, p_4, \ldots, p_{k-1}, n).
\]

By tedious calculation (or calculated with the MATLAB), we get

\[
\hat{\varphi} - \hat{\psi} = (a-b) \left\{ 2a^4 + 7a^3b + 13a^2b^2 + 2a^2n^2 + 7ab^3 + 13abn^2 + 2b^4 + 2b^2n^2 + \sum_{i=3}^{k-1} (a^2 + b^2)n_i^2 \\
+ \sum_{i=3}^{k-1} \left( 4n(a+b)n_i^2 + \sum_{j \neq i}^{k-1} \left( 8n(a+b) + 3ab + 6nn_j + 4n \sum_{\ell \neq i, j}^{k-1} n_{\ell} \right) n_i n_j \right) \right\}
\]
\[
- 4a^3n + 20a^2bn + 20ab^2n + 4b^3n + \sum_{i=3}^{k-1} \left( abn_i + 4(a^2 + ab + b^2) \sum_{j \neq i}^{k-1} n_j \right) n_i \\
+ \sum_{i=3}^{k-1} \sum_{j \neq i}^{k-1} \left( 4(a+b)n_j + 4n_i^2 + n_i n_j + \sum_{\ell \neq i, j}^{k-1} 4n_{\ell}^2 \right) n_i n_j + (3a + 3b) \sum_{i=3}^{k-1} n_i^3 + o(n^4) \right\}.
\]

Let \( Q \) and \( R \) be the expressions in the first and second square brackets, respectively. Since \( |x - \frac{n}{k-1}| < \theta n \) for \( x \in \{a, b\} \) and \( |n_i - \frac{n}{k-1}| < \theta n \) for \( 3 \leq i \leq k-1 \), we have

\[
Q/n^4 > (4k^3 - 26k^2 + 42k) \left( \frac{1}{k-1} - \theta \right)^3 + 17 \left( \frac{1}{k-1} - \theta \right)^2 + (3k^2 - 17k + 55) \left( \frac{1}{k-1} - \theta \right)^4
\]

and

\[
R/n^4 < (4k^3 - 23k^2 + 22k + 33) \left( \frac{1}{k-1} + \theta \right)^4 + 48 \left( \frac{1}{k-1} + \theta \right)^3.
\]

Let \( f(k, \theta) = Q/n^4 - R/n^4 \). Then

\[
f(k, 0) = \frac{4k^3 - 26k^2 + 42k}{(k-1)^3} + \frac{17}{(k-1)^2} + \frac{3k^2 - 17k + 55}{(k-1)^4} = \frac{4k^3 - 23k^2 + 22k + 33}{(k-1)^4} - \frac{48}{(k-1)^3}
\]

\[
= \frac{4k^4 - 24k^3 + 111k^2 - 163k + 87}{(k-1)^4}.
\]

It is easy to check that \( f(k, 0) \geq f(4, 0) > 0 \) when \( k \geq 4 \). By the continuity of \( f(k, \theta) \) with respect to \( \theta \), we have \( f(k, \theta) > 0 \) when \( \theta \) is small enough. So \( \hat{\varphi} > \hat{\psi} \) when \( n \) is large enough, which is a contradiction. The proof of the claim is completed.

\[\square\]
6 Concluding Remarks

In this article, we show that $H$ is strictly $F$-Turán good for graph $F$ with $\chi(F) = 3$ and a color-critical edge and bipartite graph $H$ with matching number $\left\lfloor \frac{|V(H)|}{2} \right\rfloor$ (Theorem 3.3). This result confirms Conjecture 1.1 proposed by Gerbner, C. Palmer [15]. But for Conjecture 1.2 it is far from being resolved. By Theorem 3.1 it is sufficient to show that every $P_\ell$ ($\ell \geq 2$) has the weak T-property. We leave this as an open problem. It has been shown that $P_2$ (Simonovits [25]), $P_3$ (Gerbner and Palmer [15]), $P_4$ (Murphy and Nir [21]), $P_5$ (Qian et al [23]), and $P_6$ (Theorem 1.5) have the weak T-property.

7 Acknowledgment

We thank Professor Dániel Gerbner for providing us a more general version of Theorem 3.1.

References

[1] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient testing of large graphs, Combinatorica 20(4)(2000), 451-476.
[2] N. Alon, C. Shikhelman, Many $T$ copies in $H$-free graphs, J. Combin. Theory Ser. B, 121(2016), 146-172.
[3] P. Erdős, On the number of complete subgraphs contained in certain graphs. Magyar Tud. Akad. Mat. Kut. Int. Közl., 7 (1962), 459-464.
[4] P. Erdős, Some recent results on extremal problems in graph theory, in: Theroy of Graphs International Symp. Rome, 1966, pp. 118-123.
[5] P. Erdős, P. Frankl, and V. Rödl, The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, Graphs Combin., 2 (1986), 113-121.
[6] P. Erdős and M. Simonovits, A limit theorem in graph theory. Studia Sci. Math. Hungar., 1 (1966), 51-57.
[7] B. Ergemlidze, E. Győri, A. Methuku and N.Salia, A note on the maximum number of triangles in a $C_5$-free graph, J. Graph Theory, 90(2019), 227-230.
[8] J. Fox, A new proof of the graph removal lemma, Ann. of Math., 174(2011), 561-579.
[9] D. Gerbner, Generalized Turán problems for small graphs. Discussiones Mathematicae Graph Theory, 2021, 10.7151/dmgt.2388.
[10] D. Gerbner, On Turán-good graphs. Discrete Mathematics. 344 (2021), 112445. 10.1016/j.disc.2021.112445.
[11] D. Gerbner, A non-aligning variant of generalized Turán problems, arXiv:2109.02181v1, 2021.
[12] D. Gerbner, Generalized Turán problems for double stars, arXiv:2112.11144v2, 2022.
[13] D. Gerbner, E. Győri, A. Methuku and M. Vizer, Generalized Turán problems for even cycles, Journal of Combinatorial Theory, Series B. 145 (2019), 169-213.
[14] D. Gerbner, C. Palmer, Counting copies of a fixed subgraph in $F$-free graphs, European Journal of Combinatorics, 82(2019), pp 103001 DOI: 10.1016/J.EJC.2019.103001
[15] D. Gerbner, C. Palmer, Some exact results for generalized Turán problems, arXiv: 2006.03756v1 (2020).
[16] L. Gishboliner and A. Shapira, A generalized Turán problem and its applications, Int. Math. Res. Not. IMRN, (11): 3417-3452, 2020.
[17] E. Győri, J. Pach and M. Simonovits, On the maximal number of certain subgraphs in $K_r$-free graphs, Graphs and Combinatorics, 7 (1991), 31-37.
[18] E. Győri, R. Wang, S. Woolfson, Extremal problems of double stars, arXiv:2109.01536v1, 2021.
[19] P. Kővári, V. T. Sós and P. Turán, On a problem of zarankiewicz, Colloquium, Mathematicum, 33 (1954), 50-57.
[20] J. Ma and Y. Qiu, Some sharp results on the generalzied Turán numbers, European J Combin., 84(2020), 103026.
[21] K. Murphy, JD Nir, Paths of length three are $K_{r+1}$-Turán-good, Electron J Combin., 28(4) (2021), #P4.34
[22] V. Nikiforov, Some inequalities for the largest rigenvalue of a graph, Combin. Probab. Comput., 11(2002), 179-189.
[23] B. Qian, C. Xie, G. Ge, Some results on $k$-Turán-good graphs, Discrete Mathematics, 344(9)(2021), 112509. DOI:10.1016/J.DISC.2021.112509
[24] I. Z. Ruzsa and E. Szemeredi, Triple systems with no six points carrying three triangles, in Combinatorics(Keszthely, 1976), Coll. Mth. Soc. J. Bolyai 18, Volume II, 939-945.
[25] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, (1968), 279-319.
[26] M. Simonovits, Paul Erdős influence on extremal graph theory, in: The mathematics of Paul Erdős, II, 148-192, Algorithms Combin., 14, Springer, Berlin, 1997.
[27] A. A. Zykov. On some properties of linear complexes. Matematicheskii Sbornik, 66 (1949), 163-188.

**Appendix:** Expressions of $\sharp\phi_2$ and $\sharp\phi_3$

Let $\phi_2$ be the number of embeddings destroyed when $v_2$ is embedded to $v$. Then at least one of $\phi_2(v_1)$ and $\phi_2(v_3)$ must be in $V_2$. We count the total number $\sharp\phi_2$ of this kind embeddings by dividing them into twenty-five cases according to the images of $v_1, v_3, v_4$ and $v_5$:
(1) \( \phi_2(v_1) \in V_2, \phi_2(v_4) \in V_2, \phi_2(v_5) \in V_1 \) and \( \phi_2(v_3) \in V_2 \) or \( V_1 \) for some \( 3 \leq i \leq k - 1 \);
(2) \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_2, \phi_2(v_4) \in V_1 \) for some \( 3 \leq i \leq k - 1 \) and \( \phi_2(v_5) \in V_1, V_2 \) or \( V_j \) for \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_4) \in V_1 \) and \( \phi_2(v_5) \in V_2, V_i \) or \( V_j \) for \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1 \) and \( \phi_2(v_5) \in V_1, V_2, V_i \) or \( V_j \) for some \( 3 \leq j \leq k - 1 \);
(3) \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_4) \in V_1 \) and \( \phi_2(v_5) \in V_2, V_i \) or \( V_j \) for \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1 \) and \( \phi_2(v_5) \in V_1, V_2, V_i \) or \( V_j \) for some \( 3 \leq j \leq k - 1 \);
(4) \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_4) \in V_2 \) and \( \phi_2(v_5) \in V_1, V_i \) or \( V_j \) for \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1 \) and \( \phi_2(v_5) \in V_1, V_2, V_i \) or \( V_j \) for some \( 3 \leq j \leq k - 1 \);
(5) \( \phi_2(v_1) \in V_2, \phi_2(v_3) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_4) \in V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \) and \( \phi_2(v_5) \in V_1, V_2, V_i \) or \( V_j \) for some \( 3 \leq \ell \neq i, j, 3 \leq \ell \leq k - 1 \);
(6) \( \phi_2(v_1) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_3) \in V_2, \phi_2(v_4) \in V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \) and \( \phi_2(v_5) \in V_1, V_2, V_i \) or \( V_j \) for some \( 3 \leq \ell \neq i, j, 3 \leq \ell \leq k - 1 \) (the number of the destroyed embeddings is the same as in the case (5));
(7) \( \phi_2(v_1) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_3) \in V_2, \phi_2(v_4) \in V_1 \) and \( \phi_2(v_5) \in V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \);
(8) \( \phi_2(v_1) \in V_i \) for some \( 3 \leq i \leq k - 1, \phi_2(v_3) \in V_2, \phi_2(v_4) \in V_1 \) and \( \phi_2(v_5) \in V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \). Therefore, we have

\[
\# \phi_2 = b(b - 1)a \left[ (b - 2)(n - 2 - b) + \sum_{i=3}^{k-1} n_i(n - 4 - n_i) \right]
+ \sum_{i=3}^{k-1} b(b - 1)n_i \left[ a(n - 4 - a) + (b - 2)(n - 2 - b) + \sum_{j=3}^{k-1} n_j(n - 4 - n_j) \right]
+ \sum_{i=3}^{k-1} bn_i a \left[ (b - 1)(n - 3 - b) + (n_i - 1)(n - 3 - n_i) + \sum_{j=3}^{k-1} n_j(n - 4 - n_j) \right]
+ \sum_{i=3}^{k-1} bn_i(b - 1) \left[ a(n - 4 - a) + (n_i - 1)(n - 3 - n_i) + \sum_{j=3}^{k-1} n_j(n - 4 - n_j) \right]
+ 2 \cdot 2 \sum_{3 \leq i < j \leq k - 1} bn_i n_j \left[ a(n - 4 - a) + (b - 1)(n - 3 - b) + (n_i - 1)(n - 3 - n_i) \right]
+ 2 \cdot 6 \sum_{3 \leq i < j < \ell \leq k - 1} bn_i n_j n_\ell \left( n - 4 - n_\ell \right)
+ \sum_{i=3}^{k-1} n_i b(n_i - 1) \left[ (b - 1)(n - 3 - b) + (n_i - 1)(n - 3 - n_i) + \sum_{j=3}^{k-1} n_j(n - 4 - n_j) \right]
+ \sum_{i=3}^{k-1} n_i b(n_i - 1) \left[ a(n - 4 - a) + (b - 1)(n - 3 - b) + \sum_{j=3}^{k-1} n_j(n - 4 - n_j) \right].
\]

Similarly, if \( \phi_3(v_5) = v \), then at least one of \( \phi_3(v_2), \phi_3(v_4) \) must be in \( V_2 \). We also can count the total number of this kind of destroyed embeddings by dividing them into twenty-five cases according to the images of \( v_2, v_4, v_3 \) and \( v_1 \):
(1) \( \phi_3(v_2) \in V_2, \phi_3(v_4) \in V_2, \phi_2(v_5) \in V_1 \) and \( \phi_2(v_6) \in V_2 \) or \( V_i \) for some \( 3 \leq i \leq k - 1 \);
(2) \( \phi_3(v_2) \in V_2, \phi_3(v_4) \in V_2, \phi_3(v_5) \in V_i \) for some \( 3 \leq i \leq k - 1 \) and \( \phi_3(v_6) \in V_1, V_2 \) or \( V_j \) for some \( 3 \leq j \leq k - 1 \) and \( j \neq i \);
(3) \( \phi_3(v_2) \in V_2, \phi_3(v_4) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_3(v_5) \in V_1 \) and \( \phi_3(v_6) \in V_2, V_i \) or \( V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \);
(4) \( \phi_3(v_2) \in V_2, \phi_3(v_4) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_3(v_5) \in V_2 \) and \( \phi_3(v_6) \in V_1, V_i \) or \( V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \);
(5) \( \phi_3(v_2) \in V_2, \phi_3(v_4) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_3(v_5) \in V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \) and \( \phi_3(v_6) \in V_1, V_2 \) or \( V_i \) for some \( \ell \neq i, j, 3 \leq \ell \leq k - 1 \);
(6) \( \phi_3(v_2) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_3(v_4) \in V_2, \phi_3(v_5) \in V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \) and \( \phi_3(v_6) \in V_1, V_i \) or \( V_j \) for some \( \ell \neq i, j, 3 \leq \ell \leq k - 1 \) (the number of the destroyed embeddings is the same as in the case (5));
(7) \( \phi_3(v_2) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_3(v_4) \in V_2, \phi_3(v_5) \in V_1 \) and \( \phi_3(v_6) \in V_2, V_i \) or \( V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \);
(8) \( \phi_3(v_2) \in V_i \) for some \( 3 \leq i \leq k - 1 \), \( \phi_3(v_4) \in V_2, \phi_3(v_5) \in V_i \) and \( \phi_3(v_6) \in V_1, V_2 \) or \( V_j \) for some \( j \neq i, 3 \leq j \leq k - 1 \).

\[
\zeta \phi_3 = b(b - 1)a \left[ (b - 2)(n - 2 - b) + \sum_{i=3}^{k-1} n_i(n - 3 - b) \right] \\
+ \sum_{i=3}^{k-1} b(b - 1)n_i \left[ (a(n - 3 - b) + (b - 2)(n - 2 - b) + \sum_{j=3}^{k-1} n_j(n - 3 - b)) \right] \\
+ \sum_{i=3}^{k-1} b n_i a \left[ (b - 1)(n - 3 - b) + (n_i - 1)(n - 4 - b) + \sum_{j=3}^{k-1} n_j(n - 4 - b) \right] \\
+ \sum_{i=3}^{k-1} b n_i(b - 1) \left[ a(n - 3 - b) + (n_i - 1)(n - 3 - b) + \sum_{j=3}^{k-1} n_j(n - 3 - b) \right] \\
+ 2 \cdot 2 \sum_{3 \leq i < j \leq k - 1} b n_i n_j \left[ (a(n - 4 - b) + (b - 1)(n - 3 - b) + (n_i - 1)(n - 4 - b)) \right] \\
+ 2 \cdot 6 \sum_{3 \leq i < j < \ell \leq k - 1} b n_i n_j n_\ell(n - 4 - b) \\
+ \sum_{i=3}^{k-1} n_i b \left[ (b - 1)(n - 4 - n_i) + (n_i - 1)(n - 3 - n_i) + \sum_{j=3}^{k-1} n_j(n - 4 - n_i) \right] \\
+ \sum_{i=3}^{k-1} n_i(b n_i - 1) \left[ a(n - 3 - n_i) + (b - 1)(n - 3 - n_i) + \sum_{j=3}^{k-1} n_j(n - 3 - n_i) \right].
\]