MODULI SPACE OF Λ-MODULES ON PROJECTIVE DELIGNE-MUMFORD STACKS

HAO SUN

Abstract. In this paper, we define the Λ-quotient functor on a Deligne-Mumford stack over an algebraic space. We prove that the Λ-quotient functor is representable by an algebraic space. We also define the moduli problem of Λ-modules on a projective Deligne-Mumford stack and construct its moduli space, which is a quasi-projective scheme.

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1. Introduction

Let $X$ be a scheme, and let $\Lambda$ be a sheaf of graded algebras over $X$. A $\Lambda$-module is a coherent sheaf equipped with a $\Lambda$-structure. Simpson studied the $\Lambda$-modules on smooth projective varieties over $\mathbb{C}$ [27]. The $\Lambda$-module is derived from the Higgs bundle which was first defined and studied by Hitchin [15]. Usually, a Higgs bundle over a smooth projective variety is considered as a pair, which includes a vector bundle and a Higgs field. Similarly, $\Lambda$-modules can also be defined in this way. A $\Lambda$-module over a smooth projective variety is also a pair $(F, \Phi)$, where $F$ is a coherent sheaf and $\Phi : \Lambda \to \mathcal{E}nd(F)$ gives a $\Lambda$-structure on $F$.

People constructed the moduli space $M(X)$ of $p$-semistable coherent sheaves [16] and the moduli space $\mathcal{M}_\Lambda(X)$ of $p$-semistable $\Lambda$-modules [27] on smooth projective varieties decades ago. Afterwards, the moduli space $M(\mathcal{X})$ of coherent sheaves on (projective) Deligne-Mumford stacks was founded [23], where $\mathcal{X}$ is a Deligne-Mumford stack.

Therefore, the only mysterious object left in the above diagram is $\mathcal{M}_\Lambda(\mathcal{X})$, the moduli space of $p$-semistable $\Lambda$-modules on Deligne-Mumford stacks. Simpson constructed the moduli space of $p$-semistable $\Lambda$-modules on a smooth “projective” Deligne-Mumford stack $\mathcal{X}$ over $\mathbb{C}$ by using a simplicial resolution of $\mathcal{X}$ [28]. A smooth “projective” Deligne-Mumford stack $\mathcal{X}$ here means that a Deligne-Mumford stack admits a smooth projective coarse moduli space and a surjective étale morphism $Y \to \mathcal{X}$ such that $Y$ is a smooth projective variety over $\mathbb{C}$.

This paper aims at working on a more general type of Deligne-Mumford stacks, projective Deligne-Mumford stacks over an algebraic space, and constructing the moduli space of $\Lambda$-modules on these Deligne-Mumford stacks. The terminology projective Deligne-Mumford stacks in this paper is different from Simpson’s definition. Roughly speaking, a projective Deligne-Mumford stack is a tame Deligne-Mumford stack over an algebraic space such that its coarse moduli space is projective over the algebraic space.

Our initial motivations grew out of our interest in the moduli space of parabolic Higgs bundles. It is well-known that there is a correspondence between parabolic Higgs bundles and Higgs bundles on orbifolds [5, 9, 22], and this correspondence can be extended to twisted parabolic Higgs bundles and twisted Higgs bundles on orbifolds [10]. Moreover, orbifolds has a natural structure as Deligne-Mumford stacks. Therefore, the moduli space of parabolic Higgs bundles is a special case of the moduli space of Higgs bundles over Deligne-Mumford stacks to a certain extend.

As we mentioned above, Simpson studied this problem a decade ago. His approach is based on the choice of a special simplicial resolution of the Deligne-Mumford stack and the existence of the moduli space of Higgs bundles on smooth projective varieties over $\mathbb{C}$. More precisely, we suppose that there exists a simplicial resolution $Y_{\bullet} = \cdots Y_1 \Rightarrow Y_0 \to \mathcal{X}$ of the Deligne-Mumford stack $\mathcal{X}$, where $Y_i$ are smooth projective varieties for $i \geq 0$. Then, the existence of the moduli space of $p$-semistable $\Lambda$-bundles on each $Y_i$ gives us the moduli space of $p$-semistable $\Lambda$-modules on the simplicial resolution $Y_{\bullet}$, and therefore the moduli space of $p$-semistable $\Lambda$-modules on $\mathcal{X}$. In summary, Simpson’s construction is based on the moduli space of Higgs bundles on smooth projective varieties (schemes). We refer the reader to [28] for more details.

In our case, such a resolution may not exist, and therefore Simpson’s approach may not work. As shown in the diagram, there is another possible way left for us. We can try to construct the moduli space of $\Lambda$-modules $\mathcal{M}_\Lambda(\mathcal{X})$ based on the existence of the moduli space of coherent sheaves $M(\mathcal{X})$. Now we review some results of the moduli space of coherent sheaves over Deligne-Mumford stacks.

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Let \( S \) be an algebraic space over an algebraically closed field \( k \), and let \( \mathcal{X} \) be a Deligne-Mumford stack over \( S \). The category (or stack) of coherent sheaves over \( \mathcal{X} \) is covered by quotient functors, therefore it is equivalent to study quotient functors. Let \( \mathcal{G} \) be a coherent sheaf over \( \mathcal{X} \). The quotient functor \( \widetilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) over \( \mathcal{X} \), which is also known as a moduli problem, has been defined and studied by Olsson and Starr. The quotient functor \( \widetilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) is proved to be representable by an algebraic space \( \text{Quot}(\mathcal{G}, \mathcal{X}) \) [25, Theorem 1.1], which gives the existence of the moduli space of coherent sheaves on \( \mathcal{X} \). Furthermore, when \( S \) is an affine scheme and \( \mathcal{X} \) is tame, each connected component of \( \text{Quot}(\mathcal{G}, \mathcal{X}) \) is an \( S \)-projective scheme [25, Theorem 1.5]. The connected components of \( \text{Quot}(\mathcal{G}, \mathcal{X}) \) are parameterized by integer polynomials. Based on the result that the connected components of \( \text{Quot}(\mathcal{G}, \mathcal{X}) \) are \( S \)-projective schemes, Nironi studied the moduli problem of \( pE \)-semistable coherent sheaves on a projective Deligne-Mumford stack \( \mathcal{X} \) (over an affine scheme \( S \) with coarse moduli space \( \pi : \mathcal{X} \to X \)) using the geometric invariant theory [23], where \( E \) is a generating sheaf on \( \mathcal{X} \). The \( pE \)-semistability is defined by modified Hilbert polynomials \( P_E \),

\[
P_E(F, m) = \chi(\mathcal{X}, F \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(m)), \quad m \gg 0,
\]

where \( F \) is a coherent sheaf on \( \mathcal{X} \). With a good choice of the generating sheaf \( E \), the \( pE \)-semistability gives us the semistability of parabolic Higgs bundles on the coarse moduli space \( X \). Clearly, the semistability depends on the choice of the generating sheaf \( E \), and this is the reason why we call it \( pE \)-semistability. However, we omit the subscript \( E \) and use the terminology \( p \)-semistability for simplicity in the main body of the paper. We apply their techniques and results to constructing the moduli space of \( \Lambda \)-modules on a Deligne-Mumford stack \( \mathcal{X} \) (over an algebraic space \( S \)) first, and then studying the \( \mathcal{X} \) the moduli space of \( pE \)-semistable \( \Lambda \)-modules on a projective Deligne-Mumford stack over an affine scheme. In summary, we consider two moduli problems in this paper:

1. the quotient functor \( \text{Quot}_\Lambda(\mathcal{G}, \mathcal{X}) \) of \( \Lambda \)-modules (also called the \( \Lambda \)-quotient functor), where \( \mathcal{X} \) is a separated and locally finitely-presented Deligne-Mumford stack over an algebraic space \( S \),

2. the moduli problem \( \mathcal{M}_\Lambda^ss(\mathcal{E}, \mathcal{O}_X(1), P) \) of \( pE \)-semistable \( \Lambda \)-modules with the modified Hilbert polynomial \( P \) over \( \mathcal{X} \), where \( \mathcal{X} \) is a projective Deligne-Mumford stack over an affine scheme \( S \), \( \mathcal{E} \) is a generating sheaf on \( \mathcal{X} \), \( \mathcal{O}_X(1) \) is a polarization on the coarse moduli space \( X \) and \( P \) is an integer polynomial.

Compared with the second moduli problem \( \mathcal{M}_\Lambda^ss(\mathcal{E}, \mathcal{O}_X(1), P) \), the setup of the first problem is more general, in which the Deligne-Mumford stack \( \mathcal{X} \) is not even a tame stack. In fact, the study of \( \mathcal{M}_\Lambda^ss(\mathcal{E}, \mathcal{O}_X(1), P) \) is based on the result of \( \text{Quot}(\mathcal{G}, \mathcal{X}) \). The prerequisite sections for studying the first moduli problem in §5 are §3.1 and §4.1, while the prerequisite sections for the second one in §6 are §3.2-§3.8, §4 and §5.4-§5.5. In conclusion,

1. Deligne-Mumford stacks considered in §3.1, §4.1 and §5.1-§5.3 are separated and locally finitely-presented over an algebraic space \( S \).

2. We consider projective (or quasi-projective) Deligne-Mumford stacks over an affine scheme \( S \) in §3.2-§3.8, §4, §5.4-§5.5 and §6.

Here is the structure of the paper.

In §2, we review the definitions and some properties of tame Deligne-Mumford stacks and projective Deligne-Mumford stacks. As a tame (or projective) Deligne-Mumford stack \( \mathcal{X} \), the natural map \( \mathcal{X} \to X \) to its coarse moduli space \( X \) induces an exact functor \( \pi_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X) \), where \( \text{QCoh} \) is the category (or stack) of quasi-coherent sheaves. The functor \( \pi_* \) may not be injective. If there exists an injective exact functor \( F : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X) \), then \( \text{QCoh}(\mathcal{X}) \) can be probably considered as a closed (or locally closed) subset of \( \text{QCoh}(X) \). In §2.2 and §2.3, we introduce the functor

\[
F_E : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X)
\]

where \( \mathcal{E} \) is a generating sheaf. This functor \( F_E \) is proved to be an injective exact natural transformation for quotient functors [25, Proposition 6.2], and the injectivity of \( F_E \) also holds for quotient functors of \( \Lambda \)-modules (see Lemma 5.8). This functor plays an important role when we study the moduli space
of $pE$-semistable coherent sheaves (see §3) and the moduli space of $pE$-semistable $\Lambda$-modules (see §6). In §2.5, we give the definition of moduli problems and representabilities we consider in this paper. A moduli problem is defined as a sheaf over the category of schemes over an algebraic space with respect to the big étale topology (or fppf topology). This definition is equivalent to consider a moduli problem as a category fibered in groupoids (CFG) satisfying the effective descent conditions [8, 24]. Given a moduli problem, it is important to understand whether there exists a coarse moduli space or a fine moduli space [14]. Furthermore, since we study the moduli problem related to coherent sheaves in this paper, we also give the definitions of co-representability and universal co-representability [16].

In §3, we first review a general result that the quotient functor $\text{Quot}(\mathcal{G}, \mathcal{X})$ is representable by an algebraic space $\text{Quot}(\mathcal{G}, \mathcal{X})$, where $\mathcal{X}$ is a Deligne-Mumford stack over an algebraic space $S$ (see [25, Theorem 1.1] or Theorem 3.1). Afterwards, we give the definition of saturations, modified Hilbert polynomials, $pE$-stability condition, Harder-Narasimhan filtrations and Jordan-Hölder filtrations. Saturations (see Corollary 3.5) and modified Hilbert polynomials (see §3.3) of coherent sheaves over $\mathcal{X}$ are preserved under the functor $F_{E}$, while the $pE$-stability and two filtrations are not preserved (see Remark 3.6). This is the reason why we have to study the $pE$-stability conditions in detail. Next, we review properties of boundedness of $pE$-semistable coherent sheaves over a projective Deligne-Mumford stack in §3.5 and §3.6. Note that there are two distinct properties called the boundedness. Boundedness I is related to the regularity and the existence of a universal family, while Boundedness II is about the upper bound of the global sections of $pE$-semistable coherent sheaves. Langer studied Boundedness II of $p$-semistable coherent sheaves on schemes in positive characteristic [18]. Based on the geometric invariant theory, we give the construction of the moduli space of $pE$-semistable coherent sheaves over a projective Deligne-Mumford stack. Most of the materials in this section is included in [23, 25], but the construction of the moduli space we give is slightly different from that in [23, §6]. If the reader find a proof of a statement in this section, it means that we have not seen it in any reference.

In §4, we give the definition of sheaves of graded algebras $\Lambda$ and $\Lambda$-modules. Sheaves of graded algebras over projective varieties are introduced in [27, §2], and we generalize the definition to Deligne-Mumford stacks. Next, we define the $pE$-semistability of $\Lambda$-modules, $\Lambda$-Harder-Narasimhan filtrations and $\Lambda$-Jordan-Hölder filtrations. In §4.3, we prove the Boundedness II of $\Lambda$-modules. Note that Boundedness I of $\Lambda$-modules is not proved in this section, which is proved in §5.5, because the proof of this property depends on the representability of the $\Lambda$-quotient functor.

In §5, we consider the $\Lambda$-quotient functor

$$\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, \mathcal{X}) : (\text{Sch}/S)^{\text{op}} \to \text{Set}.$$ 

For each $S$-scheme $T$, $\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, \mathcal{X})(T)$ is defined as the set of $\mathcal{O}_{\mathcal{X}_{T}}$-module quotients $\mathcal{G}_{T} \to \mathcal{F}_{T}$ such that

1. $\mathcal{F}_{T} \in \widetilde{\text{Quot}}(\mathcal{G})(T)$,
2. $\mathcal{F}_{T}$ is a $\Lambda_{T}$-module.

The first main result of this paper is given as follows.

**Theorem 1.1** (Theorem 5.1). Let $S$ be an algebraic space, which is locally of finite type over an algebraically closed field $k$, and let $\mathcal{X}$ be a separated and locally finitely-presented Deligne-Mumford stack over $S$. The $\Lambda$-quotient functor $\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, \mathcal{X})$ is represented by a separated and locally finitely presented algebraic space.

We apply a theorem by Artin (see [3, Theorem 5.3] or Theorem 5.2) to proving this result. The theorem by Artin lists all necessary conditions, under which a moduli problem is representable by an algebraic space. These conditions are locally of finite presentation, integrability, separation, deformation theory and obstruction theory. There are many good references about the first three conditions. We refer the reader to [8, 12, 13] for more details. The infinitesimal deformation theory of Hitchin pairs was studied by Biswas and Ramanan [6]. They construct a two-term complex and prove that the first hypercohomology group of the two-term complex is exactly the tangent space of the moduli space of
Hitchin pairs over a smooth projective curve. Based on this idea, we construct the deformation and obstruction theory for \( \Lambda \)-modules and prove that the theory satisfies all of the conditions in the Artin’s theorem. In this paper, the deformation and obstruction theory follows from Artin’s definition (see [3, §5] or §5.2.4). We make a brief review about the Artin’s theorem and necessary backgrounds in §5.2. We give the statement of the main result Theorem 5.1 in §5.1, and the proof of this theorem is included in §5.3, where we check that \( \Lambda \)-quotient functors satisfy all of the conditions in the Artin’s theorem. In §5.4, we consider the case that \( S \) is an affine scheme and \( \mathcal{X} \) is a projective Deligne-Mumford stack over \( S \). Under this condition, we prove that the functor \( F_\rho : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X) \) induces a natural transformation

\[
F_\rho : \widetilde{\text{Quot}}_\Lambda(\mathcal{G}, \mathcal{X}) \to \widetilde{\text{Quot}}_{F_\rho(\Lambda)}(F_\rho(\mathcal{G}), X),
\]

which is a monomorphism (see Lemma 5.8). With respect to this property, we prove the following theorem.

**Theorem 1.2** (Theorem 5.11). Let \( S \) be an affine scheme and let \( \mathcal{X} \) be a projective (resp. quasi-projective) stack over \( S \). The \( \Lambda \)-quotient functor \( \widetilde{\text{Quot}}_\Lambda(\mathcal{G}, \mathcal{X}, P) \) with respect to a given integer polynomial \( P \) is represented by a projective (resp. quasi-projective) \( S \)-scheme.

Theorem 5.11, Boundedness I of \( \Lambda \)-modules (see Corollary 5.14) and Boundedness II of \( \Lambda \)-modules (see Proposition 4.7) will be used to construct the moduli space of \( p_\mathcal{E} \)-semistable \( \Lambda \)-modules in §6.

In §6, we focus on the moduli problem \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \) of \( p_\mathcal{E} \)-semistable \( \Lambda \)-modules on projective Deligne-Mumford stacks. The version of this moduli problem on smooth projective varieties over \( \mathbb{C} \) is studied in [27]. We use a similar approach to construct the moduli space \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \) of \( p_\mathcal{E} \)-semistable \( \Lambda \)-modules on projective Deligne-Mumford stacks, and prove that this moduli space universally co-represents the moduli problem \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \).

**Theorem 1.3** (Theorem 6.7).

1. There exists a natural morphism

\[
\mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \to \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P)
\]

such that \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \) universally co-represents \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \).

2. The geometric points of \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \) represent the \( S \)-equivalent classes of \( p_\mathcal{E} \)-semistable \( \Lambda \)-modules with modified Hilbert polynomial \( P \).

3. \( \mathcal{M}_\Lambda^{s}(\mathcal{E}, \mathcal{O}_X(1), P) \) is a coarse moduli space of \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \).

4. The points of \( \mathcal{M}_\Lambda^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \) represent isomorphism classes of \( p_\mathcal{E} \)-stable \( \Lambda \)-modules.

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### 2. Preliminaries

#### 2.1. Tame Algebraic Stacks

Let \( k \) be an algebraically closed field, and let \( S \) be an algebraic space, which is locally of finite type over an algebraically closed field \( k \). An algebraic stack \( \mathcal{X} \) over \( S \) is a morphism, which is also considered as a family of algebraic stacks over \( S \). Let \( \mathcal{X} \) be an algebraic stack over \( S \) such that such that \( \mathcal{X} \) is locally of finite presentation over \( S \) with finite diagonal, where finite diagonal means that the natural map \( I_\mathcal{X} \to \mathcal{X} \) from the inertia stack \( I_\mathcal{X} \) to \( \mathcal{X} \) is finite. Under these conditions, the algebraic stack \( \mathcal{X} \) has a coarse moduli space \( X \) [24, Theorem 11.1.2]. Denote by \( \pi : \mathcal{X} \to X \) the natural morphism. Note that there is a natural morphism \( \rho : X \to S \).

An algebraic stack \( \mathcal{X} \), which has a coarse moduli space \( \pi : \mathcal{X} \to X \), is **tame** if the induced functor

\[
\pi_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X)
\]
is exact, where QCoh(∗) is the category of quasi-coherent sheaves over ∗. The category QCoh(∗) has a natural stack structure [24]. If char(k) = 0, the functor π∗ is always exact and the nontrivial case comes from the positive characteristic. In [1], the authors studied the tame stack in detail and proved several equivalent conditions in [1, Theorem 3.2]. We list some of the conditions as follows:

- The algebraic stack X is tame.
- Let k′ be an algebraically closed field with a morphism Spec(k′) → S. Let ξ be an object in X(Spec(k′)). Then the automorphism group scheme Autξ(k′) → Spec(k′) is linearly reductive.
- There exists an fpfp (or surjective étale) cover of the moduli space X′ → X, a finite and finitely presented scheme U over X′ and a linearly reductive group scheme G → X′ acting on U together with an isomorphism

\[ X ×_X X' \cong [U/G] \]

### 2.2. Two Functors: \( F_\mathcal{E} \) and \( G_\mathcal{E} \)

Let \( \mathcal{X} \) be a tame algebraic stack. Let \( \mathcal{E} \) be a locally free sheaf over \( \mathcal{X} \). We define two functors \( F_\mathcal{E} : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(\mathcal{X}) \) and \( G_\mathcal{E} : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(\mathcal{X}) \) as follows

\[ F_\mathcal{E}(\mathcal{F}) = \pi^* \text{Hom}_X(\mathcal{E}, \mathcal{F}), \quad G_\mathcal{E}(\mathcal{F}) = \pi^* F \otimes \mathcal{E}, \]

where \( \mathcal{F} \in \text{QCoh}(\mathcal{X}) \) and \( F \in \text{QCoh}(\mathcal{X}) \). The functor \( F_\mathcal{E} \) is exact since \( \pi^* \) is exact and \( \mathcal{E}^\vee \) is a locally free sheaf, but the functor \( G_\mathcal{E} \) may not be exact. If \( \pi : \mathcal{X} \to X \) is flat, then the functor \( G_\mathcal{E} \) is exact. The compositions of the above two functors

\[ G_\mathcal{E} \circ F_\mathcal{E} : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(\mathcal{X}), \]
\[ F_\mathcal{E} \circ G_\mathcal{E} : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(\mathcal{X}), \]

can be written in the following way

\[ G_\mathcal{E} \circ F_\mathcal{E}(\mathcal{F}) = \pi^* \pi^* \text{Hom}_X(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}, \]
\[ F_\mathcal{E} \circ G_\mathcal{E}(\mathcal{F}) = \pi^* \text{Hom}_X(\mathcal{E}, \pi^* F \otimes \mathcal{E}). \]

We define the following morphisms

\[ \theta_\mathcal{E}(\mathcal{F}) : G_\mathcal{E} \circ F_\mathcal{E}(\mathcal{F}) \to \mathcal{F}, \]
\[ \varphi_\mathcal{E}(\mathcal{F}) : F \to F_\mathcal{E} \circ G_\mathcal{E}(\mathcal{F}). \]

The morphism \( \theta_\mathcal{E}(\mathcal{F}) \) is exactly the adjunction morphism left adjoint to the identity morphism

\[ \pi^*(\mathcal{F} \otimes \mathcal{E}^\vee) \xrightarrow{\text{id}} \pi^* (\mathcal{F} \otimes \mathcal{E}^\vee). \]

If \( \mathcal{F} \) is a quasi-coherent sheaf on \( \mathcal{X} \), then the following composition is the identity

\[ F_\mathcal{E}(\mathcal{F}) \xrightarrow{\varphi_\mathcal{E}(\mathcal{F})} F_\mathcal{E} \circ G_\mathcal{E}(\mathcal{F}) \xrightarrow{\theta_\mathcal{E}(\mathcal{F})} F_\mathcal{E}(\mathcal{F}). \]

Similarly, let \( F \) be a coherent sheaf on \( X \), then the following composition is also the identity

\[ G_\mathcal{E}(\mathcal{F}) \xrightarrow{\theta_\mathcal{E}(\mathcal{F})} G_\mathcal{E} \circ F_\mathcal{E}(\mathcal{F}) \xrightarrow{\varphi_\mathcal{E}(\mathcal{F})} G_\mathcal{E}(\mathcal{F}). \]

This property is proved in [23, Lemma 2.9].

### 2.3. Generating Sheaves

With the same setup as in §2.2, a locally free sheaf \( \mathcal{E} \) is a generator for \( \mathcal{F} \in \text{QCoh}(\mathcal{X}) \), if the morphism

\[ \theta_\mathcal{E}(\mathcal{F}) : \pi^* \pi^* \text{Hom}_X(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \]

is surjective. A locally free sheaf \( \mathcal{E} \) is a generating sheaf of \( \mathcal{X} \), if it is a generator for every quasi-coherent sheaf on \( \mathcal{X} \).

If \( \mathcal{X} \) is a tame separated Deligne-Mumford stack of the form \( \mathcal{X} = [Y/G] \), where \( Y \) is a scheme and \( G \) is a finite group acting on \( Y \), then there exists a generator sheaf \( \mathcal{E} \) of \( \mathcal{X} \) [25, Proposition 5.2]. Olsson and Starr also proved a more general result about the existence of a generator sheaf of a tame, separated and finitely presented Deligne-Mumford stack \( \mathcal{X} \) over \( S \), where the stack \( \mathcal{X} = [Y/GL_{m,S}] \) can be written as a global quotient [25, Theorem 5.7].
A locally free sheaf \( \mathcal{E} \) on \( \mathcal{X} \) is \( \pi \)-very ample, if the representation of the stabilizer group at any geometric point of \( \mathcal{X} \) contains every irreducible representation. Nironi showed that a locally free sheaf \( \mathcal{E} \) on a tame Deligne-Mumford stack \( \mathcal{X} \) is a generating sheaf if and only if \( \mathcal{E} \) is \( \pi \)-very ample [23, Proposition 2.7]. This property follows from the proof of [25, Proposition 5.2].

The existence of a generating sheaf \( \mathcal{E} \) helps us to define a monomorphism of the quotient functors [25, Lemma 6.1]

\[
F_\mathcal{E} : \widetilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \to \widetilde{\text{Quot}}(F_\mathcal{E}(\mathcal{G}), X),
\]

where \( \widetilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) is the quotient functor over \( \mathcal{X} \) with respect to the given coherent sheaf \( \mathcal{G} \) and \( \widetilde{\text{Quot}}(F_\mathcal{E}(\mathcal{G}), X) \) is the quotient functor over \( X \). The quotient functors \( \widetilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) and \( \widetilde{\text{Quot}}(F_\mathcal{E}(\mathcal{G}), X) \) are represented by algebraic spaces \( \text{Quot}(\mathcal{G}, \mathcal{X}) \) and \( \text{Quot}(F_\mathcal{E}(\mathcal{G}), X) \) respectively [25, Theorem 1.1]. The monomorphism \( F_\mathcal{E} \) can be improved to be a closed embedding of the corresponding algebraic spaces [25, Theorem 4.4]

\[
\text{Quot}(\mathcal{G}, \mathcal{X}) \hookrightarrow \text{Quot}(F_\mathcal{E}(\mathcal{G}), X).
\]

This property allows us to study \( \text{Quot}(\mathcal{G}, \mathcal{X}) \) as a closed algebraic subspace of \( \text{Quot}(F_\mathcal{E}(\mathcal{G}), X) \). We will review these properties in §3.

### 2.4. Projective Deligne-Mumford Stacks

Projective Deligne-Mumford stacks are the main object we study in this paper. As its name, a projective Deligne-Mumford stack is not only a stack, but also inherits some good properties from a projective scheme. We briefly review the definition of some properties of the projective Deligne-Mumford stacks. Readers can find those materials in [17, 23, 25].

Let \( k \) be an algebraically closed field and let \( S \) be an algebraic space over \( k \). A Deligne-Mumford stack \( \mathcal{X} \) over \( k \) is a \emph{projective stack} (resp. quasi-projective stack), if \( \mathcal{X} \) is a tame separated global quotient and its coarse moduli space \( X \) is a projective (quasi-projective) scheme. Kresch proved that the following conditions are equivalent of a Deligne-Mumford stack \( \mathcal{X} \) in characteristic zero [17, Theorem 5.3, Lemma 5.4]:

- \( \mathcal{X} \) is projective (quasi-projective).
- \( \mathcal{X} \) has a projective (quasi-projective) moduli space \( X \) and there exists a generating sheaf of \( \mathcal{X} \).
- \( \mathcal{X} \) has a closed (locally closed) embedding in a smooth projective Deligne-Mumford stack.

Now let \( \mathcal{X} \) be a tame separated Deligne-Mumford stack over \( S \) which can be written as a global quotient. The stack \( \mathcal{X} \) is a \emph{projective Deligne-Mumford stack over} \( S \), if the morphism \( \mathcal{X} \to X \) factors through \( \mathcal{X} \to X \) and the morphism \( X \to S \) is a projective morphism. We also say that \( \mathcal{X} \) is a \emph{family of projective stacks over} \( S \). If \( \mathcal{X} \) is a projective stack over \( S \), each fiber \( \mathcal{X}_s \) over a geometric point \( s \) of \( S \) is a projective stack over \( k \).

### 2.5. Moduli Problem

Let \( S \) be an algebraic space, which is locally of finite type over an algebraically closed field \( k \). A moduli problem \( \widetilde{F} \) is a functor

\[
\widetilde{F} : (\text{Sch}/S)^{\text{op}} \to \text{Set},
\]

where \( (\text{Sch}/S)^{\text{op}} \) is the category of schemes over \( S \) with respect to the fppf topology or the big étale topology. The functor \( \widetilde{F} \) sends a scheme \( T \) to a set of isomorphism classes of some objects. This is the classical definition of a moduli problem.

Nowadays, people prefer to consider a moduli problem as a category \( \mathcal{M} \) fibered over \( \text{Sch}/S \) in groupoids. In other words, a moduli problem over \( \text{Sch}/S \) is a pair \((\mathcal{M}, \widetilde{F})\), where \( \mathcal{M} \) is a category fibered over \( \text{Sch}/S \) and \( \widetilde{F} : \mathcal{M} \to (\text{Sch}/S) \) is a functor, such that \( \widetilde{F}^{-1}(T) \) is a groupoid for every \( T \in \text{Sch}/S \). There is a natural way to construct a functor \( \widetilde{F} : (\text{Sch}/S)^{\text{op}} \to \text{Set} \) from the pair \((\mathcal{M}, \widetilde{F})\). Given \( T \in \text{Sch}/S \), define \( \widetilde{F}(T) = \widetilde{F}^{-1}(T) \). With respect to this construction, a moduli problem is a presheaf over \( \text{Sch}/S \) with respect to the fppf or big étale topology. In this paper, we requires that the moduli problem \((\mathcal{M}, \widetilde{F})\) satisfies the effective descent conditions. It is well-known that if \( \widetilde{F} \) is a \emph{category fibered in groupoids} (CFG) satisfying the effective descent conditions, then the corresponding functor
$	ilde{F}$ is a sheaf. There are many good references about the above constructions and properties [2, 8, 30]. In this paper, we still use the classical definition, a contravariant functor, of a moduli problem, which is a sheaf over Sch/S. We will give some examples, which are closely related to the moduli problems we consider in this paper, at the end of this subsection.

The representability is a very important property of a moduli problem. This property is also the main object we study in this paper. Let $\tilde{F} : (\text{Sch}/S)^{op} \to \text{Set}$ be a moduli problem.

These are the definitions about the representability of $\tilde{F}$ we consider in this paper.

1. If there is an $S$-algebraic space (or an $S$-scheme) $F$ such that $\tilde{F}$ is isomorphic to $\text{Hom}(\bullet, F)$, then we say that $F$ is a fine moduli space for the moduli problem, or the moduli problem $\tilde{F}$ is represented by $F$.

2. A moduli problem $\tilde{F}$ is co-represented by an $S$-algebraic space (or an $S$-scheme) $F$ if there is a morphism $\alpha : \tilde{F} \to \text{Hom}(\bullet, F)$ such that for any $S$-scheme $T$ and any morphism $\alpha' : \tilde{F} \to \text{Hom}(\bullet, T)$, there is a unique morphism $\text{Hom}(\bullet, F) \to \text{Hom}(\bullet, T)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{F} & \xrightarrow{\alpha} & \text{Hom}(\bullet, F) \\
\downarrow^{\alpha'} & & \exists ! \\
\text{Hom}(\bullet, T) & &
\end{array}
$$

3. A moduli problem $\tilde{F}$ is universally co-represented by $F$, if there is a morphism $\alpha : \tilde{F} \to \text{Hom}(\bullet, F)$ such that for any morphism $\beta : T \to F$, the fiber product $\text{Hom}(\bullet, T) \times_{\text{Hom}(\bullet, F)} \tilde{F}$ is co-represented by $T$.

4. We say that $F$ is a coarse moduli space of the moduli problem $\tilde{F}$, if there is a morphism $\alpha : \tilde{F} \to \text{Hom}(\bullet, F)$ such that

(a) $F$ co-represents $\tilde{F}$;
(b) the map $\alpha_S : \tilde{F}(S) \to \text{Hom}(S, F)$ is a bijection.

2.5.1. Moduli Problem of Coherent Sheaves over Deligne-Mumford Stacks. Let $\mathcal{X}$ be a separated and locally finitely-presented Deligne-Mumford stack over $S$. We define

$$
\overline{\mathcal{M}}(\mathcal{X}) : (\text{Sch}/S)^{op} \to \text{Set}
$$

the moduli problem of coherent sheaves as follows. For each $S$-scheme $T$, we define $\mathcal{X}_T$ as $\mathcal{X} \times_S T$ and $G_T$ the pullback of $G$ to $\mathcal{X}_T$. Define $\overline{\mathcal{M}}(\mathcal{X})(T)$ to be the set of coherent sheaves over $\mathcal{O}_{\mathcal{X}_T}$ such that

1. $\mathcal{F}_T$ is a coherent $\mathcal{O}_{\mathcal{X}_T}$-module;
2. $\mathcal{F}_T$ is flat over $T$;
3. the support of $\mathcal{F}_T$ is proper over $T$.

This moduli problem has a natural structure as an algebraic stack, which means that the moduli problem is a sheaf over Sch/S. Moreover, Nironi proved that $\overline{\mathcal{M}}(\mathcal{X})$ can be covered by the quotient functors [23, §2], and the quotient functors are represented by algebraic spaces [25, Theorem 1.1]. Therefore the moduli problem $\overline{\mathcal{M}}(\mathcal{X})$ is also represented by algebraic spaces. We will discuss this result in §3.1.

2.5.2. Moduli Problem of $\mathcal{L}$-twisted Hitchin Pairs over Deligne-Mumford Stacks. Let $\mathcal{X}$ be a separated and locally finitely-presented Deligne-Mumford stack over $S$. We fix a line bundle (invertible sheaf) $\mathcal{L}$ over $\mathcal{X}$, which is considered as the bundle for twisting. Let $\mathcal{F}$ be a coherent sheaf over $\mathcal{X}$. An $\mathcal{L}$-twisted Higgs field $\Phi$ on the coherent sheaf $\mathcal{F}$ is a homomorphism

$$
\Phi : \mathcal{F} \to \mathcal{F} \otimes \mathcal{L}.
$$
An \( \mathcal{L} \)-twisted Hitchin pair over \( \mathcal{X} \) is a pair \((\mathcal{F}, \Phi)\), where \( \mathcal{F} \) is a coherent sheaf over \( \mathcal{X} \) and \( \Phi \) is an \( \mathcal{L} \)-twisted Higgs field. We define

\[
\tilde{M}_H(\mathcal{X}, \mathcal{L}) : (\text{Sch}/S)^{\text{op}} \to \text{Set}
\]

the moduli problem of \( \mathcal{L} \)-twisted Hitchin pairs over \( \mathcal{X} \) as follows. For each \( S \)-scheme \( T \), we define the following set

\[
\tilde{M}_H(\mathcal{X}, \mathcal{L})(T) = \{ (\mathcal{F}_T, \Phi_T) \mid \mathcal{F}_T \in \tilde{M}(\mathcal{X})(T), \Phi_T : \mathcal{F}_T \to \mathcal{F}_T \otimes p_X^* \mathcal{L} \},
\]

where \( p_X : \mathcal{X} \times_S T \to \mathcal{X} \) is the natural projection. This moduli problem is also an algebraic stack [8] and is proved by the author that the moduli problem \( \tilde{M}_H(\mathcal{X}, \mathcal{L}) \) is represented by an algebraic space [29]. In fact, the twisted Hitchin pairs is a special case of the \( \Lambda \)-modules, where \( \Lambda \) is a sheaf of (differential) graded algebras (see §4.1).

3. Moduli Space of Coherent Sheaves on Deligne-Mumford Stacks

Quotient functors of coherent sheaves on a Deligne-Mumford stack \( \mathcal{X} \) are proved to be represented by algebraic spaces [25, Theorem 1.1], which is called the quotient algebraic spaces in this paper. Under some necessary conditions, a quotient algebraic space is a scheme, which is called a quotient scheme. In this case, a quotient scheme of coherent sheaves over \( \mathcal{X} \) can be embedded into a quotient scheme over the coarse moduli space \( \tilde{X} \) of \( \mathcal{X} \) [25, §6]. This property enables us to construct the moduli space of \( p \)-semistable coherent sheaves on \( \mathcal{X} \).

In §3.1, we review the result of the representability of quotient functors over a Deligne-Mumford stack. In the rest of this section, we work on a projective Deligne-Mumford stack with a given generating sheaf \( \mathcal{E} \) on \( \mathcal{X} \) and a given polarization \( \mathcal{O}_X(1) \) on \( X \). We give the definition of saturations (§3.2), modified Hilbert polynomials, \( p \)-stability condition (§3.3), Harder-Narasimhan filtrations and Jordan-Hölder filtrations (§3.4), and we review some properties of boundedness (§3.5 and §3.6). The moduli space of \( p \)-semistable coherent sheaves over \( \mathcal{X} \) is constructed as the GIT quotient of a quotient scheme [23, Theorem 6.22]. In this section, we summarize main results in constructing the moduli space of \( (p \)-semistable) coherent sheaves on a Deligne-Mumford stack [23, 25] and give a different proof for some known results.

3.1. Quotient Functors. Let \( S \) be an algebraic space, which is locally of finite type over an algebraically closed field \( k \), and let \( \mathcal{X} \) be a separated and locally finitely-presented Deligne-Mumford stack over \( S \). Denote by \( \text{Sch}/S \) the category of \( S \)-schemes with respect to the big étale topology. We take an \( \mathcal{O}_\mathcal{X} \)-module \( \mathcal{G} \), where an \( \mathcal{O}_\mathcal{X} \)-module is exactly a coherent sheaf, and define the moduli problem

\[
\tilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) : (\text{Sch}/S)^{\text{op}} \to \text{Set}
\]

as follows. For each \( S \)-scheme \( T \), define \( \mathcal{X}_T \) as \( \mathcal{X} \times_S T \) and \( \mathcal{G}_T \) the pullback of \( \mathcal{G} \) to \( \mathcal{X}_T \). Define \( \tilde{\text{Quot}}(\mathcal{G}, \mathcal{X})(T) \) to be the set of \( \mathcal{O}_{\mathcal{X}_T} \)-module quotients \( \mathcal{G}_T \to \mathcal{F}_T \) such that

1. \( \mathcal{F}_T \) is a locally finitely-preserved quasi-coherent \( \mathcal{O}_{\mathcal{X}_T} \)-module;
2. \( \mathcal{F}_T \) is flat over \( T \);
3. the support of \( \mathcal{F}_T \) is proper over \( T \).

The moduli problem \( \tilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) is called the quotient functor. The quotient functor \( \tilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) has a natural stack structure. In other words, \( \tilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) is a sheaf with respect to the étale topology of \( S \).

Artin proved that the quotient functor \( \tilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) is represented by a separated and locally finitely-presented algebraic space over \( S \) when \( \mathcal{X} \) is an algebraic space [3]. Olsson and Starr generalized this result to Deligne-Mumford stacks.

**Theorem 3.1** (Theorem 1.1 in [25]). With respect to the above notation, the quotient functor \( \tilde{\text{Quot}}(\mathcal{G}, \mathcal{X}) \) is represented by an algebraic space which is separated and locally finitely presented over \( S \).
Denote by Quot(\(G, X\)) the algebraic space representing the quotient functor \(\widehat{\text{Quot}}(G, X)\). The algebraic space Quot(\(G, X\)) is called the quotient algebraic space in this paper. If there is no ambiguity, we would like to omit \(X\), and use the notations \(\widehat{\text{Quot}}(G)\) for the quotient functor and Quot(\(G\)) for the quotient algebraic space.

Now let \(X\) be a projective Deligne-Mumford stack. We choose a polarization \(\mathcal{O}_X(1)\) on \(X\) and a generating sheaf \(\mathcal{E}\) on \(X\). The functor \(F_E\) induces a natural transformation 
\[
F_E : \widehat{\text{Quot}}(G, X) \to \widehat{\text{Quot}}(F_E(G), X),
\]
which is proved to be a monomorphism [25, Lemma 6.1]. It is well-known that the quotient functor \(\widehat{\text{Quot}}(F_E(G), X)\) is representable by disjoint union of schemes, which are parameterized by integer polynomials (Hilbert polynomials). Therefore, we have the following theorem.

**Theorem 3.2** (Theorem 1.5 and Theorem 4.4 in [25]). Let \(S\) be an affine scheme, and let \(X\) be a projective (resp. quasi-projective) Deligne-Mumford stack. The connected components of \(\text{Quot}(G, X)\) are projective (resp. quasi-projective) \(S\)-schemes, which are parameterized by integer polynomials.

Based on the above theorem, Nironi constructed a smooth atlas of Coh(\(X\)). We briefly review his construction as follows. We define the following quotient functor \(\widetilde{\text{Q}}(N, m)\)
\[
\widetilde{Q}(N, m) := \text{Quot}(\mathcal{E}^\oplus N \otimes \pi^* \mathcal{O}_X(-m), X, S),
\]
where \(N\) is a non-negative integer and \(m\) is an arbitrary integer. Denote by \(Q(N, m)\) the algebraic space representing the quotient functor \(\widetilde{Q}(N, m)\), which can be considered as a disjoint union of projective schemes. There is a natural morphism
\[
\mathcal{U}_{N, m} : \widetilde{Q}(N, m) \longrightarrow \text{Coh}(X),
\]
where Coh(\(X\)) is the stack of coherent sheaves over \(X\). Define the open subscheme \(Q^0(N, m)\) of \(Q(N, m)\), which parameterizing the isomorphism classes of coherent sheaves in \(Q(N, m)\), such that for any \(S\)-scheme \(T\) and any point \(F \in Q(N, m)\),

1. The higher derived functors \(R^i \rho_{T*}(F_{E_T}(F_T)(m))\) vanish for all positive integers \(i\), where \(\rho : X \to S\) and \(\rho_T : X_T \to T\). When \(i = 0\), \(R^0 \rho_{T*}(F_{E_T}(F_T)(m))\) is a locally free sheaf of constant rank \(N\).
2. Denote by \(E_{N, m}(\mu)\) the composition of the following morphisms
\[
E_{N, m}(\mu) : \mathcal{O}_{X_T}^\oplus N \to F_{E_T} \circ G_{E_T}(\mathcal{O}_{X_T}^\oplus N) \to F_{E_U}(F_T)(m).
\]

The pushforward \(\rho_{T*}E_{N, m}\) is an isomorphism of locally free \(\mathcal{O}_T\)-modules of the same rank.

Note that the above two conditions are both open conditions.

**Theorem 3.3** (§2 in [23]). The functor \(Q^0_{N, m}\) is represented locally by a finite type scheme. The morphism
\[
Q^0_{N, m} \subseteq Q_{N, m} \to \text{Coh}(X)
\]
is smooth for any \(N, m\), and the morphism
\[
\bigsqcup_{N, m} Q^0_{N, m} \subseteq \bigsqcup_{N, m} Q_{N, m} \to \text{Coh}(X)
\]
is surjective.

This theorem gives a smooth atlas of the stack of coherent sheaves Coh(\(X\)), and therefore, studying the moduli problem of coherent sheaves is equivalent to work on the quotient functors.
3.2. Pure Sheaves and Saturations. In this subsection, we discuss pure sheaves over a projective Deligne-Mumford stack, of which the definition is similar to that of pure sheaves over a scheme [16, §1.1] and the materials can be found in [23, §3]. We also define the saturation of a subsheaf \( \mathcal{F}' \subseteq \mathcal{F} \) and prove that the saturation is preserved under the functor \( F_\mathcal{E} \).

Let \( \mathcal{X} \) be a projective (or quasi-projective) stack with moduli space \( X \) over an algebraically closed field \( k \). We fix a polarization \( \mathcal{O}_X(1) \) on \( X \) and a generating sheaf \( \mathcal{E} \) on \( \mathcal{X} \). Let \( \mathcal{F} \) be a coherent sheaf on \( \mathcal{X} \). The support of \( \mathcal{F} \) is defined to be the closed substack associated to the ideal

\[
0 \to \mathcal{I} \to \mathcal{O}_X \to \text{End}_{\mathcal{O}_X}(\mathcal{F}).
\]

The dimension of the support \( \mathcal{I} \) is defined as the dimension of the coherent sheaf \( \mathcal{F} \). Let

\[
Y \times_\mathcal{X} Y \xrightarrow{\cong} Y \xrightarrow{i} \mathcal{X}
\]

be an \( \text{étale} \) presentation of \( \mathcal{X} \). By the flatness of the maps \( s, t, \pi' \), we have

\[
\dim(\mathcal{F}) = \dim(\pi'^* \mathcal{F}).
\]

A coherent sheaf \( \mathcal{F} \) is pure of dimension \( d \), if for every nontrivial subsheaf \( \mathcal{F}' \), the support of \( \mathcal{F}' \) is of dimension \( d \).

Similar to the case of schemes, for any coherent sheaf \( \mathcal{F} \) over \( \mathcal{X} \), we have the filtration

\[
0 \subseteq T_0(\mathcal{F}) \subseteq \cdots \subseteq T_d(\mathcal{F}) = \mathcal{F},
\]

where \( T_i(\mathcal{F})/T_{i-1}(\mathcal{F}) \) is pure of dimension \( i \) or zero. This filtration is the torsion filtration of \( \mathcal{F} \) and \( T_{d-1}(\mathcal{F}) \) is the torsion part of \( \mathcal{F} \). We use \( \mathcal{F}_\text{tor} := T_{d-1}(\mathcal{F}) \) as the notation of the torsion part of \( \mathcal{F} \).

Given a projective Deligne-Mumford stack \( \mathcal{X} \), the pureness of a coherent sheaf \( \mathcal{F} \) over \( \mathcal{X} \) is preserved by the functor \( F_\mathcal{E} \) as explained in the following corollary.

**Corollary 3.4** (Corollary 3.7 in [23]). Let \( \mathcal{X} \) be a projective Deligne-Mumford stack, and let \( \mathcal{F} \) be a coherent sheaf over \( \mathcal{X} \). The torsion filtration of \( \mathcal{F} \)

\[
0 \subseteq T_0(\mathcal{F}) \subseteq \cdots \subseteq T_d(\mathcal{F}) = \mathcal{F},
\]

is sent to the torsion filtration of \( F_\mathcal{E}(\mathcal{F}) \) under the functor \( F_\mathcal{E} \). In other words, \( F_\mathcal{E}(T_i(\mathcal{F})) = T_i(F_\mathcal{E}(\mathcal{F})) \).

Given a coherent sheaf \( \mathcal{F} \) over \( \mathcal{X} \), let \( \mathcal{F}' \) be a subsheaf of \( \mathcal{F} \). The saturation of \( \mathcal{F}' \) is the minimal subsheaf \( \mathcal{F}'_{\text{sat}} \) containing \( \mathcal{F}' \) such that \( \mathcal{F}/\mathcal{F}'_{\text{sat}} \) is pure of dimension \( \dim(\mathcal{F}) \) or zero. The saturation of a given subsheaf \( \mathcal{F}' \subseteq \mathcal{F} \) is exactly the kernel of the surjection

\[
\mathcal{F} \to \mathcal{F}/\mathcal{F}' \to (\mathcal{F}/\mathcal{F}')/T_{d-1}(\mathcal{F}/\mathcal{F}').
\]

If \( \mathcal{F}' = \mathcal{F}'_{\text{sat}} \), then we say that \( \mathcal{F}' \) is saturated. The saturation of a subsheaf is also preserved by the functor \( F_\mathcal{E} \) as proved in the following corollary.

**Corollary 3.5.** Let \( \mathcal{F} \) be a sheaf over \( \mathcal{X} \) and let \( \mathcal{F}' \) be a subsheaf of \( \mathcal{F} \). The saturation \( \mathcal{F}'_{\text{sat}} \) of \( \mathcal{F}' \) is preserved by \( F_\mathcal{E} \), i.e.

\[
F_\mathcal{E}(\mathcal{F}'_{\text{sat}}) = (F_\mathcal{E}(\mathcal{F}'))_{\text{sat}}.
\]

**Proof.** There are two natural projections

\[
j : \mathcal{F} \to \mathcal{F}/\mathcal{F}', \quad p : \mathcal{F}/\mathcal{F}' \to (\mathcal{F}/\mathcal{F}')/T_{d-1}(\mathcal{F}/\mathcal{F}').
\]

Let \( i : \mathcal{F} \to (\mathcal{F}/\mathcal{F}')/T_{d-1}(\mathcal{F}/\mathcal{F}') \) be the composition of the above two projections. By the definition of the saturation, we have the following short exact sequence

\[
0 \to \mathcal{F}'_{\text{sat}} \to \mathcal{F} \xrightarrow{i} (\mathcal{F}/\mathcal{F}')/T_{d-1}(\mathcal{F}/\mathcal{F}') \to 0.
\]

Thus we have the following diagram
From the above diagram, we have the following short exact sequence

\[ 0 \to F' \to F'_{sat} \to T_{d-1}(F/F') \to 0. \]

We apply the functor \( F_E \) to the above sequence and we have

\[ 0 \to F_E(F') \to F_E(F'_{sat}) \to F_E(T_{d-1}(F/F')) \to 0. \]

The first row is the short exact sequence by applying the functor \( F_E \) to the previous one and the second row is the short exact sequence for the sheaf \( F_E(F') \). The two objects in the first column are the same. In the third column, we have \( F_E(T_{d-1}(F/F')) \cong T_{d-1}(F_E(F/F')) \) by Corollary 3.4. Thus we have

\[ F_E(F'_{sat}) = (F_E(F'))_{sat}. \]

\[ \square \]

### 3.3. Modified Hilbert Polynomials and \( p \)-Stability Conditions.

With respect to the same conditions and notations in §3.2, let \( F \) be a coherent sheaf on \( X \). The modified Hilbert polynomial \( P_{E}(F, m) \) is defined as

\[ P_{E}(F, m) = \chi(X, F \otimes E^\vee \otimes \pi^*O_X(m)), \quad m \gg 0. \]

Since the functor \( \pi_* : \text{QCoh}(X) \to \text{QCoh}(X) \) is exact, the modified Hilbert polynomial can be written as the classical Hilbert polynomial for the coherent sheaf \( F_E(F)(m) \) over \( X \),

\[ P_{E}(F, m) = \chi(X, F_E(F)(m)), \quad m \gg 0. \]

Based on this property, the modified Hilbert polynomial is additive on short exact sequences. Also, if \( F \) is pure of dimension \( d \), the function \( P_{E}(F, m) \) is a polynomial (with respect to the variable \( m \)) and we can write it in the following way

\[ P_{E}(F, m) = \sum_{i=0}^{d} \alpha_{E,i}(F) \frac{m^i}{i!}. \]

We use the notation \( P_{E}(F) \) for the modified Hilbert polynomial of \( F \). Given an integer polynomial \( P \), if we claim that \( P \) is the modified Hilbert polynomial of \( F \), then it means that \( P = P_{E}(F) \).

The reduced modified Hilbert polynomial \( p_{E}(F) \) is a monic polynomial with rational coefficients defined as

\[ p_{E}(F) = \frac{P_{E}(F)}{\alpha_{E,d}(F)}. \]
The slope of a coherent sheaf \( \mathcal{F} \) of dimension \( d \) is
\[
\mu_\varepsilon(\mathcal{F}) = \frac{\alpha_{\varepsilon,d-1}(\mathcal{F})}{\alpha_{\varepsilon,d}(\mathcal{F})}.
\]
The rank of \( \mathcal{F} \) is
\[
\text{rk}(\mathcal{F}) = \frac{\alpha_{\varepsilon,d}(\mathcal{F})}{\alpha_{\varepsilon,d}(\mathcal{O}_X)}.
\]
A coherent sheaf \( \mathcal{F} \) is \( \varepsilon \)-semistable (resp. \( \varepsilon \)-stable), if for every proper subsheaf \( \mathcal{F}' \subseteq \mathcal{F} \) we have
\[
p_\varepsilon(\mathcal{F}') \leq p_\varepsilon(\mathcal{F}) \quad (\text{resp. } p_\varepsilon(\mathcal{F}') < p_\varepsilon(\mathcal{F})).
\]
If there is no ambiguity, we use the terminologies \( \varepsilon \)-semistable sheaf and \( \varepsilon \)-stable sheaf for simplicity.

Now let \( \mathcal{X} \) be a projective Deligne-Mumford stack over an algebraic space \( S \). Fixing an integer polynomial \( P \), we define a new moduli problem \( \widetilde{\text{Quot}}(\mathcal{G}, P) : (\text{Sch}/S)^{\text{op}} \to \text{Set} \), which can be considered as a sub-functor of the quotient functor \( \text{Quot}(\mathcal{G}) \) with respect to the given polynomial \( P \). Given an \( S \)-scheme \( T \), we define \( \text{Quot}(\mathcal{G}, P)(T) \) to be the set of \( \mathcal{O}_{\mathcal{X}_T} \)-module quotients \( (\mathcal{G}_T \to \mathcal{F}_T) \) such that
\[
(1) \quad (\mathcal{G}_T \to \mathcal{F}_T) \in \widetilde{\text{Quot}}(\mathcal{G})(T);
\]
(2) for each point \( t \in T \), the modified Hilbert polynomial of \( (\mathcal{F}_T) \), is \( P \).

The functor \( \widetilde{\text{Quot}}(\mathcal{G}, P) \) is also represented by an algebraic space by Theorem 3.1. Denote this algebraic space by \( \text{Quot}(\mathcal{G}, P) \). If \( S \) is an affine scheme, the algebraic space \( \text{Quot}(\mathcal{G}, P) \) is a projective (resp. quasi-projective) \( S \)-scheme, which is a connected component of \( \text{Quot}(\mathcal{G}) \) (see Theorem 3.2). Furthermore, any connected component of \( \text{Quot}(\mathcal{G}) \) is of the form \( \text{Quot}(\mathcal{G}, P) \) [25, §4].

### 3.4. Harder-Narasimhan Filtrations and Jordan-Hölder Filtrations.

The construction of the Harder-Narasimhan filtration and the Jordan-Hölder filtration of a coherent sheaf over a scheme is well-known [16, §1.3 and §1.5]. Since the functor \( F_\varepsilon \) is exact and the modified Hilbert polynomial \( P_\varepsilon \) is additive for short exact sequences, the construction of these two filtrations can be generalized to coherent sheaves over a projective Deligne-Mumford stack \( \mathcal{X} \). We give the definition and some results about these two filtrations in this subsection. Details can be found in [16, §1.3 and §1.5] and [23, §3.4].

**Harder-Narasimhan Filtrations.** Let \( \mathcal{F} \) be a pure sheaf of dimension \( d \) on \( \mathcal{X} \). A Harder-Narasimhan Filtration of \( \mathcal{F} \) is an increasing filtration
\[
0 = \text{HN}_0(\mathcal{F}) \subseteq \text{HN}_1(\mathcal{F}) \subseteq \cdots \subseteq \text{HN}_l(\mathcal{F}) = \mathcal{F},
\]
satisfying the following two conditions:

1. The factors \( \text{gr}_i^{\text{HN}}(\mathcal{F}) = \text{HN}_i(\mathcal{F})/\text{HN}_{i-1}(\mathcal{F}) \) are \( \varepsilon \)-semistable sheaves of dimension \( d \) for \( 1 \leq i \leq l \).
2. Denote by \( p_i(m) \) the reduced modified Hilbert polynomial \( p_\varepsilon(\text{gr}_i^{\text{HN}}(\mathcal{F}), m) \) such that
\[
p_{\max}(\mathcal{F}) := p_1 > \cdots > p_l =: p_{\min}(\mathcal{F}).
\]

For every pure coherent sheaf \( \mathcal{F} \) on \( \mathcal{X} \), there is a unique Harder-Narasimhan filtration of \( \mathcal{F} \) [23, Theorem 3.22]. The proof of the Harder-Narasimhan filtration on \( \mathcal{X} \) is similar to the case over a scheme [16, §1.3]. The key to construct the Harder-Narasimhan filtration of \( \mathcal{F} \) is to prove the existence and uniqueness of the destabilizing subsheaf of \( \mathcal{F} \). We only give the definition of the destabilizing subsheaf, and we refer the reader to [23] for the proof of the Harder-Narasimhan filtration. Let \( \mathcal{F} \) be a purely \( d \)-dimensional sheaf on \( \mathcal{X} \). There is a subsheaf \( \mathcal{F}_{de} \subseteq \mathcal{F} \) such that

1. For all subsheaves \( \mathcal{F}' \subseteq \mathcal{F} \) we have \( p(\mathcal{F}_{de}) \geq p(\mathcal{F}') \).
2. If \( p_\varepsilon(\mathcal{F}_{de}) = p_\varepsilon(\mathcal{F}') \), we have \( \mathcal{F}' \subseteq \mathcal{F}_{de} \).

The subsheaf \( \mathcal{F}_{de} \) is called the destabilizing subsheaf of \( \mathcal{F} \). Note that \( \mathcal{F}_{de} \) is \( \varepsilon \)-semistable, saturated and uniquely determined.
Jordan-Hölder Filtrations. Let $\mathcal{F}$ be a semistable sheaf on $\mathcal{X}$ with reduced modified Hilbert polynomial $p_\mathcal{E}(\mathcal{F})$. A Jordan-Hölder Filtration of $\mathcal{F}$ is an increasing filtration

$$0 = \text{JH}_0(\mathcal{F}) \subseteq \text{JH}_1(\mathcal{F}) \subseteq \cdots \subseteq \text{JH}_l(\mathcal{F}) = \mathcal{F}$$

such that the factor $\text{gr}_i^{\text{JH}}(\mathcal{F}) = \text{JH}_i(\mathcal{F})/\text{JH}_{i-1}(\mathcal{F})$ is stable with reduced modified Hilbert polynomial $p_\mathcal{E}(\mathcal{F}, m)$ for $1 \leq i \leq l$.

For every semistable sheaf $\mathcal{F}$ on $\mathcal{X}$, there is a Jordan-Hölder filtration of $\mathcal{F}$ and the graded sheaf $\text{gr}_i^{\text{JH}}(\mathcal{F}) := \bigoplus_j \text{gr}_j^{\text{JH}}(\mathcal{F})$ does not depend on the choice of the Jordan-Hölder filtration [23, Theorem 3.23].

Two $p$-semistable sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ with the same reduced modified Hilbert polynomial are $S$-equivalent if the graded sheaves of the Jordan-Hölder filtrations are isomorphic, i.e.

$$\text{gr}_i^{\text{JH}}(\mathcal{F}_1) \cong \text{gr}_i^{\text{JH}}(\mathcal{F}_2).$$

**Remark 3.6.** Unlike the pureness and the saturation, the Harder-Narasimhan filtration and the Jordan-Hölder filtration are not preserved by the functor $F_\mathcal{E}$. More precisely, let

$$0 \subseteq \text{HN}_0(\mathcal{F}) \subseteq \text{HN}_1(\mathcal{F}) \subseteq \cdots \subseteq \text{HN}_l(\mathcal{F}) = \mathcal{F}$$

be the Harder-Narasimhan filtration of the sheaf $\mathcal{F}$. The filtration

$$0 \subseteq F_\mathcal{E}(\text{HN}_0(\mathcal{F})) \subseteq F_\mathcal{E}(\text{HN}_1(\mathcal{F})) \subseteq \cdots \subseteq F_\mathcal{E}(\text{HN}_l(\mathcal{F})) = F_\mathcal{E}(\mathcal{F})$$

is not the Harder-Narasimhan filtration of $F_\mathcal{E}(\mathcal{F})$ in general. The same argument holds for the Jordan-Hölder filtration. Therefore the functor $F_\mathcal{E}$ does not preserve the $p$-semistability (resp. $p$-stability). In other words, if $\mathcal{F}$ is $p$-semistable (resp. $p$-stable), the coherent sheaf $F_\mathcal{E}(\mathcal{F})$ may not be $p$-semistable (resp. $p$-stable). A careful discussion is in [23, Remark 3.24].

3.5. Boundedness of Coherent Sheaves I. In this subsection, we first review the definition and some properties of the boundedness of coherent sheaves over a scheme. Then we extend these properties to coherent sheaves over a projective Deligne-Mumford stack. We use the notation $X$ for a scheme over $S$ in the first part of this subsection. In the second part of this subsection, $X$ will be the coarse moduli space of the given projective Deligne-Mumford stack $\mathcal{X}$. Some materials can be found in [16, §1.7] and [23, §4].

**Boundedness over Schemes.** Let $\mathcal{F}$ be a coherent sheaf over a scheme $X$. The sheaf $\mathcal{F}$ is $m$-regular if we have $H^q(X, \mathcal{F}(m - i)) = 0$. Denote by $\text{reg}(\mathcal{F})$ is least integer $m$ such that $\mathcal{F}$ is $m$-regular. The number of $\text{reg}(\mathcal{F})$ is called the regularity of $\mathcal{F}$, or more precisely, the Mumford-Castelnuovo regularity of $\mathcal{F}$. If $\mathcal{F}$ is a $m$-regular coherent sheaf on $X$, for $n \geq m$, we have

- $\mathcal{F}$ is $n$-regular;
- $H^q(\mathcal{F}(n)) \otimes H^0(\mathcal{O}_X(1)) \rightarrow H^q(\mathcal{F}(n + 1))$ is surjective;
- $\mathcal{F}(n)$ is generated by global sections.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two coherent sheaves over $X$. If $\mathcal{F}_1$ is $m_1$-regular, then there is an lower bound of the regularity for the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$.

**Lemma 3.7** (Corollary 3.4 in [7]). Let $M$ be an $m$-regular finitely generated graded $R$-module and let $N$ be an $n$-regular finitely generated graded $R$-module such that $\dim \text{Tor}_1^R(M, N) \leq 1$. Then $M \otimes N$ is $(m + n)$-regular.

This result can be easily generalized to the coherent sheaves. We omit the proof for the following lemma.

**Lemma 3.8.** Let $\mathcal{F}_1$ be an $m$-regular coherent sheaf over $X$ and let $\mathcal{F}_2$ be an $n$-regular coherent sheaf over $X$ such that $\dim \text{Tor}_1^R(\mathcal{F}_1, \mathcal{F}_2) \leq 1$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2$ is $(m + n)$-regular.
Now we will give the definition of the boundedness. A family $\overline{\mathcal{F}}$ of isomorphism classes of coherent sheaves over $X$ is **bounded**, if there is an $S$-scheme $T$ of finite type and a coherent $\mathcal{O}_X$-sheaf $\overline{\mathcal{G}}$ such that

$$\overline{\mathcal{F}} \subseteq \{ \overline{\mathcal{G}}_t \mid t \text{ a closed point in } T \}.$$ 

There are several properties equivalent to the property of the boundedness of the family $\overline{\mathcal{F}}$. We list them as follows:

- The family $\overline{\mathcal{F}}$ is bounded.
- The set of Hilbert polynomials $\{ P(F) \mid F \in \overline{\mathcal{F}} \}$ is finite and there is a non-negative integer $m$ such that $\text{reg}(F) \leq m$ for every $F \in \overline{\mathcal{F}}$. In other words, for any $F \in \overline{\mathcal{F}}$, the coherent sheaf $F$ is $m$-regular.
- The set of Hilbert polynomials $\{ P(F) \mid F \in \overline{\mathcal{F}} \}$ is finite and there is a coherent sheaf $G$ such that all $F \in \overline{\mathcal{F}}$ admit surjective morphisms $G \to F$.
- The coherent sheaves in $\overline{\mathcal{F}}$ have the same Hilbert polynomial $P$. There are constants $C_i$, $i = 0, \ldots, d = \deg(P)$ such that for every $F \in \overline{\mathcal{F}}$ there is an $F$-regular sequence of hyperplane sections $H_1, \ldots, H_d$ such that $h^0(F|_{\cap H_i}) = C_i$. This property is known as the **Kleiman Criterion** [16, Theorem 1.7.8].

By the Kleiman Criterion, we have the following theorem.

**Theorem 3.9** (Theorem 3.3.7 in [16]). Let $f : X \to S$ be a projective morphism of schemes of finite type over $k$. Let $\mathcal{O}_X(1)$ be an $f$-ample line bundle. We fix a polynomial of degree $d$ and a rational number $\mu_0$. Then the family of purely $d$-dimensional sheaves with Hilbert polynomial $P$ on the fibers of $f$ such that the maximal slope $\mu_{\text{max}} \leq \mu_0$ is bounded. In particular, the family of semistable sheaves with Hilbert polynomial $P$ is bounded.

**Boundedness over Stacks.** Now we come to the case of projective Deligne-Mumford stacks. Let $\mathcal{X}$ be a projective Deligne-Mumford stack over $S$ and denote by $X$ its coarse moduli space. Let $\mathcal{E}$ be a generating sheaf of $\mathcal{X}$. Let $\mathcal{F}$ be a coherent sheaf over $\mathcal{X}$. The regularity of $\mathcal{F}$ (with respect to $\mathcal{E}$) is defined to be the regularity of $F_\mathcal{E}(\mathcal{F})$, i.e.

$$\text{reg}_\mathcal{E}(\mathcal{F}) := \text{reg}(F_\mathcal{E}(\mathcal{F})).$$

If there is no ambiguity, a coherent sheaf $\mathcal{F}$ is $m$-regular if $F_\mathcal{E}(\mathcal{F})$ is $m$-regular.

With respect to this definition, we have the following lemma about the regularity of the tensor product of two coherent sheaves over $\mathcal{X}$.

**Lemma 3.10.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two coherent sheaves over $\mathcal{X}$. Let $m_i$ be the regularity of $\mathcal{F}_i$ for $i = 1, 2$. If $\dim \text{Tor}_1(F_\mathcal{E}(\mathcal{F}_1), F_\mathcal{E}(\mathcal{F}_2)) \leq 1$, then there is an integer $m$ such that $\mathcal{F}_1 \otimes \mathcal{F}_2$ is $m$-regular.

**Proof.** Since $\mathcal{F}_i$ is $m_i$-regular, $i = 1, 2$, we have $F_\mathcal{E}(\mathcal{F}_i)$ is $m_i$-regular. By Lemma 3.8, we know that $F_\mathcal{E}(\mathcal{F}_1) \otimes F_\mathcal{E}(\mathcal{F}_2)$ is $(m_1 + m_2)$-regular. Note that $\mathcal{E}$ is a locally free sheaf. We have

$$F_\mathcal{E}(\mathcal{F}_1) \otimes F_\mathcal{E}(\mathcal{F}_2) \cong F_\mathcal{E}(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \pi_*\mathcal{E}^\vee.$$ 

Since $\mathcal{E}$ is a locally free sheaf, there is an integer $m'$ such that $\pi_*\mathcal{E}$ is $m'$-regular. Therefore, by Lemma 3.8, there is an integer $m$ such that $F_\mathcal{E}(\mathcal{F}_1 \otimes \mathcal{F}_2)$ is $m$-regular.

A family $\overline{\mathcal{F}}$ of coherent sheaves on $\mathcal{X}$ is **bounded**, if there is an $S$-scheme $T$ of finite type and a coherent sheaf $\overline{\mathcal{G}}$ on $\mathcal{X} \times S T$ such that

$$\overline{\mathcal{F}} \subseteq \{ \overline{\mathcal{G}}_t \mid t \text{ a closed point in } T \}.$$ 

There are some properties equivalent to the property of the boundedness in the version of stack [23, §4]. We list some of them as follows:

- The family $\overline{\mathcal{F}}$ is bounded.
The set of modified Hilbert polynomials $P_E(F)$ for $F \in \mathfrak{F}$ is finite and there is a non-negative integer $m$ such that $F$ is $m$-regular for every $F \in \mathfrak{F}$.

The set of modified Hilbert polynomials $\{P_E(F) | F \in \mathfrak{F}\}$ is finite and there is a coherent sheaf $\mathcal{G}$ such that all $F \in \mathfrak{F}$ admit surjective morphisms $\mathcal{G} \to F$.

The family $F_E(\mathfrak{F})$ is bounded.

The above equivalent conditions tell us that if we want to prove the family $\mathfrak{F}$ of $p$-semistable coherent sheaves over $\mathcal{X}$ with the same modified Hilbert polynomial $P$ is bounded, it is equivalent to prove that the corresponding family $F_E(\mathfrak{F})$ over $X$ is bounded. By Theorem 3.9, if we can prove that the maximal slope $\mu_{\text{max}}(F_E(\mathfrak{F}))$ of the family $F_E(\mathfrak{F})$ of coherent sheaves is bounded, then the family $\mathfrak{F}$ is bounded.

Let $F$ be a pure $p$-semistable sheaf on $\mathcal{X}$. We choose an integer $\tilde{m}$ such that $\pi_*\text{End}_{\mathcal{O}_X}(E)(\tilde{m})$ is generated by global sections. Nironi proved the following inequality [23, Proposition 4.24]

$$\mu_{\text{max}}(F_E(\mathfrak{F})) \leq \mu_E(F) + \tilde{m} \deg(\mathcal{O}_X(1)).$$

This inequality together with Theorem 3.9 gives us the following result.

**Theorem 3.11** (Theorem 4.27 in [23]). Let $f : \mathcal{X} \to S$ be a family of projective stacks with a family of moduli spaces $X \to S$ over an algebraically closed field $k$. Let $E$ be a generating sheaf of $\mathcal{X}$, and let $O_X(1)$ be an $f$-ample line bundle. We fix an integral polynomial $P$ of degree $d$ and a rational number $\mu_0$. Then the family $\mathfrak{F}$ of purely $d$-dimensional sheaves with modified Hilbert polynomial $P$ on the fibers of $f$ such that the maximal slope $\mu_{\text{max}}(F) \leq \mu_0$ is bounded. In particular, the family of $p$-semistable purely $d$-dimensional sheaves with modified Hilbert polynomial $P$ is bounded.

### 3.6. Boundedness of Coherent Sheaves II

In this subsection, we review the result of the upper bound for the number of global sections of $p$-semistable sheaves on $\mathcal{X}$.

**Lemma 3.12** (Corollary 4.30 [16]). For any pure $p$-semistable sheaf $F$ of dimension $d$ on a projective stack $\mathcal{X}$, we have

$$h^0(\mathcal{X}, F \otimes \mathcal{E}^\vee) \leq \begin{cases} r\left(\frac{\mu_E(F) + \tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}}{d} \right), & \text{if } \mu_{\text{max}}(F) \geq \frac{d+1}{2} - r^2 - m \\
0, & \text{if } \mu_{\text{max}}(F) < \frac{d+1}{2} - r^2 - m \end{cases}$$

where $r$ is the rank of $F_E(\mathfrak{F})$, $\tilde{m}$ is an integer such that $\pi_*\text{End}_{\mathcal{O}_X}(E)(\tilde{m})$ is generated by global sections and $f(r) = -1 + \sum_{i=1}^{r} \frac{1}{i}$. 

**Corollary 3.13.** There is an integer $B$, which only depends on the rank $r$ of $F_E(\mathfrak{F})$, the integer $\tilde{m}$, the degree of $\mathcal{O}_X(1)$ and the dimension $d$ such that for any $p$-semistable sheaf $F$ of pure dimension $d$ on $\mathcal{X}$, we have

$$h^0(\mathcal{X}, F \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(m)) = h^0(\mathcal{X}, F_E(\mathfrak{F})(m))$$

$$\leq \begin{cases} r\left(\frac{\mu_E(F) + B + m}{d} \right)^d, & \text{if } \mu_{\text{max}}(F) \geq \frac{d+1}{2} - r^2 - m \\
0, & \text{if } \mu_{\text{max}}(F) < \frac{d+1}{2} - r^2 - m \end{cases}$$

**Proof.** We prove this corollary by induction on $m$. When $m = 0$, Lemma 3.12 tells us that

$$h^0(\mathcal{X}, F \otimes \mathcal{E}^\vee) \leq \begin{cases} r\left(\frac{\mu_E(F) + \tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}}{d} \right), & \text{if } \mu_{\text{max}}(F) \geq \frac{d+1}{2} - r^2 \\
0, & \text{if } \mu_{\text{max}}(F) < \frac{d+1}{2} - r^2 \end{cases}$$

Let $B$ be the integer $\tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}$. We have

$$r\left(\frac{\mu_E(F) + \tilde{m} \deg(\mathcal{O}_X(1)) + r^2 + f(r) + \frac{d-1}{2}}{d} \right) = \frac{r(\mu_E(F) + B)!}{d!(\mu_E(F) + B - d)!}$$

$$\leq \frac{r(\mu_E(F) + B)^d}{d!}. $$
Thus the inequality holds when $m = 0$. For the inductive step, the approach is the same as [27, Lemma 1.5] and [27, Corollary 1.7].

\begin{remark}
Let $\tilde{\mathcal{S}}^{ss}(P)$ be the set (or family) of $p$-semistable coherent sheaves of pure dimension $d$ on $\mathcal{X}$ with the modified Hilbert polynomial $P$. Note that the slope $\mu_{\mathcal{E}}(\mathcal{F})$ is uniquely determined by the modified Hilbert polynomial $P$, where $\mathcal{F} \in \tilde{\mathcal{S}}^{ss}(P)$. Thus, there is an upper bound for the set

$$\{h^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^*O_X(m)) \mid \mathcal{F} \in \tilde{\mathcal{S}}^{ss}(P)\}.$$  

\end{remark}

### 3.7. Geometric Invariant Theory
In this subsection, we make a brief review about the geometric invariant theory (GIT), which will be used to construct the moduli space of $p$-semistable coherent sheaves and the moduli space of $p$-semistable $\Lambda$-modules over a projective stack $\mathcal{X}$. There are many very good references about this topic [16, 21].

Let $G$ be an affine algebraic group over an algebraically closed field $k$ acting on a projective $k$-scheme $X$. Given an action $\sigma : G \times X \to X$, a pair $(Y, \pi)$ is called a geometric quotient of $X$ with respect to $G$ if

1. $\phi \circ \sigma = \phi \circ p_2$, where $p_2 : G \times X \to X$ is the natural projection,
2. $\phi$ is surjective and submersive,
3. the image of $\Psi = (\sigma, p_2) : G \times X \to X \times X$ is $X \times_Y X$,
4. $O_Y \cong (\phi_* (O_X))^G$.

We say that a geometric quotient $(Y, \phi)$ is universal if for any morphism $Y' \to Y$, the pair $(Y', \phi')$ is a geometric quotient of $X \times_Y Y'$ with respect to $G$, where $\phi'$ is induced by $\phi$. A geometric quotient $(Y, \phi)$ is good if the action $\sigma$ is closed and $\Psi$ is separated.

Let $\mathcal{L}$ be a $G$-linearized ample line bundle on $X$. A point $x \in X$ is semistable with respect to a given $G$-linearized ample line bundle $\mathcal{L}$ if there is an integer $n$ and an $G$-invariant global section $s \in H^0(X, \mathcal{L}^\otimes n)^G$ such that $s(x) \neq 0$. A point $x$ is stable (with respect to $\mathcal{L}$) if it is semistable, the stabilizer $G_x$ is finite and the $G$-orbit of $x$ is closed in the open set of all semistable points in $X$. Denote by $X^{ss}$ (resp. $X^s$) the set of all semistable points (resp. stable points).

\begin{theorem}[Theorem 1.10 in [21]]
Let $X$ be a projective scheme and let $G$ be a reductive group. Let $\mathcal{L}$ be a $G$-linearized ample line bundle on $X$. Then there is a projective scheme $Y$ and a morphism $\pi : X^{ss} \to Y$ such that $\pi$ is a universal good geometric quotient for the $G$-action. There is an open subset $Y^s \subseteq Y$ such that $X^s = \pi^{-1}(Y^s)$ and $\pi : X^s \to Y^s$ is a universal geometric quotient. Finally, there is a positive integer $m$ and a very ample line bundle $\mathcal{M}$ on $Y$ such that $\mathcal{L}^\otimes m |_{X^{ss}} \cong \pi^{-1}_*(\mathcal{M})$.
\end{theorem}

At the end of this section, we want to review Luna’s Étale Slicing Theorem. We refer the reader to [16, 21] for more details.

\begin{theorem}[Luna’s Étale Slicing Theorem]
Let $G$ be a reductive group acting on a finite type scheme $X$. Let $X \to X/G$ be the universal good geometric quotient. Let $x \in X$ be a point with a closed $G$-orbit. Then there exists a $G_x$-invariant locally closed subscheme $C \subseteq X$ passing through $x$, where $G_x$ is the stabilizer of $x$, such that the multiplication $C \times G \to X$ induces a $G$-equivariant étale morphism $C \times_{G_x} G \to X$.
\end{theorem}

### 3.8. Moduli Space of Coherent Sheaves
In this subsection, we study the moduli problem of $p$-semistable coherent sheaves over a projective Deligne-Mumford stack $\mathcal{X}$, and construct the moduli space $\mathcal{M}^{ss}(\mathcal{E}, O_X(1), P)$ of this moduli problem. The existence of the moduli space has been proved by Nironi (see [23, §5 and §6]), but we construct the moduli space in a slightly different way and we also explore some properties of smooth points in this moduli space $\mathcal{M}^{ss}(\mathcal{E}, O_X(1), P)$.

Let $S$ be a noetherian scheme of finite type, and let $\mathcal{X}$ be a projective (or quasi-projective) Deligne-Mumford stack with coarse moduli space $\pi : \mathcal{X} \to X$ over $S$. We choose a polarization $O_X(1)$ on $X$ and a generating sheaf $\mathcal{E}$ on $\mathcal{X}$. Let $P$ be an integer polynomial (as modified Hilbert polynomial), and $d$ is the degree of $P$, which is a positive integer (as pure dimension).
We consider the moduli problem

\[ \tilde{M}^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) : \text{(Sch}/S)^{op} \to \text{Set} \]

such that given an \( S \)-scheme \( T \), \( \tilde{M}^{ss}(\mathcal{E}, \mathcal{O}_X(1), P)(T) \) is the set of \( T \)-flat families of \( p \)-semistable sheaves on \( X \otimes_T T \) of pure dimension \( d \) with modified Hilbert polynomial \( P \) with respect to the following equivalence relation “\( \sim \)”. Let \( F_T, F'_T \in \tilde{M}^{ss}(\mathcal{E}, \mathcal{O}_X(1), P)(T) \) be two elements. We say \( F_T \sim F'_T \) if and only if \( F_T \cong F'_T \otimes p^*L \) for some \( L \in \text{Pic}(T) \).

The moduli problem \( \tilde{M}^{ss}(\mathcal{E}, \mathcal{O}_X(1), P) \) is defined for the \( p \)-semistable coherent sheaves. Similarly, we can define a moduli problem \( \tilde{M}^s(\mathcal{E}, \mathcal{O}_X(1), P) \) for the \( p \)-stable coherent sheaves. In this section, we will show that these two moduli problems are co-represented by projective (resp. quasi-projective) \( S \)-schemes.

We first consider the quotient functor \( \tilde{\text{Quot}}(\mathcal{G}, P) \). When \( S \) is an affine scheme, \( \tilde{\text{Quot}}(\mathcal{G}, P) \) is represented by a projective \( S \)-scheme \( \text{Quot}(\mathcal{G}, P) \) (see Theorem 3.2), which parameterizes quotients \( \mathcal{G} \to \mathcal{F} \to 0 \) with modified Hilbert polynomial \( P \). Moreover, by Theorem 3.11 there is a positive integer \( m \) such that for any element \((G \to F) \in \text{Quot}(\mathcal{G}, P), F \) is \( m \)-regular. Therefore there is a natural embedding

\[ \psi_m : \text{Quot}(\mathcal{G}, P) \hookrightarrow \text{Grass}(H^0(\mathcal{X}, F_\mathcal{E}(\mathcal{G})(m)), P(m)), \]

which is a closed embedding as \( m \) increasing. Let \( \mathcal{L} \) be the canonical invertible sheaf on the Grassmannian \( \text{Grass}(H^0(\mathcal{X}, F_\mathcal{E}(\mathcal{G})(m)), P(m)) \). Denote by \( \mathcal{L}_m \), which is a very ample invertible sheaf on \( \text{Quot}(\mathcal{G}, P) \), the pullback of \( \mathcal{L} \) by the embedding \( \psi_m \). Over a point \( G \to F \), the line bundle \( \mathcal{L}_m \) is exactly the invertible sheaf \( \mathcal{O}^{(m)}(\mathcal{X}, F(m)) \).

Now we go back to the family of \( p \)-semistable sheaves. By Theorem 3.11, we know that the family \( \widetilde{\mathcal{S}}^{ss}(P) \) of purely \( d \)-dimensional \( p \)-semistable coherent sheaves with modified Hilbert polynomial \( P \) is bounded. Thus we can find an integer \( m \) such that for any element \((\mathcal{F} \in \widetilde{\mathcal{S}}^{ss}(P), \mathcal{F} \) is \( m \)-regular. Moreover, by Remark 3.14, we can choose a positive integer \( N \) large enough such that for any \( \mathcal{F} \in \widetilde{\mathcal{S}}^{ss}(P) \), we have

\[ P(N) \geq P_\mathcal{F}(\mathcal{F}, m) = h^0(X, F_\mathcal{E}(\mathcal{F})(m)). \]

Let \( V \) be the linear space \( S^{\oplus P(N)} \). Note that

\[ V \cong H^0(X, F_\mathcal{E}(\mathcal{F})(N)). \]

Let \( \mathcal{G} \) be the coherent sheaf \( \mathcal{E} \otimes \pi^*\mathcal{O}_X(-N) \). The above discussion tells us that any coherent sheaf \( \mathcal{F} \in \widetilde{\mathcal{S}}^{ss}(P) \) corresponds to a surjection \([\mathcal{V} \otimes \mathcal{G} \to \mathcal{F}]\) together with an isomorphism

\[ V \cong H^0(X, F_\mathcal{E}(\mathcal{F})(N)). \]

With respect to the above discussion, we consider the quotient scheme \( \text{Quot}(V \otimes \mathcal{G}, P) \). This quotient scheme parameterizes pairs \((\mathcal{F}, \alpha)\) such that \( \mathcal{F} \in \text{Quot}(V \otimes \mathcal{G}, P) \) and \( \alpha : V \to H^0(X, F_\mathcal{E}(\mathcal{F})(N)) \) is a morphism. The morphism \( \alpha \) is induced by the quotient map \( V \otimes \mathcal{G} \to \mathcal{F} \). Thus the family \( \widetilde{\mathcal{S}}^{ss}(P) \) can be considered as a subset of \( \text{Quot}(V \otimes \mathcal{G}, P) \). More precisely, let \( Q_1^{ss} \subseteq \text{Quot}(V \otimes \mathcal{G}, P) \) be the set of purely \( d \)-dimensional \( p \)-semistable sheaves with a fixed modified Hilbert polynomial \( P \). Therefore \( Q_1^{ss} \) is bounded. Let \( Q_2^{ss} \) be the open set in \( Q_1^{ss} \), which parameterizes pairs \((\mathcal{F}, \alpha)\), where \( \alpha : V \to H^0(X, F(\mathcal{F})) \) is an isomorphism. The open set \( Q_2^{ss} \subseteq \text{Quot}(V \otimes \mathcal{G}, P) \) is exactly the family \( \widetilde{\mathcal{S}}^{ss}(P) \). With the same approach, we can construct \( Q_3^{ss} \subseteq \text{Quot}(V \otimes \mathcal{G}, P) \) including all \( p \)-stable sheaves.

As we discussed above for the quotient scheme, there is natural embedding

\[ \psi_N : \text{Quot}(V \otimes \mathcal{G}, P) \hookrightarrow \text{Grass}(H^0(X, F_\mathcal{E}(V \otimes \mathcal{G})(N)), P(N)), \]

where \( N \) is a large enough positive integer. We use the same notation \( \mathcal{L}_N \) for the line bundle over the scheme \( \text{Quot}(V \otimes \mathcal{G}, P) \). Note that there is a natural group action \( \text{SL}(V) \) on \( \text{Quot}(V \otimes \mathcal{G}, P) \), which induces an action on the line bundle \( \mathcal{L}_N \). Given a group action \( \text{SL}(V) \) on \( \text{Quot}(V \otimes \mathcal{G}, P) \) and an ample line bundle \( \mathcal{L}_N \) over \( \text{Quot}(V \otimes \mathcal{G}, P) \), the GIT quotient of \( \text{Quot}(V \otimes \mathcal{G}, P) \) is well-defined. Denote by \( \text{Quot}^{ss}(V \otimes \mathcal{G}, P) \) (resp. \( \text{Quot}^s(V \otimes \mathcal{G}, P) \)) the set of semistable (resp. stable) points in \( \text{Quot}(V \otimes \mathcal{G}, P) \).
with respect to the group action SL(V) and the line bundle $\mathcal{L}_N$. There is a strong relation between the semistability and $p$-semistability. A classical version about this relation over schemes is proved in [16, §4.4] and [27, Theorem 1.19].

**Theorem 3.17** ([§6 in [23]]). Let $P$ be an integral polynomial with degree $d$. There exists a large integer $N$ such that a point $\mathcal{F}$, which is a coherent sheaf, in $\text{Quot}(V \otimes \mathcal{G}, P)$ is semistable (resp. stable) with respect to the action of $\text{GL}(V)$ and the line bundle $\mathcal{L}_N$, if and only if $\mathcal{F}$ is $p$-semistable (resp. $p$-stable) coherent sheaf of pure dimension $d$ and the map $V \rightarrow H^0(X, F_\mathcal{E}(\mathcal{F}))$ is an isomorphism.

This theorem tells us that although $Q^s_2$ is defined as the set of $p$-semistable coherent sheaves, any point in $Q^s_2$ is semistable. Let

$$\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P) := Q^s_2/\text{SL}(V)$$

be the universal good geometric quotient with respect to the group action $\text{SL}(V)$ and line bundle $\mathcal{L}_N$.

**Theorem 3.18.** With respect to the situation above, we have

1. There exists a natural morphism

$$\widetilde{M}^s(\mathcal{E}, \mathcal{O}_X(1), P) \rightarrow \mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$$

such that $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ universally co-represents $\widetilde{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$. The points in the moduli space $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ represent the $S$-equivalent classes of $p$-semistable sheaves.

2. The moduli space $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is a projective $S$-scheme.

3. $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is a coarse moduli space of $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$.

4. If $x \in \mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is a point such that $Q^s_2$ is smooth at the inverse image of $x$, then $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is smooth at $x$.

**Proof.** Nironi proved (1) and (2) in [23, Theorem 6.22] and pointed out that $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is not a coarse moduli space.

Now we will prove the other two statements. By Theorem 3.15, there is an open subset $\mathcal{M}^s$ of $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ such that its preimage via the map $Q^s_2/\text{SL}(V)$ is the set of stable points. By Theorem 3.17, a point in $Q^s_2$ is stable if and only if it is $p$-stable. Therefore the open set $\mathcal{M}^s$ is exactly $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$. To prove that the set $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is a coarse moduli space, we only have to check that there is a bijection

$$\widetilde{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)(S) \rightarrow \text{Hom}(S, \mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)).$$

Clearly, two $p$-stable sheaves $\mathcal{F}_1, \mathcal{F}_2$ are $S$-equivalent if and only if $\mathcal{F}_1 \cong \mathcal{F}_2$. Therefore the bijection is directly induced from morphism $\widetilde{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)(S) \rightarrow \text{Hom}(S, \mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P))$. This finishes the proof of (3).

The proof of the last statement is similar to the classical case [16, §4], and we will use Luna’s Étale Slicing Theorem to prove this statement. Let $x$ be a point in $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$. It is easy to check the stabilizer of $x$ in the group $\text{GL}(V)$ is exactly $Z(\text{GL}(V))$, the center of $\text{GL}(V)$. By Luna’s Étale Slicing Theorem, $Q^s_2$ is a principal $\text{PGL}(V)$-bundle over $\mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$. Moreover, there is a locally closed subset $C$ of $x$ such that $C \times \text{PGL}(V) \rightarrow Q^s_2$ is an étale morphism, and the induced morphism $C/Z(\text{GL}(V)) \rightarrow \mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is also étale. Therefore, by the property of the étale morphism, if the inverse image of $x$ in $Q^s_2$ is smooth, then $x \in \mathcal{M}^s(\mathcal{E}, \mathcal{O}_X(1), P)$ is also a smooth point. □

4. $Λ$-Modules over Projective Deligne-Mumford Stacks

4.1. Graded Algebras and $Λ$-Modules over Projective Deligne-Mumford Stacks. A graded ring $R$ is a ring together with a direct sum decomposition $R = R_0 \oplus R_1 \oplus \ldots$ such that $R_i R_j \subseteq R_{i+j}$ for $i, j \geq 0$. A graded $R$-module $M$ is an $R$-module with a direct sum decomposition $M = \bigoplus M_i$ such that $R_i M_j \subseteq M_{i+j}$ for all $i, j$. A graded $R$-algebra $M$ is a graded $R$-module such that $M_i M_j \subseteq M_{i+j}$.
for all \(i, j\). With respect to the above definitions of graded structures, a sheaf of graded algebras over a stack can be defined in a similar way as in [27, §2].

Let \(S\) be an algebraic space, which is locally of finite type over an algebraically closed field \(k\). Let \(\mathcal{X}\) be a separated and locally finitely-presented Deligne-Mumford stack over \(S\).

A **sheaf of graded algebras** over \(\mathcal{X}\) is a sheaf of \(\mathcal{O}_X\)-algebras \(\Lambda\) with a filtration \(\Lambda_0 \subseteq \Lambda_1 \subseteq \ldots\) satisfying the following conditions.

1. \(\Lambda\) has both left and right \(\mathcal{O}_X\)-module structures.
2. \(\Lambda = \varprojlim \Lambda_i\) and \(\Lambda_i \cdot \Lambda_j \subseteq \Lambda_{i+j}\).
3. There is a natural morphism \(\mathcal{O}_X \to \Lambda\), of which the image is \(\Lambda_0\).
4. The graded sheaf \(\text{Gr}_i(\Lambda) = \Lambda_i / \Lambda_{i-1}\) is a \(\mathcal{O}_X\)-module for \(i \geq 1\).
5. The left and right \(\mathcal{O}_X\)-module structures on \(\text{Gr}_i(\Lambda)\) are equal. In other words, there is an isomorphism such that \(\text{Gr}_i(\Lambda)_l \cong \text{Gr}_i(\Lambda)_r\).
6. \(\text{Gr}(\Lambda) := \bigoplus_{i=0}^{\infty} \text{Gr}_i(\Lambda)\) is generated by \(\text{Gr}_1(\Lambda)\). More precisely, the morphism of sheaves
   \[
   \text{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \text{Gr}_1(\Lambda) \to \text{Gr}_i(\Lambda)
   \]
   is surjective.

Now we give a brief description of the sheaf of graded algebras in local chart. Let \(U \to \mathcal{X}\) be a local chart of \(\mathcal{X}\), where \(U = \text{Spec}(A)\) is an affine scheme and the morphism is étale. In this local chart, \(\Lambda(U)\) is a graded algebra over \(A\) such that

1. \(\Lambda(U)\) has both right and left structures.
2. \(\Lambda(U) = \varprojlim \Lambda_i(U)\) and \(\Lambda_i(U) \cdot \Lambda_j(U) \subseteq \Lambda_{i+j}(U)\).
3. \(\Lambda_0(U) \cong A\).
4. The graded sheaf \(\text{Gr}_i(\Lambda(U)) = \Lambda_i(U) / \Lambda_{i-1}(U)\) is a coherent \(A\)-module, \(i \geq 1\).
5. The left and right \(A\)-module structures on \(\text{Gr}_i(\Lambda(U))\) are equal.
6. \(\text{Gr}(\Lambda(U)) := \bigoplus_{i=0}^{\infty} \text{Gr}_i(\Lambda(U))\) is generated by \(\text{Gr}_1(\Lambda(U))\). More precisely, the morphism of sheaves
   \[
   \text{Gr}_1(\Lambda(U)) \otimes_A \cdots \otimes_A \text{Gr}_1(\Lambda(U)) \to \text{Gr}_i(\Lambda(U))
   \]
   is surjective.

A **\(\Lambda\)-sheaf** \(\mathcal{F}\) is a sheaf on \(\mathcal{X}\) together with a left \(\Lambda\)-action. A **coherent \(\Lambda\)-sheaf** (resp. **quasi-coherent \(\Lambda\)-sheaf**) \(\mathcal{F}\) is a \(\Lambda\)-sheaf such that \(\mathcal{F}\) is coherent (resp. quasi-coherent) with respect to the \(\mathcal{O}_X\)-structure. A coherent \(\Lambda\)-sheaf is also called a \(\Lambda\)-**module**. Similarly, a coherent sheaf is called a \(\mathcal{O}_X\)-**module**.

There are several ways to understand “an action of \(\Lambda\)”. Usually an action of \(\Lambda\) on \(\mathcal{F}\) means that we have a morphism

\[
\Lambda \to \text{End}(\mathcal{F}).
\]

Equivalently, this morphism can be interpreted as

\[
\Lambda \otimes \mathcal{F} \to \mathcal{F}.
\]

Sometimes we use the notation \((\mathcal{F}, \Phi)\) for a \(\Lambda\)-module, where \(\mathcal{F}\) is a coherent sheaf and \(\Phi : \Lambda \to \text{End}(\mathcal{F})\) is the action of \(\Lambda\) on \(\mathcal{F}\).

By condition (6), the graded sheaf \(\text{Gr}(\Lambda)\) is generated by \(\text{Gr}_1(\Lambda)\), which is a coherent sheaf. Therefore the sheaf \(\Lambda\) is also generated by \(\Lambda_1\). Now given an action of \(\Lambda\) on \(\mathcal{F}\), it gives a unique action of \(\text{Gr}_1(\Lambda)\) on \(\mathcal{F}\), and we have an injective map

\[
\text{Hom}(\Lambda, \text{End}(\mathcal{F})) \to \text{Hom}(\text{Gr}_1(\Lambda), \text{End}(\mathcal{F})).
\]
If $\text{Gr}_1(\Lambda)$ is locally free, a morphism $\text{Gr}_1(\Lambda) \to \mathcal{E}nd(\mathcal{F})$ induces a morphism

$$\mathcal{F} \to \mathcal{F} \otimes \text{Gr}_1(\Lambda)^*.$$  

Note that a morphism $\text{Gr}_1(\Lambda) \to \mathcal{E}nd(\mathcal{F})$ may not induce a well-defined a morphism $\Lambda \to \mathcal{E}nd(\mathcal{F})$.

Let $\mathcal{F}$ be a $\Lambda$-sheaf (resp. $\Lambda$-module) on $\mathcal{X}$. We say that $\mathcal{F}'$ is a $\Lambda$-subsheaf (resp. $\Lambda$-submodule) of $\mathcal{F}$, if $\mathcal{F}'$ is a subsheaf (resp. submodule) of $\mathcal{F}$ as the $\mathcal{O}_\mathcal{X}$-sheaf (resp. $\mathcal{O}_\mathcal{X}$-module) and $\mathcal{F}'$ is preserved under the action of $\Lambda$. The set of $\Lambda$-subsheaves of $\mathcal{F}$ can be obtained by tensoring every subsheaf of $\mathcal{F}$ by $\Lambda$, i.e.

$$\text{SubSf}_\Lambda(\mathcal{F}) = \Lambda \otimes_{\mathcal{E}nd(\mathcal{F})} \text{SubSf}(\mathcal{F}),$$

where $\text{SubSf}(\mathcal{F})$ is the set of subsheaves of $\mathcal{F}$, i.e.

$$\text{SubSf}(\mathcal{F}) = \{ \mathcal{F}' | \mathcal{F}' \subseteq \mathcal{F} \}.$$  

Here are some properties of a $\Lambda$-module $\mathcal{F}$.

- The torsion part $\mathcal{F}_{\text{tor}}$ of $\mathcal{F}$ is preserved by $\Lambda$. Thus $\mathcal{F}_{\text{tor}}$ is a $\Lambda$-module.
- Let $\mathcal{F}'$ be a $\Lambda$-submodule of $\mathcal{F}$. Then $\mathcal{F}/\mathcal{F}'$ is a $\Lambda$-module.
- Let $\mathcal{F}'$ be a $\Lambda$-submodule of $\mathcal{F}$. The saturation $\mathcal{F}^\text{sat}_\mathcal{F}$ of $\mathcal{F}'$ is a $\Lambda$-module.

Now we give some examples of sheaves of graded algebras.

**Sheaf of Differential Operators.** Let $D_X$ be the sheaf of differential operators over $\mathcal{X}$. Clearly, $D_X$ has a natural graded structure, of which the filtration $(D_X)_i$ is the sheaf of differential operators with order $\leq i$. A derivation $d$ on $D_X$ is a map $d : D_X \to D_X$ such that

1. $d(ab) = (da)b + (-1)^a a(db)$, where $\bar{a}$ is the order of $a$,
2. $d((D_X)_i) \subseteq (D_X)_{i-1}$,
3. $d^2 = 0$.

A basic example of a derivation is the Lie bracket $d_{\mathcal{F}}(\bullet) := [\frac{\partial}{\partial x}, \bullet]$.

This definition can be extended to any sheaf of graded algebras $\Lambda$. A derivation $d$ of $\Lambda$ is a map $d : \Lambda \to \Lambda$ such that

1. $d(ab) = (da)b + (-1)^a a(db)$,
2. $d(\Lambda_i) \subseteq \Lambda_{i-1}$,
3. $d^2 = 0$.

Let $v \in \text{Gr}_1(\Lambda)$. There is a natural derivation $d_v$ defined by the commutator $d_v(a) := [v, a] = va - av$. Now we consider the class of $v$ in $\text{Gr}_1(\Lambda)$. We use the same notation $v$ for the corresponding class in $\text{Gr}_1(\Lambda)$. There is a unique morphism $\sigma : \text{Gr}_1(\Lambda) \to \text{Hom}(\Omega^1_{\mathcal{X}}, \mathcal{O}_\mathcal{X})$. The morphism $\sigma$ is called the **symbol** of $\Lambda$.

Denote by $\Theta_\Lambda$ be the set of derivations of $\Lambda$. Note that $\Theta_\Lambda$ has a natural structure of sheaves. Let $\mathcal{F}$ be a coherent sheaf. A connection $\nabla$ on $\mathcal{F}$ is a $\mathcal{O}_\mathcal{X}$-morphism $\nabla : \Theta_\Lambda \to \mathcal{E}nd(\mathcal{F})$ satisfying the following conditions.

1. $\nabla_{f \theta}(s) = f \nabla_\theta(s)$,
2. $\nabla_\theta(fs) = \theta(f)s + f \nabla_\theta(s)$.
3. $\nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$.

Note that $d_v$ is also a derivation for $v \in \text{Gr}_1(\Lambda)$. Thus $\nabla_{d_v}$ is a homomorphism of $\mathcal{F}$, which induces an action of $v$ on $\mathcal{F}$. Thus a connection $\nabla$ gives us a well-defined $\Lambda$-action on $\mathcal{F}$, i.e. a $\Lambda$-sheaf $\mathcal{F}$.

**$\mathcal{L}$-Twisted Higgs Bundle.** Now we consider the example, the $\mathcal{L}$-twisted Higgs bundle. Let $\mathcal{L}$ be a locally free sheaf over $\mathcal{X}$. The sheaf of graded algebras corresponding to $\mathcal{L}$ is defined as $\Lambda_\mathcal{L} := \text{Sym}^\bullet(\mathcal{L}^*)$. Note that $\text{Gr}_1(\Lambda_\mathcal{L}) = \mathcal{L}^*$. In this case, a morphism $\Lambda_\mathcal{L} \to \mathcal{E}nd(\mathcal{F})$ is uniquely determined by a map $\text{Gr}_1(\Lambda_\mathcal{L}) \to \mathcal{E}nd(\mathcal{F})$. If we start with a morphism $\text{Gr}_1(\Lambda_\mathcal{L}) \to \mathcal{E}nd(\mathcal{F})$, this morphism will give us a well-defined map

$$\text{Gr}_1(\Lambda_\mathcal{L}) \otimes \mathcal{F} \to \mathcal{F}.$$
and then,
\[ F \to F \otimes (\text{Gr}_1(\Lambda_L))^* \Rightarrow F \to F \otimes L. \]
The induced map \( F \to F \otimes L \) is exactly an \( L \)-twisted Higgs field.

4.2. \( p \)-Semistability of \( \Lambda \)-modules. In this subsection, we study the \( p \)-semistability of \( \Lambda \)-modules and prove some properties of Harder-Narasimhan filtrations and Jordan-Hölder filtrations for \( \Lambda \)-modules.

Let \( X \) be a projective stack and with moduli space \( X \) over \( S \). We fix a polarization \( O_X(1) \) on \( X \) and a generating sheaf \( E \) on \( X \). With respect to the polarizations \( O_X(1) \) and \( E \), we have the reduced modified Hilbert polynomial \( p_{E}(F) \) of a coherent sheaf \( F \) over \( X \) (see §3.3).

Let \( \Lambda \) be a sheaf of graded algebras over \( O_X \). A \( \Lambda \)-module \( F \) is pure of dimension \( d \), if the underlying structure as an \( O_X \)-module is pure of dimension \( d \).

A \( \Lambda \)-module \( F \) is \( p \)-semistable (resp. \( p \)-stable), if \( F \) is pure and for any \( \Lambda \)-subsheaf \( F' \subseteq F \) with \( 0 < \text{rk}(F') < \text{rk}(F) \), we have
\[ p_{E}(F', m) \leq p_{E}(F, m), \quad m \gg 0, \quad (\text{resp. } <). \]

\( \Lambda \)-Harder-Narasimhan Filtrations. Let \( F \) be a purely \( d \)-dimensional \( \Lambda \)-sheaf on \( X \). A destabilizing \( \Lambda \)-subsheaf \( F_\text{de}^\Lambda \) is a \( \Lambda \)-subsheaf of \( F \) such that

1. For all \( \Lambda \)-subsheaves \( F' \subseteq F \) we have \( p_{E}(F_\text{de}^\Lambda) \geq p_{E}(F') \).
2. If \( p_{E}(F_\text{de}^\Lambda) = p_{E}(F') \), we have \( F' \subseteq F_\text{de}^\Lambda \).

Lemma 4.1. For any purely \( d \)-dimensional \( \Lambda \)-sheaf \( F \) on \( X \), there is a unique destabilizing \( \Lambda \)-subsheaf \( F_\text{de}^\Lambda \).

Proof. The proof of this lemma is similar to [16, Lemma 1.3.5]. We only give the construction of the destabilizing \( \Lambda \)-subsheaf.

Consider the set of non-trivial \( \Lambda \)-subsheaves of the given \( \Lambda \)-sheaf \( F \)
\[ \text{SubSf}_\Lambda(F) = \{ F' \mid F' \text{ is a } \Lambda \text{-subsheaf of } F \}. \]
We can define a partial-order on the set \( \text{SubSf}_\Lambda(F) \) as follows. Let \( F_1 \) and \( F_2 \) be two \( \Lambda \)-subsheaves of \( F \). We say that \( F_1 \leq F_2 \), if \( F_1 \subseteq F_2 \) and \( p_{E}(F_1) \leq p_{E}(F_2) \). We take a maximal subsheaf \( F' \subseteq F \) with respect to the partial order relation such that the coefficient \( \alpha_{E \cdot d}(F') \) is minimal among all such maximal subsheaves. This subsheaf \( F' \) is exactly the destabilizing \( \Lambda \)-subsheaf of \( F \).

Lemma 4.2. Let \( F \) be a purely \( d \)-dimensional \( \Lambda \)-sheaf on \( X \). We have
\[ F_\text{de}^\Lambda = (\Lambda \otimes \text{End}(F)) F_\text{de}, \]
where \( F_\text{de} \) is the destabilizing sheaf of \( F \).

Proof. We will prove that the sheaf \( (\Lambda \otimes \text{End}(F)) F_\text{de} \) satisfies the conditions of a destabilizing \( \Lambda \)-sheaf, and then by the uniqueness of the destabilizing \( \Lambda \)-subsheaf, we will prove this lemma.

Clearly, the sheaf \( \Lambda \otimes \text{End}(F) F_\text{de} \) is a \( \Lambda \)-subsheaf of \( F \). Thus its saturation \( (\Lambda \otimes \text{End}(F)) F_\text{de} \) is also a \( \Lambda \)-subsheaf of \( F \). Given any \( \Lambda \)-subsheaf \( F' \) of \( F \), it is also a subsheaf of \( F \). Thus we have
\[ p_{E}(F') \leq p_{E}(F_\text{de}). \]
This inequality implies that
\[ p_{E}(F') = p_{E}(\Lambda \otimes \text{End}(F)) F') \leq p_{E}(\Lambda \otimes \text{End}(F)) F_\text{de}). \]
We also have the following inequality about the saturation
\[ p_{E}(\Lambda \otimes \text{End}(F)) F_\text{de}) \leq p_{E}((\Lambda \otimes \text{End}(F)) F_\text{de})). \]
Thus we have
\[ p_{E}(F') \leq p_{E}((\Lambda \otimes \text{End}(F)) F_\text{de})). \]
This finishes the proof of the first condition of the destabilizing \( \Lambda \)-subsheaf.
Now we assume that \( p_\mathcal{E}(F') = p_\mathcal{E}((\Lambda \otimes_{\text{End}(F)} F_{\text{de}})_{\text{sat}}) \). As a subsheaf of \( \mathcal{F} \), we have
\[
p_\mathcal{E}(F') \leq p_\mathcal{E}(F_{\text{de}}).
\]
If \( p_\mathcal{E}(F') = p_\mathcal{E}(F_{\text{de}}) \), then \( F' \) is a subsheaf of \( F_{\text{de}} \). Thus \( F' \) is a \( \Lambda \)-subsheaf of \( (\Lambda \otimes_{\text{End}(F)} F_{\text{de}})_{\text{sat}} \). If \( p_\mathcal{E}(F') < p_\mathcal{E}(F_{\text{de}}) \), then
\[
p_\mathcal{E}(F') = p_\mathcal{E}(\Lambda \otimes_{\text{End}(F)} F') < p_\mathcal{E}(\Lambda \otimes_{\text{End}(F)} F_{\text{de}}) \leq p_\mathcal{E}((\Lambda \otimes_{\text{End}(F)} F_{\text{de}})_{\text{sat}}).
\]
This contradicts the assumption that \( p_\mathcal{E}(F') = p_\mathcal{E}((\Lambda \otimes_{\text{End}(F)} F_{\text{de}})_{\text{sat}}) \).

Now we give the definition of the \( \Lambda \)-Harder-Narasimhan filtration. Let \( \mathcal{F} \) be a \( \Lambda \)-module of pure dimension \( d \) on \( \mathcal{X} \). A \emph{\( \Lambda \)-Harder-Narasimhan filtration} of \( \mathcal{F} \) is an increasing filtration
\[
0 = HN_0(\mathcal{F}) \subseteq HN_1(\mathcal{F}) \subseteq \cdots \subseteq HN_l(\mathcal{F}) = \mathcal{F},
\]
such that

1. The subsheaves \( HN_i(\mathcal{F}) \) are \( \Lambda \)-modules for \( 1 \leq i \leq l \).
2. The factors \( \text{gr}_i^{HN}(\mathcal{F}) = HN_i(\mathcal{F})/HN_{i-1}(\mathcal{F}) \) are \( p \)-semistable \( \Lambda \)-modules for \( 1 \leq i \leq l \) of dimension \( d \).
3. Denote by \( p_i \) the reduced modified Hilbert polynomial \( p_\mathcal{E}(\text{gr}_i^{HN}(\mathcal{F})) \) such that
\[
p_{\text{max}}(\mathcal{F}) := p_1 > \cdots > p_l =: p_{\text{min}}(\mathcal{F}).
\]

\section*{Proposition 4.3.}
Let \( \mathcal{F} \) be a \( \Lambda \)-module of pure dimension \( d \) on \( \mathcal{X} \). There is a unique \( \Lambda \)-Harder-Narasimhan filtration of \( \mathcal{F} \).

\textbf{Proof.} The existence of the \( \Lambda \)-Harder-Narasimhan filtration is proved by induction. In the base step, we take \( HN_{l+1}(\mathcal{F}) = \mathcal{F}_{\text{de}}^\Lambda \). Then we consider the quotient sheaf \( \mathcal{F}/\mathcal{F}_{\text{de}}^\Lambda \), which is also a \( \Lambda \)-sheaf. By induction, we can assume that there is a \( \Lambda \)-Harder-Narasimhan filtration of \( \mathcal{F}/\mathcal{F}_{\text{de}}^\Lambda \). Thus we have a \( \Lambda \)-Harder-Narasimhan filtration of \( \mathcal{F} \) by taking the preimage of the \( \Lambda \)-Harder-Narasimhan filtration of \( \mathcal{F}/\mathcal{F}_{\text{de}}^\Lambda \).

The proof of the uniqueness of the \( \Lambda \)-Harder-Narasimhan filtration is exactly the same as the proof of the classical case [16, Theorem 1.3.4].

\section*{\( \Lambda \)-Jordan-Hölder Filtrations.}
Let \( \mathcal{F} \) be a \( p \)-semistable \( \Lambda \)-module on \( \mathcal{X} \) with the reduced modified Hilbert polynomial \( p_\mathcal{E}(\mathcal{F}) \). A \emph{\( \Lambda \)-Jordan-Hölder Filtration} of \( \mathcal{F} \) is an increasing filtration
\[
0 \subseteq JH_0(\mathcal{F}) \subseteq JH_1(\mathcal{F}) \subseteq \cdots \subseteq JH_l(\mathcal{F}) = \mathcal{F}
\]
such that the factors \( \text{gr}_i^{JH}(\mathcal{F}) = JH_i(\mathcal{F})/JH_{i-1}(\mathcal{F}) \) are \( p \)-stable with respect to the reduced modified Hilbert polynomial \( p_\mathcal{E}(\mathcal{F}) \) for \( 1 \leq i \leq l \).

\section*{Proposition 4.4.}
Let \( \mathcal{F} \) be a \( \Lambda \)-semistable sheaf on \( \mathcal{X} \). There is a \( \Lambda \)-Jordan-Hölder filtration of \( \mathcal{F} \), and the graded sheaf \( \text{gr}^{JH}(\mathcal{F}) := \bigoplus_i \text{gr}_i^{JH}(\mathcal{F}) \) does not depend on the choice of the Jordan-Hölder filtration.

\textbf{Proof.} The proof of the existence and uniqueness (up to isomorphism between the graded sheaves) is the same as [16, Proposition 1.5.2].

Two \( p \)-semistable \( \Lambda \)-sheaves \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) with the same reduced modified Hilbert polynomial are called \emph{\( S \)-equivalent} if the graded sheaves of the \( \Lambda \)-Jordan-Hölder filtrations are isomorphic, i.e. \( \text{gr}^{JH}(\mathcal{F}_1) \cong \text{gr}^{JH}(\mathcal{F}_2) \).
4.3. Boundedness of $\Lambda$-modules II. Let $\tilde{S}^n_d(P)$ be the set of $p$-semistable $\Lambda$-modules of pure dimension $d$ with the modified Hilbert polynomial $P$. In this subsection, we study the upper bound the following set

$$\{h^0(\mathcal{X}, F \otimes \mathcal{E}^r \otimes \pi^*\mathcal{O}_X(m)) \mid F \in \tilde{S}^n_d(P)\}.$$ 

We will prove that there is an upper-bound for the above set (see Proposition 4.7).

**Lemma 4.5.** Let $F$ be a $\Lambda$-module of pure dimension $d$ and rank $r$ over $\mathcal{X}$. Let $F'$ be a subsheaf of $F$, not necessarily preserved by $\Lambda$. Denote by $F'_r$ the image of the morphism $\Lambda_r \otimes_{\mathcal{O}_X} F' \rightarrow F$. Then $F'_r$ is a $\Lambda$-subsheaf of $F$.

**Proof.** The proof of this lemma is the same as [27, Lemma 3.2].

**Lemma 4.6.** There is an integer $m$ such that $\text{Gr}_1(\Lambda) \otimes \pi^*\mathcal{O}_X(m)$ is generated by global sections. For any $p$-semistable $\Lambda$-module $F$ of pure dimension $d$ and rank $r$, and any subsheaf $F' \subseteq F$, we have

$$\mu_F(F') \leq \mu_F(F) + mr.$$

**Proof.** We know that the functor $F_{\mathcal{E}}$ preserves coherent sheaves and the generating sheaf $\mathcal{E}$ is a locally free sheaf. The sheaf $F_{\mathcal{E}}(\text{Gr}_1(\Lambda))$ is therefore a coherent sheaf over $\mathcal{X}$. We can find an integer $m_1$ such that $F_{\mathcal{E}}(\text{Gr}_1(\Lambda) \otimes \pi^*\mathcal{O}_X(m_1))$ is generated by global sections. By the exactness of the functor $\pi_* : \text{Coh}(\mathcal{X}) \rightarrow \text{Coh}(X)$, we have

$$H^i(\mathcal{X}, \text{Gr}_1(\Lambda) \otimes \mathcal{E}^r \otimes \pi^*\mathcal{O}_X(m_1)) \cong H^i(X, F_{\mathcal{E}}(\text{Gr}_1(\Lambda) \otimes \pi^*\mathcal{O}_X(m_1))).$$

Thus the coherent sheaf $\text{Gr}_1(\Lambda) \otimes \mathcal{E}^r \otimes \pi^*\mathcal{O}_X(m_1)$ is generated by global sections. Since $\mathcal{E}$ is a locally free sheaf, there is an integer $m_2$ such that the sheaf $\mathcal{E}^r \otimes \pi^*\mathcal{O}_X(m_2)$ is generated by global sections. Thus there is an integer $m$ such that the tensor product $\text{Gr}_1(\Lambda) \otimes \pi^*\mathcal{O}_X(m)$ is generated by global sections by Lemma 3.10.

The proof of the second part of this lemma is similar to [27, Lemma 3.3]. We include the proof here for completeness. Note that a $p$-semistable $\Lambda$-module may not be a $p$-semistable coherent sheaf. Denote by $\mathcal{G}$ the destabilizing sheaf of $F$ (not the $\Lambda$-destabilizing sheaf). By the definition of the destabilizing sheaf, we only have to prove that $\mu_{\mathcal{E}}(\mathcal{G}) \leq \mu_{\mathcal{E}}(F) + mr$. Let $\mathcal{G}_i$ be the image of $\Lambda_i \otimes \mathcal{G}$ in $F$. By the definition of sheaves of graded algebras, we have the following surjections of coherent sheaves

$$\Lambda_i \otimes \mathcal{G}_i \rightarrow \mathcal{G}_{i+1} \rightarrow 0$$

for $1 \leq i \leq r$. The above surjections induce

$$\text{Gr}_1(\Lambda) \otimes (\mathcal{G}_i/\mathcal{G}_{i-1}) \rightarrow \mathcal{G}_{i+1}/\mathcal{G}_i \rightarrow 0$$

for $1 \leq i \leq r$. We know that the coherent sheaf $\text{Gr}_1(\Lambda) \otimes \pi^*\mathcal{O}_X(m)$ is generated by global sections. Thus we have the following surjective map

$$V \otimes \pi^*\mathcal{O}_X(-m) \rightarrow \text{Gr}_1(\Lambda) \rightarrow 0,$$

where $V = H^0(\mathcal{X}, \text{Gr}_1(\Lambda) \otimes \pi^*\mathcal{O}_X(m))$. This surjection induces the following one

$$V \otimes_{\mathcal{O}_X} (\mathcal{G}_i/\mathcal{G}_{i-1}) \otimes \pi^*\mathcal{O}_X(-m) \rightarrow \mathcal{G}_{i+1}/\mathcal{G}_i \rightarrow 0.$$

We take a quotient $\mathcal{O}_X$-module $Q_i$ of $\mathcal{G}_i$ with smallest reduced modified Hilbert polynomial of any quotient $\mathcal{O}_X$-module, $1 \leq i \leq r$. We discuss the quotient $Q_{i+1}$ in the following two cases.

1. If $Q_{i+1}$ has a nontrivial subsheaf which is a quotient of $\mathcal{G}_i$, then $p_{\mathcal{E}}(Q_i) \leq p_{\mathcal{E}}(Q_{i+1})$. This inequality induces that $\hat{\mu}_{\mathcal{E}}(Q_i) \leq \hat{\mu}_{\mathcal{E}}(Q_{i+1})$.

2. If $Q_{i+1}$ is a quotient of $V \otimes_{\mathcal{O}_X} (\mathcal{G}_i/\mathcal{G}_{i-1}) \otimes \pi^*\mathcal{O}_X(-m)$, we have

$$p_{\mathcal{E}}(Q_i \otimes \pi^*\mathcal{O}_X(-m)) \leq p_{\mathcal{E}}(Q_{i+1}).$$

Equivalently, we have $p_{\mathcal{E}}(Q_i, n - m) \leq p_{\mathcal{E}}(Q_{i+1}, n)$. Thus we have $\hat{\mu}_{\mathcal{E}}(Q_i) - m \leq \hat{\mu}_{\mathcal{E}}(Q_{i+1})$. 


In conclusion, we always have
\[ \hat{\mu}_E(Q_i) - m \leq \hat{\mu}_E(Q_{i+1}). \]

Taking the sum over \( i \), we have
\[ \hat{\mu}_E(Q_0) \leq \hat{\mu}_E(Q_r) + mr. \]

Now we consider the polynomial \( p_E(Q_r) \). By the definition of \( Q_r \), we have \( p_E(Q_r) \leq p_E(G_r) \). Moreover, we have \( p_E(G_r) \leq p_E((G_r)_{sat}) \), where \( (G_r)_{sat} \) is the saturation of \( G_r \). By Lemma 4.5, the sheaf \( G_r \) is an \( \Lambda \)-module. Thus we have
\[ p_E(Q_r) \leq p_E(G_r) \leq p_E((G_r)_{sat}) \leq p_E(F). \]

The above inequalities of reduced modified Hilbert polynomials imply the following inequalities of slopes
\[ \hat{\mu}_E(Q_r) \leq \hat{\mu}_E(G_r) \leq \hat{\mu}_E((G_r)_{sat}) \leq \hat{\mu}_E(F). \]

Note that \( \hat{\mu}_E(Q_0) \) is exactly \( \hat{\mu}_E(G) \). We have
\[ \hat{\mu}_E(G) = \hat{\mu}_E(Q_0) \leq \hat{\mu}_E(Q_r) + mr \leq \hat{\mu}_E(F) + mr. \]

\[ \square \]

**Proposition 4.7.** Let \( X \) be a projective stack with polarizations \( E, O_X(1) \). Let \( \tilde{S}_A^m(P) \) be the set of \( p \)-semistable \( \Lambda \)-sheaves of pure dimension \( d \) with the modified Hilbert polynomial \( P \). There is an upper-bound for the set
\[ \{ h^0(X, F \otimes E^\vee \otimes \pi^*O_X(m)) \mid \tilde{F} \in \tilde{S}_A^m \}. \]

The upper-bound depends on the rank \( r \) of \( F_E(\tilde{F}) \), the integer \( \tilde{m} \) such that \( \pi_*End_{O_X}(E)(\tilde{m}) \) is generated by global sections, the degree of \( O_X(1) \), the sheaf of graded algebras \( \Lambda \) and the dimension \( d \).

Note that the dimension \( d \) and the rank \( r \) are determined by the given integer polynomial \( P \).

**Proof.** Let \( F \) be an element in \( \tilde{S}_A^m \). Although \( F \) is a \( p \)-semistable \( \Lambda \)-module, the coherent sheaf \( F \) may not be \( p \)-semistable as an \( O_X \)-module. Thus we take the Harder-Narasimhan filtration
\[ 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = F \]
of \( F \). Denote by \( gr^HN_i(F) = F_i/F_{i-1} \) the quotient sheaf, where \( 1 \leq i \leq l \). Let \( r_i \) be the rank of \( gr^HN_i(F) \). By Corollary 3.13, we know that there is an integer \( B \) such that
\[ h^0(X, F_i/F_{i-1} \otimes E^\vee \otimes \pi^*O_X(m)) \leq r_i (\hat{\mu}_E(F_i/F_{i-1}) + m + B_i)^d. \]

By Lemma 4.6, we can find an integer \( b_i \), which only depends on \( \Lambda \) and \( r \), such that
\[ \hat{\mu}_E(F_i/F_{i-1}) \leq \hat{\mu}_E(F) + b_i. \]

Take \( B = \sup\{b_i + B_i, 1 \leq i \leq l\} \). We have the following inequality
\[ h^0(X, F_i/F_{i-1} \otimes E^\vee \otimes \pi^*O_X(m)) \leq r_i (\hat{\mu}_E(F) + m + B)^d. \]

Thus we have
\[ h^0(X, F \otimes E^\vee \otimes \pi^*O_X(m)) \leq \sum_{i=1}^l h^0(X, F_i/F_{i-1} \otimes E^\vee \otimes \pi^*O_X(m)) \leq r (\hat{\mu}_E(F) + m + B)^d d!, \]
where \( r \) is the rank of \( F_E(\tilde{F}) \). This finishes the proof of this Lemma. \[ \square \]
5. Λ-Quotient Functors

In this section, we define the Λ-quotient functor and prove one of the main results in this paper that the Λ-quotient functor is represented by an algebraic space (Theorem 5.1). The method of proving this property is based on a theorem by Artin [3, Theorem 5.3]. This theorem states that a functor is representable by an algebraic space if the functor satisfies a series of conditions. Therefore, proving the representability of the Λ-quotient functor is equivalent to check these conditions in the theorem by Artin.

In §5.2, we review the Artin's theorem (Theorem 5.2) and some necessary backgrounds about the conditions listed in the theorem. The proof of the representability of the Λ-quotient functor is discussed in §5.3. After we finish the proof of Theorem 5.1, we consider the case that X is a projective (resp. quasi-projective) Deligne-Mumford stack on an affine scheme S in §5.4. In this case, the Λ-quotient functor is represented by a projective (or quasi-projective) scheme. Olsson and Starr considered these problems for quotient functors [25, §6], and we extend their approach to the Λ-quotient functor. At the end of this section, we prove the boundedness of the family of p-semistable Λ-modules (Corollary 5.14 in §5.5).

5.1. Definitions and Results. Let S be an algebraic space, which is locally of finite type over an algebraically closed field k, and let X be a separated and locally finitely-presented Deligne-Mumford stack over S. Denote by Sch/S the category of S-schemes with respect to the big étale topology or fppf topology. Let Λ be a sheaf of graded algebras. We take a Λ-module F, the action of Λ on F induces a Gr1-module structure. A Λ-structure on F gives us all possible Λ-structures on F. Thus a Λ-module F can be equipped with distinct Λ-structures, which are defined by the action of Λ on F. As an example, when Λ acts trivially on F, the action of Λ on F is the same as the OX-module structure. In other words, the Λ-quotient functor is a sheaf with respect to the big étale topology of Sch/S.

Given a coherent sheaf F, F can be equipped with distinct Λ-structures, which are defined by the action of Λ on F. As an example, when Λ acts trivially on F, the action of Λ on F is the same as the OX-module structure. A Λ-structure on F is given by a morphism Λ → End(F). The set of all morphisms

\[ \text{Hom}_{O_X}(Λ, \text{End}(F)) \cong \text{Hom}_{O_X}(Λ \otimes F, F) \]

gives us all possible Λ-structures on F. Thus a Λ-module F is a pair (F, Φ), where F is a coherent sheaf and Φ : Λ ⊗ F → F is a morphism. Based on the discussion above, QuotΛ(G)(T) is the set of pairs (F_T, Φ_T) such that

(1) F_T ∈ Quot(G)(T);
(2) Φ_T : Λ_T ⊗ F_T → F_T is a O_X-T-module-morphism.

By definition, we know that Λ is a sheaf of graded algebras, which may not be a coherent sheaf. When constructing the moduli space and proving some properties, a coherent sheaf is a better option. Note that Gr1(Λ) is a coherent sheaf and Gr1(Λ) contains all generators of Λ. Therefore a Λ-structure on F induces a Gr1(Λ)-structure on F. We have the following injective map

\[ \text{Hom}_{O_X}(Λ, \text{End}(F)) \hookrightarrow \text{Hom}_{O_X}(\text{Gr1}(Λ), \text{End}(F)) \]
as we discussed in §4.1. Thus we work on the morphism \( \text{Gr}_1(\Lambda) \to \mathcal{E}nd(\mathcal{F}) \) sometimes.

The goal of this section is to prove the following theorem.

**Theorem 5.1.** The \( \Lambda \)-quotient functor \( \widetilde{\text{Quot}}_\Lambda(\mathcal{G}) \) is represented by a separated and locally finitely presented algebraic space.

Denote by \( \text{Quot}_\Lambda(\mathcal{G}) \) the algebraic space representing \( \widetilde{\text{Quot}}_\Lambda(\mathcal{G}) \).

### 5.2. A Theorem by Artin

Before we prove Theorem 5.1, we review some properties of a moduli problem and a theorem by Artin [3, Theorem 5.3]. We will use the Artin’s theorem to prove Theorem 5.1. In this subsection, we always assume that \( S \) is an algebraic space, which is locally of finite type over an algebraically closed field \( k \), and a moduli problem

\[
\widetilde{\mathcal{F}} : (\text{Sch}/S)^\text{op} \to \text{Set}
\]

is a presheaf over \( \text{Sch}/S \) with respect to the big étale topology or fppf topology, which can be considered as a category fibered in groupoids.

#### 5.2.1. Locally of Finite Presentation

Let \( \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}} \) be a morphism of moduli problems, which is considered as a morphism between CFG. We say that the morphism is locally of finite presentation, if for every filtered colimit of \( \mathcal{O}_S \)-algebras \( A = \lim_{\to} A_i \) and every commutative diagram

\[
\begin{align*}
\text{Spec} A & \longrightarrow \widetilde{\mathcal{F}} \\
\text{lim Spec} A_i & \longrightarrow \widetilde{\mathcal{G}}
\end{align*}
\]

there exists a unique dashed arrow lifting the morphism \( \lim \text{Spec} A_i \to \widetilde{\mathcal{G}} \). A moduli problem \( \widetilde{\mathcal{F}} \) is locally of finite presentation if the morphism \( \widetilde{\mathcal{F}} \to S \) is locally of finite presentation.

#### 5.2.2. Integrability

There are various definitions of integrability, which is called effectivity sometimes. Most of the definitions of integrability are similar, and the difference comes the condition on the map

\[
\widetilde{\mathcal{F}}(\bar{A}) \to \lim_{\leftarrow} \widetilde{\mathcal{F}}(\bar{A}/m^{n+1}), \quad n \geq 1.
\]

The one we take in this paper comes from [3]. We refer the readers to [8, 13] for the other definitions of integrability.

Let \( \widetilde{\mathcal{F}} : (\text{Sch}/S)^{\text{op}} \to \text{Set} \) be a moduli problem. Let \( \bar{A} \) be a complete noetherian local \( \mathcal{O}_S \)-algebra and denote by \( m \) the maximal ideal of \( \bar{A} \). We prefer to use the notation \( F(\bar{A}) \) instead of \( F(\text{Spec}(\bar{A})) \).

Given a positive integer \( n \), as a contravariant functor (for \( S \)-schemes), we have a natural map \( \widetilde{F(\bar{A})} \to \widetilde{F(\bar{A}/m^{n+1})} \). Thus we have a canonical map

\[
\widetilde{F(\bar{A})} \to \lim_{\leftarrow} \widetilde{F(\bar{A}/m^{n+1})}.
\]

Let \( (\mathcal{F}_n)_{n \geq 1} \) be an element in \( \lim_{\leftarrow} \widetilde{F(\bar{A}/m^{n+1})} \). If there is an element \( \mathcal{F}' \in \widetilde{F(\bar{A})} \) such that \( \mathcal{F}' \) induces \( \mathcal{F}_1 \in \widetilde{F(\bar{A}/m^2)} \), then we say that the map \( \widetilde{F(\bar{A})} \to \lim_{\leftarrow} \widetilde{F(\bar{A}/m^{n+1})} \) has a dense image.

The moduli problem \( \widetilde{\mathcal{F}} \) is integrable if for every complete noetherian local ring \( \bar{A} \), the canonical map \( \widetilde{F(\bar{A})} \to \lim_{\leftarrow} \widetilde{F(\bar{A}/m^{n+1})} \) is injective and the image is dense in \( \lim_{\leftarrow} \widetilde{F(\bar{A}/m^{n+1})} \).
5.2.3. Homogeneity. An infinitesimal extension of $S$-schemes is a closed embedding $T \hookrightarrow T'$ such that the ideal sheaf $\mathcal{I}_{T/T'}$ is nilpotent. A well-known example is the ring of dual numbers $\mathbb{D} = \mathbb{Z}[\epsilon]/(\epsilon^2)$. There is a natural embedding $T \rightarrow T[\epsilon] := T \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{D})$, of which the ideal is nilpotent.

Let $T \hookrightarrow T'$ be an infinitesimal extension of $S$-schemes and let $f : T \rightarrow R$ be an affine $S$-morphism. Then there is a universal $S$-scheme $R'$ completing the following diagram [31, §2.1]

\[
\begin{array}{ccc}
T & \longrightarrow & T' \\
\downarrow f & & \downarrow \\
R & \longrightarrow & R'
\end{array}
\]

A CFG $\tilde{F}$ is homogeneous if for each diagram above, the natural morphism

$$\tilde{F}(T) \rightarrow \tilde{F}(T') \times_{\tilde{F}(R')} \tilde{F}(R)$$

is an equivalence of categories. The homogeneity is also called the Schlessinger condition [26].

Let $f : \tilde{F} \rightarrow \tilde{F}'$ be a morphism of moduli problems. We say that $f$ is homogeneous, or that $X$ is homogeneous over $Y$, if for any $S$-scheme $T$ and any morphism $T \rightarrow \tilde{F}'$, the fiber product $\tilde{F} \times_{\tilde{F}'} T$ is homogeneous. This property is also called the relative homogeneity.

Now we consider an example. Let $S$ be an algebraic space, which is locally of finite type over an algebraically closed field $k$ and let $X$ be a separated and locally finitely-presented Deligne-Mumford stack over $S$. Let $\mathcal{E}$ and $\mathcal{F}$ be two quasi-coherent sheaves over $X$. Denote by

$$\mathcal{H}om(\mathcal{E}, \mathcal{F}) : (\text{Sch}/S)^{op} \rightarrow \text{Set}$$

the moduli problem of homomorphisms between $\mathcal{E}$ and $\mathcal{F}$ such that

$$\mathcal{H}om(\mathcal{E}, \mathcal{F})(T) := \text{Hom}(\mathcal{E}_T, \mathcal{F}_T)$$

for every $S$-scheme $T$. The moduli problem $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is representable by algebraic spaces locally of finite type [19, Proposition 2.3], which also implies that $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ satisfies the Schlessinger condition.

5.2.4. Deformation Theory. An infinitesimal extension of an $\mathcal{O}_S$-algebra $A$ is a surjective map of $\mathcal{O}_S$-algebras $A' \rightarrow A$ such that the kernel $M = \ker(A' \rightarrow A)$ is a finitely generated nilpotent ideal. Let $\tilde{F} : (\text{Sch}/S)^{op} \rightarrow \text{Set}$ be a functor. Let $A_0$ be a noetherian $\mathcal{O}_S$-domain. A deformation situation is defined as a triple

$$(A' \rightarrow A \rightarrow A_0, M, \xi)$$

where $A' \rightarrow A \rightarrow A_0$ is a diagram of infinitesimal extensions, $M = \ker(A' \rightarrow A)$ a finite $A_0$-module and $\xi \in \tilde{F}(A_0)$. We have a natural map $\tilde{F}(A) \rightarrow \tilde{F}(A_0)$. Let $\xi$ be an element in $\tilde{F}(A_0)$. Denote by $\tilde{F}_\xi(A)$ the set of elements in $\tilde{F}(A)$ whose image is $\xi \in \tilde{F}(A_0)$.

The deformation theory we consider in this paper is described in [3, Definition 5.2]. A deformation theory for $\tilde{F}$ consists of the following data and conditions

1. A functor associates to every triple $(A_0, M, \xi)$ an $A_0$-module $D = D(A_0, M, \xi)$, and to every map of triples $(A_0, M, \xi) \rightarrow (B_0, N, \eta)$ an linear map $D(A_0, M, \xi) \rightarrow D(B_0, N, \eta)$.

2. For every deformation situation, there is an operation of the additive group of $D(A_0, M, \xi)$ on $\tilde{F}_\xi(A')$ such that two elements are in the same orbit under the operation if and only if they have the same image in $\tilde{F}_\xi(A)$, where $\tilde{F}_\xi(A')$ is the subset of $\tilde{F}(A')$ of elements whose image in $\tilde{F}(A_0)$ is $\xi$.

5.2.5. Artin’s Theorem.

Theorem 5.2 (Theorem 5.3 in [3]). Let $\tilde{F}$ be a functor on $(\text{Sch}/S)^{op}$. Given a deformation theory for $\tilde{F}$, then $\tilde{F}$ is represented by a separated and locally of finite type algebraic space over $S$, if the following conditions hold:

1. $\tilde{F}$ is a sheaf with respect to the fppf-topology (or big étale topology).
(2) $\tilde{F}$ is locally of finite presentation.
(3) $\tilde{F}$ is integrable.
(4) $\tilde{F}$ satisfies the following conditions of separation.
   (a) Let $A_0$ be a geometric discrete valuation ring, which is a localization of a finite type $O_S$-algebra with residue field of finite type over $O_S$. Let $K, k$ be its fraction field and residue field respectively. If $\xi, \eta \in \tilde{F}(A_0)$ induce the same element in $\tilde{F}(K)$ and $\tilde{F}(k)$, then $\xi = \eta$.
   (b) Let $A_0$ be an $O_S$-integral domain of finite type. Let $\xi, \eta \in \tilde{F}(A_0)$. Suppose that there is a dense set $\mathcal{S}$ in $\text{Spec}(A_0)$ such that $\xi = \eta$ in $\tilde{F}(k(s))$ for all $s \in \mathcal{S}$. Then $\xi = \eta$ on a non-empty open subset of $\text{Spec}(A_0)$.

(5) The deformation theory satisfies the following conditions.
   (a) The module $D = D(A_0, M, \xi)$ commutes with localization in $A_0$ and is a finite module when $M$ is free of rank one.
   (b) The module operates freely on $F_\xi(A')$ when $M$ is of length one.
   (c) Let $A_0$ be an $O_S$-integral domain of finite type. There is a non-empty open set $U$ of $\text{Spec}(A_0)$ such that for every closed point $s \in U$, we have

\[ D \otimes_{A_0} k(s) = D(k, M \otimes_{A_0} k(s), \xi_s) \]

(6) Suppose that we have a deformation situation $(A' \to A \to A_0, M, \xi)$.
   (a) Let $A_0$ be of finite type and $M$ of length one. Let

\[
\begin{array}{ccc}
B' & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & A_0
\end{array}
\]

be a diagram of infinitesimal extensions of $A_0$ with $B' = A' \times_A B$. If $b \in \tilde{F}(B)$ is an element lying over $\xi$ whose image $a \in \tilde{F}(A)$ can be lifted to $\tilde{F}(A')$, then $b$ can be lifted to $\tilde{F}(B')$. This condition is the homogeneity.
   (b) $A_0$ is a geometric discrete valuation ring with fraction field $K$ and $M$ free of rank one. Denote by $A_K, A'_K$ the localizations of $A, A'$ respectively. If the image of $\xi$ in $\tilde{F}(A_K)$ can be lifted to $\tilde{F}(A'_K)$, then its image in $\tilde{F}(A_0 \times_K A_K)$ can be lifted to $\tilde{F}(A_0 \times_K A'_K)$.
   (c) With the same notations as in (6)(b), let $M$ be a free module of rank $n$ and $\xi \in \tilde{F}(A)$. Suppose that for every one-dimensional quotient $M_K$ of $M_K$ the lifting of $\xi_K$ to $\tilde{F}(A'_K)$ is obstructed, where $A'_K \to A_k \to A_K$ is the extension determined by $M_K$. Then there is a non-empty open set $U$ of $\text{Spec}(A_0)$ such that for every quotient $\epsilon : M \to M^*$ of length one with support in $U$, the lifting of $\xi$ to $\tilde{F}(A^*)$ is obstructed, where $A' \to A^* \to A$ denotes the resulting extension.

We want to remind the reader that Olsson and Starr applied Theorem 5.2 to prove that the quotient functor $\text{Quot}(\mathcal{G})$ is represented by an algebraic space (see Theorem 3.1 or [25, Theorem 1.1]). Therefore the quotient functor $\text{Quot}(\mathcal{G})$ satisfies all of the properties in Theorem 5.2. To prove that the $\Lambda$-quotient functor is represented by an algebraic space, we have to check that the functor $\text{Quot}_\Lambda(\mathcal{G})$ satisfies all of the conditions in the Artin’s theorem.

5.3. **Proof of Theorem 5.1.** We use the notation $\mathcal{M} := \text{Quot}_{\Lambda}(\mathcal{G})$ for the $\Lambda$-quotient functor and $\mathcal{M}' := \text{Quot}(\mathcal{G})$ for the usual quotient functor. It is easy to check that $\mathcal{M}$ is a sheaf with respect to the fppf (or big étale topology), and we omit the proof here. We refer the reader to [8] for details.

We know that the quotient functor $\mathcal{M}'$ is represented by an algebraic space. Therefore $\mathcal{M}'$ satisfies ALL conditions in Theorem 5.2. The proofs of *locally of finite presentation* (§5.3.1), *integrability* (§5.3.2) and *separation* (§5.3.3) depend on the corresponding properties of $\mathcal{M}'$. In §5.3.4, we construct
the deformation theory of $\mathcal{M}$ and prove that the deformation theory satisfies the conditions listed in Theorem 5.2(4). The obstruction theory is discussed in §5.3.5.

5.3.1. Locally of Finite Presentation. Let $A := \lim A_n$ be the colimit of $\mathcal{O}_S$-algebras $A_n$. There are natural maps $\mathcal{M}(A_n) \to \mathcal{M}(A)$ for $n \geq 1$. These maps induce the following one

$$\lim \mathcal{M}(A_n) \to \mathcal{M}(A).$$

To prove that $\mathcal{M}$ is of locally of finite presentation, we have to show that the above map is bijective.

Let $(\mathcal{F}_n, \Phi_n)_{n \geq 1}$ be an element in $\lim \mathcal{M}(A_n)$, where $\mathcal{F}_n \in \mathcal{M}'(A_n)$ and $\Phi_n : \Lambda \otimes \mathcal{F}_n \to \mathcal{F}_n$. Recall that $\mathcal{M}'$ is locally of finite presentation. Thus there exists a unique $\mathcal{F} \in \mathcal{M}'(A)$ corresponding to $(\mathcal{F}_n)_{n \geq 1}$. By [11, (8.2.5)], the map

$$\lim \text{Hom}(\Lambda \otimes \mathcal{F}_n, \mathcal{F}_n) \to \text{Hom}(\Lambda \otimes \mathcal{F}, \mathcal{F})$$

is also bijective. Therefore we can find a unique element $(\mathcal{F}, \Phi)$ corresponding to the given element $(\mathcal{F}_n, \Phi_n)_{n \geq 1} \in \lim \mathcal{M}(A_n)$. This finishes the proof.

5.3.2. Integrability. Let $\bar{A}$ be a complete noetherian local $\mathcal{O}_S$-algebra and denote by $\mathfrak{m}$ the maximal ideal of $\bar{A}$. After changing the base, we consider $X$ as a stack over $\bar{A}$. Let $\hat{\mathcal{X}}$ be the stack $\hat{\mathcal{X}} := \lim \leftarrow X(\bar{A}/\mathfrak{m}^n+1)$. There is a natural morphism $j : \hat{\mathcal{X}} \to X$. Let $\mathcal{F}$ be a coherent sheaf over $X$. Denote by $\hat{\mathcal{F}}$ the sheaf $j^*\mathcal{F}$. Before we prove the integrability, we first review the following lemma.

**Lemma 5.3** (Lemma 2.2 in [25]). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent sheaves over $X$ with proper support over $\bar{A}$. For every integer $n$, the map

$$\text{Ext}^n_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2) \to \lim \text{Ext}^n_{\mathcal{O}_{\bar{A}}}(\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2)$$

is an isomorphism.

Now we will prove that $\mathcal{M}$ is integrable. In other words, the map

$$\mathcal{M}(\bar{A}) \to \lim \mathcal{M}(\bar{A}/\mathfrak{m}^{n+1})$$

is injective and has a dense image.

Let $(\mathcal{F}, \Phi) \in \mathcal{M}(\bar{A})$, where $\Phi : \Lambda \otimes \mathcal{F} \to \mathcal{F}$. We will find a unique element $(\mathcal{F}_n, \Phi_n)_{n \geq 1}$ in $\lim \mathcal{M}(\bar{A}/\mathfrak{m}^{n+1})$ corresponding to the given pair $(\mathcal{F}, \Phi)$. Recall that $\mathcal{M}'$ satisfies all of the conditions in Theorem 5.2. Thus the morphism

$$\mathcal{M}'(\bar{A}) \to \lim \mathcal{M}'(\bar{A}/\mathfrak{m}^{n+1})$$

is an injection. We take an element $\mathcal{F} \in \mathcal{M}'(\bar{A})$, which corresponds to a unique element $(\mathcal{F}_n) \in \lim \mathcal{M}'(\bar{A}/\mathfrak{m}^{n+1})$. By Lemma 5.3, we have

$$\text{Hom}(\text{Gr}_1(\Lambda) \otimes \mathcal{F}, \mathcal{F}) \to \lim \text{Hom}(\text{Gr}_1(\Lambda) \otimes \mathcal{F}_n, \mathcal{F}_n)$$

is bijective. This bijection induces that the map

$$\text{Hom}(\Lambda \otimes \mathcal{F}, \mathcal{F}) \to \lim \text{Hom}(\Lambda \otimes \mathcal{F}_n, \mathcal{F}_n)$$

is also a bijection. As a result, $\Phi$ corresponds to a unique map $(\Phi_n)_{n \geq 1} \in \lim \text{Hom}(\Lambda \otimes \mathcal{F}_n, \mathcal{F}_n)$. Therefore the natural map

$$\mathcal{M}(\bar{A}) \to \lim \mathcal{M}(\bar{A}/\mathfrak{m}^{n+1})$$

is injective.
Now we will prove that the map \( M(\bar{A}) \to \lim M(\bar{A}/m^{n+1}) \) has a dense image. The proof is similar to that of the injectivity. We take an element
\[
((F_n, \Phi_n))_{n \geq 1} \in \lim M(\bar{A}/m^{n+1}),
\]
where \( F_n \in M(\bar{A}/m^{n+1}) \) and \( \Phi_n : A \otimes F_n \to F_n \). Since \( M' \) is integrable, we can find an element \( F \in M'(\bar{A}) \) such that \( F \) induces \( F_1 \in M'(\bar{A}/m^2) \). Let \( (F'_n)_{n \geq 1} \in \lim M'(\bar{A}/m^{n+1}) \) be the element corresponding to \( F \), where \( F'_1 = F_1 \). By Lemma 5.3, we know that the map
\[
\text{Hom}(\Lambda \otimes \mathcal{F}, \mathcal{F}) \to \lim \text{Hom}(\Lambda \otimes F'_n, F'_n)
\]
is bijective. Let \((\Phi'_n)_{n \geq 1}\) be an element in \( \lim \text{Hom}(\Lambda \otimes F'_n, F'_n) \) such that \( \Phi'_1 = \Phi_1 \). The element \((\Phi'_n)_{n \geq 1}\) corresponds to a unique map \( \Phi : \Lambda \otimes F' \to F' \). In conclusion, given an element \(((F_n, \Phi_n))_{n \geq 1} \), we find a pair \((F, \Phi)\) which induces \((F_1, \Phi_1)\).

5.3.3. Separation. The quotient functor \( M'(\mathcal{G}) \) satisfies the separation condition (4). If \( F_1, F_2 \in M'(\mathcal{G})(A_0) \) induce the same element in \( M'(\mathcal{G})(K) \) and \( M'(\mathcal{G})(k) \), then \( F_1 = F_2 \). Let \( \xi = (F_1, \Phi_1), \eta = (F_2, \Phi_2) \) be two elements in \( M(\mathcal{G}) \). If they induce the same element in \( M(\mathcal{G})(K) \) and \( M(\mathcal{G})(k) \), we have \( F_1 = F_2 \). This also implies that \( \text{Hom}(\Lambda, \mathcal{End}(F_1)) = \text{Hom}(\Lambda, \mathcal{End}(F_2)) \). Therefore, \( \Phi_1 = \Phi_2 \), and \( \xi = \eta \). This finishes the proof for condition (4a).

The proof of condition (4b) is the same, and we omit the proof here.

5.3.4. Deformation Theory. In this section, we calculate the \( A_0 \)-module \( M_\xi(A_0[M]) \) and prove that this module is the deformation theory for \( M \).

Let us consider a special case first. Let \( A' = A_0[M] := A_0 \oplus M \). Let \((\mathcal{F}, \Phi)\) be an element in \( M(A_0) \), where \( \Phi : \mathcal{F} \otimes \Lambda_{\text{Spec}(A_0)} \to \mathcal{F} \). Equivalently, \( \Phi \) can be considered as a morphism \( \Phi : \Lambda_{\text{Spec}(A_0)} \to \mathcal{End}(\mathcal{F}) \). For simplicity, we use \( \Lambda \) for the sheaf \( \Lambda_{\text{Spec}(A_0)} \) in this section. With respect to the above notation, the morphism \( \Phi \) is
\[
\Phi : \Lambda \otimes \mathcal{F} \to \mathcal{F} \text{ or } \Phi : \Lambda \to \mathcal{End}(\mathcal{F}).
\]
Define \( \mathcal{F}' = \mathcal{F} \times_{\text{Spec}(A_0)} \text{Spec}(A') \). Abusing the notation, we write \( \mathcal{F}' \) as \( \mathcal{F} \oplus \mathcal{F}[M] \). For a section \( s \) of \( \mathcal{End}(\mathcal{F})[M] \), the corresponding automorphism of \( \mathcal{F}' \) is denoted by \( 1 + s \). Moreover, if \( v + w \) is a section of \( \mathcal{End}(\mathcal{F}') \), where \( v \) is a section of \( \mathcal{End}(\mathcal{F}) \) and \( w \) is a section of \( \mathcal{End}(\mathcal{F})[M] \), we have
\[
\rho(1 + s)(v + w) = v + w + \rho(s)(v),
\]
where \( \rho \) is the natural action of \( \mathcal{End}(\mathcal{F}) \) on itself. The deformation complex \( C_M^\bullet(\mathcal{F}; \Phi) \) is defined as follows
\[
C_M^\bullet(\mathcal{F}; \Phi) : C_M^0(\mathcal{F}) = \mathcal{End}(\mathcal{F})[M] \xrightarrow{\epsilon(\Phi)} C_M^1(\mathcal{F}) = \text{Hom}(\Lambda, \mathcal{End}(\mathcal{F})[M]),
\]
and the map \( \epsilon(\Phi) \) is given by
\[
(\epsilon(\Phi)(s))(\lambda) = -\rho(s)(\Phi(\lambda)),
\]
where \( s \in \mathcal{End}(\mathcal{F})[M] \) and \( \lambda \in \Lambda \) are sections. If there is no ambiguity, we omit the notations \( M, \mathcal{F}, \Phi \) in the complex \( C_M^\bullet(\mathcal{F}; \Phi) \) and use the following notation
\[
C^\bullet : C^0 = \mathcal{End}(\mathcal{F})[M] \xrightarrow{\epsilon(\Phi)} C^1 = \text{Hom}(\Lambda, \mathcal{End}(\mathcal{F})[M])
\]
for the deformation complex.

Now we are ready to calculate \( M_\xi(A_0[M]) \). The following proposition is a generalization of Theorem 2.3 in [6].

**Proposition 5.4.** Let \( \xi = (\mathcal{F}, \Phi) \) be a \( \Lambda \)-module in \( M(A_0) \). The set \( M_\xi(A_0[M]) \) is isomorphic to the hypercohomology group \( H^1(C^\bullet) \), where \( C^\bullet \) is the complex
\[
C^\bullet : C^0 = \mathcal{End}(\mathcal{F})[M] \xrightarrow{\epsilon(\Phi)} C^1 = \text{Hom}(\Lambda, \mathcal{End}(\mathcal{F})[M]),
\]
where the map $e(\Phi)$ is defined as above.

**Proof.** Let $U = \{U_i = \text{Spec}(A_i)\}_{i \in I}$ be an étale covering of $X$ by affine schemes, where $I$ is the index set. The covering $U$ of $X$ also gives an étale covering $\{U_i \times_S \text{Spec}(A_0)\}$ of $X \times_S \text{Spec}(A_0)$. Define $U_i[M] = U_i \times_S \text{Spec}(A_0[M])$. Set

$$\mathcal{E}nd(\mathcal{F})[M]|_{U_i[M]} = C_i^0,$$

$$\text{Hom}(\Lambda, \mathcal{E}nd(\mathcal{F})[M])|_{U_i[M]} = C_i^1,$$

where $C_i^0$ and $C_i^1$ are $A_0$-modules. Similarly, modules $C_{ij}^0$ (resp. $C_{ij}^1$) are restrictions of $C^0$ (resp. $C^1$) to $U_{ij}[M] = U_i[M] \cap U_j[M]$. We consider the following Čech resolution of $C^\bullet$:

\[
\begin{array}{ccccccccc}
0 & 0 & C^0 & e(\Phi) & C^1 & 0 \\
\downarrow & & \downarrow d_0^0 & & \downarrow d_0^1 & \\
0 & \sum C_{ij}^0 & e(\Phi) & \sum C_{ij}^1 & 0 \\
\downarrow d_1^0 & & \downarrow d_1^1 & \\
0 & \sum C_{ij}^0 & e(\Phi) & \sum C_{ij}^1 & 0 \\
\downarrow d_2^0 & & \downarrow d_2^1 & \\
\vdots & & \vdots & \\
\end{array}
\]

We calculate the first hypercohomology $\mathbb{H}^1(C^\bullet)$ from the above diagram. Let $Z$ be the set of pairs $(s_{ij}, t_i)$, where $s_{ij} \in C_{ij}^0$ and $t_i \in C_i^1$ satisfying the following conditions:

1. $s_{ij} + s_{jk} = s_{ik}$ as elements of $C_{ijk}^0$.
2. $t_i - t_j = e(\Phi)(s_{ij})$ as elements of $C_{ij}^1$.

Let $B$ be the subset of $Z$ consisting of elements $(s_i - s_j, e(\Phi)(s_i))$, where $s_i \in C_i^0$. By the definition of the hypercohomology, we have

$$\mathbb{H}^1(C^\bullet) = Z/B.$$

We will prove that for each element in $\mathbb{H}^1(C^\bullet)$, it corresponds to a unique $\Lambda$-module on $X \times \text{Spec}(A_0[M])$, of which the restriction to $X \times_S \text{Spec}(A_0)$ is $(\mathcal{F}, \Phi)$. In other words, there is a bijective map between $\mathbb{H}^1(C^\bullet)$ and $\mathcal{M}_\xi(A_0[M])$.

We first prove that there is a natural map $\mathbb{H}^1(C^\bullet) \to \mathcal{M}_\xi(A_0[M])$. Given an element $(s_{ij}, t_i) \in Z$, we want to construct a $\Lambda$-module $(\mathcal{F}', \Phi')$ on $X \times_S \text{Spec}(A_0[M])$ such that

$$\mathcal{F}'|_{X \times_S \text{Spec}(A_0)} \cong \mathcal{F}, \quad \Phi'|_{X \times_S \text{Spec}(A_0)} \cong \Phi.$$

We first give the construction of $\mathcal{F}'$. For each $U_i[M]$, there is a natural projection $\pi : U_i[M] \to U_i \times_S \text{Spec}(A_0)$. Take the sheaf $\mathcal{F}'_i = \pi^*(\mathcal{F}|_{U_i \times_S \text{Spec}(A_0)})$. By the first condition of $Z$, we can identify the restrictions of $\mathcal{F}'_i$ and $\mathcal{F}'_{ij}$ to $U_{ij}[M]$ by the isomorphism $1 + s_{ij}$ of $\mathcal{F}'_{ij}$. Therefore we get a well-defined quasi-coherent sheaf $\mathcal{F}'$ on $X \times_S \text{Spec}(A_0[M])$ such that the restriction of $\mathcal{F}'$ to $X \times_S \text{Spec}(A_0)$ is $\mathcal{F}$.

Now we want to construct a morphism $\Phi' : \Lambda \to \text{End}(\mathcal{F}')$. Note that

$$\mathcal{E}nd(\mathcal{F}') \cong \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{F}[M]).$$

We also know that the morphism $\Phi'$ satisfies

$$\Phi'|_{X \times \text{Spec}(A_0)} = \Phi.$$
Thus on each affine set $U_i[M]$, we define the following morphism
\[ \Phi_i + t_i : \Lambda \to \mathcal{E}nd(\mathcal{F}^i), \]
where $\Phi_i$ is the restriction of $\Phi$ to the open set $U_i \times_S \text{Spec}(A_0)$. By the second condition of the pair $(s_{ij}, t_i)$, i.e. $t_i - t_j = e(\Phi)(s_{ij})$, we have
\[ e(\Phi_i + t_i)(1 + s_{ij}) = \Phi_j + t_j. \]
Therefore the associated Hitchin pair $(\mathcal{F}^i, \Phi^i)$ can be glued together to give a global homomorphism $\Phi' : \Lambda \to \mathcal{E}nd(\mathcal{F}')$. Given an element $(s_{ij}, t_i)$ in $Z$, we construct a $\Lambda$-module $(\mathcal{F}', \Phi')$ in $\mathcal{M}_\xi(A_0[M])$.

Let $(s_{ij}, t_i)$ be an element in $B$. In other words, $s_{ij} = s_i - s_j$ and $t_i = e(\Phi)(s_i)$. The identification of $\mathcal{F}'_i \cong \mathcal{F}'_j$ on $U_{ij}[M]$ is given by the isomorphism
\[ 1 + s_{ij} = 1 + (s_i - s_j). \]

Consider the following diagram
\[
\begin{array}{ccc}
\mathcal{F}'_{ij} & \xrightarrow{1+s_{ij}} & \mathcal{F}'_{ij} \\
\downarrow^{1+s_{ij}} & & \downarrow^{1+s_{ij}} \\
\mathcal{F}'_{ij} & \xrightarrow{1+s_{ij}} & \mathcal{F}'_{ij}
\end{array}
\]

The commutativity of the above diagram implies that $E'$ is trivial. Similarly, we have
\[ e(\Phi_i + t_i)(1 + s_i) = \Phi_i. \]

Therefore the associated Hitchin pair $(\mathcal{F}', \Phi')$ is isomorphic to $(\pi^*\mathcal{F}, \pi^*\Phi)$. In other words, for any element in $B$, the corresponding $\Lambda$-structure is trivial.

The above construction gives a well-defined map from $\mathbb{H}^1(C^*)$ to $\mathcal{M}_\xi(A_0[M])$.

Now we will construct the inverse map from $\mathcal{M}_\xi(A_0[M])$ to $\mathbb{H}^1(C^*)$. Let $(\mathcal{F}', \Phi') \in \mathcal{M}_\xi(A_0[M])$ be a $\Lambda$-module over $X \times_S \text{Spec}(A_0[M])$ such that
\[ (\mathcal{F}'|_{X \times_S \text{Spec}(A_0)}, \Phi'|_{X \times_S \text{Spec}(A_0)}) = \xi = (\mathcal{F}, \Phi). \]

We still use the covering $\{U_i[M]\}_{i \in I}$ of $X \times_S \text{Spec}(A_0[M])$ to work on this problem locally. Clearly, $\mathcal{F}'_i = \mathcal{F}'|_{U_i[M]}$ is the pull-back of $\mathcal{F}'_{U_i \times_S \text{Spec}(A_0)}$. The coherent sheaf $\mathcal{F}'$ can be obtained by gluing $\mathcal{F}'_i$ together. Thus the automorphism $1 + s_i + s_j$ of $\mathcal{F}'_{ij}$ over the intersection $U_{ij}[M]$ should satisfy the condition $s_{ij} + s_{jk} = s_{ik}$ on $U_{ijk}[M]$, where $s_{ij}$ is an element in $\mathcal{E}nd(\mathcal{F})[M]|_{U_{ij}[M]}$. Now we consider the morphism
\[ \Phi' : \Lambda \to \mathcal{E}nd(\mathcal{F}') \cong \mathcal{E}nd(\mathcal{F}) \oplus \mathcal{E}nd(\mathcal{F})[M], \]
which is given by $\Phi_i + t_i$ on the local chart $U_i \times \text{Spec}(A_0)$, where
\[ \Phi_i = \Phi|_{U_i \times \text{Spec}(A_0)} : \Lambda \to \mathcal{E}nd(\mathcal{F})|_{U_i \times \text{Spec}(A_0)} \]
and
\[ t_i : \Lambda \to \mathcal{E}nd(\mathcal{F})[M]|_{U_i \times \text{Spec}(A_0)}. \]

By the compatibility condition of $\Phi_i + t_i$ on $U_{ij}[M]$, we have
\[ e(\Phi_i + t_i)(1 + s_{ij}) = \Phi_j + t_j. \]

This gives us
\[ e(\Phi)(s_{ij}) = t_i - t_j. \]

Therefore, $(s_{ij}, t_i) \in Z$.

The above discussion gives us a well-defined map from $\mathcal{M}_\xi(A_0[M])$ to $\mathbb{H}^1(C^*)$. It is easy to check that these two maps are inverse to each other. We finish the proof of this proposition. \qed
\textbf{Corollary 5.5.} The deformation complex 
\[ C^\bullet : C^0 = \mathcal{E}nd(\mathcal{F})[M] \xrightarrow{\varepsilon(\Phi)} C^1 = \text{Hom}(\Lambda, \mathcal{E}nd(\mathcal{F})[M]) \]
has the following long exact sequence
\[ 0 \rightarrow H^0(C^\bullet) \rightarrow H^0(\mathcal{X} \times S \text{Spec}(A_0), C^0) \rightarrow H^0(\mathcal{X} \times S \text{Spec}(A_0), C^1) \]
\[ \rightarrow H^1(C^\bullet) \rightarrow H^1(\mathcal{X} \times S \text{Spec}(A_0), C^0) \rightarrow H^1(\mathcal{X} \times S \text{Spec}(A_0), C^1) \rightarrow \cdots . \]

\text{Proof.} This long exact sequence follows directly from the definition of hypercohomology (see [6]). \qed

\textbf{Corollary 5.6.} Let 0 → M₁ → M₂ → M₃ → 0 be a short exact sequence for finitely generated A₀-modules. We have a long exact sequence for hypercohomology
\[ \cdots \rightarrow H^i(C^\bullet_{M_1}) \rightarrow H^i(C^\bullet_{M_2}) \rightarrow H^i(C^\bullet_{M_3}) \rightarrow H^{i+1}(C^\bullet_{M_1}) \rightarrow \cdots . \]

Now we fix a A-module ξ ∈ \mathcal{M}_ξ(A₀[M]) and consider the corresponding deformation complex
\[ C^\bullet : C^0 = \mathcal{E}nd(\mathcal{F})[M] \xrightarrow{\varepsilon(\Phi)} C^1 = \text{Hom}(\Lambda, \mathcal{E}nd(\mathcal{F})[M]). \]
We will check that the deformation theory of \mathcal{M} with respect to the triple (A₀, M, ξ) is given by the A₀-module H¹(C^\bullet).

(5a) It is well-known that the A₀-module H¹(\mathcal{X} \times S \text{Spec}(A₀)) commutes with localization in A₀. Thus the A₀-module H¹(C^\bullet) also commutes with localization in A₀ by applying the Five Lemma to the long exact sequence in Corollary 5.5. Now let M be a free A₀-module of rank one. This case is exactly the infinitesimal deformation and we use the notation A₀[ε] := A₀[M]. By the finiteness theorem of cohomology over algebraic spaces \cite{24, 7.5}, the modules H¹(\mathcal{X} \times S \text{Spec}(A₀), C^i) are finitely generated for 0 ≤ i, j ≤ 1. Thus H¹(C^\bullet) \cong \mathcal{M}_ξ(A₀[ε]) is also a finitely generated module by the long exact sequence in Corollary 5.5.

(5b) We assume that A = A₀ and A' = A₀[M]. We will define an action D = \mathcal{M}_ξ(A₀[M]) on itself and show that this action is free. By Proposition 5.4, we know that
\[ \mathcal{M}_ξ(A₀[M]) \cong H¹(C^\bullet) = Z/B, \]
where Z is the set of pairs (s₁, t₁) such that s₁ ∈ C₀₁ and t₁ ∈ C₁₁ satisfy the following conditions
(a) s₁ + s₁k = s₁k as elements of C₀₁,
(b) t₁ − t₁ = e(Φ)(s₁) as elements of C₁₁.
There is a natural action of Z on itself
\[ (s'_₁, t'_₁)(s₁, t₁) := (s₁ + s₁, t'_₁ + t₁), \]
where (s₁, t₁) ∈ Z. This action can be naturally extended to a well-defined action of Z/B on itself, which is also a free action. Therefore we define a free action D = \mathcal{M}_ξ(A₀[M]) on itself.

(5c) The condition of (5c) is a local property. We may assume that \mathcal{X} is an algebraic space and S is an affine scheme Spec(A₀). Before prove the condition (5c), we first review the following lemma.

\textbf{Lemma 5.7} (Lemma 6.8, 6.9 in \cite{3}). Let \mathcal{X} be an algebraic space of finite type over an affine scheme S = Spec(A₀), where A₀ is an integral domain. Let \mathcal{F}, \mathcal{G} be two coherent sheaves on \mathcal{X}, and we fix a non-negative integer q. Then there is a non-empty open set U of S such that for each s ∈ U, the canonical map is an isomorphism
\[ \text{Ext}^q_{\mathcal{X}}(\mathcal{F}, \mathcal{G})_s \cong \text{Ext}^q_{\mathcal{X}_s}(\mathcal{F}_s, \mathcal{G}_s), \]
and
\[ H^q(\mathcal{X}, \mathcal{F}) \otimes_B k(s) \cong H^q(\mathcal{X}_s, \mathcal{F}_s). \]
By the above lemma, we can find a non-empty open set $U$ of $\text{Spec}(A_0)$ such that

$$H^i(\mathcal{X}, C^j)_s \cong H^i(\mathcal{X}_s, C^j_s), \quad i \geq 0, j = 1, 2,$$

for $s \in U$. Thus we have

$$\mathbb{H}^1(C^*_s) \cong \mathbb{H}^1(C^*_s)$$

by applying the Five Lemma to the long exact sequence in Corollary 5.5, where $C^*_s$ is the restriction of the complex $C^*$ to the point $s$. We finish the proof of the condition (5c).

5.3.5. Obstruction Theory. Fix a deformation situation $(A' \to A \to A_0, M, \xi)$, where $M$ is a free $A_0$-module of rank $n$. For any quotient $\epsilon : M \to M^*$, let $A' \to A^*$ be the quotient of $A'$ defined by $M^*$.

$$\begin{array}{ccc}
M & \rightarrow & A' \\
\downarrow & & \downarrow \\
M^* & \rightarrow & A^* \\
\end{array}$$

We can define the deformation situation $(A^* \to A \to A_0, M^*, \xi)$. For any element $(\mathcal{F}^*, \Phi^*) \in \mathcal{M}_\xi(A^*)$, we want to lift it to a well-defined element in $\mathcal{M}_\xi(A')$. The obstruction for this lifting property comes from the vanishing of the second hypercohomology group $\mathbb{H}^2(C^*_{\ker})$. By Corollary 5.6, we have a long exact sequence for the hypercohomology groups

$$\cdots \rightarrow \mathbb{H}^1(C^*_{\ker}) \rightarrow \mathbb{H}^1(C^*_M) \rightarrow \mathbb{H}^1(C^*_{M^*}) \rightarrow \mathbb{H}^2(C^*_{\ker}) \rightarrow \cdots .$$

Such a lifting exists if and only if the morphism $\mathbb{H}^1(C^*_M) \rightarrow \mathbb{H}^1(C^*_{M^*})$ is surjective. Thus the vanishing of the second hypercohomology $\mathbb{H}^2(C^*_{\ker})$ is necessary and sufficient for the existence of such a lifting.

(6a) The condition (6a) is exactly the homogeneity of the functor $\mathcal{M}$. Note that there is a natural forgetful functor

$$\mathcal{M} \to \mathcal{M}', \quad (\mathcal{F}, \Phi) \mapsto \mathcal{F}.$$ 

The quotient functor $\mathcal{M}'$ is homogeneous [25]. If the forgetful functor is relatively homogeneous, we can prove that the functor $\mathcal{M}$ is homogeneous [8, Lemma 10.18]. We know that the fiber of the forgetful functor at a sheaf $\mathcal{F}$ is $\mathcal{HOM}(\Lambda \otimes \mathcal{F}, \mathcal{F})$, which is homogeneous as we discussed in §5.2.3. Therefore the forgetful functor is relatively homogeneous, and the moduli problem $\mathcal{M}$ is homogeneous.

(6b) There is a natural map

$$i : \mathcal{X}_{A'K} \rightarrow \mathcal{X}_{A_0 \times K A'_K},$$

which is induced by the natural inclusion $A_0 \times K A'_K \to A'\kappa$. The induced functor $i_*$ is left exact on the category of quasi-coherent sheaves, and denote by $i^*$ the left adjoint of the functor $i_*$. We have the following isomorphisms

$$\text{Ext}^q_{A'K}(i^*\mathcal{F}_1, \mathcal{F}_2) \cong \text{Ext}^q_{A_0 \times K A'_K}(\mathcal{F}_1, i_*\mathcal{F}_2), \quad q \geq 0,$$

for quasi-coherent sheaves $\mathcal{F}_1$ over $\mathcal{X}_{A_0 \times K A'_K}$ and $\mathcal{F}_2$ over $\mathcal{X}_{A'K}$. Given a coherent sheaf $\mathcal{F}$ over $\mathcal{X}$, we consider the deformation complex

$$C^* : C^0 = \mathcal{E}nd(\mathcal{F})[M] \xrightarrow{c(\Phi)} C^1 = \text{Hom}(\Lambda, \mathcal{E}nd(\mathcal{F})[M]).$$

We have

$$H^q(\mathcal{X}_{A'K}, C^0_{A'K}) \cong H^q(\mathcal{X}_{A_0 \times K A'_K}, C^0_{A_0 \times K A'_K}),$$

$$H^q(\mathcal{X}_{A'K}, C^1_{A'K}) \cong H^q(\mathcal{X}_{A_0 \times K A'_K}, C^1_{A_0 \times K A'_K}).$$

Therefore,

$$\mathbb{H}^2(C^*_{A'K}) \cong \mathbb{H}^2(C^*_{A_0 \times K A'_K})$$
by applying \textit{Five Lemma} to the long exact sequence in Corollary 5.5. It follows that the obstructions of lifting elements to \( \mathcal{M}(A_0 \times_K A'_K) \) and lifting elements to \( \mathcal{M}(A'_K) \) are the same.

(6c) The proof of the condition (6c) is similar to that of the condition (5c). The difference is that we worked on the deformation \( \mathbb{H}^1(C^\bullet) \) in the condition (5c), while the condition (6c) focuses on the obstruction \( \mathbb{H}^2(C^\bullet) \). We use the same notation as in the statement of (5c). Let \( \xi \in \mathcal{M}(A) \). Suppose that for every one-dimensional quotient \( M_K \to M'_K \), there is a non-trivial obstruction to lift \( \xi_K \in \mathcal{M}(A_K) \) to \( \mathcal{M}(A'_K) \), where \( A'_K \) is the extension defined by \( M'_K \). We want to prove that there exists an open subset \( U \subseteq \text{Spec}(A_0) \) such that \( \xi \) cannot be lifted to \( \mathcal{M}(A'^\ast) \).

Let \( N \) be the kernel of \( M \to M'^\ast \), and we consider the deformation complex \( C^\bullet_N \)

\[ C^\bullet_N : C^0_N = \mathcal{E}nd(F)[N] \xrightarrow{\sigma(\Phi)} C^1_N = \text{Hom}(A, \mathcal{E}nd(F)[N]). \]

By Lemma 5.7, we can choose an open set \( U \) of \( S \) such that

\[ H^q(\mathcal{X}, C^0_N) \cong H^q(\mathcal{X}', (C^0_N)_s), \quad q \geq 0, \]

for \( s \in U \). This implies that

\[ \mathbb{H}^2(C^0_N) \cong \mathbb{H}^2((C^0_N)_s) \]

for \( s \in U \). Thus for every quotient \( M \to M'^\ast \) of length one with support in \( U \), the lifting of \( \xi \) to \( F(A'^\ast) \) is obstructed. This finishes the proof of the condition (6c).

5.4. A-Quotient Functors on Projective Deligne-Mumford stacks. In this subsection, let \( S \) be an affine scheme. Suppose that \( \mathcal{X} \) is a projective (resp. quasi-projective) Deligne-Mumford stack over \( S \). Let \( \mathcal{E} \) be a generating sheaf of \( \mathcal{X} \). We fix a polynomial \( P \), which is considered as a modified Hilbert polynomial and a coherent sheaf \( \mathcal{G} \) over \( \mathcal{X} \). We define the quotient functor

\[ \widetilde{\text{Quot}}_A(\mathcal{G}, P) : (\text{Sch}/S)^{\text{op}} \to \text{Set} \]

for \( A \)-modules with respect to the given polynomial \( P \) as follows. Let \( T \in (\text{Sch}/S) \) be an \( S \)-scheme. The set \( \widetilde{\text{Quot}}_A(\mathcal{G}, P)(T) \) contains the coherent sheaves \( \mathcal{F}_T \) such that

1. \( \mathcal{F}_T \in \text{Quot}(\mathcal{G}, P)(T) \) (see §3.3),
2. \( \mathcal{F}_T \) is a \( \Lambda_T \)-module,
3. \( \mathcal{F}_T \) is a projective (resp. quasi-projective) \( \mathcal{X} \)-scheme.

We will prove that if \( S \) is an affine scheme, the functor \( \widetilde{\text{Quot}}_A(\mathcal{G}, P) \) is represented by a projective (resp. quasi-projective) \( S \)-scheme in this subsection.

Recall that the functor \( F_E : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X) \) is an exact functor (see §2.2). This functor can be generalized to a natural transformation

\[ \widetilde{\text{Quot}}_A(\mathcal{G}, \mathcal{X}, P) \to \text{Quot}_{F_E}(\mathcal{G}, X, P), \]

where \( \widetilde{\text{Quot}}_A(\mathcal{G}, \mathcal{X}, P) \) is the quotient functor of \( \Lambda \)-modules with modified Hilbert polynomial \( P \) over \( \mathcal{X} \) and \( \text{Quot}_{F_E}(\mathcal{G}, X, P) \) is the quotient functor of \( F_E(\Lambda) \)-modules with Hilbert polynomial \( P \) over \( X \). Note that \( F_E(\Lambda) \) is still a sheaf of graded algebras, and the modified Hilbert polynomial is fixed under the functor \( F_E \). We still use the same notation \( F_E \) for this natural transformation.

\textbf{Lemma 5.8.} \textit{The natural transformation}

\[ F_E : \widetilde{\text{Quot}}_A(\mathcal{G}, \mathcal{X}, P) \to \text{Quot}_{F_E}(\mathcal{G}, X, P) \]

\textit{is a monomorphism.}

\textit{Proof.} We have to prove that for each \( S \)-scheme \( T \), the morphism

\[ F_E(T) : \widetilde{\text{Quot}}_A(\mathcal{G}, \mathcal{X}, P)(T) \to \text{Quot}_{F_E}(\mathcal{G}, X, P)(T) \]

is an injection. We will omit \( T \) for simplicity.
Let \((\mathcal{F}, \Phi : \Lambda \otimes \mathcal{F} \to \mathcal{F})\) be an element in \(\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, \mathcal{X}, P)\). We have
\[
(F_{\mathcal{E}}(\mathcal{F}), F_{\mathcal{E}}(\Phi)) : F_{\mathcal{E}}(\Lambda \otimes \mathcal{F}) \to F_{\mathcal{E}}(\mathcal{F}) \in \widetilde{\text{Quot}}_{F_{\mathcal{E}}(\Lambda)}(F_{\mathcal{E}}(\mathcal{G}), X, P)
\]
under the transformation \(F_{\mathcal{E}}\). The natural transformation \(F_{\mathcal{E}}\) is a monomorphism when restricting to the quotient functor \(\text{Quot}(\mathcal{G}, \mathcal{X}, P)\), i.e. the morphism
\[
F_{\mathcal{E}} : \widetilde{\text{Quot}}(\mathcal{G}, \mathcal{X}, P) \to \widetilde{\text{Quot}}(F_{\mathcal{E}}(\mathcal{G}), X, P)
\]
is an injection [25, Lemma 6.1]. Thus the coherent sheaf \(\mathcal{F}\) corresponds to a unique coherent sheaf \(F_{\mathcal{E}}(\mathcal{F})\). Now we will show that \(F_{\mathcal{E}}\) is also an injection for the morphism \(\Phi : \Lambda \otimes \mathcal{F} \to \mathcal{F}\), and we only have to prove that if

\[
F_{\mathcal{E}}(\Phi) : F_{\mathcal{E}}(\Lambda \otimes \mathcal{F}) \to F_{\mathcal{E}}(\mathcal{F})
\]
is trivial, i.e. \(\ker(F_{\mathcal{E}}(\Phi)) \cong F_{\mathcal{E}}(\Lambda \otimes \mathcal{F})\), then \(\Phi\) is also a trivial morphism. Note that there is a short exact sequence

\[
0 \to \ker(\Phi) \to \Lambda \otimes \mathcal{F} \xrightarrow{\Phi} \mathcal{F}.
\]
Applying the exact functor \(F_{\mathcal{E}}\) to the above sequence, we have

\[
0 \to F_{\mathcal{E}}(\ker(\Phi)) \cong F_{\mathcal{E}}(\Phi) \to F_{\mathcal{E}}(\mathcal{F} \otimes \Lambda) \to F_{\mathcal{E}}(\mathcal{F}).
\]
Since \(F_{\mathcal{E}}\) is an exact functor, \(\ker(F_{\mathcal{E}}(\Phi)) \cong F_{\mathcal{E}}(\mathcal{F} \otimes \Lambda)\) if and only if \(\ker(\Phi) \cong \mathcal{F} \otimes \Lambda\). Therefore the functor \(F_{\mathcal{E}}\) is also an injection for the morphism. \(\square\)

**Remark 5.9.** Recall that \(\pi : \mathcal{X} \to X\) is the natural map to the coarse moduli space \(X\), and \(\pi_* : \text{QCoh}(\mathcal{X}) \to \text{QCoh}(X)\) is an exact functor. The exact functor induces a natural transformation \(\pi_* : \text{Quot}(\mathcal{G}, \mathcal{X}, P) \to \text{Quot}(\pi_*(\mathcal{G}), X, P)\). Note that this natural transformation is not injective in general. This is also one of the reasons why people introduces the generating sheaf \(\mathcal{E}\) and define the functor \(F_{\mathcal{E}}\).

**Lemma 5.10.** The monomorphism \(F_{\mathcal{E}}\) is relatively representable by schemes, and \(F_{\mathcal{E}}\) is a \(F_{\mathcal{E}}\) is a finitely-presented finite monomorphism.

**Proof.** The proof of this lemma is the same as [25, Proposition 6.2] by applying Lemma 5.8. \(\square\)

In the above lemma, if \(\mathcal{G}\) does not necessarily have proper support over \(S\), the natural transformation \(F_{\mathcal{E}}\) is a finitely-presented quasi-finite monomorphism.

By Lemma 5.10, the functor

\[
F_{\mathcal{E}} : \widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, \mathcal{X}, P) \to \widetilde{\text{Quot}}_{F_{\mathcal{E}}(\Lambda)}(F_{\mathcal{E}}(\mathcal{G}), X, P)
\]
is relatively representable by schemes, and the moduli problem \(\widetilde{\text{Quot}}_{F_{\mathcal{E}}(\Lambda)}(F_{\mathcal{E}}(\mathcal{G}), X, P)\) is representable by projective (resp. quasi-projective) scheme. Therefore, we have the following theorem.

**Theorem 5.11.** Let \(S\) be an affine scheme and let \(\mathcal{X}\) be a projective (resp. quasi-projective) stack over \(S\). The \(\Lambda\)-quotient functor \(\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, P)\) is represented by a projective (resp. quasi-projective) \(S\)-scheme.

Denote by \(\text{Quot}_{\Lambda}(\mathcal{G}, P)\) the space representing \(\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G}, P)\).

**Corollary 5.12.** Let \(S\) be an affine scheme and let \(\mathcal{X}\) be a projective (resp. quasi-projective) stack over \(S\). Then the connected components of \(\text{Quot}_{\Lambda}(\mathcal{G})\) are projective (resp. quasi-projective) \(S\)-schemes.

**Proof.** The connected components of \(\widetilde{\text{Quot}}_{\Lambda}(\mathcal{G})\) are parameterized by integer polynomials (as modified Hilbert polynomials). This corollary follows from Theorem 5.11 immediately. \(\square\)
5.5. Boundedness of Λ-modules I. In this subsection, we will prove that the family of p-semistable Λ-modules of pure dimension d with a given modified Hilbert polynomial P is bounded.

Let $f : \mathcal{X} \to T$ be a family of projective stacks with a family of moduli spaces $X \to T$ over an algebraically closed field $k$, where $T$ is a scheme. Let $\mathcal{E}$ be a generating sheaf of $\mathcal{X}$, and let $\mathcal{O}_X(1)$ be an $f$-ample line bundle. We fix an integer polynomial $P$ of degree $d$ and a rational number $\mu_0$.

**Corollary 5.13.** Let $\mathfrak{F}$ be the family of purely d-dimensional Λ-modules with modified Hilbert polynomial $P$ on the fibers of $f : \mathcal{X} \to T$ such that the maximal slope $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F})) \leq \mu_0$. The family $\mathfrak{F}$ is bounded.

**Proof.** By Lemma 5.10, it is equivalent to consider the family $F_{\mathcal{E}}(\mathfrak{F})$ over $X$. The family $F_{\mathcal{E}}(\mathfrak{F})$ is bounded by [27, §3]. Therefore the family $\mathfrak{F}$ is bounded. □

**Corollary 5.14.** The family of p-semistable Λ-modules of pure dimension d with a given modified Hilbert polynomial P on $\mathcal{X}$ is bounded.

**Proof.** By Corollary 5.13, we only have to show that the slope $\hat{\mu}_{\max}(F_{\mathcal{E}}(\mathcal{F}))$ is bounded. The property of boundedness of the slope is given by Lemma 4.6. This finishes the proof of this corollary. □

6. Moduli Space of Λ-modules on Projective Deligne-Mumford Stacks

Let $S$ be an affine scheme, and let $\mathcal{X}$ be a projective stack over $S$. Let $\mathcal{E}$ be a generating sheaf over $\mathcal{X}$ and let $\mathcal{O}_X(1)$ be a polarization over $X$, the coarse moduli space of $\mathcal{X}$. By Theorem 5.11, $\text{Quot}_\Lambda(G, P)$ is a projective scheme. Based on this result, we study the moduli problem of p-semistable Λ-modules over $\mathcal{X}$

$$\widetilde{M}_\Lambda^p(\mathcal{E}, \mathcal{O}_X(1), P) : (\text{Sch}/S)^{\text{op}} \to \text{Set}.$$ Given an $S$-scheme $T$, $\widetilde{M}_\Lambda^p(\mathcal{E}, \mathcal{O}_X(1), P)(T)$ is the set of $T$-flat families of p-semistable Λ-modules on $\mathcal{X} \otimes_S T$ of pure dimension $d$ with modified Hilbert polynomial $P$ with respect to the following equivalence relation “~”. Let $(\mathcal{F}_T, \Phi_T), (\mathcal{F}_T', \Phi_T') \in \widetilde{M}_\Lambda^p(\mathcal{E}, \mathcal{O}_X(1), P)(T)$ be two elements. We say $(\mathcal{F}_T, \Phi_T) \sim (\mathcal{F}_T', \Phi_T')$ if and only if $\mathcal{F}_T \cong \mathcal{F}_T' \otimes p^*L$ and $\Phi_T \cong \Phi_T' \otimes 1_{p^*L}$ for some $L \in \text{Pic}(T)$.

**Lemma 6.1.** Given a polynomial $P$, there is a positive integer $N_0$ depending on $\Lambda$, $\mathcal{E}$ and $P$ such that for any $m \geq N_0$ and any p-semistable Λ-module $\mathcal{F}$ with Hilbert polynomial $P$ on $\mathcal{X}$, we have

1. $H^i(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(m))$ is locally free of rank $P(m)$ and $H^i(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(m)) = 0$ for $i > 0$.
2. The map
   $$H^0(\mathcal{X}, \mathcal{F} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(m)) \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-m) \to \mathcal{F} \to 0$$
   is surjective.

**Proof.** By Corollary 5.14, we know that the family of p-semistable Λ-modules of pure dimension $d$ with a given modified Hilbert polynomial is bounded. Note that a Λ-module is also a $\mathcal{O}_X$-module. Thus there is an integer $N_0$ such that when $m \geq N_0$, for any element $\mathcal{F}$ in this family, the Λ-module $\mathcal{F}$ is $m$-regular. By the equivalent conditions of boundedness and regularity in §3.5, the integer $m$ satisfies the requirements in the lemma. □

The p-semistability of coherent sheaves and Λ-modules over a projective scheme are open conditions [16, 27]. This statement can be directly generalized to Λ-modules over $\mathcal{X}$.

**Lemma 6.2.** Given an $S$-scheme $T$, let $X_T \to T$ be a family of projective stacks over $T$ and let $\mathcal{F}_T$ be a family of Λ-modules on $X_T$. There is an open subset $T^{ss} \subseteq T$ such that $\mathcal{F}_t$ is p-semistable if and only if $t \in T^{ss}$. The same argument holds for $p$-stable Λ-modules.
Now we will prove that the family $\tilde{\mathcal{M}}^s_{\Lambda}(P)$ of $p$-semistable $\Lambda$-modules of pure dimension $d$ with the modified Hilbert polynomial $P$ is quasi-projective. The idea is that we can choose a quotient scheme such that the quotient scheme contains the family $\tilde{\mathcal{M}}^s_{\Lambda}(P)$ as a locally closed subset. Based on this property, the family $\tilde{\mathcal{M}}^s_{\Lambda}(P)$ is representable by a quasi-projective scheme. We will first give the idea of the construction and then prove the statements in detail (see Proposition 6.3).

By Lemma 6.1, we can take an integer $m$ such that for any $\Lambda$-module $(\mathcal{F}, \Phi) \in \tilde{\mathcal{M}}^s_{\Lambda}(P)$, the coherent sheaf $\mathcal{F}$ is $m$-regular. Moreover, by Proposition 4.7, we can choose an integer $N$ such that for any $(\mathcal{F}, \Phi) \in \tilde{\mathcal{M}}^s_{\Lambda}(P)$, we have

$$P(N) \geq P_\mathcal{E}(\mathcal{F}, m) = h^0(X, F_\mathcal{E}(\mathcal{F})(m)).$$

Let $V$ be the linear space $S^{P(N)}$. Let $\mathcal{G}$ be the coherent sheaf $\mathcal{E} \otimes \pi^* \mathcal{O}_X(-N)$. The above discussion tells us that each $\Lambda$-module $(\mathcal{F}, \Phi) \in \tilde{\mathcal{M}}^s_{\Lambda}(P)$ corresponds to a surjection $[V \otimes \mathcal{G} \to \mathcal{F}]$ together with an isomorphism $V \cong H^0(X, F_\mathcal{E}(\mathcal{F})(N))$. Note that this correspondence does not take the $\Lambda$-structure into account. Therefore the quotient scheme $\text{Quot}(V \otimes \mathcal{G}, \mathcal{P})$ is so small that it cannot cover all $\Lambda$-modules. We have to find a larger quotient scheme which can cover all $\Lambda$-modules of pure dimension $d$ with Hilbert polynomial $P$.

Let $k$ be a positive integer. We consider the quotient scheme $\text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, \mathcal{P})$. Given an element $\mathcal{F} \in Q'$, we assume that the quotient map $q : \Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F}$ factors through a morphism $\Phi_k : \Lambda_k \otimes \mathcal{F} \to \mathcal{F}$. In other words, we have the following commutative diagram

$$\Lambda_k \otimes V \otimes \mathcal{G} \xrightarrow{q} \mathcal{F} \xrightarrow{\Phi_k} \Lambda_k \otimes \mathcal{F}$$

where $q' : V \otimes \mathcal{G} \to \mathcal{F}$ is a quotient map in $\text{Quot}(V \otimes \mathcal{G}, \mathcal{P})$. If a quotient map $q : \Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F}$ has this factorization property, we say that $q$ admits a factorization. Suppose that $q$ admits a factorization. We will show that the map $\Phi_k$ will give a $\Lambda$-structure on $\mathcal{F}$ under some good conditions. On the other hand, given a $\Lambda$-module $(\mathcal{F}, \Phi)$, the coherent sheaf $\mathcal{F}$ is included in $\text{Quot}(V \otimes \mathcal{G}, \mathcal{P})$ by Lemma 6.1 and the morphism $\Phi : \Lambda \otimes \mathcal{F} \to \mathcal{F}$ induces a map $\Phi_k : \Lambda_k \otimes \mathcal{F} \to \mathcal{F}$ naturally. Note that the map $\Phi$ is also uniquely determined by $\Phi_k$. Therefore a $\Lambda$-module corresponds to an element in $\text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, \mathcal{P})$ uniquely.

On the other hand, for each element $[\Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F}] \in \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, \mathcal{P})$, we have a natural map

$$V \otimes \mathcal{G} \to \Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F}.$$  

This induces a natural morphism $\alpha : V \to H^0(X, F_\mathcal{E}(\mathcal{F})(N))$. There is an open subset of $\text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, \mathcal{P})$ such that this open subset parameterizes pairs $(\mathcal{F}, \alpha)$, where $\mathcal{F} \in \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, \mathcal{P})$ and $\alpha : V \to H^0(X, F_\mathcal{E}(\mathcal{F})(N))$ is an isomorphism.

In summary, let $N$ be a large enough integer, and we want to construct a subset of $\text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, \mathcal{P})$ such that elements $q : \Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F}$ in this subset satisfy the following conditions

- The quotient map $q$ admits a factorization and induces a unique $\Lambda$-structure on $\mathcal{F}$.
- $V \cong H^0(X, F_\mathcal{E}(\mathcal{F})(N))$.

Here is the formal setup of this problem. We fix a polynomial $P$. Let $N_0$ be the positive integer determined by Lemma 6.1. We choose integers $m, N$ as discussed above, and we consider the following moduli problem

$$\tilde{Q}^s_{\Lambda} : (\text{Sch}/S)^{op} \to \text{Set},$$

and for each $S$-scheme $T$, $\tilde{Q}^s_{\Lambda}(T)$ is the set of pairs $(\mathcal{F}_T, \alpha_T)$ such that

1. $\mathcal{F}_T$ is a $p$-semistable $\Lambda$-module with the modified Hilbert polynomial $P$ on $X_T$, 
2. $\alpha_T : V_T \cong H^0(X_T, F_\mathcal{E}(\mathcal{F}_T)(N))$ is an isomorphism.
Proposition 6.3. The functor $\tilde{Q}_\Lambda^{ss}$ is representable by a quasi-projective scheme $Q_\Lambda^{ss}$ over $S$.

Proof. Let $V$ be the linear space $S^p(N)$ and let $G$ be the coherent sheaf $E \otimes \pi^*\mathcal{O}_X(-N)$. We fix a positive integer $k$. Denote by $Q_1$ the quotient scheme $\text{Quot}(\Lambda_k \otimes V \otimes G, P)$. For each $S$-scheme $f : T \to S$, the set $Q_1(T)$ parameterizes the isomorphism classes of quotients

$$f^*(\Lambda_k \otimes V \otimes G) \to \mathcal{F}_T \to 0$$

with modified Hilbert polynomial $P$, where $\mathcal{F}_T$ is a coherent sheaf over $X_T$.

There exists an open subscheme $Q_2 \subseteq Q_1$ such that any quotient map $q \in Q_2(T)$ admits a factorization. More precisely, let $q_T : f^*(\Lambda_k \otimes V \otimes G) \to \mathcal{F}_T$ be a quotient map in $Q_2(T)$. The map $q_T$ can be factored in the following way

$$f^*(\Lambda_k \otimes V \otimes G) \xrightarrow{\phi} \mathcal{F}_T$$

where $q_T^* : f^*(V \otimes G) \to \mathcal{F}_T$ is a quotient map in $\text{Quot}(V \otimes G, P)(T)$. As we discussed early in this section, a quotient map $q_T$ admitting a factorization gives us a quotient map $q_T^* : f^*(V \otimes G) \to \mathcal{F}_T$ and a $f^*(\Lambda_k)$-structure on the coherent sheaf $\mathcal{F}_T$.

If a quotient map is in $Q_2$, then the coherent sheaf will have a $\Lambda_k$-structure. However, this may not give a $\Lambda$-structure for the coherent sheaf. We will explore the conditions, under which a coherent sheaf with a $\Lambda_k$-structure is a $\Lambda$-module.

Let $q_T : f^*(\Lambda_k \otimes V \otimes G) \to \mathcal{F}_T$ be a point in $Q_2(T)$. Denote by $q_T^* : f^*(V \otimes G) \to \mathcal{F}_T$ the quotient map in the factorization of $q_T$. Let $\mathcal{K}$ be the kernel of the quotient map

$$0 \to \mathcal{K} \to f^*(V \otimes G) \xrightarrow{q_T^*} \mathcal{F}_T \to 0.$$ 

The quotient map $q_T$ induces the morphism $f^*(\Lambda_1 \otimes V \otimes G) \to \mathcal{F}_T$, which gives us the following map

$$f^*(\Lambda_1) \otimes \mathcal{K} \to f^*(\Lambda_1 \otimes V \otimes G) \to \mathcal{F}_T.$$ 

There exists a closed subscheme $Q_3 \subseteq Q_2$ such that the induced map $f^*(\Lambda_1) \otimes \mathcal{K} \to \mathcal{F}_T$ is trivial.

Now let $q_T$ be a quotient map in $Q_3$. By the discussion above, the quotient map $q_T$ induces the following one

$$f^*(\Lambda_1 \otimes V \otimes G) \to \mathcal{F}_T.$$ 

Therefore we have the following factorization

$$f^*(\Lambda_1 \otimes V \otimes G) \xrightarrow{\phi_1} \mathcal{F}_T$$

For each positive integer $j$, we have a morphism

$$f^*(\Lambda_1 \otimes \cdots \otimes \Lambda_1) \otimes \mathcal{F}_T \to \mathcal{F}_T,$$

which is induced by the morphism $\phi_1 : f^*(\Lambda_1) \otimes \mathcal{F}_T \to \mathcal{F}_T$. Denote by $\mathcal{K}_j$ the kernel of the surjection

$$\Lambda_1 \otimes \cdots \otimes \Lambda_1 \to \Lambda_j \to 0.$$ 

This gives us a well-defined map

$$f^*(\mathcal{K}_j) \otimes \mathcal{F}_T \to \mathcal{F}_T.$$
Therefore given a positive integer \( j \), there exists a closed subscheme \( Q_{4,j} \subseteq Q_3 \) such that \( q_T \in Q_{4,j}(T) \) if the corresponding map \( f^*(\mathcal{K}_j) \otimes F_T \to F_T \) is trivial. Denote by \( Q_{4,\infty} \), the intersection of all of these closed subschemes \( Q_{4,j}, \ j \geq 1 \). The conditions for \( Q_3 \) and \( Q_{4,\infty} \) guarantee that a coherent sheaf \( F \) with a \( \Lambda_k \) structure is also a \( \Lambda \)-module.

The above discussion tells us that a quotient map \( [q_T : f^*(\Lambda_k \otimes V \otimes \mathcal{G}) \to F_T] \in Q_{4,\infty} \) gives a \( f^*(\Lambda) \)-structure on \( F_T \). This structure induces a \( f^*(\Lambda) \)-structure on \( F_T \). We know that \( \Lambda \) generates \( \Lambda \). Thus a \( f^*(\Lambda) \)-structure will give us a \( f^*(\Lambda) \)-structure on \( F_T \), which will induce a \( f^*(\Lambda) \)-structure. Note that this \( f^*(\Lambda) \)-structure may not be the same as the previous one. However, there is a closed subset \( Q_5 \subseteq Q_{4,\infty} \) such that these two structures are the same.

After that, let \( Q_6 \subseteq Q_5 \) be the open subset such that if \( F \in Q_6 \), then we have \( V \cong H^0(X, F_{\mathcal{E}}(\mathcal{F}))(N) \).

Finally, by Lemma 6.2, there is an open subset \( Q_{5}^* \subseteq Q_6 \) such that \( F \) is a \( p \)-semistable \( \Lambda \)-module if and only if \( F \in Q_{5}^* \).

With the same argument as in §3.8, there is a natural embedding

\[ \psi_N : \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \to \text{Grass}(H^0(X, F_{\mathcal{E}}(\Lambda_k \otimes V \otimes \mathcal{G}))(N), P(N)), \]

Let \( \mathcal{L}_N \) be the pullback of the canonical invertible bundle over the Grassmannian, and \( \mathcal{L}_N \) is an ample line bundle on \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \). There is a natural group action \( SL(V) \) on \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \), which induces an action on the line bundle \( \mathcal{L}_N \). Given a group action \( SL(V) \) on \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) and an ample line bundle \( \mathcal{L}_N \) over \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \), semistable (resp. stable) points of \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) are well-defined. Denote by \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) the set of semistable points in \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) with respect to the group action \( SL(V) \) and the line bundle \( \mathcal{L}_N \).

Before we prove that \( Q_{5}^* \) is included in \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P) \), we review the following lemma first.

**Lemma 6.4** (Lemma 1.15 [27]). Let \( X \) be a smooth projective scheme. We take a coherent sheaf \( G \) over \( X \) and a finite dimensional vector space \( V \). Fix a polynomial \( P \). Suppose \( V \otimes G \to F \to 0 \) is a point in \( \text{Quot}(V \otimes G, X, P) \). For any subspace \( V' \subseteq V \), let \( F' \) be the subsheaf of \( F \) generated by \( V' \otimes G \). We can take a large enough positive integer \( N \) such that the reduced \( P \)-semistable points of \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) are well-defined. Denote by \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) the set of \( p \)-semistable points in \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) with respect to the group action \( SL(V) \) and the line bundle \( \mathcal{L}_N \).

This lemma can be generalized to sheaves on projective Deligne-Mumford stacks directly.

**Lemma 6.5.** There is a large enough integer \( N \) such that the subscheme \( Q_{5}^* \subseteq \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \) is included in \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P) \).

**Proof.** Let \( [q : \Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F}] \) be a point in \( \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \). We have to prove that if the point \( q \) is \( p \)-semistable \( \Lambda \)-module, then \( q \) is also semistable with respect to the invertible sheaf \( \mathcal{L}_N \) and the action of \( SL(V) \). We have a natural monomorphism

\[ \text{Quot}(\Lambda_k \otimes V \otimes \mathcal{G}, P) \to \text{Quot}(F_{\mathcal{E}}(\Lambda_k \otimes V \otimes \mathcal{G}, P), P) \]

\[ [\Lambda_k \otimes V \otimes \mathcal{G} \to \mathcal{F} \to F_{\mathcal{E}}(\Lambda_k \otimes V \otimes \mathcal{G} \to F_{\mathcal{E}}(\mathcal{F})), \]

which is induced by the exact functor \( F_{\mathcal{E}} \) [25, Lemma 6.1]. Let \( V' \) be a subspace of \( V \). Denote by \( \mathcal{F}' \) the subsheaf of \( \mathcal{F} \) generated by \( V' \). Let \( \mathcal{F}'_{\text{sat}} \) be the saturation of \( \mathcal{F} \). Note that \( \mathcal{F}'_{\text{sat}} \) is a \( \Lambda \)-module and its \( \Lambda \)-structure coincide with the one induced from \( \Lambda_k \otimes \mathcal{F}'_{\text{sat}} \to \mathcal{F}'_{\text{sat}} \) by the property of \( Q_5 \). Therefore \( \mathcal{F}'_{\text{sat}} \) is a \( \Lambda \)-submodule of \( \mathcal{F} \).

In summary, we have \( V' \subseteq H^0(X, F_{\mathcal{E}}(\mathcal{F}'_{\text{sat}}))(N) \subseteq H^0(X, F_{\mathcal{E}}(\mathcal{F}))(N) \). This gives us the following inequalities

\[ \frac{\dim V'}{r(\mathcal{F}'_{\text{sat}})} \leq \frac{H^0(F_{\mathcal{E}}(\mathcal{F}')_{\text{sat}})(N)}{r(\mathcal{F}'_{\text{sat}})} \leq \frac{P(N)}{r(\mathcal{F})} \].


Now we are working on the coherent sheaves $F_{\xi}(F)$ and $F_{\xi}(F_{\text{sat}})$ on $X$, and the rest of the proof follows from Simpson's argument [27, Lemma 4.3].

**Lemma 6.6.** Given any point $q \in Q^s_{\Lambda}^s$, the closure of any $SL(V)$-orbit of $q$ in $Q^s_{\Lambda}^s$ is contained in $Q^s_{\Lambda}^s$, where the closure is taken in Quot$^{ss}(\Lambda_k \otimes V \otimes G, P)$.

**Proof.** Based on the Hilbert-Mumford Criterion [21, Theorem 2.1], we need to find the limit point of any one-parameter subgroup action on a given point in $Q^s_{\Lambda}^s$. Let $\varphi : G_m \to SL(V)$ be an one parameter-subgroup. The vector space $V$ can be decomposed as $V = \bigoplus V_\alpha$, where $t \cdot v_\alpha = t^\alpha v_\alpha$ for $v_\alpha \in V_\alpha$. Therefore we can define a filtration $V_{\beta} := \bigoplus_{\alpha \geq \beta} V_\alpha$ of $V$. Let $q_X : \Lambda_k \otimes V \otimes G \to F$ be a point in $Q$. The filtration $F_{\beta}$ of $F$ is defined as

$$F_{\beta} := q_X(\Lambda_k \otimes V_{\beta} \otimes G),$$

and the graded part is

$$F_{\beta} := q_X(\Lambda_k \otimes V_{\beta} \otimes G).$$

With respect to the one-parameter subgroup $\varphi$ and the point $q_X$, the limit point is

$$[q_{\varphi} := \Lambda_k \otimes V \otimes G \to F'] \in \text{Quot}^{ss}(\Lambda_k \otimes V \otimes G, P),$$

where $F' = \bigoplus_{\beta} F_{\beta}$. We have to prove that the point $q_{\varphi}$ is contained in $Q^s_{\Lambda}^s$. Now let $H_{\beta}$ (resp. $H_{\geq \beta}$) be the saturation of $F_{\beta}$ (resp. $F_{\geq \beta}$). We know that $H_{\beta}$ is a $\Lambda$-submodule of $F$ (see §4.1), where the $\Lambda$-structure of $H_{\beta}$ is induced from that of $F$. Therefore, proving $q_{\varphi} \in Q^s_{\Lambda}^s$ is equivalent to prove that

- $H_{\beta}$ is a $p$-semistable $\Lambda$-submodule of $F$.
- $F' \cong \bigoplus_{\beta} H_{\beta}$.

Now we will prove the above statements. There is a natural map $F_{\beta} \to H_{\beta}$, and the image of this map is denoted by $F_{\beta}$. Therefore, the composed map

$$\Lambda_k \otimes V \otimes G \to F \to F_{\beta} \to H_{\beta}$$

induces $V \to H^0(X, F_{\xi}(J_{\beta}))(N)$. Let $J_{\beta}$ be the kernel of $V \to H^0(X, F_{\xi}(J_{\beta}))(N)$. We have

$$\dim(J_{\beta}) \geq P(N) - h^0(X, F_{\xi}(J_{\beta}))(N).$$

Denote by $J_{\beta}$ the subsheaf of $F$ generated by the image of $\Lambda_k \otimes J_{\beta} \otimes G$. Note that $J_{\beta}$ maps zero in $H_{\beta}$ and also in $J_{\beta}$. This implies that

$$r(J_{\beta}) \leq r(F) - r(J_{\beta}).$$

Since $q_{\varphi} \in \text{Quot}^{ss}(\Lambda_k \otimes V \otimes G, P)$, by Lemma 6.4, we have

$$\frac{\dim(J_{\beta})}{r(J_{\beta})} \leq \frac{P(N)}{r(F)}.$$

Combing the above three inequalities, we have

$$\frac{P(N)}{r(F)} \leq \frac{h^0(X, F_{\xi}(J_{\beta}))(N)}{r(J_{\beta})}.$$

We will prove that $H_{\geq \beta}$ is a $p$-semistable $\Lambda$-submodule of $F$ with the same reduced modified Hilbert polynomial by induction on $\beta$. Suppose that the statement holds for $H_{\geq \beta+1}$. Then $F/H_{\geq \beta+1}$ is a $p$-semistable $\Lambda$-module. Note that $J_{\beta} \subseteq H_{\beta} \subseteq F/H_{\geq \beta+1}$, which implies that $r(H_{\beta}) = r(J_{\beta})$. On the other hand, $H_{\beta}$ is a $p$-semistable $\Lambda$-submodule, and we have

$$\frac{h^0(X, F_{\xi}(H_{\beta}))(N)}{r(H_{\beta})} \leq \frac{P(N)}{r(F)}.$$
Together with the inequality \( \frac{P(N)}{\mu(F)} \leq \frac{h^0(X, F_E(\mathcal{F}_\beta)(N))}{\mu(\mathcal{F}_\beta)} \) we discussed above, we have

\[ h^0(X, F_E(\mathcal{F}_\beta)(N)) = h^0(X, F_E(\mathcal{H}_\beta)(N)). \]

Thus \( \mathcal{H}_\beta \) has the same reduced modified Hilbert polynomial as \( F \) and \( F/\mathcal{H}_{\beta+1} \). This implies that the reduced modified Hilbert polynomial of \( \mathcal{H}_{\beta} \) is the same as \( F \) and \( \mathcal{H}_{\beta} \) is p-semistable. This finishes the proof. Note that the above proof also tells us that

- \( \mathcal{H}_\beta \) is a p-semistable \( \Lambda \)-submodule of \( F \).
- \( h^0(X, F_E(\mathcal{F}_\beta)(N)) = h^0(X, F_E(\mathcal{H}_\beta)(N)). \)

Since the integer \( N \) is large enough, \( F_E(\mathcal{F}_\beta)(N) \) and \( F_E(\mathcal{H}_\beta)(N) \) are generated by global sections. By Lemma 5.8, the equality \( h^0(X, F_E(\mathcal{F}_\beta)(N)) = h^0(X, F_E(\mathcal{H}_\beta)(N)) \) implies that \( \mathcal{F}_\beta = \mathcal{H}_\beta \). Recall that \( \mathcal{F}_\beta \) is defined as the image of \( F_\beta \) in \( \mathcal{H}_\beta \). Therefore we have \( F_\beta \cong \mathcal{H}_\beta \), and

\[ F' = \bigoplus_\beta F_\beta \cong \bigoplus_\beta \mathcal{H}_\beta. \]

This finishes the proof of this lemma. \( \square \)

Let \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P)/\text{SL}(V) \) be the GIT quotient. As a quasi-projective subscheme of \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P) \), the image of \( Q^s_\Lambda \) in \( \text{Quot}^s(\Lambda_k \otimes V \otimes \mathcal{G}, P)/\text{SL}(V) \) is also quasi-projective, and the image is the GIT quotient of \( Q^s_\Lambda \) by Lemma 6.6.

Let

\[ M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) = Q^s_\Lambda/\text{SL}(V) \]

be the GIT quotient, and let \( \tilde{M}^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \) be the set for stable points in \( M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \). The discussion in this section proves the following theorem.

**Theorem 6.7.**

1. There exists a natural morphism

\[ \tilde{M}^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \to M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \]

such that \( M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \) universally co-represents \( \tilde{M}^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \).

2. The geometric points of \( M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \) represent the S-equivalent classes of p-semistable \( \Lambda \)-modules with modified Hilbert polynomial \( P \).

3. \( M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \) is a coarse moduli space of \( \tilde{M}^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \).

4. The points of \( M^s_\Lambda(\mathcal{E}, \mathcal{O}_X(1), P) \) represent isomorphism classes of p-stable \( \Lambda \)-modules.

**Example 6.8.** Let \( \mathcal{X} \) be a projective Deligne-Mumford stack over an affine scheme \( S \). Let \( \mathcal{L} \) be a fixed line bundle over \( \mathcal{X} \). Similar to \( \S 2.5.2 \), we can define the moduli problem of p-semistable \( \mathcal{L} \)-twisted Hitchin pairs \( M^s_H(\mathcal{X}, \mathcal{L}) \). In \S 4.1, \( \mathcal{L} \)-twisted Hitchin pairs can be considered as \( \Lambda \)-modules. Therefore, the moduli space of p-semistable \( \mathcal{L} \)-twisted Hitchin pairs \( M^s_H(\mathcal{X}, \mathcal{L}) \) exists and universally co-represents the moduli problem \( M^s_H(\mathcal{X}, \mathcal{L}) \).

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