Gauge fields and particle-like formations associated with shear-free null congruences

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Twistors encode the space-time background geometry and, on the other hand, relate to the structure of null geodesic congruences (NGC). Particularly, the Kerr theorem in twistor terms describes the complete set of shear-free NGC of the flat space: it asserts that any 2-spinor $\xi_A^{'}$ being a solution of the equation

$$\Pi(\xi_A^{'}, iX\xi_A^{'}\xi_A^{'}) = 0, \quad (A, A', \ldots = 0, 1)$$

(and, consequently, a function of the space-time coordinates $X^{AA'}$) gives rise to a shear-free NGC with tangent vector $k_\mu = \xi_A\xi_A^{'}$. And that, moreover, any (analytical) shear-free NGC may be obtained via this construction. Here $\Pi$ is an arbitrary homogeneous (analytical) functions of three complex variables – components of the projective null twistor $W = \{\xi, iX\xi\}$ of the Minkowski space.

Shear-free NGC constitute one of the two classes of solutions to complex eikonal equation with respect to the (ambi)twistor structure of the latter. Besides, any (analytical) congruence defines the strength of a null Maxwell field and relates to the solutions of free massless equations. Defining equations of shear-free NGC follow from (1) and have the form

$$\xi_A^{'}, \xi_B^{'}, \partial_{AA'} \xi_B^{'} = 0.$$ (2)

These equations appear also in the framework of noncommutative analysis as the conditions of differentiability of functions taking values in the algebra of complex quaternions (biquaternions) $\mathbb{B}$. Remarkably, only on the base of eqs. (2) (or of their general solution represented by the Kerr constraint) a self-consistent nonlinear field theory has been developed. Correspondent approach has been therein called algebrodynamics.

Specifically, in the framework of the analysis over $\mathbb{B}$-algebra one comes to the following invariant matrix differentiability conditions (generalized Cauchy-Riemann equations)

$$d\xi = \Phi dX\xi, \quad \leftrightarrow \quad \partial_{AA'} \xi_B^{'} = \Phi_{B'A} \xi_A^{'}.$$ (3)
After elimination of the auxiliary complex field $\Phi_{A'}$, eqs. (3) reduce to the overdetermined system of four equations
\[ \xi^{A'} \partial_{AA'} \xi_{B'} = 0, \tag{4} \]
from which, evidently, the shear-free condition (2) does follow. Moreover, for the scaling-invariant ratio of spinor components eqs. (2) and (4) turn to be equivalent, and thus for any shear-free congruence some field $\Phi(X)$ can be defined (up to a gauge transformation, see below). Thus, the structures represented by eqs. (1), (2), (3), (4) are in fact equivalent.

Geometrically, eqs. (3) can be thought of as conditions for the spinor field $\xi$ to be parallel with respect to effective affine connection $\Omega = \Phi dX$. In the 4-vector representation it gives rise to exceptional geometry with Weyl nonmetricity and skew symmetric torsion. By this, both the Weyl 4-vector and the torsion pseudotrace are proportional to each other and to complex field $\Phi_{A'}$. Following Weyl’s idea, the quantities $\Phi_{A'}$ can be identified as the components of 4-potential of a complex gauge field. Indeed, integrability conditions of eqs. (3) are precisely the conditions of self-duality of matrix connection $\Omega$. From this and the Bianchi identities it follows that complex Maxwell and $SL(2, \mathbb{C})$ Yang-Mills equations are satisfied on the solutions of (3), for the trace and the trace-free parts of connection $\Omega$ respectively.

The gauge group for these fields is, however, restricted (as well as the residual rescaling group of the shear-free equations in the form (4)). Specifically, only the following “weak” gauge transformations preserve the form of systems (3) and (4):

\[ \xi^{A'} \mapsto \hat{\xi}^{A'} = \alpha \xi^{A'}, \quad \phi_{A'A} \mapsto \hat{\phi}_{A'A} = \phi_{A'A} + \partial_{A'A} \ln \alpha, \tag{5} \]

where the gauge parameter $\alpha$ can’t depend on the space-time coordinates explicitly but only through the components of the transforming twistor $W$ (i.e. of the spinor $\xi$ and its counterpart $\tau = iX \xi$).

Correspondent field strengths allow for explicit representation through the (second order) derivatives of the spinor field or, still more, for a nice representation in twistor variables. But the most interesting feature of this field is, perhaps, the property of quantization of electric charge of any bounded field singularity, in particular of Coulomb-like type. Theorem of quantization which establishes also the existence of minimal “elementary” charge has been proved in [12, 10] and is based on the over-determinancy of shear-free equations and on the self-duality of complex field strength (for more details about the concept of singular sources of Maxwell field and the charge quantization see [6, 16]).

On these grounds, one can identify (bounded) singularities of effective Maxwell field with particle-like formations. With respect to the initial shear-free NGC they are nothing but the caustics of the congruence and, in view of the generating Kerr
constraint \( \Pi \), defined by the condition

\[
\frac{d\Pi}{dG} = 0
\]

\( G \) being the ratio of spinor components on which the homogeneous function \( \Pi \) depends essentially). For a given \( \Pi \), one can eliminate the projective spinor \( G \) from the joint system of eqs. (1) and (6) and come to a constraint on coordinates \( f(X) = 0 \). This defines (at a fixed moment of time) the shape of singular locus of the congruence and the associated fields and governs also their time evolution. The function \( f(X) \) obeys the complex eikonal equation 11. A number of interesting examples of quite nontrivial distributions and dynamics of singularities (which can be point-, string- or even 2D-membrane-like and undergo various bifurcations simulating transmutations and even annihilations of particles) has been studied in our works 6,7,9,10,11. By all, the above presented algebraic method to obtain enormously complicated singular solutions of Maxwell and Yang-Mills equations (or, at least, the structure of their singular loci) seems very powerful and has, perhaps, no analogues in literature.

There exists a simple generalization of the scheme to the curved space. In fact, a Riemannian metric of the Kerr-Schild type

\[
g_{\mu\nu} = \eta_{\mu\nu} + H k_\mu k_\nu
\]

may be defined for any starting shear-free NGC \( k_\mu = \xi_A \xi_A' \) in Minkowski space \( \eta_{\mu\nu} \) (\( H = H(X) \) being a scalar function of coordinates). Remarkably, the properties of the congruence to be null, geodesic and shear-free remain invariant under such deformations of space-time geometry, and curvature singularities of metrics co-incide with those of the electromagnetic field strengths \( F_{AB} \) being represented by the same condition (6). If the ray of the congruence is a principal direction of \( F_{AB} \) which satisfies thus the constraint \( F_{AB} \xi_A \xi_B = 0 \), then the electromagnetic field remains self-dual in the Kerr-Schild space 13. Note that precisely such Maxwell fields (generally independent from those above considered) have been explicitly constructed for shear-free congruences in 11,18,19. Together with correspondent metric (and under the proper choice of the factor \( H(X) \)) they can satisfy, moreover, the Einstein-Maxwell electrovacuum system.

In particular, this is the case of the Kerr-Newman stationary solution generated by the Kerr congruence with the twofold structure and the singular ring-caustic. Remarkably, this ring (as the source of electromagnetic and gravitational fields) possesses the value of gyromagnetic ratio inherent for Dirac’s fermion, and B. Carter, A. Ya. Burinskii and others even considered it as a possible model of electron. In the framework of our algebrodynamical scheme this is still more natural since the electric charge is here necessarily unit, “elementary” in modulus. E. T. Newman proposed to interpret ”matter” as elementary singularities and systematically studied them in the framework of complex space-time representation.
More successively, one has to study the shear-free NGC directly on the Riemannian background. For a wide class of special shear-free congruences (SSFC) the tangent vector $k^\mu$ is a repeated principal direction of the Weyl curvature tensor. SSFC preserve many remarkable properties of shear-free NGC of the flat space. In particular, for any SSFC its equations, under the proper gauge, reduce to the form (4) (with evident account of the Levi-Civita connection, $\partial \leftrightarrow \nabla$). The gauge field $\Phi$ can also be defined in a similar manner, and the SSFC spinor $\xi$ is again parallel with respect to effective Weyl-Cartan connection of the above presented type. However, the self-duality conditions are generally broken in the presence of curvature term, and therefore, an effective geometrical source appear in the Maxwell equations for the associated field $\Phi$. In asymptotically flat spaces the presence of this source doesn’t damage the Coulombian asymptotics and thus the quantization rule for admissible values of electric charge is inherited from the flat space.

To conclude, the algebrodynamical approach based on shear-free NGC equations or on related twistor and biquaternionic structures leads to a peculiar self-consistent dynamics of associated fields and to complicated particle-like distributions of the congruence fields which manifest nontrivial interactions and quantum-like properties. The approach leads also to new conjectures about the nature of physical time, the light-born matter and to naturally multivalued structure of primordial field entities (see and references therein).

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