Variational problems without having any non-trivial Lie variational symmetries

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Abstract

In this paper we construct variational problems without Lie non-trivial variational symmetry and solving them using new class of symmetries ($\mu$-symmetry) which introduced by Guiseppe Gaeta and Paola Morando (2004). The central object in this paper is horizontal one-form $\mu$ on first order jet space $J^1M$.

Key words: standard symmetry, variational problem, $\mu$-symmetry, differential invariant, variational symmetry.

1 Introduction

Hidden symmetries defined as symmetries that are lost (Type I) or gained (Type II) as the order of an ODE is reduced or as the number of variable of a PDE is reduced. Hidden symmetries are difficult to evaluate since there are no general direct method for determining them. There are several approach that we can use to investigate hidden symmetries and gain them.

In 2001, Muriel and Romero introduced $\lambda$-symmetries to evaluate Type I hidden symmetries of ODEs [MuRo-2001]. Guiseppe Gaeta and Pola Morando expanded this approach to scalar PDEs and PDEs systems. They constructed equations without Lie point symmetries too [Ge-Mo-2004]. This equations have no obvious order reduction (in ODE case) and variable reduction (in scalar PDEs case and PDEs systems) which can be reduce using $\mu$-symmetries.

In this paper we construct equations without Lie non-trivial symmetries using [Ge-Mo-2004]. you can assume these equations are Euler-Lagrangian of some variational problems (with necessary condition) and construct the variational problems have this equations as Euler-Lagrangian, using direct method (This is inverse problem in variational calculus). For such variational problems we can’t solve them using Lie symmetry method (Lie classical method), so solve them using this new class of symmetries ($\mu$-symmetries).

2 $\mu$-symmetry on scalar PDEs and PDEs systems

The starting point will be a discussion of some of the foundational results about $\mu$-symmetry. In this section we recall these results rather briefly. Reader can consult [Ge-Mo-2004] to gain complete information about this symmetries.

Let $\mu = \lambda_i dx_i$ be horizontal one-form on first order jet space $(J^1M, \pi, M)$ and compatible with contact structure $\varepsilon$ on $J^kM$ for $k \geq 2$, i.e.

$$d\mu \in J(\varepsilon).$$

(2.1)

where $J(\varepsilon)$ is Cartan ideal generated by contact structure.

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Theorem 1 (See [Ge-Mo-2004]) Condition (2.1) is equivalent to \( D_i \lambda_j - D_j \lambda_i = 0 \). Where \( D_i \) is total derivative w.r.t. \( x_i \).

Let \( X := \xi^i \partial_{x_i} + \phi(x,u) \partial_u \) be a vector field on total space \( M \) and \( q = 1 \). (i.e. the number of dependent variable is one, on the other hand we discuss these concepts in scalar PDEs framework). We define \( Y := X + \Psi_j \partial_{u_j} \) on \( k \)-th order jet space \( J^k M \) as \( \mu \)-prolong of \( X \) if its coefficient satisfy the \( \mu \)-prolongation formula

\[
\Psi_{J,i} = (D_i + \lambda_i)\Psi_J - u_{J,m}(D_i + \lambda_i)\xi^m. \tag{2.2}
\]

Remark 2 If we set \( \mu = 0 \) in (2.2) then we gain ordinary prolongation of \( X \), so we can assume ordinary prolong as \( 0 \)-prolong in \( \mu \)-prolong framework.

We can show connection between ordinary prolong and \( \mu \)-prolong in follow theorem.

Theorem 3 (See [Ge-Mo-2004]) Let \( X := \xi^i \partial_{x_i} + \Phi(x,u) \partial_u \) be a vector fields on first order jet space \( J^1 M \) and \( Y = X + \Psi_j \partial_{u_j} \) be \( \mu \)-prolong of \( X \) and \( X^{(k)} = X + \phi \partial_{u_j} \) be ordinary prolong of \( X \). Then we have \( \Psi_j = \Phi_j + F_j \), where \( F_j \) satisfy the recursion relation (with \( F_0 = 0 \)): \( F_{j,i} = (D_i + \lambda_i)F_j + \lambda_i D_i Q \); where \( Q \) is Lie characteristic.

This theorem provide an economics way of computing \( \mu \)-prolongation of \( X \) if we knew already its ordinary prolongation.

3 Variational problems and Lie standard reduction method

Variational problem is finding the extremals (maxima and/or minima) of a functional

\[
\ell_\alpha(L) = \sum_J (D)_J \partial_{u_j} \alpha, \quad \alpha = 1, 2, ..., q \tag{3.3}
\]

over some space of functions \( u = f(x), x \in \Omega \). For such problems we can define Euler-Lagrangian operators as

\[
E_\alpha = \sum_J (D)_J \partial_{u_j} \alpha, \quad \alpha = 1, 2, ..., q \tag{3.4}
\]

Symmetry on variational problems is motivated by following theorem of the calculus of variational problems.

Theorem 4 (See [St-1989, Va-2003, Ol-1995]) The smooth extremals \( u = f(x) \) of variational problem with Lagrangian \( L(x,u^{(n)}) \) must be satisfied in the systems of its related Euler-Lagrange equations.

\[
E_\alpha(L) = \sum_J (D)_J \frac{\partial L}{\partial u_j} \tag{3.5}
\]

Now we describe our approach in Lie classical method to find extremals of variational problems on open connected domain \( \Omega \) (see [St-1989, Ab-1996, Va-2003, Ol-1995]). In first step we compute Euler-Lagrangian equations of problems, next we characterize symmetry group of this equations and solve this equations using Lie symmetry method, finally we check this solutions in original problems. Now, what we can do when Euler-Lagrangian equation has no standard Lie symmetry? In this paper we gain variational problems without Lie non-trivial variational symmetry and show how we can solve such problems. Main theorem of our approach is following:

Theorem 5 (See [Ol-1986, Ol-1995]) \( G \) is variational symmetry of variational problem (3.3) if and only if it is Lie symmetry of its Euler-Lagrangian equation (3.4).
4 Euler-Lagrangian equations without Lie non-trivial symmetry

In first step we characterize equations (scalar PDEs) without Lie symmetry. For this purpose we consider \( X \) be vector field on first order jet space \( J^1M \), then determine general scalar PDEs with no Lie non-trivial symmetry which admit \( X \) as \( \mu \)-symmetry.

Consider the vector field \( X = x^2 \partial_x + t \partial_t + u \partial_u \). We have \( Q = u - x^2 u_x - tu_t \). The corresponding coordinates \((y, v)\) and the parametric coordinate \( \sigma \) in \( M = (x, t, u) \) can be chosen as \( \sigma = t \), \( y = te^{1/x} \) and \( v = u/t \). The corresponding inverse change of variables is \( x = -\ln(y) \), \( t = \sigma \), \( u = v \sigma \). Hence, the function \( v = v(\sigma, y) \), is \( X \)-invariant if and only if \( v_\sigma = 0 \). The partial derivations of \( u \) express in the partial derivatives of \( v = v(\sigma, y) \) as

\[
\begin{align*}
\mu_x &= -y^2 \ln y \sigma^{-1} v_y, \\
u_t &= v + \sigma v_\sigma + y v_y.
\end{align*}
\]  

(4.6)

The above can be inverted to give

\[
\begin{align*}
v_y &= \frac{(2 \ln t + 1/x)^2}{t^2 e^{2/x}} u_x, \\
v_\sigma &= \frac{1}{x} \left[ u_t - \frac{u}{t} + \frac{(2 \ln t + 1/x)^2}{te^{1/x}} u_x \right] ;
\end{align*}
\]

(4.7)

Similar above we have this expressions for second order derivatives.

\[
\begin{align*}
u_{xx} &= -y \ln^2 (y/\sigma) v_{yy} + \frac{2y^2 \ln^2 (y/\sigma) + 2 \ln (y\sigma) \ln (y/\sigma) v_y}{\ln^4 (y\sigma)} v_y, \\
u_{xt} &= -y \ln^2 (y/\sigma) v_y - y^2 \ln^2 (y/\sigma) v_{yy}, \\
u_{tt} &= v \sigma + v_\sigma + 2 y v_\sigma^2 + \frac{y}{\sigma} v_y + \frac{y^2}{\sigma} v_{yy},
\end{align*}
\]

(4.8)

As before object in this computation is horizontal one-form \( \mu \) on one order jet space. In this case since independent variables is two-dimensional as a result we have: \( \mu = \lambda dx + \tau dt \).

Let us come to the second \( \mu \)-prolongation of \( X \). (Standard prolongation will be ordinary by setting \( \lambda = \tau = 0 \). For this computation we can use (2.2) or recursion relation in theorem 3. Hence if we show \( \mu \)-prolongation of \( X \) as

\[
Y = X + \Psi^t \partial_u + \Psi^x \partial_u + \Psi^{xt} \partial_{u_{xx}} + \Psi^{xt} \partial_{u_{xx}} + \Psi^{tt} \partial_{u_{tt}}
\]

(4.9)

Then we have

\[
\begin{align*}
\Psi^x &= (1 - 2x) + \lambda Q, \\
\Psi^t &= \tau Q, \\
\Psi^{xx} &= (1 - 4x) u_{xx} - 2 u_x + 2 \lambda (D_x Q) + [\lambda^2 + (D_x \lambda)] Q \\
\Psi^{xt} &= -2 x u_{xt} + [\lambda (D_t Q) + \tau (D_x Q)] + (1/2) [2 \lambda \tau + (D_t \lambda) + (D_x \tau)] Q, \\
\Psi^{tt} &= -u_{tt} + 2 \tau (D_t Q) + [\tau^2 + (D_t \tau)] Q.
\end{align*}
\]

(4.10)

We consider two simplest case for \( \mu \) instead of general case.

**Case I**: \( \tau = 0 \) and \( \lambda \) is real number.

In this case by substituting this \( \mu \) in above we find,

\[
\begin{align*}
\Psi^x &= (1 - 2x) + \lambda (u - x^2 u_x - tu_t), \\
\Psi^t &= 0, \\
\Psi^{xx} &= (1 - 4x) u_{xx} - 2 u_x + 2 [2x - 1] u_x + x^2 u_{xx} + tu_{xt}] + \lambda^2 (u - x^2 u_x - tu_t), \\
\Psi^{xt} &= -2 u_{xt} - \lambda (x^2 u_{xt} + tu_{tt}), \\
\Psi^{tt} &= -u_{tt}.
\end{align*}
\]

(4.11)

Now if we take \((y, v)\) as invariants of order zero, \( \xi_1, \xi_2 \) invariants of order one and \((\eta_1, \eta_2, \eta_3)\) invariants of order two then we find

\[
\begin{align*}
y &= te^{1/x}, \\
v &= \frac{u}{x}, \\
\xi_1 &= \ln 2 - \frac{1}{2} \ln(-x^2 \lambda (S_1 - u S_2)^2) + \frac{1}{2 S_2} (\lambda - S_2) \ln \left( \frac{S_1 + u S_2}{S_3 - S_2} \right)
\end{align*}
\]

(4.12)
\[ \xi_2 = u_t, \quad \eta_1 = \ln 2 - \frac{1}{2} \ln(-3S_4 + uS_4) + \frac{1}{2} \left( \frac{\lambda^2}{2} - 1 \right) \ln \left( \frac{S_4 - uS_4}{S_3 + uu} \right), \tag{4.12} \]
\[ \eta_2 = \frac{1}{2} \lambda^2 u_x^2 - \frac{1}{2} u_x^2 + u_{xx} - tu_{ux}, \quad \eta_3 = tu_{ttx}, \]

Where \( S_1, S_2, S_3 \) and \( S_4 \) are respectively
\[ S_1 = 2\lambda^2 u_x - 2 + 4x + 2mu_x + \lambda u, \quad S_2 = \sqrt{\lambda(4x^2 + \lambda)}, \quad S_4 = \sqrt{-4 + 16x - 8\lambda x^2 + \lambda^2}, \tag{4.13} \]
\[ S_3 = 2u_{xx} - 8u_{xx} + 4\lambda^2 u_{xx} + 4u_x + 8\lambda u_x - 4\lambda u_x + 4\lambda u_{xx} + 2\lambda^2 x^2 u_x + 2\lambda^2 uu_t - \lambda^2 u. \]

**Theorem 6** Consider the equation \( \Delta := F(y, v, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3) \) with arbitrary smooth function \( F \). Let \( \lambda \) be a real constant. Then
i) The equation \( \Delta \) admits the vector field \( X \) as a \( \mu \)-symmetry with \( \mu = \lambda dx \).
ii) For \( (\partial F/\partial \xi_1)^2 + (\partial F/\partial \eta_1)^2 + (\partial F/\partial \eta_2)^2 \neq 0 \), \( X \) is not an ordinary symmetry of \( \Delta \).

**Proof.** i) As mentioned Lie point symmetry method, PDE equation admit \( X \) as \( \mu \)-symmetry when we can rewrite it in terms of \( X \)-invariants [Ge-Mo-2004]. Hence equation \( \Delta \) admits \( X \) as \( \mu \)-symmetry with \( \mu = \lambda dx \).

ii) Using (4.12), we conclude, \( \xi_1, \eta_1 \) and \( \eta_2 \) depend on \( \mu \) in solution space \( (I_x) \). So if \( F \) depend on this arguments then \( X \) is not ordinary symmetry of \( \Delta \).

**Case II:** \( \lambda = 0 \) and \( \tau \) is real number

Now by substituting this equation in (4.9), we have
\[ \Psi^x = (1 - 2x), \quad \Psi^t = \tau(u - x^2u_x - tu_t), \quad \Psi^{xx} = (1 - 4x)u_{xx} - 2u_x, \]
\[ \Psi^{xt} = -2xu_x + \tau(u_x - 2xu_x - x^2u_{xx} - tu_{xx}), \]
\[ \Psi^{tt} = -u_{xx} + 2\tau(u_t - x^2u_t - u_t - tu_{xt}) + \tau^2(u - x^2u_x - tu_t), \tag{4.14} \]
So we find
\[ y = \tau e^{t/z}, \quad v = \frac{u}{z}, \quad \xi_1 = -\frac{1}{2} \frac{(S_1 + uu)S_2^2}{\lambda} \frac{(1 - t(S_1 + uu))^2}{2x - 1}, \]
\[ \xi_2 = -\frac{1}{2} \ln \left( -\frac{1}{4} \frac{S_2^2}{\tau} \right) - \frac{1}{2} \ln \left( \frac{S_1 - uu}{S_1 + uu} \right) - \frac{1}{2} \frac{\tau^2}{S_2^2} \ln \left( \frac{S_1 - uu}{S_1 + uu} \right), \tag{4.15} \]
\[ \eta_1 = \frac{1}{2} (4x - 1)u_x^2 + 4uu_xu_{xx} - t^2, \]
\[ \eta_2 = \frac{1}{2} \tau uu_{xx}^2 - \tau uu_xu_{xx} + \tau xxu_{xx}u_{xx} + 2\tau uu_xu_{xx} - \frac{1}{2} t^2 + xu_{xx}^2, \]
\[ \eta_3 = \frac{1}{2} \left( \frac{\tau^2}{S_2} - 1 \right) \ln \left( -\frac{S_3 - uu}{S_3 + uu} \right) - \frac{1}{2} \ln \left( -\frac{1}{2} \tau (S_3 + uu) \right). \]

Where \( S_1, S_2, S_3 \) and \( S_4 \) are respectively,
\[ S_1 = 2\tau uu_t + 2\tau xx^2u_x - \tau u \quad S_2 = \sqrt{\tau(\tau + 4t)} \tag{4.16} \]
\[ S_3 = -4\tau uu_{tt} + 2uu_x + 4\tau xx^2u_{xx} + 2\tau^2 uu_t - u\tau^2, \quad S_4 = \sqrt{\tau(\tau^3 + 8t)}, \]

Similar preceding theorem, we have

**Theorem 7** Consider the equation \( \Delta := F(y, v, \xi_1, \xi_2, \eta_1, \eta_2, \eta_3) \) with arbitrary smooth function \( F \). Let \( \tau \) be a real constant. Then
i) The equation \( \Delta \) admits \( X \) as \( \mu \)-symmetry with \( \mu = \lambda x \).
ii) For \( (\partial F/\partial \xi_2)^2 + (\partial F/\partial \eta_2)^2 + (\partial F/\partial \eta_3)^2 \neq 0 \), \( X \) is not ordinary symmetry of \( \Delta \).
Using such a procedure we can construct scalar PDEs with \( \mu \)-symmetries. If we set \( X = \sum \xi^i \partial x^i + \partial_u \) and apply the mentioned procedure, then we get \((p+1)\)-PDEs without Lie non-trivial symmetries which have \( X \) as \( \mu \)-symmetry. We can solve this equations similar to Lie standard symmetry method using \( X \) as new symmetry.

Now we express step II for construct our favorite variational problems.

5 \( \mu \)-symmetry on variational problems

Characterizing systems of differential equations which are the Euler-Lagrange equations for some variational problems, is known as the inverse problem in the calculus of variations (see [Ol-1995, Va-2003]). There are different approaches to solve or investigate inverse problem ([Ol-1986, Va-2003]). In order to keep the scope manageable, we use direct method in this paper.

In this section we construct two examples of variational problems which their Euler-Lagrange equations have no Lie standard symmetry. For this purpose first we assume \( \Delta \) be equations without Lie symmetry then find appropriate variational problem which have \( \Delta \) as Euler-Lagrange equation.

**Example 8** Consider the equation

\[
\frac{du}{dx} = \left[ (x + x^2)e^u \right]_x \tag{5.17}
\]

This equation appear in page 182 of P.J. Olver [Ol-1995] as an equation which can be integrated by quadratures, but lacks non-trivial symmetries. Muriel and Romero in [Mu-Ro-2001] solve this equation by using \( \lambda \)-symmetry with \( \lambda = \left[ (x + x^2)e^u \right]_u \) and \( X = \partial_u \). Let this equation be Euler-Lagrange equation of some second order variational problem, so we have

\[
\frac{\partial L}{\partial x} + u_x \frac{\partial L}{\partial u} + u_{xx} \frac{\partial L}{\partial u_{xx}} = u_{xx} - (1 + 2x)e^u - (x + x^2)u_x e^u, \tag{5.18}
\]

Where by solving this equation we find following Lagrangian:

\[
L(x, u, u_x, u_{xx}) = -x^2 e^u - e^u_x + xu_{xx} + F(u_{xx}, u_x - xu_{xx}, u + \frac{1}{2}x^2 u_{xx} - xu_x); \tag{5.19}
\]

Where \( F \) is an arbitrary function.

So we have following proposition using (theorem 5) and corollary 7.4 in [Ol-1995]:

**Proposition 9** The following variational problem and any variational problem with lagrangian \( \hat{L} = L + \operatorname{Div} \xi \) with arbitrary smooth function \( \xi \) have no Lie non-trivial variational symmetry, and its Euler-Lagrange equation has \( \lambda \)-symmetry with \( \lambda = \left[ (x + x^2)e^u \right]_u \) and \( X = \partial_u \).

\[
\ell(u) = \int_{\Omega} (-x^2 e^u - e^u_x + xu_{xx} + F(u_{xx}, u_x - xu_{xx}, u + \frac{1}{2}x^2 u_{xx} - xu_x)) \, dx, \tag{5.20}
\]

where \( F \) is an arbitrary function.

**Example 10** Consider this equation

\[
8(u_x + 1)u_{xx} - 24(u_x^2 + 2u_x + 24u + 1)u_x + x^3 u_5 + (5x^2 + 8x)u_4 + (7x + 32)u_3 + 3u^2 = 0, \tag{5.21}
\]

Muriel and Romero in [Mu-Ro-2001] prove that this equation has no Lie non-trivial symmetry. Now we use direct method to find some second variational problem with property of equation (5.11) which is its Euler-Lagrange equation.
\[
\frac{\partial L}{\partial x} + u \frac{\partial L}{\partial u} + u_{xx} \frac{\partial L}{\partial u_x} + u_{xxx} \frac{\partial L}{\partial u_{xx}} = \\
= 8(u_x + 1) u_{xx} - 24(x u_x^2) - 2(u^2 x^2 + 2u_x + 24u + 1) u_x + x^3 u^5 + (5x^2 + 8x) u^4 + (7x + 32) \cdot u^3 + 3u^2 
\] (5.22)

= 0,

By solving this equation we have:

\[
L(x, u, u_x, u_{xx}) = F(u_x, u - xu_x, u_{xx}) + \frac{1}{504} x^9 u_x^5 + \frac{1}{56} x^8 u_x^4 u + \left( \frac{1}{21} u_x^4 - \frac{1}{14} u_x u_x^3 \right) x^7 \\
+ \left( -\frac{1}{3} u_x^3 u + \frac{4}{15} u_x^4 + \frac{1}{6} u_x^2 u_x^3 \right) x^6 + \left( -\frac{5}{12} u_x^3 - \frac{8}{9} u_x^4 u + u_x^2 - \frac{1}{4} u_x u_x^2 \right) x^5 \\
+ \left( \frac{1}{4} u^5 + \frac{5}{12} u_x^2 u - \frac{5}{3} u_x u_x^3 + 4u_x^2 u_x^2 \right) x^4 + \left( -\frac{16}{3} u_x^3 u^3 + u_x^2 - \frac{25}{6} u_x u_x^2 + \frac{5}{3} u_x^4 + 32u_x^2 u \right) x^3 \\
+ \left( 4u^4 - 3u_x u + \frac{7}{2} u^3 + 12u_x^2 - 48u_x u^2 \right) x^2 + \left( 32u^3 + 3u^2 - 48uu_x - 4u_x^2 + (8u_{xx} - 2) u_x + 8u_{xx} \right) x
\]

where \( F \) is an arbitrary function.

As a result above, (Theorem 5) and Corollary 7.4 in [Ol-1995], we can find this proposition,

**Proposition 11** Variational problem (5.12) and any Variational problem with \( \hat{L} = L + \text{Div} \xi \) with arbitrary function \( \xi \) have no Lie nontrivial variational symmetries and its Euler-Lagrangian equation has \( X = u \partial_x \) as \( \lambda \)-symmetry with \( \lambda = x/u^2 \).

**Conclusion**

In this paper first we construct Euler-Lagrange equations with no Lie non-trivial symmetry, next we find related variational problems without any Lie non-trivial variational symmetries. Finally we solve these variational problems using \( \mu \)-symmetry.

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