Solvability for Infinite Systems of Fractional Differential Equations in Banach Sequence Spaces $\ell_p$ and $c_0$

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Abstract. This paper is devoted to an infinite system of nonlinear fractional differential equations in the Banach spaces $c_0$ and $\ell_p$ with $p \geq 1$. Existence results are obtained, by using the theory of measure of noncompactness and a new generalization of Darbo’s fixed point theorem. Some examples are also included to show the efficiency of our results.

1. Introduction

The interest for studying fractional differential equations is based on the fact that the theory of fractional differential equations has been applied to various fields such as physics, chemistry, engineering and heat conduction in material with memory, see for example [14, 15]. Indeed, by applying the theory of fractional differential equations, we can find numerous applications in economics, geology, viscoelastic materials, bioengineering, fluid mechanics, chaotic dynamics and polymer science, etc. [4, 10, 13, 17, 20]. In recent years, ordinary and partial functional differential equations have been developed by the fractional calculus techniques and equations of fractional order are more general compared with integer order. The problem of the existence of solutions for fractional differential equations plays a significant role in the investigation of these types of equations and it is important to apply original studies in our investigations[1, 2, 5, 7, 12].

The theory of infinite systems of differential equations can be regarded as a particular case of differential equations in Banach spaces, where the infinite system can be represented as an ordinary differential equation. Recently, Mursaleen et al. in [11] studied a three point infinite system of fractional differential equations

$$
\begin{aligned}
D^\alpha u_i(t) &= f_i(t, u(t)), \quad t \in (0, T), \\
u_i(0) &= 0, \quad u_i(T) = au_i(\xi), \quad i = 1, 2, \ldots
\end{aligned}
$$

where $D^\alpha$ is the standard Riemann-Liouville fractional derivative of order $1 < \alpha < 2$ and $\xi \in (0, T)$ with $a\xi^{\alpha-1} < T^{\alpha-1}$. By using the theory of measure of noncompactness and condensing operators they established the existence of solutions in sequence spaces $c_0$ and $\ell_p$. In [12] Mursaleen and Rizvi studied existence results

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for the solution of infinite system of second order differential equations in Banach sequence spaces $c_0$ and $\ell_1$ using the idea of Meir-Keeler condensing operators.

Motivated by the above papers, the aim of this work is to study the existence of solutions of the following infinite system of fractional differential equations

$$D^\alpha_0, u(t) = F_t(t, h_i(t, u(t)), (Gu)(t) \int_0^t g_i(s, u(s))ds), \quad t \in [0, T]$$

supplemented with three point boundary conditions

$$u_i(0) = 0, \quad u_i(T) = au_i(\xi), \quad i = 0, 1, 2, \ldots,$$

where $\xi \in [0, T], F_i : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_i, g_i : [0, T] \times D \to \mathbb{R}$ $(D \in \{c_0, \ell_p\})$ are continuous functions and $G : A \to C(I, \mathbb{R}), A \in \{C(I, \ell_p), C(I, c_0)\}$ is a continuous operator.

The paper is organized as follows. In Section 2, we recall some essential concepts and results which are used in the main results. In the next two sections, by applying a generalization of Darbo fixed point theorem together with the technique of measure of noncompactness the existence of solutions is studied, in sequence space $\ell_p$ in Section 3 and in sequence space $c_0$ in Section 4. Examples illustrating the obtained results are also presented.

2. Preliminaries

In this section, we firstly introduce some notations and definitions which are used throughout this paper. For a bounded subset $S$ of a metric space $X$, Kuratowski [9] defined the function $\alpha(S)$, known as Kuratowski measure of noncompactness, by the formula

$$\alpha(S) = \inf \left\{ \delta > 0 : S = \bigcup_{i=1}^n S_i, \, \text{diam}(S_i) \leq \delta \, \text{for} \, 1 \leq i \leq n < \infty \right\}.$$

Another useful measure of noncompactness is the so called Hausdorff measure of noncompactness defined as

$$\chi(S) = \inf \{\varepsilon > 0 : S \text{ has finite } \varepsilon \text{-net in } X\}.$$

Banas and Goebel [3] have presented some basic properties of the Hausdorff measure of noncompactness $\chi$. Now we assume that $E$ is a real Banach space with norm $\|\cdot\|$ and zero element $\theta$. If $X$ is a nonempty subset of $E$, the closure and the closed convex hull of $X$ will be denoted by $\overline{X}$ and $\text{Conv}(X)$, respectively. Moreover, let us denote by $M_E$ the family of all nonempty and bounded subsets of $E$ and by $N_E$ its subfamily consisting of all relatively compact sets.

Definition 2.1. [3] A mapping $\mu : M_E \to [0, \infty)$ is called a measure of noncompactness if it satisfies the following conditions:

1. The family $\text{Ker} \mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\text{Ker} \mu \subseteq N_E$.
2. $X \subseteq Y \implies \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(X)$.
4. $\mu(\text{Conv}(X)) = \mu(X)$.
5. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6. If $\{X_n\}$ is a sequence of closed sets from $M_E$ such that $X_{n+1} \subseteq X_n$ for $n = 1, 2, \ldots$ and $\lim_{n \to \infty} \mu(X_n) = 0$, then $\bigcap_{n=1}^\infty X_n$ is nonempty.
Theorem 2.2. (Darbo [6]). Let $Q$ be a nonempty, closed, bounded and convex subset of a Banach space $E$ and $F : Q \rightarrow Q$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $f(X) \leq kf(X)$ for any nonempty subset $X$ of $Q$, where $\mu$ is a measure of noncompactness defined in $Q$. Then $F$ has a fixed point in $Q$.

Samadi and Ghaemi [18, 19] proved some generalizations of Darbo fixed point theorem. Also, the first author [17] extended Darbo fixed point theorem and presented the following result which is basic for our main results.

Theorem 2.3. Let $C$ be a nonempty bounded, closed and convex subset of a Banach space $E$. Assume $T : C \rightarrow C$ is a continuous operator satisfying

$$\theta((\mu(X)) + f(\mu(T(X)))) \leq f(\mu(X))$$

for all nonempty subset $X$ of $C$, where $\mu$ is an arbitrary measure of noncompactness defined in $E$, $F : (0, \infty) \rightarrow \mathbb{R}$, $\theta : (0, \infty) \rightarrow (0, \infty)$ and $(\theta, f) \in \Delta$. Then $T$ has a fixed point in $C$.

In Theorem 2.3, $\Delta$ is the set of all pairs $(\theta, f)$ satisfying the following:

$$(\Delta_1) \quad \theta(t_n) \rightarrow 0 \text{ for each strictly increasing sequence } \{t_n\};$$

$$(\Delta_2) \quad f' \text{ is strictly increasing function};$$

$$(\Delta_3) \quad \text{for each sequence } \{s_n\} \text{ of positive numbers, } \lim_{n \to \infty} s_n = 0 \text{ if and only if } \lim_{n \to \infty} f(s_n) = -\infty.$$  

$$(\Delta_4) \quad \text{If } \{t_n\} \text{ is a decreasing sequence such that } t_n \rightarrow 0 \text{ and } \theta(t_n) \leq f(t_n) - f(t_{n+1}), \text{ then we have } \sum_{n=1}^{\infty} t_n < \infty.$$  

The following essential definitions and auxiliary facts in fractional calculus will be needed in our main results.

Definition 2.4. [8] The fractional order integral of the function $y \in L^1([a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^\alpha_y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} y(s)ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.5. [8] The Riemann-Liouville derivative of order $\alpha$ with the lower limit zero for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$$D^\alpha_0 f(t) = \frac{1}{\Gamma(n - \alpha)} \times \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{n+1-\alpha}}ds, \quad t > 0, n - 1 < \alpha < n.$$  

The following lemma is the main tool in our investigation.

Lemma 2.6. [11] Let $f \in C([0, T], \mathbb{R})$ be a given function and $1 < \alpha < 2$. Then the unique solution of

$$D^\alpha_0 u(t) = f(t), \quad u(0) = 0, \quad u(T) = au(\xi), \xi \in [0, T]$$

is given by

$$u(t) = \int_0^T k(t, s)f(s)ds,$$

where $k(t, s)$ is the Green’s function, given by $k(t, s) = \begin{cases} k_1(t, s), & 0 \leq t \leq \xi, \\
 k_2(t, s), & \xi \leq t \leq T, \end{cases}$ and

$$k_1(t, s) = \begin{cases} (t - s)^{\alpha - 1}(T^{\alpha - 1} - a\xi^{\alpha - 1}) - t^{\alpha - 1}[(T - s)^{\alpha - 1} - a(\xi - s)^{\alpha - 1}], & 0 \leq s \leq t, \\
 -t^{\alpha - 1}[(T - s)^{\alpha - 1} - a(\xi - s)^{\alpha - 1}], & t \leq s \leq \xi, \\
 -(t(T - s)^{\alpha - 1}), & \xi \leq s \leq T, \end{cases}$$

$$k_2(t, s) = \begin{cases} (t - s)^{\alpha - 1}(T^{\alpha - 1} - a\xi^{\alpha - 1}) - t^{\alpha - 1}[(T - s)^{\alpha - 1} - a(\xi - s)^{\alpha - 1}], & 0 \leq s \leq \xi, \\
 (t - s)^{\alpha - 1}(T^{\alpha - 1} - a\xi^{\alpha - 1}) - ((T - s))^{\alpha - 1}, & \xi < s \leq t, \end{cases}$$
3. Solution in sequence space $\ell_p$

In this section we investigate the solution of the infinite system (1)-(2) in the sequence space $\ell_p$, the space of all absolutely $p$-summable series

$$\ell_p = \{ x \in \omega : \sum_{n=1}^{\infty} |x_n|^p < \infty \}, \quad 1 \leq p < \infty,$$

where $\omega$ is the space of all complex sequences $x = \{x_n\}_{n=1}^{\infty}$. Clearly, $\ell_p$ is a Banach space with norm

$$\|x\|_{\ell_p} = \|(x_n)\|_{\ell_p} = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Let us denote by $M_{\ell_p}$ the families of all nonempty bounded subsets of $\ell_p$. It is well known that in the space $(\ell_p, \| \cdot \|_{\ell_p})$, the Hausdorff measure of noncompactness $\chi$ is defined by the formula [3]:

$$\chi(B) = \lim_{n \to \infty} \left\{ \sup_{x \in B} \left\{ \sum_{k \geq n} |x_k|^p \right\}^{\frac{1}{p}} \right\},$$

where $B \in M_{\ell_p}$ and $x(t) = (x_i(t))_{i=1}^{\infty} \in \ell_p$.

By applying Theorem 2.3, the existence of solutions for the infinite system (1)-(2) is studied in the Banach space $(\ell_p, \| \cdot \|_{\ell_p})$. We list the following conditions:

(H1) The functions $F_i : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_i : [0, T] \times \ell_p \to \mathbb{R}$ are continuous and there exists a positive real number $\tau > 0$ such that:

$$|F_i(t, x_1, x_2) - F_i(t, y_1, y_2)|^p \leq e^{-\tau}(|x_1 - y_1|^p + |x_2 - y_2|^p),$$

$$|h_i(t, u(t))|^p \leq e^{-\tau}|u_i(t)|^p,$$

$$|h_i(t, u(t)) - h_i(t, v(t))|^p \leq e^{-\tau}|u_i(t) - v_i(t)|^p,$$

for $t \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $u(t) = (u_i(t)), v(t) = (v_i(t)) \in \ell_p$ where $i = 1, 2, 3, \ldots$.

(H2) The function $t \to F_i(t, 0, 0)$ is bounded on $[0, T]$ such that:

$$N_1 = \sum_{i=1}^{\infty} \int_{0}^{T} |F_i(s, 0, 0)|^p ds, \quad \lim_{n \to \infty} \sum_{k \geq n} \int_{0}^{T} |F_k(s, 0, 0)|^p ds = 0.$$

(H3) $G : C(I, \ell_p) \to C(I, \mathbb{R})$ is a continuous operator such that:

$$|(Gx)(t) - (Gu)(t)| \leq \|x(t) - u(t)\|_{\ell_p}, \quad |(Gx)(t)| \leq a + b\|x(t)\|_{\ell_p},$$

for $x, u \in C(I, \ell_p)$ and $t \in [0, T]$, where $a, b$ are positive real numbers.

(H4) $g : [0, T] \times \ell_p \to \mathbb{R}$ are continuous and there exist continuous functions $b_i : [0, T] \to \mathbb{R}$ such that

$$|g_i(s, u(s))| \leq |b_i(s)|, \quad q = \sup\{|b_i(s)|; s \in [0, T], i \geq 1\},$$

for $u \in C(I, \ell_p)$. Furthermore the series $\sum_{i=1}^{\infty} b_i(t)$ is uniformly convergent with

$$b(t) = \sum_{i=1}^{\infty} b_i(t), \quad B = \sup\{|b(t)|; t \in [0, T]\}.$$
(H₅) There exist $B_k$ such that

$$B_k = \sup \left\{ \sum_{i \geq k} \left| \int_0^T g_i(s, u(s))ds \right|^p : s \in [0, T] \right\}.$$ 

Also, as $k \to \infty$, $B_k \to 0$ and $\sup_k B_k = B_0$.

(H₆) There exists a positive solution $r_0$ of the inequality

$$\left(2^p T^\frac{p}{2} M^p N_1 + 2^p T^\frac{p}{2} M^p e^{-2T + T} + e^{-T} (a + br^p) B_0 T^2 T^\frac{p}{2} M^p \right)^{\frac{1}{p}} \leq r.$$ 

Moreover, assume that $(T^\frac{p}{2} T^\frac{p}{2} M^p T^2)^{\frac{1}{p}} < 1$.

**Remark 3.1.** For each $i \in \mathbb{N}$, the infinite system (1)-(2) has a solution if and only if the integral equation

$$u_i(t) = \int_0^T k(t, s)F_i(s, h_i(s, u(s))), (Gu)(s) \int_0^T g_i(s, u(s))ds ds,$$

has a solution, where $k_i(t, s) = k(t, s)$ described in Lemma 2.6.

In the following we put $M = \max_{t \in [0, T]} |k(t, s)|$.

**Theorem 3.2.** Under the assumptions (H₁) − (H₆), infinite system (1)-(2) has at least one solution $u(t) = \{u_i(t)\}_{i=1}^\infty$ such that $u(t) \in \ell_p$ for all $t \in [0, T]$.

**Proof.** Let us consider the operator $F$ defined on the space $C(I, \ell_p)$ by the formula

$$(Fu)(t) = \left( \int_0^T k(t, s)F_i(s, h_i(s, u(s))), (Gu)(s) \int_0^T g_i(s, u(s))ds ds \right),$$

for all $t \in [0, T]$, where $C(I, \ell_p)$ is the space of all continuous functions on the interval $[0, T]$ with values in the space $\ell_p$ and equipped with the norm

$$\|u\| = \sup_{t \in [0, T]} \|u(t)\|_{\ell_p}.$$

Let $u$ be an arbitrary element of $C(I, \ell_p)$ where $u(t) = \{u_i(t)\}_{i=1}^\infty \in \ell_p$ for all $t \in [0, T]$. Keeping in mind our assumptions, for any $t \in I$ we deduce that

$$\|(Fu)(t)\|_{\ell_p}^p = \sum_{i=1}^\infty \left| \int_0^T k(t, s)F_i(s, h_i(s, u(s))), (Gu)(s) \int_0^T g_i(s, u(s))ds ds \right|^p \leq \sum_{i=1}^\infty \int_0^T |k(t, s)|^p |F_i(s, h_i(s, u(s))), (Gu)(s) \int_0^T g_i(s, u(s))ds ds |^p ds \left( \int_0^T ds \right)^\frac{p}{2},$$
where $q > 1$ is a positive number such that $\frac{1}{p} + \frac{1}{q} = 1$. Consequently,

\[
\|(Fu)(t)\|_{l_p}^p \\
\leq T^\frac{p}{q} \sum_{i=1}^\infty \int_0^T |k(t, s)|^p |F_i(s, h_i(s, u(s)), (Gu)(s) \int_0^T g_i(s, u(s))ds)|^p ds \\
\leq 2^p T^\frac{p}{q} M^p \sum_{i=1}^\infty \int_0^T |F_i(s, h_i(s, u(s)), (Gu)(s) \int_0^T g_i(s, u(s))ds - F_i(s, 0, 0)|^p ds \\
+ 2^p M^p T^\frac{p}{q} \sum_{i=1}^\infty \int_0^T |F_i(s, 0, 0)|^p ds \\
\leq 2^p T^\frac{p}{q} M^p \sum_{i=1}^\infty \int_0^T \left( e^{-\tau} |h_i(s, u(s))|^p + e^{-\tau} \left| (Gu)(s) \int_0^T g_i(s, u(s))ds \right|^p \right) ds + 2^p T^\frac{p}{q} M^p N_1 \\
\leq 2^p T^\frac{p}{q} M^p e^{-\tau} \|u\|^p_T + e^{-\tau} (a + b\|u\|^p) B_0 T^2 2^p T^\frac{p}{q} M^p + 2^p T^\frac{p}{q} M^p N_1.
\]

From the above estimate we have

\[
\|Fu\| \leq \left( 2^p T^\frac{p}{q} M^p e^{-\tau} \|u\|^p_T + e^{-\tau} (a + b\|u\|^p) B_0 T^2 2^p T^\frac{p}{q} M^p + 2^p T^\frac{p}{q} M^p N_1 \right)^\frac{1}{p}.
\]

Now, we show that $F$ is continuous on $[0, T]$. Let $t_0 \in [0, T]$ and $\epsilon > 0$ be arbitrary. In view of the continuity of $k(t, s)$ there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies that

\[
|k(t, s) - k(t_0, s)| < e^\frac{\epsilon}{2^p T^\frac{p}{q} N_1 + 2^p T^\frac{p}{q} e^{-2\tau} \|u\|^p_T + e^{-\tau} (a + b\|u\|^p) B_0 T^2 2^p T^\frac{p}{q} M^p}.
\]

From (5) and (7) we get

\[
\|(Fu)(t) - (Fu)(t_0)\|_{l_p} < \epsilon.
\]

Moreover, due to (6) we conclude that $F$ is bounded in the classical supremum norm on $C(I, l_p)$ and transforms $C(I, \ell_p)$ into itself. Due to the last inequality we conclude that $F$ maps the ball $\overline{B}_r$ into itself where $r_0$ is the existing constant in the assumption $(H_b)$ and

\[
\overline{B}_r = \{ u \in C(I, \ell_p); \|u\|_{C(I, \ell_p)} \leq r, u(0) = 0, u(T) = au(\xi) \}.
\]

Next we show that $F$ is continuous on the ball $\overline{B}_r$. Let $u, v \in \overline{B}_r$ and $\epsilon > 0$ such that $\|u - v\|_{C(I, \ell_p)} < \epsilon$. For all $t \in [0, T]$, we have

\[
\|(Fu)(t) - (Fv)(t)\|_{l_p}^p \\
= \sum_{i=1}^\infty \left[ \int_0^T k(t, s) F_i(s, h_i(s, u(s)), (Gu)(s) \int_0^T g_i(s, u(s))ds) \\
- F_i(s, h_i(s, v(s)), (Gu)(s) \int_0^T g_i(s, v(s))ds) \right] ds \right|^p \\
\leq T^\frac{p}{q} \int_0^T \sum_{i=1}^\infty \left( |k(t, s)|^p |F_i(s, h_i(s, u(s)), (Gu)(s) \int_0^T g_i(s, u(s))ds) \\
- F_i(s, h_i(s, v(s)), (Gu)(s) \int_0^T g_i(s, v(s))ds)|^p \right) \left| (Gu)(s) \int_0^T g_i(s, v(s))ds \right| ds \\
\leq T^\frac{p}{q} M^p \int_0^T \sum_{i=1}^\infty \left( e^{-\tau} |h_i(s, u(s)) - h_i(s, v(s))|^p \right) \left| (Gu)(s) \int_0^T g_i(s, v(s))ds \right| ds.
\]
where the continuity of $\delta$ and $\delta_1$ implies that

$$\lim_{L \to \infty} = \left( \lim_{L \to \infty} \right)^{\frac{1}{p}} \leq T^2 M^p \int_0^T \left[ e^{-\frac{1}{p}} \left| \sum_{i=1}^k \frac{1}{|\|s_i|\|_c^{(L, \epsilon)}} \right| (Gv)(s) \right] ds + 2^p e^{-\frac{1}{p}} |Gv)(s)| \left( \frac{\int_0^T \sum_{i=1}^k |g_i(s, v(s)) - g_i(s, u(s))| ds}{p} \right)^{\frac{1}{p}} + 2^p e^{-\frac{1}{p}} \left( \frac{\int_0^T \sum_{i=1}^k |b_i(s)| ds}{p} \right)^{\frac{1}{p}} ds.

As a consequence of Lebesgue dominated convergence theorem, from the above inequality and applying the continuity of $g$ on $[0, T] \times \ell_p$, we insert that

$$\| (Fu)(t) - (Fv)(t) \|_{\ell_p}^{\frac{1}{p}} \leq T^2 M^p T \left( e^{-\frac{1}{p}} + 2^p e^{-\frac{1}{p}} (a + b\|v\|)^{\frac{1}{p}} \right) \left( \int_0^T \delta_1(\epsilon) ds \right)^\frac{1}{p} + 2^p e^{-\frac{1}{p}} (\varepsilon TB) \left( \int_0^T \delta_1(\epsilon) ds \right)^\frac{1}{p},$$

where

$$\delta_1(\epsilon) = \sup \{|g_i(t, v) - g_i(t, u)|: u, v \in \ell_p, \|u - v\|_{\ell_p, (L, \epsilon)} \leq \epsilon, t \in I, i = 1, 2, 3, \ldots\},$$

and $\delta_1(\epsilon) \to 0$ as $\epsilon \to 0$. Hence $F$ is continuous on the ball $B_n$. Now let $X$ be a nonempty subset of $B_n$. Then, taking into account our assumptions, for arbitrary fixed $t \in I$ we have

$$\chi_{\ell_p}(FX(t)) = \lim_{n \to \infty} \left\{ \sup_{u \in X} \left( \sum_{t \in \mathbb{Z}} |Fu(t)|^{\frac{1}{p}} \right)^{\frac{1}{p}} \right\} \leq \lim_{n \to \infty} \left\{ \sup_{u \in X} \left( \sum_{t \in \mathbb{Z}} \left| k(t, s) \times F_i(s, h_i(s, u(s)), (Gu)(s) \right) \int_0^T g_i(s, u(s)) ds \right|^{\frac{1}{p}} \right\} \leq (T^2)^{\frac{1}{2}} 2^p \lim_{n \to \infty} \left\{ \sup_{u \in X} \left( \sum_{t \in \mathbb{Z}} \int_0^T |k(t, s)|^{\frac{1}{p}} F_i(s, h_i(s, u(s)), (Gu)(s) \int_0^T g_i(s, u(s)) ds \right) ds \right\}^{\frac{1}{2}}$$

$$\leq (T^2)^{\frac{1}{2}} 2^p \lim_{n \to \infty} \left\{ \sup_{u \in X} \left( \sum_{t \in \mathbb{Z}} \int_0^T \left( e^{-\frac{1}{p}} \left| h_i(s, u(s)) \right|^{\frac{1}{p}} + e^{-\frac{1}{p}} \left| (Gu)(s) \right| g_i(s, u(s)) ds \right) \right) ds \right\}^{\frac{1}{2}}.$$
Let us observe that the system (8) is a special case of system (1)-(2) if we put $\in$.  

Example 3.3. Now, we investigate the following infinite system of fractional differential equations

$$D^{5/4}u_n(t) = \left\{ \begin{array}{cl}
\frac{\left(e^{-t-n}\right)^{3/2}}{2} \sin\left(\frac{e^{-t-n}}{2}\right) \cos\left(|u_n(t)|\right) \\
+ \cos\left(\frac{1}{1+|x(t)|_{\ell_p}}\right) \int_0^T \arctan\left(\frac{\frac{1}{2}e^{-s}}{8 + |u_n(s)|}\right) ds \\
\end{array} \right. \quad t \in [0, T],$$

$$u_n(0) = 0, \quad u_n(T) = \sqrt[3]{2}u_n\left(\frac{T}{3}\right), \quad n = 1, 2, 3, \ldots$$

Let us observe that the system (8) is a special case of system (1)-(2) if we put

$$F_n(t, x, y) = \frac{\left(e^{-t-n}\right)^{3/2}}{2} \sin\left(x + y\right), \quad (Gx)(t) = \cos\left(\frac{1}{1+|x(t)|_{\ell_p}}\right),$$

$$h_n(t, u(t)) = \frac{\left(e^{-t-n}\right)^{3/2}}{2} \cos\left(|u_n(t)|\right), \quad g_n(s, u(s)) = \arctan\left(\frac{\frac{1}{2}e^{-s}}{8 + |u_n(s)|}\right).$$

Suppose $t \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $u, v \in \ell_p$. From the definition of $F_n$ and $h_n$, we conclude that

$$|F_n(t, x_1, y_1) - F_n(t, x_2, y_2)|^p \leq e^{-t}\left(|x_1 - y_1|^p + |x_2 - y_2|^p\right),$$

$$|h_n(t, u(t))|^p \leq e^{-t}|u_n(t)|^p,$$

$$|h_n(t, u(t)) - h_n(t, v(t))|^p \leq e^{-t}|u_n(t) - v_n(t)|^p.$$  

Consequently $F$ and $h$ satisfy the assumption (H$_1$). Moreover, $|F_n(s, 0, 0)| = 0$ and condition (H$_2$) is clearly satisfied.

On the other hand the function

$$(Gx)(t) = \cos\left(\frac{1}{1+|x(t)|_{\ell_p}}\right)$$
verifies assumption \((H_3)\) with \(a = 1\) and \(b = 0\). To justify assumption \((H_4)\), let \(s \in [0, T]\) and \(u \in C(I, \ell_p)\). Then we have
\[
|g_n(s, u(s))| = \left| \arctan \left( \frac{\frac{1}{2}e^{-s}(8 + |u_n(s)|)}{2e^{s}} \right) \right| < \frac{e^{-s}}{2^n} = b_n(s) < e^{-s},
\]
\[
q = \sup \{ |b_n(s)| : n \geq 1, s \in [0, T] \} \leq 1.
\]

Furthermore, applying the above inequality we infer that the series \(\sum_{n=1}^{\infty} b_n(t)\) is uniformly convergent on \(I\). Again we have
\[
B_k = \sup \left\{ \sum_{n \geq k} \left| \int_0^T g_n(s, u(s)) ds \right| : s \in [0, T] \right\} \leq \sup \left\{ \left( 1-e^{-T} \right) \sum_{n \geq k} \frac{1}{2^n} \right\}.
\]
As \(k \to \infty\) we get \(\sum_{k \geq n} \frac{1}{2^n} \to 0\). Thus, \(B_k \to 0\). Finally the existing inequality in assumption \((H_4)\) has the form
\[
2^{p}T^\frac{q}{p}e^{-2^{-q}pT} + e^{-r}B_0 T2^{p}T^\frac{q}{p}M^p \leq r^p.
\]

Thus, for the number \(r_0\) we can take \(r_0 = e^{-r}B_0 T2^{p}T^\frac{q}{p}M^p \left( 1 - 2^{p}T^\frac{q}{p}M^p e^{-2^{-q}pT} \right)\). Consequently all conditions of Theorem 3.2 are satisfied and thus the system of fractional differential equation (8) has at least one solution in the space \(C(I, \ell_p)\).

4. Solution in sequence space \(c_0\)

Now we investigate the existence of solutions for the infinite system (1)-(2) in the space \(c_0\), the space of sequences converging to zero, equipped with the norm \(\|x\|_c = \sup \{|x_i| : i = 1, 2, \ldots\}\). Let us denote by \(M_{c_0}\) the families of all nonempty bounded subsets of \(c_0\). For the Banach space \((c_0, \|\cdot\|_c)\), the Hausdorff measure of noncompactness \(\chi\) is given by (cf. [3]):
\[
\chi(B) = \lim_{n \to \infty} \left\{ \sup_{k \leq n} \left\{ \max \{|u_k|\} \right\} \right\},
\]
where \(B \in M_{c_0}\) and \(x(t) = (x_i(t))_{i=1}^{\infty} \in c_0\).

We need the following assumptions in the sequel:

(A1) The functions \(F_n : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and \(h_n : [0, T] \times c_0 \to \mathbb{R}\) are continuous functions and there exist positive real numbers \(\tau > 0\) such that
\[
|F_n(t, x_1, x_2) - F_n(t, x_1', x_2')| \leq e^{-\tau}(|x_1 - y_1| + |x_2 - y_2|),
\]
\[
|h_n(t, u(t))| \leq e^{-\tau} \sup_{i \geq n} |u_i(s)|, i \geq n,
\]
\[
|h_n(t, u(t)) - h_n(t, v(t))| \leq e^{-\tau} \sup_{i \geq n} |u_i(s) - v_i(s)|, i \geq n.
\]
for \(t \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}\) and \(u(t) = (u_i(t)), v(t) = (v_i(t)) \in c_0\) where \(i = 1, 2, 3, \ldots\).

(A2) The function \(t \to F_n(t, 0, 0)\) is bounded on \([0, T]\) i.e
\[
M_1 = \sup \{|F_n(t, 0, 0) : t \in [0, T], i \geq 1\}.
\]
Moreover, \(\lim_{i \to \infty} F_n(t, 0, 0) = 0\).

(A3) \(G : C(I, c_0) \to C(I, \mathbb{R})\) is a continuous operator such that
\[
\|G(x)(t) - (Gu)(t)\| \leq \|x(t) - u(t)\|_{c_0}, \quad \|G(x)(t)\| \leq a + b\|x(t)\|_{c_0},
\]
for \(x, u \in C(I, c_0), t \in [0, T]\), where \(a, b\) are positive real numbers.
(A₄) \( g_n : [0, T] \times c_0 \to \mathbb{R} \) is continuous and there exist continuous functions \( b_i : [0, T] \to \mathbb{R} \) such that

\[
|g_n(s, u(s))| \leq |b_n(s)|, \quad q = \sup\{|b_n(s)|; s \in [0, T]\},
\]

for all \( s \in [0, T] \) and \( u \in C(I, c_0) \). Moreover, \( \lim_{n \to \infty} \int_0^T |b_n(s)|\,ds = 0 \).

(A₅) There exists a positive solution \( r_0 \) of the inequality

\[
e^{-2t}r + e^{-t}(a + br)qT + M_1 \leq r.
\]

Moreover, assume that \( TM < 1 \).

**Theorem 4.1.** Under assumptions (A₁) – (A₅), infinite system (1)-(2) has at least one solution \( u \) such that \( u(t) \in c_0 \) for all \( t \in [0, T] \).

**Proof.** Let us define the operator \( F \) on the space \( C(I, c_0) \) by

\[
(Fu)(t) = \left( \int_0^T k(t, s)F_\nu\left(t, h_n(s, u(s)), (Gu)(s) \int_0^T g_n(s, u(s))\,ds\right)\,ds \right)
\]

for all \( t \in [0, T] \), where \( C(I, c_0) \) is the space of all continuous functions on the interval \([0, T]\) with values in space \( c_0 \) and equipped with the norm \( \|u\| = \sup\{|u(t)|; t \in [0, T]\} \). We show that \((Fu) (t) \in c_0 \). For arbitrarily fixed \( t \in [0, T] \), we have

\[
\|(Fu)(t)\|_{c_0} \\
= \sup_{n \geq 1} \left| \int_0^T k(t, s)F_n\left(s, h_n(s, u(s)), \int_0^T g_n(s, u(s))\,ds\right)\,ds \right| \\
\leq \sup_{n \geq 1} \int_0^T |k(t, s)|\left|F_n\left(s, h_n(s, u(s)), \int_0^T g_n(s, u(s))\,ds\right) - F_n(s, 0, 0)\right|\,ds \\
+ |F_n(s, 0, 0)|\,ds \leq \sup_{n \geq 1} \int_0^T |k(t, s)|\left(e^{-t}|h_n(s, u(s))| + e^{-t}|(Gu)(s) \int_0^T g_n(s, u(s))\,ds\right)\,ds \\
+ |F_n(s, 0, 0)|\,ds \leq e^{-t} \sup_{n \geq 1} \int_0^T M|h_n(s, u(s))|\,ds \\
+ Me^{-t} \sup_{n \geq 1} \left|\int_0^T g_n(s, u(s))\,ds\right| + M \sup_{n \geq 1} |F_n(s, 0, 0)| \\
\leq Me^{-2t} \int_0^T \sup\{|u_i(s)|; i \geq n\}ds + Me^{-t}(a + b)\|u\|qT + MM_1T.
\]

Consequently,

\[
\|(Fu)\| \leq Te^{-2t}M\|u\| + Me^{-t}(a + b)\|u\|qT + MM_1T.
\]

We show that \( Fu \) continuously transforms the interval \([0, T]\) to the space \( c_0 \). Let \( t_0 \in [0, T] \) and \( \epsilon > 0 \) be arbitrary. By the continuity \( k(t, s) \), there exists \( \delta > 0 \) such that \( |t - t_0| < \delta \) implies that

\[
|k(t, s) - k(t_0, s)| < \frac{\epsilon}{Te^{-2t}r + e^{-t}(a + br)qT + TM_1}.
\]

Consequently, from (10) and (12) we conclude that \( \|(Fu)(t) - (Fu)(t_0)\|_{c_0} < \epsilon \). Moreover, as a consequence of (11), the function \( F \) maps the ball \( B_{r_0} \) into itself where \( r_0 \) is the existing constant in assumption (A₅) and

\[
B_{r_0} = \{ u \in C(I, c_0); \|u\|_{C(I, c_0)} \leq r, u(0) = 0, u(T) = au(\xi) \}. 
\]
Now, we prove that $F$ is a continuous operator on $\overline{B}_r$. To do this, let us fix $\varepsilon > 0$ and take arbitrary $u, v \in \overline{B}_r$ such that $\|u - v\|_{C(I,c_0)} < \varepsilon$. Then, for $t \in [0, T]$, we have

$$
\|(Fu)(t) - (Fv)(t)\|_{c_0} \leq \sup_{n \geq 1} \int_0^T |k(t, s)| e^{-t} \sup_{i \geq n} |u_i(s) - v_i(s)| ds + e^{-t}|(Gu)(s) - (Gv)(s)| \int_0^T |g_n(s, u(s))| ds
$$

$$
+ e^{-t}|(Gv)(s)| \int_0^T |g_n(s, (u,v)) - g_n(s, v)| ds,
$$

where

$$
\omega_n^T(g, \varepsilon) = \sup\{|g(t, u) - g(t, v)| : t \in [0, T], u, v \in c_0, \|u - v\|_{C(I,c_0)} < \varepsilon\}.
$$

Moreover, in light of the continuity of $g_i$ on $[0, T] \times c_0$, we have $\omega_n^T(g, \varepsilon) \to 0$. By applying this remark and the previous inequality, the continuity of $F$ is followed. Now let $X$ be a nonempty subset of $\overline{B}_r$. In view of the formula (9) and our assumptions, we have

$$
\chi_{c_0}(FX(t)) = \lim_{n \to \infty} \sup_{u \in X} \left\{ \max_{i \geq n} \int_0^T |k(t, s)| e^{-t} |h_i(s, u(s))| ds + e^{-t}|(Gu)(s)| \int_0^T |g_n(s, u(s))| ds \right\}
$$

$$
\leq \lim_{n \to \infty} \sup_{u \in X} \left\{ \max_{i \geq n} \int_0^T |k(t, s)| e^{-t} |h_i(s, u(s))| ds + e^{-2t}|(Gu)(s)| \int_0^T |g_n(s, u(s))| ds + |F_i(0, 0)| ds \right\}
$$

$$
\leq \lim_{n \to \infty} \sup_{u \in X} \left\{ \max_{i \geq n} \int_0^T |k(t, s)| e^{-t} \sup_{i \geq n} |u_i(s)| ds + e^{-t}(a + b||u||_{c_0}) \int_0^T |b_i(s)| ds + \sup_{i \geq n} |F_i(0, 0)| ds \right\}.
$$

Consequently,

$$
\chi_{C(I,c_0)}(FX(t)) \leq TM e^{-2t} \sup_{n \to \infty} \sup_{t \in X} \left\{ \max_{i \geq n} |u_i(t)| \right\}.
$$

As $M, T < 1$, by passing to logarithms, we have

$$
2t + \ln(\chi_{C(I,c_0)}(FX(t)) \leq \chi_X(FX(t)).
$$

Thus all conditions of Theorem 2.3 hold true with $f(t) = \ln(t)$ and $\theta(t) = 2t$ and Theorem 2.3 implies that $F$ has a fixed point in the space $C(I,c_0)$, which is a solution of the system (1) – (2). The proof is complete. □
Thus the operator $G$ satisfies condition (A) with $a = 1, b = 0$. In this example

$$g_n(s, u(s)) = \arctan\left(\frac{e^{-(s+1)2^{-n}}}{8 + \sum_{k=n}^{\infty} |u_k|/(1+k^{10})}\right)$$

verifies condition (A) with $b_n(s) = \frac{c^2}{2^2}$ and $q = \frac{1}{2}$. Inconsequently, the existent inequality in condition (A) has the form

$$e^{-2s}r + e^{-s}r \leq r.$$

Obviously, the last inequality has a positive solution. Thus all conditions of Theorem 4.1 are satisfied and thus the system (13) has at least one solution in the space $C(I, c_0)$. 

---

**Example 4.2.** In order to show the applicability of Theorem 4.1, the fractional differential system

$$D^{\alpha}u(t) = e^{-t-n-1} \sum_{k=2}^{n} \frac{|u_k(t)|}{(k^2 + 1)10^n} + \sqrt{\frac{1}{100} + \frac{1}{1 + \sum_{k=2}^{n} \frac{|u_k(t)|}{(1+k^{10})}}} \int_0^t g_n(s, u(s))ds$$

$$u_n(0) = 0, \quad u_n(T) = \frac{1}{2} \ln\left(\frac{7}{3}\right), \quad n = 1, 2, 3, \ldots$$

is included, where

$$F_n(t, x, y) = e^{-t-n-1} \sqrt{x_1 + \sqrt{y_1}} + \sqrt{x_2 + \sqrt{y_2}}, \quad h_n(t, u(t)) = e^{-t-n-1} \sum_{k=2}^{n} \frac{|u_k(t)|}{(k^2 + 1)10^n}$$

$$(Gx)(t) = \frac{1}{100} \times \frac{1}{1 + \sum_{k=2}^{n} \frac{|u_k(t)|}{(1+k^{10})}}, \quad g_n(s, u(s)) = \arctan\left(\frac{e^{s+2^{-n}}}{8 + \sum_{k=n}^{\infty} |u_k|/(1+k^{10})}\right).$$

For all $x_1, x_2, y_1, y_2 \in \mathbb{R}, u(t) \in c_0$ and $t \in [0, T]$, we have

$$|F_n(t, x_1, y_1) - F_n(t, x_2, y_2)| \leq e^{-t-n-1} \left|\sqrt{x_1 + \sqrt{y_1}} - \sqrt{x_2 + \sqrt{y_2}}\right|$$

$$\leq e^{-t} \left|\sqrt{x_1 + \sqrt{y_1}} - \sqrt{x_2 + \sqrt{y_2}}\right|$$

$$\leq e^{-t} \left|x_1 - x_2 + \sqrt{y_1} - \sqrt{y_2}\right|$$

$$\leq e^{-t} \left|x_1 - x_2 + |y_1 - y_2|\right|.$$
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