Linear differential systems over the quaternion skew field.

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Abstract

A basic theory on the first order right and left linear quaternion differential systems (LQDS) is given systematic in this paper. To proceed the theory of LQDS we adopt the theory of column-row determinants recently introduced by the author. In this paper, the algebraic structure of their general solutions are established. Determinantal representations of solutions of systems with constant coefficient matrices and sources vectors are obtained in both cases when coefficient matrices are invertible and singular. In the last case, we use determinantal representations of the quaternion Drazin inverse within the framework of the theory of column-row determinants.

Numerical examples to illustrate the main results are given.

Keywords Linear quaternion differential equation; Linear quaternion differential system; Quaternion matrix; Drazin inverse; Cramer rule; Noncommutative determinant; Column determinant; Row determinant.

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1 Introduction

Recently, considerable attention has been paid to the quaternion differential equations with real variables (QDEs) which have many applications in fluid mechanics (e.g. [1,2]), quantum mechanics (see e.g. [3,4]), Frenet frame in differential geometry [5], the attitude orientation and spatial rigid body dynamics [6,7], etc. Though QDEs have many applications, theoretical aspects of QDEs has been considered in few papers.

Leo and Ducati [8] solved some simple second order quaternionic differential equations. Campos and Mawhin [9] studied the existence of periodic solutions for the quaternionic Riccati equation.

Żołdek [10] given the complete description of dynamics of the Riccati equation. Wilczynski proved the existence of two periodic solutions of quaternionic Riccati equations in [11], and considered some sufficient conditions for the existence of at least one periodic solution of the quaternionic polynomial equations in [12,13]. Gasull et al. [14] proved the existence of periodic orbits, homoclinic loops, invariant tori for a one-dimensional quaternionic autonomous homogeneous differential equation. Zhang [15] studied the global structure of the quaternion Bernoulli equations. Recently, Cai and Kou [16] achieved the Laplace transform approach to solve the linear quaternion differential equations.

Even fewer papers are devoted to systems of the linear quaternion differential equations. We note the three papers from the arXiv [17,18], where the authors
studied the basic theory of LQDS such as the fundamental matrix, the algebraic structure of solutions, etc. Through the non-commutativity of the quaternion algebra, the construction of the basic theory of linear systems of quaternion differential equations has much more complicated in comparing to usual linear systems. Difficulties arise already in determining the quaternion determinant.

It’s well-known that there are several approaches to the definition of a determinant of matrices with noncommutative entries (which are also defined as noncommutative determinants). The first approach is an axiomatic defining. Let \( M(n, R) \) be the ring of \( n \times n \) matrices with entries in a ring \( R \).

**Definition 1.1.** [21][22] Let a functional \( d : M(n, R) \to R \) satisfy the following three axioms.

1. \( d(A) = 0 \) if and only if the matrix \( A \) is singular.
2. \( d(AB) = d(A) \cdot d(B) \) for \( \forall B \in M(n, R) \).
3. If the matrix \( A' \) is obtained from \( A \) by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then \( d(A') = d(A) \).

Then \( d \) is called the determinant of \( A \in M(n, R) \).

But it turns out [21], if a determinant functional satisfies Axioms 1, 2, 3, then it takes on a value in a commutative subset of the ring. The famous examples of such determinant are the determinants of Diedonné [23] and Study [24].

Another way of defining is constructive. A noncommutative determinant is constructed by similar to the usual determinant as the alternative sum of \( n! \) products of entries of a matrix but by specifying a certain ordering of coefficients in each term. The Caley determinant [25] has the such type but without success in the implementation of any of the axioms. Moore [26] was the first who achieved the fulfillment of the main Axiom 1 by such definition of a noncommutative determinant. This is done not for all square matrices over a ring but rather only Hermitian matrices. Later, Dyson [27] gave some natural generalizations, described the theory in more modern terms, and represented Moore’s determinant in terms of permutations as follows,

\[
\text{Mdet} \ A = \sum_{\sigma \in S_n} |\sigma| a_{n_{11}} a_{n_{12}} \cdots a_{n_{1l}} a_{n_{21}} a_{n_{22}} \cdots a_{n_{r1}} a_{n_{r1}} \\
\]

The disjoint cycle representation of the permutation \( \sigma \in S_n \) is written in the normal form,

\[
\sigma = (n_{11} \ldots n_{1l_1})(n_{21} \ldots n_{2l_2})\cdots(n_{r1} \ldots n_{rl_r}),
\]

where \( n_{ij} < n_{jm} \) for all \( i = 1, \ldots, r \) and \( m > 1 \), and \( n_{11} > n_{21} > \ldots > n_{r1} \). Dyson has emphasized to the need for expansion of the definition of Moore’s
determinant to arbitrary square matrices. Chen has offered the following decision of this problem in [28]. He has defined the determinant of a square matrix \( A = (a_{ij}) \in M(n, \mathbb{H}) \) over the quaternion skew field \( \mathbb{H} \) by putting,

\[
\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{n_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_s n_1} \cdot \ldots \cdot a_{n_r k_2} \cdot \ldots \cdot a_{k_l n_r},
\]

\( \sigma = (n_1 i_2 \ldots i_s) \ldots (n_r k_2 \ldots k_l), \)

\( n_1 > i_2, i_3, \ldots, i_s; \ldots; n_r > k_2, k_3, \ldots, k_l, \)

\( n = n_1 > n_2 > \ldots > n_r \geq 1. \)

Chen’s determinant does not satisfy Axiom 1 and cannot be expanded by cofactors along an arbitrary row or column with the exception of the \( n \)th row. Through by using this determinant, a determinantal representation of an inverse matrix over the quaternion skew field has been obtained.

Returning to the main topic, we note that in [17–19] it has been used the determinants of Caley and Chen, and the double determinant is used that is actually the determinant of Study. In [20], the same authors has already used the determinant of the complex adjoint matrix \( \chi_A \) to a quaternion matrix \( A \). This indicates the complexity of the successful choice of a quaternion determinant.

In this paper we explore linear systems of quaternion differential equations applying the theory of column-row determinants introduced by the author in [29,30]. By this theory, for a quaternion square matrix of order \( n \) is defining \( n \) column determinants and \( n \) row determinants that give the complete expansion of Moore’s determinant from Hermitian to arbitrary square matrices. Currently, the theory of column-row determinants is active developing. Within the framework of column-row determinants, determinantal representations of various kind of generalized inverses and (generalized inverses) solutions of quaternion matrix equations recently have been derived as by the author (see, e.g. [31–35]) so by other researchers (see, e.g. [36–39]).

Throughout the paper, we denote the real number field by \( \mathbb{R} \), the set of all \( m \times n \) matrices over the quaternion algebra

\[
\mathbb{H} = \{ a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \}
\]

by \( \mathbb{H}^{m \times n} \), the identity matrix with the appropriate size by \( \mathbf{I} \). Let \( M(n, \mathbb{H}) \) be the ring of \( n \times n \) quaternion matrices. For \( A \in \mathbb{H}^{n \times n} \), the symbols \( A^* \) stands for the conjugate transpose (Hermitian adjoint) matrix of \( A \). The matrix \( A = (a_{ij}) \in \mathbb{H}^{n \times n} \) is Hermitian if \( A^* = A \).

The main goals of the paper are to give a systematic basic theory on linear systems of quaternion differential equations, establish the algebraic structure of their general solutions, provide algorithms for finding their solutions.

The paper is organized as follows. We start with some basic concepts and also some new results from the theories of quaternion-valued differential equations, row and column determinants, the theory on quaternion vector spaces and eigenvalues of quaternion matrices, and give an algorithm obtaining eigenvalues of quaternion normal matrices in Section 2. In Section 3, we consider first order right and left linear quaternion differential systems, establish the algebraic
structure of their general solutions and, in particular, for systems with constant coefficients, and give determinantal representations of solutions of systems with constant coefficient matrices and sources vectors when coefficient matrices are invertible or singular. In Sections 2 and 3, we show numerical examples to illustrate the main results.

2 Preliminaries.

2.1 Background for quaternion-valued differential equations (QDE)

Consider a quaternion-valued function of real variable, \( f : \mathbb{R} \to \mathbb{H} \) (\( t \in \mathbb{R} \) is a real variable), such that \( f(t) = f_0(t) + f_1(t)i + f_2(t)j + f_3(t)k \). The first derivative of a quaternionic function \( f(t) \) with respect to the real variable \( t \) denote by,

\[
\frac{df(t)}{dt} = \frac{df_0(t)}{dt} + \frac{df_1(t)}{dt}i + \frac{df_2(t)}{dt}j + \frac{df_3(t)}{dt}k.
\]

It is easy to prove the following proposition on properties of the derivative of quaternionic functions.

**Proposition 2.1.** If \( q, r : \mathbb{R} \to \mathbb{H} \) and \( s : \mathbb{R} \to \mathbb{H} \) are differentiable, then \((q \pm r)(t), qr(t)\) and, for any integer \( n \geq 1 \), \( q^n \) are differentiable, and

\[
(q \pm r)'(t) = q'(t) \pm r'(t),
\]

\[
(qr)'(t) = q'(t)r(t) + q(t)r'(t),
\]

\[
[q^n(t)]' = \sum_{j=0}^{n-1} q'(t)q'(t)q^{n-j}(t).
\]

We need the exponential of \( q \in \mathbb{H} \) that can be defined by putting,

\[
e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}.
\]

From the definition, we evidently have the following properties.

**Proposition 2.2.**

1. If \( q, r \in \mathbb{H} \) are such that \( qr = rq \), then \( e^{qr} = e^q + e^r \).

2. If \( q : \mathbb{R} \to \mathbb{H} \) is differentiable and \( q'(t)q(t) = q(t)q'(t) \), then

\[
[e^q(t)]' = [e^q(t)]q'(t).
\]

Consider the following statement that we shall need below.

**Proposition 2.3.** Let \( q_l, l = 1, 2, 3 \), is one from the fundamental quaternion units, i.e. \( q_l \in \{i, j, k\} \) for all \( l = 1, 2, 3 \). Then

\[
e^{q_l}q_m = \begin{cases} q_m e^{q_l}, & \text{if } m = l, \\ q_m e^{-q_l}, & \text{if } m \neq l, \end{cases}
\]

where \( m, l \in \{1, 2, 3\} \).
Proof. If \( m = l \), let \( q_m = i \), then evidently by Definition 1, \( i \) and \( e^i \) are the commuting elements. The same is true for the other two quaternion units, \( j \) and \( k \).

If \( m \neq l \), we put \( q_m = j \), \( q_l = i \). Then

\[
e^i j = \sum_n \frac{j^n}{n!} j = \left\{ \begin{array}{l} \sum_k \frac{(-1)^k}{(2k+1)!} j, \\
\sum_k \frac{j}{(2k+1)!} j,
\end{array} \right.

= \left\{ \begin{array}{l} j \sum_k \frac{(-1)^k}{(2k+1)!}, \\
j \sum_k \frac{j}{(2k+1)!},
\end{array} \right.

= j \sum_n \frac{(-1)^n i^n}{n!} = je^{-i}.
\]

Similarly, we obtain for the all other combinations of the quaternion units. □

If \( f_i(t) \) for all \( i = 0, \ldots, 3 \) is integrable on \([a, b] \subset \mathbb{R}\), then \( f(t) \) is integrable as well, and

\[
\int_a^b f(t) dt = \int_a^b f_0(t) dt + \int_a^b f_1(t) dt i + \int_a^b f_2(t) dt j + \int_a^b f_3(t) dt k.
\]

In [7], the linear quaternion differential equations,

\[
q'(t) = a(t)q(t),
\]

and

\[
q'(t) = q(t)a(t),
\]

with the initial condition \( q(t_0) = q_0 \) have been considered and the following proposition has been derived.

**Proposition 2.4.** Let \( q(t) = \Phi_l(t)q_0 \) and \( q(t) = q_0\Phi_r(t) \) be solutions of (2) and (3), respectively. If

\[
a(t) \int_{t_0}^t a(\tau) d\tau = \int_{t_0}^t a(\tau) d\tau \ a(t),
\]

then

\[
\Phi_l(t) = \Phi_r(t) = e^{\int_{t_0}^t a(\tau) d\tau}.
\]

If \( a \) is constant, then \( \int_{t_0}^t a d\tau = a(t - t_0) \), and \( \Phi_l(t) = \Phi_r(t) = e^{a(t-t_0)} \).

The similar result has been obtained in [9] as well. Moreover, in [9], the following nonhomogeneous differential equations corresponding to (2) has been considered,

\[
q'(t) = a(t)q(t) + f(t),
\]

where \( f : [0, T] \to \mathbb{H} \) and \( a : [0, T] \to \mathbb{H} \). It has been shown if the condition (4) is satisfied, then the solutions of (6) are given by

\[
q(t) = e^{\int_{t_0}^t a(\tau) d\tau} \left( q(0) + \int_0^t e^{\int_s^t (-a(\tau)) d\tau} f(s) ds \right), \quad (t \in [0, T]).
\]
In the special case when \( a \) is constant and \( q(0) = 0 \), then the solutions of (\( \text{[4]} \)) are given by

\[
q(t) = e^{at} \left( \int_0^t e^{-as} f(s) ds \right), \quad (t \in [0,T]).
\]

**Remark 2.1.** The condition (\( \text{[4]} \)) too narrow the set of solvable linear quaternion differential equations. So, we propose to consider the two different chain rules for computing the derivative of the composition of two or more quaternion-valued functions, namely, left and right chain rules as follows.

**Definition 2.1.** Let \( q_i \) are quaternion-valued differentiable functions for all \( i = 1, \ldots, n \) and \( Q(t) = q_1 (\ldots (q_n) \ldots) (t) \). Then the left and right chain rules are defined by putting respectively in the Lagrange notation,

\[
Q'_l(t) = q_{i_1} \ldots q_{i_n}(t) = q_1 (q_2 \ldots q_{n-1}(t)) \cdot q_n(t) \cdots q_n(t) (t),
\]

\[
Q'_r(t) = q_{n_{i,1}} \ldots q_{n_{i,n-1}}(t) = q_1 q_2 \ldots q_{n-1}(t) \cdot q_n(t) (t) \cdots q_n(t) (t).
\]

By Definition 2.1 considering the left linear equation (\( \text{[2]} \)) due only to the left chain rule and the right linear equation (\( \text{[3]} \)) through the right chain rule, we respectively obtain (\( \text{[5]} \)) without a need to perform the condition (\( \text{[4]} \)). We also have change the property of the derivative of the quaternion exponent in Proposition 2.2 such that

\[
\left[ e^{q(t)} \right]' = q'(t) \left[ e^{q(t)} \right], \quad \left[ e^{q(t)} \right]' = e^{q(t)} q'(t),
\]

without a need to commutativity \( q(t) \) and \( q'(t) \).

### 2.2 Elements of the theory of column and row determinants

Suppose \( S_n \) is the symmetric group on the set \( I_n = \{1, \ldots, n\} \). For \( A = (a_{ij}) \in M(n, \mathbb{H}) \) we define \( n \) row determinants and \( n \) column determinants as follows.

**Definition 2.2.** [29] The \( i \)th row determinant of \( A = (a_{ij}) \in M(n, \mathbb{H}) \) is defined for all \( i = 1, \ldots, n \) by putting

\[
r_{d} \sigma \cdot A = \sum_{\sigma \in S_n} (-1)^{n-\tau} (a_{i_i k_1} a_{i_{k_1+1} i_{k_2}} \ldots a_{i_{k_{t-1}} i_{k_{t+1}}} \ldots a_{i_{k_{t+r-1}} i_{k_{r+1}}}),
\]

where \( \sigma = (i_{k_1} i_{k_2+1} \ldots i_{k_{t]]) \ldots i_{k_r} i_{k_1+1} \ldots \), with conditions \( i_{k_2} < i_{k_3} < \ldots < i_{k_r} \) and \( i_{k_s} < i_{k_{t+s}} \) for \( t = 2, \ldots, r \) and \( s = 1, \ldots, i_{l_t} \).

**Definition 2.3.** [29] The \( j \)th column determinant of \( A = (a_{ij}) \in M(n, \mathbb{H}) \) is defined for all \( j = 1, \ldots, n \) by putting

\[
c_{d} \tau \cdot A = \sum_{\tau \in S_n} (-1)^{n-\tau} (a_{j_{k_1+1} i_{k_2} \ldots a_{j_{k_r+1} i_{k_{r+1}}}} \ldots a_{j_{k_{t-1}+1} i_{k_{t}}},
\]

where \( \tau = (j_{k_1+1} \ldots j_{k_r} \ldots) \ldots (j_{k_{t-1}+1} i_{k_{t}}) \ldots (j_{k_{t-1}+1} j_{k_{t}+1}).
\]
with conditions, $j_{k_2} < j_{k_3} < \ldots < j_{k_c}$ and $j_{k_t} < j_{k_t+s}$ for $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

Suppose $A^{ij}$ denotes the submatrix of $A$ obtained by deleting both the $i$th row and the $j$th column. Let $a_{ij}$ be the $j$th column and $a_{ij}$ be the $i$th row of $A$. Suppose $A_{ij}(b)$ denotes the matrix obtained from $A$ by replacing its $j$th column with the column $b$, and $A_{ij}(b)$ denotes the matrix obtained from $A$ by replacing its $i$th row with the row $b$.

The following theorem has a key value in the theory of the column and row determinants.

**Theorem 2.1.** If $A = (a_{ij}) \in M(n, \mathbb{H})$ is Hermitian, then $rdet_A = \cdots = rdet_n A = \cdots = cdet_1 A = \cdots = cdet_n A \in \mathbb{R}$.

Due to Theorem 2.1 we can define the determinant of a Hermitian matrix $A \in M(n, \mathbb{H})$. By definition, we put $\det A := rdet_A = cdet_A$ for all $i = 1, \ldots, n$. The determinant of a Hermitian matrix has properties similar to a usual determinant. They are completely explored in [29, 30] by its row and column determinants and can be summarized by the following theorems.

**Theorem 2.2.** If the $i$th row of a Hermitian matrix $A \in M(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $a_{ij} = c_1 a_{i1} + \cdots + c_k a_{ik}$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{i, i_l\} \subset I_n$, then

$$rdet_A_i \cdot (c_1 a_{i1} + \cdots + c_k a_{ik}) = cdet_A_i \cdot (c_1 a_{i1} + \cdots + c_k a_{ik}) = 0.$$  

**Theorem 2.3.** If the $j$th column of a Hermitian matrix $A \in M(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $a_{ij} = a_{ij} c_1 + \cdots + a_{ij} c_k$, where $c_l \in \mathbb{H}$ for all $l = 1, \ldots, k$ and $\{j, j_l\} \subset J_n$, then

$$cdet_A_{ij} \cdot (a_{ij} c_1 + \cdots + a_{ij} c_k) = rdet_A_{ij} \cdot (a_{ij} c_1 + \cdots + a_{ij} c_k) = 0.$$  

The determinant of a Hermitian matrix also has a property of expansion along arbitrary rows and columns using row and column determinants of submatrices. So, we were able to get determinantal representations of an inverse as follows.

**Theorem 2.4.** If $A \in M(n, \mathbb{H})$ is Hermitian and $\det A \neq 0$, then there exist its unique right $(RA)^{-1}$ and unique left inverse $(LA)^{-1}$, where $(RA)^{-1} = (LA)^{-1} = A^{-1}$, and they possess the following determinantal representations,

$$(RA)^{-1} = \frac{1}{\det A} \begin{bmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\ R_{12} & R_{22} & \cdots & R_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{bmatrix},$$

$$(LA)^{-1} = \frac{1}{\det A} \begin{bmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\ L_{12} & L_{22} & \cdots & L_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ L_{1n} & L_{2n} & \cdots & L_{nn} \end{bmatrix},$$
where \( \text{det} \mathbf{A} = \sum_{j=1}^{n} a_{ij} \cdot R_{ij} = \sum_{i=1}^{n} L_{ij} \cdot a_{ij} \),

\[
R_{ij} = \begin{cases} 
- \text{rdet} \mathbf{A}^{ij}(\mathbf{a}_i) , & i \neq j; \\
\text{rdet} \mathbf{A}_i^i , & i = j,
\end{cases} \\
L_{ij} = \begin{cases} 
- \text{cdet} \mathbf{A}^{ji}(\mathbf{a}_j) , & i \neq j; \\
\text{cdet} \mathbf{A}_j^j , & i = j,
\end{cases}
\]

and \( \mathbf{A}^{ij}(\mathbf{a}_i) \) is obtained from \( \mathbf{A} \) by both replacing the \( j \)th column with the \( i \)th column and deleting the \( i \)th row and column, \( \mathbf{A}_i^i(\mathbf{a}_j) \) is obtained by replacing the \( i \)th row with the \( j \)th row, and then by deleting both the \( j \)th row and column, respectively, \( I_n = \{1, \ldots, n\} \), \( k = \min\{I_n \setminus \{i\}\} \) for all \( i, j = 1, \ldots, n \).

**Theorem 2.5.** [29] The right linearly dependence of columns of \( \mathbf{A} \in \mathbb{H}^{m \times n} \) or the left linearly dependence of rows of \( \mathbf{A}^* \) is the necessary and sufficient condition for \( \text{det} \mathbf{A}^* \mathbf{A} = 0 \).

**Theorem 2.6.** If \( \mathbf{A} \in \mathbb{M}(n, \mathbb{H}) \), then \( \text{det} \mathbf{AA}^* = \text{det} \mathbf{A}^* \mathbf{A} \).

**Definition 2.4.** For \( \mathbf{A} \in \mathbb{M}(n, \mathbb{H}) \), the double determinant of \( \mathbf{A} \) is defined by putting, \( \text{ddet} \mathbf{A} := \text{det} \mathbf{AA}^* = \text{det} \mathbf{A}^* \mathbf{A} \).

**Theorem 2.7.** [30] The necessary and sufficient condition of invertibility of \( \mathbf{A} \in \mathbb{M}(n, \mathbb{H}) \) is \( \text{ddet} \mathbf{A} \neq 0 \). Then there exists \( \mathbf{A}^{-1} = (\mathbf{L} \mathbf{A})^{-1} = (\mathbf{R} \mathbf{A})^{-1} \), where

\[
(\mathbf{L} \mathbf{A})^{-1} = (\mathbf{A}^* \mathbf{A})^{-1} = \frac{1}{\text{ddet} \mathbf{A}} \begin{bmatrix} L_{11} & L_{21} & \cdots & L_{n1} \\
L_{12} & L_{22} & \cdots & L_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
L_{1n} & L_{2n} & \cdots & L_{nn} \end{bmatrix}
\]

\[
(\mathbf{R} \mathbf{A})^{-1} = \mathbf{A}^* (\mathbf{A} \mathbf{A}^*)^{-1} = \frac{1}{\text{ddet} \mathbf{A}^*} \begin{bmatrix} R_{11} & R_{21} & \cdots & R_{n1} \\
R_{12} & R_{22} & \cdots & R_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
R_{1n} & R_{2n} & \cdots & R_{nn} \end{bmatrix}
\]

and

\[
\mathbf{L}_{ij} = \text{cdet} \mathbf{A}^* j (\mathbf{a}_i^*), \quad \mathbf{R}_{ij} = \text{rdet} \mathbf{A} \mathbf{A}^* i (\mathbf{a}_j^*),
\]

for all \( i, j = 1, \ldots, n \).

Moreover, the following criterion of invertibility of a quaternion matrix can be obtained.

**Theorem 2.8.** [30] If \( \mathbf{A} \in \mathbb{M}(n, \mathbb{H}) \), then the following statements are equivalent.

i) \( \mathbf{A} \) is invertible, i.e. \( \mathbf{A} \in \mathbb{GL}(n, \mathbb{H}) \);

ii) rows of \( \mathbf{A} \) are left-linearly independent;

iii) columns of \( \mathbf{A} \) are right-linearly independent;

iv) \( \text{ddet} \mathbf{A} \neq 0 \).
Due to Theorems 2.4 and 2.7, we evidently obtain the following Cramer rules.

**Theorem 2.9.** \[30\] Let
\[ A \cdot x = b \quad (7) \]
be a right system of linear equations with a matrix of coefficients \( A \in M(n, \mathbb{H}) \), a column of constants \( b = (b_1, \ldots, b_n) \in \mathbb{H}^{n \times 1} \), and a column of unknowns \( x = (x_1, \ldots, x_n)^T \).

(i) If \( A \) is Hermitian and \( \det A \neq 0 \), then the solution to the linear system \( (7) \) is given by components
\[ x_j = \frac{\text{cdet}_j A \cdot (b)}{\det A}, \]
where \( \text{cdet}_j A \) denotes the component of the determinant of \( A \).

(ii) If \( A \) is an arbitrary matrix and \( \text{ddet} A \neq 0 \), then the solution to the linear system \( (7) \) is given by components
\[ x_j = \frac{\text{cdet}_j (A^* A) \cdot (f)}{\text{ddet} A}, \quad j = 1, \ldots, n, \]
where \( f = A^* b \).

**Theorem 2.10.** \[30\] Let
\[ x \cdot A = b \quad (8) \]
be a left system of linear equations with a matrix of coefficients \( A \in M(n, \mathbb{H}) \), a row of constants \( b = (b_1, \ldots, b_n) \in \mathbb{H}^{1 \times n} \), and a row of unknowns \( x = (x_1, \ldots, x_n) \).

(i) If \( A \) is Hermitian and \( \det A \neq 0 \), then the solution to \( (8) \) is given by components
\[ x_i = \frac{\text{rdet}_i A \cdot (b)}{\det A}. \]

(ii) If \( \text{ddet} A \neq 0 \), then the solution to the linear system \( (8) \) is given by components
\[ x_i = \frac{\text{rdet}_i (A A^*) \cdot (z)}{\text{ddet} A}, \quad i = 1, \ldots, n, \]
where \( z = b A^* \).

For determinantal representations of quaternion generalized inverses, we are needed by the following notations. Let \( \alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\} \) and \( \beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\} \) be subsets of the order \( 1 \leq k \leq \min \{m, n\} \). By \( A_{\alpha}^{\beta} \) denote the submatrix of \( A \) determined by the rows indexed by \( \beta \) and the columns indexed by \( \alpha \). Then \( A_{\alpha}^{\beta} \) denotes the principal submatrix determined by the rows and columns indexed by \( \alpha \). If \( A \in M(n, \mathbb{H}) \) is Hermitian, then by \( |A_{\alpha}^{\beta}| \) denote the corresponding principal minor of \( \det A \). For \( 1 \leq k \leq n \), the collection of strictly increasing sequences of \( k \) integers chosen from \( \{1, \ldots, n\} \) is denoted |
by \( L_{k,n} := \{ \alpha : \alpha = (\alpha_1, \ldots, \alpha_k), 1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq n \} \). For fixed \( i \in \alpha \) and \( j \in \beta \), let \( I_{r,m} \{ i \} := \{ \alpha : \alpha \in L_{r,m}, i \in \alpha \} \), \( J_{r,n} \{ j \} := \{ \beta : \beta \in L_{r,n}, j \in \beta \} \).

The Drazin inverse for an arbitrary square matrix \( A \) over \( \mathbb{H} \) is defined to be the unique matrix \( X \) that satisfying the following equations \[ 40 \],

\[
XAX = X; \ AX = XA; A^{k+1}X = A^k.
\]

where \( k = \text{Ind} A \) is the smallest positive number such that \( \text{rank} A^{k+1} = \text{rank} A^k \). It is denoted by \( X = A^D \).

Denote by \( a_{ij}^{(m)} \) and \( a_{ij}^{(n)} \) the \( j \)th column and the \( i \)th row of \( A^m \), and by \( \hat{a}_s \), the \( s \)th column of \( (A^{2k+1})^*A^k =: \hat{A} = (\hat{a}_{ij}) \in \mathbb{H}^{n \times n} \) and the \( t \)th row of \( A^k(A^{2k+1})^* =: \check{A} = (\check{a}_{ij}) \in \mathbb{H}^{n \times n} \), respectively, for all \( s, t = 1, \ldots, n \).

**Theorem 2.11.** \[ 32 \] If \( A \in M(n, \mathbb{H}) \) with \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \), then the Drazin inverse \( A^D \) possess the determinantal representations,

\[
a_{ij}^D = \sum_{s=1}^n \left( \sum_{a \in I_{r,n}(s)} \text{rdet}_a \left( (A^{2k+1})^* (A^{2k+1})_{a,t} (\hat{a}_{ij}) \right) \right) a_{ij}^{(k)}
\]

and

\[
a_{ij}^D = \sum_{s=1}^n \left( \sum_{\alpha \in I_{r,n}(s)} \text{rdet}_s \left( (A^{2k+1}) (A^{2k+1})^*_{s,i} (\check{a}_{ij}) \right) \right) a_{ij}^{(k)}
\]

In the special case, when \( A \in M(n, \mathbb{H}) \) is Hermitian, we can obtain simpler determinantal representations of the Drazin inverse.

**Theorem 2.12.** \[ 32 \] If \( A \in M(n, \mathbb{H}) \) is Hermitian with \( \text{Ind} A = k \) and \( \text{rank} A^{k+1} = \text{rank} A^k = r \), then the Drazin inverse \( A^D = (a_{ij}^D) \in \mathbb{H}^{n \times n} \) possess the following determinantal representations,

\[
a_{ij}^D = \frac{\sum_{\beta \in J_{r,n} \{ i \}} \text{cdet}_t \left( (A^{k+1})_{a,j} (A^k) \right) \beta}{\sum_{\beta \in J_{r,n} \{ i \}} \left| (A^{k+1}) \beta \right|}
\]

or

\[
a_{ij}^D = \frac{\sum_{\alpha \in I_{r,n} \{ j \}} \text{rdet}_j \left( (A^{k+1}) \alpha (A^k) \right) \alpha}{\sum_{\alpha \in I_{r,n} \{ j \}} \left| (A^{k+1}) \alpha \right|}
\]
2.3 Some provisions of quaternion matrices and pre-Hilbert spaces.

Due to quaternion-scalar multiplying on the right, quaternionic column-vectors form a right vector $\mathbb{H}$-space, and, by quaternion-scalar multiplying on the left, quaternionic row-vectors form a left vector $\mathbb{H}$-space. Moreover, we define right and left quaternionic vector spaces, denoted by $\mathcal{H}_r$ and $\mathcal{H}_l$, respectively, with corresponding $\mathbb{H}$-valued inner products $\langle \cdot , \cdot \rangle$ which satisfy, for every $\alpha, \beta \in \mathbb{H}$, and $x, y, z \in \mathcal{H}_r(\mathcal{H}_l)$, the relations:

1. $\langle x, y \rangle = \langle y, x \rangle$;
2. $\langle x, x \rangle \geq 0 \in \mathbb{R}$ and $\|x\|^2 := \langle x, x \rangle = 0 \iff x = 0$;
3. $\langle \alpha x + y \beta, z \rangle = \langle x, z \rangle \alpha + \langle y, z \rangle \beta$ when $x, y, z \in \mathcal{H}_r$;
4. $\langle x, y \alpha + z \beta \rangle = \overline{\langle x, y \rangle} + \overline{\beta} \langle x, z \rangle$ when $x, y, z \in \mathcal{H}_r$;

It can be achieved by putting $\langle x, y \rangle_r = \overline{y}_1 x_1 + \cdots + \overline{y}_n x_n$ for $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathcal{H}_r$, and $\langle x, y \rangle_l = x_1 \overline{y}_1 + \cdots + x_n \overline{y}_n$ for $x, y \in \mathcal{H}_l$.

The right vector spaces $\mathcal{H}_r$ possess the Gram-Schmidt process which takes a nonorthogonal set of linearly independent vectors $S = \{v_1, \ldots, v_k\}$ for $k \leq n$ and constructs an orthogonal (or orthonormal) basis $S' = \{u_1, \ldots, u_k\}$ that spans the same $k$-dimensional subspace of $\mathcal{H}_r$ as $S$. To $\mathcal{H}_r$, the following projection operator is defined by

$$\text{proj}_{u}^r(v) := \frac{\langle u, v \rangle_r}{\langle u, u \rangle_r},$$

which orthogonally projects the vector $v$ onto the line spanned by the vector $u$. Then, the Gram-Schmidt process works as follows,

$$u_1 = v_1, \quad e_1 = \frac{u_1}{\|u_1\|},
$$
$$u_k = v_k - \sum_{j=1}^{k-1} \text{proj}_{u_j}^r(v_k), \quad e_k = \frac{u_k}{\|u_k\|}.$$

The sequence $u_1, \ldots, u_k$ is the required system of orthogonal vectors, and the normalized vectors $e_1, \ldots, e_k$ form an orthonormal set.

The Gram-Schmidt process for the left vector spaces $\mathcal{H}_l$ can be realized by the same algorithm but with the projection operator

$$\text{proj}_{u}^l(v) := \frac{\langle u, v \rangle_l}{\langle u, u \rangle_l}.$$
Clear, that columns of $U$ form a system of normalized vectors in $H_r$, rows of $U^*$ is a system of normalized vectors in $H_l$.

The vector norms $\|x\|_r = \sqrt{\langle x, x \rangle_r}$ and $\|x\|_l = \sqrt{\langle x, x \rangle_l}$ on $H_r$ and $H_l$, respectively, define the corresponding induced matrix norms on the space $\mathbb{H}^{n \times n}$ of all $n \times n$ matrices as follows:

$$\|A\|_r = \sup\{\|Ax\|_r : x \in \mathbb{H}^{n \times 1}, \|x\|_r = 1\},$$
$$\|A\|_l = \sup\{\|xA\|_l : x \in \mathbb{H}^{1 \times n}, \|x\|_l = 1\}.$$

Since $\|x\|_r = \|x^T\|_l$, then $\|A\|_r = \|A\|_l$ for any $A \in \mathbb{H}^{n \times n}$. Moreover, we define the norm for any $A \in \mathbb{H}^{n \times n}$ by putting, $\|A\| : = \|A\|_r = \|A\|_l$.

**Remark 2.2.** In fact $H_r$ and $H_l$ are the right and left quaternion pre-Hilbert spaces, respectively. By introducing their completeness along the every fundamental quaternion unit we can define the right and left quaternion Banach spaces.

We define the exponential of a quaternionic square matrix as well.

**Definition 2.6.** The exponential of a square matrix $A \in \mathbb{H}^{n \times n}$ is the infinite sum

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$ (13)

Since for any $k \in \mathbb{N}$,

$$\|I + \frac{1}{k!}A^2 + \cdots + \frac{1}{k!}A^k\| \leq \|I\| + \|A\| + \frac{1}{2!}\|A^2\| + \cdots + \frac{1}{k!}\|A^k\| \leq e^{\|A\|},$$

then the series (13) is converge absolutely.

From Definition 2.6, the following properties of the quaternionic matrix exponential evidently follow.

**Theorem 2.13.** Let $A \in \mathbb{H}^{n \times n}$.

(i) If $0$ is the $n \times n$ zero matrix and $I$ is the $n \times n$ unit matrix, then $e^0 = I$ and $e^I = eI$.

(ii) $e^A = \lim_{k \to \infty} \left(I + \frac{1}{k!}A\right)^k$.

(iii) $(e^A)^T = e^{(A^T)}$, $(e^A)^* = e^{(A^*)}$.

(iv) For all nonnegative integers $k$ holds $A^k e^A = e^A A^k$.

(v) If $AB = BA$, then $A e^B = e^B A$ and $e^A e^B = e^B e^A = e^{A+B}$.

(vi) If $s, t \in \mathbb{H}$, then $e^{As} e^{At} = e^{A(s+t)}$.

(vii) If $A$ is invertible, then $(e^A)^{-1} = e^{(A^{-1})}$.

(viii) If $D = \text{diag}[d_1, \ldots, d_n]$, where $d_i \in \mathbb{H}$ for all $i = 1, \ldots, n$, then $e^D = \text{diag}[e^{d_1}, \ldots, e^{d_n}]$. 12
(ix) If $A$ is diagonalizable with an invertible matrix $P$ and a diagonal matrix $D$ satisfying $A = PD P^{-1}$, then $e^A = Pe^D P^{-1}$.

(x) If $A$ is Hermitian, then $\det e^A = \text{tr} A$.

Proof. The proofs of the statements (i)-(ix) can be easily expanded from real matrix to quaternion matrices.

We only prove the point (x). Since $A$ is Hermitian, then $e^A$ is Hermitian as well, and we can define $\det e^A$. By statement (ii) of this theorem,

$$\det e^A = \det \left( \lim_{k \to \infty} \left( I + \frac{1}{k} A \right)^k \right) = \lim_{k \to \infty} \det \left( I + \frac{1}{k} A \right)^k .$$

Using Definition 2.7, Theorem 2.18 and properties of limits, we obtain

$$\det e^A = \lim_{k \to \infty} \left( 1 + \frac{\text{tr} A}{k} + O(k^{-2}) \right)^k = \lim_{k \to \infty} \left( 1 + \frac{\text{tr} A}{k} \right)^k = e^{\text{tr} A} .$$

Here the big O notation is used.

2.4 Eigenvalues of quaternion matrices

Due to the noncommutativity of quaternions, there are two types of eigenvalues. A quaternion $\lambda$ is said to be a left eigenvalue of $A \in M(n, \mathbb{H})$ if

$$A \cdot x = \lambda \cdot x$$

for some nonzero column-vector $x$ with quaternion components. Similarly, $\lambda$ is a right eigenvalue if

$$A \cdot x = x \cdot \lambda$$

for some nonzero quaternionic column-vector $x$. Then, the set $\{ \lambda \in \mathbb{H} | Ax = \lambda x, x \neq 0 \in \mathbb{H}^n \}$ is called the left spectrum of $A$, denoted by $\sigma_l(A)$. The right spectrum is similarly defined by putting, $\sigma_r(A) := \{ \lambda \in \mathbb{H} | Ax = x \lambda, x \neq 0 \in \mathbb{H}^n \}$.

The theory on the left eigenvalues of quaternion matrices has been investigated, in particular, in [41–43]. The theory on the right eigenvalues of quaternion matrices is more developed (see, e.g. [44–49]). We consider this is a natural consequence of the fact that quaternionic vector-columns form a right vector space for which left eigenvalues from (14) seem to be ”exotic”. Left eigenvalues may appear natural in the equation

$$x A = \lambda x .$$

Since $x A = \lambda x$ if and only if $A^* x^* = x^* \lambda$, then the theory of such ”natural” left eigenvalues from (14) seem to be identical to the theory of right eigenvalues from (15). Similarly, the theory of right eigenvalues from $x A = x \lambda$ is identical to the theory of left eigenvalues from (14).
Now, we present the some known results from the theory of right eigenvalues that will be applied hereinafter. Due to the above, henceforth, we will avoid the "right" specification.

In particular, it’s well known that if $\lambda$ is a nonreal eigenvalue of $A$, so is any element in the equivalence class containing $[\lambda]$, i.e. $[\lambda] = \{x|x = u^{-1}\lambda u, u \in \mathbb{H}, ||u|| = 1\}$.

**Theorem 2.14.** Any quaternion matrix $A \in M(n, \mathbb{H})$ has exactly $n$ eigenvalues which are complex numbers with nonnegative imaginary parts.

**Proposition 2.5.** Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues for $A \in M(n, \mathbb{H})$, no two of which are conjugate, and let $v_1, \ldots, v_n$ be corresponding eigenvectors. Then $v_1, \ldots, v_n$ are (right) linearly independent.

Moreover, similarly to the complex case, the following theorem can be proved.

**Theorem 2.15.** A matrix $A \in M(n, \mathbb{H})$ is diagonalizable if and only if $A$ has a set of $n$ right-linearly independent eigenvectors. Furthermore, if $\lambda_i, v_i$, for $i = 1, \ldots, n$, are eigenpairs of $A$, then

$$A = PDP^{-1},$$

where $P = [v_1, \ldots, v_n]$, $D = \text{diag} [\lambda_1, \ldots, \lambda_n]$.

**Proof.** ($\Rightarrow$) Since $A$ is diagonalizable, there exist an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. Then,

$$D = P^{-1}AP.$$  

Since $D \in M(n, \mathbb{H})$ is diagonal, it has a right-linearly independent set of $n$ right eigenvectors given by the column vectors of the identity matrix, i.e.,

$$De_i = e_id_{ii}, \quad D = \text{diag} [d_{11}, \ldots, d_{nn}], \quad I = [e_1, \ldots, e_n].$$

So, the pair $d_{ii}, e_i$ is an eigenvalue-eigenvector pair of $D$ for all $i = 1, \ldots, n$. Due to $e_id_{ii} = d_{ii}e_i$ and using (18), we have

$$e_id_{ii} = d_{ii}e_i = De_i = P^{-1}APe_i.$$  

By multiplying the extreme members of (19) by $P$ on the left, we obtain

$$A(Pe_i) = (Pe_i)d_{ii}.$$  

It means that the vectors $v_i = Pe_i$ are right eigenvectors of $A$ with eigenvalue $d_{ii}$ for all $i = 1, \ldots, n$. Since the matrix $P = [v_1, \ldots, v_n]$ is invertible, then, by Theorem 2.8 the eigenvectors $v_1, \ldots, v_n$ is right-linearly independent. ($\Leftarrow$)

Let $\lambda_i, v_i$ be eigenvalue-eigenvector pairs of $A$ for $i = 1, \ldots, n$. Consider the matrix $P = [v_1, \ldots, v_n]$. Computing the product, we obtain

$$AP = A[v_1, \ldots, v_n] = [Av_1, \ldots, Av_n] = [\lambda_1v_1, \ldots, \lambda_nv_n].$$
Since the eigenvector set \( \{v_1, \ldots, v_n\} \) is right-linearly independent, then, by Theorem 2.16, \( P \) is invertible. There exists \( P^{-1} \), and

\[
P^{-1}AP = P^{-1}[v_1\lambda_1, \ldots, v_n\lambda_n] = [P^{-1}v_1\lambda_1, \ldots, P^{-1}v_n\lambda_n].
\]

Since \( P^{-1}P = I \), then \( P^{-1}v_i = e_i \) for all \( i = 1, \ldots, n \). So,

\[
P^{-1}AP = [e_1\lambda_1, \ldots, e_n\lambda_n] = \text{diag} [\lambda_1, \ldots, \lambda_n]
\]

Denoting \( D = \text{diag} [\lambda_1, \ldots, \lambda_n] \), we conclude that \( P^{-1}AP = D \), or equivalently, \( A = PDP^{-1} \).

**Corollary 2.1.** \([46]\) If \( A \in M(n, \mathbb{H}) \) has \( n \) non-conjugate eigenvalues, then it can be diagonalized in the sense that there is a \( P \in GL_n(\mathbb{H}) \) for which \( PAP^{-1} \) is diagonal.

Those eigenvalues \( h_1 + k_1i, \ldots, h_n + k_ni \), where \( k_i \geq 0 \) and \( h_t, k_t \in \mathbb{R} \) for all \( t = 1, \ldots, n \), are said to be the standard eigenvalues of \( A \).

**Theorem 2.16.** \([47]\) Let \( A \in M(n, \mathbb{H}) \). Then there exists a unitary matrix \( U \) such that \( U^*AU \) is an upper triangular matrix with diagonal entries \( h_1 + k_1i, \ldots, h_n + k_ni \) which are the standard eigenvalues of \( A \).

**Corollary 2.2.** \([48]\) Let \( A \in M(n, \mathbb{H}) \) with the standard eigenvalues \( h_1 + k_1i, \ldots, h_n + k_ni \). Then \( \sigma_r = [h_1 + k_1i] \cup \cdots \cup [h_n + k_ni] \).

**Corollary 2.3.** \([48]\) \( A \in M(n, \mathbb{H}) \) is normal if and only if there exists an unitary matrix \( U \in M(n, \mathbb{H}) \) such that

\[
U^*AU = \text{diag} \{\lambda_1, \ldots, \lambda_n\}, \tag{20}
\]

where \( \lambda_i = h_i + k_i i \in \mathbb{C} \) is standard eigenvalues for all \( i = 1, \ldots, n \). \( A \) is Hermitian if and only if \( k_i = 0 \) and \( \lambda_i = h_i \in \mathbb{R} \).

From Theorem 2.16 and Corollaries 2.2 and 2.3, the following proposition evidently follows.

**Theorem 2.17.** An arbitrary matrix \( A \in M(n, \mathbb{H}) \) is diagonalizable if and only if it is similar to some normal matrix.

**Proof.** (\( \Rightarrow \)) Let \( A \in M(n, \mathbb{H}) \) is diagonalizable. Then by (17), there exists an invertible matrix \( P \) such that \( A = PDP^{-1} \), where \( D = \text{diag} \{\lambda_1, \ldots, \lambda_n\} \) and \( \lambda_i = h_i + k_ii \in \mathbb{C} \) is standard eigenvalues for all \( i = 1, \ldots, n \). Let \( N \in M(n, \mathbb{H}) \) be a normal matrix such that by (20) \( UNU^* = \text{diag} \{\lambda_1, \ldots, \lambda_n\} \), where \( U \) is some unitary matrix. Then, we have

\[
A = PNU^*P^{-1} = (PU)N(PP)^{-1}.
\]

(\( \Leftarrow \)) Let \( A \) is similar to some normal matrix \( N \in M(n, \mathbb{H}) \). It means there exists an invertible matrix \( T \) such that \( A = T^{-1}NT \). Since \( N = U^*\text{diag} \{\lambda_1, \ldots, \lambda_n\}U \), then

\[
A = T^{-1}U^*\text{diag} \{\lambda_1, \ldots, \lambda_n\}U = (UT)^{-1}\text{diag} \{\lambda_1, \ldots, \lambda_n\}(UT).
\]

\( \square \)
Right (15) and left (14) eigenvalues are in general unrelated [50], but it is not for Hermitian matrices. Suppose $A \in M(n, \mathbb{H})$ is Hermitian and $\lambda \in \mathbb{R}$ is its right eigenvalue, then $A \cdot x = x \cdot \lambda = \lambda \cdot x$. This means that all right eigenvalues of a Hermitian matrix are its left eigenvalues as well. For real left eigenvalues, $\lambda \in \mathbb{R}$, the matrix $\lambda I - A$ is Hermitian.

**Definition 2.7.** If $t \in \mathbb{R}$, then for a Hermitian matrix $A$ the polynomial $p_A(t) = \det(tI - A)$ is said to be the characteristic polynomial of $A$.

The roots of the characteristic polynomial of a Hermitian matrix are its real left eigenvalues, which are its right eigenvalues as well. We can prove the following theorem by analogy to the commutative case (see, e.g. [51]).

**Theorem 2.18.** If $A \in M(n, \mathbb{H})$ is Hermitian, then $p_A(t) = t^n - d_1 t^{n-1} + d_2 t^{n-2} - \ldots + (-1)^n d_n$, where $d_1 = \text{tr} A$, $d_k$ is the sum of principle minors of $A$ of order $k$ for all $1 < k < n$, and $d_n = \det A$.

### 2.5 An algorithm for obtaining eigenvalues of normal quaternion matrices

Using Theorem 2.18 we can obtain eigenvalues not only of Hermitian matrices but of normal matrices as well. Moreover, if it is known a normal matrix which is similar to some quaternionic matrix $A$, then we also can find its eigenvalues. Let us derive eigenvalues and eigenvectors of a quaternion normal matrix $N \in M(n, \mathbb{H})$. Then the following algorithm can be considered.

**Step 1:** Find eigenvalues of the corresponding Hermitian matrix $N^*N$ by solving roots of its characteristic polynomial, $\det(N^*N - \lambda I) = 0$. Let the eigenvalues of $N^*N$ be $\lambda_1, \ldots, \lambda_n$, where $\lambda_i \in \mathbb{R}$ for all $i = 1, \ldots, n$. The associated eigenvectors are computed as the solutions to the equation $(N^*N - \lambda I)v = 0$.

**Step 2:** Conduct the unitary diagonalization of $N^*N$. We determine all its own subspaces and choose orthonormal basis in each of them for example by the Gram-Schmidt orthogonalization process. We obtain orthonormal system of vectors $u_1, \ldots, u_n$. Since the matrix is diagonalizable, the union of all these bases is a basis of the whole space. Construct the unitary matrix $U$ which columns are $u_1, \ldots, u_n$.

**Step 4:** Since the unitary matrix $U$ is applicable for diagonalization of $N$ as well, we can find a diagonal matrix $D = U^*NU$, where $D = \text{diag}\{\mu_1, \ldots, \mu_n\}$, $\mu_i = h_i + k_i i \in \mathbb{C}$ is standard eigenvalues for all $i = 1, \ldots, n$ such that $\lambda_i = \overline{\mu}_i \mu_i$.

**Step 5:** Let for some quaternionic matrix $A$ it be given an invertible matrix $T$ such that $A = TNT^{-1}$. Then by Theorem 2.17 $\mu_i$ for all $i = 1, \ldots, n$ are also eigenvalues of $A$, and by Theorem 2.15 columns of $TU$ are its corresponding eigenvectors.

We illustrate this algorithm by the following example.
Example 2.1. Consider the normal matrix \( N = \begin{bmatrix} 2 & 0 & i + j \\ 0 & i & 0 \\ i - j & 0 & 2 \end{bmatrix} \). Its corresponding Hermitian matrix is
\[
M = N^*N = \begin{bmatrix} 6 & 0 & 4j \\ 0 & 1 & 0 \\ -4j & 0 & 6 \end{bmatrix}.
\]

Find the eigenvalues of \( M \) which are the roots of the characteristic polynomial
\[
p(\lambda) = \det \begin{bmatrix} \lambda - 6 & 0 & 4j \\ 0 & \lambda - 1 & 0 \\ -4j & 0 & \lambda - 6 \end{bmatrix} = \lambda^3 - 13\lambda^2 + 32\lambda - 20 \Rightarrow \begin{cases} \lambda_1 = 10, \\ \lambda_2 = 1, \\ \lambda_3 = 2. \end{cases}
\]

By computing the associated eigenvectors and after their orthonormalization, we obtain the unitary matrix \( U \) whose columns are these eigenvectors,
\[
U = \begin{bmatrix} 0.5 - 0.5j & 0 & 0.5 + 0.5j \\ 0 & 1 & 0 \\ 0.5 + 0.5j & 0 & 0.5 - 0.5j \end{bmatrix}.
\]

Finally, we have
\[
D = U^*NU = \begin{bmatrix} 1 + i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 3 + i \end{bmatrix} \Rightarrow \begin{cases} \mu_1 = 1 + i, \\ \mu_2 = i, \\ \mu_3 = 3 + i. \end{cases} \tag{21}
\]

Moreover, consider the matrix
\[
A = \begin{bmatrix} 1 - 2.5i - 0.5j + k & 4 + 3j + 2.5k \\ 1.5 - i - j - 0.5k & 2 + 1.5i - 3j + 3k \\ 0.5 - i + j - 0.5k & 3 - i - 0.5j \end{bmatrix}.
\]

There exists the matrix
\[
T = \begin{bmatrix} -k & j & 2 \\ i & k & i \\ -j & 1 & i \end{bmatrix}
\]
such that its inverse is
\[
T^{-1} = \begin{bmatrix} -0.5 + 0.5k & -0.5i + j & -0.5i \\ 0.5i - 0.5j & -1.5 & 0.5 + k \\ 0 & -0.5i + 0.5j & -0.5i - 0.5j \end{bmatrix},
\]
and \( A = TNT^{-1} \). Then by Theorem 2.17 the eigenvalues of \( A \) are (21), and its corresponding eigenvectors are the columns of the matrix
\[
TU = \begin{bmatrix} 1 - 0.5i + j - 0.5k & j & 1 + 0.5i - j - 0.5k \\ i & k & i \\ -0.5 + 0.5i - 0.5j + 0.5k & 1 & -0.5 + 0.5i - 0.5j - 0.5k \end{bmatrix}.
\]
3 Systems of quaternion linear differential equations

3.1 Definitions

Consider a matrix valued function $A(t) = (a_{ij}(t)) \in \mathbb{H}^{n \times n} \otimes \mathbb{R}$, where $a_{ij}(t)$ are quaternion-valued functions with the real variable $t$ for all $i, j = 1, \ldots, n$. Then

$$\frac{dA(t)}{dt} = \left( \frac{da_{ij}(t)}{dt} \right)_{n \times n}, \quad \int_a^b A(t)dt = \left( \int_a^b a_{ij}(t)dt \right)_{n \times n}.$$ 

Over the quaternion skew field, we can consider the following system of linear differential equations.

**Definition 3.1.** An $n \times n$ first order right linear quaternion differential system is the equation

$$x' = A(t)x + b(t), \quad (22)$$

where $A(t) \in \mathbb{H}^{n \times n} \otimes \mathbb{R}$ is the coefficient matrix, $b(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix} \in \mathbb{H}^{1 \times n} \otimes \mathbb{R}$ is the given column-vector, $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is the unknown column-vector. An $n \times n$ first order left linear quaternion differential system is the equation

$$x' = xA(t) + b(t), \quad (23)$$

where $b(t) = (b_1(t) \cdots b_n(t)) \in \mathbb{H}^{1 \times n} \otimes \mathbb{R}$ is the given row-vector, $x(t) = (x_1(t) \cdots x_n(t))$ is the unknown row-vector.

The systems (22) and (23) are called nonhomogeneous when there exists $t \in \mathbb{R}$ such that $b(t) \neq 0$, and homogeneous when the source vector $b \equiv 0$, i.e., respectively,

$$x' = A(t)x, \quad (24)$$
$$x' = xA(t). \quad (25)$$

**Remark 3.1.** By the definition of the matrix-vector product, Eq. (22) can be written as

$$\begin{cases} x'_1 = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_1(t), \\
\vdots \\
x'_n = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_n(t), \end{cases}$$

$$18$$
and Eq. (23) can be written as

\[
\begin{align*}
  x_1' &= x_1 a_{11}(t) + \cdots + x_n a_{n1}(t) + b_1(t), \\
  \vdots \\
  x_n' &= x_1 a_{1n}(t) + \cdots + x_n a_{nn}(t) + b_n(t).
\end{align*}
\]

**Definition 3.2.** Solutions of the linear differential systems (22) and (23) are, respectively, column-vector and row-vector valued functions \( \mathbf{x}(t) \) that satisfy every differential equation in the systems.

**Definition 3.3.** **Initial Value Problems** for right and left quaternion linear differential systems are, respectively, the following: Given a matrix valued function \( A(t) = (a_{ij}(t)) \in \mathbb{H}^{n \times n} \otimes \mathbb{R} \), and a quaternion vector valued function \( \mathbf{b}(t) \), a real constant \( t_0 \), and a vector \( \mathbf{x}_0 \), find a quaternion vector valued function \( \mathbf{x}(t) \) that is a solution of

\[
\begin{align*}
  \mathbf{x}' &= A(t) \mathbf{x} + \mathbf{b}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{26}
\end{align*}
\]

or

\[
\begin{align*}
  \mathbf{x}' &= \mathbf{x} A(t) + \mathbf{b}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{27}
\end{align*}
\]

Similarly to real linear differential equations, we can proved the following theorem about existence and uniqueness of solutions to the initial value problems.

**Theorem 3.1.** *If the functions \( A(t) \) and \( \mathbf{b}(t) \) are continuous on an open interval \( I \in \mathbb{H} \), and if \( \mathbf{x}_0 \) is any constant vector (column or row, respectively) and \( t_0 \) is any constant in \( I \), then there exist only one function \( \mathbf{x}(t) \), defined an interval \( \tilde{I} \in I \) with \( t_0 \in \tilde{I} \), that is a solution of the initial value problems (26) or (27), respectively.*

**3.2 General solutions of homogenous systems.**

**Definition 3.4.** A set of quaternion column-vector functions \( \{ \mathbf{x}_1(t), \ldots, \mathbf{x}_n(t) \} \) is called right linearly dependent on an interval \( I \in \mathbb{R} \) if for all \( t \in I \) there exist constant quaternions \( q_1, \ldots, q_n \), \( (q_i \in \mathbb{H} \text{ for all } i = 1, \ldots, n) \), not all of them zero, such that it holds,

\[
\mathbf{x}_1(t) q_1 + \cdots + \mathbf{x}_n(t) q_n = \mathbf{0}.
\]

Similarly, a set of quaternion row-vector functions \( \{ \mathbf{x}_1(t), \ldots, \mathbf{x}_n(t) \} \) is called left linearly dependent on an interval \( I \in \mathbb{R} \) if under the same conditions,

\[
q_1 \mathbf{x}_1(t) + \cdots + q_n \mathbf{x}_n(t) = \mathbf{0}.
\]

These sets are called right (left) linearly independent on \( I \) if they are not right (left) linearly dependent.
Theorem 3.2. If the column-vector valued functions \( \mathbf{x}_1, \mathbf{x}_2 \) are solutions of the homogenies system \((24)\), i.e., \( \mathbf{x}_1' = A(t) \mathbf{x}_1 \) and \( \mathbf{x}_2' = A(t) \mathbf{x}_2 \), then any right linear combination \( \mathbf{x} = \mathbf{x}_1 a + \mathbf{x}_2 b \), for all \( a, b \in \mathbb{H} \) is also a solution of \((24)\).

Proof. Indeed, since the derivative of a vector valued function is a linear operation, we get
\[
\mathbf{x}' = (\mathbf{x}_1 a + \mathbf{x}_2 b)' = \mathbf{x}_1' a + \mathbf{x}_2' b.
\]
Replacing the differential equation on the right-hand side above,
\[
\mathbf{x}' = A(t) \mathbf{x}_1 a + A(t) \mathbf{x}_2 b.
\]
Since the matrix-vector product is a linear operation, then
\[
A \mathbf{x}_1 a + A \mathbf{x}_2 b = A(\mathbf{x}_1 a + \mathbf{x}_2 b).
\]
Hence,
\[
\mathbf{x}' = A(\mathbf{x}_1 a + \mathbf{x}_2 b) = A \mathbf{x}.
\]
This establishes the theorem.

The following theorem can be proved similarly.

Theorem 3.3. If the row-vector valued functions \( \mathbf{x}_1, \mathbf{x}_2 \) are solutions of the homogenies system \((25)\), i.e., \( \mathbf{x}_1' = \mathbf{x}_1 A(t) \) and \( \mathbf{x}_2' = \mathbf{x}_1 A(t) \), then any left linear combination \( \mathbf{x} = a \mathbf{x}_1 + b \mathbf{x}_2 \), for all \( a, b \in \mathbb{H} \) is also solution of \((25)\).

We have the following theorems about right an left homogenies systems.

Theorem 3.4. If \( \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \) is a right linearly independent set of solutions of \((24)\), where \( A \) is a continuous matrix valued function, then there exist constant quaternions \( q_1, \ldots, q_n \), \( (q_i \in \mathbb{H} \text{ for all } i = 1, \ldots, n) \) such that every solution \( \mathbf{x} \) of \((24)\) can be written as the right linear combination
\[
\mathbf{x}(t) = \mathbf{x}_1(t) q_1 + \cdots + \mathbf{x}_n(t) q_n. \quad (28)
\]

Proof. By Theorem 3.2, the right linear combination \( \mathbf{x}(t) = \mathbf{x}_1(t) q_1 + \cdots + \mathbf{x}_n(t) q_n \) is a solution of \((24)\) as well. We now must prove that, in the case that \( \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \) is right linearly independent, every solution of \((24)\) is included in this linear combination.

Let \( \mathbf{x} \) be any solution of the differential equation \((24)\). Due to uniqueness statement in Theorem 3.1, this is the unique solution that at \( t_0 \) takes the value \( \mathbf{x}(t_0) \). This means that the initial data \( \mathbf{x}(t_0) \) parameterizes all solutions to the differential equation \((24)\). Then, we shall find the constants \( q_1, \ldots, q_n \) as solutions of the algebraic linear system,
\[
\mathbf{x}(t_0) = \mathbf{x}_1(t_0) q_1 + \cdots + \mathbf{x}_n(t_0) q_n.
\]
Introducing the notation
\[
\mathbf{X}(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)], \quad \mathbf{q} = \begin{bmatrix}
q_1 \\
\vdots \\
q_n
\end{bmatrix},
\]

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the algebraic linear system has the form
\[ x(t_0) = X(t_0)q. \]

This algebraic system has a unique solution \( q \) for every source \( x(t_0) \) when the matrix \( X(t_0) \) is invertible. By Theorem 2.7, the necessary and sufficient condition of invertibility of \( X(t_0) \in M(n, \mathbb{H}) \) is \( \text{det} X(t_0) \neq 0 \). By Theorem 2.5, it is equivalent that \( \{x_1, \ldots, x_n\} \) is right linearly independent.

**Theorem 3.5.** If \( \{x_1, \ldots, x_n\} \) is a left linearly independent set of solutions of (24), where \( A \) is a continuous matrix valued function, then there exist constant quaternions \( q_1, \ldots, q_n, (q_i \in \mathbb{H} \text{ for all } i = 1, \ldots, n) \) such that every solution \( x \) of (24) can be written as the left linear combination
\[ x(t) = q_1 x_1(t) + \cdots + q_n x_n(t). \]

**Proof.** The proof is similar to the proof of Theorem 3.4. □

So, we obtain the following definitions.

**Definition 3.5.** Let \( x_i(t) \in \mathbb{H}^{1 \times n} \otimes \mathbb{R} \) be a quaternion column-vector valued function for all \( i = 1, \ldots, n \).

1. The set \( \{x_1, \ldots, x_n\} \) is a fundamental set of solutions of (24) if it is a set of right-linearly independent column-vectors which are solutions of (24).

2. The general solution of the homogeneous equation (24) denotes any quaternion column-vector valued function \( x_{\text{gen}} \) that can be written as a right linear combination
\[ x_{\text{gen}}(t) = x_1(t)q_1 + \cdots + x_n(t)q_n, \]
where \( \{x_1, \ldots, x_n\} \) is a fundamental set of solutions of (24), while \( q_1, \ldots, q_n \) are arbitrary quaternion constants.

3. A solution matrix \( X_r(t) = [x_1(t), \ldots, x_n(t)] \) is called a fundamental matrix of (24) if the set \( \{x_1, \ldots, x_n\} \) is a fundamental set.

**Definition 3.6.** Let \( x_i(t) \in \mathbb{H}^{n \times 1} \otimes \mathbb{R} \) be a quaternion row-vector valued function for all \( i = 1, \ldots, n \).

1. The set \( \{x_1, \ldots, x_n\} \) is a fundamental set of solutions of (25) if it is a set of left-linearly independent row-vectors which are solutions of (25).

2. The general solution of (25) denotes any quaternion row-vector valued function \( x_{\text{gen}} \) that can be written as a left linear combination
\[ x_{\text{gen}}(t) = q_1 x_1(t) + \cdots + q_n x_n(t), \]
where \( \{x_1, \ldots, x_n\} \) is a fundamental set of solutions of (25), while \( q_1, \ldots, q_n \) are arbitrary quaternion constants.
3. A solution matrix $X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is called a fundamental matrix of (25) if the set $\{x_1, \ldots, x_n\}$ is a fundamental set.

**Remark 3.2.** From the above definitions, it follows that the general solutions of (24) are (25), respectively, can be represented as

$$x_{\text{gen}}(t) = X_r(t)q,$$  \hspace{1cm} (30)

$$x_{\text{gen}}(t) = qX_l(t).$$ \hspace{1cm} (31)

Moreover, for given Initial Value Problems, $x(t_0) = x^0$, we have, respectively,

$$x_{\text{gen}}(t) = X_r(t)X_r^{-1}(t_0)x^0,$$ \hspace{1cm} (32)

$$x_{\text{gen}}(t) = x^0X_l^{-1}(t_0)X_l(t).$$ \hspace{1cm} (33)

### 3.3 General solutions of non-homogenous systems.

Firstly, we note that by simple checking can be proved the following lemma.

**Lemma 3.1.** If $u^{NH}$ is a solution of the right nonhomogenous system (22), and $v^H$ is a solution of the right homogenous system (24), then $u^{NH} + v^H$ is a solution of (22). Similarly, we have to left systems (23)-(25).

Using the representations (30) and (31), respectively, for $v^H$, we evidently obtain the following theorem.

**Theorem 3.6.** For any solutions $x^{NH}$ of nonhomogenous systems (22) and (23), they can be expressed, respectively, as follows

$$x^{NH}(t) = X_r(t)q + u^{NH},$$

where $u^{NH} \in \mathbb{H}^{n \times 1}$ is an arbitrary solution of (22), and $q \in \mathbb{H}^{n \times 1}$ is a constant quaternionic column-vector;

$$x^{NH}(t) = qX_l(t) + u^{NH},$$

where $u^{NH} \in \mathbb{H}^{1 \times n}$ is an arbitrary solution of (23), and $q \in \mathbb{H}^{1 \times n}$ is a constant quaternionic row-vector.

The following theorem represent the general solution of the right nonhomogenous system (22).

**Theorem 3.7.** The general solution of (22) is given by

$$x^{NH}(t) = X_r(t)q + X_r(t) \int_{t_0}^{t} X_r^{-1}(s)b(s)dt,$$

where $t_0 \in I \in \mathbb{R}$, $q \in \mathbb{H}^{n \times 1}$ is a constant quaternionic column-vector.
This theorem is completely proved in [19] using the Chen determinant. We prove the following theorem about the general solution of the left nonhomogeneous system (23) within the framework of the theory of column-row determinants.

**Theorem 3.8.** The general solution of (23) is given by

\[
x_{NH}^{gen}(t) = qX_l(t) + \int_{t_0}^{t} b(s)X_l^{-1}(s)dt X_l(t),
\]

where \( t_0 \in I \subset \mathbb{R}, \ q \in \mathbb{H}^{1 \times n} \) is a constant quaternionic row-vector.

**Proof.** By (31) the general solution of (25) is

\[
x^{gen}(t) = q X_l(t),
\]

where \( q \in \mathbb{H}^{1 \times n} \) is a constant quaternionic row-vector. Let us find a solution of

\[
x' = x(t)A(t) + b(t),
\]

in the form,

\[ x(t) = q(t)X_l(t), \]

where \( q(t) \in \mathbb{H}^{1 \times n} \otimes \mathbb{R} \) is a quaternionic row-vector function. Differentiating (30), we have

\[
x'(t) = [q(t)X_l(t)]' = q'(t)X_l(t) + q(t)X_l'(t). \tag{37}
\]

Substituting (37) and (30) in (35), we obtain

\[
q'(t)X_l(t) + q(t)X_l'(t) = q(t)X_l(t)A(t) + b(t). \tag{38}
\]

Since \( X_l(t) \) is a solution of the corresponding homogenous system (25), then \( X_l'(t) = X_l(t)A(t) \). Therefore, \( q'(t)X_l(t) = b(t) \) which implies

\[
q'(t) = b(t)X_l^{-1}(t). \tag{39}
\]

Integrating (39) over \([t_0, t]\), we have

\[
q(t) = \int_{t_0}^{t} b(s)X_l^{-1}(s)dt + q, \tag{40}
\]

where \( q \in \mathbb{H}^{1 \times n} \) is a constant quaternionic row-vector. Substituting (40) in (36), we obtain the general solution of (23) representing by (34).

Finally, we must verify that (34) is a solution to (22). Differentiating (34), we obtain

\[
[x_{NH}^{gen}(t)]' = qX_l'(t) + b(t)X_l^{-1}(s)X_l(t) + \int_{t_0}^{t} b(s)X_l^{-1}(s)dt X_l'(t) =
\]

\[
(qX_l(t) + \int_{t_0}^{t} b(s)X_l^{-1}(s)dt X_l(t)) A(t) + b(t) = x_{NH}^{gen}(t)A(t) + b(t).
\]

The proof is complete. \( \Box \)
3.4 Quaternionic linear systems of differential equations with constant coefficients.

Let $A \in \mathbb{H}^{n \times n}$ be a constant matrix. Using properties of the exponential of a quaternion matrix and similar to the real case, we can prove the following theorems about the initial value problems with left and right quaternionic homogeneous linear systems of differential equations.

**Theorem 3.9.** If $A \in \mathbb{H}^{n \times n}$, $t_0 \in \mathbb{R}$ is an arbitrary constant, and $x_0 \in \mathbb{H}^{1 \times n}$ is any constant quaternionic row-vector, then the initial value problem for the unknown quaternionic row-vector valued function $x$ given by

$$x' = xA, \quad x(t_0) = x_0.$$  

has a unique solution given by the formula

$$x = x_0 e^{A(t-t_0)} \quad \text{(41)}$$

**Proof.** Rewrite the given equation and multiply it on the right by $e^{-At}$,

$$x'e^{-At} - Ax e^{-At} = 0$$

Using the properties of the matrix exponential, we have

$$(xe^{-At})' = 0.$$  

Integrating in the last equation above, and denoting by $q$ a constant row $n$-vector, we get

$$xe^{-At} = q.$$  

Since $(e^{-At})^{-1} = e^{At}$, then $x = q e^{At}$. Evaluating at $t = t_0$ we get the constant vector $q = x_0 e^{-At_0}$ and the solution formula,

$$x(t) = x_0 e^{-At_0} e^{At}$$

Taking account that $e^{-At_0} e^{At} = e^{A(t-t_0)}$, we finally obtain (41).

The following theorem can be proved similarly.

**Theorem 3.10.** If $A \in \mathbb{H}^{n \times n}$, $t_0 \in \mathbb{R}$ is an arbitrary constant, and $x_0 \in \mathbb{H}^{n \times 1}$ is any constant quaternionic column-vector, then the initial value problem for the unknown quaternionic column-vector valued function $x$ given by

$$x' = Ax, \quad x(t_0) = x_0.$$  

has a unique solution given by the formula

$$x = e^{A(t-t_0)} x_0 \quad \text{(42)}$$

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Remark 3.3. Comparing both corresponding pairs of the solution formulas and gives the following relation,
\[ e^{\mathbf{A}(t-t_0)} = \mathbf{X}_r(t)\mathbf{X}_r^{-1}(t_0) = \mathbf{X}_r^{-1}(t_0)\mathbf{X}_r(t) \]

Theorem 3.11. If \( \mathbf{A} \in \mathbb{H}^{n \times n} \) is diagonalizable, then the right system \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) has a general solution
\[ \mathbf{x}_{gen}(t) = \mathbf{v}_1 e^{\lambda_1 t}q_1 + \cdots + \mathbf{v}_n e^{\lambda_n t}q_n, \] (43)
where \( \{ \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)} \} \) is a set of linearly independent column eigenvectors and corresponding eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \). (They can be obtained as standard eigenvalues, i.e. \( \lambda_i \in \mathbb{C} \) for all \( i = 1, \ldots, n \).) Furthermore, every initial value problem with \( \mathbf{x}(t_0) = \mathbf{x}_0 \) has a unique solution for every initial condition \( \mathbf{x}_0 \in \mathbb{H}^{n \times 1} \), where the constants \( q_1, \ldots, q_n \) are a solution of the algebraic linear system
\[ \mathbf{x}_0 = \mathbf{v}_1 e^{\lambda_1 t}q_1 + \cdots + \mathbf{v}_n e^{\lambda_n t}q_n, \] (44)
and this solution is given by
\[ \mathbf{x}(t) = \mathbf{X}_r(t)\mathbf{X}_r^{-1}(t_0)\mathbf{x}_0 \] (45)
where \( \mathbf{X}_r(t) = [\mathbf{v}_1 e^{\lambda_1 t}, \ldots, \mathbf{v}_n e^{\lambda_n t}] \) is a fundamental matrix of the system.

Proof. Since the coefficient matrix \( \mathbf{A} \) is diagonalizable, there exist an invertible matrix \( \mathbf{P} \) and a diagonal matrix \( \mathbf{D} \) such that \( \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \). Introduce this expression into the given equation and left-multiplying it by \( \mathbf{P}^{-1} \),
\[ \mathbf{P}^{-1}\mathbf{x}' = \mathbf{P}^{-1}(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\mathbf{x}. \]
Since \( \mathbf{A} \) is constant, so is \( \mathbf{P} \) and \( \mathbf{D} \). Then, \( \mathbf{P}^{-1}\mathbf{x}' = (\mathbf{P}^{-1}\mathbf{x})' \), and
\[ (\mathbf{P}^{-1}\mathbf{x})' = \mathbf{D}(\mathbf{P}^{-1}\mathbf{x}). \]

Define the new variable \( \mathbf{y} = \mathbf{P}^{-1}\mathbf{x} \). The differential equation is now given by
\[ \mathbf{y}'(t) = \mathbf{D}\mathbf{y}(t). \] (46)

Transform the initial condition by \( \mathbf{P}^{-1}\mathbf{x}(t_0) = \mathbf{P}^{-1}\mathbf{x}_0 \), and put \( \mathbf{y}_0 = \mathbf{P}^{-1}\mathbf{x}_0 \). We get the following initial condition, \( \mathbf{y}(t_0) = \mathbf{y}_0 \). Then from (46), we have the system every equation of which can be involving by Proposition 2.4 as follows
\[
\begin{cases}
\mathbf{y}_1'(t) = \lambda_1 \mathbf{y}_1(t) \\
\vdots \\
\mathbf{y}_n'(t) = \lambda_n \mathbf{y}_n(t)
\end{cases}
\Rightarrow \begin{cases}
\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{q}_1 \\
\vdots \\
\mathbf{y}_n(t) = e^{\lambda_n t}\mathbf{q}_n
\end{cases}
\Rightarrow \mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{q}_1 \\
\vdots \\
e^{\lambda_n t}\mathbf{q}_n \end{bmatrix}. 
\]

Now, transform \( \mathbf{y} \) back to \( \mathbf{x} \),
\[ \mathbf{x} = \mathbf{P}\mathbf{y} = [\mathbf{v}_1 \cdots \mathbf{v}_n] \begin{bmatrix} e^{\lambda_1 t}\mathbf{q}_1 \\
\vdots \\
e^{\lambda_n t}\mathbf{q}_n \end{bmatrix} = \mathbf{v}_1 e^{\lambda_1 t}q_1 + \cdots + \mathbf{v}_n e^{\lambda_n t}q_n. \]
where \( v_i = v_{i,j} \) is the \( i \)th column of \( P \).

So, we obtain \( 13 \). Evaluating it at \( t_0 \) we get \( 14 \).

If we choose fundamental solutions of \( x' = Ax \) to be

\[
\{ x_1(t) = v_1 e^{\lambda_1 t}, \ldots, x_n(t) = v_n e^{\lambda_n t} \},
\]

then the associated fundamental matrix is \( X_r(t) = [v_1 e^{\lambda_1 t}, \ldots, v_n e^{\lambda_n t}] \) and the general solution can be writing as \( x = X_r(t)q \), where \( q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \). Now, from the initial condition, \( x_0 = x(t_0) = X_r(t_0)q \), follows \( q = X_r^{-1}(t_0)x_0 \), which makes sense, since \( X_r(t) \) is an invertible matrix for all \( t \), where it is defined. Using this formula for the constant vector \( q \) gives \( 15 \).

This completes the proof.

**Theorem 3.12.** If \( A \in \mathbb{H}^{n \times n} \) is diagonalizable, then the left system \( x' = xA \) has a general solution, \( x_{\text{gen}}(t) = q_1 e^{\lambda_1 t}v_1 + \cdots + q_n e^{\lambda_n t}v_n \), where \( \{v_1^{(1)}, \ldots, v_n^{(n)}\} \) is a set of linearly independent row eigenvectors and corresponding eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \). Furthermore, every initial value problem with \( x(t_0) = x_0 \) has a unique solution for every initial condition \( x_0 \in \mathbb{H}^{1 \times n} \), where the constants \( q_1, \ldots, q_n \) are a solution of the algebraic linear system, \( x_0 = q_1 e^{\lambda_1 t}v_1 + \cdots + q_n e^{\lambda_n t}v_n \), and this solution is given by \( x = x_0 X_r^{-1}(t_0)X_r(t) \), where \( X_r(t) = \begin{bmatrix} e^{\lambda_1 t}v_1 \\ \vdots \\ e^{\lambda_n t}v_n \end{bmatrix} \) is a fundamental matrix of the system.

**Proof.** The proof is similar to the proof of Theorem 3.11 by using left eigenvalues of \( A \) in the sense of 16.

Now, consider the solution formulas of an initial value problems for nonhomogeneous right and left linear systems.

**Theorem 3.13.** If \( A \in \mathbb{H}^{n \times n} \) is constant and the quaternionic column \( n \)-vector valued function \( b(t) \) is continuous, then the initial right value problem

\[
x'(t) = Ax(t) + b(t), \quad x(t_0) = x_0
\]

has a unique solution for every initial condition \( t_0 \in \mathbb{R} \) and \( x_0 \in \mathbb{H}^{n \times 1} \) given by

\[
x = e^{A(t-t_0)}x_0 + e^{At} \int_{t_0}^t e^{-A\tau} b(\tau) d\tau.
\]

**Proof.** The proof is similar to the proof of the same theorem in the complex or real case.

Rewrite the given equation as \( x' = Ax + b \), and multiply it on the left by \( e^{-At} \),

\[
e^{-At}x' - e^{-At}Ax = e^{-At}b
\]
Since $e^{-At}A = A e^{-At}$ and using the formulas for the derivative of an exponential and product, then we obtain

$$(e^{-At}x)' = e^{-At}b$$

Integrating on the interval $[t_0, t]$ the last equation above gives

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-At}b(\tau) d\tau$$

By reorder terms and using that $(e^{-At})^{-1} = e^{At}$, we have

$$x(t) = e^{At}e^{-At_0}x_0 + e^{At} \int_{t_0}^{t} e^{-At}b(\tau) d\tau.$$

Taking in account $e^{At}e^{-At_0} = e^{A(t-t_0)}$, from this we finally obtain (48).

**Theorem 3.14.** If $A \in \mathbb{H}^{n \times n}$ is constant and the quaternionic row $n$-vector valued function $b$ is continuous, then the initial left value problem

$$x'(t) = x(t)A + b(t), \quad x(t_0) = x_0$$

has a unique solution for every initial condition $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{H}^{1 \times n}$ given by

$$x = x_0 e^{A(t-t_0)} + \int_{t_0}^{t} b(\tau) e^{-A\tau} d\tau e^{At}.$$

**Proof.** The proof is similar to the proof of Theorem 3.13.

**Remark 3.4.** The general solutions of the equations (47) and (49) are, respectively,

$$x(t) = e^{At} \int e^{-At}b(t) dt,$$

$$x(t) = \int b(t) e^{-At} dt e^{At}.$$

While there is no simple algorithm to directly calculate eigenvalues for general matrices, there are numerous special classes of matrices where eigenvalues can be directly calculated.

**Example 3.1.** Consider the right linear system

$$x'(t) = Ax(t) + b(t),$$

where the coefficient matrix $A$ from Example 2.1, i.e.

$$A = \begin{bmatrix} 1 - 2.5i - 0.5j + k & 4 + 3j + 2.5k & 2 - 2i - j - 2.5k \\ 1.5 - i - j - 0.5k & 2 + 1.5i - 3j + 3k & 2 + 2.5i + j - k \\ 0.5 - i + j - 0.5k & 3 - i - 0.5j & 1 + i - 1.5j - 2k \end{bmatrix},$$


and the source column-vector
\[ \mathbf{b}(t) = \begin{bmatrix} it \\ -kt \\ jt \end{bmatrix}. \]

By Example 2.1 there exists the invertible matrix
\[ \mathbf{V} = \begin{bmatrix} 1 - 0.5i + j - 0.5k & j & 1 + 0.5i - j - 0.5k \\ i & k & i \\ -0.5 + 0.5i - 0.5j + 0.5k & 1 & -0.5 + 0.5i - 0.5j - 0.5k \end{bmatrix}, \]
such that \( A = \mathbf{V} \mathbf{D} \mathbf{V}^{-1} \), where \( \mathbf{D} = \text{diag}[1 + i, i, 3 + i] \) and
\[ \mathbf{V}^{-1} = \begin{bmatrix} -0.25 + 0.25i - 0.25j + 0.25k & -0.25 - 0.5i + 0.75j & -0.25 - 0.5i - 0.25j \\ 0.5i - 0.5j & -1.5 & 0.5 + k \\ -0.25 - 0.25i + 0.25j + 0.25k & 0.25 - 0.5i + 0.75j & 0.25 - 0.5i - 0.25j \end{bmatrix}. \]

We shall find the general solution of (51) by (50). Since, \( e^{-At} = \mathbf{V} e^{-\mathbf{D}t} \mathbf{V}^{-1} \), where
\[ e^{-\mathbf{D}t} = \begin{bmatrix} e^{(-1-i)t} & 0 & 0 \\ 0 & e^{-it} & 0 \\ 0 & 0 & e^{(-3-i)t} \end{bmatrix}, \]
then
\[ \mathbf{x}(t) = \mathbf{V} e^{\mathbf{D}t} \mathbf{V}^{-1} \int \mathbf{V} e^{-\mathbf{D}t} \mathbf{V}^{-1} \mathbf{b}(t)dt = \mathbf{V} e^{\mathbf{D}t} \int e^{-\mathbf{D}t} \mathbf{V}^{-1} \mathbf{b}(t)dt. \]

Further,
\[ e^{-\mathbf{D}t} \mathbf{V}^{-1} \mathbf{b}(t) = e^{-\mathbf{D}t} \begin{bmatrix} (-i - 0.5j)t \\ (-0.5 - i + 0.5j + 2k)t \\ (0.5 - i - k)t \end{bmatrix} = \begin{bmatrix} e^{(-1-i)t}(-i - 0.5j)t \\ e^{-it}(-0.5 - i + 0.5j + 2k)t \\ e^{(-3-i)t}(0.5 - i - k)t \end{bmatrix}, \]
and by Proposition 2.3
\[ \int e^{-\mathbf{D}t} \mathbf{V}^{-1} \mathbf{b}(t)dt = \begin{bmatrix} \int e^{(-1-i)t}tdt(-i - 0.5j) \\ \int e^{-it}tdt(-0.5 - i + 0.5j + 2k) \\ \int e^{(-3-i)t}tdt(0.5 - i - k) \end{bmatrix} = \]
\[ \begin{bmatrix} (e^{(-1-i)t}(t(-0.5 + 0.5i) + 0.5i) + g_1(-i - 0.5j))(-i - 0.5j) \\ (e^{-it}(ti + 1) + g_2)(-0.5 - i + 0.5j + 2k) \\ (e^{(-3-i)t}(t(-0.3 + 0.1i) - 0.08 + 0.06i)) + g_3)(0.5 - i - k) \end{bmatrix} = \]
\[ \begin{bmatrix} e^{(-1-i)t}(t(0.5 + 0.5i + 0.25j + 0.5 - 0.25k) + 0.5 - 0.25k) + g_1(-i - 0.5j) \\ e^{-it}(t(1 - 0.5i - 2j + 0.5k) - 0.5 - i + 0.5j + 2k) + g_2(-0.5 - i + 0.5j + 2k) \\ e^{(-3-i)t}(t(-0.05 + 0.35i + 0.1j + 0.3k) + 0.02 + 0.11i + 0.06j + 0.08k) + g_3(0.5 - i - k) \end{bmatrix}, \]
where \( g_n \in \mathbb{R} \) is arbitrary for all \( n = 1, 2, 3 \).
By the direct matrix multiplication, we have
\[ V e^{D t} = \begin{bmatrix}
(1 - 0.5i + j - 0.5k) e^{(1+i)t} & j e^{it} & (1 + 0.5i - j - 0.5k) e^{(3+i)t} \\
\frac{i e^{(1+i)t}}{e^{it}} & k e^{it} & \frac{i e^{(3+i)t}}{e^{it}} \\
(-0.5 + 0.5i - 0.5j + 0.5k) e^{(1+i)t} & e^{it} & (-0.5 + 0.5i - 0.5j - 0.5k) e^{(3+i)t}
\end{bmatrix}. \]

If we put \( g_n = 0 \) for all \( n = 1, 2, 3 \), then, finally, we obtain the following partial solution of (51):
\[ x(t) = \begin{bmatrix} 2.4 + 0.7i + 1.2j + 0.1k \\
-1.35 + 2.45i - 0.55j + 1.35k \\
0.75 - 0.45i - 2.25j + 1.65k
\end{bmatrix} t + \begin{bmatrix} -0.06 + 1.57i - 0.18j + 0.71k \\
-2.11 + 0.02i - 0.83j - 0.44k \\
-0.6 - 0.57i + 0.3j + 2.49k
\end{bmatrix}.
\]

The correctness of the result can easily be verified by substituting it in (51).

3.5 Determinantal representations of solutions of right and left linear systems with constant coefficient matrices and sources vectors

3.5.1 The case with invertible coefficient matrices

If \( A \) is invertible, then
\[ \int e^{-A t} dt = -A^{-1} e^{-A t} + G, \]
where \( G \) is an arbitrary \( n \times n \) matrix. Then for the right nonhomogeneous system, \( x'(t) = Ax(t) + b \), we have the following general solution and solution of the right initial problem, respectively,
\[ x(t) = e^{At} \int e^{-A t} dt b = -A^{-1} b + e^{At} G b, \]
\[ x(t) = -A^{-1} b + e^{A(t-t_0)} x_0 b. \]

For the left nonhomogeneous system, \( x'(t) = x(t) A + b \), we evidently obtain the following general solution and solution of the right initial problem, respectively,
\[ x(t) = b \int e^{-A t} dt e^{A t} = -b A^{-1} + b G e^{A t}, \]
\[ x(t) = -b A^{-1} + b x_0 e^{A(t-t_0)}. \]

If \( G ≡ 0 \) or (that is equivalent) \( x_0 ≡ 0 \), then the partial solution of the right nonhomogeneous system, \( x(t) = -A^{-1} b, x = \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix}, \) due to Theorem 2.9, possess the following determinantal representations for all \( i = 1,\ldots,n \):

(i) \( x_i = -\frac{\text{det}_A \begin{bmatrix} b \end{bmatrix}}{\text{det} A} \) when \( A \) is Hermitian;
(ii) \( x_i = -\frac{\text{cdet}(A^* A)}{\text{ddet} A} \) when \( A \) is arbitrary.

Similarly, if \( G \) or \( x_0 \) is the zero matrix or the zero row-vector, respectively, then the partial solution of the left nonhomogeneous system, \( x(t) = -bA^{-1} \), \( x = [x_1, \ldots, x_n] \), due to Theorem 2.10 possess the following determinantal representations for all \( i = 1, \ldots, n \):

(i) \( x_i = -\frac{\text{rdet}(A)}{\text{ddet} A} \) when \( A \) is Hermitian;

(ii) \( x_i = -\frac{\text{rdet}(A^* A)}{\text{ddet} A} \) when \( A \) is arbitrary.

3.5.2 The case with non-invertible coefficient matrices

If \( A \) is non-invertible, then due to [52] the following theorem can be expended to quaternion matrices.

**Theorem 3.15.** If \( A \in \mathbb{H}^{n \times n} \) has index \( k \), then

\[
\int e^{-At} dt = -A^D e^{-At} + (I - AA^D) t \left[ I - \frac{A^2}{2} t^2 + \cdots + \frac{(-1)^{k-1} A^{k-1}}{k!} t^{k-1} \right] + G. \tag{52}
\]

**Proof.** Differentiating Eq. (52) and using the series expansion for \( e^{-At} \), we obtain

\[
e^{-At} = A^D A \left( I - At + \frac{A^2}{2} t^2 + \cdots + \frac{(-1)^k A^k}{k!} t^k + \cdots \right) + (I - AA^D) - \\
(I - AA^D) At + (I - AA^D) \frac{A^2}{2} t^2 + \cdots + (I - AA^D) \frac{(-1)^{k-1} A^{k-1}}{(k-1)!} t^{k-1}. \tag{53}
\]

Since \( A^D A = AA^D \), from (53) it follows the identity,

\[
e^{-At} = A^D A \left( I - At + \frac{A^2}{2} t^2 + \cdots + \frac{(-1)^k A^k}{k!} t^k + \cdots \right) - \\
A^D A \left( I - At + \frac{A^2}{2} t^2 + \cdots + \frac{(-1)^k A^k}{k!} t^k + \cdots \right) + \\
I - At + \frac{A^2}{2} t^2 + \cdots + \frac{(-1)^k A^k}{k!} t^k + \cdots
\]

It completes the proof.

**Remark 3.5.** Note that from (52), we have the following identity,

\[
e^{-At}(I - AA^D) = (I - AA^D) \left[ I - At + \frac{A^2}{2} t^2 + \cdots + \frac{(-1)^{k-1} A^{k-1}}{(k-1)!} t^{k-1} \right].
\]
Similarly, we have
\[ e^{At}(I - AA^D) = (I - AA^D) \left[ I + At + \frac{A^2}{2!} t^2 + \cdots + \frac{A^{k-1}}{(k-1)!} t^{k-1} \right]. \]  

(54)

Consider the right nonhomogeneous system
\[ x' (t) = Ax(t) + b, \]
where \( A \in \mathbb{H}^{n \times n} \) is singular and \( \text{Ind} A = k \). Due to Eqs. (52) and (54), we have the following general solution and solution of the right initial problem, respectively,
\[ x(t) = \left\{ -A^D + (I - AA^D) t \left[ I + At + \frac{A^2}{2!} t^2 + \cdots + \frac{A^{k-1}}{(k-1)!} t^{k-1} \right] + e^{At} G \right\} b, \]

(55)

\[ x(t) = \left\{ -A^D + (I - AA^D) t \left[ I + At + \frac{A^2}{2!} t^2 + \cdots + \frac{A^{k-1}}{(k-1)!} t^{k-1} \right] + e^{At} \left( t - t_0 \right) x_0 \right\} b. \]

If we put \( G = 0 \), then the following partial solution of (55) is obtained,
\[ X(t) = -A^D b + (b - A^D Ab) t + \frac{1}{2} (Ab - A^D A^2 b) t^2 + \cdots \frac{1}{k!} (A^{k-1} b - A^D A^k b) t^k. \]

(56)

**Theorem 3.16.** If \( A \in \mathbb{H}^{n \times n} \) has index \( k \) and rank \( A^{k+1} = \text{rank} A^k = r \leq n \), then the partial solution (56), \( x(t) = (x_i(t)) \), possess the following determinantal representation,

(i) when \( A \in \mathbb{H}^{n \times n} \) is Hermitian,
\[ x_i = -\frac{\beta}{\beta \in J_{r_1}, \beta} \left[ (A^{k+1})_i^{\beta} \right] \sum_{\beta \in J_{r_1}} \text{cdet}_t \left( A_i^{k+1} \left( b_i^{(k)} \right) \right)^{\beta} \]
\[ + \frac{1}{2!} \left( b_i^{(1)} \right)_i \left[ (A^{k+1})_i^{\beta} \right] \sum_{\beta \in J_{r_1}} \left[ (A^{k+1})_i^{\beta} \right] \sum_{\beta \in J_{r_1}} \text{cdet}_t \left( A_i^{k+1} \left( b_i^{(k+2)} \right) \right)^{\beta} \]
\[ + \frac{1}{k!} \left( b_i^{(k-1)} \right)_i \left[ (A^{k+1})_i^{\beta} \right] \sum_{\beta \in J_{r_1}} \left[ (A^{k+1})_i^{\beta} \right] \sum_{\beta \in J_{r_1}} \text{cdet}_t \left( A_i^{k+1} \left( b_i^{(2k)} \right) \right)^{\beta} \]

(57)
where $A^l \mathbf{b} := \tilde{b}^{(l)} = (\tilde{b}^{(l)}_i) \in \mathbb{R}^{n \times 1}$ for all $l = k, \ldots, 2k$;

(ii) when $A$ is arbitrary,

\begin{equation}
\begin{aligned}
\sum_{s=1}^{n} a_{is}^{(k)} \sum_{\beta \in J_{r,n}(s)} \text{cdet}_s \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{s} \left( \tilde{d}^{(0)} \right) \right)_{\beta} + \\
\sum_{\beta \in J_{r,n}} \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{\beta} \right) \\
\sum_{s=1}^{n} a_{is}^{(k)} \sum_{\beta \in J_{r,n}(s)} \text{cdet}_s \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{s} \left( \tilde{d}^{(1)} \right) \right)_{\beta} + \\
\sum_{\beta \in J_{r,n}} \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{\beta} \right) \\
\sum_{s=1}^{n} a_{is}^{(k)} \sum_{\beta \in J_{r,n}(s)} \text{cdet}_s \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{s} \left( \tilde{d}^{(2)} \right) \right)_{\beta} + \\
\sum_{\beta \in J_{r,n}} \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{\beta} \right) \\
\sum_{s=1}^{n} a_{is}^{(k)} \sum_{\beta \in J_{r,n}(s)} \text{cdet}_s \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{s} \left( \tilde{d}^{(k)} \right) \right)_{\beta} + \\
\sum_{\beta \in J_{r,n}} \left( \begin{pmatrix} (A^{2k+1})^* (A^{2k+1}) \end{pmatrix}_{\beta} \right)
\end{aligned}
\end{equation}

where $(A^{2k+1})^* A^{k+l} \mathbf{b} := \tilde{d}^{(l)} = (\tilde{d}^{(l)}_i) \in \mathbb{R}^{n \times 1}$ for all $l = 0, \ldots, k$ and for all $i = 1, \ldots, n$.

Proof. (i) Using the determinantal representation of $A^D$ by (11), we obtain the following determinantal representation of $A^D A^m \mathbf{b} := (y_i)$,

\begin{equation}
\begin{aligned}
y_i = \sum_{s=1}^{n} a_{is}^{(m)} \sum_{t=1}^{n} a_{st}^{(m)} b_{tj} = \sum_{\beta \in J_{r,n}(s)} \text{cdet}_i \left( A_{i, i}^{k+1} (a_{s}^{(k)}) \right)_{\beta} \sum_{t=1}^{n} a_{st}^{(m)} b_{tj} = \\
\sum_{\beta \in J_{r,n}(s)} \text{cdet}_i \left( A_{i, i}^{k+1} (a_{s}^{(k+m)}) \right)_{\beta} \cdot b_{tj} = \sum_{\beta \in J_{r,n}(s)} \text{cdet}_i \left( A_{i, i}^{k+1} (\tilde{b}_{i, j}^{(k+m)}) \right)_{\beta}
\end{aligned}
\end{equation}

for all $i = 1, \ldots, n$ and $m = 1, \ldots, k$. From this, it follows (57).

(ii) The proof of (58) is similar to the proof of (57) by using the determinantal representation of $A^D$ by (11).

\qed
For the left nonhomogeneous system

\[ x'(t) = x(t)A + b, \]

where \( A \in \mathbb{H}^{n \times n} \) is singular and \( \text{Ind} A = k \), we evidently obtain the following general solution and solution of the left initial problem, respectively,

\[
x(t) = b \left\{ -A^D + (I - AA^D) \left[ I + At + \frac{A^2}{2!} t^2 + \cdots + \frac{A^{k-1}}{(k-1)!} t^{k-1} \right] + Ge^{At} \right\},
\]

\[
(59)
\]

\[
x(t) = b \left\{ -A^D + (I - AA^D) \left[ I + At + \frac{A^2}{2!} t^2 + \cdots + \frac{A^{k-1}}{(k-1)!} t^{k-1} \right] e^{(t-t_0)A} \right\}.
\]

If we put \( G = 0 \), then we obtain the following partial solution of \( (59) \),

\[
X(t) = -bA^D + (b - bAA^D)t + \frac{1}{2}(bA - bA^2 A^D)t^2 + \ldots + \frac{1}{k!}(bA^{k-1} - bA^k A^D)t^k.
\]

\[
(60)
\]

**Theorem 3.17.** If \( A \in \mathbb{H}^{n \times n} \) has index \( k \) and \( \text{rank } A^{k+1} = \text{rank } A^k = r \leq n \), then the partial solution \( (60) \), \( X(t) = (x_i(t)) \), possess the following determinantal representation,

(i) when \( A \in \mathbb{H}^{n \times n} \) is Hermitian,

\[
x_i = -\sum_{\alpha \in I_{r,n}(i)} \frac{r \text{det}_i \left( A_i^{k+1} \left( \tilde{b}^{(k)} \right) \right)_\alpha}{\sum_{\alpha \in I_{r,n}} |(A_i^{k+1})_\alpha|} + \sum_{\alpha \in I_{r,n}(i)} \frac{r \text{det}_j \left( A_j^{k+1} \left( \tilde{b}^{(k+1)} \right) \right)_\alpha}{\sum_{\alpha \in I_{r,n}} |(A_j^{k+1})_\alpha|} t + \ldots
\]

\[
+ \frac{1}{2} \left( b_{ij}^{(1)} - \sum_{\alpha \in I_{r,n}(i)} \frac{r \text{det}_i \left( A_i^{k+1} \left( \tilde{b}^{(k+2)} \right) \right)_\alpha}{\sum_{\alpha \in I_{r,n}} |(A_i^{k+1})_\alpha|} \right) t^2 + \ldots
\]

\[
+ \frac{1}{k!} \left( b_{ij}^{(k-1)} - \sum_{\alpha \in I_{r,n}(i)} \frac{r \text{det}_i \left( A_i^{k+1} \left( \tilde{b}^{(2k)} \right) \right)_\alpha}{\sum_{\alpha \in I_{r,n}} |(A_i^{k+1})_\alpha|} \right) t^k,
\]

where \( bA^l =: \tilde{b}^{(l)} = (\tilde{b}_{ij}^{(l)}) \in \mathbb{H}^{1 \times n} \) for all \( l = k, \ldots, 2k \);
when \( A \in \mathbb{H}^{n \times n} \) is arbitrary.

\[
x_i = \sum_{s=1}^{n} \left( \sum_{\alpha \in I_r, n} \text{rdet}_s \left( \left( A^{2k+1} (A^{2k+1})^* \right)_s (\tilde{d}(0)) \right)_\alpha \right) a_s^{(k)} + \sum_{\alpha \in I_r, n} \left| (A^{2k+1} (A^{2k+1})^*)_\alpha \right| \sum_{s=1}^{n} \left( \sum_{\alpha \in I_r, n} \text{rdet}_s \left( \left( A^{2k+1} (A^{2k+1})^* \right)_s (\tilde{d}(1)) \right)_\alpha \right) a_s^{(k)} + \sum_{\alpha \in I_r, n} \left| (A^{2k+1} (A^{2k+1})^*)_\alpha \right| \sum_{s=1}^{n} \left( \sum_{\alpha \in I_r, n} \text{rdet}_s \left( \left( A^{2k+1} (A^{2k+1})^* \right)_s (\tilde{d}(2)) \right)_\alpha \right) a_s^{(k)} + \sum_{\alpha \in I_r, n} \left| (A^{2k+1} (A^{2k+1})^*)_\alpha \right| \sum_{s=1}^{n} \left( \sum_{\alpha \in I_r, n} \text{rdet}_s \left( \left( A^{2k+1} (A^{2k+1})^* \right)_s (\tilde{d}(l)) \right)_\alpha \right) a_s^{(k)} + \sum_{\alpha \in I_r, n} \left| (A^{2k+1} (A^{2k+1})^*)_\alpha \right| \sum_{s=1}^{n} \left( \sum_{\alpha \in I_r, n} \text{rdet}_s \left( \left( A^{2k+1} (A^{2k+1})^* \right)_s (\tilde{d}(k)) \right)_\alpha \right) a_s^{(k)}
\]

where \( b A^{k+l} (A^{2k+1})^* =: \tilde{D}^{(l)} = (\tilde{d}_{ij}^{(l)}) \in \mathbb{H}^{1 \times n} \) for all \( l = 1, \ldots, k \) and for all \( i = 1, \ldots, n \).

**Proof.** The proof is similar to the proof of Theorem 3.10 by using the determinant representation of the Drazin inverse \([12]\) for the case (i) and \([11]\) for the case (ii), respectively.

### 3.6 An example

Let us consider the matrix equation

\[
x' + Ax = b, \tag{61}
\]

where

\[
A = \begin{bmatrix} 1 & k & -i \\ -k & 2 & j \\ i & -j & 1 \end{bmatrix}, \quad b = \begin{bmatrix} j \\ -k \\ i \end{bmatrix}.
\]

Since \( A \) is Hermitian, \( A^2 = \begin{bmatrix} 3 & 4k & -3i \\ -4k & 6 & 4j \\ 3i & -4j & 3 \end{bmatrix}, \) \( \det A = \det A^2 = 0, \) and

\[
det \begin{bmatrix} 1 & k \\ -k & 2 \end{bmatrix} = 1, \quad \det \begin{bmatrix} 3 & 4k \\ -4k & 6 \end{bmatrix} = 2, \text{ then } Ind A = 1 \text{ and } r = \text{ rank } A = 2. \]
shall find the solutions \((x_i) \in \mathbb{H}^{3 \times 1}\) by (57),

\[
x_i = \frac{\sum_{\beta \in J_{2,3}} \text{cdet}_i \left( A^2_i \left( \hat{b}^{(1)}(i) \right) \right)^{\beta}}{\sum_{\beta \in J_{2,3}} \left| A^2 \right|^\beta} + \left( b_i - \frac{\sum_{\beta \in J_{2,3}} \text{cdet}_i \left( A^2_i \left( \hat{b}^{(2)}(i) \right) \right)^{\beta}}{\sum_{\beta \in J_{2,3}} \left| A^2 \right|^\beta} \right) t
\]

for all \(i = 1, 2, 3\). We have, \(\sum_{\beta \in J_{2,3}} \left| A^2 \right|^\beta = 4\),

\[
\hat{b}^{(1)} = Ab = \begin{bmatrix} 2 + j \\ i - 3k \\ 2i + k \end{bmatrix}, \quad \hat{b}^{(2)} = A^2 b = \begin{bmatrix} 7 + 3j \\ 4i - 10k \\ 7i + 3k \end{bmatrix}.
\]

Therefore,

\[
x_1 = \frac{1}{4} \left( \text{cdet}_1 \begin{bmatrix} 2 + j \\ i - 3k \\ 2i + k \end{bmatrix} + \text{cdet}_1 \begin{bmatrix} 2 + j \\ -3i \\ 3 \end{bmatrix} \right) + \left( j - \frac{1}{4} \left( \text{cdet}_1 \begin{bmatrix} 7 + 3j \\ 4i - 10k \\ 6 \end{bmatrix} + \text{cdet}_1 \begin{bmatrix} 7 + 3j \\ -3i \\ 3 \end{bmatrix} \right) \right) t = \frac{1}{4} (2j + 0) + \left( j - \frac{1}{4} (2 + 2j + 0) \right) t = 0.5j + (-0.5 + 0.5j)t;
\]

\[
x_2 = \frac{1}{4} \left( \text{cdet}_2 \begin{bmatrix} 3 \\ 2 + j \\ -4k \\ i - 3k \end{bmatrix} + \text{cdet}_1 \begin{bmatrix} i - 3k \\ 4j \\ 2i + k \\ 3 \end{bmatrix} \right) + \left( -k - \frac{1}{4} \left( \text{cdet}_2 \begin{bmatrix} 3 \\ 7 + 3j \\ 4i - 10k \\ 3i \end{bmatrix} + \text{cdet}_1 \begin{bmatrix} 4i - 10k \\ 3i + 3k \\ 3 \end{bmatrix} \right) \right) t = -0.5i - 0.5k;
\]

\[
x_3 = \frac{1}{4} \left( \text{cdet}_2 \begin{bmatrix} 3 \\ 2 + j \\ 3i \\ 2i + k \end{bmatrix} + \text{cdet}_2 \begin{bmatrix} 6 \\ i - 3k \\ 2i + k \end{bmatrix} \right) + \left( i - \frac{1}{4} \left( \text{cdet}_2 \begin{bmatrix} 3 \\ 7 + 3j \\ 3i \\ 7i + 3k \end{bmatrix} + \text{cdet}_1 \begin{bmatrix} 6 \\ 4i - 10k \\ 7i + 3k \end{bmatrix} \right) \right) t = 0.5k + (0.5i - 0.5k)t.
\]

Note that we used Maple with the package CLIFFORD in the calculations.

### 4 Conclusion

A basic theory on first order right and left linear quaternion differential systems (LQDS) is considered in this paper. We adopted the theory of column-row determinants for quaternion matrix to proceed the theory of LQDS. The algebraic structure of their general solutions are established. Determinantal representations of solutions of systems with constant coefficient matrices and sources
vectors are obtained in both cases when coefficient matrices are invertible and singular. In the last case, we use determinantal representations of the quaternion Drazin inverse within the framework of the theory of column-row determinants.

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References

[1] J.D. Gibbon, D.D. Holm, R.M. Kerr, I. Roulstone, Quaternions and particle dynamics in the Euler fluid equations, Nonlinearity, 19 (2006) 1969-1983.

[2] V.N. Roubtsov, I. Roulstone, Holomorphic structures in hydrodynamical models of nearly geostrophic flow, Proc. R. Soc. London, Ser. A 457 (2001) 1519-1531.

[3] S.L. Adler, Quaternionic quantum mechanics and quantum fields, Oxford University Press, New York, 1995.

[4] S. Leo, G. Ducati, Delay time in quaternionic quantum mechanics, J. Math. Phys. 53(2) (2012) 022102-8 pp.

[5] A. Handson, H. Hui, Quaternion frame approach to streamline visualization, IEEE Tran. Vis. Comput. Grap. 1 (1995) 164-172.

[6] F. Udwadia, A. Schtittle, An alternative derivation of the quaternion equations of motion for rigid-body rotational dynamics, J. Appl. Mech. 77(4) (2010) 044505-4 pp.

[7] S. Gupta, Linear quaternion equations with application to spacecraft attitude propagation. IEEE Aerospace conference proceedings 1 (1998) 69-76.

[8] S. Leo, G. Ducati, Solving simple quaternionic differential equations, J. Math. Phys. 44(5) (2003) 2224-2233.

[9] J. Campos, J. Mawhin, Periodic solutions of quaternionic-values ordinary differential equations, Annali di Matematica 185 (2006) S109-S127.

[10] H. Żołdek, Classification of diffeomorphisms of S4 induced by quaternionic Riccati equations with periodic coefficients, Topol. Methods Nonlinear Anal. 33 (2) (2009) 205-215.

[11] P. Wilczynski, Quaternionic valued ordinary differential equations. The Riccati equation, J. Differential Equations 247 (2009) 2163-2187.

[12] P. Wilczynski, Planar nonautonomous polynomial equations. II. Coinciding sectors, J. Differential Equations 246 (7) (2009) 2762-2787.

[13] P. Wilczynski, Quaternionic-valued ordinary differential equations II. Coinciding sectors, J. Differential Equations 252 (2012) 4503-4528.
[14] A. Gasull, J. Llibre, X. Zhang, One-dimensional quaternion homogeneous polynomial differential equations, J. Math. Phys. 50 (8) (2009) 082705-17 pp.

[15] X. Zhang, Global structure of quaternion polynomial differential equations, Comm. Math. Phys. 303 (2) (2011) 301-316.

[16] Z.F. Cai, K.I. Kou, Laplace transform: a new approach in solving linear quaternion differential equations, Math. Meth. Appl. Sci. (2017), Doi: 10.1002/mma.4415.

[17] K.I. Kou, Y.H. Xia, Linear quaternion differential equations: basic theory and fundamental results (i). Available from: http://arxiv.org/abs/1510.02224 [Accessed on October 8, 2015].

[18] K.I. Kou, W.K. Liu, Y.H. Xia, Linear quaternion differential equations: basic theory and fundamental results (ii). Available from: http://arxiv.org/abs/1602.01660 [Accessed on February 4, 2016].

[19] Y.H. Xia, H. Huang, K.I. Kou, An algorithm for solving linear non-homogeneous quaternion-valued differential equations. Available from: http://arxiv.org/abs/1602.08713 [Accessed on February 28 2016].

[20] D.Cheng,K.I. Kou, Y.H. Xia, Linear quaternion-valued Dynamic Equations on Time Scales Feb 2016. Available from: http://arxiv.org/abs/1607.00105v1 [Accessed on July 1 2016].

[21] H.Aslaksen, Quaternionic determinants, Math. Intellig. 18(3) (1996) 57–65.

[22] N.Cohen, S. De Leo, The quaternionic determinant, Elec. J. Lin. Alg. 7 (2000) 100-111.

[23] J. Dieudonne, Les determinants sur un corps non-commutatif, Bull. Soc. Math. France 71 (1943) 27–45.

[24] E. Study, Zur Theorie der linearen Gleichungen, Acta Math. 42 (1920) 1–61.

[25] A. Cayley, On certain results relating to quaternions, Philos. Mag. 26 (1845) 141145. Reprinted in The collected mathematical papers vol. 1, 123-126, Cambridge Univ. Press, 1889.

[26] E. H. Moore, On the determinant of an hermitian matrix of quaternionic elements, Bull. Amer. Math. Soc. 28 (1922) 161–162.

[27] F. J. Dyson, Quaternion determinants, Helv. Phys. Acta 45 (1972) 289–302.

[28] L. Chen, Definition of determinant and Cramer solution over the quaternion field, Acta Math. Sinica (N.S.) 7 (1991) 171–180.
[29] I.I. Kyrchei, Cramer’s rule for quaternion systems of linear equations, Fundam. Prikl. Mat. 13(4) (2007) 67-94.

[30] I.I. Kyrchei, The theory of the column and row determinants in a quaternion linear algebra, in: Albert R. Baswell (Eds.), Advances in Mathematics Research 15, Nova Sci. Publ., New York, pp. 301-359 (2012).

[31] I.I. Kyrchei, Determinantal representations of the Moore-Penrose inverse over the quaternion skew field and corresponding Cramer’s rules, Linear Multilinear Algebra, 59 (2011) 413-431.

[32] I.I. Kyrchei, Determinantal representations of the Drazin inverse over the quaternion skew field with applications to some matrix equations, Appl. Math. Comput. 238 (2014) 193-207.

[33] I.I. Kyrchei, Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations, Linear Algebra Appl. 438(1) (2013) 136–152.

[34] I.I. Kyrchei, Determinantal representations of the W-weighted Drazin inverse over the quaternion skew field, Appl. Math. Comput. 264 (2015) 453–465.

[35] I.I. Kyrchei, Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse, Appl. Math. Comput. 309 (2017) 1-16.

[36] A. Kleyn, I. Kyrchei, Relation of row-column determinants with quasideterminants of matrices over a quaternion algebra, In: I.I. Kyrchei (Ed.), Advances in Linear Algebra Research, Nova Sci. Publ., New York, pp. 299-324 (2015).

[37] G.J. Song, C.Z. Dong, New results on condensed Cramer’s rule for the general solution to some restricted quaternion matrix equations, J. Appl. Math. Comput. 53 (2017) 321–341.

[38] G.J. Song, Bott-Duffin inverse over the quaternion skew field with applications, J. Appl. Math. Comput. 41 (2013) 377–392.

[39] G.J. Song, Characterization of the W-weighted Drazin inverse over the quaternion skew field with applications, Electron. J. Linear Algebra 26 (2013) 1–14.

[40] M.P. Drazin, Pseudoinverse in associative rings and semigroups, Am. Math. Monthly 65 (1958) 506-514.

[41] L. Huang, W. So, On left eigenvalues of a quaternionic matrix, Linear Algebra Appl. 323 (2001) 105-116.

[42] W. So, Quaternionic left eigenvalue problem, Southeast Asian Bull. Math. 29 (2005) 555-565.
[43] R. M. W. Wood, Quaternionic eigenvalues, Bull. Lond. Math. Soc. 17 (1985) 137-138.

[44] J.L. Brenner, Matrices of quaternions, Pac. J. Math. 1 (1951) 329-335.

[45] E. Macías-Virgós, M.J. Pereira-Sáez, A topological approach to left eigenvalues of quaternionic matrices, Linear Multilinear Algebra 62(2) (2014) 139-158.

[46] A. Baker, Right eigenvalues for quaternionic matrices: a topological approach, Linear Algebra Appl. 286 (1999) 303-309.

[47] T. Dray, C. A. Manogue, The octonionic eigenvalue problem, Adv. Appl. Clifford Algebr. 8(2) (1998) 341-364.

[48] F. Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21-57.

[49] D.R. Farenick, B.A.F. Pidkowich, The spectral theorem in quaternions, Linear Algebra Appl. 371 (2003) 75-102.

[50] F. O. Farid, Q.W. Wang, F. Zhang, On the eigenvalues of quaternion matrices, Linear Multilinear Algebra 59(4) (2011) 451-473.

[51] P. Lancaster, M. Tismenitsky, Theory of matrices, Acad. Press., New York, 1969.

[52] S. L. Campbell and C.D. Meyer, Generalized inverse of linear transformations, Corrected reprint of the 1979 original. Dover Publications, Inc., New York, 1991.