Analysis of MDI High-Degree Mode Frequencies and their Rotational Splittings

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Abstract
Here we present a detailed analysis of solar acoustic mode frequencies and their rotational splittings for modes with degree up to 900. They were obtained by applying spherical harmonic decomposition to full-disk solar images observed by the Michelson Doppler Imager onboard the Solar and Heliospheric Observatory spacecraft. Global helioseismology analysis of high-degree modes is complicated by the fact that the individual modes cannot be isolated, which has limited so far the use of high-degree data for structure inversion of the near-surface layers \((r > 0.97R_\odot)\). In this work, we took great care to recover the actual mode characteristics using a physically motivated model which included a complete leakage matrix. We included in our analysis the following instrumental characteristics: the correct instantaneous image scale, the radial and non-radial image distortions, the effective position angle of the solar rotation axis and a correction to the Carrington elements. We also present variations of the mode frequencies caused by the solar activity cycle. We have analyzed seven observational periods from 1999 to 2005 and correlated their frequency shift with four different solar indices. The frequency shift scaled by the relative mode inertia is a function of frequency alone and follows a simple power law, where the exponent obtained for the \(p\) modes is twice the value obtained for the \(f\) modes. The different solar indices present the same result.

Keywords: Helioseismology, Observations; Instrumental Effects; Oscillations, Solar; Solar Cycle, Observations

1. Introduction
The central frequencies of solar acoustic modes, which are obtained using spherical harmonic decomposition, have been successfully used to determine the solar interior structure to as close as 21 Mm to the solar surface \((r < 0.97R_\odot)\) using modes with angular degrees \(\ell \leq 300\) \((\text{e.g., Gough, Kosovichev, Toomre, } et al., 1996)\). The inclusion of high-degree modes \((\text{i.e., up to } \ell = 1000)\) has the potential to improve dramatically the inference of the sound speed and the adiabatic exponent \((\Gamma_1)\) in

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the outermost 2 to 3% of the solar radius, allowing to construct localized kernels as close to the solar surface as 1.75 Mm (Rabello-Soares, Basu, Christensen-Dalsgaard, et al., 2000). The effects of the equation of state, through the ionization of hydrogen and helium, are felt most strongly in the outer layers of the Sun, making this shallow region of particular interest. Furthermore, dynamical effects of convection, and the processes that excite and damp the solar oscillations, are predominantly concentrated in this region. Although the spatial resolution of the modern helioseismic instruments allows us to observe oscillation modes up to \( \ell \leq 300 \) and higher, only a small fraction of them are currently used. Unfortunately, analysis of high-degree data is complicated by the fact that the individual modes cannot be isolated (e.g., Rabello-Soares, Korzennik and Schou, 2001).

The solar structure is not static, but changes over the solar cycle. It is well known that the mode frequencies change with solar activity. It seems that the responsible mechanism is restricted to the outer layers of the Sun (Libbrecht and Woodard, 1990), where the high-degree modes are confined. However, at the moment, there is no general agreement in the precise physical mechanism that gives rise to the frequency variation. It is likely a product of the change in the subphotospheric small-scale magnetic field strength with the solar activity cycle (e.g., Goldreich, Murray, Willette, et al., 1991). Accordingly to Dziembowski and Goode (2004), the frequency shift is easily explained in terms of a variation in the turbulent velocities associated with the magnetic field variation rather than the sole direct effect of the magnetic field itself. Li, Basu, Sofia, et al. (2003), using models of the structure and evolution of the Sun, found that turbulence near the surface of the Sun plays a major role in solar variability, and only a model that includes a magnetically modulated turbulent mechanism can agree with the observed correlation between the frequency shift and the solar cycle. In such a dynamic model, the evolution of the subsurface layers of the Sun through the activity cycle plays an important role.

The frequencies of the global modes give the radius and latitude \((r, \theta)\) part of the structure. While, local helioseismic techniques such as ring-diagram analysis (Hill, 1988) allow the determination of the three-dimensional structure of the Sun, allowing the study of localized areas in the solar surface, such as those in active regions. Large variations of the mode frequencies observed in and near sunspots in comparison to magnetically quiet regions are well known to be correlated with variations in the average surface magnetic field between the corresponding regions. (e.g., Rajaguru, Basu and Antia, 2001 and Rabello-Soares, Bogart and Basu, 2007). Whether the frequencies are changed directly by the magnetic field or indirectly through an associated change in the solar structure (such as a pressure change) is still a matter of debate. A detailed analysis of the frequency-shift characteristics will hopefully help understand their physical origin.

Basu, Antia and Bogart (2004), using ring-diagram analysis, found that the sound speed is lower in the immediate subsurface layers of an active region than of a magnetically quiet region, while the opposite is true for depths below about 7 Mm. However, Basu and Antia (2002), using global analysis, have found no observable structural changes in the inner layers of the Sun below a depth of 21 Mm associated with the magnetic-activity induced frequency shifts. They were, however, unable to get closer to the solar surface due to the lack of high-degree modes in their mode set. High-degree global analysis is important to complete the picture of the near-surface layers. Besides, the determination of high-degree frequencies using different methods allows us to check the results against each other giving confidence in the
results and avoiding systematic errors. We should point out that, although the high-degree modes have short lifetimes (one–ten hours for $100 \leq \ell \leq 600$ accordingly to Burtseva, Kholikov, Serebryanskiy, et al., 2007) propagating only locally, they are averaged over most of the solar surface using spherical harmonic decomposition (over a relatively long time series) and thus can still be called global analysis.

In the traditional global helioseismology data-analysis methodology, a time series of full-disk Doppler solar images is decomposed into spherical harmonic coefficients, characterized by its degree ($\ell$) and its azimuthal order ($m$). Each coefficient time series is Fourier transformed, and the order of the radial wave function ($n$) gets separated in the frequency domain. However, a spherical harmonic decomposition is not orthonormal over less than the full sphere — i.e., the solar surface that can be observed from a single view point— resulting in what is referred as spatial leakage. At low and intermediate degrees, most of these leaks are separated in the frequency domain from the target mode (except for some $m$ leaks) and individual modes can be identified and fitted. However, at high degrees, the spatial leaks lie closer in frequency (due to a smaller mode separation) and, at high frequency, the modes become wider (as the mode lifetimes get smaller), resulting in the overlap of the target mode with the spatial leaks that merges individual peaks into ridges (see Figure 1 in Rabello-Soares, Korzennik and Schou, 2001). The characteristics of the resulting ridge (central frequency, amplitude, etc.) do not correspond to those of the underlying target mode. This has so far hindered the estimation of unbiased mode parameters at high degrees.

To recover the actual mode characteristics, we need a very good estimation of the relative amplitude of the spatial leaks present in a given $(\ell, m)$ power spectrum, also known as the leakage matrix, which in turn requires a very good knowledge of the instrumental properties (Rabello-Soares, Korzennik and Schou, 2001). In our previous papers (Rabello-Soares, Korzennik and Schou, 2001 and Korzennik, Rabello-Soares and Schou, 2004, hereafter KRS), we described in detail the large influence of the instrumental properties on the amplitude of the leaks and as a consequence in the determination of unbiased high-degree mode parameters.

In the following, we will first describe the data used in this analysis and the ridge-to-mode correction applied to them (Sections 2 and 3). In Section 4, we will discuss the influence on the mode parameters of each of the instrumental properties that were included in the spherical harmonic decomposition of the solar images. We then analyze in Section 5 the characteristics of the high-degree mode frequencies and their rotational splittings obtained in this work. Finally, in Section 6, we analyze the frequency variation induced by the solar cycle.

2. Observations and Ridge-Parameter Extraction

The data used in this work consist of full-disk Dopplergrams obtained at a one-minute cadence by the Michelson Doppler Imager (MDI) onboard the Solar and Heliospheric Observatory (SOHO). We have used two distinct sets of data. One while MDI was operating in its 4" resolution mode, allowing the detection of oscillation modes up to $\ell \approx 1500$, which we will call from now on the high-$\ell$ data set. This is the so-called Dynamics Program observing mode, which is available every year for two to three months. The second one (hereafter referred to as the medium-$\ell$ data set) using MDI Structure Program that provides almost continuous coverage year round. In
Table 1. Details of the analyzed time series. The corresponding relative mean values of solar UV spectral irradiance and their standard deviation are also listed as an indication of the solar activity level.

| Year | Starting Date | Duration | Solar Index | Starting Date |
|------|---------------|----------|-------------|---------------|
|      | high-ℓ set    |          | rel. to max. (in %) | medium-ℓ set  |
| 1999 | Mar. 13       | 77 days  | 40 ± 13     | Feb. 03       |
| 2000 | May 27        | 45 days  | 69 ± 11     | Apr. 10       |
| 2001 | Feb. 28       | 90 days  | 61 ± 14     | Jan. 23       |
| 2002 | Feb. 23       | 72 days  | 80 ± 8      | Mar. 31       |
| 2003 | Oct. 18       | 38 days  | 51 ± 17     | Oct. 28       |
| 2004 | Jul. 04       | 65 days  | 36 ± 11     | Aug. 11       |
| 2005 | Jun. 25       | 67 days  | 30 ± 9      | May 26        |

For the high-ℓ set, we computed the spherical harmonic decomposition of the MDI images for modes with 100 ≤ ℓ ≤ 900. The resulting time series were Fourier transformed in small segments (4096 minutes) whose spectra were averaged to produce an averaged power spectrum with a low but adequate frequency resolution to fit the ridge while reducing the realization noise. The number of averaged spectra varies with the year, from 12 segments in 2003, to 30 in 2001. Most of the known instrumental effects relevant to the high-degree analysis were included in the spatial decomposition and will be described in Section 4. The peaks in each (ℓ, m) spectrum were then fitted using an asymmetric Lorentzian profile with an additive background term, given by ten to the power of a second-degree polynomial in frequency (Equation 5 in KRS). Since the number of segments used in the average of each (ℓ, m) spectrum is large enough, its χ² distribution can be approximate by a Gaussian distribution and a least-square fitting was used. The fitted Lorentzian profile is characterized by the following parameters: frequency (νn,ℓ,m), amplitude (A_n,ℓ,m), width (γ_n,ℓ,m) and, asymmetry (α_n,ℓ,m). The asymmetric profile used is equivalent to the one defined by Nigam and Kosovichev (1998) where their asymmetric parameter is equal to \( \alpha/(2 - \alpha) \). The frequency splittings were parametrized in terms of Clebsch-Gordan coefficients up to a6 (Ritzwoller and Lavely, 1991). The number of modes analysed was reduced without loss of information to the work described in this paper and hence was easy to handle. The fitting was carried out only every tenth ℓ and only for some 50 equally spaced m values at each ℓ. The central frequency, i.e. the frequency free of splitting effects, is taken to be the frequency given by \( m = 0 \) in the splitting parametrization. Since the even splitting coefficients are zero on average, it is the same as the mean frequency (averaged over \( m \)). Notice that the fitted parameters are ridge parameters and do not correspond to the associated mode parameters as discussed in the introduction. In this study, high-ℓ time series available from 1999 until 2005 were used and their properties are listed in Table 1.

The focus of this work are modes with ℓ > 300 and therefore it is centered on the analysis of the high-ℓ set. We used however the results obtained by one of us...
Figure 1. Coverage, in an $\ell - \nu$ diagram, of the medium-$\ell$ (red) and high-$\ell$ (black) mode parameters for 2005. The coverage is very similar for all epochs.

analyzing the medium-$\ell$ time series (Schou, 1999) to compare with and complement our analysis. Using 72-day long time intervals, the central frequency and splitting coefficients for a given $(n, \ell)$ mode were determined directly by fitting symmetric Lorentzian profiles to its power spectra (Schou, 1999). Every $p$ mode up to $\ell \approx 200$ (up to $\ell \approx 300$ for the $f$ modes) and every $m$ were fitted. The 72-day time series that best overlap in time with the high-$\ell$ time series were selected and are listed in Table 1. Only modes with $\ell \geq 20$ were used. The mode coverage of both analyses is illustrated in Figure 1.

In the medium-$\ell$ analysis, the instrumental effects described in Section 4 were not taken into account, and neither were the distortion of the eigenfunctions by the solar differential rotation or the horizontal component of the oscillation (both described in Section 3). However, in the frequency and degree ranges of the medium-$\ell$ analysis, the spatial leaks are well separated from the target mode and individual modes can be identified and fitted, making the above-mentioned effects not as crucial as they are for the high-$\ell$ analysis. Another relevant difference between the medium- and high-$\ell$ analysis is that the medium-$\ell$ power spectra were fitted using symmetric profiles, which is well known to lead to systematic errors in the frequency measurements (e.g. Toutain, Appourchaux, Fröhlich, et al., 1998; Basu and Antia, 2000). Larson and Schou (2008) have recently re-analyzed the medium-$\ell$ time series and reported that several of these physical effects result in highly significant changes in the mode parameters. Their Figure 1 shows the total correction to be applied to the medium-$\ell$ frequency and splitting coefficients, $a_1$ and $a_2$, used here. Their Figure 2 illustrates the frequency changes due to each of these effects. The first two panels correspond to the distortion of the eigenfunctions by the solar differential rotation and the horizontal component of the leakage matrix respectively. The third, forth, and fifth panels correspond to the instrumental effects described here in Sections 4.1, 4.2, and 4.3 respectively. The panels at the bottom show the difference obtained between fitting
symmetric and asymmetric Lorentzians. Although these corrections are significant, they correspond to very small variations in the results presented in this paper and do not affect our conclusions, as it will be discussed later (Section 5).

3. Ridge-to-Mode Correction

Our methodology to recover the mode characteristics from the ridges observed at high-degree and high-frequency power spectra consists in generating and fitting a sophisticated model of the underlying modes that contribute to the ridge power distribution and deduce the offset ($\Delta^o$) between the ridge properties and the target mode (Korzennik, 1998). A synthetic power spectrum is computed for each $$(\ell, m)$$ mode consisting of several asymmetric Lorentzians for each $n$: one for the target mode $$(\ell, m)$$ and one for each spatial leak $$(\ell', m')$$ with a relative amplitude given by the leakage matrix. It has the same frequency resolution as the high-$\ell$ set power spectra and it is generated for the same set of $$(\ell, m)$$ modes.

We used the complete leakage matrix (i.e., radial and horizontal components) where the horizontal-to-vertical displacement ratio is given by Christensen-Dalsgaard (2003): $GM_\odot L/(R^2_\odot \omega^2_n,\ell)$, where $G$ is the gravitational constant, $M_\odot$ is the solar mass, $R_\odot$ is the solar radius, $\omega$ is the cyclic frequency ($\omega = 2\pi \nu$), and $L^2 = \ell (\ell + 1)$. For each $$(\ell, m)$$ synthetic power spectrum, we have taken into account the contribution of the spurious modes $$(\ell', m')$$ that obey the following expressions: $|m' - m| \leq 10$ and $|\ell' - \ell| \leq \ell_d$, where $\ell_d = 12$ for $\ell \leq 600$ and $\ell_d$ is equal to $0.02 \ell$ rounded to the closest integer for $\ell > 600$. Spurious modes with $$(\ell', m')$$ values that differ from those of the power spectrum $$(\ell, m)$$ by more than the amount specified in the equations above have a very small relative amplitude and their contribution to the synthetic power spectrum profile is negligible, i.e., their inclusion or not does not affect the fitted parameters of the peaks in the power spectrum (Rabello-Soares, Korzennik and Schou, 2005). We also included in our model the distortion of the eigenfunctions by the solar differential rotation as described in Woodard (1989), using the values given by Schou, Antia, Basu, et al. (1998) for the solar rotation. The coefficients in this superposition become negligible when $|\ell - \ell'| \geq \ell_c$ where $\ell_c = 10$ for $\ell \leq 400$ and the next even integer to $3 + 0.02 \ell$ for $\ell > 400$ (Rabello-Soares, Korzennik and Schou, 2005).

The profile resulting from the overlap of several profiles of nearby spatial leaks is reasonably well modeled by a single profile when the ratio of the mode width to their separation (given by $\partial \nu / \partial \ell$) becomes large. The difference is smaller than the rms of the observed residuals to the fit, 5%–7% at all degrees, which are dominated by the realization noise (KRS). The synthetic power spectrum is then fitted following the methodology applied to the high-$\ell$ set (Section 2), providing the ridge central frequency $\nu^{\text{model}}_{n,\ell}$ and its splittings $a^{\text{model}}_{i_{n,\ell}}$ ($i = 1, 6$). The offset is given by the difference between the modeled ridge and the target mode parameter given by the input value of the parameter used to generate the synthetic power spectra. For a given mode $$(n, \ell)$$, the offsets in central frequency can be then written as:

$$\Delta^o \nu_{n,\ell} = \nu^{\text{model}}_{n,\ell} - \nu^{\text{input}}_{n,\ell}.$$  \hspace{1cm} (1)

The offsets in each of the fitted $a$-coefficients are obtained in the same manner. Realistic input values based on observed mode parameters were used (as described...
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in KRS). The input linewidth is given by the square root of $\gamma_{\text{mode}}^2 + \gamma_{WF}^2$, where $\gamma_{\text{mode}}$ is an estimate of the mode linewidth and $\gamma_{WF}$ is the width of the window function in our case given by the length of the high-$\ell$ set time string (i.e., 4096 minutes). This expression was obtained by Korzennik (1990) assuming that the observed power spectrum is a convolution of the “true” power spectrum (i.e., the one that would have been obtained with an infinite time series), with the power spectrum of the window function both represented by Gaussian profiles, which is adequate for the purpose at hand.

If the leakage matrix is correct and complete, our simulations should adequately estimate the parameter offsets ($\Delta^\circ$) and we would be able to obtain corrected mode parameters from the observed ridge. Thus in the case of the central frequency, we would have:

$$\nu_{n,\ell}^{\text{corrected}} = \nu_{n,\ell}^{\text{observed}} - \Delta^\circ \nu_{n,\ell}.$$  (2)

The corrected mode $a$-coefficients are obtained as in Equation (2).

Figure 2 shows the estimated offsets for the central frequency and the splitting coefficients $a_1$, $a_2$, and $a_3$. They are primarily a function of frequency. Except for $a_2$, the offsets are quite large in comparison with the observed fitting uncertainties of the corresponding parameter which, for the central frequency, $a_1$ and $a_3$ coefficients, are usually smaller than 0.5 $\mu$Hz, 1 nHz, and 0.8 nHz respectively. The width and amplitude offsets will be described in a future paper. Accordingly to Korzennik (1998), the asymmetry of the ridge seems to be the same as the asymmetry of the underlying mode (see also KRS).

Another methodology to recover the mode characteristics from the observed ridges is to fit a sum of Lorentzians or asymmetric profiles to a given ridge: one for the target mode and one for each spatial leak that has a significant amplitude (Reiter, Rhodes, Kosovichev, et al., 2002; Reiter, Kosovichev, Rhodes, et al., 2003; Reiter, Rhodes, Kosovichev, et al., 2004). Such an approach is likely to be particularly appropriate in the transition region between well-resolved modes and ridges, where the spatial leaks start to overlap with the target mode but do not yet blend fully into a ridge. The reasoning behind the method used here is that, when the spatial leaks are completely blended into a ridge, there is not enough information in the power spectra to justify modeling the individual spatial leaks as part of the fitting of a given observed spectrum. The difference in the profile of a sum of overlapped asymmetric profiles and a single asymmetric profile is much smaller than the observed fitting residuals. This method is significantly less computationally demanding since the profiles fitted are much simpler. As a result we can more easily check its reliability by using different leakage matrices or time series produced with a different spatial decomposition, and quantitatively validating the corrections; as described in Section 4 and KRS. This validation step ensures the reliability of the method and allow us to estimate a quantitative upper limit on any residual bias, a crucial step in the intricate high-$\ell$ analysis. Note that the method described by Reiter et al. also relies on a very good estimation of the leakage matrix to obtain unbiased mode parameters. Indeed, the same leakage matrices have been used for both methods and it is likely that both the random and systematic errors will be quite similar where the modes are fully blended into ridges.
Figure 2. Estimated offsets for the central frequency ($\nu$) and splitting coefficients ($a_1$, $a_2$, and $a_3$). The error bars correspond to the uncertainties when fitting the synthetic power spectra. Modes with $n \leq 5$ are in color with the $f$ modes in red.

4. Influence of Instrumental Properties on the High-Degree Power Spectra

In the determination of unbiased high-degree mode parameters, it is necessary to have a good knowledge of the properties of the instrument used to collect the data. The instrumental characteristics must be taken into account either in the image spatial decomposition or in the leakage matrix calculation to obtain a correct estimation of the amplitude of the spatial leaks. In a continuing effort to infer unbiased estimates of high-degree mode parameters using the high-resolution observations from the MDI Dynamics Program, the following instrumental properties were included in the spatial decomposition of the high-$\ell$ data set one at a time and their effect on the observed power spectra analyzed: (i) the correct instantaneous image scale, (ii) the radial image distortion, (iii) the non-radial image distortion, (iv) the effective $P$ angle, and (v) a correction to the Carrington elements. Once a given instrumental property is analyzed, it is incorporated in the analysis from then on in the paper. For example,
in the analysis of the radial image distortion (Section 4.2), the correct image scale (Section 4.1) was used and, in the non-radial image distortion analysis (Section 4.3), the correct image scale and the radial image distortion were included.

4.1. Image Scale

Variations in the amount of defocus of the instrument have a direct influence in the image scale at the detector\(^1\). Although the MDI instrument has been very stable over the more than 11 years that it has been observing the Sun, continuous exposure to solar radiation has increased the instrument front window absorption resulting in a continuous small increase of the instrument defocus. Moreover the change of the front-window temperature due to the satellite orbit around the Sun also adds a small annual variation in the image defocus. The instrument has however an adjustable focus with nine possible positions chosen to best suit various science needs, resulting unfortunately in abrupt jumps in the image defocus every time that a new position is chosen and which are responsible for the largest variations (see Figure 5 in Rabello-Soares, Korzennik and Schou, 2001). The average size change (at the solar limb) per focus step is \(0.529 \pm 0.002\) pixels (Kuhn, Bush, Emilio, \textit{et al.}, 2004). Due to these different time-varying focus variation, the image scale must be continuously estimated and the correct value used in the spatial decomposition. The image scale is obtained by measuring the observed image radius, which is defined as the inflection point in the radial limb-darkening function. The \texttt{FNDLMB} routine in the GONG Reduction and Analysis Software Package (GRASP) was used. It is available from the National Solar Observatory, Tucson, AZ, U.S.A..

To show the influence of the image scale in the high-\(\ell\) mode parameters, we compared the observed ridge parameters obtained using two different spatial analyses of the 1999 time series. In one of the analyses, the time-varying image scale was obtained for the actual observational period (\textit{i.e.}, the correct image scale) and used in the spatial decomposition. In the other one, a constant value obtained from observations taken in early 1996 (at the beginning of the mission), \textit{i.e.}, the wrong value for the 1999 time series, was used. It is 0.27\% larger on average than the actual 1999 time-varying image scale. A variation in the observed ridge parameter due to a change in the data analysis corresponds to an identical variation in its offset (\(\Delta^\circ\)) in order to obtain the correct mode parameter (see Equation 2). Figure 3 shows the corresponding offset variation. The frequency offsets obtained using the correct image scale are systematically larger than the ones using a slightly larger image scale and their difference increases with frequency (Figure 3). There is no indication of a degree dependence. The variation in the frequency offset is larger for the \(f\) modes than for \(p\) modes (respectively upper and lower branches seen at frequencies smaller than 3 mHz in Figure 3). This suggests that the image scale correction has a strong effect on the horizontal component of the leakage matrix. The \(a_1\) coefficients obtained using the correct image scale are systematically larger by 1.44 \(\pm\) 0.02 nHz than using an image scale 0.27\% larger on average. Their difference is plotted in Figure 3 (bottom panel) arbitrarily against degree instead of frequency. The effect on the other parameters is very small and their mean difference is listed in Table 2. The 2000 data set was used to calculate the values in Table 2, except for the variations due to the effect

\(^1\)Image scale is the ratio between the size of the solar image at the detector and its actual size.
of the image scale where the 1999 epoch was used. The values in the table should not depend significantly on the observational period used since we are comparing variations in the analysis of the same time series. The variations in the frequency offset (Figure 3 top) are quite large in comparison with their absolute values shown in Figure 2. The theoretical offset estimation in Figure 2 corresponds to the case where the correct values for all instrumental effects were taken into account in the spherical harmonic decomposition.

The red points in Figure 3 were calculated taking into account the image scale error in the leakage matrix calculation instead of in the image spatial decomposition. The leakage matrix is calculated assuming that a constant and 0.27% larger image scale than the actual value was used in the image spatial decomposition. The variation in the parameter offsets estimated from the synthetic power spectra using Equation (1) matches very well the observations in most cases. For modes above 4 mHz, the frequency changes were underestimated by this method. This could be because we did not take into account the image-scale temporal variation, only the average difference for the entire observing period, while the spatial decomposition was carried out using a instantaneous image scale estimation.

Note that the smearing of the image represented by the point-spread function (PSF) is not taken into account nor its variation with the amount of defocus. Our preliminary analysis indicated that including a simple model of the azimuthally averaged estimation of the PSF in the leakage matrix calculation has a very small effect on the offsets. It affected mostly the frequency offset and only by less than 0.2 μHz (Rabello-Soares, Korzennik and Schou, 2006). Recently we found that the observed ridge frequencies obtained for the observational periods where MDI was set to a large defocus, i.e., 1996 to 1998, are larger than the ones obtained for the other periods where the instrument was nearly in focus, after correcting for the solar-cycle variation; their maximum difference is of the order of a few μHz (Rabello-Soares, Korzennik and Schou, 2008). A possible explanation is that an azimuthally averaged estimation of the PSF is not a good approximation to the true PSF of the instrument, which is known to depend on the azimuth angle (Schou and Bogart, 1998) with a phase that changes with focus position (e.g., KRS). Unfortunately, there is not a good estimate of the PSF for the MDI Dynamics Program at the moment. We plan to use an approximation to the azimuthally-varying PSF and analyze its influence in the mode parameter determination in the future.

4.2. Radial Image Distortion

The ray-trace model of the MDI optical configuration predicts a radial distortion (∆r) which depends on the distance from the CCD center (r):

\[
\frac{\Delta r}{r} = b (r^2 - \langle r \rangle^2),
\]

where \( b = 7 \times 10^{-9} \) pixels\(^{-2} \) and \( \langle r \rangle \) is the observed image mean radius (Kuhn, Bush, Emilio, et al., 2004). The distortion causes the apparent solar radius to be larger by \( \approx 0.17\% \) (\( \approx 0.8 \) pixels or 17 μm). Thus, the second term in Equation (3) was added to ensure that the distorted and undistorted images have the same mean radius.

As in Figure 3, Figure 4 shows the offset variation corresponding to the difference in the observed ridge frequencies when including this distortion in the spatial decomposition. Similarly to the image scale, the frequency offset increases with frequency.
Figure 3. Difference in the frequency and the $a_1$-coefficient offsets using, in the spatial decomposition for the 1999 high-$\ell$ set, the correct time-varying image scale ("corr") and a constant value, which is 0.27% larger on average than the actual values ("wrong"). The errors are given by the fitting uncertainties. The estimation of this effect when it is included in the leakage matrix calculation instead (see text for details) is shown in red.

The similarity is expected, since the radial distortion changes the image scale by an amount that is a function of the distance from the CCD center $r$ (Equation 3). The difference in the ridge width also increases with frequency from $-0.064 \pm 0.004 \mu\text{Hz}$ at $\nu < 2.5 \text{ mHz}$ to $0.28 \pm 0.04 \mu\text{Hz}$ at $\nu > 4.5 \text{ mHz}$, in the sense including minus not including the distortion, but it is very small in comparison with the fitting uncertainties and barely significant. The effect on the other parameters is small and it is listed in Table 2. The ridge modeling of this effect, introducing the radial distortion in the leakage matrix, agrees well with the observations (red circles in Figure 4).

4.3. Non-Radial Image Distortion

Solar images as observed by MDI have a nearly elliptical shape with a difference between the semi-major and the semi-minor axis of about 0.6 pixels. This is consistent
with a small tilt ($\approx 2^\circ$) in the detector around an axis that is rotated $56^\circ$ from the detector’s horizontal $x$-axis and it introduces a non-radial image distortion. In KRS, we estimated this distortion using different observational methods. Unfortunately, the distortion varies by as much as 35% depending on the data used, with the correspondent CCD tilt varying from $1.71^\circ$ to $2.6^\circ$. Kuhn, Bush, Emilio, et al. (2004) also estimated the non-radial distortion using yet another observational method – the Mercury transit across the Sun on 7 May 2003 – and found a $3.3^\circ$ tilt. Although their equations to estimate the distortion have the same general form as ours (KRS), there are other differences besides the tilt angle between the two calculations. Their estimation of the distortion varies by 14% in relation to our estimation using a $2.6^\circ$ tilt. These discrepancies in the estimated distortion using different observational methods might be attributed to a number of reasons such as the inaccuracy of our simple model for the non-circular shape of the solar images or the influence of an optical aberration (an asymmetric PSF, for example).
To analyze the effect of the non-radial distortion on the power spectra, we included in the spherical harmonic decomposition our estimation that has the largest tilt angle (2.6°), which corresponds to the distortion that better reproduces the solar limb shape. Although, to the moment, we were unable to determine precisely the non-radial distortion pattern, our estimation provides an improvement to the analysis and it will be incorporated from then on in the paper. Fortunately, it has an overall small effect on the observed spectra, as can be inferred from Table 2, and we can safely extrapolate that small variations from this distortion pattern can only correspond to variations in the parameters smaller than the differences shown in the table which were obtained comparing with using no correction for the non-radial distortion, and most likely negligible.

4.4. Position Angle P

The roll angle of the SOHO spacecraft is maintained such that the effective position angle\(^2\) (\(P_{\text{eff}}\)) of the MDI images should always be zero. However, a 0.2° difference has been measured by intercomparing MDI and GONG images obtained in 1999 and 2000 (Cliff Toner, private communication). Beck and Giles (2005) estimated a 0.07° difference, assuming that there is no equator-crossing flow, after re-gaining contact with the SOHO spacecraft (in 1999) and 0.1° before losing contact. No noticeable effect is seen in the observed ridge parameters after including a 0.2° correction (Table 2). We do not see the 3 nHz variation in the \(a_1\)-coefficient offset that was predicted by our ridge model using a slightly higher correction of 0.25° (KRS). This is probably because we did not accurately model the \(P\)-angle correction in the leakage matrix calculation, but added a very simple approximation of its effect.

\(^2\)It is the position angle of the northern extremity of the solar rotation axis with respect to MDI detector y-axis.
4.5. Carrington Elements

Accordingly to Beck and Giles (2005), the standard values used for the two angles specifying the orientation of the solar rotation axis \((i, \Omega)\), known as the Carrington elements, are off by \(\Delta i = 0.095^\circ \pm 0.002^\circ\) and \(\Delta \Omega = 0.17^\circ \pm 0.1^\circ\). This introduces a time-varying correction in the calculation of the rotation axis projection in the plane of the sky, i.e. the position angle, \(P\), and the roll angle, \(B_0\). This correction in the \(P\) angle will be on top of the one mentioned in the previous section. Introducing a correction of \(\Delta i = 0.1^\circ\) and \(\Delta \Omega = 0.1^\circ\) in the image spatial decomposition has no significant effect on the observed ridge parameters (Table 2).

5. The High-Degree Mode Parameters

Here we analyze the frequencies and splitting coefficients obtained using the high-\(\ell\) data set. The mode parameters were obtained after including the five known instrumental effects in the spatial decomposition of the high-\(\ell\) data sets described in Section 4, fitting their observed power spectra using an asymmetric Lorentzian profile (Section 2) and applying the ridge-to-mode correction to the fitted ridge parameters (Section 3).

First, in Section 5.1, the high-\(\ell\) data set mode frequencies and splitting coefficients are compared with the values obtained by the medium-\(\ell\) analysis. Figure 1 shows the region of common modes between the two sets. This comparison is done to check the goodness of the estimation of the mode parameters from the observed ridge in the high-\(\ell\) data set at these high medium-\(\ell\) common modes. Then, the estimated high-\(\ell\) frequencies and splitting coefficients at \(\ell \geq 100\) are analyzed (Sections 5.2 and 5.3).

5.1. Medium- and High-\(\ell\) Set Comparison

Figure 5 shows the differences between the mode frequency and the splitting coefficients obtained using the 72-day medium-\(\ell\) data set described in Section 2 and using the high-\(\ell\) set both observed during 2004 (Table 1). The high-\(\ell\) set used in this comparison was analyzed as described in Sections 2, 3, and including all instrumental effects analyzed in the Section 4. In the medium-\(\ell\) mode range, the spatial leaks are well separated from the target mode and individual modes can be identified and fitted. By decreasing the observed high-\(\ell\) set frequency resolution using a short time string (4096 minutes), and thus increasing the width of the window function, we force the width of the spatial leaks to increase. The spatial leaks in the high-\(\ell\) power spectra now overlap with the target mode forming a ridge at \(\ell\) as low as 100\(^5\) and the ridge-to-mode correction described in Section 3 is applied. To increase the number of common modes, the high-\(\ell\) power spectra used in the comparison were fitted for every \(\ell\) (and not every tenth \(\ell\)). The set of modes used in this comparison consists of 420 \(p\) modes with \(100 \leq \ell < 200\) and \(1.7 < \nu < 3.5\) mHz and 60 \(f\)-modes with \(230 < \ell < 300\) and \(1.5 < \nu < 1.8\) mHz observed during 2004.

\(^3\) is the angle between the plane of the ecliptic and the solar Equator and \(\Omega\) is the angle between the crossing point of the solar Equator with the ecliptic and the Vernal Equinox.

\(^4\) \(B_0\) is the heliographic latitude of the central point of the disk and presents an annual variation.

\(^5\) An even shorter time series, 2048 minutes, was used to check the results at the lowest high-\(\ell\) modes.
Figure 5. Central frequency and splitting coefficients difference between the medium- and high-ℓ sets observed during 2004 (Table 1). The differences are in the sense high- minus medium-ℓ set. Modes with $n \leq 5$ are in color where the $f$ modes are in red.

The mode frequencies agree within $1 \mu$Hz (Figure 5 top panel). Their difference varies with frequency having a maximum at $2 \text{ mHz}$ and decreasing to zero (or near zero) at $1.6$ and $3 \text{ mHz}$. There is no indication of a degree dependence. Although $1 \mu$Hz is a small value, it is almost one order of magnitude larger than the fitting uncertainty of these modes (0.12 $\mu$Hz on average) and hence undesirable. This seems to indicate that the frequency offsets $\Delta^c \nu$ (Figure 2) are underestimated by $\approx 35\%$ for modes with frequencies in the range 1.7 to 2.3 mHz, this amount decreases to $10\%$ around 1.6 mHz and zero in the interval between 2.6 and 3.2 mHz where the estimation is correct. There is some indication that at $3.4 \text{ mHz}$ the offsets are again underestimated (by $\sim 30\%$). Unfortunately, there are no common modes at higher frequencies.

A variation in the mode frequency that is purely a function of frequency does not affect the outcome of the solar structure inversion, since it is removed together with the near-surface errors in the physics of the solar model (see Section 5.2). The instrumental effect that is probably causing the frequency offset to be underestimated at
certain frequencies might be the instrumental PSF which is not included in the image spatial decomposition or in the leakage matrix calculation (see Section 4.1). Besides the frequency dependence, there is a puzzling frequency difference ($\approx 0.5 \mu$Hz) in the $p_2$ modes (green circles in Figure 5 top panel) with respect to the adjacent $n$ values.

The difference in the $a_1$ coefficients between the medium and high-$\ell$ set is negligible except for $n = 0, 1$ modes (second panel in Figure 5). Their mean difference normalized by the fitting uncertainties is, in units of $\sigma$, $-0.2$ for $n \geq 2$, 2 for $n = 1$, and 3 for the $f$ modes. The horizontal-to-vertical displacement ratio decreases exponentially with $n$. Thus, the differences observed only for $n = 0, 1$ modes could be an indication of an unaccounted instrumental effect that has a stronger effect on the horizontal component of the leakage matrix than on its vertical component. Note also that the $f$ modes in the medium-$\ell$ set are fitted using a different frequency interval around the peak in the power spectrum than the $p$ modes (Schou, 1999).

There are small differences between the medium- and high-$\ell$ set for the splitting coefficients $a_2$ and $a_3$ (third and bottom panels in Figure 5). Their mean differences normalized by the fitting uncertainties of the high-$\ell$ power spectra are $a_2 = 1$ and $a_3 = -1$ in units of $\sigma$.

In conclusion, the estimation of the mode frequency and splitting coefficients from the observed ridge in the high-$\ell$ data set at these moderate-degree values is, in general, quite good. However, there is still room for improvement, specially for the central frequency and the $f$-mode $a_1$ splitting coefficient.

The corrections in the medium-$\ell$ mode parameters calculated by Larson and Schou (2008), described in Section 2, do not have a large influence in our results. Their effect in Figure 5, including the variation between fitting symmetric and asymmetric profiles, is small in comparison with the difference between the medium- and high-$\ell$ sets. The mean total correction in the medium-$\ell$ frequency and splitting coefficients $a_1$, $a_2$ and $a_3$ to be applied to the values used here are: $0.130 \pm 0.004 \mu$Hz, $-0.244 \pm 0.009$ nHz, $(1.78 \pm 0.08) \times 10^{-3}$ nHz and $-0.137 \pm 0.006$ nHz respectively (Larson and Schou, 2008). The average was calculated over the 420 medium- and high-$\ell$ common modes. The largest correction is in the central frequency. The mean frequency variation between fitting symmetric and asymmetric profiles is $0.043 \pm 0.002 \mu$Hz for the common modes (Larson and Schou, 2008). The improved medium-$\ell$ analysis will decrease slightly the differences between medium- and high-$\ell$ frequencies showed in the top panel of Figure 5. However, it does not change the overall behavior of the frequency differences.

5.2. The Central Frequency

Figure 6 shows the difference between the high-$\ell$ set mode frequencies ($100 \leq \ell \leq 900$) and their theoretical value calculated from Christensen-Dalsgaard’s model S (Christensen-Dalsgaard, Dappen, Ajukov, et al., 1996) as a function of degree for different $n$ values. The high-$\ell$ set used in this comparison was analyzed as described in Sections 2, 3, and including all instrumental effects analyzed in the Section 4. The mode range is shown in Figure 1. The medium-$\ell$ set frequencies are also plotted as a reference. In the absence of any acceptable theory to describe the physics of the layers near the solar photosphere, the difference between the observed and theoretical frequencies due to the near-surface errors in the model are well known to be quite large (e.g., Christensen-Dalsgaard and Berthomieu, 1991). The general trend is such that the observed $p$-mode frequencies are smaller than their theoretical prediction
and, at a high enough degree, the differences increase with degree and with frequency. Accordingly to the results obtained by the high-\(\ell\) set, this difference can be as large as 60 \(\mu\)Hz. In fact, for a given \(n\), the frequency differences increase almost linearly with degree with a slope that also increases linearly with \(n\). The high-\(\ell\) set analysis gives consistent frequencies for all observational periods listed in Table 1, where the differences in the frequencies obtained at the different observational periods can be explained by their well known variation with solar cycle activity and it is described in detail in Section 6.

In order to compensate for the frequency shifts due to the near-surface errors in the model, an unknown function \(F_{\text{surf}}\), the so-called surface term \(F_{\text{surf}}\) is usually added to the equation governing helioseismic inversions:

\[
F_{\text{surf}} = Q_{n,\ell} \times \left(\delta\nu_{n,\ell}/\nu_{n,\ell}\right)_{\text{surf}},
\]

where \(Q_{n,\ell}\) is the mode inertia normalized by the inertia of a radial mode of the same frequency. Christensen-Dalsgaard, Thompson and Gough (1989) from the asymptotic theory of solar \(p\) modes pointed out that low- and moderate-degree modes propagate nearly vertically near the surface; thus, their behavior in this region is essentially independent of degree and depends only on frequency. This however does not hold for high-degree modes as can be inferred from Figure 6. A second-order asymptotic approximation, where the surface term is a function not only of frequency but also of \(L/\nu_{n,\ell}\) (Brodsky and Vorontsov, 1993), must be used as shown by Di Mauro, Christensen-Dalsgaard, Rabello-Soares, et al. (2002).

Figure 6 also shows for comparison the frequency difference applying the ring-analysis technique obtained by Rabello-Soares, Bogart and Basu (2007). A magnetically quiet region (15° in diameter) observed by the GONG+ network on 4 July 2005 during Carrington rotation 2031 centered at 115° in longitude and 3° south in latitude was tracked for 8192 minutes at the appropriate photospheric rotation rate. The region crossed the central meridian at the middle of its tracking interval. The tracked region was mapped to a plane using Postel’s projection and its power spectra, given by the three-dimensional Fourier transform of the temporal series of images, were fitted using the 13-parameter model of Basu and Antia (1999). The wavenumber \(k\) can be identified with the degree of a spherical harmonic mode of global oscillations by \(L = kR_\odot\). As the oscillations in a plane-parallel geometry are only discrete in radial order, \(\ell\) does not need to be an integer. Each “mode” is obtained by fitting a region of power spectrum that has significant overlap with those covered by neighboring “modes”, making them not strictly independent. To confirm that the frequencies obtained for this particular solar region correspond to typical values, they were compared with the frequencies obtained for 13 additional quiet regions, with latitudes ranging from -12° to +12° (Rabello-Soares, Bogart and Basu, 2007). Their difference is smaller than 3 \(\mu\)Hz for \(n \leq 5\). Part of this variation is due to the fact that the projection of the spherical solar surface onto a flat detector introduces some foreshortening that depends on the distance of the region from disk center, which can introduce systematic errors in determining the mode characteristics.

The frequency differences (in relation to the theoretical value) obtained using ring analysis present the same trend as using spherical harmonic decomposition, except for the \(f\) modes. In most cases, the ring-analysis frequencies are smaller than the ones obtained by the high-\(\ell\) set. For \(p\) modes, their difference in the sense global minus ring analysis varies between -4 and 6 \(\mu\)Hz for \(n \leq 5\) and it could be as large as 20 \(\mu\)Hz for \(n > 5\). Due to the small size of the region analyzed, the ring-analysis power spectra have low spatial resolution doing poorly at medium degree and at
Figure 6. Difference between observed frequencies and their theoretical value as a function of degree for different $n$ values using the high-$\ell$ set data obtained in 2004 (in black). The differences are in the sense observed minus theoretical. For comparison, the results for the medium-$\ell$ set (red crosses) and from ring-diagram analysis (green stars) are also plotted. The fitting uncertainties are given by the error bars. Note the different scale for the $f$ modes.

For the $f$ modes, the frequency differences (in relation to the theoretical value) obtained using ring and global analysis are different. The medium-$\ell$ $f$-mode frequency is on average $0.9 \pm 0.1 \mu$Hz larger than the theoretical values. The $f$-mode frequency obtained by ring analysis for $\ell \leq 450$ is on average $1.1 \pm 0.6 \mu$Hz larger than the theoretical frequency. For $\ell > 450$, it decreases sharply with degree and it can be
described by a fifth degree polynomial\textsuperscript{6}. At $\ell = 880$, it is 13 $\mu$Hz smaller than the theoretical values. Such a sharp decrease in the $f$-mode frequency in relation to the theoretical values similar to the one observed for the $p$-modes was already reported by other authors (Duvall, Kosovichev and Murawski, 1998 and references within). The frequencies of the $f$ modes, contrary to the $p$ modes, depend only weakly on the hydrostatic structure of the model (e.g. Gough, 1993) and a possible explanation is that the $f$-mode frequencies are reduced by granulation (Murawski and Roberts, 1993). However, the $f$-mode frequency obtained by global analysis presents a different trend in relation to the theoretical frequency. For $\ell < 710$, it is only $2.7 \pm 0.5 \mu$Hz on average larger than the theoretical frequency. At larger $\ell$, it decreases linearly with degree with a slope of $-0.021 \pm 0.001 \mu$Hz; it is zero at $\ell = 850$. The global-analysis results suggest a much weaker interaction between the $f$ modes and the granulation than predicted by the ring analysis.

The error bars in Figure 6, given by the fitting uncertainties are too small to be seen, except for the ring-analysis frequencies at $n \geq 6$. The observed frequency errors obtained using the medium-$\ell$ and high-$\ell$ sets and ring analysis are in the range $0.007 - 0.4 \mu$Hz, $0.06 - 0.6 \mu$Hz and $0.2 - 15 \mu$Hz respectively. The ratio between the global- and ring-analysis fitting uncertainties varies from 0.9 to 33. The ring-analysis uncertainties are similar to those of global analysis at low $n$ ($n \leq 2$) and high $\ell$ ($\ell > 700$). They become much higher than the global-analysis uncertainties as $n$ increases and as $\ell$ decreases. The frequency is not a parameter in the model used to fit the ring-analysis power spectra (see Basu and Antia, 1999), the observational error estimated for the fitted width is used instead, following Rajaguru, Basu and Antia (2001).

5.3. The Rotational Splitting Coefficients

High-$\ell$ splittings can be used to infer the solar rotation rate in the outermost layers of the solar convection zone. At these layers, there is a steep gradient of the rotation, making it a very interesting region to study. This was first suggested by the fact that different surface markers such as sunspots, faculae, H$\alpha$ filaments, and supergranular network present different rotation rates (e.g. Snodgrass, 1992) which has been interpreted by assuming that the markers are anchored at different depths (Foukal, 1972). The solar rotation rate determined by helioseismology (using modes with $\ell \leq 300$) has a local maximum at about $0.95 R_\odot$ (35 Mm below the solar surface) and decreases fast towards the surface (Howe, Christensen-Dalsgaard, Hill, \textit{et al.}, 2007). Using $f$-mode splittings with degree between 117 and 300, Corbard and Thompson (2002) estimated a constant radial gradient of solar angular velocity of around $-400 \text{ nHz}/R_\odot$ in the outer 15 Mm between the Equator and 30° latitude. Above 30° it decreases in absolute magnitude to zero around 50°. The addition of high-degree splittings to this analysis will help to constrain the determination of the solar rotation profile both closer to the solar surface and at higher latitudes.

Figure 7 shows a smooth variation with degree of the high-$\ell$ set $a_1$ coefficients as expected, implying a good estimation of the mode splittings. For a given $n$, the $p$-mode $a_1$ coefficients decrease with degree until $\ell \approx 300$ by 1 – 2% (which

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\textsuperscript{6}The polynomial in $\ell$ has the following coefficients in units of $\mu$Hz: $c_0 = 350 \pm 80$, $c_1 = -2.7 \pm 0.7$, $c_2 = 0.008 \pm 0.002$, $c_3 = (-1.3 \pm 0.3) \times 10^{-5}$, $c_4 = (1 \pm 0.2) \times 10^{-8}$ and $c_5 = (-3.2 \pm 0.7) \times 10^{-12}$. 

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corresponds to 4 – 6 nHz) where it becomes approximately constant. And the f-mode $a_1$-coefficients decrease almost linearly with degree from $\ell \approx 300$ to 900 by 2.5% (or 9 nHz). For a given $n$, the $a_1$ variation with $\ell$ shown in Figure 7 presents the same trend if plotted against the location of the mode lower turning point (given by $L/\nu$), indicating the expected decrease of the solar rotation with solar radius near the solar surface. The p-mode $a_1$-coefficients, especially for $2 \leq n \leq 5$, present a sharp decrease near $\ell = 200$ corresponding to a turning point of depth of $\approx 30$ Mm, which is at the beginning of the solar rotation decrease.

Although the medium-$\ell$ $a_3$-coefficient decreases slightly in absolute value with degree (from -7.8 nHz at $\ell \approx 50$ to -7.3 nHz at $\ell \approx 300$), the high-$\ell$ $a_3$-coefficient mean value (-8.6 ± 1.2 nHz) is larger in absolute value than the mean medium-$\ell$ value (-7.5 ± 0.3 nHz). The high-$\ell$ $a_3$-coefficient does not show any variation with degree. However, it has a small absolute value at low frequency ($\nu \leq 2.2$ mHz), which is similar to the medium-$\ell$ mean: -7.4±0.5 nHz.

While the odd coefficients provide information about the internal solar rotation rate, the even coefficients arise from latitudinal structural variation, centrifugal distortion, and magnetic fields. The $a_2$-coefficients obtained with the high-$\ell$ set are on average zero ($1500 < \nu < 5200$ $\mu$Hz). The $a_2$-coefficients obtained with the medium-$\ell$ set are on average zero for $\nu < 2$ mHz. At higher frequencies, they start to decrease with frequency, where for $\nu = 3550 \pm 50$ $\mu$Hz, its mean value is -0.08 ± 0.03 nHz.

6. Frequency Variation Over the Solar Cycle

The correlation between solar acoustic mode frequencies and the magnetic activity cycle is well established and has been substantially studied during the last and current solar cycles (see for example Chaplin, Elsworth, Miller, et al., 2007, Dziembowski and Goode, 2005 and references therein). However, its physical origin is still a matter of debate and the detailed analysis of the frequency shift characteristics are likely to contribute to solving this problem. The data sets used in this work and listed in Table 1 cover a considerable part of solar cycle 23. They also cover a large degree range ($20 \leq \ell \leq 900$) when combining the medium- and high-$\ell$ sets. In a previous work (Rabello-Soares, Korzennik and Schou, 2008), we used these two data sets to analyze the characteristics of the frequency variation along the solar cycle. Here we improved our previous analysis by calculating the probability that a linear relationship indeed exists between the frequency shift of a given $(n, \ell)$ mode and the solar-activity index. Although the medium- and high-$\ell$ sets are fitted with symmetric and asymmetric profiles respectively, the resulting difference in frequency does not change on average over the solar cycle (Larson and Schou, 2008).

The frequency variation with the solar cycle is given by the frequency difference between 1999 – 2004 epochs with respect to the 2005 epoch, the one with the lowest activity index in our data set. For each $(n, \ell)$ mode, their frequency shifts were fitted assuming a linear relationship with the solar activity index with a zero intercept and using a weighted least-squares minimization. For a detailed description, see Rabello-Soares, Korzennik and Schou (2008). Four different solar activity indices commonly used in the literature and that are available in the period 1999 – 2005 were used in this analysis: the solar UV spectral irradiance\(^7\) (given by the NOAA

\(^{7}\)http://www.ngdc.noaa.gov/stp/SOLAR/ftpSolaruv.html
Figure 7. The $a_1$ rotational-splitting coefficients as a function of degree for the 2004 data set, where the ones obtained with the medium-ℓ set are in red and with the high-ℓ set in black. The fitting uncertainties are given by the error bars.

Mg II core-to-wing ratio), the Magnetic Plage Strength Index (MPSI$^8$, Mt. Wilson Observatory), the solar-radio 10.7-cm flux$^9$ (National Research Council of Canada) and the sunspot number (SSN$^{10}$, SIDC, RWC, Belgium). The minimum-to-maximum solar cycle frequency shift ($\delta\nu_{n,\ell}$) was estimated by multiplying the slope of the linear fit by the corresponding solar index variation between the maximum and minimum of cycle 23. The mean activity level for each epoch in relation to the maximum reached in January 2002 is listed in Table 1, given by the solar UV spectral irradiance.

In an attempt to include in the analysis of the solar-cycle induced frequency shifts only modes that are, in fact, correlated with the solar activity, Rabello-Soares, Korzennik and Schou (2008) rejected modes whose Pearson correlation coefficient

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$^8$http://www.astro.ucla.edu/~obs/intro.html

$^9$http://www.ngdc.noaa.gov/stp/SOLAR/ftpsolarradio.html

$^{10}$http://sidc.oma.be
is smaller than 0.8, or whose slope uncertainty is larger than 20% of its absolute value. However, the correlation coefficient cannot be used directly to indicate the probability that a linear relationship exists between two observed quantities, i.e., if, indeed, there exists a physical linear relation between them. A small correlation coefficient might indicate only a small slope in their linear relation. A better approach is to calculate the probability that the data points represent a sample derived from an uncorrelated parent population. We estimated the probability $P_u(r)$ that a random sample of $N$ uncorrelated data points would yield a linear-correlation coefficient as large as or larger than the observed absolute value of $r$ (Bevington and Robinson, 1992). If $P_u(r)$ is very small, it would indicate that it is very improbable that they are linearly uncorrelated. Thus, the probability is high that the frequency shift of a particular $(n, \ell)$ mode and the solar-activity index used are correlated and the linear fit is justified.

For each $(n, \ell)$ mode and solar-activity index $i$, we calculated the Pearson correlation coefficient $r(n, \ell; i)$ and the probability $P_u(r) \equiv P_u(n, \ell; i)$. Figure 8 shows the probability $P_u(n, \ell; i)$ as function of the mode frequency. It is very similar for all four solar-activity indices. Modes with frequency around 3 mHz have the highest probability of being correlated. It decreases for modes with frequency smaller than $\approx 2.5$ mHz or larger than $\approx 4.5$ mHz. There is no noticeable dependence of $P_u$ with $\ell$ or $\nu/\ell$.

Libbrecht and Woodard (1990) were the first ones to suggest that the solar-cycle frequency variation $(\delta \nu_{n,\ell})$ is linearly proportional to the inverse mode inertia $(I_{n,\ell})$ using observations obtained at the Big Bear Solar Observatory. The observed frequency shift is larger for higher-frequency and for higher-$\ell$ modes. As pointed out by Libbrecht and Woodard (1990), these modes are more sensitive to surface perturbations, because they have, respectively, higher upper- and lower-reflection points in the Sun. This indicates that the dominant structural changes during the solar cycle, inasmuch as they affect $p$-mode frequencies, occur near the solar surface. Chaplin, Appourchaux, Elsworth, et al. (2001) obtained a similar shift at low
frequencies ($\nu \leq 2.5$ mHz): $\delta \nu_{n,\ell} \propto 1/I_{n,\ell}$. However, at higher frequencies, they found: $\delta \nu_{n,\ell} \propto \nu^\alpha/I_{n,\ell}$, where $\alpha = 1.91 \pm 0.03$. They used data sets with a much higher duty cycle obtained by the ground-based networks, GONG and BiSON, and the SOHO satellite (VIRGO/LOI).

The value of $\alpha$ depends upon the physical mechanism responsible for affecting the acoustic mode frequencies during the solar cycle. Several authors (Gough, 1990; Libbrecht and Woodard, 1990; Goldreich, Murray, Willette, et al., 1991) estimated $\alpha = 3$ assuming modifications in the thermal structure of the Sun in a thin layer at the photosphere. On the other hand, Gough (1990) pointed out that if the frequency shifts are caused by variations in the efficacy of the convection during the solar cycle, $\alpha \approx -1$.

Rabello-Soares, Korzennik and Schou (2008) showed that the scaled frequency shift can be described with a simple power law at all frequencies:

$$\delta \nu^{n,\ell}_{n,\ell} = C \gamma (\nu_{n,\ell})^{\gamma} Q_{n,\ell},$$

(4)

where $Q_{n,\ell}$ is the mode inertia, normalized by the inertia of a radial mode of the same frequency, $I_{\ell=0}(\nu_{n,\ell})$, calculated from Christensen-Dalsgaard’s model S (see Christensen-Dalsgaard, Dappen, Ajukov, et al., 1996). The $f$ and $p$ modes were fitted independently of one another. Chaplin, Appourchaux, Elsworth, et al. (2001) also plotted the frequency shift scaled by the normalized mode inertia (upper right-hand panel of their Figure 1). However, they chose to fit the frequency shift scaled by the mode inertia instead. The multiplication of the frequency shift by $Q_{n,\ell}$ normalizes the shift to its expected radial equivalent.

As in Rabello-Soares, Korzennik and Schou (2008), Equation (4) was fitted to estimate $\gamma_p$ (for the $p$ modes) and $\gamma_f$ ($f$ modes), using a weighted least-squares minimization (thin green line and black dashed line in Figure 9 respectively). Only modes that have a probability of 0.1% or less that its variation is linearly uncorrelated with the solar index were included in the present analysis. They represent 80% of the high-$\ell$ set modes and 65% of the medium-$\ell$ set modes. The frequency ranges used in the fitting of the medium- and high-$\ell$ $p$ modes and high-$\ell$ $f$ modes are: 2.5 – 4.1 mHz, 2.8 – 4.9 mHz and 1.8 – 3.0 mHz respectively. It seems that there is a step in the $p$-mode frequency shift around 2.3 mHz and only modes with $\nu \geq 2.5$ mHz were included in the fitting of the medium-$\ell$ set. $p$-modes with $\nu \leq 2.3$ mHz seem to have a similar slope to the fitted high-frequency modes, but a different y-intercept. This is illustrated by the thick green short line in Figure 9. The fact that the $f$ modes are affected by the solar cycle in a different way than the $p$ modes is expected since they have very different properties. The $f$-mode is essentially a surface wave and its frequency is less likely to be influenced by the solar stratification than the $p$-mode frequency. We repeated the analysis for all four solar indices. The results are plotted in Figure 10. The exponents obtained using the different solar indices agree within their fitting uncertainty. The weighted mean from the four solar indices is $\gamma_f = 1.29 \pm 0.07$ for the $f$ modes and $\gamma_p = 3.60 \pm 0.01$ for the $p$ modes. $\gamma_p$ was calculated after averaging the values obtained for the medium- and high-$\ell$ sets (given by the circles in Figure 10). Only $f$ modes obtained with the high-$\ell$ set were used to estimate $\gamma_f$. There are only a few modes (seven) in the medium-$\ell$ set with a probability of 0.1% or less of being uncorrelated, they have a very small frequency shift ($0.05 \pm 0.01 \mu$Hz on average), and by consequence they were excluded from the
Figure 9. Frequency shift multiplied by the normalized mode inertia as a function of frequency obtained, in this case, using the sunspot number as a proxy of solar activity. The long green continuous line is the fit to the all $p$ modes with $\nu \geq 2.5$ mHz and the dashed black line is to the $f$ modes in the high-$\ell$ set. The thick green short line is the fit to $p$ modes with $\nu \leq 2.3$ mHz using the same slope as the long green line but a different $y$-intercept. The blue line corresponds to a frequency shift given by $\alpha = 0$ with an arbitrarily chosen $y$-intercept.

The $f$ modes in the medium-$\ell$ set have very small frequencies ($\nu < 1.5$ mHz) and, accordingly to Figure 8, low-frequency modes are not well correlated with the solar cycle. High-$\ell$ $f$-modes with $\nu \leq 1.8$ mHz are also not well correlated with the solar cycle and were excluded from the analysis. Their frequency shifts show a steep increase from values similar to those of the well-correlated medium-$\ell$ $f$-modes at 1.7 mHz to values similar to the diamonds in Figure 9 at 1.8 mHz.

In Rabello-Soares, Korzennik and Schou (2008) only the solar UV spectral irradiance was used as a proxy for the solar-cycle index to calculate the exponent $\gamma$. The value obtained here for the $p$ modes, using the same solar index but a better criteria for selecting the modes included in the analysis, is the same as before: $\gamma_p = 3.63 \pm 0.02$. However, the value obtained for the $f$ modes ($\gamma_f = 1.33 \pm 0.2$) is 20% smaller than our previous result. The distinct mode selection criteria account for this difference. As the number of $f$ modes is small (45), the fitting of $\gamma_f$ is more sensitive to the mode selection.

To compare the above mentioned values of $\alpha$ obtained by Chaplin, Appourchaux, Elsworth, et al. (2001) with our results, $\nu_{n,\ell}^\alpha$ was divided by $I_{x=0}(\nu_{n,\ell})$ and then fitted by Equation (4) to estimate the corresponding $\gamma$ exponent. The $\alpha$ values obtained at the two frequency intervals $1.6 < \nu < 2.5$ mHz ($\alpha = 0$) and 2.5 $\nu < 3.9$ mHz ($\alpha = 1.91$) correspond to $\gamma = 7.59 \pm 0.18$ and $\gamma = 3.58 \pm 0.03$ respectively. Accordingly to the authors, the division of the data into two parts has been made on a purely “artificial” basis. Rabello-Soares, Korzennik and Schou (2008) showed that this imposed “breakpoint” can be explained by the fact that $\log(\nu_{n,\ell}^\alpha I_{n,\ell})$ has a quadratic dependence on $\log(\nu_{n,\ell})$, with an inflection point at 2.59 mHz (see Figure 6 in Rabello-Soares, Korzennik and Schou, 2008). Chaplin, Appourchaux, Elsworth, et al. (2001) used 10.7-cm radio flux as the solar activity index. Their results at the high frequency interval agrees well with ours (represented
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Figure 10. Exponent $\gamma$ obtained fitting Equation (4) to the $f$ (top) and $p$ (bottom) modes for four different solar indices. In the top panel, the exponents were obtained using only the high-$\ell$ set. In the bottom panel, the diamonds and stars were obtained using the medium- and high-$\ell$ sets respectively and their weighted average is given by the circles. The square is the $\gamma$ corresponding to the $\alpha$ value quoted in Chaplin et al. (2001) for the high-frequency interval. The fitting uncertainties are given by the error bars. The dashed lines are the weighted average between the four solar indices.

by a square in Figure 10 bottom panel). However, at low frequency, their result is twice as large as the one obtained here (blue line in Figure 9). The difference could be due to the mode selection used to obtain our results. Dziembowski and Goode (2005) used also a simple frequency difference including all modes. They fitted $\delta\nu_{n,\ell} I_{n,\ell}$ using truncated Legendre polynomial series and fitted the $f$ and $p$ modes independently of one another. For $p$ modes, their fitting agrees with Chaplin, Appourchaux, Elsworth, et al. (2001) thus disagreeing with ours at low frequency. For $f$ modes, in the frequency range 1.37 – 1.60 mHz, the corresponding $\gamma_f$ (inferring from their Fig. 2) is five times large as our determination at $\nu > 1.8$ mHz, and it is similar to the low-frequency $\gamma_p$ obtained by Chaplin, Appourchaux, Elsworth, et al. (2001). Dziembowski and Goode (2005) noted that the frequency shifts normalized by $I_{n,\ell}$ present an opposite trend for the $f$ and $p$ modes. They increase with increasing frequency for $p$-modes and decrease for $f$ modes. This opposite trend disappears when the frequency shifts are normalized by $Q_{n,\ell}$ instead.

Several authors have reported a sharp decrease of the frequency shifts at high frequency. The positive shifts suddenly drop to zero, and become negative reaching absolute values much larger than the positive shifts at moderate frequency. The falloff
was reported to happen around 3.7 mHz by Jefferies (1998) – using \( 100 \leq \ell \leq 250 \) modes - and Salabert, Fossat, Gelly, et al. (2004) – using \( \ell \leq 3 \) modes, which is supported by Libbrecht and Woodard (1990) determinations (\( 5 \leq \ell \leq 60 \)). However, Howe et al. (2002) do not see any falloff for \( \nu \leq 4 \) mHz modes (\( \ell \leq 300 \)) and Rhodes, Reiter and Schou (2002) see the drop at 5 mHz (using \( \ell \leq 1000 \) modes). The exact frequencies where the frequency shifts become zero or negative or reach a maximum negative value also varies widely between the publications. This sharp decrease in the frequency shift has been associated to an increase in the chromospheric temperature (e.g., Goldreich, Murray, Willette, et al., 1991 and Jain and Roberts, 1996). From Figure 8, high-frequency modes have a large probability \( P_u \) of being uncorrelated with the solar cycle, at least until 5.2 mHz. Note that \( P_u(r) \) is calculated for the absolute value of \( r \), estimating the probability of being correlated or anti-correlated with solar cycle. In our analysis, the frequency shift drops sharply around 4.6 mHz and there are a few modes (≈ ten) with a negative frequency shift with frequencies between 5 and 5.35 mHz. However, only modes with frequency smaller than 4.9 mHz obey the mode selection criteria (i.e., \( P_u \leq 0.1\% \)) and were included in Figure 9. The frequency shift drop from 4.6 to 4.9 mHz is not seen in Figure 9 because of the mode-inertia normalization.

As first suggested by Libbrecht and Woodard (1990), the most significant sources of frequency shift must be localized near the solar surface since the frequency shifts seem to be independent of \( \ell \). Consider a small perturbation in the solar equilibrium model localized near the solar surface and the corresponding changes in the mode frequency, \( \delta \nu_{n,\ell} \). For modes extending substantially more deeply than the region of the perturbation, the eigenfunctions are nearly independent of \( \ell \) at fixed frequency in that region (see Figure 8 and the associated discussion in Christensen-Dalsgaard and Berthomieu, 1991). From this, it can be inferred that if the quantity \( \delta \nu_{n,\ell}Q_{n,\ell} \) is independent of \( \ell \) at fixed \( \nu \) for a given set of modes, then the perturbation is probably largely localized outside the radius given by the maximum lower turning point of the set of modes considered (Christensen-Dalsgaard and Berthomieu, 1991). In the case of the frequency shift induced by the solar cycle, \( \delta \nu_{n,\ell}Q_{n,\ell} \) is independent of \( \ell \) at fixed \( \nu \) for all modes analyzed here, where \( \ell \leq 900 \) and \( \nu/L > 12 \) \( \mu \)Hz. Thus, the perturbation causing the frequency shift is probably localized in the region 4 Mm below the solar surface or less. Using the same reasoning, the thickness of the near-surface region where the uncertainties in the physics of the model are confined can be inferred. In this case, the set of modes where \( \delta \nu_{n,\ell}Q_{n,\ell} \) is independent of \( \ell \) at fixed \( \nu \) is such that \( \nu/L > 12 \) \( \mu \)Hz and \( \ell \leq 370 \) using the high-\( \ell \) frequency determination (Section 5.2). Including higher-degree modes, \( \delta \nu_{n,\ell}Q_{n,\ell} \) is not a function of frequency alone anymore (see, for example, Figure 3 in Rabello-Soares, Basu, Christensen-Dalsgaard, et al., 2000). As a result, the frequency uncertainties in the model are probably largely restricted to the layers 18 Mm below the surface.

7. Conclusion

In the determination of unbiased high-degree mode parameters, the instrumental characteristics must be taken into account in the image spatial decomposition or in the leakage matrix calculation itself to obtain a correct estimation of the relative amplitude of the spatial leaks. Among the instrumental characteristics analyzed here, the image scale is the one that affects the parameter determination the most. The
image scale is the ratio of the image dimensions observed on the CCD detector and the dimensions in the actual Sun. An error in the image scale introduces an error in the estimated central frequency which increases with the mode frequency. A 0.27% error in the image scale would shift the estimated central frequency by as much as 11 µHz at 5 mHz. The radial distortion also has an important effect which is expected since it is similar to an image scale error. An instrumental property not taken into account here (due to a lack of a good estimation) that could have an important effect on the measured parameters is an azimuthally varying PSF.

The applied ridge-to-mode correction recovers frequencies at moderate degree that differ from the assumed corrected values by 1 µHz or less depending on the mode frequency. The fitting uncertainty of the recovered frequencies is in the range 0.07 – 0.18 µHz. The agreement for the $a_1$, $a_2$, and $a_3$ coefficients is very good, except maybe for $f$- and $p_1$-mode $a_1$ coefficients, their mean difference with respect to the assumed correct values is, respectively, three and two normalized by the ridge fitting uncertainties.

At high degree, the differences between our frequency determination and theoretical frequencies for the $p$ modes shows the same general variation with degree as the results obtained with ring analysis. For $n \leq 5$, the global and ring analysis agree within 6 µHz. The high-degree $f$-mode frequencies obtained using ring analysis, like previous observations (Duvall, Kosovichev and Murawski, 1998 and references within), are substantially lower than the theoretical frequencies. Surprisingly, the $f$-mode high-$\ell$ set frequencies agree well with the model frequencies (within 3 µHz) whereas the ring-analysis frequency differences can be as large as 13 µHz for $\ell > 700$ modes. The implications of the high-$\ell$ frequencies and splitting coefficients on the solar structure and rotation will be addressed in a future paper.

As noted by other authors for low- and moderate-degree modes (e.g., Libbrecht and Woodard, 1990), the frequency shift induced by the solar cycle scales well with the mode inertia. We extended this analysis to high-degree modes and found out that scaling with the mode inertia normalized by the inertia of a radial mode of the same frequency follows a simple power law (given by Equation 4) with one exponent at all frequency ranges, where the $f$ and $p$ modes are fitted independently of one another.

The exponents obtained using four different solar indices agree within their fitting uncertainty, where: $\bar{\gamma}_f = 1.29 \pm 0.07$ and $\bar{\gamma}_p = 3.60 \pm 0.01$. The $f$-mode exponent is less than half of the $p$-mode value. The fundamental mode of solar oscillations has essentially the character of surface gravity waves and, contrary to the $p$ modes, it is essentially incompressible and independent of the hydrostatic structure of the Sun. Hence, it is not a surprise that these different types of modes have different exponents. Due to their different properties, it is also very likely that different physical effects are responsible for their frequency variation. Accordingly to Dziembowski and Goode (2005), for the $f$ modes, the dominant cause of frequency shift is the variation of the subphotospheric magnetic field. For the $p$ modes, it is the decrease in the radial component of the turbulent velocity in the outer layers during the increase in solar activity, which is accompanied by a decrease in temperature (due to a decrease in the efficiency of convective transport). At low frequency ($\nu < 2.3$ mHz), the $p$-mode frequency shifts have a different behavior than at high-$\nu$: a step (with the same exponent $\gamma_p$) or, as found by other authors, an exponent twice as large as the one at high-$\nu$. Low-frequency modes have a large probability of being uncorrelated with the solar cycle, which was not taken into account in the case where a large exponent.
was estimated. The $f$-mode frequency shifts also have a different behavior around 1.7 mHz; they increase abruptly by an order of magnitude.

Modes with frequency around 3 mHz have the smallest probability that their variation is linearly uncorrelated with the solar index, while modes with $\nu < 2.5$ mHz or $\nu > 4.5$ mHz have the largest probability of being uncorrelated. A large probability ($P_u$) of being uncorrelated does not necessarily mean that a given mode is not physically correlated with the solar cycle, instead it could be due to uncertainties in the measurements, a low signal-to-noise ratio. The logarithm of $P_u$ is well correlated with the logarithm of the relative uncertainty of the estimated frequency shift $\delta\nu^e$, the Pearson correlation coefficient is 0.71 for medium-$\ell$ modes and 0.54 for high-$\ell$ modes. If a given mode has a large probability of being linearly uncorrelated, it is expected that the linear fitting of its frequency shifts will have a large uncertainty, hence the high correlation coefficient between $P_u$ and the estimated frequency shift uncertainty. However, it raises the question of what could be the physical process that would make those modes less sensitive to solar activity. For a given $\ell$, the upper reflection point for lower-frequency modes is deeper in the Sun than for high-frequency modes. If the perturbation layer causing the frequency shift is above the upper turning point of the mode, it would not be affected by the solar cycle. Accordingly to model S of Christensen-Dalsgaard, the depth of the upper turning point increases sharply with decreasing frequency below 2.3 mHz. The upper turning point for a radial mode with $\nu = 2$ mHz is 0.5 Mm deeper in the Sun than a three-millihertz mode (from Figure 2 in Chaplin, Appourchaux, Elsworth, et al., 2001). At high-frequency, the observed frequency shift seems to suddenly drop to zero. However, there is no agreement on the exact frequency that this happens, the observed values range from 3.7 to 5 mHz. The frequency-shift falloff is explained by an increase of chromospheric temperature and magnetic field at solar maximum (Jain and Roberts, 1996 and references within). In the presence of an inclined magnetic field, high-frequency modes tunnel through the temperature minimum and are particularly sensitive to changes in the chromosphere, which are expected to be well correlated with solar activity.

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