Nilpotent and Absolutely Anticommuting Symmetries in the Freedman-Townsend Model: Augmented Superfield Formalism

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Abstract: We derive the off-shell nilpotent and absolutely anticommuting Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations, corresponding to the (1-form) Yang-Mills (YM) and (2-form) tensorial gauge symmetries of the four (3 + 1)-dimensional (4D) Freedman-Townsend (FT) model, by exploiting the augmented version of Bonora-Tonin’s (BT) superfield approach to BRST formalism where the 4D flat Minkowskian theory is generalized onto the (4, 2)-dimensional supermanifold. One of the novel observations is the fact that we are theoretically compelled to go beyond the horizontality condition (HC) to invoke an additional set of gauge-invariant restrictions (GIRs) for the derivation of the full set of proper (anti-)BRST symmetries. To obtain the (anti-)BRST symmetry transformations, corresponding to the tensorial (2-form) gauge symmetries within the framework of augmented version of BT-superfield approach, we are logically forced to modify the FT-model to incorporate an auxiliary 1-form field and the kinetic term for the antisymmetric (2-form) gauge field. This is also a new observation in our present investigation. We point out some of the key differences between the modified FT-model and Lahiri-model (LM) of the dynamical non-Abelian 2-form gauge theories. We also briefly mention a few similarities.

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1 Introduction

The (super)string theories (which are at the forefront of research in theoretical high energy physics) encompass in their ever-widening folds the quantum aspects of gravity as well as gauge theories and, hence, represent one of the promising and leading candidates for the unification of all fundamental interactions of nature (see, e.g. [1-3]). The $p$-form ($p = 1, 2, 3, ...$) gauge fields are an integral part of the excitation spectrum of quantized (super)string theories. These gauge fields are interesting in their own right as far as their associated field theories are concerned. We have applied the superfield approach to BRST formalism (see, e.g. [4-9]) to study $p = 1, 2, 3$ form (non-)Abelian theories and obtained their proper (anti-)BRST, (anti-)co-BRST and bosonic symmetries, to prove some of them to be the field theoretic models for the Hodge theory [10-15]. The latter field theories, for instance, are: 2D free (non-)Abelian 1-form gauge theories, 2D Abelian 1-form gauge theory in interaction with Dirac fields, 4D free 2-form and 6D free 3-form Abelian gauge theories. In fact, the higher $p$-form ($p = 2, 3, ...$) gauge theories have provided a fertile ground for the germination of new theoretical ideas as far as the study of their quantum field theoretic aspects is concerned.

There are a couple of widely well-known models for the non-Abelian 2-form gauge theories in the physical four (3 + 1)-dimensions of spacetime which are topologically massive because of their coupling with the non-Abelian 1-form gauge theories. These are celebrated Freedman-Townsend (FT) model [16] and Lahiri-model (LM) [17]. In the former case, the non-Abelian 2-form $B^{(2)} = [(dx^\mu \wedge dx^\nu)/2! B_{\mu\nu}]$ field $B_{\mu\nu}$ is an auxiliary field but it is a dynamical field in the case of the latter (because there is a kinetic term for the 2-form field).

The purpose of our present investigation is to apply the augmented version of superfield formalism (see, e.g. [7-9]) to study the FT-model of massive topological gauge field theory and point out some novel features associated with it. In this context, there are two novel observations that are worth pointing out. First, we are theoretically forced to go beyond the horizontality condition to invoke some appropriate GIRs to obtain the full set of proper (anti-) BRST symmetries. Second, we are logically compelled to modify the FT-model to incorporate an auxiliary vector field and a kinetic term for the 2-form gauge field (see, subsection 4.1) for the application of the above augmented superfield formalism [7-9].

In our present investigation, first of all, we exploit the key ingredients of the augmented superfield formalism to derive the (anti-)BRST symmetries corresponding to the (1-form) YM gauge symmetries by using theoretical tricks that are distinctly different from the ones used in our previous work [18]. In particular, in our present endeavor, the GIRs on the superfields are totally different from our earlier paper [18]. It was a challenging problem to derive the proper (anti-)BRST symmetries, corresponding to the non-Yang-Mills (NYM) tensorial gauge symmetries, for the FT-model within the framework of augmented BT-superfield formalism. To achieve that goal, in our present investigation, we demonstrate that we are theoretically compelled to write the topological mass term of the original FT-model in a different manner so that we could get a spacetime derivative on the $B_{\mu\nu}$ field. We have accomplished this goal in our present endeavor (see, subsection 4.1) which is an important requirement for the application of superfield formalism to derive the NYM symmetries. This artifact enforces us to incorporate a kinetic term for $B_{\mu\nu}$ field (which is an auxiliary field in the original FT-model) thereby turning it to be a dynamical field.
For any arbitrary $p$-form (non-)Abelian gauge theory, it is a common folklore to incorporate a gauge invariant kinetic term for the basic $p$-form gauge field. Thus, for the modified version of FT-model, we have to obtain a gauge-invariant curvature tensor $H_{\mu\nu\eta}$ which is derived from the 3-form $H^{(3)} = [(dx^\mu \wedge dx^\nu \wedge dx^\eta)/3!] H_{\mu\nu\eta}$. A simple covariant derivative on $B_{\mu\nu}$ does not do the job. Thus, we are forced to incorporate an auxiliary field ($K_{\mu}$) and demand a specific type of (N)YM transformations on it so that we could obtain a gauge-invariant $H_{\mu\nu\eta}$. This goal has also been achieved in our present endeavor. Ultimately, we have obtained a modified version of Lagrangian density for the original FT-model which respects YM as well as tensorial (NYM) gauge symmetries. The mathematical form of the curvature tensor $H_{\mu\nu\eta}$ is similar to LM with a different definition for the covariant derivative and curvature tensor for the 1-form gauge field (cf. subsection 4.1).

Even though the appearance of our modified version of Lagrangian density for the FT-model is similar to the one for the dynamical non-Abelian 2-form gauge theory of LM, there are distinct differences in the topological mass term that are incorporated in both these theories. Furthermore, the covariant derivatives and gauge-field curvature tensors are different in both these theories. In the former, the covariant derivative is defined in terms of both 1-form fields $A_\mu$ and $\phi_\mu$ whereas, in the latter case, it is w.r.t. $A_\mu$ field only. One of the important features of the modified version of FT-model is the fact that the modified form of the topological mass term remains invariant under both (1-form) YM and tensorial (NYM) gauge symmetries even though it looks completely different from the original FT-model and LM. This is a completely new observation in our present investigation which is very gratifying as far as symmetry properties of our present theory are concerned.

Our present endeavor is motivated by the following key factors. First, it is urgent for us to apply the superfield formalism to the description of FT-model because we have performed a similar kind of analysis for the dynamical non-Abelian 2-form theory (LM) in our previous work [19]. This is essential for the sake of comparison and deep understanding. Second, our attempt yields some novel observations in the context of FT-model which enriches our overall insights and understanding of the non-Abelian 2-form gauge theory. Third, our exercise leads to the derivation of the topological mass term which looks completely different from LM and it turns out to respect both YM and tensorial (NYM) gauge symmetries. Four, the 4D non-Abelian 2-form field has also relevance in the context of (super)string theory and related extended objects. Hence, the study of its field theoretic aspects is important. Finally, our present attempt is our modest step towards our main goal of providing a unitary, consistent and renormalizable non-Abelian 2-form theory whose precise construction, even now, is an outstanding problem in the realm of quantum field theory.

The material of our present investigation is organized as follows. Our Sec. 2 is devoted to a brief sketch of the (1-form) YM and (2-form) tensorial gauge symmetries of the FT-model in the Lagrangian formulation. In Sec. 3, we derive the proper (anti-)BRST symmetry transformations (corresponding to the YM symmetries) by exploiting our superfield formulation which (as far as the basic inputs are concerned) is completely different from the theoretical tricks of an earlier work [18]. Our Sec. 4 deals with the derivation of proper (anti-)BRST symmetries, corresponding to the (2-form) tensorial gauge symmetry, within the framework of our augmented superfield formalism where we are theoretically compelled to modify the FT-model by incorporating an auxiliary 1-form vector field and the kinetic term for the 2-form gauge field (where the curvature tensor for the latter depends on the
form field). Finally, in Sec. 5, we make some concluding remarks.

Our Appendix A deals with some intermediate steps that are needed in the proof of the tensorial gauge symmetry invariance of the modified FT-model [cf. (25), (28)]. In our Appendix B, we provide some explicit explanation for the ghost fields, needed in the theory, corresponding to the 1-form fields \( A_\mu \) and \( \phi_\mu \).

General convention and notations: Throughout the whole body of our text, we shall denote the (anti-)BRST symmetry transformations by \( s_{(a)b} \) corresponding to YM and NYM gauge symmetries. We shall focus only on the internal symmetries of the theory and treat the 4D Minkowskian flat (i.e. \( \eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1) \)) spacetime in the background with the flat metric \( \eta_{\mu\nu} \). Thus, we shall not discuss about any spacetime symmetries of our present 4D theory.

2 Preliminaries: (Super)gauge and (anti-)BRST

Symmetries in the Lagrangian Formalism

Let us begin with the four \((3+1)\)-dimensional \((4D)\) Lagrangian density of the FT-model of topologically massive gauge theory where the 1-forms \((A^{(1)} = dx^\mu A_\mu \cdot T, \phi^{(1)} = dx^\mu \phi_\mu \cdot T)\) and 2-form \([B^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu) B_{\mu\nu} \cdot T]\) gauge fields are merged together through the celebrated \((B \wedge F)\) term. The explicit form of the Lagrangian density is:

\[
\mathcal{L}_{(0)} = -\frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \frac{m^2}{2} \phi^\mu \cdot \phi_\mu + \frac{m}{4} \varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} \cdot F_{\eta\kappa},
\]

where the 2-form \(F^{(2)} = dA^{(1)} + i (A^{(1)} \wedge A^{(1)})\) defines the curvature tensor \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - (A_\mu \times A_\nu)\) for the 1-form gauge field \(A_\mu = A_\mu \cdot T\). The other curvature tensor is

\[
F_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} - (A_\mu \times \phi_\nu) - (\phi_\mu \times A_\nu),
\]

where \(f_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu - (\phi_\mu \times \phi_\nu)\) is the curvature tensor for the additional 1-form \(\phi^{(1)} = dx^\mu (\phi_\mu \cdot T)\) field \(\phi_\mu\) and the 2-form field \(B_{\mu\nu}\) is an auxiliary field in the theory as it has no explicit kinetic term. It is clear from (1) that, in the natural units \((\hbar = c = 1)\), the mass dimension of \((A_\mu, \phi_\mu, B_{\mu\nu})\) is one. As a consequence, the parameter \(m\), in the Lagrangian density (1), has the dimension of mass (in our present 4D gauge theory).

The above Lagrangian density (1) respects \((\delta_g \mathcal{L}_{(0)} = 0)\) the following local, continuous and infinitesimal (1-form) Yang-Mills (YM) gauge symmetry transformations \((\delta_g)\)

\[
\begin{align*}
\delta_g \phi_\mu &= - (\phi_\mu \times \Omega), & \delta_g A_\mu &= D_\mu \Omega, & \delta_g F_{\mu\nu} &= - (F_{\mu\nu} \times \Omega), \\
\delta_g F_{\mu\nu} &= - (F_{\mu\nu} \times \Omega), & \delta_g B_{\mu\nu} &= - (B_{\mu\nu} \times \Omega), \\
\delta_g f_{\mu\nu} &= - (f_{\mu\nu} \times \Omega) + (\phi_\mu \times \partial_\nu \Omega) - (\phi_\nu \times \partial_\mu \Omega),
\end{align*}
\]

\(^*\)We adopt here the convention and notations such that the background 4D flat Minkowskian metric has the signature \((+1,-1,-1,-1)\) and totally antisymmetric 4D Levi-Civita tensor \(\varepsilon_{\mu\nu\rho\kappa}\) is chosen to satisfy \(\varepsilon_{\mu\nu\rho\kappa} \varepsilon^{\mu\nu\rho\kappa} = -4!, \varepsilon_{\mu\nu\rho\kappa} \varepsilon^{\mu\nu\kappa\rho} = -3!\delta_\rho, \text{etc.}\), and \(\varepsilon_{0123} = +1\). We also choose the dot and cross products \(P \cdot Q = P^a Q^a, (P \times Q) = f^{abc} P^a Q^b T^c\) in the SU(N) Lie algebraic space where the generators \(T^a\) satisfy \([T^a, T^b] = i f^{abc} T^c\) with \(a, b, c, \ldots = 1, 2, \ldots, N^2 - 1\). Here the structure constants \(f^{abc}\) are chosen to be totally antisymmetric in \(a, b, c\) for the semi-simple SU(N) Lie algebra (see, e.g. [20] for details).
where $\Omega = \Omega \cdot T$ is the infinitesimal SU(N)-valued local YM Lorentz scalar gauge parameter and the covariant derivative $D_\mu \Omega = \partial_\mu \Omega - (A_\mu \times \Omega)$. There is yet another continuous symmetry in the theory. We call this symmetry as the non-Yang-Mills (NYM) or tensorial gauge symmetry ($\delta_t$). Under this symmetry, the relevant fields of the theory transform as:

$$\delta_t F_{\mu \nu} = \delta_t f_{\mu \nu} = \delta_t \phi_\mu = \delta_t F_{\mu \nu} = 0, \quad \delta_t B_{\mu \nu} = -(\bar{D}_\mu \Lambda_\nu - \bar{D}_\nu \Lambda_\mu),$$

where $\bar{D}_\mu \Lambda_\nu = \partial_\mu \Lambda_\nu - (A_\mu \times \Lambda_\nu) - (\phi_\mu \times \Lambda_\nu) \equiv \partial_\mu \Lambda_\nu - (A_\mu + \phi_\mu) \times \Lambda_\nu$ and $\Lambda_\mu = \Lambda_\mu \cdot T$ is an infinitesimal, local and continuous Lorentz vector gauge parameter.

It is straightforward to note that the Lagrangian density (1) transforms to a total spacetime derivative (i.e. $\delta_t L_0 = \partial_\mu [(m/2) \varepsilon^{\mu
u\rho\sigma} \Lambda_\nu \cdot F_{\rho\sigma}]$) under the above (2-form) tensorial gauge transformations (4) where the Bianchi identity $(\bar{D}_\mu F_{\nu\eta} + \bar{D}_\nu F_{\eta\mu} + \bar{D}_\eta F_{\mu\nu} = 0)$ plays an important role in the proof of the symmetry property of the above Lagrangian density ($L_0$). We shall see that, in our Sec. 4, we have to modify the original Lagrangian density (1) by incorporating an auxiliary 1-form ($K^{(1)} = dx^\mu K_\mu$) vector field $K_\mu$ and kinetic term ($\frac{1}{12} H^{\mu\nu\eta} \cdot H_{\mu\nu\eta}$) for the 2-form ($B_{\mu \nu}$) field so as to obtain the (anti-)BRST symmetry transformations corresponding to the tensorial gauge symmetry transformations (4) within the framework of augmented version of BT-superfield formalism. The latter would be fruitfully applicable if and only if a derivative term (of some variety) exists for the $B_{\mu \nu}$ field in the starting Lagrangian density (see, e.g. Sec. 4 for details).

3 (Anti-)BRST Symmetries Corresponding to the YM Gauge Symmetry: Superfield Approach

For the paper to be self-contained, we derive here the (anti-)BRST symmetry transformations, corresponding to the (1-form) YM gauge symmetry in an alternative way than the theoretical trick adopted in [18]. It can be readily noted that $\delta_g (\phi_\mu \cdot \phi^\mu) = 0, \delta_g (F_{\mu \nu} \cdot F^{\mu \nu}) = 0$. Thus, when we generalize our ordinary 4D theory onto the $(4, 2)$-dimensional supermanifold, we demand the following restrictions on the (super)fields:

$$\Phi_\mu(x, \theta, \bar{\theta}) \cdot \Phi^\mu(x, \theta, \bar{\theta}) = \phi_\mu(x) \cdot \phi^\mu(x),$$
$$\bar{F}_{MN}(x, \theta, \bar{\theta}) \cdot \bar{F}^{MN}(x, \theta, \bar{\theta}) = F_{\mu \nu}(x) \cdot F^{\mu \nu}(x),$$

where the super 2-form $\bar{F}^{(2)} = \frac{1}{2!} (dZ^M \wedge dZ^N) \bar{F}_{MN}(x, \theta, \bar{\theta})$ defines the (anti)symmetric super-curvature tensor $\bar{F}_{MN}(x, \theta, \bar{\theta})$. Here the superspace variable $Z^M = (x^\mu, \theta, \bar{\theta})$ characterizes the $(4, 2)$-dimensional supermanifold. The explicit form of the super 2-form is

$$\bar{F}^{(2)} = \bar{d} \left( \bar{A}^{(1)} + \bar{\Phi}^{(1)} \right) \wedge \bar{d} \left( \bar{A}^{(1)} + \bar{\Phi}^{(1)} \right),$$

where $\bar{A}^{(1)} = dZ^M (A_M \cdot T)$ and $\bar{\Phi}^{(1)} = dZ^M (\Phi_M \cdot T)$ are the super 1-form connections which are the generalizations of the ordinary 1-forms $A^{(1)} = dx^\mu (A_\mu \cdot T)$ and $\phi^{(1)} = dx^\mu (\phi_\mu \cdot T)$, respectively, and $\bar{d}$ (with $\bar{d}^2 = 0$) is the supersymmetric generalization of the ordinary exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) onto the $(4, 2)$-dimensional supermanifold.

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1We shall see that, in the proof of the (anti-)BRST invariance, we shall not use the Bianchi identity.
Within the framework of our augmented superfield formalism [7-9], we require that the gauge invariant quantities should be independent of the Grassmannian variables (i.e. “soul” coordinates) \( \theta \) and \( \bar{\theta} \). This is why, we have taken the GIRs in the above equation (5).

The above generalizations, from the ordinary 4D Minkowskian spacetime manifold onto the (4, 2)-dimensional supermanifold, can be explicitly expressed as (see, e.g. [18]):

\[
A^{(1)} \rightarrow \tilde{A}^{(1)} = dx^\mu \tilde{B}_\mu(x, \theta, \bar{\theta}) + d\theta \tilde{F}(x, \theta, \bar{\theta}) + d\bar{\theta} \tilde{F}(x, \theta, \bar{\theta}),
\]

\[
\Phi^{(1)} \rightarrow \tilde{\Phi}^{(1)} = dx^\mu \tilde{\Phi}_\mu(x, \theta, \bar{\theta}),
\]

(7)

where \( \tilde{B}_\mu(x, \theta, \bar{\theta}), \tilde{F}(x, \theta, \bar{\theta}), \tilde{\Phi}_\mu(x, \theta, \bar{\theta}) \) are the superfields corresponding to the 4D local fields \( A_\mu(x), C(x), \bar{C}(x), \phi_\mu(x) \) of the 4D (anti-)BRST invariant local field theory. This becomes explicit from the following expansions (see, e.g. [18]):

\[
\tilde{B}_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta \tilde{R}_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} P_\mu(x),
\]

\[
\tilde{\Phi}_\mu(x, \theta, \bar{\theta}) = \phi_\mu(x) + \theta \tilde{S}_\mu(x) + \bar{\theta} S_\mu(x) + i \theta \bar{\theta} T_\mu(x),
\]

\[
\tilde{F}(x, \theta, \bar{\theta}) = C(x) + \theta B_1(x) + \bar{\theta} B_1(x) + i \theta \bar{\theta} s(x),
\]

\[
\tilde{\Phi}(x, \theta, \bar{\theta}) = C(x) + \theta \tilde{B}_2(x) + \bar{\theta} B_2(x) + i \theta \bar{\theta} \bar{s}(x),
\]

(8)

In the above, the limiting case \((\theta, \bar{\theta}) = 0\) leads to retrieving of the 4D local fields from the superfields. On the r.h.s., we have \((R_\mu, \tilde{R}_\mu, \bar{s}, \tilde{s}, S_\mu, \tilde{S}_\mu)\) as the fermionic and \((P_\mu, T_\mu, B_1, \tilde{B}_1, B_2, \tilde{B}_2)\) as the bosonic secondary fields. In addition, we require the generalization of the auxiliary field \(B_{\mu\nu}(x)\) onto the (4, 2)-dimensional supermanifold as \(\tilde{B}_{\mu\nu}(x, \theta, \bar{\theta})\) which has the following expansion along the Grassmannian \((\theta, \bar{\theta})\) directions:

\[
\tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}) = B_{\mu\nu}(x) + \theta \tilde{R}_{\mu\nu}(x) + \bar{\theta} R_{\mu\nu}(x) + i \theta \bar{\theta} S_{\mu\nu}(x),
\]

(9)

where \(R_{\mu\nu}\) and \(\tilde{R}_{\mu\nu}\) are the fermionic secondary fields and \(S_{\mu\nu}\) is a bosonic secondary field. It is self-coincident that the fields \(R_{\mu\nu}, \tilde{R}_{\mu\nu}\) and \(S_{\mu\nu}\) are all antisymmetric in \(\mu\) and \(\nu\) because the basic field \(B_{\mu\nu}\) is antisymmetric.

The analogue of HC of the usual (non-)Abelian 1-form gauge theory [4,5], requires that the Grassmannian components of the following super curvature 2-form:

\[
\tilde{F}^{(2)} = \frac{1}{2!} (dZ^M \wedge dZ^N) \tilde{F}_{MN}
\]

\[
\equiv \tilde{d} (\tilde{A}^{(1)} + \tilde{\phi}^{(1)}) + i (\tilde{A}^{(1)} + \tilde{\phi}^{(1)}) \wedge (\tilde{A}^{(1)} + \tilde{\phi}^{(1)}),
\]

(10)

has to be set equal to zero. For this purpose, one has to compute the accurate expansion

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Footnote: It should be noted that we have taken the generalization of (anti-)ghost fields \((\tilde{C})C\) onto the (4, 2)-dimensional supermanifold only in the expansion of \(A^{(1)}(x, \theta, \bar{\theta})\) but not in \(\tilde{\Phi}^{(1)}(x, \theta, \bar{\theta})\). This is due to the fact that there are only one set of (anti-)ghost fields in the theory which are associated with the 1-form \(A^{(1)} = dx^\mu(A_\mu \cdot T)\) YM gauge potential \(A_\mu = A_\mu \cdot T\). We have discussed about this issue in our Appendix B for clarification (where similar ghost-structure has been taken into account for the 1-form fields).
for the r.h.s. of $\tilde{F}^{(2)}$ in (10). This expression can be explicitly written as:

$$\tilde{F}^{(2)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) \left( \partial_\mu (\tilde{B}_\nu + \tilde{\Phi}_\nu) - \partial_\nu (\tilde{B}_\mu + \tilde{\Phi}_\mu) + i [\tilde{B}_\mu + \tilde{\Phi}_\mu, \tilde{B}_\nu + \tilde{\Phi}_\nu] \right)$$

$$(dx^\mu \wedge d\theta) \left( \partial_\mu \tilde{F} - \partial_\theta (\tilde{B}_\mu + \tilde{\Phi}_\mu) + i [\tilde{B}_\mu + \tilde{\Phi}_\mu, \tilde{F}] \right) + \left( [\theta^\mu \wedge [\theta] (\partial_\mu \tilde{F}) - \partial_\theta (\tilde{B}_\mu + \tilde{\Phi}_\mu) + i [\tilde{B}_\mu + \tilde{\Phi}_\mu, \tilde{F}] \right) - (d\theta \wedge d\tilde{\theta}) \left( \partial_{\tilde{\theta}} \tilde{F} + \partial_\theta \tilde{F} + i \{ \tilde{F}, \tilde{F} \} \right)$$

$$- (d\theta \wedge d\tilde{\theta}) \left( \partial_{\tilde{\theta}} \tilde{F} + \frac{i}{2} \{ \tilde{F}, \tilde{F} \} \right) - (d\theta \wedge d\tilde{\theta}) \left( \partial_\theta \tilde{F} + \frac{i}{2} \{ \tilde{F}, \tilde{F} \} \right). \quad (11)$$

On equating the coefficients of $(dx^\mu \wedge d\theta), (dx^\mu \wedge d\tilde{\theta}), (d\theta \wedge d\theta), (d\theta \wedge d\tilde{\theta}), (d\theta \wedge d\tilde{\theta})$ equal to zero, we obtain the following five relationships:

$$\partial_\mu \tilde{F} - \partial_{\tilde{\theta}} (\tilde{B}_\mu + \tilde{\Phi}_\mu) + i [\tilde{B}_\mu + \tilde{\Phi}_\mu, \tilde{F}] = 0,$$

$$\partial_\nu \tilde{F} - (\tilde{B}_\nu + \tilde{\Phi}_\nu) + i [\tilde{B}_\nu + \tilde{\Phi}_\nu, \tilde{F}] = 0,$$

$$\partial_{\tilde{\theta}} \tilde{F} + \frac{i}{2} \{ \tilde{F}, \tilde{F} \} = 0, \quad \partial_{\tilde{\theta}} \tilde{F} + \frac{i}{2} \{ \tilde{F}, \tilde{F} \} = 0,$$

$$\partial_\theta \tilde{F} + \partial_{\tilde{\theta}} \tilde{F} + i \{ \tilde{F}, \tilde{F} \} = 0. \quad (12)$$

However, the above relations (existing for the set of superfields (cf. 8)) are not good enough for the derivations of all the secondary fields (of the expansions (8)) in terms of the basic and auxiliary fields of the ordinary 4D (anti-)BRST invariant Lagrangian density.

The additional gauge invariants restrictions (GIRs) are now required to supplement (and compliment) the analogue of the HC (12) to obtain all the set of secondary fields in terms of the basic and auxiliary fields of the 4D local version of (anti-)BRST invariant theory. The top relationship (which is a GIR) of equation (5) serves this purpose. In fact, taking the help of expansions in (8), it can be checked that $\tilde{F}_{\mu}(x, \theta, \tilde{\theta}) \cdot \tilde{F}^\mu(x, \theta, \tilde{\theta}) = \phi_{\mu}(x) \cdot \phi^\mu(x)$, leads to the following very important relationships

$$S_\mu \cdot \phi^\mu = 0, \quad \tilde{S}_\mu \cdot \phi^\mu = 0, \quad i \phi_\mu \cdot T^\mu = -S_\mu \cdot \tilde{S}^\mu. \quad (13)$$

At this stage, it can be checked that the top two relations of (12) lead to (among other important relations) the following

$$\bar{R}_\mu + \bar{S}_\mu = D_\mu \bar{C} - (\phi_\mu \times \bar{C}), \quad R_\mu + S_\mu = D_\mu C^\prime - (\phi_\mu \times C^\prime),$$

$$P_\mu + T_\mu = D_\mu B_2 - (\phi_\mu \times B_2) + i (R_\mu + S_\mu) \times C$$

$$\equiv -D_\mu \bar{B}_1 + (\phi_\mu \times \bar{B}_1) - i (\bar{R}_\mu + \bar{S}_\mu) \times C. \quad (14)$$

The comparison between (13) and (14) leads to

$$R_\mu = D_\mu C, \quad \bar{R}_\mu = D_\mu \bar{C}, \quad S_\mu = -(\phi_\mu \times C), \quad \bar{S}_\mu = -(\phi_\mu \times \bar{C}),$$

$$T_\mu = -(\phi_\mu \times B_2) - i (\phi_\mu \times C) \times \bar{C} \equiv (\phi_\mu \times \bar{B}_1) - i (\phi_\mu \times \bar{C}) \times C. \quad (15)$$
The final expressions, for the secondary fields that emerge from (12) and (13), are

$$R_\mu = D_\mu C, \quad \bar{R}_\mu = D_\mu \bar{C}, \quad S_\mu = -(\phi_\mu \times C), \quad \bar{S}_\mu = -(\phi_\mu \times \bar{C}),$$

$$T_\mu = -(\phi_\mu \times B) - i (\phi_\mu \times C) \times \bar{C} \equiv (\phi_\mu \times B) - i (\phi_\mu \times \bar{C}) \times C,$$

$$P_\mu = D_\mu B + i (D_\mu C \times \bar{C}) \equiv -D_\mu \bar{B} - i (D_\mu \bar{C} \times C),$$

$$B = -\frac{i}{2} (C \times C), \quad s = -(\bar{B} \times C), \quad \bar{s} = (B \times \bar{C}),$$

$$\bar{B} = -\frac{i}{2} (\bar{C} \times \bar{C}), \quad B + \bar{B} = -i (C \times \bar{C}). \quad (16)$$

where we have identified: $\bar{B}_1 = \bar{B}, \ B_2 = B$. Thus, all the secondary fields in the expansions of $\bar{B}_\mu, \bar{F}, \bar{F}$ and $\bar{\Phi}_\mu$ have been determined in terms of the basic and auxiliary fields of the 4D local (anti-)BRST invariant theory. One of the key signatures of a non-Abelian theory (within the framework of the BRST formalism) is the existence of the Curci-Ferrari (CF) condition $(B + \bar{B} + i (C \times \bar{C}) = 0)$ which is present in (16).

The above (anti-)BRST invariant (i.e. $s_{ab}[B + \bar{B} + i (C \times \bar{C})] = 0$) CF-condition emerges when we set the coefficient of $(d\theta \land d\bar{\theta})$ equal to zero. We would like to lay emphasis on the fact that one of the key signatures of a gauge theory, within the framework of the BRST formalism, is the existence of the (anti-)BRST invariant CF-type condition. This observation is as significant as the characterization of a gauge theory in terms of the existence of first-class constraints in the language of Dirac’s prescription. It is trivial to note that the Abelian 1-form gauge theory is characterized by the existence of a trivial CF-type condition $(B + \bar{B} = 0)$ within the framework of BRST formalism. We have also been able to demonstrate the mathematical origin for the CF-type condition in the language of gerbes (see, e.g. [21,22] for details) which formally proves the independent existence of the nilpotent anti-BRST symmetry (and corresponding conserved charge) vis-à-vis the nilpotent BRST symmetry (and corresponding conserved BRST charge).

As a consequence of the relations (16), it can be readily seen that we have the following explicit expansions of the superfields:

$$\bar{B}_{\mu}^{(h,g)}(x, \theta, \bar{\theta}) = A_\mu(x) + \theta (D_\mu \bar{C}(x)) + \bar{\theta} (D_\mu C(x)) + \theta \bar{\theta} \left[ i D_\mu B - (D_\mu C \times \bar{C})(x) \right],$$

$$\equiv A_\mu(x) + \theta (s_{ab} A_\mu(x)) + \bar{\theta} (s_b A_\mu(x)) + \theta \bar{\theta} (s_b s_{ab} A_\mu(x)),$$

$$\bar{\Phi}_{\mu}^{(h,g)}(x, \theta, \bar{\theta}) = \phi_\mu(x) + \theta \left[ - (\phi_\mu \times \bar{C})(x) \right] + \bar{\theta} \left[ -(\phi_\mu \times C)(x) \right] + \theta \bar{\theta} \left[ -i (\phi_\mu \times B) \right]$$

$$+ (\phi_\mu \times C) \times \bar{C}(x),$$

$$\equiv \phi_\mu(x) + \theta (s_{ab} \phi_\mu(x)) + \bar{\theta} (s_b \phi_\mu(x)) + \theta \bar{\theta} (s_b s_{ab} \phi_\mu(x)),$$

$$\bar{F}^{(h,g)}(x, \theta, \bar{\theta}) = C(x) + \theta (i \bar{B}(x)) + \bar{\theta} \left[ \frac{1}{2} (C \times C)(x) \right] + \theta \bar{\theta} \left[ -i (\bar{B} \times C)(x) \right],$$

$$\equiv C(x) + \theta (s_{ab} C(x)) + \bar{\theta} (s_b C(x)) + \theta \bar{\theta} (s_b s_{ab} C(x)),$$

$$\bar{F}^{(h,g)}(x, \theta, \bar{\theta}) = \bar{C}(x) + \theta \left[ \frac{1}{2} (\bar{C} \times \bar{C})(x) \right] + \bar{\theta} (i B(x)) + \theta \bar{\theta} \left[ i (B \times \bar{C})(x) \right],$$

$$\equiv \bar{C}(x) + \theta (s_{ab} \bar{C}(x)) + \bar{\theta} (s_b \bar{C}(x)) + \theta \bar{\theta} (s_b s_{ab} \bar{C}(x)), \quad (17)$$

where the superscripts $(h,g)$ denote the expansion of the superfields after the application
of HC and GIRs (cf. (12), (5)). The above expansions establish the mappings

\[ s_{b} \leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta}, \quad s_{ab} \leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta}, \]

\[ (s_{b} s_{ab} + s_{ab} s_{b}) \leftrightarrow \left( \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \right), \tag{18} \]

which demonstrate the automatic nilpotency \((s_{a}^{2}b = 0)\) and absolute anticommutativity \((s_{b}s_{ab} + s_{ab}s_{b} = 0)\) of the (anti-)BRST symmetry transformations \(s_{(a)b}\) because it is straightforward to note that \((\lim_{\theta \to 0} \partial_{\theta})^{2} = 0\), \((\lim_{\theta \to 0} \partial_{\theta})^{2} = 0\) and \(\partial_{\theta} \partial_{\theta} + \partial_{\theta} \partial_{\theta} = 0\).

We have still not been able to deduce precisely the proper (anti-)BRST symmetry transformations for the auxiliary field \(B_{\mu \nu}(x)\). This can be derived from yet another GIR on the (super)fields of the \((4, 2)\)-dimensional supermanifold as illustrated below

\[
\tilde{B}_{\mu \nu}(x, \theta, \bar{\theta}) \cdot \tilde{\Phi}_{\eta}^{(h,g)}(x, \theta, \bar{\theta}) = B_{\mu \nu}(x) \cdot \phi_{\eta}(x).
\]

The above choice have been made because it can be checked that \(\delta_{g}(B_{\mu \nu} \cdot \phi_{\eta}) = 0\) which shows the gauge invariance of the combination \((B_{\mu \nu} \cdot \phi_{\eta})(x)\). The substitution from (17) and (9) leads to the following relationships:

\[
\tilde{R}_{\mu \nu} = -(B_{\mu \nu} \times \bar{C}), \quad R_{\mu \nu} = -(B_{\mu \nu} \times C), \quad S_{\mu \nu} = -i \left( B_{\mu \nu} \times C \right) \times \bar{C} - \left( B_{\mu \nu} \times B \right),
\]

which implies the following expansion

\[
\tilde{B}_{\mu \nu}^{(h,g)}(x, \theta, \bar{\theta}) = B_{\mu \nu}(x) + \theta \left[ -(B_{\mu \nu} \times \bar{C})(x) \right] + \bar{\theta} \left[ -(B_{\mu \nu} \times C)(x) \right] + \theta \bar{\theta} \left( [(B_{\mu \nu} \times C) \times \bar{C} - i(B_{\mu \nu} \times B)](x) \right),
\]

\[
\equiv B_{\mu \nu}(x) + \theta(s_{a}B_{\mu \nu}(x)) + \bar{\theta} \left( s_{b}B_{\mu \nu}(x) \right) + \theta \bar{\theta} \left( s_{ab}B_{\mu \nu}(x) \right) \tag{21}
\]

Thus, ultimately, we have the following (anti-)BRST symmetry transformations

\[
s_{ab}A_{\mu} = D_{\mu} \bar{C}, \quad s_{ab}C = \frac{1}{2} \left( \bar{C} \times \bar{C} \right), \quad s_{ab}B = -(B \times \bar{C}),
\]

\[
s_{ab}\bar{B} = 0, \quad s_{ab}\phi_{\mu} = -(\phi_{\mu} \times \bar{C}), \quad s_{ab}B_{\mu \nu} = -(B_{\mu \nu} \times \bar{C}),
\]

\[
s_{b}A_{\mu} = D_{\mu} C, \quad s_{b}C = \frac{1}{2} \left( C \times C \right), \quad s_{b}\bar{C} = iB, \quad s_{b}\bar{B} = -(\bar{B} \times C),
\]

\[
s_{b}B = 0, \quad s_{b}\phi_{\mu} = -(\phi_{\mu} \times C), \quad s_{b}B_{\mu \nu} = -(B_{\mu \nu} \times C), \tag{22}
\]

which are nilpotent of order two (i.e. \(s_{a}^{2}b = 0\)) and absolutely anticommuting in nature (i.e. \(s_{b}s_{ab} + s_{ab}s_{b} = 0\)). The latter property is valid only due to the Curci-Ferrari condition \(B + \bar{B} + i(C \times \bar{C}) = 0\) which is present in our equation (16). To be precise, our whole 4D (anti-)BRST invariant theory is defined on a hyper-surface in the 4D flat Minkowskian spacetime which is described by the CF-field equation \(B + \bar{B} + i(C \times \bar{C}) = 0\).

\[\text{We precisely mean by notation } s_{b} \leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \text{ in (18) as: } \frac{\partial}{\partial \theta} \Omega^{(h,g)}(x, \theta, \bar{\theta}) \mid_{\theta=0} \leftrightarrow s_{b} \Omega(x) \text{ where } \Omega^{(h,g)}(x, \theta, \bar{\theta}) \text{ is the generic superfield obtained after the application of HC and GIRs and } \Omega(x) \text{ is the 4D generic ordinary field of the Lagrangian density of the 4D (anti-)BRST invariant theory.} \]
We note that (anti-)BRST transformations of the Nakanishi-Lautrup auxiliary fields \((B, \bar{B})\), in the above, have been derived from the requirements of the nilpotency and absolute anticommutativity properties. We lay emphasis on the fact that our method of derivation of the (anti-)BRST symmetries corresponding to the (1-form) YM gauge symmetries [cf. (3)] is totally different from the method adopted in our earlier work (see, e.g. [18] for details). Finally, we mention that the following nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for the curvature tensors of our theory, namely;

\[
\begin{align*}
    s_b F_{\mu\nu} &= -(F_{\mu\nu} \times C), \\
    s_b \bar{F}_{\mu\nu} &= -(\bar{F}_{\mu\nu} \times \bar{C}), \\
    s_b f_{\mu\nu} &= -(f_{\mu\nu} \times C) + \phi_\mu \times \partial_\nu C - \phi_\nu \times \partial_\mu C, \\
    s_{ab} F_{\mu\nu} &= -(F_{\mu\nu} \times C), \\
    s_{ab} \bar{F}_{\mu\nu} &= -(\bar{F}_{\mu\nu} \times \bar{C}), \\
    s_{ab} f_{\mu\nu} &= -(f_{\mu\nu} \times \bar{C}) + \phi_\mu \times \partial_\nu \bar{C} - \phi_\nu \times \partial_\mu \bar{C}, \\
    s_b s_{ab} F_{\mu\nu} &= -(F_{\mu\nu} \times C) \times \bar{C} - i(F_{\mu\nu} \times B), \\
    s_b s_{ab} \bar{F}_{\mu\nu} &= -(\bar{F}_{\mu\nu} \times C) \times \bar{C} - i(\bar{F}_{\mu\nu} \times B), \\
    s_b s_{ab} f_{\mu\nu} &= (f_{\mu\nu} \times C) \times \bar{C} - (\phi_\mu \times \partial_\nu C) \times \bar{C} + (\phi_\nu \times \partial_\mu C) \times \bar{C} \\
    &\quad - i(f_{\mu\nu} \times B) - (\phi_\mu \times C) \times \partial_\nu \bar{C} - (\phi_\nu \times C) \times \partial_\mu \bar{C} \\
    &\quad + i(\phi_\mu \times \partial_\nu B) - i(\phi_\nu \times \partial_\mu B),
\end{align*}
\]

are also true due to the (1-form) YM transformations (22) on the basic gauge fields \(A_\mu\) and \(\phi_\mu\). This can be checked by using the definitions of \(f_{\mu\nu}, F_{\mu\nu}\) and \(\bar{F}_{\mu\nu}\) (cf. Sec. 2).

We close this section with the following remarks. First, one can obtain the coupled (but equivalent) (anti-)BRST invariant Lagrangian densities, corresponding to the proper (anti-)BRST symmetry transformations (22), by exploiting the standard tricks of BRST formalism. This has been accomplished in our earlier work for LM (see, e.g. [18] for details). Second, the generators of transformations (22) can be obtained by exploiting the Noether theorem. These (anti-)BRST charges (and their novel features) have been obtained and discussed in our earlier work [18]. Finally, the ghost-scale symmetry in the theory can be discussed in a straightforward manner which leads to the existence of a ghost charge. We have been able to show the existence of the standard BRST algebra in our earlier work [18]. In the forthcoming section, we shall discuss about the (NYM) tensorial gauge symmetry and corresponding proper (i.e. nilpotent and absolutely anticommuting) (anti-)BRST symmetries.

4 Tensorial (anti-)BRST Symmetry Transformations: Superfield Formalism

In this section, first of all, we modify the FT-Lagrangian (1) in order to derive the off-shell nilpotent (anti-)BRST symmetry transformations corresponding to the tensorial (2-form) gauge transformations (4) in our subsection 4.1. This modification allows us to derive the proper (anti-)BRST symmetry transformations by exploiting the tricks and techniques of the augmented version of BT-superfield formulation in our subsection 4.2.
4.1 Modified Version of FT-Model

As pointed out earlier, the 2-form field $B_{\mu \nu}(x)$ is an auxiliary field in the original FT-model [cf. (1)]. This observation does not allow us to apply the basic techniques of BT-superfield formulation to derive the (anti-)BRST symmetry transformations (for the Lagrangian density (1)) corresponding to the tensorial (2-form) gauge symmetry transformations (4) because we can not apply the celebrated HC. Thus, we modify the Lagrangian density (1) in the following fashion:

$$L^{(0)}_{(M)} = -\frac{1}{4} F^{\mu \nu} \cdot F_{\mu \nu} + \frac{m^2}{2} \phi_\mu \cdot \phi^\mu - \frac{m}{2} \varepsilon^{\mu \nu \eta \kappa} (A_\nu + \phi_\nu) \cdot [\tilde{D}_\mu B_{\eta \kappa}] + \frac{1}{2} (A_\mu + \phi_\mu) \times B_{\eta \kappa},$$

(24)

where we have dropped the total spacetime derivative term from the topological mass term $[(m/4) \varepsilon^{\mu \nu \eta \kappa} F_{\mu \nu} \cdot B_{\eta \kappa}]$ of the Lagrangian density (1) of the original FT-model so as to get a covariant derivative term on the antisymmetric tensor field $B_{\mu \nu}$. This observation theoretically compels us to incorporate the kinetic term for this field in our theory, too. Thus, we propose the following modified Lagrangian density $L_{(M)}$ corresponding to the original Lagrangian density (1) of the FT-model, namely:

$$L_{(M)} = \frac{1}{12} H^{\mu \nu \eta} \cdot H_{\mu \nu \eta} - \frac{1}{4} F^{\mu \nu} \cdot F_{\mu \nu} + \frac{m^2}{2} \phi_\mu \cdot \phi^\mu - \frac{m}{2} \varepsilon^{\mu \nu \eta \kappa} (A_\nu + \phi_\nu) \cdot [\tilde{D}_\mu B_{\eta \kappa} + \frac{1}{2} (A_\mu + \phi_\mu) \times B_{\eta \kappa}],$$

(25)

where the curvature 3-form ($H^{(3)} = [dx^\mu \wedge dx^\nu \wedge dx^\eta/3!] H_{\mu \nu \eta}$) defines the curvature tensor for antisymmetric non-Abelian antisymmetric tensor gauge ($B_{\mu \nu}$) field as

$$H_{\mu \nu \eta} = [\tilde{D}_\mu B_{\nu \eta} + \tilde{D}_\nu B_{\mu \eta} + \tilde{D}_\eta B_{\mu \nu} - (K_\mu \times F_{\nu \eta}) -(K_\nu \times F_{\mu \eta})].$$

(26)

In the above, the 1-form ($K^{(1)} = dx^\mu K_\mu \cdot T$) field $K_\mu$ is a compensating auxiliary field so as to make the curvature tensor as a gauge-invariant quantity (i.e. $\delta_t H_{\mu \nu \eta} = 0$). It is elementary to check that under the following tensorial (2-form) gauge transformations

$$\delta_t B_{\mu \nu} = -(\tilde{D}_\mu A_\nu - \tilde{D}_\nu A_\mu), \quad \delta_t K_\mu = -A_\mu,$$

$$\delta_t A_\mu = \delta_t \phi_\mu = \delta_t F_{\mu \nu} = \delta_t F_{\mu \nu} = 0,$$

(27)

the curvature tensor $H_{\mu \nu \eta}$ remains invariant. Now, it is straightforward to check that the modified FT-Lagrangian density (25) respects both the (1-form) YM gauge transformations (3) as well as the (2-form) tensorial gauge transformations (27) because

$$\delta_g L_{(M)} = 0, \quad \delta_t L_{(M)} = -\partial_\mu \left[ \frac{m}{2} \varepsilon^{\mu \nu \eta \kappa} (A_\nu + \phi_\nu) \cdot [(A_\eta + \Phi_\eta) \times \Lambda_\kappa] \right],$$

(28)

which imply that the action integral $S = \int d^4 x L_{(M)}$ remains invariant for the physically well-defined fields that vanish off at infinity. It is to be noted that, in the above derivation,
we have taken $\delta_g K_\mu = -(K_\mu \times \Omega)$ in addition to (3) in the (1-form) YM gauge symmetry transformations. We would like to emphasize that the antisymmetry properties of $\varepsilon_{\mu\nu\eta\kappa}$ are heavily used in the computation of $\delta_t L_{(M)}$. For the sake of reader’s convenience, we mention a few intermediate steps in our Appendix A in the explicit proof of (28) which is quite straightforward.

We note that the curvature tensor (26) is similar in appearance as the following curvature tensor [17] of the Lahiri-model (LM) of the dynamical 4D non-Abelian 2-form theory, namely;

$$H^{(L)}_{\mu\nu\eta} = D_\mu B_{\nu\eta} + D_\nu B_{\eta\mu} + D_\eta B_{\mu\nu} - (K_\mu \times F_{\nu\eta})$$

$$- (K_\nu \times F_{\eta\mu}) - (K_\eta \times F_{\mu\nu}),$$

(29)

where the superscript $(L)$ denotes the curvature tensor for the LM and $D_\mu B_{\nu\eta} = \partial_\mu B_{\nu\eta} - (A_\mu \times B_{\nu\eta})$. Thus, there is a difference in the definition of the covariant derivative $\tilde{D}_\mu$ (which includes both the 1-form fields $A_\mu$ and $\phi_\mu$) and the covariant derivative $D_\mu$ that incorporates only the non-Abelian 1-form field $A_\mu$. The other key difference is the topological mass term for the FT-model [cf. (24)] which is totally different in its appearance vis-à-vis the topological term of the LM. In fact the latter is equal to $[(m/4)\varepsilon^{\mu\nu\eta\kappa} B_{\mu\nu} \cdot F_{\eta\kappa}]$. Furthermore, the transformation properties of the topological terms, under the tensorial gauge symmetry transformations, are totally different in both these models (see, e.g. [17,19] for details). The common feature is the observation that the Lagrangian density of both the models respect the 1-form YM as well as the 2-form tensorial gauge symmetries (see, e.g. [17,19]). We would also like to point out that, in the proof of (28), we do not need any help from the Bianchi identity (which is the case for the tensorial symmetry invariance of the original FT-model (cf. Sec. 2)).

4.2 (Anti-)BRST Symmetries Corresponding to NYM Transformations: Superfield Approach

From the structure of the Lagrangian density (25) and the definition of $H_{\mu\nu\eta}$ [cf. (26)], it is clear that the modified version of FT-model is very similar to Lahiri-model [17] for which we have already performed thorough analysis within the framework of augmented superfield formalism [19]. In what follows, we discuss some of the relevant issues that are needed for our paper to be self-contained. The method of calculation for FT-model would be same as in [19]. In this section, we shall only pin-point some of the relevant steps in the accurate derivation of the proper (anti-)BRST symmetries corresponding to the tensorial gauge symmetry transformations.

We note that $\delta_t A_\mu = \delta_t f_{\mu\nu} = \delta_t \phi_\mu = \delta_t F_{\mu\nu} = 0$ [cf. (4)]. As a consequence, these objects are invariant quantities under the tensorial gauge symmetry transformations. Exploiting the key arguments of the augmented version of BT-superfield approach, it is evident that the counterparts of $A_\mu(x)$, $\phi_\mu(x)$, $f_{\mu\nu}$, $F_{\mu\nu}$, $\mathcal{F}_{\mu\nu}$ on the $(4, 2)$-dimensional supermanifold must be independent of the “soul” coordinate $\theta$ and $\bar{\theta}$. Thus, we have the following relations:

$$\tilde{A}^{(1)} = A^{(1)}, \quad \tilde{\Phi}^{(1)} = \Phi^{(1)}, \quad \tilde{f}^{(2)} = f^{(2)}, \quad \tilde{F}^{(2)} = F^{(2)}, \quad \tilde{\mathcal{F}}^{(2)} = \mathcal{F}^{(2)},$$

(30)
where all the notations have been explained in our Sec. 3. As a consequence of the above equality, we obtain the following relationships:

\[ \partial_\mu \left( \tilde{B}_{\nu}^{(g)} + \tilde{\Phi}_{\nu}^{(g)} \right) - \partial_\nu \left( \tilde{B}_{\mu}^{(g)} + \tilde{\Phi}_{\mu}^{(g)} \right) = \mathcal{F}_{\mu \nu}(x), \]

\[ + i \left[ \left( \tilde{B}_{\mu}^{(g)} + \tilde{\Phi}_{\mu}^{(g)} \right), \left( \tilde{B}_{\nu}^{(g)} + \tilde{\Phi}_{\nu}^{(g)} \right) \right] = \mathcal{F}_{\mu \nu}(x), \]

\[ \tilde{B}_{\mu}^{(g)}(x, \theta, \bar{\theta}) = A_\mu(x), \quad \partial_\mu \tilde{B}_{\nu}^{(g)} - \partial_\nu \tilde{B}_{\mu}^{(g)} + i \left[ \tilde{\Phi}_{\mu}^{(g)}, \tilde{\Phi}_{\nu}^{(g)} \right] = f_{\mu \nu}(x), \]

where the superscript \((g)\) stands for the superfields obtained after the GIRs \((30)\). It is clear from the above equation that the gauge fields \(A_\mu\) and \(\phi_\mu\) and their corresponding curvature tensors \(F_{\mu \nu}\), \(f_{\mu \nu}\) and \(\mathcal{F}_{\mu \nu}\) do not change at all under the proper (anti-)BRST transformations corresponding to the tensorial gauge symmetry transformations.

Now the crucial observation is the fact that \(\delta_i \mathcal{H}_{\mu \nu \eta} = 0\) which shows that the curvature tensor is a gauge invariance quantity. As a consequence, within the framework of augmented superfield formalism, we shall obtain the equality of \(\mathcal{H}^{(3)} = H^{(3)}\) where the r.h.s. and l.h.s. of this HC are explicitly expressed as:

\[ H^{(3)} = dB^{(2)} + i \left\{ (A^{(1)} + \phi^{(1)}) \wedge B^{(2)} - B^{(2)} \wedge (A^{(1)} + \phi^{(1)}) \right\} \]

\[ + i \left( K^{(1)} \wedge \mathcal{F}^{(2)} - \mathcal{F}^{(2)} \wedge K^{(1)} \right), \]

\[ \tilde{H}^{(3)} = \tilde{d} \tilde{B}^{(2)} + i \left\{ (A^{(1)} + \phi^{(1)}) \wedge \tilde{B}^{(2)} - \tilde{B}^{(2)} \wedge (A^{(1)} + \phi^{(1)}) \right\} \]

\[ + i \left( \tilde{K}^{(1)} \wedge \mathcal{F}^{(2)} - \mathcal{F}^{(2)} \wedge \tilde{K}^{(1)} \right). \]  

We point out that we have taken the ordinary \(A^{(1)}\) and \(\phi^{(1)}\) in \(\tilde{H}^{(3)}\) because of the fact that restrictions \((30)\) are true. The ordinary curvature 2-form \(\mathcal{F}^{(2)} = d \left( A^{(1)} + \Phi^{(1)} \right) + i \left( A^{(1)} + \Phi^{(1)} \right) \wedge \left( A^{(1)} + \Phi^{(1)} \right)\) and its supersymmetric version \(\tilde{F}^{(2)}\) (cf. \((6)\)) have already been defined in Sec. 3 and \(\tilde{K}^{(1)}\) and \(\tilde{B}^{(2)}\) are as follows

\[ \tilde{K}^{(1)} = dx^\mu \tilde{K}_\mu(x, \theta, \bar{\theta}) + d\theta \tilde{F}_1(x, \theta, \bar{\theta}) + d\bar{\theta} \tilde{F}_1(x, \theta, \bar{\theta}), \]

\[ \tilde{B}^{(2)} = \frac{1}{2!} (dZ^M \wedge dZ^N) \tilde{B}_{MN}(x, \theta, \bar{\theta}) \]

\[ \equiv \frac{1}{2!} (dx^\mu \wedge dx^\nu) \tilde{B}_{\mu \nu}(x, \theta, \bar{\theta}) + (dx^\mu \wedge d\theta) \tilde{F}_\mu(x, \theta, \bar{\theta}) \]

\[ + (dx^\mu \wedge d\bar{\theta}) \tilde{F}_\mu(x, \theta, \bar{\theta}) + (d\theta \wedge d\bar{\theta}) \tilde{\Phi}(x, \theta, \bar{\theta}) \]

\[ + (d\theta \wedge d\bar{\theta}) \tilde{\beta}(x, \theta, \bar{\theta}) + (d\bar{\theta} \wedge d\theta) \tilde{\bar{\beta}}(x, \theta, \bar{\theta}), \]  

where the expansions have been taken along the Grassmannian directions of the \((4, 2)\)-dimensional supermanifold. The crucial difference between \(H^{(3)}_{(L)}\) of LM (see, e.g. [17]) and \(H^{(3)}\) of our discussion is the replacement of \(A_\mu\) by \((A_\mu + \phi_\mu)\) and \(F_{\mu \nu}\) by \(\mathcal{F}_{\mu \nu}\). This is why, one observes that the equation \((32)\) is the analogue of \(H^{(3)}_{(L)}\) of LM [17] but with the replacements: \(A^{(1)} \to (A^{(1)} + \phi^{(1)})\) and \(\tilde{F}^{(2)} \to \tilde{\mathcal{F}}^{(2)}\).
The supermultiplet superfields in (33) can be expanded along the Grassmannian directions $\theta$ and $\bar{\theta}$ of $(4, 2)$-dimensional supermanifold as follows:

\[
\begin{align*}
\bar{B}_{\mu
u}(x, \theta, \bar{\theta}) &= B_{\mu
u}(x) + \theta \bar{R}_{\mu
u}(x) + \bar{\theta} R_{\mu
u}(x) + i \theta \bar{\theta} S_{\mu
u}(x), \\
\bar{F}_{\mu}(x, \theta, \bar{\theta}) &= C_{\mu}(x) + \theta \bar{B}_{\mu}(x) + \bar{\theta} B_{\mu}(x) + i \theta \bar{\theta} S_{\mu}(x), \\
\bar{K}_{\mu}(x, \theta, \bar{\theta}) &= K_{\mu}(x) + \theta \bar{P}_{\mu}(x) + \bar{\theta} P_{\mu}(x) + i \theta \bar{\theta} Q_{\mu}(x), \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \phi(x) + \theta \bar{f}_{1}(x) + \bar{\theta} f_{1}(x) + i \theta \bar{\theta} b_{1}(x), \\
\bar{\beta}(x, \theta, \bar{\theta}) &= \beta(x) + \theta \bar{f}_{2}(x) + \bar{\theta} f_{2}(x) + i \theta \bar{\theta} b_{2}(x), \\
\bar{\bar{\beta}}(x, \theta, \bar{\theta}) &= \bar{\beta}(x) + \theta f_{3}(x) + \bar{\theta} \bar{f}_{3}(x) + i \theta \bar{\theta} b_{3}(x), \\
\bar{\bar{F}}_{1}(x, \theta, \bar{\theta}) &= C_{1}(x) + i \theta \bar{R}(x) + i \bar{\theta} R(x) + i \theta \bar{\theta} s_{1}(x), \\
\bar{\bar{\bar{F}}}_{1}(x, \theta, \bar{\theta}) &= \bar{C}_{1}(x) + i \theta \bar{S}(x) + i \bar{\theta} S(x) + i \theta \bar{\theta} \bar{s}_{1}(x),
\end{align*}
\]

(34)

where all the secondary fields on the r.h.s. would be expressed in terms of the basic and auxiliary fields of the ordinary 4D BRST invariant Lagrangian density due to HC ($\bar{H}^{(3)} = H^{(3)}$). In the above, we have $(\bar{R}_{\mu
u}, R_{\mu
u}, \bar{P}_{\mu}, P_{\mu}, \bar{S}_{\mu}, S_{\mu}, \bar{f}_{1}, f_{1}, \bar{f}_{2}, f_{2}, \bar{f}_{3}, f_{3}, \bar{s}_{1}, s_{1})$ and $(S_{\mu
u}, Q_{\mu}, \bar{B}_{\mu}(1), B_{\mu}(2), b_{1}, b_{2}, b_{3}, R, \bar{R}, S, \bar{S})$ secondary fields as the set of fermionic and bosonic fields, respectively. We further note that, in the limit $[(\theta, \bar{\theta}) = 0]$, we obtain the ordinary 4D basic and auxiliary fields of our (anti-)BRST invariant theory which are nothing but $B_{\mu
u}, C_{\mu}, \bar{C}_{\mu}, K_{\mu}, \phi, \beta, \bar{\beta}, C_{1}, \bar{C}_{1}$ where the set $(B_{\mu\nu}, K_{\mu}, \phi, \beta, \bar{\beta})$ are the bosonic fields and the set $(C_{\mu}, \bar{C}_{\mu}, C_{1}, \bar{C}_{1})$ are the fermionic fields.

Taking the explicit forms of $\bar{H}^{(3)}$ and $H^{(3)}$ from (32), we can proceed along the same lines as our earlier work on the augmented superfield approach to LM (see, e.g. [19] for details) and determine all the above fermionic and bosonic secondary fields in terms of ordinary basic and auxiliary fields. We skip here the details of these calculations and point out the final expression for the (anti-)BRST transformations as (see, e.g. [19]):

\[
\begin{align*}
\text{sab}B_{\mu\nu} &= -(\bar{D}_{\mu}C_{\nu} - \bar{D}_{\nu}C_{\mu}) + \bar{C}_{1} \times F_{\mu\nu}, & \text{sab}C_{\mu} &= -\bar{D}_{\mu}\bar{\beta}, \\
\text{sab}C_{\mu} &= B_{\mu}, & \text{sab}B_{\mu} &= \bar{D}_{\mu}\rho, & \text{sab}C_{1} &= iB_{1}, & \text{sab}\phi &= -\rho, \\
\text{sab}\bar{C}_{1} &= -\bar{\beta}, & \text{sab}B_{1} &= -i\rho, & \text{sab}K_{\mu} &= \bar{D}_{\mu}\bar{C}_{1} - \bar{C}_{\mu}, & \text{sab}\bar{\beta} &= -\lambda, \\
\text{sab}[A_{\mu}, \phi_{\mu}, F_{\mu\nu}, f_{\mu\nu}, F_{\mu\nu}, H_{\mu\nu\eta}, \bar{\beta}, \bar{B}_{1}, \rho, \lambda, \bar{B}_{\mu}] &= 0,
\end{align*}
\]

(35)

\[
\begin{align*}
\text{sab}B_{\mu\nu} &= -(\bar{D}_{\mu}C_{\nu} - \bar{D}_{\nu}C_{\mu}) + C_{1} \times F_{\mu\nu}, & \text{sab}C_{\mu} &= -\bar{D}_{\mu}\bar{\beta}, \\
\text{sab}C_{\mu} &= B_{\mu}, & \text{sab}B_{\mu} &= \bar{D}_{\mu}\rho, & \text{sab}C_{1} &= iB_{1}, & \text{sab}B_{1} &= -\bar{D}_{\mu}\lambda, \\
\text{sab}K_{\mu} &= \bar{D}_{\mu}C_{1} - C_{\mu}, & \text{sab}\phi &= \lambda, & \text{sab}C_{1} &= -\bar{\beta}, & \text{sab}\bar{\beta} &= \rho, \\
\text{sab}[A_{\mu}, \phi_{\mu}, F_{\mu\nu}, f_{\mu\nu}, F_{\mu\nu}, H_{\mu\nu\eta}, \bar{\beta}, B_{1}, \rho, \lambda, B_{\mu}] &= 0.
\end{align*}
\]

(36)

A close and careful look at the above transformations shows that these transformations are exactly same as the ones derived in the case of LM except we have the following replacements (see, e.g. [19] for details)

\[
D_{\mu} \rightarrow \bar{D}_{\mu}, \quad F_{\mu\nu} \rightarrow F_{\mu\nu}.
\]

(37)
In other words, we observe that the (anti-)BRST symmetry transformations, corresponding to the (2-form) tensorial (NYM) gauge symmetry transformations, for the modified FT-model and LM of dynamical 2-form non-Abelian gauge theory are connected with each other through the above replacements (cf. (37)). It is clear that the LM is a limiting case ($\phi_\mu = 0$) of the modified version of FT-model at the quantum level where the nilpotent and absolutely anticommuting (anti-)BRST symmetries exist.

We further note that the (anti-)BRST symmetry transformations are off-shell nilpotent ($s_a^2 = 0$) of order two and they are absolutely anticommuting ($s_b s_a + s_a s_b = 0$) on the constraint hyper-surface defined by the following field equations:

$$B_\mu + \bar{B}_\mu + D_\mu \phi = 0, \quad B + \bar{B} + i(C \times \bar{C}) = 0, \quad B_1 + \bar{B}_1 - i\phi = 0.$$  \quad (38)

The above hyper-surface is embedded in the 4D flat Minkowskian spacetime manifold and it is described by the above Curci-Ferrari (CF) type restrictions (38). We note that these field equations are found to be (anti-)BRST invariant. The CF-condition ($B + \bar{B} + i(C \times \bar{C}) = 0$) remains invariant under the (anti-)BRST symmetry corresponding to the (1-form) YM gauge symmetries and the other two CF-type conditions are invariant under the nilpotent (anti-)BRST symmetry transformations corresponding to the NYM gauge symmetries (cf. (35, 36)). As pointed out earlier, the CF-type conditions are the signatures of a gauge theory when it is discussed within the framework of BRST formalism which primarily imply the independent nature of anti-BRST symmetries.

By exploiting the standard tricks of the (anti-)BRST symmetries, one can obtain the (anti-) BRST invariant Lagrangian densities (see, e.g. [19] for details):

$$\mathcal{L}_{B_1} = \frac{1}{12} H^{\mu \nu \eta} \cdot H_{\mu \nu \eta} - \frac{1}{4} F^{\mu \nu} \cdot F_{\mu \nu} + \frac{m^2}{2} \phi_\mu \cdot \phi^\mu$$

$$- \frac{m}{2} \varepsilon^{\mu \nu \kappa} (A_\nu + \phi_\nu) \cdot \left[ D_\mu B_{\eta \kappa} + \frac{1}{2} (A_\mu + \phi_\mu) \times B_{\eta \kappa} \right] + B_\mu \cdot B_\mu$$

$$+ \frac{i}{2} B^{\mu \nu} \cdot (\bar{B}_\lambda \times \mathcal{F}_{\mu \nu}) + (\bar{D}_\mu B^{\mu \nu} + \bar{D}^\alpha \phi) \cdot \bar{B}_\nu + \bar{D}_\mu \bar{\beta} \cdot \bar{D}^\beta$$

$$+ \frac{1}{2} \left[ (\bar{D}_\mu \bar{C}_\nu - \bar{D}_\nu \bar{C}_\mu) - \bar{C}_1 \times \mathcal{F}_{\mu \nu} \right] \cdot \left[ (\bar{D}^\mu C^\nu - \bar{D}^\nu C^\mu) - C_1 \times \mathcal{F}^{\mu \nu} \right]$$

$$+ \rho \cdot (\bar{D}_\mu C^\mu - \lambda) + (\bar{D}_\mu \bar{C}^\mu - \rho) \cdot \lambda,$$

$$\mathcal{L}_{\bar{B}_1} = \frac{1}{12} H^{\mu \nu \eta} \cdot H_{\mu \nu \eta} - \frac{1}{4} F^{\mu \nu} \cdot F_{\mu \nu} + \frac{m^2}{2} \phi_\mu \cdot \phi^\mu$$

$$- \frac{m}{2} \varepsilon^{\mu \nu \kappa} (A_\nu + \phi_\nu) \cdot \left[ \bar{D}_\mu B_{\eta \kappa} + \frac{1}{2} (A_\mu + \phi_\mu) \times B_{\eta \kappa} \right] + B_\mu \cdot B_\mu$$

$$- \frac{i}{2} B^{\mu \nu} \cdot (B_1 \times \mathcal{F}_{\mu \nu}) - (\bar{D}_\mu B^{\mu \nu} - \bar{D}^\alpha \phi) \cdot B_\nu + \bar{D}_\mu \bar{\beta} \cdot \bar{D}^\beta$$

$$+ \frac{1}{2} \left[ (\bar{D}_\mu \bar{C}_\nu - \bar{D}_\nu \bar{C}_\mu) - \bar{C}_1 \times \mathcal{F}_{\mu \nu} \right] \cdot \left[ (\bar{D}^\mu C^\nu - \bar{D}^\nu C^\mu) - C_1 \times \mathcal{F}^{\mu \nu} \right]$$

$$+ \rho \cdot (\bar{D}_\mu C^\mu - \lambda) + (\bar{D}_\mu \bar{C}^\mu - \rho) \cdot \lambda,$$  \quad (39)

which remain invariant under the (2-form) tensorial nilpotent (anti-)BRST symmetry trans-
formations \( s_{ab} \) listed in (35, 36) because we have

\[
\begin{align*}
\mathcal{L}_{B_1} &= -\partial_{\mu} \left[ \frac{m}{2} \varepsilon^{\mu\nu\kappa}(A_{\nu} + \phi_{\nu}) \cdot \{(A_{\eta} + \Phi_{\eta}) \times \bar{C}_{\kappa} + \bar{C}_1 \times \mathcal{F}_{\eta\kappa}\} \\
&+ \rho \cdot \bar{B}^{\mu} + \lambda \cdot \bar{D}^{\mu} \bar{B} + (\bar{D}^{\mu} \bar{C}^{\nu} - \bar{D}^{\nu} \bar{C}^{\mu}) \cdot \bar{B}_{\nu} - (\bar{C}_1 \times \mathcal{F}^{\mu\nu}) \cdot \bar{B}_\nu \right], \\
\mathcal{L}_{B_1} &= \partial_{\mu} \left[ -\frac{m}{2} \varepsilon^{\mu\nu\kappa}(A_{\nu} + \phi_{\nu}) \cdot \{(A_{\eta} + \Phi_{\eta}) \times C_{\kappa} + C_1 \times \mathcal{F}_{\eta\kappa}\} \\
&+ \rho \cdot \bar{D}^{\mu} \beta + \lambda \cdot B^{\mu} + (\bar{D}^{\mu} C^{\nu} - \bar{D}^{\nu} C^{\mu}) \cdot B_{\nu} - (C_1 \times \mathcal{F}^{\mu\nu}) \cdot B_{\nu} \right].
\end{align*}
\]

We close this section with the remark that (anti-)BRST charges, ghost conserved charge, etc., can be computed by exploiting the standard tricks of Noether’s theorem exactly like what we have done [19] for the LM. Similarly, the standard BRST algebra can be computed along the same line as performed in [19].

5 Conclusions

The central results of our present investigation are the derivations of proper (anti-) BRST symmetries corresponding to the (N)YM gauge symmetries of the FT-model. We have derived the (anti-)BRST symmetries, corresponding to the (1-form) YM gauge symmetries by exploiting the theoretical tricks that are distinctly different from the ones adopted in our previous endeavor [18]. The derivation of the proper (anti-)BRST symmetries for the NYM gauge symmetries was a challenging problem for us (within the framework of the augmented version of BT-superfield formalism). In the accomplishment of the latter goal, we have been theoretically compelled to incorporate an auxiliary field and the kinetic term for the 2-form gauge field. As a consequence, the 2-form field becomes dynamical (even though this field happens to be an auxiliary field in the original FT-model). This is a new observation in our present investigation (where the Lagrangian density is modified due to theoretical needs.

We are theoretically forced to go beyond the HC to derive the full set of proper off-shell nilpotent (anti-)BRST symmetry transformations by invoking the appropriate GIRs in the context of (1-form) YM gauge symmetry transformations. This is also a novel observation. However, in the case of the derivation of proper (anti-)BRST symmetry transformations (corresponding to the (2-form) tensorial (NYM) gauge symmetry transformations), we invoke only the HC (i.e. \( \mathcal{H}^{(3)} = H^{(3)} \)) for the modified version of the FT-model where the curvature tensor \( \mathcal{H}_{\mu
u\eta} \) is defined in (26). The computations are similar in texture and contents as is the case with the dynamical non-Abelian 2-form gauge theory of LM which we have derived within the framework of superfield formalism [19]. Thus, we have quoted these results with suitable modifications in our present endeavor.

It is worthwhile to point out that the modified version of FT-model, at the classical level, is essential only because of the fact that we wish to derive the proper (anti-)BRST symmetry transformations corresponding to the tensorial gauge symmetry transformations (4) within the framework of the augmented version of BT-superfield formalism where HC (i.e. \( \mathcal{H}^{(3)} = H^{(3)} \)) plays a very crucial role (cf. (32)). The original FT-model has been studied in an excellent set of papers [23-27] and its different aspects have been discussed within
the framework of BRST, Batalin-Vilkovisky (antifield-antibracket), ’t Hooft-Veltman, etc., formalisms. In these works [23-27], the Batalin-Vilkovisky formalism has been applied to obtain the proper (anti-)BRST symmetries, one-loop unitarity of this theory has been proven by using the Ward-identities and ’t Hooft-Veltman procedure, quantization of the charged 2-form and string fields have been performed, similarities and differences with the Witten’s string field theory have been pointed out. None of these formalisms, however, have applied the strength of HC to derive the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations which is the central theme of our present investigation within the framework of BT-superfield formalism. This is why, in these works [23-27], there has been no need of the modification of the FT-model which have been performed in our present work. The outcome of our present endeavor should be seen in the emergence of the new “mass” term in the Lagrangian density (cf. (25)) which is found to be invariant under the YM and NYM gauge symmetries.

The differences between FT-model and LM have been pointed out in our present investigation. The modified version of FT-model becomes dynamical non-Abelian 2-form gauge theory similar to the dynamical theory considered in the case of LM. However, the structure of the topological mass term is totally different in both these cases. The common feature is the observation that, in both these theories, the topological mass term remains invariant under both the appropriate (N)YM symmetry transformations of the respective theories. We have also demonstrated that, at the quantum level (where the (anti-)BRST symmetries are valid), the LM is a liming case of the modified version of FT-model when we apply the augmented version of BT-superfield approach to BRST formalism. It would be a nice idea to apply our augmented version of BT-superfield formalism to other non-Abelian higher $p$-form ($p \geq 3$) gauge theories and study their novel features. We plan to pursue this direction of investigation in our future endeavors [28].

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Appendix A

Here we provide some of the intermediate steps to show that the modified FT-Lagrangian density (25) transforms to a total spacetime derivative under the tensorial (2-form) gauge symmetry transformations (27). First of all, it can be checked that $\delta_t H_{\mu\nu\eta} = 0$ because of the following basic definition:

$$[\tilde{D}_\mu, \tilde{D}_\nu] \Lambda_\eta = - (\mathcal{F}_{\mu\nu} \times \Lambda_\eta), \quad \tilde{D}_\mu \Lambda_\eta = \partial_\mu \Lambda_\eta - (A_\mu + \phi_\mu) \times \Lambda_\eta, \quad (41)$$

where $[\tilde{D}_\mu, \tilde{D}_\nu] = \tilde{D}_\mu \tilde{D}_\nu - \tilde{D}_\nu \tilde{D}_\mu$ is the commutator. Next, we note that topological term transforms under $\delta_t$ as:

$$\delta_t \left[ - \frac{m}{2} \varepsilon^{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot \tilde{D}_\mu B_{\eta\kappa} - \frac{m}{4} \varepsilon^{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot [(A_\mu + \Phi_\mu) \times B_{\eta\kappa}] \right]. \quad (42)$$
where there are two terms in the square bracket. Using equation (26) and the antisymmetric property of \( \varepsilon_{\mu\nu\eta\kappa} \), we observe that the first term transforms as:

\[
\delta_t \left[ -\frac{m}{2} \varepsilon_{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot \tilde{D}_\eta B_{\eta\kappa} \right] = -\frac{m}{2} \varepsilon_{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot (F_{\mu\eta} \times \Lambda_\kappa). \tag{43}
\]

In the computation of the transformation on the second term, we note that \( \delta_t B_{\eta\kappa} = \tilde{D}_\eta \Lambda_\kappa - \tilde{D}_\kappa \Lambda_\eta \) which leads to a single term due to presence of the Levi-Civita tensor. That is to say, we have the following explicit transformation:

\[
\delta_t \left[ -\frac{m}{4} \varepsilon_{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot [(A_\mu + \Phi_\mu) \times B_{\eta\kappa}] \right] = +\frac{m}{2} \varepsilon_{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot [(A_\mu + \Phi_\mu) \times \tilde{D}_\eta \Lambda_\kappa]. \tag{44}
\]

The above expression can be written as the sum of two terms due to the fact that: \( \tilde{D}_\eta \Lambda_\kappa = \partial_\eta \Lambda_\kappa - (A_\eta + \Phi_\eta) \times \Lambda_\kappa \). Finally, we find that it can be expressed as follows:

\[
\partial_\eta \left[ \frac{m}{2} \varepsilon_{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot [(A_\mu + \Phi_\mu) \times \Lambda_\kappa] \right] \\
+ \frac{m}{2} \varepsilon_{\mu\nu\eta\kappa} (A_\nu + \Phi_\nu) \cdot (F_{\mu\eta} \times \Lambda_\kappa). \tag{45}
\]

Adding (43) and (45), it is straightforward to see that, under the tensorial gauge symmetry transformations, the modified Lagrangian density for the FT-model transforms to a total spacetime derivative which has been mentioned in the main body of our text [cf. (28)] thereby making the action integral \( S = \int d^4 x \mathcal{L}_M \) invariant.

**Appendix B**

In this Appendix, we provide arguments in favor of our expressions in (7) where we have associated ghost fields only with the 1-form \( (A^{(1)} = dx^\mu A_\mu) \) potential \( A_\mu \) but we have not associated any ghost fields with 1-form field \( \phi_\mu \). Towards this goal in mind, let us have the general expansions for the 1-form \( \phi^{(1)} = dx^\mu \phi_\mu \), too. In other words, we have the following generalizations on the \((4, 2)\)-dimensional supermanifold:

\[
\begin{align*}
A^{(1)} &\rightarrow \tilde{A}^{(1)} = dx^\mu \tilde{B}_\mu(x, \theta, \bar{\theta}) + d\theta \tilde{F}_1(x, \theta, \bar{\theta}) + d\bar{\theta} \tilde{F}_2(x, \theta, \bar{\theta}), \\
\phi^{(1)} &\rightarrow \tilde{\phi}^{(1)} = dx^\mu \tilde{\phi}_\mu(x, \theta, \bar{\theta}) + d\theta \tilde{F}_1(x, \theta, \bar{\theta}) + d\bar{\theta} \tilde{F}_2(x, \theta, \bar{\theta}). \tag{46}
\end{align*}
\]

Now, we have the expansions of the superfields along the Grassmannian directions \((\theta, \bar{\theta})\) of the \((4, 2)\)-dimensional supermanifolds as listed below

\[
\begin{align*}
\tilde{B}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} P_\mu(x), \\
\tilde{\phi}_\mu(x, \theta, \bar{\theta}) &= \phi_\mu(x) + \theta \bar{S}_\mu^{(1)}(x) + \bar{\theta} S_\mu^{(1)}(x) + i \theta \bar{\theta} T_\mu(x), \\
\tilde{F}_1(x, \theta, \bar{\theta}) &= C_1(x) + \theta \bar{B}_3(x) + \bar{\theta} B_3(x) + i \theta \bar{\theta} s_3(x), \\
\tilde{F}_2(x, \theta, \bar{\theta}) &= C_2(x) + \theta \bar{R}_1(x) + \bar{\theta} R_1(x) + i \theta \bar{\theta} s_4(x), \\
F_1(x, \theta, \bar{\theta}) &= \tilde{C}_1(x) + \theta \bar{B}_4(x) + \bar{\theta} B_4(x) + i \theta \bar{\theta} s_3(x), \\
F_2(x, \theta, \bar{\theta}) &= \tilde{C}_2(x) + \theta \bar{S}_1(x) + \bar{\theta} S_1(x) + i \theta \bar{\theta} s_4(x). \tag{47}
\end{align*}
\]
where the bosonic fields \((A_\mu, \phi_\mu, P_\mu, T_\mu, B_3, B_4, \bar{B}_4, \bar{R}_1, R_1, \bar{S}_1, S_1)\) and the fermionic fields \((C_1, \bar{C}_1, C_2, \bar{C}_2, R_\mu, \bar{R}_\mu, S_\mu^{(1)}, \bar{S}_\mu^{(1)}, s_3, \bar{s}_3, s_4, \bar{s}_4)\) in the whole expansions match showing the validity of SUSY in the theory. It will be noted that the secondary fields on the r.h.s. of expansions in (47) are functions of the 4D spacetime coordinate \((x^\mu)\) only.

We exploit the following (HC) i.e. \(\tilde{F}^{(2)} = F^{(2)}\), namely;

\[
\bar{d} \left( \bar{A}^{(1)} + \bar{\Phi}^{(1)} \right) + i \left( \bar{A}^{(1)} + \bar{\Phi}^{(1)} \right) \wedge \left( \bar{A}^{(1)} + \bar{\Phi}^{(1)} \right) = d \left( A^{(1)} + \phi^{(1)} \right) + i \left( A^{(1)} + \phi^{(1)} \right) \wedge \left( A^{(1)} + \phi^{(1)} \right). 
\]

(48)

It will be noted that the r.h.s. of the 2-form contains only the spacetime differentials \((dx^\mu \wedge dx^\nu)\) but the l.h.s. incorporates \((dx^\mu \wedge dx^\nu), (dx^\mu \wedge d\theta), (dx^\mu \wedge d\bar{\theta}), (d\theta \wedge d\bar{\theta})\). Due the above equality (48), we have to set the coefficients of \((dx^\mu \wedge d\theta), (dx^\mu \wedge d\bar{\theta}), (d\theta \wedge d\bar{\theta})\) and \((d\theta \wedge d\bar{\theta})\) equal to zero. These restrictions yield the following useful relationships:

\[
\begin{align*}
\partial_\mu (\bar{F}_1 + \bar{F}_2) - \partial_\nu (\bar{B}_\mu + \bar{\Phi}_\mu) + i \left[ \bar{B}_\mu + \bar{\Phi}_\mu, F_1 + F_2 \right] &= 0, \\
\partial_\mu (F_1 + F_2) - \partial_\nu (\bar{B}_\mu + \bar{\Phi}_\mu) + i \left[ \bar{B}_\mu + \bar{\Phi}_\mu, F_1 + F_2 \right] &= 0, \\
\partial_\theta (F_1 + F_2) + \partial_\bar{\theta} (\bar{F}_1 + \bar{F}_2) + i \left\{ F_1 + F_2, \bar{F}_1 + \bar{F}_2 \right\} &= 0, \\
\partial_\theta (\bar{F}_1 + \bar{F}_2) + i \frac{1}{2} \left\{ F_1 + F_2, \bar{F}_1 + \bar{F}_2 \right\} &= 0, \\
\partial_\bar{\theta} (F_1 + F_2) + i \frac{1}{2} \left\{ F_1 + F_2, F_1 + F_2 \right\} &= 0.
\end{align*}
\]

(49)

The above restrictions would play very important roles in the determination of the fermionic (anti-)BRST symmetries as we briefly mention below.

In these relationships [cf. (49)], if we substitute the super-expansions (47), we obtain the following (anti-)BRST symmetry transformations (that are nothing but the coefficient of the Grassmannian variables \((\theta, \bar{\theta})\)), namely;

\[
\begin{align*}
s_{ab}A_\mu &= D_\mu (C_1 + C_2), \\
s_{ab}\phi_\mu &= -\left[ \phi_\mu \times (C_1 + C_2) \right], \\
s_{ab}B_4 + S_1 &= -\left[ (B_4 + S_1) \times (\bar{C}_1 + \bar{C}_2) \right], \\
s_{ab}\bar{B}_4 + \bar{S}_1 &= \frac{1}{2} \left[ (C_1 + C_2) \times (C_1 + C_2) \right], \\
s_{ab}B_{\mu\nu} &= -\left[ B_{\mu\nu} \times (C_1 + C_2) \right], \\
s_{ab}\phi_\mu &= -\left[ \phi_\mu \times (C_1 + C_2) \right], \\
s_{b}(C_1 + C_2) &= \frac{1}{2} \left[ (C_1 + C_2) \times (C_1 + C_2) \right], \\
s_{b}(\bar{C}_1 + \bar{C}_2) &= i(B_4 + S_1), \\
s_{b}(\bar{B}_3 + \bar{R}_1) &= -\left[ (\bar{B}_3 + \bar{R}_1) \times (C_1 + C_2) \right], \\
s_{b}(B_4 + \bar{S}_1) &= 0, \\
s_{b}B_{\mu\nu} &= -\left[ B_{\mu\nu} \times (C_1 + C_2) \right].
\end{align*}
\]

(50)

The above equation shows that if we take \(B_4 + S_1 = B, \bar{B}_3 + \bar{R}_1 = \bar{B}, C_1 + C_2 = C\) and \(\bar{C}_1 + \bar{C}_2 = \bar{C}\), we shall obtain all the (anti-)BRST symmetry transformations that have been obtained in our earlier work [18]. Furthermore, the above identifications also yield the celebrated CF-condition \(B + \bar{B} + i(C \times \bar{C}) = 0\) for the 4D non-Abelian 1-form
gauge theory. In particular, we wish to mention that it is the third relationship from the top and/or bottom in (49) that yields the (anti-)BRST invariant CF-condition. In other words, the expressions (written in (7)) are good enough to yield the proper (i.e. off-shell nilpotent and absolutely anticommuting) (1-form) YM (anti-)BRST symmetry transformations for our present theory which have been quoted in Sec. 2 (and derived in Sec. 3) in explicit fashion.

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