Enumerating the Number of Connected Vertices Labeled Graph of Order Six with Maximum Ten Loops and Containing No Parallel Edges

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Abstract
A graph G (V, E) is said to be a connected graph if for every two vertices on the graph there exist at least a path connecting them, otherwise, the graph is disconnected. Two edges or more that connect the same pair of vertices are called parallel edges. Given n vertices and m edges, m ≥ 1, there are many graphs that can be formed, either connected or disconnected. In this research, we find that the formula to count the number of connected vertices labeled graphs of order six with m edges that containing maximum ten loops and no parallel edges N(G(h₅,m)) = \sum_{t=5}^{15} N(G(h₅,m,t)), where t ≤ m; and N(G(h₅,m₅)) = 1296(m³), N(G(h₅,m₆)) = 1980 (m⁻¹), N(G(h₅,m₇)) = 3330 (m⁻²), N(G(h₅,m₈)) = 4620 (m⁻³), N(G(h₅,m₉)) = 6660 (m⁻⁴), N(G(h₅,m₁₀)) = 2640 (m⁻⁵), N(G(h₅,m₁₁)) = 1155 (m⁻⁶), N(G(h₅,m₁₂)) = 420 (m⁻⁷), N(G(h₅,m₁₃)) = 150 (m⁻⁸), N(G(h₅,m₁₄)) = 15 (m⁻⁹), N(G(h₅,m₁₅)) = (m⁻¹⁰)

Keywords
graph, disconnected, vertices, labeled, loops

1. INTRODUCTION

There is no doubt that many real-life problems can be represented using graph theory. Many branches of science use graph theory applications include chemistry, biology, computer science, economics, engineering, and others. For example, in a transportation problem, the cities can be represented by vertices while the roads that connect the cities can be represented by edges. Moreover, the edge in the graph can be assigned a number that can represent non-structural information such as cost, time, distance, and others. By representing the transportation network into a graph, the situation can be easily visualized. Some applications of graph theory are given for example: in networks design (Hsu and Lin, 2008), in cryptography (Al Etaiwi (2014); Priyadarshini (2015)), in phylogenesis (Mathur and Adlakha (2016); Deka (2015); Brandes and Cornelsen (2009)), etc.

In 1847 G.R. Kirchoff, in order to solve the linear equations that gave the current around each circuit and each branch of and electrical networks, developed the theory of tree (Vasudev, 2006). One historical work related to graph enumeration was done by Cayley in 1857 who enumerated the isomer of hydrocarbon \( C_n H_{2n+2} \) using the concept of tree (Cayley (1874)). Harary and Enumeration (1977) discussed the basic idea for graph enumeration. The method for enumerating trees and forest was given by Bona (2007), and the use of generating function for enumeration is given by Stanley (1997); Stanley (1999) and Agnarsson and Greenlaw (2006)); and introduction to various combinatorial counting techniques is given by Wilf (1994).

Graphically, given n vertices and m edges, there are many graphs that can be formed. The number of disconnected vertices labeled graphs of order five with maximum six 3-parallel edges was observed Efendi et al. (2018), and in 2019 how to enumerate the number of connected vertices labeled graph of order five with maximum five parallel edges and containing no loops is discussed Wamiliana et al. (2019). In this article we will discuss the formula to enumerate the number of connected vertices labeled graphs of order six with maximum ten loops and containing no parallel edges.

The paper is organized as follows: after Introduction is given in Section 1, Construction and Pattern Obtained is discussed in Section II. Result and Discussion are given in Section III, follows by Conclusion in Section IV.
2. OBSERVATION AND INVESTIGATION

The following diagram shows the procedure to find the formula to enumerate the number of connected vertex labeled graphs of order six with maximum ten loops and containing no parallel edges.

Given \( n \) vertices, \( n = 6 \), and \( m \) the number of edges, \( 5 \leq m \leq 25 \), there are a lot of graphs that can be formed. In this study, we construct connected vertex labeled graphs using \( n \) and \( m \) given. Moreover, the graphs constructed contain no parallel edges and may contain at most ten loops. Note that isomorphism graphs are counted as one. There are many graphs that are able to be obtained, but in this paper we only provide some patterns of graphs that can be obtained due to space limitation.

Those graphs in the first row of Figure 2 are some connected graphs that can be formed if we are given \( n = 6 \) and \( m = 5 \), and in the second row are examples of connected graphs that can be created if we are given \( n = 6 \) and \( m = 6, 7, 8, 9 \). Note that the graph is vertices labeled. Therefore, the position of the labeled is taken into account. However, isomorphic graphs are counted as one. For example, if we interchange label \( v_1 \) to \( v_6 \) in the left-most first-row picture in Figure 2, we have different graphs.

3. RESULTS AND DISCUSSION

As already stated in Section 2, in flow diagram for finding the formula, we start by observing the simplest form of the graph under consideration (\( n = 6, m = 5 \) and \( t = 5 \)), where \( m \) is the total number of edges, \( t \) is the number of edges that connect different pairs of vertices. Then, we construct all possible patterns according to that requirement. The four graphs on the first row in Figure 2 are some patterns that are able to be created according to this condition. Note that the graphs that are constructed must be connected graphs. Moreover, if two or more graphs created are isomorphic then those graphs are counted as one, for example, if we interchange \( v_6 \) and \( v_2 \) in the left-most first-row in Figure 2, then those two graphs are isomorphic and counted as one. After the observation process, the next step is calculating the number of graphs for every pattern. For example, for the pattern in the left-most first-row picture in Figure 2, we only get six graphs. Next step is grouping the graphs created. By grouping the graphs obtained in term of \( m \) and \( t \) (loops are not contributing to \( t \)), the number of graphs obtained can be put in the following table:

| \( t \) | \( \binom{6}{1} \) | \( \binom{6}{2} \) | \( \binom{6}{3} \) | \( \binom{6}{4} \) | \( \binom{6}{5} \) |
|-------|----------------|----------------|----------------|----------------|----------------|
| 5     | 6              | 15             | 20             | 15             | 6              |
| 6     | 21             | 35             | 35             | 21             | 6              |
| 7     | 56             | 84             | 84             | 56             | 6              |
| 8     | 126            | 210            | 210            | 126            | 6              |
| 9     | 252            | 462            | 462            | 252            | 6              |
| 10    | 252            | 462            | 462            | 252            | 6              |
| 10    | 126            | 210            | 210            | 126            | 6              |
| 11    | 35             | 56             | 56             | 35             | 6              |
| 12    | 15             | 20             | 20             | 15             | 6              |
| 13    | 5              | 6              | 6              | 5              | 6              |

Note: \( G(\ell)_{n,m,t} \) as connected graph of order six with \( m \) edges and \( t \) number of edges that connect different pair of vertices and contains no parallel edges (loops allowable), \( N(G(\ell)_{n,m,t}) \) as the number of graphs of order six with \( m \) edges and \( t \) number of edges that connect different pair of vertices and contains no parallel edges (loops allowable).

For \( t = 5 \): the sequence 1, 6, 21, 56, 126, 252, 462, 792, 1287, 2002, 3003 is observed.
Table 1. The number of connected vertices labeled graph of order six with a maximum ten loops and containing no parallel edges.

| m  | t   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 5  | 1296| 676 | 1296| 3330| 4620| 6660|
| 6  | 7776| 1980| 7776| 1980| 7776| 1980|
| 7  | 27216| 11880| 3330| 4620| 6660|
| 8  | 72576| 19980| 3330| 4620| 6660|
| 9  | 163296| 999990| 3330| 4620| 6660|
| 10 | 326592| 1000000| 3330| 4620| 6660|
| 11 | 598752| 1000000| 3330| 4620| 6660|
| 12 | 1026432| 1000000| 3330| 4620| 6660|
| 13 | 1667952| 1000000| 3330| 4620| 6660|
| 14 | 2594592| 1000000| 3330| 4620| 6660|
| 15 | 3891888| 1000000| 3330| 4620| 6660|
| 16 | 5945940| 1000000| 3330| 4620| 6660|
| 17 | 13873860| 1000000| 3330| 4620| 6660|
| 18 | 19999980| 1000000| 3330| 4620| 6660|
| 19 | 2548260| 1000000| 3330| 4620| 6660|
| 20 | 30030000| 1000000| 3330| 4620| 6660|

By observing the numbers in every row, Table 1 can be put in the following table:

The first sequence of numbers shows the numbers that appear in the first column that multiply by 1296. The second sequence shows the difference of two consecutive numbers in the first sequence, while the third sequence shows the difference of two consecutive numbers in the second sequence, and so on, until the last sequence which is formed a fixed difference.

Result 1: Given \( n = 6 \), \( 5 \leq m \leq 25 \), \( t = 5 \), the number of connected graphs of order six with \( m \) edges and \( t \) number of edges that connect different pair of vertices and contains no parallel edges (loops allowable) is \( N(G(t)_6,m) = 1296 \) \((m)^5\) .

Proof: From the sequence of numbers above we can see that the fixed differences occur on the fifth level. Thus the sequence can be represented by polynomial of order five:

\[ Q_5(m) = A_5 m^5 + A_4 m^4 + A_3 m^3 + A_2 m^2 + A_1 m + A_0 \]

By Substituting \( m = 5, 6, 7, 8, 9, 10 \) to the polynomial we get the following system of equations:

\[ 1296 = 3125A_5 + 625A_4 + 125A_3 + 25A_2 + 5A_1 + A_0 \] (1)

\[ 7776 = 7776A_5 + 1296A_5 + 216A_3 + 36A_2 + 6A_1 + A_0 \] (2)

\[ 27216 = 16807A_5 + 2401A_5 + 343A_5 + 49A_5 + 7A_5 + A_0 \] (3)

By solving that system of linear equations we get \( A_5 = \frac{54}{5}, A_4 = -108, A_3 = 378, A_2 = -540, A_1 = \frac{1296}{5} \) and \( A_0 = 0 \).

Therefore

\[ Q_5(m) = \frac{54}{5} m^5 - 108 m^4 + 378 m^3 - 540 m^2 + \frac{1296}{5} \]

\[ = \frac{54}{5} m(m^4 - 10m^3 + 35m^2 - 50m + 24) \]

\[ = \frac{54}{5} m(m - 1)(m - 2)(m - 3)(m - 4) \]
Table 2. Another form of Table 1

| m   | t     |
|-----|-------|
| 5   | 1 x 1296 |
| 6   | 1 x 1980 |
| 7   | 21 x 1296 |
| 8   | 21 x 1980 |
| 9   | 21 x 1980 |
| 10  | 21 x 1980 |
| 11  | 21 x 1980 |
| 12  | 21 x 1980 |
| 13  | 21 x 1980 |
| 14  | 21 x 1980 |
| 15  | 21 x 1980 |

Result 2: Given n=6, 6≤m≤25, t=6, the number of connected graphs of order six with m edges and t number of edges that connect different pair of vertices and contains no parallel edges (loops allowable), \( N(G(t)_{6,m-t}) \) is 1980 \( \times C_5^{(m-1)} \)

Proof: From the sequence of numbers above we can see that the fixed differences occur on the fifth level. Thus the sequence can be represented by polynomial of order five: \( Q_5(m) = A_5 m^5 + A_4 m^4 + A_3 m^3 + A_2 m^2 + A_1 m + A_0 \)

By substituting \( m=6,7,8,9,10,11 \) to the polynomial we get the following system of equations:

\[
\begin{align*}
1 & \quad 6 \quad 21 \quad 56 \quad 252 \quad 462 \quad 792 \quad 1287 \quad 2002 \quad 3003 \\
5 & \quad 15 \quad 35 \quad 70 \quad 126 \quad 210 \quad 330 \quad 495 \quad 715 \quad 1001 \\
10 & \quad 20 \quad 35 \quad 56 \quad 84 \quad 120 \quad 165 \quad 220 \quad 286 \\
15 & \quad 25 \quad 36 \quad 45 \quad 55 \quad 66 \\
5 & \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \\
1 & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
\end{align*}
\]

By solving that system of linear equations we get \( A_5 = \frac{33}{2}, A_4 = \frac{(-495)}{2}, A_3 = \frac{2805}{2}, A_2 = \frac{(-7425)}{2}, A_1 = 4521, A_0 = -1980 \)

Therefore

\[
Q_5(m) = \frac{33}{2} m^5 + \frac{(-495)}{2} m^4 + \frac{2805}{2} m^3 + \frac{(-7425)}{2} m^2 + 4521 m - 1980
\]

\[
N(G(t)_{6,m-t}) = \frac{165}{5} (m^5 - 15 m^4 + 85 m^3 - 225 m^2 + 274 m + 120)
\]

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Among those graphs obtained, there are graphs that may contain maximum ten loops and no parallel edges. For t = 10, N(G(t), m, 10) = 2640 \binom{m-5}{5}.

d. for t = 11, N(G(t), m, 11) = 1155 \binom{m-6}{6}.

e. for t = 12, N(G(t), m, 12) = 420 \binom{m-7}{7}.

g. for t = 13, N(G(t), m, 13) = 150 \binom{m-8}{8}.

h. for t = 14, N(G(t), m, 14) = 15 \binom{m-9}{9}.
i. for t = 15, N(G(t), m, 15) = \binom{m-10}{10}.

4. CONCLUSIONS

From the discussion above we can conclude that if there is given n vertices (n=6), m edges (5\leq m \leq 25), and t (t is the number of edges that connect different pairs of vertices), then we can construct many connected graphs G(V,E), where n=|V| and m=|E|. Among those graphs obtained, there are graphs that may contain maximum ten loops. The formula for counting the number of connected graphs that contained maximum ten loops are as follow: Given n = 6, 5\leq m \leq 25, N(G(t), m, t) the number of connected graphs of order six with m edges that containing maximum ten loops and no parallel edges, N(G(t), m) is:

\[ N(G(t), m) = \sum_{i=5}^{m} N(G(t), m, i), \text{ where } i \leq m; \text{ and } N(G(t), m, n) = 1296(n^4), N(G(t), m, 6) = 1980(n^5), N(G(t), m, 7) = 3330 \binom{m-2}{5}, N(G(t), m, 8) = 4620 \binom{m-3}{6}, N(G(t), m, 9) = 6660 \binom{m-4}{7}, N(G(t), m, 10) = 2640 \binom{m-5}{8}, N(G(t), m, 11) = 1155 \binom{m-6}{9}, N(G(t), m, 12) = 420 \binom{m-7}{10}, N(G(t), m, 13) = 150 \binom{m-8}{11}, N(G(t), m, 14) = 15 \binom{m-9}{12}, N(G(t), m, 15) = \binom{m-10}{13}. \]

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