ABOUT INTEGRABILITY OF ALMOST COMPLEX STRUCTURES ON STRICTLY NEARLY KÄHLER 6-MANIFOLDS

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Abstract. We show that any almost complex structure, positively tamed with \( \omega \) on nearly Kähler 6-manifold \((M, g, J, \omega)\) is not integrable.

1. Introduction. Let \((M, g, J, \omega)\) be an almost Hermitian manifold, where \( g \) is a Riemannian metric, \( J \) an almost complex structure compatible with \( g \) and \( \omega \) is Kähler form, \( \omega(X, Y) = g(JX, Y) \). Nearly Kähler manifold is an almost Hermitian manifold \((M, g, J, \omega)\) with the property that \( (\nabla_X J)X = 0 \) for all tangent vectors \( X \), where \( \nabla \) denotes the Levi-Civita connection of \( g \). If \( \nabla_X J \neq 0 \) for any non-zero vector field \( X \), \((M, g, J, \omega)\) is called strictly nearly Kähler. Nearly Kähler geometry comes from the concept of weak holonomy introduced by A. Gray in 1971 [1], this geometry corresponds to weak holonomy \( U(n) \). The class of nearly Kähler manifolds appears naturally as one of the sixteen classes of almost Hermitian manifolds described by the Gray-Hervella classification [2]. Nagy P.-A. has proved [3] that every compact simply connected nearly Kähler manifold \( M \) is isometric to a Riemannian product \( M_1 \times \ldots \times M_k \), such that for each \( i \), \( M_i \) is a nearly Kähler manifold belonging to the following list: Kähler manifolds, naturally reductive 3-symmetric spaces, twistor spaces over compact quaternion-Kähler manifolds with positive scalar curvature, and nearly Kähler 6-manifold. This is one of the reasons of interest to the nearly Kähler 6-manifolds. In case of dimension 6 there exists several equivalent conditions defining a strictly nearly Kähler structures \((g, J, \omega)\) on \( M \). For example, conditions

(i) \((\nabla_X J)Y\) is skew-symmetric with respect to \( X, Y \) and non-zero;

(ii) The form \( \nabla \omega \) is non-zero, totally skew-symmetric and \( \nabla_X \omega = \frac{1}{2} \epsilon_X d\omega, \forall X \in \Gamma(TM) \);

(iii) The structure group of \( M \) admits a reduction to \( SU(3) \), that is, there is \((3,0)\)-form \( \Omega \) with \(|\Omega| = 1 \), and

\[
d\omega = 3\lambda \text{Re} \, \Omega, \quad d\text{Im} \, \Omega = -2\lambda \omega^2
\]

where \( \lambda \) is a non-zero real constant, are equivalent and define strictly nearly Kähler manifold (see [4]).

By (iii), for strictly nearly Kähler structure \((g, J, \omega)\) the \( d\omega \neq 0 \) and of type \((3,0)+(0,3)\), so \( J \) is not integrable. It is natural to ask about the possibility of

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"neighborhood" of nearly Kähler structure $J$ with integrable one. It is known [5]
that the almost complex structure, underlying a strictly nearly Kähler 6-manifold
cannot be compatible with any symplectic form, even locally. In this article we give
answer on question: do there exist integrable almost complex structure, compatible
with $\omega$?

2. The main result. Denote by:

- $A^+$ – the space of all positive oriented almost complex structures (a.c.s.) on $M$.
- $A^+_\omega$ – the set of all almost complex structures on $M$, tamed by form $\omega$.

$A^+_\omega = \{ I \in A^+ : \omega(IX, IY) = \omega(X, Y), \forall X, Y \in \Gamma(TM) ; \omega(X, IX) > 0, \forall X \neq \emptyset \}$

$AO^+_g$ – the set of all $g$-orthogonal positive oriented almost complex structures.

$AO^+_g = \{ I \in A^+ : g(IX, IY) = g(X, Y), \forall X, Y \in \Gamma(TM) \}$

Lemma. For arbitrary a.c.s. $I \in A^+_\omega$ endomorphism $J + I$ is non-degenerate.

Proof. Suppose that there exists the a.c.s. $I \in A^+_\omega$, for which $J + I$ is degenerate. Then $(J + I)x(X) = 0$ for some $x \in M, X \in \Gamma(TM)$. Therefore $\omega_x(X, JX) = -\omega(X, IX)$, what contradicts to positivity of $\omega(X, JX)$ and $\omega(X, IX)$. Lemma is proved.

As a consequence of lemma for each a.c.s. $I \in A^+_\omega$ one can find endomorphism:

$$K = (J + I)^{-1}(I - J) = (1 - IJ)^{-1}(1 + IJ)$$

It is easy to show, that for $K$:

1. $KJ = -JK$;
2. $g(KX, Y) = g(X, KY), \forall X, Y \in \Gamma(TM)$;
3. $1 - K^2 > 0$

Properties 1 and 2 give additionally that $\omega(KX, Y) = -\omega(X, KY)$. On the other hand, any endomorphism $K$, with properties 1-3 defines a.c.s. $I = (1 - K)J(1 - K)^{-1} \in A^+_\omega$.

Theorem. Any almost complex structure $I \in A^+_\omega$ is non-integrable.

Proof. Let $X \in \Gamma(TM)$ – is field of eigenvectors of $1 - K^2$ with eigenvalue $\lambda$. Then

$$(1 - K^2)JX = J(1 - K^2)X = \lambda JX, \text{ and } (1 - K^2)(X - iJX) = \lambda(X - iJX).$$

So one can define $g$-orthogonal basis $\nu^1, \nu^2, \nu^3$ in the space of (1,0)-forms, which are eigenvectors of $1 - K^2$ in $T^*M \otimes \mathbb{C}$.

Form $\omega(\nu^k, \nu^l) = g(J\nu^k, \nu^l) = ig(\nu^k, \nu^l) = 0$, for $k, l = 1, 2, 3$ so $\omega$ is of type $(1,1)$ in the basis $(\nu, \bar{\nu})$.

Let define forms $\theta^k = (1 - K)\nu^k, k = 1, 2, 3$ in $T^*M \otimes \mathbb{C}$. One can see, that

$I\theta^k = (1 - K)J(1 - K)^{-1}(1 - K)\nu^k = i(1 - K)\nu^k = i\theta^k$. Therefore, $\theta^1, \theta^2, \theta^3$ – are linear independent $(1,0)$-forms, with respect of $I$.

Find value of form $\omega$ on $\theta^k, \theta^l$ for arbitrary $k, l = 1, 2, 3$:

$$\omega(\theta^k, \theta^l) = \omega((1 - K)\nu^k, (1 - K)\nu^l) = \omega(\nu^k, (1 - K^2)\nu^l) = \lambda\omega(\nu^k, \nu^l) = 0.$$ 

Therefore form $\omega$ has type $(1,1)$ in basis $(\theta, \bar{\theta})$. 

Find $d\omega^{(3,0)}$ in $(\theta, \theta)$:

$$d\omega = \nu^1 \wedge \nu^2 \wedge \nu^3 + \nu^1 \wedge \nu^2 \wedge \nu^3 =$$

$$(1-K)^{-1} \theta^1 \wedge (1-K)^{-1} \theta^2 \wedge (1-K)^{-1} \theta^3 + (1-K)^{-1} \frac{1}{\theta^1} \wedge (1-K)^{-1} \frac{1}{\theta^2} \wedge (1-K)^{-1} \frac{1}{\theta^3}$$

Let’s calculate the $d\omega^{(3,0)}$. In local frame $(\nu, \overline{\nu})$ the matrix of operator $1-K$ is

$$\begin{pmatrix} 1 & V \\ V^T & 1 \end{pmatrix},$$

where $V^T = V, 1 - VV^T > 0$. It is easy to check, that

$$\begin{pmatrix} 1 & V \\ V^T & 1 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} \Lambda^{-1} & -V\Lambda^{-1} \\ -V\Lambda^{-1} & \Lambda^{-1} \end{pmatrix},$$

where $\Lambda$ is diagonal matrix of eigenvalues of $1-K^2$, corresponding to $(\nu^1, \nu^2, \nu^3)$. Therefore

$$d\omega^{(3,0)} = \frac{1}{\sqrt{\det(1-K^2)}} \theta^1 \wedge \theta^2 \wedge \theta^3 \neq 0$$

As $\omega$ is of type (1,1), then $d\omega^{(3,0)} \neq 0$ gives $d^2 \neq 0$ for $I$, and shows non-integrability of this structure. Theorem is proved.

**Remark.** The proof of the above theorem shows that really we needn’t in nearly Kähler structure, we just use, that $d\omega^{(3,0)+(0,3)} \neq 0$, and $d\omega^{(1,2)+(2,1)} = 0$.

**3. Almost complex structures on $S^6$.** It is known that $S^6$ admits the set of nearly Kähler structures. All of them are orthogonal with respect to round metric $g_0$, $G_2$ invariant and form the space $\mathbb{R}P^7 = SO(7)/G_2$. All $g_0$ orthogonal almost complex structures on $S^6$ are not integrable [6]. For Riemannian manifold $(M, g_0)$ admitting almost complex structure [7] the space $\mathcal{A}^+$ is a smooth locally trivial fiber bundle over the space $\mathbb{P}^+_{g_0}$, with fiber $\mathcal{A}_{\mathcal{J}_{g_0}}$ over $\mathcal{J}_{g_0} \in \mathbb{P}^+_{g_0}$, where $\omega_{\mathcal{J}}(X, Y) = g_0(JX, Y)$.

So the above theorem let us to enlarge the number of non integrable almost complex structures by the structures in fibers over the nearly Kähler ones.

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