Estimates of eigenspaces and eigenvalues of a matrix

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Abstract

We discuss techniques for rigorous estimations of eigenspaces and eigenvalues of a matrix. We give two kinds of results. In the first one, which is a generalization of Gerschgorin theorems, we consider blocks on the diagonal and provide bounds for eigenvectors. Second one is based on ideas from the dynamics of hyperbolic dynamical systems. We introduce the notion of dominated matrix and for such matrices we present a theorem which allow us to rigorously estimate eigenspaces and eigenvalues. In particular, we can deal with clusters of eigenvalues and their eigenspaces.

Key words and phrases: eigenvectors, eigenvalues, Gerschgorin theorem, cone condition, spectrum of the matrix

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1 Introduction

Assume that our task is to find rigorous bounds for eigenvalues (all or some of them) and their corresponding eigenspaces of matrix $M \in \mathbb{R}^{n \times n}$. First, one usually applies some iterative scheme, for example QR-algorithm, to obtain matrix $A$ which is similar to $M$. In the best case this might be a matrix close to the diagonal one. Then one would like to apply some abstract theorem to $A_1$ to infer the rigorous bounds on the eigenvalues and the eigenspaces. In this paper we give several new theorems in this direction.

The main question we try to address can be stated as follows. Assume that we have a matrix $A$ which has a block structure

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix},$$

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where $A_{ij}$ are matrices and $A_{ii}$ are square matrices. Suppose that the blocks on the diagonal 'dominate' the off-diagonal blocks and we want to obtain efficient computable bounds for the spectrum of $A$ and eigenspaces of $A$ in terms of the spectra of diagonal blocks $A_{ii}$ and norms of the off-diagonal blocks.

When the blocks are one-dimensional the answer to our question with respect to the spectra is given by the Gerschgorin theorems and its modifications, for example the Brauer ovals [1, 6]. Apparently no direct information about the eigenspaces is obtained from this approach. However, an examination of the standard proof of the Gerschgorin theorem shows that in the case of the isolated eigenvalue of multiplicity one, we can also obtain bounds for the corresponding eigenvector.

The results of our paper are of two types. The first type is the generalization of the Gerschgorin theorems to include multidimensional blocks to obtain the bounds for eigenvalues and eigenvectors. This is the content of Section 2. The other type of results is based on the ideas coming from the hyperbolic dynamics [5]. Using forward and backward invariant cones we were able to give good bounds for eigenvalues and eigenspaces.

To describe the second method in more details we introduce some notations. Let $\|x\| := \max_j |x_j|$. For $x = (x_1, \ldots, x_k, \ldots, x_n) \in \mathbb{R}^n$ we set $\|x\|_k = \max_{i \leq k} |x_i|, \|x\|_k = \max_{i > k} |x_i|$. For linear map $A: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ we define extension and contraction constants:

$$
\mathcal{A}(+ \mathbb{R}^n) = \inf\{ R \in \mathbb{R}^+ : \|Ax\| \leq R \cdot \|x\| \text{ for all } x \in \mathbb{R}^n : \|Ax\|_k \geq \|Ax\|_k \},
$$

$$
\mathcal{A}(- \mathbb{R}^n) = \sup\{ R \in \mathbb{R}^+ : \|Ax\| \geq R \cdot \|x\| \text{ for all } x \in \mathbb{R}^n : \|Ax\|_k \leq \|Ax\|_k \}.
$$

We say that $A$ is dominating if $\mathcal{A}(+ \mathbb{R}^n) < \mathcal{A}(\mathbb{R}^n)$.

**Main Result [simplified version of Theorem 4.3].** Let $A: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ be dominating. Then there exists a unique direct sum decomposition $F_1 \oplus F_2 = \mathbb{R}^n$ of $A$-invariant subspaces $F_1, F_2$ such that

$$
\sigma(A|_{F_1}) \subset \overline{B}(0, \mathcal{A}(+ \mathbb{R}^n)), \quad \sigma(A|_{F_2}) \subset \mathbb{C} \setminus B(0, \mathcal{A}(\mathbb{R}^n)).
$$

Moreover, we have:

1) $\dim F_1 = k, \quad \dim F_2 = n-k$,

2) $F_1 \subset \{ x \in \mathbb{R}^n : \|x\|_k \geq \|x\|_{k-1} \}$ and $F_2 \subset \{ x \in \mathbb{R}^n : \|x\|_k \leq \|x\|_{k-1} \}$,

3) $\|A|_{F_1}\| \leq \mathcal{A}(+ \mathbb{R}^n)$ and $\|A|_{F_2}\|^{-1} \leq \mathcal{A}(\mathbb{R}^n)$.

As an application of our method we considered the case of the isolated eigenvalue. Assume that

$$
A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
$$

where $a_{11} \in \mathbb{C}$, $A_{12} \in \mathbb{C}^{1 \times (n-1)}$, $A_{21} \in \mathbb{C}^{(n-1) \times 1}$ and $A_{22} \in \mathbb{C}^{(n-1) \times (n-1)}$ are such that $a_{11}$ does not belong to the spectrum of $A_{22}$. From the implicit function theorem it follows that, if $\|A_{12}\|$ and $\|A_{21}\|$ are sufficiently small, then $A$ has an eigenvalue close to $a_{11}$ and this eigenvalue and the corresponding eigenvector depend analytically on $A$. Hence there exist analytic functions $\lambda(A) \in \mathbb{C}$ and $v(A) \in \mathbb{C}^n$, such that

$$
Av(A) = \lambda(A)v(A).
$$

Observe that if $A_{21} = 0$ or $A_{12} = 0$, then $\lambda(A) = a_{11}$. Therefore we expect the following behavior

$$
\lambda(A) = a_{11} + O(\|A_{12}\|, \|A_{21}\|),
$$

$$
v(A) = (1, 0, \ldots, 0)^T + O(\|A_{21}\|).
$$

Using our approach we obtain the following bounds

$$
|\lambda(A) - a_{11}| \leq 2\|A_{12}\| \cdot \|A_{21}\| \cdot \|(A_{22} - a_{11}I_{\mathbb{C}^{n-1}})^{-1}\|,
$$

$$
\|v(A) - (1, 0, \ldots, 0)^T\| \leq 2\|A_{21}\| \cdot \|(A_{22} - a_{11}I_{\mathbb{C}^{n-1}})^{-1}\| \cdot \|(1, 0, \ldots, 0)^T\|.
$$
provided \( A_{22} - a_{11} I_{C^{n-1}} \) is invertible and \( \|(A_{22} - a_{11} \cdot I_{C^{n-1}})^{-1}\|^{-2} - 4\|A_{12}\|\|A_{21}\| > 0 \). This is the content of Theorem 5.3. Observe that our bounds satisfy (1,2).

In Section 6 we compare the proposed method with the Gerschgorin theorem in the case of the isolated eigenvalue. It turns out that when the rescalings are allowed in the application of the Gerschgorin theorem, there are examples when one approach works and the other one don’t.

When multiple eigenvalues or clusters of very close eigenvalues are present then the new method still works, while the one based on the classical Gerschgorin theorem fails. This appears to be main strength of our approach.

### 1.1 Notation

By \( \mathbb{R} \) and \( \mathbb{C} \) we denote the sets of real, and complex numbers. The spectrum \( \sigma(A) \) of a square matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) we define the collection of all eigenvalues of \( A \), i.e.

\[
\sigma(A) := \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is singular} \}.
\]

By \( I_{C^n} \) we mean the identity matrix of size \( n \), while \( I \) denotes the interval \([J^−1, 1K^] \). For \( \varepsilon > 0 \) we put \( B_{C}(0, \varepsilon) := \{ z \in \mathbb{C} : |z| < \varepsilon \} \).

### 2 Generalization of Gerschgorin Theorem

The goal of this section is to generalize the Gerschgorin theorem for the matrices defined in terms of multi-dimensional blocks. The classical Gerschgorin theorem is recovered when one-dimensional blocks are considered. Moreover, we show also how from proof of the Gerschgorin theorem the information on the location of eigenvectors can be obtained

**Definition 2.1.** For a linear map (a matrix) \( B : V \to V \), where \( V \) is a finite dimensional normed space, we define \( m(B) \) by

\[
m(B) = \sup \{ \lambda \in \mathbb{R} \mid \forall x \in V \|Ax\| \geq \lambda \|x\| \}. \tag{3}
\]

It is easy to see that \( m(B) = 0 \) iff \( B \) is not invertible, and for \( B \) invertible holds \( m(B) = \frac{1}{\|B^{-1}\|} \).

The basic question we would like to answer is: assume that our matrix \( A \) is close to block-diagonal matrix \( D \), then how close is are spectra of \( A \) and \( D \).

Let \( V = \bigoplus_{i=1}^n V_i \), where \( V_i \) are finite dimensional vector spaces over \( \mathbb{C} \), and \( A : V \to V \) be decomposed into blocks \( A_{ij} : V_j \to V_i \), \( i, j = 1, 2, \ldots, n \), so that for \( v = v_1 + \cdots + v_n \), where \( v_i \in V_i \) holds

\[
A(v_1 + \cdots + v_n) = \sum_i \sum_j A_{ij} v_j. \tag{4}
\]

We define generalized Gerschgorin disks \( G_i(A) \) for the block matrix \( A \) by

\[
R_i(A) = \sum_{j,j \neq i} \|A_{ij}\|,
\]

\[
G_i(A) = \{ \lambda \in \mathbb{C} \mid m(A_{ii} - \lambda I_{V_i}) \leq R_i(A) \}, \quad i = 1, \ldots, n,
\]

where \( I_{V_i} \) is an identity map on \( V_i \). If \( A \) is known from the context, then we will usually drop \( A \) and write just \( R_i \) and \( G_i \).

Theorem below is a generalization of the Gerschgorin theorem

**Theorem 2.2.** [2, 6]

\[
\sigma(A) \subset \bigcup_{i=1}^n G_i. \tag{5}
\]

It is easy to see that if \( \dim V_i = 1 \) for all \( i \), then we recover the classical Gerschgorin theorem.
Proof. The proof is a simple modification of the proof of the Gerschgorin theorem.

Let \((\lambda, v)\), where \(v = v_1 + \cdots + v_n\), be an eigenpair for \(A\). We can assume that for some \(i_0\) \(\|v_{i_0}\| = 1\) and \(\|v_i\| \leq 1\) for \(i = 1, \ldots, n\). We will show that \(\lambda \in G_{i_0}\). Indeed, we have

\[
A_{i_0i_0} v_{i_0} + \sum_{j,j \neq i_0} A_{i_0j} v_j = \lambda v_{i_0},
\]

\[
(A_{i_0i_0} - \lambda) v_{i_0} = - \sum_{j,j \neq i_0} A_{i_0j} v_j.
\]

Hence, we obtain

\[
m(A_{i_0i_0} - \lambda) \leq \|(A_{i_0i_0} - \lambda) v_{i_0}\| \leq \sum_{j,j \neq i_0} \|A_{i_0j}\| \cdot \|v_j\| \leq \sum_{j,j \neq i_0} \|A_{i_0j}\|.
\]

Therefore \(\lambda \in G_{i_0}\). \(\square\)

Also other Gerschgorin theorems can be analogously generalized and the proofs are the same as in the classical case.

So we obtain

**Theorem 2.3.** [2, 6] Assume that \(J \subset \{1, \ldots, n\}\) is such that

\[
\left(\sum_{j \in J} G_j\right) \cap \left(\sum_{j \not\in J} G_j\right) = \emptyset.
\]

Then the number of eigenvalues of \(A\) (counting with multiplicities) contained in \(\left(\sum_{j \in J} G_j\right)\) is equal to \(\sum_{j \in J} \dim V_j\).

The Brauer theorem [1] about the location of eigenvalues using the Cassini ovals can be also easily generalized to obtain the following

**Theorem 2.4.**

\[
\sigma(A) \subset \bigcup_{i,j=1,\ldots,n} \{\lambda \in \mathbb{C} \mid m(A_{ii} - \lambda) \cdot m(A_{jj} - \lambda) \leq R_i R_j\}.
\]

2.1 Improving the estimates through scaling

One of the easiest ways to improve the estimation of the eigenvalues from the Gerschgorin theorem is through the scaling the basis of our domain. This approach is well known and can be found in the original article of Gerschgorin [2].

Assume, that we have matrix \(A \in \mathbb{C}^{n \times n}\) and let \(x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n\) such that \(x_i > 0\) for all \(i \in \{1, \ldots, n\}\). With this vector \(x\) we define the matrix \(X \in \mathbb{R}^{n \times n}\) with the elements of \(x\) on the leading diagonal, and 0 elsewhere. Note, that the matrix \(X\) is nonsingular and matrix \(X^{-1}AX\) is similar to \(A\) therefore \(\sigma(X^{-1}AX) = \sigma(A)\). If \(A = [a_{ij}]_{1 \leq i,j \leq n}\), then \(X^{-1}AX = \left[\frac{a_{ij}x_j}{x_i}\right]_{1 \leq i,j \leq n}\) and

\[
G_i = B\left(a_{ii}, \sum_{j \neq i} \frac{|a_{ij}| x_j}{x_i}\right) \text{ for } i = 1, \ldots, n.
\]

2.2 Localization of eigenvalues

Now we give a theorem about the location of the eigenvectors based on the Gerschgorin argument.

**Theorem 2.5.** Assume that for some \(j \in \{1, \ldots, n\}\)

\[
G_j \cap G_k = \emptyset, \text{ for } k = 1, 2, \ldots, n, k \neq j.
\]

Then if \(v = (v_1 + \cdots + v_n)\) is an eigenvector corresponding to \(\lambda \in G_j\), then \(\|v_k\| \leq \|v_j\|\) for \(k = 1, \ldots, n\).
Proof. To show that \( \|v_k\| \leq \|v_j\| \) we will reason by the contradiction. Assume that for some \( i \neq 0 \) holds \( \|v_i\| \geq \|v_k\| \) for \( k = 1, \ldots, n \) and \( \|v_i\| > \|v_j\| \). We will apply the basic argument from the generalized Gerschgorin theorem (Theorem 2.2) to prove that \( \lambda \in G_1 \). This will lead to a contradiction, because \( \lambda \in G_j \), hence \( \lambda \in G_j \cap G_i \neq \emptyset \).

We have
\[
\lambda v_i = A_{ii} v_i + \sum_{k \neq i} A_{ik} v_k
\]
\[
(\lambda - A_{ii}) v_i = \sum_{k \neq i} A_{ik} v_k
\]
\[
m(\lambda - A_{ii}) |v_i| \leq \sum_{k \neq i} \|A_{ik}\| |v_k|
\]
\[
m(\lambda - A_{ii}) \leq \sum_{k \neq i} \|A_{ik}\| \|v_k\| \leq \sum_{k \neq i} \|A_{ik}\|
\]
hence \( \lambda \in G_1 \). We obtained the contradiction. This finishes the proof. \( \Box \)

Let us discuss another method to estimate an eigenvector which first uses the Gerschgorin theorem to find an accurate eigenvalue and only later solve the equation for the eigenvector.

**Theorem 2.6.** Let \( A = [a_{ij}]_{1 \leq i, j \leq n} \in \mathbb{C}^{n \times n} \) be given in the block form by
\[
A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.
\]
Assume that \( G_1(A) \) has an empty intersection with other Gerschgorin disk (we either use one-dimensional blocks or single \((n-1)\)-dimensional one). Then \( G_1(A) \) contains a unique eigenvalue \( \lambda_1 \) and its eigenvector is given by \( v_1 = (1, \nu)^t \), where
\[
\|(0, \nu)\| \leq \max_{\lambda \in G_1(A)} \| (A_{22} - \lambda)^{-1} \| \cdot \| A_{21} \|.
\]

**Proof.** From Theorems 2.3 and 2.5 we know that there exists unique eigenvalue \( \lambda_1 \in G_1 = \overline{B}(a_{11}, \|A_{12}\|) \) and eigenvector \( x = (1, x_2, \ldots, x_n) = (1, \nu) \) corresponding to \( \lambda_1 \), where \( \|(0, \nu)\| \leq 1 \). We have
\[
\begin{align*}
\begin{cases}
a_{11} + A_{12} \nu = \lambda_1 \\
A_{21} + A_{22} \nu = \lambda_1 \nu.
\end{cases}
\end{align*}
\]

Hence we obtain
\[
\|\nu\| \leq \|(A_{22} - \lambda_1)^{-1}\| \cdot \| A_{21} \|.
\]

\( \Box \)

### 2.3 Computation of \( m(A) \)

In our generalization of Gerschgorin theorem we encounter the problem of the computation of \( m(C) \) for some quadratic matrix \( C \). This might not be an easy task. However, we expect these theorems to be applied in the situation when the matrix \( C \) is close to the diagonal one or some other canonical form. In such situation one can produce computationally cheap and reasonable estimate for \( m(A) \).

**Lemma 2.7.** Consider the norm \( \| \|_{\infty} \). Then for any square matrix holds
\[
\min_i \{|a_{ii}| - \sum_{j, j \neq i} |a_{ij}| \} \leq m(A).
\]

**Proof.** If \( \min_i \{|a_{ii}| - \sum_{j, j \neq i} |a_{ij}| \} \leq 0 \), then there is nothing to prove.

Therefore we can assume that
\[
0 < S := |a_{11}| - \sum_{j, j \neq 1} |a_{1j}| \leq (|a_{ii}| - \sum_{j, j \neq i} |a_{ij}|), \quad i = 1, \ldots, n.
\]

5
Let us take any \( x \in \mathbb{C}^n \), such that \( \|x\| = 1 \). Let \( i \) be such that \( |x_i| = 1 \). We have

\[
|(Ax)_i| \geq (|a_{ii}| |x_i| - \sum_{j,j \neq i} |a_{ij}| \cdot |x_j|) \geq (|a_{ii}| - \sum_{j,j \neq i} |a_{ij}|) \geq S > 0.
\]

Hence

\[
\|Ax\| \geq S. \tag{10}
\]

Let us now use the above lemma to compare our generalized Gerschgorin theorem with the Gerschgorin theorem in a case of isolated eigenvalue. So we want the first Gerschgorin disk to have an empty intersection with the remaining Gerschgorin disks. We will use \( \| \cdot \|_\infty \)-norm.

To apply Theorem 2.3 we will use the splitting \( \mathbb{R} \oplus \mathbb{R}^{n-1} \), so we will have two generalized Gerschgorin disks

\[
G_1(A) = \mathcal{B}(a_{11}, \|A_{12}\|_\infty) = \mathcal{B}(a_{11}, \sum_{j,j \neq 1} |a_{ij}|),
\]

\[
G_2(A) = \{ \lambda \in \mathbb{C} \mid m(A_{22} - \lambda) \leq \max_{j=2,\ldots,n} |a_{j1}| \}. 
\]

From Lemma 2.7 it follows that

\[
G_2(A) \subset \{ \lambda \in \mathbb{C} \mid \min_{i=2,\ldots,n} (|a_{ii} - \lambda| - \sum_{j,j \neq 1,i} |a_{ij}|) \leq \max_{j=2,\ldots,n} |a_{j1}| \}
\]

\[
= \{ \lambda \in \mathbb{C} \mid \exists i = 2,\ldots,n \ |a_{ii} - \lambda| \leq \sum_{j,j \neq 1,i} |a_{ij}| + \max_{j=2,\ldots,n} |a_{j1}| \}.
\]

So we see that \( G_1(A) \cap G_2(A) = \emptyset \) if the following condition hold for any \( i = 2,\ldots,n \)

\[
|a_{11} - a_{ii}| > \sum_{j,j \neq 1} |a_{ij}| + \sum_{j,j \neq 1,i} |a_{ij}| + \max_{j=2,\ldots,n} |a_{j1}|. \tag{11}
\]

If we will use the classical Gerschgorin theorem, i.e. blocks are one-dimensional, then to have \( G_1 \cap G_i = \emptyset \) for \( i \neq 1 \)

\[
|a_{11} - a_{ii}| > R_1 + R_i = \sum_{j \neq 1} |a_{ij}| + \sum_{j \neq i} |a_{ij}|. \tag{12}
\]

Observe that in both cases we have the same Gerschgorin disk \( G_1 \), so the bound for the first eigenvalue will be the same, provided we have nonempty intersections with other disks. Comparing (11) and (12) we see that these are very similar, however we see that (12) is weaker, so the standard Gerschgorin theorem wins over our generalized Gerschgorin theorem combined with Lemma 2.7 in terms of the applicability. Let us remark that the result might not be optimal when computing \( m(A) \) through Lemma 2.7.

The next example shows that we might have the strong inequality in Lemma 2.7.

**Example 2.8.** Consider the norm \( \| \cdot \|_\infty \). Let

\[
A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 100 \end{bmatrix}.
\]

We have

\[
\min_i |a_{ii}| - \sum_{j,j \neq i} |a_{ij}| = 0.5. \tag{13}
\]

To compute \( m(A) \) observe that

\[
\det(A) = 99.75, \quad A^{-1} = \frac{1}{\det A} \begin{bmatrix} 100 & -0.5 \\ -0.5 & 1 \end{bmatrix}, \quad \|A^{-1}\| = \frac{1}{\det A} 100.5 \approx 1.
\]

Hence

\[
m(A) = \|A^{-1}\|^{-1} \approx 1. \tag{14}
\]
Lemma 2.9. Let $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $A \in \mathbb{C}^{n \times n}$ be given. Let $A$ be decomposed into $A = J + E$ where $J$ is a diagonal matrix and $E$ equals zero on the diagonal. Assume that $J - z \cdot I_{\mathbb{C}^n}$ is invertible and $\|(J - z \cdot I_{\mathbb{C}^n})^{-1}\|^{-1} - \|E\| > 0$. Then

$$\|(A - z \cdot I_{\mathbb{C}^n})^{-1}\|^{-1} \geq \|(J - z \cdot I_{\mathbb{C}^n})^{-1}\|^{-1} - \|E\|.$$ 

Proof. It is well-known that for an invertible operator $B$ we have

$$(B - C)^{-1} = \sum_{n=0}^{\infty} (B^{-1}C)^n B^{-1} \quad \text{for} \quad C \in \mathbb{C}^{n \times n} : \|C\| < 1/\|B^{-1}\|.$$ 

Hence, if $\|C\| < 1/\|B^{-1}\|$, then

$$\|(B - C)^{-1}\| \leq \frac{\|B^{-1}\|}{1 - \|B^{-1}\| \cdot \|C\|},$$

so we obtain

$$\|(B - C)^{-1}\|^{-1} \geq \frac{1}{\|B^{-1}\|} \left(1 - \|B^{-1}\| \cdot \|C\|\right) = \frac{1}{\|B^{-1}\|} - \|C\|. \quad (15)$$

From (15) applied to $B = J - z I_{\mathbb{C}^n}$ and $C = -E$ we get assertion of the lemma. \hfill $\square$

2.4 Summary

We presented two type of results: we gave the bound for the eigenvectors (but not on the invariant subspaces corresponding to multiple or very close eigenvalues) and we generalized the Gerschgorin theorem to block matrices. This second topic requires computation of $m(B)$ for matrices, which might be difficult in the higher dimensional case. Probably this reduces the appeal of our generalization.

3 Cones and dominating maps

In this section we introduce the basic concepts and tools of our method of invariant cones to locate the eigenspaces and bound the spectrum for matrices. For this end we modify the concept of cones from [4]. Our approach is strongly motivated by the methods from the theory of hyperbolic dynamical systems, in particular by the results of Newhouse [5], who obtained conditions for hyperbolic splittings on compact invariant sets for a diffeomorphism in terms of its induced action on a cone-field and its complement.

Definition 3.1. By a cone-space we understand a finite dimensional Banach space $E$ with seminorms $\cdot \langle \cdot \rangle$ (we call it contracting), $\langle \cdot \rangle$ (which we call expanding) such that

$$\|\|x\|\| := \max(\langle x\rangle, \langle x \rangle)$$

defines an equivalent norm on $E$. By the $r$-norm for $r > 0$ on the cone-space $E$ we take

$$\|\|x\|\|_r := \max(\langle x\rangle, r \cdot \langle x \rangle).$$

Definition 3.2. Let $E$ be a cone-space. We define the $r$-contracting cone in $E$ by

$$\langle E \rangle_r := \{x \in E : \langle x \rangle \geq r \langle x \rangle\},$$

and the $r$-expanding cone in $E$ by

$$\langle E \rangle_r := \{x \in E : \langle x \rangle \leq r \langle x \rangle\}.$$
Note that

\[ E = \langle E \rangle_{r} \cup \langle E \rangle_{r}. \]  

In the same way we define \( r \)-contracting cone and \( r \)-expanding cone in subspace of \( E \). If \( r = 1 \) we will omit the subscript \( r \), in particular we speak of contracting cone. We introduce the scaling by \( r \) of semi-norms to have a better control over size of the cones (see Figure 1), which will consequently allow us to better locate the eigenvectors.

(a) The contracting cone in \( \mathbb{R} \times \mathbb{R} \).

(b) The 2-contracting cone in \( \mathbb{R} \times \mathbb{R} \).

(c) The expanding cone in \( \mathbb{R} \times \mathbb{R} \).

(d) The 2-expanding cone in \( \mathbb{R} \times \mathbb{R} \).

Figure 1: The cones in the cone-space \( \mathbb{R} \times \mathbb{R} \).

If \( E \) has a fixed product structure \( E = E_{1} \times E_{2} \), we introduce a natural cone-space structure on \( E \) by defining seminorms

\[ \langle x \rangle := \| x_{1} \|, \quad \langle r \rangle := \| x_{2} \| \quad \text{for} \quad x = (x_{1}, x_{2}) \in E_{1} \times E_{2}. \]

In the proof of our main result, Theorem 4.3, the following proposition will play a crucial role.

**Proposition 3.3.** Let \( E = E_{1} \times E_{2} \) be a cone-space and let \( r > 0 \) be given. Assume that we have direct sum decomposition \( E = V_{1} \oplus V_{2} \) such that

\[ V_{1} \subset \langle E \rangle_{r} \quad \text{and} \quad V_{2} \subset \langle E \rangle_{r}. \]

Then \( \dim V_{1} = \dim E_{1} \) and \( \dim V_{2} = \dim E_{2} \).

**Proof.** Let \( n := \dim E_{1} \) and \( m := \dim E_{2} \). First we show that \( \dim V_{1} \leq n \). For an indirect proof, assume that \( \dim V_{1} > n \). Then there exist linearly independent vectors \( v_{1}, \ldots, v_{n+1} \in V_{1} \). Obviously \( v_{i} = (w_{i}, z_{i}) \) for \( i \in \{1, \ldots, n+1\} \) and unique \( w_{i} \in E_{1}, z_{i} \in E_{2} \). Since \( w_{1}, \ldots, w_{n+1} \in E_{1} \) and \( \dim E_{1} = n \) there exist a set of \( n + 1 \) scalars, \( \alpha_{1}, \ldots, \alpha_{n+1} \), not all zero, such that

\[ \alpha_{1}w_{1} + \ldots + \alpha_{n+1}w_{n+1} = 0. \]
Note that

\[ z := \alpha_1 z_1 + \ldots + \alpha_{n+1} z_{n+1} \neq 0, \]

because otherwise the vectors \( v_1, \ldots, v_{n+1} \) would not be linearly independent. Consequently we obtain

\[
(0, z) = \left( \sum_{i=1}^{n+1} \alpha_i w_i, \sum_{i=1}^{n+1} \alpha_i z_i \right) \in V_1 \subseteq \mathcal{E}_r',
\]

and thus \( r\|z\| \leq \|0\| \), which implies that \( z = 0 \). We get a contradiction with the fact the sequence of vectors \( v_1, \ldots, v_{n+1} \) is linearly independent.

The proof that \( \dim V_2 \leq m \) is analogous. Finally, since \( \dim E = n + m \) and \( \dim V_1 \leq n \), \( \dim V_2 \leq m \) we obtain

\[
\dim V_1 = n, \quad \text{and} \quad \dim V_2 = m.
\]

\[ \Box \]

By an \textit{operator} we mean a linear mapping between cone-spaces \( E \) and \( F \). We denote the space of all operators by \( \mathcal{L}(E, F) \). If \( F = E \), we denote \( \mathcal{L}(E, E) \) by \( \mathcal{L}(E) \).

Let \( A \in \mathcal{L}(E, F) \). We define

\[
\langle A \rangle_r := \inf \{ R \in \mathbb{R}_+ \mid \| A x \|_r \leq R \| x \|_r \text{ for all } x \in E : Ax \in \mathcal{F}_r \}, \quad (17)
\]

\[
\langle A \rangle_r := \sup \{ R \in \mathbb{R}_+ \mid \| A x \|_r \geq R \| x \|_r \text{ for all } x \in E : x \in \langle E \rangle_r \}. \quad (18)
\]

\textbf{Remark 3.4.} Observe, that

\[
\| A x \|_r \leq \langle A \rangle_r \| x \|_r \quad \text{for} \quad x \in A^{-1} \mathcal{F}_r,
\]

\[
\| A x \|_r \geq \langle A \rangle_r \| x \|_r \quad \text{for} \quad x \in \langle E \rangle_r.
\]

The above definitions of \( \langle A \rangle_r \) and \( \langle A \rangle_r \) are modifications of analogous notions in [5], where \( \langle A \rangle \) is called the expansion rate and \( 1/\langle A \rangle \) is the co-expansion rate. Using those rates we can generalize the classical dominating maps which are relevant to our research.

\textbf{Definition 3.5.} We say that \( A \in \mathcal{L}(E, F) \) is \( r \)-\textit{dominating}, if

\[
\langle A \rangle_r < \langle A \rangle_r.
\]

By \( \mathcal{D}_r(E, F) \) we denote the set of all \( A \in \mathcal{L}(E, F) \) which are \( r \)-dominating. If \( F = E \), we denote the space \( \mathcal{D}_r(E, E) \) by \( \mathcal{D}_r(E) \).

\textbf{Observation 3.6.} Let \( \tilde{E} \subset E \), \( \tilde{F} \subset F \) be subspaces and let \( A \in \mathcal{L}(E, F) \) be such that \( A(\tilde{E}) \subset \tilde{F} \). Then \( A|_{\tilde{E}} \in \mathcal{L}(\tilde{E}, \tilde{F}) \) and

\[
\langle A \rangle_{\tilde{E}} \leq \langle A \rangle_r \quad \text{and} \quad \langle A \rangle_{\tilde{E}} \leq \langle A \rangle_{\tilde{E}}.
\]

Moreover, if \( A \in \mathcal{D}_r(E, F) \) then \( A \in \mathcal{D}_r(\tilde{E}, \tilde{F}) \).

\textbf{Proof.} It is a consequence of (17), (18) and Definition 3.5. \( \Box \)

It turns out that \( r \)-cones are invariant for \( r \)-dominant operators.

\textbf{Theorem 3.7.} Let \( A \in \mathcal{D}_r(E, F) \) and let \( v \in E \) be arbitrary. Then

\[
v \in \langle E \rangle_r \quad \Rightarrow \quad Av \in \langle F \rangle_r,
\]

\[
Av \in \mathcal{F}_r \quad \Rightarrow \quad v \in \langle E \rangle_r.
\]

\textbf{Proof.} The proof is a simple modification of the proof of [4, Proposition 2.1]. \( \Box \)

As a consequence of the above theorem we obtain that composition of \( r \)-dominating maps is \( r \)-dominating. Moreover, we get estimate for expansion and contraction rates.
Proposition 3.8. Let $A \in \mathcal{D}_r(F,G)$ and $B \in \mathcal{D}_r(E,F)$. Then $A \circ B \in \mathcal{D}_r(E,G)$ and
\[
\langle A \circ B \rangle_r \leq A_{(r)} \cdot B_{(r)}, \quad (A \circ B)_r \geq (A)_r \cdot (B)_r.
\] (19)

Proof. To prove the first inequality from (19), consider an $x \in E$ such that $(A \circ B)(x) \in G_{(r)}$. From (17) and Theorem 3.7 we know that $Bx \in F_{(r)}$, and thus we have
\[
\|A \circ B(x)\|_r \leq \|A_{(r)} \cdot Bx\|_r \leq \|A_{(r)}\|_{\mathcal{B}} \cdot \|x\|_r.
\]
Hence
\[
\|A \circ B\|_{(r)} \leq \|A_{(r)}\|_{\mathcal{B}}\|B\|_{(r)}.
\]
Using (18) and Theorem 3.7, we obtain the second inequality from (19).

As a simple consequence of (19) we obtain $A \circ B \in \mathcal{D}_r(E,G)$.

In the remainder of this section we show how to estimate $\|A_{(r)}\|_{\mathcal{B}}$. Consider two conespaces $E = E_1 \times E_2$ and $F = F_1 \times F_2$. Let $A : E \to F$ be an operator given in the matrix form by
\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.
\]

By
\[
\|A\|_{(r)} := \max\left(\|A_{11}\| + \frac{1}{r}\|A_{12}\|, r\|A_{21}\| + \|A_{22}\|\right)
\]
we define the $r$-norm of operator $A$, where $\cdot$ is an operator norm. Observe that it satisfies
\[
\|Ax\|_r \leq \|A\|_{(r)} \cdot \|x\|_r \quad \text{for} \quad x \in E.
\]
Note that in general it is not (except for the case when $E_1$ is one dimensional) the operator norm for $\|\cdot\|_r$.

Theorem 3.9. Let $A = [A_{ij}]_{1 \leq i,j \leq 2} \in \mathcal{L}(E_1 \times E_2, F_1 \times F_2)$ and $r \in (0, \infty)$ be given.

1) We have
\[
\langle A \rangle_r \leq \|A_{11}\| + \frac{1}{r}\|A_{12}\|.
\]

2) Additionally, if $A_{22}$ is invertible, then
\[
\|A\|_{(r)} \geq \|A_{11}\|^{-1} - r\|A_{21}\| = m(A_{22}) - r\|A_{21}\|.
\]

Proof. For the proof of the first inequality, we take $x = (x_1, x_2) \in E_1 \times E_2$ such that $Ax \in F_{(r)}$. From Definition 3.2 we have
\[
\|A_{11}x_1 + A_{12}x_2\| \geq r\|A_{21}x_1 + A_{22}x_2\|,
\]
and therefore
\[
\|Ax\|_r = \max(\|A_{11}x_1 + A_{12}x_2\|, r\|A_{21}x_1 + A_{22}x_2\|) \geq \|A_{11}x_1 + A_{12}x_2\| \leq \|A_{11}\| \cdot \|x_1\| + \frac{1}{r}\|A_{12}\| \cdot r\|x_2\| \leq \left(\|A_{11}\| + \frac{1}{r}\|A_{12}\|\right) \cdot \|x\|_r.
\]

For the proof of the second inequality, suppose that $x = (x_1, x_2) \in \langle E \rangle_r$, where $x_1 \in E_1$, $x_2 \in E_2$. Then
\[
\|x_1\| \leq r\|x_2\| = \|x\|_r.
\]
We know that
\[
\|A_{22}x_2\| \geq \|A_{22}^{-1}\|^{-1}\|x_2\| \geq 0.
\]
Finally, we obtain
\[
\|Ax\|_r \geq r\|A_{21}x_1 + A_{22}x_2\| \geq r\|A_{22}^{-1}\|^{-1}\|x_2\| \geq \left(\|A_{22}^{-1}\|^{-1} - r\|A_{21}\|\right) \cdot \|x\|_r.
\]

Example 3.10. Let us verify that the matrix $A \in \mathcal{L}(\mathbb{C} \times \mathbb{C}, \mathbb{C} \times \mathbb{C})$, $A = \begin{bmatrix} 2 & 1.5 \\ 1 & 5 \end{bmatrix}$ is dominating.

By Theorem 3.9 we have $\|A\| \leq 3.5 < 4 \leq \langle A \rangle$, and therefore $A$ is dominating.

4 Localization of eigenspaces based on cones and dominating maps

In this section we show that the eigenspaces of the $r$-dominating operator $A$ lie in the corresponding $r$-cones. Moreover, we can estimate $\sigma(A)$ with the help of $\|A\|$, $\langle A \rangle$.

Lemma 4.1. Let $A \in \mathcal{D}_r(E)$. Then

$$\lambda \in \sigma(A) \iff |\lambda| \in [0, \|A\|] \cup \langle A \rangle, \infty).$$

(24)

Moreover $[0, \|A\|] \cap \langle A \rangle, \infty) = \emptyset$.

Proof. Since $A \in \mathcal{D}_r(E)$ we get $[0, \|A\|] \cap \langle A \rangle, \infty) = \emptyset$.

Now we show implication (24). Let $\lambda$ be an eigenvalue of $A$ and let $x \in E$ be a corresponding eigenvector. By (16) we know that $x \in \langle E \rangle$. We consider two cases. First suppose that $x \in \langle E \rangle$. Since $x$ is an eigenvector, $Ax = \lambda x$, and thus $A \in \langle E \rangle$. By (17) we get

$$|\lambda| \leq \langle A \rangle.$$

Now suppose that $x \in \langle E \rangle$. By (18) we get

$$|\lambda| \geq \langle A \rangle,$$

which completes the proof.

Let $E$ be a finite dimensional vector space over the field $\mathbb{C}$ and let operator $A : E \to E$ be given. One can easily deduce from the Jordan theorem (see also [3, Appendix to Chapter 4] for the general case) that if $\sigma(A) = \sigma_1 \cup \sigma_2$ then there is a unique direct sum decomposition $E = E_{\sigma_1} \oplus E_{\sigma_2}$ such that $A(E_{\sigma_1}) \subset E_{\sigma_1}$, $A(E_{\sigma_2}) \subset E_{\sigma_2}$ and $\sigma(A|_{E_{\sigma_1}}) = \sigma_1$, $\sigma(A|_{E_{\sigma_2}}) = \sigma_2$. For $0 < c < d$ we define

$$E_{\leq c} := E_{\{\lambda : |\lambda| \leq c\}} \quad \text{and} \quad E_{\geq d} := E_{\{\lambda : |\lambda| \geq d\}}.$$

Theorem 4.2. Let $E$ be a finite dimensional cone-space and let $A \in \mathcal{D}_r(E)$. Then there is a direct sum decomposition $E = E_{\leq \|A\|} \oplus E_{\geq \langle A \rangle}$ which satisfies

$$E_{\leq \|A\|} \subset \langle E \rangle, E_{\geq \langle A \rangle} \subset \langle E \rangle.$$

Proof. From Lemma 4.1 and the comments preceding our theorem we obtain a decomposition of $E$ into $A$-invariant subspaces

$$E = E_{\leq \|A\|} \oplus E_{\geq \langle A \rangle},$$

such that

$$\sigma(A|_{E_{\leq \|A\|}}) = \{\lambda : |\lambda| \in [0, \|A\|]\} \quad \text{and} \quad \sigma(A|_{E_{\geq \langle A \rangle}}) = \{\lambda : |\lambda| \in \langle A \rangle, \infty)\}.$$

Now we show $E_{\leq \|A\|} \subset \langle E \rangle$. Consider an arbitrary $x \in E_{\leq \|A\|}$. The case when $x = 0$ is obvious. Assume that $x \neq 0$. Without any loss of the generality we can assume that $\|x\| = 1$. For an indirect proof, assume that $x \notin \langle E \rangle$. Then by (16) we get $x \notin \langle E \rangle$. Let $\varepsilon > 0$ be arbitrary. From the fact that $x \in E_{\leq \|A\|}$, we know that

$$\lim_{m \to +\infty} \sqrt{\|A^m x\|^2} = \sup \sigma(A|_{E_{\leq \|A\|}}) \leq \langle A \rangle.$$

(25)

Note that inequality (25) holds for all norms. For all $x \in E_{\leq \|A\|}$ we obtain

$$\lim_{m \to +\infty} \sqrt{\|A^m x\|^2} \leq \langle A \rangle.$$

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and thus there exists an $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$

$$m \geq M \Rightarrow \sqrt{\|A^mx\|} \leq \|A\|_r + \varepsilon.$$ 

Since $x \in \langle E \rangle_r$ and from Theorem 3.7 we obtain

$$x \in \langle E \rangle_r \Rightarrow Ax \in \langle E \rangle_r \Rightarrow \cdots \Rightarrow A^mx \in \langle E \rangle_r.$$ 

Using (18) and Remark 3.4 we get

\[
\|Ax\| \geq \langle A \rangle_r \|x\|, \\
\|A^2x\| = \|A(Ax)\| \geq \langle A \rangle_r \|Ax\| \geq \langle A \rangle_r^2 \|x\|, \\
\vdots \\
\|A^mx\| \geq \langle A \rangle_r^m \|x\|.
\]

Finally we have

$$\langle A \rangle_r = \sqrt{\langle A \rangle_r^m \|x\|} \leq \sqrt{\|A^mx\|} \leq \langle A \rangle_r + \varepsilon.$$ 

Since $\varepsilon$ was arbitrary, we get a contradiction with the fact that $A$ is $r$-dominating.

Analogously, to prove inclusion $E_{\geq \langle A \rangle_r} \subset \langle E \rangle_r$, assume that $x \in E_{\geq \langle A \rangle_r}$, and $x \notin \langle E \rangle_r$.

Then $x \in \langle E \rangle_r$. Since $\sigma(A|_{E_{\geq \langle A \rangle_r}}) = \sigma_{\geq \langle A \rangle_r} = \{ \lambda : |\lambda| \geq \langle A \rangle_r \}$ and $0 \notin \sigma_{\geq \langle A \rangle_r}$, we know that $A|_{E_{\geq \langle A \rangle_r}} : E_{\geq \langle A \rangle_r} \to E_{\geq \langle A \rangle_r}$ is invertible. Let $\varepsilon > 0$ be arbitrary. Using the fact that $x \in E_{\geq \langle A \rangle_r}$, by dual result (25), we know that

$$\limsup_{m \to +\infty} \sqrt[m]{\|A|_{E_{\geq \langle A \rangle_r}}^{-m} x\|} \leq \langle A \rangle_r^{-1},$$

and thus there exists an $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$

$$m \geq M \Rightarrow \sqrt[m]{\|A|_{E_{\geq \langle A \rangle_r}}^{-m} x\|} \leq \langle A \rangle_r^{-1} + \varepsilon. \quad (26)$$

From the Observation 3.6 and Theorem 3.7 we get

$$x \in E_{\geq \langle A \rangle_r} \Rightarrow A|_{E_{\geq \langle A \rangle_r}}^{-1} x \in E_{\geq \langle A \rangle_r} \Rightarrow \cdots \Rightarrow A|_{E_{\geq \langle A \rangle_r}}^{-m} x \in E_{\geq \langle A \rangle_r} \Rightarrow \cdots$$

and from (17) and Remark 3.4 we have

\[
\|x\| \leq \langle A \rangle_r^{-1} \|x\|, \\
\|A|_{E_{\geq \langle A \rangle_r}}^{-1} x\| \leq \langle A \rangle_r^{-2} \|x\|, \\
\vdots \\
\|A|_{E_{\geq \langle A \rangle_r}}^{-m} x\| \leq \langle A \rangle_r^{-m+1} \|x\|.
\]

Hence

$$\|x\| \leq (\langle A \rangle_r)^{m} \|A|_{E_{\geq \langle A \rangle_r}}^{-m} x\|. \quad (27)$$

Finally from the Observation 3.6 and (26), (27) we obtain

$$\langle A \rangle_r \geq \langle A \rangle_r^{-m} \geq \sqrt{\langle A \rangle_r^{-m} \|x\|} \geq \frac{1}{\langle A \rangle_r^{-1} + \varepsilon} = \langle A \rangle_r \cdot \frac{1}{1 + \varepsilon \cdot \langle A \rangle_r},$$

which gives a contradiction with the fact that $A$ is $r$-dominating. \hfill \Box

Now we are ready to state the main result on the eigenspaces and eigenvalue location using our method of cones and dominating maps.

**Theorem 4.3.** Let $E = E_1 \times E_2$ be a finite dimensional cone-space and let $A \in \mathcal{D}_r(E)$. Then there exists a unique direct sum decomposition $E = F_1 \oplus F_2$ of $A$-invariant subspaces $F_1$, $F_2$ such that

$$\sigma(A|_{F_1}) \subset \overline{B}(0, \langle A \rangle_r), \quad \sigma(A|_{F_2}) \subset \mathbb{C} \setminus B(0, \langle A \rangle_r).$$

Moreover, we have:
1) \( \dim F_1 = \dim E_1, \ \dim F_2 = \dim E_2, \)
2) \( F_1 \subset \mathcal{E}(r) \) and \( F_2 \subset \langle E \rangle \),
3) \( \| A_{F_1} \| \leq \mathcal{M}_r \) and \( \| (A_{F_2})^{-1} \| \leq (A)_r^{-1} \).

**Proof.** From Theorem 4.2 we know that exists a unique direct sum decomposition \( E = E_{\leq \mathcal{M}_r} \oplus E_{\geq (A)_r} \) which satisfies

\[
E_{\leq \mathcal{M}_r} \subset \mathcal{E}(r), \quad E_{\geq (A)_r} \subset \langle E \rangle.
\]

We take \( F_1 = E_{\leq \mathcal{M}_r} \) and \( F_2 = E_{\geq (A)_r} \). By Proposition 3.3 we obtain \( \dim F_1 = \dim E_1 \) and \( \dim F_2 = \dim E_2 \).

Now we show that \( \sigma(A_{F_1}) \subset \mathcal{B}(0, \mathcal{M}_r) \). Let \( x \in F_1 \) be an eigenvector of \( A \) and let \( \lambda \) be the eigenvalue of \( A \) corresponding to \( x \). Since \( x \) is an eigenvector \( (Ax = \lambda x) \) and \( F_1 \subset \mathcal{E}(r) \) therefore \( Ax \in \langle E \rangle_r \). By (17) we obtain that \( |\lambda| \leq \mathcal{M}_r \), so we get \( \sigma(A_{F_1}) \subset \mathcal{B}(0, \mathcal{M}_r) \).

Now suppose that \( x \in F_2 \). Since \( F_2 \subset \langle E \rangle_r \) and by (18) we get \( |\lambda| \geq (A)_r \). Hence \( \sigma(A_{F_2}) \subset \mathbb{C} \setminus B(0, (A)_r) \).

The inequalities of item 3) we obtain from (17) and (18).

As a direct consequence of the above theorem we obtain the following conclusion.

**Corollary 4.4.** Let \( r \in (0, \infty) \) and \( n \in \mathbb{N} \). Assume that an operator \( A \in \mathcal{D}_r(\mathbb{C} \times \mathbb{C}^{n-1}) \) is given. Then there exists unique eigenvalue \( \lambda \) of \( A \) such that \( |\lambda| \leq \mathcal{M}_r \) and the eigenspace corresponding to \( \lambda \) is one-dimensional. The unique (after rescaling) eigenvector \( x \) corresponding to the eigenvalue \( \lambda \) satisfies

\[
x \in (1,0,\ldots,0)^T + \{0\} \times \mathcal{B}(0,1/r)^{n-1} \subset (1,0,\ldots,0)^T + \frac{1}{r} \cdot (0,\mathbb{I},\ldots,\mathbb{I})^T + \frac{1}{r} \cdot (0,\mathbb{I},\ldots,\mathbb{I})^T i.
\]

**Proof.** It is a direct consequence of Theorem 4.3 and Definition 3.2.

Because at the origin of our approach based on cones and dominating maps is the theory of hyperbolic dynamical systems, so our method should be well suited to locate the eigenspaces and eigenvalues of products of many matrices. In the example below we contrast our approach with a naive approach, which tries to diagonalize a matrix obtained as a product of many matrices. The essential feature of this example is that the matrices we multiply are known with some accuracy only.

**Example 4.5.** Let the matrices \( A_i \in \mathcal{L}(\mathbb{R} \times \mathbb{R}) \), \( i \in \{1, \ldots, 15\} \) be such that

\[
A_i = \begin{bmatrix} [0,0.5] & \epsilon \mathbb{I} \\ \epsilon \mathbb{I} & [1.5,2] \end{bmatrix},
\]

where \( \epsilon = 0.01 \) and \( \mathbb{I} = [-1,1] \). Consider the matrix \( B := A_{15} \cdot \ldots \cdot A_1 \).

From Theorem 3.9 we obtain that \( A_i \in \mathcal{D}(\mathbb{R} \times \mathbb{R}) \) and

\[
\langle A_i \rangle_{\leq 0.5 + \epsilon, \langle A_i \rangle_{\geq 1.5 - \epsilon}}.
\]

From Theorem 3.7 and Proposition 3.8 we conclude that \( B \in \mathcal{D}(\mathbb{R} \times \mathbb{R}) \) and

\[
\langle B \rangle_{\leq \langle A_{15} \rangle \cdot \ldots \cdot A_1}, \quad \langle B \rangle_{\geq \langle A_{15} \rangle \cdot \ldots \cdot A_1}.
\]

From Theorem 4.3 we obtain that eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( B \) such that

\[
|\lambda_1| \leq (0.5 + \epsilon)^{15} \quad \text{and} \quad |\lambda_2| \geq (1.5 - \epsilon)^{15}.
\]

Now, a naive method will ask first for a computation of \( B \). Using interval arithmetic we obtained

\[
B \in \begin{bmatrix} [-1.45687, 1.45693], [-218.543, 218.544] \\ [-218.543, 218.544], [433.611, 32782.94] \end{bmatrix}.
\]

However, there exists matrix \( B_1 \) within the bounds given above, which has both eigenvalues larger than 1. For example, let us consider

\[
B_1 = \begin{bmatrix} 1 & 100 & 521 \\ -100 & \end{bmatrix}.
\]

This matrix have the eigenvalues \( \lambda_1 = 21 \) and \( \lambda_2 = 501 \). Consequently, this means that none of the methods applied to the product matrix will not give us the expected estimation \( |\lambda_1| < 1 \) and \( |\lambda_2| > 1 \).
5 Estimations of the eigenvalues and eigenvectors

In this section we develop computable estimates for the eigenvalues and eigenspaces based on the results from the previous section.

Lemma 5.1. Let $A \in \mathcal{L}(E_1 \times E_2)$ be given such that

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$  

If $A_{22}$ is invertible, $d = \|A_{22}^{-1}\|^{-1} - \|A_{11}\| > 0$ and $\Delta := d^2 - 4\|A_{12}\|\|A_{21}\| > 0$ then

$$A \in \mathcal{D}_r(E_1 \times E_2) \quad \text{for} \quad \begin{cases} r \in \left( \frac{d - \sqrt{\Delta}}{2\|A_{21}\|}, \frac{d + \sqrt{\Delta}}{2\|A_{21}\|} \right) & \text{if } \|A_{21}\| \neq 0 \\ r \in \left( \frac{\|A_{12}\|}{d}, \infty \right) & \text{if } \|A_{21}\| = 0 \end{cases}.$$  

Proof. Let $a := \|A_{12}\|$, $b := \|A_{11}\|$ and $c := \|A_{21}\|$. Making use of Theorem 3.9 it suffices to show that

$$b + \frac{a}{r} < (d + b) - cr.$$  

Multiplying both sides of the above inequality by the positive number $r$ we get the inequality

$$cr^2 - dr + a < 0. \quad (28)$$

If $c = 0$ then we get $r > \frac{a}{d}$. Suppose now that $c \neq 0$. Since from our assumption follows that $\Delta > 0$ we see inequality (28) is satisfied for

$$r \in \left( \frac{d - \sqrt{\Delta}}{2c}, \frac{d + \sqrt{\Delta}}{2c} \right).$$

\[\square\]

Remark 5.2. Let $A$ be an operator, which satisfies the assumptions of Lemma 5.1 (in particular $\Delta > 0$). Let $a := \|A_{12}\|$, $b := \|A_{11}\|$ and $c := \|A_{21}\| \neq 0$. It is easy to see, that

$$\frac{d - \sqrt{\Delta}}{2c} < \frac{d}{2c} < \frac{d + \sqrt{\Delta}}{2c} < \frac{d}{c}.$$  

Therefore, if $A$ satisfies the assumptions of Lemma 5.1 and $\|A_{21}\| \neq 0$ and we want to find possibly largest $r$ for which $A$ is $r$-dominating, then we can take $r = \frac{d}{\|A_{21}\|}$. With this choice we have $r < r_{\max} < 2r$, where $r_{\max}$ is the supremum the set of $r$'s obtained in the above lemma, therefore we might not be optimal, but we obtain easily manageable expression.

We present now our main result on the location of an isolated eigenvalue and its eigenspace.

Theorem 5.3. Let $A = [a_{ij}]_{1 \leq i,j \leq n} \in \mathcal{L}(\mathbb{C} \times \mathbb{C}^{n-1})$ be given in the block from by

$$A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} = a_{11}$. Assume that $A_{22} - a_{11} \cdot I_{\mathbb{C}^{n-1}}$ is invertible and $\|A_{22} - a_{11} \cdot I_{\mathbb{C}^{n-1}}\|^{-2} - 4\|A_{12}\|\|A_{21}\| > 0$. Then

1) there exists a unique eigenvalue $\lambda$ of $A$ which satisfies

$$|\lambda - a_{11}| \leq 2\|A_{12}\| \cdot \|A_{21}\| \cdot \|A_{22} - a_{11} \cdot I_{\mathbb{C}^{n-1}}\|^{-1};$$

2) the eigenspace corresponding to $\lambda$ is one-dimensional and there exist unique $\delta_2, \ldots, \delta_n \in \mathbb{C}$,

$$\|(0, \delta_2, \ldots, \delta_n)^T\| \leq 2\|A_{21}\| \cdot \|A_{22} - a_{11} \cdot I_{\mathbb{C}^{n-1}}\|^{-1} \cdot \|(1,0,\ldots,0)^T\|$$

such that $(1, \delta_2, \ldots, \delta_n)^T$ is the eigenvector corresponding to $\lambda$.  

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Proof. It is easy to see that if \( A_{21} = 0 \), then theorem holds. Therefore we will assume that \( \| A_{21} \| > 0 \).

In order to apply Lemma 5.1 to matrix \( A - a_{11}I \), we set \( a := \| A_{12} \| \), \( c := \| A_{21} \| \) and \( d := \| (A_{22} - a_{11}I_{C^n}^{-1})^{-1} \|^{-1} \). By Lemma 5.1 and Remark 5.2 we get \( A - a_{11}I \in D_{d/2c}(C \times C^{n-1}) \), and from Corollary 4.4 and Theorem 3.9 we conclude that there exists a unique eigenvalue \( \lambda \) of \( A \) which satisfies
\[
|\lambda - a_{11}| \leq \| A - a_{11}I \| \leq \frac{2ac}{d}.
\]

From Theorem 4.3 (second point) we know that eigenspace, which contains eigenvector corresponding to the \( \lambda \), lies in \( JC \times C^{n-1} \). Hence (see Definition 3.2) we obtain unique \( \delta_2, \ldots, \delta_n \in C \), \( \| (0, \delta_2, \ldots, \delta_n)^T \| \leq 2\| A_{21} \| \cdot \| (A_{22} - a_{11}I_{C^n}^{-1})^{-1} \| \cdot \| (1, 0, \ldots, 0)^T \| \) such that \( (1, \delta_2, \ldots, \delta_n)^T \) is the eigenvector corresponding to \( \lambda \).

Now we present results about the location of the eigenspaces.

**Theorem 5.4.** Let \( k, n \in \mathbb{N} \) such that \( 0 \leq k \leq n \) and \( A \in \mathcal{L}(C^k \times C^{n-k}) \) be given in the block from by
\[
A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
\]
where \( A_{11} \in \mathcal{L}(C^k) \), \( A_{12} \in \mathcal{L}(C^k, C^{n-k}) \), \( A_{21} \in \mathcal{L}(C^{n-k}, C^k) \) and \( A_{22} \in \mathcal{L}(C^{n-k}) \). Assume that \( A_{22} \) is invertible, \( d := \| A_{22} \|^{-1} - \| A_{11} \| > 0 \) and \( d^2 - 4\| A_{12} \|\| A_{21} \| > 0 \). Then there exists a unique direct sum decomposition \( C^k \times C^{n-k} = F_1 \oplus F_2 \), such that \( F_1 \) and \( F_2 \) are \( A \)-invariant subspaces \( F_1 \), \( F_2 \), \( \dim F_1 = k \), \( \dim F_2 = n-k \) and
\[
F_1 \subset \left\{ (x_1, x_2) \in C^k \times C^{n-k} : \| x_2 \| \leq \frac{2\| A_{21} \| d}{\| A_{12} \|} \| x_1 \| \right\}, \tag{29}
\]
\[
F_2 \subset \left\{ (x_1, x_2) \in C^k \times C^{n-k} : \frac{2\| A_{21} \| d}{\| A_{12} \|} \| x_1 \| \leq \| x_2 \| \right\}.
\]

Moreover, we have
\[
\sigma(A|_{F_1}) \subset B \left( 0, \| A_{11} \| + \frac{2\| A_{12} \| \cdot \| A_{21} \|}{d} \right), \quad \sigma(A|_{F_2}) \subset C \setminus B \left( 0, \| A_{21}^{-1} - \frac{d}{2} \right). \tag{30}
\]

**Proof.** Let \( c := \| A_{21} \| \). If \( c = 0 \), the assertion holds. Assume that \( c \neq 0 \). By Lemma 5.1 we get \( A \in D_{d/2c}(C^k \times C^{n-k}) \), and from Theorem 4.3 we know that exists a direct sum decomposition \( C^k \times C^{n-k} = F_1 \oplus F_2 \) such that \( \dim F_1 = k \), \( \dim F_2 = n-k \) and \( F_1 \), \( F_2 \) are invariant. The properties (29) and (30) are consequences of Theorem 4.3 and Theorem 3.9 and Definition 3.2, respectively.

**Corollary 5.5.** We use the same notation and decomposition of the matrix \( A \) as in Theorem 5.4. Assume that for some \( z \in C \) matrices \( A_{11} - zI_{C^k}, A_{22} - zI_{C^{n-k}} \) are invertible and \( d := \| A_{22} - zI_{C^{n-k}}^{-1} \|^{-1} - \| A_{11} - zI_{C^k} \| > 0 \), \( d^2 - 4\| A_{12} \|\| A_{21} \| > 0 \). Then there exists a unique direct sum decomposition \( C^k \times C^{n-k} = F_1 \oplus F_2 \) into \( A \)-invariant subspaces \( F_1 \), \( F_2 \) such that \( \dim F_1 = k \), \( \dim F_2 = n-k \) and
\[
F_1 \subset \left\{ (x_1, x_2) \in C^k \times C^{n-k} : \| x_2 \| \leq \frac{2\| A_{21} \| d}{\| A_{12} \|} \| x_1 \| \right\},
\]
\[
F_2 \subset \left\{ (x_1, x_2) \in C^k \times C^{n-k} : \frac{2\| A_{21} \| d}{\| A_{12} \|} \| x_1 \| \leq \| x_2 \| \right\},
\]

Moreover, we have
\[
\sigma(A|_{F_1}) \subset B \left( z, \| A_{11} - zI_{C^k} \| + \frac{2\| A_{12} \| \cdot \| A_{21} \|}{d} \right),
\]
\[
\sigma(A|_{F_2}) \subset C \setminus B \left( z, \| (A_{22} - zI_{C^{n-k}}^{-1}) \|^{-1} - \frac{d}{2} \right).
\]
5.1 Examples

We consider now two examples illustrating the power of our method in the situation when the matrix is block-diagonally dominating.

Example 5.6. Consider the matrix $A \in \mathcal{L}(\mathbb{C}^2 \times \mathbb{C}^2)$ be given by

$$
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
0 & 0.15 & 0.11 & 0.02 \\
0.2 & 0 & 0.1 & 0.05 \\
0.01 & 0.025 & 0 & 1.5 \\
0.15 & 0.05 & 1 & 0
\end{bmatrix}.
$$

We have $\|A_{11}\|_\infty = 0.2$, $\|A_{12}\|_\infty = 0.15$, $\|A_{21}\|_\infty = 0.2$. From Theorem 5.4 ($d = \|(A_{22}^{-1} - A_{11})\|_\infty - \|A_{11}\|_\infty = 1 - 0.2 = 0.8 > 0$ and $d^2 - 4\|A_{12}\|_\infty \|A_{21}\|_\infty = 0.52 > 0$) we know that there exist eigenspaces $F_1$ and $F_2$, which satisfy

$$
F_1 \subset \{ (x_1, x_2) \in \mathbb{C}^2 \times \mathbb{C}^2 : \|x_2\| \leq 0.5\|x_1\| \},
$$

$$
F_2 \subset \{ (x_1, x_2) \in \mathbb{C}^2 \times \mathbb{C}^2 : \|x_1\| \leq 2\|x_2\| \}.
$$

and $\sigma(A_{F_1}) \subset \overline{B}(0,0.275)$, $\sigma(A_{F_2}) \subset \mathbb{C} \setminus B(0,0.6)$ (see Figure 2(b)).

![Figure 2: Gerschgorin and our circles with approximate eigenvalues in Example 5.6.](image)

Observe that when using the Gerschgorin theorem with one-dimensional blocks with scalings, as described in Section 2.1, we will not be able to separate the spectrum of $A$, because the centers of Gerschgorin circles are located at zero.

Now we discuss what happens when we use the generalized Gerschgorin theorems from Section 2. First rescale the matrix $A$ by $X = \begin{bmatrix} 3 & 0 \\
0 & 1
\end{bmatrix}$ (as we shall see, such $X$ will be enough to get a disjoint generalized Gerschgorin disks) to get

$$
\tilde{A} = X^{-1}AX = \begin{bmatrix}
A_{11} & \frac{1}{3}A_{12} \\
A_{21} & A_{22}
\end{bmatrix}.
$$

We use the generalized Gerschgorin Theorems 2.2 and 2.3 applied to the above block decomposition, and obtain the generalized Gerschgorin disks:

$$
G_1(\tilde{A}) = \{ m(A_{11} - \lambda) \leq \frac{1}{3}\|A_{12}\|_\infty \},
$$

$$
G_2(\tilde{A}) = \{ m(A_{22} - \lambda) \leq 3\|A_{21}\|_\infty \}.
$$
We want to show that $G_1(\tilde{A}) \cap G_2(\tilde{A}) = \emptyset$. Let us check that $G_1(\tilde{A}) \subset \overline{B}(0, 0.25)$. We have

$$(A_{11} - \lambda)^{-1} = \frac{1}{\lambda^2 - 0.03} \begin{bmatrix} -\lambda & -0.15 \\ -0.2 & -\lambda \end{bmatrix},$$

so we get

$$m(A_{11} - \lambda) = \|(A_{11} - \lambda)^{-1}\|_\infty^{-1} = \frac{|\lambda^2 - 0.03|}{0.2 + |\lambda|}.$$ 

For $\lambda \in G_1(\tilde{A}) \subset \mathbb{C}$ we have

$$\frac{|\lambda^2 - 0.03|}{0.2 + |\lambda|} \leq 0.05.$$ 

Performing simple mathematical operations and changing the coordinate system to polar we obtain

$$r^4 - r^2 \left( \frac{3 \cos(2\varphi)}{50} + \frac{1}{400} \right) - \frac{r}{1000} + \frac{1}{1250} \leq 0, \quad r = |\lambda| \in [0, \infty), \quad \varphi \in [0, 2\pi).$$ 

Solving the above inequality we get

$$\sup r = \frac{1}{40} \left( 1 + \sqrt{65} \right) \leq \frac{1}{4}.$$ 

This means that $G_1(\tilde{A}) \subset \overline{B}(0, 0.25)$. Now we show that $\lambda \notin G_2(\tilde{A})$ for an arbitrary $\lambda \in \overline{B}(0, 0.25)$.

Indeed we have

$$(A_{22} - \lambda)^{-1} = \frac{1}{\lambda^2 - 1.5} \begin{bmatrix} -\lambda & -1.5 \\ -1 & -\lambda \end{bmatrix}.$$ 

It is easy to see that for $\lambda \in \overline{B}(0, 0.25)$ we have

$$\|(A_{22} - \lambda)^{-1}\|_\infty \leq \frac{1.5 + 0.25}{1.5 - 0.25^2} < 1.22.$$ 

Hence

$$m(A_{22} - \lambda) > 0.8, \quad \lambda \in G_1(\tilde{A}) \subset \overline{B}(0, 0.25).$$ 

Finally, we get $G_1(\tilde{A}) \cap G_2(\tilde{A}) = \emptyset$ (see Figure 3) and therefore we obtain from Theorems 2.2 and 2.3 that two eigenvalues belong to $G_1(\tilde{A})$ while the remaining two eigenvalues are inside $G_2(\tilde{A})$. As one can see, we get better estimation for eigenvalues close to 0 from generalized Gerschgorin theorem with scaling $r = 3$, than from Theorem 5.4 but by generalized Gerschgorin theorem we can not get eigenspaces.
Figure 3: Generalized Gerschgorin circles: \( G_1(\tilde{A}) \) – greater circles and \( G_2(\tilde{A}) \) – smaller ones in Example 5.6 (compare Fig. 2(b)).

The next example shows that we can use the generalized Gerschgorin theorem and approach based on cones to find the eigenvectors, while classical Gerschgorin theorem can not be used because assumption of this theorem are not satisfied.

**Example 5.7.** Consider the matrix given by

\[
A = \begin{bmatrix}
0 & 0.01 & 0.1 \\
0.2 & 0 & 1 \\
0.15 & 2 & 0
\end{bmatrix}.
\]

We will use three approaches that are discussed in this paper, but first we introduce a block decomposition of \( A \).

We decompose \( \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}^2 \), with the norm on \( \mathbb{R}^2 \) being \( \| \cdot \|_\infty \) norm. Then \( A \) can written in the block form \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \), where \( A_{11} = \begin{bmatrix} 0 \end{bmatrix} \), \( A_{12} = \begin{bmatrix} 0.01 & 0.1 \end{bmatrix} \), \( A_{21} = \begin{bmatrix} 0.2 \\ 0.15 \end{bmatrix} \) and \( A_{22} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \). The block \( A_{22} \) is invertible and \( A_{22}^{-1} = \begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix} \). We have

\[
\| A_{12} \|_\infty = 0.11, \quad \| A_{21} \|_\infty = 0.2, \quad \| A_{22}^{-1} \| = \| (A_{22} - A_{11} \cdot I_{\mathbb{R}^2})^{-1} \| = 1.
\]

- First, we use Theorem 5.3 applied to the above block decomposition.
  We calculate \( \| A_{22}^{-1} \|_\infty \leq 4 \| A_{12} \|_\infty \| A_{21} \|_\infty = 1 - 4 \cdot 0.11 \cdot 0.2 = 0.912 > 0 \). Hence
  \[
  | \lambda | < 2 \| A_{12} \|_\infty \cdot \| A_{21} \|_\infty \cdot \| (A_{22} - A_{11} \cdot I_{\mathbb{R}^2})^{-1} \|_\infty = 4.4 \cdot 10^{-2}
  \]
  and we know by Theorem 5.3, that there exists only one eigenvalue of \( A \) which satisfies this inequality. Moreover, we know that \( (1, \delta_2, \delta_3)^T \) is the eigenvector corresponding to \( \lambda \), where \( \delta_2, \delta_3 \in \mathbb{C} \),

  \[
  | \delta_i | \leq 2 \| A_{21} \|_\infty \cdot \| (A_{22} - A_{11} \cdot I_{\mathbb{R}^2})^{-1} \|_\infty = 0.4 \quad \text{for } i = 2, 3.
  \]

- We use the generalized Gerschgorin Theorems 2.2 and 2.5 applied to matrix

\[
\tilde{A} = \begin{bmatrix} A_{11} & 1/4 A_{12} \\ 4 A_{21} & A_{22} \end{bmatrix}.
\]
which is a special case of the scaling matrix $A$ (see Subsection 2.1). Our generalized Gerschgorin disks are

$$G_1(\tilde{A}) = \mathcal{B}(0, \frac{1}{4}\|A_{12}\|_{\infty}) = \mathcal{B}(0, 0.0275),$$

$$G_2(\tilde{A}) = \{m(A_{22} - \lambda) \leq 4\|A_{21}\|_{\infty}\}.$$

We want to show that $G_1(\tilde{A}) \cap G_2(\tilde{A}) = \emptyset$. We can avoid an explicit computation of $G_2(\tilde{A})$ and just show that for any $\lambda \in G_1(\tilde{A})$ holds $\lambda \notin G_2(\tilde{A})$.

Indeed we have

$$(A_{22} - \lambda)^{-1} = \frac{1}{\lambda^2 - 2} \begin{bmatrix} -\lambda & -1 \\ -2 & -\lambda \end{bmatrix}.$$ 

It is easy to see that for $\lambda \in G_1(\tilde{A})$ we have

$$\|(A_{22} - \lambda)^{-1}\|_{\infty} \leq \frac{2 + 0.0275}{2 - 0.0275^2} < 1.015,$$

hence

$$m(A_{22} - \lambda) > 0.98, \quad \lambda \in G_1(\tilde{A}).$$

We see that $G_1(\tilde{A}) \cap G_2(\tilde{A}) = \emptyset$, therefore we obtain the following bounds for the eigenvalue and the eigenvector $(1, \delta_2, \delta_3)$ from Theorems 2.2 and 2.5

$$|\lambda| \leq 0.0275, \quad |\delta_i| \leq 1.$$

As one can see, using the generalized Gerschgorin theorems we obtain better estimation of eigenvalue $\lambda$ close to zero but from Theorem 5.3 we get better estimation of the eigenvector corresponding to $\lambda$. Note that in both cases we did not use the optimal scaling of matrix $A$ (see Remark 5.2 and Subsection 2.1). Even if we take optimal scaling of the matrix $A$ we cannot get better estimation of eigenvector using the Gerschgorin theorem and its generalization (see Theorem 2.5). For more details about optimal scaling see the next section.

- The classical Gerschgorin theorem ignoring the block decompositions given above.

It is easy to see that all Gerschgorin circles have common center and the same happens when we use scalings, therefore we cannot separate the simple eigenvalues close to 0 from the other ones.

## 6 Comparisons in the case of the isolated eigenvalue

The goal of this section is to compare the strength of Theorem 5.3 with the Gerschgorin theorem with rescaling of the basis, when trying to estimate an isolated eigenvalue and corresponding eigenvector. Throughout this section we will use the $\| \cdot \|_{\infty}$ norm.

Let $A \in \mathcal{L}(\mathbb{C} \oplus \mathbb{C}^{n-1})$ be given and for $r \in (0, \infty)$ the scaling matrix $X$ be given as follows (written in blocks corresponding to our decomposition)

$$X = \begin{bmatrix} r & 0 \\ 0 & I \end{bmatrix}.$$ 

We obtain

$$\tilde{A}_r = X^{-1}AX = \begin{bmatrix} A_{11} & \frac{1}{r}A_{12} \\ rA_{21} & A_{22} \end{bmatrix}.$$ 

In the following discussion we restrict ourselves to such rescaling.

In the proof of Theorem 5.3 the method based on cones and dominating maps is used. We apply Theorem 4.2 to the matrix $A - a_{11}$ to estimate the size of the eigenvalue, $\lambda_1$, close to 0. We look for possibly large parameter $r$, such that $A - a_{11}$ is $r$-dominating and then we obtain

$$|a_{11} - \lambda_1| \leq \|A_{12}\| \leq \frac{\|A_{12}\|}{r}.$$
This is exactly \( G_1 \) obtained from the Gershgorin theorem for \( \tilde{A}_r \).

Let \( (1, \delta_2, \ldots, \delta_n) \) be the eigenvector corresponding to \( \lambda_1 \). We obtain from Theorem 4.2 the bound \( 1 \geq r\| (\delta_2, \ldots, \delta_n) \| \), while from Theorem 2.5 applied to \( \tilde{A} \) after returning to the original base we \( |\delta_1| \leq 1/r \). Hence the result is the same both from cones and the Greschgorin Theorem.

The subsequent optimization with respect of \( r \) performed in the proof of the Theorem 5.3 to obtain the formula can be also repeated by suitable rescaling using the original Gerschgorin theorem as long \( G_1(\tilde{A}_r) \) is disjoint from other Gerschgorin disks for \( \tilde{A}_r \). Therefore both approaches differ only with the range of \( r \)'s over which the optimization can be performed. In fact we are only interested in the upper bound for \( r \) in both methods.

The example below demonstrate that it is possible to use the Gerschgorin theorem to isolate and estimate the eigenvector and eigenvalue, while assumptions of Theorem 5.3 are not satisfied.

**Example 6.1.** Let \( A \in \mathcal{L}(\mathbb{C} \times \mathbb{C}^2) \) be given by the formula

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 2 & 0 \\ 50 & 0 & 100 \end{bmatrix}.
\]

The classical Gerschgorin disks are

\[
G_1 = \overline{B}(0,1), \quad G_2 = \overline{B}(2,0.5), \quad G_3 = \overline{B}(100,50).
\]

It is clear that they are mutually disjoint, hence from the Gerschgorin theorem there exists an eigenvalue \( \lambda \), \( |\lambda| < 1 \).

Now, we look at our Theorem 5.3 to estimate the eigenvalue close to 0. We have \( A_{11} = 0 \) and

\[
\|A_{12}\|_\infty = 1, \quad \|A_{21}\|_\infty = 50, \quad \|(A_{22} - A_{11} \cdot I_{\mathbb{C}^2})^{-1}\|_\infty = 0.5, \\
\|(A_{22} - A_{11} \cdot I_{\mathbb{C}^2})^{-1}\|_\infty^2 - 4\|A_{12}\| \cdot \|A_{21}\|_\infty = 4 - 200 < 0.
\]

Therefore assumptions of Theorem 5.3 are not satisfied.

Observe that also assumptions of the generalized Gerschgorin Theorems 2.2 and 2.5 are not satisfied. Our generalized Gerschgorin disks are

\[
G_1(A) = \overline{B}(0,1), \\
G_2(A) = \{m(A_{22} - \lambda) \leq 50\}.
\]

We have

\[
(A_{22} - \lambda)^{-1} = \frac{1}{(100 - \lambda)(2 - \lambda)} \begin{bmatrix} 100 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix},
\]

hence

\[
m(A_{22} - \lambda) = \min\{|2 - \lambda|, |100 - \lambda|\}.
\]

It is easy to see that \( G_1(A) \cap G_2(A) \neq \emptyset \), therefore assumptions of Theorems 2.2 and 2.5 are not satisfied.

**Better analysis.** We should look for the largest \( r \) such that \( \tilde{A}_r \) is 1-dominating and when using the Gerschgorin theorem such that \( G_1(\tilde{A}_r) \) have empty intersection with others Gerschgorin circles for \( \tilde{A}_r \).

For Gerschorin disks we need to have the following inequalities

\[
\frac{1}{r} < 2 - r/2, \quad \frac{1}{r} < 100 - 50r.
\]

We obtain \( r = 1 + \sqrt{\frac{20}{99}} \approx 2 \). Hence we obtain bound \( |\lambda_1| \leq \approx 1/2 \).

For the approach based on cones we need to find largest \( r \), such that \( \tilde{A}_r \) is 1-dominating. Using Theorem 3.9 we obtain the following condition

\[
\frac{1}{r}\|A_{12}\| = \frac{1}{r} < \|A_{22}^{-1}\|^{-1} - r\|A_{21}\| = 2 - 50r.
\]

Easy computations show that no such \( r \) exists in this case. Similar effect we get if we use the generalized Gerschgorin theorem.
In the following two examples due to the complicated mathematical calculations we will not apply the generalized Gerschgorin theorem (in both examples assumptions of Theorems 2.2 and 2.5 are satisfied). Building on Example 2.8 from Section 2.3 we can construct a matrix such that the matrix \( A - a_{11} \) will be 1-dominating, while there will be no isolation of the first Gerschgorin disk.

**Example 6.2.** Let \( A \in \mathcal{L}(\mathbb{C} \times \mathbb{C}^2) \) be given by the formula

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0.75 \\ \epsilon_1 & 1 \\ \epsilon_2 & 0.5 \\ & 100 \end{bmatrix}.
\]

Where \( \epsilon_1, \epsilon_2 \) are sufficiently small. Observe that \( G_1(A) \cap G_2(A) = \overline{B}(0, 0.75) \cap \overline{B}(1, 0.5 + \epsilon_1) \neq \emptyset \), hence the Gerschgorin Theorem 2.3 does not give us that \( \lambda_1 \in G_1(A) \).

It is easy to see that \( A - a_{11} \) will be 1-dominating. Indeed from Theorem 3.9 and Example 2.8, where \( m(A_{22}) \) was estimated, we have

\[
\lambda(A) = \frac{3}{4r} < 0.5 - r/10.
\]

Hence \( A \) is 1-dominating and Theorem 4.3 implies that \( \lambda_1 \in G_1(A) \).

**Better analysis:** We set \( \epsilon_1 = \epsilon_2 = 0.1 \). We optimize by rescaling by \( r \). The Gerschgorin disks approach leads to the following inequalities

\[
\frac{3}{4r} < 0.5 - r/10.
\]

There is no such \( r \) for which this holds.

The approach based on cones requires that

\[
\frac{1}{r} \| A_{12} \| = \frac{3}{4r} < \| A_{22} \|^{-1} - \| A_{21} \| \approx 1 - \frac{r}{10}.
\]

We obtain

\[
\sup \ r \approx 5 + \sqrt{\frac{35}{2}}.
\]

The following example illustrates the case of the matrix \( A \) for which both methods discussed above can be applied.

**Example 6.3.** We put

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0.4 & -0.5 \\ 0.2 & 1.5 & 0.4 \\ -0.1 & 0.3 & 2 \end{bmatrix}.
\]

First, by Theorem 5.3 we estimate the eigenvalue close to 0. We have \( a_{11} = 0 \) and

\[
\| A_{12} \|_\infty = \frac{9}{10}, \quad \| A_{21} \|_\infty = \frac{1}{5}, \quad \| (A_{22} - a_{11} \cdot I_{\mathbb{C}^2})^{-1} \|_\infty = \frac{5}{6},
\]

\[
\| (A_{22} - a_{11} \cdot I_{\mathbb{C}^2})^{-1} \|^{-2} - 4\| A_{12} \| \cdot \| A_{21} \|_\infty = \frac{18}{25} > 0.
\]

Therefore assumptions of Theorem 5.3 are satisfied and we obtain that the eigenvalue \( \lambda \) close to 0 satisfies \( |\lambda| \leq 0.3 \).

Now we use the Gerschgorin theorems to estimate the eigenvalue close to 0. The Gerschgorin disks are

\[
G_1(A) = \overline{B}(0, 0.9), \quad G_2(A) = \overline{B}(1.5, 0.6) \quad \text{and} \quad G_3(A) = \overline{B}(2, 0.4).
\]

It is easy to see that \( G_1(A) \cap G_2(A) \neq \emptyset \), but if we rescale the matrix \( A \) (with \( r = \frac{7}{7} \)), we obtain the matrix

\[
\hat{A}_r = \begin{bmatrix} 0 & \frac{2}{7} & -\frac{5}{7} \\ \frac{7}{25} & \frac{3}{2} & \frac{2}{7} \\ -\frac{7}{25} & \frac{3}{2} & 2 \end{bmatrix}.
\]
and consequently $G_1(\tilde{A}_r) \cap G_2(\tilde{A}_r) = \emptyset$ and $G_1(\tilde{A}_r) \cap G_3(\tilde{A}_r) = \emptyset$. Hence from the Gerschgorin theorem there exists an eigenvalue $\lambda$ such that $|\lambda| \leq \frac{9}{14}$.

**Better analysis:** We look for the largest $r$ for each method, which allows us to obtain the best estimation for the eigenvalue $\lambda$ close to 0.

For Gerschgorin disks we need to solve the following inequalities

\[
\frac{3}{2} > \frac{9}{10r} + \frac{3r}{5}, \quad 2 > \frac{9}{10r} + \frac{3r}{10}
\]

We obtain $\sup r = \frac{3}{2}$. Hence we obtain bound $|\lambda| \leq \frac{3}{2}$.

The cone based approach requires

\[
\frac{1}{r} \|A_{12}\| = \frac{9}{10r} < \|A^{-1}_{22}\| - r \|A_{21}\| = \frac{6}{5} - \frac{r}{5}.
\]

We obtain $\sup r = \frac{3}{2} (2 + \sqrt{2})$. Hence we obtain the bound $|\lambda| \leq \frac{3}{14} (2 - \sqrt{2}) \leq \frac{9}{11}$. By doing the same calculations as above for the transpose of the matrix $A$ we obtain

\[
\sup r = \frac{1}{14} \left(15 + \sqrt{141}\right) \approx 1.9196 \quad \text{and} \quad |\lambda| \leq 0.1563,
\]

from the classical Gerschgorin theorem, and for cone based we get

\[
\sup r = \frac{1}{115} \left(144 + \sqrt{12801}\right) \approx 2.23601, \quad |\lambda| \leq 0.1342.
\]

As one can see the use of cone based approach gives us better estimation of the eigenvalue close to zero than the classical Gerschgorin theorem with rescaling.

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