Geometric optics in the presence of axion-like particles in curved space-time

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We present a concise derivation of geometric optics in the presence of axionic fields in a curved space-time. Whenever light can be described via geometric optics (the eikonal approximation), the only difference to the situation without axionic field is the phenomenon of achromatic birefringence. Consequently, redshift of light and distance estimates based on propagating light rays, as well as shear and magnification due to gravitational lensing are not affected by the interaction of light with an axionic field.

The study of light propagation in the presence of an axion-like field in an arbitrary space-time geometry is of interest for constrains on axionic dark matter and the search for the QCD axion. The latter would solve the strong CP problem, but its coupling to photons, remains a free parameter. Typically, photon frequencies in astronomical observations and optical laboratory experiments are much larger than the typical frequency of axionic oscillations, which is \( \nu_a = 242 \times 10^{-21} \text{eV} \) nHz for ultra-light axion-like particles and \( \nu_a = 242 \times 10^{-22} \text{eV} \) MHz for the QCD axion. Here, \( m_\text{a} \) denotes the mass of the axionic field. If the corresponding photon wavelength being well below all typical length scales of the considered application, in particular the scale of gradients in the axionic field and in the structure of space-time, the propagation of light can be described in the limit of geometric optics (the so-called eikonal approximation, (see e.g., [3]).

Of late, ultra-light axion-like particles are being probed through observations of distant astrophysical sources [e.g., 8, 10–13]. Therefore, photon propagation in non-Euclidean space-time and their interaction with axionic fields needs to be established. In this communication we derive the most general equations of geometric optics in the presence of axionic fields in a curved space-time. Our result generalises [14, 15] to arbitrary space-times and we simplify the derivation performed in [16]. Another generalisation to light coupled to an axionic field and a cold non-magnetized plasma in Minkowski space-time is presented in [17].

The signature of the metric tensor \( g_{\mu\nu} \) is chosen to be \((- , + , + , +)\) and we use the Heaviside-Lorentz system with \( c_0 = \mu_0 = c = \hbar = 1\). The action

\[
S = \int d^4x \sqrt{-g} \mathcal{L}
\]

of the electromagnetic field coupled to an axion-like particle (ALP) in a curved space-time is given by the Lagrangian (our sign convention for the axion-photon coupling follows the one of [14])

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu a \partial_\nu a + \frac{g_{a\gamma}}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} - V,
\]

where \( F_{\mu\nu} \) denotes the electromagnetic field strength tensor and \( F^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} / 2 \) its dual tensor. \( a \) is an axionic field and \( V \equiv V(\alpha) \) its potential. Variation with respect to the gauge field \( A_\alpha \) gives rise to the equation of motion (using \( \nabla_\mu \sqrt{-g} = 0 \), where \( \nabla_\mu \) denotes the covariant derivative),

\[
\nabla_\mu F^{\mu\alpha} - g_{a\gamma} \epsilon^{\mu\nu\rho\alpha} \partial_\mu A_\nu \partial_\rho a = 0.
\]

Making use of \([\nabla_\mu, \nabla_\nu] A^\mu = R^\alpha_{\mu\nu} A^\alpha\), with \( R^\alpha_{\mu\nu\rho} \) denoting the Ricci tensor, and in Lorenz gauge, \( \nabla_\mu A^\mu = 0 \), we obtain

\[
\Box A^\alpha - R^\alpha_{\mu\nu} A^\mu - g_{a\gamma} \epsilon^{\mu\nu\rho\alpha} \partial_\mu A_\nu \partial_\rho a = 0.
\]

Using the rules for exchanging covariant derivatives one can see that this equation and the Lorenz gauge are invariant under the residual gauge freedom \( A^\alpha \rightarrow A^\alpha + \partial^\alpha f \) with \( \Box f = 0 \).

The equations of geometric optics follow from the ansatz \( A^\alpha = \text{Re}[A^\alpha \exp(\psi)] \) and the assumption that large gradients in time and space are described by the real phase \( \psi \), while all other changes are summarised by a complex, slowly evolving amplitude \( A^\alpha \). We define the wave vector

\[
k_\mu \equiv \partial_\mu \psi,
\]

justified by the resulting equations of motion. An observer characterised by the space-time velocity \( u^\mu \) measures the circular frequency \( \omega = -k_\mu u^\mu \). A direct consequence of the definition of the wave vector is

\[
\nabla_\mu k_\nu = \nabla_\nu k_\mu,
\]

which reflects the property that the vector field \( k_\mu \) is normal to the electromagnetic wave front, which defines a hypersurface of space-time.

Inserting the ansatz into the equation of motion in equation (4) and the gauge condition, the terms of order \( k^2 \) and \( k \) become,

\[
k_\mu k_\mu = 0,
\]

\[
2 k_\mu \nabla_\mu A^\alpha + A^\alpha \nabla_\mu k_\mu - g_{a\gamma} \epsilon^{\mu\nu\rho\alpha} k_\mu A_\nu \partial_\rho a = 0,
\]

\[
k_\mu A^\mu = 0,
\]

respectively. The leading term in equation (7) is the null-condition. Acting on it with a covariant derivative and using equation (8), we find the equivalent to the geodesic equation

\[
k_\mu \nabla_\mu k_\alpha = 0.
\]
Thus, in the limit of geometric optics and in the presence of an axionic field, light rays remain to be described by null geodesics in curved space-time.

We further contract equation (5) with $A_{\alpha}$, define $A^2 = A_{\alpha} A^{\alpha}$, and obtain,

$$k^\mu \nabla_\mu A + \frac{1}{2} A \nabla_\mu k^\mu = 0,$$

(11)

which tells us how the light intensity $I = \omega^2 A^2$ (see below) changes along the light path and shows that the presence of an axionic field does not affect the observed intensity of a light source.

Finally, let us introduce the normalised, space-like (complex) polarisation vector $\epsilon^\mu$, i.e. $\epsilon^\mu \epsilon_\mu^* = 1$ and $\epsilon_\mu k^\mu = 0$, and write $A_s^2 = A^2$ With the residual gauge freedom to set $\epsilon_{\mu} \epsilon_{\mu} = 0$, this leads to the phenomenon of birefringence.

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To see that most easily, let us introduce a orthonormal basis of vectors $\{u, e_1, e_2, n\}$, consisting of a time-like vector field $u^\mu$, describing a congruence of observers with $u_\mu u^\mu = -1$, a space-like vector field $n^\mu = (k^\mu - \omega n^\mu)/\omega$, pointing from the source to the family of observers, and $n^\mu n_\mu = 1$, and the space-like linear polarisation basis $e_i^\mu$, $i = 1, 2$, which spans a normal space to $n^\mu$. We can use the residual gauge freedom to set $\epsilon_\mu n^\mu = \epsilon_\mu u^\mu = 0$. Then, $\epsilon_\mu = \epsilon_i e_j^\mu$, with $e_i^\mu e_j^\mu = \delta^\mu_j$ and $|\epsilon_1|^2 + |\epsilon_2|^2 = 1$. With $k^\mu \nabla_\mu e_\alpha = 0$ (we parallel transport the basis vectors), we can now see that

$$\left(\partial_\mu + \partial_n\right) e_i - \frac{1}{2} \varepsilon_{\alpha \gamma} \left[\left(\partial_\mu + \partial_n\right) a\right] e^{\mu j m} e_j = 0.$$  

(13)

This result is obtained from equation (12) by expressing the wave vector in terms of the base vectors and inserting a decomposition of unity in spatial and time-like components $\delta^\beta_\alpha = h^\beta_\alpha - w^\beta u_\beta$, where $h^\beta_\alpha$ denotes the spatial projection operator, i.e. using $\delta^\beta_\alpha \partial_\alpha = h^\beta_\alpha \partial_\alpha = -u_\alpha \partial_\alpha$. The second term in equation (14) generates a rotation of the polarisation components. Introducing the coefficients of circular polarisation $\varepsilon_{L,R} = (\varepsilon_1 \pm i \varepsilon_2)/\sqrt{2}$, respectively, we obtain two ordinary differential equations

$$\varepsilon'_{L,R} = \pm \frac{1}{2} \varepsilon_{\alpha \gamma} a^\gamma \varepsilon_{L,R} = 0,$$

(14)

where the prime denotes a derivative along the line of light propagation. The solution to that equation is

$$\varepsilon_{L,R}(x^\mu) = \exp(\pm \Delta) \varepsilon_{L,R}(x^\mu),$$

(15)

with $\Delta = \varepsilon_{\alpha \gamma} [a(x^\mu) - a' (x^\mu)]/2$ is the rotation angle of the plane of linearly polarized light. Thus, in the limit of geometric optics, 

birefringence due to axionic fields is achromatic, and depends on the coupling $\varepsilon_{\alpha \gamma}$ and the values of the axionic field at emission and observation. Moreover, the curvature of space-time does not affect the

birefringence angle $\Delta$, when expressed with respect to a parallel transported basis of linear polarisation.

Chromatic and curvature effects show up when we go beyond the eikonal approximation, where we would actually return to a wave optics description of the system and no longer talk about light rays. Note that there are no non-linear couplings in this limit. It was shown in [12, 17] that beyond the limit of geometric optics, higher orders in the coupling lead to spectral distortions and refractive phenomena show up.

Finally, we can reformulate our results in the language of Stokes parameters. In the limit of geometric optics and using the same gauge fixing as above, the electric field reads $E_\alpha = F_{\alpha \beta} u^\beta = \Re \left[ \omega A_\alpha \exp(i \psi) \right]$. From that we find the four Stokes parameters as follows:

$$I = \omega^2 A^2,$$

(16)

$$Q = \frac{I}{1 + r^2} \left[ (1 - r^2) \cos(2 \Delta) - 2r \sin(2 \Delta) \right],$$

(17)

$$U = \frac{I}{1 + r^2} \left[ 2r \cos(2 \Delta) + (1 - r^2) \sin(2 \Delta) \right],$$

(18)

$$V = -2 \frac{I}{1 + r^2} \left[ \cos(2 \Delta) + (1 - r^2) \sin(2 \Delta) \right].$$

(19)

This implies that Stokes $I$ remains unchanged when photons propagate in axionic field.

Let us give the example of a monochromatic and linearly polarised source. For such a source, we can write $\epsilon_2(x_e) = r \epsilon_1(x_e)$, with $r$ real, which gives

$$Q = \frac{I}{1 + r^2} \left[ (1 - r^2) \cos(2 \Delta) - 2r \sin(2 \Delta) \right],$$

(20)

$$U = \frac{I}{1 + r^2} \left[ 2r \cos(2 \Delta) + (1 - r^2) \sin(2 \Delta) \right].$$

(21)

Here, $Q^2 + U^2 = I^2$ and $V = 0$.

One could imagine to set up a table top experiment based on finding birefringence of a polarised laser beam in vacuum, which could probe the mass range of the QCD axion. In astrophysics the situation of the simple example is found in the case of the face-on observation of proto-planetary discs [10]. Due to the galactic distance scale, ultra-light axions can be probed in that case. For other sources with continuous emission, the relations for the Stokes parameters may be more complicated, but the underlying principle of birefringence remains the same.

We have shown that axionic dark matter does neither affect the geodesics of light, nor its intensity and therefore the Etherington theorem [18], the relation between angular and luminosity distance, and the Sachs equations [14], which describe the propagation of light bundles in curved space-times, also hold true in the presence of axion-photon interaction and in the limit of geometric optics. Thus, cosmic distance estimation, based on parallaxes, standard candles or standard rulers are not affected. Also the photon-axion interaction cannot change the redshift of a source. Our result further implies, that the presence of axionic dark matter does not modify the conclusions on gravitational lensing as long as shape and magnification are concerned.

On the other hand, axionic dark matter gives rise to an additional achromatic birefringence that adds to
the chromatic birefringence introduced by the cosmic magneto-ionic medium [20, 21]. The investigation of the interplay of a magnetized plasma with light propagation in an axionic field in flat space-time [17] shows that we should also expect additional effects of the order of the plasma frequency squared times the axion-photon coupling, which are subdominant. Thus, the achromatic nature of axionic birefringence might provide a smoking gun in the search for ultra-light axionic dark matter on cosmological and astrophysical length and time scales.

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