On the Number of Heterochromatic Trees in Nice and Beautiful Colorings of Complete Graphs

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Abstract
We introduce classes of edge-colourings of the complete graph—that we call nice and beautiful—and study how many heterochromatic spanning trees appear under such colourings. We prove that if the colouring is nice, there is at least a quadratic number of different heterochromatic trees; and if the colouring is beautiful there is an exponential number of different such trees.

Keywords Heterochromatic spanning trees · Graceful colourings · Nice colourings

1 Introduction
Let $G = (V, E)$ be a simple graph of order $n+1$ and size $m \geq n$. Consider an injective labeling $\gamma : V \rightarrow \{0, \ldots, m\}$ of its vertices and assign to each edge $e = uv$ the number $\gamma'(e) = |\gamma(u) - \gamma(v)|$. If every edge is assigned to a different number, we say that $\gamma$ is a graceful labeling, and if such a labeling exists we call $G$ a graceful graph. The long-standing Graceful Tree Conjecture—also known as Ringel–Kotzig–Rosa conjecture [6, 8, 9], or RKR-conjecture for short—says that all trees are graceful graphs.

An $m$-edge-colouring of a graph $G$ is an assignment of colours to the edges of $G$ that uses $m$ colours. Equivalently, an $m$-edge-colouring of a graph $G$ is a partition $E = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m$ of the edge set of $G$; the sets $C_1, C_2, \ldots, C_m$ are called the colour classes. A subgraph $H$ of an edge-coloured graph $G$ is heterochromatic if all edges in $H$ have different colours.

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If we label the vertices of the complete graph $K_{n+1}$ with the numbers $\{0, 1, \ldots, n\}$ and assign to each edge the absolute value of the difference between the labels of its vertices, we get a colouring of the edges with $n$ colours which we call the graceful colouring of the complete graph—we refer to the numbers in $[n] = \{1, 2, \ldots, n\}$ as colours when assigned to the edges of the complete graph. In terms of this colouring, the RKR-conjecture says that there is an heterochromatic copy of each tree in the graceful colouring of the complete graph; that is, for each tree of order $n + 1$, there is an isomorphic copy of it, inside the gracefully coloured $K_{n+1}$, which uses $n$ different colours in its edges. This “beautiful” colouring has several properties which we use along the paper to prove the existence of “many” heterochromatic trees.

To begin with, observe that in the graceful colouring the colour $c \in [n]$ is used exactly $n - c + 1$ times; so the $n$ colour classes have sizes $n, n - 1, \ldots, 2, 1$, respectively. An $n$-colouring of the edges of the complete graph $K_{n+1}$ with such a property is called a nice colouring. For example, consider the following recursive “Stellar” colouring of a complete graph: start with a vertex; then, at each step, add a new vertex and colour all edges joining it to the previously added vertices with the same and new colour—that is, at each step, add a monochromatic star, see Fig. 1. Clearly this is a nice colouring; furthermore, it is easy to see that this nice colouring has a heterochromatic copy of each tree of size $n$.

The graceful and stellar colourings of the complete graph $K_{n+1}$ share many common properties: in both cases all monochromatic subgraphs are acyclic and $K_{n+1}$ can be decomposed into heterochromatic stars to mention just a couple. In Sect. 3 we show for every nice edge colouring of the complete graph $K_{n+1}$, there are $\Omega(n^2)$ different heterochromatic spanning trees.

Later, in Sect. 4 we define the class of beautiful edge-colourings of $K_{n+1}$, which includes the class of nice colourings, and prove that for any such a colouring, the graph $K_{n+1}$ has $\Omega(2^n)$ different heterochromatic spanning trees.

Fig. 1 Stellar colouring of $K_6$
The existence of heterochromatic trees in edge colourings has been studied by various authors. In particular Suzuki [10] and Akbari and Alipour [1] characterise independently those edge colourings that contain heterochromatic trees; viz.

**Theorem A** (Suzuki) Let $C$ be an edge colouring of a graph $G$ with $n$ vertices. There is a heterochromatic spanning tree of $G$ if and only if for any set of $r$ colours of $C$ with $1 \leq r \leq n - 2$, the graph obtained from $G$ by removing all edges of $G$ coloured with any of these $r$ colours has at most $r + 1$ connected components.

**Theorem B** (Akbari, Alipour) Let $G$ be an edge-coloured graph. Then $G$ has a heterochromatic spanning tree if and only if for every partition of $V(G)$ into $t$ parts, with $1 \leq t \leq |V(G)|$, there are at least $t - 1$ edges with distinct colours whose ends lie in different parts.

Other results concerning the complete graph have been given too. For example, colourings given by perfect matchings of the complete graph $K_{2n}$ were studied by Brualdi and Hollingsworth [3].

- If $C$ is an edge colouring of the complete graph $K_{2n}$ given by $2(n - 1)$ disjoint perfect matchings, then $K_{2n}$ contains two edge-disjoint heterochromatic trees.

An upper bound for the minimum integer $k$ such that for every edge $k$-colouring of the complete graph $K_n$ there exists a heterochromatic spanning tree, was found by Bialostocki and Voxman [4].

- If $C$ is an edge colouring of the complete graph $K_n$ with at least $\left(\frac{n-2}{2}\right) + 2$ colours, then $K_n$ has a heterochromatic spanning tree.

This last proposition can be generalised in various directions. For example, Arocha and Neumann [2] generalised Bialostocki’s result for arbitrary graphs and Montellano-Ballesteros and Rivera-Campo [7] gave the corresponding result for matroids.

- Let $G$ be a simple connected graph with at least $m \geq 2$ edges. If $C$ is an edge colouring of $G$ with exactly $m - \tau(G) + 2$ colours, then $G$ has a heterochromatic spanning tree.

- Let $M$ be a matroid with $m$ elements and rank at least 2. If $C$ is a colouring of the elements of $M$ with at least $m - \tau(M) + 2$ colours, then $G$ has a heterochromatic basis.

Were, $\tau(G)$ (respectively $\tau(M)$) denote the size of the smallest set of edges of $G$ (elements of $M$) which contains at least two edges (elements) of each spanning tree of $G$ (each basis of $M$).

### 2 Preliminary Results

Let $n \geq 0$ be an integer and $G$ be a graph with $n + 1$ vertices. For an $n$-edge-colouring of $G$ let $M_1 = (E(G), \mathcal{I}_1)$ and $M_2 = (E(G), \mathcal{I}_2)$ be matroids with ground set $E(G)$ and independence sets $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, where $X \in \mathcal{I}_1$ if the subgraph $G[X]$ of $G$...
induced by $X$ is acyclic and $X \in I_2$ if $G[X]$ is heterochromatic. A common
independent set in $M_1$ and $M_2$ is the edge set of a heterochromatic forest of $G$.

For $X \subset E(G)$ we denote by $w(X)$ and $c(X)$, respectively, the number of
connected components of the spanning subgraph of $G$ with edge set $X$ and the
number of colour classes contained in $X$.

The following lemma will be used in the proofs of our main results. A proof of
the lemma can be obtained using Suzuki’s Theorem but, for the sake of
completeness, we present an alternative proof that uses Edmond’s Matroid
Intersection Theorem [5].

**Lemma 1** Let $C$ be an $n$-edge-colouring of a graph $G$ with $n+1$ vertices. If
$w(X) + c(X) \leq n + 1$ for all $X \subset E(G)$, then $G$ has a heterochromatic spanning
tree.

**Proof** Let $X \subset E(G)$. Then $r_{M_1}(X) = n + 1 - w(X)$ and $r_{M_2}(E(G) \setminus X) = n - c(X)$. Therefore

$$r_{M_1}(X) + r_{M_2}(E(G) \setminus X) = (n + 1 - w(X)) + (n - c(X))$$

$$= (2n + 1) - (w(X) + c(X))$$

$$\geq (2n + 1) - (n + 1)$$

$$= n.$$

By Edmonds’ Matroid Intersection Theorem, $E(G)$ contains a set $X'$ with size $n$
which is independent in both matroids $M_1$ and $M_2$. This implies that the subgraph of
$G$ induced by $X'$ is a heterochromatic spanning tree of $G$. \qed

3 Heterochromatic Spanning Trees in Cute and Nice Colourings
of Graphs

An $n$-edge-colouring of the complete graph $K_{n+1}$ is a **nice colouring** if the $n$ colour
classes have sizes $1, 2, \ldots, n$. Let $G$ be a graph with $n + 1$ vertices and $1 + \binom{n}{2}$
edges. An $n$-edge-colouring of $G$ is a **cute** edge-colouring of $G$ if the sizes of the $n$
colour classes are $1, 1, 2, \ldots, n-1$.

With Lemma 1 in hand, we prove the following theorems concerning cute and
nice colourings.

**Theorem 1** Let $G$ be a graph with $n + 1$ vertices and $1 + \binom{n}{2}$ edges. If $C$ is a cute
$n$-edge-colouring of $G$, then $G$ has a heterochromatic spanning tree.

**Proof** Let $C_1, C_2, \ldots, C_n$ be the colour classes of $C$. Without loss of generality we
assume $|C_1| = 1$ and $|C_i| = i - 1$ for $i = 2, 3, \ldots, n$. Let $X \subset E(G)$ and denote by
$E_1, E_2, \ldots, E_{w(X)}$ the sets of edges of the connected components of the spanning
subgraph of $G$ with edge set $X$. Then
Since the number of edges of a graph is maximum when all edges lie in one connected component.

On the other hand if $C_{i_1}, C_{i_2}, \ldots, C_{i_{\tau(X)}}$ are the colour classes contained in $X$, then

$$|X| = \sum_{i=1}^{n} |X \cap C_i|$$

$$\geq \sum_{j=1}^{c(X)} |C_i|$$

$$\geq 1 + 1 + 2 + \cdots + (c(X) - 1)$$

$$= 1 + \binom{c(X)}{2}.$$

Therefore

$$\left(\frac{n - 2 - w(X)}{2}\right) \geq 1 + \binom{c(X)}{2}.$$

This implies

$$\left(\frac{n + 2 - w(X)}{2}\right) > \binom{c(X)}{2}$$

which in turn gives $n + 2 - w(X) > c(X)$ and therefore $w(X) + c(X) \leq n + 1$. By Lemma 1, $G$ has a heterochromatic spanning tree.

The following remark shows that the condition in Theorem 1 can only guarantee the existence of one heterochromatic spanning tree, see Fig. 2.

Fig. 2 Cute colouring of a graph $G$ with exactly one heterochromatic spanning tree $T$
**Remark 1** For each tree $T$ with $n + 1$ vertices, there is a spanning supergraph $G(T)$ of $T$ with $1 + \binom{n}{2}$ edges and a cute edge-coloring of $G(T)$ for which $T$ is the unique heterochromatic spanning tree.

**Proof** Let $T$ be a tree with vertices $v_1, v_2, \ldots, v_{n+1}$. Without loss of generality assume that for $i = 1, 2, \ldots, n+1$, the subgraph $T_i$ of $T$ induced by $v_1, v_2, \ldots, v_i$ is a tree. For $i = 2, 3, \ldots, n+1$ let $v_i^-$ be the unique vertex of $T_{i-1}$ adjacent to $v_i$ in $T_i$. Then $E(T) = \{v_i^- v_i : i = 2, 3, \ldots, n+1\}$.

Let $G(T)$ be the supergraph of $T$ with edge set \[ E(G(T)) = \{v_i v_j : 1 \leq i \leq j \leq n\} \cup \{v_{n+1}^- v_{n+1}\}; \]

clearly $G(T)$ is a spanning supergraph of $T$ with $1 + \binom{n}{2}$ edges.

Let $C$ be the edge-colouring of $G(T)$ with colour classes $C_1, C_2, \ldots, C_n$ given by: $C_1 = \{v_1 v_2\} = \{v_2^- v_2\}$ and for $k = 2, 3, \ldots, n$, $C_k = \{v_{k+1}^- v_{k+1}\} \cup \{v_i v_k : i = 1, 2, \ldots, k-1, v_i \neq v_k^-\}$. Notice that $C$ is a cute edge-colouring of $G(T)$ since $|C_1| = 1$ and $|C_k| = 1 + (k - 2) = k - 1$ for $k = 2, 3, \ldots, n$. We claim that $T$ is the only spanning tree of $G(T)$ which is heterochromatic.

Let $H$ be a heterochromatic spanning tree of $G(T)$. Edges $v_2^- v_2$ and $v_3^- v_3$ are the unique edges in $C_1$ and $C_2$, respectively, therefore they must be edges of $H$. This implies that $T_1$ and $T_2$ are subtrees of $H$. Assume $T_k$ is a subtree of $H$; since $C_k = \{v_{k+1}^- v_{k+1}\} \cup \{v_i v_k : i = 1, 2, \ldots, k-1, v_i \neq v_k^-\}$ and all edges $v_i v_k : i = 1, 2, \ldots, k-1$ have both ends in $T_k$, the only possible edge in $C_k$ that lies in $H$ is edge $v_{k+1}^- v_{k+1}$. Therefore $T_{k+1}$ is a subtree of $H$. This inductive argument shows that $T = T_{n+1}$ is a subtree of $H$ which implies $H = T$. \[\square\]

Unlike the stellar colouring, not every nice colouring of $K_n$ contains a heterochromatic copy of every tree with $n$ vertices, see Fig. 3. Nevertheless every nice colouring of $K_n$ produces many different heterochromatic spanning trees.

![Fig. 3](image-url) A nice colouring of $K_6$ with no heterochromatic spanning tree $T$ with $\Delta(T) \geq 4$
Theorem 2  Let G be a complete graph with n + 1 vertices. If C is a nice n-edge-colouring of G, then G has at least \( \left\lfloor \frac{n+1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil \) different heterochromatic spanning trees.

Proof  Let \( C_1, C_2, \ldots, C_n \) be the colour classes of C. Without loss of generality assume \( |C_i| = i \) for \( i = 1, 2, \ldots, n \).

Let \( e_1 \) be an edge in \( C_{\left\lfloor \frac{n+1}{2} \right\rfloor} \). We claim G has at least \( \left\lfloor \frac{n+1}{2} \right\rfloor \) different heterochromatic spanning trees containing the edge \( e_1 \).

First choose an arbitrary subset \( W_1 \) of \( C_n \) with size \( \left\lfloor \frac{n+1}{2} \right\rfloor \) and let \( G_{1,1} \) be the subgraph of G with edge set

\[
E(G_{1,1}) = \left( E(G) \setminus (C_{\left\lfloor \frac{n+1}{2} \right\rfloor} \cup C_i) \right) \cup \{e_1\} \cup W_1
\]

When restricted to the graph \( G_{1,1} \) colouring C is a cute edge-colouring since \( |C_1| = |\{e_1\}| = 1, \) \( |C_i| = i \) for \( i \neq C_{\left\lfloor \frac{n+1}{2} \right\rfloor} \) and \( |W_1| = \left\lfloor \frac{n+1}{2} \right\rfloor \). By Theorem 1, \( G_{1,1} \) has a heterochromatic spanning tree \( T_{1,1} \).

Assume \( T_{1,1}, T_{1,2}, \ldots, T_{1,t} \) are different heterochromatic spanning trees of G containing edge \( e_1 \). For \( i = 1, 2, \ldots, t \) let \( f_i \) be the edge in \( T_i \) having colour \( n \). If \( t < \left\lfloor \frac{n+1}{2} \right\rfloor \), then we can choose \( W_{t+1} \subset C_n \) with size \( \left\lfloor \frac{n+1}{2} \right\rfloor \) containing none of the edges \( f_1, f_2, \ldots, f_t \). Let \( G_{1,t+1} \) be the subgraph of G with edge set

\[
E(G_{1,t+1}) = \left( E(G) \setminus (C_{\left\lfloor \frac{n+1}{2} \right\rfloor} \cup C_i) \right) \cup \{e_1\} \cup W_{t+1}
\]

As with the case of \( G_{1,1} \), when restricted to \( G_{1,t+1} \), colouring C is a cute edge-colouring. By Theorem 1, \( G_{1,t+1} \) has a heterochromatic tree \( T_{1,t+1} \). Notice that \( T_{1,t+1} \neq T_{1,j} \) for \( i = 1, 2, \ldots, t \) since \( f_i \) is an edge of \( T_{1,i} \) and not an edge of \( T_{1,t+1} \).

To end the proof, repeat the previous argument for each edge \( e_2, e_3, \ldots, e_{\left\lfloor \frac{n+1}{2} \right\rfloor} \notin C_{\left\lfloor \frac{n+1}{2} \right\rfloor} \) obtaining different heterochromatic trees \( T_{i,j} \) of G with \( i = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \) and \( j = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \).

4 Heterochromatic Spanning Trees in Beautiful Colourings of Complete Graphs

Let \( C \) be a nice \( n \)-edge-colouring of \( K_{n+1} \), \( \{C_1, C_2, \ldots, C_n\} \) be its colour classes, where for each \( i = 1, \ldots, n \), \( |C_i| = i \). We denote by \( G_i \) the subgraph of \( K_{n+1} \) induced by \( C_i \).

The edge-coloring \( C \) will be called beautiful if for every colour class \( C_i \) we have that \( G_i \) is acyclic and there is a partition \( \{V_1, V_2\} \) of \( V(K_{n+1}) \), with \( |V_2| = \left\lfloor \frac{n+1}{2} \right\rfloor \geq \left\lceil \frac{n+1}{2} \right\rceil = |V_1| \), such that:

(i)  The subgraph of \( K_{n+1} \) induced by all the colour classes \( C_i \), with \( i \equiv n \) (mod 2), is isomorphic to the complete bipartite graph with parts \( \{V_1, V_2\} \).

(ii)  For every colour class \( C_i \), with \( i \equiv n \) (mod 2), we have that \( |V(G_i) \cap V_1| = \left\lfloor \frac{|V(G_i)|}{2} \right\rfloor \) and \( |V(G_i) \cap V_2| = \left\lceil \frac{|V(G_i)|}{2} \right\rceil \).  

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(iii) For every colour class $C_i$, with $i \neq n \pmod{2}$, we have that $|C_i \cap E(K_{n+1}[V_1])| = \lfloor \frac{j}{2} \rfloor$ and $|C_i \cap E(K_{n+1}[V_2])| = \lceil \frac{j}{2} \rceil$.

Theorem 3 Let $n \geq 2$ be an integer and $C$ be a beautiful $n$-edge coloring of $K_{n+1}$. Then in $K_{n+1}$ there are $2^{\lfloor \frac{n-1}{2} \rfloor}$ heterochromatic spanning trees.

Proof The proof will be done as follows. Consider a spanning subgraph $G$ of $K_{n+1}$ obtained in the following way:

1. For each colour class $C_i$, with $i \geq 2$ and $i \neq n \pmod{2}$, choose one part $V_j$ of the partition $\{V_1, V_2\}$ of $V(K_{n+1})$, and let $Y_i = C_i \cap E(K_{n+1}[V_j])$.
2. For $i = 1$ and for $i \geq 2$, with $i \equiv n \pmod{2}$, let $Y_i = C_i$.
3. Let $E(G) = \bigcup_{i=1}^n Y_i$.

Observe that $C$ induces an $n$-edge coloring of $G$. We claim that $G$ contains an heterochromatic spanning tree.

Let $X$ be a set of edges of $G$ and $A$ be the set of colour classes $Y_i$, with $i \neq n \pmod{2}$, such that $Y_i \subseteq X$; let $B$ be the set of colour classes $Y_i$, with $i \equiv n \pmod{2}$ such that $Y_i \subseteq X$, and suppose that $|A| = r$ and $|B| = s$. Observe that $c(X) = r + s$.

Let $Y_i \in A$ and $Y_j \in B$ be of maximal size, respectively. Assume first $n$ is even. Then $i_0$ is odd, and since $|A| = r$ and $Y_i$ has maximal size, we see that $i_0 \geq 2r - 1$ and the size of $Y_i$ is at least $\lfloor \frac{2r-1}{2} \rfloor$ if $Y_i$ is contained in $V_1$, and size at least $\lceil \frac{2r-1}{2} \rceil$ if $Y_i$ is contained in $V_2$.

Similarly, we see that $i_1$ is even, and since $|B| = s$ and $Y_i$ has maximal size, we see that $i_1 \geq 2s$ and therefore the size of $Y_i$ is at least $2s$. Moreover, since $G_{i_0}$ is acyclic, $|V(G_{i_0})| \geq 2s + \omega_1$, where $\omega_1$ is the number of connected components of $G_{i_0}$, and therefore, by iii), $|V_1 \cap V(G_{i_0})| = \lfloor \frac{2s+\omega_1}{2} \rfloor$ and $|V_2 \cap V(G_{i_0})| = \lceil \frac{2s+\omega_1}{2} \rceil$.

If $Y_i$ is contained in $V_1$, let \( \{x_1, \ldots, x_{\lfloor \frac{2s+\omega_1}{2} \rfloor} \} = V_2 \cap V(G_{i_0}) \) and for each $x_i$, choose an edge $e_i \in Y_i$ such that $x_i$ is incident to $e_i$. Let $H$ be the subgraph of $G$ induced by $Y_i$ and $\{e_1, \ldots, e_{\lfloor \frac{2s+\omega_1}{2} \rfloor} \}$. Graph $H$ has size at least $\lfloor \frac{2s-1}{2} \rfloor + \lfloor \frac{2s+\omega_1}{2} \rfloor \geq r + s$, and since $G_{i_0}$ is acyclic and each of the vertices $\{x_1, \ldots, x_{\lfloor \frac{2s+\omega_1}{2} \rfloor} \}$ has degree 1 in $H$, it follows that $H$ is acyclic. Thus, $|V(H)| \geq r + s + \omega_1$, where $\omega_1$ is the number of connected components of $H$, and therefore $\omega(X) \leq n + 1 - (r + s + \omega_1) + \omega_1 = n + 1 - (r + s)$. Since $c(X) = r + s$, by Lemma 1 we see that $G$ contains an heterochromatic spanning tree.

If $Y_i$ is contained in $V_2$, let $\{y_1, \ldots, y_{\lfloor \frac{2s+\omega_1}{2} \rfloor} \} = V_1 \cap V(G_{i_0})$ and for each $y_i$, choose an edge $e_i \in Y_i$ such that $y_i$ is incident to $e_i$. Let $H$ be the subgraph of $G$ induced by $Y_i$ and $\{e_1, \ldots, e_{\lfloor \frac{2s+\omega_1}{2} \rfloor} \}$. Observe that since $Y_i$ is contained in $V_2$, $Y_i$ has size at least $\lfloor \frac{2r-1}{2} \rfloor$, thus $H$ has size at least $\lfloor \frac{2r-1}{2} \rfloor + \lceil \frac{2s+\omega_1}{2} \rceil \geq r + s$. From here, as in the previous case, we see that $G$ contains an heterochromatic spanning tree.

For the case when $n$ is odd, it follows that $i_0$ is even, and since $|A| = r$ and $Y_i$ has maximal size, we see that $i_0 \geq 2r$ and the size of $Y_i$ is at least $r$ (either if it is
contained in \(V_1\) or \(V_2\). Similarly, \(i_1\) is odd, and since \(|B| = s\) and \(Y_{i_1}\) has maximal size, we see that \(i_1 \geq 2s - 1\) and therefore the size of \(Y_{i_1}\) is at least \(2s - 1\). Moreover, since \(G_{i_1}\) is acyclic, \(|V(G_{i_1})| \geq 2s - 1 + \omega_1\), where \(\omega_1\) is the number of connected components of \(G_{i_1}\), and therefore, by \(ii\), \(|V_1 \cap V(G_{i_1})| = \frac{2s-1+\omega_1}{2}\) and \(|V_2 \cap V(G_{i_1})| = \frac{2s-1+\omega_1}{2}\). From here, in an analogous way as in the case when \(n\) is even, we see that \(G\) contains an heterochromatic spanning tree and the claim follows.

Since there are \(\frac{n-1}{2}\) colour classes \(C_i\), with \(i \geq 2\) and \(i \equiv n \mod 2\), it follows there are \(2^{\frac{n+1}{2}}\) different ways to obtain a subgraph \(G\) which, by our claim, contains an heterochromatic spanning tree. Moreover, given any pair \(G_1, G_2\) of these type of subgraphs, by construction, there is at least one colour class \(C_i\), with \(i \geq 2\) and \(i \equiv n \mod 2\), such that \((E(G_1) \cap C_i) \cap (E(G_2) \cap C_i) = \emptyset\). Thus, the heterochromatic spanning trees in \(G_1\) and in \(G_2\) are different, and from here, the result follows.

**Corollary 1** Let \(n \geq 2\) be an integer and \(C\) be the graceful colouring of \(K_{n+1}\). Then in \(K_{n+1}\) there are \(2^{\frac{n+1}{2}}\) heterochromatic spanning trees.

**Proof** By Theorem 3 we only need to show that \(C\) is beautiful. Let \(V(K_{n+1}) = \{v_0, v_1, \ldots, v_n\}\), and, for each \(i \in \{1, 2, \ldots, n\}\), let \(D_i = \{v_t, v_s : |t-s| = i\}\), that is, \(D_i\) denotes the set of edges coloured with \(i \in \{1, 2, \ldots, n\}\). Since \(D_i\) contains \(n+1-i\) edges, we see that \(D_i = C_{n+1-i}\) and, it is not difficult to see that, \(G_{n+1-i}\) is acyclic.

Assume first \(n\) is even. Let \(V_2 = \{v_0, v_2, \ldots, v_n\}\) and \(V_1 = \{v_1, v_3, \ldots, v_{n-1}\}\). It is easy to see that the subgraph induced by \(D_i \cup D_{i+1} \cup \ldots \cup D_{n-1}\) (that is, the subgraph induced by the union of the colour classes \(C_{n+1-i}\), with \((n+1-i) \equiv n \mod 2\)) is the complete bipartite graph with partite sets \(V_1\) and \(V_2\). Hence (i) holds. Moreover, given \(D_j\), with \(j\) odd, \(V_2 \cap V(G_{n+1-j}) = \{v_0, \ldots, v_{n-1}\} \cup \{v_{1+j}, \ldots, v_n\}\) and \(V_1 \cap V(G_{n+1-j}) = \{v_1, \ldots, v_{n-j}\} \cup \{v_{j}, \ldots, v_{n-1}\}\). Considering the cases whenever \(j \leq n-j\) or not, it is not hard to see that (ii) holds.

Finally, given an even integer \(2 \leq j \leq n\), it is not hard to see that \(D_j\) satisfies \(|D_j \cap E(K_{n+1}[V_2])| = \frac{n+2-j}{2}\) and \(|D_j \cap E(K_{n+1}[V_1])| = \frac{n-j}{2}\). Thus, for each colour class \(C_{n+1-j}\), with \((n+1-j) \not\equiv n \mod 2\), we have that \(|C_{n+1-j} \cap E(K_{n+1}[V_2])| = \frac{n+2-j}{2}\) and \(|C_{n+1-j} \cap E(K_{n+1}[V_1])| = \frac{n-j}{2}\), and (iii) holds.

For the case where \(n\) is odd, let \(V_2 = \{v_0, v_2, \ldots, v_{n-1}\}\) and \(V_1 = \{v_1, v_3, \ldots, v_{n}\}\). As in the case where \(n\) is even, we can show that \(D_i\) (and so \(C_{n+1-i}\), with \(1 \leq i \leq n\), satisfy the statements (i), (ii), and (iii)). Therefore, \(C\) is beautiful and the corollary follows.

**5 Further Research; Heterochromatic Trees in Edge-Colourings of Bipartite Graphs**

Analogous results can be found in other classes of graphs beside the complete graph using exactly the same techniques. For example, we can proceed we bipartite graphs as follows. A \((2m-1)\)-edge-colouring of the complete bipartite graph \(K_{m,n}\) is a nice edge-colouring if the colour classes have sizes \(1, 1, 2, 2, \ldots, m-1, m-1, m\). Let
$G_{m,m}$ be a spanning subgraph of $K_{m,m}$ with $1 + 2\binom{m}{2}$ edges. A $(2m - 1)$-edge colouring of $G_{m,m}$ is a *cute* edge-colouring if the chromatic classes have sizes $1, 1, 1, 2, 2, \ldots, m - 1, m - 1$.

With the same technics as in the previous section we can prove the following results:

**Theorem 4** If $C$ is a cute edge-colouring of a subgraph $G_{m,m}$ of $K_{m,m}$ with $1 + 2\binom{m}{2}$ edges, then $G$ has a heterochromatic spanning tree.

**Remark 2** For each tree $T$ with $2m$ vertices there is a spanning bipartite supergraph $F(T)$ with $1 + 2\binom{m}{2}$ edges and a cute edge-colouring of $F(T)$ for which $T$ is the unique heterochromatic spanning tree.

**Theorem 5** If $C$ is a nice edge-colouring of a complete bipartite graph $K_{m,m}$, then $K_{m,m}$ contains at least $\binom{m}{2}\binom{m+2}{2}$ heterochromatic spanning trees if $m$ is even and at least $(\frac{m+1}{2})^2$ heterochromatic spanning trees if $m$ is odd.

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**References**

1. Akbari, S., Alipour, A.: Multicolored trees in complete graphs. J. Graph Theory 54, 221–232 (2006)
2. Arocha, J., Neumann-Lara, V.: Personal communication
3. Brualdi, R.A., Hollingsworth, S.: Multicolored trees in complete graphs. J. Comb. Theory Ser. B 68, 310–313 (1996)
4. Bialostocki, A., Voxman, W.: On the anti-Ramsey numbers for spanning trees. Bull. Inst. Comb. Appl. 32, 23–26 (2001)
5. Edmonds, J.: Submodular functions, matroids, and certain polyhedra. 1970 Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta) pp. 69–87. Gordon and Breach, New York (1969)
6. Kotzig, A.: On certain vertex valuations of finite graphs. Util. Math. 4, 261–290 (1973)
7. Montellano-Ballesteros, J.J., Rivera-Campo, E.: On the heterochromatic number of hypergraphs associated to geometric graphs an to matroids. Graphs Comb. 29, 1517–1522 (2013)
8. Ringel, G.: Problem 25, Theory of Graphs and its Applications (Proc. Sympos. Smolenice 1963, Nakl. CSAV, Praha, 1964), 162
9. Rosa, A.: On certain valuations of the vertices of a graph theory of graphs (Proc. Internat. Symposium, Rome), vol. 1967, pp. 349–355. Gordon and Breach, N. Y. and Dunod Paris (1966)
10. Suzuki, K.: A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph. Graphs Comb. 22, 261–269 (2006)

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