Superintegrability with third-order integrals in quantum and classical mechanics

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Abstract

We consider here the coexistence of first- and third-order integrals of motion in two dimensional classical and quantum mechanics. We find explicitly all potentials that admit such integrals, and all their integrals. Quantum superintegrable systems are found that have no classical analog, i.e. the potentials are proportional to $\hbar^2$, so their classical limit is free motion.
1 Introduction

In classical mechanics, an n-dimensional Hamiltonian system is called Liouville integrable if it allows $n$ functionally independent integrals of motion in involution (including the Hamiltonian), that is

$$\{H, X_i\} = 0, \quad \{X_i, X_j\} = 0, \forall i, j. \quad (1.1)$$

The Hamiltonian $H = H(x_1, \ldots, x_n, p_1, \ldots, p_n)$ and the integrals of motion $X_i = X_i(x_1, \ldots, x_n, p_1, \ldots, p_n)$ must be well defined functions on phase space $\mathbb{R}^{2n}$. The system is superintegrable if it allows more than $n$ functionally independent integrals, $n$ of them in involution. The best known superintegrable systems in $n$ dimensions are the harmonic oscillator $V = \omega r^2$ and the Coulomb potential $V = \frac{\alpha}{r}$, both of them allowing $2n - 1$ independent integrals of motion, the maximal number possible for an interacting system. Bertrand’s theorem ([2, 3]) tells us that these are the only rotationally invariant systems for which all finite trajectories are closed, a fact intimately related to their maximal superintegrability.

In quantum mechanics, a Hamiltonian system is said to be integrable if there exists a set $\{X_i\}$ of $n$ well defined, algebraically independent operators (including the Hamiltonian) that commute pairwise. It is superintegrable if it possesses further independent operators, $\{Y_j\}$ that commute with the Hamiltonian. The $Y_j$ do not necessarily commute with each other, nor with the $X_i$.

The definition of the independence of quantum operators is not unique, and this may give rise to different types of quantum superintegrability. A good working definition, which may be appropriate for applications in quantum mechanics, soliton theory and for instance in the study of the Huygens principle, is that operators are considered independent unless one of them can be expressed as a polynomial in the others ([4, 5, 16, 19]). The fact that commuting operators can be useful even if they are functionally dependent in the classical limit was clearly demonstrated by Hietarinta ([16, 17, 18, 19]). This definition is in itself not quite satisfactory since it ignores more general polynomial or functional relations between integrals. This may lead to important differences between classical and quantum integrability. Moreover, it is not appropriate for nonpolynomial integrals. Finding an appropriate and rigorous definition of the independence of quantum operators is not an easy problem, but it is worth investigating as wrong or ambiguous definitions may give rise to incorrect results. For a discussion of related problems, see e.g. [18] and [30].

Previous systematic searches for superintegrable systems concentrated on integrals of motion of at most second order in momenta ([3, 4, 13, 23, 31]). This “quadratic superintegrability” has been shown to be related to multiseparability of the Schroedinger or Hamilton-Jacobi equations. More recently, it was related to generalized symmetries ([28]) and exact solvability ([29]).

Quadratic superintegrability has been considered in spaces of nonzero con-
stant curvature ([23, 20]) and of nonconstant curvature ([21]). For superintegrable systems in $n$ dimensions see ([23]).

The purpose of this article is to start a systematic search for superintegrable systems with higher order integrals of motion. We consider a two-dimensional real Euclidian space with a one-particle Hamiltonian:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + V(x, y).$$

We request the existence of two additional integrals of motion, one of first order in the momenta and the other of third order.

The classical and quantum mechanical cases will be treated separately. When second order integrals of motion are considered, classical and quantum integrable and superintegrable potentials coincide. For third order integrals this is no longer the case (as was pointed out by Hietarinta in [16]). For integrable systems with third or higher order integrals in classical mechanics, see also [7, 11, 12, 15, 20, 22, 25].

2 Conditions for the existence of a third order invariant in classical mechanics

We are looking for a classical integral of motion that is a polynomial in the momenta with coefficients depending on the spatial coordinates, i.e.

$$X = \sum_{j,k} f_{jk}(x, y)p_1^j p_2^k,$$

that Poisson-commutes with the Hamiltonian:

$$0 = \{H, X\},$$

$$H = \frac{p_x^2 + p_y^2}{2} + V(x, y). \quad (2.1)$$

We can simplify our search by using the fact that equation (2.1) implies that $X$ is a constant over any trajectory:

$$\frac{dX}{dt} = \frac{\partial X}{\partial q_i} \dot{q}_i + \frac{\partial X}{\partial p_i} \dot{p}_i = 0 \quad (2.2)$$

with

$$\dot{p}_i = -V_{q_i}(q_1, q_2),$$

$$\dot{q}_i = p_i. \quad (2.3)$$

If we write explicitly $X$ in (2.2), we find
\[ \sum_{j+k=1}^{n} \left( \frac{\partial f_{jk}}{\partial x} p_1^{j+1} p_2^k + \frac{\partial f_{jk}}{\partial y} p_1^j p_2^{k+1} - f_{jk} V_x j p_1^{j-1} p_2^k - f_{jk} V_y k p_1^j p_2^{k-1} \right) = 0. \]

(2.4)

Since the monomials \( p_1^m p_2^n \)'s form a basis, the coefficients for each \((a, b)\) must vanish separately, thus (2.4) gives relations between the \( f_{ij} \) with odd and even \( i + j \) separately. If we are looking for an integral of odd (even) degree in the momenta, the even (odd) terms will play no role and we can without loss of generality consider only integrals that have terms only of odd (even) parity. Moreover, we may notice, in (2.3), that the terms of leading order in the \( p_i \)'s imply a relation independent of \( V \) between the \( f_{i,j} \) with \( i + j = n \). This allows us to find immediately the form of the leading order terms, so the integral of motion in the third-order case takes the form

\[ X = \sum_{i+j+k=3} A_{ijk} p_1^i p_2^j L^k + g_1(x, y)p_1 + g_2(x, y)p_2, \]

(2.5)

where the \( A_{ijk} \) are arbitrary real constants.

The requirement \( \frac{dX}{dt} = 0 \) and the Hamilton equations (2.3) yield four equations

\[ 0 = g_1 V_x + g_2 V_y, \]  
\[ (g_1)_x = 3 f_1(y) V_x + f_2(x, y) V_y, \]  
\[ (g_2)_y = f_3(x, y) V_x + 3 f_4(x) V_y, \]  
\[ (g_1)_y + (g_2)_x = 2 (f_2(x, y) V_x + f_3(x, y) V_y), \]

(2.6) (2.7) (2.8) (2.9)

where

\[ f_1(y) = -A_{300} y^3 + A_{210} y^2 - A_{120} y + A_{030}, \]
\[ f_2(x, y) = 3 A_{300} x y^2 - 2 A_{210} x y + A_{201} y^2 + A_{120} x - A_{111} y + A_{021}, \]
\[ f_3(x, y) = -3 A_{300} x^2 y + A_{210} x^2 + 2 A_{201} x y + A_{111} x - A_{102} y + A_{012}, \]
\[ f_4(x) = A_{300} x^3 + A_{201} x^2 + A_{102} x + A_{003}. \]

Requiring that equations (2.7), (2.8), and (2.9) be compatible, we obtain a linear compatibility condition for the potential, namely

\[ 0 = -f_3 V_{xxx} + (2 f_2 - 3 f_4) V_{xyy} + (-3 f_1 + 2 f_3) V_{xyy} - f_2 V_{yyy} + 2 (f_2 y - f_3 x) V_{xx} + 2 (-3 f_1 y + f_2 x + f_3 y - 3 f_4 x) V_{xy} + 2 (f_2 y + f_3 x) V_{yy} + (-3 f_1 y + 2 f_2 x - f_3 x) V_{xx} + (-f_2 y + 2 f_3 x - 3 f_4 x) V_{yy}. \]

(2.10)
Requiring that all four equations (2.6), (2.7), (2.8), (2.9) be compatible, we obtain further third-order equations for the potential, this time nonlinear ones. They are limit case (for $\bar{h} \to 0$) of the corresponding quantum compatibility conditions (3.7) to (3.9) given below.

These conditions, together with (2.10), form an overdetermined system for the potential $V(x, y)$. The solution space will hence be rather restricted. Indeed, in 1935, Drach ([7]) posed the problem of finding classical Hamiltonian systems with one third-order integral. In a complex Euclidean space $E_2(C)$ he found 10 such potentials, each one depending on arbitrary constants, not however on arbitrary functions. We recall that in the case of second order integrals, one obtains four families of potentials, each of them depending on two arbitrary functions of one variable ([13, 31]). They are the four most general potentials that allow separation of variables in cartesian, polar, parabolic and elliptic coordinates, respectively.

3 Conditions for the existence of a third order invariant in quantum mechanics

Here we are interested in the existence of third-order operators, i.e.

$$X = \sum_{i+j=0}^{3} P_{ij}(x, y)p_i^1p_2^j,$$

$$p_1 = -i\bar{h}\partial_x, \quad p_2 = -i\bar{h}\partial_y,$$

that commute with the Hamiltonian. An equivalent way of writing this operator is

$$X = \sum_{i+j=0}^{3} \{P_{ij}(x, y), p_1^ip_2^j\}.$$

Here the bracket means the anticommutator:

$$\{f, p_1^ip_2^j\} = f[p_1^ip_2^j] + p_1^ip_2^jf.$$

Each of these anticommutators can be expressed as

$$\{f, p_1^ip_2^j\} = \Re[f, p_1^ip_2^j] + i\{\Im[f], p_1^ip_2^j\}.$$

Hence we can write the operator $X$ in the form

$$X = X^+ + iX^-,$$

where $X^+$ and $X^-$ are self-adjoint operators. As the Hamiltonian itself is self-adjoint, $X^\dagger = X^+ - iX^-$ must also commute, as well as $X^+$ and $X^-$. These
two last operators commute under the same conditions, so we may restrict our
search without loss of generality to self-adjoint operators. This turns out to be
quite useful in view of the following result

\textbf{Proposition 3.1.} For each self-adjoint integral of motion of order \( n \), there
exists one integral of order \( n \) with definite parity, i.e.

\[
X_n = \sum_{j=0}^{[\frac{n}{2}]} \sum_{k=0}^{n-2j-k} \{ P_{n-2j,k}(x,y), p_1^{k} p_2^{n-2j-k} \},
\]

where \( P \) is a real function.

\textit{Proof.} This is simply due to the fact that we have a real Hamiltonian and
a purely imaginary momentum operator, so terms of even order, which are
real, must commute independently of the terms of odd order, which are purely
imaginary.

\[\square\]

In the case \( n = 3 \) we restrict ourselves to third-order integrals of the form

\[
X_3 = \sum_{i+j=3} \{ f_{ij}(x,y), p_1^{i} p_2^{j} \} + \{ g_1(x,y), p_1 \} + \{ g_2(x,y), p_2 \}.
\]

Requesting

\[
0 = [H, X],
\]

\[
H = \frac{1}{2m} (p_1^2 + p_2^2) + V(x,y),
\]

we find a set of 15 differential equations, of which the first nine can be explicitly
solved to give

\[
X = \sum_{i+j+k=3} A_{ijk} \{ L_3^{i}, p_1^{j} p_2^{k} \} + \{ g_1(x,y), p_1 \} + \{ g_2(x,y), p_2 \}.
\]

where the \( A_{ijk} \) are arbitrary real constants. So far this is similar to the classical
case.

\textbf{Remark 1.} The argument used in demonstrating proposition 3.1 can be gener-
alyzed to any expression involving the anticommutators of self-adjoint operators
homogeneous in the \( p_i \)'s, for example to terms of the form \( \{ L_3^{i}, p_1^{j} p_2^{k} \} \), as long
as the coefficients of the \( p_i \)'s are real.

6
We could get rid of the $\hbar$ and $m$ factors by a dilation of the undetermined functions,

\[
V(x, y) = \frac{\hbar^2}{2m} \tilde{V}(x, y),
\]

\[
g_1(x, y) = \hbar^2 g'_1(x, y),
\]

\[
g_2(x, y) = \hbar^2 g'_2(x, y).
\]

This is equivalent to setting $\hbar$ and $m$ equal to one, which we could do, but we prefer to keep track of the dependence on $\hbar$ (while setting $m = 1$), in order to see the classical limit.

We are left with a set of 6 equations, two of which are consequences of the other four, so that, as in the classical case, we have to solve four equations:

\[
0 = g_1 V_x + g_2 V_y - \frac{\hbar^2}{4} \left( f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yyy} + 8A_{300}(xV_y - yV_x) + 2(A_{210}V_x + A_{201}V_y) \right),
\]

\[
(g_1)_x = 3f_1(y)V_x + f_2(x, y)V_y \equiv h_1,
\]

\[
(g_2)_y = f_3(x, y)V_x + 3f_4(x)V_y \equiv h_2,
\]

\[
(g_1)_y + (g_2)_x = 2(f_2(x, y)V_x + f_3(x, y)V_y) \equiv h_3.
\]

Equations (3.4) to (3.6) are the same as in the classical case, however equation (3.3) differs from equation (2.6) by the terms proportional to $\hbar^2$. Both in the classical and quantum cases we can eliminate $g_1$ and $g_2$ and obtain compatibility conditions for the potentials.

We shall write these in a unified manner for both cases. One such compatibility condition is the third-order linear equation (2.10). To write three more conditions we introduce the notation

\[
\phi_1 = V_y V_x,
\]

\[
\phi_2 = -\hbar^2 \frac{f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yyy} + 8A_{300}(xV_y - yV_x) + 2(A_{210}V_x + A_{201}V_y)}{4V_x}.
\]

and use $h_1, h_2$ and $h_3$ introduced above. In the classical case we have $\phi_2 = 0$. The three (independent) nonlinear compatibility conditions are

\[-\phi_{2x} + \left( \frac{\phi_1 (h_3 \phi_1 + h_2 \phi_2^2 + \phi_1 \phi_{2y} + \phi_{2x} + h_4)}{\phi_{1x} + \phi_1 \phi_{1y}} \right)_x = h_4, \quad (3.7)\]
\[
\left( \frac{\phi_1^2 h_5 + \phi_1 \phi_2 y + \phi_1 h_6 + \phi_2 x + h_4}{\phi_{1x} + \phi_1 \phi_{1y}} \right)_y = -h_5, \quad (3.8)
\]

\[
h_4 (\phi_{1xy} + \phi_2^2 y) + h_5 (\phi_1^2 \phi_{1xy} - \phi_{1x}^2 - 2\phi_1 \phi_{1x} \phi_{1y}) + h_6 (\phi_1 \phi_{1xy} - \phi_{1x} \phi_{1y}) - (h_{4y} + \phi_1 h_{5x}) (\phi_{1x} + \phi_1 \phi_{1y}) = -\phi_{2x} (\phi_{1xy} + \phi_2^2 y) + \phi_{2y} (\phi_{1x} \phi_{1y} - \phi_{1xy} \phi_1) + \phi_{2xy} (\phi_{1x} + \phi_1 \phi_{1y}). \tag{3.9}
\]

In the quantum case these are fifth order equations for the potential. They can be used to express \(\phi_{2xy}, \phi_{2xx}\) and \(\phi_{2yy}\) in terms of \(\phi_{2x}, \phi_{2y}\) and \(\phi_2\). In the classical case we set \(\phi_2 = 0\), but the equations remain independent. They can be used to determine \(\phi_{1xy}, \phi_{1xx}\) and \(\phi_{1yy}\) in terms of \(\phi_{1x}, \phi_{1y}\) and \(\phi_1\). The nonlinear equations for \(V(x,y)\) are third order ones in the classical case.

In deriving these equations we have assumed

\[
\phi_{1x} + \phi_1 \phi_{1y} \neq 0, \\
\phi_{1xy} \phi_1 - \phi_{1x} \phi_{1y} \neq 0. \tag{3.10}
\]

The cases when the above conditions do not hold must be considered separately. This will actually be the case for potentials considered in this article.

We also mention the interesting fact, already noticed by Hietarinta, that a classical integrable potential is also quantum integrable, if and only if it respects the compatibility condition

\[
f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yyy} + 8A_{300}(xV_y - yV_x) + 2(A_{210}V_x + A_{201}V_y) = 0. \tag{3.11}
\]

In that case the equations are invariant under a simultaneous dilation of the potential and the \(g_i\)'s. Thus any potential that is a solution to both \((2.6)\) to \((2.9)\) and \((3.11)\) can be multiplied by an arbitrary factor, which can be used to "absorb" the \(h^2\) factor so the solution does not vanish in the classical limit.

Even if a classical superintegrable potential does not satisfy this relation, there could exist corresponding quantum superintegrable systems. In that case, though, the equations are not invariant under a dilation of the potential as in the previous case, so terms that do not satisfy both \((3.11)\) and \((2.6)\) to \((2.9)\) must be proportional to \(h^2\) and vanish in the classical limit.

We will show that condition \((3.11)\) cannot be the consequence of equations \((3.4), (3.5)\) and \((3.6)\) for in that case the classical and quantum integrable potentials would be the same.

4 Superintegrable systems with one third order and one first order integral
4.1 Integral of first-order
A potential \( V(x, y) \) allows an integral that is of first order in the momenta if and only if it is invariant under either rotations or translations. Thus the potential must satisfy
\[
aL_3 V + bp_1 V + cp_2 V = 0.
\]
Without loss of generality, we can take the potential and first order integral to be one of the following:

- \( a \neq 0 \):
  \[ V = V(r), \quad X = L_3 \]
- \( a = 0, b^2 + c^2 \neq 0 \):
  \[ V = V(x), \quad X = p_2. \]

4.2 Quantum and classical superintegrable potentials invariant under rotations.
Compatibility conditions obtained from equations (2.6) to (2.9) or (3.3) to (3.6) leave us with only two possibilities, namely
\[
V = \frac{\alpha}{r},
\]
\[
V = \omega^2 r^2
\]

The Coulomb potential and the harmonic oscillator, which are the best-known superintegrable potentials in any dimension. In addition to angular momentum \( L_3 \), the Coulomb potential in \( E_2 \) allows two second order integrals, namely the components of the Laplace-Runge-Lenz vector:
\[
X^C_1 = \{L_3, p_1\} - \frac{2\alpha y}{r};
\]
\[
X^C_2 = \{L_3, p_2\} + \frac{2\alpha x}{r}.
\]

The harmonic oscillator, in addition to angular momentum, allows two second order integrals which are the components of a quadrupole tensor:
\[
X^h_1 = -\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \omega^2 x^2 - \omega^2 y^2;
\]
\[
X^h_2 = -p_1 p_2 + 2\omega^2 xy.
\]

Commuting (or Poisson commuting) second order integrals, we in general find third order integrals.

The third order integrals obtained for these potentials are indeed direct consequences of integrals at order one and two.
4.3 Classical superintegrable potentials invariant under translation.

In the classical case, the remaining equations are readily solved. If we set $V_y = 0$, equations (2.6) to (2.9) simplify to

\[ 0 = g_1 \]
\[ 0 = A_{300} = A_{210} = A_{120} = A_{030} \]
\[ (g_2)_y = f_3(x, y)V_x \]
\[ (g_2)_x = 2f_2(x, y)V_x \]

We can at once set $A_{021}$ and $A_{003}$ to 0, for they correspond to trivial constants of motion, $p_3^2$ and $H_{p_2}$, that can be subtracted from the constant (2.7).

The compatibility condition between the two last equations forces one of the three following conditions to be satisfied (up to a translation in $x$).

\[ V = ax \] (4.1)
\[ V = \frac{a}{x^2} \] (4.2)
\[ A_{201} = A_{111} = A_{102} = A_{012} = 0 \] (4.3)

The first two potentials correspond to superintegrable systems that have one first and at least one second order integral. Their third order integrals can be obtained by commutation of these lower-order ones.

The last conditions forbids the existence of a nontrivial third-order commuting operator for any other potentials than (4.1) and (4.2).

4.4 Quantum superintegrable potentials invariant under translation.

Here the situation is more interesting. Equations (3.3) to (3.6) reduce to

\[ 0 = g_1 V_x - \frac{\hbar^2}{4} \left( f_1 V_{xxx} - 8yA_{300}V_x + 2A_{210}V_x \right) \] (4.4)
\[ (g_1)_x = 3f_1(y)V_x \] (4.5)
\[ (g_2)_y = f_3(x, y)V_x \] (4.6)
\[ (g_1)_y + (g_2)_x = 2\left( f_2(x, y)V_x \right) \] (4.7)

The linear compatibility condition leads to two equations (since coefficients of $y^0$ and of $y^1$ must vanish separately), namely

\[ 0 = (A_{210}x^2 + A_{111}x + A_{012})V_{xxx} + 4(2A_{210}x + A_{111})V_{xx} + 12A_{210}V_x; \]
\[ 0 = (3A_{300}x^2 + 2A_{201}x + A_{102})V_{xxx} + 4(6A_{300}x + 2A_{201})V_{xx} + 36A_{300}V_x. \] (4.8)
The two equations are similar and easy to solve, but it turns out their only solutions that also satisfy (4.4) to (4.7) are again the potentials $V = ax$ and $V = a/x^2$. Their third-order integrals in general are direct consequences of lower-order commuting operators, that is they can be obtained by commuting their second order integrals. In the $V = a/x^2$ case, we find three third order integrals,

$$X_1 = \{L_3, p_2\} + a\{2y^2/x^2, p_2\}$$

$$X_2 = \{L_3, p_1p_2\} - a\{4y/x^2, p_2\}$$

$$X_3 = p_1^2p_2 - a\{4/x^2, p_2\}$$

The integrals $X_2$ and $X_3$ can be obtained by commuting $X_1$ with the first-order integral $p_2$.

In the particular case $V = \hbar^2/x^2$, we find four more integrals, again related to each other by commutation with $p_2$:

$$X_4 = L_3^2 + \hbar^2\{6y^2/x + 2x, p_2\} + \hbar^2\{-3y^3/x^2 - 2y, p_1\}$$

$$X_5 = \{L_3, p_1\} - \hbar^2\{4y/x, p_2\} + \hbar^2\{6y^2/x^2 + 1, p_1\}$$

$$X_6 = \{L_3, p_1^2\} - \hbar^2\{7/x, p_2\} + \hbar^2\{-3y/x^2, p_1\}$$

$$X_7 = p_1^3 + \hbar^2\{3/x^2, p_1\}$$

In this case we find nine third order integrals, two of which are trivial ($Hp_2$ and $p_2^3$), and four are purely quantum integrals. In the classical limit they correspond to integrals of the free motion. Only the first three can be associated with the corresponding classical integrals of $V = a/x^2$.

The most interesting potentials are obtained by setting all the $A_{ijk}$ involved in (4.8) equal to 0. The expressions for $f_1, f_2, f_3, f_4$ greatly simplify and equations (4.4) to (4.7) can be solved directly. The nonlinear compatibility condition for these four equations reduces to

$$\hbar^2 V'(x)^2 = 4V(x)^3 + \alpha V(x)^2 + \beta V(x) + \gamma, \quad (4.9)$$

where the $\alpha, \beta, \gamma$ are arbitrary real integration constants. Equation (4.9) is the well-known equation for elliptic functions which can be rewritten as

$$\hbar^2 V'(x)^2 = 4(V(x) - A_1)(V(x) - A_2)(V(x) - A_3). \quad (4.10)$$
The constants $A_i$ are either all real, or one of them is real and the other two are complex conjugated. If all three constants are real, we obtain either finite or singular potentials of the form

$$V_1 = (\hbar \omega)^2 k^2 sn^2(\omega x, k),$$
$$V_2 = \frac{(\hbar \omega)^2}{sn^2(\omega x, k)},$$

respectively.

If we have e.g. $A_3 = A_2^*$ and $Im A_2 \neq 0$, we obtain the singular potential

$$V_3 = \frac{(\hbar \omega)^2}{2(en(\omega x, k) + 1)}$$

(throughout we have $0 \leq k \leq 1$, $\omega \in \mathbb{R}$).

The special cases with $k = 0$ or $k = 1$, which arise when two roots coincide, can be expressed in terms of elementary functions. The most interesting example is the "soliton" potential,

$$V_{1a} = \frac{(\hbar \omega)^2}{\cosh^2(\omega x)},$$

obtained by setting $k = 1$ in $V_1$. If we set $k = 0$, or $k = 1$ in $V_2$, we get a singular periodic, or nonperiodic potential, respectively, namely

$$V_{2a} = \frac{(\hbar \omega)^2}{\sin^2(\omega x)},$$
$$V_{2b} = \frac{(\hbar \omega)^2}{\sinh^2(\omega x)}.$$

For all these potentials $\omega$ is an arbitrary constant, hence there exist potentials of arbitrary amplitude for all nonzero values of $\hbar$.

Finally, if all roots coincide, we reobtain the known superintegrable potential

$$V_4 = \frac{\hbar^2}{x^2},$$

which explains the extra integrals found previously for that potential.

The other potentials $V_1, V_2, V_3$ also satisfy

$$\frac{\hbar^2}{4} \frac{V_{xxx}}{V_x} - 3V = \alpha, \quad \alpha = A_1 + A_2 + A_3,$$
a consequence of (4.9).

The two nontrivial integrals of motion for all these potentials can be written as

\[
X_1 = \{ L_3, p_1^2 \} + \{ (\alpha - 3V(x))y, p_1 \} + \{ -\alpha x + 2xV(x) + \int V(x)dx, p_2 \}
\]

\[
X_2 = p_1^3 + \frac{1}{2} (3V(x) - \alpha, p_1). 
\]

(4.12)

The second integral can be trivially obtained by the commutation of the first one with \( p_2 \).

5 Conclusion

We have found all potentials in two-dimensional Euclidian space \( E_2 \) that allow one first- and at least one third-order integral of motion. In the classical case the result provides no new superintegrable potentials; all the potentials found allow second order integrals and the third order integrals are consequences of the second order ones. In the case of quantum mechanics the result is quite different. Any potential satisfying the elliptic function equation (4.9) will be superintegrable in the above sense, i.e. will allow the first-order integral \( p_2 \) and two nontrivial third order integrals. All those ”behave well” in the classical limit, that is they are proportional to \( \hbar^2 \) and therefore their classical limit is the (superintegrable) free motion.

No new superintegrable systems are found for rotationally invariant potentials \( V(r) \), neither in the classical, nor in the quantum case. Thus all potentials found above are of the form \( V = V(x) \), i.e. are actually one-dimensional. The problem however remains two-dimensional as the kinetic energy and the integrals of motion also involve the \( y \) direction.

There is also an interesting link with soliton theory ([1]). All new superintegrable potentials obtained above are also translationally invariant solutions of the Korteweg-de Vries equation. The same potentials occur in the rational, trigonometric and elliptic Calogero-Moser-Sutherland models ([6]).

The difference between classical and quantum integrable and superintegrable systems with higher order symmetries makes the systematic search for such systems very interesting. First of all, Drach’s study of classical integrable systems should be completed. His systems are really complex ones and most of them do not exist in real Euclidian space. Moreover it is not clear how complete his list is. On the other hand, Rañada ([23]) has shown that 7 out of 10 Drach potentials are ”reducible” in the sense that they are superintegrable and allow two second-order integrals. The third order integral found by Drach is the Poisson commutator of the second-order ones.

The problem of classifying quantum systems with third-order integrals remains open and the conditions of Section 3 provide the means for finding all
such systems.

Work is in progress on superintegrable systems in two-dimensional Euclidian space with one second and one third-order invariant, as well as with two third order ones.

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