Moments of permutation statistics and central limit theorems

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Consider the set, \( \Pi_n \), of all partitions of \([n] := \{1, 2, \ldots, n\}\).

Their number is \( B_n \) - the \( n \)-th Bell number. For example, \( B_3 = 5 \):
\[
\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}, \{1, 2, 3\}.
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Let $X_1(\lambda) :=$ the number of blocks of size 1 in $\lambda \in \Pi_n$.

Can we find $M(X_1; n)$, where $M(f; n) := \sum_{\lambda \in \Pi_n} f(\lambda)$?
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Let $X_1(\lambda) :=$ the number of blocks of size 1 in $\lambda \in \Pi_n$.
Can we find $M(X_1; n)$, where $M(f; n) := \sum_{\lambda \in \Pi_n} f(\lambda)$?

Answer: $M(X_1; n) = nB_{n-1}$. 

Let $cr_2(\lambda) :=$ the number of 2-crossings in $\lambda \in \Pi_n$, i.e., numbers $i_1 < i_2 < j_1 < j_2$, such that $i_1, j_1$ and $i_2, j_2$ are in two different blocks. For instance, $cr_2(\{\{1\}, \{2\}, \{3\}\}) = 2$.

Can we find $M(cr_2; n)$?

$M(cr_2; n) = 14(5B_n - 2 + (2n + 9)B_n + 1 + (2n + 1)B_n),$ (Kasraoui, 2013 [6]).
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Can we find $M(cr_2; n)$?

$M(cr_2; n) = \frac{1}{4}(-5B_{n+2} + (2n + 9)B_{n+1} + (2n + 1)B_n)$ (Kasraoui, 2013 [6]).
Chern, Diaconis, Kane and Rhoades [4] found that

\[ M(X_1^2; n) = nB_{n-1} + (n^2 - n)B_{n-2}. \]

and

\[ M(cr_2^2; n) = \frac{1}{144} (225B_{n+4} - (180n + 814)B_{n+3} + (36n^2 + 156n + 489)B_{n+2} + \\
(72n^2 + 72n - 260)B_{n+1} + (36n^2 + 24n - 23)B_{n}). \]
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**Theorem 1 (CDKR)**

*For a family of set partition statistics, the moments can be written as linear combinations of shifted Bell numbers, where the coefficients are polynomials in \( n \).*
Khare, Lorentz and Yan [7] developed the same approach on the set of perfect matchings (set partitions with blocks of size 2) on $[2m]$.

**Theorem 2 (KLY)**

*For a family of statistics on perfect matchings, the moments can be written as linear combinations of double factorials with constant coefficients.*
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**Theorem 2 (KLY)**

*For a family of statistics on perfect matchings, the moments can be written as linear combinations of double factorials with constant coefficients.*

For example, they found:

$$
\sum_{M \in \mathcal{M}_{2m}} cr_2^2(M) = \binom{2m}{4} T_{2m-4} + 12 \binom{2m}{6} T_{2m-6} + 70 \binom{2m}{8} T_{2m-8},
$$

where $T_{2m} = |\mathcal{M}_{2m}| = (2m - 1)(2m - 3) \cdots 3 \cdot 1 = (2m - 1)!!$. 
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Goal: Develop the same approach for permutations!
· permutation - ordering of the numbers in \([n]\).

· \(S_n\) - the set of permutations of size \(n\).
  
  \textit{Example:} \(4172365 \in S_7\).

· Let \(A(\pi) := \{(u, v) \mid u = \pi_i, v = \pi_j, i < j\}\) be the arc set of \(\pi\).
  
  \textit{Example:} \(A(312) = \{(3, 1), (3, 2), (1, 2)\}\).

· \(\text{red}(s_1s_2 \cdots s_k) := p_1 \cdots p_k \in S_k\), where \(p_i < p_j\) iff \(s_i < s_j\).
  
  \textit{Example:} \(\text{red}(6253) = 4132\).
Definition

(i) A permutation pattern $P$ of size $k$ is a tuple $P = (P, C(P), D(P))$, where $P = p_1 \cdots p_k \in S_k$ and $C(P) \subseteq [k - 1], D(P) \subseteq [k - 1]$.

(ii) An occurrence of the pattern $P = (p_1 p_2 \cdots p_k, C(P), D(P))$ of size $k$ in $\sigma \in S_n$ is a tuple $t = (t_1, t_2, \ldots, t_k)$ with $t_i \in [n]$, such that:

a) $t_1 < t_2 < \cdots < t_k$.

b) $(t_i, t_j) \in A(\sigma)$, if and only if $(i, j) \in A(P)$.

c) if $i \in C(P)$, then the positions of $t_{p_i}$ and $t_{p_i+1}$ in $\sigma$ are consecutive.

d) if $i \in D(P)$, then $t_{i+1} = t_i + 1$.

Write $t \in_\sigma P$, if $t$ is an occurrence of $P$ in $\sigma$.

Examples:

1. $P = 132 = (132, \emptyset, \emptyset)$ [classical patterns].
   
   $t = (3, 4, 5) \in_\sigma 31524$, since $\text{red}(354) = 132$. 
Definition

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   a) $t_1 < t_2 < \cdots < t_k$.
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   d) if $i \in D(P)$, then $t_{i+1} = t_i + 1$.

Write $t \in_{P} \sigma$, if $t$ is an occurrence of $P$ in $\sigma$.

Examples:

2. $P = \underline{3214} = (3214, \{1\}, \emptyset)$ [vincular patterns].
   $t = (2, 3, 5, 7) \in_{P} 4536217$, since $\text{red}(5327) = 3214$
   and the positions of $t_3 = 5$ and $t_2 = 3$ are consecutive.
Definition

(i) A permutation pattern \( \underline{P} \) of size \( k \) is a tuple \( \underline{P} = (P, C(\underline{P}), D(\underline{P})) \), where \( P = p_1 \cdots p_k \in S_k \) and \( C(\underline{P}) \subseteq [k - 1], D(\underline{P}) \subseteq [k - 1] \).

(ii) An occurrence of the pattern \( \underline{P} = (p_1p_2 \cdots p_k, C(\underline{P}), D(\underline{P})) \) of size \( k \) in \( \sigma \in S_n \) is a tuple \( t = (t_1, t_2, \ldots, t_k) \) with \( t_i \in [n] \), such that:

a) \( t_1 < t_2 < \cdots < t_k \).

b) \((t_i, t_j) \in A(\sigma)\), if and only if \((i, j) \in A(\underline{P})\).

c) if \( i \in C(\underline{P}) \), then the positions of \( t_{p_i} \) and \( t_{p_i+1} \) in \( \sigma \) are consecutive.

d) if \( i \in D(\underline{P}) \), then \( t_{i+1} = t_i + 1 \).

Write \( t \in \_\sigma \) if \( t \) is an occurrence of \( \underline{P} \) in \( \sigma \).

Examples:

3. \( \underline{P} = \frac{1234}{4312} = (4312, \{2\}, \{3\}) \) [bivincular patterns].

\( t = (1, 3, 5, 6) \in \underline{P} \) 625143, since \( \text{red}(6513) = 4312 \), the positions of \( t_3 = 5 \) and \( t_1 = 1 \) are consecutive, and \( t_4 = 6 = t_3 + 1 \).
simple statistic: a pattern \( P \) of size \( k \) and a valuation function \( Q(t, w) = Q_1(t)Q_2(w) \), where \( Q_1, Q_2 \in \mathbb{Z}[y_1, \ldots, y_k, m] \).

\[
f(\sigma) = f_{P,Q}(\sigma) := \sum_{t \in P\sigma} Q(t, \sigma^{-1}(t)) = \sum_{t \in P\sigma} Q_1(t)Q_2(\sigma^{-1}(t)).
\]

\( f \) is of degree \( d(f) := 2k + \deg(Q) \).

statistic: a finite linear combination of simple statistics.
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statistic: a finite linear combination of simple statistics.

Examples (simple statistic):

1. $\text{cnt}_P(\sigma) := f_{P, 1}(\sigma) = \sum_{t \in \sigma} 1$, for any pattern $P$, e.g., 21, 1324, $123$, $123\overline{312}$. 
**Family of statistics**

**simple statistic**: a pattern $P$ of size $k$ and a valuation function $Q(t, w) = Q_1(t)Q_2(w)$, where $Q_1, Q_2 \in \mathbb{Z}[y_1, \ldots, y_k, m]$.

$$f(\sigma) = f_{P,Q}(\sigma) := \sum_{t \in P \sigma} Q(t, \sigma^{-1}(t)) = \sum_{t \in P \sigma} Q_1(t)Q_2(\sigma^{-1}(t)).$$

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Examples (simple statistic):

1. $\text{cnt}_P(\sigma) := f_{P,1}(\sigma) = \sum_{t \in P \sigma} 1$, for any pattern $P$, e.g., 21, 1324, $\underline{123}$, $\underline{312}$.

2. $\text{drops}(\sigma) := \sum_{\sigma_i > \sigma_{i+1}} \sigma_i - \sigma_{i+1} = \sum_{(t_1, t_2) \in 2_1 \sigma} t_2 - t_1$.

$P = 21$, $Q(t, w) = Q_1(t)Q_2(w)$, where $Q_1(t) = Q_1(t_1, t_2) = t_2 - t_1$ and $Q_2(w) = 1$. 
**simple statistic**: a pattern $P$ of size $k$ and a valuation function $Q(t, w) = Q_1(t)Q_2(w)$, where $Q_1, Q_2 \in \mathbb{Z}[y_1, \ldots, y_k, m]$.

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$f$ is of degree $d(f) := 2k + \deg(Q)$.

**statistic**: a finite linear combination of simple statistics.

Example (statistic):

$$\text{peakSqSum}(\sigma) := \sum_{\sigma(i-1) < \sigma(i) > \sigma(i+1)} \sigma(i)^2 = \sum_{(t_1,t_2,t_3) \in 132} t_3^2 + \sum_{(t_1,t_2,t_3) \in 231} t_3^2.$$

This is a sum of the simple statistics $f_{132,t_3^2}$ and $f_{231,t_3^2}$.
Theorem 3

Let $f_{P,Q}$ be a simple statistic of degree $m$, where $|P| = k$, $|C(P)| = c$ and $|D(P)| = d$. Then

$$M(f_{P,Q}, n) = R(n)(n - k)!,$$

where $R(x)$ is a polynomial of degree no more than $m - c - d$. Equivalently for $n \geq k$,

$$M(f_{P,Q}, n) = \begin{cases} 
0 & n < k \\
\sum_{i=0}^{m-c-d} c_i(n - k + i)! & n \geq k
\end{cases},$$

for some constants $c_i \in \mathbb{Q}$. 
Simple statistics:

1. $\text{cnt}_{1324}$.

   \[
   M(\text{cnt}_{1324}, n) = \frac{1}{24} n! - \frac{1}{6} (n+1)! + \frac{1}{8} (n+2)! - \frac{1}{36} (n+3)! + \frac{1}{576} (n+4)! .
   \]

   In fact, $M(\text{cnt}_P, n) = \frac{1}{k!} \binom{n}{k} n!$ for any classical pattern $P$ of size $k$.

   Express rising factorials in terms of falling factorials to get

   \[
   M(\text{cnt}_P, n) = \frac{1}{k!} \binom{n}{k} n! = \frac{(-1)^k}{k!} n! + \sum_{j=1}^{k-1} \frac{(-1)^{k-j}}{(j!)^2 (k-j)!} (n+j)! + \frac{1}{(k!)^2} (n+k)! .
   \]

2. Descent drop.

   \[
   M(\text{drops}, n) = -\frac{1}{2} (n+1)! + \frac{1}{6} (n+2)! .
   \]
Theorem 4

For any statistic $f$ of degree $m$, there is a positive integer $L \leq \frac{m}{2}$, such that for all $n \geq L$,

$$M(f, n) = U(n)(n - L)!,$$

where $U(n)$ is a polynomial of degree no more than $m + L$. Equivalently, if $n \geq L$,

$$M(f, n) = \sum_{-L \leq i \leq m} \alpha_i(n + i)!,$$

for some constants $\alpha_i \in \mathbb{Q}$.

Example: Sum of peak squares.

$$M(\text{peakSqSum}, n) = (n + 1)! - \frac{5}{4}(n + 2)! + \frac{1}{5}(n + 3)!.$$
Higher moments of statistics

Goal: Show that the higher moments of statistics are also statistics!
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Key observation: The union of two (or more) occurrences of a pattern $\sigma$ in $\pi$ is an occurrence of another pattern in $\pi$.

Example:

$\pi = 516243$

$\sigma = 132$
Goal: Show that the higher moments of statistics are also statistics!

Key observation: The union of two (or more) occurrences of a pattern $\sigma$ in $\pi$ is an occurrence of another pattern in $\pi$.

Example:

$\pi = 516243$
$\sigma = 132$

$(1, 4, 6) \in_{132} 516243,$
$(2, 3, 4) \in_{132} 516243$
Goal: Show that the higher moments of statistics are also statistics!

Key observation: The union of two (or more) occurrences of a pattern $\sigma$ in $\pi$ is an occurrence of another pattern in $\pi$.

Example:

$\pi = 516243$

$\sigma = 132$

$(1, 4, 6) \in_{132} 516243,$

$(2, 3, 4) \in_{132} 516243$

________________________

$(1, 2, 3, 4, 6) \in_{15243} 516243$
Let $P_1$, $P_2$ and $P_3$ be patterns of sizes $k_1$, $k_2$ and $k_3$, respectively. A *merge* of $P_1$ and $P_2$ onto $P_3$ is a pair of increasing functions $m_1 : [k_1] \rightarrow [k_3]$ and $m_2 : [k_2] \rightarrow [k_3]$ with certain properties. Denote a merge by $m_1, m_2 : P_1, P_2 \rightarrow P_3$. 
Let $P_1$, $P_2$ and $P_3$ be patterns of sizes $k_1$, $k_2$ and $k_3$, respectively.

A merge of $P_1$ and $P_2$ onto $P_3$ is a pair of increasing functions $m_1 : [k_1] \rightarrow [k_3]$ and $m_2 : [k_2] \rightarrow [k_3]$ with certain properties.

Denote a merge by $m_1, m_2 : P_1, P_2 \rightarrow P_3$.

Example:

$m_1(1) = 1, m_2(1) = 2,$
$m_1(2) = 4, m_2(2) = 3,$
$m_1(3) = 5, m_2(3) = 4.$

Then $m_1, m_2 : 132, 132 \rightarrow 15243.$
Lemma 1

Let $P_1$ and $P_2$ be two patterns. For any $\sigma \in S_n$, there is a one-to-one correspondence between the following sets.

$$\{(s_1, s_2) : s_1 \in_{P_1} \sigma, s_2 \in_{P_2} \sigma\} \leftrightarrow \{s_3 \in_{P_3} \sigma \mid m_1, m_2 : P_1, P_2 \rightarrow P_3\}$$
Lemma 1

Let $P_1$ and $P_2$ be two patterns. For any $\sigma \in S_n$, there is a one-to-one correspondence between the following sets.

$$\{ (s_1, s_2) : s_1 \in P_1 \sigma, s_2 \in P_2 \sigma \} \leftrightarrow \{ s_3 \in P_3 \sigma \mid m_1, m_2 : P_1, P_2 \to P_3 \}$$

Using Lemma 1, we prove that the product of two simple statistics is a statistic:

$$f_{P_1, Q_1} (\sigma) g_{P_2, Q_2} (\sigma) = \sum_{s_1 \in P_1 \sigma} Q_1(s_1) Q'_1(\sigma^{-1}(s_1)) \sum_{s_2 \in P_2 \sigma} Q_2(s_2) Q'_2(\sigma^{-1}(s_2))$$

(by Lemma 1)

$$= \sum_{P_3} \left( \sum_{s_3 \in P_3 \sigma} \left( \sum_{m_1, m_2 : P_1, P_2 \to P_3} Q_{m_1, m_2, Q_1, Q_2}(s_3) Q'_{m_1, m_2, Q_1, Q_2}(\sigma^{-1}(s_3)) \right) \right) = \sum_{P_3} f_{P_3, \tilde{Q}}.$$
Theorem 5

Let $f$ be any statistic of degree $m$. Then, for any positive integer $r$, the $r$-th moment of $f$ is given by

$$M(f^r, n) = \sum_{-I \leq i \leq J} \alpha_i(n + i)!,$$

where $I$ and $J$ are constants that satisfy $-I \geq \frac{-rm}{2}$, $J \leq mr$ and $n \geq I$, and the $\alpha_i$’s are rational constants.
Corollary 1

If $P$ is a vincular pattern of size $k$, such that $|C(P)| = c$, then

$$M(\text{cnt}_P^r, n) = \sum_{0 \leq i \leq r(k-c)} \alpha_i (n + i)!,$$

for $n \geq rk$.

Zeilberger [8] showed that if $P$ is a classical pattern of size $k$, then $\mathbb{E}(\text{cnt}_P^r)$ for a random permutation of size $n$, is a polynomial in $n$ of degree $rk$. Corollary 1 is a generalization.
1. Second moment of the number of double ascents.

\[ M(\text{cnt}_{123}^2, n) = -\frac{1}{12}n! - \frac{1}{15}(n + 1)! + \frac{1}{36}(n + 2)!. \]

2. Second moment of \( \text{cnt}_{123}^3 \).

\[ M(\text{cnt}_{123}^3, n) = \frac{1}{2}n! - \frac{9}{28}(n + 1)! + \frac{29}{672}(n + 2)! + \frac{11}{10080}(n + 3)! - \frac{1}{45360}(n + 4)!. \]
Corollary 2

Let $P$ be a pattern of size $k$ with $|C(P)| = c$, $|D(P)| = d$. Then,

$$M(\text{cnt}^r_P, n) = \sum_{\tilde{k}, \tilde{c}, \tilde{d}} w_{\tilde{k}, \tilde{c}, \tilde{d}}^{(r)} \left( \frac{n - \tilde{c}}{\tilde{k} - \tilde{c}} \right) \left( \frac{n - \tilde{d}}{\tilde{k} - \tilde{d}} \right) (n - k)!,$$

where $w_{\tilde{k}, \tilde{c}, \tilde{d}}^{(r)}$ is the number of ways to merge $r$ copies of $P$ and get a pattern $P^r$ of size $k$, with $|C(P^r)| = \tilde{c}$, $|D(P^r)| = \tilde{d}$ and where $k \leq \tilde{k} \leq rk$, $c \leq \tilde{c} \leq rc$ and $d \leq \tilde{d} \leq rd$. 

Next goal: Apply Corollary 2 to some simple patterns.
Corollary 2

Let $P$ be a pattern of size $k$ with $|C(P)| = c$, $|D(P)| = d$. Then,

$$M(\text{cnt}^r_P, n) = \sum_{\tilde{k}, \tilde{c}, \tilde{d}} w^{(r)}_{\tilde{k}, \tilde{c}, \tilde{d}} \binom{n - \tilde{c}}{k - \tilde{c}} \binom{n - \tilde{d}}{\tilde{k} - \tilde{d}} (n - k)!,$$

where $w^{(r)}_{\tilde{k}, \tilde{c}, \tilde{d}}$ is the number of ways to merge $r$ copies of $P$ and get a pattern $P^r$ of size $k$, with $|C(P^r)| = \tilde{c}$, $|D(P^r)| = \tilde{d}$ and where $k \leq \tilde{k} \leq rk$, $c \leq \tilde{c} \leq rc$ and $d \leq \tilde{d} \leq rd$.

Next goal: Apply Corollary 2 to some simple patterns.
**Theorem 6**

Let \( \text{des} := \text{cnt}_{21} \). Consider a random permutation in \( S_n \). Then, for any \( r \geq 2 \),

\[
\mathbb{E}(\text{des}') = \sum_{m=2}^{\min(n,2r)} \sum_{u=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m - u - w)^r \right) \left( \sum_{q_1 + \cdots + q_u = m} \binom{m}{q_1, \ldots, q_u} \right) \frac{(n-(m-u))}{m!}.
\]
Theorem 6

Let \( \text{des} := \text{cnt}_{21} \). Consider a random permutation in \( S_n \). Then, for any \( r \geq 2 \),

\[
\mathbb{E}(\text{des}^r) = \min(n, 2r) \sum_{m=2}^{\lceil \frac{m}{2} \rceil} \left( \sum_{u=1}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \left( \sum_{q_1+\ldots+q_u=m \atop q_i \geq 2} \binom{m}{q_1,\ldots,q_u} \right) \frac{(n-(m-u))}{m!}.
\]

Sketch of proof:

Let \( P^r \) be a pattern of size \( m \), obtained after a merge of \( r \) copies of 21.

Every such \( P^r \) with \( u \) segments has \( |C(P^r)| = m - u \).

Thus \( M(\text{cnt}_{P^r}, n) = \left( \frac{n-(m-u)}{u} \right) m! \) and

\[
w^{(r)}_{\frac{m}{2},m-u} = \sum_{u=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \sum_{q_1+\ldots+q_u=m \atop q_i \geq 2} \binom{m}{q_1,\ldots,q_u}.
\]
Moments of minimal descents

Theorem 7

Let $\adj := \cnt_{12}$. Consider a random permutation in $S_n$. Then, for any $r \geq 1$,

$$
\mathbb{E}(\adj^r) = \sum_{m=2}^{\min(n,2r)} \sum_{u=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left( \sum_{w=0}^{m-u} (-1)^w \binom{m-u}{w} (m-u-w)^r \right) \binom{m-u-1}{u-1} u! \frac{(n-(m-u))^2}{n(m)}.
$$

We will use this result to prove a limit theorem for $\adj$. 
Central limit theorems for $\text{cnt}_P$

$\text{cnt}_P$ has Normal distribution, when $n \to \infty$:

i. True, when $P$ is a classical pattern (Bóna, 2007 [1]).

ii. True, when $P$ is a vincular pattern (Hofer, 2017 [5]).

iii. Not true for an arbitrary bivincular pattern
    (last proof by Corteel, Louchard and Pemantle, 2004 [3])
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**i.** True, when $P$ is a classical pattern (Bóna, 2007 [1])

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We will reprove **i.** and **iii.** and give a lemma that would imply **ii.**
Bóna [1] uses the method of dependency graphs to obtain that the following theorem implies the CLT for an arbitrary classical pattern.

**Theorem 8**

*Let* $X_n := \text{cnt}_\sigma$ *be the number of occurrences of a classical pattern* $\sigma \in S_k$ *in a random permutation of size* $n$. *Then, there exists* $c > 0$, *such that for all* $n$,

$$\text{Var}(X_n) \geq cn^{2k-1}.$$
Bóna [1] uses the method of *dependency graphs* to obtain that the following theorem implies the CLT for an arbitrary classical pattern.

**Theorem 8**

Let \( X_n := \text{cnt}_\sigma \) be the number of occurrences of a classical pattern \( \sigma \in S_k \) in a random permutation of size \( n \). Then, there exists \( c > 0 \), such that for all \( n \),

\[
\text{Var}(X_n) \geq cn^{2k-1}.
\]

**Sketch of proof:**

Use Corollary 2 to obtain that

\[
\text{Var}(X_n) = \mathbb{E}(X_n^2) - \mathbb{E}^2(X_n) = \left[ a_\sigma(2k) \frac{\binom{n}{2k}}{(2k)!} + a_\sigma(2k - 1) \frac{\binom{n}{2k-1}}{(2k-1)!} + O(n^{2k-2}) \right] - \frac{\binom{n}{k}^2}{(k!)^2},
\]

where \( a_\sigma(r) \) is the number of ways to merge two copies of \( \sigma \) and get a pattern of size \( r \).
Sketch of proof (cont.)

Note that \( a_\sigma(2k) = \left( \begin{array}{c} 2k \\ k \end{array} \right)^2 \) and simplify to get:

\[
\text{Var}(X_n) \geq cn^{2k-1} \iff a_\sigma(2k - 1) > \left( \begin{array}{c} 2k - 1 \\ k \end{array} \right)^2.
\]
Sketch of proof (cont.)

Note that \( a_\sigma(2k) = \binom{2k}{k}^2 \) and simplify to get:

\[
\text{Var}(X_n) \geq cn^{2k-1} \iff a_\sigma(2k-1) > \binom{2k-1}{k}^2.
\]

Lemma 2 (Burstein and Hästö, [2, Lemma 4.3])

For any classical pattern \( \sigma \in S_k \),

\[
a_\sigma(2k - 1) > \binom{2k-1}{k}^2.
\]
Interpretation of the lemma

\[ A_{\sigma,\sigma'}(2k - 1) := \{(\pi, x, y) \mid \pi \in S_{2k-1}, x, y \in \text{subs}(\pi), \ \text{red}(x) = \sigma, \ \text{red}(y) = \sigma', \ |x \cap y| = 1\}, \]

where \( \text{subs}(\pi) \) denotes the set of the subsequences of the permutation \( \pi \).

\( a_{\sigma}(2k - 1) \), is the number of triples in the set \( A_{\sigma,\sigma}(2k - 1) \).

Example: \( A_{312,312}(5) \) contains \((54213, 523, 413)\).
Interpretation of the lemma

\[ A_{\sigma, \sigma'}(2k - 1) := \{ (\pi, x, y) \mid \pi \in S_{2k-1}, \ x, y \in \text{subs}(\pi), \ \text{red}(x)=\sigma, \ \text{red}(y)=\sigma', \ |x \cap y| = 1 \}, \]

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Example: \( A_{312,312}(5) \) contains \((54213, 523, 413)\).

\[
\begin{array}{|c|c|c|c|c|}
\hline
5 & 4 & 2 & 1 & 3 \\
\hline
5 & 2 & 3 &   &   \\
\hline
4 & 1 & 3 &   &   \\
\hline
\end{array}
\]

Theorem 9

If \( a_{\sigma, \sigma'}(2k - 1) := |A_{\sigma, \sigma'}(2k - 1)| \), then Lemma 2 is equivalent to

\[ a_{\sigma}(2k - 1) > \mathbb{E}(a_{\sigma, \sigma'}(2k - 1)), \]

for any fixed \( \sigma \in S_k \) and \( \sigma' \) chosen uniformly at random in \( S_k \).
Theorem 10 (Hofer, [5])

Let $X_n = \text{cnt}_\sigma$ be the number of occurrences of a vincular pattern $\sigma$ with $j$ blocks, in a random permutation of size $n$. Then, there exists $c > 0$, such that for all $n$,

$$\text{Var}(X_n) \geq cn^{2j-1}.$$
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Let $b_{\sigma}(m, j')$ be the number of merges of two copies of $\sigma$, where the resulting pattern is of size $m$ and has $j'$ blocks.

Example: $\sigma = \underline{431 \ 52}$. Below is a merge of two copies of $\sigma$.

The resulting pattern, $\underline{6531 \ 84 \ 72}$, has size $m = 8$ and $j' = 3$ blocks:

| 6 | 5 | 3 | 1 | 8 | 4 | 7 | 2 |
|---|---|---|---|---|---|---|---|
| 6 | 5 | 3 |   | 8 | 4 |   |   |
|   | 5 | 3 | 1 |   |   | 7 | 2 |

Merge of two copies of the pattern $\underline{431 \ 52}$. 
Theorem 10 (Hofer, [5])

Let $X_n = \text{cnt}_\sigma$ be the number of occurrences of a vincular pattern $\sigma$ with $j$ blocks, in a random permutation of size $n$. Then, there exists $c > 0$, such that for all $n$,

$$\text{Var}(X_n) \geq cn^{2j-1}.$$ 

Theorem 11

Theorem 10 is equivalent to

$$\sum_{l=1}^{M_{\sigma}} (2k)_l b_{\sigma}(2k-l, 2j-1) > \binom{2k}{k} \binom{2j-1}{j} j,$$

for any vincular pattern $\sigma$ with $j$ blocks, where $M_{\sigma}$ is the maximal size of a block of $\sigma$. 

Recall that \( \text{adj} \) denotes \( \text{cnt}_{\text{12}}^{21} \).

We use Theorem 7 and the method of moments to prove the following.

**Theorem 12**

\( \text{adj} \) converges in distribution to \( \text{Po}(1) \).

**Sketch of proof:**

Show that \( \lim_{n \to \infty} E(\text{adj}^r) = B_r \), where \( B_r \) is the \( r \)-th Bell number.
Recall that $\text{adj}$ denotes $\text{cnt}_{T_2}$. \[21\]

We use Theorem 7 and the method of moments to prove the following.

**Theorem 12**

$\text{adj}$ converges in distribution to Po(1).

**Sketch of proof:**

Show that $\lim_{n \to \infty} \mathbb{E}(\text{adj}^r) = B_r$, where $B_r$ is the $r$-th Bell number.

Therefore, $\text{cnt}_P$ does not necessarily converge to a Normal distribution, when $P$ is a bivincular pattern.
1) Can we find a combinatorial proof of Lemma 2 and the corresponding fact for vincular patterns?
Further questions

1) Can we find a combinatorial proof of Lemma 2 and the corresponding fact for vincular patterns?

2) Can we adapt the approach of Chern et al. to other combinatorial structures, e.g., trees, polyominoes, etc.?
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Back in 2004, Zeilberger suggested some other structures in the abstract of [8]:

“...This would be hopefully followed by sequels applied to other combinatorial objects like graph-colorings, Boolean functions, and Random Walks...”
Further questions

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Thanks for the attention!
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