KHOVANOV HOMOLOGY AND EMBEDDED GRAPHS

AHMAD ZAINY AL-YASRY

ABSTRACT. We construct a cobordism group for embedded graphs in two different ways, first by using sequences of two basic operations, called “fusion” and “fission”, which in terms of cobordisms correspond to the basic cobordisms obtained by attaching or removing a 1-handle, and the other one by using the concept of a 2-complex surface with boundary is the union of these knots. A discussion given to the question of extending Khovanov homology from links to embedded graphs, by using the Kauffman topological invariant of embedded graphs by associating family of links and knots to a such graph by using some local replacements at each vertex in the graph. This new concept of Khovanov homology of an embedded graph constructed to be the sum of the Khovanov homologies of all the links and knots.

1. INTRODUCTION

The construction of the cobordism group for links and for knots and their relation is given in [6]. We then consider the question of constructing a similar cobordism group for embedded graphs in the 3-sphere. We show that this can actually be done in two different ways, both of which reduce to the same notion for links. The first one comes from the description of the cobordisms for links in terms of sequences of two basic operations, called “fusion” and “fission”, which in terms of cobordisms correspond to the basic cobordisms obtained by attaching or removing a 1-handle. We define analogous operations of fusion and fission for embedded graphs and we introduce an equivalence relation of cobordism by iterated application of these two operations.

The second possible definition of cobordism of embedded graphs is a surface (meaning here 2-complex) in $S^3 \times [0, 1]$ with boundary the union of the given graphs. While for links, where cobordisms are realized by smooth surfaces, these can always be decomposed into a sequence of handle attachments, hence into a sequence of fusions and fissions, in the case of graphs not all cobordisms realized by 2-complexes can be decomposed as fusions and fissions, hence the two notions are no longer equivalent. The idea of categorification the Jones polynomial is known by Khovanov Homology for links which is a new link invariant introduced by Khovanov [9],[1]. For each link L in $S^3$ he defined a graded chain complex, with grading preserving differentials, whose graded Euler characteristic is equal to the Jones polynomial of the link L. The idea of Khovanov Homology for graphs arises from the same idea of Khovanov homology for links by the categorifications the chromatic polynomial of graphs. This was done by L. Helme-Guizon and Y. Rong [5], for each graph G, they defined a graded chain complex whose graded Euler characteristic is equal to the chromatic polynomial of G. In our work we try to recall, the Khovanov homology for links and graphs.

We discuss the question of extending Khovanov homology from links to embedded graphs. This is based on a result of Kauffman that constructs a topological invariant of embedded graphs in the 3-sphere by associating to such a graph a family of links and knots obtained using some local replacements at each vertex in the graph. He showed that it is a topological invariant by showing that

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the resulting knot and link types in the family thus constructed are invariant under a set of Reidemeister moves for embedded graphs that determine the ambient isotopy class of the embedded graphs. We build on this idea and simply define the Khovanov homology of an embedded graph to be the sum of the Khovanov homologies of all the links and knots in the Kauffman invariant associated to this graph. Since this family of links and knots is a topologically invariant, so is the Khovanov homology of embedded graphs defined in this manner. We close this paper by giving an example of computation of Khovanov homology for an embedded graph using this definition.

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2. Knots and Links Cobordism Groups

A notion of knot cobordism group and link cobordism group can be given by using cobordism classes of knots and links to form a group [2], [6]. The link cobordism group splits into the direct sum of the knot cobordism group and an infinite cyclic group which represents the linking number, which is invariant under link cobordism [6]. In this part we will give a survey about both knot and link cobordism groups. In a later part of this work we will show that the same idea can be adapted to construct a graph cobordism group.

2.1. Knot cobordism group. We recall the concept of cobordism between knots introduced in [2]. Two knots $K_1$ and $K_2$ are called knot cobordic if there is a locally flat cylinder $S$ in $S^3 \times [0, 1]$ with boundary $\partial S = K_1 \cup -K_2$ where $K_1 \subset S^3 \times \{0\}$ and $K_2 \subset S^3 \times \{1\}$. We then write $K_1 \sim K_2$. The critical points in the cylinder are assumed be minima (birth), maxima (death), and saddle points. In the birth point at some $t_0$ there is a sudden appearance of a point. The point becomes an unknotted circle in the level immediately above $t_0$. At a maxima or death point, a circle collapses to a point and disappearance from higher levels.

\[ \text{Figure 1. Death and Birth Points} \]

For the saddle point, two curves touch and rejoin as illustrated in figure (2).

These saddle points are of two types: negative if with increasing $t$ the number of the cross sections decreases and positive if the number increases.

A transformation from one cross section to another is called negative hyperbolic transformation if
there is only one saddle point between the two cross sections and if the number of components decreases. We can define analogously a positive hyperbolic transformation.

**Definition 2.1.** \[6\] We say that two knots $K_1$ and $K_2$ are related by an elementary cobordism if the knot $K_2$ is obtained by $r - 1$ negative hyperbolic transformations from a split link consisting of $K_1$ together with $r - 1$ circles.

What we mean by split link is a link with $n$ components $(K_i, i = 1, ..., n)$ in $S^3$ such that there are mutually disjoint $n$ 3-cells $(D_i, i = 1, ..., n)$ containing $K_i, i = 1, 2, ..., n$

**Lemma 2.2.** \[6\] Two knots are called knot cobordic if and only if they are related by a sequence of elementary cobordisms

It is well known that the oriented knots form a commutative semigroup under the operation of composition $\#$. Given two knots $K_1$ and $K_2$, we can obtain a new knot by removing a small arc from each knot and then connecting the four endpoints by two new arcs. The resulting knot is called the composition of the two knots $K_1$ and $K_2$ and is denoted by $K_1 \# K_2$.

Notice that, if we take the composition of a knot $K$ with the unknot $\bigcirc$ then the result is again $K$.

**Lemma 2.3.** The set of oriented knots with the connecting operation $\#$ forms a semigroup with identity $\bigcirc$.

Fox and Milnor \[2\] showed that composition of knots induces a composition on knot cobordism classes $[K] \# [K']$. This gives an abelian group $G_K$ with $[\bigcirc]$ as identity and the negative is $-[K] = [-K]$, where the $-K$ denotes the reflected inverse of $K$.

**Theorem 2.4.** The knot cobordism classes with the connected sum operation $\#$ form an abelian group, called the knot cobordism group and denoted by $G_K$.

**2.2. Link cobordism group.** \[6\] For links, the conjunction operation $\&$ between two links gives a commutative semigroup. $L_1 \& L_2$ is a link represented by the union of the two links $l_1 \cup l_2$ where $l_1$ represents $L_1$ and $l_2$ represents $L_2$ with mutually disjoint 3-cells $D_1$ and $D_2$ contain $l_1$ and $l_2$ respectively. Here “represents” means that we are working with ambient isotopy classes $L_i$ of links (also called link types) and the $l_i$ are chosen representatives of these classes. In the following we loosely refer to the classes $L_i$ also simply as links, with the ambient isotopy equivalence implicitly understood. The zero of this semigroup is the link consisting of just the empty link. The link cobordism group is constructed using the conjunction operation and the cobordism classes. We recall below the definition of cobordism of links.

Let $L$ be a link in $S^3$ containing $r$-components $k_1, ..., k_r$ with a split sublink $L_s = k_1 \cup k_2 \cup ... \cup k_t, t \leq r$.
of $L$. Define a knot $\hat{K}$ to be $k_1 + k_2 + \ldots + k_t + \partial B_{r+1} + \partial B_{r+2} + \ldots + \partial B_r$ where $\{B_{r+1}, B_{r+2}, B_{r+3}, \ldots, B_r\}$ are disjoint bands in $S^3$ spanning $L_s$ [6]. The operation $+$ means additions in the homology sense. Put $L_1 = L_s \cup k_1 \cup k_2 \cup \ldots \cup k_r$ and $L_2 = \hat{K} \cup k_1 \cup k_2 \cup \ldots \cup k_r$. Now, the operation of replacing $L_1$ by $L_2$ is called fusion and $L_2$ by $L_1$ is called fission.

**Figure 3.** Band

**Definition 2.5.** [6] Two links will be called link cobordic if one can be obtained from the other by a sequence of fusions and fissions. This equivalence relation is denoted by $\simeq$. $[L]$ denotes the link cobordism class of $L$.

**Theorem 2.6.** [6] The link cobordism classes with the conjunction operation form an abelian group, called the link cobordism group and denoted by $G_L$.

**Proof.** For two cobordism classes $[L_1]$ and $[L_2]$ the multiplication between them is well defined and given by

$$[L_1] \& [L_2] = [L_1 \& L_2]$$

The zero of this operation is the class $[\bigcirc]$ which is the trivial link of a countable number of components. The negative of $[L]$ is $-[L] = [-L]$, where $-L$ denoted the reflected inverse of $L$. \hfill \square

**Lemma 2.7.** For any link $L$, a conjunction $L \& -L$ is link cobordic to zero.

To study the relation between the knot cobordism group $G_K$ and link cobordism group $G_L$ define a natural mapping $f : G_K \rightarrow G_L$ which assigns to each knot cobordism class $[k]$ the corresponding link cobordism class $[L]$ where $L$ is the knot $k$ viewed as a one-component link. We claim that $f$ is a homomorphism. $f$ is well defined from the following lemma

**Lemma 2.8.** [6] Two knots are link cobordic if and only if they are knot cobordic.
Now, $K_1 \# K_2$ is a fusion of $K_1 \& K_2$ then $K_1 \# K_2$ is cobordic to $K_1 \& K_2$, therefore $f$ is a homomorphism. Again by using the lemma 2.8 if a knot is link cobordic to zero then it is also knot cobordic to zero, and hence $\ker(f)$ consists of just $0$.

**Lemma 2.9.** $f$ is an isomorphism of $G_K$ onto a subgroup of $G_L$.

**Theorem 2.10.** $[G_K]$ is a direct summand of $G_L$ and $G_L$ is a subgroup of $G_L$ whose elements have total linking number zero. The other summand is isomorphic to the additive group of integers.

3. Graphs and Cobordisms

3.1. **Graph cobordism group.** In this section we construct cobordism groups for embedded graphs by extending the notions of cobordisms used in the case of links.

**Definition 3.1.** Two graphs $E_1$ and $E_2$ are called cobordic if there is a surface $S$ have the boundary $\partial S = E_1 \cup -E_2$ with $E_1 = S \cap (S^3 \times \{0\})$, $E_2 = S \cap (S^3 \times \{1\})$ and we set $E_1 \sim E_2$. Here by ”surfaces” we mean 2-dimensional simplicial complexes that are PL-embedded in $S^3 \times [0,1]$. $[E]$ denotes the cobordism class of the graph $E$.

By using the graph cobordism classes and the conjunction operation $\&$, we can induce a graph cobordism group. $E_1 \& E_2$ is a graph represented by the union of the two graphs $E_1 \cup E_2$ with mutually disjoint 3-cells $D_1$ and $D_2$ containing (representatives of) $E_1$ and $E_2$, respectively. Here again we do not distinguish in the notation between the ambient isotopy classes of embedded graphs (graph types) and a choice of representatives.

**Lemma 3.2.** The graph cobordism classes in the sense of Definition 3.1 with the conjunction operation $\&$ form an abelian group called the graph cobordism group and denoted by $G_E$.

**Proof.** For two cobordism classes $[E_1]$ and $[E_2]$ the operation between them is given by

$$[E_1] \& [E_2] = [E_1 \& E_2].$$

This operation is well defined. To show that : Suppose $E_1 \sim F_1$, for two graphs $E_1$ and $F_1$. Then there exists a surface $S_1$ with boundary $\partial S_1 = E_1 \cup -F_1$. Suppose also, $E_2 \sim F_2$, for two graphs $E_2$ and $F_2$. Then there exists a surface $S_2$ with boundary $\partial S_2 = E_2 \cup -F_2$. We want to show that $E_1 \& E_2 \sim F_1 \& F_2$, i.e. we want to find a surface $S$ with boundary $\partial S = (E_1 \& E_2) \cup -(F_1 \& F_2)$.

Define the cobordism $S$ to be $S_1 \cup S_2$ where $S_1 \cup S_2$ represents $S_1 \cup S_2$ with mutually disjoint 4-cells $D_1 \times [0,1]$ and $D_2 \times [0,1]$, containing $S_1$ and $S_2$ respectively with $D_1 \times \{0\}$ containing $E_1$, $D_2 \times \{0\}$ containing $E_2$, and $D_2 \times \{1\}$ containing $F_2$. The boundary of $S$ is given by

$$\partial S = \partial (S_1 \cup S_2) = \partial S_1 \cup \partial S_2 = \partial S_1 \cup \partial S_2 = (E_1 \cup -F_1) \cup (E_2 \cup -F_2) = (E_1 \& E_2) \cup -(F_1 \& F_2)$$

Then the operation is well defined. The zero of this operation is the class $[\emptyset]$, which is the trivial graph of a countable number of components. The negative of $[E]$ is $-[E] = [-E]$, where $-E$ denotes the reflected inverse of $E$. \qed

3.2. **Fusion and fission for embedded graphs.** We now describe a special kind of cobordisms between embedded graphs, namely the basic cobordisms that correspond to attaching a 1-handle and that give rise to the analog in the context of graphs of the operations of fusion and fission described already in the case of links. Let $E$ be a graph containing $n$-components with a split subgraph $E_s = G_1 \cup G_2 \cup G_3 \ldots \cup G_n$. We can define a new graph $\hat{E}$ to be $G_1 + G_2 + G_3 + \ldots + G_t + \partial B_{t+1} + \partial B_{t+2} + \ldots + \partial B_n$ where $\{B_{t+1}, B_{t+2}, B_{t+3}, \ldots, B_n\}$ are disjoint bands in $S^3$ spanning $E_s$. The operation $+$ means addition in the homology sense. Put $E_1 = E_s \cup G_{t+1} \cup G_{t+2} \cup \ldots \cup G_n$ and $E_2 = \hat{E} + G_{t+1} + G_{t+2} + \ldots + G_n$. Now, The operation of replacing $E_1$ by $E_2$ is called fusion and $E_2$ by $E_1$ is called fission.
Notice that, in order to make sure that all resulting graphs will still have at least one vertex, one needs to assume that the 1-handle is attached in such a way that there is at least an intermediate vertex in between the two segments where the 1-handle is attached, as the figure (4) above.

**Remark 3.3.** Unlike the case of links, a fusion and fission for graphs does not necessarily change the number of components. For example see the figure (5) below.

We can use the operations of fusion and fission described above to give another possible definition of cobordism of embedded graphs.

**Definition 3.4.** Two graphs will be called graph cobordic if one can be obtained from the other by a sequence of fusions and fissions. We denote this equivalence relation by \( \simeq \), and by \([E]\) the graph cobordism class of \(E\).

Thus we have two corresponding definitions for the graph cobordism group. One can check from the definition of fusion and fission that they gives the existence of a cobordism (surface) between two graphs \(E_1\) and \(E_2\).

**Lemma 3.5.** Two graphs \(E_1\) and \(E_2\) that are cobordant in the sense of Definition 3.4 are also cobordant in the sense of Definition 3.7. The converse, however, is not necessary true.

**Proof.** As we have seen, a fusion/fission operation is equivalent to adding or removing a band to a graph and this implies the existence of a saddle cobordism given by the attached 1-handle, as illustrated in figure (2). By combining this cobordism with the identity cobordism in the region...
outside where the 1-handle is attached, one obtains a PL-cobordism between \( E_1 \) and \( E_2 \). This shows that cobordism in the sense of Definition 3.4 implies cobordism in the sense of Definition 3.1. The reason why the converse need not be true is that, unlike what happens with the cobordisms given by embedded smooth surfaces used in the case of links, the cobordisms of graphs given by PL-embedded 2-complexes are not always decomposable as a finite set of fundamental saddle cobordism given by a 1-handle. Thus, having a PL-cobordism (surface in the sense of a 2-complex) between two embedded graphs \( E_1 \) and \( E_2 \) does not necessarily imply the existence of a finite sequence of fusions and fissions.

**Lemma 3.6.** The graph cobordism classes in the sense of Definition 3.4 with the conjunction operation form an abelian group called the graph cobordism group and denoted by \( G \).

**Proof.** The proof is the same as the proof on lemma 3.2 since fusion and fission are a special case of cobordisms.

The result of Lemma 3.5 shows that there are different equivalence classes \( [E_1] \neq [E_2] \) in \( G \) that are identified \( [E_1] = [E_2] \) in \( G \). Thus, the number of cobordism classes when using Definition 3.1 is smaller than the number of classes by the fusion/fission method of Definition 3.4.

4. **Khovanov Homology**

In the following we recall a homology theory for knots and links embedded in the 3-sphere. We discuss later how to extend it to the case of embedded graphs.

4.1. **Khovanov Homology for links.** In recent years, many papers have appeared that discuss properties of Khovanov Homology theory, which was introduced in [9]. For each link \( L \in S^3 \), Khovanov constructed a bi-graded chain complex associated with the diagram \( D \) for this link \( L \) and applied homology to get a group \( Kh_{i,j}(L) \), whose Euler characteristic is the normalized Jones polynomial.

\[
\sum_{i,j} (-1)^j q^i \dim(Kh_{i,j}(L)) = J(L)
\]

He also proved that, given two diagrams \( D \) and \( D' \) for the same link, the corresponding chain complexes are chain equivalent, hence, their homology groups are isomorphic. Thus, Khovanov homology is a link invariant.

4.2. **The Link Cube.** Let \( L \) be a link with \( n \) crossings. At any small neighborhood of a crossing we can replace the crossing by a pair of parallel arcs and this operation is called a resolution. There are two types of these resolutions called 0-resolution (Horizontal resolution) and 1-resolution (Vertical resolution) as illustrated in figure (6).

![Figure 6. 0 and 1- resolutions to each crossing](image)

We can construct a \( n \)-dimensional cube by applying the 0 and 1-resolutions \( n \) times to each crossing.
to get $2^n$ pictures called smoothings (which are one dimensional manifolds) $S_\alpha$. Each of these can be indexed by a word $\alpha$ of $n$ zeros and ones, i.e. $\alpha \in \{0,1\}^n$. Let $\xi$ be an edge of the cube between two smoothings $S_{\alpha_1}$ and $S_{\alpha_2}$, where $S_{\alpha_1}$ and $S_{\alpha_2}$ are identical smoothings except for a small neighborhood around the crossing that changes from 0 to 1-resolution. To each edge $\xi$ we can assign a cobordism $\Sigma_\xi$ (orientable surface whose boundary is the union of the circles in the smoothing at either end)

$$\Sigma_\xi : S_{\alpha_1} \longrightarrow S_{\alpha_2}$$

This $\Sigma_\xi$ is a product cobordism except in the neighborhood of the crossing, where it is the obvious saddle cobordism between the 0 and 1-resolutions. Khovanov constructed a complex by applying a 1 + 1-dimensional TQFT (Topological Quantum Field Theory) which is a monoidal functor, by replacing each vertex $S_\alpha$ by a graded vector space $V_\alpha$ and each edge (cobordism) $\Sigma_\xi$ by a linear map $d_\xi$, and we set the group $CKh(D)$ to be the direct sum of the graded vector spaces for all the vertices and the differential on the summand $CKh(D)$ is a sum of the maps $d_\xi$ for all edges $\xi$ such that $\text{Tail}(\xi) = \alpha$ i.e.

$$d^i(v) = \sum_\xi \text{sign}(-1)d_\xi(v) \quad (4.1)$$

where $v \in V_\alpha \subseteq CKh(D)$ and $\text{sign}(-1)$ is chosen such that $d^2 = 0$.

An element of $CKh^{i,j}(D)$ is said to have homological grading $i$ and $q$-grading $j$ where

$$i = |\alpha| - n_\text{−} \quad (4.2)$$

$$j = \text{deg}(v) + i + n_\text{−} + n_\text{+} \quad (4.3)$$

for all $v \in V_\alpha \subseteq CKh^{i,j}(D)$, $|\alpha|$ is the number of 1’s in $\alpha$, and $n_\text{−}, n_\text{+}$ represent the number of negative and positive crossings respectively in the diagram $D$.

4.3. **Properties.** [14], [10] Here we give some properties of Khovanov homology.

**Proposition 4.1.**

1. If $D'$ is a diagram obtained from $D$ by the application of a Reidemeister moves then the complexes $(CKh^{*,*}(D))$ and $(CKh^{*,*}(D'))$ are homotopy equivalent.

2. For an oriented link $L$ with diagram $D$, the graded Euler characteristic satisfies

$$\sum (-1)^i \text{qdim}(CH^{i,*}(L)) = J(L) \quad (4.4)$$

where $J(L)$ is the normalized Jones Polynomials for a link $L$ and

$$\sum (-1)^i \text{qdim}(CH^{i,*}(D)) = \sum (-1)^i \text{qdim}(CH^{i,*}(D))$$

3. Let $L_{\text{odd}}$ and $L_{\text{even}}$ be two links with odd and even number of components then $Kh^{*,\text{even}}(L_{\text{odd}}) = 0$ and $Kh^{*,\text{odd}}(L_{\text{even}}) = 0$

4. For two oriented link diagrams $D$ and $D'$, the chain complex of the disjoint union $D \sqcup D'$ is given by

$$CKh(D \sqcup D') = CKh(D) \otimes CKh(D'). \quad (4.5)$$

5. For two oriented links $L$ and $L'$, the Khovanov homology of the disjoint union $L \sqcup L'$ satisfies

$$Kh(L \sqcup L') = Kh(L) \otimes Kh(L').$$

6. Let $D$ be an oriented link diagram of a link $L$ with mirror image $D^m$ diagram of the mirror link $L^m$. Then the chain complex $CKh(D^m)$ is isomorphic to the dual of $CKh(D)$ and

$$Kh(L) \cong Kh(L^m).$$
4.4. Links Cobordisms. Let $S \subset S^3 \times [0,1]$ be a compact oriented surface with boundary $\partial S = L_0 \cup -L_1$. We assume that the $L_0$ and $L_1$ are links. In this section we want to recall how one constructs a linear map between the homologies of the boundary links by following Khovanov [9]. The first idea is, we can decompose $S$ into elementary subcobordisms $S_t$ for finitely many $t \in [0,1]$ with

$$S_t = S \cap S^3 \times [0,t]$$

and

$$\partial S_t = L_{t-1} \cup -L_t$$

where $L_{t-1}$ and $L_t$ are one dimensional manifolds, not necessary links. Using a small isotopy we can obtain that they are links for some $t \in [0,1]$. Here we assume that $S$ is a smooth embedded surface. A smooth embedded surface $S$ can be represented by a one parameter family $D_t, t \in [0,1]$ of planar diagrams of oriented links $L_t$ for finitely many $t \in [0,1]$ and this representation is called a movie $M$. Between any two consecutive clips of a movie the diagrams will differ by one of the “Reidemeister moves” or “Morse moves”. The Reidemeister moves are the first moves in figure (9) and the Morse moves are given in figure (7).

These two types of moves will be called local moves. This means that between any two consecutive diagrams there is a local move either of Reidemeister or of Morse type. The necessary condition is that the projection diagram $D_0$ in the first clip in $M$ should be the projection of the link $L_0$ and the projection diagram in the $D_1$ in the last clip of the movie $M$ should be the projection of the link $L_1$ (boundary of $S$). Notice that the orientation of $S$ induces an orientation on all intersection links $L_t$. To show that, let $v$ be a tangent vector to $L_t$. Then orient $v$ in the positive direction if $(v,w)$ gives the orientation of $S$ where $w$ is the tangent vector to $S$ in the direction of increasing of $t$. Khovanov...
constructed a chain map between complexes of two consecutive diagrams that changed by a local move, hence a homomorphisms between their homologies. The composition of these chain maps defines a homomorphism between the homology groups of the diagrams of the boundary links.

5. HOMOLOGY THEORIES FOR EMBEDDED GRAPHS

In this part we will present a method to extend Khovanov homology from links to embedded graphs $G \subset S^3$. Our construction is obtained by using Khovanov homology for links, applied to a family of knots and links associated to an embedded graph. This family is obtained by a result of Kauffman [8] as a fundamental topological invariant of embedded graphs obtained by associating to an embedded graph $G$ in three-space a family of knots and links constructed by some operations of cutting graphs at vertices. Before we give this construction, we motivate the problem of extending Khovanov homology to embedded graphs by recalling another known construction of a homology theory, graph homology, which is defined for graphs.

5.1. Graph homology. We recall here the construction and some basic properties of graph homology. As we discuss below, graph homology can be regarded as a categorification of the chromatic polynomial of a graph, in the same way as Khovanov homology gives a categorification of the Jones polynomial of a link. A construction is given for a graded homology theory for graphs whose graded Euler characteristic is the Chromatic Polynomial of the graph [5]. Laure Helm-Guizon and Yongwu Rong used the same technique to get a graded chain complex. Their construction depends on the edges in the vertices of the cube $\{0,1\}^n$ whose elements are connected subgraphs of the graph $G$. In this subsection we recall the construction of Laure Helm-Guizon and Yongwu Rong.

5.1.1. Chromatic Polynomial. Let $G$ be a graph with set of vertices $V(G)$ and set of edges $E(G)$. For a positive integer $t$, let $\{1,2,\ldots,t\}$ be the set of $t$-colors. A coloring of $G$ is an assignment of a $t$-color to each vertex of $G$ such that vertices that are connected by an edge in $G$ always have different colors. Let $P_G(t)$ be the number on $t$-coloring of $G$, i.e., is the number of vertex colorings of $G$ with $t$ colors (in a vertex coloring two vertices are colored differently whenever they are connected by an edge $e$), then $P_G(t)$ satisfies the Deletion-Contraction relation

$$P_G(t) = P_{G-e}(t) + P_{G/e}(t)$$

For an arbitrary edge $e \in E(G)$ we can define $G-e$ to be the graph $G$ with deleted edge $e$, and by $G/e$ the graph obtained by contracting edge $e$ i.e. by identifying the vertices incident to $e$ and deleting $e$. In addition to that $P_{K_n}(t) = t^n$ where $K_n$ is the graph with $n$ vertices and $n$ edges. $P_G(t)$ is called Chromatic Polynomial. Another description can be give to $P_G(t)$, let $s \subset E(G)$, define $G_s$ to be the graph whose vertex set is the same vertex set of $G$ with edge set $s$. Put $k(s)$ the number of connected components of $G_s$. Then we have

$$P_G(t) = \sum_{s \subset E(G)} (-1)^{|s|} t^{k(s)}$$

5.1.2. Constructing $n$-cube for a Graph. First we want to give an introduction to the type of algebra that we will use it in our work later.

Definition 5.1. Let $\mathcal{V} = \bigoplus_i V_i$ be a graded $\mathbb{Z}$-module where $\{V_i\}$ denotes the set of homogenous elements with degree $i$, and the graded dimension of $\mathcal{V}$ is the power series

$$qdim \mathcal{V} = \sum_i q^{i \dim \mathbb{Q}(V_i \otimes \mathbb{Q})}$$

We can define the tensor product and directed sum for the graded $\mathbb{Z}$-module as follows:
Theorem 5.2. [5] Let $\mathcal{V}$ and $\mathcal{W}$ be a graded $\mathbb{Z}$-modules, then $\mathcal{V} \otimes \mathcal{W}$ and $\mathcal{V} \oplus \mathcal{W}$ are both graded $\mathbb{Z}$-module with

1. $\text{qdim}(\mathcal{V} \oplus \mathcal{W}) = \text{qdim}(\mathcal{V}) + \text{qdim}(\mathcal{W})$
2. $\text{qdim}(\mathcal{V} \otimes \mathcal{W}) = \text{qdim}(\mathcal{V}) \cdot \text{qdim}(\mathcal{W})$

Let $G$ be a graph with edge set $E(G)$ and $n = |E(G)|$ represents the cardinality of $E(G)$. We need first to order the edges in $E(G)$ and denote the edges by $\{e_1, e_2, \ldots, e_n\}$. Consider the $n$-dimensional cube $\{0, 1\}^n$ [5], (see the figure (8)).

Each vertex can be indexed by a word $\alpha \in \{0, 1\}^n$. This vertex $\alpha$ corresponded to a subset $s = s_{\alpha}$ of $E(G)$. This is the set of edges of $G$ that are incident to the chosen vertex. Then $e_i \in s_{\alpha}$ if and only if $\alpha_i = 1$. Define $|\alpha| = \sum \alpha_i$ (height of $\alpha$) to be the number of $1$'s in $\alpha$ or equivalently the number of edges in $s_{\alpha}$. We associate to each vertex $\alpha$ in the cube $\{0, 1\}^n$, a graded vector space $V_{\alpha}$ as follows [5]. Let $V_{\alpha}$ be a graded free $\mathbb{Z}$-module with 1 and $x$ basis elements with degree 0 and 1 respectively, then $V_{\alpha} = \mathbb{Z} \oplus \mathbb{Z}x$ with $\text{qdim}(V_{\alpha}) = 1 + q$ and hence, $\text{qdim}(V_{\alpha} \otimes k) = (1 + q)^k$.

Consider $G_{s_\alpha}$, the graph with vertex set $V(G)$ and edge set $s_{\alpha}$. Replace each component of $G_{s_\alpha}$ by a copy of $V_{\alpha}$ and take the tensor product over all components.

Define the graded vector space $\mathcal{V}_{\alpha} = V_{\alpha}^{\otimes k}$ where $k$ is the number of the components of $G_{s_\alpha}$. Set the vector space $\mathcal{V}$ to be the direct sum of the graded vector space for all the vertices. The differential map $d^i$, defined by using the edges of the cube $\{0, 1\}^n$. We can label each edge of $\{0, 1\}^n$ by a sequence of $\{0, 1, *\}^n$ with exactly one *. The tail of the edge labeled by $* = 0$ and the head by $* = 1$. To define the differential we need first to define $\text{Per-edge}$ maps between the vertices of the cube $\{0, 1\}^n$.

These maps is defined to be a linear maps such that every square in the cube $\{0, 1\}^n$ is commutative. Define the $\text{Per-edge}$ map $d_\xi : \mathcal{V}_{\alpha_1} \longrightarrow \mathcal{V}_{\alpha_2}$ for the edge $\xi$ with tail $\alpha_1$ and head $\alpha_2$ as follows: Take $\mathcal{V}_{\alpha_i} = V_{\alpha_i}^{\otimes k_i}$ for $i = 1, 2$ with $k_i$ is the number of the connected components of $G_{s_{\alpha_i}}$. Let $e$ be the edge and $s_{\alpha_2} = s_{\alpha_1} \cup \{e\}$, then there are two possible cases. First one (easy case): $d_\xi$ will be the identity map if the edge $e$ joins a component $r$ of $G_{s_{\alpha_1}}$ to itself. Then $k_1 = k_2$ with a natural correspondence between
the components of $G_{s_1}$ and $G_{s_2}$. Second one: if $e$ joins two different components of $G_{s_1}$, say $r_1$ and $r_2$, then $k_2 = k_1 - 1$ and the components of $G_{s_2}$ are $r_1 \cup r_2 \cup \{e\} \cup \ldots \cup r_{k_1}$. Define $d_2$ to be the identity map on the tensor factor coming from $r_3, r_4, \ldots, r_{k_1}$. Also define $d_2$ on the remaining tensor factor to be the multiplication map $V_{\alpha} \otimes V_{\alpha} \longrightarrow V_{\alpha}$ sending $x \otimes y$ to $xy$. The differential $d^i : V^i \longrightarrow V^{i+1}$ is given by

$$d^i = \sum_{|\xi| = i} \text{sign}(\xi) d_\xi$$

Where sign$(\xi)$ is chosen so that $d^2 = 0$.

Theorem 5.3. \cite{14, 5} The following properties hold for graph homology.
- The graded Euler characteristic for the graph homology given by

$$\sum_{i,j} (-1)^i q^i \dim(Kh^{i,j}(G)) = P^G(t)$$

where $P^G(t)$ is the chromatic polynomial.
- In graph homology a short exact sequence

$$0 \longrightarrow CKh^{i-1,j}(G/e) \longrightarrow CKh^{i,j}(G) \longrightarrow CKh^{i,j}(G - e) \longrightarrow 0$$

can be constructed by using the deletion-contraction relation for a given edge $e \in G$. This gives a long exact sequence

$$\cdots \longrightarrow Kh^{i-1,j}(G/e) \longrightarrow Kh^{i,j}(G) \longrightarrow Kh^{i,j}(G - e) \longrightarrow \cdots$$

5.2. Kauffman’s invariant of Graphs. We give now a survey of the Kauffman theory and show how to associate to an embedded graph in $\mathbb{S}^3$ a family of knots and links. We then use these results to give our definition of Khovanov homology for embedded graphs. In \cite{8} Kauffman introduced a method for producing topological invariants of graphs embedded in $\mathbb{S}^3$. The idea is to associate a collection of knots and links to a graph $G$ so that this family is an invariant under the expanded Reidemeister moves defined by Kauffman and reported here in figure (9).

He defined in his work an ambient isotopy for non-rigid (topological) vertices. (Physically, the rigid vertex concept corresponds to a network of rigid disks each with (four) flexible tubes or strings emanating from it.) Kauffman proved that piecewise linear ambient isotopies of embedded graphs in $\mathbb{S}^3$ correspond to a sequence of generalized Reidemeister moves for planar diagrams of the embedded graphs.

Theorem 5.4. \cite{8} Piecewise linear (PL) ambient isotopy of embedded graphs is generated by the moves of figure (9), that is, if two embedded graphs are ambient isotopic, then any two diagrams of them are related by a finite sequence of the moves of figure (9).

Let $G$ be a graph embedded in $\mathbb{S}^3$. The procedure described by Kauffman of how to associate to $G$ a family of knots and links prescribes that we should make a local replacement as in figure (10) to each vertex in $G$. Such a replacement at a vertex $v$ connects two edges and isolates all other edges at that vertex, leaving them as free ends. Let $r(G, v)$ denote the link formed by the closed curves formed by this process at a vertex $v$. One retains the link $r(G, v)$, while eliminating all the remaining unknotted arcs. Define then $T(G)$ to be the family of the links $r(G, v)$ for all possible replacement choices,

$$T(G) = \bigcup_{v \in V(G)} r(G, v).$$

For example see figure (11).
FIGURE 9. Generalized Reidemeister moves by Kauffman

FIGURE 10. Local replacement to a vertex in the graph G
Theorem 5.5. Let $G$ be any graph embedded in $S^3$, and presented diagrammatically. Then the family of knots and links $T(G)$, taken up to ambient isotopy, is a topological invariant of $G$.

For example, in the figure the graph $G_2$ is not ambient isotopic to the graph $G_1$, since $T(G_2)$ contains a non-trivial link.

5.3. Definition of Khovanov homology for embedded graphs. Now we are ready to speak about a new concept of Khovanov homology for embedded graphs by using Khovanov homology for the links (knots) and Kauffman theory of associate a family of links to an embedded graph $G$, as described above.

Definition 5.6. Let $G$ be an embedded graph with $T(G) = \{L_1, L_2, \ldots, L_n\}$ the family of links associated to $G$ by the Kauffman procedure. Let $Kh(L_i)$ be the usual Khovanov homology of the link $L_i$ in this family. Then the Khovanov homology for the embedded graph $G$ is given by

$$Kh(G) = Kh(L_1) \oplus Kh(L_2) \oplus \ldots \oplus Kh(L_n)$$

Its graded Euler characteristic is the sum of the graded Euler characteristics of the Khovanov homology of each link, i.e. the sum of the Jones polynomials,

$$\sum_{i,j,k} (-1)^j q^i \dim(Kh^{i,j}(L_k)) = \sum_k J(L_k). \tag{5.1}$$

We show some simple explicit examples.

Example 5.7. In figure $T(G_1) = \{\bigcirc\bigcirc, \bigcirc\} \text{ then for } Kh(\bigcirc) = \mathbb{Q}$

$$Kh(G_1) = Kh(\bigcirc\bigcirc) \oplus Kh(\bigcirc)$$

Now, from proposition no.5

$$Kh(G_1) = Kh(\bigcirc) \otimes Kh(\bigcirc)$$

Another example comes from $T(G_2) = \{\bigcirc\bigcirc, \bigcirc\} \text{ then}$

$$Kh(G_2) = Kh(\bigcirc\bigcirc) \oplus Kh(\bigcirc)$$

Since $Kh^{0,0}(\bigcirc) = \mathbb{Q}$, and from [14]
\[ Kh(\bigotimes) = \] 
\[
\begin{array}{cccc}
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & -2 & -1 & 0 \\
  -1 & & & \\
  -2 & & & \\
  -3 & & & \\
  -4 & \mathbb{Q} & & \\
  -5 & \mathbb{Q} & & \\
  -6 & \mathbb{Q} & & \\
\end{array}
\]

Then,

\[ Kh(G_2) = \] 
\[
\begin{array}{cccc}
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & -2 & -1 & 0 \\
  -1 & & & \mathbb{Q} \oplus \mathbb{Q} \\
  -2 & & & \mathbb{Q} \\
  -3 & & & \\
  -4 & \mathbb{Q} & & \\
  -5 & \mathbb{Q} & & \\
  -6 & \mathbb{Q} & & \\
\end{array}
\]

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