A statistical mechanical approach to restricted integer partition functions

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Abstract. The main aim of this paper is twofold: (1) suggesting a statistical mechanical approach to the calculation of the generating function of restricted integer partition functions which count the number of partitions—a way of writing an integer as a sum of other integers under certain restrictions. In this approach, the generating function of restricted integer partition functions is constructed from the canonical partition functions of various quantum gases. (2) Introducing a new type of restricted integer partition functions corresponding to general statistics which is a generalization of Gentile statistics in statistical mechanics; many kinds of restricted integer partition functions are special cases of this restricted integer partition function. Moreover, with statistical mechanics as a bridge, we reveal a mathematical fact: the generating function of restricted integer partition function is just the symmetric function which is a class of functions being invariant under the action of permutation groups. Using this approach, we provide some expressions of restricted integer partition functions as examples.

Keywords: quantum gases, anyons and fractional statistical models, rigorous results in statistical mechanics
A statistical mechanical approach to restricted integer partition functions

Contents

1. Introduction .................................................. 3
   1.1. The integer partition function ......................... 3
   1.2. The number of microstates in statistical mechanics  3

2. The integer partition function and the symmetric function:
   a brief review .............................................. 5
   2.1. Partitions .................................................. 5
   2.2. The unrestricted integer partition function .......... 6
   2.3. Restricted integer partition functions ............... 6
   2.4. Symmetric functions and $S$-functions ............... 6

3. The generating function of restricted integer partition functions
   counting the integer partitions with length $N$ .......... 7
   3.1. The generating function of $P^{(e)}(E; N)$ ........... 7
   3.2. The generating function of $Q^{(e)}(E; N)$ .......... 8
   3.3. The generating function of $P^{(q)}(E; N)$ .......... 8

4. The restricted integer partition function corresponding
   to general statistics ..................................... 10
   4.1. The restricted integer partition function $P^{(e)}_{(q)}(E; N)$
       and general statistics ................................ 10
   4.2. The two-variable generating function of $P^{(e)}_{(q)}(E; N)$ 10
   4.3. Restricted integer partition functions as special cases of $P^{(e)}_{(q)}(E; N)$ .... 11
       4.3.1. Restricted integer partition functions counting partitions with length $l(\lambda) = N$ .... 11
       4.3.2. Restricted integer partition functions counting partitions with length $l(\lambda) \leq N$ .... 12
       4.3.3. Restricted integer partition functions counting partitions without restrictions on the length $l(\lambda)$ ........ 13
       4.3.4. A relation between the $S$-function and the Gauss polynomial: the restricted integer partition function $p^{(1,2,3,\ldots,n)}(E; N)$ ........ 14
       4.3.5. The generating function of restricted integer partition functions $P^{(0,1,2,\ldots)}(E; N), Q^{(0,1,2,\ldots)}(E; N), P^{(1^2,2^2,\ldots)}(E; N), \text{and } Q^{(1^2,2^2,\ldots)}(E; N)$ and quantum gases 15

5. Calculating $P^{(e)}(E; N), \text{ and } P^{(e)}_{(q)}(E; N)$ from the
   generating functions: examples .......................... 16
   5.1. Expressions of $P^{(e)}(E; N), Q^{(e)}(E; N),\text{ and } P^{(e)}_{(q)}(E; N)$ for $N = 2, 3, 4,$
       and 5 ...................................................... 16

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1. Introduction

The problem of integer partition functions is important in both statistical mechanics and mathematics [1, 2]. The restricted integer partition function counts the number of partitions which are ways of writing an integer as a sum of other integers under certain restrictions.

1.1. The integer partition function

In mathematics, a partition of an integer \( E \) is a way of writing \( E \) as a sum of other integers [1]. The integer partition function of the integer \( E \) counts the number of partitions of \( E \). There are two kinds of integer partition functions: the unrestricted integer partition function \( P(E) \) and the restricted integer partition function \( P(E|\text{restrictions}) \). The unrestricted integer partition function \( P(E) \) counts the number of all possible partitions [3] and the restricted integer partition function \( P(E|\text{restrictions}) \) counts the number of partitions under certain restrictions [1].

1.2. The number of microstates in statistical mechanics

In statistical mechanics, the macrostate of a system can be specified completely in terms of state variables such as the total energy \( E \) and the total particle number \( N \), and a microstate is a way in which a macrostate can be realized [4, 5]. The number of microstates, or, the state density, \( \Omega(E,N) \) counts the number of microstates of a given macrostate specified by the total energy \( E \) and the total particle number \( N \) [4, 5]. For a non-interacting system, e.g. an ideal quantum gas, the microstate is a way in which the total energy \( E \) is distributed among the \( N \) particles. That is, the microstate is a representation of \( E \) in terms of the sum of \( N \) single-particle energies \( \varepsilon_i \), where \( \varepsilon_i \) is the eigenvalue of the single-particle state [4]. Therefore, the number of microstates of an ideal quantum gas \( \Omega(E,N) \) counts the ways of representing \( E \) as a sum of \( N \) single-particle energies \( \varepsilon_i \). In statistical mechanics, various kinds of quantum statistics are distinguished by the maximum occupation number. The maximum occupation number is the maximum number that particles are allowed to occupy a single-particle state. For Bose–Einstein statistics, there is no restriction on the maximum occupation number, but for Fermi–Dirac statistics, the maximum occupation number is 1. Gentile statistics is a generalization of Bose–Einstein and Fermi–Dirac statistics, whose maximum...
An occupation number is an arbitrary integer \( q \) [6–12]. General statistics is a generalization of Gentile statistics, whose maximum occupation number of different quantum states takes on different values [13]. In a word, the number of microstates \( \Omega(E, N) \) of an ideal quantum gas is the number of representations of \( E \) in terms of the sum of \( N \) single-particle energies \( \varepsilon_i \) with a constraint that each \( \varepsilon_i \) repeats no more than a given time, such as 1 for Fermi gases, \( q \) for Gentile gases, \( \infty \) for Bose gases, and so on.

Comparing the restricted integer partition function in mathematics and the number of microstates in statistical mechanics, we can see that the number of microstates of an ideal quantum gas is closely related to the restricted integer partition function that counts the partition under the following restrictions: (1) the number of elements (summands) is \( N \), (2) the element (summands) belongs to a set \( \{ \varepsilon_1, \varepsilon_2, \ldots \} \), and (3) the element (summand) repeats no more than a given time.

In this paper, first, by resorting to the canonical partition function of quantum ideal gases in statistical mechanics [14], we construct the generating function of restricted integer partition functions corresponding to ideal Bose, Fermi, and Gentile gases, respectively. The result shows that the generating functions for these restricted integer partition functions are symmetric functions which are invariant under the action of the permutation group and can be represented as linear combinations of the \( S \)-function which is an important class of symmetric functions [15, 16]. We also calculate the exact expression of the restricted integer partition function from the generating function as examples. Second, based on general statistics which is a generalization of Gentile statistics [13], we introduce a new type of restricted integer partition functions and show that a number of restricted integer partition functions are special cases of such kind of restricted integer partition functions.

The relation between restricted integer partition functions and statistical mechanics has been discussed in [17–27]. The restricted integer partition function that counts partitions with elements belonging to nature numbers corresponds to the number of microstates, or, the state density, of the system consisting of linear simple-harmonic oscillators in statistical mechanics [18–21], the restricted integer partition function that counts partitions with distinct elements corresponds to ideal Fermi gases, the restricted integer partition function that count partitions with elements repeating no more than \( \infty \) times corresponds to ideal Bose gases [18–20], and the restricted integer partition function that counts partitions with elements repeating no more than \( q \) times corresponds to ideal Gentile gases with maximum number \( q \) [21], etc. The results given by statistical mechanics are used to solve the integer partition function problems and vise versa. For example, Bohr and Kalckar use the integer partition function to calculate the density of energy levels in heavy nuclei [17], fluctuations in one- and three-dimensional traps are discussed by resorting to the theory of restricted integer partition functions [28], the problem of integer partition functions is addressed using the microcanonical approach in statistical mechanics [23], the quantum statistical approach is used to estimate the expression of some restricted integer partition functions [18, 19, 21] and to estimate the number of restricted plane partitions [24], etc.

Some authors discuss the relation between the thermodynamics quantity of ideal systems and the symmetric functions, which are both invariant under permutations. For example, the canonical partition function for a parastatistical system can
be expressed as sums of $S$-functions [29], the partition function of an ideal gas can be represented in terms of symmetric functions, such as the elementary and the complete symmetric polynomial [30], the partition function of the six vertex model is equal to a factorial Schur function which is a generalization of $S$-functions [31], and there is a close correspondence between the partition function of ideal quantum gases and certain symmetric polynomials [32, 33]. Our previous work shows that the canonical partition function for quantum ideal gases, such as ideal Bose, Fermi, and Gentile gases, can be represented as linear combinations of the $S$-function [14]. In [34, 35], the author introduces the symmetric function and the plane partition function. There are also discussions on restricted integer partition functions [1, 2, 36–39].

In this paper, based on the exact canonical partition function in statistical mechanics, we construct the generating function for a series of restricted integer partition functions. The result reveals a relation between the restricted integer partition function and the symmetric function.

Especially, in this paper we introduce a type of restricted integer partition functions based on general statistics in statistical mechanics, which is a generalization of Bose–Einstein, Fermi–Dirac, and Gentile statistics. Many restricted integer partition functions are special cases of this kind of restricted integer partition functions. The restricted integer partition function introduced in this paper enables us to consider a number of restricted integer partition functions in a unified framework.

In section 2, a brief review of the concept of the integer partition function and the symmetric function is given. In section 3, the exact generating function of restricted integer partition functions corresponding to ideal Bose, Fermi, and Gentile gases is given. In section 4, we introduce a kind of restricted integer partition functions corresponding to general statistics in statistical mechanics. In section 5, we calculate some expressions of restricted integer partition functions from the generating functions as examples. Conclusions are given in section 6. In the appendix, some expressions of generating functions are given.

2. The integer partition function and the symmetric function: a brief review

In this section, we give a brief review on the mathematical concept of the integer partition function and the symmetric function.

2.1. Partitions

The partition of an integer $E$, denoted by $(\lambda)$, is a representation of $E$ in terms of other positive integers which sum up to $E$. For example, the partitions for 3 are $(\lambda) = (3)$, $(\lambda)^' = (2, 1)$, and $(\lambda)^'' = (1^3)$, where $1^3$ means 1 is appearing 3 times. The summand in the partition is called the element (or the part), denoted by $\lambda_i$. The number of elements is called the length, denoted by $l(\lambda)$. The number of elements is called the length, denoted by $l(\lambda)$. For example, the elements of the partition $(\lambda) = (2, 1)$ are $\lambda_1 = 2$ and $\lambda_2 = 1$, and the length is $l(2,1) = 2$. In a partition $(\lambda)$, the elements are always arranged in descending order: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$. 

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2.2. The unrestricted integer partition function

The number of all partitions is called the unrestricted integer partition function, denoted by \( P(E) \), e.g. \( P(3) = 3 \). A famous expression of \( P(E) \) is given by Ramanujan [3]. The generating function of \( P(E) \), \( \sum_E P(E) z^E = \prod_i 1/(1 - z^i) \), is given in [40, caput XVI].

2.3. Restricted integer partition functions

If one counts the partitions under certain restrictions, then the number of partitions is called the restricted integer partition function, denoted by \( P(E| \text{restrictions}) \) [1]. For example, \( P(3| \text{elements are even}) = 0 \) since no partitions meet the restriction and \( P(3| l(\lambda) = 2) = 1 \) since the corresponding partition is \( (\lambda) = (2,1) \).

Many mathematicians, such as Gupta [41], Erdős [42], and Andrews [1, 2] studied the problem of integer partitions. Many studies devote to the restricted integer partition function. For example, the generating function of \( P(E) \) elements belong to \( \{1, 2, 3 \ldots, n\} \) and \( l(\lambda) \leq N \) is the Gauss polynomial [1, 36]. A recursive relation of \( P(E) \) elements belong to \( \{1, 2, 3 \ldots, n\} \) is considered in [1]. \( P(E| l(\lambda) = N) \) and \( P(E| l(\lambda) = N, \text{elements repeat no more than 1 time}) \) are considered in [2, 37, 38]. An approximate expression of \( P(E) \) elements belong to \( \{a_1, a_2, \ldots, a_k\} \) is given in [39].

2.4. Symmetric functions and \( S \)-functions

The symmetric function \( f(x_1, x_2, \ldots, x_n) \) is invariant under the action of the permutation group \( S_n \); that is, for \( \sigma \in S_n \), \( \sigma f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n) \). The \( S \)-function \( \lambda(x_1, x_2, \ldots) \), also known as the Schur function, is an important kind of the symmetric function, because it forms the base of the symmetric function space, i.e. a symmetric function can be expanded in a linear combination of the \( S \)-function. The \( S \)-function is closely related to the representation theory of permutation group and the unitary group [15, 16]. There are also some other symmetric functions, e.g. the elementary symmetric polynomial, the complete symmetric polynomial, etc [16].

The \( S \)-function is defined as [15, 16]

\[
(\lambda)(x_1, x_2, \ldots) = \sum_{\lambda'} \frac{g^{(\lambda)}(\lambda')}{N!} \chi_{(\lambda')}(1) \prod_{m=1}^{k} \left( \sum_{i} x_i^m \right)^{a_{(\lambda'),m}},
\]

(2.1)

where \( N \) is the sum of elements in \( \lambda \), \( \sum_{(\lambda)'} \) indicates summation over all partitions \( (\lambda)' \) of \( N \), \( g_{(\lambda)} \) is defined as \( g_{(\lambda)} = N! \left( \prod_{j=1}^{N} j^{a_{(\lambda),j}} \right) \) with \( a_{(\lambda),m} \) counting the times of the number \( m \) appeared in \( \lambda \), and \( \chi_{(\lambda)'} \) is the simple characteristic of the permutation group of order \( N \) [43].
3. The generating function of restricted integer partition functions counting the integer partitions with length $N$

The generating function for a restricted integer partition function $P(E|\text{restrictions})$ is defined as [1, 41]

$$Z(z) = \sum_{E=0}^{\infty} P(E|\text{restrictions}) z^E.$$  

(3.1)

In this section, we construct the generating functions for three restricted integer partition functions: the restricted integer partition functions corresponding to ideal Bose, Fermi, and Gentile gases. These three restricted integer partition functions are important [17–26]. The generating function is an effective way to calculate the partition function, especially in some cases that the generating function can be obtained relatively easy. Our starting point is the canonical partition function of these three quantum gases in statistical mechanics [14].

For a set $\{\varepsilon\} = \{\varepsilon_1, \varepsilon_2, \ldots\}$ consisting of integers with $0 < \varepsilon_1 < \varepsilon_2 < \cdots$, we consider the following three restricted integer partition functions:

$$P^{(\varepsilon)}(E; N) \equiv P\{E|\text{elements belong to } \{\varepsilon\} \text{ and } l(\lambda) = N\},$$

(3.2)

$$Q^{(\varepsilon)}(E; N) \equiv P\{E|\text{elements belong to } \{\varepsilon\}, l(\lambda) = N,$$

(3.3)

and elements repeat no more than 1 time) ,

$$P_q^{(\varepsilon)}(E; N) \equiv P\{E|\text{elements belong to } \{\varepsilon\}, l(\lambda) = N,$$

(3.4)

and elements repeat no more than $q$ times)

with $q = 2, 3, \ldots, N - 1$.

In the following, for convenience we drop the superscript $\{\varepsilon\}$ when $\{\varepsilon\} = \{1, 2, 3, 4, \ldots\}$. For example, $P(E; N) \equiv P^{(1,2,3,4\ldots)}(E; N)$, $Q(E; N) \equiv Q^{(1,2,3,4\ldots)}(E; N)$, and $P_q(E; N) \equiv P_q^{(1,2,3,4\ldots)}(E; N)$.

The restricted integer partition function $P^{(\varepsilon)}(E; N)$, $Q^{(\varepsilon)}(E; N)$, and $P_q^{(\varepsilon)}(E; N)$ correspond to ideal Bose, Fermi, and Gentile gases with the maximum occupation number $q$, respectively [17–26].

In the present paper, however, we construct the generating functions for $P^{(\varepsilon)}(E; N)$, $Q^{(\varepsilon)}(E; N)$, and $P_q^{(\varepsilon)}(E; N)$. Starting from the generating function, we calculate some expressions of $P(E; N)$, $Q(E; N)$, and $P_q(E; N)$ as examples.

3.1. The generating function of $P^{(\varepsilon)}(E; N)$

**Theorem 1.** The generating function of $P^{(\varepsilon)}(E; N)$ is

$$\sum_{E=0}^{\infty} P^{(\varepsilon)}(E; N) z^E = (N)(z^{\varepsilon_1}, z^{\varepsilon_2}, \ldots),$$

(3.5)
where \((N) (x_1, x_2, \ldots)\) is the S-function given by equation (2.1) corresponding to the partition \((\lambda) = (N)\).

**Proof.** For an ideal Bose gas, the maximum occupation number is \(\infty\), so the number of microstates \(\Omega_B (E, N)\) in the macrostate, denoted by \((N, E)\), is the number of representations of \(E\) in terms of \(N\) single-particle energies \(\varepsilon_i\) which sum up to \(E\). By the definition of \(P^{(\varepsilon)} (E; N)\), equation (3.2), one directly arrives at

\[
P^{(\varepsilon)} (E; N) = \Omega_B (E, N) .
\]

The canonical partition function of an ideal Bose gas is [14]

\[
Z_B (\beta, N) = \sum_{E=0}^{\infty} \Omega_B (E, N) e^{-\beta E} = (N) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots) .
\]

Substituting equation (3.6) into equation (3.7) and equaling \(z\) and \(e^{-\beta}\) give equation (3.5).

\section{3.2. The generating function of \(Q^{(\varepsilon)} (E; N)\)}

**Theorem 2.** The generating function of \(Q^{(\varepsilon)} (E; N)\) is

\[
\sum_{E=0}^{\infty} Q^{(\varepsilon)} (E; N) z^E = (1^N) (z^{\varepsilon_1}, z^{\varepsilon_2}, \ldots),
\]

where \((1^N) (x_1, x_2, \ldots)\) is the S-function given by equation (2.1) corresponding to the partition \((\lambda) = (1^N)\).

**Proof.** For an ideal Fermi gas, the maximum occupation number is 1, so the number of microstates \(\Omega_F (E, N)\) in the macrostate \((N, E)\) is the number of representations of \(E\) in terms of \(N\) distinct single-particle energies \(\varepsilon_i\) which sum up to \(E\). By the definition of \(Q^{(\varepsilon)} (E; N)\), equation (3.3), one directly arrives at

\[
Q^{(\varepsilon)} (E; N) = \Omega_F (E, N) .
\]

The canonical partition function of an ideal Fermi gas is [14]

\[
Z_F (\beta, N) = \sum_{E=0}^{\infty} \Omega_F (E, N) e^{-\beta E} = (1^N) (e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots) .
\]

Substituting equation (3.9) into equation (3.10) and setting \(z = e^{-\beta}\) give equation (3.8).

\section{3.3. The generating function of \(P_q^{(\varepsilon)} (E; N)\)}

Before going on, we first define the order of partitions [14]. An integer \(N\) has many partitions, we arrange the partitions in the following order: \((\lambda), (\lambda)^{\prime}, \ldots\), when \(\lambda_1 > \lambda_1^{\prime}\), \((\lambda), \ldots\)

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(\lambda)'$, when \(\lambda_1 = \lambda'_1\) but \(\lambda_2 > \lambda'_2\); and so on. One keeps comparing \(\lambda_i\) and \(\lambda'_i\) until all the partitions of \(N\) are arranged in the prescribed order. Let \((\lambda)_J\) denote the \(J\)th partition of \(N\) and \(\lambda_{J,i}\) denote the \(i\)th element in the partition \((\lambda)_J\). More details can be found in, e.g. [14].

**Theorem 3.** The generating function of \(P_q^{(\varepsilon)}(E; N)\) is

\[
\sum_{E=0}^{\infty} P_q^{(\varepsilon)}(E; N) z^E = \sum_{I=1}^{P(N)} Q^I(q) (\lambda)_I (z^{\varepsilon_1}, z^{\varepsilon_2}, \ldots),
\]

(3.11)

where \((\lambda)_I(x_1, x_2, \ldots)\) is the \(S\)-function given by equation (2.1) corresponding to the \(I\)th partition of \(N\) and the coefficient

\[
Q^I(q) = \sum_{K=1}^{P(N)} (k_K^I)^{-1} \Gamma^K(q)
\]

(3.12)

with \(\Gamma^K(q)\) satisfying

\[
\Gamma^K(q) = 0 \text{ when } \lambda_{K,1} > q,
\]

\[
\Gamma^K(q) = 1 \text{ when } \lambda_{K,1} \leq q,
\]

(3.13)

and \((k_K^I)^{-1}\) satisfying

\[
\sum_{I=1}^{P(N)} (k_K^I)^{-1} k_L^I = \delta_K^L
\]

(3.14)

with \(k_L^I\) the Kostka number [16].

**Proof.** For an ideal Gentile gas, the maximum occupation number is \(q\). Thus the number of microstates \(\Omega_q(E, N)\) in the macrostate \((N, E)\) is the number of representations of \(E\) in terms of \(N\) single-particle energies \(\varepsilon_i\) which sum up to \(E\) and each \(\varepsilon_i\) repeats no more than \(q\) times. By the definition of \(P_q^{(\varepsilon)}(E; N)\), one directly arrives at

\[
P_q^{(\varepsilon)}(E; N) = \Omega_q(E, N).
\]

(3.15)

The canonical partition function of an ideal Gentile gas is [14]

\[
Z_q(\beta, N) = \sum_{E=0}^{\infty} \Omega_q(E, N) e^{-\beta E} = \sum_{I=1}^{P(N)} Q^I(q) (\lambda)_I \left(e^{-\beta \varepsilon_1}, e^{-\beta \varepsilon_2}, \ldots\right).
\]

(3.16)

Substituting equation (3.15) into equation (3.16) and equaling \(z\) and \(e^{-\beta}\) give equation (3.11). ■

In a word, the main result of this section is that the generating function of the restricted integer partition function, equations (3.5), (3.8) and (3.11), is a symmetric function and can be written as linear combinations of the \(S\)-function.
4. The restricted integer partition function corresponding to general statistics

In this section, based on general statistics in which the maximum occupation number of a state can take on unrestricted integer values or infinity and the maximum occupation numbers of different states may be different [13], we introduce a restricted integer partition function. General statistics is a generalization of quantum statistics, including Bose–Einstein, Fermi–Dirac, and Gentile statistics. Bose–Einstein, Fermi–Dirac, and Gentile statistics are special cases of general statistics, so the restricted integer partition function corresponding to Bose–Einstein, Fermi–Dirac, and Gentile statistics are special cases of the restricted integer partition functions corresponding to general statistics. This means that a number of restricted integer partition functions can be considered in a unified framework.

4.1. The restricted integer partition function $P_{\{q\}}^{\{\varepsilon\}}(E; N)$ and general statistics

For $\{q\} = \{q_1, q_2, \ldots\}$ with $q_i$ an integer, the restricted integer partition function $P_{\{q\}}^{\{\varepsilon\}}(E; N)$ is defined as

$$P_{\{q\}}^{\{\varepsilon\}}(E; N) \equiv P(E \mid \text{elements belong to } \{\varepsilon\} , \ l(\lambda) = N, \ \text{and the element } \varepsilon_i \text{ repeats no more than } q_i \text{ times}) \quad (4.1)$$

Theorem 4. The restricted integer partition function $P_{\{q\}}^{\{\varepsilon\}}(E; N)$ is the number of microstates $\Omega_{\{q\}}(E, N)$ of an ideal general-statistics gas with maximum occupation numbers $\{q\} = \{q_1, q_2, \ldots\}$, i.e.

$$P_{\{q\}}^{\{\varepsilon\}}(E; N) = \Omega_{\{q\}}(E, N) \quad (4.2)$$

Proof. For an ideal general-statistics gas, the maximum occupation number of the state $\varepsilon_i$ is $q_i$ [13], so the number of microstates $\Omega_{\{q\}}(E, N)$ in the macrostate $(N, E)$ is the number of representations of $E$ in terms of $N$ single-particle energies $\varepsilon_i$ which sum up to $E$ and $\varepsilon_i$ repeats no more than $q_i$ times. By the definition of $P_{\{q\}}^{\{\varepsilon\}}(E; N)$, equation (4.1), one directly arrives at equation (4.2). \hfill \blacksquare

4.2. The two-variable generating function of $P_{\{q\}}^{\{\varepsilon\}}(E; N)$

In this section, we give the two-variable generating function of $P_{\{q\}}^{\{\varepsilon\}}(E; N)$.

Some kinds of restricted integer partition functions need to be described by two-variable generating functions [1]:

$$\Xi(z, x) = \sum_{N, E=0}^{\infty} P(\text{restrictions on } l(\lambda) \text{ and other restrictions}) \ z^{E} x^{N}, \quad (4.3)$$

where $N$ is the length $l(\lambda)$ of the partition.

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The two-variable generating function of the restricted integer partition function $P_{\{q\}}^{(e)} (E; N)$ defined by equation (4.1) is then
\[ 
\Xi (z, x) = \sum_{N,E=0}^{\infty} P_{\{q\}}^{(e)} (E; N) z^E x^N. 
\] (4.4)

**Theorem 5.** The two-variable generating function of the restricted integer partition function $P_{\{q\}}^{(e)} (E; N)$ is
\[ 
\sum_{N,E=0}^{\infty} P_{\{q\}}^{(e)} (E; N) z^E x^N = \prod_i \frac{1 - (z^{\epsilon_i} x)^{q_i + 1}}{1 - z^{\epsilon_i} x}. 
\] (4.5)

**Proof.** The grand canonical partition function of a general-statistics gas is [13]
\[ 
\Xi (\alpha, \beta) = \sum_{E,N=0}^{\infty} \Omega_{\{q\}} e^{-\beta E} e^{-\alpha N} = \prod_i \frac{1 - \left[ (e^{-\beta})^{\epsilon_i} e^{-\alpha} \right]^{q_i + 1}}{1 - (e^{-\beta})^{\epsilon_i} e^{-\alpha}}. 
\] (4.6)

Substituting equation (4.2) into equation (4.6) and setting $z = e^{-\beta}$ and $x = e^{-\alpha}$ give equation (4.5).

### 4.3. Restricted integer partition functions as special cases of $P_{\{q\}}^{(e)} (E; N)$

A series of restricted integer partition functions are special cases of the restricted integer partition function $P_{\{q\}}^{(e)} (E; N)$ defined by equation (4.1). We show that starting from the restricted integer partition function $P_{\{q\}}^{(e)} (E; N)$, one can obtain many restricted integer partition functions.

In the following, we consider three types of restricted integer partition functions: the restricted integer partition functions counting partitions with length $l(\lambda) = N$, with length $l(\lambda) \leq N$, and without restrictions on the length $l(\lambda)$.

#### 4.3.1. Restricted integer partition functions counting partitions with length $l(\lambda) = N$.

In this section, we consider the following three restricted integer partition functions: $P^{(e)} (E; N)$, defined in equation (3.2) corresponding to Bose–Einstein statistics, $Q^{(e)} (E; N)$, defined in equation (3.3) corresponding to Fermi–Dirac statistics, and $P_{q}^{(e)} (E; N)$, defined in equation (3.4) corresponding to Gentile statistics.

These three restricted integer partition functions, $P^{(e)} (E; N)$, $Q^{(e)} (E; N)$, and $P_{q}^{(e)} (E; N)$, are three special cases of the restricted integer partition function corresponding to general statistics $P_{\{q\}}^{(e)} (E; N)$ defined by equation (4.1):
\[ 
P^{(e)} (E; N) = P_{\{q_{i} = \infty\}}^{(e)} (E; N), 
\] (4.7)
\[ 
Q^{(e)} (E; N) = P_{\{q_{i} = 1\}}^{(e)} (E; N), 
\] (4.8)
\[ P_{q}^{(\varepsilon)} (E; N) = P_{\{q=\varepsilon\}}^{(\varepsilon)} (E; N). \] (4.9)

One can see from equations (4.7)–(4.9) that the restricted integer partition functions \( P^{(\varepsilon)} (E; N) \), \( Q^{(\varepsilon)} (E; N) \), and \( P_{q}^{(\varepsilon)} (E; N) \) can be obtained by setting \( q_i = \infty \), \( q_i = 1 \), and \( q_i = q \) in the restricted integer partition function \( P_{\{q\}}^{(\varepsilon)} (E; N) \) given by equation (4.1). The two-variable generating functions of these three restricted integer partition functions can be then obtained by setting \( q_i = \infty \), \( q_i = 1 \), and \( q_i = q \) in the two-variable generating functions (4.5), respectively:

\[
\sum_{E,N} P^{(\varepsilon)} (E; N) z^{E} x^{N} = \prod_{i=1}^{\lambda} \left( 1 - z^{\varepsilon_i} x \right), \quad \sum_{E,N} Q^{(\varepsilon)} (E; N) z^{E} x^{N} = \prod_{i=1}^{\lambda} \left( 1 + z^{\varepsilon_i} x \right), \quad \text{and} \quad \sum_{E,N} P_{q}^{(\varepsilon)} (E; N) z^{E} x^{N} = \prod_{i=1}^{\lambda} \left[ 1 - (z^{\varepsilon_i} x)^{q+1} \right] / (1 - z^{\varepsilon_i} x). \]

These results agree with the result in [2]. In [2], two-variable generating functions of \( P^{(\varepsilon)} (E; N) \), \( Q^{(\varepsilon)} (E; N) \), and \( P_{q}^{(\varepsilon)} (E; N) \) are obtained by recognizing that the exponent in the expansion of the two-variable generating of \( x \) will keep count of how many parts are used in each partition.

### 4.3.2. Restricted integer partition functions counting partitions with length \( l_{(\lambda)} \leq N \)

The restricted integer partition function counting partitions with length \( l_{(\lambda)} \leq N \) is a kind of important restricted integer partition functions [1, 2]. In this section, we consider the following restricted integer partition function:

\[
p_{\{q\}}^{(\varepsilon)} (E; N) \equiv P \left( E \right) \text{ elements belong to } \{\varepsilon\}, \ l_{(\lambda)} \leq N, \text{ and the element } \varepsilon_i \text{ repeats no more than } q_i \text{ times}.
\] (4.10)

There are three important special cases of the restricted integer partition function \( p_{\{q\}}^{(\varepsilon)} (E; N) \):

\[
p^{(\varepsilon)} (E; N) \equiv P \left( E \right) \text{ elements belong to } \{\varepsilon\} \text{ and } l_{(\lambda)} \leq N,
\] (4.11)

\[
q^{(\varepsilon)} (E; N) \equiv P \left( E \right) \text{ elements belong to } \{\varepsilon\}, \ l_{(\lambda)} \leq N, \text{ and elements repeat no more than 1 time},
\] (4.12)

\[
p_{q}^{(\varepsilon)} (E; N) \equiv P \left( E \right) \text{ elements belong to } \{\varepsilon\}, \ l_{(\lambda)} \leq N, \text{ and elements repeat no more than } q \text{ times}.
\] (4.13)

The restricted integer partition function \( p_{\{q\}}^{(\varepsilon)} (E; N) \) is a special case of the restricted integer partition function corresponding to general statistics \( P_{\{\infty, q\}}^{(\varepsilon)} (E; N) \) defined by equation (4.1):

\[
p_{\{q\}}^{(\varepsilon)} (E; N) = P_{\{\infty, q\}}^{(\varepsilon)} (E; N),
\] (4.14)

where \( \{0, \varepsilon\} \) denotes \( \{0, \varepsilon_1, \varepsilon_2, \ldots\} \) and \( \{\infty, q\} \) denotes \( \{\infty, q_1, q_2, \ldots\} \). Then the three special cases of \( p_{\{q\}}^{(\varepsilon)} (E; N) \) are

\[
p^{(\varepsilon)} (E; N) = P_{\{0, q\}}^{(\varepsilon)} (E; N),
\] (4.15)

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J. Stat. Mech. (2018) 053111
A statistical mechanical approach to restricted integer partition functions

\[ q^{(\varepsilon)}(E; N) = P^{(0,\varepsilon_1,\varepsilon_2,\ldots)}(E; N), \]  

(4.16)

\[ p^{(\varepsilon)}_q(E; N) = P^{(0,\varepsilon_1,\varepsilon_2,\ldots)}(E; N). \]  

(4.17)

The two-variable generating function of the restricted integer partition function \( p^{(\varepsilon)}_q(E; N) \) defined by equation (4.10) can be obtained by substituting equation (4.14) into equation (4.5):

\[ \sum_{E,N=0}^{\infty} p^{(\varepsilon)}_q(E; N) z^E x^N = \frac{1}{1 - x} \prod_{i=1}^{\infty} \frac{1 - (z^{\varepsilon_i} x)^q}{1 - z^{\varepsilon_i} x}. \]  

(4.18)

The two-variable generating functions of the three special cases of the restricted integer partition function \( p^{(\varepsilon)}_q(E; N) \) are then

\[ \sum_{N,E=0}^{\infty} p^{(\varepsilon)}(E; N) z^E x^N = \frac{1}{1 - x} \prod_{i=1}^{\infty} \frac{1}{1 - z^{\varepsilon_i} x}. \]  

(4.19)

\[ \sum_{N,E=0}^{\infty} q^{(\varepsilon)}(E; N) z^E x^N = \frac{1}{1 - x} \prod_{i=1}^{\infty} (1 + z^{\varepsilon_i} x), \]  

(4.20)

\[ \sum_{N,E=0}^{\infty} p^{(\varepsilon)}_q(E; N) z^E x^N = \frac{1}{1 - x} \prod_{i=1}^{\infty} \frac{1 - (z^{\varepsilon_i} x)^q}{1 - z^{\varepsilon_i} x}. \]  

(4.21)

Additionally, besides the two-variable generating function given by equation (4.19), we can also obtain a one-variable generating function of the restricted partition integer function \( p^{(\varepsilon)}_q(E; N) \):

\[ \sum_E p^{(\varepsilon)}_q(E; N) z^E = (N) (1, z^{\varepsilon_1}, z^{\varepsilon_2}, \ldots). \]  

(4.22)

This can be achieved by substituting equation (4.15) into equation (3.5).

4.3.3. Restricted integer partition functions counting partitions without restrictions on the length \( l(\lambda) \). The restricted integer partition function counting partitions without restrictions on \( l(\lambda) \) is another kind of important restricted integer partition functions [1, 2]. In this section, we consider the following restricted integer partition function:

\[ P^{(\varepsilon)}_{(q)}(E) \equiv P(E) \text{ elements belong to } \{\varepsilon\} \]  

and the element \( \varepsilon_i \) repeats no more than \( q_i \) times).

(4.23)

There are three important special cases of the restricted integer partition function \( P^{(\varepsilon)}_{(q)}(E) \):

\[ P^{(\varepsilon)}(E) \equiv P(E) \text{ elements belong to } \{\varepsilon\}, \]  

(4.24)

\[ Q^{(\varepsilon)}(E) \equiv P(E) \text{ elements belong to } \{\varepsilon\} \]  

and elements repeat no more than 1 time),

(4.25)
A statistical mechanical approach to restricted integer partition functions

\[ P_{q^i}^\{z\} (E) \equiv P (E| \text{elements belong to } \{z\} \text{ and elements repeat no more than } q \text{ times}) . \]  

(4.26)

The restricted integer partition function can be represented by \( P_{q^i}^\{z\} (E; N) \) as

\[ P_{q^i}^\{z\} (E) = \sum_N P_{q^i}^\{z\} (E; N) . \]  

(4.27)

The three special cases of \( P_{q^i}^\{z\} (E) \) then can be represented as

\[ P^\{z\} (E) = \sum_N P^\{z\} (E; N) , \]  

(4.28)

\[ Q^\{z\} (E) = \sum_N Q^\{z\} (E; N) , \]  

(4.29)

\[ P^\{z\} (E) = \sum_N P^\{z\} (E; N) . \]  

(4.30)

The generating function of the restricted integer partition function \( P_{q^i}^\{z\} (E) \) can be obtained by setting \( x = 1 \) in equation (4.5):

\[ \sum_{E=0}^\infty P_{q^i}^\{z\} (E) z^E = \prod_i \frac{1 - (z^{\varepsilon_i})^{q_i+1}}{1 - z^{\varepsilon_i}} . \]  

(4.31)

Moreover, by setting \( q_i = \infty \), \( q_i = 1 \), and \( q_i = q \) in equation (4.31), we can obtain the generating functions of restricted integer partition function \( P^\{z\} (E) \), \( Q^\{z\} (E) \), and \( P^\{z\} (E) \): \( \sum_E P^\{z\} (E) z^E = \prod_i 1 / (1 - z^{\varepsilon_i}) \), \( \sum_E Q^\{z\} (E) z^E = \prod_i (1 + z^{\varepsilon_i}) \), and \( \sum_{E=0}^\infty P^\{z\} (E) z^E = \prod_i \left[ 1 - (z^{\varepsilon_i})^{q_i+1} \right] / (1 - z^{\varepsilon_i}) \). The generating functions of \( P^\{z\} (E) \), \( Q^\{z\} (E) \), and \( P^\{z\} (E) \) agree with the result in [1, 2] in which the generating functions is obtained by recognizing that the exponent in the expansion of the generating function will count the number of partitions.

4.3.4. A relation between the S-function and the Gauss polynomial: the restricted integer partition function \( p^{(1,2,\ldots,n)} (E; N) \). In this section, we give a relation between the S-function and the Gauss polynomial by inspection of the restricted integer partition function \( p^{(1,2,\ldots,n)} (E; N) = P (E| \text{elements belong to } \{1,2,3,\ldots,n\} \text{ and } l(\lambda) \leq N) \), a special case of the restricted integer partition function \( p^\{z\} (E; N) \) defined in equation (3.2) with \( \{z\} = \{1,2,3,\ldots,n\} \).

**Theorem 6.** A relation between the S-function \( (N) (1, z, z^2, \ldots, z^n) \) and the Gauss polynomial \( G (n, N; x) = \prod_{i=1}^{N+n} (1 - x^i) / \left[ \prod_{\mu=1}^n (1 - x^\mu) \prod_{v=1}^N (1 - x^v) \right] \) is

\[ (N) (1, z, z^2, \ldots, z^n) = G (n, N; z) . \]  

(4.32)
Proof. The generating function of \( p_{\{1,2,3,\ldots,n\}} (E;N) \), equation (4.22) with \( \{\varepsilon\} = \{1,2,3,\ldots,n\} \), can be written as
\[
\sum_{E} p_{\{1,2,3,\ldots,n\}} (E;N) z^E = (N) (1, z, z^2, \ldots, z^n),
\]
(4.33)
i.e. the generating function of \( p_{\{1,2,3,\ldots,n\}} (E;N) \) is the \( S \)-function \((N) (1, z, z^2, \ldots, z^n)\). It is also shown in [1, 36] that the generating function of \( p_{\{1,2,3,\ldots,n\}} (E;N) \) is a Gauss polynomial:
\[
\sum_{E=0}^{mn} p_{\{1,2,3,\ldots,n\}} (E, m) z^E = G (N, n; z).
\]
(4.34)
Comparing equations (4.33) and (4.34), we arrive at equation (4.32) directly. □

4.3.5. The generating function of restricted integer partition functions \( P^{(0,1,2,\ldots)} (E;N) \), \( Q^{(0,1,2,\ldots)} (E;N) \), \( P^{{1,2^2,\ldots}} (E;N) \), and \( Q^{{1^2,2^2,\ldots}} (E;N) \) and quantum gases. In this section, we give the generating function of restricted integer partition functions on the nature number. The restricted integer partition function is closely related to a one-dimensional ideal quantum gas in an external harmonic-oscillator potential.

The restricted integer partition function \( P^{(0,1,2,\ldots)} (E;N) \) and \( Q^{(0,1,2,\ldots)} (E;N) \) corresponding to a one-dimensional ideal quantum gas in an external harmonic-oscillator potential with ground-state energy equaling to 0, while the restricted integer partition function \( P (E;N) \) and \( Q (E;N) \) corresponding to one-dimensional ideal quantum gases in an external harmonic-oscillator potential with nonzero ground-state energy [18–21].

Setting \( \{\varepsilon\} = \{0,1,2,\ldots\} \) and using equations (3.5) and (3.8) give the generating function of \( P^{(0,1,2,\ldots)} (E;N) \) and \( Q^{(0,1,2,\ldots)} (E;N) \):
\[
\sum_{E=0}^{\infty} P^{(0,1,2,\ldots)} (E;N) z^E = \prod_{i=1}^{N} \frac{1}{(1 - z^i)},
\]
(4.35)
\[
\sum_{E=0}^{\infty} Q^{(0,1,2,\ldots)} (E;N) z^E = z^{N(N-1)/2} \prod_{i=1}^{N} \frac{1}{(1 - z^i)}.
\]
(4.36)
We can also obtain the generating function of the restricted integer partition function \( P (E;N) \) and \( Q (E;N) \). Setting \( \{\varepsilon\} = \{1,2,\ldots\} \) and using equations (3.5) and (3.8) give
\[
\sum_{E} P (E;N) z^E = z^N \prod_{i=1}^{N} 1/ (1 - z^i) \quad \text{and} \quad \sum_{E} Q (E;N) z^E = z^{N(N+1)/2} \prod_{i=1}^{N} 1/ (1 - z^i).
\]
The generating functions of the restricted integer partition function \( P (E;N) \) and \( Q (E;N) \) agree with the result in [41]. In [41], the generating functions of the restricted integer partition function \( P (E;N) \) and \( Q (E;N) \) are obtained by selecting the coefficient of \( z^E x^N \) in the expansion of \( \prod_{i=1}^{N} 1/ (1 - z^i x) \) and \( \prod_{i=1}^{N} (1 + z^i x) \), respectively.

Now we consider the restricted integer partition function on square numbers.

The restricted integer partition functions \( P^{{1^2,2^2,\ldots}} (E;N) \) and \( Q^{{1^2,2^2,\ldots}} (E;N) \) are actually the state density of an ideal quantum gas in a one-dimensional periodic box:
$P^{1,2^2,...}(E;N)$ for ideal Bose gases and $Q^{1,2^2,...}(E;N)$ for ideal Fermi gases. We can calculate the exact state density from the exact generating function of $P^{1,2^2,...}(E;N)$ and $Q^{1,2^2,...}(E;N)$, equations (3.5) and (3.8) with $\{\varepsilon\} = \{1^2, 2^2, \ldots\}$.

For $N=2$, the restricted integer partition functions $P^{1,2^2,...}(E;2)$ and $Q^{1,2^2,...}(E;2)$ are the state density of two quantum particles.

Setting $\{\varepsilon\} = \{1^2, 2^2, \ldots\}$ and using equations (3.5) and (3.8) with $N=2$ give the exact generating function of $P^{1,2^2,...}(E;N)$ and $Q^{1,2^2,...}(E;N)$,

$$\sum_{E=0}^{\infty} P^{1,2^2,...}(E;2) z^E = \frac{1}{8} \theta_3^2(0,z) - \frac{1}{2} \theta_3(0,z) + \frac{3}{8},$$

(4.37)

where $\theta_3(u,q)$ is the elliptic theta function [44], and

$$\sum_{E=0}^{\infty} Q^{1,2^2,...}(E;2) z^E = \frac{1}{8} \theta_3^2(0,z) - \frac{1}{8}. $$

(4.38)

5. Calculating $P^{\{\varepsilon\}}(E;N)$, $Q^{\{\varepsilon\}}(E;N)$, and $P^\{\varepsilon\}_q(E;N)$ from the generating functions: examples

The generating function of the restricted integer partition function, equation (3.1), is indeed a result of the $Z$-transform performed on the restricted integer partition function [45]. Thus, we can obtain the restricted integer partition function by applying an inverse $Z$-transform on the generating function [1]:

$$P(E|\text{conditions}) = \frac{1}{2\pi i} \oint_C Z(z) z^{-n-1} dz,$$

(5.1)

where $Z(z)$ is the generating function defined in equation (3.1).

The exact generating functions of $P^{\{\varepsilon\}}(E;N)$, $Q^{\{\varepsilon\}}(E;N)$, and $P^\{\varepsilon\}_q(E;N)$ are given in equations (3.5), (3.8) and (3.11), respectively. One can obtain an explicit expression of the generating function once $N$ and the set $\{\varepsilon\}$ are determined. For example, the generating function for restricted integer partition functions on nature numbers and square numbers can be obtained by setting $\{\varepsilon\} = \{1,2,3,\ldots\}$ and $\{\varepsilon\} = \{1^2, 2^2, 3^2, \ldots\}$, respectively. Therefore, by substituting the generating functions (3.5), (3.8), and (3.11) into equation (5.1), one can in principle obtain the exact expression of the restricted integer partition functions $P^{\{\varepsilon\}}(E;N)$, $Q^{\{\varepsilon\}}(E;N)$, and $P^\{\varepsilon\}_q(E;N)$.

5.1. Expressions of $P^{\{\varepsilon\}}(E;N)$, $Q(E;N)$, and $P^\{\varepsilon\}_q(E;N)$ for $N=2, 3, 4, \text{ and } 5$

In this section, using equation (5.1), we give the exact expressions of $P^{\{\varepsilon\}}(E;N)$, $Q(E;N)$, and $P^\{\varepsilon\}_q(E;N)$ for $N=2, 3, 4, \text{ and } 5$ as examples. Moreover, in the appendix, we list the expressions of the generating functions of $P^{\{\varepsilon\}}(E;N)$, $Q^{\{\varepsilon\}}(E;N)$, and $P^\{\varepsilon\}_q(E;N)$ for $N=2, 3, 4, 5, \text{ and } 6$. The expressions of the generating functions of
A statistical mechanical approach to restricted integer partition functions

$P(E; N)$, $Q(E; N)$, and $P_q(E; N)$ can be obtained by setting $\{\varepsilon\} = \{1, 2, 3, \ldots\}$ in the expression of generating functions listed in appendix.

$N = 2$. The generating function of $Q(E; 2)$ by equation (3.8) reads

$$\sum_E Q(E; 2) z^E = \frac{z^3}{(z - 1)^2 (1 + z)}.$$  \hspace{1cm} (5.2)

Substituting equation (5.2) into equation (5.1) gives

$$Q(E; 2) = \frac{E}{2} - \frac{1}{4} (-1)^E - \frac{3}{4}.$$  \hspace{1cm} (5.3)

$N = 3$. The generating function for $Q(E; 3)$ by equation (3.8) reads

$$\sum_E Q(E; 3) z^E = \frac{z^6}{(z - 1)^3 (1 + z)(1 + z + z^2)}.$$  \hspace{1cm} (5.4)

Substituting equation (5.4) into equation (5.1) gives

$$Q(E; 3) = \frac{1}{72} \left[ 6E^3 - 36E + 16 \cos \left( \frac{2E\pi}{3} \right) + 9 (-1)^E + 47 \right].$$  \hspace{1cm} (5.5)

The generating function of $P_2(E; 3)$ by equation (3.11) reads

$$\sum_E P_2(E; 3) z^E = \frac{[z (z - 1) - 1] z^4}{(z - 1)^3 (z + 1)(1 + z + z^2)}.$$  \hspace{1cm} (5.6)

Substituting equation (5.6) into equation (5.1) gives

$$P_2(E; 3) = \frac{1}{72} \left[ 6E^3 - 32 \cos \left( \frac{2E\pi}{3} \right) - 9 (-1)^E - 31 \right].$$  \hspace{1cm} (5.7)

$N = 4$. The generating function of $Q(E; 4)$ by equation (3.8) reads

$$\sum_E Q(E; 4) z^E = \frac{z^{10}}{(z - 1)^4 (1 + z)^2 (1 + z^2)(1 + z + z^2)}.$$  \hspace{1cm} (5.8)

Substituting equation (5.8) into equation (5.1) gives

$$Q(E; 4) = \frac{1}{288} \left[ 2E^4 - 30E^2 + 135E + 9E (-1)^E - 45 (-1)^E 
- 18 (-i)^E - 18iE - 32U_E \left( \frac{1}{2} \right) - 175 \right],$$  \hspace{1cm} (5.9)

where $U_E(x)$ is the Chebyshev polynomials of second kind [46]. The generating function of $P_2(E; 4)$ by equation (3.11) reads

$$\sum_E P_2(E; 4) z^E = \frac{(1 + z^2 - z^4) z^6}{(z - 1)^4 (z + 1)^2 (1 + z^2)(1 + z + z^2)}.$$  \hspace{1cm} (5.10)

Substituting equation (5.10) into equation (5.1) gives
\[
P_2(E; 4) = \frac{1}{288} \left[ 2E^3 + 6E^2 - 105E + 9E (-1)^E + 9 (-1)^E 
+ 18 (-i)^E + 18i^E + 64U_E \left( -\frac{1}{2} \right) + 179 \right].
\] (5.11)

The generating function of \( P_3(E; 4) \) by equation (3.11) reads
\[
\sum_E P_3(E; 4) z^E = \frac{(1 + z - z^3 - z^4 + z^5) z^5}{(z - 1)^4 (z + 1)^2 (1 + z^2) (1 + z + z^2)}.
\] (5.12)

Substituting equation (5.12) into equation (5.1) gives
\[
P_3(E; 4) = \frac{1}{288} \left[ 2E^3 + 6E^2 - 9E + 9 (-1)^E E 
- 63 (-1)^E - 54 (-i)^E - 54i^E - 32U_E \left( -\frac{1}{2} \right) - 85 \right].
\] (5.13)

\( N = 5 \). The generating function of \( Q(E; 5) \) by equation (3.8) reads
\[
\sum_E Q(E; 5) z^E = \frac{z^{15}}{(z - 1)^5 (1 + z)^2 (1 + z^2) (1 + z + z^2) (1 + z + z^2 + z^3 + z^4)}.
\] (5.14)

Substituting equation (5.14) into equation (5.1) gives
\[
Q(E; 5) = \frac{1}{86400} \left[ 30E^4 - 900E^3 + 9300E^2 - 38250E + 6912 \cos \left( \frac{2}{5} E \pi \right) 
+ 5400 \cos \left( \frac{E}{2} \pi \right) + 6400 \cos \left( \frac{3}{5} E \pi \right) + 6912 \cos \left( \frac{4}{5} E \pi \right) + 10125 \cos (E \pi) 
- 1350 \cos (E \pi) E - 5400 \sin \left( \frac{E}{2} \pi \right) + 10125 i \sin (E \pi) - 1350 i \sin (E \pi) E + 50651 \right].
\] (5.15)

The generating function of \( P_2(E; 5) \) by equation (3.11) reads
\[
\sum_E P_2(E; 5) z^E = \frac{(-1 - z + z^5) z^9}{(z - 1)^5 (z + 1)^2 (z^2 + 1) (z^2 + z + 1) (1 + z + z^2 + z^3 + z^4)}.
\] (5.16)

Substituting equation (5.16) into equation (5.1) gives
\[
P_2(E; 5) = \frac{1}{86400} \left[ 30E^4 + 300E^3 - 6900E^2 - 26550E - 1350 (-1)^E E 
+ 7425 (-1)^E + 6912 \cos \left( \frac{2}{5} E \pi \right) - 5400 \cos \left( \frac{E}{2} \pi \right) + 6400 \cos \left( \frac{3}{5} E \pi \right) 
+ 6912 \cos \left( \frac{4}{5} E \pi \right) + 5400 \sin \left( \frac{E}{2} \pi \right) + 6400 \sqrt{3} \sin \left( \frac{2}{3} E \pi \right) - 22249 \right].
\] (5.17)

The generating function for \( P_3(E; 5) \), equation (3.11), reads
\[
\sum_E P_3(E; 5) z^E = \frac{(-1 - z + z^5 + z^6 - z^8) z^7}{(z - 1)^5 (z + 1)^2 (z^2 + 1) (z^2 + z + 1) (1 + z + z^2 + z^3 + z^4)}.
\] (5.18)

Substituting equation (5.18) into equation (5.1) gives

\[\text{https://doi.org/10.1088/1742-5468/aabfc9}\]
A statistical mechanical approach to restricted integer partition functions

\[ P_4(E; 5) = \frac{1}{86 400} \left[ 30E^4 + 300E^3 + 300E^2 - 23 850E + 6912 \cos \left( \frac{3\pi}{5}E \right) + 16 200 \cos \left( \frac{2\pi}{5}E \right) \right. \\
- 3200 \cos \left( \frac{2\pi}{3}E \right) + 6912 \cos \left( \frac{4\pi}{5}E \right) + 7425 \cos (\pi E) - 3200 \cos \left( \frac{2\pi}{5}E \right) \\
- 116 200 \sin \left( \frac{\pi}{5}E \right) - 3200 \sqrt{3} \sin \left( \frac{2\pi}{5}E \right) + 7425 \sin (\pi E) - 3200 \cos \left( \frac{2\pi}{3}E \right) \\
\left. + 16 200 \cos \left( \frac{4\pi}{5}E \right) - 7425 \cos (\pi E) - 1350 \cos (\pi E) - 19 129 \right\]. \tag{5.19}

The generating function of \( P_4(E; 5) \) by equation (3.11) reads

\[ \sum_E P_4(E; 5) z^E = \frac{(-1 - z + 2z^2 - z^3 - z^4 + z^5) z^6}{(z - 1)^5 (z + 1)^2 (z^2 + 1) (1 + z + z^2 + z^3 + z^4)}. \tag{5.20} \]

Substituting equation (5.20) into equation (5.1) gives

\[ P_4(E; 5) = \frac{1}{86 400} \left[ 30E^4 + 300E^3 + 300E^2 - 2250E + 5400 \sin \left( \frac{\pi}{5}E \right) - 3200 \sqrt{3} \sin \left( \frac{2\pi}{3}E \right) \\
- 675i (2E + 5) \sin (\pi E) - 5400 \cos \left( \frac{\pi}{2}E \right) - 3200 \cos \left( \frac{2\pi}{3}E \right) \\
- 27 648 \cos \left( \frac{2\pi}{5}E \right) - 27 648 \cos \left( \frac{4\pi}{5}E \right) - 675 (2E + 5) \cos (\pi E) - 19 129 \right\]. \tag{5.21} \]

Moreover, we also calculate the expression of \( P(E, N) \) by the generating function. In [1, 2, 41], the expression of \( P(E, N) \) is obtained by the recursive method.

The generating function of \( P(E; 2) \) by equation (3.5) reads

\[ \sum_E P(E; 2) z^E = \frac{z^2}{(z - 1)^2 (1 + z)}. \tag{5.22} \]

Substituting equation (5.22) into equation (5.1) gives

\[ P(E; 2) = \frac{E}{2} + \frac{1}{4} (-1)^E - \frac{1}{4}. \tag{5.23} \]

The generating function of \( P(E; 3) \) by equation (3.5) reads

\[ \sum_E P(E; 3) z^E = -\frac{z^3}{(z - 1)^3 (1 + z) (1 + z + z^2)}. \tag{5.24} \]

Substituting equation (5.24) into equation (5.1) gives

\[ P(E; 3) = \frac{1}{72} \left[ 6E^2 + 16 \cos \left( \frac{2\pi}{3}E \right) - 9 (-1)^E - 7 \right]. \tag{5.25} \]

The generating function of \( P(E; 4) \) by equation (3.5) reads

\[ \sum_E P(E; 4) z^E = \frac{z^4}{(z - 1)^4 (1 + z)^2 (1 + z^2) (1 + z + z^2)}. \tag{5.26} \]

Substituting equation (5.26) into equation (5.1) gives

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A statistical mechanical approach to restricted integer partition functions

Table 1. Some explicit results of $P(E; N)$, $Q(E; N)$, and $P_q(E; N)$ for different $E$.

| $E$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|-----|----|----|----|----|----|----|----|----|----|-----|
| $Q(E; 2)$ | 4 | 9 | 14 | 19 | 24 | 29 | 34 | 39 | 44 | 49 |
| $P(E; 2)$ | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| $Q(E; 3)$ | 4 | 24 | 61 | 114 | 184 | 271 | 374 | 494 | 631 | 784 |
| $P_2(E; 3)$ | 8 | 33 | 74 | 133 | 208 | 299 | 408 | 533 | 674 | 833 |
| $P(E; 3)$ | 8 | 33 | 75 | 133 | 208 | 300 | 408 | 533 | 675 | 833 |
| $Q(E; 4)$ | 1 | 23 | 108 | 297 | 632 | 1154 | 1906 | 2928 | 4263 | 5952 |
| $P_2(E; 4)$ | 6 | 58 | 197 | 465 | 904 | 1556 | 2461 | 3663 | 5202 | 7120 |
| $P_3(E; 4)$ | 9 | 63 | 206 | 477 | 920 | 1574 | 2484 | 3688 | 5231 | 7152 |
| $P(E; 4)$ | 9 | 64 | 206 | 487 | 920 | 1575 | 2484 | 3689 | 5231 | 7153 |
| $Q(E; 5)$ | 0 | 7 | 84 | 377 | 1115 | 2611 | 5260 | 9542 | 16019 | 25337 |
| $P_2(E; 5)$ | 2 | 57 | 312 | 995 | 2419 | 4980 | 9157 | 15512 | 24692 | 37425 |
| $P_3(E; 5)$ | 5 | 80 | 370 | 1106 | 2599 | 5246 | 9525 | 16000 | 25315 | 38201 |
| $P_4(E; 5)$ | 6 | 83 | 376 | 1114 | 2610 | 5259 | 9541 | 16018 | 25336 | 38224 |
| $P(E; 5)$ | 7 | 84 | 377 | 1115 | 2611 | 5260 | 9542 | 16019 | 25337 | 38225 |

The generating function of $P(E; 4)$ by equation (5.27) reads

$$P(E; 4) = \frac{1}{288} \left[ 2E^3 + 6E^2 - 9E + 9(-1)^E E + 9(-1)^E + 18(-i)^E + 18iE - 32U_E \left( -\frac{1}{2} \right) - 13 \right].$$

The generating function of $P(E; 5)$ by equation (3.11) reads

$$\sum_E P(E; 5) z^E = \frac{z^5}{(z - 1)^5 (1 + z)^2 (1 + z^2) (1 + z + z^2) (1 + z + z^2 + z^3 + z^4)}.$$  

Substituting equation (5.28) into equation (5.1) gives

$$P(E; 5) = \frac{1}{86400} \left[ 30E^4 + 300E^3 + 300E^2 - 2250E - 1350E(-1)^E + 3375(-1)^E + 5400 \sin \left( \frac{E}{2}\pi \right) - 3200\sqrt{3}\sin \left( \frac{2}{3}E\pi \right) - 5400 \cos \left( \frac{E}{2}\pi \right) - 3200 \cos \left( \frac{2}{3}E\pi \right) + 6912 \cos \left( \frac{2}{5}E\pi \right) + 6912 \cos \left( \frac{4}{5}E\pi \right) - 1849 \right].$$

It is worthy to note that here, though there are imaginary units in the expressions, the results are real. To illustrate this, we list some explicit results of $P(E; N)$, $Q(E; N)$, and $P_q(E; N)$ for different $E$ in Table 1.

The results of $P(E; N)$ given by equations (5.23), (5.25), (5.27) and (5.29) coincide with the results obtained by the recursive method [1, 2, 41].

6. Conclusions

In this paper, starting from the canonical partition function of various kinds of quantum ideal gases, we obtain the generating function of some restricted integer partition functions that count the number of integer partitions with length $N$. We calculate the exact

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expression of restricted integer partition functions from the corresponding generating functions which are constructed by resorting to statistical mechanics.

We introduce a new type of restricted integer partition functions. The restricted integer partition functions introduced in the present paper corresponds to general statistics which is a generalization of Gentile statistics proposed in [13]. Many kinds of integer partition functions are special cases of this restricted integer partition function. This allows us to consider a number of restricted partition functions in a unified framework.

We also obtain a relation between the integer partition function and the symmetric function. Concretely, we show that the generating function of the restricted integer partition functions corresponding to ideal Bose, Fermi, and Gentile gases are symmetric functions and can be expressed as linear combinations of the $S$-function which is an important class of the symmetric function.

We also provides some expressions of restricted integer partition functions, by use of the approach suggested in the paper, as examples.

The generating function of the restricted integer partition functions obtained in the present paper is obtained from canonical partition functions which is from the canonical ensemble. It is in principle possible to calculate various restricted integer partition functions from other statistical ensembles. For example, the thermodynamic quantities calculated in the canonical ensemble can also be obtained in the grand canonical ensemble to some extent [47, 48]. Furthermore, the canonical partition function is nothing but a kind of spectral functions which are functions of the eigenvalue of a system and various spectral functions can be achieved from each other [49–51]. This in principle allows us to start with other statistical ensembles and thermodynamic quantities. All the information of a thermodynamic system is embodied in the corresponding mechanical system [52, 53], which inspires us to find relations between partitions and mechanics. Moreover, the result provided in the present paper may be useful in the additive and diophantine problems [37, 54–56].

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**Appendix. The expression of the generating function of $P^{\{\epsilon\}}(E;N)$, $Q^{\{\epsilon\}}(E;N)$, and $P_q^{\{\epsilon\}}(E;N)$**

In this appendix, we express the generating function of $P^{\{\epsilon\}}(E;N)$ and $Q^{\{\epsilon\}}(E;N)$ in terms of the determinant of certain matrices and list the explicit expressions of the generating function of $P_q^{\{\epsilon\}}(E;N)$ for $N = 3, 4, 5, 6$. The details of the calculation can be found in [14].

**A.1. The matrix form of expressions of $P^{\{\epsilon\}}(E;N)$ and $Q^{\{\epsilon\}}(E;N)$**

The generating functions of $P^{\{\epsilon\}}(E;N)$ and $Q^{\{\epsilon\}}(E;N)$ are given in equations (3.5) and (3.11) in terms of $S$-functions. However, the $S$-functions $(N)(x_1, x_2, \ldots)$ and
A statistical mechanical approach to restricted integer partition functions

\((1^N, x_1, x_2, \ldots)\) can be represented as the determinant of a matrix \([15, 16]\). In this section, we express the generating function of \(P^{(e)}(E; N)\) and \(Q^{(e)}(E; N)\) in matrices forms.

\[
\sum_E Q^{(e)}(E, N) z^E = \frac{1}{N!} \det \begin{pmatrix}
    p(z) & 1 & 0 & \ldots & 0 \\
    p(z^2) & p(z) & 2 & \ldots & 0 \\
    p(z^3) & p(z^2) & p(z) & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    p(z^N) & p(z^{N-1}) & \ldots & \ldots & p(z)
\end{pmatrix}, \quad (A.1)
\]

where \(p(z) = \sum_{e \in \{e\}} z^e\).

\[
\sum_E P^{(e)}(E, N) z^E = \frac{1}{N!} \det \begin{pmatrix}
    p(z) & -1 & 0 & \ldots & 0 \\
    p(z^2) & p(z) & -2 & \ldots & 0 \\
    p(z^3) & p(z^2) & p(z) & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    p(z^N) & p(z^{N-1}) & \ldots & \ldots & -(N-1)
\end{pmatrix}. \quad (A.2)
\]

**A.2. The explicit expression of \(P^{(e)}(E; N)\) for \(N = 3, 4, 5,\) and 6**

In this section, we list the explicit expressions of \(P^{(e)}(E; N)\) for \(N = 3, 4, 5,\) and 6. The details of the calculation can be found in [14].

\[
\sum_E P^{(e)}(E, 3) z^E = \frac{1}{6} p(z)^3 + \frac{1}{2} p(z) p(z^2) - \frac{2}{3} p(z^3). \quad (A.3)
\]

\[
\sum_E P^{(e)}(E, 4) z^E = \frac{1}{24} p(z)^4 + \frac{1}{4} p(z)^2 p(z^2) + \frac{1}{8} p(z^2)^2 - \frac{2}{3} p(z^3) p(z) + \frac{1}{4} p(z^4). \quad (A.4)
\]

\[
\sum_E P^{(e)}(E, 4) z^E = \frac{1}{24} p(z)^4 + \frac{1}{4} p(z)^2 p(z^2) + \frac{1}{8} p(z^2)^2 + \frac{1}{3} p(z^3) p(z) - \frac{3}{4} p(z^4). \quad (A.5)
\]

\[
\sum_E P^{(e)}(E, 5) z^E = \frac{1}{120} p(z)^5 + \frac{1}{12} p(z)^3 p(z^2) + \frac{1}{8} p(z) p(z^2)^2 \\
- \frac{1}{3} p(z)^2 p(z^3) - \frac{1}{3} p(z^2) p(z^3) + \frac{1}{4} p(z) p(z^4) + \frac{1}{5} p(z^5). \quad (A.6)
\]

\[
\sum_E P^{(e)}(E, 5) z^E = \frac{1}{120} p(z)^5 + \frac{1}{12} p(z)^3 p(z^2) + \frac{1}{8} p(z) p(z^2)^2 \\
+ \frac{1}{6} p(z)^2 p(z^3) + \frac{1}{6} p(z^2) p(z^3) - \frac{3}{4} p(z) p(z^4) + \frac{1}{5} p(z^5). \quad (A.7)
\]

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A statistical mechanical approach to restricted integer partition functions

\[ \sum_{E} P_{4}^{(z)} (E, 5) z^{E} = \frac{1}{120} p(z)^{5} + \frac{1}{12} p(z)^{3} p(z^{2}) + \frac{1}{8} p(z) p(z^{2})^{2} \]
\[ \quad + \frac{1}{6} p(z)^{2} p(z^{3}) + \frac{1}{6} p(z^{2}) p(z^{3}) + \frac{1}{4} p(z) p(z^{4}) - \frac{4}{5} p(z^{5}). \quad (A.8) \]

\[ \sum_{E} P_{2}^{(z)} (E, 6) z^{E} = \frac{1}{6} p(z)^{6} + \frac{1}{48} p(z)^{4} p(z^{2}) + \frac{1}{16} p(z)^{2} p(z^{2})^{2} + \frac{1}{48} p(z^{2})^{3} \]
\[ \quad - \frac{1}{9} p(z)^{3} p(z^{3}) - \frac{1}{3} p(z) p(z^{2}) p(z^{3}) + \frac{2}{9} p(z^{3})^{2} \]
\[ \quad + \frac{1}{8} p(z)^{2} p(z^{4}) + \frac{1}{8} p(z^{2}) p(z^{4}) + \frac{1}{5} p(z^{5}) p(z) - \frac{1}{5} p(z^{6}) \quad (A.9) \]

\[ \sum_{E} P_{3}^{(z)} (E, 6) z^{E} = \frac{1}{6} p(z)^{6} + \frac{1}{48} p(z)^{4} p(z^{2}) + \frac{1}{16} p(z)^{2} p(z^{2})^{2} + \frac{1}{48} p(z^{2})^{3} \]
\[ \quad + \frac{1}{18} p(z)^{3} p(z^{3}) + \frac{1}{6} p(z) p(z^{2}) p(z^{3}) + \frac{1}{18} p(z^{3})^{2} \]
\[ \quad - \frac{3}{8} p(z)^{2} p(z^{4}) - \frac{3}{8} p(z^{2}) p(z^{4}) + \frac{1}{5} p(z^{5}) p(z) + \frac{6}{5} p(z^{6}) \quad (A.10) \]

\[ \sum_{E} P_{4}^{(z)} (E, 6) z^{E} = \frac{1}{6} p(z)^{6} + \frac{1}{48} p(z)^{4} p(z^{2}) + \frac{1}{16} p(z)^{2} p(z^{2})^{2} + \frac{1}{48} p(z^{2})^{3} \]
\[ \quad + \frac{1}{18} p(z)^{3} p(z^{3}) + \frac{1}{6} p(z) p(z^{2}) p(z^{3}) + \frac{1}{18} p(z^{3})^{2} \]
\[ \quad + \frac{1}{8} p(z)^{2} p(z^{4}) + \frac{1}{8} p(z^{2}) p(z^{4}) - \frac{4}{5} p(z^{5}) p(z) + \frac{1}{6} p(z^{6}), \quad (A.11) \]

\[ \sum_{E} P_{5}^{(z)} (E, 6) z^{E} = \frac{1}{6} p(z)^{6} + \frac{1}{48} p(z)^{4} p(z^{2}) + \frac{1}{16} p(z)^{2} p(z^{2})^{2} + \frac{1}{48} p(z^{2})^{3} \]
\[ \quad + \frac{1}{18} p(z)^{3} p(z^{3}) + \frac{1}{6} p(z) p(z^{2}) p(z^{3}) + \frac{1}{18} p(z^{3})^{2} \]
\[ \quad + \frac{1}{8} p(z)^{2} p(z^{4}) + \frac{1}{8} p(z^{2}) p(z^{4}) + \frac{1}{5} p(z^{5}) p(z) - \frac{5}{6} p(z^{6}) \quad (A.12) \]

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