HYPERBOLIC FIBERED SLICE KNOTS WITH RIGHT-VEERING MONODROMY

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Abstract. We construct a hyperbolic fibered slice knot with right-veering monodromy, which disproves a conjecture posed in [HKK+21].

1. INTRODUCTION

We give a negative answer to Question 8.2 posed in [HKK+21].

Question 1.1. If $K$ is a hyperbolic fibered slice knot, does the fractional Dehn twist coefficient (FDTC) of the monodromy vanish?

We construct a hyperbolic slice fibered knot $K'$ with positive FDTC.

1.1. Motivation. The authors [HKK+21] observes that many low-crossing slice fibered knots have zero FDTC. Any $(p, 1)$–cable of slice fibered knot is still slice fibered, whereas FDTC of $(p, 1)$–cable equals to $1/p$. The authors therefore ask the above question about hyperbolic fibered slice knots.

Baldwin, Ni and Sivek [BNS22, Corollary 1.7] prove the following related proposition in terms of the $\tau$–invariant in Heegaard Floer homology:

Proposition 1.2. If $K \subset S^3$ is a fibered knot with thin knot Floer homology such that $\tau(K) < g(K)$, then FDTC vanishes.

The $\tau$–invariant vanishes for slice knots. Proposition 1.2 explains the case for low-crossing fibered slice knots because many of those are thin.

We have an immediate corollary:

Corollary 1.3. The knot Floer homology of $K'$ is not thin.

1.2. Organization. We follow the recipe of Kazez and Roberts [KR13] to construct hyperbolic fibered knots with positive FDTC. The search for ribbon knot is inspired by the work of Hitt and Silver [HS91]. In section 2 we review Nielsen-Thurston classification of surface automorphism and examples from Kazez and Roberts. We construct our example $K'$ in section 3.

2. MONODROMIES OF FIBERED KNOTS IN $S^3$

2.1. Surface automorphism. We first recall Nielsen-Thurston classification of surface automorphisms:

Theorem 2.1. [CCB88, Thu88] Let $S$ be an oriented hyperbolic surface with geodesic boundary, and let $h \in Aut(S, \partial S)$. Then $h$ is freely isotopic to either
(1) a pseudo-Anosov homeomorphism $\phi$ that preserves a pair of geodesic laminations $\lambda^s$ and $\lambda^t$,
(2) a periodic homeomorphism $\phi$ such that $\phi^n = id$ for some $n$,
(3) a reducible homeomorphism $h'$ that preserves a maximal collection of simple closed geodesic curves in $S$. To avoid overlap, we consider $h$ reducible only when it is not periodic.

In particular, we only regard $h$ as reducible only if it is not periodic to avoid overlap. Let $\Phi : S \times [0,1] \to S$ be an isotopy from $h$ to its Thurston representative $\phi$. Considering the restriction of $\Phi$ to the boundary $\partial S$, we have a homeomorphism:

$$\Phi_\partial : \partial S \times [0,1] \to \partial S \times [0,1]$$

defined by $\Phi_\partial(x,t) = (\Phi_t(x),t)$. The fractional Dehn twist coefficient $c(h)$ can be defined as the winding number of the arc $\Phi_\partial(\theta \times [0,1])$. Nielson-Thurston classification guarantees that $c(h) \in \mathbb{Q}$.

Thurston proved that a fibered knot is hyperbolic if and only if its monodromy is pseudo-Anosov. Fractional Dehn twist coefficient is closely related to the following notion of right-veeringness.

**Definition 2.2.** [HKM07] A homeomorphism $h \in Aut(S, \partial S)$ is called right-veering if for every based point $x \in \partial S$ and every properly embedded arc $\alpha$ starting at $x$, $h(\alpha)$ is to the right of $\alpha$, after isotoping $h(\alpha)$ so that it intersects $\alpha$ minimally. Similarly, $h$ is called left-veering if $h(\alpha)$ is to the left of $\alpha$.

**Proposition 2.3.** [HKM07] $h$ is right-veering if and only if $c(h) > 0$ for every component of $\partial S$, and $h$ is left-veering if and only if $c(h) < 0$ for every component of $\partial S$.

If $c(h) = 0$, one can find two arcs such that one is moved by $h$ to the right and the other to the left. The significance of right-veeringness is highlighted by the following theorem of Honda, Kazez and Matić:

**Theorem 2.4.** [HKM09] Every open book that is compatible with a tight contact structure is right-veering.

A large source of examples of reducible right-veering homeomorphism comes from the class of fibered cable knots. Indeed, if $h$ is the monodromy of a fibered $(p,q)$–cable knot $K_{p,q}$ with Seifert surface $S$, then $c(h) = 1/pq$ and $h$ is reducible. Let $\{C_i\}$ be the collection of curves preserved by $h'$. $\{C_i\}$ partitions $S$ into subsurfaces $\{S_j\}$ permuted by $h'$. Let $S_0$ be the subsurface containing $\partial S = K_{p,q}$, then $h'|_{S_0}$ is periodic. Kazez and Roberts characterize the monodromy $h$ of a fibered knot $K$ in $S^3$ in the following theorem:

**Theorem 2.5.** [KR13]

(1) If $h$ is periodic, then $K$ is the unknot or a $(p,q)$–torus knot.
(2) If $h$ has a reducible Thurston representative $h'$ with periodic $h'|_{S_0}$, then $K$ is a $(p,q)$–cable knot, and $c(h) = 1/pq$.
(3) [Gab97] If $h$ is either pseudo-Anosov or reducible with $h'|_{S_0}$ pseudo-Anosov. Then either $c(h) = 0$ or $c(h) = 1/r$, where $2 \leq |r| \leq 4g(K) - 2$.

**Corollary 2.6.** $c(h) = 0$ or $1/r$ for some integer $r$, $|r| \geq 2$. In particular, $|c(h)| \leq 1/2$.

In particular, the $(2,1)$–cable of a fibered knot in $S^3$ has its monodromy attaining maximum FDTC. We review hyperbolic case in the next section.
2.2. Stallings’ twist and \((2,1)\)-cable. Let \(U\) be an unknot properly embedded in a surface \(F\). We say \(U\) is untwisted relative to \(F\) if \(U\) bounds a disk transverse to \(F\) along \(U\). A Stallings’ twist \cite{Sta78} is a surgery along such an untwisted \(U\). Kazez and Roberts apply Stallings’ twist on \((2,1)\)-cables to produce hyperbolic fibered knots with maximum FDTC \(= 1/2\).

Let \((S, h)\) be an open book decomposition of \(S^3\) with connected binding \(K\), where \(h\) is pseudo-Anosov and \(c(h) = 0\). Let \(K_{2,1}\) be the \((2,1)\)-cable of \(K\). The fibered surface \(\Sigma\) of \(K_{2,1}\) can be viewed as the union of two copies \(S_0, S_1\) of \(S\) connected by a 1-handle. Let \(H\) be the monodromy of this new open book.

We choose a simple closed curve \(C\) in \(\Sigma\) such that \(C_0 = C \cap S_0\) and \(C_1 = C \cap S_1\) are two essential arcs. Moreover, we require \(C_i\) to be nonseparating in \(S_i\). Let \(T_C\) be the right-handed Dehn twist along \(C\) and \(H' = T_C \circ H\).

**Theorem 2.7.** \cite{KR13} \(H'\) is pseudo-Anosov and \(c(H') = 1/2\).

3. RIBBON FIBERED KNOT

We are ready to construct a hyperbolic ribbon fibered knot with positive FDTC. Let \(K\) be the knot \(10_{153}\) from Rolfsen’s knot table. \(K\) is a hyperbolic ribbon fibered knot with 3-genus 3. Figure 1 is a ribbon diagram for \(10_{153}\).

![Figure 1. A ribbon diagram for the knot \(K = 10_{153}\)](image)

Let \(h\) denote the monodromy. According to \cite{CL}, \(h\) can be presented as described in Figure 2. One can see that \(h\) is neither right-veering nor left-veering by choosing different endpoints of \(\gamma\). Therefore, \(c(h) = 0\).
Figure 2. Monodromy of the fibered knot 10_{153}. $h$ can be presented as the word \textbf{abcBEGhcd}, where $x$ denotes a right-handed Dehn twist about $x$ and $X$ denotes a left-handed Dehn twist about $x$. A word is read from right to left so that $aB$ means perform a left-handed Dehn twist about $b$ then perform a right-handed Dehn twist about $a$.

A Seifert surface $S$ of $K$ can be obtained by Seifert’s algorithm as explained in Figure 3. The genus of $F$ is 3 so that $F$ is the fibered surface.

Figure 3. The surface obtained by Seifert’s algorithm has genus 3. $c$ is a non-separating properly embedded arc on the surface.

Let $K_{2,1}$ be the $(2,1)$–cable of $K$ (Figure 4). The twisted band connecting the two copies of $K$ is added at $p$. $c$ is a nonseparating properly embedded arc on the fibered surface of $K$. $K_{2,1}$ is also fibered whose fibered surface $\Sigma$ can be obtained by connecting two copies of $S$. 
with the same twisted band at \( p \). Then define a simple closed curve \( C \) to be a band sum of the two copies of \( c \) along an arc running across the twisted band.

![Figure 4. (2, 1)—cable of \( K \).](image)

Let \( T_C \) denote the right-handed Dehn twist along \( C \), and denote the resulting fibered knot \( K' \). By [KR13, Corollary 4.6], the monodromy \( T_C \circ H \) is pseudo-Anosov and right-veering with \( c(T_C \circ H) = \frac{1}{2} \).

Recall that \( K = 10_{153} \) is a ribbon knot, so is \( K_{2,1} \). \( C \) is an unknotted untwisted curve. Performing a right-handed Dehn twist along \( C \) has the same effect on \((S^3, K_{2,1})\) applying a \((-1)\)—surgery along \( C \). The resulting manifold is still \( S^3 \) and we have a new knot \( K' \). \( C \) winds around two copies of a ribbon band (Figure 5).

![Figure 5. The curve \( C \) winds around two copies of a ribbon band. This figure shows one of the ribbon. The other ribbon is on the other copy from the \((2, 1)\)—cable.](image)
Figure 6 illustrates the effect of \((-1)\)-surgery along \(C\) to the ribbon bands. The resulting knot \(K'\) is still a ribbon knot.

**Theorem 3.1.** \(K'\) is a hyperbolic ribbon fibered knot with \(\text{FDTC} = 1/2\). Hence, the monodromy is right-veering.

![Diagram of knot and surgeries](image)

**Figure 6.** The effect of \((-1)\)-surgery along \(C\) after isotopy.

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