On the origin of families of fermions and their mass matrices

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(Dated: July 2, 2018)

Abstract

We are proposing a new way of describing families of quarks and leptons, using the approach unifying all the internal degrees of freedom, proposed by one of us. Spinors, living in $d = 1 + 13$–dimensional space, carry in this approach only the spin and interact with only the gravity through vielbeins and two kinds of the spin connection fields - the gauge fields of the Poincaré group ($p^a$, $S^{ab}$) and the second kind of the Clifford algebra objects ($\tilde{S}^{ab}$). All the quarks and the leptons of one family appear in one Weyl representation of a chosen handedness of the Lorentz group, if analyzed with respect to the Standard model gauge groups: the right handed (with respect to $SO(1,3)$) weak chargeless quarks and leptons and the left handed weak charged quark and leptons. A part of the starting Lagrange density of a Weyl spinor in $d = 1 + 13$ transforms right handed quarks and leptons into left handed quarks and leptons manifesting as the Yukawa couplings of the Standard model. The second kind of Clifford algebra objects generates families of quarks and leptons and contributes to diagonal and off diagonal Yukawa couplings.

The approach predicts an even number of families, treating leptons and quarks equivalently. In this paper we investigate within this approach the appearance of the Yukawa couplings within one family of quarks and leptons as well as among the families (without assuming any Higgs fields). We present the mass matrices for four families and investigate whether our way of generating families might explain the origin of families of quarks and leptons as well as their observed properties - the masses and the mixing matrices. Numerical results are presented in the paper following this one.
I. INTRODUCTION

The Standard model of the electroweak and strong interactions (extended by the inclusion of the massive neutrinos) fits well all the existing experimental data. It assumes around 25 parameters and constraints, the origin of which is not yet understood. Questions like: Why has Nature chosen $SU(3) \times SU(2) \times U(1)$ to describe the charges of spinors and $SO(1, 3)$ to describe the spin of spinors?, Why are the left handed spinors weak charged, while the right handed spinors are weak chargeless?, Where do the Yukawa couplings (together with the weak scale and the families of quarks and leptons) come from?, and many others, remain unanswered.

The advantage of the approach, unifying spins and charges\[1, 2, 3, 4, 5, 6, 7, 8, 9, 10\], is that it might offer possible answers to the open questions of the Standard electroweak model. We demonstrated in references\[6, 8, 9, 10\] that a left handed $SO(1, 13)$ Weyl spinor multiplet includes, if the representation is interpreted in terms of the subgroups $SO(1, 3)$, $SU(2)$, $SU(3)$ and the sum of the two $U(1)$’s, all the spinors of the Standard model - that is the left handed $SU(2)$ doublets and the right handed $SU(2)$ singlets of (with the group $SU(3)$ charged) quarks and (chargeless) leptons. Right handed neutrinos - weak and hyper chargeless - are also included. In the gauge theory of gravity (in our case in $d = (1 + 13)$-dimensional space), the Poincaré group is gauged, leading to spin connections and vielbeins, which determine the gravitational field\[2, 8, 11\]. There are vielbein fields and spin connection fields, which might manifest - after the appropriate compactification (or some other kind of making the rest of d-4 space unobservable at low energies) - in the four dimensional space-time as all the gauge fields of the known charges, as well as the Yukawa couplings within each family. No additional Higgs field is needed to generate masses of families and to “dress” the right handed spinors with the weak charge. It is a part of the starting Lagrangean in $d \geq 1 + 13$, which manifests in $d = 1 + 3$ as the Yukawa coupling and does what the Higgs does in the Standard model. If assuming a second kind of the Clifford algebra objects, the corresponding gauge fields manifest as the Yukawa couplings among families (contributing also to Yukawa couplings within each family).

In the refs.\[3, 10, 14, 15, 16, 17\] it was shown, that the approach unifying spins and charges might explain the Yukawa couplings if an appropriate break of both symmetries, connected with the two kinds of the Clifford algebra objects, appears. An even number of
families is predicted, in particular, the fourth family of quarks and leptons might appear under certain conditions at low energies in agreement with [18].

The approach seems to have, like all the Kaluza-Klein-like theories, a very serious disadvantage, namely that there might not exist any massless, mass protected spinors, which are, after the break of symmetries, chirally coupled to the desired (Kaluza-Klein) gauge fields [12]. This would mean that there are no observable spinors at low energies. Since the idea that it is only one internal degree of freedom - the spin - and the Kaluza-Klein idea that the gravity is the only gauge field, are beautiful and attractive, we have tried hard to find any example, which would give hope to Kaluza-Klein-like theories by demonstrating that a kind of a break of symmetries leads to massless, mass protected spinors, chirally coupled to the Kaluza-Klein gauge fields, observable at low energies. We discuss in ref. [13] such a case - a toy model of a spinor, living in $d(= 1 + 5)$-dimensional space, which breaks into a finite disk with the boundary, which allows spinors of only one handedness. Although not yet realistic, the toy model looks promising.

In the present paper we analyze how do families of quarks and leptons, and accordingly also the Yukawa couplings, appear within the approach unifying spins and charges. We comment on the type of contributions to the Yukawa couplings and discuss some general properties of the mass matrices, which follow from the assumptions of the approach, trying to find out whether the approach could show a possible answer to the questions: What is the origin of the families of quarks and leptons?, What does determine the Yukawa couplings?, Why only the left handed quarks and leptons carry the weak charge?.

Since we do not know, which way of breaking the starting symmetries of the approach is the appropriate one and since results of the investigation drastically depend on the way of breaking symmetries and might as well depend on non adiabatic processes following the break of symmetries, this paper (and also the paper which follows this one and represent some numerical investigations) can only be understood as an attempt to see whether the approach unifying spins and charges has a chance to explain the origin of families of quarks and leptons and their properties and to which extend might it help to understand the appearance of families and the Yukawa couplings.

We are not (yet) performing the calculations of breaking the symmetry $SO(1,13)$ to $SO(1,7) \times U(1) \times SU(3)$ within our approach. (Some very rough estimates can be found in ref. [19].) The break of symmetries influences both kinds of gauge fields, although we can
not yet tell indeed in which way. Therefore, we can not tell the strength of the fields which appear in the Yukawa couplings as "the vacuum expectation values" and which lead further to $SO(1, 3) \times U(1) \times SU(3)$. We only can evaluate (after making some assumptions) several relations among the spin connection fields. Using then these very preliminary relations and the known experimental data, we can make a prediction for the number of families at "physical energies" and discuss properties of quarks and leptons within this approach: their masses and mixing matrices. Accordingly the results can be taken only as a first step in analyzing properties of families of quarks and leptons within the approach unifying spins and charges, which might offers a mechanism for generating families and correspondingly the Yukawa couplings. We shall present some numerical results in the paper, following this one.

In Sect II of this paper we present the action for a Weyl spinor in $(1 + 13)$-dimensional space within our approach and suggest a break of the symmetry $SO(1, 13)$ to $SO(1, 7) \times SO(6)$ and further. We assume that the break of $SO(1, 13)$ to $SO(1, 7) \times U(1) \times SU(3)$ does lead to massless Weyl spinors with the $U(1) \times SU(3)$ charges. The main point of this paper is to demonstrate that while one Weyl spinor representation of $SO(1, 13)$, if analyzed with respect to subgroups $SO(1, 3) \times SU(2) \times U(1) \times SU(3)$, contains all the spinors needed in the Standard model (the right handed weak chargeless quarks and leptons and left handed weak charged quarks and leptons), the starting action for a Weyl spinor, which carries only (two kinds of) the spin and no charges and interacts with only the gravitational field, includes the Yukawa couplings, which transform the right handed weak chargeless spinors into left handed weak charged spinors and contribute to the mass terms just as it is suggested by the Standard model, without assuming the existence of a Higgs weak charged doublet. There are, namely, the generators of the Lorentz transformations within the group $SO(1, 7)$ in our model ($S^{0s}, s = 7, 8$, for example), which take care of what in the Standard model the Higgs doublet, together with $\gamma^0$, does.

In Subsect II A we comment on a possible break of the starting symmetry $SO(1, 13)$ the internal symmetry of which is connected by $S^{ab}$, while in Subsect II B of Sect II we discuss properties of the group $SO(1, 13)$ in terms of subgroups, which appear in the Standard electroweak model. In the same section, Subsect II C we present briefly the technique [14, 15], which turns out to be very helpful when discussing spinor representations, since it allows to generate as well as present spinor representations and families of spinor representations
in a very transparent way. In particular, the technique helps to point out very clearly how do the Yukawa couplings appear in our approach. In Subsect II D we comment on the appearance of families within our technique [2, 15].

In Sect III we discuss in details within our approach the appearance of the Yukawa couplings within one family, while in Sect IV we discuss the number of families as well as the Yukawa couplings among the families.

In Sect V we present, after making several assumptions and simplifications, a possible explicit expression for the mass matrices for four families of quarks and leptons in terms of the spin connection fields.

II. WEYL SPINORS IN \( d = (1 + 13) \) MANIFESTING FAMILIES OF QUARKS AND LEPTONS IN \( d = (1 + 3) \)

We start with a left handed Weyl spinor in \((1 + 13)\)-dimensional space. A spinor carries no charges, only two kinds of spins and interacts accordingly with only gauge gravitational fields - with spin connections and vielbeins. We assume two kinds of the Clifford algebra objects defining two kinds of the generators of the Lorentz algebra and allow accordingly two kinds of gauge fields[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. One kind is the ordinary gauge field (gauging the Poincaré symmetry in \( d = 1 + 13 \)). The corresponding spin connection field appears for spinors as a gauge field of \( S^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) \), where \( \gamma^a \) are the ordinary Dirac operators. The contribution of these fields to the mass matrices manifests in only the diagonal terms (connecting right handed weak chargeless quarks or leptons with left handed weak charged partners within one family of spinors).

The second kind of gauge fields is in our approach responsible for the appearance of families and consequently for the Yukawa couplings among families of spinors (contributing also to diagonal matrix elements) and will be used in this paper to explain the origin of the families of quarks and leptons. The corresponding spin connection fields appear for spinors as a gauge field of \( \tilde{S}^{ab} (\tilde{S}^{ab} = \frac{1}{2}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)) \) with \( \tilde{\gamma}^a \), which are the Clifford algebra objects[2, 15], like \( \gamma^a \), but anticommute with \( \gamma^a \).

Accordingly we write the action for a Weyl (massless) spinor in \( d(= 1 + 13) \) - dimensional
space as follows:

\[ S = \int d^4x \mathcal{L} \]

\[ \mathcal{L} = \frac{1}{2} (\bar{\mathbf{E}} \gamma^a p_0 \mathbf{p}_a \psi + h.c.) + \frac{1}{2} (E \bar{\mathbf{E}} \gamma^a f^a_0 p_0 \mathbf{p}_a \psi + h.c.), \]

\[ p_0 = p_0 + \frac{1}{2} S^{ab} \omega_{aba} - \frac{1}{2} \bar{S}^{ab} \bar{\omega}_{aba}. \]

(1)

Here \( f^a_0 \) are vielbeins (inverse to the gauge field of the generators of translations \( e^a_\alpha \), \( e^a_\alpha f^b_a = \delta^*_b \), \( e^a_\alpha f^\beta_a = \delta^*_\beta \)), with \( E = \text{det}(e^a_\alpha) \), while \( \omega_{aba} \) and \( \bar{\omega}_{aba} \) are the two kinds of the spin connection fields, the gauge fields of \( S^{ab} \) and \( \bar{S}^{ab} \), respectively, corresponding to the two kinds of the Clifford algebra objects \([10, 14]\), namely \( \gamma^a \) and \( \bar{\gamma}^a \), with the property \( \{\gamma^a, \bar{\gamma}^b\}_+ = 0 \), which leads to \( \{S^{ab}, \bar{S}^{cd}\}_- = 0 \). We shall discuss the properties of these two kinds of \( \gamma^a \)'s in Subsects. [I]C and [I]D.

To see that one Weyl spinor in \( d = (1 + 13) \) with the spin as the only internal degree of freedom, can manifest in four-dimensional "physical" space as the ordinary \( (SO(1, 3)) \) spinor with all the known charges of one family of quarks and leptons of the Standard model, one has to analyze one Weyl spinor (we make a choice of the left handed one) representation in terms of the subgroups \( SO(1, 3) \times U(1) \times SU(2) \times SU(3) \). We shall do this in Subsect. [I]B of this section. (The reader can see this analyses in several references, like the one in [10].)

To see that the Yukawa couplings are the part of the starting Lagrangean of Eq. (1), we rewrite the Lagrangean in Eq. (1) as follows:

\[ \mathcal{L} = \bar{\psi} \gamma^m (p_m - \sum_{A,i} g^A \tau^A_i A^A_m) \psi + \sum_{s=7,8} \bar{\psi} \gamma^s p_0 s \psi + \text{the rest.} \]

(2)

Index \( A \) determines the charge groups \( (SU(3), SU(2) \) and the two \( U(1)'s \), index \( i \) determines the generators within one charge group. \( \tau^A_i \) denote the generators of the charge groups (expressible in terms of \( S^{st}_m \)), \( s, t \in 5, 6, \ldots, 14 \), while \( A^A_m, m = 0, 1, 2, 3 \), denote the corresponding gauge fields (expressible in terms of \( \omega_{stm} \)).

The second term can be rewritten in terms of the kinetic part \( \bar{\psi} \gamma^0 \gamma^s p_s \psi \) and the part \( -\bar{\psi} \gamma^0 \gamma^s S^{tt}_s f^s_\omega \omega_{tt'}psi - \bar{\psi} \gamma^0 \gamma^s \bar{S}^{tt'} \bar{f}^s_\omega \bar{\omega}_{tt'} \psi \), which looks like a mass term (also the kinetic term, if nonzero, contributes to the mass term), since \( f^s_\omega \omega_{tt'} \) and \( f^s_\omega \bar{\omega}_{tt'} \), \( s, t \in 5, 6, 7, 8, \sigma \in (5), (6), (7), (8) \), behave in \( d(= 1 + 3) \) - dimensional space like scalar fields, while the operator \( \gamma^0 \gamma^s, s = 7, 8 \), for example, transforms a right handed weak chargeless spinor (for example
$e_R$) into a left handed weak charged spinor (in this case to $e_L$), without changing the spin in $d = 1 + 3$ (Subsect II C, Eq.(12) and the third and the fifth row of Table II or the fourth and the sixth row of the same table) - just what the Yukawa couplings with the Higgs doublet included do in the Standard model formulation. The reader will find the detail explanation in Subsects II C, II D. It should be pointed out that no Higgs weak charge doublet is needed here, as $S^0_s, s = 7, 8$ does its job.

One can always rewrite the Lagrangean from Eq.(1) in the way of Eq.(2). The question is, of course, what are the terms, which are in Eq.(2) written under ”the rest” and whether they can be assumed as negligible at ”low energy world”. We have no proof that any break of symmetry, presented in Subsect II A leads to such an effective Lagrangean, which would after the first break (or several successive breaks) of the starting symmetry of $SO(1, 13)$ manifest any massless spinors, which would then, after further breaks, manifest in the ”physical space” the masses corresponding to the Yukawa couplings of Eq.(2), while all the rest terms are negligibly small. We just assume instead, that we start with the Lagrangean of Eq.(2) and then study properties of the system, described by such a Lagrange density at ”physical” energies.

We also would like to point out that the fact that the generators of families of spinors $\tilde{S}^{ab}$ and the generators of the Lorentz transformations of spinors $S^{ab}$ commute ($\{\tilde{S}^{ab}, S^{ab}\}_- = 0$), suggests that most of properties of quarks and leptons must be the same within this approach. There are namely only the generators of families, which define off diagonal elements of the Yukawa couplings. But they do not at all distinguish among quarks and leptons. Since also in the diagonal matrix elements differ quarks and leptons in only one parameter times the identity, the question arises: What is then the reason for so different mixing matrices of quarks and leptons as observed? Might it be that there are the nonperturbative effects (like in the hadron case when quarks ”dress nonadiabatically” into the clouds of quarks and antiquarks and the gluon field before forming a hadron) which are responsible for so different properties of quarks and leptons? Could instead be that very peculiar breaks of symmetries cause the difference in off diagonal matrix elements for quarks and leptons? Or one must take the appearance of the Majorana fermions into account? The approach by itself gives different off diagonal elements of mass matrices for $u$-quarks and $d$-quarks, and for $\nu$ and electrons (although it still relates them). We shall discuss this point later in this paper, as well as in the paper following this one.
A. Break of symmetries

There are several ways of breaking the group $SO(1, 13)$ down to subgroups of the Standard model. (One of) the most probable breaks, suggested by the approach unifying spins and charges, is the following one

$$\begin{align*}
SO(1, 13) & \\
\downarrow & \\
SO(1, 7) \otimes SU(3) \otimes U(1) & \\
\downarrow & \\
SO(1, 3) \otimes U(1) \otimes SU(3)
\end{align*}$$

We start from a massless left handed Weyl spinor in $d = 1 + 13$. We assume that the first break of symmetries leads again to massless spinors in $d = 1 + 7$, chirally coupled with the $SU(3)$ and $U(1)$ charge to the corresponding fields, which follow from the spin connection and vielbein fields in $d = 1 + 13$. (The reader can find more about this kind of breaking the starting symmetry in ref. [19].) We have no justification for such an assumption (except that we have shown on one toy model [13] that in that very special case such an assumption is justified). And we have no calculation, which would help to guess the strength of the "vacuum expectation values" of the fields. The Yukawa like terms themselves then break further the symmetry, ending up with the "physical" degrees of freedom. An additional non yet solved problem is, how does the break of symmetries influences the part $\tilde{S}^{st}_{st}$. 

B. Spin and charges of one left handed Weyl representation of $SO(1,13)$

We discuss in this subsection the properties of one Weyl spinor representation when analysing the representation in terms of subgroups of the group $SO(1,13)$.

The group $SO(1, 13)$ of the rank 7 has as possible subgroups the groups $SO(1, 3)$ (the "complexified" $SU(2) \times SU(2)$), $SU(2), SU(3)$ and the two $U(1)$’s, with the sum of the ranks of all these subgroups equal to 7. These subgroups are candidates for describing the spin, the weak charge, the colour charge and the two hyper charges, respectively (only one is needed in the Standard model). The generators of these groups can be written in terms
of the generators $S^{ab}$ as follows
\[ \tau^{Ai} = \sum_{a,b} c^{Ai}_{\;\;ab} S^{ab}, \]
\[ \{\tau^{Ai}, \tau^{Bj}\} = i\delta^{AB} f^{Aijk} \tau^{Ak}. \]  
(3)

We could count the two $SU(2)$ subgroups of the group $SO(1, 3)$ in the same way as the rest of subgroups. Instead we shall use $A = 1, 2, 3, 4$, to represent only the subgroups describing charges and $f^{Aijk}$ to describe the corresponding structure constants. Coefficients $c^{Ai}_{\;\;ab}$, with $a, b \in \{5, 6, ..., 14\}$, have to be determined so that the commutation relations of Eq.(3) hold.

The weak charge ($SU(2)$ with the generators $\tau^{11}$) and one $U(1)$ charge (with the generator $\tau^{21}$) content of the compact group $SO(4)$ (a subgroup of $SO(1, 13)$) can be demonstrated when expressing
\[ \tau^{11} := \frac{1}{2}(S^{58} - S^{67}), \quad \tau^{12} := \frac{1}{2}(S^{57} + S^{68}), \quad \tau^{13} := \frac{1}{2}(S^{56} - S^{78}), \quad \tau^{21} := \frac{1}{2}(S^{56} + S^{78}). \]  
(4)

To see the colour charge and one additional $U(1)$ content in the group $SO(1, 13)$ we write $\tau^{3i}$ and $\tau^{41}$, respectively, in terms of the generators $S^{ab}$
\[ \tau^{31} := \frac{1}{2}(S^{912} - S^{1011}), \quad \tau^{32} := \frac{1}{2}(S^{911} + S^{1012}), \quad \tau^{33} := \frac{1}{2}(S^{910} - S^{1112}), \]
\[ \tau^{34} := \frac{1}{2}(S^{914} - S^{1013}), \quad \tau^{35} := \frac{1}{2}(S^{913} + S^{1014}), \quad \tau^{36} := \frac{1}{2}(S^{1114} - S^{1213}), \]
\[ \tau^{37} := \frac{1}{2}(S^{1113} + S^{1214}), \quad \tau^{38} := \frac{1}{2\sqrt{3}}(S^{910} + S^{1112} - 2S^{1314}), \]
\[ \tau^{41} := -\frac{1}{3}(S^{910} + S^{1112} + S^{1314}). \]  
(5)

To reproduce the Standard model groups one must introduce the two superpositions of the two $U(1)$'s generators as follows
\[ Y = \tau^{41} + \tau^{21}, \quad Y' = \tau^{41} - \tau^{21}. \]  
(6)

The above choice of subgroups of the group $SO(1, 13)$ manifests the Standard model charge structure of one Weyl spinor of the group $SO(1, 13)$, with one additional hyper charge.

We may very similarly proceed also with the generators $\tilde{S}^{ab}$ by assuming that a kind of a break makes the starting $SO(1, 13)$ group to manifest in terms of some $\tilde{\tau}^{\tilde{Ai}}$ like
\[ \tilde{\tau}^{\tilde{Ai}} = \sum_{a,b} c^{\tilde{Ai}}_{\;\;ab} \tilde{S}^{ab}, \]
\[
\{\bar{\tau}^{\bar{A}i}, \bar{\tau}^{\bar{B}j}\}_- = i\delta^{\bar{A}\bar{B}} \bar{f}^{\bar{A}ijk} \bar{\tau}^{\bar{A}k}.
\]  
(7)

We shall try to guess the way of breaking through the comparison of the results with the experimental data in the paper following this one.

C. Spinor representation in terms of Clifford algebra objects

In this subsection we briefly present our technique \[14\] for generating spinor representations in any dimensional space. The advantage of this technique is simplicity in using it and transparency in understanding detailed properties of spinor representations. We also show how families of spinors enter into our approach \[2, 15\].

We start by defining two kinds of the Clifford algebra objects, \(\gamma^a\) and \(\tilde{\gamma}^a\), with the properties

\[
\{\gamma^a, \gamma^b\} = 2\eta^{ab}; \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 0.
\]  
(8)

The operators \(\tilde{\gamma}^a\) are introduced formally as operating on any Clifford algebra object \(B\) from the left hand side, but they also can be expressed in terms of the ordinary \(\gamma^a\) as operating from the right hand side as follows

\[
\tilde{\gamma}^a B := i(\sigma^a) B \gamma^a,
\]  
(9)

with \((-)^{n_B} = +1\) or \(-1\), when the object \(B\) has a Clifford even or odd character, respectively.

Accordingly two kinds of generators of the Lorentz transformations follow, namely \(S^{ab} := (i/4)(\gamma^a\gamma^b - \gamma^b\gamma^a)\) and \(\tilde{S}^{ab} := (i/4)(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\), with the property \(\{S^{ab}, \tilde{S}^{cd}\}_- = 0\).

We define a basis of spinor representations as eigen states of the chosen Cartan subalgebra of the Lorentz algebra \(SO(1, 13)\), with the operators \(S^{ab}\) and \(\tilde{S}^{ab}\) in the two Cartan subalgebra sets, with the same indices in both cases.

By introducing the notation

\[
\begin{align*}
{\sigma}^{ab}(\pm i) &:= \frac{1}{2}(\gamma^a \mp i\gamma^b), \\
{\sigma}^{ab}(\pm) &:= \frac{1}{2}(1 \pm \gamma^a\gamma^b), \quad \text{for } \eta^{aa}\eta^{bb} = -1,
\end{align*}
\]

\[
\begin{align*}
{\sigma}^{ab}(\pm i) &:= \frac{1}{2}(\gamma^a \pm i\gamma^b), \\
{\sigma}^{ab}(\pm) &:= \frac{1}{2}(1 \pm i\gamma^a\gamma^b), \quad \text{for } \eta^{aa}\eta^{bb} = 1,
\end{align*}
\]  
(10)

it can be shown that

\[
\begin{align*}
S^{ab} \{\sigma\}^{ab}(k) & = \frac{k}{2} \{\sigma\}^{ab}(k), & S^{ab} \{\sigma\}^{ab}[k] & = \frac{k}{2} \{\sigma\}^{ab}[k], \\
\tilde{S}^{ab} \{\sigma\}^{ab}(k) & = \frac{k}{2} \{\sigma\}^{ab}(k), & \tilde{S}^{ab} \{\sigma\}^{ab}[k] & = -\frac{k}{2} \{\sigma\}^{ab}[k].
\end{align*}
\]  
(11)
The above binomials are all "eigen vectors" of the generators $S^{ab}$, as well as of $\tilde{S}^{ab}$.

We further find

$$\gamma^a_{ab}(k) = \eta^{aa} [-k], \quad \gamma^b_{ab}(k) = -i k [ab],$$

$$\gamma^a_{ab}[k] = (-k), \quad \gamma^b_{ab}[k] = -i k \eta^{aa} (ab) \quad (12)$$

and

$$\tilde{\gamma}^a_{ab}(k) = -i \eta^{aa} [ab], \quad \tilde{\gamma}^b_{ab}(k) = -k [ab],$$

$$\tilde{\gamma}^a_{ab}[k] = i \langle ab \rangle, \quad \tilde{\gamma}^b_{ab}[k] = -k \eta^{aa} (ab) \quad (13)$$

Using the following useful relations

$$ab^\dagger(k) = \eta^{aa} (-k), \quad [ab][k] = [k], \quad (14)$$

we may define

$$< ab^\dagger | ab \rangle = 1 = < ab^\dagger | ab \rangle \quad (15)$$

We shall later make use of the relations

$$ab_{ab}(k)[k] = 0, \quad ab_{ab}(-k)[k] = \eta^{aa} [k], \quad [ab][k][k] = [k][k], \quad [k][k] = 0,$$

$$ab_{ab}(k)[k] = 0, \quad [ab][k][k] = (k), \quad (ab)[ab][k] = (k)[k], \quad [k][k] = 0, \quad (16)$$

as well as the relations, following from Eqs. (13,16),

$$ab_{ab}(k)[k] = 0, \quad (ab)(-k)[k] = -i \eta^{aa} [k], \quad (ab)[k][-k] = 0,$$

$$ab_{ab}[k] = i (ab), \quad (ab)[k][k] = 0, \quad (ab)[k][k] = 0,$$

$$ab_{ab}(k)[k] = -i \eta^{aa} [k]. \quad (17)$$

Here

$$(\pm i) = \frac{1}{2} (\tilde{\gamma}^a \mp \tilde{\gamma}^b), \quad (\pm 1) = \frac{1}{2} (\tilde{\gamma}^a \pm i \tilde{\gamma}^b),$$

$$[\pm i] = \frac{1}{2} (1 \pm \tilde{\gamma}^a \tilde{\gamma}^b), \quad [\pm 1] = \frac{1}{2} (1 \pm i \tilde{\gamma}^a \tilde{\gamma}^b). \quad (18)$$

The reader should notice that $\gamma^a$'s transform the binomial $(ab)$ into the binomial $[ab]$ whose "eigen value" with respect to $S^{ab}$ changes sign, while $\tilde{\gamma}^a$'s transform the binomial $(ab)$ into $[ab]$ with unchanged "eigen value" with respect to $S^{ab}$. We define the operators
of handedness of the group $SO(1,13)$ and of the subgroups $SO(1,3), SO(1,7), SO(6)$ and $SO(4)$ as follows

$$
\Gamma^{(1,13)} = i2^7 S^{03} S^{12} S^{56} \ldots S^{13} 14, \quad \Gamma^{(1,3)} = -i2^2 S^{03} S^{12},$$
$$
\Gamma^{(1,7)} = -i2^4 S^{03} S^{12} S^{56} S^{78}, \quad \Gamma^{(1,9)} = i2^5 S^{03} S^{12} S^{9} 10 S^{11} 12 S^{13} 14,$$
$$
\Gamma^{(6)} = -2^3 S^{9} 10 S^{11} 12 S^{13} 14, \quad \Gamma^{(4)} = 2^2 S^{56} S^{78}. \quad (19)
$$

We shall represent one Weyl left handed spinor as products of binomials $(k)$ or $[k]$, which are "eigen vectors" of the members of the Cartan subalgebra set. We make the following choice of the Cartan subalgebra set of the algebra $S^{ab}$

$$
S^{03}, S^{12}, S^{56}, S^{78}, S^{9} 10, S^{11} 12, S^{13} 14. \quad (20)
$$

We are now prepared to make a choice of a starting basic vector of one Weyl representation of the group $SO(1,13)$, which is the eigen state of all the members of the Cartan subalgebra (Eq. (20)) and is left handed ($\Gamma^{(1,13)} = -1$)

$$
\begin{array}{cccccccc}
03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\
(+i)(+)|(+)(+)|(+)(-)(-)|\psi = \\
(\gamma^0 - \gamma^3)(\gamma^1 + i\gamma^2)|(\gamma^5 + i\gamma^6)(\gamma^7 + i\gamma^8)||| (\gamma^9 + i\gamma^{10})(\gamma^{11} - i\gamma^{12})(\gamma^{13} - i\gamma^{14})|\psi.
\end{array} \quad (21)
$$

The signs "|" and "||" are to point out the $SO(1,3)$ (up to |), $SO(1,7)$ (up to ||) and $SO(6)$ (after ||) substructure of the starting basic vector of the left handed multiplet of $SO(1,13)$, which has $2^{14/2-1} = 64$ vectors. Here $|\psi\rangle$ is any vector, which is not transformed to zero and therefore we shall not write down $|\psi\rangle$ any longer. One easily finds that the eigen values of the chosen Cartan subalgebra elements of $S^{ab}$ and $\tilde{S}^{ab}$ (Eq. (20)) are $(+i/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2)$ and $(+i/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2)$, respectively. This state has with respect to the operators $S^{ab}$ the following properties: With respect to the group $SO(1,3)$ is a right handed spinor ($\Gamma^{(1,3)} = 1$) with spin up ($S^{12} = 1/2$), it is weak chargeless (it is an $SU(2)$ singlet - $\tau^{13} = 0$) and it carries a colour charge (it is the member of the $SU(3)$ triplet with $(\tau^{33} = 1/2, \tau^{38} = 1/(2\sqrt{3}))$, it has $\tau^{21} = 1/2$ and $\tau^{41} = 1/6$ and correspondingly the two hyper charges equal to $Y = 2/3$ and $Y' = -1/3$, respectively. We further find according to Eq. (19) that $\Gamma^{(4)} = 1$ (the handedness of the group $SO(4)$, whose subgroups are $SU(2)$ and $U(1)$), $\Gamma^{(1,7)} = 1$ and $\Gamma^{(6)} = -1$. The starting
vector (Eq. 21) can be recognized in terms of the Standard model subgroups as the right handed weak chargeless $u$-quark carrying one of the three colours.

To obtain all the basic vectors of one Weyl spinor, one only has to apply on the starting basic vector of Eq. (21) the generators $S^{ab}$. All the quarks and the leptons of one family of the Standard model appear in this multiplet (together with the corresponding anti quarks and anti leptons). We present in Table I all the quarks of one particular colour (the right handed weak chargeless $u_R$, $d_R$ and left handed weak charged $u_L$, $d_L$, with the colour $(1/2, 1/(2\sqrt{3}))$ in the Standard model notation). They all are members of one $SO(1,7)$ multiplet.

| $i$ | $|\bar{\psi}_i|$ | $\Gamma^{(1,3)}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{21}$ | $\tau^{33}$ | $\tau^{38}$ | $Y$ | $Y'$ |
|-----|-----------------|------------|-------|--------|--------|--------|--------|-----|-----|
| 1   | $u_R^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | 1     | 1/2 | 1   | 0   | 1/2 | 1/2 | 1/(2\sqrt{3}) | 1/6 | 2/3 | -1/3 |
| 2   | $u_R^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | 1     | -1/2 | 1   | 0   | 1/2 | 1/2 | 1/(2\sqrt{3}) | 1/6 | 2/3 | -1/3 |
| 3   | $d_R^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | 1     | 1/2 | 1   | 0   | -1/2 | 1/2 | 1/(2\sqrt{3}) | 1/6 | -1/3 | 2/3 |
| 4   | $d_R^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | 1     | -1/2 | 1   | 0   | -1/2 | 1/2 | 1/(2\sqrt{3}) | 1/6 | -1/3 | 2/3 |
| 5   | $d_L^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | -1    | 1    | -1   | -1/2 | 0  | 1/2 | 1/(2\sqrt{3}) | 1/6 | 1/6 | 1/6 |
| 6   | $d_L^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | -1    | -1/2 | -1   | -1/2 | 0  | 1/2 | 1/(2\sqrt{3}) | 1/6 | 1/6 | 1/6 |
| 7   | $d_L^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | -1    | 1    | -1   | -1/2 | 0  | 1/2 | 1/(2\sqrt{3}) | 1/6 | 1/6 | 1/6 |
| 8   | $u_L^{13}$      | 03         | 12    | 56     | 78     | 9 10 11 12 13 14 | -1    | -1/2 | -1   | -1/2 | 0  | 1/2 | 1/(2\sqrt{3}) | 1/6 | 1/6 | 1/6 |

Table I. The 8-plet of quarks - the members of $SO(1,7)$ subgroup, belonging to one Weyl left handed $(\Gamma^{(1,13)} = -1 = \Gamma^{(1,7)} \times \Gamma^{(6)})$ spinor representation of $SO(1,13)$. It contains the left handed weak charged quarks and the right handed weak chargeless quarks of a particular colour ($(1/2, 1/(2\sqrt{3}))$). Here $\Gamma^{(1,3)}$ defines the handedness in $(1+3)$ space, $S^{12}$ defines the ordinary spin (which can also be read directly from the basic vector), $\tau^{13}$ defines the weak charge, $\tau^{21}$ defines the $U(1)$ charge, $\tau^{33}$ and $\tau^{38}$ define the colour charge and $\tau^{41}$ another $U(1)$ charge, which together with the first one defines $Y$ and $Y'$. The reader can find the whole Weyl representation in the ref. [10].

In Table II we present the leptons of one family of the Standard model. All the leptons belong to the same multiplet with respect to the group $SO(1,7)$. They are colour chargeless
and differ accordingly from the quarks in Table I in the second \(U(1)\) charge and in the colour charge. The quarks and the leptons are equivalent with respect to the group \(SO(1,7)\).

Table II. The 8-plet of leptons - the members of \(SO(1,7)\) subgroup, belonging to one Weyl left handed (\(\Gamma^{(1,13)} = -1 = \Gamma^{(1,7)} \times \Gamma^{(6)}\)) spinor representation of \(SO(1,13)\), is presented. It contains the left handed weak charged leptons and the right handed weak chargeless leptons, all colour chargeless. The two 8-plets in Table I and II are equivalent with respect to the groups \(SO(1,7)\). They only differ in properties with respect to the group \(SU(3)\) and \(U(1)\) and consequently in \(Y\) and \(Y'\).

### D. Appearance of families

While the generators of the Lorentz group \(S^{ab}\), with a pair of \((ab)\), which does not belong to the Cartan subalgebra (Eq. 20), transform one vector of one Weyl representation into another vector of the same Weyl representation, transform the generators \(\tilde{S}^{ab}\) (again if the pair \((ab)\) does not belong to the Cartan set) a member of one family into the same member of another family, leaving all the other quantum numbers (determined by \(S^{ab}\)) unchanged. This is happening since the application of \(\gamma^a\) (from the left) changes the operator \((+)\) (or the operator \((+i)\)) into the operator \([-]\) (or the operator \([-i]\), respectively), while the operator \(\bar{\gamma}^a\) (which is understood, up to a factor \(\pm i\), as the application of \(\gamma^a\) from the
right hand side) changes \((-+)\) (or \(+(+i)\)) into \([+\) (or into \([+i]\), respectively), without changing the ”eigen values” of the Cartan subalgebra set of the operators \(S^{ab}\). Below, as an example, the application of \(\tilde{S}^{01}\) on the state of Eq. (21) (up to a constant) is presented:

\[
\begin{array}{cccc}
03 & 12 & 56 & 78 \\
(+i)(+) & (++) & | & (+(−)(−) \\
03 & 12 & 56 & 78 \\
[+i][+] & (++) & | & (+(−)(−) \\
\end{array}
\]

(22)

One can easily see that both vectors of (22) describe a right handed \(u\)-quark of the same colour. They are equivalent with respect to the operators \(S^{ab}\). They only differ in properties, determined by the operators \(\tilde{S}^{ab}\) and accordingly also with respect to the Cartan subalgebra set \((\tilde{S}^{03}, \tilde{S}^{12}, \ldots, \tilde{S}^{13}14)\). The first row vector has the following ”eigen values” of this Cartan subalgebra set \((i/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2)\), while the corresponding ”eigen values” of the vector in the second row are \((-i/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2, -1/2)\). Therefore, the operators \(\tilde{S}^{ab}\) can be used to generate families of quarks and leptons of the Standard model.

We present here some useful relations, concerning families

\[
\begin{align*}
\tilde{S}^{ac}(k)[l] &= \frac{i}{2} \eta^{aa} \eta^{cc} [k][l], & \tilde{S}^{ad}(k)[l] &= \frac{1}{2} \eta^{aa} [k][l], \\
\tilde{S}^{bc}(k)[l] &= \frac{1}{2} k \eta^{cc} [k][l], & \tilde{S}^{bd}(k)[l] &= -\frac{i}{2} k [k][l], \\
\tilde{S}^{ac}(k)[l] &= -\frac{i}{2} (k)(l), & \tilde{S}^{ad}(k)[l] &= \frac{1}{2} l \eta^{cc} (k)(l), \\
\tilde{S}^{bc}(k)[l] &= \frac{1}{2} k \eta^{aa} (k)(l), & \tilde{S}^{bd}(k)[l] &= \frac{1}{2} k \eta^{aa} \eta^{cc} (k)(l), \\
\tilde{S}^{ac}(k)[l] &= -\frac{1}{2} k \eta^{aa} (k)(l), & \tilde{S}^{ad}(k)[l] &= \frac{1}{2} l \eta^{cc} (k)(l), \\
\tilde{S}^{bc}(k)[l] &= \frac{i}{2} \eta^{cc} (k)[k], & \tilde{S}^{bd}(k)[l] &= \frac{1}{2} l \eta^{cc} (k)[k], \\
\tilde{S}^{ac}(k)[l] &= -\frac{1}{2} k \eta^{aa} \eta^{cc} (k)[k], & \tilde{S}^{ad}(k)[l] &= \frac{i}{2} k \eta^{aa} \eta^{cc} (k)[k].
\end{align*}
\] (23)

III. MASS MATRICES IN THE APPROACH UNIFYING SPINS AND CHARGES

- TERMS WITHIN EACH FAMILY

We are now prepared to look at the terms, which manifest as the Yukawa couplings in our approach unifying spins and charges. Let us at first neglect the terms \(\tilde{S}^{ab} \tilde{\omega}_{abc}\) (Eqs. (12)) to
which $\tilde{S}^{ab}$ contribute and see how does the ordinary Poincaré gauging gravity contribute to the Yukawa couplings. It contributes to the matrix elements within families only.

Let us look at the Lagrange density (Eq. [2]) for one Weyl spinor of a particular handedness - say the left handed one ($\Gamma(1+7) = -1$) - and of all possible $SU(3) \times U(1)$ charges. The Lagrange density (Eq. [2]) manifests couplings of a spinor with the colour, the weak and the hyper charges fields and it also manifests the mass term

$$L = \bar{\psi} \gamma^m (p_m - \sum_{A,i} g^A \tau^{Ai} A_m^{Ai}) \psi +$$

$$+ \bar{\psi}^+ \gamma^0 \gamma^s (p_s - \sum_{A,i} g^A \tau^{Ai} A_s^{Ai}) \psi + \text{terms with } \tilde{S}^{ab} \tilde{\omega}_{abc} + \text{the rest.} \quad (24)$$

We use the notation $f^\alpha_a p_\alpha = p_a$. Since for simplicity we assume that in the "physical space" there is no gravity, it follows: $f^\mu_m = \delta^\mu_m$. We also (just) assume that the term of the type $\tilde{\tau}^{Ai} \tilde{A}_m^{Ai}$ is negligible at "physical energies". We recognize that the terms with $\gamma^0 \gamma^7$ or $\gamma^0 \gamma^8$ transform a right handed weak chargeless spinors with the spin $1/2$ (like it is the $u_R$ quark from the first row in Table I or the $e_R$ electron from the third row in Table II) into a left handed weak charged spinors with the spin $1/2$ (in our example into the $u_L$ quark from the seventh’s row in Table I or the $e_L$ electron from the fifth’s row in Table II).

We can rearrange the first term in the Lagrangean of Eq. [24], with $m \in \{0, 1, 2, 3\}$, to manifest the Standard model structure

$$L_f = \bar{\psi} \gamma^m \{p_m - \frac{g}{2} (\tau^+ W^+_m + \tau^- W^-_m) + \frac{g^2}{\sqrt{g^2 + g'^2}} Q' Z_m +$$

$$+ \frac{g g'}{\sqrt{g^2 + g'^2}} Q A_m +$$

$$+ \sum_i g^3 \tau^{3i} A_m^{3i} + g Y' A_m^{Y'} \} \psi, \quad (25)$$

with

$$Q = \tau^{13} + Y = S^{56} + \tau^{41},$$

$$Q' = \tau^{13} - (\frac{g'}{g})^2 Y = \frac{1}{2} (1 - (\frac{g'}{g})^2) S^{56} - \frac{1}{2} (1 + (\frac{g'}{g})^2) S^{78} - (\frac{g'}{g})^2 \tau^{41}. \quad (26)$$

We assume that (due to an appropriate break of symmetries) the term $A_m^{Y'}$ is non observable at "physical" energies (not yet).

We rearrange the mass term of the Lagrange density of Eq. [24] in a similar way as the "dynamical" part of the Lagrange density in the "physical space" (the first term of this
equation), so that the fields $A_{s}^{4i}, s \in \{5, 6, 7, 8\}$, instead of $\omega_{it's}$, appear. The charge $Q$ is conserved, as seen in Eq.(26), if we assume that no terms with either $\gamma^5$ or $\gamma^6$ or $\tau^{3i}$ contribute to the mass term.

Since all the operators in Eq.(24) are to be applied on right handed spinors, which are weak chargeless objects (as seen from Table I and II), the part with $\sum_i \tau^{1i}A_{s}^{4i}$ contributes zero to mass matrices and we shall leave it out. We also expect, that at the "observable" energies the contribution of the components $p_s$ of momenta, with $s = 5, 6, \ldots$, are negligible. Accordingly we neglect also this term.

What then stays in the Lagrange density for the mass terms of spinors if the terms with $\tilde{S}_{ab}$ are not yet taken into account, is as follows

$$- L_Y = -\psi^+ \gamma^0 \sum_{s=7,8} \gamma^s (YA^Y_s + Y'A'^Y_{s'}) \psi$$

$$+ \text{terms with } \tilde{S}_{ab} \tilde{\omega}_{abc}. \quad (27)$$

These terms distinguish among the spinors: they are different for quarks than for leptons, as well as different for the $u$ quarks than for the $d$ quarks and different for the electrons than for the neutrinos, since according to Table I and II, different spinors carry different values of the two hyper charges. The first two terms in Eq.(27) contribute to mass matrices within one family only. The expression for the mass terms (the Yukawa couplings in the Standard model language) within one family (Eq.(27)) can be further rewritten, if introducing the following superposition of operators

$$\left(\gamma^7 \pm i\gamma^8\right) = 2^{78}. \quad (28)$$

It then follows

$$- L_Y = -\psi^+ \gamma^0 \sum_{y=Y,Y'} \left\{ \left(\left(+\right)yA_y^+ + \left(-\right)yA_y^-\right) \psi \right\}$$

$$+ \text{terms with } \tilde{S}_{ab} \tilde{\omega}_{abc}, \quad (29)$$

with $A_y^\pm = -(A_y^+ \mp iA_y^-)$ and $y = Y, Y'$. According to Eq.(16), saying that

$$\left(\left(+\right)(+), \left(+\right)(-), \left(-\right)(+), \left(-\right)(-); \left(+\right), \left(-\right)\right) = 0, \quad (30)$$

we conclude, after reading also Table I and Table II, that the term with the fields $A_{s}^Y, y = Y, Y'$, contributes only to the masses of the $d$–quarks and the electrons, while $A_{s}^u, y = Y, Y'$, contributes only to the masses of the $u$–quarks and the neutrinos.
IV. MASS MATRICES IN THE APPROACH UNIFYING SPINS AND CHARGES

- TERMS WITHIN AND AMONG FAMILIES

We have seen in Subsect.IID that while the operators $S^{ab}$ transform the members of one Weyl representation among themselves, the operators $\tilde{S}^{ab}$ transform one member of a family into the same member of another family, changing nothing but the family index. Each spinor basic vector has accordingly two indices: one index tells to which family a spinor belongs, another index tells which member of a particular family a spinor represents.

There are two types of terms in the Lagrange density of Eq.(1), which contribute to the mass matrices. We have studied in the previous Sect.III only the terms, determined by the generators of the Lorentz group ($S^{ab}$) and the corresponding gauge fields. After making a few assumptions we ended up with quite a simple expression for the contribution to the masses of quarks and leptons within one family.

The assumption, that there are two kinds of gauge fields connected with two kinds of the generators of the Lorentz transformations, is new\[2, 7, 8, 9\] and requires accordingly additional cautions, when using it. On the other hand, the idea of the existence of two kinds of Clifford algebra objects leads to the concept of families and is therefore too exciting not to be used to try to describe families of quarks and leptons and to see what can the approach say about families of quarks and leptons.

As already said, the two kinds of generators are in our approach accompanied by the two kinds of gauge fields, gauging $S^{ab}$ and $\tilde{S}^{ab}$, respectively. We shall assume that breaks of the symmetries of the Poincaré group in $d = 1 + 13$ influence this additional spin connection field as well. Since we do not know, how these breaks could occur for any of these two kinds of degrees of freedom, the calculations which will follow can be understood only as an attempt to study these degrees of freedom, that is families of quarks and leptons, within the proposed approach and not yet as a severe prediction of properties of families of this approach.

On the other side, many a property of families, presented in this paper, not connected with the break of symmetries, can follow from any concept of construction of families, if operators for generating families commute with the generators of spins and charges and if the generators are accompanied by gauge fields. What our proposal might offer in addition to the general concept of families is the prediction of the number of families and possible
relations among matrix elements of mass matrices due to connected effects followed by breaks of symmetries.

Our approach suggests an even number of families. Namely: The number of all the orthogonal basic states is in our approach for a particular $d$ equal to $2^d$. Since we start with one Weyl spinor and neither $S^{ab}$ nor $\tilde{S}^{ab}$ change the Clifford oddness or evenness of basic states, we stay with $2^{d-2}$ basic states. Each Weyl representation has $2^{d/2-1}$ members. For any $d$ there are accordingly $2^{d/2-1}$ families. After the assumption that some spontaneous breaks of symmetries lead to massless spinors in $d = (1 + 7)$- dimensional space, in which spinors carry the $SU(3)$ and $U(1)$ charges (the two $U(1)$’s, one from $SO(6)$ and another from $SO(1, 7)$, being connected), there are accordingly only $2^{8/2-1} = 8$ families at most, seen at low energies. The Yukawa-like terms further break symmetry, leading to $Q = S^{56} + \tau_{41}$ and $SO(1, 3), SU(3)$ as conserved quantities and to massive spinors in the ”physical” space with $d = (1 + 3)$.

Since the application of $\tilde{S}^{ab}$ generates the equivalent representations with respect to the Lorentz group ($\{S^{ab}, \tilde{S}^{cd}\} = 0$ and accordingly $\tilde{S}^{ab} (k) = \tilde{S}^{ab} [-k]$), it follows that if we know properties with respect to $\tilde{S}^{ab}$ for one of basic states of a Weyl spinor, we know them for the whole Weyl spinor - up to the influence of the break of symmetries and up to the fact that the contribution to the Yukawa couplings of the two kinds of generators are in our approach related.

To demonstrate properties of families we shall make use of the first state in Table I and Table II. The two tables differ only in the $SU(3)$ and $U(1)$ charges (both kinds originate in $SO(6)$) and these two charges do not concern the $SO(1, 7)$ part. Accordingly we shall tell only families, connected with $SO(1, 7)$.

\begin{align*}
I. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} & V. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} \\
II. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} & VI. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} \\
III. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} & VII. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} \\
IV. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} & VIII. & \begin{array}{c|c|c|c}
03 & 12 & 56 & 78 \\
(\cdot i)(\cdot) & (\cdot)(\cdot) & | & \ldots \\
\end{array} \\
\end{align*}

The diagonal terms, to which $S^{st, \omega}_{st's'}$ contribute, depend on a state of the Weyl representation, as we have commented in Sect. III and distinguish between quarks and leptons, as well as between the $u$ and the $d$ quarks and between the electrons and the neutrinos. The diagonal and the off diagonal elements, to which $\tilde{S}^{ab, \tilde{\omega}}_{abs}$ contribute, distinguish between
the $u$ and the $d$ quarks and between the electrons and the neutrinos, due to the factor $(\pm)^{78}$ (determined by the ordinary $\gamma^a$ operators), but do not distinguish between the quarks and the leptons. However, the way of breaking symmetries might influence the commutation relations among both kinds of generators. (We could, in some rough estimation, take these effects into account for instance just by putting an additional index to gauge fields $\tilde{\omega}_{abc}$. We shall discuss such possibilities in the paper which will present the numerical results.)

The diagonalization of the $u$ ($\nu$) mass matrix leads accordingly to a different transformation matrix than the diagonalization of $d$ ($e$) mass matrix. The mixing matrix for quarks is correspondingly not the unit matrix (as expected, if it should agree with the experimental data) but might not differ from the mixing matrix for leptons.

Let us now write down the whole expression for the Yukawa couplings, with $\tilde{S}^{ab}\tilde{\omega}_{abc}$ included

$$- \mathcal{L}_Y = \psi^\dagger \gamma^0 \gamma^5 p_{0a} \psi = \psi^\dagger \gamma^0 \{(+)^{78} p_{0+} + (-)^{78} p_{0-}\} \psi,$$

with

$$p_{0\pm} = (p_\gamma \mp i p_8) - \frac{1}{2} S^{ab} \omega_{a\pm} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{a\pm},$$

$$\omega_{a\pm} = \omega_{a7} \mp i \omega_{a8}, \quad \tilde{\omega}_{a\pm} = \tilde{\omega}_{a7} \mp i \tilde{\omega}_{a8}. \quad (33)$$

We shall rewrite diagonal matrix elements, to which $\tilde{S}^{ab}$ contribute, in a similar way as we did in the previous sections for the contribution of $S^{ab}$. We therefore introduce the appropriate superposition of the operators $\tilde{S}^{ab}$

$$\tilde{N}_3^\pm : = \frac{1}{2}(\tilde{S}^{12} \pm i \tilde{S}^{63}),$$

$$\tilde{\tau}^{13} : = \frac{1}{2}(\tilde{S}^{56} - \tilde{S}^{78}),$$

$$\tilde{Y} = \tilde{\tau}^{41} + \tilde{\tau}^{21}, \quad \tilde{Y}' = \tilde{\tau}^{41} - \tilde{\tau}^{21},$$

$$\tilde{\tau}^{21} : = \frac{1}{2}(\tilde{S}^{56} + \tilde{S}^{78}), \quad \tilde{\tau}^{41} : = -\frac{1}{3}(\tilde{S}^{910} + \tilde{S}^{1112} + \tilde{S}^{1314}). \quad (34)$$

We allow also terms with $\tilde{S}^{mn}, m, n = 0, 1, 2, 3$, which in diagonal matrix elements of a mass matrix appear as $\tilde{N}_3^\pm$. Taking into account that

$$-\frac{1}{2} S^{st} \omega_{st\pm} = YA^Y_{\pm} + Y'A^Y'_{\pm} + \tau^{13} A^{13}_{\pm},$$

$$-\frac{1}{2} \tilde{S}^{st} \tilde{\omega}_{st\pm} = \tilde{Y} \tilde{A}^Y_{\pm} + \tilde{Y}' \tilde{A}^Y'_{\pm} + \tilde{\tau}^{13} \tilde{A}^{13}_{\pm},$$

$$-\frac{1}{2} S^{mn} \omega_{mn\pm} = \tilde{N}^{+3} \tilde{A}^{+3}_{\pm} + \tilde{N}^{-3} \tilde{A}^{-3}_{\pm}, \quad (35)$$

20
with the pairs \((m, n) = (0, 3), (1, 2); (s, t) = (5, 6), (7, 8)\), belonging to the Cartan sub algebra and \(\Omega_\pm = \Omega_7 \mp i\Omega_8\), where \(\Omega_7, \Omega_8\) stay for any of the above fields, we find

\[
\begin{align*}
A_{13}^\pm &= - (\omega_{56\pm} - \omega_{78\pm}), \\
A_Y^\pm &= - \frac{1}{2} \left( A_{41}^{\pm} + (\omega_{56\pm} + \omega_{78\pm}) \right), \\
A_Y'^\pm &= - \frac{1}{2} \left( A_{41}^{\pm} - (\omega_{56\pm} + \omega_{78\pm}) \right), \\
\tilde{A}_{13}^\pm &= - (\tilde{\omega}_{56\pm} - \tilde{\omega}_{78\pm}), \\
\tilde{A}_Y^\pm &= - \frac{1}{2} (\tilde{A}_{41}^{\pm} + (\tilde{\omega}_{56\pm} + \tilde{\omega}_{78\pm}) ), \\
\tilde{A}_Y'^\pm &= - \frac{1}{2} (\tilde{A}_{41}^{\pm} - (\tilde{\omega}_{56\pm} + \tilde{\omega}_{78\pm}) ), \\
\tilde{A}_{N_3}^{\mp} &= - (\tilde{\omega}_{12\pm} - i \tilde{\omega}_{03\pm}), \\
\tilde{A}_{N_3}'^{\mp} &= - (\tilde{\omega}_{12\pm} + i \tilde{\omega}_{03\pm}),
\end{align*}
\]

(36)

where the fields \(A_y^\pm, y = 13, 41, Y, Y'\), and \(\tilde{A}_y^\pm, \tilde{y} = \tilde{N}_3^+, \tilde{N}_3^-, 13, 41, \tilde{Y}, \tilde{Y}'\), are uniquely expressible with the corresponding spin connection fields. Let us repeat that \(\omega_{abc} = f^\alpha_c \omega_{ab\alpha}\) and \(\tilde{\omega}_{abc} = f^\alpha_c \tilde{\omega}_{aba}\).

The operators, which contribute to non diagonal terms in mass matrices, are superpositions of \(\tilde{S}^{\alpha\beta}\) and can be written in terms of nilpotents

\[
\begin{align*}
\tilde{S}^{\alpha\beta} &= \sum_{(a,b)} \frac{1}{2} (\tilde{\omega}_{ab\pm} + \tilde{\omega}_{ab\mp}),
\end{align*}
\]

(37)

with indices \((ab)\) and \((cd)\) which belong to the Cartan sub algebra indices (Eq. (20)). We may write accordingly

\[
\begin{align*}
\sum_{(a,b)} \frac{1}{2} \sum_{(ac),(bd)} \tilde{S}^{ab} \tilde{\omega}_{ab\pm} &= - \sum_{(ac),(bd), k,l} \tilde{S}^{ab} \tilde{\omega}_{ab\pm} \\
&= \sum_{(ac),(bd), k,l} \frac{1}{2} \sum_{(ac),(bd), k,l} \tilde{S}^{ab} \tilde{\omega}_{ab\pm},
\end{align*}
\]

(38)

where the pair \((a, b)\) in the first sum runs over all the indices, which do not characterise the Cartan sub algebra, with \(a, b = 0, \ldots, 8\), while the two pairs \((ac)\) and \((bd)\) denote only the Cartan sub algebra pairs (for \(SO(1,7)\) we only have the pairs \((03), (12); (03), (56); (03), (78); (12), (56); (12), (78); (56), (78)\)); \(k\) and \(l\) run over four possible values so that \(k = \pm i\), if \((ac) = (03)\) and \(k = \pm 1\) in all other cases, while \(l = \pm 1\). The fields \(\tilde{A}_k^{\pm}((ac),(bd))\) can then be expressed by \(\tilde{\omega}_{ab\pm}\) as follows

\[
\begin{align*}
\tilde{A}_k^{\pm}((ab),(cd)) &= \frac{i}{2} (\tilde{\omega}_{ac\pm} - \frac{i}{r} \tilde{\omega}_{bc\pm} - i \tilde{\omega}_{ad\pm} - \frac{1}{r} \tilde{\omega}_{bd\pm}),
\end{align*}
\]

(39)
\[
\begin{align*}
\tilde{A}_\pm^-(ab), (cd)) & = -\frac{i}{2}(\tilde{\omega}_{ac\pm} + \frac{i}{r}\tilde{\omega}_{bc\pm} + i\tilde{\omega}_{ad\pm} - \frac{1}{r}\tilde{\omega}_{bd\pm}), \\
\tilde{A}_\pm^+(ab), (cd)) & = -\frac{i}{2}(\tilde{\omega}_{ac\pm} + \frac{i}{r}\tilde{\omega}_{bc\pm} - i\tilde{\omega}_{ad\pm} + \frac{1}{r}\tilde{\omega}_{bd\pm}), \\
\tilde{A}_\pm^- (ab), (cd)) & = -\frac{i}{2}(\tilde{\omega}_{ac\pm} - \frac{i}{r}\tilde{\omega}_{bc\pm} + i\tilde{\omega}_{ad\pm} + \frac{1}{r}\tilde{\omega}_{bd\pm}),
\end{align*}
\]

with \( r = i \), if \((ab) = (03)\) and \( r = 1 \) otherwise. We simplify the index \( kl \) in the exponent of fields \( \tilde{A}^{kl\pm((ac),(bd))} \) to \( \pm \), omitting \( i \).

Any break of symmetries in the \( S^{ab} \) sector would cause relations among the corresponding \( \tilde{\omega}_{ab\pm} \). Namely, if \(-\frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\pm} = \tilde{\tau}^{Ai}A^{Ai}_\pm \) is not just a unitary transformation of basic states, but means due to a break of symmetries that, let us say, a particular \( A^{A'i}_\pm = 0 \), then this can only happen, if \( \tilde{\omega}_{ab\pm} \) are related.

The Lagrange density, representing the mass matrices of fermions (the Yukawa couplings in the Standard model) (Eq.(32)), can be rewritten as follows

\[
\mathcal{L}_Y = \psi^+ \gamma^0 \left\{ \begin{array}{l}
78 \begin{cases}
(+) & \sum_{y=Y,Y'} \sum_{\tilde{y}=\tilde{N}_3, \tilde{\tau}^{13}, \tilde{Y}, \tilde{Y'}} yA_y^\dagger + \tilde{y}\tilde{A}_{\tilde{y}}^\dagger \\
(-) & \sum_{y=Y,Y'} \sum_{\tilde{y}=\tilde{N}_3, \tilde{\tau}^{13}, \tilde{Y}, \tilde{Y'}} yA_y^\dagger + \tilde{y}\tilde{A}_{\tilde{y}}^\dagger \\
(+) & \sum_{\{(ac),(bd)\}, k,l} \frac{ac}{bd} (\bar{k})(\bar{l}) \tilde{A}^{kl\pm((ac),(bd))} \\
(-) & \sum_{\{(ac),(bd)\}, k,l} \frac{ac}{bd} (\bar{k})(\bar{l}) \tilde{A}^{kl\pm((ac),(bd))} \end{cases} \psi,
\end{array} \right.
\]

with pairs \((ac),(bd)\), which run over all the members of the Cartan sub algebra, while \( k = \pm i \), if \((ac) = (03)\), otherwise \( k = \pm 1 \) and \( l = \pm 1 \). The terms \( \tilde{A}^{kl\pm((ac),(bd))} \) are expressible in terms of \( \tilde{\omega}_{ab\pm} \) as presented in Eq.(39), while any break of symmetries relates \( \tilde{\omega}_{ab\pm} \) in a very particular way. We omitted the term with \( \tau^{13} \), as well as the terms \( p_{\pm} \), since, as already explained in the previous section, the first one when being applied on the right handed spinors contributes zero, while for the second ones we assume that at low energies their contribution is negligible.

The mass matrix, which follows from these Lagrange density and depends strongly on all possible breaks of symmetries, is in general not Hermitean.

Let us now repeat the assumptions we have made up to now. They are either the starting assumptions of our approach unifying spins and charges, or we made them to be able to connect the starting Lagrange density at low energies with the observable phenomena.
a.i. We use the approach, unifying spins and charges, which assumes, that in \( d = 1 + 13 \) massless spinors carry two types of spins: the ordinary (in \( d = 1 + 13 \)) one, which we describe by \( S^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) \) and the additional one, described by \( \tilde{S}^{ab} = \frac{1}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) \). The two types of the Clifford algebra objects anti commute (\( \{\gamma^a, \tilde{\gamma}^b\} = 0 \)). Spinors carry no charges in \( d = 1 + 13 \). The operators \( S^{ab} \) determine (after an appropriate break of symmetries) at low energies the ordinary spin in \( d = 1 + 3 \) and all the known charges, while \( \tilde{S}^{ab} \) generate families of spinors. Accordingly spinors interact with only the gravitational fields, the gauge fields of the Poincaré group \((p_\alpha, S^{ab})\), and the gauge fields of the operators \( \tilde{S}^{ab} \) (\( p_{0a} = f^\alpha_a - \frac{1}{2}(S^{cd}_{cda} + \tilde{S}^{cd}_{cda}) \)).

a.ii. The break of symmetries of \( SO(1,13) \) into \( SO(1,7) \times SU(3) \times U(1) \) occurs in a way that only massless spinors in \( d = 1 + 7 \) with the charge \( SU(3) \) and \( U(1) \) survive, with the one \( U(1) \) from \( SO(1,7) \) and the next \( U(1) \) from \( SO(6) \) aligned, while \( S^{56} \) does not contribute to the Yukawa-like terms, so that \( Q = \tau^4 + S^{56} \) is conserved in \( d = 1 + 3 \).

a.iii. The break of symmetries influences both: the Poincaré symmetry and the symmetry described by \( \tilde{S}^{ab} \), it might be that to some extend in a similar way. The study of both kinds of breaking symmetries stays as an open problem.

a.iv. The terms which include \( p_s, s = 5, \ldots, 14 \), do not contribute at low energies.

V. AN EXAMPLE OF MASS MATRICES FOR FOUR FAMILIES

Let us make, for simplicity, two further assumptions besides the four (a.i-a.iv.) ones, presented at the end of Sect.IV:

b.i. There are no terms, which would in Eq.\( \ref{56} \) transform \( [\pm] \) into \( \pm \). This assumption (which could also be understood as a break of symmetry, which requires that terms of the type \( \tilde{S}^{5a}\tilde{\omega}_{5ab} \) and \( \tilde{S}^{6a}\tilde{\omega}_{6ab} \) are negligible and might be a part of requirement a.iii.) leaves us with only four families of quarks and leptons. (This assumption might be justified with a break of symmetry in the \( \tilde{S}^{ab} \) sector from \( SO(1,7) \) to \( SO(1,5) \times U(1) \), with all the contributions of the terms \( \tilde{S}^{5a}\tilde{\omega}_{5ab} \) and \( \tilde{S}^{6a}\tilde{\omega}_{6ab} \) equal to zero.)

b.ii. The rough estimation will be done on ”a tree level”.

Since we do not know either how does the break of symmetries occur or how does the break influence the strength of the fields \( \omega_{abc} \) and \( \tilde{\omega}_{abc} \), we can not really say, to which extend are the above assumptions justified. For none of them we have a justification. Also
the nonperturbative effects could be very strong and the tree level might not mean a lot. But yet a simplified version can help us to understand to what conclusions might the proposed approach lead with respect to families of quarks and leptons and their properties.

Our approach (which predicts an even number of families) suggests that under the assumptions a. and b. there are the following four families of quarks and leptons

\[
\begin{align*}
I. & \ (03)_{12} \ (56)_{78} \\
II. & \ (03)_{12} \ (56)_{78} \\
III. & \ (03)_{12} \ (56)_{78} \\
IV. & \ (03)_{12} \ (56)_{78}
\end{align*}
\]

We see from Table I (and II) that due to the properties of the nilpotents \( \pm \) (Eq.16), to the \( u \) quark (and to the \( \nu \) lepton) mass matrix only the operator \( - \) (accompanied by the fields \( A_-, \tilde{A}_- \)) contributes, while to the \( d \) quark (and to the \( e \) lepton) mass matrix only \( + \) (accompanied by the fields \( A_+, \tilde{A}_+ \)) contributes. This means that the off diagonal matrix elements of the Yukawa couplings are different for \( u \)-quarks (\( \nu \)) and for \( d \)-quarks (\( e \)), although still related, while the quarks have the same off diagonal matrix elements as the corresponding leptons (unless some breaks of symmetries do not destroys this symmetry).

Assuming that after the appropriate breaks of symmetries the fields contributing to the Yukawa couplings obtain some nonzero expectation values (which are in general related in a very particular way) and integrating the Lagrange density \( L_Y \) over the coordinates and the internal (spin) degrees of freedom, we end up with the mass matrices for four families of quarks and leptons (Eq.41), whose structure is presented in Table III.
Table III. The mass matrices for four families of quarks and leptons in the approach unifying spins and charges, obtained under the assumptions a.i.- a.iv. and b.i.- b.ii.. The values $A_{I}^{l}$, $l = I, II, III, IV$, and $\tilde{A}_{I}^{lm}((ac), (bd)); l, m = \pm$, determine matrix elements for the $u$ quarks and the neutrinos, the values $A_{I}^{l'}$, $l' = I, II, III, IV$, and $\tilde{A}_{I}^{lm}((ac), (bd)); l, m = \pm$, determine the matrix elements for the $d$ quarks and the electrons. Diagonal matrix elements are different for quarks than for leptons and distinguish also between the $u$ and the $d$ quarks and between the $\nu$ and the $e$ leptons (Eqs [10], [36]. They also differ from family to family. Non diagonal matrix elements distinguish among families and among ($u, \nu$) and ($d, e$). The presented matrix should be understood as a very preliminary estimate of the mass matrices of quarks and leptons.

The explicit forms of the diagonal matrix elements for the above choice of assumptions in terms of $\omega_{abc}$, $\tilde{\omega}_{abc}$ and $\tilde{A}_{I, \pm}^{41}$ is as follows

$$A_{u}^{I} = \frac{2}{3}A_{-} + \frac{1}{3}A_{+} + \tilde{\omega}_{-},$$
$$A_{u}^{I'} = -A_{-} + \tilde{\omega}_{-},$$
$$A_{d}^{I} = -\frac{1}{3}A_{-} + \frac{2}{3}A_{+} + \tilde{\omega}_{+},$$
$$A_{d}^{I'} = -A_{+} + \tilde{\omega}_{+},$$
$$A_{u}^{II} = A_{u}^{I} + (i\tilde{\omega}_{03} + \tilde{\omega}_{12}), \quad A_{u}^{II'} = A_{u}^{I'} + (i\tilde{\omega}_{03} + \tilde{\omega}_{12}),$$
\[ A_{d}^{II} = A_{d}^{I} + (i\omega_{03} + \omega_{12}^+), \quad A_{e}^{II} = A_{e}^{I} + (i\omega_{03} + \omega_{12}^+), \]
\[ A_{d}^{III} = A_{d}^{I} + (i\omega_{03} - \omega_{78}^-), \quad A_{e}^{III} = A_{e}^{I} + (i\omega_{03} - \omega_{78}^-), \]
\[ A_{d}^{IV} = A_{d}^{I} + (i\omega_{03} + \omega_{78}^+), \quad A_{e}^{IV} = A_{e}^{I} + (i\omega_{03} + \omega_{78}^+), \]
\[ A_{u}^{II} = A_{u}^{I} + (\omega_{12}^+ - \omega_{78}^-), \quad A_{\nu}^{II} = A_{\nu}^{I} + (\omega_{12}^+ - \omega_{78}^-), \]
\[ A_{u}^{IV} = A_{u}^{I} + (\omega_{12}^+ + \omega_{78}^+), \quad A_{\nu}^{IV} = A_{\nu}^{I} + (\omega_{12}^+ + \omega_{78}^+), \]

(42)

with \(-\omega_{\pm}^I = \frac{1}{2}(i\omega_{03} + \omega_{12}^\pm + \omega_{56}^\pm + \omega_{78}^\pm + \frac{1}{3}A_{I}^{41})\). The explicit forms of non diagonal matrix elements are written in Eq. (39). As already stated, the break of symmetries, which is not taken into account in Table III, would strongly relate “vacuum expectation values” of \(\tilde{\omega}_{ab\pm}\).

To evaluate briefly the structure of mass matrices we make one further assumption:

b.iii. Let the mass matrices be real and symmetric (while all the \(\omega_{abc}\) and \(\tilde{\omega}_{abc}\) are assumed to be real).

We then obtain for the u quarks (and neutrinos) the mass matrices as presented in Table IV.

| \(u\) | \(I_{R}\) | \(II_{R}\) | \(III_{R}\) | \(IV_{R}\) |
|------|--------|--------|--------|--------|
| I_{L} | \(A_{u}^{I}\) | \(\tilde{A}_{u}^{++}(03), (12) = \frac{1}{2}(\omega_{327} + \omega_{018})\) | \(\tilde{A}_{u}^{++}(03), (78) = \frac{1}{2}(\omega_{387} + \omega_{078})\) | \(-\tilde{A}_{e}^{++}(12), (78) = \frac{1}{2}(\omega_{277} + \omega_{187})\) |
| II_{L} | \(\tilde{A}_{u}^{--}(03), (12) = \frac{1}{2}(\omega_{327} + \omega_{018})\) | \(A_{u}^{II} = A_{u}^{I} + (\omega_{127} - \omega_{038})\) | \(\tilde{A}_{u}^{-+}(12), (78) = -\frac{1}{2}(\omega_{277} - \omega_{187})\) | \(-\tilde{A}_{e}^{+-}(03), (78) = \frac{1}{2}(\omega_{387} - \omega_{078})\) |
| III_{L} | \(\tilde{A}_{u}^{--}(03), (78) = \frac{1}{2}(\omega_{387} + \omega_{078})\) | \(-\tilde{A}_{u}^{+-}(12), (78) = -\frac{1}{2}(\omega_{277} - \omega_{187})\) | \(A_{u}^{III} = A_{u}^{I} + (\omega_{787} - \omega_{038})\) | \(\tilde{A}_{e}^{+-}(03), (12) = -\frac{1}{2}(\omega_{327} - \omega_{018})\) |
| IV_{L} | \(\tilde{A}_{u}^{--}(12), (78) = \frac{1}{2}(\omega_{277} + \omega_{187})\) | \(-\tilde{A}_{u}^{+-}(03), (78) = \frac{1}{2}(\omega_{387} - \omega_{078})\) | \(\tilde{A}_{e}^{++}(03), (12) = -\frac{1}{2}(\omega_{277} - \omega_{187})\) | \(A_{e}^{IV} = A_{e}^{I} + (\omega_{127} + \omega_{787})\) |
Table IV. The mass matrix of four families of the $u$-quarks (and neutrinos) obtained within the approach unifying spins and charges and under the assumptions a.i.-a.iv., b.i.-b.iii. Neutrinos and $u$-quarks distinguish in $A_u^I \neq A_u^I$. The break of symmetries, not yet taken into account, would relate $\tilde{\omega}_{ab7,8}$ and would accordingly reduce the number of free parameters.

The corresponding mass matrix for $d$-quarks (and electrons) is presented in Table V.

| $d-$ | $I_R$ | $II_R$ | $III_R$ | $IV_R$ |
|------|-------|--------|---------|--------|
| $I_L$ | $A_d^I$ | $\tilde{A}_d^+((03),(12)) = \frac{1}{2}(\tilde{\omega}_{327} - \tilde{\omega}_{018})$ | $-\tilde{A}_d^+((03),(78)) = -\frac{1}{2}(\tilde{\omega}_{387} - \tilde{\omega}_{078})$ | $\tilde{A}_d^+((12),(78)) = -\frac{1}{2}(\tilde{\omega}_{277} + \tilde{\omega}_{187})$ |
| $II_L$ | $\tilde{A}_d^-((03),(12)) = \frac{1}{2}(\tilde{\omega}_{327} - \tilde{\omega}_{018})$ | $A_d^I = A_d^I + (\tilde{\omega}_{127} + \tilde{\omega}_{038})$ | $-\tilde{A}_d^+((12),(78)) = -\frac{1}{2}(\tilde{\omega}_{277} - \tilde{\omega}_{187})$ | $\tilde{A}_d^+((03),(78)) = -\frac{1}{2}(\tilde{\omega}_{387} + \tilde{\omega}_{078})$ |
| $III_L$ | $-\tilde{A}_d^-((03),(78)) = -\frac{1}{2}(\tilde{\omega}_{387} - \tilde{\omega}_{078})$ | $\tilde{A}_d^+((12),(78)) = \frac{1}{2}(\tilde{\omega}_{277} - \tilde{\omega}_{187})$ | $A_d^{III} = A_d^I + (\tilde{\omega}_{787} + \tilde{\omega}_{038})$ | $\tilde{A}_d^+((03),(12)) = \frac{1}{2}(\tilde{\omega}_{127} + \tilde{\omega}_{327})$ |
| $IV_L$ | $-\tilde{A}_d^-((12),(78)) = -\frac{1}{2}(\tilde{\omega}_{277} + \tilde{\omega}_{187})$ | $\tilde{A}_d^+((03),(78)) = \frac{1}{2}(\tilde{\omega}_{387} + \tilde{\omega}_{078})$ | $-\tilde{A}_d^+((03),(12)) = -\frac{1}{2}(\tilde{\omega}_{018} + \tilde{\omega}_{327})$ | $A_d^I = A_d^I + (\tilde{\omega}_{127} + \tilde{\omega}_{787})$ |

Table V. The mass matrix of four families of the $d$-quarks and electrons. The quarks and the leptons distinguish in this approximation in $A_d^I \neq A_e^I$. Other comments are the same as in Table IV.

The relation between the mass matrix of $u$-quarks and the mass matrix of neutrinos is, under the assumptions and simplifications made during deriving both tables, as follows: All the off-diagonal elements are for the neutrinos the same as for the $u$-quarks, while the diagonal matrix elements depend on the eigen values of $Y$ and $Y'$. Accordingly, both $A_\alpha^I, \alpha = u, \nu$, can be understood as independent parameters, expressible in terms of $A^Y, A'^Y$, and $\tilde{\omega}^I$. 
Similarly, the relation between the mass matrix of \( d \)-quarks and the mass matrix for electrons is, under the same assumptions and simplification as used for finding the expressions for the mass matrices for the \( u \)-quarks and the neutrinos, as follows: All the off-diagonal elements are the same for both - the \( d \)-quarks and the electrons, while the diagonal matrix elements distinguish in the eigen values of \( Y \) and \( Y' \). Again, both \( A_{\beta}^{I}, \beta = d, e \), can be understood as independent parameters, expressible in terms of \( A_{+}^{Y}, A_{+}^{Y'}, \) and \( \tilde{\omega}^{I}_{+} \).

The requirement about reality and symmetry of mass matrices, relates \( A_{+}^{Y} = A_{-}^{Y}, A_{+}^{Y'} = A_{-}^{Y'}, \tilde{\omega}^{I}_{+} = \frac{1}{2} \tilde{\omega}_{038} + \omega, \tilde{\omega}^{I}_{-} = -\frac{1}{2} \tilde{\omega}_{038} + \omega, \) where \( \tilde{\omega} = \tilde{\omega}_{127} + \tilde{\omega}_{567} + \tilde{\omega}_{787} + \frac{1}{3} \tilde{A}_{41}^{41} \) and \( \tilde{A}_{41}^{41} \) is the real part of either \( \tilde{A}_{41}^{41} \) or \( \tilde{A}_{41}^{41} \). The same assumption relates also off diagonal elements for \( u \)-quarks and \( d \)-quarks (or neutrinos and electrons), as seen from both tables, so that there are 13 free parameters, which determine \( 4 \times 4(4 + 1)/2 \) mass matrix elements, and from these mass matrices \( 4 \times 4 \) masses of quarks and leptons and \( 2(4(4 + 1)/2 - 1) \) elements of the two mixing matrices should follow.

Further break of symmetries would further relate the \( \tilde{\omega}_{ab\pm} \) fields, reducing strongly the number of free parameters on Table IV and Table V. A very peculiar boundary conditions could - when breaking symmetries - even cause differences in off diagonal matrix elements of quarks and leptons. Also could the nonperturbative effects beyond the ”tree level” be responsible for the differences observed in the measured properties of quarks and leptons or for what in many references are trying to achieve with additional Higgs fields. We did not take into account any Majorana fermions.

Most of the above assumptions were proposed to be able to make a rough estimation of properties of the mass matrices, predicted by the approach unifying spins and charges.

We shall present the calculations with the parameters presented in Tables IV and V in the paper, which will follow this one.

**VI. CONCLUDING DISCUSSIONS**

In this paper we discuss about a possible origin of the families of quarks and leptons and of their Yukawa couplings as proposed by the approach unifying spins and charges [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

The approach assumes that a Weyl spinor of a chosen handedness carries in \( d(= 1 + 13) \) - dimensional space nothing but two kinds of spin degrees of freedom. One kind belongs to the
Poincaré group in $d = 1 + 13$, another one generates families. The idea of generating families with the second kind of the Clifford algebra objects (which commute with the generators of the Lorentz transformations for spinors) is new, as it is new also the idea that there are the generators of the Lorentz transformations (accompanied by the spin connection fields in $d > 4$) which are (together with $\gamma^0$) responsible for the Yukawa couplings within a family, transforming a right handed weak chargeless quark or lepton into a left handed weak charged one. Spinors interact with only the gravitational fields, manifested by vielbeins and spin connections, the gauge fields of the momentum $p_\alpha$ and the two kinds of the generators of the Lorentz group $S^{ab}$ and $\tilde{S}^{ab}$, respectively.

To derive the mass matrices from the starting Lagrangean - that is to calculate the Yukawa couplings of the Standard model - no additional (Higgs) field is needed. In order to make a simple and transparent evaluation of properties of the mass matrices and consequently some estimations and rough predictions for the masses and mixing matrices for quarks and leptons, observed at ”physical” energies, we made several assumptions, approximations and simplifications, not necessary all of them are ”physical” (and some of them should soon be relaxed in further studies):

i. The break of symmetries of the group $SO(1,13)$ into $SO(1,7) \times SU(3) \times U(1)$ occurs in a way that only massless spinors in $d = 1 + 7$ with the charge $SU(3) \times U(1)$ survive. And yet the two $U(1)$ charges, following from $SO(6)$ and $SO(1,7)$, respectively, are related. (Our work on the compactification of a massless spinor in $d = 1 + 5$ into $d = 1 + 3$ and a finite disk gives us some hope that this assumption might be fulfilled[13].) The requirement that the terms with $S^{5a}, S^{6a}$ do not contribute to the mass term at low energies, assures that the charge $Q = \tau^{41} + S^{56}$ is conserved.

ii. The break of symmetries influences the Poincaré symmetry and the symmetries described by $\tilde{S}^{ab}$. But it is assumed that the gauge symmetries connected with $\tilde{S}^{ab}$ do not manifest as gauge fields (additional to the known charge gauge fields) in $d = 1 + 3$. It is also assumed that there are no terms, which would in Eq.(31) transform (++) into [++] (which can be explained by the related break of symmetries in $S^{ab}$ and $\tilde{S}^{ab}$ sektor). This assumption reduces the number of families by a factor 2. It really means that the group $SO(1,7)$, whose generators are $\tilde{S}^{ab}$, is broken into $SO(1,5) \times U(1)$ in a way that terms $\tilde{S}^{5a} \tilde{\omega}_{5ab}$ and $\tilde{S}^{6a} \tilde{\omega}_{6ab}$ bring no contribution to the mass matrices. Otherwise no additional break of symmetry was taken into account.
We also assume that terms which include the components $p_s, s = 5, \ldots, 14$ of the momentum $p^a$ do not contribute at low energies to the mass matrices. We leave for further studies to find out how do different ways of breaking symmetries of the Poincaré group in $d = 1 + 13$ and the $SO(1, 13)$ group of $\tilde{S}_{ab}$ influence the mass matrices.

iii. We make estimations on a ”tree level”.

iv. We assume the mass matrices to be real and symmetric.

Our starting Weyl spinor representation of a chosen handedness in $d(= 1 + 13)$—dimensional space manifests, if analyzed in terms of the subgroups $SO(1, 3), SU(3), SU(2)$ and two $U(1)'s$ (the sum of the ranks of the subgroups is the same as the rank of the starting group) of the group $SO(1, 13)$, the spin and all the charges of one family of quarks and leptons. It includes left handed weak charged quarks and leptons and right handed weak chargeless quarks and leptons in the same representation and does accordingly answer one of the open questions of the Standard model: Why only the left handed fermions carry the weak charge while the right handed ones are weak chargeless, how can it at all happen that handedness, which concerns only the spin (in $d = 1 + 3$), is so strongly related to a (weak) charge?

We use our technique [14, 15] to present spinor representations in a transparent way so that one easily sees how does a part of the covariant derivative of a spinor in $d = 1 + 13$ manifest in $d = 1 + 3$ as Yukawa couplings. We use the same technique to represent also families of spinors. Since the starting action in $d = 1 + 13$ manifests in $d = 1 + 3$ the (even number) of families and the Yukawa couplings, it offers a possible answer to the questions, why families of quarks and leptons and the corresponding Yukawa couplings manifest in nature.

We found the off diagonal mass matrix elements of the quarks and the leptons strongly related. We expect that these relations might very probably turn out to be too strong (since there are in our case the same off diagonal and diagonal matrix elements, which determine the orthogonal rotations of the matrices for the $u$—quarks and neutrinos and the $d$—quarks and electron into the diagonal forms and consequently also the corresponding mixing matrices for quarks are the same as for leptons, while the experimental data shows quite a difference in the mixing matrices of quarks and leptons). We suspect that some particular breaks of symmetries (with the help of very peculiar boundary conditions added) might be responsible in our approach for the differences between quarks and leptons in off diagonal and also those
diagonal matrix elements, which are generated by the family generators. Or might the reason for the difference be in the Majorana like neutrinos (as suspected in many references), which are not treated in these studies.

If the symmetry of mass matrices as presented in this paper for four families breaks further, the relations among the parameters determining the mass matrices follow, reducing the presented number of independent parameters $\tilde{\omega}_{abcd}$. In particular, an exact break of the symmetry of $SO(1,5)$ in the $S^{ab}$ sector into $SU(3) \times U(1)$ would manifest in decoupling of the fourth family from the first three (by relating $\tilde{\omega}_{abc}$ so that the corresponding matrix elements would be zero), while the break of $SO(1,5)$ into $SU(2) \times SU(2) \times U(1)$ would manifest in decoupling of the first two families from the second two families.

We treat the quarks and the leptons in an equivalent way, with no Majorana neutrinos included.

Not all the above assumptions and simplifications are needed in order to be able to estimate mass matrices of quarks and leptons with not too much effort. And in addition, it might happen that this too simplified estimates lead to unrealistic conclusions. (In particular, the $CP$ violation can under the assumption v. hardly be possible.) One also can not expect, that ”a tree level” estimate is good enough to evaluate properties of quarks and leptons. Nonperturbative effects might strongly influence the results and they might even be a very strong reason for the difference in properties of the families of the $u$-quarks, the $d$-quarks and both kinds of leptons. Corrections bellow the tree level might bring also the contributions, which several references try to simulate by more than one Higgs field.

For the four predicted families of quarks and leptons we present the explicit expressions for the mass matrices in the above mentioned approximations.

Since in our approach the generators of the Poincaré group and of those, generating families, commute, many a property of mass matrices, presented in this paper, would be true also for, let us say, models, in which the generators of the Poincaré group and those of generating families, commute. However, in our case breaks of symmetries in the two sectors are related and these relations might be very important for the properties of quarks and leptons.

On the other hand we should find the explanation why the additional gauge fields, connected with the $\tilde{S}^{ab}$ sector does not manifest in $d = 1 + 3$.

We shall present numerical estimates for the Yukawa couplings after relating our results
with the known experimental data in the paper[23], following this one, together with further discussions of the properties of families of quarks and leptons as following from the approach unifying spins and charges, in order to be able to see whether this approach shows the right way beyond the Standard model of electroweak and colour interactions.

Acknowledgments

It is a pleasure to thank all the participants of the workshops entitled "What comes beyond the Standard model", taking place at Bled annually in July, starting at 1998, for many very fruitful discussions, in particular to H.B. Nielsen.

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[24] Latin indices $a, b, ..., m, n, ..., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, ..., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions 0,1,2,3 ($m, n, ..$ and $\mu, \nu, ..$), indices from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

[25] We shall from now on simplify the notation from $g^A A^a_i$ to $A^A_i$. 

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