Calculation of the Cherenkov light yield from low energetic secondary particles accompanying high-energy muons in ice and water with Geant4 simulations

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Abstract

In this work we investigate and parameterize the amount and angular distribution of Cherenkov photons, which are generated by low-energy secondary particles (typically $\lesssim 500$ MeV), which accompany a muon track in water or ice. These secondary particles originate from small energy loss processes. We simulate these processes with Geant-4 and investigate the contributions of the different energy loss processes as a function of the muon energy and the maximum transferred energy. For the calculation of the angular distribution we have developed a generic transformation method, which allows us to derive the angular distribution of Cherenkov photons for an arbitrary distribution of track directions and their velocities.

Keywords: neutrino telescope, Cherenkov light, Geant4, muon propagation

1. Introduction

Muons, which originate from charged current interactions of muon neutrinos are a primary detection channel for high-energy neutrino telescopes such as IceCube, Baikal or Antares [1], [2], [3]. These muons propagate with a speed $v$ close to the vacuum speed of light $c$ through the detection medium, which is water or ice. In these media the refraction index is typically $n \approx 1.33$. The phase velocity of light is given by $c_{\text{med}} = c/n$ [4]. Particles with a speed $v > c_{\text{med}}$ will emit optical Cherenkov light, which is detected by photo-detectors.

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The Cherenkov light is emitted into a cone, which half opening angle $\theta_c$ is given by
\[
\cos (\theta_c) = \frac{1}{n\beta}
\]
Here, $\beta = v/c$ is the Lorentz factor of the particle [5]. The number of emitted photons per unit track $x$ and wavelength interval is given by the Frank-Tamm formula [5, 6]
\[
\frac{d^2N}{dx d\lambda} = \frac{2\pi\alpha z^2}{\lambda^2} \cdot \sin^2(\theta_c)
\]
For optical wavelengths, these photons can propagate through an optically transparent medium and can be detected by photo-detectors. Note, that the following discussion focuses on the injected light yield and propagation effects such as absorption, scattering [7] or chromatic dispersion [8, 4] are not considered.

A relativistic track ($\beta = 1$) in water or ice ($n \approx 1.33$) produces typically $N_0 \approx 250$ cm$^{-1}$ optical photons in a wavelength interval between 300 nm and 500 nm, which is a typical sensitive region of photo-detectors used in neutrino telescopes like IceCube [9].

The Cherenkov angle for a relativistic track ($\beta = 1$) is $\theta_{c,0} = \arccos(1/n) \approx 41^\circ$. For smaller particle velocities the opening angle of the cone shrinks according to equation [1] and the number of photons decreases according to the factor $\sin^2(\theta_c)$ in equation [2].

The Cherenkov threshold is given by $v \leq c_{med}$, which is equivalent to $\beta \leq \frac{1}{n}$ and $\theta_c = 0$. In terms of kinetic energy the Cherenkov threshold is
\[
E_c \geq m \cdot \left(1 - \frac{1}{n^2} \right) - 1
\]
In water and ice, the threshold is $E_{c,\mu} \approx 55$ MeV for muons and $E_{c,e^-} \approx 0.26$ MeV for electrons.

During propagation through the medium muons suffer energy loss by ionization of atoms and stochastic radiative processes such as bremsstrahlung, pair production and photonuclear interaction [5, 11]. Charged secondary particles with an energy above the Cherenkov threshold will also emit Cherenkov light and contribute to the total light yield. Secondary electrons and positrons, for which the Cherenkov threshold is relatively low will considerably enhance the brightness of the muon track. In this picture, a muon, which propagates through water or ice can be considered to be accompanied by a glowing halo of secondary particles. This is illustrated in figure [1] Shown are all charged secondary particles. Many of these
Figure 1: Visualization of a 5 TeV muon, $\mu^+$, which propagates a distance of 40 m through water (figure from [10]). Shown are all generated charged secondary particles (red for negative and blue for positive charge) as the result of a Geant4 simulation. Neutral particles, like photons are not shown. No lower energy threshold has been applied to the Geant4 simulation.

particles, in particular those close to the muon track, have an energy sufficiently above the Cherenkov threshold.

For high energies of TeV and above, radiative processes dominate the energy loss. These occur stochastically and can lead to a large, catastrophic energy loss. Therefore, large local fluctuations of the density of secondary tracks occur along the path of the muon. This can also be seen in figure [1].

For neutrino telescopes such stochastic energy losses have to be considered and can be accurately simulated by muon propagation programs like MMC or MUM [11, 12]. Usually, because of computing limitations, only catastrophic energy losses bigger than a predefined energy threshold $E_{\text{max}}$ are simulated stochastically whereas the large number of small energy losses are approximated as continuous losses. A typical value of $E_{\text{max}} \approx 0.5$ GeV has been established in [13] and is used to date in IceCube as a reasonable compromise between accuracy and computing resources.

The amount of additional Cherenkov light due to energy losses below $E_{\text{max}}$ has been parameterized in [13]. The relative amount of Cherenkov radiating track
length $l_{sec}$ from secondary processes per unit muon track $l_{\mu}$ was found as

$$\frac{l_{sec}}{l_{\mu}} = 0.172 + 0.032 \cdot \ln (E/\text{GeV})$$ (4)

for $E_{\text{max}} = 0.5 \text{ GeV}$ for $e^+$, $e^-$ and $E_{\text{max}} = 1 \text{ GeV}$ for $\gamma$. This calculation has been done with an older version of the Geant toolkit (3.16, [14]) and the reduction in light yield due to particles with $\beta < 1$ was neglected.

In this paper these calculations are repeated and extended using the Geant4 toolkit [16]. With today’s increased computing power, the accuracy of the simulations can be improved substantially. Muons are simulated with higher statistics and the Frank-Tamm factor $\sin^2(\theta_c)$ in equation 2 is used to calculate the correct light yield. The light yield is investigated as a function of the muon energy $E_\mu$ and the chosen $E_{\text{max}}$. Furthermore, the angular distribution of the additional Cherenkov photons is calculated and parameterized.

For the calculation of the angular distribution an improved transformation method is used, which was originally developed in [13]. It is described in Appendix A. The advantage of this transformation method is that Cherenkov photons do not need to be generated and propagated during the Geant4 simulation.

2. The Monte Carlo method

The Geant4 (GEometry ANd Tracking) toolkit is designed for the Monte Carlo simulation of particle-interactions with matter in high-energy physics [16]. It is object-oriented and programmed in the C++ language in contrast to its predecessors [14], which were programmed in FORTRAN-77.

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Figure 2: Basic principle of the simulation

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1 An initial attempt of this work has been started in [15].
The simulation principle of this work is illustrated in figure 2. For the calculation of the yield of photons, the muon is propagated through the simulation medium and secondary particles are created, which again produce further particles. For each secondary track segment $i$ we store the length $l_i$, the Lorentz factor $\beta_i$ and the inclination angle $\alpha_i$ of the direction with respect to the muon direction. In the following $\alpha_i$ is called zenith angle. Summing over all track segments allows to calculate the Cherenkov photon yield and the corresponding angular distribution.

![Figure 3: Geometrical layout of the simulation](image)

For the simulations we have defined a volume, which is filled with water or ice. The geometry is a cylinder with a total length of 40 m and a radius of 30 m, as shown in figure 3. The dimensions are chosen such that all secondary particles are well confined within the geometry and fully tracked. The cylinder is logically divided into two cylindrical sub-volumes I and II. The first sub-volume has a length of 10 m. The used media properties are given in Appendix B.

The simulated muon is injected from the bottom center into this volume with its initial momentum pointing into positive z-direction. It is killed once it reaches the end of this first sub-volume, after having propagated 10 m. A propagation length of 10 m is chosen as a compromise to optimize simulation speed between an approximately constant muon energy along the muon’s track and to allow for a higher probability of rare processes per event. The average energy loss of the muon is small compared to the here considered muon energies, substantially above 100 GeV and therefore the muon energy can be considered approximately constant.

However, if only the first sub-volume would be considered the total additional track length from secondary particles would be underestimated, because the injected the muon is not accompanied by secondary particles from upstream processes. In order to compensate for this effect, secondary particles are, unlike the muon, allowed to enter the second sub-volume and are fully considered in the simulation. By this method the missing “spill-in” secondaries are compensated by
additional “spill-out” secondaries.

For the here described simulation it is important to simulate all particles with energies above the Cherenkov threshold. Details on the simulated physics processes are given in [Appendix B]. In Geant4 some electromagnetic processes require production thresholds to avoid infrared divergences \[17\]. These production thresholds are specified as a cut-in-range threshold, using the `SetCuts()` method of `G4VUserPhysicsList`. Here, particles are tracked if their mean expected range is larger than this cut-in-range threshold. For each material and particle type, this cut-in-range is transformed into a corresponding energy threshold. Here, a cut-in-range of 100 µm is chosen. This corresponds to a kinetic energy threshold of \(E_{\text{cut},e^\pm} \approx 80\) keV for electrons, which is well below the Cherenkov threshold \(E_{c,e^\pm} \approx 264\) keV. Once produced, all secondary particles are tracked until they stop.

In order not to falsely include catastrophic energy losses an upper energy threshold \(E_{\text{max}}\) is implemented. Technically in the implementation, secondary particles are only created, if they correspond to a process by which the muon loses less than \(E_{\text{max}}\) in energy. In this work we use the default value \(E_{\text{max}} = 0.5\) GeV but also investigate the effect of different values.

Electro-magnetic cross-sections in Geant4 are certified up to a maximum muon energy of 10 TeV. In order to extend the simulations beyond this limit, the cross-sections are extrapolated by Geant4 up to 100 TeV (see [Appendix B]).

Special care was taken about multiple scattering. In order to avoid unphysical biases in the resulting angular distributions, we simulated each single scattering process individually (see [Appendix B]).

For \(\beta = 1\), the number of emitted Cherenkov photons is proportional to the length of the track and can be calculated using equation 2. For \(\beta < 1\) the yield is smaller and proportional to the factor

\[
\sin^2(\theta_c) = 1 - \cos^2(\theta_c) = 1 - \frac{1}{\beta^2 \cdot n^2} \quad (5)
\]

In order to properly account for the smaller yield, the length of each track segment \(l\) can be scaled with the *Frank-Tamm factor*

\[
\hat{l} = \frac{\sin^2(\theta_c)}{\sin^2(\theta_{c,0})} \cdot l \quad \text{with} \quad \sin^2(\theta_{c,0}) = 1 - \frac{1}{n^2} \quad (6)
\]

The value \(\hat{l}\) thus corresponds to the equivalent length of a relativistic track with the same photon yield as the track length \(l\).

The use of the equivalent length \(\hat{l}\) instead of an explicit calculation of photons has the advantage that the here presented results can be rescaled to slightly different indices of refraction, and are independent on the assumed wavelength interval.
3. Cherenkov photon yield from secondary tracks

3.1. The maximum energy threshold

An important parameter for this study is the energy $E_{\text{max}}$ above which processes are considered catastrophic. Low-energy processes below $E_{\text{max}}$ occur rather frequently and can be considered quasi-continuous for the typical scales and spacing of sensors in large neutrino telescopes. As a first step, we investigate the occurrence of these small energy losses as a function of $E_{\text{max}}$.

![Graphs showing the occurrence of charged particles tracks along a 5 TeV muon track.](image)

**Figure 4:** Examples for the occurrence of charged particles tracks along a 5 TeV muon track, which is propagated 10 m through water. Shown is the track length summed over all charged tracks along the direction of the muon with a bin size of 10 cm. In addition to the 10 cm muon track in each bin, all secondary tracks have been counted if they originate from an energy loss process below $E_{\text{max}}$ and if their energy is above the Cherenkov threshold. From left to right four individually simulated muons with rising $E_{\text{max}}$ (0.2 GeV, 0.5 GeV, 2.5 GeV, 5 GeV) are shown.
The effect of $E_{\text{max}}$ is illustrated in figure 4. It shows for four randomly chosen muon tracks the occurrence of secondary particle tracks along the path for those particles, which originate from energy losses below $E_{\text{max}}$. It is obvious that the stochastic nature of catastrophic losses increases with rising $E_{\text{max}}$. For $E_{\text{max}} = 0.2$ GeV rarely bins with large individual losses, e.g. with more than a factor 2 of the muon light yield occur. These losses seem to be largely uncorrelated and distributed randomly and almost continuously along the track. However, for $E_{\text{max}} = 2.5$ GeV and $E_{\text{max}} = 5$ GeV large local fluctuations are observed and the muon light yield can be exceeded by a factor 10 or more, for several subsequent bins.

Figure 5: Probability distribution of the occurrence of secondary track lengths for 10 cm bins. The values $l_{\text{add}}$ exclude the muon track length. Each 10 cm bin in figure 4 has been histogramed for a total of 10 simulated muons. The figures show the distributions for different $E_{\text{max}} = 0.2$ GeV, 0.5 GeV, 2.5 GeV, 5.0 GeV. The rightmost bin is the overflow bin. An exponential function $f(l_{\text{add}}) = \text{const} \cdot \exp(-l_{\text{add}}/\tau)$ is fitted to the data from 0 cm to 10 cm.
The distributions of the size of these fluctuations are shown in figure 5. For small $E_{\text{max}} \leq 0.5 \text{ GeV}$, the deviation of the tails from an exponential distribution become smaller. This indicates that the occurrence of additional track lengths from low-energy processes can be approximated by a Poissonian process for step sizes of the order of 10 cm. The typical value $\tau \approx 3 \text{ cm}$ corresponds to an additional yield of $\sim 30\%$ more Cherenkov photons relative to the bare muon track.

For larger step-length $l \gg 10 \text{ cm}$, e.g. of the order of meters, the exponentially distributed additional track length can be approximated continuously by a Gaussian distribution. According to the central limit theorem, the mean and variance are then

$$\langle l_{\text{add}} \rangle = \frac{l_\mu}{10 \text{ cm}} \cdot \tau \quad \text{and} \quad \sigma_{l_{\text{add}}} = \sqrt{\frac{l_\mu}{10 \text{ cm}} \cdot \tau} \quad (7)$$

For $E_{\text{max}} \geq 0.5 \text{ GeV}$ the distributions of track lengths show an increasingly non-exponential tail to large $l_{\text{add}}$ and large local fluctuations appear. The fitted $\tau$ remains reasonably constant for different cut-off energies, but the difference between the mean value of $l_{\text{add}}$ and the fitted $\tau$ increases with the cut-off energy. Therefore, the occurrence of secondary tracks above $E_{\text{max}} \geq 0.5 \text{ GeV}$ cannot be considered Poissonian and quasi-continuously. In the following the value $E_{\text{max}} = 0.5 \text{ GeV}$ is used as default, unless noted otherwise.

3.2. Total secondary track yield

The total amount of additional Cherenkov photons depends (see eq.2) on the Lorentz factor $\beta$ of each secondary track.

A large number of muons is propagated as described in section 2 for 10 m through ice and the track length $l_{\text{add}}$, additional to the muon, is recorded and investigated as a function of $\beta$.

Figure 6 shows the distributions $\Delta l_{\text{add}}/\Delta \beta$ for different $E_\mu$ and $E_{\text{max}}$. Most of the secondary track-segments have a large velocity $\beta \approx 1$. Interestingly, the shape of the distributions are remarkably invariant for different $E_\mu$ and $E_{\text{max}}$ and only the total $l_{\text{add}}$ changes. The application of the Frank-Tamm factor (eq.6) shows that the majority of the additional Cherenkov photons is strongly dominated by secondary track-segments with $\beta \geq 0.99$.

In order to understand the weak dependence on the primary energy, figure 7 shows the distributions $\Delta l_{\text{add}}/\Delta \beta$ for the different energy loss processes. It becomes obvious that the processes below $E_{\text{max}}$ are dominated by ionization, followed by pair-production, which contributes about $5\% - 25\%$. The nuclear-interaction process contributes to about $0.1\%$ and bremsstrahlung even less. At high energies these processes usually contribute significantly to the muon energy loss [11]. However, they usually result in large energy transfers, which rarely fall below $E_{\text{max}}$. 

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Figure 6: Velocity distribution of secondary track lengths. Shown is the additional track length $l_{\text{add}}$ per 10 m muon track versus the Lorentz factor $\beta$ for bins of 0.002 in $\beta$. The top figures a) and b) show the distributions for different muon energies with $E_{\text{max}} = 0.5$ GeV and the bottom figures c) and d) the distributions for different $E_{\text{max}}$ for a 10 TeV muon. In the left figures each track segment enters with its physical length, whereas in the figures on the right side each length is weighted with the Frank-Tamm factor (eq. 6) and the distribution corresponds to $\tilde{l}_{\text{add}}$, which is proportional to the effective Cherenkov photon yield.

Therefore, bremsstrahlung, being the hardest energy loss process, has the smallest contribution to the energy losses below $E_{\text{max}}$.

This can be verified when investigating the energy dependence of the amount of secondary tracks for the different processes. This is shown in figure 8. In the left figure the maximum energy cut has not been applied and the radiative processes show the expected linear increase with energy while ionization only weakly depends on energy. The application of $E_{\text{max}}$ in the right figure results in a change of the relative strength and ionization becomes the most important process with only a
Figure 7: Velocity distribution of secondary track lengths. Shown is the additional track length $\hat{l}_{\text{add}}$ per 10 m muon track versus the Lorentz factor $\beta$ for bins of 0.002 in $\beta$ for $E_\mu = 10$ TeV and $E_{\text{max}} = 0.5$ GeV. The contributions of different secondary energy loss processes are shown. The Frank-Tamm factor (eq. 6) has been applied.

Figure 8: Additional track length without (left) and with (right) a maximum energy cut $E_{\text{max}} = 0.5$ GeV as a function of the muon energy. The solid lines show the result without and the dashed lines with application of the Frank-Tamm factor.

weak ($\sim 3\%$) increase with energy. Bremsstrahlung and photo-nuclear interactions are marginally relevant and hence the weak energy dependence can be identified as a result of an increasing contribution of the pair-production process.

The weak energy dependence is thus a consequence of the strong suppression of radiative processes.
Figure 9: Parametrization of the additional track length as function of $E_\mu$. Shown left are the separately parameterizations of the two strongest processes ionisation, and pair production for $E_{max} = 0.5 \text{ GeV}$ The right figure shows the parameterization of the total for different values of $E_{max}$. Each parameterization is done for $l$ without and $\hat{l}$ with the Frank-Tamm factor.

### 3.3. Parameterization of the secondary Cherenkov photon yield

In the following we parameterize the (effective) additional track length $l_{\text{add}}$ and $\hat{l}_{\text{add}}$ as a function of the muon energy for different $E_{max}$ with the expression

$$
\frac{l_{\text{add}}}{l_\mu} = \lambda_0 + \kappa \cdot \ln \left( \frac{E_\mu}{\text{GeV}} \right) \quad \text{and} \quad \frac{\hat{l}_{\text{add}}}{l_\mu} = \hat{\lambda}_0 + \hat{\kappa} \cdot \ln \left( \frac{E_\mu}{\text{GeV}} \right)
$$

The result is shown in figure 9. The parameterization result for different processes and $E_{max}$ are summarized in table C.3 in Appendix C.

As a typical value the amount of Cherenkov photons is increased by about 38% for $E_\mu = 10 \text{ TeV}$, when the Frank-Tamm factor is applied.

As expected the effect of pair production and the slope of the energy dependency increases with increasing $E_{max}$. Changing $E_{max}$ by a factor 5 from 0.2 GeV to 1 GeV leads to a change of the additional track length by typically a factor 2.

The effect of the Frank-Tamm factor reduces the number of additional photons by about 10 – 12%. The reduction is largely independent of $E_\mu$ and $E_{max}$.

The dependency on the index of refraction is small. Though the total number of generated photons by the muon and all secondaries increases according to equation 2 the fractional amount of secondary photons relative to the muon changes only because of a slightly changed Cherenkov threshold. However, only very few Cherenkov photons are created by secondary particles close to the Cherenkov threshold anyway.

For the comparison with the result in [13], (see eq.4), one has to consider that the older calculation used different values $E_{max} = 1 \text{ GeV}$ for photons and $E_{max} =
0.5 GeV for electrons and positrons. However, in table C.3 it can be seen that that parameterization is consistently between our parameterizations for $E_{\text{max}} = 0.5$ GeV and $E_{\text{max}} = 1$ GeV.

### 3.4. Water versus ice

The same simulations have been repeated using water ($\rho = 1$ g cm$^{-3}$) and seawater ($\rho = 1.039$ g cm$^{-3}$) instead of ice ($\rho = 0.91$ g cm$^{-3}$). For these simulations we have used pure water and ocean water, which contains small traces of other elements (for details see table B.1, Appendix B).

![Figure 10: Distribution of angular directions $\alpha$ (left) and Lorentz factors $\beta$ (right) of secondary particles in pure water, sea water and ice. The simulated parameters are $E_\mu = 10$ TeV and $E_{\text{max}} = 500$ MeV in water and ice. Shown at the bottom are also the relative deviations $(i - w)/(i + w)$ of the two types of water ($w$) relative to ice ($i$) (Red: relative deviation ice/pure water; Blue: ratio ice/sea water)](image)

As shown in figure 10, the number of photons does not change strongly. Naively, one could expect that the number of photons increases proportionally to the density change, that this assumption is wrong can be explained by two canceling effects: Though the number of secondary tracks per unit track of the muon increases with the density ratio, the length of these tracks decreases by the same ratio, because of larger energy losses and fewer emitted Cherenkov photons. Hence, the resulting number of photons in water is similar to the number of photons in ice within $\lesssim 1\%$, comparable to the uncertainties of the chemical composition. We conclude, that the here presented parameterizations, though simulated for ice, can be applied to water without a density correction.

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[^1]: Note, that the density of South Pole ice at the depth of the IceCube detector is $\rho = 0.9216$ g cm$^{-3}$
4. Angular distribution

The angular distribution of secondary Cherenkov photons with respect to the muon axis depends on two parameters: the zenith angles $\alpha$ of the secondary tracks (see figure 2) and on the Lorentz factors $\beta$ of these tracks (see equation 6).

![Figure 11: Density distribution of the relative track length versus the zenith angle $\alpha$ and Lorentz factor $\beta$ for a 10 TeV muon. The vertical color codes corresponds to the histogrammed lengths $\hat{l}$ for a 10 m muon with the Frank-Tamm factor applied.](image)

An example of the density distribution of secondary track length relative to these parameters $\frac{\Delta^2 \hat{l}_{\text{add}}}{\Delta \alpha \Delta \beta}$ is shown in figure 11. Most tracks are emitted into forward direction with large values of $\beta$.

In figure 12 the dependence of the angular distribution $\frac{\Delta^2 \hat{l}_{\text{add}}}{\Delta \alpha}$ on the parameter $E_{\text{max}}$ and the muon energy $E_{\mu}$ is shown. The corresponding figures for the parameter $\beta$ have been shown in figure 6. It can be seen that the angular distribution depends only weakly on the muon energy. Also the Frank-Tamm factor is only a small correction to the overall shape. The parameter $E_{\text{max}}$ has a larger effect. Not only the total track length increases with larger $E_{\text{max}}$, but the additional tracks are pointing preferably into forward direction. This corresponds to the larger boost of higher-energy secondaries.

When assuming azimuthal symmetry, the angular distribution of Cherenkov photons can be calculated according to the procedure described in Appendix A, if the Lorentz factor and zenith angle $\alpha$ for each track segment is known.

An advantage of the transformation method is that Cherenkov photons do not need to be generated and propagated during the Geant4 simulation, which greatly reduces the computing time. Furthermore, during the Geant4 simulation no specific
Figure 12: Histograms of the additional track length per 10 m muon track as a function of the zenith angle $\alpha$ of the track. The top figures show the results for $E_{\text{max}} = 500$ MeV and different $E_{\mu}$ and the bottom figures the results for $E_{\mu} = 10$ TeV and different $E_{\text{max}}$. The left figures show the physical track length and the Frank-Tamm factor has been applied additionally to the tracks in the right figures.

wavelength interval needs to be assumed and the transformation can be applied later with different values of $n$ on the same data set. Disadvantage of the method is the assumption of azimuthal symmetry for the occurrence of track elements. This is a good assumption, e.g. for high statistics of small secondary tracks, which are produced with random orientation along the muon track. However, for a single low-energy electron of a few MeV multiple scattering will lead to a specific path, which is not symmetric in azimuth.

The calculated angular distributions are shown in figure 13. The distribution depends only weakly on the muon energy where higher-energy muons produce a slightly more pronounced Cherenkov peak. The dependence on $E_{\text{max}}$ is stronger. Correspondingly to the enhancement of tracks into forward direction (see figure 12) also the Cherenkov peak becomes more pronounced. For larger $E_{\text{max}}$ the relative contribution of pair production increases whereas the contribution from ionization remains constant.

When looking at the individual energy loss processes large differences are ob-
Figure 13: The angular distribution of secondary Cherenkov photons for different $E_\mu$ with $E_{\text{max}} = 500$ MeV (left) and for different $E_{\text{max}}$ with $E_\mu = 10$ TeV (right). Shown are the normalized angular distribution per photon and steradian.

Figure 14: The angular distribution of Cherenkov photons (left) and the corresponding angular distribution of secondary track length (right) for the different processes and $E_{\text{max}} = 500$ MeV and $E_\mu = 10$ TeV. For the Cherenkov photons the normalized angular distributions per photon and steradian are shown. For the track length the Frank-Tamm factor has been applied.

served. The distributions of Cherenkov photons and the corresponding distributions of track length are shown in figure 14. Bremsstrahlung and pair production exhibit a similarly sharp distribution. The ionization process has a less pronounced ("chopped off") Cherenkov peak. This is due to the fact, that kinematics impose a minimum angle $\alpha$ as a function of $E_{\text{max}}$, which is of the order of a few degrees (see equation B.1). The Cherenkov light from nuclear interactions is largely featureless with approximately a factor 2 more photons into forward than into backward direction. Correspondingly to the maximum energy threshold $E_{\text{max}} = 0.5$ GeV, the considered momentum transfers are not large compared to the transversal fermi motion of interacting nuclear partons. Hence large scattering angles appear and the
forward direction is less pronounced compared to electro-magnetic interactions.

Figure 15: Example of the parameterization of the angular distribution for $E_\mu = 10\text{ TeV}$ and $E_{\text{max}} = 500\text{ MeV}$. Left is the asymmetric parameterization function and right the symmetric parameterization.

The angular distribution has been parameterized with two functions, in the following called asymmetric and simple parameterization. The asymmetric parameterization function is

$$
\frac{1}{N} \frac{dN}{d\Omega} (x) = a \begin{cases} 
  e^{bx} & \text{if } -1 \leq x < 0, \\
  e^{cx} & \text{if } 0 \leq x < 0.4, \\
  e^{0.4(c-d)x} e^{0.6e(f-0.6)x} e^{-(f-x)} & \text{if } 0.4 \leq x < 0.6 \\
  e^{0.4(c-d)x} e^{0.6e(f-0.6)x} e^{-h(x-f)} & \text{if } 0.6 \leq x < f, \\
  e^{0.4(c-d)x} e^{0.6e(f-0.6)x} e^{-h(x-f)} & \text{if } f \leq x \leq 1,
\end{cases}
$$

with $x = \cos \phi$ and $f = \cos(\theta_{c,0}) = 1/n$, where $n$ is the index of refraction. This function fits the parts left and right of the Cherenkov peak individually. It is a piecewise continuous function consisting of an exponential function from -1.0 to 0.0, from 0.0 to 0.4 and from 0.4 to 0.6. Around the Cherenkov peak a function $\exp ax^b$ is fitted on both sides.

The simple parameterization function is

$$
\frac{1}{N} \frac{dN}{d\Omega} (x) = ae^{b[\cos(\theta_{c,0}) - x]},
$$

with $x = \cos \phi$ and $\cos(\theta_{c,0}) = 1/n$, where $n$ is the index of refraction. This parameterization is fit with only three parameters and describes the data reasonably well but slightly worse than the asymmetric parameterization. Note, that the parameter $a$ could be eliminated by normalization of the resulting distributions. Examples of the fits are shown in figure 15. The resulting fit parameters for different $E_\mu$ and $E_{\text{max}}$ for the asymmetric and the simple fit are listed in Appendix C.
Figure 16: Relative deviation of the asymmetric parameterization (left) and the simple parameterization (right) from the data. The deviations are shown for parameterizations of different $E_\mu$ and $E_{\text{max}} = 500$ MeV (see tables C.4 and C.5).

Figure [16] shows the relative residuals of the parameterizations for different $E_\mu$. The accuracy of the asymmetric parameterization is for all angles within a few percent. The simple parameterization is less accurate with relative deviations up to 10%. The accuracy is found to be very similar for different energies.

Figure 17: Relative deviation of the parameterization for $E_\mu = 3$ TeV from the data for different $E_\mu$ and $E_{\text{max}} = 500$ MeV. Left is the asymmetric parameterization (left) and right the simple parameterization (see tables C.4 and C.5).

Figure [17] shows the difference of different energies with a parameterization of fixed energy. It becomes obvious that without the proper treatment of the energy dependence, the good accuracy of the asymmetric parameterization would be substantially degraded. On the other hand for the simple fit, the energy dependence is smaller than the overall uncertainty and the parameterization could be used approximately for a fixed energy.
Figure 18: Parameters of the asymmetric fit versus energy (left). Markers represent the parameters and the solid lines are best fit lines. The right figure shows the relative deviations of the parameterization when the energy dependence is used from the linear fits.

The energy dependence of each parameter $p_i$ of the asymmetric parameterization has been fit with a linear ansatz

$$p_i = \lambda_0 + \lambda_1 \cdot \ln \left( \frac{E_\mu}{\text{GeV}} \right).$$

(11)

Figure 18 (left) shows the fit energy dependence of the parameters of the asymmetric parameterization. The numerical fit parameters are given in table C.6 in Appendix C. Figure 18 (right) shows the relative deviation of the data from the parameterization with this linear ansatz. The parameterization error related to the energy dependence is substantially smaller than the overall uncertainty.

5. Summary and Conclusions

In this paper we have focused on a very specific subject: the parameterization of Cherenkov light from low-energy secondary particles, which accompany high-energy muons due to energy loss in water and ice.

While high-energy catastrophic energy losses are usually treated stochastically by muon propagation codes, a similar treatment of the low-energy tail for these processes would require a large amount of CPU. This is unnecessary because these processes are almost continuous. For this we present a parameterization of the total Cherenkov light light yield and its angular distribution.

The total amount of Cherenkov light from these low-energy processes is of the order of a few ten percent of the light from the bare muon and small compared to catastrophic processes. However, the proper consideration of this light is required
for determining a correct energy scale e.g. for high-energy neutrino telescopes, in particular to achieve systematic uncertainties of the order of a few percent. As an example for the IceCube neutrino telescope, the parametrization [13] is used. That parametrization is based on an out-dated Geant3.16 simulation and does not include the correct Frank-Tamm factor for the photon yield. Furthermore, no parameterization for the angular distribution of Cherenkov photons is provided. Here, we present a substantially improved parametrization of the total photon yield and the angular distribution, based on Geant4.

In the first step of these simulations it is verified that the light yield from low-energy processes increases logarithmically with the muon energy for a fixed high-energy cut-off $E_{\text{max}}$. It is found to be weakly dependent on $E_{\text{max}}$. A reasonable value for the transition between continuous and stochastic treatment is $E_{\text{max}} \approx 0.5$ GeV, similar to the findings in [13].

We find that the proper treatment of the Frank-Tamm factor leads to a small reduction of roughly 10% of the additional photon yield. We find this reduction largely independent on $E_{\mu}$ and $E_{\text{max}}$.

A more detailed investigation reveals, that the Cherenkov light yield is dominated by ionization processes, followed by pair production while bremsstrahlung and nuclear interactions are only marginally contributing.

For the calculation of the angular Cherenkov light distribution, we present an improved version of a transformation method introduced in [13]. This method allows to calculate the photon distribution from a distribution of track directions and their velocities. The resulting angular distribution of photons depends only weakly on the energy but more strongly on $E_{\text{max}}$. We are presenting a simple 3-parameter fit and a more complex 8 parameter fit to these distributions and the energy dependence of these parameters.

For the interpretation of the accuracy of these results it has to be considered that the additional light yield corresponds to only a few ten percent of the light yield of a bare muon track. Hence, the errors of this calculation will have a small effect on the calculation of the total light yield for a muon track. For high muon energies, $E_{\mu} \gtrsim 1$ TeV, catastrophic processes dominate the light yield and outshine the muon as well as the small secondary tracks, which are considered here.

The accuracy of the here presented calculations are limited by the accuracy of muon propagation code in Geant4. For high-energies, $E_{\mu} > 1$ TeV, the cross sections in this code are poorly constrained by experimental data and rely on the proper calculation of electromagnetic processes. The typical accuracy of these calculations is a few % in the energy loss, see e.g. [11]. For our calculations we have identified ionization as the dominating process and this process has comparably smaller errors than other radiative processes. Hence, we estimate the total
systematic uncertainty of the light yield of the order of 1\% and similar the differential error for the energy dependence, because the light yield increases only logarithmically with $E_\mu$.

For the angular distribution, we find that the relative accuracy of the parametrization is about 10\% for the simple and about 2 – 3\% for the asymmetric parameterization.

Corresponding to the 10 m length of the muon track its energy is not constant but that uncertainty is small. E.g. at 100 GeV the mean energy loss is about 1 GeV corresponding to 1\% uncertainty, which could be corrected for.

For the application of the here presented results it has to be considered that an additional uncertainty appears because the low-energy secondary processes are assumed to appear quasi-continuously and the direction of secondary tracks is azimuthally symmetric. This is addressed in section 3.1 where it is found that the chosen value $E_{\text{max}}$ is a compromise with respect to computing efficiency. A residual non-Poissonian component has to be considered. However, the here used step-length of 10 cm is rather small compared to the length of a muon track in large neutrino telescopes of several hundred m. Obviously, a systematic error related to the non-continuous light emission and non-azimuthal symmetry will be largely suppressed for reasonably-long muon tracks. Furthermore, during propagation the Cherenkov photons are largely affected by scattering and change their direction.

In the here presented calculations an index of refraction of $n = 1.33$ is used which corresponds to a specific Cherenkov angle. If required, a different index could be applied to the here presented parameterizations, because the light yield is calculated in units of equivalent track length and the resulting uncertainties due to the velocity distribution of secondary tracks close to the Cherenkov threshold are small.

No specific wavelength-interval, e.g. related to the sensitivity of e.g. photosensors or the material transparency, has been assumed for the here presented calculations.

**Appendix A. Transformation method for the calculation of the angular distribution of Cherenkov photon from an angular distribution of track segments**

Goal of the transformation-method is to calculate the angular distribution of the resulting Cherenkov photons $\frac{dN}{d\phi}$ with respect to the direction of the injected muon track. For the muon-track, which is aligned with the z-axis one expects a pure cone and all photons are emitted under the Cherenkov angle

$$\frac{dN}{d\phi} = N \cdot \delta(\theta_c)$$
where $\theta_c$ is defined by equation 1 and $\delta(x)$ is the delta function. However, for inclined track-segments, e.g. for secondary electrons, one expects that the Cherenkov cone is inclined with respect to the $z$-axis and the photons are emitted in a broader angular region with respect to the $z$-axis. If the track-segment has a smaller velocity $\beta < 1$, the number of photons is reduced according to equation 2 and the Cherenkov angle $\theta_c$ shrinks.

During the Geant-simulations we have stored the distribution of all track segments $l_i$ with respect to the rapidity $\beta$ and the polar angle $\alpha$ to the $z$-axis. The $z$-axis is defined by the direction of the injected muon. This procedure therefore results in a 2-dimensional histogram (e.g. see figure 11):

$$\frac{d^2 l}{d\alpha d\beta}$$

The Cherenkov photons will be distributed randomly along the rim of the Cherenkov cone with a total circumference $S$ (see figure A.19). The relative fraction of the cone’s circumference $\frac{\Delta s}{S}$, which points into the direction $\phi \pm \Delta \phi$ is proportional to the fraction of photons emitted into that direction. Idea of the transformation algorithm is to calculate for all track elements and directions the fraction of photons, which are emitted into that direction. The method had been initially developed in [13], however, there the calculations were done assuming $\beta = 1$ for all tracks. Here, we extend the method to variable values of $\beta$ in order to properly account for the changing Cherenkov angle and photon yield for each track-segment (eq.2).

The geometry is illustrated in figure A.19. Note, that for the following calculation we use radians instead of degrees.

The rim of the cone, which is inclined by the angle $\alpha$ will have zenith angles between

$$\phi_{\text{min}} \leq \phi \leq \phi_{\text{max}}$$

with

$$\phi_{\text{min}} = |\alpha - \theta_c| \quad \text{and} \quad \phi_{\text{min}} = \alpha + \theta_c$$

except for $\phi > \pi - \theta_c$ where

$$\phi_{\text{max}} = 2 \cdot \pi - \alpha - \theta_c$$

As we are only interested in the relative fraction of the rim, we normalize the neck of the rim $F \equiv 1$. Then, the length of the cone is $\cos(\theta_c)$, the radius of the circular rim is $b = \sin(\theta_c)$ and therefore

$$S = 2\pi \cdot b = 2\pi \sin(\theta_c)$$
The track element $ds$, which points towards the direction $\phi$ can be represented by an azimuth angle $\psi$ in the plane of the cone’s rim.

$$ds = \sin(\theta_c) \cdot d\psi$$  \hspace{1cm} (A.5)

The fraction of the rim is then

$$\frac{ds(\alpha, \beta)}{S} = \frac{2 \cdot \frac{ds}{d\psi} \cdot d\psi}{2\pi \sin(\theta_c)} = \frac{1}{\pi} \cdot d\psi$$  \hspace{1cm} (A.6)
The factor 2 arises if we only allow for angles \(0 \leq \psi \leq \pi\). Because of the symmetry of the cone the elements \(ds'\) for \(\psi' = 2\pi - \psi\) point into the same direction.

In the following express the angle \(\psi\) as a function of the known angles \(\theta_c(\beta), \alpha(l_i)\) and the target \(\phi\). The cosine law (see figure [A.19] (C),(D)) results in

\[
\cos(\psi) = \frac{b^2 + c^2 - a^2}{2bc}
\]  

(A.7)

From the figures [A.19] (C),(D) one can obtain the identities

\[
b = \sin(\theta_c) \quad \tan(\alpha) = \frac{c}{\cos(\theta_c)} \quad a^2 = d^2 + v^2
\]  

(A.8)

and further

\[
\sin(\phi) = \frac{d}{F} = d, \quad v = |u - w|, \quad \cos(\phi) = \frac{u}{F} = u, \quad \cos \alpha = \frac{\cos \theta_c}{w}
\]  

(A.9)

resolving for \(a, b, c\) and inserting into equation [A.7] gives

\[
\cos(\psi) = \frac{\sin^2(\theta_c) + \tan^2(\alpha) \cdot \cos^2(\theta_c) - \sin^2(\phi) - \left(\cos(\phi) - \frac{\cos(\theta_c)}{\cos(\alpha)}\right)^2}{2 \cdot \sin(\theta_c) \cdot \tan(\alpha) \cdot \cos(\theta_c)}
\]  

(A.10)

which can be simplified to

\[
\cos(\psi) = \frac{\cot(\theta_c)}{\sin(\alpha)} \cdot \left(\frac{\cos(\phi)}{\cos(\theta_c)} - \cos(\alpha)\right)
\]  

(A.11)

With this formula the \(\Delta \Psi\)-range, which corresponds to the bin boundaries of the source histogram \(\frac{d\psi}{d\alpha d\beta}\) and target histogram \(\frac{dN}{d\phi}\) can be calculated.

For practical reasons it is convenient not to calculate the number of photons explicitly but rather to use a relative light yield with respect to a relativistic track. The relative photon yield for a track \(\beta < 1\) relative to \(N_0\) (for \(\beta = 1\)) is

\[
\frac{N}{N_0} = \frac{\sin^2(\theta_c)}{\sin^2(\theta_c,0)}
\]  

(A.12)

In the following we will use

\[
\hat{I}(\phi) \equiv l \cdot \frac{\sin^2(\theta_c)}{\sin^2(\theta_c,0)} \cdot \frac{d\psi}{\pi}
\]  

(A.13)

which is the fractional part of the track length \(l\), which illuminates the direction \(\phi\) rescaled to the photon-yield relative to a relativistic track.

This particular choice is advantageous, because it is independent of a specific wavelength interval for the Cherenkov photons. The here presented results can be rescaled to arbitrary wavelength intervals.

The numerical algorithm proceeds as follows:
1. A loop over all bins in the 2-dimensional source-histogram $\frac{d^2l}{d\alpha d\beta}$ gives the track length $l_{ij}$, which corresponds to a certain bin $\beta_i$ and $\alpha_j$.

2. The Cherenkov angle $\theta_c(\beta_i, n)$ is calculated according to equation A.1.

3. The target range $\phi_{\text{min}}(\theta_c, \alpha_j)$ and $\phi_{\text{max}}(\theta_c, \alpha_j)$ are calculated according to equation A.2 and A.3.

4. For all bins $\phi_k$ of the target histogram $\frac{d\hat{l}}{d\phi}$, for which $\phi_{\text{min}} < \phi_k < \phi_{\text{max}}$ we can calculate the $\Psi$-range, which corresponds to the bin boundaries according to equation A.11

$$\Delta\psi \equiv \Psi(\theta_c, \alpha_j, \text{min}(\phi_k, \phi_{max})) - \Psi(\theta_c, \alpha_j, \text{max}(\phi_k, \phi_{min})) \quad (A.14)$$

5. The target histogram bin $\frac{d\hat{l}}{d\phi}(\phi_k)$ is incremented with the value

$$\Delta\hat{l}(\phi_k) = l_{ij} \cdot \frac{\Delta\psi}{\pi} \cdot \frac{\sin^2(\theta_c)}{\sin^2(\theta_c, 0)} \quad (A.15)$$

The final value $\hat{l}_k$ of the target histogram then corresponds to the equivalent relativistic track length, emitting the total number of photons as observed under this zenith.

6. The procedure is numerically improved by the interpolation of the values $\beta$ and $\alpha$ within the bins of the input histogram.

Examples of the transformation are shown in figure A.20. For a track into direction (z) (top figures) the light cone is aligned with the z-axis and all photons are emitted under a fixed angle. A relativistic track of unitlength 1 will produce a distribution which is peaked at the Cherenkov angle and the integral corresponds to 1 unitlength (top left). For smaller values of $\beta$ the opening angle of the cone shrinks and the total light yield is reduced (top right). For inclined tracks the photons are distributed over a large angular region (bottom left). Also here for smaller values of $\beta$ the opening angle of the cone shrinks and the total light yield is reduced (bottom right).

**Appendix B. Geant4 configuration parameters used for this study**

In this chapter a summary of the defined media properties and physics processes is given.
Figure A.20: Examples of calculated effective light distribution. Shown is the resulting distribution: $d\hat{I}/d\phi$ for a single track of unit length with different directions $\alpha$ and velocities $\beta$ and a refraction index $n = 1.33$.

| Medium   | Density $\frac{g}{cm^3}$ | Element      | Fraction of mass | Isotope abundance |
|----------|---------------------------|--------------|------------------|-------------------|
| Ice      | 0.910                     | Hydrogen     | 88.81%           |                   |
|          |                           | Oxygen       | 11.19%           |                   |
| Water    | 1.000                     | Hydrogen     | 88.81%           |                   |
|          |                           | $H^1$        | 99.9%            |                   |
|          |                           | $H^2$        | 0.01%            |                   |
|          |                           | Oxygen       | 11.19%           |                   |
|          |                           | $O^{16}$     | 99.76%           |                   |
|          |                           | $O^{17}$     | 0.04%            |                   |
|          |                           | $O^{18}$     | 0.20%            |                   |
| Sea water| 1.039                     | Hydrogen     | 10.74%           |                   |
|          |                           | Oxygen       | 85.41%           |                   |
|          |                           | Chlorine     | 2.37%            |                   |
|          |                           | Sodium       | 1.32%            |                   |
|          |                           | Magnesium    | 0.15%            |                   |

Table B.1: Composition of ice, water and sea water as used in the Geant4-simulation. Water is defined as G4_WATER from the NIST database.
Appendix B.1. Materials

In Geant4 macroscopic properties of matter are described by `G4Material` and the atomic properties are described by `G4Element`. A material can consist of multiple elements and therefore represent a chemical compound, mixture as well as pure materials.

For the performed simulations, three different materials were used: Ice, water and sea water. The properties are summarized in table B.1.

Unless noted otherwise, the value $n = 1.33$ is used for the index of refraction.

Appendix B.2. Physicslist

All physics processes, which are used during the simulation must be registered in `G4VUserPhysicsList`. For these simulations a customized version of the list `G4EmStandardPhysics_option3` is used. Multiple scattering has been replaced by single scattering and the process `G4MuNuclearInteraction` has been added. In table B.2 all registered processes are summarized.

Single scattering is used in favor of multiple scattering because the step size in multiple scattering is not limited. This leads to an overestimated track length in certain directions. This is observed in particular for $\delta$-electrons from the ionisation process. The zenith angle $\alpha$ of the quasi free electron is kinematically closely related to the velocity $\beta$ and the muon energy $E_\mu$

$$\cos \alpha = \frac{1 - \sqrt{1 - \beta^2}}{\beta} \frac{E_\mu + m_e}{\sqrt{E_\mu^2 - m_\mu^2}}.$$  \hspace{1cm} (B.1)

The directions of these lower-energy electrons is quickly smeared due scattering. However, the process multiple scattering leads to an overestimated track length for the initial direction and hence the distribution of Cherenkov photons is biased.

The default maximum energy for the cross section tables and the calculation of $dE/dx$ in Geant4 is $10 \text{ TeV}$. However, muon processes can be extrapolated theoretically up to $10 \text{ PeV}$ and several high-energy corrections are implemented in Geant4, e.g. LPM suppression in gamma conversion and nuclear EM form-factors in single scattering models [19].

Appendix C. Parameterization results of the light yield and angular Cherenkov light distribution

\footnote{The finite width of the distribution results from the finite bin size of the source histogram and becomes slightly larger for smaller $\beta$.}
| Particle | Process | Model |
|----------|---------|-------|
| $\gamma$ | G4PhotoElectricEffect | G4PEEffectFluoModel |
| | G4ComptonScattering | G4KleinNishinaModel |
| | G4GammaConversion | |
| $e^-$ | G4CoulombScattering | |
| | G4eIonisation | |
| | G4eBremsstrahlung | |
| $e^+$ | G4CoulombScattering | |
| | G4eIonisation | |
| | G4eBremsstrahlung | |
| | G4eplusAnnihilation | |
| $\mu^+, \mu^-$ | G4CoulombScattering | |
| | G4MuIonisation | |
| | G4MuBremsstrahlung | |
| | G4MuPairProduction | |
| | G4MuNuclearInteraction | |
| $p^+, p^-, K^+, K^-, p^+$ | G4CoulombScattering | |
| | G4hIonisation | |
| | G4hBremsstrahlung | |
| | G4hPairProduction | |
| $\alpha, \text{He}^{4+}$ | G4ionIonisation | |
| | G4CoulombScattering | |
| | G4NuclearStopping | |
| all unstable particles | G4Decay | |

Table B.2: Physics processes of most important particles used in the simulation. If no model is specified the default model is used. For hadrons and ions that are not listed Coulomb scattering and ionisation are defined.
| Process | $n$ | $E_{\text{max}}$/GeV | $\hat{\lambda}_0$ | $\hat{\kappa}$ | $\lambda_0$ | $\kappa$ |
|--------|-----|----------------------|-------------------|----------------|-------------|---------|
| total  | 1.33| 0.2                  | 0.1969            | 0.0081         | 0.2329      | 0.0087  |
| ion    |     |                      | 0.2117            | 0.0024         | 0.2486      | 0.0026  |
| pair   |     |                      | -0.0150           | 0.0058         | -0.0158     | 0.0061  |
| brems  |     |                      | 0.0002            | 0              | 0.0002      | 0       |
| nucl   |     |                      | 0.0000            | 0              | 0.0000      | 0       |
| total  | 1.33| 0.5                  | 0.1880            | 0.0206         | 0.2235      | 0.0219  |
| ion    |     |                      | 0.2489            | 0.0030         | 0.2879      | 0.0033  |
| pair   |     |                      | -0.0626           | 0.0175         | -0.0651     | 0.0186  |
| brems  |     |                      | 0.0004            | 0              | 0.0004      | 0       |
| nucl   |     |                      | 0.0005            | 0              | 0.0005      | 0       |
| total  | 1.33| 1.0                  | 0.1028            | 0.0445         | 0.1334      | 0.0472  |
| ion    |     |                      | 0.2712            | 0.0043         | 0.3114      | 0.0046  |
| pair   |     |                      | -0.1686           | 0.0400         | -0.1783     | 0.0423  |
| brems  |     |                      | 0.0008            | 0              | 0.0009      | 0       |
| nucl   |     |                      | 0.0010            | 0              | 0.0011      | 0       |
| total  | 1.33| 0.5                  | 0.1842            | 0.0204         | 0.2196      | 0.0218  |
| ion    |     |                      | 0.2247            | 0.0030         | 0.2837      | 0.0032  |
| pair   |     |                      | -0.0612           | 0.0174         | -0.0648     | 0.0185  |
| brems  |     |                      | 0.0004            | 0              | 0.0004      | 0       |
| nucl   |     |                      | 0.0005            | 0              | 0.0005      | 0       |
| total  | 1.36| 0.5                  | 0.1916            | 0.0207         | 0.2268      | 0.0220  |
| ion    |     |                      | 0.2528            | 0.0030         | 0.2914      | 0.0087  |
| pair   |     |                      | -0.0619           | 0.0176         | -0.0654     | 0.0186  |
| brems  |     |                      | 0.0004            | 0              | 0.0004      | 0       |
| nucl   |     |                      | 0.0005            | 0              | 0.0005      | 0       |
| total  | 1.33| 0.5-1.0              | -                 | -              | 0.172       | 0.032   |

Table C.3: Parameterization of the additional light yield for different $n$ and $E_{\text{max}}$. 

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| $E_{\text{max}}$ MeV | $E_{\text{primary}}$ GeV | a    | b    | c    | d    | e    | g    | h    | i    |
|---------------------|------------------------|------|------|------|------|------|------|------|------|
| 200                 | 1000                   | 0.0325 | 2.090 | 2.486 | 3.412 | 5.910 | 1.134 | 4.639 | 1.169 |
|                     | 3000                   | 0.0320 | 2.090 | 2.498 | 3.438 | 5.846 | 1.104 | 4.535 | 1.124 |
|                     | 10000                  | 0.0317 | 2.090 | 2.500 | 3.469 | 5.530 | 1.046 | 4.344 | 1.060 |
|                     | 30000                  | 0.0313 | 2.090 | 2.514 | 3.497 | 5.451 | 1.024 | 4.309 | 1.035 |
| 500                 | 1000                   | 0.0288 | 2.075 | 2.527 | 3.665 | 4.997 | 0.806 | 4.058 | 0.782 |
|                     | 3000                   | 0.0281 | 2.079 | 2.543 | 3.720 | 4.953 | 0.765 | 4.084 | 0.742 |
|                     | 10000                  | 0.0275 | 2.077 | 2.550 | 3.768 | 4.881 | 0.723 | 4.102 | 0.703 |
|                     | 30000                  | 0.0269 | 2.076 | 2.567 | 3.822 | 4.892 | 0.706 | 4.163 | 0.687 |
| 1000                | 1000                   | 0.0264 | 2.071 | 2.557 | 3.855 | 4.711 | 0.629 | 4.105 | 0.613 |
|                     | 3000                   | 0.0254 | 2.074 | 2.580 | 3.927 | 4.767 | 0.589 | 4.224 | 0.575 |
|                     | 10000                  | 0.0246 | 2.073 | 2.588 | 4.007 | 4.820 | 0.561 | 4.343 | 0.550 |
|                     | 30000                  | 0.0239 | 2.071 | 2.606 | 4.081 | 4.857 | 0.540 | 4.437 | 0.532 |

Table C.4: Parameters of the asymmetric parameterization, with $n = 1.33$ ($f = 1/n \approx 0.7519$)

| $E_{\text{max}}$ MeV | $E_{\text{primary}}$ GeV | a       | b       | c       |
|---------------------|------------------------|---------|---------|---------|
| 200                 | 1000                   | 0.4036  | -3.075  | 0.7299  |
|                     | 3000                   | 0.4147  | -3.118  | 0.7183  |
|                     | 10000                  | 0.4282  | -3.163  | 0.7038  |
|                     | 30000                  | 0.4367  | -3.195  | 0.6967  |
| 500                 | 1000                   | 0.5597  | -3.531  | 0.6033  |
|                     | 3000                   | 0.5983  | -3.626  | 0.5840  |
|                     | 30000                  | 0.6660  | -3.776  | 0.5554  |
|                     | 10000                  | 0.6381  | -3.711  | 0.5656  |
| 1000                | 1000                   | 0.7403  | -3.899  | 0.5261  |
|                     | 3000                   | 0.8266  | -4.049  | 0.5024  |
|                     | 10000                  | 0.9084  | -4.175  | 0.4837  |
|                     | 30000                  | 0.9778  | -4.278  | 0.4709  |

Table C.5: Parameters of the simple parameterization, with $n = 1.33$ ($f = 1/n \approx 0.7519$)
Table C.6: Parameterization of energy dependence of fit parameters for $n = 1.33$ and $E_{\text{max}} = 500$ MeV for the asymmetric parameterization (left) and the simple parameterization (right).

| $p_l$ | $\lambda_0$ | $\lambda_1$ |
|-------|-------------|-------------|
| a     | 0.03257     | -0.00055    |
| b     | 2.07658     | 0           |
| c     | 2.45248     | 0.01092     |
| d     | 3.35320     | 0.04536     |
| e     | 5.22537     | -0.03425    |
| g     | 1.00817     | -0.02997    |
| h     | 3.85166     | 0.02905     |
| i     | 0.97332     | -0.02845    |

| $p_l$ | $\lambda_0$ | $\lambda_1$ |
|-------|-------------|-------------|
| a     | 0.34485492  | 0.03144662  |
| b     | -3.04159806 | -0.07192665 |
| c     | 0.69937264  | -0.01420755 |
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