Basic canonical brackets in the gauge field theoretic models for the Hodge theory

S. Gupta, R. Kumar and R. P. Malik

Physics Department, Centre of Advanced Studies, Banaras Hindu University, Varanasi - 221 005, India

DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi - 221 005, India

We deduce the canonical brackets for a two (1 + 1)-dimensional (2D) free Abelian 1-form as well as a four (3 + 1)-dimensional (4D) 2-form gauge theory by exploiting the beauty and strength of the continuous symmetries of the Becchi-Rouet-Stora-Tyutin (BRST) invariant Lagrangian densities that respect, in totality, six continuous symmetries. These symmetries entail upon these models to become the field theoretic examples for the Hodge theory. Taken together, these symmetries enforce the existence of exactly the same canonical brackets amongst the creation and annihilation operators that appear in the canonical method of quantization for the normal mode expansion of the basic fields of these theories. In other words, we provide an alternative to the canonical method of quantization for our present gauge field theoretic models for the Hodge theory where the continuous symmetries play a decisive role. We conjecture that our method of quantization would be valid for any arbitrary gauge field theoretic model for the Hodge theory in any arbitrary dimension of spacetime.

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I. INTRODUCTION

Symmetry principles, through the ages, have helped physicists to unravel some of the deepest mysteries of the nature. It is well-known, for instance, that the local symmetries govern interactions. They lead to the conservation laws in the realm of classical and quantum systems and dictate selection rules in the context of the latter. In our present investigation, we establish that the continuous symmetries of the gauge field theoretic models for the Hodge theory lead to the derivation of basic (anti)commutators amongst the creation and annihilation operators that are at the heart of the covariant canonical quantization of a gauge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism.

In the canonical method of quantization (for a given field theoretic model), a triplet of central ideas are exploited. These are the spin-statistics theorem, definition of the canonical conjugate momentum and normal ordering. First, using the spin-statistics theorem, we distinguish between the bosonic and fermionic field variables. Second, we compute the canonical conjugate momenta corresponding to the field variables from the Lagrangian density of a given field theoretic system and define the (graded) Poisson brackets at the classical level. The latter are upgraded to the (anti)commutators at the quantum level. Ultimately, in terms of the normal mode expansions of the fields and their corresponding momenta, the above (anti)commutators are converted into the (anti)commutators amongst the creation and annihilation operators and, then, the basic brackets of the theory ensue. The physical quantities of interest (e.g. Hamiltonian, conserved charges, etc.), expressed in terms of the creation and annihilation operators are, finally, normal ordered to make physical sense (so that the unwanted infinities do not appear in the theory).

In our present investigation, we shall utilize the virtues of spin-statistic theorem and normal ordering but we shall not take the help of the definition of canonical conjugate momentum in our central goal of obtaining the correct basic brackets amongst the creation and annihilation operators of our present gauge field theoretic models (i.e. 2D and 4D free Abelian 1-form and 2-form gauge theories, respectively) for the Hodge theory within the framework of BRST
formalism. Rather, we shall exploit the beauty and strength of the continuous symmetries (and their generators) to obtain the correct (anti)commutators amongst the creation and annihilation operators of our present theories which incorporate fermionic as well as bosonic field operators. In fact, it is the strength of all the six continuous symmetry transformations for these models (of Hodge theory) that entails upon the basic (anti)commutators to emerge in a very natural fashion (and match with that of the canonical formalism).

In our present paper, we demonstrate that the (anti-)BRST, (anti-)co-BRST, ghost and a bosonic symmetry transformations of a free 2D Abelian 1-form and a free 4D Abelian 2-form gauge theories (which happen to be the field theoretic models for the Hodge theory\(^{[2–6]}\)) imply the existence of canonical brackets that are required for the covariant canonical quantization of the above theories within the framework of BRST formalism\(^{[7–10]}\). We emphasize that all the above symmetries, taken together, lead to the derivation of the one and the same non-vanishing canonical brackets [see, equations (34) and (60) below] that are also derived by exploiting the usual canonical method from the Lagrangian density (see, Sec. V below). Thus, it is clear that the multi-faceted usefulness of the continuous symmetries enforce the existence of covariant canonical brackets, too, for a class of field theories (i.e. free 2D Abelian 1-form and 4D Abelian 2-form gauge theories) that turn out to be the physical models for the Hodge theory due to their various symmetries.

The prime factors that have propelled us to pursue our present investigation are as follows. First and foremost, it is very exciting to note that a set of continuous symmetries dictate the basic canonical brackets of a given class of theories that are models for the Hodge theory. Second, our present work has the potential to be generalized to the case of any arbitrary field theoretic model for the Hodge theory. Third, the derivation of the basic brackets from the symmetry consideration (even though algebraically more involved) is more beautiful than the usual mathematical derivation of the same by exploiting the definition of canonical momenta from a given Lagrangian density. Finally, our present work adds yet another glittering feather in the crown of the theoretical versatility of symmetry principles because we derive here the basic canonical brackets for 2D and 4D theories by exploiting the latter.

The contents of our paper are organized as follows. In Sec. II, we discuss various continuous symmetry properties of the Lagrangian densities for the 2D free Abelian 1-form and 4D free Abelian 2-form gauge theories. Our Sec. III is devoted to the derivation of the conserved charges which, in turn, are expressed in terms of the creation and annihilation operators. Sec. IV deals with the derivation of the canonical (anti)commutators by exploiting the basic tenets of symmetry principles. Our Sec. V focuses on the derivation of the canonical brackets from the first principles applied to the Lagrangian densities of the two theories which are models for the Hodge theory. Finally, in Sec. VI, we make some concluding remarks.

In our first Appendix A, we prove the uniqueness of the basic canonical brackets [cf. (34) below] by deforming it consistently. We take into account the ghost number consideration and spin-statistics theorem in our deformation procedure. We perform this exercise only for the free 2D Abelian 1-form theory for the sake of brevity but it can be extended to the case of free 4D theory in a straightforward manner. Our second Appendix B contains all the (anti)commutators that emerge from the (anti-) dual BRST, (anti-)BRST, bosonic and ghost symmetry generators for the (anti-)BRST invariant model of the free 4D Abelian 2-form gauge theory.

**Notations and conventions:** We adopt here the convention such that the D-dimensional flat Minkowskian metric \(\eta_{\mu\nu}\) is endowed with signatures \((+1, -1, -1, \ldots)\) and \(P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i\) is the dot product between two non-null vectors \(P_\mu\) and \(Q_\mu\) where the Einstein summation convention is taken into account. Here the Greek indices \(\mu, \nu, \ldots = 0, 1, 2, \ldots, (D - 1)\) and Latin indices \(i, j, \ldots = 1, 2, \ldots, (D - 1)\). The 2D \(F_{\mu\nu}\) has only electric field as its non-vanishing component (i.e. \(F_{01} = -\varepsilon_{\mu\nu} \partial_\mu A_\nu = E\)). We take 2D Levi-Civita tensor \((\varepsilon_{\mu\nu})\) with the choice \(\varepsilon_{01} = +1 = -\varepsilon^{01}\) and it obeys \(\varepsilon_{\mu\nu} \varepsilon^{\mu\nu} = -2!, \varepsilon_{\mu\nu} \varepsilon^{\mu\lambda} = \delta_\nu^\lambda\). We also have the d’Alembertian operator \(\Box = \partial_0^2 - \partial_i^2\). For the 4D theory, the totally antisymmetric Levi-Civita tensor \((\varepsilon_{\mu\nu\rho\kappa})\) is such that \(\varepsilon_{0123} = +1 = -\varepsilon^{0123}\), \(\varepsilon_{\mu\nu\rho\kappa} \varepsilon^{\mu\nu\rho\kappa} = -4!\), \(\varepsilon_{\mu\nu\rho\kappa} \varepsilon^{\mu\nu\rho\kappa} = -3! \delta_\nu^\rho\), etc. and the component \(\varepsilon_{0ijk} = \varepsilon_{ijk}\) is taken as the 3D Levi-Civita tensor. We also have the 4D d’Alembertian operator as \(\Box = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\).

### II. LAGRANGIAN FORMALISM: CONTINUOUS SYMMETRIES

In this section, we discuss various continuous symmetries of the 2D and 4D Abelian 1-form and 2-form gauge theories within the framework of Lagrangian formalism.
A. 2D Abelian 1-form theory: BRST approach

We begin with the (anti-)BRST invariant Lagrangian density for a free 2D Abelian 1-form gauge theory in the Feynman gauge (see, e.g. [2]):

\[ L_{b_1} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left( \partial \cdot A \right)^2 - i \partial_\mu \bar{C} \partial^\mu C \]
\[ \equiv \frac{1}{2} E^2 - \frac{1}{2} \left( \partial \cdot A \right)^2 - i \partial_\mu \bar{C} \partial^\mu C. \] (1)

In the above, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the curvature tensor derived from the 2-form \( F^{(2)} = d A^{(1)} \equiv \frac{1}{2} (dx^\mu \wedge dx^\nu) F_{\mu\nu} \)
where \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative and connection 1-form \( A^{(1)} = dx^\mu A_\mu \) defines the vector potential \( A_\mu \). The gauge-fixing term \([ - \frac{1}{2} \left( \partial \cdot A \right)^2 ]\) owes its origin to the co-exterior derivative \( \delta = - * d * \) because \( \delta A^{(1)} \equiv - * d * A^{(1)} = + (\partial \cdot A) \). Here the (*) operation is the Hodge duality defined on the 2D Minkowski spacetime manifold. The fermionic (anti-)ghost fields (\( \bar{C} \)) invariant under the (anti-)BRST symmetry transformations (2).

The Lagrangian density (1) respects the on-shell (i.e. \( \Box C = 0, \Box \bar{C} = 0 \)) nilpotent (i.e. \( s_{(a)b}^2 = 0 \)) (anti-)BRST symmetry transformations \((s_{(a)b})\) as

\[ s_b A_\mu = \partial_\mu C, \quad s_b C = 0, \quad s_b \bar{C} = -i (\partial \cdot A), \quad s_b E = 0, \]
\[ s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} \bar{C} = 0, \quad s_{ab} C = +i (\partial \cdot A), \quad s_{ab} E = 0, \] (2)

where the physical (gauge-invariant) electric field \((E)\), owing its origin to exterior derivative \( d = dx^\mu \partial_\mu \), remains invariant under the (anti-)BRST symmetry transformations (2).

We have the following on-shell (\( \Box C = 0, \Box \bar{C} = 0 \)) nilpotent \((s_{(a)d})\), continuous and infinitesimal (anti-)co-BRST symmetry transformations \(s_{(a)d}\) (see, e.g. Ref. [2] and references therein)

\[ s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad s_d C = 0, \quad s_d \bar{C} = -i E, \quad s_d (\partial \cdot A) = 0, \]
\[ s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C, \quad s_{ad} C = 0, \quad s_{ad} \bar{C} = +i E, \quad s_{ad} (\partial \cdot A) = 0, \] (3)

that leave the Lagrangian density (1) quasi-invariant because it transforms to a total spacetime derivative.[2] It should be noted here that the gauge-fixing term \((\partial \cdot A)\), owing its origin to the co-exterior derivative, remains invariant under the on-shell nilpotent (anti-)co-BRST transformations.

A bosonic symmetry \((s_\omega)\) (as the anticommutator \(\{s_b, s_d\} \equiv -\{s_{ab}, s_{ad}\} = s_\omega\)) leads to the following transformations[2]

\[ s_\omega A_\mu = \partial_\mu E - \varepsilon_{\mu\nu} \partial^\nu (\partial \cdot A), \quad s_\omega E = \Box (\partial \cdot A), \quad s_\omega C = 0, \quad s_\omega \bar{C} = 0, \quad s_\omega (\partial \cdot A) = \Box E, \] (4)

under which the Lagrangian density (1) transforms to a total spacetime derivative (see, e.g. Ref. [2]). Furthermore, we have an infinitesimal version of the ghost-scale symmetry transformation \((s_g)\) in the theory, namely;

\[ s_g A_\mu = 0, \quad s_g C = +C, \quad s_g \bar{C} = -\bar{C}, \quad s_g E = 0, \quad s_g (\partial \cdot A) = 0, \] (5)

which is derived from the scale transformations \((C \to e^{+\Lambda} C, \bar{C} \to e^{-\Lambda} \bar{C}, A_\mu \to e^0 A_\mu)\) where the infinitesimal scale parameter \(\Lambda\) is spacetime independent and, for the sake of brevity, it has been set equal to one in the above transformations. It is, thus, crystal clear that we have a set of six continuous symmetries in the theory. Two of these (i.e. \(s_g, s_\omega\)) are bosonic in nature and rest of them (i.e. \(s_{(a)b}, s_{(a)d}\)) are fermionic \((s_{(a)b}^2 = 0, s_{(a)d}^2 = 0)\). The latter property is nothing but the nilpotency of order two that is associated with the (anti-)BRST and (anti-)co-BRST symmetry transformations which provide the analogues of the cohomological operators \(d\) and \(\delta\). Similarly, the bosonic transformation \(s_\omega\) [cf. (4)] provides the physical realization of the Laplacian operator \(\Delta = (d + \delta)^2 = \{d, \delta\}\) (see, e.g. [2] for details).
We start off with the (anti-)BRST invariant Lagrangian density of the 4D free Abelian 2-form gauge theory in the Feynman gauge as (see, e.g. Ref. [4])

\[ \tilde{\mathcal{L}}_{B_2} = \frac{1}{2} \left( \partial^\nu B_{2\mu} - \partial_\mu \phi_1 \right) \left( \partial_\nu B^{\mu} - \partial^\mu \phi_1 \right) - \frac{1}{2} \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} \partial^\rho B^{\mu\lambda} - \partial_\mu \phi_2 \right) \left( \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} - \partial^\mu \phi_2 \right) + \left( \partial_\mu C_\nu - \partial_\nu C_\mu \right) \left( \partial^{\mu} C^{\nu} \right) - \frac{1}{2} \left( \partial \cdot \tilde{C} \right) \left( \partial \cdot C \right) - \partial_\mu \tilde{\beta} \partial^\mu \beta, \]

(6)

where $B_{\mu\nu}$ is antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$) tensor gauge field derived from the 2-form $B^{(2)} = \frac{1}{2} \left( dx^\mu \wedge dx^\nu \right) B_{\mu\nu}$, $\phi_1$ and $\phi_2$ are massless scalar and pseudo-scalar fields. The Lorentz vector fields ($\bar{C}_\mu C^\mu$) are the fermionic (anti-)ghost fields (with $C_\mu^2 \neq \bar{C}_\mu \bar{C}_\mu = 0$, $C_\mu \bar{C}_\nu + \bar{C}_\nu \bar{C}_\mu = 0$, etc.) and $(\tilde{\beta})\beta$ are the bosonic (anti-)ghost fields which are the Lorentz scalars. These (anti-)ghost fields are required for the validity of unitarity in the theory. The bosonic (anti-)ghost fields $(\tilde{\beta})\beta$ are the ghost-for-ghost fields in the theory because of the stage-one reducibility that exists in it.

The above Lagrangian density (6) remains quasi-invariant (because it transforms to a total spacetime derivative) under the following on-shell nilpotent ($\tilde{s}_{(a)b}^2 = 0$) as well as absolutely anticommuting ($\tilde{s}_{b} \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_{b} = 0$) (anti-)BRST symmetry transformations $\tilde{s}_{(a)b}$ (see, e.g. Ref. [3] for details)

\[
\tilde{s}_{ab} B_{\mu\nu} = (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad \tilde{s}_{ab} C_\mu = -\partial_\mu \tilde{\beta}, \quad \tilde{s}_{ab} \phi_1 = \frac{1}{2} \left( \partial \cdot \bar{C} \right), \quad \tilde{s}_{ab} \phi_2 = 0,
\]

(7)

\[
\tilde{s}_{b} B_{\mu\nu} = (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad \tilde{s}_{b} C_\mu = \partial_\mu \beta, \quad \tilde{s}_{b} \phi_1 = \frac{1}{2} \left( \partial \cdot C \right), \quad \tilde{s}_{b} \phi_2 = 0.
\]

(8)

It can be checked that, under the above (anti-)BRST symmetry transformations, the total kinetic term remains invariant. The kinetic term is the second term of Lagrangian density (6) which is the generalization of the original Lagrangian density $\mathcal{L}_0 = \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho}$ where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ owes its origin to the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) because $dB^{(2)} = \frac{1}{2} \left( dx^\mu \wedge dx^\nu \wedge dx^\sigma \right) H_{\mu\nu\rho}$. The Lagrangian density (6) is also endowed with the on-shell nilpotent [i.e. $(\tilde{s}_{(a)b}^2 = 0)$] (anti-)co-BRST [or (anti-)dual BRST] symmetry transformations, under which the total gauge-fixing term remains invariant. These on-shell nilpotent (anti-)co-BRST symmetry transformations are [4]

\[
\tilde{s}_{ad} B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \partial^\rho \bar{C}^\sigma, \quad \tilde{s}_{ad} C_\mu = -\left( \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho B^{\nu\sigma} - \partial_\mu \phi_2 \right), \quad \tilde{s}_{ad} \phi_1 = \partial_\mu \beta,
\]

\[
\tilde{s}_{ad} \phi_2 = \frac{1}{2} \left( \partial \cdot C \right), \quad \tilde{s}_{ad} \bar{C}_\mu = -\left( \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho \bar{C}^{\nu\sigma} - \partial_\mu \phi_2 \right), \quad \tilde{s}_{ad} \left( \tilde{\beta}, \phi_1, \partial^\mu B_{\mu\nu} \right) = 0,
\]

(9)

\[
\tilde{s}_{d} B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \partial^\rho \bar{C}^\sigma, \quad \tilde{s}_{d} C_\mu = \left( \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\rho B^{\nu\sigma} - \partial_\mu \phi_2 \right), \quad \tilde{s}_{d} \bar{C}_\mu = -\partial_\mu \tilde{\beta},
\]

\[
\tilde{s}_{d} \phi_2 = \frac{1}{2} \left( \partial \cdot \bar{C} \right), \quad \tilde{s}_{d} \tilde{\beta} = -\left( \frac{1}{2} \partial \cdot \bar{C} \right), \quad \tilde{s}_{d} \left( \tilde{\beta}, \phi_1, \partial^\mu B_{\mu\nu} \right) = 0.
\]

(10)

The above symmetry transformations leave the Lagrangian density (6) quasi-invariant because, once again, the Lagrangian density transforms to a total spacetime derivative. The (anti-)BRST and (anti-)co-BRST symmetry transformations obey the two key properties: (i) the nilpotency of order two (i.e. $\tilde{s}_{(a)b}^2 = 0$, $\tilde{s}_{(a)d}^2 = 0$), and (ii) the absolute anticommutativity (i.e. $\tilde{s}_{b} \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_{b} = 0$, $\tilde{s}_{ab} \tilde{s}_{ad} + \tilde{s}_{ad} \tilde{s}_{d} = 0$). The former property (i.e. nilpotency) shows the fermionic nature of the (anti-)BRST as well as (anti-)co-BRST symmetry transformations whereas the latter property (i.e. absolute anticommutativity) implies that the BRST and anti-BRST as well as co-BRST and anti-co-BRST symmetries are linearly independent of one-another, respectively.

The (anti-)co-BRST symmetry transformations, we re-emphasize, leave the total gauge-fixing term of the theory invariant as can be seen from (9) and (10). The gauge-fixing term for the theory is the first term of the Lagrangian density (6) and it owes its origin to the co-exterior derivative $(\delta)$ because $\delta B^{(2)} = dx^\mu \left( \partial^\nu B_{\nu\mu} \right)$ yields the gauge-fixing term $(\partial^\nu B_{\nu\mu})$ for the gauge field $B_{\mu\nu}$. The total first term of (6) is the generalization of $(\partial^\nu B_{\nu\mu})$ which is derived from $\delta B^{(2)} = - * d * B^{(2)}$. Here $(*)$ is the Hodge duality operation of differential geometry on 4D spacetime manifold.
We note that the anticommutator of the above two symmetries (i.e. \( \{ \tilde{s}_b, \tilde{s}_d \} = \tilde{s}_s = - (\tilde{s}_{ab}, \tilde{s}_{ad}) \)) leads to a bosonic symmetry in the theory\([4]\)

\[
\tilde{s}_s B_{\mu \nu} = \varepsilon_{\mu \nu \eta \sigma} \partial^\eta (\partial_\sigma B^{\mu \sigma}) + \frac{1}{2} \varepsilon_{\nu \xi \sigma \eta} \partial_\mu (\partial_\xi B^{\nu \eta}) - \frac{1}{2} \varepsilon_{\mu \xi \sigma \eta} \partial_\nu (\partial_\xi B^{\mu \eta}), \\
\tilde{s}_s C_\mu = -\frac{1}{2} \partial_\mu (\partial \cdot C), \quad \tilde{s}_s \tilde{C}_\mu = \frac{1}{2} \partial_\mu (\partial \cdot \tilde{C}), \quad \tilde{s}_s (\beta, \tilde{\beta}, \phi_1, \phi_2) = 0.
\] (11)

Under the above symmetry transformations, the Lagrangian density (6) changes to a total spacetime derivative. As a consequence, the action integral \( S = \int dx \mathcal{L}_g \) remains invariant (i.e. \( \delta S = 0 \)), for the physically well-defined fields. In addition to the above continuous symmetries, the Lagrangian density (6) also respects the following infinitesimal ghost symmetry transformations (\( \tilde{s}_g \)):

\[
\tilde{s}_g C_\mu = +\Sigma C_\mu, \quad \tilde{s}_g \tilde{C}_\mu = -\Sigma \tilde{C}_\mu, \quad \tilde{s}_g \beta = +2\Sigma \beta, \quad \tilde{s}_g \tilde{\beta} = -2\Sigma \tilde{\beta}, \quad \tilde{s}_g (B_{\mu \nu}, \phi_1, \phi_2) = 0, \quad (12)
\]

where \( \Sigma \) is a global scale parameter. Thus, our 4D free Abelian 2-form gauge theory is also endowed with six continuous symmetries which render this model to be a field theoretic model for the Hodge theory\([4-6]\) because the symmetry transformations (9), (10) and (11) provide the physical realizations of the cohomological operators \( d, \delta \) and \( \Delta \) of differential geometry (see, e.g. Ref. [11-13] for details), respectively.

### III. CONSERVED CHARGES IN TERMS OF THE CREATION AND ANNIHILATION OPERATORS.

In this section, we express the conserved charges of the 2D Abelian 1-form and 4D 2-form theories in terms of the normal ordered creation and annihilation operators.

#### A. 2D Abelian 1-form theory: conserved charges

The continuous symmetry transformations [cf. (2)-(5)], according to Noether’s theorem, lead to the derivation of the conserved currents \( (\partial_\mu J_\nu^r = 0) \). These, in turn, provide us the expressions for the conserved charges (i.e. \( Q_r = \int dx J_\nu^r, \ r = b, ab, d, ad, g, \omega \)). These charges, for our present theory are, (see, e.g. Ref. [2]).

\[
Q_b = \int dx \left[ \partial_\mu (\partial \cdot A) C - (\partial \cdot A) \tilde{C} \right], \quad Q_d = \int dx \left[ E \tilde{C} - \tilde{E} C \right], \\
Q_{ab} = \int dx \left[ \partial_\mu (\partial \cdot A) C - (\partial \cdot A) \tilde{C} \right], \quad Q_{ad} = \int dx \left[ E \tilde{C} - \tilde{E} C \right], \\
Q_\omega = \int dx \left[ \partial_\mu (\partial \cdot A) E - \dot{E} (\partial \cdot A) \right], \quad Q_g = i \int dx \left[ C \tilde{C} + \tilde{C} C \right], \quad (13)
\]

where the dot, on a generic field \( \Phi \), denotes the time derivative [i.e. \( \dot{\Phi} = (\partial \Phi)/(\partial t) \)]. We lay emphasis on the fact that these conserved charges have been computed from the Noether conserved current where the concept of canonical momentum plays no role at all. In fact, it is the action principle (i.e. \( \delta S = 0 \)) that plays a decisive role in the above derivations of the conserved Noether currents.

It is evident, from the Lagrangian density (1), that the basic fields of the theory satisfy the following Euler-Lagrange equations of motion:

\[
\Box A_\mu = 0, \quad \Box C = 0, \quad \Box \tilde{C} = 0. \quad (14)
\]

The normal mode expansions of these fields, in the phase space of our present theory, are listed below (see, e.g. Ref. [14])

\[
A_\mu (x, t) = \int \frac{dk}{\sqrt{2\pi} 2k_0} \left[ a_\mu (k) e^{+ik \cdot x} + a_\mu^\dagger (k) e^{-ik \cdot x} \right], \\
C(x, t) = \int \frac{dk}{\sqrt{2\pi} 2k_0} \left[ c(k) e^{+ik \cdot x} + c^\dagger (k) e^{-ik \cdot x} \right], \\
\tilde{C}(x, t) = \int \frac{dk}{\sqrt{2\pi} 2k_0} \left[ \tilde{c}(k) e^{+ik \cdot x} + \tilde{c}^\dagger (k) e^{-ik \cdot x} \right]. \quad (15)
\]
where the 2-vector \( k_\mu = (k_0, k_1 = k) \) is the momentum vector and \( a^\dagger_\mu(k), c^\dagger(k) \) and \( \bar{c}(k) \) are the creation operators for a photon, a ghost and an anti-ghost quanta, respectively. The non-dagger operators \( a_\mu(k), c(k) \) and \( \bar{c}(k) \) stand for the corresponding annihilation operators for a single quantum, respectively.

Plugging in these expansions in the expressions for the charges in (13), we obtain the following from of the conserved charges, namely;

\[
Q_b = -\int dk k^\mu \left[ a^\dagger_\mu(k) c(k) + c^\dagger(k) a_\mu(k) \right], \quad Q_d = -\int dk \varepsilon^{\mu\nu} k_\mu \left[ c^\dagger(k) a_\nu(k) + a^\dagger_\nu(k) \bar{c}(k) \right],
\]

\[
Q_{ab} = -\int dk k^\mu \left[ a^\dagger_\mu(k) \bar{c}(k) + \bar{c}(k) a_\mu(k) \right], \quad Q_{ad} = -\int dk \varepsilon^{\mu\nu} k_\mu \left[ c^\dagger(k) a_\nu(k) + a^\dagger_\nu(k) c(k) \right],
\]

\[
Q_{\omega} = i \int dk \varepsilon^{\mu\nu} k_\mu k^\rho \left[ a^\dagger_\rho(k) a_\nu(k) - a^\dagger_\nu(k) a_\rho(k) \right] \equiv i \int dk k^2 \varepsilon^{\mu\nu} a^\dagger_\mu(k) a_\nu(k),
\]

\[
Q_g = -\int dk \left[ c^\dagger(k) c(k) + c(k) \bar{c}(k) \right],
\]

where the normal ordering has been taken into account so that all the creation operators are kept towards the left. This ordering renders the above charges physically sensible. In the above, we have also taken into account the expression for the Dirac \( \delta \)-function as: \( \delta(k-k') = (1/2\pi) \int dx e^{x(k-k')} \) in the explicit computations of charges [in equation (16)]. Thus, we have already exploited one of the key requirements (i.e. normal ordering) of the quantization scheme, we have proposed to follow.

**B. 4D Abelian 2-form theory: conserved charges**

In this subsection, first of all, we calculate the conserved charges corresponding to the continuous symmetries of the Lagrangian density (6). We have seen, in the previous section, that the 4D free Abelian 2-form gauge theory respects six continuous symmetry transformations. According to the Noether theorem, these continuous symmetries [cf. (7)-(12)] lead to the derivation of six conserved (i.e. \( \partial_\mu J^\mu_r = 0 \)) currents \( (J^\mu_r, \ r = b, ab, ad, g, \omega) \) corresponding to each symmetry transformation. The conservation laws \( (\partial_\mu J^\mu_r = 0) \) can be proven by exploiting the Euler-Lagrange equations of motion [cf. (23) below] derived from our starting Lagrangian density (6). The zero component \( (J^0_r) \) of these conserved currents lead to the derivation of the following conserved (i.e. \( \tilde{Q}_r = 0 \)) charges (i.e. \( \tilde{Q}_r = \int d^3x \ J^0_r \) as listed below:

\[
\tilde{Q}_b = \int d^3x \left[ (\partial^i \tilde{C}^0 - \partial^0 \tilde{C}^i)(\partial_1 \beta) - \frac{1}{2} (\partial^0 \beta)(\partial \cdot \tilde{C}) - \frac{1}{2} (\partial_i B^{i0} - \partial^0 \phi_1)(\partial \cdot C) \right] + H^{0ij}(\partial_i C_j) - \epsilon_{ijk}(\partial_i \phi_2)(\partial^j \tilde{C}^k) + (\partial^0 \tilde{C}^i - \partial^i \phi_1)(\partial \cdot B_{0i} + \partial^j B_{ji} - \partial_i \phi_1),
\]

\[
\tilde{Q}_{ab} = \int d^3x \left[ (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0)(\partial_1 \tilde{\beta}) + \frac{1}{2} (\partial^0 \tilde{\beta})(\partial \cdot C) - \frac{1}{2} (\partial_i B^{i0} - \partial^0 \phi_1)(\partial \cdot C) \right] + H^{0ij}(\partial_i C_j) - \epsilon_{ijk}(\partial_i \phi_2)(\partial^j \tilde{C}^k) + (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0)(\partial \cdot B_{0i} + \partial^j B_{ji} - \partial_i \phi_1),
\]

\[
\tilde{Q}_d = \int d^3x \left[ \epsilon_{ijk} (\partial_i \tilde{C}^k)(\partial_0 B^{0i} + \partial_i B^{i0} - \partial^i \phi_1) + \frac{1}{2} (\partial \cdot C)(\partial_0 \tilde{\beta} - (\partial_0 \beta))(\partial^i \phi_1) \right] + \frac{1}{2} \left( - \epsilon_{ijk} \partial_0 B^{jk} + 2 \epsilon_{ijk} \partial_i B^{0k} \right)(\partial^0 \tilde{C}^0 - \partial^0 \phi_2) + \frac{1}{4} \epsilon_{ijk} (\partial^j B^{i0})(\partial \cdot C) + (\partial_1 \phi_2)(\partial^0 \tilde{C}^i - \partial^i \phi_1),
\]

\[
\tilde{Q}_{ad} = \int d^3x \left[ \epsilon_{ijk} (\partial_i \tilde{C}^k)(\partial_0 B^{0i} + \partial_i B^{i0} - \partial^i \phi_1) + \frac{1}{2} (\partial \cdot C)(\partial_0 \tilde{\beta} - (\partial_0 \beta))(\partial^i \tilde{C}^0) \right] + \frac{1}{2} \left( - \epsilon_{ijk} \partial_0 B^{jk} + 2 \epsilon_{ijk} \partial_i B^{0k} \right)(\partial^0 \tilde{C}^0 - \partial^0 \phi_2) + \frac{1}{4} \epsilon_{ijk} (\partial^j B^{i0})(\partial \cdot C) + (\partial_1 \phi_2)(\partial^0 \tilde{C}^i - \partial^i \phi_1),
\]
\[ \tilde{Q}_\omega = \int d^3x \left[ \frac{1}{2} \partial_\mu (\partial^\mu C^i - \partial^\mu \bar{C}^i) + \frac{1}{2} \partial_\mu (\partial^\mu \bar{C}^i - \partial^\mu C^i) + \epsilon_{ijm} H^{0ij} \partial_\mu (\partial^\mu B^{0m}) \right. \\
- \epsilon_{ijk} \partial^j (\partial_0 B^{0k} + \partial_0 B^{ik}) (\partial^0 \phi_1) - \frac{1}{2} \epsilon_{ijk} \partial_0 (\partial^0 B^{ik}) (\partial^0 B^{0i} - \frac{1}{2} \epsilon_{ijk} \partial_i (\partial^0 B^{ik}) (\partial_0 B^{0l} + \partial_m B^{0i}) \\
- \frac{1}{2} \epsilon_{ijk} (\partial_0 B^{jk}) \partial^i (\partial_0 B^{0i} + \partial_l B^{ik}) + \frac{1}{2} \epsilon_{ijk} (\partial^i B^{0k}) \partial_0 (\partial_0 B^{0i} - \frac{1}{2} \epsilon_{ijk} \partial_i (\partial^0 B^{0k}) \partial_0 (\partial_0 B^{0i} + \partial_l B^{ik}) \\
+ \epsilon_{ijk} (\partial^0 B^{0k}) \partial^j (\partial_0 B^{0i} + \partial_l B^{ik}) + \frac{1}{2} \epsilon_{ijk} \partial^j \partial^0 (\partial_0 B^{0i} + \partial_m B^{0i}) + (\partial_\mu \phi_2) \partial_\mu H^{0ij} \right], \\
(21) \\
\tilde{Q}_g = \int d^3x \left[ (\partial^0 C^i - \partial^0 \bar{C}^i) \bar{C}_i + (\partial^0 \bar{C}^i - \partial^0 C^i) C_i - \frac{1}{2} (\partial^\mu C^0) C_0 - \frac{1}{2} (\partial^\mu \bar{C}^0) \bar{C}_0 - 2 \beta (\partial^\mu \bar{\beta}) + 2 \bar{\beta} (\partial^\mu \beta) \right], \\
(22) \\
\text{where } H^{0ij} = \partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}. \text{ The above conserved charges turn out to be the generators for the corresponding symmetry transformations [cf. (7)-(12)].}

The Euler-Lagrange equations of motion for the relevant fields of the 4D theory, derived from the Lagrangian density (6), are as follows

\[ \Box B_{\mu \nu} = 0, \quad \Box \phi_1 = 0, \quad \Box \phi_2 = 0, \quad \Box \beta = 0, \quad \Box \bar{\beta} = 0, \]
\[ \Box C_\mu = \frac{3}{2} \partial_\mu (\partial \cdot C), \quad \Box \bar{C}_\mu = \frac{3}{2} \partial_\mu (\partial \cdot \bar{C}). \]

(23)

We choose the gauge condition for the Lorentz vector fermionic (anti-)ghost fields such that \((\partial \cdot C) = (\partial \cdot \bar{C}) = 0\). With these gauge conditions, the last two equations in (23) reduce to the following simple and explicit form:

\[ \Box C_\mu = 0, \quad \Box \bar{C}_\mu = 0. \]

(24)

As a consequence, we can express the basic fields of our present theory in terms of the creation and annihilation operators as follows (see, e.g. [4]):

\[ B_{\mu \nu}(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( b_{\mu \nu}(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + b_{\mu \nu}^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]
\[ C_\mu(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( c_\mu(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + c_\mu^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]
\[ \bar{C}_\mu(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( \bar{c}_\mu(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + \bar{c}_\mu^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]
\[ \beta(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( b(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + b^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]
\[ \bar{\beta}(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( \bar{b}(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + \bar{b}^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]
\[ \phi_1(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( f_1(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + f_1^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]
\[ \phi_2(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \left( f_2(\vec{k}) e^{+ i \vec{k} \cdot \vec{x}} + f_2^\dagger(\vec{k}) e^{- i \vec{k} \cdot \vec{x}} \right), \]

(25)

where \(k_\mu(= k_0, k_i)\) is the momentum 4-vector. The non-dagger operators \(b_{\mu \nu}, c_\mu, \bar{c}_\mu, b, \bar{b}, f_1, f_2\) and the dagger operators \(b_{\mu \nu}^\dagger, c_\mu^\dagger, \bar{c}_\mu^\dagger, b^\dagger, \bar{b}^\dagger, f_1^\dagger, f_2^\dagger\), appearing in the above normal mode expansions of the basic fields of the theory, are the annihilation and creation operators, respectively. It is interesting to note that the choice of the above gauge conditions [i.e. \((\partial \cdot C) = (\partial \cdot \bar{C}) = 0\)] imply the following relationships in the phase space [when we take the above expansions for the fields \(C_\mu(x)\) and \(\bar{C}_\mu(x)\)]:

\[ k^\mu c_\mu(\vec{k}) = k^\mu c_\mu^\dagger(\vec{k}) = 0, \quad k^\mu \bar{c}_\mu(\vec{k}) = k^\mu \bar{c}_\mu^\dagger(\vec{k}) = 0, \]

(26)
along with \(k^2 = k^\mu k_\mu = 0\) due to \(\Box C_\mu(x) = \Box \bar{C}_\mu(x) = 0\). To be specific, it is clear, from the equations of motion (22) and (23), that all the fields are massless because \(k^2 = 0\) and the gauge conditions \((\partial \cdot C) = (\partial \cdot \bar{C}) = 0\) imply (26).
IV. CANONICAL BRACKETS: SYMMETRY CONSIDERATIONS

The subject matter of our present section is to derive the canonical (anti)commutators amongst the creation and annihilation operators for the free Abelian 2D 1-form and 4D 2-form gauge theories without using the definition of the canonical conjugate momenta.

A. Basic brackets for the 2D Abelian 1-form theory: symmetry principles

According to the common folklore in quantum field theory, the conserved charges (13) (that are derived due to the presence of continuous symmetries in the theory) generate the continuous symmetry transformations (see. e.g. Ref. [14]), as

\[ s_r \Phi = \pm i [\Phi, Q_r]_\pm, \quad r = b, a, a_d, a_{ad}, g, \]

(27)

where \( \Phi \) is the generic field of the theory and \( Q_r \) are the conserved charges of the theory [cf. (13)]. The (+)— signs, as the subscripts on the square bracket, correspond to the (anti)commutators for the generic field \( \Phi \) being (fermionic) bosonic in nature. Obviously, we have used here the spin-statistics theorem to differentiate between the bosonic and fermionic fields which is an important ingredient of our quantization scheme. Quantum mechanically, this implies the use of suitable brackets [i.e. (anti)commutators] for the quantization scheme. The (+)— signs in front of the expression on the r.h.s. (i.e. \( \pm i [\Phi, Q_r]_\pm \)) need explanation. The pertinent points regarding the choice of a specific sign (in front of the bracket) are:

1. for \( s_r = s_b, s_{ab}, s_d, s_{ad}, \) only the negative sign would be taken into account (i.e. \( s_b A_\mu = -i [A_\mu, Q_b] \), \( s_b \bar{C} = -i \{ \bar{C}, Q_b \} \), etc.), and

2. for \( s_r = s_g, s_\omega, \) the negative sign would be taken into account for the bosonic field and the positive sign would be chosen for the fermionic field (e.g. \( s_g A_\mu = -i [A_\mu, Q_g] \), \( s_g C = +i [C, Q_g] \), \( s_g \bar{C} = +i [\bar{C}, Q_g] \), etc.).

At this juncture, let us take an example (i.e. \( s_b A_\mu = \partial_\mu C \)) to make it clear that the symmetry principles dictate the structure of the canonical brackets. This aspect of our derivation is the key feature of our quantization scheme which is completely different from the canonical method of quantization scheme. Mathematically, this symmetry transformation can be expressed as\[ [14]\]

\[ s_b A_\mu = -i [A_\mu, Q_b] = \partial_\mu C. \]

(28)

Now taking the normal mode expansions for \( A_\mu \) and \( C \) [from equation (15)], it is clear that we have the following relationships

\[ [Q_b, a_\mu(k)] = k_\mu c(k), \quad [Q_b, a_\mu^\dagger(k)] = -k_\mu c^\dagger(k). \]

(29)

Plugging in the expression for \( Q_b \) in terms of the creation and annihilation operators [cf. (16)], we obtain

\[ a_\mu(k), a_\mu^\dagger(k') = \eta_{\mu\nu} \delta(k - k'), \quad a_\mu^\dagger(k), a_\mu(k') = 0, \quad a_\mu^\dagger(k), c(k') = 0, \quad a_\mu(k), c^\dagger(k') = 0, \quad c(k), c^\dagger(k') = 0. \]

(30)

In exactly similar fashion, the following BRST transformations

\[ s_b C = -i \{ C, Q_b \} = 0 \Longrightarrow \{ Q_b, c(k) \} = 0, \quad \{ Q_b, c^\dagger(k) \} = 0, \]

\[ s_b \bar{C} = -i \{ \bar{C}, Q_b \} = -i (\partial \cdot A) \Longrightarrow \{ Q_b, \bar{c}(k) \} = +i k^\mu a_\mu(k), \quad \{ Q_b, \bar{c}^\dagger(k) \} = -i k^\mu a_\mu^\dagger(k), \]

(31)

lead to the derivation of the following brackets

\[ \{ c^\dagger(k), \bar{c}(k') \} = -i \delta(k - k'), \quad \{ c(k), \bar{c}^\dagger(k') \} = +i \delta(k - k'), \quad \{ c(k), \bar{c}(k') \} = 0, \]

\[ \{ c^\dagger(k), \bar{c}^\dagger(k') \} = 0, \quad [a_\mu(k), \bar{c}(k')] = 0, \quad [a_\mu(k), \bar{c}^\dagger(k')] = 0, \quad [a_\mu^\dagger(k), \bar{c}(k')] = 0, \quad [a_\mu^\dagger(k), \bar{c}^\dagger(k')] = 0. \]

(32)

It is worthwhile to point out that the following statements are true, namely:
1. the above brackets have been derived by taking into account (see, e.g. Ref. [14] for details) only the on-shell nilpotent BRST symmetry transformations [i.e. $s_A A_\mu = \partial_\mu C$, $s_B C = 0$, $s_C = -i (\partial \cdot A)$], and

2. the expressions on the r.h.s. of equations (29) and (31) enforce, in a definite manner, the choice of the (anti)commutators in (30) and (32) when we use the expression for $Q_b$ from (16).

We would like to emphasize that the above exercise can be performed with all the six conserved charges listed in (13). The relevant (anti)commutators, emerging out from this algebraic exercise, are as follows

\[
\begin{align*}
[Q_{ab}, a_\mu(k)] &= +k_\mu \bar{c}(k), & [Q_{ab}, a_\mu^\dagger(k)] &= -k_\mu c^\dagger(k), \\
[Q_{ab}, c(k)] &= -i k^\mu a_\mu(k), & [Q_{ab}, c^\dagger(k)] &= +i k^\mu a_\mu^\dagger(k), \\
[Q_{ab}, \bar{c}(k)] &= 0, & {Q_{ab}, \bar{c}^\dagger(k)} &= 0, & {Q_{ab}, c(k)} &= 0, & {Q_{ab}, c^\dagger(k)} &= 0, \\
[Q_{ad}, a_\mu(k)] &= -\varepsilon_{\mu\nu\kappa} k^\nu \bar{c}(k), & [Q_{ad}, a_\mu^\dagger(k)] &= +\varepsilon_{\mu\nu\kappa} k^\nu c^\dagger(k), \\
[Q_{ad}, c(k)] &= -i \varepsilon^\nu k_\nu a_\nu(k), & [Q_{ad}, c^\dagger(k)] &= +i \varepsilon^\nu k_\nu a_\nu^\dagger(k), \\
[Q_{ad}, \bar{c}(k)] &= -\varepsilon_{\mu\nu\kappa} k^\nu \bar{c}(k), & [Q_{ad}, \bar{c}^\dagger(k)] &= +\varepsilon_{\mu\nu\kappa} k^\nu c^\dagger(k), \\
[Q_{ad}, c(k)] &= +i \varepsilon^\nu k_\nu a_\nu(k), & [Q_{ad}, c^\dagger(k)] &= -i \varepsilon^\nu k_\nu a_\nu^\dagger(k), \\
[Q_{ad}, \bar{c}(k)] &= 0, & {Q_{ad}, \bar{c}^\dagger(k)} &= 0, & {Q_{ad}, c(k)} &= 0, & {Q_{ad}, c^\dagger(k)} &= 0, \\
[Q_{\omega}, a_\mu(k)] &= -i k^\nu \varepsilon_{\mu\nu\kappa} a_\kappa(k), & [Q_{\omega}, a_\mu^\dagger(k)] &= -i k^\nu \varepsilon_{\mu\nu\kappa} (a^\kappa)^\dagger(k) \\
[Q_{\omega}, c(k)] &= 0, & [Q_{\omega}, c^\dagger(k)] &= 0, & [Q_{\omega}, \bar{c}(k)] &= 0, & [Q_{\omega}, \bar{c}^\dagger(k)] &= 0.
\end{align*}
\]

The outcome of all the above (anti)commutators, with the help of the normal mode expansions (15) and the expressions for the charges in (16), lead to the following non-vanishing basic brackets

\[
\begin{align*}
[a_\mu(k), a_\nu^\dagger(k')] &= \eta_{\mu\nu} \delta(k - k'), & \{c(k), c^\dagger(k')\} &= +i \delta(k - k'), & \{c^\dagger(k), \bar{c}(k')\} &= -i \delta(k - k'),
\end{align*}
\]

within the framework of BRST formalism. All the rest of the (anti)commutators turn out to be zero. To summarize, we have already utilized all the ingredients of our quantization scheme (without any use of the definition of canonically conjugate momenta anywhere in our discussions). To re-emphasize, the ingredients of our method are (i) the spin-statistics theorem, (ii) the normal ordering, and (iii) the definition of a generator for a given continuous symmetry transformation.

**B. Basic brackets for the 4D Abelian 2-form theory: symmetry principles**

In this subsection, we shall, once again, exploit the ideas of spin-statistics theorem normal ordering and symmetry generators to deduce the correct canonical brackets amongst the creation and annihilation operators of the relevant fields for the 4D Abelian 2-form theory. In contrast to the 2D theory where the BRST transformations were taken as example, we shall take here (for the sake of generality) the dual-BRST charge to find out the canonical brackets from the symmetry principle. In this regard, let us focus on the dual-BRST transformations ($\hat{s}_d$):

\[
\hat{s}_d B_{\mu\nu} = -i [ B_{\mu\nu}, \hat{Q}_d] = \varepsilon_{\mu\nu\rho\kappa} \partial^\rho \hat{C}^\kappa,
\]

where we have used the definition of a generator in terms of the dual-BRST charge $\hat{Q}_d$. The above equation has primarily two explicit independent transformations:

\[
\begin{align*}
\hat{s}_d B_{0i} &= -i [B_{0i}, \hat{Q}_d] = \epsilon_{ijk} \partial^j \hat{C}^k,
\hat{s}_d B_{ij} &= -i [B_{ij}, \hat{Q}_d] = \epsilon_{ijk} (\partial^0 \hat{C}^k - \partial^k \hat{C}^0),
\end{align*}
\]

Let us calculate the first commutation relation corresponding to the $B_{0i}$ component. To this end in mind, we express the r.h.s. (i.e $\epsilon_{ijk} \partial_j \hat{C}^k$) in terms of the creation and annihilation operators that are present in the normal mode expansion of $\hat{C}_\mu(x, t)$ as follows [cf. (25)]

\[
\begin{align*}
\epsilon_{ijk} (\partial^j \hat{C}^k) &= i \int \frac{d^3k}{(2\pi)^3 \cdot 2k_0} \epsilon_{ijk} k^j \left( \hat{c}^\dagger(k) e^{-i\kappa \cdot \vec{x}} - \hat{c}(k) e^{+i\kappa \cdot \vec{x}} \right) \\
&= \frac{i}{2} \int \frac{d^3k}{(2\pi)^3 \cdot 2k_0} \epsilon_{ijk} k^j \left[ \left( \hat{c}^\dagger(k) e^{-i\kappa \cdot \vec{x}} - \hat{c}(k) e^{+i\kappa \cdot \vec{x}} \right) + \left( \hat{c}^\dagger(k) e^{-i\kappa \cdot \vec{x}} - \hat{c}(k) e^{+i\kappa \cdot \vec{x}} \right) \right].
\end{align*}
\]
Here we have taken \( t = 0 \), for the sake of simplicity, because \( \tilde{Q}_d \) is a conserved charge. The reason for breaking the above expression, into two similar terms will become clear, later on, when we shall compare the exponentials. In the second term of the above equation, changing \( \vec{k} \rightarrow -\vec{k} \) and re-arranging the terms, we obtain

\[
\epsilon_{ijk}(\partial^j \tilde{C}^k) = \frac{i}{2} \int \frac{d^3 k}{(2\pi)^3 \cdot 2k_0} \epsilon_{ijk} k^j \left[ \left( \tilde{c}^k(\vec{k}) + \tilde{c}^k(\vec{k}) \right) e^{-i\vec{k} \cdot \vec{x}} - \left( \tilde{c}^k(\vec{k}) + \tilde{c}^k(\vec{k}) \right) e^{i\vec{k} \cdot \vec{x}} \right]. \tag{38}
\]

Now, on the l.h.s. of the first equation of (36), we re-express \( B_{0i}(\vec{x}) \) in terms of creation and annihilation operators as follows (at \( t = 0 \)):

\[
-i [B_{0i}(\vec{x}), \tilde{Q}_d] = -i \left[ \frac{d^3 p}{2} \epsilon_{ijk} p^j \left( \tilde{c}^k(\vec{p}) + \tilde{c}^k(\vec{p}) \right) \delta^{(3)}(\vec{k} - \vec{p}) \right], \tag{40}
\]

\[
[B_{0i}(\vec{k}), \tilde{Q}_d] = \frac{1}{2} \epsilon_{ijk} k^j \left( \tilde{c}^k(\vec{k}) + \tilde{c}^k(\vec{k}) \right) \delta^{(3)}(\vec{k} - \vec{p}), \tag{41}
\]

Comparing the exponentials from the r.h.s. of the equations (38) and (39), we obtain the following relationships, namely:

\[
[b_{0i}(\vec{k}), \tilde{Q}_d] = -\left( \frac{1}{2} \epsilon_{ijk} k^j \left( \tilde{c}^k(\vec{k}) + \tilde{c}^k(\vec{k}) \right) \right) \]

\[
= -\frac{1}{2} \int \frac{d^3 p}{2} \epsilon_{ijk} p^j \left( \tilde{c}^k(\vec{p}) + \tilde{c}^k(\vec{p}) \right) \delta^{(3)}(\vec{k} - \vec{p}), \tag{40}
\]

\[
[b_{0i}(\vec{k}), \tilde{Q}_d] = \frac{1}{2} \epsilon_{ijk} k^j \left( \tilde{c}^k(\vec{k}) + \tilde{c}^k(\vec{k}) \right) \]

\[
= \frac{1}{2} \int \frac{d^3 p}{2} \epsilon_{ijk} p^j \left( \tilde{c}^k(\vec{p}) + \tilde{c}^k(\vec{p}) \right) \delta^{(3)}(\vec{k} - \vec{p}), \tag{41}
\]

where in the last steps of the above equations (40) and (41), we have used the property of the Dirac \( \delta \)-function [i.e. \( \int d^3 k f(\vec{k}) \delta^{(3)}(\vec{k} - \vec{p}) = f(\vec{p}) \)]. The relevant part of \( \tilde{Q}_d \) [which has non-vanishing commutation relations with \( B_{0i}(\vec{x}) \)] is as follows [cf. (19)]

\[
\tilde{Q}_d^{(r_1)} = \int d^3 y \epsilon_{ijk} (\partial^j \tilde{C}^k)(\partial_0 B_{0i}). \tag{42}
\]

This is due to the fact that the canonical conjugate momenta, corresponding to \( B_{0i} \), is \( \Pi_{(B)}^{0i} = \frac{1}{2}(\partial_0 B_{0i} + \partial_j B^{ji} - \partial^i \phi_1) \) [cf. (69) below] which gives the non-zero value for the commutator. Thus, the bracket which would contribute to the evaluation of the above commutator is \( [B_{0i}(\vec{x}), \tilde{B}^{0j}(\vec{y})] \) because \( B_{0i} \) commutes with the spatial derivatives on \( B_{1j} \) and \( \phi_1 \) which are present in the expression for \( \Pi_{(B)}^{0i} \). The correct form of the existing bracket \( [B_{0i}(\vec{x}), \tilde{B}^{0j}(\vec{y})] \) is explicitly written in (70). It is also self-evident that \( B_{0i} \) would commute with the rest of the terms of \( \tilde{Q}_d \). The above equation for \( \tilde{Q}_d^{(r_1)} \) can be expressed in terms of the creation and annihilation operators that appear in the normal mode expansions of the basic fields [cf. (25)]. The relevant expression \( \tilde{Q}_d^{(r_1)} \) for \( \tilde{Q}_d \) that contribute the commutator is

\[
\tilde{Q}_d^{(r_1)} = \int \frac{d^3 y d^3 p d^3 q}{(2\pi)^3 \cdot 2p_0 \cdot 2q_0} \epsilon_{ijk}(\vec{p} \cdot \vec{q} \cdot \vec{y}) \left( \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(\vec{q}) e^{-i(\vec{p} \cdot \vec{y})} - \tilde{c}^k(\vec{p}) (b_{0i}^{\dagger}(\vec{q}) e^{-i(\vec{p} \cdot \vec{y})} - \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(\vec{q}) e^{-i(\vec{p} \cdot \vec{y})} + \tilde{c}^k(\vec{p}) (b_{0i}^{\dagger}(\vec{q}) e^{-i(\vec{p} \cdot \vec{y})} \right) \]

\[
\int \frac{d^3 y d^3 p d^3 q}{(2\pi)^3 \cdot 2p_0 \cdot 2q_0} \epsilon_{ijk}(\vec{p} \cdot \vec{q} \cdot \vec{y}) \left( \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(\vec{q}) e^{-i(\vec{p} \cdot \vec{y})} - \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(\vec{q}) e^{-i(\vec{p} \cdot \vec{y})} \right) \]

After integrating the above equation with respect to \( d^3 p \) and \( d^3 q \), we obtain

\[
\tilde{Q}_d^{(r_1)} = -\frac{1}{2} \int d^3 p \epsilon_{ijk} p^j \left( \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(-\vec{p}) + \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(-\vec{p}) - \tilde{c}^k(\vec{p}) b_{0i}^{\dagger}(-\vec{p}) \right), \tag{44}
\]

where we have used the standard definition of the Dirac \( \delta \)-function: \( \delta^{(3)}(\vec{p} - \vec{q}) = \int \frac{d^3 y}{(2\pi)^3} e^{+i(\vec{p} \cdot \vec{y})} \) and the property \( \int d^3 q f(\vec{q}) \delta^{(3)}(\vec{p} - \vec{q}) = f(\vec{p}) \). In the first two terms of above equation changing \( \vec{p} \rightarrow -\vec{p} \) and collecting the coefficients of \( b_{0i}^{\dagger}(\vec{p}) \) and \( (b_{0i}^{\dagger}(\vec{p}) \), we obtain the following explicit expression for \( \tilde{Q}_d^{(r_1)} \):

\[
\tilde{Q}_d^{(r_1)} = \frac{1}{2} \int d^3 p \epsilon_{ijk} p^j \left( \left( \tilde{c}^k(\vec{p}) + \tilde{c}^k(\vec{p}) \right) b_{0i}^{\dagger}(\vec{p}) + \left( \tilde{c}^k(\vec{p}) + \tilde{c}^k(\vec{p}) \right) (b_{0i}^{\dagger}(\vec{p}) \right), \tag{45}
\]
Substituting the above expression for $\hat{Q}_d^{(r_1)}$ in (40), we have following relationship that contains the two existing commutators

$$[b_{0i}(\vec{k}), \hat{Q}_d^{(r_1)}] = \frac{1}{2} \int d^3p \left( \epsilon_{ijk} p^j \left( (\vec{c}^k)^(\dagger)(\vec{p}) + \vec{c}^k(-\vec{p}) \right) \right) [b_{0i}(\vec{k}), b^{\dagger}_{0l}(\vec{p})]$$

$$+ \left( \vec{c}^k(\vec{p}) + (\vec{c}^k)^(\dagger)(-\vec{p}) \right) [b_{0i}(\vec{k}), (b^{\dagger}_{0l})^(\dagger)(\vec{p})].$$

(46)

We point out that in the computation of the above brackets, we have, actually, six basic brackets (i.e. $[b_{0i}(\vec{k}), (\vec{c}^k)^(\dagger)(\vec{p})], [b_{0i}(\vec{k}), (\vec{c}^k)(\vec{p})], [b_{0i}(\vec{k}), \vec{c}^k(-\vec{p})], [b_{0i}(\vec{k}), \vec{c}^k(\vec{p})], [b_{0i}(\vec{k}), b^{\dagger}_{0l}(\vec{p})], [b_{0i}(\vec{k}), (b^{\dagger}_{0l})^(\dagger)(\vec{p})]$). However, four of them, amongst the (anti-)ghost creation/annihilation operators and the bosonic annihilation operator $b_{0i}$, are zero because the (anti-)ghost fields are decoupled from the rest of our present theory. This statement can also be verified by the fact that, under the ghost symmetry transformations, the field $B_{\mu\nu}$ does not transform (i.e. $\delta_{\vec{g}}B_{\mu\nu} = i [B_{\mu\nu}, \hat{Q}_d] = 0$). Exploiting the bracket $[B_{\mu\nu}, \hat{Q}_d] = 0$ and taking the help from equations (22) and (25), it is clear that $[b_{0i}(\vec{k}), \vec{c}^k(\vec{p})] = 0, [b_{0i}(\vec{k}), (\vec{c}^k)^(\dagger)(\vec{p})] = 0, [b_{ij}(\vec{k}), \vec{c}^k(\vec{p})] = 0, [b_{ij}^\dagger(\vec{k}), (\vec{c}^k)^(\dagger)(\vec{p})] = 0$, etc. Now, comparing the r.h.s. of (40) and (46), we obtain following useful commutators, namely:

$$[b_{0i}(\vec{k}), b^{\dagger}_{0l}(\vec{p})] = 0, \quad [b_{0i}(\vec{k}), (b^{\dagger}_{0l})^(\dagger)(\vec{p})] = -\delta^i_l \delta^{(\dagger)}(\vec{k} - \vec{p}).$$

(47)

It is straightforward to check that, if we substitute the value of $\hat{Q}_d^{(r_1)}$ [cf. (45)] in the equation (41), instead of equation (40), we shall obtain exactly the same set of canonical commutation relations as given in (47). We have not normal ordered the terms in (45). However, the results, obtained in (47), would remain unaffected by the normal ordering in (45) (for the expression for the contributing part $\hat{Q}_d^{(r_1)}$ of $\hat{Q}_d$).

Let us now concentrate on the second commutation relation corresponding to the $B_{ij}$ component of $B_{\mu\nu}$ field [cf. (36)]

$$\delta_{\vec{d}}B_{ij} = -i [B_{ij}, \hat{Q}_d] = \epsilon_{ijl}(\partial^0\vec{C}^l - \partial^l\vec{C}^0)$$

(48)

The l.h.s. of the above equation can be expanded in terms of the creation and annihilation operators in the following manner (at $t = 0$)

$$\epsilon_{ijl}(\partial^0\vec{C}^l - \partial^l\vec{C}^0) = i \int \frac{d^3k}{(2\pi)^3 \cdot 2k_0} \epsilon_{ijl} \left( (k^0\vec{c}^l(\vec{k}) - k^l\vec{c}^0(\vec{k})) e^{-i\vec{k}\cdot\vec{x}} - (k^0(\vec{c}^l)^(\dagger)(\vec{k}) - k^l(\vec{c}^0)^(\dagger)(\vec{k})) e^{i\vec{k}\cdot\vec{x}} \right)$$

$$- \left( (k^0\vec{c}^l(\vec{k}) - k^l\vec{c}^0(\vec{k}) - k^l\vec{c}^0(\vec{k}) - k^l(\vec{c}^0)^(\dagger)(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \right).$$

(49)

On the l.h.s. of equation (48), in the expression for commutator, we express $B_{ij}(\vec{x})$ (at $t = 0$) in terms of the creation and annihilation operators as follows

$$-i [B_{ij}(\vec{x}), \hat{Q}_d] = -i \int \frac{d^3k}{(2\pi)^3 \cdot 2k_0} \left( [b_{ij}(\vec{k}), \hat{Q}_d] e^{-i\vec{k}\cdot\vec{x}} + [b_{ij}^\dagger(\vec{k}), \hat{Q}_d] e^{i\vec{k}\cdot\vec{x}} \right).$$

(50)

Comparing the exponentials from the equations (49) and (50), we obtain

$$[b_{ij}(\vec{k}), \hat{Q}_d] = -\frac{1}{2} \epsilon_{ijl} \left( (k^0\vec{c}^l(\vec{k}) - k^l\vec{c}^0(\vec{k}) - k^l\vec{c}^0(\vec{k}) - k^l(\vec{c}^0)^(\dagger)(\vec{k}) \right)$$

$$= -\frac{1}{2} \int d^3p \epsilon_{ijl} \delta^{(3)}(\vec{k} - \vec{p}) \left( p^0\vec{c}^l(p) - p^0(\vec{c}^l)^(\dagger)(p) - p^l\vec{c}^0(p) - p^l(\vec{c}^0)^(\dagger)(p) \right),$$

(51)

$$[b_{ij}^\dagger(\vec{k}), \hat{Q}_d] = \frac{1}{2} \epsilon_{ijl} \left( (k^0(\vec{c}^l)^(\dagger)(\vec{k}) - k^0(\vec{c}^l)^(\dagger)(\vec{k}) - k^l\vec{c}^0(\vec{k}) - k^l\vec{c}^0(\vec{k}) \right)$$

$$= \frac{1}{2} \int d^3p \epsilon_{ijl} \delta^{(3)}(\vec{k} - \vec{p}) \left( p^0(\vec{c}^l)^(\dagger)(p) - p^0(\vec{c}^l)^(\dagger)(p) - p^l(\vec{c}^0)(p) - p^l(\vec{c}^0)^(\dagger)(p) \right).$$

(52)

We note that, on the r.h.s. of the above brackets, we have used the property of Dirac $\delta$-function. The relevant part of $\hat{Q}_d$ [which has non-vanishing commutation relations with $B_{ij}(\vec{x})$] can be given as [cf. (19)]

$$\hat{Q}_d^{(r_2)} = -\frac{1}{2} \int d^3y \epsilon_{lmn}(\partial^0 B^{mn})(\partial^l \vec{C} - \partial^l\vec{C}^0),$$

(53)
because the canonical conjugate momenta w.r.t. $B_{ij}$ is $\Pi_{ij}^{(2)} = \frac{1}{2} (H^{0ij} - \epsilon^{ijk} \partial_k \phi_2)$ [cf. (69) below] where $H^{0ij} = \partial^0 B_{ij} + \partial^i B_{0j} + \partial^j B_{0i}$. We, once again, point out that the spatial derivatives of $B_{ij}^0$, $B_{ij}$ and $\phi_2$ commute with $B_{ij}$. As a consequence, the canonical bracket which gives the non-zero value is $[B_{ij}(x), \tilde{B}^m(y)]$ [cf. (70) below].

Now re-expressing $\tilde{Q}_d^{(r2)}$ in terms of normal mode expansions of basic fields

$$\tilde{Q}_d^{(r2)} = -\frac{1}{2} \int \frac{d^3y}{(2\pi)^3} \frac{d^3q}{2\pi^2} \epsilon_{lmm} p^l \left[ q^0 (b_{mn}^0)(\vec{p})(c^l_q + q^l b_{mn}^0(\vec{p}) c^0_q) e^{-i(\vec{p}+\vec{q})} - q^0 b_{mn}^0(\vec{p})(c^l_q + q^l b_{mn}^0(\vec{p}) c^0_q) e^{-i(\vec{p}-\vec{q})} + q^0 (b_{mn}^0)(\vec{p})(c^l_q + q^l b_{mn}^0(\vec{p}) c^0_q) e^{i(\vec{p}-\vec{q})} + q^0 (b_{mn}^0)(\vec{p})(c^l_q + q^l b_{mn}^0(\vec{p}) c^0_q) e^{i(\vec{p}+\vec{q})}\right].$$

Integrating over $d^3y$ and $d^3q$ in the above equation, we obtain the following expression for the relevant part of $\tilde{Q}_d$, in terms of the creation and annihilation operators, as

$$\tilde{Q}_d^{(r2)} = -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \epsilon_{lmm} p^l \left[ p^0 b_{mn}^0(\vec{p})(c^l(-\vec{p}) - (\vec{c}^l)(\vec{p})) - p^0 (b_{mn}^0)(\vec{p})(c^l(-\vec{p}) - (\vec{c}^l)(\vec{p})) + p^l b_{mn}^0(\vec{p})(c^l(-\vec{p}) + (\vec{c}^l)(\vec{p})) + p^l (b_{mn}^0)(\vec{p})(c^0(-\vec{p}) + (\vec{c}^0)(\vec{p}))\right].$$

Substituting the above expression for $\tilde{Q}_d^{(r2)}$ in the l.h.s. of the equation (51), we obtain

$$[b_{ij}(\vec{k}), \tilde{Q}_d^{(r2)}] = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \epsilon_{lmm} \left[ b_{ij}(\vec{k}), b_{mn}^0(\vec{p})\right](\vec{c}^l(-\vec{p}) - (\vec{c}^l)(\vec{p}) - p^0 b_{ij}(\vec{k}), (b_{mn}^0)(\vec{p})(\vec{c}^l(-\vec{p}) - (\vec{c}^l)(\vec{p})) + p^l b_{ij}(\vec{k}), (b_{mn}^0)(\vec{p})(\vec{c}^0(-\vec{p}) + (\vec{c}^0)(\vec{p})).$$

We note that in the above equation, the canonical brackets of $B_{ij}$ with the (anti-)ghost creation/annihilation operators are again zero because of our earlier argument that the (anti-)ghost fields are decoupled from the rest of the present theory. Now rearranging the various terms, we have the following expression for the above commutator

$$[b_{ij}(\vec{k}), \tilde{Q}_d^{(r2)}] = -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \epsilon_{lmm} \left[ b_{ij}(\vec{k}), b_{mn}^0(\vec{p})\right](\vec{c}^l(-\vec{p}) - (\vec{c}^l)(\vec{p}) + p^l (\vec{c}^0(-\vec{p}) + p^l (\vec{c}^0)(\vec{p})) - [b_{ij}(\vec{k}), (b_{mn}^0)(\vec{p})](\vec{p}^0 b_{ij}(\vec{k}), b_{mn}^0(\vec{p})\vec{c}^l(-\vec{p}) - p^0 (\vec{c}^l)(\vec{p}) + p^l (\vec{c}^0)(\vec{p}) - p^l (\vec{c}^0(-\vec{p}) - p^l (\vec{c}^0)(\vec{p})).$$

Comparing the r.h.s. of (49) and (57), we get the following basic useful commutation relations amongst the creation and annihilation operators

$$[b_{ij}(\vec{k}), b_{mn}^0(\vec{p})] = 0, \quad [b_{ij}(\vec{k}), (b_{mn}^0)(\vec{p})] = -\left(\delta^m_\mu \delta^n_\nu - \delta^m_\nu \delta^n_\mu\right) \delta^{(3)}(\vec{k} - \vec{p}).$$

It is worthwhile to mention that, instead of substituting the value of $\tilde{Q}_d$ [cf. (55)] into equation (51), if we had substituted its value into equation (52), we would have obtained exactly the same set of canonical commutation relations as listed in (58).

At this juncture, we point out that the above computations can also be performed with other dual-BRST symmetry transformations ($\tilde{s}_d$) listed in (10) by exploiting the symmetry principles. The relevant (anti)commutators, that emerge from the above exercise, are as follows

$$[\tilde{Q}_d, f_2(\vec{k})] = 0, \quad [\tilde{Q}_d, f_1(\vec{k})] = 0, \quad [\tilde{Q}_d, \bar{b}(\vec{k})] = 0, \quad [\tilde{Q}_d, \bar{b}^\dagger(\vec{k})] = 0,$$

$$[\tilde{Q}_d, f_2(\vec{k})] = \frac{1}{2} k^\mu e_\mu(\vec{k}), \quad [\tilde{Q}_d, f_3(\vec{k})] = -\frac{1}{2} k^\mu c_\mu^l(\vec{k}), \quad [\tilde{Q}_d, b(\vec{k})] = -\frac{1}{2} k^\mu c_\mu(\vec{k}),$$

$$[\tilde{Q}_d, \bar{b}^\dagger(\vec{k})] = \frac{1}{2} k^\mu e_\mu^l(\vec{k}), \quad [\tilde{Q}_d, \bar{b}_\mu(\vec{k})] = k_\mu \bar{b}(\vec{k}), \quad [\tilde{Q}_d, \bar{b}_\mu^\dagger(\vec{k})] = -k_\mu \bar{b}^\dagger(\vec{k}),$$

$$[\tilde{Q}_d, e_\mu(\vec{k})] = k_\mu f_2(\vec{k}) - \frac{1}{2} \varepsilon_{\mu
u\rho\sigma} k^\nu b_{\rho\sigma}(\vec{k}), \quad [\tilde{Q}_d, e_\mu^l(\vec{k})] = -k_\mu f_2^l(\vec{k}) + \frac{1}{2} \varepsilon_{\mu
u\rho\sigma} k^\nu (b_{\rho\sigma}^l(\vec{k})).$$

(59)
The above brackets, along with (40), (41), (51) and (52), lead to the derivation of complete set of non-vanishing canonical (anti)commutation relations among the creation and annihilation operators (at t = 0) as

\[
[b(\vec{k}), b^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k} - \vec{k}'), \quad [b(\vec{k}), b^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k} - \vec{k}'),
\]

\[
[f_1(\vec{k}), f_1^\dagger(\vec{k}')] = -\delta^{(3)}(\vec{k} - \vec{k}'), \quad \{c_0(\vec{k}), (\bar{c}^0)\dagger(\vec{k}')\} = -2\delta^{(3)}(\vec{k} - \vec{k}'),
\]

\[
\{\bar{c}_0(\vec{k}), (c^0)\dagger(\vec{k}')\} = 2\delta^{(3)}(\vec{k} - \vec{k}'), \quad \{c_i(\vec{k}), \bar{c}^j(\vec{k}')\} = -\delta_i^j\delta^{(3)}(\vec{k} - \vec{k}'),
\]

\[
\{c_i(\vec{k}), (\bar{c}^j)\dagger(\vec{k}')\} = \delta_i^j\delta^{(3)}(\vec{k} - \vec{k}'), \quad [\bar{f}_2(\vec{k}), f_2^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k} - \vec{k}'),
\]

\[
[b_0(\vec{k}), (b^{0\dagger})\dagger(\vec{k}')] = -\delta_i^j\delta^{(3)}(\vec{k} - \vec{k}'), \quad [b_i(\vec{k}), (b^{mn})\dagger(\vec{k}')] = -\delta_i^m\delta^m_j - \delta_i^n\delta^j_n\delta^{(3)}(\vec{k} - \vec{k}'). \tag{60}
\]

for the 4D free Abelian 2-form gauge theory in the realm of BRST formalism. The rest of the (anti)commutation relations are zero. We, once again, emphasize that, in the derivation of the above basic brackets, we have not used the definition of canonical momenta anywhere in our whole discussion.

We have, so far, exploited only the dual-BRST symmetry transformations and corresponding charge. Similar exercise can also be performed with the rest of all the continuous symmetries (and their corresponding charges) present in our 4D free Abelian 2-form gauge theory which lead to exactly the same set of non-vanishing (anti)commutation relations (amongst the creation and annihilation operators) as listed in (60). For the sake of completeness, we have quoted all the (anti)commutators with charges \(Q_b, Q_w\) and \(Q_g\) in our Appendix B.

V. USUAL CANONICAL METHOD: LAGRANGIAN FORMALISM

Taking the help of the definition of canonical conjugate momenta, we obtain the canonical basic brackets for the free 2D Abelian 1-form and 4D Abelian 2-form gauge theories in our present section.

A. Basic brackets for the 2D Abelian 1-form theory: canonical approach

It is evident that the canonical conjugate momenta, from the Lagrangian density (1), for the basic fields of the theory, are

\[
\Pi^\mu = \frac{\partial L_{(b)}}{\partial (\partial_\mu A_\mu)} = -F^{0\mu} - \eta^{0\mu}(\partial \cdot A),
\]

\[
\Pi_{(C)} = \frac{\partial L_{(b)}}{\partial (\partial_\mu \bar{C})} = +i \bar{C}, \quad \Pi_{(\bar{C})} = \frac{\partial L_{(b)}}{\partial (\partial_\mu C)} = -i \bar{C}. \tag{61}
\]

where we have used the convention of left-derivative for the differentiation w.r.t. the fermionic fields \(C\) and \(\bar{C}\). As a consequence, we have the following non-vanishing canonical brackets:

\[
[A_\mu(x, t), \Pi_{(\nu)}(x', t)] = i \eta_{\mu\nu} \delta(x - x'),
\]

\[
\{\bar{C}(x, t), \Pi_{(\bar{C})}(x', t)\} = i \delta(x - x') \Rightarrow \{\bar{C}(x, t), \bar{C}(x', t)\} = -\delta(x - x'),
\]

\[
\{C(x, t), \Pi_{(C)}(x', t)\} = i \delta(x - x') \Rightarrow \{C(x, t), C(x', t)\} = \delta(x - x'). \tag{62}
\]

All the rest of the brackets are zero. The top entry, in the above, implies the following commutators in terms of the components of the 2D gauge field \(A_\mu\) and the corresponding conjugate momenta, namely:

\[
[A_0(x, t), (\partial \cdot A)(x', t)] = -i \delta(x - x'), \quad [A_i(x, t), E_j(x', t)] = i \delta_{ij} \delta(x - x'). \tag{63}
\]

The above form of commutators would be useful later.

To simplify, the rest of our computations, we re-express the normal mode expansions of the basic fields [cf. (15)], as (see, e.g. Ref. [13])

\[
A_\mu(x, t) = \int dk \left[f^*(k, x) \ a_\mu(k) + f(k, x) \ a_\mu^\dagger(k)\right],
\]

\[
C(x, t) = \int dk \left[f^*(k, x) \ c(k) + f(k, x) \ c^\dagger(k)\right],
\]

\[
\bar{C}(x, t) = \int dk \left[f^*(k, x) \ \bar{c}(k) + f(k, x) \ \bar{c}^\dagger(k)\right]. \tag{64}
\]
where the new functions:
\[
f(k, x) = \frac{e^{-ikx}}{\sqrt{2\pi 2k_0}} \quad f^*(k, x) = \frac{e^{ikx}}{\sqrt{2\pi 2k_0}}.
\] (65)
form an orthonormal set because they satisfy\cite{15}
\[
\begin{align*}
\int dx \ f^*(k, x) i {\delta}_0^i f(k', x) &= \delta(k - k'), \\
\int dx \ f^*(k, x) i {\delta}_0^i f^*(k', x) &= 0, \quad \int dx \ f(k, x) i {\delta}_0^i f(k', x) = 0.
\end{align*}
\] (66)
where we have taken into account the following definition
\[
A_{\mu}^0 B = A(\partial_\mu B) - (\partial_\mu A) B,
\] (67)
for the operator $i {\delta}_0^i$ between two non-zero variables A and B. Using the above relations, it is straightforward to check that
\[
\begin{align*}
c(k) &= \int dx \ [C(x, t) i {\delta}_0^i f(k, x)], \quad c^1(k) = \int dx \ [f^*(k, x) i {\delta}_0^i C(x, t)], \\
\bar{c}(k) &= \int dx \ [\bar{C}(x, t) i {\delta}_0^i f(k, x)], \quad \bar{c}^1(k) = \int dx \ [f^*(k, x) i {\delta}_0^i \bar{C}(x, t)], \\
\alpha_\mu(k) &= \int dx \ [A_\mu(x, t) i {\delta}_0^i f(k, x)], \quad a_\mu^1(k) = \int dx \ [f^*(k, x) i {\delta}_0^i A_\mu(x, t)].
\end{align*}
\] (68)

Thus, we have expressed the creation and annihilation operators in terms of the fields and the orthonormal functions $f(k, x)$ and $f^*(k, x)$.

At this stage, a few comments are in order. First and foremost, it is straightforward to check that only the canonical brackets (34) survive in the explicit computation. Second, there exist six anticommutators from the four fermionic operators $c(k)$, $c^1(k)$, $\bar{c}(k)$, $\bar{c}^1(k)$. Out of which, four would be zero because of the orthonormality relations (66) and because of the fact that $C^2 = \bar{C}^2 = 0$, \{C(x, t), C(x', t)\} = 0, \{C(x, t), \bar{C}(x', t)\} = 0. Third, there exist three basic commutators from the annihilation operator $a_\mu(k)$ and creation operator $a_\mu^1(k)$. Out of which, two would turn out to be zero because the commutation relations in (63) can be recast in the form $[A_\mu(x, t), \dot{A}_\nu(x', t)] = -i \eta_{\mu\nu} \delta(x - x')$ due to the fact that (i) $\dot{A}_0 = (\partial \cdot A) + \partial_\mu A_\mu$ and $\dot{A}_i = E_i + \partial_i A_0$, and (ii) the spatial derivative of the gauge field $\dot{A}_\mu$ commutes with itself.

It is straightforward to check that the canonical brackets of (62) and (63) [that are derived from the Lagrangian density (1)] lead to the derivation of the same brackets that are listed in (34). Thus, we conclude that the basic canonical brackets [cf. (34)] between the creation and annihilation operators of the bosonic and fermionic fields of the theory can be derived from (i) the continuous symmetry considerations, and (ii) by exploiting the definition of momenta from the Lagrangian density of the theory. In other words, the non-vanishing basic brackets amongst the creation and annihilation operators can be derived from the usual canonical approach as well as from the symmetry principles. The former turn out to be one and the same.

B. Basic brackets for the 4D Abelian 2-form theory: canonical approach

The canonical conjugate momenta for all the dynamical fields that are present in the Lagrangian density (6), for the 4D Abelian 2-form theory, are listed below
\[
\begin{align*}
\Pi_{(\phi_1)} &= \dot{\phi}_1 + \partial_\mu B_{\mu}, \quad \Pi_{(\phi_2)} = -\dot{\phi}_2 + i \frac{1}{2} \epsilon^{ijk} \partial_j B_{jk}, \quad \Pi_{(\beta)} = -\dot{\beta}, \quad \Pi_{(\bar{\beta})} = -\dot{\bar{\beta}}, \\
\Pi_0^0(C) &= \frac{1}{2} (\partial \cdot \bar{C}), \quad \Pi_i^0(C) = -(\partial^0 \bar{C}^i - \partial^i \bar{C}^0), \quad \Pi_0(C) = (\partial^0 \bar{C}^i - \partial^i \bar{C}^0), \\
\Pi_i^0(\bar{C}) &= -\frac{1}{2} (\partial \cdot C), \quad \Pi_{(B)}^0 = \frac{1}{2} (\partial_\mu B_{\mu} + \partial_j B_{ij} - \partial_\mu \phi_1), \quad \Pi_{(B)}^{ij} = \frac{1}{2} H^{0ij} - \frac{1}{2} \epsilon^{ijk} (\partial_k \phi_2),
\end{align*}
\] (69)
where $\Pi_0^0(C)$ and $\Pi_i^0(C)$ are the momenta corresponding to $C_0$ and $C_i$ components of $C_\mu$ field as well as $\Pi_0^{ij}(B)$ and $\Pi_{(B)}^{ij}$ are the momenta corresponding to $B_{0i}$ and $B_{ij}$ field components of gauge field $B_{\mu\nu}$. In the above, we have
adopted the convention of the left-derivative for the fermionic fields. The basic dynamical fields of the theory and their corresponding conjugate momenta obey the following non-vanishing (equal-time) canonical (anti)commutation relations:

\[
\begin{align*}
[\phi_1(\vec{x}, t), \phi_1(\vec{y}, t)] &= i \delta^{(3)}(\vec{x} - \vec{y}), & [\phi_2(\vec{x}, t), \phi_2(\vec{y}, t)] &= -i \delta^{(3)}(\vec{x} - \vec{y}), \\
[\beta(\vec{x}, t), \beta(\vec{y}, t)] &= -i \delta^{(3)}(\vec{x} - \vec{y}), & [\tilde{\beta}(\vec{x}, t), \tilde{\beta}(\vec{y}, t)] &= -i \delta^{(3)}(\vec{x} - \vec{y}), \\
\{C_0(\vec{x}, t), \tilde{C}^0(\vec{y}, t)\} &= 2i \delta^{(3)}(\vec{x} - \vec{y}), & \{C_1(\vec{x}, t), \tilde{C}^j(\vec{y}, t)\} &= -i \delta^j \delta^{(3)}(\vec{x} - \vec{y}), \\
\{\tilde{C}_0(\vec{x}, t), \tilde{C}^0(\vec{y}, t)\} &= -2i \delta^{(3)}(\vec{x} - \vec{y}), & \{\tilde{C}_1(\vec{x}, t), \tilde{C}^j(\vec{y}, t)\} &= i \delta^j \delta^{(3)}(\vec{x} - \vec{y}), \\
\left[ B_{0\mu}(\vec{x}, t), \tilde{B}^\mu(\vec{y}, t) \right] &= i \delta^\mu \delta^{(3)}(\vec{x} - \vec{y}), & \left[ B_{1j}(\vec{x}, t), \tilde{B}^{kj}(\vec{y}, t) \right] &= i (\delta^k \delta^j - \delta^j \delta^k) \delta^{(3)}(\vec{x} - \vec{y}).
\end{align*}
\]

All the rest of the (anti)commutators are zero. Now our aim is to calculate the (anti)commutation relations amongst the creation and annihilation operators of the theory. In order to simplify our computations, we re-express the normal mode expansions of the basic dynamical fields [cf. (25)] in terms of the orthonormal functions \( f(\vec{k}, x) \) and \( f^*(\vec{k}, x) \) [cf. (65) for 2D case] as (see, e.g. [15] for details)

\[
\begin{align*}
B_{\mu\nu}(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) b_{\mu\nu}(\vec{k}) + f(\vec{k}, x) b^\dagger_{\mu\nu}(\vec{k}) \right], \\
C_\mu(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) c_{\mu}(\vec{k}) + f(\vec{k}, x) c_{\mu}^\dagger(\vec{k}) \right], \\
\tilde{C}_\mu(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) \tilde{c}_{\mu}(\vec{k}) + f(\vec{k}, x) \tilde{c}_{\mu}^\dagger(\vec{k}) \right], \\
\beta(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) b(\vec{k}) + f(\vec{k}, x) b^\dagger(\vec{k}) \right], \\
\tilde{\beta}(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) \tilde{b}(\vec{k}) + f(\vec{k}, x) \tilde{b}^\dagger(\vec{k}) \right], \\
\phi_1(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) f_1(\vec{k}) + f(\vec{k}, x) f_1^\dagger(\vec{k}) \right], \\
\phi_2(\vec{x}, t) &= \int d^3 k \left[ f^*(\vec{k}, x) f_2(\vec{k}) + f(\vec{k}, x) f_2^\dagger(\vec{k}) \right].
\end{align*}
\]

Using the above relationships, we can express all the creation and annihilation operators in terms of the orthonormal functions and the basic fields of the 4D theory. We can check that the following expressions are true, namely;

\[
\begin{align*}
b_{\mu\nu}(\vec{k}) &= \int d^3 x \, B_{\mu\nu}(x) i_{\delta_0}^{\mu\nu} f(\vec{k}, x), & b_{\mu\nu}^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu\nu} B_{\mu\nu}(x), \\
c_{\mu}(\vec{k}) &= \int d^3 x \, C_{\mu}(x) i_{\delta_0}^{\mu} f(\vec{k}, x), & c_{\mu}^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu} C_{\mu}(x), \\
\tilde{c}_{\mu}(\vec{k}) &= \int d^3 x \, \tilde{C}_{\mu}(x) i_{\delta_0}^{\mu} f(\vec{k}, x), & \tilde{c}_{\mu}^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu} \tilde{C}_{\mu}(x), \\
b(\vec{k}) &= \int d^3 x \, \beta(x) i_{\delta_0}^{\mu} f(\vec{k}, x), & b^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu} \beta(x), \\
\tilde{b}(\vec{k}) &= \int d^3 x \, \tilde{\beta}(x) i_{\delta_0}^{\mu} f(\vec{k}, x), & \tilde{b}^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu} \tilde{\beta}(x), \\
f_1(\vec{k}) &= \int d^3 x \, \phi_1(x) i_{\delta_0}^{\mu} f(\vec{k}, x), & f_1^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu} \phi_1(x), \\
f_2(\vec{k}) &= \int d^3 x \, \phi_2(x) i_{\delta_0}^{\mu} f(\vec{k}, x), & f_2^\dagger(\vec{k}) &= \int d^3 x \, f^*(\vec{k}, x) i_{\delta_0}^{\mu} \phi_2(x).
\end{align*}
\]

It is straightforward to verify that, using the above relationships (72), one can obtain exactly same (non-)vanishing (anti)commutation relations, as listed in (60) (amongst the creation and annihilation operators for the 4D free Abelian 2-form gauge theory). Thus, we have derived the correct (non-)vanishing basic brackets from the usual canonical method as well as by exploiting the basic ideas of symmetry principles.
VI. CONCLUSIONS

In our present investigation, the key ideas that have been exploited for the quantization scheme are the spin-statistic theorem, normal ordering (in the expressions for the conserved charges) and the central concepts of the continuous symmetry transformations (and their corresponding generators). The last ingredient of the above quantization scheme is the novel one and it differs from the standard method of canonical quantization scheme where the (graded) Poisson brackets (defined with the help of conjugate momenta) at the classical level are promoted to the (anti)commutators at the quantum level. In both the quantization schemes, we use the virtues of spin-statistics theorem and normal ordering.

One of the most beautiful observations in our present endeavor is the emergence of the one and the same set of non-vanishing basic canonical brackets [cf. (34)] from all the continuous symmetry transformations present in the free 2D Abelian 1-form gauge theory. These basic brackets are found to be unique and any other alternatives/deformations to them would not work with all the continuous symmetries (see, Appendix A for details). Even though the continuous symmetry transformations (and their corresponding generators) look completely different, the hidden basic brackets (34) [that emerge from the application of the standard definition of a generator for a given continuous symmetry (cf. (27))] are exactly the same. We have repeated the same exercise for the free 4D Abelian 2-form gauge theory and obtained their non-vanishing (anti)commutators in (60) by exploiting our main ingredients of quantization scheme (that have been mentioned several times).

The above key observation ensures that the continuous symmetry transformations of a gauge field theoretic model for the Hodge theory encode in their folds the basic (anti)commutators corresponding to the fermionic and bosonic fields of this theory. To the best of our knowledge, our method of derivation of the basic brackets in (34) and (60) is a novel observation in the realm of the quantization scheme for a special class (i.e. 2D Abelian 1-form and 4D Abelian 2-form) gauge theories which provide the physical realizations for the de Rham cohomological operators of differential geometry in the language of their symmetry properties and corresponding generators. In our recent work, we have shown that any arbitrary p-form Abelian gauge theory would provide a field theoretic model for the Hodge theory in $D = 2p$ dimensions of spacetime and our method of quantization will be applicable to these models. We conjecture that our method of quantization scheme will be true for any arbitrary model for the Hodge theory in any arbitrary dimension of spacetime.

Our method of quantization is completely different from the derivation of canonical brackets by considering the equations of motion for the quantum mechanical system of harmonic oscillator by Wigner. In our approach, we do not use the equation of motion anywhere except in the proof of nilpotency of the (anti-)BRST and (anti-)dual BRST symmetries, respectively. Rather, we exploit the ideas of symmetry principle and the definition of generators in our approach. In fact, it is the existence of six continuous symmetries of our present models for the Hodge theory that entails upon the emergence of basic brackets (34) and (60) uniquely.

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Appendix A

Here we describe very briefly the uniqueness of the canonical brackets (taken as the (anti)commutation relations) amongst the creation and annihilation operators which have been derived in the equation (34). Exploiting the general definition (27) of the generator and the transformations (2) and (3), it can be checked that the following alternative
Choose the appropriate brackets from \((A1)\). It is evident that in the above equation, we express the ghost charge in terms of the creation and annihilation operators \([cf. (16)]\) and \((anti)commutators in (A1) are not consistent in proving the sanctity of equation (A2) in its exact form. To corroborate the above assertions, let us take a couple of explicit examples with \(Q_\omega\) and \(Q_g\) which generate the bosonic and ghost symmetry transformations. For instance, the following two brackets from (33)

\[
\begin{align*}
[Q_\omega, a_\mu(k)] = -ik^2 \varepsilon_{\mu\nu} a^\nu(k), & \quad [Q_g, c(k)] = ic(k), \\
\end{align*}
\]

(A2)

must be satisfied by the alternative brackets given in (A.1). The l.h.s. of the first commutator (i.e. \([Q_\omega, a_\mu(k)] = -ik^2 \varepsilon_{\mu\nu} a^\nu(k)\)) of the above equation (A.2) implies the following commutator:

\[
[Q_\omega, a_\mu(k)] = i \int \text{d}k' k'^2 \varepsilon_{\rho\nu} \left( a_\rho(k') [a_\nu(k), a_\mu(k)] + [a_\rho(k'), a_\mu(k)] a_\nu(k') \right).
\]

(A3)

Using the appropriate commutators from (A1), we obtain

\[
[Q_\omega, a_\mu(k)] = -ik^2 \varepsilon_{\mu\nu} a^\nu(k) - ik^2 \varepsilon_{\rho\nu} A(k) (a^\rho)^\dagger(k) a_\mu(k) a^\nu(k).
\]

(A4)

It is obvious that the r.h.s. of the above equation does not match with the required result of (33). Thus, the brackets (A1) are not consistent. They can be consistent if and only if \(A(k) = 0\) which, ultimately, implies the uniqueness of (34).

To check the sanctity and preciseness of the brackets (A1), now let us take the second commutator (i.e. \([Q_g, c(k)] = ic(k)\)) of the above equation (A.2). The l.h.s. of the commutator is as follows

\[
[Q_g, c(k)] = - \int \text{d}k' \left( - \{c^\dagger(k'), c(k)\} \ c(k') + c^\dagger(k') \ \{c(k'), c(k)\} \right.
\]

\[
\left. - \ \{c^\dagger(k'), c(k)\} \ \bar{c}(k') + c^\dagger(k') \ \{\bar{c}(k'), c(k)\} \right).
\]

(A5)

In the above equation, we express the ghost charge in terms of the creation and annihilation operators \([cf. (16)]\) and choose the appropriate brackets from (A1). It is evident that \(\{c(k), c(k')\} = 0, \ \{c(k), c(k')\} = 0\) [cf. (A.1)]. With these inputs, it can be checked that the following is true, namely:

\[
[Q_g, c(k)] = ic(k) + c^\dagger(k) A(k) c(k) \bar{c}(k).
\]

(A6)

The above equation is in conflict with the requirement of the r.h.s. of equation (A2) [unless \(A(k) = 0\)]. Thus, the (anti)commutators in (A1) are not consistent in proving the sanctity of equation (A2) in its exact form.
In this Appendix, we collect all the (anti)commutators that are generated by the BRST charge (\(\tilde{\mathcal{Q}}\)) and the ghost charge (\(\mathcal{Q}\)). We wish to lay emphasis on the fact that we have taken only two brackets in equation (A2). However, as it turns out, all the rest of the relevant brackets are found to be inconsistent. Hence, the canonical (anti)commutators of equation (34) are unique in the sense that they are consistent with all the six continuous symmetries and corresponding conserved charges. In general, there might exist many kinds of deformations like (A1). However, we conjecture that the consistency with all the six continuous symmetries would always lead to the derivation of non-vanishing basic brackets as (34).

### Appendix B

In this Appendix, we collect all the (anti)commutators that are generated by the BRST charge (\(\tilde{\mathcal{Q}}\)), bosonic charge (\(\mathcal{Q}\)) and the ghost charge (\(\mathcal{Q}\)) for the 4D free Abelian 2-form theory. These brackets are as follows

\[
[\tilde{\mathcal{Q}}_b, f_2(\vec{k})] = 0, \quad [\tilde{\mathcal{Q}}_b, b^\dagger_2(\vec{k})] = 0, \quad [\tilde{\mathcal{Q}}_b, b(\vec{k})] = 0, \quad [\tilde{\mathcal{Q}}_b, b^\dagger(\vec{k})] = 0,
\]

\[
[\tilde{\mathcal{Q}}_b, f_1(\vec{k})] = -\frac{1}{2} k^\mu c_{(\vec{k})}^\mu, \quad [\tilde{\mathcal{Q}}_b, f_1^\dagger(\vec{k})] = +\frac{1}{2} k^\mu c_{(\vec{k})}^\mu, \quad [\tilde{\mathcal{Q}}_b, b(\vec{k})] = +\frac{1}{2} k^\mu c_{(\vec{k})}^\mu,
\]

\[
[\mathcal{Q}, b(\vec{k})] = k_\mu k^\mu c_{(\vec{k})}^\nu, \quad [\mathcal{Q}, c_{(\vec{k})}^\mu] = + k_\mu c_{(\vec{k})}^\mu, \quad [\mathcal{Q}, c_{(\vec{k})}^\mu] = - k_\mu c_{(\vec{k})}^\mu,
\]

\[
{\tilde{\mathcal{Q}}} c(\vec{k}) = i \varepsilon_{\mu\nu\sigma\eta} k^\mu k^\nu b(\vec{k}) + \frac{i}{2} \varepsilon_{\mu\nu\sigma\eta} k^\mu k^\nu b(\vec{k}) + \frac{i}{2} \varepsilon_{\mu\nu\sigma\eta} k^\mu k^\nu b(\vec{k}),(\vec{k})
\]

\[
[\tilde{\mathcal{Q}}_b, b(\vec{k})] = 0, \quad [\tilde{\mathcal{Q}}_b, b^\dagger(\vec{k})] = 0, \quad [\mathcal{Q}, b(\vec{k})] = 0, \quad [\mathcal{Q}, b^\dagger(\vec{k})] = 0,
\]

\[
[\tilde{\mathcal{Q}}_b, f_1(\vec{k})] = 0, \quad [\tilde{\mathcal{Q}}_b, f_1^\dagger(\vec{k})] = 0, \quad [\mathcal{Q}, f_1(\vec{k})] = 0, \quad [\mathcal{Q}, f_1^\dagger(\vec{k})] = 0,
\]

The (anti)commutators with the anti-co-BRST, anti-BRST charges can be computed in a straightforward manner from the corresponding (anti)commutation relations with the co-BRST (see, e.g. Sec. IV A) and BRST charges.

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