Floquet topological phases with fourfold degenerate edge modes in a driven spinful Creutz ladder

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Floquet engineering has the advantage of generating new phases with large topological invariants and many edge states by simple driving protocols. In this work, we propose an approach to obtain Floquet edge states with fourfold degeneracy and even-integer topological characterizations in a spinful Creutz ladder model, which is realizable in current experiments. Putting the ladder under periodic quenches, we found rich Floquet topological phases in the system, which belong to the symmetry class CII. Each of these phases is characterized by a pair of even integer topological invariants \((w_0, w_\pi) \in \mathbb{Z} \times \mathbb{Z}\), which can take arbitrarily large values with the increase of driving parameters. Under the open boundary condition, we further obtain multiple quartets of topological edge states with quasienergies zero and \(\pi\) in the system. Their numbers are determined by the bulk topological invariants \((w_0, w_\pi)\) due to the bulk-edge correspondence. Finally, we propose a way to dynamically probe the Floquet topological phases in our system by measuring a generalized mean chiral displacement. Our findings thus enrich the family of Floquet topological matter, and put forward the detection of their topological properties.

I. INTRODUCTION

Floquet topological states of matter have attracted great attention in the past decade (see Refs. \textsuperscript{1}–\textsuperscript{9} for reviews). These intrinsically nonequilibrium phases could appear in systems subject to time-periodic driving fields \textsuperscript{10–18}. Theoretically, various types of topological phases that are unique to periodically driven systems have been discovered \textsuperscript{9}–\textsuperscript{24}, and symmetry classification schemes for Floquet topological matter have also been proposed \textsuperscript{25–29}. Experimentally, Floquet topological phases have been observed in solid state materials \textsuperscript{30}, cold atoms in optical lattices \textsuperscript{31–33}, photonic \textsuperscript{34–36} and phononic \textsuperscript{37–39} systems. On application side, Floquet states have the potential of realizing setups with many topological transport channels \textsuperscript{40–42} and creating new schemes of topological quantum computations \textsuperscript{43–55}. In recent years, the study of Floquet topological matter has also been extended to higher-order topological models \textsuperscript{46–50} and non-Hermitian systems \textsuperscript{51–55}.

In the engineering of Floquet topological matter, one of the essential idea is that the period driving fields could induce long-range hopping and interactions on top of the non-driven system, leading to new phases with large topological invariants and many topologically protected edge states \textsuperscript{11,12}. This idea has been successfully applied to obtain a series of Floquet topological insulating and superconducting phases in one- and two-dimensional systems, both Hermitian \textsuperscript{40–42,44–63} and non-Hermitian \textsuperscript{51–54}, yielding large quantized Floquet-Thouless pumps \textsuperscript{59,62} and edge-state transport coefficients \textsuperscript{40,42}. The Floquet Majorana edge modes obtained following this idea are further applied in a spatiotemporal proposal of topological quantum computing \textsuperscript{43–45}.

According to the periodic table of topological insulators and superconductors \textsuperscript{64,65}, systems belong to the symmetry classes AIII, BDI and CII in one-dimension (1d) are characterized by integer topological invariants, and therefore could be engineered to obtain phases with large topological numbers and many edge modes by the Floquet method. Indeed, the studies in Refs. \textsuperscript{12,51–54,60,61} are all based on model systems in either the symmetry class AIII or BDI. However, systems belong to the symmetry class CII in 1d, which are characterized by even integer topological invariants \((2\mathbb{Z})\) and fourfold degenerate edge modes are seldomly been considered in the Floquet engineering. One possible reason for such a bias is that a 1d model in the CII class has at least four Floquet quasienergy bands, and therefore its theoretical treatment is more complicated than those two-band candidates in AIII and BDI classes. Nevertheless, when Floquet driving fields are applied, the \(2\mathbb{Z}\)-characterization of CII-class models imply that they could support topological phases with even more edge modes under the open boundary condition (OBC), and the properties of these new Floquet phases certainly deserve a detailed study.

Motivated by the above considerations, in this work we propose a 1d Floquet system in the symmetry class CII, whose topological phases are characterized by a pair of even integer topological invariants \((2\mathbb{Z} \times 2\mathbb{Z})\), and also featured by quartets of topological edge modes with fourfold degeneracy at both quasienergies zero and \(\pi\). We start by introducing an experimentally realizable quasi-1d ladder model of spin-1/2 fermions, which belongs to the symmetry class CII of the periodic table \textsuperscript{64,65}. Our Floquet system is then obtained by applying periodic quenches to the physical parameters of the ladder. Thanks to the...
driving fields, rich Floquet topological phases are found in this periodically quenched ladder model. A pair of topological winding numbers are then introduced to characterize these new phases, which can in principle take arbitrarily large even integer values for the model we considered. Under the OBC, we further obtain many quartets of Floquet topological zero and π edge modes, whose numbers are determined by the bulk topological invariants we introduced. This establishes the bulk-edge correspondence of 1d Floquet systems in the CII symmetry class. Finally, we propose to detect the Floquet topological phases in our system by measuring the chiral displacements, a pair of dynamical winding numbers can be constructed, which are equal to the topological invariants of the system. In Sec. [VI] we summarize our results and discuss potential future directions.

II. THE MODEL

In this section, we introduce the model that will be explored in this work. Our construction is based on the Creutz ladder (CL) model, which is originally proposed as a toy model to study chiral fermions in lattice gauge theory [66,68]. It is a quasi-1d lattice formed by two coupled chains, populated with spinless fermions and subject to a perpendicular magnetic field. In early studies, it was already identified that for certain values of the magnetic flux, the model belongs to the symmetry class AIIB and there are chiral symmetry protected edge modes at the boundaries of the ladder [67]. In later investigations, various modified versions of the CL are proposed [69–79], and also realized experimentally in atom-optical setups like photonic lattices [80,81] and cold atoms [82,83]. Intriguing topological features of the CL are reflected in the formation of AB cages [81] and the quantized Thouless pumping [84].

One version of the modified CL model concerns the case with spin-1/2 fermions and spin-orbit couplings in the ladder. Recently, such kinds of spinful CL (SCL) have been investigated in several studies [84,86]. It was found that the existence of spin degrees of freedom could not only modify the symmetry classification of the CL (e.g., from AIIB to CII), but also induce new topological transport phenomena. However, in these studies only a few topological phases of the SCL with small winding numbers were identified, and the richness of topological states in the SCL has not been uncovered. In this study, we will reveal the possible new topological phases that can appear in the SCL with the help of Floquet engineering.

To do so, we first introduce our version of the SCL model, which is schematically shown in Fig. [1]. The model can be described by the Hamiltonian

$$H_0 = H_1 + H_2,$$

where the two components $H_1$ and $H_2$ are given by:

$$H_1 = \sum_n J_x |n⟩⟨n+1| + H.c. \sigma_0 \otimes \tau_z$$

$$- \sum_n iV |n⟩⟨n+1| - H.c. \sigma_y \otimes \tau_0, \quad (2)$$

$$H_2 = \sum_n J_y |n⟩⟨n| \sigma_y \otimes \tau_x$$

$$+ \sum_n iJ_d |n⟩⟨n+1| - H.c. \sigma_z \otimes \tau_x. \quad (3)$$

Here $n$ is the unit cell index. Each unit cell of the ladder contains two sublattices $A$ and $B$. $J_x$ and $J_y$ are intercell and intracell hopping amplitudes along the $x$ and $y$ directions of the ladder. $J_d$ is the diagonal hopping amplitude between different sublattices in adjacent unit cells, with the prefactor $i$ originated from the magnetic flux perpendicular to the plaquettes of the ladder. $V$ is the strength of spin-orbit coupling between electrons of different spins in the same sublattice of different unit cells. The Pauli matrix $\sigma_x, \sigma_y, \sigma_z$ and $\tau_x, \tau_y, \tau_z$ act on the spin-1/2 and sublattice degrees of freedom, respectively, and $\sigma_0, \tau_0$ both represent the 2 by 2 identity matrix. In the lattice representation, it is straightforward to see that our SCL Hamiltonian $H_0$ belongs to the symmetry class CII of topological insulators. It possesses the time reversal symmetry (TRS) $\mathcal{T} = i\sigma_y \otimes \tau_0 K$ with $\mathcal{T}^2 = -1$, particle hole symmetry (PHS) $\mathcal{C} = \sigma_x \otimes \tau_y K$ with $\mathcal{C}^2 = -1$, and chiral symmetry (CS) $S = \mathcal{C} \mathcal{T} = -\sigma_z \otimes \tau_y$ with $S^2 = 1$, in the sense that $\mathcal{T}H_0\mathcal{T}^{-1} = H_0$, $\mathcal{C}H_0\mathcal{C}^{-1} = -H_0$ and $SH_0S = -H_0$. The topological phases of $H_0$ are thus characterized by a winding number $w$, which can only take even integer values ($2\mathbb{Z}$) due to the symmetry constraints. Furthermore, under the OBC, fourfold degenerate edge modes are expected to appear when the SCL is in a topologically nontrivial phase. The exact number of these edge modes is determined by the bulk topological invariant $w$.

We now incorporate our Floquet engineering approach, with the purpose of realizing new topological phases with

![FIG. 1. Schematic diagram of the spinful Creutz ladder. The backward (forward) image corresponds to the spin ↑ (↓) copy of the ladder. Each unit cell contains two sublattices A and B. The magnetic flux (⊗) is perpendicular to each plaquette (shaded square). The nearest neighbor hopping amplitudes along x, y and diagonal directions of the ladder are denoted by $J_x$, $J_y$ and $J_d$, respectively. The spin-orbit interaction $V$ couples particles with opposite spins in adjacent unit cells.](image-url)
large winding numbers and many degenerate edge modes in the SCL. We consider a simple scheme, in which the SCL is piecewise quenched within each driving period. Such a periodically quenched SCL (PQSCL) is described by the Hamiltonian:

\[
H(t) = \begin{cases} 
H_1 & t \in [jT, jT + T/2) \\
H_2 & t \in [jT + T/2, jT + T) 
\end{cases},
\]

where \( j \in \mathbb{Z} \), \( t \) is time and \( T \) is the driving period. Within each driving period, the Hamiltonians \( H_1 \) and \( H_2 \) are given by Eqs. \( \mathbf{2} \) and \( \mathbf{3} \). The Floquet operator of the PQSCL, which governs its dynamics over a complete driving period \( T \) [e.g., from \( t = jT + 0^- \) to \( (j+1)T + 0^- \)], is then given by

\[
U = T e^{-i \int_0^T H(t) dt} = e^{-\frac{i}{\hbar} H_2} e^{-\frac{i}{\hbar} H_1},
\]

where \( T \) is the time ordering operator. In the second equality, we have set \( \hbar = T = 1 \) and choose the unit of energy to be \( \hbar/T \). Under the OBC, the quasienergy spectrum \( \varepsilon \) and Floquet eigenstates of the PQSCL are obtained by solving the eigenvalue equation \( U |\psi\rangle = e^{-i \varepsilon} |\psi\rangle \). Under the PBC, we can perform a Fourier transform and obtain the Floquet operator in the momentum representation as

\[
U(k) = e^{-i h_2(k)} e^{-i h_1(k)},
\]

where

\[
h_1(k) = J_x \cos k \sigma_0 \otimes \tau_z + V \sin k \sigma_y \otimes \tau_0,
\]

\[
h_2(k) = \frac{J_y}{2} \sigma_0 \otimes \tau_x - J_d \sin k \sigma_z \otimes \tau_y,
\]

and \( k \in [-\pi, \pi] \) is the quasimomentum. In the basis of Floquet-Bloch eigenstates, \( U(k) \) can also be expressed as

\[
U(k) = \sum_{\ell=1,2} \sum_{\eta = \pm} e^{-i \varepsilon^\ell_\eta(k)} |\varepsilon^\ell_\eta(k)\rangle \langle \varepsilon^\ell_\eta(k)|,
\]

where \( \varepsilon^\ell_\eta(k) \) is the quasienergy dispersion, and \( |\varepsilon^\ell_\eta(k)\rangle \) are the corresponding Floquet eigenstates. The indices \( \ell = 1, 2 \) count the two Floquet bands whose quasienergies are in the range \( (0, \pi) \). Though the Floquet operator \( U(k) \) describes a four-band model, its analytical diagonalization is complicated due to the presence of spin-orbit coupling. Instead, the quasienergy spectrum and Floquet eigenstates of \( U(k) \) can be easily found by numerical calculations. In the next section, we will unravel the richness of the bulk spectrum and topological properties of the PQSCL described by \( U(k) \).

### III. BULK TOPOLOGICAL PROPERTIES

In this section, we focus on the characterization of bulk Floquet topological phases of the PQSCL. It is clear that the Floquet operator \( U(k) \) does not explicitly possess the TRS, PHS and CS of the non-driven SCL Hamiltonian \( H_0 \). The classification of Floquet operators like \( U(k) \) in 1d then rely on the introduction of a pair of symmetric time frames \( \mathbf{37} \). Upon similarity transformations, the bulk Floquet operator \( U(k) \) in Eq. \( \mathbf{1} \) can be expressed in these time frames as

\[
U_1(k) = e^{-\frac{i}{\hbar} h_1(k)} e^{-i \theta_{h_2(k)}(k)} e^{-\frac{i}{\hbar} h_1(k)}, \quad U_2(k) = e^{-\frac{i}{\hbar} h_2(k)} e^{-i \theta_{h_1(k)}(k)} e^{-\frac{i}{\hbar} h_2(k)},
\]

where \( h_1(k) \) and \( h_2(k) \) are given by Eqs. \( \mathbf{7} \) and \( \mathbf{8} \). It is clear that the Floquet operators \( U_1(k) \), \( U_2(k) \) and \( U_1(k) \) share the same quasienergy spectrum. In the meantime, the Floquet operators \( U_1(k) \) and \( U_2(k) \) both possess the TRS \( T = i \sigma_y \otimes \tau_0 \), PHS \( C = \sigma_x \otimes \sigma_y \) and CS \( S = TC^* = -\sigma_z \otimes \sigma_y \), in the sense that

\[
TU_\alpha(k)T^\dagger = U_\alpha^*(-k), \quad CU_\alpha(k)C^! = U_\alpha(-k), \quad SU_\alpha(k)S = U_\alpha^*(k),
\]

for \( \alpha = 1, 2 \). Moreover, since we have \( TT^* = -1 \), \( CC^* = -1 \) and \( SS^2 = 1 \), the Floquet operators \( U_1(k) \) and \( U_2(k) \) belong to the symmetry class CII according to the periodic table of Floquet topological insulators \( \mathbf{26} \). Therefore, the topological phases of Floquet operator \( U(k) \) are characterized by a pair of winding numbers, which can only take even integer values \( (2\mathbb{Z} \times 2\mathbb{Z}) \).

To obtain these topological invariants, we first introduce winding numbers \( w_1 \) and \( w_2 \) for the Floquet operators \( U_1(k) \) and \( U_2(k) \) in symmetric time frames, i.e.,

\[
w_\alpha = \int_{-\pi}^\pi \frac{dk}{4\pi i} \text{Tr}[S \mathcal{Q}_\alpha(k) \partial_k \mathcal{Q}_\alpha(k)]
\]

for \( \alpha = 1, 2 \). Here \( S \) is the chiral symmetry operator. The \( \mathbb{Q} \)-matrix \( \mathcal{Q}_\alpha(k) \) is defined as

\[
\mathcal{Q}_\alpha(k) = \sum_{\ell=1,2} \sum_{\eta = \pm} \eta |\varepsilon^\ell_\eta(k)\rangle \langle \varepsilon^\ell_\eta(k)|,
\]

where \( |\varepsilon^\ell_\eta(k)\rangle \) is the Floquet eigenstate of \( U_\alpha(k) \) with quasienergy \( \varepsilon^\ell_\eta(k) = \eta \varepsilon^\ell_\eta(k) \), and \( \ell, \eta = 1, 2 \) are the indices of the two Floquet bands, whose quasienergies are in the range \( (0, \pi) \). \( \mathcal{Q}_\alpha(k) \) can thus be viewed as a “flat-band version” of the Floquet effective Hamiltonian of \( U_\alpha(k) \), in the sense that all the positive (negative) quasienergies are set to \( 1 \) (\( -1 \)). It is clear that such a band flattening procedure does not change the topological properties of \( U_\alpha(k) \), so long as its Floquet spectrum is gapped at the quasienergies zero and \( \pi \) during the flattening process.

In terms of the winding numbers \( w_1 \) and \( w_2 \), we can define the invariants that characterize the Floquet topological phases of the PQSCL. They are given by

\[
w_0 = \frac{w_1 + w_2}{2}, \quad w_\pi = \frac{w_1 - w_2}{2}.
\]
According to the symmetry classification scheme of our Floquet system, we have $(w_0, w_\pi) \in 2\mathbb{Z} \times 2\mathbb{Z}$, and their values specify all possible Floquet topological phases that can appear in the PQSCL described by the Floquet operator $U(k)$ in Eq. (6).

In the remaining part of this section, we verify Eq. (25) for the PQSCL numerically, and then construct its Floquet topological phase diagram. To do so, we first introduce a pair of gap characteristic functions, defined as

$$\Delta_0 \equiv \min_{k \in [-\pi, \pi]} |\cos[\varepsilon_\ell(k)] - 1|,$$  \hspace{1cm} (16)

$$\Delta_\pi \equiv \min_{k \in [-\pi, \pi]} |\cos[\varepsilon_\ell(k)] + 1|,$$  \hspace{1cm} (17)

where $\varepsilon_\ell(k)$ is the quasienergy of the eigenstate $|\varepsilon_\ell^\pm(k)\rangle$ of $U_\alpha(k)$ [see Eq. (4)]. It is clear that due to the CS and the $2\pi$-periodicity of the quasienergy, the Floquet spectrum of PQSCL will become gapless at the quasienergy zero ($\pi$) if $\Delta_0 = 0$ ($\Delta_\pi = 0$). A topological phase transition accompanied by the quantized change of winding number $w_0$ ($w_\pi$) is then expected when the gap function $\Delta_0$ ($\Delta_\pi$) vanishes.

In Figs. 2(a) and (b), we present the (rescaled) gap functions $\Delta_0$ (blue dotted lines), $\Delta_\pi$ (red dash-dotted lines) and the winding numbers $w_0$ (yellow solid lines), $w_\pi$ (purple dashed lines) with respect to the change of diagonal hopping amplitude $J_d$ and spin-orbit coupling strength $V$, respectively. In both panels, we observe that the winding number $w_0$ or $w_\pi$ indeed shows a quantized jump every time when $\Delta_0 = 0$ or $\Delta_\pi = 0$, reflecting the existence of Floquet topological phase transitions in the PQSCL. Furthermore, with the increase of $J_d$, we obtain topological phases with larger and larger winding numbers $(w_0, w_\pi)$. Such a trend will continue, and we could in principle find Floquet topological phases in the PQSCL, whose winding numbers $(w_0, w_\pi)$ could take arbitrarily large even integers. To the best of our knowledge, this is the first proposal of a topological insulator in CII class with unlimited winding numbers under the framework of Floquet engineering.

To fully reveal the Floquet topological phases that can appear in the PQSCL model, we present its topological phase diagram versus the system parameters $(J_d, V)$ in Fig. 3 for a typical situation. The other system parameters are chosen as $J_x = 0.5\pi$, $J_y = 0.6\pi$, and $V = 0.2\pi$ ($J_d = 2.5\pi$) for panel (a) [(b)]. The black lines separating different regions are the boundaries between different topological phases, obtained from Eqs. (16) and (17).
In the previous section, we have shown that the PQSCL model belongs to the symmetry class CII, and each of its Floquet topological phases is characterized by a pair of even integer winding numbers \((n_0, n_\pi)\) in \(2\mathbb{Z} \times 2\mathbb{Z}\). When the bulk Floquet spectrum of PQSCL is gapped at the quasienergies zero and \(\pi\), we could further obtain zero and \(\pi\) Floquet edge modes under the OBC, and their numbers \((n_0, n_\pi)\) should be related to the topological invariants \((\omega_0, \omega_\pi)\) as \(n_0 = 2|\omega_0|\) and \(n_\pi = 2|\omega_\pi|\), which describe nothing but the bulk-edge correspondence of 1d Floquet topological insulators in the CII class.

More generally, one can also introduce the topological invariants of the system directly under the OBC. To do so, we first define the non-commutative winding number in the symmetric time frame \(\alpha \, (= 1, 2)\) under the OBC as

\[
\tilde{w}_\alpha \equiv \frac{1}{2N_b} \text{Tr}_b(SQ_\alpha|Q_\alpha, \hat{n}|).
\]

Here \(S\) is the chiral symmetry operator, and \(\hat{n} = \sum_{n=1}^{N} n|n⟩⟨n|\sigma_0 \otimes \tau_0\) is the position operator of the unit cell. The total number of unit cells of the ladder is \(N = N_b + 2N_c\), with \(N_b\) and \(N_c\) being the number of cells in the bulk and edge intervals of the ladder, respectively. In terms of the unit cell coordinate \(n\), we have \(n \in [N_c + 1, N_c + N_b]\) for the bulk interval and \(n \in [1, N_c] \cup [N - N_c + 1, N]\) for the left and right edge intervals. The trace \(\text{Tr}_b(\cdot)\) in Eq. (18) is taken only over the degrees of freedom in the bulk interval \(n \in [N_c + 1, N_c + N_b]\). In numerical calculations, \(N_c\) should be chosen large enough in order to avoid boundary effects. The open boundary Q-matrix \(Q_\alpha\) in Eq. (18) is given by

\[
Q_\alpha = \sum_{n=1}^{N_b} \sum_{\eta=\pm} \eta|\tilde{\varepsilon}_{\alpha n}^\eta⟩⟨\tilde{\varepsilon}_{\alpha n}^\eta|,
\]

for the two time frames \(\alpha = 1, 2\), where the Floquet eigenstates \(|\tilde{\varepsilon}_{\alpha n}^\eta⟩\) satisfy the eigenvalue equation \(U_\alpha|\varepsilon_{\alpha n}^\eta⟩ = e^{-i\tilde{\varepsilon}_{\alpha n}^\eta} e^{|\varepsilon_{\alpha n}^\eta|}|\varepsilon_{\alpha n}^\eta⟩\) for all the quasienergy eigenvalues \(\{\varepsilon_{1,2N}\} \in (0, \pi)\). Physically, \(Q_\alpha\) can be understood as a flattened effective Floquet Hamiltonian of the system, whose positive and negative quasienergies are set to 1 and \(-1\), respectively. With \((\tilde{w}_1, \tilde{w}_2)\) defined in Eq. (18), we can construct a pair of topological invariants for the system under the OBC as

\[
\tilde{w}_0 = \frac{\tilde{w}_1 + \tilde{w}_2}{2}, \quad \tilde{w}_\pi = \frac{\tilde{w}_1 - \tilde{w}_2}{2}.
\]

Since in a given time frame \(\alpha \, (= 1, 2)\), early studies \[88\] have established that \(w_\alpha = \tilde{w}_\alpha\), we could finally state the bulk-edge correspondence of our PQSCL model as:

\[
(n_0, n_\pi) = (2|\omega_0|, 2|\omega_\pi|) = (2\tilde{w}_0, 2\tilde{w}_\pi).
\]

Note that the second equality holds even when the system possesses disorder, assuming that the disorder does not break the TRS, PHS and CS of the model. Furthermore, since \(\omega_0\) and \(\omega_\pi\) are both even integers, the values of \(n_0\) and \(n_\pi\) must be integer multiples of four. This implies that under the OBC, the Floquet zero and \(\pi\) edge modes always appear as quartets in the PQSCL, with their four-fold degeneracy being protected by the symmetries of the system.

To demonstrate the edge states and bulk-edge correspondence of the PQSCL model, we present its quasienergy spectrum under the OBC for two typical situations. In Fig. (4a), we show the Floquet spectrum \(\varepsilon\) of the system versus the diagonal hopping amplitude \(J_d\). The other system parameters are chosen as \(J_x = 0.5\pi\), \(J_y = 0.6\pi\), \(V = 0.2\pi\), and the ladder contains a number of \(N = 200\) unit cells. From Fig. (4a), we see that across each topological phase transition shown in the phase diagram Fig. 3, the spectrum gap closes and reopens at the quasienergy zero or \(\pi\), accompanied by the emergence of new Floquet zero or \(\pi\) edge modes. The numbers of these edge modes \(n_0\) and \(n_\pi\) are denoted explicitly for each phase in the figure, and their values are consistent with the bulk-edge correspondence as described by Eq. (21). Notably, with the increase of \(J_d\), the system undergoes
a series of topological phase transitions, with more and more edge modes appear at both zero and $\pi$ quasienergies. Therefore, we can in principle obtain arbitrarily many quartets of zero and $\pi$ Floquet edge modes by simply tuning the diagonal hopping amplitude $J_d$ of the ladder. The observation of these edge modes in atom-optical quantum simulators could not only verify the topological nature of the PQSCL model, but also demonstrate the power of Floquet engineering in the realization of new phases with large topological invariants.

In Fig. 4(b), we show the Floquet spectrum $\varepsilon$ of the PQSCL versus the spin-orbit coupling strength $V$. The other system parameters are fixed at $J_x = 0.5\pi$, $J_y = 0.6\pi$, $J_d = 2.5\pi$, and the number of unit cells $N = 200$. Again, we observe Floquet zero and $\pi$ edge modes in each of the topologically nontrivial phases, with their numbers $(n_0, n_\pi)$ being related to the winding numbers $(w_0, w_\pi)$ through the bulk-edge relation Eq. (21). Across each topological phase transition (i.e., gap closing and reopening) point, the number of edge modes at quasienergies zero or $\pi$ also appear or disappear as quartets, as predicted by our theory. Note that the numbers of edge states $(n_0, n_\pi)$ change non-monotonically with the increase of $V$, with no tendency of raising as compared with the case of $(n_0, n_\pi)$ versus $J_d$.

To sum up, we have established the relation between the bulk topological invariants $(w_0, w_\pi)$ of the PQSCL and the number of its fourfold degenerate edge modes $(n_0, n_\pi)$ at both quasienergies zero and $\pi$ under the OBC, as given by Eq. (21). In the next section, we further construct a dynamical observable, which could help us to detect the topological invariants and phase transitions of the PQSCL model.

V. DYNAMICAL PROBES

The mean chiral displacement (MCD) is an observable, which could directly produce the topological winding numbers of a system with chiral symmetry in its dynamical evolution [90, 92]. It is obtained by measuring the long-time average of chiral displacement operator $S\hat{n}$, where $S$ is the unitary and Hermitian operator that defines the chiral symmetry of the system, and $\hat{n}$ is the unit cell position operator. The MCD was first proposed for 1d non-driven systems in the symmetry classes AII and BDI [90], and later extended to Floquet systems [61, 91], non-Hermitian systems [52, 53] and systems in two dimensions [16]. In this work, we further extend the MCD to 1d Floquet systems in the symmetry class CII, and employ it to probe the topological phases of the PQSCL dynamically.

We define the MCD as the long-time average of the chiral displacement operator $S\hat{n}$ in a given symmetric time frame. For a 1d Floquet system in the symmetry class CII, it can be expressed as

$$c_\alpha = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \langle SU^m_\alpha \hat{n} U^m_\alpha \rangle_0.$$  \hspace{1cm} (22)

where $\alpha = 1, 2$ is the index of the symmetric time frame, $U_\alpha$ is the corresponding Floquet operator, $M$ is the total number of driving periods (in the unit $T = 1$), $S$ is the chiral symmetry operator and $\hat{n}$ is the unit cell position operator. $\langle \cdots \rangle_0 \equiv \text{Tr}(\rho \cdots)$ performs an average over the initial state $\rho_0$. For our PQSCL, $\rho_0$ can be chosen as the an incoherent summation of eigenmodes occupying each spin and sublattice degrees of freedom in the central unit cell $(n = 0)$ of the lattice, i.e.,

$$\rho_0 = \sum_{\sigma = \uparrow, \downarrow} \sum_{s = A, B} |0\sigma s\rangle \langle 0\sigma s|.$$ \hspace{1cm} (23)

Note here that the initial state $\rho_0$ is a mixed state. In experiments, one can simply execute the dynamics for each component state $|0\sigma s\rangle \langle 0\sigma s|$ of $\rho_0$ separately, and then sum up their contributions to $c_\alpha$. Taking the periodic boundary condition, we can express $c_\alpha$ in momentum representation as

$$c_\alpha = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \text{Tr}[SU^m_\alpha(k) i\partial_k U^m_\alpha(k)],$$ \hspace{1cm} (24)

where $k \in [-\pi, \pi]$ is the quasimomentum, and $\alpha = 1, 2$ are the indices of two time frames. For our PQSCL, $U_1(k)$ and $U_2(k)$ are given by Eqs. (10) and (11), respectively. With the help of $c_1$ and $c_2$, we can construct a pair of dynamical winding numbers $c_0$ and $c_\pi$, defined as

$$c_0 = \frac{c_1 + c_2}{2}, \hspace{1cm} c_\pi = \frac{c_1 - c_2}{2}.$$ \hspace{1cm} (25)

Then it can be shown that the dynamical winding numbers $(c_0, c_\pi)$ are equal to the topological invariants $(w_0, w_\pi)$ of the PQSCL (see Appendix A for a proof), i.e.,

$$w_0 = c_0, \hspace{1cm} w_\pi = c_\pi.$$ \hspace{1cm} (26)

The relations in Eqs. (24) to (26) establish the dynamical characterization of topological phases for 1d Floquet systems in the CII symmetry class, and provide a way for the detection of their topological invariants in cold atom and photonic setups.

In Figs. 5(a) and 5(b), we numerically demonstrate the relations in Eq. (26) for the PQSCL model in two typical situations. In Fig. 5(a), we present the values of $w_0$ (blue solid line), $w_\pi$ (red dashed line), $c_0$ (black circles) and $c_\pi$ (magenta squares) versus the diagonal hopping amplitude $J_d$ of the PQSCL. Other system parameters are chosen to be $J_x = 0.5\pi$, $J_y = 0.6\pi$, $V = 0.2\pi$, and the MCDs are averaged over $M = 100$ driving periods to obtain the dynamical winding numbers. We observe that
the behavior of dynamical winding numbers \((c_0, c_\pi)\) follow closely with the theoretical predictions of topological invariants \((w_0, w_\pi)\), as presented in Eq. (26). Notably, across each topological phase transition point, the dynamical winding numbers \((c_0, c_\pi)\) show the same quantized jumps as \((w_0, w_\pi)\). Therefore, \((c_0, c_\pi)\) provide nice dynamical probes to the topological invariants and Floquet topological phase transitions for our PQSCL model. In Fig. 5(b), we further show the topological invariants \(w_0\) (blue solid line), \(w_\pi\) (red dashed line), and dynamical winding numbers \(c_0\) (black circles) and \(c_\pi\) (magenta squares) versus the spin-orbit coupling strength \(V\) in panels (a) and (b), respectively. The system parameters are chosen as \((J_x, J_y, V) = (0.5\pi, 0.6\pi, 0.2\pi)\) for panel (a) and \((J_x, J_y, J_d) = (0.5\pi, 0.6\pi, 2.5\pi)\) for panel (b). In both panels, \((w_0, w_\pi)\) are calculated according to Eqs. (13) and (15), whereas \((c_0, c_\pi)\) are computed numerically following Eqs. (24) and (25), with the total number of driving periods being \(M = 100\). The ticks along the horizontal axis correspond to the local minimum of bulk gap functions \((\Delta_0, \Delta_\pi)\), which are obtained from Eqs. (16) and (17).

FIG. 5. The topological invariants \(w_0\) (blue solid lines), \(w_\pi\) (red dashed lines), and dynamical winding numbers \(c_0\) (black circles), \(c_\pi\) (magenta squares) versus the diagonal hopping amplitude \(J_d\) and spin-orbit coupling strength \(V\) in panels (a) and (b), respectively. The system parameters are chosen as \((J_x, J_y, V) = (0.5\pi, 0.6\pi, 0.2\pi)\) for panel (a) and \((J_x, J_y, J_d) = (0.5\pi, 0.6\pi, 2.5\pi)\) for panel (b). In both panels, \((w_0, w_\pi)\) are calculated according to Eqs. (13) and (15), whereas \((c_0, c_\pi)\) are computed numerically following Eqs. (24) and (25), with the total number of driving periods being \(M = 100\). The ticks along the horizontal axis correspond to the local minimum of bulk gap functions \((\Delta_0, \Delta_\pi)\), which are obtained from Eqs. (16) and (17).

Practically, we have checked that the quantization of \((c_0, c_\pi)\) is already good enough for the number of evolution periods to be as few as \(M = 15\). This should be already within reach in experimental platforms like photonic and cold atom systems, where the MCDs have already been measured for other lattice models [99, 92]. Combining these facts with the existing realizations of the Creutz ladder model [80–83], we expect that our PQSCL and its topological properties should also be realizable and detectable in similar experimental setups.

VI. SUMMARY

In this work, we proposed a periodically quenched spinful Creutz ladder model, and investigated its Floquet topological phases. The model belongs to the symmetry class CII in the periodic table of topological matter. Its phases are characterized by a pair of winding numbers \((w_0, w_\pi)\), which can only take even integer values \((2\pi \times 2\pi)\). We established the phase diagram of the model and found rich Floquet topological phases in the CII class, which possess large and even integer topological winding numbers. Under the open boundary condition, we further obtained multiple quartets of Floquet topological edge modes at both zero and \(\pi\) quasienergies, with their numbers being determined by the bulk topological invariants \((w_0, w_\pi)\). Finally, we showed that the topological phases of the system can be probed by measuring a generalized mean chiral displacement in two chiral symmetric time frames, and established the relation between the resulting dynamical winding numbers and the topological invariants of the system. Our discoveries thus introduced a new member to the zoo of Floquet topological phases, which is featured by even integer topological invariants, fourfold degenerate edge modes, and also within reach in various atom-optical quantum simulators.

In recent years, the study of Floquet topological matter has been extended to non-Hermitian domain, where the interplay between driving fields and gain/loss or nonreciprocal effects could lead to unique Floquet non-Hermitian topological phases with large topological invariants [91–94]. It is expected that by adding non-Hermitian effects to our driven Creutz ladder, Floquet phases with quadruple edge modes could also be induced even in the non-Hermitian regime, and the full characterization of these new phases is an interesting direction to explore in future studies. On application side, it is known that Majorana edge modes could appear in the Creutz ladder with superconducting pairings. Therefore, by adding superconducting pairing terms to our driven Creutz ladder, we might be able to obtain many quartets of Floquet Majorana zero and \(\pi\) edge modes. Whether the fourfold degeneracy of these Majorana modes could give more room for the braiding operations in the recently proposed Floquet topological quantum computing [43] is also an interesting topic to explore in the future.
where we have used the fact that $\langle \varepsilon_{\eta}^\prime(k)|S|\varepsilon_{\eta}^\prime(k)\rangle \propto \delta_{\eta m} \delta_{\eta,-\eta}$. Similarly, we can express the MCD in Eq. (24) in the time frame $\alpha$ as:

$$\alpha \alpha \alpha = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{\ell, \eta} \left[ e^{-im^{\prime} \eta^2(k)} \mathcal{S}|\varepsilon_{\eta}^\prime(k)\rangle \langle \varepsilon_{\eta}^\prime(k)| \right] \times \partial_\ell [e^{-im^{\prime} \eta^2(k)} |\varepsilon_{\eta}^\prime(k)\rangle \langle \varepsilon_{\eta}^\prime(k)|],$$

(A3)

where $\ell = 1, 2$ and $\eta = \pm$. Taking the trace explicitly in the quasienergy eigenbasis, we can further express $c_\alpha$ as:

$$c_\alpha = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{\ell, \eta} \left[ e^{-im^{\prime} \eta^2(k)} |\varepsilon_{\eta}^\prime(k)\rangle \langle \varepsilon_{\eta}^\prime(k)| \mathcal{S} \right]$$

(A4)

It is clear that the term in the second line of Eq. (A4) vanishes, since $\langle \varepsilon_{\eta}^\prime(k)|S|\varepsilon_{\eta}^\prime(k)\rangle \propto \langle \varepsilon_{\eta}^\prime(k)|S|\varepsilon_{\eta}^\prime(k)\rangle = 0$. The term in the fourth line of Eq. (A4) carries an oscillating phase factor $e^{-im^{\prime} \eta^2(k)}$. So it will also vanish under the long-time average $\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \cdots$. The only contribution to the value of $c_\alpha$ left after taking the long-time average comes from the term in the third line of Eq. (A4), which is independent of the driving period index $m$. Working out the sum over $m$, we end with:

$$c_\alpha = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sum_{\ell, \eta} \left[ e^{-im^{\prime} \eta^2(k)} |\varepsilon_{\eta}^\prime(k)\rangle \langle \varepsilon_{\eta}^\prime(k)| \partial_\ell [e^{-im^{\prime} \eta^2(k)} |\varepsilon_{\eta}^\prime(k)\rangle \langle \varepsilon_{\eta}^\prime(k)|] \right].$$

(A5)

Therefore, comparing Eqs. (A2) with (A5), we conclude that the winding number and MCD in the symmetric time frame $\alpha$ satisfy $w_\alpha = c_\alpha$ for $\alpha = 1, 2$. Finally, according to the definitions of topological invariants $(w_0, w_\pi) = \left( \frac{w_1 + w_2}{2}, \frac{w_1 - w_2}{2} \right)$ and dynamical winding numbers $(c_0, c_\pi) = \left( \frac{c_1 + c_2}{2}, \frac{c_1 - c_2}{2} \right)$ in the main text, we arrive at the desired relationship between the topological and dynamical winding numbers of the PQSCL, i.e., $(w_0, w_\pi) = (c_0, c_\pi)$ as given by Eq. (26) in the main text.

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