ON THE CLASSIFICATION OF THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF GLOBALLY STABLE SCALAR DIFFERENTIAL EQUATIONS WITH RESPECT TO STATE–INDEPENDENT STOCHASTIC PERTURBATIONS

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Abstract. In this paper we characterise the global stability, global boundedness and recurrence of solutions of a scalar nonlinear stochastic differential equation. The differential equation is a perturbed version of a globally stable autonomous equation with unique equilibrium where the diffusion coefficient is independent of the state. We give conditions which depend on the rate of decay of the noise intensity under which solutions either (a) tend to the equilibrium almost surely, (b) are bounded almost surely but tend to zero with probability zero, (c) or are recurrent on the real line almost surely. We also show that no other types of asymptotic behaviour are possible. Connections between the conditions which characterise the various classes of long-run behaviour and simple sufficient conditions are explored, as well as the relationship between the size of fluctuations and the strength of the mean reversion and diffusion coefficient, in the case when solutions are a.s. bounded.

1. Introduction

In this paper, we characterise the global asymptotic stability of the unique equilibrium of a scalar deterministic ordinary differential equation when it is subjected to a stochastic perturbation independent of the state.

We fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \geq 0}, \mathbb{P})$. Let $B$ be a standard one–dimensional Brownian motion which is adapted to $(\mathcal{F}(t))_{t \geq 0}$. We consider the stochastic differential equation

$$dX(t) = -f(X(t)) \, dt + \sigma(t) \, dB(t), \quad t \geq 0; \quad X(0) = \xi \in \mathbb{R}. \quad (1.1)$$

We suppose that

$$f \in C(\mathbb{R}; \mathbb{R}); \quad xf(x) > 0, \quad x \neq 0; \quad f(0) = 0. \quad (1.2)$$

and that $\sigma$ obeys

$$\sigma \in C([0, \infty); \mathbb{R}). \quad (1.3)$$

These conditions ensure the existence of a continuous adapted process which obeys (1.1) on $[0, \infty)$, and we will refer to any such process as a solution. We do not rule out the existence of more than one process, but part of our analysis will show that all solutions share the same asymptotic properties. Hypotheses such as local Lipschitz continuity or monotonicity can be imposed in order to guarantee that there is a unique such solution.

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In the case when \( \sigma \) is identically zero, it follows under the hypothesis (1.2) that any solution \( x \) of
\[
x'(t) = -f(x(t)), \quad t > 0; \quad x(0) = \xi,
\] obeys
\[
\lim_{t \to \infty} x(t; \xi) = 0 \text{ for all } \xi \in \mathbb{R}.
\] (1.5)
Clearly \( x(t) = 0 \) for all \( t \geq 0 \) if \( \xi = 0 \). The question naturally arises: if any solution \( x \) of (1.4) obeys (1.5), under what conditions on \( f \) and \( \sigma \) does any solution \( X \) of (1.1) obey
\[
\lim_{t \to \infty} X(t, \xi) = 0, \quad \text{a.s. for each } \xi \in \mathbb{R}.
\] (1.6)
The convergence phenomenon captured in (1.6) for any solution of (1.1) is often called almost sure global convergence (or global stability for the solution of (1.4)), because solutions of the perturbed equation (1.1) converge to the zero equilibrium solution of the underlying unperturbed equation (1.4).

It was shown in Chan and Williams [9] that if \( f \) is strictly increasing with \( f(0) = 0 \) and \( \lim_{x \to \infty} f(x) = \infty \), \( \lim_{x \to -\infty} f(x) = -\infty \), (1.7) then any solution \( X \) of (1.1) obeys (1.6) holds if \( \sigma \) obeys
\[
\lim_{t \to \infty} \sigma^2(t) \log t = 0.
\] (1.8)
Moreover, Chan and Williams also proved, if \( t \mapsto \sigma^2(t) \) is decreasing to zero, that if the solution \( X \) of (1.1) obeys (1.6), then \( \sigma \) must obey (1.8). These results were extended to finite–dimensions by Chan in [8]. The results in [9, 8] are motivated by problems in simulated annealing.

In Appleby, Gleeson and Rodkina [5], the monotonicity condition on \( f \) and (1.7) were relaxed. It was shown if \( f \) is locally Lipschitz continuous and obeys (1.2), and in place of (1.7) also obeys
\[
\text{There exists } \phi > 0 \text{ such that } \phi := \liminf_{|x| \to \infty} |f(x)|,
\] then any solution \( X \) of (1.1) obeys (1.6) holds if \( \sigma \) obeys
\[
\lim_{t \to \infty} \sigma^2(t) \log t = 0.
\] (1.8)
Moreover, Chan and Williams also proved, if \( t \mapsto \sigma^2(t) \) is decreasing to zero, that if the solution \( X \) of (1.1) obeys (1.6), then \( \sigma \) must obey (1.8). Moreover, it was also shown, without monotonicity on \( \sigma \), that if
\[
\lim_{t \to \infty} \sigma^2(t) \log t = +\infty,
\] (1.10) then the solution \( X \) of (1.1) obeys
\[
\limsup_{t \to \infty} |X(t, \xi)| = +\infty, \quad \text{a.s. for each } \xi \in \mathbb{R}.
\] (1.11)
Furthermore, it was shown that the condition (1.8) could be replaced by the weaker condition
\[
\lim_{t \to \infty} \int_0^t e^{-2(t-s)} \sigma^2(s) \log_2 \int_0^t \sigma^2(s) e^{2s} \, ds = 0
\] (1.12) and that (1.12) and (1.8) are equivalent when \( t \mapsto \sigma^2(t) \) is decreasing. In fact, it was even shown that if \( \sigma^2 \) is not monotone decreasing, \( \sigma \) does not have to satisfy (1.8) in order for \( X \) to obey (1.6).

In this paper, we improve upon the results in [5] and [9, 8] in a number of directions. First, we show that neither the Lipschitz continuity of \( f \) nor the condition (1.9) is needed in order to guarantee that any solution \( X \) of (1.1) obeys (1.6). Moreover, we give necessary and sufficient conditions for the convergence of solutions which do not require the monotonicity of \( \sigma^2 \). One of our main results shows that
if \( f \) obeys (1.2) and \( \sigma \) is also continuous, then any solution \( X \) of (1.1) obeys (1.6) if and only if
\[
S'(\epsilon) := \sum_{n=0}^{\infty} \sqrt{\int_{n}^{n+1} \sigma^2(s) \, ds} \exp \left( -\frac{\epsilon^2}{2} \frac{1}{\int_{n}^{n+1} \sigma^2(s) \, ds} \right) < +\infty, \quad \text{for every } \epsilon > 0,
\]
and it is even shown that if (1.13) does not hold, then \( \mathbb{P}[X(t) \to 0 \text{ as } t \to \infty] = 0 \) for any \( \xi \in \mathbb{R} \) (Theorem 9). Another significant development from [5] and [9, 8] is a complete classification of the asymptotic behaviour of (1.1) in terms of the data, rather than merely satisfactory sufficient conditions. In Theorem 7, we show that when \( f \) obeys (1.7), that any solution is either (a) convergent to zero with probability one, (b) bounded but not convergent to zero, with probability one, or (c) recurrent on \( \mathbb{R} \) with probability one, according as to whether \( S'(\epsilon) \) is always finite, sometimes finite, or never finite, for \( \epsilon > 0 \). Apart from classifying the asymptotic behaviour, the novel feature here is that bounded but non–convergent solutions are examined.

Although the condition (1.13) is necessary and sufficient for \( X \) to obey (1.6), it may prove to be a little unwieldy for use in some situations. For this reason we deduce some sharp sufficient conditions for \( X \) to obey (1.6). If \( f \) obeys (1.2) and \( \sigma \) is continuous and obeys (1.12), then any solution \( X \) of (1.1) obeys (1.6) (Theorem 6). In the spirit of Theorem 9, we also establish converse results in the case when \( \sigma^2 \) is monotone (Theorem 11), and demonstrate that the condition (1.12) is hard to relax if we require \( X \) to obey (1.6). The relationship between the conditions which characterise the asymptotic behaviour, and which involve \( S'(\epsilon) \), and sufficient conditions are explored in several results, notably in Proposition 3 and 4.

Also, in the case when solutions are bounded, we analyse the relationships between the deterministic bounds on solutions and the drift and diffusion coefficients. In particular, in Propositions 6, 7, and 8, we demonstrate the bounds on any solution increase with greater noise intensity, and with weaker mean reversion.

These results are proven by showing that the stability of (1.1) is equivalent to the asymptotic stability of a process \( Y \) which is the solution of an affine SDE with the same diffusion coefficient \( \sigma \) (Proposition 5, especially part (A)). A classification of the asymptotic behaviour \( Y \) has already been achieved in [2, 4], and the relevant results of [2] are listed in Section 3. The proof of part (a) of Proposition 5 uses the additional condition that \( \sigma \notin L^2(0, \infty) \); the case when \( \sigma \in L^2(0, \infty) \) is easier, uses different methods, and is dealt with separately in Theorem 1. Essentially, in the case when \( \sigma \notin L^2(0, \infty) \) the recurrence of one–dimensional standard Brownian motion forces solutions to return to an arbitrarily small neighbourhood of the origin infinitely often. Then, if the noise fades sufficiently quickly so that the affine SDE is convergent to zero, the difference \( Z := X - Y \) obeys a perturbed version of the ordinary differential equation (1.4) where the perturbation fades to zero asymptotically, and by virtue of the recurrence property, there exist arbitrarily large time when \( Z \) is arbitrarily close to zero. By considering an initial value problem for \( Z \) starting at such times, deterministic methods can then be used to show that \( Z \) tends to zero, and hence that \( X \) tends to zero. A similar method is employed in Theorem 7 to establish an upper bound on \( |X| \) when \( Y \) is bounded, but does not tend to zero. Establishing that solutions of (1.1) is unbounded, or obeys certain lower bounds, is generally achieved by writing a variation of constants formula for \( X \) in terms on \( Y \), and then using the known asymptotic behaviour of \( Y \) to force a contradiction.

Although many parts of the analysis in this paper apply to finite–dimensional equations, we do not pursue this question here. The major reason for doing has
already been mentioned above. An important ingredient in establishing the asymptotic stability in the case when $\sigma \notin L^2(0, \infty)$ is the fact that process $M$ defined by $M(t) = \int_0^t \sigma(s) \, dB(s)$ can be considered as a one-dimensional Brownian motion on $[0, \infty)$, and therefore returns to the origin infinitely often. For the one-dimensional SDE, this causes the solution to return to an arbitrarily small neighbourhood of the equilibrium infinitely often, and once other stochastic terms fade, ensures convergence to the equilibrium.

Once we turn to consider higher dimensional equations, the corresponding stochastic process $M(t) = \int_0^t \sigma(s) \, dB(s)$ in higher dimensions may start to inherit some properties of finite-dimensional Brownian motion, for dimension greater than or equal to three. However, this means that $M$ can be transient, obeying

$$\lim_{t \to \infty} |M(t)| = +\infty, \text{ a.s.}$$

In this situation, therefore, one can no longer use the scalar method of proof to show that the solution of the finite dimensional SDE returns to an arbitrarily small neighbourhood of the equilibrium infinitely often. At this moment, it is unclear to the authors whether this is merely a technical problem, or if it presages different asymptotic behaviour, with solutions losing stability more readily than in the scalar case.

Other interesting questions which can be attacked by means of the methods in this paper include an analysis of local stability, where there are a finite number of equilibria of the underlying deterministic dynamical system (1.4). Some work in this direction has been conducted in a discrete-time setting in [1]. Numerical methods under monotonicity methods have been studied in [3]. We expect that the sharper information on the asymptotic behaviour of the linear SDE will lead to improved results for the corresponding nonlinear equations.

Section 2 deals with preliminary results, including the proof that solutions of (1.1) exist. Results for an auxiliary affine SDE, proven in [3], are recapitulated in Section 3, along with some new results for the stability of affine equations. Section 4 considers general results, including the classification of the almost sure behaviour of solutions under the additional assumption (1.9) on $f$. Section 5 considers the characterisation of asymptotic stability using only the assumption (1.2). Proofs of many results are deferred to the end of the paper, and these proofs are presented in Sections 6, 7, 8, 9 and 10.

2. Preliminaries

2.1. Notation. In advance of stating and discussing our main results, we introduce some standard notation. We denote the maximum of the real numbers $x$ and $y$ by $x \vee y$ and the minimum of $x$ and $y$ by $x \wedge y$. Let $C(I; J)$ denote the space of continuous functions $f : I \to J$ where $I$ and $J$ are intervals contained in $\mathbb{R}$. We denote by $L^1(0, \infty)$ the space of Lebesgue integrable functions $f : [0, \infty) \to \mathbb{R}$ such that $\int_0^\infty |f(s)| \, ds < +\infty$.

2.2. Remarks on existence and uniqueness of solutions of (1.1). There is an extensive theory regarding the existence and uniqueness of solutions of stochastic differential equations under a variety of regularity conditions on the drift and diffusion coefficients. Perhaps the most commonly quoted conditions which ensure the existence of a strong local solution are the Lipschitz continuity of the drift and diffusion coefficients. However, in this paper, we would like to establish our asymptotic results under weaker hypotheses on $f$. We do not concern ourselves greatly with relaxing conditions on $\sigma$, because $\sigma$ being continuous proves sufficient to ensure the existence of solutions in many cases.
The existence of a unique solution of
\[ dX(t) = f(X(t)) \, dt + \sigma(t, X(t)) \, dB(t) \quad (2.1) \]
can be asserted in the case when \( |\sigma(t, x)| \geq c > 0 \) for some \( c > 0 \) for all \((t, x)\) and \( f \) being bounded, so no continuity assumption is required on \( f \). However, assuming such a lower bound on \( \sigma \) would not natural in the context of this paper: for asymptotic stability results, we would typically require that \( \lim \inf_{t \to \infty} \sigma^2(t) = 0 \). Moreover, \( f \) being bounded excludes the important category of strongly mean-reverting functions \( f \) that have been investigated for this stability problem in [9] and [5].

One of the aims of this paper and of [5] is to relax monotonicity assumptions on \( \sigma \) which are required in [9]. Therefore, although we are often interested in functions \( \sigma \) which tend to zero in some sense, we do not want to exclude the cases when \( \sigma(t) = 0 \) for all \( t \) in a given interval (or indeed union of intervals). Our analysis will show that in these cases, the behaviour of \( \sigma \) on the intervals where it is nontrivial can give rise to solutions of (1.1) obeying (1.6) or (1.11). However, on those time intervals \( I \) for which \( \sigma \) is zero, the process \( X \) obeys the differential equation
\[ X'(t) = -f(X(t)), \quad t \in I \]
where \( X(\inf I) \) is a random variable. On such an interval, it is conceivable that a lack of regularity in \( f \) could give rise to multiple solutions of the ordinary equation (and hence the SDE (1.1)), so our most general existence results which make assertions about the existence of solutions (but say nothing about unicity of solutions), and which use the weakest hypotheses on \( f \) that we impose in this work, do not appear to be especially conservative.

For these reasons, we prove that there is a continuous and adapted process which obeys (1.1) by using very elementary methods, rather than by appealing to a result from the substantial body of sophisticated theory concerning the existence of solutions of (2.1). Of course, these methods are of very limited utility in establishing existence and uniqueness for more general equations of the form (2.1): our method of proof works because the diffusion coefficient is independent of the state. In fact, our method of proof gives a weaker conclusion for the existence of solutions of (2.1) in the case when \( f \) is bounded and \( \sigma(t, x) = \sigma(t) \) for all \((t, x)\) and \( |\sigma(t)| \geq c > 0 \). Our result states that there is a solution of (1.1) (or (2.1)) while existing results guarantee the existence of a unique solution under the assumption that \( f \) be continuous. Despite these general limitations, however, our proof does ensure existence of solutions for all the problems that are of concern in this paper, while existing results cannot always be applied without the imposition of additional hypotheses.

When \( f \) obeys (1.2) and \( \sigma \) obeys (1.3), we now demonstrate that there exists a continuous and adapted process \( X \) which satisfies (1.1). The existence of a local solution is ensured by the continuity of \( f \) and \( \sigma \), while the fact that any such solution is well-defined for all time follows from the mean-reverting condition \( xf(x) > 0 \) for \( x \neq 0 \) which is part of (1.2). In the paper, the spirit of our approach is to show that any solution of (1.1) has the stated asymptotic properties, even though multiple solutions exist, without paying particular concern as to whether solutions are unique.

**Proposition 1.** Suppose that \( f \) obeys (1.2) and \( \sigma \) obeys (1.3). Then there exists a continuous adapted process \( X \) which obeys (1.1) on \([0, \infty)\), a.s.

The proof is postponed to Section 6. In order to ensure that solutions of (1.1) are unique, it is often necessary to impose additional regularity properties on \( f \). One common and mild assumption which ensures uniqueness is that
\[ f \text{ is locally Lipschitz continuous.} \quad (2.2) \]
See e.g., [12]. Another assumption which guarantees the uniqueness of the solution is that the drift coefficient $-f$ obeys a one-sided Lipschitz condition. More precisely, imposing such an assumption on $f$ implies

There exists $K \in \mathbb{R}$ such that $(f(x) - f(y))(x - y) \geq -K|x - y|^2$ for all $x, y \in \mathbb{R}$.  

(2.3)

It is to be noted that if $f$ is non-decreasing, it obeys (2.3), because the righthand side is non-negative, and we can choose $K = 0$. Since non-decreasing functions do not have to be Lipschitz continuous, we see that in general (2.3) does not imply (2.2), so these additional assumptions can be used to cover different situations.

**Proposition 2.** Suppose that $f$ obeys (1.2) and (2.3) and that $\sigma$ obeys (1.3). Then there exists a unique continuous adapted process $X$ which obeys (1.1) on $[0, \infty)$ a.s.

Again the proof is deferred to Section 6.

In the proof of Proposition 1, and elsewhere throughout the paper, it is helpful to introduce the following processes and notation. Consider the affine stochastic differential equation

$$dY(t) = -Y(t)\,dt + \sigma(t)\,dB(t), \quad t \geq 0; \quad Y(0) = 0.$$  

(2.4)

Since $\sigma$ is continuous, there is a unique continuous adapted process which obeys (2.4), and we identify the a.s. event $\Omega_Y$ on which this solution is defined:

$$\Omega_Y = \{ \omega \in \Omega : \text{there is a unique continuous adapted process } Y \text{ for which the realisation } Y(\cdot, \omega) \text{ obeys (2.4)} \}.$$  

(2.5)

It is also helpful throughout the paper to identify the event $\Omega_X$ on which the continuous adapted process $X$ obeys (1.1), so we therefore define

$$\Omega_X = \{ \omega \in \Omega : \text{the continuous adapted process } X \text{ is such that the realisation } X(\cdot, \omega) \text{ obeys (1.1)} \}.$$  

(2.6)

By virtue of Proposition 1, $\Omega_X$ is an almost sure event.

2.3. Preliminary asymptotic results. We first consider hypotheses on the data i.e., on $\sigma$ under which any solution $X$ of (1.1) obeys (1.6). We note that when $\sigma \in L^2(0, \infty)$, we have $X$ obeys (1.6). However, we cannot apply directly the semimartingale convergence theorem of Lipster–Shiryaev (see e.g., [11] Theorem 7, p.139) to the non-negative semimartingale $X^2$, because it is not guaranteed that $\mathbb{E}[X^2(t)] < +\infty$ for all $t \geq 0$. The proof of the following theorem, which is deferred to the next section, uses the ideas of [11] Theorem 7, p.139] heavily, however.

**Theorem 1.** Suppose that $f$ satisfies (1.2). Suppose that $\sigma$ obeys (1.3) and $\sigma \in L^2(0, \infty)$. If $X$ is any solution of (1.1), then $X$ obeys (1.6).

The proof is relegated to Section 7.1. Our next result shows that if, on the contrary, $\sigma \not\in L^2(0, \infty)$, we can only guarantee that $X$ visits a neighbourhood of the equilibrium infinitely often.

**Theorem 2.** Suppose that $f$ obeys (1.2), and that $\sigma$ obeys (1.3) and $\sigma \not\in L^2(0, \infty)$. Then any solution $X$ of (1.1) obeys $\liminf_{t \to \infty} |X(t)| = 0$ a.s.

Again the proof is postponed to Section 7.1.
3. Linear Equation

We start by recalling results concerning the asymptotic behaviour of the related affine stochastic differential equation (2.4). These were presented in Appleby, Cheng, and Rodkina [2]. There is a unique continuous adapted processes which obeys (2.4). We note that

\[ Y(t) = e^{-t} \int_0^t e^{s\sigma(s)} dB(s), \quad t \geq 0. \]  

(3.1)

Let \( \Phi : \mathbb{R} \to [0, 1] \) be the distribution function of a standard normal random variable, so that

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad x \in \mathbb{R}. \]  

(3.2)

We interpret \( \Phi(-\infty) = 0 \) and \( \Phi(\infty) = 1 \). Define the sequence \( \theta : \mathbb{N} \to [0, \infty) \) such that

\[ \theta^2(n) = \int_n^{n+1} \sigma^2(s) ds. \]  

(3.3)

Let \( \epsilon > 0 \) and consider the sum

\[ S(\epsilon) = \sum_{n=0}^{\infty} \left\{ 1 - \Phi\left( \frac{\epsilon}{\theta(n)} \right) \right\}. \]  

(3.4)

This summation is difficult to evaluate directly, because \( \Phi \) is not known in closed form. However, it can be shown that \( S(\epsilon) \) is finite or infinite according as to whether the sum

\[ S'(\epsilon) := \sum_{n=0}^{\infty} \theta(n) \exp\left( -\frac{\epsilon^2}{2 \theta^2(n)} \right) \]  

(3.5)

is finite or infinite, where we interpret the summand to be zero in the case where \( \theta(n) = 0 \).

**Lemma 1.** \( S(\epsilon) \) given by (3.4) is finite if and only if \( S'(\epsilon) \) given by (3.5) is finite.

**Proof.** We note by e.g., [10, Problem 2.9.22],

\[ \lim_{x \to \infty} \frac{1 - \Phi(x)}{x \cdot e^{-x^2/2}} = \frac{1}{\sqrt{2\pi}}. \]  

(3.6)

If \( S(\epsilon) \) is finite, then \( 1 - \Phi(\epsilon/\theta(n)) \to 0 \) as \( n \to \infty \). This implies \( \epsilon/\theta(n) \to \infty \) as \( n \to \infty \). Therefore by (3.6), we have

\[ \lim_{n \to \infty} \frac{1 - \Phi(\epsilon/\theta(n))}{\theta(n)/\epsilon \cdot \exp(-\epsilon^2/(2\theta^2(n)))} = \frac{1}{\sqrt{2\pi}}. \]  

(3.7)

Since \( (1 - \Phi(\epsilon/\theta(n)))_{n \geq 1} \) is summable, it therefore follows that the sequence

\( (\theta(n)/\epsilon \cdot \exp(-\epsilon^2/(2\theta^2(n))))_{n \geq 1} \)

is summable, so \( S'(\epsilon) \) is finite, by definition.

On the other hand, if \( S'(\epsilon) \) is finite, and we define \( \phi : [0, \infty) \to \mathbb{R} \) by

\[ \phi(x) = \begin{cases} 
  x \exp(-1/(2x^2)), & x > 0, \\
  0, & x = 0,
\end{cases} \]

we have that \( (\phi(\theta(n)/\epsilon))_{n \geq 1} \) is summable. Therefore \( \phi(\theta(n)/\epsilon) \to 0 \) as \( n \to \infty \). Then, as \( \phi \) is continuous and increasing on \( [0, \infty) \), we have that \( \theta(n)/\epsilon \to 0 \) as \( n \to \infty \), or \( \epsilon/\theta(n) \to \infty \) as \( n \to \infty \). Therefore (3.7) holds, and thus \( (1 - \Phi(\epsilon/\theta(n)))_{n \geq 1} \) is summable, which implies that \( S(\epsilon) \) is finite, as required. \( \square \)
Clearly, there are three possibilities; either (A) $S'(\epsilon)$ is finite for all $\epsilon > 0$; (B) $S'(\epsilon)$ is infinite for all $\epsilon > 0$ or (C) $S'(\epsilon)$ is finite for some $\epsilon > 0$ and infinite for others. In the last case, we notice that as $\Phi$ is increasing, we have that $\epsilon \mapsto S'(\epsilon)$ is non-increasing, there must exist an $\epsilon' > 0$ such that $S'(\epsilon)$ is finite for all $\epsilon > \epsilon'$ and $S'(\epsilon) = +\infty$ for all $\epsilon < \epsilon'$.

By virtue of this observation, it can be seen that the following result characterises the asymptotic behaviour of $Y$. Roughly speaking, if $S'(\epsilon)$ is always finite, $Y$ tends to zero with probability one; if $S'(\epsilon)$ can be finite or infinite, it fluctuates asymptotically within interval of the form $[-c, c]$, while if $S'(\epsilon)$ is always infinite, $Y$ is recurrent on $\mathbb{R}$.

**Theorem 3.** Suppose that $\sigma$ obeys (1.3) and that $Y$ is the unique continuous adapted process which obeys (2.4). Let $\theta$ be defined by (3.3) and $S'(\cdot)$ by (3.5).

(A) If $\theta$ is such that

$$S'(\epsilon) \text{ is finite for all } \epsilon > 0,$$

then

$$\lim_{t \to \infty} Y(t) = 0, \quad \text{a.s.}$$

(B) If $\theta$ is such that there exists $\epsilon' > 0$ such that

$$S'(\epsilon) \text{ is finite for all } \epsilon > \epsilon',$$

$$S'(\epsilon) = +\infty \text{ for all } \epsilon < \epsilon',$$

then the event $\Omega_1$ defined by

$$\Omega_1 := \{ \omega \in \Omega : Y(t, \omega) < +\infty \}.$$  \hspace{1cm} (3.11)

is almost sure and there exist deterministic $0 < Y \leq Y < +\infty$ defined by

$$Y := \inf_{\omega \in \Omega_1} \limsup_{t \to \infty} |Y(t, \omega)| > 0,$$

$$Y := \sup_{\omega \in \Omega_1} \limsup_{t \to \infty} |Y(t, \omega)| > 0.$$  \hspace{1cm} (3.12)

(C) If $\theta$ is such that

$$S'(\epsilon) = +\infty \text{ for all } \epsilon > 0,$$

then

$$\limsup_{t \to \infty} |Y(t)| = +\infty, \quad \liminf_{t \to \infty} |Y(t)| = 0, \quad \text{a.s.}$$

In Theorem 3 no monotonicity conditions are imposed on $\sigma$. The form of Theorem 3 is inspired by those of [9, Theorem 1] and [7, Theorem 6, Corollary 7].

**Remark 1.** The existence of deterministic bounds on $|Y|$ in (3.12) and (3.13) in part (B) was established as part of Theorem 1 in [2]. Moreover, it was established as part of the proof that explicit bounds on $\overline{Y}$ and $\underline{Y}$ can be given in terms of the critical value of $\epsilon = \epsilon'$ in (3.10). The estimates given by the analysis in [2] are

$$\underline{Y} \geq \underline{y} := \frac{e^{-1}}{1 + e^{-1}} \epsilon', \quad \overline{Y} \leq \left( \frac{1}{1 - e^{-1}} + e \right) \epsilon' =: \overline{y}.$$  \hspace{1cm} (3.16)

Hence we have $0.2689 \epsilon' \leq \limsup_{t \to \infty} |Y(t)| \leq 4.3003 \epsilon'$, a.s.

It remains an open question as to whether in general the explicit bounds $\overline{y}$ and $\underline{y}$ on $\overline{Y}$ and $\underline{Y}$ can be improved. In part of Theorem 4 in which case (B) holds, it can be shown by an independent argument that $\underline{y} = \overline{y} = \epsilon'$ and therefore that $\overline{Y} = \overline{Y} = \epsilon'$. 
Remark 2. If \( \sigma \) obeys (1.3) and \( \sigma \in L^2(0, \infty) \), and \( Y \) is the solution of (2.4), then \( Y \) obeys \( \lim_{t \to \infty} Y(t) = 0 \) a.s. by Theorem 1. Moreover, if \( \sigma \in L^2(0, \infty) \), then \( \sigma \) obeys (3.9). If \( \sigma \) obeys either (3.10) or (3.14), then \( \sigma \not\in L^2(0, \infty) \).

The condition that \( S'(\epsilon) \) is finite or infinite can be difficult to check. However, in the case when

There exists \( L \in [0, \infty] \) such that
\[
L = \lim_{t \to \infty} \sigma^2(t) \log t \tag{3.17}
\]
each of the conditions (3.8), (3.10) and (3.14) is possible according as to whether the limit \( L \) is zero, non-zero and finite, or infinite. In this case therefore, the asymptotic behaviour of any solution of (1.1) can be classified completely.

Proposition 3. Suppose that \( \sigma \in C([0, \infty); \mathbb{R}) \) obeys (3.17) and that \( S' \) is defined by (3.5).

\( \text{(A)} \) If \( L = 0 \), then \( S' \) obeys (3.8).
\( \text{(B)} \) If \( L \in (0, \infty) \), then \( S' \) obeys (3.10).
\( \text{(C)} \) If \( L = \infty \), then \( S' \) obeys (3.14).

Scrutiny of the proof reveals that we can replace the condition (3.17) with the weaker condition
\[
L = \lim_{n \to \infty} \theta^2(n) \log n \tag{3.18}
\]
and still obtain the same trichotomy in Proposition 3. The proof of Proposition 3 is postponed to Section 8.

The conditions of Theorem 3 can be quite difficult to check in practice. In [2], easily-checked sufficient conditions on \( \sigma \) for which \( Y \) is bounded, stable or unstable, are developed. These results are extended slightly here, and will also be used to analyse the nonlinear equation (1.1). For this reason, they are stated afresh here.

In the case when \( \sigma \in L^2(0, \infty) \) we have that \( Y \) tends to zero. Therefore, we confine attention to the case where \( \sigma \not\in L^2(0, \infty) \). In this case, we can define a number \( T > 0 \) such that \( \int_0^t e^{2s} \sigma^2(s) \, ds > e^t \) for \( t > T \) and so one can define a function \( \Sigma : [T, \infty) \to [0, \infty] \) by
\[
\Sigma(t) = \left( \int_0^t e^{-2(t-s)} \sigma^2(s) \, ds \right)^{1/2} (\log t)^{1/2}, \quad t \geq T. \tag{3.19}
\]

Our main result in this direction can now be stated. Apart from part (C) it appears in [2, Theorem 3.2].

Theorem 4. Suppose that \( \sigma \) obeys (1.3) and that \( Y \) is the unique continuous adapted process which obeys (2.4). Let \( \Sigma \) be given by (3.19).

\( \text{(A)} \) If \( \lim_{t \to \infty} \Sigma^2(t) = 0 \) then
\[
\lim_{t \to \infty} Y(t) = 0, \quad \text{a.s.} \tag{3.20}
\]
\( \text{(B)} \) If \( \liminf_{t \to \infty} \Sigma^2(t) = L < +\infty \) then
\[
\limsup_{t \to \infty} |Y(t)| \geq \sqrt{2L}, \quad \text{a.s.} \tag{3.21}
\]
\( \text{(C)} \) If \( \limsup_{t \to \infty} \Sigma^2(t) = L < +\infty \) then
\[
\limsup_{t \to \infty} |Y(t)| \leq \sqrt{2L}, \quad \text{a.s.} \tag{3.22}
\]
\( \text{(D)} \) If \( \lim_{t \to \infty} \Sigma^2(t) = L < +\infty \) then
\[
\limsup_{t \to \infty} |Y(t)| = \sqrt{2L}, \quad \text{a.s.} \tag{3.23}
\]
(E) If \( \lim_{t \to \infty} \Sigma^2(t) = +\infty \) then
\[
\limsup_{t \to \infty} |Y(t)| = +\infty, \quad \text{a.s.} \quad (3.24)
\]

The proof of part (C) uses the methods of [2, Theorem 3.2], so is not given. It is now clear that part (D) is merely a corollary of parts (B) and (C). Parts (A) and (E) may also be thought of as limiting cases of part (D) as \( L \to 0 \) and \( L \to \infty \), respectively. We note that when \( \sigma \) obeys (3.17), then \( \Sigma^2(t) \to L \) as \( t \to \infty \), so that in part (D), we have from the proof of part (B) of Proposition 3 that \( S' \) obeys (3.10) with \( \epsilon' = \sqrt{2L} \) and by (3.23), that \( Y = \sqrt{2L} = \epsilon' \) in (3.12) and (3.13). This strengthens the general estimates given on \( Y \) and \( Y' \) in (3.16).

Theorem 3 gives necessary and sufficient conditions in terms of the sequence \( \theta \) for \( Y \) to exhibit certain types of asymptotic behaviour, while Theorem 4 gives sufficient conditions in terms of the function \( \Sigma \). In the next result, we explore the relationship between \( \Sigma \) and \( \theta \), and the conditions in Theorems 3 and 4. One consequence of this analysis is to give simpler sufficient conditions equivalent to those in part (A) of Theorem 4 under which \( Y \) tends to zero.

Proposition 4. Suppose that \( \Sigma \) is given by (3.19), that \( \theta \) is given by (3.3), and that \( \Theta \) is given by
\[
\Theta^2(n) = \sum_{j=0}^{n-1} e^{-2(n-j)} \theta^2(j), \quad n \geq 1. \quad (3.25)
\]

(i) The following statements are equivalent:
(A) \( \lim_{t \to \infty} \Sigma^2(t) = 0 \);
(B) \( \lim_{n \to \infty} \Sigma^2(n) = 0 \);
(C) \( \lim_{n \to \infty} \theta^2(n) \log n = 0 \).
Moreover, all imply that \( S'(\epsilon) < +\infty \) for all \( \epsilon > 0 \).

(ii) The following statements are equivalent:
(A) \( \limsup_{t \to \infty} \Sigma^2(t) \in (0, \infty) \);
(B) \( \limsup_{n \to \infty} \Sigma^2(n) \in (0, \infty) \);
(C) \( \limsup_{n \to \infty} \theta^2(n) \log n \in (0, \infty) \).
Moreover, all imply that there exists \( \epsilon' > 0 \) such that \( S'(\epsilon) < +\infty \) for all \( \epsilon > \epsilon' \).

(iii) The following statements are equivalent:
(A) \( \liminf_{t \to \infty} \Sigma^2(t) \in (0, \infty) \);
(B) \( \liminf_{n \to \infty} \Sigma^2(n) \in (0, \infty) \);
(C) \( \liminf_{n \to \infty} \Theta^2(n) \log n \in (0, \infty) \).
Moreover, all imply that \( \liminf_{n \to \infty} \theta^2(n) \log n \in [0, \infty) \).

(iv) The following statements are equivalent:
(A) \( \lim_{t \to \infty} \Sigma^2(t) = +\infty \);
(B) \( \lim_{n \to \infty} \Sigma^2(n) = +\infty \);
(C) \( \lim_{n \to \infty} \Theta^2(n) \log n = +\infty \).
Moreover, all imply that \( \limsup_{n \to \infty} \theta^2(n) \log n = \infty \).

Once again, the proof is relegated to Section 8.

4. Nonlinear Equation

In this section we explore the asymptotic behaviour of the nonlinear differential equation (1.1). In the first part of this section, we establish a connection between the solution of (2.4) and solutions of (1.1). This enables us to state the main results of the paper, which appear, together with interpretation and examples, in the second part of this section.
4.1. Connection between the linear and nonlinear equation. In our first result, we show that knowledge of the pathwise asymptotic behaviour of \( Y(t) \) as \( t \to \infty \) enables us to infer a great deal about the asymptotic behaviour of \( X(t) \) as \( t \to \infty \). Indeed, we show in broad terms that \( X \) inherits the asymptotic behaviour exhibited by \( Y \), when \( f \) obeys (1.2).

**Proposition 5.** Suppose that \( f \) satisfies (1.2) and that \( \sigma \) obeys (1.3). Let \( X \) be any solution of (1.1), and \( Y \) the solution of (2.4), and suppose that the a.s. events \( \Omega_X \) and \( \Omega_Y \) are defined as in (2.4) and (2.3) respectively.

(A) Suppose that there is an a.s. event defined by 
\[
\{ \omega \in \Omega_Y : \lim_{t \to \infty} |Y(t, \omega)| = 0 \}.
\]
Then \( \lim_{t \to \infty} X(t) = 0 \) a.s.

(B) Suppose that the event \( \Omega_1 \) defined by (3.11) is almost sure. Then the event
\[
\Omega_2 = \Omega_1 \cap \Omega_X
\]
is almost sure, and there exists a positive and deterministic \( \overline{X} \) given by
\[
\overline{X} = \inf_{\omega \in \Omega_2} \limsup_{t \to \infty} |X(t, \omega)|.
\]

(C) Suppose that there is an a.s. event defined by
\[
\{ \omega \in \Omega_Y : \limsup_{t \to \infty} |Y(t, \omega)| = +\infty \}.
\]
Then \( \limsup_{t \to \infty} |X(t)| = +\infty \) a.s.

In the proof of part (B), we can even determine an explicit lower bound for \( \overline{X} \). If the event \( \Omega_1 \) is defined by (3.11), we may define as in (3.12) and (3.13) the deterministic numbers \( 0 < \underline{Y} \leq \overline{Y} < +\infty \). For any \( f \) obeying (1.2) it can be shown that there is function \( y \mapsto \underline{a}(y) = g(f, y) \) which, for \( y \geq 0 \), obeys
\[
2\overline{a} + \max_{|x| \leq \underline{a}} |f(x)| = y.
\]
This leads to the estimate
\[
\underline{X} \geq g(f, \underline{Y}),
\]
where \( \underline{Y} \) is given by (3.12). Moreover, as it transpires that \( g(f, \cdot) \) is an increasing function, by (3.16), we can estimate \( \underline{X} \) explicitly according to
\[
\underline{X} \geq g(f, y),
\]
where \( y \) is given explicitly by (3.16).

An interesting implication of part (C) is that an arbitrarily strong mean–reverting force (as measured by \( f \)) cannot keep solutions of (1.1) within bounded limits if the noise perturbation is so intense that a linear mean–reverting force cannot keep solutions bounded. Therefore, the system will run “out of control” (in the sense of becoming unbounded) however strongly the function \( f \) pushes it back towards the equilibrium state.

4.2. Main results. Due to Theorem 3, we can readily use Proposition 5 to characterise the asymptotic behaviour of solutions of (1.1).

**Theorem 5.** Suppose that \( \sigma \) obeys (1.3), \( f \) obeys (1.2) and that \( X \) is any continuous adapted process which obeys (1.1). Let \( \theta \) be defined by (3.3) and \( S'(\cdot) \) by (4.5).

(A) If \( \theta \) is such that (3.8) holds, then
\[
\lim_{t \to \infty} X(t) = 0, \quad \text{a.s.}
\]
(B) If \( \theta \) is such that (3.10) holds, then there exists an almost sure event \( \Omega_2 = \Omega_1 \cap \Omega_X \), and a deterministic \( \underline{X} > 0 \) defined by (1.2) such that

\[
\underline{X} = \inf_{\omega \in \Omega_2} \limsup_{t \to \infty} |X(t, \omega)| > 0.
\]

Moreover, \( \underline{X} \) obeys

\[
\underline{X} \geq \underline{g}(f, Y),
\]

where \( \underline{g}(f, \cdot) \) is the unique solution of (1.3), and \( \underline{Y} \) is defined by (3.12).

Furthermore,

\[
\liminf_{t \to \infty} |X(t)| = 0, \quad \text{a.s.}
\]

(C) If \( \theta \) is such that (3.8) holds, then

\[
\limsup_{t \to \infty} |X(t)| = +\infty, \quad \liminf_{t \to \infty} |X(t)| = 0, \quad \text{a.s.}
\]

Proof. If \( \theta \) is such that (3.8) holds, then from Theorem 3 we have \( \lim_{t \to \infty} Y(t) = 0 \), a.s. Taking this together with Proposition 5 part (A) holds. If \( \theta \) is such that (3.11) holds, or if \( \theta \) is such that (3.14) holds, then taken together with Theorem 3 and Proposition 5 we have that the first part (B) and of (C) is true. For the second part of (B) and (C), we recall that if (3.10) or (3.14) hold, Remark 2 implies that \( \sigma \notin L^2(0, \infty) \). In this case, we already know that \( \liminf_{t \to \infty} |X(t)| = 0 \), a.s. by Theorem 2.

The formula (4.3), which is established in the proof of part (B) of Proposition 5 relates the lower bound on the large fluctuations \( \underline{X} \) to the size of the diffusion coefficient \( \sigma \) and the nonlinearity in \( f \). Thus, we may view \( \underline{X} = \underline{g}(f, Y) = \underline{g}(f, \sigma) \), because \( \underline{Y} \) depends on \( \sigma \) but not on \( f \). It is clear that the larger the diffusion coefficient, the larger the value of \( \underline{Y} \). We now show for fixed \( f \) that \( \underline{X} \) is increasing and that \( \underline{g}(f, y) \to \infty \) as \( y \to \infty \). Moreover, we show for fixed \( y \) that \( \underline{g}(f_1, y) \geq \underline{g}(f_2, y) \) if

\[
|f_2(x)| \geq |f_1(x)|, \quad x \in \mathbb{R}.
\]

These ordering results seem to make intuitive sense, as we would expect weaker mean reversion and a larger diffusion coefficient to lead to larger fluctuations in \( X \).

**Proposition 6.** Suppose that \( f \) obeys (1.2). Let \( \underline{g} \) be the unique solution of (1.3).

Then

(i) \( y \mapsto \underline{g}(f, y) \) is increasing and \( \lim_{y \to \infty} \underline{g}(f, y) = +\infty \), \( \lim_{y \to 0^+} \underline{g}(f, y) = 0 \).

(ii) If \( f_1 \) and \( f_2 \) are functions that obey (1.2) and also satisfy (1.5), then

\[
\underline{g}(f_1, y) \geq \underline{g}(f_2, y).
\]

Proof. Define \( h_f : [0, \infty) \to [0, \infty) \) according to

\[
h_f(x) := \max \{|f(y)|, \quad x \geq 0. \quad (4.6)
\]

Then \( h_f \) is increasing and continuous, and obeys the limits \( \lim_{x \to \infty} h_f(x) = +\infty \) and \( \lim_{x \to 0^+} h_f(x) = 0 \). By (1.3), \( h_f(\underline{g}(f, y)) = y \). Therefore

\[
\underline{g}(f, y) = h_f^{-1}(y), \quad y \geq 0. \quad (4.7)
\]

Hence \( y \mapsto \underline{g}(f, y) \) is increasing. Finally, \( \lim_{y \to \infty} \underline{g}(f, y) = \infty \) and \( \lim_{y \to 0^+} \underline{g}(f, y) = \lim_{y \to 0^+} h_f^{-1}(y) = 0 \).

To prove part (ii), note by (1.5) that

\[
h_{f_1}(\underline{g}(f_1, y)) = y = h_{f_2}(\underline{g}(f_2, y)) = 2\underline{g}(f_2, y) + \max_{|u| \leq \underline{g}(f_2, y)} |f_2(u)|
\]

\[
\geq 2\underline{g}(f_2, y) + \max_{|u| \leq \underline{g}(f_2, y)} |f_1(u)| = h_{f_1}(\underline{g}(f_2, y)).
\]

Since \( h_{f_1} \) is an increasing function, we have \( \underline{g}(f_1, y) \geq \underline{g}(f_2, y) \) as required. \( \square \)
Just as the conditions of Theorem 3 can be quite difficult to check in practice for $Y$, the same is true for the conditions of Theorem 5 on $\theta$ for $X$. As in Theorem 4 and because of Proposition 6, we can supply easily checked sufficient conditions on $\sigma$ for which $X$ is bounded, stable or unstable.

In the case when $\sigma \in L^2(0,\infty)$ we have that $X$ tends to zero. Therefore, we make a further hypothesis on the statement of Proposition 5 shows that part (B) does not rule out the possibility of (1.1) do not tend to zero but are nonetheless bounded. In the case when (2.8) for which we have some definite, as before, the function $\Sigma : [T,\infty) \rightarrow [0,\infty)$ by (3.19).

**Theorem 6.** Suppose that $f$ obeys (2.2) and that $\sigma$ obeys (3.3). Let $X$ be any solution of (1.1). Let $\Sigma$ be given by (3.19).

(A) If $\lim_{t \to \infty} \Sigma^2(t) = 0$ then $\lim_{t \to \infty} X(t) = 0$ a.s.

(B) If there exists $L \in (0,\infty)$ such that $\liminf_{t \to \infty} \Sigma^2(t) = L$, then there exists an almost sure event $\Omega_2 = \Omega_1 \cap \Omega_X$, and a deterministic $\bar{X} > 0$ defined by (4.10) such that $\bar{X} = \inf_{\omega \in \Omega_2} \limsup_{t \to \infty} |X(t, \omega)| > 0$. Moreover, $\bar{X} \geq \varphi(f, \bar{Y})$, where $\varphi(f, \cdot)$ is the unique solution of (4.13), and $\bar{Y}$ is defined by (3.12).

(C) If $\lim_{t \to \infty} \Sigma^2(t) = +\infty$ then $\limsup_{t \to \infty} |X(t)| = +\infty$ a.s.

**Proof.** If $\lim_{t \to \infty} e^{-2t} \log t \int_0^t e^{2s} \sigma^2(s) ds = 0$ then $\lim_{t \to \infty} Y(t) = 0$ from Theorem 4. Combining this with Proposition 5, we get $\lim_{t \to \infty} X(t) = 0$ proving part (A). Similarly, parts (B) and (C) follow from parts (B) and (E) of Theorem 3 and Proposition 5.

We finish this Section by giving a sufficient condition on $f$ for which solutions of (1.1) do not tend to zero but are nonetheless bounded. In the case when $\sigma$ is such that either parts (A) or (C) apply, we have unambiguous information about the asymptotic behaviour of solutions: either almost all sample paths tend to zero, or almost all sample paths exhibit unbounded fluctuations. However, scrutiny of the statement of Proposition 5 shows that part (B) does not rule out the possibility that $\limsup_{t \to \infty} |X(t)| = +\infty$ with positive probability (or even almost surely).

We make a further hypothesis on $f$, under which this is impossible, and $X$ is forced to be bounded. The hypothesis is

$$\lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty. \quad (4.8)$$

An estimate on the lower bound $\bar{X}$ in case (B) is given in (4.3), which is found as part of the proof of Proposition 5. $\bar{X}$ is given in terms of $f$ and $\sigma$. Similarly, an estimate can be determined for the upper bound. Towards this end, we introduce functions which are a type of generalised inverse of $f$ by defining the functions $f^-$ and $f^+$ by

$$f^+(x) = \sup \{z : f(z) = x\}, \quad x \geq 0, \quad (4.9)$$

$$f^-(x) = \inf \{z : f(z) = x\}, \quad x \leq 0. \quad (4.10)$$

These functions are well-defined if $f$ obeys (2.2) and (3.3). We notice also that if $f$ is increasing, then $f^\pm$ are exactly the inverse of $f$.

We may therefore define for any $f$ the function $y \mapsto \varphi(f, y)$ by

$$\varphi(f, y) = 2y + \max (f^+(y), -f^-(-y)), \quad y \geq 0. \quad (4.11)$$

The main conclusion of the following theorem is that an explicit upper bound can be found for $\limsup_{t \to \infty} |X(t)|$. In fact, it can be shown that if $\bar{Y}$ obeys (3.13), then

$$\limsup_{t \to \infty} |X(t, \omega)| \leq \varphi(f, \bar{Y}), \quad \text{for each } \omega \in \Omega_2, \quad (4.12)$$

where $\Omega_2$ is given by (4.1).
We are finally in a position to state the main result of this section.

**Theorem 7.** Suppose that \( \sigma \) obeys (3.3), \( f \) obeys (1.2) and (1.8). Suppose that \( X \) is any continuous adapted process which obeys (1.1). Let \( \theta \) be defined by (3.3) and \( S'() \) by (3.3).

(A) If \( \theta \) is such that (3.3) holds, then \( \lim_{t \to \infty} X(t) = 0 \), a.s.

(B) If \( \theta \) is such that (3.10) holds, then there exists an almost sure event \( \Omega_2 = \Omega_1 \cap \Omega_X \) where \( \Omega_1 \) defined in (3.11), and deterministic \( 0 < \underline{X} \leq X < +\infty \) such that

\[
\underline{X} = \inf_{\omega \in \Omega_2} \limsup_{t \to \infty} |X(t, \omega)|, \quad \bar{X} = \sup_{\omega \in \Omega_2} \limsup_{t \to \infty} |X(t, \omega)|. \tag{4.13}
\]

Moreover,

\[
\underline{X} \geq \underline{z}(f, \bar{Y}),
\]

where \( \underline{z}(f, \cdot) \) is the unique solution of (1.3), and \( \bar{Y} \) is defined by (3.12), and

\[
\bar{X} \leq \bar{\pi}(f, \bar{Y}),
\]

where \( \bar{\pi}(f, \cdot) \) is defined by (1.11) and \( \bar{Y} \) is defined by (3.13). Furthermore,

\[
\liminf_{t \to \infty} |X(t)| = 0, \quad \text{a.s.}
\]

(C) If \( \theta \) is such that (3.14) holds, then

\[
\limsup_{t \to \infty} |X(t)| = +\infty, \quad \liminf_{t \to \infty} |X(t)| = 0, \quad \text{a.s.}
\]

We prove part (B) only, as the results of parts (A) and (C) follow from Theorem 6. Therefore, under the additional hypothesis that \( f \) obeys (1.8), it follows from Theorem 5 and 7 that either (i) solutions tend to zero with probability one, when \( \sigma \) obeys (3.8), (ii) solutions fluctuate within finite bounds with probability one, when \( \sigma \) obeys (3.10) or (iii) solutions fluctuate unboundedly with probability one, when \( \sigma \) obeys (3.14). In the second case, part (B) of Theorem 7 can be restated as

\[
\underline{z}(f, \bar{Y}) \leq \limsup_{t \to \infty} |X(t)| \leq \bar{\pi}(f, \bar{Y}), \quad \text{a.s.,}
\]

and moreover we have weaker but explicit estimates on these deterministic bounds given by

\[
0 < \underline{z}(f, \bar{Y}) \leq \limsup_{t \to \infty} |X(t)| \leq \bar{\pi}(f, \bar{Y}) < +\infty, \quad \text{a.s.,}
\]

where \( \bar{Y} \) are given by (3.10).

It is interesting to determine the effect of weaker mean reversion and an increasing diffusion coefficient on the upper bound of the large deviations of \( X \), given by \( \bar{\pi}(f, \bar{Y}) \), just as we did for the lower bound on the size of the largest fluctuations in Proposition 6 given by \( \underline{z}(f, \bar{Y}) \). As before, it can be shown that weaker mean reversion and increasing diffusion coefficients increase the bound \( \bar{\pi} \). Also, if the effect of the diffusion coefficient alone is negligible (so that \( \bar{Y} \to 0 \)), or unboundedly large (so that \( \bar{Y} \to \infty \)), we see that cases (A) and (C) in Theorem 7 can be viewed as limiting cases of the asymptotic behaviour described in case (B). These properties of the bounds are established in the following result.

**Proposition 7.** Suppose that \( f \) obeys (1.2) and (1.8). Let \( \bar{\pi} \) be given by (1.11).

Then

(i) \( y \mapsto \bar{\pi}(f, y) \) is increasing and \( \lim_{y \to \infty} \bar{\pi}(f, y) = +\infty \), \( \lim_{y \to 0} \bar{\pi}(f, y) = 0 \).

(ii) If \( f_1 \) and \( f_2 \) are functions that obey (1.2) and (1.8), and also satisfy (1.9), then \( \bar{\pi}(f_1, y) \geq \bar{\pi}(f_2, y) \).

The proof is relegated to the final section. We finish the section with an example which shows how estimates of \( \underline{X} \) and \( \bar{X} \) can be obtained in practice.
Example 8. We see how these estimates on the fluctuations behave for a specific class of examples. Suppose that \( f(x) = x^n \) where \( n \) is an odd integer and that \( \sigma^2(t) \log t \to L \in (0, \infty) \) as \( t \to \infty \). Then by Theorem 4 it follows that
\[
\limsup_{t \to \infty} |Y(t)| = \sqrt{2L} \quad \text{a.s.}
\]
so we have \( Y = \sqrt{2L} \). Since \( f \) is increasing we have for \( x \geq 0 \) that
\[
f^+(x) = f^{-1}(x) = x^{1/n}, \quad f^-(x) = f^{-1}(-x) = -x^{1/n}, \quad \max_{|y| \leq x} |f(y)| = x^n
\]
so that \( \varphi(L) = \varphi(f, Y) \) and \( \varpi(L) = \varpi(f, Y) \) satisfy
\[
2\varphi + \varphi^n = \sqrt{2L}, \quad \varpi = 2\sqrt{2L} + (\sqrt{2L})^{1/n}.
\]
From this, we readily see that
\[
\lim_{L \to 0^+} \varphi(L) = \frac{1}{2}, \quad \lim_{L \to 0^+} \frac{\varpi(L)}{(\sqrt{2L})^{1/n}} = 1,
\]
and that
\[
\lim_{L \to \infty} \frac{\varphi(L)}{(\sqrt{2L})^{1/n}} = 1, \quad \lim_{L \to \infty} \frac{\varpi(L)}{2(\sqrt{2L})} = 1.
\]
Notice that \( \lim_{L \to 0^+} \varphi(L) = 0 \) and \( \lim_{L \to \infty} \varphi(L) = \infty \).

It is clear that these asymptotic bounds are widely spaced, because
\[
\lim_{L \to 0^+} \frac{\varpi(L)}{\varphi(L)} = \lim_{L \to \infty} \frac{\varpi(L)}{\varphi(L)} = +\infty.
\]
It would be an interesting question to determine whether either of these bounds is satisfactory, but we do not pursue this here. We suspect that the upper bound \( \varpi(L) \) as \( L \to \infty \) is very conservative, however, as it does not take into account the strong mean reversion of \( f \).

5. Asymptotic Stability

It should be remarked that one consequence of Theorem 5 is that sample paths of \( X \) tend to zero with non–zero probability if and only if \( \theta \) obeys (3.5), in which case almost all sample paths tend to zero. Therefore, we have the following immediate corollary of Theorem 5.

Theorem 9. Suppose \( f \) obeys (1.2) and that \( \sigma \) obeys (1.4). Let \( X \) be any solution of (1.1). Let \( \theta \) be defined by (3.3) and let \( \Phi \) be given by (3.2). Then the following are equivalent:

- (A) \[
\sum_{n=1}^{\infty} \theta(n) \exp \left( \frac{1}{2} \frac{\epsilon^2}{\theta^2(n)} \right) < +\infty, \quad \text{for every } \epsilon > 0.
\]
- (B) \[
\lim_{t \to \infty} X(t, \xi) = 0 \quad \text{with positive probability for some } \xi \in \mathbb{R}.
\]
- (C) \[
\lim_{t \to \infty} X(t, \xi) = 0 \quad \text{a.s. for each } \xi \in \mathbb{R}.
\]

Part (A) refines part of [5 Proposition 3.3]. Also, if \( X(t) \to 0 \) as \( t \to \infty \), it does so a.s., and so \( \theta \) obeys (3.5). Therefore, \( Y(t) \to 0 \) as \( t \to \infty \). This forces \( \liminf_{t \to \infty} \Sigma^2_2(t) = 0 \), for else we would have \( \limsup_{t \to \infty} |Y(t)| > 0 \) a.s., as essentially pointed out by [5 Proposition 3.3].

It should also be noted that no monotonicity conditions are required on \( \sigma \) in order for this result to hold, and that a.s. global stability is independent of the form of \( f \). The conditions and form of Theorem 5 and 9 are inspired by those of [9 Theorem 1] and by [7 Theorem 6, Corollary 7].

An interesting fact of Theorem 9 is that it is unnecessary for \( \sigma(t) \to 0 \) as \( t \to \infty \) in order for \( X \) to obey (1.5). In fact, we can even have \( \limsup_{t \to \infty} |\sigma(t)|^2 = \infty \) and still have \( X(t) \to 0 \) as \( t \to \infty \) a.s. Some examples are supplied in [5].
Let \((\mathbf{1.8})\) implies \(\lim_{t \to \infty} \Sigma(t) = 0\), that \((\mathbf{1.10})\) implies \(\lim_{t \to \infty} \Sigma(t) = \infty\), and finally that \(\lim \inf_{t \to \infty} \sigma^2(t) \log t > 0\) implies that \(\lim \inf_{t \to \infty} \Sigma(t) > 0\). The next result is therefore an easy corollary of Theorem 4 or of Proposition 5 and Proposition 9.

**Theorem 10.** Suppose that \(f\) satisfies \((\mathbf{1.2})\), and that \(\sigma\) obeys \((\mathbf{1.3})\). Let \(X\) be any solution of \((\mathbf{1.4})\).

(i) If \(\sigma\) obeys \(\lim_{t \to \infty} \sigma^2(t) \log t = 0\), then \(X\) obeys \((\mathbf{1.6})\).

(ii) If \(\sigma\) obeys \(\lim \inf_{t \to \infty} \sigma^2(t) \log t \in (0, \infty)\), then \(\mathbb{P}[\lim_{t \to \infty} X(t) = 0] = 0\) and \(\lim \inf_{t \to \infty} |X(t)| = 0\), a.s.

(iii) If \(\sigma\) obeys \(\lim_{t \to \infty} \sigma^2(t) \log t = \infty\), then

\[
\limsup_{t \to \infty} |X(t)| = \infty, \quad \liminf_{t \to \infty} |X(t)| = 0, \quad \text{a.s.}
\]

Part (i) is part of \([5\text{ Proposition 3.3(a)}]\). Part (iii) is \([5\text{ Lemma 3.7}]\). In \([9]\), Chan and Williams have proven in the case when \(t \to \sigma^2(t)\) is decreasing, that \(Y\) obeys \((\mathbf{3.20})\) if and only if \(\sigma\) obeys \((\mathbf{1.3})\). Our final result is a corollary of this observation and Theorem 9, and also of \([5\text{ Theorem 3.8}]\). A stronger result than Theorem 10 can be stated if the sequence \(\sigma\) in \((\mathbf{1.3})\) is decreasing: in this case, \(\lim_{n \to \infty} \theta^2(n) \log n = 0\) is equivalent to \((\mathbf{1.6})\).

**Theorem 11.** Suppose that \(f\) satisfies \((\mathbf{1.2})\). Suppose that \(\sigma\) obeys \((\mathbf{1.3})\) and \(t \to \sigma^2(t)\) is decreasing. Let \(X\) be any solution of \((\mathbf{1.4})\). Then the following are equivalent:

(A) \(\sigma\) obeys \(\lim_{t \to \infty} \sigma^2(t) \log t = 0\);

(B) \(\lim_{t \to \infty} X(t, \xi) = 0\) a.s. for each \(\xi \in \mathbb{R}\).

The remark preceding this result points to the importance of the condition \(\theta(n)^2 \log n \to 0\) as \(n \to \infty\), as indeed does Proposition 4 part (a). We now supply an example in which \(\theta(n)^2 \log n \to 0\) as \(n \to \infty\), but \(t \to \sigma^2(t)\) has “spikes” which prevents it from satisfying the condition \(\lim_{t \to \infty} \sigma^2(t) \log t = 0\).

**Example 12.** Consider the decomposition of \([0, \infty)\) into a union of disjoint intervals

\([0, \infty) = \bigcup_{k=0}^\infty \{I_k \cup J_k \cup K_k\}\),

where \(\epsilon_k \in (0, 1/2)\) for each \(k \geq 0\) and

\(I_k = [k, k + \epsilon_k], \quad J_k = (k + \epsilon_k, k + 1 - \epsilon_k), \quad K_k = [k + 1 - \epsilon_k, k + 1], \quad k \in \mathbb{N}\).

Let \((l_k)_{k \geq 0}\) and \((\epsilon_k)_{k \geq 0}\) be positive sequences and consider the function \(\sigma: [0, \infty) \to [0, \infty)\) defined by

\[
\sigma^2(t) = \begin{cases} 
  l_k - \frac{l_k - q_k}{\epsilon_k} (t - k), & t \in [k, k + \epsilon_k], \\
  q_k, & t \in (k + \epsilon_k, k + 1 - \epsilon_k), \\
  l_{k+1} + \frac{l_k - q_k}{\epsilon_k} (t - k - 1), & t \in [k + 1 - \epsilon_k, k + 1).
\end{cases}
\]

Then \(t \mapsto \sigma^2(t)\) is continuous. If \(\theta\) is defined by \((\mathbf{3.3})\), then

\[
\theta^2(k) = q_k (1 - \epsilon_k) + \frac{1}{2} l_{k+1} + l_k.
\]

Notice also that \(\sigma^2(k) = l_k\). Suppose \(q_k \log k \to 0, \quad \epsilon_k (l_k + l_{k+1}) \log k \to 0\) but \(\lim \sup_{k \to \infty} l_k \log k > 0\). Then \(\theta^2(k) \log k \to 0\) as \(k \to \infty\), but

\[
\limsup_{k \to \infty} \sigma^2(t) \log t \geq \limsup_{k \to \infty} \sigma^2(k) \log k = \limsup_{k \to \infty} l_k \log k > 0.
\]

Concrete examples of sequences for which these conditions hold include

\[q_k = \frac{1}{k+1}, \quad \epsilon_k = \frac{1}{k+3}, \quad l_k = 1,\]
or
\[ q_k = \frac{1}{k+1}, \quad \epsilon_k = \frac{1}{(k+3)^2}, \quad l_k = k. \]

6. Proof of Existence Results from Section 2.2

6.1. Proof of Proposition 1. Consider the affine stochastic differential equation (2.4). Since \( \sigma \) is continuous, there is a unique continuous adapted process which obeys (2.4). Let \( \Omega_Y \) be the a.s. event defined by (2.5) on which \( Y \) is defined. Now, for each \( \omega \in \Omega_Y \), define the function
\[ \phi(t, x, \omega) = -f(x + Y(t, \omega)) + Y(t, \omega), \quad t \geq 0. \]

Since \( f \) is continuous, and the sample path \( t \mapsto Y(t, \omega) \) is continuous, \( (t, x) \mapsto \phi(t, x, \omega) \) is continuous. Consider now the differential equation
\[ z'(t, \omega) = \phi(t, z(t, \omega), \omega), \quad t > 0; \quad z(0, \omega) = \xi. \]

By the continuity of \( \phi \) in both arguments, by the Peano existence theorem, there exists a continuous local solution \( t \mapsto z(t, \omega) \) for each \( \omega \in \Omega_Y \) and \( 0 \leq t < \tau_\omega \). Presently, it will be shown that \( \tau_\omega(\omega) = +\infty \) a.s. on \( \Omega_Y \).

Moreover, as \( Y \) is adapted to \( (\mathcal{F}_B(t))_{t \geq 0} \), \( z \) is also adapted to \( (\mathcal{F}_B(t))_{t \geq 0} \). Now consider the process \( X \) defined on \( \Omega_Y \) by \( X(t) = z(t) + Y(t) \) for \( t \in [0, \tau_\omega] \). By construction it is continuous and adapted. Furthermore, we have for \( t \in [0, \tau_\omega) \)
\[ X(t, \omega) = z(t, \omega) + Y(t, \omega) \]
\[ = \xi + \int_0^t \phi(s, z(s, \omega), \omega) \, ds + \int_0^t -Y(s, \omega) \, ds + \left( \int_0^t \sigma(s) \, dB(s) \right)(\omega) \]
\[ = \xi + \int_0^t \left\{ -f(z(s, \omega) + Y(s, \omega)) + Y(s, \omega) \right\} \, ds + \int_0^t -Y(s, \omega) \, ds \]
\[ + \left( \int_0^t \sigma(s) \, dB(s) \right)(\omega) \]
\[ = \xi + \int_0^t -f(X(s, \omega)) \, ds + \left( \int_0^t \sigma(s) \, dB(s) \right)(\omega) \]
\[ = \left( \xi + \int_0^t -f(X(s)) \, ds + \int_0^t \sigma(s) \, dB(s) \right)(\omega). \]

Hence \( X(\cdot, \omega) \) obeys (1.1) for each \( \omega \in \Omega_Y \) on the interval \([0, \infty)\). The proof that \( \tau_\omega \) is infinite a.s. was given in the Appendix of [5].

6.2. Proof of Proposition 2. The proof is inspired by an observation in e.g., [10]. Note first by Proposition 1 that the continuity of \( f \) together with (1.2) guarantees the existence of a continuous adapted process which obeys (1.1). Suppose therefore that \( X_1 \) and \( X_2 \) are any two solutions of (1.1). Then
\[ d(X_1(t) - X_2(t)) = (-f(X_1(t)) + f(X_2(t))) \, dt, \]
and by Itô’s rule we have that
\[ d(X_1(t) - X_2(t))^2 = -2(X_1(t) - X_2(t)) (f(X_1(t)) - f(X_2(t))) \, dt, \quad t \geq 0. \]
Since \( X_1(0) = X_2(0) = \xi \), we have
\[ (X_1(t) - X_2(t))^2 = -2 \int_0^t (X_1(s) - X_2(s)) (f(X_1(s)) - f(X_2(s))) \, ds, \quad t \geq 0. \]

Since \( f \) obeys (2.3), we have
\[ (X_1(t) - X_2(t))^2 \leq 2K \int_0^t (X_1(s) - X_2(s))^2 \, ds, \quad t \geq 0. \]
If $K \leq 0$, we can conclude automatically that $X_1(t) = X_2(t)$ for all $t \geq 0$ a.s., and that therefore the solution is unique. If $K > 0$, by applying Gronwall’s inequality to the non-negative continuous function $t \mapsto (X_1(t) - X_2(t))^2$, we conclude that $X_1(t) = X_2(t)$ for all $t \geq 0$ a.s., and once again we have uniqueness.

7. Proofs of Preliminary Results

7.1. Proof of Theorem 1

By Itô’s rule, we have

$$X^2(t) = \xi^2 - \int_0^t 2X(s)f(X(s))\,ds + \int_0^t \sigma^2(s)\,ds + \int_0^t 2X(s)\sigma(s)\,dB(s), \quad t \geq 0. \tag{7.1}$$

Since $xf(x) \geq 0$ for all $x \in \mathbb{R}$ and $\sigma \in L^2(0, \infty)$, we have

$$X^2(t) \leq \xi^2 + \int_0^\infty \sigma^2(s)\,ds + 2\int_0^t X(s)\sigma(s)\,dB(s), \quad t \geq 0.$$ 

Define $M$ to be the local martingale given by $M(t) = \int_0^t X(s)\sigma(s)\,dB(s)$ for $t \geq 0$. Suppose that there is an event $A = \{\omega : \lim_{t \to \infty} \langle M(t) \rangle = +\infty\}$ such that $\mathbb{P}[A] > 0$. Then a.s. on $A$ we have $\lim_{t \to \infty} M(t) = -\infty$, which implies that $\lim_{t \to \infty} X^2(t) = -\infty$ a.s. on $A$, which is absurd. Therefore $\lim_{t \to \infty} \langle M(t) \rangle < +\infty$ a.s., so it follows that $\lim_{t \to \infty} M(t) =: M(\infty)$ exists a.s. and is a.s. finite. Therefore we have that $t \mapsto |X(t)|$ is a.s. bounded. Since $X^2(t) \geq 0$, it follows from (7.1) that

$$\int_0^t 2X(s)f(X(s))\,ds = \xi^2 - X^2(t) + \int_0^t \sigma^2(s)\,ds + 2M(t) \leq \xi^2 + \int_0^\infty \sigma^2(s)\,ds + 2M(t).$$

Therefore, as $xf(x) \geq 0$ for all $x \in \mathbb{R}$ and $M(t) \to M(\infty)$ as $t \to \infty$ (where $M(\infty)$ is finite), we have that

$$\lim_{t \to \infty} \int_0^t X(s, \omega)f(X(s, \omega))\,ds =: I(\omega) \in (0, \infty)$$

Therefore, as $t \mapsto |X(t)|$ is a.s. bounded, and all the terms on the righthand side of (7.1) have finite limits as $t \to \infty$, it follows that there is $L = L(\omega) \in [0, \infty)$ such that $\lim_{t \to \infty} X^2(t, \omega) = L(\omega)$ for all $\omega$ in an a.s. event $A$, say. By continuity this means that there is an a.s. event $A = \{\omega : X(t, \omega)^2 \to L \in [0, \infty) \text{ as } t \to \infty\}$ such that $A = A_+ \cup A_- \cup A_0$ where

$$A_+ = \{\omega : X(t, \omega) \to \sqrt{L(\omega)} \in (0, \infty) \text{ as } t \to \infty\},$$

$$A_- = \{\omega : X(t, \omega) \to -\sqrt{L(\omega)} \in (-\infty, 0) \text{ as } t \to \infty\},$$

and $A_0 = \{\omega : X(t, \omega) \to 0 \text{ as } t \to \infty\}$. Suppose that $\omega \in A_+$. Then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s, \omega)f(X(s, \omega))\,ds = \sqrt{L(\omega)}f(\sqrt{L(\omega)}) > 0, \tag{7.2}$$

by continuity of $X$, $f$ and the fact that $xf(x) > 0$ for $x \neq 0$. Since the last two terms on the righthand side of (7.1) have finite limits as $t \to \infty$, (7.2) implies that for $\omega \in A_+$ that

$$0 \leq \lim_{t \to \infty} \frac{X^2(t, \omega)}{t} = -2\sqrt{L(\omega)}f(\sqrt{L(\omega)}) < 0,$$

a contradiction. Therefore $\mathbb{P}[A_+] = 0$. A similar argument yields $\mathbb{P}[A_-] = 0$. Since $\mathbb{P}[A] = 1$, we must have $\mathbb{P}[A_0] = 1$, as required.
7.2. Proof of Theorem 2 Let $A_1 = \{\omega : \liminf_{t \to \infty} X(t, \omega) > 0\}$ and suppose that $\mathbb{P}[A_1] > 0$. In particular, for $\omega \in A_1$, define $(0, \infty] \ni c(\omega) = \liminf_{t \to \infty} X(t, \omega)$ Then there exists $T_1(\omega) > 0$ such that $X(t, \omega) > 0$ for all $t > T_1(\omega)$. Hence for $t \geq T_1(\omega)$, we have

$$X(t) = X(0) - \int_0^{T_1} f(X(s)) \, ds - \int_{T_1}^t f(X(s)) \, ds + \int_0^t \sigma(s) \, dB(s)$$

Since $\sigma \notin L^2(0, \infty)$ it follows that $\liminf_{t \to \infty} \int_0^t \sigma(s) \, dB(s) = -\infty$ a.s. Therefore a.s. on $A_1$ we have

$$c(\omega) = \liminf_{t \to \infty} X(t, \omega) \leq X(0) - \int_0^{T_1} f(X(s)) \, ds + \liminf_{t \to \infty} \int_0^t \sigma(s) \, dB(s) = -\infty,$$

a contradiction. Hence $\mathbb{P}[A_1] = 0$, so $\liminf_{t \to \infty} X(t) \leq 0$ a.s. To prove that $\limsup_{t \to \infty} X(t) \geq 0$ a.s., define $X_-(t) = -X(t)$, $f_-(x) = -f(-x)$, $\sigma_-(t) = -\sigma(t)$. Then

$$dX_-(t) = -f_-(X_-(t)) \, dt + \sigma_-(t) \, dB(t), \quad t \geq 0.$$ 

By the same argument as above, it can be shown that $\liminf_{t \to \infty} X_-(t) \leq 0$ a.s., which yields $\limsup_{t \to \infty} X(t) \geq 0$ a.s. Combining this with $\liminf_{t \to \infty} X(t) \leq 0$ a.s. yields the required result.

8. Proofs of Proposition 3 and 4

8.1. Proof of Proposition 3 By (3.6) we have

$$\lim_{x \to \infty} \frac{\log(1 - \Phi(x)) - \log x^{-1} + x^2/2}{x^2/2} = \log \left(1/\sqrt{2\pi}\right),$$

and so

$$\lim_{x \to \infty} \frac{\log(1 - \Phi(x))}{x^2/2} = -1.$$ 

Suppose that $\theta(n) \to 0$ as $n \to \infty$, we have for $\epsilon > 0$ that

$$\lim_{n \to \infty} \frac{\log(1 - \Phi(\epsilon/\theta(n)))}{\epsilon^2/(2\theta^2(n))} = -1.$$ 

Thus

$$\lim_{n \to \infty} \frac{\log(1 - \Phi(\epsilon/\theta(n)))}{\log n} = \lim_{n \to \infty} \frac{\log(1 - \Phi(\epsilon/\theta(n)))}{\epsilon^2/(2\theta^2(n))} \cdot \frac{\epsilon^2/(2\theta^2(n))}{\log n}$$

$$= -\frac{\epsilon^2}{2} \lim_{n \to \infty} \frac{1}{\theta^2(n) \log n}. \quad (8.1)$$

In cases (A) and (B), we have that $\theta^2(n) := \int_n^{n+1} \sigma^2(s) \, ds$ obeys

$$\lim_{n \to \infty} \theta^2(n) \log n = L, \quad (8.2)$$

and in each case $\theta(n) \to 0$ as $n \to \infty$. Therefore (8.1) holds in both case (A) and case (B). To prove part (A), note that when $L = 0$, from (8.2) and (8.1), we have

$$\lim_{n \to \infty} \frac{\log(1 - \Phi(\epsilon/\theta(n)))}{\log n} = -\infty$$

for every $\epsilon > 0$, so by (3.4), we have $S(\epsilon) < +\infty$ for every $\epsilon > 0$. Therefore, by Lemma 1, $S'$ obeys (3.8), as required.

To prove part (B), note that when $L \in (0, \infty)$, from (8.2) and (8.1), we have

$$\lim_{n \to \infty} \frac{\log(1 - \Phi(\epsilon/\theta(n)))}{\log n} = -\frac{\epsilon^2}{2L}.$$
If $\epsilon > \sqrt{2L}$, then by (3.1) we have $S(\epsilon) < +\infty$, and thus by Lemma 1, $S'(\epsilon) < +\infty$.

On the other hand, if $\epsilon < \sqrt{2L}$, by (3.3) we have that $S(\epsilon) = +\infty$, and so by Lemma 1, $S'(\epsilon) = +\infty$. Therefore (3.10) holds with $e' = \sqrt{2L}$.

In case (C), suppose that there exists $e' > 0$ such that $S'(e') < +\infty$. Then by Lemma 1, we have that $S(e') < +\infty$. Hence by considering the last term in the sum on the left hand side of (8.3), we get that $1 - \Phi(e'/\theta(n)) \to 0$ as $n \to \infty$. This implies that $\theta(n) \to 0$ as $n \to \infty$. Thus, we have that (8.1) holds.

Now, because $\sigma^2(t) \log t \to 0$ as $t \to \infty$, we have that $\theta^2(n) \log n \to 0$ as $n \to \infty$.

Therefore, using this fact and (8.1), we have that

$$\lim_{n \to \infty} \frac{\log(1 - \Phi(e'/\theta(n)))}{\log n} = 0.$$  

Therefore, it follows from (3.4) that $S(\epsilon) = +\infty$, a contradiction. Therefore, we must have that $S'(\epsilon) = +\infty$ for every $\epsilon > 0$, which is (3.11), as claimed.

8.2. Proof of Proposition 4. For $n \leq t < n+1$, we have that $\Sigma^2(t) \leq e^{2\Sigma^2(n+1)}$ and $\Sigma^2(t) \geq e^{-2\Sigma^2(n)}$. Thus it is easy to see that $\Sigma^2(t) \to 0$ as $t \to \infty$ if and only if the sequence $\{\Sigma^2(n)\}_{n \geq 0}$ converges to zero.

Writing

$$\Sigma^2(n) = e^{-2n} \sum_{j=0}^{n-1} \int_j^{j+1} e^{2\sigma^2(s)} ds \cdot \log n,$$

and using (8.2), we readily get the double inequality

$$\Theta^2(n) \cdot \log n \leq \Sigma^2(n) \leq e^2 \Theta^2(n) \cdot \log n. \quad (8.3)$$

Hence by considering the last term in the sum on the left hand side of (8.3), we get $\Sigma^2(n) \geq e^{-2\theta^2(n-1)} \log n$, so $\theta^2(n-1) \log(n-1) \leq e^2 \Sigma^2(n)$. Therefore, if $\Sigma^2(n) \to 0$, we have that $\theta^2(n) \log n \to 0$ as $n \to \infty$.

On the other hand, if $\theta^2(n) \log n \to 0$ as $n \to \infty$, for every $\epsilon > 0$ there exists an integer $N(\epsilon) \geq 1$ such that $\theta^2(n) \log n < \epsilon$ for all $n \geq N(\epsilon)$. Thus for $n \geq N(\epsilon)+1$, by (8.3), we have

$$\Sigma^2(n) \leq e^2 \sum_{j=0}^{N(\epsilon)-1} \frac{e^{2\theta^2(j)}}{e^{2n}/\log n} + e^2 \sum_{j=N(\epsilon)}^{n-1} \frac{e^{2j}/\log j}{e^{2n}/\log n},$$

so

$$\limsup_{n \to \infty} \Sigma^2(n) \leq e^2 \limsup_{n \to \infty} \sum_{j=0}^{n-1} \frac{e^{2j}/\log j}{e^{2n}/\log n}.$$  

Since $x \mapsto e^{2x}/\log x$ is increasing on $[2, \infty)$ we have that

$$\sum_{j=0}^{n-1} e^{2j}/\log j \leq \sum_{j=2}^{n-1} \int_j^{j+1} e^{2x}/\log x dx = \int_2^n e^{2x}/\log x dx.$$  

By l'Hôpital's rule

$$\lim_{t \to \infty} \frac{\int_2^n e^{2x}/\log x dx}{e^{2t}/\log t} = \lim_{t \to \infty} \frac{1}{2 - 1/(t \log t)} = \frac{1}{2},$$

so

$$\limsup_{n \to \infty} \sum_{j=2}^{n-1} \frac{e^{2j}/\log j}{e^{2n}/\log n} \leq \frac{1}{2}.$$  

Hence $\limsup_{n \to \infty} \Sigma^2(n) \leq e^2/2$. Since $\epsilon > 0$ is arbitrary, we have $\Sigma^2(n) \to 0$ as $n \to \infty$, as required.

Since for $t \in [n, n+1)$ we have $\Sigma^2(t) \leq e^2 \Sigma^2(n+1)$ and $\Sigma^2(t) \geq e^{-2} \Sigma^2(n)$, it follows that $\limsup_{t \to \infty} \Sigma^2(t) \in (0, \infty)$ implies $\limsup_{n \to \infty} \Sigma^2(n) < +\infty$. If $\limsup_{n \to \infty} \Sigma^2(n) = 0$, then $\limsup_{t \to \infty} \Sigma^2(t) = 0$, a contradiction. Therefore we
have \( \limsup_{t \to \infty} \Sigma^2(t) \in (0, \infty) \) implies \( \limsup_{n \to \infty} \Sigma^2(n) \in (0, \infty) \). On the other hand, if \( \limsup_{n \to \infty} \Sigma^2(n) = L \in (0, \infty) \), we see immediately that
\[
0 < e^{-2}L \leq \limsup_{t \to \infty} \Sigma^2(t) \leq e^2L < +\infty,
\]
so the first two statements are equivalent. By (8.3), we have \( \limsup_{n \to \infty} \Sigma^2(n) = L \in (0, \infty) \) implies that
\[
\limsup_{n \to \infty} \Theta^2(n) \log n \leq L, \quad \limsup_{n \to \infty} \Theta^2(n) \log n \geq L/e^2.
\]
By (8.25), it follows that \( \Theta^2(n) = e^2 \Theta^2(n+1) - \Theta^2(n) \). Hence \( \limsup_{n \to \infty} \Sigma^2(n) = L \in (0, \infty) \) implies that
\[
\limsup_{n \to \infty} \log n \leq e^2 \limsup_{n \to \infty} \Theta^2(n+1) \log n + \limsup_{n \to \infty} \Theta^2(n) \log n \leq (1 + e^2)L.
\]
To see that \( \limsup_{n \to \infty} \Theta^2(n) \log n \rightarrow 0 \) as \( n \rightarrow \infty \), suppose not. Then \( \Theta^2(n) \log n \rightarrow 0 \) as \( n \rightarrow \infty \), and thus by part (i), we have that \( \Sigma^2(n) \rightarrow 0 \) as \( n \rightarrow \infty \), which is false by hypothesis. Thus \( \limsup_{n \to \infty} \Sigma^2(n) = L \in (0, \infty) \) implies \( \limsup_{n \to \infty} \Theta^2(n) \log n \in (0, \infty) \).

To prove part (iii), note that for \( n \leq t < n+1 \), we have \( e^{-2} \Sigma^2(n) \leq \Sigma^2(t) \leq e^{-2} \Sigma^2(n+1) \). Hence \( \liminf_{t \to \infty} \Sigma^2(t) \in (0, \infty) \) implies \( \liminf_{t \to \infty} \Sigma^2(n) < +\infty \), suppose not, then \( \liminf_{t \to \infty} \Sigma^2(t) = +\infty \), and thus we must have that \( \liminf_{n \to \infty} \Sigma^2(t) \in (0, \infty) \) implies \( \liminf_{n \to \infty} \Sigma^2(n) < +\infty \).

Let \( \liminf_{n \to \infty} \Sigma^2(t) \in (0, \infty) = L \), then
\[
0 < e^{-2}L \leq \liminf_{n \to \infty} \Sigma^2(t) \leq e^2L < \infty
\]
i.e. \( \liminf_{t \to \infty} \Sigma^2(t) \in (0, \infty) \). Thus (A) and (B) are equivalent.

By (8.3), \( \liminf_{n \to \infty} \Sigma^2(n) \in (0, \infty) \), if and only if \( \liminf_{n \to \infty} \Theta^2(n) \log n \in (0, \infty) \). Also \( \Sigma^2(n) \geq e^{-2} \Theta^2(n-1) \log n \) implies \( \liminf_{n \to \infty} \Theta^2(n-1) \log n \leq \liminf_{n \to \infty} e^{2} \Sigma^2(n) < +\infty \), with \( \liminf_{n \to \infty} e^{2} \Sigma^2(n) \log n \geq 0 \) by hypothesis, we finish the proof for (iii).

To prove (iv), note that for \( n \leq t < n+1 \), we have \( e^{-2} \Sigma^2(n) \leq \Sigma^2(t) \leq e^{-2} \Sigma^2(n+1) \). Hence \( \lim_{t \to \infty} \Sigma^2(t) = \infty \) if and only if \( \lim_{n \to \infty} \Sigma^2(n) = \infty \). Again from (8.3), we have \( \lim_{n \to \infty} \Theta^2(n) \log n = \infty \) implies \( \lim_{n \to \infty} \Sigma^2(n) = \infty \) and vice versa. To prove that all imply that \( \limsup_{n \to \infty} \Theta^2(n) \log n = \infty \), suppose \( \limsup_{n \to \infty} \Theta^2(n) \log n = L' < \infty \), then from \( \Theta^2(n) \log n = e^2 \Theta^2(n+1) - \Theta^2(n) \), we have \( \limsup_{n \to \infty} \Theta^2(n) \log n = \limsup_{n \to \infty} [e^2 \Theta^2(n+1) - \Theta^2(n)] < \infty \). Therefore \( \limsup_{n \to \infty} \Theta^2(n) \log n < \infty \), which is a contradiction.

9. Proof of Proposition 5

9.1. Proof of Part (A) of Proposition 5. In the case when \( \sigma \in L^2(0, \infty) \), we have that each of the events \( \{ \omega : \lim_{t \to \infty} Y(t, \omega) = 0 \} \) and \( \{ \omega : \lim_{t \to \infty} X(t, \omega) = 0 \} \) are a.s. by Theorem 4.

Suppose now that \( \sigma \notin L^2(0, \infty) \). Define
\[
\Omega_c = \Omega_X \cap \Omega_Y,
\]
where \( \Omega_X \) is given by (2.6) and \( \Omega_Y \) is defined by (2.5). Define for each \( \omega \in \Omega_c \) the realisation \( z(\cdot, \omega) \) by \( z(t, \omega) = X(t, \omega) - Y(t, \omega) \) for \( t \geq 0 \). Then \( z(\cdot, \omega) \) is in

we have that
Thus $T: [0 \rightarrow \mu R t T$ by (9.3), a contradiction. Hence $\lim \sup_{t \rightarrow \infty} |X(t, \omega)| = 0$.

Therefore $A_2$ is an a.s. event by hypothesis. Since $\sigma \not\in L^2(0, \infty)$, $A_3$ is an a.s. event by Theorem 3. Thus the event $A_4$ defined by $A_4 = A_2 \cap A_3$ is almost sure. Fix $\omega \in A_4$. Since $Y(t, \omega) \rightarrow 0$ as $t \rightarrow \infty$ and $\lim \inf_{t \rightarrow \infty} |X(t, \omega)| = 0$, it follows that

$$\lim_{t \rightarrow \infty} |z(t, \omega)| \leq \lim_{t \rightarrow \infty} |X(t, \omega)| + |Y(t, \omega)| = \lim_{t \rightarrow \infty} |X(t, \omega)| + \lim_{t \rightarrow \infty} |Y(t, \omega)| = 0.$$ 

Let $\eta \in (0, 1)$. We next show that $\lim \sup_{t \rightarrow \infty} |z(t, \omega)| \leq \eta$. Since $f$ is continuous on $\mathbb{R}$, it is uniformly continuous on $[-2, 2]$. Therefore, there exists a function $\mu: [0, \infty) \rightarrow [0, \infty)$ such that $\mu(0) = 0$, $\mu(\nu) \rightarrow 0$ as $\nu \downarrow 0$, and for which for every $\nu \in [0, 4]$ is defined by $\mu(\nu) = \max_{|x|\nu \leq 2, |y| \leq \nu} |f(x) - f(y)|$.

Thus $\mu$ is a modulus of continuity of $f$ on $[-2, 2]$. Let $\epsilon > 0$ be so small that

$$\epsilon < \frac{\eta}{4} \mu(\epsilon) < f(\eta) \wedge |f(\eta)|.$$ 

Then for $u \in [\eta - \epsilon, \eta + \epsilon] \subset (0, 2)$ we have $|f(u) - f(\eta)| \leq \mu(\epsilon)$, so $f(u) \geq f(\eta) - \mu(\epsilon) > \epsilon$. Therefore

$$\epsilon < \inf_{u \in (\eta - \epsilon, \eta + \epsilon)} f(u). \quad (9.2)$$ 

On the other hand for $u \in [-\eta - \epsilon, -\eta + \epsilon] \subset (-2, 0)$ we have $|f(u) - f(\eta)| \leq \mu(\epsilon)$, so $f(u) \leq f(\eta) + \mu(\epsilon) < \epsilon$.

Therefore

$$- \epsilon > \sup_{u \in (\eta - \epsilon, \eta + \epsilon)} f(u). \quad (9.3)$$ 

Since $Y(t, \omega) \rightarrow 0$ as $t \rightarrow \infty$, there exists $T_1(\epsilon, \omega) > 0$ such that $|Y(t, \omega)| < \epsilon$ for all $t > T_1(\epsilon)$. Suppose that $\lim \sup_{t \rightarrow \infty} |z(t, \omega)| > \eta$. Since $\lim \inf_{t \rightarrow \infty} |z(t, \omega)| = 0$, we may therefore define $T_2(\epsilon, \omega) = \inf \{ t > T_1(\epsilon, \omega) : |z(t, \omega)| = \eta/2 \}$. Also define $T_3(\epsilon, \omega) = \inf \{ t > T_2(\epsilon, \omega) : |z(t, \omega)| = \eta \}$.

In the case when $z(T_3(\epsilon, \omega), \omega) = \eta$, we have that $z'(T_3(\epsilon, \omega), \omega) \geq 0$. Since $|Y(T_3(\epsilon, \omega), \omega)| < \epsilon$ we have

$$0 \leq z'(T_3(\epsilon, \omega), \omega) = -f(z(T_3(\epsilon, \omega)) + Y(T_3(\epsilon, \omega), \omega)) + Y(T_3(\epsilon, \omega), \omega)$$

$$= -f(\eta + Y(T_3(\epsilon, \omega), \omega)) - Y(T_3(\epsilon, \omega), \omega)$$

$$< -f(\eta + Y(T_3(\epsilon, \omega), \omega)) + \epsilon \leq - \inf_{|z - \eta| < \epsilon} f(u) + \epsilon < 0,$n

by (9.2), a contradiction. On the other hand, in the case when $z(T_3(\epsilon, \omega), \omega) = -\eta$, we have that $z'(T_3(\epsilon, \omega), \omega) \leq 0$. Since $|Y(T_3(\epsilon, \omega), \omega)| < \epsilon$ we have

$$0 \geq z'(T_3(\epsilon, \omega), \omega) = -f(z(T_3(\epsilon, \omega)) + Y(T_3(\epsilon, \omega), \omega)) + Y(T_3(\epsilon, \omega), \omega)$$

$$= -f(-\eta + Y(T_3(\epsilon, \omega), \omega)) + Y(T_3(\epsilon, \omega), \omega)$$

$$> -f(-\eta + Y(T_3(\epsilon, \omega), \omega)) - \epsilon \geq - \sup_{|z + \eta| < \epsilon} f(u) - \epsilon > 0,$n

by (9.3), a contradiction. Hence $T_3(\epsilon, \omega)$ does not exist for any $\omega \in A_4$. Hence $\lim \sup_{t \rightarrow \infty} |z(t, \omega)| \leq \eta$. Since $\eta > 0$ is arbitrary, we make the limit as $\eta \downarrow 0$ to obtain $\lim \sup_{t \rightarrow \infty} |z(t, \omega)| = 0$. Since $X = Y + z$, and $Y(t, \omega) \rightarrow 0$ as $t \rightarrow \infty$, we have that $X(t, \omega) \rightarrow 0$ as $t \rightarrow \infty$, and because this is true for each $\omega$ in the a.s. event $A_4$, the result has been proven.
9.2. Proof of Part (C) of Proposition 5. Let the a.s. event $\Omega_5$ be as defined in (9.1). Define $\Omega_3 = \{ \omega \in \Omega : \limsup_{t \to \infty} |Y(t, \omega)| = +\infty \}$ which is a.s. by hypothesis. Define $F(t) = X(t) - f(X(t))$ for $t \geq 0$. Then (9.1) can be rewritten as

$$dX(t) = \{-X(t) + F(t)\} \ dt + \sigma(t) dB(t), \quad t \geq 0,$$

so by variation of constants we get

$$X(t) = X(0)e^{-t} + \int_0^t e^{-(t-s)}F(s)ds + Y(t), \quad t \geq 0.$$ 

Rearranging and taking absolute values gives

$$|Y(t)| \leq |X(t)| + |X(0)|e^{-t} + \int_0^t e^{-(t-s)}|F(s)|ds, \quad t \geq 0. \quad (9.4)$$

Define $A_5 = \{ \omega \in \Omega_X : \sup_{t \geq 0} |X(t, \omega)| < +\infty \}$ and suppose that $\mathbb{P}[A_5] > 0$. Define $A_4 = A_5 \cap \Omega_3$. Then $\mathbb{P}[A_6] = \mathbb{P}[A_5] > 0$. Let $\omega \in A_6$ and define $X_1(\omega) = \sup_{t \geq 0} |X(t, \omega)|$. Then $|X(t, \omega)| \leq X_1(\omega)$ for all $t \geq 0$. Since $f$ is continuous, for all $y \geq 0$, there exists $\overline{f}(y) < +\infty$ such that

$$\max_{|x| \leq M} |f(x)| = \overline{f}(y). \quad (9.5)$$

Therefore $|f(X(t, \omega))| \leq \overline{f}(X_1(\omega))$ for all $t \geq 0$. Hence by (9.4), for each $\omega \in A_6$, we have that for all $t \geq 0$

$$|Y(t, \omega)| \leq X_1(\omega) + X_1(\omega) + \int_0^t e^{-(t-s)}(X_1(\omega) + \overline{f}(X_1(\omega))) ds$$

$$\leq 3X_1(\omega) + \overline{f}(X_1(\omega)).$$

Since $\limsup_{t \to \infty} |Y(t, \omega)| = +\infty$ for each $\omega \in A_6 \subseteq \Omega_3$, we have a contradiction, so therefore we must have $\mathbb{P}[A_6] = 0$. This, taken together with continuity the continuity of $X$, gives $\limsup_{t \to \infty} |X(t)| = +\infty$ a.s., proving part (C) of Proposition 5.

9.3. Proof of Part (B) of Proposition 5. Define $\Omega_2 = \Omega_1 \cap \Omega_e$. Then by hypothesis, for every $\omega \in \Omega_2$ we have that there is a finite and positive $Y^*(\omega)$ such that

$$Y^*(\omega) = \limsup_{t \to \infty} |Y(t, \omega)|.$$ 

By definition $\underline{Y} \leq Y^*(\omega) \leq \overline{Y}$. Define for $\omega \in \Omega_2$

$$X^*(\omega) = \limsup_{t \to \infty} |X(t, \omega)|,$$

where $X^*(\omega) = 0$ and $X^*(\omega) = +\infty$ are admissible values. By (9.4), we have

$$Y^*(\omega) \leq X^*(\omega) + \limsup_{t \to \infty} \int_0^t e^{-(t-s)}|F(s, \omega)|ds \leq X^*(\omega) + \limsup_{t \to \infty} |F(t, \omega)|.$$ 

By the definition of $\overline{f}$, $F$ and $X^*$ we have

$$\limsup_{t \to \infty} |F(t, \omega)| \leq X^*(\omega) + \overline{f}(X^*(\omega)).$$

Since $\overline{f}$ is defined by (9.5) and $h_f$ by (1.6), we obtain

$$Y^*(\omega) \leq 2X^*(\omega) + \overline{f}(X^*(\omega)) = h_f(X^*(\omega)).$$

By Proposition 6, $h_f$ is an increasing function, so we have $X^*(\omega) \geq h_f^{-1}(Y^*(\omega))$.

Now by the definition of $X^*$, $\underline{X}$ and the fact that $h_f^{-1}$ is increasing, we have

$$\underline{X} = \inf_{\omega \in \Omega_2} X^*(\omega) \geq \inf_{\omega \in \Omega_2} h_f^{-1}(Y^*(\omega)) = h_f^{-1} \left( \inf_{\omega \in \Omega_2} Y^*(\omega) \right).$$
Thus

\[ Y \geq x \]

as \( h_f^{-1} \) is increasing,

\[ X \geq h_f^{-1} \left( \inf_{\omega \in \Omega_2} Y^*(\omega) \right) \geq h_f^{-1}(Y) = z(f, Y), \]

using (1.7) at the last step. Notice lastly that part (i) of Proposition 6 implies that \( z(f, Y) > 0 \) because \( Y > 0 \), by hypothesis.

10. Proofs of Theorem 7 and Proposition 7

10.1 Preliminary results. The asymptotic estimate (4.12) in Theorem 7 is shown by first establishing the estimate

\[ \limsup_{t \to \infty} |X(t, \omega)| \leq \max(x_+(\overline{Y}), x_-(\overline{Y})) + \overline{Y}, \quad \text{for each } \omega \in \Omega_2 \quad (10.1) \]

where we define \( x_+, x_- : [0, \infty) \to \mathbb{R} \) by

\[ x_+(y) = \sup \{ x > 0 : \min_{a \in [-y, y]} f(x + a) = y \}, \quad y \geq 0, \quad (10.2) \]

\[ -x_-(y) = \inf \{ x < 0 : \max_{a \in [-y, y]} f(x + a) = -y \}, \quad y \geq 0. \quad (10.3) \]

We prefer the estimate in (4.12) in part because the estimate on the right hand side of (10.1) is difficult to analyse in general, due to the complexity of \( x_+ \) and \( x_- \). Moreover, there is no loss of sharpness in the estimate in (4.12) relative to (10.1) in the case when \( f \) is increasing. To see this, first note that when \( f \) is increasing on \( \mathbb{R} \), it can readily be seen that \( x_+(y) = y + f^{-1}(y) \) and \( x_-(y) = y - f^{-1}(-y) \).

Therefore, if we grant that (10.1) holds, it follows that

\[ \limsup_{t \to \infty} |X(t, \omega)| \leq 2 \overline{Y} + \max(f^{-1}(\overline{Y}), -f^{-1}(\overline{Y})), \quad \text{for each } \omega \in \Omega_2. \]

Therefore, if we define

\[ \overline{x}(f, y) = 2y + \max(f^{-1}(y), -f^{-1}(-y)), \quad (10.4) \]

it can be seen that

\[ \limsup_{t \to \infty} |X(t, \omega)| \leq \overline{x}(f, Y), \quad \text{for each } \omega \in \Omega_2. \]

On the other hand, \( \overline{x}(f) \) defined in (10.4) is equal to \( \overline{x}(f) \) defined in (4.11) when \( f \) is increasing, because \( f^{-}(x) = f^{-1}(x) \) for \( x \leq 0 \) and \( f^{+}(x) = f^{-1}(x) \) for \( x \geq 0 \), where \( f^+ \) and \( f^- \) are defined in (4.9) and (4.10).

Therefore, the second stage in proving the asymptotic estimate (4.12) reduces to showing that

\[ y + \max(x_+(y), x_-(y)) \leq \overline{x}(f, y), \quad y \geq 0, \quad (10.5) \]

and accordingly, we start the proof of Theorem 7 by first establishing (10.5).

Lemma 2. Suppose that \( f \) obeys (1.2) and (1.8). Then the functions \( f^+ \) and \( f^- \) given by (4.9) and (4.10) are well-defined and with \( x_+, x_- \) and \( \overline{x} \) defined by (10.2), (10.3) and (4.11) respectively, we have (10.5).

Proof. Let \( z > x + f^+(x) \). Suppose \( u \in [-x, x] \). Then \( z + u > f^+(x) \). By the definition of \( f^+ \) we have \( f(u) > x \) for all \( u > f^+(x) \). Therefore, for each \( z > x + f^+(x) \), we have \( f(z + u) > x \) for all \( u \in [-x, x] \). Hence

\[ \min_{u \in [-x, x]} f(z + u) > x, \quad \text{for all } z > x + f^+(x). \]

Since \( x_+(y) = \sup \{ x > 0 : \min_{u \in [-y, y]} f(x + u) = y \} \), we have that

\[ y + f^+(y) \geq x_+(y). \quad (10.6) \]
Let $x > 0$. Let $z < -x + f^-(z)$, for all $a < f^-(z)$. Suppose $u \in [-x, x]$. Then $z + u < f^-(z)$. By the definition of $f^-$ we have $f(a) < -x$ for all $a < f^-(z)$. Therefore, for each $z < -x + f^-(z)$, we have $f(z + u) < -x$ for all $u \in [-x, x]$. Hence
\[
\max_{u \in [-x, x]} f(z + u) < -x, \quad \text{for all } z < -x + f^-(z).
\]
Since $-x(y) = \inf\{x > 0 : \max_{u \in [-y, y]} f(x + u) = -y\}$, we have that $-y + f^-(y) \leq -x(y)$, so
\[
y - f^-(y) \geq x(y).
\] (10.7)
Hence by (4.11), (10.6), (10.7), for any $y \geq 0$ we have
\[
\mathcal{P}(f, y) = 2y + \max(f^+(y), -f^-(y)) = y + \max(y + f^+(y), y - f^-(y)) \geq y + \max(x(y), -x(y)),
\]
which is (10.5).
\[\square\]

10.2. Proof of Theorem 7. We start with a lemma.

**Lemma 3.** Let $f$ obey (1.2) and (4.8). Suppose that $p$ is a continuous function such that
\[
\limsup_{t \to \infty} |p(t)| \leq \mathcal{P}.
\]
Suppose that $z$ is any continuous solution of
\[
z'(t) = -f(z(t)) + p(t), \quad t > 0; \quad z(0) = \xi.
\]
Then
\[
\limsup_{t \to \infty} |z(t)| \leq \max(x_+(\mathcal{P}), -x_-(\mathcal{P})) \leq \mathcal{P} + \max(f^+(\mathcal{P}), -f^-(\mathcal{P})�,
\]
where $x_+$ is defined by (10.2), $x_-$ by (10.3) and $f^+$ by (4.3), (4.10). Moreover, if $x(t) = z(t) + p(t)$ for $t \geq 0$, and $\mathcal{P}$ is defined by (4.11), then
\[
\limsup_{t \to \infty} |x(t)| \leq \mathcal{P}(f, \mathcal{P}).
\]

**Proof.** For every $\eta > 0$, there exists $T(\eta) > 0$ such that for $t \geq T(\eta)$ we have $|p(t)| \leq \mathcal{P} + \eta$.

The bound on $p$ yields the estimate
\[
z(t) - \mathcal{P} - \eta \leq z(t) + p(t) \leq z(t) + \mathcal{P} + \eta, \quad t \geq T(\eta).
\]
Since $f(x) \to \infty$ as $x \to \infty$, for every $\eta > 0$ there exists $\tilde{x}_+(\eta) > \eta$ such that
\[
\min_{a \in [-\mathcal{P} - \eta, \mathcal{P} + \eta]} f(x + a) \geq \mathcal{P} + 2\eta, \quad \text{for all } x \geq \tilde{x}_+(\eta).
\]
Note that $x_+$ defined by (10.2) obeys
\[
\min_{a \in [-\mathcal{P}, \mathcal{P}]} f(x + a) \geq \mathcal{P}, \quad \text{for all } x \geq x_+(\mathcal{P}).
\] (10.8)
Also as $f(x) \to -\infty$ as $x \to -\infty$, for every $\eta > 0$ there exists an $\tilde{x}_-(\eta) > \eta$ such that
\[
\max_{a \in [-\mathcal{P} - \eta, \mathcal{P} + \eta]} f(x + a) \leq -\mathcal{P} - 2\eta, \quad \text{for all } x \leq -\tilde{x}_-(\eta).
\]
Note that $x_-$ defined by (10.2) obeys
\[
\max_{a \in [-\mathcal{P}, \mathcal{P}]} f(x + a) \leq -\mathcal{P}, \quad \text{for all } x \leq x_-(\mathcal{P}).
\] (10.9)
Let $x(\eta) = \max(\tilde{x}_+(\eta), \tilde{x}_-(\eta))$.
Suppose that there is $t_1(\eta) > T(\eta)$ such that $z(t_1) > \tilde{x}_+(\eta)$. If not, it follows that
\[
z(t) \leq \tilde{x}_+(\eta) \text{ for all } t \geq T(\eta)
\]
and we have that \( \limsup_{t \to \infty} z(t) \leq \tilde{x}_+(\eta) \), which implies that \( \limsup_{t \to \infty} z(t) \leq x_+(\bar{p}) \). We will show that there exists a \( t_2(\eta) > t_1(\eta) \) such that \( z(t_2) = \tilde{x}_+(\eta) \) and moreover for all \( t \geq t_2(\eta) \) that \( z(t) \leq \tilde{x}_+(\eta) \). This implies that \( \limsup_{t \to \infty} z(t) \leq \tilde{x}_+(\eta) \) or indeed that \( \limsup_{t \to \infty} z(t) \leq x_+(\bar{p}) \).

By the definition of \( t_1 \) we have \( z(t_1) + \eta(\tilde{t}_1) > 0 \) and

\[
 z'(t_1) = -f(z(t_1) + p(t_1)) + p(t_1) \leq - \min_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(z(t_1) + a) + \bar{p} + \eta \leq -\eta.
\]

Then we have either that \( z(t) > \tilde{x}_+(\eta) \) for all \( t \geq t_1(\eta) \) or that there is a minimal \( t_2(\eta) > t_1(\eta) \) such that \( z(t_2) = \tilde{x}_+(\eta) \). In the former case for every \( t \geq t_1(\eta) \) we have

\[
 z'(t) = -f(z(t) + p(t)) + p(t) \leq - \min_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(z(t) + a) + \bar{p} + \eta \leq -\eta.
\]

Since \( \eta > 0 \), we may define \( t_3 = (z(t_1) - \tilde{x}_+(\eta))/\eta + t_1 + 1 \). Then \( z(t_3) \leq z(t_1) - \eta(t_3 - t_1) < \tilde{x}_+(\eta) \), a contradiction. Therefore, there exists a \( t_2 > t_1 \) such that \( \eta > 0 \), we may define \( t_3 = (z(t_1) - \tilde{x}_+(\eta))/\eta + t_1 + 1 \). Then \( z(t_3) \leq z(t_1) - \eta(t_3 - t_1) < \tilde{x}_+(\eta) \), a contradiction. Therefore, there exists a \( t_2 > t_1 \) such that \( z(t_2) = \tilde{x}_+(\eta) \). Now

\[
 z'(t_2) = -f(z(t_2) + p(t_2)) + p(t_2) \leq - \min_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(\tilde{x}_+(\eta) + a) + \bar{p} + \eta \leq -\eta.
\]

Then either there exists a minimal \( t_3(\eta) > t_2(\eta) \) such that \( z(t_3) = \tilde{x}_+(\eta) \) or we have that \( z(t) < \tilde{x}_+(\eta) \) for all \( t > t_2(\eta) \). In the former case, we must have \( z'(t_3) \geq 0 \).

But once again we have

\[
 z'(t_3) = -f(z(t_3) + p(t_3)) + p(t_3) \leq - \min_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(\tilde{x}_+(\eta) + a) + \bar{p} + \eta \leq -\eta,
\]

a contradiction. Thus we have \( z(t) < \tilde{x}_+(\eta) \) for all \( t > t_2(\eta) \), which implies that \( \limsup_{t \to \infty} z(t) \leq x_+(\bar{p}) \).

Suppose that there is \( t_1(\eta) > T(\eta) \) such that \( z(t_1) < -\tilde{x}_-(\eta) \). If not, it follows that

\[
 z(t) \geq -\tilde{x}_-(\eta) \quad \text{for all} \quad t \geq T(\eta)
\]

and we have that \( \liminf_{t \to \infty} z(t) \geq -\tilde{x}_-(\eta) \), which implies that \( \liminf_{t \to \infty} z(t) \geq -x_-(\bar{p}) \). We will show that there is a \( t_2(\eta) > t_1(\eta) \) such that \( z(t_2) = -\tilde{x}_-(\eta) \) and moreover that for all \( t \geq t_2(\eta) \) that \( z(t) \geq -\tilde{x}_-(\eta) \). This will imply that \( \liminf_{t \to \infty} z(t) \geq -\tilde{x}_-(\eta) \) or that \( \liminf_{t \to \infty} z(t) \geq -x_-(\bar{p}) \).

By the definition of \( t_1 \) we have \( z(t_1) + \eta(\tilde{t}_1) < 0 \) and

\[
 z'(t_1) = -f(z(t_1) + p(t_1)) + p(t_1) \geq - \max_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(z(t_1) + a) - \bar{p} - \eta \geq \eta.
\]

Then we have either that \( z(t) < -\tilde{x}_-(\eta) \) for all \( t \geq t_1(\eta) \) or that there is a minimal \( t_2(\eta) > t_1(\eta) \) such that \( z(t_2) = -\tilde{x}_-(\eta) \). In the former case for every \( t \geq t_1(\eta) \) we have

\[
 z'(t) = -f(z(t) + p(t)) + p(t) \geq - \max_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(z(t) + a) - \bar{p} - \eta \geq \eta.
\]

Since \( \eta > 0 \), we may define \( t_3 = (z(t_1) + \tilde{x}_-(\eta))/\eta + t_1 + 1 \). Then \( z(t_3) \geq z(t_1) + \eta(t_3 - t_1) > -\tilde{x}_-(\eta) \), a contradiction. Therefore, there exists a \( t_2 > t_1 \) such that \( z(t_2) = -\tilde{x}_-(\eta) \). Now

\[
 z'(t_2) = -f(z(t_2) + p(t_2)) + p(t_2) \geq - \max_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(\tilde{x}_-(\eta) + a) - \bar{p} - \eta \geq \eta.
\]

Then either there exists a minimal \( t_3(\eta) > t_2(\eta) \) such that \( z(t_3) = -\tilde{x}_-(\eta) \) or we have that \( z(t) > -\tilde{x}_-(\eta) \) for all \( t > t_2(\eta) \). In the former case, we must have \( z'(t_3) \leq 0 \). But once again we have

\[
 z'(t_3) = -f(z(t_3) + p(t_3)) + p(t_3) \geq - \max_{a \in [-\bar{p} - \eta, \bar{p} + \eta]} f(\tilde{x}_-(\eta) + a) - \bar{p} - \eta \geq \eta,
\]
a contradiction. Thus we have $z(t) > -\bar{x}_-(\eta)$ for all $t > t_2(\eta)$, which implies that 
\[
\liminf_{t \to \infty} z(t) \geq -x_-(\overline{\mathcal{P}}),
\]
and so $\limsup_{t \to \infty} |z(t)| \leq \max(x_+(\overline{\mathcal{P}}), x_-(\overline{\mathcal{P}}))$, as required.

We have thus shown that 
\[
\limsup_{t \to \infty} z(t) \leq x_+(\overline{\mathcal{P}}), \quad \liminf_{t \to \infty} z(t) \geq -x_-(\overline{\mathcal{P}}),
\]
and so $\limsup_{t \to \infty} |z(t)| \leq \max(x_+(\overline{\mathcal{P}}), x_-(\overline{\mathcal{P}}))$, as required.

Since $\limsup_{t \to \infty} |p(t)| \leq \overline{\mathcal{P}}$, it follows that 
\[
\limsup_{t \to \infty} |x(t)| \leq \overline{\mathcal{P}} + \max(x_+(\overline{\mathcal{P}}), x_-(\overline{\mathcal{P}})).
\]
Therefore using Lemma 2 (specifically (10.5)), we have 
\[
\limsup_{t \to \infty} |x(t)| \leq \overline{\mathcal{P}} + \max(x_+(\overline{\mathcal{P}}), x_-(\overline{\mathcal{P}})) \leq \mathcal{F}(f, \overline{\mathcal{P}}),
\]
which is precisely the final estimate required.

\[\square\]

10.3. **Proof of Theorem** [7] Let $\Omega_\epsilon$ be the event defined in (9.1). Then for every $\omega \in \Omega_\epsilon$ we may define $z(t, \omega) := X(t, \omega) - Y(t, \omega)$ for $t \geq 0$ where $Y$ is the solution of (2.4). Then $z(0) = X(0)$ and each sample path of $z$ is in $C^1(0, \infty)$ with 
\[
z'(t, \omega) = -f(z(t, \omega) + Y(t, \omega)) + Y(t, \omega), \quad t > 0.
\]
If $\theta$ obeys (3.10a) and (3.10b), it follows from part (B) of Theorem 3 that there exists an a.s. event $\Omega_0$, defined by (3.11), such that there is a finite, positive and deterministic $\overline{Y}$ satisfying (3.13) i.e.
\[
\overline{Y} = \sup_{\omega \in \Omega_0} \limsup_{t \to \infty} |Y(t, \omega)|.
\]
Let $\Omega_2 = \Omega_1 \cap \Omega_0$. Fix $\omega \in \Omega_2$. Then by Lemma 3 with $Y(\cdot, \omega)$ in the role of $p$, and $z(\cdot, \omega)$ in the role of $z$, we have that 
\[
\limsup_{t \to \infty} |z(t, \omega)| \leq \max(x_+(\overline{\mathcal{P}}), x_-(\overline{\mathcal{P}})).
\]
Putting $X(\cdot, \omega)$ in the role of $x$ in Lemma 3 we can infer from Lemma 3 that for $\omega \in \Omega_2$
\[
\limsup_{t \to \infty} |X(t, \omega)| \leq \mathcal{F}(f, \overline{\mathcal{P}}).
\]
Since this estimate holds for all $\omega \in \Omega_2$, we have precisely (4.12), as required.

10.4. **Proof of Proposition** [7] For a given $f$, $f^+$ and $f^-$ are non-decreasing functions. We show first that $\lim_{x \to 0} f^+(x) = 0$.

Since $f(x) \to \infty$ as $x \to \infty$, there exists $a > 0$ such that $f(x) \geq 1$ for all $x \geq a$. Let $\epsilon$ be any positive number with $\epsilon < a$. Then, as $f$ is continuous and strictly positive on $[\epsilon, a]$, it follows that there exists $x_\epsilon \in [\epsilon, a]$ such that $0 < f(x_\epsilon) = \min_{\epsilon \leq y \leq a} f(y)$. Define $\delta(\epsilon) = f(x_\epsilon)$. Then, if $0 < x < \delta(\epsilon)$, we have that $f^+(x) \leq \epsilon$.

To justify this, suppose to the contrary that $f^+(x') > \epsilon$ for some $x' \in (0, \delta(\epsilon))$. Then $f^+(x') = \sup \{z > 0 : f(z) = x' \} > \epsilon$. Now, for $f^+(x') = z' > \epsilon$, we have $f(z') \geq f(x_\epsilon) = \delta(\epsilon)$. However, by hypothesis $\delta(\epsilon) > x'$, so $f(z') > x'$. However, $z' = f^+(x') = \sup \{z > 0 : f(z) = x' \}$ implies that $f(z') = x'$, so we have a contradiction. Therefore, for every $\epsilon \in (0, a)$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $0 < x < \delta(\epsilon)$, we have that $f^+(x) \leq \epsilon$. Thus, as $f^+$ is a non-negative function, and $\epsilon \in (0, a)$ is arbitrary, this is precisely $\lim_{x \to 0} f^+(x) = 0$. The proof that $\lim_{x \to 0} f^-(x) = 0$ is similar.

Note that $\lim_{x \to \infty} f^+(x) = \lim_{x \to \infty} f^-(x) = \infty$ (by (13.8)) so it is clear that 
\[
y \mapsto \mathcal{F}(f, y)
\]
is increasing, and moreover that $\lim_{y \to \infty} \mathcal{F}(f, y) = \infty$. Also, as $\lim_{x \to 0} f^+(x) = \lim_{x \to 0} f^-(x) = 0$, we have that $\lim_{y \to 0} \mathcal{F}(f, y) = 0$, which proves part (i).
To prove part (ii), suppose first that there is $x > 0$ such that $f_1^+(x) < f_2^+(x)$. By definition, $f_1(z) > x$ for all $z > f_1^+(x)$. Since $f_1^+(x) < f_2^+(x)$, we have $f_1(f_2^+(x)) > x$. But $f_2(f_2^+(x)) = f_1(f_2^+(x))$ by (10.5). Hence $f_2(f_2^+(x)) > x$. But $f_2(f_2^+(x)) = x$, by definition, so we have the contradiction $x > x$. Hence

$$f_1^+(x) \geq f_2^+(x), \quad x > 0. \quad (10.10)$$

Suppose next there is $y < 0$ such that $f_1^-(y) > f_2^-(y)$. By definition, $f_1(z) < y$ for $z < f_1^-(y)$. Since $f_2^-(y) < f_1^-(y)$, it follows that $f_1(f_2^-(y)) < y$. By (10.5), we have $-f_2(u) \geq -f_1(u)$ for all $u < 0$. Hence with $u = f_2^-(y)$, we get $-f_2(f_2^-(y)) \geq -f_1(f_2^-(y)) > -y$. But $f_2(f_2^-(y)) = y$, by definition, so we have $-y = -f_2(f_2^-(y)) \geq -f_1(f_2^-(y)) > -y$, a contradiction. Thus we have $f_1^-(y) \leq f_2^-(y)$ for all $y < 0$, or

$$-f_1^-(y) \geq -f_2^-(y), \quad y < 0. \quad (10.11)$$

Therefore, it follows from (10.11), (10.10) and (10.11) that

$$\overline{\varphi}(f_2, y) = 2y + \max(f_2^+(y), -f_2^-(y))$$

$$\leq 2y + \max(f_1^+(y), -f_1^-(y)) = \overline{\varphi}(f_1, y),$$

as required.

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