Correlators of arbitrary untwisted operators and excited twist operators for $N$ branes at angles.

Igor Pesando

1Dipartimento di Fisica, Università di Torino
and I.N.F.N. - sezione di Torino
Via P. Giuria 1, I-10125 Torino, Italy

ipesando@to.infn.it

January 28, 2014

Abstract

We compute the generic correlator with $L$ untwisted operators and $N$ (excited) twist fields for branes at angles on $T^2$ and show that it is given by a generalization of the Wick theorem. We give also the recipe to compute efficiently the generic OPE between an untwisted operator and an excited twisted state.

keywords: D-branes, Conformal Field Theory
1 Introduction and conclusions

Since their introduction, D-branes have been very important in the formal development of string theory as well as in attempts to apply string theory to particle phenomenology and cosmology. However, the requirement of chirality in any physically realistic model leads to a somewhat restricted number of possible D-brane set-ups. An important class is intersecting brane models where chiral fermions can arise at the intersection of two branes at angles. An important issue for these models is the computation of Yukawa couplings and flavour changing neutral currents.

Besides the previous computations many other computations often involve correlators of twist fields and excited twist fields. It is therefore important and interesting in its own to be able to compute these correlators also because it is annoying to be able to compute, at least theoretically, all possible correlators involving all kinds of excited spin fields while not being able to do so with twist fields. As known in the literature [1] and explicitly shown in [2] for the case of magnetized branes these computations boil down to the knowledge of the Green function in presence of twist fields and of the correlators of the plain twist fields. In many previous papers correlators with excited twisted fields have been computed on a case by case basis without a clear global picture, see for example ([3], [4], [5])

In this technical paper we have analyzed the \( N \) excited twist fields amplitudes with \( L \) boundary vertices at tree level for open strings localized at D-branes intersections on \( R^2 \) (or \( T^2 \)) using the classical path integral approach ([1],[6]) which is more efficient than the also classical sewing approach ([7], [8]). This approach has been explored in many papers in the branes at angles setup as well as the T dual magnetic branes setup see for example ([9], [11], [12], [13], [14], [15], [16], [17], [18], [19],[20], [21], [22]). We will nevertheless follow a slightly different approach, the so called Reggeon vertex [23], which allows to compute the generating function of all correlators, in particular we will use the formulation put forward in [24].

This paper is organized as follows. In section 2 we review the geometrical framework of branes at angles and fix our conventions. In the same section we discuss carefully how to make use of the doubling trick in presence of multiple cuts and the existence of local and global constraints. In section 3 we compute the OPE of chiral and boundary vertex operators with an arbitrary excited twist operator by relying on the operator to state correspondence. We propose also a better notation for excited twist operators than that usually used which requires a new symbol for any excited twist operator. In this same section and for use in the fourth we establish also which chiral operators are best suited to obtain excited twist operators in the easiest way. Finally in section 4 we compute the generating function of correlators of \( N \) excited twist operators with \( L \) boundary operators. We do this in steps by first computing the interaction of boundary and chiral vertices with twist field operators and then computing the desired correlators by letting appropriate combinations of chiral vertex operators collide with twist fields. Our main result is the
generating function given in eq. (108) which shows that all correlators can be computed once the $N$ plain twist operators correlator together with the Green function in presence of these $N$ twists are known. This expression remains nevertheless quite formal since it requires the precise knowledge of the Green function $^1$ and its regularized versions. Therefore somewhat explicit expressions of these quantities are given in appendix E (see also appendixes C and D) and completely explicit expressions for all involved quantities in the $N=3$ case are given in appendix F. From these expressions it is clear that the computation of amplitudes, i.e. moduli integrated correlators, with (untwisted) states carrying momenta are very unwieldy because Green functions can at best be expressed as sum of product of type D Lauricella functions. This should however not be a complete surprise since in [25] it was shown that twist fields correlators in orbifold setup are connected to loop amplitudes which, up to now, have not been expressed in term of simpler functions.

2 Review of branes at angles

The Euclidean action for a string configuration is given by

\[ S = \frac{1}{4\pi \alpha'} \int d\tau_E \int_0^\pi d\sigma \left( \partial^a X^I \right)^2 = \frac{1}{4\pi \alpha'} \int_H d^2u \left( \partial_a X \partial_{\bar{a}} \bar{X} + \partial_{\bar{a}} X \partial_a \bar{X} \right) \]  

(1)

where $u \in H$, the upper half plane, $d^2u = e^{2\tau_E} d\tau_E d\sigma = \frac{du \, d\bar{u}}{2i}$ and $I = 1, 2$ or $z, \bar{z}$ so that $X = X^z = \frac{1}{\sqrt{2}}(X^1 + iX^2)$, $\bar{X} = X^{\bar{z}} = X^*$. The complex string coordinate is a map from the upper half plane to a closed polygon $\Sigma$ in $\mathbb{C}$, i.e. $X : H \rightarrow \Sigma \subset \mathbb{C}$. For example in fig. 1 we have pictured the interaction of $N=4$ branes at angles $D_t$ with $t = 1, \ldots, N$. The interaction between brane $D_t$ and $D_{t+1}$ is in $f_t \in \mathbb{C}$. We use the rule that index $t$ is defined modulo $N$. As shown in [2] given the number of twist fields $N$ there are $N-2$ different

\[ \begin{array}{c}
\sigma = \pi \\
\sigma = 0 \\
D_1 \quad D_4 \quad D_3 \quad D_2 \quad D_1
\end{array} \]

Figure 1: Map from the Minkowskian worldsheet to the target polygon $\Sigma$.  

$^1$ Note that the Green functions used in this paper are dimensionful and normalized as $\partial_a \partial_{\bar{a}} G^{IJ} (u, \bar{u}; v, \bar{v}; \{ \epsilon_i \}) = -\frac{\alpha'}{2} \delta^{IJ} \delta^2(u-v)$.  

3
sectors which correspond to the number of reflex angles (the interior angles bigger than $\pi$) as shown in figure 2 in the case $N = 6$. The intuitive reason why they are different is that we need go through the straight line, i.e. no twist, if we want to go from a reflex angles to a more usual convex one.

They are labeled by an integer $1 \leq M \leq N - 2$ given by

$$M = \sum_{t=1}^{N} \epsilon_t \tag{2}$$

where $0 < \epsilon_t < 1$ (we define also $\bar{\epsilon}_t = 1 - \epsilon_t$ for simplifying the expressions) are the twists defined as in eq. (8) and correspond to the angles measured from brane $D_t$ to brane $D_{t+1}$ when they are labeled clockwise as in figure 3. It is also possible to have the very same geometrical configuration where branes are simply labeled counterclockwise as shown in figure 4 for which we have $M_{ccw} = N - M$ when we still measure angles from brane $D_t$ to brane $D_{t+1}$. To understand the meaning we notice that if we relabel the branes as in the clockwise case, i.e. we consider the branes labeled as in figure 3 but with the angles measured as in figure 4 and we use the conventions we use for the local description, described in the next section, the interactions on worldsheet would take place in the reversed world sheet time order or in the same order but on the other boundary as shown in figure 5. Since the physics must be connected and actually the two correlators are connected by complex conjugation and exchange $\epsilon \leftrightarrow 1 - \epsilon$ we have chosen to measure them clockwise.

Figure 2: The four different cases with $N = 6$. a) $M_{ccw} = 2$ and $M = 4$ where $M_{ccw}$ is measured counterclockwise and $M_{cw}$ clockwise. b) $M_{ccw} = 3$ and $M = 3$. c) $M_{ccw} = 4$ and $M = 2$. d) $M_{ccw} = 5$ and $M = 1$. 
2.1 The local description

Locally at the interaction point \( f_t \in \partial \Sigma \subset \mathbb{C} \) the boundary conditions for the brane \( D_t \) are given by

\[
Re(e^{-i\pi \alpha t}X_{loc}|_{\sigma=0}) = Im(e^{-i\pi \alpha t}X_{loc}|_{\sigma=\pi}) - g_t = 0
\]  

(3)

where \( g_t \in \mathbb{R} \) is the distance from the line parallel to the brane going through the origin. Similarly the boundary conditions for the brane \( D_{t+1} \) is given by

\[
Re(e^{-i\pi \alpha t+1}X'_{loc}|_{\sigma=\pi}) = Im(e^{-i\pi \alpha t+1}X_{loc}|_{\sigma=0}) - g_{t+1} = 0
\]  

(4)

The interaction point is then

\[
f_t = \frac{e^{i\pi \alpha t+1}g_t - e^{i\pi \alpha t}g_{t+1}}{\sin \pi(\alpha_{t+1} - \alpha_t)}
\]  

(5)
When we write the Minkowskian string expansion as \( X(\sigma, \tau) = X_L(\tau + \sigma) + X_R(\tau - \sigma) \) the previous boundary conditions imply (and not become since they are not completely equivalent because of zero modes)

\[
X'_L \text{loc}(\xi) = e^{i2\pi \alpha t} X'_R \text{loc}(\xi), \quad X'_L \text{loc}(\xi + \pi) = e^{i2\pi \alpha t+1} X'_R \text{loc}(\xi - \pi)
\]

or in a more useful way in order to explicitly compute the mode expansion

\[
X'_L \text{loc}(\xi + 2\pi) = e^{i2\pi \epsilon t} X'_L \text{loc}(\xi), \quad X'_R \text{loc}(\xi + 2\pi) = e^{-i2\pi \epsilon t} X'_R \text{loc}(\xi)
\]

where we have defined

\[
\epsilon_t = \begin{cases} 
(\alpha_{t+1} - \alpha_t) & \alpha_{t+1} > \alpha_t \\
1 + (\alpha_{t+1} - \alpha_t) & \alpha_{t+1} < \alpha_t 
\end{cases}
\]

so that \( 0 < \epsilon_t < 1 \) and there is no ambiguity in the phase \( e^{i2\pi \epsilon t} \) entering the boundary conditions. The quantity \( \pi \epsilon t \) is the angle between the two branes \( D_t \) and \( D_{t+1} \) measured counterclockwise as shown in fig. 6. A consequence of this definition is that \( \epsilon \) becomes \( 1 - \epsilon \) when we flip the order of two branes. For example the angles in fig. 4 become those in fig. 3 when we reverse the order we count the branes, i.e. when we follow the boundary clockwise instead of counterclockwise.

We introduce as usual the Euclidean fields \( X_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}), \bar{X}_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}) \) by a worldsheet Wick rotation in such a way they are defined on the upper half plane by \( u_{\text{loc}} = e^{\tau_E + i\sigma} \in H \). The previous choice of having brane \( D_t \) at \( \sigma = 0 \) and brane \( D_{t+1} \) at \( \sigma = \pi \) implies that in the local description where the interaction point is at \( x = 0 \) \( D_t \) is mapped into \( x > 0 \) and \( D_{t+1} \) into \( x < 0 \). The boundary conditions (6) can then immediately be written as

\[
\begin{align*}
\partial X_{\text{loc}}(x_{\text{loc}} + i0^+) &= e^{i2\pi \alpha t} \bar{\partial} \bar{X}_{\text{loc}}(x_{\text{loc}} - i0^+) & 0 < x_{\text{loc}}, \\
\partial X_{\text{loc}}(x_{\text{loc}} + i0^+) &= e^{i2\pi \alpha t+1} \bar{\partial} \bar{X}_{\text{loc}}(x_{\text{loc}} - i0^+) & x_{\text{loc}} < 0
\end{align*}
\]

and similarly relations for \( \bar{X} \) which can be obtained by complex conjugation. When we add to the previous conditions the further constraints

\[
\text{Im}(e^{-i\pi \alpha t} X(x, x)) = g_t \quad x > 0
\]
we obtain a system of conditions which are equivalent to the original ones \([3,4]\).

In order to express the boundary conditions \([7]\) in the Euclidean formulation it is better to introduce the local fields defined on the whole complex plane by the doubling trick as

\[
\begin{align*}
\partial X_{loc}(z_{loc}) &= \left\{ \begin{array}{ll}
\partial_u X_{loc}(u_{loc}) & z_{loc} = u_{loc} \quad \text{with } \text{Im } z_{loc} > 0 \text{ or } z_{loc} \in \mathbb{R}^+
\end{array} \right.
\end{align*}
\]

\[
\partial \bar{X}_{loc}(z_{loc}) &= \left\{ \begin{array}{ll}
\partial_u \bar{X}_{loc}(\bar{u}_{loc}) & z_{loc} = \bar{u}_{loc} \quad \text{with } \text{Im } z_{loc} < 0 \text{ or } z_{loc} \in \mathbb{R}^+
\end{array} \right.
\]

In this way we can write eqs. \([7]\) as

\[
\partial X_{loc}(e^{i2\pi \delta}) = e^{i2\pi \epsilon_t} \partial X_{loc}(\delta), \quad \partial \bar{X}_{loc}(e^{i2\pi \delta}) = e^{-i2\pi \epsilon_t} \partial \bar{X}_{loc}(\delta) \quad (12)
\]

Notice that the two Minkowskian boundary conditions in eqs. \([7]\) can be mapped one into the other by complex conjugation while the corresponding ones in the Euclidean version given in eqs. \([12]\) are independent and each is mapped into itself by complex conjugation therefore the Euclidean classical solutions for \(X_{loc}\) and \(\bar{X}_{loc}\) are independent.

The quantization of the string with given boundary conditions yields

\[
X_{loc}(u_{loc}, \bar{u}_{loc}) = f_t + \frac{1}{2} \sqrt{2\alpha'} e^{i\pi \alpha} \sum_{n=0}^{\infty} \left[ \bar{\alpha}(t)n+\epsilon_t u_{loc} - (n+\epsilon_t) n + \epsilon_t \right]
\]

\[
+ i \frac{1}{2} \sqrt{2\alpha'} e^{i\pi \alpha} \sum_{n=0}^{\infty} \left[ -\bar{\alpha}(t)n+\epsilon_t - n + \epsilon_t \right]
\]

\[
\bar{X}_{loc}(u_{loc}, \bar{u}_{loc}) = f_t^* + \frac{1}{2} \sqrt{2\alpha'} e^{-i\pi \alpha} \sum_{n=0}^{\infty} \left[ \bar{\alpha}(t)n+\epsilon_t u_{loc} - (n+\epsilon_t) n + \epsilon_t \right]
\]

\[
+ i \frac{1}{2} \sqrt{2\alpha'} e^{-i\pi \alpha} \sum_{n=0}^{\infty} \left[ -\bar{\alpha}(t)n+\epsilon_t - n + \epsilon_t \right]
\]

\[
(13)
\]

with \(\epsilon_t = 1 - \epsilon_t\) and we can interpret \(f_t\) as the classical solution

\[
X_{loc,cl} = f_t, \quad \bar{X}_{loc,cl} = \bar{f}_t \quad (14)
\]

since it is the only solution to the equations of motion with finite euclidean action.

We find also the non trivial commutation relations \((n, m \geq 0)\)

\[
[\alpha(t)n+\epsilon_t, \alpha(t)^{\dagger}m+\epsilon_t] = (n+\epsilon_t) \delta_{m,n}, \quad [\bar{\alpha}(t)n+\epsilon_t, \bar{\alpha}(t)^{\dagger}m+\epsilon_t] = (n+\epsilon_t) \delta_{m,n} \quad (15)
\]
and the vacuum is defined in the usual way by
\[
\alpha(t)n+i\epsilon|T_0\rangle = \tilde{\alpha}(t)n+i\epsilon|T_0\rangle = 0 \quad n \geq 0 \quad (16)
\]
At first sight it may seem that the vacuum encodes the \(\epsilon\) information only but as we show in eq. \([52]\) it contains also information about \(f_t\) and \(\alpha_t\) and \(\alpha_{t+1}\) which can be extracted from the proper OPEs.

### 2.2 Global description

In the local description, where the interaction point is at \(x_{loc} = 0\), \(D_t\) is mapped into \(x_{loc} > 0\) and \(D_{t+1}\) into \(x_{loc} < 0\) this means that in the global description the world sheet interaction points are mapped on the boundary of the upper half plane so that \(x_{t+1} < x_t\). The global equivalent of the local boundary conditions given in eq.s \([9]\) which is useful for the path integral formulation where we cannot use the equations of motion is

\[
\partial_u X(u, \bar{u})|_{u=x+i0+} = e^{i2\pi\alpha_t} \partial_{\bar{u}} \bar{X}(u, \bar{u})|_{u=x+i0+} \quad x_t < x < x_{t-1} \quad (17)
\]

To the previous constraints one must also add the global equivalent to eq. \([10]\)

\[
X(x_t, \bar{x}_t) = f_t, \quad \bar{X}(x_t, \bar{x}_t) = f_t^* \quad (18)
\]

in order to get a system of boundary conditions equivalent to the original ones. When using an operatorial approach the global equivalent of eq.s \([9]\) become

\[
\partial X_L(x + i0^+) = e^{i2\pi\alpha_t} \partial \bar{X}_R(x - i0^+) \quad x_t < x < x_{t-1} \quad (19)
\]

\[
\partial \bar{X}_L(x + i0^+) = e^{-i2\pi\alpha_t} \partial X_R(x - i0^+) \quad x_t < x < x_{t-1}
\]

If we introduce the global fields defined on the whole complex plane by the doubling trick as

\[
\partial X(z) = \begin{cases} 
\partial_u X(u) & z = u \text{ with } \text{Im } z > 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \\
\partial_{\bar{u}} \bar{X}(\bar{u}) & z = \bar{u} \text{ with } \text{Im } z < 0 \text{ or } z \in \mathbb{R} - [x_N, x_1]
\end{cases}
\]

\[
\partial \bar{X}(z) = \begin{cases} 
\partial_u \bar{X}(u) & z = u \text{ with } \text{Im } z > 0 \text{ or } z \in \mathbb{R} - [x_N, x_1] \\
\partial_{\bar{u}} X(\bar{u}) & z = \bar{u} \text{ with } \text{Im } z < 0 \text{ or } z \in \mathbb{R} - [x_N, x_1]
\end{cases}
\]

\[
(20)
\]

the local boundary conditions \([7]\) can be written in the global formulation as

\[
\partial X(x_t + e^{i2\pi\delta}) = e^{i2\pi\epsilon_t} \partial X(x_t + \delta) \\
\partial \bar{X}(x_t + e^{i2\pi\delta}) = e^{-i2\pi\epsilon_t} \partial \bar{X}(x_t + \delta). \quad (21)
\]
Finally it is worth noticing the behavior of the previously introduced fields under complex conjugation when $z$ is restricted to $z \in \mathbb{C} - [-\infty, x_1]$

$$\left[\partial \chi(z)\right]^* = e^{-i2\pi\alpha_1} \partial \chi(z \to \bar{z}) = \bar{\partial} \chi(\bar{z}) = \begin{cases} \bar{\partial}_u \bar{X}(\bar{u}) & \bar{z} = \bar{u} \\ e^{-i2\pi\alpha_1} \partial_u X(u) & \bar{z} = u \end{cases}$$

$$\left[\partial \bar{X}(z)\right]^* = e^{-i2\pi\alpha_1} \partial \bar{X}(z \to \bar{z}) = \bar{\partial} \bar{X}(\bar{z}) = \begin{cases} \bar{\partial}_u \bar{X}(\bar{u}) & \bar{z} = \bar{u} \\ e^{i2\pi\alpha_1} \partial_u \bar{X}(u) & \bar{z} = u \end{cases}$$ (22)

where $\partial \chi(z \to \bar{z})$ means that the holomorphic $\partial \chi(z)$ is evaluated at $\bar{z}$.

The previous expressions also show that it is not necessary to introduce the antiholomorphic fields $\bar{\chi}$ and $\bar{\partial} \bar{X}(\bar{z})$ which it is possible to construct applying the doubling trick on $\bar{\partial}_u \bar{X}(\bar{u})$ and $\partial_u \bar{X}(u)$ respectively.

3 Twisted Fock Space and OPEs

Given the vacuum $|T\rangle$ defined as usual in eq. (16) and the expansions (13) we can immediately write a normalized basis element of the Fock space as

$$\prod_{n=0}^{\infty} \left[ \frac{1}{N_n!} \left( \frac{\alpha_{n+\epsilon}}{\sqrt{n+\epsilon}} \right)^N \frac{1}{\sqrt{n+\epsilon}} \right] |T\rangle$$ (23)

We want now to explore the state to operator correspondence. To the vacuum $|T\rangle$ we associate the twist field $\sigma_{\epsilon, f}(x)$ which depends both on the twist $\epsilon$ and on the position $f \in \mathbb{C}$ so that

$$|T\rangle = \lim_{x \to 0} \sigma_{\epsilon, f}(x) |0\rangle_{SL(2)}$$ (24)

with normalization

$$\langle T|T\rangle = 1$$ (25)

For the other states in Fock space it is however better to introduce the non normalized states

$$\prod_{n=0}^{\infty} \left( k_i \alpha_n \right)^{\epsilon} \left( k_i \alpha_n \right)^{\epsilon} \left( N_n \right)^{N_n} |T\rangle$$ (26)

with $k_\epsilon = -i\frac{1}{2} \sqrt{2\alpha} e^{i\epsilon\alpha}$ and $k_\xi = -i\frac{1}{2} \sqrt{2\alpha} e^{-i\epsilon\alpha}$ to which we make correspond the (generically non primary) chiral operators

$$\prod_{n=0}^{\infty} \left( \partial_u^{n+1} X \right)^{N_n} \left( \partial_u^{n+1} \bar{X} \right)^{N_n} \sigma_{\epsilon, f} (x)$$ (27)

This notation can be partially misleading since, for example, it is not true that (see eq. (37) for the right OPE)$^3$

$$\partial_u^2 X(u, \bar{u}) \sigma_{\epsilon, f}(x) \sim \frac{1}{(u-x)^{\#}} (\partial_u^2 X \sigma_{\epsilon, f})(x) + \ldots$$ (28)

$^2$ In this section we have dropped the dependence on $t$ as much as possible to simplify the notation e.g. $\epsilon_t \to \epsilon$.

$^3$ Notice that on shell $\partial_u^2 X(u, \bar{u}) = \frac{1}{2} \partial_u^2 X_L (u)$.
and therefore the operator \[^{(27)\text{ }}\] is just a way to write the operator which corresponds to the state \[^{(26)\text{ }}\] under the operator to state correspondence. The advantage of this notation is that it clearly shows which state corresponds to which operator and that it is consistent with the usual untwisted state to operator correspondence for which

\[
\prod_{n=1}^{\infty} \left( k \epsilon_n \alpha_n \right)^{N_n} \left( k \epsilon_n \alpha_n \right)^{\bar{N}_n} |0\rangle_{SL(2)} \leftrightarrow \prod_{n=1}^{\infty} \left( \partial_n^X \right)^{N_n} \left( \partial_n^\bar{X} \right)^{\bar{N}_n} (u, \bar{u})
\]

(29)

We have in particular, as we soon show, that

\[
\partial_u X(u, \bar{u}) \sigma_{\epsilon, f}(x) \sim (u - x)^{\epsilon - 1} (\partial_u X \sigma_{\epsilon, f})(x)
\]

\[
\partial_u \bar{X}(u, \bar{u}) \sigma_{\epsilon, f}(x) \sim (u - x)^{\bar{\epsilon} - 1} (\partial_u \bar{X} \sigma_{\epsilon, f})(x)
\]

(30)

so that the excited twist operators \((\partial_u X \sigma_{\epsilon, f})(x)\) and \((\partial_u \bar{X} \sigma_{\epsilon, f})(x)\) are what usually written as \(\tau(x)\) and \(\bar{\tau}(x)\). This observation also hints to another reason why to use the notation \[^{(27)\text{ }}\] which is to avoid the proliferation of new symbols, one for each excited twist operator.

Finally notice that almost all boundary operators can be recovered from chiral ones, for example when \(x_t < x < x_{t-1}\) and with the help of the boundary conditions \[^{(17)\text{ }}\] we get

\[
\partial_x (X(x, x)) = \partial_u X(u, \bar{u})|_{u=x} + \bar{\partial}_u X(u, \bar{u})|_{u=x} = \partial_u X(u, \bar{u})|_{u=x} + e^{i2\pi \alpha_t} \partial_u \bar{X}(u, \bar{u})|_{u=x}
\]

(31)

where both \(\partial_u X(u, \bar{u})|_{u=x}\) and \(\partial_u \bar{X}(u, \bar{u})|_{u=x}\) are chiral operators computed on the boundary. A notable exception is however \(e^{ikX(x,x)+ikX(x,x)}\) which is intrinsically non chiral since it cannot be expressed off shell using chiral operators. In particular from the previous eq. \[^{(31)\text{ }}\] it follows that boundary operators have more complex OPEs, such as

\[
\partial_x X(u, \bar{u})|_{u=x_1} \sigma_{\epsilon, f}(x_2) \sim (x_1 - x_2)^{\epsilon - 1} (\partial_u X \sigma_{\epsilon, f})(x_2) + (x_1 - x_2)^{\bar{\epsilon} - 1} e^{i2\pi \alpha_t} (\partial_u \bar{X} \sigma_{\epsilon, f})(x_2).
\]

(32)

3.1 Chiral OPEs from local Fock space

Let us now see how to compute OPEs of a chiral operator with an excited twist field using the local operatorial formalism. We will make several examples to make clear the way to proceed.

We start considering the simplest example, i.e. the OPE \(\partial_u X(u, \bar{u}) \sigma_{\epsilon, f}(x)\).
In order to find this OPE we compute

$$\lim_{x \to 0} \partial_u X(u, \bar{u}) \sigma_{\epsilon, f}(x)|0\rangle_{SL(2)} = \partial_{u_{loc}} X_{loc}(u_{loc})|T\rangle$$

(33)

where $\partial_u X_{loc}(u_{loc})$ is the operator in the twisted Fock space which corresponds to the abstract operator $\partial_u X(u, \bar{u})$ and we can identify $u_{loc} = u - x$. Now using the explicit expansion we get

$$\partial_{u_{loc}} X_{loc}(u_{loc})|T\rangle = u_{loc}^{-1} \kappa_{\epsilon} \alpha_{\epsilon}^1|T\rangle + \ldots$$

(34)

from which we deduce not only the leading order of the OPE (30) but also the higher order terms

$$\partial_u X(u, \bar{u}) \sigma_{\epsilon, f}(x) = (u - x)^{\epsilon - 1} (\partial_u X \sigma_{\epsilon, f})(x) + (u - x)^{\epsilon} (\partial_u^2 X \sigma_{\epsilon, f})(x) + \ldots$$

(35)

As a second example we consider the OPE $\partial_u^2 X(u, \bar{u}) \sigma_{\epsilon, f}(x)$. Proceeding as before and using the fact that the local Fock space operator which corresponds to $\partial_u^2 X(u, \bar{u})$ is $\partial_{u_{loc}}^2 X_{loc}(u_{loc})$ we can find

$$\partial_{u_{loc}}^2 X_{loc}(u_{loc})|T\rangle = u_{loc}^{\epsilon - 2} (\kappa_{\epsilon} \alpha_{\epsilon}^1|T\rangle + u_{loc}^{\epsilon - 1} \kappa_{\epsilon} \alpha_{\epsilon+1}^1|T\rangle + \ldots$$

(36)

so that we can deduce the OPE

$$\partial_u^2 X(u, \bar{u}) \sigma_{\epsilon, f}(x) = (u - x)^{\epsilon - 2} (\kappa_{\epsilon} \alpha_{\epsilon}^1|T\rangle + (u - x)^{\epsilon - 1} \kappa_{\epsilon} \alpha_{\epsilon+1}^1|T\rangle + \ldots$$

(37)

which shows clearly what stated before about the wrongness of eq. [28]. Obviously eq. (37) is compatible with eq. (36) since the former can be obtained from the latter by taking the derivative $\partial_u$.

In the previous examples the local Fock space operator has the same functional form of the abstract one but as discussed in [26] for the T dual configuration of branes with magnetic field this is not always the case. The correct mapping, which is redaired in section 3.3, is given by

$$\prod_{n=1}^{\infty} \left( \partial_u^N X \right)^{N_n} \left( \partial_u^N \bar{X} \right)^{N_n} (u, \bar{u}) \leftrightarrow \prod_{n=1}^{\infty} \frac{\partial^{N_n}}{\partial c_n^{N_n}} \frac{\partial^{N_n}}{\partial c_n^{N_n}} S_{c, \bar{c}}(c, \bar{c})$$

(38)

---

4 One could wonder whether eq. (33) should actually written using the quantum fluctuation, i.e. $\lim_{x \to 0} \partial_u X(u, \bar{u}) \sigma_{\epsilon, f}(x)|0\rangle_{SL(2)} = \partial_{u_{loc}} X_{loc}(u_{loc})|T\rangle$. The answer is no because CFT has no meaningful way of splitting $X$ into the classical and quantum part, moreover if we assume the previous expression we naively get $\lim_{x \to 0} \partial_u X(u, \bar{u}) \sigma_{\epsilon, f}(x)|0\rangle_{SL(2)} = \lim_{x \to 0} \partial_u (X_{cl} + X_q)(u, \bar{u}) \sigma_{\epsilon, f}(x)|0\rangle_{SL(2)} = \partial_u (X_{cl} + X_{loc,q})(u_{loc})|T\rangle$ and while $\partial_u X_{cl}$ has the expected singularity for $u \to x_t$ it contains also information on the location of the other twist fields, explicitly $\partial_u X_{cl} \sim (u - x_t)^{\epsilon - 1} R(u, \{x_{f\neq t}\})$ which would imply that full excited twists would contain information on these locations, e.g. $\partial_u X_{\sigma_{\epsilon_t, f_t}}(x_t) = \{R(x_t, \{x_{f\neq t}\}) \sigma_{\epsilon_t, f_t}(x_t) + (\partial_u X_q \sigma_{\epsilon_t, f_t})(x_t)$.
where the chiral generating function $S_c(c, \bar{c})$ is given by

$$
S_c(c, \bar{c}) = e^{\sum_{n=1}^{\infty} \left[ \bar{c}_n \partial_{u_{loc}}^n X_{loc}(u_{loc}, \bar{u}_{loc}) + c_n \partial_{\bar{u}_{loc}}^n \bar{X}_{loc}(u_{loc}, \bar{u}_{loc}) \right]}
$$

 exp \left\{ \sum_{n,m=1}^{\infty} \bar{c}_n c_m \partial_{u_{loc}}^n \partial_{\bar{u}_{loc}}^m \partial_{v_{loc}}^{n+m} |_{v_{loc}=u_{loc}} \Delta_c^{2 \varepsilon}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc}; \epsilon) \right\}

(39)

where $\cdots$ is the normal ordering and

$$
\Delta_c^{2 \varepsilon}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc}; \epsilon) = G_N^{2 \varepsilon}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc}; \{0, \epsilon, \infty, \bar{\epsilon}\}) - G_U^{2 \varepsilon}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc})
$$

(40)

is the regularized Green function with the twist $\sigma_{2 \varepsilon}$ at $x_{loc} = 0$ and the anti-twist $\sigma_{2 \varepsilon}$ in $x_{loc} = \infty$ and $G_U^{2 \varepsilon}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc})$ is the Green function for the untwisted string with boundary conditions corresponding to $D_t$ (see appendix A for more details). It is worth discussing how $\Delta_c$ has to be interpreted either as a kind of generating function or as a difference of Green functions, one of which associated to a couple of twist fields. This point of view is important in order to avoid confusions which could arise when considering the role of $\Delta_c$ in correlators with many twist fields. The answer is given in the derivation of eq. (39), which shows that $\Delta_c$ is a difference of Green functions.

As an example of the consequences of the previous expression (39) we can compute the local operator which corresponds to the energy-momentum tensor $T(u) = -\frac{2}{\alpha'} \partial_u X \partial_u \bar{X}$. We find the local operator

$$
T_{loc}(u_{loc}) = -\frac{2}{\alpha'} : \partial_{u_{loc}} X_{loc} \partial_{\bar{u}_{loc}} \bar{X}_{loc} : -\frac{2}{\alpha'} \partial_{u_{loc}} \partial_{\bar{u}_{loc}} \Delta_c^{2 \varepsilon}|_{v_{loc}=u_{loc}}
$$

$$
= -\frac{2}{\alpha'} : \partial_{u_{loc}} X_{loc} \partial_{\bar{u}_{loc}} \bar{X}_{loc} : \frac{\epsilon \bar{\epsilon}}{2} \frac{1}{u_{loc}^2}
$$

(41)

Then we can compute the OPE $T(u)\sigma_{2 \varepsilon}(x)$ from

$$
\left[ -\frac{2}{\alpha'} : \partial_{u_{loc}} X_{loc} \partial_{\bar{u}_{loc}} \bar{X}_{loc} : -\frac{\epsilon \bar{\epsilon}}{2} \frac{1}{u_{loc}^2} \right] |T\rangle = -\frac{\epsilon \bar{\epsilon}}{2} \frac{1}{u_{loc}^2} |T\rangle + \frac{1}{u_{loc}} \alpha_{2 \varepsilon} \bar{\alpha}_{2 \varepsilon} |T\rangle + O(1)
$$

(42)

to be

$$
T(u)\sigma_{2 \varepsilon}(x) \sim \frac{\epsilon \bar{\epsilon}}{2} \frac{1}{(u - x)^2} \sigma_{2 \varepsilon}(x) + \frac{1}{(u - x)^2} \frac{1}{k_{\epsilon} k_{\bar{\epsilon}}} \left( \partial_u X \partial_u \bar{X} \sigma_{2 \varepsilon}(x) \right) + O(1)
$$

(43)

---

5 For the chiral correlators there is no difference in using the untwisted Green function for $D_t$ or $D_{t+1}$ since the $G_U^{2 \varepsilon}$ is the only piece of the untwisted Green function $G_U^{2 \varepsilon}$ which contributes and it is insensitive to the angle at which the brane is rotated, this is not anymore true for the boundary correlators as we discuss later.
from which we read both the conformal dimension of $\sigma_{e,f}$ and that $k_i k_t \partial_v \sigma_{e,f} = (\partial_u X \partial_u \tilde{X} \sigma_{e,f}) (x)$. It is noteworthy that the double pole comes from the extra piece $\partial \partial \Delta_c$ in eq. (41) which would not be present in a naive state to operator correspondence.

As a last example to make clear the algorithm we consider the more complex OPE $(\partial^2_u X \partial_u X \partial_u \tilde{X})(u) (\partial^2_u \tilde{X} \sigma_{e,f}) (x)$ at the leading order. First we compute the operatorial realization of $(\partial^2_u X \partial_u X \partial_u \tilde{X})(u)$ to be

$$
\begin{align*}
& : (\partial^2_{u,loc} X_{loc} \partial_{u,loc} X_{loc} \partial_{u,loc} \tilde{X}_{loc})(u) : \\
& + \partial^2_{u,loc} \partial_{v,loc} \Delta_c^{zz} |_{v_{loc}=u_{loc}} \partial_{u,loc} X_{loc} + \partial_{u,loc} \partial_{v,loc} \Delta_c^{zz} |_{v_{loc}=u_{loc}} \partial^2_{u,loc} \tilde{X}_{loc} \\
= & : (\partial^2_{u,loc} X_{loc} \partial_{u,loc} X_{loc} \partial_{u,loc} \tilde{X}_{loc})(u) - \frac{k_i k_t (1-\epsilon)(2-\epsilon)}{2 u_{loc}^3} \partial_{u,loc} X_{loc} - \frac{k_i k_t (1-\epsilon)}{2 u_{loc}^2} \partial^2_{u,loc} X_{loc} \\
& \text{(44)}
\end{align*}
$$

then we associate the state $k_i \alpha_\epsilon |T_{\epsilon+2} \rangle$ to the excited twist $(\partial^2_u \tilde{X} \sigma_{e,f}) (x)$ then an easy computation gives

$$
(\partial^2_u X \partial_u X \partial_u \tilde{X})(u) (\partial^2_u \tilde{X} \sigma_{e,f}) (x) \sim (k_i k_t)^2 \frac{(1-\epsilon)^2}{2(u-x)^{1-\epsilon}} \sigma_{e,f}(x) \quad \text{(45)}
$$

where the leading order contribution comes from the terms linear in $X_{loc}$.

### 3.2 Boundary OPEs from local Fock space

In the computation of the interaction of twisted states with untwisted ones quite often we are not interested in chiral operators but in boundary operators such as $e^{ikX(x,x)+ik\tilde{X}(x,x)}$. For this case we must extend the analysis given in the previous section. The correct mapping is then given by

$$
e^{ikX+i\tilde{X}} \prod_{n=1}^{\infty} \left( \partial^2_{\tilde{X}} X^n \partial_{\tilde{X}}^2 \tilde{X}^{n-1} \right) (x+i0^+, x-i0^+) \leftrightarrow \prod_{n=1}^{\infty} \frac{\partial^{N_n}}{\partial \tilde{X}^{N_n}} \frac{\partial^{\tilde{N}_n}}{\partial \tilde{X}^{\tilde{N}_n}} S(c, \bar{c}) \bigg|_{c_0=0, c_n \geq 1 = 0} \quad \text{(46)}$$

where the generating function $S(c, \bar{c})$ is given by

$$S(c, \bar{c}) = : \sum_{n=0}^{\infty} \left[ \tilde{c}_n \partial^n_{x_{loc}} X_{loc}(x_{loc}+i0^+, x_{loc}-i0^+) + c_n \partial^n_{x_{loc}} \tilde{X}_{loc}(x_{loc}+i0^+, x_{loc}-i0^+) \right] :$$

$$
\begin{align*}
\exp \left\{ \sum_{n,m=0}^{\infty} \tilde{c}_n \ c_m \ \partial^n_{x_1} \partial^m_{x_2} \Delta_{loc}^{zz}(x_1; x_2; \epsilon) |x_1=x_2=x_{loc} \rangle \right\} \\
\exp \left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} c_n \ c_m \ \partial^n_{x_1} \partial^m_{x_2} \Delta_{loc}^{zz}(x_1; x_2; \epsilon) |x_1=x_2=x_{loc} \rangle \right\} \\
\exp \left\{ \frac{1}{2} \sum_{n,m=0}^{\infty} \tilde{c}_n \ c_m \ \partial^n_{x_1} \partial^m_{x_2} \Delta_{loc}^{zz}(x_1; x_2; \epsilon) |x_1=x_2=x_{loc} \rangle \right\}
\end{align*}
$$

\quad \text{(47)}
where : \cdots : is the normal ordering and we defined

\[ \Delta_{\text{bou}}^{IJ}(x_1; x_2; \epsilon) = \begin{cases} G^{IJ}(x_1+i0^+, x_1-i0^+; x_2+i0^+, x_2-i0^+; \{0, \epsilon; \infty, \epsilon\}) & x_1, x_2 > 0 \\ -G^{IJ}_{U(t)}(x_1+i0^+, x_1-i0^+; x_2+i0^+, x_2-i0^+) & x_1, x_2 < 0 \\ G^{IJ}(x_1+i0^+, x_1-i0^+; x_2+i0^+, x_2-i0^+; \{0, \epsilon; \infty, \epsilon\}) & x_1, x_2 > 0 \\ -G^{IJ}_{U(t+1)}(x_1+i0^+, x_1-i0^+; x_2+i0^+, x_2-i0^+) & x_1, x_2 < 0 \end{cases} \] (48)

with \( G^{IJ}(u_\text{loc}, \bar{u}_\text{loc}, v_\text{loc}, \bar{v}_\text{loc}; \{0, \epsilon; \infty, \epsilon\}) \) the Green function\(^6\) with the twist \( \sigma_\epsilon, f \) at \( x_\text{loc} = 0 \) and the anti-twist \( \sigma_\epsilon, f \) in \( x_\text{loc} = \infty \) and \( G^{IJ}_{U(t)}(u_\text{loc}, \bar{u}_\text{loc}, v_\text{loc}, \bar{v}_\text{loc}) \) is the Green function of the untwisted string with both ends on \( D_t \) and the need to distinguish \( x > 0 \) from \( x < 0 \) is due to the different boundary conditions of the twisted string in this ranges. As shown in eq. (126) in appendix A all \( \Delta_{\text{bou}}^{IJ} \) are equal to a real symmetric function \( \Delta_{\text{bou}}(x_1, x_2) \) up to phases which combine to allow the previous generating function to be written as

\[ S(c, \bar{c}) = : \exp \left\{ \sum_{n=0}^{\infty} \left[ \bar{c}_n \partial^n_{x_\text{loc}} X_{\text{loc}}(x_\text{loc} + i0^+, x_\text{loc} - i0^+) + c_n \partial^n_{x_\text{loc}} \bar{X}_{\text{loc}}(x_\text{loc} + i0^+, x_\text{loc} - i0^+) \right] \right\} : \exp \left\{ \sum_{n,m=0}^{\infty} c_n ||D_t|| D_m |D_t| \partial^n_{x_1} \partial^m_{x_2} \Delta_{\text{bou}}(x_1; x_2; \epsilon) \right\}_{|x_1 = x_2 = x_\text{loc}} \] (49)

when \( x_\text{loc} > 0 \) with

\[ c_n ||D_t| = \frac{e^{-i\pi \alpha t} c_n}{\sqrt{2}} + e^{i\pi \alpha t} \bar{c}_n. \] (50)

When \( x_\text{loc} < 0 \) we get a very similar expression but with the substitution \( c_n ||D_t| \rightarrow \bar{c}_n ||D_{t+1}| \) since in this case the vertex is on the brane \( D_{t+1} \).

Given the previous results we can now compute the operatorial realization of the the vertex \( e^{ik \cdot X(x,x)} \) to be, similarly to the results (26, 27)

\[
\left\{ \begin{array}{l}
|x_\text{loc}| \delta k^2 ||D_t| e^{-\frac{1}{2} R^2(\epsilon_t) \alpha' k^2 ||D_t|} : e^{i(k X_{\text{loc}}(x_{\text{loc}}, x_{\text{loc}}) + k \bar{X}_{\text{loc}}(x_{\text{loc}}, x_{\text{loc}}))} : x_\text{loc} > 0 \ [x_t < x < x_{t-1}] \\
|x_\text{loc}| -\alpha' k^2 ||D_{t+1}| e^{-\frac{1}{2} R^2(\epsilon_t) \alpha' k^2 ||D_{t+1}|} : e^{i(k X_{\text{loc}}(x_{\text{loc}}, x_{\text{loc}}) + k \bar{X}_{\text{loc}}(x_{\text{loc}}, x_{\text{loc}}))} : x_\text{loc} < 0 \ [x_{t+1} < x < x_t] 
\end{array} \right. 
\] (51)

where \( R^2(\epsilon_t) = 2\psi(1) - \psi(\epsilon_t) - \psi(\bar{\epsilon}_t) > 0 \), \( \psi(z) = \frac{d \ln \Gamma(z)}{dz} \) being the digamma function. Notice that we have not required the momentum \( k \) to be tangent to the brane since the normal part gives simply a phase due to the boundary conditions nevertheless, as commented before, the explicit expression for \( \Delta_{\text{bou}}^{IJ} \) implies that only the momentum parallel to the \( D_t \) brane \( k^i ||D_t| = \frac{e^{-i\pi \alpha t} k^i + e^{i\pi \alpha t} k^i}{\sqrt{2}} \)

enters the form factor \( e^{-\frac{1}{2} R^2(\epsilon_t) \alpha' k^2 ||D_t|} \).

\( ^6 \)We normalize the Green function such that \( \partial \bar{\partial} G(u_\text{loc}, \bar{u}_\text{loc}, v_\text{loc}, \bar{v}_\text{loc}) = -\frac{\alpha'}{2} \delta^2(u_\text{loc} - v_\text{loc}). \)
Using the previous operatorial realization we can then obtain the OPEs

\[ e^{i(kX(x,x)+kX(x,x))}\sigma_{\epsilon_{t}f_{t}}(x_{t}) \sim (x-x_{t})^{-\alpha'k^{2}_{D_{t}}}e^{-\frac{1}{2}R^{2}(\epsilon_{t})\alpha'k^{2}_{D_{t}}}e^{i(kf_{t}+kf_{t})} \]

\[ \left[ \sigma_{\epsilon_{t}f_{t}}(x_{t}) - (x-x_{t})^{\epsilon_{t}}\frac{\sqrt{2}e^{-i\alpha k_{\parallel}D_{t}}(\partial_{\mu}X\sigma_{\epsilon_{t}f_{t}})(x_{t})}{\epsilon_{t}} \right. \]

\[ \left. -(x-x_{t})^{\epsilon_{t}}\frac{\sqrt{2}e^{-i\alpha k_{\parallel}D_{t}}(\partial_{\mu}\bar{X}\sigma_{\epsilon_{t}f_{t}})(x_{t}) + \ldots}{\epsilon_{t}} \right] \quad x_{t} < x < x_{t-1} \]

and similarly for the \( x_{\text{loc}} < 0 \) case which corresponds to \( x_{t+1} < x < x_{t} \) with the substitution \( k_{\parallel}D_{t} \rightarrow k_{\parallel}D_{t+1} \) and \( (x-x_{t})^{\epsilon_{t}} \rightarrow (x_{t}-x)^{\epsilon_{t}} \). The previous OPE justify writing not only \( \sigma_{\epsilon_{t}}(x_{t}) \) but \( \sigma_{\epsilon_{t}f_{t}}(x_{t}) \) since the \( f_{t} \) and \( f_{t} \) can be computed using the phases of the leading order terms with different momenta. Moreover the phases \( e^{-i\alpha k_{\parallel}D_{t}} \) and \( e^{-i\alpha k_{\parallel}D_{t+1}} \) can be extracted from the projections \( k_{\parallel}D_{t}, k_{\parallel}D_{t+1} \) of momentum \( k \).

In a similar way we can compute the operator associated to \( \partial_{x}X(x,x) \) \( e^{ikX(x,x)} \) to be

\[ |x_{\text{loc}}|^{-\alpha'k^{2}_{D_{t}}}e^{-\frac{1}{2}R^{2}(\epsilon_{t})\alpha'k^{2}_{D_{t}}} : \left( \partial_{x_{\text{loc}}}X_{\text{loc}}(x_{\text{loc}},x_{\text{loc}}) - \frac{ie^{i\alpha k_{\parallel}D_{t}}}{\sqrt{2}x_{\text{loc}}} \right) e^{i(kX_{\text{loc}}(x_{\text{loc}},x_{\text{loc}})+kX_{\text{loc}}(x_{\text{loc}},x_{\text{loc}}))} : \]

(53)

when \( x_{\text{loc}} > 0 \) \( [x_{t} < x < x_{t-1}] \) and similarly for \( x_{\text{loc}} < 0 \) \( [x_{t+1} < x < x_{t}] \).

It is worth stressing that the term proportional to \( \frac{i}{x_{\text{loc}}}e^{i\alpha k_{\parallel}D_{t}} \) is fundamental in getting the right interaction among a gluon and twisted matter \( (28,26,27) \) when the amplitude is computed in operatorial formalism.

### 3.3 Short derivation of the generating function \( S \)

We will now quickly review the motivations to write down the generating vertex \( (47) \) and why it works.

Our aim is to get some hints on how to regularize the contact divergences that appear in the path integral computation of amplitudes in presence of twist fields. We consider the boundary case since all the others can be treated in an analogous manner. In the untwisted case the operatorial generating function is simply the Sciuto-Della Selva- Saito vertex \( (23) \)\(^7\)

\[ \mathcal{S}_{U}(c,\bar{c}) = :e^{\sum_{n=0}^{\infty} [c_{n}\partial_{x_{\text{loc}}}X_{U,\text{loc}}(x_{\text{loc}}+i0^{+},x_{\text{loc}}-i0^{+})+\bar{c}_{n}\partial_{x_{\text{loc}}}X_{U,\text{loc}}(x_{\text{loc}}+i0^{+},x_{\text{loc}}-i0^{+})]} : \]

(54)

Dropping the \( \text{loc} \) indication we can rewrite it as

\[ \mathcal{S}_{U}[J] = :e^{\int dx_{a}J_{1}(x_{a})X_{U}^{1}(x_{a},x_{a})} : \]

(55)

\(^7\) Because of the boundary conditions not all terms at the exponent are independent but this does not change the arguments in this section.
where we have introduced the current \( J_I(x_a) \) which must be set to \( J_I(x_a) = \sum_{n=0}^{\infty} c_n G^I_n(x_a - x) \) when we want to reproduce the original vertex but can also be taken more general as we do in the following.

We can now compute the OPE of two such generating functions (vertices)

\[
S_U[J_1] S_U[J_2] = e^{ \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b) }\ : \ S_U[J_1] S_U[J_2] : \\
e^{ \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b) } S_U[J_1 + J_2]
\]

which is valid for generic currents as long as they have compact support and the points in the support of \( J_I \) have bigger absolute values than those in the support of \( J_2 \) so that the operatorial product is radial ordered. Now the non operatorial term \( e^{ \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b) } \) can be roughly understood as the generating function for the OPE coefficients and we want to check that it is reproduced when using the generating function (47) which is defined in the twisted Fock space. In particular we consider the generating function with a more general current given by

\[
S_T[J] = e^{ \frac{1}{2} \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) \Delta^{IJ}_{bou}(x_a, x_b) } : e^{ \int dx_a J_I(x_a) X^I(x_a, x_a) } :
\]

It is then immediate to compute the product

\[
S_T[J_1] S_T[J_2] = e^{ \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b) } : S_T[J_1] S_T[J_2] :\\
e^{ \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b) } S_T[J_1 + J_2]
\]

where we have added and subtracted \( \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b) \) to the exponent in order to complete the prefactor in eq. (57) in the case where \( J = J_1 + J_2 \) and used the symmetry \( \Delta^{IJ}_{bou}(x_a, x_b) = \Delta^{IJ}_{bou}(x_b, x_a) \).

Having verified that the generating function (47) gives vertices with the same OPEs as the untwisted ones we can exam the reason which leads to it ([25, 29, 20]). In operatorial formalism the normal ordered vertex (55) for the untwisted case can be obtained from a regularized non normal ordered generating function by a multiplicative renormalization as

\[
S_U[J] = \lim_{\eta \to 0^+} N(\eta) \ S_{U,ren}[J, \eta]
\]

where the regularized generating function is defined by a point splitting as

\[
S_{U,ren}[J, \eta] = e^{ \int dx_a J_I(x_a) [X^I_{U}^{(+) \eta}(x_a, x_a) + \overline{X^I_{U}^{(-) \eta}}(x_a \epsilon - \eta, x_a \epsilon - \eta)]}
\]

without normal ordering and the multiplicative renormalization is given by the inverse of the factor we get by normal ordering the regularized generating function

\[
N(\eta) = e^{ - \frac{1}{2} \int dx_a \int dx_b J_I(x_a) J_2 J(x_b) G^{IJ}_{U,bou}(x_a, x_b \epsilon - \eta) }
\]

Now the generating function for the twisted string is defined in an analogous way by regularizing the non normal ordered generating function and then
renormalizing in a minimal way using the same renormalization factor as in
the untwisted case \([61]\), explicitly

\[
S_T[J] = \lim_{\eta \to 0^+} N(\eta) \ S_{T,\text{reg}}[J, \eta]
\]  

with

\[
S_{T,\text{reg}}[J, \eta] = e^{\int dx_a J(x_a) [X^I(+) (x_a, x_a) + X^I(\eta) (x_a e^{-\eta}, x_a e^{-\eta})]}
\]  

### 3.4 Getting excited twists

We are interested in excited twist states hence we would now write a kind of
SDS vertex which generates these states \([26]\). The main observation is then
that

\[
\frac{\partial^{n-1}}{\partial u_{\text{loc}}} \left[ u_{\text{loc}} \alpha \partial_{u_{\text{loc}}} X_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}) \right] = (n-1)! \ k_{\epsilon} \alpha_{n-1+\epsilon} + O(u_{\text{loc}})
\]

therefore a normal ordered products of these operators gives directly an ex-
cited twist state, e.g.

\[
\lim_{u_{\text{loc}} \to 0} : \frac{\partial^{n-1}}{\partial u_{\text{loc}}} \left[ u_{\text{loc}} \alpha \partial_{u_{\text{loc}}} X_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}) \right] \frac{\partial^{m-1}}{\partial u_{\text{loc}}} \left[ u_{\text{loc}} \alpha \partial_{u_{\text{loc}}} \bar{X}_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}) \right] : |T\rangle
\]

\[
= k_{\epsilon} k_{\epsilon} (n-1)! (m-1)! \alpha_{n-1+\epsilon} \alpha_{m-1+\epsilon} |T\rangle = \left( \partial^n X \partial^m \bar{X} \sigma_{\epsilon, f} \right) (0)|0\rangle_{\text{SL}(2)}
\]

The generating function for products of these operators is obviously

\[
T(d, \bar{d}) = \lim_{u_{\text{loc}} \to 0} : \exp \left\{ \sum_{n=1}^{\infty} \left[ \partial_n \frac{\partial^{n-1}}{\partial u_{\text{loc}}} \left[ u_{\text{loc}} \alpha \partial_{u_{\text{loc}}} X_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}) \right] + d_n \frac{\partial^{n-1}}{\partial u_{\text{loc}}} \left[ u_{\text{loc}} \alpha \partial_{u_{\text{loc}}} \bar{X}_{\text{loc}}(u_{\text{loc}}, \bar{u}_{\text{loc}}) \right] \right] : \right\}
\]

\[
\left[ \prod_{n=1}^{\infty} \left( \partial^n X \right)^{N_n} \left( \partial^n \bar{X} \right)^{\bar{N}_n} \sigma_{\epsilon, f} \right] (0)|0\rangle_{\text{SL}(2)} \leftrightarrow \lim_{u_{\text{loc}} \to 0} \lim_{\bar{u}_{\text{loc}} \to 0} \frac{\partial^{N_n}}{\partial d_n} \frac{\partial^{\bar{N}_n}}{\partial \bar{d}_n} \partial d_{\text{reg}} T(d, \bar{d}) \bigg|_{d=d=0}
\]

Comparing with eq. \([39]\) we realize that there is not the exponent quadratic
in \(d\), this means that the abstract operator corresponding to e.g. eq. \([65]\) is
not simply \(\lim_{u \to x} \partial_u^{n-1} [(u-x)^{\epsilon} \partial_u X] \partial_u^{m-1} [(u-x)^{\epsilon} \partial_u \bar{X}] \) but it is

\[
\lim_{u \to x} \left( \partial_u^{n-1} [(u-x)^{\epsilon} \partial_u X] \partial_u^{m-1} [(u-x)^{\epsilon} \partial_u \bar{X}] - \partial_u^{n-1} \partial_u^{m-1} [(u-x)^{\epsilon} (v-x)^{\epsilon} \partial_u \partial_v \Delta_{c}^{(x,v)} (u-x, \bar{u}-\bar{x}; v-x, \bar{v}-\bar{x}; \epsilon)]_{v=u} \right)
\]
in fact computing its OPE with the twist field \( \sigma_{e,f}(x) \) as in section 3.1 we get
\[
\left( \partial_u^{m-1} \left[ (u - x)^\ell \partial_u X \right] \partial_v^{n-1} \left[ (u - x)^\ell \partial_v \bar{X} \right] - \partial_u^{n-1} \partial_v^{m-1} \left[ (u - x)^\ell (v - x)^\ell \partial_u \partial_v \Delta_{\bar{c}}^{\bar{z}} \right] \right) \sigma_{e,f}(x)
\sim \left( \partial^m X \partial^n \bar{X} \sigma_{e,f} \right)(x) + O(u - x)
\] (69)

Notice that this OPE explains why we can use \( u - x \) as argument of \( \Delta_{\bar{c}}^{\bar{z}} \) and not another function which behaves as \( u - x + O(u - x)^2 \). We conclude therefore that the generating function for the abstract operators which give excited twists is given by
\[
\mathcal{T}_{abs}(d, \bar{d}) = \exp \left\{ \sum_{n=1}^{\infty} \left[ \tilde{d}_n \partial_u^{n-1} \left[ (u - x)^\ell \partial_u X(u, \bar{u}) \right] + \partial_u^{n-1} \left[ (u - x)^\ell \partial_u \bar{X}(u, \bar{u}) \right] \right] \right\}
\exp \left\{ - \sum_{n,m=1}^{\infty} \tilde{d}_n \tilde{d}_m \partial_u^{n-1} \partial_v^{m-1} \left[ (u - x)^\ell (v - x)^\ell \partial_u \partial_v \Delta_{\bar{c}}^{\bar{z}} (u - x, \bar{u} - x; v - x, \bar{v} - x; \epsilon) \right] \right\}
\] (70)

Explicitly this means that
\[
\left[ \prod_{n=1}^{\infty} \left( \partial_u^{n} X \right)^{N_n} \left( \partial_v^{n} \bar{X} \right)^{\bar{N}_n} \sigma_{e,f} \right](x) = \lim_{u \to x} \prod_{n=1}^{\infty} \frac{\partial^{N_n}}{\partial d_n^{N_n}} \frac{\partial^{\bar{N}_n}}{\partial \bar{d}_n^{\bar{N}_n}} \mathcal{T}_{abs}(d, \bar{d}) \bigg|_{d = \bar{d} = 0} \sigma_{e,f}(x)
\] (71)

### 4 The path integral approach

The classic method 11 to compute twists correlators is by the path integral
\[
\langle \sigma_{e_1, f_1}(x_1) \ldots \sigma_{e_N, f_N}(x_N) \rangle = \int_{\mathcal{M}(x_t, e_t, f_t)} D X e^{-S_E}
\] (72)
where \( \mathcal{M}(x_t, e_t, f_t) \) is the space of string configurations satisfying the boundary conditions [19] and [18]. Since the integral is quadratic we can then efficiently separate the classical fields from the quantum fluctuations as
\[
X(u, \bar{u}) = X_{cl}(u, \bar{u}; \{x_t, e_t\}) + X_q(u, \bar{u}; \{x_t, e_t\})
\] (73)
where \( X_{cl} \) satisfies the previous boundary conditions [17] [18] while \( X_q \) satisfies the same boundary conditions but with all \( f_t = 0 \). After this splitting we obtain
\[
\langle \sigma_{e_1, f_1}(x_1) \ldots \sigma_{e_N, f_N}(x_N) \rangle = \mathcal{N}(x_t, e_t) e^{-S_{E,cl}(x_t, e_t, f_t)}
\] (74)
where the factor \( \mathcal{N}(x_t, e_t) \) is the quantum contribution. In particular when all \( f_t \) are equal, i.e. \( f_t = f \) the classical action \( S_{E,cl}(x_t, e_t, f) = f \) is zero since the branes are the boundary of a zero area polygon therefore we can identify
\[
\mathcal{N}(x_t, e_t) = \langle \sigma_{e_1, f_1 = f}(x_1) \ldots \sigma_{e_N, f_N = f}(x_N) \rangle
\] (75)
which is also true for the quantum fluctuation for which $f = 0$. Our aim is now to compute correlators with both (excited) twist field operators and untwisted operators $V_{\xi_i}(x_i)$ ($i = 1 \ldots L$) associated with the untwisted state $\xi_i$ of which the ones in eq. (30) are a particular case. We start with correlators with plain twist field operators which can be computed as

$$\langle \sigma_{\epsilon_1 f_1}(x_1) \ldots \sigma_{\epsilon_N f_N}(x_N) \prod_{i=1}^L V_{\psi_i}(x_i) \rangle = \int_{\mathcal{M}(\{x_t, \epsilon_t, f_t\})} \mathcal{D}X \prod_{i=1}^L V_{\psi_i}(x_i) \ e^{-S_E}$$

(76)

To do so we notice that it is by far easier not to compute the previous correlator but to compute the generating function of all the correlators, i.e. the Reggeon vertex, in the form of the previous path integral (72) plus linear sources

$$V_{N+L}(J_i) = \lim_{\{\eta_i\} \to 0} \int_{\mathcal{M}(\{x_t, \epsilon_t, f_t\})} \mathcal{D}X \ e^{-S_E+\sum_{i=1}^L \int dx \ J_i(x) X^I(x,x)}$$

(77)

where $J_i(x) = \sum_{n=0}^\infty c(i)n! \partial^n_x \delta(x - x_i)$ since all untwisted operators can be obtained by taking derivatives with respect to the coefficients $c(i)n!$ as explained in the previous section. This starting point is very similar to (30), (24) where it was recognized that the generator for all closed (super)string amplitudes is a quadratic path integrals. The idea in the previous papers is that the appropriate boundary condition for R and/or NS sector can be obtained simply by inserting linear sources with the desired boundary conditions. Because of this assumption the quantum fluctuations are the same for all the amplitudes: from the purely NS to the mixed ones. It was later realized that this prescription misses a proper treatment of quantum fluctuations [31] and that when this part is considered the amplitudes factorize correctly [8].

4.1 Boundary correlators with non excited twists on $\mathbb{R}^2$

Our strategy is therefore to compute the path integral (77) by properly defining it in order to regularize the divergences which arise as usual because of current self interactions. Taking inspiration from what is done for deriving the generating function we first regularize the $\delta$ functions in the currents, a step which corresponds to the point splitting in operatorial formalism and then subtract the self interaction of the untwisted string. We are therefore led to consider the path integral

$$V_{N+L}(J_i) = \lim_{\{\eta_i\} \to 0} \int_{\mathcal{M}(\{x_t, \epsilon_t, f_t\})} \mathcal{D}X \ e^{-S_E} \times \prod_{i=1}^L \left[ e^{-\frac{1}{2} \int dx_a \int dx_b J_{iI}(x_a, \eta_i) J_{jJ}(x_b, \eta_i) G_{ij}^{IJ}(x_a, x_b) e^{\int dx \ J_{iI}(x, \eta_i) X^I(x,x)} \right]$$

(78)
where the regularized currents are defined as

$$J_{iI}(x, \eta_i) = \sum_{n=0}^{\infty} c_{(i)nI} \frac{\partial^{n} t_i}{x_{i}} \delta(x - x_i; \eta_i)$$  \hspace{1cm} (79)$$

with $\delta(x - x_i; \eta_i)$ a regularization of the $\delta$ such that $\lim_{\eta_i \to 0} \delta(x - x_i; \eta_i) = \delta(x - x_i)$ and $G_{U(t_i), \text{bou}}^{IJ}(x_a, x_b)$ is the untwisted Green function with boundary conditions which depends on the brane on which the point $x_i$ is. This dependence is the reason why we have written $U(t_i)$.

It is then immediate to compute the previous path integral by using the splitting of $X$ into quantum and classical part (73) and get

$$V_{N+L}(J_i) = \langle \sigma_{\epsilon_1, f_1}(x_1) \ldots \sigma_{\epsilon_N, f_N}(x_N) \rangle$$

$$\times \prod_{i=1}^{L} \int e^{\sum_{n=0}^{\infty} c_{(i)nI} \frac{\partial^{n} t_i}{x_{i}} X_{ij}(x_i)}$$

$$\times \prod_{i=1}^{L} \int e^{\frac{1}{2} \sum_{m=0}^{\infty} c_{(i)mJ} \frac{\partial^{m} t_i}{x_{i}} \Delta^{IJ}_{(N,M), \text{bou}}(x_i, \hat{x}_i; \{x_{i}, \epsilon_i\}) |_{\hat{x}_i = x_i}$$

$$\times \prod_{1 \leq i < j \leq L} e^{\sum_{n=0}^{\infty} c_{(i)nI} \sum_{m=0}^{\infty} c_{(j)mJ} \frac{\partial^{m} t_i}{x_{i}} \Delta^{IJ}_{(N,M), \text{bou}}(x_i, x_j; \{x_{i}, \epsilon_i\})}$$  \hspace{1cm} (80)$$

where we have introduced the boundary Green function

$$G_{(N,M), \text{bou}}^{IJ}(x_i, x_j; \{x_{i}, \epsilon_i\}) = G_{(N,M)}^{IJ}(x_i + i0^+, x_i - i0^+; x_j + i0^+, x_j - i0^+; \{x_{i}, \epsilon_i\})$$  \hspace{1cm} (81)$$

in presence of $N$ twist fields $\sigma_{\epsilon_t, f_t = 0}(x_t)$ (with $f_t = 0$ since these terms come from the quantum fluctuations) in the sector $M = \sum_{t=1}^{N} \epsilon_t$ and the regularized Green function

$$\Delta^{IJ}_{(N,M), \text{bou}}(x_i, \hat{x}_i; \{x_{i}, \epsilon_i\}) = G_{(N,M), \text{bou}}^{IJ}(x_i, \hat{x}_i; \{x_{i}, \epsilon_i\}) - G_{U(t_i), \text{bou}}^{IJ}(x_i, \hat{x}_i)$$  \hspace{1cm} (82)$$

As it happened for the $N = 2$ boundary Green function and its regularized version all the components of the boundary Green function are proportional to real symmetric functions $G_{(N,M), \text{bou}}(x_i, x_j)$ and $\Delta_{(N,M), \text{bou}}(x_i, x_j)$ respectively up to phases. Using the result from the appendix (D) where the explicit expressions for $G_{(N,M), \text{bou}}(x_i, x_j)$ and $\Delta_{(N,M), \text{bou}}(x_i, x_j)$ are given we can then write

$$V_{N+L}(J_i) = \langle \sigma_{\epsilon_1, f_1}(x_1) \ldots \sigma_{\epsilon_N, f_N}(x_N) \rangle$$

$$\times \prod_{i=1}^{L} \int e^{\sum_{n=0}^{\infty} c_{(i)nI} \frac{\partial^{n} t_i}{x_{i}} X_{ij}(x_i)}$$

$$\times \prod_{i=1}^{L} \int e^{\sum_{m=0}^{\infty} c_{(i)mI} \frac{\partial^{m} t_i}{x_{i}} \Delta_{(N,M), \text{bou}}(x_i, \hat{x}_i; \{x_{i}, \epsilon_i\}) |_{\hat{x}_i = x_i}$$

$$\times \prod_{1 \leq i < j \leq L} e^{\sum_{n=0}^{\infty} c_{(i)nI} \sum_{m=0}^{\infty} c_{(j)mI} \frac{\partial^{m} t_i}{x_{i}} \Delta_{(N,M), \text{bou}}(x_i, x_j; \{x_{i}, \epsilon_i\})}$$  \hspace{1cm} (83)$$
where $c_{n∥D_{t_{i}}}$ are defined as in eq. [50] and $t_{i}$ is chosen by the brane on which $x_{i}$ lies, i.e. $x_{t_{i}+1} < x_{i} < x_{t_{i}}$.

### 4.1.1 Some consequences

There are now two immediate consequences. The first and more trivial is that all correlators with just one derivative vertex have contribution only from the classical solution $X_{cl}$ [5], i.e.

$$\langle \sigma_{\epsilon_{1},f_{1}}(x_{1}) \ldots \sigma_{\epsilon_{N},f_{N}}(x_{N}) \partial_{x}^{n} X_{cl}(x,x) \rangle = \partial_{x}^{n} X_{cl}(x,x) \langle \sigma_{\epsilon_{1},f_{1}}(x_{1}) \ldots \sigma_{\epsilon_{N},f_{N}}(x_{N}) \rangle$$  \hspace{1cm} (84)

The second and more interesting is that all correlators can be essentially computed with Wick theorem plus classical contributions plus self interactions which are absent in Wick theorem. This implies that a string in presence of not excited twist fields (defects) is free and can be quantized almost in the usual manner.

### 4.1.2 Some examples

As a first example we want to compute the correlator of a tachyon with $N$ twist fields

$$\langle e^{ik_{I}X_{I}(x,x)} \prod_{t=1}^{N} \sigma_{\epsilon_{t},f_{t}}(x_{t}) \rangle \bigg|_{c_{1}=ik,\bar{c}_{1}=ik} = \prod_{t=1}^{N} \sigma_{\epsilon_{t},f_{t}}(x_{t}) \bigg|_{c_{1}=ik,\bar{c}_{1}=ik} e^{i k_{I} X_{cl}(x,x) + i k_{I} X_{cl}(x,x)} e^{-k_{I}^{2} D_{t} \Delta(N,M,bou(i))(x,x)}$$  \hspace{1cm} (85)

which shows that untwisted matter sees a kind of form factor of the interacting twisted matter in accord with the result from OPE and what discussed in [26] for the case of a stringy instanton. Again a priori we can take $k^{I}$ not parallel to the brane $D_{t}$ but the explicit form of the Green function implies a projection in the direction parallel to $D_{t}$. It is interesting to compare the previous result with what we can get using the OPE (52) in the previous correlator. Using the OPE at the leading order we get

$$\langle e^{ik_{I}X_{I}(x,x)} \prod_{t=1}^{N} \sigma_{\epsilon_{t},f_{t}}(x_{t}) \rangle \sim (x - x_{t})^{-\alpha' k_{I}^{2} D_{t}} e^{-\frac{1}{2} R^{2}(\epsilon_{t})} e^{i(k_{f t} + k_{f t})} \prod_{t=1}^{N} \sigma_{\epsilon_{t},f_{t}}(x_{t})$$  \hspace{1cm} (86)

Now using the obvious behavior

$$X_{cl}(u, \bar{u}) \sim_{u \rightarrow x_{t}} f_{t} + O((u - x_{t})^{\epsilon_{t}}) + O([u - x_{t}]^{\alpha'})$$  \hspace{1cm} (87)

and the result shown in app. \[\Box\]

$$\Delta(N,M,bou(i))(x, \hat{x}) \sim_{x \rightarrow x_{t}} \alpha' \frac{1}{2} \ln |x - x_{t}|^{2} + R^{2}(\epsilon_{t}) + O(x - \hat{x}) + O(x - x_{t})$$  \hspace{1cm} (88)
in the exact expression we find the expected consistency with the OPE result.

Secondly let us consider the correlator of a gauge boson with twisted matter

\[
\langle \epsilon_I \partial_x X^I e^{ik_1 X^I(x,x_i)} \prod_{t=1}^N \sigma_{\epsilon_t,f_t}(x_t) \rangle = \langle \prod_{t=1}^N \sigma_{\epsilon_t,f_t}(x_t) \rangle \langle e^{ik \tilde{X}_d(x,x)} e^{-k^2 \parallel D_1 \Delta_{(N,M),\text{bou}(i)}(x,x)} \rangle
\]

which exhibits the same structure as the vertex (53) and the OPE it can be computed using it.

Next we consider the interaction of two tachyons with the twisted matter in order to show the Wick-like expression in a simple case

\[
\langle e^{ik_1 X^I(x_1,x_1)} e^{ik_2 X^I(x_2,x_1)} \prod_{t=1}^N \sigma_{\epsilon_t,f_t}(x_t) \rangle = V_{N+2}\mid_{\epsilon_j=ik_j,\bar{\epsilon}_j=i\bar{k}_j}
\]

\[
=\langle \prod_{t=1}^N \sigma_{\epsilon_t,f_t}(x_t) \rangle \prod_{j=1}^2 \left[ e^{ik_\tilde{X}_d(x_j,x_j)} e^{-k^2 \parallel D_{1j} \Delta_{(N,M),\text{bou}(i)}(x_j,x_j)} \right]
\]

\[
e^{-k_1 \parallel D_1 k_2 \parallel D_2 G_{(N,M),\text{bou}}(x_1,x_2)}
\]

(90)

where \(G_{(N,M),\text{bou}}(x_1,x_2)\) is the common factor of all the components of the boundary Green function \(G_{(N,M),\text{bou}}(x_1,x_2)\) and is given in eq. (146) whose explicit expression (145) implies that only the momenta parallel to the brane on which the vertex lies contributes.
Finally, we consider a more lengthy example

\[
\langle (\partial^2_x X \partial_x X \partial_x \tilde{X})(x_1)(\partial^2_x \tilde{X})(x_2) \rangle = \prod_{t=1}^{N} \sigma_{\epsilon_t, f_t}(x_t) = \left. \frac{\partial^3}{\partial \xi_{(1)} \partial \xi_{(2)} \partial \xi_{(3)}} \right|_{c_{(3)}=0} V_{N+2}
\]

\[
= \prod_{t=1}^{N} \left[ (\partial^2_x X_{cl} \partial_x X_{cl} \partial_x \tilde{X}_{cl})(x_1)(\partial^2_x \tilde{X}_{cl})(x_2) \right.
\]

\[
+ e^{i\pi(-\alpha_1 - \alpha_2)} (\partial^2_x X_{cl} \partial_x X_{cl})(x_1) \partial_{\xi_1} \partial_{\xi_2} G_{(N,M),bou}(x_1, x_2)
\]

\[
+ e^{i\pi(\alpha_1 - \alpha_2)} (\partial^2_x X_{cl} \partial_x \tilde{X}_{cl})(x_1) \partial_{\xi_1} \partial_{\xi_2} G_{(N,M),bou}(x_1, x_2)
\]

\[
+ e^{i\pi(\alpha_1 - \alpha_2)} (\partial_x X_{cl} \partial_x \tilde{X}_{cl})(x_1) \partial_{\xi_1} \partial_{\xi_2} G_{(N,M),bou}(x_1, x_2)
\]

\[
+ \frac{1}{2} \partial^2_x X_{cl}(x_1) \partial_x \tilde{X}_{cl}(x_2) \partial_{\xi_1} \partial_{\xi_2} \Delta_{(N,M),bou}(x_1, \tilde{x}_1)|_{\tilde{x}_1=x_1}
\]

\[
+ \frac{1}{2} \partial_x X_{cl}(x_1) \partial_x X_{cl}(x_2) \partial_{\xi_1} \partial_{\xi_2} \Delta_{(N,M),bou}(x_1, \tilde{x}_1)|_{\tilde{x}_1=x_1}
\]

\[
+ \frac{1}{2} e^{2i\pi \alpha_1} \partial_x \tilde{X}_{cl}(x_1) \partial_x X_{cl}(x_2) \partial_{\xi_1} \partial_{\xi_2} \Delta_{(N,M),bou}(x_1, \tilde{x}_1)|_{\tilde{x}_1=x_1}
\]

\[
+ \frac{1}{2} e^{i\pi(\alpha_1 - \alpha_2)} \partial_{\xi_1} \Delta_{(N,M),bou}(x_1, \tilde{x}_1)|_{\tilde{x}_1=x_1} \partial_{\xi_1} \partial_{\xi_2} G_{(N,M),bou}(x_1, x_2)
\]

\[
+ \frac{1}{2} e^{i\pi(\alpha_1 - \alpha_2)} \partial_{\xi_1} \Delta_{(N,M),bou}(x_1, \tilde{x}_1)|_{\tilde{x}_1=x_1} \partial_{\xi_1} \partial_{\xi_2} G_{(N,M),bou}(x_1, x_2)
\]

\[
+ \frac{1}{2} e^{i\pi(\alpha_1 - \alpha_2)} \partial_{\xi_1} \Delta_{(N,M),bou}(x_1, \tilde{x}_1)|_{\tilde{x}_1=x_1} \partial_{\xi_1} \partial_{\xi_2} G_{(N,M),bou}(x_1, x_2)
\]

\[
\bigg]\bigg)\bigg|_{x_1=x_1, x_2=x_2}
\]

4.2 Boundary correlators with non excited twists on $T^2$

The wrapping contributions have been studied for the pure twist field correlators in [32] for the N=3 case and in [14] for the case $M = N - 2$ and there is not any difference among the different $M$ values therefore the results obtained there are valid. Let us anyhow quickly review them. Given a minimal $N$-polygon in $T^2$ with vertices $\{f_t\}$, i.e. with all vertices in the fundamental cell, we can consider non minimal polygons which wrap the $T^2$. These can be easier described as polygons which have vertices $\{\tilde{f}_t\}$ in the covering $\mathbb{R}^2$ where $T^2 \equiv \mathbb{R}^2/\Lambda$ with the lattice defined as $\Lambda = \{n_1 e_1 + n_2 e_2 | n_1, n_2 \in \mathbb{Z}\}$. These configurations give an additive contribution to the classical path integral as

\[
V_{N+M}^{(T^2)}(J_t) = \sum_{\{f_t\}} \left[ \langle \sigma_{\epsilon_1, \tilde{f}_1}(x_1) \ldots \sigma_{\epsilon_N, \tilde{f}_N}(x_N) \rangle \right] \prod_{i=1}^{M} \epsilon_{n=0} e_{n=0} \left[ \sum_{n=0}^{\infty} e_{m=0} \partial_{x_i} \partial_{x_i} \Delta_{(N,M),bou}(x_i, \tilde{x}_i)|_{\tilde{x}_i=x_i} \right]
\]

\[
\times \prod_{i=1}^{M} \left[ \sum_{n=0}^{\infty} e_{n=0} \partial_{x_i} \partial_{x_i} \Delta_{(N,M),bou}(x_i, \tilde{x}_i)|_{\tilde{x}_i=x_i} \right]
\]

\[
\times \prod_{1 \leq i < j \leq M} \left[ \sum_{n=0}^{\infty} e_{n=0} \partial_{x_i} \partial_{x_j} G_{(N,M),bou}(x_i, x_j)|_{x_i=x_j} \right]
\]

(92)
In order to determine the possible vertices \( \{ \tilde{f}_t \} \) without redundancy it is necessary to keep a vertex fixed and then expand the polygon. For definiteness we keep fixed the vertex \( \tilde{f}_1 = f_1 \) which lies at the intersection between \( D_N \) and \( D_1 \). We then move the next vertex \( f_2 \) along the \( D_1 \) brane. Explicitly we write \( \tilde{f}_2 = \tilde{f}_1 + (f_2 - f_1) + n_1 t_1 = f_2 + n_1 t_1 \) with \( n_1 \in \mathbb{Z} \) and \( t_1 \) the shortest tangent vector to \( D_1 \) which is in \( \Lambda \). We can now continue for all the other vertices for which we have \( \tilde{f}_t = \tilde{f}_{t-1} + (f_t - f_{t-1}) + n_{t-1} t_{t-1} = f_t + \sum_{k=1}^{t-1} n_k t_k \). For consistency we need requiring \( \tilde{f}_{N+1} \equiv \tilde{f}_1 = f_1 \), therefore the possible wrapped polygons are obtained from the solution of the Diophantine equation

\[
\sum_{t=1}^{N} n_t t_t = 0
\]

which cannot be solved in general terms but only on a case by case basis as discussed in [14].

4.3 Chiral correlators with non excited twists.

As a warming up for the computation of excited twist fields which we perform in the next section we consider the interaction of chiral vertices with plain twists.

We can now follow the same strategy we used in section 4.1 and compute the path integral with the insertion of an arbitrary number \( L_c \) of currents which act as generating functions for the chiral vertex operators. As done for the boundary correlators in section 4.1 we first regularize the currents and then subtract the self interaction of the untwisted string. We are therefore led to consider the path integral

\[
V_{N+L_c}(J_c) = \lim_{\{ \eta_c \} \to 0} \int_{\mathcal{M}(\{ x_t, \epsilon_t, f_t \})} DX e^{-S_E} \times \prod_{c=1}^{L_c} e^{-\frac{1}{2} \int d^2 u_a \int d^2 u_b J_{cI}(u_a, \eta_c) J_{cJ}(u_b, \eta_c) \partial_u a \partial_u b G_{U_{(t_c)}}^{IJ}(u_a, \bar{u}_a; u_b, \bar{u}_b) \int d^2 u \partial_u J_{cI}(u, \eta_c) \partial_u X^I(u, \bar{u})}
\]

where in the second line we have written the regularization factor analogous to the one used in section 4.1 which regularizes the currents in the third line. In the previous expression the regularized currents are defined as

\[
J_{cI}(u, \bar{u}, \eta_c) = \sum_{n=1}^{\infty} c(\eta_c) n I \partial_u^{n-1} \delta^2(u - u_c; \eta_c)
\]

with \( \delta^2(u - u_c; \eta_c) \) a regularization of the \( \delta^2(\cdot) \) such that \( \lim_{\eta_c \to 0} \delta^2(u; \eta_c) = \delta^2(u) \). Notice that we need using a \( \delta^2() \) in the previous expression since we use “directional” derivatives along \( u \). To subtract the self interaction of the untwisted string we used \( G_{U_{(t_c)}}^{IJ}(u_a, \bar{u}_a; u_b, \bar{u}_b) \), the untwisted Green function.
computed for an arbitrary \( t_c \) since we chose to consider currents with at least one derivative \( \partial_u \) so that only \( G^{z \bar{z}}_{U} \) give with a non vanishing contribution which is also independent on the brane \( D_t \).

Finally performing the path integral we get

\[
V_{N+L_c}(J_c) = \langle \sigma_{\epsilon_1, f_1}(x_1) \cdots \sigma_{\epsilon_N, f_N}(x_N) \rangle \\
\times \prod_{c=1}^{L_c} \left\{ e^{i \sum_{n=1}^{\infty} c_{(c) n} \hat{\theta}^{a n}_{u c} \{ \partial_u X^I(u_c, \bar{u}_c) \}} \times e^{i \frac{1}{2} \sum_{n, m=1}^{\infty} c_{(c) n} c_{(c) m} \hat{\theta}^{a n}_{u c} \hat{\theta}^{a m}_{u c} \Delta^{IJ}(N, M)(c) \{ u_c, \bar{u}_c; u_c, \bar{u}_c; \{ x_I, \epsilon_I \} \}|_{v_c=uc} \right\} \\
\times \prod_{1 \leq c < \epsilon \leq N} e^{i \sum_{n, m=1}^{\infty} d_{(c) n} d_{(c) m} \partial_u \partial_{\epsilon c} \hat{G}^{IJ(N, M)}(c) \{ u_c, \bar{u}_c; u_c, \bar{u}_c; \{ x_I, \epsilon_I \} \}}
\]

(96)

where we have written the dependence on the complex conjugate variables such as \( \bar{u} \) even if the derivatives are independent in order to be consistent with the notation used in the boundary case. The regularized chiral Green function is defined as expected as

\[
\partial_u \partial_{\epsilon c} \Delta^{IJ(N, M)(c)}(u, \bar{u}; v, \bar{v}; \{ x_I, \epsilon_I \}) = \partial_u \partial_{\epsilon c} G^{IJ(N, M)}(u, \bar{u}; v, \bar{v}; \{ x_I, \epsilon_I \}) - \partial_u \partial_{\epsilon c} G^{IJ}_{U(t)}(u, \bar{u}; v, \bar{v}; \epsilon_t)
\]

(97)

and we notice that the subtraction term is different from zero only when \( IJ = z \bar{z} \) or \( IJ = \bar{z} z \) because of the derivatives.

## 4.4 Correlators of excited twists on \( \mathbb{R}^2 \)

Finally we can compute the correlators of excited twist fields by letting the appropriate chiral currents collide with the twist fields. We follow the same strategy we used in section 4.1 and in the previous section 4.3 and compute the path integral with the insertion of one generating function (70) for each twist field. As done for the boundary correlators in section 4.1 we first regularize the \( \delta^2() \) functions in the currents and then subtract the self interaction of the untwisted string. We are therefore led to consider the path integral

\[
V_{N}(K_t) = \lim_{\{ u_t \} \to \{ x_t \} \eta \to 0} \int_{\mathcal{M}(\{ x_t, \epsilon_t, f_t \})} DX \ e^{-SE} \\
\times \prod_{t=1}^{N} \left[ e^{-\frac{1}{2} \int d^2 u a_t d^2 u_b K_{I t I}(u_a, \eta_t) K_{I t I}(u_b, \eta_t)(u_a - x_t)^i t (u_b - x_t)^j t \partial_{u a} \partial_{u b} G^{IJ}_{U(t)}(u_a, \bar{u}_a; u_b, \bar{u}_b) \right] \\
\times \int d^2 u K_{I t I}(u, \eta_t)(u - x_t)^i t \partial_{u} X^I(u, \bar{u}) \\
\times e^{-\int d^2 u a_t d^2 u_b K_{I t I}(u_a, \eta_t) K_{I t I}(u_b, \eta_t)(u_a - x_t)^i t (u_b - x_t)^j t \partial_{u a} \partial_{u b} \Delta^{I J \bar{z}}(x_I, \epsilon_I) (u_a - x_t, \bar{u}_a - x_I; u_b - x_t, \bar{u}_b - x_I; \epsilon_I) \right] 
\]

(98)

where in the second line we have written the regularization factor analogous to the one used in section 4.1 which regularizes the third line and finally in
the last line we have the necessary subtraction term which is in eq. (70). In the previous expression the regularized currents are defined as

\[ K_{tI}(u, \bar{u}, \eta_t) = \sum_{n=1}^{\infty} d_{nI} \partial_u^{n-1} \delta^2(u - u_t; \eta_t) \]  

(99)

with \( \delta^2(u - u_t; \eta_t) \) a regularization of the \( \delta^2() \) as in previous section. For writing a more compact expression we have also used \( \epsilon_{t\bar{z}} = \epsilon_t \) defined as

\[ \epsilon_{t\bar{z}} = \epsilon_t, \quad \epsilon_{t\bar{z}} = \epsilon_t \]  

(100)

Finally performing the path integral we get

\[ V_N(K_t) = \lim_{\{u_t\} \to \{x_t\}} \langle \sigma_{\epsilon_{t1}, f_1}(x_1) \ldots \sigma_{\epsilon_{tN}, f_N}(x_N) \rangle \]

\[ \times \prod_{t=1}^{N} \left\{ e^{\sum_{n,m=1}^{\infty} d_{(nI)\tau} \partial_u^{n-1} \delta^2(u - u_t; \eta_t)} \right\} \]

\[ \times e^{\frac{1}{2} \sum_{n,m=1}^{\infty} d_{(nI)\tau} d_{(mJ)\tau} \partial_u^{n-1} \partial_v^{n-1} \delta^2(u - u_t; \eta_t) \delta^2(v - v_t; \eta_t)} \]

\[ \times \prod_{1 \leq \tau < \tau' \leq N} e^{\sum_{n,m=1}^{\infty} d_{(nI)\tau} d_{(mJ)\tau} \partial_u^{n-1} \partial_v^{n-1} \delta^2(u - u_t; \eta_t) \delta^2(v - v_t; \eta_t)} \]  

(101)

where we have defined the regularized Green function at the twist fields \( t \) to be

\[ \partial_u \partial_v \Delta_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) = \]

\[ = \partial_u \partial_v \Delta_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) - \partial_u \partial_v \Delta_{e}(u - x_t, \bar{u} - x_t; v - x_t, \bar{v} - x_t; \epsilon_t) \]

\[ = \partial_u \partial_v G^{IJ}_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) - \partial_u \partial_v G^{IJ}_{(N,M)\tau=2}(u - x_t, \bar{u} - x_t; v - x_t, \bar{v} - x_t; \epsilon_t) \]  

(102)

and we used \( G^{IJ}_{U\tau}(u - x_t, v - x_t) = G^{IJ}_{U\tau}(u, v) \) to write the last line and again we have written the dependence on \( \bar{u} \) and \( \bar{v} \) even if the derivatives are independent on them for having a consistent notation. Actually because of the chiral derivatives the previous expression simplifies in two cases to

\[ \partial_u \partial_v \Delta^{x\bar{z}}_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) = \partial_u \partial_v G^{x\bar{z}}_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) \]

\[ \partial_u \partial_v \Delta^{\bar{x}z}_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) = \partial_u \partial_v G^{\bar{x}z}_{(N,M)\tau}(u, \bar{u}; v, \bar{v}; \{x_t, \epsilon_t\}) \]  

(103)

It is interesting to notice that regularized Green functions are got by subtracting the divergent part with the proper monodromy at the point of regularization which at the points where a twist field is located means \( G_{N=2} \) while in all other points means \( G_U \). In particular both \( (u - x_t)^{\epsilon_{t\tau}}(v - x_t)^{\epsilon_{t\bar{\tau}}} \partial_u \partial_v \Delta_{(N,M)\tau}^{IJ} \) and \( (u - x_t)^{\epsilon_{t\tau}}(v - x_t)^{\epsilon_{t\bar{\tau}}} \partial_v \partial_u G^{IJ}_{(N,M)\tau} \) are analytic functions at \( u = x_t \) whose explicit expression is given in appendix E.
More explicitly the previous generating function can be written as

\[ V_N(K_t) = \lim_{\{u_t\} \to \{x_t\}} \langle \sigma_{\epsilon_1, f_1} (x_1) \ldots \sigma_{\epsilon_N, f_N} (x_N) \rangle \]

\[
\times \prod_{t=1}^{N} \left\{ e^{\sum_{n,m=1}^{\infty} \tilde{d}(t)n \partial_{u_t}^{n-1}[(u_t - x_t)^{\epsilon_t} \partial_u X_{cl}(u_t)]} + d(t)n \partial_{u_t}^{n-1}[(u_t - x_t)^{\epsilon_t} \partial_u X_{cl}(u_t)] \right\}
\]

\[
\times e^{\frac{1}{2} \sum_{n,m=1}^{\infty} d(t)n \partial_{u_t}^{n-1} \partial_{v_t}^{m-1}[(u_t - x_t)^{\epsilon_t} (v_t - x_t)^{\epsilon_t} \partial_u \partial_v G_{\epsilon(M)}^{z_z}(u_t, u_t; v_t, v_t; \{x_t, \epsilon_t\})]} \right\} \right] \}
\]

\[
\times \prod_{1 \leq t < \ell \leq N} \left\{ e^{\sum_{n,m=1}^{\infty} \tilde{d}(t)n \partial_{u_t}^{n-1} \partial_{v_t}^{m-1}[(u_t - x_t)^{\epsilon_t} (v_t - x_t)^{\epsilon_t} \partial_u \partial_v G_{\epsilon(M)}^{z_z}(u_t, u_t; v_t, v_t; \{x_t, \epsilon_t\})]} \right\}
\]

\[
\times \sum_{n=1}^{\infty} \tilde{d}(t)n \partial_{u_t}^{n-1} \partial_{v_t}^{m-1}[(u_t - x_t)^{\epsilon_t} (v_t - x_t)^{\epsilon_t} \partial_u \partial_v \Delta_{\epsilon(M)}^{z_z}(u_t, u_t; v_t, v_t; \{x_t, \epsilon_t\})] \}
\]

(104)

### 4.4.1 Some examples

Let us consider the simplest non trivial correlator with one excited twist field

\[
\langle (\partial^n X \sigma_{\epsilon_t, f_t}) \prod_{i \neq t} \sigma_{\epsilon_i, f_i} \rangle = \frac{\partial}{\partial d(t)_{nz}} V_N |_{d=d=0}
\]

\[
= \left( \prod_i \sigma_{\epsilon_i, f_i} \right) \lim_{u_t \to x_t} \partial_{u_t}^{n-1}[(u_t - x_t)^{\epsilon_t} \partial_u X_{cl}(u_t)]
\]

(105)

where the limit is not strictly necessary since the expression \((u_t - x_t)^{\epsilon_t} \partial_u X_{cl}(u_t)\) is regular in \(x_t\). The explicit expression can be easily computed and it is given in appendix [F] for some cases. Nevertheless correlators with only one excited twist can be less trivial as

\[
\langle (\partial^n X \partial^m \bar{X} \sigma_{\epsilon_t, f_t}) \prod_{i \neq t} \sigma_{\epsilon_i, f_i} \rangle = \frac{\partial^2}{\partial d(t)_{nz} \partial d(t)_{nz}} V_N |_{d=d=0}
\]

\[
= \left( \prod_i \sigma_{\epsilon_i, f_i} \right) \lim_{u_t \to x_t} \left[ \partial_{u_t}^{n-1}[(u_t - x_t)^{\epsilon_t} \partial_u X_{cl}(u_t)] \partial_{u_t}^{m-1}[(u_t - x_t)^{\epsilon_t} \partial_u X_{cl}(u_t)] \right]
\]

(106)
Finally we can also consider

\[
\langle (\partial^n T \sigma_{\epsilon_1, f_i}) (\partial^m \bar{T} \sigma_{\epsilon_2, f_j}) \prod_{i \neq j, l} \sigma_{\epsilon_l, f_l} \rangle = \frac{\partial^2}{\partial d(t)_{mz} \partial d(t)_{mz}} V_N |_{d=d=0}
\]

\[
= \langle \prod_{i} \sigma_{\epsilon_i, f_i} \rangle \lim_{u_{t,i} \to u_{t,i}} \left[ \partial_{u_{t,i}}^{n-1} [(u_{t,i} - x_{i})^{\epsilon_i} \partial_u X_{\hat{c}l}(u_t)] \partial_{u_{t,i}}^{m-1} [(u_{t,i} - x_{i})^{\epsilon_i} \partial_u X_{\hat{c}l}(u_t)] \right.
\]

\[
+ \partial_{u_{t,i}}^{n-1} [ \partial_{u_{t,i}}^{m-1} [(u_{t,i} - x_{i})^{\epsilon_i} \partial_u X_{\hat{c}l}(u_t)]]
\]

\[
\left. \right] \sum_{\{ \epsilon_i, x_i \}} \left( (u_{t,i} - x_{i})^{\epsilon_i} \partial_u X_{\hat{c}l}(u_t) \right)
\]

(107)

This correlator is the correlator of eq. (4.83) of \[\text{[5]}, \text{i.e. } \langle \tau_1(x_1) \tau_2(x_2) \sigma_2(x_3) \sigma_2(x_4) \rangle \]
when we set \(N = 4, n = m = 1\) and \(\epsilon_1 = \alpha, \epsilon_2 = 1 - \alpha, \epsilon_3 = \beta \text{ and } \epsilon_4 = 1 - \beta.\)

### 4.5 Correlators of boundary operators and excited twists on \(\mathbb{R}^2\)

Finally we can assemble the results from previous section to write the generating function for correlators of boundary operators and excited twists on \(\mathbb{R}^2\) to be

\[
V_{N+L}(K_t, J_i) = \lim_{\{ u_t \} \to \{ x_t \}} \left\{ \langle \sigma_{\epsilon_1, f_1}(x_1) \ldots \sigma_{\epsilon_N, f_N}(x_N) \right\}
\]

\[
\times \prod_{i=1}^{N} \left\{ \epsilon \sum_{n=1}^{\infty} d_t(x_{i}) \partial_{x_{i}}^{n-1} [(u_{t,i} - x_{i})^{\epsilon_i} \partial_u X_{\hat{c}l}(u_t)] \right\}
\]

\[
\times \epsilon \sum_{n=0}^{\infty} c_t(x_{i}) \partial_{x_{i}}^{n} X_{\hat{c}l}(x_{i}, x_{i})
\]

\[
\times \left\{ \epsilon \sum_{n=0}^{\infty} c_t(x_{i}) \sum_{m=0}^{\infty} c_t(x_{j}) \partial_{x_{j}}^{m} \Delta^{f_1}_{(N,M)} (x_{i}, x_{j}, \{ x_{t}, \epsilon_{t} \}) \right\}
\]

\[
\times \prod_{1 \leq i < j \leq N} \left\{ \epsilon \sum_{n=0}^{\infty} c_t(x_{i}) \sum_{m=0}^{\infty} c_t(x_{j}) \partial_{x_{j}}^{m} \Delta^{f_1}_{(N,M)} (x_{i}, x_{j}, \{ x_{t}, \epsilon_{t} \}) \right\}
\]

\[
\times \prod_{1 \leq i \leq N} \left\{ \epsilon \sum_{n=0}^{\infty} c_t(x_{i}) \sum_{m=0}^{\infty} c_t(x_{j}) \partial_{x_{j}}^{m} \Delta^{f_1}_{(N,M)} (x_{i}, x_{j}, \{ x_{t}, \epsilon_{t} \}) \right\}
\]

(108)

where the last line is the interaction between the twist fields and the boundary operators. The generating function for correlators on \(T^2\) can be formally easily obtained as done in section 4.2 by summing over all possible wrapping contributions as in eq. \[\text{[92]}\]
4.6 Correlators of bulk operators and excited twists on $\mathbb{R}^2$

We can now make an educated guess of the generating function of the correlators of bulk operators and excited twists. As long as the bulk vertex operators do not involve momenta there is no doubt on the result since the bulk field can be written as the product of a chiral vertex times an antichiral vertex therefore the generating function is nothing else but the product of the generating function for the chiral part times the generating function for the antichiral part times the obvious interaction of the chiral current with the antichiral. What requires an educated guess is when the bulk vertex operators involve momenta since in this case we know that the local description requires to separate the right moving from the left moving part and normal order them separately, i.e. to the abstract vertex $e^{ik_1X^i(u,\tilde{u})}$ corresponds the local version $e^{ik_1X^i_{(loc)}(u_{loc})}$ :: $e^{ik_1X^i_{(loc)}R(\tilde{u}_{loc})}$ : up to cocycles ($[33], [26]$ but see also $[34]$) Hence we guess that in the presence of momenta the twisted Green function must be split in its chiral-chiral, chiral-antichiral and so on pieces, i.e. $G = G_{(LL)} + G_{(LR)} + G_{(RL)} + G_{(RR)}$. This is obviously consistent with the case where derivatives are present since applying a derivative like $\partial_u$ of the Green function is actually projecting it on the chiral piece. We therefore guess that the generating function of the correlators of $L_c$ bulk operators and $N$ excited twists read up to phases due to cocycles

$$V_{N+L_c}(K_t, J_c) = \lim_{\{u_c\} \to \{x_t\}} \langle \sigma_{e_1, f_1}(x_1) \ldots \sigma_{e_N, f_N}(x_N) \rangle$$

$$\times \prod_{c=1}^{L_c} \left\{ e^{\sum_{n=1}^{\infty} q_{(L_c)\alpha} \partial_{\alpha} \cdot \frac{\partial^m}{\partial_{\alpha}^{m-1}} [(u_t-x_t)^{c}\partial_{\alpha} X^i_t(u_{\hat{t}}, \tilde{u}_{\hat{t}})]} \right\}$$

$$\times \prod_{c=1}^{L_c} \left\{ e^{\sum_{n=0}^{\infty} q_{(R_c)\alpha} \partial_{\alpha} \cdot \frac{\partial^m}{\partial_{\alpha}^{m-1}} [(u_t-x_t)^{c}\partial_{\alpha} \Delta_{(N,M)tl}(u_{c}, \tilde{u}_{c}; \{x_t, \epsilon_t\})]|_{u_t=\hat{u}_c}} \right\}$$

$$\times \prod_{c=1}^{L_c} \left\{ e^{\sum_{n=0}^{\infty} q_{(R_c)\alpha} \partial_{\alpha} \cdot \frac{\partial^m}{\partial_{\alpha}^{m-1}} [(u_t-x_t)^{c}\partial_{\alpha} \Delta_{(N,M),(LL,c)}(u_{c}, \tilde{u}_{c}; \{x_t, \epsilon_t\})]|_{\tilde{u}_c=\hat{u}_c}} \right\}$$

$$\times \prod_{c=1}^{L_c} \left\{ e^{\sum_{n=0}^{\infty} q_{(R_c)\alpha} \partial_{\alpha} \cdot \frac{\partial^m}{\partial_{\alpha}^{m-1}} [(u_t-x_t)^{c}\partial_{\alpha} \Delta_{(N,M),(RR,c)}(u_{c}, \tilde{u}_{c}; \{x_t, \epsilon_t\})]|_{\tilde{u}_c=\hat{u}_c}} \right\}$$
Using the previous generating function it would be interesting deriving the boundary state with $N$ twist fields. This could be done as in [35] and would be an interesting generalization of the boundary state with open string interactions derived in [36]. This boundary state could be used as in [37] to derive useful information about the long distance spacetime geometry generated by branes at angles.

4.7 Rewriting the Reggeon vertex using auxiliary Fock spaces

In the previous section we have given the explicit form of the generating function for correlators of boundary operators and excited twists. Traditionally and for sewing a different expression is used where to any operator insertion, i.e. external leg is associated an auxiliary Fock space. If we associate to any twist operator an auxiliary Fock space with vacuum $|T_i\rangle$ and we identify\(^8\)

\[
(n-1)!d_{(t)n} \leftrightarrow \frac{-2}{\alpha'} k_{\epsilon_t} \frac{\alpha(t,\text{aux})_{n-\epsilon_t}}{n-\epsilon_t}, \quad (n-1)!d_{(t)n} \leftrightarrow \frac{-2}{\alpha'} k_{\epsilon_t} \frac{\alpha(t,\text{aux})_{n-\epsilon_t}}{n-\epsilon_t},
\]

and we do the same for any boundary vertex operator to which we associate an auxiliary Fock space with vacuum $|0_a, p_{(j)} = 0\rangle$ and we identify ($n \geq 0$)

\[
n!c_{(j)n} \leftrightarrow \frac{-2}{\alpha'} k_{\epsilon_t} \alpha_{(j,\text{aux})_{n}}, \quad n!c_{(j)n} \leftrightarrow \frac{-2}{\alpha'} k_{\epsilon_t} \alpha_{(j,\text{aux})_{n}}\]

\(^8\) The normalization is chosen in the usual way such that applying the map $d$ to auxiliary operators on eq. [36] we have $\langle T_{\text{aux}} | T(d \rightarrow \ddot{\alpha}_{\text{aux}}, d \rightarrow \alpha_{\text{aux}}) \lim_{u_{\text{aux}} \rightarrow 0} \partial_{u_{\text{aux}}}^n \{u_{\text{aux}} \ddot{\partial}_{u_{\text{aux}}} X_{\text{loc}}(u_{\text{aux}}, \ddot{u}_{\text{aux}})\} | T_{\text{aux}} \rangle = \partial_{\dot{u}_{\text{loc}}}^n \{\dot{u}_{\text{loc}} \ddot{\partial}_{\dot{u}_{\text{loc}}} X_{\text{loc}}(u_{\text{loc}}, \ddot{u}_{\text{loc}})\}$.
we can write the generating function as a usual Reggeon vertex as

\[
\langle V_{N+L} \rangle = \prod_{t=1}^{N} \langle T_t \rangle \prod_{i=1}^{L} \langle 0, x(i) = 0 \rangle 
\]

\[
\prod_{t=1}^{N} \left[ \frac{2}{\eta^2} \int_{w=x(t)} \frac{dz}{2\pi i} \chi(t, aux)(z-x(t)) \chi(t, aux)(w-x(t)) \partial_z \partial_w \hat{G}_{(N,M)}^{\hat{z}}(z;\bar{z};w,\bar{w}) \right]
\]

\[
\prod_{i,j=1}^{L} \left[ \frac{1}{\eta^2} \int_{w=x(i)} \frac{dz}{2\pi i} \chi(i, aux)(z-x(i)) \partial_w \hat{G}_{(N,M)}^{\hat{z}}(z;\bar{z};w,\bar{w}) \right]
\]

\[
\prod_{i=1}^{L} \prod_{t=1}^{N} \left[ \frac{1}{\eta^2} \int_{w=x(t)} \frac{dz}{2\pi i} \chi(t, aux)(z-x(t)) \chi(t, aux)(w-x(t)) \partial_z \partial_w \hat{G}_{(N,M)}^{\hat{z}}(z;\bar{z};w,\bar{w}) \right]
\]

where we have used the doubled fields \( \chi_f(z) \) defined as in eq. \[111 \] for the twisted case and in the usual way for the untwisted one. Notice that the terms \( t = \hat{t} \) and \( i = j \) must be regularized by properly computing the two contour integrals as discussed in \[23 \].
Then we can compute the untwisted Green functions even if care must be taken in order to deal with the non trivial commutators

\[ [\alpha(t)_n, \alpha(t)_m^\dagger] = n\delta_{m,n}, \quad [\tilde{\alpha}(t)_n, \tilde{\alpha}(t)_m^\dagger] = n\delta_{m,n}, \quad [x^1, p^1] = i \]

where \( x^1, p^1 \) are the zero mode position and momentum of string \( X^1 = \frac{1}{\sqrt{2}} (e^{-i\pi\alpha X} + e^{i\pi\alpha X}) \) with \( NN \) boundary condition. The vacuum is defined in the usual way by

\[ \alpha(t)_n |0_t\rangle = \tilde{\alpha}(t)_n |0_t\rangle = p^1 |0_t\rangle = 0 \quad n \geq 1 \]

even if care must be taken in order to deal with the \( DD \) zero modes. Then we can compute the untwisted Green functions

\[ G_{U(t)}^{zz}(u_{loc}, u_{loc}; v_{loc}, v_{loc}) = [X_{loc}^{(+)} (u_{loc}, u_{loc}), X_{loc}^{(-)} (v_{loc}, v_{loc})] = k_e^2 \ln |u_{loc} - v_{loc}|^2 \]

\[ G_{U(t)}^{zz}(u_{loc}, u_{loc}; v_{loc}, v_{loc}) = [X_{loc}^{(+)} (u_{loc}, \tilde{u}_{loc}), X_{loc}^{(-)} (v_{loc}, \tilde{v}_{loc})] = k_{e \tilde{e}} \ln |u_{loc} - \tilde{v}_{loc}|^2 \]

\[ G_{U(t)}^{zz}(u_{loc}, u_{loc}; v_{loc}, v_{loc}) = [X_{loc}^{(+)} (u_{loc}, \tilde{u}_{loc}), X_{loc}^{(-)} (v_{loc}, \tilde{v}_{loc})] = k_e k_{\tilde{e}} \ln |u_{loc} - \tilde{v}_{loc}|^2 \]

A Details and useful formula for the untwisted string and \( N = 2 \) case

We start considering the untwisted string associated to the \( D_t \) brane, i.e. the string with both ends on \( D_t \). This has boundary conditions

\[ Re(e^{-i\pi\alpha \partial_y X_{loc}}|y=0) = Im(e^{-i\pi\alpha X_{loc}}|y=0) - g_t = 0 \]

in the upper half plane \( H \) and has the expansion

\[ X_{loc}(u_{loc}, \bar{u}_{loc}) = e^{i\pi\alpha t} \left[ x^1 + ig_t - i2\alpha' p^1 \ln |u_{loc}| \right] + \frac{1}{2}\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[ \frac{\tilde{\alpha}(t)_n}{n} u_{loc}^{-n} - \frac{\alpha(t)_n}{n} u_{loc}^n \right] \]

\[ + \frac{1}{2} \sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[ -\frac{\tilde{\alpha}(t)_n}{n} \bar{u}_{loc}^{-n} + \frac{\alpha(t)_n}{n} \bar{u}_{loc}^n \right] \]

\[ \tilde{X}_{loc}(u_{loc}, \bar{u}_{loc}) = e^{-i\pi\alpha t} \left[ x^1 - ig_t - 2\alpha' ip^1 \ln |u_{loc}| \right] + \frac{1}{2}\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[ \frac{\tilde{\alpha}(t)_n}{n} u_{loc}^{-n} - \frac{\alpha(t)_n}{n} u_{loc}^n \right] \]

\[ + \frac{1}{2} \sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[ \frac{\tilde{\alpha}(t)_n}{n} \bar{u}_{loc}^{-n} + \frac{\alpha(t)_n}{n} \bar{u}_{loc}^n \right] \]
where $k_{t\epsilon} = -i \frac{1}{2} \sqrt{2\alpha'} e^{i \pi a t}$ and $k_{t\epsilon} = -i \frac{1}{2} \sqrt{2\alpha'} e^{-i \pi a t}$ as in the main text. Notice that $G_{U(t)}^{zz}$ does not feel that the brane is rotated while both $G_{U(t)}^{zz}$ and $G_{U(t)}^{zz}$ do because of the phase in $k_{t\epsilon}^2$ and $k_{t\epsilon}^2$.

In a similar way we can compute the $N = 2$ twisted Green functions

$$G_{N=2}^{zz}(u, \bar{u}; v, \bar{v}; \{0, \epsilon; \infty, \bar{\epsilon}\}) = [X_{loc}^{(+)}(u_{loc}, \bar{u}_{loc}), X_{loc}^{(-)}(v_{loc}, \bar{v}_{loc})]$$

$$= -k_{t\epsilon} \left[ 1 + \frac{1}{\epsilon} \left( \frac{u_{loc}}{v_{loc}} \right)^{\epsilon} 2F_1(1, \epsilon; 1 + \epsilon; \frac{v_{loc}}{u_{loc}}) + \frac{1}{\epsilon} \left( \frac{\bar{u}_{loc}}{\bar{v}_{loc}} \right)^{\epsilon} 2F_1(1, \bar{\epsilon}; 1 + \bar{\epsilon}; \frac{\bar{v}_{loc}}{\bar{u}_{loc}}) \right]$$

$$G_{N=2}^{zz}(u, \bar{u}; v, \bar{v}; \{0, \epsilon; \infty, \bar{\epsilon}\}) = [\tilde{X}_{loc}^{(+)}(u_{loc}, \bar{u}_{loc}), \tilde{X}_{loc}^{(-)}(v_{loc}, \bar{v}_{loc})]$$

$$= -k_{t\epsilon} \left[ 1 + \frac{1}{\epsilon} \left( \frac{\bar{u}_{loc}}{\bar{v}_{loc}} \right)^{\epsilon} 2F_1(1, \bar{\epsilon}; 1 + \bar{\epsilon}; \frac{\bar{v}_{loc}}{\bar{u}_{loc}}) + \frac{1}{\epsilon} \left( \frac{u_{loc}}{v_{loc}} \right)^{\epsilon} 2F_1(1, \epsilon; 1 + \epsilon; \frac{v_{loc}}{u_{loc}}) \right]$$

$$G_{N=2}^{zz}(u, \bar{u}; v, \bar{v}; \{0, \epsilon; \infty, \bar{\epsilon}\}) = [X_{loc}^{(+)}(u_{loc}, \bar{u}_{loc}), \tilde{X}_{loc}^{(-)}(v_{loc}, \bar{v}_{loc})]$$

$$= G_{N=2}^{zz}(v, \bar{v}; u, \bar{u}; \{0, \epsilon; \infty, \bar{\epsilon}\}) = G_{N=2}^{zz}(v, \bar{v}; u, \bar{u}; \{0, \epsilon; \infty, \bar{\epsilon}\})$$

where we have used

$$2F_1(1, \epsilon; 1 + \epsilon; x) = \sum_{n=0}^{\infty} \frac{\epsilon}{n + \epsilon} x^n \quad |x| < 1$$

as follows from the general expression for the hypergeometric function $2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!c_n} x^n$ with $(a)_n = \Gamma(a + n)/\Gamma(a)$ the Pochhammer symbol. They have the following symmetry properties

$$G_{N=2}^{I\bar{J}}(u, \bar{u}; v, \bar{v}; \{0, \epsilon; \infty, \bar{\epsilon}\}) = G_{N=2}^{I\bar{J}}(v, \bar{v}; u, \bar{u}; \{0, \epsilon; \infty, \bar{\epsilon}\}) = G_{N=2}^{\bar{I}J}(v, \bar{v}; u, \bar{u}; \{0, \epsilon; \infty, \bar{\epsilon}\})$$

which follow from the hypergeometric transformation properties in particular

$$2F_1(1, \epsilon; 1 + \epsilon; x) = \frac{x}{\epsilon} 2F_1(1, \epsilon; 1 + \epsilon; 1/x).$$

For future use f.x. in eq. (145) we notice that

$$g(u, v; \epsilon) = (k_{t\epsilon}, k_{t\epsilon})^{-1} \partial_u \partial_v G_{U(t)}^{zz} = z^{-\epsilon} w^{-\epsilon} \frac{\bar{\epsilon} z + \epsilon w}{(u - v)^2}$$

$$l(u, v; \epsilon) = \partial_u \partial_v G_{U(t)}^{zz} = 0$$

$$h(u, v; \epsilon) = \partial_u \partial_v G_{U(t)}^{zz} = 0$$

(121)

In order to write $\Delta^{I\bar{J}}$, the regularized Green function, in a more compact and transparent way we introduce the quantity

$$D(x; \epsilon) = \sum_{n=0}^{\infty} \frac{x^{n+\epsilon}}{n + \epsilon} - \sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{x^\epsilon}{\epsilon} 2F_1(1, \epsilon; 1 + \epsilon; x) + \log(1 - x) \quad |\arg(x)| < \pi$$

(122)
which can be expanded around $x = 1$ as

$$D(x; \epsilon) = \psi(1) - \psi(\epsilon) - \sum_{n=1}^{\infty} \left( \frac{\epsilon - 1}{n} \right) \frac{(x - 1)^n}{n}$$  \hspace{1cm} (123)$$

where $\psi(x) = d \log \Gamma(x)/dx$ is the digamma function. All of this expansion but the constant term can be easily obtained by computing $D'(x; \epsilon)$ and the integrating on $x$. To get the constant term is necessary to use the

$$2F_1(a, b; a + b; x) = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [2\psi(n + 1) - \psi(a + n) - \psi(b + n) - \log(1 - x)] (1 - x)^n$$

|arg(1 - x)| < \pi, \quad |1 - x| < 1  \hspace{1cm} (124)$$

We can now write the boundary $\Delta_{11}^{LJ}$ defined as

$$\Delta_{11}^{LJ}(x_1, x_2; \epsilon) = \left\{ \begin{array}{ll} G_{N=2}^{LJ}(x_1 + i0^+, x_1 - i0^+, x_2 + i0^+, x_2 - i0^+; \{0, \epsilon; x_1, x_2\}) & x_1, x_2 > 0 \\ -G_{U(1)}^{LJ}(x_1 + i0^+, x_1 - i0^+, x_2 + i0^+, x_2 - i0^+; \{0, \epsilon; x_1, x_2\}) & x_1, x_2 < 0 \\ G_{N=2}^{LJ}(x_1 + i0^+, x_1 - i0^+, x_2 + i0^+, x_2 - i0^+; \{0, \epsilon; x_1, x_2\}) & x_1, x_2 > 0 \\ -G_{U(1)}^{LJ}(x_1 + i0^+, x_1 - i0^+, x_2 + i0^+, x_2 - i0^+; \{0, \epsilon; x_1, x_2\}) & x_1, x_2 < 0 \end{array} \right.$$  \hspace{1cm} (125)$$

as

$$\Delta_{11}^{zz}(x_1, x_2; \epsilon) = \left\{ \begin{array}{ll} e^{i2\pi \alpha_t} \Delta_{11}(x_1, x_2) & x_1, x_2 > 0 \\ e^{i2\pi \alpha_{t+1}} \Delta_{11}(x_1, x_2) & x_1, x_2 < 0 \end{array} \right.$$  \hspace{1cm} (126)$$

where we have defined the common factor

$$\frac{-2}{\alpha_t} \Delta_{11}(x_1, x_2) = D \left( \frac{x_1}{x_2}; \epsilon \right) + D \left( \frac{x_1}{x_2}; \bar{\epsilon} \right) + \log(x_1^2) \sim \log(x_1^2) + R^2(\epsilon_t)$$  \hspace{1cm} (127)$$

We have to consider the cases $x_1, x_2 > 0$ and $x_1, x_2 < 0$ because, for example, $\frac{v_{loc}}{\bar{v}_{loc}} = \frac{x_2}{x_1}$ when $x_1, x_2 > 0$ and $\frac{v_{loc}}{\bar{v}_{loc}} = \frac{|x_2|}{|x_1|} e^{i2\pi}$ when $x_1, x_2 < 0$ and this gives rise to different phases. This is issue is not present for $\Delta_{11}^{zz}$ because $\frac{v_{loc}}{\bar{v}_{loc}} = \frac{x_2}{x_1}$ independently on $x_1, x_2 > 0$ or $x_1, x_2 < 0$. Notice that these phases are fundamental for projecting an arbitrary momentum $(k, \bar{k})$ in the direction parallel to the $D_5$ brane as shown explicitly in section 4.1.

In a similar way we define the regularized Green function with the twist $\sigma_{\epsilon, f}$ at $x_{loc} = 0$ and the anti-twist $\sigma_{\epsilon, f}$ in $x_{loc} = \infty$ used in chiral operators correlators. In particular because of the fact that there are at least $\partial_{\epsilon} \partial_{\bar{\epsilon}}$ the only piece which contributes is

$$\Delta_{11}^{zz}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc}; \epsilon) = \left[ G_{N=2}^{zz}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc}; \{0, \epsilon; \infty, \bar{\epsilon}\}) - G_{U(1)}^{zz}(u_{loc}, \bar{u}_{loc}; v_{loc}, \bar{v}_{loc}; \{0, \epsilon; \infty, \bar{\epsilon}\}) \right]$$

$$= -\frac{\alpha'}{2} \left[ D \left( \frac{v_{loc}}{u_{loc}}; \epsilon \right) + D \left( \frac{v_{loc}}{u_{loc}}; \bar{\epsilon} \right) + \log |u_{loc}|^2 \right]$$  \hspace{1cm} (128)$$

which is again independent on the phase of $k_{\epsilon, \bar{\epsilon}}$. 

34
B  Classical solutions

In this appendix we would like to summarize the results of the previous work [22] (see also [15] and [14]). Defined the anharmonic ratio for a complex variable $z \in \mathbb{C}$ to be

$$\omega = \frac{z - x_2}{z - x_{N}} \frac{x_1 - x_N}{x_1 - x_2}$$

(129)

so that $\omega_1 = 1$, $\omega_2 = 0$ and $\omega_N = -\infty$, a basis of the derivatives of zero modes of the two dimensional laplacian satisfying the boundary conditions (17) is

$$\frac{\partial}{\partial \omega} X_n(\omega) = \prod_{t=1}^{N-1} |\omega - \omega_t| - \epsilon_t \omega_n^n, \quad 0 \leq n \leq N - M - 2$$

$$\frac{\partial}{\partial \omega} \bar{X}_r(\omega) = \prod_{t=1}^{N-1} |\omega - \omega_t| - \epsilon_t \omega_r^r, \quad 0 \leq r \leq M - 2$$

(130)

so that we can write the classical solution

$$X_{cl}(u, \bar{u}; \{x_t, \epsilon_t, f_t\}) = f_N + \sum_{n=0}^{N-M-2} a_n(\omega_t) \int_{-\infty, \omega \in \mathbb{H}^+}^{\omega_t} d\omega \frac{\partial}{\partial \omega} X_n(\omega) + \sum_{r=0}^{M-2} b_r(\omega_t) \left[ \int_{-\infty, \omega \in \mathbb{H}^+}^{\omega_t} d\omega \frac{\partial}{\partial \omega} \bar{X}_r(\omega) \right]^*$$

(131)

which satisfies also the global constraints (18). The real coefficients $e^{-i\pi \alpha_1} a_n(\omega_t)$ and $e^{-i\pi \alpha_1} b_r(\omega_t)$ are fixed by the constraints

$$X_{cl}(x_{t+1}, \bar{x}_{t+1}) - X_{cl}(x_t, \bar{x}_t) = f_{t+1} - f_t \quad t = 2, \ldots N - 1$$

(132)

The “reality” of the coefficients can be easily seen once we introduce the following functions which are real on the real axis

$$\frac{\partial}{\partial \omega} X_n(\omega) = \prod_{t=1}^{N-1} |\omega - \omega_t| - \epsilon_t \omega_n^n, \quad 0 \leq n \leq N - M - 2$$

$$\frac{\partial}{\partial \omega} \bar{X}_r(\omega) = \prod_{t=1}^{N-1} |\omega - \omega_t| - \epsilon_t \omega_r^r, \quad 0 \leq r \leq M - 2$$

(133)

and their integrals (which can be expressed using the type D Lauricella functions)

$$I_{t,n}^{(N)}(\epsilon) = \int_{\omega_t + 1}^{\omega_t} d\omega \frac{\partial}{\partial \omega} X_n(\omega)$$

$$I_{t,r}^{(N)}(\epsilon) = \int_{\omega_t + 1}^{\omega_t} d\omega \frac{\partial}{\partial \omega} \bar{X}_r(\omega)$$

(134)
so that we can write the constraints as
\[
(-1)^{l-1} \sum_{n=0}^{N-M-2} a_n I_{l,n}^{(N)}(\epsilon) + \sum_{r=0}^{M-2} b_r I_{l,r}^{(N)}(\epsilon) = e^{-i\pi\alpha_1} [e^{-i\pi\alpha_1}(f_l - f_{l+1})]
\]
(135)

where the quantity between square brackets on the right hand side is real.
Finally the classical action can be written as
\[
S_{cl} = \frac{1}{8\pi\alpha'} \left[ \sum_{n,m=0}^{N-M-2} (e^{-i\pi\alpha_1}a_n) (e^{-i\pi\alpha_1}a_m) \sum_{l=1}^{N-2} \sum_{i=t+1}^{N-1} \sin \left( \pi \sum_{u=t+1}^{l} \epsilon_u \right) I_{l,n}^{(N)}(\epsilon) I_{l,m}^{(N)}(\epsilon) + \sum_{s,t=0}^{M-2} (e^{-i\pi\alpha_1}b_r(e^{-i\pi\alpha_1}b_s) \sum_{l=1}^{N-2} \sum_{i=t+1}^{N-1} \sin \left( \pi \sum_{u=t+1}^{l} \epsilon_u \right) I_{l,r}^{(N)}(\epsilon) I_{l,s}^{(N)}(\epsilon) \right]
\]
(136)

C Green function

In (2) following previous works on the subjects we defined the derivatives of the Green function on the whole complex plane \( \mathbb{C} \) using the doubling trick as
\[
g_{(N,M)}(z, w; \{x_t\}) = -\frac{2}{\alpha'} \frac{\partial \chi_q(z) \partial \chi_q(w) \sigma_{e_1,f}(x_1) \ldots \sigma_{e_N,f}(x_N)}{\langle \sigma_{e_1,f}(x_1) \ldots \sigma_{e_N,f}(x_N) \rangle}
\]
\[
h_{(N,M)}(z, w; \{x_t\}) = -\frac{2}{\alpha'} \frac{\partial \chi_q(z) \partial \chi_q(w) \sigma_{e_1,f}(x_1) \ldots \sigma_{e_N,f}(x_N)}{\langle \sigma_{e_1,f}(x_1) \ldots \sigma_{e_N,f}(x_N) \rangle}
\]
\[
l_{(N,M)}(z, w; \{x_t\}) = -\frac{2}{\alpha'} \frac{\partial \chi_q(z) \partial \chi_q(w) \sigma_{e_1,f}(x_1) \ldots \sigma_{e_N,f}(x_N)}{\langle \sigma_{e_1,f}(x_1) \ldots \sigma_{e_N,f}(x_N) \rangle}
\]
(137)

with expansions
\[
g_{(N,M)}(z, w; \{x_t\}) = \frac{\partial \omega_z \partial \omega_w}{\partial z \partial w} \frac{1}{(\omega_z - \omega_w)^2} \sum_{n=0}^{N-M} \sum_{s=0}^{M} a_n \delta(\omega_{t\neq 1,2,N}) \partial_{\omega} \chi^{(n)}(\omega_z) \partial_{\omega} \chi^{(s)}(\omega_w)
\]
\[
\frac{1}{(z - w)^2} \sum_{n=0}^{N-M} \sum_{s=0}^{M} a_n \delta(\omega_{t\neq 1,2,N}) \partial_{\omega} \chi^{(n)}(\omega_z) \partial_{\omega} \chi^{(s)}(\omega_w)
\]
\[
h_{(N,M)}(z, w; \{x_t\}) = e^{-i2\pi\alpha_1} \frac{\partial \omega_z \partial \omega_w}{\partial z \partial w} \sum_{r,s=0}^{M-2} b_{rs} \delta(\omega_{t\neq 1,2,N}) \partial_{\omega} \chi^{(r)}(\omega_z) \partial_{\omega} \chi^{(s)}(\omega_w)
\]
\[
l_{(N,M)}(z, w; \{x_t\}) = e^{i2\pi\alpha_1} \frac{\partial \omega_z \partial \omega_w}{\partial z \partial w} \sum_{n,m=0}^{N-M-2} c_{nm} \delta(\omega_{t\neq 1,2,N}) \partial_{\omega} \chi^{(n)}(\omega_z) \partial_{\omega} \chi^{(m)}(\omega_w)
\]
(138)

where we have used the anharmonic ratio as defined in eq. (129) and we have extended the range of definition from \( N - M - 2 \) to \( N - M \) for \( \chi^{(n)} \) and
from \( M - 2 \) to \( M \) for \( \mathcal{X}^{(s)} \) in order to write in a more compact way \( g_{(N,M)} \) and have therefore used hatted indexes. The previous quantities are subject to the constraints

\[
\left[ \prod_{t=1}^{N-1} \frac{(\omega_z - \omega_t)}{(\omega_z - \omega_t)^{t-1}} \left( \sum_{\bar{s}=0}^{N-M} a_{\bar{s}\bar{s}}(\omega_{t\neq1,2,N})\omega_z^{\bar{s}} \omega_t^{\bar{s}} \right) \right] |_{\omega_u = \omega_z} = 1
\]

because \( g \) must have only a double pole with coefficient 1 and

\[
\int_{x_{t+1}}^{x_t} dx \ g_{(N,M)}(x + i0^+, w) + e^{i2\pi\alpha_1} h_{(N,M)}(x - i0^+, w) = 0
\]

\[
\int_{x_{t+1}}^{x_t} dx \ l_{(N,M)}(z, x - i0^+) + e^{i2\pi\alpha_1} g_{(N,M)}(z, x + i0^+) = 0
\]

due to the boundary conditions. This set of equations is an overdetermined but consistent one as discussed in [22]. Using the previous quantities we wrote that the Green function in presence of \( N \) twists is given by

\[
\frac{-2}{\alpha'} G_{(N,M)}^{zz}(u, \bar{u}; v, \bar{v}; \{x_t\}) = \int_t^u \int_t^u \frac{du'}{H} \int_t^u \frac{dv'}{H} g_{(N,M)}(u', v'; \{x_t\})
\]

\[
+ e^{-i2\pi\alpha_1} \int_t^u \frac{du'}{H} \int_t^u \frac{dv'}{H} l_{(N,M)}(u', v'; \{x_t\})
\]

\[
+ e^{i2\pi\alpha_1} \int_{x_{t+1}}^u \frac{du'}{H} \int_{x_{t+1}}^u \frac{dv'}{H} h_{(N,M)}(u', v'; \{x_t\})
\]

\[
+ \int_t^u \int_{x_{t+1}}^u \frac{du'}{H} \int_{x_{t+1}}^u \frac{dv'}{H} g_{(N,M)}(v', \bar{u}; \{x_t\})
\]

\[
(141)
\]

and

\[
\frac{-2}{\alpha'} G_{(N,M)}^{zz}(u, \bar{u}; v, \bar{v}; \{x_t\}) = \int_t^u \int_t^u \frac{du'}{H} \int_t^u \frac{dv'}{H} l_{(N,M)}(u', v'; \{x_t\})
\]

\[
+ e^{i2\pi\alpha_1} \int_{x_{t+1}}^u \frac{du'}{H} \int_{x_{t+1}}^u \frac{dv'}{H} g_{(N,M)}(u', \bar{v}; \{x_t\})
\]

\[
+ e^{i4\pi\alpha_1} \int_{x_{t+1}}^u \frac{du'}{H} \int_{x_{t+1}}^u \frac{dv'}{H} h_{(N,M)}(u', \bar{v}; \{x_t\})
\]

\[
(142)
\]
and

\[-\frac{2}{\alpha'} G^{zz}_{(N,M)}(u, \bar{u}; v, \bar{v}; \{x_t\}) = \int_{x_{t_1}; u' \in H}^u du' \int_{x_{t_2}; v' \in H}^u dv' h_{(N,M)}(u', v'; \{x_t\}) + e^{-i2\pi \alpha_1} \int_{x_{t_1}; u' \in H}^u du' \int_{x_{t_2}; v' \in H}^u dv' g_{(N,M)}(\bar{u}', u'; \{x_t\}) + e^{-i2\pi \alpha_1} \int_{x_{t_1}; u' \in H}^u du' \int_{x_{t_2}; v' \in H}^u dv' g_{(N,M)}(\bar{u}', v'; \{x_t\}) + e^{-i4\pi \alpha_1} \int_{x_{t_1}; u' \in H}^u du' \int_{x_{t_2}; v' \in H}^u dv' l_{(N,M)}(\bar{u}', \bar{v}'; \{x_t\}) \]

(143)

where the normalization is needed in order to match the singularity of the untwisted Green function. The arbitrariness of the lower integration limit is due to the constraints which allow to change $t_3$ ans similarly for $x_{t_2}$. We would now justify this result since it is important in computing correlators involving momenta. The reason why we fixed the lower integration limit to one of the twist location is because we want

\[G^{IJ}(x_{t_1}, x_{t_1}; v, \bar{v}; \{x_t\}) = G^{IJ}(u, \bar{u}; x_{t_1}, x_{t_1}; \{x_t\}) = 0 \quad (144)\]

as follows from the boundary condition in the case of the quantum fluctuation where $f_t \to 0$.

**D  Boundary Green functions and their regularized version $\Delta_{(N,M), bou}$**

For the computation of the boundary correlators it is interesting and useful to notice that all the components of the Green function are proportional, analogously to eqs we have

\[G^{zz}_{(N,M), bou}(x_1, x_2) = e^{i\pi(\alpha_1 + \alpha_2)} G_{(N,M), bou}(x_1, x_2)\]

\[G^{zz}_{(N,M), bou}(x_1, x_2) = e^{-i\pi(\alpha_1 + \alpha_2)} G_{(N,M), bou}(x_1, x_2)\]

\[G^{zz}_{(N,M), bou}(x_1, x_2) = e^{i\pi(\alpha_1 - \alpha_2)} G_{(N,M), bou}(x_1, x_2)\]

\[G^{zz}_{(N,M), bou}(x_1, x_2) = e^{-i\pi(\alpha_1 - \alpha_2)} G_{(N,M), bou}(x_1, x_2)\]

(145)

---

9 This is obvious for $G^{zz}_{(N,M)}$ but for $G^{zz}_{(N,M)}$ there would seem to be a mismatch of phases since $\partial_u \partial_{\bar{v}} G_{U(t)}^{zz} = e^{i2\pi \alpha_1} \frac{2}{\alpha'(u-v)}$ has a phase which depends on the brane while naively $\partial_u \partial_{\bar{v}} G_{(N,M)}^{zz} = e^{i2\pi \alpha_1} \frac{2}{\alpha'(u-v)} g_{(N,M)}(u, \bar{v}) \sim e^{i2\pi \alpha_1} \frac{2}{\alpha'(u-v)}$. The issue is solved by noticing that the singularity is only there when $u, \bar{v} \to x \in \mathbb{R}$ and therefore $g_{(N,M)}(x + i\delta_1, x - i\delta_2)$ ($\delta_{1,2} > 0$) has the singularity with the required phase, explicitly for $x_1 < x < x_{t_1}$ $g_{(N,M)}(x + i\delta_1, x - i\delta_2) = e^{i2\pi(\alpha_1 - \alpha_2)} g_{(N,M)}(x + i\delta_1, x + i\delta_2)$ since $\prod_{u=1}^{N-1} (\omega_x - \omega_u)^{-e_u} = e^{i2\pi(\alpha_1 - \alpha_2)} \prod_{u=1}^{N-1} (\omega_x + i\delta - \omega_u)^{-e_u}$.

---

38
where the point $x_1$ is on the brane $D_{t_1}$, i.e. $x_{t_1} < x_1 < x_{t_1-1}$ and similarly for $x_2$ and we have defined the common real symmetric function $G_{(N,M),bou}(x_1, x_2) = G_{(N,M),bou}(x_2, x_1)$ to be

$$\frac{-2}{\alpha^2} G_{(N,M),bou}(x_1, x_2) = e^{i\pi(N_t=N_{t2})} \sum_{r,s=0}^{N-M-2} \sum_{n,m=0}^{2} c_{nm} \left( \omega t \neq 1,2,N \right) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(m)}(\omega)$$

$$+ e^{i\pi(N_t+N_{t2})} \sum_{r,s=0}^{M-2} \sum_{n,m=0}^{2} c_{nm} \left( \omega t \neq 1,2,N \right) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(r)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

$$+ e^{i\pi(N_t+N_{t2})} \sum_{r,s=0}^{N-M-2} \sum_{n,m=0}^{2} a_{n,s} \left( \omega t \neq 1,2,N \right) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

$$+ e^{i\pi(N_t+N_{t2})} \sum_{r,s=0}^{N-M-2} \sum_{n,m=0}^{2} a_{n,s} \left( \omega t \neq 1,2,N \right) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

$$= \left( \frac{2}{\alpha^2} \right) \sum_{r,s=0}^{N-M-2} \sum_{n,m=0}^{2} a_{n,s} \left( \omega t \neq 1,2,N \right) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

$$= \sum_{r,s=0}^{N-M-2} \sum_{n,m=0}^{2} a_{n,s} \left( \omega t \neq 1,2,N \right) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

(146)

where $\partial \omega X^{(n)}(\omega)$ and $\partial \omega X^{(s)}(\omega)$ are the functions which are real when $\omega \in \mathbb{R}$ and we have introduced the integers

$$N_t = \sum_{u=1}^{t-1} \theta(\alpha_a - \alpha_{a+1})$$

$$N_t = N_t + (t-1)$$

(147)

which enter the game because of the way $\epsilon$ is defined in eq. (8).

Another interesting point is to study the behavior of $\Delta_{(N,M),bou}(x_1, x_2)$ with $x_1, x_2 \in (x_t, x_{t-1})$, i.e. they are on $D_t$ when $x_1, x_2 \to x_t$. This is an important check since we should recover both the singularity and the form factor $R^2(\epsilon t)$ of the function $\Delta_{bou}$ given in eq. (127). In the limit $x_1, x_2 \to x_t$ only the last two lines of eq. (146) contribute, if we change variables as $\omega_z = \omega t + (\omega x_1 - \omega t)y_1$ and $\omega_w = \omega t + (\omega x_2 - \omega t)y_2$ in the third line and in a similar way in the forth one we get

$$\frac{-2}{\alpha^2} G_{(N,M),bou}(x_1, x_2) \sim_{x_1, x_2 \to x_t} \sum_{n=0}^{N-M} \sum_{\hat{s}=0}^{M} a_{n,s} \left( \omega t \neq 1,2,N \right) \left[ \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \right]$$

$$\int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

$$= \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(n)}(\omega) \int_{\omega t_1}^{\omega t_2} d\omega \ \partial \omega X^{(s)}(\omega)$$

(148)

where we used eq.s (139) which imply $\sum_{\hat{n}=0}^{N-M} \sum_{\hat{s}=0}^{M} a_{\hat{n},\hat{s}} \left( \omega u \neq 1,2,N \right) \omega_t^\hat{n}+\hat{s} = 0$, $\sum_{\hat{n}=0}^{N-M} \sum_{\hat{s}=0}^{M} a_{\hat{n},\hat{s}} \left( \omega u \neq 1,2,N \right) \omega_t^\hat{n}+\hat{s}-1 = \epsilon_t \Pi_{u \neq t,N} (\omega_t - \omega_u)$ and the analogous

\textsuperscript{10}The easiest way to verify that it is symmetric is to notice that $G_{(N,M),bou}(x_1, x_2)$ is symmetric.
equation \( \sum_{n=0}^{N-M} \sum_{\delta=0}^{M} a_{\delta} (\omega_{\mu} \neq 1, 2, N) s_{\omega_{\mu}^{\delta+1}} = \epsilon t \prod_{u \neq t, N} (\omega_{t} - \omega_{u}) \). In the previous equation we have introduced also \( \hat{\gamma} = \frac{\omega_{2} - \omega_{t}}{\omega_{1} - \omega_{2}} = \frac{x_{2} - x_{t}}{x_{2} - x_{N}} / \frac{x_{1} - x_{t}}{x_{1} - x_{N}} \) where \( x_{1}, x_{2} \) are not the location of the twist fields but the points where the Green function is evaluated. This expression has to be compared with the analogous for \( N = 2 \) which can be written as

\[
\frac{-2}{\alpha'} G_{N=2}(\bar{x}_{1}, \bar{x}_{2}) = \int_{0}^{1} dy_{1} \int_{0}^{\bar{x}_{2}/\bar{x}_{1}} dy_{2} \frac{y_{1}^{-\epsilon t_{1} y_{2}^{-\epsilon t_{2}} (\bar{\epsilon}_{1} y_{1} + \epsilon_{t} y_{2}) + (y_{1} \leftrightarrow y_{2})}{(y_{1} - y_{2})^{2}} \quad (149)
\]

then we can write

\[
G_{(N, M), bou}(x_{1}, x_{2}) \sim G_{N=2}(1, \bar{y}) = G_{N=2}(\frac{x_{1} - x_{t}}{x_{1} - x_{N}}, \frac{x_{2} - x_{t}}{x_{2} - x_{N}}) \quad (150)
\]

In the limit \( x_{1}, x_{2} \to x_{t} \) we notice that \( \bar{y} = \frac{x_{1} - x_{t}}{x_{2} - x_{t}} + O(x_{1} - x_{2}) \) then we can use the previous result \([127]\) to write

\[
\Delta_{(N, M), bou}(x_{1}, x_{2}) \sim \log(x_{1}^{2}) + R^{2}(\epsilon t) + O(x_{1} - x_{2}) \quad (151)
\]

### E Functions entering excited twist fields correlators

If we look at eq. \([104]\) and the more general eq. \([108]\) we see immediately that the key quantities are

\[
\begin{align*}
(u_t - x_t)^{\epsilon t} \partial_u X_{cl}^{I}(u_t) \\
(u_t - x_t)^{\epsilon t} (v_t - x_t)^{\epsilon t} \partial_u \partial_v \Delta_{(N, M), cl}^{IJ}(u_t, v_t; \{ x_t, \epsilon_t \}) \\
(u_t - x_t)^{\epsilon t} (v_t - x_t)^{\epsilon t} \partial_u \partial_v G_{(N, M), cl}^{IJ}(u_t, v_t; \{ x_t, \epsilon_t \}) \\
(u_t - x_t)^{\epsilon t} \partial_u G_{(N, M), cl}^{IJ}(u_t, \bar{u_t}; x_j, \{ x_t, \epsilon_t \})
\end{align*}
\]

(152)

We would now like to give a more explicit expression for these quantities while a completely explicit expression is given in the next section for few cases. Actually the previous expressions except the cases with only one derivative can be written as

\[
\begin{align*}
(u - x_t)^{\epsilon t} \partial_u X_{cl}(u) &= \sum_{n=0}^{N-M-2} a_{n}(\omega_{t}) (u - x_t)^{\epsilon t} \partial_u X^{(n)}(\omega_{u}) \\
(u - x_t)^{\epsilon t} \partial_u X_{cl}(u) &= \sum_{r=0}^{M-2} b_r(\omega_{t}) (u - x_t)^{\epsilon t} \partial_u X^{(r)}(\omega_{u})
\end{align*}
\]

(153)

40
and

\[ -\frac{2}{\alpha'} (u - x_t)^{\epsilon t} (v - x_t)^{\epsilon t} \partial_u \partial_v \Delta^{zz}_{(N,M)(t)}(u, v; \{x_i, \epsilon_i\}) = \]

\[ \frac{1}{(u - v)^2} \left[ \sum_{n=0}^{N-M} \sum_{s=0}^{M} a_{ns}(\omega_{i\neq1,2,N}) (u - x_t)^{\epsilon t} \partial_{\omega} \bar{\chi}^{(n)}(\omega_u) (v - x_t)^{\epsilon t} \partial_{\omega} \bar{X}^{(s)}(\omega_v) \right. \]

\[ - \left( \epsilon_t (u - x_t) + \epsilon_t (v - x_t) \right) \]

\[ \frac{-2}{\alpha'} (u - x_t)^{\epsilon t} (v - x_t)^{\epsilon t} \partial_u \partial_v \Delta^{zz}_{(N,M)(t)}(u, v; \{x_i, \epsilon_i\}) = \]

\[ \frac{-2}{\alpha'} (u - x_t)^{\epsilon t} (v - x_t)^{\epsilon t} \partial_u \partial_v \bar{G}^{zz}_{(N,M)(t)}(u, v; \{x_i, \epsilon_i\}) = \]

\[ e^{i2\pi \alpha_1} \frac{\partial \omega_u}{\partial u} \frac{\partial \omega_v}{\partial v} \sum_{n,m=0}^{N-M-2} c_{nm}(\omega_{i\neq1,2,N}) (u - x_t)^{\epsilon t} \partial_{\omega} \chi^{(n)}(\omega_u) (v - x_t)^{\epsilon t} \partial_{\omega} \bar{X}^{(m)}(\omega_v) \]

\[ e^{-i2\pi \alpha_1} \frac{\partial \omega_u}{\partial u} \frac{\partial \omega_v}{\partial v} \sum_{r,s=0}^{M-2} b_{rs}(\omega_{i\neq1,2,N}) (u - x_t)^{\epsilon t} \partial_{\omega} \bar{X}^{(r)}(\omega_u) (v - x_t)^{\epsilon t} \partial_{\omega} \bar{X}^{(s)}(\omega_v) \]

(154)

(155)

The remaining cases which involve only one derivative and are needed for computing the interaction among untwisted vertices with momenta and
excited twists require a little more work. The easiest cases can be written as

\[
\frac{-2}{\alpha'} (u - x_t)^{\tilde{t}} \partial_u G_{(N,M)}^{zz}(t)(u, \bar{u}; x, x; \{x_i, \epsilon_i\}) =
\]

\[
e^{i2\pi\alpha_1} \frac{\partial \omega_u}{\partial u} \sum_{n,m=0}^{N-M} c_{nm}(\omega_{i\not\equiv 1,2,N}) (u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(n)}(\omega_u) \int_{0: \omega \in H} d\omega \frac{\partial_{\omega} \chi^{(m)}(\omega)}{(\omega_u - \omega)^2}
\]

\[
+ e^{i2\pi\alpha_1} \frac{\partial \omega_u}{\partial u} \sum_{n=0}^{N-M} \sum_{\tilde{n}=0}^{M} a_{\tilde{n}\tilde{t}}(\omega_{i\not\equiv 1,2,N}) (u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(\tilde{n})}(\omega_u) \int_{0: \omega \in H} d\omega \frac{\partial_{\omega} \chi^{(\tilde{t})}(\omega)}{(\omega_u - \omega)^2}
\]

then the \(IJ = z\bar{z}, \bar{z}z\) cases are

\[
\frac{-2}{\alpha'} (u - x_t)^{\tilde{t}} \partial_u G_{(N,M)}^{zz}(t)(u, \bar{u}; x, x; \{x_i, \epsilon_i\}) =
\]

\[
= \frac{\partial \omega_u}{\partial u} \sum_{n=0}^{N-M} \sum_{\tilde{n}=0}^{M} a_{\tilde{n}\tilde{t}}(\omega_{i\not\equiv 1,2,N}) (u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(\tilde{n})}(\omega_u) \int_{0: \omega \in H} d\omega \frac{\partial_{\omega} \chi^{(\tilde{t})}(\omega)}{(\omega_u - \omega)^2}
\]

\[
+ e^{i2\pi\alpha_1} \frac{\partial \omega_u}{\partial u} \sum_{n,m=0}^{N-M} c_{nm}(\omega_{i\not\equiv 1,2,N}) (u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(n)}(\omega_u) \int_{0: \omega \in H} d\omega \frac{\partial_{\omega} \chi^{(m)}(\omega)}{(\omega_u - \omega)^2}
\]

\[
= \frac{\partial \omega_u}{\partial u} \sum_{n=0}^{N-M} \sum_{\tilde{n}=0}^{M} a_{\tilde{n}\tilde{t}}(\omega_{i\not\equiv 1,2,N}) (u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(\tilde{n})}(\omega_u) \int_{0: \omega \in H} d\omega \frac{\partial_{\omega} \chi^{(\tilde{t})}(\omega)}{(\omega_u - \omega)^2}
\]

\[
+ e^{i2\pi\alpha_1} \frac{\partial \omega_u}{\partial u} \sum_{r,s=0}^{N-M} b_{rs}(\omega_{i\not\equiv 1,2,N}) (u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(r)}(\omega_u) \int_{0: \omega \in H} d\omega \frac{\partial_{\omega} \chi^{(s)}(\omega)}{(\omega_u - \omega)^2}
\]

(156)

We see therefore that all the previous cases boil down to computing the blocks

\[
(u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(\tilde{n})}(\omega_u) = (u - x_t)^{\tilde{t}} \frac{\partial \omega_u}{\partial u} \partial_{\omega} \chi^{(\tilde{n})}(\omega_u)
\]

\[
(u - x_t)^{\tilde{t}} \partial_{\omega} \chi^{(\tilde{t})}(\omega_u) = (u - x_t)^{\tilde{t}} \frac{\partial \omega_u}{\partial u} \partial_{\omega} \chi^{(\tilde{t})}(\omega_u)
\]

(158)
F Explicit formula for the $N = 3$, $M = 1$ case

In this appendix we collect all formula from [22] in order to allow a quick computation of $N = 3$ amplitudes. We start with the basis of derivatives of zero modes on the laplacian operator \( \overline{\partial} \) which in this case is

\[
\partial_\omega\chi(0)(\omega_z) = (\omega_z - 1)^{-\epsilon_1}\omega_z^{-\epsilon_2} \\
\partial_\omega\chi(r)(\omega_z) = 0
\]

(159)
since as usual $\omega_1 = 1$, $\omega_2 = 0$. We find then classical solution

\[
X^{(3,1,ew)}_{cl}(u, \bar{u}) = f_2 + \frac{e^{i\pi(1-\epsilon_2)}(f_1 - f_2)}{B(1, \epsilon_2)B(\epsilon_1, \epsilon_2)} 2F_1(\epsilon_2, 1-\epsilon_1; 1+\epsilon_2; \omega_w) \omega_w^{\epsilon_2}
\]

(160)
and its classical action

\[
S^{(3,1,ew)}_{cl}(\{\epsilon_t, f_t\}) = \frac{1}{4\pi \alpha'} \frac{1}{2} |f_1 - f_2|^2 \frac{\sin(\pi \epsilon_1) \sin(\pi \epsilon_2)}{\sin(\pi \epsilon_3)}
\]

(161)
which is nothing else but the area of the triangle delimited by the branes.

We have the twist fields correlator on $\mathbb{R}^2$

\[
\langle \prod_{t=1}^{3} \sigma_{\epsilon_t, f_t}(x_t) \rangle = \frac{k}{x_1^{1/2}(\epsilon_1 \epsilon_2 - \epsilon_3 \epsilon_3)} \frac{1}{x_1^{1/2}(\epsilon_1 \epsilon_2 \epsilon_3 - \epsilon_2 \epsilon_2 - \epsilon_3 \epsilon_3)} \frac{1}{x_2^{1/2}(\epsilon_2 \epsilon_2 + \epsilon_3 \epsilon_3 - \epsilon_1 \epsilon_1)}
\]

(162)
The derivatives of the Green function when defined on the whole complex plane minus cuts are

\[
g^{(3,1)}(z, w; \{x_t\}) = \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \frac{1}{(\omega_z - \omega_w)^2} \frac{1}{(\omega_z - 1)\epsilon_1 \omega_z^{\epsilon_2}} \frac{1}{(\omega_w - 1)\epsilon_1 \omega_w^{\epsilon_2}}
\]

\[
\left[ (1 - \epsilon_1 - \epsilon_2)\omega_z^2 + (\epsilon_1 + \epsilon_2)\omega_z \omega_w - (1 - \epsilon_2)\omega_z - \epsilon_2 \omega_w \right]
\]

\[
h^{(3,1)}(z, w; \{x_t\}) = 0
\]

\[
l^{(3,1)}(z, w; \{x_t\}) = e^{i2\pi \alpha_1} c_0 \frac{\partial \omega_z}{\partial z} \frac{\partial \omega_w}{\partial w} \frac{\partial \omega_w}{\partial w} \frac{\partial \omega_w}{\partial w}
\]

\[
= e^{i2\pi \alpha_1} c_0 x_{12} x_{1N} x_{2N} \prod_{t=1}^{N=3} (z - x_t)^{-\epsilon_t} \prod_{t=1}^{N=3} (w - x_t)^{-\epsilon_t}
\]

(163)
where

\[
c_0 = -\epsilon_1 \frac{B(\epsilon_2, \epsilon_3)}{B(\epsilon_2, \epsilon_3)}
\]

(164)
Then we can write the Green function

\[
\frac{-2}{\alpha'} G^{zz}_{(3,1)}(u, \bar{u}; v, \bar{v}; \{x_i, \epsilon_i\}) =
\]

\[
\omega_u^\epsilon_2 (\omega_u - 1)^\epsilon_1 \cdot \int_{0,\omega \in H} \omega (\omega - \omega_u)^{-1} \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

\[
+ \omega_v^\epsilon_2 (\omega_v - 1)^\epsilon_1 \cdot \int_{0,\omega \in H^-} \omega (\omega - \omega_v)^{-1} \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

\[
+ c_{00} \int_{0,\omega \in H} \omega \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1} \int_{0,\omega \in H^-} \omega \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

(165)

and

\[
\frac{-2}{\alpha'} G^{zz}_{(3,1)}(u, \bar{u}; v, \bar{v}; \{x_i, \epsilon_i\}) =
\]

\[
e^{i2\pi \alpha_1} \omega_u^\epsilon_2 (\omega_u - 1)^\epsilon_1 \cdot \int_{0,\omega \in H^-} \omega (\omega - \omega_u)^{-1} \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

\[
+ e^{i2\pi \alpha_1} \omega_v^\epsilon_2 (\omega_v - 1)^\epsilon_1 \cdot \int_{0,\omega \in H^-} \omega (\omega - \omega_v)^{-1} \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

\[
+ e^{i2\pi \alpha_1} c_{00} \int_{0,\omega \in H} \omega \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1} \int_{0,\omega \in H^-} \omega \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

(166)

and

\[
\frac{-2}{\alpha'} G^{zz}_{(3,1)}(u, \bar{u}; v, \bar{v}; \{x_i, \epsilon_i\}) =
\]

\[
e^{-i2\pi \alpha_1} \omega_u^\epsilon_2 (\omega_u - 1)^\epsilon_1 \cdot \int_{0,\omega \in H} \omega (\omega - \omega_u)^{-1} \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

\[
+ e^{-i2\pi \alpha_1} \omega_v^\epsilon_2 (\omega_v - 1)^\epsilon_1 \cdot \int_{0,\omega \in H} \omega (\omega - \omega_v)^{-1} \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

\[
+ e^{-i2\pi \alpha_1} c_{00} \int_{0,\omega \in H^-} \omega \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1} \int_{0,\omega \in H^-} \omega \omega^{-\epsilon_2} (\omega - 1)^{-\epsilon_1}
\]

(167)

which clearly shows the logarithmic singularity as \( u \rightarrow v \). The double integral of \( g_{(3,1)} \) can be expressed as the product of two single integrals by using an integration by part and then rewriting the new resulting integral. If we would not use this procedure we would obtained an integral of Lauricella function which is by far more complex. The idea is simple and amounts to write the
new integral as
\[
\frac{1}{\omega_z - \omega_w} \frac{\partial}{\partial \omega_w} \left[ \prod_{t=1}^{N-1} (\omega_w - \omega_t)^{-\epsilon_t} \sum_{n=0}^{N-M} \sum_{s=0}^{M} a_n s (\omega_{t+1,2,N}) \omega_z^s \omega_w^n \right] = \frac{\partial}{\partial \omega_w} \left[ \prod_{t=1}^{N-1} (\omega_w - \omega_t)^{-\epsilon_t} \text{Polynomial}_1(\omega_z, \omega_w) \right] + \prod_{t=1}^{N-1} (\omega_w - \omega_t)^{-\epsilon_t} \text{Polynomial}_2(\omega_w)
\] (168)

where the key point is that \( \text{Polynomial}_2(\omega_w) \) depends only on \( \omega_w \). The previous step always possible because the function \( g_{(N,M)} \) has only a double pole. Moreover a choice of which integral to do first either \( \omega_z \) or \( \omega_w \) can drastically simplify the final result. The final and simplest result is then in our case
\[
\int_0^{\omega_w} d\omega \int_0^{\omega_z} d\omega g_{(3,1)}(\omega, \hat{\omega}) = \int_0^{\omega_w} d\omega \frac{1}{\omega_z - \omega_w} \hat{\omega}^{-\epsilon_2} (\hat{\omega} - 1)^{-\epsilon_1} \omega_z^2 (\omega_z - 1)^{\epsilon_1}
\] (169)

The boundary Green function reads
\[
-\frac{2}{\alpha^t} (-1)^{N_1 + N_2} G_{(3,1),\text{bou}}(x_1, x_2; \{x_i, \epsilon_i\}) =
|\omega_{x_1}|^{\epsilon_2} |\omega_{x_1} - 1|^{\epsilon_1} \cdot \int_0^{\omega_{x_2}} d\omega (\omega - \omega_{x_1})^{-1} |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1} + |\omega_{x_2}|^{\epsilon_2} |\omega_{x_2} - 1|^{\epsilon_1} \cdot \int_0^{\omega_{x_1}} d\omega (\omega - \omega_{x_2})^{-1} |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1} + e^{i\pi(N_1 + N_2)} c_0 0 \int_0^{\omega_{x_1}} d\omega \ |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1} \cdot \int_0^{\omega_{x_2}} d\omega \ |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1}
\] (170)

where the integer \( N_i \) is defined in eq. (147) and the sign entering the definition of the Green boundary function is chosen consistently with eq.s (145). The regularized version of the boundary Green function at \( x_1 \) (remember that in this case both \( x_1 \) and \( x_2 \) are on the same brane) is
\[
-\frac{2}{\alpha^t} \Delta_{(3,1),\text{bou}(1)}(x_1, x_2; \{x_i, \epsilon_i\}) =
|\omega_{x_1}|^{\epsilon_2} |\omega_{x_1} - 1|^{\epsilon_1} \cdot \int_0^{\omega_{x_2}} d\omega \ |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1} - |\omega_{x_2}|^{-\epsilon_2} |\omega_{x_2} - 1|^{-\epsilon_1} + |\omega_{x_2}|^{\epsilon_2} |\omega_{x_2} - 1|^{\epsilon_1} \cdot \int_0^{\omega_{x_1}} d\omega \ |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1} - |\omega_{x_1}|^{-\epsilon_2} |\omega_{x_1} - 1|^{-\epsilon_1} + c_0 0 \int_0^{\omega_{x_1}} d\omega \ |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1} \cdot \int_0^{\omega_{x_2}} d\omega \ |\omega|^{-\epsilon_2} |\omega - 1|^{-\epsilon_1}
\] (171)

which is nothing else but the unregularized boundary Green function to which we have subtracted the logarithm.
We also have the basic blocks for the twisted computations which correspond to eq.s 158

\begin{align}
(u - x_1)^{\epsilon_1} \partial_u \chi^{(\tilde{\eta})}(\omega_u) &= e^{i\pi \epsilon_1} x_1^{\epsilon_N - \tilde{n}} x_1^{\epsilon_1} (u - x_2)^{-\epsilon_2 + \tilde{n}} (u - x_N)^{-\epsilon_N - \tilde{n}} \\
(u - x_2)^{\epsilon_2} \partial_u \chi^{(\tilde{\eta})}(\omega_u) &= e^{i\pi \epsilon_1} x_2^{\epsilon_N - \tilde{n}} x_2^{\epsilon_1} (u - x_1)^{-\epsilon_1} (u - x_2)^{-\epsilon_2 + \tilde{n}} (u - x_N)^{-\epsilon_N - \tilde{n}} \\
(u - x_3)^{\epsilon_3} \partial_u \chi^{(\tilde{\eta})}(\omega_u) &= e^{i\pi \epsilon_1} x_3^{\epsilon_N - \tilde{n}} x_3^{\epsilon_1} (u - x_1)^{-\epsilon_1} (u - x_2)^{-\epsilon_2 + \tilde{n}} (u - x_N)^{-\epsilon_N - \tilde{n}}
\end{align}

(172)

with \(x_{ij} = x_t - x_i\) and \(\tilde{n} = 0, 1, 2\). Similarly for \(\partial_u \chi^{(\tilde{\eta})}(\omega_u)\) which can be obtained from the previous ones with \(\epsilon \leftrightarrow \tilde{\epsilon}\) and \(\tilde{n} \leftrightarrow \tilde{r}\).

It is also a good check that in the limit \(u, v \to x_t\) the Green function gives the expected singularities. This can be obtained with a simple change of variable or, essentially in the same way, rewriting the Green function using the Lauricella functions, for example

\[
-\frac{2}{\alpha'} G_{(3,1)}^{(z,x)}(u, \tilde{u}; v, \tilde{v}; \{x_i, \epsilon_i\}) = \\
-\tilde{\epsilon}_2 (1 - \omega_u)^{\epsilon_1} \left(\frac{\omega_v}{\omega_u}\right)^{\epsilon_2} F^D_2(e_2; 1, \epsilon_1; 1 + \tilde{\epsilon}_2; \frac{\omega_v}{\omega_u}; \omega_v) \\
-\tilde{\epsilon}_2 (1 - \omega_v)^{\epsilon_1} \left(\frac{\omega_u}{\omega_v}\right)^{\epsilon_2} F^D_2(e_2; 1, \epsilon_1; 1 + \tilde{\epsilon}_2; \frac{\omega_u}{\omega_v}; \omega_v) \\
+ c_0 e^{2 \omega_u^2 \omega_v^2} F_1(e_2; \tilde{e}_1; 1 + \tilde{\epsilon}_2; \omega_u) F_1(e_2; \tilde{e}_1; 1 + \tilde{\epsilon}_2; \omega_v)
\]

(173)

where for \(Re\ c > Re\ a > 0\)

\[
F^D_n(a,b_1,\ldots,b_n,c;x_1,\ldots,x_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-x_1t)^{-b_1} \cdots (1-x_nt)^{-b_n} dt
\]

(174)

and then using the obvious relation

\[
F^D_n(a,b_1,\ldots,b_n,c;x_1,\ldots,x_n = 0) = F^D_{n-1}(a,b_1,\ldots,b_{n-1},c;x_1,\ldots,x_{n-1})
\]

(175)

to get in the limit \(u, v \to x_2\), \(\omega_v/\omega_u\) constant the desired behavior of the Green function for \(N = 2\) given in eq.s 118 upon the use of \(\frac{\omega_v}{\omega_u} \to \frac{v-x_2}{a-x_2}\). For the limit \(u, v \to x_{1,3}\) we can proceed in the same way but we have to use the relation which connects the hypergeometric computed at \(x\) to that computed in \(1/x\) or to use the symmetries (120).

References

[1] L. J. Dixon, D. Friedan, E. J. Martinec, S. H. Shenker, Nucl. Phys. B282 (1987) 13-73.
[2] I. Pesando, arXiv:1107.5525 [hep-th].
[3] T. T. Burwick, R. K. Kaiser and H. F. Muller, Nucl. Phys. B 355 (1991) 689.
[4] J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, Nucl. Phys. B 397 (1993) 379 [hep-th/9207049].
S. Stieberger, D. Jungnickel, J. Lauer and M. Spalinski, Mod. Phys. Lett. A 7 (1992) 3059 [hep-th/9204037].

[5] P. Anastasopoulos, M. D. Goodsell and R. Richter, JHEP 1310 (2013) 182 [arXiv:1305.7166 [hep-th]].

[6] J. J. Atick, L. J. Dixon, P. A. Griffin, D. Nemeschansky, Nucl. Phys. B298 (1988) 1-35.
M. Bershadsky, A. Radul, Int. J. Mod. Phys. A2 (1987) 165-178.

[7] E. Corrigan, D. B. Fairlie, Nucl. Phys. B91 (1975) 527.
J. H. Schwarz, C. C. Wu, Nucl. Phys. B72 (1974) 397.
J. H. Schwarz, Nucl. Phys. B65 (1973) 131-140.
E. Corrigan, D. I. Olive, Nuovo Cim. A11 (1972) 749-773.
P. Hermansson, B. E. W. Nilsson, A. K. Tollsten, A. Watterstam, “Derivation Of The Brink-olive Correction Factor Using The Dual Ramond Superghost Vertex,” Phys. Lett. B244 (1990) 209-214.
B. E. W. Nilsson, A. K. Tollsten, “General NSR String Reggeon Vertices From A Dual Ramond Vertex,” Phys. Lett. B240 (1990) 96.
N. Engberg, B. E. W. Nilsson, A. Westerberg, “The Twisted string vertex algorithm applied to the Z(2) twisted scalar string four vertex,” Nucl. Phys. B435 (1995) 277-294. [hep-th/9405159].
N. Engberg, B. E. W. Nilsson, P. Sundell, “An Algorithm for computing four Ramond vertices at arbitrary level,” Nucl. Phys. B404 (1993) 187-214. [hep-th/9301107].
N. Engberg, B. E. W. Nilsson, P. Sundell, “On the use of dual Reggeon vertices for untwisted and twisted scalar fields,” Int. J. Mod. Phys. A7 (1992) 4559-4583.

[8] N. Di Bartolomeo, P. Di Vecchia, R. Guatieri, “General properties of vertices with two Ramond or twisted states,” Nucl. Phys. B347 (1990) 651-686.

[9] M. Bianchi, G. Pradisi and A. Sagnotti, Phys. Lett. B 273 (1991) 389.
M. Bianchi and E. Trevigne, JHEP 0508, 034 (2005) [hep-th/0502147].
M. Bianchi and E. Trevigne, JHEP 0601 (2006) 092 [hep-th/0506080].
P. Anastasopoulos, M. Bianchi and R. Richter, arXiv:1110.5359 [hep-th].
P. Anastasopoulos, M. Bianchi and R. Richter, JHEP 1203 (2012) 068 [arXiv:1110.5424 [hep-th]].

[10] E. Kiritsis and C. Kounnas, Phys. Lett. B 320 (1994) 264 [Addendum-ibid. B 325 (1994) 536] [hep-th/9310202].
G. D’Appollonio and E. Kiritsis, Nucl. Phys. B 674 (2003) 80 [hep-th/0305081].
G. D’Appollonio and E. Kiritsis, Nucl. Phys. B 712 (2005) 433 [hep-th/0410269].

[11] I. Antoniadis and K. Benakli, Phys. Lett. B 326 (1994) 69 [hep-th/9310151].
[12] E. Gava, K. S. Narain and M. H. Sarmadi, Nucl. Phys. B 504 (1997) 214 [hep-th/9704006].
[13] J. R. David, JHEP 0010 (2000) 004 [hep-th/0007235].
J. R. David, JHEP 0107 (2001) 009 [hep-th/0012089].
[14] S. A. Abel and A. W. Owen, Nucl. Phys. B 663 (2003) 197 [hep-th/0303124].
S. A. Abel, A. W. Owen, Nucl. Phys. B 682 (2004) 183-216. [hep-th/0310257].
S. A. Abel and M. D. Goodsell, JHEP 0602 (2006) 049 [hep-th/0512072].
S. A. Abel and M. D. Goodsell, JHEP 0710 (2007) 034 [hep-th/0612110].
[15] M. Cvetic and I. Papadimitriou, Phys. Rev. D 68 (2003) 046001 [Erratum-ibid. D 70 (2004) 029903] [arXiv:hep-th/0303083].
M. Cvetic and R. Richter, Nucl. Phys. B 762 (2007) 112 [hep-th/0606001].
M. Cvetic, I. Garcia-Etxebarria and R. Richter, JHEP 1001 (2010) 005 [arXiv:0905.1694 [hep-th]].
[16] D. Lust, P. Mayr, R. Richter and S. Stieberger, Nucl. Phys. B 696 (2004) 205 [hep-th/0404134].
D. Lust, S. Stieberger and T. R. Taylor, Nucl. Phys. B 808 (2009) 1 [arXiv:0807.3333 [hep-th]].
[17] M. Bertolini, M. Billo, A. Lerda, J. F. Morales and R. Russo, Nucl. Phys. B 743 (2006) 1 [arXiv:hep-th/0512067].
[18] A. Lawrence and A. Sever, JHEP 0709 (2007) 094 [arXiv:0706.3199 [hep-th]].
[19] D. Duo, R. Russo, S. Sciuto, JHEP 0712 (2007) 042. [arXiv:0709.1805 [hep-th]].
R. Russo, S. Sciuto, JHEP 0704 (2007) 030. [hep-th/0701292].
[20] K. -S. Choi and T. Kobayashi, Nucl. Phys. B 797 (2008) 295 [arXiv:0711.4894 [hep-th]].
[21] J. P. Conlon and L. T. Witkowski, JHEP 1112 (2011) 028 [arXiv:1109.4153 [hep-th]].
J. P. Conlon, M. Goodsell and E. Palti, Fortsch. Phys. 59 (2011) 5 [arXiv:1008.4361 [hep-th]].
[22] I. Pesando, Nucl. Phys. B 866 (2013) 87 [arXiv:1206.1431 [hep-th]].
[23] S. Sciuto, Lett. Nuovo Cim. 2 (1969) 411.
A. Della Selva and S. Saito, Lett. Nuovo Cim. 4 (1970) 689.
[24] J. L. Petersen, J. R. Sidenius, A. K. Tollsten, Phys. Lett. B213 (1988) 30.
J. L. Petersen, J. R. Sidenius, A. K. Tollsten, Nucl. Phys. B317 (1989) 109.
J. L. Petersen, J. R. Sidenius, Nucl. Phys. B301 (1988) 247.
J. L. Petersen, J. R. Sidenius, A. K. Tollsten, Phys. Lett. B214 (1988) 533.
[25] S. Hamidi and C. Vafa, Nucl. Phys. B 279 (1987) 465.

[26] I. Pesando, JHEP 1106 (2011) 138 [arXiv:1101.5898 [hep-th]].
I. Pesando, Phys. Lett. B 668 (2008) 324 [arXiv:0804.3931 [hep-th]].
I. Pesando, Nucl. Phys. B 876 (2013) 1 [arXiv:1305.2710 [hep-th]].

[27] P. Di Vecchia, R. Marotta, I. Pesando and F. Pezzella, J. Phys. A 44 (2011) 245401 [arXiv:1101.0120 [hep-th]].

[28] M. Bilbo, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, JHEP 0302 (2003) 045 [hep-th/0211250].

[29] M. Berkooz, B. Durin, B. Pioline and D. Reichmann, JCAP 0410, 002 (2004) [hep-th/0407216].

[30] P. Di Vecchia, R. Nakayama, J. L. Petersen, J. R. Sidenius, S. Sciuto, Nucl. Phys. B287 (1987) 621.

[31] P. Di Vecchia, R. Madsen, K. Hornfeck, K. O. Roland, “A Vertex Including Emission Of Spin Fields,” Phys. Lett. B235 (1990) 63.
P. Di Vecchia, R. A. Madsen, K. Roland, “A vertex including emission of spin fields for an arbitrary BC system,” Nucl. Phys. B354 (1991) 154-190.

[32] D. Cremades, L. E. Ibanez and F. Marchesano, JHEP 0307 (2003) 038 [hep-th/0302105].

[33] J. Polchinski, Cambridge, UK: Univ. Pr. (1998) 402 p

[34] P. Di Vecchia, A. Liccardo, R. Marotta, I. Pesando and F. Pezzella, JHEP 0711 (2007) 100 [arXiv:0709.4149 [hep-th]].

[35] I. Pesando, JHEP 1002 (2010) 064 [arXiv:0910.2576 [hep-th]].

[36] I. Pesando, Nucl. Phys. B 793 (2008) 211 [hep-th/0310027].

[37] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Nucl. Phys. B 507 (1997) 259 [hep-th/9707068].
P. Di Vecchia, A. Liccardo, R. Marotta, I. Pesando and F. Pezzella, JHEP 0711 (2007) 100 [arXiv:0709.4149 [hep-th]].

[38] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B 269 (1986) 1.