HASSE PRINCIPLES FOR MULTINORM EQUATIONS

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Abstract. A classical result of Hasse states that the norm principle holds for finite cyclic extensions of global fields, in other words local norms are global norms. We investigate the norm principle for finite dimensional commutative étale algebras over global fields; since such an algebra is a product of separable extensions, this is often called the multinorm principle. Under the assumption that the étale algebra contains a cyclic factor, we give a necessary and sufficient condition for the Hasse principle to hold, in terms of an explicitly constructed element of a finite abelian group. This can be seen as an explicit description of the Brauer-Manin obstruction to the Hasse principle.

Keywords: multinorm, tori, Shafarevich groups, local-global principles, Brauer-Manin obstructions

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0. Introduction

Let \( k \) be a global field, and \( L \) be a finite dimensional commutative étale algebra over \( k \). We say that the Hasse norm principle holds for \( L \) if the local-global principle holds for the equation

\[
N_{L/k}(t) = c
\]

for all \( c \in k^\times \); this terminology is inspired by Hasse’s result that the norm principle holds in the case of cyclic extensions \( \text{[Ha 31, Ha3 2]} \) §I (3.11) and §II (15)). Over the years, the norm principle for separable field extensions attracted a lot of attention; it is known not to hold in general, and many positive results are also available, see for instance \( \text{[PR 94]}, \) pages 308-309 for a survey; for more recent results, see \( \text{[BN 16], [FLN 18]}, \) and the references therein.

It is natural to ask for Hasse principles in the case when \( L \) is a finite dimensional commutative étale algebra, and not just a field extension. Since \( L \) is by definition a product of separable extensions, the equation \( (0.1) \) is often called a multinorm equation.

This more general problem was also studied extensively, in particular by Hürlimann \( \text{([Hu 84])}, \) Colliot-Thélène and Sansuc (unpublished), Platonov and Rapinchuk (see \( \text{[PR 94]}, \) sections 6.3 and 9.3), Prasad and Rapinchuk \( \text{([PR 10], Section 4)}, \) Pollio and Rapinchuk \( \text{([PoR 13]), Demarche and Wei}\)

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Multinorm equations also arise when dealing with classical groups of type $A_n$ (see for instance [PR 10] Prop. 4.2). In spite of many interesting results, some quite simple cases were still open. We illustrate this, as well as our results, by the following example:

**Example.** Assume that $L$ is a product of $n$ non-isomorphic quadratic field extensions of $k$. If $n = 1$ or $n = 2$, then the Hasse principle holds for $L$ - this is clear for $n = 1$, and easy for $n = 2$ (for instance, it is a consequence of [Hu 84], Proposition 3.3). It is also well-known that it does not hold in general when $n = 3$ (see for instance [CT 14]). In the present paper, we show that the Hasse principle holds if $n \geq 4$.

To obtain this result and others, let us assume that one of the factors of $L$ is a cyclic field extension of $k$. Under this hypothesis, we construct a finite abelian group $\mathcal{H}(L)$ having the property that

$$\mathcal{H}(L) = 0 \iff \text{the Hasse principle holds for } L$$

(cf. Section 5). Assume now that $\mathcal{H}(L) \neq 0$, and that $c \in k^\times$ is such that (0.1) has a solution locally everywhere. Then we construct a homomorphism

$$\alpha_c : \mathcal{H}(L) \to \mathbb{Q}/\mathbb{Z}$$

such that

(0.1) has a solution over $k \iff \alpha_c = 0$

(see Sections 6 and 7, in particular Theorem 7.1).

These results can be summarized as follows: let $I_L$ be the idèle group of $L$. Then sending $c \in k^\times$ to $\alpha_c$ gives rise to an isomorphism

$$k^\times \cap N_{L/k}(I_L)/N_{L/k}(L^*) \to \mathcal{H}(L)^*$$

(where $\mathcal{H}(L)^*$ is the dual of $\mathcal{H}(L)$, cf. Corollary 7.12).

We also give a necessary and sufficient condition for the Hasse principle to hold when one of the factors is metacyclic (see Proposition 7.13).

The results are easy to use. To illustrate this, we consider the case where $L$ is a product of cyclic extensions; assume that $L = \prod_{i \in J} K_i$, where $K_i/k$ is a cyclic extension of degree $d_i$. Let $\mathcal{P}$ be the set of prime numbers dividing $\prod_{i \in J} d_i$.

For all $p \in \mathcal{P}$ and all $i \in J$, let $K_i(p)$ be the largest subfield of $K_i$ such that $[K_i(p) : k]$ is a power of $p$, and set $L(p) = \prod_{i \in J} K_i(p)$. Then we have

$$\mathcal{H}(L) = \bigoplus_{p \in \mathcal{P}} \mathcal{H}(L(p)),$$

(see Proposition 8.6).

For any cyclic field extension $K/k$ of prime power degree, we denote by $K_{\text{prim}}$ the unique subfield of $K$ of degree $p$ over $k$. Set

$$L(p)_{\text{prim}} = \prod_{i \in J} K_i(p)_{\text{prim}}.$$
Then we have

$$\mathcal{I}(L) = 0 \iff \bigoplus_{p \in \mathcal{P}(L)} \mathcal{I}(L(p)_{\operatorname{prim}}) = 0,$$

(cf Theorem 8.1), and

$$\mathcal{I}(L(p)_{\operatorname{prim}}) \simeq (\mathbb{Z}/p\mathbb{Z})^{m_p(L)},$$

where $\mathcal{P}(L)$ is a set of prime numbers (subset of $\mathcal{P}$), and $m_p(L)$ is a positive integer; both are determined explicitly (see Theorem 8.2).

The paper is structured as follows. Sections 1-4 contain some preliminary results, including a new proof of a proposition of Hürlimann, [Hu 84] Prop. 3.3. The group $\mathcal{I}(L)$ is defined in Section 5, and the homomorphism $\alpha_c$ in Section 6. In both sections, we start with the case where the étale algebra $L$ has a cyclic factor of prime power degree, which is the essential case. We also show how one can reduce the exponent of the prime number, using the exact sequence of Proposition 5.10 - this is then used in inductive arguments. The main result is proved in Section 7 (see Theorem 7.1). Section 8 contains the application of the above results to the special case where all the factors of the étale algebra are cyclic.

Note that the results of this paper are related to the Brauer-Manin obstruction. Indeed, for $c = 1$, the equation (1.1) yields the so-called norm-one-torus defined by $L/k$ (see 1.2 for details); we denote this torus by $T_{L/k}$. When $k$ is an algebraic number field, then one can deduce from [San81] that the only obstruction to the Hasse principle is the Brauer-Manin obstruction, and is an element of the group $\mathcal{I}^2(k, \hat{T}_{L/k})$. We show that $\mathcal{I}(L) \simeq \mathcal{I}^2(k, \hat{T}_{L/k})$ (see Proposition 5.16), hence our results provide an explicit description of the Brauer-Manin obstruction.

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1. Notation, definitions and basic facts

1.1. Weil restriction

If $f : R \to R'$ is a homomorphism of commutative rings such that $R'$ is a projective $R$-module of finite type, and if $W$ is an affine $R'$-scheme, then we denote by $R_{R'/R}W$ the Weil restriction (see for instance [O 84], Appendix 2).

1.2. Étale algebras, tori and characters

Let $k$ be a field, let $k_s$ be a separable closure of $k$ and set $\Gamma_k = \operatorname{Gal}(k_s/k)$. We fix once and for all this separable closure $k_s$, and all separable extensions of $k$ that will appear in the paper will be contained in $k_s$. We use standard notation in Galois cohomology; in particular, if $M$ is a discrete $\Gamma_k$-module and $i$ is an integer $\geq 0$, we set $H^i(k, M) = H^i(\Gamma_k, M)$. 
If $L$ is a commutative étale $k$-algebra of finite rank, we denote by $N_{L/k}$ the norm map, and set $T_{L/k} = \text{R}_{L/k}(\mathbb{G}_m)$; then $T_{L/k}$ is the $k$-torus determined by the exact sequence

$$1 \to T_{L/k} \to \text{R}_{L/k}(\mathbb{G}_m) \xrightarrow{N_{L/k}} \mathbb{G}_m \to 1.$$ (1.1)

For a $k$-torus $T$, we denote by $\hat{T} = \text{Hom}(T, \mathbb{G}_m)$ its character group. If $K/k$ is a finite separable extension, set $\Gamma_K = \text{Gal}(k_s/K)$. If moreover $M$ is a discrete $\Gamma_K$-module, set $I_{K/k}(M) = \text{Ind}_{\Gamma_k}^{\Gamma_K}(M)$.

The following lemmas will be used several times in the sequel

**Lemma 1.1.** Let $F/k$ be a separable extension of finite degree, and let $L$ be the product of $n$ copies of $F$. Then we have

(i) $T_{L/k} \simeq \text{R}_{F/k}(\mathbb{G}_m)^{n-1} \times T_{F/k}$.

(ii) $H^1(k, \hat{T}_{L/k}) \simeq H^1(k, \hat{T}_{F/k})$.

**Proof.** The isomorphism $(\text{R}_{F/k}(\mathbb{G}_m))^n \to (\text{R}_{F/k}(\mathbb{G}_m))^n$ sending $(b_1, ..., b_n)$ to $(b_1, ..., b_{n-1}, b_1b_{n-1}b_n)$ induces an isomorphism $T_{L/k} \simeq \text{R}_{F/k}(\mathbb{G}_m)^{n-1} \times T_{F/k}$. This proves (i). By (i), we have $\hat{T}_{L/k} \simeq I_{F/k}(\mathbb{Z})^{n-1} \oplus \hat{T}_{F/k}$; since $H^1(k, I_{F/k}(\mathbb{Z})) = 0$, this implies (ii).

**Lemma 1.2.** Let $K/k$ be a cyclic extension of degree $d$. Then there exists an isomorphism

$$H^1(k, \hat{T}_{K/k}) \to \mathbb{Z}/d\mathbb{Z}$$

which is functorial with respect to base change.

**Proof.** Let $\sigma$ be a generator of $\text{Gal}(K/k)$. Consider the exact sequence

$$1 \to \mathbb{G}_m \to \text{R}_{K/k}(\mathbb{G}_m) \to T_{K/k} \to 1,$$

where the map from $\text{R}_{K/k}(\mathbb{G}_m)$ to $T_{K/k}$ sends $x$ to $x/\sigma(x)$, and its dual sequence

$$0 \to \hat{T}_{K/k} \to I_{K/k}(\mathbb{Z}) \to \mathbb{Z} \to 0.$$

This exact sequence induces

$$I_{K/k}(\mathbb{Z})^{\Gamma_k} \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\cdot d} H^1(k, \hat{T}_{K/k}) \xrightarrow{\epsilon} H^1(k, I_{K/k}(\mathbb{Z})) = 0.$$

We have $I_{K/k}(\mathbb{Z})^{\Gamma_k} \simeq \mathbb{Z}$, generated by the sum of the elements of $\text{Gal}(K/k)$, and the map $\epsilon$ is multiplication by $d$; hence we obtain an isomorphism $H^1(k, \hat{T}_{K/k}) \to \mathbb{Z}/d\mathbb{Z}$ which is independent of the choice of the generator $\sigma$. 
1.3. The multinorm problem

Let $L$ be an étale $k$-algebra, and let $c \in k^\times$. Let $X_c$ be the affine $k$-variety determined by the equation $N_{L/k}(t) = c$. Then $X_c$ is a torsor under the torus $T_{L/k}$ defined in 1.2, hence defines a class $[X_c] \in H^1(k, T_{L/k})$; the variety $X_c$ has a $k$-point if and only if $[X_c] = 0$. Hence we have

$$c \in N_{L/k}(L^\times) \iff X_c(k) \neq \emptyset \iff [X_c] = 0.$$

2. A construction

Let $k$ be a field, and let $L$ be a commutative étale $k$-algebra of finite rank; assume that $L$ is not a field. We keep the notation of the previous section. The aim of this section is to introduce a $k$-torus that will play a basic role in the study of the cohomology of the torus $T_{L/k}$, and of the multinorm problem.

Let us write $L = K \times K'$, where $K$ and $K'$ are étale $k$-algebras, and set $E = K \otimes_k K'$.

The norm maps $N_{K/k} : K \to k$ and $N_{K'/k} : K' \to k$ induce $N_{E/K'} : E \to K$ and $N_{E/K} : E \to K'$. Let $f : R_E(k(G_m)) \to R_L(k(G_m))$ be defined by $f(x) = (N_{E/K}(x)^{-1}, N_{E/K'}(x))$. It is clear that the image of $f$ is contained in $T_{L/k}$. Moreover, $f$ is surjective as a map of algebraic groups (easily checked after base change to the separable closure $k_s$ of $k$).

Consider the torus $S_{K,K'}$ defined by the exact sequence

$$1 \to S_{K,K'} \to R_{E/k}(G_m) \to T_{L/k} \to 1.$$

Note that $S_{K,K'}$ also fits in the exact sequence

$$1 \to S_{K,K'} \to R_{E/k}(T_{E/K'}) \overset{N_{E/K}}{\to} T_{K/k} \to 1. \tag{2.1}$$

where $T_{E/K'}$ is defined by the exact sequence of $K'$-tori

$$1 \to T_{E/K'} \to R_{E/k'}(G_m) \overset{N_{E/K'}}{\to} G_m \to 1.$$

3. Tate-Shafarevich groups

We keep the notation of the previous sections, and assume that $k$ is a global field. Let $\Omega_k$ be the set of all places of $k$; if $v \in \Omega_k$, we denote by $k_v$ the completion of $k$ at $v$.

For any $k$-torus $T$, set $\Pi^i(k, T) = \text{Ker}(H^i(k, T) \to \prod_{v \in \Omega_k} H^i(k_v, T))$. If $M$ is a $\Gamma_k$-module, set $\Pi^i(k, M) = \text{Ker}(H^i(k, M) \to \prod_{v \in \Omega_k} H^i(k_v, M))$. Recall that by Poitou-Tate duality, we have $\Pi^2(k, T) \simeq \Pi^1(k, T)^*$. 

3.1. Hasse principle for the multinorm problem

Let $L$ be an étale $k$-algebra, and let $c \in k^\times$. If $X_c(k_v) \neq \emptyset$ for all $v \in \Omega_k$, then we have $[X_c] \in \mathfrak{H}^1(k, T_{L/k})$. In particular, the Hasse principle holds for all $c \in k^\times$ if and only if $\mathfrak{H}^1(k, T_{L/k}) = 0$.

We have the following relationship between the Tate-Shafarevich groups of the torus $T_{L/k}$, and the torus $S_{K,K'}$ defined in §2:

Lemma 3.1. We have $\mathfrak{H}^1(k, T_{L/k}) \simeq \mathfrak{H}^2(k, S_{K,K'})$.

Proof. By the definition of the torus $S_{K,K'}$, we have the exact sequence

$$1 \rightarrow S_{K,K'} \rightarrow R_{E/k}(\mathbb{G}_m) \rightarrow T_{L/k} \rightarrow 1,$$

which gives rise to the cohomology exact sequence

$$0 \rightarrow H^1(k, T_{L/k}) \rightarrow H^2(k, S_{K,K'}) \rightarrow H^2(k, R_{E/k}(\mathbb{G}_m)).$$

By the corresponding cohomology exact sequence over $k_v$ for all $v \in \Omega_k$ and the Brauer-Hasse-Noether Theorem, we have $H^2(k, R_{E/k}(\mathbb{G}_m)) = 0$, hence $\mathfrak{H}^1(k, T_{L/k}) \simeq \mathfrak{H}^2(k, S_{K,K'})$, as claimed.

We now compute the group $\mathfrak{H}^2(k, \hat{T}_{K/k})$ for a cyclic extension $K/k$ - note that by Poitou-Tate duality, this is equivalent to Hasse’s cyclic norm principle, which is the following proposition:

Proposition 3.2. Let $K/k$ be a cyclic extension. Then $\mathfrak{H}^1(k, T_{K/k}) = 0$.

Proof. We give a proof for the convenience of the reader. Let $\sigma$ be a generator of $\text{Gal}(K/k)$. Consider the exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow R_{K/k}(\mathbb{G}_m) \rightarrow T_{K/k} \rightarrow 1,$$

where the map from $R_{K/k}(\mathbb{G}_m)$ to $T_{K/k}$ sends $x$ to $x/\sigma(x)$. This sequence gives rise to an injection $H^1(k, T_{K/k}) \rightarrow H^2(k, \mathbb{G}_m)$. By the Brauer-Hasse-Noether theorem, we have $H^1(k, \hat{T}_{K/k}) = 0$, hence $\mathfrak{H}^1(k, T_{K/k}) = 0$.

Corollary 3.3. Let $K/k$ be a cyclic extension. Then $\mathfrak{H}^2(k, \hat{T}_{K/k}) = 0$.

This follows from the previous proposition, combined with Poitou-Tate duality.

4. A result of Hürlimann

Using the above lemmas, we generalize a result of Hürlimann ([Hu 84, Prop. 3.3]).

Proposition 4.1. Let $K/k$ be a cyclic extension of $k$, and let $K'/k$ be a separable extension of finite degree. Let $c \in k^\times$. Then the local-global principle holds for the multinorm equation $N_{K/k}(x)N_{K'/k}(y) = c$. 

Proof. Set $L = K \times K'$; the assertion is equivalent to the vanishing of $\text{III}^1(k, T_{L/k})$. By Lemma 3.1, we have $\text{III}^1(k, T_{L/k}) \simeq \text{III}^2(k, S_{K,K'})$. By Poitou-Tate duality we have $\text{III}^2(k, S_{K,K'}) \simeq \text{III}^1(k, \hat{S}_{K,K'})^*$, hence it suffices to prove that $\text{III}^1(k, \hat{S}_{K,K'}) = 0$. Since $K/k$ is a cyclic extension, the algebra $E = K \otimes_k K'$ is isomorphic to a product of copies of $F$, where $F/K'$ is some cyclic field extension. Set $d = [K : k]$ and $f = [F : K']$.

Consider the dual sequence of (2.1):

\[
0 \longrightarrow \hat{T}_{K/k} \overset{i}{\to} I_{K'/k}(\hat{T}_{E/K'}) \overset{\rho}{\to} \hat{S}_{K,K'} \longrightarrow 0 ,
\]

and the sequence induced by (4.1)

\[
(4.2) \quad H^1(k, \hat{T}_{K/k}) \overset{i}{\to} H^1(k, I_{K'/k}(\hat{T}_{E/K'})) \overset{\rho}{\to} H^1(k, \hat{S}_{K,K'}) \overset{\delta}{\to} H^2(k, \hat{T}_{K/k}) .
\]

We have $H^1(k, I_{K'/k}(\hat{T}_{E/K'})) \simeq H^1(K', \hat{T}_{E/K'})$; lemmas 1.1 (ii) and 1.2 imply that $H^1(K', \hat{T}_{E/K'}) \simeq H^1(K', \hat{T}_{F/K'}) \simeq \mathbb{Z}/f\mathbb{Z}$. The map $i^1$ is the natural projection from $\mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$, therefore $i^1$ is surjective. This implies that $\delta : H^1(k, \hat{S}_{K,K'}) \to H^2(k, \hat{T}_{K/k})$ is injective; moreover, $\delta$ induces an injection $\text{III}^1(k, \hat{S}_{K,K'}) \to \text{III}^2(k, \hat{T}_{K/k})$. Since $K/k$ is a cyclic extension, we have $\text{III}^2(k, \hat{T}_{K/k}) = 0$ by Corollary 3.3. The proposition then follows.

5. The group $\text{III}(K, K')$

We keep the notation of the previous sections: in particular, $L = K \times K'$, where $K$ and $K'$ are étale $k$-algebras, and $E = K \otimes K'$. In addition, we now assume that $K/k$ is a cyclic extension. Under this hypothesis, we define a finite abelian group $\text{III}(K, K')$ using the local splitting patterns of $E$, and we show that $\text{III}^1(k, T_{L/k})$ is isomorphic to the dual of $\text{III}(K, K')$.

Let $K' = \prod_{i \in \mathcal{I}} K_i$, where the $K_i/k$ are field extensions. Then we have $E = \prod_{i \in \mathcal{I}} E_i$, with $E_i = K \otimes K_i$.

If $v \in \Omega_k$ and if $A$ is a commutative $k$-algebra, set $A^v = A \otimes_k k_v$.

5.1. The prime power degree case

Suppose that $K$ is a cyclic extension of degree $p^e$, where $p$ is a prime number. We start with some notation and definitions. For each $i \in \mathcal{I}$, let $M_i$ be a cyclic extension of $K$ such that $E_i$ is isomorphic to a product of copies of $M_i$. Let $p^e_i = [M_i : K_i]$; without loss of generality, we assume that $e_i \geq e_{i+1}$ for $1 \leq i \leq m - 1$.

Let $s$ and $t$ be positive integers. For $s \geq t$, let $\pi_{s,t}$ be the canonical projection $\mathbb{Z}/p^s\mathbb{Z} \to \mathbb{Z}/p^t\mathbb{Z}$. For $x \in \mathbb{Z}/p^s\mathbb{Z}$ and $y \in \mathbb{Z}/p^t\mathbb{Z}$, we say that $x$ dominates $y$ if $s \geq t$ and $\pi_{s,t}(x) = y$; if this is the case, we write $x \succeq y$. For $x \in \mathbb{Z}/p^s\mathbb{Z}$ and $y \in \mathbb{Z}/p^t\mathbb{Z}$, let $\delta(x, y)$ be the greatest nonnegative integer $d \leq \min\{s, t\}$ such that $\pi_{s,d}(x) = \pi_{t,d}(y)$. We have $\delta(x, y) = \min\{s, t\}$ if and only if $x \succeq y$ or $y \succeq x$. 
Let \( \mathcal{I} = \{1, \ldots, m\} \). For \( a = (a_1, \ldots, a_m) \in \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \) and \( n \in \mathbb{Z}/p^i\mathbb{Z} \), let \( I_n(a) \) be the set \( \{i \in \mathcal{I} \mid n \geq a_i\} \) and let \( I(a) = (I_0(a), \ldots, I_{p^e-1}(a)) \).

Let \( \mathcal{E} \) be the set of \( p^e \)-tuples \( (I_0, \ldots, I_{p^e-1}) \), where \( I_0, \ldots, I_{p^e-1} \) are subsets of \( \mathcal{I} \) such that \( \bigcup_{0 \leq n \leq p^e-1} I_n = \mathcal{I} \). Now we characterize the image of the map \( I : \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \to \mathcal{E} \).

An element \( (I_0, \ldots, I_{p^e-1}) \in \mathcal{E} \) is said to be coherent if for all \( n_1, n_2 \in \mathbb{Z}/p^e\mathbb{Z} \) we have:

1. If \( i \in I_{n_1} \cap I_{n_2} \), then \( \pi_{e_1,e_i}(n_1) = \pi_{e_1,e_i}(n_2) \).
2. If \( i \in I_{n_1} \) and \( \pi_{e_1,e_i}(n_1) = \pi_{e_1,e_i}(n_2) \), then \( i \in I_{n_2} \).

Let \( \mathcal{E}^c \) be the subset of all coherent elements in \( \mathcal{E} \). For \( a \in \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \), it is clear that \( I(a) \) is a coherent element. Conversely for a coherent element \( (I_0, \ldots, I_{p^e-1}) \in \mathcal{E}^c \), we set \( a_i = \pi_{e_1,e_i}(n) \) for \( i \in I_n \). Note that condition (1) of the definition of a coherent element ensures that the \( a_i \)'s are well-defined. Hence \( a = (a_1, \ldots, a_m) \) is a well-defined element in \( \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \); condition (2) implies that \( I(a) = (I_0, \ldots, I_{p^e-1}) \). This shows that \( I \) is a bijection between \( \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \) and \( \mathcal{E}^c \).

If \( w \) is a place of \( K_i \), we denote by \( K_i^w \) the completion of \( K_i \) at \( w \). Given a positive integer \( 0 \leq d \leq e \) and \( i \in \mathcal{I} \), let \( \Sigma_i^d \) be the set of all places \( v \in \Omega_k \) such that at each place \( w \) of \( K_i \) above \( v \), the algebra \( K \otimes K_i^w \) is isomorphic to a product of isomorphic field extensions of degree at most \( p^d \) of \( K_i^w \). Let \( \Sigma_i = \Sigma_i^0 \), in other words, \( \Sigma_i \) is the set of all places \( v \in \Omega_k \) where \( E_i^w \) is isomorphic to a product of copies of \( K_i^w \).

Let \( (I_0, \ldots, I_{p^e-1}) \in \mathcal{E}^c \). For \( n_1 \in \mathbb{Z}/p^e\mathbb{Z} \) and \( i \in \mathcal{I} \), set \( \delta(n_1, i) = \delta(n_1, \pi_{e_1,e_i}(n_2)) \), where \( n_2 \) is an element in \( \mathbb{Z}/p^e\mathbb{Z} \) such that \( i \in I_{n_2} \). Since \( (I_0, \ldots, I_{p^e-1}) \) is coherent, \( \delta(n_1, i) \) is independent of the choice of \( n_2 \) and hence is well-defined. Note that if we let \( a = (a_1, \ldots, a_m) \) be the element in \( \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \) corresponding to \( (I_0, \ldots, I_{p^e-1}) \), then \( \delta(n_1, i) = \delta(n_1, a_i) \). For \( I_n \subset \mathcal{I} \), define

\[
\Omega(I_n) = \bigcap_{i \in I_n} \Sigma_i^{\delta(n_1, i)}.
\]

For \( I_n = \mathcal{I} \), we set \( \Omega(I_n) = \Omega_k \).

Set
\[
G = G_k(K, K') = \{ (a_1, \ldots, a_m) \in \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \mid \bigcup_{n \in \mathbb{Z}/p^e\mathbb{Z}} \Omega(I_n(a)) = \Omega_k \}.
\]

**Lemma 5.1.** The set \( G \) is a subgroup of \( \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^i\mathbb{Z} \).
Proof. Let \( a = (a_1, ..., a_m) \) and \( b = (b_1, ..., b_m) \) be elements of \( G \). By the definition of \( G \), for each \( v \in \Omega_k \), there exist some \( n, n' \in \mathbb{Z}/p^e \mathbb{Z} \) such that \( v \in \Omega(I_n(a)) \) and \( v \in \Omega(I_{n'}(b)) \). We claim that
\[
v \in \Omega(I_{n+n'}(a+b)).
\]
This is clear when \( I_{n+n'}(a+b) = \mathcal{I} \). Suppose that \( I_{n+n'}(a+b) \neq \mathcal{I} \). First note that \( \delta(n+n', a_i+b_i) \geq \min\{\delta(n, a_i), \delta(n', b_i)\} \) and that \( \min\{\delta(n, a_i), \delta(n', b_i)\} \leq e_i \) for all \( i \in \mathcal{I} \). Pick an arbitrary \( i \notin I_{n+n'}(a+b) \). Without loss of generality, we suppose that \( \min\{\delta(n, a_i), \delta(n', b_i)\} = \delta(n, a_i) \). If \( i \notin I_n(a) \), we have \( v \in \Sigma_i^{\delta(n,a_i)} \subseteq \Sigma_i^{\delta(n+n',a_i+b_i)} \); hence we have \( v \in \Omega(I_{n+n'}(a+b)) \).

If \( i \in I_n(a) \), then by definition \( \delta(n, a_i) = e_i \). We have \( \delta(n, a_i) \leq \delta(n', b_i) \) by assumption, hence \( \delta(n', b_i) \geq e_i \). But \( \delta(n', b_i) \leq e_i \), therefore we have \( \delta(n', b_i) = e_i \), and hence \( i \in I_{n'}(b) \). This implies that \( i \in I_{n+n'}(a+b) \), and this is a contradiction. This completes the proof of the lemma.

Let \( D \) be the subgroup of \( \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^e \mathbb{Z} \) generated by the diagonal element \((1, ..., 1)\), and note that \( D \) is contained in \( G \). Set
\[
\text{III}_k(K, K') = G/D.
\]

Example 5.2. Assume that \( k = \mathbb{Q} \), and that \( L = \mathbb{Q}(\sqrt{a}) \times \mathbb{Q}(\sqrt{b}) \times \mathbb{Q}(\sqrt{ab}) \), where \( a, b \) are distinct square-free integers. Set \( K = \mathbb{Q}(\sqrt{a}) \), \( K_1 = \mathbb{Q}(\sqrt{b}) \) and \( K_2 = \mathbb{Q}(\sqrt{ab}) \). Then with the above notation we have \( \mathcal{I} = \{1, 2\} \), and \( E_1 = E_2 = \mathbb{Q}(\sqrt{a}, \sqrt{b}) \), hence \( e = e_1 = e_2 = 1 \). This implies that either \( \text{III}(K, K') = 0 \), or \( \text{III}(K, K') \simeq \mathbb{Z}/2\mathbb{Z} \). Note that
\[
\text{there exists } v \in \Omega_k \text{ such that } E_1^v \text{ is a field } \iff \Sigma_1 \cup \Sigma_2 \neq \Omega_k,
\]
hence
\[
\text{III}(K, K') = 0 \iff \text{there exists } v \in \Omega_k \text{ such that } E_1^v \text{ is a field}.
\]

Set now \( a = 13, b = 17 \) : then there exists no \( v \in \Omega_k \) such that \( E_1^v \) is a field, therefore \( \text{III}(K, K') = \mathbb{Z}/2\mathbb{Z} \). Note that it is well-known that the multinorm principle fails in this case (see for instance [CT 14], Proposition 5.1).

Theorem 5.3. Suppose that \( K/k \) is a cyclic extension of degree \( p^e \), where \( p \) is a prime number. Then \( \text{III}^1(k, S_{K,K'}) \simeq \text{III}(K, K') \).

Proof. Consider the dual sequence of (2.1),
\[
(5.2) \quad 0 \rightarrow \hat{T}_{K/k} \rightarrow I_{K'/k}(\hat{T}_{E/K'}) \rightarrow \hat{S}_{K,K'} \rightarrow 0,
\]
and the exact sequence induced by (5.2),
\[
(5.3) \quad H^1(k, \hat{T}_{K/k}) \rightarrow H^1(k, I_{K'/k}(\hat{T}_{E/K'})) \rightarrow H^1(k, \hat{S}_{K,K'}) \rightarrow H^2(k, \hat{T}_{K/k}).
\]
We have \( \text{III}^2(k, \hat{T}_{K/k}) = 0 \) by Corollary 5.3 therefore \( \text{III}^1(k, S_{K,K'}) \) is in the image of \( \rho^1 \).
Note that \( H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i})) \simeq H^1(K_i, \hat{T}_{E_i/K_i}) \), and that by Lemma 1.1 (ii), we have \( H^1(K_i, \hat{T}_{E_i/K_i}) \simeq H^1(K_i, \hat{T}_{M_i/K_i}) \). Moreover, by Lemma 1.2 we have \( H^1(K_i, \hat{T}_{M_i/K_i}) \simeq \mathbb{Z}/p^{e_i} \mathbb{Z} \).

In the following we identify \( H^1(k, \hat{T}_{K/k}) \) to \( \mathbb{Z}/p^{e_i} \mathbb{Z} \) and \( H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i})) \) to \( \mathbb{Z}/p^{e_i} \mathbb{Z} \) for \( 1 \leq i \leq m \). Under this identification, the map
\[
i^1 : H^1(k, \hat{T}_{K/k}) \to H^1(k, I_{K'/k}(\hat{T}_{E_i/K_i})) = \bigoplus_{i \in I} H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i}))
\]
sends \( \mathbb{Z}/p^{e_i} \mathbb{Z} \) to \( \bigoplus_{i \in I} \mathbb{Z}/p^{e_i} \mathbb{Z} \) by the natural projections. Therefore we can rewrite the exact sequence (5.3) as follows:
\[
(5.4) \quad \mathbb{Z}/p^{e_i} \mathbb{Z} \xrightarrow{i^1} \bigoplus_{i \in I} \mathbb{Z}/p^{e_i} \mathbb{Z} \xrightarrow{\rho^1} H^1(k, \hat{S}_{K,K'}) \to H^2(k, \hat{T}_{K/k}),
\]
where \( i^1 \) is the natural projection from \( \mathbb{Z}/p^{e_i} \mathbb{Z} \) to \( \mathbb{Z}/p^{e_i} \mathbb{Z} \) for each \( i \). Note that the image of \( i^1 \) is the subgroup \( D \), and we have the exact sequence
\[
(5.5) \quad 0 \to \bigoplus_{i \in I} \mathbb{Z}/p^{e_i} \mathbb{Z} / D \xrightarrow{\rho^1} H^1(k, \hat{S}_{K,K'}) \to H^2(k, \hat{T}_{K/k}).
\]

Let \( a = (a_1, ..., a_m) \in \bigoplus_{i \in I} \mathbb{Z}/p^{e_i} \mathbb{Z} \) and \([a]\) be its image in \( \bigoplus_{i \in I} \mathbb{Z}/p^{e_i} \mathbb{Z} / D \). We claim that \( \rho^1([a]) \) is in \( \mathbb{I}^1(k, \hat{S}_{K,K'}) \) if and only if \( a \in G \).

We denote by \( a^v \) the image of \( a \) in \( \bigoplus_{i=1}^m H^1(k_v, I_{K_v/v}(\hat{T}_{E_i/K_i})) \), and by \( D_v \) the image of \( D \) in this sum.

By the exact sequence (5.5) over \( k_v \), we have \( \rho^1([a]) \in \mathbb{I}^1(k, \hat{S}_{K,K'}) \) if and only if \( a^v \in D_v \) for all places \( v \in \Omega_k \). Therefore, it suffices to prove that \( a \in G \) if and only if \( a^v \in D_v \) for all places \( v \in \Omega_k \).

Suppose that \( a \in G \), and let \( v \in \Omega_k \). Then there exists \( n \in \mathbb{Z}/p^{e_i} \mathbb{Z} \) such that \( v \in \Omega(I_n(a)) \). If \( I_n(a) = \mathcal{T} \), then clearly \( a \in D \subseteq G \). Suppose that \( I_n(a) \neq \mathcal{T} \). This implies that for each \( i \notin I_n(a) \) and for each place \( w \) of \( K_i \) above \( v \), the étale algebra \( K^w_i \otimes K \) is isomorphic to a product of field extensions of \( K^w_i \) of degree at most \( \delta(n, i) \). Let \( \delta_i = \delta(n, i) = \delta(n, a_i) \). Note that
\[
H^1(k_v, I_{K_v/v}(\hat{T}_{E_i/K_i})) = H^1(K^w_i, \hat{T}_{E_i/K_i}).
\]
We have
\[
H^1(K^w_i, \hat{T}_{E_i/K_i}) \simeq \bigoplus_{w|v} H^1(K^w_i, \hat{T}_{E_i/K_i}) \simeq \bigoplus_{w|v} \mathbb{Z}/p^{e_i,w} \mathbb{Z},
\]
where \( e_i,w \leq \delta_i \), and the localization map \( H^1(K_i, \hat{T}_{E_i/K_i}) \to H^1(K^w_i, \hat{T}_{E_i/K_i}) \) is the canonical projection \( \pi_{e_i,w} \) from \( \mathbb{Z}/p^{e_i} \mathbb{Z} \) to each component \( \mathbb{Z}/p^{e_i,w} \mathbb{Z} \). Since for all \( i \notin I_n(a) \) we have \( e_i,w \leq \delta_i \), and \( \pi_{e_i,w}(a_i) = \pi_{e_i,\delta_i}(n) \), this implies that \( a^v = (n, ..., n)^v \).

Suppose conversely that \( a^v \in D_v \) for all \( v \in \Omega_k \) and \( a \notin G \). Then \( a \notin D \), and there exists a place \( v \in \Omega_k \) such that \( v \notin \bigcup_{n \in \mathbb{Z}/p^{e_i} \mathbb{Z}} \Omega(I_n(a)) \). Since \( a^v \in D_v \), there
exists \( n' \in \mathbb{Z}/p^e\mathbb{Z} \) such that \( a' = (v^s(n'))_v \). Let \( n = \pi_{e_i} (n')_v \). As \( v \notin \Omega(I_0(a)) \), there exists \( i \notin I_n(a) \) and a place \( w \) of \( K_i \) above \( v \) such that \( K_i^w \otimes K \) is isomorphic to a product of field extensions of degree \( p^{e_i} \), with \( e_i > 0 \). Then by the definition of \( \delta_i = \delta(n, a_i) \), we have \( \pi_{e_i} (a_i) \neq \pi_{e_i} (n) \). Hence the localization \( a_i^w \) of the \( i \)-th coordinate of \( a \) is not equal to the localization of the \( i \)-th coordinate of \( (n, \ldots, n) \), which is a contradiction. Our claim then follows. Therefore, we have \( \Pi^1(k, \hat{S}_{K, K'}) \simeq \Pi(K, K') \).

**Corollary 5.4.** Suppose that \( K/k \) is a cyclic extension of degree \( p^e \), where \( p \) is a prime number. Then \( \Pi^1(k, T_{K, K'}) \simeq \Pi(K, K')^\ast \).

**Proof.** By Lemma 3.1, we have \( \Pi^1(k, T_{L/k}) \simeq \Pi^2(k, S_{K, K'}) \). Theorem 5.3 implies that \( \Pi^1(k, \hat{S}_{K, K'}) \simeq \Pi(K, K') \). By Poitou-Tate duality, we have \( \Pi^2(k, S_{K, K'}) \simeq \Pi^1(k, \hat{S}_{K, K'})^\ast \), hence the corollary is proved.

5.2. The group \( \Pi(K/K_0, K') \)

Let \( K_0 \) be the unique subfield of \( K \) such that \( [K_0 : K] = p^{e-1} \). The proof of the main theorem in the prime power case uses induction on \( e \), and the comparison of the groups \( \Pi(K, K') \) and \( \Pi(K_0, K') \). We first define a homomorphism \( F : \Pi(K_0, K') \to \Pi(K, K') \), and then determine the cokernel of \( F \), denoted by \( \Pi(K/K_0, K') \).

Note that if \( e = 1 \), then \( K_0 = K \), and hence \( \Pi(K_0, K') \) is trivial; in this case, \( \Pi(K/K_0, K') \) is the group \( \Pi(K, K') \) itself.

**The homomorphism** \( \Pi(K_0, K') \to \Pi(K, K') \).

Recall that we have \( K' = \prod_{i \in I} K_i \), that \( E_i = K \otimes K_i \), and that \( E_i \) is the product of copies of a cyclic extension of degree \( p^{e_i} \) of \( K_i \). Set \( E_i^0 = K_0 \otimes K_i \). Then \( E_i^0 \) also splits as a product of copies of a cyclic extension of \( K_i \); let us denote by \( p^{f_i} \) the degree of this extension.

**Proposition 5.5.** For all \( i \in I \), we have \( f_i \leq e_i \). If moreover \( e_i \neq 0 \), then \( e_i = f_i + 1 \).

This is an immediate consequence of the following proposition :

**Proposition 5.6.** Let \( F/k \) be a field extension, and let \( K \otimes_k F \) be a product of cyclic field extensions of \( F \) of degree \( p^{e_F} \); let \( K_0 \otimes_k F \) be a product of cyclic field extensions of \( F \) of degree \( p^{f_F} \). Then we have

(i) \( f_F \leq e_F \);
(ii) \( f_F \geq e_F - 1 \);
(iii) If \( e_F \neq 0 \), then \( e_F = f_F + 1 \).
Proof. If \( n \) is a positive integer, let us denote by \( C_n \) the cyclic group of order \( n \). Let us consider the homomorphisms

\[
\Gamma_F \rightarrow \Gamma_k \xrightarrow{\phi_K} C_{p^e} \xrightarrow{\pi} C_{p^{e-1}} \rightarrow 1,
\]

where \( \iota \) is the inclusion of \( \Gamma_F \) into \( \Gamma_k \), the homomorphism \( \phi_K : \Gamma_k \rightarrow C_{p^e} \) corresponds to the cyclic extension \( K/k \), and \( \pi : C_{p^e} \rightarrow C_{p^{e-1}} \) is the quotient of \( C_{p^e} \) by its unique subgroup of order \( p \). Note that the image of \( \phi_K \circ \iota \) is the Galois group of the cyclic factors of \( K \otimes_k F \), and hence is of order \( p^{e} \). Similarly, the image of \( \pi \circ \phi_K \circ \iota \) is the Galois group of the cyclic factors of \( K_0 \otimes_k F \), and hence is of order \( p^{e} \). Therefore we have \( f_F \leq e_F \). Moreover, if \( e_F \neq 0 \), then the image of \( \phi_K \circ \iota \) contains the unique subgroup of order \( p \) of \( C_{p^e} \), and hence \( e_F = f_F + 1 \). This completes the proof of the proposition.

For all \( i \in I \), let \( F_i : \mathbb{Z}/p^i\mathbb{Z} \rightarrow \mathbb{Z}/p^{e_i}\mathbb{Z} \) be the inclusion of the subgroup of order \( p^i \) in the group \( \mathbb{Z}/p^{e_i}\mathbb{Z} \), and set \( F_{K/K_0} = F = \bigoplus_{i \in I} F_i \).

Proposition 5.7. The map \( F : \bigoplus_{i \in I} \mathbb{Z}/p^i\mathbb{Z} \rightarrow \bigoplus_{i \in I} \mathbb{Z}/p^{e_i}\mathbb{Z} \) induces an injective homomorphism \( F : \Pi(K_0, K') \rightarrow \Pi(K, K') \).

Proof. Let us recall some notation from 5.1, for \( K \) and \( K_0 \): For all \( i \in I \) and for all positive integers \( d \), we denote by \( \Sigma(K)^d \) the set of all places \( v \in \Omega_k \) such that at each place \( w \) of \( K_i \) above \( v \), the algebra \( K \otimes K_i^w \) is isomorphic to a product copies of a cyclic extension of degree at most \( p^d \) of \( K_i^w \). Recall that

\[
G = G(K, K') = \{ a \in \bigoplus_{i \in I} \mathbb{Z}/p^{e_i}\mathbb{Z} \mid \bigcup_{n \in \mathbb{Z}/p^{e_i}\mathbb{Z}} \Omega(I_n(a)) = \Omega_k \},
\]

and that \( D \) is the diagonal subgroup of \( G \). Similarly, set

\[
G_0 = G(K_0, K') = \{ b \in \bigoplus_{i \in I} \mathbb{Z}/p^{h_i}\mathbb{Z} \mid \bigcup_{n \in \mathbb{Z}/p^{h_i}\mathbb{Z}} \Omega(I_n(b)) = \Omega_k \},
\]

and let \( D_0 \), be the diagonal subgroup of \( G_0 \). Then we have \( \Pi(K, K') = G/D \) and \( \Pi(K_0, K') = G_0/D_0 \).

Let \( b \in G_0 \), and let us show that \( F(b) \in G \). Let \( v \in \Omega_k \). Then there exists \( r \in \mathbb{Z}/p^{f_i}\mathbb{Z} \) such that \( v \in \Sigma(K_0)^{\delta(r,i)} \) for all \( i \in I \) such that \( i \notin I_r(b) \). Note that for all positive integers \( \delta \), we have \( \Sigma(K_0)^{\delta} \subset \Sigma(K)^{\delta+1} \). Set \( n = F_1(r) \in \mathbb{Z}/p^{e_i}\mathbb{Z} \); then we have \( \delta(n, F_1(b_i)) = \delta(r, b_i) + 1 \). Hence we have \( v \in \Sigma(K_i)^{\delta(r,b_i)+1} \), and therefore \( F(b) \in G \).

It is clear that \( F \) is injective.

Remark 5.8. For any subextension \( N/k \) of \( K/k \), let \( F_{K/N} : \Pi(N, K') \rightarrow \Pi(K, K') \) be the injective homomorphism obtained by successive applications of Proposition 5.7.
The group $\text{III}(K/K_0, K')$.

As we will see, the cokernel of $F$ is isomorphic to the group $\text{III}(K/K_0, K')$, defined as follows:

For all $i \in I$, set $r_i = \min\{1, e_i\}$. For all $c \in \oplus_{i \in I} \mathbb{Z}/p^{r_i} \mathbb{Z}$ and $n \in \mathbb{Z}/p \mathbb{Z}$, set $I_n^i(c) = \{i \in I \mid n \geq c_i\}$. If $I_n^i(c) \neq I$, set $\Omega(I_n^i(c)) = \cap \Sigma_i$; if $I_n^i(c) = I$, set $\Omega(I_n^i(c)) = \Omega_k$. Set

$$G(K/K_0, K') = \{c \in \oplus_{i \in I} \mathbb{Z}/p^{r_i} \mathbb{Z} \mid \bigcup_{n \in \mathbb{Z}/p^{r_i} \mathbb{Z}} \Omega(I_n^i(c)) = \Omega_k\},$$

let $D(K/K_0, K')$ be the diagonal subgroup of $G(K/K_0, K')$, and set $\text{III}(K/K_0, K') = G(K/K_0, K')/D(K/K_0, K')$.

Lemma 5.9. The projection $\pi : \oplus_{i \in I} \mathbb{Z}/p^{r_i} \mathbb{Z} \to \oplus_{i \in I} \mathbb{Z}/p^{r_i} \mathbb{Z}$ induces a homomorphism $\pi : \text{III}(K, K') \to \text{III}(K/K_0, K')$.

Proof. Let $a \in G$, and set $\overline{a} = \pi(a)$. Let us show that $\overline{a} \in G(K/K_0, K')$. Let $v \in \Omega_k$; then there exists $s \in \mathbb{Z}/p^{r_i} \mathbb{Z}$ such that $v \in \Omega(I_n(a))$. Set $n = \pi_{e_1, 1}(s)$, and let us prove that $v \in \Omega(I_n^i(\overline{a}))$. This is clear if $I_n^i(\overline{a}) = I$. Suppose that $I_n^i(\overline{a}) \neq I$. If $i \in I$ is such that $i \notin I_n^i(\overline{a})$, then we have $i \notin I_n(a)$, and therefore $v \in \Sigma_i^{(s, a_i)}$. Since $n = \pi_{e_1, 1}(s)$ and $i \notin I_n^1(\overline{a})$, we have $\delta(s, a_i) = 0$, and hence $v \in \Sigma_i$. Therefore we have $\overline{a} \in G(K/K_0, K')$, as claimed, and this completes the proof of the lemma.

Proposition 5.10. The sequence

$$0 \to \text{III}(K_0, K') \xrightarrow{F} \text{III}(K, K') \xrightarrow{\pi} \text{III}(K/K_0, K') \to 0$$

is exact.

Proof. It is clear that $F$ is injective, and that $\pi \circ F = 0$; it remains to check that $\pi$ is surjective, and that $\text{Ker}(\pi) \subset \text{Im}(F)$. Let us check the second assertion first. Let $a \in \text{III}(K, K')$ be such that $\pi(a) = 0$. Then there exists $b \in \oplus_{i \in I} \mathbb{Z}/p^{r_i} \mathbb{Z}$ such that $F(b) = a$; let us check that $b \in \text{III}(K_0, K')$. Let $v \in \Omega_k$. Then there exists $n \in \mathbb{Z}/p^{r_i} \mathbb{Z}$ such that $v \in \Omega(I_n(a))$. If $i \in I_n(a)$, then we have $\pi_{e_1, e_i}(n) = a_i$. Since $a_i = F_i(b_i)$, this implies that there exists $r \in \mathbb{Z}/p^{h_i} \mathbb{Z}$ such that $n = F_i(r)$ and $I_n(a) = I_r(b)$. Let us show that $v \in \Omega(I_r(b))$. For all $i \in I$ such that $i \notin I_n(a)$, we have $v \in \Sigma(K_i^{(s, a_i)})$. Note that $\delta(n, a_i) = \delta(r, b_i) + 1$ and $[K : K_0] = p$. Hence $v \in \Sigma(K_i^{(s, a_i)})$ implies that $v \in \Sigma(K_0)^{\delta(n, a_i)}$. Therefore we have $v \in \Omega(I_r(b))$, as claimed, and this implies that $b \in \text{III}(K_0, K')$. Let us now prove that $\pi$ is surjective. Let $\overline{a} \in \text{III}(K/K_0, K')$. For each $n \in \mathbb{Z}/p \mathbb{Z}$, let us fix a lifting $r(n) \in \mathbb{Z}/p^{r_i} \mathbb{Z}$. If $i \in I_n^1(\overline{a})$, set $a_i = \pi_{e_1, e_i}(r(n))$. Let us check that $a_i \in \mathbb{Z}/p^{r_i} \mathbb{Z}$ is well-defined. Suppose that $n_1, n_2 \in \mathbb{Z}/p \mathbb{Z}$ are such that $i \in I_n^1(\overline{a})$; then we have $\pi_{1, r_i}(n_1) = \pi_{1, r_i}(n_2)$. If $n_1 \neq n_2$, then this implies that $r_i = 0$, hence
We have $e_i = 0$. We have $\pi_{e_1,e_i}(r(n_1)) = \pi_{e_1,e_i}(r(n_2))$ in this case, hence $a_i$ is well-defined. Let us check that $a \in \mathbb{III}(K, K')$. Since $\mathbb{I}$ is in bijective correspondence with the partitions $(\mathbb{I}^n_{\mathbb{I}}(a)) = \mathbb{E}_K$. Let $v \in \mathbb{V}_K$; then there exists $n \in \mathbb{Z}/p\mathbb{Z}$ such that $v \in \Omega(\mathbb{I}^n_{\mathbb{I}}(a))$. Let $r = r(n)$; we claim that $v \in \Omega(I_r(a))$. If $I^n_{\mathbb{I}}(a) = \mathbb{I}$, then we have $I_r(a) = \mathbb{I}$, and the claim is clear. Suppose that $I^n_{\mathbb{I}}(a) \neq \mathbb{I}$. If $i \notin I_r(a)$, then we have $i \notin I^n_{\mathbb{I}}(a)$ by construction, hence $v \notin \Sigma_i$. Since $\Sigma_i \subset \Sigma_i(K)^{\delta(r,a_i)}$, the claim follows. This completes the proof of the proposition.

**The group $\mathbb{III}(K/K_0, K')$ and partitions.**

In this section we give some properties of $G(K/K_0, K')$ in terms of partitions of $\mathbb{I}$. This will be useful in the proof of the main theorem (Thm. 7.1).

**Lemma 5.11.** The set $G(K/K_0, K')$ is in bijective correspondence with the partitions $(J_0, ..., J_{p-1})$ of the set $\{i \in \mathbb{I} \mid r_i = 1\}$ such that $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_n) = \Omega_K$.

**Proof.** Recall that we have $r_i = 0$ or $1$. Set $\mathbb{I}' = \{i \in \mathbb{I} \mid r_i = 1\}$. Let $a \in G(K/K_0, K')$, and set $I^1(a) \cap \mathbb{I}' = (I^1_0(a) \cap \mathbb{I}', ..., I^1_{p-1}(a) \cap \mathbb{I}')$; note that $I^1(a) \cap \mathbb{I}'$ is a partition of $\mathbb{I}'$. Hence the set $G(K/K_0, K')$ is then in bijective correspondence with the partitions $(J_0, ..., J_{p-1})$ of $\mathbb{I}'$ such that $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_n) = \Omega_K$, as claimed.

In the sequel, we identify $G(K/K_0, K')$ with the set of these partitions. We also note a consequence for the case where $K$ is of degree $p$, in other words, if $e = 1$. If $K/k$ is of prime degree, then either $E_i$ is a field extension of $K_i$, or $E_i$ is a product of copies of $K_i$. Let $J$ be the subset of $I$ such that $E_i$ is a field extension of $K_i$ if $i \in J$, and that $E_i$ is a product of copies of $K_i$ if $i \notin J$.

**Lemma 5.12.** Assume that $K/k$ is a degree $p$ extension. Then $G(K, K')$ is in bijective correspondence with the partitions $(J_0, ..., J_{p-1})$ of $J$ such that $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_n) = \Omega_K$.

**Proof.** Since $e = 1$, we have $K_0 = k$ and $G(K/K_0, K') = G(K, K')$. Hence the lemma follows from Lemma 5.11.

Let $K_{\text{prim}}$ be the unique subfield of $K$ of degree $p$ over $k$.

**Proposition 5.13.** Assume that $K_{\text{prim}}$ is linearly disjoint from $K_i$ for all $i \in I$. Then the group $\mathbb{III}(K/K_0, K')$ is a subgroup of $\mathbb{III}(K_{\text{prim}}, K')$.

**Proof.** Note that $G(K_{\text{prim}}, K')$ is in bijection with the set of partitions $(J_0, ..., J_{p-1})$ of $I$ such that $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega_{K_{\text{prim}}}(J_n) = \Omega_k$, where $\Omega_{K_{\text{prim}}}(I_n) = \cap_{i \notin I_n} \cup_i \Sigma_i(K_{\text{prim}})$ (see Lemma 5.12).

As $K_{\text{prim}}$ is linearly disjoint from $K$, $r_i = 1$ for all $i \in I$. Hence by Lemma 5.11, the set $G(K/K_0, K')$ is in bijection with the set of partitions $(I_0, ..., I_{p-1})$ of $I$ such that $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega_K(I_n) = \Omega_k$, where $\Omega_K(I_n) = \cap_{i \notin I_n} \Sigma_i(K)$. 


Note that $\Sigma_i(K_{\text{prim}}) \supseteq \Sigma_i(K)$ for all $i \in \mathcal{I}$. Hence $G(K/K_0, K') \subset G(K_{\text{prim}}, K')$, and this implies that $\text{III}(K/K_0, K')$ is a subgroup of $\text{III}(K_{\text{prim}}, K')$.

**Proposition 5.14.** Assume that $K_{\text{prim}}$ is linearly disjoint from $K_i$ for all $i \in I$. If $\text{III}(K_{\text{prim}}, K') = 0$, then $\text{III}(K, K') = 0$.

**Proof.** By Proposition 5.13 we have $\text{III}(K/K_0, K') = 0$; hence Proposition 5.10 implies that $\text{III}(K, K') = \text{III}(K_0, K')$. Repeating this argument, we see that $\text{III}(K, K') = \text{III}(K_{\text{prim}}, K')$. But $\text{III}(K_{\text{prim}}, K') = 0$ by hypothesis, hence $\text{III}(K, K') = 0$, as claimed.

The following lemma will be useful in the sequel.

**Lemma 5.15.** Let $(I_0, \ldots, I_{p-1}) \in G(K/K_0, K')$, and let $r, r'$ be two distinct elements of $\mathbb{Z}/p\mathbb{Z}$. Set $J_r = I_r$, $J_r' = \bigcup I_n$, and $J_n = \emptyset$ if $n \neq r, r'$. Then $(J_0, \ldots, J_{p-1}) \in G(K/K_0, K')$. If moreover $(I_0, \ldots, I_{p-1}) \not\in D(K/K_0, K')$ and $I_r \neq \emptyset$, then $(J_0, \ldots, J_{p-1}) \not\in D(K/K_0, K')$.

**Proof.** Let us show that $\Omega(J_r) \cup \Omega(J_{r'}) = \Omega_k$. Let $v \in \Omega_k$ be such that $v \not\in \Omega(J_r)$. Since we have $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega(I_n) = \Omega_k$, there exists $n(v) \in \mathbb{Z}/p\mathbb{Z}$ with $n(v) \neq r$ such that $v \in \Omega(I_{n(v)})$. Since $n(v) \neq r$, we have $\Omega(I_{n(v)}) \subset \bigcap_{i \in I_r} \Sigma_i = \Omega(J_{r'})$. Therefore we have $\Omega(J_r) \cup \Omega(J_{r'}) = \Omega_k$, and hence $(J_0, \ldots, J_{p-1}) \in G(K/K_0, K')$.

Let us prove the second statement. If $(J_0, \ldots, J_{p-1}) \in D(K/K_0)$, then either $J_r = T'$ or $J_{r'} = T'$; we have $I_r = T'$ in the first case, hence $(I_0, \ldots, I_{p-1}) \in D(K/K_0, K')$, and $I_r = \emptyset$ in the second case. This completes the proof of the lemma.

### 5.3. The general case

Recall that $K/k$ is a cyclic extension of degree $d$, and let $\mathcal{P}$ be the set of prime numbers dividing $d$. For all $p \in \mathcal{P}$, let $K(p)$ be the largest subfield of $K$ such that $[K(p) : k]$ is a power of $p$ and set $d(p) = [K(p) : k]$. Set

$$\text{III}(K, K') = \bigoplus_{p \in \mathcal{P}} \text{III}(K(p), K').$$

**Proposition 5.16.** We have $\text{III}^1(k, \mathcal{S}_{K,K'}) \simeq \text{III}(K, K')$.

**Proof.** By 5.3 we have $\text{III}^1(k, \mathcal{S}_{K(p), K'}) \simeq \text{III}(K(p), K')$, hence it suffices to show that $\text{III}^1(k, \mathcal{S}_{K,K'}) \simeq \prod_{p \in \mathcal{P}} \text{III}^1(k, \mathcal{S}_{K(p), K'})$. For every $p \in \mathcal{P}$, set $E(p) = K(p) \otimes_k K'$ and $L(p) = K(p) \times K'$. The inclusion $K(p) \to K$ induces maps $\epsilon_p : T_{K(p)/k} \to T_{K/k}$, $\epsilon_p : T_{E(p)/K'} \to T_{E/K'}$ and $\epsilon_p : S_{K(p), K'} \to S_{K,K'}$. We have the commutative diagram, coming from cohomology exact sequences associated to the dual sequences of 2.13:
where the vertical maps are induced by the maps $\epsilon_p$. Set $\hat{\epsilon}^1 = \bigoplus \hat{\epsilon}_p^1$.

For all $i \in \mathbb{Z}$, let $M_i$ be a cyclic extension of $K$, such that $E_i$ is isomorphic to a product of copies of $M_i$, and let $d_i = [M_i : K_i]$. If $p$ is a prime divisor of $[K : k]$, set $E_i(p) = K(p) \otimes_k K_i$, and let $M_i(p)$ be a cyclic extension of $K_i$ such that $E_i(p)$ is isomorphic to a product of copies of $M_i(p)$; set $d_i(p) = [M_i(p) : K_i]$. Note that $d_i(p)$ is the highest power of $p$ dividing $d_i$, and that $d_i = \prod_{p \in P} d_i(p)$.

Note that $H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i})) \simeq H^1(K_i, \hat{T}_{E_i/K_i})$, and that by Lemma [1.1 (ii), we have $H^1(K_i, \hat{T}_{E_i/K_i}) \simeq H^1(K_i, \hat{T}_{M_i/K_i})$. Moreover, by Lemma [1.2 we have $H^1(K_i, \hat{T}_{M_i/K_i}) \simeq \mathbb{Z}/d_i \mathbb{Z}$. Similarly, we have $H^1(k, I_{K_i(p)/k}(\hat{T}_{E_i(p)/K_i})) \simeq \mathbb{Z}/d_i(p) \mathbb{Z}$. Note that the morphism $\hat{\epsilon}_p^1$ restricted to each $H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i}))$

is the canonical projection $\mathbb{Z}/d_i \mathbb{Z} \to \mathbb{Z}/d_i(p) \mathbb{Z}$. Hence by the Chinese remainder theorem, the morphism $\hat{\epsilon}_p^1$ restricted to each $H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i}))$

is an isomorphism. Similarly,

is also an isomorphism.

By Corollary [3.3, we have $\text{III}^2(k, \hat{T}_{K/k}) = 0$ and $\text{III}^2(k, \hat{T}_{\mathbb{Q}/k}) = 0$, hence $\text{III}^1(k, \hat{S}_{K/K'})$ and $\text{III}^1(k, \hat{S}_{K'/K'})$ are in the image of the maps $\rho^1$. Now we show that $\hat{\epsilon}^1 : \text{III}^1(k, \hat{S}_{K/K'}) \to \bigoplus_{p \in P} \text{III}^1(k, \hat{S}_{K(p), K'})$ is injective. Let $\beta \in \text{III}^1(k, \hat{S}_{K/K'})$. Suppose that $\hat{\epsilon}^1(\beta) = 0$. As $\text{III}^1(k, \hat{S}_{K/K'})$ is in the image of $\rho^1$, there is $\gamma \in H^1(k, I_{K'/k}(\hat{T}_{E/K}))$ such that $\rho^1(\gamma) = \beta$. By the commutativity of the diagram, we have $\rho^1(\hat{\epsilon}^1(\gamma)) = \hat{\epsilon}^1(\rho^1(\gamma)) = 0$. Hence there is $\zeta \in \bigoplus_{p \in P} H^1(k, \hat{T}_{K(p)/k})$ such that $\hat{\epsilon}^1(\zeta) = \hat{\epsilon}^1(\gamma)$. Then $\hat{\epsilon}^1 \circ \hat{\epsilon}^1 \circ (\hat{\epsilon}^1)^{-1}(\zeta) = \hat{\epsilon}^1(\gamma)$. As

is an isomorphism, we have $\hat{\epsilon}^1 \circ (\hat{\epsilon}^1)^{-1}(\zeta) = \gamma$ and hence $\beta = \rho^1(\gamma) = 0$. This proves the injectivity.
As $\Theta^2(k; \hat{T}_{K(p)/k}) = 0$ by Corollary 3.3 the group $\Theta^1(k, \hat{S}_{K(p), K'})$ is in the image of the maps $\rho^i$ for all $p \in \mathcal{P}$. Since $H^1(k, I_{K_i/k}(\hat{T}_{E_i/K_i})) \to \bigoplus_{p \in \mathcal{P}} H^1(k, I_{K_i/k}(\hat{T}_{E_i(p)/K_i}))$ is an isomorphism, we see that

$$\epsilon^1: \Theta^1(k, \hat{S}_{K, K'}) \to \bigoplus_{p \in \mathcal{P}} \Theta^1(k, \hat{S}_{K(p), K'})$$

is surjective. This completes the proof of the proposition.

Note that the proposition, together with Lemma 3.1, implies that $\Theta^1(K, K')$ does not depend on the decomposition of $L$ as $L = K \times K'$. We will also use the notation $\Theta^1(L) = \Theta^1(K, K')$, where $L = K \times K'$ is any decomposition of $L$ with $K/k$ a cyclic extension.

In summary, we proved

**Corollary 5.17.** We have $\Theta^1(L)^* \simeq \Theta^1(k; T_{L/k})$.

**Example 5.18.** Let $p$ and $q$ be two distinct odd prime numbers, with $p > q$. For all positive integers $n$, let $\zeta_n$ be a primitive $n$th root of unity. Let $k = \mathbb{Q}$, and

$$L = \mathbb{Q}(\zeta_p^2) \times \mathbb{Q}(\zeta_{pq}) \times \mathbb{Q}(\zeta_{q^2}).$$

Since $\mathbb{Q}(\zeta_p^2)$ and $\mathbb{Q}(\zeta_{q^2})$ are both cyclic, we can determine $\Theta^1(L)$ in two ways; this shows that the order of $\Theta^1(L)$ divides $p - 1$, and that

$$\Theta^1(L) = \Theta^1(\mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_{pq}) \times \mathbb{Q}(\zeta_{q^2})).$$

But since $\mathbb{Q}(\zeta_p)$ is a subfield of $\mathbb{Q}(\zeta_{pq})$, we have $\Theta^1(\mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_{pq}) \times \mathbb{Q}(\zeta_{q^2})) = \Theta^1(\mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_{q^2}))$. Note that by Proposition 4.1 we have $\Theta^1(\mathbb{Q}(\zeta_p) \times \mathbb{Q}(\zeta_{q^2})) = 0$, hence we have

$$\Theta^1(L) = 0.$$

6. The Brauer-Manin map

We keep the notation of the previous section: in particular, $L = K \times K'$, where $K$ is a cyclic extension of $k$ of degree $d$, $K'$ is an étale $k$-algebra, and $E = K \otimes K'$. We write $K' = \prod_{i \in I} K_i$, where the $K_i/k$ are field extensions, and $E = \prod_{i \in I} E_i$, with $E_i = K \otimes K_i$. The group $\Theta^1(L) = \Theta^1(K, K')$ is defined in the previous section.

Let $c \in k^\times$ and recall that $X_c$ is the affine $k$-variety defined by the equation

$$N_{L/k}(t) = c.$$

Assume that $X_c(k_v) \neq \emptyset$ for all $v \in \Omega_k$. In the following, we define a homomorphism $\alpha_c: \Theta^1(L) \to \mathbb{Q}/\mathbb{Z}$ such that $X_c(k) \neq \emptyset$ if and only if $\alpha_c = 0$; the map $\alpha_c$ will be called the Brauer-Manin map associated to $c$. We choose this terminology, because this map is an analog of the map given by the Brauer-Manin pairing on $X_c$. 
6.1. Local points

We start with some preliminary results. We are assuming that $\prod X_v(k_v) \neq \emptyset$; as we will see, this set contains elements satisfying certain finiteness conditions. More precisely, we introduce the notion of local points - these can be thought of as adelic points of $X_c$.

We first recall the notion of cyclic algebra. Let us choose a generator $g$ of the cyclic group $\text{Gal}(K/k)$, and let $\phi : \Gamma_k \to \mathbb{Z}/d\mathbb{Z}$ be given by the composition of the isomorphism $\text{Gal}(K/k) \to \mathbb{Z}/d\mathbb{Z}$ sending $g$ to $1$ with the surjection $\Gamma_k \to \text{Gal}(K/k)$. Let us consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \to 0$, and let $\delta : H^1(K, \mathbb{Z}/d\mathbb{Z}) \to H^2(k, \mathbb{Z})$ be the connecting homomorphism of the associated cohomology exact sequence. If $c \in k^\times$, let us denote by $c$ the corresponding element of $H^0(k, \mathbb{G}_m)$. The cup product $\delta(\phi)(c)$ is an element of $H^2(k, \mathbb{G}_m)$, and via the identification $H^2(k, \mathbb{G}_m) \simeq \text{Br}(k)$ it is mapped to the class of the cyclic algebra defined by $K$ and $c$ (see for instance [GS 06], Proposition 4.7.3). We denote this cyclic algebra by $(K, c)$.

The first observation is the following:

Lemma 6.1. Suppose that $E_i$ is isomorphic to a product of copies of a field $M_i$, and set $[M_i : K_i] = d_i$. Then for any $x \in K_i^\times$, the order of the cyclic algebra $(K, N_{K_i/k}(x))$ divides $d_i$. In particular, if $d_i = 1$, then for any $x \in K_i^\times$, the algebra $(K, N_{K_i/k}(x))$ splits.

Proof. Given $x \in K_i^\times$, consider the class of the cyclic algebra $(M_i, x) = \delta(\phi|_{K_i})(x)$, where $\phi|_{K_i} : \Gamma_{K_i} \to \mathbb{Z}/d\mathbb{Z}$ is the restriction of $\phi$ to $\Gamma_{K_i}$. Let $r$ be the order of $(M_i, x)$ in $\text{Br}(K_i)$. Since $M_i$ is of degree $d_i$ over $K_i$, we have $r|d_i$. By the projection formula ([GS 06], Prop. 3.4.10), the corestriction of $(M_i, x)$ is $(K, N_{K_i/k}(x))$. Therefore the order of $(K, N_{K_i/k}(x))$ divides $d_i$.

Let $x = (x^v) \in \prod_{v \in \Omega_k} X_c(k_v)$, and let us write $x^v = (x_0^v, x_1^v, \ldots, x_m^v)$, with $x_0^v \in K^v$ and $x_i^v \in K_i^v$ for $i \in \mathcal{I}$. Let us consider the invariant map $\text{inv} : \text{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ and set $b_i^v(x) = b_i^v(x_i^v) = \text{inv}(K^v, N_{K_i/k_v}(x_i^v)) \in \frac{1}{d_i}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Note that if $d_i$ is odd, then $b_i^v(x) = 0$ for all infinite places $v \in \Omega_k$.

We say that $x = (x_i^v) \in \prod_{v \in \Omega_k} X_c(k_v)$ is a local point of $X_c$ if for each $i \in \mathcal{I}$, we have $b_i^v(x) = 0$ for almost all $v \in \Omega_k$. The following lemma implies the existence of local points whenever $\prod_{v \in \Omega_k} X_c(k_v) \neq \emptyset$.

Lemma 6.2. Assume that $\prod_{v \in \Omega_k} X_c(k_v) \neq \emptyset$. Then there exists $x = (x_i^v) \in \prod_{v \in \Omega_k} X_c(k_v)$ such that $b_i^v(x) = 0$ for almost all $v \in \Omega_k$, and for all $i \in \mathcal{I}$.

Proof. For each $v \in \Omega_k$ such that $(K, c)^v$ is split, there exists $x_0^v \in K^v$ such that $N_{K^v/k_v}(x_0^v) = c$. Then $x = (x_0^v, 1, \ldots, 1)$ is a $k_v$-point of $X_c$, and $b_i^v(x) = 0$
for all $i \in \mathcal{I}$. Since $(K, c)^v$ is split for almost all places $v \in \Omega_k$, the lemma follows.

We now prove some properties of local points which will be used later. The next lemma is an analog of the classical reciprocity formula that appears in the classical Brauer-Manin obstruction.

**Lemma 6.3.** Let $x = (x_i^v) \in \prod_{v \in \Omega_k} X_v(k_v)$ be a local point of $X_c$. Then we have

$$\sum_{v \in \Omega_k} \sum_{i \in \mathcal{I}} b_i^v(x) = 0.$$ 

**Proof.** Let us write $x^v = (x_0^v, x_1^v, \ldots, x_m^v)$, with $x_0^v \in K^v$ and $x_i^v \in K_i^v$ for $i \in \mathcal{I}$. For all $i \in \mathcal{I}$, set $y_i^v = N_{L_i^v/k_v}(x_i^v)$. Set $y_0^v = N_{K_i^v/k_v}(x_0^v)$, and note that $y_0^v \prod_i y_i^v = c$. We have

$$\sum_{i \in \mathcal{I}} b_i^v(x) = \sum_{i \in \mathcal{I}} \text{inv}(K^v, y_i^v) = \text{inv}(K^v, \prod_{i \in \mathcal{I}} y_i^v) = \text{inv}(K^v, c/y_0^v) = \text{inv}(K^v, c).$$

Since $c \in k^\times$, the Brauer-Hasse-Noether Theorem implies that $\sum_{v \in \Omega_k} \text{inv}(K^v, c) = 0$. Hence we have $\sum_{v \in \Omega_k} \sum_{i \in \mathcal{I}} b_i^v(x) = 0$, as claimed.

**Lemma 6.4.** Let $x = (x_i^v)$ be a local point of $X_c$, and set $b_i^v = b_i^v(x) = \text{inv}(K^v, N_{K_i^v/k_v}(x_i^v))$. For all $i \in \mathcal{I}$, let $\tilde{x}_i^v \in K_i^v$ and set

$$\tilde{b}_i^v = \text{inv}(K^v, N_{K_i^v/k_v}(\tilde{x}_i^v)).$$

Suppose that for all $i \in \mathcal{I}$ we have $\tilde{b}_i^v = 0$ for almost all $v \in \Omega_k$, and that $\sum_{i \in \mathcal{I}} b_i^v = \sum_{i \in \mathcal{I}} \tilde{b}_i^v$ for all $v \in \Omega_k$. Then for all $v \in \Omega_k$, there exists $\tilde{x}_0^v \in K^v$ such that $\tilde{x} = (\tilde{x}_i^v)$ is a local point of $X_c$.

**Proof.** Let $\tilde{y}_i^v = N_{K_i^v/k_v}(\tilde{x}_i^v)$ and $y_i^v = N_{K_i^v/k_v}(x_i^v)$. Since $\sum_{i \in \mathcal{I}} b_i^v = \sum_{i \in \mathcal{I}} \tilde{b}_i^v$, the algebras $(K^v, \prod_{i \in \mathcal{I}} y_i^v)$ and $(K^v, \prod_{i \in \mathcal{I}} \tilde{y}_i^v)$ are isomorphic, hence there exists some $z \in K^v$ such that $\prod_{i \in \mathcal{I}} (\tilde{y}_i^v)^{-1} = N_{K_i^v/k_v}(z)$. Therefore $(x_0^v z, \tilde{x}_1^v, \ldots, \tilde{x}_m^v)$ is a $k_v$-point of $X_c$.

**Lemma 6.5.** Let $x = (x_i^v)$ be a local point of $X_c$, and set $b_i^v = b_i^v(x) = \text{inv}(K^v, N_{K_i^v/k_v}(x_i^v))$. Suppose that for all $i \in \mathcal{I}$, we have $\sum_{v \in \Omega_k} b_i^v = 0$. Then $X_c$ has a $k$-point.

**Proof.** By the Brauer-Hasse-Noether Theorem, for every $i \in \mathcal{I}$ there exists a central simple algebra $A_i$ over $k$ such that $\text{inv}(A_i) = b_i^v$ for all $v \in \Omega_k$. Set $y_i^v = N_{K_i^v/k_v}(x_i^v)$. Since $(K^v, y_i^v)$ splits over $K^v$ for all $v$, the algebra $A_i$ also splits over $K$. Hence there exists $\tilde{y}_i \in k$ such that $A_i$ is Brauer equivalent to $(K, \tilde{y}_i)$ (see [GS 06] Cor. 4.7.6). Since $(K, \prod_{i \in \mathcal{I}} \tilde{y}_i) \simeq (K^v, \prod_{i \in \mathcal{I}} y_i^v) \simeq (K, c)_v$, the
Brauer-Hasse-Noether Theorem implies that \((K, \prod_{v \in \mathcal{I}} \tilde{y}_v) \simeq (K, c)\), and hence
\[
\prod_{v \in \mathcal{I}} \tilde{y}_v = cN_{K/k}(w) \quad \text{for some } w \in K^\times.
\]
Moreover, we claim that the element \(\tilde{y}_v\) belongs to the group \(N_{K/k}(K^\times)N_{K_1/k}(K_1^\times)\). To see this, we note that
\[
(K, \tilde{y}_v) = (K, y_v^n) = (K, N_{K_1/k}(x_i^n)).
\]
Hence we have \(\tilde{y}_v \in N_{K/k}(J_K)N_{K_1/k}(J_1)\) where \(J_i\) is the idèle group of \(K_i\), for all \(i \in \mathcal{I}\), and \(J_K\) is the idèle group of \(K\). By Proposition 6.6, we have \(\tilde{y}_v = N_{K/k}(w_i)N_{K_1/k}(z_i)\) for some \(w_i \in K^\times\) and \(z_i \in K_1^\times\). Therefore \(\prod_{v \in \mathcal{I}} \tilde{y}_v = \prod_{v \in \mathcal{I}} N_{K/k}(w_i)N_{K_1/k}(z_i) = cN_{K/k}(w)\) and \((w^{-1}\prod_{v \in \mathcal{I}} w_i, z_1, ..., z_m)\) is a k-point of \(X_c\). This completes the proof of the lemma.

6.2. Brauer-Manin map - the prime power degree case

Now suppose that \(K\) is a cyclic extension of degree \(d = p^e\), where \(p\) is a prime. Let \(x = (x_i^n) \in \prod_{v \in \Omega_k} X_v(k_v)\) be a local point of \(X_c\). Let \(M_i\) be a cyclic extension of \(K_i\) such that the algebra \(E_i\) is isomorphic to a product of copies of \(M_i\); then the degree of \(M_i\) is \(p^{e_i}\) for some \(0 \leq e_i \leq e\). Without loss of generality, we assume that \((e_1, ..., e_m)\) is a decreasing sequence. Let us define
\[
\alpha_c : \prod(K, K') \to \mathbb{Q}/\mathbb{Z}
\]
by \(\alpha_c(a_1, ..., a_m) = \sum_{v \in \Omega_k} \sum_{i \in \mathcal{I}} a_i b_i^v(x)\), where \((a_1, ..., a_m) \in G \subseteq \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^{e_i}\mathbb{Z}\). Note that by Lemma 6.1 we have \(b_i^v(x) \in \frac{1}{p^{e_i}}\mathbb{Z}/\mathbb{Z}\). Hence \(a_i b_i^v(x)\) is well-defined. Moreover, by Lemma 6.3 the map \(\alpha_c\) vanishes on the subgroup \(D\) of \(G\); hence, the map \(\alpha_c : \prod(K, K') \to \mathbb{Q}/\mathbb{Z}\) is well-defined.

Note that the map \(\alpha_c\) is an analogue of the map given by the Brauer-Manin pairing. In the following we show that \(\alpha_c\) has a classical property of the Brauer-Manin pairing.

**Proposition 6.6.** The map \(\alpha_c : \prod(K, K') \to \mathbb{Q}/\mathbb{Z}\) is independent of the choice of the local point \(x = (x_i^n)\).

**Proof.** We use the notation of section 5.1. Let \(a \in G\) and \(I(a) = (I_0, ..., I_{p^e-1})\). If \(a \in D\), then by Lemma 6.3 we have \(\alpha_c(a) = 0\). In the following, we assume that \(a \notin D\).

By the definition of \(G\), we have \(\Omega(I_0) \cup ... \cup \Omega(I_{p^e-1}) = \Omega_k\). Given a place \(v \in \Omega_k\), there exists \(n(v) \in \mathbb{Z}/p^{e_i}\mathbb{Z}\) such that \(v \in \Omega(I_{n(v)})\). Set \(\delta_i = \delta(n(v), a_i)\) and let \(K_i^w = \prod_{w \mid v} K_i^w\), where \(K_i^w\) are field extensions of \(k_v\). Then for all \(i \notin I_{n(v)}\), the algebra \(E_i^w\) is isomorphic to a products of field extensions of \(K_i^w\) of degree at most \(p^{n(v)}\). Set \(b_i^v = b_i^v(x)\); by Lemma 6.1 we have \(b_i^v \in \frac{1}{p^{n(v)}}\mathbb{Z}/\mathbb{Z}\). By the definition of \(\delta_i\), we have \(\pi_{e_i, \delta_i}(a_i) = \pi_{e_i, \delta_i}(n(v))\). Hence for \(i \notin I_{n(v)}\), we have
\[
a_i b_i^v = \pi_{e_i, \delta_i}(a_i) b_i^v = \pi_{e_i, \delta_i}(n(v)) b_i^v = n(v) b_i^v.
\]
Hence for all \(v \in \Omega_k\), we have
\[
\sum_{i \in I} a_i b_i^v = n(v) \sum_{i \in I} b_i^v = n(v) \inv(K, c)_v,
\]
which is again independent of the \(x_i^v\)'s. Therefore, the map \(\alpha_c\) is independent of the choice of the local point, and the proposition is proved.

The map \(\alpha_c : \Pi(K, K') \to \mathbb{Q}/\mathbb{Z}\) will be called the Brauer-Manin map for \(X_c\).

Let \(K_0\) be the unique subfield of \(K\) such that \([K_0 : k] = p^{e-1}\), and set \(L_0 = K_0 \times K'\). If \(c \in k^\times\), let \(X_c^0\) be the affine \(k\)-variety determined by \(N_{L_0/k}(t) = c\). There is a natural map \(\rho : X_c \to X_c^0\) defined as \(\rho(x_0, \ldots, x_m) = (N_{K/K_0}(x_0), x_1, \ldots, x_m)\).

If \(X_c^0(k_v) \neq \emptyset\) for all \(v \in \Omega_k\), we denote by \(\alpha_c^0 : \Pi(K_0, K') \to \mathbb{Q}/\mathbb{Z}\) the corresponding Brauer-Manin map.

If \(t_i \in K_i^0\), set
\[
b_i^v(K, t_i) = \inv(K^v, N_{K_i^0/k_v}(t_i)), \quad \text{and} \quad b_i^v(K_0, t_i) = \inv(K_0^v, N_{K_i^0/k_v}(t_i)).
\]
Recall that a local point of \(X_c\) is \(x = (x_i^v) \in \prod_{v \in \Omega_k} X_c(k_v)\) such that for each \(i \in I\), we have \(b_i^v(K, x_i^v) = 0\) for almost all \(v \in \Omega_k\).

The following lemma is an analogue of the functoriality of the Brauer-Manin pairing.

**Lemma 6.7.** Assume that \(X_c(k_v) \neq \emptyset\) for all \(v \in \Omega_k\). Then we have

(i) \(X_c^0(k_v) \neq \emptyset\) for all \(v \in \Omega_k\).

(ii) \(\alpha_c \circ F = \alpha_c^0\).

**Proof.** If \(x^v \in X_c(k_v)\), then \(N_{L_i^0/L_0}(x^v) \in X_c^0(k_v)\). This proves (i). Let us check (ii). Let \(x = (x_i^v)\) be a local point of \(X_c\). Note that \(b_i^v(K_0, x_i^v) = p b_i^v(K, x_i^v)\). Let \(a \in \Pi(K_0, K')\). Then we have
\[
\alpha_c(F(a)) = \sum_{v \in \Omega_k} \sum_{i \in I} a_i(p b_i^v(K, x_i^v)) = \sum_{v \in \Omega_k} \sum_{i \in I} a_i b_i^v(K_0, x_i^v) = \alpha_c^0(a).
\]
This completes the proof of the lemma.

### 6.3. Brauer-Manin map - the general case

Recall that \(K/k\) is a cyclic extension of degree \(d\), and that \(L = K \times K'\), where \(K'\) is an étale \(k\)-algebra. We keep the notation of 5.3, in particular, \(\mathcal{P}\) is the set of prime divisors of \(d\). For all \(p \in \mathcal{P}\), we denote by \(K(p)\) the largest subfield of \(K\) of degree a power of \(p\), and we set \(L(p) = K(p) \times K'\). For all \(c \in k^\times\) and \(p \in \mathcal{P}\), we let \(X_c(p)\) be the \(T_{L(p)/k}\)-torsor defined by
\[
N_{L(p)/k}(x) = c.
\]
Let \(x = (x_i^v) \in \prod_{v \in \Omega_k} X_c(k_v)\) be a local point of \(X_c\), and let us write \(x = (x_0^v, x^v)\) with \(x_0^v \in K^v\) and \(x^v \in K'\). Then \((N_{K^v/K_0^v}(x_0^v), x^v)\) is a local point of \(X_c(p)\).
Let $\alpha(p)$ be the Brauer-Manin map of $X_c(p)$, as defined above. By Proposition $6.6$ the map $\alpha(p)$ is independent of the choice of the local point. Recall that $\text{III}(K, K') = \bigoplus_{p \in \mathcal{P}} \text{III}(K(p), K')$, and let us define $\alpha_c : \text{III}(K, K') \to \mathbb{Q}/\mathbb{Z}$ by $\alpha_c = \bigoplus_{p \in \mathcal{P}} \alpha_c(p)$. Hence $\alpha_c$ is also independent of the choice of the local point. We call $\alpha_c$ the Brauer-Manin map for $X_c$.

7. Necessary and sufficient condition

We keep the notation of the previous sections. The main theorem is the following:

**Theorem 7.1.** The affine $k$-variety $X_c$ has a $k$-point if and only if $X_c$ has a $k_v$-point at each place $v \in \Omega_k$ and $\alpha_c$ is the zero map.

7.1. The prime power degree case

We suppose that $K$ is cyclic of degree $p^e$, where $p$ is a prime number and $e \geq 1$. The proof of Theorem 7.1 uses induction on $e$.

The proof can be divided into three parts. Recall that $\rho : X_c \to X_0^c$ is the map given by $(x_0, x_1, \ldots, x_m) \mapsto (N_{K/K_0}(x_0), x_1, \ldots, x_m)$. (See §6.2.) Suppose that $X_c$ has a $k_v$-point for all $v \in \Omega_k$ and that $X_0^c$ has a $k$-point $z = (z_0, \ldots, z_m)$. In the first part we use the point $z$ to get a system of local solutions $(\tilde{x}_v^i)$ of $X_c$ such that $\sum_{v \in \Omega_k} b_v^i(K, \tilde{x}_v^i) \in \frac{1}{p} \mathbb{Z}/\mathbb{Z}$. (See Lemma 7.3-Lemma 7.5)

In Lemma 7.6-Lemma 7.9, we show that one can further modify $(\tilde{x}_v^i)$ so that $\sum_{v \in \Omega_k} b_v^i(K, \tilde{x}_v^i) = 0$. In the end we conclude our main theorem by induction on $e$ and Lemma 6.5.

We would like to mention that an alternative way to prove the theorem is to find the generators of the unramified Brauer group of $X_c$ and show that the Brauer-Manin map defined here is the evaluation on the unramified Brauer group. However, here we choose to prove the theorem in a more elementary way and keep the unramified Brauer group for our future work.

We start with some preliminary results.

Recall that $E_i = K \otimes_k K_i$.

**Lemma 7.2.** Suppose that $K$ is a cyclic extension of degree $p^e$, where $p$ is a prime and $e \geq 1$. Let $v \in \Omega_k$ and $i \in \mathcal{I}$ be such that $E_i^v$ is not isomorphic to a product of copies of $K_i^v$. Then for all $b \in \frac{1}{p} \mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$, there exists $x \in K_i^v$ such that $\text{inv}(K_i^v, N_{K_i^v/k_v}(x)) = b$.

**Proof.** Suppose that $K_i^v$ is isomorphic to a product of copies of a field extension $M$ of $k_v$, and set $[M : k_v] = p^f$. Since by hypothesis $E_i^v$ is not isomorphic to a product of copies of $K_i^v$, we have $f \geq 1$. Assume that $K_i^v \simeq \prod_{j \in J} M_{i,j}$, where $M_{i,j}$ is a field extension of $k_v$ for all $j \in J$. 


It suffices to prove that $\frac{1}{p}\mathbb{Z}/\mathbb{Z} \subseteq \text{inv}(M, N_{M_{i,j}/k_v}(M_{i,j}^v))$ for some $j \in J$; hence we may assume that $K^v$ is a field extension of $k_v$ of degree $p^e$ with $e \geq 1$, and that $K_i^v$ is a field.

Let $\text{Br}(K^v/k_v)$ be the subgroup of the Brauer group of $k_v$ split by $K^v$; this group is isomorphic to $\mathbb{Z}/p^e\mathbb{Z} \simeq k_v^\times/N((K^v)^\times)$. (See [GS 06] Cor. 4.4.10 and [CF 67] Chap. VI §1.1 Thm. 3 Cor. 2.)

For all $i \in I$, let $M_i$ be a field such that $E_i^v = K \otimes_{k_v} K_i^v$ is a product of copies of $M_i$, and set $[M_i : K_i^v] = p^{e_i}$; the hypothesis implies that $e_i \geq 1$. The corestriction map $\text{Br}(K_i^v) \to \text{Br}(k_v)$ is an injection and restricts to an injection of $\text{Br}(M_i/K_i^v)$ into $\text{Br}(K^v/k_v)$, the image being the unique subgroup of order $p^{e_i}$ of the cyclic group of order $p^e$. By the projection formula ([GS 06] Prop. 3.4.10), the image consists of cyclic algebras of the type $(K_i^v, N_{K_i^v/k_v}(z))$ with $z$ an element of $K_i^v$. Hence $\frac{1}{p}\mathbb{Z}/\mathbb{Z} \subseteq \frac{1}{p^e}\mathbb{Z}/\mathbb{Z} = \text{inv}(K^v, N_{K^v/k_v}(K_i^v)^\times)$. This completes the proof of the lemma.

Let $K_0$ be the unique subfield of $K$ such that $[K_0 : k] = p^{e-1}$.

Recall that we have $K' = \prod_{i \in I} K_i$, that $E_i = K \otimes K_i$, and that $E_i$ is the product of copies of a cyclic extension of degree $p^{e_i}$ of $K_i$. Set $E_i^0 = K_0 \otimes K_i$. Then $E_i^0$ also splits as a product of copies of a cyclic extension of $K_i$; let us denote by $p_i^f$ the degree of this extension. Recall that for all $i \in I$, we have $f_i \leq e_i$. If moreover $e_i \neq 0$, then $e_i = f_i + 1$ (cf. lemma 5.5). For all $i \in I$, the map $F_i : \mathbb{Z}/p_i^f\mathbb{Z} \to \mathbb{Z}/p_i^e\mathbb{Z}$ is the inclusion of the unique subgroup of order $p_i^f$ in the group $\mathbb{Z}/p_i^e\mathbb{Z}$, and we set $F = \bigoplus_{i \in I} F_i$.

The map $F : \bigoplus_{i \in I} \mathbb{Z}/p_i^f\mathbb{Z} \to \bigoplus_{i \in I} \mathbb{Z}/p_i^e\mathbb{Z}$ induces a homomorphism $F : \text{III}(K_0, K') \to \text{III}(K, K')$. Recall that the cokernel of $F$ is isomorphic to the group $\text{III}(K/K_0, K')$, defined in section 5.

For all $i \in I$, set $r_i = \min\{1, e_i\}$. For all $c \in \bigoplus_{i \in I} \mathbb{Z}/p_i^{r_i}\mathbb{Z}$ and $n \in \mathbb{Z}/p\mathbb{Z}$, set $I_n^i(c) = \{i \in I \mid n \geq c\}$. If $I_n^i(c) \neq I$, set $\Omega(I_n^i(c)) = \bigcap_{i \notin I_n^i(c)} \Sigma_i$; if $I_n^i(c) = I$, set $\Omega(I_n^i(c)) = \Omega_k$. Set

$$G(K/K_0, K') = \{c \in \bigoplus_{i \in I} \mathbb{Z}/p_i^{r_i}\mathbb{Z} \mid \bigcup_{n \in \mathbb{Z}/p^{r_i}\mathbb{Z}} \Omega(I_n^i(c)) = \Omega_k\},$$

let $D(K/K_0, K')$ be the diagonal subgroup of $G(K/K_0, K')$, and recall that $\text{III}(K/K_0, K') = G(K/K_0, K')/D(K/K_0, K')$.

Recall that the projection $\pi : \bigoplus_{i \in I} \mathbb{Z}/p_i^{r_i}\mathbb{Z} \to \bigoplus_{i \in I} \mathbb{Z}/p_i^{r_i}\mathbb{Z}$ induces a homomorphism $F : \text{III}(K, K') \to \text{III}(K/K_0, K')$ (cf. lemma 5.9).

If $t_i \in K_i^v$, set $b_i^v(K, t_i) = \text{inv}(K^v, N_{K_i^v/k_v}(t_i))$, and $b_i^v(K_0, t_i) = \text{inv}(K_0^v, N_{K_i^v/k_v}(t_i))$.

Recall that a local point of $X_c$ is $x = (x_i^v) \in \prod_{v \in \Omega_k} X_c(k_v)$ such that for each $i \in I$, we have $b_i^v(K, x_i^v) = 0$ for almost all $v \in \Omega_k$. 


Lemma 7.3. Let $x = (x_i^v)$ be a local point of $X_c$, and let $z = (z_i)$ be a global point of $X_c^0$. Then for all $v \in \Omega_k$, we have

$$p \sum_{i \in I} b_i^v(K, x_i^v) = p \sum_{i \in I} b_i^v(K, z_i).$$

Proof. Since $x$ is a local point of $X_c$, we have $\sum_{i \in I} b_i^v(K, x_i^v) = \text{inv}(K^v, c)$ for all $v \in \Omega_k$. Similarly, we have $\sum_{i \in I} b_i^v(K_0, z_i) = \text{inv}(K_0^v, c)$ for all $v \in \Omega_k$. Note that $\text{inv}(K_0^v, c) = p \text{inv}(K^v, c)$, and $\text{inv}(K_0^v, z_i) = p \text{inv}(K^v, z_i)$ for all $i \in I$. Hence we have

$$p \sum_{i \in I} b_i^v(K, x_i^v) = p \text{inv}(K^v, c) = \sum_{i \in I} b_i^v(K_0, z_i) = p \sum_{i \in I} b_i^v(K, z_i),$$
as claimed.

Lemma 7.4. Let $x = (x_i^v)$ be a local point of $X_c$, and assume that $X_c^0(k) \neq \emptyset$. Then there exist $\tilde{x}_i^v \in K_i^v$ such that

(i) For each $i \in I$, we have $b_i^v(K, \tilde{x}_i^v) = 0$ for almost all $v \in \Omega_k$.

(ii) For all $i \in I$, we have $\sum_{v \in \Omega_k} b_i^v(K, \tilde{x}_i^v) \in \frac{1}{p}\mathbb{Z}/\mathbb{Z}$.

(iii) For all $v \in \Omega_k$, we have

$$\sum_{i \in I} b_i^v(K, x_i^v) = \sum_{i \in I} b_i^v(K, \tilde{x}_i^v).$$

Proof. Let $z = (z_i)$ be a global point of $X_c^0$. Set $b_i^v = b_i^v(K, x_i^v)$ and $h_i^v = b_i^v(K, z_i)$. By Lemma 7.3, we have $p \sum_{i \in I} b_i^v = p \sum_{i \in I} h_i^v$. Since $b_i^v = 0$ and $h_i^v = 0$ for almost all $v \in \Omega_k$, for almost places $v \in \Omega_k$ we have $\sum_{i \in I} h_i^v = \sum_{i \in I} b_i^v$.

Suppose that there is $v \in \Omega_k$ such that $\sum_{i \in I} h_i^v \neq \sum_{i \in I} b_i^v$. If $v \in \cap \Sigma_i$, then $b_i^v = h_i^v = 0$ for all $i \in I$, which is a contradiction. Hence there exists $i \in I$ such that $v \notin \Sigma_i$. Since $p \sum_{j \in I} b_j^v = p \sum_{j \in I} h_j^v$ in $\mathbb{Q}/\mathbb{Z}$, we know that

$$\sum_{j \in I} b_j^v = \sum_{j \in I} h_j^v \in \frac{1}{p}\mathbb{Z}/\mathbb{Z}.$$  
By Lemma 7.2, there exists $\tilde{x}_i^v \in K_i^v$ such that $\text{inv}(K^v, N_{K_i^v/K_0^v}(\tilde{x}_i^v)) = h_i^v - \sum_{j \in I} (h_j^v - b_j^v)$.

Set $\tilde{h}_i^v = \text{inv}(K^v, N_{K_i^v/K_0^v}(\tilde{x}_i^v))$; for all $j \neq i$, let $\tilde{x}_j^v = z_j$, $\tilde{h}_j^v = h_j^v = b_j^v(K, z_j)$. Then we have $\sum_{j \in I} \tilde{h}_j^v = \sum_{j \in I} b_j^v$; this proves (iii).

Since $\tilde{h}_i^v = h_i^v$ for almost all $v \in \Omega_k$, (i) holds. As $z = (z_i)$ is a global point of $X_c^0$, we have $\sum_{v \in \Omega_k} b_i^v(K_0, z_i) = 0$, hence $\sum_{v \in \Omega_k} h_i^v \in \frac{1}{p}\mathbb{Z}/\mathbb{Z}$; moreover, $h_i^v - \tilde{h}_i^v \in \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ for all $i \in I$ and all $v \in \Omega_k$. Therefore we have $\sum_{v \in \Omega_k} \tilde{h}_i^v \in \frac{1}{p}\mathbb{Z}/\mathbb{Z}$, and this proves (ii).
Lemma 7.5. Assume that $X_c(k_v) \neq \emptyset$ for all $v \in \Omega_k$, and that $X^0_c(k) \neq \emptyset$. Then there exists a local point $\bar{x} = (\bar{x}_i^v)$ of $X_c$ such that for all $i \in \mathcal{I}$, we have

$$\sum_{v \in \Omega_k} b_i^v(K, \bar{x}_i^v) \in \frac{1}{p} \mathbb{Z}/\mathbb{Z}.$$ 

Proof. Let $x = (x_i^v)$ be a local point of $X_c$. By Lemma 7.4 there exist $\bar{x}_i^v \in K^v_i$ such that $b_i^v(K, \bar{x}_i^v) = 0$ for almost all $v \in \Omega_k$, that $\sum_{i \in \mathcal{I}} b_i^v(K, x_i^v) = \sum_{i \in \mathcal{I}} b_i^v(K, \bar{x}_i^v)$, and that for all $i \in \mathcal{I}$, we have $\sum_{v \in \Omega_k} b_i^v(K, \bar{x}_i^v) \in \frac{1}{p} \mathbb{Z}/\mathbb{Z}$. By Lemma 6.4 for all $v \in \Omega_k$, there exists $\bar{x}_0^v \in (K^v)^\times$ such that $(\bar{x}_0^v, \bar{x}_1^v, ..., \bar{x}_m^v) \in X_c(k_v)$. This completes the proof of the lemma.

Recall that if $X_c(k_v) \neq \emptyset$ for all $v \in \Omega_k$, and that for all $c \in K^\times$, we have a homomorphism $\alpha_c : III(K, K') \to \mathbb{Q}/\mathbb{Z}$. We now show that $\alpha_c$ induces a homomorphism $\overline{\alpha}_c : III(K/K_0, K') \to \mathbb{Q}/\mathbb{Z}$ such that $\overline{\alpha}_c \circ \pi = \alpha_c$.

Lemma 7.6. Assume that $X_c(k_v) \neq \emptyset$ for all $v \in \Omega_k$ and that $X^0_c(k) \neq \emptyset$. Then there exists a homomorphism $\overline{\alpha}_c : III(K/K_0, K') \to \mathbb{Q}/\mathbb{Z}$ such that $\overline{\alpha}_c \circ \pi = \alpha_c$.

Proof. Let $x = (x_i^v)$ be a local point of $X_c$, and set $b_i^v = \text{inv}(K^v, N_{k_0/k}(x_i^v))$ for all $i \in \mathcal{I}$ and all $v \in \Omega_k$. Let $a = (a_1, ..., a_m) \in G$. Then by Lemma 5.3

$$\pi(a) = (\overline{a}_1, ..., \overline{a}_m) \in G(K/K_0, K') \subseteq \bigoplus_{i \in \mathcal{I}} \mathbb{Z}/p^r \mathbb{Z}. $$

Note that $r_i = \min\{1, e_i\}$ by definition. If $e_i = 0$, then $c_i = r_i = 0$ and $a_i = \overline{a}_i = 0$. If not, then $r_i = 1$ and $\overline{a}_i = a \pmod{p}$. By Lemma 7.3 we may assume that

$$\sum_{v \in \Omega_k} b_i^v \in \frac{1}{p} \mathbb{Z}/\mathbb{Z}$$

for all $i \in \mathcal{I}$; hence $a_i(\sum_{v \in \Omega_k} b_i^v) = \overline{a}_i(\sum_{v \in \Omega_k} b_i^v)$. We then have

$$\alpha_c(a_1, ..., a_m) = \sum_{i \in \mathcal{I}} a_i \left( \sum_{v \in \Omega_k} b_i^v \right) = \sum_{i \in \mathcal{I}} \overline{a}_i \left( \sum_{v \in \Omega_k} b_i^v \right).$$

Hence $\alpha_c$ induces a homomorphism $\overline{\alpha}_c : III(K/K_0, K') \to \mathbb{Q}/\mathbb{Z}$, as claimed.

Lemma 7.7. Let $x = (x_i^v)$ be a local point of $X_c$, and set $b_i^v = b_i^v(K, x_i^v)$. Assume that $\alpha_c = 0$, and that $X^0_c(k) \neq \emptyset$. Let $(I_0, ..., I_{p-1}) \in G(K/K_0, K')$. Then we have

$$\sum_{i \in I_n} \sum_{v \in \Omega_k} b_i^v = 0$$

for all $n \in \mathbb{Z}/p\mathbb{Z}$.

Proof. Let $n \in \mathbb{Z}/p\mathbb{Z}$. The statement is trivial if $I_n$ is empty, and it follows from Lemma 6.3 if $I_n = I'$. Assume that $I_n$ is not empty, and $I_n \neq I'$. Let $n' \in \mathbb{Z}/p\mathbb{Z}$ such that $n' \neq n$, and set $J_n = I_n$, $J_{n'} = \bigcup_{r \neq n} I_r$, and $J_r = \emptyset$ if $r \neq n, n'$. Then by Lemma 5.1.5 we have $(J_0, ..., J_{p-1}) \in G(K/K_0, K')$. Since $X^0_c(k) \neq \emptyset$, by Lemma 7.6 there exists a homomorphism $\overline{\alpha}_c : III(K/K_0, K') \to \mathbb{Q}/\mathbb{Z}$ such that $\overline{\alpha}_c \circ \pi = \alpha_c$. By hypothesis $\alpha_c$ is the zero map, hence we have $\overline{\alpha}_c = 0$. Therefore we have

$$\sum_{r \in \mathbb{Z}/p\mathbb{Z}} \sum_{i \in J_r} \sum_{v \in \Omega_k} r b_i^v = \sum_{i \in J_n} \sum_{v \in \Omega_k} n b_i^v + \sum_{i \in J_{n'}} \sum_{v \in \Omega_k} n' b_i^v = 0.$$
By Lemma 6.3 we have \( \sum_{i \in J} \sum_{v \in \Omega_k} b_i^v = 0 \), hence \((n - n') \sum_{i \in J_n} \sum_{v \in \Omega_k} b_i^v = 0 \). Recall that \( n' \neq n \) by hypothesis, therefore we have \( \sum_{i \in J_n} \sum_{v \in \Omega_k} b_i^v = 0 \); since \( J_n = I_n \), we have \( \sum_{v \in \Omega_k} b_i^v = 0 \), as claimed.

**Lemma 7.8.** Let \( x = (x_i^v) \) be a local point of \( X_c \). Assume that \( \alpha_c = 0 \), and that \( X_c^0(k) \neq \emptyset \). Let \( (I_0, \ldots, I_{p-1}) \in G(K/K_0, K') \), and let \( n \in \mathbb{Z}/p\mathbb{Z} \). Then there exists a local point \( \tilde{x} = (\tilde{x}_i^v) \) of \( X_c \) such that \( \tilde{x}_i^v = x_i^v \) if \( i \not\in I_n \), and that \( \sum_{v \in \Omega_k} b_i^v(K, \tilde{x}) = 0 \) for all \( i \in I_n \).

**Proof.** We prove this by induction on the cardinality of \( I_n \). If \(|I_n| = 0 \) then the claim is trivial; if \(|I_n| = 1 \), then it follows from Lemma 7.7 since we have \( \sum_{v \in \Omega_k} b_i^v(x) = 0 \) for all \( i \in I_n \). Suppose that the claim is true for \(|I_n| < h \). For \(|I_n| = h \), suppose that there are nonempty disjoint subsets \( I_0^1 \) and \( I_1^1 \) of \( I_n \) satisfying \( I_0^1 \cup I_1^1 = I_n \) and \( (\cap \Sigma_i) \cup (\cap \Sigma_i) = \Omega_k \). Then consider the element \( (J_0, \ldots, J_{p-1}) \) where \( J_r = J_r \) if \( r \neq n, n + 1 \), \( J_n = I_n^0 \) and \( J_n + 1 = I_n^1 \cup I_n + 1 \). Note that \( \Omega(I_r) = \Omega(J_r) \) if \( r \neq n, n + 1 \) and that \( \Omega(I_{n + 1}) \subset \Omega(J_{n + 1}) \). Let us prove that \( (J_0, \ldots, J_{p-1}) \) represents an element of \( \Pi(K, K') \); for this, we have to check that \( \Omega(J_0) \cup \cdots \cup \Omega(J_{p-1}) = \Omega_k \). Since \( \Omega(I_r) \subset \Omega(J_r) \) if \( r \neq n \) and \( \Omega(I_0) \cup \cdots \cup \Omega(I_{p-1}) = \Omega_k \), it suffices to check that if \( v \in \Omega(I_n) \), then \( v \in \Omega(J_0) \cup \cdots \cup \Omega(J_{p-1}) \). If \( v \in \cap \Sigma_i \), then we have \( v \notin \Omega(I_{n+1}) \); otherwise, we have \( v \in \cap \Sigma_i \) because \( (\cap \Sigma_i) \cup (\cap \Sigma_i) = \Omega_k \). Hence we have \( v \in (\cap \Sigma_i) \cup (\cap \Sigma_i) = \Omega(I_{n+1}) \). Therefore \( (J_0, \ldots, J_{p-1}) \) represents an element of \( \Pi(K, K') \). Since \(|J_n| < h \), we can apply the induction hypothesis, and hence there exists a local point \( x = (x_i^v) \) such that \( x_i^v = x_i^v \) if \( i \not\in J_n = I_n^0 \), and that \( \sum_{v \in \Omega_k} b_i^v(K, x) = 0 \) for all \( i \in J_n = I_n^1 \). The same argument with \( I_n^0 \) instead of \( I_n^1 \) gives the desired result.

Assume now that \( I_n \) does not have any non-trivial subpartitions, in other words, that there are no nonempty disjoint subsets \( I_0^1 \) and \( I_1^1 \) of \( I_n \) satisfying \( I_0^1 \cup I_1^1 = I_n \) and \( (\cap \Sigma_i) \cup (\cap \Sigma_i) = \Omega_k \). Let us consider the graph with vertex set \( I_n \), and edge set \( E = \{(i, j) | \Sigma_i \cup \Sigma_j \neq \Omega_k \} \); since \( I_n \) has no non-trivial subpartitions, this graph is connected. Set \( b_i^v = b(K, x_i^v) \) and for all \( i \in I_n \), set \( d_i = \sum_{v \in \Omega_k} b_i^v \). Let us fix an ordering of \( I_n \), say \( I_n = \{i_0, \ldots, i_t \} \). Since the graph is connected, there exists a loop-free path between \( i_0 \) and \( i_1 \). Along this path, for any two adjacent vertices \( i, j \), there exists \( v \in \Omega_k \) such that \( v \not\in \Sigma_i \cup \Sigma_j \). By Lemma 7.5 we may assume that \( b_i^v \in \mathbb{Z}/p\mathbb{Z} \) for all \( i \in I_n \). Applying Lemma 7.2 by modifying \( x_i^v \) and \( x_j^v \) we can modify \( b_i^v \) to \( b_i^v - d_{i_0} \) and \( b_j^v \) to \( b_j^v + d_{i_0} \). Note that this modification does not change \( \sum_{v \in \Omega_k} b_i^v \). Therefore by Lemma 6.3, after changing also \( x_i^v \) if necessary, the modified \( (x_i^v) \) is still a local point of \( X_c \). After these modifications, we have \( \sum_{v \in \Omega_k} b_i^v = 0 \), \( \sum_{v \in \Omega_k} b_i^v = d_{i_1} + d_{i_0} \).
and all the other \( d_i \)'s remain unchanged. We repeat this process along a loop-
free path from \( i_1 \) to \( i_2 \), and we modify each adjacent pair along the path from
\( i_1 \) to \( i_2 \) by \( d_{i_0} + d_{i_1} \) and so on. At the end, we modify each adjacent pair along
the path from \( i_{t-1} \) to \( i_t \) by \( \sum_{r=0}^{t-1} d_{i_r} \). After this process, we have \( \sum_{v \in \Omega_k} b_i^v = 0 \) for
\( r = 0, \ldots, t-1 \) and \( \sum_{v \in \Omega_k} b_i^v = d_{i_t} + \sum_{r=0}^{t-1} d_{i_r} \). However, by Lemma \( \ref{lemma:7.7} \) we know
that \( \sum_{r=0}^{t} d_{i_r} = 0 \); hence, we have \( \sum_{v \in \Omega_k} b_i^v = 0 \). Moreover, only finitely many \( b_i^v \)'s
are modified, so \( b_i^v = 0 \) for almost all \( v \); the lemma then follows.

**Proposition 7.9.** Let \( x = (x_i^v) \) be a local point of \( X_c \). Assume that \( \alpha_c = 0 \),
and that \( X_c^0(k) \neq \emptyset \). Then there exists a local point \( \tilde{x} = (\tilde{x}_i^v) \) of \( X_c \) such that
for all \( i \in I \), we have
\[
\sum_{v \in \Omega_k} b_i(K, \tilde{x}_i^v) = 0.
\]

**Proof.** This follows from Lemma \( \ref{lemma:7.8} \).

**Proof of Theorem 7.1** for \( K \) of prime power degree.

It is clear that if \( X_c \) has a \( k \)-point, then \( X_c \) has a \( k_v \)-point for all \( v \in \Omega_k \)
and \( \alpha_c = 0 \). Conversely, suppose that \( X_c \) has a \( k_v \)-point for all \( v \in \Omega_k \) and
that \( \alpha_c = 0 \). Let us show that the variety \( X_c \) has a \( k \)-point. We show our
claim by induction on the exponent \( e \). Suppose that \( e = 1 \). Then \( K_0 = k \), and
\( X_c^0(k) \neq \emptyset \). By Proposition \( \ref{prop:7.9} \) there exists a local point \( x = (x_i^v) \) of \( X_c \) such that
for all \( i \in I \), we have \( \sum_{v \in \Omega_k} b_i(x_i^v) = 0 \). Lemma \( \ref{lemma:6.3} \) implies that \( X_c(k) \neq \emptyset \).

Assume now that \( e > 1 \). Since \( X_c \) has a \( k_v \)-point for all \( v \in \Omega_k \), the variety
\( X_c^0 \) also has a \( k_v \)-point for all \( v \in \Omega_k \). As \( \alpha_c \) is the zero map, by Lemma \( \ref{lemma:6.7} \)
the Brauer-Manin map \( \alpha_c \) for \( X_c^0 \) is also the zero map. Therefore \( X_c \) has a
\( k \)-point by induction hypothesis. By Proposition \( \ref{prop:7.9} \) there exists a local point
\( x = (x_i^v) \) of \( X_c \) such that for all \( i \in I \), we have \( \sum_{v \in \Omega_k} b_i(x_i^v) = 0 \). Lemma \( \ref{lemma:6.5} \)
implies that \( X_c(k) \neq \emptyset \).

### 7.2. The general case

Recall that \( K/k \) is a cyclic extension of degree \( d \), and that \( L = K \times K' \),
where \( K' \) is an arbitrary \( k \)-algebra. We keep the notation of 5.3, in particular, \( \mathcal{P} \) is the set of prime divisors of \( d \). For all \( p \in \mathcal{P} \), we denote by \( K(p) \) the largest subfield of \( K \) of order a power of \( p \), and \( L(p) = K(p) \times K' \). For all \( c \in k^\times \) and \( p \in \mathcal{P} \), the affine \( k \)-variety defined by
\[
N_{L(p)/K(p)}(x) = c.
\]
is denoted by \( X_c(p) \). Recall that there is a natural map from \( X_c \) to \( X_c(p) \)
\((\S 6.3) \). We denote by \( \alpha_c(p) \) be the Brauer-Manin map of \( X_c(p) \). Recall that
\[ \text{III}(K, K') = \bigoplus_{p \in P} \text{III}(K(p), K'), \text{ and that } \alpha_c : \text{III}(K, K') \to \mathbb{Q}/\mathbb{Z} \text{ is given by } \\
\alpha_c = \bigoplus_{p \in P} \alpha_c(p). \]

**Lemma 7.10.** Let \( c \in k^\times \). Then \( X_c \) has a \( k \)-point if and only if \( X_c(p) \) has a \( k \)-point for all \( p \in P \).

**Proof.** Let \( z \in X_c(k) \) be a \( k \)-point of \( X_c \), and let us write \( z = (x, y) \) with \( x \in K \) and \( y \in K' \). Then \((N_{K/K(p)}(x), y)\) is a \( k \)-point of \( X_c(p) \) for all \( p \in P \). Conversely, suppose that for all \( p \in P \), the \( k \)-variety \( X_c(p) \) has a \( k \)-point \((x_p, y_p) \in K(p) \times K' \). For all \( p \in P \), set \( r_p = \prod_{q \in P, q \neq p} [K(q) : k] \), and let \( s_p \in \mathbb{Z} \) such that \( \sum_{p \in P} r_p s_p = 1 \). Set \( x = \prod_{p \in P} x_p^{s_p} \), and \( y = \prod_{p \in P} y_p^{r_p s_p} \). Then \((x, y)\) is a \( k \)-point of \( X_c \).

**Remark 7.11.** The above lemma is compatible with base change to any field extension \( l \) of \( k \).

**Proof of Theorem 7.1.** Suppose that \( X_c \) has a \( k \)-point. Then by lemma 7.10, \( X_c(p) \) has a \( k \)-point for all \( p \in P \). This implies that \( \alpha_c(p) = 0 \) for all \( p \in P \), and hence \( \alpha_c = 0 \). Conversely, suppose that \( X_c \) has a \( k \)-point for all \( v \in \Omega_k \) and that \( \alpha_c = 0 \). Then \( X_c(p) \) has a \( k \)-point for all \( v \in \Omega_k \). Since \( \alpha_c = 0 \), we have \( \alpha_c(p) = 0 \) for all \( p \in P \). But \( K(p) \) is a cyclic extension of prime power degree, hence this implies that \( X_c(p) \) has a \( k \)-point for all \( p \in P \). Therefore \( X_c \) has a \( k \)-point by Lemma 7.10.

**Corollary 7.12.** Let \( I_L \) be the idèle group of \( L \). Then sending \( c \in k^\times \) to \( \alpha_c \) gives rise to an isomorphism

\[ (k^\times \cap N_{L/k}(I_L))/N_{L/k}(L^\times) \to \text{III}(L)^*. \]

**Proof.** It is clear from the definition of \( \alpha_c \) that sending \( c \in k^\times \) to \( \alpha_c \) is a homomorphism; Theorem 7.1 implies that this homomorphism is injective. That it is an isomorphism follows from the fact that \( \text{III}(L)^* \simeq \text{III}^1(k, T_{L/k}) \) (see Corollary 5.17).

**Metacyclic extensions**

In the following we apply the main theorem to the case where \( K \) is a metacyclic extension of \( k \) (recall that a metacyclic extension is a Galois extension such that all the Sylow subgroups of its Galois group are cyclic). As before, let \( X_c \) be the \( k \)-variety defined by the equation (1.1). Assume that \( K/k \) is a metacyclic extension of degree \( q = \prod_{j=1}^s p_j^{e_j} \), where \( p_j \)'s are distinct primes. Let \( q_j = p_j^{e_j} \) and \( r_j = q/q_j \). For \( 1 \leq j \leq s \), let \( G_j \) be a \( p_j \)-Sylow subgroup of \( \text{Gal}(K/k) \) and let \( F_j \) be the subfield of \( K \) fixed by \( G_j \). Note that \([F_j : k] = r_j \).
Let $X^j_c$ be $X_c \otimes_k F_j$. Then the injection $k \to F_j$ induces a natural injection of $X_c(k)$ to $X_c(F_j) = X^j_c(F_j)$.

Suppose that $X_c$ has a $k_v$-point for all $v \in \Omega_k$. Then $X^j_c$ has a $F_{j,w}$-point for all $w \in \Omega_{F_j}$. Since $F_j$ is a cyclic extension of $k$, we can define the Brauer-Manin map $\alpha_j$ for $X^j_c$. The necessary and sufficient condition for the Hasse principle for $X_c$ to hold is the following:

**Proposition 7.13.** Assume that $K$ is a metacyclic extension. Then $X_c$ has a $k$-point if and only if $X_c$ has a $k_v$-point for all $v \in \Omega_k$ and $\alpha_j = 0$ for $1 \leq j \leq s$.

**Proof.** Assume that $X_c$ has a $k_v$-point for all $v \in \Omega_k$, and that $\alpha_j = 0$ for $1 \leq j \leq s$. Then the variety $X^j_c$ has a $F_{j,w}$-point for all $w \in \Omega_{F_j}$. Since $\alpha_j = 0$ for all $1 \leq j \leq s$, by Theorem 8.2. the variety $X^j_c$ has a $F_j$-point. Let $(x_{j,i})$ be a $F_j$-point of $X^j_c$, where $x_{j,i} \in (F_j \otimes_k K_i)^\times$. Let $b_j \in \mathbb{Z}$ such that $\sum_{j=1}^{s} b_j r_{j} = 1$, and set $z_i = \prod_{j=1}^{s} N_{F_j \otimes K_i/K_i}(x_{j,i})^{b_j}$; then $(z_i)$ is a point of $X_c$. The other direction is trivial.

8. **Products of cyclic extensions**

In this section, we suppose that $L$ is a *product of cyclic extensions*, and we denote by $\mathfrak{III}(L)$ the obstruction group. In the following, we give a simple criterion for the vanishing of $\mathfrak{III}(L)$; in other words, an easy way to decide whether the Hasse principle holds for $L$.

Assume that $L = \prod_{i \in J} K_i$, where $K_i/k$ is a cyclic extension of degree $d_i$. Let $\mathcal{P}$ be the set of prime numbers dividing $\prod_{i \in J} d_i$. For all $p \in \mathcal{P}$ and all $i \in J$, let $K_i(p)$ be the largest subfield of $K_i$ such that $[K_i(p) : k]$ is a power of $p$, and set $L(p) = \prod_{i \in J} K_i(p)$.

For any cyclic field extension $K/k$ of prime power degree, we denote by $K_{\text{prim}}$ the unique subfield of $K$ of degree $p$ over $k$. Set $L(p)_{\text{prim}} = \prod_{i \in J} K_i(p)_{\text{prim}}$.

The aim of this section is to prove the following two results:

**Theorem 8.1.** Assume that $K_{\text{prim}}$ is linearly disjoint from $K_i$ for all $i \in I$.

\[ \mathfrak{III}(L) = 0 \iff \bigoplus_{p \in \mathcal{P}(L)} \mathfrak{III}(L(p)_{\text{prim}}) = 0, \]

where $\mathcal{P}(L)$ is a set of prime numbers, subset of $\mathcal{P}$.

The set $\mathcal{P}(L)$ is determined in Theorem 8.3 see below.

**Theorem 8.2.**

\[ \mathfrak{III}(L(p)_{\text{prim}}) \simeq (\mathbb{Z}/p\mathbb{Z})^{m_p(L)}, \]

where $m_p(L)$ is a positive integer.
The value of $m_p(L)$ is given in Theorem 8.3.

We start with the proof of Theorem 8.2, which amounts to treating the case where $L$ is a product of cyclic extensions of prime degree.

**Theorem 8.3.** Let $p$ be a prime number, and assume that $L$ is a product of $n$ non-isomorphic cyclic extensions of degree $p$. Then we have

(a) If $n \leq 2$, then $\Pi(L) = 0$.

(b) If $3 \leq n \leq p + 1$, then either $\Pi(L) = 0$, or $\Pi(L) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2}$.

(c) If $n \geq p + 2$, then $\Pi(L) = 0$.

Note that Theorem 8.3 implies immediately Theorem 8.2, and gives the value of the integer $m_p(L)$. Note also that the case $n = 1$ is a special case Hasse’s cyclic norm theorem, the case $n = 2$ follows from Hürlimann’s result [Hu 84] Prop. 3.3, (see also Proposition 4.1), and that the case $n = 3, p = 3$ is a result of Colliot-Thélène, cf. [CT 14], Théorème 4.1.

In order to prove Theorem 8.3, we need to come back to the definition of $\Pi(L) = \Pi(K, K')$ in the case where $K$ is cyclic of prime degree, and give a description of this group in terms of partitions.

We keep the notation of 5.1, with $e = 1$. In particular, $p$ is a prime number, and $L = K \times K'$, where $K$ is a cyclic extension of $K$ of degree $p$. Recall that $E_i = K \otimes K_i$, and note that $E_i$ is either a cyclic field extension of $K_i$ or a product of $p$ copies of $K_i$. Let $J$ be the subset of $i \in \mathcal{I}$ such that $E_i/K_i$ is a field extension, and let $r = |J|$.

Recall that $\Sigma_i$ is the set of $v \in \Omega_k$ such that $E_i^v$ is the product of $p$ copies of $K_i^v$. For all $J' \subset J$ with $J' \neq J$, set $\Omega(J') = \cap_{i \notin J'} \Sigma_i$, and let $\Omega(J) = \Omega_k$.

By lemma 5.12, the group $G(K, K')$ is in bijection with the set of partitions $(J_0, \ldots, J_{p-1})$ of $J$ such that $\bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_n) = \Omega_k$. We identify $G(K, K')$ with the set of these partitions. Note that under this identification, $D(K, K')$ corresponds to the partitions where one of the subsets is $J$, and all the others are empty; these will be called the trivial partitions of $J$.

For all $n \in \mathbb{Z}/p\mathbb{Z}$ and all $a \in (\mathbb{Z}/p\mathbb{Z})^r$, set $J_n(a) = \{i \in J \mid a_i = n\}$. Then lemma 5.12 can be reformulated as follows:

**Lemma 8.4.** $G(K, K')$ is in bijection with the set

$$\{a \in (\mathbb{Z}/p\mathbb{Z})^r \mid \bigcup_{n \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_n(a)) = \Omega_k\}.$$  

**Proof of Theorem 8.3**

Note first that (a) follows from Proposition 4.1. From now on, we assume that $n \geq 3$. Theorem 8.3, as well as a precise condition for when $\Pi(L) = 0$ in case (b), is a consequence of Proposition 8.5 below.
For any positive integer \( d \), a finite separable extension \( F \) of \( k \) is said to have local degrees \( \leq d \) if for all places \( v \in \Omega_k \), the étale algebra \( F \otimes_k k_v \) is a product of field extensions of \( k_v \) with degrees \( \leq d \).

**Proposition 8.5.** Let \( p \) be a prime number, and assume that \( L \) is a product of distinct field extensions of degree \( p \) of \( k \), at least one of which is cyclic.

Then \( \mathfrak{X}(L) \equiv 0 \iff \) the factors of \( L \) are distinct subfields of a field extension \( F/k \) of degree \( p^2 \), and all the local degrees of \( F \) are \( \leq p \).

Moreover, if \( \mathfrak{X}(L) \equiv 0 \), and if \( L \) is a product of \( n \) distinct degree \( p \) field extensions of \( k \), then \( \mathfrak{X}(L) \simeq (\mathbb{Z}/p\mathbb{Z})^{n-2} \).

**Proof.** Let \( K \) be a cyclic factor of \( L \), and let us write \( L = K \times K' \), where \( K' \) is a product of field extensions of degree \( p \) of \( k \). Suppose that \( \mathfrak{X}(L) \equiv 0 \). Then there exists a partition \( (I_0, I_1) \) of \( J \) such that \( \Omega(I_0) \cup \Omega(I_1) = \Omega_k \). Indeed, let \( (J_0, \ldots, J_{p-1}) \) be a non-trivial partition of \( J \) such that \( \bigcup_{r \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_i) = \Omega_k \).

Without loss of generality, we can assume that \( J_0 \) is not empty. Set \( I_0 = J_0 \), and let \( I_1 = \bigcup_{i \neq 0} J_i \); then we have \( \Omega(I_0) = \Omega(J_0) \), and \( \Omega(J_i) \subset \Omega(I_1) \) for all \( r \neq 0 \). Therefore \( \Omega(I_0) \cup \Omega(I_1) = \Omega_k \), as claimed. Let \( K_i \) and \( K_j \) be two distinct factors of \( K' \), and let \( K_i K_j \) be the composite of \( K_i \) and \( K_j \). For all \( v \in \Sigma_i \), we have

\[
K \otimes_k (K_i K_j)^v \simeq K \otimes_k K_i^v \otimes K_j \otimes K_i K_j \otimes K_i K_j^v,
\]

and, since \( v \in \Sigma_i \), this is isomorphic to the product of \( p \) copies of \( (K_i K_j)^v \).

Let \( i \in I_0 \) and \( j \in I_1 \). As we have \( \Omega(I_0) \cup \Omega(I_1) = \Omega_k \), the tensor product \( K \otimes_k (K_i K_j)^v \) is isomorphic to the product of \( p \) copies of \( (K_i K_j)^v \) for all \( v \in \Omega_k \). This implies that \( K \) is a subfield of \( K_i K_j \). Recall that \( K \) is cyclic, and that \( K_i, K_j \) are not isomorphic; hence we have \( K \otimes_k K_i \simeq K K_i \subset K_i K_j \). The degree of \( K_i K_j \) is at most \( p^2 \), hence we have \( K K_i = K_i K_j = K K_j \), and \( K_i \otimes_k K_j \simeq K_i K_j \) is of degree \( p^2 \) over \( k \).

Let \( i \in I_0 \), and set \( F = K K_i \); we just saw that \( F \) is independent of the choice of \( i \), and that \( F = K_i K_j \) for all \( j \in I_1 \). This shows that \( K_i \) is a subfield of \( F \) for all \( i \in J \). Since \( (I_0, I_1) \) represents a non-trivial element of \( \mathfrak{X}(L) \), for all \( v \in \Omega_k \) there exists \( i \in J \) such that \( F^v \simeq K \otimes_k K_i^v \) is isomorphic to a product of \( p \) copies of \( K_i^v \). Therefore all the local degrees of \( F \) are \( \leq p \).

Conversely, let \( F \) be a separable extension of degree \( p^2 \) of \( k \) such that all the factors of \( L \) are distinct subfields of \( F \). It suffices to prove that all non-trivial partitions \( (J_0, \ldots, J_{p-1}) \) of \( J \) satisfy \( \bigcup_{r \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_i) = \Omega_k \). Suppose that this is not the case. Let \( (J_0, \ldots, J_{p-1}) \) be a non-trivial partition of \( J \) with \( \bigcup_{r \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_i) \neq \Omega_k \). Let \( v \in \Omega_k \) with \( v \notin \bigcup_{r \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_i) \). Since \( v \notin \Omega(J_0) \), there exists \( i \notin J_0 \) such that \( iv \notin \Sigma_i \). Let \( r \in \mathbb{Z}/p\mathbb{Z} \) such that \( i \in J_r \); since \( v \notin \Omega(J_r) \), there exists \( j \notin J_r \) such that \( v \notin \Sigma_j \).

Since the degree \( p \) extensions \( K, K_i \) and \( K_j \) are distinct subfields of \( F \), we have \( F \simeq K \otimes K_i \simeq K \otimes K_j \). Note that \( [K^v : k_v] = p \), because \( v \notin \Sigma_i \). Let us
write $K_i^v$ as a product of separable extensions of $k_v$. If one of the factors $M_s$ of $K_i^v$ is such that $1 < [M_s : k_v] < p$, then $M_s$ and $K^v$ are linearly disjoint, and this contradicts the assumption that all the local degrees of $F$ are $\leq p$. Hence $K_i^v$ is either a degree $p$ field extension of $k_v$, or a product of $p$ copies of $k_v$. However, if $K_i^v$ and $K^v$ are both fields, then $E_i^v$ is a field extension of degree $p^2$ of $k_v$. Since $F^v \cong E_i^v$, this contradicts the hypothesis that all the local degrees of $F$ are $\leq p$. Therefore $K_i^v$ is a product of $p$ copies of $k_v$, and hence $F^v \cong E_i^v$ is a product of $p$ copies of $K^v$.

Set $d = [K_iK_j : k]$. Since $v \notin \Sigma_j$, the same argument shows that $K_j^v$ is a product of $p$ copies of $k_v$, hence $(K_iK_j)^v$ is a product of $d$ copies of $k_v$. Note that $(K_iK_j)^v$ is a subalgebra of $F^v$, and that $F^v$ is a product of $p$ copies of $K^v$; hence we have $d \leq p$. As $K_i$ and $K_j$ are distinct subfields of $K_iK_j$, we have $d = rp$ for some integer $r > 1$, and this leads to a contradiction.

Hence for all non-trivial partitions $(J_0, \ldots, J_{p-1})$ of $J$ we have $\bigcup_{r \in \mathbb{Z}/p\mathbb{Z}} \Omega(J_i) = \Omega_k$. This shows that $\prod(K, K') = \prod(L) \cong (\mathbb{Z}/p\mathbb{Z})^{n-2}$.

Proof of Theorem 8.1

Assume now that $L$ is a product of $n$ cyclic extensions, $L = K_1 \times \cdots \times K_n$, where $K_i/k$ is a cyclic extension of degree $d_i$, and let $J = \{1, \ldots, n\}$. Note that $\prod(L) = \prod(K_i, K'_i)$ for any $i \in J$, where $L = K_i \times K'_i$. This will be used repeatedly in the sequel.

Let $\mathcal{P}$ be the set of prime numbers dividing $d_1 \ldots d_n$. For all $p \in \mathcal{P}$ and all $i \in J$, let $K_i(p)$ be the largest subfield of $K_i$ such that $[K_i(p) : k]$ is a power of $p$, and set $L(p) = K_1(p) \times \cdots \times K_n(p)$.

Proposition 8.6. We have

$$\prod(L) = \bigoplus_{p \in \mathcal{P}(L)} \prod(L(p)).$$

Proof. This follows from Proposition 5.16 and from the fact that $L$ is a product of cyclic extensions.

Lemma 8.7. Let $p$ be a prime number, and let $K_i/k$, $i \in J$, be cyclic extensions of degree a power of $p$ of $k$. For all $i \in J$, let $N_i/k$ be a subextension of $K_i/k$. Then $\prod_{i \in J} N_i$ injects into $\prod_{i \in J} K_i$.

Proof. This follows from Proposition 5.7 and Remark 5.8 following this proposition.

Proof of Theorem 8.1 Assume that $\prod(L) = 0$. By Lemma 8.7 and Proposition 8.6 the group $\prod(L_{\text{prim}})$ injects into $\prod(L)$, hence this implies that $\prod(L_{\text{prim}}) = 0$. Conversely, suppose that $\prod(L_{\text{prim}}) = 0$. By Proposition 8.6 we may assume that $L$ is a product of extensions of degree a power of a prime $p$. Let us write $L = K \times K'$, for some cyclic field extension $K'/k$; then $L_{\text{prim}} = K_{\text{prim}} \times K'_{\text{prim}}$. Since $\prod(L_{\text{prim}}) = 0$, by Proposition 5.14 we
have \( \text{III}(K, K'_\text{prim}) = 0 \). Permuting \( K \) with one of the other cyclic factors and repeating the same procedure, we obtain \( \text{III}(L) = 0 \).

**Example 8.8.** Let \( p \) be a prime number, and let \( F/k \) be an extension with Galois group \( C_p \times C_p \), where \( C_p \) denotes the cyclic group of order \( p \). Let \( K_1, \ldots, K_{p+1} \) be the distinct subfields of degree \( p \) of \( F \). Set \( L = K_1 \times \cdots \times K_{p+1} \). Then by Proposition 8.5, we have \( \text{III}(L) = 0 \) or \( \text{III}(L) = (\mathbb{Z}/p\mathbb{Z})^{p-1} \). Moreover, we have

\[
\text{III}(L) = 0 \iff \text{there exists } v \in \Omega_k \text{ such that } F^v \text{ is a field.}
\]

- Assume first that there exists \( v \in \Omega_k \) such that \( F^v \) is a field. Then \( \text{III}(L) = 0 \), hence for all \( c \in k^\times \), we have \( X_c(k) \neq \emptyset \). In other words, we have

\[
N_{L/k}(L^\times) = k^\times
\]
in this case.

- Assume now that all the local degrees of \( F \) are \( \leq p \). Then by Proposition 8.5 we have \( \text{III}(L) = (\mathbb{Z}/p\mathbb{Z})^{p-1} \).

Let \( \Omega_i \) be the set of \( v \in \Omega_k \) such that \( K_i^v \) is split. Note that we have \( \Omega_1 \cup \cdots \cup \Omega_{p+1} = \Omega_k \). This implies that \( X_c(k^v) \neq \emptyset \) for all \( v \in \Omega_k \) and for all \( c \in k^\times \).

Let \( K = K_{p+1} \). For all \( c \in k^\times \) and for all \( v \in \Omega_k \), let us denote by \( [K, c]_v \in \mathbb{Z}/p\mathbb{Z} \) the image of \( \text{inv}(K, c)_v \) by the isomorphism \( \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}/p\mathbb{Z} \).

Then the map

\[
f : k^\times/N_{L/k}(L^\times) \rightarrow (\mathbb{Z}/p\mathbb{Z})^{p-1}
\]
given by

\[
c \mapsto (\sum_{\Omega_1} [K, c]_v, \ldots, \sum_{\Omega_{p+1}} [K, c]_v),
\]
is an isomorphism.

When \( p = 2 \), we recover a well-known result of Serre and Tate, see [CF 67], Exercise 5.2, page 360; see also [CT 14], Proposition 5.1.

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