Coloring claw-free graphs with $\Delta - 1$ colors

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Abstract

We prove that every claw-free graph $G$ that does not contain a clique on $\Delta(G) \geq 9$ vertices can be $\Delta(G) - 1$ colored.

1 Introduction

The first non-trivial result about coloring graphs with around $\Delta$ colors is Brooks’ theorem from 1941.

Theorem 1.1 (Brooks [4]). Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings.

Conjecture 1.2 (Borodin and Kostochka [3]). Every graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.

Graphs exist (see Figure 1) showing that the $\Delta \geq 9$ condition is necessary. Using probabilistic methods, Reed [19] proved the conjecture for $\Delta \geq 10^{14}$.

In [8], Dhurandhar proved the Borodin-Kostochka Conjecture for a superset of line graphs of simple graphs defined by excluding the claw, $K_5 - e$, and another graph $D$ as induced subgraphs. Kierstead and Schmerl [11] improved this by removing the need to exclude $D$. The aim of this paper is to remove the need to exclude $K_5 - e$; that is, to prove the Borodin-Kostochka Conjecture for claw-free graphs.

Theorem 5.5. Every claw-free graph satisfying $\chi \geq \Delta \geq 9$ contains a $K_\Delta$.

This also generalizes the result of Beutelspacher and Hering [1] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. The value of 9 in Theorem 5.5 is best possible since the graph with $\Delta = 8$ in Figure 1 is claw-free (both this example and the following tightness example appear in Section 11.2 of [16]).
is also optimal in the following sense. We can reformulate the statement as: every claw-free graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$. Consider a similar statement with $\Delta - 1$ replaced by $f(\Delta)$ for some $f: \mathbb{N} \to \mathbb{N}$ and $9$ replaced by $\Delta_0$. We show that $f(x) \geq x - 1$ for $x \geq \Delta_0$. Consider $G_t := K_t \ast C_5$ (here $A \ast B$ denotes the join of $A$ and $B$ and is formed from $A$ and $B$ by adding all edges with one endpoint in $A$ and the other in $B$). We have $\chi(G_t) = t + 3$, $\omega(G_t) = t + 2$, and $\Delta(G_t) = t + 4$ and $G_t$ is claw-free. Hence for $t \geq \Delta_0 - 4$ we have $t + 3 \leq \max\{t + 2, f(t + 4)\} \leq f(t + 4)$ giving $f(x) \geq x - 1$ for $x \geq \Delta_0$.

As shown in [18], the situation is very different for line graphs of multigraphs, which satisfy $\chi \leq \max\{\omega, \frac{7\Delta + 10}{8}\}$. There it was conjectured that $f(x) := \frac{5\Delta + 8}{6}$ works for line graphs of multigraphs; this would be best possible. The example $K_t \ast C_5$ is claw-free, but it is not quasi-line.

Question. What is the situation for quasi-line graphs? That is, what is the optimal $f$ such that every quasi-line graph with large enough maximum degree satisfies $\chi \leq \max\{\omega, f(\Delta)\}$.

Borodin and Kostochka conjectured (to themselves) [14] that their conjecture also holds for list coloring.

**Conjecture 1.3** (Borodin and Kostochka [14]). Every graph with $\Delta \geq 9$ satisfies $\chi_l \leq \max\{\omega, \Delta - 1\}$.

We make some progress on this conjecture for claw-free graphs, proving it for circular interval graphs and severely restricting line graph counterexamples. These two classes are the base cases of the structure theorem for quasi-line graphs of Chudnovsky and Seymour [6] that we use. Finally, we prove the following.
Theorem 5.6. If every quasi-line graph satisfying $\chi_l \geq \Delta \geq 9$ contains a $K_\Delta$, then the same statement holds for every claw-free graph.

In [10], Gravier and Maffray conjecture the following strengthening of the list coloring conjecture. Conjecture 1.3 for claw-free graphs would be an immediate consequence.

Conjecture 1.4 (Gravier and Maffray [10]). Every claw-free graph satisfies $\chi_l = \chi$.

The outline of this paper is as follows. A quasi-line graph is one in which the neighborhood of every vertex can be covered by two cliques. Quasi-line graphs are a proper subset of claw-free graphs and a proper superset of line graphs. We use a structure theorem of Chudnovsky and Seymour, which says (roughly) that every quasi-line graph is either a (i) a line graph, (ii) a circular interval graph, or (iii) the result of “pasting together” smaller quasi-line graphs. So to prove the Borodin-Kostochka conjecture for quasi-line graphs, we prove it for circular interval graphs, we recall Rabern’s proof for line graphs, and we show how to “paste together” good colorings of smaller graphs to get a good coloring of a larger graph.

If a graph $G$ is claw-free, but not quasi-line, then we show that $G$ contains a vertex $v$ with an induced $C_5$ in its neighborhood. We use the presence of this induced $K_1 \ast C_5$ to show that $G$ must contain a $d_1$-choosable subgraph (defined in Section 2). Since such a subgraph cannot appear in a vertex critical graph, this completes the proof of the Borodin-Kostochka conjecture for claw-free graphs. (In fact, this reduction from claw-free to quasi-line graphs works equally well for the list version of the Borodin-Kostochka Conjecture.)

It is likely that some of our list coloring arguments could be shortened by using Ohba’s Conjecture, which was recently proved by Noel, Reed, and Wu [17]. However, we prefer to keep this paper as self-contained as possible. On a related note, by using a lemma of Kostochka [15], we can reduce the Borodin-Kostochka Conjecture for any hereditary graph class to the case when $\Delta = 9$ (see the introduction of [7] for more details). However, this reduction does not seem to simplify any of our proofs and does not work for list coloring, so we omit it.

Now we introduce some notation and terminology that will be used through the paper. We write $K_t$ for the complete graph on $t$ vertices and $E_t$ for the edgeless graph on $t$ vertices. If $G$ is a vertex critical graph with $\chi = \Delta$, then every vertex in $G$ has degree $\Delta - 1$ or $\Delta$. We call the former vertices low and the latter vertices high. For vertices $x, y$ in $G$, we write $x \leftrightarrow y$ if $xy \in E(G)$ and $x \not\leftrightarrow y$ if $xy \not\in E(G)$. An almost complete graph is a graph $G$ for which there exists $v \in V(G)$ such that $G - v$ is a complete graph. We write diamond for $K_4 - e$ and we write paw for $K_3$ with a pendant edge, that is $P_3 + K_1$. We write chair for the graph formed by subdividing a single edge of $K_{1,3}$. All the definitions for list coloring that we use are at the start of Section 2.

2 List coloring lemmas

2.1 The main idea

We investigate the structure of vertex critical graphs with $\chi = \Delta$. Let $G$ be such a graph. All of our list coloring lemmas serve the same purpose: exclude graphs from being induced subgraphs of $G$. To see how this works, let $F$ be an induced subgraph of $G$. Since $G$ is
vertex critical, we may \((\Delta - 1)\)-color \(G - F\). After doing so, we give each \(v \in V(F)\) a list of colors \(L(v)\) by taking \(\{1, \ldots, \Delta - 1\}\) and removing all colors appearing on neighbors of \(v\) in \(G - F\). Then, as \(v\) has at most \(d_G(v) - d_F(v)\) neighbors in \(G - F\) we have \(|L(v)| \geq \Delta - 1 - (d_G(v) - d_F(v)) \geq d_F(v) - 1\). If we could properly color \(F\) from these lists, we would have a \((\Delta - 1)\)-coloring of \(G\), which is impossible. We call a graph \(F\) \(d_1\)-choosable if it can be colored from any list assignment \(L\) with \(|L(v)| \geq d_F(v) - 1\) for each \(v \in V(F)\). Then, as we just saw, no \(d_1\)-choosable graph can be an induced subgraph of \(G\). So, by finding many small \(d_1\)-choosable graphs, we can severely restrict the structure of \(G\). The next section gives the formal definitions and list coloring lemmas that we need for this application. The reader should feel free to skip this section for now and return as needed.

2.2 The details

Let \(G\) be a graph. A list assignment to the vertices of \(G\) is a function from \(V(G)\) to the finite subsets of \(\mathbb{N}\). A list assignment \(L\) to \(G\) is good if \(G\) has a proper coloring \(c\) where \(c(v) \in L(v)\) for each \(v \in V(G)\). It is bad otherwise. We call the collection of all colors that appear in \(L\), the pot of \(L\). That is \(\text{Pot}(L) := \bigcup_{v \in V(G)} L(v)\). For a subset \(A\) of \(V(G)\) we write \(\text{Pot}_A(L) := \bigcup_{v \in A} L(v)\). Also, for a subgraph \(H\) of \(G\), put \(\text{Pot}_H(L) := \text{Pot}_{V(H)}(L)\). For \(S \subseteq \text{Pot}(L)\), let \(G_S\) be the graph \([\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}]\). We also write \(G_c\) for \(G_{\{c\}}\).

For \(f: V(G) \to \mathbb{N}\), an \(f\)-assignment on \(G\) is an assignment \(L\) of lists to the vertices of \(G\) such that \(|L(v)| = f(v)\) for each \(v \in V(G)\). We say that \(G\) is \(f\)-choosable if every \(f\)-assignment on \(G\) is good. We call a graph that is \(f\)-choosable where \(f(v) := d(v) - 1\) a \(d_1\)-choosable graph.

We restate some of the results on \(d_1\)-choosable graphs from \([7]\) that we need here; we omit all of their proofs. We do prove Lemma 2.2 which is a strengthening of a special case of Lemma 2.1 (and which is not proved in \([7]\)).

We need the following list coloring lemmas from \([7]\). Given a graph \(G\) and \(f: V(G) \to \mathbb{N}\), we have a partial order on the \(f\)-assignments to \(G\) given by \(L < L'\) iff \(|\text{Pot}(L)| < |\text{Pot}(L')|\).

When we talk of minimal \(f\)-assignments, we mean minimal with respect to this partial order.

**Small Pot Lemma.** Let \(G\) be a graph and \(f: V(G) \to \mathbb{N}\) with \(f(v) < |G|\) for all \(v \in V(G)\). If \(G\) is not \(f\)-choosable, then \(G\) has a bad \(f\)-assignment \(L\) such that \(|\text{Pot}(L)| < |G|\).

The core of the Small Pot Lemma is the following. We will also prove a lemma that gets more when \(|S| = 1\).

**Lemma 2.1.** Let \(G\) be a graph and \(f: V(G) \to \mathbb{N}\). Suppose \(G\) is not \(f\)-choosable and let \(L\) be a minimal bad \(f\)-assignment. Assume \(L(v) \neq \text{Pot}(L)\) for each \(v \in V(G)\). Then, for each nonempty \(S \subseteq \text{Pot}(L)\), any coloring of \(G_S\) from \(L\) uses some color not in \(S\).

When \(|S| = 1\), we can say more. We use the following lemma in the proof that the graph \(D_8\) in Figure 3 is \(d_1\)-choosable. It should be useful elsewhere as well.

**Lemma 2.2.** Let \(G\) be a graph and \(f: V(G) \to \mathbb{N}\). Suppose \(G\) is not \(f\)-choosable and let \(L\) be a minimal bad \(f\)-assignment. Then for any \(c \in \text{Pot}(L)\), there is a component \(H\) of \(G_c\) such that \(\text{Pot}_H(L) = \text{Pot}(L)\). In particular, \(\text{Pot}_{G_c}(L) = \text{Pot}(L)\).
Proof. Suppose otherwise that we have $c \in \text{Pot}(L)$ such that $\text{Pot}_H(L) \not\subset \text{Pot}(L)$ for all components $H$ of $G_c$. Say the components of $G_c$ are $H_1, \ldots, H_t$. For $i \in [t]$, choose $\alpha_i \in \text{Pot}(L) - \text{Pot}_{H_i}(L)$. Now define a list assignment $L'$ on $G$ by setting $L'(v) := L(v)$ for all $v \in V(G) - V(G_c)$ and for each $i \in [t]$ setting $L'(v) := (L(v) - c) \cup \{\alpha_i\}$ for each $v \in V(H_i)$. Then $|\text{Pot}(L')| < |\text{Pot}(L)|$ and hence by minimality $L$ we have an $L'$-coloring $\pi$ of $G$. Plainly $Q := \{v \in V(G_c) \mid \pi(v) = \alpha_i \text{ for some } i \in [t]\}$ is an independent set. Since $c$ does not appear outside $G_c$, we can recolor all vertices in $Q$ with $c$ to get an $L$-coloring of $G$. This contradicts the fact that $L$ is bad. \hfill \blacksquare

The next two lemmas allow us to color pairs in $H$ without worrying about completing the coloring to $H$.

**Lemma 2.3.** Let $H$ be a $d_0$-choosable graph such that $G := K_1 * H$ is not $d_1$-choosable and $L$ a minimal bad $d_1$-assignment on $G$. If some nonadjacent pair in $H$ have intersecting lists, then $|\text{Pot}(L)| \leq |H| - 1$.

With the same proof, we have the following.

**Lemma 2.4.** Let $H$ be a $d_0$-choosable graph such that $G := K_1 * H$ is not $f$-choosable where $f(v) \geq d(v)$ for the $v$ in the $K_1$ and $f(v) \geq d(x) - 1$ for $x \in V(H)$. If $L$ is a minimal bad $f$-assignment on $G$, then all nonadjacent pairs in $H$ have disjoint lists.

**Lemma 2.5.** Let $A$ be a connected graph with $|A| \geq 4$ and $B$ an arbitrary graph. If $A * B$ is not $d_1$-choosable, then $B$ is $E_3 * K_{|B|-3}$ or almost complete.

**Lemma 2.6.** $K_3 * B$ is not $d_1$-choosable iff $B$ is one of the following: almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, $E_3 + K_{|B|-3}$, or $|B| \leq 5$ and $B = E_3 * K_{|B|-3}$.

**Lemma 2.7.** If $K_2 * B$ is not $d_1$-choosable, then $B$ consists of a disjoint union of complete subgraphs, together with at most one incomplete component $H$. If $H$ has a dominating vertex $v$, then $K_2 * H = K_3 * (H - v)$, so by Lemma 2.6 we can completely describe $H$. Otherwise $H$ is formed either by adding an edge between two disjoint cliques or by adding a single pendant edge incident to each of two distinct vertices of a clique. Furthermore, all graphs formed in this way are not $d_1$-choosable.

Pulling out some particular cases makes for easier application. A *chair* is formed from $K_{1,3}$ by subdividing an edge. An *antichair* is the complement of a chair.

**Lemma 2.8.** $K_2 * \text{antichair}$ is $d_1$-choosable.

**Lemma 2.9.** $K_3 * P_4$ is $d_1$-choosable.

The situation is simpler for joins with $E_2$, as shown by the next lemma.

**Lemma 2.10.** $E_2 * B$ is not $d_1$-choosable iff $B$ is the disjoint union of complete subgraphs and at most one copy of $P_3$.

We often need to handle low vertices in our proofs, which corresponds to a vertex with $|L(v)| \geq d(v)$ when we try to complete the partial coloring.
Lemma 2.11. Let \( A \) be a graph with \(|A| \geq 4\). Let \( L \) be a list assignment on \( G := E_2 * A \) such that \(|L(v)| \geq d(v) - 1\) for all \( v \in V(G) \) and each component \( D \) of \( A \) has a vertex \( v \) such that \(|L(v)| \geq d(v)\). Then \( L \) is good on \( G \).

Lemma 2.12. Let \( A \) be a graph with \(|A| \geq 3\). Let \( L \) be a list assignment on \( G := E_2 * A \) such that \(|L(v)| \geq d(v) - 1\) for all \( v \in V(G) \), \(|L(v)| \geq d(v)\) for some \( v \) in the \( E_2 \) and each component \( D \) of \( A \) has a vertex \( v \) such that \(|L(v)| \geq d(v)\). Then \( L \) is good on \( G \).

Lemma 2.13. Let \( H \) be a \( d_0 \)-choosable graph such that \( G := K_1 * H \) is not \( d_1 \)-choosable; let \( L \) be a bad \( d_1 \)-assignment on \( G \). Then

1. for any independent set \( I \subseteq V(H) \) with \(|I| = 3\), we have \( \bigcap_{v \in I} L(v) = \emptyset \);
2. for disjoint nonadjacent pairs \( \{x_1, y_1\} \) and \( \{x_2, y_2\} \) at least one of the following holds
   
   (a) \( L(x_1) \cap L(y_1) = \emptyset \);
   
   (b) \( L(x_2) \cap L(y_2) = \emptyset \);
   
   (c) \( |L(x_1) \cap L(y_1)| = 1 \) and \( L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2) \).

Let \( E_2^n \) denote the join of \( n \) copies of \( E_2 \), i.e., \( E_2^n \) is isomorphic to \( K_{2n} - E(M) \), where \( M \) is a perfect matching. The following lemma first appeared in [9]. We also prove it in [7].

Lemma 2.14. \( E_2^n \) is \( n \)-choosable.

3 Circular interval graphs

Given a set \( V \) of points on the unit circle together with a set of closed intervals \( C \) on the unit circle we define a graph with vertex set \( V \) and an edge between two different vertices if and only if they are both contained in some element of \( C \). Any graph isomorphic to such a graph is a circular interval graph. Similarly, by replacing the unit circle with the unit interval, we get the class of linear interval graphs.

Lemma 3.1. Every circular interval graph satisfying \( \chi_i \geq \Delta \geq 9 \) contains a \( K_\Delta \).

Proof. Suppose the contrary and choose a counterexample \( G \) minimizing \(|G|\). Put \( \Delta := \Delta(G) \). Then \( \chi_i(G) = \Delta \), \( \omega(G) \leq \Delta - 1 \), \( \delta(G) \geq \Delta - 1 \) and \( \chi_i(G - v) \leq \Delta - 1 \) for all \( v \in V(G) \). Since \( G \) is a circular interval graph, by definition \( G \) has a representation in a cycle \( v_1v_2\ldots v_n \). Let \( K \) be a maximum clique in \( G \). By symmetry we may assume that \( V(K) = \{v_1, v_2, \ldots, v_t\} \) for some \( t \leq \Delta - 1 \); further, if possible we label the vertices so that \( v_{t-3} \leftrightarrow v_{t+1} \) and the edge goes through \( v_{t-2}, v_{t-1}, v_t \).

Claim 1. \( v_1 \not\leftrightarrow v_{t+1} \) and \( v_2 \not\leftrightarrow v_{t+2} \) and \( v_1 \not\leftrightarrow v_{t+2} \). Assume the contrary. Clearly we cannot have \( v_1 \leftrightarrow v_{t+1} \) and have the edge go through \( v_2, v_3, \ldots, v_t \) (since then we get a clique of size \( t + 1 \)). Similarly, we cannot have \( v_2 \leftrightarrow v_{t+2} \) and have the edge go through \( v_3, v_4, \ldots, v_{t+1} \). So assume the edge \( v_1v_{t+2} \) exists and goes around the other way. If \( v_1 \leftrightarrow v_{t+1} \), then let \( G' = G \setminus \{v_1\} \) and if \( v_1 \not\leftrightarrow v_{t+1} \), then let \( G' = G \setminus \{v_1, v_{t+1}\} \). Now let \( V_1 = \{v_2, v_3, \ldots, v_t\} \) and \( V_2 = V(G') \setminus V_1 \). Let \( K' = G[V_1] \) and \( L' = G[V_2] \); note that \( K' \) and \( L' \) are each cliques of size at most \( \Delta - 2 \). Now for each \( S \subseteq V_2 \), we have \( |N_{G}(S) \cap V_1| \geq |S| \)
(otherwise we get a clique of size \( t \) in \( G' \) and a clique of size \( t + 1 \) in \( G \)). Now by Hall’s Theorem, we have a matching in \( \overline{G} \) between \( V_1 \) and \( V_2 \) that saturates \( V_2 \). This implies that \( G' \subseteq E_{2}^{\Delta - 2} \), which in turn gives \( G \subseteq E_{2}^{\Delta - 1} \). By Lemma \( 2.14 \), \( G \) is \((\Delta - 1)\)-choosable, which is a contradiction.

**Claim 2.** \( v_{t-3} \leftrightarrow v_{t+1} \) and the edge passes through \( v_{t-2}, v_{t-1}, v_{t} \). Assume the contrary. If \( t \geq 7 \), then since \( t \leq \Delta - 1 \), \( v_{4} \) has some neighbor outside of \( K \); by (reflectional) symmetry we could have labeled the vertices so that \( v_{t-3} \leftrightarrow v_{t+1} \). So we must have \( t \leq 6 \). Each vertex \( v \) that is high has either at least \( \lceil \Delta/2 \rceil \) clockwise neighbors or at least \( \lfloor \Delta/2 \rfloor \) counterclockwise neighbors. This gives a clique of size \( 1 + \lfloor \Delta/2 \rfloor \geq 6 \). If \( v_{3} \) is high, then either \( v_{3} \) has at least 4 clockwise neighbors, so \( v_{3} \leftrightarrow v_{7} \), or else \( v_{3} \) has at least 6 counterclockwise neighbors, so \( |K| \geq 7 \). Thus, we may assume that \( v_{3} \) is low; by symmetry (and our choice of labeling prior to Claim 1) \( v_{4} \) is also low. Now since \( v_{4} \) has only 3 counterclockwise neighbors, we get \( v_{4} \leftrightarrow v_{7} \) (in fact, we get \( v_{4} \leftrightarrow v_{9} \)). Thus, \( \{v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\} \) induces \( K_{3} \ast E_{2} \) with a low degree vertex in both the \( K_{3} \) and the \( E_{2} \), which contradicts Lemma \( 2.12 \).

**Claim 3.** \( v_{t-2} \not\leftrightarrow v_{t+2} \). Assume the contrary. By Claim 1 the edge goes through \( v_{t-1}, v_{t}, v_{t+1} \). If \( v_{t-3} \leftrightarrow v_{t+2} \), then \( \{v_{1}, v_{2}, v_{t-3}, v_{t-2}, v_{t-1}, v_{t}, v_{t+1}, v_{t+2}\} \) induces \( K_{4} \ast B \), where \( B \) is not almost complete; this contradicts Lemma \( 2.5 \). If \( v_{t-3} \not\leftrightarrow v_{t+2} \), then we get a \( K_{3} \ast P_{4} \) induced by \( \{v_{1}, v_{t-3}, v_{t-2}, v_{t-1}, v_{t}, v_{t+1}, v_{t+2}\} \), which contradicts Lemma \( 2.9 \).

**Claim 4.** \( v_{t-1} \not\leftrightarrow v_{t+2} \). Suppose the contrary. Now \( \{v_{1}, v_{t-3}, v_{t-2}, v_{t-1}, v_{t+1}, v_{t+2}\} \) induces \( K_{2} \ast \text{antichair} \) (with \( v_{t-1}, v_{t} \) in the \( K_{2} \)), which contradicts Lemma \( 2.8 \).

**Claim 5.** \( G \) is \((\Delta - 1)\)-choosable. Let \( S = \{v_{t-3}, v_{t-2}, v_{t-1}, v_{t}\} \). If any vertex of \( S \) is low, then \( S \cup \{v_{1}, v_{t+1}\} \) induces \( K_{4} \ast E_{2} \) with a low vertex in the \( K_{4} \), which contradicts Lemma \( 2.11 \). So all of \( S \) is high. If \( v_{t} \not\leftrightarrow v_{t+2} \), then \( \{v_{t}, v_{t-1}, \ldots, v_{t-\Delta+1}\} \) (subscripts are modulo \( n \)) induces \( K_{\Delta} \). So \( v_{t} \leftrightarrow v_{t+2} \). Since \( v_{t-1} \not\leftrightarrow v_{t+2} \) and all of \( S \) is high, we get \( v_{n} \in (\cap_{v \in \{S \setminus \{v_{t}\}\}} N(v)) \setminus N(v_{t}) \). Now we must have \( v_{n} \not\leftrightarrow v_{t+1} \) (for otherwise \( G \) is \((\Delta - 1)\)-choosable, as in Claim 1). So we get \( K_{3} \ast P_{4} \) induced by \( \{v_{t+1}, v_{t}, v_{t-1}, v_{t-2}, v_{t-3}, v_{t}, v_{n}\} \), which contradicts Lemma \( 2.9 \).

**4 Quasi-line graphs**

A graph is *quasi-line* if every vertex is bisimplicial (its neighborhood can be covered by two cliques). We apply a version of Chudnovsky and Seymour’s structure theorem for quasi-line graphs from King’s thesis \([12]\). The undefined terms will be defined after the statement.

**Lemma 4.1.** Every connected skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.

A *homogeneous pair of cliques* \((A_{1}, A_{2})\) in a graph \( G \) is a pair of disjoint nonempty cliques such that for each \( i \in [2] \), every vertex in \( G - (A_{1} \cup A_{2}) \) is either joined to \( A_{i} \) or misses all of \( A_{i} \) and \(|A_{1}| + |A_{2}| \geq 3 \). A homogeneous pair of cliques \((A_{1}, A_{2})\) is *skeletal* if for any \( e \in E(A, B) \) we have \( \omega(G[A \cup B] - e) < \omega(G[A \cup B]) \). A graph is *skeletal* if it contains no nonskeletal homogeneous pair of cliques.

Generalizing a lemma of Chudnovsky and Fradkin \([5]\), King proved a lemma allowing us to handle nonskeletal homogeneous pairs of cliques.
Lemma 4.2 (King [12]). If $G$ is a nonskeletal graph, then there is a proper subgraph $G'$ of $G$ such that:

1. $G'$ is skeletal;
2. $\chi(G') = \chi(G)$;
3. If $G$ is claw-free, then so is $G'$;
4. If $G$ is quasi-line, then so is $G'$.

It remains to define the generalization of line graphs introduced by Chudnovsky and Seymour [3]; this is the notion of compositions of strips (for a more detailed introduction, see Chapter 5 of [12]). We use the modified definition from King and Reed [13]. A strip $(H, A_1, A_2)$ is a claw-free graph $H$ containing two cliques $A_1$ and $A_2$ such that for each $i \in [2]$ and $v \in A_i$, $N_H(v) - A_i$ is a clique. If $H$ is a linear interval graph with $A_1$ and $A_2$ on opposite ends, then $(H, A_1, A_2)$ is a linear interval strip. Now let $H$ be a directed multigraph (possibly with loops) and suppose for each edge $e$ of $H$ we have a strip $(H_e, X_e, Y_e)$. For each $v \in V(H)$ define

$$C_v := \left( \bigcup \{ X_e \mid e \text{ is directed out of } v \} \right) \cup \left( \bigcup \{ Y_e \mid e \text{ is directed into } v \} \right)$$

The graph formed by taking the disjoint union of $\{ H_e \mid e \in E(H) \}$ and making $C_v$ a clique for each $v \in V(H)$ is the composition of the strips $(H_e, X_e, Y_e)$. Any graph formed in such a manner is called a composition of strips. It is easy to see that if for each strip $(H_e, X_e, Y_e)$ in the composition we have $V(H_e) = X_e = Y_e$, then the constructed graph is just the line graph of the multigraph formed by replacing each $e \in E(H)$ with $|H_e|$ copies of $e$.

It will be convenient to have notation and terminology for a strip together with how it attaches to the graph. An interval 2-join in a graph $G$ is an induced subgraph $H$ such that:

1. $H$ is a (nonempty) linear interval graph,
2. The ends of $H$ are (not necessarily disjoint) cliques $A_1, A_2$,
3. $G - H$ contains cliques $B_1, B_2$ (not necessarily disjoint) such that $A_1$ is joined to $B_1$ and $A_2$ is joined to $B_2$,
4. there are no other edges between $H$ and $G - H$.

Note that $A_1, A_2, B_1, B_2$ are uniquely determined by $H$, so we are justified in calling both $H$ and the quintuple $(H, A_1, A_2, B_1, B_2)$ the interval 2-join. An interval 2-join $(H, A_1, A_2, B_1, B_2)$ is trivial if $V(H) = A_1 = A_2$ and canonical if $A_1 \cap A_2 = \emptyset$. A canonical interval 2-join $(H, A_1, A_2, B_1, B_2)$ with leftmost vertex $v_1$ and rightmost vertex $v_1$ is reducible if $H$ is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$ or $N_H(A_2) \setminus A_2 = N_H(v_1) \setminus A_2$. We call such a canonical interval 2-join reducible because we can reduce it as follows. Suppose $H$ is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. Put $C := N_H(v_1) \setminus A_1$ and then $A'_1 := C \setminus A_2$ and $A'_2 := A_2 \setminus C$. Since $H$ is not complete $v_t \in A'_2$ and hence $H' := G[A'_1 \cup A'_2]$ is a nonempty linear interval graph that gives the reduced canonical interval 2-join $(H', A'_1, A'_2, A_1 \cup (C \cap A_2), B_2 \cup (C \cap A_2))$. 

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Lemma 4.3. If \((H, A_1, A_2, B_1, B_2)\) is an irreducible canonical interval 2-join in a skeletal vertex critical graph \(G\) with \(\chi(G) = \Delta(G) \geq 9\), then \(B_1 \cap B_2 = \emptyset\), \(|A_1|, |A_2| \leq 3\) and \(H\) is complete.

Proof. Let \((H, A_1, A_2, B_1, B_2)\) be an irreducible canonical interval 2-join in a skeletal vertex critical graph \(G\) with \(\chi(G) = \Delta(G) \geq 9\). Put \(\Delta := \Delta(G)\).

Note that, since it is vertex critical, \(G\) contains no \(K_\Delta\) and in particular \(G\) has no simplicial vertices. Label the vertices of \(H\) left-to-right as \(v_1, \ldots, v_t\). Say \(A_1 = \{v_1, \ldots, v_l\}\) and \(A_2 = \{v_R, \ldots, v_t\}\). For \(v \in V(H)\), define \(r(v) := \max \{i \in [t] \mid v \leftrightarrow v_i\}\) and \(l(v) := \min \{i \in [t] \mid v \leftrightarrow v_i\}\). These are well-defined since \(|H| \geq 2\) and \(H\) is connected by the following claim.

Claim 1. \(A_1, A_2, B_1, B_2 \neq \emptyset\), \(B_1 \not\subseteq B_2\), \(B_2 \not\subseteq B_1\) and \(H\) is connected. Otherwise \(G\) would contain a clique cutset.

Claim 2. If \(H\) is complete, then \(R - L = 1\). Suppose \(V(H) \neq A_1 \cup A_2\). Then any \(v \in V(H) \setminus A_1 \cup A_2\) would be simplicial in \(G\), which is impossible. Hence \(R - L = 1\).

Claim 3. If \(H\) is not complete, then \(r(v_L) = r(v_1) + 1\) and \(l(v_R) = l(v_t) - 1\). In particular, \(v_1, v_L\) are low and \(|A_1|, |A_2| \geq 2\). Suppose otherwise that \(H\) is not complete and \(r(v_L) \neq r(v_1) + 1\). By definition, \(N_H(v_1) \subseteq N_H(v_L)\) and \(v_1, v_L\) have the same neighbors in \(G \setminus H\). Hence if \(r(v_L) > r(v_1) + 1\), then \(d(v_L) - d(v_1) \geq 2\), impossible. So we must have \(r(v_L) = r(v_1)\) and hence \(N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1\). Thus the 2-join is reducible, a contradiction. Therefore \(r(v_L) = r(v_1) + 1\). Similarly, \(l(v_R) = l(v_t) - 1\).

Claim 4. \(|A_1|, |A_2| \leq 3\). Suppose otherwise that \(|A_1| \geq 4\). First, suppose \(H\) is complete. By Claim 2, \(V(H) = A_1 \cup A_2\). If \(v_1\) is low, then for any \(w_1 \in B_1 \setminus B_2\) the vertex set \(\{v_1, \ldots, v_4, v_t, w_1\}\) induces a \(K_4 \ast E_2\) violating Lemma 2.11. Hence \(v_1\) is high. If \(|A_2| \geq 2\) and \(|B_1 \setminus B_2| \geq 2\), then for any \(w_1, w_2 \in B_1 \setminus B_2\), the vertex set \(\{v_1, \ldots, v_4, v_t, v_i, w_1, w_2\}\) induces a \(K_4 \ast 2K_2\), which is impossible by Lemma 2.5. Hence either \(|A_2| = 1\) or \(|B_1 \setminus B_2| = 1\). Suppose \(|A_2| = 1\). Then, since \(A_1 \cup B_1\) induces a clique and \(|A_1 \cup B_1| = d(v_1)\), \(v_1\) must be low, impossible. Hence we must have \(|B_1 \setminus B_2| = 1\). Thus \(|B_1 \cap B_2| = |B_1| - 1\). Hence \(V(H) \cup B_1 \cap B_2\) induces a clique with \(|A_1| + |A_2| + |B_1| - 1 = d(v_1) = \Delta\) vertices, impossible.

Therefore \(H\) must be incomplete. By Claim 3, \(v_1\) is low. But then as above for any \(w_1 \in B_1 \setminus B_2\) the vertex set \(\{v_1, \ldots, v_4, v_t, w_1\}\) induces a \(K_4 \ast E_2\) violating Lemma 2.11. Hence we must have \(|A_1| \leq 3\). Similarly, \(|A_2| \leq 3\).

Claim 5. \(R - L = 1\). Suppose otherwise that \(R - L \geq 2\). Then by Claim 2, \(H\) is incomplete. Hence by Claim 3, \(r(v_R) = r(v_1) + 1\), \(l(v_R) = l(v_t) - 1\), \(v_1, v_L\) are low and \(|A_1|, |A_2| \geq 2\). But then \((A_1, \{v_{r(v_L)}\})\) is a non-skeletal homogeneous pair of cliques in \(G\), impossible.

Claim 6. \(B_1 \cap B_2 = \emptyset\). Suppose otherwise that we have \(w \in B_1 \cap B_2\).

Subclaim 6a. Each \(v \in V(H)\) is low, \(|B_1| = |B_2|\), \(|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1\), \(d(v) = |A_1| + |A_2| + |B_1| - 1\) for each \(v \in V(H)\) and \(H\) is complete. By Claim 5, we have \(d(v) \leq |A_1| + |A_2| + |B_1| - 1\) for \(v \in A_1\) and \(d(v) \leq |A_1| + |A_2| + |B_2| - 1\) for \(v \in A_2\). Also, as \(B_1 \not\subseteq B_2\) and \(B_2 \not\subseteq B_1\), we have \(d(w) \geq \max \{|B_1|, |B_2|\} + |A_1| + |A_2|\). So \(d(w) \geq d(v) + 1\) for any \(v \in V(H)\). This implies that each \(v \in V(H)\) is low, \(|B_1| = |B_2|\), \(|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1\), \(d(v) = |A_1| + |A_2| + |B_1| - 1\) for each \(v \in V(H)\) and hence \(H\) is complete.

Subclaim 6b. \(|B_1 \cap B_2| \leq 3\). Suppose otherwise that \(|B_1 \cap B_2| \geq 4\). Pick \(w_1 \in B_1 \setminus B_2, w_2 \in B_2 \setminus B_1\) and \(z_1, z_2, z_3, z_4 \in B_1 \cap B_2\). Then the set \(\{z_1, z_2, z_3, z_4, w_1, w_2, v_1, v_t\}\) induces a
subgraph violating Lemma 2.5. Hence \(|B_1 \cap B_2| \leq 3\).

**Subclaim 6c. Claim 6 is true.** By Subclaim 6a and Subclaim 6b we have \(3 \geq |B_1 \cap B_2| = |B_1| - 1\) and hence \(|B_1| = |B_2| \leq 4\). Suppose \(|A_1|, |A_2| \leq 2\). Then \(\Delta - 1 = d(v_1) \leq 3 + |B_1| \leq 7\), a contradiction. Hence by symmetry we may assume that \(|A_1| \geq 3\). But then for \(v_1 \in B_1 \setminus B_2\), the set \(\{v_1, v_2, v_3, v_i, w_1\}\) induces a \(K_3 \ast E_2\) violating Lemma 2.12.

**Claim 7.** \(H\) is complete. Suppose \(H\) is incomplete. By Claim 5, \(R - L = 1\). Then, by Claim 3 \(r(v_L) = r(v_1) + 1\) and \(l(v_R) = l(v_1) - 1\). Since \(v_1\) is not simplicial, \(r(v_1) \geq L + 1 = R\). Hence \(l(v_R) = 1\) and thus \(l(v_i) = 2\). Similarly, \(r(v_1) = t - 1\). So, \(H\) is \(K_t\) less an edge. But \((A_1, A_2)\) is a homogeneous pair of cliques with \(|A_1|, |A_2| \geq 2\) and hence there is an edge between \(A_1\) and \(A_2\) that we can remove without decreasing \(\omega(G[A_1 \cup A_2])\). This contradicts our assumption that \(G\) is skeletal.

\(\square\)

**Lemma 4.4.** An interval 2-join in a skeletal vertex critical graph satisfying \(\chi = \Delta \geq 9\) is either trivial or canonical.

**Proof.** Let \((H, A_1, A_2, B_1, B_2)\) be an interval 2-join in a skeletal vertex critical graph satisfying \(\chi = \Delta \geq 9\). Suppose \(H\) is nontrivial; that is, \(A_1 \neq A_2\). Put \(C := A_1 \cap A_2\). Then \((H \setminus C, A_1 \setminus C, A_2 \setminus C, C \cup B_1, C \cup B_2)\) is a canonical interval 2-join. Reduce this 2-join until we get an irreducible canonical interval 2-join \((H', A_1', A_2', B_1', B_2')\) with \(H' \leq H \setminus C\). Since \(C\) is joined to \(H - C\), it is also joined to \(H'\). Hence \(C \subseteq B_1' \cap B_2' = \emptyset\) by Lemma 4.3. Hence \(A_1 \cap A_2 = C = \emptyset\) showing that \(H\) is canonical.

\(\square\)

**Theorem 4.5.** Every quasi-line graph satisfying \(\chi \geq \Delta \geq 9\) contains a \(K_\Delta\).

**Proof.** We will prove the theorem by reducing to the case of line graphs, i.e., for every strip \((H, A_1, A_2)\) we have \(A_1 = A_2\). Suppose not and choose a counterexample \(G\) minimizing \(|G|\).

Plainly, \(G\) is vertex critical. By Lemma 4.2 we may assume that \(G\) is skeletal. By Lemma 3.1 \(G\) is not a circular interval graph. Therefore, by Lemma 4.1 \(G\) is a composition of linear interval strips. Choose such a composition representation of \(G\) using the maximum number of strips.

Let \((H, A_1, A_2)\) be a strip in the composition. Suppose \(A_1 \neq A_2\). Put \(B_1 := N_{G \setminus H}(A_1)\) and \(B_2 := N_{G \setminus H}(A_2)\). Then \((H, A_1, A_2, B_1, B_2)\) is an interval 2-join. Since \(A_1 \neq A_2\), \(H\) is canonical by Lemma 4.3. Suppose \(H\) is reducible. By symmetry, we may assume that \(N_{H}(A_1) \setminus A_1 = N_{H}(v_1) \setminus A_1\). But then replacing the strip \((H, A_1, A_2)\) with the two strips \((G[A_1], A_1, A_1)\) and \((H \setminus A_1, N_{H}(A_1) \setminus A_1, A_2)\) gives a composition representation of \(G\) using more strips, a contradiction. Hence \(H\) is irreducible. By Lemma 4.3 \(H\) is complete and thus replacing the strip \((H, A_1, A_2)\) with the two strips \((G[A_1], A_1, A_1)\) and \((G[A_2], A_2, A_2)\) gives another contradiction.

Therefore, for every strip \((H, A_1, A_2)\) in the composition we must have \(V(H) = A_1 = A_2\). Hence \(G\) is a line graph of a multigraph. But this is impossible by Lemma 6.1.

\(\square\)

## 5 Claw-free graphs

In this section we reduce the Borodin-Kostochka conjecture for claw-free graphs to the case of quasi-line graphs. We first show that a certain graph cannot appear in the neighborhood of any vertex in our counterexample.
Hence that nonadjacent $x_i$’s have a common color different than $c$. Hence, by Lemma 2.3, we have $|\text{Pot}(L)| \leq 5$. Thus we have $c \in L(y) \cap L(x_5)$. Also, $L(x_1) \cap L(x_4) \neq \emptyset$, $L(x_1) \cap L(x_3) \neq \emptyset$ and $L(x_2) \cap L(x_4) \neq \emptyset$. By Lemma 2.13, the common color in all of these sets must be $c$. Hence $c$ is in all the lists. Now consider the list assignment $L'$ where $L'(z) = L(z) - c$ for all $z \in N_6$. Then $|\text{Pot}(L')| = 4$ and since $\sum_{i=1}^{5} |L'(x_i)| = 9 > |\text{Pot}(L')|\omega(C_5)$, we see that that nonadjacent $x_i$’s have a common color different than $c$. Now applying Lemma 2.13 gives a final contradiction.

By a thickening of a graph $G$, we just mean a graph formed by replacing each $x \in V(G)$ by a complete graph $T_x$ such that $|T_x| \geq 1$ and for $x, y \in V(G)$, $T_x$ is joined to $T_y$ iff $x \leftrightarrow y$. Each such $T_x$ is called a thickening clique.

**Lemma 5.2.** Any graph $H$ with $\alpha(H) \leq 2$ such that every induced subgraph of $K_1 \ast H$ is not $d_1$-choosable can either be covered by two cliques or is a thickening of $C_5$.

**Proof.** Suppose not and let $H$ be a counterexample. Now by Lemma 5.1, $H$ does not contain a common color different than $c$. Now applying Lemma 2.13 gives a final contradiction.

By a thickening of a graph $G$, we just mean a graph formed by replacing each $x \in V(G)$ by a complete graph $T_x$ such that $|T_x| \geq 1$ and for $x, y \in V(G)$, $T_x$ is joined to $T_y$ iff $x \leftrightarrow y$. Each such $T_x$ is called a thickening clique.

**Lemma 5.2.** Any graph $H$ with $\alpha(H) \leq 2$ such that every induced subgraph of $K_1 \ast H$ is not $d_1$-choosable can either be covered by two cliques or is a thickening of $C_5$.

**Proof.** Suppose not and let $H$ be a counterexample. Now by Lemma 5.1, $H$ does not contain an induced $N_6$.

**Claim 1.** $H$ contains an induced $C_4$ or an induced $C_5$. Suppose not. Then $H$ must be chordal since $\alpha(H) \leq 2$. In particular, $H$ contains a simplicial vertex $x$. But then $\{x\} \cup N_H(x)$ and $V(H) - N_H(x) - \{x\}$ are two cliques covering $H$, a contradiction.

**Claim 2.** $H$ does not contain an induced $C_5$ together with a vertex joined to at least 4 vertices in the $C_5$. Suppose the contrary. If the vertex is joined to all of the $C_5$, then we have in $K_1 \ast H$ an induced $K_2 \ast C_5$, which is $d_1$-choosable by Lemma 2.7. If the vertex is joined to only four vertices in the $C_5$, then $K_1 \ast H$ contains an induced $K_1 \ast N_6$, which is impossible by Lemma 5.1.

**Claim 3.** $H$ contains no induced $C_4$. Suppose otherwise that $H$ contains an induced $C_4$, say $x_1x_2x_3x_4x_1$. Put $R := V(H) - \{x_1, x_2, x_3, x_4\}$. Let $y \in R$. As $\alpha(H) \leq 2$, $y$ has a neighbor in $\{x_1, x_3\}$ and a neighbor in $\{x_2, x_4\}$. If $y$ is adjacent to all of $x_1, \ldots, x_4$, then $K_1 \ast H$ contains $K_2 \ast C_4$, which is $d_1$-choosable by Lemma 2.7, and this is impossible. If $y$ is adjacent to three of $x_1, \ldots, x_4$, then $K_1 \ast H$ contains $E_2 \ast \text{paw}$, which is $d_1$-choosable by Lemma 2.10, and this is again impossible.

Thus every $y \in R$ is adjacent to all and only the vertices on one side of the $C_4$. We show that any two vertices in $R$ must be adjacent to the same or opposite side and this gives
the desired covering by two cliques. If this does not happen, then by symmetry we may suppose we have \( y_1, y_2 \in R \) such that \( y_1 \leftrightarrow x_1, x_2 \) and \( y_2 \leftrightarrow x_2, x_3 \). We must have \( y_1 \leftrightarrow y_2 \) for otherwise \( \{y_1, y_2, x_4\} \) is an independent set. But now \( x_1 y_1 y_2 x_3 x_4 x_1 \) is an induced \( C_5 \) in which \( x_2 \) has 4 neighbors.

**Claim 2.** Each \( x \) which is not on the \( C_5 \) contains an induced \( C_5 \). Thus, \( |H| \) is an independent set. But now \( H \) is a thickening of \( C_5 \). This final contradiction completes the proof.

**Figure 3: The graph \( D_8 \).**

**Lemma 5.3.** The graph \( D_8 \) is \( d_1 \)-choosable.

**Proof.** Suppose not and let \( L \) be a minimal bad \( d_1 \)-assignment on \( G := D_8 \).

**Claim 1.** \(|\text{Pot}(L)| \leq 6\). By the Small Pot Lemma, we know that \(|\text{Pot}(L)| \leq 7\). Suppose \(|\text{Pot}(L)| = 7\). Put \( \{a, b\} := \text{Pot}(L) - L(w) \).

We must have \( L(y_3) = \{a, b\} \). Otherwise we could color \( y_3 \) from \( L(y_3) - \{a, b\} \) and note that \( G - y_3 - w \) is \( d_0 \)-choosable and hence has a coloring from its lists. Then we can easily modify this coloring to use both \( a \) and \( b \) at least once. But now we can color \( w \).

If there exist distinct vertices \( u, v \in V(G) \) such that \( a \in L(u), b \in L(v) \) and \( \{u, v\} \not\subseteq \{x_2, x_3, x_4\} \), then we can color \( G \) as follows. Color \( y_3 \) arbitrarily to leave \( a \) available on \( u \) and \( b \) available on \( v \). Again, \( G - y_3 - w \) has a coloring. We can modify it to use \( a \) and \( b \), then color \( w \). Thus, \( a \) and \( b \) each appear only on some subset of \( \{y_3, x_2, x_3, x_4\} \).

If \( a \in L(x_2) \cap L(x_4) \), then we use \( a \) on \( x_2 \) and \( x_4 \) and color greedily \( y_3, x_3, y_4, x_1, x_5, w \) (actually any order will work if \( y_3 \) is first and \( w \) is last). If \( a \) appears only on \( y_3 \) and exactly one neighbor \( x_i \), then we violate Lemma \( 2.2 \) since \(|\text{Pot}_{y_3, x_i}(L)| < 7\). So now \( a \) appears precisely on either \( y_3, x_2, x_3 \) or \( y_3, x_3, x_4 \). Similarly \( b \) appears precisely on either \( y_3, x_2, x_3 \) or \( y_3, x_3, x_4 \).

If \( \{a, b\} \cap L(x_2) = \emptyset \), then we use \( a \) on \( y_3 \) and \( b \) on \( x_3 \), then greedily color \( y_4, x_4, x_5, x_1, w, x_2 \). By symmetry, we may assume that \( a \in L(x_2) \). But then since \( \{a, b\} \subseteq L(x_3) \) we have \(|\text{Pot}_{y_3, x_2, x_3}(L)| < 7\), violating Lemma \( 2.2 \). Hence \(|\text{Pot}(L)| \leq 6\).

**Claim 2.** \(|\text{Pot}(L)| \leq 5\). Suppose \(|\text{Pot}(L)| = 6\). Choose \( a \in \text{Pot}(L) - L(w) \) and \( b \in L(w) \cap L(y_3) \). Put \( H := G - y_3 - w \).

First we show that \( b \in L(x_2) \cap L(x_3) \cap L(x_4) \). If not, we use \( b \) on \( y_3 \) and \( w \), then greedily color \( x_1, x_5, y_4 \). Now we can finish by coloring last the \( x_i \) such that \( b \notin L(x_i) \).
We must have \( a \in L(y_3) \) or else we color \( x_2, x_4 \) with \( b \) and something else in \( H \) with \( a \) (since \( G_a \) contains an edge by Lemma 2.2) and finish. Now \( a \not\in L(x_1), L(x_5), L(y_4) \), for otherwise we color \( x_2, x_4 \) with \( a \), \( y_3 \) with \( a \) and then color \( x_1, x_5, y_4, x_3 \) in order using \( a \) when we can, then color \( w \). Now \( a \) is on \( y_3 \) and at least two of \( x_2, x_3, x_4 \) or else we violate Lemma 2.2. Now \( a \not\in L(x_2) \cap L(x_4) \) since otherwise we color \( x_2, x_4 \) with \( a \), then \( y_3 \) with \( b \), then greedily color \( x_1, x_5, y_4, x_3, w \). Also \( a \not\in L(x_2) \cap L(x_3) \) since then \( \{a, b\} \subseteq L(y_3) \cap L(x_2) \cap L(x_3) \) and hence \( |\text{Pot}_{y_3,x_2,x_3}(L)| < 6 \), violating Lemma 2.2. Therefore \( V(G_a) = \{y_3, x_3, x_4\} \).

Now \( |\text{Pot}_{y_3,x_3,x_4}(L)| \leq 6 \) and hence \( L(x_3) \cap L(x_4) = \{a, b\} \) for otherwise we violate Lemma 2.2. Suppose \( L(x_3) = \{a, b, c, d\} \) and \( L(x_4) = \{a, b, e, f\} \). Then by symmetry \( L(x_1) \) contains either \( c \) or \( e \). If \( c \in L(x_1) \), color \( x_1, x_3 \) with \( c, x_4 \) with \( a \), and \( y_3 \) with \( b \). Now we can greedily finish. If \( e \in L(x_1) \), color \( x_1, x_4 \) with \( e, x_3 \) with \( a \), and \( y_3 \) with \( b \); again we can greedily finish. Hence \( |\text{Pot}(L)| \leq 5 \).

Claim 3. \( L \) does not exist. Since \( |\text{Pot}(L)| \leq 5 \) we see that \( x_3, x_5 \) have two colors in common and \( x_2, x_4 \) have two colors in common as well. In fact, these sets of common colors must be the same and equal to \( L(y_3) := \{a, b\} \) or we can finish the coloring (by first coloring \( y_3 \), then invoking Lemma 2.13). Similarly, we may assume that \( a \in L(y_4) \) (if \( \{a, b\} \cap L(y_4) = \emptyset \), then we have \( L(x_2) \cap L(y_4) \cap (\text{Pot}(L) \setminus \{a, b\}) \neq \emptyset \) and we have color \( a \) on \( x_3, x_5 \), so we can color \( y_3 \) with \( b \) and then finish by Lemma 2.13). Similarly, \( L(x_1) \) contains \( a \) or \( b \). But it cannot contain \( a \) for then we could color \( y_3, y_4, x_1 \) with \( a \), and \( x_2, x_4 \) with \( b \), and then finish greedily. Say \( L(x_4) = \{a, b, c\} \). Then as no nonadjacent pair has a color in common that is in \( \text{Pot}(L) \setminus \{a, b\} \) we have \( L(x_2) = \{a, b, c\} \), then by symmetry of \( c \) and \( d \) we have \( L(x_5) = \{a, b, c\} \). Then \( L(x_3) = \{a, b, d, e\} \) and hence \( L(x_1) = \{a, b\} \), which contradicts that \( a \not\in L(x_1) \). We conclude that \( L \) cannot exist.

Lemma 5.4. Let \( H \) be a thickening of \( C_5 \) such that \( |H| \geq 6 \). Then \( K_1 \ast H \) is \( f \)-choosable, where \( f(v) \geq d(v) \) for the \( v \) in the \( K_1 \) and \( f(x) \geq d(x) - 1 \) for \( x \in V(H) \).

\textbf{Proof.} Suppose not and let \( L \) be a minimal bad \( f \)-assignment on \( K_1 \ast H \). By the Small Pot Lemma, \( |\text{Pot}(L)| \leq |H| \). Note that \( H \) is \( d_0 \)-choosable since it contains an induced diamond. Let \( x_1, \ldots, x_5 \) be the vertices of an induced \( C_5 \) in \( H \). Then \( \sum_i |L(x_i)| = \sum_i d_H(x_i) = 3|H| - 5 \geq 2|H| \geq \omega(H[x_1, \ldots, x_5])|\text{Pot}(L)| \) and hence some nonadjacent pair in \( \{x_1, \ldots, x_5\} \) have a color in common. Now applying Lemma 2.4 gives a contradiction.

We are now in a position to finish the proof of the Borodin-Kostochka Conjecture for claw-free graphs.

Theorem 5.5. Every claw-free graph satisfying \( \chi \geq \Delta \geq 9 \) contains a \( K_\Delta \).

\textbf{Proof.} Suppose not and choose a counterexample \( G \) minimizing \( |G| \). Then \( G \) is vertex critical and not quasi-line by Theorem 4.15. Hence \( G \) contains a vertex \( v \) that is not bisimplicial. By Lemma 5.2 \( G_v := G[N(v)] \) is a thickening of a \( C_5 \). Also, by Lemma 5.4 \( v \) is high. Pick a \( C_5 \) in \( G_v \) and label its vertices \( x_1, \ldots, x_5 \) in clockwise order. For \( i \in [5] \), let \( T_i \) be the thickening clique containing \( x_i \). Also, let \( S \) be those vertices in \( V(G) \setminus N(v) \setminus \{v\} \) that have a neighbor in \( \{x_1, \ldots, x_5\} \). First we establish a few properties of vertices in \( S \).

Claim 1. For \( z \in S \) we have \( N(z) \cap \{x_1, \ldots, x_5\} \in \{\{x_i, x_{i+1}\}, \{x_i, x_{i+1}, x_{i+2}\}\} \) for some \( i \in [5] \). Let \( z \in S \) and put \( N := N(z) \cap \{x_1, \ldots, x_5\} \). If \( |N| \geq 4 \), then some subset of
\{v, z\} \cup N \text{ induces the graph } E_2 \ast P_4, \text{ which is } d_1\text{-choosable by Lemma } 2.10. \text{ Hence } |N| \leq 3. \text{ Since } G \text{ is claw-free, the vertices in } N \text{ must be contiguous.}

**Claim 2.** If \( z \in S \text{ is adjacent to } x_i, x_{i+1}, x_{i+2}, \text{ then } |T_i| = |T_{i+1}| = |T_{i+2}| = 1. \) Suppose not. First, we deal with the case when \( |T_{i+1}| \geq 2. \) Pick \( y \in T_{i+1} - x_{i+1}. \) If \( y \not\leftrightarrow z, \text{ then } \{x_i, y, z, x_{i-1}\} \text{ induces a claw, impossible. Thus } y \leftrightarrow z \text{ and } \{v, z, x_i, x_{i+1}, x_{i+2}, y\} \text{ induces the graph } E_2 \ast \text{diamond, which is } d_1\text{-choosable by Lemma } 2.10. \)

Hence, by symmetry, we may assume that \( |T_i| \geq 2. \) If \( y \not\leftrightarrow z, \text{ then } \{v, x_1, \ldots, x_5, y, z\} \) induces a \( D_8 \) contradicting Lemma 5.3. Hence \( y \leftrightarrow z \) and \( \{v, z, x_i, x_{i+1}, x_{i+2}, y\} \) induces the graph \( E_2 \ast \text{paw, which is } d_1\text{-choosable by Lemma } 2.10. \)

**Claim 3.** For \( i \in [5], \text{ let } B_i \text{ be the } z \in S \text{ with } N(z) \cap \{x_1, \ldots, x_5\} = \{x_i, x_{i+1}\}. \) Then \( B_i \cup B_{i+1} \text{ and } B_i \cup T_i \cup T_{i+1} \text{ both induce cliques for any } i \in [5]. \) Otherwise there would be a claw.

**Claim 4.** \( |T_i| \leq 2 \text{ for all } i \in [5]. \) Suppose otherwise that we have \( i \) such that \( |T_i| \geq 3. \) Put \( A_i := N(x_i) \cap S. \) By Claim 2, \( A_i \subseteq B_{i-1} \cup B_i \) and \( A_i \) is joined to \( T_i. \) Thus \( T_i \) is joined to \( F_i := \{v\} \cup A_i \cup T_{i-1} \cup T_{i+1}. \) If \( A_i \neq \emptyset, \) then \( F_i \) induces a graph that is connected and not almost complete, which is impossible by Lemma 2.6. If \( A_i = \emptyset, \) then \( x_i \) must have at least \( \Delta - 2 \) neighbors in \( T_{i-1} \cup T_i \cup T_{i+1}. \) But that leaves at most one vertex for \( T_{i-2} \cup T_{i+2}, \) which is impossible.

**Claim 5.** \( G \text{ does not exist.} \) Since \( d(v) = \Delta \geq 9, \text{ by symmetry we may assume that } |T_i| = 2 \text{ for all } i \in [4]. \) As in the proof of Claim 4, we get that \( T_2 \) is joined to \( F_2. \) Since \( |T_i| \leq 2 \text{ for all } i, \) we must have \( A_i \neq \emptyset \) (for all \( i, \) but in particular for \( A_2). \) Since \( A_i \subseteq B_{i-1} \cup B_i, \) by symmetry, we may assume that \( A_2 \cap B_2 \neq \emptyset. \) Pick \( z \in A_2 \cap B_2 \) and \( y_i \in T_i - x_i \) for \( i \in [3]. \) Then \( F_2 \) has the graph in Figure 4 as an induced subgraph, but this is impossible by Lemma 2.7.

![Figure 4: K_2 joined to this graph is d_1-choosable](image)

We note that this reduction to the quasi-line case also works for the Borodin-Kostochka conjecture for list coloring; that is, we have the following result.

**Theorem 5.6.** If every quasi-line graph satisfying \( \chi_l \geq \Delta \geq 9 \text{ contains a } K_\Delta, \text{ then the same statement holds for every claw-free graph.} \)

6 Line graphs

In [18], the second author proved the Borodin-Kostochka conjecture for line graphs of multigraphs. Our aim in this section is to lay out what we can prove about the list version of the Borodin-Kostochka conjecture for line graphs (of multigraphs). Our main result in this
direction is Theorem 5.6, which says that if \( H \) has minimum degree at least 7 and \( G \) is the line graph of \( H \), then the list version of the Borodin-Kostochka Conjecture holds for \( G \).

**Theorem 6.1** (Rabern [18]). Every line graph of a multigraph satisfying \( \chi \geq \Delta \geq 9 \) contains a \( K_\Delta \).

Some of the techniques used in the proof of this theorem carry over to the Borodin-Kostochka conjecture for list coloring; unfortunately, a key part of the proof used the fan equation and we do not have that for list coloring.

**Lemma 6.2.** Fix \( t \geq 2 \) and \( j \in \{0,1\} \). Let \( B \) be the complement of a bipartite graph with \( \omega(B) < |B| - j \). Let \( L \) be a list assignment on \( G := K_t \ast B \) with \( |L(v)| \geq d(v) - j \) for each \( v \in V(K_t) \) and \( |L(v)| \geq d(v) - 1 \) for each \( v \in V(B) \). If \( G \) is not \( L \)-colorable, then:

- \( t = 3 \) and \( B \) is the disjoint union of two complete subgraphs; or,
- \( t = 2 \) and \( B \) is the disjoint union of two complete subgraphs; or,
- \( t = 2 \), \( B \) is formed by adding an edge between two disjoint complete subgraphs; or,
- \( t = 2 \), \( B \) has a dominating vertex \( v \) and \( B - v \) is the disjoint union of two complete subgraphs.

**Proof.** If \( t \geq 4 \), then by Lemma 2.5 \( B \) is almost complete and hence \( j = 0 \). But then Lemma 2.11 gives a contradiction. Hence \( t \leq 3 \).

Suppose \( t = 3 \). By Lemma 2.6, \( B \) is either almost complete or \( K_r + K_{|B|-r} \). Suppose \( B \) is almost complete. Then \( j = 0 \). Let \( z \in V(B) \) be the vertex outside of the \( |B| - 1 \) clique and \( x \in V(B) \) some nonneighbor of \( x \). Then \( |L(x)| + |L(z)| \geq d(x) + d(z) - 2 = d_B(z) + |B| + 2 \).

By the Small Pot Lemma (see Section 2), \( |Pot(L)| \leq |B| + 2 \). Hence if \( d_B(z) > 0 \), we could color \( x \) and \( z \) the same and then greedily complete the coloring to the rest of \( G \), impossible. So, \( B \) is \( K_1 + K_{|B|-1} \).

Now suppose \( t = 2 \). If \( B \) has no dominating vertex, then by Lemma 2.7 \( B \) is the disjoint union of two complete subgraphs or \( B \) is formed by adding an edge between two disjoint complete subgraphs. Otherwise \( B \) has a dominating vertex \( v \) and hence \( B = K_3 \ast B - v \).

Similarly to the \( t = 3 \) case, we conclude that \( B - v \) is the disjoint union of two complete subgraphs.

**Lemma 6.3.** Let \( H \) be a multigraph and let \( G \) be the line graph of \( H \) such that \( \omega(G) < \chi_l(G) = \Delta(G) \). Suppose we have a bad \((\Delta(G) - 1)\)-assignment \( L \) on \( G \), and that \( G \) is vertex critical with respect to \( L \). Then \( \mu(H) \leq 3 \) and no multiplicity 3 edge is in a triangle. Let \( xy \in E(G) \) have \( \mu(xy) = 2 \). Then \( xy \) is contained in at most one triangle. Moreover, this triangle is either 4-sided or 5-sided. If the triangle is 5-sided, then one of \( x \) or \( y \) has all its neighbors in the triangle and in particular has degree at most 4 in \( H \).

**Proof.** Put \( \Delta := \Delta(G) \). Let \( xy \in E(H) \) be an edge in \( H \). Let \( A \) be the set of all edges incident with both \( x \) and \( y \). Let \( B \) be the set of edges incident with either \( x \) or \( y \) but not both. Then, in \( G \), \( A \) is a clique joined to \( B \) and \( B \) is the complement of a bipartite graph. Put \( F := G[A \cup B] \).

Since \( xy \) is \( L \)-critical, we can color \( G - F \) from \( L \). Doing so leaves a list assignment \( J \) on
$F$ with $|J(v)| = \Delta - 1 - (d_G(v) - d_F(v)) = d_F(v) - 1 + \Delta - d_G(v)$ for each $v \in V(F)$. Put $j := d_G(xy)+1-\Delta$. Since $d_G(xy)+1 = |A|+|B|$ and $\Delta > \omega(G) \geq \omega(A)+\omega(B) = |A|+\omega(B)$, we have $\omega(B) < |B|-j$.

Therefore we may apply Lemma 6.2. We conclude $\mu(xy) \leq 3$. Also, if $B$ is a disjoint union of two cliques in $G$, then $xy$ is in no triangle. Now suppose $\mu(xy) = 2$. If $B$ has no dominating vertex in $G$, then $xy$ is in exactly one triangle and it is 4-sided. Otherwise, by symmetry we may assume that $B$ has a dominating vertex $xz$. Then $y$ must have all its edges to $x$ and $z$ and $y$ must have at least one edge to $z$ for otherwise we would have a $\Delta$ clique in $G$. Since $B - xz$ is the disjoint union of two cliques, we must have $\mu(xz) = 1$. Also $\mu(yz) \leq 2$ and hence $d_H(y) \leq 4$.

**Lemma 6.4.** Let $H$ be a multigraph and let $G$ be the line graph of $H$ such that $\omega(G) < \chi_l(G) = \Delta(G)$. Suppose we have a bad $(\Delta(G) - 1)$-assignment $J$ on $G$, and that $G$ is vertex critical with respect to $J$. Then $H$ cannot have triple edges $uv$ and $vw$, such that $d(u) \geq 6$, $d(w) \geq 6$, and $d(v) \geq 7$ (or $d(v) \geq 6$ and every edge incident to $v$ in $H$ is low in $G$).

**Proof.** Assume the contrary and let $H$ be a counterexample. Recall from Lemma 2.2 above that the maximum edge multiplicity of $H$ is at most 3.

Let $a_1$, $a_2$, $a_3$ be three edges incident to $u$ but not $v$; let $b_1$, $b_2$, $b_3$ be the edges incident to $u$ and $v$; let $c$ be incident to $v$ but not $u$ or $w$; let $d_1$, $d_2$, $d_3$ be incident to $v$ and $w$; let $e_1$, $e_2$, $e_3$ be incident to $w$ (but not $u$ or $v$). We use these names for both the edges of $H$ and the vertices of $G$, interchangeably.

By criticality of $G$, we can color $V(G) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3, c, d_1, d_2, d_3, e_1, e_2, e_3\}$ from $J$. Let $L$ denote the list of remaining colors on the uncolored vertices. Note that $|L(a_i)| \geq 4$, $|L(c)| \geq 4$, $|L(\alpha)| \geq 5$, $|L(b_1)| \geq 8$, and $|L(b_2)| \geq 8$. We may assume that equality holds in each case.

**Claim 1.** If there exist $\alpha \in L(a_1) \cap L(c)$, then we can color $G$ from its lists. Suppose such an $\alpha$ exists. We use $\alpha$ on $a_1$ and $c$. This saves a color on each of $b_1$, $b_2$, $b_3$. Now $|L(d_1) \setminus \{\alpha\}| + |L(d_2) \setminus \{\alpha\}| \geq 7 + 3 > 8 = |L(b_1)|$, so we can color $d_1$ and $d_2$ to save an additional color on $b_1$. Now we greedily color $e_1$, $e_2$, $e_3$, $d_2$, $d_3$, $c$, $a_2$, $a_3$, $b_2$, $b_1$.

**Claim 2.** If there exists $\alpha \in L(a_1) \cap L(d_1)$, then we can color $G$ from its lists. Suppose such an $\alpha$ exists. If $\alpha \in L(c)$, then we proceed as above. Otherwise we use $\alpha$ on $a_1$ and $d_1$. This saves a color on $b_1$, $b_2$, and $b_3$. We may assume that $\alpha \in L(b_1)$, since otherwise we can color greedily toward $b_1$. Now we get $|L(a_2) \setminus \{\alpha\}| + |L(c)| \geq 3 + 5 > 7 = |L(b_1) \setminus \{\alpha\}|$. Thus, we can color $a_2$ and $c$ to save a color on $b_1$. Afterwards we color greedily toward $b_1$.

**Claim 3.** We may assume that $L(b_1) = L(b_2) = L(b_3) = L(d_1) = L(d_2) = L(d_3)$; otherwise we can color $G$ from its lists. Suppose to the contrary that (by symmetry) there exists $\alpha \in L(d_1) \setminus L(b_1)$. If we also have $\alpha \notin L(b_2)$, then we may use $\alpha$ on $d_1$ (color $a_1$ arbitrarily) and proceed as in Claim 2. So now we have $\alpha \in L(b_2)$. By Claim 2 and symmetry, we have $\alpha \notin L(e_i)$. Thus we use $\alpha$ on $d_1$ (without reducing the $L(e_i)$). Since we have $|L(c) \setminus \{\alpha\}| + |L(a_1)| > |L(b_2) \setminus \{\alpha\}|$, we can color $c$ and $a_1$ to save a color on $b_2$. Now we color $d_2$ and $a_2$ to save a second color on $b_1$. Finally, we color greedily toward $b_1$.

**Claim 4.** We can color $G$. By Claim 2, we know that $L(a_1) \cap L(d_1) = \emptyset$. By Claim 3, we know that $L(b_1) = L(d_1)$; thus, $L(a_1) \cap L(b_1) = \emptyset$. By symmetry, we get $L(a_i) \cap L(b_j) = \emptyset$ for all $i, j \in \{1, 2, 3\}$. Now we can color the $a_i$ arbitrarily, which saves 3 colors on each of the $b_i$. Finally, we color greedily towards $b_1$. This proves the lemma. \hfill \Box
As an application of the lemma above, we show that if $H$ has minimum degree at least 7 and $G$ is the line graph of $H$, then the list version of the Borodin-Kostochka Conjecture holds for $G$. We need the following theorem, due to Borodin, Kostochka, and Woodall.

**Theorem 6.5** (Borodin, Kostochka, Woodall [2]). Let $G$ be a bipartite multigraph with partite sets $X$, $Y$, so that $V = X \cup Y$. $G$ is edge-$f$-choosable, where $f(e) := \max\{d(x), d(y)\}$ for each edge $e = xy$.

**Theorem 6.6.** Let $H$ be a multigraph with $\delta(H) \geq 7$ and let $G$ be the line graph of $H$. Then $\chi_l(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.

**Proof.** Suppose the contrary, and let $G_0$ (and $H_0$) be a counterexample with list assignment $L$. Let $G$ (and $H$) be a vertex critical subgraph with respect to $L$. It suffices to color $G$ from $G_0$. Note that $\Delta(G) = \Delta(G_0)$, since otherwise we can color $G$ from $L$ by the list version of Brooks’ Theorem. Since $G$ is $L$-critical, we have $\delta(G) \geq \Delta(G) - 1$. Thus, we have $d_H(u) \geq d_{H_0}(u) - 1$ for all $u \in V(H)$ so $\delta(H) \geq 6$. In particular, if $uv$ is high in $G$, then $d_H(u) = d_{H_0}(u)$ and $d_H(v) = d_{H_0}(v)$. Note that if $\mu_H(xy) = 3$, then in $H_0$ each of $x$ and $y$ is incident to only one triple edge (by Lemma 6.4, since $G$ is critical with respect to $L$).

**Claim.** If $xy$ is an edge of $H$ with multiplicity 3 and $d(x) = 7$, then vertex $xy$ is low in $G$.

Suppose the contrary. Since $G$ is $\chi_l$-critical, for each edge $xy \in H$, we have $d_H(x) + d_H(y) = \Delta(G) + \mu(xy) + 1$ if $xy$ is high and $d_H(x) + d_H(y) = \Delta(G) + \mu(xy)$ if $xy$ is low. Suppose that $\mu(xy) = 3$, $d(x) = 7$, and $xy$ is high. Now we get $d_{H_0}(y) = \Delta(G) - 3$. By the last sentence of the previous paragraph, we know that every edge incident to $y$ other than $xy$ has multiplicity at most 2. Let $z$ be a neighbor of $y$ other than $x$. By the degree condition above, we get $d_{H_0}(y) + d_{H_0}(z) \leq \Delta(G) + \mu_{H_0}(xy) + 1 \leq \Delta(G) + 3$. This implies that $d_{H_0}(z) \leq 6$, which is a contradiction. This proves the claim. More simply, for any vertex $x$ with $d_H(x) = 6$, we see that every edge $xy$ incident to $x$ must be low in $G$.

Now for each triple edge of $H$ that is high in $G$, we delete one of the edges; call the resulting graph $G'$ (and $H'$). Clearly, we have $\delta(H') \geq 6$. By the previous Lemma and the claim, $d_{H'}(x) \geq 7$ for every vertex $x$ incident to a triple edge or an edge $xy$ that corresponds to a high vertex in $G$. For if $xy$ is a triple edge and $d(x) = 7$, then edge $xy$ is low in $G$. Similarly, if $d(x) = 6$, then every edge $xy$ is low in $G$. Otherwise, each vertex of $H$ that is incident to a triple edge has degree at least 8 and is incident to exactly one triple edge.

Let $B$ be a maximum bipartite subgraph of $H'$. For each vertex $x \in H$, we have $d_B(x) \geq d_{H'}(x)/2$ (otherwise $B$ has more edges if we move $x$ to the other part); thus $\delta(B) \geq 3$ and $d_B(x) \geq 4$ for each vertex incident to a triple edge or an edge $xy$ that is high in $G$. Let $G_B$ denote the line graph of $B$. Since $G$ is critical with respect to $L$, we can color $G - V(G_B)$ from $L$. So to show that $G$ is $(\Delta - 1)$-choosable, it suffices to show that we can color $G_B$ when each vertex $v$ that is high in $G$ gets a list of size $d_{G_B}(v) - 1$ and each vertex $v$ that is low in $G$ gets a list of size $d_{G_B}(v)$. Consider a high vertex $xy$ in $G$; recall that $d_H(x) \geq 7$ and $d_H(y) \geq 7$, so $d_B(x) \geq 4$ and $d_B(y) \geq 4$. The degree of $xy$ in $G_B$ is $d_B(x) + d_B(y) - \mu_B(xy) - 1$. Since $\mu_B(xy) \leq 2$, we see that $xy$ has a list of size at least $d_B(x) + d_B(y) - 4 \geq \max\{d_B(x), d_B(y)\}$. Each low vertex $xy$ has a list of size at least $d_B(x) + d_B(y) - \mu(xy) - 1 \geq \max\{d_B(x), d_B(y)\}$. So by Theorem 6.5, we can color $G_B$ from its remaining lists.

**Theorem 6.6** is best possible in the sense that if we replace “$\delta(H) \geq 7$” by “$\delta(H) \geq 6$”,

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then the theorem is false. One counterexample is when $H$ is a 5-cycle in which each edge has multiplicity 3, shown in Figure 1 (d).

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