On Uniform Reductions between Direct Product and XOR Lemmas

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Abstract. There is a close connection between Direct Product and XOR lemmas in the sense that in many settings, we can prove one given the other. The known reductions that are used for the above purpose are either in the non-uniform setting or give non-matching parameters. By non-matching parameter we mean that \( k \)-wise Direct Product lemma implies \( k' \)-wise XOR lemma (and vice versa) for \( k \neq k' \). In this work, we discuss reductions between \( k \)-wise Direct Product and \( k \)-wise XOR lemmas. That is, we show that if the \( k \)-wise direct product lemma holds, then so does the \( k \)-wise XOR lemma and vice versa. We show that even though there is a perfectly uniform reduction in one direction, the reduction in the other direction requires some amount of non-uniformity. We give reductions in both directions matching information-theoretic bounds up to polynomial factors. Our techniques also give a small quantitative improvement over the known results about proving \( k \)-wise XOR lemma using \( 2^k \)-wise Direct Product lemma.

1 Introduction

The \( k \)-wise Direct Product and XOR lemmas are statements of the form: if a function \( f : \{0,1\}^n \rightarrow \{0,1\} \) is hard to compute on the average using machines with some resource bound (within certain computation model), then computing \( f^k \) and \( f^{\oplus k} \) is even harder on average using machines with comparable resources. Here, \( f^k : \{0,1\}^{nk} \rightarrow \{0,1\}^k \) denotes the \( k \)-wise direct product function that is defined as \( f^k(x_1, x_2, ..., x_k) = f(x_1)f(x_2)...f(x_k) \), where \( i \in \{0,1\}^n \). The \( k \)-wise XOR function \( f^{\oplus k} : \{0,1\}^{nk} \rightarrow \{0,1\} \) is defined as \( f^{\oplus k}(x_1, x_2, ..., x_k) = f(x_1)\oplus f(x_2)\oplus ...\oplus f(x_k) \). In more general versions of such results, function computation may be replaced with solving puzzles, attacking protocols etc. Due to wide applicability of such results in areas such as computational complexity and cryptography, there has been a lot of work [Yao82, Lev87, GNW, Imp95, IW97, STV01, JKW10] in proving such results. It has been observed that in most of these cases, if the direct product lemma holds, then so does the XOR lemma and vice versa. This is not a mere coincidence. There is a formal connection between Direct Product and XOR lemmas. A number of previous works [GNW, Ung09, VW08, Jai08, JKW10] have studied this connection. In this paper, we revisit connections between Direct Product and XOR lemmas. We study reductions between Direct Product and corresponding XOR lemmas in either direction. Most of the known reductions either work in the non-uniform setting (e.g., [GNW]) or give non-matching parameters. In this work, we discuss proofs that show that the \( k \)-wise XOR lemma holds given that the \( k \)-wise Direct Product lemma is holds and vice versa.

A function \( f : \{0,1\}^n \rightarrow \{0,1\} \) is said to be \((\delta, t)\)-hard if for any machine \( A \) running in time at most \( t \), we have

\[
\Pr_{x \in \{0,1\}^n} [A(x) = f(x)] \leq (1 - \delta).
\]

(1)

Suppose we use \( k \) such machines to compute \( f^k(x_1, ..., x_k) \), each running in time \( t \) independently and the \( i^{th} \) machine computing \( f(x_i) \), then the fraction of inputs on which the computation is
Moreover work on such relationships. Various settings [Yao82, Lev87, GNW, Imp95, IW97, BIN97, Raz98, STV01, HVV06, IJKW10, DIJK09]. The relationship between k-wise Direct Product and XOR lemmas have been studied in some of these works [GNW, Ung09, VW08, Jai08, IJKW10]. However, the nature of the relationships studied do not have the same value of k. More specifically, the hardness of f* is shown using the hardness of f**k (and vice versa) where k ≠ k'. For instance, we note the following.

- In [Jai08, LJKW10, DIJK09], it is shown that f**k is hard given that f**k is hard. The reduction crucially uses the Goldreich-Levin theorem [GL89]. Such ideas were also discussed in [GNW].
- In [VW08, Jai08, Ung09, IK10], it is shown that k-wise Direct Product lemma holds given that the k'-wise XOR lemma holds for all k' ≤ k. Relationships from a threshold version of the Direct Product lemma is discussed in [Ung09, IK10].

Our results Our study of the relationship between k-wise Direct Product lemma and k-wise XOR lemmas is summarised in the following two main theorems.

**Theorem 1 (k-wise Direct Product to k-wise XOR).** There is a probabilistic algorithm A with the following property: Let k ∈ N and let 0 < ε < 1 be such that ε > 2^{-k}. Let C' be an algorithm with running time t that computes f**k for some f : {0,1}^n → {0,1} on at least (1/2 + ε) fraction of the inputs. Given such an algorithm C', algorithm A outputs an algorithm C with running time t' that computes f**k on at least Ω(ε^k) fraction of the inputs. Moreover t' = poly(k,1/ε) · t.

**Theorem 2 (k-wise XOR to k-wise Direct Product).** There is a constant c and a probabilistic algorithm A with the following property: Let k ∈ N and let 0 < ε < 1 be such that ε > 2^{-k}. Let C' be an algorithm with running time t that computes f**k for some f : {0,1}^n → {0,1} on at least ε fraction of the inputs. Given such an algorithm C', algorithm A outputs with probability Ω(ε) an algorithm C with running time t' that computes f**k on at least 1/2 + Ω(ε^c) fraction of the inputs. Moreover t' = poly(k,1/ε) · t.

We also obtain improvements in parameters of the known theorem showing a k-wise XOR lemma using a 2k-wise direct product lemma [LJKW10, Jai08].

**Theorem 3 (2k-wise Direct Product to k-wise XOR).** There is a probabilistic algorithm A with the following property: Let k ∈ N and let 0 < ε < 1 be such that ε > 2^{-k}. Let C' be an algorithm with running time t that computes f**k for some f : {0,1}^n → {0,1} on at least (1/2 + ε) fraction of the inputs. Given such an algorithm C', algorithm A outputs an algorithm C with running time t' that computes f**k on at least Ω(ε^2/k + ε^4) fraction of the inputs. Moreover t' = poly(k,1/ε) · t.
In the reductions used in [Jai08, JKW10], the $\epsilon^4$ term is not present. So, the improvement is significant only when $\epsilon \geq 1/\sqrt{k}$. This may occur when $\delta$ in (1) is large (say $\delta = \Omega(\log k/k)$).

Note that using a pruning technique, [JKW10] also give a reduction of the following nature: “If approximately computing $f^{2k}$ is $(1 - O(\epsilon^2), t)$-hard, then $f^{2k}$ is $(1/2 - \epsilon, t')$-hard”. An algorithm $A$ approximately computes $f^{2k}(\bar{x})$ if for most $i \in \{1, ..., 2k\}$, $A(\bar{x})[i] = f^{2k}(\bar{x})[i]$.

2 Preliminaries

For any string $r \in \{0,1\}^m$, and we denote the Hamming weight (number of 1's) of $r$ by $H(r)$. The $j^{th}$ bit of such a string is denoted by $r[j]$. For any $m$-tuple, $\bar{z} = (z_1, ..., z_m)$ and any string $r \in \{0,1\}^m$, $\bar{z}[r]$ denotes the $H(r)$-tuple $(z_{r[1]}, z_{r[2]}, ..., z_{r[H(r)]})$, where $r[1] = 1, r[2] = 1, ..., r[H(r)] = 1$.

The following simple lemma will be used in a number of places in the paper.

Lemma 1. Let $n, k \in \mathbb{N}$ and $0 < \omega \leq 1$. Given any string $m \in \{0,1\}^n$ such that there are $\omega n$ 1's in the string $m$. Then

$$\Pr_{(i_1, ..., i_k) \in [n]^k} [m[i_1] \oplus ... m[i_k] = 1] = \frac{1}{2} + \frac{(1 - 2\omega)^k}{2}.$$ 

Proof. Let $S \subseteq [n]$ denote the subset of indices such that $m[i] = 1$ iff $i \in S$. We have:

$$\Pr_{(i_1, ..., i_k) \in [n]^k} [m[i_1] \oplus ... m[i_k] = 1] = \Pr_{(i_1, ..., i_k) \in [n]^k} [\{|j : i_j \in S\} \text{ is even}]$$

$$= \binom{k}{0} \omega^0 (1 - \omega)^k + \binom{k}{2} \omega^2 (1 - \omega)^k - 2 + ...$$

$$= \frac{1}{2} + \frac{(1 - 2\omega)^k}{2}.$$ 

We will use the following version of the Goldreich-Levin Theorem.

Theorem 4 (Theorem from [GL89]). There is a probabilistic algorithm $A$ with the following property. Let $x \in \{0,1\}^n$ be any string, and let $B : \{0,1\}^n \rightarrow \{0,1\}$ be any predicate such that $\Pr_{r \in \{0,1\}^n} [B(r) = (x, r)] \geq 1/2 + \gamma$, for some $\gamma > 0$. Then, given oracle access to $B$, the algorithm $A$ runs in time $\text{poly}(n, 1/\gamma)$, and outputs the string $x$ with probability at least $\Omega(\gamma^2)$.

3 Direct Product to XOR (proof of Theorems 1 and 2)

Here, we discuss the following question: If there is an algorithm that computes $f^{\oplus k}$ on at least $1/2 + \epsilon$ fraction of inputs, then does there exist a constant $d$ and an algorithm such that this algorithm computes $f^k$ on at least $\epsilon^d$ fraction of inputs? In this section we will see that such a uniform reduction is indeed possible.

3.1 Information theoretic bounds on non-uniformity

Here, we study the amount of non-uniformity required for the reduction. We continue the discussion after the following theorem.

Theorem 5. Let $\epsilon > 2^{-k}$. Let $f_1, ..., f_t : \{0,1\}^n \rightarrow \{0,1\}$ be functions such that
We prove the contrapositive. Let $M$ be computing the function $f^k$ on at least $\frac{1}{2} + \epsilon$ fraction of the inputs. Indeed, let $A = \{\bar{x} : f^k(\bar{x}) = f^k_i(\bar{x})\}$. So, $|A_i| \geq (d + 1/d) \cdot \epsilon^2 \cdot 2^{nk}$. Note that from property (1) in the theorem we have $\forall i \neq j, |A_i \cap A_j| \leq \epsilon^5 \cdot 2^{nk}$. So, we have

$$|A_1 \cup A_2 \cup \ldots \cup A_t| = |A_1| + |A_2| - |A_1 \cap A_2| - |A_3| - |A_3 \cap (A_1 \cup A_2)| + \ldots$$

$$\geq 2^{nk} \left( t \cdot (d + 1/d) \cdot \epsilon^2 - \epsilon^5 \cdot (1 + 2 + \ldots + t - 1) \right)$$

$$\geq 2^{nk} \left( t \cdot (d + 1/d) \cdot \epsilon^2 - \epsilon^5 \cdot t^2/2 \right)$$

$$> 2^{nk}$$

This gives a contradiction.

### 3.2 Proof of Theorem 6

We prove the contrapositive. Let $M$ denote the algorithm that takes $nk$ bit inputs and in time $t$ outputs a single bit such that

$$\Pr_{\bar{x}=(x_1, \ldots, x_n) \in \{0,1\}^n} [M(\bar{x}) = f^{\oplus k}(\bar{x})] \geq \frac{1}{2} + \epsilon.$$  \hspace{1cm} (2)
Given such an algorithm $M$, we will construct an algorithm $M'$ with running time $t' = t\cdot \text{poly}(k, 1/\epsilon)$ such that
\[
\Pr_{\bar{x}=(x_1, ..., x_i)\in\{(0,1)^n\}^k} \left[ M'(\bar{x}) = f^k(\bar{x}) \right] \geq \Omega(\epsilon^4).
\] (3)

We first note that $M(\bar{x}|_r)$ and $\langle f^{2k}(\bar{x}), r \rangle$ correlate well for random $2k$-tuples $\bar{x} \in \{(0,1)^{n}\}^{2k}$ and $r \in \{0,1\}^{2k}$ with $H(r) = k$.

**Lemma 2.** The following holds:
\[
\Pr_{\bar{x}\in\{(0,1)^{n}\}^{2k}, r\in\{(0,1)^{2k}\}, H(r)=k} \left[ M(\bar{x}|_r) = \langle f^{2k}(\bar{x}), r \rangle \right] \geq \frac{1}{2} + \epsilon.
\]

Proof. The lemma follows from (2). \qed

For any $\bar{x} \in \{(0,1)^{n}\}^{2k}$, let $\gamma_{\bar{x}} = \Pr_{r\in\{(0,1)^{2k}\}, H(r)=k} \left[ M(\bar{x}|_r) = \langle f^{2k}(\bar{x}), r \rangle \right] - 1/2$. From Lemma 2, we have
\[
\mathbb{E}[\gamma_{\bar{x}}] \geq \epsilon. \tag{4}
\]

We will now design an algorithm $A$ for computing $f^{\oplus 2i}$ for any $i \leq k$. Following is the description of the algorithm:

$A(x_1, ..., x_{2i})$: Given input $(x_1, ..., x_{2i})$, randomly partition the elements into two $i$-tuples $(y_1, ..., y_i)$ and $(z_1, ..., z_i)$. Randomly select $t_1, ..., t_{k-i} \in \{0,1\}^n$. Output $M(t_1, ..., t_{k-i}, y_1, ..., y_i) \oplus M(t_1, ..., t_{k-i}, z_1, ..., z_i)$

The next two lemmas analyze the above algorithm. In the next lemma, we analyze $A$ with respect to a fixed $2k$-tuple.

**Lemma 3.** For any $i \in \{0,1, ..., k\}$ and $2k$-tuple $\bar{x} \in \{(0,1)^{n}\}^{2k}$, we have
\[
\Pr_{r\in\{(0,1)^{2k}\}, H(r)=2i} \left[ A(\bar{x}|_r) = f^{\oplus 2i}(\bar{x}|_r) \right] \geq \frac{1}{2} + 2\gamma_{\bar{x}}^2.
\]

Proof. Consider a bipartite graph $G = (L, R; E)$ where the vertices on the left correspond to subsets of $\{1, ..., 2k\}$ of size $(k - i)$ and the vertices on the right correspond to subsets of size $k$. There is an edge from a vertex $P \in L$ to a vertex $Q \in R$ iff $P \subset Q$. Furthermore, let us color an edge $(P, Q)$ red if $A(\bar{x}|_Q) \neq f^{\oplus k}(\bar{x}|_Q)$ and green otherwise. Let $1/2 + \alpha_P$ be the fraction of red edges incident on a vertex $P$ on the left. Then from the definition of $\gamma_{\bar{x}}$, we know that $\mathbb{E}_P[\alpha_P] \geq \gamma_{\bar{x}}$. Let us now analyze the probability of our algorithm $A$ computing $f^{\oplus 2i}$. Consider any $2i$-size subset of $\{1, ..., 2k\}$ and let $R, R'$ denote the random partition of this subset into sets of size $i$. First note that $f^{\oplus 2i}(\bar{x}|_{R,R'})$ can be written down as
\[
f^{\oplus 2i}(\bar{x}|_{R,R'}) = f^{\oplus k}(\bar{x}|_{R,P}) \oplus f^{\oplus k}(\bar{x}|_{R',P})
\]
where $P$ is any set of size $(k - i)$ and is disjoint from $R$ and $R'$. The success probability of $A$ can be analysed as follows:
\[
\Pr_{R,R'}[A(\bar{x}|_{R,R'}) = f^{\oplus 2i}(\bar{x}|_{R,R'})] = \\
\Pr_{R,R',P} \left[ A(\bar{x}|_{R,P}) = f^{\oplus k}(\bar{x}|_{R,P}) \text{ and } A(\bar{x}|_{R',P}) = f^{\oplus k}(\bar{x}|_{R',P}) \right] \\
+ \Pr_{R,R',P} \left[ A(\bar{x}|_{R,P}) \neq f^{\oplus k}(\bar{x}|_{R,P}) \text{ and } A(\bar{x}|_{R',P}) \neq f^{\oplus k}(\bar{x}|_{R',P}) \right]
\]
The above probabilities do not change if we reverse the order of choosing these sets. Let us first pick the set \( P \) and then choose the sets \( R \) and \( R' \). In that case the above probability can be lower bounded as:

\[
\Pr_{R,R'}[A(\bar{x}|R,R') = f^{\oplus 2i}(\bar{x}|R,R')] \geq \mathbb{E}_P \left[ (1/2 + \alpha_P)^2 + (1/2 - \alpha_P)^2 \right] \\
= \mathbb{E}_P \left[ 1/2 + 2\alpha_P^2 \right] \\
\geq 1/2 + 2\mathbb{E}_P[\alpha_P]^2 \\
\geq 1/2 + 2\gamma_\bar{x}^2
\]

\[\square\]

**Lemma 4.** For any \( i \in \{0, 1, \ldots, k\} \), we have

\[
\Pr_{\bar{x} \in \{0, 1\}^n} \left[ A(\bar{x}) = f^{\oplus 2i}(\bar{x}) \right] \geq \frac{1}{2} + 2\epsilon^2.
\]

**Proof.** This follows from Lemma 3 and 4. \[\square\]

Now, consider the following algorithm \( B \) that given \( \bar{x} \in \{0, 1\}^k \) and \( r \in \{0, 1\}^k \) as inputs, attempts to compute \( \langle f^k(\bar{x}), r \rangle \).

\[
B(\bar{x}, r) : \text{Given input } \bar{x} \in \{0, 1\}^k \text{ and } r \in \{0, 1\}^k, \text{ check if } H(r) \text{ is odd. If so, output a random bit, otherwise output } A(\bar{x}|r).
\]

For any \( \bar{x} \in \{0, 1\}^n \), let \( \beta_\bar{x} = \Pr_{r \in \{0, 1\}^k} \left[ B(\bar{x}, r) = \langle f^k(\bar{x}), r \rangle \right] - 1/2 \). The next lemma gives the bound on the expectation of \( \beta_\bar{x} \) for random \( \bar{x} \in \{0, 1\}^n \).

**Lemma 5.** \( \mathbb{E}_{\bar{x} \in \{0, 1\}^n} \left[ \beta_\bar{x} \right] \geq \epsilon^2 \).

**Proof.** The proof follows from Lemma 4 and the definition of \( B \). \[\square\]

So, for any \( \bar{x} \in \{0, 1\}^n \), since \( \Pr_{r \in \{0, 1\}^k} \left[ B(\bar{x}, r) = \langle f^k(\bar{x}), r \rangle \right] = 1/2 + \beta_\bar{x} \), applying the Goldreich-Levin theorem (Theorem 4), we obtain an algorithm that with probability \( \Omega(\epsilon^2) \) outputs \( f^k(\bar{x}) \) and runs in time \( t \cdot \text{poly}(k, 1/\beta_\bar{x}) \). Using Lemma 5, we get that there is an algorithm \( M' \) correctly computes \( f^k(\bar{x}) \) for randomly chosen \( \bar{x} \in \{0, 1\}^n \) with probability \( \Omega(\epsilon^4) \). Moreover the running time of \( M' \) is at most \( t' = t \cdot \text{poly}(k, 1/\epsilon) \). This concludes the proof of the theorem. \[\square\]

### 3.3 Proof of Theorem 3

The proof of Theorem 3 is similar to that of Theorem 1 given in the previous subsection. Here, we just highlight the differences. Instead of designing an algorithm that attempts to compute \( \langle f^k(\bar{x}), r \rangle \) for \( \bar{x} \in \{0, 1\}^n \) and \( r \in \{0, 1\}^k \), we design the following algorithm \( B \) that attempts to compute \( \langle f^k(\bar{x}), r \rangle \) for \( \bar{x} \in \{0, 1\}^n \) and \( r \in \{0, 1\}^{2k} \):

\[
B(\bar{x}, r) : \text{Given input } \bar{x} \in \{0, 1\}^{2k} \text{ and } r \in \{0, 1\}^{2k}, \text{ check if } H(r) = k. \text{ If so, output } M(\bar{x}|r). \text{ If } H(r) \text{ is odd, then output a random bit, otherwise output } A(\bar{x}|r).
\]

Again, for any \( \bar{x} \in \{0, 1\}^{2k} \), let \( \beta_\bar{x} = \Pr_{r \in \{0, 1\}^{2k}} \left[ B(\bar{x}, r) = \langle f^k(\bar{x}), r \rangle \right] - 1/2 \). The next lemma gives the bound on the expectation of \( \beta_\bar{x} \) for random \( \bar{x} \in \{0, 1\}^{2k} \).
Lemma 6. \( E_{x \in \{0,1\}^{2k}}[\beta_x] \geq \Omega \left( \frac{\epsilon^2}{\sqrt{k}} \right) + \epsilon^2. \)

Proof. The second term on the RHS is the same as in the previous section. The additional term \( \Omega(\epsilon/\sqrt{k}) \) is due to the fact that \( \Pr_{r \in \{0,1\}^{2k}}[H(r) = k] = \frac{k^2}{2^m} = \Omega(1/\sqrt{k}). \) \( \square \)

Again, applying the Goldreich-Levin theorem, we get that there is an algorithm that runs in time \( t' = t \cdot \text{poly}(k, 1/\epsilon) \) and outputs \( f^{2k}(\bar{x}) \) for \( \bar{x} \in \{0,1\}^{2k} \) with probability at least \( \Omega \left( \frac{\epsilon^2}{k} + \epsilon^4 \right) \).

This concludes the proof. \( \square \)

4 XOR to Direct Product (proof of Theorem 3)

Unlike the reduction from Direct Product to XOR that was perfectly uniform, such a reduction in the reverse direction does not seem possible. This can be seen using the following simple example:

Let \( \epsilon = 1/2 \) and \( k \) be odd. Consider two functions \( f_0, f_1 \), where \( \forall x, f_0(x) = 0 \) and \( \forall x, f_1(x) = 1 \). Consider any partition of the set \( \{0,1\}^n \) into sets \( S_0 \) and \( S_1 \) such that \( |S_0| = |S_1| \) and let \( \forall s \in S_0, f^k(s) = f^k_0(s) \) and \( \forall s \in S_1, f^k(s) = f^k_1(s) \). We observe that \( \forall s \in S_0, f^k_0(s) = 0 \) and \( \forall s \in S_1, f^k_1(s) = 1 \) (since \( k \) is odd). So, any algorithm given access to just \( f^k \) will not be able to figure out whether it is should output XOR’s with respect to \( f_0 \) or \( f_1 \). In this case, 1 bit of advice is necessary to point to the correct function w.r.t. which the \( k \)-wise XOR’s should be computed.

As in the previous section, we first study information-theoretic bounds on non-uniformity and then try to match those bounds.

4.1 Information-theoretic bounds on non-uniformity

The next two theorems define an upper and lower bound on the non-uniformity required in the reduction.

Theorem 7. Let \( \epsilon > 2^{-k/12} \). Let \( f_1, \ldots, f_t \) be functions such that:

1. for all \( i \neq j \), \( \Pr_{(x_1,\ldots,x_k)\in\{0,1\}^k} \left[ f^{i\oplus k}_j(x_1,\ldots,x_k) = f^{\oplus k}_j(x_1,\ldots,x_k) \right] < \frac{1}{2} + \frac{\epsilon^6}{2}, \)
2. there is a function \( B : \{0,1\}^k \rightarrow \{0,1\}^k \) such that for all \( i \), \( \Pr_{(x_1,\ldots,x_k)\in\{0,1\}^k} \left[ f^k_i(x_1,\ldots,x_k) = B(x_1,\ldots,x_k) \right] \geq \epsilon. \)

Then \( t = O(1/\epsilon) \).

Theorem 8. Let \( \epsilon > 2^{-k/12} \). There is a function \( B : \{0,1\}^k \rightarrow \{0,1\}^k \) and \( t = \Omega(1/\epsilon) \) functions \( f_1, \ldots, f_t : \{0,1\}^n \rightarrow \{0,1\} \) such that:

1. for all \( i \neq j \), \( \Pr_{(x_1,\ldots,x_k)\in\{0,1\}^k} \left[ f^{i\oplus k}_j(x_1,\ldots,x_k) = f^{\oplus k}_j(x_1,\ldots,x_k) \right] < \frac{1}{2} + \frac{\epsilon^{12}}{2}, \)
2. for all \( i \), \( \Pr_{(x_1,\ldots,x_k)\in\{0,1\}^k} \left[ f^k_i(x_1,\ldots,x_k) = B(x_1,\ldots,x_k) \right] \geq \epsilon. \)

The proof of the above theorems are given in the Appendix. What these theorems imply is that if there is an algorithm that computes \( f^k \) on at least \( \epsilon \) fraction of the inputs, then it should be possible to construct a list of algorithms, with list size \( O(1/\epsilon) \) such that at least one of these algorithms compute \( f^{\oplus k} \) on at least \( 1/2 + \epsilon^{O(1)} \) fraction of the inputs. Furthermore, a smaller list of such algorithms is not possible.

We now give a reduction that matches the information-theoretic bounds up to polynomial factors.
4.2 Proof of Theorem 2

We will use the following main theorem of Impagliazzo et al. [IJKW10]

**Theorem 9 (Theorem 1.2 in [IJKW10]).** There is a constant $c$ and a probabilistic algorithm $A$ with the following property: Let $k \in \mathbb{N}$ and let $0 < \epsilon, \delta < 1$ be such that $\epsilon > e^{-\delta k/c}$. Let $C'$ be an algorithm that computes $f^k$ for some $f : \{0,1\}^n \to \{0,1\}$ on at least $\epsilon$ fraction of the inputs. Given such an algorithm $C'$, algorithm $A$ outputs with probability $\Omega(\epsilon)$ an algorithm $C$ that computes $f$ on at least $(1 - \delta)$ fraction of the inputs.

The following simple observation will also be used in the proof.

**Proof (Proof of Theorem 2).** The proof follows from the Theorem of Impagliazzo et al. and Lemma 1 by setting $\epsilon^{4c} = (1 - 2\delta)^k$.

Note that given an algorithm $B$ as in Theorem 2, one can run the algorithm independently $l = O(1/\epsilon)$ times to output a list of $l$ algorithms $C_1, \ldots, C_l$ such that with high probability at least one of them computes $f^\oplus_k$ on at least $(1/2 + \Omega(\epsilon^{4c}))$ fraction of the inputs.

5 Conclusion and Open Problems

Our reductions meet the optimal information theoretic bounds only up to polynomial factors. One open problem is to give reductions that match the optimal bounds. Our reduction from $k$-wise XOR to $k$-wise direct product lemma is a brute-force reduction that goes through an algorithm that computes the function $f$ itself. Due to this, we get bounds that are far from the optimal. Note the presence of the constant $c$ that results from the use of the algorithm that approximately computes $f$ (this algorithm uses an algorithm that computes $f^k$ and some advice). So, an open question is whether a direct reduction from $k$-wise XOR to $k$-wise direct product lemma is possible. Note that this is what we did in the reverse direction.

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IJKW10. Russell Impagliazzo, Ragesh Jaiswal, Valentine Kabanets, and Avi Wigderson. Uniform direct product theorems: Simplified, optimized, and derandomized. *SIAM J. Comput.*, 39(4):1637–1665, January 2010.
We first give the proof of Theorem 5. For this, we will require following theorem from [IJK09].

Theorem 10 (Theorem 42 from [IJK09]). Let \( 0 \leq \delta \leq 1 \) and \( \epsilon > (1-2\delta)^{k/2} \). Let \( f_1, \ldots, f_t : \{0,1\}^n \rightarrow \{0,1\} \) be any function satisfying the following two conditions

1. For all \( i \neq j \), \( |\{x : x \in \{0,1\}^n, f_i(x) \neq f_j(x)\}| > \delta \cdot 2^n \), and
2. there is some function \( B : (\{0,1\}^n)^k \rightarrow \{0,1\} \) such that for all \( i \),

\[
Pr_{\vec{x} \in (\{0,1\}^n)^k} [B(\vec{x}) = f_i^{\otimes k}(\vec{x})] \geq 1/2 + \epsilon/2
\]

Then \( t \leq \frac{1}{\epsilon^2 (1-2\delta)^k} \).

The following simple lemma will also be used.

Lemma 7. Let \( 0 \leq \delta \leq 1/4 \) and \( \epsilon = \sqrt{2} \cdot (1-2\delta)^{k/2} \). Let \( f \) and \( g \) be two functions such that

\[
|\{x : x \in \{0,1\}^n, f(x) \neq g(x)\}| \geq \delta \cdot 2^n.
\]

Then

\[
Pr_{\vec{x} \in (\{0,1\}^n)^k} [f^{k}(\vec{x}) = g^{k}(\vec{x})] \leq \sqrt{\epsilon}.
\]

Proof. The proof follows from the simple observation that for any \( \vec{x} \), \( f^{k}(\vec{x}) \) and \( g^{k}(\vec{x}) \) will match only when none of the elements of \( \vec{x} \) are in the set \( \{x : x \in \{0,1\}^n, f(x) \neq g(x)\} \). This probability is at most \( (1-\delta)^k \leq 2^{1/4} (1-2\delta)^{1/4} = \sqrt{\epsilon} \) (since \( \delta \leq 1/4 \)).

Proof (Proof of Theorem 5). The proof follows by setting \( \epsilon = \sqrt{2} \cdot (1-2\delta)^{k/2} \) and using Lemma 7 with Theorem 10.
For proving Theorem 6 we will use the following theorem from Impagliazzo et al. [IJK09].

**Theorem 11 (Theorem 52 in [IJK09]).** Let $N = 2^n$, $m = N\binom{N}{k}$, $\epsilon > \max(\min(2^{-1/256}, 2^{-N/256}), k^2/N \leq o(\epsilon))$, and $\delta < 1/4$. There exists a function $B : \{0, 1\}^n \rightarrow \{0, 1\}$ and $t = \Omega(1/\epsilon^2)$ functions $f_1, ..., f_t$ such that the following holds:

1. For all $i \neq j$, $\text{Pr}_{x \in \{0, 1\}^n} [f_i(x) \neq f_j(x)] \geq \delta$, and
2. For all $i$, $\text{Pr}_{x \in \{0, 1\}^n} [B(x) = f_i^{\oplus k}(x)] \geq 1/2 + \Omega(\epsilon)$.

**Proof (Proof of Theorem 6).** The proof follows from the above theorem and Lemma 1 by setting $\epsilon = (1 - \delta/5)^k$.

**B  k-wise Direct Product to k-wise XOR**

The ideas of the proofs in this section have been borrowed from [IJK09, Jai08].

**Proof (Proof of Theorem 7).** We will use $\epsilon = 2 \cdot (1 - \delta)^{k/2}$. Given functions $f_1, ..., f_t$, let $B$ be the function that satisfies the conditions in the lemma. We will argue that $t \leq \frac{2}{\epsilon}$. For the sake of contradiction, assume that $t > \frac{2}{\epsilon}$. For any $i \in [t]$ let $A_{f_i} = \{\bar{x} : \bar{x} \in \{0, 1\}^n \text{ and } f_i^{\oplus k}(\bar{x}) \neq f_i^{\oplus k}(\bar{x})\}$. So, we know that $\forall i, |A_{f_i}| \geq \epsilon \cdot 2^{nk}$. We now claim that for any $i \neq j$, $\text{Pr}_{x \in \{0, 1\}^n} [f_i(x) \neq f_j(x)] > \delta$. This is because otherwise we will have $\text{Pr}_{x \in \{0, 1\}^n} [f_i^{\oplus k}(\bar{x}) = f_j^{\oplus k}(\bar{x})] \geq 1/2 + (1 - 2\delta)^k/2 \geq 1/2 + \epsilon^6/2$ (since $\epsilon > 2^{-k/12}$).

Now we claim that for any $i \neq j$, we have $|A_{f_i} \cap A_{f_j}| < (1 - \delta)^k = (\epsilon^2/4)^k \cdot 2^{nk}$. We will consider the sets $A_{f_i}$ in a sequence. For any $j \leq 2/\epsilon$, we have $|A_{f_j} \cap (A_{f_1} \cup ... \cup A_{f_{j-1}})| < \frac{2}{\epsilon} \cdot \frac{2}{\epsilon} \cdot 2^{nk} < \frac{2}{\epsilon} \cdot 2^{nk}$. On the other hand, we know that $\forall j, |A_{f_j}| \geq \epsilon \cdot 2^{nk}$. So, each $A_{f_j}$ contains at least $\epsilon/2$ fraction of elements that are not contained in $A_{f_1} \cup ... \cup A_{f_{j-1}}$. So, $|A_{f_1} \cup ... \cup A_{f_{2k}}| > 2^{nk}$ which gives a contradiction.

**Proof (Proof of Theorem 8).** Let $N = 2^n >> k$. For any $\delta < 1/2$, there exists at least $l = 2^{N/16}$ functions $g_1, ..., g_l$ such that for any $i \neq j$, $|\{x : g_i(x) \neq g_j(x)\}| \geq (\delta/2) \cdot N$. This means that for all $i \neq j$, $\text{Pr}_{x \in \{0, 1\}^n} [g_i^{\oplus k}(\bar{x}) = g_j^{\oplus k}(\bar{x})] \leq \frac{1}{2} + (1-\delta)^k$. We construct $B$ in the following manner: Since $\epsilon > 2^{-k/4}$ we have $t = 1/\epsilon < l$. Consider $f_1 = g_1, f_2 = g_2, ..., f_t = g_t$. We partition the set $(\{0, 1\}^n)^k$ into $t$ equal size subsets. For any tuple $\bar{x}$ in the $i^{th}$ subset we set $B(\bar{x}) = f_i^{\oplus k}(\bar{x})$. Finally, the theorem follows by setting $\epsilon = (1 - \delta)^{k/12}$.