K-THEORY OF C*-ALGEBRAS FROM ONE-DIMENSIONAL GENERALIZED SOLENOIDS

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Abstract. We compute the $K$-groups of $C^*$-algebras arising from one-dimensional generalized solenoids. The results show that Ruelle algebras from one-dimensional generalized solenoids are one-dimensional generalizations of Cuntz-Krieger algebras.

1. Introduction

Ian Putnam and David Ruelle have developed a theory of $C^*$-algebras for certain hyperbolic dynamical systems ([16, 17, 18, 21]). These systems include Anosov diffeomorphisms, topological Markov chains and some examples of substitution tiling systems. The corresponding $C^*$-algebras are modeled as reduced groupoid $C^*$-algebras for various equivalence relations.

This paper is concerned with $C^*$-algebras of an orientable one-dimensional generalized solenoid $(X, f)$, where $X$ has local canonical coordinates which are contracting and expanding directions for $f$. Naively speaking, Williams’s orientable generalized solenoids are higher dimensional analogues of topological Markov chains ([23, 24]). We consider the principal groupoids of stable and unstable equivalence on $(X, f)$, denoted $G_s(X, f)$ and $G_u(X, f)$, respectively. We give them topologies and Haar systems ([16, 17]) so that we may build their reduced groupoid $C^*$-algebras $S(X, f)$ and $U(X, f)$, respectively, as in [19]. The homeomorphism $f: X \to X$ induces automorphism of $G_s(X, f)$ and $G_u(X, f)$, and we form semi-direct products $G_s \rtimes \mathbb{Z}$ and $G_u \rtimes \mathbb{Z}$. Their groupoid $C^*$-algebras are denoted $R_s(X, f)$ and $R_u(X, f)$, respectively, and are called the Ruelle algebras ([17, 18]). In the case of topological Markov chains, the Ruelle algebras are the Cuntz-Krieger algebras, and the stable and unstable equivalence algebras are the corresponding $AF$-subalgebras of the Cuntz-Krieger algebras.

An important tool in the study of $C^*$-algebras is $K$-theory. Giordano, Herman, Putnam and Skau showed that almost complete information about the orbit structure of Cantor systems is encoded by the $K$-theory of their associated $C^*$-algebras ([5, 6]). And Kirchberg and Phillips showed in their recent papers ([8, 14]) that nuclear, purely infinite, separable, simple $C^*$-algebras are classified by their $K$-theory.

In this paper, we compute the $K$-groups of the unstable equivalence algebras and the Ruelle algebras of 1-solenoids to answer the questions posed in [17, §4]. We show that the unstable equivalence algebra of a 1-solenoid $(X, f)$ with an adjacency matrix $M$ is strongly Morita equivalent to the crossed product of a natural Cantor system of $(X, f)$ by $\mathbb{Z}$ so that its $K_0$-group is order isomorphic to the dimension group of $M$ and its $K_1$-group is $\mathbb{Z}$. Then we use the Pimsner-Voiculescu exact
sequence, the Universal Coefficient Theorem and Spanier-Whitehead duality to obtain that the $K_0$-groups of Ruelle algebras are isomorphic to $\mathbb{Z} \oplus \{ \Delta_M / \text{Im}(\text{Id} - \delta_M) \}$ and the $K_1$-groups are $\mathbb{Z} \oplus \text{Ker}(\text{Id} - \delta_M)$. Thus $C^*$-algebras from one-dimensional generalized solenoids are one-dimensional analogues of the Cuntz-Krieger algebras.

The outline of the paper is as follow: In section 2, we recall the axioms of one-dimensional generalized solenoids and their ordered group invariants. In section 3, we review the definitions of Smale spaces, and show that orientable one-dimensional solenoids are Smale spaces. Then we observe that the stable equivalence algebras are strongly Morita equivalent to inductive limit systems of $C^*$-algebras, and that the $K$-theory of the unstable equivalence algebras are determined by the adjacency matrices of one-dimensional generalized solenoids. In section 4, we compute $K$-groups of unstable and stable Ruelle algebras, and show that they are $*$-isomorphic to each other by the classification theorem of Kirchberg-Phillips.

2. ONE-DIMENSIONAL SOLENOIDS

We review the properties of one-dimensional generalized solenoids of Williams which will be used in later sections. As general references for the notions of one-dimensional generalized solenoids and their ordered group invariants we refer to [23, 24, 25, 26].

One-dimensional generalized solenoids. Let $X$ be a finite directed graph with vertex set $V$ and edge set $E$, and $f : X \to X$ a continuous map. We define some axioms which might be satisfied by $(X, f)$ ([25]).

Axiom 0. (Indecomposability) $(X, f)$ is indecomposable.
Axiom 1. (Nonwandering) All points of $X$ are nonwandering under $f$.
Axiom 2. (Flattening) There is $k \geq 1$ such that for all $x \in X$ there is an open neighborhood $U$ of $x$ such that $f^k(U)$ is homeomorphic to $(-\epsilon, \epsilon)$.
Axiom 3. (Expansion) There are a metric $d$ compatible with the topology and positive constants $C$ and $\lambda$ with $\lambda > 1$ such that for all $n > 0$ and all points $x, y$ on a common edge of $X$, if $f^n$ maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \geq C\lambda^n d(x, y)$.
Axiom 4. (Nonfolding) $f^n|_{X - V}$ is locally one-to-one for every positive integer $n$.
Axiom 5. (Markov) $f(V) \subseteq V$.

Let $\overline{X}$ be the inverse limit space

$$\overline{X} = X \xleftarrow{f} X \xleftarrow{f} \cdots = \{ (x_0, x_1, x_2, \ldots) \in \prod_{0}^{\infty} X | f(x_{n+1}) = x_n \},$$

and $\overline{f} : \overline{X} \to \overline{X}$ the induced homeomorphism defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (f(x_0), f(x_1), f(x_2), \ldots) = (f(x_0), x_0, x_1, \ldots).$$

Remark 2.1. Williams’ construction ([24, 6.2]) gives a (unique) measure $\mu_0$ for which there is a constant $\lambda > 1$ such that $\mu_0(X) = 1$ and $\mu_0(f(I)) = \lambda \mu_0(I)$ for every small interval $I \subset X$. Define $d_0(x_0, y_0)$ to be the measure of the smallest interval from $x_0$ to $y_0$ in $X$, and

$$d(x, y) = \sum_{i=0}^{\infty} \lambda^{-i} d_0(x_i, y_i).$$
for $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$ in $\overline{X}$. Then $(\overline{X}, d)$ is a compact metric space.

Let $Y$ be a topological space and $g: Y \to Y$ a homeomorphism. We call $Y$ a 1-dimensional generalized solenoid or 1-solenoid and $g$ a solenoid map if there exist a directed graph $X$ and a continuous map $f: X \to X$ such that $(X, f)$ satisfies all six Axioms and $(\overline{X}, f)$ is topologically conjugate to $(Y, g)$. We call a point $x \in X$ a non-branch point if $x$ has an open neighborhood which is homeomorphic to an open interval, and branch point otherwise. An elementary presentation $(X, f)$ of a 1-solenoid is such that $X$ is a wedge of circles and $f$ leaves the unique branch point of $X$ fixed.

**Proposition 2.2** ([24, 5.2]). For each 1-solenoid $(\overline{X}, f)$, there exists an integer $m$ such that $(\overline{X}, f^m)$ has an elementary presentation.

Suppose that $(X, f)$ is a presentation of a 1-solenoid. Since the inverse limit spaces of $(X, f)$ and $(X, f^n)$ are homeomorphic ([1]) for every positive integer $n$, for the purpose of computing invariants of the space $X$ there is no loss of generality in replacing $(X, f)$ with $(X, f^n)$ where $n = m \cdot k$ is a positive integer such that $(\overline{X}, f^m)$ has an elementary presentation $(Y, g)$ and for every $y \in Y$ there is an open set $U_y$ such that $g^k(U_y)$ is an open interval. Hence we can assume that every point $x \in X$ has a neighborhood $U_x$ such that $f(U_x)$ is an interval.

Recall that a continuous map $\gamma: [0, 1] \to G$, a directed graph, is orientation preserving if $e^{-1} \circ \gamma: I \to [0, 1]$ is increasing for every interval $I \subset [0, 1]$ such that $\gamma(I)$ is a subset of a directed edge $e$. A continuous map $\phi: G_1 \to G_2$ between two directed graphs is orientation preserving if, for every orientation preserving map $p: [0, 1] \to G_1$, the map $\phi \circ p: [0, 1] \to G_2$ is orientation preserving ([24]).

When we can give a direction to each edge of $X$ so that the connection map $f: X \to X$ is orientation preserving, we call $(X, f)$ an orientable presentation. For a 1-solenoid $Y$ with a solenoid map $g$, if there exists an orientable presentation $(X, f)$ then $Y$ is called an orientable 1-solenoid.

**Standing Assumption.** In this paper, we always assume that $(X, f)$ is an orientable elementary presentation such that every point $x \in X$ has a neighborhood $U_x$ such that $f(U_x)$ is an interval.

**Notation 2.3.** Suppose that $(X, f)$ is a presentation of a 1-solenoid, and that $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the edge set of the directed graph $X$. For each edge $e_i \in \mathcal{E}$, we can give $e_i$ the partition $\{I_{i,j}\}$, $1 \leq j \leq l(i)$, such that

1. the initial point of $I_{i,1}$ is the initial point of $e_i$,
2. the terminal point of $I_{i,j}$ is the initial point of $I_{i,j+1}$ for $1 \leq j < l(i)$,
3. the terminal point of $I_{i,l(i)}$ is the terminal point of $e_i$,
4. $f|_{\text{Int} I_{i,j}}$ is injective, and
5. $f(I_{i,j}) = e_{i,j}^{s(i,j)}$ where $e_{i,j} \in \mathcal{E}$, $s(i,j) = 1$ if the direction of $f(I_{i,j})$ agree with that of $e_{i,j}$, and $s(i,j) = -1$ if the direction of $f(I_{i,j})$ is reverse to that of $e_{i,j}$.

The wrapping rule $\tilde{f}: \mathcal{E} \to \mathcal{E}^*$ associated with $f$ is given by

\[
\tilde{f}: e_i \mapsto e_{i,1}^{s(i,1)} \cdots e_{i,l(i)}^{s(i,l(i))},
\]
and the adjacency matrix $M$ of $(\mathcal{E}, \hat{f})$ is given by
$$M(i, k) = \#\{I_{i,j} \mid f(I_{i,j}) = e_k^{-1}\}.$$

**Remark 2.4** ([24, 6.2]). The measure $\mu_0$ in remark 2.1 is given as follows: Suppose that $\lambda$ is the Perron-Frobenius eigenvalue of the adjacency matrix $M$ and that $v = (v_1, \ldots, v_n)$ is the corresponding Perron eigenvector such that $\sum_{i=1}^n v_i = 1$. For edges $e_i, e_j$ of $X$ and an interval $I$ of $e_i$ such that $f^n(I) = e_j$ and $f^n|_{\text{Int}I}$ is injective, let
$$\mu_0(e_i) = v_i \text{ and } \mu_0(I) = \lambda^{-n}v_j.$$ Then $\mu_0$ is extended to a regular Borel measure on $X$ by the standard procedure.

**Theorem 2.5** ([11, 27]). Suppose that $(\overline{X}, \overline{f})$ is a 1-solenoid. Then there exists a uniquely ergodic flow $\phi$ whose phase space is $\overline{X}$.

Suppose that $(X, f)$ is a presentation of a 1-solenoid and that $\mu_0$ is the measure given on $X$ as in remark 2.4. For a measurable set $I$ in $X$, we let $U_n(I) = \{(x_0, \ldots, x_n, \ldots) \in \overline{X} \mid x_n \in I\}$, and define a measure $\mu$ on $\overline{X}$ by
$$\mu(U_n(I)) = \mu_0(I).$$ Then $\mu$ is extended to a regular Borel measure on $\overline{X}$ in the standard way. We call this measure **Williams measure** of the flow $\phi$ on $\overline{X}$. It is not difficult to verify that $\mu$ is the unique $\phi$-invariant measure on $\overline{X}$.

A closed subset $K$ of a phase space $Y$ of a flow $\phi$ is called a **cross section** if the mapping $\phi: K \times \mathbb{R} \to Y$ defined by $(p, t) \mapsto p \cdot t$ is a local homeomorphism onto $Y$. The **return time map** $r_K: K \to K$ of a cross section $K$ is defined by $x \mapsto y = x \cdot t_x$ where $x \in K$ and $t_x$ is the smallest positive number such that $x \cdot t_x = y \in K$.

**Theorem 2.6** ([10, 2]). Suppose that $(\overline{X}, \overline{f})$ is a 1-solenoid with the corresponding adjacency matrix $M$, and that $(K, r_K)$ is a cross section with the return time map of $\overline{X}$. Then
1. $K_1(C(K) \times_{r_K} \mathbb{Z}) = \mathbb{Z}$,
2. $K_0(C(K) \times_{r_K} \mathbb{Z})$ is order isomorphic to $\Delta_M$, and
3. $K_0(C(K) \times_{r_K} \mathbb{Z})$ has a unique state.

### 3. Smale spaces and C*-algebras from solenoids

**Smale spaces** ([16, 21]). Suppose that $(Y, d)$ is a compact metric space and $\phi$ is a homeomorphism of $Y$. Assume that we have constants
$$0 < \lambda_0 < 1, \; \epsilon_0 > 0$$
and a continuous map
$$(x, y) \in \{(x, y) \in Y \times Y \mid d(x, y) \leq 2\epsilon_0\} \mapsto [x, y] \in Y$$
satisfying the following:
$$[x, x] = x, \quad [[x, y], z] = [x, z], \quad [x, [y, z]] = [x, z], \quad [\phi(x), \phi(y)] = \phi([x, y])$$
for $x, y, z \in Y$ whenever both sides of the equation are defined. For every $0 < \epsilon \leq \epsilon_0$ let
$$V^s(x, \epsilon) = \{y \in Y \mid [x, y] = y \text{ and } d(x, y) < \epsilon\}$$
$$V^u(x, \epsilon) = \{y \in Y \mid [y, x] = y \text{ and } d(x, y) < \epsilon\}.$$
We assume that
\[
d(\varphi(y), \varphi(z)) \leq \lambda_0 d(y, z) \quad y, z \in V^s(x, \epsilon),
\]
\[
d(\varphi^{-1}(y), \varphi^{-1}(z)) \leq \lambda_0 d(y, z) \quad y, z \in V^u(x, \epsilon).
\]

Then \((Y, d, \varphi)\) is called a **Smale space**.

**Groupoids** ([14, 15]). We refer to the work of Renault ([19]) for the detailed theory of topological groupoids and their associated \(C^*\)-algebras. We give two examples of groupoids.

**Examples 3.1** ([11, 1.2]). (1) **Equivalence relations.** Suppose that \(R\) is an equivalence relation on a set \(S\). We give \(R\) the following groupoid structure:
\[
(s_1, t_1) \cdot (s_2, t_2) = (s_1, t_2) \text{ if } t_1 = s_2 \text{ and } (s, t)^{-1} = (t, s).
\]

(2) **Flows.** Suppose that \(S\) is a zero dimensional space and \(r: S \to S\) is a homeomorphism. We consider the space \(S \times \mathbb{R}\) with the equivalence relation, \((s, \tau + 1) \sim (r(s), \tau)\). Let \(\Sigma = S \times \mathbb{R}/\sim\) be the quotient space and define a flow \(\phi: \Sigma \times \mathbb{R} \to \Sigma\) by \(\phi_t(s, \tau) = [(s, t + \tau)]\). Give the following groupoid structure on \(\Sigma \times \mathbb{R}\):
\[
(\sigma_1, t_1) \cdot (\sigma_2, t_2) = (\sigma_1, t_1 + t_2) \text{ if } \sigma_2 = \phi_t(\sigma_1) \text{ and } (\sigma, t)^{-1} = (\phi_t(\sigma), -t).
\]

For a Smale space \((Y, d, \varphi)\), define
\[
G_{s,0} = \{(x, y) \in Y \times Y \mid y \in V^s(x, \epsilon_0)\} \quad G_{u,0} = \{(x, y) \in Y \times Y \mid y \in V^u(x, \epsilon_0)\}
\]
and let
\[
G_s = \bigcup_{n=0}^{\infty} (\varphi \times \varphi)^{-n} (G_{s,0}) \quad G_u = \bigcup_{n=0}^{\infty} (\varphi \times \varphi)^n (G_{u,0}).
\]

Then \(G_s\) and \(G_u\) are equivalence relations on \(Y\), called **stable** and **unstable equivalence**. Each \((\varphi \times \varphi)^{-n} (G_{s,0})\), \((\varphi \times \varphi)^n (G_{u,0})\) is given the relative topology of \(Y \times Y\), and \(G_s\) and \(G_u\) are given the inductive limit topology. Then \(G_s\) and \(G_u\) are locally compact Hausdorff principal groupoids. The Haar systems \(\mu^s_x \mid x \in Y\) and \(\mu^u_x \mid x \in Y\) for \(G_s\) and \(G_u\), respectively, are described in [17, 3.c]. The groupoid \(C^*\)-algebras of \(G_s\) and \(G_u\) are denoted \(S(Y, \varphi)\) and \(U(Y, \varphi)\), respectively.

The map \(\varphi \times \varphi\) acts as an automorphism of \(G_s\) and \(G_u\). We form the semi-direct products
\[
G_s \rtimes \mathbb{Z} = \{(x, n, y) \mid n \in \mathbb{Z} \text{ and } (\mathcal{F}^n(x), y) \in G_s\}
\]
\[
G_u \rtimes \mathbb{Z} = \{(x, n, y) \mid n \in \mathbb{Z} \text{ and } (\mathcal{F}^n(x), y) \in G_u\}
\]
with groupoid operations
\[
(x, n, y) \cdot (u, m, v) = (x, n + m, v) \text{ if } y = u \text{ and } (x, n, y)^{-1} = (y, -n, x).
\]

The product topology of \(G_s \times \mathbb{Z}\) is transferred to \(G_s \rtimes \mathbb{Z}\) by the bijective map \(\eta: (x, y, n) \mapsto (x, n, \varphi(y))\). And a Haar system on \(G_s \rtimes \mathbb{Z}\) is given by \(\mu^s_x \circ \eta^{-1}\) where \(\mu^s_x\) is the Haar system on \(G_s\). The groupoid \(C^*\)-algebras \(C^*(G_s \rtimes \mathbb{Z})\) and \(C^*(G_u \rtimes \mathbb{Z})\) are denoted \(R_s(Y, \varphi)\) and \(R_u(Y, \varphi)\) and are called the **Ruelle algebras**.
Theorem 3.2 (10, 13, 17). Suppose that \((Y, \varphi)\) is a topologically mixing Smale space. Then

1. \(S(Y, \varphi)\) and \(U(Y, \varphi)\) are amenable, nuclear, separable and simple \(C^*\)-algebras, and
2. \(R_s(Y, \varphi)\) and \(R_u(Y, \varphi)\) are amenable, non-unital, nuclear, purely infinite, separable, simple and stable \(C^*\)-algebras.

For general properties of these \(C^*\)-algebras, we refer to 10, 13, 18.

Suppose that \((\overline{\mathcal{X}}, \overline{f})\) is a 1-solenoid with the metric \(d\) given in remark 2.7. Let \(\lambda_0 = \epsilon_0 = \frac{k}{n}\) and define \([\cdot, \cdot]: \mathcal{X} \times \mathcal{X} \to [0, \infty)\) by \(d(x, y) = z\) where \(z_0 = x_0\) and \(z_n\) is the unique element contained in the \(\lambda_0^n\)-neighborhood of \(y_0\) such that \(f^n(z_n) = x_0\). Then it is not difficult to show that \((\overline{\mathcal{X}}, \overline{f}, d)\) satisfies the above conditions. Therefore we have the following:

Proposition 3.3. One-dimensional generalized solenoids are Smale spaces.

Stable equivalence algebras for 1-solenoids. Suppose that \(G_s\) is the stable equivalence groupoid of a 1-solenoid \((\overline{\mathcal{X}}, \overline{f})\) and that \(S(\overline{\mathcal{X}}, \overline{f})\) is the corresponding groupoid algebra. We first repeat the structural question of Putnam (17, §4). For classical 1-solenoids, we refer to 2, 10.

Question. Can \(S(\overline{\mathcal{X}}, \overline{f})\) be written as an inductive limit?

Solenoid \((\overline{\mathcal{X}}, \overline{f})\) of \(X\). Let \(U^*_s = \{x \in U_{p} \mid x = f^n(g_p) \text{ for some } n \in \mathbb{N}\} \cap U_p\). Then \(U^*_s\) is an \(r\)-discrete, second countable, locally compact, Hausdorff groupoid with counting measure as Haar system.

Proposition 3.4 (18, §3). (1) \(G_s(p)\) is an \(r\)-discrete, second countable, locally compact, Hausdorff groupoid with counting measure as Haar system.

(2) \(S(\overline{\mathcal{X}}, \overline{f})\) is strongly Morita equivalent to \(C^*(G_s(p))\).

Now we choose \(p\) to be a fixed point of \(\overline{f}\) such that \(\pi_k(p)\) is contained in the interior of an edge \(e \in \mathcal{E}\). Since the orbits of \((\overline{\mathcal{X}}, \overline{f})\) are determined by the cofinality relation, \(x = (x_0, x_1, \ldots) \in U_p\) if and only if there is a positive integer \(n = n(x)\) such that \(x_k \in e\) for every \(k \geq n\). Then \((\overline{f} \times \overline{f}) (G_s(p)) = G_s(p)\). Let

\[
G_{s, n}(p) = \{(x, y) \in G_{s, n} \mid x, y \in U_p\} = \{(x, y) \in G_s(p) \mid f^n(x_0) = f^n(y_0)\}.
\]

Then \(G_{s, n}(p)\) is a compact open subset of \(G_s(p)\), and \(G_{s, n}(p)^0 = G_s(p)^0 = g(U_p)\). Since \(G_s(p)\) is \(r\)-discrete, the range maps \(r: G_s(p) \to G_s(p)^0\) and \(r_n = r|_{G_{s, n}(p)}\) are local homeomorphisms. Hence the Haar system of \(G_s(p)\) restricted to \(G_{s, n}(p)\) gives a Haar system for each \(G_{s, n}(p)\). Then we can express \(C^*(G_s(p))\) as an inductive limit

\[
C^*(G_{s, 1}(p)) \to C^*(G_{s, 2}(p)) \to \cdots \to C^*(G_{s, n}(p)) \to \cdots.
\]
Unstable equivalence algebras. Suppose that \((\overline{X}, \overline{f})\) is an orientable solenoid and that \(\phi\) is the flow on \(\overline{X}\) given in theorem \ref{thm:SolenoidFlow}. Then there exists a cross section with return time map \((K, r)\) such that \(\overline{X}\) is the suspension space of \((K, r)\).

**Lemma 3.5** \([\ref{ref:author2}, \text{p.59}]\). The \(C^*\)-algebra of \((\overline{X}, \mathbb{R}, \phi)\) is isomorphic to \(C(\overline{X}) \times \mathcal{R}\mathbb{R}\).

**Proposition 3.6** \([\ref{ref:author2}, \ref{ref:author7}]\). Suppose that \((\overline{X}, \overline{f})\) is an orientable solenoid, and that \((Z, r)\) is a cross section with the return time map of the flow \(\phi\). Then

1. \(U(\overline{X}, \overline{f}) \simeq C(\overline{X}) \times \mathcal{R}\mathbb{R}\) and
2. \(C(\overline{X}) \times \mathcal{R}\mathbb{R}\) is strongly Morita equivalent to \(C(K) \times \mathbb{Z}\).

**Proof.** (1). Suppose \(x = (x_0, x_1, \ldots), y = (y_0, y_1, \ldots) \in \overline{X}\) and \((x, y) \in G_u\). Then
\[
\lim_{n \to -\infty} \left( \overline{f}^n(x), \overline{f}^n(y) \right) \to 0 \quad \text{as} \quad n \to -\infty \implies d_0(x_n, y_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{and that there exists a} \ t \in \mathbb{R} \quad \text{such that} \quad y = \phi_t(x).
\]
Let \(\alpha: (\overline{X}, \mathbb{R}, \phi) \to G_u\) be given by \((x, t) \mapsto (x, \phi_t(x))\). Then it is not difficult to see that \(\alpha\) is an isomorphism. Therefore \(U(\overline{X}, \overline{f})\) is isomorphic to \(C(\overline{X}) \times \mathcal{R}\mathbb{R}\) by lemma \ref{lem:isomorphism}. And by the same argument \(C(K) \times \mathbb{Z}\) is isomorphic to the groupoid \(C^*\)-algebra of \((K, Z, r)\).

(2). Since \(\overline{X}\) is the suspension of \((K, r)\), for every \(x \in \overline{X}\) there exist unique \(z_x \in K\) and \(\tau_x \in [0, 1]\) such that \(x = \phi_{\tau_x}(z_x)\). Define \(I = \{(x, n - \tau_x) \mid x \in \overline{X}, n \in \mathbb{Z}\}\), and let \(C(I)\) be the completion of \(C_c(I)\). Then by the Theorem in \([\ref{ref:author7}, \text{§4.a}]\) \(C(I)\) is a \(C(\overline{X}) \times \mathcal{R}\mathbb{R}\)-\(C(K) \times \mathbb{Z}\) imprimitivity bimodule. For completeness, we write down the module structures and the inner products.

**Module structures.** Suppose that \(\alpha \in C_c(I), g \in C_c(\overline{X}, \mathbb{R}, \phi)\) and \(h \in C_c(K, \mathbb{Z}, r)\). Then
\[
(g \cdot \alpha)(x, n - \tau_x) = \int g(x, t) \cdot \alpha(\phi_t(x), n - \tau_x - t) \, d\mu[^x](t) \quad \text{and}
\]
\[
(\alpha \cdot h)(x, n - \tau_x) = \sum_m \alpha(x, m - \tau_x) \cdot h(r^m(z_x), n - m)
\]
give that \(C(I)\) is a left \(C(\overline{X}) \times \mathcal{R}\mathbb{R}\) and right \(C(K) \times \mathbb{Z}\) bimodule with \((\hat{g} \cdot \hat{\alpha}) \cdot \hat{h} = \hat{g} \cdot (\hat{\alpha} \cdot \hat{h})\) for every \(\hat{\alpha} \in C(I), \hat{g} \in C(\overline{X}) \times \mathcal{R}\mathbb{R}\) and \(\hat{h} \in C(K) \times \mathbb{Z}\).

**Inner products.** Define \(\langle \cdot, \cdot \rangle_L: C_c(I) \times C_c(I) \to C_c(\overline{X}, \mathbb{R}, \phi)\) and \(\langle \cdot, \cdot \rangle_R: C(I) \times C(I) \to C_c(K, \mathbb{Z}, r)\) by
\[
\langle \alpha, \beta \rangle_L(x, t) = \sum \alpha(x, m - \tau_x) \cdot \beta(x, m - \tau_x) \quad \text{and}
\]
\[
\langle \alpha, \beta \rangle_R(z, k) = \int \alpha(\phi_t(z), k - t) \cdot \beta(\phi_t(z), k - t) \, d\mu[^\phi](z)(t).
\]

Then we have the following corollary from propositions \ref{prop:isomorphism}.

**Corollary 3.7** \([\ref{ref:author2}, \ref{ref:author23}]\). (1) \(U(\overline{X}, \overline{f})\) is a simple \(C^*\)-algebra.

(2) \(K_1(U(\overline{X}, \overline{f})) = \mathbb{Z}\).

(3) \(K_0(U(\overline{X}, \overline{f}))\) is order isomorphic to \(\Delta_M\) where \(M\) is the adjacency matrix of \((\overline{X}, \overline{f})\).

Recall that the flow \(\phi\) on \(\overline{X}\) is uniquely ergodic without rest point (theorem \ref{thm:ergodic}). So \(C(\overline{X}) \times \mathcal{R}\mathbb{R}\) has the unique trace \(\tau_u\) induced by the Williams measure \(\mu\) \((\ref{ref:author23}, \text{3.3.10})\). Thus \(\tau_u^*\), the induced state on \(K_0(C(\overline{X}) \times \mathcal{R}\mathbb{R})\), is the unique state.
Proposition 3.8. Suppose that \((\mathcal{X}, \mathcal{J})\) is a 1-solenoid and that \(M\) is the corresponding adjacency matrix with the normalized Perron eigenvector \(v = (v_1, \ldots, v_n)\). Then
\[
\tau_\mu^*(K_0(U(\mathcal{X}, \mathcal{J})), K_0(U(\mathcal{X}, \mathcal{J})))_+ = \langle (\Delta_M, \Delta_M^+), v \rangle.
\]

Proof. Suppose that \(\mathcal{E}_k = \mathcal{E}\) is the edge set of the \(k\)th coordinate space of \(\mathcal{X}\). Then by proposition 2.6
\[
(K_0(U(\mathcal{X}, \mathcal{J})), K_0(U(\mathcal{X}, \mathcal{J})))_+ \cong \left( \lim_{\rightarrow} C(\mathcal{E}_k, \mathbb{Z}), \lim_{\rightarrow} C_+ (\mathcal{E}_k, \mathbb{Z}) \right) \cong (\Delta_M, \Delta_M^+).
\]

For \(g \in C(\mathcal{E}_k, \mathbb{Z})\), \(x = (x_0, \ldots, x_k, \ldots) \in \mathcal{X}\) with \(x_k = e^{2\pi i s} \in e_i \in \mathcal{E}_k\) and the canonical projection to the \(k\)th coordinate space \(\pi_k: \mathcal{X} \to X\), define \(g_k \in C(X_k, S^1)\) and \(\tilde{g} \in C(X, S^1)\) by
\[
g_k: x_k \mapsto \exp(2\pi i g_i) s \quad \text{and} \quad \tilde{g}: x \mapsto g_k \circ \pi_k(x).
\]

Then every \(\tilde{g}\) is a unitary element in \(C(\mathcal{X})\), and \(K_0(U(\mathcal{X}, \mathcal{J})) \cong K_1(C(\mathcal{X}))\) is generated by \(\tilde{g}\). If we denote \(g = (g(e_1), \ldots, g(e_n))\), then by Theorem 2.2 of [13]
\[
\tau_\mu^*(\tilde{g}) = \frac{1}{2\pi i} \int_{\mathcal{X}} \tilde{g}' \overline{\tilde{g}} \, d\mu = \int_{X_k} g' \overline{g} \, d\mu_0 = \sum_{i=1}^{n} g(e_i) \mu_0(e_i) = \sum_{i=1}^{n} g(e_i) v_i = \langle (g(e_1), \ldots, g(e_n)), v \rangle.
\]

The above proposition refines Theorem 2.2 of [13] that
\[
\tau_\mu^*(K_0(C(\mathcal{X}) \times_\phi \mathbb{R})) = \langle A_\mu, H^1(\mathcal{X}) \rangle.
\]

Corollary 3.9 ([13]). If \(p\) and \(q\) are projections in \(M_\infty (C(\mathcal{X}) \times_\phi \mathbb{R})\) such that \(\tau_\mu(p) < \tau_\mu(q)\), then \(p\) is equivalent to a subprojection of \(q\).

Lemma 3.10 ([13]). \(C(\mathcal{K}) \times_\phi \mathbb{Z}\) has real rank zero and topological stable rank one.

Since \(C(\mathcal{X}) \times_\phi \mathbb{R}\) and \(C(\mathcal{K}) \times_\phi \mathbb{Z}\) are separable algebras, they have strictly positive elements. So strong Morita equivalence of \(C(\mathcal{X}) \times_\phi \mathbb{R}\) and \(C(\mathcal{K}) \times_\phi \mathbb{Z}\) implies that they are stably isomorphic, i.e., \(C(\mathcal{X}) \times_\phi \mathbb{R} \otimes \mathcal{K}\) is \(*\)-isomorphic to \(C(\mathcal{K}) \otimes \mathcal{K}\) where \(\mathcal{K}\) is the algebra of compact operators on a separable Hilbert space. Therefore we have the following proposition.

Proposition 3.11. \(U(\mathcal{X}, \mathcal{J})\) has real rank zero and topological stable rank one.

4. Ruelle Algebras for Solenoids

We compute \(K\)-groups of Ruelle algebras for 1-solenoids to show that they are \(*\)-isomorphic.

Unstable equivalence Ruelle algebras. Suppose that \((\mathcal{X}, \mathcal{J})\) is an oriented 1-solenoid and that \(G_u \simeq (\mathcal{X}, \mathbb{R}, \phi)\) is the unstable equivalence groupoid on \(\mathcal{X}\). Recall that for \(x, y \in \mathcal{X}\) such that \(y = \phi_t(x), \ t \in \mathbb{R}\), we have \(\mathcal{J}^{-1}(y) = \phi_{t\lambda^{-1}} \circ \mathcal{J}^{-1}(x)\).
Definition 4.1 (\[14, \S 4\]). Let \(\alpha_u\) be an automorphism on \(U(X, f)\) defined by
\[
\alpha_u(g)(x, t) = \lambda^{-1} g(f^{-1}(x), t\lambda^{-1}) \quad \text{for} \quad g \in C_c(X, \mathbb{R}, \phi) \quad \text{and} \quad (x, t) \in (X, \mathbb{R}).
\]
The unstable equivalence Ruelle algebra \(R_u(X, f)\) is the crossed product
\[
R_u(X, f) = U(X, f) \rtimes_{\alpha_u} \mathbb{Z} = (C(X) \times_{\phi} \mathbb{R}) \times_{\alpha_u} \mathbb{Z}.
\]

Remarks 4.2. (1) Let \(A\) be an \(n \times n\) integer matrix and \(\Delta_A\) the dimension group of \(A\). The dimension group automorphism \(\delta_A\) of \(A\) is the restriction of \(\delta_A\) to \(A\) so that \(\delta_A(v) = Av\) (\[14, 7.5.1\]). Then \(\Delta_A/\text{Im}(\text{Id} - \delta_A)\) is isomorphic to \(\mathbb{Z}^n/(\text{Id} - A)\mathbb{Z}^n\).

(2) For \(g \in C(\mathcal{E}_k, \mathbb{Z})\), let \(g_k \in C(X_k, S^1)\) be as in the proof of proposition 3.8. The wrapping rule \(f: \mathcal{E}_{k+1} \to \mathcal{E}_k\) induces a map \(f^*: C(\mathcal{E}_k, \mathbb{Z}) \to C(\mathcal{E}_{k+1}, \mathbb{Z})\) by \(g \mapsto g \circ f\) where \((g \circ f)(e) = \sum_{i=1}^j g(e_i)\) such that \(f(e) = e_1 \cdots e_j\). Then \(g_k \circ f \circ \pi_k\) is homotopic to \((g \circ f^*)_{k+1} \circ \pi_{k+1}\) (\[14, 3.6\]).

Proposition 4.3. Suppose that \((X, f)\) is a 1-solenoid with the adjacency matrix \(M\) and corresponding dimension group automorphism \(\delta_M\). Then
\[
K_0(R_u(X, f)) \cong \mathbb{Z} \oplus \{\Delta_M/\text{Im}(\text{Id} - \delta_M)\} \quad \text{and} \quad K_1(R_u(X, f)) \cong \mathbb{Z} \oplus \ker(\text{Id} - \delta_M).
\]

Proof. We have the following Pimsner-Voiculescu exact sequence.
\[
K_0(U(X, f)) \xrightarrow{1-\alpha_u} K_0(U(X, f)) \xrightarrow{\iota_*} K_0(R_u(X, f)) \xrightarrow{1-\alpha_u} K_1(U(X, f)) \xrightarrow{\iota_*} K_1(R_u(X, f))
\]

We consider \(\alpha_u: K_0(U(X, f)) = K_0(C(X) \times_{\phi} \mathbb{R}) \to K_0(C(X) \times_{\phi} \mathbb{R})\) as the automorphism \(\delta_u: K_1(C(X)) \to K_1(C(X))\) given by the Thom isomorphism of Connes. Define \(\beta: C(X) \to C(X)\) by \(h \mapsto h \circ f^{-1}\) for \(h \in C(X)\). Then the induced automorphism \(\beta_*: K_1(C(X)) \to K_1(C(X))\) is the required isomorphism.

For \(g \in C(\mathcal{E}_k, \mathbb{Z})\), let \(\hat{g} \in C(X, S^1)\) be the induced unitary element as in the proof of proposition 3.8. Then \(\beta^{-1}(\hat{g}) = \hat{g} \circ f^{-1} = g_k \circ \pi_k \circ f = g_k \circ f \circ \pi_k\) is homotopic to \((g \circ f^*)_{k+1} \circ \pi_{k+1}\). Hence if we denote \(g\) as \((g(e_1), \ldots, g(e_n))\) \(\in \mathbb{Z}^n\), then \(g \circ f^*\) is given by \(Mg\) and the induced automorphism \(\beta_*^{-1}: K_1(C(X)) \to K_1(C(X))\) is the dimension group automorphism \(\delta_M\) of the adjacency matrix \(M\). Therefore \(\beta_*\) is the inverse of \(\delta_M\), and \(1 - \alpha_u: K_0(U(X, f)) \to K_0(U(X, f))\) is the same as \(\text{Id} - \delta_*: \Delta_M \to \Delta_M\).

Since \(K_1(U(X, f))\) is isomorphic to \(\mathbb{Z}\), \(\alpha_u: \mathbb{Z} \to \mathbb{Z}\) is trivially the identity map. Thus the six-term exact sequence is divided into the following two short exact sequences;
\[
0 \to \Delta_M/\text{Im}(\text{Id} - \delta_M^{-1}) \to K_0(R_u(X, f)) \to \mathbb{Z} \to 0
\]
and
\[
0 \to \mathbb{Z} \to K_1(R_u(X, f)) \to \ker(\text{Id} - \delta_M^{-1}) \to 0.
\]

Therefore we conclude that
\[
K_0(R_u(X, f)) \cong \mathbb{Z} \oplus \{\Delta_M/\text{Im}(\text{Id} - \delta_M^{-1})\} = \mathbb{Z} \oplus \{\Delta_M/\text{Im}(\text{Id} - \delta_M)\}
\]
and
\[
K_1(R_u(X, f)) \cong \mathbb{Z} \oplus \ker(\text{Id} - \delta_M^{-1}) = \mathbb{Z} \oplus \ker(\text{Id} - \delta_M).
\]
Examples 4.4. (1). Suppose that $X$ is the unit circle and that $f: X \to X$ is given by $z \mapsto z^n$, $n \geq 2$. Then the adjacency matrix is $(n)$, $K_0(U(X, \mathcal{J})) = \mathbb{Z}[\frac{1}{n}]$ and $K_1(U(X, \mathcal{J})) = \mathbb{Z}$. Since $\delta_{(n)}^{-1}$ is multiplication by $\frac{1}{n}$, we have $K_0(R_u(X, \mathcal{J})) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}$ and $K_1(R_u(X, \mathcal{J})) = \mathbb{Z}$. See [3] for details.

(2). Suppose that $Y$ is a wedge of two circles $a$ and $b$ and that $g: Y \to Y$ is given by $a \mapsto aab$ and $b \mapsto ab$. Then the adjacency matrix is $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. So $K_0(U(Y, \mathcal{J})) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(U(Y, \mathcal{J})) = \mathbb{Z}$. Since $1 - \alpha_{u*}: \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is an isomorphism, we obtain $K_0(R_u(Y, \mathcal{J})) = K_1(R_u(Y, \mathcal{J})) = \mathbb{Z}.$

Stable equivalence Ruelle algebras. We use $K$-theoretic duality of the Ruelle algebras and the Universal Coefficient Theorem to compute $K$-groups of $R_u(X, \mathcal{J})$.

Remark 4.5 ([20]). Let $\mathcal{N}$ be the category of separable nuclear $C^\ast$-algebras which contains the separable Type I $C^\ast$-algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{Z}$ and by $\mathbb{R}$. Then it is not difficult to verify that unstable and stable equivalence Ruelle algebras of 1-solenoids are contained in $\mathcal{N}$.

Proposition 4.6 ([17, 5.c]). Suppose that $(X, \mathcal{J})$ is a 1-solenoid. Then $R_u(X, \mathcal{J})$ is dual to $R_u(X, \mathcal{J})$ so that $K_*(R_u(X, \mathcal{J}))$ is isomorphic to $K^{*+1}(R_u(X, \mathcal{J}))$.

Proposition 4.7 ([20, 1.19]). Suppose that $(X, \mathcal{J})$ is a 1-solenoid. Then there are short exact sequences

$$0 \to \text{Ext}^1_\mathbb{Z}(K_0(R_0(X, \mathcal{J})), \mathbb{Z}) \to K^1(R_u(X, \mathcal{J})) \to \text{Hom}(K_1(R_u(X, \mathcal{J})), \mathbb{Z}) \to 0$$

$$0 \to \text{Ext}^2_\mathbb{Z}(K_0(R_0(X, \mathcal{J})), \mathbb{Z}) \to K^0(R_u(X, \mathcal{J})) \to \text{Hom}(K_0(R_u(X, \mathcal{J})), \mathbb{Z}) \to 0$$

Hence $K$-groups of the stable equivalence Ruelle algebra are determined by Ext- and Hom-groups of $K_*(R_u(X, \mathcal{J}))$. Transform $Id - M$ to the Smith form

$$
\begin{pmatrix}
 d_1 \\
 d_2 \\
 \vdots \\
 d_n
\end{pmatrix}
$$

where $d_i \geq 0$ and $d_i$ divides $d_{i+1}$ ([17, §7.4]). Then $\Delta_M/\text{Im}(Id - \delta_M)$ is isomorphic to $\oplus_{i=1}^n \mathbb{Z}_{d_i}$. The dimension of $\text{Ker}(Id - \delta_M)$ is equal to the number of zeros in the diagonal of the Smith form. Suppose $d_1 = \cdots = d_m = 0$ and $d_{m+1} \neq 0$. Then we have

$$\text{Ext}^{1}_{\mathbb{Z}}(K_0(R_u(\mathcal{J})), \mathbb{Z}) = \text{Ext}^{1}_{\mathbb{Z}}(\mathbb{Z}_{d_{m+1}} \oplus \cdots \oplus \mathbb{Z}_{d_n}, \mathbb{Z}) = \mathbb{Z}_{d_{m+1}} \oplus \cdots \oplus \mathbb{Z}_{d_n}$$

and

$$\text{Hom}(K_1(R_u(\mathcal{J})), \mathbb{Z}) = \mathbb{Z}^{m+1}.$$

Hence we have

$$K^1(R_u(\mathcal{J})) \cong \text{Hom}(K_1(R_u(\mathcal{J})), \mathbb{Z}) \oplus \text{Ext}^{1}_{\mathbb{Z}}(K_0(R_u(\mathcal{J})), \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{m} \oplus \mathbb{Z}_{d_{m+1}} \oplus \cdots \oplus \mathbb{Z}_{d_n} \cong \mathbb{Z} \oplus \{ \Delta_M/\text{Im}(Id - \delta_M) \}.$$
Recall that $K_1(R_u(X,\mathcal{F})) = \mathbb{Z} \oplus \ker(1 - \delta_M)$ is a torsion-free subgroup of $\mathbb{Z}^{n+1}$. Thus we have $\text{Ext}_0^2(K_1(R_u(X,\mathcal{F})), \mathbb{Z}) = 0$ and

$$K^0(R_u(X,\mathcal{F})) \cong \text{Hom}(K_0(R_u(X,\mathcal{F})), \mathbb{Z}).$$

Then $K_0(R_u(X,\mathcal{F})) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ implies

$$\text{Hom}(K_0(R_u(X,\mathcal{F})), \mathbb{Z}) \cong \text{Hom}(\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ker(1 - \delta_M).$$

Therefore we conclude that:

**Proposition 4.8.** Suppose that $(X,\mathcal{F})$ is a 1-solenoid. Then

$$K_0(R_u(X,\mathcal{F})) \cong \mathbb{Z} \oplus \{\Delta_M/\text{Im}(1 - \delta_M)\} \text{ and } K_1(R_u(X,\mathcal{F})) \cong \mathbb{Z} \oplus \ker(1 - \delta_M).$$

**Remark 4.9.** The isomorphisms in proposition 4.8 are unnatural as the short exact sequences in the Universal Coefficient Theorem split unnaturally.

Recall that the unstable and stable equivalence Ruelle algebras of a 1-solenoid are nuclear, purely infinite, separable, simple and stable $C^*$-algebras (proposition 2.2). Then the classification theorem of Kirchberg-Phillips implies the following proposition.

**Proposition 4.10.** $R_u(X,\mathcal{F})$ is $*$-isomorphic to $R_u(X,\mathcal{F})$.

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