Quantum four-body system in $D$ dimensions

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By the method of generalized spherical harmonic polynomials, the Schrödinger equation for a four-body system in $D$-dimensional space is reduced to the generalized radial equations where only six internal variables are involved. The problem on separating the rotational degrees of freedom from the internal ones for a quantum $N$-body system in $D$ dimensions is generally discussed.

I. INTRODUCTION

Recent years have witnessed a flurry of investigations into the arbitrary $D$-dimensional problems [1–5] in many branches of physical chemistry and chemical physics. The problems associated with the $D$-dimensional hydrogen atom [6–8], the $D$-dimensional harmonic oscillator [9–11], and the connection between the two [12–16] have been thoroughly discussed. During the past few years, with the application of dimensional scaling to the quantum theory of atomic and molecular structure, large-$D$ helium problem has also been studied by many authors [17–21]. This approach requires solving the few-body Schrödinger equation in a $D$-dimensional coordinate space and has been applied to a large number of physically interesting problems [22–25]. Due to the complexity of the problem for an $N$-body system in $D$ dimensions, so far there is no complete theoretical solution when $N > 3$.

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In our recent work [26], a new method for separating the rotational degrees of freedom from the internal ones in a few-body system was proposed. The power of this new approach is in its ability of great simplification in calculation of energy levels of a few-body system in terms of the generalized radial equations involved only internal variables, which are derived from the Schrödinger equation without any approximation. Some typical three-body system in three-dimensional space, such as a helium atom [27–29] and a positronium negative ion [30] have been solved numerically with high precision. The key to the approach is that we have found a complete set of independent eigenfunctions of angular momentum for the system, which are homogeneous polynomials in the components of Jacobi coordinate vectors and satisfy the Laplace equation, and chosen a suitable set of internal variables. Any wave function with a given angular momentum can be expanded with respect to the base functions where the coefficients, called the generalized radial functions, depend only upon the internal variables. The generalized radial equations satisfied by the generalized radial functions are easily derived owing to the nice property of the base functions [26]. This method has been generalized to the arbitrary dimensional space for a three-body system [31]. The exact interdimensional degeneracies in the system can be obtained directly from the generalized radial equations [32].

To further this study, we expect to apply this approach to an \( N \)-body system in \( D \) dimensions. As noticed in our previous paper [31], the cases with \( N < D \) are very different to the cases with \( N \geq D \). The general formulas are hard to express uniformly due to arbitrariness of \( D \) and \( N \). However, the main characters are manifested fully in a four-body system of \( D \) dimensions, but not fully in a three-body system. The four-body problems also play a fundamental role in nuclear and hypernuclear physics [33–35]. In this paper we will study the problem of separating the rotational degrees of freedom from the internal ones for a quantum four-body system in \( D \) dimensions in some detail. The general case (\( N \)-body system) will be summarized.

The plan of this paper is as follows. In Sec. II, after separating the motion of the center of mass by Jacobi coordinate vectors, we will define the generalized spherical harmonic polynomials for a four-body system in \( D \) dimensions and prove that they constitute a complete set of independent base functions for a given total orbital angular momentum in the system. Some new features in comparison with the three-body case are also dis-
cussed in this section. The generalized radial equations satisfied by the generalized radial functions are established in Sec. III. In Sec. IV, we will generalize this method to separate the rotational degrees of freedom from the internal ones for an \( N \)-body system in \( D \) dimensions. Some conclusions will be given in Sec. V.

II. THE GENERALIZED SPHERICAL HARMONIC POLYNOMIALS

For a quantum \( N \)-body system in an arbitrary \( D \)-dimensional space, we denote the position vectors and the masses of \( N \) particles by \( r_k \) and by \( m_k \), \( k = 1, 2, \ldots, N \), respectively. \( M = \sum_k m_k \) is the total mass. The Schrödinger equation for the \( N \)-body system with a pair potential \( V \), depending upon the distance of each pair of particles, \(|r_j - r_k|\), is

\[
-\frac{1}{2} \nabla^2 \Psi + V \Psi = E \Psi, \quad \nabla^2 = \sum_{k=1}^N m_k^{-1} \nabla^2_{r_k},
\]

where \( \nabla^2_{r_k} \) is the Laplace operator with respect to the position vector \( r_k \). For simplicity, the natural units \( \hbar = c = 1 \) are employed throughout this paper. The total orbital angular momentum operators \( L_{ab} \) in \( D \) dimensions are defined as [21,36]

\[
L_{ab} = -L_{ba} = -i \sum_{k=1}^N \left\{ r_{ka} \frac{\partial}{\partial r_{kb}} - r_{kb} \frac{\partial}{\partial r_{ka}} \right\}, \quad a, b = 1, 2, \ldots D,
\]

where \( r_{ka} \) denotes the \( a \)th component of the position vector \( r_k \). Now, we replace the position vectors \( r_k \) with the Jacobi coordinate vectors \( R_j \):

\[
R_0 = M^{-1/2} \sum_{k=1}^N m_k r_k, \quad R_j = \left( \frac{m_{j+1}M_j}{M_{j+1}} \right)^{1/2} \left( r_{j+1} - \sum_{k=1}^j \frac{m_k r_k}{M_j} \right), \quad 1 \leq j \leq (N-1), \quad M_j = \sum_{k=1}^j m_k, \quad M_N = M,
\]

where \( R_0 \) describes the position of the center of mass, \( R_1 \) describes the mass-weighted separation from the second particle to the first particle. \( R_2 \) describes the mass-weighted separation from the third particle to the center of mass of the first two particles, and so on. It is straightforward to illustrate that the potential \( V \) is a function of \( R_j \cdot R_k \) and is rotationally invariant.

In the center-of-mass frame, \( R_0 = 0 \). A straightforward calculation by replacement of variables shows that the Laplace operator in Eq. (1) and the total orbital angular
As discussed in Ref. [31], in $D$-dimensional space, the wave function $\Psi(x, y, z)$ with a given total angular momentum has to belong to an irreducible representation of $SO(D)$, and the angular momentum is also denoted by the representation. For a four-body system, there

\[ \sum_{j=1}^{N-1} \nabla^2 \mathbf{R}_j, \quad L_{ab} = \sum_{j=1}^{N-1} L^{(j)}_{ab} = -i \sum_{j=1}^{N-1} \left\{ R_{ja} \frac{\partial}{\partial R_{jb}} - R_{jb} \frac{\partial}{\partial R_{ja}} \right\}, \]

(4)

For a four-body system, there are three Jacobi coordinate vectors $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_3$, which will be denoted for simplicity by $\mathbf{x, y}$ and $\mathbf{z}$, respectively:

\[ \mathbf{x} = \left[ \frac{m_1 m_2}{m_1 + m_2} \right]^{1/2} \{ \mathbf{r}_2 - \mathbf{r}_1 \}, \]
\[ \mathbf{y} = \left[ \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3} \right]^{1/2} \left\{ \mathbf{r}_3 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \right\}, \]
\[ \mathbf{z} = \left[ \frac{(m_1 + m_2 + m_3 + m_4)m_4}{m_1 + m_2 + m_3} \right]^{1/2} \mathbf{r}_4. \]

Hence,

\[ \nabla^2 = \nabla_x^2 + \nabla_y^2 + \nabla_z^2, \quad L_{ab} = L^{(x)}_{ab} + L^{(y)}_{ab} + L^{(z)}_{ab} \]
\[ L^2 = \sum_{a<b=2}^D L^2_{ab}, \quad \left[ \mathbf{L}^{(x)} \right]^2 = \sum_{a<b=2}^D \left[ L^{(x)}_{ab} \right]^2, \]
\[ \left[ \mathbf{L}^{(y)} \right]^2 = \sum_{a<b=2}^D \left[ L^{(y)}_{ab} \right]^2, \quad \left[ \mathbf{L}^{(z)} \right]^2 = \sum_{a<b=2}^D \left[ L^{(z)}_{ab} \right]^2. \]

(6)

The Schrödinger equation (1) for $D \geq N = 4$ reduces to

\[ \left\{ \nabla_x^2 + \nabla_y^2 + \nabla_z^2 \right\} \Psi(x, y, z) = -2 \left\{ E - V(\xi_j, \eta_j, \zeta_j) \right\} \Psi(x, y, z), \]
\[ \xi_1 = x \cdot x, \quad \xi_2 = \eta_1 = x \cdot y, \quad \xi_3 = \zeta_1 = x \cdot z, \]
\[ \eta_2 = y \cdot y, \quad \eta_3 = \zeta_2 = y \cdot z, \quad \zeta_3 = z \cdot z. \]

(7)

where $\xi_j, \eta_j, \text{and } \zeta_j$ are internal variables. It is worth noticing that for the cases $3 = D < N$ two Jacobi coordinate vectors $\mathbf{x}$ and $\mathbf{y}$ can determine the body-fixed frame and this set of internal variables is not complete because two configurations with different directions of $\mathbf{z}$ reflecting to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$ are described by the same internal variables. As pointed in Ref. [26], the variables $\zeta_3$ has to be changed to $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$. We will further discuss this problem in Sec. IV.

Since Eq. (7) is rotational invariant, the total orbital angular momentum is conserved. As discussed in Ref. [31], in $D$-dimensional space, the wave function $\Psi(x, y, z)$ with a given total angular momentum has to belong to an irreducible representation of $SO(D)$, and the angular momentum is also denoted by the representation. For a four-body system, there
are only three Jacobi coordinate vectors so that the possible irreducible representation is described by a three-row Young pattern $[\mu, \nu, \tau]$ of $SO(D)$, or its highest weight $M = (M_1, M_2, M_3, 0, \ldots, 0)$, where

$$M_1 = \mu - \nu, \quad M_2 = \nu - \tau, \quad M_3 = \tau.$$  \hspace{1cm} (8)

We only need to consider the highest weight state $\Psi_M^{[\mu, \nu, \tau]}(x, y, z)$ because its partners can be calculated from it by the lowering operators. In this paper the highest weight state will be simply called the wave functions with the given angular momentum $[\mu, \nu, \tau]$ for simplicity.

Now we are going to find a complete set of independent eigenfunctions of total orbital angular momentum, where ”independent” means that each one in the set cannot be expressed as a combination of the remaining with coefficients only depending on the internal variables. As discussed in our previous paper [31], the spherical harmonic polynomials $\mathcal{Y}_m^l(\hat{x})$ are homogeneous polynomials in the components of $x$ of degree $l$, spanning an irreducible traceless tensor space described by the Young pattern $(l) \equiv [l, 0, 0]$. When $D > 6$, the explicit forms for some polynomials with higher weights $m$ are as follows [37]:

$$\begin{align*}
\mathcal{Y}_0^l(x) &= N_{D,l}(x_1 + ix_2)^l, \\
\mathcal{Y}_{(l-2,1,0,\ldots,0)}^l(x) &= -\sqrt{l}N_{D,l}(x_1 + ix_2)^{l-1}(x_3 + ix_4), \\
\mathcal{Y}_{(l-4,2,0,\ldots,0)}^l(x) &= \sqrt{l(l-1)/2}N_{D,l}(x_1 + ix_2)^{l-2}(x_3 + ix_4)^2,
\end{align*}$$  \hspace{1cm} (9)

where the last equality holds for $l > 1$, and $N_{D,l}$ denotes the normalization factor given in [37]. The product of two spherical harmonic polynomials $\mathcal{Y}_m^l(\hat{x})$ and $\mathcal{Y}_m'^{l'}(\hat{y})$ belongs to the direct product of two representation $(l)$ and $(l')$, which is a reducible representation. It can be reduced by the Littlewood-Richardson rule and contraction of a pair of $x_a$ and $y_a$, where the latter relates to the internal variables:

$$(l) \otimes (l') \simeq \bigoplus_{s=0}^{\min\{l,l'\}} \bigoplus_{t=0}^{\min\{l,l'\}-s} [l + l' - s - 2t, s, 0].$$  \hspace{1cm} (10)

Since a base function containing a factor depending on internal variables is not independent, only those representations $[l + l' - s, s, 0]$ $[t = 0$ in Eq. (10)] calculated by the Littlewood-Richardson rule are related to the independent base functions [31]. Calculating by the Clebsch-Gordan coefficients and removing the normalization factor, we obtain
the independent base functions for the representations \([l+l'-s,0]\), called the generalized spherical harmonic polynomial \(Q^{(l+l'-s)s}_l(x,y)\). Changing the parameters \(\mu = l + l' - s\), \(\nu = s\) and \(q = l\), we define the generalized spherical harmonic polynomial \(Q^{\mu\nu}_q(x,y)\) for the representation \([\mu,\nu]\) [31] as

\[
Q^{\mu\nu}_q(x,y) = \frac{X_{12}^{q-\nu}Y_{12}^{\mu-q}}{(q-\nu)!((\mu-q))!}(X_{12}Y_{34} - Y_{12}X_{34})^\nu, \quad 0 \leq \nu \leq q \leq \mu, \quad (11)
\]

For the product of three spherical harmonic polynomials, Eq. (10) is generalized to

\[
(l \otimes (l') \otimes (l'')) \cong \bigoplus_{r=0}^{\min\{l,l'\}} \bigoplus_{\nu=r}^{\min\{l+l'-r\}} \bigoplus_{\tau=0}^{\min\{r,l''-\nu+r\}} [l + l' + l'' - \nu - \tau, \nu, \tau] \oplus \ldots . \quad (12)
\]

The ellipsis denotes those representations related to the base functions which are not independent.

Filling the digits 1, 2 and 3 arbitrarily into a given Young pattern \([\mu,\nu,\tau]\) \((\mu \geq \nu \geq \tau)\) we obtain a young tableau. A Young tableau is called standard if the digit in every column of the tableau increases downwards and the digit in every row does not decrease from left to right. In fact, the digits ”1”, ”2” and ”3” denote the components of \(x, y,\) and \(z\), respectively. Obviously, the representation \([l + l' + l'' - \nu - \tau, \nu, \tau]\) listed in Eq. (12) corresponds to a standard Young tableau, where the number of digit ”1” in the first row is \(l\), the numbers of digit ”2” in the first and the second rows are respectively \((l' - r)\) and \(r\), and the numbers of digit ”3” in the first, second and third rows are respectively \((l'' + r - \nu - \tau)\), \(\nu - r\) and \(\tau\). The base functions in the remaining representation spaces, which correspond to non-standard Young tableaux, are not independent.

For a given pattern \([\mu,\nu,\tau]\), each standard Young tableau is determined by three parameters \(q, p\) and \(r\), where \(q\) is the number of digit ”1” in the first row, \(p\) and \(r\) are the numbers of digit ”2” in the first and the second rows, respectively. \(q, p\) and \(r\) should satisfy the constraints: \(\tau \leq r \leq q\) and \(r \leq \nu \leq q + p \leq \mu\). The number of standard Young tableaux for the given Young pattern \([\mu,\nu,\tau]\) is equal to the dimension of the representation \([\mu,\nu,\tau]\) of the SU(3) group:

\[
d_{[\mu,\nu,\tau]}(SU(3)) = \frac{1}{2}(\mu - \tau + 2)(\nu - \tau + 1)(\mu - \nu + 1). \quad (13)
\]
For a given representation \([\mu, \nu, \tau]\) of \(\text{SO}(D)\), each standard Young tableau denoted by \((q, p, r)\) corresponds to a representation space. The highest weight state in the representation space \((q, p, r)\) is the generalized spherical harmonic polynomial \(Q_{qpr}^{\mu\nu\tau}(x, y, z)\):

\[
Q_{qpr}^{\mu\nu\tau}(x, y, z) = \begin{cases} 
\frac{X_{12}^{q-p} Y_{12}^{\mu-p} T_{12}^{\nu-p} T_{13}^{\tau-p} T_{23}^{\tau}}{(q-p)! (\mu-q-p)! (\nu-q-p)!(\tau-q-p)!} & \text{when } q \geq \nu, \\
\frac{Y_{12}^{q-p} Z_{12}^{\mu-p} T_{12}^{\nu-p} T_{13}^{\tau-p} T_{23}^{\tau}}{(q+p-p)! (\mu-q-p)! (\nu-q-p)!(\tau-q-p)!} & \text{when } q < \nu,
\end{cases}
\]

\[
\tau \leq r \leq q, \quad r \leq \nu \leq q + p \leq \mu,
\]

\[
X_{12} = x_1 + ix_2, \quad Y_{12} = y_1 + iy_2, \quad Z_{12} = z_1 + iz_2,
\]

\[
X_{34} = x_3 + ix_4, \quad Y_{34} = y_3 + iy_4, \quad Z_{34} = z_3 + iz_4,
\]

\[
X_{56} = x_5 + ix_6, \quad Y_{56} = y_5 + iy_6, \quad Z_{56} = z_5 + iz_6,
\]

\[
T_{12} = X_{12} Y_{34} - X_{34} Y_{12}, \quad T_{13} = X_{12} Z_{34} - X_{34} Z_{12}, \quad T_{23} = Y_{12} Z_{34} - Y_{34} Z_{12},
\]

\[
T = X_{12} Y_{34} Z_{56} + X_{34} Y_{56} Z_{12} + X_{56} Y_{12} Z_{34} - X_{12} Y_{56} Z_{34} - X_{34} Y_{12} Z_{56} - X_{56} Y_{34} Z_{12}.
\]

(14)

It is evident that \(Q_{qpr}^{\mu\nu\tau}(x, y, z)\) do not contain a function of the internal variables as a factor, nor do their partners due to the rotational symmetry. Therefore, \(Q_{qpr}^{\mu\nu\tau}(x, y, z)\) are independent base functions for the given angular momentum described by \([\mu, \nu, \tau]\). Due to Eq. (12), the set of \(Q_{qpr}^{\mu\nu\tau}(x, y, z)\) is complete. The reason why the generalized spherical harmonic polynomial denoted by a non-standard Young tableau is not independent can be seen from the following identity:

\[
T_{23} X_{12} = T_{13} Y_{12} - T_{12} Z_{12}, \quad \begin{array}{c|c|c}
2 & 1 & 2 \\
3 & 3 & 2 \\
\end{array} \begin{array}{c|c|c}
1 & 2 & 1 \\
3 & 2 & 3 \\
\end{array} = \begin{array}{c|c|c}
1 & 3 & 1 \\
2 & 3 & 2 \\
\end{array}.
\]

(15)

This identity is similar to the Fock’s cyclic symmetry condition \([38]\). The left-hand-side of Eq. (15) corresponds to a non-standard Young tableau, and two terms in the right-hand-side correspond to two standard Young tableaux, respectively.

Since the problem on completeness of the set is very important, we are going to prove this problem by another method. On the one hand, because the base function \(Q_{qpr}^{\mu\nu\tau}(x, y, z)\) is a homogeneous polynomial of degree \(\mu + \nu + \tau\) in the components of \(x, y\) and \(z\), we calculate the number \(R_D(l)\) of base functions in the sets for the representations \([\mu, \nu, \tau]\) with \(\mu + \nu + \tau = l\). Namely, we want to calculate how many homogeneous polynomials of degree \(l\) exist in the sets of the independent base functions. We first calculate how many base functions exist in the set for a given representation \([\mu, \nu, \tau]\). The dimension of the
representation $[\mu, \nu, \tau]$ of SO($D$) is $d_D([\mu, \nu, \tau])$:

$$d_D([\mu, \nu, \tau]) = (D + 2\mu - 2)(D + \mu + \nu - 3)(D + \mu + \tau - 4)(D + 2\nu - 4)$$
$$\times (D + \nu + \tau - 5)(D + 2\tau - 6)(\mu - \nu + 2)(\mu - \nu + 1)(\nu - \tau + 1)$$
$$\times \frac{(D + \mu - 5)!(D + \nu - 6)!(D + \tau - 7)!}{(D - 2)!(D - 4)!(D - 6)!(\mu + 2)!(\nu + 1)!(\tau)!}. \quad (16)$$

Thus, the number of base functions in the set for the representation $[\mu, \nu, \tau]$ is $d_{[\mu, \nu, \tau]}(SU(3))d_D(\mu, \nu, \tau)$. Then, the number $R_D(l)$ of base functions in the sets for the representation $[\mu, \nu, \tau]$ with $\mu + \nu + \tau = l$ is:

$$R_D(l) = \sum_{\tau=0}^{[l/3]} \sum_{\nu=\tau}^{[\tau+\mu/2]} d_{[\nu-\tau, \nu, \tau]}(SU(3)) d_D(([\nu-\tau, \nu, \tau]), \quad (17)$$

where $[x]$ denotes the largest integer less than or equal to $x$.

On the other hand, the number of linear independent homogeneous polynomials of degree $l$ in the components of $x$, $y$ and $z$ is $M_D(l)$:

$$M_D(l) = \binom{l + 3D - 1}{3D - 1}. \quad (18)$$

After removing those polynomials in the form $\xi_j f(x, y, z)$, $\eta_j f(x, y, z)$ and $\zeta_j f(x, y, z)$ where $f(x, y, z)$ is a polynomial of degree $(l - 2)$, the number $M_D(l)$ reduces to $K_D(l)$:

$$K_D(l) = M_D(l) - 6M_D(l - 2) + 15M_D(l - 4) - 20M_D(l - 6)$$
$$+ 15M_D(l - 8) - 6M_D(l - 10) + M_D(l - 12)$$
$$= \{ (3D - 7)(3D - 8)(3D - 9)(3D - 10)(3D - 11)(3D - 12)$$
$$+ 12l(D - 4)[72 + (3D - 10)(3D - 11)(27D^2 - 153D + 236)]$$
$$+ 4l^2[184 + 45(D - 4)(3D - 10)(9D^2 - 57D + 98)]$$
$$+ 480l^3(D - 4)(9D^2 - 63D + 126) + 80l^4(27D^2 - 207D + 404)$$
$$+ 576l^5(D - 4) + 64l^6 \} \frac{(l + 3D - 13)!}{l!(3D - 7)!}, \quad (19)$$

where $l + 3D \geq 13$ and $K_4(0) = 1$. It is checked by MATHEMATICA that

$$R_D(l) = K_D(l). \quad (20)$$

Thus, we have proved again that $d_{[\mu, \nu, \tau]}(SU(3))$ polynomials $Q_{qpr}^{\mu\nu\tau}(x, y, z)$ construct a complete set of independent base functions for the angular momentum $[\mu, \nu, \tau]$. 

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The generalized spherical harmonic polynomial $Q_{qpr}^{μντ}(x, y, z)$ is a homogeneous polynomial of degrees $q, (p + r)$ and $(μ + ν + τ - q - p - r)$ in the components of $x, y$ and $z$, respectively. It is a simultaneous eigenfunction of $\nabla^2$, $\nabla_2^2$, $\nabla_3^2$, $\nabla_x \cdot \nabla_y$, $\nabla_x \cdot \nabla_z$, $\nabla_y \cdot \nabla_z$, and the angular momentum operators $L^2, [L^{(x)}]^2, [L^{(y)}]^2, [L^{(z)}]^2$,

\[
\nabla^2 Q_{qpr}^{μντ}(x, y, z) = \nabla_2^2 Q_{qpr}^{μντ}(x, y, z) = \nabla_3^2 Q_{qpr}^{μντ}(x, y, z) = 0,
\]

\[
\nabla_x \cdot \nabla_y Q_{qpr}^{μντ}(x, y, z) = \nabla_x \cdot \nabla_z Q_{qpr}^{μντ}(x, y, z) = \nabla_y \cdot \nabla_z Q_{qpr}^{μντ}(x, y, z) = 0,
\]

\[
L^2 Q_{qpr}^{μντ}(x, y, z) = C_2(μ, \nu, \tau)Q_{qpr}^{μντ}(x, y, z),
\]

\[
[\nabla^{(x)}]^2 Q_{qpr}^{μντ}(x, y, z) = q(q + D - 2)Q_{qpr}^{μντ}(x, y, z),
\]

\[
[\nabla^{(y)}]^2 Q_{qpr}^{μντ}(x, y, z) = (p + r)(p + r + D - 2)Q_{qpr}^{μντ}(x, y, z),
\]

\[
[\nabla^{(z)}]^2 Q_{qpr}^{μντ}(x, y, z) = (μ + ν + τ - q - p - r)(μ + ν + τ - q - p - r + D - 2)
\]

\[
\times Q_{qpr}^{μντ}(x, y, z).
\]

where $C_2(μ, \nu, \tau)$ is the Casimir calculated by a general formula (see (1.131) in Ref. [39]).

The parity of $Q_{qpr}^{μντ}(x, y, z)$ is obviously equal to $(-1)^{μ+ν+τ}$.

Now, we turn to discuss the case $D \leq 6$. As is well known, the irreducible traceless tensor space of $SO(D)$ described by a Young pattern has the following properties. It is a null space if sum of the lengths of the first two columns of the Young pattern is larger than $D$. It reduces into the selfdual and antiselfdual tensor spaces if the row number of the Young pattern is equal to $D/2$. Two representations are equivalent if their Young patterns are the same as each other except for the first column and the sum of their row numbers is equal to $D$. Those properties cause the situation for $D \leq 6$ different to that for $D > 6$.

When $D = 6$, there is no problem for the representation $[μ, ν, τ]$ with $τ = 0$, but when $τ \neq 0$, the representation is reducible. We denote the generalized spherical harmonic polynomials for the selfdual and antiselfdual representations by $Q_{qpr}^{(S)μντ}(x, y, z)$ and $Q_{qpr}^{(A)μντ}(x, y, z)$, respectively. $Q_{qpr}^{(S)μντ}(x, y, z)$ is the same as that given in Eq. (14), and $Q_{qpr}^{(A)μντ}(x, y, z)$ can be obtained from $Q_{qpr}^{(S)μντ}(x, y, z)$ by replacing $X_{56}, Y_{56}$ and $Z_{56}$ with $X'_{56}, Y'_{56}$ and $Z'_{56}$:

\[
X'_{56} = x_5 - ix_6, \quad Y'_{56} = y_5 - iy_6, \quad Z'_{56} = z_5 - iz_6.
\]

The formula (16) for the dimension of the representation $[μ, ν, τ]$ of $SO(D)$ holds for $D = 6$. 

\[\text{(21)}\]
when \( \tau = 0 \). When \( \tau \neq 0 \), \( d_D([\mu, \nu, \tau]) \) in Eq. (16) is equal to the sum of the dimensions of the selfdual and antiselfdual representations such that the equality (19) still holds for \( D = 6 \).

When \( D = 5 \), in the possible Young pattern \([\mu, \nu, \tau]\), \( \tau \) has to be 0 or 1. The representation \([\mu, \nu, 1]\) is equivalent to the representation \([\mu, \nu, 0]\). Their dimensions calculated from Eq. (16) are also the same. The generalized spherical harmonic polynomials \( Q^{\mu\nu\tau}_{qpr}(x, y, z) \) given in Eq. (14) hold for \( D = 5 \) except for \( x_6 = y_6 = z_6 = 0 \) and \( \tau = 0 \) or 1. Therefore, the equality (19) holds for \( D = 5 \).

For \( D = 3 \), two Jacobi coordinate vectors, say \( x \) and \( y \), can completely determine the body-fixed frame so that the variables \( \zeta_3 \) has to be changed as \( (x \times y) \cdot z \) in order to distinguish two configurations with different directions of \( z \). We have discussed in detail the four-body system in three dimensions in our previous paper [26].

The case of \( D = 4 \) is quite complicated because \( SO(4) \) is not a simple group. The representation \([\mu, \nu, 0]\) reduces to a direct sum of a selfdual representation \([ (S)\mu, \nu, 0]\) and an antiselfdual one \([ (A)\mu, \nu, 0]\). The generalized spherical harmonic polynomials \( Q^{(S)\mu\nu}_{qpr}(x, y, z) \) for the selfdual representations is the same as \( Q^{\mu\nu\tau}_{qpr}(x, y, z) \) with \( \tau = 0 \) given in Eq. (14). The generalized spherical harmonic polynomials \( Q^{(A)\mu\nu}_{qpr}(x, y, z) \) for the antiselfdual representation can be obtained from \( Q^{(S)\mu\nu}_{qpr}(x, y, z) \) by replacing \( X_{34} \), \( Y_{34} \) and \( Z_{34} \) with \( X'_{34} \), \( Y'_{34} \) and \( Z'_{34} \):

\[
X'_{34} = x_3 - ix_4, \quad Y'_{34} = y_3 - iy_4, \quad Z'_{34} = z_3 - iz_4. \tag{22}
\]

If \( \tau = 1 \), then \( \nu = 1 \) and the representation \([\mu, 1, 1]\) is equivalent to the representation \([\mu, 0, 0]\). The standard Young tableau is described by the parameters \( q \) and \( p \) (\( r = 1 \)), where \( q \) and \( p \) are respectively the numbers of digits "1" and "2" in the first row of the Young tableau. The generalized spherical harmonic polynomials for two representations \([\mu, \lambda, \lambda]\), \( \lambda = 0, \) or 1, are

\[
Q^{\mu\lambda\lambda}_{qp}(x, y, z) = \frac{X_1^{q-\lambda}Y_1^{r-p}Z_1^{q-p}T^\lambda}{(q-\lambda)!p!(\mu-q-p)!},
\]

\[
T = X_{34}Y_{34}'Z_{34}' + X_{34}'Y_{34}Z_{34} - X_{34}'Y_{34}Z_{34} - X_{34}Y_{34}'Z_{34} - X_{34}'Y_{34}Z_{34} - X_{34}Y_{34}Z_{34}. \tag{23}
\]

The surprising thing is that Eq. (19) does not satisfied for \( D = 4 \) and \( l \geq 6 \). For
example,

\[
\begin{array}{cccccc}
  l & 6 & 7 & 8 & 9 & 10 \\
  R_4(l) & 5346 & 10908 & 20550 & 36332 & 60996 \\
  K_4(l) & 5336 & 10836 & 20256 & 35436 & 58728 \\
\end{array}
\]

(24)

The reason is that the formula (18) for \(K_D(l)\) does not hold for \(D = 4\) and \(l \geq 6\). For \(D = 4\) we find an identity with respect to the polynomials of degree 6 checked by MATHEMATICA:

\[
\xi_1 T_{23}^2 + \eta_2 T_{13}^2 + \zeta_3 T_{12}^2 - 2\xi_2 T_{13} T_{23} + 2\xi_3 T_{12} T_{23} - 2\eta_3 T_{12} T_{13} = 0.
\]

(25)

The identity obtained from Eq. (25) by replacing \(X_{34}, Y_{34}\) and \(Z_{34}\) respectively with \(X'_{34}, Y'_{34}\) and \(Z'_{34}\) still holds. Those equalities obtained by applying the lowering operators and (or) by multiplying a factor to above two identities are also identities. Thus, the forms \(\xi_j f(x, y, z), \eta_j f(x, y, z)\) and \(\zeta_j f(x, y, z)\), where \(f(x, y, z)\) is a homogeneous polynomial of \(x, y\) and \(z\) of degree \((l - 2)\), are not independent when \(l \geq 6\). It is easy to count by MATHEMATICA that the revised \(K_4(l)\) by considering the identities coincides with \(R_4(l)\).

### III. GENERALIZED RADIAL EQUATIONS

In the preceding section we proved that \(d_{[\mu, \nu, \tau]}(SU(3))\) polynomials \(Q_{\mu \nu \tau}^{qpr}(x, y, z)\) construct a complete set of independent base functions for the angular momentum \([\mu, \nu, \tau]\). Thus, any function \(\Psi_{M}^{[\mu, \nu, \tau]}(x, y, z)\) with angular momentum \([\mu, \nu, \tau]\) in the system can be expanded with respect to the base functions \(Q_{qpr}^{\mu \nu \tau}(x, y, z)\), where the coefficients are functions of internal variables.

\[
\Psi_{M}^{[\mu, \nu, \tau]}(x, y, z) = \sum_{\mu} \sum_{\nu=q}^{\mu} \sum_{\tau}^{\min\{q, \nu\}} \psi_{qpr}^{\mu \nu \tau}(\xi_j, \eta_j, \zeta_j)Q_{qpr}^{\mu \nu \tau}(x, y, z),
\]

(26)

where the coefficients \(\psi_{qpr}^{\mu \nu \tau}(\xi_j, \eta_j, \zeta_j)\) are called the generalized radial functions. When substituting Eq. (26) into the Schrödinger equation (5), the main calculation is to apply the Laplace operator (4) to the function \(\Psi_{M}^{[\mu, \nu, \tau]}(x, y, z)\). The calculation consists of three parts. In the following, we remove the arguments \((\xi_j, \eta_j, \zeta_j)\) and \((x, y, z)\) for simplicity. The first is to apply the Laplace operator to the generalized radial functions \(\psi_{qpr}^{\mu \nu \tau}(\xi_j, \eta_j, \zeta_j),\)
which can be calculated by replacement of variables:

\[ \nabla^2 \psi_{\mu\nu\tau}^{\nu\mu\tau} = \left\{ 4\xi_1\xi_1^2 + 4\eta_2\eta_2^2 + 4\zeta_3\zeta_3^2 + 2D (\partial_{\xi_1} + \partial_{\eta_2} + \partial_{\zeta_3}) + (\xi_1 + \eta_2)\partial_{\xi_1}^2 \\
+ (\xi_1 + \zeta_3)\partial_{\zeta_3}^2 + (\eta_2 + \zeta_3)\partial_{\eta_2}^2 + 4\xi_2 (\partial_{\xi_1} + \partial_{\eta_2})\partial_{\xi_2} + 4\zeta_3 (\partial_{\xi_1} + \partial_{\zeta_3})\partial_{\zeta_3} \right\} \psi_{\mu\nu\tau}^{\nu\mu\tau} \]

(27)

where \( \partial_{\xi} \) denotes \( \partial / \partial \xi \) and so on. The second is to apply the Laplace operator to the generalized spherical harmonic polynomials \( Q_{\mu\nu\tau}^{\nu\mu\tau} \). This part is vanishing because \( Q_{\mu\nu\tau}^{\nu\mu\tau} \) satisfies the Laplace equation. The last is the mixed application

\[
2 \left\{ (\partial_{\xi_1} \psi_{\mu\nu\tau}^{\nu\mu\tau})2x + (\partial_{\xi_2} \psi_{\mu\nu\tau}^{\nu\mu\tau})y + (\partial_{\xi_3} \psi_{\mu\nu\tau}^{\nu\mu\tau})z \right\} \cdot \nabla_x Q_{\mu\nu\tau}^{\nu\mu\tau} \\
+ 2 \left\{ (\partial_{\eta_2} \psi_{\mu\nu\tau}^{\nu\mu\tau})x + (\partial_{\eta_3} \psi_{\mu\nu\tau}^{\nu\mu\tau})y + (\partial_{\zeta_3} \psi_{\mu\nu\tau}^{\nu\mu\tau})z \right\} \cdot \nabla_y Q_{\mu\nu\tau}^{\nu\mu\tau} \\
+ 2 \left\{ (\partial_{\xi_3} \psi_{\mu\nu\tau}^{\nu\mu\tau})x + (\partial_{\eta_3} \psi_{\mu\nu\tau}^{\nu\mu\tau})y + (\partial_{\zeta_3} \psi_{\mu\nu\tau}^{\nu\mu\tau})z \right\} \cdot \nabla_z Q_{\mu\nu\tau}^{\nu\mu\tau}.
\]

(28)
Hence, we obtain the generalized radial equation, satisfied by the functions $\psi_{q^p r}^{\mu \nu \tau}$:

$$\begin{align*}
x \cdot \nabla x Q_{qr}^{\mu \nu \tau} &= q Q_{qr}^{\mu \nu \tau}, \\
y \cdot \nabla y Q_{qr}^{\mu \nu \tau} &= (p + r)Q_{qr}^{\mu \nu \tau}, \\
z \cdot \nabla z Q_{qr}^{\mu \nu \tau} &= (\mu + \nu + \tau - q - p - r)Q_{qr}^{\mu \nu \tau}, \\
x \cdot \nabla_y Q_{qr}^{\mu \nu \tau} &=\begin{cases} (p + 1)(q - r) \frac{Q_{qr}^{\mu \nu \tau}}{q - \nu} - \frac{(\mu - q - p + 1)(r - \tau + 1)}{q - \nu} Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)r, & \text{when } q > \nu, \\
(q - \nu + 1)Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)r, & \text{when } q \geq \nu, \\
2(q - p - \nu + 1)(q - r + 1) \frac{Q_{qr}^{\mu \nu \tau}}{\nu - q} - \frac{(r - \tau + 1)(\mu + \nu - 2q - p)}{\nu - q} Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)r, & \text{when } q < \nu, \\
\end{cases} \\
z \cdot \nabla z Q_{qr}^{\mu \nu \tau} &=\begin{cases} (\mu - q - p + 1)(q - \nu + r - \tau) \frac{Q_{qr}^{\mu \nu \tau}}{q - \nu} - (p + 1)(\nu - r + 1)Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)r, & \text{when } q > \nu, \\
-(\nu - q + 1)Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)(r - 1), & \text{when } q \leq \nu, \\
(q - \nu + 1)Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)r, & \text{when } q \geq \nu, \\
2(q - p - \nu + 1)(q - r + 1) \frac{Q_{qr}^{\mu \nu \tau}}{\nu - q} + (r - \tau + 1)(\mu + \nu - 2q - p)Q_{qr}^{\mu \nu \tau} (q - 1)(p + 1)r, & \text{when } q < \nu, \\
\end{cases} \\
x \cdot \nabla_z Q_{qr}^{\mu \nu \tau} &= (\mu - q - p + 1)Q_{qr}^{\mu \nu \tau}, \quad \text{when } q \geq \nu, \\
(\mu - q - p + 1)Q_{qr}^{\mu \nu \tau} + (\nu - r + 1)Q_{qr}^{\mu \nu \tau}, \quad \text{when } q < \nu, \\
y \cdot \nabla_z Q_{qr}^{\mu \nu \tau} &= \begin{cases} (p + 1)Q_{q^p r}^{\mu \nu \tau} + (r - \tau + 1)Q_{qr}^{\mu \nu \tau}, & \text{when } q \geq \nu, \\
(q + p - \nu + 1)Q_{q^p r}^{\mu \nu \tau} + (r - \tau + 1)Q_{qr}^{\mu \nu \tau}, & \text{when } q < \nu, \\
\end{cases}
\end{align*}$$

Hence, we obtain the generalized radial equation, satisfied by the functions $\psi_{q^p r}^{\mu \nu \tau}(\xi, \eta, \zeta)$:

$$\begin{align*}
\nabla^2 \psi_{q^p r}^{\mu \nu \tau} + 4q \partial_\xi \psi_{q^p r}^{\mu \nu \tau} + 4(p + r) \partial_\eta \psi_{q^p r}^{\mu \nu \tau} + 4(\mu + \nu + \tau - p - q - r) \partial_\zeta \psi_{q^p r}^{\mu \nu \tau} \\
+ 2p(q - r + 1) \frac{\partial_\xi \psi_{q^p r}^{\mu \nu \tau}}{q - \nu + 1} - \frac{2(\mu - q - p)(r - \tau)}{q - \nu + 1} \partial_\xi \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r \\
+ 2(q - \nu) \partial_\xi \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r + \frac{2(\mu - q - p)(q - \nu + r - \tau + 1)}{q - \nu + 1} \partial_\xi \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r \\
- \frac{2(p - \nu - r)}{q - \nu + 1} \partial_\xi \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)(r + 1) + 2(q - \nu) \partial_\xi \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r + 2(\mu - q - p) \partial_\xi \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r \\
+ 2(q - r) \partial_\eta \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r + 2p \partial_\eta \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r + 2(r - \tau) \partial_\eta \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r \\
+ 2(\nu - r) \partial_\zeta \psi_{q^p r}^{\mu \nu \tau} + 2p \partial_\zeta \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r + 2(r - \tau) \partial_\zeta \psi_{q^p r}^{\mu \nu \tau} (q - 1)(p + 1)r \\
= -2(E - V) \psi_{q^p r}^{\mu \nu \tau}, \quad \text{for } q > \nu,
\end{align*}$$

(29)
\[ \nabla^2 \psi_{qpr} + 4 q \partial_{\xi_1} \psi_{qpr} + 4 (p + r) \partial_{\xi_2} \psi_{qpr} + 4 (\mu + r - p) \partial_{\xi_3} \psi_{qpr} \\
+ 2 p(q - r + 1) \partial_{\xi_2} \psi_{(q+1)(p-1)} - 2 (\mu - q - p)(r - \tau) \partial_{\xi_2} \psi_{(q+1)(p-1)} \\
+ 2(p + 1)(q - r) \partial_{\xi_2} \psi_{(q-1)(p+1)r} - 2 (\mu - q - p)(r - \tau) \partial_{\xi_2} \psi_{(q-1)(p+1)(r-1)} \\
+ 2 (\mu - q - p)(r - \tau + 1) \partial_{\xi_3} \psi_{(q+1)(p-1)r} - 2 p(q - r) \partial_{\xi_3} \psi_{(q+1)(p-1)(r-1)} \\
+ 2 p(q - r) \partial_{\xi_3} \psi_{(q-1)(p-1)r} - 2(r - \tau)(\mu - q - p + 1) \partial_{\xi_3} \psi_{(q-1)(p+1)(r-1)} \\
+ 2 (\mu - q - p) \partial_{\eta_3} \psi_{(q+1)(p)(r+1)} + 2(q - r) \partial_{\eta_3} \psi_{(q+1)(p)(r+1)} + 2 p \partial_{\eta_3} \psi_{(q-1)(p)(r)} \\
+ 2(r - \tau) \partial_{\eta_3} \psi_{(q-1)(p)(r-1)} = -2(E - V) \psi_{qpr}, \quad \text{for } q = \nu, \\
\text{(30b)} \]

\[ \nabla^2 \psi_{qpr} + 4 q \partial_{\xi_1} \psi_{qpr} + 4 (p + r) \partial_{\xi_2} \psi_{qpr} + 4 (\mu + \nu - \tau - p - q) \partial_{\xi_3} \psi_{qpr} \\
+ 2(\nu - q) \partial_{\xi_2} \psi_{(q+1)(p-1)r} + \frac{2(\nu + 1)(q - r)}{\nu - q + 1} \partial_{\xi_2} \psi_{(q+1)(p-1)r} \\
- \frac{2(\mu - q - p)(r - \tau)}{\nu - q + 1} \partial_{\xi_2} \psi_{(q+1)(p-1)(r-1)} - \frac{2(q + p - \nu)(q - r)}{\nu - q + 1} \partial_{\xi_2} \psi_{(q-1)(p-1)(r-1)} \\
+ \frac{2(r - \tau)(\mu + \nu - 2q - p + 1)}{\nu - q + 1} \partial_{\xi_2} \psi_{(q-1)(p-1)(r-1)} \\
+ 2 (\mu - q - p) \partial_{\eta_3} \psi_{(q+1)(p+1)r} + 2(q - r) \partial_{\eta_3} \psi_{(q+1)(p+1)(r+1)} + 2(q + p - \nu) \partial_{\eta_3} \psi_{(q+1)(p)(r)} \\
+ 2(r - \tau) \partial_{\eta_3} \psi_{(q+1)(p)(r-1)} = -2(E - V) \psi_{qpr}, \quad \text{for } q < \nu, \\
\text{(30c)} \]

where \( \nabla^2 \psi_{qpr} \) is given in Eq. (27). Only six internal variables \( \xi_1, \xi_2, \xi_3, \eta_2, \eta_3, \) and \( \zeta_3 \) are involved both in the equations and in the functions. Eq. (30) holds either for \( D > 6 \) or for \( 4 \leq D \leq 6 \). For the latter cases some selfdual representation, antiselfdual representation, or equivalent representations may occur. Especially, for a four-body system in \( D = 4 \) dimensions, the representation \([\mu, 1, 1]\) is equivalent to the representation \([\mu, 0, 0]\), but the generalized radial equations for them are decoupled. They will be coupled for the \( N \)-body system with \( N > D = 4 \).

### IV. QUANTUM \( \textit{N}-\text{BODY SYSTEM IN D DIMENSIONS} \)

It is hard to write a unified formulas of the generalized radial equations for an \( N \)-body system in arbitrary \( D \)-dimensions. However, from the study of the three-body [26,31] and four-body system, we are able to summarize the main features on separating the rotational degrees of freedom from the internal ones for an \( N \)-body Schrödinger equation in \( D \) dimensions.

First, after removing the motion of the center of mass, there are \((N - 1)\) Jacobi
coordinate vectors $\mathbf{R}_j$ for an $N$-body system. On the other hand, in an $D$-dimensional space it needs $(D - 1)$ vectors to determine the body-fixed frame. When $D \geq N$, all Jacobi coordinate vectors are used to determine the body-fixed frame, and all internal variables can be chosen as $\mathbf{R}_j \cdot \mathbf{R}_k$. The numbers of the rotational variables and the internal variables are $(N - 1)(2D - N)/2$ and $N(N - 1)/2$, respectively. When $D < N$, only $(D - 1)$ Jacobi coordinate vectors are involved to determine the body-fixed frame, and the rest can be expressed by the first $(D - 1)$ Jacobi coordinate vectors and the internal variables. The set of internal variables $\mathbf{R}_j \cdot \mathbf{R}_k$ is no longer complete because it could not distinguish two configurations, say with different $\mathbf{R}_D$ reflecting to the superplane spanned by the first $(D - 1)$ Jacobi coordinate vectors. The correct choice for the internal variables are

$$\xi_{jk} = \mathbf{R}_j \cdot \mathbf{R}_k, \quad \zeta_\alpha = \sum_{a_1 \ldots a_D} \epsilon_{a_1 \ldots a_D} R_{1a_1} \ldots R_{(D-1)a_{D-1}} R_{a a_D},$$

$$1 \leq j \leq D - 1, \quad j \leq k \leq N - 1, \quad D \leq \alpha \leq N - 1. \quad (31)$$

The numbers of the rotational variables and the internal variables are $D(D - 1)/2$ and $D(2N - D - 1)/2$, respectively.

Second, for an $N$-body system in $D$-dimensions ($D \geq N$), the angular momentum is described by an irreducible representation of $\text{SO}(D)$ denoted by an $(N - 1)$-row Young pattern $[\mu] \equiv [\mu_1, \mu_2, \ldots, \mu_{N-1}]$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{N-1}$. Due to the rotational symmetry, one only needs to discuss the eigenfunctions of angular momentum with the highest weight. The complete set of independent base functions with the highest weight consists of the eigenfunctions $Q^{[\mu]}_{(q)}(\mathbf{R}_1, \ldots \mathbf{R}_{N-1})$ identified by the standard Young tableau $(q)$. Filling the digits $1, 2, \ldots, N - 1$ arbitrarily into a given Young pattern $[\mu]$ we obtain a young tableau. A Young tableau is called standard if the digit in every column of the tableau increases downwards and the digit in every row does not decrease from left to right. Any standard Young tableau is described by a set of parameters $(q)$ which contains $(N - 1)(N - 2)/2$ parameters $q_{jk}$, $1 \leq k \leq j \leq N - 1$, denoting the number of the digit $j$ in the $k$th row in the standard Young tableau. The number of independent base functions $Q^{[\mu]}_{(q)}(\mathbf{R}_1, \ldots \mathbf{R}_{N-1})$ in the complete set is equal to the dimension $d_{[\mu]}[\text{SU}(N - 1)]$ of the irreducible representation $[\mu]$ of the SU($N - 1$) group. $Q^{[\mu]}_{(q)}(\mathbf{R}_1, \ldots \mathbf{R}_{N-1})$ is a homogeneous polynomial of degree $\sum \mu_k$ with respect to the components of $(N - 1)$ Jacobi coordinate vectors $\mathbf{R}_j$, and satisfies the generalized Laplace equations [see Eq. (20)]. The explicit form of $Q^{[\mu]}_{(q)}(\mathbf{R}_1, \ldots \mathbf{R}_{D-1})$
for the given standard Young tableau \((q)\) is very easy to write. In the Young tableau, for each column with the length \(t\), filled by digits \(j_1 < j_2 < \ldots < j_t\), \(Q^{[\mu]}_{(q)}(R_1, \ldots R_{D-1})\) contains a determinant as a factor. The \(r\)th row and \(s\)th column in the determinant is \(R_{jr(2s-1)} + i R_{jr(2s)}\) if \(D > 2(N-1)\). \(Q^{[\mu]}_{(q)}(R_1, \ldots R_{D-1})\) also contains a numerical coefficient for convenience. When \(N \leq D \leq 2(N-1)\), some selfdual representation, antiselfdual representation and equivalent representations have to be considered just like the discussion given in the end of Sec. II. When \(D < N\), only the first \((D-1)\) Jacobi coordinate vectors are involved in the base functions \(Q^{[\mu]}_{(q)}(R_1, \ldots R_{D-1})\), which are the same as those for smaller \(N = D\).

At last, when \(D \geq N\), any wave function \(\Psi^{[\mu]}_{M}(R_1, \ldots, R_{N-1})\) with the given angular momentum \([\mu]\) can be expanded with respect to the complete and independent base functions \(Q^{[\mu]}_{(q)}(R_1, \ldots, R_{N-1})\)

\[
\Psi^{[\mu]}_{M}(R_1, \ldots, R_{N-1}) = \sum_{(q)} \psi^{[\mu]}_{(q)}(\xi) Q^{[\mu]}_{(q)}(R_1, \ldots, R_{N-1}), \tag{32}
\]

where the coefficients \(\psi^{[\mu]}_{(q)}(\xi)\), called the generalized radial functions, only depends upon the internal variables. When \(D < N\), \(\psi^{[\mu]}_{(q)}(\xi)\) and \(Q^{[\mu]}_{(q)}(R_1, \ldots, R_{N-1})\) in Eq. (32) have to be replaced with \(\psi^{[\mu]}_{(q)}(\xi, \zeta)\) and \(Q^{[\mu]}_{(q)}(R_1, \ldots, R_{D-1})\), respectively. Substituting Eq. (32) into the \(N\)-body Schrödinger equation in the center-of-mass frame

\[
\sum_{j=1}^{N-1} \nabla^2_{R_j} \Psi^{[\mu]}_{M}(R_1, \ldots, R_{N-1}) = -2 \{E - V(\xi)\} \Psi^{[\mu]}_{M}(R_1, \ldots, R_{N-1}), \tag{33}
\]

one is able to obtain the generalized radial functions. The main calculation is to apply the Laplace operator to the function \(\Psi^{[\mu,\nu,\tau]}_{M}(x, y, z)\). The calculation consists of three parts. The first is to apply the Laplace operator to the generalized radial functions \(\psi^{[\mu]}_{(q)}(\xi)\), which can be calculated by replacement of variables. When \(D \geq N\) we have

\[
\nabla^2 \psi^{[\mu]}_{(q)}(\xi) = \left\{ \sum_{j=1}^{N-1} \left( 4 \xi_{jj} \partial_{\xi_{jj}}^2 + 2D \partial_{\xi_{jj}} \right) \right. \\
+ \sum_{j=1}^{N-1} \sum_{k=j+1}^{N-1} \left[ (\xi_{jj} + \xi_{kk}) \partial_{\xi_{jk}}^2 + 4\xi_{jk} \left( \partial_{\xi_{jj}} + \partial_{\xi_{kk}} \right) \partial_{\xi_{jk}} \right] \\
+ 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^{N-1} \sum_{t=k+1}^{N-1} \xi_{kt} \partial_{\xi_{jk}} \partial_{\xi_{jt}} \right\} \psi^{[\mu]}_{(q)}(\xi), \tag{34}
\]

where \(\xi_{jk} = \xi_{kj}\) and \(\partial_{\xi}\) denotes \(\partial / \partial \xi\) and so on. The second is to apply the Laplace operator to the generalized spherical harmonic polynomials. This part is vanishing because
the polynomials satisfy the Laplace equation. The last is the mixed application. When $D \geq N$ we have

$$2 \sum_{j=1}^{N-1} \left\{ \left( \partial_{\xi_j} \psi^{[\mu]}_{(q)} \right) + \sum_{j \neq k=1}^{N-1} \left( \partial_{\xi_j} \psi^{[\mu]}_{(q)} \right) R_k \right\} \cdot \nabla R_j Q^{[\mu]}_{(q)},$$

(35)

where the formulas for $R_j \cdot \nabla R_j Q^{[\mu]}_{(q)}$ and $R_k \cdot \nabla R_j Q^{[\mu]}_{(q)}$ can be calculated from the property of the polynomial $Q^{[\mu]}_{(q)}(R_1, \ldots, R_{N-1})$. When $D < N$, the internal variables have to be chosen as those given in Eq. (31) so that Eq. (34) becomes more complicated and Eq. (35) contains more terms of $\partial \zeta_{\alpha} / \partial R_j \cdot \nabla R_j Q^{[\mu]}_{(q)}$ [26].

V. CONCLUSIONS

In this paper, the problem of separating the rotational degrees of freedom from the internal ones for the Schrödinger equation of a four-body system in $D$ dimensions is studied in detail by the method of the generalized spherical harmonic polynomials. We have found a complete set of independent base functions with the given angular momentum described by an irreducible representation $[\mu, \nu, \tau]$ of $\text{SO}(D)$. This set of base functions have different form for the case $D > 6$ and $3 \leq D \leq 6$. We have provided an appropriate choice of internal variables for this system and derived the generalized radial equations depending solely on internal variables. The main features on the problem of separating the rotational degrees of freedom from the internal ones for the Schrödinger equation of a $N$-body system in $D$ dimensions is summarized.

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