Algebra of screening operators for the deformed $W_n$ algebra

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Abstract

We construct a family of intertwining operators (screening operators) between various Fock space modules over the deformed $W_n$ algebra. They are given as integrals involving a product of screening currents and elliptic theta functions. We derive a set of quadratic relations among the screening operators, and use them to construct a Felder-type complex in the case of the deformed $W_3$ algebra.

1 Introduction

The method of bosonization is known to be the most effective way of calculating the conformal blocks in conformal field theory. The basic idea in this approach is to realize the commutation relations for the symmetry algebra (such as the Virasoro or affine Lie algebras) and the chiral primary fields in terms of operators acting on some bosonic Fock spaces. Quite often, the physical Hilbert space of the theory is not the total Fock space itself, but only a subquotient of it. In this case, it is necessary to ‘project out’ the physical space from the Fock spaces by a cohomological method. In the case of the Virasoro minimal models, Felder introduced a two-sided complex

$$
\cdots \rightarrow \mathcal{F}^{(-1)} \rightarrow \mathcal{F}^{(0)} \rightarrow \mathcal{F}^{(1)} \rightarrow \mathcal{F}^{(2)} \rightarrow \cdots ,
$$

(1.1)

consisting of Fock spaces $\mathcal{F}^{(i)}$. As it turns out, the cohomology of this complex vanishes except at the 0-th degree, and the remaining non-trivial cohomology

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affords the irreducible representation of the Virasoro algebra. The primary fields realized on the Fock spaces commute with the coboundary operator $d$, and hence make sense as operators on the cohomology space. Similar resolutions have been described for representations of affine Lie algebras \[3, 4, 5\]. For an extensive review on this subject, the reader is referred to \[6\].

It has been recognized that the idea of bosonization is quite fruitful also in non-critical lattice models \[7, 8\] and massive field theory \[3\]. The present work is motivated by the recent progress along this line, on the restricted solid-on-solid models \[10, 11\]. In \[10\], the Andrews-Baxter-Forrester (ABF) model was studied. Here the counterpart of the conformal primary fields are the vertex operators (half transfer matrices) which appear in the corner transfer matrix method. Just as in conformal field theory, the bosonization discussed in \[10\] consists in two steps. The first step is to introduce a family of bosonic Fock spaces and realize the commutation relations of the vertex operators in terms of bosons. The second step is to realize the physical space of states of the model as the 0-th cohomology of a complex of the type \((1.1)\). In fact, the analogy with conformal field theory goes further. Each Fock space has the structure of a module over the deformed Virasoro algebra (DVA) discovered in \[12\], where the deformation parameter $x$ ($0 < x < 1$) is the one which enters the Boltzmann weights of the models. As was shown in \[10, 13\], the above complex is actually that of DVA modules, i.e., the operator $d$ commutes with the action of DVA. Felder’s complex \((1.1)\) is recovered in the limit $x \to 1$ (we shall refer to this as the conformal limit).

The ABF models have $sl_n$ generalizations, the former being the case $n = 2$. In the work \[11\], the first step of the bosonization was carried over to the case of general $n$. However the second step was not addressed there. The aim of the present paper is to construct an analog of the complex \((1.1)\) in the case $n = 3$. In this situation the role of DVA is played by the deformed $W_3$ algebra introduced in \[14, 15\]. We shall also construct for general $n$ a family of intertwiners of deformed $W_n$ algebras (DWA), which we expect to be sufficient to construct the complex in the general case.

In the conformal case, such a Felder-type complex for $\hat{sl}_3$ was constructed in \[4\]. Strictly speaking, \[4\] discusses representations of affine Lie algebras, while our case corresponds (in the limit) to those of $W_n$ algebras. In other words we are dealing with a coset theory rather than a Wess-Zumino-Witten theory. However the construction of the complex is practically the same for both cases.

In the case $n \geq 3$, each component $\mathcal{F}^{(i)}$ of the complex is itself a direct sum of an infinite number of Fock spaces. The coboundary operator $d$ can be viewed as a collection of maps between various Fock spaces. We call these maps the screening operators. They are given in the form of an integral of a product of screening currents, multiplied by a certain kernel function expressed in terms of elliptic theta functions. The main result of this paper is the explicit construction of these screening operators. In comparison with the conformal case, a simplifying feature is that the screening operators can be expressed as products of
more basic, mutually commuting operators. In the conformal case, such a multiplicative structure exists only ‘inside the contour integral’ (see [5] and section 6 below). It has been pointed out [15] that the screening currents satisfy the commutation relations of the elliptic algebra studied in [16]. This connection turns out to be quite helpful in finding the basic operators referred to above and their commutation relations.

Let us mention some questions that remain open. In order to ensure the nilpotency property $d^2 = 0$, the signs of the screening operators have to be chosen carefully. We have verified that this is possible for $n = 3$. In the general case there are additional complications which we have not settled yet. More importantly, in this paper we do not discuss the cohomology of the complex, though we expect the same result persists as in the conformal case. The construction of the complex in [5] is based on the one-to-one correspondence between intertwiners of Fock space modules over $\hat{sl}_n$, and the singular vectors in the Verma modules of $U_q(sl_n)$ with $q$ a root of unity. It would be interesting to search for an analog of the latter in the deformed situation.

The outline of this paper is as follows. In section 2 we prepare the notation and the setting. Also the form of the complex to be constructed is briefly explained. The construction of the screening operators for general $n$ is rather technical. To ease the reading, we first discuss in section 3 the case $n = 3$ in detail. In section 4 we introduce the screening operators in general, and state their commutation relations. In section 5 we show that they commute with the action of DWA. In section 6 we briefly discuss the CFT limit of the basic operators. The text is followed by 3 appendices. In appendix A we discuss the condition when we construct intertwiners between two Fock modules. In appendix B we list the commutation relations for the basic operators that will be used to derive the quadratic relations of the screening operators. In appendix C we outline the proof that the screening operators commute with DWA.

## 2 Preliminaries

In this section we prepare the notation to be used in the text. Throughout this paper, we fix a positive integer $r \geq n + 2$ and a real number $x$ with $0 < x < 1$.

### 2.1 Lie algebra $sl_n$

Let us fix the notation concerning the Lie algebra $sl_n$.

Let $\varepsilon_i$ ($1 \leq i \leq n$) be an orthonormal basis in $\mathbb{R}^n$ relative to the standard inner product $(\ , \ )$. We set $\bar{\varepsilon}_i = \varepsilon_i - \varepsilon_{i+1} = (1/n) \sum_{j=1}^n \varepsilon_j$. We shall denote by:

- $\omega_i = \sum_{j=1}^i \bar{\varepsilon}_j$ the fundamental weights,
- $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ the simple roots,
\[ \theta = \sum_{i=1}^{n-1} \alpha_i \] the maximal root,
\[ P = \sum_{i=1}^{n} \mathbb{Z} \varepsilon_i \] the weight lattice,
\[ Q = \sum_{i=1}^{n-1} \mathbb{Z} \alpha_i \] the root lattice,
\[ \Delta_+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n-1 \} \] the set of positive roots,
\[ W \simeq S_n \] the classical Weyl group,
\[ W \simeq W \wr Q \] the affine Weyl group.

For an element \( \gamma \in Q \), we set
\[ |\gamma| = \sum_{i=1}^{n-1} c_i \] for \( \gamma = \sum_{i=1}^{n-1} c_i \alpha_i \).

For a root \( \alpha = \varepsilon_i - \varepsilon_j \), \( r_\alpha \) signifies the corresponding reflection (often identified with the transposition \( (ij) \in S_n \)). We also write \( s_i = r_\alpha \).

### 2.2 Bosons

We recall from [11] our convention about the bosons. Let \( \beta_j^m \) be the oscillators \((1 \leq j \leq n-1, m \in \mathbb{Z} \setminus \{0\})\) with the commutation relations
\[
[\beta_j^m, \beta_k^{m'}] = \begin{cases} 
\frac{(n-1)m_x [(r-1)m_x]}{[n]_x [r]_x} \delta_{m+m',0} & (j = k), \\
-m_x \text{sgn}(j-k) \frac{[m]_x [(r-1)m_x]}{[n]_x [r]_x} \delta_{m+m',0} & (j \neq k).
\end{cases}
\]

Here the symbol \([a]_x\) stands for \((x^a - x^{-a})/(x - x^{-1})\). Define \( \beta^n_m \) by
\[
\sum_{j=1}^{n} x^{-2jm} \beta^j_m = 0.
\]

Then the commutation relations (2.1),(2.2) are valid for all \( 1 \leq j, k \leq n \).

We also introduce the zero mode operators \( P_\lambda, Q_\lambda \) indexed by \( \lambda \in P \). By definition they are \( \mathbb{Z} \)-linear in \( \lambda \) and satisfy
\[
[iP_\lambda, Q_\mu] = (\lambda, \mu) \quad (\lambda, \mu \in P).
\]

We shall deal with the bosonic Fock spaces \( \mathcal{F}_{l,k} \) \((l, k \in P)\) generated by \( \beta_{-m}^j \) \((m > 0)\) over the vacuum vectors \( |l,k\rangle \):
\[
\mathcal{F}_{l,k} = \mathbb{C}[\{\beta_{-1}^j, \beta_{-2}^j, \ldots \}_{1 \leq j \leq n}] |l,k\rangle.
\]
where
\[
\beta^j_m |l, k\rangle = 0, \quad (m > 0),
\]
\[
P_\alpha |l, k\rangle = (\alpha, \sqrt{r} l - \sqrt{r-1} k) |l, k\rangle,
\]
\[
|l, k\rangle = e^{i \sqrt{r-1} Q_l - i \sqrt{r-1} Q_k} |0, 0\rangle.
\]
In what follows we set
\[
\hat{\pi}_i = \sqrt{r(r-1)} P_{\alpha_i}.
\]
It acts on \(F_{l,k}\) as an integer,
\[
\hat{\pi}_i |F_{l,k}\rangle = (\alpha_i, r l - (r-1) k).
\]
In this paper we work on \(F_\lambda^{\text{def}} = F_{l,\lambda} (\lambda \in P)\) with a fixed value of \(l \in P\).

2.3 Screening currents
We define the screening currents \(\xi_j(u)\) \((j = 1, \cdots, n-1)\) by
\[
\xi_j(u) \equiv U_j(z) = e^{i \sqrt{r-1} Q_\alpha z^\frac{1}{2} \xi_j + \frac{1}{r} \sum_{m \neq 0} \frac{1}{m} (\beta^j_m - \beta^j_{m+1}) (x^i z)^{-m}}, \quad (2.4)
\]
where the variable \(u\) is related to \(z\) via \(z = x^{2u}\).

We shall need the following commutation relations between them.
\[
\xi_j(u) \xi_j(v) = \frac{[u - v - 1]}{[u - v + 1]} \xi_j(v) \xi_j(u), \quad (2.5)
\]
\[
\xi_j(u) \xi_{j \pm 1}(v) = \frac{[u - v + \frac{1}{2}]}{[u - v - \frac{1}{2}]} \xi_{j \pm 1}(v) \xi_j(u), \quad (2.6)
\]
\[
\xi_i(u) \xi_j(v) = \xi_j(v) \xi_i(u) \quad (|i - j| > 1). \quad (2.7)
\]
Here the symbol \([u]\) stands for the theta function satisfying
\[
[u + r] = -[u] = [-u],
\]
\[
[u + \tau] = -e^{2\pi i (u + \frac{1}{2})/r} [u] \text{ where } \tau = \frac{\pi i}{\log x}.
\]

Explicitly it is given by
\[
[u] = x^{u^2/r-u} \Theta_{x^{2u}}(x^{2u}), \quad (2.8)
\]
\[
\Theta_q(z) = (z; q)_\infty (qz^{-1}; q)_\infty (q; q)_\infty, \quad (2.9)
\]
\[
(z; q)_\infty = \prod_{i=0}^{\infty} (1 - zq^i). \quad (2.10)
\]

Quite generally we say that an operator \(X\) has weight \(\nu\) if \(X F_\lambda \subset F_{\lambda + \nu}\) for any \(\lambda\). Then \(\xi_j(u)\) has weight \(-\alpha_j\). This implies
\[
\hat{\pi}_i \xi_j(u) = \xi_j(u) (\hat{\pi}_i - (\alpha_i, \alpha_j)(1-r)). \quad (2.11)
\]
2.4 The complex

In this section we describe the form of the complex we are going to construct.

Fix an integral weight $\Lambda \in P$ satisfying
\[ (\Lambda, \alpha_i) > 0 \quad (i = 1, \ldots, n - 1), \quad (\Lambda, \theta) < r, \]  
(2.12)

Note that
\[ 0 < (\Lambda, \alpha) < r \]  
(2.13)
for any positive root $\alpha$. Consider the orbit of $\Lambda$ under the action of the affine Weyl group $W$. An element of $W\Lambda$ can be written uniquely as
\[ \lambda = t_\gamma \sigma \Lambda = \sigma \Lambda + r \gamma, \]  
(2.14)
where $\sigma \in W$ and $\gamma \in Q$. We assign a degree $\deg(\lambda) \in \mathbb{Z}$ to (2.14) by setting
\[ \deg(\lambda) = l(\sigma) - 2|\gamma|. \]  
(2.15)
Here $l(\sigma)$ denotes the length of $\sigma \in W$. (The right hand side of (2.15) is known as the modified length of $w = t_\gamma \sigma \in W$, see e.g. [4, 5].) We shall construct a complex of the form
\[ \cdots \rightarrow F^{(-1)}_{\Lambda} \rightarrow F^{(0)}_{\Lambda} \rightarrow F^{(1)}_{\Lambda} \rightarrow F^{(2)}_{\Lambda} \rightarrow \cdots \]  
(2.16)
where
\[ F^{(i)}_{\Lambda} = \bigoplus_{\lambda \in W\Lambda \atop \deg(\lambda) = i} F_{\lambda} \quad (i \in \mathbb{Z}). \]  
(2.17)
Except for $n = 2$, $F^{(i)}_{\Lambda}$ is a direct sum of an infinite number of Fock spaces.

The coboundary map $d : F^{(i)}_{\Lambda} \rightarrow F^{(i+1)}_{\Lambda}$ can be viewed as a collection of operators
\[ d_{\lambda, \lambda'} : F^{(i)}_{\Lambda} \rightarrow F^{(i+1)}_{\Lambda} \]  
(2.18)
associated with each pair $\lambda, \lambda' \in W\Lambda$ satisfying $\deg(\lambda') = \deg(\lambda) + 1$. We shall impose a restriction on the possible pair $\lambda, \lambda'$ as explained below.

For a positive root $\alpha$ and an element $\lambda \in W\Lambda$, we define an integer $m_\alpha(\lambda)$ by
\[ 0 < m_\alpha(\lambda) < r, \]  
(2.19)
\[ m_\alpha(\lambda) \equiv (\lambda, \alpha) \mod r. \]  
(2.20)
In other words, if $\lambda = t_\gamma \sigma \Lambda$, then we have
\[ m_\alpha(\lambda) = \begin{cases} (\sigma \Lambda, \alpha) & \text{if } (\sigma \Lambda, \alpha) > 0; \\ (\sigma \Lambda, \alpha) + r & \text{if } (\sigma \Lambda, \alpha) < 0. \end{cases} \]  
(2.21)
Set
\[ \lambda^\alpha = \lambda - m_\alpha(\lambda) \alpha = \begin{cases} t_\gamma r_\alpha \sigma \Lambda & \text{if } (\sigma \Lambda, \alpha) > 0; \\ t_{\gamma - \alpha} r_\alpha \sigma \Lambda & \text{if } (\sigma \Lambda, \alpha) < 0. \end{cases} \]  
(2.22)
Clearly the weight $\lambda^\alpha$ also belongs to the orbit $W\Lambda$. 
**Definition 2.1** We say that an ordered pair \((\lambda, \lambda')\) is admissible if the following hold for some positive root \(\alpha\).

\[
\lambda' = \lambda^\alpha, \quad (2.23)
\]

\[
\deg(\lambda^\alpha) = \deg(\lambda) + 1. \quad (2.24)
\]

We set \(d_{\lambda', \lambda} = 0\) if \(\lambda, \lambda'\) is not admissible. Otherwise, write \(\lambda' = \lambda^\alpha\) and

\[
d_{\lambda^\alpha, \lambda} = X_\alpha(\lambda) : \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\lambda^\alpha}. \quad (2.25)
\]

The construction of the complex is equivalent to finding an operator \((2.25)\) for each admissible pair \(\lambda, \lambda^\alpha\), so that we have \(d^2 = 0\). We shall refer to \((2.25)\) as a screening operator. We also require that the screening operators commute with the DWA generators. In practice, we find it convenient to construct the screening operators in the form

\[
X_\alpha(\lambda) = s_\alpha(\lambda)X_\alpha(\lambda)
\]

where \(s_\alpha(\lambda) = \pm 1\) is a sign factor. In Section 4 we give both \(s_\alpha(\lambda)\) and \(X_\alpha(\lambda)\) so that we have \(d^2 = 0\) for the case \(n = 3\). The general case is incomplete because we could not find a proper choice of the signs \(s_\alpha(\lambda)\).

The construction of the screening operators \((2.25)\) is based on the screening currents \((2.4)\). Let us consider the case where \(\alpha\) in \((2.25)\) is a simple root \(\alpha_j\). It turns out that the pair \(\lambda, \lambda^\alpha_j\) is admissible for any \(\lambda \in W\Lambda\) (see Lemma A.3). In this case the operator \((2.25)\) can be found as follows:

\[
X_\alpha_j(\lambda) = X_j^a \quad (a = m_{\alpha_j}(\lambda)), \quad (2.26)
\]

\[
X_j = \oint_{|z| = 1} \frac{dz}{2\pi iz} \xi_j(u) \frac{[u + \frac{1}{2} - \hat{\pi}_j]}{[u - \frac{1}{2}]} \hspace{1cm} (z = x^{2u}). \quad (2.27)
\]

Here the integration is taken over the contour \(|z| = 1\). Notice that the kernel function \(F(u) = [u + \frac{1}{2} - \hat{\pi}_j]/[u - \frac{1}{2}]\) has the quasi-periodicity

\[
F(u + \tau) = e^{2\pi i(1-\hat{\pi}_j)/\tau} F(u) \quad (2.28)
\]

which ensures that the integrand of \((2.27)\) is a single valued function in \(z\).

For \(n = 2\), \((2.27)\) exhausts the possible screening operators. For \(n \geq 3\) we must also construct operators corresponding to non-simple roots. As we shall see, they are given by similar (but more complicated) integrals over products of the screening currents.

### 3 Case \(n = 3\)

Before embarking upon the construction of the complex in general, let us first elaborate on the case \(n = 3\). Hopefully this will make clear the main points of the construction.
The following figure shows the configuration of the weights in the orbit $W\Lambda$ for $n = 3$.

Figure 1. The orbit $W\Lambda$ for $n = 3$. It forms a hexagonal lattice, consisting of three types of basic hexagons $A, B, C$ and their translates by $r$ times the root lattice $Q$.

In the figure, each vertex represents a weight $\lambda \in W\Lambda$. An arrow from $\lambda$ to $\lambda'$ indicates that the pair $(\lambda, \lambda')$ is admissible. As was mentioned before, $(\lambda, \lambda^\alpha)$ is always admissible for a simple root $\alpha = \alpha_1, \alpha_2$. For $n = 3$ there is also the 'third root' $\theta = \alpha_1 + \alpha_2$. It turns out that $(\lambda, \lambda^\theta)$ is admissible if and only if

$$\lambda = t_\gamma \sigma \Lambda \quad \text{with} \quad \sigma = s_1, s_2, s_1s_2s_1.$$

The nilpotency $d^2 = 0$ leads to two types of relations for the screening operators. The first type involves only screening operators corresponding to one simple root $\alpha_j$, and has the form

$$X_j^a X_j^{r-a} = 0.$$

For $n = 2$, this relation was proved in [13]. The same argument applies to show $X_j^r = 0$ for any $j$. 

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The second type involves the root \( \theta = \alpha_1 + \alpha_2 \), and occurs for each square inside a hexagon (see Fig.1). Let us write the operator \( X_\theta(\lambda) \) as \( X_{12}^{(a)} (a = m_\theta(\lambda)) \), indicating that it has weight \(-a\theta\). Let \( a_i = (\Lambda_\alpha, \alpha) \) \((i = 1, 2)\), \( a_0 = r - a_1 - a_2 \), and suppose \( \lambda = t_\gamma \sigma \Lambda \). If \( \sigma = s_1 \), then \( m_\theta(\lambda) = a_2 \), and the following relations must be satisfied:

\[
\begin{align*}
X_2^{r-a_1}X_1^{a_2} \pm X_{12}^{(a_2)}X_2^{a_0} &= 0, \\
X_2^{a_1}X_1^{r-a_1} \pm X_1^{a_0}X_{12}^{(a_1)} &= 0, \\
X_2^{r-a_2}X_1^{a_0} \pm X_{12}^{(a_1)}X_1^{a_1} &= 0, \\
X_1^{r-a_0}X_2^{a_0} \pm X_{12}^{(a_2)}X_1^{a_1} &= 0.
\end{align*}
\]

The operator \( X_{12}^{(a)} \) and the signs \( \pm \) in the commutation relations will be given later. The relations in the other cases \( \sigma = s_2, s_1s_2s_1 \) are obtained by permuting the upper indices \((a_0, a_1, a_2) \rightarrow (a_2, a_0, a_1)\) successively.

In the conformal case \( \tau = 1 \) (and \( n = 3 \)), the ‘third’ screening operator \( X_\theta(\lambda) \) satisfying the relations (3.1)-(3.4) was found in the work [5]. The construction in [5] is based on the observation that the screening operators \( X_r \) for the simple roots satisfy the same Serre relations as do the Chevalley generators of the quantum group \( U_q(\mathfrak{sl}_n) \), with \( q \) being a root of unity \((q^{2r} = 1 \text{ in the present notation})\). With the aid of these relations, the operator \( X_\theta(\lambda) \) was expressed as a (non-commutative) polynomial in \( X_1 \) and \( X_2 \). This method does not easily generalize to the deformed case, since it appears that there is no analog of the Serre relations between \( X_1 \) and \( X_2 \). Nevertheless there exists a family of operators, in terms of which the third screening operator can be written in a simple factorized form.

Consider an operator of weight \(-\theta\) of the form

\[
\oint z_1 \oint z_2 \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} \xi_1(u_1)\xi_2(u_2) F(u_1, u_2),
\]

with some function \( F(u_1, u_2) \) which is periodic in \( u_i \) with period \( r \). The integration is taken over \(|z_1| = |z_2| = 1\). Recall (2.11) and (2.28). In order that the integrand be single valued in \( z_i \), we demand that

\[
F(u_1 + \tau, u_2) = e^{-2\pi i \hat{\pi}_1/r} F(u_1, u_2), \quad F(u_1, u_2 + \tau) = e^{-2\pi i (\hat{\pi}_2 - 1)/r} F(u_1, u_2),
\]

where \( \tau = \pi i / \log x \). Assume further that \( F(u_1, u_2) \) is holomorphic except for possible simple poles at \( u_i = 1/2 \) and \( u_i = 1/2 + 1/2 = 0 \). (As for the last pole, we have taken into account the commutation relation (2.6).) If we regard the \( \hat{\pi}_i \)'s as constants, then the space of functions satisfying these conditions is 3 dimensional. It is straightforward to find a spanning set of such functions. This motivates us to introduce the following family of operators parameterized by \( k \):

\[
X_{12}(k) = (-1)^k \oint z_1 \oint z_2 \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} \xi_1(u_1)\xi_2(u_2) \times \frac{[u_1 + k + \frac{1}{2} - \hat{\pi}_1]}{[u_1 - \frac{1}{2}]} \frac{[u_2 - k - \frac{1}{2} - \hat{\pi}_2]}{[u_2 - \frac{1}{2}]} \frac{[u_1 - u_2 - k - \frac{1}{2}]}{[u_1 - u_2 + \frac{1}{2}]}.
\]

(3.5)
Proposition 3.1

\[ [X_{12}(k), X_{12}(l)] = 0 \text{ for any } k, l. \]  
(3.6)

\[ X_1 X_2 = X_{12}(-1), \quad X_2 X_1 = X_{12}(0), \]  
(3.7)

\[ X_1 X_{12}(k) = X_{12}(k - 1) X_1, \]  
(3.8)

\[ X_2 X_{12}(k - 1) = X_{12}(k) X_2. \]  
(3.9)

The proof of these statements will be given later in the context of general \( n \).

Notice the periodicity relation

\[ X_{12}(k + r) = (-1)^{r-1} X_{12}(k). \]  
(3.10)

Set

\[ X_{12}^{(a)}(k) = \prod_{b=1}^{a} X_{12}(k - b + 1). \]  
(3.11)

Then, for any non-negative integers \( a, b \), we have

\[ X_1^{a+b} X_2^b = X_{12}^{(b)}(-a - 1) X_1^a, \]  
(3.12)

\[ X_1^a X_2^{a+b} = X_{12}^{(a)}(-b - 1), \]  
(3.13)

\[ X_2^a X_1^{a+b} = X_{12}^{(a)}(a + b - 1), \]  
(3.14)

\[ X_2^{a+b} X_1^b = X_{12}^{(b)}(a + b - 1) X_2^a. \]  
(3.15)

As an example, let us verify the first relation. Consider first the case \( b = 1 \). From (3.7) and (3.8) we have

\[
X_1^{a+1} X_2 = X_1^a X_{12}(-1) \\
= X_1^{a-1} X_{12}(-2) X_1 \\
= \cdots \\
= X_{12}(-a - 1) X_1^a.
\]

The case of general \( b \) follows immediately from this and the definition (3.11). The other relations are derived in a similar manner.

Now set

\[ \overline{X}_{\alpha_1 + \alpha_2} (\lambda) = X_{12}^{(a)}(k), \quad a = m_{\alpha_1 + \alpha_2}(\lambda), \; k = (\lambda, \alpha_1) - 1. \]

Comparing (3.12)-(3.15) with (3.1)-(3.4) and taking (3.10) into account, we see that the desired relations are satisfied up to sign.

It remains to settle the issue about the signs \( s_\alpha(\lambda) \). From the definition, the screening operators have the periodicity

\[ \overline{X}_\alpha (\lambda + \beta) = \overline{X}_\alpha (\lambda) \quad \text{if } \beta \in r\mathbb{Z}\alpha_1 + 2r\mathbb{Z}\alpha_2. \]

Thus the signs can also be chosen according to the same periodicity. A direct verification shows that the following is one possible solution for the \( s_\alpha(\lambda) \).
Figure 2. The choice of the signs $s_\alpha(\lambda)$. We set $a = \varepsilon_r^{k_1}$, $b = \varepsilon_r^{k_2}$, $\varepsilon_r = (-1)^{r-1}$. The vertices at the top row correspond to the weights $A = \Lambda$, $B = r_1 r_2 \Lambda + \alpha_1$, $C = r_2 r_1 \Lambda + \alpha_1$, $D = \Lambda + \alpha_1 - \alpha_2$, $E = r_1 r_2 \Lambda + 2\alpha_1 - \alpha_2$, $F = r_2 r_1 \Lambda + 2\alpha_1 - \alpha_2$, with $r_j = r_{\alpha_j}$.

4 Screeninig operators

4.1 Basic operators

We are now in a position to introduce the operators which will play a basic role in the construction of screening operators for general $n$.

Let $\alpha = \alpha_i + \cdots + \alpha_{i+m}$ be a positive root. We often write it as $\alpha_{i-i+m}$. Define

$$X_\alpha(k_1, \ldots, k_m) = \oint \cdots \oint \frac{dz_j}{2\pi i z_j} \frac{\xi_i(u_i)}{u_i - \frac{1}{2}} \cdots \frac{\xi_{i+m}(u_{i+m})}{u_{i+m} - \frac{1}{2}} \times \prod_{j=i+1}^{i+m} \frac{[u_{i+1} - u_j]}{[u_{i+1} - u_j + \frac{1}{2}]} f_{\alpha}^{(k_1, \ldots, k_m)}(u_i, \ldots, u_{i+m+1}, \hat{\pi}_i, \ldots, \hat{\pi}_{i+m}). \quad (4.1)$$

Here

$$f_{\alpha}^{(k_1, \ldots, k_m)}(u_i, \ldots, u_{i+m+1}, \hat{\pi}_i, \ldots, \hat{\pi}_{i+m}) = (-1)^{k_1 + \cdots + k_m}$$

$$\prod_{l=1}^{m} \frac{[u_{i+l-1} - u_{i+l} - \frac{1}{2}]}{[u_{i+l-1} - u_{i+l}]} \prod_{l=0}^{m} [u_{i+l} - k_l + k_{l+1} - \frac{1}{2} - \hat{\pi}_{i+l}], \quad (4.2)$$

and $k_0 = -1, k_{m+1} = 0$ is implied. The integrand of (4.1) is a single-valued function in $z_j$ ($i \leq j \leq i + m$). To see this note that

$$\xi_i(u_i) \cdots \xi_{i+m}(u_{i+m}) \left( \prod_{j=i}^{i+m} \frac{z_j^{i-\hat{\pi}_j} - z_j^{-i-\hat{\pi}_j}}{z_j^{i+\hat{\pi}_j} - z_j^{-i+\hat{\pi}_j}} \right)^{\frac{i+m}{2}} \xi_{i+m} \xi_{i+m+1} \cdots \xi_{i+2m} \quad (4.3)$$

is single-valued. When $\alpha = \alpha_j$ is a simple root, (4.1) reduces to (2.26).

The basic property of (4.1) is the following commutativity.
\textbf{Theorem 4.1} For any \(k_1, \ldots, k_m, p\) we have

\[ [X_\alpha(k_1, \ldots, k_m), X_\alpha(k_1 + p, \ldots, k_m + p)] = 0. \quad (4.4) \]

In view of this, we define for a non-negative integer \(a\)

\[ X_\alpha^{(a)}(k_1, \ldots, k_m) = \prod_{b=1}^{a} X_\alpha(k_1 - b + 1, \ldots, k_m - b + 1). \quad (4.5) \]

Sometimes, we abbreviate \(X_\alpha^{(a)}(k_1, \ldots, k_m)\) to \(X_\alpha^{(a)}(k_1, \ldots, k_m)\).

\textbf{Proof of Theorem 4.1.} Using the commutation relations (2.5)-(2.7) and (2.11), we can write the product \(X_\alpha(k_1, \ldots, k_m)X_\alpha(k_1 + p, \ldots, k_m + p)\) in the form

\[ \oint d\xi d\eta \xi_i(\eta_i) \xi_i(\eta_i + 1) \cdots \xi_{i+m}(\eta_{i+m}) \frac{\xi_i(\eta_i)}{\xi_i(\eta_i + 1)} \cdots \frac{\xi_{i+m}(\eta_{i+m})}{\xi_{i+m}(\eta_{i+m} + 1)} \times F(u_i, v_i, \ldots, u_{i+m}, v_{i+m}) (z_k = x^{2u_k}, w_k = x^{2v_k}), \]

and likewise for the product in the opposite order. Symmetrizing with respect to the integration variables and equating the integrand, we are led to prove the following equality.

\[ S \left( f^{(k_1, \ldots, k_m)}_{\alpha}(u_i, \ldots, u_{i+m}; \hat{\pi}_i - (1-r), \hat{\pi}_{i+1}, \ldots, \hat{\pi}_{i+m-1}, \right. \]

\[ \left. \hat{\pi}_{i+m} - (1-r) \right) f^{(k_1+p, \ldots, k_m+p)}_{\alpha}(u_i, \ldots, u_{i+m}; \hat{\pi}_i, \ldots, \hat{\pi}_{i+m} ) \times \prod_{j=i}^{i+m} \frac{u_j - v_j - 1}{u_j - v_j} \prod_{j=i}^{i+m-1} \frac{u_j - v_j + 1}{u_j - v_j + 1} \prod_{j=i+1}^{i+m} (-1) \frac{u_j - v_j + 1}{u_j - v_j - 1} \right) \]

\[ = S \left( f^{(k_1+p, \ldots, k_m+p)}_{\alpha}(u_i, \ldots, u_{i+m}; \hat{\pi}_i - (1-r), \hat{\pi}_{i+1}, \ldots, \hat{\pi}_{i+m-1}, \right. \]

\[ \left. \hat{\pi}_{i+m} - (1-r) \right) f^{(k_1, \ldots, k_m)}_{\alpha}(u_i, \ldots, u_{i+m}; \hat{\pi}_i, \ldots, \hat{\pi}_{i+m} ) \times \prod_{j=i}^{i+m} \frac{u_j - v_j + 1}{u_j - v_j} \prod_{j=i}^{i+m-1} \frac{u_j - v_j - 1}{u_j - v_j + 1} \prod_{j=i+1}^{i+m} (-1) \frac{u_j - v_j - 1}{u_j - v_j - 1} \right). \quad (4.6) \]

Here the symbol \(S\) means the symmetrization of \((u_j, v_j)\) for each \(j = i, \ldots, i+m\). This is equivalent to

\[ A \left( \prod_{l=1}^{m} [u_{i+l-1} - u_{i+l} - k_l - \frac{1}{2}] [u_i + k_l + 3 - \hat{\pi}_l] \left( \prod_{l=1}^{m-1} [u_{i+l} - k_l + k_{l+1} - \frac{1}{2} - \hat{\pi}_{i+l}] \right) \right. \]

\[ \times \left[ u_{i+m} - k_m + \frac{1}{2} - \hat{\pi}_{i+m} \right] \prod_{l=1}^{m} [v_{i+l-1} - v_{i+l} - k_l - p - \frac{1}{2}] [v_i + k_1 + p + \frac{1}{2} - \hat{\pi}_{i}] \]

\[ \times \left[ u_{i+m} - k_m + \frac{1}{2} - \hat{\pi}_{i+m} \right] \prod_{l=1}^{m} [v_{i+l-1} - v_{i+l} - k_l - p - \frac{1}{2}] [v_i + k_1 + p + \frac{1}{2} - \hat{\pi}_{i}] \]
\[
\left( \prod_{l=1}^{m-1} [v_{i+l} - k_l + k_{l+1} - \frac{1}{2} - \hat{\pi}_{i+l}] \right) [v_{i+m} - k_m - p - \frac{1}{2} - \hat{\pi}_{i+m}]
\]
\[
\times \prod_{j=i}^{i+m} [u_j - v_j - 1] \prod_{j=1}^{i+m-1} [u_j - v_{j+1} + \frac{1}{2} \prod_{j=i+1}^{i+m} [u_j - v_{j-1} + \frac{1}{2}]]
\]
\[
= A \left( \prod_{l=1}^{m} [u_{i+l-1} - u_{i+l} - k_l - \frac{1}{2}] [u_i + k_1 + \frac{1}{2} - \hat{\pi}_i] \left( \prod_{l=1}^{m-1} [u_{i+l} - k_l + k_{l+1} - \frac{1}{2} - \hat{\pi}_{i+l}] \right) \right.
\]
\[
\times \left[ u_{i+m} - k_m - \frac{1}{2} - \hat{\pi}_{i+m} \right] \prod_{l=1}^{m} [v_{i+l-1} - v_{i+l} - k_l - p - \frac{1}{2}] [v_i + k_1 + p + \frac{3}{2} - \hat{\pi}_i]
\]
\[
\times \left( \prod_{l=1}^{m-1} [v_{i+l} - k_l + k_{l+1} - \frac{1}{2} - \hat{\pi}_{i+l}] \right) [v_{i+m} - k_m - p + \frac{1}{2} - \hat{\pi}_{i+m}]
\]
\[
\times \prod_{j=i}^{i+m} [u_j - v_j + 1] \prod_{j=1}^{i+m-1} [u_j - v_{j+1} - \frac{1}{2}] \prod_{j=i+1}^{i+m} [u_j - v_{j-1} - \frac{1}{2}].
\]
\]

(4.7)

Here the symbol \( A \) means the anti-symmetrization of \((u_{i+j}, v_{i+j})\) for each \( j = 0, \ldots, m \).

We prove this equality by induction on \( m \). The (LHS) – (RHS) is a theta function of order 4 in \( u_i \). Using the induction hypothesis, one can check that it vanishes at \( u_i = v_i, v_i \pm 1 \). Taking into account the quasi-periodicity, we conclude that it must have a factor

\[
[u_i - v_i][u_i - v_i - 1][u_i - v_i + 1][u_i + 2v_i - u_{i+1} - v_{i+1} + \frac{1}{2} - \hat{\pi}_i].
\]

This is a contradiction unless \( \text{(LHS)} - \text{(RHS)} = 0 \). \( \square \)

### 4.2 Definition of screening operators

Let us come to the definition of the screening operators. Let \((\lambda, \lambda^a)\) be an admissible pair, with \( \alpha = \alpha_{i \ldots i+m} \). We define \( \mathfrak{X}_\alpha(\lambda) : \mathcal{F}_\lambda \to \mathcal{F}_{\lambda^a} \) by the formula

\[
\mathfrak{X}_\alpha(\lambda) = X^{(a)}_\alpha(k_1, \ldots, k_m),
\]

(4.9)

where

\[
a = m_\alpha(\lambda),
\]

(4.10)

\[
k_j = (\lambda, \alpha_{i \ldots i+j-1}) - 1.
\]

(4.11)

Note that on \( \mathcal{F}_\lambda \) the operator \( \hat{\pi}_j \) has the fixed value

\[
\hat{\pi}_j |_{\mathcal{F}_\lambda} = (r l + (1 - r)\lambda, \alpha_j)
\]

\[
= (\lambda, \alpha_j) \mod r.
\]

(4.12)

More explicitly the operator \( \mathfrak{H}_j \) is given as follows.
Proposition 4.3 Notations being as in (4.2) and (4.9) (4.12), we set

\[ f_{\alpha}^{(a)}(u_{1}^{(1)}, \ldots, u_{i}^{(a)}, \ldots, u_{i+m}^{(1)}, \ldots, u_{i+m}^{(a)}) \]

\[ = S \left( \prod_{b=1}^{a} f_{\alpha}^{(b)}(k_{1}-b+1, \ldots, k_{m-b}+1)(u_{1}^{(b)}, \ldots, u_{i+m}^{(b)}; \tilde{\pi}_{i} - (a - b)(1 - r), \tilde{\pi}_{i+1}, \ldots, \tilde{\pi}_{i+m-1}, \tilde{\pi}_{i+m} - (a - b)(1 - r)) \prod_{1 \leq b \leq c \leq a \atop 1 \leq j \leq 1 + m} \frac{[u_{j}^{(b)} - u_{j}^{(c)}]}{[u_{j}^{(b)} - u_{j}^{(c)}]} \right) \]

\[ \times \prod_{1 \leq b \leq c \leq a \atop 1 \leq j \leq 1 + m-1} \frac{[u_{j}^{(b)} - u_{j+1}^{(c)} + \frac{1}{2}]}{[u_{j}^{(b)} - u_{j+1}^{(c)}]} \prod_{1 \leq b \leq c \leq a \atop i \leq j \leq 1 + m} (-1)^{[u_{j}^{(b)} - u_{j-1}^{(c)}]} \left( \frac{u_{j}^{(b)} - u_{j}^{(c)} + \frac{1}{2}}{u_{j}^{(b)} - u_{j}^{(c)}} \right) \]

\[ (4.13) \]

\[ = S \left( \prod_{b=1}^{a} f_{\alpha}^{(b)}(k_{1}-a+b, \ldots, k_{m-a+b})(u_{1}^{(b)}, \ldots, u_{i+m}^{(b)}; \tilde{\pi}_{i} - (a - b)(1 - r), \tilde{\pi}_{i+1}, \ldots, \tilde{\pi}_{i+m-1}, \tilde{\pi}_{i+m} - (a - b)(1 - r)) \prod_{1 \leq b \leq c \leq a \atop 1 \leq j \leq 1 + m} \frac{[u_{j}^{(b)} - u_{j}^{(c)}]}{[u_{j}^{(b)} - u_{j}^{(c)}]} \right) \]

\[ \times \prod_{1 \leq b \leq c \leq a \atop 1 \leq j \leq 1 + m-1} \frac{[u_{j}^{(b)} - u_{j+1}^{(c)} + \frac{1}{2}]}{[u_{j}^{(b)} - u_{j+1}^{(c)}]} \prod_{1 \leq b \leq c \leq a \atop i \leq j \leq 1 + m} (-1)^{[u_{j}^{(b)} - u_{j-1}^{(c)}]} \left( \frac{u_{j}^{(b)} - u_{j}^{(c)} + \frac{1}{2}}{u_{j}^{(b)} - u_{j}^{(c)}} \right) \],

\[ (4.14) \]

where \( S \) means the symmetrization of \((u_{j}^{(1)}, \ldots, u_{j}^{(a)})\) for each \( j = i, \ldots, i + m \). Then we have

\[ \mathfrak{X}_{\alpha}(\lambda) = \oint \cdots \oint \prod_{1 \leq b \leq c \leq a \atop i \leq j \leq 1 + m} \frac{d_{z}^{(b)}}{2 \pi i z_{j}^{(b)}} \]

\[ \times \left( \frac{\xi_{i}(u_{i}^{(1)})}{[u_{i}^{(1)} - \frac{1}{2}]} \ldots \frac{\xi_{i}(u_{i}^{(a)})}{[u_{i}^{(a)} - \frac{1}{2}]} \right) \left( \frac{\xi_{i+m}(u_{i+m}^{(1)})}{[u_{i+m}^{(1)} - \frac{1}{2}]} \ldots \frac{\xi_{i+m}(u_{i+m}^{(a)})}{[u_{i+m}^{(a)} - \frac{1}{2}]} \right) \]

\[ \times \prod_{1 \leq b \leq c \leq a \atop i \leq j \leq 1 + m} \frac{[u_{j}^{(b)} - u_{j}^{(c)}]}{[u_{j}^{(b)} - u_{j}^{(c)}] - 1} \prod_{1 \leq b \leq c \leq a \atop i \leq j \leq 1 + m} (-1)^{[u_{j}^{(b)} - u_{j-1}^{(c)}]} \left( \frac{u_{j}^{(b)} - u_{j}^{(c)} + \frac{1}{2}}{u_{j}^{(b)} - u_{j}^{(c)}} \right) \]

\[ \times f_{\alpha}^{(a)}(u_{1}^{(1)}, \ldots, u_{i}^{(a)}, \ldots, u_{i+m}^{(1)}, \ldots, u_{i+m}^{(a)}). \] \[ (4.15) \]

The function \( f_{\alpha}^{(a)}(u_{1}^{(1)}, \ldots, u_{i}^{(a)}, \ldots, u_{i+m}^{(1)}, \ldots, u_{i+m}^{(a)}) \) has the following zeros:

(i) \( u_{j}^{(b)} = \frac{1}{2} (1 \leq b \leq a, i \leq j \leq i + m) \)

(ii) \( u_{j}^{(b)} = u_{j+1}^{(c)} + \frac{1}{2} (1 \leq b, c, d \leq a, i \leq j \leq i + m - 1) \)

(iii) \( u_{j}^{(b)} - \frac{1}{2} = u_{j+1}^{(d)} (1 \leq b, c, d \leq a, i \leq j \leq i + m - 1) \)
The function
\[
\prod_{1 \leq b \leq a} \frac{f^{(a)}(u_i^{(1)}, \ldots, u_{i+m}^{(a)})}{u^{(b)}_{j} - \frac{b}{2}}
\]
is invariant under the simultaneous shift \(u_j^{(b)} \mapsto u_j^{(b)} + c\). The function
\[
f^{(a)}(u_i^{(1)}, \ldots, u_{i+m}^{(a)}) \prod_{1 \leq b \leq a} [u_j^{(b)} - u_{j+1}^{(c)}]
\]
is holomorphic everywhere.

**Proof.** The equality of (1.13) and (1.14) follows from the proof of Theorem 1.1. Substituting (1.10)-(1.12) into (1.13) and comparing with the definition (1.2), we see that the function \(f^{(a)}_{\alpha, \beta} \) has a zero at \(u_j^{(b)} = \frac{b}{2} \) for \(1 \leq b \leq a \) and \(i + 1 \leq j \leq i + m\). A similar argument applied to (1.14) shows that it is true also for \(j = i\).

For the proof of the rest of Proposition 4.3, we recall the definition of the elliptic algebra in [10]. (See also Appendix B.) The adaptation to our context goes as follows.

For \(\pi_i \in \mathbb{Z}/r\mathbb{Z} \) (1 \( \leq i \) \( \leq n-1 \)) and \(\gamma = \sum_{i=1}^{n-1} a_i \pi_i \) \((a_i \in \mathbb{Z}_{\geq 0})\) define \(F^{(\pi_1, \ldots, \pi_{n-1})}_{\gamma}\) to be the set of functions in \(|\gamma| = \sum_{i=1}^{n-1} a_i\) variables

\[
u_i^{(b)} \quad (1 \leq i \leq n-1, 1 \leq b \leq a_i)
\]
satisfying the following properties: \(f^{(a)(1)} \ldots, u_{n-1}^{(a)}\) is

\begin{enumerate}
\item[(P1)] meromorphic in \(u_i^{(b)} \in \mathbb{C}\),
\item[(P2)] symmetric in each set of variables \((u_i^{(1)}, \ldots, u_i^{(a)})\),
\item[(P3)] quasi-periodic in each variable \(u_i^{(b)}\) in the following sense:
\[
f(u_i^{(b)} + r) = -f(u_i^{(b)}),
\]
\[
f(u_i^{(b)} + \tau) = -e^{2\pi i u_i^{(b)}} e^{\gamma \alpha \pi_i - \pi_i / r} f(u_i^{(b)}),
\]
\item[(P4)] holomorphic except for (at most) simple poles at \(u_i^{(b)} = u_{i+1}^{(c)}\),
\item[(P5)] zero at one of the following:
\begin{itemize}
\item \(u_i^{(b)} = u_{i+1}^{(c)} - \frac{1}{2} = u_i^{(d)} + \frac{1}{2}\)
\item \(u_{i+1}^{(b)} = u_i^{(c)} - \frac{1}{2} = u_i^{(d)} + \frac{1}{2}\)
\end{itemize}
\end{enumerate}

Note that if \(a_i = 0\) for some \(i\) then \(\hat{\pi}_i\) does not appear in the condition for \(f\). If so, we sometimes abbreviate \(\hat{\pi}_i\) from the notation \(F^{(\pi_1, \ldots, \pi_{n-1})}_\gamma\).
Let
\[ \gamma^{(1)} = \sum_{i=1}^{n-1} a_i^{(1)} \alpha_i, \quad \gamma^{(2)} = \sum_{i=1}^{n-1} a_i^{(2)} \alpha_i, \]
and let \( \hat{\pi}'_i \) be given by
\[ \hat{\pi}'_i = \hat{\pi}_i - (\gamma^{(2)}, \alpha_i)(1 - r). \]

Then one can show \cite{16} that there exists an associative mapping (the \( \ast \)-product)
\[ F_{\gamma^{(1)}} \otimes F_{\gamma^{(2)}} \to F_{\gamma^{(1)} + \gamma^{(2)}}, \]
which sends \( f \otimes g \) to \( f \ast g \) where
\[
(f \ast g)(u_1^{(1)}, \ldots, u_1^{(a_1)}, v_1^{(1)}, \ldots, v_1^{(a_2)}, \ldots, u_{n-1}^{(1)}, \ldots, u_{n-1}^{(a_1)}, v_{n-1}, \ldots, v_{n-1}^{(2)}) = S \left( f(u_1^{(1)}, \ldots, u_{n-1}^{(a_1)}) g(v_1^{(1)}, \ldots, v_{n-1}^{(2)}) \right)
\times \prod_{1 \leq i \leq n-1} \left[ \frac{u_i^{(b)} - v_i^{(c)}}{1} \right] \prod_{1 \leq i \leq n-2} \left[ \frac{u_i^{(b)} - v_{i+1}^{(c)}}{u_i^{(b)} - v_{i+1}^{(c)}} \right] \prod_{2 \leq i \leq n-1} \left( -1 \frac{u_i^{(b)} - v_{i-1}^{(c)}}{u_i^{(b)} - v_{i-1}^{(c)}} \right).
\]

Here \( S \) means the symmetrization of \( (u_1^{(1)}, \ldots, u_1^{(a_1)}, v_1^{(1)}, \ldots, v_1^{(a_2)} \) for each \( i = 1, \ldots, n - 1 \). Because of the symmetrization the pole at \( u_i^{(b)} = v_i^{(c)} \) is canceled. The property (P5) is preserved by the \( \ast \)-product.

Suppose that \( F(u_1^{(1)}, \ldots, u_{n-1}^{(a_1)}; \hat{\pi}_1, \ldots, \hat{\pi}_{n-1}) \) is a function of the variables \( (u_1^{(1)}, \ldots, u_{n-1}^{(a_1)}) \) and \( (\hat{\pi}_1, \ldots, \hat{\pi}_{n-1}) \), and belongs to \( F_{\gamma^{(1)}} \) for any choice of \( (\hat{\pi}_1, \ldots, \hat{\pi}_{n-1}) \in Z^{n-1} \).

Note that \( (\hat{\pi}_1, \ldots, \hat{\pi}_{n-1}) \) in the expression \( F_{\gamma^{(1)}} \) is considered as an element of \( (Z/rZ)^{n-1} \). Therefore, the factor \( 1 - r \) in the shift \cite{11.13} could be simply 1. However, the function \( F \) may have dependence on \( (\hat{\pi}_1, \ldots, \hat{\pi}_{n-1}) \in Z^{n-1} \). In fact, we use \cite{11.13} in this form in \cite{11.18} below.

We define an operator \( X(F) \) by
\[ X(F) = \oint \frac{dz_1^{(1)}}{2\pi i z_1^{(1)}} \ldots \oint \frac{dz_{n-1}^{(a_1)}}{2\pi i z_{n-1}^{(a_1)}} \frac{\xi_1(u_1^{(1)})}{u_1^{(1)} - \frac{1}{2}} \ldots \frac{\xi_{n-1}(u_{n-1}^{(a_1)})}{u_{n-1}^{(a_1)} - \frac{1}{2}}. \]
From (P3) we have the following periodicity with respect to each $u_i$ of Felder’s contour in the CFT limit.

Then we have

\[ X(F)X(G) = X(f \ast g). \]

From this follows (4.15). Then, (ii) and (iii) are nothing but the condition (P5).

Let us prove (iv). Assume that $f$ has zeros at $u_i^{(b)} = \frac{1}{2} (1 \leq i \leq n-1, 1 \leq b \leq a_i^{(1)})$. Set

\[ f(u_1^{(1)}, \ldots, u_{n-1}^{(a_{n-1}^{(1)})}) = F(u_1^{(1)}, \ldots, u_{n-1}^{(a_{n-1}^{(1)})}; \hat{\pi}_1, \ldots, \hat{\pi}_{n-1}), \]

Similarly we define $X(G)$ from $G(v_1^{(1)}, \ldots, v_{n-1}^{(a_{n-1}^{(2)})}; \hat{\pi}_1, \ldots, \hat{\pi}_{n-1})$.

Set

\[ f(u_1^{(1)}, \ldots, u_{n-1}^{(a_{n-1}^{(1)})}) = F(u_1^{(1)}, \ldots, u_{n-1}^{(a_{n-1}^{(1)})}; \hat{\pi}_1, \ldots, \hat{\pi}_{n-1}), \]

\[ g(v_1^{(1)}, \ldots, v_{n-1}^{(a_{n-1}^{(2)})}) = G(v_1^{(1)}, \ldots, v_{n-1}^{(a_{n-1}^{(2)})}; \hat{\pi}_1, \ldots, \hat{\pi}_{n-1}). \]

From (P3) we have the following periodicity with respect to each $u_i^{(b)}$

\[ \tilde{f}(u_i^{(b)} + r) = f(u_i^{(b)}), \]

\[ \tilde{f}(u_i^{(b)} + \tau) = e^{2\pi i (\frac{\langle \gamma, \gamma \rangle}{2} - \hat{\pi}_i)/r} \tilde{f}(u_i^{(b)}). \]

Fix generic $(u_1^{(1)}, \ldots, u_{n-1}^{(a_{n-1}^{(1)})})$ and consider a simultaneous shift of $\tilde{f}$

\[ \tilde{f}_s(c) = \tilde{f}(u_1^{(1)} + c, \ldots, u_{n-1}^{(a_{n-1}^{(1)})} + c). \]

If

\[ \frac{(\gamma, \gamma)}{2} \equiv \sum_{i=1}^{n-1} a_i \hat{\pi}_i \mod r, \]

then $\tilde{f}_s(c)$ is doubly periodic in $c$. In Proposition 4.3 we choose $\gamma = (\lambda, \alpha)\alpha$, and then (4.22) is valid. From (P4) $\tilde{f}_s$ has no pole in $c$, and therefore it is constant. This completes the proof of Proposition 4.3. \hfill \Box

Note that the statement (iv) of Proposition 4.3 corresponds to the closeness of Felder’s contour in the CFT limit.
4.3 Quadratic relations

With the screening operators introduced above, let us examine the nilpotency property $d^2 = 0$. Unfortunately, we could not find a complete solution to this problem in the general case $n \geq 4$. In Appendix A we show the following.

**Theorem 4.4** Suppose that $\alpha, \beta \in \Delta_+$ and

$$(\lambda, \lambda^\alpha, \lambda^{\alpha,\beta})$$

is admissible, i.e., both $$(\lambda, \lambda^\alpha)$$ and $$(\lambda^\alpha, \lambda^{\alpha,\beta})$$ are admissible.

(i) If $(\alpha, \beta) = 2$, then $\alpha = \beta$ and it is a simple root.

(ii) Otherwise, there exists $\alpha' (\neq \alpha), \beta' \in \Delta_+$ such that

$$(\lambda, \lambda^{\alpha'}, \lambda^{\alpha',\beta'})$$

is admissible and

$$\lambda^{\alpha,\beta} = \lambda^{\alpha',\beta'}.$$  \hspace{1cm} (4.25)

The pair $(\alpha', \beta')$ is uniquely determined by this condition.

In the second case, we say that the set of admissible weights (4.23) and (4.24) satisfying (4.25) form a commutative square.

Consider the case (i). If $\alpha = \alpha_j$ is a simple root and $a = m_\alpha(\lambda)$, then we have $m_\alpha(\lambda^\alpha) = r - a$. Hence $d^2 = 0$ is ensured by the relation

$$X_j^{r-a} X_j^a = X_j^r = 0.$$  \hspace{1cm} (4.27)

This has been proved in

In the case (ii), we need the following.

**Theorem 4.5** For each commuting square, we have the identity of screening operators

$$X_\beta(\lambda^\alpha) X_\alpha(\lambda) = s_\lambda(\alpha, \beta; \alpha' \beta') X_\beta(\lambda^{\alpha'}) X_{\alpha'}(\lambda)$$

where $s_\lambda(\alpha, \beta; \alpha' \beta') = \pm 1$.

The sign factor arises here because of the (anti-)periodicity property of the operator $X_\alpha(k_1, \ldots, k_m),$

$$X_\alpha(k_1, \ldots, k_m)|_{k_i \to k_i + r} = \varepsilon_r X_\alpha(k_1, \ldots, k_m), \quad (\varepsilon_r = (-1)^{r+1}).$$

The precise formula for $s_\lambda(\alpha, \beta; \alpha' \beta') = \pm 1$ will be given below. To have $d^2 = 0$ we must choose the signs $s_\alpha(\lambda)$ appropriately. Theorem 4.5 reduces the problem to finding $s_\alpha(\lambda)$ satisfying

$$s_\beta(\lambda^\alpha)s_\alpha(\lambda) = -s_\lambda(\alpha, \beta; \alpha', \beta') s_{\beta'}(\lambda^{\alpha'}) s_{\alpha'}(\lambda).$$
We have no solution except for the special cases \( n = 2, 3 \).

The assertion (1.26) amounts to a number of identities of theta functions. These are derived in appendix B. Below we shall indicate which identities are used in each case.

**Case** \((\alpha, \beta) = 0\) In this case, \(\alpha' = \beta, \beta' = \alpha\) hold (see Appendix A). The assertion (4.26) is nothing but (B.24) and (B.25). We can show that \(s_\lambda(\alpha; \beta; \alpha', \beta') = 1\).

**Case** \((\alpha, \beta) = \pm 1\) From the case-by-case analysis of Appendix A, we see that altogether there are 8 cases to consider. In the following we set \(\gamma_1 = \alpha_i, \ldots, i+1\) and \(\gamma_2 = \alpha_{i+1}, \ldots, i+l+m\).

***Case A+***: \(\alpha = \gamma_1, \beta = \gamma_2, m_\alpha(\lambda) > m_\beta(\lambda)\).

***Case B+***: \(\alpha = \gamma_1, \beta = \gamma_2, m_\alpha(\lambda) < m_\beta(\lambda)\).

***Case C+***: \(\alpha = \gamma_2, \beta = \gamma_1, m_\alpha(\lambda) > m_\beta(\lambda)\).

***Case D+***: \(\alpha = \gamma_2, \beta = \gamma_1, m_\alpha(\lambda) < m_\beta(\lambda)\).

***Case A−***: \(\alpha = \gamma_1 + \gamma_2, \beta = \gamma_1\).

***Case B−***: \(\alpha = \gamma_2, \beta = \gamma_1 + \gamma_2\).

***Case C−***: \(\alpha = \gamma_1 + \gamma_2, \beta = \gamma_2\).

***Case D−***: \(\alpha = \gamma_1, \beta = \gamma_1 + \gamma_2\).

The cases \(X_+\) and \(X_−\) \((X = A, B, C, D)\) form a commutative square. We will prove this statement and show the corresponding equality of the screening operators.

Set

\[
\kappa_\alpha(\lambda) = \frac{m_\alpha(\lambda) - (\sigma \Lambda, \alpha)}{r}.
\]  

(4.28)

To get (4.26) for \(\alpha = \beta' = \gamma_1, \alpha' = \gamma_1 + \gamma_2, \beta = \gamma_2\), we use (B.30) with

\[
a = m_\alpha'(\lambda), \\
b = m_\alpha(\lambda) - m_\alpha'(\lambda), \\
k_j = (\lambda, \alpha_{i, \ldots, i+j-1}) - 1 \quad (1 \leq j \leq l \text{ or } l + 2 \leq j \leq l + m).
\]

The signature in (B.24) is given by

\[
s_\lambda(\alpha; \beta; \alpha', \beta') = \varepsilon_r^{\kappa_\alpha(\lambda)|\beta|m_\alpha'(\lambda)}.
\]  

(4.29)

To get (4.26) for \(\alpha = \gamma_1, \beta = \alpha' = \gamma_2, \beta' = \gamma_1 + \gamma_2\), we use (B.31) with

\[
a = m_\alpha'(\lambda), \\
b = m_\alpha(\lambda), \\
k_j = (\lambda, \alpha_{i, \ldots, i+j-1}) - 1 \quad (1 \leq j \leq l \text{ or } l + 2 \leq j \leq l + m).
\]
The signature in \((B.24)\) is given by
\[
s_\lambda(\alpha, \beta; \alpha', \beta') = \varepsilon_{\rho}^{\kappa_{\alpha}(\lambda)|\alpha'|m_\alpha(\lambda)}.
\] (4.30)

To get \((4.26)\) for \(\alpha = \gamma_2, \beta = \alpha' = \gamma_1, \beta' = \gamma_1 + \gamma_2\), we use \((B.28)\) with
\[
\begin{align*}
a &= m_{\alpha'}(\lambda), \\
b &= m_\alpha(\lambda), \\
k_j &= (\lambda, \alpha_{i\ldots i+j-1}) - 1 \quad (1 \leq j \leq l), \\
k_j &= (\lambda, \alpha_{i+l+1\ldots i+j-1}) - 1 \quad (l + 2 \leq j \leq l + m).
\end{align*}
\]

The signature in \((B.24)\) is given by
\[
s_\lambda(\alpha, \beta; \alpha', \beta') = \varepsilon_{\rho}^{\kappa_{\alpha'}(\lambda)|\alpha|m_\alpha(\lambda)}.
\] (4.31)

To get \((4.26)\) for \(\alpha = \beta' = \gamma_2, \beta = \gamma_1, \alpha' = \gamma_1 + \gamma_2\), we use \((B.29)\) with
\[
\begin{align*}
a &= m_{\alpha'}(\lambda), \\
b &= m_\alpha(\lambda) - m_{\alpha'}(\lambda), \\
k_j &= (\lambda, \alpha_{i\ldots i+j-1}) - 1 \quad (1 \leq j \leq l), \\
k_j &= (\lambda, \alpha_{i+l+1\ldots i+j-1}) - 1 \quad (l + 2 \leq j \leq l + m).
\end{align*}
\]

The signature in \((B.24)\) is given by
\[
s_\lambda(\alpha, \beta; \alpha', \beta') = \varepsilon_{\rho}^{(\kappa_{\alpha}(\lambda)+\kappa_{\alpha'}(\lambda))|\alpha'|m_{\alpha'}(\lambda)}.
\] (4.32)

5 Operators \(\overline{X}_\alpha(\lambda)\) as intertwiners of DWA

In this section we demonstrate that the screening operators introduced above are the intertwiner operators for the DWA. A special case of the statement has been proved \([14, 15]\) for the screening operators acting in the vacuum module, where the theta function factor becomes unit and the screening operator has a particularly simple form:
\[
\overline{X}_{\alpha_j} = \oint \frac{dz}{2\pi i z} U_j(z).
\] (5.1)

The proof of \([14, 15]\) was based on the fact that the screening currents commute with DWA generators up to a total difference. As it was explained \([13]\) on the example of the \(sl_2\) case, in dealing with general \((\alpha, \lambda)\) one needs to be sure that the additional theta function terms do not lead to nonvanishing contributions to the commutator of the DWA generators and \(\overline{X}_\alpha(\lambda)\). We prove that this property is guaranteed by the relations of the elliptic algebra of screening operators.
5.1 Deformed $W$ Algebra

The DWA can be constructed as the subalgebra in the universal enveloping algebra of the Heisenberg algebra of operators $\beta_n$ and $P_\alpha$. Let us introduce the local bosonic fields \[ \Lambda_j(z) = x^{-2\sqrt{r(r-1)}P_j} : \exp \left( \sum_{m \neq 0} \frac{(x^m - x^{-m})}{m} \beta_m z^{-m} \right) : \] (5.2)

The explicit realization of the generators of DWA in terms of these fields is given by means of the non-linear transformation:

\[ : (x^{-2z}\partial_z - \Lambda_1(z))(x^{-2z}\partial_z - \Lambda_2(z)x^2) \cdots (x^{-2z}\partial_z - \Lambda_n(z)x^{2n-2}) : = \sum_{j=0}^n (-1)^j W^{(j)}(z) x^{2(n-j)} \partial_z, \] (5.3)

where $W^{(0)}(z) \equiv 1$. In the limit $x \to 1$ this formula leads to the quantum Miura transformation describing the bosonic realization of the $W_n$-algebra with the Virasoro subalgebra central charge

\[ c = (n-1) \left( 1 - \frac{n(n+1)}{r(r-1)} \right). \]

The bosonic fields $W^{(j)}(z)$ defined via the transform (5.3) constitute an associative algebra \[14, 15\]. For instance, the commutation relations between the currents $W^{(j)}(z)$ and $W^{(1)}(z)$

\[ W^{(1)}(z) = \sum_{s=1}^n \Lambda_s(z) \] (5.4)

are described by

\[ f^{(j)}(\frac{w}{z}) W^{(1)}(z) W^{(j)}(w) - W^{(j)}(w) W^{(1)}(z) f^{(j)}(\frac{z}{w}) = -(x - x^{-1})^2 \times \]

\[ \times [r]_x[r-1]_x \{ W^{(j+1)}(xw) \delta(x^{j+1} \frac{w}{z}) - W^{(j+1)}(x^{-1}w) \delta(x^{-(j+1)} \frac{w}{z}) \}. \] (5.5)

Here $\delta(z) = \sum_{j \in \mathbb{Z}} z^j$ and

\[ f^{(j)}(z) = \exp \left( -(x - x^{-1})^2 \sum_{m > 0} \frac{z^m}{m} [rm]_x [(r-1)m]_x \frac{[(n-j)m]_x}{[nm]_x} \right). \]

It is important for us that the relation (5.5) leads \[15\] to

\[\text{The difference between our notations and those in } [14, 15] \text{ is that we are working with the zero mode } P \text{ shifted as } P \rightarrow P - \frac{1}{\sqrt{r(r-1)}} \sum_{j=1}^{n-1} \omega_j. \text{ In the conformal limit } x \to 1 \text{ this corresponds to the transform from the complex plane to the annulus. The parameters } p, t, q \text{ in } [14] \text{ are related with } x \text{ and } r \text{ as } q = x^{-2r}, t = x^{-2(r-1)}, p = x^{-2}.\]
Lemma 5.1 The Fourier modes $W_t^{(1)}$, $t \in \mathbb{Z}$ of the currents $W^{(1)}(z)\)$$

\[ W_t^{(1)} = \oint \frac{dz}{2\pi i z} z^t W^{(1)}(z) \] (5.6)

generate the whole DWA.

5.2 Intertwining property of $\overline{X}_\alpha(\lambda)$.

The main statement in this section is that the screening operators defined by (4.9) satisfy the basic property of the intertwining operators for DWA:

Proposition 5.2

\[ [W^{(j)}(z), \overline{X}_\alpha(\lambda)] = 0. \] (5.7)

According to the Lemma 5.1 the proof of the Proposition 5.2 follows from

Lemma 5.3

\[ [W^{(1)}(z), \overline{X}_\alpha(\lambda)] = 0. \] (5.8)

We will prove this fact in Appendix C. Though we have no proof, we believe that

Conjecture The screening operators (4.9) exhaust all the intertwiners for DWA.

We expect further that the irreducible representations of DWA arise as the cohomologies of the BRST complex (2.16). An important remark is that the definition of the DWA is invariant under the change $r \to 1 - r$ [14, 15]. For this reason there exists another set of intertwiners given by the second type screening operators. The construction for them is fairly obvious and we will not consider this case any more.

Finally, let us stress that the property iv) of the Theorem 4.3 which is satisfied for $\overline{X}_\alpha(\lambda)$ turns out to be essential in the proof of Lemma 5.3. For this reason arbitrary products of basic operators or basic operators itself do not commute with DWA generators and can not be treated as screening operators. It has been noted at the end of sec. 2.4 and 6 that a similar situation takes place in the CFT. The general conditions for the existence of intertwining operators are discussed in Appendix A.

6 CFT limit

The problem of finding the intertwining operators of $W$-algebras [17] has been studied for $sl_3$ case (i.e., $n = 3$) in [3], [3]. However, it was not clear how to generalize the result for arbitrary $n$. In this section we take the CFT limit
or equivalently by $x \to 1$, $i \log(z) \sim 1$ of the basic operators for DWA, and introduce the notion of basic operators for W-algebra associated to $sl_n$ algebra with $n \geq 2$. In the case $n = 3$ we recover the results of [4].

We restrict our discussion to formal level in the sense that the well-definedness of integrals in the operators is not considered. In the deformed case, there is no such problem. The integrals are well-defined because the integrands are single-valued and the contours are on the unit circle.

Our main goal is to find the formal limit of the operators (4.1) corresponding to the positive root $\alpha = \alpha_i + \cdots + \alpha_{i+m}$. Recall that these basic operators have the form:

$$X_\alpha(k_1, \ldots, k_m) = i \int \cdots \int \prod_{j=1}^{i+m} \frac{dz_j}{2\pi i z_j} U_i(z_i) \cdots U_{i+m}(z_{i+m})$$

$$\times F(u_i, \ldots, u_{i+m}),$$

(6.1)

where the function $F(u_i, \ldots, u_{i+m})$ is given by

$$F(u_i, \ldots, u_{i+m}) = (-1)^{k_1+\cdots+k_m} \prod_{j=1}^{m} \left[ \frac{u_{i+j-1} - u_{i+j} - k_j - \frac{1}{2}}{u_{i+j-1} - u_{i+j} + \frac{1}{2}} \right]$$

$$\times \prod_{j=0}^{m} \left[ \frac{u_{i+j} - k_j + k_{j+1} - \frac{1}{2} - \pi_{i+j}}{u_{i+j} - \frac{1}{2}} \right],$$

(6.2)

and again $k_0 = -1, k_{m+1} = 0$ is implied. Let us rewrite this expression into the sum of "elementary" integrals with the specific ordering of the variables on the unit circle:

$$X_\alpha(k_1, \ldots, k_m) = \sum_{\sigma \in S_{m+1}} \int \cdots \int \prod_{j=1}^{i+m} \frac{dz_j}{2\pi i z_j}$$

$$\times U_{\sigma(i)}(z_i) \cdots U_{\sigma(i+m)}(z_{i+m}) F_\sigma(u_i, \ldots, u_{i+m}) ,$$

(6.3)

where $S_{m+1}$ is the permutation group of numbers $(i, \ldots, i+m)$ and the function $F_\sigma$ is determined by the condition

$$U_i(z_{\sigma^{-1}(i)}) \cdots U_{i+m}(z_{\sigma^{-1}(i+m)}) F(z_{\sigma^{-1}(i)}, \ldots, z_{\sigma^{-1}(i+m)})$$

$$= U_{\sigma(i)}(z_i) \cdots U_{\sigma(i+m)}(z_{i+m}) F_\sigma(z_i, \ldots, z_{i+m}),$$

(6.4)

(6.5)

or equivalently by

$$F_\sigma(u_i, \ldots, u_{i+m}) = F(u_{\sigma^{-1}(i)}, \ldots, u_{\sigma^{-1}(i+m)}) \prod_{\sigma(j')-\sigma(j)=1} (-1)^{\frac{u_{j'} - u_j - \frac{1}{2}}{u_{j'} - u_j + \frac{1}{2}}}. \quad (6.6)$$

(6.7)

Now we are able to work out the conformal limit of the operators $X_\alpha(k_1, \ldots, k_m)$. 

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In what follows in this section, we will use the same notations for the CFT limits of the bosons, screening operators etc., and mention only the changes in the definitions.

First let us discuss the limit of the commutation relations for bosons and the screening current. The first one can be found directly by setting \( x \rightarrow 1 \) in the formulae (2.1)-(2.3). Namely, the oscillators \( \beta^j_m \ (1 \leq j \leq n-1, \ m \in \mathbb{Z}\{0\}) \) are defined by the commutation relations

\[
[\beta^j_m, \beta^k_{m'}] = \frac{(n-1)(r-1)}{n} \delta_{m+m',0} \quad (j = k),
\]

\[
= - \frac{1}{n} \frac{(r-1)}{r} m \delta_{m+m',0} \quad (j \neq k),
\]

while \( \beta^n_m \) is determined via the equation

\[
\sum_{j=1}^{n} \beta^j_m = 0.
\]

Note that the definitions of the bosonic Fock spaces and the zero mode operators \( P_\lambda, Q_\lambda \) remain the same as in section 2.2.

The prescription for taking the limit of the operator \( U_j(z) \ (j = 1, \ldots, n-1) \) is also very simple. We demand that together with \( x \rightarrow 1, \) the parameter \( u \) tends to a limiting value in such a way that \( z = x^{2u} \) is fixed. Therefore the screening currents of \([14, 15]\) become the screening currents of \([17]\):

\[
U_j(z) = e^{i \sqrt{\frac{\pi r}{r-1}} Q_{\alpha j} z^{\frac{1}{r^{\delta_{\alpha_j}+\frac{1}{r^\delta}}}} \cdot e^{\sum_{m \neq 0} \frac{1}{m} (\beta^j_m - \beta^{j+1}_m) z^{-m}},
\]

where \( z \in \mathbb{C} \).

Note that the product of operators \( U(z)U(\zeta) \) is defined apriori when \( |z| > |\zeta| \), and to be understood as an analytic continuation. When we compare \( U(z)U(\zeta) \) with \( U(\zeta)U(z) \) in the deformed case, there is a common domain of convergence (a neighborhood of the unit circle \( |z/\zeta| = 1 \)). However, in the CFT limit, this will shrink because poles accumulate to \( z/\zeta = 1 \). Therefore, in order to compare \( U(z)U(\zeta) \) with \( U(\zeta)U(z) \) in the CFT limit, we must specify the path of analytic continuation. In fact, we have

\[
U_i(z)U_j(\zeta) = \begin{cases} q^{-(\alpha_i,\alpha_j)}U_j(\zeta)U_i(z), & \text{if } \arg z < \arg \zeta, \\ q^{(\alpha_i,\alpha_j)}U_j(\zeta)U_i(z), & \text{if } \arg z > \arg \zeta. \end{cases}
\]

Here the complex number \( q \) is

\[
q = e^{i \pi \frac{1}{r-1}},
\]

and the left hand side means the analytic continuation from the region, \( \arg z = \arg \zeta \) and \( |z| > |\zeta| \), while the right hand side means the analytic continuation from the region, \( \arg z = \arg \zeta \) and \( |z| < |\zeta| \).
To work out the conformal limit of the theta depending part of (6.1) we parametrize the variables \( x, u \) as following

\[
x = e^{-\epsilon} \quad (\epsilon > 0), \\
u = \frac{v}{2i\epsilon} \quad (0 < \text{Re} \, v < 2\pi).
\]

Now the limit is given by \( \epsilon \to 0 \), while \( z = e^{iv} \) remains to be fixed. According to such a prescription the function (6.2) is changed to be the \( \mathcal{P}_\alpha \)-depending factor

\[
F(u_i, \ldots, u_{i+m}) \to (-1)^m q^{-m-k_1-\cdots-k_m} e^{\frac{\epsilon}{r}(1-\hat{\pi}_1-\cdots-\hat{\pi}_{i+m})}
\]

if the condition \( \text{arg}(z_j) < \text{arg}(z_{j+1}) \) holds for any \( j = i, \ldots, i+m-1 \). For a non-trivial transposition \( \sigma \in S_{m+1} \), i.e., when some of the screening currents \( U_{j+1} \) stands to the left of \( U_j \) the limit has the form

\[
F_\sigma(u_i, \ldots, u_{i+m}) \to q^{f(\sigma)} (-1)^m q^{-m-k_1-\cdots-k_m} e^{\frac{\epsilon}{r}(1-\hat{\pi}_1-\cdots-\hat{\pi}_{i+m})}
\]

where the function \( f(\sigma) \) is

\[
f(\sigma) = \sum_{\sigma(j') = \sigma(j) + 1} (2k_{\sigma(j')}) + 1.
\]

Now it is convenient to introduce the definition of an "elementary" integral \( I_{i_1\ldots i_{m+1}} \) (\( 1 \leq i_1, \ldots, i_{m+1} \leq n-1 \)) with the ordering of the variables as follows:

\[
I_{i_1\ldots i_{m+1}} = \int \cdots \int_{0 < \text{arg} \, z_1 < \cdots < \text{arg} \, z_{i+m} < 2\pi} \prod_{j=i}^{i+m} \frac{dz_j}{2\pi iz_j} \times U_{i_1}(z_{i_1}) \cdots U_{i_{m+1}}(z_{i+m}).
\]

In terms of these objects the formal limit of the basic operators (4.1) is given by the expression

\[
X_\alpha(k_1, \ldots, k_m) = q^{-m-k_1-\cdots-k_m} \sum_{\sigma \in S_{m+1}} q^{f(\sigma)} I_{\sigma(i)\ldots\sigma(i+m)}. \tag{6.19}
\]

Here, for notational convenience we have omitted the irrelevant common factor

\[
(-1)^m e^{\frac{\epsilon}{r}(1-\hat{\pi}_1-\cdots-\hat{\pi}_{i+m})}.
\]

Indeed, whereas the operator \( \mathcal{P} \) itself does not commute with screening operators, the important properties of (6.19) such as commutations with other basic operators and \( W \)-algebra generators do not depend on this factor.

Note that the quasi-periodicity (3.10) of basic operators follows from

\[
q^r = (-1)^{r-1}.
\]
The product of two integrals of the form \( I_{i_1\cdots i_m}, I_{i_{m+1}\cdots i_{m+s}} \) is given according to the definition (6.18) as follows. Let \( S_{m+s} \) be the permutation group of numbers \( \{1, \ldots, m+s\} \). Denote by \( S_{m,s} \) the set of elements \( \sigma \in S_{m+s} \) such that \( \sigma^{-1}(j) < \sigma^{-1}(j+1) \) for each \( j \neq m, m+s \). Then
\[
I_{i_1\cdots i_m} I_{i_{m+1}\cdots i_{m+s}} = \sum_{\sigma \in S_{m,s}} g_{m,s}(\sigma) I_{i_{\sigma(1)}\cdots i_{\sigma(m+s)}}
\]
where
\[
g_{m,s}(\sigma) = \sum_{\sigma(j) < \sigma(l)} (\alpha_{i_{\sigma(j)}}, \alpha_{i_{\sigma(l)}})
\]
For instance, let us work out the decomposition of the product of two basic operators (see also \([2, 5]\)):
\[
X_{12}(k) = q^{-k-1}I_{12} + q^k I_{21},
\]
\[
X_1 = I_1.
\]
into the ”elementary” integrals. In this example we will also show the validity of (3.8). The formal product of these operators is the operator of weight \(-2\alpha_1 - \alpha_2\).

It can be derived from the definition (6.20)
\[
I_{i_1i_2i_3} = I_{i_{1i_2i_3}} + q^{(\alpha_{i_1}, \alpha_{i_2})} I_{i_{1i_3i_2}} + q^{(\alpha_{i_1}, \alpha_{i_2})} I_{i_{3i_1i_2}} \quad (6.24)
\]
\[
I_{i_1i_2i_3} = I_{i_{1i_2i_3}} + q^{(\alpha_{i_1}, \alpha_{i_2})} I_{i_{1_2i_3}} + q^{(\alpha_{i_1}, \alpha_{i_2})} I_{i_{2i_1i_3}} \quad (6.25)
\]
In particular,
\[
I_{12}I_1 = (q + q^{-1})I_{112} + I_{121},
\]
\[
I_{21}I_1 = q I_{121} + (1 + q^2)I_{211},
\]
\[
I_1I_{12} = (1 + q^2)I_{112} + q I_{121},
\]
\[
I_{121} = I_{121} + (q + q^{-1})I_{211}.
\]
Using these formulae we find that
\[
X_1X_{12}(k) = q^{-k}(q + q^{-1})I_{112} + (q^k + q^{-k})I_{121} + q^k(q + q^{-1})I_{211}
\]
\[
= X_{12}(k-1)X_1.
\]
This confirms that (3.8) still holds in the CFT limit. Similarly, one can check (3.6), (3.7), (3.9) and similar properties of the basic operators in the \( n > 3 \) case.

It would be useful, however, to express the basic operators in the ”conventional” form, i.e., as non-commutative polynomials of \( X_j \). Let us first prepare the necessary notations. Introduce the bracket
\[
\{A, B\}_k = -[k]_q AB + [k + 1]_q BA, \quad (6.28)
\]
\[
\{A, B\}_0 = BA, \quad (6.29)
\]
\[
\{A, B\}_{-1} = AB, \quad (6.30)
\]
\[
\{\{A, B\}_{k_1}, C\}_{k_2} = \{A, \{B, C\}_{k_2}\}_{k_1}, \quad if\{A, C\} = 0. \quad (6.31)
\]
Now one finds that the following Lemma holds:

Lemma 6.1

\[
q^{m-k_1-\cdots-k_m} \sum_{\sigma \in S_{m+1}} q^{f(\sigma)} I_{\sigma(i)\cdots\sigma(i+m)} = q^{m-k_1-\cdots-k_{m-1}} \sum_{\sigma \in S_m} q^{f(\sigma)} \{ I_{\sigma(i)\cdots\sigma(i+m-1)}, X_m \} k_m. \tag{6.32}
\]

The proof follows from a straightforward decomposition of the right hand side into ”elementary integrals” \( I_{i_1\cdots i_{m+1}} \). Applying the equation (6.32) we arrive at the

Proposition 6.2

\[
X_{i_1\cdots i+m}(k_1, \ldots, k_m) = \{ \{ X_{i_1}, X_{i_1+1} \}_{k_1}, X_{i_1+2} \}_{k_2}, \ldots, X_{i+m} \}_{k_m}. \tag{6.33}
\]

In particular, a conformal analogue of the operators (3.3) is

\[
X_{12}(k) = -[k]_q X_1 X_2 + [k + 1]_q X_2 X_1. \tag{6.34}
\]

Note that using this representation the equality of two decompositions (6.27) and CFT analogue for (3.3) can be rewritten as \( q \)-Serre relations

\[
[k] q X_{j+1} X_j - ([k + 1]_q + [k - 1]_q) X_j X_{j+1} X_j + [k]_q X_{j+1} X_j^2 = 0,

[k] q X_{j+1}^2 X_j - ([k + 1]_q + [k - 1]_q) X_{j+1} X_j X_{j+1} + [k]_q X_j X_{j+1}^2 = 0.
\]

These equations together with the commutativity of screening currents \( U_j, U_l \) for \(|j - l| \neq 1 \) imply that in the CFT limit, the screening operators corresponding to the simple roots satisfy the relations for the nilpotent half of the quantum group \( U_q(sl_n) \).

Using the properties of the bracket (6.28)-(6.31) one can check that the operators (3.3) satisfy the basic relations (3.14)-(3.18) of Lemma B.4. which now have the form:

\[
X_{i_1\cdots i+l-1}(k_1, \ldots, k_{l-1}) X_{i+l\cdots i+l+m}(k_{l+1}, \ldots, k_{l+m}) = X_{i_1\cdots i+l+m}(k_1, \ldots, k_{l-1}, -1, k_{l+1}, \ldots, k_{l+m}), \tag{6.35}
\]

\[
X_{i_1\cdots i+l+m}(k_{i+1}, \ldots, k_{i+m}) X_{i_1\cdots i+l-1}(k_1, \ldots, k_{l-1}) = X_{i_1\cdots i+l+m}(k_1, \ldots, k_{l-1}, 0, k_{l+1}, \ldots, k_{l+m}), \tag{6.36}
\]

\[
X_i X_{i_1\cdots i+m}(k_1, k_2, \ldots, k_m) = X_{i_1\cdots i+m}(k_1 - 1, k_2, \ldots, k_m) X_i, \tag{6.37}
\]

\[
X_{i+m} X_{i_1\cdots i+m}(k_1, \ldots, k_{m-1}, k_m - 1) = X_{i_1\cdots i+m}(k_1, \ldots, k_{m-1}, k_m) X_{i+m},
\]

27
\[ X_{i \cdots i + m}(k_1, \ldots, k_m) X_{j \cdots j + l}(k'_1, \ldots, k'_l) = X_{j' \cdots j + l}(k'_1, \ldots, k'_l) \times X_{i \cdots i + m}(k_1, \ldots, k_m), \quad \text{if } i + m + 1 < j. \]  

(6.38)

(6.39)

In the case \( n = 3 \) the procedure given in section 3 may be followed in CFT limit to obtain the intertwining operators for \( W_3 \) algebra starting from the basic operators (6.34). It can be easily verified that the expression for \( \mathcal{X}_\alpha(\lambda) \) constructed in such a way coincides with the result of [5]. We also inspect directly in CFT limit the commutativity property of basic operators for the \( n = 4 \).

We believe that for general \( n \), the construction of the intertwining operators \( \mathcal{X}_\alpha(\lambda) \) for \( W \) algebra is identical for those for the DWA (4.5) with the only difference in the definition of the basic operators (6.33).

Our discussion was rather formal since we did not examine the well-definedness of the operators. To treat \( \mathcal{X}_\alpha(\lambda) \) as operators on the Fock space one must fix the contour prescription [5]. We do not discuss this problem in this paper.

7 Discussions

In this paper, we constructed the intertwining operators which commute with the action of the deformed \( W_n \)-algebra on the bosonic Fock spaces [14, 15]. In the conformal limit, the case we have discussed in this paper corresponds to representations of the \( W_n \)-algebra with the central charge

\[ c = (n - 1) \left( 1 - \frac{n(n+1)}{r(r-1)} \right) \]  

(7.1)

where \( r \) is an integer such that \( r \geq n + 2 \). In the language of solvable lattice models, they correspond to the \( sl_n \) RSOS models [1].

The main difference in our construction compared to the corresponding conformal limit, is that we need to construct basic operators which change the weight of Fock spaces by a positive root. In the conformal case, the basic operators can be expressed in terms of the operators corresponding to the simple roots. In other words, the case \( n \geq 3 \) contains a new feature which was not seen in the \( n = 2 \) case considered in [1].

One can look at this situation in the following way. It is well-known that the algebra of the screening operators in the conformal field theory is isomorphic to the algebra \( U_q(h_\lambda) \) with \( q = e^{\pi i(r-1)/r} \). An elliptic deformation of this algebra was considered in [10]. In this paper, we identified it with the algebra of the screening operators of the deformed \( W_n \)-algebra, and we derived a set of quadratic relations among the generators of that algebra. These relations can be considered as the elliptic deformation of the Serre relations.
One of our original aims was to construct the Felder-type complex for the irreducible $W_n$ modules. This was not achieved in two reasons. First of all, except for the case of $n = 3$, we cannot fix the signs so that we have $d^2 = 0$ for the coboundary operators. Secondly, even for the case $n = 3$, we have no result on the cohomology of the complex.

We considered the CFT limit of our construction. Our discussion stays in a formal level because we only considered the limit of the integrands of the screening operators.

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A Admissibility and commuting squares

A.1 Admissible pairs

Here we derive the condition for the admissibility of a pair $(\lambda, \lambda')$. We follow the notation of Section 2.4. In particular, we suppose $\lambda = t_\gamma \sigma \Lambda$ throughout this section.

Lemma A.1 The condition $(\sigma \Lambda, \alpha) \geq 0$ is equivalent to $l(r_\alpha \sigma) \geq l(\sigma)$.

Proof. Suppose that $r_\alpha = (ij)$ is the transposition and

$$\sigma = (\sigma(1), \ldots, i, \ldots, j, \ldots, \sigma(n)).$$  \hspace{1cm} (A.1)

We have

$$r_\alpha \sigma = (\sigma(1), \ldots, j, \ldots, i, \ldots, \sigma(n)).$$  \hspace{1cm} (A.2)

Therefore $l(r_\alpha \sigma) \geq l(\sigma)$ is equivalent to $i \preceq j$. Since $\alpha$ is positive and $r_\alpha = (ij)$, we have $\alpha = \varepsilon_i - \varepsilon_j$ if $i < j$ and $\alpha = \varepsilon_j - \varepsilon_i$ if $i > j$. Noting that $\sigma^{-1} \varepsilon_i = \varepsilon_{\sigma^{-1}(i)}$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$, we conclude that $l(r_\alpha \sigma) \geq l(\sigma)$ is equivalent to $\sigma^{-1} \alpha \geq 0$, and therefore to $(\sigma \Lambda, \alpha) = (\Lambda, \sigma^{-1} \alpha) \geq 0$.

Lemma A.2 $l(r_\alpha) = 2|\alpha| - 1$.

Proof. If $\alpha = \alpha_i \cdots \alpha_{i+m}$, a reduced expression of $r_\alpha$ is given by

$$r_\alpha = s_i \cdots s_{i+m-1} s_{i+m} s_{i+m-1} \cdots s_i.$$
We consider an operator $X_\alpha(\lambda)$ if and only if

$$d_\alpha(\lambda) \overset{\text{def}}{=} \deg(\lambda^\alpha) - \deg(\lambda) = 1.$$  \hspace{1cm} (A.3)

Note that

$$d_\alpha(\lambda) = \begin{cases} l(r_\alpha \sigma) - l(\sigma) & \text{if } (\sigma \Lambda, \alpha) > 0; \\ l(r_\alpha \sigma) - l(\sigma) + 2|\alpha| & \text{if } (\sigma \Lambda, \alpha) < 0. \end{cases}$$ \hspace{1cm} (A.4)

In particular, we have

$$0 < d_\alpha(\lambda) < 2|\alpha|. \hspace{1cm} (A.5)$$

**Lemma A.3** A pair $(\lambda, \lambda^\alpha)$ is admissible if and only if one of the following holds:

(i) $(\sigma \Lambda, \alpha) > 0$, and $(\sigma \Lambda, \beta) < 0$ or $(\sigma \Lambda, \gamma) < 0$ for any partition $\alpha = \beta + \gamma$ ($\beta, \gamma \in \Delta_+)$.

(ii) $(\sigma \Lambda, \alpha) < 0$, and $(\sigma \Lambda, \beta) < 0$ and $(\sigma \Lambda, \gamma) < 0$ for any partition $\alpha = \beta + \gamma$ ($\beta, \gamma \in \Delta_+)$.

In particular, $(\lambda, \lambda^\alpha)$ is always admissible for a simple root $\alpha = \alpha_j$.

**Proof.** We follow the argument in the proof of Lemma A.1. If $l(r_\alpha \sigma) > l(\sigma)$, we set $\beta = \varepsilon_i - \varepsilon_k$ and $\gamma = \varepsilon_k - \varepsilon_j$ for $k$ such that $i < k < j$. The condition $d_\alpha(\lambda) = l(r_\alpha \sigma) - l(\sigma) = 1$ is equivalent to $\sigma^{-1}(k) < \sigma^{-1}(i)$ or $\sigma^{-1}(j) < \sigma^{-1}(k)$ for any such $k$. This is equivalent to $(\sigma \Lambda, \beta) < 0$ or $(\sigma \Lambda, \gamma) < 0$, respectively. If $l(r_\alpha \sigma) < l(\sigma)$, we set $\beta = \varepsilon_j - \varepsilon_k$ and $\gamma = \varepsilon_k - \varepsilon_i$ for $k$ such that $j < k < i$. The condition $d_\alpha(\lambda) = l(r_\alpha \sigma) - l(\sigma) + 2|\alpha| = 1$ is equivalent to $\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$ for any such $k$. This is equivalent to $(\sigma \Lambda, \beta) < 0$ and $(\sigma \Lambda, \gamma) < 0$. \hfill \Box

**A.2 Commuting squares**

In this subsection we prove Theorem 4.4.

The assertion (i) follows immediately from

$$d_\alpha(\lambda) + d_\alpha(\lambda^\alpha) = 2|\alpha|. \hspace{1cm} (A.6)$$

Below we shall prove the assertion (ii) case-by-case.

Case $(\alpha, \beta) = 0$

Set $m = m_\alpha(\lambda)$ and $m' = m_\beta(\lambda)$. Recall (2.22). We have

$$(r_\alpha \sigma \Lambda, \beta) = (\sigma \Lambda, \beta). \hspace{1cm} (A.7)$$
This implies \( m_\beta(\lambda^\alpha) = m' \). Therefore, we have
\[
\lambda - \lambda^{\alpha,\beta} = m\alpha + m'\beta. \tag{A.8}
\]
Similarly, we have
\[
\lambda - \lambda^{\beta,\alpha} = m\alpha + m'\beta. \tag{A.9}
\]
This implies
\[
d_\beta(\lambda) + d_\alpha(\lambda^\beta) = \deg(\lambda^{\beta,\alpha}) - \deg(\lambda) = \deg(\lambda^{\alpha,\beta}) - \deg(\lambda) = 2. \tag{A.10}
\]
Since \( d_\beta(\lambda), d_\alpha(\lambda^\beta) > 0 \), we have
\[
d_\beta(\lambda) = d_\alpha(\lambda^\beta) = 1. \tag{A.11}
\]
Namely, \((\lambda, \lambda^{\gamma}, \lambda - m\alpha - m'\beta)\) is admissible.

Let us show the uniqueness of \(\alpha', \beta'\). Suppose that
\[
\alpha = \varepsilon_i - \varepsilon_j \quad \text{and} \quad \beta = \varepsilon_k - \varepsilon_l.
\]
We consider only the case when \(k < i < j < l\) and set \(\gamma_1 = \varepsilon_k - \varepsilon_i\) and \(\gamma_2 = \varepsilon_j - \varepsilon_i\). The other cases are similar. If
\[
(\lambda, \lambda^{\gamma_1}, \lambda - m\alpha - m'\beta) \tag{A.12}
\]
is admissible and \(\gamma \neq \alpha, \beta\), then we have
\[
\gamma = \varepsilon_k - \varepsilon_j = \gamma_1 + \alpha \quad \text{or} \quad \gamma = \varepsilon_i - \varepsilon_l = \alpha + \gamma_2 \tag{A.13}
\]
and
\[
m = m' = m_{\gamma_1+\alpha}(\lambda) = m_{\alpha+\gamma_2}(\lambda). \tag{A.14}
\]
Since \(m_\alpha(\lambda) \equiv (\sigma\Lambda, \alpha)\) and \(m_{\gamma_1+\alpha}(\lambda) \equiv (\sigma\Lambda, \gamma_1 + \alpha) \mod r\), we have \((\sigma\Lambda, \gamma_1) \equiv 0 \mod r\). This is a contradiction.

Case \((\alpha, \beta) = 1\)

Set \(m = m_\alpha(\lambda)\) and \(m' = m_\beta(\lambda^\alpha)\). We have \(\alpha, \beta\). The only way other than \(\alpha, \beta\) to write \(\lambda - \lambda^{\alpha,\beta}\) as a positive linear combination of two positive roots is
\[
\lambda - \lambda^{\alpha,\beta} = \begin{cases} m(\alpha - \beta) + (m + m')\beta & \text{if } \alpha - \beta \in \Delta_+; \\ (m + m')\alpha + m'(\beta - \alpha) & \text{if } \beta - \alpha \in \Delta_+. \end{cases} \tag{A.15}
\]
The uniqueness is then obvious from \(\alpha, \beta\). Note that
\[
(r_\beta \sigma\Lambda, \alpha - \beta) = (\sigma\Lambda, \alpha) = \begin{cases} m & \text{if } (\sigma\Lambda, \alpha) > 0; \\ m - r & \text{if } (\sigma\Lambda, \alpha) < 0, \end{cases} \tag{A.16}
\]
\[
(r_\alpha \sigma\Lambda, \beta - \alpha) = (\sigma\Lambda, \beta - \alpha) = \begin{cases} m' & \text{if } (\sigma\Lambda, \alpha - \beta) < 0; \\ m' - r & \text{if } (\sigma\Lambda, \alpha - \beta) > 0. \end{cases} \tag{A.17}
\]
Let us show that \((\lambda, \lambda^\beta, \lambda^{\beta, \alpha - \beta})\) is admissible if \(\alpha - \beta \in \Delta_+\). From (A.16) follows \(m_{\alpha - \beta}(\lambda^\beta) = m\). Let us prove
\[
m_{\beta}(\lambda) = m + m'.
\] (A.18)

If \((\sigma \Lambda, \alpha) > 0\) and \((\sigma \Lambda, \alpha - \beta) < 0\), the statement (A.18) follows from (A.16) and (A.17). The case \((\sigma \Lambda, \alpha) < 0\) and \((\sigma \Lambda, \alpha - \beta) > 0\) contradicts with Lemma A.3 because \((\lambda, \lambda^\alpha)\) is admissible. In the remaining cases, we have \((\sigma \Lambda, \beta) = m + m' - r\). From Lemma A.3 (applied to \((\lambda, \lambda^\alpha)\)), we have \(m + m' - r < 0\), and therefore (A.18). Now, we have \(\lambda^{\beta, \alpha - \beta} = \lambda^{\alpha, \beta}\). This implies (see (A.11))
\[
d_{\alpha}(\lambda) = d_{\alpha - \beta}(\lambda^\beta) = 1.
\] (A.19)

Thus we proved the admissibility of \((\lambda, \lambda^\beta, \lambda^{\beta, \alpha - \beta})\) if \(\alpha - \beta \in \Delta_+\).

Next, we show that \((\lambda, \lambda^{\beta - \alpha}, \lambda^{\beta - \alpha, \alpha})\) is admissible if \(\beta - \alpha \in \Delta_+\). From (A.17) follows \(m_{\beta - \alpha}(\lambda) = m'\). Let us prove
\[
m_{\alpha}(\lambda^{\beta - \alpha}) = m + m'.
\] (A.20)

Note that \((r_{\beta - \alpha} \sigma \Lambda, \alpha) = (\sigma \Lambda, \beta)\). If \((\sigma \Lambda, \alpha) > 0\) and \((\sigma \Lambda, \beta - \alpha) > 0\), the statement (A.20) follows from (A.16) and (A.17). The case \((\sigma \Lambda, \alpha) < 0\) and \((r_{\alpha} \sigma \Lambda, \beta) = (\sigma \Lambda, \beta - \alpha) < 0\) contradicts with Lemma A.3 because \((\lambda^\alpha, \lambda^{\alpha, \beta})\) is admissible and \((r_{\alpha} \sigma \Lambda, \alpha) = -(\sigma \Lambda, \alpha) > 0\). In the remaining cases, we have \((\sigma \Lambda, \beta) = m + m' - r\). From Lemma A.3 (applied to \((\lambda^{\alpha, \beta})\)) we have \((r_{\alpha} \sigma \Lambda, \beta - \alpha) = (\sigma \Lambda, \beta) < 0\), and therefore (A.20). Thus we proved (ii) when \((\alpha, \beta) = 1\).

Case \((\alpha, \beta) = -1\)

Set \(m = m_{\alpha}(\lambda)\) and \(m' = m_{\beta}(\lambda^\alpha)\). Because of (2.13) we have \((\sigma \Lambda, \beta) \neq 0 \mod r\) and therefore \(m \neq m'\). We have (A.8). The only way other than (A.8) to write \(\lambda - \lambda^{\alpha, \beta}\) as a positive linear combination of two positive roots is
\[
\lambda - \lambda^{\alpha, \beta} = \begin{cases} (m - m')\alpha + m'(\alpha + \beta) & \text{if } m > m'; \\ m(\alpha + \beta) + (m' - m)\beta & \text{if } m < m'. \end{cases}
\] (A.21)

Again he uniqueness is obvious from (A.21). Note that
\[
(r_{\beta} \sigma \Lambda, \alpha + \beta) = (\sigma \Lambda, \alpha) = \begin{cases} m & \text{if } (\sigma \Lambda, \alpha) > 0; \\ m - r & \text{if } (\sigma \Lambda, \alpha) < 0, \end{cases}
\] (A.22)
\[
(r_{\alpha} \sigma \Lambda, \beta) = (\sigma \Lambda, \alpha + \beta) = \begin{cases} m' & \text{if } (\sigma \Lambda, \alpha + \beta) > 0; \\ m' - r & \text{if } (\sigma \Lambda, \alpha + \beta) < 0. \end{cases}
\] (A.23)

Let us show that \((\lambda, \lambda^{\alpha + \beta}, \lambda^{\alpha + \beta, \alpha})\) is admissible if \(m > m'\). From (A.23) follows \(m_{\alpha + \beta}(\lambda) = m'\). Let us prove
\[
m_{\alpha}(\lambda^{\alpha + \beta}) = m - m'.
\] (A.24)
Note that \((r_{\alpha+\beta}\sigma\Lambda, \alpha) = -(\sigma\Lambda, \beta)\). If \((\sigma\Lambda, \alpha) \geq 0\) and \((\sigma\Lambda, \alpha + \beta) \geq 0\), we have 

\[-(\sigma\Lambda, \beta) = m - m' > 0,\]

and therefore \((A.24)\). If \((\sigma\Lambda, \alpha) < 0\) and \((\sigma\Lambda, \alpha + \beta) > 0\), we have 

\[-(\sigma\Lambda, \beta) = m - m' - r < 0,\]

and therefore \((A.24)\). If \((\sigma\Lambda, \alpha) > 0\) and \((\sigma\Lambda, \alpha + \beta) < 0\), we have 

\[-(\sigma\Lambda, \beta) = m - m' + r > r.\]

This is a contradiction.

Let us show that \((\lambda, \lambda^\beta, \lambda^{\alpha+\beta})\) is admissible if \(m < m'\). From \((A.23)\) we have 

\[m_{\alpha+\beta}(\lambda^\beta) = m.\]

Let us prove 

\[m_{\beta}(\lambda) = m' - m.\]  \((A.25)\)

If \((\sigma\Lambda, \alpha) \geq 0\) and \((\sigma\Lambda, \alpha + \beta) \geq 0\), we have \((\sigma\Lambda, \beta) = m' - m > 0\), and therefore \((A.25)\). If \((\sigma\Lambda, \alpha) < 0\) and \((\sigma\Lambda, \alpha + \beta) > 0\), we have \((\sigma\Lambda, \beta) = m' - m + r > r\).

This is a contradiction. If \((\sigma\Lambda, \alpha) > 0\) and \((\sigma\Lambda, \alpha + \beta) < 0\), we have \((\sigma\Lambda, \beta) = m' - m - r < 0\), and therefore \((A.25)\). We have completed the proof of (ii) when \((\alpha, \beta) = -1\).

### B Generalized Serre relations

We modify the relations \((2.5)\), \((2.6)\) and \((2.11)\) (keeping \((2.7)\)) as follows.

\[
\xi_i(u)\xi_i(v) = \frac{[u-v-\delta]}{[u-v+\delta]}\xi_i(v)\xi_i(u), \quad \text{\((B.1)\)}
\]

\[
\xi_i(u)\xi_j(v) = \frac{[u-v+\delta]}{[u-v-\delta]}\xi_j(v)\xi_i(u) \text{ if } |i-j|=1, \quad \text{\((B.2)\)}
\]

\[
\hat{\pi}_i\xi_j(u) = \xi_j(u)(\hat{\pi}_i - (\alpha_i, \alpha_j)\delta). \quad \text{\((B.3)\)}
\]

Here, \(\delta\) is a parameter. Note that if we set \(\delta = 0\) we have a commutative algebra.

Fix 

\[\{a_1, \ldots, a_{n-1}\} \quad (a_i \in \mathbb{Z}_0). \quad \text{\((B.4)\)}\]

Consider a function \(f\) of the variables \(u_j^{(b)}\) \((1 \leq j \leq n-1, 1 \leq b \leq a_j)\) and \(\kappa_j\) \((1 \leq j \leq n-1)\). We assume that \(f\) is symmetric in \((u_j^{(1)}, \ldots, u_j^{(a_j)})\) for each \(1 \leq j \leq n-1\). We call \(f\) a function of type \((a_1, \ldots, a_{n-1})\).

Suppose that \(f\) is of type \((a_1, \ldots, a_{n-1})\) and \(g\) is of type \((b_1, \ldots, b_{n-1})\). We define the \(*\)-product \(f*g\) of \(f\) and \(g\) to be the function of type \((a_1+b_1, \ldots, a_{n-1}+b_{n-1})\) given by

\[
(f*g)(u_1^{(1)}, \ldots, u_1^{(a_1)}, v_1^{(1)}, \ldots, v_1^{(b_1)}, \ldots, u_n^{(1)}, \ldots, u_n^{(a_n-1)}, v_n^{(1)}, \ldots, v_n^{(b_n-1)}; \kappa_1, \ldots, \kappa_{n-1}) = S\left(f(u_1^{(1)}, \ldots, u_1^{(a_1)}, \ldots, u_n^{(1)}, \ldots, u_n^{(a_n-1)}; \kappa_1 + (-2b_1 + b_2)\delta; \kappa_2 + (b_1 - 2b_2 + b_3)\delta; \ldots; \kappa_{n-2} + (b_{n-3} - 2b_{n-2})\delta; \kappa_{n-1} + (b_{n-2} - 2b_{n-1})\delta\right) \\
\times g(v_1^{(1)}, \ldots, v_1^{(b_1)}, \ldots, v_n^{(1)}, \ldots, v_n^{(b_n-1)}; \kappa_1, \ldots, \kappa_{n-1})
\]
\[
\prod_{1 \leq j \leq n-1} \prod_{1 \leq b \leq b_j} \left[ \left( u_j^{(a)} - v_j^{(b)} \right) - \delta \right] \\
\prod_{1 \leq j \leq n-2} \prod_{1 \leq b \leq b_j+1} \left[ \left( u_j^{(a)} - v_j^{(b)} \right) + \frac{\delta}{2} \right] \\
\prod_{2 \leq j \leq n-1} \prod_{1 \leq b \leq b_j} \left[ \left( u_j^{(a)} - v_{j+1}^{(b)} \right) \right] \right). 
\]

(B.5)

Here the symbol \( S \) means the symmetrization of \( (u_j^{(1)}, \ldots, u_j^{(a)}, v_j^{(1)}, \ldots, v_j^{(b)}) \) for each \( 1 \leq j \leq n-1 \).

Let \( f^{(k_1, \ldots, k_m)}_{(i+1,m)} \) be the following function of type \((a_1, \ldots, a_{n-1})\) with

\[
a_j = \begin{cases} 1 & \text{if } i \leq j \leq i + m; \\ 0 & \text{otherwise}, \end{cases}
\]

\[
f^{(k_1, \ldots, k_m)}_{(i+1,m)}(u_i, \ldots, u_{i+m}; \kappa_i, \ldots, \kappa_{i+m}) = \prod_{j=1}^{m} \frac{u_{i+j-1} - u_{i+j} - (k_j + \frac{1}{2})\delta}{u_{i+j-1} - u_{i+j}} \\
\times \prod_{j=0}^{m} \left( u_{i+j} - (k_j - k_{j+1} + \frac{1}{2})\delta - \kappa_{i+j} \right) \quad (k_0 = -1, k_{m+1} = 0). 
\]

(B.6)

If \( m = 0 \), we understand the function \( f_i \) of type \((0, \ldots, 0, 1, 0, \ldots, 0)\), to be \( [u_i^{(1)} + \frac{\delta}{2} - \kappa_i] \).

Theorem B.1

\[
f^{(k_1, \ldots, k_m)}_{(i+1,m)} \ast f^{(k_1+p, \ldots, k_m+p)}_{(i+1,m)} = f^{(k_1+p, \ldots, k_m+p)}_{(i+1,m)} \ast f^{(k_1, \ldots, k_m)}_{(i+1,m)}. 
\]

(B.7)

Proof. This is similar to Theorem 1.4. \( \Box \)

Set

\[
f_{i+1,m}[k_1, \ldots, k_m] = f^{(k_1, \ldots, k_m)}_{(i+1,m)}, \quad \text{B.8}
\]

\[
f^{(a)}_{i+1,m}[k_1, \ldots, k_m] = \ast \prod_{b=1}^{a} f_{i+1,m}[k_1-b+1, \ldots, k_m-b+1] \quad \text{B.9}
\]

where the symbol \( \ast \) in front of the usual product symbol means that this is a \( \ast \)-product. The functions \( f^{(a)}_{i+1,m}[k_1, \ldots, k_m] \) satisfy a set of quadratic relations in \( \ast \)-product. They are given below. By specialization \( \delta = 1 \), we get the relations for the screening operators (4.5) \( X^{(a)}_{i+1,m}(k_1, \ldots, k_m) \).

For the proof of the quadratic relations we prepare a lemma.

Let \( F \) be the algebra over \( \mathbb{C} \) with the \( \ast \)-product, that is generated by elements \( f_{i+1,m}[k_1, \ldots, k_m] \). The algebra \( F \) is graded

\[
F = \oplus_{(a_1, \ldots, a_{n-1})} F_{a_1, \ldots, a_{n-1}} \quad \text{B.10}
\]

where \( F_{a_1, \ldots, a_{n-1}} \) consists of the functions of type \((a_1, \ldots, a_{n-1})\).
Lemma B.2 If \( f, g \in F \) and \( f \ast g = 0 \), then \( f = 0 \) or \( g = 0 \).

Proof. Suppose that \( f \) is of type \((a_1, \ldots, a_{n-1})\) and \( g \) is of type \((b_1, \ldots, b_{n-1})\). We expand \( f \) and \( g \) in power series of \( \delta \). If both \( f \) and \( g \) are non-zero, we have

\[
\begin{align*}
  f &= f_0 \delta^{m_1} + o(\delta^{m_1}), \quad f_0 \neq 0, \\
  g &= g_0 \delta^{m_2} + o(\delta^{m_2}), \quad g_0 \neq 0,
\end{align*}
\]

for some \( m_1 \) and \( m_2 \). From \( f \ast g = 0 \) it follows that

\[
S\left(f_0(u_1^{(1)}, \ldots, u_1^{(a_1)}, \ldots, u_n^{(1)}, \ldots, u_n^{(a_{n-1})}; \kappa_1, \ldots, \kappa_{n-1})
\right.
\]
\[
\cdot g_0(v_1^{(1)}, \ldots, v_1^{(b_1)}, \ldots, v_n^{(1)}, \ldots, v_n^{(b_{n-1})}; \kappa_1, \ldots, \kappa_{n-1})\) = 0
\]

(B.11)

We will show that \( f_0 = 0 \) or \( g_0 = 0 \). Choose \( w_1, \ldots, w_{n-1} \in \mathbb{C} \) so that \( f_0 \) and \( g_0 \) are holomorphic at \( u_j^{(b)} = w_j \) and \( v_j^{(b)} = w_j \), respectively. Power series expansion in \( u_j^{(b)} - w_j \) and \( v_j^{(b)} - w_j \) reduces the problem to the case when \( f_0 \) and \( g_0 \) are symmetric polynomials in \((u_j^{(1)}, \ldots, u_j^{(a_j)})\) and \((v_j^{(1)}, \ldots, v_j^{(a_j)})\), respectively. Finally the following lemma reduces the problem to the case of the polynomial ring.

Lemma B.3 Let \( G_{a_1,\ldots,a_{n-1}} \) be the \( \mathbb{C} \)-linear space of polynomials in the variables

\[
\begin{align*}
  u_1^{(1)}, \ldots, u_1^{(a_1)}, \ldots, u_n^{(1)}, \ldots, u_n^{(a_{n-1})},
\end{align*}
\]

that are symmetric in \((u_j^{(1)}, \ldots, u_j^{(a_j)})\) for each \( 1 \leq j \leq n - 1 \). Set

\[
G = \bigoplus_{(a_1,\ldots,a_{n-1})\in\mathbb{Z}^{n-1}_{\geq 0}} G_{a_1,\ldots,a_{n-1}}.
\]

(B.12)

Define the \(*\)-product in \( G \) by

\[
 f \in G_{a_1,\ldots,a_{n-1}}, g \in G_{b_1,\ldots,b_{n-1}} \mapsto f \ast g \in G_{a_1+b_1,\ldots,a_{n-1}+b_{n-1}},
\]

where

\[
(f \ast g)(u_1^{(1)}, \ldots, u_1^{(a_1)}, u_1^{(1)}, \ldots, u_1^{(b_1)}, \ldots, u_n^{(1)}, \ldots, u_n^{(a_{n-1})}, v_1^{(1)}, \ldots, v_1^{(b_1)}, \ldots, v_n^{(1)}, \ldots, v_n^{(b_{n-1})})
\]

\[
= S\left(f(u_1^{(1)}, \ldots, u_1^{(a_1)}, \ldots, u_n^{(1)}, \ldots, u_n^{(a_{n-1})})g(v_1^{(1)}, \ldots, v_1^{(b_1)}, \ldots, v_n^{(1)}, \ldots, v_n^{(b_{n-1})})\right).
\]

There is a ring homomorphism between \( G \) and the polynomial ring of the variables

\[
(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, \ldots; \ldots; x_n^{(0)}, x_n^{(1)}, x_n^{(2)}, \ldots).
\]
Proof. For simplicity, we consider the case $n = 2$. The isomorphism is such that the subspace $G_a$ of $G$ corresponds to the space of degree $a$ homogeneous polynomials in $(x_1^{(0)}, x_1^{(1)}, x_1^{(2)}, \ldots)$. The isomorphism is given by

$$x_1^{(m_1)} \cdots x_1^{(m_n)} \mapsto S\left((u_1^{(1)})^{m_1} \cdots (u_1^{(a)})^{m_n}\right). \quad (B.13)$$

The basic relations are

**Lemma B.4**

\[
\begin{align*}
f_{i \ldots i+t-1}[k_1, \ldots, k_{t-1}] & \ast f_{i \ldots i+t+i+m}[k_{t+1}, \ldots, k_{t+m}] \\
& = f_{i \ldots i+t+m}[k_1, k_{t-1}, k_{t+1}, \ldots, k_{t+m}], \\
& \quad \quad (B.14) \\
f_{i \ldots i+t+i+m}[k_{t+1}, \ldots, k_{t+m}] & \ast f_{i \ldots i+t-1}[k_1, \ldots, k_{t-1}] \\
& = f_{i \ldots i+t+m}[k_1, k_{t-1}, 0, k_{t+1}, \ldots, k_{t+m}], \\
& \quad \quad (B.15) \\
f_i \ast f_{i \ldots i+m}[k_1, k_2, \ldots, k_m] & = f_{i \ldots i+m}[k_1 - 1, k_2, \ldots, k_m] \ast f_i, \\
& \quad \quad (B.16) \\
f_{i+m} \ast f_{i \ldots i+m}[k_1, \ldots, k_{m-1}, k_m - 1] & = f_{i \ldots i+m}[k_1, \ldots, k_{m-1}, k_m] \ast f_{i+m}, \\
& \quad \quad (B.17) \\
f_{i \ldots i+m}[k_1, \ldots, k_m] \ast f_{j \ldots j+l}[k_1', \ldots, k_l'] & = f_{j \ldots j+l}[k_1, \ldots, k_l'] \\
& \ast f_{i \ldots i+m}[k_1, \ldots, k_m], \quad \quad \text{if } i + m + 1 < j. \\
& \quad \quad (B.18)
\end{align*}
\]

The proof is straightforward.

**Lemma B.5**

\[
\begin{align*}
f_{i \ldots i+t}[k_1, \ldots, k_t] & \ast f_{i \ldots i+t+i+m}[k_1 + k_{t+1}, \ldots, k_t + k_{t+1}, k_{t+1}, k_{t+2}, \ldots, \\
& \cdots, k_{t+m}] = f_{i \ldots i+t+i+m}[k_1 + k_{t+1}, \ldots, k_t + k_{t+1}, k_{t+1} - 1, k_{t+2}, \ldots, k_{t+m}] \\
& \ast f_{i \ldots i+t}[k_1, \ldots, k_t], \\
& \quad \quad (B.19) \\
f_{i \ldots i+t+i+m}[k_{t+1}, \ldots, k_{t+m}] & \ast f_{i \ldots i+t+i+m}[k_1, \ldots, k_{t-1}, k_t - 1, k_{t+1} + k_t, \ldots, \\
& \cdots, k_{t+m} + k_t] = f_{i \ldots i+t+i+m}[k_1, \ldots, k_{t-1}, k_t, k_{t+1} + k_t, \ldots, \\
& \cdots, k_{t+m} + k_t] \ast f_{i \ldots i+t+i+m}[k_{t+1}, \ldots, k_{t+m}], \\
& \quad \quad (B.20) \\
f_{i \ldots i+t+i+m}[k_1, \ldots, k_{t-1}, k_t, k_{t+1} + k_t, \ldots, k_{t+m} + k_t] \\
& \ast f_{i \ldots i+t+i+m+p}[k_{t+1}, \ldots, k_{t+m}, -k_t, k_{t+m+2}, \ldots, k_{t+m+p}] \\
& = f_{i \ldots i+t+i+m+p}[k_{t+1}, \ldots, k_{t+m}, -k_t - 1, k_{t+m+2}, \ldots, k_{t+m+p}] \\
& \ast f_{i \ldots i+t+i+m}[k_1, \ldots, k_{t-1}, k_t - 1, k_{t+1} + k_t, \ldots, k_{t+m} + k_t]. \\
& \quad \quad (B.21)
\end{align*}
\]
The proof will be given later.
The following are simple consequences of the above.

**Proposition B.6**

\[
f_{i+1}[k_1, \ldots, k_l] \ast f^{(b)}_{i-1+i+l+m}[k_1 + k_{l+1}, \ldots, k_l + k_{l+1} + a - 1, k_{l+2}, \ldots, k_{l+m}] = \]
\[
\cdots, k_{l+1} - 1, k_{l+2}, \ldots, k_{l+m}]
\ast f^{(a)}_{i-1+i+l}[k_1, \ldots, k_l],
\]

(B.22)

\[
f^{(a)}_{i+i+l+m}[k_{l+1}, \ldots, k_{l+m}] \ast f^{(b)}_{i-i+i+l+m}[k_1, \ldots, k_{l-1}, k_l - 1, k_{l+1} + k_l, \ldots, k_{l+m} + k_l]
\ast f^{(a)}_{i+i+l}[k_1, \ldots, k_{l+m}],
\]

(B.23)

The following is also valid. However, the general case for \(a, b\) does not follow from the special case \(a = b = 1\).

**Proposition B.7**

\[
f^{(a)}_{i+i+l+m}[k_{l+1}, \ldots, k_{l+m}] \ast f^{(b)}_{i-i+i+l+m+p}[k_1, \ldots, k_{l-1}, k_l - 1, k_{l+1} + k_l, \ldots, k_{l+m} + k_l + a - 1, k_{l+1} + k_l, \ldots, k_{l+m} + k_l, k_l + 1, k_{l+m} + 2, \ldots, k_{l+m} + p]
\ast f^{(a)}_{i+i+l}[k_1, \ldots, k_{l+m} + 2, \ldots, k_{l+m} + p]
\ast f^{(a)}_{i+i+l+m}[k_{l+1}, \ldots, k_{l+m} + 2, \ldots, k_{l+m} + p]
\]

(B.25)

We use these relations for integer parameters \(k_i\). However, they are valid without this restriction because the general case follows from the integer case.

Let us derive (B.19). The other cases follow similarly. Without loss of generality one can assume that \(m = 1\). Using (B.14) and (B.13) we have

\[
f_{i+l} \ast f_{i-i+l-1}[k_1, \ldots, k_{l-1}] \ast f_{i-i+l}[k_1 + k_l, \ldots, k_{l-1} + k_l, k_l]
\ast f_{i+l}[k_1 + k_l, \ldots, k_{l-1} + k_l, k_l] \ast f_{i+l} \ast f_{i-i+l-1}[k_1, \ldots, k_{l-1}].
\]

(B.26)
Proof. Let us derive (B.28). The other cases can be proven similarly. Using (B.17) we have

\[ f_{i+1} \cdot (f_{i+i-1}[k_1, \ldots, k_{l-1}] \cdot f_{i+i-1}[k_1 + k_l, \ldots, k_{l-1} + k_l]) - f_{i+1-1}[k_1 + k_l, \ldots, k_{l-1} + k_l, k_l - 1] \cdot f_{i+i-1}[k_1, \ldots, k_{l-1}] = 0. \]  
(B.27)

Since \( f_{i+1} \) is not a zero divisor, we have (B.19).

Combining all these, in particular (B.22) and (B.23), we arrive at the formulas which we need for the quadratic relations of the screening operators.

**Proposition B.8**

\[ f_{i:i+i+1-1}[k_1, \ldots, k_l] \cdot f_{i:i+i+1-1}[k_{l+2}, \ldots, k_{l+m}] 
= f_{i:i+i+1-1}[k_1 - a, \ldots, k_l - a, -a - 1, k_{l+2}, \ldots, k_{l+m}] \cdot f_{i:i+i+1-1}[k_1, \ldots, k_l], \]  
(B.28)

\[ f_{i:i+i+1-1}[k_1, \ldots, k_l] \cdot f_{i:i+i+1-1}[k_{l+2}, \ldots, k_{l+m}] 
= f_{i:i+i+1-1}[k_{l+2}, \ldots, k_{l+m}] \cdot f_{i:i+i+1-1}[k_1, \ldots, k_l, -b - 1, k_{l+2} - b, \ldots, k_{l+m} - b], \]  
(B.29)

\[ f_{i:i+i+1-1}[k_1, \ldots, k_l, a + b - 1, k_{l+2}, \ldots, k_{l+m}] \cdot f_{i:i+i+1-1}[k_{l+2} - b, \ldots, k_{l+m} - b]. \]  
(B.30)

\[ f_{i:i+i+1-1}[k_1, \ldots, k_l, a + b - 1, k_{l+2}, \ldots, k_{l+m}] \cdot f_{i:i+i+1-1}[k_1, \ldots, k_l], \]  
(B.31)

**Proof.** Let us derive (B.28). The other cases can be proven similarly. Using (13), (B.14), (B.22), we have

\[ f_{i:i+i+1-1}[k_1, \ldots, k_l] = f_{i:i+i+1-1}[k_1 - a - b + 1, \ldots, k_l - a - b + 1] \cdot f_{i:i+i+1-1}[k_{l+2} - b + 1, \ldots, k_{l+m} - b + 1] \cdot f_{i:i+i+1-1}[k_{l+2}, \ldots, k_{l+m}] 
= f_{i:i+i+1-1}[k_1 - a - b + 1, \ldots, k_l - a - b + 1, -1, k_{l+2} - b + 1, \ldots, k_{l+m} - b + 1] \cdot f_{i:i+i+1-1}[k_{l+2}, \ldots, k_{l+m}]. \]  
(B.32)
C Commutativity with DWA.

In this section we prove the Lemma 5.3. It would be more convenient for us to use here the "multiplicative" variable $z$ instead of $u$. For this reason, let us define the theta function

$$[[z]] \equiv [u], \quad z = x^{2u}$$  \hspace{1cm} (C.1)

having the periodicity property $[[zx^2]] = -[[z]]$. Abusing the notations, let us use the same symbol $f^{(a)}_\alpha(z^{(1)}_i, \ldots, z^{(a)}_{i+m})$ for the function $f^{(a)}_\alpha(u^{(1)}_i, \ldots, u^{(a)}_{i+m})$ where $z^{(b)}_j = x^{2u^{(b)}_j}$. The screening operator $\overline{X}_\alpha(\lambda)$ in the notations (4.9)-(4.12) is given by (4.15), i.e., in the multiplicative variables,

$$\overline{X}_\alpha(\lambda) = \oint \cdots \oint \prod_{1 \leq i < j \leq i+m} \frac{dz^{(b)}_j}{2\pi iz^{(b)}_j} \cdot \left( \prod_{1 \leq i \leq a} U_i(z^{(1)}_i) \cdots U_i(z^{(a)}_i) \cdot (U_{i+m}(z^{(1)}_{i+m}) \cdots U_{i+m}(z^{(a)}_{i+m})) \cdot \prod_{1 \leq b < c} \frac{[[z^{(b)}_j/z^{(c)}_j]]}{[[z^{(b)}_j/x/z^{(c)}_j]]} \cdot \prod_{1 \leq b < c} \frac{[[z^{(b)}_j]]}{[[z^{(b)}_j/z^{(c)}_j]]} \right) \cdot f^{(a)}_\alpha(z^{(1)}_1, \ldots, z^{(a)}_i, \ldots, z^{(1)}_{i+m}, \ldots, z^{(a)}_{i+m}).$$  \hspace{1cm} (C.2)

The non-trivial couplings between the screening currents with each others and with the field $\Lambda_j(z)$ are:

$$U_j(z)U_{j+1}(w) = z^{\frac{r-1}{r}}s(w/z) : U_j(z)U_{j+1}(w) : ,$$
$$U_{j+1}(z)U_j(w) = z^{\frac{r-1}{r}}s(w/z) : U_{j+1}(z)U_j(w) : ,$$
$$U_j(z)U_j(w) = z^{2\frac{r-1}{r}}t(w/z) : U_j(z)U_j(w) : ,$$
$$\Lambda_j(z)U_j(w) = x^{-2r-1}T_j(w/z) : \Lambda_j(z)U_j(w) : ,$$
$$U_j(w)\Lambda_j(z) = x^{-2r-1}T_j(w/z) : \Lambda_j(z)U_j(w) : ,$$
$$U_j(z)\Lambda_{j+1}(w) = \hat{T}_j(w/z) : \Lambda_{j+1}(w)U_j(z) : ,$$
$$\Lambda_{j+1}(w)U_j(z) = \hat{T}_j(w/z) : \Lambda_{j+1}(w)U_j(z) : ,$$  \hspace{1cm} (C.3)

where

$$s(z) = \frac{(x^{2r-1}z; x^{2r})_\infty}{(xz; x^{2r})_\infty},$$
$$t(z) = (1 - z)\frac{(x^{2r}z; x^{2r})_\infty}{(x^{2r-2}z; x^{2r})_\infty},$$
where $\Lambda^0, \Lambda^1$. According to this Lemma we have
\begin{equation}
\Lambda_j(z) W = \Lambda_j(z) W \nabla \Lambda_j(z) \text{ and } U_j(z) \Lambda_j(z) \Lambda_j(z) \text{ must be understood as the expansion in } z/w.
\end{equation}

The screening currents commute with the generator $W^{(1)}_t$ up to a total difference [14, 15].

\textbf{Lemma C.1}

\begin{equation}
[W^{(1)}_t, U_j(z)] = (x^{-2r+2} - 1)(z x^{j-r})^t : \Lambda_j(z x^{j-r}) U_j(z) : - \{ z \rightarrow x^{2r} \}.
\end{equation}

According to this Lemma we have
\begin{align}
(x^{-2r+2} - 1)^{-1} [W^{(1)}_t, U_j(z) & \cdots U_{i+m}(z^{(a)}_{i+m})] \\
& = \sum_{i \leq j \leq i+m \atop 1 \leq a} \left( (z^{(f)}_j x^{j-r}) U_i(z^{(1)}_i) \cdots : \Lambda_j(z^{(f)}_j x^{j-r}) U_j(z^{(f)}_j) : \cdots U_{i+m}(z^{(a)}_{i+m}) \right. \\
& \left. - \{ z^{(f)}_j \rightarrow x^{2r} z^{(f)}_j \} \right).
\end{align}

By normal-ordering the operator part we obtain an expression involving various functions and operators. For our purpose it is useful to introduce the following objects. Let $0 \leq s \leq m, 1 \leq f_0, \cdots, f_s \leq a$ and $i \leq k \leq i+m-s$. (The case of our immediate interest above is $s = 0$. However, the general case will be necessary as we proceed.) We define
\begin{equation}
J^{(f_0, \ldots, f_s)}_{k+i+s} \left( f \right) = \left( \begin{array}{c} f_0 \ldots f_s \\ |w| = 1 \end{array} \right) \left( \begin{array}{c} 2\pi i w \end{array} \right) \left. \frac{d w}{j^{(f_0)}(z^{(1)}_i, \ldots, z^{(a)}_{i+m})} \right|_{z^{(f_p)}_{k+p} = x^{s-p} w} \quad (0 \leq p \leq s)
\end{equation}

where
\begin{align}
J^{(f_0)}_{k}(z^{(1)}_i, \ldots, z^{(a)}_{i+m}) &= U^{(f_0)}(z^{(1)}_i, \ldots, z^{(a)}_{i+m}) \\
& \times F(z^{(1)}_i, \ldots, z^{(a)}_{i+m}) T^{(f_0)}(z^{(1)}_i, \ldots, z^{(a)}_{i+m}) S^{(f_0)}(z^{(1)}_i, \ldots, z^{(a)}_{i+m})
\end{align}

and $U^{(f_0)}$, $F$, $T^{(f_0)}$ and $S^{(f_0)}$ are given below.
\begin{equation}
U^{(f_0)}(z^{(1)}_i, \ldots, z^{(a)}_{i+m}) = (z^{(f_0)}_k x^{k-r})^t : \Lambda_k(z^{(f_0)}_k x^{k-r}) \prod_{1 \leq b \leq a \atop i \leq j \leq i+m} U_j(z^{(b)}_j) :,
\end{equation}

\begin{equation}
F(z^{(1)}_i, \ldots, z^{(a)}_{i+m}) = \prod_{1 \leq b \leq a \atop i \leq j \leq i+m} \left[ \frac{z^{(b)}_j}{z^{(c)}_j} \right] \prod_{1 \leq b \leq a \atop i \leq j \leq i+m} \left[ \frac{z^{(b)}_j}{z^{(c)}_j} \right],
\end{equation}

\begin{align}
\tau_j(z) &= \frac{1 - z x^{r+j-2}}{1 - z x^{r-j}} , \\
\hat{\tau}_j(z) &= \frac{1 - z x^{r-j-2}}{1 - z x^{r+j}} .
\end{align}
\[ T_k^{(f_0)}(z_i^{(1)}, \ldots, z_i^{(a)}) = \prod_{1 \leq b \leq a \atop b \neq f_0} \frac{z_j^{(b)}/x^{2(\tau - 1)}}{[x z_j^{(b)} / z_j^{(c)}]} \times \prod_{1 \leq b \leq a \atop b \neq f_0} x^{-2(\tau - 1)} \tau_k(z_k^{(b)} x^{r - k} / z_k^{(f_0)}), \quad (C.9) \]

\[ S_k^{(f_0)}(z_i^{(1)}, \ldots, z_i^{(a)}) = \prod_{1 \leq b \leq a \atop b \neq f_0} \frac{z_j^{(b)}/x^{2(\tau - 1)}}{[x z_j^{(b)} / z_j^{(c)}]} \times \prod_{1 \leq b \leq a \atop b \neq f_0} \tau_{k-1}(z_k^{(f_0)} x^{r - k} / z_k^{(b)} z_{k-1}). \quad (C.10) \]

The restriction \( z_{k+p}^{(f_p)} = x^{s-p} w \) \( (0 \leq p \leq s) \) is regular except for the functions \( s(z_{k+p+1}^{(f_{p+1})} / z_{k+p}) \). For these singular terms we use the convention

\[ s(z^{-1}) \bigg|_{z=x} \overset{\text{def}}{=} \text{Res}_{z=x} s(z^{-1}) \frac{dz}{2\pi i z} = \frac{(x^{2(r-1)}; x^{2r})_{\infty}}{(x^{2r}; x^{2r})_{\infty}}. \quad (C.11) \]

Set

\[ I(s) = \sum_{k=i}^{i+m} \sum_{f_0, \ldots, f_s = 1}^{a} \left[ \cdots \left[ \prod_{1 \leq b \leq a \atop \tau_j^{(b)}(f_0, \ldots, f_s), (b,j) \neq (f_0,k), \ldots, (f_s,k+s)} \frac{dz_j^{(b)}}{2\pi i z_j^{(b)}}, \right] \right] \quad (C.12) \]

The contour for \( z_j^{(b)} \) is \( |z_j^{(b)}| = 1 \).

The induction goes as follows.

Step 1 \([W_{l}^{(1)}, X_{a}(\lambda)] = I(0)\)
Step 2 \(I(s) = I(s+1) \) \( (0 \leq s \leq m - 1)\)
Step 3 \(I(m) = 0\)

Step 1 follows from (C.2), (C.3) and (C.6)

We will show that the only poles of \( J_k^{(f_0)}(z_i^{(1)}, \ldots, z_i^{(a)}) \bigg|_{z_{k+p}^{(f_p)} = x^{s-p} w} \) \( (0 \leq p \leq s)\) between the contours \( |w| = 1 \) and \( |w| = x^{2r-s} \) are

\[ (A) \quad w = x z_{k+s+1}^{(b)} \quad (i \leq k \leq i + m - s - 1, 1 \leq b \leq a), \]
\[ (B) \quad w = x^{2r-s-1} z_{k-1}^{(b)} \quad (i + 1 \leq k \leq i + m - s, 1 \leq b \leq a). \quad (C.13) \]

In particular, Step 3 follows.

Let us abbreviate \( U_k^{(f_0)}(z_i^{(1)}, \ldots, z_i^{(a)}) \bigg|_{z_{k+p}^{(f_p)} = x^{s-p} w} \) \( (0 \leq p \leq s)\), etc., by \( U_k^{(f_0)} \bigg|_{z_s} \).
• $U^{(f_0)}_s$ has no poles. This is because the term is normal-ordered.

• $F_s$ has no poles. This is proved in Proposition 4.3.

• $T^{(f_a)}_s$ has no poles in the region

$$x^{2r-s} < |w| < 1.$$  \hfill (C.14)

(a) Consider the poles of \([s^{(b)}_j / x^2 z^{(c)}_j]^{-1}\) \(z^{(b)}_j = x^p w\). They are situated at $w = x^{2r-s-2} z^{(c)}_j \quad (\nu \in \mathbb{Z})$. However the factor $t(z^{(c)}_j / z^{(b)}_j) \big|_{z^{(b)}_j = x^p w}$ cancels the poles for $\nu \geq 0$. Therefore, there are no such poles in the region (C.14).

(b) Consider the poles of \([s^{(b)}_j / x^2 z^{(c)}_j]^{-1}\) \(z^{(c)}_j = x^p w\). The possible poles in (C.14) are $w = x^{2r-s-1} z^{(c)}_j$ or $w = x^{2r-s-1} z^{(c)}_{k+1}$. The former is canceled by $\tau_k(z^{(c)}_k x^{r-k} / x^s w)$, and the latter is canceled by the property (P5) of $f^{(a)}_\alpha$.

(c) Consider the poles of $t(z^{(c)}_j / z^{(b)}_j) \big|_{z^{(b)}_j = x^p w}$. The possible poles in (C.14) are $w = x^{2r-s-1} z^{(c)}_j$ or $w = x^{2r-s-2} z^{(c)}_{k+1}$ \((f_0 < c)\). The former is canceled by the property (P5) of $f^{(a)}_\alpha$ and the latter is canceled by $\tau_k(z^{(c)}_k x^{r-k} / x^s w)$.

(d) There is no pole of $t(z^{(c)}_j / z^{(b)}_j) \big|_{z^{(b)}_j = x^p w}$ in (C.14).

(e) There is no pole of $\tau_k(z^{(c)}_k x^{r-k} / x^s w)$ in (C.14).

• $S^{(f_0)}_s$ has two types of poles (A) and (B) of (C.13) arising from the factors $s(z^{(c)}_{k+s+1} / z^{(c)}_{k+s+1}) \big|_{z^{(c)}_{k+s+1} = w}$ and $\tilde{\tau}_{k-1}(z^{(f_0)}_k / x^{r-k} z^{(b)}_{k-1}) \big|_{z^{(b)}_k = x^p w}$, respectively.

(a) Consider the poles of $s(z^{(c)}_j / z^{(b)}_j) \big|_{z^{(b)}_j = x^p w}$. They are situated at $w = x^{p-s+1+2r} z^{(c)}_{k+p+1} \quad (\nu \in \mathbb{Z}_{\geq 0})$. The only poles in (C.14) are $w = x z^{(c)}_{k+s+1}$ \((1 \leq c \leq a)\).
(b) The pole of $\tilde{t}_{k-1}(z^{(f_0)}/x^{r-k}z^{(b)}_{k-1})$ at $w = x^{2r-s-1}z^{(b)}_{k-1}$.

(c) Consider the poles of $[[x z^{(b)}_j/z^{(c)}_{j+1}]^{-1}]_{z^{(b)}_j=x^{s-p}w}$. They are situated at $w = x^{2r+p-s-1}z^{(c)}_{k+p}$ ($\nu \in \mathbb{Z}$). The only pole in (C.14) is $w = x^{2r-s-1}z^{(c)}_k$. This is canceled by the property (P5) of $f^{(a)}_\alpha$.

(d) Consider the poles of $[[x z^{(b)}_j/z^{(c)}_{j+1}]^{-1}]_{z^{(c)}_{j+1}=x^{s-p}w}$. They are situated at $w = x^{2r+p-s+1}z^{(b)}_{k+p-1}$ ($\nu \in \mathbb{Z}$). The only pole in (C.14) is $w = xz^{(b)}_k$. This is canceled by the property (P5) of $f^{(a)}_\alpha$.

We finished checking (C.13).

We will conclude the induction by showing Step 2. We take the residues of $J^{(f_0)}_k(z^{(1)}_i, \ldots, z^{(a)}_{i+m})_{z^{(f_p)}_{k+p}=x^{s-p}w}$ (0 ≤ p ≤ s) at (A) and (B) of (C.13).

(A): We take the residue at $w = xz^{(f_{s+1})}_{k+s+1}$, and then rename $z^{(f_{s+1})}_{k+s+1}$ to $w$ in order to compare it with $J^{(f_0)}_k(z^{(1)}_i, \ldots, z^{(a)}_{i+m})_{z^{(f_p)}_{k+p}=x^{s+1-p}w}$ (0 ≤ p ≤ s+1). Except for the factor $s(z^{(f_{s+1})}_{k+s+1}/z^{(f_s)}_{k+s})$, this procedure is equivalent to changing the substitution rule $z^{(f_p)}_{k+p} = x^{s-p}w$ (0 ≤ p ≤ s) to $z^{(f_p)}_{k+p} = x^{s+1-p}w$ (0 ≤ p ≤ s + 1). The residue of $s(z^{(f_{s+1})}_{k+s+1}/w)_{z^{(f_p)}_{k+p}=x^{s+1-p}w}$ at $w = xz^{(f_{s+1})}_{k+s+1}$ gives $s(x^{-1})$ in the convention of (C.11). Thus we get the term in $J^{f_0, \ldots, f_{s+1}}_{k, \ldots, k+s+1}$ that is corresponding to the cycle $|w| = 1$.

(B): It is convenient to consider the residue of $J^{(f_1)}_{k+1}(z^{(1)}_i, \ldots, z^{(a)}_{i+m})_{z^{(f_p+1)}_{k+1+p}=x^{s-p}w}$ (0 ≤ p ≤ s) at $w = x^{2r-s-1}z^{(f_0)}_{k}$. We take the residue and then rename $z^{(f_0)}_{k}$ to $x^{-2r+s+1}w$. We will compare

(I) $J^{(f_1)}_{k+1}(z^{(1)}_i, \ldots, z^{(a)}_{i+m})_{z^{(f_p+1)}_{k+1+p}=x^{s-p}w}$ (0 ≤ p ≤ s), $z^{(f_0)}_{k}=x^{-2r+s+1}w$

and

(II) $J^{(f_0)}_k(z^{(1)}_i, \ldots, z^{(a)}_{i+m})_{z^{(f_p)}_{k+p}=x^{s+1-p}w}$ (0 ≤ p ≤ s+1)

taking care of the pole of $\tilde{t}_k(z^{(f_1)}_{k+1}/x^{-r-k-1}z^{(f_0)}_k)_{z^{(f_1)}_{k+1}=x^w}$ at $w = x^{2r-s-1}z^{(f_0)}_k$.

Let us abbreviate the restrictions (I) and (II) by $|I|$ and $|II|$, respectively. We will compute the ratios of the corresponding terms in (I) and (II).

- (U): $U^{(f_1)}_{k+1}/U^{(f_0)}_k = x^{-2(r-1)}$
This follows from the identity

\[ x^{2(r-1)} : \Lambda_{k+1}(x^{r+k}z)U_k(z) := \Lambda_k(x^{r+k}z)U_k(x^{2r}z) : \]

Set

\[ T_0 = \prod_{1 \leq b \leq a, i \leq j \leq i+m} ^{c} \left[ \frac{z_j^{(b)}}{x^2 z_j^{(c)}} \right] \]

and

\[ S_0 = \prod_{1 \leq b \leq a, i \leq j \leq i+m} \left[ \frac{xz_j^{(b)}}{z_j^{(c)} + 1} \right]. \]

- \textbf{(F)}: \left. \frac{F(T_0 S_0) - 1}{F(T_0) S_0} \right|_I = 1

The signs arising from the periodicity \([x^{2r}z] = -[z]\) cancel out.

- \textbf{(T)}: \left. \frac{T_{k+1}^{(f_1)}T_0}{T_k^{(f_0)}T_0} \right|_I = \prod_{b \neq f_1} \frac{1 - x^{2r-s-2}z_{k+1}^{(b)}/w}{1 - x^{-s}z_{k+1}^{(b)}/w} \prod_{b \neq f_0} \frac{x^{-4(r-1)} 1 - x^{2r-s-1}z_k^{(b)}/w}{1 - x^{s+1}z_k^{(b)}/w}

We used

\[ \frac{t(x^{2r}z)}{t(z)} = \frac{(1 - x^{2(r-1)}z)(1 - x^{2r}z)}{(1 - z)(1 - x^2z)}. \]

- \textbf{(S1)}:

\[ \text{Res}_{w=x^{2r-s-1}z_k^{(f_0)}} \hat{\tau}_k(x^{s+k+1-r}w/z_k^{(f_0)}) \frac{dw}{2\pi iw} = x^{2(r-1)} - 1. \]

- \textbf{(S2)}:

\[
\begin{align*}
\prod_{b \neq f_0} \hat{\tau}_k(x^{f_1}z_{k+1}^{(b)}/x^{r-k}z_k^{(b)}) & = \prod_{b \neq f_0} \hat{\tau}_k(x^{f_1}z_{k+1}^{(b)}/x^{r-k}z_k^{(b)}) \bigg|_{z_{k+1}^{(f_1)} = x^w} \\
\prod_{b} \hat{\tau}_{k-1}(z_k^{(f_0)}z_k^{(b)}/z_{k-1}) & = \prod_{b} \hat{\tau}_{k-1}(z_k^{(f_0)}z_k^{(b)}/z_{k-1}) \bigg|_{z_k^{(f_0)} = x^{s+1}w} \\
& = \prod_{b \neq f_0} x^{2(r-1)} \frac{1 - x^{-s-1}z_k^{(b)}/w}{1 - x^{2r-s-1}z_k^{(b)}/w} \prod_{b} \frac{1 - x^{-2r+s+2}w/z_k^{(b)}}{1 - x^s w/z_k^{(b)}} \text{(C.15)}
\end{align*}
\]
In the restriction (I) we must pay a special attention to the factor \( (z_k^{(f_0)})^{-\frac{1}{r}} \times s(z_{k+1}^{(f_1)}/z_k^{(f_0)}) \). We have

\[
(z_k^{(f_0)})^{-\frac{1}{r}} s(z_{k+1}^{(f_1)}/z_k^{(f_0)}) \bigg|_{z_k^{(f_0)} = x^{s+1} w} = x^{2(r-1)} (x^{s+1} w) \frac{r}{r-1} s(x^{2r-1})
\]

\[
\frac{x^{2(r-1)} (x^{s+1} w) \frac{r-1}{r}}{1 - x^{2(r-1)}} s(x^{-1})
\]

where \( s(x^{-1}) \) is in the sense of \([C,11]\). Taking this into account and collecting (S1) and (S2) we have

- (S): 
  \[
  S_{k+1}^{(f_1) S_0} \bigg|_I / S_k^{(f_0) S_0} \bigg|_I = \frac{-x^{2(r-1)} 1 - x^{s+1} z_k / w}{1 - x^{2r-s-1} z_k / w} \prod_{b \neq f_1} x^{2(r-1)} \frac{1 - x^{-s} z_{k+1}^{(b)} / w}{1 - x^{2r-s-2} z_{k+1}^{(b)} / w}.
  \]

From (U), (F), (T) and (S) we can conclude that the residues of

\[
J_{k+1}^{(f_1)}(z_1^{(1)}, \ldots, z_i^{(a)}) \bigg|_{z_k^{(f_0)} = x^{s+1} w} \bigg|_{z_{k+1}^{(f_1)} = x^{s+1} w} (0 \leq p \leq s)
\]

at \( w = x^{2r-s-1} z_k^{(f_0)} \) gives the term in \( I_{k,\ldots,k+s+1}^{(f_0,\ldots,f_{s+1})} \) that is corresponding to the cycle \( |w| = x^{2r-s-1} \). This completes the proof of the commutativity \([W_t^{(1)}, \Xi_\alpha(\lambda)] = 0\).

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