HODGE STRUCTURES OF THE MODULI SPACES OF PAIRS

VICENTE MUÑOZ

Abstract. Let $X$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Fix $n \geq 2$, $d \in \mathbb{Z}$. A pair $(E, \phi)$ over $X$ consists of an algebraic vector bundle $E$ of rank $n$ and degree $d$ over $X$ and a section $\phi \in H^0(E)$. There is a concept of stability for pairs which depends on a real parameter $\tau$. Let $M_\tau(n, d)$ be the moduli space of $\tau$-semistable pairs of rank $n$ and degree $d$ over $X$. Here we prove that the cohomology groups of $M_\tau(n, d)$ are Hodge structures isomorphic to direct summands of tensor products of the Hodge structure $H^1(X)$. This implies a similar result for the moduli spaces of stable vector bundles over $X$.

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ over the field of complex numbers. Fix $n \geq 2$ and $d \in \mathbb{Z}$. We shall denote by $M(n, d)$ the moduli space of $(S$-equivalence classes of) semistable bundles of rank $n$ and degree $d$ over $X$. The open subset consisting of stable bundles will be denoted $M^s(n, d) \subset M(n, d)$. Note that $M(n, d)$ is a projective variety, which is in general not smooth if $n$ and $d$ are not coprime. On the other hand, $M^s(n, d)$ is a smooth quasi-projective variety. If $L_0$ is a fixed line bundle of degree $d$, then we have the moduli spaces $M^s(n, L_0)$ and $M(n, L_0)$ consisting of stable and polystable bundles $E$, respectively, with determinant $\det(E) \cong L_0$.

A pair $(E, \phi)$ over $X$ consists of a bundle $E$ of rank $n$ and degree $d$ over $X$ together with a section $\phi \in H^0(E)$. There is a concept of stability for a pair which depends on the choice of a parameter $\tau \in \mathbb{R}$. This gives a collection of moduli spaces of $\tau$-polystable pairs $M_\tau(n, d)$, which are projective varieties. It contains a smooth open subset $M^s_\tau(n, d) \subset M_\tau(n, d)$ consisting of $\tau$-stable pairs. If we fix the determinant $\det(E) \cong L_0$, then we have the moduli spaces of pairs with fixed determinant, $M^s_\tau(n, L_0)$ and $M_\tau(n, L_0)$. Pairs are discussed at length in [3, 4, 8, 12].

The range of the parameter $\tau$ is an open interval $I$ split by a finite number of critical values $\tau_c$. For a non-critical value $\tau \in I$, there are no properly semistable pairs, so $M_\tau(n, d) = M^s_\tau(n, d)$ is smooth and projective. For a critical value $\tau = \tau_c$, $M_\tau(n, d)$ is in general singular at properly $\tau$-semistable points. Our first main result is the following.

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**Theorem 1.1.** Let \( \tau \in I \) be non-critical. Then the pure Hodge structures \( H^i(\mathcal{M}_\tau(n, d)) \) are isomorphic to direct summands of some tensor products of \( H^1(X) \). A similar result holds for \( H^i(\mathcal{M}_\tau(n, L_0)) \).

We recover from this a well-known result of Atiyah [2].

**Corollary 1.2.** Suppose that \( n \) and \( d \) are coprime. Then the pure Hodge structures \( H^i(M(n, d)) \) and \( H^i(M(n, L_0)) \) are isomorphic to direct summands of some tensor products of \( H^1(X) \).

Our strategy of proof is the following. First, we find more convenient to rephrase the problem in terms of triples. A triple \( (E_1, E_2, \phi) \) consists of a pair of bundles \( E_1, E_2 \) of ranks \( n_1, n_2 \) and degrees \( d_1, d_2 \), respectively, and a homomorphism \( \phi : E_2 \to E_1 \).

There is a suitable concept of stability for triples depending on a real parameter \( \sigma \). This gives rise to moduli spaces \( N_\sigma = N_\sigma(n_1, n_2, d_1, d_2) \) of \( \sigma \)-polystable triples.

There is an identification of moduli spaces of pairs and triples given by \( M(\tau(n, d)) \to N_\sigma(n_1, n_2, d_1, d_2) \), \( (E, \phi) \mapsto (E, \O, \phi) \), where \( \O \) is the trivial line bundle, and \( \sigma = (n + 1)\tau - d \). Actually, this rephrasing is a matter of aesthetic. The arguments are carried out with the moduli spaces of triples so that they could eventually be generalised to the case of triples of arbitrary ranks \( n_1, n_2 \).

The range of the parameter \( \sigma \) is an interval \( I = (\sigma_m, \sigma_M) \subset \mathbb{R} \) split by a finite number of critical values \( \sigma_c \). When \( \sigma \) moves without crossing a critical value, then \( N_\sigma(n_1, d, \O) \) remains unchanged, but when \( \sigma \), crosses a critical value, \( N_\sigma \) undergoes a birational transformation which we call a flip. This consists on removing some subvariety and inserting a different one. We give a stratification of the flip locus, and describe explicitly the strata. This allows us to prove Theorem 1.1.

For \( \sigma = \sigma_m^+ = \sigma_m + \epsilon \), \( \epsilon > 0 \) small, we have a morphism \( N_\sigma \to M(n, d) \). When \( d \) is large enough, this is a fibration over the locus \( M^s(n, d) \). This allows us to deduce Corollary 1.2 from Theorem 1.1.

For proving the main result, we introduce the notion of a mixed Hodge structure to be \( R_X \)-generated when the graded pieces are pure Hodge structures which are isomorphic to direct summands of tensor products of the Hodge structure \( H^1(X) \). We actually obtain the following.

**Theorem 1.3.** Let \( \tau \in I \) (critical or not). Then the mixed Hodge structures \( H^i(\mathcal{M}^+_\tau(n, d)) \) and \( H^i(\mathcal{M}^+_\tau(n, L_0)) \) are \( R_X \)-generated.

Let \( n \geq 2, d \in \mathbb{Z} \) (coprime or not). Then the mixed Hodge structures \( H^i(M^s(n, d)) \) and \( H^i(M^s(n, L_0)) \) are \( R_X \)-generated.

We prove this by induction on the rank, since the flip loci can be suitable described in terms of moduli spaces of lower rank. In the course of the proof, we get an explicit geometrical description of the flip loci. This can be useful for many other applications:
• Extend the results of [10] to compute the Hodge-Deligne polynomials of the moduli spaces of pairs for arbitrary rank \( n > 3 \).

• Compute the \( K \)-theory class of the moduli spaces of pairs and the moduli spaces of bundles.

• Prove the Generalized Hodge Conjecture for moduli spaces of pairs and bundles corresponding to curves \( X \) which are generic, by using the result of [1].

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2. Hodge structures

Let us start by recalling the Hodge-Deligne theory of algebraic varieties over \( \mathbb{C} \). See the original reference [7] or the nice book [13] for generalities on this.

2.1. Pure Hodge structures. Let \( H \) be a finite-dimensional vector space over \( \mathbb{Q} \). A pure Hodge structure of weight \( k \) on \( H \) is a decomposition

\[
H_{\mathbb{C}} = H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q},
\]

such that \( H^{q,p} = \overline{H}^{p,q} \), the bar denoting complex conjugation in \( H \). A Hodge structure of weight \( k \) on \( H \) gives rise to the so-called Hodge filtration \( F \) on \( H_{\mathbb{C}} \), where

\[
F^p = \bigoplus_{s \geq p} H^{s,k-s},
\]

which is a descending filtration. Note that \( \text{Gr}^p_F H_{\mathbb{C}} = F^p/F^{p+1} = H^{p,q} \).

Let \( \mathfrak{hs} \) denote the category of pure Hodge structures. This is an abelian category with tensor products. We shall denote by \( \mathbb{Q}(l) \) the Hodge structure (of weight \( 2l \)) given by the vector space \( H = (2\pi i)^l \mathbb{Q} \) with \( H^{l,l} = \mathbb{C} \). The trivial Hodge structure is \( \mathbb{Q} = \mathbb{Q}(0) \), and the Hodge structure \( \mathbb{T} = \mathbb{Q}(1) \) is known as the Tate Hodge structure. For any Hodge structure \( H \), the \( l \)-th Tate twist is \( H(l) := H \otimes \mathbb{T}^{\otimes l} = H \otimes \mathbb{Q}(l) \), which has the same underlying vector space as \( H \), but with grading \( H((l)^{p,q} = H^{p-l,q-l} \).

A pure Hodge structure \( H \) of weight \( k \) is polarizable if there exists a morphism of Hodge structures \( \theta : H \otimes H \to \mathbb{Q}(k) \), which is a non-degenerate bilinear map. Let \( \mathfrak{phs} \) denote the category whose objects are polarised Hodge structures. This is an abelian sub-category with tensor products of \( \mathfrak{hs} \). The category \( \mathfrak{phs} \) is semi-simple, that is, if \( H' \subset H \) is a sub-Hodge structure of a polarised Hodge structure \( H \), then \( H' \) is also polarisable, and there exists another sub-Hodge structure \( H'' \subset H \) such that \( H = H' \oplus H'' \).

Consider the Grothendieck group of \( \mathfrak{hs} \). This is the abelian group \( K_0(\mathfrak{hs}) \) generated by elements \([H]\), where \( H \) is a pure Hodge structure, with the relation \([H] = [H'] + \ldots\)
$[H^n]$, whenever there is an exact sequence
\[ 0 \to H' \to H \to H'' \to 0. \]
Clearly, $K_0(\mathfrak{h})$ is generated by the simple Hodge structures. A Hodge structure is simple if it does not admit proper sub-Hodge structures. Note that any polarisable Hodge structure is a direct sum of simple Hodge structures.

Let $H$ be a pure Hodge structure. Then there is a filtration $0 \subset H_1 \subset H_2 \subset \cdots \subset H_r = H$ such that $H_i = H_i/H_{i-1}$ is a simple Hodge structure. In $K_0(\mathfrak{h})$ we have that $H$ is equivalent to $\bigoplus H_i$. Note that the element $\bigoplus H_i$ is uniquely defined (up to the order of the summands). We define $\text{Gr}(H) := \bigoplus H_i$.

If $Z$ is a compact smooth projective variety (hence compact Kähler) then the cohomology of $Z$, $H^k(Z)$, admits a Hodge structure given by the Hodge decomposition of harmonic forms into $(p,q)$ types. This is pure of weight $k$ and it is polarised.

2.2. Mixed Hodge structures. Let $H$ be a finite-dimensional vector space over $\mathbb{Q}$. A mixed Hodge structure over $H$ consists of an ascending weight filtration $W$ on $H$ and a descending Hodge filtration $F$ on $H_\mathbb{C}$ such that $F$ induces a pure Hodge filtration of weight $r$ on each rational vector space $Gr^W_r H = W_r/W_{r-1}$. We define $H^{p,q} = Gr^p_F(Gr^W_{p+q} H)_\mathbb{C}$.

Let $\mathfrak{mhs}$ denote the category of mixed Hodge structures. This is an abelian category with tensor products. There is a natural map [13] (3.1) from $\mathfrak{mhs}$ to $K_0(\mathfrak{h})$,
\[ \Psi : H \mapsto \sum_r [Gr^W_r H], \] (2.1)
which sends a mixed Hodge structure to a direct sum of pure Hodge structures of different weights.

Deligne has shown [7] that, for each complex algebraic variety $Z$, the cohomology $H^k(Z)$ and the cohomology with compact support $H^k_c(Z)$ both carry natural (mixed) Hodge structures. If $Z$ is a smooth projective variety then this is the pure Hodge structure previously mentioned.

2.3. Hodge structures $R$-generated. Let $R \subset \mathfrak{h}$ be a given collection of simple Hodge structures. We define
\[ \langle R \rangle \subset K_0(\mathfrak{h}) \]
as the smallest sub-ring containing all elements of $R$, and which is closed by taking sub-objects (that is, if $\sum n_i H_i \in \langle R \rangle$, with $n_i \neq 0$, then $H_i \in \langle R \rangle$, for all $i$), and which is closed under Tate twists.

We say that a Hodge structure $H \in \mathfrak{h}$ is $R$-generated if $[H] \in \langle R \rangle$. We say that a mixed Hodge structure $H \in \mathfrak{mhs}$ is $R$-generated if $\Psi(H) \in \langle R \rangle$.

In general, we shall take a collection of polarised Hodge structures $R = \{H_1, \ldots, H_m\}$. In this case, all Hodge structures in $\langle R \rangle$ are polarisable. Actually, a simple polarised
Hodge structure $H \in \langle R \rangle$ is a sub-Hodge structure of some

$$H \subset H^{k_1}_{1} \otimes \ldots \otimes H^{k_m}_{m}(I),$$

for suitable $k_1, \ldots, k_m \geq 0$, $l \in \mathbb{Z}$. Equivalently, there is an epimorphism from a tensor product of elements of $R$ to $H$.

In particular, if $Z$ is a smooth projective variety, and $H^*(Z)$ is $R$-generated, then $H^k(Z)$ is a sub-Hodge structure of some direct sum of tensor products of Hodge structures in $R$, for each $k$.

**Lemma 2.1.** Let $Z = Y \sqcup W$. If two of the mixed Hodge structures $H^*_c(Z)$, $H^*_c(Y)$ and $H^*_c(W)$ are $R$-generated, then so is the third.

**Proof.** We can assume that $Y$ is a closed subvariety of $Z$, and then $W = Z - Y$ is open in $W$. Suppose that $H^*_c(Y)$ and $H^*_c(W)$ are $R$-generated, and let us see that $H^*_c(Z)$ is also $R$-generated. (The other cases are dealt with in an analogous way.) We have a long exact sequence

$$\ldots \to H^{k}_c(W) \to H^{k}_c(Z) \to H^{k}_c(Y) \to H^{k+1}_c(W) \to H^{k+1}_c(Z) \to \ldots$$

Since $\text{Gr}_r^W$ is an exact functor \cite{7}, we have an exact sequence for any given $r$

$$\ldots \to \text{Gr}_r^W H^{k}_c(W) \to \text{Gr}_r^W H^{k}_c(Z) \to \text{Gr}_r^W H^{k}_c(Y) \to \text{Gr}_r^W H^{k+1}_c(W) \to \ldots$$

Then we have short exact sequences

$$0 \to A_k \to \text{Gr}_r^W H^{k}_c(Z) \to B_k \to 0,$$  \hspace{1cm} (2.2)

for each $k$. Also we have exact sequences

$$0 \to B_k \to \text{Gr}_r^W H^{k}_c(Y) \to C_k \to 0,$$

$$0 \to C_k \to \text{Gr}_r^W H^{k+1}_c(W) \to A_{k+1} \to 0.$$

In $K_0(\mathfrak{h})$ we have that $\text{Gr}(\text{Gr}_r^W H^{k}_c(Y)) = \text{Gr}(B_k) \oplus \text{Gr}(C_k)$, hence $\text{Gr}(B_k)$ is also $R$-generated, being a sub-Hodge structure of an $R$-generated one. The same is true for $\text{Gr}(A_{k+1})$. This happens for all $k$. So (2.2) implies that $H^*_c(Z)$ is $R$-generated. \hfill $\square$

We can apply the above to the set $R_{triv} = \emptyset$. The Hodge structures in $\langle R_{triv} \rangle$ are the trivial Hodge structures (that is, those for which $H^{p,q} = 0$ unless $p = q$). Lemma \ref{2.1} has the following corollary.

**Corollary 2.2.** Suppose that $Z$ admits a stratification $Z = \sqcup Z_i$ where $Z_i$ is a product of affine spaces $\mathbb{A}^r$ and/or spaces of the form $\mathbb{A}^r - \mathbb{A}^s$, $0 \leq s < r$. Then $H^k_c(Z)$ is $R_{triv}$-generated.

**Proof.** This follows from Lemma \ref{2.1} and the fact that the Hodge structure of $\mathbb{A}^r$ is trivial. \hfill $\square$
Notation: Let $X$ be a smooth projective complex curve of genus $g \geq 2$, and consider the Hodge structure $H^1(X)$. This is a polarised Hodge structure, with the polarisation given by the cup product. We shall denote $R_X = \{H^1(X)\}$.

3. Moduli spaces of triples

Let $X$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. A triple $T = (E_1, E_2, \phi)$ on $X$ consists of two vector bundles $E_1$ and $E_2$ over $X$, of ranks $n_1$ and $n_2$ and degrees $d_1$ and $d_2$, respectively, and a homomorphism $\phi : E_2 \to E_1$. We shall refer to $(n_1, n_2, d_1,d_2)$ as the type of the triple, and $(n_1, n_2)$ as the rank of the triple.

For any $\sigma \in \mathbb{R}$, the $\sigma$-slope of $T$ is defined by

$$
\mu_\sigma(T) = \frac{d_1 + d_2}{n_1 + n_2} + \frac{\sigma}{n_1 + n_2}.
$$

We say that a triple $T = (E_1, E_2, \phi)$ is $\sigma$-stable if $\mu_\sigma(T') < \mu_\sigma(T)$ for any proper subtriple $T' = (E_1', E_2', \phi')$. We define $\sigma$-semistability by replacing the above strict inequality with a weak inequality. A triple $T$ is $\sigma$-polystable if it is the direct sum of $\sigma$-stable triples of the same $\sigma$-slope. We denote by

$$
N_\sigma(n_1, n_2, d_1, d_2)
$$

the moduli space of $\sigma$-polystable triples of type $(n_1, n_2, d_1, d_2)$. This moduli space was constructed in [5] and [14]. It is a complex projective variety. The open subset of $\sigma$-stable triples will be denoted by $N_\sigma^s(n_1, n_2, d_1, d_2)$.

Let $L_1, L_2$ be two bundles of degrees $d_1, d_2$ respectively. Then the moduli spaces of $\sigma$-semistable triples $T = (E_1, E_2, \phi)$ with $\det(E_1) = L_1$ and $\det(E_2) = L_2$ will be denoted

$$
N_\sigma(n_1, n_2, L_1, L_2),
$$

and $N_\sigma^s(n_1, n_2, L_1, L_2)$ is the open subset of $\sigma$-stable triples.

Let $\mu(E) = \deg(E)/\text{rk}(E)$ denote the slope of a bundle $E$, and let $\mu_i = \mu(E_i) = d_i/n_i$, for $i = 1,2$. Write

$$
\sigma_m = \mu_1 - \mu_2 ,
$$

$$
\sigma_M = \begin{cases} 
(1 + n_1 + n_2)/(n_1 - n_2) \mu_1 - \mu_2 , & \text{if } n_1 \neq n_2 , \\
\infty , & \text{if } n_1 = n_2 ,
\end{cases}
$$

and let $I$ be the interval $I = (\sigma_m, \sigma_M)$. Then a necessary condition for $N_\sigma^s(n_1, n_2, d_1, d_2)$ to be non-empty is that $\sigma \in I$ (see [6]). Note that $\sigma_m > 0$. To study the dependence of the moduli spaces on the parameter $\sigma$, we need to introduce the concept of critical value [5] [12].

Definition 3.1. The values $\sigma_c \in I$ for which there exist $0 \leq n_1' \leq n_1$, $0 \leq n_2' \leq n_2$, $d_1'$ and $d_2'$, with $n_1'n_2' \neq n_1n_2'$, such that

$$
\sigma_c = \frac{(n_1 + n_2)(d_1' + d_2') - (n_1' + n_2')(d_1 + d_2)}{n_1'n_2 - n_1n_2'},
$$

(3.1)
are called critical values. We also consider $\sigma_m$ and $\sigma_M$ (when $\sigma_M \neq \infty$) as critical values.

The interval $I$ is split by a finite number of values $\sigma_c \in I$. The stability and semistability criteria for two values of $\sigma$ lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic. When $\sigma$ crosses a critical value, the moduli space undergoes a transformation which we call a flip. We shall study the flips in some detail in the next section.

**Theorem 3.2.** For non-critical values $\sigma \in I$, $N_\sigma = N_\sigma(n, 1, d_1, d_2)$ is smooth and projective, and it consists only of $\sigma$-stable points (i.e. $N_\sigma = N_\sigma^s$). For critical values $\sigma = \sigma_c$, $N_\sigma$ is projective, and the open subset $N_\sigma^s \subset N_\sigma$ is smooth. In both cases, the dimension of $N_\sigma$ is $(n^2 - 1)(g - 1) + d_1 - nd_2$.

**Relationship with pairs.** A pair $(E, \phi)$ over $X$ consists of a vector bundle $E$ of rank $n$ and degree $d$, and $\phi \in H^0(E)$. Let $\tau \in \mathbb{R}$. We say that $(E, \phi)$ is $\tau$-stable (see [8, Definition 4.7]) if:

- For any subbundle $E' \subset E$, we have $\mu(E') < \tau$.
- For any subbundle $E' \subset E$ with $\phi \in H^0(E')$, we have $\mu(E/E') > \tau$.

The concept of $\tau$-semistability is defined by replacing the strict inequalities by weak inequalities. A pair $(E, \phi)$ is $\tau$-polystable if $E = E' \oplus E''$, where $\phi \in H^0(E')$, $(E', \phi')$ is $\tau$-stable, and $E''$ is a polystable bundle of slope $\tau$. The moduli space of $\tau$-polystable pairs is denoted by $M_\tau(n, d)$.

Interpreting $\phi \in H^0(E)$ as a morphism $\phi : O \to E$, where $O$ is the trivial line bundle on $X$, we have a map $(E, \phi) \mapsto (E, O, \phi)$ from pairs to triples. The $\tau$-stability of $(E, \phi)$ corresponds to the $\sigma$-stability of $(E, O, \phi)$, where $\sigma = (n + 1)\tau - d$ (see [5]). Therefore we have an isomorphism of moduli spaces

$$N_\sigma(n, 1, d) \cong M_\tau(n, d) \times \text{Jac} X,$$

given by $(E, L, \phi) \mapsto ((E \otimes L^*, \phi), L)$.

If $L_0$ is a line bundle of degree $d$, then $M_\tau(n, L_0)$ denotes the subspace of $M_\tau(n, d)$ consisting of pairs $(E, \phi)$ where $\det(E) \cong L_0$. Note that $N_\sigma(n, 1, L_0, O) \cong M_\tau(n, L_0)$.

4. **Flips for the moduli spaces of pairs**

The homological algebra of triples is controlled by the hypercohomology of a certain complex of sheaves which appears when studying infinitesimal deformations [6, Section 3]. Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples of types $(n'_1, n'_2, d'_1, d'_2)$ and $(n''_1, n''_2, d''_1, d''_2)$, respectively. Let $\text{Hom}(T'', T')$ denote the linear space of homomorphisms from $T''$ to $T'$, and let $\text{Ext}^1(T'', T')$ denote the linear space of equivalence classes of extensions of the form

$$0 \to T' \to T \to T'' \to 0,$$
where by this we mean a commutative diagram
\[
\begin{array}{c}
0 \longrightarrow E'_1 \longrightarrow E_1 \longrightarrow E''_1 \longrightarrow 0 \\
\phi' \quad \phi \quad \phi'' \\
0 \longrightarrow E'_2 \longrightarrow E_2 \longrightarrow E''_2 \longrightarrow 0.
\end{array}
\]

To analyze \(\text{Ext}^1(T'', T')\) one considers the complex of sheaves
\[
C^\bullet(T'', T') : (E''_1 \otimes E'_1) \oplus (E''_2 \otimes E'_2) \overset{c}{\longrightarrow} E''_2 \otimes E'_1,
\]
where the map \(c\) is defined by
\[
c(\psi_1, \psi_2) = \phi'\psi_2 - \psi_1\phi''.
\]

We introduce the following notation:
\[
\mathbb{H}^i(T'', T') = \mathbb{H}^i(C^\bullet(T'', T')).
\]

**Proposition 4.1** ([11, Proposition 3.1]). There are natural isomorphisms
\[
\begin{align*}
\text{Hom}(T'', T') &\cong \mathbb{H}^0(T'', T'), \\
\text{Ext}^1(T'', T') &\cong \mathbb{H}^1(T'', T'),
\end{align*}
\]
and a long exact sequence associated to the complex \(C^\bullet(T'', T')\):
\[
\begin{align*}
0 &\longrightarrow \mathbb{H}^0(T'', T') \longrightarrow H^0((E''_1 \otimes E'_1) \oplus (E''_2 \otimes E'_2)) \longrightarrow H^0(E''_2 \otimes E'_1) \\
&\longrightarrow \mathbb{H}^1(T'', T') \longrightarrow H^1((E''_1 \otimes E'_1) \oplus (E''_2 \otimes E'_2)) \longrightarrow H^1(E''_2 \otimes E'_1) \\
&\longrightarrow \mathbb{H}^2(T'', T') \longrightarrow 0.
\end{align*}
\]

We shall follow the following results later:

**Lemma 4.2.** Suppose that \(T', T''\) are triples such that \(T'\) is \(\sigma\)-stable and \(T''\) is \(\sigma\)-semistable. Then \(\mathbb{H}^0(T', T'') = 0\) unless \(T''\) contains a subtriple isomorphic to \(T'\), in which case \(\mathbb{H}^0(T', T'') = \mathbb{C}\).

**Lemma 4.3** ([11, Lemma 3.10]). If \(T'' = (E''_1, E''_2, \phi'')\) is an injective triple, that is \(\phi'' : E''_2 \rightarrow E''_1\) is injective, then \(\mathbb{H}^2(T'', T') = 0\).

Fix the type \((n_1, n_2, d_1, d_2)\) for the moduli spaces of triples. For brevity, write \(N_\sigma = N_\sigma(n_1, n_2, d_1, d_2)\). Let \(\sigma_c \in I\) be a critical value and set
\[
\sigma_c^+ = \sigma_c + \epsilon, \quad \sigma_c^- = \sigma_c - \epsilon,
\]
where \(\epsilon > 0\) is small enough so that \(\sigma_c\) is the only critical value in the interval \((\sigma_c^-, \sigma_c^+)\).

**Definition 4.4.** We define the flip loci as
\[
\begin{align*}
S_{\sigma_c^+} &= \{ T \in N_{\sigma_c^+}^s : T \text{ is } \sigma_c^-\text{-unstable} \} \subset N_{\sigma_c^+}^s, \\
S_{\sigma_c^-} &= \{ T \in N_{\sigma_c^-}^s : T \text{ is } \sigma_c^+\text{-unstable} \} \subset N_{\sigma_c^-}^s.
\end{align*}
\]
It follows that (see [6, Lemma 5.3])
\[ N^s_{\sigma_c^+} - S_{\sigma_c^+} = N^s_{\sigma_c} = N^s_{\sigma_c} - S_{\sigma_c}. \]

For a triple \( T \), we denote \( \lambda(T) = \frac{n_2}{n_1 + n_2} \). If \( T \in S_{\sigma_c^+} \), then there is a subtriple \( T' \subset T \) satisfying
\[ \mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) \text{, and } \lambda(T') < \lambda(T). \]
Conversely, if \( T \in S_{\sigma_c^-} \), then there exists a subtriple \( T' \subset T \) satisfying
\[ \mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) \text{, and } \lambda(T') > \lambda(T). \]
Clearly, in both cases, \( T \) is properly \( \sigma_c \)-semistable.

From now on we shall restrict to the case of triples of rank \( (n,1) \). Denote \( N_\sigma = N_\sigma(n,1,d,d_0) \), the moduli space of \( \sigma \)-semistable triples \( (E,L,\phi) \), where \( L \) is a line bundle of degree \( d_0 \), \( E \) is a bundle of rank \( n \) and degree \( d \), and \( \phi : L \to E \). Let \( \sigma_c \) be a critical value for the moduli spaces of type \( (n,1,d,d_0) \). We want to describe a stratification of \( S_{\sigma_c^\pm} \) by suitable locally closed subvarieties.

**Lemma 4.5.** Let \( \sigma_c \) be a critical value. Let \( T \in S_{\sigma_c^\pm} \). Then there is a unique filtration
\[ 0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_r = T \text{ such that } T_i/T_{i-1} \text{ is the maximal } \sigma_c \text{-polystable subtriple of } T/T_{i-1} \text{ of the same } \sigma_c \text{-slope}. \]

**Proof.** We only need to prove that there is a unique maximal \( \sigma_c \)-polystable subtriple for any given triple \( T \). If \( T_1 \) and \( T_1' \) are two \( \sigma_c \)-polystable subtriples of \( T \), then the subtriple \( T_1 + T_1' \subset T \) sits in an exact sequence
\[ 0 \to T_1 \cap T_1' \to T_1 \oplus T_1' \to T_1 + T_1' \to 0. \tag{4.2} \]
As \( \mu_{\sigma_c}(T_1 \oplus T_1') \leq \mu_{\sigma_c}(T_1 + T_1') \leq \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T_1) = \mu_{\sigma_c}(T_1') \), we have that all the triples in \( T \) have the same \( \sigma_c \)-slope. Therefore \( T_1 + T_1' \) is a quotient of the \( \sigma_c \)-polystable triple \( T_1 \oplus T_1' \), hence it is itself \( \sigma_c \)-polystable. This contradicts the maximality of either \( T_1 \) or \( T_1' \). \( \square \)

**Definition 4.6.** We shall call **standard filtration** to the filtration provided by Lemma 4.5. We shall write \( T_i = T_i/T_{i-1}, i = 1, \ldots, r \).

**Remark 4.7.** The result in Lemma 4.5 is true for triples of any type \( (n_1,n_2,d_1,d_2) \), for any \( \sigma \) and any \( T \in N_\sigma \). Note that if \( T \) is \( \sigma \)-stable then the standard filtration is \( 0 \subset T \).

**Lemma 4.8.** Let \( 0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_r = T \) be the standard filtration of \( T \), and let \( \bar{T}_i = T_i/T_{i-1}, i = 1, \ldots, r \).
\[ \begin{itemize} 
\item If \( T \in S_{\sigma_c^+} \) then \( \bar{T}_r \) is a \( \sigma_c \)-stable triple of type \( (n',1) \), and \( \bar{T}_i, 1 \leq i < r \), are triples of the form \( (F_{i,1},0,0) \), where \( F_i \) are polystable bundles all of the same slope. Moreover, \( n' > 0 \) if \( \sigma_c > \sigma_m \) and \( n' = 0 \) if \( \sigma_c = \sigma_m \).
\end{itemize} \]
• If \( T \in \mathcal{S}_{\sigma_c^{-}} \) then \( \bar{T}_1 \) is a \( \sigma_c \)-stable triple of type \((n',1)\), and \( \bar{T}_i \), \( 1 < i \leq r \), are triples of the form \((F_i,0,0)\), where \( F_i \) are polystable bundles all of the same slope. Moreover, always \( n' > 0 \).

\[ \mu(F_{ij}) = \mu_{\sigma_c}(\bar{T}_i) = \mu_c, \]

so all the bundles are of the same slope. Moreover, \( \bar{T}_r \) should be \( \sigma_c \)-polystable. It cannot have summands of rank \((n'',0)\) and \( \sigma_c \)-slope \( \mu_c \), since this would imply that \( T \) has such a quotient, and hence it cannot be \( \sigma_c^{-} \)-stable. So \( \bar{T}_r \) is \( \sigma_c \)-stable triple of the form \((E',L,\phi')\).

If \( \sigma_c > \sigma_m \) then \( \phi' \) is injective, so \( n' > 0 \). If \( \sigma_c = \sigma_m \), then \( [12, \text{Proposition } 4.10] \) tells us that all \( \sigma_c \)-semistable triples are not \( \sigma_c \)-stable, unless \( n' = 0 \).

Let \( T \in \mathcal{S}_{\sigma_c^{-}} \), and let \( 0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_r = T \) be its standard filtration. Then \( T/T_{r-1} \) is a quotient of \( T \). As \( T \) is \( \sigma_c^{+} \)-stable, \( \bar{T}_r = T/T_{r-1} \) is of rank \((n',1)\) and \( \mu_{\sigma_c}(\bar{T}_r) = \mu_{\sigma_c}(T) =: \mu_c \). Therefore, \( T_{r-1} \) is of rank \((n-n',0)\), and hence all \( \bar{T}_i \) are of rank \((n_i,0)\), for \( i < r \). So all of the triples \( \bar{T}_i \) are direct sums of triples of the form \((F_{ij},0,0)\), where \( F_{ij} \) are stable bundles of the same slope, \( 1 \leq i \leq r-1 \), \( 1 \leq j \leq k_i \). Note that

\[ \mu(F_{ij}) = \mu_{\sigma_c}(\bar{T}_i) = \mu_c, \]

so all the bundles are of the same slope. Moreover, \( \bar{T}_r \) should be \( \sigma_c \)-polystable. It cannot have summands of rank \((n'',0)\) and \( \sigma_c \)-slope \( \mu_c \), since this would imply that \( T \) has such a quotient, and hence it cannot be \( \sigma_c^{-} \)-stable. So \( \bar{T}_r \) is \( \sigma_c \)-stable triple of the form \((E',L,\phi')\).

Let \( \sigma_c > \sigma_m \) then \( \phi' \) is injective, so \( n' > 0 \). If \( \sigma_c = \sigma_m \), then \( [12, \text{Proposition } 4.10] \) tells us that all \( \sigma_c \)-semistable triples are not \( \sigma_c \)-stable, unless \( n' = 0 \).

Let \( T \in \mathcal{S}_{\sigma_c^{-}} \), and let \( 0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_r = T \) be its standard filtration. Then \( T_1 \subset \bar{T} \) must be of rank \((n',1)\), by the \( \sigma_c^{-} \)-stability of \( T \). In principle, we could say that \( T_1 \) is \( \sigma_c \)-polystable, but it is actually \( \sigma_c \)-stable, since otherwise it has a subtriple of type \((n'',0)\) but then \( T \) cannot be \( \sigma_c^{-} \)-stable. Therefore \( T/T_1 \) is of rank \((n-n',0)\), and hence all \( \bar{T}_i \) are of rank \((n_i,0)\), for \( i > 1 \). So all of the triples \( \bar{T}_i \) are direct sums of triples of the form \((F_{ij},0,0)\), where \( F_{ij} \) are stable bundles of the same slope \( \mu(F_{ij}) = \mu_{\sigma_c}(\bar{T}_i) = \mu_c =: \mu_{\sigma_c}(T) \). Moreover \( \mu_{\sigma_c}(\bar{T}_1) = \mu_c \) and \( T_1 \) is a \( \sigma_c \)-stable triple of the form \((E',L,\phi')\). Note that \( \sigma_c > \sigma_m \) (since we are dealing with \( \mathcal{S}_{\sigma_c} \)), so \( n' > 0 \).

5. Stratification of the flip loci

Let \( \sigma_c \) be a critical value. In this section we want to describe a stratification of the flip loci \( \mathcal{S}_{\sigma_c^{\pm}} \). Recall that we are dealing with the moduli spaces \( \mathcal{N}_\sigma = \mathcal{N}_\sigma(n,1,d,d_o) \).

5.1. The flip locus \( \mathcal{S}_{\sigma_c^{+}} \). Now we want to describe geometrically \( \mathcal{S}_{\sigma_c^{+}} \). We stratify \( \mathcal{S}_{\sigma_c^{+}} \) according to the types of the triples in the standard filtration. Let us fix some notation: let \( b \geq 1 \) and \( r \geq 1 \). Fix \( n' \geq 1 \) and \( d' \) such that

\[ \frac{d' + \sigma_c}{n' + 1} = \mu_{\sigma_c}(T) =: \mu_c. \]  \hspace{1cm} (5.1)

Let \( (n_1,d_1), \ldots, (n_b,d_b) \) satisfy

\[ \frac{d_i}{n_i} = \mu_c. \]  \hspace{1cm} (5.2)
Consider $a_{ij} \geq 0$, for $1 \leq i \leq b$ and $1 \leq j \leq r$, such that $(a_{1j}, \ldots, a_{bj}) \neq (0, \ldots, 0)$, for all $j$. We assume that
\[
\sum_{i,j} a_{ij} n_i + n' = n.
\] (5.3)

Write $a_j = (a_{1j}, \ldots, a_{bj})$, $1 \leq j \leq r$, and $n = \{(n_i, d_i, a_{ij}) ; 1 \leq i \leq b, 1 \leq j \leq r\}$. Consider
\[
\tilde{U}(n) = \{(E_1, \ldots, E_b) \in M^s(n_1, d_1) \times \cdots \times M^s(n_b, d_b) ; E_i \neq E_j, \text{ for } i \neq j\}. \quad (5.4)
\]

For each $(E_1, \ldots, E_b) \in \tilde{U}(n)$, set $S_i = (E_i, 0, 0)$, $1 \leq i \leq b$.

We define $X^+(n) \subset S_{\sigma_+}^+$ as the subset formed by those triples $T$ whose standard filtration is $0 = T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{r+1} = T$, such that
\[
\tilde{T}_j = T_j / T_{j-1} \cong S(a_j) := S_{a_{1j}}^1 \oplus \cdots \oplus S_{a_{bj}}^b,
\] (5.5)
for some $(E_1, \ldots, E_b) \in \tilde{U}(n)$. Note that it must be $\tilde{T}_{r+1} \in N_{\sigma_+}^s(n', 1, d', d_o)$. By Lemma 4.8, the subsets $X^+(n)$ stratify $S_{\sigma_+}$, for all possibilities for $n$ as above. That is,
\[
S_{\sigma_+}^+ = \bigsqcup_n X^+(n).
\]

To describe $X^+(n)$, first note that the presentation (5.5) is not unique. The finite group
\[
F = \{\tau \text{ permutation of } (1, \ldots, b) ; n_{\tau(i)} = n_i, a_{\tau(i)j} = a_{ij} \forall i, j \}
\] (5.6)
acts (freely) on $\tilde{U}(n)$ by permuting the bundles. Let $U(n) = \tilde{U}(n)/F$. There is a map
\[
X^+(n) \to U(n) \times N_{\sigma_+}^s(n', 1, d', d_o).
\] (5.7)
The pull-back of (5.7) under $\tilde{U}(n) \to U(n)$ is
\[
\tilde{X}^+(n) \to \tilde{M}(n) := \tilde{U}(n) \times N_{\sigma_+}^s(n', 1, d', d_o).
\]
and $F$ acts on $\tilde{X}^+(n)$ so that $X^+(n) = \tilde{X}^+(n)/F$.

We shall construct an affine bundle over $\tilde{M}(n)$ parametrizing iterated extensions, and an open subset $\tilde{X}^+(n)$ of it, parametrizing those extensions corresponding to $\sigma_+^+$-stable triples. In this way, we shall get quotient maps
\[
\tilde{X}^+(n) \to \tilde{X}^+(n) \to X^+(n).
\]
The first quotient corresponds to the automorphisms of the extensions, and the second one to permuting the order of the $S_i$'s.

To do this, we start by describing the elements in $X^+(n)$ more explicitly.

**Proposition 5.1.** Let $(E_1, \ldots, E_b, \tilde{T}_{r+1}) \in \tilde{M}(n)$. Define triples $\tilde{T}_j$ by downward recursion as follows: $\tilde{T}_{r+1} = \tilde{T}_{r+1}$ and for $1 \leq j \leq r$ define $\tilde{T}_j$ as an extension
\[
0 \to S(a_j) \to \tilde{T}_j \to \tilde{T}_{j+1} \to 0.
\] (5.8)
Let $\xi_j \in \text{Ext}^1(\tilde{T}_{j+1}, S(a_j))$ be the extension class corresponding to (5.8). Denote $T = \tilde{T}_1$. Then $T \in X^+(n)$ if and only if the following conditions are satisfied:
(1) The extension class $\xi_j \in \Ext^1(\tilde{T}_{j+1}, S(a_j)) = \prod_i \Ext^1(\tilde{T}_{j+1}, S_i)^{a_{ij}}$ lives in $\prod_i V(a_{ij}, \Ext^1(\tilde{T}_{j+1}, S_i))$, with the notation $V(k, W) = \{(w_1, \ldots, w_k) \in W^k : w_1, \ldots, w_k \text{ are linearly independent}\}$, for $W$ a vector space.

(2) Consider the map $S(a_{j+1}) \to \tilde{T}_{j+1}$ and the element $\xi'_j$ which is the image of $\xi_j$ under $\Ext^1(\tilde{T}_{j+1}, S(a_j)) \to \Ext^1(S(a_{j+1}), S(a_j))$. Then the class $\xi'_j \in \Ext^1(S(a_{j+1}), S(a_j)) = \prod_i \Ext^1(S_i, S(a_j))^{a_{ij}}$ lives in $\prod_i V(a_{i,j+1}, \Ext^1(S_i, S(a_j)))$.

Two extensions $\xi_j$ give rise to isomorphic $\tilde{T}_j$ if and only if the triples $\tilde{T}_{j+1}$ are isomorphic and the extension classes are the same up to action of the group $\GL(a_j) := \GL(a_{1j}) \times \cdots \times \GL(a_{b,j})$.

Proof. Assume that $T \in X^+(n)$. So $T$ is $\sigma_c^+$-stable. As $\tilde{T}_j$ is a quotient of $T$ with the same $\sigma_c$-slope, it is also $\sigma_c^+$-stable (if not, there would be a quotient $\tilde{T}_j \to T''$ with the same $\sigma_c$-slope, but with $\lambda(T'') = 0$; this would violate the $\sigma_c^+$-stability of $T$ since $T \to T''$).

Now let us prove (1) by downward induction on $j$. If $\xi_j$ does not live in $\prod_i V(a_{ij}, \Ext^1(\tilde{T}_{j+1}, S_i))$, then there is some $i_o$ and some linear projection $\prod_i \Ext^1(\tilde{T}_{j+1}, S_i)^{a_{ij}} \to \Ext^1(\tilde{T}_{j+1}, S_{i_o})$ under which $\xi_j$ maps to zero. This corresponds to a quotient $S(a_j) \to S_{i_o}$. Let $\tilde{S}$ be the subbundle defined as the kernel $\tilde{S} \hookrightarrow S(a_j) \to S_{i_o}$. Then the class $\xi_j$ lies in the image of $\Ext^1(\tilde{T}_{j+1}, \tilde{S}) \to \Ext^1(\tilde{T}_{j+1}, S(a_j))$. This produces an extension $0 \to \tilde{S} \to \tilde{T} \to \tilde{T}_{j+1} \to 0$, such that $\tilde{T} \subset \tilde{T}_j$ is a subtriple with $\lambda(\tilde{T}) > 0$, violating the $\sigma_c^+$-stability of $\tilde{T}_j$.

Conversely, $T$ defined as in the statement is $\sigma_c$-semistable. All $\tilde{T}_j$ are extensions of $\sigma_c$-semistable objects of the same $\sigma_c$-slope, so all of them are $\sigma_c$-semistable of the same $\sigma_c$-slope. Assume now that we know that $\tilde{T}_{j+1}$ is $\sigma_c^+$-stable. If $\tilde{T}_j$ is not $\sigma_c^+$-stable, then there should be a subtriple $\tilde{T} \subset \tilde{T}_j$ with $\lambda(\tilde{T}) > 0$ and $\mu_{\sigma_c}(\tilde{T}) = \mu_{\sigma_c}(\tilde{T}_j)$. This lies in an exact sequence

$$0 \to \tilde{S} \to \tilde{T} \to \tilde{T}'' \to 0$$

It is clear that all the triples are of the same $\sigma_c$-slope, and that $\lambda(\tilde{T}'') > 0$. Therefore the $\sigma_c^+$-stability of $\tilde{T}_{j+1}$ implies that $\tilde{T}'' = \tilde{T}_{j+1}$. Therefore the extension $\xi_j$ maps to zero under

$$\Ext^1(\tilde{T}_{j+1}, S(a_j)) \to \Ext^1(\tilde{T}_{j+1}, S(a_j)/\tilde{S}).$$

Take $i_o$ such that there is a quotient $S(a_j)/\tilde{S} \to S_{i_o}$. We see that $\xi_j$ maps to zero under $\Ext^1(\tilde{T}_{j+1}, S(a_j)) \to \Ext^1(\tilde{T}_{j+1}, S_{i_o})$, so $\xi_j \notin \prod_i V(a_{ij}, \Ext^1(\tilde{T}_{j+1}, S_i))$.

Regarding (2), assume that $T \in X^+(n)$ at let us see that the class $\xi'_j$ satisfies the condition (2) by downward induction on $j$. First suppose that $\xi'_j \notin \prod_i V(a_{i,j+1}, \Ext^1(S_i, S(a_j)))$. Then there is some $S_{i_o} \subset S(a_{j+1})$ such that $\xi_j$ maps
to zero under $\prod_i \text{Ext}^1(S(a_{j+1}), S(a_j)) \to \text{Ext}^1(S_{i_0}, S(a_j))$. The pull-back of the extension \([5.8]\) under $S_{i_0} \to S(a_{j+1}) \subset \tilde{T}_{j+1}$,

$$
\begin{align*}
0 & \to S(a_j) \to \tilde{T} \to S_{i_0} \to 0 \\
0 & \to S(a_j) \to \tilde{T}_{j} \to \tilde{T}_{j+1} \to 0,
\end{align*}
$$

is split. Therefore there is a polystable subtriple $S(a_j) \oplus S_{i_0} \subset \tilde{T}_{j}$. This violates the fact that the standard filtration of $T$ is of given type $n$.

Conversely, assume that (2) is satisfied and let us see that the standard filtration of $T$ is of type $n$. We can see this by downward induction on $j$. Assume that it is known for $j + 1$. We consider the exact sequence $S(a_j) \to \tilde{T}_{j} \to \tilde{T}_{j+1}$. If the maximal $\sigma_c$-polystable subtriple of $\tilde{T}_{j}$ is not $S(a_j)$, then there is some $i_o$ such that $S(a_j) \oplus S_{i_0} \subset \tilde{T}_{j}$. Therefore there is some inclusion $S_{i_0} \hookrightarrow \tilde{T}_{j+1}$ such that the extension class $\xi_j$ maps to zero under

$$
\text{Ext}^1(\tilde{T}_{j+1}, S(a_j)) \to \text{Ext}^1(S_{i_0}, S(a_j)).
$$

Clearly, $S_{i_0} \subset S(a_{j+1}) \subset \tilde{T}_{j+1}$. Therefore $\xi'_j$ does not satisfy (2).

Finally, let us prove the last assertion. Again the uniqueness of the standard filtration implies that if two triples $T, T'$ are isomorphic, then the associated triples $\tilde{T}_{j}, \tilde{T}'_{j}$ are isomorphic for all $j$. By downward induction, we can assume that $\tilde{T}_{j+1} = \tilde{T}'_{j+1}$. We have isomorphisms

$$
\begin{align*}
0 & \to S(a_j) \to \tilde{T}_{j} \to \tilde{T}'_{j+1} \to 0 \\
0 & \to S(a_j) \to \tilde{T}_{j} \to \tilde{T}'_{j+1} \to 0
\end{align*}
$$

As $\tilde{T}_{j}$ is $\sigma_c^+$-stable, the only automorphisms of it are multiplication by non-zero scalars. On the other hand, $\text{Aut}(S(a_j)) = \text{GL}(a_{1j}) \times \cdots \times \text{GL}(a_{bj}) = \text{GL}(a_j)$. This group acts on the space of extensions. After an action by an element of $\text{GL}(a_j)$, we can assume that the first vertical arrow of \([5.8]\) is an equality. Therefore the extensions are equivalent.

**Remark 5.2.** The description in Proposition \([5.1]\) can be applied to the critical value $\sigma_c = \sigma_m$. In this case, $N_{\sigma_m^+} = S_{\sigma_m^+}$. The only difference is that now $\tilde{T}_{r+1}$ should be of the form $L \to 0$, that is, $\tilde{T}_{r+1} \in N_{\sigma_c^+}(0, 1, 0, d_o) = \text{Jac}^{d_o}X$.

Note that there is an open stratum in $S_{\sigma_m^+}$ corresponding to $r = 1, b = 1, a_1 = (a_{11} = 1), n_o = (n, d, a_{11})$. In this case

$$
\mathcal{M}(n_o) = M^*(n, d) \times \text{Jac}^{d_o}X.
$$

It corresponds to triples $\phi : L \to E$ for which $E$ is a stable bundle. We shall denote this open stratum and its complement as

$$
\mathcal{U}_m \subset N_{\sigma_m^+}, \quad \mathcal{D}_m = N_{\sigma_m^+} - \mathcal{U}_m.
$$
5.2. The flip locus $S_{\sigma^{-}}$. There is an analogous description of $S_{\sigma^{-}}$. As before, let $b \geq 1$ and $r \geq 1$. Fix $n' \geq 1$ and $d'$ satisfying (5.1). Let $(n_1, d_1), \ldots, (n_b, d_b)$ satisfy (5.2). Consider $a_{ij} \geq 0$, for $1 \leq i \leq b$ and $2 \leq j \leq r + 1$, such that $a_j = (a_{1j}, \ldots, a_{bj}) \neq (0, \ldots, 0)$, for all $j$. We assume (5.3). Write $n = \{(n_i, d_i, a_{ij}) : 1 \leq i \leq b, 2 \leq j \leq r + 1\}$. We consider $\bar{U}(n)$ as in (5.4).

We define $X^-(n) \subset S_{\sigma^{-}}$ as the subset formed by those triples $T$ whose standard filtration is $0 = T_0 \subset T_1 \subset \cdots \subset T_{r+1} = T$, such that

$$
\bar{T}_j = T_j/T_{j-1} \cong S(a_j) := S_{a_{1j}}^{a_{1j}} \oplus \cdots \oplus S_{a_{bj}}^{a_{bj}},
$$

(5.10) for some $(E_1, \ldots, E_b) \in U(n)$, $2 \leq j \leq r + 1$. It must be $T_1 \in \mathcal{N}_{\sigma_{^{-}}}^s(n', 1, d', d_o)$. By Lemma 4.8 the subsets $X^-(n)$ stratify $S_{\sigma^{-}}$, for all possible choices of $n$. That is,

$$S_{\sigma^{-}} = \bigsqcup_n X^-(n).$$

Note again that the finite group $F$ given in (5.6) acts on $\bar{U}(n)$, and that the pull-back of the map

$$X^-(n) \to \bar{U}(n) \times \mathcal{N}_{\sigma_{^{-}}}^s(n', 1, d', d_o).$$

under $\bar{U}(n) \to U(n) = \bar{U}(n)/F$ is

$$\bar{X}^-(n) \to \mathcal{M}(n) := \bar{U}(n) \times \mathcal{N}_{\sigma_{^{-}}}^s(n', 1, d', d_o).$$

and $X^-(n) = \bar{X}^-(n)/F$.

The proof of the following result is analogous to that of Proposition 5.1.

**Proposition 5.3.** Let $(E_1, \ldots, E_b, T_1) \in \mathcal{M}(n)$. Define triples $\bar{T}_j$ by recursion as follows: $\bar{T}_1 = T_1$, and for $2 \leq j \leq r + 1$ define $\bar{T}_j$ as an extension

$$0 \to \bar{T}_{j-1} \to \bar{T}_j \to S(a_j) \to 0.$$

Let $\xi_j \in \text{Ext}^1(S(a_j), \bar{T}_{j-1})$ be the corresponding extension class. Denote $T = \bar{T}_{r+1}$. Then $T \in X^-(n)$ if and only if the following conditions are satisfied:

1. The extension class $\xi_j \in \text{Ext}^1(S(a_j), \bar{T}_{j-1}) = \prod_i \text{Ext}^1(S_i, \bar{T}_{j-1})^{a_{ij}}$ lives in $\prod_i V(a_{ij}, \text{Ext}^1(S_i, \bar{T}_{j-1}))$.
2. Consider the map $\bar{T}_{j-1} \to S(a_{j-1})$ and the element $\xi'_j$ which is the image of $\xi_j$ under $\text{Ext}^1(S(a_j), \bar{T}_{j-1}) \to \text{Ext}^1(S(a_j), S(a_{j-1}))$. Then the element $\xi'_j \in \text{Ext}^1(S(a_j), S(a_{j-1})) = \prod_i \text{Ext}^1(S(a_j), S_i)^{a_{ij-1}}$ lives in $\prod_i V(a_{ij-1}, \text{Ext}^1(S(a_j), S_i))$.

Two extensions $\xi_j$ give rise to isomorphic $\bar{T}_j$ if and only if the triples $\bar{T}_{j-1}$ are isomorphic and the extension classes are the same up to action of the group $\text{GL}(a_j) := \text{GL}(a_{1j}) \times \cdots \times \text{GL}(a_{bj})$.

**Remark 5.4.** This description is also valid for $\sigma_{= \sigma} = \sigma_M$, with no change. In this case, $\mathcal{N}_{\sigma_{= \sigma}}^s = S_{\sigma_{= \sigma}}$.
Again for \( \sigma_c = \sigma_M \), there is an open stratum, corresponding to the case \( r = 1, b = 1, (n', d') = (1, d_0), a_1 = (a_{11} = 1) \), \( n_o = (n - 1, d - d_o, 1) \). Then \( N_{\sigma_c}^s(1, 1, d_o, d_o) = \text{Jac}^{d_o}(X) \), and \((n_1, d_1) = (n - 1, d - d_o)\), so
\[
\mathcal{M}(n_0) = M^s(n - 1, d - d_o) \times \text{Jac}^{d_o}X.
\]
The elements of \( \tilde{X}^-(n_0) = X^-(n_0) \) correspond to extensions
\[
0 \to L \to L \to 0 \to 0
\]
\[
0 \to L \to E \to F \to 0,
\]
where \( F \in M^s(n - 1, d - d_o) \) and \( L \in \text{Jac}^{d_o}X \). This is the open stratum considered in [6, section 7.2], which is a dense open subset of \( \mathcal{N}_{\sigma_M}^s \).

6. \( R_X \)-Generation of the Flip Loci

Let \( F \) be a quasi-projective variety. We say that \( F \) is affinely stratified (abbreviated A.S.) if there is a stratification \( F = \sqcup F_i \) such that each \( F_i \) is a product of varieties of the form \( \mathbb{A}^r \) or \( \mathbb{A}^r - \mathbb{A}^s \), \( 0 \leq s < r \). Note that Corollary 2.2 implies that its cohomology is a trivial Hodge structure.

Let \( \sigma_c \) be a critical value, and fix a type \( n \) corresponding to it. We want to construct an affine bundle over \( \tilde{U}(n) \) parametrizing iterated extensions, and an open subset \( \tilde{X}^+(n) \) of it, parametrizing those extensions corresponding to \( \sigma_c^+ \)-stable triples. We start with
\[
M_{r+1} = \mathcal{M}(n) = \tilde{U}(n) \times \mathcal{N}_{\sigma_c}^s(n', 1, d', d_o),
\]
and define spaces \( M_j \) by downward induction as follows: \( M_j \) is the space parametrizing extensions \((5.8)\). More precisely, if \( M_{j+1} \) is already constructed, then we define a fiber bundle
\[
\pi_j : \tilde{M}_j \to M_{j+1}
\]
whose fibers are the spaces \( \text{Ext}^1(\tilde{T}_{j+1}, S(a_j)) \). Then any element in \( \tilde{M}_j \) gives rise to a triple \( \tilde{T}_j \). Let \( M_j \subset \tilde{M}_j \) be the subset of those extensions \( \xi_j \) satisfying (1) and (2) of Proposition 5.1 that is, when the corresponding triple \( \tilde{T}_j \) is \( \sigma_c^+ \)-stable. We denote
\[
p_j : M_j \to M_{j+1}
\]
the restriction of \( \pi_j \) to \( M_j \). The space we are interested in is \( \tilde{X}^+(n) := M_1 \).

**Theorem 6.1.** The fibration \( p_j : M_j \to M_{j+1} \) is a locally trivial fibration (in the Zariski topology) whose fiber \( F_j \) is A.S.

**Proof.** Fix a point in \( M_{j+1} \). This determines a \( \sigma_c^+ \)-stable triple \( \tilde{T}_{j+1} \). Note that \( \mathbb{H}^0(\tilde{T}_{j+1}, S(a_j)) = 0 \) since \( \tilde{T}_{j+1} \) is \( \sigma_c^+ \)-stable, \( \tilde{T}_{j+1} \) and \( S(a_j) \) are of the same \( \sigma_c \)-slope and \( S(a_j) \) is \( \sigma_c \)-polystable and does not contain a copy of \( \tilde{T}_{j+1} \) (see Lemma 4.2). Also \( \mathbb{H}^2(\tilde{T}_{j+1}, S(a_j)) = 0 \) by Lemma 4.3. So \( \dim \text{Ext}^1(\tilde{T}_{j+1}, S(a_j)) \) is constant, and therefore \( \pi_j : \tilde{M}_j \to M_{j+1} \) is a vector bundle, locally trivial in the Zariski topology.
We want to understand the fiber $F_j$ of $p_j : M_j \to M_{j+1}$, which consists of those extensions in $\text{Ext}^1(\bar{T}_{j+1}, S(a_j))$ satisfying (1) and (2) of Proposition 5.1. The triple $\bar{T}_{j+1}$, being an element in $M_{j+1}$, actually sits in an exact sequence

$$0 \to T_{j+2} \to T_{j+1} \to S(a_{j+1}) \to 0.$$ 

So there is an associated long exact sequence

$$0 \to \text{Hom}(S(a_{j+1}), S(a_j)) \to \text{Ext}^1(\bar{T}_{j+2}, S(a_j)) \to \text{Ext}^1(\bar{T}_{j+1}, S(a_j)) \to \text{Ext}^1(S(a_{j+1}), S(a_j)) \to 0. \tag{6.1}$$

Clearly, $\dim \text{Hom}(S(a_{j+1}), S(a_j)) = \sum_i a_{ij} a_{i,j+1}$ is constant (independent of the point in the base space $M_{j+1}$), and so the dimensions of all the spaces in (6.1) are constant.

We aim to prove that $F_j$ is A.S. Note that if we have a fibration $F \to E \to B$ in which both $F$ and $B$ are A.S., then so is $E$ (stratify $B$ by affine sets, and note that the fibration should be trivial over each one of them).

We obtain that $F_j$ is A.S. as a consequence of the following claims.

**Claim 1.** Let $a, b \geq 1$ be integers. Consider

$$M_{ab}(\mathbb{C}) = \{ f : \mathbb{C}^a \otimes \mathbb{C}^b \to \mathbb{C} \}.$$ 

Let $0 \leq r \leq \min\{a, b\}$ and $U_r = \{ f \in M_{ab}(\mathbb{C}) ; \text{rg}(f) = r \}$. Then $U_r$ is A.S. There are vector bundles $K_1, K_2 \to U_r$ (of ranks $a - r, b - r$ resp.), defined by

$$K_{1,f} = \ker(f : \mathbb{C}^a \to (\mathbb{C}^b)^*),$$

$$K_{2,f} = \ker(f : \mathbb{C}^b \to (\mathbb{C}^a)^*), \quad \text{for } f \in U_r \subset M_{ab}(\mathbb{C}).$$

**Proof.** This is well-known. Let us give a short proof for completeness. Take a full flag $0 \subset F_1 \subset F_2 \subset \ldots \subset F_a = \mathbb{C}^a$. Fix integers $0 \leq d_1 \leq \ldots \leq d_a = r$, with the condition that $d_{i+1} - d_i \leq 1$, and write $d = (d_1, \ldots, d_a)$. Consider the set

$$U_d = \{ f ; \text{rg}(f|_{F_i}) = d_j, \forall j \}.$$ 

First we see that $U_d$ is A.S. Define $U_{d,l} = \{ f : F_l \to (\mathbb{C}^b)^* ; \text{rg}(f|_{F_i}) = d_j, \forall j \leq l \}$, $1 \leq l \leq a$. Then $U_{d,l+1} \to U_{d,l}$ is a fiber bundle, whose fiber is

$$\left\{ \begin{array}{ll}
\text{Hom}(F_{l+1}/F_{l}, \text{im}(f|_{F_l})) & \text{if } d_{l+1} = d_l, \\
\text{Hom}(F_{l+1}/F_{l}, \mathbb{C}^b) - \text{Hom}(F_{l+1}/F_{l}, \text{im}(f|_{F_l})) & \text{if } d_{l+1} = d_l + 1.
\end{array} \right.$$ 

The first case is an affine space, the second is the difference of two affine spaces. So both are A.S. spaces. This implies that $U_d = U_{d,a}$ and $U_r = \bigsqcup_d U_d$ are A.S. The assertion about $K_1, K_2$ is clear. \qed
Claim 2. Let \(a, b \geq 1\) be integers and let \(V\) be a finite dimensional vector space. Consider
\[
M_{ab}(V) = \{ f : \mathbb{C}^a \otimes \mathbb{C}^b \to V \}.
\]
Let \(0 \leq r \leq a, 0 \leq s \leq b\), and define the subset
\[
U_{r,s} = \{ f \in M_{ab}(V) : \text{rg}(f : \mathbb{C}^a \to (\mathbb{C}^b)^* \otimes V) = r, \text{rg}(f : \mathbb{C}^b \to (\mathbb{C}^a)^* \otimes V) = s \}.
\]
Then \(U_{r,s}\) is A.S. There are vector bundles \(K_1, K_2 \to U_{r,s}\) (of ranks \(a-r, b-s\) resp.), defined by
\[
K_{1,f} = \ker(f : \mathbb{C}^a \to (\mathbb{C}^b)^* \otimes V),
\]
\[
K_{2,f} = \ker(f : \mathbb{C}^b \to (\mathbb{C}^a)^* \otimes V), \quad \text{for } f \in U_{r,s}.
\]

Proof. Consider a decomposition \(V = V_1 \oplus \ldots \oplus V_k\), into 1-dimensional vector subspaces \(V_i\). Let \(\tilde{V}_i = V_i \oplus \ldots \oplus V_i, 1 \leq i \leq k,\) and denote by \(\pi_i : V \to \tilde{V}_i\) the projection. For collections of integers \(0 \leq r_1 \leq \ldots \leq r_k = r\) and \(0 \leq s_1 \leq \ldots \leq s_k = s\), denote \(r = (r_1, \ldots, r_k), s = (s_1, \ldots, s_k)\), and define the subsets:
\[
U_{r,s,l} = \{ f : \mathbb{C}^a \otimes \mathbb{C}^b \to \tilde{V}_i ; \text{rg}(\pi_i \circ f : \mathbb{C}^a \to (\mathbb{C}^b)^* \otimes \tilde{V}_i) = r_i, \text{rg}(\pi_i \circ f : \mathbb{C}^b \to (\mathbb{C}^a)^* \otimes \tilde{V}_i) = s_i, \forall i \leq l \},
\]
for \(1 \leq l \leq k\), and \(U_{r,s} := U_{r,s,k}\). Let us check that \(U_{r,s,l}\) is A.S. For \(l = 1\), this is the content of Claim 1 (in case \(r_1 \neq s_1\), this set is empty). For \(l > 0\), we have a natural map
\[
U_{r,s,l+1} \to U_{r,s,l}, \quad f \mapsto \pi_l \circ f.
\]
By induction, there are vector bundles \(K_1, K_2 \to U_{r,s,l}\), of ranks \(a-r_l, b-s_l\), resp., such that \(K_{1,f_0} = \ker(f_0 : \mathbb{C}^a \to (\mathbb{C}^b)^* \otimes \tilde{V}_l), K_{2,f_0} = \ker(f_0 : \mathbb{C}^b \to (\mathbb{C}^a)^* \otimes \tilde{V}_l)\), for \(f_0 \in U_{r,s,l}\).

Consider now \(0 \leq i \leq \min\{r_{l+1} - r_l, s_{l+1} - s_l\},\) and define
\[
U_{r,s,l+1}^i = \{ f \in U_{r,s,l+1} : \text{rg}(f : K_{1,\pi_{l+1}^i} \otimes K_{2,\pi_{l+1}^i} \to \tilde{V}_{l+1}/\tilde{V}_l = \tilde{V}_{l+1}) = i \}.
\]
Clearly, we have an stratification \(U_{r,s,l+1} = \bigsqcup U_{r,s,l+1}^i\). The set
\[
B_i = \{(f_0, f_1) ; f_0 \in U_{r,s,l}, f_1 : K_{1,f_0} \otimes K_{2,f_0} \to \tilde{V}_{l+1}, \text{rg}(f_1) = i \}
\]
is a bundle over \(U_{r,s,l}\), and the fibers are A.S. by Claim 1. So \(B_i\) are A.S. There are vector bundles \(K_1, K_2 \to B_i\), defined by
\[
K_{1,(f_0,f_1)} = \ker(f_1 : K_{1,f_0} \to (K_{2,f_0})^* \otimes V_{l+1}) \subset K_{1,f_0},
\]
\[
K_{2,(f_0,f_1)} = \ker(f_1 : K_{2,f_0} \to (K_{1,f_0})^* \otimes V_{l+1}) \subset K_{2,f_0},
\]
and they are of ranks \(a-r_l-i, b-s_l-i\), resp.

There is a natural map \(U_{r,s,l+1}^i \to B_i\), which is a fiber bundle. To see this, consider \((f_0, f_1) \in B_i\), and let us find the fiber over it. Fix decompositions
\[
\mathbb{C}^a = (\mathbb{C}^a/K_{1,f_0}) \oplus (K_{1,f_0}/K_{1,f_0}f_1) \oplus K_{1,(f_0,f_1)},
\]
\[
\mathbb{C}^b = (\mathbb{C}^b/K_{2,f_0}) \oplus (K_{2,f_0}/K_{2,f_0}f_1) \oplus K_{2,(f_0,f_1)}.
\]
Then the fiber over \((f_0, f_1)\) consists of maps \(g : \mathbb{C}^a \otimes \mathbb{C}^b \to V_{l+1}\) with nine components \(g_{ij}\), according to the decompositions \([82]\). The components \(g_{11}, g_{12}, g_{21}\) can be chosen arbitrarily (thus they move in a vector space). The component
\[
g_{22} : (K_{1, f_0}/K_{1, (f_0, f_1)}) \otimes (K_{2, f_0}/K_{2, (f_0, f_1)}) \to V_{l+1}
\]
is fixed and equal to \(f_1\). The components \(g_{23}, g_{32}, g_{33}\) are zero (since they should equal \(f_1\)). Finally, the components
\[
\begin{align*}
g_{13} & : K_{1, (f_0, f_1)} \to (\mathbb{C}^b/K_{2, f_0})^* \otimes V_{l+1}, \\
g_{31} & : K_{2, (f_0, f_1)} \to (\mathbb{C}^a/K_{1, f_0})^* \otimes V_{l+1},
\end{align*}
\]
must satisfy \(\text{rg}(g_{13}) = r_{l+1} - r_l - i\) and \(\text{rg}(g_{31}) = s_{l+1} - s_l - i\). This implies that they move in spaces which are A.S., by Claim 1.

**Claim 3.** Let \(a_1, \ldots, a_k, b \geq 1\) be integers and let \(V_1, \ldots, V_k\) be finite dimensional vector spaces. Put \(\mathbf{a} = (a_1, \ldots, a_k)\), and consider
\[
M_{\mathbf{a}b} = \bigoplus_{i=1}^k \{ f_i : \mathbb{C}^{a_i} \otimes \mathbb{C}^b \to V_i \}.
\]
Let \(0 \leq r_i \leq a_i\), and write \(\mathbf{r} = (r_1, \ldots, r_k)\). Define
\[
U_{\mathbf{r}} = \{ f = (f_1, \ldots, f_k) \in M_{\mathbf{a}b} : \text{rg}(f_i : \mathbb{C}^{a_i} \to (\mathbb{C}^b)^* \otimes V_i) = r_i, \forall i, \text{rg}(f : \mathbb{C}^b \to \bigoplus_i (\mathbb{C}^{a_i})^* \otimes V_i) = b \}.
\]
Then \(U_{\mathbf{r}}\) is A.S., and \(K_i \to U_{\mathbf{r}}\), defined by \(K_i f = \ker(f : \mathbb{C}^{a_i} \to (\mathbb{C}^b)^* \otimes V_i)\), for \(f \in U_{\mathbf{r}}\), are vector bundles of ranks \(a_i - r_i\), resp.

**Proof.** The case \(k = 1\) is covered by Claim 2. So suppose \(k > 1\). Consider integers \(0 \leq p_1 \leq \ldots \leq p_k = b\), and write \(\mathbf{p} = (p_1, \ldots, p_k)\). For \(1 \leq l \leq k\), define
\[
U_{\mathbf{r}, \mathbf{p}, l} = \{ f = (f_1, \ldots, f_l) \in \bigoplus_{i=1}^l \text{Hom}(\mathbb{C}^{a_i} \otimes \mathbb{C}^b, V_i) : \text{rg}(f_i : \mathbb{C}^{a_i} \to (\mathbb{C}^b)^* \otimes V_i) = r_i, 1 \leq i \leq l, \text{rg}(f : \mathbb{C}^b \to \bigoplus_{i=1}^l (\mathbb{C}^{a_i})^* \otimes V_i) = p_l \}.
\]
and \(U_{\mathbf{r}, \mathbf{p}} = U_{\mathbf{r}, \mathbf{p}, k}\). We have a stratification \(U_{\mathbf{r}} = \bigsqcup_{\mathbf{p}} U_{\mathbf{r}, \mathbf{p}}\). Note that there is a vector bundle \(C_l \to U_{\mathbf{r}, \mathbf{p}, l}\), defined by \(C_l f_0 = \ker(f_0 : \mathbb{C}^b \to \bigoplus_{i=1}^l (\mathbb{C}^{a_i})^* \otimes V_i)\), for \(f_0 \in U_{\mathbf{r}, \mathbf{p}, l}\), of rank \(b - p_l\).

There is a fibration
\[
U_{\mathbf{r}, \mathbf{p}, l+1} \to U_{\mathbf{r}, \mathbf{p}, l}.
\]
The fiber over \(f_0 \in U_{\mathbf{r}, \mathbf{p}, l}\) consists of those \(f_{l+1} \in \text{Hom}(\mathbb{C}^b \otimes \mathbb{C}^{a_{l+1}}, V_{l+1})\) such that \(f_{l+1} : C^{a_{l+1}} \to (\mathbb{C}^b)^* \otimes V_{l+1}\) has rank \(r_{l+1}\) and \(f_{l+1}|_{C_{l, f_0}} : C_{l, f_0} \subset \mathbb{C}^b \to (\mathbb{C}^{a_{l+1}})^* \otimes V_{l+1}\) has rank \(p_{l+1} - p_l\).

Working similarly as in Claim 2, we can see that this fiber is A.S. □
Claim 4. Take $k, q \geq 1$. Let $a_1, \ldots, a_k \geq 1$ and $b_1, \ldots, b_q \geq 1$ be integers, and let $V_{it}$ be finite dimensional vector spaces, $1 \leq i \leq k, 1 \leq t \leq q$. Write $\mathbf{a} = (a_1, \ldots, a_k), \mathbf{b} = (b_1, \ldots, b_q)$ and consider

$$M_{\mathbf{a}\mathbf{b}} = \bigoplus_{i,t} \{ f_{it} : \mathbb{C}^{a_i} \otimes \mathbb{C}^{b_t} \rightarrow V_{it} \}. $$

Let $0 \leq r_i \leq a_i$, and put $\mathbf{r} = (r_1, \ldots, r_k)$. Define

$$U_{\mathbf{r}} = \{ f = (f_{it}) \in M_{\mathbf{a}\mathbf{b}} : \text{rg}(f_{it}) : \mathbb{C}^{a_i} \rightarrow \bigoplus_{t=1}^l (\mathbb{C}^{b_t})^* \otimes V_{it}) = r_{i0}, \forall i_0, $$

$$\text{rg}(f_{it}) : \mathbb{C}^{b_t} \rightarrow \bigoplus_{i=1}^k (\mathbb{C}^{a_i})^* \otimes V_{it}) = b_{t0}, \forall t_0 \}.$$ 

Then $U_{\mathbf{r}}$ is A.S., and $K_i \rightarrow U_{\mathbf{r}}$, defined by $K_i(f) = \ker((f_{it})_t : \mathbb{C}^{a_i} \rightarrow \bigoplus_{t=1}^l (\mathbb{C}^{b_t})^* \otimes V_{it})$, are vector bundles of ranks $a_i - r_i$.

Proof. Clearly the case $q = 1$ corresponds to Claim 3. For each $i$, consider integers $0 \leq r_{i1} \leq \ldots \leq r_{iq} = r_i$, and abbreviate $\varsigma = (r_{it})$. Define the sets

$$U_{\varsigma, l} = \{ f \in \bigoplus_{i \leq i \leq l} \text{Hom}(\mathbb{C}^{a_i} \otimes \mathbb{C}^{b_t}, V_{it}) ; \text{rg}(f : \mathbb{C}^{a_i} \rightarrow \bigoplus_{t=1}^l (\mathbb{C}^{b_t})^* \otimes V_{it}) = r_{is}, $$

$$1 \leq i \leq k, 1 \leq s \leq l, \text{rg}(f : \mathbb{C}^{b_t} \rightarrow \bigoplus_{i=1}^k (\mathbb{C}^{a_i})^* \otimes V_{it}) = b_{t1}, 1 \leq t \leq l \},$$

and $U_\varsigma = \bigcup_{\varsigma} U_{\varsigma, l}$. Clearly $U_{\varsigma, l+1} = U_{\varsigma, l}$ is a fiber bundle. Let $f_0 \in U_{\varsigma, l}$. Then $C_{i, f_0} = \ker(f_0 : \mathbb{C}^{a_i} \rightarrow \bigoplus_{t=1}^l (\mathbb{C}^{b_t})^* \otimes V_{it})$ define vector bundles of rank $a_i - r_{it}$, for all $i$.

The fiber of $U_{\varsigma, l+1} \rightarrow U_{\varsigma, l}$ over $f_0 \in U_{\varsigma, l}$ consists of those $\tilde{f} = (f_{i, l+1})_i \in \bigoplus_i \text{Hom}(\mathbb{C}^{a_i} \otimes \mathbb{C}^{b_{l+1}}, V_{i, l+1})$ such that

$$\tilde{f} : \mathbb{C}^{b_{l+1}} \rightarrow \bigoplus_{i} (\mathbb{C}^{a_i})^* \otimes V_{i, l+1}$$

has rank $b_{l+1}$, and

$$f_{i, l+1}|_{C_{i, f_0}} : C_{i, f_0} \subset \mathbb{C}^{a_i} \rightarrow (\mathbb{C}^{b_{l+1}})^* \otimes V_{i, l+1}$$

has rank $r_{i, l+1} - r_{it}$, for all $i$. This gives an A.S. space, as can be seen with an argument similar to the previous cases. \hfill $\Box$

Claim 5. Take $k, q \geq 1, \mathbf{a} = (a_1, \ldots, a_k), \mathbf{b} = (b_1, \ldots, b_q)$ as before. Let $V_{it}$ and $W_i$ be finite dimensional vector spaces, $1 \leq i \leq k, 1 \leq t \leq q$. Consider

$$\widetilde{M}_{\mathbf{a}\mathbf{b}} = \bigoplus_{i, t} \{ f_{it} : \mathbb{C}^{a_i} \otimes \mathbb{C}^{b_t} \rightarrow V_{it} \} \oplus \bigoplus_i \{ g_i : \mathbb{C}^{a_i} \rightarrow W_i \}, $$

and

$$\widetilde{U} = \{ f = (f_{it}, g_i) \in \widetilde{M}_{\mathbf{a}\mathbf{b}} : \text{rg}((f_{i0}, g_{i0})_t : \mathbb{C}^{a_i} \rightarrow \bigoplus_{t} (\mathbb{C}^{b_t})^* \otimes V_{i0}) \oplus W_{i0}) $$

$$= a_{i0}, \forall i_0, \text{rg}((f_{i0})_t : \mathbb{C}^{b_{i0}} \rightarrow \bigoplus_{i} (\mathbb{C}^{a_i})^* \otimes V_{i0}) = b_{t0}, \forall t_0 \}.$$ 

Then $\widetilde{U}$ is A.S.
There is a map \( \pi : \tilde{M}_{ab} \to M_{ab} \), and denote \( \tilde{U}_r = \pi^{-1}(U_r) \cap \tilde{U} \), for each \( r \), with notations as in Claim 4. Then \( \tilde{U} = \bigsqcup_r \tilde{U}_r \), and the maps \( \tilde{U}_r \to U_r \) are fibrations whose fibers are easily seen to be A.S. The fiber over \( f_0 \in U_r \) is given by \( (g_i) \in \bigsqcup_j \Hom(C^n_i, W_i) \) such that \( g_i|_{K_i, f_0} : K_i, f_0 \to W_i \) is injective for all \( i \).

Now we apply these results to the fiber \( F_j \) of our map
\[
p_j : M_j \to M_{j+1}.
\]
For a point in \( M_{j+1} \), the exact sequence (6.1) splits as a direct sum of exact sequences
\[
0 \to \Hom(S(a_{j+1}), S_i) \to \Ext^1(\bar{T}_{j+2}, S_i) \to \Ext^1(\bar{T}_{j+1}, S_i) \to \Ext^1(S(a_{j+1}), S_i) \to 0.
\]
according to \( S(a_j) = \bigoplus S_i^{a_{ij}} \). Therefore, we can set
\[
\begin{align*}
V_{it} &= \Ext^1(S_t, S_i)^*, \\
W_i &= (\Ext^1(\bar{T}_{j+2}, S_i) / \Hom(S(a_{j+1}), S_i))^*, \\
a &= a_j, \\
b &= a_{j+1}.
\end{align*}
\]
Then, with the notations of Claim 5, we have that \( \Ext^1(\bar{T}_{j+1}, S(a_j)) = \tilde{M}_{ab} \), and the extensions in \( \Ext^1(\bar{T}_{j+1}, S(a_j)) \) giving rise to triples \( \tilde{T}_j \) which are \( \sigma^+_c \)-stable are
\[
F_j = \Ext^1(\bar{T}_{j+1}, S(a_j)) \cap M_{j+1} = \tilde{U} \subset \tilde{M}_{ab},
\]
by Proposition 5.1. So the fiber of the map \( p_j \) is A.S., as required. □

**Remark 6.2.** The same description works for \( \sigma_c = \sigma_m \) with the condition \( d/n - d_2 > 2g - 2 \). Let us see this.

By Remark 5.2, the elements in \( M_{r+1} \) are of the form \( \bar{T}_{r+1} = (0, L, 0) \), so \( M_{r+1} = J\text{ae}d_c \times X \). When studying the map \( \pi_r : M_r \to M_{r+1} \), we cannot now apply Lemma 4.3 to prove the vanishing of \( \mathbb{H}^2 \). However, the condition \( d/n - d_2 > 2g - 2 \) ensures that, for any \( S(a_r) = (F_r, 0, 0) \), where \( F_r \) is a polystable bundle of slope \( d/n \), we have \( \mathbb{H}^2(\bar{T}_{r+1}, S(a_r)) = H^1(F_r \otimes L^*) = 0 \), using Proposition 4.4. With this information, we have that \( \pi_r \) is a fiber bundle, and the rest of the argument can be carried out as before.

The construction of \( \tilde{X}^-(n) \) is entirely similar. Start with
\[
M_1 = M(n) = \tilde{U}(n) \times N_{\sigma_c}^{n'}(n', 1, d', d_0).
\]
Define by induction \( M_j \) as the space parametrizing the triples \( \tilde{T}_j \) of Proposition 5.3 as follows: consider the vector bundle \( \pi_j : M_j \to M_{j-1} \) whose fibers are the spaces \( \Ext^1(S(a_j), \bar{T}_{j-1}) \). Then let \( M_j \subset M_j \) be the subset corresponding to those extensions \( \xi_j \) satisfying (1) and (2) of Proposition 5.3. The space we are interested in is \( \tilde{X}^-(n) := M_{r+1} \).

We have the following result.
Theorem 6.3. Consider the fibration $p_j : M_j \to M_{j-1}$. Then $p_j$ is a locally trivial fibration (in the Zariski topology), and the fiber $F_j$ is an A.S. space.

7. Hodge structures of moduli of triples

Let $X$ be a complex curve of genus $g \geq 2$. We aim to find information on the Hodge structures of the cohomology of the moduli spaces $\mathcal{N}_{\sigma}(n,1,d,d_0)$, for non-critical values $\sigma$, and on the moduli spaces $M(n,d)$. Consider the Hodge structure $H^1(X)$ and set

$$R_X = \{H^1(X)\}.$$  

Let us see that the Hodge structures of the moduli spaces of pairs and of bundles are $R_X$-generated.

Lemma 7.1. The cohomology of $\text{Jac} X$ is $R_X$-generated. The cohomology of $\text{Sym}^k X$ is $R_X$-generated, for any $k \geq 1$.

Proof. The first assertion is clear, since $H^*(\text{Jac} X)$ is the exterior algebra on $H^1(X)$. For the case of the symmetric products of $X$, note that Macdonald [9] proves that there is an epimorphism of rings

$$\bigwedge^* H^1(X) \otimes \mathbb{Q}[\theta] \to H^*(\text{Sym}^k X),$$  

where $\theta$ is the hyperplane class, which is defined as the class of the divisor $D = \text{Sym}^{k-1} X \subset \text{Sym}^k X$. Clearly, this is a class in $H^{1,1}(\text{Sym}^k X) \cap H^2(\text{Sym}^k X, \mathbb{Z})$, hence (7.1) is a morphism of Hodge structures. This proves that $H^*(\text{Sym}^k X)$ is $R_X$-generated.

Note that $M(1,d) = \text{Jac} X$ and $\mathcal{N}^{s}_{\sigma}(1,1,d_1,d_2) = \text{Jac} X \times \text{Sym}^{d_2-d_1} X$. So Lemma 7.1 serves as starting point for an induction (on the rank) to prove that all Hodge structures of all moduli spaces are $R_X$-generated.

Proposition 7.2. Let $n \geq 2$. Assume that all the Hodge structures $H^*_c(M^*(n',d'))$ and $H^*_c(\mathcal{N}^s_{\sigma}(n',1,d',d_0))$ are $R_X$-generated for $n' < n$. Then for any critical value $\sigma_c > \sigma_m$ and for any type $\mathbf{n}$, the Hodge structures $H^*_c(X^+(\mathbf{n}))$ are $R_X$-generated. The same happens for $\sigma_c = \sigma_m$ and for any type $\mathbf{n} \neq \mathbf{n}_0 = (n,d,1)$ ($\mathbf{n}_0$ corresponds to the stratum $\mathcal{U}_m \subset \mathcal{N}^{s}_{\sigma_m^*}$, c.f. Remark 5.2.

The same result holds for the Hodge structures $H^*_c(X^-(\mathbf{n}))$.

Proof. If all $H^*_c(M^*(n',d'))$ and $H^*_c(\mathcal{N}^s_{\sigma}(n',1,d',d_0))$ are $R_X$-generated for $n' < n$, then all $H^*_c(\tilde{\mathcal{U}}^+(\mathbf{n}))$ and $H^*_c(\mathcal{M}(\mathbf{n}))$ are also $R_X$-generated. By Theorem 6.1, $\tilde{\mathcal{X}}^+(\mathbf{n}) \to \mathcal{M}(\mathbf{n})$ is a fibration whose fiber $F$ is A.S. By Corollary 2.2, $H^*_c(F)$ is a direct sum of trivial Hodge structures. Using the Leray spectral sequence, we get that $H^*_c(\tilde{\mathcal{X}}^+(\mathbf{n}))$ is $R_X$-generated.

Now we want to see that $H^*_c(\tilde{\mathcal{X}}^+(\mathbf{n}))$ is $R_X$-generated. First note that by Poincaré duality, this is equivalent to see that $H^*(\tilde{\mathcal{X}}^+(\mathbf{n}))$ is $R_X$-generated. Now $\tilde{\mathcal{X}}^+(\mathbf{n}) \to...$
$\tilde{X}^+(n)$ is a locally trivial fibration, with fibers $GL = \prod GL(a_j)$. Therefore there is a fibration

$$\tilde{X}^+(n) \to \tilde{X}^+(n) \to PGL,$$

where $PGL$ is the classifying space for $GL$. By the Leray spectral sequence, the cohomology of $\tilde{X}^+(n)$ is generated by that of $\tilde{X}^+(n)$ and the pull-back of the cohomology in $H^*(PGL)$. Let us see that the pull-back of $H^*(PGL)$ is a trivial Hodge structure. Consider the following spaces $\hat{M}_j$. First $\hat{M}_{r+1} = M_{r+1}$, and $\hat{M}_j \to \hat{M}_{j+1}$ are fibrations with fibers $\hat{F}_j = \prod_i V(a_{ij}, \text{Ext}^1(\tilde{T}_{j+1}, S_i))$. That is, extensions as in Proposition 5.1 but only satisfying property (1). Clearly, $M_1 \subset M_1$. On the other hand, we still have an action of $GL$ on $\hat{M}_1$, which is an iterated Grassmannian bundle. By Corollary 2.2, and the usual A.S. of the Grassmannian given by Schubert cells, we have that the cohomology of $\hat{X}$ is $H^*(\tilde{X}^+(n))$ is $R_X$-generated.

Finally, $X^+(n) = \tilde{X}^+(n)/F$, with $F$ defined in 5.6. Thus we have that $H^c(\tilde{X}^+(n)) = H^c(\tilde{X}^+(n))/F$ is $R_X$-generated as well.

**Proposition 7.3.** If $H^c(X^+(n))$ are $R_X$-generated, for all critical values $\sigma_c$ and for all types $n$ (with the exception of $\sigma_c = \sigma_m$ and $n \neq n_0 = (n,d,1)$), then $H^c(M^s(n,d))$ and $H^c(N^s_\sigma(n_1,d,d_o))$, for any $\sigma$, are $R_X$-generated.

**Proof.** This is a repeated application of Lemma 4.5. For $\sigma_c > \sigma_m$, $S_{\sigma_c} = \bigcup X^\pm(n)$, so the (mixed) Hodge structures of these spaces are $R_X$-generated. The same argument works for $D_m = \bigcup_{n \neq n_0} X^+(n) \subset N^c_{\sigma_m}$.

This implies first that $N^c_{\sigma_m} = S_{\sigma_m}$ has cohomology $H^c(N^c_{\sigma_m})$ which is $R_X$-generated. Then we work inductively on $\sigma$. If $H^c(N_{\sigma})$ is $R_X$-generated for any $\sigma > \sigma_c$, then $H^c(N^c_{\sigma_c})$ is $R_X$-generated, as $N^c_{\sigma_c} = N^c_{\sigma_c} \setminus N^c_{\sigma_c}$, and then $H^c(N^c_{\sigma_c})$ is $R_X$-generated as $N^c_{\sigma_c} = N^c_{\sigma_c} \setminus S_{\sigma_c}$.

Finally $H^c(N^c_{\sigma_m})$ is $R_X$-generated. But $H^c(D_m)$ is also $R_X$-generated. So $H^c(U_m)$ is $R_X$-generated, since $U_m = N^c_{\sigma_m} - D_m$. Taking $d/n - d_o > 2g - 2$, we have a projective fibration $U_m \to M^s(n,d) \times \text{Jac}^d X$, by [12] Proposition 4.10 (see Remarks 5.2 and 6.2). So $H^c(M^s(n,d))$ is $R_X$-generated as well. □
Propositions 7.2 and 7.3 together prove Theorem 1.3, and this implies in particular Theorem 1.1 and Corollary 1.2.

**Remark 7.4.** The same kind of question can be asked about the moduli spaces of triples and of bundles of fixed determinant. Let $M_n(L_0)$ denote the moduli space of semistable bundles $E$ over $X$ of rank $n$ and satisfying $\text{det}(E) \cong L_0$, where $L_0$ is a fixed line bundle of degree $d$. Consider also the moduli space $N^\sigma_n(L_1, L_2)$ of $\sigma$-polystable triples $(E_1, L_2, \phi)$ with $E_1$ a rank $n$ bundle with $\text{det}(E_1) = L_1$, where $L_1, L_2$ are fixed line bundles of degrees $d_1, d_2$, resp.

The same techniques developed in Sections 3–7 can be used to prove that the Hodge structures of $M_n(L_0)$ and of $N^\sigma_n(L_1, L_2)$ are $R_X$-generated.

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