A Resilient Quantum Secret Sharing Scheme

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Abstract  A resilient secret sharing scheme is supposed to generate the secret correctly even after some shares are damaged. In this paper, we show how quantum error correcting codes can be exploited to design a resilient quantum secret sharing scheme, where a quantum state is shared among more than one parties.

Keywords Quantum error-correction · Resilient quantum secret sharing · Undoing quantum measurement

1 Introduction

The quantum teleportation of a single qubit was first proposed by Bennett et al. [2], using a maximally entangled bipartite quantum state. The idea of teleportation has been successfully applied in the process of quantum secret sharing. In quantum teleportation, Alice sends a qubit to a distant receiver Bob through some unitary operation involving the qubit and the entangled channel shared between them. Further results related to quantum teleportation, that can be exploited in quantum secret sharing, include results using multipartite quantum channels such as tripartite GHZ state [8], four-partite GHZ state [16], an asymmetric W state [1] and the cluster state [3]. The perfect teleportation of an arbitrary two-qubit state was proposed using quantum channels formed by the tensor product of two Bell states [19], tensor product of two orthogonal states [22], genuinely entangled five qubit state [12], five qubit cluster state [11] and six qubit genuinely entangled states [10].
The idea of Quantum Secret Sharing (QSS) of a single qubit was first due to Hilery et al. [7] using three and four qubit GHZ states. Later this process was investigated by Karlsson et al. [9] using three particle entanglement, Cleve et al. [4] using a process similar to error correction and Zheng using W state [26]. The QSS of an arbitrary two-qubit state was proposed by Deng et al. using two GHZ states [5]. QSS using cluster states was demonstrated by Nie [14], Panigrahi [13, 17] and Han [6]. Recently two qubit QSS was discussed using arbitrary pure or mixed resource states [25] and asymmetric multipartite state [24].

In this paper we do not exploit the ideas related to teleportation in quantum secret sharing. The idea of teleportation does not naturally take care of the situation when some adversarial model is considered where some shares may be disturbed. Rather, we use quantum error correcting codes to take care of the situation. There exist some works [18, 23] for building quantum secret sharing schemes using (classical or quantum) error correcting codes. However, none of these schemes addresses the resiliency issue.

An arbitrary pure single-qubit quantum state is given by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$, where $\alpha, \beta \in \mathbb{C}$. In quantum error correction scheme known as repetition code [15], the above state is encoded as $\alpha|000\rangle + \beta|111\rangle$. This encoding can correct a single bit-flip error. In [20], the authors pointed out that any measurement in the computational basis states $|0\rangle$, $|1\rangle$ causes a projection onto the $\sigma_z$ axis of the Bloch sphere and can be interpreted as an incoherent phase flip. Hence, any protocol correcting against phase flips is sufficient to reverse measurements in the computational basis. The repetition code can be modified to protect against such phase-flip errors by a simple basis change from $|0\rangle$, $|1\rangle$ to $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

The schematic diagram for undoing a quantum measurement is given in Fig. 1. This idea [20] motivated us to devise a novel quantum secret sharing protocol based on quantum error correcting codes. It is resilient in the sense that it survives when the line of Bob or Charlie or both is compromised.

1.1 Our Contribution

In light of the above discussion, let us now explain our exact proposal and its importance. In line of Fig. 1, we also like to refer to Fig. 2. At the bottom of the figure, $|\psi_i\rangle$ below each column denotes the joint 3-qubit state immediately after the application of the gates in that column. For example, $|\psi_3\rangle$ is the state after the application of three Hadamard gates in the right half of the diagram.

The single qubit state $|\psi\rangle$, as in Fig. 2, is to be distributed between three parties Alice, Bob and Charlie. Two additional $|1\rangle$ qubits are used by the distributor for this purpose. Each individual particle of the three qubit state $|\psi_3\rangle$ is distributed to Alice, Bob and Charlie.
Without loss of generalization, it is considered that when they come together to generate the secret, then Bob and Charlie will co-operate with Alice and the secret will be generated with Alice. The particle with Alice corresponding to the three-qubit state $|\psi_8\rangle$ should be same as $|\psi\rangle$ for a successful execution of the protocol.

We claim that our model is resilient. We consider that Bob or Charlie may cheat. This is pointed out by marking “Eaves” in Fig. 2 instead of ”Error” in Fig. 1. As an example, by cheating, we mean that Bob or Charlie may measure their individual qubits in certain basis to gain some information about the secret state $|\psi\rangle$. In our protocol, Alice can detect cheating with certain probabilities if either Bob or Charlie or both Bob and Charlie cheat(s). Further, the cheater will not be able to obtain any information about the secret state from the particle associated with him.

One can consider that parties other than Alice, Bob and Charlie may also be interested to gain information about the secret to disturb the protocol. However, it is enough to consider that Bob and Charlie themselves may play that adversarial role. We consider that Alice will be honest and the secret will be prepared by her in cooperation with Bob and Charlie.

Let us present an example. Consider that the cheating is done by Bob and/or Charlie and they measure in $\{|0\rangle, |1\rangle\}$ basis. Alice will not disturb the particle that is with her, but in case of cheating, she has to operate a phase gate to get back the secret. On the other hand, Alice can always measure the other two particles corresponding to Bob and Charlie. Thus, Alice can measure those two qubits in $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis to obtain relevant information regarding cheating.

To the best of our knowledge, for quantum secret sharing, there exists no such scheme in literature that can decide

- whether the cheating is done,
- whether the expected state is generated or not and
- whether the exact state could be achieved after some transformation.

2 The Proposed Secret Sharing Protocol

First, the dealer $D$ encodes the secret using the encoding circuit (left part of Fig. 2, up to $|\psi_3\rangle$). The dealer then distributes all three qubits among the participants: Alice, Bob and Charlie. Each of them applies Hadamard gate after getting the qubits. Now, one of them, say Alice, decides to generate the secret and so the other two delivers their qubits to Alice.

Alice applies two CNOT gates (as shown in Fig 2) considering her qubit as the control. Next, she applies Toffoli gates considering Bob and Charlie’s bits as control. Then she
applies Hadamard (H) and phase (Z) gates to generate the secret at her end. Alice measures the particles corresponding to Bob and Charlie in \(|00\rangle, |01\rangle, |10\rangle, |11\rangle\) basis.

- If she gets \(|11\rangle\), she decides that no cheating has been done in the process.
- If she gets \(|01\rangle\), she understands that cheating is done by Bob.
- If she gets \(|10\rangle\), she understands that cheating is performed by Charlie.
- If she gets \(|00\rangle\), she knows that both Bob and Charlie have cheated.

In all three cases related to cheating, she has to apply a phase gate to generate the secret. On the other hand we have the following cases related to cheating.

- If Bob has cheated by measuring in \(|0\rangle, |1\rangle\) basis, then Alice will observe either \(|01\rangle\) or \(|11\rangle\) with probability \(\frac{1}{2}\) each.
- If Charlie has cheated by measuring in \(|0\rangle, |1\rangle\) basis, then Alice will observe either \(|10\rangle\) or \(|11\rangle\) with probability \(\frac{1}{2}\) each.
- If both Bob and Charlie have cheated by measuring their particle in \(|0\rangle, |1\rangle\) basis individually, then Alice will observe any of \(|00\rangle, |01\rangle, |10\rangle, |11\rangle\) with probability \(\frac{1}{4}\) each.

Thus, in any case, the secret quantum state will be created perfectly. However, Alice will be able to understand whether the cheating has been done with certain probabilities.

2.1 Detailed Calculation

We show the step-by-step evolution of the joint state as the protocol runs from left to right in Fig. 2. We can write

\[ |\psi_0\rangle = (\alpha |0\rangle + \beta |1\rangle) |\rangle |1\rangle. \]

After the application of the Hadamard gate, the joint state becomes

\[ |\psi_1\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0\rangle + |1\rangle) + \beta (|0\rangle - |1\rangle) \right] |\rangle |1\rangle. \]

After the application of the two CNOT gates, the state becomes

\[ |\psi_2\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|011\rangle + |100\rangle) + \beta (|011\rangle - |100\rangle) \right]. \]

Next, the dealer applies three Hadamard gates, one on each line, yielding the state

\[ |\psi_3\rangle = \frac{1}{2} \left[ \alpha (|000\rangle - |110\rangle + |011\rangle - |101\rangle) + \beta (|100\rangle - |010\rangle - |001\rangle + |111\rangle) \right]. \]  

Now, the dealer distributes this state amongst Alice, Bob and Charlie, one particle to each. After getting the qubits, each party applies Hadamard on his/her qubit. So the resultant state becomes

\[ |\psi_4\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|011\rangle + |100\rangle) + \beta (|011\rangle - |100\rangle) \right] = |\psi_2\rangle. \]

After two CNOT gates, the state becomes

\[ |\psi_5\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|011\rangle + |111\rangle) + \beta (|011\rangle - |111\rangle) \right]. \]

Next, after application of the Toffoli gate, the state becomes

\[ |\psi_6\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (|0\rangle + |1\rangle) - \beta (|0\rangle - |1\rangle) \right] |\rangle |1\rangle. \]
After the last Hadamard gate, it reduces to

$$|\psi_7\rangle = (\alpha|0\rangle - \beta|1\rangle)|1\rangle|1\rangle.$$ 

The phase gate in the end converts this state into

$$|\psi_8\rangle = (\alpha|0\rangle + \beta|1\rangle)|1\rangle|1\rangle = |\psi_0\rangle.$$ 

Now Alice measures her ancilla in the \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} basis. In case she gets |11\rangle, she knows that the secret has been successfully generated.

3 Resiliency of the Scheme and Analysis of Cheating

In this section, we analyze what happens when Bob or Charlie or both tries to cheat to the corresponding lines. We do not consider any other eavesdropper as any cheating by other parties can also be done by Bob and Charlie themselves.

3.1 When Bob Attempts Cheating

As Bob wants to get some information about the secret, the way we consider is to measure the qubit just after he receives it. Assuming that he measures in the computational basis, he gets |0\rangle with probability $\frac{1}{2}$ and |1\rangle with probability $\frac{1}{2}$.

We have the state $|\psi_3\rangle$ in hand. Suppose Bob gets |0\rangle. From (1), we can write the remaining two-qubit state as

$$|\psi_{AC}^3\rangle = \frac{1}{\sqrt{2}} [\alpha(|00\rangle - |11\rangle) + \beta(|10\rangle - |01\rangle)].$$

After the measurement by Bob, he again replaces the measured qubit (here |0\rangle). Then each of Alice, Bob and Charlie applies Hadamard, and the joint state becomes

$$|\psi_{AB0}^4\rangle = \frac{1}{2} [\alpha(|00| + |100| + |110| + |011|) + \beta(|001| - |100| - |110| + |011|)].$$

Now, Bob and Charlie give their qubits to Alice. Alice applies CONTs considering her qubit as control. Then the overall state becomes

$$|\psi_{AB0}^5\rangle = \frac{1}{2} [\alpha(|001| + |111| + |101| + |011|) + \beta(|001| - |111| - |101| + |011|)].$$

Next, Alice operates Toffoli considering Bob and Charlie’s qubits as control. Then we get

$$|\psi_{AB0}^6\rangle = \frac{1}{2} [\alpha(|00| + |11|) (|01| + |11|) + \beta(|00| - |11|) (|01| - |11|)].$$

Now, Alice applies Hadamard on the first qubit. Thus, the resultant state becomes

$$|\psi_{AB0}^7\rangle = \frac{1}{\sqrt{2}} [(\alpha|0\rangle - \beta|1\rangle)|11\rangle + (\alpha|0\rangle + \beta|1\rangle)|01\rangle].$$

After Alice applies the last phase gate, the state becomes

$$|\psi_{AB0}^8\rangle = \frac{1}{\sqrt{2}} [(\alpha|0\rangle + \beta|1\rangle)|11\rangle + (\alpha|0\rangle - \beta|1\rangle)|01\rangle].$$

Now, Alice measures the particles corresponding to Bob and Charlie in the basis \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}. In this case, she gets |11\rangle with probability $\frac{1}{2}$ and |01\rangle with probability $\frac{1}{2}$. In the first case, she gets back the secret, but cannot detect that Bob has cheated. In
the second case, she discovers that Bob has indeed cheated and she can get back the secret by applying one more phase gate to her qubit.

Similarly, when Bob measures $|1\rangle$, the final state becomes

$$
\psi_{8}^{AB'C} = \frac{1}{\sqrt{2}} [(\alpha|0\rangle + \beta|1\rangle) |11\rangle - (\alpha|0\rangle - \beta|1\rangle) |01\rangle].
$$

(3)

As can be seen by comparing (2) and (3), we get the same result after Alice measures the particles corresponding to Bob and Charlie in computational basis.

3.2 When Charlie Attempts Cheating

In a similar manner, Charlie measures his qubit just after he receives it. Assuming that he measures in the computational basis, he gets $|0\rangle$ with probability $\frac{1}{2}$ and $|1\rangle$ with probability $\frac{1}{2}$.

Suppose Charlie obtains $|0\rangle$. From (1), we can write the remaining two-qubit state as

$$
\psi_{3}^{AB} = \frac{1}{\sqrt{2}} [\alpha (|00\rangle - |11\rangle) + \beta (|10\rangle - |01\rangle)].
$$

Now, each applies Hadamard on the respective input and the joint state becomes

$$
\psi_{4}^{ABC'} = \frac{1}{2} [\alpha (|100\rangle + |010\rangle + |011\rangle + |101\rangle) - \beta (|100\rangle - |010\rangle - |011\rangle + |101\rangle)].
$$

Now, Bob and Charlie give their qubits to Alice. Alice applies CONTs considering her qubit as control. Then she obtains the overall state as

$$
\psi_{5}^{ABC'} = \frac{1}{2} [\alpha (|111\rangle + |010\rangle + |011\rangle + |110\rangle) - \beta (|111\rangle - |010\rangle - |011\rangle + |110\rangle)].
$$

Next, Alice operates Toffoli considering Bob and Charlie’s qubits as control. Then we get

$$
\psi_{6}^{ABC'} = \frac{1}{2} [\alpha (|0\rangle + |1\rangle) (|11\rangle + |10\rangle) - \beta (|0\rangle - |1\rangle) (|11\rangle - |10\rangle)].
$$

Now, Alice applies Hadamard on the first qubit. Thus, the resultant state becomes

$$
\psi_{7}^{ABC'} = \frac{1}{\sqrt{2}} [(\alpha|0\rangle - \beta|1\rangle) |11\rangle + (\alpha|0\rangle + \beta|1\rangle) |10\rangle].
$$

After Alice applies the last phase gate, the state becomes

$$
\psi_{8}^{ABC'} = \frac{1}{\sqrt{2}} [(\alpha|0\rangle + \beta|1\rangle) |11\rangle + (\alpha|0\rangle - \beta|1\rangle) |10\rangle].
$$

(4)

Alice measures the qubits of Bob and Charlie in $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis. In this case, she gets $|11\rangle$ with probability $\frac{1}{2}$ and $|10\rangle$ with probability $\frac{1}{2}$. In the first case, she gets back the secret, but cannot detect that Charlie tried to cheat. In the second case, she knows for certain that Charlie has indeed cheated and she can get back the secret simply by applying one more phase gate to her qubit.

Similarly, when Charlie measures $|1\rangle$, the final state becomes

$$
\psi_{8}^{ABC'} = \frac{1}{\sqrt{2}} [(\alpha|0\rangle + \beta|1\rangle) |11\rangle - (\alpha|0\rangle - \beta|1\rangle) |10\rangle].
$$

(5)

As can be seen by comparing (4) and (5), we get the same result.
3.3 When both Bob and Charlie are Eavesdroppers

Assume that both Bob and Charlie measure in the computational basis. Suppose, Bob gets $|0\rangle$ and Charlie also gets $|0\rangle$. From (1), we can write the remaining state as

$$|\psi^A_3\rangle = \alpha |0\rangle + \beta |1\rangle.$$ 

After each applies Hadamard, the joint state becomes

$$|\psi^{AB'C'}_4\rangle = \frac{1}{2\sqrt{2}} \left[ \alpha (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) 
+ \beta (|000\rangle + |001\rangle + |010\rangle + |011\rangle - |100\rangle - |101\rangle - |110\rangle - |111\rangle) \right].$$

Now, Bob and Charlie give their qubits to Alice. Alice applies CONTs considering her qubit as control. It can be easily verified that after this, the overall state remains the same, i.e.,

$$|\psi^{AB'C'}_5\rangle = |\psi^{AB'C'}_4\rangle.$$

Next, Alice operates Toffoli considering Bob and Charlie’s qubits as control. Then we get the resultant state as

$$|\psi^{AB'C'}_6\rangle = \frac{1}{2\sqrt{2}} \left[ \alpha (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) 
+ \beta (|000\rangle + |001\rangle + |010\rangle - |011\rangle - |100\rangle - |101\rangle - |110\rangle + |111\rangle) \right].$$

Now, Alice applies Hadamard on the first qubit. Thus, the resultant state becomes

$$|\psi^{AB'C'}_7\rangle = \frac{1}{\sqrt{2}} \left[ (\alpha|0\rangle - \beta|1\rangle) |11\rangle + (\alpha|0\rangle + \beta|1\rangle) (|00\rangle + |01\rangle + |10\rangle) \right].$$

After Alice applies the last phase gate, the state becomes

$$|\psi^{AB'C'}_7\rangle = \frac{1}{\sqrt{2}} \left[ (\alpha|0\rangle + \beta|1\rangle) |11\rangle + (\alpha|0\rangle - \beta|1\rangle) (|00\rangle + |01\rangle + |10\rangle) \right].$$

Now, Alice measures the qubits of Bob and Charlie in $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis. In this case, she gets each basis state with probability $\frac{1}{4}$. The cases can be summarized as follows.

- If she gets $|11\rangle$, she gets back the secret, but cannot detect if anyone tried to eavesdrop.
- If she gets $|01\rangle$, she knows that Bob was eavesdropping.
- If she gets $|10\rangle$, she knows that Charlie was eavesdropping.
- If she gets $|00\rangle$, she knows that both Bob and Charlie were eavesdropping.

By similar analysis of the cases when Bob and Charlie measure the other three possible pairs of outputs, we get the same result.

4 Analysis of Arbitrary Measurements

The protocol that we discussed can withstand errors introduced due to measurement by the advisory in the computational basis. If the advisory performs the measurement in other basis, then the secret may not be recovered. In this section we first analyse what happens when the adversary measures in arbitrary basis and then discuss how our secret sharing protocol can be modified to tolerate measurement in any basis.
4.1 Analysis of Measurement in Arbitrary Basis

Let Bob (without loss of generality) measure in \( \{|\gamma\rangle, \gamma^\perp\rangle\} \) basis, where \( |\gamma\rangle = a|0\rangle + b|1\rangle \) and \( |\gamma^\perp\rangle = b^*|0\rangle - a^*|1\rangle \), with \( a, b \in \mathbb{C} \) and \( |a|^2 + |b|^2 = 1 \). Then we can write \( |0\rangle = a^*|\gamma\rangle + b|\gamma^\perp\rangle \) and \( |1\rangle = b^*|\gamma\rangle - a|\gamma^\perp\rangle \).

Thus, we can rewrite (1) as

\[
|\psi_3\rangle = \frac{1}{2} \left[ \alpha (|000\rangle - |110\rangle + |011\rangle - |101\rangle) + \beta (|100\rangle - |010\rangle - |001\rangle + |111\rangle) \right]
\]

\[
= \frac{1}{2} \left[ \alpha \left( |0\rangle(a^*|\gamma\rangle + b|\gamma^\perp\rangle) - |1\rangle(b^*|\gamma\rangle - a|\gamma^\perp\rangle) \right) + |0\rangle(b^*|\gamma\rangle - a|\gamma^\perp\rangle) - |1\rangle(a^*|\gamma\rangle + b|\gamma^\perp\rangle) \right] + \beta \left( |1\rangle(a^*|\gamma\rangle + b|\gamma^\perp\rangle) - |0\rangle(b^*|\gamma\rangle - a|\gamma^\perp\rangle) \right) .
\]

When Bob measures this state in \( \{|\gamma\rangle, \gamma^\perp\rangle\} \) basis, it can be easily seen that she gets each of \( |\gamma\rangle \) and \( |\gamma^\perp\rangle \) with probability \( \frac{1}{2} \).

Suppose Bob observes \( |\gamma\rangle \). The remaining state becomes

\[
|\psi_{3AC}^A\rangle = \frac{1}{\sqrt{2}} \left[ \alpha (a^*|00\rangle - b^*|10\rangle + b^*|01\rangle - a^*|11\rangle) + \beta (a^*|10\rangle - b^*|00\rangle - a^*|01\rangle + b^*|11\rangle) \right].
\]

After each applies Hadamard, the joint state becomes

\[
|\psi_{4AB'C}^A\rangle = \frac{1}{2} \left[ (aa^* - \beta b^* - ab^* + \beta a^*)(a + b)|00\rangle + (aa^* - \beta b^* + ab^* - \beta a^*)(a - b)|10\rangle + (aa^* - \beta b^* + ab^* - \beta a^*)(a - b)|11\rangle \right] .
\]

Now, Bob and Charlie give their qubits to Alice. Alice applies CONTs considering her qubit as control. Then the overall state becomes

\[
|\psi_{5AB'C}^A\rangle = \frac{1}{2} \left[ (aa^* - \beta b^* - ab^* + \beta a^*)(a + b)|00\rangle + (aa^* - \beta b^* + ab^* - \beta a^*)(a + b)|10\rangle + (aa^* - \beta b^* + ab^* - \beta a^*)(a - b)|11\rangle \right] .
\]

Next, Alice operates Toffoli considering Bob and Charlie’s qubits as control. Then we get

\[
|\psi_{6AB'C}^A\rangle = \frac{1}{2} \left[ (aa^* - \beta b^* - ab^* + \beta a^*)(a + b)|00\rangle + (aa^* - \beta b^* + ab^* - \beta a^*)(a + b)|01\rangle + (aa^* - \beta b^* + ab^* - \beta a^*)(a - b)|10\rangle + (aa^* - \beta b^* - ab^* + \beta a^*)(a - b)|11\rangle \right] .
\]
Now, Alice applies Hadamard on the first qubit. Thus, the resultant state becomes

\[
\psi_{7}^{ABC_{1}} = \frac{1}{\sqrt{2}} \left[ \left( (\alpha - \beta(ab^* + a^*b)) \ket{0} + (\alpha(ab^* + a^*b) - \beta) \ket{1} \right) \ket{11} 
+ \left( (\alpha(|a|^2 - |b|^2) + \beta(a^*b - ab^*)) \ket{0} 
- (\alpha(ab^* - a^*b) + \beta(|b|^2 - |a|^2)) \ket{1} \ket{11} \right) \right].
\]

After Alice applies the last phase gate, the state becomes

\[
\psi_{8}^{ABC_{1}} = \frac{1}{\sqrt{2}} \left[ \left( (\alpha - \beta(ab^* + a^*b)) \ket{0} - (\alpha(ab^* + a^*b) - \beta) \ket{1} \right) \ket{11} 
+ \left( (\alpha(|a|^2 - |b|^2) + \beta(a^*b - ab^*)) \ket{0} 
+ (\alpha(ab^* - a^*b) + \beta(|b|^2 - |a|^2)) \ket{1} \ket{01} \right) \right].
\]

Note that due to the measurement, \( \alpha \) is shifted to \( \alpha' = \alpha - \beta(ab^* + a^*b) \) and \( \beta \) is shifted to \( \beta' = \beta - \alpha(ab^* + a^*b) \). Thus, depending on the values of \( a \) and \( b \), the secret evolves.

Similar analysis holds for the case when Bob measures \( \ket{\gamma} \). Continuing with the analysis of Bob’s measurement as \( \ket{\gamma} \), we find that when Alice measures the qubits corresponding to Bob and Charlie in \( \{|00\}, \ket{11}\) basis, she gets \( \ket{11} \) with probability \( \frac{1}{2} \) and \( \ket{01} \) with probability \( \frac{1}{2} \). Note that if one takes \( a = 1, b = 0 \), then (6) coincides with (2). On the other hand, if one takes \( a = 0, b = 1 \), then (6) coincides with (3).

Next, let us consider the case \( a = b = \frac{\sqrt{2}}{2} \), i.e., the measurement basis is \( \{|+, \ket{-}\}\). In this case, when Bob measures \( \ket{\pm} \), (6) reduced to

\[
\psi_{8}^{ABC_{1}} = \frac{1}{\sqrt{2}}(\alpha - \beta) (\ket{0} - \ket{1}) \ket{11}.
\]

Upon measuring her ancilla bits in the computational basis, Alice would get wrong information that there is no eavesdropping. Moreover, the state is also not recovered properly at Alice’s end. Similar cases can be analysed considering Charlie or both.

Thus, if the participants go for a measurement in arbitrary basis, then it is not possible to reconstruct the secret with simple quantum error correcting codes. However, with codes that can correct any kind of error, it is possible to design quantum secret sharing schemes that are more resilient. Below we outline this extension.

4.2 Possible Resiliency Against Arbitrary Operations by Cheaters

In this section, we discuss remedies against not only arbitrary measurements but also arbitrary interactions by the Cheaters. The protocol described in Section 2 uses three qubit phase flip repetition code. Here, we modify the protocol with nine qubit Shor code [21]. This code is a combination of three qubit phase flip and bit flip codes. The qubit is first encoded using the phase flip code, i.e., \( \ket{0} \) is encoded as \( \ket{++} \) and \( \ket{1} \) is represented as \( \ket{--} \). Next, each of these qubits are encoded using three qubit bit flip code, i.e., \( \ket{+} \) is encoded as \( \ket{000} + \ket{111})/\sqrt{2} \) and \( \ket{-} \) is encoded as \( \ket{000} - \ket{111})/\sqrt{2} \).

Suppose, before the interaction by the adversary, the state of the system is \( \ket{\psi_{ABC}} \). Any error \( T \), including the one due to interaction, can be analyzed by expanding it in an
operator-sum representation with operation elements \( \{ E_i \} \), we would have the post-interaction state as

\[
\Upsilon |\psi_{ABC}\rangle \langle \psi_{ABC}| = \sum_i E_i |\psi_{ABC}\rangle \langle \psi_{ABC}| E_i^\dagger.
\]

As an operator, \( E_i \) can be expanded as a linear combination of the identity, the bit flip \( X \), the phase flip \( Z \) and the combined bit and phase flip \( XZ \). Measuring the error syndrome would collapse this superposition into one of the four states, namely, \( |\psi_{ABC}\rangle \), \( X |\psi_{ABC}\rangle \), \( Z |\psi_{ABC}\rangle \) and \( XZ |\psi_{ABC}\rangle \). By applying the appropriate inverse operation, the recovery is possible.

5 Conclusion

In this paper, we show how quantum error correcting codes can be suitably used to present quantum secret sharing schemes with certain kind of resiliency. Our work is not in the line of mostly studied quantum secret sharing that lends the idea from teleportation. We explain the basic idea with a simple code and point out its merits and demerits. We also outline how any arbitrary adversarial attempt can be resisted with more complicated quantum error correcting codes. Detailed study with different kinds of access structures and finding out suitable error correcting codes are important as future works in this direction.

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