Hardness Results for the Gapped Consecutive-Ones Property Problem

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Abstract. Motivated by problems of comparative genomics and paleogenomics, in \cite{6} the authors introduced the Gapped Consecutive-Ones Property Problem \((k, \delta)-C1P\): given a binary matrix \(M\) and two integers \(k\) and \(\delta\), can the columns of \(M\) be permuted such that each row contains at most \(k\) blocks of ones and no two consecutive blocks of ones are separated by a gap of more than \(\delta\) zeros. The classical C1P problem, which is known to be polynomial is equivalent to the \((1, 0)-C1P\) problem. They showed that the \((2, \delta)-C1P\) Problem is NP-complete for all \(\delta \geq 2\) and that the \((3, 1)-C1P\) problem is NP-complete. They also conjectured that the \((k, \delta)-C1P\) Problem is NP-complete for \(k \geq 2, \delta \geq 1\) and \((k, \delta) \neq (2, 1)\). Here, we prove that this conjecture is true. The only remaining case is the \((2, 1)-C1P\) Problem, which could be polynomial-time solvable.

1 Introduction

Let \(M\) be a binary matrix with \(n\) rows and \(m\) columns. A block in a row of \(n\) is a maximal sequence of consecutive entries containing 1. A gap is a sequence of consecutive zeros that separates two blocks; the size of a gap is the length of the sequence of zeros. \(M\) is said to have the Consecutive-Ones Property (C1P) if its columns can be permuted such that each row contains one block (no gap then). We call a permutation of the columns of \(M\) that witnesses this property a consecutive-ones ordering of \(M\), and the resulting matrix of such a permutation is consecutive. Testing a binary matrix \(M\) for the C1P can be done in linear time \cite{2,12}. Matrix \(M\) has the C1P if and only if a \(PQ\)-tree \cite{2} can be built for \(M\), moreover, the \(PQ\)-tree stores all consecutive-ones orderings of \(M\). The C1P has also been used in molecular biology, in relation with physical mapping \cite{1} and the reconstruction of ancestral genomes \cite{5} as follows: each column of the matrix represents a genomic marker (sequence) that is believed to have been present (up to small evolutionary changes such
as nucleotide mutations or small rearrangements) and unique in the considered ancestral genome or physical map, and each row of the matrix represents a set of markers that are believed to have been contiguous along an ancestral chromosome, and the goal is to find one (or several if possible) total orders on the markers that respect all rows (i.e., that keep all entries 1 consecutive in each row). See [5] for a comprehensive introduction to this problem. However, a common problem in such applications is that matrices obtained from experiments do not have the C1P [9].

Handling a matrix $M$ that does not have the C1P has been approached using different points of view. A first general approach consists of transforming $M$ into a matrix that has the C1P, while minimizing the modifications to $M$; such modifications can involve either in removing rows, or columns, or both, or in flipping some entries from 0 to 1 or 1 to 0. In all cases, the corresponding optimization problems have been proven NP-hard [8,11]. A second approach consists of relaxing the condition of consecutivity of the ones of each row, by allowing gaps, with some restriction to these gaps. The question is then to decide if there is an ordering of the columns of $M$ that satisfies these relaxed C1P conditions. As far as we know, the only restriction that has been considered is the number of gaps, either per row or in $M$. In [9], the authors introduced the notion of the $k$-consecutive-ones property ($k$-C1P). A binary matrix $M$ has the $k$-C1P when its set of columns can be permuted such that each row contains at most $k$ blocks. They call a permutation of the columns of $M$ that witnesses this property a $k$-consecutive-ones ordering of $M$, and the resulting matrix of such a permutation is $k$-consecutive.

In [9], the authors show that deciding if a binary matrix $M$ has the $k$-C1P is NP-complete, even if $k = 2$. Also, finding an ordering of the columns that minimizes the number of gaps in $M$ is NP-complete even if each row of $M$ has at most two ones [10].

In the present work, we follow the second approach, motivated by the problem of reconstructing ancestral genomes using max-gap clusters [5]: the restrictions to the allowed gaps are that both the number of gaps per row and the size of each gap are bounded. Formally, let $k$ and $\delta$ be two integers. A binary matrix $M$ is said to have the $(k, \delta)$-Consecutive-Ones Property, denoted by $(k, \delta)$-C1P, if its columns can be permuted such that each row contains at most $k$ blocks and no
gap larger than $\delta$. Here, we call a permutation of the columns of $M$ that witnesses this property a \((k, \delta)\)-consecutive-ones ordering of $M$, and the resulting matrix of such a permutation is \((k, \delta)\)-consecutive. In [6], we introduced this problem and gave preliminary complexity and algorithmic results. In particular we showed that the \((2, \delta)\)-C1P Problem is NP-complete for all $\delta \geq 2$ and that the \((3, 1)\)-C1P problem is NP-complete. In the present work, we settle the complexity for all possible values of $k$ and $\delta$: we show that testing for the \((k, \delta)\)-C1P is NP-complete for every $k \geq 2, \delta \geq 1$, $(k, \delta) \neq (2, 1)$. This leaves only one case open: the \((2, 1)\)-C1P Problem. Note that from an application point of view (i.e., paleogenomics and the reconstruction of ancestral genomes), answering the \((k, \delta)\)-C1P Problem for small values of both $k$ and $\delta$ is very relevant. Indeed, in most cases, it is errors in computing the initial matrix $M$ that makes it not have the C1P: these errors correspond to small gaps in some rows of this matrix. These errors are due to small overlapping genome rearrangements or mistakes in identifying proper ancestral genomic markers.

In Section 2, we introduce notations related to the gapped-C1P problem. Then, in Section 3, we state and prove our two main results. The main point in our proofs is a more general result that states that, given an arbitrary binary matrix, one can add a relatively small number of additional rows to the matrix such that the order of a chosen subset of columns must be fixed if some gaps conditions among these columns are to be respected. We believe this result can have applications in other problems related to the C1P. Finally, we conclude with some open problems and perspectives.

## 2 Notation and Conventions

First, we introduce some notation and conventions that we use in the following. We have the binary matrix $M$ on the set \(\{1, \ldots, N\}\) of columns. In the constructions used to show NP-completeness, we will divide columns of the matrix into ordered sequences of blocks $b_1, \ldots, b_m$ by designing rows enforcing the columns of each block to appear consecutive and the blocks to appear in the order $b_1, \ldots, b_m$ (or in the reversed order), i.e., for any $i < j$, column $c \in b_i$ and $d \in b_j$, $c$ appears before $d$ in any \((k, \delta)\)-consecutive ordering of $M$. 


for any $k \geq 2, \delta \geq 1$. Furthermore, the columns of a block $b_i$ will be denoted $b_i^1, \ldots, b_i^{[b_i]}$. To specify a row in the matrix $M$, we use the convention of only listing in the square brackets, the columns that contain 1 in this row. For example, $[1, 5, 8]$ represents a row with ones in columns 1, 5 and 8, and zeroes everywhere else. We will also use blocks to specify columns in the block, for example, if $b_1 = \{1, 2, 3\}$, then $[b_1, 5]$ would mean $[1, 2, 3, 5]$ and $[b_1 \setminus \{b_2^2\}, 4, 5]$ would mean $[1, 3, 4, 5]$.

Given a column $i$ in matrix $M$ and an integer $d \geq 0$, the set of columns $N_d(i) = \{i - d, \ldots, i - 1, i + 1, \ldots, i + d\}$ of $M$ is called the $d$-neighborhood of $i$.

3 Results

First, we have the following important property of matrices which have the $(k, \delta)$-C1P, for every $k \geq 2, \delta \geq 1$.

**Theorem 1.** For all $k \geq 2, \delta \geq 1$ and $n \geq 2\delta + 3$, given matrix $M$ on $N \geq n$ columns, $n(\delta + 1) - \frac{\delta(\delta + 3)}{2} - 1$ rows can be added to $M$ to force $n$ selected columns to appear consecutive and in fixed order (or the reverse order) in any $(k, \delta)$-consecutive ordering of $M$.

**Proof.** Given that $1, \ldots, N$ are the columns of $M$, let $C = \{i + 1, i + 2, \ldots, i + n\}$, for some $i \leq N - n$ be the subset of $n$ columns that we want to force to appear consecutive and in this order (or the reverse order) in any $(k, \delta)$-consecutive ordering of $M$ for any $k \geq 2, \delta \geq 1$. Throughout the proof, when the context is clear that we are referring only to the elements of $C$, we denote $C = \{1, \ldots, n\}$, and index its elements accordingly.

We add the rows $[i, j]$ to $M$, for any $1 \leq i < j \leq n$ such that $|i - j| \leq \delta + 1$. This amounts to adding $(n - (\delta + 1))(\delta + 1) + \delta + (\delta - 1) + \cdots + 2 + 1 = (n - (\delta + 1))(\delta + 1) + \frac{\delta(\delta + 1)}{2} = n\delta + n - \frac{\delta^2}{2} - \frac{3\delta}{2} - 1 = n(\delta + 1) - \frac{\delta(\delta + 3)}{2} - 1$ rows to $M$. We now show that the columns in $C$ appear in the sequence $1, \ldots, n$, or $n, \ldots, 1$ in any $(k, \delta)$-consecutive ordering of $M$. If we represent any $(k, \delta)$-consecutive ordering of $M$ by a permutation $\pi$ of the columns of $M$, i.e., $\pi(i)$ is the $i$-th column in the permuted matrix, $\pi(M)$ is the entire permuted matrix, then we have the following claim.
Claim. For any \( \pi(i), \pi(j) \in C \), if \(|\pi(i) - \pi(j)| \leq \delta + 1 \) then \(|i - j| \leq \delta + 1 \).

Proof. If \( 1 \leq \pi(i), \pi(j) \leq n \) and \(|\pi(i) - \pi(j)| \leq \delta + 1 \) then \( M \) contains a row \([\pi(i), \pi(j)]\). Hence, in the permuted matrix, \( \pi(M) \), we have a row \([i, j]\). Since \( \pi(M) \) is a \((k, \delta)\)-consecutive ordering of \( M \), there can be at most \( \delta \) zeros between columns \( i \) and \( j \) in \( \pi(M) \), and hence \(|i - j| \geq \delta + 1 \).

Note that another way of stating this claim is: For any \( \pi(i), \pi(j) \in C \), if \( \pi(j) \in N_{\delta + 1}(\pi(i)) \) then \( j \in N_{\delta + 1}(i) \).

Next, we will show that the columns in \( C \) have to appear consecutive in any \((k, \delta)\)-consecutive ordering of \( M \). Let \( i_{\min}, i_{\max} \) be the first (last) column in \( \pi(M) \) containing a column in \( C \), i.e., \( i_{\min} = \min_{c \in C} \pi^{-1}(c) \) and \( i_{\max} = \max_{c \in C} \pi^{-1}(c) \). Then this consecutiveness property can be expressed as follows.

Claim. We have that \( i_{\max} - i_{\min} = m - 1 \).

Proof. Consider an \( i \in M \) such that \( \pi(i) \) is in the middle part of \( C \), in \( C_{\text{MID}} = \{\delta + 1, \ldots, m - \delta - 1\} \neq \emptyset \) \((C_{\text{MID}} \neq \emptyset \) since \( n \geq 2\delta + 3 \). Obviously, \( i_{\min} \leq i \leq i_{\max} \). Then, for every \( d \in N_{\delta + 1}(\pi(i)) \), \( d \in C \), and hence, \( i_{\min} \leq \pi^{-1}(d) \leq i_{\max} \), and by the first claim, also \( \pi^{-1}(d) \in N_{\delta + 1}(i) \). Since permutation \( \pi \) is a one-to-one mapping from the set \( M \) to itself, and \(|N_{\delta + 1}(\pi(i))| = 2\delta + 2 \) \((|N_{\delta + 1}(i)| = 2\delta + 2 \), it follows that for each \( j \) such that \( j \in N_{\delta + 1}(i) \), there is a \( d \in N_{\delta + 1}(\pi(i)) \subseteq C \) such that \( \pi(j) = d \). Hence, for every \( i \) such that \( \pi(i) \in C_{\text{MID}} \), we have that for every \( j \in N_{\delta + 1}(i) \), \( \pi(j) \in C \). Consequently, for every such an \( i, i \in I = \{i_{\min} + \delta + 1, \ldots, i_{\max} - \delta - 1\} \).

Let \( i_1, i_2 \in I \). Let \( C_{\text{BOR}} = C \setminus C_{\text{MID}} \). Since, for all \( j \in N_{\delta + 1}(i_1) \cup N_{\delta + 1}(i_2), \pi(j) \in C \), we have that \( \pi(i_1 - \delta - 1), \ldots, \pi(i_1 - 1), \pi(i_2 + 1), \ldots, \pi(i_2 + \delta + 1) \in C_{\text{BOR}} \). Note that these \( 2\delta + 2 \) elements in \( C_{\text{BOR}} \) are distinct, even if \( i_1 = i_2 \), the case that arises when \( n = 2\delta + 3 \). By the definitions of \( C_{\text{MID}}, i_{\min} \) and \( i_{\max} \), it follows that \( \pi(i_{\min}) \) and \( \pi(i_{\max}) \) are also in \( C_{\text{BOR}} \). Hence, if either \( i_1 > i_{\min} + \delta + 1 \) or \( i_2 < i_{\max} - \delta - 1 \), then we have at least \( 2\delta + 3 \) distinct values from \( C_{\text{BOR}} \), which is a contradiction, since by the fact that \( n \geq 2\delta + 3 \), and by the definition of \( C_{\text{MID}}, |C_{\text{BOR}}| = \).
2\delta + 2. Therefore, \( i_1 = i_{\min} + \delta + 1, \ i_2 = i_{\max} - \delta - 1 \), and for all \( i \in \{i_{\min}, \ldots, i_{\max}\} \setminus I \), \( \pi(i) \in C_{\text{BOR}} \). Thus for all \( i \in I \), either \( \pi(i) \in C_{\text{MID}} \) or \( \pi(i) \not\in C \).

If there is no \( i \in \{i_{\min}, \ldots, i_{\max}\} \) such that \( \pi(i) \not\in C \), then all the elements in \( \pi(i_{\min}), \ldots, \pi(i_{\max}) \) are in \( C \), and the claim follows. Assume there is an \( i \) such that \( i \not\in C \), and let \( i_0 \) be the smallest such \( i \). Since, for all \( i \in \{i_{\min}, \ldots, i_{\max}\} \setminus I \), \( \pi(i) \in C_{\text{BOR}} \subseteq C \), it follows that that \( i_0 \in I \), where \( i_0 \neq i_1 \), by the definition of \( i_1 \). Therefore, \( i_0 > i_1 = i_{\min} + \delta + 1 \), and hence, \( \pi(i_0 - 1) \in C_{\text{MID}} \). Since \( i_0 \in N_{\delta+1}(i_0 - 1) \), it follows that \( i_0 \) must also be in \( C \), contradicting this assumption, thus the claim follows.

Now, by the previous claim, we have that the set of columns \( C \subseteq M \) are consecutive in any \((k, \delta)\)-consecutive ordering of \( M \). Given this, and the fact that any column of \( M \setminus C \) is zero in any of these rows added to \( M \) to force the columns of \( C \) to be consecutive, this set of rows is \((k, \delta)\)-consecutive for any permutation of the columns of \( M \), provided only that the columns \( C \) are consecutive somewhere in this ordering of \( M \). Hence, to prove the theorem, it is sufficient to show that in the case that \( M = C = \{1, \ldots, n\} \), the columns of \( \pi(M) \) are ordered either in increasing or decreasing order in any \((k, \delta)\)-consecutive ordering of \( M \).

We will proceed by induction on \( n \). We need the following claim.

**Claim.** If \( M = C \), then either for all \( i \in \{1, \ldots, \delta + 1, n - \delta, \ldots, n\} \), \( \pi(i) = i \) or for all \( i \in \{1, \ldots, \delta + 1, n - \delta, \ldots, n\} \), \( \pi(i) = n - i + 1 \).

**Proof.** We will show the claim by induction on \( i \). In the base case, we need to show that \( \{\pi(1), \pi(n)\} = \{1, n\} \). Assume that both \( \pi(1) \) and \( \pi(n) \not\in \{1, n\} \). Then the set \( N_{\delta+1}(\pi(1)) \cap M \) has more that \( \delta + 1 \) elements. By the first claim, for every \( d \in N_{\delta+1}(\pi(1)) \cap M \), \( \pi^{-1}(d) \in N_{\delta+1}(1) \cap M \). Since \( \pi \) is a one-to-one mapping from the set \( M \) to itself, and \( |N_{\delta+1}(\pi(1)) \cap M| > \delta + 1 \), then this implies that \( |N_{\delta+1}(1) \cap M| > \delta + 1 \). This is a contradiction, because \( |N_{\delta+1}(1)| = \delta + 1 \). Hence, either \( \pi(1) = 1 \) or \( \pi(1) = n \), and similarly, \( \pi(n) = 1 \) or \( \pi(n) = n \). Without loss of generality, we can assume that \( \pi(1) = 1 \) and \( \pi(n) = n \), and show by induction that the columns in \( \pi(M) \) are ordered in increasing order.
For the inductive step, consider an \( i \leq \delta + 1 \) and assume that 
\( \pi(j) = j \) for every \( j \in \{1, \ldots, i-1, n-i+2, \ldots, n\} \). By the induction hypothesis, \( \pi(i) \in \{i, \ldots, n-i+1\} \). Assume that \( \pi(i) > i \) and \( \pi(i) < n-i+1 \). Then the set \( N_{\delta+1}(\pi(i)) \cap M \) has more than \( \delta + i \) elements. Again, by the first claim, and the fact that \( \pi \) is a one-to-one mapping, this implies that \( |N_{\delta+1}(\pi(i)) \cap M| > \delta + i \), a contradiction.

Hence, either \( \pi(i) = i \) or \( \pi(i) = n-i+1 \). Assume that \( \pi(i) = n-i+1 \). By the induction hypothesis, \( \pi(n) = n \). Obviously, then \( |\pi(n) - \pi(i)| = |n - (n-i+1)| = i-1 \leq \delta + 1 \), and hence, by the first claim, \( |n-i| \leq \delta + 1 \). Since \( n \geq 2\delta + 3 \), and \( i \leq \delta + 1 \), then \( |n-i| = n-i \geq 2\delta + 3 - (\delta + 1) = \delta + 2 \), which is a contradiction. Thus, \( \pi(i) = i \), and similarly, \( \pi(n-i+1) = n-i+1 \).

We now proceed by induction on \( n \), to prove the theorem. For the base case, assume that \( n = 2\delta + 3 \). By the last claim, for every \( i \in M \setminus \{\delta + 2\} \), \( \pi(i) = i \) (\( \pi(i) = n-i+1 \), respectively). It then follows, by the fact that \( \pi \) is a one-to-one mapping from the set \( M \) to itself, that \( \pi(\delta+2) = \delta + 2 \).

Now, for induction, assume that \( n > 2\delta + 3 \). Since \( \delta \geq 1 \), by the last claim, either \( \pi(1) = 1 \), \( \pi(2) = 2 \) or \( \pi(1) = m \), \( \pi(2) = m-1 \). Without loss of generality, assume that \( \pi(1) = 1 \) and \( \pi(2) = 2 \).

Consider \( M' \), the matrix that results from the removal of column 1 from \( M \), and all rows \([1, i]\), for \( i = 2, \ldots, n \), from this set of rows we add to \( M \). By the induction hypothesis, \( M' \) is \((k, \delta)\)-consecutive, \( k \geq 2, \delta \geq 1 \), only for the orders \( \{2, \ldots, n\} \) and \( \{n, \ldots, 2\} \) of the columns of \( M' \). So if the columns \( M \setminus \{1\} \) are ordered \( \{2, \ldots, n\} \), since \( \pi(1) = 1 \), then the theorem holds. Otherwise, the columns \( M \setminus \{1\} \) are ordered \( \{n, \ldots, 2\} \), and thus \( \pi(2) = m \), which is a contradiction. Thus the theorem holds.

We now use this Theorem 1 to construct a reduction from 3SAT to the problem of testing for the \((k, \delta)\)-C1P to show that this problem is NP-complete for every \( k, \delta \geq 2 \).

**Theorem 2.** Testing for the \((k, \delta)\)-C1P is NP-complete for every \( k, \delta \geq 2 \).

**Proof.** Let \( \phi \) be a 3CNF formula over the \( n \) variables \( \{v_1, \ldots, v_n\} \), with \( m \) clauses \( \{C_1, \ldots, C_m\} \). We construct a matrix \( M_\phi \) with \( 2n +
$d + 5m$ columns and $n + 6m + 2d - 3$ rows, where $d = \max\{2k, 5\}$, such that $M_\phi$ has the $(k, \delta)$-C1P iff $\phi$ is satisfiable for $k, \delta \geq 2$.

In [9], the authors show that, given a 3CNF formula $\phi$, they can construct a matrix $M_\phi$ that has the $k$-C1P iff $\phi$ is satisfiable for $k \geq 2$. Our construction is very similar to this, with the extra condition that $M_\phi$ cannot have any gap larger than $\delta$.

To achieve this, we first force a subset of the columns of $M_\phi$ to be consecutive and in fixed order in any $(k, 1)$-consecutive ordering of $M_\phi$, and then we will build off of this, a construction similar to that of [9]. In particular, we impose this order on the subset $\{2n + 1, \ldots, 2n + d\}$ of the columns $\{1, \ldots, 2n + d + 5m\}$ of $M_\phi$ by adding the $d(\delta + 1) - \frac{3(\delta + 3)}{2} - 1 = 2d - 3$ rows $[i, j]$ to $M_\phi$, for any $2n + 1 \leq i < j \leq 2n + d$ such that $|i - j| \leq \delta + 1$. By Theorem 1, these $d$ columns must be in fixed order (or the reverse). We can assume the former without loss of generality.

Now we associate variable $v_i$ with block $b_i = \{2i - 1, 2i\}$, for $i = 1, \ldots, n$, imposing the same restrictions on these columns as in [9]. So for each $b_i$, we add the row $[b_i, b_{i+1}, \ldots, b_n, 2n + 1, 2n + 3, \ldots, 2n + 2k - 3, 2n + 2k - 1]$ to $M_\phi$.

Next we associate clause $C_j$ with block $B_j = \{2n + d + 5j - 4, \ldots, 2n + d + 5j\}$, for $j = 1, \ldots, m$, and add the row $[2n + d - 2k + 2, 2n + d - 2k + 4, \ldots, 2n + d - 4, 2n + d - 2, 2n + d, B_1, B_2, \ldots, B_j]$ to $M_\phi$.

Now the columns of every $(k, \delta)$-consecutive ordering of the matrix $M_\phi$ are ordered: the blocks $b_1, \ldots, b_n$, followed by the $d$ columns $2n + 1, \ldots, 2n + d$ that remain consecutive and in order, followed by blocks $B_1, \ldots, B_m$. We now add the same rows to $M_\phi$ as in [9] to associate each clause to its three variables to properly simulate 3SAT, only that within the segment of $d$ columns $2n + 1, \ldots, 2n + d$, each row takes value $[2n + 2k - 5, 2n + 2k - 3, 2n + 2k - 2, \ldots, 2n + d]$. The idea is that this segment of $d$ columns enforces $k - 2$ gaps, while each gap is of size 1.

Finally, we slightly modify the construction in the proof of Theorem 2 to show that testing for the $(k, 1)$-C1P is NP-complete for every $k \geq 3$ by reduction from 3SAT.

**Theorem 3.** Testing for the $(k, 1)$-C1P is NP-complete for every $k \geq 3$. 

Proof. Let $\phi$ be a 3CNF formula over the $n$ variables $\{v_1, \ldots, v_n\}$, with $m$ clauses $\{C_1, \ldots, C_m\}$. We construct a matrix $M_\phi$ with $2n + d + 4m$ columns and $n + 4m + 2d - 3$ rows, where $d = \{2k, 5\}$, such that $M_\phi$ has the $(k, 1)$-C1P iff $\phi$ is satisfiable for $k \geq 3$. We do this as follows.

We again associate columns $1, \ldots, 2n$ with the variables of $\phi$, and again use Theorem 1 to force the subset $\{2n + 1, \ldots, 2n + d\}$ of the columns $\{1, \ldots, 2n + d + 4m\}$ of $M_\phi$ to appear consecutive and in fixed order in any $(k, 1)$-consecutive ordering of $M_\phi$ for $k \geq 2$.

We associate each clause $C_j \in \{C_1, \ldots, C_m\}$, with block $B_j = \{2n + d + 4j - 4, \ldots, 2n + d + 4j\}$. Now, we need to introduce only three more rows to associate the clauses to their variables to properly simulate 3SAT. Suppose that clause $C_j$ contains the literal $v_\alpha$. As such, we add the row $[2\alpha, 2\alpha + 1, \ldots, 2n + 1, 2n + 3, 2n + 5, \ldots, 2n + 2k - 5, 2n + 2k - 3, 2n + 2k - 2, 2n + d, B_j^1, B_j^2, B_j^3]$ to $M_\phi$. If $v_\alpha$ is false, this forces $B_j^1$ and $B_j^2$ to be among the first three columns of block $B_j$ in any $(k, 1)$-consecutive ordering of $M_\phi$ for $k \geq 3$. Note that any other ordering of the columns of $B_j$ would introduce either a gap of size 2, or a $k$-th gap in this row. If another literal in $C_j$ is $v_\beta$, we add the row $[2\beta, 2\beta + 1, \ldots, 2n + 1, 2n + 3, 2n + 5, \ldots, 2n + 2k - 5, 2n + 2k - 3, 2n + 2k - 2, 2n + d, B_j^1, B_j^2, B_j^3]$ to $M_\phi$. If $v_\beta$ is false, this forces $B_j^1$ and $B_j^3$ to be among the first three columns of block $B_j$ in any $(k, 1)$-consecutive ordering of $M_\phi$ for $k \geq 3$. If $v_\gamma$ is the third literal of $C_j$, we add the row $[2\gamma, 2\gamma + 1, \ldots, 2n + 1, 2n + 3, 2n + 5, \ldots, 2n + 2k - 5, 2n + 2k - 3, 2n + 2k - 2, 2n + d, B_j^1, B_j^3, B_j^4]$ to $M_\phi$. If $v_\gamma$ is false, this forces $B_j^1$ and $B_j^4$ to be among the first three columns of block $B_j$ in any $(k, 1)$-consecutive ordering of $M_\phi$ for $k \geq 3$. Finally, since $B_j^3, B_j^2, B_j^3, B_j^4$ cannot simultaneously be among the first three columns of block $B_j$, we have that not all three literals can be false in any $(k, 1)$-consecutive ordering of $M_\phi$ for $k \geq 3$. It is easy to show, that if any literal in $C_j$ is true, then there is some $(k, \delta)$-consecutive ordering of the rows involving block $B_j$.

4 Conclusion

While this work improves on the most interesting open question given in [5], there still remain several open questions. The remaining open question that is most interesting now is the complexity of
deciding the \((2,1)\)-C1P for a binary matrix \(M\). Since the two NP-completeness constructions presented here force either a gap of size two, or at least two gaps of size one in any legal configuration of \(M\), if testing for the \((2,1)\)-C1P is NP-complete, it would certainly require a different type of construction.

Deciding the \(k\)-C1P, for \(k \geq 2\) has been proven NP-complete in [9], and we have shown that deciding the \((k, \delta)\)-C1P is NP-complete for \(k \geq 2, \delta \geq 1, (k, \delta) \neq (2,1)\). However, the complexity of deciding the gapped C1P when only \(\delta\) is fixed (we call this the \((*, \delta)\)-C1P) is still an interesting open question. We have a preliminary proof that deciding the \((*, \delta)\)-C1P is NP-complete for all \(\delta \geq 1\), by reducing from the version of 3SAT where each variable appears at most twice positively and once negatively.

Another natural problem is the \((k, \delta)\)-C1P Problem considered here, but with a third parameter added, namely the maximum number of entries 1 that can be present in a row of \(M\), called the degree of \(M\). This problem is motivated by the fact that in the framework described in [5], it is possible to constrain matrices used to reconstruct ancestral genomes to have a small degree. Note that with matrices of degree 2, the number of gaps can be at most 1, and the \((2, \delta)\)-C1P problem is then equivalent to the problem of deciding if the graph whose incidence matrix is \(M\) has bandwidth at most \((\delta + 1)\). For \(\delta = 1\), the graph bandwidth problem can be solved in linear time [3], while in [14] a dynamic programming algorithm with time and space complexity exponential in \(\delta\) was described. We adapted in [6] this algorithm for testing the \((k, \delta)\)-C1P for matrices of small degree, but the exponential space complexity makes it difficult to use in practice on matrices with degree greater than 3. However, deciding the \((k, \delta)\)-C1P for small values of \(k\) and \(\delta\) may become tractable if the degree of the matrix is bounded as well. The design of efficient algorithms, both in time and space, for deciding the gapped consecutive-ones property is a promising research avenue, with immediate applications in genomics.

Adding the degree of the matrix as a third parameter (we call it \(d\) here) to the problem of deciding the \((k, \delta)\)-C1P to give the new problem of deciding the \((d, k, \delta)\)-C1P then introduces more interesting open questions from a complexity theory perspective. We know that deciding the \((d, k, \delta)\)-C1P is polynomial-time solvable by the
above algorithm, and in fact, this problem where $k$ is unbounded is just the $(d, d, \delta)$-C1P, because $k \leq d$. The complexity of deciding this property when $\delta$ is unbounded, namely the $(d, k, *)$-C1P is still open. We have a preliminary proof that deciding the $(d, k, *)$-C1P, for all $d \geq 4, k \geq 3$ is NP-Complete, by a reduction from 3SAT, leaving open the complexity of deciding the $(4, 2, *)$-C1P and the $(3, 2, *)$-C1P. While this implies that this problem is intractible in general, in practice, $\delta$ and $d$ are quite small, so the design of efficient algorithms for these cases can still be a fruitful avenue of research.

From a purely combinatorial point of view, there has been a renewed interest in the characterization of non-C1P matrices in terms of forbidden submatrices introduced by Tucker [15]. It has recently been shown that this characterization could be used in the design of algorithms related to the C1P [7, 4]. The question there is the following: is there a nice characterization of non $(k, \delta)$-C1P matrices in terms of forbidden matrices?

Finally it is also natural to ask if there exists a structure that can represent all orderings that satisfy some gaps conditions related to the consecutive-ones property. Such a structure exists for the ungapped C1P: for a matrix that has the C1P, its PQ-tree represents all its valid consecutive orderings, and it can be computed in linear time [12]. This notion has even been extended to matrices that do not have the C1P through the notion of PQR-tree [13, 12]. Although the existence of such a structure with nice algorithmic properties is ruled out by the hardness of deciding the gapped C1P, it remains open to find classes of matrices such that deciding the gapped C1P is tractable, and in such case, to represent all possible orderings in a compact structure. Here again, this question is motivated both by theoretical considerations (for example representing all possible layouts of a graph of bandwidth 2), but also by computational genomics problems [5].

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