SPECIAL RELATIVISTIC SIMULATIONS OF MAGNETICALLY DOMINATED JETS IN COLLAPSING MASSIVE STARS

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ABSTRACT

We perform a series of two-dimensional magnetohydrodynamic core-collapse simulations of rapidly rotating and strongly magnetized massive stars. To study the properties of magnetic explosions for a longer time stretch of postbounce evolution, we develop a new code under the framework of special relativity including a realistic equation of state with a multiflavor neutrino leakage scheme. Our results show the generation of the magnetically dominated jets in the two ways. One is launched just after the core bounce in a prompt way and another is launched at ∼ 100 ms after the stall of the prompt shock. We find that the shock-revival occurs when the magnetic pressure becomes strong enough, due to the field wrapping, to overwhelm the ram pressure of the accreting matter. The critical toroidal magnetic fields for the magnetic shock-revival are found to be universal of ∼ 10^{15} G behind the jets. We point out that the time difference before the shock-revival has a strong correlation with the explosion energies. Our results suggest that the magnetically dominated jets are accompanied by the formation of the magnetars. Since the jets are mildly relativistic, we speculate that they might be the origin of some observed X-ray flashes.

Key words: gamma rays: bursts – methods: numerical – MHD – pulsars: general – relativity – supernovae: general

Online-only material: color figures

1. INTRODUCTION

There has been growing evidence shedding light on the relations between the high-energy astrophysical phenomena and their origins. A number of host galaxies of long-duration gamma-ray bursts (GRBs) are recently identified as metal-poor galaxies whose metallicities are lower than that of average massive star-forming galaxies (Savaglio et al. 2006; Stanek et al. 2006, and reference therein). The preponderance of short-lived massive star formation in such young galaxies, as well as the identification of SN Ib/c light curves peaking after the bursts in a few cases, has provided strong support for a massive stellar collapse origin of the long GRBs (Paczynski 1998; Galama et al. 1998; Stanek et al. 2003). The duration of the long GRBs may correspond to the accretion of debris falling into the central black hole (BH; Piro et al. 1998), which suggests the observational consequence of the BH formation as well as the supernova of neutron star formation. There is also growing observational evidence of supermagnetized neutron stars with the magnetic fields of ∼ 10^{14} – 10^{15} G, the so-called magnetars (Duncan & Thompson 1992; see Lattimer & Prakash 2007 for a recent review). The magnetic fields are determined by the measured period and derivative of period under the assumption that the spin-down is caused due to the usual magnetic dipole radiation (Zhang & Harding 2000; Harding & Lai 2006). Tentative detections of spectral features during the burst phase also indicate B ∼ 10^{15} G when interpreted as proton cyclotron lines (Gavriil et al. 2002; Ibrahim et al. 2003; Rea et al. 2003). Recently X-ray flash (XRF), which is a low-energy analog of the GRB, is receiving great attention for its possible relevance to the magnetar formations (Mazzali et al. 2006; Toma et al. 2007). A large amount of neutron-rich Ni ejected by SN2006aj associated with XRF060218 is interpreted to be the formation of such objects, not the BH after the explosion (Maeda et al. 2007a). So far a number of numerical simulations have been done in an effort to further understanding of the formation mechanisms of these compact objects such as neutron stars, magnetars, and the BHs in combination with their possible consequences like GRBs and XRFs. The leading model for the long-duration GRBs is the collapsar model (MacFadyen & Woosley 1999). In the model, the core of massive stars with significant angular momentum collapses into a BH. The neutrinos emitted from the rotation-supported accretion disk around the BH heat the matter of the funnel region of the disk to launch the GRB outflows. The relativistic flows are expected to ultimately form a fireball, which is good for the explanation of the observed afterglow (e.g., Piran 1999). In addition, it is suggested that the strong magnetic fields in the cores of the order of 10^{15} G also play an active role both for driving the magneto-driven jets and for extracting a significant amount of energy from the central engine (e.g., Usov 1992; Wheeler et al. 2000; Thompson et al. 2004; Uzdensky & MacFadyen 2007a, and see references therein).

In order to understand such scenarios, the ultimate necessity of the stellar core-collapse simulations is to perform the simulations tracing all the phases in a consistent manner starting from the stellar core collapse, core bounce, shock-stall, stellar explosion (phase 1) or BH formation and the formation of accretion disk (phase 2), energy deposition to the funnel region by neutrinos and/or magnetic fields (phase 3), to the launching of the fireballs (phase 4). Here, for convenience, we call each stage as phase 1, 2, etc. The requirement for the numerical modeling to this end is highly computationally expensive, namely the multidimensional MHD simulations not only with general relativity for handling the BH formation, but also with the multiangle neutrino transfer for treating highly anisotropic neutrino
We summarize the numerical methods in Section 2. Section 3 is devoted to the initial models. In Section 4, we show the numerical results. In Section 5, we summarize our study and discuss the implications of our model for the magnetars and the X-ray flashes. Details of the numerical scheme and the code tests are given in the appendices.

2. NUMERICAL METHODS

The results presented in this paper are calculated by the newly developed SRMHD code. The novel point of this code is that the detailed microphysical processes relevant for the stellar-core-collapse simulations are also coupled to the MHD. We briefly summarize the numerical methods in the following.

The MHD part of the code is based on the formalism of De Villiers et al. (2003). Before going to the basic equations, we write down the definition of the primary code variables. The state of the relativistic fluid element at each point in the space-time is described by its density, $\rho$; specific energy, $e$; velocity, $v^i$; and pressure, $p$. The magnetic field in the rest frame of the fluid is described by the 4-vector $\sqrt{4\pi}b^\mu = F^{\mu\nu}U_\nu$, where $F^{\mu\nu}$ is the dual of the electromagnetic field strength tensor and $U_\nu$ is the 4-velocity.

After some mathematical procedures presented in Appendix A, the basic equations of SRMHD are described as follows:

\[
\frac{\partial D}{\partial t} + \frac{1}{\sqrt{\gamma}} \partial_i \sqrt{\gamma} D v^i = 0
\]

\[
\frac{\partial E}{\partial t} + \frac{1}{\sqrt{\gamma}} \partial_i \sqrt{\gamma} E v^i = -p \frac{\partial W}{\partial t} - \frac{p}{\sqrt{\gamma}} \partial_i \sqrt{\gamma} W v^i - L_v
\]

\[
\frac{\partial S_i}{\partial t} - b^i b_i + \frac{1}{\sqrt{\gamma}} \partial_j \sqrt{\gamma} (S_i v^j - b_j b^j) = -\frac{1}{2} (\rho h (W v_k^2) - (b^i)^2) \partial_k \gamma^{jk} - (p h W^2 - b^i b^j) \partial_i \Phi - \partial_i \left( p + \frac{\|b^i\|^2}{2} \right)
\]

\[
\frac{\partial B^i}{\partial t} + \partial_j (W v^j b^i - W v^i b^j) = 0
\]

where $W = \frac{1}{\sqrt{1 - v^2}}$, $D = \rho W$, $E = e W$, and $S_i = \rho h W^2 v_i$ are the Lorentz boost factor, auxiliary variables correspond to density, energy, and momentum, respectively. Equations (2)–(4) represent the mass, energy, and momentum conservations. $L_v$ in the right-hand side of Equation (3) is a total neutrino cooling rate determined by microphysical processes which will be explained later. In Equation (4), it is noted that the relativistic enthalpy, $h = (1 + e/\rho + p/\rho + |b|^2/\rho)$ includes magnetic energy. Equation (5) is the induction equation for the magnetic fields. $B^i$ is related to that in the rest frame of fluid as $B^i = W b^i - W b_i b^j$. Here $b_i$ is a time component of the 4-vector, $b_i$. Equation (6) is the Poisson equation for the gravitational potential, $\Phi$. 

radiation from the disks. So various approximative approaches to each phase have been undertaken. As we mention below, these studies are complimentary in the sense that the different epochs are focused on, with the different initial conditions for the numerical modeling being taken.

In addition to the elaborate studies in the conventional supernova context (see recent reviews for Kotake et al. 2006; Janka et al. 2007), much attention has been paid recently to the roles of rapid rotation and magnetic fields for studying the formation of magnetars and its possible application to the collapsars (Yamada & Sawai 2004; Takiwaki et al. 2004; Kotake et al. 2004b; Sawai et al. 2005; Matt et al. 2006; Moiseenko et al. 2006; Obergaulinger et al. 2006; Nishimura et al. 2006; Burrows et al. 2007; Cerdá-Durán et al. 2007; Scheidegger et al. 2007; Komissarov & Barkov 2007). After the failed or weak explosion, the accretion to the central objects may lead to the formation of a BH (phase 2). Several general relativistic (GR) studies are on the line for the understanding of the hydrodynamics at the epoch of the BH formation, in which more massive progenitors ($>25 M_\odot$) than those of the study in phase 1 are generally employed (Shibata et al. 2006; Sekiguchi & Shibata 2007). Treating the BH as an absorbing boundary or using the fixed metric approaches, the numerical studies of phase 3 are concerned with the initiation of the outflows from the funnel region of the disk to the acceleration of the jets as a result of the neutrino heating and/or MHD processes until the jets become mildly relativistic (Koide et al. 1998; MacFadyen & Woosley 1999; Proga et al. 2003; Nishikawa et al. 2005; De Villiers et al. 2005; Kroluk et al. 2005; Hawley & Kroluk 2006; Mizuta et al. 2006; Fujimoto et al. 2006; Uzdensky & MacFadyen 2006; McKinney & Narayan 2007; Komissarov & McKinney 2007; Nagataki et al. 2007; Suwa et al. 2007a, 2007b; Barkov & Komissarov 2008). Numerical studies of phase 4 are mainly concerned with the dynamics later on, namely, the jet propagation to the breakout from the star, when the acceleration of the jets to the high Lorentz factor is expected (Stone & Hardee 2000; Aloy et al. 2000; Zhang et al. 2003; Leismann et al. 2005; Mizuta et al. 2006; McKinney 2006; Mizuno et al. 2007).

Our previous study was devoted to phase 1, in which we performed a series of two-dimensional core-collapse simulations of rotating and magnetized massive stars under the framework of the Newtonian magnetohydrodynamics (MHD; Takiwaki et al. 2004). We found that the magneto-driven jet-like shocks were launched from the protoneutron stars just after core bounce. However, at the moment, we were unable to follow the dynamics much later on until when the collimated jets reach further out from the center. The Alfvén velocity of the jet propagating into the outer layer of the iron core can be estimated by the following simple order-of-magnitude estimation,

\[
v_A = \frac{B}{\sqrt{4\pi \rho}} \sim 10^{10} \text{ cm s}^{-1} \frac{B/10^{13} \text{ G}}{\sqrt{\rho/(10^5 \text{ g cm}^{-3})}},
\]

with $\rho$ and $B$ being the typical density and magnetic field there. It can be readily inferred that the Alfvén velocity can exceed the speed of light unphysically in the Newtonian simulation. To avoid this problem we construct a new code under the framework of special relativistic MHD (SRMHD). We take a wider parametric range for the strength of the rotation than that of our previous work. By so doing, we hope to study more systematically than before how the strong magnetic fields and the rapid rotation affect the properties of the magnetic explosions.

\[
\text{Equation (5) is the induction equation for the magnetic fields.}
\]

\[
\text{where } W = \frac{1}{\sqrt{1 - v^2}}, D = \rho W, E = e W, \text{ and } S_i = \rho h W^2 v_i \text{ are the Lorentz boost factor, auxiliary variables correspond to density, energy, and momentum, respectively. Equations (2)–(4) represent the mass, energy, and momentum conservations. } L_v \text{ in the right-hand side of Equation (3) is a total neutrino cooling rate determined by microphysical processes which will be explained later. In Equation (4), it is noted that the relativistic enthalpy, } h = (1 + e/\rho + p/\rho + |b|^2/\rho) \text{ includes magnetic energy. Equation (5) is the induction equation for the magnetic fields. } B^i \text{ is related to that in the rest frame of fluid as } B^i = W b^i - W b_i b^j. \text{ Here } b_i \text{ is a time component of the 4-vector, } b_i. \text{ Equation (6) is the Poisson equation for the gravitational potential, } \Phi.\]
This newly developed code is an Eulerian code based on the finite-difference method. The numerical approach for solving the basic Equations (2)-(4), consists of the two steps, namely, the transport and the source step. These procedures are essentially the same as those of ZEUS-2D (Stone & Norman 1992). At the transport step, the second-order upwind scheme of Van Leer is implemented (van Leer 1977). To handle the numerical oscillations, we employ an artificial viscosity. In the special relativistic (SR) treatments, many forms for the compression heating are possible (Hawley et al. 1984b). In our code, we employ the form of \( \frac{\partial h}{\partial r} \sqrt{\gamma} W \nu^i \) as the compression heating, which becomes the well known artificial viscosity of von Neumann and Richtmyer (Stone & Norman 1992) under the Newtonian approximation. While not explicitly included in the above expression for the enthalpy, the contribution from the compression heating on the inertia is included in our calculations. The detailed status on the shock capturing using this term is shown in Appendix D.

The time evolution of the magnetic fields is solved by induction equation, Equation (5). In doing so, the code utilizes the so-called constrained transport method, which ensures the divergence free (\( \nabla \cdot \mathbf{B} = 0 \)) of the numerically evolved magnetic fields at all times. Furthermore, the method of characteristics (MOC) is implemented to propagate accurately all modes of MHD waves. The detailed explanation and the numerical tests are delivered in Appendix B. The self-gravity is managed by solving the Poisson equation, Equation (6) with the incomplete Cholesky decomposition conjugate gradient method.

Together with these hydrodynamic procedures, the following microphysical processes are implemented in this code. We approximate the neutrino transport by a multiflavor leakage scheme (Epstein & Pethick 1981; Rosswog & Liebendörfer 2003), in which three neutrino flavors: electron neutrino, \( \nu_\ell \); electron antineutrino, \( \bar{\nu}_\ell \); and the heavy-lepton neutrinos, \( \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau \) (collectively referred to as \( \nu_X \)), are taken into account. The included neutrino reactions are electron capture on proton and free nuclei; positron capture on neutron; photo-, pair, plasma processes (Fuller et al. 1985; Takahashi et al. 1978; Itoh et al. 1989, 1990). We added a transport equation for the lepton fraction \( Y_i \) as

\[
\frac{\partial Y_i}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \gamma} \gamma Y_i = -\gamma_i
\]

to treat their charge due to the relevant charged current reactions, whose reaction rates are collectively represented by \( \gamma_i \) here, with \( Y_e, Y_\nu, Y_\bar{\nu} \) being electron, positron, electron neutrino, and antineutrino neutrino fraction, respectively (see Epstein & Pethick 1981; Rosswog & Liebendörfer 2003; Kotake et al. 2003 for details of the estimation of \( \gamma_i \)). \( L_\nu \) in Equation (3) represents the total neutrino cooling rate which is also estimated by the scheme. As for the equation of state (EOS), we employ a realistic one based on the relativistic mean field theory (Shen et al. 1998). Since the pressure is not represented as the analytic function of density and internal energy as in the case of polytropic EOS, an iterative algorithm is employed to update the fundamental variables (see Appendix C for details).

In our two-dimensional simulations, the spherical coordinate is used with \( 300(r) \times 60(\theta) \) grid points to cover the computational domain. Axial symmetry and reflection symmetry across the equatorial plane are assumed. The radial grid is nonuniform, extending from 0 to \( 5 \times 10^8 \) cm with a finer grid near the center. The finest grid is set to \( 5 \times 10^5 \) cm. The polar grid uniformly covers from \( \theta = 0 \) to \( \theta = \frac{\pi}{4} \). This choice of the grid numbers is sufficient for the aim of this paper as will be discussed in Section 5.

Finally, we summarize the difference on the numerical approach from our previous work (Takiwaki et al. 2004). The most major development is the fully SR treatment on MHD. And for the microphysical parts the cooling terms by neutrino contain contributions from not only \( \nu_e \) but also \( \bar{\nu}_e \) and \( \nu_X \). These advances provide more reliable results on the magnetorotational core collapse.

### 3. INITIAL MODELS

We make precollapse models by taking the profiles of density, internal energy, and electron fraction distribution from a rotating presupernova model of E25 by Heger & Langer (2000). This model has a mass of \( 25 M_\odot \), at the zero-age main sequence (ZAMS), however, it loses the hydrogen envelope and becomes a Wolf–Rayet star of \( 5.45 M_\odot \) before core collapse. Our computational domain involves the whole iron core of \( 1.69 M_\odot \).

It is noted that this model seems to be a good candidate as a progenitor of the GRB since the lack of the line spectra of the ejected envelopes is reconciled with the observations of the supernovae associated with GRBs (e.g., Meszaros 2006).

Since little is known about the spatial distributions of the rotation and the magnetic fields in the evolved massive stars (see, however, Spruit 2002), we add the following rotation and magnetic field profiles in a parametric manner to the nonrotating core mentioned above. For the rotation profile, we assume a cylindrical rotation of

\[
\Omega(X, Z) = \Omega_0 \frac{X_0^2}{X^2 + Z_0^2 + Z^2},
\]

where \( \Omega \) is the angular velocity and \( X \) and \( Z \) denote distance from the rotational axis and the equatorial plane. We adopt values of the parameters, \( X_0 \) and \( Z_0 \), as \( 10^7 \) cm and \( 10^8 \) cm, respectively. The parameter, \( X_0 \) represents the degree of differential rotation. We assume the strong differential rotation as in our previous study (Takiwaki et al. 2004).

As for the initial configuration of the magnetic fields, we assume that the field is nearly uniform and parallel to the rotational axis in the core and dipolar outside. For that purpose, we consider the following effective vector potential,

\[
A_r = A_\theta = 0,
\]

\[
A_\phi = \frac{B_0}{2} \frac{r_0^3}{r^3} r \sin \theta,
\]

where \( A_r, A_\theta, A_\phi \) is the vector potential in the \( r, \theta, \phi \) direction, respectively, \( r_0 \) is the radius of the core, and \( B_0 \) is the model constant. In this study, we adopt the value of \( r_0 \) as \( 2 \times 10^8 \) cm which is approximately the size of the iron core at a precollapse stage. This vector potential can produce the uniform magnetic fields when \( r \) is small compared with \( r_0 \), and the dipole magnetic fields when \( r_0 \) is small compared with \( r \). Since the outer boundary is superposed at \( r = 4 \times 10^8 \) cm, the magnetic fields are almost uniform in the computational domain as noted in the previous work (Takiwaki et al. 2004). It is noted that this is a far better way than the loop current method for constructing the dipole magnetic fields (Symbalisty 1984), because our method produces no divergence of the magnetic fields near the loop current. We set the outflow boundary conditions for the magnetic fields at the outer boundary of the calculated regions.
We compute nine models changing the total angular momentum and the strength of magnetic fields by varying the value of $\Omega_0$ and $B_0$. The model parameters are shown in Table 1. The models are named after this combination, with the first letters, B12, B11, and B10, representing the strength of the initial magnetic field and the following letters, TW4.0, TW1.0, TW0.25, respectively. Here $T/|W|$ indicates the ratio of the rotational energy to the absolute value of the gravitational energy. The corresponding values of $\Omega_0$ are 151 rad s$^{-1}$, 76 rad s$^{-1}$, 38 rad s$^{-1}$ for TW4.0, TW1.0, TW0.25, respectively. It is noted that the value of $\Omega_0$ in Equation (8) are 151 rad s$^{-1}$, 76 rad s$^{-1}$, 38 rad s$^{-1}$ for TW4.0, TW1.0, TW0.25, respectively. The specific angular momenta range from 0 to $5 \times 10^{16}$ cm$^2$ s$^{-1}$, which are in good agreement with the requirement of the collapsar model (MacFadyen & Woosley 1999). Current stellar evolution calculations predict that the rapidly rotating massive stars with metallicity experience the so-called chemically homogeneous cores during their evolution (Yoon & Langer 2005). Such stars are considered to satisfy the requirements of the collapsar model, namely rapid rotation of the core (Woosley & Heger 2006). According to a GRB progenitor model of 350B in Woosley & Heger (2006), the magnetic field strength of the core reaches up to $\sim 10^{12}$ G and the specific angular momentum is in the order of $j_{16} \sim 1$, with which our choices for the initial magnetic field and the initial rotation rate are reconciled.

### Table 1

| $B_0$ (Gauss) | $T/|W|$ (%) |
|---------------|-------------|
| $10^{10}$ G   | B10TW0.25   |
| $10^{11}$ G   | B10TW1.0    |
| $10^{12}$ G   | B10TW4.0    |
| $10^{13}$ G   | B11TW0.25   |
| $10^{14}$ G   | B11TW1.0    |
| $10^{15}$ G   | B11TW4.0    |
| $10^{16}$ G   | B12TW0.25   |
| $10^{17}$ G   | B12TW1.0    |
| $10^{18}$ G   | B12TW4.0    |

Notes. Model names are labeled by the initial strength of magnetic fields and rotation. $T/|W|$ represents the ratio of the rotational energy to the absolute value of the gravitational energy. The corresponding values of $\Omega_0$ in Equation (8) are $151$ rad s$^{-1}$, $76$ rad s$^{-1}$, $38$ rad s$^{-1}$ for TW4.0, TW1.0, TW0.25, respectively. $B_0$ represents the strength of the poloidal magnetic fields (see Equation (10)). The corresponding values of $E_{\text{m}}/|W|$ are $2.5 \times 10^{-8}$, $2.5 \times 10^{-6}$, and $2.5 \times 10^{-4}$ for $10^{10}$ G, $10^{11}$ G, and $10^{12}$ G, respectively with $E_{\text{m}}$ being the magnetic until energy.

### Table 2

| Model Names | $T/|W_{\text{bnc}}|$ | $\rho_{\text{bnc}}$ [10$^{19}$ g cm$^{-3}$] | $W_{\text{bnc}}$ [10$^{55}$ erg] | $E_{\text{m}}/|W_{\text{bnc}}|$ | $E_{\text{m}}/|W_{\text{bnc}}|$ | $E_{\text{m}}/|W_{\text{bnc}}|$ | $A_{\text{amp}}$ | $\rho_{\text{bnc}}$ (ms) |
|-------------|----------------------|------------------------------------------|--------------------------------|----------------------|----------------------|----------------------|---------------|-----------------------|
| B12TW0.25   | 0.10                 | 2.1                                      | 1.1                           | $2.5 \times 10^{-4}$  | $1.0 \times 10^{-3}$ | 0.3                  | 100           | 245                   |
| B11TW0.25   | 0.10                 | 2.1                                      | 1.1                           | $2.5 \times 10^{-6}$  | $1.0 \times 10^{-5}$ | 0.3                  | 100           | 245                   |
| B10TW0.25   | 0.10                 | 2.1                                      | 1.1                           | $2.5 \times 10^{-8}$  | $1.0 \times 10^{-7}$ | 0.3                  | 100           | 245                   |
| B12TW1.0    | 0.18                 | 1.3                                      | 1.1                           | $2.5 \times 10^{-4}$  | $9.0 \times 10^{-3}$ | 0.07                 | 720           | 295                   |
| B11TW1.0    | 0.18                 | 1.3                                      | 1.1                           | $2.5 \times 10^{-6}$  | $7.0 \times 10^{-5}$ | 0.07                 | 610           | 295                   |
| B10TW1.0    | 0.18                 | 1.3                                      | 1.1                           | $2.5 \times 10^{-8}$  | $7.0 \times 10^{-7}$ | 0.07                 | 610           | 295                   |
| B12TW4.0    | 0.20                 | 0.095                                   | 0.68                          | $2.5 \times 10^{-4}$  | $20 \times 10^{-3}$  | 0.3                  | 800           | 477                   |
| B11TW4.0    | 0.19                 | 0.11                                    | 0.74                          | $2.5 \times 10^{-6}$  | $29 \times 10^{-5}$  | 0.1                  | 4400          | 484                   |
| B10TW4.0    | 0.19                 | 0.11                                    | 0.74                          | $2.5 \times 10^{-8}$  | $31 \times 10^{-7}$  | 0.1                  | 4400          | 484                   |

Notes. Characteristic properties before core bounce. $T/|W_{\text{bnc}}|$ is the rotational energy per gravitational energy at bounce. $\rho_{\text{bnc}}$ is the maximum density at bounce. $E_{\text{m}}/|W_{\text{bnc}}|$ and $E_{\text{m}}/|W_{\text{bnc}}|$ are the magnetic energy per the gravitational energy initially and at bounce, respectively. $\frac{E_{\text{m}}}{E_{\text{m}}}$ is the ratio of the poloidal magnetic energy to the total magnetic energy at bounce. $A_{\text{amp}}$ represents the amplification rate of magnetic energy until core bounce, which is defined as $A_{\text{amp}} \overset{\text{df}}{=} (E_{\text{m}}/|W_{\text{bnc}}|)/(E_{\text{m}}/|W_{\text{bnc}}|)$. $\rho_{\text{bnc}}$ represents the time until bounce.

### 4. RESULTS

#### 4.1. Hydrodynamics Before Core Bounce

First, we briefly mention the dynamics before core bounce, when the gross features are rather similar among the computed models. The characteristic properties are summarized in Table 2. The story before core bounce is almost the same as that of the canonical core-collapse supernovae with rapid rotation (see Kotake et al. 2003). The core begins to collapse due to electron captures and the photodissociation of the iron nuclei and eventually experiences the bounce at the subnuclear density by the additional support of the centrifugal forces. In fact, the central densities at bounce become smaller and the epoch bounce is delayed as the initial rotation rates become larger (see $\rho_{\text{bnc}}$ and $t_{\text{bnc}}$ in Table 2).

As the compression proceeds, the rotational energy increases and reaches a few $10^{22}$ erg at the moment of the bounce (see from $T/|W_{\text{bnc}}| \times W_{\text{bnc}}$ in Table 2). Given the same initial rotation rates, the values of $T/|W_{\text{bnc}}|$ do not depend on the initial field strength so much. This means that the angular momentum transfer is negligible before bounce, which is also the case for the Newtonian hydrodynamics (Yamada & Sawai 2004). At bounce, the unshocked core becomes more flattened as the initial rotation rate becomes larger (compare the panels in Figure 1). The central protoneutron stars rotate very rapidly reaching to $\sim 3000$ rad s$^{-1}$ with the typical surface magnetic fields of $\sim 10^{13}$ G to $\sim 10^{15}$ G for B10 and B12 models, respectively. From the table, it is also seen that the amplification rates of the magnetic fields ($A_{\text{amp}}$) are mainly determined by the initial rotational rates. One exception is the model B12TW4.0. Due to very rapid rotation with the highest magnetic fields initially imposed, the model bounces predominantly due to the magnetic force. As a result, the core bounce occurs earlier with the lower central density with less gravitational energy of the inner core than the models with the same initial rotation rate (see Table 2). This earlier magnetically supported bounce leads to the suppression of the amplification rate, which is exceptionally observed for this model.

In this way, the hydrodynamic properties before bounce are mainly governed by the differences of the initial rotation rates. On the other hand, the differences of the magnetic field strength begin to play an important role in the dynamics later on. We will mention them in detail in the next sections.
direction to the equatorial plane at ∼ the bottom left panel, the shock at core bounce stalls in the panel) is remarkable for the right panel. Note that the unit of the horizontal and the vertical axes is in cm.

For later convenience, we call the former and the latter models bounce promptly reaches the surface of the iron core or not.

Prompt MHD Exploding Model

From the top left panel of Figure 5, the plasma shock gradually grow due to the field wrapping, and the plasma shock stalls, the stalled shock begins to oscillate. The middle right panel), the stalled shock near the pole suddenly stalls in all directions at ∼ 10^7 cm. As shown in the right side of the top right panel of Figure 3), this magneto-driven jet does not stall and penetrate to the surface of the iron core, which is essentially the reproduction of the pioneering results in the MHD supernova simulations by LeBlanc & Wilson (1970) and its analysis by Meier et al. (1976). The speed of the head of the jet is mildly relativistic of ∼ 0.3c, with c being the speed of light (the right side of the bottom right panel of Figure 3). At 20 ms after bounce, the jet finally reaches the surface of the iron core of ∼ 10^8 cm. At this moment, the explosion energy, which will be a useful quantity for comparing the strength of the explosion among the models later, reaches 1.4 × 10^{50} erg.

Delayed MHD Exploding Model

The speed of the head of the jet is mildly relativistic of ∼ 0.3c, with c being the speed of light (the right side of the bottom right panel of Figure 3). At 20 ms after bounce, the jet finally reaches the surface of the iron core of ∼ 10^8 cm. At this moment, the explosion energy, which will be a useful quantity for comparing the strength of the explosion among the models later, reaches 1.4 × 10^{50} erg.

Delayed MHD Exploding Model

The models with weaker initial magnetic fields belong to the delayed MHD exploding model (see Figure 2). In the following, we explain their properties taking model B10TW1.0 as an example. It is noted again that this model has the same initial rotation rate as that of model B12TW1.0 of the previous section, but with 2 orders of magnitudes weaker initial magnetic fields.

In the case of model B10TW1.0, the shock wave at bounce stalls in all directions at ∼ 1.5 × 10^7 cm. As shown in the top left panel of Figure 5, the plasma β is so high that the magnetic fields play no important role before bounce. After the shock stalls, the stalled shock begins to oscillate. The middle left and the bottom left panels show the prolate and oblate phases, respectively, during the oscillations. Until ∼ 70 ms after bounce, the oscillation of the shock front continues diminishing its amplitude. The number of the oscillations is about 5 times in this duration. Without the magnetic fields, the oscillation should cease settling into the equilibrium state with the constant accretion through the stalled-shock to the center. However, during this oscillation, the magnetic fields behind the stalled shock smoothly grow due to the field wrapping, and the plasma β around the polar regions becomes low as seen from the right side of the top right panel. Soon after the toroidal magnetic fields become as high as ∼ 10^{15} G behind the stalled shock (see the middle right panel), the stalled shock near the pole suddenly

4.2. Prompt Versus Delayed MHD Exploding Model

After bounce, we can categorize the computed models into two groups, by the criterion of whether the shock generated at bounce promptly reaches the surface of the iron core or not. For later convenience, we call the former and the latter models the prompt and delayed MHD exploding model, respectively, throughout the paper. The models and the corresponding groups are shown in Figure 2. To begin with, we choose a typical model from the two groups and mention their properties in detail.

Prompt MHD Exploding Model

The models into the prompt (red blocks) or delayed (green blocks) MHD exploding model by the difference of t_{1000 km} shown in this table, which is the shock-arrival time to the radius of 1000 km after bounce.

(A color version of this figure is available in the online journal.)

| $B_0$ (Gauss) | 10^{10}G | 10^{11}G | 10^{12}G |
|---------------|----------|----------|----------|
| 122 ms        | 72 ms    | 32 ms    |
| 96 ms         | 27 ms    | 20 ms    |
| 104 ms        | 32 ms    | 25 ms    |

(A color version of this figure is available in the online journal.)

Figure 2. Classification of the computed models into the prompt (red blocks) or delayed (green blocks) MHD exploding model by the difference of t_{1000 km} shown in this table, which is the shock-arrival time to the radius of 1000 km after bounce.

(A color version of this figure is available in the online journal.)

The magnetic fields for the prompt MHD models are strong enough to power the jet already at the epoch of bounce. This is clearly shown in the top left panel, showing that the “plasma β” $\beta \equiv p / \rho / c^2$, being the ratio of the matter to the magnetic pressure, outside the unshocked core near the poles becomes very low (typically 10^{-2}). From the right side of the bottom left panel, the toroidal magnetic field strength reaches over 10^{15} G. The dynamics around the poles are strongly affected by these strong magnetic fields.

The three-dimensional plots of Figure 4 are useful for seeing how the field wrapping occurs. From the top left panel, it is seen that the field lines are strongly wound around the rotational axis. The white lines in the top right panel show the streamlines of the matter. A fallback of the matter just outside the head of the jet downward to the equator (like a cocoon) is seen. In this jet with a cocoon-like structure, the magnetic pressure is always dominant over the matter pressure (see the region where plasma β is less than 1 in the right side of the top right panel of Figure 3). This magneto-driven jet does not stall and penetrate to the surface of the iron core, which is essentially the reproduction of the pioneering results in the MHD supernova simulations by LeBlanc & Wilson (1970) and its analyses by Meier et al. (1976).
begins to propagate along the rotational axis and turns to be a collimated jet (see the bottom right panel). This revived jet does not stall in the iron core. This is the reason why we call this collimated jet (see the bottom right panel). This revived jet does begin to propagate along the rotational axis and turns to be a collimated jet (see the bottom right panel). This is the reason why we call this model the delayed MHD exploding model. The speed of the jet reaches about $5 \times 10^9$ cm s$^{-1}$ (see the bottom right panel). Also in this jet, the toroidal component of the magnetic fields is dominant over the poloidal one and a fallback of the matter is found in the outer region of the jet (cocoon) as in the case of the prompt MHD exploding model (see the bottom two panels of Figure 4). At $\sim 96$ ms after bounce, the jet reaches $\sim 10^6$ cm. The explosion energy at that time reaches $0.094 \times 10^{50}$ ergs.

As mentioned, the dynamic behaviors between the prompt and delayed MHD exploding models after bounce seem apparently different. However, there are some important similarities between them, which we discuss in the next section.

4.3. Similarities of the Prompt and Delayed MHD Exploding Models

In this section we focus on the similarities between the prompt and delayed MHD exploding models.

From Figure 6, it can be seen that the radial velocities and the magnetic fields of the jets are quite similar among the models. Typical values of the toroidal magnetic fields are $10^{14} - 10^{16}$ G and typical velocities are $10\% - 30\%$ of the speed of light. The opening angles of the jets are also similar. The width of this jet is about $8 \times 10^9$ cm when the jet reaches $7.5 \times 10^7$ cm, which means that the half opening angle of the jets is about $6^\circ$ at this time. These characteristic values of the jets are summarized in Table 3.

Detailed properties of the jets in the vicinity along the rotational axis are shown in Figures 7 and 8 to see the origin of these similarities. We fix the initial rotation rate in Figure 7 and the initial field strength in Figure 8 to see their effects separately. In Figure 7, the initial rotation rate is $T/|W| = 1.0\%$ and the different lines correspond to the difference between the initial magnetic fields from $10^{12}$ (B12) to $10^{10}$ G (B10). In Figure 8, the initial magnetic field is $10^{11}$ G and the different lines correspond to the difference in the initial rotation rates.

From the top and middle panels of Figures 7 and 8, we find that the radial profiles of the toroidal magnetic field, the plasma $\beta$ ($0.1 - 0.01$), the density, and the velocity, are rather similar behind the shock whose position can be seen from the discontinuity at $\sim 700$ km. Above all, it is surprising to see the remarkable similarity in the profiles of the toroidal magnetic fields behind the shock among the models (top left in Figures 7 and 8). The typical strength behind the shock is seen to be $\sim 10^{15}$ G. This critical strength of the toroidal magnetic field

Figure 3. Time evolution of various quantities characterizing the dynamics near the bounce for a prompt MHD exploding model (see the text in Section 4.2 of prompt MHD exploding models). This is for model B12TW1.0. In each panel, the left side represents the logarithm of density [g cm$^{-3}$]. Time in each panel is measured from the epoch of bounce. At the top panels, the right side is the logarithm of the “plasma $\beta = \frac{\text{density}}{\text{pressure}^2}$,” indicated by “Beta.” At the bottom left panel, the right side is the logarithm of toroidal component of the magnetic fields [G], indicated by “$B_{\phi}$.” At the bottom right panel, the right side is the radial velocity in unit of the speed of light, $c$. Note that the unit of the horizontal and the vertical axes of all panels are in cm.

(A color version of this figure is available in the online journal.)
for the shock-revival is estimated as follows. The matter behind the stalled-shock is pushed inward by the ram pressure of the accreting matter. This ram pressure is estimated as

\[ P = 4 \times 10^{28} \left( \frac{\rho}{10^{10} \text{ g cm}^{-3}} \right) \left( \frac{\Delta v}{2 \times 10^9 \text{ cm s}^{-1}} \right)^2 \text{erg cm}^{-3}, \]

(11)

where the typical density and the radial velocity are taken from Figure 1 and the bottom right panel of Figure 5, respectively. When the toroidal magnetic fields are amplified as large as \( \sim 10^{15} \text{ G} \) due to the field wrapping behind the shock, the resulting magnetic pressure, \( B^2/8\pi \), can overwhelm the ram pressure, leading to the magnetic shock-revival. The origin of the similarity of the jets seen in Figure 3 comes from this mechanism. We find that this process works in all the computed models. It is noted that the importance of the magnetic-shock revival was noticed also in the analytic models by Uzdensky & MacFadyen (2007a, 2007b). In addition to their expectations, our simulations show that the explosion energy becomes smaller than their estimations because the magnetic tower cannot be wider as they assumed.

From the bottom panels of Figure 8, it can be seen that the poloidal fields behind the shock front do not depend on the initial rotation rate so much given the same initial field strength, while the difference of the poloidal magnetic fields behind the shock in the bottom panels of Figure 7 simply comes from the difference in the initial field strength. This feature is observed in both the prompt and delayed models.

4.4. Dependence of Jet Arrival Times and Explosion Energies on Initial Rotation Rates and Magnetic Field Strengths

In the previous section, we discussed the similarities among the computed models. From this section, we move on to discuss the differences among them.
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Figure 5. Same as Figure 3 but for the quantities showing the dynamics near the bounce for a delayed exploding model (see the text in Section 4.2: delayed MHD exploding model). This is for model B10TW1.0. During the oscillations of the stalled shock after bounce (from middle left to bottom left), the magnetic fields behind the stalled shocks become large enough, due to the field wrapping (top and middle right), leading to the shock-revival for the formation of the magnetically dominated jet (bottom right). Note that the unit of the horizontal and the vertical axes of all panels is in cm.
(A color version of this figure is available in the online journal.)

Jet arrival time first we discuss the “jet arrival time” shown in Table 2, which is the timescale when the jet reaches the outer edge of the iron core of \( \sim 1000 \) km. As discussed in the previous section, this timescale is mainly determined as how long it takes for the magnetic fields behind the shock to become as large as the critical toroidal magnetic fields (\( \sim 10^{15} \) G) as a result of the field wrapping.
From the top left panel of Figure 9, it is seen that the strong initial magnetic fields shorten the jet arrival time. This tendency is seen in all the computed models regardless of the prompt
or delayed exploding models. When the initial magnetic fields are strong enough ($\sim 10^{-4}$ of the gravitational energy), the jet arrival times between the different initial rotational models become almost the same. In this case, the critical magnetic fields for the shock-revival are already generated by the compression before core bounce. So the strong magneto-driven jets can produce the prompt MHD explosions in a similar way. For the rapidly rotating models (the sequence of TW1.0 and TW4.0), it is seen that the decrease in the rate of the jet arrival time as a function of the initial $E_{\text{m}}/|W|$ becomes smaller when the initial $E_{\text{m}}/|W|$ is larger than $\sim 10^{-6}$ (see the kink in the panel). This is because magnetic fields that are too strong transport the angular momentum of the protoneutron star outward, leading to the suppression of the efficiency of the field wrapping after bounce.

In the top right panel of Figure 9, the dependence of the jet arrival time on the initial rotation rate is shown. By intuition,
the jet arrival time may become shorter as the initial rotation rates become larger since the field wrapping should become more efficient. The panel shows that this is true for moderately rotating models of the initial \( T/|W| \) less than 0.01, but not true for the more rapidly rotating models. This can be explained as follows. Rotation of the core that is too rapid hinders the central core’ collapse due to the stronger centrifugal forces. This feature is clearly shown in the middle left panel of Figure 9 showing the density profiles. The density near the center is \( \sim 100 \) times lower than that of the slowly rotating models. Since the angular momentum is well conserved before bounce (see Section 4.1), the inner core \( (\lesssim 20 \text{ km}) \) gains smaller angular velocities for rapidly rotating models by the weakened compression as seen in the middle right panel of Figure 9. Reflecting these aspects, the amplification rate of the magnetic fields \( \left( \frac{dE_m}{dt} / E_m \right) \) near core bounce becomes smaller for the most rapidly rotating model (TW4.0) as seen from the bottom panel of Figure 9. This suppression makes the jet arrival time almost constant or longer as the initial \( T/|W| \) becomes larger than \( \sim 0.01 \) as in the top right panel of Figure 9.

**Explosion energies** In addition to a wide variety of jet arrival times, we find a large difference in the strengths of the magnetic explosions.

As a measure of the strength, we define the explosion energy as

\[
E_{\text{exp1000km}} = \int dV e_{\text{local}} = \int dV (e_{\text{kin}} + e_{\text{int}} + e_{\text{mag}} + e_{\text{grav}}),
\]

(A color version of this figure is available in the online journal.)
being the kinetic, internal, magnetic, and gravitational energy, respectively (see Appendix A1 for their definitions in special relativity) and D represents the domain where the local energy is positive, indicating that the matter is not bound by the gravity. The explosion energy is evaluated when the jet arrives at the radius of 1000 km at the polar direction. The value of the explosion energy is summarized in Figure 10. Generally speaking, it is found that the explosion energies become larger for the prompt MHD exploding models (red) than the delayed MHD exploding models (green).

What makes the difference on the explosion energies at the shock break out from the iron cores? First, the initial strength of the magnetic field is the primary agent to affect the explosion energies. The explosion energies are larger for models with the larger initial fields as seen in Figure 11. Second, the geometry of the jets also has effects on the explosion energies. Figure 11 shows the toroidal magnetic field (left side) and the local energy (right side) in the jets from the stronger to the weak magnetic field models (from top to bottom panels) at the shock breakout. In each right panel, it is noted that the regions with the positive local energies ($e_{\text{local}} > 0$ in Equation (12)) are drawn with color scales and the regions with black are for the regions with the negative local energies. It is seen that the regions where the local energy is positive mostly coincide with the regions where the strong toroidal magnetic fields are generated. As the initial field strength becomes larger, the regions where the local energy becomes positive become larger (i.e., the jets become wider), leading to larger explosion energies. In the case of the delayed exploding model (the right panel in Figure 11), it is found that the width of the jets becomes narrower, which results in the smaller explosion energies. Although the properties of the jets just on the rotational axis are similar among the models as seen
5. SUMMARY AND DISCUSSION

We performed a series of two-dimensional MHD simulations of rotational core collapse of magnetized massive stars. The main motivation was to clarify how the strong magnetic fields and the rapid rotation of the core affect the magnetic explosions. To handle the very strong magnetic fields, we developed a new code under the framework of special relativity. A novel point is that the microphysics such as the realistic EOS and the neutrino
cooling are implemented to the SRMHD code. Due to these advantages, our computation can achieve a longer time-stretch compared to previous studies. The obtained results can be summarized as follows.

1. Magnetically powered jets are commonly found in all of the computed models. In the jets, the magnetic fields are dominated by the toroidal components as a result of the field wrapping. For the profiles and strengths of the toroidal fields behind the jets, we find a remarkable similarity. We find that the jet-like explosions occur when the magnetic pressure behind the shock becomes strong, due to the field wrapping, enough to overwhelm the ram pressure of the accreting matter. The required toroidal magnetic fields are approximately $\sim 10^{15} \, \text{G}$, which can be also understood by a simple order-of-magnitude estimation. Reflecting the similarity in the mechanism of producing jets, global properties of the jets such as the velocities ($\sim 20\%$ of the speed of light) and the half opening angle of the jets ($\sim 6^\circ$) are also found to be similar among the computed models.

2. The timescale before the onset of the magnetic shock-revival are quite different depending on the initial strengths of rotation and magnetic fields. When the initial strengths of rotation and magnetic fields are larger, the jet can be launched just after the core bounce, which we called the prompt MHD exploding models. We furthermore find that even for the model with the weaker initial field and slow rotation, the jet-like explosions can occur after sufficient field wrapping to reach the critical field strength, which we called the delayed MHD exploding models. In this case, the explosion can be delayed by about $\sim 100 \, \text{ms}$ after bounce. The explosion energy also strongly depends on the time difference before the shock-revival. The stronger initial magnetic fields make wider exploding regions, leading to the larger explosion energy. The largest MHD-driven explosion energy obtained is $\sim 10^{50} \, \text{erg}$.

In addition to the magnetic shock-revival, the neutrino-driven shock revival, namely neutrino heating from the newly born protoneutron star may energize the jets as suggested by Metzger et al. (2007). Although we treated only the neutrino coolings in our computations, we try to estimate the effect of the neutrino heating in the following way to see which one could be more important in producing the jets. We compare the energy gained by the neutrino heating to the magnetic energy. The specific neutrino heating rate due to neutrino absorptions $(\nu_e + n \rightarrow e^- + p$ and $\nu_p + p \rightarrow e^+ + n)$, which are the dominant heating processes at certain radius in the postbounce phase, can be estimated using Equation (10) of Qian & Woosley (1996). For model B11TW1.0 at $22 \, \text{ms}$ after bounce, the jet reaches $\sim 5 \times 10^7 \, \text{cm}$ and the density at the head of the jet, $\rho_{\text{jet}}$, is $\sim 1.5 \times 10^8 \, \text{g} \, \text{cm}^{-3}$. At that time, the neutrino sphere locates...
at $\sim 7 \times 10^6$ cm. The average energies and the luminosity for the electron and antielectron neutrinos are about 10 and 14 MeV, and 50 and $\sim 7 \times 10^{51}$ erg s$^{-1}$. In this setup, the heating rate due to the neutrino absorptions, $\dot{q}_{\nu}$, reads $7.7 \times 10^{22}$ MeV s$^{-1}$ g$^{-1}$. In the same way, the heating rate due to the neutrino pair annihilation ($\nu + \bar{\nu} \rightarrow e^+ + e^-$) can be estimated as $\sim 1.0 \times 10^{16}$ MeV s$^{-1}$ g$^{-1}$, which is negligible compared to the neutrino absorptions, albeit with the GR corrections (Salmonson & Wilson 1999; Asano & Fukuyama 2001, 2000), for far outside the neutrino spheres where the magnetic shock revival occurs. If the neutrino luminosity maintains during the...
uncertain, could be changed in a systematic manner like in Sawai et al. (2005, 2007); to see their effects on dynamics. While this study focused on the shock propagation in the iron cores, in which the jets become only mildly relativistic, we plan to continue to follow the dynamics later on until the jets break out of the stars (phase 4 in the introduction), in which the jets are expected to be relativistic. Very recent studies by Komissarov & Barkov (2007) and Bucciantini et al. (2007) are on this line. Our simulation can be more consistent than their studies in the sense that we start the simulations from the onset of the core collapse, and that the protoneutron stars are not excised like their models. By continuing the simulations of the jet propagations until the shock-breakout, we plan to study the possible connection between the magnetically driven jets and the origins of the XRFs in the forthcoming work (T. Takiwaki et al. 2009, in preparation).

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APPENDIX A

DERIVATION OF THE BASIC EQUATIONS FOR SRMHD

In this Appendix, we summarize formalisms on the basic equations and the numerical tests for our newly developed SRMHD code. For the formalisms, we follow the derivation of De Villiers et al. (2003) and Hawley et al. (1984). For convenience, we proceed with the derivation keeping the metric general forms, i.e., $dx^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, where $\alpha$ is the lapse function, $\beta$ is the shift vector, and $\gamma_{ij}$ is the spatial 3-metric. And we take the Minkowski metric later.

There are four fundamental MHD equations. The conservation of baryon number is

$$\partial_\mu \rho U^\mu = 0, \quad (A1)$$

where $\rho$, $U^\mu (\mu = 0, 1, 2, 3)$ are the baryon mass density and 4-velocity at each point. The conservation of the stress–energy is

$$\partial_\mu T^{\mu \nu} = 0, \quad (A2)$$

where $T^{\mu \nu}$ is the stress–energy tensor and Maxwell’s equations

$$\partial_\mu F^{\mu \nu} = 4\pi J^\nu, \quad (A3)$$

$$\partial_\mu \pi b^{\mu \nu} = 0, \quad (A4)$$

where $F^{\mu \nu}$ is the antisymmetrical electromagnetic tensor and $\pi b^{\mu \nu}$ is the magnetic induction in the rest frame of the fluid,

$$b^\mu = \frac{1}{\sqrt{\alpha}} \pi b^{\mu \nu} U_\nu. \quad (A7)$$

We adopt the ideal MHD limit and assume infinite conductivity, i.e., $F_{\mu \nu}U^\nu = 0$. Combining Equation (A5) with Equation (A7) and conditions for infinite conductivity, we obtain

$$F_{\mu \nu} = \epsilon_{\alpha \beta \mu \nu} \sqrt{4\pi} b^\alpha U^\beta. \quad (A8)$$

The orthogonality condition

$$b^\mu U_\mu = 0 \quad (A9)$$

follows directly from Equation (A7).

The induction Equation (A4) can also be rewritten by substituting the definitions,

$$\partial_\mu (U^\mu b^\beta - b^\mu U^\beta) = 0. \quad (A10)$$

By expanding this equation using the product rule and applying the orthogonality condition Equation (A9), we obtain the identity

$$U_\nu b^\mu \nabla_\nu U^\mu = 0. \quad (A11)$$

It is useful to rewrite the energy–momentum tensor as

$$T^{\mu \nu} = (\rho h^* + |b|^2) U^\mu U^\nu + \left( p + \frac{|b|^2}{2} \right) g^{\mu \nu} - b^\mu b^\nu. \quad (A12)$$

We have to expand basic equations in terms of the code variable, and transform the equation for auxiliary density, energy, and momentum functions $D = \rho W$, $E = eW$, $S_i = \rho W^2 v_i$. Finally, the set of variables $D$, $E$, $S_i$, $B_i$ will be evolved through the basic equations transformed here.

The equation of baryon conservation, Equation (A1), can be expanded in terms of the code variables easily,

$$\partial_\mu ( U^\mu D^\beta - D^\mu U^\beta) = 0. \quad (A13)$$

The equation of energy conservation is derived by contracting Equation (A2) with $U_\nu$,

$$U_\nu \nabla_\mu T^{\mu \nu} = U_\nu \nabla_\mu \left\{ (\rho h^* + |b|^2) U^\mu U^\nu \right\}$$

$$\left( p + \frac{|b|^2}{2} \right) g^{\mu \nu} - b^\mu b^\nu \right\} = 0. \quad (A14)$$

By using the identities (A11) and (A1), we obtain the local energy conservation

$$\nabla_\mu ( \rho \epsilon U^\mu ) + P \nabla_\mu U^\mu = 0. \quad (A15)$$

The orthogonality condition

$$b^\mu U_\mu = 0 \quad (A9)$$

follows directly from Equation (A7).

The induction Equation (A4) can also be rewritten by substituting the definitions,

$$\partial_\mu (U^\mu b^\beta - b^\mu U^\beta) = 0. \quad (A10)$$

By expanding this equation using the product rule and applying the orthogonality condition Equation (A9), we obtain the identity

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$$\left( p + \frac{|b|^2}{2} \right) g^{\mu \nu} - b^\mu b^\nu \right\} = 0. \quad (A14)$$

By using the identities (A11) and (A1), we obtain the local energy conservation

$$\nabla_\mu ( \rho \epsilon U^\mu ) + P \nabla_\mu U^\mu = 0. \quad (A15)$$
Applying the definition for the auxiliary energy function $E$, the energy equation is rewritten as follows:

$$\partial_t (E) + \frac{1}{\sqrt{\gamma}} \partial_r (\sqrt{\gamma} E V^r) + P \partial_t (W) + \frac{P}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} W V^i) = 0.$$  \hspace{1cm} (A16)

The momentum conservation equations follow from

$$\nabla_{\mu} T^\mu_{\nu} = \nabla_{\mu} \left\{ (\rho h^s + \|b\|^2)^2 U^\mu U_{\nu} + \left( P + \frac{\|b\|^2}{2} \right) \delta^\mu_{\nu} - b^\mu b_{\nu} \right\} = 0. \hspace{1cm} (A17)$$

This equation can be rewritten as

$$\frac{1}{\alpha \sqrt{\gamma}} \partial_t \sqrt{\gamma} S^\nu_{\mu} + \frac{1}{2 \alpha} S^a_{\mu} S^b_{\nu} \partial_r g^{ab\nu} + \partial_r \left( P + \frac{\|b\|^2}{2} \right) - \frac{1}{\alpha \sqrt{\gamma}} \partial_r \alpha \sqrt{\gamma} b^{\mu} b_{\nu} = \frac{1}{2} b_{\mu} b_{\rho} \partial_r g^{\rho\nu} = 0. \hspace{1cm} (A18)$$

To obtain the final form of the equations, multiply Equation (A18) by the lapse $\alpha$, split the $\mu$ index into its space ($i$) and time ($t$) components, and restrict $\nu$ to the spatial indices ($j$) only

$$\partial_t (S_{\nu} - \alpha b_{\rho} b_{\nu}) + \frac{1}{\sqrt{\gamma}} \partial_i \left( S^i_{\nu} V^i - \alpha b^i b_{\nu} \right) = -\frac{1}{2} \left( S^a_{\nu} S^b_{\nu} + \alpha b_{\mu} b_{\nu} \right) \partial_j g^{\mu\nu} - \alpha \partial_j \left( P + \frac{\|b\|^2}{2} \right). \hspace{1cm} (A19)$$

The $\nu$ index can be restricted to the spatial indices because the equation that arises from $\nu = t$ for the time components of momentum and magnetic fields is redundant, corresponding to the total energy conservation equation. In our formalism, we solve Equation (A16) separately for the internal energy. Taking the following metric,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \gamma_{ij} \end{pmatrix},$$

where $\gamma_{ij}$ is the spatial metric whose concrete description depends on the coordinate system. Finally, we describe our treatment of the gravity. Under the weak field limit, time–time component of the metric, $g_{tt}$, takes the form of $-\left(1 - 2\Phi\right)$ where $\Phi$ is the Newtonian gravitational potential (e.g., Shapiro & Teukolsky 1983). The third term of the momentum equation then becomes,

$$-\frac{1}{2} S^a_{\nu} S^b_{\nu} \partial_j g^{ab\nu} \approx \rho h W^2 \partial_j \Phi. \hspace{1cm} (A20)$$

Under this limit, the Einstein equation becomes the Poisson equation for the gravitational potential (see Equation (6)). Since the origin of the source term is the $tt$ component of the energy momentum tensor, we replace $\phi$ in the ordinary Newtonian limit with $\Phi_{\nu\nu}$, as we discussed in the appendix.

The validity of using the Newtonian potential in the core-collapse simulations here may be discussed by seeing the value of compactness parameter, $M(r)/M_{\odot}$, where $r$ is the radius and $M(r)$ is the enclosed mass within $r$. In the vicinity of the protoneutron star of $\sim 1.2 M_{\odot}$ with the typical size of $\sim 20$ km, the parameter is $\sim 0.18$. So the error caused by neglecting higher order metric perturbations is estimated to below $\sim 3\%$. The qualitative features found in this paper may be unchanged due to the incursion of the GR.

A.1. Energy Descriptions

We need to modify the description of energy from the Newtonian one to the SR one. The total local energy, $e_{\text{local}}$, is defined by sum of the various energy,

$$e_{\text{local}} = e_{\text{kin}} + e_{\text{int}} + e_{\text{mag}} + e_{\text{grav}},$$

where $e_{\text{kin}}$, $e_{\text{int}}$, $e_{\text{grav}}$, and $e_{\text{mag}}$ are kinetic energy, internal energy, gravitational energy, and magnetic energy, respectively. Their specific description is as follows:

$$e_{\text{kin}} = \rho W (W - 1), \hspace{1cm} (A23)$$

$$e_{\text{int}} = e W^2 + p (W^2 - 1), \hspace{1cm} (A24)$$

$$e_{\text{grav}} = - \rho h W^2 \Phi, \hspace{1cm} (A25)$$

$$e_{\text{mag}} = B^2 \left(1 - \frac{1}{2 W^2}\right) - \frac{b^2}{2 W^2}. \hspace{1cm} (A26)$$

These descriptions are used for the calculations of the explosion energy in Section 4.4.

APPENDIX B

SPECIAL RELATIVISTIC MOC

The MOC is popularly used in the MHD simulations. In this algorithm the magnetic fields are evolved along the characteristic lines of the Alfvén waves. A detailed procedure for this algorithm for the Newtonian case is given in Stone & Norman (1992). For the SR computations, we derive the solutions of the SR Alfvén waves in an analytic form,

$$\frac{DW v_i + b_i / \sqrt{\rho h}}{Dr} \bigg|_{-} = 0, \hspace{1cm} (B1)$$

$$\frac{DW v_i - b_i / \sqrt{\rho h}}{Dr} \bigg|_{+} = 0, \hspace{1cm} (B2)$$

$$\frac{D}{Dr} \left|_{-} \right. \overset{\text{def}}{=} \frac{\partial}{\partial t} + \frac{v_j - b_j / \sqrt{\rho h W}}{(1 - b^2 / \sqrt{\rho h W})} \frac{\partial}{\partial x_j}, \hspace{1cm} (B3)$$

$$\frac{D}{Dr} \left|_{+} \right. \overset{\text{def}}{=} \frac{\partial}{\partial t} + \frac{v_j + b_j / \sqrt{\rho h W}}{(1 + b^2 / \sqrt{\rho h W})} \frac{\partial}{\partial x_j}, \hspace{1cm} (B4)$$

where $W$, $\rho$, and $h$ are the Lorentz factor, density, and enthalpy respectively. $v_j$ and $b_j$ is the perpendicular component of the velocity and the magnetic field to the $x_i$ directions.

In the subroutine for solving SR MOC in the code, the velocity and the magnetic fields are updated at half-time step along the characteristics using the above equations. By giving the analytic forms, it is readily seen that the speed of the propagation is guaranteed to be below the speed of light even for the regions where the density becomes low and the magnetic fields become...
strong, which is quite important for keeping the stable numerical calculations in good accuracy.

Alfvén wave propagation The propagation of a linear Alfvén wave is a basic test problem of MHD simulation. We consider a constant background magnetic field, $B_x$, and fluid velocity, $v_x$. And we add small transverse perturbations with velocity, $v_z$ ($v_y$), and magnetic field, $B_z$ ($B_y$). In this situation $b^i = \sum v^i b_k \approx v_j b_j$, therefore the analytic solution for the Alfvén wave becomes

$$\frac{D W v_z + b_z/\sqrt{\rho h}}{D t} = 0, \quad (B5)$$

$$\frac{D W v_z - b_z/\sqrt{\rho h}}{D t} = 0, \quad (B6)$$

$$\frac{D}{D t} \left|_{-} \right. \approx \frac{\partial}{\partial t} + v_x \frac{b_y}{\sqrt{D_{W} W}} \frac{\partial}{\partial x}, \quad (B7)$$

$$\frac{D}{D t} \left|_{+} \right. \approx \frac{\partial}{\partial t} + v_x \frac{b_y}{\sqrt{D_{W} W}} \frac{\partial}{\partial x}. \quad (B8)$$

If we take

$$W v_z + b_z/\sqrt{\rho h} = 0, \quad (B9)$$

the minus mode does not propagate. We assume $B_x = 0.09$, $v_x = 0.08$ for the Newtonian Alfvén wave and $B_x = 0.9$, $v_x = 0.8$ for the relativistic Alfvén wave. We take $v_z$ as $10^{-7} v_x \sin(2\pi x)$ and $B_z$ is determined from $b_z$ in Equation (B9).

The result is shown in Figure 12. In both Newtonian and relativistic cases, the form of the wave is not changed. It indicates that the computations are successfully performed in our code. The propagated waveforms are very smooth and no oscillations are found like the ones in the previous study (De Villiers et al. 2003).

APPENDIX C

CONSERVATIVE VARIABLES AND FUNDAMENTAL VARIABLES

In our numerical code, variables such as $h W v^i$, $D$, $E$, $B^i$ are evolved. These variables are called conservative variables. It is necessary to compute fundamental variables such as $v^i$, $\rho$, $e$, $B^i$ from these conservative variables. In our computations, pressure is not described as an analytic function of energy $e$, therefore algorithms used in other GRMHD simulations (De Villiers et al. 2003) are not available here. If the value of the enthalpy, $h$, is found, the Lorentz factor, $W$, is obtained from the values of $h W v^i$ and then all the fundamental variables are obtained. To determine the value of $h$, we search the root of the equation below:

$$f(h) \overset{\text{def}}{=} (1 + e/\rho + p/\rho + |b|^{2}/\rho) - h = 0, \quad (C1)$$
Figure 13. Left and right panels show the shock tube and the wall reflection tests, respectively. From top to bottom, the Lorentz factor becomes larger (for right panels, top: $v = 0.1c \ (W = 1.01)$, middle: $v = 0.5c \ (W = 1.33)$, and bottom: $v = 0.9c \ (W = 5.26)$). Note that the pressure and the density are normalized by 100.

Shock tubes We consider the shock tube used by Sod (1978) in his comparison of finite difference scheme. First, we perform the weak shock problem and the strong shock problem in the Newtonian case (Hawley et al. 1984b). For the weak one, the initial conditions of this problem are gas with $\Gamma = 1.4$ with pressure and density $P_l = 1.0$, $\rho_l = 10^5$ to the left of $x = 0.5$ and $P_r = 0.1$, $\rho_r = 0.125 \times 10^5$ to the right. For the strong one, the initial conditions of this problem are gas with $\Gamma = 1.4$ with pressure and density $P_l = 0.67$, $\rho_l = 100$ to the left of $x = 0.5$ and $P_r = 0.67 \times 10^{-7}$, $\rho_r = 1.0$ to the right. The numerical values greatly correspond to the analytic value.

APPENDIX D
TEST PROBLEMS

At last we show some results of the numerical tests on our code. We have done three typical problems in both nonrelativistic case and relativistic case. Three problems are the shock tube problem, reflection shock problem, and magnetic shock tube problem. We present the results one by one.

where $|b|^2 = (B^2 + b_0^2) / W^2$. The fundamental variables in the equation are obtained when $h$ is assumed, as stated above. We use simple bisection method to search the root of Equation (C1) and make the error of the equation, $\Delta h / h$, below $10^{-4}$.
the left boundary. When the fluid hits the wall a shock forms and
problem involving the shock heating of cold fluid hitting a wall at
A second test presented here is the wall shock
which converted kinetic energy to internal energy.
that of the analytical solution. It is due to the artificial viscosity
Figure 14. Mildly relativistic magnetic shock tubes.
(A color version of this figure is available in the online journal.)

| Test | Variable | Left | FR | SC | CDl | CDr | FR | Right |
|------|----------|------|----|----|-----|-----|----|-------|
| Newtonian | $\rho$ | 1.00 | 0.66 | 0.84 | 0.70 | 0.25 | 0.12 | 0.13 |
| $B^x (\times 10^{-2})$ | $P(\times 10^{-4})$ | 1.00 | 0.44 | 0.73 | 0.50 | 0.50 | 0.09 | 0.10 |
| = 0.75 | $v^x (\times 10^{-3})$ | 0.00 | 0.67 | 4.6 | 5.99 | 6.02 | -2.79 | 0.00 |
| | $v^y (\times 10^{-2})$ | 0.00 | -0.25 | -1.10 | -1.58 | -1.58 | -0.20 | 0.00 |
| | $b_y (\times 10^{2})$ | 1.00 | 0.6 | -0.5 | -0.03 | -0.54 | -0.9 | -1.00 |
| Mildly relativistic | $\rho$ | 1.00 | 0.65 | 0.84 | 0.70 | 0.24 | 0.11 | 0.13 |
| $B^x (\times 10^{-1})$ | $P(\times 10^{-2})$ | 1.00 | 0.42 | 0.71 | 0.49 | 0.49 | 0.08 | 0.10 |
| = 0.75 | $v^x (\times 10^{-1})$ | 0.00 | 0.67 | 0.46 | 0.58 | 0.58 | -0.26 | 0.00 |
| | $v^y (\times 10^{-1})$ | 0.00 | -0.24 | -0.94 | -1.5 | -1.5 | -1.9 | 0.00 |
| | $b_y (\times 10^{1})$ | 1.00 | 0.56 | 0.3 | -0.52 | -0.52 | -0.88 | -1.00 |
| Relativistic | $\rho$ | 1.00 | 0.59 | 0.70 | 0.65 | 0.31 | 0.11 | 0.13 |
| $B^x$ | $P$ | 1.00 | 0.51 | 0.60 | 0.47 | 0.47 | 0.08 | 0.10 |
| = 0.75 | $v^x$ | 0.00 | 0.41 | 0.27 | 0.28 | 0.28 | -0.12 | 0.00 |
| | $v^y$ | 0.00 | -0.07 | -0.62 | -0.58 | -0.11 | -0.11 | 0.00 |
| | $b_y$ | 1.00 | 0.61 | 0.19 | -0.24 | -0.24 | -0.81 | -1.00 |

Notes. This table lists the measured values in each state of shock tube. “Left” is the initial left state for the given variables, “FR” is the value of the variable at the foot of the second fast rarefaction fan, “SC” is the value of the peak of the slow compound wave, “CDl” is the value of left of the constant discontinuity, “CDr” is the value of right of the constant discontinuity, “FR” is the value of the variable at the foot of the second fast rarefaction fan and “Right” is the initial right state for the given variables.

three types of pressure, i.e., $p_1 = 1.33$ ($W = 1.08$) for the low relativistic case, $p_1 = 6.67$ ($W = 1.28$) for the mildly relativistic case, and $p_1 = 666.7$ ($W = 3.28$) for the highly relativistic case. The results are shown in the left panels of Figure 13. For the low and mildly relativistic cases, the numerical value greatly corresponds to the analytical value. For the highly relativistic case, the velocity of the numerical comparison does not reach that of the analytical solution. It is due to the artificial viscosity which converted kinetic energy to internal energy.

Wall reflections A second test presented here is the wall shock problem involving the shock heating of cold fluid hitting a wall at the left boundary. When the fluid hits the wall a shock forms and travels to the right, separating the preshocked state composed of the initial data and the postshocked state with solution in the wall frame

$$V_S = \frac{\rho_1 W_1 V_1}{\rho_2 - \rho_1 W_1}, \quad \text{(D1)}$$

$$P_2 = \rho_2 (\Gamma - 1) (W_1 - 1), \quad \text{(D2)}$$

$$\rho_2 = \rho_1 \left[ \frac{\Gamma + 1}{\Gamma - 1} + \frac{\Gamma}{\Gamma - 1} (W_1 - 1) \right], \quad \text{(D3)}$$

where $V_S$ is the velocity of the shock front, and the preshocked and postshocked velocity were both assumed to be negligible ($v = V_2 = 0$).
The initial data are set up to be uniform across the grid with adiabatic index $\Gamma = 4/3$, preshocked density $\rho_1 = 1$, and preshocked pressure $P_1 = 10^{-6}$. And we change the velocity of the unshocked region parametrically. The result is shown in the right panels of Figure 13. For all computations, the differences between the numerical solution and the analytical one are small, however, in the case of the relativistic one, pressure of the numerical solution is bigger than the analytical one. It is also due to the artificial viscosity assumed here.

**Magnetic shock tubes** At last, we present magnetic shock tube problems (Brio & Wu 1988). We show the initial condition and the results of the computations in Table 4. And we present the mildly relativistic case in Figure 14, showing that our code can handle the various magnetosonic waves as good as the code by De Villiers et al. (2003).

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