Inertial Frames

(These remarks are largely based on Rob DiSalle’s entry on inertial frames in *The Stanford Encyclopedia of Philosophy.*)

One way of characterizing an inertial frame is to say that it is a frame of reference (or system of coordinates) in which Newton’s second law of motion and Newton’s third law of motion are both true. In such a frame, Newton’s first law, the law of inertia, must also be true. That is, a body on which no (net) forces act (a “free particle”) is not accelerated; it either remains at rest or moves in a straight line with constant speed. Such a line is an *inertial trajectory.*

The classical or Galilean principle of relativity tells us that, if we have one inertial frame, then any other frame moving with constant velocity with respect to the first is also an inertial frame. (Here, mass and force are assumed to be frame-invariant quantities.) If we have one inertial frame $F_1$
and a second inertial frame $F_2$ moving along the x-axis of $F_1$ in the positive x-direction, then the coordinate transformation connecting the two frames is the *Galilean transformation*,

\[
\begin{align*}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t.
\end{align*}
\]

One can think of inertial frames as families of parallel straight line filling all of space and time. (What justifies the assumption that frames can fill all of space and time?)

Suppose that we have two such families, $F_1$ and $F_2$. Each of these families of straight lines represents the trajectories of a family of free particles that are relatively at rest (and therefore each defines an inertial frame). Relative to each other, the frames defined by $F_1$ and $F_2$ are in uniform motion.
Each of the surfaces $S$ is a “hypersurface of absolute simultaneity” representing all of space at a given moment; evidently (given the Galilean transformations) two inertial frames will agree on which events in spacetime occur at a given time and so on which events are simultaneous.

The Galilean transformations above are functions that map points in spacetime to other points in spacetime by mapping four coordinates to four coordinates. One can write this compactly as $g: \mathbb{R}^4 \rightarrow \mathbb{R}^4$. It can be shown that the Galilean transformations map straight lines to straight lines and also map parallel lines to parallel lines. Such functions are called *affine transformations*.

Geometry is the study of properties or structures in a space left unchanged (or invariant) under various classes of
transformations. So affine geometry is the study of the invariants of affine transformations. A space endowed with just these structures or properties is called an affine space.

DiSalle sums up the discussion of inertial frames as follows: “The structure defined by the class of inertial frames can be captured in the statement that spacetime is a four-dimensional affine space, whose straight lines (geodesics) are the trajectories of particles in uniform rectilinear motion.”

One key point to remember is that, although in the figure above one family of straight lines is drawn as proceeding straight up the page (and so may be thought to be “at rest”), all families of parallel straight lines are equivalent (the geometric representation of the physical fact of classical or Galilean relativity). An affine transformation (which one can think of as like beveling a deck of cards or shearing) can translate the “slanted” family $F_2$ to be vertical lines, and then the family of lines $F_1$ will be slanted to the right.
There is another way of expressing the same ideas that will be helpful for us. Consider spacetime as a four-dimensional affine space. Choose some inertial trajectory as time axis, $T$. Then corresponding to each point along $T$ (that is, corresponding to each instant of time) are all the points of the spacetime that occur at that time. That is, there is a projection of spacetime into a sequence of three-dimensional spaces at successive times.

The relation of simultaneity “decomposes” spacetime into 3-dimensional pieces, each representing “all of space at a given time,” by projecting spacetime onto time, i.e., by identifying spacetime points that have the same time coordinates.
Similarly, one can think of the notion of “same place” as projecting spacetime onto space, i.e., by identifying spacetime points that have the same spatial coordinates; each of the trajectories thus singled out represents “a given place at all times.”

Of course, the projection of spacetime onto space is arbitrary, given classical relativity. This is the root of objections to Newton’s “absolute” space.

Here is a different sort of characterization of inertial frame from an excellent physics text, *A First Course in General Relativity* by Bernard Schutz.

“[A]n inertial observer is simply a coordinate system for spacetime, which makes an observation simply by recording the location \((x, y, z)\) and time \((t)\) of any event. This coordinate system must satisfy the following three
properties to be called \textit{inertial}:

(1) The distance between point $P_1$ (coordinates and point $P_2$ (coordinates $x_2$, $y_2$, $z_2$) is independent of time.

(2) The clocks that sit at every point ticking off the time coordinate $t$ are synchronized, and all run at the same rate.

(3) The geometry of space at any constant time $t$ is Euclidean.

Notice that this definition does not mention whether the observer accelerates or not. That will come later. It will turn out that only an unaccelerated observer can keep his clocks synchronized.” (3)

Side note. Often nowadays in discussions of relativity it proves to be convenient to write the coordinate indices as superscripts rather than subscripts. Thus one might find point $P_2$ above indicated by $(x^2, y^2, z^2)$. There is some risk of confusion between coordinate labels and the raising of variables to powers, but context should clarify what is intended.