Deformations of the canonical commutation relation and Lorentz symmetry violation

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Deformations of the canonical commutation relation and, independently, Lorentz symmetry violations are two common features in many candidate models for quantum gravity. Despite that, overlap between both has not been explored yet. In this brief paper, we take the first steps on this direction. At the nonrelativistic level and in the realm of commutative spacetime coordinates, a large class of both isotropic and non isotropic modifications of the canonical commutation relation is shown to produce signals experimentally indistinguishable from those predicted by the Standard Model Extension, the common framework for studying Lorentz-violating phenomena beyond the Standard Model.

Introduction — Many tentative approaches to formulate a consistent quantum theory of gravity suggest there is a fundamental length scale in nature; among these are string theories, loop quantum gravity, and noncommutative geometry [1, 2]. Model-independent arguments indicate this fundamental scale may be revealed as departures from the Heisenberg uncertainty principle, but cannot specify them completely [3]. Phenomenologically, implementation at the purely kinematic level of such departures can be done by modifying, or deforming as commonly said, the canonical commutator \([x_i, p_j] = i\hbar \delta_{ij}\) [4, 5]. This approach to generalized uncertainty principles (GUP) began with investigations on the possible existence of a finite resolution for position measurements \([4, 5]\) and has since grown far beyond, encompassing now, to name some, investigations on naturally occurring momentum cut-off at high energies [6–8], existence of a classical regime at the Planck scale [9, 10], curved momentum space [11], and quantum decoherence process [12].

It may be argued the so far most studied deformation of the canonical commutator is that proposed by Kempf [13],

\[
[x_i, p_j] = i\hbar [(1 + \beta p^2)\delta_{ij} + 2\beta p_i p_j],
\]

which predicts minimal uncertainty of \(\hbar/\sqrt{1+\beta}\) for the measurement of any spatial Cartesian coordinate, and has the nice feature of \([x_i, x_j] = 0 + O(\beta^2)\). Analysis of the Lamb shift on the hydrogen \(\alpha\) level sets \(\beta \lesssim 10^{-7} \xi^2_p/\hbar^2 \sim 0.1\) GeV\(^{-2}\) (in natural units) [14, 15] and places this sort of GUP prediction in the regime of nuclear physics. Although other commutator deformations usually lead to very different experimental predictions, bounds on specific models can be given wider context within a framework encompassing general isotropic GUP (iGUP) models [16].

To the best of our knowledge, attention so far has been given only to spatial isotropic deformations. Nevertheless, it is not clear whether spacetime necessarily exhibits such symmetric behavior at energy scales relevant for quantum gravitational phenomena. Assuming it may not, here we report on some first steps toward extending the study of deformed commutators to non isotropic scenarios. The possibilities are many, but for an illustration consider some anisotropic versions of Kempf’s model:

\[
[x_i, p_j] = i\hbar \left\{ \begin{array}{l}
[1 + (\beta \cdot p)^2] \delta_{ij} + \cdots , \\
(1 + p \cdot \beta \cdot p) \delta_{ij} + \cdots , \\
\cdots , \end{array} \right.
\]

Ignoring differing dimensionalities, we see the isotropic deformation \(\beta p^2\) is generalized to more complex structures coupling the momentum to a vector, as in \((\beta \cdot p)^2\), or to a second-rank tensor, as in \(p \cdot \beta \cdot p\), among others, but the basic idea is that the scalar nature of the deformation parameter \(\beta\) is generalized to that of a background tensor \(\beta\) effectively introducing preferred directions in space. It turns out, such spatial anisotropy is a hallmark of so-called Lorentz symmetry violations, signals of which have been intensively studied in the last decades within the Standard Model Extension (SME) framework [17–19] and experimentally searched [20] as candidate signatures of quantum gravity [21, 22]. Despite both GUP and Lorentz-violating (LV) phenomena being possible encounters on the road to a consistent quantum theory of gravity, their overlap has not been explored yet.

Next, we establish the context of this paper regarding GUP and the SME framework. Then, after a brief review of key aspects of iGUP models, we show that, at the level of effective physics, predictions from a large class of iGUP Hamiltonians are indistinguishable from those of the nonrelativistic SME Hamiltonian for fermions. This result highlights that iGUP does not necessarily originate from Lorentz-invariant physics and motivates consideration of more general, non isotropic GUP models. We tackle this case in the second half of the paper, comparing its predictions to those of the SME and extracting realistic bounds on non isotropic GUP parameters.

The SME and GUP — Even though the SME is based on conventional commutators, its Lagrangian is constructed to contain all possible terms (Lorentz-invariant or not) correcting the Standard Model (SM) plus General Relativity (GR) Lagrangian at the classical level. Although the discussed approach to GUP models modifies quantum mechanics at its heart, any realistic prediction must come at the level of effective physics as small corrections to the SM plus GR as well. Expectation, therefore, is that any GUP-based prediction is contained in the SME framework. Full exploration of this claim requires a quantum field theoretical approach

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entailing predictions of deformed canonical commutators at the nonrelativistic limit, currently unavailable and beyond our present scope. Instead, we focus on showing that at least a class of nonrelativistic GUP models, specified next, predicts Hamiltonians equivalent to those of the nonrelativistic SME fermion sector.

In flat spacetime, SME coefficients controlling interactions beyond the Standard Model are usually taken as constants in both space and time, behaving as fixed background tensors. This assumption enforces invariance under space-time translations at the fundamental level and, thus, energy-momentum conservation; hence, we consider only nonrelativistic GUP models featuring the same, which are those with commuting position operators ($[x_i, x_j] = 0$).

At last, analysis of the SME fermion sector reveals that, at the level of effective physics, isotropic corrections to the nonrelativistic Hamiltonian are necessarily spin-dependent when accompanying odd powers on the momentum — cf. Eqs. (108) and (113) of [23]. In contrast, a long-standing challenge has been formulating GUP models for spin or within a sensible relativistic regime (where spin manifests naturally); thus, we are lead to consider only commutator deformations as universal phenomena, restricting the most general deformation of the canonical commutation relations ($[x_i, x_j] = 0$).

Isotropic GUP models — As far as the momenta are kept conventional, $[p_i, p_j] = 0$, demanding spatial isotropy restricts the most general deformation of the canonical commutation relation to be

$$[x_i, p_j] = i\hbar[f(p)\delta_{ij} + g(p)p_ip_j],$$

with $f$ and $g$ real functions depending on $p \equiv |p|$ and deformation parameters (e.g., $\beta$). A far-reaching consequence is that position measurements no longer necessarily commute,

$$[x_i, x_j] = i\hbar c(p)\varepsilon_{ijk}L_k,$$

with $L_k \equiv f^{-1}e_{ijk}x_ip_j$ satisfying the conventional angular momentum algebra and being identified as the generator of three-dimensional rotations. We call $c(p)$ the commutativity function, found to be

$$c(p) = g(f - p\partial_p f) - p^{-1}f\partial_p f,$$

because requiring $f$ and $g$ such that $c = 0$ ensures position operators that commute with each other; call this a commutative GUP model. Operationally, depending on the specific deformations $f$ and $g$, position measurements cannot be made arbitrarily precise and, as a consequence, position eigenstates no longer form a basis for physical states [6, 24, 29]; formally $x_i$ is no longer self-adjoint, but merely symmetric [24]. Since this possibility is excluded for momentum measurements, the deformed algebra always finds a Hilbert space representation on momentum wave functions,

$$\psi(p) = \langle p|\psi\rangle,$$

with momentum represented by the conventional multiplicative operator and position represented by an unconventional derivative operator

$$x_i\psi(p) = i\hbar(f\partial_{p_i} + gp_i\cdot\partial_p + \gamma(p)i\psi(p),$$

with $\gamma(p)$ unobservable and adjustable to ensure $x_i$ is a symmetric operator.

One particular consequence of $[x_i, x_j] \neq 0$ is the loss of translation invariance at the fundamental level, even though rotational symmetry is retained. Here we are not interested in this feature. The escape is demanding the commutativity condition $c = 0$, enforcing

$$gp = \frac{f\partial_p f}{f - p\partial_p f} \iff [x_i, x_j] = 0.$$  

In this case, there is a generator of translations, $T = p/f(p)$, such that $[x_i, T_j] = i\hbar\delta_{ij}$ holds. Although it no longer be identified as the momentum operator, we can find a wave function representation on its eigenvectors [6],

$$\tilde{\psi}(\rho) = \langle \rho|\psi\rangle$$

where $T_i|\rho\rangle = \rho_i|\rho\rangle$, such that

$$x_i\tilde{\psi}(\rho) = i\hbar\partial_{\rho_i}\tilde{\psi}(\rho),$$

$$p_i\tilde{\psi}(\rho) = p_i|\rho\rangle \iff \frac{p_i}{f(p)} = \rho_i,$$

where $p_i$ satisfies $p_i \rightarrow \rho_i$ for $\rho_i \rightarrow 0$. At least for the one-dimensional case, it is known the model features nonvanishing minimum position uncertainty if and only if $p_i(\rho)$ is defined only for $\rho_i$ belonging to a restricted interval [27]. Perturbatively, it is convenient to express $p_i(\rho)$ as a power series on $\rho$, so that the free fermion Hamiltonian splits into a conventional kinetic part $p^2/2m_\psi$ and a perturbation $\delta H$. Restricting to $p_i(\rho^2) = \rho_i(1 + \sum \alpha_n\rho^n)$ ($n$ even $\geq 0$), the perturbation reads

$$\delta H_{\text{GUP}} = \frac{2\alpha_0 + \alpha_0^2}{2m_\psi} \rho^2 + \sum_{n=4, 6, \ldots}^{\infty} \sum_{k=0, 2, \ldots}^{n-2} \frac{\overline{\pi}_{n-k-2}\overline{\pi}_k}{2m_\psi} \rho^n$$

with $\overline{\pi}_0 = 1 + \alpha_0$ and $\overline{\pi}_n = \alpha_n$ for $n > 0$. Since $\alpha_0$ is dimensionless, it must depend on deformation parameters such that $\alpha_0 \rightarrow 0$ if these parameters vanish. To illustrate an interesting consequence of $\alpha_0 \neq 0$, consider there is a single parameter $\beta$ such that $[\beta] = [\rho]^{-2}$ and $\alpha_0 = \text{const.} \times \beta$. Dimensional analysis reveals the constant may depend on the particle’s mass, allowing deformations of the canonical commutator to dependent on the particle species; one possibility is through a species-dependent mass rescaling of $\hbar$ (see commutator (9) on [28]). In the spirit of commutator deformations as universal phenomena, $\alpha_0 = 0$ is often set — and we remark $\alpha_0$ above is then identified as $\beta$ in the context of Kempf’s model mentioned before. Here $\alpha_0 \neq 0$ is chosen instead for greater generality.

Isotropic nonrelativistic SME coefficients are to be related with each $\alpha_n$ above, but one may wish as well for a relation to the coefficients of the series expansion $1 + \sum \alpha_n\rho^n$ ($n$ even $\geq 0$) of $f(p^2)$ appearing directly in the deformed canonical commutator. The first relations are found to be $\alpha_0 = \alpha_0$, $\alpha_2 = \alpha_2$ and $\alpha_4 = \alpha_4 - 2\alpha_2^2\alpha_0^{-3}$ [16].
Comparison to the isotropic nonrelativistic SME Hamiltonian — The free-fermion sector of the SME has been studied in great details on [23]. There, the full Lorentz-violating perturbation Hamiltonian is structured in Eq. (80) in Cartesian basis and (88) in spherical basis. Neglecting spin-dependent perturbations, the nonrelativistic isotropic limit is conveniently given in spherical basis by (113), here written as

$$\delta H_{LV} = \sum_{n=0,2,\ldots}^{\infty} (\hat{\alpha}_n^{NR} - \hat{\beta}_n^{NR}) \rho^n,$$

where $\hat{\alpha}_n^{NR}$ and $\hat{\beta}_n^{NR}$ are isotropic Lorentz-violating coefficients of mass dimension $1 - n$, the first associated to CPT-violating physics as well. Originally, this expression involves $p$ instead of $\rho$, but since the SME is based on conventional quantum mechanics, we can identify both as the same quantity here. If Lorentz symmetry is not an exact symmetry of nature, the above expression for $\delta H$ is to be understood as holding only in preferred inertial reference frames while invalid in others, as sequences of boosts generally induce a relative rotation among frames. As a consequence, if any GUP has its origin in high-energy Lorentz-violating phenomena, its isotropic formulation predicts the subset of rotation-invariant effects only.

Isotropic GUP models, as formulated by deformations of the canonical commutator, respect charge, parity and time reversal symmetries and, therefore, relate only to LV-coefficients $c^{iNR}_n$ above. Among these, $c^{i0}_0$ equally rescales all energy eigenvalues in the preferred inertial frame where this isotropic SME limit holds, but induces anisotropic signals in other frames [29]. Since iGUP models are constructed under the premise of spatial isotropy, there is no GUP parameter analogous to $c^{i0}_0$ or, equivalently, one could say it is already absorbed in the definition of the energy eigenvalues. Finally, the overlap between $\delta H_{iGUP}$ and $\delta H_{LV}$ corresponds to identifications $2\alpha_0 + \alpha_0' = -2m_0 c^{2NR}$ and

$$\sum_{k=0,2,\ldots}^{n-2} \rho_{n-k-2} = -2m_0 c^{2NR}_n \quad (n \text{ even } \geq 4),$$

confirming predictions from commutative iGUP models based on deformed commutators with $f = f(p^2)$ are contained in the SME framework indeed.

Before proceeding to the case of non isotropic GUP, it is illustrative checking current experimental bounds on iGUP and LV models are in harmony using the above relations. Concerning LV-coefficients, there is only one experimental bound available for $c^{iNR}_2$ in the electron sector coming from an observed $1.8 \sigma$ difference in the theoretical and experimental values of the positronium $1S-2S$ frequency [30]:

$$c^{iNR}_2 \approx (4.5 \pm 2.4) \times 10^{-6} \text{ GeV}^{-1},$$

where $\alpha \approx 1/137$ is the fine-structure constant and $m_e = \frac{1}{2} \times 0.511$ MeV is the Positronium reduced mass, suggesting experimental reach of about $10^{-5}$ GeV$^{-1}$ for $c^{iNR}_2$ and $10^5$ GeV$^{-3}$ for $c^{NR}_4$. Taken independently, it set the following constraints on iGUP parameters:

$$|\alpha_0| \lesssim 10^{-8}, \quad |\alpha_2| \lesssim 10^2 \text{ GeV}^{-2} \sim 10^{40} \ell_p^2/h^2.$$  

Notice the bound on $\alpha_2$ we derive here is consistent with other spectroscopic bounds from dedicated analysis of iGUP models, in particular the best available ($0.1 \text{ GeV}^{-2}$) mentioned in the beginning of this paper. Additionally, the constraint on $\alpha_0$ is, as far as we know, the first to be reported. Conversely, the best dedicated bound on $\alpha_2$ suggests

$$c^{NR}_4 \lesssim 10^2 \text{ GeV}^{-3},$$

which reduces by three orders of magnitude current upper limit estimates on this LV-coefficient.

Non isotropic GUP models — The commutativity of momentum operators is an automatic feature in the isotropic scenario, but this is not so for the anisotropic case. Nevertheless, we assume $[p_i, p_j] \equiv 0$ for a less drastic departure from isotropy. By all means, giving up spatial isotropy still allows the canonical commutator to have virtually any tensor structure depending on what sort of anisotropies are considered. What we propose here is the following anisotropic deformation:

$$[x_i, p_j] = i\hbar [f(p)\delta_{ij} + g_i(p)p_j],$$

with $f$ and $g_i$ real. It by no means exhausting all possibilities, but has its advantages. First, the isotropic case is immediately recovered setting $f(p) \rightarrow f(p)$ and $g_i(p) \rightarrow g(p)p_i$. Second, as we are interested only in models with commutative position operators, this proposal is found to allow for a very direct generalization of the commutativity condition [31]; namely,

$$g_i = \frac{f \partial_{p_i} f}{f - p \cdot \partial_p f} \leftrightarrow [x_i, x_j] = 0,$$

which can be verified by straightforward calculation noticing the deformed algebra can be realized on momentum wave function representation setting

$$x_i \tilde{\psi}(p) = i\hbar (f \partial_{p_i} + g_i p \cdot \partial_p + \gamma_i) \tilde{\psi}(p),$$

with $\gamma_i(p)$ unobservable and freely adjustable to ensure $x_i$ is symmetric, similarly to the isotropic case. From here and on, we consider this commutative case only.

Function $f(p)$ is dimensionless, hence it depends on momentum only through the combination $\beta_{i_1 i_2 \ldots i_n} p_{i_1} p_{i_2} \cdots p_{i_n}$, with $\beta_{i_1 i_2 \ldots i_n}$ a generic background field of dimension $[p]^{-n}$. Considering the behavior of $i\partial_{p_i}$ and $p$ under the discrete transformations of parity and time reversal, we infer that coupling to background tensors of even rank preserve both symmetries, whereas coupling to those of odd rank violate them; further inspection also reveals $g_i(p)$ is an odd function under parity or time reversal transformations. The theory is therefore allowed to violate discrete symmetries at the fundamental level and not only by external potentials (e.g., particle decay).
Almost identically to the isotropic case, the anisotropically deformed algebra finds a representation on eigenvectors of the translation operator $T_i$ and expressions for $x_i$ and $p_i$ are structurally the same as before except for $p_i = f(p) = p_i$ since $f$ is allowed to be direction-dependent. Generally, such dependence prevents solving exactly for $p_i(p)$ and a perturbative solution may be needed, but for any particular model featuring a single anisotropy, an exact solution is always attainable in principle; in the end of the day, one is ready to write down the Hamiltonian.

Due to the complexity of explicitly writing down a general expression for $f(p)$, to the rest of this paper we concentrate on a specific model, though we remark our methods apply to any other as long as $f(p)$ contains only even powers on momentum. Hence, consider one possible anisotropic version of Kempf’s model:

$$ [x_i, p_j] = i\hbar[(1 + p \cdot \beta \cdot p)\delta_{ij} + g_i(p)p_j], \quad \text{(19)} $$

with commutativity of position operators imposing

$$ g_i = 2(p \cdot \beta)i \frac{1 + p \cdot \beta \cdot p}{1 - p \cdot \beta \cdot p} \quad \text{(20)} $$

From the 6 linearly independent (LI) components of $\beta$, the isotropic case is recovered restricting to the only scalar component, $\beta_{ij} \rightarrow \delta_{ij}$. In this case, note $g_i \approx 2(p \cdot \beta)$, reduces to $g \approx 2\beta$ as expected.

Finding $p_i(p)$ is immediate after solving $p_i = f(p) = p_i$ first for $p \cdot \beta \cdot p$ then picking the right solution demanding $p_i \rightarrow p_i$ for $\beta \rightarrow 0$. One finds

$$ p_i(p) = \frac{2p_i}{1 + \sqrt{1 - 4p \cdot \beta \cdot p}} \quad \text{(21)} $$

An interesting feature of this model is the restriction $p \cdot \beta \cdot \rho \leq 1/4$ on translation eigenvalues or, equivalently, $p \cdot \beta \cdot \rho \leq 1$ on momentum eigenvalues, effectively setting a direction-dependent upper limit to the momentum (as opposed to a single upper limit on its absolute value as in isotropic models [2, 10]). Although this behavior motivates further investigation, its beyond our present scope since we are mostly concerned with relating this model to LV-models. For this intent, we write the free fermion perturbation Hamiltonian to first order on $\beta$:

$$ \delta H_{AGUP} = \rho \cdot \beta \cdot \rho \frac{\rho^2}{m_\psi} = \sum_{\ell m} \beta_{4\ell m} Y_\ell^m(\hat{\rho})\rho^4 \quad \text{(22)} $$

where $Y_\ell^m(\hat{\rho})$ denotes the spherical harmonic function of degree $\ell$ and order $m$; the unit vector $\hat{\rho} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ defines the usual spherical polar angles; and we also define a spherical coefficient $\beta_{4\ell m}$ corresponding to the Cartesian $\beta_{ij}$, aiming further comparison to LV-coefficients. The subindex 4 is a reminder $\beta_{4\ell m}$ comes along with $\rho^4$, also $\ell = \{0, 2\}$ to ensure $\beta_{4\ell m}$ has 6 LI components as $\beta_{ij}$ does.

Comparison to the nonrelativistic SME Hamiltonian —

The nonrelativistic SME Hamiltonian is now considered on its generally anisotropic version, Eq. (108) of [23]. Its spin-independent part (109) is given in terms of spherical LV-coefficients due to their simpler rotation properties and in our notation reads

$$ \delta H_{LV} = \sum_{n \ell m} (n_{\ell m} - e_{\ell m}) Y_\ell^m(\hat{\rho})\rho^n \quad \text{(23)} $$

where $n_{\ell m}$ and $e_{\ell m}$ are the relevant coefficients controlling Lorentz violation at this level. Note $\ell = m = 0$ components correspond, up to a factor of $\sqrt{4\pi}$, to isotropic coefficients $n_{\ell m}$ and $e_{\ell m}$ discussed before, sharing corresponding behavior under CPT. The summation ranges are $n \geq 0$, $\ell = n, n-2, n-4, \ldots \geq 0$, and $m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$. For both coefficients the number of independent components is $\frac{1}{2}(n+1)(n+2)$ and the mass dimension is $1 - d$. Coefficients of LV-coefficients to anisotropic GUP parameters for our generalization of Kempf’s model begins retaining only $n_{\ell m}$ above to preserve invariance under CPT transformation. Next, remember the anisotropy $\beta_{ij}$ has 6 LI components and comes along with $\rho^4$ in $\delta H_{AGUP}$. In contrast, $e_{\ell m}$ has 6 LI components, but in $\delta H_{LV}$ couples to $\rho^2$ instead. On the other hand, $n_{\ell m}$ couples to $\rho^4$ as desired, although having 15 LI components among $\ell = 4$ (9 LI), $\ell = 2$ (5 LI), and $\ell = 0$ (1 LI). This suggests $\beta_{ij}$ is completely contained in $e_{\ell m}$ restricted to $\ell = \{0, 2\}$. The exact correspondence is

$$ \beta_{4\ell m} = m_\psi e_{\ell m}^{\text{NR}} \quad \text{for } \ell = \{0, 2\}, \quad \text{(24)} $$

revealing Hamiltonian predictions from our generalization of Kempf’s model to anisotropic commutator deformations are contained in the SME framework as well. Although no direct bound on the LV-coefficient $e_{\ell m}$ is currently available, we can still estimate upper limits for components of $\beta_{ij}$ after the above identification. First we note the nonrelativistic coefficient $n_{\ell m}$ is a mix of relativistic coefficients $c_{n\ell m}^{(d)}$ with different mass dimensions ($4 - d$) related by Eq. (111) of [23]. Same arguments as before indicate all components of $\beta_{4\ell m}$ are contained in such mixture restricted to coefficients with $n = 4$ only, namely

$$ \beta_{4\ell m} = \sum_{d=4,6,8,\ldots} m_\psi^{d-6} c_{4\ell m}^{(d)} \quad \text{for } \ell = \{0, 2\} \quad \text{(25)} $$

In passing, because $d$ is higher than 4, we remark only nonrenormalizable operators in Lorentz-violating quantum field theory produce nonrelativistic effects mimicking this particular deformation of the canonical commutator, be it isotropic or not.

To proceed and estimate the experimental sensitivity to $\beta_{ij}$, we assume the least suppressed contribution to $\beta_{4\ell m}$ comes from the spherical coefficient $c_{4\ell m}^{(6)}$. At the present, high precision measurements of the 1S-2S hydrogen transition frequency offers the best constraint on components of the corresponding Cartesian coefficient $c_{\ell m}^{(6)\mu\nu\rho\sigma}$ defined on Eq. (27) of Ref. [23]. Since the laboratory frame is noninertial due to Earth’s rotation, any fixed background field
would appear time-dependent, making comparison of measurements on different time periods unattainable. To overcome this issue, bounds on spatial anisotropies are widely reported in the canonical Sun-centered frame [20, 32] with coordinates \((T, X, Y, Z)\), which in the timescale of laboratory experiments can be taken as inertial. In this frame, searches for annual variations of the 1S–2S frequency generally leads

\[
\eta \leq 10^{-4} \text{ GeV}^{-2}
\]

where \(J = \{X, Y, Z\}\) — the exact bound differs for each \(J\) [30], but as we look for an estimate on \(\beta_{ij}\), the weakest suffices. It turns out, this anisotropic coefficient combination in the Sun-centered frame relates to the isotropic combination \(c_{2}^{\text{eff}} = c_{0011}^{(6)} + c_{0022}^{(6)} + c_{0033}^{(6)}\) in the laboratory frame (cf. Eq. (98) of [23] and Table XVIII of [33]), revealing experimental reach of \(10^{-8} \text{ GeV}^{-2}\) since, in the laboratory frame, there is no suppression due to Earth’s orbital speed of about \(10^{-4}\) in natural units. Finally, since \(c_{\text{eff}}^{(6)\mu\nu\rho\sigma}\) is symmetric under exchange of any pair of indices \(i\) and \(j\) and only spatial indices of are relevant to \(\beta_{ij}\), we identify \(c_{\text{eff}}^{(6)ijij}\) with \(\beta_{ij}\) and, therefore,

\[
\beta_{11} + \beta_{22} + \beta_{33} \lesssim 10^{-8} \text{ GeV}^{-2} \sim 10^{30} \ell_{p}^{2}/\hbar^{2}. \tag{27}
\]

For the rotationally invariant limit \(\beta_{ij} \rightarrow \beta \delta_{ij}\) this bound represents an improvement by a factor of \(10^{10}\) over the most stringent spectroscopic bound available in the literature.

Notice the above combination is isotropic only on the laboratory frame at a uniquely specified position in space and time while being anisotropic at any other as Earth rotates and moves around the Sun. To translate it into a bound in the Sun-centered frame, we first identify \(c_{\text{eff}}^{(6)\mu\nu\rho\sigma}\) with \(\beta^{\mu\nu\rho\sigma} = \beta^{ij} \delta_{ij}\), and enclosing parentheses mean symmetrization on all indices. Under this assumption, we arrive at

\[
\beta^{TJ} \lesssim 10^{-4} \text{ GeV}^{-2} \sim 10^{34} \ell_{p}^{2}/\hbar^{2}. \tag{28}
\]

This bound places effects of this particular parameter in the realm of nuclear physics and its very derivation has a far-reaching significance. Even though starting from a non-relativistic model for anisotropic GUP, we are still able to derive bounds on parameters that would otherwise naturally appear only in a relativistic formulation — something missing even for isotropic GUP models. This was possible basically because \(\beta_{ij}\) is fully contained in SME coefficients and because boosting to the Sun-centered frame mix spatial and temporal indices.

Conclusions and Perspectives — Physics of anisotropic GUP models offers a fresh venue for investigation of possible quantum gravitational phenomena. Here we reported the first steps on formulating such models in the case \([x_{i}, x_{j}] = [p_{i}, p_{j}] = 0\) with deformations based on even powers on momentum. We investigated its overlap with Lorentz-violating models in flat spacetime in the SME framework, showing both predicts equivalent effective physics. The challenge of formulating GUP models in the relativistic domain remains open, but new insights are provided: to be consistent with the SME, any nonrelativistic limit of GUP models containing odd powers on momentum must be spin-dependent; and relativistic anisotropic GUP parameters can be probed even though the model is formulated on a nonrelativistic regime.

There are at least two interesting directions for future investigations. One is the generalization of our results to GUP models with \([x_{i}, x_{j}] \neq 0\), where translation symmetry is broken and connection with the SME may require its formulation in curved spacetimes [12, 34, 35]. The other is the actual meaning of an anisotropic canonical commutator: in particular, as in the generalization \(1 + \beta p^{2} \rightarrow 1 + p \cdot \beta \cdot p\) of Kempf’s model, we may expect anisotropic commutators to generalize allow for relative directions between momentum and anisotropy where \([x_{i}, p_{j}] \rightarrow 0\) at sufficiently high momenta. Although the idea of a classical regime at the Planck scale has been explored in the past years, linking it to preferred directions in space, and possibly time, has not been considered and may introduce a range of interesting phenomena that might help to sort out models producing sensible physics.

Acknowledgments — I wish to thank Alan Kostelecky for very useful remarks during the conceptual stage of this work.

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