Boundary Element Solution of Electromagnetic Fields for Non-Perfect Conductors at Low Frequencies and Thin Skin Depths

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A novel boundary element formulation for solving problems involving eddy currents in the thin skin depth approximation is developed. It is assumed that the time-harmonic magnetic field outside the scatterers can be described using the quasistatic approximation. A two-term asymptotic expansion with respect to a small parameter characterizing the skin depth is derived for the magnetic and electric fields outside and inside the scatterer, which can be extended to higher order terms if needed. The introduction of a special surface operator (the inverse surface gradient) allows the reduction of the computational complexity of the solution. A two-term asymptotic expansion with respect to a small parameter characterizing the skin depth is derived for the magnetic and electric fields outside and inside the scatterer, which can be extended to higher order terms if needed. The introduction of a special surface operator (the inverse surface gradient) allows the reduction of the computational complexity of the solution. A method to compute this operator is developed. The obtained formulation operates only with scalar quantities and requires the computation of surface integral operators that are customary in boundary element (method of moments) solutions to the Laplace equation. The formulation can be accelerated using the fast multipole method. The resulting method is much faster than solving the vector Maxwell equations. The obtained solutions are compared with the Mie solution for scattering from a sphere, and the error of the solution is studied. Computations for much more complex shapes of different topologies, including for magnetic and electric field cages used in testing, are also performed and discussed.

Index Terms—Asymptotic methods, boundary element method, boundary integral equations, computational electromagnetics, eddy currents, method of moments.

I. INTRODUCTION

MANY systems of practical interest consist of conductors and dielectric materials. The modeling of time-varying electromagnetic fields in such systems is an important engineering problem where scattering from antennas, buildings, and various other objects of arbitrary shape must be computed. In many cases, the conductors can be modeled as perfect electric conductors, and this approximation is widely used. However, there are a number of situations where this approximation is invalid. When eddy currents appear due to finite conductivity (“non-perfectness” of the conductor), which is caused by the diffusion of the magnetic field into the conductor and must be accounted for in the modeling of the fields.

Scattering from a non-perfect conductor can be modeled using Ohm’s law and Maxwell’s equations for the electromagnetic fields inside and outside of the conductor, with the fields coupled together by transmission boundary conditions on the surface of the conductor [1]. For time-harmonic electromagnetic fields, Ohm’s law leads to the concept of complex electric permittivity, and the well-known boundary integral equations for Maxwell’s equations [2] can be used. The complex electric permittivity of a conductor results in a complex-valued wavenumber, and the reciprocal of the imaginary part of this number defines the skin depth, \( \delta \), or the depth of penetration of the magnetic field inside the conductor. Several numerical challenges with the boundary integral solvers appear in this case where the skin depth is much smaller than the characteristic length of the scatterer, \( a \) (i.e., \( \delta \ll a \)). Indeed, the size of the boundary elements, \( \Delta \), should be much smaller than the skin depth (i.e., \( \Delta \ll \delta \)) to enable a valid discrete representation of the continuous electromagnetic fields. Such a requirement may lead to huge numbers of boundary elements and drastically increase the computational complexity of the problems and even make them practically unsolvable.

In this article, we consider an approximation that is not applicable either at very high or at very low frequencies. We assume that the wavelength in the air or some other carrier dielectric medium is much larger than the wavelength, so the displacement current in that medium can be neglected. Combined with the thin skin depth approximation this creates opportunities for a simplified but computationally efficient approach, which is based on perturbation theory.

The perturbation approach with respect to small parameter \( \delta/a \) is not new. An asymptotic method was first introduced by Rytov [3] and also can be found in Jackson’s book [1]. Mitzner [4] derived a boundary integral equation for this, which enables computing the scattered fields from bodies with thin skin depths with no additional assumptions imposed on the carrier media. In the last two decades, a number of authors proposed the use of boundary element methods for solving problems with eddy currents [5]–[18]. It can be noted that in many studies (see [5], [8]), the boundary integral equations for eddy current simulations are derived from the general boundary integral equations for Maxwell’s equations in terms of vector quantities that require special surface basis functions (such as the RWG-basis [19]) and substantially complicates...
the method. There exist studies treating the problem in 2-D approximation and using isogeometric discretization requiring a relatively low number of boundary elements [15]. In these studies, the surface impedance boundary conditions (SIBC) were applied for the external solvers. In the latter and some other studies, a quasistatic approximation for the magnetic field is accepted. In the 3-D case considered in [10], only the magnetic field is computed using the zero-order approximation for the internal problem. In the magnetostatic approximation for the external field, the solution to the scattering problem can be reduced to computing only one scalar quantity, the surface magnetic potential [10]. Combinations of the boundary element method and finite element method, as well as fast multipole accelerations, can also be found [11]. Mathematical aspects of the problem are considered in several studies (see [14], [17]).

We develop and demonstrate a new approach to solve low-frequency electromagnetic problems in the case of thin skin depth inside conductors. This approach is free of the limitation, \( \Delta \ll \delta \), allowing for much smaller numerical problem sizes. The approach reduces to the solution of a few electrostatic and magnetostatic problems for scalar potentials, and it is much more efficient compared with conventional boundary element (or method of moments) Maxwell solvers. This is achieved as the problem can be expressed via scalar potentials for the external problem and standard methods for the solution of the scalar Laplace equations can be applied. This presents a new boundary integral solution for the Maxwell equations in the absence of the displacement current and, generally, the divergence constraint vector Laplace equation (DCVLE) [21] (the electric field satisfies this equation), which can be used in a number of different problems. Moreover, for large problems, the method can be accelerated using the fast multipole method (FMM) for the Laplace equation [22], [23], which makes it scalable and suitable for parallelization on large computational clusters [24]. In this article, a two-term asymptotic solution is obtained, which can be extended to higher orders if needed. The unique feature of the present method is the use of special surface operators, such as the delta function, \( \mu_1 \sigma_1 \), which makes it scalable and suitable for parallelization on large computational clusters [24]. In this article, a two-term asymptotic solution is obtained, which can be extended to higher orders if needed. The unique feature of the present method is the use of special surface operators, such as the delta function, \( \mu_1 \sigma_1 \), which makes it scalable and suitable for parallelization on large computational clusters [24]. 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when the conductor is composed of several materials with different properties or there are several different scatterers in the domain, as will arise in some of our application problems.

III. METHOD

A. Boundary Integral Formulation for External Problem

First, consider the external problem in the domain $V_1$. It is clear here that we are in the magnetostatic regime

$$
H^{(sc)}_1 = -\nabla \psi^{(sc)}_1, \quad \nabla^2 \psi^{(sc)}_1 = 0.
$$

(12)

The electric field, however, cannot be written this way because the curl of $E^{(sc)}_1$ is not zero. Instead, by the general Helmholtz decomposition, this field can be represented as

$$
E^{(sc)}_1 = -\nabla \phi^{(sc)}_1 + i\omega \mu_1 \nabla \times C^{(sc)}_1, \quad \nabla^2 \phi^{(sc)}_1 = 0
$$

(13)

where $C^{(sc)}_1$ is some vector field. Substituting (13) into (2), we see that $C^{(sc)}_1$ satisfies the equation

$$
\nabla \times \nabla \times C^{(sc)}_1 = H^{(sc)}_1 = -\nabla \psi^{(sc)}_1.
$$

(14)

Denoting

$$
\phi^{(sc)}_1 = \nabla \cdot C^{(sc)}_1 + \psi^{(sc)}_1
$$

(15)

and using the identity, $\nabla^2 = \nabla(\nabla \cdot) - \nabla \times \nabla \times$, we obtain

$$
\nabla^2 C^{(sc)}_1 = \nabla \phi^{(sc)}_1.
$$

(16)

As any particular solution of this equation can be used, we use

$$
C^{(sc)}_1(r) = -\int_{V_1} G(r, r') \nabla_r \phi^{(sc)}_1(r') dV.
$$

(17)

Here and below, integration over volume and surface is taken over variable $r'$ ($dV = dV(r')$, $dS = dS(r')$), $G(r, r')$ is the free-space Green’s function for the Laplace equation

$$
G(r, r') = \frac{1}{4\pi |r - r'|}, \quad \nabla^2 G(r, r') = -\delta(r - r')
$$

(18)

and $\delta(r - r')$ is the delta function in 3-D space. Taking the divergence of both sides of (17) and using (15), we obtain an integral equation for the unknown scalar function $\phi^{(sc)}_1$

$$
\phi^{(sc)}_1 - \psi^{(sc)}_1 = -\nabla \cdot \int_{V_1} G(r, r') \nabla_r \phi^{(sc)}_1(r') dV.
$$

(19)

The right-hand side of this equation can be transformed using Green’s identity as follows:

$$
-\nabla \cdot \int_{V_1} G(r, r') \nabla_r \phi^{(sc)}_1(r') dV = -\int_{V_1} \nabla G(r, r') \cdot \nabla \phi^{(sc)}_1(r') dV
$$

(20)

$$
= -\int_{V_1} \nabla G(r, r') \cdot \nabla \phi^{(sc)}_1(r') dV
$$

(21)

$$
= \int_{V_1} \nabla G(r, r') \cdot \nabla \phi^{(sc)}_1(r') dV
$$

(22)

$$
= \int_{V_1} \nabla G(r, r') \cdot \nabla \phi^{(sc)}_1(r') dV + \left[ \nabla^2 G(r, r') + \delta(r - r') \right] \phi^{(sc)}_1(r') dV
$$

(23)

$$
= \int_{V_1} \phi^{(sc)}_1(r') \frac{\partial G(r, r')}{\partial n(r')} dS + a(r) \phi^{(sc)}_1(r)
$$

(24)

where the normal is directed from $V_2$ to $V_1$ (which causes a negative sign to appear in front of the surface integral in the last expression), $a(r) = 1$ for points internal to $V_1$ (i.e., $r \in V_1$), and $a(r) = 1/2$ for points on $S$ (i.e., $r \in S$). In the latter case, the surface integral is singular and should be treated in terms of its principal value (p.v.). We now introduce the following notation for single- and double-layer potentials:

$$
L[s](r) = \int_S s(r') G(r, r') dS
$$

(25)

$$
M[s](r) = \text{p.v.} \int_S s(r') \frac{\partial G(r, r')}{\partial n(r')} dS.
$$

(26)

This shows that the unknown variable $\phi^{(sc)}_1$ on the surface can be found by solving the boundary integral equation

$$
M[\phi^{(sc)}_1](r) + \int_S \phi^{(sc)}_1(r') \nabla G(r, r') dV + \int_{V_1} \phi^{(sc)}_1(r') \nabla G(r, r') dV = \int_{V_1} G(r, r') \nabla \phi^{(sc)}_1(r') dV.
$$

(27)

Note that $\nabla \times C^{(sc)}_1$ can then be expressed via the scalar, $\phi^{(sc)}_1$. Indeed, using (17), we have

$$
C^{(sc)}_1(r) = -\int_{V_1} G(r, r') \nabla_r \phi^{(sc)}_1(r') dV
$$

(28)

$$
= -\int_{V_1} \nabla_r \left[ G(r, r') \phi^{(sc)}_1(r') \right] dV
$$

(29)

$$
+ \int_{V_1} \phi^{(sc)}_1(r') \nabla_r G(r, r') dV
$$

= \int_{V_1} G(r, r') \phi^{(sc)}_1(r') n(r') dS
$$

(30)

$$
- \int_{V_1} \phi^{(sc)}_1(r') \nabla_r G(r, r') dV
$$

$$
= \int_{V_1} G(r, r') \phi^{(sc)}_1(r') n(r') dS
$$

(31)

Taking the curl and plugging the result into (13), we obtain

$$
E^{(sc)}_1 = -\nabla \phi^{(sc)}_1 + i\omega \mu_1 \nabla \times L[\phi^{(sc)}_1].
$$

(32)

This equation shows that only the boundary value of the scalar function $\phi^{(sc)}_1$ is needed to compute the rotational part of $E^{(sc)}_1$, and (27) provides a method to determine this function using the magnetic potential. Note that, for the Laplace equation, the operator $M + (1/2)I$ in (27) is degenerate. However, this problem can be solved by applying the additional condition that the average of $\phi^{(sc)}_1$ over the surface is zero (or in the case of several objects, the average over each object surface is zero). In fact, any constant added to $\phi^{(sc)}_1$ does not affect the value of $\nabla \times L[\phi^{(sc)}_1]$.

B. Asymptotic Expansions for Thin Skin Depth

Now, consider the internal problem in the domain $V_2$. The magnetic field here satisfies the vector Helmholtz equation

$$
\nabla^2 H_2 + \frac{2i}{\omega} H_2 = 0.
$$

(33)

At small $\delta$, the field will be nonzero only in some vicinity of the surface, so we separate the $\nabla$ operator into parts related to differentiation along and normal to the surface

$$
\nabla = \nabla_s + \frac{\partial}{\partial n}.
$$

(34)
The Laplacian can then be written as
\[ \nabla^2 = \nabla \cdot \left( \nabla + \frac{\partial}{\partial n} \right) \]

\[ = \nabla^2_s + \nabla \cdot \left( \frac{n}{\partial n} + (n \cdot \nabla) \frac{\partial}{\partial n} \right) \]

\[ = \nabla^2_s + 2k^2 \frac{\partial}{\partial n} + \frac{\partial^2}{\partial n^2} \]
\]

where \( \nabla^2_s \) is the surface Laplace–Beltrami operator and \( \kappa = \frac{1}{2} \nabla \cdot n \)

is the mean surface curvature. Let us introduce curvilinear coordinates, \((\zeta, \eta, \zeta')\), fitted to the surface so that the surface corresponds to \( \zeta = 0 \) and \( \zeta' \) is directed opposite to the external normal to the center, i.e., \( \partial / \partial \zeta = - \partial / \partial n \). Since the electromagnetic field changes only within the skin depth from the value on the surface to zero, we have
\[ E_2(r) = E_2(\zeta', \eta', \zeta), \quad H_2(r) = H_2(\zeta', \eta', \zeta') \]

\[ \zeta' = \frac{\zeta}{a}, \quad \eta' = \eta, \quad \zeta'' = \frac{\zeta'}{\delta} \]

so we obtain
\[ \frac{\partial^2}{\partial \zeta'^2} H_2 + 2i H_2 = 2\kappa' \frac{\partial}{\partial \zeta'} H_2 - \frac{\partial^2}{\partial \zeta^2} H_2 \]

\[ \nabla' = a \nabla, \quad \kappa' = \kappa a \]

where \( \nabla' \) is the dimensionless surface del operator in the coordinates marked by primes, and \( \kappa' \) is the dimensionless surface curvature, which is assumed to be of the order of unity.

Furthermore, we consider expansions over the small parameter \( \delta / a \) of the form
\[ H_2 = H_2^{(0)} + \frac{\delta}{a} H_2^{(1)} + \left( \frac{\delta}{a} \right)^2 H_2^{(2)} + \left( \frac{\delta}{a} \right)^3 H_2^{(3)} + \cdots \]

(44)

and, similarly, for all other internal and external fields. Note that the expansions in this form were considered earlier by several authors (see [12], [20]) who also provided related expressions of the differential operators in curvilinear coordinates.

Substituting (44) into (42) and collecting terms of the same power of \( \delta / a \), we obtain the following recurrence relations:
\[ \frac{\partial^2}{\partial \zeta'^2} H_2^{(0)} + 2i H_2^{(0)} = 0 \]

(45)

\[ H_2^{(0)} \bigg|_{\zeta' = 0} = H_2^{(0)}_{23}, \quad H_2^{(0)} \bigg|_{\zeta' = \infty} = 0 \]

(46)

\[ \frac{\partial^2}{\partial \zeta'^2} H_2^{(1)} + 2i H_2^{(1)} = 2\kappa' \frac{\partial}{\partial \zeta'} H_2^{(0)} \]

(47)

\[ H_2^{(1)} \bigg|_{\zeta' = 0} = H_2^{(1)}_{23}, \quad H_2^{(1)} \bigg|_{\zeta' = \infty} = 0 \]

(48)

\[ \frac{\partial^2}{\partial \zeta'^2} H_2^{(j)} + 2i H_2^{(j)} = 2\kappa' \frac{\partial H_2^{(j-1)}}{\partial \zeta'} - \nabla' H_2^{(j-2)} \]

\[ \quad j = 2, 3 \ldots \]

(49)

In this article, we focus on the first-order approximation, which requires the first two terms. The boundary conditions relate the fields to the conditions on the surface, marked by the subscript \( S \) and to the conditions far from the surface, where the field should decay to zero. Solutions to (45) and (47) are
\[ H_2^{(0)} = H_2^{(0)}_{23} e^{-(1-i)\zeta'}, \]

(50)

\[ H_2^{(1)} = \left( H_2^{(1)}_{23} + \kappa' H_2^{(0)} e^{-(1-i)\zeta'} \right) \]

(51)

1) Zero-Order Approximation: The zero-order approximation corresponds to the case of vanishing skin depth, i.e., \( \delta = 0 \). This means that the scattered field in the zero-order approximation is equal to the field scattered by a perfect conductor. Thus, the magnetic field can be found from the solution to the Laplace equation with the Neumann boundary conditions
\[ H_1^{(sc)(0)}(r) = -\nabla \Psi_1^{(sc)(0)}, \quad \nabla^2 \Psi_1^{(sc)(0)} = 0 \]

(52)

\[ \left. \frac{\partial \Psi_1^{(sc)(0)}}{\partial n} \right|_S = -\left. \frac{\partial \Psi_1^{(0)}}{\partial n} \right|_S. \]

(53)

This shows that the normal component of the total external field is zero, so, from the boundary conditions in (8), this means that the normal component of \( H_2^{(0)} \) on the surface is also zero. The tangential component can be found from the solution to (52)
\[ H_2^{(0)} = -n \times n \times H_2^{(0)} = -n \times n \times \left( H_1^{(in)} + H_1^{(sc)} \right) \bigg|_S \]

(54)

\[ = -\nabla \left( \Psi_1^{(in)} + \Psi_1^{(sc)} \right). \]

(55)

Using this value, we obtain \( H_2^{(0)} \) from (50).

As soon as \( \Psi_1^{(sc)(0)} \) is available, the auxiliary function \( \phi_1^{(sc)(0)} \) responsible for the rotational part of the scattered electric field can be found from the equation
\[ M \left[ \phi_1^{(sc)(0)}(r) \right] + \frac{1}{2} \phi_1^{(sc)(0)}(r) = \Psi_1^{(sc)(0)}(r), \quad r \in S. \]

(56)

Since the electric field inside a perfect conductor is zero,
\[ E_2^{(0)} = 0 \]

(57)

the tangential components of the incident and scattered fields on the surface are simply related, and according to (32), we have
\[ -n \times n \times E_1^{(in)(0)} = n \times n \times E_1^{(sc)(0)} \]

(58)

\[ = n \times n \times \left( -\nabla \Psi_1^{(sc)(0)} + i \omega \mu_1 \nabla \times L \left[ n \Psi_1^{(sc)(0)} \right] \right) \]

(59)

\[ = -n \times n \times \nabla \Psi_1^{(sc)(0)} + i \omega \mu_1 n \times n \times \nabla \times L \left[ n \Psi_1^{(sc)(0)} \right] \]

(60)

\[ = \nabla \Psi_1^{(sc)(0)} + i \omega \mu_1 n \times n \times \nabla \times L \left[ n \Psi_1^{(sc)(0)} \right], \quad r \in S. \]

(61)

The problem here is to obtain \( \Psi_1^{(sc)(0)} \) from its surface gradient. Such a problem is solvable, and we introduce the inverse surface del operator \( \nabla^{-1} \) and propose an algorithm for computing it, later on in this article. Using this, we get
\[ \Phi_1^{(sc)(0)}(r) \]

(56)

\[ = \nabla^{-1} \left[ -n \times n \times E_1^{(in)(0)} + i \omega \mu_1 \nabla \times L \left[ n \Psi_1^{(sc)(0)} \right] \right], \quad r \in S. \]

(62)
This determines the boundary conditions for the Dirichlet problem. After the Laplace equation is solved with these boundary conditions, we have \( \Phi^{(sc)}_1(\mathbf{r}) \) for any point in \( V_1 \), and we can compute the scattered electric field using (32)

\[
\mathbf{E}^{(sc)(0)}_1(\mathbf{r}) = -\nabla \Phi^{(sc)(0)}_1(\mathbf{r}) + i\omega \mu_1 \mathbf{n} \times L[\mathbf{n}\Phi^{(sc)(0)}_1(\mathbf{r})], \quad \mathbf{r} \in V_1.
\]

(63)

2) First-Order Approximation: The first-order approximation can be constructed by considering Faraday’s law, which, using the primed coordinates in (41) and (43), takes the form

\[
\frac{1}{a} \nabla' \times \mathbf{E}^0_2 - \frac{1}{\delta} \mathbf{n} \times \frac{\partial \mathbf{E}^0_2}{\partial \zeta'} = i\omega \mu_2 \mathbf{H}^0_2.
\]

(64)

Plugging in the first-order approximation, we get

\[
\frac{1}{a} \nabla' \times \left( (\mathbf{E}^{(0)}_2 + \frac{\delta}{a} \mathbf{E}^{(1)}_2) \right) - \frac{1}{\delta} \mathbf{n} \times \left( \frac{\partial \mathbf{E}^{(0)}_2}{\partial \zeta'} + \frac{\delta}{a} \frac{\partial \mathbf{E}^{(1)}_2}{\partial \zeta'} \right) = i\omega \mu_2 \left( \mathbf{H}^{(0)}_2 + \frac{\delta}{a} \mathbf{H}^{(1)}_2 \right).
\]

(65)

Using (57) and collecting remaining terms near the same powers of \( \delta \), we obtain

\[
\mathbf{n} \times \frac{\partial \mathbf{E}^{(2)}_1}{\partial \zeta'} = -i\omega \mu_2 \mathbf{a} \mathbf{H}^{(0)}_2
\]

(66)

\[
\mathbf{H}^{(1)}_2 = \frac{1}{i\omega \mu_2 a} \nabla' \times \mathbf{E}^{(1)}_2.
\]

(67)

Since \( \mathbf{E}_2 \) is tangential on the surface [see (6)], we have

\[
\frac{\partial \mathbf{E}^{(1)}_2}{\partial \zeta'} = -\mathbf{n} \times \mathbf{E}^{(1)}_2 + \frac{\partial \mathbf{E}^{(0)}_2}{\partial \zeta'} = -i\omega \mu_2 a \mathbf{n} \times \mathbf{H}^{(0)}_2
\]

(68)

\[
= -\frac{\omega \mu_2 a}{\delta} \frac{\partial}{\partial \zeta'} \left( \mathbf{n} \times \mathbf{H}^{(0)}_2 \right).
\]

(69)

Using (50), we get

\[
\mathbf{E}^{(1)}_2 = \mathbf{E}^{(1)}_{2S} e^{-(i-\zeta') c'}, \quad \mathbf{E}^{(1)}_{2S} = \frac{1}{2} - i\omega \mu_2 a \left( \mathbf{n} \times \mathbf{H}^{(0)}_{2S} \right).
\]

(70)

Now having this result, we can use (67) to determine the normal component of the internal magnetic field on the surface

\[
\mathbf{n} \cdot \mathbf{H}^{(1)}_{2S} = \frac{1}{i\omega \mu_2 a} - \mathbf{n} \cdot \nabla' \times \mathbf{E}^{(1)}_{2S}.
\]

(71)

This determines the boundary conditions for the Neumann problem to solve for the scattered magnetic field via (7)

\[
\left. \frac{\partial \psi^{(sc)(1)}_1}{\partial n} \right|_S = -\mathbf{n} \cdot \mathbf{H}^{(sc)(1)}_1 = -\mu_2 \mathbf{n} \cdot \mathbf{H}^{(1)}_{2S}
\]

(72)

\[
= -\frac{1}{i\omega \mu_2 a} \mathbf{n} \cdot \nabla' \times \mathbf{E}^{(1)}_{2S}.
\]

(73)

This then gives us the tangential component of the internal magnetic field via (8)

\[
-\mathbf{n} \times \mathbf{H}^{(1)}_{2S} = -\mathbf{n} \times \mathbf{H}^{(sc)(1)}_1 = -\nabla' \psi^{(sc)(1)}_1 \bigg|_S.
\]

(74)

All together, the internal magnetic field on the surface is

\[
\mathbf{H}^{(1)}_{2S} = -\mathbf{n} \times \mathbf{H}^{(1)}_{2S} + \mathbf{n} \cdot \mathbf{H}^{(1)}_{2S}.
\]

(75)

According to (51), we now have the first-order approximation for the internal problem.

The last thing to determine in the first-order approximation is the scattered electric field. The steps to get this from the known magnetic potential and the boundary conditions are similar to those for the zero-order approximation. First, we determine the auxiliary potential

\[
M \left[ \psi^{(sc)(1)}_1(\mathbf{r}) \right] = \frac{1}{2} \psi^{(sc)(1)}_1(\mathbf{r}) = \psi^{(sc)(1)}_1(\mathbf{r}).
\]

(76)

Then, we calculate the surface electric potential

\[
\Phi^{(sc)(1)}_1 = \nabla' \left[ -\mathbf{E}^{(1)}_{2S} \right] - \mathbf{n} \times \mathbf{E}^{(0)}_2 \times \mathbf{n} \times \mathbf{H}^{(0)}_{2S} + \frac{\omega \mu_1}{\delta} \mathbf{n} \times \nabla L[\mathbf{n}\Phi^{(sc)(1)}_1(\mathbf{r})], \quad \mathbf{r} \in S.
\]

(77)

Finally, we solve the Dirichlet problem to determine the scattered field

\[
\mathbf{E}^{(sc)(1)}_1 = -\nabla \Phi^{(sc)(1)}_1 + i\omega \mu_1 \mathbf{n} \times L[\mathbf{n}\Phi^{(sc)(1)}_1].
\]

(78)

C. Analytical Solution for the Sphere

It is not difficult to construct an analytical solution, which can be used for tests. Perhaps, the simplest solution is the solution for a sphere of radius \( a \) illuminated by a plane wave. Assume that the plane wave propagates in the \( x \)-direction and is polarized in the \( z \)-direction for the electric field and the \( y \)-direction for the magnetic field

\[
\mathbf{E}^{(in)} = E_0 i e^{ikx}, \quad \mathbf{H}^{(in)} = H_0 i e^{ikx}, \quad k_1 = \frac{\omega}{c_1}
\]

(79)

where \( k_1 \) and \( c_1 \) are the wavenumber and the speed of light in medium 1. In the low-frequency approximation, \( k_1 \to 0 \), this corresponds to the following exact solution of (1) and (2):

\[
\mathbf{E}^{(in)} = i \mathbf{E}_0, \quad \mathbf{H}^{(in)} = H_0 \mathbf{E}_0.
\]

(80)

\[
\mathbf{H}^{(in)} = H_0 \delta \mathbf{e}_z, \quad \psi^{(in)} = -H_0 z.
\]

(81)

Note that, in (1) and (2), while \( \mathbf{H}^{(in)} \) can be a constant, \( \mathbf{E}^{(in)} \) cannot be a constant in the presence of the magnetic field and non-zero frequency (according to Faraday’s law \( \nabla \times \mathbf{E}^{(in)} \neq 0 \)). Let us introduce spherical coordinates referenced to the center of the sphere

\[
(x, y, z) = r \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta.
\]

(82)
conditions
\[ E^{(sc)(0)}_1 = -E_0 a^3 (i_\frac{1}{r^3} - i_\frac{3y}{r^5}) + i \omega \mu_1 H_0 \]
\[ \times \left[ \frac{a^3}{2r^2} i_\times i_r - \frac{a^5}{2} \left( \frac{y i_x + x i_y}{r^3} - \frac{z x y}{r^6} i_z \right) \right] \] (87)
\[ E^{(0)}_2 = 0 \] (88)
\[ E^{(sc)(1)}_1 = -i \omega \mu_1 c_1 \frac{H_0 a^3}{2r^2} i_r i_z \] (89)
\[ E^{(1)}_2 = \frac{3(1-i)}{4} \omega \mu_2 a H_0 (i_r \times i_z) e^{-(1-i)(a-r)/\delta} \] (90)
\[ \psi^{(sc)(0)}_1 = -H_0 \frac{a^3}{2r^2} z \] (91)
\[ H^{(sc)(0)}_1 = H_0 \frac{a^3}{2r^2} (i_\times i_r - i_3 i_z) \] (92)
\[ H^{(0)}_2 = \frac{3}{2} H_0 (i_\times i_r - i_3) e^{-(1-i)(a-r)/\delta} \] (93)
\[ \psi^{(sc)(1)}_1 = -c_1 \psi^{(sc)(0)}_1 \] (94)
\[ H^{(sc)(1)}_1 = -c_1 H^{(sc)(0)}_1 \] (95)
\[ c_1 = \frac{3(1+i) \mu_2}{2 \mu_1} \] (96)

Apart from this solution, which reflects the approach of this article, there exists an exact solution of the full problem, which is the Mie solution [25]. To compare that solution to the present one, the external and internal wavenumbers in that problem should be set to
\[ k_1 = \frac{\omega \mu_1 H_0}{E_0}, \quad k_2 = (1+i) \sqrt{\frac{\omega \mu_2 \sigma_2}{2}} = 1 + i \frac{1}{\delta}. \] (97)

Figs. 2 and 3 compare the Mie solution and the analytical, first-order (two-term) approximation from (87) to (96) for the imaginary part of the magnetic and electric fields, respectively. The computations were carried out for a copper ball in air at 100 kHz (\(a = 1 \text{ mm}, \delta/a = 0.2061\)). The lines show the magnetic and electric field lines. It is seen that the two-term solution is, qualitatively, similar to the Mie solution (the maximum relative error in the domain shown is roughly 9%). The colors show the magnitude of the field (the imaginary and real parts are both taken into account), which achieves its maximum in a relatively narrow zone near the boundary. It is also seen that the magnitude of the electric field is substantially smaller inside the sphere than outside, which is due to \(E^{(0)}_2 = 0\).

It is also seen that the internal magnetic field has a non-zero normal component, which is a manifestation of the first-order term, \(H^{(0)}_1\) (recall that \(H^{(0)}_1\) has only a tangential component). On the other hand, the internal electric field is tangential to the surface, which is also clearly seen in the Mie solution.

Fig. 4 illustrates the error of the two-term solution for the imaginary part of the magnetic field computed at the surface point, \(z = a\). According to (92), (94), and (96), we have
\[ \text{Im}\left\{H^{(sc)}_1\right\}_{z=a} = \delta \frac{\omega}{a} \text{Im}\left\{H^{(sc)(1)}_1\right\}_{z=a} = \frac{3 \delta}{2} \frac{\mu_2}{a \mu_1} H_0 k_2 \] (98)

Values were compared for copper (\(\mu_2/\mu_1 = 1\), i.e., non-magnetic) and stainless steel (\(\mu_2/\mu_1 = 4\), i.e., slightly magnetic) balls in the range, \(0.01 \leq \delta/a \leq 1\). According to the expansions, the relative error should be \(O(\delta/a)\). The graph shows that this holds. Moreover, for \(\mu_2/\mu_1 = 1\), the asymptotic constant in \(O(\delta/a)\) is close to one, but, for
\[ \frac{\mu_2}{\mu_1} = 4, \] it is about 4. This means that the residual should be written rather as \( O((\delta/a)(\mu_2/\mu_1)) \) than \( O(\delta/a) \).

The reason for factor \( \mu_2/\mu_1 \) can be understood by consideration of subsequent approximations. Denoting the normal and tangential components of the fields with subscripts \( n \) and \( t \), we have the following orders of magnitude:

\[
\begin{align*}
\mathbf{H}^{(j+1)}_{2n} &\sim \mathbf{H}^{(j+1)}_{2\tan}, \\
\mathbf{H}^{(j+1)}_{n} &\sim (\mu_2/\mu_1)\mathbf{H}^{(j+1)}_{2n} \\
\mathbf{H}^{(j+1)}_{1n} &\sim (\mu_2/\mu_1)\mathbf{H}^{(j+1)}_{2n},
\end{align*}
\]  

(100)

(101)

Here, the second relations (100)–(101) are due to the boundary conditions (8)–(9) \((\mathbf{H}^{(m,j+1)}_{n}) = 0, \ j = 0, 1, \ldots \). The first relation in (100) follows from Faraday’s law. Indeed, the normal and tangential projections of (64) show that

\[
\begin{align*}
\mathbf{H}^{(j+1)}_{2n} &= \frac{1}{i\omega \mu_2 \alpha} \mathbf{n} \times \nabla \times \mathbf{E}^{(j+1)}_{2} \\
\mathbf{H}^{(j+1)}_{2\tan} &= -\frac{1}{i\omega \mu_2 \alpha} \mathbf{n} \times \frac{\partial \mathbf{E}^{(j+1)}_{2}}{\partial \zeta}.
\end{align*}
\]  

(102)

(103)

The first relation in (101) reflects the fact that in magnetostatics the normal component of the magnetic field completely determines the full field and so its tangential component (the Neumann problem for the Laplace equation). Hence, relations (100) and (101) show that \( \mathbf{H}^{(j+1)}_{2n} \sim (\mu_2/\mu_1)\mathbf{H}^{(j+1)}_{2n} \) and \( \mathbf{H}^{(j+1)}_{2\tan} \sim (\mu_2/\mu_1)\mathbf{H}^{(j+1)}_{2n}, j = 0, 1, \ldots \). This shows that, indeed, if we expand up to the \( n \)th approximation, the residual can be estimated as \( O((\delta/a)^{n+1}(\mu_2/\mu_1)^{n+1}) \).

This estimate shows that for ferromagnetic materials with large \( \mu_2/\mu_1 \) ratios, the obtained solution is applicable only at very small values of \( \delta/a \). For example, the error for carbon steel \((\mu_2/\mu_1 = 100, \text{i.e., very magnetic}) \) is on the order of one at \( \delta/a \approx 10^{-2} \), which is consistent with the above mentioned observation.

### IV. Numerical Simulations

To implement the first-order (two-term), low-frequency approximation described in this article, we need several numerical tools. First, solvers for the Laplace equation with the Dirichlet and Neumann boundary conditions are needed. A solver for the auxiliary equation in (27) is needed as well. The solution of the Laplace equation can be represented in the form of a single-layer potential

\[
\Phi(r) = L[s](r).
\]  

(104)

Thus, the problem is to determine the single-layer density \( s \) from the boundary conditions. For the Dirichlet problem, this reduces to solving of boundary integral equation

\[
L[s](r) = \Phi(r), \quad r \in S.
\]  

(105)

For the Neumann problem, the boundary integral equations turns to

\[
L'[s](r) - \frac{1}{2}s(r) = \frac{\partial}{\partial n}\Phi(r), \quad r \in S
\]  

(106)

\[
L'[s](r) = p.v. \int_{S} s(r') \frac{\partial G(r, r')}{\partial n(r')} dS.
\]  

(107)

### B. Computation of the Inverse Surface Gradient

The BEM that we used for the examples in this article is based on the center panel approximation, that is, the solution on the boundary is piecewise constant on each panel, and the boundary conditions are enforced at the panel centers. One step in the low-frequency approximation is to compute the inverse surface gradient, \( \phi = \nabla^{-1}\phi \). In other words, given a surface gradient, \( \mathbf{v} \), at the panel centers of a triangular mesh, we want to determine the potential, \( \phi \), on the panels such that

\[
\nabla \phi = \mathbf{v}.
\]  

If \( \phi_1 \) and \( \phi_2 \) are the values of the potential at neighboring vertices, \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), of the mesh, then the directional gradient between them along the edge, \( \mathbf{l} \), can be approximated as

\[
\frac{1}{|\mathbf{l}|} \cdot \mathbf{v} = \frac{1}{|\mathbf{l}|} \cdot \nabla \phi = \frac{\partial \phi - \phi_1}{\partial |\mathbf{l}|}, \quad \mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1.
\]  

(108)

Ideally, in this formula, \( \mathbf{v} \) should be evaluated at the point, \( \mathbf{x} = (\mathbf{x}_1 + \mathbf{x}_2)/2 \), but, since \( \mathbf{v} \) are available only at the panel centers, we compute \( \mathbf{v} \) simply as \( \mathbf{v} = (\mathbf{v}_1 + \mathbf{v}_2)/2 \), where \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are the values of \( \mathbf{v} \) on the faces sharing \( \mathbf{l} \). Hence, each edge produces a linear equation

\[
\phi_2 - \phi_1 = \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_1) \cdot (\mathbf{v}_1 + \mathbf{v}_2).
\]  

(109)

Since the number of edges in the mesh is larger than the number of vertices, this forms an overdetermined system, which can be solved using least squares. However, this problem does not have a unique solution. Any constant can be added to \( \phi \), and its gradient will still equal \( \mathbf{v} \). Thus, the obtained system can be poorly conditioned. For a simple connective surface, an additional constraint setting the average of \( \phi \) over the surface to zero can be added to the system. This constraint has an intuitive explanation: if the scattered field is generated by an external field, it should not have any monopole component, meaning that \( \phi \) should average to zero over the surface. The equation enforcing this additional constraint is

\[
\sum_{j=1}^{N_v} w_j \phi_j = 0, \quad w_j = \frac{1}{3} \sum_{i} A_{ij}^{(i)}
\]  

(110)

where \( w_j \) is the weight (surface area) associated with the \( j \)th vertex, \( A_{ij}^{(i)} \) are the areas of the triangles sharing the \( j \)th vertex, and \( N_v \) is the total number of vertices. For multi-connective surfaces (several objects), each object should be supplied by a similar condition, as the rank of the system is \( N_v - M \), where \( M \) is the number of single connective surfaces constituting \( S \). Note also that, technically, it is simpler to assign some value to some vertex, say \( \phi_1 = 0 \), find a solution, and then correct all of the values to a given average. In any event, as soon as the values at the vertices are determined, the values at
Fig. 5. Relative $L_2$-norm errors for the pairs of solutions for $\text{Im}\{H_{\text{sc}}\}$ on the surface obtained by different methods in the text to the solution listed second in the legend. Computations performed for the copper ($\mu_2/\mu_1 = 1$) and stainless steel ($\mu_2/\mu_1 = 4$) balls of radius $a = 1$ mm. The dashed and dashed–dotted straight lines show linear dependence.

Fig. 6. Imaginary parts of the scattered and internal fields (copper, $f = 0.1$ MHz, $\delta = 0.2061$ mm). The pear shape shown in the figure was generated as a convex hull of two spheres of radii 1 and 0.3 mm whose centers are separated by the distance 2 mm.

the panel centers can be found using a simple vertex-to-face interpolation

$$\phi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3).$$  \hspace{1cm} (111)

We also mention that this method can naturally be extended to the case when the BEM uses vertex collocation. In this case, it becomes even simpler.

C. Examples

We computed and analyzed several cases using the method, some of which are briefly described in the following. In all examples, the incident field was generated in air, which is a dielectric with the permittivity $\varepsilon_1 = 8.85 \times 10^{-12}$ F/m and the permeability $\mu_1 = 1.257 \times 10^{-6}$ H/m. In the cases illustrated in Figs. 5–12, the incident electric field is a plane wave of unit intensity (e.g., $E_0 = 1$ V/m), which results in the value $H_0 = ((\varepsilon_1/\mu_1))^{1/2}E_0 = 2.65 \times 10^{-3}$ A/m. Also, in these figures, the pictures show the internal fields inside the scatterers and the scattered fields outside.

1) Sphere: First, we computed the benchmark case for the sphere and compared it with the analytical solution described earlier and also with the Mie solution. The obtained results show errors on the order of a few percent for low discretization, and the accuracy of the solution increases for higher discretization, consistent with the BEM accuracy described in our recent article [23]. In the tests, we used $\delta/a$ in the range 0.01–1 and meshes with $10^5$–$10^6$ faces. A typical example is shown in Fig. 5, which provides cross-estimation of the error in $\text{Im}\{H_{\text{sc}}\}$ (which is the same as $\text{Im}\{H_1\}$). In contrast to Fig. 4, we evaluated the relative error not at the single surface...
error is computed as a ratio of the $L_2$-norm of the difference of solutions and the $L_2$-norm of the reference solution. The error check was performed for copper ($\mu_2/\mu_1 = 1$) and the stainless steel ($\mu_2/\mu_1 = 4$) balls.

The results show that the error between the analytical solution and the BEM does not depend neither on $\delta/a$, nor $\mu_2/\mu_1$ due to the fact that the compared terms are simply proportional to $(\mu_2/\mu_1)(\delta/a)$ [see (92), (94), and (96)] and, in all cases, is about 5%, which is smaller than the error between the analytical and the Mie solutions. Note that this 5% can be reduced using, say, linear panel approximation instead of constant panel BEM used in this article, but there is no substantial need for this as this error is smaller than the intrinsic errors of the model.

Comparison of the BEM or analytical solution with the Mie solution shows three regions for the behavior of the residual. At relatively low and high $\delta/a$, the errors are related to the model errors, while, in some medium range, the relative error behaves approximately as $(\mu_2/\mu_1)(\delta/a)$, which is also consistent with Fig. 4. While the deviation from the linear dependence at larger $\delta/a$ can be simply explained by the fact that the model is developed for the thin skin depth, the increase of the error at low $\delta/a$ is also due to another model assumption that the external field can be computed using magnetostatics. Indeed, this LF assumption is valid when $k_1 a \ll 1$ with the error of approximation $O(k_1 a)$.

In terms of the considered relative error for the imaginary part of the scattered field, this produces the error

$$
\epsilon_2^{(LF)} \sim \frac{k_1 a}{(\mu_2/\mu_1)(\delta/a)}. \quad (112)
$$

In our example, we used $a = 10^{-3}$ m, and $\delta/a = 0.01$ corresponds to $f \approx 42$ MHz for copper and 440 MHz for stainless steel, in which $k_1 = 2\pi f/c_1 \approx 0.9$ m$^{-1}$ and 0.09 m$^{-1}$, respectively. Thus, (112) predicts 9% of the error for copper and 23% for stainless steel, which is in an excellent agreement with that in Fig. 4. Note that we did not see this LF effect on Fig. 4 since the error was estimated for a single point, which is not representative for the entire surface. Also, using (11), we can rewrite (112) in the form

$$
\epsilon_2^{(LF)} \sim \frac{2}{c_1 \sigma_2 \mu_2 (\mu_2/\mu_1)(\delta/a)^2} \frac{1}{a} \quad (113)
$$

which shows that, for fixed materials, $\delta/a$ can be made as small as desired by increasing the object size $a$. Thus, Fig. 4 can be also considered as the limiting case of very large $a$.

2) Non-Spherical Shapes of Different Topologies: Next, we used simple, non-spherical shapes of different topologies, including a pear-shaped body and a torus, which are bodies of rotation. Figs. 6 and 7 illustrate the imaginary part of the scattered and internal fields in response to the incident field given by (84). The pear shape was generated as the convex hull of two spheres of radii 1 and 0.3 mm with centers separated by 2 mm. The major and minor radii of the torus are 2.5 and 1 mm, respectively. The material for both shapes was copper, and the frequency of the field was 0.1 MHz, so $\delta = 0.2061$ mm. Note that on this and the following...
figures showing the field lines, there can be some asymmetry introduced by the plotting software (which automatically determines the position of the field lines to fill the picture more or less evenly). In these pictures, the shades of gray show the magnitude of the field on a logarithmic scale (this is also applicable to the figures in the next subsection). The mesh used contains 6514 faces for the pear shape and 6480 faces for the torus.

One peculiarity of the pear shape is that the curvature changes significantly, which, according to (51), has an effect on the internal magnetic field. The internal field is computed well within the skin depth. Closer to the center of the body, the field lines of the asymptotic solution are substantially distorted. However, since the magnitude of the field decays exponentially toward the center, such deviations from the true shape are not so important. It is seen that while the magnetic field penetrates the body surface smoothly (at \( \mu_2 = \mu_1 \)), the magnitudes of the electric field inside and outside the conductor are substantially different. Also, the internal electric field, and so the electric current, is tangential to the surface. Fig. 8 shows the real and imaginary parts of the electric field on the surface. It is remarkable that, in the present example, the magnitudes of the real and imaginary parts differ by a factor of \( 10^5 \). Nevertheless, the present method handles this situation and produces robust results: the error in each component is on the order of 1%.

3) Multi-Connective Domains: To illustrate that the method works for domains consisting of several disconnected objects (multiply connective domains), we conducted computations for several spheres with different material properties. As it was mentioned earlier, the modifications of the code here are related to the fact that the rank deficiency of the matrices, \( M + (1/2)I \), and the inverse surface gradient is exactly the number of the disconnected objects in the domain, so additional equations, such as specifying the zero average over each object, should be added. The results obtained show that this approach works well, and there were no difficulties to compute such cases.

Note that, in the case of multiply-connected domains, one can treat \( \sigma_2 \) and \( \mu_2 \) as piecewise constants on \( S \), which may take different values in different connected regions. Indeed, separated objects do not interact directly, but only via the carrier medium, for which the boundary conditions depend on the properties of the scatterer and can be different for different scatterers.

In the cases illustrated in Figs. 9 and 10, the spheres are of the same radius (\( a = 1 \) mm) and placed at the same distance apart. In Fig. 9, both spheres are made of copper \( (f = 0.1 \) MHz, \( \delta/a = 0.2061) \). Fig. 10 is computed at \( f = 100 \) MHz for spheres of different conductivities (on the left) and also of different permeabilities (on the right). Here, we have \( \delta/a = 0.0326, 0.0228, \) and \( 0.0209 \) for titanium, lead, and stainless steel, respectively. Since \( \mu_2/\mu_1 = 1 \) for both titanium and lead, the scattered magnetic field is stronger for larger \( \delta/a \) and “pushed” from the titanium to the lead sphere, which is seen by the turn points of the magnetic field lines, which happens at \( z > 1.5 \) mm, not at \( z = 1.5 \) mm as it should be for the balls of the same properties (see Fig. 9). However, in the case of the stainless steel \( (\mu_2/\mu_1 = 4) \), the field is “pushed” in the opposite direction, i.e., from the stainless steel to the titanium, and turn points are located here at \( z < 1.5 \) mm. The explanation here is that the scattered field is proportional to \( (\mu_2/\mu_1) (\delta/a) \), not \( (\delta/a) \) alone.

4) Real-World Examples: Apart from these canonical and simple shapes, we can also use the methods in this article to solve real-world problems. The only purpose of the cases shown in the following is to demonstrate that the results obtained are consistent with the physical intuition and see what kind of potential problems with the developed method one can face. The Army Research Laboratory (ARL) has two facilities for generating low-frequency electric and magnetic fields for sensor calibration and characterization, as well as for hardware-in-the-loop experiments. The electric-field cage, constructed in 2006, generates a uniform, single-axis electric field to a high degree of accuracy [26]. It is composed of two large parallel plates separated by \( 4.2 \) m with \( 20 \) equally space “guard tubes” between them to control the fringing fields. The guard tubes are made of aluminum and are \( 2 \) in thick (cross-sectional radius of \( 2.54 \) cm). The mesh of the electric-field cage, as shown in Fig. 11, contains 120000 faces (20 guard tubes with 6000 faces per guard tube). For testing purposes, we illuminated the electric-field cage with the incident field in (84) and (85). The magnetic and electric fields for the cage are shown in Fig. 12. Computations were performed at \( 1 \) kHz \( (\delta = 2.6 \) mm). The magnetic-field cage, constructed in 2017, generates a uniform, three-axis magnetic field, and is used for similar purposes. It is a coil system, similar to a Helmholtz or Merritt coil, with six coils in the \( x \)-direction and two coils in the \( y \)- and \( z \)-directions (carbon steel, \( \mu_2/\mu_1 = 100, \) and \( \sigma_2 = 6.99 \cdot 10^6 \) S/m). The size, spacing, and drive currents were optimized to account for the steel walls in the lab and produce a highly accurate field. The mesh of the magnetic-field cage is shown in Fig. 13. For the incident field, we used the Biot–Savart law to compute the fields produced by the six coils in the \( x \)-direction. This provides, for straight segments \( C_j \) connecting end points \( r_j^{(1)} \) and \( r_j^{(2)} \) and forming closed
loops \( C \), the following expressions for the electric and magnetic fields:

\[
\mathbf{H}^{(i)}(\mathbf{r}) = \sum_{j=1}^{N} \mathbf{H}^{(i)}_{ij}(\mathbf{r}) \tag{114}
\]

\[
\mathbf{E}^{(i)}(\mathbf{r}) = \sum_{j=1}^{N} \mathbf{E}^{(i)}_{ij}(\mathbf{r}), \quad \mathbf{C} = \bigcup_{j=1}^{N} \mathbf{C}_j \tag{115}
\]

\[
\mathbf{H}^{(i)}_{ij}(\mathbf{r}) = \frac{I_j}{4\pi} \int_{\mathbf{C}_j} d\mathbf{r'} \times \left( \frac{\mathbf{r} - \mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|^3} \right) \tag{116}
\]

\[
\mathbf{E}^{(i)}_{ij}(\mathbf{r}) = \frac{i\omega \mu_0 I_j}{4\pi} \int_{\mathbf{C}_j} \frac{d\mathbf{r'}}{|\mathbf{r} - \mathbf{r'}|} \tag{117}
\]

\[
\mathbf{r}^{(m)} = \mathbf{r} - \mathbf{r}^{(m)}, \quad r_j^{(m)} = |\mathbf{r}_j^{(m)}|, \quad m = 1, 2 \tag{119}
\]

\[
e_j = \frac{\mathbf{r}^{(2)}_j - \mathbf{r}^{(1)}_j}{|\mathbf{r}^{(2)}_j - \mathbf{r}^{(1)}_j|}. \tag{120}
\]

Here, \( I_j \) is the current in the \( j \)th line element. Fig. 14 illustrates the incident and scattered magnetic fields in the cage at frequency 60 Hz. As discussed earlier, because the steel walls have a high permeability (\( \mu_2/\mu_1 = 100 \)), this will cause the error to be approximately 100 times higher than if they were non-magnetic. It should be noted that, though, for such a frequency, we have \( \delta = 2.5 \) mm, which is much smaller than the wall thickness \( d = 10 \) cm, and while the skin depth criterion is satisfied, the accuracy of computations for this case is questionable due to large ratio \( \mu_2/\mu_1 \), so parameter \( (\mu_2/\mu_1)(\delta/d) \) is not small, while \( (\mu_2/\mu_1)(\delta/l) \), where \( l \) is the characteristic length of the walls (meters), is small. This is an example of a problem that would benefit from expanding to the second- or third-order approximation, which would allow us to investigate much lower frequencies in the presence of highly magnetic materials. Such a study is beyond the scope of this article and, hopefully, will be conducted in future. Computations with this cage are shown in Fig. 14.

5) Domain Inversion: In all of the cases considered earlier, the conductor occupied a finite domain and was surrounded by an infinite dielectric. In practice, however, there are many cases where the opposite situation holds: the incident field is generated in a finite dielectric surrounded by an infinite conductor. One such case is an antenna emitting in Earth’s waveguide between the ground and the ionosphere. To reduce the scale of the problem, we considered a rectangular box 500 km wide, 500 km long, and 100 km high. We assumed that both the ground and the ionosphere were uniform conductors \( (\sigma_2 = 1 \) mS/m for the ground and \( \sigma_2 = 0.1 \) mS/m for the ionosphere) and non-magnetic \( (\mu_2/\mu_1 = 1) \). At 100 Hz, the wavelength in air is 3000 km, while the skin depths in the ground and ionosphere are 1.6 and 5 km, respectively. Thus, the specified domain size, material properties, and frequency satisfy the conditions for which the present approximation is derived. Note that modeling the ionosphere with a constant isotropic conductivity is questionable since the Hall conductivity introduces anisotropy, especially in the polar regions, where it is strongest [27]. Near the equator, though, the Hall conductivity plays a much lesser role, so we assumed that the Pederssen and parallel conductivities are of the same order. In addition, we assumed that the thickness of the most conductive layer of the ionosphere is larger than 5 km. In any event, the case considered here is not designed for accurate predictions, but rather as an illustration that the present method can be used to consider such problems.

As the normal to the surface is now directed outside of the domain occupied by air, the sign near the 1/2 in (27) and (106) should be flipped, and that’s it! Particularly, \( M + (1/2)I \) becomes \( M - (1/2)I \), which is now non-singular. We also note that the boundary conditions on the open sides of the rectangular domain can be naturally handled by the accepted single-layer representation of the potentials. In this case, we simply specify zero charge on those surfaces.
In the computations, the antenna was modeled as a Hertzian dipole

\[
\mathbf{E}_1^{(in)} = \mathbf{p} G_1(r - r_s) + \frac{1}{k_1^2} \nabla [\mathbf{p} \cdot \nabla G_1(r - r_s)] \tag{121}
\]

\[
\mathbf{H}_1^{(in)} = \frac{1}{io\mu_1} \nabla G_1(r - r_s) \times \mathbf{p} \tag{122}
\]

\[
G_1(r) = \frac{e^{ik_1r}}{4\pi r}, \quad r = |r|, \quad k_1 = \frac{\omega}{c_1} \tag{123}
\]

where \(r_s\) are the coordinates of the dipole, \(\mathbf{p}\) is the dipole moment, and \(G_1\) is the free-space Green function for the Helmholtz equation with wavenumber \(k_1\) corresponding to the speed of light \(c_1\). The surface of the computational domain was discretized using 7488 triangles. Fig. 15 shows the magnitude of the surface current, \(|\sigma_2 \mathbf{E}_2|\), and the field lines on the ground and ionosphere surfaces for different values of \(\mathbf{p}\). The antenna is located at the center of the ground surface at a height of 1 km. The plotted cases correspond to the dipole, monopole, and mixed type of antennas.

V. Conclusion

The effects of eddy currents are well known, but their computation can be challenging due to the need for very high-resolution meshes in the full-wave Maxwell solvers. The method developed in this article is substantially simpler than the full-wave solver and produces physically meaningful results. Comparisons with the exact Mie solution for the sphere show that the computational errors stay within the error bounds of the method and are mostly determined by the errors of the approximation rather than the errors of the boundary element method or the FMM. While this article provides a framework of how to construct a general asymptotic expansion, only the zero- and first-order approximations have been explicitly derived. This is sufficient for a number of practically important problems when solutions for perfect conductors need to be corrected to account for the imperfectness of real conductors. However, if the accuracy of the first-order (two-term) approximation of this article is insufficient, higher-order approximations could be performed. Such work would be especially useful for highly magnetic materials, when the accuracy of the approximation is determined by the parameter, \((\mu_2/\mu_1)(\delta/a)\), rather than \(\delta/a\) alone.

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