Klein-Gordon equations for energy-momentum of relativistic particle in rapidity space

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Abstract

The notion of four- rapidity is defined as a four-vector with one time-like and three space-like coordinates. It is proved, the energy and momentum defined in the space of four-rapidity obey Klein-Gordon equations constrained by the classical trajectory of a relativistic particle. It is shown, for small values of a proper mass influence of the constraint is weakened and the classical motion gains features of a wave motion.

1 Introduction

The goal of this paper is to prove that energy and momentum defined in the space of four- rapidity obey the differential equation similar the Klein-Gordon equation. This method is based on a key-formula connecting the fraction with an exponential function the argument of which is proportional the difference between numerator and denominator. The energy and momentum can be defined either as functions of the hyperbolic angle, or as functions of the circular angle. In a covariant formulation we arrive to concept of rapidity expressed as a four-vector the time-like part of which is presented by the hyperbolic angle. The circular angle is extended into three quantities
functionally depending of the hyperbolic angle. It is shown, the energy and 
momentum defined in a such way satisfy the Klein-Gordon equations written 
in four-dimensional space with Minkowskii signature.

2 Key-formulae linking an exponential function with ratio of two quantities

2.1 Parametrization of relativistic evolution with respect to hyper-
bolic angle.

The dynamical variables of the relativistic particle, the energy $p_0$, the 
momentum $p$ and the proper mass satisfy a mass-shell equation [1]

$$p_0^2 = p^2 + m^2,$$  \hspace{1cm} (2.0)

where the speed of light $c$ taken in unit $c = 1$.

Our construction is based on the Key-formula which establishes some nat-
ural interrelation between the ratio of a pair of quantities and an exponential 
function.

Consider general form of the complex number given by unit $e$ obeying 
the quadratic equation [2]:

$$e^2 - 2p_0 e + p^2 = 0,$$ \hspace{1cm} (2.1)

with distinct positive real roots $x_1, x_2$, so that,

$$2p_0 = x_1 + x_2, \quad p^2 = x_1 x_2.$$ \hspace{1cm} (2.2)

The coefficients $p_0, p^2$ are real numbers and $p_0^2 > p^2$. The solutions of equation 
(2.1) are defined by

$$x_1 = p_0 + m, \quad x_2 = p_0 - m, \quad m = \sqrt{p_0^2 - p^2}.$$ \hspace{1cm} (2.3)

The normal form of the matrix obeying equation (2.1) is given by

$$E = \begin{pmatrix} 0 & -p^2 \\ 1 & 2p_0 \end{pmatrix}.$$ \hspace{1cm} (2.4)
Consider an evolution generated by matrix $E$. Write the Euler formula
\[
\exp(E\phi) = E \, g_1(\phi) + I \, g_0(\phi),
\] (2.5)
$I$ is a unit matrix. Diagonal form of this matrix equation consists of two equations
\[
\exp(x_2\phi) = x_2 \, g_1(\phi) + g_0(\phi), \quad \exp(x_1\phi) = x_1 \, g_1(\phi) + g_0(\phi).
\] (2.6)
Notice, the modified cosine-sine functions $g_0(\phi), g_1(\phi)$ depend also of coefficients $p_0, p^2$. Form the following ratio
\[
\exp((x_2 - x_1)\phi) = \frac{x_2 \, g_1(\phi) + g_0(\phi)}{x_1 \, g_1(\phi) + g_0(\phi)} = \frac{x_2 + D}{x_1 + D},
\] (2.7)
where
\[
D = \frac{g_0(\phi)}{g_1(\phi)}.
\] (2.8)
Let $\phi = \phi_0$ be the point where $g_0(\phi_0) = 0$. Then,
\[
\exp((x_2 - x_1)\phi_0) = \frac{x_2}{x_1}.
\] (2.9)
From (2.3) it follows
\[
m = \frac{1}{2}(x_2 - x_1), \quad p_0 = \frac{1}{2}(x_1 + x_2), \quad p^2 = x_1x_2.
\] (2.10)
Let $\phi = \phi_0$ be an initial point. Then according with (2.9) we conclude that the roots and the coefficients of equation (2.1) $x_1, x_2$ and $p_0, p$ are functions of $\phi_0$, however the difference $2m = x_2 - x_1$ does not depend of $\phi_0$:
\[
x_2(\phi_0) = \exp(m\phi_0) \frac{m}{\sinh(m\phi_0)}, \quad x_1(\phi_0) = \exp(-m\phi_0) \frac{m}{\sinh(m\phi_0)},
\] (2.11)
Use these formulae in (2.9). Then,
\[
\exp((x_2 - x_1)\phi_0) = \frac{p_0 + m}{p_0 - m}.
\] (2.12)
Consequently, we have the following dependence $p_0, p$ of $\phi_0$:
\[
p_0(\phi_0) = m \coth(m\phi_0), \quad p(\phi_0) = \frac{m}{\sinh(m\phi_0)}.
\] (2.13)
In Refs.\[3, 4, 5\] formula (2.9) has been denominated as *Key-formula*.

2.1 Parametrization of evolution with respect to periodic angle.

Now, consider general complex algebra with generator \(e\) obeying the quadratic equation

\[
e^2 - 2pe + p_0^2 = 0.
\]  
(2.14)

which differs from (2.1) by transposition of the coefficients \(p_0\) and \(p\). Since

\[p_0^2 > p^2,\]

two solutions of equation (2.14) are given by complex conjugated numbers:

\[
y_2 = p + im, \quad y_1 = p - im, \quad m = +\sqrt{p_0^2 - p^2}.
\]  
(2.15)

Exponential function at solutions of this equation is defined by expansions

\[
\exp(y_2\theta) = y_2 f_1(\theta) + f_0(\theta), \quad \exp(y_1\theta) = y_1 f_1(\theta) + f_0(\theta),
\]  
(2.16)

where functions \(f_0(\theta), f_1(\theta)\) depend of coefficients \(p, p_0^2\). Form the following ratio

\[
\exp(i2m\theta) = \frac{y_2 f_1(\theta) + f_0(\theta)}{y_1 f_1(\theta) + f_0(\theta)} = \frac{y_2 + F}{y_1 + F}
\]

\[
= \frac{p + im + F}{p - im + F},
\]  
(2.17)

where

\[
F = \frac{f_0}{f_1}.
\]  
(2.18)

Let \(\theta = \theta_0\) be the initial point where \(f_0(m\theta_0) = 0\). Then, formula (2.17) is reduced into the following relationship

\[
\exp(i2m\theta_0) = \frac{p(\theta_0) + im}{p(\theta_0) - im}.
\]  
(2.19)

From this formula it follows

\[
p(\theta_0) = m \cot(m\theta_0), \quad p_0(\theta_0) = m \frac{1}{\sin(m\theta_0)}.
\]  
(2.20)

The roots \(y_1, y_2\) also are functions of \(\theta_0\),

\[
y_1 = \exp(-im\theta_0) \frac{m}{\sin(m\theta_0)}, \quad y_2 = \exp(im\theta_0) \frac{m}{\sin(m\theta_0)}.
\]  
(2.21)
Thus, we obtained two representations for the energy-momentum. The first one is done via hyperbolic trigonometric functions,

\[ p_0(\phi) = m \coth(m\phi), \quad p(\phi) = m \frac{1}{\sinh(m\phi)}, \quad (2.22) \]

and the other one is defined by ordinary periodic trigonometric functions

\[ p_0(\theta) = m \frac{1}{\sin(m\theta)}, \quad p(\chi) = m \cot(m\theta). \quad (2.23) \]

In the both representations the arguments of the trigonometric functions are proportional to mass \(m\).

Since formulae (2.22) and (2.23) are related to same physical quantities, we come to the next relationships between hyperbolic and periodic trigonometric functions

\[ \tanh(m\phi) = \sin(m\theta), \text{ or } \sinh(m\phi) = \tan(m\theta). \quad (2.24) \]

Notice, when \(m = 0\), \(\phi = \theta\).

The relationships between \(\phi\) and \(\theta\) can be presented also as follows

\[ \exp(m\phi) = \frac{1 + \sin(m\theta)}{1 - \sin(m\theta)} = \frac{1 + \tan \frac{m\theta}{2}}{1 - \tan \frac{m\theta}{2}}, \quad (2.25a) \]

\[ \exp(im\theta) = \frac{1 + i \sinh(m\phi)}{1 - i \sinh(m\phi)} = \frac{1 + i \tanh \frac{m\phi}{2}}{1 - i \tanh \frac{m\phi}{2}}. \quad (2.25b) \]

Also, it is important to notice that the differential relationship between variables \(\theta\) and \(\phi\) coincides with the definition of the velocity:

\[ \frac{d\theta}{d\phi} = \frac{dr}{dt} = \frac{p}{p_0}. \quad (2.27) \]

These formulae express a general interrelation between periodic and hyperbolic trigonometry. Let \(\triangle ABC\) be a right-angled triangle with right angle at \(C\). Denote the sides by \(a = BC, b = AC\), the hypotenuse \(AB\) by \(c\). If we make a geometrical motion by moving the point \(A\) along line \(AC\), then this motion changes the sides \(c, b\), but remains invariant the side \(a\). The angle \(A\)
can be used in quality of parameter this evolution. In accordance with the
\textit{Key-formula} (2.19) we write
\begin{equation}
\frac{b + ia}{b - ia} = \exp(2i\alpha), \quad b = a \cot(a\theta), \quad c = a\frac{1}{\sin(a\theta)},
\end{equation}
(2.28)

It is easily seen that \( a\theta = A \). On the other hand, In accordance with \textit{Key-formula} (2.12) we have
\begin{equation}
\frac{c + a}{c - a} = \exp(2a\phi), \quad c = a \coth(a\phi), \quad b = a\frac{1}{\sinh(a\phi)}.
\end{equation}
(2.29)

\section{3 Pythagoras theorem and two dimensional Fermi-like oscillator}

For the sake of convenience in this section let us use for derivatives short notations
\begin{equation}
\frac{d}{d\phi} = d, \quad \frac{d}{d\theta} = \partial.
\end{equation}

Then differentiating formula (2.22) and (2.23) we come to the following system of differential equations
\begin{equation}
dp_0 = -p^2, \quad dp = -pp_0, \quad \partial p_0 = -pp_0, \quad \partial p = -p_0^2.
\end{equation}
(3.1)

The operators \( d \) and \( \partial \) do not commute. Introduce two dimensional vector of a state by
\begin{equation}
\Phi(p_0, p) = \begin{pmatrix} p_0 \\ p \end{pmatrix}.
\end{equation}
(3.2)

Calculate actions of the operators \( d^2 - \partial^2 \) and \( d\partial - \partial d \) on this vector:
\begin{equation}
(d^2 - \partial^2)\Phi(p_0, p) = m^2 \Phi(p_0, p),
\end{equation}
(3.3)
\begin{equation}
(\partial d - d\partial)\Phi(p_0, p) = m^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi(p_0, p).
\end{equation}
(3.4)

Introduce operators
\begin{equation}
a^- = d - \partial, \quad a^+ = d + \partial,
\end{equation}
with following commutation and anti-commutation rules

\[
\frac{1}{2}(a^- a^+ + a^+ a^-) = m^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(3.5)

\[
a^- a^+ - a^+ a^- = 2m^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(3.6)

It is seen, we deal with some kind of two dimensional Fermi-like oscillator with Hamilton operator

\[
H = \frac{1}{2}(a^- a^+ - a^+ a^-),
\]

(3.7)

and with anti-commutation relation given by

\[
(a^- a^+ + a^+ a^-) = 2m^2,
\]

(3.8)

acting on the state \(\Phi_0(p_0, p)\). The Hamilton operator possesses with two eigenvalues

\[
H \Phi_n = E_n \Phi_n, \quad n = 1, 2,
\]

(3.9)

where

\[
E_1 = +m^2, \quad E_2 = -m^2,
\]

so that,

\[
a^+ \Phi_1 = \Phi_2, \quad a^- \Phi_2 = \Phi_1,
\]

and, due to anti-commutation relations (3.8),

\[
a^\pm H + Ha^\pm = 0.
\]

4 Klein-Gordon equations for energy-momentum of classical relativistic particle in the space of rapidity

It is seen that equation (3.3) is nothing else than two dimensional Klein-Gordon equation. Comparing this equation with two dimensional Klein-Gordon equation written in terms of space-time coordinates we come to conclusion that the parameter \(\phi\)- is a time-like parameter, whereas the parameter
\( \theta \) is an analogue of a space coordinate. In order to pass to the Klein-Gordon equation in four-dimensional Minkowski space with signature \((+ - - -)\) we shall extend the parameter \( \theta \) till to three dimensional vector. In this way we arrive to covariant formulation of evolution equations.

The momentum is a spatial part of the four-vector energy-momentum with components \( p_k, k = 1, 2, 3. \) Now, instead of \( \phi \) we will use the letter \( \rho_0, \) and \( \theta \) has to be replaced by spatial part of four-vector of rapidity containing components \( \rho_1, \rho_2, \rho_3. \)

In these variables the evolution equations have to be written in a Lorentz-covariant form. The evolution equations we shall extend as follows. The single variable \( p \) is replaced by the components of three-vector of momentum, \( p_k, k = 1, 2, 3. \) The square \( p^2 \) means \( p^2 = -p^k p_k. \) In this way we arrive to the following set of equations

\[
\begin{align*}
(a) \quad d^0 p_0 &= -p^k p_k, \\
(b) \quad d^0 p_k &= p_k p^0, \quad k = 1, 2, 3. \\
\end{align*}
\]

\( (4.1) \)

\[
\begin{align*}
(a) \quad \partial_k p_0 &= p_k p_0, \\
(b) \quad \partial^k p_k &= -p^2_0. \\
\end{align*}
\]

(4.2)

Hereafter we use the following notations for derivatives

\[
\partial^k = \frac{\partial}{\partial \rho_k}, \quad d^0 = \frac{d}{d \rho_0},
\]

and adopt, so-called, a summation convention, according to which any repeated index in one term, once up, once down, implies summation over all its values.

Remember, however, that there exist some functional dependence between \( \rho_0 \) and \( \rho_k, k = 1, 2, 3, \) so that the spatial variables are functions of the time-like parameter, i.e., \( \rho_k = \rho_k(\rho_0), k = 1, 2, 3. \) This means, the full derivative with respect to \( \rho_0 \) is

\[
d^0 p_0 = -p^k p_k = \frac{d \rho_k}{d \rho_0} \frac{\partial}{\partial \rho_k} p_0. \quad (4.3)
\]

On making use of equations \( (4.1)-(4.2), \) we get

\[
d^0 p_0 = p^2 = -p^k p_k = p_k \frac{d \rho_k}{d \rho_0} p_0. \quad (4.4)
\]
In order to provide this equality we have to take

\[ p_k \frac{dp_k}{d\rho_0} = \frac{p^2}{p_0}. \] (4.5)

Our purpose is to complete the evolution equations (4.1)-(4.2) with an equation containing the following derivative

\[ \frac{\partial}{\partial \rho_n} p_k. \]

For that reason let us re-write equation (4.1b) as follows

\[ \frac{d}{d\rho_0} p_k = \frac{dp_n}{d\rho_0} \partial_{\rho_n} p_k = p_k p_0. \]

In order to provide this equality we have to suppose that

\[ \frac{\partial}{\partial \rho_n} p_k = p_k p_0 \frac{p^2}{p^2}. \] (4.6)

One may easily check that formula (4.6) is in accordance with (4.1) and (4.2).

**Equations with second order derivatives.**

Firstly, calculate the second order derivatives of \( p_0 \) and \( p \) with respect to time-like variable \( \rho_0 \). We have,

\[ \frac{d}{d\rho_0} \frac{d}{d\rho_0} p_0 = -2p^k p_k p_0 = 2p^2 p_0. \] (4.7)

Secondly, calculate action of the Laplace operator on \( p_0 \). Define the Laplace operator by

\[ \Delta = \frac{\partial}{\partial \rho_k} \frac{\partial}{\partial \rho^k}. \]

By taking into account (4.2a) we obtain

\[ \partial^k \partial_k p_0 = -p_0^2 p_0 + p^k p_k p_0 = -p_0^3 - p^2 p_0. \] (4.8)

Joining this equation with (4.7) we come to Klein-Gordon equation for \( p_0 \):

\[ d^0 d_0 p_0 + \partial^k \partial_k p_0 = -m^2 p_0. \] (4.9)
Now calculate action of operator $\Delta$ on $p_k$ by using formulae (4.6) and (4.2b).

$$\partial^n \partial_n p_k = \partial^n \left( p_k p_n \frac{p_0^2}{p^2} \right)$$

$$= \left( \partial^n \left( p_k p_n \right) \right) \frac{p_0^2}{p^2} + p_k p_n \partial^n \left( \frac{p_0^2}{p^2} \right)$$

$$= \frac{p_0^2}{p^2} \left( p^n p_n \right) \frac{p_0^2}{p^2} + p_k \partial^n \left( \frac{p_0^2}{p^2} \right)$$

$$= -2 \frac{p_0^4}{p^2} p_k + p_k p_n \partial^n \left( \frac{p_0^2}{p^2} \right)$$

$$= -2 \frac{p_0^4}{p^2} p_k - 2 p_k p^n p_n \frac{p_0^2 m^2}{p^4}$$

$$= -2 \frac{p_0^4}{p^2} p_k + 2 p_k \frac{p_0^2 m^2}{p^2}$$

$$= \left( -2 \frac{p_0^4}{p^2} p_k + 2 p_k \frac{p_0^4}{p^2} \right) = -2 p_k p_0^2.$$

Joining this result with

$$\frac{d}{d\rho^0} \frac{d}{d\rho_0} p_k = p_k p_0^2 - p_k p^n p_n = p_k p_0^2 + p_k p^2,$$

we come to analogue of Klein-Gordon equation for $p_k$:

$$\Delta p_k + d^0 d_0 p_k = -m^2 p_k.$$

**Comparison with Klein-Gordon equation used in relativistic quantum mechanics.**

From formula (4.5)

$$p_k \frac{dp^k}{d\rho_0} = \frac{p^2}{p_0},$$

we may conclude that

$$\frac{dp^k}{d\rho_0} = v^k + M^{kl} p_l,$$
where

\[ v^k = \frac{dx^k}{dx^0} \]

is the velocity with respect to coordinate time, \( M^{kl} = -M^{lk} \) is an arbitrary anti-symmetric tensor.

In the relativistic quantum mechanics the Klein-Gordon equation is obtained simply by using some conventional receipt according to which components of four-momentum in the mass-shell equation are replaced by corresponding differential operators as follows [6]

\[
p_k = -i\hbar \frac{\partial}{\partial x^k}, \quad p_0 = i\hbar \frac{\partial}{\partial x^0}.
\]

So, we come to the following correspondence

\[
\hbar \rho^\mu \Rightarrow x^\mu, \quad \frac{\partial}{\partial \rho^\mu} = \hbar \frac{\partial}{\partial x^\mu}.
\]

### 5 Concluding remarks

1. We considered two ways of description an evolution constrained by Pythagoras formula. The first one is given by the hyperbolic angle, and the second one, by the periodic angle. The both angles are proportional to the fixed side of the right angled triangle. In the case of relativistic mechanics, the hypotenuse is the energy, the fixed side is the mass and the moving side is the momentum.

2. The derivative of the periodic angle with respect to the hyperbolic angle is equal to a ratio of the moving side to the hypotenuse, in the case of relativistic mechanics, this ratio is the velocity

\[
\frac{d\theta}{d\phi} = \frac{p}{p_0} = \frac{v}{c}.
\]

3. This relationship prompts us to conclude that the hyperbolic angle \( \phi \) is the time-like parameter, whereas the periodic angle \( \theta \) is the space-like parameter.
4. The evolution equations admit an extension to the case of three (or more) dimensions, however, in this case we could not find an explicit expression for the momenta.

5. Near the point where the fixed side of the triangle (mass) becomes infinitesimal and according to the Klein-Gordon equation it is conjectured that this motion will display features of the wave motion.

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