Andreev reflection occurs at a normal-superconductor interface when an incident electron/hole on the normal side is reflected as a hole/electron [1]. It is a current-carrying process which, within a mean-field description, conserves the quasiparticle but not the charge current. The Andreev reflection of an incident hole is equivalent to the emission of a singlet electron pair from the superconductor into the normal metal [2]. It has been observed in the emission of a singlet electron pair from the superconductor to the normal metal [2].

Andreev reflection is to create a potential step that forces a difference in the atom densities on each side of the step. At a given temperature, the densities may be such that the gas is normal on the left and superfluid on the right. However, since in a normal boson gas the chemical potential stays below zero kinetic energy, transport near equilibrium is possible only if the condensate is confined. The $v$ component represents propagation below the chemical potential, so it can only be evanescent on the N side within that scenario [3, 10]. As Andreev reflection from a confined condensate is ruled out, one is led to consider the alternative case of a decaying condensate.

We define the normal (or non-superfluid) region as that in which the condensate flows faster than the local speed of sound; in our case, the outgoing coherent beam. For simplicity, we assume the interaction coupling strength to be zero on the N side, which does not change the essential physics. In general, transport through an interface separating subsonic from supersonic flow poses a new paradigm in superfluid transport [11].

Like for fermions, particle current is not conserved within a (non-selfconsistent) mean-field description of Andreev processes [12, 13]. In both cases the condensate is responsible for the loss or gain of particles. However, there are several important differences with respect to the fermion case. The eigenvalue problem whose solutions are the Bogoliubov quasiparticles is non-Hermitian. This results in a peculiar normalization condition $\nu \equiv \int (|u|^2 - |v|^2) dx = 1$ for the quasiparticle wave functions [14]. We will see that the conservation of $\nu$ yields unconventional relations between the scattering amplitudes. A counterintuitive but straightforward consequence is that, unlike for fermions, both particle-like and hole-like excitations carry an increase in the density (with respect to the Bogoliubov vacuum), since $\rho \sim |u|^2 + |v|^2$. The existence of these differences suggests that bosonic quasiparticles define a rich novel class of quantum transport problems.

The need for a confined condensate to generate Andreev reflection can be better appreciated from a study of the Bogoliubov – de Gennes (BdG) equations for bosons [12, 16]:

\begin{align*}
\frac{\partial}{\partial t} \psi &= i \Delta \psi + \phi \\
\phi &= i \sum n (u_n \gamma_n - v_n \gamma_n^\dagger)
\end{align*}
The right (left) going wave numbers are $\delta \mu > 0$, i.e. a decaying condensate. Labels are explained after Eq. (1) in main text. Elastic conversion between propagating branches (Andreev reflection) requires $\delta \mu > 0$, i.e. a decaying condensate. The condensate wave function $\Psi$ satisfies the time-dependent Gross-Pitaevskii equation

$$i\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{L}(x) & -n_0(x)g(x) \\ n_0(x)g(x) & -\mathcal{L}^*(x) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (1)$$

$$\mathcal{L}(x) \equiv -\frac{1}{2} D_x^2 + V(x) + 2n_0(x)g(x) - \mu$$

Here, $\hbar = m = 1$ units have been used, $n_0(x)$ is the condensate density, $g(x)$ is the (piecewise constant) coupling strength, $D_x \equiv \partial_x - i\epsilon' \phi(x)$, and $\mu$ is the condensate chemical potential. The scattering channels are defined by the plane wave solutions $(u, v \sim e^{i(kx - ct)})$ in the flat region $V(x) = V_0$. The dispersion relations in the non-superfluid region $(g(x)n_0(x) = 0)$ are shown in Fig. 1. In the confined case ($\delta \mu \equiv \mu - V_0 < 0$), the left Fig. 1 shows that, away from the interface, asymptotic propagation with $\epsilon > 0$ is only possible for the $u$ (particle-like) component, while the $v$ (hole-like) component can only be evanescent. A decaying ($\delta \mu > 0$) condensate exhibits a richer scenario. The right Fig. 1 shows a crossing of the $u$- and $v$-branches. This permits elastic conversion from a particle into a “hole”, in formal analogy with fermionic Andreev reflection. For a given energy within the allowed crossing range $\epsilon \in (-\delta \mu, \delta \mu)$, there are four propagating waves. The right (left) going wave numbers are $k_N^+$ and $-k_N^-$ ($k_S^+$ and $k_S^-$), where $k_N^\pm \equiv \sqrt{2(\delta \mu \pm \epsilon)}$. The group velocities are those of a quadratic dispersion relation, $w_N^2 = k_N^2$. Thus, for an incident particle from the normal side with velocity $w_i = \sqrt{2(\delta \mu + \epsilon)}$, there are two outgoing channels with velocity $w_f = w_i$ (normal reflection) or $w_f^2 + w_r^2 = 4 \delta \mu$ (Andreev reflection). The prediction of this novel form of propagation on the normal side is the main result of this paper.

On the superfluid side, excitations travel with wavevector $k_S^+ \equiv \sqrt{2(\epsilon^2 + \mu^2 - \mu)}$. Here, $\mu = gn_+$, with $n_+ \equiv \lim_{x \to +\infty} n_0(x)$. At low energies, the speed $w_S = d\epsilon/dk_S^+$ approaches the speed of sound $c = gn_+$. An Andreev transmission process occurs for energies $\epsilon \in (0, \delta \mu)$ when a phonon incident from $S$ is transmitted into $N$ as a hole-like ($u = 0$) quasiparticle. The other transmission channel is particle-like ($v = 0$) and travels in $N$ at a different velocity ($w_N^+ \text{ vs. } w_N^-$). Remarkably, the outgoing hole-like scattering channels which characterize an Andreev process may exist despite its anomalous normalization $\nu < 0$. We note, however, that the conservation of $\nu$ applies to the total scattering state: a particular channel may have $\nu < 0$ whereas the total state retains the standard normalization $\nu > 0$.

Bogoliubov quasiparticles are generally defined against the background of a particular condensate wave function. We consider a one-atom potential of the type shown in Fig. 2 which provides a model for a leaking condensate. The condensate wave function $\Psi$ satisfies the time-dependent Gross-Pitaevskii equation.
\[ i\partial_t \Psi = -\frac{1}{2} \partial_{xx} \Psi + V(x)\Psi + g(x)|\Psi|^2\Psi , \quad (2) \]

\[ V(x) = Z \delta(x) + V_0 \theta(-x) . \quad (3) \]

In the asymptotic regions it admits a solution of the type \( \Psi(x \to \pm \infty) \sim e^{i (q - \mu t)} \). In the previous discussion on scattering channels, we have proved that the solutions to the BdG equations (1) do not have complex eigenvalues. In other words, the thick barrier limit the exact stationary solution resembles a solution which has been proven to be stable. In general, the condensed atom density on the N side is uniform and equal to \( n_- = |j|/\sqrt{2\delta\mu} \), with atom velocity \( w = -\sqrt{2\delta\mu} \), as determined by simple energy considerations. In this thick barrier limit, the resulting current is \( j \simeq -\mu^2 \sqrt{2\delta\mu}/gZ^2 \). This solution applies to an interface separating a supersonic (normal) from a subsonic (superfluid) region (11).

Next we write the general form of the scattering states within the approximation of a flowless condensate. For an atom incident from the N side with \( |\epsilon| < \delta\mu \) we have

\[
\Psi_N \equiv \begin{cases} 
(\psi_0) \left[ \chi^+ + r_+ (\chi^+)^* \right] + (\psi_0) r_\epsilon \chi^- , & x \to -\infty \\
(\psi_0) r_\epsilon \chi^- + (\psi_0) r_\epsilon e^{-k_S^-} x , & x \to \infty 
\end{cases}
\]

(4)

where \( \chi^\pm \equiv e^{ik_S^\pm x}/\sqrt{w_S^\pm} \), \( \chi^\pm \equiv e^{i\epsilon x}/\sqrt{w_S^\pm} \) and \( k_S^- \equiv \sqrt{2\alpha + \mu^2 + |\epsilon|} \) and \( w_\alpha , w_0 \) are such that \( w_\alpha^2 + w_0^2 = \sqrt{2\alpha + \mu^2 + |\epsilon|} \) and \( w_\alpha , w_0 \) are such that \( w_\alpha^2 + w_0^2 = \sqrt{2\alpha + \mu^2 + |\epsilon|} / \mu \). For an incoming phonon from the S side with energy \( 0 < \epsilon < \delta\mu \), we have

\[
\Psi_S \equiv \begin{cases} 
(\psi_0) t_n (\chi^+)^* + (\psi_0) t_n \chi^- , & x \to -\infty \\
(\psi_0) \left[ \chi^+ + r_+ \chi^- \right] + (\psi_0) r_\epsilon e^{-k_S^-} x , & x \to \infty 
\end{cases}
\]

(5)

The subindices stand for normal \((n)\), Andreev \((a)\), phonon \((p)\) and evanescent \((e)\), while \( r \) and \( t \) represent reflection and transmission coefficients. Equations (11) and (15) do not include the case \( \epsilon > \delta\mu \), which shows no Andreev processes, or that of an incoming channel with \( \nu < 0 \), which does not seem realizable unless the beam comes from another Bose condensate.

Conservation laws may be obtained from the generalized Wronskian

\[ W_{ij} \equiv u_i D_x^* u_j^* - u_j D_x u_i + v_i D_x v_j^* - v_j D_x^* v_i , \quad (6) \]

which is constant for any two stationary solutions of Eq. (11) with same energy \( \epsilon \). When applied to \( \psi_N \) and \( \psi_S \), the constancy of (6) generates the following relations between the scattering amplitudes:

\[
|r_n|^2 - |r_a|^2 + \text{sgn}(\epsilon) |t_p|^2 = |t_n|^2 - |t_a|^2 + |r_p|^2 = 1 \\
r_n t_n^* + r_a t_a^* + t_p - r_a t_a = 0
\]

(7)

The negative signs here can be traced back to the non-Hermitian character of the effective Hamiltonian in the bosonic BdG equations (1). This represents a major difference with respect to the fermionic Bogoliubov problem and is a source of difficulties in the interpretation of the particle-hole transformation for bosons.

If we assume a step-like profile for the condensate density, the scattering amplitudes can be found analytically. Here we note some important trends in the low transparency limit \( (Z/\sqrt{\mu}) \gg 1 \): The Andreev reflection amplitude \( |r_a| \) scales like \( Z^{-1/2} \), which indicates that, like for superconductors (18), an Andreev reflection process involves the transmission of two atoms. This reflects the pairing correlations which exist in the quantum depletion cloud of the condensate and which are expressed through the structure of the Bogoliubov transformation and the BdG equations (16). Analogously, the scaling \( |t_a| \sim Z^{-1} \) suggests that for large \( Z \) the corresponding scattering processes involve the transmission of one atom.

More information on conservation laws can be obtained from the continuity equation

\[ \partial_t |u|^2 + \partial_x j_u = \partial_t |v|^2 + \partial_x j_v = 2 n_0(x) g(x) \text{Im}(uv^*) , \quad (8) \]

where \( j_x \equiv \text{Im}(s^* D_x s) \), which tells us simultaneously about the lack of conservation of the atom mass current (density \( |u|^2 \) current \( j_u \)) and the conservation of the quasiparticle current (density \( |u|^2 - |v|^2 \), current \( j_u - j_v \)), the latter being also expressed by the constancy of the Wronskian (6). Both mass density and current are defined as their total values in the presence of the excitation of wave function (4) minus the corresponding values of the Bogoliubov vacuum, which includes the zero-temperature depletion cloud (8) and thus a fluctuating atom flux. We note that, while the \( u \) component carries the current in the usual way, the \( v \) does it in the opposite one. Another important feature is that, unlike for its fermionic counterpart (18), the mass current has the same sign as the group velocity in all channels (the mass density is always positive).

As in the case of superconductivity (13, 18), we interpret the non-conservation of atom number as the exchange of atoms with the Bogoliubov vacuum. The net balance of extracted atoms per unit time can be calculated by subtracting, from the current at a point deep
in the superfluid region, the current at any point in the normal region, $\Delta I \equiv j(x_+) - j(x_-)$. We obtain

$$\Delta I = |r_a|^2 + |r_n|^2 + \frac{|t_p|^2}{u_0^2 + v_0^2} - 1 \approx -2 \frac{\text{sgn}(\varepsilon) k_N^+ k_S^+}{1 + \varepsilon^2/\mu^2} \frac{1}{Z^2} \varepsilon/\delta \mu.$$  

For the second equality, which gives the current to leading order in $Z^{-1}$, we have used [7] and the exact results for the scattering amplitudes [17]. Thus, for an atom incident from the N side, the scattering process which follows is such that atoms are lost to the Bogoliubov vacuum if $\varepsilon > 0$, while they are extracted from that vacuum if $\varepsilon < 0$. From the continuity equation [8] it is clear that the transfer of atoms between the condensate and the quasiparticle field takes place exclusively on the S side, since $g(x) = 0$ in N.

The generalization of the above discussion to the case of a three-dimensional planar or two-dimensional linear interface yields qualitative differences with respect to the superconducting case. Invoking the conservation of energy and parallel momentum, we find that for $\varepsilon \in (-\delta \mu, \delta \mu)$ the beam angles (see Fig. 3) satisfy the relation

$$(1 - \varepsilon/\delta \mu) \tan^2 \alpha_i - (1 + \varepsilon/\delta \mu) \tan^2 \alpha_n = 2 \varepsilon/\delta \mu. \quad (10)$$

This result leads to two conclusions: (a) If $0 < \varepsilon < \delta \mu$, no Andreev reflection occurs for $\tan \alpha_i < \sqrt{\varepsilon/2(\delta \mu - \varepsilon)}$ and one has $\alpha_n < \alpha_i$. (b) If $-\delta \mu < \varepsilon < 0$ Andreev reflection is possible, but only satisfying $\alpha_n > \alpha_i$ and $\tan \alpha_n \geq \sqrt{-\varepsilon/2(\delta \mu + \varepsilon)}$.

The observation of bosonic Andreev reflection poses a challenge. A tunable coupling constant can be achieved through the handling of Feshbach resonances [19] or, in the 1D limit, by increasing the width of the confining channel. Such long channels can form in the vicinity of a planar semiconductor chip. A potential barrier at the interface can be introduced with a properly focused, blue-detuned laser. The crucial measurement of escaping velocities may rely on precise Bragg spectroscopy or on the interference between incoming and outgoing beams. The 2D/3D case presents greater potential flexibility because anomalous reflection can occur in a range of different directions. Confinement by a linear interface in 2D could be achieved with a laser beam. For instance, $^{87}$Rb atoms may be confined in a waveguide with transverse trapping frequency of 500 Hz and experience a delta-barrier of $Z = 5\sqrt{\mu}$ if they are exposed to a narrow Gaussian laser of 1 $\mu$m waist, blue-detuned by 10 nm to the $5^2S_1/2-5^2P_{1/2}$ transition [20] (780.24 nm, linewidth 12$\pi$ Hz, saturation intensity 1.64 mW cm$^{-2}$). With a step barrier such that $\delta \mu = 0.5 \mu$, bulk superfluid linear densities in the range 0.2–20 $\mu$m$^{-1}$ place the system in the low-density mean-field region, where the present estimate applies. For sufficient density range we obtain 1D coherence-length 7–0.7 $\mu$m, laser intensity 80–800 mW cm$^{-2}$, and 10–1000 $\mu$m s$^{-1}$ for the Andreev velocity scale ($\sim 2\sqrt{\delta \mu}$), close to the superfluid sound speed of 70–700 $\mu$m s$^{-1}$. For these parameters the emitted condensate linear density is 0.04 times the bulk S density, while the corresponding condensate escape velocity is the same as the speed of sound in S, the equality being due to our choice of $\delta \mu$.

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