Differential-Geometric Decomposition of Flat Nonlinear Discrete-Time Systems *

Bernd Kolar *, Markus Schöberl, Johannes Diwold

Institute of Automatic Control and Control Systems Technology, Johannes Kepler University, Linz, Austria

Abstract

We prove that every flat nonlinear discrete-time system can be decomposed by coordinate transformations into a smaller-dimensional subsystem and an endogenous dynamic feedback. For flat continuous-time systems, no comparable result is available. The advantage of such a decomposition is that the complete system is flat if and only if the subsystem is flat. Thus, by repeating the decomposition at most $n - 1$ times, where $n$ is the dimension of the state space, the flatness of a discrete-time system can be checked in an algorithmic way. If the system is flat, then the algorithm yields a flat output which only depends on the state variables and their forward-shifts. Again, no comparable result for flat continuous-time systems is available. The algorithm requires in each decomposition step the construction of state- and input transformations, which are obtained by straightening out certain vector fields or distributions with the float-box theorem or the Frobenius theorem. Thus, from a computational point of view, only the calculation of flows and the solution of algebraic equations is needed. We illustrate our results by two examples.

Key words: Differential-geometric methods; Discrete-time systems; Nonlinear control systems; Feedback linearization; Difference flatness; Normal forms.

1 Introduction

The concept of flatness has been introduced by Fliess, Lévine, Martin and Rouchon in the 1990s for nonlinear continuous-time systems (see e.g. [4], [5], and [6]). Flat continuous-time systems have the characteristic feature that all system variables can be expressed by a flat output and its time derivatives. They form an extension of the class of static feedback linearizable systems, and can be linearized by an endogenous dynamic feedback. The reason for the ongoing popularity of flat systems lies in the fact that the knowledge of a flat output allows an elegant systematic solution to motion planning problems as well as the design of tracking controllers. However, in contrast to the static feedback linearization problem, which has been solved in [11] and [9], there still exist no efficiently verifiable necessary and sufficient conditions for flatness, and the construction of flat outputs is a challenging problem.

For nonlinear discrete-time systems, flatness can be defined analogously to the continuous-time case. The main difference is that time derivatives are replaced by forward-shifts. To distinguish both concepts, often the terms differential flatness and difference flatness are used (see e.g. [25]). Like in the continuous-time case, flat discrete-time systems form an extension of the class of static feedback linearizable systems, and can be linearized by an endogenous dynamic feedback (see e.g. [13]). The static feedback linearization problem for discrete-time systems has already been studied and solved in several papers using different mathematical frameworks, see [7], [10], and [1]. There exist verifiable necessary and sufficient conditions, which give rise to an algorithm for the calculation of a linearizing output. The more general dynamic feedback linearization problem, which includes flatness as a special case, has been studied for discrete-time systems e.g. in [1] and [2]. In particular, [2] addresses the difference between linearization by endogenous and exogenous dynamic feedback for discrete-time systems. However, like in the continuous-time case, no efficiently verifiable necessary and sufficient conditions are available. Thus, the construction of flat outputs is also a difficult problem.
In practical applications, flat outputs often have some physical meaning, see e.g. [6]. Therefore, the construction of flat outputs is – like the construction of Lyapunov functions – often based on physical considerations. A possible more systematic approach is to transform the system into a decomposed form, where the complete system is flat if and only if a smaller-dimensional subsystem is flat. Repeating this decomposition with the subsystem may then lead after several steps to a flat output. Such methods have been developed with different types of decompositions for continuous-time systems in [21], [22], [23], and [24], and they were transferred to discrete-time systems in [15] and [16] (see also [14]). The fundamental question is, however, under which conditions such decompositions exist, and whether every flat system allows a decomposition or not. For continuous-time systems, this question is a very difficult one. For discrete-time systems, in contrast, the situation is completely different. We present a simple geometric proof that a flat discrete-time system can always be transformed by state- and input transformations into a subsystem and an endogenous dynamic feedback. This type of decomposition has been studied in [15] both in a differential-geometric and an algebraic framework, but without a proof that for flat systems the decomposition is always possible. In the present paper, we focus on the geometric framework. For a further discussion in the algebraic framework, see [12]. The advantage of the geometric approach is that the decompositions can be constructed systematically in special coordinates, and that the proof for the existence of a decomposition of flat systems becomes particularly simple. As a consequence of the latter result, the flatness of discrete-time systems can be checked in an algorithmic way. If the system is flat, then a repeated decomposition will yield a flat output after at most \( n - 1 \) steps, where \( n \) denotes the dimension of the state space. Since the constructed flat output only depends on the state variables, we obtain the additional result that every flat discrete-time system has a flat output which is independent of the inputs and their forward-shifts.

The paper is organized as follows: In Section 2 we recall the definition of difference flatness and give an overview of some important properties of flat discrete-time systems. In Section 3 we discuss the decomposition of discrete-time systems into a subsystem and an endogenous dynamic feedback by means of coordinate transformations. We give geometric conditions for the existence of such a decomposition, and show that for flat systems these conditions are always satisfied. In Section 4 we present an algorithm for the calculation of flat outputs, which is based on a repeated application of the decomposition of Section 3. Furthermore, we show that every flat discrete-time system has a flat output which only depends on the state variables. We illustrate our results by two examples in Section 5.

2 Discrete-Time Systems and Flatness

In this contribution we consider discrete-time systems

\[ x^{i,+} = f^i(x,u), \quad i = 1, \ldots, n \]  

in state representation with \( \dim(x) = n, \dim(u) = m \), and smooth functions \( f^i(x,u) \). Geometrically, such a system can be interpreted as a map \( f \) from a manifold \( \mathcal{X} \times \mathcal{U} \) with coordinates \( (x, u) \) to a manifold \( \mathcal{X}^+ \) with coordinates \( x^+ \). We assume throughout the paper that the system meets

\[ \text{rank}(\partial_{(x,u)}f) = n, \]

which means that the map \( f \) is a submersion and therefore locally surjective. Since this assumption is necessary for accessibility (see e.g. [8]) and consequently also for flatness, it is no restriction. To achieve the desired decompositions, we will use state- and input transformations

\[ \bar{x}^i = \Phi^i_x(x), \quad i = 1, \ldots, n \]
\[ \bar{u}^j = \Phi^j_u(x,u), \quad j = 1, \ldots, m, \]

and it should be noted that the variables \( x^+ \) are transformed of course in the same way as the variables \( x \). The transformed system is given by

\[ \bar{x}^{i,+} = \Phi^i_x(x^+) \circ f(x,u) \circ \hat{\Phi}(\bar{x}, \bar{u}), \quad i = 1, \ldots, n, \]

with the inverse \( (x,u) = \hat{\Phi}(\bar{x}, \bar{u}) \) of (2). The superscript \( + \) is only used to denote the forward-shift of the state variables \( x \). For the inputs and flat outputs we also need higher forward-shifts, and use a subscript in brackets instead. For instance, \( u_{[\alpha]} \) denotes the \( \alpha \)-th forward-shift of \( u \). To keep formulas short and readable, we also use the Einstein summation convention. Furthermore, we want to emphasize that all our results are local. This is due to the use of the inverse- and the implicit function theorem, the flow-box theorem, and the Frobenius theorem, which allow only local results. We also assume that all functions are smooth in order to avoid mathematical subtleties.

In the following, we summarize the concept of difference flatness, which is the discrete-time counterpart of differential flatness for continuous-time systems. Roughly speaking, the main difference is that time derivatives are replaced by forward-shifts. Since many results can be shown in a similar way to the continuous-time case, we omit detailed proofs. Analogously to the static feedback linearization problem for discrete-time systems, we define flatness around an equilibrium

\[ x_0^i = f^i(x_0,u_0), \quad i = 1, \ldots, n \]

of the system (1). The reason is that even in one time step the state of a discrete-time system can move far
away from the initial state, regardless of the input values. Thus, in order not to lose localness, we consider a suitable neighborhood of an equilibrium. To introduce the concept of difference flatness, we need a space with coordinates \((x, u, u_1, u_2, \ldots)\). On this space we have the forward-shift operator \(\delta_{xu}\), which acts on a function \(g\) according to the rule

\[
\delta_{xu}(g(x, u, u_1, u_2, \ldots)) = g(f(x, u, u_1, u_2, u_3, \ldots).
\]

A repeated application of \(\delta_{xu}\) is denoted by \(\delta^n_{xu}\). In this framework, an equilibrium \((x_0, u_0, u_1, \ldots)\), and flatness of discrete-time systems can be defined as follows.

**Definition 1** The system \((1)\) is said to be flat around an equilibrium \((x_0, u_0)\), if the \(n + m\) coordinate functions \(x\) and \(u\) can be expressed locally by an \(m\)-tuple of functions

\[
y^j = \varphi^j(x, u, u_1, \ldots, u_q), \quad j = 1, \ldots, m \quad (4)
\]

and their forward-shifts

\[
y[1] = \delta_{xu}(\varphi(x, u, u_1, \ldots, u_q)) \\
y[2] = \delta^2_{xu}(\varphi(x, u, u_1, \ldots, u_q)) \\
\vdots
\]

up to some finite order. The \(m\)-tuple \((4)\) is called a flat output.

If \((4)\) is a flat output, then the \(m(\beta + 1)\) functions \(\varphi, \delta_{xu}(\varphi), \delta^2_{xu}(\varphi), \ldots, \delta^\beta_{xu}(\varphi)\) are functionally independent for arbitrary \(\beta \geq 0\). Therefore, the representation of \(x\) and \(u\) by the flat output and its forward-shifts is unique, and it has the form

\[
x^i = F^i_x(y_{[0, R-1]}), \quad i = 1, \ldots, n \\
y^j = F^j_u(y_{[0, R]}), \quad j = 1, \ldots, m. \quad (5)
\]

The multi-index \(R = (r_1, \ldots, r_m)\) contains the number of forward-shifts of each component of the flat output which is needed to express \(x\) and \(u\), and \(y_{[0, R]}\) is an abbreviation for \(y\) and its forward-shifts up to order \(R\). Written in components,

\[
y_{[0, R]} = (y_{[0, r_1]}, \ldots, y_{[0, r_m]})
\]

with

\[
y^j_{[0, r_j]} = (y^j_1, y^j_2, \ldots, y^j_{r_j}), \quad j = 1, \ldots, m.
\]

With the forward-shift operator \(\delta_y\) in coordinates \((y, y_1, y_2, \ldots)\), which acts on a function \(h\) according to the rule

\[
\delta_y(h(y, y_1, y_2, \ldots)) = h(y_1, y_2, y_3, \ldots), \quad (6)
\]

the parametrization of arbitrary forward-shifts \(u^j_{[a]}\) of \(u\) follows from \((5)\) as

\[
u^j_{[a]} = \delta^n_y(F^j_u(y_{[0, R]})), \quad j = 1, \ldots, m.
\]

It is a well-known fact that the parametrization \(F_x\) of the state only depends on \(y_{[0, R-1]}\), and that the highest forward-shifts \(y_{[R]} = (y_{[r_1]}, \ldots, y_{[r_m]})\) that are required in \((5)\) only appear in the parametrization \(F_u\) of the input. It is also not hard to show that the map \((x, u) = F(y_{[0, R]})\) given by \((5)\) is a submersion, i.e., that the rows of its Jacobian matrix are linearly independent. Likewise, the map

\[
y = \varphi(x, u, u_1, \ldots, u_q) \\
y^1 = \delta_{xu}(\varphi(x, u, u_1, \ldots, u_q)) \\
y^2 = \delta^2_{xu}(\varphi(x, u, u_1, \ldots, u_q)) \\
\vdots
\]

\[
y_{[R]} = \delta^R_{xu}(\varphi(x, u, u_1, \ldots, u_q))
\]

is also a submersion. This is a simple consequence of the already mentioned functional independence of the flat output and its forward-shifts. If the system \((1)\) is static feedback linearizable and \(y = \varphi(x)\) is a linearizing output, then the submersion \((5)\) becomes a diffeomorphism, and its inverse is given by \((7)\). In this case, the parametrization \((5)\) can be used as a coordinate transformation which transforms the system \((1)\) into the discrete-time Brunovsky normal form.

If we substitute the parametrization \((5)\) into the identity

\[
\delta_{xu}(x^i) = f^i(x, u), \quad i = 1, \ldots, n,
\]

we get the important identity

\[
\delta_y(F^i_x(y_{[0, R-1]})) = f^i \circ F(y_{[0, R]}), \quad i = 1, \ldots, n. \quad (8)
\]

Because of \((8)\), it is obvious that \(F_x\) can indeed only depend on \(y_{[0, R-1]}\). Otherwise, \(\delta_y(F_x)\) would depend on forward-shifts of \(y\) that are not contained in \(y_{[0, R]}\). A further fundamental consequence of the identity \((8)\) and the special form of the forward-shift operator \((6)\) is that the system equations \((1)\) do not impose any restrictions on the feasible trajectories

\[
y^j(k), \quad j = 1, \ldots, m \quad (9)
\]
of the flat output (4). That is, for every trajectory (9) of the flat output there exists a uniquely determined solution \((x(k), u(k))\) of the system (1) such that the equations
\[
y^j(k) = \varphi^j(x(k), u(k), u(k + 1), \ldots, u(k + q)),
\]
j = 1, \ldots, m are satisfied identically. The trajectories \(x(k)\) and \(u(k)\) of state and input are determined by \(y(k)\) and its forward-shifts via the parametrization (5). Thus, just like in the case of differentially flat continuous-time systems, there is a one-to-one correspondence between solutions of the system (1) and arbitrary trajectories of the flat output.

3 Decomposition of Flat Systems

In this section we deal with a transformation of the system (1) into a certain decomposed form, which can be interpreted as a splitting into a subsystem and an endogenous dynamic feedback. This decomposed form has the property that the complete system is flat if and only if the subsystem is flat.

**Lemma 2** A system of the form
\[
x_i^{1, +} = f_{1i}^{1}(x_1, x_2, u_1), \quad i_1 = 1, \ldots, n - m_2
\]
\[
x_{i_2}^{2, +} = f_{2i}^{2}(x_1, x_2, u_1, u_2), \quad i_2 = 1, \ldots, m_2
\]
with \(\dim(u_2) = \dim(x_2) = m_2\) and \(\text{rank}(\partial_u f) = \dim(u) = m\) is flat if and only if the subsystem
\[
x_i^+ = f_1(x_1, x_2, u_1)
\]
with the \(m\) inputs \((x_2, u_1)\) is flat.

**Proof.**

**Flatness of (11) ⇒ Flatness of (10):** If \(y\) is a flat output of the subsystem (11), then the system variables \(x_1, x_2,\) and \(u_1\) of this subsystem can be expressed as functions of \(y\) and its forward-shifts. Because of the regularity of the Jacobian matrix \(\partial_{u_2} f_2\), which is an immediate consequence of rank(\(\partial_u f) = \dim(u)\) and the structure of (10), the implicit function theorem allows to express \(u_2\) as function of \(x_1, x_2, u_1,\) and \(x_2^{+}\). Consequently, \(u_2\) can also be expressed as a function of \(y\) and its forward-shifts, and \(y\) is a flat output of the complete system (10).

**Flatness of (10) ⇒ Flatness of (11):** Because of the regularity of \(\partial_{u_2} f_2\), we can perform an input transformation
\[
\hat{u}_2^{j_2} = f_2^{j_2}(x_1, x_2, u_1, u_2), \quad j_2 = 1, \ldots, m_2
\]
such that (10) takes the simpler form
\[
x_i^{1, +} = f_{1i}^{1}(x_1, x_2, u_1), \quad i_1 = 1, \ldots, n - m_2
\]
\[
x_{i_2}^{2, +} = \hat{u}_2^{j_2}, \quad i_2 = 1, \ldots, m_2.
\]

If
\[
y = \varphi(x_1, x_2, u_1, \hat{u}_2, u_{1, [1]}, \hat{u}_{2,[1]}, \ldots, u_{1, [q]}, \hat{u}_{2, [q]})
\]
is a flat output of (12), then by substituting \(\hat{u}_2^{j_2} = f_2^{j_2}(x_1, x_2, u_1, u_2)\) and \(u_{2,[\alpha]} = f_{2,[\alpha+1]}(x_1, x_2, u_1, u_2)\), \(\alpha \geq 1\) we immediately get a flat output of the subsystem (11).

Note that the Jacobian matrix \(\partial_{[x_2,u_1]} f_1\) does not necessarily have rank \(m\). Thus, the subsystem (11) may have redundant inputs. In this case, a flat output of the subsystem (11) contains components of \(x_2\).

**Remark 3** The structure of (10) and
\[
\text{rank} \left( \begin{bmatrix} \partial_{u_1} f_1 & 0 \\ \partial_{u_2} f_1 & \partial_{u_2} f_2 \end{bmatrix} \right) = \text{rank}(\partial_u f) = m
\]

imply \(\text{rank}(\partial_{u_1} f_1) = m - m_2 = \dim(u_1)\). As a consequence, the redundant inputs of the subsystem (11) can always be found among the variables \(x_2\).

The equations
\[
x_{i_2}^{2, +} = f_2^{i_2}(x_1, x_2, u_1, u_2), \quad i_2 = 1, \ldots, m_2
\]
of (10) can be interpreted as an endogenous dynamic feedback for the subsystem (11). This is in accordance with the fact that applying or removing an endogenous dynamic feedback has no effect on the flatness of a system.

Our next objective is to derive necessary and sufficient differential-geometric conditions for the existence of a transformation of the system (1) into the decomposed form (10). To formulate these conditions, we use the notion of \(f\)-related vector fields. For completeness, we briefly explain the basics. More details can be found in [3]. By \(f_\ast : T(\mathcal{X} \times \mathcal{U}) \to T(\mathcal{X}^+ \times \mathcal{U})\) we denote the tangent map of \(f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}^+ \times \mathcal{U}\), and by \(f_{2p} \ast : T_p(\mathcal{X} \times \mathcal{U}) \to T_{f^p(\mathcal{X})}(\mathcal{X}^+ \times \mathcal{U})\) we denote the tangent map of \(f\) at some point \(p \in \mathcal{X} \times \mathcal{U}\). If
\[
v = v_2^i(x, u)\partial_{x_i} + v_1^i(x, u)\partial_{u_i}
\]
is a vector field on \(\mathcal{X} \times \mathcal{U}\), then the vector \(f_{2p} \ast(v_p)\) at \(f(p) \in \mathcal{X}^+\) is called the pushforward of the vector \(v_p\) at \(p \in \mathcal{X} \times \mathcal{U}\) by \(f\). However, since \(f\) is only a submersion and not a diffeomorphism, the vector field \(v\) does not necessarily induce a well-defined vector field on \(\mathcal{X}^+\).

The problem is that the inverse image \(f^{-1}(q)\) of a point \(q \in \mathcal{X}^+\) is an \(m\)-dimensional submanifold of \(\mathcal{X} \times \mathcal{U}\), and it may happen that for a pair of points \(p_1\) and \(p_2\) on this submanifold we get \(f_{2p_1} \ast(v_{p_1}) \neq f_{2p_2} \ast(v_{p_2})\). In other words, the vector at the point \(f(p_1) = f(p_2) = q\) may be not unique. If, however, there exists a vector field
\[
w = w^i(x^+)\partial_{x_i^+}
\]
on $\mathcal{X}^+$ such that for all $q \in \mathcal{X}^+$ and $p \in f^{-1}(q) \subset \mathcal{X} \times \mathcal{U}$ we have $f_{*p}(v_p) = w_q$, then the vector fields $v$ and $w$ are said to be $f$-related and we write $w = f_*(v)$. In components, $f$-relatedness means

$$w^i(x^+) \circ f(x, u) = \partial_{x^i} f^* v^k_+(x, u) + \partial_{w^i} f^* v^k_0(x, u),$$

$i = 1, \ldots, n$. Since we assume that $f$ is a submersion and therefore locally surjective, the vector field (14) determined by a given vector field (13) is unique if it exists. Moreover, as a submersion, the map $f$ induces a fibration (foliation) of the manifold $\mathcal{X} \times \mathcal{U}$ with $m$-dimensional fibres (leaves). Thus, we will adopt some terminology used for fibre bundles (see e.g. [20]), and call vector fields (13) on $\mathcal{X} \times \mathcal{U}$ that are $f$-related to a vector field (14) on $\mathcal{X}^+$ “projectable”. Similarly, we will call a distribution $D$ on $\mathcal{X} \times \mathcal{U}$ “projectable” if it admits a basis that consists of projectable vector fields. Since we deal particularly with involutive distributions, we will also make use of the fact that the Lie brackets $[v_1, v_2]$ and $[w_1, w_2]$ of two pairs $v_1, w_1$ and $v_2, w_2$ of $f$-related vector fields are again $f$-related, i.e.,

$$f_*(v_1, v_2) = (w_1, w_2).$$

For this reason, the pushforward $f_*D$ of an involutive projectable distribution is again an involutive distribution.

Checking whether a vector field or distribution is projectable or not becomes very simple if we use coordinates on $\mathcal{X} \times \mathcal{U}$ that are adapted to the fibration. Adapted coordinates can be introduced by a transformation of the form

$$x^i_+ = f^i(x, u), \quad i = 1, \ldots, n,$$

$$\xi^j = h^j(x, u), \quad j = 1, \ldots, m,$$

(15)

where the $m$ functions $h^j(x, u)$ must be chosen in such a way that (15) is a (local) diffeomorphism. Thus, the Jacobian matrix

$$\begin{bmatrix}
\partial_x f & \partial_u f \\
\partial_x h & \partial_u h
\end{bmatrix}$$

must be regular. Because of the linear independence of the rows of the Jacobian matrix of a submersion, this is always possible. With coordinates $(x^+, \xi)$ on $\mathcal{X} \times \mathcal{U}$, the map $f$ takes the simple form $f = \text{pr}_1$. All points of $\mathcal{X} \times \mathcal{U}$ with the same value of $x^+$ belong to the same fibre and are mapped to the same point of $\mathcal{X}^+$, regardless of the value of the fibre coordinates $\xi$. The vector field (13) in adapted coordinates has in general the form

$$v = a^i(x^+, \xi) \partial_{x_+^i} + b^j(x^+, \xi) \partial_{\xi^j},$$

(16)

and because of $f = \text{pr}_1$ an application of the tangent map $f_*$ to (16) yields

$$f_*(v) = a^i(x^+, \xi) \partial_{x_+^i}.$$  

(17)

Obviously, (17) is a well-defined vector field on $\mathcal{X}^+$ if and only if the functions $a^i$ are independent of the coordinates $\xi$. In this case, (17) corresponds to the vector field (14).

With these mathematical preliminaries, we can formulate conditions for the existence of a transformation of the system (1) into the form (10).

**Theorem 4** Consider a system (1) with $\text{rank}(\partial_u f) = m$. There exists a coordinate transformation

$$\begin{align}
(\bar{x}_1, \bar{x}_2) &= \Phi(u)(x) \\
(\bar{u}_1, \bar{u}_2) &= \Phi(u)(x, u)
\end{align}$$

(18a)

with $\text{dim}(\bar{u}_2) = \text{dim}(\bar{x}_2) = m_2$ such that in transformed coordinates the system has the form

$$\begin{align}
\bar{x}_1^+ &= \bar{f}_1(\bar{x}_1, \bar{x}_2, \bar{u}_1) \\
\bar{x}_2^+ &= \bar{f}_2(\bar{x}_1, \bar{x}_2, \bar{u}_1, \bar{u}_2)
\end{align}$$

(19)

if and only if on $\mathcal{X} \times \mathcal{U}$ there exists an $m_2$-dimensional projectable and involutive subdistribution $D \subset \text{span}\{\partial_u\}$.

**Proof.**

**Sufficiency:** Since $D$ is involutive and $D \subset \text{span}\{\partial_u\}$, because of the Frobenius theorem there exists an input transformation (18b) with $\text{dim}(\bar{u}_2) = m_2$ such that $D = \text{span}\{\partial_{\bar{u}_2}\}$. Furthermore, since $D \subset \text{span}\{\partial_u\}$ is projectable and the Jacobian matrix $\partial_u f$ has full rank, the pushforward $f_*D$ is a well-defined $m_2$-dimensional involutive distribution on $\mathcal{X}^+$. Thus, because of the Frobenius theorem there exists a state transformation (18a) with $\text{dim}(\bar{x}_2) = m_2$ such that $f_*D = \text{span}\{\partial_{\bar{x}_2}\}$. In these coordinates, the transformed map $\bar{f}(\bar{x}, \bar{u})$ has the form (19). This can be seen as follows: Let $f_1$ and $f_2$ denote the $x_1$- and $x_2$-components of $f$. Then the (pointwise defined) pushforwards of the vector fields $\partial_{\bar{u}_2}$, $j_2 = 1, \ldots, m_2$ are given by

$$f_*(\partial_{\bar{u}_2}) = \partial_{\bar{x}_1^+} f_1 \partial_{\bar{x}_1^+} + \partial_{\bar{x}_2^+} f_2 \partial_{\bar{x}_2^+}.$$  

Since by construction $f_*(\partial_{\bar{u}_2}) \in f_*D = \text{span}\{\partial_{\bar{x}_2^+}\}$, we immediately get

$$\partial_{\bar{u}_2} f_1 = 0, \quad i_1 = 1, \ldots, n - m_2, \quad j_2 = 1, \ldots, m_2,$$

which shows that the functions $f_1^i$ are independent of $\bar{u}_2$.

**Necessity:** To prove necessity, assume that there exists a coordinate transformation (18) such that (1) takes the form (19). Because of $\text{rank}(\partial_{\bar{u}_2} f_2) = m_2$, there exists a

Note again that state transformations are performed simultaneously for $x$ and $x^+$. 

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where the first $n - m$ equations are independent of all inputs. For systems that are only flat but not static feedback linearizable, a transformation into the form (20) is in general not possible. However, we will show that a flat system can always be transformed into the form (19) with $m_2 \geq 1$. That is, in the “worst case” with $m_2 = 1$ there exists at least a decomposition

$$
\bar{x}^1, + = \bar{f}^1(\bar{x}, \bar{u}^1, \ldots, \bar{u}^{m-1})
$$

$$
\vdots
$$

$$
\bar{x}^{n-1}, + = \bar{f}^{n-1}(\bar{x}, \bar{u}^1, \ldots, \bar{u}^{m-1})
$$

$$
\bar{x}^{n, +} = \bar{f}^n(\bar{x}, \bar{u}^1, \ldots, \bar{u}^{m-1}, \bar{u}^m)
$$

(21)

where the first $n-1$ equations are independent of $\bar{u}^m$. To keep the proof of this remarkable feature of flat systems as short as possible, it is convenient to rewrite the conditions of Theorem 4 for the case $m_2 = 1$ in terms of $f$-related vector fields instead of distributions.

**Corollary 5** A system (1) with $\text{rank}(\partial_w f) = m$ can be transformed into the form (21) if and only if there exists a pair of vector fields

$$
v = v^j(x, u)\partial_w^j
$$
on $X \times U$ and

$$
w = w^i(x^+)\partial_{x^+}^i
$$
on $X^+$ which are $f$-related, i.e., that satisfy

$$
w^i \circ f(x, u) = (\partial_{x^+} f^i(x, u)) v^j(x, u), \quad i = 1, \ldots, n.
$$

(22)

---

**Proof.**

**Sufficiency:** Because of the flow-box theorem, there exists an input transformation

$$
\bar{u}^j = \Phi^j_w(x, u), \quad j = 1, \ldots, m
$$

which transforms the vector field $v$ into the form

$$
v = \partial_{\bar{u}^m},
$$

and a state transformation

$$
\bar{x}^i = \Phi_w^i(x), \quad i = 1, \ldots, n
$$

which transforms the vector field $w$ into the form

$$
w = \partial_{\bar{x}^{n, +}}.
$$

In these new coordinates, condition (22) has the form

$$
\delta_i = (\partial_w \bar{f}(\bar{x}, \bar{u})) \delta_m, \quad i = 1, \ldots, n.
$$

Because of $\partial_w \bar{f}(\bar{x}, \bar{u}) = 0$ for $i = 1, \ldots, n-1$, the functions $\bar{f}^1, \ldots, \bar{f}^{n-1}$ are independent of $\bar{u}^m$.

**Necessity:** If the system is in the form (21), we can perform an input transformation $\bar{u}^m = \bar{f}^m(\bar{x}, \bar{u})$ such that we get

$$
\bar{x}^1, + = \bar{f}^1(\bar{x}, \bar{u}^1, \ldots, \bar{u}^{m-1})
$$

$$
\vdots
$$

$$
\bar{x}^{n-1}, + = \bar{f}^{n-1}(\bar{x}, \bar{u}^1, \ldots, \bar{u}^{m-1})
$$

$$
\bar{x}^{n, +} = \bar{u}^m.
$$

In these coordinates, it is obvious that the vector fields $v = \partial_{\bar{u}^m}$ and $w = \partial_{\bar{x}^{n, +}}$ are $f$-related. □

The concept of the proofs of Theorem 4 and Corollary 5 is of course almost identical. The difference is that in the proof of Corollary 5 we straighten out vector fields with the flow-box theorem, whereas in the proof of Theorem 4 we straighten out distributions with the Frobenius theorem. The connection between the distributions of Theorem 4 and the vector fields of Corollary 5 is obviously given by

$$
D = \text{span}\{v\} \quad \text{and} \quad f_* D = \text{span}\{w\}.
$$

In the following, we prove the main result of the paper.
Theorem 6 A flat system (1) with rank(∂uf) = m can be transformed into the form (21), i.e., (19) with \( m_2 = 1 \).

Proof. The proof is based on the identity (8). Differentiating both sides of (8) with respect to \( y_{[r,s]} \) for some arbitrary \( s \in \{1, \ldots, m\} \) gives

\[
\partial_{y_{[r,s]}} \left( \delta_y (F_2^j) \right) = (\partial_{x,v} f^i \circ F) \partial_{y_{[r,s]}} F_2^j, \quad i = 1, \ldots, n.
\]

Since \( \delta_y \) only substitutes variables, shifting and differentiating with respect to \( y_{[r,s]} \) is equivalent to first differentiating with respect to \( y_{[r,s]-1} \) and shifting afterwards. Thus, we get the equivalent identity

\[
\delta_y \left( \partial_{y_{[r,s]-1}} F_2^j \right) = (\partial_{x,v} f^i \circ F) \partial_{y_{[r,s]}} F_2^j, \quad i = 1, \ldots, n. \tag{23}
\]

Now let us consider this identity in coordinates \((x, u, u_1, \ldots)\). Substituting (7) into (23) gives the identity

\[
\delta_{xu} (\tilde{w}^i (x, u, u_1, \ldots)) = (\partial_{x,v} f^i) \tilde{v}^j (x, u, u_1, \ldots). \tag{24}
\]

The functions \( \tilde{w}^i (x, u, u_1, \ldots) \) and \( \tilde{v}^j (x, u, u_1, \ldots) \) of (24) are obtained by substituting (7) into (23) and using \( \partial_{y_{[r,s]-1}} F_2^j \) and \( \partial_{y_{[r,s]}} F_2^j \) of (23). Note also that substituting (7) into \( \partial_{x,v} f^i \circ F \) yields just \( \partial_{x,v} f^i \), and that we have to replace the shift operator \( \delta_y \) in \( y \)-coordinates by the shift operator \( \delta_{xu} \) in \( (x, u) \)-coordinates.

Evaluating the expression \( \delta_{xu} (\tilde{w}^i (x, u, u_1, \ldots)) \) on the left-hand side of (24) yields

\[
\tilde{w}^i (f(x, u), u_1, u_2, \ldots) = (\partial_{x,v} f^i) \tilde{v}^j (x, u, u_1, \ldots). \tag{25}
\]

This identity holds (locally) for all values of \( x, u, u_1, \ldots \). Thus, if we evaluate (25) at any particular point of our underlying space with concrete numerical values of the coordinates \((x, u, u_1, u_2, \ldots)\), we still get a valid identity. The same is of course true if we evaluate (25) on a subspace by setting only some of the coordinates to numerical values. For our purpose, it is beneficial to evaluate the identity on a subspace determined by

\[
\begin{align*}
  u_1 &= c_1 \\
  u_2 &= c_2 \\
  \vdots
\end{align*}
\]

with arbitrary numerical values for the forward-shifts of \( u \) appearing in (25). By doing so, we get the relation

\[
\tilde{w}^i (f(x, u), c_1, c_2, \ldots) = (\partial_{x,v} f^i) \tilde{v}^j (x, u, c_1, \ldots). \tag{26}
\]

With

\[
\tilde{w}^i (x^+, c_1, c_2, \ldots) = w^i (x^+, c_1, c_2, \ldots) \tag{27}
\]

and

\[
w^i (x, u) = \tilde{v}^j (x, u, c_1, \ldots) \tag{28}
\]

this can be written as

\[
w^i (x^+ \circ f (x, u)) = (\partial_{x,v} f^i) w^j (x, u), \quad i = 1, \ldots, n, \tag{29}
\]

which is just condition (22). Thus, the vector fields

\[
v = v^j (x, u) \partial_{x,v} \tag{29}
\]

on \( \mathcal{X} \times \mathcal{U} \) and

\[
w = w^i (x^+) \partial_{x^+,v} \tag{30}
\]

on \( \mathcal{X}^+ \) are \( f \)-related. Applying Corollary 5 completes the proof. \( \square \)

Remark 7 Let us summarize the idea of the proof once more. Starting with the identity (8), which is a basic property of every flat system, it is always possible to construct a pair of \( f \)-related vector fields (29) and (30). The choice for the numerical values \( c_1, c_2, \ldots \), which are used for the construction of these vector fields, is of course not unique, and for different choices we get in general different pairs of \( f \)-related vector fields. However, as soon as we have any pair of \( f \)-related vector fields, no matter how they were constructed, we can straighten them out by the flow-box theorem and get a state transformation and an input transformation which transforms the system into the decomposed form (21), cf. Corollary 5. The transformed system equations (21) are just as general as the original ones, and of course not restricted to input sequences with the numerical values (26) used for the construction of the vector fields (29) and (30). Substituting numerical values (26) into the identity (25) is just a useful operation to construct the vector fields (29) and (30), but does not restrict the validity of the transformed system equations (21) obtained by straightening these vector fields out.

As a consequence of Theorem 6, the existence of a decomposed form (21) is a necessary condition for flat discrete-time systems. Based on similar ideas as in the proof of Theorem 6, it has been shown in [17] that for flat continuous-time systems

\[
\dot{x} = f^i (x, u), \quad i = 1, \ldots, n
\]

otherwise, (25) would not hold any more. A choice which is sufficiently close to the value of \( u_0 \) from the equilibrium (3) is always possible.
there always exists a transformation \( \bar{u} = \Phi_u(x, u) \) into the so-called partial affine input form (PAI-form)

\[
\dot{x}^i = a^i(x, \bar{u}^1, \ldots, \bar{u}^{m-1}) + b^i(x, \bar{u}^1, \ldots, \bar{u}^{m-1}) \bar{u}^m, \quad i = 1, \ldots, n,
\]

where \( \bar{u}^m \) appears in an affine way. This PAI-form is closely related to the well-known ruled manifold necessary condition derived in [19] for flat continuous-time systems. Thus, the existence of the decomposed form (21) for flat discrete-time systems can be interpreted as discrete-time counterpart to the existence of a PAI-form (31) for flat continuous-time systems.

4 Calculation of Flat Outputs

We show in this section that a repeated application of the results of Section 3 gives rise to an algorithm, which allows to check the flatness of a discrete-time system (1) with \( \text{rank}(\partial_u f) = m \) in at most \( n - 1 \) steps. If the system is flat, then the algorithm provides a flat output. Otherwise, it stops and we can conclude that the system is not flat. Roughly speaking, the idea is as follows: If the system (1) is flat, then Theorem 6 guarantees that it can be transformed into form (19) with an at most \( (n - 1) \)-dimensional subsystem \( \bar{x}_1 = f_1(\bar{x}_1, \bar{x}_2, \bar{u}_1) \). Because of Lemma 2 this subsystem is also flat, and therefore Theorem 6 guarantees that the subsystem can again be transformed into form (19). Repeating this procedure reduces the problem of checking the flatness of the original system (1) to the problem of checking the flatness of smaller and smaller subsystems. Obviously, for a system (1) with \( \text{dim}(x) = n \) we can perform at most \( n - 1 \) such decomposition steps. If in some step we encounter a subsystem with the same number of input and state variables, then we can read off a flat output of this subsystem (the state variables), and the original system is also flat. Otherwise, if we find a subsystem which does not allow a further decomposition, then Theorem 6 implies that this subsystem is not flat. Therefore, the original system (1) cannot be flat either.

What we have not mentioned in this brief sketch of the basic idea is the fact that there may appear subsystems with redundant inputs, i.e., where the Jacobian matrix with respect to the inputs of the subsystem does not have full rank (see Lemma 2 and Remark 3). In this case, we have to eliminate these redundant inputs with a suitable coordinate transformation, before we can apply Theorem 4 to construct a decomposition of the subsystem.

Remark 8 This effect is well-known from static feedback linearization, see e.g. [18]. For instance, if a static feedback linearizable system is transformed into the form (20), it may happen that \( \text{rank}(\partial_{\bar{u}} f_1) < m \).

The elimination of redundant inputs is, however, very easy: For a system (1) with \( \text{rank}(\partial_u f) = \hat{m} < m \) there always exists an input transformation \( \hat{u} = \hat{\Phi}_u(x, u) \) with \( \text{dim}(\hat{u}) = \hat{m} \) that eliminates \( m - \hat{m} \) redundant inputs \( \ddot{u} \). If \( u = \hat{\Phi}_u(x, \hat{u}, \ddot{u}) \) denotes the inverse input transformation, then the transformed system is of the form

\[
\dot{x}_i^+ = f_i^i(\hat{u}, \ddot{u}), \quad i = 1, \ldots, n
\]

with

\[
f_i(x, \hat{u}) = f_i(x, \hat{\Phi}_u(x, \hat{u}, \ddot{u})), \quad i = 1, \ldots, n
\]

and \( \text{rank}(\partial_{\hat{u}} f) = \hat{m} \). The following lemma establishes an important connection between a flat output of the transformed system (32) with \( \hat{m} \) inputs and the original system (1) with \( m \) inputs.

Lemma 9 Consider a system (1) with \( \text{rank}(\partial_u f) = m < \hat{m} \), and an input transformation \( \hat{u} = \hat{\Phi}_u(x, u) \) with \( \text{dim}(\hat{u}) = \hat{m} \) that eliminates \( m - \hat{m} \) redundant inputs \( \ddot{u} \). If an \( \hat{m} \)-tuple \( \hat{y} \) is a flat output of the transformed system (32) with \( \hat{m} \) inputs \( \hat{u} \), then the \( m \)-tuple \( y = (\hat{y}, \ddot{u}) \) is a flat output of the original system (1) with \( m \) inputs \( u \).

Proof. Since \( \hat{y} \) is a flat output of the transformed system (32), \( x \) and \( \hat{u} \) can be expressed as functions of \( \hat{y} \) and its forward-shifts. Because of \( \hat{y} = (\hat{y}, \ddot{u}) \), the inverse input transformation \( u = \hat{\Phi}_u(x, \hat{u}, \ddot{u}) \) shows immediately that the input \( u \) of the original system (1) can be expressed by \( y \) and its forward-shifts. Thus, eliminated redundant inputs are candidates for components of a flat output.

Now we can describe the algorithm in detail. To enhance the readability, every step is divided into three subtasks: (A) checking whether the (sub-)system is flat by a simple dimension argument, (B) checking whether a decomposition is possible, (C) performing the decomposition. For the decompositions, we use the more general formulation of Theorem 4 with distributions, instead of the 1-dimensional special case of Corollary 5 with vector fields. To keep the notation as simple as possible, after every decomposition step the state and input of the remaining subsystem are renamed again as \( x \) and \( u \).

Algorithm 10 Start with the original system (1) and the first decomposition step with \( k = 1 \). We assume that the original system meets \( \text{rank}(\partial_u f) = m \), i.e., has no redundant inputs.

Decomposition Step \( k \geq 1 \):

(A) If \( \text{dim}(x) > \text{dim}(u) \), go to (B). Otherwise, \( y_k = x \) is a flat output of the system considered in the \( k \)-th decomposition step. A flat output of the original system is given by \( y = (y_k, \ldots, y_1) \). This follows immediately from a \( (k - 1) \)-fold application of Lemma 2 and Lemma 9.
(B) Transform the input vector fields $\partial_u$ into adapted coordinates (15), and check whether there exists a projectable and involutive subdistribution $D \subset \text{span}\{\partial_u\}$. In case of a positive result, go to (C). In case of a negative result, according to Theorem 6 the system considered in the $k$-th decomposition step is not flat. By a $(k-1)$-fold application of Lemma 2, the original system cannot be flat either.

In the case of a positive result, continue with item (A) of the next decomposition step.

Second, in item (C) the distribution $D$ and its pushforward $f_*D$ have to be straightened out by an input transformation and a state transformation. Since straightening out involutive distributions by the Frobenius theorem requires the solution of (nonlinear) ODEs, this task is typically considerably more difficult than the construction of the distributions in item (B). However, for the calculation of a linearizing output of a static feedback linearizable system it is also necessary to straighten out a sequence of distributions by the Frobenius theorem. Thus, from a computational point of view, the construction of a flat output is essentially of the same complexity as the construction of a linearizing output of a static feedback linearizable system. The main difference is that we have to solve additionally algebraic equations to determine a suitable subdistribution $D \subset \text{span}\{\partial_u\}$, whereas in the static feedback linearization problem we always work with the complete distribution $D = \text{span}\{\partial_u\}$.

Remark 11 Note that in general the choice of an involutive distribution $D$ that meets the conditions of Theorem 4, or equivalently a pair of vector fields that meets the conditions of Corollary 5, is not unique. Thus, the decomposition (19) is not uniquely determined.

The algorithm is in fact a generalization of the transformation of static feedback linearizable systems into a triangular form which is discussed in [18]. The transformation into this triangular form can be interpreted as a repeated application of the decomposition (20), and yields a linearizing output. For the calculation of flat outputs, we simply have to replace the decomposition (20) by the more general decomposition (19). However, it is important to emphasize that the decompositions we perform in each of the steps are typically not unique, and that different decompositions might lead to different flat outputs. This is in accordance with the fact that flat outputs (of multi-input systems) are never unique. It is also obvious that every flat output which is obtained by the suggested algorithm can only depend on $x$ and $u$ but not on forward-shifts of $u$. This is a simple consequence of the fact that we do not introduce any additional variables. Since the algorithm yields (in principle) a flat output for every flat discrete-time system, we can conclude that every flat discrete-time system must have a flat output which only depends on $x$ and $u$. By a closer inspection, we get an even stronger result.

In fact, the formulation of Theorem 6 guarantees already the existence of a decomposition for flat systems. However, according to Theorem 4, this is equivalent to the existence of an at least 1-dimensional projectable and involutive subdistribution $D \subset \text{span}\{\partial_u\}$.\footnote{In fact, the formulation of Theorem 6 guarantees already the existence of a decomposition for flat systems. However, according to Theorem 4, this is equivalent to the existence of an at least 1-dimensional projectable and involutive subdistribution $D \subset \text{span}\{\partial_u\}$.

Note that, as discussed in Lemma 9, adding $y_k$ to a flat output of (34) yields a flat output of (33).}
Theorem 12  Every flat discrete-time system (1) with \( \text{rank}(\partial_u f) = m \) has a flat output of the form \( y = \varphi(x) \), which is independent of the input \( u \) and its forward-shifts.

Proof. Suppose the algorithm terminates after \( k = \bar{k} \) steps. Then the constructed flat output is of the form

\[ y = (y_\bar{k}, \ldots, y_1), \]

where \( y_\bar{k} \) consists of the state variables of the last subsystem, and \( y_{\bar{k}-1}, \ldots, y_1 \) are eliminated redundant inputs of the subsystems constructed in the first \( \bar{k} - 1 \) decomposition steps. Thus, input variables of the original system could only appear in the components \( y_{\bar{k}-1}, \ldots, y_1 \). However, as discussed in Remark 3 and item (C) of the algorithm, due to the full rank of the Jacobian matrix \( \partial_u f \) the redundant inputs of the subsystems (33) can always be found among the (transformed) state variables of the complete system. Thus, the flat output depends indeed only on the state variables. \( \Box \)

This result is also remarkable, since Theorem 12 does not have a counterpart for flat continuous-time systems.

5 Examples

In this section, we illustrate our results with two examples.

5.1 An Academic Example

In the following, we demonstrate the algorithm for the calculation of flat outputs with the system

\[
\begin{align*}
\dot{x}_1 &= \frac{x_2^2 + x_3^2 + 3x_4^2}{u_1^2 + 2u_2^2 + 1} \\
\dot{x}_2 &= x_3^2 + 1 + u_1^2 + 2u_2 - 3 + x_4 - 3u_2^2 \\
\dot{x}_3 &= u_1 + 2u_2 \\
\dot{x}_4 &= x_3^2 + 1 + u_2^2 
\end{align*}
\]

(36)

All coordinate transformations that we will perform are defined in a neighborhood of the equilibrium \( (x_0, u_0) = (0, 0) \).

In the first step, we have to check the existence of a projectable involutive subdistribution \( D \subset \text{span}\{\partial_u \xi\} \). For this purpose, we introduce adapted coordinates (15) on \( X \times U \). After the transformation

\[
\begin{align*}
\xi^1 &= x_1 \\
\xi^2 &= x_3 \\
\xi^3 &= x_4 \\
\end{align*}
\]

the vector fields \( \partial_{\xi^1} \) and \( \partial_{\xi^2} \) are given by

\[-\frac{x_1^1}{x_2^1 + 1} \partial_{x^1} + \xi^1_2 \partial_{x^2} + \partial_{x^3} + \]

and

\[-2 \frac{x_2^1}{x_2^1 + 1} \partial_{x^1} + (2\xi^1_2 + 1) - \partial_{x^2} + 2\partial_{x^3} + \partial_{x^4}.
\]

Because of the presence of the fibre coordinates \( \xi^1 \) and \( \xi^2 \), neither \( \partial_{\xi^1} \) nor \( \partial_{\xi^2} \) itself is projectable. However, the linear combination \( -2\partial_{\xi^1} + \partial_{\xi^2} \) reads in adapted coordinates as

\[-3\partial_{x^2} + \partial_{x^4},
\]

and is hence a projectable vector field. Since there is no other possibility (besides a scaling), the distribution \( D = \text{span}\{-2\partial_{\xi^1} + \partial_{\xi^2}\} \) is uniquely determined, and the pushforward yields \( f_* D = \text{span}\{-3\partial_{\bar{x}_2} + \partial_{\bar{x}_4}\} \). Because of \( \text{dim}(D) < 2 \), the system (36) cannot be static feedback linearizable. Now we straighten out the involutive distributions \( D \) and \( f_* D \) by input-and state transformations. The input transformation

\[
\begin{align*}
\bar{u}^1 &= u^1 + 2u^2 \\
\bar{u}^2 &= u^2 
\end{align*}
\]

gives \( D = \text{span}\{\partial_{\bar{u}^2}\} \), and the state transformation

\[
\begin{align*}
\bar{x}^1 &= x^1 \\
\bar{x}^2 &= x^2 + 3x^4 \\
\bar{x}^3 &= x^3 \\
\bar{x}^4 &= x^4 
\end{align*}
\]

gives \( f_* D = \text{span}\{\partial_{\bar{x}_4}\} \). Accordingly, the transformed system reads

\[
\begin{align*}
\bar{x}^1 &= \frac{x^2 + x^3}{u^1 + 1} \\
\bar{x}^2 &= \bar{x}^1(\bar{x}^3 + 1)\bar{u}^1 + \bar{x}^4 \\
\bar{x}^3 &= \bar{u}^1 \\
\bar{x}^4 &= \bar{x}^1(\bar{x}^3 + 1) + \bar{u}^2,
\end{align*}
\]

where the first three equations are independent of \( \bar{u}^2 \). Now consider the subsystem

\[
\begin{align*}
\bar{x}^1 &= \frac{x^2 + x^3}{u^1 + 1} \\
\bar{x}^2 &= \bar{x}^1(\bar{x}^3 + 1)\bar{u}^1 + \bar{x}^4 \\
\bar{x}^3 &= \bar{u}^1
\end{align*}
\]

(37)

with the inputs \( (\bar{x}^4, \bar{u}^1) \). Because of \( \text{rank}(\partial_{\bar{x}^4, \bar{u}^1}) = 2 \), where by \( \bar{f} \) we refer to the system (37), there are no redundant inputs. Thus, \( y_1 = \{\} \) is empty.
In the second step, the complete input distribution of the system (37) is projectable (this can be verified again by introducing adapted coordinates) and we can choose $D = \text{span}\{\partial_{\bar{x}^4}, \partial_{y_1}\}$, which is clearly involutive. Since this distribution is already straightened out, we need no input transformation, i.e., we can simply set
\[
\bar{x}^4 = \bar{x}^4 \\
\bar{u}^1 = \bar{u}^1.
\]

The pushforward of $D$ is given by
\[
\tilde{f}_s D = \text{span}\{\partial_{\bar{x}^2+}, -\bar{x}^{3,+} \partial_{\bar{x}^1,+} + \partial_{\bar{x}^3,+}\},
\]
and the state transformation
\[
\bar{x}^1 = \bar{x}^1(\bar{x}^3 + 1) \\
\bar{x}^2 = \bar{x}^2 \\
\bar{x}^3 = \bar{x}^3
\]
yields $\tilde{f}_s D = \text{span}\{\partial_{\bar{x}^2+}, \partial_{\bar{x}^3+}\}$. In new coordinates, the system reads
\[
\bar{x}^{1,+} = \bar{x}^2 + \bar{x}^3 \\
\bar{x}^{2,+} = \bar{x}^1 \bar{u}^1 + \bar{x}^4 \\
\bar{x}^{3,+} = \bar{u}^1,
\]
where the first line is independent of both inputs $\bar{x}^4$ and $\bar{u}^1$. The subsystem
\[
\bar{x}^{1,+} = \bar{x}^2 + \bar{x}^3 \tag{38}
\]
with the inputs $(\bar{x}^2, \bar{x}^3)$ meets $\text{rank}(\partial_{[\bar{x}^2, \bar{x}^3]} \tilde{f}) = 1 < 2$. Thus, there exists a redundant input. The elimination of a redundant input is obviously not unique. Possible choices are e.g. the transformations
\[
\hat{\hat{\bar{x}}} = \bar{x}^2 + \bar{x}^3 \\
y_2 = \bar{x}^2
\]
or
\[
\hat{\hat{\bar{x}}} = \bar{x}^2 + \bar{x}^3 \\
y_2 = \bar{x}^3.
\]
In both cases, the transformed system (38) reads
\[
\bar{x}^{1,+} = \hat{\hat{\bar{x}}} \tag{39}
\]

In the third step, we finally have a system with the same number of input- and state variables. Thus, a flat output of (39) is given by $y_1 = \bar{x}^1$. Adding the redundant input $y_2$ yields a flat output of (38), which is also a flat output of the complete system (36). In original coordinates, the flat output $y = (\bar{x}^1, \bar{x}^3)$ is given by $y = (x^1(x^3 + 1), x^2 + 3x^4)$, and the flat output $y = (\bar{x}^1, \bar{x}^3)$ is given by $y = (x^1(x^3 + 1), x^3)$.

For the flat output
\[
y = (x^1(x^3 + 1), x^2 + 3x^4),
\]
the map (5) is given by
\[
x^1 = \frac{y^1}{y_{[1]} - y^2 + 1} \\
x^2 = 3y^1(y_{[2]} - y_{[3]}) + y^2 - 3y_{[1]}^2 \\
x^3 = y_{[1]}^2 - y^2 \\
x^4 = y_{[1]}^3 - y_{[2]}^2 + y_{[1]}^2 \\
u^1 = 2y^1 + 2y_{[1]}(y_{[3]} - y_{[2]}) + y_{[2]}^2 - y_{[1]}^2 - 2y_{[2]}^2 \\
u^2 = -y^1 + y_{[1]}(y_{[2]}^2 - y_{[3]}) + y_{[2]}^2.
\]
That is, there appear forward-shifts of $y^1$ and $y^2$ up to the orders $r_1 = 3$ and $r_2 = 2$. In the following, we shall use this example to illustrate the method that we have applied in the proof of Theorem 6 to show that every flat system allows a decomposition (21). Since the 1-dimensional distributions $D$ and $f_s D$ in the first decomposition step of the system (36) are unique, the method of Theorem 6 must yield exactly the same decomposition. Substituting (7) into
\[
\partial_{y_{[2]}} F^i_x, \quad i = 1, \ldots, 4
\]
and
\[
\partial_{y_{[3]}} F^j_u, \quad j = 1, 2
\]
yields
\[
\hat{\hat{\bar{w}}}^1 = 0 \\
\hat{\hat{\bar{w}}}^2 = 3x^3(x^3 + 1) \\
\hat{\hat{\bar{w}}}^3 = 0 \\
\hat{\hat{\bar{w}}}^4 = -x^1(x^3 + 1)
\]
and
\[
\hat{\hat{\bar{v}}}^1 = 2(x^2 + x^3 + 3x^4) \\
\hat{\hat{\bar{v}}}^2 = -(x^2 + x^3 + 3x^4).
\]
Since the functions $\hat{\hat{\bar{w}}}$ are independent of $u$ and the functions $\hat{\hat{\bar{v}}}$ are independent of forward-shifts of $u$, we directly get $\hat{w}^i = \hat{\hat{\bar{w}}}^i$ and $\hat{v}^i = \hat{\hat{\bar{v}}}^i$. It can be checked easily that the condition (22) is indeed satisfied. Therefore, the pair of vector fields
\[
v = 2(x^2 + x^3 + 3x^4)\partial_{y_1} - (x^2 + x^3 + 3x^4)\partial_{y_2}
\]
and
\[
w = 3x^{1,+}(x^{3,+} + 1)\partial_{y_2} - x^{1,+}(x^{3,+} + 1)\partial_{y_4}
\]
is $f$-related. Because of
\[ \text{span}\{v\} = \text{span}\{-2\partial_{u^3} + \partial_{u^2}\} = D \] (40)
and
\[ \text{span}\{w\} = \text{span}\{-3\partial_{x^2} + \partial_{x^1}\} = f_* D, \] (41)

these vector fields span exactly the same distributions that we have constructed in the first decomposition step in adapted coordinates.

Remark 13. It should be noted that the case $w^i = \tilde{w}^i$ and $v^i = \tilde{v}^i$ is a special one and does not hold in general. With the more sophisticated flat output $y = (x^1(x^3 + 1) + e^{u^1+2u^2}, x^3)$, we would get functions $\tilde{w}^i$ and $\tilde{v}^i$ that also depend on forward-shifts of $u$. After setting these forward-shifts to constant values as shown in (27) and (28), the resulting $f$-related vector fields
\[ v = e^{c_1+2c_3}(2\partial_{u^1} - \partial_{u^2}) \]
and
\[ w = e^{c_1+2c_3}(3\partial_{x^2} - \partial_{x^1}) \]
span again the same distributions (40) and (41), independent of the chosen values $c_1$ and $c_3$. This is a consequence of the fact that system (36) possesses only a 1-dimensional projectable subdistribution $D \subseteq \text{span}\{\partial_u\}$.

5.2 A Wheeled Mobile Robot

As a second example, we consider the exact discretization of the kinematic model
\[ \begin{align*}
\dot{x}^1 &= \sin(x^3)u^1 \\
\dot{x}^2 &= \cos(x^3)u^1 \\
\dot{x}^3 &= u^2
\end{align*} \] (42)

of a wheeled mobile robot, which is also discussed in the context of dynamic feedback linearization in [2]. The variables $x^1$ and $x^2$ describe the position of the center of the axle, and $x^3$ its orientation. The control inputs are the translatory velocity $u^1$ and the angular velocity $u^2$. It is well-known that the continuous-time system (42) is flat, and a flat output is given by $y = (x^1, x^2)$, i.e., by the position of the axle.

With the assumption that the inputs $u^1$ and $u^2$ are constant between sampling instants, the system (42) can be solved analytically, and an exact discrete-time model is given by
\[ \begin{align*}
x^{1,+} &= x^1 + 2u^1\psi(u^2)\cos(\gamma(x^3, u^2)) \\
x^{2,+} &= x^2 + 2u^1\psi(u^2)\sin(\gamma(x^3, u^2)) \\
x^{3,+} &= x^3 + Tu^2
\end{align*} \] (43)

with
\[ \psi(u^2) = \begin{cases} 
\sin\left(\frac{T}{2}u^2\right), & \text{if } u^2 \neq 0 \\
\frac{T}{2}, & \text{if } u^2 = 0 
\end{cases} \]
and
\[ \gamma(x^3, u^2) = x^3 + \frac{T}{2}u^2, \]
see [2]. As shown in [2], the system (43) can be linearized by an exogenous dynamic feedback. In the following, we prove that the system is not flat, and can thus indeed not be linearized by an endogenous dynamic feedback. Before we apply our algorithm, we perform the input transformation
\[ \bar{u}^1 = 2u^1\psi(u^2) \]
\[ \bar{u}^2 = \gamma(x^3, u^2) \]
to obtain the simpler system representation
\[ \begin{align*}
x^{1,+} &= x^1 + \bar{u}^1\cos(\bar{u}^2) \\
x^{2,+} &= x^2 + \bar{u}^1\sin(\bar{u}^2) \\
x^{3,+} &= -x^3 + 2\bar{u}^2.
\end{align*} \]

Now let us check the existence of a projectable involutive subdistribution $D \subseteq \text{span}\{\partial_u\}$. After introducing adapted coordinates
\[ \begin{align*}
x^{1,+} &= x^1 + \bar{u}^1\cos(\bar{u}^2) \\
x^{2,+} &= x^2 + \bar{u}^1\sin(\bar{u}^2) \\
x^{3,+} &= -x^3 + 2\bar{u}^2. \\
\xi^1 &= x^3 \\
\xi^2 &= \bar{u}^1
\end{align*} \]
on $X \times U$, the vector fields $\partial_{\xi^1}$ and $\partial_{\xi^2}$ are given by
\[ \cos\left(\frac{x^{3,+}+\xi^1}{2}\right)\partial_{x^{1,+}} + \sin\left(\frac{x^{3,+}+\xi^1}{2}\right)\partial_{x^{2,+}} + \partial_{\xi^2} \]
and
\[ -\xi^2\sin\left(\frac{x^{3,+}+\xi^1}{2}\right)\partial_{x^{1,+}} + \xi^2\cos\left(\frac{x^{3,+}+\xi^1}{2}\right)\partial_{x^{2,+}} + 2\partial_{x^{3,+}}. \]

With a normalized basis of the form (A.4), it can be observed that there does not exist any projectable linear combination of these vector fields (there does not exist any linear combination where the coefficients of $\partial_{x^{1,+}}$, $\partial_{x^{2,+}}$ and $\partial_{x^{3,+}}$ are independent of $\xi^1$ and $\xi^2$). Thus, the algorithm stops already in the first step with a negative result. Since every flat discrete-time system possesses an at least 1-dimensional projectable subdistribution $D \subseteq \text{span}\{\partial_u\}$, the exact discretization (43) of the wheeled mobile robot (42) is not flat.

An Euler discretization, in contrast, would preserve the flatness and even the flat output $y = (x^1, x^2)$ of
the continuous-time system (42). In fact, the Euler discretization
\[
\begin{align*}
x_{1}^{+} &= x_{1} + T \sin(x_{3})u_{1} \\
x_{2}^{+} &= x_{2} + T \cos(x_{3})u_{1} \\
x_{3}^{+} &= x_{3} + Tu_{2}
\end{align*}
\]
is already in the decomposed form (10) with \( m_{2} = 1 \). Thus, the flat output \( y = (x_{1}, x_{2}) \) can be read off directly from the system equations. However, it is important to emphasize that in general also an Euler discretization does not necessarily preserve the flatness of continuous-time systems. In case of the mobile robot (42), the flatness and the particular flat output are preserved because of the special triangular structure of the system.

### 6 Conclusion

We have shown that every flat discrete-time system can be decomposed by state- and input transformations into a subsystem and an endogenous dynamic feedback. This remarkable feature can be considered as discrete-time counterpart to the existence of a PAI-form (31) for flat continuous-time systems, which is closely related to the well-known ruled-manifold necessary condition. In contrast to the PAI-form or the ruled-manifold criterion, such a decomposition directly gives rise to an algorithm which allows to check the flatness of a discrete-time system in at most \( n - 1 \) steps. If the system is flat, then the algorithm yields a flat output which only depends on the state variables. Consequently, every flat discrete-time system has a flat output which does not depend on the inputs and their forward-shifts. Compared to the complexity of the flatness problem in the continuous-time case, these results represent a fundamental simplification. From a computational point of view, it would nevertheless be desirable to avoid the coordinate transformations that have to be performed in each of the steps. Thus, current research is concerned with the development of a coordinate-independent test for flatness. More precisely, the idea is to separate the test for flatness from the calculation of a flat output, similar to the test for static feedback linearizability. Furthermore, motivated by the existence of flat outputs which only depend on the state variables, future work will address the question whether there exist suitable normal forms for flat discrete-time systems.

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### A Appendix

The purpose of this section is to illustrate a computationally efficient construction of projectable linear combinations of the input vector fields \( \partial_{u} \).

In adapted coordinates (15), the \( m \) input vector fields \( \partial_{u} \) are of the form
\[
\begin{align*}
v_{1} = a_{1}^{1}(x^{+}, \xi)\partial_{x_{1}^{+}} + \ldots + a_{n}^{1}(x^{+}, \xi)\partial_{x_{n}^{+}} \\
&+ b_{1}^{1}(x^{+}, \xi)\partial_{\xi_{1}} + \ldots + b_{m}^{1}(x^{+}, \xi)\partial_{\xi_{m}} \\
&\vdots \\
v_{m} = a_{1}^{m}(x^{+}, \xi)\partial_{x_{1}^{+}} + \ldots + a_{n}^{m}(x^{+}, \xi)\partial_{x_{n}^{+}} \\
&+ b_{1}^{m}(x^{+}, \xi)\partial_{\xi_{1}} + \ldots + b_{m}^{m}(x^{+}, \xi)\partial_{\xi_{m}}.
\end{align*}
\] (A.1)

Now we have to check whether there exists a linear combination
\[
c^{1}(x^{+}, \xi)v_{1} + \ldots + c^{m}(x^{+}, \xi)v_{m}
\] (A.2)
which is of the form
\[
a^{i}(x^{+})\partial_{x_{i}^{+}} + b^{i}(x^{+}, \xi)\partial_{\xi_{i}},
\]
i.e., projectable. The criterion is that the resulting coefficients \( c^{i}(x^{+}, \xi)a_{k}^{i}(x^{+}, \xi) \) of the linear combination (A.2) in the directions \( \partial_{x_{i}^{+}} \), \( i = 1, \ldots, n \) must be independent of \( \xi \), i.e., they must satisfy
\[
\partial_{\xi_{i}}(c^{i}(x^{+}, \xi)a_{k}^{i}(x^{+}, \xi)) = 0
\] (A.3)
for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). To avoid the partial derivatives of the unknown coefficients \( c^{i}(x^{+}, \xi) \), it is beneficial to use a normalized basis
\[
\begin{align*}
v_{1} &= \partial_{x_{1}^{+}} \\
&+ a_{m+1}^{1}(x^{+}, \xi)\partial_{x_{m+1}^{+}} + \ldots + a_{n}^{1}(x^{+}, \xi)\partial_{x_{n}^{+}} \\
&+ b_{1}^{1}(x^{+}, \xi)\partial_{\xi_{1}} + \ldots + b_{m}^{1}(x^{+}, \xi)\partial_{\xi_{m}} \quad (A.4)
\end{align*}
\]
\[
\begin{align*}
v_{m} &= \partial_{x_{m}^{+}} \\
&+ a_{m+1}^{m}(x^{+}, \xi)\partial_{x_{m+1}^{+}} + \ldots + a_{n}^{m}(x^{+}, \xi)\partial_{x_{n}^{+}} \\
&+ b_{1}^{m}(x^{+}, \xi)\partial_{\xi_{1}} + \ldots + b_{m}^{m}(x^{+}, \xi)\partial_{\xi_{m}}
\end{align*}
\]
for the distribution spanned by the vector fields (A.1). Up to a renumbering of the state variables, this can always be achieved by suitable linear combinations\(^9\) Because of the \( m \times m \) identity matrix in the coefficients of the normalized basis (A.4), the equations (A.3) with \( i = 1, \ldots, m \) imply that all coefficients \( c_{1}, \ldots, c_{m} \) must

---

\(^9\) Because of \( \text{rank}(\partial_{u}f) = m \), the matrix formed by the coefficients \( a_{k}^{i} \) of (A.1) has full rank \( m \).
be independent of $\xi$. Consequently, the remaining equations of (A.3) with $i = m + 1, \ldots, n$ simplify to the $(n-m)m$ algebraic equations

$$c^k(x^+^0)\partial^j_\xi a^k_j(x^+, \xi) = 0, \quad i = m+1, \ldots, n, \quad j = 1, \ldots, m.$$ 

References

[1] E. Aranda-Bricaire, Ú. Kotta, and C.H. Moog. Linearization of discrete-time systems. *SIAM Journal on Control and Optimization*, 34(6):1999–2023, 1996.

[2] E. Aranda-Bricaire and C.H. Moog. Linearization of discrete-time systems by exogenous dynamic feedback. *Automatica*, 44(7):1707–1717, 2008.

[3] W.M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, Orlando, 2nd edition, 1986.

[4] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *Comptes rendus de l’Académie des sciences. Série I, Mathématique*, 315:619–624, 1992.

[5] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of non-linear systems: introductory theory and examples. *International Journal of Control*, 61(6):1327–1361, 1995.

[6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Transactions on Automatic Control*, 44(5):922–937, 1999.

[7] J.W. Grizzle. Feedback linearization of discrete-time systems. In A. Bensoussan and J.L. Lions, editors, *Analysis and Optimization of Systems*, volume 83 of *Lecture Notes in Control and Information Sciences*, pages 273–281. Springer, Berlin, 1986.

[8] J.W. Grizzle. A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM Journal on Control and Optimization*, 31(4):1026–1044, 1993.

[9] L. Hunt and R. Su. Linear equivalents of nonlinear time varying systems. In *Proceedings 5th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, pages 119–123, 1981.

[10] B. Jakubczyk. Feedback linearization of discrete-time systems. *Systems & Control Letters*, 9(5):411–416, 1987.

[11] B. Jakubczyk and W. Respondek. On linearization of control systems. *Bull. Acad. Polonaise Sci. Ser. Sci. Math.*, 28:517–522, 1980.

[12] A. Kaldmäe. *Advanced Design of Nonlinear Discrete-time and Delayed Systems*. PhD thesis, Tallinn University of Technology, 2016.

[13] A. Kaldmäe and Ú. Kotta. On flatness of discrete-time nonlinear systems. In *Proceedings 9th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, pages 588–593, 2013.

[14] B. Kolar. *Contributions to the Differential Geometric Analysis and Control of Flat Systems*. Shaker Verlag, Aachen, 2017.

[15] B. Kolar, A. Kaldmäe, M. Schöberl, Ú. Kotta, and K. Schlacher. Construction of flat outputs of nonlinear discrete-time systems in a geometric and an algebraic framework. *IFAC-PapersOnLine*, 49(18):796–801, 2016.

[16] B. Kolar, M. Schöberl, and K. Schlacher. A decomposition procedure for the construction of flat outputs of discrete-time nonlinear control systems. In *Proceedings 22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, pages 775–782, 2016.

[17] B. Kolar, M. Schöberl, and K. Schlacher. Properties of flat systems with regard to the parameterization of the system variables by the flat output. *IFAC-PapersOnLine*, 49(18):814–819, 2016.

[18] H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, New York, 1990.

[19] P. Rouchon. Necessary condition and genericity of dynamic feedback linearization. *Journal of Mathematical Systems, Estimation, and Control*, 4(2):1–14, 1994.

[20] D.J. Saunders. *The Geometry of Jet Bundles*. Cambridge University Press, Cambridge, 1989.

[21] K. Schlacher and M. Schöberl. Construction of flat outputs by reduction and elimination. In *Proceedings 7th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, pages 666–671, 2007.

[22] K. Schlacher and M. Schöberl. A jet space approach to check Pfaffian systems for flatness. In *Proceedings 52nd IEEE Conference on Decision and Control (CDC)*, pages 2576–2581, 2013.

[23] M. Schöberl. *Contributions to the Analysis of Structural Properties of Dynamical Systems in Control and Systems Theory - A Geometric Approach*. Shaker Verlag, Aachen, 2014.

[24] M. Schöberl and K. Schlacher. On an implicit triangular decomposition of nonlinear control systems that are 1-flat - a constructive approach. *Automatica*, 50:1649–1655, 2014.

[25] H. Sira-Ramirez and S.K. Agrawal. *Differentially Flat Systems*. Marcel Dekker, New York, 2004.