Self-testing entangled measurements in quantum networks

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Self-testing refers to the possibility of characterizing an unknown quantum device based only on the observed statistics. Here we develop methods for self-testing entangled quantum measurements, a key element for quantum networks. Our approach is based on the natural assumption that separated physical sources in a network should be considered independent. This provides a natural formulation of the problem of certifying entangled measurements. Considering the setup of entanglement swapping, we derive a robust self-test for the Bell-state measurement, tolerating noise levels up to ~5%. We also discuss generalizations to other entangled measurements.

Introduction.—The advent of quantum communication paves the way towards the development of quantum networks, where local quantum processors exchange information and entanglement via quantum links [1, 2]. It is therefore important, though challenging, to devise certification methods for ensuring the correct functioning of such a complex structure. The first step consists of certifying the correct operation of the building blocks of the quantum network, e.g. sources producing entanglement and nodes performing local quantum operations. A simple architecture of a quantum network is that of a quantum repeater [3]. Here several independent sources distribute entangled photon pairs between distant nodes. Typically, the latter perform entangled (or joint) quantum measurements—where the measurement eigenstates are entangled—on photons coming from different sources, which enables the distribution of entanglement between distant nodes, initially uncorrelated. Such a network thus features two basic ingredients: sources of entangled states and entangled quantum measurements. In this work we focus on the latter, i.e. certifying entangled measurements.

The problem of certifying sources of entangled states has already attracted considerable attention. In particular, “self-testing” techniques have been developed, which represent the strongest possible form of certification. Based on the degree of violation of a Bell inequality [4, 5], one can certify that the (uncharacterized) entangled state prepared in an experiment is close to a desired target (ideal) state [7]. For instance, the maximal violation of the well-known CHSH Bell inequality [6] implies that the underlying state is essentially a two-qubit maximally entangled state [8–12]. Importantly, such certification does not rely on a previous characterization of the local measurement devices and, therefore, is fully device-independent (i.e. black-box certification). Self-testing methods have been developed for a wide range of entangled quantum states [13–17]. Crucially, these methods could be made robust to noise [18–20], which makes them relevant in an experimental context. More recently, self-testing methods were developed for the certification of channels [23], and adapted to prepare-and-measure set-ups [24].

On the other hand, the problem of certifying entangled quantum measurements has received much less attention. A few works demonstrated that the use of an entangled measurement can be guaranteed from statistics in specific cases. This involved either the maximal violation of a Bell inequality [25, 26], an assumption on the dimension of the systems [27–30], or the use of causal models [31]. Importantly, however, none of these methods gives a precise characterization of the entangled measurement; they simply certify the mere fact that some of the measurement operators are entangled.

In this work we present self-testing methods tailored to entangled quantum measurements. After formalizing the problem, we present a self-test of the Bell-state measurement (BSM), arguably the paradigmatic example of an entangled measurement and a key ingredient in many quantum information protocols [32]. Specifically, we show that observing particular statistics in the entanglement swapping scenario necessarily implies that the performed measurement is equivalent to the BSM. After discussing generalizations to other entangled measurements, we derive a noise-robust self-test for the BSM. All these results are device independent but require the natural assumption that the two separated physical sources are independent. We conclude with some open questions.

Formalizing the problem.—Previous works have developed methods for self-testing entangled states, as well as sets of local measurements. For instance, observing the maximal quantum violation of the CHSH Bell inequality implies that the local measurements are essentially a pair of anti-commuting Pauli observables [10, 33]. Hence, what is certified in this case is how two measurements relate to each other, but not what they are individually.

In the present work, we focus on a different problem, namely to self-test a single measurement featuring entangled eigenstates. For clarity, we first formalize the problem without considering the specific structure of the eigenstates. Let \( \mathcal{P} = \{P^d_A\}_{d=1}^D \) be the “ideal” \( d \)-outcome measurement acting on a Hilbert space \( \mathcal{H}_A \) and \( \mathcal{F} = \{F^d_A\}_{d=1}^D \) be the “real” measurement acting on
$\mathcal{H}_A$. Our goal is formalize the notion that $\mathcal{P}$ and $\mathcal{F}$ are in some sense equivalent. In the standard tomographic (device-dependent) setting we would simply require that all the measurement operators are the same (implying that $\mathcal{H}_{A'} = \mathcal{H}_A$). Clearly, this cannot work in the device-independent setting as, for instance, one cannot even certify that two Hilbert spaces have the same dimension.

In the device-independent scenario the best we can hope for - what we achieve here for the BSM - is to certify that $\mathcal{F}$ is at least as powerful as $\mathcal{P}$, i.e. that $\mathcal{F}$ can be used to simulate $\mathcal{P}$. We say that $\mathcal{F}$ is capable of simulating $\mathcal{P}$ if there exists a completely positive unital map $\Lambda : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_{A'})$ such that

$$\Lambda(F^A_j) = P^j_{A'}$$

for all $j$ and let us justify this definition by providing an explicit simulation procedure. Note that the dual map $\Lambda^\dagger : \mathcal{L}(\mathcal{H}_{A'}) \to \mathcal{L}(\mathcal{H}_A)$ is completely positive and trace-preserving, i.e. it is a quantum channel. Given an unknown state $\sigma$ acting on $\mathcal{H}_{A'}$, we would like to obtain the statistics produced under the ideal measurement $\mathcal{P}$. It suffices to apply the channel $\Lambda^\dagger$ to $\sigma$ and perform the real measurement $\mathcal{F}$. Indeed, the probability of observing the outcome $j$ is given by

$$\Pr[j] = \text{Tr} \left( \Lambda^\dagger(\sigma) F^A_j \right) = \text{Tr} \left( \sigma P^j_{A'} \right),$$

matching the statistics of the ideal measurement. It is important that the quantum channel $\Lambda^\dagger$ is universal, i.e. it does not depend on the input state $\sigma$.

The second key aspect of our problem is the fact that the measurement eigenstates are entangled. Clearly, this is only meaningful given that there is a well-defined bipartition for the measurement device. This point is addressed in a very natural way in the context of quantum networks. Consider as in Fig. 1 a network featuring three observers (Alice, Bob, and Charlie), and two separated sources: the first source distributes a quantum system to Alice and Bob (represented by a state on $\mathcal{H}_A \otimes \mathcal{H}_B_1$), while the second source distributes a system to Bob and Charlie (given by a state on $\mathcal{H}_B_2 \otimes \mathcal{H}_C$). It is natural to assume that, due to their separation, the two sources are independent from each other, an assumption also made in recent works discussing Bell nonlocality in networks (see e.g. [34, 35]). Hence, Bob receives two well-defined physical systems (one from Alice and one from Charlie), which ensures that his measurement device features a natural bipartition, specifically $\mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$.

The problem of self-testing an entangled measurement can now be formalized as follows. Given an ideal measurement for Bob $\mathcal{P} = (P^j_{B_1 B_2})_{j=1}^d$ acting on $\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$ and a measurement $\mathcal{F} = (F^j_{B_1 B_2})_{j=1}^d$ acting on $\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$ we say that $\mathcal{F}$ is capable of simulating $\mathcal{P}$ if there exist completely positive unital maps $\Lambda_{B_1} : \mathcal{L}(\mathcal{H}_{B_1}) \to \mathcal{L}(\mathcal{H}_{B_1})$ and $\Lambda_{B_2} : \mathcal{L}(\mathcal{H}_{B_2}) \to \mathcal{L}(\mathcal{H}_{B_2})$ such that

$$(\Lambda_{B_1} \otimes \Lambda_{B_2})(P^j_{B_1 B_2}) = P^j_{B_1 B_2}$$

for all $j$. Next we look at specific scenarios and show that observing certain statistics allows us to self-test an entangled measurement, i.e. conclude that the real measurement applied in the experiment is capable of simulating some specific ideal measurement.

**Self-testing the Bell-state measurement.**—Let us start by presenting a simple procedure for self-testing the BSM. The four Bell states (maximally entangled two-qubit states) are given by

$$|\Phi^0\rangle := |\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Phi^1\rangle := |\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\Phi^2\rangle := |\psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\Phi^3\rangle := |\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

and the BSM corresponds to $\Phi = (\Phi^b)_{b=0}^3$ with $\Phi^b := |\Phi^b\rangle\langle \Phi^b|$. Our certification procedure relies on the task of entanglement swapping [36], see Fig. 1. The goal is to generate entanglement between two initially uncorrelated parties (Alice and Charlie) by using an additional party (Bob) who is independently entangled with each of them. Specifically, Alice and Bob share a maximally entangled state $|\phi^+\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B_1$, and similarly Bob and Charlie share $|\phi^+\rangle_{BC} \in \mathcal{H}_B_2 \otimes \mathcal{H}_C$. When Bob performs the BSM and obtains outcome $b$, the state of Alice and Charlie is projected to $\Phi^b_{AC}$. That is, for each outcome $b$, Alice and Charlie now share one of the four Bell states. If the outcome $b$ is communicated to (say) Alice she can apply a local unitary operation on her qubit, so that she now shares with Charlie a specific Bell state.

Our self-testing procedure is based on the observation that for every outcome of Bob, the conditional state shared between Alice and Charlie can be self-tested and, moreover, we can choose their local measurements to be independent of $b$. If Alice and Charlie perform the local
measurements \( A_0 := \sigma_z, A_1 := \sigma_x, C_0 := (\sigma_z + \sigma_x)/\sqrt{2}, \)
\( C_1 := (\sigma_x - \sigma_z)/\sqrt{2}, \) their statistics conditioned on \( b \) will
maximally violate some CHSH inequality. More specifically, we will observe \( \text{CHSH}_b = 2\sqrt{2}, \)
where
\[
\text{CHSH}_0 := (A_0C_0) + (A_0C_1) + (A_1C_0) - (A_1C_1),
\]
\[
\text{CHSH}_1 := (A_0C_0) + (A_0C_1) - (A_1C_0) + (A_1C_1),
\]
\[
\text{CHSH}_2 := -\text{CHSH}_1, \quad \text{CHSH}_3 := -\text{CHSH}_0.
\]
It turns out that observing these statistics necessarily implies that Bob performs a BSM, according to the
definition given in Eq. (2).

**Theorem 1.** Let the initial state shared by Alice, Bob
and Charlie be of the form
\[
\tau_{AB_1B_2C} = \tau_{AB_1} \otimes \tau_{B_2C}
\]
and let \( \mathcal{B} := (B_{b_1B_2})_{b_1=0}^3 \) be a four-outcome measurement
acting on \( \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}. \) If there exist measurements for
Alice and Charlie such that the resulting statistics conditioned on \( b \) exhibit the maximal violation of the CHSH
inequality, then there exist completely positive and unital
maps \( \Lambda_{B_1} : \mathcal{L}(\mathcal{H}_{B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A'}), \Lambda_{B_2} : \mathcal{L}(\mathcal{H}_{B_2}) \rightarrow \mathcal{L}(\mathcal{H}_{C'}) \)
for \( \mathcal{A}' = \mathcal{C}' = 2 \) such that
\[
(\Lambda_{B_1} \otimes \Lambda_{B_2})(B_{b_1B_2}) = \Phi_{\mathcal{A}'\mathcal{C}'}
\]
for \( b \in \{0, 1, 2, 3\}. \)

While a complete proof is given in Appendix B, we only briefly sketch the argument here. From now on, it is
important to distinguish the ideal system (denoted with primes) from the real system (without primes). Let \( p_b \)
be the probability of Bob observing the outcome \( b \) and let \( \tau_{\mathcal{A}\mathcal{C}'}^b \)
be the normalized state between Alice and Charlie conditioned on that particular outcome, i.e.
\[
p_b \tau_{\mathcal{A}\mathcal{C}'}^b = \text{Tr}_{B_1B_2} \left[ \mathbb{1}_{\mathcal{A}\mathcal{C}'} \otimes B_{b_1B_2}^b (\tau_{AB_1} \otimes \tau_{B_2C}) \right].
\]
Since every conditional state \( \tau_{\mathcal{A}\mathcal{C}'}^b \) maximally violates some CHSH inequality, the standard self-testing result
[11, 20] tells us that for each \( b \) there exist local extraction channels that produce a maximally entangled state of
two qubits. In fact, since the extraction channels are always constructed from the local observables which do
not depend on \( b \), there exists a single pair of extraction channels \( \Gamma_A : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_{A'}), \Gamma_C : \mathcal{L}(\mathcal{H}_C) \rightarrow \mathcal{L}(\mathcal{H}_{C'}), \)
which always produces the “correct” maximally entangled
state, i.e.
\[
(\Gamma_A \otimes \Gamma_C)(\tau_{\mathcal{A}\mathcal{C}'}^b) = \Phi_{\mathcal{A}'\mathcal{C}'}.
\]
Since applying these extraction channels commutes with the measurement performed by Bob, we can formally
construct the state \( \sigma_{A'\mathcal{B}_1} := (\Gamma_A \otimes \mathbb{1}_{\mathcal{B}_1})(\tau_{AB_1}). \) Since this
is a positive semidefinite operator satisfying \( \sigma_{A'} = \mathbb{1}/2, \) it can be rescaled to become the Choi state of a unital
map from \( \mathcal{L}(\mathcal{H}_{B_1}) \) to \( \mathcal{L}(\mathcal{H}_{A'}). \) More specifically, we
choose the Choi state of \( \Lambda_{B_1} \) to be \( 2\sigma_{A'\mathcal{B}_1}^\dagger, \) where \( ^\dagger \)
denotes the transpose in the standard basis. Similarly, we define \( \sigma_{B_2C'} := (\mathbb{1}_{\mathcal{B}_2} \otimes \Gamma_C)(\tau_{B_2C}) \) and choose the Choi
state of \( \Lambda_{B_2} \) to be \( 2\sigma_{B_2C'}^\dagger. \) The final result of the theorem
follows from a straightforward computation, in which we
show that the state \( \sigma_{A'\mathcal{C}'} \) shared between Alice and Charlie
after Bob measured \( B_{b_1B_2}^b \) is by definition the image of \( B_{b_1B_2}^b \)
by the renormalized Choi map associated to \( \sigma_{A'\mathcal{B}_1}^\dagger \otimes \sigma_{B_2C'}^\dagger \) (see Appendix F).

Before discussing self-testing of the BSM in the noisy
case, we present two natural generalizations of Theorem
1.

**Generalizations.**—The first extension is a self-testing of the “tilted” Bell-state measurement (tilted BSM), featuring
four partially entangled two-qubit states as eigenstates
\[
|\phi^+_\theta\rangle = c_\theta|00\rangle + s_\theta|11\rangle, \quad |\phi^-_\theta\rangle = s_\theta|00\rangle - c_\theta|11\rangle,
|\psi^+_\theta\rangle = c_\theta|01\rangle + s_\theta|10\rangle, \quad |\psi^-_\theta\rangle = s_\theta|01\rangle - c_\theta|10\rangle,
\]
where \( 0 < \theta \leq \pi/4 \) and \( c_\theta = \cos \theta, s_\theta = \sin \theta. \) The
self-test is again based on entanglement swapping, with initially two shared maximally entangled states. The
difference is that Bob’s measurement now prepares partially
entangled states for Alice and Charlie, which they can
self-test [14, 15] via the maximal violation of the tilted
CHSH inequalities [37]; see Appendix C for details.

Our second generalization is for a three-qubit entangled
measurement, with eight eigenstates given by the
GHZ states
\[
|\text{GHZ}^+_0\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}, \quad |\text{GHZ}^+_1\rangle = \frac{|011\rangle \pm |100\rangle}{\sqrt{2}},
|\text{GHZ}^+_2\rangle = \frac{|101\rangle \pm |010\rangle}{\sqrt{2}}, \quad |\text{GHZ}^+_3\rangle = \frac{|110\rangle \pm |001\rangle}{\sqrt{2}}.
\]
The self-testing procedure involves a star network of 4
observers. The central node (Rob) shares a maximally
entangled state with each of the three other observers.
For each of the 8 measurement outcomes, Rob’s measure-
ment prepares a GHZ state shared by the three other
observers, which can be self-tested [13, 20] via the max-
imal violation of the Mermin Bell inequalities [38]; see
Appendix C for details.

**Robust self-testing of the Bell-state measurement.**—So
far, we have shown that the BSM can be self-tested in
the noiseless case, i.e. when Alice and Charlie observe
the maximal CHSH violation. However, from a practical
point of view, it is of course crucial to investigate whether
such a result can be made robust to noise. In this section,
we derive a noise-robust version of Theorem 1.

Recall that given the ideal measurement \( \mathcal{F} \) acting on
\( \mathcal{H}_A \) and the real measurement \( \mathcal{F} \) acting on \( \mathcal{H}_A \) we say that the real
measurement \( \mathcal{F} \) is capable of simulating the ideal measurement \( \mathcal{F} \) if there exists a completely positive
unital map \( \Lambda : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A) \) such that \( \Lambda(p'_A) = p'_A, \)
for all \( j. \) Since in the device-independent setting one cannot certify non-projective measurements, let us from
now on assume that $\mathcal{P}$ is a projective measurement and we define the quality of $\mathcal{F}$ as a simulation of $\mathcal{P}$ as

$$Q(\mathcal{F}, \mathcal{P}) := \frac{1}{|A'|} \max_{\Lambda} \sum_{j=1}^{d} \langle \Lambda(F_{A}'^{j}), P_{A'}^{j} \rangle,$$

where $|A'|$ is the dimension of the ideal Hilbert space $\mathcal{H}_{A'}$, $\langle \cdot, \cdot \rangle$ is the Hilbert-Schmidt inner product and the maximization is taken over completely positive unital maps $\Lambda : \mathcal{L}(\mathcal{H}_{A}) \rightarrow \mathcal{L}(\mathcal{H}_{A'})$. The quantity $Q(\mathcal{F}, \mathcal{P})$ is well-defined as long as $\mathcal{F}$ and $\mathcal{P}$ have the same number of outcomes and $Q(\mathcal{F}, \mathcal{P}) \in [0, 1]$ (see Appendix D). Moreover, since $Q(\mathcal{F}, \mathcal{P}) = 1$ iff $\mathcal{F}$ is capable of simulating $\mathcal{P}$, it is justified to think of $Q$ as a measure of how good the simulation is. This definition naturally generalizes to the case where $\mathcal{F}$ and $\mathcal{P}$ act jointly on two subsystems as

$$Q(\mathcal{F}, \mathcal{P}) := \frac{1}{|B'_1| \cdot |B'_2|} \max_{\Lambda_{B_1}, \Lambda_{B_2}} \sum_{j=1}^{d} \langle \Lambda_{B_1} \otimes \Lambda_{B_2}(F_{B_1}^{j} \otimes F_{B_2}^{j}), P_{B_1}^{j} \otimes P_{B_2}^{j} \rangle,$$

where the maximization is taken over completely positive unital maps $\Lambda_{B_1} : \mathcal{L}(\mathcal{H}_{B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{B_1'})$ and $\Lambda_{B_2} : \mathcal{L}(\mathcal{H}_{B_2}) \rightarrow \mathcal{L}(\mathcal{H}_{B_2'})$. Since we are interested in certifying entangled measurements, we assume that the ideal measurement $\mathcal{F}$ contains at least one entangled operator. The threshold value $Q_{\text{sep}}(\mathcal{P})$, above which we can conclude that the real measurement is entangled, is simply the largest value of $Q$ achievable when the real measurement is separable, i.e.

$$Q_{\text{sep}}(\mathcal{P}) := \frac{1}{|B'_1| \cdot |B'_2|} \max_{\mathcal{F} \in \mathcal{M}_{\text{sep}}} \sum_{j=1}^{d} \langle F_{B_1}^{j} \otimes F_{B_2}^{j}, P_{B_1}^{j} \otimes P_{B_2}^{j} \rangle,$$

where $\mathcal{M}_{\text{sep}}$ is the set of separable measurements acting on $\mathcal{H}_{B'_1} \otimes \mathcal{H}_{B'_2}$. Since $\mathcal{P}$ contains some entangled measurement operators, we have $Q_{\text{sep}}(\mathcal{P}) < 1$ and clearly exceeding this threshold guarantees that at least one measurement operator of $\mathcal{F}$ is entangled. For the special case of rank-1 projective measurements a simple to evaluate upper bound on $Q_{\text{sep}}(\mathcal{P})$ can be derived in terms of the Schmidt coefficients of the measurement operators (see Appendix D). For the BSM this bound turns out to be tight and we conclude that $Q_{\text{sep}}(\Phi) = \frac{1}{2}$.

Let us now state the robust version of Theorem 3 and sketch the proof.

**Theorem 2.** Let the initial state shared by Alice, Bob and Charlie be of the form

$$\tau_{A'B_1B_2C} = \tau_{A'B_1} \otimes \tau_{B_2C}$$

and let $\mathcal{B} := (B_1^{b}, B_2^{b})_{b=0}^{3}$ be a four-outcome measurement acting on $\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$. Let $\tau_{B_1} = \mathcal{T}_{A} \tau_{A'B_1}$, $\tau_{B_2} = \mathcal{T}_{C} \tau_{B_2C}$ be the marginal states and $p_{b} := \langle B_1^{b} \otimes \tau_{B_2}, B_1^{b} \otimes B_2^{b} \rangle$ be the probability of Bob observing outcome $b$. Suppose that the statistics of Alice and Charlie conditioned on that outcome give the violation of $\beta_{b}$ of the

**CHSH$_{b}$ inequality and that the average violation satisfies**

$$\beta_{\text{ave}} := \sum_{b} p_{b} \beta_{b} > 2.$$ 

If we define $q := g(\beta_{\text{ave}})$ for

$$g(x) := \frac{1}{2} + \frac{1}{2} \left( \frac{x - x^*}{2 \sqrt{2} - x^*} \right),$$

where $x^* := (16 + 14\sqrt{2})/17$, then the quality of the real measurement $\mathcal{B}$ as a simulation of the Bell-state measurement $\Phi$ satisfies

$$Q(\mathcal{B}, \Phi) \geq \frac{1}{2(1 + \eta^*)} \min_{\eta \in [0, \eta^*]} \left[ \frac{2q - 1}{\sqrt{1 - v^2}} + \frac{1}{1 + v} \right],$$

where $\eta^* := 2\sqrt{q(1 - q)}$.

The final bound, plotted as a function of $\beta_{\text{ave}}$ in Fig. 2, is non-trivial for $\beta_{\text{ave}} \geq 2.689$ (corresponding to $\sim 5\%$ of noise) which certifies that the measurement is entangled. As the proof is rather technical, we give a brief overview below, but defer a formal argument to Appendix E. The proof follows the argument given for the exact case until Eq. (3), which in the noisy case must be replaced by an approximate statement. The standard construction of extraction channels [20] yields channels $\Gamma_{A, \tau}$ such that the fidelity between the extracted state and the corresponding Bell state satisfies

$$F((\Gamma_{A} \otimes \Gamma_{C})(\tau_{A'C'}^{b}), \Phi_{A'C'}^{b}) \geq g(\beta_{b})$$

for all $b$. As before we define $\sigma_{A'B_1} := (\Gamma_{A} \otimes \mathbb{1}_{B_1})(\tau_{A'B_1})$ and $\sigma_{B_2C'} := (\mathbb{1}_{B_2} \otimes \Gamma_{C})(\tau_{B_2C})$, which allows us to write

$$p_{b} F((\Gamma_{A} \otimes \Gamma_{C})(\tau_{A'C'}^{b}), \Phi_{A'C'}^{b}) \geq p_{b} g(\beta_{b}).$$

As the marginals of $\sigma_{A'}$ and $\sigma_{C'}$ are no longer guaranteed to be uniform, $\sigma_{A'B_1}$ and $\sigma_{B_2C'}$ cannot be rescaled to become Choi states of unital channels. A more complicated construction yields $\lambda_{AB_1}^{T}$ and $\lambda_{BC'}^{T}$, with uniform marginals on subsystems $A'$ and $C'$, but their closeness to $\sigma_{A'B_1}$ and $\sigma_{B_2C'}$ depends on the bias of the marginals.
σ_A and σ_C'. Fortunately, this bias can be estimated from the observed Bell violation. Applying the unital channels corresponding to λ_A'B_1 and λ_B'C' yields

$$\langle (A_B \otimes A_B')(B_{B_1B_2}^b), \Phi_{A'C'}^b \rangle = \langle \lambda_{A'B_1}^T \otimes \lambda_{B_2C'}^T, \Phi_{A'C'}^b \otimes B_{B_1B_2}^b \rangle,$$

which we can relate to the observed Bell violation through Eq. (4). In Appendix E.2 we explain how the same approach can be used to derive robust results for the GHZ measurement discussed before.

Conclusions.—We discussed the problem of self-testing entangled measurements in a quantum network. In particular, we developed a self-test for the Bell-state measurement. This result (Theorem 1), intimately connects the problem of self-testing entangled measurements with that of self-testing entangled states, which we have illustrated with two other examples. It would be interesting to understand the generality of this connection, and see if higher-dimensional entangled measurements can also be self-tested, based e.g. on the results of Ref. [17]. Another intriguing question is whether all entangled measurements (where all eigenstates are pure and entangled) can be self-tested. Note that Theorem 1 cannot be directly extended to measurements where the eigenstates feature a different amount of entanglement.

Moreover, we developed a robust self-test for the BSM. This opens the possibility to experimentally certify an entangled measurement in the device-independent setting, which represents the strongest form of certification. Our analysis allows for some level of noise (≈ 5%), but this appears to be slightly too demanding for current experiments; see e.g. Ref. [40] where a fidelity of 84% for the swapped state was reported. It would thus be highly desirable to develop more robust methods, and we expect that our bounds can be improved. Moreover, for photonic experiments, one could develop self-tests for partial BSM [41].

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APPENDIX A: THE FORMAL SWAP ISOMETRY

In this appendix, we introduce the formal swap gate $S_{X,Z}$ and swap channel $\Gamma_{X,Z}$ defined for two operators $X,Z$ of a Hilbert space $\mathcal{H}$. Note that these gates perform a swap only under some conditions (e.g., anti-commutation on the support of an input state) given in Lemma 1 stated below. In the following, $\mathcal{H}'$ is a qubit Hilbert space, $S_{X,Z}$ maps any state $|\psi'\rangle \otimes |\psi\rangle \in \mathcal{H}' \otimes \mathcal{H}$ into $\mathcal{H}' \otimes \mathcal{H}$ (we write again $S_{X,Z}$ the corresponding maps over density matrices) and $\Gamma_{X,Z}$ maps any operator $\rho \in \mathcal{L}(\mathcal{H})$ into $\mathcal{L}(\mathcal{H}')$. This two transformations are introduced in Figure 3 and read

$$S_{X,Z}(|0\rangle|\psi\rangle) = \frac{1}{2}(|0\rangle(|I + Z)|\psi\rangle + |1\rangle X(|I - Z)|\psi\rangle),$$

$$S_{X,Z}(|1\rangle|\psi\rangle) = \frac{1}{2}(|0\rangle X(|I + Z)|\psi\rangle + |1\rangle(|I - Z)|\psi\rangle),$$

$$\Gamma_{X,Z}(\rho) = \text{Tr}_H(S_{X,Z}(|0\rangle\langle 0| \otimes \rho)).$$

This formal swap idea was already introduced in [18] to test-self states. Here, we introduce a slightly different operator which simplifies the formulation of Lemma 1.

Let $X', Z'$ be the usual Pauli matrices over the qubit space $\mathcal{H}'$. We have:

**Lemma 1.** Assume that $X^2 = Z^2 = I$ and $X,Z$ anti-commute over the support of a state $|\psi\rangle \in \mathcal{H}$. Acting with $X$ (resp. $Z$) before applying $S_{X,Z}$ is equivalent to an action of $X'$ (resp. $Z'$) after applying $S_{X,Z}$, i.e.

$$S_{X,Z} : X \cdot |\psi'\rangle \otimes |\psi\rangle = X' \cdot S_{X,Z} : |\psi'\rangle \otimes |\psi\rangle,$$

$$S_{X,Z} : Z \cdot |\psi'\rangle \otimes |\psi\rangle = Z' \cdot S_{X,Z} : |\psi'\rangle \otimes |\psi\rangle.$$  

The conjugation of $X,Z$ with $X$ (resp. $Z$) in the definition of $S_{X,Z}$, which maps $X$ into $-X$ (resp. $Z$ into $-Z$) is equivalent to an action of $X' \otimes X$ (resp. $Z' \otimes Z$) after applying $S_{X,Z}$, i.e.

$$S_{X,Z} : |\psi'\rangle \otimes |\psi\rangle = Z' \otimes Z \cdot S_{X,Z} : |\psi'\rangle \otimes |\psi\rangle,$$

$$S_{X,Z} : |\psi'\rangle \otimes |\psi\rangle = X' \otimes X \cdot S_{X,Z} : |\psi'\rangle \otimes |\psi\rangle.$$  

Fig. 3. (a) Swap gate $S_{X,Z}$ constructed out of two operators $X,Z$ which anti-commute over the support of a state $|\psi\rangle \in \mathcal{H}$. $S_{X,Z}$ has two entries, a qubit $|\psi'\rangle \in \mathcal{H}'$ and a state $|\psi\rangle \in \mathcal{H}$. $H'$ is the Hadamard gate. (b) Swap isometry $\Gamma_{X,Z}$. It corresponds to the swap gate in which the qubit $|\psi'\rangle$ is initialized to $|0\rangle$ and the output state in $\mathcal{H}$ is traced out.

**Proof.** By linearity, we can restrict ourselves to $|\psi\rangle \in \{ |0\rangle, |1\rangle \}$. Then, this can directly obtained from Eq. (5) and (6) by adding $X$ or $Z$ in front of $|\psi'\rangle \otimes |\psi\rangle$ or substituting $X$ into $-X$ or $Z$ into $-Z$ and using the anti-commutation rules.

Remark that Lemma 1 still holds when $|\psi\rangle$ is replaced by a density matrix, possibly defined over a larger Hilbert space.

APPENDIX B: SELF-TESTING OF THE BELL-STATE MEASUREMENT

We now come to the proof of the main theorem of our letter. Let us introduce the formal Pauli matrices

$$Z_A = A_0, \quad X_A = A_1,$$

$$Z_C = \frac{C_0 + C_1}{\sqrt{2}}, \quad X_C = \frac{C_0 - C_1}{\sqrt{2}},$$

$$Z_C = r(Z_C^r)r(Z_C^c)^{-1}, \quad X_C = r(X_C^r)r(X_C^c)^{-1},$$

where for a Hermitian operator $O^*$, $O = r(O^*)$ is the regularized operator i.e. the same operator in which all zero eigenvalues have been replaced by 1. We have:

**Theorem 3.** Let the initial state shared by Alice, Bob and Charlie be of the form

$$\tau_{AB_1B_2C} = \tau_{AB_1} \otimes \tau_{B_2C},$$

and let $\mathcal{B} := (B_{b_1}b_{b_2})^{3}_{b=0}$ be a four-outcome measurement acting on $\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$. If there exist measurements for Alice and Charlie such that the resulting statistics conditioned on $b$ exhibit the maximal violation of the CHSH$_b$
inequality, then there exist completely positive and unital maps \( \Lambda_B : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A'), \Lambda_B : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_C') \) for \(|A'| = |C'| = 2\) such that

\[
(\Lambda_B \otimes \Lambda_B)(B^b_{B_1B_2}) = \Phi^b_{A'C'}
\]

for \( b \in \{0, 1, 2, 3\} \).

The proof is in two steps. We first prove that the Hilbert spaces of both Alice and Charlie can be replaced by qubit Hilbert spaces. In that case, after Bob’s measurement, Alice and Charlie share one of the four Bell states. Then we choose the Choi state of \( \Lambda_B \) and \( \Lambda_B \) to be proportional to the shared state between Alice/Bob and Bob/Charlie, and show that they satisfy the desired properties in order to extract the BSM.

Step 1. Let \( \Gamma^0_A = \Gamma_{X_A,z_A} \) and \( \Gamma^0_C = \Gamma_{X_C,z_C} \). We introduce the reduced states

\[
\begin{align*}
\sigma_{A'B_1} &:= \Gamma^0_A(\tau_{A'B_1}), \\
\sigma_{AC}^b &:= \Gamma^0_A(\tau_{AC}) \\
\sigma_{AC}^{b'} &:= \Gamma^0_A(\tau_{AC})
\end{align*}
\]

where \( \tau_{AC}^b \) is the state shared between Alice and Charlie after Bob measured \( b \in \{0, \ldots, 3\} \). Then, we have

\[
\sigma_{AC}^{b'} = \Phi_{A'C'}^b. \tag{12}
\]

Proof. We first prove the case \( b = 0 \), which corresponds to the test of a maximally entangled state of two qubits [18]. We briefly sketch it here for completeness. After Bob measured \( b = 0 \), the state \( \tau_{AC} \) maximally violates the CHSH inequality. Hence \( A, X_A \) and \( Z_C, C \) anti-commute and square to identity over the support of \( \tau_{AC} \) (e.g. see [15]). Hence we can apply Lemma 1 respectively to \( (Z_A, X_A) \) and \( (Z_C, C) \) which introduces the two qubit Hilbert spaces \( \mathcal{H}_A', \mathcal{H}_C' \) and the maps

\[
\begin{align*}
S^0 & = S_{X_A,z_A}, \\
S^0_C & = S_{X_C,z_C}, \\
\Gamma^0_A & = \Gamma_{X_A,z_A}, \\
\Gamma^0_C & = \Gamma_{X_C,z_C}.
\end{align*}
\]

We write \( \mathcal{H}_{AC} := \mathcal{H}_A \otimes \mathcal{H}_C \) and \( \mathcal{H}_{AC}' := \mathcal{H}_A' \otimes \mathcal{H}_C' \). Let

\[
W_0 = A_0 C_0 + A_0 C_1 + A_1 C_0 - A_1 C_1 \tag{13}
\]

be the Bell operator acting over \( \mathcal{H}_{AC} \) and

\[
W'_0 = A'_0 C'_0 + A'_0 C'_1 + A'_1 C'_0 - A'_1 C'_1 \tag{14}
\]

the ideal Bell operator acting over \( \mathcal{H}_{AC}' \). Eq. (8) and (9) show that

\[
S^0_A \otimes S^0_C(W_0|00\rangle\langle00| \otimes \tau_{AC}^0) = W_0 S^0_A \otimes S^0_C(|00\rangle\langle00| \otimes \tau_{AC}^0).
\]

Hence \( \sigma_{AC}^0 = (\Gamma^0_A \otimes \Gamma^0_C)(\tau_{AC}^0) \) maximally violates CHSH. It is straightforward to show that the eigenvalue 2\(\sqrt{2}\) of the operator \( W'_0 \) is non-degenerated, with associated eigenvector \( \Phi_{A'C'}^0 \). Hence \( S^0_A \otimes S^0_C(|00\rangle\langle00| \otimes \tau_{AC}^0) \) is a product state between \( \mathcal{H}_{AC} \) and \( \mathcal{H}_{AC}' \), and \( \sigma_{AC}^0 = \Phi_{A'C'}^0 \).

Let us now prove Step 1 for \( b = 1 \), the other cases being similar. Post selecting the statistics over \( b = 1 \), we have a maximal violation of CHSH. Hence, as CHSH is linked to CHSH by the relabeling \( A_1 \rightarrow -A_1 \), considering

\[
\begin{align*}
S^1_A & = S_{-X_A, z_A}, \\
S^0_C & = S^0_C, \\
\Gamma^0_A & = \Gamma_{-X_A, z_A}, \\
\Gamma^0_C & = \Gamma^0_C,
\end{align*}
\]

we can exploit the proof for \( b = 0 \) (with \( X_A \) replaced by \( -X_A \), which gives that \( S^1_A \otimes S^0_C(|00\rangle\langle00| \otimes \tau_{AC}^0) \) is a product state and

\[
(\Gamma^1_A \otimes \Gamma^1_C)(\tau_{AC}^1) = \Phi_{A'C'}^1.
\]

With Lemma 1, we have

\[
\begin{align*}
\sigma_{A'C'}^b & = (\Gamma^0_A \otimes \Gamma^0_C)(\tau_{AC}) \\
& = \text{Tr}_{\mathcal{H}_AC}(S_A^0 \otimes S_C^0(|00\rangle\langle00| \otimes \tau_{AC}^0)) \\
& = Z_A' \text{Tr}_{\mathcal{H}_AC}(Z_A S_A^0 \otimes S_C^1(|00\rangle\langle00| \otimes \tau_{AC}^1)Z_A) \\
& = Z_A' \Phi_{AC}^{b'} Z_A' = \Phi_{AC}^{b'}.
\end{align*}
\]

\hfill \Box

Step 2. Let \( \Lambda_B : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A') \) and \( \Lambda_B : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_C') \) be respectively the Choi-Jamiołkowski maps associated to the operator \( 2\sigma_{A'B_1} \) and \( 2\sigma_{B_2C'} \). These maps are unital and

\[
\Lambda_B(\mathbb{I}) = \Phi_{A'C'}, \tag{15}
\]

which proves Theorem 3.

Proof. \( \Lambda_B \) is unital if it maps \( \mathbb{I}_B \) to \( \mathbb{I}_A' \). By definition of the Choi-Jamiołkowski map, we have

\[
\begin{align*}
\Lambda_B(\mathbb{I}_B) & = \text{Tr}_{B_1B_2}(\sigma_{A'B_1} \otimes \sigma_{B_2C'}) \\
& = \sum_b \text{Tr}_{B_1B_2C'}(B^b_{B_1B_2} \sigma_{A'B_1} \otimes \sigma_{B_2C'}) \\
& = \frac{1}{3} \sum_b \text{Tr}_{C'}(\sigma_{A'C'}^b) = \mathbb{I}_{A'},
\end{align*}
\]

where we used that \( \sum_b B^b_{B_1B_2} = \mathbb{I} \) and \( \sigma_{AC}^b = \Phi_{AC}^b \).

Moreover, according to the definition of the Choi-Jamiołkowski isomorphism (see Appendix V), the last statement is equivalent to Equation 12: we find \( \Lambda_B(\Phi_{A'C'}^0) = \Phi_{A'C'}^0 \).

\hfill \Box

APPENDIX C: GENERALIZATION

This result of Theorem 1 can be generalized to other entangled measurements. A common way to self-test a state is to construct extraction channels (here \( \Gamma^0_A, \Gamma^0_C \)
out of the local measurement operators. As we show in the following, it is often possible self-test a family of states which form a basis of the considered Hilbert space by relabeling the measurement operators. If the corresponding extraction channels are all linked together in a specific way, the proof of Theorem 1 can be generalized to self-test the measurement in the corresponding basis. Here, we show explicitly that this is the case for the tilted BSM and the GHZ measurement.

I. TILTED BELL-STATE MEASUREMENT

The proof of our main result can be extended to the case of the tilted Bell-state measurement (tilted BSM), which is a measurement in the basis:

\[ \phi^+_\theta = |00\rangle + \theta |11\rangle, \quad \phi^-_\theta = |00\rangle - \theta |11\rangle, \]

where \( \theta \in \{0, \pi/4 \} \) and \( \theta = \cos \theta, \ \theta = \sin \theta \). Remark that for \( \theta = \pi/4 \), we recover the usual Bell states. We call them \( \phi^\pm_\theta \), for \( b = 0, \cdots, 3 \), keeping the same ordering. We consider again an entanglement swapping scenario in which Alice/Bob and Bob/Charlie share a maximally entangled state \( \phi^+ \), Bob now performs the tilted BSM. Alice and Charlie perform the ideal local measurements \( A'_0 := \sigma_z, \ A'_1 := \sigma_x, \ C'_0 := (\cos \mu \sigma_x + \sin \mu \sigma_z)/\sqrt{2}, \ C'_1 := (\cos \mu \sigma_z - \sin \mu \sigma_x)/\sqrt{2} \) where \( \tan(\mu) = \sin(2\theta) \). Then, their statistics conditioned on \( b \) will maximally violate a version of the tilted CHSH inequality. For \( b = 0 \), we have that

\[ \text{CHSH}^\theta_0 := +\eta\langle A_0 \rangle + \langle A_0 C_0 \rangle + \langle A_0 C_1 \rangle - \langle A_1 C_0 \rangle, \]

where \( \eta = 2/\sqrt{1 + 2\tan^2(2\theta)} \). The other variants are \( \text{CHSH}^\theta_0 \), obtained with the symmetries introduced for the CHSH case: \( \text{CHSH}^\theta_0 \) is obtained from \( \text{CHSH}^\theta_0 \) with \( A_1 \rightarrow -A_1 \), \( \text{CHSH}^\theta_0 := -\text{CHSH}^\theta_0 \) and \( \text{CHSH}^\theta_0 := -\text{CHSH}^\theta_0 \).

The formulation of the self-testing result and the proof can directly be deduced from the CHSH case, where the anticommuting operators are defined in [15].

II. GHZ MEASUREMENT

The GHZ measurement features 8 eigenstates, given by the eight GHZ states

\[ |\text{GHZ}^{0,\pm}\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}, \quad |\text{GHZ}^{A,\pm}\rangle = \frac{|011\rangle \pm |100\rangle}{\sqrt{2}}, \]

\[ |\text{GHZ}^{B,\pm}\rangle = \frac{|101\rangle \pm |010\rangle}{\sqrt{2}}, \quad |\text{GHZ}^{C,\pm}\rangle = \frac{|110\rangle \pm |001\rangle}{\sqrt{2}}. \]

Note that here we use for convenience the labels \( 0, A, B, C \). In the ideal protocol, Alice, Bob and Charlie independently share maximally entangled states \( \phi^+ \) with a central party Rob and measure \( A'_0 = B'_0 = C'_0 = X' \) and \( A'_1 = B'_1 = C'_1 = Y' \). Rob measures in the GHZ basis. The considered Bell expression is the Mermin Inequality, of maximal violation 4, give by

\[ \text{Mer}^{0,+} := |\langle A_0 B_0 C_0 \rangle - |\langle A_0 B_1 C_0 \rangle - |\langle A_1 B_0 C_0 \rangle |. \]

The other used symmetries of the Mermin inequality are \( \text{Mer}^r \) for \( r = (P, \pm) \) with \( P \in \{0, A, B, C\} \). More precisely, \( \text{Mer}^{A,+} \) is obtained from \( \text{Mer}^{0,+} \) with \( A_1 \rightarrow -A_1 \) and similarly for \( \text{Mer}^{B,+} \) and \( \text{Mer}^{C,+} \), and \( \text{Mer}^{0,-} := -\text{Mer}^{P,+} \). In this ideal scenario, there is a maximal violation of the inequality \( \text{Mer}^r \) conditioned on Rob result \( r = (P, \pm) \). We first introduce the formal Pauli matrices

\[ X_A = A_0, \quad X_B = B_0, \quad X_C = C_0, \]

\[ Y_A = A_1, \quad Y_B = B_1, \quad Y_C = C_1, \]

\[ Z_A = -iX_A Y_A, \quad Z_B = -iX_B Y_B, \quad Z_C = -iX_C Y_C. \]

We have the following theorem:

**Theorem 3.** Let the initial state shared by Alice, Bob, Charlie and Rob be of the form

\[ \tau = \tau_{AR} \otimes \tau_{BR} \otimes \tau_{CR} \]

and let \( \mathcal{R} := (R_{AR} R_{RB} R_{RC}) \) be an eight-outcome measurement acting on \( \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \). If there exist measurements for Alice, Bob, Charlie and Rob such that the resulting statistics conditioned on \( \tau \) exhibit the maximal violation of the Mermin inequality, then there exist completely positive and unital maps \( \Lambda_{AR} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A'), \Lambda_{RB} : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_B'), \Lambda_{RC} : \mathcal{L}(\mathcal{H}_C) \rightarrow \mathcal{L}(\mathcal{H}_C') \) for \( |A'| = |B'| = |C'| = 2 \) such that

\[ \langle A_{AR} \otimes A_{RB} \otimes A_{RC} \rangle (R_{AR} R_{RB} R_{RC}) = \text{GHZ}^r_{A'B'C'} \]

for \( r = (P, \pm) \) with \( P \in \{0, A, B, C\} \).

The proof is in similar to the previous one, in two steps. We introduce the notation \( \tau_0 = (+, 0) \).

**Step 1.** For \( P = A, B, C \), let \( \Gamma^P_0 = \Gamma_{XP,ZP} \). We introduce the reduced states

\[ \sigma_{PRP} := \Gamma^P_0 (\tau_{PRP}), \]

\[ \sigma_{A'B'C'} := (\Gamma^A_0 \otimes \Gamma^B_0 \otimes \Gamma^C_0) (\tau_{AC}), \]

\[ = 8 \text{Tr}(P_{AR} R_{RB} R_{RC} (\sigma^A_{A'R} \otimes \sigma^B_{B'R} \otimes \sigma^C_{C'R})), \]

where \( \tau_{AC} \) is the state shared between Alice, Bob and Charlie after Rob measured \( \tau \). Then, we have:

\[ \sigma_{A'B'C'} = \text{GHZ}^r_{A'B'C'} \]

(17)

**Proof.** For result \( r = r_0 \), the proof is similar to the CHSH case. Considering the square of the Bell operator associated to Mermin, on can show that for any party \( P \), the formal Pauli matrices \( X_P, Y_P \) anti-commutes and square to identity over the support of \( \tau \). This implies that \( X_P, Z_P \)
anti-commutes and square to identity over the support of $\tau$. Moreover, in the rest of the derivation, $Y_P$ can always be replaced with $-iZ_P X_P$ over the support of $\tau$. Then, the maximal violation of Mer$^\otimes$ can be used to prove that $(S_A^r \otimes S_B^r \otimes S_C^r)(\langle 000|000 \rangle \otimes \tau_{ABC}^r)$ is a product state between $\mathcal{H}_{ABC} := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\mathcal{H}_{A'B'C'} := \mathcal{H}_A' \otimes \mathcal{H}_B' \otimes \mathcal{H}_C'$, and

$$\left( \Gamma_A^r \otimes \Gamma_B^r \otimes \Gamma_C^r \right)(\tau_{ABC}^r) = \text{GHZ}_{A'B'C'}^r.$$  

Let us now prove Step 1 for any $r = (P, \pm)$. For $P = 0, A, B, C$, we introduce the operators

$$T_{P,+}^r = X_P, \quad T_{P,-}^r = T_{P,+}^r Z_A Z_B Z_C,$$

where $T^r = \mathbb{I}$. A straightforward calculation shows that for $x = 0, 1$, Mer$^r$ is formally linked to Mer$^0$ by the transformation $A_x \rightarrow T^r A_x T^r = e_A^x A_x$ with the anti-commutation rule $X_A Z_A = -Z_A X_A$, which define a sign $e_A^x = \pm 1$, and similarly transformation for $B, C$ (which define $e_B^y, e_C^z = \pm 1$ for $y, z = 0, 1$). Hence, considering

$$S_P^r = S_{r,x}^r X_P X_P^r X_P^r Z_P,$$

for $P = 0, A, B, C$ and $t = 0, 1$, we can exploit the proof for $r = (0, +)$ to obtain

$$\left( \Gamma_A^r \otimes \Gamma_B^r \otimes \Gamma_C^r \right)(\tau_{ABC}^r) = \text{GHZ}_{A'B'C'}^r.$$

Then with Lemma 1, basic computations similar to the CHSH case show that

$$\left( \Gamma_A^r \otimes \Gamma_B^r \otimes \Gamma_C^r \right)(\tau_{ABC}^r) = T^r \left( \Gamma_A^r \otimes \Gamma_B^r \otimes \Gamma_C^r \right)(\tau_{ABC}^r) T^r = T^r \text{GHZ}_{A'B'C'}^r = \text{GHZ}_{A'B'C'}^r.$$  

**Step 2.** For $P \in \{A, B, C\}$, let $\Lambda_{RP} : \mathcal{L}(\mathcal{H}_{RP}) \rightarrow \mathcal{L}(\mathcal{H}_{P})$ be the Choi-Jamiolkowski map associated to the operator $2\sigma_{PR}$. This map is unital and

$$\Lambda_{RA} \otimes \Lambda_{RB} \otimes \Lambda_{RC} (R_{RA} R_{RB} R_{RC}) = \text{GHZ}_{A'B'C'}^r$$

for $r = (P, \pm$) with $P \in \{0, A, B, C\}$, which proof Theorem 3.

Proof. The proof is exactly similar to the one of Step 2 of Theorem 3. We first prove unitality and then show that the final statement is no more than a rewriting of Eq. (17).

**III. FURTHER GENERALIZATION**

The two previous examples demonstrate that the method used to self-test the BSM can be generalized to other entangled measurements on qubits. We expect that our method directly generalizes to a basis created out of $N$-qubit or higher dimensional systems. Our proof relies on two steps. First, a self-test result of a state $|\Phi^r\rangle$ in which a tensor product of local extraction map $\Gamma^r$ is constructed out of the measurement operators of the parties. Second, symmetries which allow to self-test a full basis of state $|\Phi^r\rangle$ with the same measurement operators and such that the corresponding extraction maps $\Gamma^r$ can be linked to $\Gamma^0$ in a proper way.

We expect that recent work developed methods for self-testing different classes of states, such as [22], can be exploited to generalize our result.

**APPENDIX D: BASIC PROPERTIES OF THE QUALITY MEASURE**

In this appendix we prove some basic properties of the quality measure defined in the main text and for completeness let us first restate the definition. We consider two measurements with $d$ outcomes: the ideal, projective measurement $\mathcal{F} = (P_A^j)_{j=1}^d$ acting on $\mathcal{H}_A$ and the real (not necessarily projective) measurement $\mathcal{F} = (F_A^j)_{j=1}^d$ acting on $\mathcal{H}_A$. The quality of $\mathcal{F}$ as a simulation of $\mathcal{P}$ is given by

$$Q(\mathcal{F}, \mathcal{P}) := \frac{1}{|\mathcal{A}'|} \max_{\Lambda} \sum_{j=1}^d \langle \Lambda(F_A^j), P_A^j \rangle,$$  

where $|\mathcal{A}'|$ is the dimension of the Hilbert space $\mathcal{H}_A'$ and the maximization is taken over completely positive unital maps $\Lambda : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A')$. Let us start by showing that $Q(\mathcal{F}, \mathcal{P}) \in [0, 1]$ and examining the extremal cases.

**Proposition 1.** We always have

$$Q(\mathcal{F}, \mathcal{P}) \geq \frac{1}{|\mathcal{A}'|} \sum_{j=1}^d \text{Tr}(F_A^j) \cdot \text{Tr}(P_A^j).$$

Moreover, if the right-hand side vanishes, we must have $Q(\mathcal{F}, \mathcal{P}) = 0$.

**Proof.** The bound comes from the map

$$\Lambda(X) := \frac{\text{Tr}(X)}{|\mathcal{A}|} \cdot \mathbb{1}_{\mathcal{A}'},$$

which is easily checked to be completely positive and unital. If the right-hand side vanishes, we must have

$$\text{Tr}(F_A^j) \cdot \text{Tr}(P_A^j) = 0$$

for every $j$, which implies that for every $j$ either $F_A^j = 0$ or $P_A^j = 0$. In such a case every term on the right-hand side of Eq. (19) must vanish regardless of the choice of $\Lambda$.

Note that if $\mathcal{P}$ is not only projective, but also rank-1 (which implies $d = |\mathcal{A}'|$), this lower bound simplifies to $Q(\mathcal{F}, \mathcal{P}) \geq \frac{1}{d}$. 

\[\boxed{}\]
Proposition 2. We always have $Q(\mathcal{F}, \mathcal{P}) \leq 1$. Moreover, if $Q(\mathcal{F}, \mathcal{P}) = 1$, then there exists a completely positive unital map $\Lambda$ such that

$$\Lambda(F^j_A) = P^j_A$$

for every $j$.

Proof. For a fixed map $\Lambda$ let $Q^j_{\mathcal{A}'} := \Lambda(F^j_A)$. Since the map is completely positive we have $Q^j_{\mathcal{A}'} \geq 0$ for all $j$ and since it is unital we have

$$\sum_{j=1}^{d} Q^j_{\mathcal{A}'} = \sum_{j=1}^{d} \Lambda(F^j_A) = \Lambda(1_{A'}) = 1_{A'},$$

which together implies that $Q^j_{\mathcal{A}'} \leq 1_{A'}$ for every $j$. Therefore,

$$\frac{1}{|A'|} \sum_{j=1}^{d} \left| \langle Q^j_{\mathcal{A}'}, P^j_{A'} \rangle \right| \leq \frac{1}{|A'|} \sum_{j=1}^{d} \left| \langle 1_{A'}, P^j_{A'} \rangle \right| = 1.$$ (20)

Since this bound holds for every completely positive unital map $\Lambda$, we immediately obtain $Q(\mathcal{F}, \mathcal{P}) \leq 1$.

If $Q(\mathcal{F}, \mathcal{P}) = 1$, there exists a map $\Lambda$ such that the resulting operators $Q^j_{\mathcal{A}'}$ saturate the upper bound given in Eq. (20). This means that the equality

$$\langle Q^j_{\mathcal{A}'}, P^j_{A'} \rangle = \langle 1_{A'}, P^j_{A'} \rangle$$

holds for every $j$, which implies that $Q^j_{\mathcal{A}'} \geq P^j_{A'}$ for all $j$. Finally, the relation

$$1_{A'} = \sum_{j=1}^{d} Q^j_{\mathcal{A}'} \geq \sum_{j=1}^{d} P^j_{A'} = 1_{A'}$$

forces all these inequalities to hold as equalities. \hfill \Box

If $\mathcal{F}$ and $\mathcal{P}$ act jointly on two subsystems the quality of simulation is given by

$$Q(\mathcal{F}, \mathcal{P}) := \frac{1}{|B'_1| \cdot |B'_2|} \max_{\Lambda_{B_1}, \Lambda_{B_2}} \sum_{j=1}^{d} \langle (\Lambda_{B_1} \otimes \Lambda_{B_2})(F^j_{B_1B_2}), P^j_{B'_1B'_2} \rangle$$

where the maximization is taken over completely positive unital maps $\Lambda_{X} : \mathcal{L}(\mathcal{H}_X) \to \mathcal{L}(\mathcal{H}_X)$ for $X = B_1, B_2$. Proposition 2 straightforwardly extends to the bipartite setting, so let us state it without a proof.

Proposition 3. We always have $Q(\mathcal{F}, \mathcal{P}) \leq 1$. Moreover, if $Q(\mathcal{F}, \mathcal{P}) = 1$, then there exist completely positive unital maps $\Lambda_{B_1} : \mathcal{L}(\mathcal{H}_{B_1}) \to \mathcal{L}(\mathcal{H}_{B'_1})$ and $\Lambda_{B_2} : \mathcal{L}(\mathcal{H}_{B_2}) \to \mathcal{L}(\mathcal{H}_{B'_2})$ such that

$$(\Lambda_{B_1} \otimes \Lambda_{B_2})(F^j_{B_1B_2}) = P^j_{B'_1B'_2}$$

for every $j$.

The separability threshold is given by

$$Q_{sep}(\mathcal{P}) := \frac{1}{|B'_1| \cdot |B'_2|} \max_{j \in \mathcal{M}_{sep}} \sum_{j=1}^{d} \langle F^j_{B'_1B'_2}, P^j_{B'_1B'_2} \rangle,$$

where $\mathcal{M}_{sep}$ is the set of separable measurements acting on $\mathcal{H}_{B'_1} \otimes \mathcal{H}_{B'_2}$.

Proposition 3 implies that if $\mathcal{P}$ is an entangled measurement, we must have $Q_{sep}(\mathcal{P}) < 1$, but computing an explicit upper bound is not entirely trivial. In the following proposition we compute an explicit upper bound for measurements composed of rank-1 projectors.

Proposition 4. Let $\mathcal{P}$ be a rank-1 projective measurement given by

$$P^j_{B'_1B'_2} = |e_j \rangle \langle e_j|_{B'_1B'_2}$$

and let the Schmidt decomposition of $|e_j \rangle_{B'_1B'_2}$ be

$$|e_j \rangle_{B'_1B'_2} = \sum_{l} \alpha_{j,l} |a_{j,l} \rangle_{B'_1} |b_{j,l} \rangle_{B'_2}.$$ Then,

$$Q_{sep}(\mathcal{P}) \leq \alpha_{max}^2,$$

where $\alpha_{max} := \max_{j,l} \alpha_{j,l}$ is the largest Schmidt coefficient.

Proof. For fixed channels $\Lambda_{A}$ and $\Lambda_{B}$ let $Q^j_{B'_1B'_2} := (\Lambda_{A} \otimes \Lambda_{B})(F^j_{AB})$. These are still separable operators, i.e. we can write them as

$$Q^j_{B'_1B'_2} = \sum_{k} \lambda_{j,k} |\psi_{j,k} \rangle \langle \psi_{j,k}|_{B'_1B'_2}$$

for some product states $|\psi_{j,k} \rangle$. Moreover, they satisfy

$$\sum_{j=1}^{d} Q^j_{B'_1B'_2} = 1_{B'_1B'_2}.$$ Then, we obtain

$$\sum_{j=1}^{d} \langle Q^j_{B'_1B'_2}, P^j_{B'_1B'_2} \rangle = \sum_{j=1}^{d} \sum_{k} \lambda_{j,k} |\langle e_j | \psi_{j,k} \rangle|^2$$

$$\leq \alpha_{max}^2 \sum_{j=1}^{d} \sum_{k} \lambda_{j,k}$$

$$= \alpha_{max}^2 \sum_{j=1}^{d} \text{Tr} (Q^j_{B'_1B'_2})$$

$$= \alpha_{max}^2 \cdot \text{Tr}(1_{B'_1B'_2})$$

where we have used the fact that the overlap between a product state and an entangled state cannot exceed the square of the largest Schmidt coefficient. Since the bound does not depend on the specific choice of maps, it holds universally. \hfill \Box
Clearly, the estimate above is rather crude, but it can be tight, e.g. for the BSM we obtain the value of $\frac{1}{7}$ which turns out to be correct.

A tighter bound can be obtained if we take into account the individual Schmidt coefficients of the ideal projectors. Clearly,

$$\sum_{j=1}^{d} \langle Q_{B_1'B_2'}, P_{B_1'B_2'}^j \rangle \leq \sum_{j=1}^{d} \alpha_{j,\text{max}}^2 \lambda_{j,k}$$

$$= \sum_{j=1}^{d} \alpha_{j,\text{max}}^2 \text{Tr} \left( Q_{B_1'B_2'}^j \right),$$

where $\alpha_{j,\text{max}} := \max_{j,l} \alpha_{j,l}$. In the last step we must determine the choice of traces $\text{Tr} \left( Q_{B_1'B_2'}^j \right)$ that maximizes the right-hand side of this expression. Let us order the outcomes such that the coefficients $\alpha_{j,\text{max}}$ are non-increasing. The bound stated in Proposition 4 corresponds to assigning all the trace to $j = 1$. However, since each individual term is upper-bounded by $\text{Tr} P_{B_1'B_2'}^j = 1$, it suffices to set $\text{Tr} Q_{B_1'B_2'} = \alpha_{1,\text{max}}^2$ and then distribute the remaining trace over the other terms. For $j \geq 2$ the optimal choice is given by

$$\text{Tr} Q_{B_1'B_2'}^j = \min \left\{ \alpha_{j,\text{max}}^{-2} \left| B_1' \right\rangle \left\langle B_2' \right| - \sum_{k=1}^{j-1} \alpha_{k,\text{max}}^{-2} \left| B_1' \right\rangle \left\langle B_2' \right| \right\}.$$  

It is easy to verify that the resulting upper bound is non-trivial as long as there are some entangled projectors and in some cases it can be tight. If we choose a measurement composed of two product states and two Bell states $(|00\rangle, |11\rangle, |\Phi^+\rangle, |\Phi^-\rangle)$, we obtain the value of $\frac{3}{4}$ which turns out to be tight.

The measure given in Eq. (19) captures how well the real measurement $\mathcal{T}$ simulates the ideal measurement $\mathcal{F}$. Since the measure is simply a sum over terms corresponding to all possible measurement outcomes, one might be tempted to think that in order to certify the quality of a single measurement operator, it would suffice to look at the relevant term. This is, however, not quite true as shown by the following example. Suppose we want to certify that $A_{\eta}$ is capable of simulating a rank-1 projector $P_{\eta}^0$. While the upper bound

$$\langle \Lambda(F_A^0), P_{\eta}^0 \rangle \leq \langle \Pi_{\eta}, P_{\eta}^0 \rangle = 1$$

still holds, saturating it does not allow us to conclude that $\Lambda(F_A^0) = P_{\eta}^0$. In particular, another valid solution is given by $\Lambda(F_A^0) = \Pi_{\eta}$. In order to construct a measure which is maximized iff $\Lambda(F_A^0) = P_{\eta}^0$, one must include an extra component, e.g. the trace of the resulting operator. Indeed, the conditions

$$\langle \Lambda(F_A^0), P_{\eta}^0 \rangle = 1 \quad \text{and} \quad \text{Tr} \left( \Lambda(F_A^0) \right) = 1$$

are sufficient to conclude $\Lambda(F_A^0) = P_{\eta}^0$. In particular, this means that in the bipartite case the entanglement of a single measurement operator cannot be inferred by looking only at $\langle \Lambda(F_{B_1'B_2'}^0), P_{B_1'B_2'}^0 \rangle$. For instance, if $P_{B_1'B_2'}^0$ is a rank-1 entangled projector, the maximal value of

$$\langle \Lambda(F_{B_1'B_2'}^0), P_{B_1'B_2'}^0 \rangle = 1$$

can be achieved by a separable measurement operator $F_{B_1'B_2}$, e.g. $F_{B_1'B_2} = \mathbb{I}_{B_1'B_2}$.

### APPENDIX E: NOISE TOLERANT RESULTS

#### IV. PROOF OF THEOREM 2

In this appendix we provide a complete proof of Theorem 2. We begin by proving three auxiliary lemmas. The first one concerns an arbitrary two-qubit state. For a Hermitian operator $X$ we denote its spectrum by $\text{spec}(X)$.

**Lemma 2.** Let $\rho_{AB}$ be a two-qubit state and let $\Phi_{AB}$ some pure maximally entangled state. If $F(\rho_{AB}, \Phi_{AB}) \geq c$ for some $c \in [\frac{1}{2}, 1]$, then

$$\text{spec}(\rho_{A}) \subseteq \left[ 1 - \frac{\eta}{2}, \frac{1 + \eta}{2} \right]$$

for $\eta := 2\sqrt{c(1-c)}$.

**Proof.** Note that if $c = 1$, then $\rho_{AB} = \Phi_{AB}$ and we necessarily have $\text{spec}(\rho_{A}) = \{ \frac{1}{2} \}$. For $c \in [\frac{1}{2}, 1)$ we find the trade-off by solving a semidefinite program in which we constrain the fidelity with the maximally entangled state and maximize the expectation value of some single-qubit Pauli observable. This gives the upper bound on the spectrum of $\rho_{A}$, whereas the lower bound follows from normalization. Without loss of generality we can assume the maximally entangled state to be $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and the Pauli observable to be $\sigma_z$. Then, the primal problem reads

$maximize \quad \langle \sigma_z, \rho_{A} \rangle$

subject to

$$\langle \Phi_{AB}, \rho_{AB} \rangle \geq c,$$

$$\langle \mathbb{I}, \rho_{AB} \rangle = 1,$$

$$\text{Tr}(\rho_{AB}) \geq 0.$$  

Computing the dual leads to

-minimize $\lambda_1 - \lambda_2 c$

subject to

$$\lambda_1 \mathbb{I} \geq \lambda_2 \Phi_{AB}^{+} + \sigma_z \otimes \mathbb{I}$$

over $\lambda_1 \in \mathbb{R}$, $\lambda_2 \geq 0$.

For $c \in [1/2, 1)$ the assignment

$$\lambda_1 = \sqrt{c} \frac{1}{1-c} \quad \text{and} \quad \lambda_2 = \frac{2c - 1}{\sqrt{c(1-c)}}$$

constitutes a valid solution to the dual and the corresponding value equals $2\sqrt{c(1-c)}$.  

$\square$
To see that this bound cannot be improved, note that it is saturated by partially entangled states $|\psi^\theta\rangle := \cos \theta |00\rangle + \sin \theta |11\rangle$ for $\theta \in [0, \pi/4]$ (the fidelity equals $c = (1 + \sin 2\theta)/2$, the spectrum of the reduced state is $\{\cos^2 \theta, \sin^2 \theta\}$).

In the second lemma we prove an operator inequality for an arbitrary qubit-qudit state.

**Lemma 3.** Let $\nu_{AB}$ be a qubit-qudit state such that

$$\text{spec}(\nu_A) = \left\{ \frac{1 - \eta}{2}, \frac{1 + \eta}{2} \right\}$$

for some $\eta \in [0, 1)$. Moreover, define

$$\mu_{AB} := (\nu_A^{-1/2} \otimes I) \nu_{AB} (\nu_A^{-1/2} \otimes I).$$

Then, the operator inequality

$$\mu_{AB} \geq s(\eta) \nu_{AB} - t(\eta) \frac{1}{2} \otimes \nu_B$$

holds for

$$s(\eta) := \frac{2}{\sqrt{1 - \eta^2}} \quad \text{and} \quad t(\eta) := \frac{4}{\sqrt{1 - \eta^2}} - \frac{4}{1 + \eta}.$$  

Computing $\mu_{AB}$ gives

$$\mu_{AB} = (\nu_A^{-1/2} \otimes I) \nu_{AB} (\nu_A^{-1/2} \otimes I) = \frac{1}{2} \nu_A^{-1} \otimes E_0 + \frac{2}{\sqrt{1 - \eta^2}} \left[ \sigma_x \otimes E_z + \sigma_y \otimes E_y \right] + \frac{2}{1 - \eta^2} (-\eta \sigma_z \otimes E_z).$$

The operator inequality (21) is equivalent to

$$\mu_{AB} - s(\eta) \nu_{AB} + t(\eta) \frac{1}{2} \otimes \nu_B \geq 0.$$  

Writing out the left-hand side gives

$$\mu_{AB} - s(\eta) \nu_{AB} + t(\eta) \frac{1}{2} \otimes \nu_B = \frac{1}{2} \nu_A^{-1} \otimes E_0 + \left[ \frac{-s(\eta) + t(\eta)}{2} E_0 \right] + \sigma_z \otimes \left[ \frac{-\eta E_0 + E_z}{1 - \eta^2} - \frac{s(\eta) E_z}{2} \right].$$

To show positivity it suffices to analyse each block separately. Positivity of the $|1\rangle\langle 1|$ block is clear (it is a sum of
two positive semidefinite operators), but the \(|0\rangle\langle 0|\) block requires more work. Since for \(\eta \in [0,1)\) we have

\[
\frac{1}{1 + \eta} - \frac{s(\eta)}{2} \leq 0,
\]

we apply the bound \(E_z \leq E_0\) to obtain

\[
\left( \frac{1}{1 + \eta} - \frac{s(\eta)}{2} \right) (E_0 + E_z) + \frac{t(\eta) E_0}{2} \geq \left( \frac{2}{1 + \eta} - \frac{s(\eta)}{2} \right) E_0 = 0.
\]

\[\square\]

The last lemma is a simple generalization of the CHSH self-testing result from Ref. [20], which shows that the same extraction channels can be used for different variants of the CHSH inequality.

**Lemma 4.** For \(b \in \{0,1,2,3\}\) let \(\tau_{AC}^b\) be arbitrary normalized states acting on \(\mathcal{H}_A \otimes \mathcal{H}_C\). For observables \(A_0, A_1\) acting on \(\mathcal{H}_A\) and \(C_0, C_1\) acting on \(\mathcal{H}_C\) define the Bell operators

\[
W_0 := A_0 \otimes C_0 + A_0 \otimes C_1 + A_1 \otimes C_0 - A_1 \otimes C_1,
\]

\[
W_1 := A_0 \otimes C_0 + A_0 \otimes C_1 - A_1 \otimes C_0 + A_1 \otimes C_1,
\]

\[
W_2 := -W_1,
\]

\[
W_3 := -W_0.
\]

and let \(\beta_b := \text{Tr}(W_b \tau_{AC}^b)\) be the corresponding Bell value. Then, there exist quantum channels \(\Gamma_A : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A')\) and \(\Gamma_C : \mathcal{L}(\mathcal{H}_C) \rightarrow \mathcal{L}(\mathcal{H}_C')\), where \(|A'| = |C'| = 2\), such that for \(b \in \{0,1,2,3\}\)

\[
F((\Gamma_A \otimes \Gamma_C)(\tau_{AC}^b), \Phi_{AC'}^b) \geq g(\beta_b),
\]

where

\[
g(x) := \frac{1}{2} + \frac{1}{2} \cdot \frac{x - x^*}{2\sqrt{2} - x^*}
\]

for \(x^* := (16 + 14\sqrt{2})/17\).

**Proof.** The original result proves only the statement corresponding to \(b = 0\). However, since the extraction channels depend only on the observables, one might expect that the same choice works equally well for other variants of the CHSH inequality. Indeed, if we keep precisely the same extraction channels and write down the operator inequalities corresponding to \(b = 1, 2, 3\), we realize they are all unitarily equivalent to the \(b = 0\) case, which leads to analogous self-testing statements. \[\square\]

We are now ready to prove the main theorem.

**Theorem 2.** Let the initial state shared by Alice, Bob and Charlie be of the form

\[
\tau_{AB_1B_2C} = \tau_{AB_1} \otimes \tau_{B_2C}
\]

and let \(B := (B_{b_1}^{b_2})_{b_1,b_2=0} \) be a four-outcome measurement acting on \(\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}\). Let \(\tau_{B_1} = \text{Tr}_A \tau_{AB_1}\),

\[
\tau_{B_2} = \text{Tr}_C \tau_{B_2C}
\]

be the marginal states and \(p_b := (\tau_{B_1} \otimes \tau_{B_2}, B_{b_1}^{b_2})\) be the probability of Bob observing outcome \(b\). Suppose that the statistics of Alice and Charlie conditioned on that outcome give the violation of \(\beta_b\) of the CHSH inequality and that the average violation satisfies \(\beta_{\text{ave}} := \sum_b p_b \beta_b > 2\). If we define \(q := g(\beta_{\text{ave}})\) for

\[
g(x) := \frac{1}{2} + \frac{1}{2} \cdot \frac{x - x^*}{2\sqrt{2} - x^*},
\]

where \(x^* := (16 + 14\sqrt{2})/17\), then the quality of the real measurement \(B\) as a simulation of the Bell-state measurement \(\Phi\) satisfies

\[
Q(B, \Phi) \geq \frac{1}{2(1 + \sqrt{s}/\sqrt{1 - s})} \cdot \left[ \frac{2q - 1}{\sqrt{1 - s^2}} + \frac{1}{1 + s} \right],
\]

where \(s := 2\sqrt{q(1 - q)}\).

**Proof.** Recall that \(\tau_{AC}^b\) is defined as

\[
p_b \tau_{AC}^b = \text{Tr}_{B_1B_2} \left( (\mathbb{1}_{AC} \otimes B_{b_1}^{b_2}) \tau_{AB_1} \otimes \tau_{B_2C} \right).
\]

Lemma 4 guarantees the existence of completely positive trace-preserving maps \(\Gamma_A : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A')\) and \(\Gamma_C : \mathcal{L}(\mathcal{H}_C) \rightarrow \mathcal{L}(\mathcal{H}_C')\), where \(|A'| = |C'| = 2\), such that

\[
F((\Gamma_A \otimes \Gamma_C)(\tau_{AC}^b), \Phi_{AC'}^b) \geq g(\beta_b),
\]

where \(\beta_b\) is the violation of the CHSH inequality between Alice and Charlie conditioned on Bob observing the outcome \(b\). Define

\[
\sigma_{A'B_2} := (\Gamma_A \otimes \mathbb{1}_{B_2})(\tau_{AB_1}),
\]

\[
\sigma_{B_2C'} := (\mathbb{1}_{B_1} \otimes \Gamma_C)(\tau_{B_2C})
\]

Since \(\Phi_{AC'}^b\) are pure states and applying the channels \(\Gamma_A, \Gamma_C\) commutes with the measurement performed on \(B_1B_2\) we have

\[
p_b F((\Gamma_A \otimes \Gamma_C)(\tau_{AC}^b), \Phi_{AC'}^b)
\]

\[
= p_b ((\Gamma_A \otimes \Gamma_C)(\tau_{AC}^b), \Phi_{AC'}^b)
\]

\[
= \langle \sigma_{A'B_1} \otimes \sigma_{B_2C'}, \Phi_{AC'}^b \otimes B_{b_1}^{b_2} \rangle,
\]

which, in particular, implies that

\[
\langle \sigma_{A'B_1} \otimes \sigma_{B_2C'}, \Phi_{AC'}^b \otimes B_{b_1}^{b_2} \rangle \geq p_b g(\beta_b).
\]

(23)
Recall that our goal is to construct a pair of unital maps \( \Lambda_B : \mathcal{L}(\mathcal{H}_{B_1}) \to \mathcal{L}(\mathcal{H}_{A'}) \) and \( \Lambda_B : \mathcal{L}(\mathcal{H}_{B_2}) \to \mathcal{L}(\mathcal{H}_{C'}) \) for which we can prove a lower bound on
\[
\langle (\Lambda_B \otimes \Lambda_B) (B_{B_1}^b B_{B_2}^b), \Phi_{A'C'}^b \rangle.
\]
If \( \lambda_{A'B_1} \) and \( \lambda_{B_2'C'} \) denote the Choi states of the maps \( \Lambda_B \) and \( \Lambda_B \), respectively, we have
\[
\langle (\Lambda_B \otimes \Lambda_B) (B_{B_1}^b B_{B_2}^b), \Phi_{A'C'}^b \rangle = \text{Tr}_{B_1 B_2} \left[ (\lambda_{A'B_1} \otimes \lambda_{B_2'C'}) (\mathds{1}_{A'C'} \otimes (B_{B_1}^b B_{B_2}^b)\) \right]
\]
and therefore
\[
\langle (\Lambda_B \otimes \Lambda_B) (B_{B_1}^b B_{B_2}^b), \Phi_{A'C'}^b \rangle = \langle \lambda_{A'B_1} \otimes \lambda_{B_2'C'}, \Phi_{A'C'}^b \otimes (B_{B_1}^b B_{B_2}^b) \rangle = \langle \lambda_{A'B_1}^b \otimes \lambda_{B_2'C'}^b, \Phi_{A'C'}^b \otimes B_{B_1}^b B_{B_2}^b \rangle,
\]
where in the second step we have used the fact that the Bell states are invariant under transposition (in the standard basis). The similarity between this expression and Eq. (23) suggests that the Choi states \( \lambda_{A'B_1} \) and \( \lambda_{B_2'C'} \) should be constructed from \( \sigma_{A'B_1} \) and \( \sigma_{B_2'C'} \), respectively. The only remaining difficulty is the fact that the marginals \( \sigma_A \) and \( \sigma_C \) are not necessarily proportional to \( \mathds{1} \). Let us first show how to bound the non-uniformity of these marginals from the observed Bell violations. Let us parametrize the marginal of \( \sigma_A \) by \( \eta_A \in [0,1] \) such that
\[
\text{spec}(\sigma_A) = \left\{ \frac{1 - \eta_A}{2}, \frac{1 + \eta_A}{2} \right\}.
\]
Let \( \sigma_{A'C'}^b := (\Gamma_A \otimes \Gamma_C)(\rho_{AC}^b) \) and note that
\[
F(\sigma_{A'C'}^b, \Phi_{A'C'}^b) = \langle \sigma_{A'C'}^b, \Phi_{A'C'}^b \rangle \geq g(\beta_b).
\]
Let \( U_{A'C'}^b \) be a local unitary acting on \( \mathcal{H}_{C'} \) such that \( (\mathds{1}_{A'} \otimes U_{A'C'}^b) |\Phi^0_{A'C'} \rangle = |\Phi^b_{A'C'} \rangle \) and define
\[
\sigma_{A'C'}^b := \sum_b p_b (\mathds{1}_{A'} \otimes U_{A'C'}^b) \sigma_{A'C'}^b (\mathds{1}_{A'} \otimes U_{A'C'}^b),
\]
where the summation goes over \( b \in \{0,1,2,3\} \). It is easy to verify that
\[
F(\sigma_{A'C'}^b, \Phi_{A'C'}^0) = \langle \sigma_{A'C'}^b, \Phi_{A'C'}^0 \rangle = \sum_b p_b \langle \sigma_{A'C'}^b, \Phi_{A'C'}^b \rangle \geq \sum_b p_b g(\beta_b) = g(\beta_{ave}) = q,
\]
where we have used the fact that the function \( g \) is linear. Moreover,
\[
\sigma_{A'} = \sum_b p_b \sigma_{A'}^b = \sigma_{A'},
\]
which implies that the two have the same spectrum. The lower bound given in Eq. (25) plugged into Lemma 2 implies that \( \eta_A \leq \eta^* \) for
\[
\eta^* := 2\sqrt{q(1-q)}.
\]
It is easy to check that for \( \beta_{ave} > 2 \), we have \( \eta^* < 1 \), i.e. the reduced state \( \sigma_{A'} \) is full-rank. By symmetry the same bound applies to \( \eta_{C'} \).

We are now ready to define the Choi states of the channels \( \Lambda_B \) and \( \Lambda_B \). Let
\[
\lambda_{A'B_1}^b := (\sigma_{A'B_1}^{-1/2} \mathds{1}) \sigma_{A'B_1} (\sigma_{A'B_1}^{-1/2} \mathds{1}) = \frac{2}{1 + \eta_{C'}} \sigma_{B_2'C'} + \sigma_{B_2'} \otimes \left( \mathds{1} - \frac{2}{1 + \eta_{C'}} \sigma_{C'C'} \right),
\]
which are easily verified to be valid Choi states. For this particular choice we have
\[
\lambda_{A'B_1}^b \otimes \lambda_{B_2'C'}^b \geq \frac{2}{1 + \eta_{C'}} \lambda_{B_2'C'}^b \geq \frac{1}{2} \sum_b p_b \langle \tau_{B_1} \otimes \tau_{B_2} B_{B_1}^b B_{B_2}^b \rangle = \frac{p_b}{2}.
\]
Combining these two results yields
\[
\langle \lambda_{A'B_1}^b \otimes \lambda_{B_2'C'}^b, \Phi_{A'C'}^b \otimes B_{B_1}^b B_{B_2}^b \rangle \geq \frac{p_b}{2(1 + \eta_{C'})} [4s(\eta_A) g(\beta_b) - t(\eta_A)],
\]
which implies
\[
Q(\mathcal{B}, \Phi) \geq \frac{1}{4} \sum_{b=0}^{3} \langle \lambda_{A'B_1}^b \otimes \lambda_{B_2'C'}^b, \Phi_{A'C'}^b \otimes B_{B_1}^b B_{B_2}^b \rangle \geq \frac{4s(\eta_A) \sum_b p_b g(\beta_b) - t(\eta_A)}{8(1 + \eta_{C'})} = \frac{4s(\eta_A) q - t(\eta_A)}{8(1 + \eta_{C'})}.
\]
This bound still depends on \( \eta_A \) and \( \eta_{C'} \) and in order to remove this dependence we must minimize over \( \eta_A, \eta_{C'} \in [0, \eta^*] \). Since for \( q \geq \frac{1}{2} \) the numerator is strictly positive, the minimization over \( \eta_{C'} \) reduces to simply setting \( \eta_{C'} = \eta^* \), which leads to the main result of the theorem. \( \square \)
V. NOISE TOLERANCE FOR THE GHZ MEASUREMENT

In the previous section, we have shown that Theorem 1, in which we self-test a BSM in the case of ideal statistics, can be turned into Theorem 2, which is noise tolerant. We explicitly presented a generalization of Theorem 1 to tilted BSM and GHZ measurement and discussed further generalization to other cases. In the following, we argue that Theorem 3 (self-testing of the GHZ measurement) can be made noise tolerant using the same approach. We expect this method generalizes to other cases.

Our method is based on existing results about the self-test of the GHZ state $\text{GHZ}^{0,+}$ with Mermin inequality. For a tripartite state $\tau_{ABC}$ maximally violating Mermin inequality, it yields a product channel $\Gamma = \Gamma_A \otimes \Gamma_B \otimes \Gamma_C$ which maps $\tau_{ABC}$ to $\text{GHZ}^{0,+}$. We first prove in Step 1 that after Rob obtained result $r$, the post measured state is mapped to the appropriate version of the GHZ state with $\Gamma$. Without loss of generality, we assume that $\Gamma_A, \Gamma_B, \Gamma_C$ are locally applied by Alice, Bob, Charlie before Rob’s measurement: now these three parties have a qubit each and share Bob, Charlie before Rob’s measurement: now these three we assume that the state $\sigma$ with $\sigma \approx \text{GHZ}^{0,+}$.

Then, in Step 2, we introduce the Choi-Jamiołkowski isomorphism $\Lambda = \Lambda_{R_A} \otimes \Lambda_{R_B} \otimes \Lambda_{R_C}$ associated to the state $\sigma_{A'R_A} \otimes \sigma_{B'R_B} \otimes \sigma_{C'R_C}$ (up to normalization). By construction the measurement operators of Rob $R_{R_A,R_B,R_C}$ are mapped to the post measured state shared between, Alice, Bob, Charlie: $\Lambda(R_{R_A,R_B,R_C}^r) = \sigma_{A'B'C'}$. As the marginal states $\sigma_A', \sigma_B', \sigma_C'$ are maximally mixed, the channel $\Lambda$ is unital, which proves Theorem 3.

In the noisy case, the proof has the same structure. As the self-test of $\text{GHZ}^{0,+}$ with Mermin inequality is noise tolerant, for a tripartite state $\tau_{ABC}$ and sufficiently high violation of the Mermin inequality, there exist channels $\Gamma = \Gamma_A \otimes \Gamma_B \otimes \Gamma_C$ such that $\Gamma(\tau_{ABC}) \approx \text{GHZ}^{0,+}$. The rest of the proof directly applies. In particular, $\Gamma(\tau_{ABC}) \approx \text{GHZ}^{0,+}$ and if $\Lambda$ is the channel corresponding to $\sigma_{A'R_A} \otimes \sigma_{B'R_B} \otimes \sigma_{C'R_C}$ through the Choi-Jamiołkowski isomorphism, we have $\Lambda(R_{R_A,R_B,R_C}^r) = \sigma_{A'B'C'}$. However, as the marginal states $\sigma_{A'}, \sigma_{B'}, \sigma_{C'}$ are no longer necessarily maximally mixed, $\Lambda$ may not be unital. Hence, we have to introduce new states $\lambda_{A'R_A}, \lambda_{B'R_B}, \lambda_{C'R_C}$ associated to a Choi-Jamiołkowski isomorphism $\Lambda = \Lambda_{R_A} \otimes \Lambda_{R_B} \otimes \Lambda_{R_C}$ such that:

- $\lambda_{A'}, \lambda_{B'}, \lambda_{C'}$ are maximally mixed.
- $\lambda_{A'R_A}, \lambda_{B'R_B}, \lambda_{C'R_C}$ are close to $\sigma_{A'R_A} \otimes \sigma_{B'R_B} \otimes \sigma_{C'R_C}$.

When this is the case, we have $\Lambda(R_{R_A,R_B,R_C}^r) \approx \lambda_{A'R_A} \otimes \lambda_{B'R_B} \otimes \lambda_{C'R_C} \approx \text{GHZ}'$ with $\Lambda$ unital, which proves the noisy variant of the theorem.

To find the new states $\lambda_{A'R_A}, \lambda_{B'R_B}, \lambda_{C'R_C}$, we can use the two constructions given in Eq. (26). Their distance to $\lambda_{A'R_A}, \lambda_{B'R_B}, \lambda_{C'R_C}$ can be controlled in an analogous manner once we have established a bound on the bias of the marginals and this can be achieved by grouping two parties together. If we want to estimate the bias of Alice’s marginal, we group Bob and Charlie together and we observe that now we have a maximally entangled two-qubit state between $A$ and $BC$, which allows us to use Lemma 2. This proves a robust version of Theorem 3.

APPENDIX F: Choi-Jamiołkowski Isomorphism

We recall here the definition of the Choi-Jamiołkowski isomorphism. Let $\rho_{AB}$ acting over $\mathcal{H}_A \otimes \mathcal{H}_B$ be a (possibly not normalized) bipartite state. Then its associated Choi-Jamiołkowski isomorphism $\Gamma : \mathcal{H}_B \rightarrow \mathcal{H}_A$ is defined by the identity

$$\forall \sigma, \Gamma(\sigma) = Tr_B(\mathbb{1} \otimes \sigma^T \cdot \rho_{AB}).$$

We have the following proposition, which can directly be deduced from the definition.

**Proposition 5.** Let $\rho_k$ acting over $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\Gamma^k : \mathcal{H}_k \rightarrow \mathcal{H}_k$ be the associated Choi-Jamiołkowski isomorphism. Let $\Omega$ be an operator of $\mathcal{H}_B := \bigotimes_k \mathcal{H}_k$. Then

$$\bigotimes_k \Gamma^k(\Omega) = Tr_B \left( \Omega^T \cdot \bigotimes_k \rho_k \right).$$

**Proof.** We introduce a decomposition $\Omega = \sum \bigotimes_k \omega_{k,i}$ where $\omega_{k,i}$ is an operator of $\mathcal{H}_k$ and apply the definition of the Choi map:

$$\bigotimes_k \Gamma^k(\Omega) = \bigotimes_k \Gamma^k(\sum \bigotimes_i \omega_{i,l}) = \bigotimes_k \Gamma^k(\omega_{i,l})$$

$$= \sum_i Tr_{\mathcal{H}_k}(\mathbb{1}_{A_k} \otimes \omega_{k,i}^T \cdot \rho_k)$$

$$= \sum_i Tr_{\mathcal{H}} \left( \bigotimes_k \mathbb{1}_{A_k} \otimes \omega_{k,i}^T \cdot \rho_k \right)$$

$$= Tr_{\mathcal{H}} \left( \bigotimes_k \mathbb{1}_{A_k} \otimes \Omega^T \cdot \rho_k \right).$$

$\square$