Quantum Chaos of the Bose-Fermi Kondo model at the intermediate temperature

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(Dated: March 23, 2021)

We study the quantum chaos in the Bose-Fermi Kondo model in which the impurity spin interacts with conduction electrons and a bosonic bath at the intermediate temperature in the large \(N\) limit. The out-of-time-ordered correlator (OTOC) is expected to have an exponential growth after a typical time scale \(\tau_{cr}\), indicating the absence of the butterfly effect. Usually, it is convenient to define the “regulated” OTOC\textsuperscript{14–19} for the OTOCs numerically. In section V we introduce the BFKM in the large-\(N\) limit. In section III we use the Keldysh method to derive the self-consistent equations for Green’s functions and solve it with the help of fast-Fourier transformation method. In section IV the OTOCs are calculated based on the Bethe-Salpeter equation in the large-\(N\) limit and the Lyapunov exponent is extracted from the OTOCs numerically. In section VI we demonstrate the results. Finally we summarize the results and give conclusions in section VII.

\section{I. INTRODUCTION}

Recently, the study on quantum chaos in the many-body physics has drawn intensive interest\textsuperscript{1–9}. Quantum chaos can be diagnosed by the so-called out-of-time-ordered correlators (OTOCs)\textsuperscript{11,12}. The behaviors of OTOCs have been investigated both theoretically and experimentally\textsuperscript{13–19}. The OTOC was first introduced in the context of superconductivity\textsuperscript{13} and then generalized to study the information scrambling in the black hole close to the horizon. Usually, it is convenient to define the “regulated” OTOC\textsuperscript{14–19}.

\begin{equation}
\mathcal{C}(t) = \text{Tr}\{ \sqrt{\rho} [\hat{W}(t), \hat{V}(0)] \sqrt{\rho} [\hat{W}(t), \hat{V}(0)] \}. \tag{1}
\end{equation}

where \(\rho = e^{-\beta H}\) is the thermal density at the temperature \(T = 1/\beta\). And \(\hat{W}\) and \(\hat{V}\) are local operators. In a chaotic system, the OTOC is expected to have an exponential growth \(\mathcal{C}(t) \propto e^{\lambda_L t}\) at the intermediate time, where \(\lambda_L\) is called the Lyapunov exponent. Given a perturbation the initial quantum entanglement will spread across all system after a typical time scale \(\tau_{cr} \sim \lambda_L^{-1}\). During this process the initial information is lost and the system goes into the state of thermalization. Under some reasonable conditions, the Lyapunov exponent \(\lambda_L\) is proven to have an upper bound \(\lambda_L \leq 2\pi k_B T/\hbar\)\textsuperscript{20} and saturates in the models with gravity duals. The most-celebrated \(0 + 1\) SYK\textsubscript{4} model with random all to all interactions\textsuperscript{21–25} is a concrete example.

The Kondo model describes the systems in which the impurity spin strongly interacts with conduction electrons. Recently, the information scrambling in the two-channel and one-channel Kondo model has been investigated by mapping them onto the Majorana resonant level models\textsuperscript{26}. Their results show that the OTOC for the impurity spin in two channel Kondo model is temperature independent and saturates to 1/4 at late time, while the OTOC in one channel Kondo model vanishes at late time, indicating the absence of the butterfly effect. The Bose-Fermi Kondo model (BFKM) in which the impurity spin interacts with both conduction electron and bosonic bath has rich physical properties, specially the non-Fermi liquid state. From RG analysis\textsuperscript{20–22}, it contains several nontrivial fixed points. At low temperature and energy limit, a conformal symmetry can emerge at some fixed points. This model is an important system to study the dual of the gravity and many-body physics. Based on non-Fermi liquid behavior and the emergent conformal symmetry, one may expect there are highly chaotic behaviors in this model.

In this paper, we calculate the Lyapunov exponent in the BFKM in the large-\(N\) limit. Our calculation shows that there are three types diagrams which have the most important contributions to the OTOCs: two one-rung and one two-rung ladder diagrams. In order to extract the Lyapunov exponent \(\lambda_L\) from the Bethe-Salpeter equation, the equal-spaced discretization in energy domain has to been taken. As a consequence of the shortcoming of the method, we can only study quantum chaos at the intermediate temperature region instead of the low temperature region \(T \ll T_K\) where \(T_K\) is the bare Kondo temperature. At the intermediate temperature region we find that \(\lambda_T\) decreases monotonically as increasing temperature and the chaotic property will lost at high temperature for given \(J_K\). Moreover, for fixed \(T\) and \(J_K\), our numerical results show the chaotic properties are lost as the coupling between impurity and bosonic bath \(g\) increases, violating the expectation that the most chaotic behavior occurs at the quantum critical point with the non-Fermi liquid nature.

The paper is organized as follows. In section II we introduce the BFKM in the large-\(N\) limit. In section III we use the Keldysh method to derive the self-consistent equations for Green’s functions and solve it with the help of fast-Fourier transformation method. In section IV the OTOCs are calculated based on the Bethe-Salpeter equation in the large-\(N\) limit and the Lyapunov exponent is extracted from the OTOCs numerically. In section V we demonstrate the results. Finally we summarize the results and give conclusions in section VI.
II. REVIEW OF THE BOSE-FERMI KONDO MODEL

We will start from a system with N fermion flavors. The Hamiltonian can be cast as
\[ \hat{H} = \sum_{k, \sigma, \alpha} E_k c_{k\sigma\alpha}^\dagger c_{k\sigma\alpha} + \sum_k \epsilon_k \Phi_k^\dagger \Phi_k \]
\[ + \frac{J_K}{N} \sum_{\alpha=1}^M S_s \cdot s_{\alpha} + \sum_k g \sqrt{N} \sum \Phi (\Phi_k + \Phi_k^\dagger), \]
(2)
where \( c_{k\sigma\alpha}^\dagger (c_{k\sigma\alpha}) \) is the creation (annihilation) operator of the conduction electron with channel index \( \alpha = 1, \ldots, M \) and spin \( \sigma = 1, \ldots, N \). The conduction electrons at the impurity site transform under the fundamental representation \( s_{\alpha} = \sum_{k, \sigma} c_{k\sigma\alpha}^\dagger \Phi_{\sigma\alpha}(i = 1, \ldots, N^2 - 1) \) and couple to the impurity spin with interaction strength \( J_K \). It is convenient to rewrite the impurity spin with \( N \) components pseudo-fermion \( f_{\alpha} \) as \( S_{\alpha} = \sum_{\sigma, \sigma'} f_{\sigma''}^{\dagger} s_{\alpha} f_{\sigma'} \) by taking antisymmetric representation with constraint \( \sum_{\sigma} f_{\sigma''}^{\dagger} f_{\sigma'} = Q \) which can be absorbed into action by introducing the Lagrange multiplier \( \mu \). In this paper, we consider the case with particle-hole symmetry, requiring \( Q = N/2 \). \( g \) is the interaction strength with bosonic bath \( \Phi_k \) with \( N^2 - 1 \) independent components which comes from the spin or magnetic fluctuation. The ratio between \( M \) number and \( N \) is denoted as \( \kappa \), which is taken as \( \kappa = 1/2 \) in the following of the paper. The density of state of conduction electron \( A_{\Lambda}(\omega) \) around Fermi surface can be approximately as \( A_{\Lambda}(\omega) = \sum_k \delta(\omega - E_k) = \rho_0 \omega < D/2, \) where \( D = 1/\rho_0 \) is the band width. The Greens' function for boson bath in the imaginary time is denoted as \( G_{\Phi}(\tau) \equiv = -\langle T\Phi(\tau)\Phi(0) \rangle \). The bosonic spectral \( A_{\Phi}(\omega) = -\text{Im} G_{\Phi}(\omega + i\eta)/\pi = \sum_k [\delta(\omega - \epsilon_k) - \delta(\omega + \epsilon_k)] \) is considered as the sub-Ohmic bosonic spectrum, namely
\[ A_{\Phi}(\omega) = |\omega|^{1-\epsilon} \text{sign}(\omega), \]
(3)
for \( |\omega| < A \). And the parameter \( \epsilon \) is at the range \( [0,1) \). In this paper we set \( \hbar = 1 \) and \( k_B = 1 \).

For the pure multichannel Kondo model, the previous study proves there exists a nontrivial intermediate fixed point \( MCK \) between the trivial local moment fixed point \( LM \) and strong coupling limit \( J_K \rightarrow \infty \) with conformal symmetry. When coupling to the bosonic bath, the other two fixed points can appear. One of them is the critical local moment point \( LM' \) and another one is the unstable critical fixed point \( C \). The difference of RG flows between BFKM and Kondo model leads to different chaotic behaviors between them.

III. GREEN’S FUNCTIONS IN REAL TIME

In order to derive the Green’s functions in the real time, it is convenient to rewrite the model in the Keldysh time contour with backward and forward time evolution, which is denoted by the sign “-“ and “+“ respectively. Then field \( \psi \) regardless of boson or fermion can be split into two parts as \( \psi(t) = (\psi^+, \psi^-) \) based on its causal position. After performing Keldysh rotation, the Green’s functions are written as
\[ G_{F} = \left( \begin{array}{cc} G_{FR} & G_{FA}^R \\ 0 & G_{FA} \end{array} \right), \quad G_{B}(t) = \left( \begin{array}{cc} G_{BA} & G_{FR}^R \\ G_{BA}^R & 0 \end{array} \right), \]
(4)
where \( R \) (A) represents the retarded (advanced) Green’s function respectively. The Keldysh part, which is denoted by \( K \), is related to the retarded and advanced part by the fluctuation-dissipation theorem in the frequency domain,
\[ G_{F}^{K}(\omega) = \tanh(\frac{\omega}{2T})(G_{FR}(\omega) - G_{FA}(\omega)), \]
\[ G_{B}^{K}(\omega) = \coth(\frac{\omega}{2T})(G_{FR}(\omega) - G_{BA}(\omega)). \]
(5)
(6)
Therefore, the noninteracting action can be written as
\[
S_0 = \int d\omega \left\{ \sum_{\sigma,\sigma'} \sum_k \left( \bar{\Psi}_{\sigma}(\omega, k) G^{-1}_{\sigma}(\omega, E_k) \Psi_{\sigma}(\omega, k) + \sum_{\sigma'} \bar{\Phi}_{\sigma\sigma'}(\omega, k) D^{-1}_{\sigma\sigma'}(\omega, E_k) \Phi_{\sigma\sigma'}(\omega, k) + \sum_{\sigma} \bar{F}_{\sigma}(\omega) G^{-1}_{0}\bar{F}_{\sigma}(\omega) \right) \right\},
\]
where the fields are written in Keldysh space as \( \bar{\Psi}_{\sigma}(k) = (c_{1,\sigma\alpha}^k, c_{2,\sigma\alpha}^k) \), \( \bar{\Phi}(k) = (\Phi_{1,\kappa}, \Phi_{2,\kappa}) \) and \( \bar{F}_{\sigma} = (f_{1\sigma}, f_{2\sigma}) \). The Green’s function for conduction electron and bosonic bath are given by
\[
G^R_{\sigma}(\omega, k) = (G^A_{\sigma}(\omega, k))^* = \frac{1}{\omega + i\eta - E_k},
\]
\[
D^R_{\sigma}(\omega, k) = (D^A_{\sigma}(\omega, k))^* = \sum_{\sigma,\epsilon k} \frac{1}{\omega + i\eta - \epsilon},
\]
and the bare Green’s function for impurity is given by
\[
G^R_{0}(\omega) = \frac{1}{\omega + i\eta - \lambda},
\]
where \( \lambda \) is the saddle point of the auxiliary field \( \mu \) to force the conservation of impurity electrons and it can be taken as zero because of the particle-hole symmetry \( f_\sigma \leftrightarrow f_\sigma^\dagger \). For the interaction between the bosonic bath and impurity, the action is
\[
S_{\Phi, F} = \frac{g}{\sqrt{2N}} \sum_{\sigma,\alpha} \int dt \left\{ \bar{F}_{\sigma}(t) \gamma_1 F_{\sigma,\alpha}(t) (\Phi_{1,\sigma\alpha} + \bar{\Phi}_{1,\sigma\alpha}) + \bar{F}_{\sigma}(t) \gamma_2 F_{\sigma,2\alpha}(t) (\Phi_{2,\sigma\alpha} + \bar{\Phi}_{2,\sigma\alpha}) \right\} + H.C.
\]
For interaction between conduction electron and impurity, we introduce \( M \)-flavors Hubbard-Stratovich fields \( B_\alpha \), which leads to
\[
S_{\Phi, B, F} = \frac{1}{\sqrt{2N}} \sum_{\sigma,\alpha} \int dt \left\{ \bar{F}_{\sigma}(t) \gamma_1 \Psi_{\sigma\alpha} B_{1\alpha} + \bar{F}_{\sigma}(t) \gamma_2 \Psi_{\sigma\alpha} B_{2\alpha} \right\} + H.C.
\]
where \( B_\alpha = (B_{1\alpha}^k, B_{2\alpha}^k) \). The matrix \( \gamma_1 \) and \( \gamma_2 \) are given as following,
\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
and the bare propagator \( D_0 \) for \( B_\alpha \) is
\[
D_0^R(\omega) = (D_0^A(\omega))^* = -\frac{1}{J_K}.
\]
Therefore, the partition function is
\[
Z = \int D[\Phi, F, B, \lambda] e^{i(S_0 + S_{\Phi, F} + S_{\Phi, B, F})}.
\]
Due to interaction, the Green’s functions will be renormalized and the self-energy for fermions or bosons has the following structure,
\[
\Sigma_F = \begin{pmatrix} \Sigma_F^R & \Sigma_F^K \\ 0 & \Sigma_F^K \end{pmatrix}, \quad \Sigma_B = \begin{pmatrix} \Sigma_B^R & \Sigma_B^K \\ 0 & \Sigma_B^K \end{pmatrix}.
\]
(15)

The self-energies are obtained by taking into account the most relevant Feynman diagrams in the large-N limit, shown in Fig. 2. Therefore it is straightforward to obtain the following self-consistent equations,
\[
(G^R(\omega))^{-1} = \omega - \lambda - \Sigma^R(\omega) - \Sigma^R(\omega),
\]
\[
i\Sigma^R_\sigma(t) = \frac{g^2}{2} \left( D^K(t) G^R(t) + D^K_0(t) G^K(t) \right),
\]
\[
i\Sigma^B_\sigma(t) = \frac{K}{2} \left( D^K(t) G^R(t) + D^K_0(t) G^K(t) \right),
\]
\[
(D^R(\omega))^{-1} = -1/J_K - \Pi^R(\omega),
\]
\[
i\Pi^R(t) = \frac{1}{2} \left( -G^K(-t) G^R(t) - G^K(\omega) G^K(t) \right).
\]
Here \( D^R_\sigma(t) = \int \frac{d\omega}{2\pi} D^R_\sigma(\omega) e^{-i\omega t} \) and \( G^R(\omega) = \int \frac{d\omega'}{2\pi} G^R(\omega') e^{-i\omega' t} \). The Green’s function for conduction electron and bosonic bath is obtained by using the Kramers-Kronig relation: \( G^R(\omega) = \int d\omega' A_c(\omega') \frac{\omega' - \omega}{\omega + i\eta - \omega} \) and \( D^R(\omega) = \int d\omega' A_b(\omega') \frac{\omega' - \omega}{\omega + i\eta + \omega} \).

To obtain the impurity and bosonic Green’s functions, we numerically solve the self-consistent equations by the fast-Fourier transformation (FFT) method. In practice, the electron spectral is taken as a Gaussian function \( A_c(\omega) = e^{-\omega^2/\pi} / \pi \). In the Fig. 3 we plot the impurity and bosonic spectral functions respectively for different temperatures \( T \) and bosonic coupling \( g \) while fixing the Kondo coupling \( J_K \pi / D = 1.0 \). From Fig. 3 one
can observe amplitudes of impurity and bosonic spectral functions both decrease as increasing the bosonic bath coupling g.

IV. OUT OF TIME CORRELATOR AND BETHE-SALPETER EQUATION

It is convenient to evaluate the retarded “regulated” squared anti-commutator defined as

$$C(t_1, t_2) = \frac{\theta(t_1)\theta(t_2)}{N^2} \sum_{\sigma, \sigma'} \text{Tr} \left( \sqrt{\rho} \{ f_{\sigma}(t_1), f_{\sigma'}^\dagger(0) \} \right),$$

(21)

where $\rho = \exp(-\beta H)$ is the thermal density matrix.

It is clear that the OTOC is defined in the two-copied Keldysh contours separated by the imaginary time $i\beta/2$ as shown in Fig. 4. Here we denote each Keldysh contour with indices $s = (u, d)$. Therefore the fermionic or bosonic field $\psi$ in the two-copied Keldysh contours is generalized to $\tilde{\psi} = (\psi^d_{u,c}, \psi^d_{u,q}, \psi^d_{d,c}, \psi^d_{d,q})$ after performing the Keldysh rotation for each time fold. Moreover the Green’s function in each time fold remains the same, while the interloop Green’s function $G_{s\bar{s}}$ or $D_{s\bar{s}}$ ($d = u$ and $\bar{d} = d$) has the following structure,

$$G_{s\bar{s}} = \begin{pmatrix} 0 & G^K_{u\bar{d}}(\omega) \\ 0 & 0 \end{pmatrix}, \quad D_{s\bar{s}}(t) = \begin{pmatrix} D^K_{s\bar{s}}(\omega) & 0 \\ 0 & 0 \end{pmatrix},$$

(22)

here the component $G^K_{u\bar{d}}$ or $D^K_{s\bar{s}}$ has the generalized fluctuation-dissipation theorem,

$$G^K_{u\bar{d}}(\omega) = (G^K_{d\bar{u}}(\omega))^*, \quad D^K_{u\bar{d}}(\omega) = 2i \frac{\text{Im} G^K(\omega)}{\cosh(\frac{\omega}{2T})},$$

(23)

and

$$D^K_{u\bar{d}}(\omega) = D^K_{d\bar{u}}(\omega) = 2i \frac{\text{Im} D^K(\omega)}{\sinh(\frac{\omega}{2T})}. $$

(24)

In the augmented Keldysh space, the OTO correlator $C$ can be rewritten as

$$C(t_1, t_2) = -\frac{\theta(t_1)\theta(t_2)}{N^2} \sum_{\sigma, \sigma'} \left( \int \text{d} \mu \text{d} \lambda \theta(\mu - t_1)\theta(\mu - t_2) \int e^{iS_{\bar{a}K}} \right) \langle \rho \{ f_{\sigma}(t_1), f_{\sigma'}^\dagger(0) \} \rangle,$$

(25)

where $\langle \ldots \rangle_{aK} = \int \text{d} \Phi, \text{d} F, \text{d} B, \text{d} \lambda e^{iS_{\bar{a}K}}$ is the average in the augmented Keldysh contours. In large-N limit, vertex correction is ignored, hence we use bare vertex in Bethe-Salpeter equation as illustrated in Fig. 5 to evaluate the quantum chaos in OTOC. The diagram contains two types of ladder diagrams, the first type are the two one-rung diagrams with $\Phi$ field connecting the upper and down worlds, and the second type is the two-rungs diagram with conduction electron fields linking the different worlds. Here are Feynman rules for these diagrams: (i) the rail lines in the upper world represents the advanced Green’s functions; (ii) the rail lines sited in the down world are retarded Green’s functions; (iii) the rungs connecting two worlds corresponds to the $G^K_{u\bar{d}}$ or $D^K_{u\bar{d}}$.

FIG. 5. (Color online) Diagrammatically representations of the Bethe-Salpeter equations.

Since there is no dissipation, the time translation symmetry holds. Hence we can calculate the following Fourier transformation,

$$C(t_1, t_2) = \frac{1}{N} \int \frac{d\Omega d\omega}{(2\pi)^2} e^{-i\Omega(t_1 - t_2) - i\omega t} C(\Omega, \omega),$$

(26)

where we introduce the center of mass time separation $t = (t_1 + t_2)/2$. Following the aforementioned rules to calculate the OTOC, the zero order for $C(\Omega, \omega)$ is

$$C_0(\Omega, \omega) = G^R(\Omega + \frac{\omega}{2})G^A(\Omega - \frac{\omega}{2}) = A_\omega(\Omega),$$

(27)

and $A_\omega(\Omega)$ is a positive real number due to $G^K(\Omega + \omega/2) = (G^K(\Omega - \omega/2))^*$. Summing up the ladder diagrams, we can obtain the Bethe-Salpeter equation,

$$C(\Omega, \omega) = A_\omega(\Omega) \left\{ 1 + \int \frac{d\Omega'}{2\pi} \left( K_{1,\omega}(\Omega, \Omega') + K_{2,\omega}(\Omega, \Omega') \right) \right\} \times C(\Omega', \omega),$$

(28)

followed by one-rung kernel $K_{1,\omega}$,

$$K_{1,\omega} = \frac{i g^2}{2} D^K_{\Phi,\bar{d}}(\Omega' - \Omega) + \frac{i g^2}{2} D^K_{\Phi,d}(\Omega - \Omega'),$$

(29)
and one two-rungs kernel $\mathcal{K}_{2,\omega}$,

$$
\mathcal{K}_{2,\omega}(\Omega, \Omega') = \frac{\kappa}{4} \int \frac{d\omega'}{2\pi} \left( D^{R}(\omega' + \omega/2) \times G_{c,ud}^{K}(\Omega - \omega') G_{c,du}^{K}(\Omega' - \omega') D^{A}(\omega' - \omega/2) \right). \tag{30}
$$

Finally, one can get the following irreducible ladder diagrams after dropping the irrelevant inhomogeneous term in Eq. 28

$$
\mathcal{C}(\Omega, \omega) = A_{\omega}(\Omega) \int \frac{d\Omega'}{2\pi} \tilde{\mathcal{K}}_{\omega}(\Omega, \Omega') \mathcal{C}(\Omega', \omega). \tag{31}
$$

Here $\tilde{\mathcal{K}}_{\omega} = \mathcal{K}_{1,\omega} + \mathcal{K}_{2,\omega}$. We clarify that the above equation we obtained contains leading order contributions in the large-N limit. While for finite N, the vertex correction might be relevant especially for strong coupling and one need take into account higher order diagrams. The Lyapunov exponent $\lambda_{L}$ corresponds to the positive solution of $-i\omega$ to make the integral kernel in the Bethe-Salpeter equation an unite eigenvalue of the Lyapunov exponent at the fixed Kondo coupling $J_{K}$. The existence of positive solution will signal the chaotic behavior, and on the other hand, it implies the absence of the chaotic properties. The details of the numerical calculation of Lyapunov exponent can be found in Appendix.

\section{V. NUMERICAL RESULTS}

In this section, we give the numerical results of the Lyapunov exponent at the $O(1)$ order at the intermediate temperature in the large $N$ limit. From the Eq. 30 and Eq. 31, it can be found that the eigenvalue $\lambda_{i}$ of the kernel matrix $A_{\omega}K_{\omega}$ at $g = 0$ is proportional to the square of Kondo coupling $J_{K}$ at the weak coupling limit, $\lambda_{i} \propto J_{K}^{2}$. Hence the condition for the existence of unit eigenvalue for given $-i\omega$ cannot be satisfied because of $\lambda_{i} \propto J_{K}^{2} \ll 1$, implying there is no chaotic behavior at the weak coupling limit for the pure multichannel Kondo model. This observation can be confirmed by the following numerical results.

In Fig. 6, we plot the ratio $\lambda_{L}/2\pi T$ as a function of temperature $T\pi/D$ for the BKFM at $g\pi/D = 0$. The black solid, the blue dashed, the red dashed and the purple dot-dashed curves correspond to different Kondo coupling $J_{K}$, with $J_{K}=1.5$, $1.2$, $0.8$ and $0.6$. Insert: $\lambda_{L}/2\pi T$ as a function of $J_{K}$ for fixed temperature $T\pi/D = 0.1$.

In Fig. 7, we plot the ratio $\lambda_{L}/2\pi T$ as a function of temperature $T\pi/D$ for (a,b) and as a function of boson coupling $g$ for (c,d).

Now we introduce the coupling between impurity and bosonic bath $g$. By increasing $g$, the systems will go...
through the overscreened multichannel Kondo phase to a critical local moment phase which is separated by an unstable fixed point with critical coupling \( g_c \). Generally near the quantum critical point, the non-Fermi liquid will arise and the conformal symmetry emerges, leading to the larger chaotic behavior with \( \lambda_L \sim T \) than other regions away from critical point. To check this argument, we plot the \( \lambda_L/2\pi T \) as functions of temperature at different \( g\pi/D \) and as functions of \( g\pi/D \) at a fixed temperature \( T\pi/D = 0.1 \) as illustrated in Fig. 7. In Fig. 7 (a,b) we can obtain the two main observations: (1) For sub-ohmic case \( \epsilon = 0.5 \), the Lyapunov exponent decreases as growing bosonic coupling \( g \) at the same parameters \((J_K, T)\). This shares the same behavior to the ohmic case \( \epsilon = 0 \) although they have quiet different RG flows. This fact indicate the violation of the above argument. (2) Butterfly effect is stronger at the ohmic case than the sub-ohmic one, by comparing Fig. 7 (a) to Fig. 7 (b). In order to investigate the behavior of Lyapunov exponent when crossing critical point, we performed a detailed \( g \) dependence calculation for various \( J_k \) at fixed temperature as shown in Fig. 7 (c,d). One can clearly see the Lyapunov exponent is indeed monotonically decreasing as increasing \( g \), and is finally vanishing at finite \( g \). This monotonous behavior is consistent with the result that residual entropy for this model increases monotonously from MCK phase to LM' phase. This behavior of residual entropy violate the \( g \)-theorem which demand the entropy should decrease along RG trajectories if conformal invariance is presented. As the original model accually break conformal symmetry in sub-ohmic case, one may not expect the largest chaotic behavior at critical point similar to impurity entropy case. From numerical perspective, the reason of the decreasing of \( \lambda_L \) with \( g \) can be attribute to the suppressive effect of magnitude of spectral functions of impurity and auxiliary boson as shown in Fig. 7.

VI. CONCLUSIONS

In this paper, we derive the Bethe-Salpeter equation for our defined impurity OTO correlator for BFKM in large-N limit. We find the biggest contribution comes from two one-rung and one two-rungs diagram whose upper and down world lines are connected by bosonic bath and the conduction electrons respectively. The numerical calculation at the intermediate temperature shows that the Lyapunov exponent \( \lambda_L \) decreases with increasing temperature and finally vanishle at a finite temperature, which is different with the one and two-channel Kondo models in which impurity OTOC is temperature-independent. We also observe the system has no butterfly effect below a typical Kondo coupling \( J_K \) for finite temperature. When coupled to bosonic bath, the monotonously decrease of \( \lambda_L \) do not obey general argument that the highest chaotic behavior occurs at the quantum critical point, but is consistent with the behavior of impurity entropy and violation of \( g \)-theorem in the model.

VII. ACKNOWLEDGMENTS

We thank Boyang Liu and Pengfei Zhang for very helpful discussions. The work is supported by the Hong Kong Research Grants Council, GRF 17304719, CRF C6026-16W and C6005-17G.

Appendix A: Numerical technique to solve the saddle point equations

The Fourier transformation method is used to solve the self-consistent equations and also the integral kernel \( A_\epsilon(K) \) in the main text. We discrete the frequency and time domain as

\[
\Omega_d = \frac{2\pi f_s}{N} \left[ \frac{1}{N} - \frac{3}{2N}, \ldots, \frac{N - 3}{2N}, \frac{N - 1}{2N} \right] \quad (A1)
\]

\[
T_d = \frac{1}{f_s} \left[ \frac{1}{2}, \frac{1}{2} - \frac{1}{2}, \ldots, \frac{N - 3}{2N}, \frac{N - 1}{2N} \right] \quad (A2)
\]

The Fourier transformation \( G(t) = \int \frac{d\omega}{2\pi} G(\omega) e^{-i\omega t} \) and \( G(\omega) = \int_0^{+\infty} dt G(\omega) e^{i\omega t} \) for Green’s functions can be performed by the FFT algorithm. We iteratively solve the equations and obtain the self-consistent solutions if the error \( \text{max}(|G(\omega) - \hat{G}(\omega)|) \) where \( G \) and \( \hat{G} \) belong to two nearest iterative steps is less than \( 10^{-6} \). In practice, the total point \( N_t = 2^{21} + 1 \) and \( f_s = 4 \) is used. The cutoff for the frequency is \( \omega_c = 4\pi \) which is is four times of band width \( D \). Moreover, in our numerical calculations the spectral function of the bosonic bath is taken as

\[
A_\Phi(\omega) = \begin{cases} 
|\omega|^{1-c}\text{sign}(\omega), |\omega| < \Lambda \\
|\omega|^{1-c}\text{sign}(\omega) e^{-c(|\omega| - \Lambda)}, |\omega| \geq \Lambda 
\end{cases} \quad (A3)
\]

here energy cutoff \( \Lambda = 0.05 \) and \( c = 15 \).

Appendix B: Numerical method for the Lyapunov exponent

To numerically calculate the Lyapunov, we first discreate the frequency to transform the integral equation to a linear algebra equation where the integral kernel \( A_\epsilon(\Omega) K_{\epsilon'}(\Omega, \Omega') \) becomes a matrix. The Lyapunov exponent corresponds to the positive value \(-i\omega\) which leads to existence of unite eigenvalue of the kernel matrix. Here the energy is discretized in range \((-5, 5)\) with total number \( N_{\text{size}} = 800 \). The consistency of the result for \( N_{\text{size}} \) is checked with smaller interval. In Fig. 8 we illustrates the evolution of \( E_0 = \min|\lambda| \) where \( \lambda_i \) is the eigenvalue of the integral kernel of the Bethe-Salpeter equation Eq. 51.
FIG. 8. (Color online) Plot the magnitude of $E_0$ as a function of $-i\omega/T$ in the positive axis at the given $J_K\pi/D = 1.2$ for $g\pi/D = 0.2$ (a), 0.4 (b), 1.0 (c) and 3.0 (d).

Appendix C: Bethe-Salpeter equation at the low temperature limit

Argument in the main text that $\lambda_i \propto J_K^2$ at the intermediate temperature when absence of bosonic bath cannot be applied the case at the low temperature limit. For $g = 0$ case, the flow diagram tells us that the system will flow to $MCK$ point at any finite $J_K$. Then we can apply the scaling ansatz:

$$\text{Im } G(\omega) = -M_f|\omega|^{\alpha_f - 1}, \quad (C1)$$
$$\text{Im } D(\omega) = -M_B|\omega|^{\alpha_B - 1} \quad (C2)$$

for fixed point Green’s functions at zero temperature\cite{note1}. Insert these relations into self-consistent equations one can obtain the relation for amplitudes and exponents:

$$\alpha_f = 1 - \alpha_B = \frac{1}{1 + \kappa}, \quad (C3)$$
$$M_cM_fM_B \propto \alpha_f \tan\left(\frac{\pi\alpha_f}{2}\right) \quad (C4)$$

for $MCK$ fixed point at $g = 0$ case. Where $M_c$ is the amplitudes prefactor for condition electron. This means that at zero temperature, the solution of Bethe-Salpeter Eq. [28] will not dependent on the value of $J_K$, i.e., the eigenvalue $\lambda_i$ will saturate to the same value for any finite $J_K$, similar to the entropy results\cite{note1}.

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