GAUSSIAN ITERATIVE ALGORITHM AND INTEGRATED AUTOMORPHISM EQUATION FOR RANDOM MEANS

JUSTYNA JARCZYK
Institute of Mathematics, University of Zielona Góra
Szafrana 4a, PL-65-516 Zielona Góra, Poland

WITOLD JARCZYK∗
Institute of Mathematics and Informatics
The John Paul II Catholic University of Lublin
Konstantynów 1h, PL-20-708 Lublin, Poland

Abstract. Gauss-type iterates for random means are considered and their limit behaviour is studied. Among others the invariance of the limit with respect to the given random mean-type mapping $M$ is established under some relatively weak assumptions. The algorithm is applied to prove the existence and uniqueness of solutions $\varphi$ of the equation

$$\varphi(x) = \int_{\Omega} \varphi(M(x, \omega)) \, dP(\omega)$$

in the class of (deterministic) means in $p$ variables.

Introduction. A mean in $p$ variables, on a real interval $I$, is any function $M : I^p \to I$ with values lying between the minimal and maximal coordinate of the argument:

$$\min \{x_1, \ldots, x_p\} \leq M(x) \leq \max \{x_1, \ldots, x_p\}$$

for all $x = (x_1, \ldots, x_p) \in I^p$. We generalize this classical notion by making the mean dependent on a random parameter $\omega$ running through a set $\Omega$ endowed with a probability $P$.

The paper consists of two main parts. In the first one (see Section 2) we show how to iterate mappings $M = (M_1, \ldots, M_p) : I^p \times \Omega \to I^p$ whose coordinates $M_1, \ldots, M_p$ are random means and then we establish a version of the so-called Gaussian algorithm for such mappings. It describes a possible limit behaviour of iterative sequences $(M^n)_{n \in \mathbb{N}}$ and leads to an interesting invariance question.

The second part (see Section 3) shows how to utilize the Gaussian algorithm to study the existence and uniqueness of solutions of the integrated functional equation

$$\varphi(x) = \int_{\Omega} \varphi(M(x, \omega)) \, dP(\omega).$$

in the class of means $\varphi : I^p \to I$.

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∗ Corresponding author: Witold Jarczyk.
1. Preliminaries. We start with the definitions of random-valued function and random mean. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Given a separable metric space $X$ any function $f : X \times \Omega \to X$ which is measurable with respect to the product $\sigma$-algebra $\mathcal{B} \otimes \mathcal{A}$, where $\mathcal{B}$ stands for the $\sigma$-algebra of Borel subsets of $X$, is called a random-valued function (cf. [3], also [9]).

Fix an interval $I$ of reals and a positive integer $p$. Denote by $\mathcal{B}$ the $\sigma$-algebra of Borel subsets of $I^p$. A random mean on the interval $I$ is any function $M : I^p \times \Omega \to I$ which is $\mathcal{B} \otimes \mathcal{A}$-measurable and such that $M (\cdot, \omega)$ is a mean for a.a. $\omega \in \Omega$. It is said to be continuous if $M (\cdot, \omega)$ is continuous for a.a. $\omega \in \Omega$.

Having $p$ random means $M_1, \ldots, M_p$ on $I$ we come to the random mean-type mapping $M := (M_1, \ldots, M_p) : I^p \times \Omega \to I^p$. Clearly, it is a random-valued function. Thus, making use of the below definition of iterates of random-valued functions (see [3]), we can iterate random mean-type mappings. For an important example of a continuous random mean-type see [12, Example 2.3]

Given a random-valued function $f : X \times \Omega \to X$ we put $\Omega^\infty := \Omega^\mathbb{N}$ and define iterates $f^i : X \times \Omega^\infty \to X$ by the equalities

$$f^1(x, \omega) = f(x, \omega_1)$$

and

$$f^{i+1}(x, \omega) := f (f^i(x, \omega), \omega_{i+1})$$

requested for all $x \in X$, $\omega = (\omega_1, \omega_2, \ldots) \in \Omega^\infty$ and $i \in \mathbb{N}$. It is an easy observation that

$$f^{i+1}(x, \omega) = f^i (f (x, \omega_1), (\omega_2, \omega_3, \ldots))$$

for all $x \in X$, $\omega \in \Omega^\infty$ and $i \in \mathbb{N}$. Moreover, it can be also observed that the iterates of a random-valued function are again random-valued.

Take any random mean $M : I^p \times \Omega \to I$ and put

$$\Omega_0 := \{ \omega \in \Omega : M (\cdot, \omega) \text{ is a mean on } I \}.$$ 

Then $P (\Omega_0) = 1$ and $M|_{I^p \times \Omega_0}$ is a random mean on $I$ such that $M (\cdot, \omega)$ is a mean for all $\omega \in \Omega_0$. Since $P^\infty (\Omega_0) = 1$ (here and in what follows $P^\infty$ stands for the product measure on $\Omega^\infty$), then, iterating the restriction $M|_{I^p \times \Omega_0}$ of any mean-type mapping $M : I^p \times \Omega \to I^p$, we obtain a sequence of random means that is equal to $(M^i)_{i \in \mathbb{N}}$ almost everywhere with respect to $P^\infty$. For that reason, when carrying out “countable” operations on a random mean-type mapping $(M_1, \ldots, M_p)$ (e.g. iteration), we may assume that $M_i (\cdot, \omega)$ are, in fact, means for $i = 1, \ldots, p$ and for all $\omega \in \Omega$. In what follows we actually do it. Under this assumption each random mean-type mapping is a parametrized mean-type mapping in the sense of the paper [12]. In particular, any random mean, is in fact, a parametrized mean. Using results of [12] we immediately obtain the facts of the next section describing the Gauss algorithm for random means. They will be used in Sec. 3 to prove the main results of that paper devoted to the integrated functional equation 1. Here $M : I^p \times \Omega \to I^p$ is a given random mean-type mapping and we are interested in solutions $\varphi$ being means on the interval $I$.

We complete that section with the following immediate consequence of Lemma 2.2 from [12].

Remark 1. All iterates of a continuous random mean-type mapping are continuous.
2. Gaussian algorithm for random mean-type mappings. Origins of the phenomenon, known today as the Gaussian algorithm, can be found in the Lagrangean treatise [15] written at the end of 18th century and then in Gaussian work [10] published a hundred years later (see also [11], where the results of [10] were reminded by Geppert in 1927). For some other references the reader is referred to the survey [13].

Here we formulate four results describing the Gaussian iterates (and their limit behaviour) of random mean-type mappings. Remembering that iterating such continuous mappings we may assume that, in fact, $M(\cdot, \omega)$ is continuous for all $\omega \in \Omega$ and, consequently, most of the assertions of that section immediately follows from [12, Theorems 3.1, 3.3 and 3.5].

Given a random mean-type mapping $M : I^p \times \Omega \to I^p$ and a positive integer $n$ the $i$-th coordinate of the $n$-th iterate $M^n : I^p \times \Omega^\infty \to I^p$ is denoted by $M_{i,n}$:

$$M^n = (M_{1,n}, \ldots, M_{p,n}).$$

Moreover, we put

$$K^-_n(x, \omega) := \min \{M_{1,n}(x, \omega), \ldots, M_{p,n}(x, \omega)\}$$

and

$$K^+_n(x, \omega) := \max \{M_{1,n}(x, \omega), \ldots, M_{p,n}(x, \omega)\}$$

for all $x \in I^p$ and $\omega \in \Omega^\infty$.

**Theorem 2.1.** Let $M : I^p \times \Omega \to I^p$ be a random mean-type mapping. Then

(i) for every $n \in \mathbb{N}$ the functions $M_{1,n}, \ldots, M_{p,n}$ and $K^-_n, K^+_n$ are random means;

(ii) $K^-_n(\cdot, \omega) \leq K^-_{n+1}(\cdot, \omega) \leq K^+_n(\cdot, \omega) \leq K^+_n(\cdot, \omega)$ for all $n \in \mathbb{N}$ and a.a. $\omega \in \Omega^\infty$;

(iii) the functions $K^-, K^+ : I^p \times \Omega^\infty \to I$, defined by

$$K^-(\cdot, \omega) := \lim_{n \to \infty} K^-_n(\cdot, \omega)$$

and

$$K^+(\cdot, \omega) := \lim_{n \to \infty} K^+_n(\cdot, \omega)$$

for a.a. $\omega \in \Omega^\infty$, are $M$-invariant random means:

$$K^-(\cdot, \omega) = K^-(M(\cdot, \omega_1), (\omega_2, \omega_3, \ldots))$$

for a.a. $\omega \in \Omega^\infty$

and

$$K^+(\cdot, \omega) = K^+(M(\cdot, \omega_1), (\omega_2, \omega_3, \ldots))$$

for a.a. $\omega \in \Omega^\infty$, and

$$K^-_n(\cdot, \omega) \leq K^+_n(\cdot, \omega)$$

for a.a. $\omega \in \Omega^\infty$;

(iv) if $L : I^p \times \Omega^\infty \to I$ is an $M$-invariant random mean, then

$$K^-_n(\cdot, \omega) \leq L(\cdot, \omega) \leq K^+_n(\cdot, \omega)$$

for a.a. $\omega \in \Omega^\infty$.

**Proof.** For (i) and (ii) see assertions (i) and (ii), respectively, of [12, Theorem 3.3] where the continuity assumption is not used at all.

(iii) Assertions (i) and (ii) readily yield the existence of $K^-$ and $K^+$. Now using an induction, one can check that

$$M_{i,n+1}(x, \omega) = M_{i,n}(M(x, \omega_1), (\omega_2, \omega_3, \ldots))$$

whence

$$K^-_{n+1}(x, \omega) = K^-_n(M(x, \omega_1), (\omega_2, \omega_3, \ldots))$$

and

$$K^+_{n+1}(x, \omega) = K^+_n(M(x, \omega_1), (\omega_2, \omega_3, \ldots))$$

for all $x \in I^p$ and $\omega \in \Omega^\infty$. 

Now using
for all \( x \in I^p, \omega \in \Omega^\infty \) and for every \( n \in \mathbb{N} \). This gives the \( M \)-invariance of \( K^- \) and \( K^+ \). The rest of (iii) follows easily.

(iv) Fix any \( n \in \mathbb{N} \). Then, using induction, for a.a. \( \omega \in \Omega^\infty \) we get

\[
L (\cdot, \omega) = L(M(\cdot, \omega_1), (\omega_2, \omega_3, \ldots)) = L(M^1(\cdot, \omega), (\omega_2, \omega_3, \ldots))
\]

\[
= L(M^n(\cdot, \omega), (\omega_{n+1}, \omega_{n+2}, \ldots)).
\]

Thus, since \( L \) is a random mean, for a.a. \( \omega \in \Omega^\infty \) the number \( L(\cdot, \omega) \) lies between the minimum and maximum of the set

\[
\{M_{1,n}(\cdot, \omega), \ldots, M_{p,n}(\cdot, \omega)\},
\]

that is the inequalities

\[
K^- (\cdot, \omega) \leq L (\cdot, \omega) \leq K^+ (\cdot, \omega)
\]

hold for all \( n \in \mathbb{N} \) and for a.a. \( \omega \in \Omega^\infty \) and the assertion follows. \( \square \)

The next result is an immediate consequence of [12, Theorem 3.1].

**Theorem 2.2.** Let \( M : I^p \times \Omega \to I^p \) be a random mean-type mapping. Assume that the sequence \((M^n(\cdot, \omega))_{n \in \mathbb{N}}\) is pointwise convergent for a.a. \( \omega \in \Omega^\infty \) and let \( K = (K_1, \ldots, K_p) : I^p \times \Omega^\infty \to I^p \) be defined for a.a. \( \omega \in \Omega^\infty \) by

\[
K(\cdot, \omega) := \lim_{n \to \infty} M^n(\cdot, \omega).
\]

Then \( K \) is a random mean-type mapping and its coordinates \( K_1, \ldots, K_p \) are \( M \)-invariant random means. Moreover, there is an essentially unique \( M \)-invariant random mean if and only if \( K_1(\cdot, \omega) = \ldots = K_n(\cdot, \omega) \) for a.a. \( \omega \in \Omega^\infty \). It is \( K := K_1 \).

Here and in what follows the phrase “essentially unique \( M \)-invariant random mean” indicates that \( K(\cdot, \omega) = L(\cdot, \omega) \) for a.a. \( \omega \in \Omega^\infty \) whenever \( L \) is any \( M \)-invariant random mean.

The assumption of the convergence of the sequence \((M^n)_{n \in \mathbb{N}}\) is generally hardly satisfied and verified. A good solution is to assume additionally that \( \Omega \) is a compact topological space and \( M \) is a continuous function. In the next result we assume less, namely that \( M = (M_1, \ldots, M_p) \) is a continuous random mean-type mapping, i.e. \( M_i(\cdot, \omega) \) is a continuous random mean for a.a. \( \omega \in \Omega \). In particular, we still not need any topology in \( \Omega \).

**Theorem 2.3.** Let \( M : I^p \times \Omega \to I^p \) be a continuous random mean-type mapping. Then

(i) for every \( n \in \mathbb{N} \) the random means \( M_{1,n}, \ldots, M_{p,n} \) and \( K^-_n, K^+_n \) are continuous;

(ii) for a.a. \( \omega \in \Omega^\infty \) the means \( K^-(\cdot, \omega) \) and \( K^+(\cdot, \omega) \) are lower and upper semi-continuous, respectively;

(iii) if \( K^-(\cdot, \omega) = K^+(\cdot, \omega) \) for a.a. \( \omega \in \Omega^\infty \), then \( K := K^- \) is a continuous random mean and

\[
\lim_{n \to \infty} M^n(\cdot, \omega) = (K(\cdot, \omega), \ldots, K(\cdot, \omega)) \text{ for a.a. } \omega \in \Omega^\infty;
\]

(iv) there is an essentially unique \( M \)-invariant random mean if and only if \( K^-(\cdot, \omega) = K^+(\cdot, \omega) \) for a.a. \( \omega \in \Omega^\infty \); it is \( K := K^- \).

All the assertions follow from [12, Theorem 3.3]. Contrary to the deterministic case, where the continuity of the given mean-type mapping \( M : I^p \to I^p \), satisfying some additional weak assumptions, forces the crucial equality \( K^- = K^+ \) (see [16]
where \( p = 2 \), also [17] for the general case), the situation where we deal with random mean-type mappings is more complicated. Nevertheless, the desired equality \( K^- (\cdot, \omega) = K^+ (\cdot, \omega) \) held for a.a. \( \omega \in \Omega^\infty \) may be obtained under some stronger assumptions. The last result of the present section follows from [12, Theorem 3.5] and Theorem 2.3 (iv).

**Theorem 2.4.** Assume that \( \Omega \) is endowed with a compact topology and let \( \mathbf{M} = (M_1, \ldots, M_p) : I^p \times \Omega \rightarrow I^p \) be a continuous mapping such that \( M_i (\cdot, \omega) \) is a mean for all \( i = 1, \ldots, p \) and all \( \omega \in \Omega \). Assume that

\[
\begin{align*}
\text{if } \min \{ M_1 (x, \omega), \ldots, M_p (x, \omega) \} &= \min \{ x_1, \ldots, x_p \} \\
\text{and } \max \{ M_1 (x, \omega), \ldots, M_p (x, \omega) \} &= \max \{ x_1, \ldots, x_p \} \\
\text{then } x_1 = \ldots = x_p \text{ for all } x = (x_1, \ldots, x_p) \in I^p \text{ and } \omega \in \Omega.
\end{align*}
\]

(2)

Then

(i) there is a random mean \( K : I^p \times \Omega^\infty \rightarrow I \) such that

\[
\lim_{n \rightarrow \infty} M_{i,n} = K, \quad i = 1, \ldots, p,
\]

uniformly on every compact subset of \( I^p \times \Omega^\infty \);

(ii) the function \( K \) is continuous;

(iii) \( K \) is an essentially unique \( \mathbf{M} \)-invariant random mean and

\[
K(\cdot, \omega) = K^- (\cdot, \omega) = K^+ (\cdot, \omega) \quad \text{for a.a. } \omega \in \Omega^\infty.
\]

(3)

**Remark 2.** We say that a random mean \( M : I^p \times \Omega \rightarrow I \) is strict if for a.a. \( \omega \in \Omega \) the inequalities

\[
\min \{ x_1, \ldots, x_p \} < M (x, \omega) < \max \{ x_1, \ldots, x_p \}
\]

hold for all \( x \in I^p \) such that \( \min \{ x_1, \ldots, x_p \} < \max \{ x_1, \ldots, x_p \} \). It is easy to check (see [12, Remark 4.6] for details) that for every random mean-type mapping the assumption of strictness of all but one its coordinates enforces condition 2.

3. **Integrated automorphism equation for random means.** Equation 1 is of the form

\[
\varphi (x) = \int_{\Omega} \varphi (f(x, \omega)) dP(\omega),
\]

(4)

that is \( \varphi (x) = \mathbb{E} (\varphi (f(x, \cdot))) \). The literature devoted to equations of that type is really vast. A pretty large number of mathematicians investigated equation 4 assuming different forms of the given function \( f \), acting in various classes of solutions \( \varphi \) and, first of all, taking diverse issues.

Problems connected with equations 4 appear in many contexts. In some cases the classical convolution Choquet-Deny equation for probability measures (see [7], also [18]), i.e. the equation \( \mu = \mu * \sigma \) is a special case of 4 with \( f \) given by \( f(x, \omega) = x + \xi (\omega) \).

The study of the so-called archetypal equation

\[
\varphi (x) = \int_{\mathbb{R}^2} \varphi (a(x - b)) d\mu (a, b),
\]

(5)

where \( \mu \) is a given probability measure on \( \mathbb{R}^2 \), was started by Derfel [8] in 1989. It is equivalent to equation 4 with \( f \) defined by \( f(x, \omega) = \alpha (\omega) (x - \beta (\omega)) \); here \( (\alpha, \beta) : \Omega \rightarrow \mathbb{R}^2 \) is a random vector with a distribution \( \mu \). A number of deep results connected with equation 5 are due to Bogachev, Derfel and Molchanov (see [5] and [6]). Some of them result in theory of functional-differential equations, including
the famous pantograph equation. For quite recent results devoted to the archetypal equation the reader is referred to the paper [19] by Sudzik.

Studies of equation 4 are also conducted by a group around Baron, consisting of Kapica, Morawiec, also the second author (see, e.g. [1], [2], [14], [4]). In all of them a mix of iterative and probabilistic methods is used. Equation 4 is also closely related to some refinement type equation.

Here we deal with equation 1, that is 4 where \( f \) is a given random mean-type mapping \( \mathbf{M} : I_p \times \Omega \to I_p \). We are looking for solutions \( \varphi \) being a mean on the interval \( I \), that is \( \varphi : I_p \to I \).

Given a random mean-type mapping \( \mathbf{M} : I_p \times \Omega \to I_p \) denote by \( \phi(\mathbf{M}) \) the set of means \( \varphi : I_p \to I \) satisfying equation 1. Keeping in mind the denotation introduced at the beginning of Section 2 we formulate the below theorem describing the structure of the set \( \phi(\mathbf{M}) \). In what follows the expectations \( \kappa^-, \kappa^+ : I_p \to I \) of \( K^-, K^+ : I_p \times \Omega^\infty \to I \), given by

\[
\kappa^-(x) = \int_{\Omega^\infty} K^-(x, \omega) dP^\infty(\omega), \quad \kappa^+(x) = \int_{\Omega^\infty} K^+(x, \omega) dP^\infty(\omega),
\]

respectively, play a fundamental role.

**Theorem 3.1.** Let \( \mathbf{M} : I_p \times \Omega \to I_p \) be a random mean-type mapping. Then

(i) \( \phi(\mathbf{M}) \) is a non-void convex subset of the linear space of all functions mapping \( I_p \) into \( \mathbb{R} \);

(ii) \( \kappa^-, \kappa^+ \in \phi(\mathbf{M}) \);

(iii) \( \kappa^- = \inf \phi(\mathbf{M}) \) and \( \kappa^+ = \sup \phi(\mathbf{M}) \).

(In the set of all real functions defined on \( I_p \) we introduce the common linear operations and order.)

**Proof.** (i) Convexity of \( \phi(\mathbf{M}) \) is obvious. Its non-emptiness is a direct consequence of (ii).

(ii) We consider the case of \( \kappa^- \) only. Given an \( x \in I_p \) the function \( K^-(x, \cdot) \) is \( \mathcal{A}^\infty \)-measurable and, since \( K^- \) is a random mean according to Theorem 2.1 (iii), for a.a. \( \omega \in \Omega^\infty \) we have

\[
\min\{x_1, \ldots, x_p\} \leq K^-(x, \omega) \leq \max\{x_1, \ldots, x_p\},
\]

so the function \( K^-(x, \cdot) \) is \( P^\infty \)-integrable. Moreover, by the Fubini theorem and by the \( \mathbf{M} \)-invariance of the random mean \( K^- \) we have

\[
\int_{\Omega} \kappa^-(\mathbf{M}(x, \omega)) dP(\omega)
\]

\[
= \int_{\Omega} \left( \int_{\Omega^\infty} K^-(\mathbf{M}(x, \omega_1), (\omega_2, \omega_3, \ldots)) dP^\infty(\omega_2, \omega_3, \ldots) \right) dP(\omega_1)
\]

\[
= \int_{\Omega^\infty} \left( \int_{\Omega} K^-(\mathbf{M}(x, \omega_1), (\omega_2, \omega_3, \ldots)) dP(\omega_1) \right) dP^\infty(\omega_2, \omega_3, \ldots)
\]

\[
= \int_{\Omega^\infty} K^-(x, \omega) dP(\omega_1) dP^\infty(\omega_2, \omega_3, \ldots)
\]

\[
= \int_{\Omega} K^-(x, \omega) dP^\infty(\omega) = \kappa^-(x)
\]
for every $x \in I^p$.

(iii) Take any $\varphi \in \phi(M)$ and fix an arbitrary $x \in I^p$. Then

$$\varphi(x) = \int_{\Omega} \varphi(M(x, \omega)) dP(\omega) = \int_{\Omega^\infty} \varphi(M^1(x, \omega)) dP^\infty(\omega).$$

Using an induction we see that for every $n \in \mathbb{N}$

$$\varphi(x) = \int_{\Omega^\infty} \varphi(M^n(x, \omega)) dP^\infty(\omega) \geq \int_{\Omega^\infty} K_n^-(x, \omega)dP^\infty(\omega)$$

since $\varphi$ is a mean and its values are not less than the minimal coordinate of the argument. Consequently, recalling the $P^\infty$-integrability of $K_n^-(x, \cdot)$ and making use of the dominated convergence theorem, we have

$$\varphi(x) = \int_{\Omega^\infty} K^-(x, \omega)dP^\infty(\omega) = \kappa^-(x).$$

Similarly, we show that $\varphi(x) \leq \kappa^+(x)$. This together with (ii) gives the assertion (iii).

Theorem 3.1 guarantees the existence of solutions of equation 1 in the class of means. The following corollary characterizes the uniqueness of such solutions.

**Corollary 1.** Let $M : I^p \times \Omega \rightarrow I^p$ be a random mean-type mapping. Equation 1 has exactly one solution $\varphi : I^p \rightarrow I$ being a mean if and only if for every $x \in I^p$ we have

$$K^-(x, \cdot, \omega) = K^+(x, \cdot, \omega) \text{ for a.a. } \omega \in \Omega^\infty.$$  

(6)

**Proof.** By Theorem 3.1 the uniqueness of the desired solution is equivalent to the equality $\kappa^- = \kappa^+$ which, clearly, is the same as condition 6 held for all $x \in I^p$. □

That desired uniqueness can be proved under stronger assumptions imposed already in Theorem 2.4.

**Theorem 3.2.** Assume that $\Omega$ is endowed with a compact topology and let $M = (M_1, \ldots, M_p) : I^p \times \Omega \rightarrow I^p$ be a continuous mapping such that $M_i(\cdot, \omega)$ is a mean for all $i = 1, \ldots, p$ and all $\omega \in \Omega$. Assume condition 2. Then there is a continuous random mean $K : I^p \times \Omega^\infty \rightarrow I$ such that condition 3 holds and there is exactly one mean $\varphi : I^p \rightarrow I$ satisfying equation 1. This solution is continuous and defined by the formula

$$\varphi(x) = \int_{\Omega^\infty} K(x, \omega)dP^\infty(\omega).$$

**Proof.** The desired function $K$ exists on account of Theorem 2.4. Since 3 implies condition 6 then the uniqueness part follows from Corollary 1. The formula for $\varphi$ is a simple consequence of the equality $\varphi = \kappa^-$ and condition 3. □

We conclude the paper with a special case when the set $\Omega$ is finite. Without loss of generality we may assume that $\Omega = \{1, 2, \ldots, k\}$ for some positive integer $k$, the $\sigma$-algebra $\mathcal{A}$ is $2^\Omega$ and $P = (p_1, \ldots, p_k)$ is a probability vector, i.e. $p_1, \ldots, p_k \in (0, +\infty)$ and $p_1 + \ldots + p_k = 1$. Then we come to the following result.

**Theorem 3.3.** Let $M = (M_1, \ldots, M_p) : I^p \times \{1, \ldots, k\} \rightarrow I$ be a continuous random mean-type mapping satisfying condition 2. Then there is a continuous random mean $K : I^p \times \{1, \ldots, k\}^N \rightarrow I$ such that

$$K(\cdot, \omega) = K^-(\cdot, \omega) = K^+(\cdot, \omega) \text{ for a.a. } \omega \in \{1, \ldots, k\}^N.$$
and there is exactly one mean \( \varphi : I^p \to I \) satisfying equation
\[
\varphi(x) = \sum_{j=1}^{k} p_j \varphi(M(x, j)).
\]
This solution is continuous and defined by the formula
\[
\varphi(x) = \int_{\{1, \ldots, k\}^\mathbb{N}} K(x, \omega) dP^\infty(\omega).
\]

Proof. We have two important consequences of the finiteness of \( \Omega := \{1, \ldots, k\} \).
Assuming, by default, that it is endowed with the discrete topology we see that \( \Omega \)
is compact and the mapping \( M : I^p \times \Omega \to I^p \) is continuous. Consequently, the
assertion follows from Theorems 2.4 and 3.2. \( \square \)

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E-mail address: j.jarczyk@wmie.uz.zgora.pl
E-mail address: wjarczyk@kul.lublin.pl