Research Article

Estimation Using Suggested EM Algorithm Based on Progressively Type-II Censored Samples from a Finite Mixture of Truncated Type-I Generalized Logistic Distributions with an Application

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In this paper, the identifiability property has been studied for a suggested truncated type-I generalized logistic mixture model which is denoted by (TTIGL). A suggested form of the EM algorithm has been applied on type-II progressive censored samples to obtain the maximum likelihood estimates (MLEs) of the parameters, survival function (SF), and hazard rate function (HRF) of the studied mixture model. Monte Carlo simulation algorithm has been applied to study the behavior of the mean squares errors (MSE’s) of the estimates. Also, a comparative study is conducted between the suggested EM algorithm and the ordinary algorithm of maximizing the likelihood function, which depends on the differentiation of the log likelihood function. The results of this paper have been applied on a real dataset as an application.

1. Introduction

The progressive type-II censored model is a very important model in the field of reliability and life testing (see [1]). This censoring model can be shown as follows.

Consider a lifetime test in which \( n \) identical units are tested. \( R_i \) surviving units are removed randomly from the experiment once the \( i^{th} \) failure has occurred, \( 1 \leq i \leq r \). Thus, if the number of observed failures is \( r \), then \( R_1 + R_2 + \cdots + R_r \) units are progressive censored, and hence \( n = R_1 + R_2 + \cdots + R_r \) and \( X_{1:n} < X_{2:n} < \cdots < X_{r:n} \) describe the progressive censored failure times, where \( M = (R_1, R_2, \ldots, R_r) \) and \( \sum_{i=1}^{r} R_i = n - r \). The likelihood function (LF) based on type-II progressive censored data \( \mathbf{x} = (x_{1:n}, x_{2:n}, \ldots, x_{r:n}) \) which can be written for simplicity as \( \mathbf{x} = (x_1, x_2, \ldots, x_r) \) is given by

\[
L(\theta; \mathbf{x}) = c \prod_{i=1}^{r} f_{\theta}(x_i)[S_{\theta}(x_i)]^{R_i},
\]

where \( c = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - R_1 - R_2 - \cdots - R_{r-1} - r + 1) \) (see [1]). The functions \( f_{\theta}(x_i) \) and \( S_{\theta}(x_i) \) are the probability density function (PDF) and the survival function (SF) of the studied distribution at a value \( x_i \).

The importance of the mixture models appear in the theoretical and applied fields when the population under study is heterogeneous. For details about mixture models, see [2–5].

The contributions of this paper are suggesting an EM algorithm suitable for estimation based on progressively type-II censored samples from a finite mixture of distributions, studying the identifiability property of a finite mixture of TTIGL distributions, and finally using a real dataset as an application.

A random variable \( X \) is said to have a finite mixture of certain distributions with PDF’s \( f_j(x; \theta_j), j = 1, 2, \ldots, k \) and vector of parameters \( \theta_j, j = 1, 2, \ldots, k \) if its PDF is given by
\[ f_\Theta(x) = \sum_{j=1}^{k} p_j f_j(x; \Theta_j), \]  

(2)

where

\[ \Theta = (\theta_1, \theta_2, \ldots, \theta_k, \theta_1, \theta_2, \ldots, p_k), \]  

(3)

\[ \sum_{j=1}^{k} p_j = 1 \text{ and } 0 \leq p_j \leq 1. \]

The cumulative distribution function (CDF) and SF are

\[ F_\Theta(x) = \sum_{j=1}^{k} p_j F_j(x; \Theta_j), \]

(4)

\[ S_\Theta(x) = \sum_{j=1}^{k} p_j S_j(x; \Theta_j). \]

This paper is organized as follows. In Section 2, a suggested form of EM algorithm is introduced to compute the MLEs of the parameters of a finite mixture of distributions based on progressively type-II censored samples. In Section 3, the identifiability property of the finite mixture of TTIGL distributions is studied using Chandra’s theorem in [6]. In Section 4, the suggested form of the EM algorithm is applied on a finite mixture of TTIGL distributions. In Section 5, the main results are introduced. Finally, concluding remarks are introduced in Section 6.

\[ Q(\Theta, \Theta^0) = \sum_{i=1}^{n} \sum_{j=1}^{k} \left\{ \log(p_j) (1 + R_i) E_U[U_{ij}; X_i = x_i] \right\}_{\theta^0} + \log(f_j(x_i; \Theta_j^0)) E_U[U_{ij}; X_i = x_i] |_{\theta^0} + R_i \log[S_j(x_i; \Theta_j^0)] E_U[U_{ij}; X_i = x_i] |_{\theta^0}. \]

(7)

where

\[ E_U[U_{ij}|x_i^*] |_{\theta^0} = \begin{cases} \frac{p_j^0 f_j(x_i; \Theta_j^0)}{\sum_{i=1}^{k} p_i^0 f_i(x_i; \Theta_i^0)}, & t^* = x_i, \\ \frac{p_j^0 S_j(x_i; \Theta_j^0)}{\sum_{i=1}^{k} p_i^0 S_i(x_i; \Theta_i^0)}, & x_i < t^* < x_{i+1}, \end{cases} \]

(8)

\[ \hat{\Theta}_j = \arg \max_{\theta_j} \left[ \sum_{i=1}^{n} \left( p_j^0 f_j(x_i; \Theta_j^0) \log(f_j(x_i; \Theta_j^0)) \right) + \frac{R_i p_j^0 S_j(x_i; \Theta_j^0) \log(S_j(x_i; \Theta_j^0))}{\sum_{i=1}^{k} p_i^0 S_i(x_i; \Theta_i^0)} \right], \]

(10)

where \( j = 1, 2, \ldots, k. \)

The E and M steps should be repeated until the value \( |L(\Theta^{t+1}; x) - L(\Theta^t; x)| \) become a small amount. In this case, \( \Theta^t \) will be the MLE of \( \Theta \), denoted by \( \hat{\Theta} \).
3. Identifiability of the Finite Mixture of TTIGL Distributions

We can say that the random variable $X \sim \text{Mixture TTIGL}(\beta, \gamma, \alpha)$ distributions with parameters $\theta_j = (\beta_j, \gamma_j, \alpha_j)$, $j = 1, 2, \ldots, k$, if its PDF is given as below:

$$f_{\theta}(x) = \sum_{j=1}^{k} p_j f_j(x; \theta_j),$$  \hfill (11)

where $\theta = (\theta_1, \theta_2, \ldots, \theta_k, p_1, \ldots, p_k)$ and for $j = 1, 2, \ldots, k$,

$$\theta_j = (\beta_j, \gamma_j, \alpha_j),$$

$$f_j(x; \theta_j) = \frac{\alpha_j}{\gamma_j (1 - 2^{-\gamma_j})} \exp \left[ \frac{(x - \beta_j)}{\gamma_j} \right] \left[ 1 + \exp \left[ \frac{(x - \beta_j)}{\gamma_j} \right] \right]^{-\alpha_j},$$

$$\text{if } x > \beta_j, (\beta_j \geq 0, \gamma_j > 0, \alpha_j > 0); \quad p_j > 0, \quad \sum_{j=1}^{k} p_j = 1.$$ \hfill (12)

AL-Hussaini and Ateya [8, 9] studied the estimation problem under a finite mixture of TTIGL $(0, \gamma, \alpha)$ distributions using the classical and Bayes methods based on complete class type-I censoring scheme. Ateya and Alharthi in [10] studied the estimation problem under a finite mixture of modified Weibull distributions under type-I, type-II, and type-II progressive censoring schemes using the ordinary likelihood method which depends on the differentiation of the LF with respect to the parameters and without studying the identifiability property. Also, Ateya and Alharthi in [11] studied the estimation problem under the same mixture model under type-I and type-II censoring schemes using the EM algorithm without studying the type-II progressive censoring case. Ateya in [12] studied the identifiability property and the estimation problem using EM algorithm under a finite mixture of generalized exponential distribution under type-I and type-II censoring schemes without studying type-II progressive censoring case. For more details about TTIGL distribution and its mixtures, see [8, 9, 13–15]. In our study, we will take $\beta = 0$, and then the vector of parameters will be $\theta = (\gamma, \alpha)$.

For a value $x_i$ of the random variable $T$, let

$$w_{ji} = 1 + e^{-\epsilon_j(x_i, \theta_j)},$$

$$\epsilon_j(x_i, \theta_j) = 1 - w_{ji}^{-\gamma_j},$$

$$\eta_j(x_i, \theta_j) = w_{ji}^{-\alpha_j} - 1,$$ \hfill (13)

So, PDF (12) and its corresponding SF and HRF can be written in the following forms (with $\beta = 0$):

$$f_j(x_i; \theta_j) = \frac{\alpha_j}{\gamma_j (1 - 2^{-\gamma_j})} \epsilon_j(x_i, \theta_j) \eta_j(x_i, \theta_j),$$

$$x_i > 0, \ (\gamma, \alpha) > 0;$$

$$S_j(x_i; \theta_j) = (1 - 2^{-\gamma_j})^{-\epsilon_j(x_i, \theta_j)},$$

$$h(x_i; \theta_j) = \frac{\alpha_j}{\gamma_j} \eta_j(x_i, \theta_j).$$ \hfill (14)

In the next section, we will write $\epsilon_j(x_i; \theta_j), \eta_j(x_i, \theta_j)$ instead of $\epsilon_j(x_i; \theta_j), \eta_j(x_i, \theta_j)$ and $\epsilon_j, \eta_j$ instead of $\epsilon_j, \eta_j$.

The SF and HRF of the finite mixture can be written as

$$S_{\theta}(x) = \sum_{j=1}^{k} p_j S_j(x; \theta_j),$$ \hfill (15)

and

$$h_{\theta}(x) = \frac{f_{\theta}(x)}{S_{\theta}(x)}.$$ \hfill (16)

Note that the SF of a finite mixture of distributions is a finite mixture of the SFs corresponding to the distributions, but this is not true with respect to the HRF.

It is very important to know that the statistical inference problem for the parameters in case of the mixture distributions cannot be discussed before proving the identifiability property (see [5]).

The identifiability property has been explored by a variety of authors [12, 16–27].

In this section, the identifiability property of the suggested mixture has been proved using Chandra’s theorem in [6].

**Theorem 1** (see [6]). Let $\Phi$ be the class of all CDFs with elements $F_1, F_2, \ldots, F_k$ and let $M: F_1 \rightarrow \phi_1$ be a linear mapping with domain $D_{\phi_1}$. Assume that there is a total ordering (\leq) of $\Phi$ such that

1. $F_1 \leq F_2, \ (F_1, F_2 \in \Phi)$ implies $D_{\phi_1} \subseteq D_{\phi_2}$.
2. For each $F_1 \in \Phi$, there exists $\phi_1$ in the closure of $T_1 = \{\phi: \phi(s) \neq 0\}$ such that $\lim_{s \rightarrow 1, \phi_1} \phi(s)/\phi_1(s) = 0$, for all $F_1 < F_2, \ (F_1, F_2 \in \Phi)$.

Then, this class is identifiable relative to $\Phi$.

**Proposition 1.** The class of all finite mixtures of TTIGL $(0, \gamma, \alpha)$ distributions is identifiable.

**Proof.** Let $X \sim \text{TTIGL}(0, \gamma, \alpha)$, with $PDF$ given by (12). Define the transform $\phi_1(s)$ as a moment generating function of the TTIGL $(0, \gamma, \alpha)$, which can be written as

$$\phi_1(s) = \phi_X(s) = \frac{\alpha_i}{\gamma_i (1 - 2^{-\gamma_i})} \int_0^{\infty} e^{sx} \exp \left[ \frac{x}{\gamma_i} \right] \left[ 1 + \exp \left[ \frac{x}{\gamma_i} \right] \right]^{-(\alpha_i + s)} dx$$

$$= \frac{\alpha_i}{(1 - 2^{-\gamma_i})} B(1-s\gamma_i, \alpha_i + s\gamma_i), s < \frac{1}{\gamma_i},$$ \hfill (17)

$$\phi_2(s) = \phi_X(s) = \frac{\alpha_2}{\gamma_2 (1 - 2^{-\gamma_2})} \int_0^{\infty} e^{sx} \exp \left[ \frac{x}{\gamma_2} \right] \left[ 1 + \exp \left[ \frac{x}{\gamma_2} \right] \right]^{-(\alpha_2 + s)} dx$$

$$= \frac{\alpha_2}{(1 - 2^{-\gamma_2})} B(1-s\gamma_2, \alpha_2 + s\gamma_2), s < \frac{1}{\gamma_2}.$$ \hfill (18)
where

$$B_p(a, b) = \int_0^p u^{a-1} (1 + u)^{-(a+b)} \, du, \quad a > 0, \ b > 0,$$

is the incomplete beta (type-II) function.

Also, the CDF of TTIGL(0, γ_1, α_1) and TTIGL(0, γ_2, α_2) can be written in the following forms:

$$F_1(x; γ_1, α_1) = \frac{1 + \exp[-x/γ_1]}{1 - 2^{-α_1}}, \quad x > 0,$$  \hspace{1cm} (20)

and

$$F_2(x; γ_2, α_2) = \frac{1 + \exp[-x/γ_2]}{1 - 2^{-α_2}}, \quad x > 0. \hspace{1cm} (21)$$

From (17) and (18), we can see that $D_{θ_1} = (0, 1/γ_1)$ and $D_{θ_2} = (0, 1/γ_2)$.

Now, we will make sure that the two conditions of the previous theorem are met.

### 5. Main Results

In this section, Monte Carlo simulation algorithm has been applied to make a comparison between the suggested EM algorithm and the ordinary method using the (MSE’s) criterion. In the end of this section, the suggested EM algorithm is applied on a real dataset as an application.

#### 5.1. Simulated Results

In this section, a type-II progressive censored sample from a mixture of two TTIGL(0, γ, α) distributions has been generated for different schemes $M = (R_1, R_2, \ldots, R_r)$ as follows:

(1) Generate independent random variates $u_1, u_2, \ldots, u_r$ from $U(0, 1)$.

(2) Define $w_i = u_i / \sum_{j=i}^r u_j, \ i = 1, 2, \ldots, r$.

(3) Define $v_i = 1 - \prod_{j=i}^r w_j, \ i = 1, 2, \ldots, r$, which represent a progressive type-II censored sample from $U(0, 1)$.

(4) Generate a random variate $y_i, i = 1, 2, r$ from $U(0, 1)$.

(5) If $y_i \leq p$, generate from $F_1(x; γ_1, α_1)$ using $v_i$; otherwise, generate from $F_2(x; γ_2, α_2)$ using $v_i$.

(6) Based on the generated type-II progressive censored sample and for different schemes $M = (R_1, R_2, \ldots, R_r)$ where $\sum_{i=1}^r R_i = n - r$, the MLE’s of the all parameters and functions have been obtained using the suggested form of the EM algorithm and also using the ordinary algorithm for maximizing the likelihood function.

(7) Over $m$ samples, the MSE’s of all estimates have been computed based on the suggested EM and ordinary algorithms of estimation.
Table 1: Average MLE’s and MSE’s of $p, \alpha_1, \alpha_2, \gamma_1,$ and $\gamma_2$.

| $(n, r)$ | $\hat{p}$ MSE($\hat{p}$) | $\hat{\alpha}_1$ MSE($\hat{\alpha}_1$) | $\hat{\alpha}_2$ MSE($\hat{\alpha}_2$) | $\hat{\gamma}_1$ MSE($\hat{\gamma}_1$) | $\hat{\gamma}_2$ MSE($\hat{\gamma}_2$) |
|----------|--------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| (30, 20) | 0.2349                   | 2.0183                          | 3.6175                          | 1.8835                          | 2.1175                          |
|          | 0.0149                   | 0.4402                          | 0.9134                          | 0.0916                          | 0.0607                          |
| (30, 25) | 0.2678                   | 2.2121                          | 3.8586                          | 1.8054                          | 2.1344                          |
|          | 0.0102                   | 0.3476                          | 0.6085                          | 0.0499                          | 0.0496                          |
| (30, 30) | 0.28                     | 2.6862                          | 4.3513                          | 1.7149                          | 2.1034                          |
|          | 0.0036                   | 0.2601                          | 0.5618                          | 0.0241                          | 0.0339                          |
| (40, 20) | 0.2415                   | 2.1137                          | 3.5517                          | 1.7917                          | 2.2105                          |
|          | 0.0217                   | 0.4714                          | 1.0214                          | 0.1042                          | 0.1105                          |
| (50, 20) | 0.2391                   | 2.1053                          | 3.7703                          | 1.7295                          | 2.2215                          |
|          | 0.0372                   | 0.5004                          | 1.1508                          | 0.1505                          | 0.1751                          |
| (60, 20) | 0.2446                   | 2.0913                          | 3.7905                          | 1.6806                          | 2.305                           |
|          | 0.0513                   | 0.6013                          | 1.2201                          | 0.2115                          | 0.1905                          |

Table 2: Average MLE’s and MSE’s of $S(1.3)$ and $h(1.3)$.

| $n$    | $\hat{S}(1.3)$ MSE($\hat{S}(1.3)$) | $\hat{h}(1.3)$ MSE($\hat{h}(1.3)$) |
|--------|-----------------------------------|-----------------------------------|
| (30, 20) | 0.8585                             | 0.1629                            |
|        | 0.0010                             | 0.0014                            |
| (30, 25) | 0.8330                             | 0.1868                            |
|        | 0.0009                             | 0.0013                            |
| (30, 30) | 0.8289                             | 0.1855                            |
|        | 0.0007                             | 0.0009                            |
| (40, 20) | 0.8215                             | 0.1773                            |
|        | 0.0091                             | 0.0053                            |
| (50, 20) | 0.8152                             | 0.1733                            |
|        | 0.0151                             | 0.0082                            |
| (60, 20) | 0.8226                             | 0.1705                            |
|        | 0.0217                             | 0.0105                            |

By combining the two real datasets, the new ordered real dataset will be 0.1, 0.2, 0.3, 0.7, 0.8, 0.8, 0.9, 1.1, 1.2, 1.3, 1.5, 1.8, 1.8, 1.9, 1.9, 1.9, 2.0, 2.1, 2.2, 2.3, 2.3, 2.4, 2.5, 2.6, 2.6, 2.7, 2.7, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.5, 3.9, 4.0, 4.2, 4.2, 4.5, 4.7, 5.3, 5.6, 6.2, 6.3, 6.6, 8.8, 9.5, 9.6, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.5, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, and 38.5.

In this paper, the combined real dataset of size $n = 160$ is analyzed using a mixture of two TTIGL$(0, \gamma, \alpha)$. The estimated parameters of the mixture model, the associated $(K - S)$ test statistic, and the $(P$ value) are summarized in Table 5. It is clear that the computed $(K - S)$ test statistic is less than the critical value for $(K - S)$ test statistic, under significance level of 0.05, which is equal to 0.108; also, the computed $(P$ value) is greater than the chosen significance level (0.05) which means that the suggested mixture model fits the combined real dataset quite well.
For more illustration, Figure 1 shows the histogram of the real data and the fitted PDF of the suggested mixture model computed at the estimated parameters.

Also, Figure 2 shows the fitted CDF and the empirical CDF of the suggested mixture model, where the dotted curve represents the empirical CDF curve and the continuous curve represents the fitted CDF curve computed at the estimated parameters.

Three type-II progressive censored samples are generated from the combined real dataset using the schemes $M_1 = (1^{80})$, $M_2 = (1^{60}, 0^{20}, 1^{10})$, and $M_3 = (1^{50}, 0^{40}, 1^{10})$. The three samples, respectively, are as follows:

Sample (1): 0.1, 0.3, 0.8, 0.9, 1.2, 1.5, 1.8, 1.9, 2.0, 2.2, 2.3, 2.5, 2.6, 2.7, 2.9, 3.1, 3.2, 3.4, 3.5, 3.6, 4.0, 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.9, 5.0, 5.3, 5.6, 5.7, 6.1, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.0, 8.5, 8.6, 8.6, 8.8, 8.9, 9.5, 9.6, 9.8, 10.7, 10.9, 11.0, 11.1, 11.2, 11.9, 12.3, 12.5, 12.9, 13.0, 13.2, 13.6, 13.7, 14.1, 15.4, 16.0, 17.3, 18.1, 18.4, 19.0, 20.6, 21.4, 23.0, 28.0, and 33.1

### Table 3: MSE's under ordinary and EM algorithms of the MSE's of $p, \alpha_1, \alpha_2, \gamma_1$, and $\gamma_2$. 

| $(n, r)$ | Method          | MSE's ($\hat{p}$) | MSE's ($\hat{\alpha}_1$) | MSE's ($\hat{\alpha}_2$) | MSE's ($\hat{\gamma}_1$) | MSE's ($\hat{\gamma}_2$) |
|---------|-----------------|-------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| (30, 20)| Ordinary algorithm | 0.0215            | 0.5108                    | 1.0112                    | 0.1007                    | 0.1019                    |
|         | EM algorithm    | 0.0149            | 0.4402                    | 0.9134                    | 0.0916                    | 0.0607                    |
| (30, 25)| Ordinary algorithm | 0.0203            | 0.4116                    | 0.7808                    | 0.0727                    | 0.0545                    |
|         | EM algorithm    | 0.0102            | 0.3476                    | 0.6085                    | 0.0499                    | 0.0496                    |
| (30, 30)| Ordinary algorithm | 0.0116            | 0.3195                    | 0.6115                    | 0.0448                    | 0.0408                    |
|         | EM algorithm    | 0.0036            | 0.2601                    | 0.5618                    | 0.0241                    | 0.0339                    |
| (40, 20)| Ordinary algorithm | 0.0331            | 0.5217                    | 1.1229                    | 0.1317                    | 0.2005                    |
|         | EM algorithm    | 0.0217            | 0.4714                    | 1.0214                    | 0.1042                    | 0.1105                    |
| (50, 20)| Ordinary algorithm | 0.0407            | 0.5901                    | 1.2205                    | 0.2007                    | 0.2702                    |
|         | EM algorithm    | 0.0372            | 0.5004                    | 1.1508                    | 0.1505                    | 0.1751                    |
| (60, 20)| Ordinary algorithm | 0.0602            | 0.6317                    | 1.3065                    | 0.3177                    | 0.3401                    |
|         | EM algorithm    | 0.0513            | 0.6013                    | 1.2201                    | 0.2115                    | 0.1905                    |

### Table 4: MSE's under ordinary and EM algorithms of the MLE's of $S(1.3)$ and $h(1.3)$.

| $n$      | Method          | MSE ($\hat{R}(1.3)$) | MSE ($\hat{h}(1.3)$) |
|---------|-----------------|-----------------------|-----------------------|
| (30, 20)| Ordinary algorithm | 0.0316                | 0.0119                |
|         | EM algorithm    | 0.0010                | 0.0014                |
| (30, 25)| Ordinary algorithm | 0.0105                | 0.0104                |
|         | EM algorithm    | 0.0009                | 0.0013                |
| (30, 30)| Ordinary algorithm | 0.0091                | 0.0084                |
|         | EM algorithm    | 0.0007                | 0.0009                |
| (40, 20)| Ordinary algorithm | 0.0416                | 0.0152                |
|         | EM algorithm    | 0.0091                | 0.0053                |
| (50, 20)| Ordinary algorithm | 0.0522                | 0.0204                |
|         | EM algorithm    | 0.0151                | 0.0082                |
| (60, 20)| Ordinary algorithm | 0.0801                | 0.0308                |
|         | EM algorithm    | 0.0217                | 0.0105                |

### Table 5: Estimates of the parameters of the suggested mixture model and associated $(K - S)$ test statistic based on real dataset.

| Model              | $\hat{p}$   | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\gamma}_1$ | $\hat{\gamma}_2$ | $K - S$ | $P$ value |
|--------------------|-------------|-------------------|-------------------|-------------------|-------------------|---------|-----------|
| Mixture of two TTIGL | 0.4672      | 3.0065            | 2.7158            | 5.9059            | 3.8595            | 0.0407  | 0.8336    |

For more illustration, Figure 1 shows the histogram of the real data and the fitted PDF of the suggested mixture model computed at the estimated parameters.

Also, Figure 2 shows the fitted CDF and the empirical CDF of the suggested mixture model, where the dotted curve represents the empirical CDF curve and the continuous curve represents the fitted CDF curve computed at the estimated parameters.

Three type-II progressive censored samples are generated from the combined real dataset using the schemes $M_1 = (1^{80})$, $M_2 = (1^{60}, 0^{20}, 1^{10})$, and $M_3 = (1^{50}, 0^{40}, 1^{10})$. The three samples, respectively, are as follows:

Sample (1): 0.1, 0.3, 0.8, 0.9, 1.2, 1.5, 1.8, 1.9, 2.0, 2.2, 2.3, 2.5, 2.6, 2.7, 2.9, 3.1, 3.2, 3.4, 3.5, 3.6, 4.0, 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.9, 5.0, 5.3, 5.6, 5.7, 6.1, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.0, 8.5, 8.6, 8.6, 8.8, 8.9, 9.5, 9.6, 9.8, 10.7, 10.9, 11.0, 11.1, 11.2, 11.9, 12.3, 12.5, 12.9, 13.0, 13.2, 13.6, 13.7, 14.1, 15.4, 16.0, 17.3, 18.1, 18.4, 19.0, 20.6, 21.4, 23.0, 28.0, and 33.1

**Figure 1:** The histogram of the real data and the fitted PDF of the suggested mixture model.
carried out between the suggested EM algorithm and the ordinary algorithm for maximizing the LF, and it is found that the suggested EM algorithm is better than the ordinary algorithm which can be interpreted as follows: the accuracy of the estimates using the ordinary algorithm decreases in case of high number of parameters (like our case). Finally, the results of the paper are applied on simulated and real data.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] N. Balakrishnan and R. Aggarwala, Progressive Censoring: Theory, Methods, and Applications, Springer, Berlin, Germany, 2000.
[2] Y. K. Bozidar, C. Gauss, M. O. Edwin, and A. R. F. Marcelino, “A new extended mixture normal distribution,” Mathematical Communications, vol. 22, pp. 53–73, 2017.
[3] G. J. McLachlan and D. Peel, Finite Mixture Models, Wiley, New York, USA, 2000.
[4] S. Kumar C and L. Manju, “A note on logistic mixture distributions,” Biostatistics and Biometrics Open Access Journal, vol. 2, no. 5, pp. 555–598, 2017.
[5] D. M. Titterington, A. F. M. Smith, and U. E. Makov, Statistical Analysis of Finite Mixture Distributions, Wiley, New York, USA, 1985.
[6] S. Chandra, “On mixture of probability distributions,” Scandinavian Journal of Statistics, vol. 4, pp. 105–112, 1977.
[7] G. J. Krishnan and T. Krishnan, The EM Algorithm and Extensions, Wiley, New York, USA, 1997.
[8] E. K. Al-Hussaini and S. F. Ateya, “Maximum likelihood estimations under a mixture of truncated type I generalized logistic components model,” J. Statist. Th. Appl., vol. 2, pp. 47–60, 2003.
[9] E. K. Al-Hussaini and S. F. Ateya, “Bayes estimations under a mixture of truncated type I generalized logistic components model,” J. Statist. Th. Appl., vol. 4, pp. 183–208, 2005.
[10] S. F. Ateya and A. S. Alharthi, “Maximum likelihood estimation under a finite mixture of modified Weibull distributions based on censored data with application,” JASS, vol. 20, pp. 231–239, 2014a.
[11] S. F. Ateya and A. S. Alharthi, “Estimation under a finite mixture of modified Weibull distributions based on censored data via EM algorithm with application,” Journal of Statistical Theory and Applications, vol. 13, no. 3, pp. 196–204, 2014b.
[12] S. F. Ateya, “Maximum likelihood estimation under a finite mixture of generalized exponential distributions based on censored data,” Statistical Papers, vol. 55, no. 2, pp. 311–325, 2014.
[13] A. Al-An gyr, “Truncated Logistic Distributions as Lifetime Models,” M. Sc. Thesis, Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia, 1997.
[14] E. K. Al-Hussaini, G. R. Al-Dayan, and A. Al-An gyr, “Bayesian prediction bounds under the truncated type I
generalized logistic model,” *J. Egyptian Math. Soc.* vol. 14, pp. 55–67, 2006.

[15] S. F. Ateya and A. E.-B. A. Ahmad, “Inferences based on generalized order statistics under truncated type-I generalized logistic distribution,” *Statistics*, vol. 47, no. 1, pp. 141–155, 2013.

[16] K. E. Ahmad, “Identifiability of finite mixtures using a new transform,” *Annals of the Institute of Statistical Mathematics*, vol. 40, no. 2, pp. 261–265, 1988.

[17] K. E. Ahmad and E. K. Al-Hussaini, “Remarks on the non-identifiability of mixtures of distributions,” *Annals of the Institute of Statistical Mathematics*, vol. 34, no. 3, pp. 543–544, 1982.

[18] A. S. Al-Moisheer, “A mixture of two burr type III distributions: identifiability and estimation under type II censoring,” *Mathematical Problems in Engineering*, vol. 2016, Article ID 7035279, 12 pages, 2016.

[19] G. Menges, “Three essays in econometrics,” *Statistische Hefte*, vol. 4, no. 1, pp. 1–37, 1963.

[20] N. C. Mohanty, “On the identifiability of finite mixture of Laguerre distributions,” *IEEE Transactions on Information Theory*, vol. 18, pp. 514–515, 1872.

[21] C. E. G. Otiniano, C. R. Gonçalves, and C. C. Y. Dorea, “Mixture of extreme-value distributions: identifiability and estimation,” *Communications in Statistics - Theory and Methods*, vol. 46, no. 13, pp. 6528–6542, 2017.

[22] G. L. M. Pezzott, L. E. B. Salasar, J. G. Leite, and F. Louzada-Neto, “A note on identifiability and maximum likelihood estimation for a heterogeneous capture-recapture model,” *Communications in Statistics - Theory and Methods*, vol. 49, no. 21, pp. 5273–5293, 2020.

[23] R. R. Rennie, “On the interdependence of the identifiability of multivariate mixtures and the identifiability of the marginal mixtures,” *Sankhya A*, vol. 34, pp. 449–452, 1972.

[24] H. Teicher, “Identifiability of mixtures,” *The Annals of Mathematical Statistics*, vol. 32, no. 1, pp. 244–248, 1961.

[25] H. Teicher, “Identifiability of finite mixtures,” *The Annals of Mathematical Statistics*, vol. 34, no. 4, pp. 1265–1269, 1963.

[26] H. Teicher, “Identifiability of mixtures of product measures,” *The Annals of Mathematical Statistics*, vol. 38, no. 4, pp. 1300–1302, 1967.

[27] S. J. Yakowitz and J. D. Spragins, “On the identifiability of finite mixtures,” *The Annals of Mathematical Statistics*, vol. 39, no. 1, pp. 209–214, 1968.

[28] G. S. Rao, “Estimation of stress-strength reliability from truncated type-I generalised logistic distribution,” *International Journal of Mathematics in Operational Research*, vol. 7, no. 4, pp. 372–381, 2015.

[29] D. K. Al-Mutairi, M. E. Ghitany, and D. Kundu, “Inferences on stress-strength reliability from Lindley distributions,” *Communications in Statistics - Theory and Methods*, vol. 42, no. 8, pp. 1443–1463, 2013.