ON THE ISOMORPHISM CLASS OF $q$-GAUSSIAN $W^*$-ALGEBRAS FOR INFINITE VARIABLES

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ABSTRACT. Let $M_q(H_R)$ be the $q$-Gaussian von Neumann algebra associated with a separable infinite dimensional real Hilbert space $H_R$ where $-1 < q < 1$. We show that $M_q(H_R) \not\cong M_0(H_R)$ for $-1 < q \neq 0 < 1$. The $C^*$-algebraic counterpart of this result was obtained recently in [BCKW22]. Using ideas of Ozawa we show that this non-isomorphism result also holds on the level of von Neumann algebras.

1. Introduction

Von Neumann algebras of $q$-Gaussian variables originate from the work of Bożejko and Speicher [BoSp91] (see also [BKS97]). To a real Hilbert space $H_R$ and a parameter $-1 < q < 1$ it associates a von Neumann algebra $M_q(H_R)$. At parameter $q = 0$ this assignment $H_R \mapsto M_q(H_R)$ is known as Voiculescu’s free Gaussian functor. The dependence of $q$ of these von Neumann algebras has been an intriguing and very difficult problem. A breakthrough result in this direction was obtained by Guionnet-Shlyakhtenko [GuSh14] who showed that for finite dimensional $H_R$ for a range of $q$ close to $0$ all von Neumann algebras $M_q(H_R)$ are isomorphic. The range for which isomorphism is known decreases as the dimension $H_R$ becomes larger. The Guionnet-Shlyakhtenko approach is based on free analogues of (optimal) transport techniques. Their result also relies on existence and power series estimates of conjugate variables obtained by Dabrowski [Dab14]. In fact the free transport techniques provide even an isomorphism result of underlying $q$-Gaussian $C^*$-algebras.

In case $H_R$ is infinite dimensional the isomorphism question of $q$-Gaussian algebras was addressed by Nelson and Zeng [NeZe18]. They showed that for mixed $q$-Gaussians for which the array $(q_{ij})_{ij}$ of commutation coefficients decays fast enough to 0 one obtains isomorphism of mixed $q$-Gaussian $C^*$- and von Neumann algebras. However, the isomorphism question for the original (non-mixed) $q$-Gaussians remained open, see Questions 1.1 and 1.2 of [NeZe18]. In [BCKW22] we showed that on the level of $C^*$-algebras there exists a non-isomorphism result. In the current note we improve on this result: we show that for an infinite dimensional separable real Hilbert space $H_R$ and $-1 < q < 1, q \neq 0$ we have $M_q(H_R) \not\cong M_0(H_R)$. This then fully answers Questions 1.1 and 1.2 of [NeZe18] and provides a stark contrast to the results of Guionnet-Shlyakhtenko for finite dimensional $H_R$.

The distinguishing property of $M_q(H_R)$ and $M_0(H_R)$ is a variation of the Akemann-Ostrand property that was suggested in a note by Ozawa [Oza10] (see also [DKP22]) and which we shall call $W^*$-AO. We formally define it in Definition 2.1. The most important novelty is that we quotient $B(L^2(M))$ by the $C^*$-algebra $K_M$ which is much larger than the ideal of compact operators on $L^2(M)$. This larger quotient turns out to provide von Neumann algebraic descriptions of the Akemann-Ostrand property [Oza10]. We use this to distinguish $M_q(H_R)$ and $M_0(H_R)$.

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2. Preliminaries

$\mathcal{B}(X,Y)$ denotes the bounded operators between Banach spaces $X \to Y$. $\mathcal{K}(X,Y)$ denotes the compact operators, meaning that they map the unit ball to a relatively compact set. We set $\mathcal{B}(X) := \mathcal{B}(X,X)$ and $\mathcal{K}(X) := \mathcal{K}(X,X)$.

The algebraic tensor product (vector space tensor product) is denoted by $\otimes_{\text{alg}}$ and $\otimes_{\text{min}}$ is the minimal tensor product of $C^*$-algebras. $\otimes$ is used for tensor products of elements.

We refer to [Tak79] as a standard reference on von Neumann algebras. For a von Neumann algebra $M$ we denote by $(M, L^2(M), J, L^2(M) +)$ its standard form. For $x \in M$ we write $x^{op} := Jx^*J$ which is the right multiplication with $x$ on the standard space. For a finite von Neumann algebra $M$ with trace $\tau$ we have $M \subseteq L^2(M)$ where $L^2(M)$ is the completion of $M$ with respect to the inner product $\langle x, y \rangle = \tau(y^*x)$. Therefore every $T \in \mathcal{B}(L^2(M))$ determines a map $Q_0(T) \in \mathcal{B}(M, L^2(M)) : x \mapsto T(x)$. Set

$$Q_1 : \mathcal{B}(L^2(M)) \to \mathcal{B}(M, L^2(M))/\mathcal{K}(M, L^2(M)) : T \mapsto Q_0(T) + \mathcal{K}(M, L^2(M)).$$

$Q_1$ is clearly continuous and we define the closed left-ideal $\mathcal{K}^L_M = \ker(Q_1)$ and the hereditary $C^*$-subalgebra $\mathcal{K}_M := (\mathcal{K}^L_M)^* \cap \mathcal{K}^L_M$ of $\mathcal{B}(L^2(M))$ (see also [Oza10]). We let $\mathcal{M}(\mathcal{K}_M) \subseteq \mathcal{B}(L^2(M))$ be the multiplier algebra of $\mathcal{K}_M$; indeed this multiplier algebra is faithfully represented on $L^2(M)$ by [Lan95, Proposition 2.1]. Then $\mathcal{K}_M$ is an ideal in the $C^*$-algebra $\mathcal{M}(\mathcal{K}_M)$. We have $M \subseteq \mathcal{M}(\mathcal{K}_M)$ and $M^{op} \subseteq \mathcal{M}(\mathcal{K}_M)$.

2.1. A von Neumann version of the Akemann-Ostrand property.

**Definition 2.1.** Let $M$ be a finite von Neumann algebra. We say that $M$ has $W^*\mathcal{A}\mathcal{O}$ if the map

$$(2.1) \quad \theta : M \otimes_{\text{alg}} M^{op} \to \mathcal{M}(\mathcal{K}_M)/\mathcal{K}_M : a \otimes b^{op} \mapsto ab^{op} + \mathcal{K}_M,$$

is continuous with respect to the minimal tensor norm and thus extends to a $*$-homomorphism $M \otimes_{\text{min}} M^{op} \to \mathcal{M}(\mathcal{K}_M)/\mathcal{K}_M$.

We recall the following from [Oza10] Section 4. Let $\Gamma$ be a discrete group and let $\mathcal{L}(\Gamma)$ and $\mathcal{R}(\Gamma)$ be the left and right group von Neumann algebra respectively acting on $l^2(\Gamma)$. In this case $L^2(\mathcal{L}(\Gamma)) \simeq l^2(\Gamma)$ as bimodules with the natural left and right actions of $\mathcal{L}(\Gamma)$ and $\mathcal{R}(\Gamma)$ on $l^2(\Gamma)$. We have $J\delta_s = \delta_{s-1}$ which extends to an antilinear isometry on $l^2(\Gamma)$. Then $\mathcal{R}(\Gamma) = J\mathcal{L}(\Gamma)J$.

Assume $\Gamma$ is icc so that $\mathcal{L}(\Gamma)$ and $\mathcal{R}(\Gamma)$ are factors, i.e. $\mathcal{L}(\Gamma) \cap \mathcal{R}(\Gamma) = C1$. The map

$$\pi : C^*(\mathcal{L}(\Gamma), \mathcal{R}(\Gamma)) \to \mathcal{B}(l^2(\Gamma) \otimes l^2(\Gamma)) : a \otimes b^{op} \mapsto a \otimes b^{op}, \quad a \in \mathcal{L}(\Gamma), b^{op} \in \mathcal{R}(\Gamma).$$

is a well-defined $*$-homomorphism by Takesaki’s theorem on minimality of the spatial tensor product. In [Oza10] Section 4, Theorem 4, Ozawa showed the following theorem.

**Theorem 2.2.** Let $\Gamma$ be an exact icc group such that the $*$-homomorphism

$$C^*_r(\Gamma) \otimes_{\text{alg}} C^*_r(\Gamma)^{op} \to \mathcal{B}(l^2(\Gamma) \otimes l^2(\Gamma)) : a \otimes b^{op} \mapsto ab^{op} + \mathcal{K}(l^2(\Gamma)),$$

is continuous with respect to $\otimes_{\text{min}}$. Then $\mathcal{L}(\Gamma)$ has $W^*\mathcal{A}\mathcal{O}$.

**Proof.** By [Oza10] Section 4, Theorem 4 we have

$$\ker(\pi) = \mathcal{K}(\mathcal{L}(\Gamma)) \cap C^*(\mathcal{L}(\Gamma), \mathcal{R}(\Gamma)).$$

Therefore,

$$\mathcal{L}(\Gamma) \otimes_{\text{min}} \mathcal{R}(\Gamma) \cong \mathcal{K}(\mathcal{L}(\Gamma), \mathcal{R}(\Gamma))/\mathcal{K}(\mathcal{L}(\Gamma)) \cap C^*(\mathcal{L}(\Gamma), \mathcal{R}(\Gamma)),$$

which concludes the theorem. □

**Remark 2.3.** It follows that if $\Gamma$ is an icc group that is bi-exact (or said to be in class $\mathcal{S}$, see [BrOz08 Section 15]) then $\mathcal{L}(\Gamma)$ has $W^*\mathcal{A}\mathcal{O}$. 
2.2. $q$-Gaussians. Let $-1 < q < 1$ and let $H_\mathbb{R}$ be a real Hilbert space with complexification $H := H_\mathbb{R} \oplus iH_\mathbb{R}$. Set the symmetrization operator $P_q^k$ on $H^{\otimes k}$,  
\begin{equation}
P_q^k(\xi_1 \otimes \ldots \otimes \xi_n) = \sum_{\sigma \in S_k} q^{i(\sigma)} \xi_{\sigma(1)} \otimes \ldots \otimes \xi_{\sigma(n)},
\end{equation}
where $S_k$ is the symmetric group of permutations of $k$ elements and $i(\sigma) := \# \{(a, b) \mid a < b, \sigma(b) < \sigma(a)\}$ the number of inversions. The operator $P_q^k$ is positive and invertible $[\text{BoSp01}]$. Define a new inner product on $H^{\otimes k}$ by  
\begin{equation}
\langle \xi, \eta \rangle_q := \langle P_q^k \xi, \eta \rangle,
\end{equation}
and call the new Hilbert space $H_q^{\otimes k}$. Set the Hilbert space direct sum $F_q(H) := \mathbb{C} \Omega \oplus (\oplus_{k=1}^{\infty} H_q^{\otimes k})$ where $\Omega$ is a unit vector called the vacuum vector. For $\xi \in H$ let  
\begin{equation}
l_q(\xi)(\eta_1 \otimes \ldots \otimes \eta_k) := \xi \otimes \eta_1 \otimes \ldots \otimes \eta_k, \quad l_q(\xi) \Omega = \xi,
\end{equation}
and then $l_q^*(\xi) = l_q(\xi^*)^\ast$. These ‘creation’ and ‘annihilation’ operators are bounded and extend to $F_q(H)$. We define a von Neumann algebra by the double commutant
\begin{equation}
M_q(H_\mathbb{R}) := \{l_q(\xi) + l_q^*(\xi) \mid \xi \in H_\mathbb{R}\}''.
\end{equation}
Then $\tau_\Omega(x) := \langle x\Omega, \Omega \rangle$ is a faithful tracial state on $M_q(H_\mathbb{R})$ which is moreover normal. Now $F_q(H)$ is the standard form Hilbert space of $M_q(H_\mathbb{R})$ and $Jx \Omega = x^+ \Omega$. For vectors $\xi_1, \ldots, \xi_k \in H$ there exists a unique operator $W_q(\xi_1 \otimes \ldots \otimes \xi_k) \in M_q(H_\mathbb{R})$ such that
\begin{equation}
W_q(\xi_1 \otimes \ldots \otimes \xi_k) \Omega = \xi_1 \otimes \ldots \otimes \xi_k.
\end{equation}
These operators are called Wick operators. It follows that $W_q(\xi)^{op} \Omega = \xi$.

\textbf{Remark 2.4.} Let $\mathbb{F}_\infty$ be the free group with countably infinitely many generators. $\mathbb{F}_\infty$ is icc and exact $[\text{BrOz08}]$ and hence Theorem 2.2 applies. We conclude that $\mathcal{L}(\mathbb{F}_\infty)$ has $W^*\text{AO}$. We have that $\mathcal{L}(\mathbb{F}_\infty) \simeq \Gamma_0(H_\mathbb{R})$ with $H_\mathbb{R}$ a separable infinite dimensional real Hilbert space (see $[\text{DNV92} \text{ Theorem 2.6.2}]$) and so $\Gamma_0(H_\mathbb{R})$ has the $W^*\text{AO}$. 

3. Non-isomorphism of $q$-Gaussian von Neumann algebras

The following theorem provides a necessary condition for $W^*\text{AO}$.

\textbf{Theorem 3.1.} Let $M$ be a finite von Neumann algebra with finite normal faithful tracial state $\tau$. Suppose there exists a unital von Neumann subalgebra $B \subseteq M$ and infinitely many subspaces $M_i \subseteq M, i \in \mathbb{N}$ that are left and right $B$-invariant and mutually $\tau$-orthogonal in the sense that $\tau(y^* x) = 0$ for $x \in M_i, y \in M_j, i \neq j$. Suppose moreover that there exists $\delta > 0$ and finitely many operators $b_j, c_j \in B$, with $\sum_j b_j \otimes c_j^{op}$ non-zero, such that for every $i \in \mathbb{N} \setminus \{i\}$ we have
\begin{equation}
\|Q_0(\sum_j b_j c_j^{op})\|_{B(M_i, L^2(M_i))} \geq (1 + \delta)\|\sum_j b_j \otimes c_j^{op}\|_{B_{\text{min}} B^{op}}.
\end{equation}

Then $M$ does not have $W^*\text{AO}$.

\textbf{Proof.} Let $X$ be the set of finite rank operators $x \in B(L^2(M))$ such that there exists $L_x \subseteq I$ finite with $\ker(x)^\perp \subseteq \oplus_{i \in I} L^2(M_i)$. Take $x \in X$ and choose $k \in \Gamma(I)$. Then,
\begin{equation}
\|Q_0(\sum_j b_j c_j^{op} + x)\|_{B(M, L^2(M))} \geq \|Q_0(\sum_j b_j c_j^{op} + x)\|_{B(M_k, L^2(M_k))} = \|Q_0(\sum_j b_j c_j^{op})\|_{B(M_k, L^2(M_k))} \geq (1 + \delta)\|\sum_j b_j \otimes c_j^{op}\|_{B_{\text{min}} B^{op}}.
\end{equation}
The operators in $X$ are norm dense in $\mathcal{K}(L^2(M))$ and by [Oza10, Section 2, Proposition] we have that $Q_0(K(L^2(M)))$ is dense in $Q_0(K^L_M)$ in the norm of $\mathcal{B}(M, L^2(M))$. As $Q_0$ is contractive $Q_0(X)$ is dense in $Q_0(K^L_M)$. It therefore follows that for any $x \in K^L_M$ we have
$$\|Q_0(\sum_j b_j e_j^p + x)\|_{\mathcal{B}(M, L^2(M))} \geq (1 + \delta) \| \sum_j b_j \otimes e_j^p\|_{B \otimes_{\min} B^{op}}.$$ Since $Q_0$ is contractive for every $x \in K^L_M$ we have,
$$\| \sum_j b_j e_j^p + x\|_{\mathcal{B}(L^2(M))} \geq (1 + \delta) \| \sum_j b_j \otimes e_j^p\|_{B \otimes_{\min} B^{op}}.$$ Hence, certainly for the Banach space quotient norm we have
$$\| \sum_j b_j e_j^p + K_M \|_{\mathcal{B}(L^2(M))/K_M} \geq (1 + \delta) \| \sum_j b_j \otimes e_j^p\|_{B \otimes_{\min} B^{op}}.$$ As the left hand side norm is the norm of the $C^*$-quotient $M(K_M)/K_M$ this concludes the proof (see [Lam95, Proposition 2.1] as in the preliminaries).

The proof of the following theorem essentially repeats its $C^*$-algebraic counterpart from [BCKW22, Theorem 3.3].

**Theorem 3.2.** Assume $\dim(H_\mathbb{R}) = \infty$ and $-1 < q < 1, q \neq 0$. Then the von Neumann algebra $M_q(H_\mathbb{R})$ does not have $W^*\text{AO}$.

**Proof.** Let $d \geq 2$ be such that $q^2d > 1$. Let
$$M := M_q(\mathbb{R}^d \oplus H_\mathbb{R}), \quad B := M_q(\mathbb{R}^d \oplus 0).$$ Let $\{f_i\}_i$ be an infinite set of orthogonal vectors in $0 \oplus H_\mathbb{R}$ such that $\|W_q(f_i)\| = 1$. Let $M_{q,i} := BW_q(f_i)B$ which is a $B$-$B$ invariant subset of $M$. Then $M_{q,i}$ and $M_{q,j}$ are $\tau_\mathbb{R}$-orthogonal if $i \neq j$. For $k \in \mathbb{N}$ let
$$\mathcal{B}(k) = \{W_q(\xi) \mid \xi \in (\mathbb{R}^d \oplus 0)^{\otimes k}\}.$$ It is proved in [BCKW22, Equation (3.2)] that for $b, c \in \mathcal{B}(k)$ we have
$$\langle bW_q(f_i)c\Omega, f_i\rangle_q = \langle be^{op}f_i, f_i\rangle_q = q^k \langle be^{op}\Omega, \Omega\rangle_q.$$ Then for finitely many $b_j, e_j \in \mathcal{B}(k)$ we have
(3.2)
$$\|Q_0(\sum_j b_j e_j^p)\|_{\mathcal{B}(M_{q,i}, L^2(M_{q,i}))} \geq \| \sum_j b_j W_q(f_i)c_j\|_{L^2(M_{q,i})} \geq |\{\sum_j b_j W_q(f_i)c_j\Omega, f_i\rangle_q | = | \sum_j q^k \langle b_j \Omega c_j, \Omega\rangle_q |.$$ Now let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d \oplus 0$ and for $j = (j_1, \ldots, j_k) \in \{1, \ldots, d\}^k$ let $e_j = e_{j_1} \otimes \ldots \otimes e_{j_k}$. Let $J_k$ be the set of all such multi-indices of length $k$. So $\#J_k = d^k$. Set $\xi_j = (P^k_q)^{-\frac{1}{2}}e_j$ so that $\langle \xi_j, \xi_j\rangle_q = \langle P^k_q \xi_j, \xi_j\rangle = 1$. Now (3.2) yields that for all $k \geq 1$ and all $i$,
$$\|Q_0(\sum_{j \in J_k} W_q(\xi_j)^* e_{\xi_j}^{op})\|_{\mathcal{B}(M_{q,i}, L^2(M_{q,i}))} \geq \sum_{j \in J_k} q^k \langle W_q(\xi_j)^* \Omega W_q(\xi_j), \Omega\rangle_q = \sum_{j \in J_k} q^k \langle \Omega W_q(\xi_j), W_q(\xi_j)\Omega\rangle_q.$$ From [Nou04, Proof of Theorem 2] (or see [BCKW22, Proof of Theorem 3.3]) we find,
$$\| \sum_{j \in J_k} W_q(\xi_j)^* \otimes W_q(\xi_j)^{op}\|_{B \otimes_{\min} B^{op}} \leq \left( \prod_{i=1}^\infty (1 - q^i)^{-1}\right)3(k+1)^2 d^{k/2}.$$ Therefore, as $q^2d > 1$ there exists $\delta > 0$ such that for $k$ large enough we have for every $i$,
$$\|Q_0(\sum_{j \in J_k} W_q(\xi_j)^* e_{\xi_j}^{op})\|_{\mathcal{B}(M_{q,i}, L^2(M_{q,i}))} \geq (1 + \delta) \| \sum_{j \in J_k} W_q(\xi_j)^* \otimes W_q(\xi_j)^{op}\|_{B \otimes_{\min} B^{op}}.$$
Hence the assumptions of Theorem 3.1 are witnessed which shows that $W^*\text{AO}$ does not hold. \hfill \Box

**Corollary 3.3.** Let $H_R$ be an infinite dimensional real separable Hilbert space. The von Neumann algebras $\Gamma_0(H_R)$ and $\Gamma_q(H_R)$ with $-1 < q < 1, q \neq 0$ are non-isomorphic.

**Proof.** This is a consequence of Theorem 3.2 and Remark 2.4 as $W^*\text{AO}$ is preserved under isomorphism. \hfill \Box

**References**

[BCKW22] M. Borst, M. Caspers, M. Klisse, M. Wasilewski, *On the isomorphism class of $q$-Gaussian $C^*$-algebras for infinite variables*, to appear in Proceedings of the American Mathematical Society, arXiv: 2202.13640.

[BoSp91] M. Bożejko, R. Speicher, *An example of a generalized Brownian motion*, Comm. Math. Phys. **137** (1991), no. 3, 519–531.

[BKS97] M. Bożejko, B. Kümmerer, R. Speicher, *q-Gaussian processes: non-commutative and classical aspects*, Comm. Math. Phys. **185** (1997), no. 1, 129–154.

[BrOs08] N. Brown, N. Ozawa, *$C^*$-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, 88.

[Da14] Y. Dabrowski, *A free stochastic partial differential equation*, Ann. Inst. Henri Poincaré Probab. Stat. **50** (2014), no. 4, 1404–1455.

[DKP22] C. Ding, S. Kunnawalkam Elayavalli, J. Peterson, *Properly Proximal von Neumann Algebras*, arXiv:2204.00517.

[DNV92] K. Dykema, A. Nica, D. Voiculescu, *Free Random Variables*, in the series CRM Monograph Series (volume 1), American Mathematical Society, Providence (1992).

[GuSh14] A. Guionnet, D. Shlyakhtenko, *Free monotone transport*, Invent. Math. **197** (2014), no. 3, 613–661.

[Lan95] E. Lance, *Hilbert $C^*$-modules, A toolkit for operator algebraists*. London Mathematical Society Lecture Note Series, 210. Cambridge University Press, Cambridge, 1995. x+130 pp.

[NeZe18] B. Nelson, Q. Zeng, *Free monotone transport for infinite variables*, Int. Math. Res. Not. IMRN 2018, no. 17, 5486–5535.

[Nou04] A. Nou, *Non injectivity of the $q$-deformed von Neumann algebra*, Math. Ann. **330** (2004), no. 1, 17–38.

[Oza10] N. Ozawa, *A comment on free group factors*, Noncommutative harmonic analysis with applications to probability II, 241—245, Banach Center Publ., 89, Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[Tak79] M. Takesaki, *Theory of operator algebras. I*. Springer-Verlag, New York-Heidelberg, 1979. vii+415 pp.