ON THE NON-VANISHING OF GENERALIZED KATO CLASSES FOR ELLIPTIC CURVES OF RANK 2

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Abstract. We prove the first cases of a conjecture by Darmon–Rotger on the non-vanishing of generalized Kato classes attached to elliptic curves $E$ over $\mathbb{Q}$ of rank 2. Our method also shows that the non-vanishing of generalized Kato classes implies that the $p$-adic Selmer group of $E$ is 2-dimensional. The main novelty in the proof is a formula for the leading term at the trivial character of an anticyclotomic $p$-adic $L$-function attached to $E$ in terms of the derived $p$-adic height of generalized Kato classes and an enhanced $p$-adic regulator.

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1. Introduction

1.1. Motivating question. Let $E$ be an elliptic curve over $\mathbb{Q}$ (hence modular by [BCDT01]), and let $L(E, s)$ be its Hasse–Weil $L$-series. A major advance towards the Birch and Swinnerton-Dyer conjecture was the proof by Gross–Zagier [GZ86] and Kolyvagin [Kol88] of the implication

$$\text{ord}_{s=1} L(E, s) = 1 \implies \text{ord}_{s=1} L(E, s) = \text{rank}_\mathbb{Z} E(\mathbb{Q})$$

(1.1)

The proof of (1.1) resorts to choosing an imaginary quadratic field $K$ for which the construction of Heegner points on $E$ (over ring class extensions of $K$) becomes available and such that $\text{ord}_{s=1} L(E/K, s) = 1$. By the Gross–Zagier formula, the basic Heegner point $y_K \in E(K)$ is then non-torsion, which by Kolyvagin’s work implies that $E(K)$ has rank 1. Since $y_K$ descends to $E(\mathbb{Q})$ precisely when $L(E, s)$ vanishes to odd order at $s = 1$, the above implication follows.

A more recent major advance is Skinner’s converse [Ski14] to the theorem of Gross–Zagier and Kolyvagin, taking the form of the implication

$$\text{rank}_\mathbb{Z} E(\mathbb{Q}) = 1 \text{ and } \#\text{III}(E/\mathbb{Q})[p^\infty] < \infty \implies \text{ord}_{s=1} L(E, s) = 1$$

(1.2)

for certain primes $p$ of good ordinary reduction for $E$. The proof of (1.2) uses progress [Wan19] towards an Iwasawa main conjecture over an auxiliary imaginary quadratic field $K$ as before.
which under the hypotheses of [L2] implies $y_K \notin E(Q)_{\text{tors}}$ by the $p$-adic Gross–Zagier formula of [BDP13], yielding the conclusion by the classical Gross–Zagier formula.

It is natural to wonder about the extension of these results for elliptic curves $E/Q$ of rank 2. Since a stumbling block in this setting is the lack of a systematic construction of algebraic points on $E$ playing the role of Heegner points, a most urgent question to ask might be the following:

**Question 1.1.** Let $E$ be an elliptic curve over $Q$ of rank 2, and choose an imaginary quadratic field $K$ with

$$\text{ord}_{s=1} L(E/K, s) = \text{ord}_{s=1} L(E, s) = 2. \tag{1.3}$$

Can one use $K$ to produce explicit nonzero classes in the $p$-adic Selmer group $\text{Sel}(Q, V_p E)$ for suitable primes $p$?

Here $\text{Sel}(Q, V_p E)$ denotes the inverse limit under the multiplication-by-$p$ maps of the usual $p^n$-descent Selmer groups $\text{Sel}_{p^n}(E/Q) \subset H^1(Q, E[p^n])$ tensored with $Q_p$, thus sitting in the exact sequence

$$0 \to E(Q) \otimes Z_p \to \text{Sel}(Q, V_p E) \to T_p \text{III}(E/Q) \otimes Z_p Q_p \to 0,$$

where $T_p \text{III}(E/Q)$ should be trivial, since $\text{III}(E/Q)$ is expected to be finite.

In this paper, for good ordinary primes $p$, we provide an affirmative answer to Question 1.1 with condition [L3] replaced by an algebraic counterpart:

$$\text{rank}_Z E(K) = \text{rank}_Z E(Q) = 2 \quad \text{and} \quad \# \text{III}(E/Q)[p^\infty] < \infty.$$  

Moreover, we prove analogues of the implications

$$y_K \notin E(Q)_{\text{tors}} \implies \dim_{Q_p} \text{Sel}(Q, V_p E) = 1$$

and

$$\text{rank}_Z E(Q) = 1 \quad \text{and} \quad \# \text{III}(E/Q)[p^\infty] < \infty \implies y_K \notin E(Q)_{\text{tors}}$$

appearing in the course of [L1] and [L2], respectively, in the rank 2 setting, with $y_K$ replaced by certain generalized Kato classes in $\text{Sel}(Q, V_p E)$.

### 1.2. A conjecture of Darmon–Rotger for rank 2 elliptic curves.

Following their spectacular work [DR17a] on the Birch and Swinnerton-Dyer conjecture for elliptic curves twisted by certain degree four Artin representations, Darmon–Rotger formulated in [DR16] a non-vanishing criterion for the generalized Kato classes introduced in [DR17a]. In this paper, we consider the special case of their conjectures concerned with elliptic curves of rank 2.

Let $E/Q$ be an elliptic curve of conductor $N$, and let $K$ be an imaginary quadratic field of discriminant prime to $N$. Fix a prime $p > 2$ of good ordinary reduction for $E$, and assume that $p = \mathfrak{p}
\mathfrak{p}$ splits in $K$. Let $\chi : G_K = \text{Gal}(\overline{Q}/K) \to C^\times$ be a ring class character of conductor prime to $Np$ with $\chi(\overline{\mathfrak{p}}) \neq \pm 1$, and set $\alpha := \chi(\overline{\mathfrak{p}})$, $\beta := \chi(p)$.

Let $f \in S_2(\Gamma_0(N))$ be the newform associated with $E$ by modularity, so that $L(E, s) = L(f, s)$, and let $g$ and $h$ be the weight 1 theta series of $\chi$ and $\chi^{-1}$, respectively. As explained in [DR16] (in which $g$ and $h$ can be more general weight 1 eigenforms), attached to the triple $(f, g, h)$ and the prime $p$ one has four generalized Kato classes

$$\kappa(f, g_\alpha, h_{\alpha^{-1}}), \kappa(f, g_\alpha, h_{\beta^{-1}}), \kappa(f, g_\beta, h_{\alpha^{-1}}), \kappa(f, g_\beta, h_{\beta^{-1}}) \in H^1(Q, V_{fgh}), \tag{1.4}$$

where $V_{fgh} \simeq V_p E \otimes V_g \otimes V_h$ is the tensor product of the $p$-adic representations associated to $f$, $g$, and $h$. The class $\kappa(f, g_\alpha, h_{\alpha^{-1}})$ arises as the $p$-adic limit

$$\kappa(f, g_\alpha, h_{\alpha^{-1}}) = \lim_{\ell \to 1} \kappa(f, g_\ell, h_\ell)$$

Question 3.1. Let $E$ be an elliptic curve over $Q$ of rank 2, and choose an imaginary quadratic field $K$ with

$$\text{ord}_{s=1} L(E/K, s) = \text{ord}_{s=1} L(E, s) = 2. \tag{1.3}$$

Can one use $K$ to produce explicit nonzero classes in the $p$-adic Selmer group $\text{Sel}(Q, V_p E)$ for suitable primes $p$?

Here $\text{Sel}(Q, V_p E)$ denotes the inverse limit under the multiplication-by-$p$ maps of the usual $p^n$-descent Selmer groups $\text{Sel}_{p^n}(E/Q) \subset H^1(Q, E[p^n])$ tensored with $Q_p$, thus sitting in the exact sequence

$$0 \to E(Q) \otimes Z_p \to \text{Sel}(Q, V_p E) \to T_p \text{III}(E/Q) \otimes Z_p Q_p \to 0,$$

where $T_p \text{III}(E/Q)$ should be trivial, since $\text{III}(E/Q)$ is expected to be finite.

In this paper, for good ordinary primes $p$, we provide an affirmative answer to Question 1.1 with condition [L3] replaced by an algebraic counterpart:

$$\text{rank}_Z E(K) = \text{rank}_Z E(Q) = 2 \quad \text{and} \quad \# \text{III}(E/Q)[p^\infty] < \infty.$$  

Moreover, we prove analogues of the implications

$$y_K \notin E(Q)_{\text{tors}} \implies \dim_{Q_p} \text{Sel}(Q, V_p E) = 1$$

and

$$\text{rank}_Z E(Q) = 1 \quad \text{and} \quad \# \text{III}(E/Q)[p^\infty] < \infty \implies y_K \notin E(Q)_{\text{tors}}$$

appearing in the course of [L1] and [L2], respectively, in the rank 2 setting, with $y_K$ replaced by certain generalized Kato classes in $\text{Sel}(Q, V_p E)$.

### 1.2. A conjecture of Darmon–Rotger for rank 2 elliptic curves.

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Let $f \in S_2(\Gamma_0(N))$ be the newform associated with $E$ by modularity, so that $L(E, s) = L(f, s)$, and let $g$ and $h$ be the weight 1 theta series of $\chi$ and $\chi^{-1}$, respectively. As explained in [DR16] (in which $g$ and $h$ can be more general weight 1 eigenforms), attached to the triple $(f, g, h)$ and the prime $p$ one has four generalized Kato classes

$$\kappa(f, g_\alpha, h_{\alpha^{-1}}), \kappa(f, g_\alpha, h_{\beta^{-1}}), \kappa(f, g_\beta, h_{\alpha^{-1}}), \kappa(f, g_\beta, h_{\beta^{-1}}) \in H^1(Q, V_{fgh}), \tag{1.4}$$

where $V_{fgh} \simeq V_p E \otimes V_g \otimes V_h$ is the tensor product of the $p$-adic representations associated to $f$, $g$, and $h$. The class $\kappa(f, g_\alpha, h_{\alpha^{-1}})$ arises as the $p$-adic limit

$$\kappa(f, g_\alpha, h_{\alpha^{-1}}) = \lim_{\ell \to 1} \kappa(f, g_\ell, h_\ell)$$
as \((g_{\ell}, h_{\ell})\) runs over the classical weight \(\ell \geq 2\) specializations of Hida families \(g\) and \(h\) passing through the \(p\)-stabilizations
\[
g_{\alpha} := g(q) - \beta g(q^p), \quad h_{\alpha^{-1}} := h(q) - \beta^{-1} h(q^p),
\]
in weight 1, and where \(\kappa(f, g_{\alpha}, h_{\alpha^{-1}})\) is obtained from the \(p\)-adic étale Abel–Jacobi image of certain higher-dimensional Gross–Kudla–Schoen diagonal cycles [GK92, GS95] on triple products of modular curves.

One of the main results of [DR17a] is an explicit reciprocity law (just stated for \(\kappa(f, g_{\alpha}, h_{\alpha^{-1}})\) here) of the form
\[
(1.5) \quad \exp^*(\text{res}_p(\kappa(f, g_{\alpha}, h_{\alpha^{-1}}))) = (\text{nonzero constant}) \cdot L(f \otimes g \otimes h, 1),
\]
whereby the classes \((1.4)\) land in the Bloch–Kato Selmer group \(\text{Sel}(\mathbb{Q}, V_{fgh}) \subset H^1(\mathbb{Q}, V_{fgh})\) precisely when the triple product \(L(f \otimes g \otimes h, s)\) vanishes at \(s = 1\); the main conjecture of [DR16] went further to predict that these classes span a non-trivial subspace of \(\text{Sel}(\mathbb{Q}, V_{fgh})\) precisely when \(L(f \otimes g \otimes h, s)\) vanishes to order exactly 2 at \(s = 1\).

Since for our specific \(g\) and \(h\) we have the factorization
\[
(1.6) \quad L(f \otimes g \otimes h, s) = L(E, s) \cdot L(E^K, s) \cdot L(E/K, \chi^2, s),
\]
where \(E^K\) is the \(K\)-quadratic twist of \(E\), arising from the decomposition
\[
(1.7) \quad V_{fgh} \simeq (V_P \otimes \text{Ind}_{K}^{\mathbb{Q}}) \oplus (V_P \otimes \text{Ind}_{K}^{\mathbb{Q}} \chi^2),
\]
the cases of the main conjecture of Darmon–Rotger concerned with elliptic curves of rank 2 may be stated as follows, where we let
\[
(1.8) \quad \kappa_{\alpha, \alpha^{-1}}, \kappa_{\alpha, \beta^{-1}}, \kappa_{\beta, \alpha^{-1}}, \kappa_{\beta, \beta^{-1}} \in H^1(\mathbb{Q}, V_P E)
\]
be the natural image of the classes \((1.4)\) under the projection \(H^1(\mathbb{Q}, V_{fgh}) \to H^1(\mathbb{Q}, V_P E)\).

**Conjecture 1.2** (Darmon–Rotger). Assume that \(L(E^K, 1)\) and \(L(E/K, \chi^2, 1)\) are both nonzero. Then the following are equivalent:

1. The classes \((1.4)\) span a non-trivial subspace of \(\text{Sel}(\mathbb{Q}, V_P E)\).
2. \(\dim_{\mathbb{Q}_p} \text{Sel}(\mathbb{Q}, V_P E) = 2\).
3. \(\text{rank}_{\mathbb{Q}_p} E(\mathbb{Q}) = 2\).
4. \(\text{ord}_{s=1} L(E, s) = 2\).

**Remark** 1.3. Of course, the equivalence of (2) \(\iff\) (3) amounts to the finiteness of \(\text{III}(E/\mathbb{Q})[p^\infty]\), and the equivalence (3) \(\iff\) (4) is the rank 2 case of the Birch–Swinnerton-Dyer conjecture.

Conjecture \((1.4)\) is a special case of [DR16] Conj. 3.2] and testing the predicted non-vanishing criterion for \((1.8)\) experimentally presented an “interesting challenge” at the time of its formulation (see [loc.cit., §4.5.3]). As an application of the main results of this paper, numerical examples supporting this conjecture will be presented in [5].

### 1.3. Main results.

Let \(\tilde{\rho}_{E,p} : G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}_{\mathbb{F}_p}(E[p])\) be the residual Galois representation associated to \(E\), and write
\[
N = N^+ N^-
\]
with \(N^+\) (resp. \(N^-\)) divisible only by primes which are split (resp. inert) in \(K\). Consider the strict Selmer group defined by
\[
\text{Sel}_{\text{str}}(\mathbb{Q}, V_P E) := \ker\left( \text{Sel}(\mathbb{Q}, V_P E) \stackrel{\log_p}{\longrightarrow} \mathbb{Q}_p \right),
\]
where \(\log_p\) denotes the composition of the restriction map \(\text{loc}_p : \text{Sel}(\mathbb{Q}, V_P E) \to E(\mathbb{Q}_p) \otimes \mathbb{Q}_p\) with the formal group logarithm \(E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \to \mathbb{Q}_p\).
Theorem A. Assume that $L(E^K, 1)$ and $L(E/K, \chi^2, 1)$ are both nonzero, and that
- $\bar{\rho}_{E,p}$ is irreducible,
- $N$ is square-free,
- $\bar{\rho}_{E,p}$ is ramified at every prime $q | N$.

Then $\kappa_{\alpha, \alpha - 1} = 0$ and the following statements are equivalent:

(i) The class $\kappa_{\alpha, \alpha - 1}$ is a non-trivial element in Sel($\mathbb{Q}, V_pE$).

(ii) $\text{dim}_{\mathbb{Q}_p}\text{Sel}^{\text{str}}(\mathbb{Q}, V_pE) = 1$.

Remark 1.4. The hypotheses in Theorem A imply in particular that $E$ has root number $+1$, and either of the statements (i) or (ii) implies that $L(E, 1) = 0$ by [Kat04]. Thus the elliptic curves in Theorem A all satisfy (1.9) $\text{ord}_s L(E, s) \geq 2$.

On the other hand, if the root number of $E$ is $+1$ and $\bar{\rho}_{E,p}$ is irreducible and ramified at some prime $q$, by [BFH90] and [Vat03] there exist infinitely many imaginary quadratic fields $K$ and ring class characters $\chi$ of prime-power conductor such that the following hold:
- $q$ is inert in $K$,
- every prime factor of $N/q$ splits in $K$,
- $L(E^K, 1) \neq 0$ and $L(E/K, \chi^2, 1) \neq 0$.

Therefore, by Theorem A the generalized Kato classes (1.3) provide an explicit construction of non-trivial Selmer classes for rank 2 elliptic curves analogous to the construction of Heegner classes for rank 1 elliptic curves. The tables of §6 exhibit numerical examples satisfying the rank part of the BSD conjecture and the hypotheses of Theorem A, yielding the first instances of non-trivial generalized Kato classes for rank 2 elliptic curves.

Remark 1.5. Another construction of non-trivial classes in Sel($\mathbb{Q}, V_pE$) for elliptic curves $E/\mathbb{Q}$ satisfying (1.9) will appear in forthcoming work by Skinner–Urban (see [SU06, Urb13]). Their construction of Selmer classes is completely different from that of generalized Kato classes, and it would be very interesting to compare the two constructions.

We obtain immediately from Theorem A the following result towards Conjecture 1.2.

Corollary B. Let the hypotheses be as in Theorem A. If $\text{rank}_{\mathbb{Z}}F_E(\mathbb{Q}) = 2$ and $\text{III}(E/\mathbb{Q})[p^\infty]$ is finite, then the generalized Kato classes $\kappa_{\alpha, \alpha - 1}$ and $\kappa_{\beta, \beta - 1}$ are both non-zero and generate the strict Selmer group $\text{Sel}^{\text{str}}(\mathbb{Q}, V_pE)$.

The above corollary has the flavor of a rank 2 analogue of Skinner’s converse to Kolyvagin’s theorem [Skil4]. In the opposite direction, Theorem A also yields the following rank 2 analogue of Kolyvagin’s theorem in terms of generalized Kato classes.

Corollary C. Let the hypotheses be as in Theorem A. Then the implication

$$\kappa_{\alpha, \alpha - 1} \neq 0 \implies \text{dim}_{\mathbb{Q}_p}\text{Sel}(\mathbb{Q}, V_pE) = 2$$

holds.

1.4. Outline of the proofs. We conclude the Introduction with a sketch of the proof of the implication (ii) $\implies$ (i) in Theorem A, establishing the non-vanishing of $\kappa_{E,K} := \kappa_{\alpha, \alpha - 1} \in H^1(\mathbb{Q}, V_pE)$.

- Step 1: Euler system construction of Bertolini–Darmon theta elements.

Denote by $\Gamma_\infty$ the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of $K$. Building on generalizations of Gross’ explicit form of Waldspurger’s special value formula [Wal85, Gros87], one can construct a $p$-adic $L$-function $\Theta_f/K \in \mathbb{Z}_p[\Gamma_\infty]$ interpolating “square-roots” of the central critical values $L(E/K, \phi, 1)$, as $\phi$ runs over finite order characters of $\Gamma_\infty$ (see [BD96, CHIS]).
The element $\Theta_{f/K}$ has been widely studied in the literature, but its place in Perrin-Riou’s vision [PR00, LZ14], whereby $p$-adic $L$-functions ought to arise as the image of families of special cohomology classes under generalized Coleman power series maps, remained mysterious.

Letting $\kappa(f, gh) = \{\kappa(f, g_\ell, h_\ell)\}_\ell$ be the $p$-adic family of diagonal cycle classes giving rise to $\kappa(f, g_\alpha, h_{\alpha-1})$ in the limit at $\ell \to 1$, in [4] we prove that

$$\text{Col}^\eta(\text{loc}_p(\kappa(f, gh))) = \Theta_{f/K} \cdot (\text{nonzero constant}),$$

where $\text{Col}^\eta$ is a generalized Coleman power series map defined in terms of an anticyclotomic variant of Perrin-Riou’s big exponential map. The proof of (1.10) combines a refinement of the explicit reciprocity law of Darmon–Rotger [DR17a] with a factorization of the $p$-adic triple product $L$-function [Hsi19].

- **Step 2: Leading coefficient formula and derived $p$-adic heights.**

Viewing (1.10) as an identity in the power series ring $\mathbb{Z}_p[T] \simeq \mathbb{Z}_p[\Gamma_\infty]$, its value at $T = 0$ recovers the implication

$$L(E, 1) = 0 \implies \kappa_{E,K} \in \text{Sel}(\mathbb{Q}, V_{pE}).$$

To further deduce the non-vanishing of $\kappa_{E,K}$ we consider the leading coefficient of (1.10) at $T = 0$. To that end, let

$$\text{Sel}(K, V_{pE}) = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset \cdots \supset S^{(\infty)}$$

be the filtration defined by Bertolini–Darmon [BD93] and Howard [How04], and the associated derived anticyclotomic $p$-adic height pairings

$$h^{(r)}: S^{(r)} \times S^{(r)} \to \mathbb{Q}_p.$$  

From the standard properties of $h^{(r)}$, one can easily see that if

$$\text{Sel}(\mathbb{Q}, V_{pE}) = \text{Sel}(K, V_{pE}) \quad \text{and} \quad \dim_{\mathbb{Q}_p} \text{Sel}(\mathbb{Q}, V_{pE}) = 2,$$

as we have under the hypotheses of Theorem A, the filtration (1.11) reduces to

$$\text{Sel}(\mathbb{Q}, V_{pE}) = S^{(1)} = S^{(2)} = \cdots = S^{(r)} \quad \text{and} \quad S^{(r+1)} = S^{(r+2)} = \cdots = S^{(\infty)} = \{0\}$$

for some $r \geq 2$. Setting

$$\rho := \text{ord}_{T=0} \Theta_{f/K}(T),$$

one can deduce that $r \geq \rho$ from the work of Skinner–Urban [SU14]: in particular, $\text{Sel}(\mathbb{Q}, V_{pE}) = S^{(\rho)}$. Based on the explicit Rubin-style formula for derived $p$-adic heights established in §3, we prove that for any basis $(P, Q)$ of $\text{Sel}(\mathbb{Q}, V_{pE})$, the $\rho$-th derived $p$-adic height $h^{(\rho)}(P, Q)$ is non-zero, and

$$\kappa_{E,K} = \frac{\bar{\theta}_{f/K}^\rho(P, Q)}{h^{(\rho)}(P, Q)} \cdot (P \otimes \log_p Q - Q \otimes \log_p P) \cdot (\text{explicit nonzero constant in } \overline{\mathbb{Q}}),$$

where $\bar{\theta}_{f/K} := (\frac{d}{dx})^\rho \Theta_{f/K}(T)|_{T=0}$ is the leading coefficient of $\Theta_{f/K}$ (see Corollary 5.1 for the precise statement). This in particular implies the non-vanishing of $\kappa_{E,K}$.

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2. Derived $p$-adic height pairings

In this section, we review the definition of the derived $p$-adic height pairings in [How04].
2.1. **Notation and definitions.** Let \( p \) be a prime, let \( K \) be a number field, and let \( \Sigma \) be a finite set of places of \( K \) containing the archimedean places and the places above \( p \). Let \( K_\Sigma \) be the maximal algebraic extension of \( K \) unramified outside \( \Sigma \) and set \( G_{K,\Sigma} = \text{Gal}(K_\Sigma/K) \). Let \( K_\infty/K \) be a \( \mathbb{Z}_p \)-extension in \( \Sigma \), and assume that all primes above \( p \) are totally ramified in \( K_\infty \). Denote by \( K_n \) be the subfield with of \( K_\infty \) with \([K_n:K] = p^n\), and let \( \Gamma_n = \text{Gal}(K_n/K) \) and \( \Gamma_\infty = \text{Gal}(K_\infty/K) \). Let \( \Lambda = \mathbb{Z}_p[\Gamma_\infty] \) and let \( \kappa_\Lambda : G_{K,\Sigma} \to \text{Gal}(K_\infty/K) \to \Lambda^{\Sigma} \) be the tautological character \( \kappa_\Lambda(\sigma) = \sigma|_{K_\infty} \). Let \( \nu : \Gamma_\infty \to \Gamma_\infty \) be the involution \( \gamma \mapsto \gamma^{-1} \), and for any \( \Lambda \)-module \( M \) and \( k \in \mathbb{Z} \), let \( M\{k\} \) be the \( G_{K,\Sigma} \)-module \( M \) on which \( G_{K,\Sigma} \) acts via \( \kappa_\Lambda^k \).

Let \( \mathcal{O} \) be a local ring finitely generated over \( \mathbb{Z}_p \), let \( \mathfrak{m} \) be the maximal ideal of \( \mathcal{O} \), and put \( \Lambda_{\mathcal{O}} = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O} \). Denote by \( \text{Mod}_\mathcal{O} \) the category of \( \mathcal{O}[G_{K,\Sigma}] \)-modules finite free over \( \mathcal{O} \). For \( T \) an object of \( \text{Mod}_\mathcal{O} \) we let \( T_\Lambda = T \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}} \{ -1 \} \) be the \( G_{K,\Sigma} \)-module \( T \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}} \) twisted by \( \kappa_\Lambda^{-1} \).

Let \( K \) be the localization of \( \Lambda_{\mathcal{O}} \) at the prime \( \mathfrak{m} \Lambda_{\mathcal{O}} \), and set \( P = K/\Lambda_{\mathcal{O}} \). Likewise we define \( T_K = T \otimes_{\mathcal{O}} K \{-1\} \) and \( T_P = T \otimes_{\mathcal{O}} P \{-1\} \). We shall denote the limits

\[
H^1(K_\infty, T) := \lim_n H^1(K_n, T), \quad \hat{H}^1(K_\infty, T) := \lim_n \hat{H}^1(K_n, T)
\]

with respect to the restriction and corestriction maps, respectively. Let

\[ \text{pr}_{K_n} : \hat{H}^1(K_\infty, T) \to H^1(K_n, T) \]

be the canonical projection map. Throughout we shall make use of the identification

\[ H^1(K, T_\Lambda) = \hat{H}^1(K_\infty, T) \]

deduced from \([\text{How04}] \) Lem. 1.4] and Shapiro’s lemma. Let \( T^* = \text{Hom}(T, \mathcal{O}(1)) \) and denote by \( e : T \times T^* \to \mathcal{O}(1) \) the canonical \( G_{K,\Sigma} \)-equivariant perfect pairing, which uniquely extends to a perfect \( G_{K,\Sigma} \)-equivariant pairing

\[ e_\Lambda : T_\Lambda \times T_\Lambda^* \to \Lambda_{\mathcal{O}}(1) \]

classified by

\[ e_\Lambda(t \otimes \lambda_1, s \otimes \lambda_2) = \lambda_1 \lambda_2 e_\Lambda(t, s) \]

for all \( \lambda_1, \lambda_2 \in \Lambda_{\mathcal{O}} \).

For any place \( v \) of \( K \) and any finite extension \( L \) of \( K_v \), let \( \text{inv}_L : H^2(L, \mathcal{O}(1)) \simeq \mathcal{O} \) be the invariant map and let \( \langle \cdot, \cdot \rangle_L : H^1(L, T) \times H^1(L, T^*) \to \mathcal{O} \) be the perfect pairing \( \langle z, w \rangle_L := \text{inv}_L(z \cup w) \), and define the bilinear pairing

\[ \langle \cdot, \cdot \rangle_{K_\infty,v} : H^1(K_v, T_\Lambda) \times H^1(K_v, T_\Lambda^*) \to H^2(K_v, \Lambda_{\mathcal{O}}(1)) \simeq \mathcal{O} \]

by \( \langle z, w \rangle_{K_\infty,v} = \text{inv}_v(e_\Lambda(z \cup w)) \). Fix a topological generator \( \gamma \) of \( \Gamma_\infty \) and set

\[ g_n := \gamma^{p^n} - 1 \in \Lambda. \]

Thus \( \Lambda_{\mathcal{O}}/(g_n) \simeq \mathcal{O}[\Gamma_n] \), and if \( z = (z_n) \in H^1(K_v, T_\Lambda) = \lim_n H^1(K_{n,v}, T) \) and \( w = (w_n) \in H^1(K_v, T_\Lambda^*) = \lim_n H^1(K_{n,v}, T^*) \), then

\[ \langle z, w \rangle_{K_\infty,v} \mod g_n = \sum_{\tau \in \Gamma_n} (z_n^{-1}, w_n)_{K_n,v} \tau. \]

(2.1)

Let \( \mathcal{F} = \{ H^1_{T}(K_v, T_K) \}_{v \in \Sigma} \) be a Selmer structure on \( T_K \), namely a choice of \( K \)-submodule \( H^1_{T}(K_v, T_K) \subset H^1(K_v, T_K) \) for every \( v \in \Sigma \), and let \( H^1_{T}(K_v, T_P) \) be the image of the natural map \( H^1_{T}(K_v, T_K) \to H^1(K_v, T_P) \) induced by the quotient \( K \to P \). Define the Selmer module \( H^1_{T}(K, T_P) \) to be the kernel of the map

\[ H^1(G_{K,\Sigma}, T_P) \to \prod_{v \in \Sigma} H^1(K_v, T_P)/H^1_{T}(K_v, T_P). \]
2.2. Abstract Rubin formula. In this subsection, we suppose that \( m = 0 \) for some \( m > 0 \), namely that \( \mathcal{O} \) is Artinian. By [How04, Lem. 1.2], we then have

\[
K = \bigcup_{n=0}^{\infty} \Lambda_{\mathcal{O}} \frac{1}{g_n}.
\]

Moreover, by [How04, Lem. 1.5] and Shapiro’s lemma, there is a natural isomorphism

\[
\eta_\gamma: H^1(K, T_P) = \lim_{\longleftarrow} H^1(K, T_{\Lambda} \otimes \Lambda \mathcal{O} g_n^{-1}/\Lambda_{\mathcal{O}}) \simeq \lim_{\longleftarrow} H^1(K, T_{\Lambda}/g_n T_{\Lambda}) = H^1(K_\infty, T).
\]

By definition, for \( z = \{z_n\} \in \tilde{H}^1(K_\infty, T) \) we have

\[
\eta_\gamma(z) = \frac{z}{\gamma - 1} \in H^1(K, T).
\]

For each \( n \), let \( H^1(J, T) \) be the Selmer module consisting of classes \( s \in H^1(K, T) \) such that \( J \in \{H^1(K, T) \text{ such that the image of } s \text{ in } H^1(K_\infty, T) \text{ belongs to } \eta_\gamma(H^1(J, T))\} \). Thus

\[
H^1(J, T) = \lim_{\longleftarrow} H^1(J, T) = \eta_\gamma(H^1(J, T)).
\]

Let \( J \) be the augmentation ideal of \( \Lambda_{\mathcal{O}} \), i.e., the principal ideal of \( \Lambda_{\mathcal{O}} \) generated by \( \gamma - 1 \), and for \( r > 0 \) put

\[
Y^{(r)}_T := H^1(J, T)[r] \cap J_{r-1}^{-1} H^1(K, T).
\]

This defines a decreasing filtration \( Y^{(1)}_T \supset Y^{(2)}_T \supset Y^{(2)}_T \supset \cdots \).

Let \( F_{\Sigma} = \{H^1_{\Sigma}(K_v, T^K_v)\}_{v \in \Sigma} \) be the Selmer structure on \( T^K_v \) such that \( H^1_{\Sigma}(K_v, T^K_v) \) and \( H^1_{\Sigma}(K_v, T^K_v) \) are orthogonal complements under local Tate duality for every \( v \in \Sigma \), and let

\[
[-,-]_{CT}: H^1_{\Sigma}(K, T_P) \times H^1_{\Sigma}(K, T_P) \to P
\]

be the \( \Lambda_{\mathcal{O}} \)-adic Cassels–Tate pairing of [How04, Thm. 1.8]. The \( r \)-th derived height pairing

\[
h^{(r)}_s(-,-): Y^{(r)}_T \times Y^{(r)}_T \to J^r/J^{r+1}
\]

in [How04, Def. 2.2] is defined by

\[
h^{(r)}_s(z, w) := (\gamma - 1)^{r} \cdot [\eta^{-1}_\gamma(z), \eta^{-1}_\gamma(w)]_{CT}
\]

writing \( z = (\gamma - 1)^{r-1} u \) with \( u \in H^1_{\Sigma}(K, T_P) \). Note that \( [\eta^{-1}_\gamma(u), \eta^{-1}_\gamma(w)]_{CT} \in (\gamma - 1)^{-1} \Lambda/\Lambda \), so that \( h^{(r)}_s(z, w) \) belongs to \( J^r/J^{r+1} \).

The following is a restatement of [How04, Thm. 2.5], which can be viewed as an abstract generalization of Rubin’s formula [Rub94, Thm. 3.2(ii)] (cf. [Nek06, Prop. 11.5.11]).

**Proposition 2.1.** Let \( z \in Y^{(r)}_T \) and \( w \in Y^{(r)}_T \). Suppose that there exist \( z \in H^1(K, T_{\Lambda}) \) and \( w_\Sigma = (w_v) \in \bigoplus_{v \in \Sigma} H^1_{\Sigma}(K_v, T^K_v) \) such that \( \text{pr}_K(z) = z \) and \( \text{pr}_{K_v}(w_v) = \text{loc}_v(w) \). Then

\[
h^{(r)}_s(z, w) = -\sum_{v \in \Sigma} (z, w_v)_{K_{\Lambda_v}} \pmod{J^{r+1}}.
\]

**Proof.** Let \( y = \eta^{-1}_\gamma(z) \in H^1_{\Sigma}(K, T_P) \) and \( t = \eta^{-1}_\gamma(w) \in H^1_{\Sigma}(K, T_P) \). Choose cochains \( \tilde{y} \in C^1(G_{K,\Sigma}, T^K_K) \) and \( \tilde{t} \in C^1(G_{K,\Sigma}, T^K_K) \) lifting \( s \) and \( t \), respectively, let \( \epsilon_0 \in C^2(G_{K,\Sigma}, P(1)) \) be such that \( d\epsilon_0 = d\tilde{y} \cup \tilde{t} \), and choose \( \ell_\Sigma \in \bigoplus_{v \in \Sigma} Z^1(G_{K_v, T^K_v}) \) lifting \( \text{loc}_v(t) \in \bigoplus_{v \in \Sigma} Z^1(K_v, T^K_v) \). According to the definition of the Cassels–Tate pairing [How04, (2), p. 1321], we find that

\[
h^{(r)}_s(z, w) = (\gamma - 1)^2 \cdot [y, t]_{CT} = (\gamma - 1)^2 \cdot \text{inv}_\Sigma(\text{loc}_\Sigma(\tilde{y}) \cup \tilde{t} - \text{loc}_\Sigma(\epsilon_0)).
\]
Let \( \tilde{z} \in \check{Z}^1(G_{K,\Sigma}, T_{\lambda}) \) and \( \tilde{w}_\Sigma \in \bigoplus_{v \in \Sigma} \check{Z}^1(G_{K_v}, T_{\lambda}) \) be cocycles representing \( z \) and \( w_\Sigma \). Then \( \check{y} = \tilde{z}/(\gamma - 1) \) and \( \tilde{t}_\Sigma = \tilde{w}_\Sigma/(\gamma - 1) \) are liftings of \( z \) and \( t_\Sigma \), and using (2.3) with \( \epsilon_0 = 0 \) (note that \( d\tilde{z} = 0 \)), we obtain

\[
\begin{align*}
    h^{(r)}(z, w) &= (\gamma - 1)^2 \cdot \text{inv}_\Sigma(e_{\Lambda}(\frac{\text{loc}_\Sigma(z)}{\gamma - 1} \cup \frac{\tilde{w}_\Sigma}{\gamma - 1})) \\
    &= -\text{inv}_\Sigma(e_{\Lambda}(\text{loc}_\Sigma(z) \cup w_\Sigma)) = -\sum_{v \in \Sigma} (z, w_v)_{K_v,v} \pmod{J^{r+1}}.
\end{align*}
\]

This completes the proof. \( \square \)

2.3. Derived \( p \)-adic heights for elliptic curves. Let \( E \) be an elliptic curve over \( K \) with good ordinary reduction at every place above \( p \). Let \( T = \varprojlim_k E[p^k] \) be the \( p \)-adic Tate module of \( E \), and take \( \Sigma \) to consist of the archimedean places, the places above \( p \), and the places of bad reduction of \( E \). Let \( T_k = E[p^k] \), and consider the modules \( Y_{T_k}^{(r)} \) defined in (2.4) taking for \( F \) the Selmer structure in [How04 Def. 3.2]. Since \( T_k^\dagger = T_k \) and \( F^\perp = F \) by the Weil pairing, the discussion of (2.2) yields a derived height pairing \( h_{\mathbb{Z}/p^k \mathbb{Z}} \). The constructions of \( Y_{T_k}^{(r)} \) and \( h_{\mathbb{Z}/p^k \mathbb{Z}} \) are clearly compatible under the quotient map \( \mathbb{Z}/p^{k+1} \mathbb{Z} \to \mathbb{Z}/p^k \mathbb{Z} \), and in the limit they define

\[
Y_T^{(r)} := \varprojlim_k Y_{T_k}^{(r)}, \quad h^{(r)} := \varprojlim_k h_{\mathbb{Z}/p^k \mathbb{Z}}^{(r)}.
\]

According to [How04 Lem. 4.1] there is canonical isomorphism

\[
(2.6) \quad Y_T^{(1)} \otimes_{\mathbb{Z}_p} Q_p \simeq \text{Sel}(K, V_p E).
\]

Letting \( S_p^{(r)}(E/K) \) be the subspace of \( \text{Sel}(K, V_p E) \) spanned by the image of \( Y_T^{(r)} \subset Y_T^{(1)} \) under the isomorphism (2.6), we have \( S_p^{(1)}(E/K) = \text{Sel}(K, V_p E) \) and for every \( r > 0 \) we obtain the \( r \)-th derived \( p \)-adic height pairing

\[
h^{(r)}: S_p^{(r)}(E/K) \times S_p^{(r)}(E/K) \to J^r/J^{r+1} \otimes_{\mathbb{Z}_p} Q_p,
\]

where \( J \) is the augmentation ideal of \( \Lambda = \mathbb{Z}_p[[\Gamma_\infty]] \).

Corollary 2.2. Let \( z, w \in S_p^{(r)}(E/K) \). Suppose that there exist a global class \( z \in \check{H}^1(K_\infty, T) \) and local classes \( w_v \in \varprojlim_n H^1_{\text{fin}}(K_{n,v}, T) \) for every \( v \mid p \) such that \( \text{pr}_{K_v}(z) = z \) and \( \text{pr}_{K_v}(w_v) = \text{loc}_v(z) \). Then

\[
h^{(r)}(z, w) = -\sum_{v \mid p} \langle \text{loc}_v(z), w_v \rangle_{K_v, v} \pmod{J^{r+1}}.
\]

Proof. This follows from Proposition 2.1 and the fact that \( H^1(K_{n,v}, T) \otimes Q_p = 0 \) for \( v \nmid p \). \( \square \)

3. Perrin-Riou’s theory for Lubin–Tate formal groups

In this section we explicitly compute the derived \( p \)-adic height pairings for elliptic curves via Perrin-Riou’s big exponential maps.

3.1. Preliminaries. We begin by reviewing the generalization of Perrin-Riou’s theory [PR94] to Lubin–Tate formal groups developed in [Kob15]. Throughout we fix a completed algebraic closure \( \mathbb{C}_p \) of \( Q_p \). Let \( Q_p^\ur \subset \mathbb{C}_p \) be the maximal unramified extension of \( Q_p \) and let \( Fr \in \text{Gal}(Q_p^\ur/Q_p) \) be the absolute Frobenius. Let \( F/Q_p \) be a finite unramified extension and let \( \mathcal{O} = \mathcal{O}_F \) be the valuation ring of \( F \). Put

\[
R := \mathcal{O}[X].
\]
Let $\mathcal{F} = \text{Spf } R$ be a relative Lubin–Tate formal group of height one defined over $\mathcal{O}$, and for each $n \in \mathbb{Z}$ set $\mathcal{F}^{(n)} := \mathcal{F} \times_{\text{Spec } \mathcal{O}, \mathbb{F}_p} \text{Spec } \mathcal{O}$. The Frobenius morphism $\varphi_{\mathcal{F}} \in \text{Hom}(\mathcal{F}, \mathcal{F}^{(-1)})$ induces a homomorphism $\varphi_{\mathcal{F}} : R \to R$ defined by

$$\varphi_{\mathcal{F}}(f) := f^{\text{Fr}} \circ \varphi_{\mathcal{F}},$$

where $f^{\text{Fr}}$ is the conjugate of $f$ by Fr. Let $\psi_{\mathcal{F}}$ be the left inverse of $\varphi_{\mathcal{F}}$ satisfying

$$\varphi_{\mathcal{F}} \psi_{\mathcal{F}}(f) = p^{-1} \sum_{x \in \mathcal{F}[p]} f(X \oplus_{\mathcal{F}} x).$$

Let $F_\infty = \bigcup_{n=1}^{\infty} F(\mathcal{F}[p^n])$ be the Lubin–Tate $\mathbb{Z}_p^\times$-extension associated with the formal group $\mathcal{F}$, and for every $n \geq -1$, let $F_n$ be the subfield of $F_\infty$ with $\text{Gal}(F_n/F) \simeq (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ (so $F_{-1} = F$). Letting $G_\infty = \text{Gal}(F_\infty/F)$, there is a unique decomposition $G_\infty = \Delta \times \Gamma_\infty^F$, where $\Delta \simeq \text{Gal}(F_0/F)$ is the torsion subgroup of $G_\infty$ and $\Gamma_\infty^F \simeq \mathbb{Z}_p$.

For every $a \in \mathbb{Z}_p^\times$, there is a unique formal power series $[a](X) \in R$ such that

$$[a]^{\text{Fr}} \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}} \circ [a] \quad \text{and} \quad [a](X) \equiv aX \pmod{X^2}.$$ 

Letting $\varepsilon_{\mathcal{F}} : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times$ be the Lubin–Tate character, we let $\sigma \in G_\infty$ act on $R$ by

$$\sigma \cdot f(X) := f([\varepsilon_{\mathcal{F}}(\sigma)](X)),$$

thus making $R$ into an $\mathcal{O}[G_\infty]$-module.

**Lemma 3.1.** $R^{\varphi=p=0}$ is free of rank one over $\mathcal{O}[G_\infty]$.

**Proof.** This is a standard fact. See [Kob18] Prop. 5.4. \qed

Let $L \subset \mathbb{C}_p$ be a finite extension over $\mathbb{Q}_p$, and let $V$ be a finite-dimensional $L$-vector space on which $G_{\mathbb{Q}_p}$ acts as a continuous $L$-linear crystalline representation. Let $D(V) = D_{\text{cris}, \mathbb{Q}_p}(V)$ be the filtered $\varphi$-module associated with $V$ over $\mathbb{Q}_p$ and set

$$\mathcal{D}_\infty(V) := D(V) \otimes_{\mathbb{Z}_p} R^{\varphi=p=0} \simeq D(V) \otimes_{\mathbb{Z}_p} \mathcal{O}[G_\infty].$$

Let $d : R \to \Omega_R$ be the standard derivation. Fix an invariant differential $\omega_{\mathcal{F}} \in \Omega_R$, and let $\log_{\mathcal{F}} \in R \hat{\otimes} \mathbb{Q}_p$ be the logarithm map satisfying $\log_{\mathcal{F}}(0) = 0$ and $d \log_{\mathcal{F}} = \omega_{\mathcal{F}}$. Let also $\partial : R \to R$ be defined by $df = \partial f \cdot \omega_{\mathcal{F}}$.

Let $\epsilon = (\epsilon_n) \in T_p\mathcal{F} = \lim_{\longrightarrow n} \mathcal{F}^{(n+1)}[p^{n+1}]$ be a basis of the $p$-adic Tate module of $\mathcal{F}$, where the inverse limit is with respect to the maps $\varphi_{\mathcal{F}}^{-(n+1)} : \mathcal{F}^{(n+1)}[p^{n+1}] \to \mathcal{F}^{(n)}[p^n]$. Following [Kob18] p. 42, we associate to $\epsilon$ and $\omega_{\mathcal{F}}$ a $p$-adic period $t_\epsilon \in B_{\text{cris}}^+$ for $\mathcal{F}$ as follows. For each $n$, there exists a unique isomorphism $\varphi_n^\dagger : \mathcal{F}^{(n)} \to \mathcal{F}$ such that

$$\varphi_{\mathcal{F}}^{-(n+1)} \circ \varphi_{\mathcal{F}}^{-(n)} \circ \varphi_{\mathcal{F}}^{-(n-1)} \circ \cdots \circ \varphi_{\mathcal{F}}^{-1} \circ \varphi_{\mathcal{F}} \circ \varphi_{\mathcal{F}} = [p^n] \circ \varphi_n^\dagger.$$

Put $w_n := \varphi_n^\dagger((\epsilon_n-1) \in \mathcal{F}[p^n])$, so that $[p](w_n) = w_{n-1}$ by definition. Let $A_{\text{inf}} = A_{\text{inf}}(\mathcal{O}_{\mathbb{C}_p} / \mathcal{O}_F)$ and $\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ be as defined in [Fon94] §1.2.2. It is not difficult to show that there is a unique sequence $(\tilde{w}_n)$ of elements in $\mathcal{F}(A_{\text{inf}})$ such that $[p](\tilde{w}_n) = \tilde{w}_{n-1}$ and $\theta(\tilde{w}_n) = w_n$, and we set $t_\epsilon := \log_{\mathcal{F}}(\tilde{w}_n) \in B_{\text{cris}}^+$. This $p$-adic period $t_\epsilon$ satisfies

$$D_{\text{cris}, F}(\varepsilon_{\mathcal{F}}) = F t_\epsilon^{-1}, \quad \varphi_{t_\epsilon} = \varepsilon_{\mathcal{F}} \cdot \omega_{\mathcal{F}}.$$

Fix an extension $\varepsilon_{\mathcal{F}} : \text{Gal}(F_\infty / \mathbb{Q}_p) \to L^\times$ of the Lubin–Tate character $\varepsilon_{\mathcal{F}}$, and for each $j \in \mathbb{Z}$ let $V(j) := V \otimes_L \varepsilon_{\mathcal{F}}^j$ denote the $j$-th Lubin–Tate twist of $V$. By definition, $D_{\text{cris}, F}(V(j)) = D(V) \otimes_{\mathbb{Q}_p} F t_\epsilon^{-1}$. Define the derivation $d_\varepsilon : \mathcal{D}(V(j)) \to \mathcal{D}(V(j-1))$ by

$$d_\varepsilon f := \eta_{t_\epsilon} \otimes \partial g, \quad \text{where } f = \eta \otimes g \in D_{\text{cris}, F}(V(j)) \otimes_{\mathcal{O}} R^{\varphi=p=0},$$

where $\varepsilon_{\mathcal{F}}$ is the uniformizer in $F$ such that $\varphi_{\mathcal{F}}^\dagger(\varepsilon_{\mathcal{F}}) = \varepsilon_{\mathcal{F}} \cdot \omega_{\mathcal{F}}$.
and the map
\[ \tilde{\Delta} : D_\infty(V) \to \bigoplus_{j \in \mathbb{Z}} \mathcal{D}_{\text{cris},F}(V\langle-j\rangle) / (1 - \varphi) \]
by \( f \mapsto (\partial^j f(0)t^j_x \mod (1 - \varphi)) \).

**Remark 3.2.** When \( F = \mathbb{G}_m \), we have \( F_\infty = F(\zeta_\infty) \), the corresponding Lubin–Tate character is the \( p \)-adic cyclotomic character \( \varepsilon_{\text{cycl}} : G_\mathbb{Q} \to \mathbb{Z}_p^\times \), \( \varphi_{\mathbb{G}_m}(f) = f^{Fr}((1 + X)^p - 1) \), and \( \psi_{\mathbb{G}_m}(f) \) is given by the unique power series such that
\[ \varphi_{\mathbb{G}_m} \psi_{\mathbb{G}_m}(f) = p^{-1} \sum_{\gamma = 1} f(\zeta(1 + X) - 1). \]

If we take \( \omega_{\mathbb{G}_m} \) to be the invariant differential \( (1 + X)^{-1}dX \), then \( \varphi = (1 + X) \frac{d}{dX} \) and \( \log_{\mathbb{G}_m} \) is the usual logarithm \( \log(1 + X) \). In the following, we fix a sequence \( \{\zeta_n\}_{n=1,2,3,...} \) of primitive \( p^n \)-th roots of unity with \( \zeta_{p^n} = \zeta_p \), and let \( t \in D(V) \) be the \( p \)-adic period corresponding to \( (\zeta_p, 1) \in T_p \mathbb{G}_m \).

### 3.2. Perrin-Riou’s big exponential map and the Coleman map.

For a finite extension \( K \) over \( \mathbb{Q}_p \), let
\[ \exp_{K,V} : D(V) \otimes \mathbb{Q}_p K \to H^1(K, V) \]
be Bloch–Kato’s exponential map [BK90, §3]. In this subsection, we recall the main properties of Perrin-Riou’s map \( \Omega_{V,h} \) interpolating \( \exp_{F, (V_j)} \) as \( j \) runs over non-negative integers \( j \).

Let \( V^* := \text{Hom}_L(V, L(1)) \) be the Kummer dual of \( V \) and denote by
\[ [\cdot, \cdot]_V : D(V^*) \otimes K \times D(V) \otimes K \to K \otimes \mathbb{Q}_p L \]
the \( K \)-linear extension of the de Rham pairing
\[ (\cdot, \cdot)_{dR} : D(V^*) \times D(V) \to L. \]
Let \( \exp^\star_{K,V} : H^1(K, V) \to D(V) \otimes K \) be the dual exponential map characterized uniquely by
\[ \text{Tr}_{K/Q_p}([x, \exp^\star_{K,V}(y)]_V) = ([\exp^\star_{K,V}, (x), y]_V, \]
for all \( x \in D(V^*) \otimes K, y \in H^1(K, V) \).

Choose a \( G_{\mathbb{Q}_p} \)-stable \( \mathcal{O}_L \)-lattice \( T \subset V \), and let
\[ \hat{H}^1(F_\infty, T) = \lim_n H^1(F_n, T), \quad \hat{H}^1(F_\infty, V) = \hat{H}^1(F_\infty, T) \otimes \mathbb{Q}_p \]
be the Iwasawa cohomology \( \mathbb{Z}_p[G_\infty] \)-modules associated with \( V \). We denote by
\[ \text{Tw}_j : \hat{H}^1(F_\infty, V) \simeq \hat{H}^1(F_\infty, V(j)) \]
the twisting map by \( \zeta_j^\tau \). For a non-negative real number \( r \) and any subfield \( K \) in \( \mathbb{C}_p \), we put
\[ \mathcal{H}_{r,K}(X) = \left\{ \sum_{n \geq 0, \tau \in \Delta} c_{n, \tau} \cdot \tau \cdot X^n \in K[\Delta] \mid \sup_n |c_{n, \tau}|_p n^{-r} < \infty \text{ for all } \tau \in \Delta \right\}, \]
where \( |\cdot|_p \) is the normalized valuation of \( K \) with \( |p|_p = p^{-1} \). Let \( \gamma \) be a topological generator of \( \Gamma_\infty \), and denote by \( \mathcal{H}_{r,K}(G_\infty) \) the ring of elements \( \{ f(\gamma - 1) : f \in \mathcal{H}_{r,K}(X) \} \), so in particular
\[ \mathcal{H}_{0,K}(G_\infty) = \mathcal{O}_K[G_\infty] \otimes \mathbb{Q}_p. \]
Put
\[ \mathcal{H}_{\infty,K}(G_\infty) = \bigcup_{r \geq 0} \mathcal{H}_{r,K}(G_\infty). \]

For \( n \geq -1 \), we define a map
\[ \Xi_{n,V} : D(V) \otimes \mathbb{Q}_p \mathcal{H}_{\infty,F}(X) \to D(V) \otimes \mathbb{Q}_p F_n \]
For simplicity, we shall write \( H^{\lambda} \) for all \( \lambda \in \Omega_{V,h} \) which is \( \tilde{\lambda} \)-linear and characterized by the following interpolation property. Let \( \exp_{F_n,V(j)}(\Xi_{n,V(j)}(d^{-j}G)) \in H^1(F_n,V(j)) \), and if \( j \leq -h \), then

\[
\exp_{F_n,V(j)}(\pr_{F_n}((T_{V,j} \circ \Omega^e_{V,h}(g))) = \frac{1}{(-h-j)!} \cdot \Xi_{n,V(j)}(d^{-j}G)) \in D(V(j)) \otimes_{Q_p} F_n,
\]

where \( G \in D(V) \otimes_{Q_p} \mathcal{H}_{h,F}(X) \) is a solution of the equation

\[
(1 - \varphi \otimes \varphi_{F}) G = g.
\]

Moreover, if \( D_{[s]} \) is a \( \varphi \)-invariant \( Q_p \)-subspace of \( D(V) \) such that all eigenvalues of \( \varphi \) on \( D_{[s]} \) have \( p \)-adic valuation \( s \), then \( \Omega^e_{V,h} \) maps \( (D_{[s]} \otimes R^{\psi=0})^{\tilde{\lambda}=0} \) into \( \tilde{H}^1(F_{\infty},T) \otimes_{\tilde{\lambda}} \mathcal{H}_{s+h,F}(G_{\infty}) \).

**Proof.** In the case \( \mathcal{F} = G_m \), the construction of \( \Omega^e_{V,h} \) and its interpolation property at integers \( j \geq 1 - h \) is due to Perrin-Riou [PR94] §3.2.3 Théorème, §3.2.4(i)], while the interpolation formula at integers \( j \leq -h \) is a consequence of the “explicit reciprocity formula” proved by Colmez [Col98] Thm. IX.4.5]. Their methods can be adapted to general relative Lubin–Tate formal groups of height one. Furthermore, if \( \Omega_{V,h} \) and the interpolation at \( j \geq 1 - h \), and in [Zha05 Thm. 6.2] for the explicit reciprocity formula.

To introduce the Coleman map, we further assume the following hypothesis:

\[
\mathcal{D}_{\infty}(V) = D_{\infty}(V).
\]

For simplicity, we shall write \( \mathcal{H}_K \) for \( \mathcal{H}_{K}(G_{\infty}) \) in the sequel. We let

\[
[-, -]_V : D(V^*) \otimes_{Q_p} \mathcal{H}_F \times D(V) \otimes_{Q_p} \mathcal{H}_F \rightarrow L \otimes_{Q_p} \mathcal{H}_F
\]

be the pairing defined by

\[
[\eta_1 \otimes \lambda_1, \eta_2 \otimes \lambda_2]_V = \langle \eta_1, \eta_2 \rangle_{dR} \otimes \lambda_1 \lambda_2
\]

for all \( \lambda_1, \lambda_2 \in \mathcal{H}_F \). For any \( e \in R^{\psi=0} \) and \( e \) a generator of \( T_p \mathcal{F} \), there is unique \( O_L[G_{\infty}] \)-linear Coleman map \( \text{Col}^e_\mathcal{F} : \tilde{H}^1(F_{\infty},V^*) \rightarrow D(V^*) \otimes_{Q_p} \mathcal{H}_F \) characterized by

\[
\text{Tr}_{F/F_p}(\text{Col}^e_\mathcal{F}(z), \eta)_V = \langle z, \Omega_{V,h}(\eta \otimes e) \rangle_{F_{\infty}} \in L \otimes_{Q_p} \mathcal{H}_F
\]

for all \( \eta \in D(V) \).

Let \( \mathcal{Q} \) be the completion of \( Q_p \) in \( C_p \), let \( W \) be the ring of integers of \( \mathcal{Q} \), and set \( F_n^w = F_n Q_p^w \). Let \( \sigma_0 \in \text{Gal}(F_{\infty}^w/Q_p) \) be such that \( \sigma_0 Q_p^w = F_r \) is the absolute Frobenius. Fix an
isomorphism \( \rho : \hat{G}_m \simeq \mathcal{F} \) defined over \( \mathcal{W} \) and let \( \rho : \mathcal{W}[T] \simeq R \otimes_{\mathcal{O}} \mathcal{W} \) be the map defined by \( \rho(f) = f \circ \rho^{-1} \). Then we have
\[
\varphi_{\mathcal{F}} \circ \rho = \rho^{Fr} \circ \varphi_{\hat{G}_m}.
\]

Let \( e \in R^{\psi,x=0} \) be a generator over \( \mathcal{O}[G_{\infty}] \) and write \( \rho(1+X) = h_e \cdot e \) for some \( h_e \in \mathcal{W}[G_{\infty}] \). This implies that \( e(0) \in \mathcal{O}^x \). Now we fix \( \epsilon = (\epsilon_n)_{n=0,1,2,...} \) to be the generator of \( T_p \mathcal{F} \) given by
\[
\epsilon_n = \rho^{Fr}_{-(n+1)} (\zeta_p^{n+1} - 1) \in \mathcal{F}^{(n+1)}[p^{n+1}].
\]

Let \( \eta \in D(V) \) be such that \( \varphi \eta = \alpha \eta \) and of slope \( s \) (i.e. \( |\alpha|_p = p^{-s} \)). For every \( z \in \hat{H}^1(F_{\infty}, V^*) \), we define
\[
\text{Col}^n(z) := \sum_{j=1}^{[F:Q_p]} \left[ \text{Col}^n_{\mathcal{F}}(z^{0,\ell} \cdot \eta), h_e \cdot \sigma_0^j \in \mathcal{H}^{s+h,LQ}(\hat{G}_{\infty}) \right]
\]
where \( \hat{G}_{\infty} := \text{Gal}(F_{\infty}/Q_p) \), and \([-,-] : D(V^*) \otimes \mathcal{H}_{Q} \times D(V) \to \mathcal{H}_{LQ} \) is the image of \([-,-]_V \) under the natural map \( L \otimes Q_p \), \( \mathcal{H}_{Q} \to \mathcal{H}_{LQ} \).

For any integer \( j \), put
\[
z_{-j,n} := \text{pr}_{F_{n}}(\text{Tw}_{-j}(z)) \in \hat{H}^1(F_{n}, V^*(1-j)).
\]
We say that a finite order character \( \chi \) of \( \hat{G}_{\infty} \) has conductor \( p^{n+1} \) if \( n \) is the smallest integer \( \geq -1 \) such that \( \chi \) factors through \( \text{Gal}(F_{n}/Q_p) \).

**Theorem 3.4.** Suppose that \( \text{Fil}^{-1} D(V) = D(V) \) and let \( h = 1 \). Let \( \psi \) be a \( p \)-adic character of \( \hat{G}_{\infty} \) such that \( \psi = \chi \mathcal{F} \) with \( \chi \) a finite order character of conductor \( p^{n+1} \). If \( j < 0 \), then
\[
\text{Col}^n(z)(\psi) = \frac{(-1)^{j-1}}{(-j-1)!} \prod_{\tau \in \text{Gal}(F_{n}/Q_p)} \chi^{-1}(\tau) \left[ \log_{F_n,V^*(1-j)} z_{-j,n} \otimes t^{-j}, \varphi^{-(n+1)} \eta \right] \]
If \( j \geq 0 \), then
\[
\text{Col}^n(z)(\psi) = j!(1)^{j} \prod_{\tau \in \text{Gal}(F_{n}/Q_p)} \chi^{-1}(\tau) \left[ \exp_{F_n,V^*(1-j)} z_{-j,n} \otimes t^{-j}, \varphi^{-(n+1)} \eta \right]
\]
Here \( \tau(\psi) \) is the Gauss sum defined by
\[
\tau(\psi) := \sum_{\tau \in \text{Gal}(F_{n}/F_{ur})} \psi_{\text{cyc}}^{-1}(\tau \sigma_0^{n+1}) \zeta_p^{n+1}.
\]

**Proof.** This follows from the explicit reciprocity formula in Theorem 3.3 and the computation and in [Kob28 Thm. 5.10] (cf. [LZ13 Thm. 4.15]).

### 3.3. The derived \( p \)-adic heights and the Coleman map

Let \( E \) be an elliptic curve over \( Q \) with good ordinary reduction at \( p \), and let \( V = T_p E \otimes_{\mathcal{O}_p} L \) with \( L \) a finite extension of \( Q_p \). We have \( \text{Fil}^{-1} D(V) = D(V) \) and \( V^* = V \). Let \( \omega_E \) be the Néron differential of \( E \), regarded as an element in \( D(H^1_{et}(E/Q_p)) \). We fix an embedding \( \iota_p : Q \hookrightarrow C_p \), and for any subfield \( H \subset Q \), let \( \hat{H} \) denote the completion of \( \iota_p(H) \) in \( C_p \).
Let $K$ be an imaginary quadratic field in which $p = p\mathbb{F}$ splits, with $p$ the prime of $K$ above $p$ induced by $\iota_p$. Let $K_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of $K$, and set $\Gamma_\infty = \text{Gal}(K_\infty/K)$ and $\hat{\Gamma}_\infty = \text{Gal}((\hat{K}_\infty/Q_p))$. For any integer $c > 0$ let $H_c$ be the ring class field of $K$ of conductor $c$, and choosing $c$ to be prime to $p$, put $F = H_c$. Let $\xi \in K$ be a generator of $p[F:Q_p]$ and let $F_\infty$ be the Lubin–Tate $\mathbb{Z}_p$-extension over $F$ associated with $\xi/\overline{\xi}$. By [Kob18 Prop. 3.7] we have $F_\infty = \bigcup_{n=0}^{\infty} H_{p^n}$, and hence $F_\infty$ is a finite extension of $\hat{K}_\infty$. Moreover, hypothesis (3.2) holds since $D(V)^{\varphi}\mathbb{Q}_p = \{0\}$ for any $j \in \mathbb{Z}$, given that the $\varphi$-eigenvalues of $D(V)$ are $p$-Weil numbers while $\xi/\overline{\xi}$ is a $1$-Weil number.

Let $\alpha_p \in \mathbb{Z}_p^\times$ be the $p$-adic unit eigenvalue of the Frobenius map $\varphi$ acting on $D(V)$, and let $\eta \in D(V) = D(H^1_\phi(E/F, Q_p)) \otimes D(L(1))$ be a $\varphi$-eigenvector of slope $-1$ such that

$$\varphi \eta = p^{-1} \alpha_p \cdot \eta$$

and hence $\langle \eta, \omega \rangle_{dR} = 1$.

Let $e \in R^{\varphi = 0}$ be a generator over $\mathcal{O}_F[[G_\infty]]$ such that $e(0) = 1$. Applying the big exponential map $\Omega_{V,1}$ in Theorem 3.3 we define

$$(3.4) \quad w^\eta = \Omega_{V,1}(\eta \otimes e) \in \hat{H}^1(F_\infty, V).$$

The following lemma is a standard fact.

**Lemma 3.5.** We have

$$\text{pr}_F(w^\eta) = \exp_{F,V} \left(\frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} \eta\right) \in H^1(F, V).$$

**Proof.** Let $g = \eta \otimes e$ and let $G(X) \in D(V) \otimes \mathcal{H}_1, Q(X)$ such that $(1 - \varphi \otimes \varphi_F)G = g$. Then we have

$$G(e_0) = \eta \otimes e(e_0) - \eta + (1 - \varphi)^{-1} \eta.$$ 

The equation $\psi_F e(X) = 0$ implies

$$\sum_{\zeta \in F^{Fr^{-1}}[p]} e^{\zeta} = 0.$$ 

It follows that

$$\text{Tr}_{F_0/F}(G^{Fr^{-1}}(e_0)) = \sum_{\tau \in \text{Gal}(F_0/F)} \eta \otimes e(e_0^\tau) - \eta + (1 - \varphi)^{-1} \eta = \frac{p \varphi - 1}{1 - \varphi} \eta,$$

and hence

$$\text{pr}_F(w^\eta) = \text{cor}_{F_0/F}(\Xi_0, V(G)) = \exp_{F,V} \text{Tr}_{F_0/F} \left(\frac{p^{-1} \varphi^{-1}(G^{Fr^{-1}}(e_0))}{1 - \varphi} \eta\right) = \exp_{F,V} ((1 - p^{-1} \varphi^{-1})(1 - \varphi)^{-1} \eta).$$

This completes the proof. \qed

**Lemma 3.6.** Let $Q_p^\text{cyc}$ be the cyclotomic $\mathbb{Z}_p^\times$-extension of $Q_p$. Let $\sigma_{\text{cyc}} \in \text{Gal}(F^\text{ur}_\infty/Q_p)$ be the Frobenius such that $\sigma_{\text{cyc}}|Q_p^\text{cyc} = 1$ and $\sigma_{\text{cyc}}|Q_p^\text{ur} = \text{Fr}$. For each $z \in \hat{H}^1(\hat{K}_\infty, V)$, we have

$$\langle z, \text{cor}_{F_\infty/K_\infty} (w^\eta) \rangle_{K_\infty} = \text{pr}_{K_\infty} (\text{Col}^\eta(z)) \sum_{i=1}^{[F:Q_p]} \frac{\sigma_{\text{cyc}}^i|K_\infty}{[F_\infty : K_\infty] : h^e_{\text{cyc}}} \in \mathcal{W}[\hat{K}_\infty] \otimes Q_p.$$ 

**Proof.** We first recall that for every $e \in (R \otimes \mathcal{W})^{\varphi_F = 0}$, the big exponential map $\Omega_{V,1}(\eta \otimes e)$ in Theorem 3.3 is given by

$$\Omega_{V,1}(\eta \otimes e) = (\exp_{F_n,V}(\Xi_n, V(G^\eta)))_{n=0,1,2,...},$$
where $G_e \in D(V) \otimes \mathcal{H}_1 \mathcal{Q}(X)$ is a solution of $(1 - \varphi \otimes \varphi_\mathcal{Q}) G_e = \eta \otimes \epsilon$. By the definition of $G_e$, we verify that
\begin{equation}
\Xi_{n,V}(G_e) = p^{-(n+1)}(\varphi^{-(n+1)} \otimes 1) G_e^{\text{Fr}^{-(n+1)}}(\epsilon_n)
\end{equation}

\[= \sum_{m=0}^{\infty} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes e^\text{Fr}^{-(n+1)}(\epsilon_{n-m})
\]

\[= \sum_{m=0}^{n+1} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes e^\text{Fr}^{-(n+1)}(\epsilon_{n-m}) + p^{-(n+1)}(1 - \varphi \otimes \text{Fr}^{-(n+1)}(\eta \otimes \epsilon(0))).
\]

Put $z_n = \text{pr}_K(z)$ and $\hat{G}_n = \text{Gal}(F_n/F)$. Following the computation in [Kob18, Thm. 5.10], we find that $[\text{pr}_K(\text{Col}_e(z)), \eta]$ is given by
\begin{equation}
\sum_{m=0}^{\infty} \sum_{\gamma \in \hat{G}_n} \exp_{\hat{K}_n,V}(z_n^{-1}\sigma_0^{n+1-m}) \gamma, \sum_{\tau \in \hat{G}_n} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes e^\text{Fr}^{-(n+1)}(\epsilon_{n-m}) \tau \sigma_0^{n+1-m} \tau|_{K_n}.
\end{equation}

On the other hand,
\begin{equation}
\text{pr}_K((z, \text{cor}_{F_n/K_n}(w^n)))|_{K_n} = \frac{1}{[F_n : K_n]} \sum_{j=1}^{[F_n : Q_p]} \text{pr}_K((z^{\sigma_0^j}, w^n)|_{F_n}) \sigma_0^j|_{K_n},
\end{equation}

and $\text{pr}_K((z^{\sigma_0^j}, w^n)|_{F_n})$ equals
\begin{equation}
\sum_{\gamma \in \hat{G}_n} \exp_{\hat{K}_n,V}(z_n^{-1}\sigma_0^{n+1-m}) \gamma, \sum_{\tau \in \hat{G}_n} (p\varphi)^{-(n+1)} \varphi^m \eta \otimes e^\text{Fr}^{-(n+1)}(\epsilon_{n-m}) \tau \sigma_0^{n+1-m} \tau|_{K_n}
\end{equation}

From this, it follows immediately that
\begin{equation}
\text{pr}_K((z, \text{cor}_{F_n/K_n}(w^n)))|_{K_n} = \frac{1}{[F_n : K_n]} \sum_{j=1}^{[F_n : Q_p]} \text{pr}_K((z^{\sigma_0^j}, w^n)) \sigma_0^j|_{K_n}
\end{equation}

On the other hand, by definition,
\begin{equation}
\text{Col}^p(z) = \sum_{j=1}^{[F_n : Q_p]} \text{Col}_{g_\rho}(z^{\sigma_0^j}, \eta) \sigma_0^j
\end{equation}

with $g_\rho = \rho(1+X)$. From [36] with $e = g_\rho$ and the fact that $g_\rho^{\sigma_0^{m-n-1}}(\epsilon_{n-m}) = \zeta_{p^{m-n-1}} \in Q_p^{\text{cyc}}$, we deduce that
\begin{equation}
\text{Col}_{g_\rho}(z^{\sigma_0^j}, \eta) = \text{Col}_{g_\rho}(z^{\sigma_0^j}, \eta),
\end{equation}
so \((\text{Col}^0(z))^\sigma = \text{Col}^0(z) \cdot \sigma^\text{cyc}\). Now the lemma follows from [3,7]. □

Now we give a formula for the derived \(p\)-adic heights over \(K\) in terms of the Coleman map over \(F_\infty\). For every prime \(v\) of \(K\) above \(p\), let \(H^1_{\text{fin}}(K_v, V) \subset H^1(K_v, V)\) be the Bloch–Kato finite subspace, and set

\[
\log_{\omega_E,v} = \langle \log_{K_v,V}(-), \omega_E \otimes t^{-1} \rangle_{dR} : H^1_{\text{fin}}(K_v, V) \to L.
\]

Since \(p\) is a prime of good reduction for \(E\), by [BK90, Cor. 3.8.4] we have \(H^1_{\exp}(K_v, V) = H^1_{\text{fin}}(K_v, V)\), where \(H^1_{\exp}(K_v, V) \subset H^1(K_v, V)\) is the image of \(\exp_{K_v,V}\). For the ease of notation, we write \(\text{Col}^p(-)\) for \(\text{pr}_K(\text{Col}^p(-))\) in what follows.

**Proposition 3.7.** Let \(z, x \in S^r_p(E/K) \otimes Q_pL\), and suppose that there exists \(z \in \hat{H}^1(\hat{K}_\infty, V)\) such that \(\text{pr}_K(z) = x\). If \(\text{Col}^p(\text{loc}_v(z)) \in J^r\mathcal{W}[\hat{\Gamma}_\infty] \otimes Q_p\) for some \(v \in \{p, \overline{p}\}\), then

\[
h^r_z(x, z) = \frac{-1}{1 - \alpha_p} \cdot \frac{1}{1 - \alpha_p^{-1}} \left[ F : Q_p \right]^{-1} \left( \langle \log_{\omega_E,p}(-), \omega_E \otimes t^{-1} \rangle_{dR} \right) \bmod J^{\alpha_p} \mathcal{W} \otimes Q_p,
\]

where \(\overline{x}\) and \(\overline{z}\) are the complex conjugates of \(x\) and \(z\).

**Proof.** Let \(w_p := \text{cor}_{F_\infty/\hat{K}_\infty}(w^n) \in \hat{H}^1(\hat{K}_\infty, V)\). Since \(\dim_{Q_p} H^1_{\text{fin}}(Q_p, V) = 1\), we can write

\[
\text{loc}_p(x) = c \cdot \text{pr}_{Q_p}(w_p) = c \cdot \text{cor}_{F/Q_p}(\text{pr}_F(w^n))
\]

for some \(c \in Q_p\). By Lemma 3.5,

\[
\langle \log_{Q_p,V}(\text{loc}_p(x)), \omega_E \otimes t^{-1} \rangle_{dR} = c[F : Q_p] \cdot \left( \frac{1 - \varphi^{-1}}{1 - \varphi} \eta, \omega_E \otimes t^{-1} \right)_{dR}.
\]

Since \(\varphi \eta = p^{-1} \alpha_p \cdot \eta\), this shows that

\[
c = \frac{1}{1 - \alpha_p^{-1}} \cdot [F : Q_p] \cdot \log_{\omega_E,p}(x).
\]

Applying Corollary 2.2, we find that that

\[
h_z^r(x, z) = \left( -1 - p^{-1} \alpha_p \right) \cdot \left( 1 - \alpha_p^{-1} \right)^{-1} \cdot [F : Q_p]^{-1}
\]

\[
\times \left( \langle \log_{\omega_E,p}(x), \langle \text{loc}_p(z), w_p \rangle_{K_\infty} + \log_{\omega_E,p}(\overline{z}) \cdot \langle \text{loc}_p(\overline{z}), w_p \rangle_{K_\infty} \right) \bmod J^{\alpha_p} \mathcal{W} \otimes Q_p,
\]

Since \(\rho(1 + X) = h_x \cdot e\) and \(e(0) = 1\), we find that \(1 = e(0) \cdot (h_x|_{\gamma = 1})\) and hence \(h_x \equiv 1 \pmod J\). The assertion now follows from the above equation and Lemma 4.6. □

4. Euler system construction of theta elements

In this section we prove Theorem 4.1, recovering the square-root anticyclotomic \(p\)-adic \(L\)-functions of Bertolini–Darmon [BD90] (in the definite case) as the image of a \(p\)-adic family of diagonal cycles [DR17a] under the Coleman map of [3,2].

4.1. Ordmary \(\Lambda\)-adic forms. Fix a prime \(p > 2\). Let \(\mathbb{P}\) be a normal domain finite flat over \(\Lambda := \mathcal{O}/[1 + p\mathbb{Z}_p]\), where \(\mathcal{O}\) is the ring of integers of a finite extension \(L/Q_p\). We say that a point \(x \in \text{Spec } \mathcal{O}(\mathbb{Q}_p)\) is **locally algebraic** if its restriction to \(1 + p\mathbb{Z}_p\) is given by \(x(\gamma) = \gamma^{k_x} \epsilon_x(\gamma)\) for some integer \(k_x\), called the **weight** of \(x\), and some finite order character \(\epsilon_x : 1 + p\mathbb{Z}_p \to \mu_{p^\infty}\); we say that \(x\) is **arithmetic** if it has weight \(k_x \geq 2\). Let \(\mathbb{X}^\circ_{ar}\) be the set of arithmetic points.
Fix a positive integer $N$ prime to $p$, and let $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character modulo $Np$. Let $S^0(N, \chi, \mathbb{I})$ be the space of ordinary $\mathbb{I}$-adic cusp forms of tame level $N$ and branch character $\chi$, consisting of formal power series

$$ f(q) = \sum_{n=1}^{\infty} a_n(f) q^n \in \mathbb{I}[q] $$

such that for every $x \in \mathbb{X}_\mathbb{I}^+$ the specialization $f_x(q)$ is the $q$-expansion of a $p$-ordinary cusp form $f_x \in S_{k, \chi}(Np^{r_x+1}, \chi^2 q^{-k_x} \epsilon_x)$. Here $r_x \geq 0$ is such that $\epsilon_x(1+p)$ has exact order $p^{r_x}$, and $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mu_{p-1}$ is the Teichmüller character.

We say that $f \in S^0(N, \chi, \mathbb{I})$ is a primitive Hida family if for every $x \in \mathbb{X}_\mathbb{I}^+$ we have that $f_x$ is an ordinary $p$-stabilized newform (in the sense of [Hsi19 Def. 2.4]) of tame level $N$. Given a primitive Hida family $f \in S^0(N, \chi, \mathbb{I})$, and writing $\chi = \chi' \chi_p$ with $\chi'$ a Dirichlet modulo $N$ (resp. $p$), there is a primitive $f^i \in S^0(N, \chi_p \mathbb{I}, \mathbb{I})$ with Fourier coefficients

$$ a_{\ell}(f^i) = \begin{cases} \chi'(\ell) a_{\ell}(f) & \text{if } \ell \nmid N, \\ a_{\ell}(f) \chi^2 \chi_p(\ell)^{-1} & \text{if } \ell \mid N, \end{cases} $$

having the property that for every $x \in \mathbb{X}_\mathbb{I}^+$ the specialization $f_x^i$ is the $p$-stabilized newform attached to the character twist $f_x \otimes \chi$.

By [Hid86] (cf. [Wil88 Thm. 2.2.1]), attached to every primitive Hida family $f \in S^0(N, \chi, \mathbb{I})$ there is a continuous $\mathbb{I}$-adic representation $\rho_f : G_{Q} \to \text{GL}_2(\text{Frac} \mathbb{I})$ which is unramified outside $Np$ and such that for every prime $\ell \nmid Np$,

$$ \text{tr} \rho_f(\text{Frob}_\ell) = a_{\ell}(f), \quad \det \rho_f(\text{Frob}_\ell) = \chi \omega^2(\ell) \ell^{a_{\ell}(f) - 1}, $$

where $\ell \in \mathbb{I}^\times$ is the image of $\omega^{-1}(\ell)$ under the natural map $1 + p\mathbb{Z}_p \to \mathcal{O}[1 + p\mathbb{Z}_p]^\times = \Lambda^\times \to \mathbb{I}^\times$. In particular, letting $\langle \varepsilon_{\text{cyc}} \rangle : G_{Q} \to \mathbb{I}^\times$ be defined by $\langle \varepsilon_{\text{cyc}} \rangle(\sigma) = \langle \varepsilon_{\text{cyc}}(\sigma) \rangle_1$, it follows that $\rho_f$ has determinant $\chi_1 \varepsilon_{\text{cyc}}^{-1}$, where $\chi_1 : G_{Q} \to \mathbb{I}^\times$ is given by $\chi_1 := \sigma:\varepsilon_{\text{cyc}}^{-2}\langle \varepsilon_{\text{cyc}} \rangle_1$, with $\sigma$ the Galois character sending $\text{Frob}_\ell \mapsto \chi(\ell)^{-1}$. Moreover, by [Wil88 Thm. 2.2.2] the restriction of $\rho_f$ to $G_{Q_p}$ is given by

$$ \rho_f|_{G_{Q_p}} \sim \begin{pmatrix} \psi_f & * \\ 0 & \psi_f^{-1} \chi_1^{-1} \varepsilon_{\text{cyc}}^{-1} \end{pmatrix} \quad (4.1) $$

where $\psi_f : G_{Q_p} \to \mathbb{I}^\times$ is the unramified character with $\psi_f(\text{Frob}_p) = a_p(f)$.

4.2. Triple product $p$-adic $L$-function. Let

$$ (f, g, h) \in S^0(N_f, \chi_f, \mathbb{I}_f) \times S^0(N_g, \chi_g, \mathbb{I}_g) \times S^0(N_h, \chi_h, \mathbb{I}_h) $$

be a triple of primitive Hida families. Set

$$ \mathcal{R} := \mathbb{I}_f \hat{\otimes} \mathbb{I}_g \hat{\otimes} \mathbb{I}_h, $$

which is a finite extension of the three-variable Iwasawa algebra $\mathcal{R}_0 := \Lambda \hat{\otimes} \Lambda \hat{\otimes} \Lambda$, and define the weight space $\mathcal{X}_{\mathcal{R}}^f$ for the triple $(f, g, h)$ in the $f$-dominated unbalanced range by

$$ \mathcal{X}_{\mathcal{R}}^f := \left\{ (x, y, z) \in \mathcal{X}_f^+ \times \mathcal{X}_g^+ \times \mathcal{X}_h^+ : k_x \geq k_y + k_z \text{ and } k_x \equiv k_y + k_z \pmod{2} \right\}, $$

where $\mathcal{X}_g^+$ (and similarly $\mathcal{X}_h^+$) is the set of locally algebraic points in Spec $\mathbb{I}_g(\overline{\mathbb{Q}}_p)$ for which $g_q(g)$ is the $q$-expansion of a classical modular form.

For $\phi \in \{ f, g, h \}$ and a positive integer $N$ prime to $p$ and divisible by $N_\phi$, define the space of $A$-adic test vectors $S^0(N, \phi, \mathbb{I}_\phi)_{\phi}$ to be the $\mathbb{I}_\phi$-submodule of $S^0(N, \chi_\phi, \mathbb{I}_\phi)$ generated by $\{ \phi(q^d) \}$, as $d$ ranges over the positive divisors of $N/N_\phi$.

For the next result, let $N := \text{lcm}(N_f, N_g, N_h)$, and consider the following hypothesis:

$$ (\Sigma^-) \quad \text{for some } (x, y, z) \in \mathcal{X}_{\mathcal{R}}^f, \text{ we have } \varepsilon_q(f_x^0, g_y^0, h_z^0) = +1 \text{ for all } q \mid N. $$
Here \( \varepsilon_q(f^0_x, g^0_y, h^0_z) \) denotes the local root number of the Kummer self-dual twist of the Galois representations attached to the newforms \( f^0_x, g^0_y, \) and \( h^0_z \) corresponding to \( f_x, g_y, \) and \( h_z, \) respectively.

**Theorem 4.1.** Assume that the residual representation \( \bar{\rho}_f \) satisfies

\[
(CR) \quad \bar{\rho}_f \text{ is absolutely irreducible and } p\text{-distinguished},
\]

and that, in addition to \((\Sigma^-),\) the triple \((f, g, h)\) satisfies

\[
(ev) \quad \chi_f \chi_g \chi_h = \omega^{2a} \text{ for some } a \in \mathbb{Z},
\]

\[
(sq) \quad \gcd(N_f, N_g, N_h) \text{ is square-free.}
\]

Then there exist \( \Lambda \)-adic test vectors \( (\tilde{f}^*, \tilde{g}^*, \tilde{h}^*) \) and an element

\[
\mathcal{L}_f^p(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*) \in \mathcal{R}
\]

such that for all \((x, y, z) \in \mathcal{X}_R\) of weight \((k, \ell, m):\)

\[
\nu(x, y, z)(\mathcal{L}_p^f(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*)^2) = \frac{\Gamma(k, \ell, m)}{2^n(k, \ell, m)} \cdot \frac{\mathcal{E}(f_x, g_y, h_z)^2}{\mathcal{E}_0(f_x)^2 \cdot \mathcal{E}_1(f_x)^2} \cdot \prod_{q \mid N} c_q \cdot \frac{L(f^0_x \otimes g^0_y \otimes h^0_z, c)}{\pi^{2(k-2)} \cdot \|f^0_x\|^2},
\]

where:

- \( c = (k + \ell + m - 2)/2, \)
- \( \Gamma(k, \ell, m) = (c - 1)! \cdot (c - m)! \cdot (c - \ell)! \cdot (c + 1 - \ell - m)!, \)
- \( \alpha(k, \ell, m) \in \mathcal{R} \) is a linear form in the variables \( k, \ell, m, \)
- \( \mathcal{E}(f_x, g_y, h_z) = (1 - \beta_{f_x, \alpha_g, \alpha_h})(1 - \beta_{f_x, \beta_g, \beta_h})(1 - \beta_{f_x, \gamma_g, \gamma_h})(1 - \beta_{f_x, \delta_g, \delta_h}), \)
- \( \mathcal{E}_0(f_x) = (1 - \beta_{f_x, \alpha_g}), \quad \mathcal{E}_1(f_x) = (1 - \beta_{f_x, \beta_g}), \)

and \( \|f^0_x\|^2 \) is the Petersson norm of \( f^0_x \) on \( \Gamma_0(N_f). \)

**Proof.** See [Hsi19] Thm. A. More specifically, the construction of \( \mathcal{L}_p^f(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*) \) under hypotheses \((CR), (ev),\) and \((sq)\) is given in [Hsi19] \S 3.6 (where it is denoted \( \mathcal{L}_F^f \)), and the proof of its interpolation property assuming \((\Sigma^-)\) is contained in [Hsi19] \S 7.

4.3. **Triple tensor product of big Galois representations.** Let \((f, g, h)\) be a triple of primitive Hida families with \( \chi_f \chi_g \chi_h = \omega^{2a} \) for some \( a \in \mathbb{Z}. \) For \( \phi \in \{f, g, h\}, \) let \( V_{\phi} \) be the natural lattice in \( \text{Frac} \mathbb{L}_\phi \) realizing the Galois representation \( \rho_\phi \) in the étale cohomology of modular curves (see [Oht04]), and set

\[
\mathbb{V}_{fgh} := V_f \otimes V_g \otimes V_h.
\]

This has rank 8 over \( \mathcal{R}, \) and by hypothesis its determinant can be written as det \( \mathbb{V}_{fgh} = \chi^2 \varepsilon_{\text{cyc}} \) for a \( p\)-ramified Galois character \( \chi \) taking the value \((-1)^a \) at complex conjugation. Similarly as in [How07] Def. 2.1.3], we define the **critical twist**

\[
\mathbb{V}_{fgh}^\dagger := \mathbb{V}_{fgh} \otimes \chi^{-1}.
\]

More generally, for any multiple \( N \) of \( N_\phi \) one can define Galois modules \( V_{\phi}(N) \) by working in tame level \( N \); these split non-canonically into a finite direct sum of the \( \mathbb{L}_\phi \)-adic representations \( V_{\phi} \) (see [DR17a] \S 1.5.3]), and they define \( \mathbb{V}_{fgh}^\dagger(N) \) for any \( N \) divisible by \( \text{lcm}(N_f, N_g, N_h). \)

If \( f \) is a classical specialization of \( \bar{f} \) with associated \( p\)-adic Galois representation \( V_f, \) we let \( \mathbb{V}_{f,gh} \) be the quotient of \( \mathbb{V}_{fgh} \) given by

\[
\mathbb{V}_{f,gh} := V_f \otimes V_g \otimes V_h.
\]

Denote by \( \mathbb{V}_{f,gh}^\dagger \) the corresponding quotient of \( \mathbb{V}_{fgh}^\dagger, \) and by \( \mathbb{V}_{f,gh}^\dagger(N) \) its level \( N \) counterpart.
4.4. Theta elements and factorization. We recall the factorization proven in [Hsi19, §8]. Let $f \in S_2(pN_f)$ be a $p$-stabilized newform of tame level $N_f$ defined over $O$, let $f^\circ \in S_2(N_f)$ be the associated newform, and let $\alpha_p = \alpha_p(f) \in O^\times$ be the $U_p$-eigenvalue of $f$. Let $K$ be an imaginary quadratic field of discriminant $D_K$ prime to $N_f$. Write

$$N_f = N^+N^-$$

with $N^+$ (resp. $N^-$) divisible only by primes which are split (resp. inert) in $K$, and choose an ideal $\mathfrak{R}^+ \subset O_K$ with $O_K/\mathfrak{R}^+ \simeq \mathbb{Z}/N^+\mathbb{Z}$.

We assume that $pO_K = \mathfrak{p}O_K$ splits in $K$, with $\mathfrak{p}$ the prime of $K$ above $p$ induced by our fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}_p$. Let $\Gamma_\infty = \text{Gal}(K_\infty/K)$ be the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of $K$, fix a topological generator $\gamma \in \Gamma_\infty$, and identity $O[[\Gamma_\infty]]$ with the one-variable power series ring $O[[T]]$ via $\gamma \mapsto 1 + T$. For any prime-to-$p$ ideal $\mathfrak{a}$ of $K$, let $\sigma_\mathfrak{a}$ be the image of $\mathfrak{a}$ in the Galois group of the ray class field $K(p^{\infty})/K$ of conductor $p^{\infty}$ under the geometrically normalized reciprocity law map.

**Theorem 4.2.** Let $\chi$ be a ring class character of $K$ of conductor $cO_K$ with values in $O$, and assume that:

1. $(pN_f, cD_K) = 1$,
2. $N^-$ is the square-free product of an odd number of primes,
3. $\tilde{\rho}_f$ is absolutely irreducible and $p$-distinguished,
4. if $q|N^-$ is a prime with $q \equiv 1 \pmod{p}$, then $\tilde{\rho}_f$ is ramified at $q$.

Then there exists a unique element $\Theta_{f/K, \chi}(T) \in O[[T]]$ such that for every $p$-power root of unity $\zeta \in \mathbb{Q}_p$:

$$\Theta_{f/K, \chi}(\zeta - 1)^2 = \frac{p^n}{\alpha_p^{2n}} \cdot \mathcal{E}_p(f, \chi, \zeta)^2 \cdot \frac{L(f^\circ/K \otimes \chi \epsilon, 1)}{(2\pi)^2 \cdot \Omega_{f^\circ, N^-}} \cdot u_K^2 \cdot D_K \chi \epsilon_{\zeta} \cdot \epsilon_{\sigma_0^+} \cdot \epsilon_p,$$

where:

- $n \geq 0$ is such that $\zeta$ has exact order $p^n$,
- $\epsilon_{\zeta} : \Gamma_\infty \to \mu_{p^n}$ be the character defined by $\epsilon_{\zeta}(\gamma) = \zeta$,
- $\mathcal{E}_p(f, \chi, \zeta) = \left\{ \begin{array}{ll} (1 - \alpha_p^{-1}(\mathfrak{p})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{array} \right.$
- $\Omega_{f^\circ, N^-} = 4 \cdot \|f^\circ\|_{\Gamma_0(N_f)}^2 \cdot \eta_{f^\circ, N^-}$ is the Gross period of $f^\circ$,
- $\sigma_0^+ \in \Gamma_\infty$ is the image of $\mathfrak{R}^+$ under the geometrically normalized Artin’s reciprocity map,
- $u_K = |O_K^\times|/2$, and $\epsilon_p = \{\pm 1\}$ is the local root number of $f^\circ$ at $p$.

**Proof.** See [BD96] for the first construction, and [CH18, Thm. A] for the stated interpolation property. \hfill \Box

When $\chi$ is the trivial character, we write $\Theta_{f/K, \chi}(T)$ simply as $\Theta_{f/K}(T)$. Suppose now that $f$ is the specialization of a primitive Hida family $f \in S^a(N_f, \ell)$ with branch character $\chi_f = 1$ at an arithmetic point $x_1 \in \mathcal{X}_f^+$ of weight 2. Let $\ell \nmid pN_f$ be a prime split in $K$, and let $\chi$ be a ring class character of $K$ of conductor $\ell^{2m}O_K$ for some even $m > 0$. Set $C = D_K\ell^{2m}$ and let

$$g = \theta_{\chi}(S_2) \in S^b(C, \omega^{-1}) \otimes \eta_{K/Q}, O[S_2], \quad h = \theta_{\chi^{-1}}(S_3) \in S^c(C, \omega^{-1}) \otimes \eta_{K/Q}, O[S_3])$$

be the primitive CM Hida families constructed in [Hsi19, §8.3], where $\eta_{K/Q}$ is the quadratic character associated to $K$. The $p$-adic triple product $L$-function of Theorem [11] for this triple $(f, g, h)$ is an element in $\mathcal{R} = \mathbb{I}[S_2, S_3]$; in the following we let

$$L_p^f(\tilde{f}^*, g^*h^*) \in O[[S]]
denote the restriction to the “line” \( S = S_2 = S_3 \) of its image under the specialization map at \( x_1 \).

Let \( \mathbb{K}_\infty \) be the \( \mathbb{Z}_p \)-extension of \( K \), and let \( K_p^\infty \) denote the \( p \)-ramified \( \mathbb{Z}_p \)-extension in \( \mathbb{K}_\infty \), with Galois group \( \Gamma_p^\infty = \text{Gal}(K_p^\infty/K) \). Let \( \gamma_p \in \Gamma_p^\infty \) be a topological generator, and for the formal variable \( T \) let \( \Psi_T : \text{Gal}(\mathbb{K}_\infty/K) \to \mathcal{O}[T]^{\times} \) be the universal character defined by

\[
\Psi_T(\sigma) = (1 + T)^{l(\sigma)}, \quad \text{where } \sigma|_{K_p^\infty} = l_p(\sigma).
\]

Denoting by the superscript the action of the non-trivial automorphism of \( K/\mathbb{Q} \), the character \( \Psi_T^{1-c} \) factors through \( \Gamma_\infty \) and yields an identification \( \mathcal{O}[[\Gamma_\infty]] \simeq \mathcal{O}[T] \) corresponding to the topological generator \( \gamma_1^{1-c} \in \Gamma_\infty \). Let \( p^b \) be the order of the \( p \)-part of the class number of \( K \). Hereafter, we shall fix \( v \in \overline{\mathbb{Z}_p^\times} \) such that \( v = \varepsilon_{\text{cyc}}(\gamma_p^p) \in 1 + p\mathbb{Z}_p \). Let \( K(\chi, \alpha_p)/K \) (resp. \( K(\chi)/K \)) be the finite extension obtained by adjoining to \( K \) the values of \( \chi \) and \( \alpha_p \) (resp. the values of \( \chi \)).

**Proposition 4.3.** Set \( T = v^{-1}(1 + S) - 1 \). Then

\[
\mathcal{L}_p^f(\tilde{f}^*, \tilde{g}^* \tilde{h}^*) = \pm \Psi_T^{1-c}(\sigma_{\omega_1}) \cdot \Theta_{f/K}(T) \cdot C_{f,\chi} \cdot \sqrt{L_{\text{alg}}(f/K \otimes \chi^2, 1)},
\]

where \( C_{f,\chi} \in K(\chi, \alpha_p)^\times \) and

\[
L_{\text{alg}}(f/K \otimes \chi^2, 1) := \frac{L(f/K \otimes \chi^2, 1)}{\pi^2 \Omega_{f,c,N^-}} \in K(\chi).
\]

**Proof.** This is the factorization formula of [HS19, Prop. 8.1] specialized to \( S = S_2 = S_3 \), using the interpolation property of \( \Theta_{f/K,\chi^2}(T) \) at \( \zeta = 1 \).

**Remark 4.4.** The factorization of Proposition 4.3 reflects the decomposition of Galois representations

\[
\mathcal{V}_{f,gh}^\dagger = (V_f(1) \otimes \text{Ind}_K^\mathbb{Q} \Psi_T^{1-c}) \oplus (V_f(1) \otimes \text{Ind}_K^\mathbb{Q} \chi^2).
\]

**4.5. Euler system construction of theta elements.** For the rest of the paper, assume that \( f, g = \theta_\chi(S) \), and \( h = \theta_{\chi^{-1}}(S) \) are as in [4.3], viewing the latter two in \( S^0(C, \omega^{-1} \eta_K/\mathbb{Q}, \mathcal{O}[S]) \).

Keeping the notations from [4.3] by DR16, §1 there exists a class

\[
\kappa(f, gh) \in H^1(Q, \mathcal{V}_{f,gh}^\dagger(N))
\]

constructed from twisted diagonal cycles on the triple product of modular curves of tame level \( N \) (we shall briefly recall the construction of this class in Theorem 4.10 below), where we may take \( N = \text{lcm}(N_f, C) \).

Every triple of test vectors \( \tilde{F} = (\tilde{f}, \tilde{g}, \tilde{h}) \) defines a Galois-equivariant projection

\[
\text{pr}_{\tilde{F}} : H^1(Q, \mathcal{V}_{f,gh}^\dagger(N)) \to H^1(Q, \mathcal{V}_{f,gh}^\dagger)
\]

and we let

\[
\kappa(\tilde{f}, \tilde{g}, \tilde{h}) := \text{pr}_{\tilde{F}}(\kappa(f, gh)) \in H^1(Q, \mathcal{V}_{f,gh}^\dagger).
\]

Since \( \Psi_T^{1-c} \) gives the universal character of \( \Gamma_\infty = \text{Gal}(K_\infty/K) \), by [4.3] and Shapiro’s lemma we have the equalities

\[
H^1(Q, \mathcal{V}_{f,gh}^\dagger) = H^1(Q, V_f(1) \otimes \text{Ind}_K^\mathbb{Q} \Psi_T^{1-c}) \oplus H^1(Q, V_f(1) \otimes \text{Ind}_K^\mathbb{Q} \chi^2) = \hat{H}^1(K_\infty, V_f(1)) \oplus H^1(K, V_f(1) \otimes \chi^2).
\]

Let \( g \) and \( h \) be the weight 1 eigenform \( \theta_\chi \) and \( \theta_{\chi^{-1}} \), respectively, so that the specialization of \( (g, h) \) at \( T = 0 \) (\( \Leftrightarrow S = v - 1 \)) is a \( p \)-stabilization of the pair \( (g, h) \).
Lemma 4.5. Assume that $L(f \otimes g \otimes h, 1) = 0$ and that $L(f/K \otimes \chi^2, 1) \neq 0$. Then for every choice of test vectors $F = (\tilde{f}, \tilde{g}, \tilde{h})$ we have:

1. $\kappa(\tilde{f}, \tilde{g}, \tilde{h}) \in \tilde{H}^1(K_{\infty}, V_f(1))$.
2. $\text{loc}_{\tilde{\kappa}}(\kappa(\tilde{f}, \tilde{g}, \tilde{h})) = 0 \in \tilde{H}^1(K_{\infty, \tilde{\kappa}}, V_f(1))$.

Proof. Let $\kappa = \kappa(\tilde{f}, \tilde{g}, \tilde{h})$ and for every $? \in \{f, g, h\}$, let $V_?V_f$ be the rank one subspace of $V_f$ fixed by the inertia group at $p$. By (4.7), in order to prove (1) it suffices to show that some specialization of $\kappa$ has trivial image in $H^1(K, V_f(1) \otimes \chi^2)$. Let

$$\kappa_{f, gh} := \kappa|_{S=E} \in H^1(Q, V_{fgh}) = H^1(K, V_f(1)) \oplus H^1(K, V_f(1) \otimes \chi^2),$$

where $V_{fgh} := V_f(1) \otimes V_g \otimes V_h$. As noted in [DR17a, p. 634], the Selmer group $\text{Sel}(Q, V_{fgh}) \subset H^1(Q, V_{fgh})$ is given by

$$\text{Sel}(Q, V_{fgh}) = \ker \left( H^1(Q, V_{fgh}) \xrightarrow{\partial_p, \text{loc}_{\tilde{\kappa}}} H^1(Q_p, V_f^{-1}(1) \otimes V_g \otimes V_h) \right),$$

where $\partial_p$ is the natural map induced by the projection $V_f \twoheadrightarrow V_f^{-1} := V_f/F^0V_f$, and so

$$\text{Sel}(Q, V_{fgh}) = \text{Sel}(K, V_f(1)) \oplus \text{Sel}(K, V_f(1) \otimes \chi^2).$$

The implications $L(f \otimes g \otimes h, 1) = 0 \Rightarrow \kappa_{f, gh} \in \text{Sel}(Q, V_{fgh})$ and $L(f/K \otimes \chi^2, 1) \neq 0 \Rightarrow \text{Sel}(K, V_f(1) \otimes \chi^2) = 0$, which follow from [DR17a, Thm. C] and [CH15, Thm. 1], respectively, thus yield assertion (1).

We proceed to prove (2). We know that the local class $\text{loc}_p(\kappa)$ belongs to $H^1(Q_p, F^+V_{fgh})$, where

$$F^+V_{fgh} := (F^0V_f(1) \otimes F^0V_g \otimes V_h + F^0V_f(1) \otimes V_g \otimes F^0V_h + V_f(1) \otimes F^0V_g \otimes F^0V_h) \otimes \chi^{-1}$$

is a rank four subspace of $V_{fgh}^\dagger$ (see [DR17a, Cor. 2.3]). In our case where $(g, h) = (\theta_{\chi, \theta_{\chi^{-1}}}, 1)$, we have

$$F^+V_{fgh} = V_f(1) \otimes \Psi_T^{-c} + F^0V_f(1) \otimes (\chi^2 \otimes \chi^{-2}),$$

where $\Psi_T$ is viewed as a character of $G_{Q_p}$ via the embedding $K \hookrightarrow Q_p$ induced by $p$. From part (1) of the lemma, it follows that

$$\text{loc}_p(\kappa) = (\text{loc}_p(\kappa), \text{loc}_p(\tilde{\kappa})) \in H^1(K_p, V_f(1) \otimes \Psi_T^{-c}) \oplus \{0\} \subset H^1(K_p, V_f(1) \otimes \Psi_T^{-c} \oplus H^1(K_p, V_f(1) \otimes \Psi_T^{1-c}) = H^1(Q_p, V_f(1) \otimes \text{Ind}_K^Q \Psi_T^{1-c}).$$

We thus conclude that $\text{loc}_p(\tilde{\kappa}) = 0$, and hence $\text{loc}_{\tilde{\kappa}}(\kappa) = 0$. \hfill \square

From now on, assume that $f^0 \in S_2(N_f)$ is the newform corresponding to an elliptic curve $E/Q$ with good ordinary reduction at $p$. In particular, $V_f(1) \simeq V_pE$, and under the conditions in Lemma 4.3 we have the class $\kappa(\tilde{f}, \tilde{g}, \tilde{h}) \in \tilde{H}^1(K_{\infty}, V_pE \otimes L)$.

The following key theorem is a variant of the “explicit reciprocity law” of [DR17a, Thm. 5.3] in our setting in terms of the Coleman map constructed in 3.2.

Theorem 4.6 (Darmon–Rotger). Assume that $L(f \otimes g \otimes h, 1) = 0$ and that $L(f/K \otimes \chi^2, 1) \neq 0$. Then $\text{loc}_{\tilde{\kappa}}(\kappa(\tilde{f}, \tilde{g}, \tilde{h})) = 0$ and

$$L_f^0(\tilde{f}, \tilde{g}, \tilde{h}) = \frac{\alpha_p}{2} \cdot (1 - \alpha_p^{-1}a_p(g)a_p(h)^{-1}) \cdot \text{Col}^0(\text{loc}_p(\kappa(\tilde{f}, \tilde{g}, \tilde{h})))$$

where $\tilde{F}^* = (\tilde{f}^*, \tilde{g}^*, \tilde{h}^*)$ is the triple of test vectors from Theorem 4.7.
Consider the triple product of modular curves over $\mathbb{Q}$:

$$W_{s,s} := X_0(Np) \times X_s \times X_s,$$

where $X_0(Np)$ and $X_s$ are the classical modular curves attached to the congruence subgroups $\Gamma_0(Np)$ and $\Gamma_1(Np^s)$, respectively, and the model for the latter is the one for which the cusp $\infty$ is defined over $\mathbb{Q}$. The group $G_{s}(N) := (\mathbb{Z}/Np^s\mathbb{Z})^\times$ acts on $X_s$ by the diamond operators $\langle a; b \rangle$ ($a \in (\mathbb{Z}/N\mathbb{Z})^\times$, $b \in (\mathbb{Z}/p^s\mathbb{Z})^\times$), and we let

$$W_s := W_{s,s}/D_s$$

be the quotient of $W_{s,s}$ by the action of the subgroup $D_s \subset G_{s}(N) \times G_{s}(N)$ consisting of elements of the form ($\langle a; b \rangle; (a; b^{-1})$). Let $\Delta_{s,s,s} \in \operatorname{CH}^2(W_{s,s})/(\mathbb{Q}(\xi_s))$ be the class in the Chow group defined by the "twisted diagonal cycle" defined in (41), and let $\Delta_s \in \operatorname{CH}^2(W_s)/(\mathbb{Q}(\xi_s))$ denote its natural image under the projection $p_s : W_{s,s} \to W_s$. By Proposition 1.4, after applying the correspondence $\varepsilon_{s,s}$ in (47) the cycle $\Delta_{s,s,s}$ becomes null-homologous, and so

$$\Delta_s := \varepsilon_{s,s}(\Delta_{s,s,s}) \in \operatorname{CH}^2(W_s)/(\mathbb{Q}(\xi_s)),$$

letting $\varepsilon_{s,s}$ still denote the linear endomorphism of $\operatorname{CH}^2(W_s)$ defined by the above correspondence. Let $\varepsilon_s : G_{Q} \to (\mathbb{Z}/p^s\mathbb{Z})^\times$ be the mod $p^s$ cyclotomic character, and let $X^\dagger_s$ be the twist of $X_s$ by the cocycle $\sigma \in G_{Q} \mapsto \langle 1; \varepsilon_s(\sigma) \rangle$. By Proposition 1.6, we may alternatively view

$$\Delta_s \in \operatorname{CH}^2(W^\dagger_s)/(\mathbb{Q}),$$

where $W^\dagger_s$ the quotient of $W^\dagger_{s,s} := X_0(Np) \times X_s \times X^\dagger_s$ be a diamond action defined as before.

Consider the $p$-adic étale Abel–Jacobi map

$$\operatorname{AJ}_{et} : \operatorname{CH}^2(W^\dagger_s)/(\mathbb{Q}) \to H^1(\mathbb{Q}, \operatorname{H}^3_{et}(W^\dagger_s/\mathbb{Q}, \mathbb{Z}_p)(2)).$$

Let $e_{ord} = \lim_n U_p^n$ be Hida’s ordinary projector. Set

$$V^\dagger_{s,s} := H^1_{et}(X_0(Np)/\mathbb{Q}, \mathbb{Z}_p) \otimes e_{ord}(H^1(X_s/\mathbb{Q}, \mathbb{Z}_p)(1)) \otimes e_{ord}(H^1(X^\dagger_s/\mathbb{Q}, \mathbb{Z}_p)(1)),$$

and let $V^\dagger_s := (V^\dagger_{s,s})_{D_s}$ denote the $D_s$-co-invariants. Let $\varpi_2 : X_{s+1} \hookrightarrow X_s$ be the degeneracy map given by $\tau \mapsto p\tau$ on the complex upper half plane, which naturally defines

$$(\varpi_2)_s = (1, \varpi_2, \varpi_2)_s : V^\dagger_{s+1,s+1} \to V^\dagger_{s,s}.$$

Let $\tilde{\kappa}_s \in H^1(\mathbb{Q}, V^\dagger_{s,s})$ denote the image of $\operatorname{AJ}_{et}(\Delta_s)$ under the composite map

$$H^1(\mathbb{Q}, H^3_{et}(W^\dagger_s/\mathbb{Q}, \mathbb{Z}_p)(2)) \xrightarrow{\varepsilon_s \circ pr_{s,s}} H^1(\mathbb{Q}, H^3_{et}(W^\dagger_s/\mathbb{Q}, \mathbb{Z}_p)(2)) \xrightarrow{(1, e_{ord} \circ pr_{1,1})} H^1(\mathbb{Q}, V^\dagger_{s,s})(D_s)(2)),$$

where the first arrow is defined by Lemma 1.8, and $pr_{1,1}$ is the projection onto the $(1, 1, 1)$-component in the Künneth decomposition for $H^3_{et}(W^\dagger_s/\mathbb{Q}, \mathbb{Z}_p)$. By Proposition 1.8, we have $$(\varpi_2)_s(\tilde{\kappa}_{s+1}) = (1, U_p, 1)(\tilde{\kappa}_s),$$ and hence we obtain the compatible family

$$\kappa_\infty := \lim_s (1, U_p, 1)^{-s}(\tilde{\kappa}_s) \in H^1(\mathbb{Q}, V^\dagger_{s,s})$$

with limit with respect to the maps induced by (41)(11). The triple $(f, g, h)$ defines a natural projection $\varpi_{f,g,h} : V^\dagger_{s,s} \to V^\dagger_{f,g,h}(N)$, and following Definition 1.15 one sets

$$\kappa(f, gh) := \varpi_{f,g,h}(\kappa_\infty) \in H^1(\mathbb{Q}, V^\dagger_{f,g,h}(N))$$.
this is the class in \( \mathfrak{H}_c \). Now, to prove the equality (4.9) in the theorem, it suffices to show that both sides agree at infinitely many points. Let \( x \in \mathcal{X}_+^\times \) have weight 2 with \( \zeta := \epsilon_x(1+p) \in \mu_{p-\infty} \) a primitive \( p^s \)-th root of unity, and set

\[
\kappa(f, g, h_x) := \kappa(f, gh)\big|_{T=\zeta^1-1}.
\]

Directly from the definitions (cf. Proposition 2.5), we have

\[
\kappa(f, g, h_x) = a_p(g_x)^{-s} \cdot \varpi_f g, h_x(AJ_{\Delta_x}) \in H^1(\mathcal{Q}, V_{f g, h_x}(N)),
\]

where \( V_{f g, h_x}(N) \) is the \((f, g, h_x)\)-isotypical component of \( \mathcal{H} \), and \( \varpi_f g, h_x \) is the projection to that component. By Corollary 2.3 and (77), the image of \( \kappa(f, g, h_x) \) in the local cohomology group \( H^1(\mathcal{Q}, V_{f g, h_x}(N)) \) lands in the Bloch–Kato finite subspace \( \mathcal{H}_{\mathrm{fin}}(\mathcal{Q}, V_{f g, h_x}(N)) \subset H^1(\mathcal{Q}, V_{f g, h_x}(N)) \), and so we may consider the image \( \log_p \kappa(f, g, h_x) \) of this restriction under the Bloch–Kato logarithm map

\[
\log_p : \mathcal{H}_{\mathrm{fin}}(\mathcal{Q}, V_{f g, h_x}(N)) \to (\mathfrak{Fil}^0 D_{f g, h_x}(N))^\vee,
\]

where \( D_{f g, h_x}(N) \) by the de Rham comparison isomorphism, we have

\[
D_{f g, h_x}(N) \cong H^1_{\mathrm{DR}}(X_0(Np)/\mathcal{Q}, Q_p(\zeta)/\mathcal{Q})(1)[g_x] \times H^1_{\mathrm{DR}}(X_0/\mathcal{Q}, Q_p(\zeta)(1)[h_x],
\]

As in p. 639, attached to the test vectors \((f, g, h_x)\) one has the de Rham classes \((\eta_f, \omega_{g_x}^0, \omega_{h_x}^0)\), and comparing Proposition 2.10 and Corollary 2.11 we deduce from (4.12) that

\[
\langle \log_p \kappa(f, g, h_x), \eta_f \otimes \omega_{g_x}^0 \otimes \omega_{h_x}^0 \rangle_{\mathrm{DR}} = a_p(g_x)^{-s} \cdot \langle \Delta_p(\Delta_x), \eta_f \otimes \omega_{g_x}^0 \omega_{h_x}^0 \rangle_{\mathrm{DR}} = \epsilon(x, g, h_x) \cdot \sigma_x^{-1} \cdot \alpha_p^{-1} \cdot a_p(g_x)^{-s} \cdot \alpha_p(h_x)^{-s} \cdot \tau_p(f, g \Delta h^t).
\]

Taking \((f, g, h)\) to be the test vectors \( \Phi^* \) from Theorem 4.1 above, the construction in [Hsi19, §3.6] yields \( \mathcal{Z}_{\mathcal{H}}(f, g h) = f^\vee \cdot (g^{-1} h^t) \). Since by construction \( \mathcal{H} \) specializes at \( x \) to \( \mathcal{H}^t_x \), we thus see as in the proof of Theorem 4.16 that

\[
\langle \log_p \kappa(f, g, h_x), \eta_f \otimes \omega_{g_x}^0 \otimes \omega_{h_x}^0 \rangle_{\mathrm{DR}} = \mathcal{E}(f, g, h_x) \cdot \sigma_x^{-1} \cdot \alpha_p^{-1} \cdot a_p(g_x)^{-s} \cdot \alpha_p(h_x)^{-s} \cdot \mathcal{Z}_{\mathcal{H}}(f, g h^t)(x).
\]

On the other hand, letting \( \psi_x := \psi_T|_{T=\zeta^1-1} \), we obtain that \((g_x, h_x)\) is a pair of theta series attached to the characters \((\chi \psi_x^1, \chi \lambda^1)\) of \( G_K \) with \( a_p(g_x) = \chi \psi_x^1(\sigma) \) and \( a_p(h_x) = \chi^{-1} \psi_x^{-1}(\sigma) \). Moreover, we have

\[
\epsilon_x|_{\mathcal{G}_K} = \psi_x^1 + \epsilon_{\sigma} \cdot \psi_x^1, \quad \psi_x^{-1} = \phi_x^1 \cdot \psi_x^{-1}
\]

for some finite order character \( \phi_x \) of \( \mathrm{Gal}(F_{\infty}/\mathcal{Q}_p) \), viewing the character in the left-hand side of this equality as character on \( \mathrm{Gal}(F_{\infty}/F) \) by composition with \( \mathrm{Gal}(F_{\infty}/F) \subset \mathrm{Gal}(F_{\infty}/\mathcal{Q}_p) \to \mathrm{Gal}(K_{\infty,p}/K_p) \subset \Gamma_{\infty} \). Setting \( \eta = \eta_f \otimes t^{-1} \) and \( z_x = \log_p(\kappa(f, g^p h^t)) \), we thus see that

\[
\langle \log_p \kappa(f, g, h_x), \eta_f \otimes \omega_{g_x}^0 \otimes \omega_{h_x}^0 \rangle_{\mathrm{DR}} = \langle \log_p(z_x) \otimes t, \eta \rangle_{\mathrm{DR}} = g(x) \cdot \alpha_p(a_p(g_x)^{-s} \cdot \alpha_p(h_x)^{-s} \cdot \mathcal{Z}_{\mathcal{H}}(f, g h^t)(x),
\]

using Theorem 3.3 with \( j = -1 \) for the last equality. Comparing (4.13) with (4.14) and letting \( s \) vary, the result follows.

We can now immediately deduce the following key cohomological construction of \( \Theta_{f/K} \):
**Theorem 4.7.** With notations and assumptions as in Theorem 4.6, we have

\[
\text{Col}^q(\text{loc}_p(\kappa(f^*, \tilde{g}^* h^*))) = \pm \Psi_T^{\omega - 1} (\alpha_p) \cdot \Theta_{f/K}(T) \cdot \sqrt{L_{\text{alg}}(E/K \otimes \chi^2, 1)} \cdot \frac{2C_{f, \chi}}{\alpha_p (1 - \alpha_p \bar{\chi}(\mathbf{F})^2)},
\]

where \(C_{f, \chi} \in K(\chi, \alpha_p)^\times\) is the non-zero algebraic number as in Proposition 4.3

**Proof.** Note that \(\alpha_p(g)a_p(h)^{-1} = \chi(\mathbf{F})^2\). The theorem thus follows immediately from Proposition 4.3 and Theorem 4.6. \(\square\)

4.6. **Generalized Kato classes.** Set \(\alpha = \chi(\mathbf{F})\), and denote by \((g_a, h_{a-1})\) the weight 1 forms obtained by specializing the Hida families \((g, h)\) at \(S = v - 1\). Thus \(g_a\) (resp. \(h_{a-1}\)) is the \(p\)-stabilization of the theta series \(g = \theta_\chi\) (resp. \(h = \theta_{\chi^{-1}}\)) having \(U_p\)-eigenvalue \(\alpha\) (resp. \(\alpha^{-1}\)). By specialization, the \(O[S]-\text{adic class in}\) (4.3) yields the class

\[
\kappa(f, g_a, h_{a-1}) := \kappa(f, \tilde{g}^* h^*)|_{S=v-1} \in H^1(Q, V_{fgh}),
\]

where \(V_{fgh} := V_f \otimes V_g \otimes V_h\). Setting \(\beta = \chi(p)\) and alternatively changing the roles of \(p\) and \(\mathbf{F}\) in the construction \(g\) and \(h\) we thus obtain the four \textit{generalized Kato classes}

\[
(4.15) \quad \kappa(f, g_a, h_{a-1}), \kappa(f, g_{a-1}, h_{a}), \kappa(f, g_{a-1}, h_{a}), \kappa(f, g_{a}, h_{a-1}) \in H^1(Q, V_{fgh}).
\]

From now on, we assume that \(\alpha \neq \pm 1\), so that the four classes (4.15) are \textit{a priori} distinct. Recall that \(f\) is the \(p\)-stabilization of the newform associated to an elliptic curve \(E/Q\), so that \(V_f(1) \simeq V_pE\), and let \(\kappa_{\alpha, a-1}, \kappa_{\alpha, a}, \kappa_{\beta, a-1}, \kappa_{\beta, a} \in H^1(K, V_pE \otimes L)\) be the image of the classes (4.15) under the map \(H^1(Q, V_{fgh}) \to H^1(K, V_pE \otimes L)\) induced by (1.7).

**Corollary 4.8.** Assume that \(L(E/K, 1) = 0\) and that \(L(f/K \otimes \chi^2, 1) \neq 0\). Then:

1. \(\kappa_{\alpha, a-1}, \kappa_{\beta, a-1} \in \text{Sel}(K, V_pE \otimes L)\).
2. \(\kappa_{\alpha, a} = \kappa_{\beta, a-1} = 0\).

**Proof.** By the factorization (1.6), the inclusions in part (1) follow from the proof of Lemma 4.5. To see part (2), we make use of the 3-variable generalized Kato class

\[
\kappa := \kappa(f, g, h')(S_1, S_2, S_2) \in H^1(Q, \Psi_{fgh'}^\dagger),
\]

defined in \[\text{DR17A} \quad \S 3.7 (119)\] attached to the triple \(f = f(S_1), g = \theta_\chi(S_2)\) and \(h' = \theta_\chi(S_3)\). Thus \(\kappa(f, g_a, h_{a-1})\) is the specialization \(\kappa((1 + p)^2 - 1, v - 1, v - 1)\). Let

\[
\kappa' := \kappa((1 + p)^2 - 1, v(1 + T) - 1, v(1 + T)^{-1} - 1) \in H^1(Q, \Psi_{fgh'}^\dagger),
\]

where \(\Psi_{fgh'}^\dagger \simeq V_pE \otimes (\text{Ind}_K^Q \chi^2 \oplus \text{Ind}_K^Q \Psi_{T}^{1-c})\). As in Lemma 4.5 by \[\text{DR17B} \quad \text{Prop. 3.28}\] the class \(\text{loc}_p(\kappa')\) belongs to \(H^1(Q_p, F^+\Psi_{fgh'}^\dagger)\), where

\[
F^+\Psi_{fgh'}^\dagger = V_pE \otimes \chi^{-2} + F^0V_pE \otimes (\Psi_{T}^{1-c} \oplus \Psi_{T}^{1-c}).
\]

It follows that the projection \(\kappa_F'\) of \(\kappa'\) into \(\hat{H}^1(K_\infty, V_pE)\) is crystalline at \(p\), and hence \(\kappa_F'\) is a Selmer class for \(V_pE\) over the anticyclotomic \(\mathbf{Z}_p\)-extension \(K_\infty/K\). Since the space of such universal norms is trivial by Cornut–Vatsal \[\text{CV05}\] (the sign of \(E/K\) is +1 in our case), this shows that \(\kappa_F' = 0\) and therefore \(\kappa(f, g_a, h_{a-1}) = \kappa_{a, a-1} = 0\). The vanishing of \(\kappa_{a, a-1}\) is shown in the same manner. \(\square\)
5. Proof of the main theorem

Proof of Theorem A. Let $V = V_p E \otimes_{Q_p} L$ and $S = \text{Sel}(K, V)$. Let $S^{(r)} := S_p(E/K) \otimes_{Q_p} L$ be the subspaces in $\mathcal{S}_3$ and $S^{(\infty)}$ be the subspace of anticyclotomic universal norms. By \cite[Thm. 4.2]{How04} we have the filtration
\begin{equation}
S = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset S^{(r+1)} \supset \cdots \supset S^{(\infty)},
\end{equation}
and $S^{(r+1)}$ is the null space of the $r$-th derived height pairing $h^{(r)}: S^{(r)} \times S^{(r)} \rightarrow J^r/J^{r+1} \otimes L$. In addition, by \cite[Remark 1.12, Thm. 4.2]{How04} we have for every $x, y \in S^{(r)}$
\begin{equation}
\tag{5.2}
h^{(r)}(x, y) = (-1)^{r+1} h^{(r)}(y, x); \quad h^{(r)}(x^\tau, y^\tau) = (-1)^r h^{(r)}(x, y),
\end{equation}
where $\tau$ denotes the complex conjugation. Since $L(E^P, 1) \neq 0$, we have $\text{Sel}(Q, V_p E^P) = \{0\}$ by Kolyvagin’s work \cite{Ko88} (or Kato’s \cite{Kat04}), and so letting $S^+$ be the subspace of $S$ fixed by $\tau$, this shows
\begin{equation}
S = S^+.
\end{equation}
By \cite[(5.2)]{5.2}, this implies that $h^{(1)}$ is identically zero, and so $S = S^{(2)}$; by the same argument, $S^{(r)} = S^{(r+1)}$ for every odd $r \geq 1$. On the other hand, $S^{(\infty)} = \{0\}$ by Cornut–Vatsal \cite{CV05}.

We first prove the implication (ii) \Rightarrow (i). Suppose that $\dim_{Q_p} \text{Sel}_{str}(Q, V_p E) = 1$. By $p$-parity \cite{Nek01}, this implies that $\dim_{Q_p} \text{Sel}(Q, V_p E) = 2$ and the composite map
\begin{equation}
\log_{\omega_{E,p}}: S = \text{Sel}(K, V) \xrightarrow{\log_p} H^1_{\text{fin}}(K_p, V) \xrightarrow{3.3} L
\end{equation}
is nonzero. Under our hypotheses we have
\begin{equation}
\dim_L S = \dim_{Q_p} \text{Sel}(Q, V_p E) = 2.
\end{equation}
In view of \cite[(5.2)]{5.2}, we thus find that \cite[(5.1)]{5.1} reduces to
\begin{equation}
S = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset \cdots \supset S^{(\infty)} = \{0\}
\end{equation}
for some (even) integer $r \geq 2$ and the derived $p$-adic height $h^{(r)}$ is a non-degenerate pairing on $S^{(r)}$.

Let $X_{\infty}$ be the Pontryagin dual of $\text{Sel}_{p=\infty}(E/K_{\infty})$, which is known to be $\Lambda$-torsion \cite{BD05}. Let $J \subset \Lambda$ be the augmentation ideal, and fix a pseudo-isomorphism
\begin{equation}
X_{\infty} \cong M \oplus M', \quad \text{with} \quad M \cong \mathbb{Z}/J^{e_1} \oplus \mathbb{Z}/J^{e_2} \oplus \cdots
\end{equation}
with $M'$ a torsion $\Lambda$-module having characteristic ideal prime to $J$. By \cite[Cor. 4.3(c)]{How04} we have $e_i = \dim_L (S^{(i)}/S^{(i+1)})$; letting $L_p \subset \Lambda$ be a generator of the principal ideal $\text{char}(X_{\infty})$, combining \cite[(5.3)]{5.3} and \cite[(5.4)]{5.4} this shows that
\begin{equation}
\ord_L L_p = 2r.
\end{equation}

On the other hand, by our hypotheses on $\tilde{p}_{E,p}$ the divisibility in the Iwasawa main conjecture due to Skinner–Urban \cite{SU14} (see \cite[§3.6.3]{loc.cit.}) implies that $(\Theta^2_{f/K}) \supset (L_p)$, and so
\begin{equation}
s^{\rho} \geq \rho := \ord_f(\Theta^2_{f/K}).
\end{equation}
Let $\tilde{\theta}_{f/K}$ be the leading coefficient of $\Theta^2_{f/K}$ defined by
\begin{equation}
\tilde{\theta}_{f/K} := \Theta^2_{f/K}(T) \pmod{J^p/J^{p+1}} \in J^p/J^{p+1}.
\end{equation}
From \cite[(5.3)]{5.3} and \cite[(5.5)]{5.5} we see that $S = S^{(\rho)}$. Thus combining the derived $p$-adic height formula in Proposition 3.7, Theorem 4.7, and part (2) of Lemma 4.5 we deduce that for every $x \in S^{(\rho)} = S$ we have
\begin{equation}
\tag{5.6}
h^{(\rho)}(\kappa_{\alpha, \alpha^{-1}}, x) = \frac{1 - p^{-1} \alpha_p}{1 - \alpha_p} \cdot \tilde{\theta}_{f/K} \cdot \log_{\omega_{E,p}}(x) \cdot C,
\end{equation}
where \( \alpha_p \) is the \( p \)-adic unit root of \( X^2 - a_p(E)X + p = 0 \) and \( C \) is a non-zero algebraic number with \( C^2 \in K(\chi, \alpha_p)^\times \). Since \( \tilde{\theta}_{f/K} \neq 0 \) and as noted above our hypotheses imply that the map \( \log_{\omega_{E,p}} \) is non-zero, we see that \( r = \rho \) and the non-vanishing of \( \kappa_{\alpha,\alpha^{-1}} \) follows.

Now we proceed to establish the implication (i) \( \Rightarrow \) (ii). Suppose that \( \kappa_{\alpha,\alpha^{-1}} \neq 0 \). We shall prove that \( \dim_L S = 2 \) and \( \log_{\omega_{E,p}} \) is a non-zero map. Consider again the filtration \( \Theta \). Then the combination of Proposition 5.4 Theorem 4.7 and part (2) of Lemma 4.5 shows that the class \( \kappa_{\alpha,\alpha^{-1}} \) belongs to \( S^{(\alpha)} \) with \( \rho = \ord_f(\Theta_{f/K}) \); in particular, \( S^{(\alpha)} \neq 0 \). With notations as in (5.3), this implies that \( e_{r_0} \neq 0 \) for some even \( r_0 \geq \rho \), and so \( e_{r_0} \geq 2 \) by (5.2). On the other hand, we have

\[
2\rho \geq e_1 + 2e_2 + \cdots + re_r + \cdots
\]

according to the divisibility in the Iwasawa main conjecture due to Bertolini–Darmon [BD05] (see also [PW11]). This implies that \( e_\rho = 2 \) and \( e_r = 0 \) if \( r \neq \rho \). We thus conclude that \( \dim_L S = 2 \) and \( h^{(\rho)} \) is non-degenerate on \( S^{(\alpha)} = S \). In light of (5.6), this shown that the map \( \log_{\omega_{E,p}} \) is non-zero, yielding the proof of the implication (i) \( \Rightarrow \) (ii).

The following is an immediate consequence of the height formula (5.6):

**Corollary 5.1.** The class \( \kappa_{\alpha,\alpha^{-1}} \mod \mathbb{Q}^\times \) depends only on \( K \), not on the auxiliary choice of ring class character \( \chi \). Moreover, as elements in \( E(\mathbb{Q}) \otimes_{\mathbb{Z}} L \), we have

\[
\kappa_{\alpha,\alpha^{-1}} = C \cdot \frac{1 - \rho^{-1} \alpha_p}{1 - \alpha_p} \cdot \frac{\tilde{\theta}_{f/K}}{h^{(\rho)}(P, Q)} \cdot (P \otimes \log_p Q - Q \otimes \log_p P)
\]

for any basis \( (P, Q) \) of \( E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \).

This suggests that the refined conjecture [DR16, Conj. 3.12] in this case should follow from the \( p \)-adic Birch–Swinnerton-Dyer formula of [BD98, Conj. 4.3].

### 6. Numerical examples

In this section, we give examples of elliptic curves of rank 2 having non-trivial generalized Kato classes. To be more precise, we consider elliptic curves \( E/\mathbb{Q} \) with

\[
\ord_{s=1} L(E, s) = \text{rank}_\mathbb{Z} E(\mathbb{Q}) = 2
\]

and conductor \( N \in \{ q, 2q \} \) with \( q \) an odd prime. We take a square-free integer \( -\Delta < 0 \) such that \( K = \mathbb{Q}(\sqrt{-\Delta}) \) has class number one, \( q \) is inert in \( K \), and \( L(E/K, 1) \neq 0 \), and take a prime \( p > 3 \) of good ordinary prime for \( E \) which is split in \( K \) and such that \( E[p] \) is an irreducible \( G_\mathbb{Q} \)-module. For every triple \( (E, p, -\Delta) \), letting \( f \in S_2(\Gamma_0(N)) \) be the newform associated to \( E \), we give numerical examples where the associated theta element

\[
\Theta_{E/K}(T) = \Theta_{f/K}(T) \in \mathbb{Z}_p[[T]]
\]

vanishes to order exactly 2 at \( T = 0 \). When that is the case, by the work of Bertolini–Darmon [BD95, BD05] on the anticyclotomic Iwasawa main conjecture (see [BD05, Cor. 3]), it follows that \( \text{Im}(E/K)[p^\infty] \) is finite. Moreover, the residual representation \( E[p] \) must ramify at \( N^- = q \) by [Rib90, Thm. 1.1] and we checked that \( E[p] \) is irreducible, either by [Maz78] for \( p \geq 11 \) or checking that the elliptic curves we consider have no rational \( m \)-isogeny for \( m > 3 \) according to Cremona’s tables. Thus for every ring class character \( \chi \) with \( L(E/K, \chi^2, 1) \neq 0 \) (as one can always find by virtue of [Vat03, Thm. 1.4], as extended in [CH13, Thm. D]), the examples below provide instances where the generalized Kato class \( \kappa_{E,K} \) is a nonzero class in the 2-dimensional \( \text{Sel}(\mathbb{Q}, V_p E) \) by virtue of Corollary B.

To explain these numerical examples, we prepare some notation. Let \( B/\mathbb{Q} \) be the definite quaternion algebra of discriminant \( q \). Let \( R \) be an Eichler order of level \( N/q \) and \( \Cl(R) \) be

\footnote{As extended by Pollack–Weston [PW11] to allow for weaker hypotheses.}
the class group of \( R \). Let \( f_E : \text{Cl}(R) \to \mathbb{Z} \) be the \((p\text{-adically normalized})\) Hecke eigenfunction associated with \( f \) by the Jacquet–Langlands correspondence. Fix an optimal embedding \( \mathcal{O}_K \hookrightarrow R \) and an isomorphism \( i_p : R \otimes \mathbb{Z}_p \cong M_2(\mathbb{Z}_p) \) such that \( i_p(K) \) lies in the subspace of diagonal matrices. For \( a \in \mathbb{Z}_p^* \) and an integer \( n \), put
\[
 r_n(a) = i_p^{-1}( \begin{pmatrix} 1 & a p^{-n} \\ 0 & 1 \end{pmatrix} ) \in \hat{B}^x, \quad \hat{B} := B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}. 
\]

Consider the sequence \( \{ P_n^a \}_{n=0,1,...} \) of right \( R \)-ideals defined by \( P_n^a := (r_n(a)R) \cap B \). The images of these ideals \( P_n^a \) in \( \text{Cl}(R) \) are usually referred to Gross points of level \( p^n \). Letting \( u = 1 + p \), we define the \( n \)-th theta element \( \Theta_{E/K,n}(T) \in \mathbb{Z}_p[T] \) by
\[
 \Theta_{E/K,n}(T) := \frac{1}{\alpha_p^{n+1}} \sum_{i=0}^{p^n-1} \sum_{a \in \mathbb{Z}_{p-1}} (\alpha_p \cdot f_E(P_n^{aw}) - f_E(P_n^{aw+1})) (1 + T)^i. 
\]

By the definition of theta elements in \cite[§2.7]{BD96}, if \( K \) has class number one, we then have
\[
 \Theta_{E/K}(T) \equiv \Theta_{E/K,n}(T) \pmod{(1 + T)^{p^n - 1}}. 
\]

Since \((p^n,(1+T)p^n-1) \subset (p^n,Tp)\) and \( p > 2 \), to check the vanishing \( \Theta_{E/K}(T) \) to exact order \( 2 \) at \( T = 0 \), it suffices to compute \( \Theta_{E/K,n}(T) \) for sufficiently large \( n \). The following examples were obtained by implementing the Brandt module package in SAGE.

| \( E \) | \( p \) | \( -\Delta \) | \( \Theta_{E/K,2}(T) \mod (p^2,Tp) \) |
|-------|-----|------|-----------------|
| 389a1 | 11  | -2   | \( 107^2 + 69T^3 + T^4 + 103T^5 + 106T^6 + 66T^7 + 11T^8 + 55T^9 + 110T^{10} \) |
| 433a1 | 11  | -7   | \( 87^2 + 22T^3 + 86T^4 + 7T^5 + 10T^6 + 12T^7 + 29T^8 + 88T^9 + 48T^{10} \) |
| 446c1 | 7   | -3   | \( 22T^2 + 27T^3 + 3T^4 + 167T^5 + 11T^6 \) |
| 563a1 | 5   | -1   | \( 18T^2 + 9T^3 + 5T^4 \) |
| 643a1 | 5   | -1   | \( T^2 + 21T^4 \) |
| 709a1 | 11  | -2   | \( 27T^2 + 114T^3 + 3T^4 + 14T^5 + 36T^6 + 15T^7 + 42T^8 + 44T^9 + 91T^{10} \) |
| 718b1 | 5   | -19  | \( 3T^2 + 20T^3 + 12T^4 \) |
| 794a1 | 7   | -3   | \( 47T^2 + 23T^3 + 8T^4 + 24T^5 + 7T^6 \) |
| 997b1 | 11  | -2   | \( 71T^2 + 41T^3 + 83T^4 + 197T^5 + 114T^6 + 111T^7 + 101T^8 + 46T^9 + 102T^{10} \) |
| 997c1 | 11  | -2   | \( 54T^2 + 38T^3 + 36T^4 + 81T^5 + 82T^6 + 18T^7 + 72T^8 + 95T^9 + 4T^{10} \) |
| 1034a1| 5   | -19  | \( 22T^2 + 4T^3 + 6T^4 \) |
| 1171a| 5   | -1   | \( 6T^2 + 6T^3 + 20T^4 \) |
| 1483a1| 13  | -1   | \( 128T^2 + 148T^3 + 127T^4 + 162T^5 + 30T^6 + 149T^7 + 141T^8 + 97T^9 + 49T^{10} + 13T^{11} + 29T^{12} \) |
| 1531a1| 5   | -1   | \( 16T^2 + 7T^3 + 21T^4 \) |
| 1613a1| 17  | -2   | \( 128T^2 + 165T^3 + 224T^4 + 287T^5 + 140T^6 + 211T^7 + 147T^8 + 160T^9 + 59T^{10} + 122T^{11} + 195T^{12} + 43T^{13} + 207T^{14} + 214T^{15} + 285T^{16} \) |
| 1627a1| 13  | -1   | \( 101T^2 + 151T^3 + 58T^4 + 104T^5 + 3T^6 + 165T^7 + 128T^8 + 63T^9 + 17T^{10} + 55T^{11} + 166T^{12} \) |
| 1907a1| 13  | -1   | \( 72T^2 + 131T^3 + 32T^4 + 142T^5 + 84T^6 + 104T^7 + 90T^8 + 105T^9 + 38T^{10} + 92T^{11} + 116T^{12} \) |
| 1913a1| 7   | -3   | \( 41T^2 + 167T^3 + 28T^4 + 23T^5 + 14T^6 \) |
| 2027a1| 13  | -1   | \( 54T^2 + 128T^3 + 93T^5 + 83T^6 + 161T^7 + 113T^8 + 133T^9 + 49T^{10} + 151T^{11} + 13T^{12} \) |
| $E$   | $p$ | $-\Delta$ | $\Theta_{E/K,3}(T) \mod (p^3, T^p)$                               |
|-------|-----|-----------|---------------------------------------------------------------|
| 571b1 | 5   | $-1$      | $100T^2 + 100T^3 + 15T^4$                                     |
| 1621a1| 11  | $-2$      | $1089T^2 + 807T^4 + 9867^5 + 586T^6 + 1098T^7 + 772T^8 + 228T^9 + 1296T^{10}$ |

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