Classification of the automorphisms of the noncommutative torus among the (chaotic and non-chaotic) shallow ones and the non-chaotic complex ones

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Adopting the measure of quantum complexity, the quantum logical depth, previously introduced by the author the automorphisms of the noncommutative torus are classified among the (chaotic and non-chaotic) shallow ones and the non-chaotic complex ones.
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I. ACKNOWLEDGEMENTS

I would like to thank Giovanni Jona-Lasinio and Vittorio de Alfaro for many useful suggestions. Of course nobody among them has any responsibility to any (eventual) mistake contained in these pages.
II. INTRODUCTION

No term in Science is used in a plethora of completely different meanings as the term complexity.
Indeed the locution Science of Complex Systems is very often used as a label intentionally left undefined in order to comprehend any kind of research program.
A mathematical precise measure of complexity, called logical depth, was introduced by Charles Bennett [1] and generalized by myself to quantum systems [2].
A great intellectual achievement of such a notion consists in the fact that it clearly distinguishes between complexity and randomness: a random system is no complex.
Since, by Brudno theorem [3], [4], [5], the dynamical entropy of a classical dynamical system is equal to the algorithmic information [6] (that despite being usually also called algorithmic complexity or Kolmogorov complexity [7] in a misleading way is not a measure of complexity) of almost every trajectory symbolically codified and since by definition a dynamical system is chaotic if it has positive dynamical entropy it follows that a complex dynamical system is not chaotic.
The generalization of Brudno Theorem to the quantum case recently performed by Fabio Benatti, Tyll Kruger, Markus Muller, Rainer Siegmund-Schultze and Arleta Szkola [8], [9] implies that also at the quantum level quantum chaos implies no quantum complexity.
This is a rather astonishing fact since quantum chaoticity and quantum complexity are very often erroneously used as synonymous.
In this paper we show this fact by considering the example consisting in the automorphisms of the noncommutative torus and classifying them among the (chaotic and non-chaotic) shallow ones and the non-chaotic complex ones.
III. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF STRINGS AND SEQUENCES OF CBITS

I will follow from here and beyond the notation of my Phd-thesis [10]; consequentially, given the binary alphabet \( \Sigma := \{0, 1\} \), I will denote by \( \Sigma^* \) the set of all the strings on \( \Sigma \) (i.e. the set of all the strings of cbits), by \( \Sigma^\infty \) the set of all the sequences on \( \Sigma \) (i.e. the set of all the sequences of cbits) and by \( CHAITIN - RANDOM(\Sigma^\infty) \) its subset consisting of all the Martin-Löf-Solovay-Chaitin random sequences of cbits.

I will furthermore denote strings by an upper arrow and sequences by an upper bar, so that I will talk about the string \( \vec{x} \in \Sigma^* \) and the sequence \( \bar{x} \in \Sigma^\infty \); \( |\vec{x}| \) will denote the length of the string \( \vec{x} \), \( x_n \) will denote the \( n \)th-digit of the string \( \vec{x} \) or of the sequence \( \bar{x} \) while \( \vec{x}_n \) will denote its prefix of length \( n \).

I will, finally, denote by \( <_l \) the lexicographical-ordering relation over \( \Sigma^* \) and by \( \text{string}(n) \) the \( n \)th-string in such an ordering.

Fixed once and for all a universal Chaitin computer \( U \), let us recall the following basic notions:

given a string \( \vec{x} \in \Sigma^* \) and a natural number \( n \in \mathbb{N} \):

Definition III.1
canonical program of \( \vec{x} \):

\[
\vec{x}^* := \min_{<_l} \{ \vec{y} \in \Sigma^* : U(\vec{y}) = \vec{x} \} \tag{3.1}
\]

Definition III.2
\( \vec{x} \) is \( n \)-compressible:

\[
|\vec{x}^*| \leq |\vec{x}| - n \tag{3.2}
\]

Definition III.3
\( \vec{x} \) is \( n \)-incompressible:

\[
|\vec{x}^*| > |\vec{x}| - n \tag{3.3}
\]

Definition III.4
halting time of the computation with input \( \vec{x} \):

\[
T(\vec{x}) := \begin{cases} \text{number of computational steps after which } U \text{ halts on input } \vec{x}, & \text{if } U(\vec{x}) = \downarrow \\ +\infty, & \text{otherwise} \end{cases} \tag{3.4}
\]

We have at last all the ingredients required to introduce the notion of logical depth as to strings.

Given a string \( \vec{x} \in \Sigma^* \) and two natural number \( s, t \in \mathbb{N} \):

Definition III.5
logical depth of \( \vec{x} \) at significance level \( s \):

\[
D_s(\vec{x}) := \min \{ T(\vec{y}) : U(\vec{y}) = \vec{x}, \vec{y} \text{ } s\text{-incompressible} \} \tag{3.5}
\]

Definition III.6
\( \vec{x} \) is \( t \)-deep at significance level \( s \):

\[
D_s(\vec{x}) > t \tag{3.6}
\]
Definition III.7

\( \vec{x} \) is t-shallow at significance level \( s \):

\[
D_s(\vec{x}) \leq t
\]  

(3.7)

I will denote the set of all the t-deep strings as \( t - DEEP(\Sigma^*) \) and the set of all the t-shallow strings as \( t - SHALLOW(\Sigma^*) \).

Exactly as it is impossible to give a sharp distinction among Chaitin-random and regular strings while it is possible to give a sharp distinction among Martin-Löf-Solovay-Chaitin-random and regular sequences, it is impossible to give a sharp distinction among deep and shallow strings while it is possible to give a sharp distinction among deep and shallow sequences.

Given a sequence \( \vec{x} \in \Sigma^\infty \):

Definition III.8

\( \vec{x} \) is strongly deep:

\[
\text{card}\{n \in \mathbb{N} : D_s(\vec{x}(n)) > f(n)\} < \aleph_0 \quad \forall s \in \mathbb{N}, \forall f \in \text{REC} - \text{MAP}(\mathbb{N}, \mathbb{N})
\]

(3.8)

where, following once more the notation adopted in [10], \( \text{REC} - \text{MAP}(\mathbb{N}, \mathbb{N}) \) denotes the set of all the (total) recursive functions over \( \mathbb{N} \).

To introduce a weaker notion of depth it is necessary to fix the notation as to reducibilities and degrees:

denoted the Turing reducibility by \( \leq_T \) and the polynomial time Turing reducibility by \( \leq_{PT} \) [11] let us recall that there is an intermediate constrained-reducibility among them: the one, called recursive time bound reducibility, in which the halting-time is constrained to be not necessarily a polynomial but a generic recursive function; since recursive time bound reducibility may be proved to be equivalent to truth-table reducibility (I demand to [12], [6] for its definition and for the proof of the equivalence) I will denote it by \( \leq_{tt} \).

A celebrated theorem proved by Peter Gacs in 1986 [13] states that every sequence is computable by a Martin Löf-Solovay-Chaitin-random sequence:

Theorem III.1

Gacs’ Theorem:

\[
\vec{x} \leq_T \vec{y} \quad \forall \vec{x} \in \Sigma^\infty, \forall \vec{y} \in \text{CHAITIN} - \text{RANDOM}(\Sigma^\infty)
\]

(3.9)

This is no more true, anyway, if one adds the constraint of recursive time bound, leading to the following:

Definition III.9

\( \vec{x} \) is weakly deep:

\[
\exists \vec{y} \in \text{CHAITIN} - \text{RANDOM}(\Sigma^\infty) : \neg(\vec{x} \leq_{tt} \vec{y})
\]

(3.10)

I will denote the set of all the strongly-deep binary sequences by \( STRONGLY - DEEP(\Sigma^\infty) \) and the set of all the weakly-deep binary sequences as \( WEAKLY - DEEP(\Sigma^\infty) \).

Shallowness is then once more defined as the opposite of depth:

Definition III.10

strongly-shallow sequences of cbits:

\[
STRONGLY - SHALLOW(\Sigma^\infty) := \Sigma^\infty - (STRONGLY - DEEP(\Sigma^\infty))
\]

(3.11)

Definition III.11

weakly-shallow sequences of cbits:

\[
WEAKLY - SHALLOW(\Sigma^\infty) := \Sigma^\infty - (WEAKLY - DEEP(\Sigma^\infty))
\]

(3.12)

Weakly-shallow sequences of cbits may also be characterized in the following useful way [1]:
Theorem III.2

Alternative characterization of weakly-shallow sequences of cbits:

\[ \bar{x} \in WEAKLY - SHALLOW(\Sigma^\infty) \iff \exists \mu_{\text{recursive}} : \bar{x} \in \mu - RANDOM(\Sigma^\infty) \]  

(3.13)

where, following once more the notation of [10], \( \mu - RANDOM(\Sigma^\infty) \) denotes the set of all the Martin-Löf random sequences w.r.t. the measure \( \mu \).

As to sequences of cbits, the considerations made in section II may be thoroughly formalized through the following:

Theorem III.3

Weak-shallowness of Martin Löf - Solovay - Chaitin random sequences:

\[ CHAITIN - RANDOM(\Sigma^\infty) \cap WEAKLY - DEEP(\Sigma^\infty) = \emptyset \]  

(3.14)

PROOF:

Since the Lebesgue measure \( \mu_{\text{Lebesgue}} \) is recursive and by definition:

\[ CHAITIN - RANDOM(\Sigma^\infty) = \mu_{\text{Lebesgue}} - RANDOM(\Sigma^\infty) \]  

(3.15)

the thesis immediately follows by the theorem [III.2]
IV. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF CLASSICAL DYNAMICAL SYSTEMS

Since much of the fashion about complexity is based on a spread confusion among different notions, starting from the basic difference among *plain Kolmogorov complexity* $K$ and *algorithmic information* $I$, much care has to be taken.

Let us start from the following notions by Brudno:

**Definition IV.1**

*Brudno algorithmic entropy of* $\bar{x} \in \Sigma^\infty$:

$$B(\bar{x}) := \lim_{n \to \infty} \frac{K(\bar{x}(n))}{n}$$

(4.1)

At this point one could think that considering the asymptotic rate of *algorithmic information* instead of *plain Kolmogorov complexity* would result in a different definition of the algorithmic entropy of a sequence.

That this is not the case is the content of the following:

**Theorem IV.1**

$$B(\bar{x}) = \lim_{n \to \infty} \frac{I(\bar{x}(n))}{n}$$

(4.2)

**Proof:**

It immediately follows by the fact that [14]:

$$|I(\bar{x}(n)) - K(\bar{x}(n))| \leq o(n)$$

(4.3)■

**Definition IV.2**

$\bar{x} \in \Sigma^\infty$ is *Brudno-random*:

$$B(\bar{x}) > 0$$

(4.4)

I will denote the set of all the Brudno random binary sequences by $BRUDNO(\Sigma^\infty)$.

One great source of confusion in a part of the literature arises from the ignorance of the following basic result proved by Brudno himself [3]:

**Theorem IV.2**

*Brudno randomness is weaker than Chaitin randomness*:

$$BRUDNO - RANDOM(\Sigma^\infty) \supset CHAITIN - RANDOM(\Sigma^\infty)$$

(4.5)

as we will see in the sequel of this section.

Following the analysis performed in [10] (to which I demand for further details) I will recall here some basic notion of Classical Ergodic Theory:

*given a classical probability space* $(X, \mu)$:

**Definition IV.3**

*endomorphism of* $(X, \mu)$:

$$T : HALTING(\mu) \to HALTING(\mu) \text{ surjective :}$$

$$\mu(A) = \mu(T^{-1}A) \quad \forall A \in HALTING(\mu)$$

(4.6)

where $HALTING(\mu)$ is the halting-set of the measure $\mu$, namely the $\sigma$-algebra of subsets of $X$ on which $\mu$ is defined.
Definition IV.4

classical dynamical system:
a triple \((X, \mu, T)\) such that:

- \((X, \mu)\) is a classical probability space
- \(T : HALTING(\mu) \to HALTING(\mu)\) is an endomorphism of \((X, \mu)\)

Given a classical dynamical system \((X, \mu, T)\):

Definition IV.5

\((X, \mu, T)\) is ergodic:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^k(B)) = \mu(A) \mu(B) \quad \forall A, B \in HALTING(\mu)
\]

Definition IV.6

\(n\)-letters alphabet:

\[
\Sigma_n := \{0, \ldots, n-1\}
\]

Clearly:

\[
\Sigma_2 = \Sigma
\]

Definition IV.7

finite measurable partition of \((X, \mu)\):

\[
A = \{A_0, \ldots, A_{n-1}\} \quad n \in \mathbb{N}:
\]

- \(A_i \in HALTING(\mu)\)
- \(i = 0, \ldots, n-1\)
- \(A_i \cap A_j = \emptyset \quad \forall i \neq j\)
- \(\mu(X - \bigcup_{i=0}^{n-1} A_i) = 0\)

I will denote the set of all the finite measurable partitions of \((X, \mu)\) by \(\mathcal{P}(X, \mu)\).

Given two partitions \(A = \{A_i\}_{i=0}^{n-1}, B = \{B_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)\):

Definition IV.8

A is a coarse-graining of \(B\) (\(A \preceq B\)):

- every atom of \(A\) is the union of atoms by \(B\)

Definition IV.9

coarsest refinement of \(A = \{A_i\}_{i=0}^{n-1}\) and \(B = \{B_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)\):

\[
A \vee B \in \mathcal{P}(X, \mu)
\]

\[
A \vee B := \{A_i \cap B_j \mid i = 0, \ldots, n-1 \text{ and } j = 0, \ldots, m-1\}
\]

Clearly \(\mathcal{P}(X, \mu)\) is closed both under coarsest refinements and under endomorphisms of \((X, \mu)\).

Let us observe that, beside its abstract, mathematical formalization, the definition IV.7 has a precise operational meaning.

Given the classical probability space \((X, \mu)\) let us suppose to make an experiment on the probabilistic universe it describes using an instrument whose distinguishing power is limited in that it is not able to distinguish events belonging to the same atom of a partition \(A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)\).
Consequentially the outcome of such an experiment will be a number:
\[ r \in \Sigma_n \]  
(4.12)
specifying the observed atom \( A_r \) in our coarse-grained observation of \((X, \mu)\).

I will call such an experiment an operational observation of \((X, \mu)\) through the partition \( A \).

Considered another partition \( B = \{B_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu) \) we have obviously that the operational observation of \((X, \mu)\) through the partition \( A \lor B \) is the conjunction of the two experiments consisting in the operational observations of \((X, \mu)\) through the partitions, respectively, \( A \) and \( B \).

Consequentially we may consistently call an operational observation of \((X, \mu)\) through the partition \( A \) more simply an \( A \)-experiment.

The experimental outcome of an operational observation of \((X, \mu)\) through the partition \( A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu) \) is a classical random variable having as distribution the stochastic vector \( (\mu(A_0), \ldots, \mu(A_{n-1})) \) whose classical probabilistic information, i.e. its Shannon entropy, I will call the entropy of the partition \( A \), according to the following:

**Definition IV.10**

entropy of \( A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu) \):
\[
H(A) := H\left(\begin{array}{c}
\mu(A_0) \\
\vdots \\
\mu(A_{n-1})
\end{array}\right)
\]  
(4.13)

It is fundamental, at this point, to observe that, given an experiment, one has to distinguish between two conceptually different concepts:

1. the uncertainty of the experiment, i.e. the amount of uncertainty on the outcome of the experiment before to realize it
2. the information of the experiment, i.e. the amount of information gained by the outcome of the experiment

As lucidly observed by Patrick Billingsley [15], the fact that in Classical Probabilistic Information Theory both these concepts are quantified by the Shannon entropy of the experiment is a consequence of the following:

**Theorem IV.3**
The soul of Classical Information Theory:

information gained = uncertainty removed  
(4.14)

Theorem [IV.3] applies, in particular, as to the partition-experiments we are discussing.

Let us now consider a classical dynamical system \( \text{CDS} := (X, \mu, T) \).

The T-invariance of \( \mu \) implies that the partitions \( A = \{A_i\}_{i=0}^{n-1} \) and \( T^{-1}A := \{T^{-1}A_i\}_{i=0}^{n-1} \) have equal probabilistic structure. Consequentially the \( A \)-experiment and the \( T^{-1}A \)-experiment are replicas, not necessarily independent, of the same experiment, made at successive times.

In the same way the \( \vee_{k=0}^{n-1} T^{-k}A \)-experiment is the compound experiment consisting of \( n \) repetitions \( A, T^{-1}A, \ldots, T^{-(n-1)}A \) of the experiment corresponding to \( A \in \mathcal{P}(X, \mu) \).

The amount of classical information per replication we obtain in this compound experiment is clearly:
\[
\frac{1}{n} H(\vee_{k=0}^{n-1} T^{-k}A)
\]

It may be proved [16] that when \( n \) grows this amount of classical information acquired per replication converges, so that the following quantity:
\[
h(A, T) := \lim_{n \to \infty} \frac{1}{n} H(\vee_{k=0}^{n-1} T^{-k}A)
\]  
(4.15)
exists.

In different words, we can say that \( h(A, T) \) gives the asymptotic rate of acquired classical information per replication of the \( A \)-experiment.

We can at last introduce the following fundamental notion originally proposed by Kolmogorov for \( K \)-systems and later extended by Yakov Sinai to arbitrary classical dynamical systems [17], [18], [19], [20], [16]:

\[
H_{\mu}(A) := \frac{1}{n} H(\vee_{k=0}^{n-1} T^{-k}A)
\]
Definition IV.11

Kolmogorov-Sinai entropy of CDS:

\[ h_{CDS} := \sup_{A \in \mathcal{P}(X,\mu)} h(A,T) \]  \hspace{1cm} (4.16)

By definition we have clearly that:

\[ h_{CDS} \geq 0 \]  \hspace{1cm} (4.17)

Definition IV.12

CDS is chaotic:

\[ h_{CDS} > 0 \]  \hspace{1cm} (4.18)

The definition [IV.12] shows explicitly that the concept of classical-chaos is an information-theoretic one: a classical dynamical system is chaotic if there is at least one experiment on the system that, no matter how many times we insist on repeating it, continues to give us classical information.

That such a meaning of classical chaoticity is equivalent to the more popular one as the sensible (i.e. exponential) dependence of dynamics from the initial conditions is a consequence of Pesin’s Theorem stating (under mild assumptions) the equality of the Kolmogorov-Sinai entropy and the sum of the positive Lyapunov exponents.

This inter-relation may be caught observing that:

• if the system is chaotic we know that there is an experiment whose repetition definitely continues to give information: such an information may be seen as the information on the initial condition that is necessary to furnish more and more with time if one want to keep the error on the prediction of the phase-point below a certain bound

• if the system is not chaotic the repetition of every experiment is useful only a finite number of times, after which every further repetition doesn’t furnish further information

Let us now consider the issue of symbolically translating the coarse-grained dynamics following the traditional way of proceeding described in the second section of [5]: given a number \( n \in \mathbb{N} : n \geq 2 \) let us introduce the following:

Definition IV.13

\( n \)-adic value:

the map \( v_n : \Sigma_n^\infty \mapsto [0,1] : \)

\[ v_n(\bar{x}) := \sum_{i=1}^{\infty} \frac{x_i}{n^i} \]  \hspace{1cm} (4.19)

the more usual notation:

\( (0.x_1 \cdots x_m \cdots)_{n} := v_n(\bar{x}) \bar{x} \in \Sigma_n^\infty \)  \hspace{1cm} (4.20)

and the following:

Definition IV.14

\( n \)-adic nonterminating natural positional representation:

the map \( r_n : [0,1] \mapsto \Sigma_n^\infty : \)

\[ r_n((0.x_1 \cdots x_i \cdots)_n) := \bar{x} \]  \hspace{1cm} (4.21)

with the nonterminating condition requiring that the numbers of the form \( (0.x_1 \cdots x_i(n-1))_{n} = (0.\cdots (x_i+1)\bar{0})_{n} \) are mapped into the sequence \( x_1 \cdots x_i(n-1) \).

Given \( n_1, n_2 \in \mathbb{N} : \min(n_1, n_2) \geq 2 \):
Definition IV.15

change of basis from $n_1$ to $n_2$:
the map $cb_{n_1,n_2} : \Sigma^\infty_{n_1} \to \Sigma^\infty_{n_2}$:

$$cb_{n_1,n_2}(\bar{x}) := r_{n_2}(v_{n_1}(\bar{x})) \quad (4.22)$$

It is important to remark that [6]:

Theorem IV.4

Basis-independence of randomness:

$$CHAITIN - RANDOM(\Sigma^\infty_{n_2}) = cb_{n_1,n_2}(CHAITIN - RANDOM(\Sigma^\infty_{n_1})) \quad \forall n_1, n_2 \in \mathbb{N} : \min(n_1, n_2) \geq 2 \quad (4.23)$$

Considered a partition $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

Definition IV.16

symbolic translator of CDS w.r.t. $A$:

$$\psi_A : X \to \Sigma^\infty_n : \quad \psi_A(x) := i : x \in A_i \quad (4.24)$$

In this way one associates to each point of $X$ the letter, in the alphabet having as many letters as the number of atoms of the considered partition, labeling the atom to which the point belongs.

Concatenating the letters corresponding to the phase-point at different times one can then codify $k \in \mathbb{N}$ steps of the dynamics:

Definition IV.17

$k$-point symbolic translator of CDS w.r.t. $A$:

$$\psi^{(k)}_A : X \to \Sigma^k_n : \quad \psi^{(k)}_A(x) := \prod_{j=0}^{k-1} \psi_A(T^j x) \quad (4.25)$$

and whole orbits:

Definition IV.18

orbit symbolic translator of CDS w.r.t. $A$:

$$\psi^{(\infty)}_A : X \to \Sigma^\infty_n : \quad \psi^{(\infty)}_A(x) := \prod_{j=0}^{\infty} \psi_A(T^j x) \quad (4.26)$$

The asymptotic rate of acquisition of plain Kolmogorov complexity of the binary sequence obtained translating symbolically the orbit generated by $x \in X$ through the partition $A \in \mathcal{P}(X, \mu)$ is clearly given by:

Definition IV.19

$$B(A, x) := B(cb_{\text{card}(A),2}(\psi^{(\infty)}_A(x))) \quad (4.27)$$

We saw in the definition IV.11 that the Kolmogorov-Sinai entropy was defined as $K(A, x)$ computed on the more probabilistically-informative $A$-experiment; in the same way the Brudno algorithmic entropy of $x$ is defined as the value of $B(A, x)$ computed on the more algorithmically-informative $A$-experiment:
Definition IV.20

Brudno algorithmic entropy of (the orbit starting from) $x$:

$$B_{\text{CDS}}(x) := \sup_{A \in \mathcal{P}(X,\mu)} B(cb_{\text{card}(A),2}(\psi_{A}^{(\infty)}(x)))$$  \hspace{1cm} (4.28)

Refering to [10] for further details, let us recall that, as it is natural for different approaches of studying a same object, the probabilistic approach and the algorithmic approach to Classical Information Theory are deeply linked:

the partial map $D_1 : \Sigma^* \mapsto \Sigma^*$ defined by:

$$D_1(\bar{x}) := \bar{x}^*$$  \hspace{1cm} (4.29)

is by construction a prefix-code of pure algorithmic nature, so that it would be very reasonable to think that it may be optimal only for some ad hoc probability distribution, i.e. that for a generic probability distribution $P$ the average code word length of $D_1$ w.r.t. $P$:

$$L_{D_1,P} = \sum_{\bar{x} \in \text{HALTING}(D_1)} P(\bar{x})I(\bar{x})$$  \hspace{1cm} (4.30)

won’t achieve the optimal bound, the Shannon information $H(P)$, stated by the corner stone of Classical Probabilistic Information, i.e. the following celebrated:

Theorem IV.5

Classical noiseless coding theorem:

$$H(P) \leq L_P \leq H(P) + 1$$  \hspace{1cm} (4.31)

(where $L_P$ is the minimal average code word length allowed by the distribution $P$)

Contrary, the deep link between the probabilistic-approach and the algorithmic-approach makes the miracle: under mild assumptions about the distribution $P$ the code $D_1$ is optimal as it is stated by the following:

Theorem IV.6

Link between Classical Probabilistic Information and Classical Algorithmic Information:

HP:

$P$ recursive classical probability distribution over $\Sigma^*$

TH:

$$\exists c_P \in \mathbb{R}_+: 0 \leq L_{D_1,P} - H(P) \leq c_P$$  \hspace{1cm} (4.32)

With an eye at the theorem [IV.1] it is then natural to expect that such a link between classical probabilistic information and classical algorithmic information generates a link between the asymptotic rate of acquisition of classical probabilistic information and the asymptotic rate of acquisition of classical algorithmic information of the coarse grained dynamics of CDS observed by repetitions of the experiments for which each of them is maximal.

Refering to [4] for further details such a reasoning, properly formalized, proves the following:

Theorem IV.7

Brudno theorem:

HP:

CDS ergodic
Let us now consider the **algorithmic approach to Classical Chaos Theory** strongly supported by Joseph Ford, whose objective is the characterization of the concept of chaoticity of a classical dynamical system as the algorithmic-randomness of its symbolically-translated trajectories.

To require such a condition for all the trajectories would be too restrictive since it is reasonable to allow a chaotic dynamical system to have a countable number of periodic orbits.

Let us then introduce the following two notions:

**Definition IV.21**

*CDS is strongly algorithmically-chaotic:*

\[
\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_{\text{card}(A), 2}(\psi_A^{(\infty)}(x)) \in \text{CHAITIN} - \text{RANDOM}(\Sigma^\infty)
\]  

(4.34)

**Definition IV.22**

*CDS is weak algorithmically-chaotic:*

\[
\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_{\text{card}(A), 2}(\psi_A^{(\infty)}(x)) \in \text{BRUDNO} - \text{RANDOM}(\Sigma^\infty)
\]  

(4.35)

The difference between the definition [IV.21] and the definition [IV.22] follows by the theorem [IV.2].

Clearly the theorem [IV.7] implies the following:

**Corollary IV.1**

\[
\text{chaoticity} = \text{weak algorithmic chaoticity} < \text{strong algorithmic chaoticity}
\]

that shows that the algorithmic approach to Classical Chaos Theory is equivalent to the usual one only in weak sense.

The plethora of wrong statements found in a part of the literature caused by the ignorance of corollary [IV.1] is anyway of little importance if compared with the complete misunderstanding of the difference existing among the concepts of *chaoticity* and *complexity* for classical dynamical systems; with this regards the analysis made in section II may be now thoroughly formalized introducing the following natural notions:

**Definition IV.23**

*CDS is strongly-complex:*

\[
\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_{\text{card}(A), 2}(\psi_A^{(\infty)}(x)) \in \text{STRONGLY - COMPLEX}(\Sigma^\infty)
\]  

(4.36)

**Definition IV.24**

*CDS is weakly complex:*

\[
\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_{\text{card}(A), 2}(\psi_A^{(\infty)}(x)) \in \text{WEAKLY - COMPLEX}(\Sigma^\infty)
\]  

(4.37)

One has that:

**Theorem IV.8**

**Weak-shallowness of chaotic dynamical systems:**

\[
\text{CDS chaotic } \Rightarrow \text{CDS weakly-shallow}
\]

**PROOF:**

The thesis immediately follows combining the theorem [III.3] with the definition [IV.23] and the definition [IV.24].
V. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF STRINGS AND SEQUENCES OF QUBITS

The idea that the physical complexity of a quantum object has to be measured in terms of a quantum analogue of Bennett’s notion of logical depth has been first proposed by Michael Nielsen [21], [22]. Unfortunately, beside giving some general remark about the properties he thinks such a notion should have, Nielsen have not given a mathematical definition of it.

The first step in this direction consists, in my opinion, in considering that, such as the notion of classical-logical-depth belongs to the framework of Classical Algorithmic Information Theory, the notion of quantum-logical-depth belongs to the framework of Quantum Algorithmic Information Theory [10].

One of the most debated issues in such a discipline, first discussed by its father Karl Svozil [23] and rediscovered later by the following literature [24], [25], [26], [27], [28], [10], is whether the programs of the involved universal quantum computers have to be strings of cbits or strings of qubits. As I have already noted in [10], anyway, it must be observed that, owing to the natural bijection among the computational basis $E^\star$ of the Hilbert space of qubits’ strings and $\Sigma^\star$, one can always assume that the input is a string of qubits while the issue, more precisely restated, is whether the input has (or not) to be constrained to belong to the computational basis.

So, denoted by $H_2 := \mathbb{C}^2$ the one-qubit’s Hilbert space (endowed with its orthonormal computational basis $E_2 := \{|i\rangle, i \in \Sigma\}$), denoted by $H_2^\otimes n := \bigotimes_{k=0}^n H_2$ the $n$-qubits’ Hilbert space, (endowed with its orthonormal computational basis $E_n := \{|x\rangle, x \in \Sigma^n\}$), denoted by $H_2^\otimes^\star := \bigoplus_{n=0}^\infty H_2^\otimes n$ the Hilbert space of qubits’ strings (endowed with its orthonormal computational basis $E_\star := \{|x\rangle, x \in \Sigma^\star\}$) and denoted by $H_2^\otimes^\infty := \bigotimes_{n\in\mathbb{N}} H_2$ the Hilbert space of qubits’ sequences (endowed with its orthonormal computational rigged-basis $E_\infty := \{|x\rangle, x \in \Sigma^\infty\}$), one simply assumes that, instead of being a classical Chaitin universal computer, $U$ is a quantum Chaitin universal computer, i.e. a universal quantum computer whose input, following Svozil’s original position on the mentioned issue, is constrained to belong to $E_\star$ and is such that, w.r.t. the natural bijection among $E_\star$ and $\Sigma^\star$, satisfies the usual Chaitin constraint of having prefix-free halting-set.

The definition of the logical depth of a string of qubits is then straightforward:

given a vector $|\psi\rangle \in H_2^\otimes^\star$ and a string $\vec{x} \in \Sigma^\star$:

**Definition V.1**

canonical program of $|\psi\rangle$:

$$|\psi\rangle^\star := \min_{\vec{y} \in \Sigma^\star} \{ U(|\vec{y}\rangle) = |\psi\rangle \} \quad (5.1)$$

**Definition V.2**

halting time of the computation with input $|\vec{x}\rangle$:

$$T(\vec{x}) := \begin{cases} \text{number of computational steps after which } U \text{ halts on input } \vec{x}, & \text{if } U(\vec{x}) = 1 \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.2)$$

**Definition V.3**

logical depth of $|\psi\rangle$ at significance level $s$:

$$D_s(|\psi\rangle) := \min\{ T(\vec{y}) : U(|\vec{y}\rangle) = |\psi\rangle, \vec{y} \text{-s-incompressible} \} \quad (5.3)$$

**Definition V.4**

\[ ^1 \text{as it should be obvious, the unusual locution } \text{rigged-basis} \text{ I am used to adopt is simply a shortcut to denote that such a "basis" has to be intended in the mathematical sense it assumes when } H_2^\otimes^\infty \text{ is considered as endowed with a suitable rigging, i.e. as part of a suitable rigged Hilbert space } S \subset H_2^\otimes^\infty \subset S' \text{ as described in [29], [31].} \]
\(|\psi>|\) is t-deep at significance level s:

\[ D_s(|\psi>|) > t \]  (5.4)

**Definition V.5**

\(|\psi>|\) is t-shallow at significance level s:

\[ D_s(|\psi>|) \leq t \]  (5.5)

I will denote the set of all the t-deep strings of qubits as \(t - DEEP(\mathcal{H}_2^{\otimes \infty})\).

Let us observe that a sharp distinction among depth and shallowness of qubits’ strings is impossible; this is nothing but a further confirmation of the fact, so many times shown and analyzed in [10], that almost all the concepts of Algorithmic Information Theory, both Classical and Quantum, have a clear, conceptually sharp meaning only when sequences are taken into account.

The great complication concerning sequences of qubits consists in that their mathematically-rigorous analysis requires to give up the simple language of Hilbert spaces passing to the more sophisticated language of noncommutative spaces; indeed, as extensively analyzed in [10] adopting the notion of noncommutative cardinality therein explicitly introduced, the fact that the correct noncommutative space of qubits’ sequences is the hyperfinite \(I_1\) factor \(R\):

**Definition V.6**

noncommutative space of qubits’ sequences:

\[
\Sigma_{NC}^{\infty} := \bigotimes_{n=0}^{\infty}(M_2(\mathbb{C}), \tau_{\text{unbiased}}) = R
\]  (5.7)

and not the noncommutative space \(\mathcal{B}(\mathcal{H}_2^{\otimes \infty})\) of all the bounded linear operators on \(\mathcal{H}_2^{\otimes \infty}\) (that could be still managed in the usual language of Hilbert spaces) is proved by the fact that, as it must be, \(\Sigma_{NC}^{\infty}\) has the continuum noncommutative-cardinality:

\[
\text{card}_{NC}(\Sigma_{NC}^{\infty}) = \aleph_1
\]  (5.8)

while \(\mathcal{B}(\mathcal{H}_2^{\otimes \infty})\) has only the countable noncommutative cardinality:

\[
\text{card}_{NC}(\Sigma_{NC}^{\infty}) = \aleph_0
\]  (5.9)

While the definition [III.8] of a strongly-deep sequence of cbits has no natural quantum analogue, the definition of a weakly-deep sequence of qubits is straightforward.

Even though the space of an algorithmically random sequences of qubits, whose characterization I demand to [10], let us observe that the equality between truth-table reducibility and recursive time bound reducibility existing as to Classical Computation may be naturally imposed to Quantum Computation in the following way:

given two arbitrary mathematical quantities x and y:

**Definition V.7**

\(x\) is quantum-truth-table reducible to \(y\):

\[ x \leq_Q y := x\text{ is U-computable from } y\text{ in bounded U-computable time} \]  (5.10)

Given a sequence of qubits \(\tilde{a} \in \Sigma_{NC}^{\infty}\):

---

2 Following Miklos Redei’s many remarks [31, 32] mentioned in [10], about how Von Neumann considered his classification of factors as a theory of noncommutative cardinalities though he never thought, as well as Redei, that the same \(\aleph\)’s symbolism of the commutative case could be adopted.

3 Following the notation of [10] the infinite tensor product of copies of a generic quantum probability space \((A, \omega)\) is defined as:

\[
\otimes_{n=1}^{\infty} (A, \omega) := \pi_{\otimes_{n=1}^{\infty} \omega}(\otimes_{n=1}^{\infty} A)^{\prime}\prime
\]  (5.6)

where \(\pi_{\phi}\) denotes the Gelfand-Naimark-Segal representation associated to a state \(\phi\).
Definition V.8

\( \bar{a} \) is weakly-deep:

\[ \exists \bar{b} \in \text{RANDOM}(\Sigma_{NC}^\infty) : \neg (\bar{a} \leq_{Q} \bar{b}) \] (5.11)

Denoted the set of all the weakly-deep sequences of qubits as \( \text{WEAKLY-DEEP}(\Sigma_{NC}^\infty) \):

Definition V.9

set of all the weakly-shallow sequences of qubits:

\[ \text{WEAKLY-SHALLOW}(\Sigma_{NC}^\infty) := \Sigma_{NC}^\infty \setminus (\text{WEAKLY-DEEP}(\Sigma_{NC}^\infty)) \] (5.12)

It is natural, at this point, to conjecture that an analogue of the theorem III.2 exists in Quantum Algorithmic Information Theory too:

Conjecture V.1

Alternative characterization of weakly-shallow sequences of qubits:

\[ \bar{a} \in \text{WEAKLY-SHALLOW}(\Sigma_{NC}^\infty) \iff \exists \omega \in S(\Sigma_{NC}^\infty) U - \text{computable} : \bar{a} \in \omega - \text{RANDOM}(\Sigma_{NC}^\infty) \] (5.13)

where \( \omega - \text{RANDOM}(\Sigma_{NC}^\infty) \) denotes the set of all the \( \omega \)-random sequences of qubits w.r.t. the state \( \omega \in S(\Sigma_{NC}^\infty) \) to be defined generalizing the definition of \( \text{RANDOM}(\Sigma_{NC}^\infty) \) to states different by \( \tau_{\text{unbiased}} \) along the lines indicated in [10] as to the definition of the laws of randomness \( L_{\text{RANDOMNESS}}(\Sigma_{NC}^\infty, \omega) \) of the noncommutative probability space \( (\Sigma_{NC}^\infty, \omega) \).

As to sequences of qubits, the considerations made in the section II may be thoroughly formalized, at the prize of assuming the conjecture V.1 as an hypothesis, through the following:

Theorem V.1

Weak-shallowness of random sequences of qubits:

HP:

Conjecture [V.1] holds

TH:

\[ \text{RANDOM}(\Sigma_{NC}^\infty) \cap \text{WEAKLY-DEEP}(\Sigma_{NC}^\infty) = \emptyset \] (5.14)

PROOF:

Since the unbiased state \( \tau_{\text{unbiased}} \) is certainly U-computable and by definition:

\[ \text{RANDOM}(\Sigma_{NC}^\infty) = \tau_{\text{unbiased}} - \text{RANDOM}(\Sigma_{NC}^\infty) \] (5.15)

the assumption of the conjecture [V.1] as an hypothesis immediately leads to the thesis ■
VI. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF QUANTUM DYNAMICAL SYSTEMS

As we have seen in section IV the Kolmogorov-Sinai entropy $h_{KS}(CDS)$ of a classical dynamical system $CDS := (X, \mu, T)$ has a clear physical information-theoretic meaning that we can express in the following way:

1. an experimenter is trying to obtain information about the dynamical evolution of CDS performing repeatedly on the system a given experiment $exp \in EXPERIMENTS$,

2. $h(exp, CDS)$ is the asymptotic rate of acquisition of classical information about the dynamics of CDS that he acquires replicating exp

3. $h_{KS}(CDS)$ is such an asymptotic rate, computed for the more informative possible experiment:

\[ h_{KS}(CDS) = \sup_{exp \in EXPERIMENTS} h(exp, CDS) \]  

(6.1)

Let us now pass to analyze quantum dynamical systems.

Given a quantum dynamical system $QDS$ the physical information-theoretical way of proceeding would consist in analyzing the same experimental situation in which an experimenter is trying to obtain information about the dynamical evolution of QDS performing repeatedly on the system a given experiment $exp \in EXPERIMENTS$:

1. to define $h(exp, QDS)$ as the asymptotic rate of acquisition of information about the dynamics of QDS that he acquires replicating the experiment exp

2. to define the dynamical entropy of QDS as such an asymptotic rate, computed for the more informative possible experiment:

\[ h_{d.e.}(QDS) = \sup_{exp \in EXPERIMENTS} h(exp, QDS) \]  

(6.2)

Definition VI.1

dynamical entropy of QDS:

$QDS$ is chaotic:

\[ h_{d.e.}(QDS) > 0 \]  

(6.3)

The irreducibility of Quantum Information Theory to Classical Information Theory, caused by the fact that the theorem IV.3 doesn’t extend to the quantum case owing to the existence of some non-accessible information about a quantum system (as implied by the Gr"{o}nwald-Lindblad-Holevo Theorem) and the consequent irreducibility of the qubit to the cbit [33], [10], would then naturally lead to the physical issue whether the information acquired by the experimenter is classical or quantum, i.e. if $h_{d.e.}(QDS)$ is a number of cbits or a number of qubits.

Such a physical approach to quantum dynamical entropy was performed first by G"{o}ran Lindblad [34] and later refined and extended by Robert Alicki and Mark Fannes resulting in the so called Alicki-Lindblad-Fannes entropy [35].

Many attempts to define a quantum analogue of the Kolmogorov-Sinai entropy pursued, instead, a different purely mathematical approach consisting in generalizing noncommutatively the mathematical machinery of partitions and coarsest-refinements underlying the definition IV.11 obtaining mathematical objects whose (eventual) physical meaning was investigated subsequently.

This was certainly the case as to the Connes-Narnhofer-Thirring entropy, the entropy of Sauvageot and Thouvenot and Voiculescu’s approximation entropy [36], [37], [38], [39].

As to the Connes-Narnhofer-Thirring entropy, in particular, the noncommutative analogue playing the role of the classical partitions are the so called Abelian models whose (eventual) physical meaning is rather obscure since, as it has been lucidly shown by Fabio Benatti in [40], they don’t correspond to physical experiments performed on the system, since even a projective-measurement (i.e. a measurement corresponding to a Projection Valued Measure) cannot, in general, provide an abelian model, owing to the fact that its reduction-formula corresponds to a decomposition of the state of the system if and only if the measured observable belongs to the centralizer of the state of the system.
In [41] we have shown how, in the framework of Deformation Quantization, the quantum dynamical entropy may be defined very naturally simply as the Kolmogorov-Sinai entropy of the quantum flow.

Such a definition of the quantum dynamical entropy, anyway, has the defect of being applicable only to quantum dynamical systems that can be obtained quantizing suitable classical Hamiltonian dynamical systems.

It may be worth observing, by the way, that the non-existence of an agreement into the scientific community as to the correct quantum analogue of the Kolmogorov-Sinai entropy and hence on the definition of quantum chaoticity shouldn’t surprise, such an agreement lacking even for the well more basic notion of quantum ergodicity. Zelditch’s quantum ergodicity [42] (more in the spirit of the original Von Neumann’s quantum ergodicity [43] to which it is not anyway clear if it reduces exactly as to quantum dynamical systems of the form \((A, \omega, \alpha)\) with \(\text{card}_{NC}(A) \leq N_0\) and \(\alpha \in \mathbb{IN}(A)\)) differing from Thirring’s quantum ergodicity [44] adopted both in [40] and in [35].

Returning, now, to the physical approach based on the definition \(\text{h_{d.e.(QDS)}}\) the mentioned issue whether the dynamical entropy \(h_{d.e.}\) is a measure of classical information or of quantum information (i.e. if it is a number of cbits or qubits) is of particular importance as soon as one tries to extend to the quantum domain Joseph Ford’s algorithmic approach to Chaos Theory seen in the section IV:

1. in the former case, in fact, one should define quantum algorithmic chaoticity by the requirement that almost all the trajectories, symbolically codified in a suitable way, belong to \(\text{BRUDNO}(\Sigma^\infty)\) for quantum weak algorithmic chaoticity and to \(\text{CHAITIN} - \text{RANDOM}(\Sigma^\infty)\) for quantum strong algorithmic chaoticity

2. in the latter case, instead, one should define quantum algorithmic chaoticity by the requirement that almost all the trajectories, symbolically codified in a suitable way, belong to \(\text{RANDOM}(\Sigma^\infty)\)

In any case one would then be tempted [2] to conjecture the existence of a Quantum Brudno Theorem stating that the quantum chaoticity is equivalent to quantum algorithmic chaoticity, a task partially performed by Fabio Benatti, Tyll Krüger, Markus Müller, Rainer Siegmund-Schultze and Arleta Szkola [8,9].

The mentioned issue whether the dynamical entropy \(h_{d.e.}(QDS)\) is a measure of classical information or of quantum information (i.e. if it is a number of cbits or qubits) is of great importance also as to the definition of a deep quantum dynamical system (i.e. a physically-complex quantum dynamical system):

1. in the former case, in fact, one should define a strongly (weakly) - deep quantum dynamical system as a quantum dynamical system such that almost all its trajectories, symbolically codified in a suitable way, belong to \(\text{STRONGLY} - \text{DEEP}(\Sigma^\infty)\)(\(\text{WEAKLY} - \text{DEEP}(\Sigma^\infty)\)).

2. in the latter case, instead, one should define a weakly-deep quantum dynamical system as a quantum dynamical system such that almost all its trajectories, symbolically codified in a suitable way, belong to \(\text{WEAKLY} - \text{DEEP}(\Sigma^\infty)\).

In any case, by the Quantum Brudno Theorem and by the theorem [11.3] or by the theorem [V.1] one would be almost certainly led to a quantum analogue of theorem [IV.8] stating that a chaotic quantum dynamical system is weakly-shallow, i.e. is not physically complex.

In particular, given a noncommutative space (i.e a Von Neumann algebra) \(A\), let us denote by by \(\text{Aut}_{\text{chaotic}}(A)\) the set of the chaotic automorphisms of \(A\), by \(\text{Aut}_{\text{complex}}(A)\) the set of the deep automorphisms of \(A\) and by \(\text{Aut}_{\text{shallow}}(A)\) the set of the shallow automorphisms of \(A\).

Then:

**Theorem VI.1**

Quantum chaoticity implies quantum shallowness:

\[
\text{Aut}_{\text{chaotic}}(A) \cap \text{Aut}_{\text{complex}}(A) = \emptyset
\]  

**PROOF:**

- Let us suppose that the quantum dynamical entropy is a measure of classical information. Then the thesis follows combining the Quantum Brudno Theorem with the theorem [IV.8]

- Let us suppose, instead that the quantum dynamical entropy is a measure of quantum information. Then the thesis follows combining the Quantum Brudno Theorem with the theorem [V.1]
VII. COMPLEX AND SHALLOW AUTOMORPHISMS OF THE NONCOMMUTATIVE TORUS

Let us consider the family of quantum dynamical systems of the form $QAT(\theta, C) := (T^2_\theta, \tau, \alpha_C)$ where:

$$T^2_\theta := \{ \sum_{r,s \in \mathbb{Z}} a_{r,s} u^r v^s : \{a_{r,s}\} \in S(\mathbb{Z}^2) \}$$  \hspace{1cm} (7.1)

where:

$$S(\mathbb{Z}^2) := \{ \{a_{r,s}\}_{r,s \in \mathbb{Z}} : \sup_{r,s \in \mathbb{Z}} (1 + r^2 + s^2)^k |a_{r,s}|^2 < \infty \ \forall k \in \mathbb{N} \}$$  \hspace{1cm} (7.2)

is the space of the rapidly decreasing double sequences and where $u$ and $v$:

$$vu = e^{2\pi i \theta} uv$$  \hspace{1cm} (7.3)

are the generators of the torus algebra:

$$A^2_\theta := C(T^2) \bowtie \mathbb{Z}$$  \hspace{1cm} (7.4)

defined as the cross-product of $C(T^2)$ and $\mathbb{Z}$ w.r.t. to the following automorphism of $C(T^2)$:

$$\alpha(f)(t) := f(t + \theta)$$  \hspace{1cm} (7.5)

$\tau$ is the tracial state over $T^2_\theta$:

$$\tau(\sum_{r,s \in \mathbb{Z}} a_{r,s} u^r v^s) := a_{00}$$  \hspace{1cm} (7.6)

while $\alpha_C$ is the automorphism of the noncommutative probability space $(T^2_\theta, \tau)$ of the form:

$$u \mapsto u^a v^b \hspace{0.5cm} v \mapsto u^c v^d \hspace{0.5cm} C := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$  \hspace{1cm} (7.7)

Since the equation (7.3) implies that:

$$A^2_\theta \equiv A^2_{\theta + n} \ \forall n \in \mathbb{Z}$$  \hspace{1cm} (7.8)

$$A^2_\theta \equiv A^2_{1-\theta}$$  \hspace{1cm} (7.9)

the inequivalent (i.e. non $\ast$-isomorphic) torus algebras are parametrized by $\theta \in [0, \frac{1}{2})$.

Furthermore since:

$$\theta \in \mathbb{N} \Rightarrow A^2_\theta \equiv C(T^2)$$  \hspace{1cm} (7.10)

$$\theta \in \mathbb{Q} \Rightarrow Z(A^2_\theta) = C(T^2)$$  \hspace{1cm} (7.11)

$$\theta \notin \mathbb{Q} \Rightarrow Z(A^2_\theta) = \{ \lambda I \mid \lambda \in \mathbb{C} \}$$  \hspace{1cm} (7.12)

we will assume from here and beyond that $\theta \in [0, \frac{1}{2}] \cap (\mathbb{R} - \mathbb{Q})$ in order to deal with a factor.

In the following we will assume that the quantum dynamical entropy of the automorphisms of the noncommutative torus is their Connes-Narnhofer-Thirring entropy or their Alicki-Fannes-Lindblad entropy:

$$h(\alpha_C) \in \{h_{CNT}(\alpha_C), h_{AFL}(\alpha_C)\} \ \forall C \in SL(2, \mathbb{Z})$$  \hspace{1cm} (7.13)

We remind that:

$$h_{CNT}(\alpha_C) := \sup_{N \in F(T^2_\theta)} h_+(\alpha_C, N)$$  \hspace{1cm} (7.14)
\[ h_r(\alpha_C, N) := \lim_{n \to \infty} H_r(N, \alpha_C(N) \cdots \alpha_C^{n-1}(N)) \]  
\( (F(T^2_\theta)) \) being the set the finite dimensional subalgebras of \( T^2_\theta \) and where the \( k \)-subalgebra entropy functional \( H_\theta(N_1, \cdots, N_k) \) is defined in a very technical way about which we demand to \([40]\) and, furthermore, we remind that:

\[ h_{AFL}(\alpha_C) := \sup_{X \in P(T^2_{\theta,0})} h(\tau, \alpha_C, X) \]  
\( (P(T^2_{\theta,0})) \) being the set of the operational partitions of the \( \alpha_C \)-invariant \( * \)-subalgebra \( T^2_{\theta,0} \) of \( T^2_\theta \), with \( S(\rho[X]) \) being the Von Neumann entropy of the correlation matrix \( \rho[X]_{ij} := \tau(x^*_ix_i) \) of the operational partition of unit \( X := (x_1, \cdots, x_k) \).

**Remark VII.1**

Since the notions of *quantum algorithmic chaoticity* and of *quantum deepness* involve a symbolical codification of the dynamics let us analyse how such a symbolical codification is performed.

Let us observe, first of all, that in the Alicki-Fannes-Lindblad approach at each step an observer performs on the system the measurement corresponding to an operational partition of unity \( X := \{x_1, \cdots, x_k\} \):

\[ \sum_{i=1}^k x^*_ix_i = \mathbb{1} \]  
\( (7.19) \)

Such a measurement will give one among the possible results labelled by \( \{1, \cdots, k\} \).

In this way to the coarse-grained observation of the quantum dynamics through the operational partition \( X \) it results naturally associated a sequence \( \bar{x} \in \Sigma^\infty_k \), i.e. the sequence \( \bar{x} := cb_{k,2}(\bar{x}') \in \{0,1\}^\infty \).

Let us observe, anyway, that since an operational partition is not necessarily orthogonal so that, in general, \( x^*_ix_j \neq 0 \) \( i \neq j \), the information obtained by the measurement associated to the operational partition \( X = (x_1, \cdots, x_k) \) is higher than the Shannon entropy:

\[ S(\bar{P}) = -\sum_{i=1}^k p_i \log_2 p_i \]  
\( (7.20) \)

where \( \bar{P} = (p_1, \cdots, p_k) \) is the probability vector for the possible outcomes.

Let us observe that the measurement associated to \( X \) corresponds to a coarse-grained observation of the underlying quantum probability space \( (T^2_\theta, \tau) \) as the \( k^2 \) dimensional quantum probability space \( (M_k(\mathbb{C}), \tau[X]) \) where \( \tau[X] \) is the normal state over \( M_k(\mathbb{C}) \) with density matrix \( \rho[X]_{ij} \), as induced by the coarse-graining completely positive unity preserving map \( \Gamma_X : M_k(\mathbb{C}) \mapsto T^2_\theta \):

\[ \Gamma_X([a_{ij}]) := \sum_{i,j=1}^k a_{ij}x^*_ix_j \]  
\( (7.21) \)

Let us now consider the coarse-grained observation of the dynamics.

We have, at this purpose, to recall some notion about operator algebras.

Given an \( n \in \mathbb{N} \) let us denote by \( \Sigma_{NC_n} := M_n(\mathbb{C}) \) the noncommutative alphabet of \( n \) letters and let us introduce the following:
Definition VII.1

sequences over $\Sigma_{NC}$:

$$\Sigma_{NC}^\infty := \otimes_{n=1}^\infty (\Sigma_{NC}, \tau_n) \quad (7.22)$$

where $\tau_n$ is the tracial state over $\Sigma_{NC}$.

Let us now recall that [46], [47]:

Theorem VII.1

$$\Sigma_{NC}^\infty_{n_1} = \Sigma_{NC}^\infty_{n_2} = R \quad \forall n_1, n_2 \in \mathbb{N}_+ \quad (7.23)$$

where we have identified $*$-isomorphic $W^*$-algebras.

Let us now observe that the orbit associated to an initial condition $a \in M_k(\mathbb{C})$ will be $\tilde{a} \in \otimes_{n=1}^\infty (M_k(\mathbb{C}), \tau[X])$.

We have clearly that:

$$\tau[X] = \tau_k \Rightarrow \otimes_{n=1}^\infty (M_k(\mathbb{C}), \tau[X]) = \Sigma_{NC}^\infty \quad (7.24)$$

where $\tau_k$ denotes the tracial state over $M_k(\mathbb{C})$ and where, following the notation of [10], [2], $\Sigma_{NC} := \Sigma_{NC;2}$ is the one qubit alphabet, i.e. the binary noncommutative alphabet.

An explicit computation of such dynamical entropies shows that [40], [35], [9]:

Theorem VII.2

$$h_{KS}(C) = h_{CNT}(\alpha_C) = h_{AFL}(\alpha_C) = \begin{cases} \log \lambda, & \text{if } Tr(C) > 2; \\ 0, & \text{if } Tr(C) \leq 2. \end{cases} \quad (7.25)$$

where $\lambda$ is the greatest eigenvalue of the matrix $C$.

Let us now assume that the quantum dynamical entropy of an automorphism of the noncommutative torus is its Connes-Narnhofer-Thirring entropy or its Alicki-Fannes-Lindblad entropy:

$$h_{d.e}(C) \in \{h_{CNT}(\alpha_C), h_{AFL}(\alpha_C)\} \quad \forall C \in SL(2, \mathbb{Z}) \quad (7.26)$$

Then obviously:

Corollary VII.1

$$Aut_{chaotic}(T^2_\theta) = \{\alpha_C : Tr(C) > 2\} \quad (7.27)$$

Furthermore:

Theorem VII.3

$$\{\alpha_C : Tr(C) > 2\} \cap Aut_{complex}(T^2_\theta) = \emptyset \quad (7.28)$$

PROOF:

Then thesis immediately follows combining the theorem VII.2 with the theorem VII.1.
Example VII.1

The classical and quantum Arnold cats are not complex:

Let us consider in particular the automorphism of the torus $T^2$ identified by the matrix:

$$C_{cat} := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

(7.29)

usually called in the literature the Arnold cat map.

Since $Tr(C_{cat}) = 3 > 2$ and $\lambda = \frac{3 + \sqrt{5}}{2}$ it follows that:

$$h_{KS}(C_{cat}) = h_{CNT}(\alpha_{C_{cat}}) = h_{AFL}(\alpha_{C_{cat}}) = \log \left( \frac{3 + \sqrt{5}}{2} \right)$$

(7.30)

By the theorem [VII.3] it follows not only that such a classical dynamical system is not complex but also that the quantum dynamical system $\alpha_{C_{cat}}$, usually called the quantum Arnold cat map, is not complex too.
### VIII. NOTATION

| Symbol | Description |
|--------|-------------|
| ∀      | for every (universal quantificator) |
| ∀ − µ − a.e. | for µ − almost every |
| ∃      | exists (existential quantificator) |
| ∃!     | exists and is unique |
| x = y  | x is equal to y |
| x := y | x is defined as y |
| x ≡ y  | x is equivalent to y |
| ¬p     | negation of p |
| Σ      | binary alphabet {0, 1} |
| Σn     | n-letters’ alphabet |
| Σ*     | strings on the alphabet Σn |
| Σ∞     | sequences on the alphabet Σn |
| ⃗x     | string |
| | length of the string ⃗x |
| | lexicographical ordering on Σ* |
| | n\textsuperscript{th} string in lexicographic order |
| | sequence |
| | concatenation operator |
| | n\textsuperscript{th} digit of the string ⃗x or of the sequence ¯x |
| | prefix of length n of the string ⃗x or of the sequence ¯x |
| | canonical string of the string ⃗x |
| | plain Kolmogorov complexity of the string ⃗x |
| | algorithmic information of the string ⃗x |
| | U halts on input ⃗x |
| | logical-depth of ⃗x at significance level s |
| | t-deep strings of cbits |
| | t-shallow strings of cbits |
| | (total) recursive functions over N |
| | Turing reducibility |
| | polynomial time Turing reducibility |
| | truth-table reducibility |
| | Martin L"of - Solovay - Chaitin random sequences of cbits |
| | halting set of the measure µ |

| Symbol | Description |
|--------|-------------|
| string(n) | string |
| |  |
| x_n    |  |
| ⃗x(n)  |  |
| ⃗x^*   |  |
| K(⃗x) |  |
| I(⃗x) |  |
| U(⃗x) ↓ |  |
| D_s(⃗x) |  |
| t − DEEP(Σ*) |  |
| t − SHALLOW(Σ*) |  |
| REC − MAP(N, N) |  |
| ≤_T |  |
| ≤_P |  |
| ≤_T |  |
| CHAITIN − RANDOM(Σ∞) | Martin L"of - Solovay - Chaitin random sequences of cbits |
| HALTING(µ) | halting set of the measure µ |
Lebesgue measure

Martin L"of random sequences of cbits w.r.t. $\mu$

Brudno algorithmic entropy of the sequence $\bar{x}$

Brudno random sequences of cbits

strongly-deep sequences of cbits

strongly-shallow sequences of cbits

weakly-deep sequences of cbits

weakly-shallow sequences of cbits

(finite, measurable) partitions of $(X, \mu)$

coarse-graining relation on partitions

Kolmogorov-Sinai entropy of CDS

symbolic translator w.r.t. A

k-point symbolic translator w.r.t. A

orbit symbolic translator w.r.t. A

change of basis from $n_1$ to $n_2$

Brudno algorithmic entropy of $x$'s orbit in CDS

Shannon entropy of the distribution $P$

average code-word length w.r.t. the code $D$ and the distribution $P$

minimal average code-word length w.r.t. the distribution $P$

one-qubit's Hilbert space

n-qubits' Hilbert space

computational basis of $H_2^n$

Hilbert space of qubits' strings

computational basis of $H_2^\#$

Hilbert space of qubits' sequences

computational rigged-basis of $H_2^\#$

bounded linear operators on $\mathcal{H}$

canonical program of $|\psi>$

logical depth of $|\psi>$ at significance level $s$

t-deep strings of qubits

t-shallow strings of qubits

states over the noncommutative space $A$

center of the noncommutative space $A$

automorphisms of the noncommutative space $A$

chaotic automorphisms of the noncommutative space $A$

complex automorphisms of the noncommutative space $A$

shallow automorphisms of the noncommutative space $A$

inner automorphisms of the noncommutative space $A$

noncommutative cardinality of the noncommutative space $A$

unbiased noncommutative probability distribution

noncommutative space of qubits' sequences

sequences over the noncommutative alphabet with n letters

hyperfinite $II_1$ factor

random sequences of qubits

quantum truth-table reducibility

weakly-deep sequences of qubits

weakly-shallow sequences of qubits

laws of randomness of $(A, \omega)$

random sequences of qubits w.r.t. $\omega$

dynamical entropy of the quantum dynamical system QDS

Connes-Narnhofer-Thirring entropy of the quantum dynamical system QDS

Alicki-Fannes-Lindlad entropy of the quantum dynamical system QDS

cross-product of A and G induced by $\alpha$

torus algebra

noncommutative torus

matrix of the Arnold cat map

id est

exempli gratia
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