PRIMITIVE DIVISORS, DYNAMICAL ZSIGMONDY SETS, AND VOJTA’S CONJECTURE

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ABSTRACT. A primitive prime divisor of an element $a_n$ of a sequence $(a_m)_{m \geq 0}$ is a prime $p$ that divides $a_n$, but does not divide $a_m$ for all $m < n$. The Zsigmondy set $\mathcal{Z}$ of the sequence is the set of $n$ such that $a_n$ has no primitive prime divisors. Let $f : X \to X$ be a self-morphism of a variety, let $D$ be an effective divisor on $X$, and let $P \in X$, all defined over $\overline{\mathbb{Q}}$. We consider the Zsigmondy set $\mathcal{Z}(X,f,P,D)$ of the sequence defined by the arithmetic intersection of the $f$-orbit of $P$ with $D$. Under various assumptions on $X$, $f$, $D$, and $P$, we use Vojta’s conjecture with truncated counting function to prove that the set of points $f^n(P)$ with $n \in \mathcal{Z}(X,f,P,D)$ is not Zariski dense in $X$.

INTRODUCTION

Let $\mathcal{A} = (a_n)_{n \geq 0}$ be a sequence of integers. A prime $p$ is called a primitive prime divisor for $a_n$ if $p \mid a_n$, but $p \nmid a_m$ for all $0 \leq m < n$ with $a_m \neq 0$.\(^1\) The existence of primitive prime divisors in various types of sequences has been much studied, both for its intrinsic interest and for numerous applications.

The Zsigmondy set of $\mathcal{A}$ is

$$\mathcal{Z}(\mathcal{A}) = \{n \geq 0 : a_n \text{ has no primitive prime divisors}\}.$$ 

A typical result, due to Zsigmondy [31], is that if $u > v > 0$ are coprime integers, then the Zsigmondy set of the sequence $(u^n - v^n)_{n \geq 0}$ is finite, and indeed $\mathcal{Z}(u^n - v^n) \subseteq \{1, 2, 6\}$.

\(^1\)The classical definition does not include the requirement that $a_m \neq 0$, but including this condition allows for cleaner statements of theorems. Note that if some $a_m = 0$, then $a_n$ is divisible by every prime, so if we didn’t exclude such $a_m$, then $a_n$ with $n > m$ would never have a primitive prime divisor.
In this note we investigate primitive prime divisors in sequences associated to dynamical systems. In this setting, we will use a strong version of a conjecture of Paul Vojta (Conjecture 8) to show that certain Zsigmondy sets are small. We start by stating a special case of our main theorem that does not require any technical definitions or notation. In the statement, we assume that points in \( \mathbb{P}^N(\mathbb{Q}) \) are always written in normalized form, i.e., as 
\[
[x_0, \ldots, x_N] \text{ with } x_0, \ldots, x_N \in \mathbb{Z} \text{ and } \gcd(x_0, \ldots, x_N) = 1.
\]

**Theorem 1.** Assume that Vojta’s conjecture with truncated counting function [27, Conjecture 22.5] is true for \( \mathbb{P}^N \). Let \( f : \mathbb{P}^N \to \mathbb{P}^N \) be a morphism of degree \( d \geq 3 \) defined over \( \mathbb{Q} \). Let \( F(X_0, \ldots, X_N) \in \mathbb{Z}[X_0, \ldots, X_N] \) be a non-constant homogeneous polynomial such that the locus \( F = 0 \) in \( \mathbb{P}^N \) is a reduced normal-crossings hypersurface. Assume further that
\[
\deg(F) > (N + 1) \left(1 + \frac{1}{d-2}\right).
\]
Let \( P \in \mathbb{P}^N(\mathbb{Q}) \), and let
\[
\mathcal{P} = (F(f^n(P)))_{n \geq 0} \subset \mathbb{Z}
\]
be the sequence of values of \( F \) on the points in the \( f \)-orbit of \( P \). Then the set
\[
\{f^n(P) : n \in \mathbb{Z}(\mathcal{P})\}
\]
is not Zariski dense in \( \mathbb{P}^N \).

Primitive divisors and Zsigmondy sets for dynamically defined sequences have been studied by many mathematicians. In Section 1 we briefly recount some of this history, which started with the 19th-century work of Bang [3] and Zsigmondy [31]. We mention in particular a recent paper by Gratton, Nguyen, and Tucker [8] in which they prove a strengthened version of Theorem 1 for \( \mathbb{P}^1 \) under the assumption that the ABC conjecture of Masser and Oesterlé is true. Their paper, as well as a paper of Yasufuku [30] in which he uses Vojta’s conjecture to study integrality of points in orbits, were the inspirations for the present note, since Vojta’s conjecture (with truncated counting function) is a natural higher-dimensional analogue of the ABC conjecture.

In order to describe our general version of Theorem 1, we reformulate it geometrically. The zero-locus of the polynomial \( F \) defines an ample effective divisor \( D \), and for most primes \( p \), the condition that \( p \) divide \( F(f^n(P)) \) is equivalent to the condition that when we reduce modulo \( p \), we have
\[
\widetilde{f^n(P)} \mod p \in \widetilde{D} \mod p,
\]
i.e., the reduction of the point $f^n(P)$ modulo $p$ lies on the reduction of the divisor $D$ modulo $p$. So $p$ is a primitive divisor of $F(f^n(P))$ if and only if

$$\text{ord}_p(f^n(P)) \in \text{ord}_p(D)$$

and

$$\text{ord}_p(f^m(P)) \notin \text{ord}_p(D) \quad \text{for all } 1 \leq m < n.$$  

With this formulation, we can define Zsigmondy sets relative to effective divisors for orbits of points on arbitrary varieties defined over arbitrary global fields, although there will be a small amount of ambiguity due to the necessity of choosing an integral model. However, different models lead to Zsigmondy sets that differ in finitely many elements, so our finiteness statements will be independent of the chosen model.

We now state a special case of our main result, using notation that is described in detail in Sections 2, 3, and 4. For a generalization and proof of Theorem 2, see Theorem 12 and Remark 13 in Section 5.

**Theorem 2.** Let $(X_K, f_K, H)$ be a polarized dynamical system of degree $d \geq 3$ with rank $\text{NS}(X_K) = 1$, let $P \in X_K(K)$ with $h_{f,H}(P) > 0$, and let $D_K$ be an ample effective reduced normal crossings divisor on $X_K$ satisfying

$$\deg(D_K) > \frac{d-1}{d-2} \cdot \deg(-\kappa_X). \quad (2)$$

Further, assume that Vojta’s conjecture (Conjecture 8) is true for the variety $X_K$ and the divisor $D_K$. Then

$$\left\{ f^n(P) : n \in \mathbb{Z}(O_f(P), D) \right\} \quad (3)$$

is not Zariski dense in $X_K$. More precisely, except for finitely many points, the set (3) is contained in a proper closed subvariety $Y \subset X$ that does not depend on $P$.

**Remark 3.** We comment briefly on the conditions (1) and (2) which require $\deg(F)$ and $\deg(D_K)$ to be sufficiently large. In some sense, these conditions are necessary, since for example the theorem is false on $\mathbb{P}^1$ with $\deg(F) = 2$ if we take $F(X,Y) = XY$ and $f([[X,Y]]) = [X^d, Y^d]$. The authors of [8] eliminate this problem on $\mathbb{P}^1$ by replacing $F$ with $F \circ f^j$ for some fixed $j$, thereby increasing the degree. This works provided that the function $F \circ f^j$ acquires enough roots, which will be true unless $F$ and $f$ are of a very special form. In our higher dimensional setting, we can do the same thing, but we do not have a good description of the “bad” $(f,F)$ pairs. Thus in Theorem 1 we can
replace (1) with the condition that there exists some \( j \geq 0 \) such that \( \{ F \circ f^j = 0 \} \) is a reduced normal crossings hypersurface and \( F \) satisfies

\[
\deg F > \frac{d-1}{d-2} \cdot \frac{N+1}{d^j}.
\]

However, we do not know a nice characterization of the pairs \( (F, f) \) that ensure the existence of such a \( j \).

In the final two sections we consider related problems and generalizations. Section 6 discusses the problem of primitive divisors in algebraic groups, where iteration of a map may be replaced by the powering map to create sequences that are denser, and hence less likely to have primitive divisors. In this situation, even assuming Vojta’s conjecture, we explain why we are unable to prove anything significant about the Zsigmondy set. Finally, in Section 7, we replace points and divisors, which have dimension 1 and codimension 1, respectively, with arbitrary subvarieties that have (arithmetically) complementary codimensions. Again, even assuming Vojta’s conjecture, we have no theorems, but we take the opportunity to describe a general framework and to raise some questions.

1. A Brief History of Primitive Prime Divisors

As noted earlier, the history of primitive divisors started in the 19th century with work of Bang [3] and Zsigmondy [31], who studied the Zsigmondy set of \( u^n - v^n \). The Bang–Zsigmondy sequences are examples of divisibility sequences, that is, sequences \( (a_n)_{n \geq 1} \) such that \( a_n \mid a_{nd} \) for all \( n, d \geq 1 \). They are associated to the multiplicative group \( \mathbb{G}_m(\mathbb{Q}) \).

Lucas sequences are generalizations attached to twisted multiplicative groups. The study of primitive divisors in Lucas sequence was completed in 2001 by Bilu, Hanrot, and Voutier [5], who proved that a Lucas sequence has primitive divisors for each term with \( n > 30 \). The existence of primitive divisors has many applications, for example in the original proof of Wedderburn’s theorem.

Elliptic divisibility sequences (EDS) are divisibility sequences with \( \mathbb{G}_m \) replaced by an elliptic curve. The properties of EDS were first studied formally by Ward [29]. The author [19] gave an ineffective proof of the existence of primitive divisors in EDS, and there are many papers giving quantitative and/or effective estimates; see for example [11, 13, 24, 28]. Poonen [18] used EDS primitive divisors to resolve certain cases of Hilbert’s 10th problem in number fields.

More generally, one can define divisibility sequences associated to a non-torsion point \( P \) in any (commutative) algebraic group \( G \). In
Section 6 we discuss this general construction, which was originally described in [21, Section 6].

One can also define divisibility sequences in a dynamical setting. For example, let \( f(x) \in \mathbb{Q}(x) \) be a rational function, let \( \alpha \in \mathbb{Q} \) be a wandering point, and let \( \beta \in \mathbb{Q} \) be a periodic point. Then the sequence of numerators of \( f^n(\alpha) - \beta \) satisfies a strong divisibility property, and subject to appropriate conditions on \( f \), Ingram and the author [12] proved that the Zsigmondy set of this sequence is finite. This was applied by Voloch and the author [23] to prove a local-global criterion for dynamics on \( \mathbb{P}^1 \).

The situation regarding dynamical primitive divisors becomes much more difficult if the target point \( \beta \) is a wandering point, since the resulting sequence has no obvious divisibility properties. This was left as an open question in [12], and even with the assumption that the ABC conjecture is true, it required considerable ingenuity by Gratton, Nguyen, and Tucker [8] to prove the existence of primitive divisors for dynamical systems on \( \mathbb{P}^1 \) with wandering target point. The present paper replaces \( \mathbb{P}^1 \) with an arbitrary variety and replaces the wandering target point with a wandering target divisor satisfying some geometric conditions. And in order to prove anything in this higher dimensional setting, we require a strong form of Vojta’s conjecture in place of the ABC conjecture.

2. Primitive Prime Divisors and Zsigmondy Sets

We begin by setting some notation that will remain fixed throughout this article. We start with our basic objects of study.

- \( K \) a number field or a characteristic 0 one-dimensional function field.
- \( X_K \) a smooth projective variety defined over \( K \).
- \( D_K \) an effective divisor on \( X_K \).

In order to define primitive prime divisors and Zsigmondy sets, we first choose an integral model for \( X_K \).

- \( R \) a Dedekind domain in \( K \) whose fraction field is \( K \).
- \( X_R \) a scheme that is smooth and proper over Spec \( R \) and with generic fiber \( X_K \), i.e., satisfying \( X_R \times_R K = X_K \).
- \( D_R \) the closure of \( D_K \) in \( X_R \).

We call \( X_R \) a model of \( X_K \). We note that \( X_K \) always has a model, since \( X_K \) is smooth and projective over \( K \), so we can take \( R \) to be the ring of \( S \)-integers in \( K \) for a sufficiently large finite set of places \( S \). Of course, the variety \( X_K \) has many different models, but as we will see, the associated Zsigmondy sets are equal up to finite alteration.
Since $X_R$ is proper over $\text{Spec } R$, a point $Q \in X_K(K)$ defines a point $Q_R \in X_R(R)$, i.e., the point $Q$ induces a morphism 
$$\sigma_Q : \text{Spec } R \rightarrow X_R.$$ 
We write $|D_K|$ for the support of $D_K$. If $Q \notin |D_K|$, then $\sigma_Q^*(D_R)$ is an effective divisor on $\text{Spec } R$, which is simply a formal sum of points in $\text{Spec } R$, 
$$\sigma_Q^*(D_R) = \sum_{p \in \text{Spec } R} n_p(Q,D)p.$$ 

**Definition.** The (global) arithmetic intersection ideal of $Q$ and $D$ is the ideal 
$$(Q \cdot D)_R = \prod_{p \in \text{Spec } R} p^{n_p(Q,D)}.$$  

**Remark 4.** In the notation of Theorem 1, if we let $D_K = \{F = 0\} \in \text{Div}(\mathbb{P}_K^N)$, then the ideal $(Q \cdot D)_Z$ is the ideal in $Z$ generated by the integer $F(Q)$.

**Definition.** Let $Q = \{Q_n\}_{n \geq 0}$ be a sequence of points in $X(K) \setminus |D_K|$. We say that $p \in \text{Spec } R$ is a primitive (prime) divisor of $Q_n$ relative to $D$ if 
$$p \mid (Q_n \cdot D)_R \quad \text{and} \quad p \nmid (Q_m \cdot D)_R \quad \text{for all } 0 \leq m < n.$$ 

The Zsigmondy set of $Q$ relative to $D$ is 
$$\mathcal{Z}(Q,D) = \{n \geq 0 : Q_n \text{ has no primitive divisors relative to } D\}.$$ 

The definition of primitive divisors and the Zsigmondy set clearly depend on our choice of the model $X_R$. We now show that $\mathcal{Z}(Q,D)$ is well-defined up to alteration by finitely many elements. This shows in particular that the statement of Theorem 12, which is our main result, is independent of the choice of a model for $X_K$.

**Proposition 5.** Let $\mathcal{Z} = \mathcal{Z}(Q,D)$ and $\mathcal{Z}' = \mathcal{Z}'(Q,D)$ be Zsigmondy sets of $Q$ relative to $D$ that are defined using models $X_R$ and $X'_R$ of $X_K$, say associated to rings $R$ and $R'$, respectively. Then the set difference 
$$\mathcal{Z} \setminus (\mathcal{Z} \cap \mathcal{Z}') \text{ is finite.}$$ 

**Proof.** Each point in $\text{Spec } R$ corresponds to a distinct valuation of $K$, and all but finitely many (equivalence classes) of valuations of $K$ come from $\text{Spec } R$. The same is true of $\text{Spec } R'$, so we can find a model $X''_R$ for $X_K$ over a ring $R''$ such that 
$$X_R'' \hookrightarrow X_R \quad \text{and} \quad X_R'' \hookrightarrow X'_R.$$ 

$$\text{Spec}(R'') \hookrightarrow \text{Spec}(R) \quad \text{and} \quad \text{Spec}(R'') \hookrightarrow \text{Spec}(R').$$
By definition, for each non-negative integer \( n \notin \mathbb{Z} \), we can find a prime \( p \in \text{Spec} R \) that is a primitive divisor of \( Q_n \). Since each prime in \( \text{Spec} R \) is a primitive divisor for at most one \( Q_n \), and since \( \text{Spec}(R) \setminus \text{Spec}(R') \) is finite, we can find an \( N \) so that for every \( n \geq N \) with \( n \notin \mathbb{Z} \), there is a prime \( p \in \text{Spec}(R') \) that is a primitive divisor of \( Q_n \).

Similarly, we can find an \( N' \) so that for every \( n \geq N' \) with \( n \notin \mathbb{Z}' \), there is a prime \( p \in \text{Spec}(R') \) that is a primitive divisor of \( Q_n \). So if we let \( \mathbb{Z}'' \) be the Zsigmondy set of \( Q \) relative to \( D \) using the model \( X'' \), and if we set \( N'' = \max\{N, N'\} \), then we have proven that

\[
\{n \in \mathbb{Z}'' : n \geq N''\} = \{n \in \mathbb{Z}' : n \geq N''\} = \{n \in \mathbb{Z} : n \geq N''\}.
\]

This gives (5), which completes the proof of Proposition 5. □

3. Height Functions and Vojta’s Conjecture

In this section we discuss height functions and set the notation required to state Vojta’s conjecture.

- \( M_K \): a complete set of inequivalent absolute values on \( K \). We write \( M_K^0 \), respectively \( M_K^\infty \), for the set of non-archimedean, respectively archimedean, absolute values in \( M_K \).
- \( N_p \): the number of elements in the residue field associated to a place \( p \in M_K^0 \).
- \( h_D \): a Weil height on \( X_K \) for the divisor \( D \) relative to the field \( K \).
- \( \lambda_{D,v} \): a \( v \)-adic local height on \( (X \setminus |D|)(K_v) \) for the divisor \( D \), relative to the field \( K_v \). If \( D \) is effective and \( v \in M_K^0 \), we assume that \( \lambda_{D,v} \) is chosen to be a non-negative function.

**Remark 6.** Having fixed a model \( X_R \), we can use intersection theory to specify a local height \( \lambda_{D,p} \) for \( p \in \text{Spec} R \subset M_K^0 \) via the formula

\[
\lambda_{D,p}(Q) = \text{ord}_p(Q \cdot D)_R \cdot \log N_p,
\]

where we recall that \( (Q \cdot D)_R \) is the arithmetic intersection ideal (4). (Cf. [15, Chapter 11, Section 5].) We will assume henceforth that for \( p \in \text{Spec} R \), the local height \( \lambda_{D,p} \) has been chosen in this way.

We further assume that our absolute values and local and global heights are chosen to satisfy

\[
h_D(P) = \sum_{v \in M_K} \lambda_{D,v}(P) \quad \text{for all } P \in X_K(K) \setminus |D|.
\]

For basic material on height functions and their normalizations, see for example [10, §§B.1–B.3] or [15, Chapters 3 and 4].
Definition. An effective divisor $D \in \text{Div}(X)$ is a normal crossings divisor if $D = \sum_{i=1}^{r} n_i D_i$, where the $D_i$ are irreducible subvarieties and the variety $\bigcup_{i=1}^{r} |D_i|$ has normal crossings. If all of the $n_i = 1$, then $D$ is a reduced divisor. We write $D_{\text{red}} = \sum_{i=1}^{r} D_i$ for the reduced divisor associated to $D$.

Vojta’s original conjectures [26] used arithmetic analogues of the proximity and counting functions of classical Nevanlina theory. Later, Vojta took stronger results in Nevanlinna theory involving truncated counting functions and transplanted them into an arithmetic setting. This led to arithmetic conjectures that are natural generalizations of the ABC-conjecture of Masser and Oesterlé [17] and of Szpiro’s conjecture [25].

Definition. Let $S \subset M_K$ be a finite set of places containing $M_K^\infty$. The (arithmetic) truncated counting function is

$$N^{(1)}_S(D, P) = \sum_{p \notin S} \min \{\lambda_{D,p}(P), \log N_p\} = \sum_{p \notin S, n_p(P,D) \geq 1} \log N_p.$$ 

It is well-defined for $P \in X(K) \setminus |D|$.

Remark 7. It is clear from the definition that

$$N^{(1)}_S(D, P) = N^{(1)}_S(D_{\text{red}}, P),$$

since $\lambda_{D,p}(P) = 0$ if and only if $\lambda_{D_{\text{red}},p}(P) = 0$.

Conjecture 8 (Vojta’s conjecture with truncated counting function [27, Conjecture 22.5]). Let $S \subset M_K$ be a finite set of places containing $M_K^\infty$, let $D$ be a normal crossings divisor on $X$, let $\kappa_X$ be a canonical divisor on $X$, and let $H$ be an ample line bundle on $X$. All heights are taken relative to the number field $K$.

For all $\epsilon > 0$ there is a proper Zariski closed set

$$Y = Y(X, D, H, \epsilon) \subset X$$

such that for all $C \geq 0$, the inequality

$$N^{(1)}_S(D, P) \geq h_{\kappa_X+D}(P) - \epsilon h_H(P) - C$$

holds for all but finitely many $P \in (X \setminus Y)(K)$.

4. Dynamical Systems

We study the dynamical system given by iteration of the $K$-morphism

$$f_K : X_K \to X_K.$$
We denote the $f_K$-orbit of a point $P \in X_K(K)$ by

$$O_f(P) = \{ P, f_K(P), f_K^2(P), f_K^3(P), \ldots \}.$$  

For the remainder of this article, we assume without further comment that the Dedekind domain $R$ and model $X_R$ have been chosen so that $f_K$ extends to an $R$-scheme morphism

$$f_R : X_R \rightarrow X_R.$$  

We call $(X_R, f_R)$ a model for $(X_K, f_K)$. Having normalized our local heights using intersection theory on $X_R$ via formula (6), we see that they behave functorially,

$$\lambda_{D, p}(f_K(P)) = \lambda_{f_R^* D, p}(P). \quad (7)$$

This follows from the projection formula

$$(f_K(Q), D)_R = (Q, f_K^* D)_R,$$

which is the arithmetic analogue of [9, Appendix A, Formula A4].

Using the projection formula (7) in the definition of the truncated counting function yields

$$N_S^{(1)}(D, f_K(P)) = N_S^{(1)}(f_K^* D, P).$$

More generally, for any $j \geq 0$ we have

$$N_S^{(1)}(D, f_K^j(P)) = N_S^{(1)}((f_K^j)^* D)_{\text{red}, P}. \quad (8)$$

**Definition.** A polarized dynamical system (defined over $K$) is a triple $(X_K, f_K, H)$ consisting of a smooth projective variety $X_K$, a $K$-morphism $f_K : X_K \rightarrow X_K$, and an ample divisor $H \in \text{Div}(X_K) \otimes \mathbb{R}$ with the property that

$$f_K^* H \sim dH \quad \text{for some } d > 1.$$  

Here $\sim$ indicates linear equivalence in $\text{Pic}(X) \otimes \mathbb{R}$. We call $d$ the degree of the polarized dynamical system.

**Example 9.** Let $f_K : \mathbb{P}^N_K \rightarrow \mathbb{P}^N_K$ be a morphism of degree at least two, and let $H$ be any ample divisor on $\mathbb{P}^N_K$. Then $(\mathbb{P}^N_K, f_K, H)$ is a polarized dynamical system of degree equal to $\text{deg}(f)$.

The following well-known result summarizes the theory of canonical heights for polarized dynamical systems.

**Theorem 10.** Let $(X_K, f_K, H)$ be a polarized dynamical system of degree $d$. Then for all $P \in X(K)$, the limit

$$\hat{h}_{f, H}(P) = \lim_{n \rightarrow \infty} d^{-n} h_H(f_K^n(P))$$

exists and has the following properties:
(a) $\hat{h}_{f,H}(P) = h_H(P) + O(1)$, where the $O(1)$ depends on $X_K$, $f_K$, and the choice of height function $h_H$, but is independent of $P$.

(b) $\hat{h}_{f,H}(f_K(P)) = d h_{f,H}(P)$.

(c) If $K$ is a number field, then $\hat{h}_{f,H}(P) = 0$ if and only if $P$ is preperiodic, i.e., has finite forward orbit.

Proof. See for example [7] or [22, §3.4]. We note that (c) is also true over function fields if $f_K$ is not isotrivial, but the proof is much more difficult; see [2, 4].

Definition. If the Néron-Severi group $NS(X_K)$ of the variety $X_K$ has rank 1, then we fix a degree map

$$\deg: NS(X_K) \otimes \mathbb{R} \sim \mathbb{R}$$

with the property that $\deg(H) > 0$ for ample divisors $H$. The degree map is well-defined up to multiplication by a positive constant.

Height functions have many functorial properties; see for example [10, Theorem B.3.2]. We will need the following special case of one of these properties.

**Proposition 11.** Assume that $NS(X_K)$ has rank 1, let $H \in \text{Div}(X_K)$ be an ample divisor, and let $D \in \text{Div}(X_K)$ be an arbitrary divisor. Then for every $\epsilon > 0$ we have

$$|(\deg H)h_D - (\deg D)h_H| \leq \epsilon h_H + O(1).$$

Proof. The assumption on $NS(X_K)$ implies that the divisor class of $(\deg H)D$ is algebraically equivalent to the divisor class of $(\deg D)H$ in $NS(X) \otimes \mathbb{R}$, since they both have the same degree. We let $E = (\deg H)D - (\deg D)H$ and apply [10, Theorem B.3.2](f), which says that since $E$ is algebraically equivalent to 0, we have $h_E = o(h_H)$. □

5. **Primitive Divisors in Dynamical Systems and Vojta’s Conjecture**

In this section we use Vojta’s conjecture to prove our main theorem, which says that certain dynamically defined Zsigmondy sets are small.

**Theorem 12.** Let $(X_K, f_K, H)$ be a polarized dynamical system of degree $d$, and let $D_K$ be an ample effective divisor on $X_K$. Make the following assumptions:

- $NS(X_K)$ has rank 1.
- $P \in X(K)$ is a point with $\hat{h}_{f,H}(P) > 0$. 


There is an integer \( j \geq 0 \) such that the divisor 
\[ \Delta_j := ((f^j)^* D)^\text{red} \]
is a normal crossings divisor whose degree satisfies
\[ \frac{\deg \Delta_j}{d^j} > \frac{\deg D}{d - 1} + \frac{\deg(-\kappa_X)}{d^j}. \tag{9} \]

Vojta’s conjecture (Conjecture 8) is true for the variety \( X \) and the divisor \( ((f^j)^* D)^\text{red} \). We let \( Y_{f,D,j} \subseteq X \) denote the exceptional set in Vojta’s conjecture. Then the Zsigmondy set of the \( f \)-orbit of \( P \) relative to \( D \) has the property that
\[ \{ f^n(P) : n \in \mathbb{Z}(\mathcal{O}_f(P), D) \} \subset Y_{f,D} \cup \{ \text{finite set} \}. \]

Remark 13. Taking \( j = 0 \) in Theorem 12, condition (9) reduces to the assumption that \( D \) itself is a reduced normal crossings divisor satisfying
\[ \deg D > \frac{d - 1}{d - 2} \cdot \deg(-\kappa_X). \]
This is the version of Theorem 12 that we stated in the introduction.

Remark 14. If the divisor \((f^j)^* D\) in Theorem 12 is already reduced, then in \( \text{NS}(X) \otimes \mathbb{R} \) we have
\[ \Delta_j = ((f^j)^* D)^\text{red} = (f^j)^* D \equiv \frac{\deg D}{\deg H} (f^j)^* H \equiv \frac{\deg D}{\deg H} d^j H \equiv d^j D. \]
After a little bit of algebra, the degree condition (9) then becomes
\[ \deg(D) > \frac{d - 1}{d - 2} \cdot \frac{\deg(-\kappa_X)}{d^j}. \]
Hence in Theorem 12 it suffices to assume that \((f^j)^* D\) is reduced for some
\[ j > \log_d \left( \frac{d - 1}{d - 2} \cdot \frac{\deg(-\kappa_X)}{\deg(D)} \right). \]

Remark 15. It would be interesting to weaken some of the hypotheses in Theorem 12. For example, what happens if \( \text{NS}(X_K) \) has rank larger than 1? Or we could take a dynamical system that is not quite polarized. For example, suppose that \( f_K \) is an automorphism, and suppose that there are nef divisors \( D^+ \) and \( D^- \) that are eigendivisors for \( f_K \) and \( f_K^{-1} \), respectively, such that \( D^+ + D^- \) is ample. This is the situation for certain K3 surfaces [20]. Or we could take an affine morphism \( f : \mathbb{A}^N \to \mathbb{A}^N \) that does not extend to a morphism of \( \mathbb{P}^N \). In
In this setting, we mention that if $f$ is a regular automorphism, then there is a good theory of canonical heights [14, 16] that might be useful.

**Proof of Theorem 12.** To ease notation, we write $f$ for $f_K$, and having chosen an ample divisor $H$, we normalize our degree function to satisfy $\deg(H) = 1$. We fix a model $X_R$ for $X_K$, and when applying Vojta’s conjecture, we take $S$ to be the finite set of places

$$S = M_K \setminus \text{Spec}(R).$$

We observe that a prime $p \in \text{Spec} R$ is a primitive divisor for $f^n(P)$ relative to $D$ if and only if

$$\text{ord}_p(f^n(P), D) > 0,$$

and

$$\text{ord}_p(f^m(P), D) = 0 \text{ for all } 0 \leq m < n.$$

In view of the definition

$$\lambda_{D,p}(Q) = \text{ord}_p(Q \cdot D) \cdot \log N_p$$

of our local height functions (Remark 6), we see that $f^n(P)$ will have a primitive divisor relative to $D$ if the following sum is strictly positive:

$$B_n := \sum_{p \in \text{Spec} R} \lambda_{D,p}(f^n P) \quad \text{for all } \lambda_{D,p}(f^m P) = 0 \text{ for all } m < n$$

We take a point $f^n(P) \notin Y_{f,D,j}$ which is not one of the finitely many exceptions in Vojta’s conjecture (Conjecture 8) and compute

$$B_n \geq \sum_{p \in \text{Spec} R} \log N_p$$

$$\geq \sum_{p \in \text{Spec} R} \log N_p - \sum_{p \in \text{Spec} R} \log N_p$$

$$= N_S^{(1)}(D, f^n P) - \sum_{p \in \text{Spec} R} \log N_p$$

$$\geq N_S^{(1)}(D, f^n P) - \sum_{m=0}^{n-1} \sum_{p \in \text{Spec} R} \lambda_{D,p}(f^m P)$$

$$\geq N_S^{(1)}(D, f^n P) - \sum_{m=0}^{n-1} h_D(f^m P) - O(1)$$

since $D$ is effective, $\lambda_{D,v}$ is bounded below for all $v \in M_K$. 


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\[ N_S^{(1)} \left( ((f^j)^*)D^\text{red}, f^{n-j}(P) \right) - \sum_{m=0}^{n-1} h_D(f^mP) - O(1) \quad \text{from (8)}, \]

\[ N_S^{(1)}(\Delta_j, f^{n-j}(P)) - \sum_{m=0}^{n-1} h_D(f^mP) - O(1) \]

using our notation \( \Delta_j = ((f^j)^*)D^\text{red} \),

\[ \geq h_{\kappa_X + \Delta_j} (f^{n-j}(P)) - \epsilon h_H(f^{n-j}(P)) - \sum_{m=0}^{n-1} h_D(f^mP) - O(1) \]

from Vojta’s conjecture (Conjecture 8),

\[ \geq (\deg(\kappa_X + \Delta_j)) - 2\epsilon h_H(f^{n-j}(P)) \]

\[ - \sum_{m=0}^{n-1}(\deg D + \epsilon) h_H(f^mP) - O(1) \quad \text{from Proposition 11}, \]

\[ \geq (\deg(\kappa_X + \Delta_j)) - 2\epsilon \hat{h}_{f,H}(f^{n-j}(P)) \]

\[ - \sum_{m=0}^{n-1}(\deg D + \epsilon) \hat{h}_{f,H}(f^mP) - O(n) \quad \text{from Theorem 10(a)}, \]

\[ = (\deg(\kappa_X + \Delta_j)) - 2\epsilon d^{n-j} \hat{h}_{f,H}(P) \]

\[ - \sum_{m=0}^{n-1}(\deg D + \epsilon) d^m \hat{h}_{f,H}(P) - O(n) \quad \text{from Theorem 10(b)}, \]

\[ \geq \left( \frac{\deg \kappa_X + \deg \Delta_j - 2\epsilon}{d^j} - \frac{\deg D + \epsilon}{d - 1} \right) d^n \hat{h}_{f,H}(P) - O(n). \]

Using the assumed bound (9) on \( \deg \Delta_j \), we see that if we choose \( \epsilon \) sufficiently small, then there is a constant \( \kappa > 0 \) that does not depend on \( n \) such that

\[ B_n \geq \kappa d^n \hat{h}_{f,H}(P) - O(n). \]

Since we have also assumed that \( \hat{h}_{f,H}(P) > 0 \), this proves that the sum \( B_n \) defined by (10) is positive for all sufficiently large \( n \) such that \( f^n(P) \notin Y_{f,D,j} \) and \( f^n(P) \) is not one of the finitely many exceptions in Vojta’s conjecture. It follows that all such \( f^n(P) \) have a primitive divisor relative to \( D \), and hence they are not in the Zsigmondy set \( Z(O_f(P), D) \). This completes the proof that there are only finitely many \( n \in Z(O_f(P), D) \) such that \( f^n(P) \notin Y_{f,D,j}. \) \( \square \)
Classical Zsigmondy problems are associated with algebraic groups. Thus let $G_K$ be a (commutative) algebraic group, let $G_R$ be a model for $G_K$ over $R$, and let $\gamma \in G_K(K)$. In this setting, one can study the Zsigmondy set associated to the sequence of ideals $a_n = a_n(\gamma) \subset R$ defined by the property that $a_n$ is the smallest ideal with the property that $\gamma^n \equiv 1 \pmod{a_n}$ in $G_R(R)$.

The sequence $(a_n)$ is a divisibility sequence, i.e., if $m \mid n$, then $a_m \mid a_n$. See [21, Section 6], and especially [21, Proposition 8], for the general set-up. If $G_K = \mathbb{G}_m$, then we obtain classical divisibility sequences such as $u^n - v^n$, while if $G_K$ is an elliptic curve, then we obtain classical elliptic divisibility sequences [29]. In both cases, Zsigmondy sets are finite.

The situation for $G_K = \mathbb{G}_m^2$ is very different. In this case we obtain sequences such as $a_n = \gcd(u^n - 1, v^n - 1)$, for which Ailon and Rudnick conjecture that there are infinitely many $n$ such that $a_n = 1$, so in particular the Zsigmondy set is infinite [1]. In the opposite direction, Bugeaud, Corvaja, and Zannier [6] use deep methods to prove that $\lim_{n \to \infty} n^{-1} \log a_n = 0$; and the author [21] showed that Vojta’s conjecture implies a far-reaching generalization of the results in [6].

More generally, if $\dim(G_K) \geq 2$, then excluding certain obvious degenerate situations, one expects that the Zsigmondy set of the sequence $(a_n)$ should be infinite. Intuitively, this should be true because $R$-valued points in $G_R(R)$ are subschemes of dimension 1, so two such points are unlikely to intersect in $G_R$ if $\dim(G_R) \geq 3$. (Note that $\dim(G_R) = \dim(G_K) + 1$, since Spec($R$) has dimension 1.)

The underlying reason that the sequence $(a_n)$ is a divisibility sequence is because the target point, namely the identity element of $G$, is fixed by the $n^{th}$-power maps that are being applied to $\gamma$. If we instead choose a target point $\beta \in G$ that does not have finite order and define $b_n$ as the smallest ideal such that $\gamma^n \equiv \beta \pmod{b_n}$ in $G_R(R)$, then $(b_n)$ will not be a divisibility sequence, and even in the case that $\dim(G) = 1$ it is a difficult question to determine if the Zsigmondy set of $(b_n)$ is finite.

But in general, if we want to associate to the sequence of points $\gamma^n$ a Zsigmondy set that has a Zariski non-density property, then we should pair the point $\gamma$, which has dimension 1 in $G_R$, with a divisor $D$. 
which has codimension 1. This is what we did in Section 2. We thus fix an ample effective divisor $D_K \in \text{Div}(G_K)$, we let $\Gamma = (\gamma^n)_{n \geq 0}$, and we consider the Zsigmondy set $Z(\Gamma, D)$. Taking $G = \mathbb{G}_m$ and $D_K = (1)$, or taking $G$ an elliptic curve and $D_K = (O)$, we recover the classical divisibility sequences whose Zsigmondy sets are finite. For higher dimensional $G$, if the divisor $D_K$ is not fixed by the powering map in the group, it seems a difficult question to characterize $Z(\Gamma, D)$, even if one assumes Vojta’s conjecture. The difficulty comes from two facts. First, the sequence of ideals $(\gamma^n, D)_R$ is not a divisibility sequence, so when checking if $\gamma^n$ has a primitive prime divisor, we must exclude all primes dividing terms with $m < n$. (If it were a divisibility sequence, we would only need to consider the terms with $m \mid n$.)

Second, the terms in the sequence $\log N_{K/Q}(\gamma^n, D)_R$ grow only polynomially in $n$, so the magnitude of the terms with $m < n$ overwhelms the $n$th term. This is in contrast to the dynamical setting, where the sequence $\log N_{K/Q}(f^n(P), D)_R$ grows exponentially, which allows the $n$th term to overwhelm the terms with $m < n$. We thus raise the following question, which has an affirmative answer for one-dimensional groups and $D_K = (1)$, but for which there is currently no good evidence pro or con in higher dimensions.

**Question 16.** Let $G_K$ be a commutative algebraic group, let $D_K$ be an ample effective normal crossings divisor, let $\gamma \in G_K(K)$ be a wandering point, and let $\Gamma = (\gamma^n)_{n \geq 0}$. Is it true that the set

$$\{\gamma^n : n \notin Z(\Gamma, D)\}$$

is not Zariski dense in $G_K$?

7. **Primitive Divisors for Cycles Having Complementary Codimension**

As noted in Section 6, one reason that we expect Zsigmondy sets $Z(Q, D)$ to be small is because the $R$-points $Q_n$ have dimension 1 in $X_R$ and the divisor $D_R$ has codimension 1 in $X_R$, so as the size of $Q_n$ grows, the intersection ideal of $Q_n$ and $D_R$ should grow. This suggests taking other sequences of subschemes of $X_R$ having complementary codimensions.

**Definition.** Let $K$, $X_K$, $R$, and $X_R$ be as defined in Section 2. Let $\xi_K$ and $\eta_K$ be equidimensional subvarieties of $X_K$ such that

$$\xi_K \cap \eta_K = \emptyset \quad \text{and} \quad \text{codim}(\xi_K) + \text{codim}(\eta_K) = \dim(X_K) + 1. \quad (11)$$
Let $\xi_R$ and $\eta_R$ be the closures in $X_R$ of $\xi_K$ and $\eta_K$, respectively. For $p \in \text{Spec}(R)$, we set

$$i_p(\xi, \eta) = 1 \text{ if } (\tilde{\xi}_R \mod p) \cap (\tilde{\eta}_R \mod p) \neq \emptyset,$$

and otherwise we set $i_p(\xi, \eta) = 0$. In other words, $i_p(\xi, \eta)$ is the indicator function for whether $\xi$ and $\eta$ intersect when they are reduced modulo $p$.

**Definition.** Let $X = (\xi_n)_{n \geq 0}$ and $N = (\eta_n)_{n \geq 0}$ be sequences of equidimensional subvarieties of $X_K$ satisfying (11). We say that $p \in \text{Spec}(R)$ is a primitive (prime) divisor of the pair $(\xi_n, \eta_n)$ if

$$i_p(\xi_n, \eta_n) = 1 \quad \text{and} \quad i_p(\xi_m, \eta_m) = 0 \quad \text{for all } 0 \leq m < n.$$

Then the Zsigmondy set of the sequences $X$ and $N$ is

$$\mathcal{Z}(X, N) = \{ n \geq 0 : \text{the pair } (\xi_n, \eta_n) \text{ has no primitive divisors} \}.$$

We make a specific conjecture in the dynamical setting. This conjecture can certainly be generalized, but it is not entirely clear how to characterize the “degenerate” cases, so we are content with the following modest conjecture.

**Conjecture 17.** Let $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a morphism of degree $d \geq 2$. Let $\xi$ and $\eta$ be smooth irreducible subvarieties of $\mathbb{P}^N_K$ with the following properties:

- $\text{codim}(\xi) + \text{codim}(\eta) = N + 1$.
- $f^n(\xi) \cap \eta = \emptyset$ for all $n \geq 0$.
- For any infinite sequence of integers $n_i \geq 0$, the union of the varieties $f^{n_i}(\xi)$ is Zariski dense in $\mathbb{P}^N$.

Let $\mathcal{X} = (f^n(\xi))$ be the sequence consisting of the $f$-orbit of $\xi$, and let $\mathcal{N} = (\eta)$ be the constant sequence. Then the Zsigmondy set $\mathcal{Z}(\mathcal{X}, \mathcal{N})$ is finite.

**Remark 18.** In the setting of Conjecture 17, one might consider a second morphism $g : \mathbb{P}^N_K \to \mathbb{P}^N_K$ and replace the constant sequence $\mathcal{N} = (\eta)$ with the moving sequence $\mathcal{N} = (g^n(\eta))$. Then the question of whether $\mathcal{Z}(\mathcal{X}, \mathcal{N})$ is finite will depend in some complicated way on how $f$ and $g$ interact dynamically. This should lead to all sorts of interesting questions.

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