A VARIANT OF THE PRIME NUMBER THEOREM

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Abstract. Let \( \Lambda(n) \) be the von Mangoldt function, and let \( [t] \) be the integral part of real number \( t \). In this note, we prove that for any \( \varepsilon > 0 \) the asymptotic formula

\[
\sum_{n \leq x} \Lambda\left(\left[\frac{x}{n}\right]\right) = x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + O_e(x^{9/19+\varepsilon}) \quad (x \to \infty)
\]

holds. This improves a recent result of Bordellès, which requires \( \frac{97}{203} \) in place of \( \frac{9}{19} \).

1. Introduction

The prime number theorem is a basic result in number theory and has many applications. Denoting by \( \Lambda(n) \) the von Mangoldt function, the prime number theorem states, in strong form, as follows: there is a constant \( c > 0 \) such that for \( x \to \infty \), we have

\[
\sum_{n \leq x} \Lambda(n) = x + O(x \exp\{-c(\log x)^{3/5}(\log \log x)^{-1/5}\})
\]

and

\[
\sum_{n \leq x} \Lambda(n) = x + O_e(x^{1/2+\varepsilon}) \iff \text{Riemann Hypothesis},
\]

where \( \varepsilon \) is an arbitrarily small positive constant. Clearly it is also interesting to study the distribution of prime numbers in different sequences of integers such as the arithmetic progressions, the Beatty sequence \( ([\alpha n + \beta])_{n \geq 1} \), the Piatetski-Shapiro sequence \( ([n^c])_{n \geq 1} \), etc., where \( [t] \) denotes the integral part of the real number. For example, Banks and Shparlinski [2, Corollary 5.6] proved the following result: Let \( \alpha \) and \( \beta \) be fixed real numbers with \( \alpha > 0 \), irrational and of finite type, then there is a positive constant \( c = c(\alpha, \beta) \) such that

\[
\sum_{n \leq x} \Lambda([\alpha n + \beta]) = x + O(x \exp\{-c(\log x)^{3/5}(\log \log x)^{-1/5}\})
\]
as \( x \to \infty \). About works related to the Piatetski-Shapiro prime number theorem, we refer the reader to see [12, 8, 13]. On the other hand, Bordellès-Dai-Heyman-Pan-Shparlinski [4] established an asymptotic formula of

\[
S_f(x) := \sum_{n \leq x} f\left(\left[\frac{x}{n}\right]\right),
\]

under some simple assumptions of \( f \). Subsequently, Wu [16] and Zhai [17] improved their results independently. In particular, applying [16, Theorem 1.2(i)] or [17, Theorem 1] to the von Mangoldt function \( \Lambda(n) \), we have

\[
S_{\Lambda}(x) = x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + O(x^{1/2+\varepsilon})
\]

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for \( x \geq 1 \). With the help of the Vaughan identity and the method of one-dimensional exponential sum, Ma and Wu \([11]\) broke the \( \frac{1}{2} \)-barrier:

\[
S_\Lambda(x) = x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + O(x^{35/71 + \varepsilon}) \quad (x \geq 1).
\]

Very recently Bordellès \([3\text{, Corollary 1.3}]\) sharpened the exponent \( \frac{35}{71} \) to \( \frac{97}{203} \) by using a result of Baker on 2-dimensional exponential sums \([1\text{, Theorem 6}]\). The aim of this short note is to propose a better exponent by establishing an estimate on 3-dimensional exponential sums (see Proposition 3.1 below).

**Theorem 1.** For any \( \varepsilon > 0 \), we have

\[
S_\Lambda(x) = x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + O(x^{9/19 + \varepsilon}) \quad (x \to \infty).
\]

For comparison, we have \( \frac{97}{203} \approx 0.4778 \) and \( \frac{9}{19} \approx 0.4736 \). In fact, Bordellès established a more general result (see \([3\text{, Theorem 1.1}]\)):

\[
S_\Lambda(x) = x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + O(x^{\frac{14(\kappa+1)}{29\kappa-\lambda+30} + \varepsilon}) \quad (x \to \infty),
\]

where \((\kappa, \lambda)\) is an exponent pair satisfying \( \kappa \leq \frac{1}{6} \) and \( \lambda^2 + \lambda + 3 - \kappa(5 + 9\kappa - \lambda) > 0 \). The exponent \( \frac{97}{203} \) comes from the choice of \((\kappa, \lambda) = (\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon)\) — Bourgain’s new exponent pair \([5\text{, Theorem 6}]\). We note that under the exponent pair hypothesis (i.e. \((\varepsilon, \frac{1}{2} + \varepsilon)\) is an exponent pair, see \([7]\)), Bordellès’ (1.3) only gives the exponent \( \frac{28}{59} \approx 0.4745 \), which is larger than our constant \( \frac{9}{19} \approx 0.4736 \).

Some related works on the quantity (1.1) can be found in \([10, 15]\).

## 2. Preliminary Lemmas

In this section, we shall cite three lemmas, which will be needed in the next section. The first one is \([6\text{, Lemma 1}]\).

**Lemma 2.1.** Let \( \alpha \in \mathbb{R}^* \) and \( \beta \in \mathbb{R}^* \). For \( M \geq 1 \), \( N \geq 1 \) and \( \Delta > 0 \), define

\[
\mathcal{D}(M, N; \Delta) := \left\{ (m, \tilde{m}, n, \tilde{n}) \in \mathbb{N}^4 : m, \tilde{m} \sim M; n, \tilde{n} \sim N; \left| \left( \frac{m}{\tilde{m}} \right)^\alpha - \left( \frac{n}{\tilde{n}} \right)^\beta \right| \leq \Delta \right\},
\]

where \( m \sim M \) means that \( M < m \leq 2M \). Then we have

\[
\mathcal{D}(M, N) \ll_{\alpha, \beta} MN \log(2MN) + \Delta(MN)^2
\]

uniformly for \( M \geq 1 \), \( N \geq 1 \) and \( \Delta > 0 \), where the implied constant depends on \( \alpha \) and \( \beta \).

The second one is the Vaughan identity \([13\text{, formula (3)}]\).

**Lemma 2.2.** There are six real arithmetical functions \( \alpha_k(n) \) verifying \( |\alpha_k(n)| \ll n^\varepsilon \) for \((n \geq 1, 1 \leq k \leq 6)\) such that, for all \( D \geq 100 \) and any arithmetical function \( g \), we have

\[
\sum_{D < d < 2D} \Lambda(d)g(d) = S_1 + S_2 + S_3 + S_4,
\]
Lemma 2.3. Let $x > 1$ and $H > 1$, we have

$$S_1 := \sum_{m \leq D^{1/3}} \alpha_1(m) \sum_{D < mn \leq 2D} g(mn),$$

$$S_2 := \sum_{m \leq D^{1/3}} \alpha_2(m) \sum_{D < mn \leq 2D} g(mn) \log n,$$

$$S_3 := \sum_{D^{1/3} < m, n \leq D^{2/3}} \sum_{D < mn \leq 2D} \alpha_3(m) \alpha_4(n) g(mn),$$

$$S_4 := \sum_{D^{1/3} < m, n \leq D^{2/3}} \sum_{D < mn \leq 2D} \alpha_5(m) \alpha_6(n) g(mn).$$

The sums $S_1$ and $S_2$ are called as type I, $S_3$ and $S_4$ are called as type II.

Theorem 2.3. Let $\psi(t) := \{t\} - \frac{1}{2}$, where $\{t\}$ means the fractional part of real number $t$. For $x > 1$ and $H > 1$, we have

$$\psi(x) = - \sum_{1 < |h| \leq H} \Phi\left(\frac{h}{H + 1}\right) e\left(\frac{hx}{H + 1}\right) + R_H(x),$$

where $e(t) := e^{2\pi i t}$, $\Phi(t) := \pi t (1 - |t|) \cot(\pi t) + |t|$ and the error term $R_H(x)$ satisfies

$$|R_H(x)| \leq \frac{1}{2H + 2} \sum_{0 < |h| \leq H} \left(1 - \frac{|h|}{H + 1}\right) e(hx).$$

3. Multiple exponential sums

Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta \in \mathbb{R}$ be some constants. For $X > 0$, $H > 1$, $M \geq 1$ and $N \geq 1$, define

$$(3.1) \quad S_\delta = S_\delta(H, M, N) := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_{h, m} b_n e\left(X \frac{M^\beta N^\gamma}{H^\alpha} \frac{h^\alpha}{m^\beta n^\gamma + \delta}\right),$$

where the $a_{h, m}$ and $b_n$ are complex numbers such that $|a_{h, m}| < 1$ and $|b_n| < 1$. When $\delta = 0$, this sum has been studied by Heath-Brown [8] and Fouvry-Iwaniec [6]. The aim of this section is to prove an estimate of $S_\delta$ by adapting and refining Heath-Brown’s approach (see also [9]). The following proposition will play a key role in the proof of Theorem 4.

Proposition 3.1. Under the previous notation, for any $\varepsilon > 0$ we have

$$(3.2) \quad S_\delta \ll (XHMN)^{1/2} + (HM)^{1/2} N + HM N^{1/2} + X^{-1/2} H M N) X^\varepsilon,$$

$$(3.3) \quad S_\delta \ll ((X^\kappa H^{2+\kappa} M^{2+\kappa} N^{1+\kappa+\lambda})^{1/(2+2\varepsilon)} + H M N^{1/2}$$

$$+(HM)^{1/2} N + X^{-1/2} H M N) X^\varepsilon$$

uniformly for $M \geq 1$, $N \geq 1$, $H \leq M^{\beta-1} N^\gamma$ and $|\delta| \leq 1/\varepsilon$, where $(\kappa, \lambda)$ is an exponent pair and the implied constant depends on $(\alpha, \beta, \gamma, \varepsilon)$ only.

Proof. Obviously we have

$$h^\alpha m^{-\beta} \leq C_1 H^\alpha M^{-\beta} =: \Xi \quad (h \sim H, m \sim M),$$
where the $C_j = C_j(\alpha, \beta, \gamma)$ denotes some positive constant depending on $(\alpha, \beta, \gamma)$ at most. Let $K \geq 1$ be a parameter to be chosen later. Introducing the following set

$$T_k := \{(h, m) : h \sim H, m \sim M, \Xi(k-1) < h^\alpha m^{-\beta} K \leq \Xi k\},$$

we can write

$$S_\delta = \sum_{n \sim N} \sum_{k \in K} \sum_{(h, m) \in T_k} a_{h, m} e\left(\frac{X^\beta N^\gamma}{H^\alpha} \frac{h^\alpha}{m^\beta n^\gamma + \delta}\right).$$

By the Cauchy-Schwarz inequality, we derive

$$|S_\delta|^2 \leq NK \sum_{n \sim N} \sum_{k \in K} \sum_{(h, m) \in T_k} |S(h, h', m, m')|,$$

where

$$S(h, h', m, m') := \sum_{n \sim N} e\left(\frac{X^\beta N^\gamma}{H^\alpha} \frac{h^\alpha}{m^\beta n^\gamma + \delta} - \frac{h'^\alpha}{m^\beta n'^\gamma + \delta}\right).$$

Noticing that $(h, m), (h', m') \in T_k$ imply $|h^\alpha m^{-\beta} - h'^\alpha m'^{-\beta}| \leq \Xi K^{-1}$, we can write

$$|S_\delta|^2 \leq NK \sum_{h, h' \sim H, m, m' \sim M} \left|S(h, h', m, m')\right| \leq NK(S_\delta^1 + S_\delta^2),$$

where

$$S_\delta^1 := \sum_{h, h' \sim H, m, m' \sim M} \left|S(h, h', m, m')\right|,$$

$$S_\delta^2 := \sum_{h, h' \sim H, m, m' \sim M} \left|S(h, h', m, m')\right| \quad \Xi(HM)^{-1} < |h^\alpha m^{-\beta} - h'^\alpha m'^{-\beta}| \leq \Xi K^{-1}.$$

We have made the convention that $S_\delta^2 = 0$ if $K \leq HM$. Since

$$\left|\frac{h^\alpha}{m^\beta} - \frac{h'^\alpha}{m'^\beta}\right| \leq \frac{\Xi}{HM} \iff \left|\frac{h^\alpha}{h'^\alpha} - \frac{m^\beta}{m'^\beta}\right| \leq \frac{C_2}{\Xi} \iff \left|\frac{h^\alpha}{m^\beta} - \frac{h'^\alpha}{m'^\beta}\right| \leq \frac{C_2}{HM},$$

Lemma 2.11 implies that the number of $(h, h', m, m')$ verifying $|h^\alpha m^{-\beta} - h'^\alpha m'^{-\beta}| \leq \Xi(HM)^{-1}$ is $\ll \mathcal{D}(H, M; C_2/HM) \ll \Xi HMX^\varepsilon$. Thus we have trivially

$$S_\delta^1 \ll \varepsilon HMX^\varepsilon.$$

Next we bound $S_\delta^2$. We write

$$\frac{h^\alpha}{m^\beta n^\gamma + \delta} - \frac{h'^\alpha}{m'^\beta n'^\gamma + \delta} = \frac{1}{n^\gamma} \left(\frac{h^\alpha}{m^\beta} - \frac{h'^\alpha}{m'^\beta}\right) - \frac{\delta}{n^\gamma} \left(\frac{h^\alpha}{m^\beta n^\gamma + \delta} - \frac{h'^\alpha}{m'^\beta n'^\gamma + \delta}\right) =: f(n).$$

Since $H \leq M^{-1} N^\gamma$, we have

$$\frac{h^\alpha}{m^\beta n^\gamma + \delta} - \frac{h'^\alpha}{m'^\beta n'^\gamma + \delta} \leq \frac{C_3 \Xi}{M^\beta N^\gamma} \ll \frac{\Xi}{HM} (h, h' \sim H; m, m' \sim M),$$

where $C_3$ is a positive constant depending on $(\alpha, \beta, \gamma)$.
Therefore for \((h, h', m, m')\) verifying \(\Xi(HM)^{-1} < |h^\alpha m^{-\beta} - h'^\alpha m'^{-\beta}| \leq \Xi K^{-1}\), the first member on the right-hand side of (3.8) dominates the second one. Split \((\Xi(HM)^{-1}, \Xi K^{-1})\) into dyadic intervals \((\Delta, 2\Delta\Xi)\) with \(1/HM \leq \Delta \leq 1/K\). Take \(K = \max\{2C_4X/N, 1\}\) such that for \((h, h', m, m')\) verifying \(\Xi \Delta < |h^\alpha m^{-\beta} - h'^\alpha m'^{-\beta}| \leq 2\Xi\Delta\) we have
\[
\max_{n \sim N} |f'(n)| = C_4X\Delta N^{-1} \leq \frac{1}{2}.
\]
By Kusmin-Landau’s inequality and Lemma 2.1 we have
\[
S_\delta^2 \ll_{\varepsilon} X^\varepsilon \max_{4/HM \leq \Delta \leq 1/K} \Delta(HM)^2(X\Delta N^{-1})^{-1} \ll_{\varepsilon} X^{-1+\varepsilon}(HM)^2N.
\]
Combining (3.7) and (3.9) with (3.6) gives us
\[
|S_\delta|^2 \ll_{\varepsilon} (HMN^2K + X^{-1}(HMN)^2)X^\varepsilon \ll_{\varepsilon} (XHMN + HMN^2 + (HM)^2 N + X^{-1}(HMN)^2)X^\varepsilon,
\]
which is equivalent to (3.2).

Next we prove (3.4). If \(X \leq HM\), then (3.2) implies that
\[
S_\delta \ll_{\varepsilon} ((HM)^{1/2}N + HNM^{1/2} + X^{-1/2}HMN)X^\varepsilon.
\]
Now we can suppose that \(X \geq HM\). Applying the exponent pair \((\kappa, \lambda)\) to \(S(h, h', m, m')\) and using Lemma 2.1, we can derive that
\[
S_\delta^2 \ll_{\varepsilon} X^\varepsilon \max_{4/HM \leq \Delta \leq 1/K} \Delta(HM)^2 ((X\Delta N^{-1})^\kappa N^\lambda + (X\Delta N^{-1})^{-1}) \ll_{\varepsilon} X^\varepsilon (X^\kappa H^2 M^{2N^{-\kappa}} + X^{-1}H^2 M^2N).
\]
Combining (3.7) and (3.11) with (3.6) gives us
\[
|S_\delta|^2 \ll_{\varepsilon} (X^\kappa H^2 M^{2N^{1-\kappa}} + HMN^2K)X^\varepsilon
\]
for all \(K \in [1, HM]\), where we have removed the term \(X^{-1}H^2 M^2N^2K\) (\(\leq HMN^2K\) since we have suppose that \(X \geq HM\)). Noticing that this estimate is trivial if \(K \geq HM\), we can optimise the parameter \(K\) over \([1, \infty)\) to get
\[
|S_\delta|^2 \ll_{\varepsilon} ((X^\kappa H^2 M^{2N^{1-\kappa}} + HMN^2K)^{1/(1+\kappa)} + HMN^2)X^\varepsilon.
\]
Combining this with (3.10), we obtain (3.4).}

4. A Key Inequality

The aim of this section is to prove the following proposition, which will play a key role for the proof of Theorem 1. Define
\[
\mathcal{S}_\delta(x, D) := \sum_{d \sim D} \Lambda(d) \psi\left(\frac{x}{d + \delta}\right).
\]

Proposition 4.1. Let \(\delta \notin -\mathbb{N}\) be a fixed constant. Then we have
\[
\mathcal{S}_\delta(x, D) \ll (x^{2\kappa}D^{3+\lambda})^{1/(4\kappa+4)} + D^{5/6}
+ (x^{3\kappa'}D^{-2\kappa'+2\lambda'+1})^{1/(3\kappa'+3)} + (x^{3\kappa'}D^{-5\kappa'+2\lambda'+1})^{1/3})x^\varepsilon.
\]
uniformly for \(x \geq 3\) and \(1 \leq D \leq x^{2/3}\), where \((\kappa, \lambda)\) and \((\kappa', \lambda')\) are exponent pairs. In particular, uniformly for \(x^{6/13} \leq D \leq x^{2/3}\) we have
\[
\mathcal{S}_\delta(x, D) \ll_{\varepsilon} (x^2D^7)^{1/12}x^\varepsilon.
\]
Proof. We apply the Vaughan identity (2.1) with \( g(d) = \psi \left( \frac{x}{d^{\alpha}} \right) \) to write

(4.4) \[ \mathcal{S}_\delta(x, D) = \mathcal{S}_{\delta,1} + \mathcal{S}_{\delta,2} + \mathcal{S}_{\delta,3} + \mathcal{S}_{\delta,4}, \]

where

\[
\mathcal{S}_{\delta,1} := \sum_{m \leq D^{1/3}} \alpha_1(m) \sum_{D < mn \leq 2D} \psi \left( \frac{x}{mn + \delta} \right),
\]

\[
\mathcal{S}_{\delta,2} := \sum_{m \leq D^{1/3}} \alpha_2(m) \sum_{D < mn \leq 2D} \psi \left( \frac{x}{mn + \delta} \right) \log n,
\]

\[
\mathcal{S}_{\delta,3} := \sum_{D^{1/3} < m, n \leq D^{2/3}} \alpha_3(m) \alpha_4(n) \psi \left( \frac{x}{mn + \delta} \right),
\]

\[
\mathcal{S}_{\delta,4} := \sum_{D^{1/3} < m, n \leq D^{2/3}} \alpha_5(m) \alpha_6(n) \psi \left( \frac{x}{mn + \delta} \right).
\]

Firstly we estimate \( \mathcal{S}_{\delta,3} \). In view of Lemma 2.3, we can write

(4.5) \[ \mathcal{S}_{\delta,3} = - \frac{1}{2\pi i} \sum_{H'} \sum_{M} \sum_{N} (\mathcal{S}^\uparrow_{\delta,3}(H', M, N) + \overline{\mathcal{S}^\downarrow_{\delta,3}(H', M, N)}) + \sum_{M} \sum_{N} \mathcal{S}^\downarrow_{\delta,3}(M, N), \]

where \( MN \asymp D \) (i.e. \( D \ll MN \ll D \)), \( \alpha(h) := \frac{H'}{h} \Phi \left( \frac{h}{H+1} \right) \ll 1 \) and

\[
\mathcal{S}^\uparrow_{\delta,3}(H', M, N) := \frac{1}{H'} \sum_{h \sim H'} \sum_{m \sim M} \sum_{n \sim N} \sum_{D < mn \leq 2D} \alpha(h) \alpha_3(m) \alpha_4(n) e \left( \frac{hx}{mn + \delta} \right),
\]

\[
\mathcal{S}^\downarrow_{\delta,3}(M, N) := \sum_{m \sim M} \sum_{n \sim N} \alpha_3(m) \alpha_4(n) R_H \left( \frac{x}{mn + \delta} \right).
\]

Firstly we bound \( \mathcal{S}^\uparrow_{\delta,3}(H', M, N) \). We remove the extra multiplicative condition \( D < mn \leq 2D \) at the cost of a factor \( \log x \). On the other hand, in view of the symmetry of the variables \( m \) and \( n \), we can suppose that

(4.6) \[ D^{1/3} \leq M \leq D^{1/2} \leq N \leq D^{2/3}. \]

By Proposition 3.1 with \( \alpha = \beta = \gamma = 1 \) and \( (X, H, M, N) = (xH'/MN, H', M, N) \), we have

(4.7) \[ \mathcal{S}^\uparrow_{\delta,3}(H', M, N) \ll x \left( x^\varepsilon M^2 N^{1+\lambda} \right)^{1/(2+2\varepsilon)} + MN^{1/2} + M^{1/2} N + (x^{-1} DH')^{1/2} x^\varepsilon, \]

provided \( H' \leq H \leq N \).

Secondly we bound \( \mathcal{S}^\downarrow_{\delta,3}(M, N) \). Using (2.2), we have

\[
\mathcal{S}^\downarrow_{\delta,3}(M, N) \ll x^\varepsilon \sum_{m \sim M} \sum_{n \sim N} \left| R_H \left( \frac{x}{mn + \delta} \right) \right| \leq \frac{x^\varepsilon}{H} \sum_{m \sim M} \sum_{n \sim N} \sum_{0 \leq |h| \leq H} \left( 1 - \frac{|h|}{H + 1} \right) e \left( \frac{hx}{mn + \delta} \right) \ll x^\varepsilon \left( DH^{-1} + \max_{1 \leq H' \leq H} \left| \mathcal{S}^\downarrow_{\delta,3}(H', M, N) \right| \right),
\]

where

\[
\mathcal{S}^\downarrow_{\delta,3}(H', M, N) := \frac{1}{H} \sum_{m \sim M} \sum_{n \sim N} \sum_{h \sim H'} \left( 1 - \frac{|h|}{H + 1} \right) e \left( \frac{hx}{mn + \delta} \right).
\]
Clearly we can bound $\tilde{\mathcal{S}}_{\delta,3}(H', M, N)$ in the same way as $\mathcal{S}_{\delta,3}(H', M, N)$ and obtain

$$
\mathcal{S}_{\delta,3}(M, N) \ll_{\varepsilon} (D H^{-1} + (x^6 M^2 N^{1+\lambda})^{1/(2+2\epsilon)}) + MN^{1/2} + M^{1/2} N + (x^{-1} D H)^{1/2}) x^\varepsilon,
$$

provided $H \leq N$. Combining (4.7) and (4.8) with (4.5) and using (4.6), we find that

$$
\mathcal{S}_{\delta,3} \ll_{\varepsilon} (D H^{-1} + (x^6 M^2 N^{1+\lambda})^{1/(2+2\epsilon)}) + MN^{1/2} + M^{1/2} N + (x^{-1} D H)^{1/2}) x^\varepsilon,
$$

provided $H \leq D^{1/2} (\leq N)$. Optimising $H$ over $[1, D^{1/2}]$, it follows that

$$
\mathcal{S}_{\delta,3} \ll_{\varepsilon} ((x^6 D^{3+\lambda})^{1/(4+4\epsilon)} + D^{5/6} + (x^{-1} D^2)^{1/3}) x^\varepsilon.
$$

Clearly the same estimate also holds for $\mathcal{S}_{\delta,4}$.

On the other hand, let $(\kappa', \lambda')$ be an exponent pair, then [11 (5.16)] gives us

$$
\mathcal{S}_{\delta,j} \ll ((x^3 D^{2-2\epsilon'+2\lambda'+1})^{1/(3\epsilon'+3)} + x^{\epsilon'} D^{(-5\epsilon'+2\lambda'+1)/3} + x^1 D^2)x^\varepsilon.
$$

for $j = 1, 2$. Inserting (4.10) and (4.11) into (4.3) and using the fact that

$$
\max\{(x^{-1} D^2)^{1/3}, x^{-1} D^2\} \leq D^{5/6}
$$

for $1 \leq D \leq x^{2/3}$, we get (4.2).

Taking $(\kappa, \lambda) = (\kappa', \lambda') = (\frac{1}{2}, \frac{1}{2})$ in (4.2), we find that

$$
\mathcal{S}_\delta(x, D) \ll \left((x^2 D^{7})^{1/12} + D^{5/6} + (x^3 D^2)^{1/9} + (x^3 D^{-1})^{1/6}\right) x^\varepsilon,
$$

which implies (4.4), since the last three terms can be absorbed by the first one provided $x^{6/13} \leq D \leq x^{2/3}$.

\section{5. Proof of Theorem [11]}

Let $N \in [x^{6/13}, x^{1/2}]$ be a parameter which can be chosen later. First we write

$$
\sum_{n \leq x} \Lambda\left(\left[\frac{x}{n}\right]\right) := S_1(x) + S_2(x)
$$

with

$$
S_1(x) := \sum_{n \leq N} \Lambda\left(\left[\frac{x}{n}\right]\right), \quad S_2(x) := \sum_{N < n \leq x} \Lambda\left(\left[\frac{x}{n}\right]\right).
$$

We have trivially

$$
S_1(x) \ll_{\varepsilon} N x^\varepsilon.
$$

Next we bound $S_2(x)$. Putting $d = [x/n]$, then $x/n - 1 < d \leq x/n \Leftrightarrow x/(d+1) < n \leq x/d$. Thus we can write

$$
S_2(x) = \sum_{d \leq x/N} \Lambda(d) \sum_{x/(d+1) < n \leq x/d} 1
$$

$$
= \sum_{d \leq x/N} \Lambda(d) \left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{x}{d+1} + \psi\left(\frac{x}{d+1}\right)\right)
$$

$$
= x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + \mathcal{R}_1(x) - \mathcal{R}_0(x) + O(N),
$$

where $\mathcal{R}_0(x)$ is the main term and $\mathcal{R}_1(x)$ is the error term.
where we have used the following bounds
\[
x \sum_{d > x/N} \frac{\Lambda(d)}{d(d+1)} \ll \varepsilon N x^\varepsilon, \quad \sum_{d \leq N} \Lambda(d) \left( \psi\left( \frac{x}{d+1} \right) - \psi\left( \frac{x}{d} \right) \right) \ll \varepsilon N x^\varepsilon
\]
and defined
\[
R_\delta(x) = \sum_{N < d \leq x/N} \Lambda(d) \psi\left( \frac{x}{d+\delta} \right).
\]
Writing \(D_j := x/(2^j N)\), we have \(x^{6/13} \leq N \leq D_j \leq x/N \leq x^{7/13}\) for \(0 \leq j \leq \frac{\log(x^2/N)}{\log 2}\) since \(x^{6/13} \leq N \leq x^{1/2}\). Thus we can apply (4.3) of Proposition 4.1 to get
\[
|R_\delta(x)| \leq \sum_{0 \leq j < \log(x^2/N)/\log 2} |\hat{S}_\delta(x, D_j)| \ll \sum_{0 \leq j < \log(x^2/N)/\log 2} (x^2 D_j^7)^{1/12} x^\varepsilon \ll (x^9 N^{-7})^{1/12} x^\varepsilon.
\]
Putting this into (5.3) and taking \(N = x^{9/19}\), we find that
\[
S_2(x) = x \sum_{d \geq 1} \frac{\Lambda(d)}{d(d+1)} + O(x^{9/19 + \varepsilon}).
\]
Inserting (5.2) with \(N = x^{9/19}\) and (5.4) into (5.1), we get the required result.

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