Some topological issues for ferromagnets and fluids

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Abstract

We analyze the canonical structure of a continuum model of ferromagnets and clarify known difficulties in defining a momentum density. The moments of the momentum density corresponding to volume-preserving coordinate transformations can be defined, but a non-singular definition of the other moments requires an enlargement of the phase space which illuminates a close relation to fluid mechanics. We also discuss the nontrivial connectivity of the phase space for two and three dimensions and show how this feature can be incorporated in the quantum theory, working out the two-dimensional case in some detail.
1 Introduction

It has been known for some time that there is a problem in defining a suitable momentum density for ferromagnets, as noted in [1, 2]. The difficulty has to do with the global topology of the space of fields [1]. One can certainly define an adequate momentum density in the spin wave approximation, and even express it in terms of globally defined variables. But such an expression does not preserve the rotational symmetry of the problem [2]. A nonsingular expression which preserves the rotational symmetry is not possible. In this paper, we shall consider the continuum model of ferromagnets and analyze the canonical structure of the theory.

We show how these results can be understood and clarified by standard canonical quantization of the continuum action for the ferromagnet. We show that it is possible to define the moments of the momentum density corresponding to volume-preserving coordinate transformations, but the other moments are not unique and have singularities. Our resolution of the singularities is based on enlarging the phase space. We note that the phase space for the ferromagnet is obtained from the standard phase space for fluid mechanics by restricting to the incompressible case. In other words, as far as the canonical structure is concerned, the ferromagnet may be viewed as an incompressible fluid (with a Hamiltonian which is quite different from the fluid Hamiltonian).

Since volume-preserving moments include the total momentum and total angular momentum, we may ask why we should need a momentum density, why we need to address the singularities. One simple reason has to do with excitations which can be locally generated, say, by applying a focused beam of radiation to a small region in the ferromagnetic lattice. Because of the localized nature, it is the conservation law expressed in terms of the divergence of the momentum density which is relevant when the radiation is absorbed and magnetic excitations are generated. This example shows that our analysis is more than just formalistic improvement.

We also discuss the connectivity of the phase space for two and three dimensions. Generally, nontrivial connectivity can lead to inequivalent quantum theories as, for example, in the case of $\theta$-vacua in Chromodynamics. The present situation is however more complicated, since the nontrivial connectivity involves all the phase space variables, which do not commute among themselves in the quantum theory. By working out the two-dimensional case in some detail, we show how one can incorporate this feature in the quantum theory. Ferromagnetic systems with Skyrmion excitations, the quantum Hall ferromagnet being a specific example, can provide a physical context for our considerations [3]. The $\theta$-term determines the statistics of the Skyrme solitons. A full quantum treatment of these excitations will require our formalism, since the noncommutativity of the phase space variables involved in
the definition of the $\theta$-term must be taken into account.

In section 2, we discuss the canonical structure of the ferromagnet. Section 3 is devoted to a similar analysis for fluids and its relation to the ferromagnet. General topological observations are made in section 4. The quantization of the two-dimensional case including the effects of the nontrivial connectivity of the phase space is done in section 5.

2 The canonical structure of the ferromagnet

The degrees of freedom for the ferromagnet are the spin variables $S_a(I)$ obeying the commutation rules

$$[S_a(I), S_b(J)] = i\epsilon_{abc}S_c(I) \delta_{IJ}$$

(1)

Here $a, b, c = 1, 2, 3$ and the indices $I, J$ label the lattice sites. The action for the ferromagnet should be such that canonical quantization leads to the commutation rules given above. For a single spin, this is well known. The action can be taken as

$$S = -in \int dt \ Tr(\sigma_3 g^{-1} \dot{g}) - \int dt H$$

(2)

Here $g$ is an element of the group $SU(2)$. It may be taken to be of the form $\exp(itt_a \theta^a)$ where $t_a = \frac{1}{2} \sigma_a$ and $\sigma_a$ are the Pauli matrices. The action (2) is invariant under $g \rightarrow g \exp(\frac{i}{2} \sigma_3 \varphi)$ modulo surface terms. Therefore, the theory is defined on the two-dimensional sphere $SU(2)/U(1) = S^2$. The spin variables are given by $\text{Tr}(\sigma_3 g^{-1} t_a)$. The naive continuum limit of this action is thus given by

$$S = -i \int \frac{d^3xdv}{v} \ Tr(\sigma_3 g^{-1} \dot{g}) - \int dt H(S)$$

(4)

Here $n$ will turn out to be an integer with $\frac{1}{2}n$ giving the maximal spin or $j$-value of the spin variables. We shall consider spin-$\frac{1}{2}$ variables from now on, so that $n = 1$ for all equations below. The results can be trivially extended to arbitrary values of $n$.

For several independent spins, we may generalize this action as

$$S = -i \sum I \int dt \ Tr(\sigma_3 g^{-1} \dot{g}) - \int dt H$$

(3)

The naive continuum limit of this action is thus given by

$$S = -i \int \frac{d^3xdv}{v} \ Tr(\sigma_3 g^{-1} \dot{g}) - \int dt H(S)$$

(4)

$v$ is the spatial volume corresponding to a single spin, determined by the unit cell of the lattice. Notice that this action (4) is invariant under $g \rightarrow g \exp(\frac{i}{2} \sigma_3 \varphi)$ modulo surface terms, where $\varphi$ is now a function of $x$, so that the dynamical variables are in $SU(2)/U(1) = S^2$. 

3
Since this is a first order action, \( g \) will denote the phase space coordinates and with the above observation, we can identify the phase space \( \mathcal{P} \) as

\[
\mathcal{P} = \left\{ \text{set of all maps } \mathbb{R}^3 \to SU(2)/U(1) \right\}
\]  

(5)

We now consider the canonical quantization of this action. The surface term in \( \delta \mathcal{S} \) resulting from the partial integration over the time-variable \( t \) is

\[
\mathcal{A} = -i \int \frac{d^3x}{v} \text{Tr}(\sigma_3 g^{-1} \delta g)
\]  

(6)

This is the canonical or symplectic potential of the theory. Explicitly, if \( \theta^a \) are the group parameters

\[
g^{-1} \delta g \equiv -it_a \left( \tilde{E}^a_i \right) \delta \theta^i, \quad \delta gg^{-1} \equiv -it_a E^a_i \delta \theta^i
\]  

(7)

and

\[
\mathcal{A} = - \int \frac{d^3x}{v} \tilde{E}^3_i \delta \theta^i \equiv \int \mathcal{A}_i(x) \delta \theta^i(x)
\]

\[
\equiv \mathcal{A}_I \delta \theta^I
\]  

(8)

where we have introduced a composite index \( I = (i, x) \) to avoid clutter in notation.

The standard formula

\[
\delta e^A = \int_0^1 d\gamma \ e^{\gamma A} \delta \hat{A} e^{(1-\gamma)A}
\]  

(9)

may be used to determine \( \tilde{E}^a_i \) and \( E^a_i \) in terms of the group parameters.

It may be useful at this stage to recall some general features of the canonical formalism \[4\]. Consider a general symplectic potential

\[
\mathcal{A} = \mathcal{A}_I \delta \xi^I = \int d^3x \mathcal{A}_i(x) \delta \xi^i(x)
\]  

(10)

The functional curl of \( \mathcal{A} \) gives the canonical two-form or symplectic structure \( \Omega_{ij}(x, x') \) whose inverse will give the fundamental Poisson brackets. Explicitly

\[
\Omega_{IJ} = \delta_I \mathcal{A}_J - \delta_J \mathcal{A}_I
\]

\[
\Omega_{ij}(x, x') = \frac{\delta}{\delta \xi^i(x)} \mathcal{A}_j(x') - \frac{\delta}{\delta \xi^j(x')} \mathcal{A}_i(x)
\]  

(11)

where we have again used the composite index notation \( I = (i, x) \) and \( J = (j, x') \) and \( \delta_I = \delta/\delta \xi^i(x) \).
With the understanding that the variations are antisymmetrized, we may write \( \Omega \) as a differential two-form

\[
\Omega = \delta A = \frac{1}{2} \Omega_{IJ} \delta \xi^I \wedge \delta \xi^J \\
= \frac{1}{2} \int \Omega_{ij}(x, x') \delta \xi^i(x) \wedge \delta \xi^j(x')
\]  
(12)

Notice that from the definition of \( \Omega \), we have the identity

\[
\delta_I \Omega_{JK} + \delta_J \Omega_{KI} + \delta_K \Omega_{IJ} = 0
\]  
(13)

We now turn to canonical transformations and their generators. Let \( \xi^I \to \xi^I + a^I(\xi) \) be an infinitesimal transformation of the canonical variables. This transformation is canonical if it preserves the canonical structure \( \Omega \). The change in \( \Omega \) arises from two sources, due to the \( \xi \)-dependence of the components \( \Omega_{IJ} \) and due to the fact that \( \Omega_{IJ} \) transforms under change of phase space coordinate frames. (The latter is the change due to the frame factors \( \delta \xi^I \wedge \delta \xi^J \) in writing \( \Omega \) as a differential form.) The total change is

\[
\Delta \Omega_{IJ} = \delta_I \alpha_J - \delta_J \alpha_I
\]
(14)

where we have used (13). (This change is also the so-called Lie derivative of \( \Omega \).) More explicitly, expanding out the composite notation,

\[
\Delta \Omega_{ij}(x, x') = \left[ \frac{\delta}{\delta \xi^i(x)} \alpha_j(x') - \frac{\delta}{\delta \xi^j(x')} \alpha_i(x) \right]
\]
\[
\alpha_i(x) = \int d^3x' a^k(x') \Omega_{ki}(x', x)
\]  
(15)

The transformation \( \xi^I \to \xi^I + a^I(\xi) \) may be represented by a vector field \( V = a^I \delta_I = \int_x a^I(x)(\delta/\delta \xi^I(x)) \). The expression \( \alpha_I = a^K \Omega_{KI} \) is then a one-form or covariant vector corresponding to the contraction of \( V \) with \( \Omega \), often denoted by \( V \wedge \Omega \). In other words

\[
V \wedge \Omega = a^K \Omega_{KI} \delta \xi^I
\]
(16)

The change in \( \Omega \) is thus the curl of \( V \wedge \Omega \), which is denoted by \( \delta(V \wedge \Omega) \) in the language of differential forms.

Equation (14) shows that the transformation \( \xi^I \to \xi^I + a^I \) will preserve \( \Omega \), and hence be a canonical transformation, if

\[
\delta_I \alpha_J - \delta_J \alpha_I = 0
\]  
(17)
This condition may be solved as
\[ \alpha_I = -\delta_I G \]  
for some function $G$ of the phase space variables. The function $G$ defined by this equation is the generator of the canonical transformation. (Equation (17) is the general requirement for a transformation to be canonical. Equation (18) is a necessary and sufficient condition locally on the phase space. If the phase space has nontrivial topology, the vanishing of $\Delta \Omega$ may have more general solutions. Even though locally all solutions look like (18), the $G$ so-defined may not exist globally on the phase space. We will return to this question in more detail later.)

The inverse of $\Omega$ is defined by $(\Omega^{-1})^{IJ} \Omega_{JK} = \delta^I_K$ which expands out as
\[ \int_V d^3 x'(\Omega^{-1})^{ij}(x, x') \Omega_{jk}(x', x'') = \delta^i_k \delta^{(3)}(x - x'') \]  
(19)

Using the inverse of $\Omega$, the Poisson bracket of functions $F, G$ on phase space is defined as
\[ \{F, G\} = (\Omega^{-1})^{IJ} \delta_I F \delta_J G \]
\[ = \int d^3 x \ d^3 x' \ (\Omega^{-1})^{ij}(x, x') \frac{\delta F}{\delta \xi_i(x)} \frac{\delta G}{\delta \xi_j(x')} \]  
(20)

We may also rewrite equation (18) as $a^I = (\Omega^{-1})^{IJ} \delta_J G$. This shows that the change of any function $F$ under a canonical transformation generated by $G$ is given by
\[ \delta F = a^I \delta_I F = (\Omega^{-1})^{IJ} \delta_J G \delta_I F \]
\[ = \{F, G\} \]  
(21)

The change of any variable, so long as the transformation is canonical, is given by the Poisson bracket of the variable with the generating function for the transformation.

We will now apply these results to the ferromagnet. With the understanding that the variations are antisymmetrized, as emphasized by the wedge symbol, the symplectic form $\Omega$ for the ferromagnet can be written as the differential two-form
\[ \Omega = \delta A = i \int \frac{d^3 x}{v} \text{Tr}(\sigma_3 g^{-1} \delta g \wedge g^{-1} \delta g) \]
\[ = \frac{1}{2} \int \Omega_{ij}(x, x') \delta \theta^i(x) \wedge \delta \theta^j(x') \]
\[ = \frac{1}{2} \int \frac{d^3 x}{v} \epsilon_{ab3} \tilde{E}_i^a \tilde{E}_j^b \delta \theta^i \wedge \delta \theta^j \]
\[ \Omega_{ij}(x, x') = \frac{1}{v} \epsilon_{ab3} \tilde{E}_i^a \tilde{E}_j^b \delta^{(3)}(x - x') \]  
(22)
The potential $A$ is not invariant under $g \rightarrow g \exp(it_3 \varphi)$ but undergoes the transformation
\[ A\left(g e^{it_3 \varphi}\right) = A(g) + \delta \int \frac{d^3x}{v} \varphi \] (23)
Since $A$ changes by a gradient term, its curl $\Omega$ is invariant under this transformation. Thus while $A$ depends on all three $SU(2)$ variables, $\Omega$ can be written in terms of $SU(2)/U(1)$ variables only.

Consider the left translations of $g$ given by $\delta g = it_a \xi^a g$, where $\xi^a$ denote infinitesimal changes of the group parameters. These transformations may be represented by the vector field
\[ V(\xi) = -\int d^3x \xi^a (E^{-1})^i_a \frac{\delta}{\delta \theta_i} \] (24)
For this transformation to be canonically generated, we must have $\delta (V(\xi) | \Omega) = 0$. where $V | \Omega$ denotes the interior contraction of $V$ with $\Omega$. In the present case, the topology of the phase space $P$ is such that this requirement is equivalent to $V(\xi) | \Omega = -\delta G$. $G$ is the canonical generator of the transformation. Explicitly we find
\[ V(\xi) | \Omega = -\delta \phi(\xi) \]
\[ \phi(\xi) = \int \frac{d^3x}{v} \phi_a \xi^a \]
\[ \phi_a \equiv \text{Tr}(g \sigma_3 g^{-1} t_a) \] (25)
Thus the left translations of $g$ can indeed be canonically generated and the generating function is $\phi_a$. Notice that $\phi_a$ are invariant under $g \rightarrow g e^{it_3 \varphi}$ and so are defined entirely in terms of the phase space variables. Indeed $\phi_a$ may be taken as giving coordinates on $S^2 = SU(2)/U(1)$ since $\phi_a \phi_a = 1$. Since $\phi_a$ give a coordinatization of the phase space, all physical observables may be taken as functions of $\phi_a$.

A general parametrization of $g$, which is valid everywhere except near one of the poles, is
\[ g = \frac{1}{\sqrt{1 + \bar{z}z}} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{1}{2}} & 0 \\ 0 & e^{-i\frac{1}{2}} \end{pmatrix} \] (26)
The variables $\phi_a$ in this parametrization are
\[ \phi_1 = -\frac{z + \bar{z}}{1 + \bar{z}z}, \quad \phi_2 = i \frac{\bar{z} - z}{1 + \bar{z}z} \]
\[ \phi_3 = \frac{1 - \bar{z}z}{1 + \bar{z}z} \] (27)
In these coordinates, the symplectic two-form $\Omega$ is given as
\[ \Omega = 2i \int \frac{d^3x}{v} \frac{\delta \bar{z} \wedge \delta z}{(1 + \bar{z}z)^2} \] (28)
which leads to the definition of Poisson brackets as
\[
\{ F, G \} = i \frac{v}{2} \int d^3x \ (1 + \bar{z}z)^2 \left( \frac{\delta F}{\delta \bar{z}} \frac{\delta G}{\delta z} - \frac{\delta F}{\delta z} \frac{\delta G}{\delta \bar{z}} \right)
\] (29)
This immediately leads to the relations
\[
\{ \phi_a(x), \phi_b(y) \} = - \epsilon_{abc} \phi_c(x) \ \delta^{(3)}(x - y)
\] (30)
This shows that we can identify the spin variables as \( S_a(x) = - \phi_a(x) \). In the quantum theory, by the usual correspondence rules, the commutation rules for these can be written as
\[
[S_a(x), S_b(y)] = i \epsilon_{abc} S_c(x) \ \delta^{(3)}(x - y)
\] (31)
in agreement with the continuum version of the rules (1). We see that the action (4) does indeed lead to the correct commutation rules. As mentioned before, all physical observables are functions of \( \phi_a \) or \( S_a(x) \). In particular \( H \) is a function of \( S_a \) and their gradients.

Notice that the expression for \( \Omega \) in (28) is the area element on the two-dimensional sphere. The corresponding potential \( \mathcal{A} \) is thus the potential for a monopole with its attendant Dirac string singularities. With \( \phi^a \) providing a coordinatization of a two-sphere, equations (28, 31) show that we have the usual description of the ferromagnet. Thus our considerations, as mentioned in the introduction, can indeed be applied to physical situations such as the quantum Hall ferromagnet.

We now turn to the momentum density \( J_i \) which is expected to generate the coordinate transformation \( x^i \rightarrow x^i + a^i(x) \), or \( g \rightarrow g + a^i \partial_i g \). We may write the corresponding vector field as
\[
V(a) = \int d^3x \ a^i \frac{\partial \theta^k}{\partial x^i} \frac{\delta}{\delta \theta^k(x)}
\] (32)
Contracting this with \( \Omega \) we get,
\[
V(a) \Omega = - \delta \int \frac{d^3x}{v} \ a^i \left[ -i \ Tr(\sigma_3 g^{-1} \partial_i g) \right] + i \int \frac{d^3x}{v} \ (\partial_i a^i) \ Tr(\sigma_3 g^{-1} \delta g)
\] (33)
We see that \( V(a) \Omega \) cannot be taken as the variation of some functional because of the second term on the right hand side in (33). In particular, \( \delta(V(a) \Omega) \) is not zero. This proves that the transformation \( g \rightarrow g + a^i \partial_i g \) cannot be generated as a canonical transformation. Therefore, one cannot define a momentum density for the ferromagnet. However, the same equation shows that transformations for which \( \partial_i a^i = 0 \) can be canonically generated since the second term in (33) vanishes and the corresponding generator is given by
\[
J(a) = - i \int \frac{d^3x}{v} \ a^i \ Tr(\sigma_3 g^{-1} \partial_i g)
\] (34)
The condition \( \partial_i a^i = 0 \) corresponds to the volume-preserving transformations. Thus, our conclusion is that only volume preserving transformations can be canonically implemented for the ferromagnet. The total momentum and the orbital angular momentum are included in this set.

As mentioned in the introduction, this peculiarity of the momentum density for the ferromagnet has been noticed and analyzed in a different way before. The canonical two-form \( \Omega \) is the area element on the two-sphere \( SU(2)/U(1) \) which is the target of the mappings \( \phi_a : \mathbb{R}^3 \to S^2 \) which constitute the phase space. As such it has the same form as the magnetic field of a Dirac monopole on a sphere of fixed radius.

Using the explicit form of the \( \phi_a \) as given in (27), it is easy to show that \( \Omega \), the symplectic two-form given in (28), may be written as

\[
\Omega = \frac{1}{2} \int \frac{d^3 x}{v} \epsilon_{abc} \phi_a \delta \phi_b \wedge \delta \phi_c. \tag{35}
\]

Recalling that the \( \phi_a \) provide a coordinatization of the phase space \( S^2 \), this shows that \( \Omega \) is indeed an area element on \( S^2 \). Consequently, the symplectic potential \( A \) is nothing but the potential due to a magnetic monopole located at the center of \( S^2 \).

The evaluation of a potential for such a field has the well known Dirac string singularity. Our potential \( A \) does not show this because it is defined on a larger space; this is evident from its lack of invariance under \( g \to ge^{i \epsilon} \). The difficulty in defining the momentum density is related to this. One can define a momentum density by enlarging the phase space. This is best seen by comparison with the Lagrangian for fluid motion, to which we turn now.

### 3 Fluid mechanics

In three spatial dimensions, one can parametrize the local velocity of a fluid as

\[
v_i = \partial_i \chi + \alpha \partial_i \beta \tag{36}
\]

This is the standard Clebsch parametrization and can be used to write a Lagrangian for fluid mechanics [5, 6]. With zero viscosity, the appropriate action is

\[
S = \int dt \ d^3 x \left[ \rho \left( \dot{\chi} + \alpha \dot{\beta} \right) - \frac{1}{2} \rho (\partial \chi + \alpha \partial \beta)^2 \right] \tag{37}
\]

The first term identifies the canonically conjugate pairs of variables \((\rho, \chi), (\rho \alpha, \beta)\). It is easily verified that the equations of motion as the Euler equation and the equation of continuity. It has been noted for sometime that one can write the Clebsch parametrization as [6]

\[
v_i = -i \operatorname{Tr}(\sigma_3 g^{-1} \partial_i g) \tag{38}
\]
where $g$ is an element of $SU(2)$. In fact with the parametrization (26) one can verify this explicitly with the identifications $\chi \leftrightarrow \chi$, $z = r \exp(i\beta)$, $\alpha = 2r^2/(1 + r^2)$. Thus we may write the action for fluids as

$$S = -i \int d^3x \rho \text{Tr}(\sigma_3 g^{-1} \dot{g}) - \int dt H$$

(39)

There is a subtle specialization in this replacement since the variables in the Clebsch parametrization (36) are not necessarily compact, but are replaced by the $SU(2)$ variables which are compact. In the quantum theory, this leads to quantization of $\int \rho$ and the vorticity. We are therefore considering fluids with an underlying particle structure and which have quantized vorticity. (One may even make a case that this is a better modelling of actual fluids.)

The phase space $\mathcal{P}_F$ for the fluid is thus the set of all maps of the form $\mathbb{R}^3 \to \mathbb{R}^4$ where $\rho \in \mathbb{R}_+$ and $g \in SU(2) = S^3$ form the four-dimensional space $\mathbb{R}^4$. The canonical one-form and two-form for the action (39) are easily found to be

$$\mathcal{A} = -i \int d^3x \rho \text{Tr}(\sigma_3 g^{-1} \delta g)$$

$$\Omega = -i \int d^3x \left[ \delta \rho \text{Tr}(\sigma_3 g^{-1} \delta g) - \rho \text{Tr}(\sigma_3 g^{-1} \delta g \ g^{-1} \delta g) \right]$$

(40)

The contraction of $V(\xi)$ of (24) with this $\Omega$ leads to $V(\xi) \Omega = -\delta \int \rho \phi_a \xi^a$. Thus the generator of left translations of $g$ is now given by $\rho \phi_a$. The variable $S_a(x) = -\rho \phi_a$ obeys the Poisson bracket relations

$$\{S_a(x), S_b(y)\} = \epsilon_{abc} S_c(x) \delta^{(3)}(x - y)$$

(41)

The transformation $g \to g + a^i \partial_i g$ has to be augmented by the transformation of $\rho$. By considering the contraction of $V(a)$ with $\Omega$, we can see that the transformation which can be implemented canonically is given by

$$V(a) + \tilde{V}(a) = V(a) + \int d^3x \partial_i (\rho a^i) \frac{\delta}{\delta \rho}$$

(42)

For this vector field, we find

$$(V(a) + \tilde{V}(a)) \Omega = -\delta \int d^3x a^i \left[ -i \rho \text{Tr}(\sigma_3 g^{-1} \partial_i g) \right]$$

(43)

which identifies the momentum density as

$$J_i = -i \rho \text{Tr}(\sigma_3 g^{-1} \partial_i g)$$

(44)
The complete set of Poisson bracket relations can then be worked out as

\[
\begin{align*}
\{S(\xi), S(\xi')\} &= S(\xi \times \xi') \\
\{\rho(f), S(\xi)\} &= 0 \\
\{\rho(f), \rho(h)\} &= 0 \\
\{\rho(f), g(x)\} &= -ig(x) \, t_3 \, f(x) \\
\{J(a), \rho(f)\} &= \rho(a \cdot \partial f) \\
\{J(a), g(x)\} &= -a \cdot \partial g(x) \\
\{J(a), J(b)\} &= J(a \cdot \partial b - b \cdot \partial a)
\end{align*}
\]

(45)

\[(\xi \times \xi')^a = \epsilon^{abc} \xi^b \xi^c.\]

A brief aside on the quantum theory will be useful at this stage. In the quantum theory, the above bracket relations become commutation rules for the corresponding operators. Consider the unitary transformation given by \(U = \exp(-2\pi i C), \ C = \int \rho.\) Since this corresponds to \(f = 2\pi,\) we see that \(S(\xi)\) and \(J(a)\) are invariant under \(U.\) The only variable which changes is \(g,\) with \(U^\dagger gU = g \exp(i\pi \sigma_3) = -g.\) Since all the physical variables involve even numbers of \(g\)'s, we see that \(U\) can be taken as the identity operator. Thus the angular nature of the variable \(\chi\) in \(g\) leads to the quantization condition \(\int \rho = N\) for some integer \(N.\) This is part of the quantization condition alluded to earlier.

It is clear from the choice of fluid variables and the bracket relations (45) that, as far the canonical structure is concerned, the ferromagnet can be considered as an incompressible fluid for which \(\rho\) is a constant, to be set to \(1/v.\) Thus a reduction of the phase space \(P_F\) of fluid mechanics by the constraint \(\rho \approx 1/v\) will lead to the ferromagnet. This shows why only volume-preserving transformations \(x^i \to x^i + a^i\) can be canonically implemented, since the density has been fixed in the reduction of the phase space.

The bracket relations (45) show that \(\rho\) is the generator of transformations of the form \(g \to ge^{it_3 \varphi}.\) This may be regarded as a ‘gauge’ transformation, to be factored out to go to the physical phase space \(P\) for the ferromagnet. In the canonical setting, factoring out the gauge transformations \(g \to ge^{it_3 \varphi}\) is achieved by setting \(\rho \approx 1/v\) and also imposing a conjugate, gauge-fixing constraint and then defining Dirac brackets. The variables \(S_a\) are not affected since they are invariant under the gauge transformations (i.e., \(\rho\) commutes with them) and so they project down unaltered in form to \(P.\) \(J(a)\) are not ‘gauge-invariant’, and so the form of \(J(a)\) will be changed by the projection to the phase space \(P.\) One has to choose a gauge in which the projected \(J(a)\) are defined. There will be no unique expression for the projected \(J(a).\) Another related difficulty is that any gauge-fixing has singularities \([1].\) This is most easily seen by noting that the curl of \(J_i\) is well defined even after we set \(\rho\) to a constant,

\[
dJ = \frac{1}{2}(\partial_i J_j - \partial_j J_i) \, dx^i \wedge dx^j = i\text{Tr}(\sigma_3 g^{-1} dg \wedge g^{-1} dg) \tag{46}
\]
The right hand side is invariant under $g \rightarrow ge^{i\tau \varphi}$; it is the expression for the field of a monopole. If we can solve this equation for $J_i$ expressing it entirely in terms of $SU(2)/U(1)$ variables, there is no difficulty with defining everything on $\mathcal{P}$. But solving (46) requires a gauge choice for $J_i$ and will involve the Dirac string. There is no gauge choice which can avoid this singularity. A patchwise, nonsingular solution is possible if we can move the string around by gauge transformations, which is effectively what is achieved by keeping the $(\rho, \chi)$-degrees of freedom.

In conclusion, for the ferromagnet, a canonical generator of coordinate transformations can be defined only if we restrict to volume-preserving transformations. Secondly, the ferromagnet may be regarded as an incompressible fluid. This analogy allows one to define the generator of arbitrary coordinate transformations (or momentum density) in an enlarged phase space $\mathcal{P}_F$. Any choice of constraints which allows us to go back to the ferromagnet will have singularities and lead to singular and nonunique expressions for the momentum density.

### 4 Topology of the phase space

It is well known that the topology of the phase space can lead to quantization conditions and to the existence of multivalued wave functions, etc [7]. We will now consider how such possibilities could arise for the ferromagnet.

We start by considering the quantization of a general classical theory formulated in the following manner. We first consider arbitrary complex valued functions on the full phase space $\mathcal{P}$ of the theory which are also square integrable over the full phase space; these are the prequantum wavefunctions. (Strictly speaking, the prequantum wavefunctions are sections of a line bundle on the phase space, the curvature of the bundle being given by the symplectic two-form.) One can then obtain an explicit representation of classical observables as operators on such functions, with the commutator algebra reproducing the Poisson bracket relations. Then one can impose a condition, the so-called polarization condition, which restricts the prequantum wavefunctions to half of the phase space degrees of freedom. This restricted set defines the Hilbert space of the quantum theory. What we have outlined is the geometric quantization of the theory. In this approach, since the prequantum wavefunctions are functions on the phase space, double-valuedness or multivaluedness requires that there be noncontractible curves on the phase space. In other words, we need $\Pi_1(\mathcal{P}) \neq 0$.

For a $d$-dimensional ferromagnet, the phase space $\mathcal{P}$ consists of maps from space $\mathbb{R}^d$ to $SU(2)/U(1)$. We will be interested in the cases $d = 2, 3$. In the ground state of the ferromagnet we have a fixed magnetization, $\langle \phi_a(x) \rangle = \delta_{a3}$. For excitations in a large enough
sample, we can assume that we have the condition \( \phi_a(x) = \delta_{a3} \) at the boundary of space. The set of maps \( \mathbb{R}^3 \to S^2 \) with this boundary condition is equivalent to the set of maps \( S^d \to S^2 \). Thus

\[
P = \{ \text{set of all maps} : S^d \to S^2 \} \quad (47)
\]

We now turn to the connectivity of this space. The set of maps \( S^d \to S^2 \) can be classified by the homotopy group \( \Pi_d(S^2) = \mathbb{Z} \) for \( d = 2, 3 \). Thus the phase space \( P \) consists of a series of mutually disconnected components, each connected component being labelled by the winding number corresponding to the elements of \( \Pi_d(S^2) \). As noted before, we can use \( \phi_a, \ a = 1, 2, 3 \), as coordinates for the target space two-sphere, with \( \phi_a \phi_a = 1 \). A point on \( P \) is thus given by a map \( \phi_a(x) : \vec{x} \to S^2 \). For most of what follows, we can restrict attention to one of the connected components of \( P \), say the topologically trivial one \( P_0 \) which admits the (ground state) configuration \( \phi_a(x) = \delta_{a3} \).

Consider now a closed curve in \( P_0 \) which starts with \( \phi_a(x) = \delta_{a3} \) goes through a sequence of configurations, keeping \( \phi_a \to \delta_{a3} \) as \( |\vec{x}| \to \infty \), and finally returning to \( \phi_a(x) = \delta_{a3} \). Parametrizing the curve by \( \lambda, \ 0 \leq \lambda \leq 1 \), we see that this loop in \( P_0 \) is described by \( \phi_a(x, \lambda) \) with \( \phi_a \to \delta_{a3} \) at the boundary of the region \((x, \lambda)\). It is thus a map from \( S^{d+1} \) to the target space \( S^2 \). The homotopy classes of such closed loops give \( \Pi_1(P_0) \); in our case, this is given by \( \Pi_{d+1}(S^2) \) from the argument given here. Now \( \Pi_3(S^2) = \mathbb{Z} \) and \( \Pi_4(S^2) = \mathbb{Z}_2 \), so that we can expect multivalued prequantum wavefunctions for two spatial dimensions and double-valued wavefunctions for three spatial dimensions. One can expect this to lead to arbitrary statistics or anyons for certain excitations in two spatial dimensions and fermionic excitations (as well as bosons) in three dimensions. (The topology associated with \( \Pi_3(S^2) = \mathbb{Z} \) has been used to construct anyons before, but there are differences in our case, since our discussion involves the phase space and not the configuration space. The use of \( \Pi_4(S^2) = \mathbb{Z}_2 \) is new, as far as we know, although \( \Pi_4(S^3) = \mathbb{Z}_2 \) has been used for the configuration space of Skyrmions.

The techniques for incorporating topological features of the configuration space into the quantum theory are standard by now. But there are some unusual elements in the analysis when it is the phase space which shows such nontrivial connectivity. We will therefore discuss how this can be done, taking the two-dimensional case as an illustrative example. In this case, \( \Pi_1(P_0) = \Pi_3(S^2) = \mathbb{Z} \). This shows that there is a vector potential or one-form on the phase space which is closed or has zero curl, but which cannot be written as a phase space gradient or \( \delta \) of some functional. This flat potential can lead to inequivalent representations of the basic commutation algebra of operators. We will now carry out the quantization incorporating this feature.
5 The flat potential for two spatial dimensions

We start with the construction of the flat potential in the case of two spatial dimensions. Since the relevant topology is \( \Pi_3(S^2) = \mathbb{Z} \), the associated invariant is the Hopf invariant \[8\]. We first construct it on the classical phase space \( \mathcal{P} \). This consists of maps from \( S^2 \) to \( S^2 \) and falls into mutually disjoint sectors labelled by the winding number

\[
Q \equiv \int d^2x \, J_0(x) = \frac{1}{8\pi} \int d^2x \, \epsilon^{ij} \epsilon_{abc} \phi_a \partial_i \phi_b \partial_j \phi_c = \frac{i}{4\pi} \int \text{Tr}(\sigma_3 g^{-1} dg \, g^{-1} dg) \quad (48)
\]

The associated current is given by

\[
J^\alpha = \frac{1}{8\pi} \epsilon^{\alpha\mu\nu} \epsilon_{abc} \phi_a \partial_\mu \phi_b \partial_\nu \phi_c = \frac{i}{4\pi} \epsilon^{\alpha\mu\nu} \text{Tr}(\sigma_3 g^{-1} \partial_\mu g \, g^{-1} \partial_\nu g) \quad (49)
\]

The Hopf invariant can now be defined as

\[
S_{\text{Hopf}} = \int A_\mu J^\mu \quad (50)
\]

where the gauge potential \( A_\mu \) is defined by the Chern-Simons equation

\[
\partial_\mu A_\nu - \partial_\nu A_\mu = \epsilon_{\mu\nu\alpha} J^\alpha \quad (51)
\]

For our purpose, it is best to solve this in the gauge where \( \nabla_i A_i = 0 \) which works out as

\[
A_0(x) = -\epsilon_{ik} \nabla_k \int d^2y \, G(x, y) J_i(y) \\
A_i(x) = \epsilon_{ik} \nabla_k \int d^2y \, G(x, y) J_0(y) \quad (52)
\]

where the \( G(x, y) \) is the Green’s function

\[
G(x, y) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik \cdot (x-y)}}{k^2} \quad (53)
\]

We then get the Hopf invariant as

\[
S_{\text{Hopf}} = \epsilon_{ij} \int J_0(x) \nabla_i G(x, y) J_j(y) = \frac{i}{2\pi} \int d^2x \, d^2y \, d\lambda \, J_0(x) G(x, y) \text{Tr}[g^{-1} \nabla_i g, \sigma_3] g^{-1} \partial_\lambda g = \frac{1}{2\pi} \int d^2x \, d^2y \, d\lambda \, J_0(x) \nabla_i G(x, y) \nabla_i \phi_a g^a \quad (54)
\]
where $\partial_\lambda g g^{-1} = -i\theta^a t_a$, $t_a = \sigma_a / 2$. Notice that this is in the form of the integral along the $\lambda$-direction of a potential

$$
C(\xi) = \int d^2 y \, C_a \xi_a = \frac{1}{2\pi} \int d^2 x d^2 y \, J_0(x) \nabla_i G(x, y) \nabla_i \phi_\xi \xi^a
$$

(55)

(Here $\lambda$ denotes the parameter of a curve in the phase space, along which we are integrating. In writing the Hopf invariant as a term in the action, as is done for some theories, this parameter can be taken as time. We have in fact used this for notational simplicity in writing the currents in (49-52).)

Let $L_a$ denote the left-variation of $g$ given by

$$
L_a(x) g(y) = t_a \delta^{(2)}(x - y) g
$$

Since $C$ itself arises from the left-variation of the Hopf term, it is easily verified that it obeys the flatness condition

$$
\delta_\xi C(\xi') - \delta_{\xi'} C_\xi + C(\xi \times \xi') = 0
$$

(56)

where $\delta_\xi = \int d^2 x \, \xi^a(x) L_a(x)$ and $(\xi \times \xi')^a = \epsilon^{abc} \xi^b \xi'^c$.

We now discuss how this classically available flat potential can be incorporated in the quantum theory. This is not easily done, because $C$ involves the phase space coordinates and so becomes an operator in any representation in general. (This does not happen in $\sigma$-models where $SU(2)/U(1)$ is just the coordinate part of the phase space; in that case, all quantities appearing in the expression for $C$ commute among themselves and $C$ is realized as a $c$-number function in the Schrödinger representation.) There are two ways to take account of $C$ in our theory. The first involves the use of coherent states on the coset $SU(2)/U(1)$ so that the difference between the prequantum wavefunctions and the true wavefunctions is essentially a holomorphicity requirement. The second approach will involve writing an operator version of the potential (55). We will discuss both these approaches briefly.

In the first approach, the wavefunctions are taken to be functionals of $g$, of the form $\Psi[g]$. On these, we can define right translations $R_a$ and left translations $L_a$. As functional differential operators, these are given by

$$
L_a = i \left( E^{-1} \right)_a^i \frac{\delta}{\delta \theta^i}, \quad R_a = i \left( \tilde{E}^{-1} \right)_a^i \frac{\delta}{\delta \theta^i}
$$

(57)

In these expressions, $\theta^i$ denote the group parameters. The action of these translations on $g$ is given by $L_a g = t_a g$, $R_a g = g t_a$.

Corresponding to the symplectic potential (3), namely, $A = -(i/v) \int d^2 x \, \text{Tr}(\sigma_3 g^{-1} \delta g)$, treated as a vector potential, we can define covariant derivatives on the phase space as

$$
R_\pm = R_\pm, \quad R_3 = R_3 - (1/v).
$$

The condition that the phase space is really $SU(2)/U(1)$ and
not $SU(2)$ is expressed by $\tilde{R}_3 \Psi[g] = 0$. The polarization or holomorphicity condition can be taken as $\tilde{R}_+ \Psi[g] = 0$.

The prequantum operators corresponding to $\phi_a$ (or $S_a = -\phi_a/v$) will be given as functional differential operators acting on functions of $g$. From the general rules they are seen to be given in terms of the left translations operators $L_a$ as

$$\hat{\phi}_a \Psi[g] = -v L_a \Psi[g] = -iv \left( E^{-1} \right)^i_a \frac{\delta}{\delta \phi^i} \Psi[g]$$

(58)

where $\delta gg^{-1} = it_a E^a_i d\phi^i$, $L_a g = t_a g$. The prequantum operators can be used as the true quantum operators only if the polarization condition commutes with them. This is the case for us, since left and right translations commute in general. Thus we may use the differential operators (58) as the quantum version of $\phi_a$ for the coherent state representation for the $\Psi$'s defined by the conditions

$$R_3 \Psi[g] = 0$$

$$R_+ \Psi[g] = 0$$

(59)

We can now obtain a different representation of the commutation rules (30) by adding a term proportional to the flat potential $C$ to the left translation operators, i.e.,

$$\hat{\phi}_a = -v (L_a + k A_a)$$

(60)

where $k$ is a parameter which labels the statistics of the particles involved. Clearly, since $L_a$ is the left translation operator, the commutation rules (30) are satisfied by virtue of the flatness condition (56). (Notice that we can write $C_a = -i L_a S_{\text{Hopf}}$.) We can thus use this representation for various observables, in particular, in the Hamiltonian to construct the Schrödinger equation for the wavefunctions. The physics of the problem will clearly depend on the choice of the representation or the parameter $k$ and amounts to a different quantization of the theory. However, there is a difficulty which has to do with the polarization condition. While $R_a$ and $L_b$ commute and so the use of $L_b$ for $\hat{\phi}_b$ preserve the conditions (59), this is not the case once we use the modified representation (60). But this can be taken care of by using a modified version of the right translations as well. By taking the right variation of the Hopf invariant, we get a potential

$$\tilde{C}_a(y) = \frac{1}{2\pi} \int d^2 x \, J_0(x) \nabla_i G(x, y) \text{Tr} \left( g^{-1} \nabla_i g [\sigma_3, t_a] \right)$$

$$= -i R_a S_{\text{Hopf}}$$

(61)

This is a flat potential for right translations and, using this, we obtain a new representation for the covariant derivatives as $\mathcal{R}_a + k \tilde{C}_a$. It is now easy to see that

$$[L_a + k C_a, \tilde{R}_b + k \tilde{C}_b] = 0$$

(62)
since $\mathcal{C}_a \sim L_a S_{\text{Hopf}}$, $\tilde{\mathcal{C}}_a \sim R_a S_{\text{Hopf}}$. Thus, instead of (59), we can use the conditions

$$
\begin{align*}
(R_3 + k \tilde{C}_3 - \frac{1}{v}) \Psi[g] &= 0 \\
(R_+ + k \tilde{C}_+) \Psi[g] &= 0
\end{align*}
$$

(63)

The general solution to this is of the form

$$
\Psi[g] = \exp\left(ik \int d\lambda \left[\tilde{C}_+ 2i \text{Tr}(g^{-1} \partial_\lambda g \tilde{t}_-) + \tilde{C}_3 2i \text{Tr}(g^{-1} \partial_\lambda g t_3)\right]\right) \Phi[g]
$$

(64)

where $\Phi$ obeys the old conditions (59). Notice that we are integrating the flat potentials $\tilde{C}_a$ along two of the group directions; if we did this for all three, we would simply have $e^{ikS_{\text{Hopf}}}$. We may therefore write

$$
\Psi[g] = \exp (ik S_{\text{Hopf}} - ik \Theta) \Phi[g]
$$

(65)

where

$$
\Theta = 2i \int d\lambda \tilde{C}_- \text{Tr}(g^{-1} \partial_\lambda g t_+)
$$

(66)

The action of $\hat{\phi}_a$ now simplifies as

$$
(L_a + k \mathcal{C}_a) \Psi = L_a e^{-ik \Theta} \Phi[g]
$$

(67)

Thus the effect of the extra term is the phase factor $e^{-ik \Theta}$.

We have thus obtained a coherent state approach to the quantization where the multivaluedness due to $\Pi_1(\mathcal{P}) = \mathbb{Z}$ can be taken account of explicitly by a phase factor, the formulation being compatible with the polarization requirement on the wavefunctions.

The second way of incorporating the effect of $\mathcal{A}_a$ requires the operator version of (55). The operator corresponding to the topological charge density may be defined by the commutation rule

$$
[\phi_a(x), J_0(f)] = -i \frac{v}{4\pi} \epsilon^{ij} \nabla_i f \nabla_j \phi_a(x)
$$

(68)

where $J_0(f) = \int d^2 x J_0(x) f(x)$. We see easily that the topological charge commutes with $\phi_a(x)$, and hence with any local observable, so that it is superselected, as expected for a topological charge. We can also check that the expression

$$
J_0 = \frac{1}{8\pi} \epsilon^{ij} \epsilon_{abc} \nabla_i \phi_b \phi_a \nabla_j \phi_c
$$

(69)

can be used as the operator expression with the ordering indicated. If singularities involved in the commutator with $\phi_a(x)$ are regulated in an angular symmetric way, this does lead to the commutation rule (68).
Consider now the expression for $C_a$; we write this as

\[ C_a(x) = \frac{1}{2\pi} \int d^2y \ J_0(y) \nabla_i G_{\text{Reg}}(y, x) \nabla_i \phi_a(x) \]

\[ G_{\text{Reg}}(x, y) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip \cdot (x-y)}}{p^2} e^{-p^2/M^2} \tag{70} \]

As the regulator parameter $M \to \infty$ this reverts to the expression \([55]\). The regularization is angular symmetric; there is no ordering ambiguity in the expression for $C_a$ with this regulator. We can now define an operator $\Lambda$ by the commutation relation

\[ [\phi_a(x), \Lambda] = -iv \ C_a(x) \tag{71} \]

In terms of $\Lambda$ we can construct $U = \exp(-ik\Lambda)$. The new representation is then given by $\phi_a = U^\dagger \phi_a U$. Even though formally of the form of a unitary operator, $U$ is, in general, not unitary for arbitrary $k$ and hence this representation is inequivalent to the previous one with $k = 0$.

\section{Conclusion}

We have addressed two specific problems in this paper, both of which are related to topological issues in the continuum version of the ferromagnet. The first one has to do with the definition of the momentum density. Only moments of the momentum density corresponding to volume-preserving transformations can be defined. A more general definition is possible, in a way free of singularities, only by enlarging the phase space. What is interesting is that the enlarged phase space is mathematically very closely related to the phase space for fluid dynamics. The connection between the ferromagnet and fluid dynamics that we have emphasized should enable one to apply the powerful techniques developed in the latter discipline to the former and lead, in general to a successful cross-fertilization of both fields.

The second problem deals with the fact that in two and three dimensions the connectivity of the phase space is nontrivial. This can lead to nontrivial statistics for excitations. But the relevant topological term involves all the phase space coordinates which are mutually noncommuting. There are subtleties due to this fact, and therefore, unlike the case of the usual sigma models, a full quantum treatment requires additional considerations. We have given one way of handling this situation.

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References

[1] F.D.M. Haldane, Phys. Rev. Lett. 57, 1488 (1986).

[2] R. Balakrishnan and A.R. Bishop, Phys. Rev. Lett. 55, 537 (1985).

[3] S.L. Sondhi et al., Phys. Rev. B 47, 16419 (1993); K. Moon et al., Phys. Rev. B 51, 5138 (1995); R. Ray, Phys. Rev. B 60, 14154 (1999).

[4] see, for example, V.I. Arnol’d, Mathematical Methods of Classical Mechanics (Springer-Verlag, New York, 1989); V. Guillemin and S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press (1990).

[5] A. Clebsch, J. Reine Angew. Math. 56, 1 (1859); H. Lamb, Hydrodynamics (Cambridge University Press, Cambridge UK, 1932) p. 248; C. C. Lin in International School of Physics E. Fermi (XXI), G. Careri ed. (Academic Press, New York, 1963); for a recent review, see R. Jackiw, Lectures on Fluid Dynamics (Springer, New York, 2002).

[6] R. Jackiw and S-Y. Pi, Phys. Rev. D61, 105015 (2000); R. Jackiw, V. P. Nair and S.-Y. Pi, Phys. Rev. D62, 085018 (2000).

[7] N. Woodhouse, Geometric Quantization, Oxford University Press (1980); A.P. Balachandran, G. Marmo, B.S. Skagerstam and A. Stern, Classical Topology and Quantum States, World Scientific, Singapore (1991).

[8] F. Wilczek, Phys.Rev.Lett. 49, 957 (1982); F. Wilczek and A. Zee, Phys.Rev.Lett. 51, 2250 (1983); M. Bowick, D. Karabali and L.C.R. Wijewardhana, Nucl.Phys. B271, 417 (1986); G. Semenoff and P. Sodano, Nucl.Phys. B328, 753 (1989); for a recent review, see F. Wilczek, Fractional Statistics and Anyon Superconductivity, World Scientific, Singapore (1990).