Extending the weak cosmic censorship conjecture to the charged Buchdahl star by employing the gedanken experiments

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Abstract. In this paper, we wish to investigate the weak cosmic censorship conjecture (WCCC) for the non black hole object, Buchdahl star and test its validity. It turns out that the extremal limit for the star is over-extremal for black hole, $Q^2/M^2 \leq 9/8 > 1$; i.e., it could have $9/8 \geq Q^2/M^2 > 1$. By carrying out both linear and non-linear perturbations, we establish the same result for the Buchdahl star as well. That is, as for black hole it could be overcharged at the linear perturbation while the result is overturned when the non-linear perturbations are included. Thus WCCC is always obeyed by the Buchdahl star.

Keywords: Compact objects, massive stars, WCCC, gravity
1 Introduction

In general relativity, Penrose first proposed the singularity theorem in 1965 [1] which was further generalized into the powerful Penrose-Hawking singularity theorems [2]. The emergence of singularity marks the limit of validity of a theory, and so is the case for Einstein’s theory of gravitation, General relativity (GR). Although it is the most successful theory, yet it is incomplete as indicated by occurrence of singularities [2]. Later on, in 1969 Penrose proposed [3] the weak cosmic censorship conjecture (WCCC) stating that gravitational collapse would always result in forming an event horizon of black hole which would hide the singularity inside. That is, there is a cosmic censor prohibiting occurrence of naked singularity in gravitational collapse. A singularity would therefore be always hidden behind the black hole horizon and would never be visible to outside observer. However it is a conjecture and there is no theorem proving its validity. There has been intense activity of testing this conjecture by constructing counter examples [see, e.g. 4–13]. Despite this, the jury is still out whether naked singularities do or do not form in gravitational collapse. That is, the question of validity of WCCC remains open.

Another aspect of the question is, could an existing black hole horizon of charged and/or rotating black hole be destroyed by over-extremalizing (i.e., \(Q^2 > M^2\) for a charged black hole). That is, could a black hole be over-extremalized so that singularity is laid bare – naked. The question was first addressed by Wald [14]. He showed that an extremal black hole could never be extremalized by a test particle accretion. Further it was also shown that a non-extremal black hole cannot be converted into an extremal one [15]. What happens is that as extremality approaches, the parameter window for accreting particle hitting the horizon pinches off. That is once a horizon is formed, it cannot be destroyed and WCCC cannot be violated.

Later, Hubeny [16] showed that a black hole can indeed be overcharged in a discontinuous process involving linear order perturbations. That is, extremality cannot be attained but it could perhaps be jumped over in a discrete process. The above experiment was also applied to test the WCCC for Kerr and Kerr-Newman black holes [17, 18].
The horizon of black hole can be destroyed provided in-falling particle adds enough electric charge/angular momentum to black hole’s charge/angular momentum. After that, a large number of works have been done in various contexts [see, e.g. 19–28] initially without including backreaction effects. Later, it was also shown that inclusion of backreaction effects did not alter the earlier result [see,e.g. 29–34]. Also it is worth noting that the above experiment was extended to the various black hole solutions, i.e.asymptotically AdS black holes [35–39].

Although this experiment gave rise to intense discussion on violation of the WCCC, Sorce and Wald [40] once again put it all to rest by showing that when second order perturbations are taken into account, the WCCC that was violated at the linear order would always be restored. So a new version of the gedanken experiment has been proposed, with inclusion non-linear perturbations [34, 41–47], the linear order result of WCCC violation is always overturned. The new version of the gedanken experiment has been applied to black holes in higher dimensions. It was already shown that higher dimensional charged black hole could be overcharged for linear order perturbations [48]. However, five dimensional charged rotating black hole having single rotation cannot be over-extremilized when rotation parameter dominates over charge [49]. However it turns out that in \( D \geq 6 \) [46, 50–52] it cannot be overcharged even at the linear order accretion. Note that the gedanken experiment has also been recently extended to RN-AdS black hole [53] and RN-dS black hole surrounded by dark matter field [54] under the non-linear order perturbation.

In this paper we wish to extend this WCCC analysis to the charged Buchdahl star. It is defined [55, 56] by gravitational potential, \( \Phi(r) = 4/9 \) while black hole by \( \Phi(r) = 1/2 \). The Buchdahl compactness limit is given by \( \Phi(r) \leq 4/9 \). The Buchdahl star is the most compact non black hole object having the limiting compactness bound, and it is the most compact stable object [57, 58]. Thus, it should be pertinent and interesting to extend the weak censorship conjecture to Buchdahl star by applying a new version of the gedanken experiments. Note that the characterization, \( \Phi(r) = 1/2, 4/9 \) respectively for black hole (BH) and Buchdahl star (BS) is universal irrespective of object being uncharged and non-rotating or otherwise. It is remarkable that BS shares almost all BH properties [59] including, in particular, the extremal limit which for the charged case is \( Q^2/M^2 > 9/8 > 1 \). So an extremal BS is over-extremal relative to BH. If quantum mechanical corrections are included, it might be possible to have a solution with the horizon, allowing charge \( Q \) that is slightly greater than \( M (Q < \sqrt{2}M) \) as shown in [60]. However, inclusion of quantum corrections are beyond the scope of the present study. We shall study both linear and non-linear perturbations of BS in the context of WCCC and show that the result is the same as for BH. That is, it is possible to violate WCCC at the linear order which is then overturned when non-linear perturbations are included.

The one critical difference for the Buchdhal star is that its boundary is timelike as against the null horizon for black hole. For the former, perturbations could be reflected back or flow out as timelike boundary could be crossed both ways. This would further restrict the overcharging process and hence would not however conflict with the result
that the Buchdahl star cannot be overcharged.

The paper is organized as follows: In section 2 we briefly discuss the Buchdahl star metric and its properties. In section 3 we describe variational identities and Einstein-Maxwell theory for linear and non-linear variational inequalities. In section 4 we build up linear and non-linear order perturbations inequalities for studying overcharging of Buchdahl star. Finally, we end with a discussion in section 5.

2 The Buchdahl star space-time metric

The spherically symmetric Reissner-Nordström metric describes gravitational field of a charged static object whether black hole or otherwise. It is given by

\[ ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2d\Omega^2, \]  

with the line element of 2-sphere \( d\Omega^2 \) and

\[ F(r) = 1 - 2\Phi(r), \]  

where \( \Phi(r) = (M - Q^2/2r)/r \) with \( M \) and \( Q \) corresponding to mass and electric charge of BS. A black hole is defined by the condition \( \Phi(r = r_+) = 1/2 \) that gives the black hole horizon, while Buchdahl star by \( \Phi(r = r_{BS}) = 4/9 \) that gives the surface, radius of the star [55, 56]. The Buchdahl compactness bound is in general given by \( \Phi(r) \leq 4/9, [58] \). BS is defined by the equality indicating the most compact non black hole object.

The BS condition,

\[ \Phi(r = r_{BS}) = (M - Q^2/2r)/r = 4/9, \]  

gives

\[ \frac{M}{r_{BS}} = \frac{8/9}{1 + \sqrt{1 - (8/9)\beta^2}}, \]  

or BS surface radius is given by

\[ r_{BS} = \frac{9M}{8}(1 + \sqrt{1 - (8/9)\beta^2}), \]  

where \( \beta^2 = Q^2/M^2 \). Here and in what follows, \( r_{BS} \) would indicate BS surface radius. We would use the same symbol, \( r_+ = r_{BS} \), as for BH, but it would be clear from the context, in particular the appearance of the factor 8/9 would clearly identify BS. Thus the extremal limit for BS is defined by \( \beta^2 = 9/8 > 1 \), which is over extremal for BH \( (r_+ = M(1 + \sqrt{1 - \beta^2}) \). It is interesting to note that BS could be overcharged relative to BH.

The electromagnetic potential is given by

\[ A = -\frac{Q}{r}dt. \]
Because of spherical symmetry, it is straightforward to compute area, surface gravity and electric potential at any \( r \), and so at the BS boundary as well. In particular, the surface gravity is given by

\[
k = \frac{F'(r)}{2} \bigg|_{r=r_{BS}} = \frac{M}{r_{BS}^2} \sqrt{1 - \left(\frac{8}{9}\right) \beta^2}.
\] (2.7)

Note that for BH surface gravity, write \((8/9) \beta^2 \rightarrow \beta^2\) and \(r_{BS} \rightarrow r_+\) in the above expression. In the next section we study variational identities and perturbation inequalities for linear and non-linear order perturbations for testing WCCC for the Buchdahl star.

3 Variational identities and Einstein-Maxwell theory

We now adapt the Iyer-Wald formalism to derive perturbation inequalities for linear and non-linear order perturbations under the spherically symmetric perturbation. This formalism bases on a diffeomorphism covariant theory for manifold \( \mathcal{M} \) of any given dimension. This theory can be identified by the Lagrangian \( L \) including \( g_{\alpha\beta} \) and \( \psi \) that respectively describe the gravitational field and other fields that exist in the environment surrounding the object [40, 61]. So, it is then possible to consider \( \phi = (g_{ab}, \psi) \) as dynamical fields around compact objects. A variation of the Lagrangian \( L \) including all these properties can be written as follows:

\[
\delta L = E \delta \phi + d\Theta(\phi, \delta \phi).
\] (3.1)

Here \( E \) describes the equations of motion (EOM) that satisfies \( E = 0 \), whereas \( \Theta \) is referred to as the symplectic potential. Let us then write the symplectic current including the symplectic potential \((n-1)\)-form as

\[
\omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi).
\] (3.2)

Next we are able to define the Noether current \((n-1)\)-form taking into account the dynamical field, \( \phi \), with an arbitrary vector, \( \xi^a \). So we have

\[
J_\xi = \Theta(\phi, L_\xi \phi) - \xi \cdot L.
\] (3.3)

For \( dJ_\xi = 0 \) the Noether current \((n-1)\)-form can be defined by [62]

\[
J_\xi = dQ_\xi + C_\xi.
\] (3.4)

In the above equation \( Q_\xi \) is referred to as the Noether charge, whereas \( C_\xi = \xi^a C_a \) marks the above mentioned theory’s constraint and can be considered to be zero when \( dJ_\xi = 0 \) continues to hold good for EOM.
We then obtain the linear variational identity for given vector, $\xi^a$, on the basis of Eqs. (3.3) and (3.4). For that at the Cauchy surface, $\Xi$, we can write

$$\int_{\partial \Xi} \delta Q_\xi - \xi \cdot \Theta(\phi, \delta \phi) = \int_\Xi \omega(\phi, \delta \phi, L_\xi^\phi)$$

$$- \int_\Xi \xi \cdot E \delta \phi - \int_\Xi \delta C_\xi. \quad (3.5)$$

From the above equation the variation term for $\xi^a$ is the first term on the right-hand side $\int_\Xi \omega(\phi, \delta \phi, L_\xi^\phi)$ that no longer exists as the vector $\xi^a$ is taken to be Killing and represents a symmetry of the dynamical field, $\phi$. As long as $\xi^a$ behaves as Killing, it then satisfies EOM, i.e. $E = L_\xi^\phi = 0$. Following the linear variational identity, one can then have the non-linear one at the same surface

$$\int_{\partial \Xi} \delta^2 Q_\xi - \xi \cdot \delta \Theta(\phi, \delta \phi) = \int_\Xi \omega(\phi, \delta \phi, \mathcal{L}_\xi \delta \phi)$$

$$- \int_\Xi \xi \cdot E \delta \phi - \int_\Xi \delta^2 C_\xi. \quad (3.6)$$

For the Killing vector, $\zeta^a$, Eq. (3.5) then yields

$$\int_{\partial \Xi} \delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi) = - \int_\Xi \delta C_\zeta. \quad (3.7)$$

Here we note that in the above equation one can assume that $\zeta^a$ is vector field for which the dynamical field $\phi$ represents the exterior solution of a stationary Buchdahl star. If this is the case $\zeta^a = \zeta^a(t)$ can then be regarded as the timelike Killing vector field that always satisfies $E = L_\zeta^\phi = 0$ for the EOM. In what follows, we shall restrict ourselves to $\zeta^a$ vector. We further represent the Cauchy surface’s boundaries that can start from spatial infinity and continues up to the bifurcation surface, $B$, at the other end. It is worth noting here that the surface will be of bifurcate type for a near extremal Buchdahl star. So Eq. (3.7) consists of two boundaries, i.e. spatial infinity at one end and the bifurcation surface at the other, and that is given by

$$\int_{\partial \Xi} \delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi) = \int_\infty \delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi)$$

$$- \int_B \delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi). \quad (3.8)$$

Collecting the above results Eq. (3.5) can be obtained for the linear variational identity in the following form

$$\delta M = \int_B [\delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta \phi)] - \int_\Xi \delta C_\zeta. \quad (3.9)$$

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where $\delta M$ is referred to as the contribution to the boundary integral at spatial infinity, i.e.

$$
\int_\infty \delta Q_\xi - \zeta \cdot \Theta(\phi, \delta \phi) = \delta M.
$$

(3.10)

What is done in the above is also similar to the case for the non-linear variational identity. Thus, we have

$$
\delta^2 M = \int_B |\delta^2 Q_\xi - \zeta \cdot \delta \Theta(\phi, \delta \phi)|
- \int_\Sigma \zeta \cdot \delta E \delta \phi - \int_\Sigma \delta^2 C_\xi + E_\Xi(\phi, \delta \phi).
$$

(3.11)

In the above expression, the energy $E_\Xi(\phi, \delta \phi)$ can be identified by $\delta \phi$ at the Cauchy surface $\Xi$. We will further discuss the terms on the right hand side of Eq. (3.11).

For our purpose we further consider Einstein-Maxwell theory for studying linear and non-linear order perturbations for over-extremalizing Buchdahl star. We write the Lagrangian in the form

$$
L = \frac{e}{16\pi} \left( R - F^{a\beta} F_{a\beta} \right),
$$

(3.12)

where $e$ and $F_{a\beta}$ respectively refer to the volume element and Faraday tensor of electromagnetic field. For the dynamical fields, $\phi = (g_{ab}, A_a)$, we write

$$
E(\phi) \delta \phi = -e \left( \frac{1}{2} T^{ab} \delta g_{ab} + j^a \delta A_a \right),
$$

(3.13)

with the stress-energy tensor defined by

$$
T_{ab} = \frac{1}{8\pi} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) - T^{EM}_{ab},
$$

(3.14)

and the current

$$
j^a = \frac{1}{4\pi} \nabla_b F^{ab}.
$$

(3.15)

In what follows, we shall write the symplectic potential $\Theta$ that is formed from two parts, i.e. electromagnetic and gravity parts. So it takes the following form as

$$
\Theta_{ijk}(\phi, \delta \phi) = \frac{1}{16\pi} \epsilon_{aijk} g^{ab} g^{cd} (\nabla_d \delta g_{bc} - \nabla_b \delta g_{cd})
- \frac{1}{4\pi} \epsilon_{aijk} F^{ab} \delta A_b,
$$

(3.16)

with Levi-Civita tensor $\epsilon_{aijk}$. The Einstein-Maxwell theory allows the symplectic current to have the form as

$$
\omega_{ijk} = \frac{1}{4\pi} \left[ \delta_2 (\epsilon_{aijk} F^{ab}) \delta_1 A_b - \delta_1 (\epsilon_{aijk} F^{ab}) \delta_2 A_b \right]
+ \frac{1}{16\pi} \epsilon_{aijk} \omega^a.
$$

(3.17)
This clearly shows that it consists of the electromagnetic and gravity parts. The part relevant to the gravity, \( w^i \), can be written in the following form

\[
w^i = p_{ijkab} \left( \delta_2 g^i_{jk} \nabla h \delta_1 g_{ab} - \delta_1 g^i_{jk} \nabla h \delta_2 g_{ab} \right),
\]

(3.18)

where \( p_{ijkab} \) is given by

\[
p_{ijkab} = g^{ia} g^{bj} g^{kh} - \frac{1}{2} g^{ih} g^{ja} g^{bk} - \frac{1}{2} g^{ij} g^{kh} g^{ab} - \frac{1}{2} g^{jk} g^{ia} g^{bh} + \frac{1}{2} g^{jk} g^{ih} g^{ab}.
\]

(3.19)

Here, we adopt the following conditions

\[
L_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a,
\]

\[
\nabla_a A_b = F_{ab} + \nabla_b A_a,
\]

(3.20)

for the Noether current \( J_\xi \) to have the following form

\[
(J_\xi)_{ijk} = \frac{1}{8\pi} \epsilon_{aijk} \nabla_b (\nabla^{[b} g^{a]}_{c}) + \epsilon_{aijk} T^a_{b\xi} g^{cb} + \frac{1}{4\pi} \epsilon_{aijk} \nabla_c (F^{ca} A_b \xi^b) + \epsilon_{aijk} A_b j^a \xi^b.
\]

(3.21)

Also the Noether charge \( Q_\xi \) (see Eq. (3.4)) can be identified by

\[
(Q_\xi)_{ijk} = -\frac{1}{16\pi} \epsilon_{ijkab} \nabla^a \xi^b - \frac{1}{8\pi} \epsilon_{ijkab} T^{ab}_{\xi} A_c \xi^c,
\]

(3.22)

and the constraint of the theory

\[
(C_\xi)_{ijk} = \epsilon_{aijk} (T^a_{\xi} + A_k j^a).
\]

(3.23)

4 Perturbation inequalities and gedanken experiment

In this section, our aim is to show, as for the charged black hole, whether the results for linear and non-linear order perturbations could also be established for charged Buchdahl star. For that we adopt the new gedanken experiment, proposed by Sorce and Wald [40] involving non-linear perturbations. For this experiment, we shall consider a one-parameter family of field \( \phi(\alpha) \) perturbation for the given spacetime. In doing so, we have

\[
G_{ab}(\alpha) = 8\pi \left[ T^{GR}_{ab}(\alpha) + T^{EM}_{ab}(\alpha) \right],
\]

(4.1)

\[
\nabla_b F^a(\alpha) = 4\pi j^a(\alpha).
\]

(4.2)

Note that \( \phi(0) \) satisfies \( T_{ab}(0) = 0 \) and \( j^a(0) = 0 \), i.e., infalling particles are supposed to cross the BS boundary. As discussed earlier, the following hypersurface \( \Sigma = \Sigma_1 \cup H \) has
been taken for this family of perturbation \( \phi(\alpha) \), where \( \Xi \) involves a region that begins from the so-called bifurcation surface \( B \) at one end and continues up to the BS boundary portion \( H \) so that it turns spacelike when it reaches \( \Xi_1 \) at the other end. It approaches asymptotical flatness at infinity. The spacetime geometry representing Buchdahl star is assumed to be linearly stable under the one parameter family of field \( \phi(\alpha) \) perturbation. Note that we further work with Gaussian null coordinates at \( H \), where one can write

\[
\int_B \delta Q_\zeta(\alpha) = \frac{k}{8\pi} A_B(\alpha),
\]

where \( A_B \) refers to the bifurcate surface area [40]. This perturbation however does vanish on the bifurcation surface \( B \) as a consequence of hypersurface’s property. Regardless of this fact, it turns out that the spacetime geometry tends to new perturbed state with \( M(\alpha) \) and \( Q(\alpha) \) at very late time due to the dynamical field perturbed by in-falling matter sources.

Following the perturbation Eq. (4.3) and recalling Eq. (3.9) we further derive inequality for the linear order perturbation

\[
\delta M = \int_B \left[ \delta Q_\zeta - \zeta \cdot \Theta(\phi, \delta\phi) \right] - \int_\Xi \delta C_\zeta. \quad (4.4)
\]

It turns out that the first term in the above equation vanishes at the bifurcation surface, and so it remains

\[
\int_\Xi \delta C_\zeta = \int_H \epsilon_{aijk} \zeta^b_1 \left( \delta T^a_b + A_b \delta j^a \right). \quad (4.5)
\]

By imposing the condition \( \Phi^+ = -\zeta^b A_b|_H \) with \( \int_H \delta(\epsilon_{aijk} j^a) = \delta Q \), Eq. (4.4) takes the form as

\[
\delta M - \Phi^+ \delta Q = - \int_H \epsilon_{aijk} \zeta^b_1 \delta T^{ab}, \quad (4.6)
\]

where the volume element is defined by \( \epsilon_{aijk} = -4k^i[a \tilde{e}_{ijk}] \) at the BS boundary portion \( H \). For that it is straightforward to obtain the null energy condition satisfying

\[
\delta T_{ab} k^a k^b \geq 0. \quad (4.7)
\]

This, in turn, leads to the following form for linear order perturbation

\[
\delta M - \Phi^+ \delta Q \geq 0. \quad (4.8)
\]
In a similar way, for non-linear perturbation, one obtains

\[ \delta^2 M = \int_B \left[ \delta^2 Q \zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi) \right] - \int_{z} \zeta \cdot \delta E \delta \phi \\
- \int_\Sigma \zeta \cdot \delta E + E_\Sigma(\phi, \delta \phi) \\
= \int_B \left[ \delta^2 Q \zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi) \right] + E_H(\phi, \delta \phi) \\
- \int_H \zeta \cdot \delta E \delta \phi - \int_H \epsilon_{aijb} \delta^2 T^a_k + A_b \delta^2 \eta^a \\
= \int_B \left[ \delta^2 Q \zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi) \right] + E_H(\phi, \delta \phi) \\
+ \int_H \epsilon_{ijk} \zeta^a \delta^2 T^a_k + \Phi + \delta^2 Q. \tag{4.9} \]

In the above equation, we used the gauge condition \( \zeta^a A_a = 0 \) on the portion \( H \) with the timelike Killing vector field \( \zeta^a \) always tangent to \( H \). We then apply \( \delta^2 T^a_k k^a k^d \geq 0 \) to write

\[ \delta^2 M - \Phi + \delta^2 Q = \int_B \left[ \delta^2 Q \zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi) \right] + E_H(\phi, \delta \phi). \tag{4.10} \]

One can then further consider \( \phi(\alpha)^{BS} \) as a perturbed field as a consequence of matter falling into the Buchdahl star. After matter is absorbed, Buchdahl star parameters become

\[ M(\alpha) = M + \alpha \delta M \text{ and } Q(\alpha) = Q + \alpha \delta Q. \tag{4.11} \]

From the above equation, we shall choose \( \delta M \) and \( \delta Q \) as the linear order perturbations, as shown in Eq. (4.8). Let us then compute rest of the terms on the right hand side of Eq. (4.10) for the perturbation field \( \phi^{BS} \), according to which \( \delta^2 M = \delta^2 Q_B = \delta E = E_H(\phi, \delta \phi^{BS}) = 0 \) always. Therefore, for this perturbation field we apparently have

\[ \delta^2 M - \Phi + \delta^2 Q = \int_B \left[ \delta^2 Q \zeta - \zeta \cdot \delta \Theta(\phi, \delta \phi^{BS}) \right]. \tag{4.12} \]

From the above equation, the term \( \int_B \zeta \cdot \delta \Theta(\phi, \delta \phi^{BS}) \) vanishes since \( \zeta^a = 0 \) continues to hold at the bifurcation surface \( B \). Hence, for the perturbation field, taking into account Eq. (4.3) the non-linear variational inequality is given by

\[ \delta^2 M - \Phi + \delta^2 Q \geq -\frac{k}{8\pi} \delta^2 A^{BS}. \tag{4.13} \]

Following this new version of the gedanken experiment as alluded above, we investigate overcharging of a nearly extremal Buchdahl star, being an analogue of extremal black hole. We recall Eq. (5) indicating extremality by \( \beta^2 = 9/8 \) and over-extremality by \( \beta^2 > 9/8 \). The latter means there exists no boundary radius for BS.
For BH, it then exposes the singularity—bare it naked while for BS, it would mean that the star would probably disperse away. This is perhaps because since it is already overcharged, repulsive contribution due to charge would override attraction due to mass.

We first explore an extremal BS; i.e.,

\[
M^2 - \frac{8}{9}Q^2 = 0. \tag{4.14}
\]

Note that when a particle falls into BS, its parameters would change to \( M + \delta M \) and \( Q + \delta Q \). An extremal BS can be overcharged if and only if the following inequality

\[
\delta M < \frac{8}{9} \frac{Q}{M} \delta Q, \tag{4.15}
\]

holds good. However, for an extremal BS the corresponding electromagnetic potential is

\[
\Phi_{\text{rbs}} = \frac{8}{9} \frac{Q}{M}. \tag{4.16}
\]

Thus, as mentioned earlier, when the null energy condition is satisfied, the linear order perturbation inequality (i.e. Eq. (4.8)) becomes

\[
\delta M - \frac{8}{9} \frac{Q}{M} \delta Q \geq 0. \tag{4.17}
\]

The above two inequalities are in clear contradiction, hence an extremal Buchdahl star can not be overcharged via the gedanken experiment.

Let us then consider a near extremal Buchdahl star and apply the gedanken experiment afresh allowing for second order perturbations. We consider one parameter family of perturbations, \( g(\alpha) \), and write

\[
g(\alpha) = M(\alpha)^2 - \frac{8}{9}Q(\alpha)^2. \tag{4.18}
\]

Now the question is what happens when higher order perturbations are included, would the linear order result hold good or not? For a near extremal BS, we write from Eq. (4.18) \( g(0) = M^2 \epsilon^2 = M^2(1 - (8/9)Q^2/M^2)^{1/2} \) with \( \epsilon \neq 0 \) and \( \alpha \neq 0 \), the function \( g(\alpha) \) can be expanded up to second order in both \( \epsilon \) and \( \alpha \) as

\[
g(\alpha) = M^2 \epsilon^2 + \left[ 2M \delta M - \frac{16}{9} Q \delta Q \right] \alpha \\
+ \left[ M \delta^2 M - \frac{8}{9} Q \delta^2 Q + \delta M^2 - \frac{8}{9} \delta Q^2 \right] \alpha^2 \\
+ O(\alpha^3, \alpha^2 \epsilon, \alpha \epsilon^2, \epsilon^3). \tag{4.19}
\]

Here the second and third terms respectively correspond to the linear and non-linear order perturbations.
Following [16, 40], we define the minimum possible value of $\delta M$ given by the requirement that particle falls into the star which is given by

$$\delta M \geq \frac{Q}{r_{BS}} \delta Q = \frac{8}{9} \frac{Q}{M} \delta Q \left(1 - \epsilon\right) + O(\epsilon^2),$$  \hspace{1cm} (4.20)

This refers to the minimum energy (i.e. the optimal choice of linear order perturbation) for a charged particle to enter the BS. For the above minimal energy it is straightforward to obtain linear order perturbation of $g(\alpha)$ in the following form

$$g(\alpha) = M^2 \epsilon^2 + 2M(8/9)^{1/2} \delta Q \left[ \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} - \sqrt{1 - \epsilon^2} \right] \alpha$$

$$= M^2 \epsilon^2 - 2(8/9)^{1/2} M \delta Q \epsilon \alpha + O(\alpha^2).$$  \hspace{1cm} (4.21)

From Eq. (4.21) it is obvious that $g(\alpha) < 0$ is attainable for the linear order perturbation if and only if the following condition is satisfied

$$\delta Q > \frac{9}{16} \frac{M^2}{Q} \frac{\epsilon}{\alpha} = \frac{M}{2(8/9)^{1/2}} \frac{\epsilon}{\alpha}.$$  \hspace{1cm} (4.22)

Hence, it is possible to overcharge BS by linear order perturbation as is the case for charged black hole.

Next let us include the second order perturbations using the optimal choice for the linear order Eq. (4.20). For the non-linear part, i.e., the third term of Eq. (4.19) can be written as follows:

$$\left[ M \delta^2 M \Phi - \frac{8}{9} Q \delta^2 Q + \delta M^2 - \frac{8}{9} \delta Q^2 \right] \alpha^2$$

$$= M \left[ \delta^2 M - \Phi^+ \delta^2 Q + \frac{1}{M} \left( \frac{8}{9} \right)^{1/2} \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} \delta Q \right]^2$$

$$- \frac{8}{9} \left( \frac{\delta Q^2}{M} \right) \alpha^2 = M \left[ \delta^2 M - \Phi^+ \delta^2 Q + O(\epsilon) \right] \alpha^2.$$  \hspace{1cm} (4.23)

Taking into account Eq. (4.13) $\delta^2 M - \Phi^+ \delta^2 Q$ in the above equation can be written as follows:

$$\delta^2 M - \Phi^+ \delta^2 Q \geq - \frac{k}{8\pi} \delta^2 A_{BS},$$  \hspace{1cm} (4.24)

with the BS boundary area $A$ and the surface gravity $k$, as shown in Eq. (2.7). It is then straightforward to obtain the surface gravity, $k$, in the following form

$$k = \frac{F'(r)}{2} \bigg|_{r=r_+}$$

$$= \frac{M(8/9)^2}{(M + M\epsilon)^2} - \frac{(1 - \epsilon^2)(8/9)^2}{(M + M\epsilon)^3}$$

$$= \frac{64}{81M^3} \epsilon + O(\epsilon^2).$$  \hspace{1cm} (4.25)
Following the above procedure we then obtain the analytic form for $\delta^2 A^{BS}$ as follows:

$$\delta^2 A^{BS} = \frac{9\pi}{4 (9 M^2 - 8 Q^2)^{3/2}} \left[ 27 M^3 (9 \delta M^2 - 4 \delta Q^2) 
+ 9 M^2 (9 \delta M^2 - 4 \delta Q^2) \sqrt{9 M^2 - 8 Q^2} 
+ 8 Q^2 \left( -9 \sqrt{9 M^2 - 8 Q^2} \delta M^2 
+ 24 Q \delta M \delta Q + 4 \sqrt{9 M^2 - 8 Q^2} \delta Q \right) 
- 324 M Q^2 \delta M^2 \right]$$

$$= -\frac{9\pi (1 + \epsilon)}{e^3} \left[ 2(e - 1) \sqrt{\frac{2}{e + 1}} - 1 \sqrt{1 - e^2} 
+ 2 - (4 - 3\epsilon) e \right] \delta Q^2 = -\frac{9\pi (1 + \epsilon)}{e} \delta Q^2 + O(e).$$

(4.26)

Given the above equations for $k$ and $\delta^2 A^{BS}$, the second order inequality yields

$$\delta^2 M - \Phi_+ \delta^2 Q \geq -\frac{k}{8\pi} \delta^2 A^{BS}$$

$$= \frac{8 (1 + \epsilon)}{9 M} \delta Q^2 + O(e).$$

(4.27)

Taking into account all the above we rewrite Eq. (4.19) in the following form

$$g(\alpha) = M^2 \epsilon^2 - 2 \left(\frac{8}{9}\right)^{1/2} M \delta Q \epsilon \alpha + \left(\frac{8}{9}\right) \delta Q^2 \alpha^2$$

$$+ O(\alpha^3, \alpha^2 \epsilon, \alpha \epsilon^2, \epsilon^3)$$

$$= \left(M \epsilon - \left(\frac{8}{9}\right)^{1/2} \delta Q \alpha\right)^2 + O(\alpha^3, \alpha^2 \epsilon, \alpha \epsilon^2, \epsilon^3).$$

(4.28)

This clearly shows that $g \geq 0$ always. Thus when second order perturbations are included, BS can never be overcharged and WCCC would be always respected. As for BH, WCCC may be violated at the linear order, which would though be always overturned at the non-linear order. Non-linear perturbations thus always favour WCCC for both BH and BS.

5 Discussion

In this paper, the main aim is to extend WCCC to a non black hole compact object like BS. This question is motivated by the fact that the latter does share almost all the
properties with the former [55, 59]. It is indeed the most compact non-black hole object [56].

What was then expected was that, as for BH, WCCC may be violated at the linear order which would however be restored when non-linear perturbations were included. This is precisely what we have been able to bear out for the Buchdahl star by a detailed analysis paralleling the BH case.

The one critical difference between BH and BS is that boundary of the former is a null surface from which nothing could come out while for the latter it is timelike allowing both way crossing. What it would mean for accretion process being considered for overcharging of BS is that matter might also have some outflow. This would further hinder the overcharging effort. It therefore appears that timelike boundary of BS may in fact further work counter to the accretion process of overcharging. This would have been very pertinent, had the case been otherwise; i.e., accretion process helped overcharging.

One may ask that this analysis could have been carried out at any radius \( r \). No, that couldn’t have been the case because the question of overcharging arises only when the extremal limit is defined. The extremal limit is defined only for BH and BS as characterized respectively by \( \Phi(r) = 1/2, 4/9 \). It is however true that the analysis proceeds as if the boundary is null. The pertinent point is that its non-null character does not go contrary to the result, instead it goes in line. It should however be appreciated that even though BS boundary is not null, it is very close to the BH horizon and hence for all practical purposes it is almost as compact an object as BH. It should therefore be not surprising if it shares the properties with BH.

There is one very distinct difference between the linear and non-linear case. For the former, the WCCC violation requires the condition Eq. (4.22) while there is no such condition as shown in Eq. (4.28) for the non-linear case. That is, WCCC would always be respected without any constraints on the parameters.

Another related question is, could a non-extremal BS be extremalized; i.e., converting non-extremal into extremal. This is not possible for BH as shown in [15]. This was because as extremality is approached, the parameter window for infalling particles pinches off prohibiting extremalization. It would therefore be expected that the same should be the case for BS which we would next intend to examine in a separate investigation.

Like many other properties, BS shares yet another property with BH, in the context of the weak cosmic censorship conjecture, which may be violated at the linear order perturbations but is always restored at the non-linear order.

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References

[1] R. Penrose, Gravitational Collapse and Space-Time Singularities, *Phys. Rev. Lett.* 14 (1965) 57.
[2] S.W. Hawking and R. Penrose, The Singularities of Gravitational Collapse and Cosmology, *Proc. R. Soc. Lond. A* 314 (1970) 529.
[3] R. Penrose, Gravitational Collapse: the Role of General Relativity, *Riv. Nuovo Cimento* 1 (1969).
[4] D. Christodoulou, Gravitational collapse, *Ann. N. Y. Acad. Sci.* 470 (1986) 147.
[5] P.S. Joshi, Global aspects in gravitation and cosmology (1993).
[6] P.S. Joshi, Gravitational collapse: The story so far, *Pramana* 55 (2000) 529 [gr-qc/0006101].
[7] R. Goswami, P.S. Joshi and P. Singh, Quantum Evaporation of a Naked Singularity, *Phys. Rev. Lett.* 96 (2006) 031302 [gr-qc/0506129].
[8] T. Harada, H. Iguchi and K. Nakao, Physical Processes in Naked Singularity Formation, *Progress of Theoretical Physics* 107 (2002) 449 [gr-qc/0204008].
[9] Z. Stuchlík and J. Schee, Observational phenomena related to primordial Kerr superspinars, *Class. Quantum Grav.* 29 (2012) 065002.
[10] R.S.S. Vieira, J. Schee, W. Kluzniak, Z. Stuchlík and M. Abramowicz, Circular geodesics of naked singularities in the Kehagias-Sfetsos metric of Hořava’s gravity, *Phys. Rev. D* 90 (2014) 024035 [1311.5820].
[11] Z. Stuchlík and J. Schee, Optical effects related to Keplerian discs orbiting Kehagias-Sfetsos naked singularities, *Class. Quantum Grav.* 31 (2014) 195013 [1402.2891].
[12] B. Giacomazzo, L. Rezzolla and N. Stergioulas, Collapse of differentially rotating neutron stars and cosmic censorship, *Phys. Rev. D* 84 (2011) 024022 [1105.0122].
[13] P.S. Joshi, The Story of Collapsing Stars: Black Holes, Naked Singularities, and the Cosmic Play of Quantum Gravity (2015), 10.1093/acprof:oso/9780199686766.001.0001.
[14] R. Wald, Gedanken experiments to destroy a black hole., *Ann. Phys. (N.Y.)* 82 (1974) 548.
[15] N. Dadhich and K. Narayan, On the third law of black hole dynamics, *Phys. Lett. A* 231 (1997) 335 [gr-qc/9704070].
[16] V.E. Hubeny, Overcharging a black hole and cosmic censorship, *Phys. Rev. D* 59 (1999) 064013 [gr-qc/9808043].
[17] T. Jacobson and T.P. Sotiriou, Overspinning a Black Hole with a Test Body, *Phys. Rev. Lett.* 103 (2009) 141101 [0907.4146].
[18] A. Saa and R. Santarelli, Destroying a near-extremal Kerr-Newman black hole, *Phys. Rev. D* 84 (2011) 027501 [1105.3950].
[19] S. Shaymatov, M. Patil, B. Ahmedov and P.S. Joshi, Destroying a near-extremal Kerr black hole with a charged particle: Can a test magnetic field serve as a cosmic censor?, *Phys. Rev. D* 91 (2015) 064025 [1409.3018].
[20] M. Bouhmadi-López, V. Cardoso, A. Nerozzi and J.V. Rocha, Black holes die hard: Can one spin up a black hole past extremality?, *Phys. Rev. D* 81 (2010) 084051 [1003.4295].
[21] J.V. Rocha and R. Santarelli, Flowing along the edge: Spinning up black holes in AdS spacetimes with test particles, *Phys. Rev. D* 89 (2014) 064065 [1402.4840].
[22] S. Jana, R. Shaikh and S. Sarkar, Overcharging black holes and cosmic censorship in Eddington-inspired Born-Infeld gravity, Phys. Rev. D 98 (2018) 124039 [1808.09656].

[23] Y. Song, M. Zhang, D.-C. Zou, C.-Y. Sun and R.-H. Yue, Destroying a Near-Extremal Kerr-Newman-AdS Black Hole with Test Particles, Commun. Theor. Phys. 69 (2018) 694 [1705.01676].

[24] K. Düztaş, Can test fields destroy the event horizon in the Kerr-Taub-NUT spacetime?, Class. Quantum Grav. 35 (2018) 045008 [1710.06610].

[25] K. Düztaş and M. Jamil, String analogues of Reissner-Nordström black holes cannot be overcharged, Mod. Phys. Lett. A 34 (2019) 1950248 [1812.06966].

[26] K. Düztaş, M. Jamil, S. Shaymatov and B. Ahmedov, Testing Cosmic Censorship Conjecture for Extremal and Near-extremal (2+1)-dimensional MTZ Black Holes, Class. Quantum Grav. 37 (2020) 175005 [1808.04711].

[27] S.-J. Yang, J. Chen, J.-J. Wan, S.-W. Wei and Y.-X. Liu, Weak cosmic censorship conjecture for a Kerr-Taub-NUT black hole with a test scalar field and particle, Phys. Rev. D 101 (2020) 064048 [2001.03106].

[28] S.-J. Yang, J.-J. Wan, J. Chen, J. Yang and Y.-Q. Wang, Weak cosmic censorship conjecture for the novel 4D charged Einstein-Gauss-Bonnet black hole with test scalar field and particle, Eur. Phys. J. C 80 (2020) 937 [2004.07934].

[29] P. Zimmerman, I. Vega, E. Poisson and R. Haas, Self-force as a cosmic censor, Phys. Rev. D 87 (2013) 041501 [1211.3889].

[30] J.V. Rocha and V. Cardoso, Gravitational perturbation of the BTZ black hole induced by test particles and weak cosmic censorship in AdS spacetime, Phys. Rev. D 83 (2011) 104037 [1102.4352].

[31] S. Isoyama, N. Sago and T. Tanaka, Cosmic censorship in overcharging a Reissner-Nordström black hole via charged particle absorption, Phys. Rev. D 84 (2011) 124024 [1108.6207].

[32] M. Colleoni and L. Barack, Overspinning a Kerr black hole: The effect of the self-force, Phys. Rev. D 91 (2015) 104024 [1501.07330].

[33] Z. Li and C. Bambi, Destroying the event horizon of regular black holes, Phys. Rev. D 87 (2013) 124022 [1304.6592].

[34] S. Shaymatov, Magnetized Reissner-Nordström black hole restores cosmic censorship conjecture, Int. J. Mod. Phys. Conf. Ser. 49 (2019) 1960020.

[35] B. Gwak and B.-H. Lee, Thermalization of three-dimensional black holes via charged particle absorption, Phys. Rev. D 73 (2011) 041501 [1211.3889].

[36] J. Natário, L. Queimada and R. Vicente, Test fields cannot destroy extremal black holes, Class. Quantum Grav. 33 (2016) 175002 [1601.06809].

[37] J. Natário and R. Vicente, Test fields cannot destroy extremal de Sitter black holes, Gen. Relativ. Gravit. 52 (2020) 5 [1908.09854].

[38] Y. Zhang and S. Gao, Testing cosmic censorship conjecture near extremal black holes with cosmological constants, Int. J. Mod. Phys. D 23 (2014) 1450044 [1309.2027].

[39] B. Gwak and B.-H. Lee, Thermodynamics of three-dimensional black holes via charged particle absorption, Phys. Lett. B 755 (2016) 324 [1610.08215].
[40] J. Sorce and R.M. Wald, Gedanken experiments to destroy a black hole. II. Kerr-Newman black holes cannot be overcharged or overspun, Phys. Rev. D 96 (2017) 104014 [1707.05862].

[41] J. An, J. Shan, H. Zhang and S. Zhao, Five-dimensional Myers-Perry black holes cannot be overspun in gedanken experiments, Phys. Rev. D 97 (2018) 104007 [1711.04310].

[42] B. Gwak, Weak cosmic censorship conjecture in Kerr-(anti-)de Sitter black hole with scalar field, J. High Energy Phys. 09 (2018) 81 [1807.10630].

[43] B. Ge, Y. Mo, S. Zhao and J. Zheng, Higher-dimensional charged black holes cannot be over-charged by gedanken experiments, Phys. Lett. B 783 (2018) 440 [1712.07342].

[44] B. Ning, B. Chen and F.-L. Lin, Gedanken experiments to destroy a BTZ black hole, Phys. Rev. D 100 (2019) 044043 [1902.00949].

[45] Y.-L. He and J. Jiang, Weak cosmic censorship conjecture in Einstein-Born-Infeld black holes, Phys. Rev. D 100 (2019) 124060 [1912.05217].

[46] S. Shaymatov, N. Dadhich and B. Ahmedov, The higher dimensional Myers-Perry black hole with single rotation always obeys the Cosmic Censorship Conjecture, Eur. Phys. J. C 79 (2019) 585 [1809.10457].

[47] J. Jiang, Static charged Gauss-Bonnet black holes cannot be overcharged by the new version of gedanken experiments, Phys. Lett. B 804 (2020) 135365.

[48] K.S. Revelar and I. Vega, Overcharging higher-dimensional black holes with point particles, Phys. Rev. D 96 (2017) 064010 [1706.07190].

[49] S. Shaymatov, N. Dadhich, B. Ahmedov and M. Jamil, Five-dimensional charged rotating minimally gauged supergravity black hole cannot be over-spun and/or over-charged in non-linear accretion, Eur. Phys. J. C 80 (2020) 481 [1908.01195].

[50] S. Shaymatov and N. Dadhich, Weak cosmic censorship conjecture in the pure Lovelock gravity, JCAP 2022 (2022) 060 [2008.04092].

[51] S. Shaymatov and N. Dadhich, On overspinning of black holes in higher dimensions, Phys. Dark Universe 31 (2021) 100758 [2004.09242].

[52] S. Shaymatov, N. Dadhich and B. Ahmedov, Six-dimensional Myers-Perry rotating black hole cannot be overspun, Phys. Rev. D 101 (2020) 044028 [1908.07799].

[53] X.-Y. Wang and J. Jiang, Examining the weak cosmic censorship conjecture of RN-AdS black holes via the new version of the gedanken experiment, JCAP 2020 (2020) 052 [2011.03938].

[54] S. Shaymatov, B. Ahmedov and M. Jamil, Testing the weak cosmic censorship conjecture for a Reissner-Nordström-de Sitter black hole surrounded by perfect fluid dark matter, Eur. Phys. J. C 81 (2021) 588.

[55] N. Dadhich, Maximum force for black holes and Buchdahl stars, Phys. Rev. D 105 (2022) 064044 [2201.10381].

[56] A. Alho, J. Natário, P. Pani and G. Raposo, Compactness bounds in General Relativity, arXiv e-prints (2022) arXiv:2202.00043 [2202.00043].

[57] S. Hod, Upper bound on the gravitational masses of stable spatially regular charged compact objects, Phys. Rev. D 98 (2018) 064014 [1903.10530].

[58] N. Dadhich, Buchdahl compactness limit and gravitational field energy, JCAP 2020 (2020) 035 [1903.03436].
[59] S. Chakraborty and N. Dadhich, *Universality of the Buchdahl sphere*, arXiv e-prints (2022) arXiv:2204.10734 [2204.10734].

[60] R. Casadio, O. Micu and D. Stojkovic, *Horizon wave-function and the quantum cosmic censorship*, Phys. Lett. B 747 (2015) 68 [1503.02858].

[61] V. Iyer and R.M. Wald, *Some properties of the Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D 50 (1994) 846 [gr-qc/9403028].

[62] V. Iyer and R.M. Wald, *Comparison of the Noether charge and Euclidean methods for computing the entropy of stationary black holes*, Phys. Rev. D 52 (1995) 4430 [gr-qc/9503052].