Abstract

Let \( l : V(G) \rightarrow \mathbb{N} \) be a labeling of the vertices of a graph \( G \) by positive integers. Define \( c(u) = \sum_{v \in N(u)} l(v) + d(u) \), where \( d(u) \) denotes the degree of \( u \) and \( N(u) \) denotes the open neighborhood of \( u \). In this paper we introduce a new labeling called \( d \)-lucky labeling and study the same as a vertex coloring problem. We define a labeling \( l \) as \( d \)-lucky if \( c(u) \neq c(v) \), for every pair of adjacent vertices \( u \) and \( v \) in \( G \). The \( d \)-lucky number of a graph \( G \), denoted by \( \eta_d(G) \), is the least positive \( k \) such that \( G \) has a \( d \)-lucky labeling with \( \{1, 2, ..., k\} \) as the set of labels. We obtain \( \eta_d(G) = 2 \) for hypercube network, butterfly network, benes network, mesh network, hypertree and X-tree.

Keywords: Lucky labeling; \( d \)-lucky labeling; benes network; butterfly network; mesh network.

1. Introduction

Graph coloring is one of the most studied subjects in graph theory. It is an assignment of labels called colors to the elements of a graph, subject to certain constraints. Karonski, Luczak and Thomason\(^2\) initiated the study of proper labeling. The rule of using colors originates from coloring the countries of a map, where each face is colored exactly. In its simplest outline, vertex coloring or proper labeling is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. The problem of proper labeling offers numerous variants and established great significance at recent times, for example see\(^1,2,6\). Graph coloring is used in various research areas of computer science such as networking, image segmentation, clustering, image capturing and data mining.
There is a spectrum of labeling procedures that are available in the literature, leading to proper vertex coloring of graphs. For a mapping \( f : V(G) \to \{1,2,\ldots,k\} \), a proper vertex coloring is obtained through Lucky labeling\(^9,^{12}\), Vertex-labeling by product\(^1\), Vertex-labeling by gap, Vertex-labeling by degree and Vertex-labeling by maximum\(^5\). For a mapping \( f : E(G) \to \{1,2,\ldots,k\} \), a proper vertex coloring is obtained through Edge-labeling by sum\(^{11}\), Edge-labeling by product\(^5\) and Edge-labeling by gap\(^5\). In this paper we introduce a new labeling called \( d\)-lucky labeling and compute the \( d\)-lucky number of certain networks.

2. Some special classes of graphs with \( \eta_{dl}(G) = 2 \)

We begin with the definition of \( d\)-lucky labeling. For a vertex \( u \) in a graph \( G \), let \( N(u) = \{v \in V(G) / uv \in E(G)\} \) and \( N[u] = N(u) \cup \{u\} \).

**Definition 2.1** Let \( l : V(G) \to \{1,2,\ldots,k\} \) be a labeling of the vertices of a graph \( G \) by positive integers. Define \( c(u) = \sum_{v \in N(u)} l(v) + d(u) \), where \( d(u) \) denotes the degree of \( u \). We define a labeling \( l \) as \( d\)-lucky if \( c(u) \neq c(v) \), for every pair of adjacent vertices \( u \) and \( v \) in \( G \). The \( d\)-lucky number of a graph \( G \), denoted by \( \eta_{dl}(G) \), is the least positive \( k \) such that \( G \) has a \( d\)-lucky labeling with \( \{1,2,\ldots,k\} \) as the set of labels.

**Definition 2.2** The vertex set \( V \) of \( Q_n \) consists of all binary sequence of length \( n \) on the set \( \{0,1\} \), that is, \( V = \{x_1x_2\ldots x_n \in \{0,1\}, i=1,2,\ldots,n\} \). Two vertices are linked by an edge if and only if \( x \) and \( y \) differ exactly in one coordinate, that is, \( \sum_{i=1}^{n} |x_i - y_i| = 1 \). In terms of cartesian product, \( Q_n \) is defined recursively as follows. \( Q_1 = K_2, Q_n = Q_{n-1} \times Q_1 = K_2 \times K_2 \times \ldots \times K_2, \ n \geq 2. \)

**Theorem 2.3** The \( n\)-dimensional hypercube network \( Q_n \) admits \( d\)-lucky labeling and \( \eta_{dl}(Q_n) = 2 \).

![Fig. 1. \( d\) -lucky labeling of hypercube network, \( Q_4 \).](image)

A most popular bounded - degree derivative network of the hypercubes is called a butterfly network.

**Definition 2.4** [10] The \( n\)-dimensional butterfly network, denoted by BF\((n)\), has a vertex set \( V = \{(x;i) : x \in V(Q_n), 0 \leq i \leq n\} \). Two vertices \( (x;i) \) and \( (y;j) \) are linked by an edge in BF\((n)\) if and only if \( j = i + 1 \) and either

1. \( x = y \), or
2. \( x \) differs from \( y \) in precisely the \( j^{th} \) bit.

For \( x = y \), the edge is said to be a straight edge. Otherwise, the edge is a cross edge. For fixed \( i \), the vertex
\((x,i)\) is a vertex on level \(i\).

**Definition 2.5** [10] The topological structure of a mesh network is defined as the Cartesian product \(P_l \times P_m\) denoted by \(M(l,m)\), where \(P_l\) and \(P_m\) denotes and undirected path on \(l\) and \(m\) vertices respectively.

**Remark 2.6** The mesh \(M(l,m)\) has \(lm\) vertices and \(2(lm) - (l + m)\) edges, where \(l,m \geq 2\) and \(l,m\) denotes rows and columns of \(M(l,m)\) respectively.

**Theorem 2.7** The \(n\)-dimensional butterfly network \(BF(n)\) admits \(d\)-lucky labeling and \(\eta_{dl}(BF(n)) = 2\).

**Proof.** Label the vertices in consecutive levels of \(BF(n)\) as 1 and 2 alternately, beginning from level 0. We note that every edge \(e = (u, v)\) in \(BF(n)\) has one end at level \(i\) and the other end at level \(i+1\) or level \(i-1\) (if it exists), \(0 \leq i \leq n\).

Case 1: Suppose \(u\) is in level 0, then \(u\) is incident on one cross edge and one straight edge with the other ends at level 1. Since \(l(u) = 1\) and each member of \(N(u)\) is labeled 2, we have \(c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6\), where \(d(u)\) is the degree of \(u\). Since \(l(v) = 2\) and each member of \(N(v)\) is labeled 1, we have \(c(v) = \sum_{u \in N(v)} l(u) + d(v) = 8\). Thus \(c(u) \neq c(v)\). The same argument holds good when \(u\) is in level \(n\).

Case 2: Suppose \(u\) is in level \(i\), \(i\) is even, \(0 < i < n\) and \(V\) is in level \(i+1\). Then \(u\) is incident on one cross edge and one straight edge with the other ends at level \(i+1\) and also incident on one cross edge and one straight edge with the other ends at level \(i-1\). Since \(l(u) = 2\), each member of \(N(u)\) is labeled 1. Therefore, we have \(c(u) = \sum_{v \in N(u)} l(v) + d(u) = 8\). Further \(l(v) = 1\) and each member of \(N(v)\) is labeled 2. Therefore, we have \(c(v) = \sum_{u \in N(v)} l(u) + d(v) = 12\). Thus \(c(u) \neq c(v)\). A similar argument shows that \(c(u) \neq c(v)\) if \(V\) is in level \(i-1\). The case when \(i\) is odd is also similar. See Figure 2(a) for \(d\)-lucky labeling of \(BF(3)\) with \(c(u)\) listed within paranthesis for any \(u \in V\). Hence \(n\)-dimensional butterfly network admits \(d\)-lucky labeling.

**Theorem 2.8** The mesh network denoted by \(M(l,m)\) admits \(d\)-lucky labeling and \(\eta_{dl}(M(l,m)) = 2\).

**Proof.** Let \(G\) be a mesh \(M(l,m)\), where \(l,m \geq 2\). Then \(G\) admits \(d\)-lucky labeling and \(\eta_{dl}(G) = 2\). Label the vertices in row \(i\), \(i\) even, as 1 and 2 alternately, beginning with label 1 from left to right. Label all the vertices in row \(i\), \(i\) odd, as 2. Edges with both ends in the same row are called horizontal edges. Edges with one end in row \(i\)
and the other end in row \((i + 1)\) or row \((i - 1)\) are called vertical edges.

Case 1: Suppose \(u\) and \(v\) are in row 1, where \(d(u) = 2\), then \(u\) has one horizontal edge and one vertical edge incident at it. If \(l(u) = 2\), by labeling of \(G\) the adjacent vertices on the horizontal and vertical edges incident with \(u\) are labeled as 2 and 1 respectively. We have \(c(u) = \sum_{v \in N(u)} f(v) + d(u) = 5\). On the other hand, if \(v\) and each member of \(N(v)\) is labeled 2, we have \(c(v) = \sum_{u \in N(v)} f(u) + d(v) = 9\). Thus \(c(u) \neq c(v)\). A similar argument holds when \(l(u) = 1\) or when \(u\) is in row \(n\).

Case 2: Suppose \(u\) and \(v\) are in row \(i\) even, where \(3 = l(u)\) and \(4 = l(v)\), \(u\) has two vertical edges in rows \(i - 1\) and \(i + 1\) and one horizontal edge with the other end in row 1 incident with it. Since \(l(u) = 1\), each member of \(N(u)\) is labeled 2. Therefore, we have \(c(u) = \sum_{v \in N(u)} f(v) + d(u) = 9\). On the other hand, suppose \(v\) is in row \(i\), then \(v\) has two vertical edges in row \(i - 1\) and \(i + 1\) and two horizontal edges with the other end in row \(i\) incident with it. Since \(l(v) = 2\), each member of \(N(v)\) in the horizontal row is labeled 1 and \(N(v)\) in vertical column is labeled 2. Therefore, we have \(c(v) = \sum_{u \in N(v)} f(u) + d(v) = 10\). The vertex sums are distinct. A similar argument holds when \(v\) is in row \(n - 1\).

Case 3: Suppose \(u\) and \(v\) are in row \(i\) odd, where \(3 = l(u)\) and \(4 = l(v)\), \(u\) has two vertical edges in rows \(i - 1\) and \(i + 1\) and one horizontal edge with the other end in row \(i\) incident with it. Since \(l(u) = 2\), by labeling of \(G\) the adjacent vertices on the horizontal and vertical edges incident with \(u\) are labeled as 2 and 1 respectively. Therefore, we have \(c(u) = \sum_{v \in N(u)} f(v) + d(u) = 7\). On the other hand, suppose \(v\) is in row \(i\), then \(v\) has two vertical edges in row \(i - 1\) and \(i + 1\) and two horizontal edges with the other end in row \(i\) incident with it. Since \(l(v) = 2\), each member of \(N(v)\) is labeled 2. Therefore, we have \(c(v) = \sum_{u \in N(v)} f(u) + d(v) = 12\). The vertex sums are distinct. (For illustration, see Figure 2(b), \(d\)-lucky labeling of \(M(5,6)\) mesh network with \(c(u)\) listed within parenthesis for any \(u \in V\)). Hence the mesh network admits \(d\)-lucky labeling with \(\eta_{dl}(G) = 2\).

**Definition 2.9** The \(n\)-dimensional benes network consists of back-to-back butterfly, denoted by \(BB(n)\). The \(BB(n)\) has \(2n + 1\) levels, each with \(2^n\) vertices. The first and last \(n + 1\) levels in the \(BB(n)\) form two \(BF(n)\)s respectively, while the middle level in \(BB(n)\) is shared by these butterfly networks. The \(n\)-dimensional benes network has \((n + 1)2^{n+1}\) vertices and \(n2^{n+2}\) edges. It has only 2-degree vertices and 4-degree vertices, and thus, is eulerian.

**Theorem 2.10** The \(n\)-dimensional benes network \(BB(n)\) admits \(d\)-lucky labeling and \(\eta_{dl}(BB(n)) = 2\).

**Proof.** Label the vertices in consecutive levels of \(BB(n)\) as 1 and 2 alternately, beginning from level 0. We note that every edge \(e = (u, v)\) in \(BB(n)\) has one end at level \(i\) and the other end at level \(i + 1\) or level \(i - 1\) (if it exists), \(0 \leq i \leq n\).
Case 1: Suppose $u$ is in level 0, then $u$ is incident on one cross edge and one straight edge with the other ends at level $i = 1$. Since $l(u) = 2$ and each member of $N(u)$ is labeled 1, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 4$, where $d(u)$ is the degree of $u$. Since $l(v) = 1$ and each member of $N(v)$ is labeled 2, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 12$. Thus $c(u) \neq c(v)$. The same argument holds good when $u$ is in level $n$.

Case 2: Suppose $u$ is in level $i$, $i$ is even, $0 < i < n$ and $v$ is in level $i + 1$. Then $u$ is incident on one cross edge and one straight edge with the other ends at level $i + 1$ and also incident on one cross edge and one straight edge with the other ends at level $i - 1$. Since $l(u) = 2$, each member of $N(u)$ is labeled 1. Therefore, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 8$. Further $l(v) = 1$ and each member of $N(v)$ is labeled 2. Therefore, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 12$. Thus $c(u) \neq c(v)$. A similar argument shows that $c(u) \neq c(v)$ if $v$ is in level $i-1$. The case when $i$ is odd is also similar. See Figure 3 for $d$-lucky labeling of $BB(3)$ with $c(u)$ listed within parenthesis for any $v$. Hence $n$-dimensional benes network admits $d$-lucky labeling.

**Definition 2.11** [13] A hypertree is an interconnection topology for incrementally expandable multicomputer systems, which combines the easy expandability of tree structures with the compactness of the hypercube; that is, it combines the best features of the binary tree and the hypercube. The basic skeleton of a hypertree is a complete binary tree $T_r$. Here the nodes of the tree are numbered as follows: The root node has label 1. The root is said to be at level 0. Labels of left and right children are formed by appending 0 and 1, respectively to the labels of the parent node. Here the children of the nodes $x$ are labeled as $4x$ and $4x + 1$. Additional links in a hypertree are horizontal and two nodes in the same level of the tree are joined if their label difference is $2^{l-2}$. We denote an $r$-level hypertree as $HT(r)$. It has $2^{r+1} - 1$ vertices and $3(2^r - 1)$ edges.

**Definition 2.12** An $X$-tree $XT_n$ is obtained from complete binary tree on $2^{n+1} - 1$ vertices of length $2^n - 1$, and adding paths $P_i$ left to right through all the vertices at level $i$; $1 \leq i \leq n$.

**Theorem 2.13** The $r$-level hypertree $HT_r$ admits $d$-lucky labeling and $\eta_{dl}(HT(r)) = 2$.

![Fig.4. $d$-lucky labeling of hypertree, $HT(3)$.](image)

**Theorem 2.14** The X-tree $XT_r$ admits $d$-lucky labeling and $\eta_{dl}(XT_r) = 2$.
3. Conclusion

A new labeling called d-lucky labeling is defined and the graph which satisfies the d-lucky labeling is called a d-lucky graph. d-lucky labeling of some special classes of graphs like hypercube networks, butterfly networks, benes network, mesh network, hypertree and X-tree are investigated.

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