An $A_5$ Model of Four Lepton Generations

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Abstract

We study the lepton sector of a four generations model based on the discrete flavor group $A_5$. The best features of the three family $A_4$ model survive, including the tribimaximal pattern of three generation neutrino mixings. At leading order the three light neutrino mass relations of $m_{\nu_1} = m_{\nu_3}$ and $m_{\nu_2} = 0$ are predicted. The splitting of the neutrino masses can be naturally obtained as a result of the breaking of $A_5$ down to $A_4$ and a degenerate spectrum is preferred in our model. The electron mass is zero at tree level, but calculable through quantum corrections in our $A_5$ model.

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I. INTRODUCTION

Recently there has been considerable efforts to explore the possibility of a fourth generation of fermions in the standard model (SM). In particular these studies of fourth generation include its constraints by current electroweak precision data \[1–4\], its interplay with Higgs physics \[5\], CP violation effects in \(B\)-physics \[6\] and accessibility of the fourth generation quarks \[7\] and leptons \[8\] at the Large Hadron Collider (LHC). For a recent review of fourth generation physics, see for example Ref. \[9\].

In this paper, we will focus mainly on the fourth generation of leptons and comment only briefly on inclusion of the extra quark family toward the end. If we assume the flavor symmetry of the lepton sector is described correctly at tree level by an \(A_4\) model \[10–17\], we then need to reconcile the prospects of a heavy fourth generation of leptons with the results known for the neutrinos in the first three generations. This requires that we extend the successful results of \(A_4\) models, such as tribimaximal mixing (TBM), to the four generation model. The simplest approach is to demand a model where the \(A_4\) model can be minimally embedded. The simplest extension of this type is based on \(A_5\) where surprisingly many new features arise, some of which will be immediately testable in the next round of neutrino experiments or at the LHC.

In the literature there has been numerous discussions of three generation lepton flavor symmetry using the \(A_5\) group. It is well known that the tribimaximal value of the solar mixing angle \(\theta_{12}^{\text{TBM}} = \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) = 35.26^\circ\), which is very close to the experimental best-fit value of \(34.43^\circ\). In Refs. \[18, 19\] the solar mixing angle is related to the golden ratio \(\phi = (1 + \sqrt{5})/2\) \[20\] via \(\theta_{12} = \tan^{-1} \left( \frac{1}{\phi} \right) = 31.72^\circ\) which is \(2\sigma\) below the experimental best-fit value. The icosahedral group \(I\), isomorphic to the permutation group \(A_5\), has also been utilized in lepton flavor symmetry in Refs. \[21–23\] (also see recent reviews in \[24, 25\]) where the golden ratio also shows up naturally. This suggests a possible connection to \(A_5\) as shown in Ref. \[26\]. Another interesting parameterization involving the golden ratio is \(\theta_{12} = \frac{\pi}{\phi} = \cos^{-1} \left( \frac{1}{\phi} \right)\) and \(\theta_{\text{cabibbo}} = \frac{\pi}{12}\) introduced in Ref. \[27\] for both the lepton and quark sectors. This might have connection with the \(A_5\) group as well.

In this paper we propose the four generation lepton sector fit into the context of an \(A_5 \times Z_2 \times Z_3\) symmetry where \(A_5\) subsequently breaks into \(A_4\), with the goal of retaining the tribimaximal mixing as well as the lepton mass pattern (three hierarchical charged leptons
and three tiny neutrino masses). In Section 2 we begin with a brief introduction of the
discrete group \( A_5 \) (for more details see Appendix A) and then utilize the structure to the
products of its irreducible representations to construct a four generation lepton model. In
Section 3 we work out the lepton mass spectrum and discuss how the tribimaximal mixing is
embedded in this model. Some of the rich flavor physics phenomenologies of this model are
also discussed. Finally, we conclude in Section 4 with some comments on collider physics
implications. We also comment on the natural extension of our \( A_5 \) model to the double
icosahedral group \( I' \) as the flavor symmetry to describe four generations of both quarks and
leptons.

II. THE MODEL

There are five irreducible representations (irreps) of \( A_5 \): one singlet \( 1 \), two triplets, \( 3 \) and
\( 3' \), one quartet \( 4 \) and one quintet \( 5 \). The character table and the multiplication rules of the
five irreducible representations are given in Appendix A.

Our model is based on \( A_5 \) with an additional \( Z_2 \times Z_3 \) symmetry to avoid unwanted
terms in the Lagrangian. We assign the four generations of left-handed lepton doublets to
\( A_5 \times Z_2 \times Z_3 \) as follows:

\[
\begin{pmatrix}
\nu_4 \\
l_4
\end{pmatrix}_L, \quad
\begin{pmatrix}
\nu_3 \\
l_3
\end{pmatrix}_L, \quad
\begin{pmatrix}
\nu_2 \\
l_2
\end{pmatrix}_L, \quad
\begin{pmatrix}
\nu_1 \\
l_1
\end{pmatrix}_L
\]

, \( L_{L4}(4,1,\omega) \)

(1)

where \( \begin{pmatrix}
\nu_i \\
l_i
\end{pmatrix}_L \) with \( i = 1, 2, 3, 4 \) represents the left-handed lepton doublets of the \( i^{th} \) family
and \( \omega = e^{i2\pi/3} \) is the cubic root of unity. For the right-handed sector we introduce the
charged lepton fields

\[
l_{R5}(5,-1,+1) \quad l_{R5}^c(3,-1,+1) \quad \text{and} \quad l_{R1}^{(1)}(1,-1,+1) \quad l_{R1}^{(2)}(1,-1,+1)
\]

(2)

which transform as a \( 5 \), a \( 3 \), and two trivial singlets \( 1 \) under \( A_5 \) symmetry respectively. Here
\( c \) stands for charge conjugation. The right-handed neutrinos are assigned as one quintet \( 5 \)
and a trivial singlet

\[
N_{R5}(5,+1,+1) \quad \text{and} \quad N_R^{(1)}(1,+1,+1).
\]

(3)
We have implicitly embedded the right-handed charged leptons $\mu_R, e_R$ in the 5 dimensional irreducible representation (irrep) of $A_5$.

For the Higgs sector we introduce three $A_5$ quartet and one $A_5$ triplet scalars. The first quartet $S_4$ is a gauge singlet which transform as $(4, +1, +1)$ under $A_5 \times Z_2 \times Z_3$. The other two quartets are electroweak doublets with $Z_2 \times Z_3$ charges and are written as $H_4(4, +1, \omega^2)$ and $H'_4(4, -1, \omega^2)$. The $A_5$ triplet $\Phi_3(3, +1, \omega^2)$ is also an electroweak doublet and has nontrivial $Z_3$ charge. The most general renormalizable Lagrangian of the Yukawa couplings among these fields and Majorana mass terms for the neutrinos that are invariant under both the standard model gauge group and the $A_5 \times Z_2 \times Z_3$ discrete flavor symmetry is

$$L_{\text{Yukawa}} = \frac{1}{2} M_1 N_R^{(1)} N_R^{(1)} + \frac{1}{2} M_5 N_{R5} N_{R5} + Y_{S1}(S_4 N_{R5} N_{R5}) + Y_{S2}(S_4(l_{R5})^c l_{R5}^-)$$

$$+ Y_1(L_{L4} N_R^{(1)} H_4) + Y_2(L_{L4} N_{R5} H_4) + Y_3(L_{L4} N_{R5} \Phi_3)$$

$$+ Y_4(L_{L4} l_{R5} H'_4) + Y_5(L_{L4} l_{R1} H'_4) + Y_6(L_{L4} l_{R1} H'_4) + \text{H.c.}$$

\[ (4) \]

Suppose the scalar $S_4$ develops a VEV $\langle S_4 \rangle = (V_S, 0, 0, 0)$. It will break the discrete group $A_5$ into $A_4$, causing the irreps of $A_5$ to decompose as

$$A_5 \rightarrow A_4$$

$$1 \rightarrow 1$$

$$3 \rightarrow 3$$

$$3' \rightarrow 3$$

$$4 \rightarrow 1 + 3$$

$$5 \rightarrow 1' + 1'' + 3$$

Hence the lepton fields are decomposed as

$$L_{L4} \rightarrow L_{L1}(1, +1, \omega) + L_{L3}(3, +1, \omega)$$

$$l_{R5} \rightarrow l_{R3}(3, -1, +1) + l_{R1'}(1', -1, +1) + l_{R1''}(1'', -1, +1)$$

and

$$N_{R5} \rightarrow N_{R3}(3, +1, +1) + N_{R}^{(2)}(1', +1, +1) + N_{R}^{(3)}(1'', +1, +1)$$

Here we assume the right-handed $A_4$ triplet $l_{R3}$ is combined with the lepton $(l)^R_{R3}$ shown in Eq. (2) to form a vector field $l_{L+R,3}$ so as to insure gauge anomaly cancellation. Similarly,
the Higgs scalars decompose as

\[ S_1 \rightarrow S_1(1, +1, +1) + S_3(3, +1, +1), \]
\[ H_4 \rightarrow H_1(1, +1, \omega^2) + H_3(3, +1, \omega^2), \]
\[ H'_4 \rightarrow H'_1(1, -1, \omega^2) + H'_3(3, -1, \omega^2), \]

and

\[ \Phi_3 \rightarrow \Phi_3(3, +1, \omega^2). \]

After this stage of \( A_5 \rightarrow A_4 \) breaking via \( \langle S_4 \rangle \), the Yukawa Lagrangian can be written as

\[ L_{Yukawa}^{A_4} = \frac{1}{2} M_1 N_R^{(1)} N_R^{(1)} + \frac{1}{2} M_5 (N_R^{(2)} N_R^{(2)} + N_R^3 N_R^3) + Y_{S1} [(V_S + S_1)(N_R^{(2)} N_R^{(3)} + N_R^3 N_R^3) + S_3 (N_R^3 N_R^3)] + Y_{S2} [(V_S + S_1)(l_{R3})^c l_{R3} + S_3 (l_{R3})^c (l_{R1} + l_{R1} + l_{R1})] + Y_1 [L_{L1} N_R^{(1)} H_1 + L_{L3} N_R^{(1)} H_3] + Y_2 [L_{L1} N_R^3 H_3 + L_{L3} N_R^3 (H_1 + H_3) + L_{L3} (N_R^{(2)} + N_R^{(3)}) H_3] + Y_3 [L_{L1} N_R^3 \Phi_3 + L_{L3} N_R^3 \Phi_3 + L_{L3} (N_R^{(2)} + N_R^{(3)}) \Phi_3] + Y_4 [L_{L1} l_{R3} H'_3 + L_{L3} l_{R3} (H'_1 + H'_3) + L_{L3} (l_{R1} + l_{R1} + l_{R1}) H'_3] + Y_5 [L_{L1} l_{R1}^{(1)} H'_1 + L_{L3} l_{R1}^{(1)} H'_3] + Y_6 [L_{L1} l_{R1}^{(2)} H'_1 + L_{L3} l_{R1}^{(2)} H'_3] + H.c. \]  \( (5) \)

For convenience, the multiplication rules for \( A_4 \) are summarized in Appendix B.

III. PHENOMENOLOGY

We begin with the mass spectrum of the leptons.

A. Charged Lepton Masses

After the subsequent breaking of the \( A_4 \) and SM gauge symmetries due to the VEVs of the scalar fields \( S_3, H_1, H_3, H'_1 \) and \( H'_3 \), the charged lepton mass terms are given by

\[ L_l = Y_{S2} [(V_S + \langle S_1 \rangle)(l_{R3})^c l_{R3} + \langle S_3 \rangle (l_{R3})^c (l_{R3} + l_{R1} + l_{R1})] + Y_4 [L_{L1} l_{R3} \langle H'_3 \rangle + L_{L3} l_{R3} (\langle H'_1 \rangle + \langle H'_3 \rangle) + L_{L3} (l_{R1} + l_{R1} + l_{R1}) \langle H'_3 \rangle] + Y_5 [L_{L1} l_{R1}^{(1)} \langle H'_1 \rangle + L_{L3} l_{R1}^{(1)} \langle H'_3 \rangle] + Y_6 [L_{L1} l_{R1}^{(2)} \langle H'_1 \rangle + L_{L3} l_{R1}^{(2)} \langle H'_3 \rangle]. \]  \( (6) \)
We take the VEVs of $H'_3$ and $H'_1$ to be

$$
\langle H'_3 \rangle = (V'_{31}, V'_{32}, V'_{33}) \quad \text{and} \quad \langle H'_1 \rangle = V'_1,
$$

which yields the $7 \times 7$ charged lepton mass matrix of the form

$$
M_l = \begin{pmatrix}
Y_5 V'_1 & Y_6 V'_1 & 0 & 0 & Y_4 V'_{31} & Y_4 V'_{32} & Y_4 V'_{33} \\
Y_5 V'_{31} & Y_6 V'_{31} & Y_4 V'_{31} & Y_4 V'_{31} & Y_4 V'_1 & Y_4 V'_{33} & Y_4 V'_{32} \\
Y_5 V'_{32} & Y_6 V'_{32} & \omega Y_4 V'_{32} & \omega^2 Y_4 V'_{32} & Y_4 V'_{33} & Y_4 V'_1 & Y_4 V'_{31} \\
Y_5 V'_{33} & Y_6 V'_{33} & \omega^2 Y_4 V'_{33} & \omega Y_4 V'_{33} & Y_4 V'_{32} & Y_4 V'_1 & Y_4 V'_{31} \\
0 & 0 & Y_{S2}(S'_3)_1 & Y_{S2}(S'_3)_1 & Y_{S2}(V_S + \langle S_1 \rangle) & Y_{S2}(S'_3)_3 & Y_{S2}(S'_3)_2 \\
0 & 0 & \omega Y_{S2}(S'_3)_2 & \omega^2 Y_{S2}(S'_3)_2 & Y_{S2}(S'_3)_3 & Y_{S2}(V_S + \langle S_1 \rangle) & Y_{S2}(S'_3)_1 \\
0 & 0 & \omega^2 Y_{S2}(S'_3)_3 & \omega Y_{S2}(S'_3)_3 & Y_{S2}(S'_3)_2 & Y_{S2}(S'_3)_1 & Y_{S2}(V_S + \langle S_1 \rangle)
\end{pmatrix}
$$

written in the left-handed and right-handed charged leptons bases given by $(L_{L1}, L_{L3} = (L_{L31}, L_{L32}, L_{L33}), l'_{R3} = (l'_{R31}, l'_{R32}, l'_{R33}))$ and $(l^{(1)}_{R1}, l^{(2)}_{R1}, l_{R1'}, l_{R1''}, l_{R31}, l_{R32}, l_{R33})^T$ respectively. Note that the first two columns of the mass matrix are proportional to each other, hence the determinant of this matrix is zero, Det$(M_l) = 0$, which implies it has a zero eigenvalue to be identified as the electron mass. We thus predicts the electron is massless at tree level in our model. This is because the two $A_5$ singlet right-handed charged leptons (see Eq. (2)) have the same Yukawa couplings structure due to the $A_5$ multiplication rules and this result is independent of the values of VEVs. The construction provides another example of electron-muon universality proposed in Ref [28], where a class of models was devised to evaluate the small ratio $m_e/m_\mu$. Under the assumption that the $A_5$ breaking scale is much higher than those for the breaking of $A_4$ and the SM gauge symmetries, namely $V_S \gg \langle S_1 \rangle, \langle S_3 \rangle, \langle H'_1 \rangle$, and $\langle H'_3 \rangle$, we can treat the vector field $l_{L+R,3}$ as being decoupled from the four chiral lepton generations, while their mixings can be ignored at leading order. If we set $V'_{31} = V'_{32} = V'_{33} = V'$ for simplicity, then we obtain the following charged lepton masses for the four generations

$$
m_e = 0 \quad , \quad m_\mu = \sqrt{3} Y_4 V',
$$

$$
m_\tau = \frac{1}{2} \left[ (Y_5 V'_1 + (Y_6 - Y_4) V') - \sqrt{(Y_5 V'_1 + (Y_6 - Y_4) V')^2 + 4 Y_4 (Y_5 V'_1 + 3 Y_6 V') V'} \right],
$$

and

$$
m_{\tau'} = \frac{1}{2} \left[ (Y_5 V'_1 + (Y_6 - Y_4) V') + \sqrt{(Y_5 V'_1 + (Y_6 - Y_4) V')^2 + 4 Y_4 (Y_5 V'_1 + 3 Y_6 V') V'} \right].
$$
FIG. 1: A 1-loop diagram which gives the electron a finite small mass through new vector leptons.

A 95% C.L. lower mass limit for the heavy charged leptons is set around 100 GeV \[29\] which can be used to give bounds on the unknown parameters in Eq. (9). A finite electron mass can be generated via a 1-loop diagram through the intermediate vector fermions as illustrated in Fig. 1. The result is similar to the many earlier attempts to calculate electron mass \[28, 30–32\] or fermion spectrum \[33, 34\]. The electron mass is estimated to be

\[ m_e \sim \frac{Y_4^2 m_H^2}{16\pi^2 m_{L+R,3}} \approx \frac{Y_4^2 m_H^2}{16\pi^2 Y_{S2} V_S}, \tag{10} \]

where we ignore both mixings in lepton and scalar sectors and \( m_H \) represents the common mass scale of scalars for simplicity. By using Eq. (9) we have the relation \( m_e/m_\mu \approx \frac{Y_4 m_H^2}{16\pi^2 Y_{S2} V_S} \approx \frac{\alpha}{\pi} \). We should point out that the result depends sensitively on several parameters as in many previous works \[28, 30–33\]; however, a recent study in Refs. \[34\] indicates that the parameters could be reduced considerably.

Note that the charged lepton are given masses by the VEV \( \langle H'_4 \rangle \) which is not directly related to the neutrino masses, so we can calculate the mass eigenstates of charged leptons by using the biunitary transformations to diagonalize the mass matrix \( M_l \). The rotation matrix of left-handed fields can be absorbed into the redefinition of Higgses \( H'_1 \) and \( H'_3 \), while the rotation of the right-handed charged leptons will not affect the tribimaximal mixings for neutrinos at leading order.

B. Neutrino Masses

We next turn to the neutrino masses. The full neutrino mass matrix in the model is a \( 10 \times 10 \) matrix, written as

\[
M_\nu = \begin{pmatrix}
M_{\nu_L}^T (4 \times 4) & 0 \\
M_D^T (6 \times 4) & M_N (6 \times 6)
\end{pmatrix}
\tag{11}
\]
From Eq. (5) we obtain the $6 \times 6$ right-handed neutrino mass matrix in the $(N_{R}^{(1)}, N_{R}^{(2)}, N_{R}^{(3)}, N_{R31}, N_{R32}, N_{R33})$ basis

$$M_{NR} = \begin{pmatrix}
M_1 & 0 & 0 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2M_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2M_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2M_2 \\
\end{pmatrix}, \quad (12)$$

where we define $M_2 = \frac{1}{2}M_5 + Y_{S1}V_S$. Since the $A_5$ breaking scale $V_S$ is assumed to be higher than the subsequent breaking scales of the $A_4$ and SM gauge symmetries, we ignore the effects of $\langle S_3 \rangle$ in the following discussion of the derivation of tribimaximal mixings and we will treat these effects as perturbations when we calculate the neutrino mass spectrum.

The Dirac mass terms of the neutrino sector are given by

$$L_{Dirac} = Y_1 \left[ L_{L1}N_{R}^{(1)}H_1 + L_{L3}N_{R}^{(1)}H_3 \right] + Y_2 L_{L3}N_{R3}H_1 + L_{L1}N_{R3} \left[ Y_2 H_3 + Y_3 \Phi_3 \right] + L_{L3}N_{R3} \left[ Y_2 H_3 + Y_3 \Phi_3 \right] + L_{L3}(N_{R}^{(2)} + N_{R}^{(3)}) \left[ Y_2 H_3 + Y_3 \Phi_3 \right]. \quad (13)$$

We see that the linear combination of the two $A_4$ triplet fields, $H_3$ and $\Phi_3$, contribute to the Dirac neutrino masses. One can decouple the fourth generation neutrino from the three light generations and satisfy the conditions of tribimaximal mixings by assuming the VEVs of $H_3$ are relatively small compared to the VEVs of $\Phi_3$ as we will demonstrate below. Hence the Dirac matrix of the neutrino sector $M_D$ is given by

$$\nu_L M_D N_R^T = $$

$$\left( \begin{array}{cccc}
\nu_{L1} & \nu_{L31} & \nu_{L32} & \nu_{L33}
\end{array} \right) \left( \begin{array}{cccc}
Y_1 v_{H_1} & 0 & 0 & Y_3 v_{\phi_1} \\
0 & Y_3 v_{\phi_1} & Y_3 v_{\phi_2} & Y_3 v_{\phi_3} \\
0 & \omega Y_3 v_{\phi_2} & \omega^2 Y_3 v_{\phi_2} & Y_3 v_{\phi_3} \\
0 & \omega^2 Y_3 v_{\phi_3} & \omega Y_3 v_{\phi_3} & Y_3 v_{\phi_1} \\
\end{array} \right) \left( \begin{array}{c}
N_{R}^{(1)} \\
N_{R}^{(2)} \\
N_{R}^{(3)} \\
N_{R31} \\
N_{R32} \\
N_{R33}
\end{array} \right), \quad (14)$$

where we set the VEVs $\langle \Phi_3 \rangle = (v_{\phi_1}, v_{\phi_2}, v_{\phi_3})$ and $\langle H_1 \rangle = v_{H_1}$. 

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The left-handed Majorana mass matrix $M_{\nu L}$ is obtained by the seesaw mechanism,

$$M_{\nu L} = M_D M_{NR}^{-1} M_D^T,$$

thus the ten components of the $4 \times 4$ symmetric matrix can be written as

$$M_{\nu L}(\nu_{L1},\nu_{L1}) = \frac{Y_1^2 v_{H_1}^2 + Y_3^2 (v_{\phi_1}^2 + v_{\phi_2}^2 + v_{\phi_3}^2)}{2M_2},$$

$$M_{\nu L}(\nu_{L1},\nu_{L31}) = \frac{Y_2 Y_3 v_{\phi_1} v_{H_1} + 2Y_3^2 v_{\phi_1} v_{\phi_2}}{2M_2},$$

$$M_{\nu L}(\nu_{L1},\nu_{L32}) = \frac{Y_2 Y_3 v_{\phi_3} v_{H_1} + 2Y_3^2 v_{\phi_1} v_{\phi_2}}{2M_2},$$

$$M_{\nu L}(\nu_{L31},\nu_{L31}) = \frac{Y_2^2 v_{H_1} + Y_3^2 (4v_{\phi_1}^2 + v_{\phi_2}^2 + v_{\phi_3}^2)}{2M_2},$$

$$M_{\nu L}(\nu_{L31},\nu_{L32}) = \frac{2Y_2 Y_3 v_{H_1} v_{\phi_3} - Y_3^2 v_{\phi_1} v_{\phi_2}}{2M_2},$$

$$M_{\nu L}(\nu_{L31},\nu_{L33}) = \frac{2Y_2 Y_3 v_{H_1} v_{\phi_2} - Y_3^2 v_{\phi_1} v_{\phi_3}}{2M_2},$$

$$M_{\nu L}(\nu_{L32},\nu_{L32}) = \frac{Y_2^2 v_{H_1} + Y_3^2 (v_{\phi_1}^2 + 4v_{\phi_2}^2 + v_{\phi_3}^2)}{2M_2},$$

$$M_{\nu L}(\nu_{L32},\nu_{L33}) = \frac{2Y_2 Y_3 v_{H_1} v_{\phi_3} - Y_3^2 v_{\phi_2} v_{\phi_3}}{2M_2},$$

$$M_{\nu L}(\nu_{L33},\nu_{L33}) = \frac{Y_2^2 v_{H_1} + Y_3^2 (v_{\phi_1}^2 + v_{\phi_2}^2 + 4v_{\phi_3}^2)}{2M_2}.$$

We can diagonalize the $4 \times 4$ symmetric mass matrix $M_{\nu L}$ by writing it in the form

$$M_{\nu L} = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix},$$

in the $(\nu_{\tau'}, \nu_\tau, \nu_\mu, \nu_e)$ basis, and then using the unitary transformation,

$$M_{\text{diag}} = U_{TBM}^{4g} M_{\nu L} U_{TBM}^{4gT},$$

where the 4 generations mixing matrix in the tribimaximal mixing limit can be expressed as

$$U_{TBM}^{4g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} \end{pmatrix}.$$
There are six conditions corresponding to the vanishing off diagonal terms, viz.

\[ E = H , \quad G = I , \quad (20) \]
\[ G + J = E + F , \quad (21) \]
\[ B = C = D = 0 . \quad (22) \]

Eq. (20) implies \( v_{\phi_1} = v_{\phi_2} \) and the solution for Eq. (21) is

\[ Y_3(v_{\phi_1} - v_{\phi_3})[Y_3(2v_{\phi_1} + 3v_{\phi_3}) - 2Y_2v_{H_1}] = 0 . \quad (23) \]

Together with the conditions from Eq. (22) we have only one consistent solution, \( v_{\phi_1} = v_{\phi_2} = v_{\phi_3} \equiv v \). Thus we write

\[ \langle \Phi_3 \rangle = (v, v, v) \quad (24) \]

and the constraint from Eq. (22) becomes

\[ v_{H_1} = -\frac{2Y_3}{Y_2}v . \quad (25) \]

This leads to the four neutrino masses

\[ m_{\nu_4} = \frac{Y_1^2 v_{H_1}^2}{M_1} + \frac{3Y_3^2 v^2}{2M_2}, \]
\[ m_{\nu_3} = \frac{15Y_3^2 v^2}{2M_2}, \]
\[ m_{\nu_2} = 0, \]
\[ m_{\nu_1} = m_{\nu_3}. \quad (26) \]

The parameters in the first term of \( m_{\nu_4} \) all differ from those in the expressions for the other three SM neutrinos, hence they can be used to easily satisfy the experimental bound \( m_{\nu_4} > M_Z/2 \). At leading order the three light neutrino masses are \( m_{\nu_1} = m_{\nu_3} \) and \( m_{\nu_2} = 0 \), which are phenomenological unacceptable. However, the inclusion of perturbative effects on the masses due to the heavy Majorana neutrinos through the VEVs of \( S_3 \) will correct the above three light neutrinos masses and allow a good fit to the oscillation data. We note that this perturbative effect is from the assumed \( A_5 \) symmetry and its subsequent spontaneous break down to \( A_4 \). The situation is similar to the proposal in Ref. [35], in which the neutrino masses can be split by the small VEVs of several heavy Higgs triplets which are assigned to different representations of the \( A_4 \) group. Therefore, taking the VEVs
of $S_3$, to be $\langle S_3 \rangle = (\delta_1, \delta_2, \delta_3)$, the perturbations to the right-handed Majorana mass matrix, as shown in Eq. (12) are

$$M_{NR} = \begin{pmatrix}
M_1 & 0 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 & 0 \\
0 & 0 & 2M_2 & Y_{S1}\delta_3 & Y_{S1}\delta_2 \\
0 & 0 & 0 & Y_{S1}\delta_3 & 2M_2 & Y_{S1}\delta_1 \\
0 & 0 & 0 & Y_{S1}\delta_2 & Y_{S1}\delta_1 & 2M_2
\end{pmatrix}. \quad (27)$$

The three light neutrino masses become

$$m_{\nu_3} \approx \frac{6Y_3^2 v^2}{M_2} + \frac{3}{2M_2 + Y_{S1}\delta_2} Y_3^2 v^2,$$

$$m_{\nu_2} \approx -\frac{3Y_3^2 v^2}{2M_2} + \frac{2Y_3^2 v^2}{2M_2 - Y_{S1}\delta_2} + \frac{(2M_2 + Y_{S1}\delta_2)Y_3^2 v^2}{M_2(2M_2 - Y_{S1}\delta_2)},$$

$$m_{\nu_1} \approx \frac{9Y_3^2 v^2}{2M_2} + \frac{2Y_3^2 v^2}{2M_2 - Y_{S1}\delta_2} + \frac{(2M_2 + Y_{S1}\delta_2)Y_3^2 v^2}{M_2(2M_2 - Y_{S1}\delta_2)},$$

for $\delta_1 \neq 0$ and $\delta_2 = \delta_3 = 0$. Similarly, for the cases of $(0, \delta_2, 0)$ and $(0, 0, \delta_3)$ we have

$$m_{\nu_3} \approx \frac{9Y_3^2 v^2}{M_2} + \frac{6M_2}{4M_2^2 - Y_3^2 Y_{S1}\delta_2^2} + \frac{3}{2M_2 - Y_{S1}\delta_2} Y_3^2 v^2,$$

$$m_{\nu_2} \approx -\frac{3Y_3^2 v^2}{2M_2} + \frac{2Y_3^2 v^2}{2M_2 + Y_{S1}\delta_2} + \frac{(2M_2 + Y_{S1}\delta_2)Y_3^2 v^2}{M_2(2M_2 + Y_{S1}\delta_2)},$$

$$m_{\nu_1} \approx \frac{9Y_3^2 v^2}{2M_2} + \frac{2Y_3^2 v^2}{2M_2 - Y_{S1}\delta_2} + \frac{(2M_2 + Y_{S1}\delta_2)Y_3^2 v^2}{M_2(2M_2 - Y_{S1}\delta_2)} + \frac{(2M_2 + Y_{S1}\delta_2)Y_3^2 v^2}{2M_2(4M_2^2 - Y_3^2 Y_{S1}\delta_2^2)}, \quad (29)$$

and

$$m_{\nu_3} \approx \frac{3Y_3^2 v^2}{M_2} + \frac{9Y_3^2 v^2}{2M_2 - Y_{S1}\delta_3},$$

$$m_{\nu_2} \approx -\frac{3Y_3^2 v^2}{2M_2} + \frac{2Y_3^2 v^2}{2M_2 + Y_{S1}\delta_3} + \frac{Y_3^2 v^2(2M_2 + Y_{S1}\delta_3)^2}{8M_2^3},$$

$$m_{\nu_1} \approx \frac{6Y_3^2 v^2}{M_2} + \frac{Y_3^2 v^2}{2M_2 + Y_{S1}\delta_3} + \frac{3Y_3^2 v^2(2M_2 + Y_{S1}\delta_3)}{4M_2^2} + \frac{Y_3^2 v^2(2M_2 + Y_{S1}\delta_3)^2}{8M_2^3}, \quad (30)$$
respectively. Note that the above three sets of equations reduce to Eq. (26) when \( \delta_i \to 0 \) as they should. In these three cases we have the decoupling fourth neutrino mass which should be required to satisfy

\[
m_{\nu_4} = \frac{Y_1^2 v_{H_1}^2}{M_1} + \left( \frac{2}{2M_2 + Y_{S1}\delta_i} + \frac{1}{2M_2} \right) Y_3^2 v^2 > M_Z/2.
\]

By assuming the new physics to be around \( \sim 1 \) TeV, and taking \( v_{H_1} = 220 \) GeV, \( v = 10 \) GeV, \( Y_{S1}\delta_{i=1,2,3} = 400 \) GeV, \( M_1 = 500 \) GeV, \( M_2 = 10^8 \) GeV, \( Y_1 = 1 \), and \( Y_3 = 10^{-3} \), we have

\[
\begin{align*}
m_{\nu_3} &\approx 0.75 \times 10^{-2} \text{ eV}, & m_{\nu_2} &\approx 5.4 \times 10^{-6} \text{ eV}, & m_{\nu_1} &\approx 0.35 \times 10^{-2} \text{ eV}; \\
m_{\nu_3} &\approx 0.75 \times 10^{-2} \text{ eV}, & m_{\nu_2} &\approx 6.0 \times 10^{-9} \text{ eV}, & m_{\nu_1} &\approx 0.70 \times 10^{-2} \text{ eV}; \\
m_{\nu_3} &\approx 0.75 \times 10^{-2} \text{ eV}, & m_{\nu_2} &\approx 6.0 \times 10^{-15} \text{ eV}, & m_{\nu_1} &\approx 0.85 \times 10^{-2} \text{ eV};
\end{align*}
\]

and

\[
m_{\nu_4} \approx 96.8 \text{ GeV},
\]

which satisfies the current 95% CL heavy neutrino mass limit, \( m_{\nu_4} > 90.3 \) GeV for Dirac coupling and \( m_{\nu_4} > 80.5 \) GeV for Majorana coupling \[31\]. Here the splitting of the fourth generation neutrino from the three active neutrinos is caused by the hierarchical spectrum \( M_1, M_2 \) of right-handed Majorana fields, so it can be argued that the masses of \( N_{R1}^{(1)} \) and \( N_{R5} \) have different origins, as shown in Eqs. (4) and (12). We note that \( m_{\nu_2} \) is too small to accommodate the oscillation data. However, the three perturbations \( \delta_{1,2,3} \) are different non-zero quantities, and in general they can be varied at the same time independently. We find that the inclusion of a VEV for \( H_3 \) and the relaxation of the condition \( v_{\phi_1} = v_{\phi_2} = v_{\phi_3} \) will alter the tribimaximal mixings and can lead to masses in the range \( 10^{-1} - 10^{-3} \) eV for the three light neutrinos for certain choices of parameters. \[1\] Therefore, we anticipate that higher order corrections will provide enough degrees of freedom to fit the experimental data and a degenerate mass spectrum of neutrino masses is preferred in the present model.

\[1\] A thorough parameter space scan to obtain realistic neutrino mass spectrum is thus quite interesting but it is outside the scope of this work.
C. Some Phenomenology

The model contains many new sources of lepton flavor violation due to the extra scalars and fermions, and is similar to other flavor symmetry models. For example, lepton flavor violating processes in the $A_4$ models are studied in ref. [36].

Here we address the presence of the extra contribution to the muon anomalous magnetic moment and leptonic rare decay processes ($\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$) due to the existence of heavy vectorlike field $l_{L+R,3}$ which does not appear in other non-Abelian discrete symmetry models. The current limit of the muon anomalous $g - 2$ is [37]

$$\Delta a_\mu = (290 \pm 90) \times 10^{-11},$$

which is a $3.2\sigma$ deviation between SM calculations and experiment [38]. The leading contribution to the muon anomalous magnetic moment in our model is showed in Fig. 2 and yields

$$\Delta a_\mu \approx \frac{Y_4 Y_{S2} m_\mu m_{L+R,3}}{12\pi^2} \frac{1}{m_H^2} F(z),$$

where $z = \left(\frac{m_{L+R,3}}{m_\mu}\right)^2$. Note that we have ignored all the mixing factors in both Higgs and lepton sectors. The function $F(z)$ is defined as

$$F(z) = -\frac{3}{2(1 - z)^2} \left[ 3 - 4z + z^2 + 2 \ln z \right].$$

Similar diagrams lead to rare radiative decays of $\mu \rightarrow e\gamma$ and $\tau \rightarrow \mu(e)\gamma$. The stringent bounds for these decays are [37]

$$B(\mu \rightarrow e\gamma) < 1.2 \times 10^{-11}, \quad B(\tau \rightarrow e\gamma) < 1.1 \times 10^{-7}, \quad B(\tau \rightarrow \mu\gamma) < 4.5 \times 10^{-8}$$

at 90% C.L. If the photon is on shell, we can write the invariant amplitude for the decays $l_i \rightarrow l_i\gamma$ in the form

$$A_{l_i l_i\gamma} = \frac{ie}{8\pi} \epsilon_\mu^*(q) \bar{u}_i(p - q) \left[ \sigma^{\mu\nu} q_\nu(c_1 P_L + c_2 P_R) \right] u_j(p),$$
where the matrices $P_L$ and $P_R$ are defined by $P_L = (1 - \gamma_5)/2$, $P_R = (1 + \gamma_5)/2$ and $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$. The coefficients $c_1$ and $c_2$ were calculated in [39] and are

$$c_1 = \frac{Y_4 Y_S m_\mu}{\pi m_H^2} H(z) + \frac{Y_2^2 m_{L+R,3}}{\pi m_H^2} G(z)$$

and

$$c_2 = \frac{Y_4 Y_S m_\mu}{\pi m_H^2} H(z) + \frac{Y_4^2 m_{L+R,3}}{\pi m_H^2} G(z) \quad (39)$$

with

$$H(z) = -\frac{1}{6(1-z)} \left[ 1 + \frac{3}{1-z} - \frac{6}{(1-z)^2} - \frac{6z}{(1-z)^3 \ln z} \right], \quad (40)$$

$$G(z) = -\frac{1}{1-z} \left[ 1 + \frac{2}{1-z} + \frac{2}{(1-z)^2 \ln z} \right]. \quad (41)$$

The mixings between the charged leptons are ignored as before. If we take $m_H = 1$ TeV as the new physics scale and the limit of $m_{L+R,3} \approx m_H$, the anomalous muon magnetic moment and the branching ratio of $\mu \rightarrow e\gamma$ becomes

$$\Delta a_\mu \approx 5.1 \times 10^{-13} \quad (42)$$

and

$$B(\mu \rightarrow e\gamma) \approx \frac{\alpha}{6\pi G_F^2 m_\mu^2 m_H^2} (Y_4^2 + Y_4^4) \approx 9.7 \times 10^{-11} \quad (43)$$

respectively. Here we have used the relation $m_\mu = \sqrt{3} Y_4 V'$ shown in Eq. [9] and assumed the VEV of Higgs $V' \sim M_W$ and couplings $Y_{S2} = Y_4$. Similarly the estimate of the branching ratio for the $\tau \rightarrow \mu\gamma$ yields

$$B(\tau \rightarrow \mu\gamma) \approx 3.4 \times 10^{-13}, \quad (44)$$

which is well below the current experimental bound given in Eq. [37].

**IV. DISCUSSION AND CONCLUSIONS**

We minimally extend the three generations $A_4$ lepton flavor symmetry to a four generations $A_5$ lepton flavor symmetry. We find that the tribimaximal pattern in three generation neutrino mixings survives, and all the mass bounds on SM leptons can be satisfied. Notably, the electron is predicted to be massless at tree level, hence one must calculate $m_e$ via quantum corrections. If the masses of the extra heavy leptons are within the reach of the LHC, we should be able to test the model. For a sequential fourth generation of leptons, the
LHC with \( \sqrt{s} = 7 \text{ TeV} \) and an integrated luminosity of 1 fb\(^{-1} \) of data can exclude fourth generation charged leptons with masses up to 250 GeV \[8\]. It may be worth while to repeat this analysis for the \( A_5 \) model where mixings between the sequential fourth generation and the extra vector-like leptons are allowed. The lepton flavor violating phenomenology in this model is very rich and needed to be studied further. Before we close, we note that \( A_4 \) extended in the binary tetrahedral group \( T' \) can provide a model of both the quark and lepton sectors \[40\]. The three generation \( T' \) model has calculable Cabibbo angle as well as other attractive features \[40\]. There is an analogous extension of the \( A_5 \) to the binary icosahedral group \( I' \), where the attractive features of the \( T' \) model can be utilized and four families of quarks and leptons can be accommodated simultaneously \[41\].

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**Appendix A: Discrete Symmetry Group \( A_5 \) and the Icosahedron**

\[\text{FIG. 3: The regular icosahedron.}\]

\( A_5 \) is a discrete symmetry group of even permutation of five objects. Its order, the number of elements, is equal to \( (5!)/2 = 60 \), which can be divided into 5 distinct conjugate classes. A
regular icosahedron (see Fig. 3) consists of 20 equilateral triangles, 30 edges and 12 vertices. The rotations of the icosahedron can be classified into five types, including 0 or 2π rotation (the identity), π rotations about the midpoint of each edge, 2π/5 and 4π/5 rotations about an axis through each vertex, and rotations by 2π/3 about axes through the center of each face. These five types of rotations form five conjugate classes denoted by

$$C_1, C_{15}, C_{12}, C'_{12}, C_{20}.$$  \hspace{1cm} (A1)

Hence the icosahedral symmetry group $I$ also has 60 elements. One can show that the icosahedron symmetry group $I$ and the even permutation group $A_5$ are isomorphic. The orthogonality relations for $A_5$ imply $\sum_{i=1}^{5} n_i^2 = 60$, where $n_i$ denotes the dimensionality of the irreducible representation $R_i$, and we know that the number of classes of a discrete group is equal to its number of irreducible representation. Therefore, we have five irreducible representations (irreps): one trivial singlet 1, two triplets 3 and 3', one quartet 4 and one quintet 5. The character table and multiplication rules for $A_5$ are shown in Table I and II respectively.

The sixty elements of $A_5$ can be generated by two elements, $s$ and $t$, which satisfy $s^2 = t^5 = (t^2stst^{-1}st^{-1})^5 = e$. The explicit matrix representations of $s$ and $t$ for each irrep can
be written down in terms of the golden ratio $\phi = \frac{1 + \sqrt{5}}{2}$:

\[1: s \rightarrow 1, \quad t \rightarrow 1, \quad (A2)\]

\[3: s = \frac{1}{2} \begin{pmatrix} -1 \phi^2 & \frac{1}{\phi} \\ \phi^2 & 1 \phi & 1 \\ \frac{1}{\phi} & 1 - \phi \end{pmatrix}, \quad t = \frac{1}{2} \begin{pmatrix} 1 \phi^2 & \frac{1}{\phi} \\ -\phi^2 & \frac{1}{\phi} & 1 \\ \frac{1}{\phi} & -1 \phi \end{pmatrix}, \quad (A3)\]

\[3': s = \frac{1}{2} \begin{pmatrix} -\phi^2 & \frac{1}{\phi} \\ \phi^2 & 1 \phi \end{pmatrix}, \quad t = \frac{1}{2} \begin{pmatrix} -\phi^2 & \frac{1}{\phi} \\ \phi^2 & 1 \phi \end{pmatrix}, \quad (A4)\]

\[4: s = \frac{1}{4} \begin{pmatrix} -1 & -1 & -3 & -\sqrt{5} \\ -1 & 3 & 1 & -\sqrt{5} \\ -3 & 1 & -1 & \sqrt{5} \\ -\sqrt{5} & -\sqrt{5} & \sqrt{5} & -1 \end{pmatrix}, \quad t = \frac{1}{4} \begin{pmatrix} -1 & 1 & -3 & \sqrt{5} \\ -1 & -3 & 1 & \sqrt{5} \\ 3 & 1 & 1 & \sqrt{5} \\ \sqrt{5} & -\sqrt{5} & -\sqrt{5} & -1 \end{pmatrix}, \quad (A5)\]

\[5: s = \frac{1}{2} \begin{pmatrix} \frac{1 - 3\phi}{4} & \frac{\phi^2}{2} & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{4\phi} \\ \frac{\phi^2}{2} & 1 & 0 & \frac{\sqrt{5}}{2\phi} \\ 0 & -1 & 1 & -\frac{\sqrt{3}\phi}{2} \\ \frac{\sqrt{5}}{2\phi} & \frac{\sqrt{5}}{2\phi} & -\frac{\sqrt{3}\phi}{2} & -\frac{3\phi - 1}{4} \end{pmatrix}, \quad t = \frac{1}{2} \begin{pmatrix} \frac{1 - 3\phi}{4} & \frac{\phi^2}{2} & -\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{4\phi} \\ \frac{\phi^2}{2} & 1 & 0 & \frac{\sqrt{5}}{2\phi} \\ 0 & -1 & 1 & -\frac{\sqrt{3}\phi}{2} \\ \frac{\sqrt{5}}{2\phi} & \frac{\sqrt{5}}{2\phi} & -\frac{\sqrt{3}\phi}{2} & -\frac{3\phi - 1}{4} \end{pmatrix}, \quad (A6)\]

|     | 1 | 3 | 3' | 4 | 5 |
|-----|---|---|----|---|---|
| $C_1$ | 1 | 3 | 3 | 4 | 5 |
| $C_{15}$ | 1 | -1 | -1 | 0 | 1 |
| $C_{20}$ | 1 | 0 | 0 | 1 | -1 |
| $C_{12}$ | 1 | $\phi$ | $1 - \phi$ | -1 | 0 |
| $C'_{12}$ | 1 | $1 - \phi$ | $\phi$ | -1 | 0 |

**TABLE I:** Character table of $A_5$ where $\phi = \frac{1 + \sqrt{5}}{2}$ is the golden ratio.
Table II: Multiplication rules for the $A_5$ discrete group.

Appendix B: $A_4$ Multiplication Rules

$A_4$ is a discrete symmetry corresponding to the even permutation of four objects. It has four irreducible representations: three in-equivalent one-dimensional representations $(1, 1', 1'')$ and a three-dimensional representation $3$. The multiplication rules are given in Table III.

Table III: Multiplication rules for the $A_4$ discrete group.

Given two triplets denoted by $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, the decomposition of $3 \otimes 3$ can be constructed explicitly as

\[
(a \otimes b)_1 = a_1b_1 + a_2b_2 + a_3b_3, \\
(a \otimes b)_{1'} = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3, \\
(a \otimes b)_{1''} = a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3, \\
(a \otimes b)_{3_a} = (a_2b_3 + a_3b_2, a_3b_1 + a_1b_3, a_1b_2 + a_2b_1), \\
(a \otimes b)_{3_a} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1),
\]

where $\omega$ is the cube root of unity, $\omega = e^{2\pi i/3}$. 

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Appendix C: Higgs potential

In this Appendix we discuss the Higgs potential and its minimization in our model. The most general form of the Higgs potential containing the scalar fields $S_4$, $H_4$, $H'_4$ and $\Phi_3$, invariant under the discrete $A_5 \times Z_2 \times Z_3$ symmetries is given by

$$V = V(S_4) + V(H_4) + V(H'_4) + V(\Phi_3) + V(S_4, H_4) + V(S_4, H'_4)$$

$$+ V(S_4, \Phi_3) + V(H_4, H'_4) + V(H_4, \Phi_3) + V(H'_4, \Phi_3) + V(H_4, H'_4, \Phi_3)$$

where the individual terms are

$$V(S_4) = m_s^2 S_4^2 + \mu_s (S_4^2) S_4 + \lambda_s^2 (S_4^2) S_4,$$

$$V(H_4) = \mu^2 (H_4^2) H_4 + \lambda^2 (H_4^2) H_4 H_4,$$

$$V(H'_4) = \mu^2 (H'_4^2) H'_4 + \lambda^2 (H'_4^2) H'_4 H'_4,$$

$$V(\Phi_3) = \mu^2 (\Phi_3^2) \Phi_3 + \lambda^2 (\Phi_3^2) \Phi_3,$$

$$V(S_4, H_4) = \delta^{HS} (H_4^2 H_4^2) S_4 + \lambda^{HS} (H_4^2 H_4^2) S_4,$$

$$V(S_4, H'_4) = \delta^{HS} (H'_4^2 H'_4^2) S_4 + \lambda^{HS} (H'_4^2 H'_4^2) S_4,$$

$$V(S_4, \Phi_3) = \delta^{PS} (\Phi_3^2) S_4 + \lambda^{PS} (\Phi_3^2) S_4,$$

$$V(H_4, H'_4) = \lambda^{HH'} (H_4^2 H'_4) + \lambda^{HH'} (H'_4^2 H_4) + \lambda^{HH'} (H_4^2 H'_4) + \lambda^{HH'} (H'_4^2 H_4)$$

$$+ \left[ \lambda^{HH'} (H_4^2 H'_4) + \lambda^{HH'} (H'_4^2 H_4) + \text{H.c.} \right],$$

$$V(H_4, \Phi_3) = \lambda^{H\Phi} (H_4^2 \Phi_3) + \lambda^{H\Phi} (H_4^2 \Phi_3)$$

$$+ \left[ \lambda^{H\Phi} (H_4^2 \Phi_3) + \text{H.c.} \right],$$

$$V(H'_4, \Phi_3) = \lambda^{H\Phi} (H'_4^2 \Phi_3) + \lambda^{H\Phi} (H'_4^2 \Phi_3)$$

$$+ \left[ \lambda^{H\Phi} (H'_4^2 \Phi_3) + \text{H.c.} \right],$$

$$V(H_4, H'_4, \Phi_3) = \lambda^{HH\Phi} (H_4^2 \Phi_3 \gamma (H'_4^2 H_4) + \lambda^{HH\Phi} (H'_4^2 \Phi_3 \gamma (H'_4^2 H_4)$$

$$+ \lambda^{HH\Phi} (H'_4^2 \Phi_3 \gamma (H'_4^2 H_4) + \text{H.c.}}$$

Here we have introduced the notations $\alpha = 1, 3, 3', 4, 5$; $\beta = 1, 3, 5$; and $\gamma = 3', 4, 5$ respectively. $A_5$ is broken down to $A_4$ in the first stage of symmetry breaking, where it is not difficult to see that one can always choose the VEVs of the scalar field $S_4$ to the direction $\langle S_4 \rangle = (V_S, 0, 0, 0)$, which can be the global minimum of $V(S_4)$. The result of this alignment is that the $A_5$ breaking is responsible for the right-handed Majorana masses of
$N_{R5}$ (Eq. (12)) and the remaining three components of $S_4$, which form $S_3$ under $A_4$, will generate perturbations of active neutrino masses (Eq. (27)-(30)). As pointed out above, the scalar fields will decompose as $4 \rightarrow 1 + 3$ and $3 \rightarrow 3$ after $A_5$ breaks into $A_4$. One will obtain the $A_4$ symmetry potential with the collective coefficients $\delta'$s and $\lambda'$s of the decomposed fields ($S_1, S_3, H_1, H_3, H'_1, H'_3$ and $\Phi_3$) from Eqs. (C2)-(C12). Now we turn to the minimization of the $A_4$ potential.

In our discussion all vacua can be accommodated by the large parameter space in the Higgs potential except the ad hoc vacuum alignment of $\Phi_3$ shown in Eq. (24). The reason is the existence of the interaction terms between the scalar fields $S_4, H_4, H'_4$ and $\Phi_3$ in $V(S_4, \Phi_3), V(H_4, \Phi_3), V(H'_4, \Phi_3)$ and $V(H_4, H'_4, \Phi_3)$. These terms will produce more independent equations derived from the extremum conditions than the unknown VEVs. The authors of Refs. [11] and [13] showed how to deal with the vacuum alignment problem under the non-Abelian group symmetry. Here we could extend the model with an extra spacial dimension $y$, via the method introduced in the first paper in [11]. The fields are localized at the boundaries $y = 0$ and $y = L$ as shown in Fig. 4 and the configurations will realize the needed vacuum. There are non-local effects involving both branes. A detailed explanation of this possibility is beyond the scope of this paper.

FIG. 4: Fifth dimension and locations of the field configuration.

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