Spatial and Topological Interdiction for Transmission Systems

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Abstract—This paper presents novel formulations and algorithms for the N-k interdiction problem in transmission networks. In particular, it models two new classes of N-k attacks: (i) Spatial N-k attacks where the attack is constrained to be within a specified distance of a bus chosen by an attacker and (ii) Topological N-k attacks where the attack is constrained to connected components. These two specific types of N-k attacks compute interdiction plans designed to better model localized attacks, such as those induced by natural disasters or physical attacks. We formulate each of these problems as bilevel, max-min optimization problems and present an algorithm to solve these formulations. Detailed case studies analyzing the behavior of these interdiction problems and comparing them to the traditional worst-case N-k interdiction problem are also presented.

Index Terms—N-k interdiction, power grids, Stackelberg game, coordinated attacks, penalty method

I. INTRODUCTION

The electric power transmission grid plays an important and critical role in sustaining the socioeconomic systems that modern society depends on. Recent events, including natural disasters and intentional physical attacks on the grid, illustrate the need for methods that identify small sets of components whose failure leads to significant system impacts. One model that identifies such sets is the N-k interdiction problem [22]. The interdiction problem seeks to identify k components in the system whose simultaneous or near-simultaneous failure causes the worst case disruption to the grid, where disruption is typically measured through load shed. This problem is often modeled as a bilevel, Stackelberg game (see [7]) with an attacker and a defender. The attacker’s and defender’s actions are sequential and the attacker has a perfect model of how the defender will optimally respond to an attack. The objective of the attacker is to identify k components whose loss maximizes the minimum load that a defender must shed in response to the simultaneous or near-simultaneous failure of those components. In this problem, N and k refer to the total number of components and the number of attacked components, respectively. The number of possible N-k contingencies, even for small values of k, makes complete enumeration computationally intractable.

The motivation of the paper stems from the fact that the current adversarial models in the literature assume the adversary or the attacker is all powerful and can interdict different parts of the system simultaneously, even if they have large geographic separation. The focus of this paper is to develop novel interdiction formulations that model a resource constrained adversary that is more reflective of reality and illustrate the value of modeling such adversaries by comparing them to the traditional models in the literature. In particular, we formulate models where the attacker is constrained either topologically or spatially as such scenarios encompass many practical situations such as coordinated physical attacks, hurricanes and earthquakes. We stress that the techniques presented in this paper are general and are extendable to other formulations of resource constrained adversaries.

Over the last ten years, there has been considerable progress in modeling and developing algorithm to solve the deterministic and probabilistic variants of the N-k problem with approaches ranging from exact (computing an optimal N-k interdiction) to heuristic. Here, we present an overview of the literature for the deterministic N-k interdiction problem and refer interested readers to [24] for detailed literature review of probabilistic variants. We start by reviewing the literature on variants of the N-k interdiction problem that use the DC power flow equations and develop exact approaches for solving the problem. The first work is by Salmeron et al. [21] where a bilevel formulation for the problem is presented. A Benders decomposition algorithm based on this formulation was developed in [1] and tested on small instances. The inner problem in [21] is replaced by its dual in [19] and is approximated using KKT conditions in [3]. The first work to systematically develop a decomposition algorithm to solve the N-k problem based on Benders decomposition on large test instances is [22]. The algorithm developed in [22] was tested on a “U.S. Regional Grid” with 5000 buses, 5000 lines, and 500 gen-

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1We do not assume a natural event is intelligent. In this context, k is used to model the expected number of components that fail during a natural event. The interdiction problem determines the components that could fail under that assumption and lead to the worst outcome.
erators. The works in [1], [22] measure disruption in terms of long-term power shedding, ignoring short-term shedding resulting from cascading outages immediately after the attack, and use a DC power flow model to compute this long-term power shedding. More recently, [5] develops computationally efficient algorithms to solve a minimum cardinality variant of the N-k problem, where the objective is to find a minimum cardinality attack with a throughput less than a pre-specified bound. They also formulate and solve a nonlinear continuous version of the problem where the attacker is allowed to change the transmission line parameters to disrupt the system instead of removing transmission lines. The authors in [11] study the N-k problem where the system operator is allowed to use both load shedding and line switching as defensive operations via a Benders decomposition algorithm.

The literature has also developed interdiction models that use AC power flow formulations and heuristic approaches that are scalable to larger instances. For example, [17] and [12], [20] have used the AC power flow equations and approximations to these equations, respectively, in interdiction modeling. Examples of heuristic approaches include [6] and [2]. Finally, there is literature on the related problem of contingency identification (see [10], [14], [15], [23] and references therein). In particular, [15] develops a heuristic approach to identify multiple contingencies that can initiate cascades on large transmission systems, and [10] uses current injection-based methods and “line outage distribution factors” to identify high consequence contingencies. This work in the literature focuses on using a variety of “criticality” measures, based on the DC power flow model, for the different components in the system that aid in identifying these contingencies. To the best of our knowledge, there is no work in the literature that considers interdiction problems with spatial proximity or topological connectivity restrictions. The contributions of this paper can be summarized as follows:

- The paper develops novel models of the N-k interdiction problem based on topological connectivity and spatial proximity.
- presents a generic constraint-generation algorithm that works for any variant of the N-k interdiction problem on power grids.
- corroborates the effectiveness and scalability of the algorithm on standard test instances, and,
- examines the results from the case studies in comparison with the traditional N-k interdiction problem.

### II. Problem Formulation

This section presents a bilevel, mixed-integer linear program for a general N-k interdiction problem and then specializes the formulation to model topological connectivity and spatial proximity restrictions. For ease of exposition, we assume that only transmission lines are interdicted, though the formulations and algorithms extend to the general case where any component in the transmission system can be interdicted. We first present the nomenclature and terminology that we use throughout the rest of the paper. Unless otherwise stated, all the values are in per-unit (pu).

**Sets:**
- \( \mathbb{N} \) - set of buses (nodes) in the network
- \( \mathbb{N}(i) \) - set of nodes connected to bus \( i \) by an edge
- \( \mathcal{E} \) - set of edges (lines) in the network
- \( \mathcal{E}(i) \) - set of edges in \( \mathcal{E} \) connected to bus \( i \) and oriented from \( i \)
- \( \mathcal{E}^r(i) \) - set of edges in \( \mathcal{E} \) connected to bus \( i \) and oriented to \( i \)

**Variables:**
- \( \theta_i \) - phase angle at bus \( i \)
- \( p_l^d \) - active power generated at bus \( i \)
- \( \ell_i \) - percent active power (load) shed at bus \( i \)
- \( x_{ij} \) - binary interdiction variable for line \( (i, j) \)
- \( \delta_{ij} \) - auxiliary flow variable for each line \( (i, j) \)
- \( y_i \) - auxiliary binary variable for bus \( i \in \mathbb{N} \)
- \( x \) - vector of interdiction variables \( x_{ij} \)

**Constants:**
- \( p_i^d \) - active power demand at bus \( i \)
- \( b_{ij} \) - susceptance of line \( (i, j) \)
- \( (0, \mathbf{p}_l) \) - bounds for active power generated at bus \( i \)
- \( t_{ij} \) - thermal limit of line \( (i, j) \)
- \( k \) - number of interdicted components
- \( D \) - planar distance limit

For a value \( \cdot \), we use the notation \((\cdot)^+ \cdot^−\) as shorthand for \(\max\{0, (\cdot)\}\) and \(-\min\{0, (\cdot)\}\), respectively. Using the above set of notations, the N-k interdiction problem is formulated as follows:

\[
\begin{align*}
\mathcal{F} & \max_{\mathbf{x} \in \mathcal{X}} \eta(\mathbf{x}) \\
\text{(LSx)} & \quad \eta(\mathbf{x}) = \min \sum_{i \in \mathbb{N}} p_i^d \ell_i \\
\text{subject to:} & \quad (2a) \\
& \quad p_i^d - (1 - \ell_i) p_i^d = \sum_{(i,j) \in \mathcal{E}(i)} p_{ij} - \sum_{(j,i) \in \mathcal{E}^r(i)} p_{ji} \quad \forall i \in \mathbb{N}, \\
& \quad 0 \leq p_i^d \leq \mathbf{p}_i, \quad \forall i \in \mathbb{N}, \quad (2b) \\
& \quad -M x_{ij} \leq p_{ij} + b_{ij} (\theta_i - \theta_j) \leq M x_{ij} \quad \forall(i,j) \in \mathcal{E} \quad (2c) \\
& \quad -t_{ij} (1 - x_{ij}) \leq p_{ij} \leq t_{ij} (1 - x_{ij}) \quad \forall(i,j) \in \mathcal{E} \quad (2d) \\
& \quad 0 \leq \ell_i \leq 1 \quad \forall i \in \mathbb{N} \quad (2f) \\
\end{align*}
\]

where \( M = \sum_{i \in \mathbb{N}} p_i^d \). For the traditional N-k interdiction problem, the set \( \mathcal{X} \) is given by \( \{ x : \sum_{(i,j) \in \mathcal{E}} x_{ij} = k \} \).
Additional restrictions on the set $X$ enforce topological connectivity or spatial proximity restrictions on the $N$-k interdiction plans. The inner problem is a linear program for a fixed interdiction plan. It formulates a minimum load shedding problem using the DC power flow constraints. Constraints (2d) and (2e) are functions of the interdiction variables $x$. We remark that this model does not shed load that is co-located with a generator because the model only interdicts lines. To achieve this effect, buses can be split into separate generator and load bus with no geographic distance. Alternatively, bus interdiction variables can be included in the formulation to shed loads that are co-located with generators in the same bus.

**A. Spatial $N$-k Interdiction**

This model enforces the attacker to choose $k$ lines and a bus $i$ such that the $k$ lines are within a planar distance of $\frac{D}{2}$ units from the bus $i$. This enforces the interdicted lines to be within a planar distance of $D$ units of one another. Instances of this kind of an interdiction plan include natural disasters which are localized, localized cyber attacks, and physical attacks. To formulate the problem, we use $\varphi_{ij}$ to denote the set of all buses within a planar distance, $\frac{D}{2}$, of line $(i, j)$. Given $\varphi_{ij}$ for each line $(i, j)$, the set $X$ for spatial $N$-k interdiction is modeled as:

$$\sum_{(i,j)\in\mathcal{E}} x_{ij} \leq k, \quad \sum_{i\in N} x_{i} = 1 \quad (3a)$$

$$x_{ij} \leq \sum_{n\in\varphi_{ij}} x_{i}, \quad \forall (i, j) \in \mathcal{E}, \quad (3b)$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in \mathcal{E}, \quad x_{i} \in \{0, 1\} \quad \forall i \in N. \quad (3c)$$

Constraints (3a) ensure (i) at most $k$ components are interdicted and (ii) exactly one bus is selected. Unlike the traditional interdiction, the spatial interdiction problem may not contain a feasible interdiction plan with exactly $k$ lines removed. This may occur when $D$ is very small and there are less than $k$ components of the transmission system in the region of impact. Without loss of generality, if the location of the event’s bus is known, then the binary variable for that bus can explicitly be set to one. Finally, constraints (3b) ensure that all the interdicted components lie within the circle of radius $\frac{D}{2}$ units centered around the chosen bus. Alternately, this condition can be restated as every interdicted component is within a distance of $D$ units from each other and within a distance of $\frac{D}{2}$ units from the chosen bus.

**B. Topological $N$-k Interdiction**

The spatial $N$-k interdiction assumes that a damaging event’s region of impact is circular and aims at finding the worst-case $N$-k attack within this circular footprint. But damaging events like tornadoes, hurricanes, etc. may not be circular or even regularly shaped. Furthermore, spatial data might not be available in many cases. Hence, as a proxy, the topological $N$-k interdiction considers connected components, i.e., it computes a worst-case connected $N$-k attack. To formulate the connectivity constraints, we use a single-commodity flow formulation [18] by introducing a “super sink” bus ($\kappa$) that is connected to every bus in the network. $\kappa$ serves as a source of $k+1$ units of flow that are carried to all the buses in the network that are incident on at least one interdicted line.

Flow variables $\delta_{ij}$ for each $(i, j) \in \mathcal{E}$ determine the amount of flow carried on the interdicted edges. The set $X$ for the topological $N$-k interdiction is modified as follows:

$$\sum_{(i,j)\in\mathcal{E}} x_{ij} = k, \quad \sum_{i\in N} x_{ki} = 1, \quad (4a)$$

$$x_{ij} \leq y_{i}, \quad x_{ij} \leq y_{j} \quad \forall (i, j) \in \mathcal{E}, \quad (4b)$$

$$\delta_{ki} \leq (k+1) \cdot x_{ki} \quad \forall i \in N, \quad (4c)$$

$$\delta_{ij} \leq k \cdot x_{ij} \quad \forall (i, j) \in \mathcal{E}, \quad (4d)$$

$$\sum_{(j,i)\in\mathcal{E}^r(i)} \delta_{ji} - \sum_{(i,j)\in\mathcal{E}(i)} \delta_{ij} = y_{i} - \delta_{ki} \quad \forall i \in N, \quad (4e)$$

$$x_{ij} \in \{0, 1\} \quad \forall (i, j) \in \mathcal{E}, \quad (4f)$$

$$-k \leq \delta_{ij} \leq k \quad \forall (i, j) \in \mathcal{E}, \quad (4g)$$

$$y_{i} \in \{0, 1\} \quad \forall i \in N. \quad (4h)$$

Constraint (4a) ensures that the number of interdicted lines is $k$ and that a dummy edge originating from $\kappa$ is chosen to ensure flow is delivered to the buses incident on interdicted lines. Constraints (4b) enforce the auxiliary binary variables $y_i$ and $y_j$ to take value 1 if the line $(i, j)$ is interdicted. The constraints from (4c)–(4e) are single-commodity flow constraints and ensure that the interdicted lines form a connected component. Intuitively, the single commodity flow formulation ensures that each bus incident on an interdicted line receives one unit of flow from $\kappa$. In the next section, we develop a penalty-based constraint-generation algorithm to solve the bilevel problems to optimality.

**III. SOLUTION METHODOLOGY**

This section describes a constraint-generation algorithm that works directly with the bilevel structure of the interdiction problem and does not dualize the inner problem. We begin by first presenting a reformulation of the inner problem $LS(x)$.

**A. Reformulation of the interdiction problem**

The reformulation we propose assumes that $LS(x)$ is feasible for all $x \in X$. This is a reasonable assumption because the operator can shed all load in the system to make the inner
problem feasible. The reformulation depends on the following property of linear programs.

**Lemma 1.** Consider a linear program of the following form

\[
(P_1): \min \{c \cdot z : Az \leq b, z \geq 0\}
\]

that has a finite optimum. Let \(\alpha\) denote the vector of dual variables that correspond to the constraints \(Az \leq b\). Let \(\bar{\alpha}\) be a bound on the optimal dual variables \(\alpha^*\). Consider the following optimization problem:

\[
(P_2): \min \{c \cdot z + \bar{\alpha} \cdot (Az - b)^+ : z \geq 0\}
\]

Then the optimal values of \((P_1)\) and \((P_2)\) are identical.

**Proof.** The proof follows by taking the dual of \((P_1)\), adding the valid inequality \(\alpha \leq \bar{\alpha}\) and taking the dual again. \(\Box\)

For a fixed interdiction plan \(x \in X\), we use \((\mu_{ij}^1(x), \mu_{ij}^2(x))\) and \((\pi_{ij}^1(x), \pi_{ij}^2(x))\) to denote the optimal dual variables for constraints \((2d)\) and \((2e)\), respectively. The notation \((\bar{\mu}_{ij}^1(x), \bar{\mu}_{ij}^2(x))\) and \((\bar{\pi}_{ij}^1(x), \bar{\pi}_{ij}^2(x))\) denotes the upper bounds of of the dual variables over each \(x \in X\). By Lemma 1, the inner problem is equivalent to

\[
\eta^*(x) = \min \sum_{i \in N} p_i^d \ell_i + \sum_{(i,j) \in E} \left\{ \begin{array}{l}
\bar{\mu}_{ij}^1(x)[p_{ij} + b_{ij}(\theta_i - \theta_j) - M x_{ij}]^+ + \\
\bar{\mu}_{ij}^2(x)\left[ M x_{ij} - p_{ij} - b_{ij}(\theta_i - \theta_j) \right]^- + \\
\bar{\pi}_{ij}^1(x)\left[ p_{ij} - t_{ij}(1 - x_{ij}) \right]^+ + \\
\bar{\pi}_{ij}^2(x)\left[ t_{ij}(1 - x_{ij}) - p_{ij} \right]^- 
\end{array} \right. 
\]

subject to: Eqs \((2b), (2c), (2f)\),

\[-M \leq p_{ij} + b_{ij}(\theta_i - \theta_j) \leq M \quad \forall (i, j) \in E, \quad (7b)\]

\[-t_{ij} \leq p_{ij} \leq t_{ij} \quad \forall (i, j) \in E. \quad (7c)\]

This reformulation is a variant of “penalty methods” used to solve non-linear optimization problems \([4]\). Since \(x_{ij} \in \{0, 1\}\), the objective in \((7a)\) is equivalent to

\[
\eta^*(x) = \min \sum_{i \in N} p_i^d \ell_i + \sum_{(i,j) \in E} \left\{ \begin{array}{l}
\bar{\mu}_{ij}^1(x)[p_{ij} + b_{ij}(\theta_i - \theta_j)]^+ \cdot (1 - x_{ij}) + \\
\bar{\mu}_{ij}^2(x)\left[ p_{ij} + b_{ij}(\theta_i - \theta_j) \right]^+ - (1 - x_{ij}) + \\
\bar{\pi}_{ij}^1(x)\left[ p_{ij} + b_{ij}(\theta_i - \theta_j) \right]^+ + \\
\bar{\pi}_{ij}^2(x)\left[ p_{ij} + b_{ij}(\theta_i - \theta_j) \right]^+ - \cdot (1 - x_{ij}) \end{array} \right. 
\]

Our proposed constraint generation approach is based on estimating the dual upper bounds \(\mu_{ij}^1(x), \mu_{ij}^2(x), \pi_{ij}^1(x), \pi_{ij}^2(x)\). A similar approach was used in \([9]\) where the authors found a constant upper bound of 1 for every dual variable; their approach is valid for the special case they consider where the inner problem is a standard network flow problem. In this paper, we propose to use a piece-wise constant upper bound one pair of dual variables, \(\mu_{ij}^1(x), \mu_{ij}^2(x)\) that is based on the following observation.

**Lemma 2.** For any interdiction plan \(x \in X\) where \(X\) models a traditional, topological or spatial interdiction problem, the optimal dual values, \(\mu_{ij}^1(x)\) and \(\mu_{ij}^2(x)\) satisfy

\[
\mu_{ij}^1(x) = \mu_{ij}^2(x) = 0, \quad \text{whenever} \quad x_{ij} = 1. \quad (9)
\]

**Proof.** The proof follows by applying complementary slackness condition to the constraints \((2d)\). Whenever \(x_{ij} = 1\), the value of \(p_{ij} + b_{ij}(\theta_i - \theta_j)\) must lie in \((-M, M)\), i.e., the constraints are never tight. Therefore, the corresponding dual variables must be zero. \(\Box\)

Exploiting the property in Lemma 2, the objective in Eq. \((8)\) now reduces to

\[
\eta^*(x) = \min \sum_{i \in N} p_i^d \ell_i + \sum_{(i,j) \in E} \left( \bar{\pi}_{ij}^1(x)p_{ij}^1 + \bar{\pi}_{ij}^2(x)p_{ij}^2 \right) x_{ij}. 
\]

Choosing constant dual upper bound values of \((\bar{\pi}_{ij}^1(x), \bar{\pi}_{ij}^2(x))\), for \((\pi_{ij}^1(x), \pi_{ij}^2(x))\), respectively, we obtain the following equivalent formulation of the full interdiction problem:

\[
(T_1) \quad \max \eta \quad \text{subject to:} 
\]

\[
\eta \leq \sum_{i \in N} p_i^d \ell_i(x) + \sum_{(i,j) \in E} \left( \bar{\pi}_{ij}^1(x)p_{ij}^1 + \bar{\pi}_{ij}^2(x)p_{ij}^2 \right) x_{ij} \quad \text{for} \quad x \in X \quad (10a) \]

\[
\text{subject to:} \ Eqs (2b), (2c), (2f), (7b), (7c). \quad (10b) \]

where, \(\ell_i(x), p_{ij}^1(x),\) and \(p_{ij}^2(x)\) are the optimal load shed values, \(i \in N\), and the positive and negative parts of the active power flow on each transmission line of the network for a fixed \(N\)-k attack defined by \(x\).

**B. Constraint generation algorithm**

The formulation in Eq. \((10)\) is solved using a constraint generation algorithm that alternates between solving the outer maximization problem in Eq. \((1)\) and the inner problem \(LS(x)\). Constraints of the form \((10b)\) are generated at each iteration. It first relaxes all the constraints in Eq. \((10b)\) from \((10)\) and solves the resulting problem to obtain an upper bound to the optimal objective value of the interdiction problem. This solution obtained is then used to solve the inner problem i.e., a minimum load shedding problem with the components in the current solution removed. The optimal load shedding factors and active power flow on each line are then used to add a cut of the form \((10b)\). The constraint-generation algorithm generates the constraints in \((10b)\) dynamically until an optimality tolerance is reached. At each iteration of the constraint-generation algorithm, in addition to the constraint in step 10 of Algorithm 1, we add the following no-good cut in Eq. \((11)\). These no-good cuts eliminate the selection of the same interdiction plan in the outer problem after it
is selected once. The cuts have been observed to improve the convergence behaviour of the algorithm for the traditional interdiction problems [22].

\[ \sum_{(i,j) \in E} \hat{x}_{ij} \cdot x_{ij} \leq k - 1. \]  
\hspace{1cm} (11)

For the sake of clarity, a pseudo-code of the algorithm is given in Algorithm 1.

### Algorithm 1 Constraint-generation algorithm

**Input:** optimality tolerance, \( \varepsilon > 0 \)

**Output:** \( x^* \in X \), an \( \varepsilon \)-optimal \( N-k \) attack

1. initial problem: \( \mathcal{F}_1 \) without constraint (10b)
2. \( \eta^* \leftarrow -\infty \) \( \triangleright \) lower bound on the optimal obj. value
3. \( \eta^n \leftarrow +\infty \) \( \triangleright \) upper bound on the optimal obj. value
4. \( \hat{x} \) any initial \( N-k \) attack
5. solve inner problem for \( \hat{x} \)
6. \( \eta(\hat{x}) \leftarrow \) the load shed for the attack defined by \( \hat{x} \)
7. \( \ell_i(\hat{x}) \leftarrow \) active load shed factor for \( \hat{x} \) and \( \forall i \in \mathcal{N} \)
8. \( \pi_{ij}(\hat{x}) \leftarrow \) active power through line \( (i,j) \in \mathcal{E} \)
9. if \( \pi_{ij}(\hat{x}) \geq \eta \) then \( \eta^* \leftarrow \pi_{ij}(\hat{x}) \) and \( x^* \leftarrow \hat{x} \)
10. add \( \eta \leq \eta(\hat{x}) + \sum_{(i,j) \in \mathcal{E}} (\pi_{ij}^1(\hat{x}) + \pi_{ij}^2(\hat{x})) \cdot x_{ij} \) to \( \mathcal{F}_1 \) and resolve the outer problem
11. update \( \hat{x} \), and set \( \eta^n \) using solution from Step 10
12. if \( \eta^n - \eta^* \leq \varepsilon \eta^* \) then \( (x^*, \eta^*) \) is the \( \varepsilon \)-optimal solution to the interdiction problem, stop
13. Add no-good cut (11)
14. return to step: 5

### C. Choosing bounds on the dual variables

We also note that tight upper bounds on the optimal dual values over all \( N-k \) attacks i.e., \( \pi_{ij}^1 \) and \( \pi_{ij}^2 \) for each line \( (i,j) \in \mathcal{E} \) are essential to obtain a reasonable convergence behaviour. A trivial and valid value for the dual upper bounds for any line \( (i,j) \in \mathcal{E} \) is the total load in the system i.e., \( \pi_{ij}^1 = \pi_{ij}^2 = \sum_{d \in \mathcal{N}} p_d^t \). Intuitively, the tightest value of the dual upper bounds specify the minimum amount of load that can be served by increasing the thermal limit of the corresponding line by one unit over all \( N-k \) attacks. Given this interpretation, the total load is a very conservative choice for the \( \pi_{ij}^1 \) and \( \pi_{ij}^2 \). In the subsequent paragraphs, we focus on (i) finding lines for which \( \pi_{ij}^1 = 0 \) or \( \pi_{ij}^2 = 0 \) and (ii) heuristic choices of dual upper bounds.

Let \( \mathcal{E} \) denote the lines \( (i,j) \in \mathcal{E} \) in the transmission network that are never congested (i.e., the constraint (7c) is never tight) for any feasible interdiction plan \( x \in X \). Then, for any \( (i,j) \in \mathcal{E} \), we know that \( \pi_{ij}^1 = \pi_{ij}^2 = 0 \). The lines for which \( p_{ij} < t_{ij} \) for any \( x \in X \) have \( \pi_{ij}^1 = 0 \) and lines that always satisfy \( p_{ij} > -t_{ij} \) for any \( x \in X \), \( \pi_{ij}^2 = 0 \). It remains to compute the lines that satisfy these conditions for any interdiction plan.

To that end, for each line, we solve the following pair of linear programs which aid in finding the lines that are never congested for any feasible \( x \in X \):

\[ \max \pm p_{ij} \text{ subject to:} \]

Eq. (2b) - (2f), \( x \in X \) and \( 0 \leq x_{ij} \leq 1 \).

If the maximum value of \( p_{ij} \) (\(-p_{ij}\)) is less than \( t_{ij} \), then \( \pi_{ij}^1 \) \( (\pi_{ij}^2) \) takes a value zero i.e., the line is never congested in any interdiction plan \( x \in X \). For the remaining lines, we set \( \pi_{ij}^1 = \pi_{ij}^2 = \sum_{d \in \mathcal{N}} p_d^t \). This choice of dual upper bounds can still be weak for systems with a large total load value. Hence, we also examine heuristic choices of the dual upper bounds. In particular, we use \( \pi_{ij}^1 = \pi_{ij}^2 = 1 \) for all lines \( (i,j) \in \mathcal{E} \). This is a valid bound for the transportation model, i.e., the inner problem \( \text{LS}(\hat{x}) \) without the constraints (2d). Intuitively, these upper bounds have the following interpretation: if a line \( (i,j) \in \mathcal{E} \) is removed from a network, then the maximum amount of load that is shed is the absolute value of the power flowing on the line. This is not always true because of Braess’ paradox [22]. Nevertheless, if the total load shed induced by the removal of any subset of lines \( \mathcal{E} \) in the network does not incur more than \( \sum_{(i,j) \in \mathcal{E}} |p_{ij}| \) load shed, then \( \pi_{ij}^1 = \pi_{ij}^2 = 1 \) is a valid set of upper bounds for the optimal dual variables over all feasible interdiction plans \( x \in X \). Indeed, we see in our experiments in the next section that the choice \( \pi_{ij}^1 = \pi_{ij}^2 = 1 \) is valid for the test cases we consider.

### IV. Case Studies

In this section, we present two case studies to demonstrate the computational effectiveness of the constraint-generation algorithm in computing an optimal \( N-k \) interdiction plan. The case studies are performed on two PGLib-OPF v18.08 [25]

API test cases: the IEEE single-area RTS96 with 24 buses and the geolocated WECC 240 with 240 buses. The IEEE single-area RTS96 test system is artificially geolocated in the state of Utah. The \( k \) values for both the spatial and the topological interdiction problems are varied from 2 to 6. All of the interdiction formulations and algorithms were implemented in Julia v0.6 using the optimization modeling layer JuMP.jl v0.18 [13] and PowerModels v0.8 [8]. Furthermore, an optimality tolerance of \( \varepsilon = 1\% \) was used as a termination criteria.

#### A. Computational performance and choice of dual bounds

Fig. 1 shows the number of iterations the constraint-generation algorithm needs to converge to an \( \varepsilon \)-optimal solution on the RTS96 single-area system with valid and heuristic
dual upper bounds for the spatial and topological $N$-$k$ interdiction problems. For this set of experiments on the RTS96 test system $D$ was set to 10 km. From Fig. 1 it is clear that the computational performance of the algorithm is superior when the heuristic dual upper bounds are used. Despite the lack of a proof of validity for these bounds, they still yield optimal solutions to the interdiction problems. Hence, throughout the rest of the paper, heuristic dual upper bounds are used for all experiments.

Table I shows the $\varepsilon$-optimal load shed values, computation time, number of iterations, and the relative optimality gap for the traditional, spatial (with $D = 10$ km) and topological interdiction problems for varying values of $k$. Table II shows the same for the WECC 240 test system. These tables corroborates the effectiveness of the algorithm in computing $\varepsilon$-optimal solutions with very little computational effort.

### B. Value of realistic attacker models

From Tables I and II we observe that the load shed obtained using the traditional $N$-$k$ interdiction formulation is much higher than those obtained from the spatial and the topological interdiction problems. This is because the traditional $N$-$k$ permits the attacker to interdict components in the system that are geographically far apart in the transmission system. This however is less realistic and as shown by the results can significantly overestimate the required load shed values.

This phenomenon is demonstrated in the spatial interdiction model by examining the impact of parameter $D$ on the load shed values. To that end, we let $D$ take any value from 100 km to 1000 km in steps of 100 km. Fig. 2 shows the load shed values for various values of $D$ and $k$. The load shed for fixed values of $k$ is observed to quickly increase with $D$ (and will converge to the load-shed value for the traditional $N-k$ formulation when $D$ is sufficiently large), thus establishing the importance of a more accurate attacker model. Fig. 3 shows the number of iterations needed by the constraint-generation algorithm to converge to an $\varepsilon$-optimal solution for the spatial $N$-$k$ interdiction problem. The plot suggests that the difficulty of the problem does not increase or decrease substantially with the value of the parameter $D$ and for a fixed value of $k$.

The final set of plots in Fig. 4 shows locations where the load is shed for the WECC 240 test system by the traditional (see Fig. 4a), spatial with $D = 500$ km (see Fig. 4b), and topological (see Fig. 4c) interdiction models, respectively for $k = 6$. As observed from Fig. 4, the traditional interdiction problem has the ability to compute $N$-$k$ attacks where the components interdicted have a large geographical separation. As a result, the attack produces a load shed in geographically different parts of the system. Both the spatial and the topological $N$-$k$ models result in $N$-$k$ attacks where the load shed is more localized to a particular region indicative of the fact that the topological $N$-$k$ model serves as a proxy for the spatial interdiction problem in the absence of spatial grid data.

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**Table I**

| $k$ | load shed (p.u) | time (sec.) | iterations | opt. gap (%) |
|-----|----------------|-------------|------------|--------------|
| Traditional $N$-$k$ interdiction |
| 2   | 4.0            | 4.42        | 21         | 0.00         |
| 3   | 7.37           | 3.45        | 15         | 0.50         |
| 4   | 11.05          | 3.43        | 11         | 0.33         |
| 5   | 14.21          | 2.96        | 10         | 0.86         |
| 6   | 15.96          | 3.43        | 13         | 0.00         |
| Spatial $N$-$k$ interdiction |
| 2   | 4.0            | 2.76        | 20         | 0.00         |
| 3   | 6.3            | 2.56        | 20         | 0.00         |
| 4   | 8.0            | 2.44        | 19         | 0.70         |
| 5   | 9.59           | 2.97        | 25         | 0.00         |
| 6   | 11.0           | 2.38        | 19         | 0.00         |
| Topological $N$-$k$ interdiction |
| 2   | 4.0            | 4.39        | 11         | 0.00         |
| 3   | 6.29           | 4.10        | 11         | 0.24         |
| 4   | 7.72           | 5.31        | 19         | 0.00         |
| 5   | 11.05          | 4.85        | 11         | 0.00         |
| 6   | 11.05          | 20.09       | 35         | 0.00         |
TABLE II
RESULTS COMPARING THE DIFFERENT INTERDICTION MODELS ON THE WECC 240 TEST SYSTEM. AS k GETS LARGER, THE EFFECT OF CONSTRAINING THE ATTACKER GROWS CONSIDERABLY. THE TRADITIONAL APPROACH FOR DETERMINING THE INTERDICTION CAN BE ALMOST 200% MORE THAN A CONSTRAINED ATTACKER. INTERESTINGLY, CONSTRAINING THE ATTACKER ALSO HAS THE SIDE EFFECT OF REDUCING THE NUMBER OF ITERATIONS THAT ARE REQUIRED FOR CONVERGENCE.

| k  | load shed (p.u) | time (sec.) | iterations | opt. gap (%) |
|----|----------------|-------------|------------|--------------|
| 2  | 219.19         | 3.71        | 14         | 0.00         |
| 3  | 331.8          | 4.9         | 19         | 0.00         |
| 4  | 418.89         | 4.64        | 16         | 0.06         |
| 5  | 482.22         | 7.48        | 24         | 0.80         |
| 6  | 556.65         | 5.63        | 18         | 0.77         |

Spatial N-k interdiction

| k  | load shed (p.u) | time (sec.) | iterations | opt. gap (%) |
|----|----------------|-------------|------------|--------------|
| 2  | 192.22         | 3.03        | 5          | 0.00         |
| 3  | 222.65         | 3.59        | 7          | 0.00         |
| 4  | 233.99         | 5.22        | 11         | 0.57         |
| 5  | 255.73         | 4.98        | 10         | 0.94         |
| 6  | 273.68         | 7.15        | 16         | 0.00         |

Topological N-k interdiction

| k  | load shed (p.u) | time (sec.) | iterations | opt. gap (%) |
|----|----------------|-------------|------------|--------------|
| 2  | 121.26         | 5.52        | 5          | 0.40         |
| 3  | 211.26         | 5.52        | 4          | 0.00         |
| 4  | 222.49         | 9.72        | 5          | 0.88         |
| 5  | 233.4          | 41.64       | 12         | 0.00         |
| 6  | 332.03         | 21.27       | 6          | 0.00         |

Fig. 2. Load shed, in p.u., for the spatial interdiction problem for varying values of k and D on the WECC 240 test system.

Fig. 3. Number of iterations of the constraint-generation algorithm for the spatial interdiction problem for varying values of k and D on the WECC 240 test system.

V. CONCLUSION

This paper presents two N-k interdiction models that constrain the capabilities of the attacker to better model localized interdiction. These models use spatial proximity and topological connectivity to formulate localized interdiction. A general constraint-generation algorithm was developed to handle a broad class of interdiction problems and was demonstrated in these localized interdiction formulations and the traditional interdiction models. Case studies on the IEEE RTS 96 and WECC 240 systems show the computational effectiveness of the algorithm and the effectiveness of the model in identifying attacks that are localized.

There are number of potentially interesting future directions for this work. First, new models of constraining the attack model to better reflect the capabilities and outcomes of extreme events, such as those based on stochastic N-k models could be developed [24]. Second, these localized models could be connected to AC power flow models, such as the convex relaxations developed in [16], to improve the realism of the chosen attacks. Finally, it would also be interesting to develop methods for deriving stronger valid bounds on the dual variables.

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Fig. 4. Map of the buses where the load is shed for the (a) traditional, (b) spatial, and (c) topological interdiction problems with $k = 6$. The radius of the each of the circles centered around a bus represents the percentage of total load shed in that bus. The interdicted lines are shown in red. All the buses on which the interdicted lines are incident are highlighted using red boxes. We also remark that some of the interdicted lines in all the three figures connect buses that are very close to each other and hence is not highlighted in red, and hence the reason for highlighting the buses on which the lines are incident on. For the spatial interdiction problem, the value of $D$ was set to 500 km.

REFERENCES

[1] Rogelio E Alvarez. Interdicting electrical power grids. PhD thesis, Naval Postgraduate School, Monterey, California, 2004.
[2] José M Arroyo and Francisco J Fernández. Application of a genetic algorithm to $N$-$k$ power system security assessment. *International Journal of Electrical Power & Energy Systems*, 49:114–121, 2013.
[3] José M Arroyo and Francisco D Galiana. On the solution of the bilevel programming formulation of the terrorist threat problem. *IEEE Transactions on Power Systems*, 20(2):789–797, 2005.
[4] Dimitri P Bertsekas. *Nonlinear programming*. Athena scientific Belmont, 1999.
[5] Daniel Bienstock and Abhinav Verma. The $N$-$k$ problem in power grids: New models, formulations, and numerical experiments. *SIAM Journal on Optimization*, 20(5):2352–2380, 2010.
[6] Vicki M Bier, Eli R Gratz, Naraphorn J Haphuriwat, Wairimu Magua, Gerald Brown, Matthew Carlyle, Javier Salmerón, and Kevin Wood. Defending critical infrastructure. *Interfaces*, 36(6):530–544, 2006.
[7] Carleton Coffrin, Russell Bent, Kaarthik Sundar, Yeesian Ng, and Miles Lubin. Powermodels.jl: An open-source framework for exploring power system security assessment. In 2018 Power Systems Computation Conference (PSCC), pages 1–8, June 2018.
[8] Kaarthik Sundar, Carleton Coffrin, Harsha Nagarajan, and Russell Bent. Worst-case interdiction formulations. In *IEEE Transactions on Network Science and Engineering*, 3(3):132–146, 2016.
[9] Alexis L Motto, José M Arroyo, and Francisco D Galiana. A mixed-integer LP procedure for the analysis of electric grid security under disruptive threat. *IEEE Transactions on Power Systems*, 20(3):1357–1365, 2005.
[10] Ali Pinar, Juan Meza, Vaibhav Donde, and Bernard Lesieutre. Optimizing strategies for the vulnerability analysis of the electric power grid. *SIAM Journal on Optimization*, 20(4):1786–1810, 2010.
[11] Andrés Delgado, José Manuel Arroyo, and Natalia Alguacil. Analysis of electric grid interdiction with line switching. *IEEE Transactions on Power Systems*, 25(2):633–641, 2010.
[12] Vaibhav Donde, Vanessa López, Bernard Lesieutre, Ali Pinar, Chao Yang, and Juan Meza. Severe multiple contingency screening in electric power systems. *IEEE Transactions on Power Systems*, 23(2):406–417, 2008.
[13] Lain Dunning, Joey Huchette, and Miles Lubin. Jump: A modeling language for mathematical optimization. *SIAM Review*, 59(2):295–320, 2017.
[14] Mark K Enns, John J Quada, and Bert Sackett. Fast linear contingency analysis. *IEEE Transactions on Power Apparatus and Systems*, PAS-101(4):783–791, 1982.
[15] Margaret J Eppstein and Paul DH Hines. A “random chemistry” algorithm for identifying collections of multiple contingencies that initiate cascading failure. *IEEE Transactions on Power Systems*, 27(3):1698–1705, 2012.
[16] Hassan Hijazi, Carleton Coffrin, and Pascal Van Hentenryck. Convex quadratic relaxations for mixed-integer nonlinear programs in power systems. *Mathematical Programming Computation*, 9(3):321–367, 2017.
[17] Taedong Kim, Stephen J Wright, Daniel Bienstock, and Sean Harnett. Analyzing vulnerability of power systems with continuous optimization formulations. *IEEE Transactions on Network Science and Engineering*, 3(3):132–146, 2016.
[18] Thomas L Magnanti and Laurence A Wolsey. Optimal trees. *Handbooks in operations research and management science*, 7:503–615, 1995.
[19] Ali Pinar, Juan Meza, Vaibhav Donde, and Bernard Lesieutre. Stochastic network interdiction. *Operations Research*, 46(2):184–197, 1998.
[20] Alexis L Motto, José M Arroyo, and Francisco D Galiana. A mixed-integer LP procedure for the analysis of electric grid security under disruptive threat. *IEEE Transactions on Power Systems*, 20(3):1357–1365, 2005.
[21] Javier Salmeron, Kevin Wood, and Ross Baldick. Analysis of electric grid security under terrorist threat. *IEEE Transactions on Power Systems*, 19(2):905–912, 2004.
[22] Javier Salmeron, Kevin Wood, and Ross Baldick. Worst-case interdiction analysis of large-scale electric power grids. *IEEE Transactions on Power Systems*, 24(1):96–104, 2009.
[23] Saleh Soltan and Gil Zussman. Quantifying the effect of $k$-line failures in power grids. In *Power and Energy Society General Meeting (PESGM)*, 2016, pages 1–5. IEEE, 2016.
[24] Kaarthik Sundar, Carleton Coffrin, Harsha Nagarajan, and Russell Bent. Probabilistic $N$-$k$ failure-identification for power systems. *Networks*, 71(3):302–321, 2018.
[25] The IEEE PES Task Force on Benchmarks for Validation of Emerging Power System Algorithms. PGLib Optimal Power Flow Benchmarks. Published online at https://github.com/power-grid-lib/pglib-optf. Accessed: November 4, 2017.