Entropy production away from the equilibrium

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For a system moving away from equilibrium, we express the entropy production via a two-point correlation function for any time and any distance from equilibrium. The long-time limit gives the sum of the Lyapunov exponents for a general dynamical system expressed via the formula of a Green-Kubo type.

I. INTRODUCTION

We discuss here how systems go away from the equilibrium under the action of external factors. An appropriate way to describe the process is to study the evolution of the density \( n \) which satisfies the continuity equation

\[
\frac{\partial n}{\partial t} + \text{div}(n \mathbf{v}) = 0 .
\]

Here \( n(t, \mathbf{r}) \) is either a phase-space density or just plain density in space and \( \mathbf{v}(t, \mathbf{r}) \) is either a velocity field defining the phase space dynamics or the velocity field in space. We consider a compact phase space and assume that the total mass is conserved: \( \int n(t, \mathbf{r}) \, d\mathbf{r} = 1 \). Equilibrium is generally characterized by smooth measures like microcanonical distribution over the energy surface in the phase space or just uniform density in space when the flow is incompressible. Equilibrium state realizes the maximum of the Gibbs entropy

\[
S = -\int n(\mathbf{r}) \ln n(\mathbf{r}) d\mathbf{r}
\]

and is given by \( n = \text{const} = 1/V \). This state is not stationary if \( \mathbf{v}(t, \mathbf{r}) \) is compressible that is \( \omega = \text{div} \mathbf{v} \neq 0 \). If the system is initially at equilibrium then a compressible flow move it away from it. The density field \( n(\mathbf{r}, t) \) is getting inhomogeneous and the Gibbs entropy starts to decrease. This decrease can be interpreted as the entropy flux from the system to the environment that provides for compressibility of the velocity field (for a closed Hamiltonian system \( \omega = 0 \)). Our main interest here is in the entropy production rate which can be expressed via the velocity divergence:

\[
\frac{dS}{dt} = \int d\mathbf{r} n(\mathbf{r}, t) \omega(\mathbf{r}, t) = \int \omega(\mathbf{q}(t, \mathbf{x}), t) \frac{d\mathbf{x}}{V} ,
\]

The main feature of our approach is the application of Lagrangian description with trajectories \( \mathbf{q}(t, \mathbf{x}) \) defined by \( \dot{\mathbf{q}} = \mathbf{v}(\mathbf{q}, t) \) and \( \mathbf{q}(0, \mathbf{x}) = \mathbf{x} \). Our main result derived below is the formula for the entropy production rate at an arbitrary time in terms of the two-point correlation function along the Lagrangian trajectories:

\[
\frac{d\omega^2}{dt^2} = -\int \omega(\mathbf{x}, 0) \omega(\mathbf{q}(t, \mathbf{x}), t) \frac{d\mathbf{x}}{V} \equiv \langle \omega(0) \omega(t) \rangle
\]

\[
\frac{dS}{dt} = -\int_0^t dt' \langle \omega(0) \omega(t') \rangle dt'.
\]

The formula holds either for time-independent \( \mathbf{v}(\mathbf{r}) \) or for a statistically steady \( \mathbf{v}(t, \mathbf{r}) \). Near equilibrium, (1.2) is a well-known Green-Kubo formula (see, e.g. (5.7) in [1]). It is remarkable that the entropy production is expressed via the pair correlation function not only near equilibrium but in any state.

Equation (1.2) gives \( \frac{dS}{dt} \leq 0 \) at short times which is physically quite natural since the entropy has nowhere to go but down. When the velocity field is steady (or statistically steady) the density field may tend to a non-equilibrium steady state (see [1, 2] and the references therein). Examples of such states are Sinai-Ruelle-Bowen measures in hyperbolic dynamical systems [3, 4]. The steady state is characterized by the limiting entropy production rate \( \lim_{t \to \infty} \dot{S} = \sum \lambda_i \). Here \( \sum \lambda_i \) is the sum of Lyapunov exponents \( \lambda_i \) if the latter exist. This sum is generally non-positive [3, 4]. We observe that it obeys the formula that looks exactly the same as the Green-Kubo formula:

\[
\sum \lambda_i = -\int_0^\infty \langle \omega(0) \omega(t) \rangle dt \leq 0.
\]

If the non-equilibrium steady state density \( \mu \) is smooth then its Gibbs entropy is finite and necessarily \( \lim_{t \to \infty} \dot{S} = \sum \lambda_i = 0 \). It is often the case that \( \mu \) is singular, the simplest example is steady \( \mathbf{v}(\mathbf{r}) \) in one dimensional case where
n accumulates in the stagnation points. Generally for such states \( \dot{S} = \sum \lambda_i < 0 \) that does not contradict stationarity since \( S = -\infty \) where the measure is singular. Constant flux of entropy from the system to the environment is an important feature of these states.

We see that the entropy production has the same sign both at the beginning and at the end of the evolution. How it behaves in between may depend on the system. In particular, it follows from (1.4) that when two-point correlation function of the velocity divergence does not change sign, the entropy production rate \( \dot{S} \) monotonically decreases from 0 in equilibrium to its (negative) minimum realized in the non-equilibrium steady state with singular density \( \mu \). Entropy decrease corresponds to the increase of order in the system.

II. GENERAL CONSIDERATIONS

We consider first a smooth dynamical system defined by a steady velocity field \( \mathbf{v} \):

\[
\frac{dx}{dt} = \mathbf{v}(x),
\]

where \( x \) is the phase space coordinate of the system. For most part we assume that the phase space is the finite volume \( V \) region in \( d \)-dimensional Euclidean space even though generalizations are possible. Velocity field defines the flow \( \mathbf{q}(t, r) \) in the phase space by

\[
\frac{\partial \mathbf{q}(t, r)}{\partial t} = \mathbf{v}(\mathbf{q}(t, r)), \quad \mathbf{q}(0, r) = r.
\]

Let us define

\[
W_{ij}(t, r) \equiv \frac{\partial q_i(t, r)}{\partial r_j}.
\]

An important relation between \( \mathbf{q}(t, r) \) and \( \mathbf{v}(r) \) holds

\[
\mathbf{v}(\mathbf{q}(t, r)) = W(t, r)\mathbf{v}(r),
\]

which allows, in particular, to describe the evolution of the functions \( f(\mathbf{q}(t, r)) \) defined on trajectories. Using (2.3) we have for any differentiable \( f \)

\[
\frac{d}{dt}f(\mathbf{q}(t, r)) = (\mathbf{v}(r) \cdot \nabla_r)f(\mathbf{q}(t, r)) = -\omega(r)f(\mathbf{q}(t, r)) + \nabla_r \cdot [\mathbf{v}(r)f(\mathbf{q}(t, r))].
\]

We now integrate it over the space:

\[
\frac{d}{dt} \int \frac{d\mathbf{r}}{V} f(\mathbf{q}(t, r)) = -\int \frac{d\mathbf{r}}{V} \omega(\mathbf{r})f(\mathbf{q}(t, r)) \equiv -\langle \omega(0)f(t) \rangle,
\]

\[
\int \frac{d\mathbf{r}}{V} f(\mathbf{r}) - \int \frac{d\mathbf{r}}{V} f(\mathbf{q}(t, r)) = \int_0^t \langle \omega(0)f(t') \rangle dt',
\]

where we assumed that the space integral of the last term in (2.4) vanishes because it can be written as an integral over the boundary. This is true in the periodic case or in the case where the normal component of velocity vanishes (it might be interesting to consider the cases where boundary is important as well). We defined the correlation function

\[
\langle g(0)f(t) \rangle \equiv \int \frac{d\mathbf{r}}{V} g(\mathbf{r})f(\mathbf{q}(t, r)).
\]

We shall see below that our main result (1.4) is a particular case of (2.5). Integrating the equation (2.6) over time we have

\[
\int \frac{d\mathbf{r}}{V} f(\mathbf{r}) - \frac{1}{t} \int_0^t dt' \int \frac{d\mathbf{r}}{V} f(\mathbf{q}(t', \mathbf{r})) = \int_0^t \langle \omega(0)f(t') \rangle dt' - \frac{1}{t} \int_0^t t' \langle \omega(0)f(t') \rangle dt'.
\]

Let us consider what may happen with Equations (2.6) and (2.8) when \( t \to \infty \). From Equation (2.6) we observe that provided \( \int_0^t \langle \omega(0)f(t) \rangle dt \) exists there must also exist a finite limit of the spatial average

\[
\lim_{t \to \infty} \int \frac{d\mathbf{r}}{V} f(\mathbf{q}(t, r)) = \int \frac{d\mathbf{r}}{V} f(\mathbf{r}) - \int_0^\infty \langle \omega(0)f(t) \rangle dt.
\]
If we choose the initial state for \( \mu \) as a constant, \( n_0(r) \equiv n(t = 0, r) = 1/V \), then (2.9) can be written as follows:

\[
\lim_{t \to \infty} \int \frac{dr}{V} f(r) n(t, r) = \int \frac{dr}{V} f(r) - \int_0^\infty \langle \omega(0) f(t) \rangle dt. \tag{2.10}
\]

This suggests that the finiteness of temporal correlations i.e. the existence of the integrals \( \int_0^\infty \langle \omega(0) f(t) \rangle dt \) for continuous \( f \) is equivalent to the existence of the limiting non-equilibrium state characterized by the probability measure \( \mu_{\text{lim}} = \lim_{t \to \infty} n(r, t) \). Note that the difference of the non-equilibrium state measure \( \mu_{\text{lim}} \) and equilibrium measure \( E \) satisfies

\[
\mu_{\text{lim}}(f) - E(f) = - \int_0^\infty \langle \omega(0) f(t) \rangle dt, \quad \mu_{\text{lim}}(f) = \int f d\mu_{\text{lim}}, \quad E(f) = \int \frac{dr}{V} f(r). \tag{2.11}
\]

Let us now consider Equation (2.8). It involves a weaker limit [1]

\[\mu_{\text{av}} = \lim_{t \to \infty} \frac{1}{t} \int_0^t n(r, t') dt'. \]

The existence of this limit is equivalent to the convergence of the subtracted correlation integral

\[\mu_{\text{av}}(f) - E(f) = - \lim_{t \to \infty} \left[ \int_0^t \langle \omega(0) f(t') \rangle dt' - \frac{1}{t} \int_0^t t' \langle \omega(0) f(t') \rangle dt' \right]. \tag{2.12}\]

We have \( \mu_{\text{av}}(f) = \mu_{\text{lim}}(f) \) where \( \int_0^\infty \langle \omega(0) f(t) \rangle dt \) exists.

A limit of the type used in \( \mu_{\text{av}} \) appears where one considers the sum of Lyapunov exponents \( \sum \lambda_i(r) \) (for the discussion of Lyapunov exponents \( \lambda_i \) see [7, 8]). That sum determines the growth rate of an infinitesimal volume initially located at \( r \)

\[
\sum \lambda_i(r) = \lim_{t \to \infty} \frac{\ln \det W(t, r)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \omega(q(t', r)) dt'. \tag{2.13}
\]

We observe that \( \sum \lambda_i \) is represented as a time-average of a function on the phase space. This is a unique combination of \( \lambda_i \) representable in such a form. Using Equation (2.8) we have

\[
\int \frac{dr}{V} \sum \lambda_i(r) = - \lim_{t \to \infty} \left[ \int_0^t \langle \omega(0) \omega(t') \rangle dt' - \frac{1}{t} \int_0^t t' \langle \omega(0) \omega(t') \rangle dt' \right] = - \int_0^\infty \langle \omega(0) \omega(t) \rangle dt, \tag{2.14}
\]

where the last equality holds provided the integral exists. We used \( E(\omega) = 0 \) assuming that the integral over the boundary vanishes. The above formula holds for systems whose stationary measure is arbitrarily far from the equilibrium one and nevertheless it has a remarkable resemblance to the Green-Kubo formula holding near equilibrium. Note that it suggests that \( \int dr \sum \lambda_i \leq 0 \) always. This will be proved in the next section. Here we note that \( \int dr \sum \lambda_i < 0 \) signifies that \( \mu \) is singular. Therefore, the criterium of singularity of the non-equilibrium measure is \( \int_0^\infty \langle \omega(0) \omega(t) \rangle dt > 0 \).

The above relations are simplified for systems satisfying the SRB theorem that guarantees the equality between temporal average and average with respect to the limiting measure for any continuous \( f \):

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(q(t', r)) dt' = \int f d\mu_{\text{SRB}}. \tag{2.15}
\]

The above limit holds for almost all \( r \) in the sense of the usual volume in Euclidean space and \( \mu_{\text{SRB}} \) is called the SRB measure [3, 4]. If the theorem holds we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \langle g(0) f(t') \rangle dt' = E(g) \mu(f), \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t \langle \omega(0) f(t') \rangle dt' = 0.
\]

Note that \( \mu_{\text{SRB}} \) satisfies the relation (2.12) satisfied by \( \mu_{\text{av}} \). For SRB-theorem systems \( \sum \lambda_i(r) \) is constant almost everywhere and we have

\[
\sum \lambda_i = - \int_0^\infty \langle \omega(0) \omega(t) \rangle dt. \tag{2.16}
\]

We now apply the relations of this section for the investigation of the evolution of the entropy which possesses some unique properties.
III. ENTROPY

The evolution of the entropy is closely related to the evolution of infinitesimal volumes in the phase space. Indeed, let us associate with each fluid element initially located at \( \mathbf{r} \) the density of its Gibbs entropy \[\text{(1.2)}\]:

\[
s(\mathbf{r}, t) = -\ln n(\mathbf{q}(t, \mathbf{r}), t), \quad S(t) = \int d\mathbf{r} n_0(\mathbf{r}) s(\mathbf{r}, t).
\]

where \( n_0(\mathbf{r}) = n(\mathbf{r}, t = 0) \). The same quantity \( \omega(\mathbf{q}(t, \mathbf{r})) \) determines the time derivative of both the entropy density and the logarithm of the ratio of infinitesimal volumes \( v(t)/v(0) \):

\[
\frac{\partial s}{\partial t} = \omega(\mathbf{q}(t, \mathbf{r})) = \frac{\partial \ln \det W(t, \mathbf{r})}{\partial t}.
\]

We can now relate the volume fraction of the trajectories that during time\( t \) change their entropy density by some \( \Delta \) to the volume fraction of the trajectories with the change \( -\Delta \) under the time-reversed evolution. We have

\[
\int d\mathbf{r} \delta \left( -\ln \det \partial_j q_i(-t, \mathbf{r}) + \Delta \right) \det \partial_j q_i(-t, \mathbf{r}) = \int d\mathbf{r} \delta \left( -\ln \det \partial_j q_i(t, \mathbf{r})|_{\mathbf{r}=\mathbf{q}(-t, \mathbf{r})} - \Delta \right) \det \partial_j q_i(-t, \mathbf{r})
\]

\[
= \int d\mathbf{r} \delta \left( -\ln \det \partial_j q_i(t, \mathbf{r}) - \Delta \right) = \int d\mathbf{r} \delta \left( \Delta s(t, \mathbf{r}) - \Delta \right) = \int d\mathbf{r} \delta \left( \Delta s(t, \mathbf{r}) + \Delta \right).
\]

It follows that the measures of \( \Delta s(t, \mathbf{r}) \equiv s(0, \mathbf{r}) - s(t, \mathbf{r}) \) obey

\[
\frac{\int d\mathbf{r} \delta (\Delta s(t, \mathbf{r}) - \Delta)}{\int d\mathbf{r} \delta (\Delta s(t, \mathbf{r}) + \Delta)} = e^{\Delta}.
\]

In the case of time-reversible dynamics, \( \int d\mathbf{r} \delta (\Delta s(-t, \mathbf{r}) + \Delta) = \int d\mathbf{r} \delta (\Delta s(t, \mathbf{r}) + \Delta) \) and

\[
\frac{\int d\mathbf{r} \delta (\Delta s(t, \mathbf{r}) - \Delta)}{\int d\mathbf{r} \delta (\Delta s(t, \mathbf{r}) + \Delta)} = e^{\Delta},
\]

which says that, starting from equilibrium, the probability to have the entropy change \( \Delta \) is by factor \( \exp(\Delta) \) larger than the probability to have the entropy change \( -\Delta \). That relation was first established by Evans and Searles \[9\]. The probability is defined here with respect to Lebesgue measure (the volume fraction occupied by respective trajectories) and the relation is valid for any time. Recasting the above equation in the form

\[
\frac{\int d\mathbf{r} \delta (\epsilon(t, \mathbf{r}) - a)}{\int d\mathbf{r} \delta (\epsilon(t, \mathbf{r}) + a)} = \exp [at], \quad \epsilon(t, \mathbf{r}) = \Delta s(t, \mathbf{r}) |t|
\]

one can compare it to the Gallavotti-Cohen formula \[10\]:

\[
\frac{\int d\mu \delta (\epsilon(t, \mathbf{r}) - a)}{\int d\mu \delta (\epsilon(t, \mathbf{r}) + a)} = \exp [at],
\]

that holds at large \( t \). Probability in \[3.8\] is defined with respect to the SRB measure. Comparison between \[3.3\] and \[3.9\] is discussed in \[11\].
Let us now consider more closely the density of the entropy production rate \( \partial_t s(t, r) = \omega(q(t, r)) \). According to the general formula (2.1) we have

\[
\partial_t^2 s(t, r) = -\omega(r)\omega(q(t, r)) + \nabla_r \cdot [v(r)\omega(q(t, r))].
\] (3.10)

Both \( \omega(q(t, r)) \) and its time derivative can have arbitrary sign. However, the above formula for \( \partial_t^2 s(t, r) \) suggests that the global quantities like space integrals can exhibit some general properties. The relation (3.9) for the entropy density implies an analogous relation for the entropy

\[
S(t) - S(0) = \int_0^t dt' \int dr n_0(r) \omega(q(t', r)), \quad \lim_{t \to \infty} \frac{S(t) - S(0)}{t} = \sum \lambda_i.
\] (3.11)

Note that for systems with \( r \)-independent \( \sum \lambda_i(r) \) the last expression equals \( \sum \lambda_i \). Generally, considering the case \( n_0(r) = \text{const} \) we find

\[
\sum \lambda_i \leq 0,
\] (3.12)

which is expected from the Equation (2.6). Indeed since among all the densities having a given normalization the maximal \( S \) corresponds to a constant density (see Appendix A) and the evolution conserves normalization, we have \( S(t) - S(0) \leq 0 \) with equality holding only if the flow brings the density back to the constant at some time \( t \). The relation (3.12) was demonstrated before [5, 6]. Another interesting quantity is the entropy production rate that can be written as

\[
\Gamma(t) \equiv \frac{dS}{dt} = \int n_0(r)\omega(q(t|r))dr = \int d\omega(r,t)\omega(r)
\] (3.13)

For an arbitrary \( n_0(r) \), the entropy production rate can have any sign at small \( t \). The most interesting case is the case of \( n_0(r) = \text{const} \) where the evolution of \( n(r,t) \) is that from equilibrium to the non-equilibrium steady state characterized by the SRB measure. In this case we have

\[
\Gamma(t) = \int \frac{dr}{V} \omega(q(t|r)).
\] (3.14)

Clearly \( \Gamma(0) = 0 \). Under the condition of sufficiently fast temporal decay of correlations (existence of \( \int_0^\infty t dt \langle \omega(0)\omega(t) \rangle \)) it follows from the analysis of the Sect. II that

\[
\lim_{t \to \infty} \Gamma(t) = \sum \lambda_i \leq 0.
\] (3.15)

The case \( \sum \lambda_i = 0 \) is the case where the steady non-equilibrium state \( \mu \) does not exchange entropy with the environment. We wish to consider \( \sum \lambda_i < 0 \). Then \( \Gamma(t = \infty) < 0 \). We also should have \( \Gamma(t) < 0 \) at small \( t \) since at these times \( n(t, r) \) goes away from the constant giving the maximum to \( S \). It is natural then to ask what flows correspond to \( \Gamma(t) \) being a monotonic decreasing function of \( t \) reaching its minimum in the non-equilibrium steady state. This property cannot be general: if the flow brings back \( n(t, r) \) close to its original constant value (not exactly the same to have \( \int dr \sum \lambda_i < 0 \) then \( \int_0^t \Gamma(t')dt' \approx 0 \) and \( \Gamma(t) \) changes sign. To understand better the evolution of \( \Gamma(t) \) consider

\[
\frac{d\Gamma}{dt} = \int d\omega(r, t' \omega(q(t'|r))) = -\int d\omega(r, t' \omega(q(t|r)) + \frac{d\omega(r, t)\cdot [v(r)\omega(q(t|r))]},
\] (3.16)

where we used the formula (3.10). We observe that for \( v \) such that \( v \cdot \nabla \omega < 0 \) everywhere we have \( \Gamma(t) < 0 \) and the conjecture on the monotonic decrease of \( \Gamma \) holds. Note that the opposite case \( v \cdot \nabla \omega > 0 \) would violate the possibility to integrate by parts \( \int v \cdot \nabla \omega = -\int \omega^2 < 0 \).

We now concentrate on \( n_0(r) = 1/V \) case where

\[
\frac{d\Gamma}{dt} = -\langle \omega(0)\omega(t) \rangle, \quad \Gamma(t) = -\int_0^t \langle \omega(0)\omega(t') \rangle dt'.
\] (3.17)

Clearly at small \( t \) we have \( \Gamma(t) < 0 \). The above relation allows us to introduce the criterion of the generalization of the principle of maximal generation of entropy that holds near equilibrium. If the system possesses positive everywhere correlation function \( \langle \omega(0)\omega(t) \rangle \) (that implies, in particular \( \sum \lambda_i < 0 \) then on its way from equilibrium to the non-equilibrium steady state the system gives more and more entropy to the environment with the rate that reaches maximum in the steady state.

Let us summarize our results on the entropy production rate \( \Gamma(t) \). The first question is the sign-definiteness of \( \Gamma(t) \). We argued that even for \( n_0(r) = 1/V \) such assertion cannot be made generally. It seems that the correct language to consider this question and the question of the generalization of the maximum entropy production principle to the strongly non-equilibrium situations is to consider \( \Gamma(t) \). We have shown that the latter is equal to minus the value of an auto-correlation function at time difference \( t \).
IV. GENERALIZATION FOR A TIME-DEPENDENT VELOCITY WITH STATIONARY STATISTICS

We generalize the above results to the case of time-dependent velocity \( v(t, r) \). The phase-space trajectories \( q(t|t', r) \) depend now on both the initial time \( t' \) and the final time \( t \) and are characterized by

\[
\frac{\partial q(t|t', r)}{\partial t} = v(q(t|t', r), t), \quad q(t'|t', r) = r. \tag{4.1}
\]

We introduce

\[
W_{ij}(t|t', r) = \frac{\partial q_{i}(t|t', r)}{\partial r_{j}}.
\]

To establish the generalization of (3.6) we use the same proof replacing the function \( q(-t, r) \) by \( q(0|t, r) \). For example we have \( W \)

\[
\frac{\partial q_{i}(0|t, r)}{\partial r_{j}} = \left( \frac{\partial q_{i}(0|t, r)}{\partial r_{j}} \right)^{-1}. \tag{4.2}
\]

We find the identity

\[
\frac{\int dr \delta (-\ln \det \partial_{j} q_{i}(0|t, r)/t - a)}{\int dr \delta (-\ln \det \partial_{j} q_{i}(0|t, r)/t + a)} = \exp[at] \tag{4.3}
\]

that holds for any \( v(t, r) \). If \( v(t, r) \) has stationary statistics, with the average defined by the spatial integration, then one can replace \( q_{i}(0|t, r) \) by \( q_{i}(-t|0, r) \) in the denominator of the above formula and the formula \( \int \) results. Moreover if the statistics is time-reversible we arrive at the equation

\[
\frac{\int dr \delta (\epsilon(t, r) - a)}{\int dr \delta (\epsilon(t, r) + a)} = \exp[at], \tag{4.4}
\]

completing the generalization.

Next we consider the entropy production rate for the case \( n_{0}(r) = 1/V \)

\[
\Gamma(t) = \int \omega(t, q(t, r)) \frac{dr}{V}. \tag{4.5}
\]

Direct time differentiation of the above equation is not very useful in this case: we have no control over \( \partial_{t} \omega \). The idea is to manipulate the integral recasting it in the form more suitable for the time-differentiation. Changing variables \( y = q(0|t, r) \) we find

\[
\Gamma(t) = \int \omega(t, y) \exp \left[ -\int_{0}^{t} \omega(q(t'|t, y), t') dt' \right] \frac{dy}{V}. \tag{4.6}
\]

We now assume stationarity that is we assume that the above average is equal to

\[
\Gamma(t) = \int \omega(y, 0) \exp \left[ -\int_{0}^{t} \omega(q(t'|y), t') dt' \right] \frac{dy}{V}. \tag{4.7}
\]

This allows us to avoid the appearance of \( \partial_{t} \omega \) in \( \Gamma \). We have

\[
\frac{d\Gamma}{dt} = -\int \omega(y, 0) \omega(q(-t|y), -t) \exp \left[ -\int_{0}^{t} s(q(t'|y), t') dt' \right] \frac{dy}{V} = -\langle \omega(0) \omega(t) \rangle, \tag{4.8}
\]

where the last equality is obtained by getting back in the integral to the variable \( r = q(t|0, y) \). The corresponding expression for the sum of Lyapunov exponents follows. This completes the generalization of the main results derived for the steady velocity field to the case of the time-dependent velocity with stationary statistics. If integrating over space does not guarantee stationarity, i.e. coincidence of \( W \) and \( W \), and one needs to add average over velocity ensemble, then \( \int \) and \( \int \) are also valid after such ensemble average.
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APPENDIX A: PROPERTIES OF THE GIBBS ENTROPY

We make a general remark on the entropy properties which are not related to the dynamics. We use the fact that
\[ \ln x - 1 + 1/x \geq 0 \] (here \( x > 0 \)) with the equality reached only at \( x = 1 \). It follows that for any two normalized functions \( n \) and \( n' \) we have

\[ -\int n \ln \left( \frac{n}{n'} \right) \, dr \leq 0, \quad (A1) \]

with the equality reached iff \( n = n' \). The above inequality says that entropy considered as expectation value over some probability distribution \( n \) of minus logarithm of arbitrary other distribution \( n' \) is minimal where \( n' = n \). An interesting conclusion follows for a compact space with volume \( V \). We have for any distribution \( n \)

\[ \int n \ln (nV) \, dr \geq 0, \quad (A2) \]

with the equality reached iff \( n \) is uniform. A compact space is analogous to the probability space consisting of finite number of events in that one can define the entropy that is bounded from above by the maximal value reached at the uniform distribution. We also have that

\[ \int \frac{dr}{V} \ln (nV) \leq 0. \quad (A3) \]

In particular, using the above equations for \( n = n(r,t) \) we find from Eq. (A2) that the sum of forward in time Lyapunov exponents is non-positive while the last equation implies the same conclusion about the sum of backward in time Lyapunov exponents.