TIME GAUGE FIXING AND HILBERT SPACE
IN QUANTUM STRING COSMOLOGY

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ABSTRACT
Recently the low-energy effective string theory has been used by Gasperini and Veneziano to elaborate a very interesting scenario for the early history of the universe (“birth of the universe as quantum scattering”). Here we investigate the gauge fixing and the problem of the definition of a global time parameter for this model, and we obtain the positive norm Hilbert space of states.

PACS: 04.60.Ds, 98.80.Hw

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1. Introduction.

In the customary quantum gravity approach to the origin of the universe [1], use is made of the Wheeler-DeWitt (WDW) equation whose solutions with appropriate boundary conditions describe the “tunneling from nothing”.

This fundamental approach has assumed a renewed interest since the classical string cosmology [2] describes the formation of a Friedmann-Robertson-Walker (FRW) universe with essentially the present characteristics as evolving from the string perturbative vacuum. The transition from an initial “pre-big bang” phase to the present one is represented in quantum string cosmology by a scattering and reflection of the WDW wave function in superspace [3,4].

Now the WDW equation has the usual problems of measure and definition of inner product; in the present case Gasperini and Veneziano (in the following GV) have surmounted the ambiguities of the differential representation of operators using the symmetry of the classical action, which is the right recipe, see e.g. [5].

In view of the renewed interest in the matter we have given a closer look at the determination of a hermitian Hamiltonian and a positive norm Hilbert space. A Hilbert space is requested in quantum mechanics; the sole WDW equation gives wave functions, but no inner product. Establishing a positive norm Hilbert space of states is an exercise in the gentle art of finding a time gauge such that evolution is described by a unitary operator. Now, this requires the model to be taken seriously, since the time determined by fixing the gauge is defined as a suitable function of the canonical coordinates of the problem under consideration. This may be thought of as inadequate to the complexity of gravitational systems, but these are the rules of quantum mechanics applied to minispace; there is no alternative.

The two approaches to gauge fixing are 1) quantisation of the constraint (Dirac method) followed by gauge fixing, or 2) reduction of the canonical space by introducing a classical time gauge fixing condition and use of the constraint, followed by quantisation in the reduced space if the reduced Hamiltonian is hermitian [6]. With a proper gauge condition the two methods give coincident results. In this way we obtain the definition of the norm.

2. Conventions and Definitions.

We gather here the necessary formulae so that the paper is self–consistent. The definitions and results are essentially as in [3,4] with a few changes in notation and normalization.

We start from the usual four dimensional low-energy effective string action [7]

\[
S = \frac{1}{2\lambda_s^2} \int_V d^4x \sqrt{-g} e^{-2\Phi} (R + 4\partial_\mu \Phi \partial^\mu \Phi - \Lambda) ,
\]  

(2.1)
where $\Phi$ is the dilaton, $\Lambda > 0$ is the cosmological constant, and $\lambda_s$ is the fundamental string-length parameter. In (2.1) we use for the Ricci scalar the conventions of Landau-Lifshits [8]. The metric is assumed to be spatially homogeneous and isotropic:

$$ds^2 = -N^2(t)dt^2 + a^2(t) \omega^p \otimes \omega^p.$$  \hspace{1cm} (2.2)

Here $N$ is the lapse function and the scale factor $a$ is positive by definition. The $\omega^p$'s are the 1-forms that satisfy the Maurer-Cartan structure equation

$$d\omega^p = \frac{k}{2} \epsilon_{pqr} \omega^q \wedge \omega^r,$$ \hspace{1cm} (2.3)

where $k = 0, \pm 1$. Accordingly the dilaton field is assumed to depend only on time. Let us define

$$\gamma = \ln a, \quad \varphi = \Phi - \frac{3}{2} \ln a.$$ \hspace{1cm} (2.4)

$\varphi$ is usually called the “shifted dilaton field” [3]. It is advisable to use the Lagrange multiplier

$$\mu = N e^{2\varphi}.$$ \hspace{1cm} (2.5)

Indeed using (2.2-5) in (2.1) one has [3,4] (here and throughout the paper we neglect inessential surface terms)

$$S = \frac{V}{2\lambda_s^2} \int dt \left[ 3 \frac{\dot{\gamma}^2}{\mu} - 4 \frac{\dot{\varphi}^2}{\mu} + \mu e^{-4\varphi} \left( 6ke^{-2\gamma} - \Lambda \right) \right],$$ \hspace{1cm} (2.6)

where $V$ is the spatial volume element with $a = 1$ and dots represent differentiation with respect to $t$. In the following we will set $V/2\lambda_s^2 = 1$. In canonical form the action (2.6) becomes

$$S = \int dt \left\{ \dot{\gamma} \gamma + \dot{\varphi} \varphi - H \right\},$$ \hspace{1cm} (2.7)

where

$$p_\gamma = 6 \frac{\dot{\gamma}}{\mu}, \quad p_\varphi = -8 \frac{\dot{\varphi}}{\mu},$$ \hspace{1cm} (2.8)

are respectively the conjugate momenta of $\gamma$ and $\varphi$, and

$$H = \mu H = \mu \left[ \frac{p_\gamma^2}{12} - \frac{p_\varphi^2}{16} + \Lambda e^{-4\varphi} - 6ke^{-2\gamma} \right].$$ \hspace{1cm} (2.9)

Here $H$ is the generator of time-reparametrizations (gauge transformations); we will simply call it the “Hamiltonian” of the system. The Lagrange multiplier $\mu$ enforces the constraint

$$H = 0.$$ \hspace{1cm} (2.10)
which expresses the invariance under time-reparametrization. The gauge transformations generated by $H$ are

\begin{align*}
\delta q_i &= \epsilon \frac{\partial H}{\partial p_i} = \epsilon [q_i, H]_{\mu}, \\
(2.11a) \\
\delta p_i &= -\epsilon \frac{\partial H}{\partial q_i} = \epsilon [p_i, H]_{\mu}, \\
(2.11b) \\
\delta l &= \frac{d\epsilon}{dt}, \\
(2.11c)
\end{align*}

where $q_i = \{\gamma, \varphi\}$, $p_i = \{p_{\gamma}, p_{\varphi}\}$. Throughout the paper we will set $k = 0$ in (2.9), i.e. we will consider only flat spacetimes. The case of $k = \pm 1$ will be considered elsewhere. As a warm up exercise, now we illustrate the procedure on the simple case of null cosmological constant, corresponding to the D’Alembert Hamiltonian.

### 3. The D’Alembert Case.

The case $\Lambda = 0$ corresponds to a string with critical dimension [2]. Taking $k = 0$ and $\Lambda = 0$ in (2.9) the Hamiltonian becomes

\[ H = \mu H = \mu \left[ \frac{p_{\gamma}^2}{12} - \frac{p_{\varphi}^2}{16} \right]. \]

The finite gauge transformations (2.11) can be integrated explicitly. The result is

\begin{align*}
\gamma &= \gamma_0 + \frac{p_{\gamma}}{6} \tau, \\
(3.2a) \\
p_{\gamma} &= \text{constant}, \\
(3.2b) \\
\varphi &= -\frac{p_{\varphi}}{8} \tau, \\
(3.2c) \\
p_{\varphi} &= \text{constant}, \\
(3.2d) \\
\tau &= \int_{t_0}^{t} \mu(t') \, dt', \quad \mu(t) > 0. \\
(3.2e)
\end{align*}

where $\gamma_0$, $p_{\gamma}$, and $p_{\varphi}$ are gauge invariant quantities. We can define the new variables (action-angle variables)

\begin{align*}
\xi &= 6 \frac{\gamma}{p_{\gamma}}, \\
(3.3a) \\
p_{\xi} &= \frac{1}{12} p_{\gamma}^2, \\
(3.3b)
\end{align*}
that will be used later. The Poisson relation of $\xi$ and $p_\xi$ is

$$\{\xi, p_\xi\}_\rho = 1. \quad (3.4)$$

Thus $\{\varphi, p_\varphi, \xi, p_\xi\}$ form a complete set of canonically conjugate variables. Note that $p_\varphi$ and $p_\xi$ are gauge invariant quantities and $\xi$ transforms by gauge transformations as

$$\xi \rightarrow \bar{\xi} = \xi + \tau. \quad (3.5)$$

This is the reason for the interest in $\xi$. Eq. (3.5) suggests that $\xi$ is a proper variable to fix the gauge and obtain a unitary evolution in the gauge fixed space (see later). We call the set $\{\varphi, p_\varphi, \xi, p_\xi\}$ “hybrid” variables because they are not the maximal gauge invariant choice of canonical coordinates. Indeed we can identify a maximal set of gauge invariant canonical variables (we will refer to them as “Shanmugadhasan” variables, see [9])

\begin{align*}
x &= \varphi + \frac{3}{4} \gamma p_\varphi, \\
px &= p_\varphi, \\
y &= 6 \gamma p_\gamma, \\
p_y &= H = \frac{1}{12} p_\gamma^2 - \frac{1}{16} p_\varphi^2,
\end{align*}

(3.6a) - (3.6d)

which is a set of canonically conjugate variables.

The variables $x$ and $p_x$ are gauge invariant and thus generate rigid invariance transformations. Of course the meaning of gauge invariant variables is transparent in the case of $x$: it is the initial value of $\varphi$. These variables and the functions $f(x, p_x)$ are the observables: “The set of the observables is isomorphic to the set of functions of the initial data” [10].

For sake of completeness, we write the generating function of the canonical transformation $\{\gamma, p_\gamma; \varphi, p_\varphi\} \rightarrow \{x, p_x; y, p_y\}$.

$$F = -\frac{3\gamma^2}{y} + \frac{4}{y} (\varphi - x)^2. \quad (3.7)$$

Each set, $\{\varphi, p_\varphi; \xi, p_\xi\}$ or $\{x, p_x; y, p_y\}$, can be used in the quantisation program and leads to identical results, both in the Dirac method (quantise before constraining) and in the reduced method (constrain before quantising). Let us first quantise in the hybrid variables.

\textbf{a) Quantisation in Hybrid Variables}
Let us start by the Dirac method. Wave functions are solutions of the WDW equation. Now, the gauge has to be fixed \([5,6,11]\) in the scalar product of solutions of the WDW equation. Let us define the scalar product as

\[
(\Psi_2, \Psi_1) = \int d[\alpha] \Psi_2^* \delta(\Theta) \Delta_{FP} \Psi_1 ,
\]

where \(\Theta(\alpha_i) = 0\) is the gauge fixing identity, \(\alpha_i (i = 1, 2)\) are the canonical coordinates, and \(\Delta_{FP}\) is the Faddeev-Popov (FP) determinant. \(d[\alpha]\) is the off-shell measure and is of course defined in the unconstrained phase space.

Now the first problem is the choice of the variables and of the measure. We require the measure to be gauge invariant and invariant with respect to the rigid symmetries of the system. The choice \(d[\alpha] = dp_{\xi} dp_{\phi}\) is gauge invariant and invariant under rigid transformations generated by \(p_{\phi}\) and \(p_{\xi}\), however it is not suitable for fixing the gauge. The suitable measure is

\[
d[\alpha] = dp_{\phi} d\xi ,
\]

which is gauge invariant and invariant under rigid transformations generated by \(p_{\phi}\) and \(p_{\xi}\). Furthermore it is expressed in function of \(\xi\). This allows to enforce the gauge fixing procedure.

In this representation \(\{\xi, p_{\phi}\}\) are differential operators. We have

\[
\hat{p}_{\xi} \rightarrow -i\partial_{\xi} , \quad \hat{\phi} \rightarrow i\partial_{p_{\phi}} , \quad \hat{\xi} \rightarrow \xi , \quad \hat{p}_{\phi} \rightarrow p_{\phi} .
\]

Thus the WDW equation is

\[
\left( -i\partial_{\xi} - \frac{1}{16} p_{\phi}^2 \right) \Psi(\xi, p_{\phi}) = 0 .
\]

The solutions of (3.11) that are eigenstates of \(\hat{p}_{\phi}\) with eigenvalue \(k\) are

\[
\Psi_k(p_{\phi}, \xi) = C(k)\delta(p_{\phi} - k)e^{ik^2\xi/16} .
\]

Now we have to fix the gauge. There is a class of viable gauges for which there are no Gribov copies and the FP determinant \(\Delta_{FP}\) is invariant under gauge transformations. This can be proved as in \([5]\). Let us simply choose \(\xi\) as time, i.e. take

\[
\Theta(\xi, p_{\phi}) = \xi - t
\]

(\(t\) is the gauge fixed time parameter); then \(\Delta_{FP} = 1\). This gauge is unique and finally the gauge fixed scalar product is

\[
(\Psi_2, \Psi_1) = \int dp_{\phi} \Psi_2^*(p_{\phi}, t)\Psi_1(p_{\phi}, t) .
\]
of course a positive definite Hilbert space. Note that the seemingly obvious choice for the gauge fixing $\Theta' \equiv \gamma - t = 0$ (or also $a - t = 0$) leads to the non positive definite scalar product usual in the Klein-Gordon case ($\Delta'_{FP} = p_\gamma$); it does not allow a first quantization interpretation and needs reinterpretation as a second quantized field.

The gauge fixed functions in the representation $\{\varphi, \xi = t\}$ read

$$\Psi_k(\varphi, t) = \frac{1}{\sqrt{2\pi}} e^{ik\varphi + ik^2t/16},$$

(3.15)

obviously orthonormal in the Fourier transformed gauge fixed measure $d\varphi$.

Let us discuss now the reduced method [11]. We impose the gauge identity $\xi - t = 0$ that gives the effective Hamiltonian

$$H_{\text{eff}} = -\frac{1}{16} \hat{p}_\varphi^2.$$  

(3.16)

The gauge identity implies $\mu = 1$ since from the definition of $\xi$ and the classical general solution of the gauge equations it follows $\xi = \tau + \text{const.}$. The Schrödinger equation is

$$i\partial_t \psi(\xi, p_\varphi) = -\frac{1}{16} \hat{p}_\varphi^2 \psi(\xi, p_\varphi).$$

(3.17)

The stationary eigenfunctions of $\hat{p}_\varphi$ coincide with (3.15) and are orthonormal in the reduced space measure. This proves the equivalence of the two quantization procedures.

b) Quantization in Shanmugadhasan Variables

We can quantize the system also in the Shanmugadhasan representation. Performing the canonical transformation to the new variables the action becomes

$$S = \int dt \{ \dot{x} p_x + \dot{y} p_y - \mu p_y \}.$$  

(3.18)

Let us first quantize the system by the Dirac method. The first step is the determination of the measure in the inner product (3.8). The requirement of invariance of the measure under the rigid transformations generated by $p_x$ or $x$ and the gauge transformation generated by $p_y$ selects $d[\alpha] = dx dy$ (equivalently $d[\alpha] = dp_x dy$), where $-\infty < x, y, p_x < \infty$. The measure $d[\alpha] = dp_x dp_y$ cannot be chosen since the gauge fixing function must contain $y$. So, consider the measure $d\mu = dx dy$: the conjugate variables $p_x$ and $p_y$ are represented as

$$\hat{p}_x \rightarrow -i\partial_x, \quad \hat{p}_y \rightarrow -i\partial_y, \quad \hat{x} \rightarrow x, \quad \hat{y} \rightarrow y,$$

(3.19)

and the WDW equation becomes

$$-i\partial_y \Psi(x, y) = 0.$$  

(3.20)
The solutions of (3.20) that are eigenfunctions of $\hat{p}_x$ with eigenvalue $k$ are

$$\Psi_k(x) = C(k)e^{ikx}.$$  \hfill (3.21)

Now we introduce the gauge fixing. The convenient gauge is

$$\Theta(x, y) = y - t.$$  \hfill (3.22)

Obviously this gauge is unique and $\Delta_{FP} = 1$. The wave functions (3.21) are of course orthonormal (choosing $C(k) = (2\pi)^{-1/2}$) in the inner product so defined.

Let us now quantize the system by the alternative method of reducing first the phase space by a canonical identity. Again the gauge fixing condition is $y = t$ which determines the Lagrange multiplier as $\mu = 1$. Using the constraint $H = 0$ and the gauge fixing condition, the effective Hamiltonian on the gauge shell becomes $H_{\text{eff}} = -\hat{p}_y = 0$. The reduced space Schrödinger equation just tells that fixed gauge wave functions do not depend on $y$. Diagonalizing $\hat{p}_x$ we obtain again the wave functions (3.21). The two quantization methods give the identical gauge fixed positive norm Hilbert space.

We have seen that the quantization of the system can be successfully completed both in hybrid and Shanmugadhasan variables. The two quantization procedures are equivalent. Further, the sets of physical wave functions (3.15) and (3.21) coincide when represented in the same variables. Let us discuss this point.

In order to relate the two representations (3.10) and (3.19) we need the generating function $F$ of the canonical transformation between the Shanmugadhasan and the hybrid variables:

$$F(\varphi, \xi; p_x, p_y) = \varphi p_x + \xi p_y + \frac{1}{16}\xi p_x^2.$$ \hfill (3.23)

The relation between the wave functions in the two representations is given by

$$\Psi(\xi, \varphi) = \int dp_x dp_y e^{iF(\varphi, \xi; p_x, p_y)}\Psi(p_x, p_y).$$ \hfill (3.24)

Substituting in (3.24) the Fourier transform of the wave functions (3.21)

$$\Psi(p_x, p_y) = \delta(p_x - k)\delta(p_y),$$ \hfill (3.25)

it is straightforward to obtain (3.15). This proves the equivalence between the hybrid and Shanmugadhasan representation.

In the Shanmugadhasan variables the reduced Hamiltonian coincides with the original $H$ and vanishes. The reason is that after the time gauge fixing we are left with gauge invariant variables; hence inner products and matrix elements are
purely algebraic relations because all operators are built from classical constant of the motion. The wave functions contain one less variable because there is no dependence on the gauge fixed time.

On the contrary, the gauge fixed wave functions for hybrid variables evolve with time, and the reduced Hamiltonian does not vanish, so these variables seem to contain more physics. However the physical content is the same. The time dependence expresses the fact that the hybrid observables are function of time and of the observable gauge invariant quantities.

Let us conclude this section noting that since the Hamiltonian (3.1) is essentially symmetric for \( \{ \gamma, p_\gamma \} \leftrightarrow \{ \varphi, p_\varphi \} \), both in the classical and the quantum treatment one can use the \( \{ \varphi, p_\varphi \} \) degrees of freedom to define the time.

4. Non Vanishing Cosmological Constant.

This case corresponds to the case treated in [3,4] by GV. The Hamiltonian is

\[
\mathcal{H} = \mu \left[ \frac{p_\gamma^2}{12} - \frac{p_\varphi^2}{16} + \Lambda e^{-4\varphi} \right].
\] (4.1)

Again the gauge equations generated by \( H \) are integrable. We have

\[
\begin{align*}
\gamma &= \gamma_0 + \frac{p_\gamma}{6} \tau, \quad \text{(4.2a)} \\
p_\gamma &= \text{constant}, \quad \text{(4.2b)} \\
e^{2\varphi} &= \pm \frac{\sqrt{\Lambda}}{\omega} \sinh(\omega \tau), \quad \text{(4.2c)} \\
p_\varphi &= -4\omega \coth(\omega \tau), \quad \text{(4.2d)} \\
\tau &= \int_{t_0}^{t} \mu(t') \, dt', \quad \mu(t) > 0, \quad \text{(4.2e)}
\end{align*}
\]

where

\[
\omega = \pm \sqrt{\frac{p_\varphi^2}{16} - \Lambda e^{-4\varphi}}. \quad \text{(4.3)}
\]

In (4.2c) the two signs correspond to the sign of \( \tau \). \( \gamma_0, p_\gamma \) and \( \omega \) are gauge invariant. On the constraint \( H = 0, p_\gamma = \pm \sqrt{12|\omega|} \). The choice of positive \( p_\gamma \) corresponds to the choice of a pre-big bang accelerated expansion \( \tau > 0 \) and a post-big bang decelerating expansion \( \tau < 0 \) at the basis of the string cosmology (see e.g. [4]).

Note that \( \gamma \) and \( p_\gamma \) transform very simply for gauge transformations; formulae (3.2a,b) hold. This fact will be exploited later. Let us connect the GV gauge parameter \( t_{GV} \) to our gauge parameter \( \tau \). The two parameters are related by

\[
dt_{GV} = e^{-2\varphi} \, d\tau, \quad \text{(4.4)}
\]
that is
\[
\sinh(|\omega|\tau) \sinh(t_{GV}\sqrt{\Lambda}) = -1. \tag{4.5}
\]
The use of \(\tau\) is suggested by the simplicity of Eqs. (4.2) with the choice (2.5) of the Lagrange multiplier. From these equations it is easy to obtain the on-shell solutions of GV [4] that we report for completeness:

- Pre-big bang regime, \(t_{GV} < 0\):
  \[
  a = a_0 \left[ \tanh \left( -\frac{t_{GV}\sqrt{\Lambda}}{2} \right) \right]^{-1/\sqrt{3}},
  \quad (4.6a)
  \]
  \[
  2(\varphi - \varphi_0) = -\ln \left[ \sinh \left( -t_{GV}\sqrt{\Lambda} \right) \right];
  \]

- Post-big bang regime, \(t_{GV} > 0\):
  \[
  a = a_0 \left[ \tanh \left( \frac{t_{GV}\sqrt{\Lambda}}{2} \right) \right]^{1/\sqrt{3}},
  \quad (4.6b)
  \]
  \[
  2(\varphi - \varphi_0) = -\ln \left[ \sinh \left( t_{GV}\sqrt{\Lambda} \right) \right].
  \]

As in the D’Alembert case, we can define “hybrid” and Shanmugadhasan variables. The hybrid variables are \(\{\varphi, p_\varphi, \xi, p_\xi\}\) defined as in section 3. The Shanmugadhasan canonical set is \(\{w, p_w, z, p_z\}\) defined by

\[
w \equiv \omega, \quad (4.7a)
\]
\[
p_w = -12\omega \frac{\gamma}{p_\gamma} - 2 \text{arcth} \left( \frac{4\omega}{p_\varphi} \right), \quad (4.7b)
\]
\[
z \equiv \xi = 6 \frac{\gamma}{p_\gamma}, \quad (4.7c)
\]
\[
p_z \equiv H = \frac{1}{12} p_\gamma^2 - \frac{1}{16} p_\varphi^2 + \Lambda e^{-4\varphi}. \quad (4.7d)
\]

All variables are gauge invariant except \(z\) (\(\delta z = \epsilon\)); \(w\) and \(p_w\) generate rigid symmetry transformations. Let us quantize now the system along the lines of section 3.

a) Quantization in Shanmugadhasan Variables

Performing the canonical transformation to the Shanmugadhasan variables the action becomes

\[
S = \int dt \{ \dot{w} p_w + \dot{z} p_z - \mu p_z \}. \quad (4.8)
\]
Let us quantize first the system by the Dirac method. The requirement of invariance of the measure under the rigid transformations generated by \( w \) or \( p_w \) and the gauge transformation generated by \( p_z \) selects the measures \( d[\alpha] = dw \) or equivalently \( d[\alpha] = dp_w dz \), where \(-\infty < w, z, p_w < \infty\). Given for instance the first one, we have the representation of the conjugate variables as differential operators:

\[
\hat{p}_w \rightarrow -i\partial_w, \quad \hat{p}_z \rightarrow -i\partial_z, \quad \hat{w} \rightarrow w, \quad \hat{z} \rightarrow z. \tag{4.9}
\]

The WDW equation becomes

\[
-\partial_z \Psi(w, z) = 0. \tag{4.10}
\]

The solutions of (4.10) that are eigenfunctions of \( \hat{w} \) with eigenvalues \( k \) are

\[
\Psi_k(w) = C(k)\delta(w - k). \tag{4.11}
\]

The gauge can be fixed as

\[
\Theta(w, z) = z - t = 0. \tag{4.12}
\]

(\( t \) is thus the fixed gauge time). So the scalar product is defined as

\[
(\Psi_2, \Psi_1) = \int dw \Psi_2^*(w)\Psi_1(w). \tag{4.13}
\]

Choosing \( C(k) = 1 \), the eigenfunctions (4.11) are orthonormal in the gauge fixed measure above.

Let us now quantize the system by the reduced method. Again the gauge fixing condition is \( z = t \). As for the case of section 3, this choice determines the Lagrange multiplier as \( \mu = 1 \). Using the constraint \( H = 0 \) and the gauge fixing condition, the effective Hamiltonian on the gauge shell becomes \( H_{\text{eff}} = -\hat{p}_z = 0 \) (typical of the Shanmugadhasan choice of coordinates). The wave functions do not depend on \( z \) and all matrix elements are of purely algebraic nature. Diagonalizing \( \hat{w} \) we obtain again the wave functions (4.11) in the reduced Hilbert space. As in the D’Alembert case, this proves the equivalence of the Dirac and reduced quantization methods in the representation used.

b) Quantization in Hybrid Variables

Let us begin using the Dirac method. As in the case of section 3 we have to choose the representation and establish the measure. Quite analogously, the right measure is (3.9). In this case it is better to work in the Fourier transformed space, so

\[
d[\alpha] = d\varphi d\xi. \tag{4.14}
\]

Note that (4.14) is not gauge invariant nor invariant under rigid transformations. However it is related to (3.9) by a Fourier transformation.
In the representation \{ξ, ϕ\} the conjugate variables to ξ and ϕ are differential operators. We have
\[ \hat{p}_ξ \rightarrow -i∂_ξ, \quad \hat{p}_ϕ \rightarrow -i∂_ϕ, \quad \hat{ξ} \rightarrow ξ, \quad \hat{ϕ} \rightarrow ϕ. \] (4.15)
The WDW equation is
\[ \left(-i∂_ξ + \frac{1}{16}∂_ϕ^2 + Λe^{-4ϕ}\right)Ψ(ξ, ϕ) = 0. \] (4.16)
The solutions of (4.16) that are eigenstates of \(\hat{ω}\) with eigenvalue \(k\) are of the form
\[ Ψ_k(ϕ, ξ) = A(k)J_{2ik}e^{ik^2ξ}, \] (4.17)
where \(Z\) is a generic linear combination of Bessel functions. The choice
\[ Ψ_k(ϕ, ξ) = A(k)J_{2ik}e^{ik^2ξ} \] (4.18)
has been selected by GV as representing the reflection of the WDW wave function correspondent to the birth of a decelerating expanding universe.

Now we have to fix the gauge. Using \(Θ = ξ - t\) the definition of the inner product is
\[ (Ψ_2, Ψ_1) = \int dϕΨ_2^*(ϕ, t)Ψ_1(ϕ, t) = \int dz \frac{z}{z}Ψ_2^*(z, t)Ψ_1(z, t). \] (4.19)
where \(z = 2\sqrt{Λ}e^{2ϕ}\). Note that the choice \(γ = t\) does not yield a positive definite norm.

The two sets of real orthonormal functions in the gauge fixed measure (4.19):
\[ χ^{(1)}_k(z, t) = \sqrt{\frac{k \cosh(\pi k)}{2 \sinh(\pi k)}} \left[ e^{-πkH^{(1)}_{2ik}(z)} + e^{πkH^{(2)}_{2ik}(z)} \right] e^{ik^2t}, \] (4.20a)
\[ χ^{(2)}_k(z, t) = i \sqrt{\frac{k \sinh(\pi k)}{2 \cosh(\pi k)}} \left[ e^{-πkH^{(1)}_{2ik}(z)} - e^{πkH^{(2)}_{2ik}(z)} \right] e^{ik^2t}. \] (4.20b)
Let us discuss now the reduced method. The gauge ξ = t gives the effective Hamiltonian
\[ H_{\text{eff}} = -\hat{ω}^2 = -\frac{\hat{p}_ϕ^2}{16} + Λe^{-2ϕ}, \] (4.21)
and the Schrödinger equation coincides with (4.16). The stationary Schrödinger equation is
\[ \left[ \frac{1}{16}∂_ϕ^2 + Λe^{-4ϕ}\right]Ψ(ϕ) = EΨ(ϕ), \quad E < 0. \] (4.22)
and its solutions are those of Eq. (4.17) where \( k = \sqrt{-E} \) and they can be chosen orthonormal as in (4.20).

Acknowledgments.

We thank Maurizio Gasperini and Gabriele Veneziano for interesting discussions.

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