Form Factors for Generalized Grey Brownian Motion

José Luís da Silva  
CIMA, University of Madeira, Campus da Penteada,  
9020-105 Funchal, Portugal.  
Email: luis@uma.pt

Ludwig Streit  
BiBoS, Universitat Bielefeld, Germany,  
CIMA, Universidade da Madeira, Funchal, Portugal  
Email: streit@uma.pt

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Abstract
In this paper we investigate the form factors of paths for a class of non Gaussian processes. These processes are characterized in terms of the Mittag-Leffler function. In particular, we obtain a closed analytic form for the form factors, the Debye function, and can study their asymptotic decay.

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In recent years fractional Brownian motion and processes related to fractional dynamics have become an object of intensive study. From the mathematical point of view these processes in general lack both the Markov and the semimartingale property, so that many of the classical methods from stochastic analysis do not apply, making their analysis more challenging. On the other hand, these processes are capable of modeling systems with long-range self interaction and memory effects. In 1992 Schneider introduced the notion of grey Brownian motion [Sch92] in order to solve the fractional-time diffusion equation where the time derivative is a Caputo-Djrbashian derivative of fractional order. In the 90’s Mainardi and coauthors started a systematic study of fractional differential equations, see for example [Mai10] and references therein, and introduced the generalized grey Brownian motion (ggBm for short), and the corresponding fractional-time differential equations of its density. More recently, Grothaus et al. [GJRS15], developed an infinite dimensional analysis with respect to (non Gaussian) measures of Mittag-Leffler type. In addition, in [GJ16] the Green function of the fractional-time heat equation is constructed by extending the fractional Feynman-Kac from Schneider [Sch92].

Form factors, aka structure factors of geometrical objects, play an important role in crystallography. More recently they have been put to use to encode geometric features of fractal and/or random objects\(^1\). In particular the form factor S of random paths \(\{X(t)|t \in [0,n]\}\) is given by

\[
S^X(k) := \frac{1}{n^2} \int_0^n dt \int_0^n ds \mathbb{E}(e^{i(k,\Delta X(t)-\Delta X(s))})
\]  

In this paper we intend to compute and discuss the form factors of ggBm trajectories; we proceed as follows. In Section 2 we introduce the background

\(^1\)Form factors of random paths play a central role in the theoretical and experimental analysis of polymer conformations, see e.g. [Ham16], [Ter02]}
needed later on, first, the functional setup and the Mittag-Leffler functions (and related functions). We then define the Mittag-Leffler measures $\mu_\beta$, $0 < \beta < 1$ on the space of vector-valued generalized functions and the generalized grey Brownian motion $\text{ggBm}$. In Section 3 we investigate the form factors for three families of the class $\text{ggBm}$. In addition, we exhibit their asymptotic decay for large values of the argument, as well as the equation relating the end-to-end length and the radius of gyration for the class $\text{ggBm}$.

2 Generalized grey Brownian motion in arbitrary dimensions

2.1 Prerequisites

Let $d \in \mathbb{N}$ and $L^2_d$ be the Hilbert space of vector-valued square integrable measurable functions

$$L^2_d := L^2(\mathbb{R}) \otimes \mathbb{R}^d.$$ 

The space $L^2_d$ is unitarily isomorphic to a direct sum of $d$ identical copies of $L^2 := L^2(\mathbb{R})$, (i.e., the space of real-valued square integrable measurable functions with Lebesgue measure). Any element $f \in L^2_d$ may be written in the form

$$f = (f_1 \otimes e_1, \ldots, f_d \otimes e_d), \quad (2)$$

where $f_i \in L^2(\mathbb{R})$, $i = 1,\ldots,d$ and $\{e_1,\ldots,e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. The inner product in $L^2_d$ is given by

$$(f, g)_0 = \sum_{k=1}^d (f_k, g_k)_{L^2} = \sum_{k=1}^d \int_{\mathbb{R}} f_k(x) g_k(x) \, dx,$$

where $g = (g_1 \otimes e_1, \ldots, g_d \otimes e_d)$, $f_k \in L^2$, $k = 1,\ldots,d$, $f$ as given in (2). The corresponding norm in $L^2_d$ is given by

$$|f|^2_0 := \sum_{k=1}^d |f_k|^2_{L^2} = \sum_{k=1}^d \int_{\mathbb{R}} f_k^2(x) \, dx.$$

As a densely imbedded nuclear space in $L^2_d$ we choose $S_d := S(\mathbb{R}) \otimes \mathbb{R}^d$, where $S(\mathbb{R})$ is the Schwartz test function space. An element $\varphi \in S_d$ has the form

$$\varphi = (\varphi_1 \otimes e_1, \ldots, \varphi_d \otimes e_d), \quad (3)$$
where \( \varphi_i \in S(\mathbb{R}) \), \( i = 1, \ldots, d \). Together with the dual space \( S'_d := S'(\mathbb{R}) \otimes \mathbb{R}^d \) we obtain the basic nuclear triple

\[
S_d \subset L^2_d \subset S'_d.
\]

The dual pairing between \( S'_d \) and \( S_d \) is given as an extension of the scalar product in \( L^2_d \) by

\[
\langle f, \varphi \rangle_0 = \sum_{k=1}^d (f_k, \varphi_k)_{L^2},
\]

where \( f \) and \( \varphi \) as in (2) and (3), respectively. In \( S'_d \) we invoke the Borel \( \sigma \)-algebra generated by the cylinder sets.

We define the operator \( M_{\alpha/2} \) on \( S(\mathbb{R}) \) by

\[
M_{\alpha/2} \varphi := \begin{cases} 
K_{\alpha/2} D_{-}^{(\alpha-1)/2} \varphi, & \alpha \in (0, 1), \\
\varphi, & \alpha = 1, \\
K_{\alpha/2} I_{-}^{(\alpha-1)/2} \varphi, & \alpha \in (1, 2),
\end{cases}
\]

with the normalization constant \( K_{\alpha/2} := \sqrt{\alpha \sin(\alpha \pi/2) \Gamma(\alpha)} \). \( D_{-} \), \( I_{-} \) denote the left-side fractional derivative and fractional integral of order \( r \) in the sense of Riemann-Liouville, respectively:

\[
(D_{-}^r f)(x) = \frac{1}{\Gamma(1-r)} \frac{d}{dx} \int_{-\infty}^x f(t)(x-t)^{-r} dt, \\
(I_{-}^r f)(x) = \frac{1}{\Gamma(r)} \int_x^\infty f(t)(t-x)^{r-1} dt, x \in \mathbb{R}.
\]

We refer to [SKM93] or [KST06] for the details on these operators. It is possible to obtain a larger domain of the operator \( M_{\alpha/2} \) in order to include the indicator function \( \mathbb{1}_{[0,t]} \) such that \( M_{\alpha/2} \mathbb{1}_{[0,t]} \in L^2 \), for the details we refer to Appendix A in [GJ16]. We have the following

**Proposition 1** (Corollary 3.5 in [GJ16]). *For all \( t, s \geq 0 \) and all \( 0 < \alpha < 2 \) it holds that*

\[
\left( M_{\alpha/2} \mathbb{1}_{[0,t]}, M_{\alpha/2} \mathbb{1}_{[0,s]} \right)_{L^2} = \frac{1}{2} \left( t^\alpha + s^\alpha - |t-s|^\alpha \right).
\]

The Mittag-Leffler function was introduced by G. Mittag-Leffler in a series of papers [ML03, ML04, ML05]. In Section 3 we also need its generalization originally due to R. Agarwal [Aga53].
Definition 2 (Mittag-Leffler function). 1. For $\beta > 0$ the Mittag-Leffler function $E_\beta$ (MLf for short) is defined as an entire function by the following series representation

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C};$$

where $\Gamma$ denotes the gamma function.

2. For any $\rho \in \mathbb{C}$ the generalized Mittag-Leffler function (gMLf for short) is an entire function defined by its power series

$$E_{\beta,\rho}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \rho)}, \quad z \in \mathbb{C}.$$

Note the relation $E_{\beta,1}(z) = E_\beta(z)$ and $E_1(z) = e^z$ for any $z \in \mathbb{C}$.

We have the following asymptotic for the gMLf.

Proposition 3 (cf. [GKMS14, Section 4.7]). Let $0 < \beta < 2$, $\alpha \in \mathbb{C}$ and $\delta$ be such that

$$\frac{\beta \pi}{2} < \delta < \min\{\pi, \beta \pi\}.$$

Then, for any $m \in \mathbb{N}$, the following asymptotic formulas hold:

1. If $|\arg(z)| \leq \delta$, then

$$E_{\beta,\alpha}(z) = \frac{1}{\beta} z^{(1-\alpha)/\beta} \exp(z^{1/\beta}) - \sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\alpha - \beta n)} + O(|z|^{-m-1}), \quad |z| \to \infty.$$

2. If $\delta \leq |\arg(z)| \leq \pi$, then

$$E_{\beta,\alpha}(z) = -\sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\alpha - \beta n)} + O(|z|^{-m-1}), \quad |z| \to \infty.$$
where $2\Psi_2$ is the Fox-Wright function (also called generalized Wright function [KST02], [GKMS14, Appendix F, eq. (F.2.14)] and [MP07]) given for $x, a_i, c_i \in C$ and $b_i, d_i \in \mathbb{R}$ by

$$2\Psi_2((a_1, b_1), (a_2, b_2), (c_1, d_1), (c_2, d_2) \mid x) = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + b_1 n) \Gamma(a_2 + b_2 n) x^n}{\Gamma(c_1 + d_1 n) \Gamma(c_2 + d_2 n) n!}.$$ 

In particular, when $\rho = \alpha$ and $\gamma = \beta$, eq. (8) yields

$$\int_0^1 t^{\alpha-1} (1-t)^{\sigma-1} E_{\beta,\alpha}(xt^\beta) \, dt = \Gamma(\sigma) E_{\beta,\alpha+\sigma}(x). \quad (9)$$

Both integrals (8) and (9) will be used in Section 3 below.

### 2.2 The Mittag-Leffler measure

The Mittag-Leffler measures $\mu_\beta$, $0 < \beta \leq 1$ are a family of probability measures on $S'_d$ whose characteristic functions are given by the Mittag-Leffler functions, see Definition 2. Using the Bochner-Minlos theorem, see [GV68], [Hid70], the following definition makes sense.

**Definition 4** (cf. [GJRS15]). For any $\beta \in (0, 1]$ the Mittag-Leffler measure is defined as the unique probability measure $\mu_\beta$ on $S'_d$ by fixing its characteristic functional

$$\int_{S'_d} e^{i\langle w, \varphi \rangle_0} \, d\mu_\beta(w) = E_\beta \left( -\frac{1}{2} |\varphi|_0^2 \right), \quad \varphi \in S_d. \quad (10)$$

**Remark 5.**

1. The measure $\mu_\beta$ is also called grey noise (reference) measure, cf. [GJRS15] and [GJ16].

2. The range $0 < \beta \leq 1$ ensures the complete monotonicity of $E_\beta(-x)$, see Pollard [Pol48], i.e., $(-1)^n E^{(n)}_\beta(-x) \geq 0$ for all $x \geq 0$ and $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. In other words, this is sufficient to show that

$$S_d \ni \varphi \mapsto E_\beta \left( -\frac{1}{2} |\varphi|_0^2 \right) \in \mathbb{R}$$

is a characteristic function in $S_d$. 

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We consider the complex Hilbert space of square integrable measurable functions defined on $S'_d$, $L^2(\mu_\beta) := L^2(S'_d, \mathcal{B}, \mu_\beta)$, with scalar product defined by

$$(F, G)_{L^2(\mu_\beta)} := \int_{S'_d} F(w) \overline{G}(w) d\mu_\beta(w), \quad F, G \in L^2(\mu_\beta).$$

The corresponding norm is denoted by $\| \cdot \|_{L^2(\mu_\beta)}$. It follows from (10) that all moments of $\mu_\beta$ exist and we have

Lemma 6. For any $\varphi \in S_d$ and $n \in N_0$ we have

$$\int_{S'_d} \langle w, \varphi \rangle_0^{2n+1} d\mu_\beta(w) = 0,$$

$$\int_{S'_d} \langle w, \varphi \rangle_0^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)} |\varphi|^2_0.$$

In particular, $\| \langle \cdot, \varphi \rangle \|_{L^2(\mu_\beta)}^2 = \frac{1}{\Gamma(\beta + 1)} |\varphi|^2_0$ and by polarization for any $\varphi, \psi \in S_d$ we obtain

$$\int_{S'_d} \langle w, \varphi \rangle_0 \langle w, \psi \rangle_0 d\mu_\beta(w) = \frac{1}{\Gamma(\beta + 1)} \langle \varphi, \psi \rangle_0.$$

2.3 Generalized grey Brownian motion

For any test function $\varphi \in S_d$ we define the random variable

$$X^\beta(\varphi) : S'_d \rightarrow \mathbb{R}^d, \quad w \mapsto X^\beta(\varphi, w) := \left( \langle w_1, \varphi_1 \rangle, \ldots, \langle w_d, \varphi_d \rangle \right).$$

The random variable $X^\beta(\varphi)$ has the following properties which are a consequence of Lemma 6 and the characteristic function of $\mu_\beta$ given in (10).

Proposition 7. Let $\varphi, \psi \in S_d$, $k \in \mathbb{R}^d$ be given. Then

1. The characteristic function of $X^\beta(\varphi)$ is given by

$$E(e^{ik \cdot X^\beta(\varphi)}) = E_\beta \left( -\frac{1}{2} \sum_{j=1}^d k_j^2 |\varphi|^2_{L^2} \right). \quad (11)$$

2. The characteristic function of the random variable $X^\beta(\varphi) - X^\beta(\psi)$ is

$$E(e^{ik \cdot (X^\beta(\varphi) - X^\beta(\psi))}) = E_\beta \left( -\frac{1}{2} \sum_{i=1}^d k_j^2 |\varphi - \psi|^2_{L^2} \right). \quad (12)$$
3. The expectation of the $X^\beta(\varphi)$ is zero and
\[
\|X^\beta(\varphi)\|^2_{L^2(\mu_\beta)} = \frac{1}{\Gamma(\beta + 1)}|\varphi|^2_0.
\] (13)

4. The moments of $X^\beta(\varphi)$ are given by
\[
\int_{S'_d} |X^\beta(\varphi, w)|^{2n+1} d\mu_\beta(w) = 0,
\]
\[
\int_{S'_d} |X^\beta(\varphi, w)|^{2n} d\mu_\beta(w) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)}|\varphi|^{2n}_0.
\]

Remark 8. $X^\beta$ is a "generalized process", with white noise as a special case: $\mu_1$ is the Gaussian white noise measure [GV68].

The property (13) of $X^\beta(\varphi)$ gives the possibility to extend the definition of $X^\beta$ to any element in $L^2_d$, in fact, if $f \in L^2_d$, then there exists a sequence $(\varphi_k)_{k=1}^\infty \subset S_d$ such that $\varphi_k \rightarrow f$, $k \rightarrow \infty$ in the norm of $L^2_d$. Hence, the sequence $(X^\beta(\varphi_k))_{k=1}^\infty \subset L^2(\mu_\beta)$ forms a Cauchy sequence which converges to an element denoted by $X^\beta(f) \in L^2(\mu_\beta)$.

We define $\mathbb{I}_{[0,t)} \in L^2_d$, $t \geq 0$, by
\[
\mathbb{I}_{[0,t)} := (\mathbb{I}_{[0,t)} \otimes e_1, \ldots, \mathbb{I}_{[0,t)} \otimes e_d)
\]
and consider the process $X^\beta(\mathbb{I}_{[0,t)}) \in L^2(\mu_\beta)$ such that the following definition makes sense.

Definition 9. For any $0 < \alpha < 2$ we define the process
\[
S'_d \ni w \mapsto B^{\beta,\alpha}(t, w) := \left(\langle w, (M^\alpha/2 \mathbb{I}_{[0,t)}) \otimes e_1 \rangle, \ldots, \langle w, (M^\alpha/2 \mathbb{I}_{[0,t)}) \otimes e_d \rangle \right)
\]
\[
= \left(\langle w_1, M^\alpha/2 \mathbb{I}_{[0,t)}, \ldots, \langle w_d, M^\alpha/2 \mathbb{I}_{[0,t)} \rangle \right), t > 0 \ \ (14)
\]
as an element in $L^2(\mu_\beta)$ and call this process $d$-dimensional generalized grey Brownian motion (ggBm), in short
\[
B^{\beta,\alpha}(t) = X^\beta \left(M^\alpha/2 \mathbb{I}_{[0,t)} \right).
\]

Remark 10. 1. The $d$-dimensional ggBm $B^{\beta,\alpha}$ exist as a $L^2(\mu_\beta)$-limit and hence the map $S'_d \ni \omega \mapsto \langle \omega, M^\alpha/2 \mathbb{I}_{[0,t)} \rangle$ yields a version of ggBm, $\mu_\beta$-a.s., but not in the pathwise sense.

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2. For any fixed $0 < \alpha < 2$ one can show by the Kolmogorov-Centsov continuity theorem that the paths of the process are $\mu_\beta$-a.s. continuous, cf. [GJ16, Prop. 3.8].

3. Below we mainly deal with expectation of functions of ggBm, therefore the version of ggBm defined above is sufficient.

**Proposition 11.** For any $0 < \alpha < 2$, the process $B^{\beta,\alpha} := \{B^{\beta,\alpha}(t), t \geq 0\}$, is $\alpha/2$ self-similar with stationary increments.

**Proof.** Given $k = (k_1, k_2, \ldots, k_n) \in \mathbb{R}^n$, we have to show that for any $0 < t_1 < t_2 < \ldots < t_n$ and $a > 0$:

$$E \left( \exp \left( i \left\langle \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,a t_j]) \right\rangle \right) \right) = E \left( \exp \left( ia^{\alpha/2} \left\langle \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,t_j]) \right\rangle \right) \right).$$

(15)

It follows from (11) that eq. (15) is equivalent to

$$E_\beta \left( -\frac{1}{2} \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,a t_j]) \right)^2 = E_\beta \left( -\frac{1}{2} a^{\alpha/2} \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,t_j]) \right)^2.$$

Because of the complete monotonicity of $E_\beta$, the above equality reduces to

$$\left| \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,a t_j]) \right|^2_{L^2} = a^{\alpha} \left| \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,t_j]) \right|^2_{L^2},$$

which is easy to show, taking into account (4). A similar procedure may be applied in order to prove the stationarity of the increments. Hence, for any $h \geq 0$, we have to show that

$$E \left( \exp \left( i \sum_{j=1}^{n} k_j (B^{\beta,\alpha}(t_j + h) - B^{\beta,\alpha}(h)) \right) \right) = E \left( \exp \left( i \sum_{j=1}^{n} k_j B^{\beta,\alpha}(t_j) \right) \right).$$

The above procedure reduces this equality to check the following

$$\left| \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([h,t_j+h]) \right|^2_{L^2} = \sum_{j=1}^{n} k_j M^{\alpha/2}_- \mathbb{1}([0,t_j]) \right|^2_{L^2},$$

which is verified as before. \qed
Remark 12. The family \{B^{\beta,\alpha}(t), \ t \geq 0, \ \beta \in (0,1], \ \alpha \in (0,2]\} forms a class of \alpha/2-self-similar processes with stationary increments which includes:

1. for \beta = \alpha = 1, the process \{B^{1,1}(t), \ t \geq 0\}, standard \textit{d}-dimensional Bm.

2. for \beta = 1 and 0 < \alpha < 2, \{B^{1,\alpha}(t), \ t \geq 0\}, \textit{d}-dimensional fBm with Hurst parameter \alpha/2.

3. for \alpha = 1, \{B^{\beta,1}(t), \ t \geq 0\} a 1/2-self-similar non Gaussian process with

\[ E\left(e^{i(k,B^{\beta,1}(t))}\right) = E_{\beta}\left(-\frac{|k|^2}{2}t\right), \quad k \in \mathbb{R}^d. \quad (16) \]

4. for 0 < \alpha = \beta < 1, the process \{B^{\beta}(t) := B^{\beta,\beta}(t), \ t \geq 0\}, \beta/2 self-similar and called \textit{d}-dimensional grey Brownian motion (gBm for short). The characteristic function of \(B^{\beta}(t)\) is given by

\[ E\left(e^{i(k,B^{\beta}(t))}\right) = E_{\beta}\left(-\frac{|k|^2}{2}t^{\beta}\right), \quad k \in \mathbb{R}^d. \quad (17) \]

For \textit{d} = 1, gBm was introduced by W. Schneider in [Sch90, Sch92].

5. For other choices of the parameters \beta and \alpha we obtain, in general, non Gaussian processes.

3 Form factors for different classes of ggBm

In this section we investigate the form factors of ggBm and two particular cases introduced in Subsection 2.3, cf. Remark 12. To begin with we note that, given a \textit{d}-dimensional stochastic process \(X\), the form factor associated to \(X\) is the function defined by

\[ S^X(k) := \frac{1}{n^2} \int_0^n dt \int_0^n ds E\left(e^{i(k,X(t)-X(s))}\right), \quad k \in \mathbb{R}^d, \ n \in \mathbb{N} \quad (18) \]

which, in case \(X\) is \(\nu\)-self-similar, reduces to

\[ S^X(k) = \int_0^1 dt \int_0^1 ds E\left(e^{in\nu(k,X(t)-X(s))}\right). \quad (19) \]
This function encodes in particular, to lowest order in $k$, the root-mean-square radius of gyration (or simply radius of gyration) of $X$, defined by

$$
(R_g^X)^2 := \frac{1}{2} \frac{1}{n^2} \int_0^n dt \int_0^n ds \, E \left( \left| X(t) - X(s) \right|^2 \right)
$$

play an important role in the study of random path conformations. For $\nu$-self-similar $d$-dimensional processes $X$, the radius of gyration obeys a general relation to the mean square end-to-end length

$$
(R_e^X)^2 := E \left( \left| X(n) - X(0) \right|^2 \right)
$$

of the paths, namely

$$
\frac{(R_e^X)^2}{(2\nu + 1)(2\nu + 2)} = (R_g^X)^2. \tag{20}
$$

In particular, for a $d$-dimensional Bm $B$ ($\nu = 1/2$) we have the celebrated equality, see for example [Ter02, Section 2.4]

$$
\frac{(R_e^B)^2}{6} = (R_g^B)^2. \tag{21}
$$

In the following subsections we compute explicitly these notions for the different classes of ggBm.

### 3.1 The form factor for ggBm $B^{\beta,\alpha}$

The most general case of the two parameter family $B^{\beta,\alpha} = \{B^{\beta,\alpha}(t), t \geq 0\}$ with $0 < \beta < 1$ and $0 < \alpha < 2$ is now prepared to be investigated. It corresponds to a $d$-dimensional ggBm process which is $\alpha/2$-self-similar with stationary increments, cf. Section 2.3. The characteristic function of the increment of ggBm is given by

$$
E \left( e^{i(k,B^{\beta,\alpha}(t)) - B^{\beta,\alpha}(s))} \right) = E_{\beta} \left( -\frac{|k|^2}{2} |t - s|^\alpha \right). \tag{22}
$$

Denote by $S^{\beta,\alpha} := S^{B^{\beta,\alpha}}$ the structure factors for the ggBm process $B^{\beta,\alpha}$. Then, according to (19) $S^{\beta,\alpha}$ is given by

$$
S^{\beta,\alpha}(k) := \int_0^1 dt \int_0^1 ds \, E \left( e^{i\frac{1}{2}(k,B^{\beta,\alpha}(t)) - B^{\beta,\alpha}(s))} \right), \quad k \in \mathbb{R}^d, \ n \in \mathbb{N}
$$
and it follows from (22) that $S^{\beta,\alpha}$ is equal to

$$S^{\beta,\alpha}(k) = \int_0^1 dt \int_0^1 ds \, E_{\beta} \left( -\frac{n^\alpha |k|^2}{2} |t-s|^\alpha \right) = 2 \int_0^1 dt \int_0^t ds \, E_{\beta} \left( -\frac{n^\alpha |k|^2}{2} |t-s|^\alpha \right).$$

A change of variables $\tau = t - s$ reduces $S^{\beta,\alpha}$ to the following integral

$$S^{\beta,\alpha}(k) = 2 \int_0^1 d\tau \, (1-\tau) E_{\beta} \left( -\frac{n^\alpha |k|^2}{2} \tau^\alpha \right).$$

Using eq. (8) the above integral can be computed and the explicit expression for $S^{\beta,\alpha}$ is given in terms of the Fox-Wright function

$$S^{\beta,\alpha}(k) = 2 \Psi_2 \left( \begin{array}{c} (1, \alpha), (1, 1) \\ (1, \beta), (3, \alpha) \end{array} \bigg| -\frac{n^\alpha |k|^2}{2} \right)$$

or by its Taylor series as

$$S^{\beta,\alpha}(k) = 2 \sum_{j=0}^{\infty} \frac{\Gamma(\alpha j + 1) \Gamma(j + 1) \left(-\frac{n^\alpha |k|^2}{2}\right)^j}{\Gamma(\beta j + 1) \Gamma(\alpha j + 3) j!} = 2 \sum_{j=0}^{\infty} \frac{1}{\Gamma(\beta j + 1)(\alpha j + 1)(\alpha j + 2)} \left(-\frac{n^\alpha |k|^2}{2}\right)^j.$$

For fixed $\beta, \alpha$ the form factor depends only on $y^2 = \frac{n^\alpha |k|^2}{2}$ via the so-called Debye function $f_D$:

$$S^{\beta,\alpha}(k) = : f_D(y; \beta, \alpha),$$

For the process $B^{\beta,\alpha}$ we thus have

$$f_D(y; \beta, \alpha) = 2 \sum_{j=0}^{\infty} \frac{1}{\Gamma(\beta j + 1)(\alpha j + 1)(\alpha j + 2)} \left(-y^2\right)^j.$$

Remark 13. Among possible limit cases of the Debye function $f_D$ we point out:

1. for $\beta = \alpha = 1$, clearly we obtain the Debye function $f_D(\cdot; 1, 1)$ for Bm, see (29).
2. for any $0 < \alpha < 2$ fixed, the limit $\beta \to 1$ is

$$f_D(y; 1, \alpha) = 2 \sum_{n=0}^{\infty} \frac{(-y^2)^n}{n!(\alpha n + 2)(\alpha n + 1)}$$

$$= \frac{2(y^2)^{-1/\alpha}}{\alpha} \gamma \left( \frac{1}{\alpha}, y^2 \right) - \frac{2(y^2)^{-1/\alpha}}{\alpha} \gamma \left( \frac{2}{\alpha}, y^2 \right), \quad (24)$$

which coincides with the Debye function for fBm with Hurst parameter $H = \alpha/2$, and $\gamma$ is the incomplete gamma function.

In Figure 1 we present the plots (linear scale in left and log-log scale right) of the Debye function $f_D$ for different values of $\beta$ and $\alpha$. The asymptotic in general is harder to obtain for the Fox-Wright function. In [Bra64] and [KST02] there are studies on the asymptotic for certain classes of special functions which include the Fox-Wright function.

The radius of gyration for ggBm is obtained by expanding the form factor to lowest order

$$(R_{g}^{\beta, \alpha})^2 = \frac{n^\alpha}{\Gamma(\beta + 1)(\alpha + 1)(\alpha + 2)}$$

and with

$$(R_{e}^{\beta, \alpha})^2 = E\left( (B^{\beta, \alpha}(n))^2 \right) = \frac{n^\alpha}{\Gamma(\beta + 1)}$$

we have

$$\frac{(R_{e}^{\beta, \alpha})^2}{(\alpha + 1)(\alpha + 2)} = (R_{g}^{\beta, \alpha})^2.$$
3.2 The form factor for gBm $B^\beta$

For $\alpha = \beta$ the above specializes to gBm, with the Debye function

$$f_D(y; \beta, \beta) = 2 \sum_{j=0}^{\infty} \frac{1}{\Gamma(\beta j + 1)(\beta j + 1)(\beta j + 2)} (-y^2)^j = 2E_{\beta,3}(-y^2).$$

Remark 14. The two limit cases ($\beta \to 1$ and $\beta \to 0$) of the Debye function $f_D(\cdot; \beta, \beta)$ are then as follows.

1. For $\beta \to 1$ we obtain

$$f_D(y; 1, 1) = 2 \sum_{j=0}^{\infty} \frac{(-y^2)^j}{(j+2)!} = \frac{2}{y^4} \left(e^{-y^2} - 1 + y^2\right),$$

i.e., the Debye function for Bm, cf. [Ham16, Ch. 28].

2. The other limit $\beta \to 0$ is given by

$$\lim_{\beta \to 0} f_D(y; \beta, \beta) = \frac{1}{1 + y^2}, \quad |y| < 1.$$  

Remark 15. In Figure 2 we show the plots of the Debye function $f_D(\cdot; \beta, \beta)$ for different values of $\beta$.

1. For large values of $y$ the function $f_D(\cdot; 1, 1)$ falls off as $2y^{-2}$ (Bm case), right plot upper continuous line.

2. On the other hand, all the other cases $0 < \beta < 1$, the function $f_D(\cdot; \beta, \beta)$ falls according to the asymptotic of the Mittag-Leffler function $E_{\beta,3}$ given in (7). In that case, we have

$$f_D(y; \beta, \beta) = 2 \sum_{j=1}^{N-1} \frac{(-y^2)^j}{\Gamma(3 - \beta j)} \sim \frac{2}{\Gamma(3 - \beta)} y^{-2}. $$

In terms of the right plot in Figure 2, this corresponds to the lower lines.

The gBm is a $\beta/2$—self-similar process, so the relation between the end-to-end length and the radius of gyration for gBm is given by

$$\frac{(R_\beta^e)^2}{(\beta + 1)(\beta + 2)} = (R_\beta^g)^2.$$

(27)
3.3 The form factor for $B^{\beta,1}$

In this case we have

$$f_D(y;\beta,1) = 2 \sum_{j=0}^{\infty} \frac{(-y^2)^j}{\Gamma(\beta j + 1)(j + 1)(j + 2)}$$

$$= 2 \sum_{j=0}^{\infty} \frac{(-y^2)^j}{\Gamma(\beta j + 1)(j + 1)} - 2 \sum_{j=0}^{\infty} \frac{(-y^2)^j}{\Gamma(\beta j + 1)(j + 2)}.$$  \hspace{1cm} (28)

**Remark 16.** The two limit cases ($\beta \to 1$ and $\beta \to 0$) of the Debye function $f_D(\cdot;\beta,1)$ follows from (28). Namely

1. For $\beta \to 1$ we have

$$f_D(y;1,1) = \frac{2}{y^4} \left( e^{-y^2} - 1 + y^2 \right)$$  \hspace{1cm} (29)

which coincides with the Debye function for Bm.

2. The other limit $\beta \to 0$ is obtained as

$$\lim_{\beta \to 0} f_D(y;\beta,1) = 2 \sum_{j=0}^{\infty} \frac{(-y^2)^j}{(j + 1)(j + 2)}$$

$$= \frac{2}{y^4} \left( -y^2 + (1 + y^2) \ln(1 + y^2) \right).$$  \hspace{1cm} (30)

**Remark 17.** In Figure 3 we present the plots of the Debye function $f_D(y;\beta,1)$ for different values of $\beta$. 

Figure 2: The Debye function $f_D^\beta = f_D(\cdot;\beta,\beta)$ for gBm. Left linear scale; right logarithmic scale.
Figure 3: The Debye function $f_{D}^{\beta,1} = f_{D}(\cdot; \beta, 1)$. Left linear scale; right logarithmic scale.

1. As it follows from (29) for large values of $y$ the function $f_{D}(\cdot; 1, 1)$ falls as $2y^{-2}$, right plot lower dash-dot line.

2. The other limit case $\beta \to 0$, eq. (30) implies that $\lim_{\beta \to 0} f_{D}(y; \beta, 1)$ falls as $(-2 + 4 \ln(y)) y^{-2}$, right plot upper continuous line.

3. In the intermediate cases $f_{D}(y; \beta, 1)$ decays as $(k_1 + k_2 \ln(y)) y^{-2}$ where $k_1, k_2$ are constants. As an example, for $\beta = \frac{1}{3},$

   $$f_{D}(y; 1/3, \beta) \approx (-0.827976 + 2.95395 \ln(y)) y^{-2}, \ y \to \infty,$$

   dash and dotted lines in the right hand plot.

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