Fully developed relativistic turbulence

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We use a simple model consisting of energy-momentum tensor conservation and a Maxwell-Cattaneo equation for its viscous part to study nonlinear phenomena in a real relativistic fluid. We focus on new types of behavior without nonrelativistic equivalents, such as an entropy cascade driven by fluctuations in the tensor degrees of freedom of the theory. We write down the von Kármán-Howarth equations for this kind of turbulence, and consider the correlations corresponding to fully developed turbulence.
I. INTRODUCTION

Recent developments in relativistic heavy ion collisions [1] and cosmology [2, 3] have driven attention to the nonlinear dynamics of relativistic fluids on very short time scales, even less than the fluid relaxation time [4]. It is expected that this nonlinear phenomena will present at least some aspects of turbulence, such as the presence of energy cascades [5–10]. However, very little is known about relativistic hydrodynamic turbulence.

Recently, it has been proposed that relativistic turbulence proceeds not through an energy cascade, but an entropy one [11] (see also [12–14]). If confirmed, this insight will mean a substantial step forward in our understanding of the subject. Our goal is to provide a further illustration of the issues at hand by studying relativistic turbulence under the assumption of homogeneity and isotropy.

Probably one of the obstacles in the development of the field is lack of agreement on what are exactly the equations of motion for a real relativistic fluid. There are two large families of theories, the so-called first order theories where the dynamical variables are the same for real and ideal fluids [15–22], and the second order theories where the set of variables for a real fluid is larger than for an ideal one. In a typical second order theory, the viscous part of the energy momentum tensor is regarded as a hydrodynamic variable on its own. Theories of this class are Israel-Stewart [23–27], Divergence type Theories [28–38], DNMR [39, 40, 42–47] and Anisotropic Hydrodynamics [48–55]. We shall concentrate on second order theories because we have in mind applications such as reheating after inflation and the generation of primordial gravitational waves, where spin 2 degrees of freedom in the fluid play a central role [2, 3]. These degrees of freedom are easily incorporated in a second order theory, while not so in first order ones.

Even after making this first choice, a large number of alternative models remain. We shall study turbulent behavior in a minimal model where the equations of motion are energy-momentum conservation and a Maxwell-Cattaneo like equation for the viscous part thereof [56–59]. As in nonrelativistic turbulence, we shall keep only quadratic terms in the equations of motion. The motivation for this model is further discussed in Appendix A. See also [37, 60, 61].

Following [11], our interest is in phenomena with no nonrelativistic equivalents. Therefore we shall focus on the, maybe unrealistic, case where turbulence is driven by the tensor degrees of freedom in the theory. Our goal is to discuss what are the relevant correlation functions for the relativistic turbulent fluctuations, what constraints are placed on them by homogeneity and isotropy, what are the relevant von Kármán - Howarth equations [64–69], and finally whether there is something like “fully developed” relativistic turbulence.

It ought to be noted that we are considering fluids very far from equilibrium. The thermal fluctuations of a relativistic real fluid are discussed in [38].

This comment is organized as follows. We present our model in next section; as said, further motivation is given in Appendix A. The model is such that positive entropy production is built in; nevertheless, we provide a direct check in section III, whereby we identify the relevant entropy flux, whose scale invariance will signal an entropy cascade. Then in section IV we analyze the fluid correlation functions under the assumption of homogeneity and isotropy. In sections V and VI we discuss the relativistic von Kármán - Howarth equations and the would be fully developed turbulence. We conclude with some final remarks.

II. THE MODEL

To investigate the nonlinear behavior of relativistic real fluids, we shall assume a minimal model, consisting of the conservation law for the energy-momentum tensor and a Maxwell-Cattaneo equation for the viscous part thereof.

Let us analyze these currents more closely. As usual, we may decompose

\[ T^{\mu\nu} = \rho \left[ u^\mu u^\nu + \frac{1}{3} \Delta^{\mu\nu} + \Pi^{\mu\nu} \right] \]  

(1)

where \( \rho = \rho_0 e^\delta \) is the energy density, and we assume the conformal equation of state \( p = \rho/3 \). \( u^\mu \) is the Landau-Lifshitz fluid four velocity obeying \( u^2 = -1 \), and

\[ \Delta_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \]  

(2)

is the orthogonal projector. The viscous energy-momentum tensor \( \Pi^{\mu\nu} \) is assumed to be dimensionless, transverse and traceless \( \Pi^\mu_\mu = \Pi^{\mu\nu} u_\nu = 0 \).
Energy conservation leads to

\[ \dot{x} + \frac{4}{3} \theta + \Pi^{\rho \sigma} u_{\rho,\sigma} = 0 \]

\[ \frac{4}{3} u^\nu + \frac{1}{3} \Delta^\nu \delta_{\rho} + \Pi^\mu_{\nu} + \Pi^{\mu \nu} \delta_{\mu} - u^\nu \Pi^{\rho \sigma} u_{\rho,\sigma} = 0 \]

(3)

where \( \dot{x} = u^\mu,_{\mu} \), and

\[ \theta = u^\mu_{;\mu} \]

(4)

The Maxwell-Cattaneo equation of motion for \( \Pi^{\mu \nu} \) reads (see Appendix (A))

\[ 0 = \dot{\Pi}^{\mu \nu} + \left[ \frac{1}{\lambda} - \frac{10}{21} \right] \Pi^{\mu \nu} + \frac{4}{15} \left[ \Delta^{\nu \rho} u_{\rho}^{\mu} + \Delta^{\rho \nu} u_{\rho}^{\mu} - \frac{2}{3} \Delta^{\nu \rho} \theta \right] + \frac{6}{7} \left[ \Pi^{\rho \sigma} u_{\rho}^{\mu} + \Pi^{\mu \rho} u_{\rho}^{\sigma} - \frac{2}{3} \Delta^{\rho \sigma} u_{\rho,\sigma} \right] + \frac{1}{4} [u^{\nu} \Pi^{\rho \rho} + u^{\rho} \Pi^{\nu \rho}] \delta_{\rho,\mu} - \frac{1}{7} \left[ \Delta^{\mu \sigma} \Pi^{\nu \rho} + \Delta^{\nu \sigma} \Pi^{\mu \rho} - \frac{2}{3} \Delta^{\nu \rho} \Pi^{\rho \sigma} \right] u_{\rho,\sigma} \]

(5)

We now define

\[ u^\mu = u^0 v^\mu \]

(6)

where \( v^\mu = (1, v^k/c) \) and \( u^0 = (1 - v^2/c^2)^{-1/2} \). Then

\[ u^{\mu, \nu} = u^0 v^{\mu, \nu} + \frac{u^0}{c^2} v^\nu v^\lambda_{;\lambda} v^{\lambda, \nu} = u^0 [v^{\mu, \nu} + v^\mu b^\nu] \]

\[ \theta = u^0 [v^\lambda_{;\lambda} + a] \]

(7)

where \( b^\nu = u^0 v^\lambda_{;\lambda} v^{\lambda, \nu} / c^2 \), \( a = v_\rho b^\rho \). Also call \( X^\nu = v^\mu X_{;\mu} \). Observe that \( \dot{u}^\nu = u^0 u_{;\nu} = u^0 \left[ u^{\nu} + a v^\nu \right] \) and \( u^0 v^2 = -1 \). Also

\[ v_\nu \Pi^{\mu \nu} + \Pi^{\alpha \sigma} v_{\rho,\sigma} = 0 \]

(8)

We may decompose \( \Pi^{\mu \nu} \) as

\[ \Pi^{0i} = \frac{v_i}{c} \Pi^{ij} \]

\[ \Pi^{00} = \frac{v_i v_j}{c^2} \Pi^{ij} \]

(9)

and

\[ \Pi^{ij} = \frac{v_i v_j}{c^2} \Pi^{ij} \]

(10)

This suggests writing

\[ \Pi^{ij} = \tilde{\Pi}^{ij} \left[ 1 + \frac{1}{3} \delta^{ij} \Pi^{kl} \right] \]

(11)

\( \tilde{\Pi}^{ij} = 0 \). Then

\[ \frac{v_i v_j}{c^2} \Pi^{ij} = \frac{v_i v_j}{c^2} \tilde{\Pi}^{ij} + \frac{1}{3c^4} v^2 v_i v_j \Pi^{ij} \]

(12)

and

\[ \Pi^{ij} = \tilde{\Pi}^{ij} \left[ 1 + \frac{1}{3c^2} \delta^{ij} \frac{v_k v_l \tilde{\Pi}^{kl}}{1 - \frac{4}{3} \frac{v^2}{c^2}} \right] \]

(13)

We introduce these variables in the equations of motion keeping only quadratic nonlinearities. Observe that \( b^\nu \) is already quadratic in \( v^\mu \) and \( a \) is cubic. So we have
\[ 0 = c \delta' + \frac{4}{3} v^j_j + \Pi^{jk} v_{j,k} \]
\[ 0 = \frac{4}{3} c v^j_j - \frac{4}{9} v^j_j v_{k,j} + \frac{1}{3} c^2 \delta^j_j + (v_k \Pi^{jk})_{,t} + c^2 \Pi^{jk} + c^2 \Pi^{jk} \delta^j_j \]
\[ 0 = c \Pi^{jk} + \left[ \frac{c}{\lambda} - \frac{10}{21} v^j_j \right] \Pi^{jk} + \frac{4}{15} \left[ v^{j,k} + v^{j,j} - \frac{2}{3} \delta^{j,k} v^j_j \right] \]
\[ + \frac{6}{7} \left( \Pi^{kl} v^j_l + \Pi^{kl} v^j_l - \frac{2}{3} \delta^{j,k} \Pi^{lm} v_{l,m} \right) - \frac{1}{7} \left( \Pi^{kl} v^j_l + \Pi^{kl} v^j_l - \frac{2}{3} \delta^{j,k} \Pi^{lm} v_{l,m} \right) \]
\[ - \frac{1}{5} \left( v^j \left( \frac{1}{3} \delta^j_j + \Pi^{jl} \right) + v^k \left( \frac{1}{3} \delta^j_j + \Pi^{kl} \right) - \frac{2}{3} \delta^{j,k} v_m \left( \frac{1}{3} \delta^m_m + \Pi^{ml} \right) \right) \]  

(14)

In the only quadratic term still containing time derivatives, we use the linear equations to get

\[ 0 = c \delta' + \frac{4}{3} v^j_j + \Pi^{jk} v_{j,k} \]
\[ 0 = c v^j_j - \frac{1}{5} v^j_j v_{k,j} + \frac{1}{4} c^2 \delta^j_j - \frac{3c}{4\lambda} v_k \Pi^{jk} - \frac{1}{5} v_k \left( v^{j,k} + v^{k,j} \right) + \frac{3}{4} c^2 \Pi^{jk} + \frac{9}{16} c^2 \Pi^{jk} \delta^j_j \]
\[ 0 = c \Pi^{jk} + \left[ \frac{c}{\lambda} - \frac{10}{21} v^j_j \right] \Pi^{jk} + \frac{4}{15} \left[ v^{j,k} + v^{j,j} - \frac{2}{3} \delta^{j,k} v^j_j \right] \]
\[ + \frac{6}{7} \left( \Pi^{kl} v^j_l + \Pi^{kl} v^j_l - \frac{2}{3} \delta^{j,k} \Pi^{lm} v_{l,m} \right) - \frac{1}{7} \left( \Pi^{kl} v^j_l + \Pi^{kl} v^j_l - \frac{2}{3} \delta^{j,k} \Pi^{lm} v_{l,m} \right) \]
\[ - \frac{1}{5} \left( v^j \left( \frac{1}{3} \delta^j_j + \Pi^{jl} \right) + v^k \left( \frac{1}{3} \delta^j_j + \Pi^{kl} \right) - \frac{2}{3} \delta^{j,k} v_m \left( \frac{1}{3} \delta^m_m + \Pi^{ml} \right) \right) \]  

(15)

### III. ENTROPY

In this model, the entropy flux

\[ S^\mu = S (\rho, x, y) u^\mu \]

where \( x = \text{tr} \, \Pi^2, \ y = \text{tr} \, \Pi^3 \) so

\[ S^\mu_\mu = \dot{S} + S \theta \]  

(17)

On dimensional grounds, \( S = \rho^{3/4} s (x, y) \), so

\[ S^\mu_\mu = \rho^{3/4} \left\{ s \left[ \frac{3}{4} \delta + \theta \right] + 2 \Pi^{\nu \rho} \Pi_{\nu \rho} \frac{\partial s}{\partial x} + 3 \Pi^{\nu \rho} \Pi_{\nu \rho} \frac{\partial s}{\partial y} \right\} \]
\[ = \rho^{3/4} \left\{ - \frac{3}{4} \Pi_{\mu \nu} u_{\mu, \nu} s + 2 \Pi^{\nu \rho} \Pi_{\nu \rho} \frac{\partial s}{\partial x} + 3 \Pi^{\nu \rho} \Pi_{\nu \rho} \frac{\partial s}{\partial y} \right\} \]  

(18)

Write to order \( \Pi^{\mu \nu}_{\mu} \)

\[ s = s_0 e^{-\alpha x - \beta y} \]

(19)

Then, keeping only terms quadratic in \( \Pi^{\mu \nu} \)

\[ S^\mu_\mu = \rho^{3/4} s_0 \left\{ \frac{3}{4} \Pi^{\mu \nu} u_{\mu, \nu} + 2 \alpha \left[ \frac{1}{\lambda} - \frac{10}{21} \theta \right] x + \frac{8}{15} \Pi^{\rho \sigma} u_{\rho, \sigma} + \frac{10}{7} \Pi^{2 \rho \sigma} u_{\rho, \sigma} \right\} \]
\[ + \frac{8}{5} \beta \left[ \Pi^{2, \rho \sigma} u_{\rho, \sigma} - \frac{1}{3} \theta x \right] e^{-\alpha x - \beta y} \]  

(20)

so, setting

\[ \alpha = \frac{45}{64} \]
\[ \beta = -\frac{25}{14} \alpha \]  

(21)
We get positive entropy production

$$S_{\mu}^\nu = \frac{45}{32\lambda^3/4}s_0 \Pi^{\rho\nu}\Pi_{\mu\nu}$$

(22)

In the incompressible limit where the scalar degrees of freedom are frozen, to be discussed more fully in next section, the mean entropy flux is given by

$$cS_{\mu}^k = (u^0v^k)_{,k} \approx -s \left[ \frac{1}{c^2} \epsilon + \frac{\alpha}{\tau} \right]$$

(23)

where

$$\epsilon = \langle v^k (v^j v_j) \rangle_{,k}$$

$$\tau^{-1} = \langle v^k (\bar{\Pi}^i \bar{\Pi}_i) \rangle_{,k}$$

(24)

They are defined by the limits

$$\epsilon = \lim_{x \rightarrow z} \frac{\partial}{\partial x} \langle v^j (v^k v_j) (z) \rangle$$

$$\tau^{-1} = \lim_{x \rightarrow z} \frac{\partial}{\partial x} \langle \bar{\Pi}^i (v^k \bar{\Pi}_i) (z) \rangle$$

(25)

where one must take the derivative first, then the limit (in the opposite order we get zero). $\epsilon$ corresponds to the nonrelativistic cascade; although one may use these equations to work out the relativistic corrections to the Kolmogorov spectrum, we shall not discuss it further. On the other hand, $\tau^{-1}$ corresponds to a new kind of cascade with no nonrelativistic analog.

**IV. HOMOGENEOUS ISOTROPIC FLOWS**

We shall now investigate the structure of the velocity and viscous energy momentum tensor correlations in a statistically homogeneous, isotropic flow.

It is convenient to separate the equations of motion into scalar, vector and tensorial equations. For this end, we define

$$v_j = \phi_j + V_j$$

$$\bar{\Pi}_{ij} = \psi_{ij} - \frac{1}{3} \delta_{ij} \Delta \psi + q_{i,j} + q_{j,i} + Q_{ij}$$

(26)

where

$$V^j = q^j = Q^j_{,k} = Q^j = 0$$

(27)

The linearized equations of motion decouple, so we have a theory with scalar degrees of freedom $S_\alpha$, vector degrees of freedom $V_\alpha$ and a tensorial degree of freedom $\bar{\Pi}$. These modes propagate at different velocities \[62, 63\]. For our model we get \[3\sqrt{2}/5c\] for the scalar modes, \[1/\sqrt{5}c\] for vector modes, and 0 for tensor modes. In other words, the tensor sector are the “slow” degrees of freedom, and there is an incompressible limit where scalar modes are frozen, and vector modes are slaved to tensor ones. We shall work within this regime in what follows, namely, we shall disregard the scalar degrees of freedom and assume that $\bar{\Pi}^{jk}$ contains tensor degrees of freedom only. Then the EOMs become

$$0 = v^j - \frac{3c}{4\lambda} v_k \bar{\Pi}^{jk} - \frac{1}{5} v^k \left[ v_{,j}^{,k} + v^{k,j} \right]$$

$$0 = \bar{\Pi}^{jk} + \frac{6}{7} \bar{\Pi}^{jl} - \frac{2}{3} \bar{\Pi}^{kl} = \frac{2}{3} \bar{\Pi}^{lm} v_{i,m}$$

$$- \frac{1}{7} \left[ \bar{\Pi}^{kl} v^l_{,j} + \bar{\Pi}^{kl} v^j_{,l} + 2 \bar{\Pi}^{lm} v_{i,m} - \frac{2}{3} \bar{\Pi}^{lm} \bar{\Pi}^{jk} \right]$$

(28)

The correlation functions most relevant to our discussion are the two point function

$$\langle \bar{\Pi}^{ij} (x,t) \bar{\Pi}^{kl} (y,t) \rangle = \int \frac{dp}{(2\pi)^3} \delta^{ij} e^{ip(x-y)} F^{ijkl} (p,t)$$

(29)
and

$$
\langle \tilde{\Pi}^i (x, t) \tilde{\Pi}^j (y, t) v^k (z, t) \rangle = \int \frac{d^3p d^3q}{(2\pi)^6} e^{i[p(x-z)+q(y-z)]} G^{ijk} (p, q, t) \tag{30}
$$

Both are severely restricted by the isotropy assumption. $F^{ijkl} (p, t)$ has to be symmetric in $(i, j)$, $(k, l)$ and under the exchange of these pairs by one another. Also $p_i F^{ijkl} = F^{ijkl} = 0$. So we must have

$$
F^{ijkl} (p, t) = \frac{1}{2} [h_p^{ik} h_p^{jl} + h_p^{il} h_p^{jk} - h_p^{ij} h_p^{kl}] f (p, t) \tag{31}
$$

where $p = |p|$ and

$$
h_p^{ij} = \delta^{ij} - \frac{p^i p^j}{p^2} \tag{32}
$$

Observe that $f (p, t)$ may be obtained as the Fourier transform of the fully contracted correlation

$$
\langle \tilde{\Pi}^i (x, t) \tilde{\Pi}^j (y, t) \rangle = 2 \int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} f (p, t) \tag{33}
$$

$G^{ijk}$ obeys $p_i G^{ijk} = q_j G^{ijk} = (p + q)_k G^{ijk} = 0$ and $G^{ijk} (p, q, t) = G^{ijk} (q, p, t)$. Therefore

$$
G^{ijk} (p, q, t) = h_p^{il} h_q^{jm} h_{(p+q)}^{kn} g^{lmn} (p, q, t) \tag{34}
$$

where

$$
g^{lmn} (p, q, r) = g_1 q^p p^m (p - q)^n + g_2 q^p \delta^{mn} + g_2 T p^m q^n + g_3 (p - q)^n \delta^{lm} \tag{35}
$$

The coefficients are functions of $|p|$, $|q|$ and $pq = p q$. We observe the contractions

$$
G^{ijk} = h_{(p+q)}^{kn} (p - q)_n \left[ -g_1 (pq) \left( 1 - \frac{(pq)^2}{p^2 q^2} \right) - \frac{1}{2} g_2 \frac{(pq)}{p^2} \left( 1 + \frac{(pq)}{q^2} \right) \right]
$$

$$
+ \frac{1}{2} g_2 T \frac{(pq)}{q^2} \left( 1 + \frac{(pq)}{p^2} \right) + g_3 \left( 1 + \frac{(pq)^2}{p^2 q^2} \right) \right] \tag{36}
$$

$$
G^{ikj} = h_{q_m p_m}^{pm} \left[ -\frac{1}{2} g_1 \Delta \left( 1 + \frac{(pq)}{p^2} \right) - \frac{1}{2} g_2 \frac{q^2}{p^2} \frac{(pq)}{p^2} \left( 1 + \frac{(pq)}{p^2} \right) \left( 1 + \frac{(pq)}{q^2} \right) \right]
$$

$$
+ 2 g_2 T \left( 1 - \frac{\Delta}{8 p^2} \right) + g_3 \frac{(pq)}{(p+q)^2} \left( 1 + \frac{(pq)}{p^2} \right) \right] \tag{36}
$$

$$
G^{ij} = h_p^{il} q_i \left[ \frac{1}{2} g_1 \Delta \left( 1 + \frac{(pq)}{q^2} \right) + 2 g_2 \frac{(pq)}{p^2} \left( 1 - \frac{\Delta}{8 q^2} \right) \right]
$$

$$
- \frac{1}{2} g_2 T \frac{p^2}{(p+q)^2} \left( 1 + \frac{(pq)}{p^2} \right) \left( 1 + \frac{(pq)}{q^2} \right) - 2 g_3 \frac{(pq)}{(p+q)^2} \left( 1 + \frac{(pq)}{q^2} \right) \right] \tag{36}
$$

where

$$
\Delta = (p - q)^k h_{(p+q)}^{kn} (p - q)_n = \frac{4 p^2 q^2}{(p+q)^2} \left( 1 - \frac{(pq)^2}{p^2 q^2} \right) \tag{37}
$$

V. VON KÁRMÁN-HOWARTH EQUATIONS

The above correlations are further constrained by the relativistic von Kármán-Howarth equations, namely
\[- \left[ \frac{d}{dt} + \frac{2c}{\lambda} \right] \langle \Pi^{jk} (x, t) \Pi^{jk} (z, t) \rangle = \langle \Pi^{jk} (x, t) \left( v^{k} \Pi^{jk} \right) (z, t) \rangle \]

\[+ \langle \Pi^{jk} (z, t) \left( v^{k} \Pi^{jk} \right) (x, t) \rangle + \frac{12}{7} \langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^{l} \right) (z, t) \rangle + \frac{12}{7} \langle \Pi^{jk} (z, t) \left( \Pi^{jl} v^{l} \right) (x, t) \rangle \]

\[- \frac{2}{7} \langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^{l} \right) (z, t) \rangle - \frac{2}{7} \langle \Pi^{jk} (z, t) \left( \Pi^{jl} v^{l} \right) (x, t) \rangle \]

(38)

Now

\[\langle \Pi^{jk} (x, t) \left( v^{k} \Pi^{jk} \right) (z, t) \rangle = (-i) \int \frac{d^{3}p}{(2\pi)^{3}} e^{ip(x-z)} p_{k} \int \frac{d^{3}q}{(2\pi)^{3}} G^{jk} (p, q, t) \]

(39)

It is clearly symmetric in \((x, z)\). Similarly

\[\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^{l} \right) (z, t) \rangle = (-i) \int \frac{d^{3}p}{(2\pi)^{3}} e^{ip(x-z)} p_{k} \int \frac{d^{3}q}{(2\pi)^{3}} G^{kj} (p, q, t) \]

(40)

\[\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^{l} \right) (z, t) \rangle = (-i) \int \frac{d^{3}p}{(2\pi)^{3}} e^{ip(x-z)} \int \frac{d^{3}q}{(2\pi)^{3}} q_{k} G^{kjj} (p, q, t) \]

(41)

so the von Kármán-Howarthing equation in momentum domain reads

\[\frac{1}{2} \left[ \frac{d}{dt} + \frac{2c}{\lambda} \right] f (p, t) = i \int \frac{d^{3}q}{(2\pi)^{3}} \left[ p_{k} G^{ijk} (p, q, t) + \frac{12}{7} p_{k} G^{kj} (p, q, t) - \frac{2}{7} q_{k} G^{kjj} (p, q, t) \right] \]

(42)

In terms of the decomposition eqs. (41) and (42)

\[p_{k} G^{ijk} = \frac{1}{4} \frac{\Delta}{p^{2}q^{2}} \left[ -2g_{1} \left( pq \right) \left( p^{2}q^{2} - \left( pq \right)^{2} \right) + 2g_{3} \left( p^{2}q^{2} + \pi \right)^{2} - g_{2} \left( pq \right) \left( q^{2} + pq \right) + g_{2}^{2} \left( pq \right) \left( p^{2} + pq \right) \right] \]

\[p_{k} G^{kj} = -\frac{1}{4} \frac{\Delta}{p^{2}q^{2}} \left[ 2g_{1} \left( p^{2}q^{2} - \left( pq \right)^{2} \right) \left[ p^{2} + pq \right] + g_{2} \left( q^{2}p^{2} + \left( pq \right) p^{2} + \left( pq \right) q^{2} + \left( pq \right)^{2} \right] \right] \]

\[q_{k} G^{kjj} = \frac{1}{4} \frac{\Delta}{p^{2}q^{2}} \left[ 2g_{1} \left( p^{2}q^{2} - \left( pq \right)^{2} \right) \left( q^{2} + pq \right) + g_{2} \left( p^{2}q^{2} + 4q^{2} + 4q^{2} \left( pq \right) \left( pq \right)^{2} \right) \right] \]

\[\quad - g_{2}^{2} \left( q^{2}p^{2} + \left( pq \right) p^{2} + \left( pq \right) q^{2} + \left( pq \right)^{2} \right) - 2g_{3} \left( pq \right) \left[ q^{2} + pq \right] \]

(43)

where \((pq) = p_{i}q^{i}\)

VI. FULLY DEVELOPED TURBULENCE

Fully developed turbulence is the case where the dynamically generated dimensionful variable \(\tau\) defined in eq. (25) is scale independent,

\[\frac{\partial}{\partial x^{k}} \langle \Pi^{ij} (x) \left( v^{k} \Pi^{ij} \right) (z) \rangle = \tau^{-1} = \text{constant} \]

(44)

whereby

\[p_{k} \int \frac{d^{3}q}{(2\pi)^{3}} G^{ijk} (p, q, t) = (-i) \tau^{-1} \delta (p) \]

(45)

or else

\[\int \frac{d^{3}q}{(2\pi)^{3}} G^{ijk} (p, q, t) = (-i) \tau^{-1} \frac{p^{k}}{p^{2}} \delta (p) \]

(46)
These equations are the relativistic counterpart to the Kolmogorov 4/5 law.

Since $\tau$ has dimensions of time, the simplest dimensionally correct ansatz for $\langle \Pi^{ij} (x) \Pi^{kl} (y) v^m (z) \rangle$ would be linear in $r = x - z$ and $s = y - z$. Imposing the relevant symmetry, tracelessness and divergencelessness constraints, we obtain the most general form

$$
\langle \Pi^{ij} (x) \Pi^{kl} (y) v^m (z) \rangle = \frac{A}{\tau} \left\{ G^{(ij),(kl),m} (r) + G^{(kl),(ij),m} (s) \right\}
$$

where

$$
G^{(ij),(kl),m} (r) = r^i \left[ \delta^{jk} \delta^{lm} + \delta^{jl} \delta^{km} - 6 \delta^{lm} \delta^{kl} \right] + r^j \left[ \delta^{ik} \delta^{lm} + \delta^{il} \delta^{km} - 6 \delta^{km} \delta^{il} \right] + r^k \left[ 8 \delta^{ij} \delta^{lm} + \delta^{ij} \delta^{km} \right] + r^m \left[ -6 \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} + 8 \delta^{ij} \delta^{kl} \right]
$$

which reduces to

$$
\langle \Pi^{ij} (x) \Pi^{kl} (y) v^k (z) \rangle = \frac{7A}{\tau} \left\{ -r^i \delta^{jk} - r^k \delta^{ij} + 4r^j \delta^{ik} - 8s^i \delta^{jk} + 4s^j \delta^{ik} \right\}
$$

Fourier transforming, we obtain a representation eqs. (34) and (35) as

$$
g_1 = 0
$$

$$
g_2 = \frac{28}{4\pi p^3} \delta (p) \left( \delta^\prime (q) - \frac{2}{q} \delta (q) \right)
$$

$$
g_3 = \frac{7}{4\pi} \left[ \frac{1}{q^2} \left( \delta^\prime (q) - \frac{2}{q} \delta (q) \right) \delta (p) - \frac{1}{p^2} \left( \delta^\prime (p) - \frac{2}{p} \delta (p) \right) \delta (q) \right]
$$

We therefore find

$$
\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^l \right) (z, t) \rangle = \langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^l \right) (z, t) \rangle = -30 \frac{7A}{\tau}
$$

But

$$
\langle \Pi^{jk} (x, t) \left( v^l \Pi^{lk} \right) (z, t) \rangle = 0
$$

There is a similar situation in nonrelativistic incompressible turbulence, because K41 theory requires $\langle v^l (x + r) (v_j v^k) (x) \rangle \propto r^k$, but an actual computation shows that $\langle v^l (x + r) (v_j v^k) (x) \rangle = O (r^3)$ when $r \to 0$.

This shows that the ansatz eq. (49) is too naive and must be modified, and eq. (50) suggests the modification must be to include a non zero $g_1$. Then the following situation arises. $g_1$ contributes to $\langle \Pi^{jk} (x, t) \left( v^l \Pi^{lk} \right) (z, t) \rangle$ only through the part of $(pq) \Delta g_1$ which is even in $q$. If $g_1$ is chosen such that there is no odd part, namely $(pq) \Delta g_1$ is an odd function of $q$, then its contribution to $\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^l \right) (z, t) \rangle$ is the same as to $\langle \Pi^{jk} (x, t) \left( v^l \Pi^{lk} \right) (z, t) \rangle$, and the same holds for $\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^l \right) (z, t) \rangle$, but for the sign. So if $g_1$ is chosen in this way, leaving $g_2$ and $g_3$ as in eq. (50), then we may assert that

$$
\langle \Pi^{jk} (x, t) \left( v^l \Pi^{lk} \right) (z, t) \rangle = -\tau^{-1}
$$

$$
\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^l \right) (z, t) \rangle = -\left( 1 + 210 A \right) \tau^{-1}
$$

$$
\langle \Pi^{jk} (x, t) \left( \Pi^{jl} v^l \right) (z, t) \rangle = \left( 1 - 210 A \right) \tau^{-1}
$$

and the von Karman-Howarth equation becomes

$$
\frac{1}{2} \left[ \frac{d}{dt} + \frac{2c_0}{\lambda} \right] \langle \Pi^{jk} (x, t) \Pi^{jk} (z, t) \rangle = (3 + 100A) \frac{1}{\tau}
$$
so we get
\[
\langle \Pi^j_k (x, t) \Pi^{j_k} (z, t) \rangle \approx \frac{\lambda}{c_T} \tag{54}
\]
Comparing with eq. (22) we recognize this as a relationship between entropy production and transport
\[
\langle S^\mu_\mu \rangle = \frac{45}{32} \rho^{3/4} s_0 (3 + 100A) \frac{1}{c_T} \tag{55}
\]

VII. FINAL REMARKS

In this comment we have intended to show that the richer dynamics of real relativistic fluids vis a vis their nonrelativistic counterparts allows for new kinds of turbulent phenomena, and have provided a schematic description of how turbulence driven by tensor degrees of freedom may look like in the simple case of homogeneous, isotropic, fully developed turbulence.

Admittedly these patterns of flow seem very unlikely to be observed in actual experiments (for the more prevalent case of thermal fluctuations see [38]). However, if they exist at all, they may influence the behavior of a relativistic fluid, not unlike a nonthermal fixed point affects the whole renormalization group flow in nonequilibrium field theory [70–72].

Our long term goal is to assemble a turbulence toolkit to assist in the development of phenomenological models of reheating after inflation [73]. This is certainly the most violent and far from equilibrium process in the history of the Universe. As suggested some time ago [74], turbulence may provide a qualitative understanding of this process, which complements full scale numerical simulations already under way [73].

Appendix A: Motivating eq. (5)

We consider the hydrodynamical description for a gas of massless particles obeying a Boltzmann equation with an Anderson-Witting collision term [76–78]
\[
p^\mu f_\mu = \frac{1}{\lambda} u^\mu p_\mu [f - f_0] \tag{A1}
\]
where \( f_0 \) is an equilibrium 1pdf
\[
f_0 = e^{\beta_0 p_\mu} \tag{A2}
\]
and
\[
u_\mu [T^{\mu\nu} - T_0^{\mu\nu}] = 0 \tag{A3}
\]
thus enforcing EMT conservation.

Our primary concern is that the hydrodynamic description should enforce energy-momentum conservation and positive entropy production. One simple way to achieve the second requirement is to postulate an ansatz [37]
\[
f = e^{\sum_A \zeta_A \varphi_A (p, x)} \tag{A4}
\]
where the \( \varphi_A (p, x) \) are known functions of position and momentum, and the \( \zeta_A \) will be the hydrodynamic variables. The equations of motion are obtained from the moments of the kinetic equation
\[
\int \frac{2 d^4 p}{(2\pi)^3} \delta (-p^2) \theta (p^0) \varphi_A (p, x) \left\{ p^\mu f_\mu - \frac{1}{\lambda} u^\mu p_\mu [f - f_0] \right\} = 0 \tag{A5}
\]
Then positive entropy production follows from the kinetic theory \( H \) theorem. The equations of motion are conservation laws for currents
\[
A^\Lambda_\mu = \int \frac{2 d^4 p}{(2\pi)^3} \delta (-p^2) \theta (p^0) \varphi_A (p, x) p^\mu f_\mu \tag{A6}
\]
By choosing $\varphi^0 = p^\mu$, $\zeta^0 = \beta^0$, we enforce energy momentum conservation as one of the equations of motion. Our second choice will be $\varphi^1 = p^\mu p^\nu/(-U_\mu p^\mu)$. $U^\mu$ is a fiducial unit vector to be identified with $u^\mu$ after deriving the equations of motion, and we assume $\zeta_\mu^\mu = U_\mu \zeta^{\mu\nu} = 0$ for a conformal fluid. This choice has been very successful in reproducing flows in Bjorken and Gubser backgrounds [37]. We get the currents

$$T^{\mu\nu} = \frac{\partial \Phi^\mu}{\partial \beta^\nu} = \frac{2}{(2\pi)^3} \int d^4p \, \delta(-p^2) \, \theta(p^0) \, p^\mu p^\nu f$$

$$A^{\mu\nu\rho} = \frac{\partial \Phi^\mu}{\partial \zeta^\nu_{\rho}} = \frac{2}{(2\pi)^3} \int d^4p \, \delta(-p^2) \, \theta(p^0) \, p^\mu \frac{p^\nu p^\rho}{(-U_\theta p^\theta)} f$$

$$S^\mu = \frac{2}{(2\pi)^3} \int d^4p \, \delta(-p^2) \, \theta(p^0) \, p^\mu [1 - \ln f] f$$

(A7)

$T^{\mu\nu}$ and $A^{\mu\nu\rho}$ are symmetric and traceless on any pair of indexes, and $u_\mu A^{\mu\nu\rho} = -T^{\nu\rho}$. The fluid equations are

$$\frac{T^{\mu\nu}}{\zeta^\nu} = 0$$

$$S^{\lambda\chi_{\nu\rho}} \left[ A^{\mu\nu\rho} - K^{\mu\nu\rho\sigma} U_{\sigma,\mu} + \frac{1}{\chi} T^{\nu\rho} \right] = 0$$

(A8)

where

$$S^{\lambda\chi_{\nu\rho}} = \frac{1}{2} \left[ \Delta_{\lambda\chi} \Delta_{\lambda\rho} + \Delta_{\lambda\rho} \Delta_{\lambda\chi} - \frac{2}{3} \Delta_{\lambda\chi} \Delta_{\lambda\rho} \right]$$

(A9)

$(\Delta_{\mu\nu}$ as in eq. (2)) and

$$K^{\mu\nu\rho\sigma} = \frac{2}{(2\pi)^3} \int d^4p \, \delta(-p^2) \, \theta(p_0) \, p^\mu \frac{p^\nu p^\rho p^\sigma}{(-U_\theta p^\theta)^2} f$$

(A10)

The fluid equations imply

$$S^{\mu}_{\nu\rho} = \frac{1}{\chi} \zeta_{\nu\rho} T^{\nu\rho}$$

(A11)

and so the Second Law is enforced as long as

$$\zeta_{\nu\rho} T^{\nu\rho} \geq 0$$

(A12)

Although we know that positive entropy production follows from the kinetic theory $H$ theorem, we may also make a direct check. In a frame where $\zeta_{ij} = \text{diag}(\zeta_+ + \zeta_-, \zeta_+ - \zeta_-, -2\zeta_+)$ and writing

$$p^i = p (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

(A13)

we have

$$\zeta_{\nu\rho} T^{\nu\rho} = \tilde{\zeta} \cdot \nabla J (\tilde{\zeta})$$

(A14)

where $\tilde{\zeta} = (\zeta_+, \zeta_-)$ and

$$J = \int \frac{p^2 dp}{(2\pi)^3} e^{-p/T} \int_0^\pi d\theta \, \sin \theta \, e^{p \zeta_+ (1 - 3 \cos^2 \theta)} \int_{-\pi}^\pi d\varphi \, e^{p \zeta_- \sin^2 \theta \cos 2\varphi}$$

(A15)

If $\tilde{\zeta} = \zeta \tilde{\zeta}$, $\tilde{\zeta}^2 = 1$, then

$$\zeta_{\nu\rho} T^{\nu\rho} = \zeta \frac{d}{d\zeta} J (\zeta \tilde{\zeta})$$

(A16)

and the positivity follows from

$$\frac{d}{d\zeta} J (0) = 0$$

$$\frac{d^2}{d\zeta^2} J (\zeta \tilde{\zeta}) \geq 0$$

(A17)
Let us analyze the hydrodynamic currents more closely. As usual, we may decompose the energy-momentum tensor as in eq. (1). Then, the requirement that

\[ u_\mu A^{\mu\nu\rho} = -T^{\nu\rho} \]  

(A18)

Implies

\[ A^{\mu\nu\rho} = \rho \left\{ u_\mu u_\nu u_\rho + \frac{1}{3} \left( u_\mu \Delta^{\nu\rho} + u_\nu \Delta^{\mu\rho} + u_\rho \Delta^{\mu\nu} \right) + u_\mu \Pi^{\nu\rho} + u_\nu \Pi^{\mu\rho} + u_\rho \Pi^{\mu\nu} \right\} \]  

(A19)

Energy conservation leads to eqs. (4), and

\[ S^{\nu\rho\lambda\chi} A^\mu_{\lambda \chi \mu} = \rho \left\{ \Pi^{\nu\rho} - \frac{1}{3} \theta \Pi^{\nu\rho} + \frac{1}{3} \left( \Delta^{\mu\rho} u_\mu + \Delta^{\mu\nu} u_\mu - \frac{2}{3} \Delta^{\nu\rho} \theta \right) + \Pi^{\mu\rho} u_\mu + \Pi^{\rho\mu} u_\nu \right\} + \frac{2}{3} \Delta^{\nu\rho} \Pi^{\mu\sigma} u_{\mu,\sigma} + \frac{1}{4} \left[ u_\nu \Pi^{\mu\rho} + u_\rho \Pi^{\nu\mu} \right] \delta_{\mu, \nu} \right\} \]  

(A20)

Where we have used the equations for \( \dot{\rho} \) and \( \dot{u}^\nu \) disregarding quadratic terms in \( \Pi^{\mu\nu} \).

It only remains to compute \( K^{\mu\nu\rho\sigma} \) up to linear terms in \( \Pi^{\mu\nu} \). Let us write the generic form

\[ K^{\mu\nu\rho\sigma} = A u_\mu u_\nu u_\rho u_\sigma + B [u_\mu u_\nu \Delta^{\rho\sigma} + u_\mu u_\rho \Delta^{\nu\sigma} + u_\mu u_\sigma \Delta^{\rho\nu} + u_\nu u_\rho \Delta^{\mu\sigma} + u_\rho u_\sigma \Delta^{\mu\nu} + u_\sigma u_\rho \Delta^{\mu\nu} + u_\nu u_\sigma \Delta^{\rho\mu} + u_\nu u_\rho \Delta^{\mu\sigma} + u_\rho u_\sigma \Delta^{\mu\nu}] + \rho \left\{ \Pi^{\mu\rho} + \Pi^{\rho\mu} + \Pi^{\rho\mu} + \Pi^{\mu\rho} + \Pi^{\mu\rho} + \Pi^{\mu\rho} + \Pi^{\mu\rho} + \Pi^{\mu\rho} \right\} \]  

(A21)

The condition that \( u_\mu K^{\mu\nu\rho\sigma} = -A^{\nu\rho\sigma} \) reduces this to

\[ K^{\mu\nu\rho\sigma} = \rho \left\{ u_\mu u_\nu u_\rho u_\sigma + \frac{1}{3} \rho \left[ u_\mu u_\nu \Delta^{\rho\sigma} + u_\mu u_\rho \Delta^{\nu\sigma} + u_\mu u_\sigma \Delta^{\rho\nu} + u_\nu u_\rho \Delta^{\mu\sigma} + u_\sigma u_\rho \Delta^{\mu\nu} + u_\nu u_\sigma \Delta^{\rho\mu} + u_\rho u_\sigma \Delta^{\mu\nu} + u_\nu u_\rho \Delta^{\mu\sigma} + u_\rho u_\sigma \Delta^{\mu\nu} \right] + \frac{1}{3} \rho \left[ u_\mu u_\nu \Pi^{\rho\sigma} + u_\mu u_\rho \Pi^{\nu\sigma} + u_\mu u_\sigma \Pi^{\rho\nu} + u_\nu u_\rho \Pi^{\mu\sigma} + u_\sigma u_\rho \Pi^{\mu\nu} + u_\nu u_\sigma \Pi^{\rho\mu} + u_\rho u_\sigma \Pi^{\mu\nu} + u_\nu u_\rho \Pi^{\mu\sigma} + u_\rho u_\sigma \Pi^{\mu\nu} \right] + \frac{1}{4} \left[ u_\nu \Pi^{\mu\rho} + u_\rho \Pi^{\nu\mu} \right] \right\} \]  

(A22)

Tracelessness implies

\[ 0 = -\frac{1}{3} \rho \Delta^{\rho\sigma} + 5D \Delta^{\rho\sigma} \]

\[ 0 = -\rho + 7E \]  

(A23)

We therefore may write

\[ K^{\mu\nu\rho\sigma} = \rho \left\{ u_\mu u_\nu u_\rho u_\sigma + \frac{1}{3} \rho \left[ u_\mu u_\nu \Delta^{\rho\sigma} + u_\mu u_\rho \Delta^{\nu\sigma} + u_\mu u_\sigma \Delta^{\rho\nu} + u_\nu u_\rho \Delta^{\mu\sigma} + u_\sigma u_\rho \Delta^{\mu\nu} + u_\nu u_\sigma \Delta^{\rho\mu} + u_\rho u_\sigma \Delta^{\mu\nu} + u_\nu u_\rho \Delta^{\mu\sigma} + u_\rho u_\sigma \Delta^{\mu\nu} \right] + \frac{1}{3} \rho \left[ u_\mu u_\nu \Pi^{\rho\sigma} + u_\mu u_\rho \Pi^{\nu\sigma} + u_\mu u_\sigma \Pi^{\rho\nu} + u_\nu u_\rho \Pi^{\mu\sigma} + u_\sigma u_\rho \Pi^{\mu\nu} + u_\nu u_\sigma \Pi^{\rho\mu} + u_\rho u_\sigma \Pi^{\mu\nu} + u_\nu u_\rho \Pi^{\mu\sigma} + u_\rho u_\sigma \Pi^{\mu\nu} \right] + \frac{1}{15} \left[ \Delta^{\rho\sigma} \Pi^{\mu\nu} + \Delta^{\mu\rho} \Pi^{\nu\sigma} + \Delta^{\nu\sigma} \Pi^{\mu\rho} + \Delta^{\mu\rho} \Pi^{\nu\sigma} + \Delta^{\nu\sigma} \Pi^{\mu\rho} + \Delta^{\rho\sigma} \Pi^{\mu\nu} \right] \right\} \]  

(A24)

This contributes to the equations of motion a term

\[ S^\chi_{\lambda \mu \nu \rho \sigma} K^{\chi \lambda \rho \sigma} u_{\rho, \sigma} = \rho \left\{ \frac{1}{15} \left[ \Delta^{\rho\sigma} u_\mu + \Delta^{\mu\sigma} u_\nu - \frac{2}{3} \Delta^{\nu\rho} \theta \right] \right\} + \frac{1}{7} \left[ \Pi^{\rho\sigma} u_\mu + \Pi^{\rho\mu} u_\nu + \Delta^{\mu\sigma} \Pi^{\nu\rho} u_{\rho, \sigma} + \Delta^{\nu\sigma} \Pi^{\rho\mu} u_{\rho, \sigma} + \Pi^{\nu\rho} \theta - \frac{4}{3} \Delta^{\nu\rho} \Pi^{\mu\sigma} u_{\rho, \sigma} \right\} \]  

(A25)

We thus obtain eq. (5).
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