CLASSIFICATION OF THE \(d\)-REPRESENTATION-Finite TRIVIAL EXTENSIONS OF QUIVER ALGEBRAS

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ABSTRACT. We give a complete classification of all \(d\)-representation-finite symmetric Nakayama algebras, and of all \(d\)-representation-finite trivial extensions of path algebras of quivers, over an arbitrary field.

1. INTRODUCTION

Let \(k\) be a field. We denote by \(T(A)\) the trivial extension algebra of a finite-dimensional \(k\)-algebra \(A\), and by \(kQ\) the path algebra of a quiver \(Q\). The purpose of this note is to prove the following two results.

**Theorem 1.1.** Let \(Q\) be a connected acyclic quiver, and \(d\) an integer greater than or equal to 2. Then \(T(kQ)\) is \(d\)-representation-finite if and only if one of the following holds.

(a) The quiver \(Q\) is of Dynkin type \(A_3\) or \(A_6\), and \(d = 2\);
(b) \(Q\) is of Dynkin type \(A_n\), and \(d = 2n - 1\);
(c) \(Q\) is of Dynkin type \(D_4\), and \(d = 4\).

**Theorem 1.2.** Let \(B\) be a ring-indecomposable symmetric Nakayama algebra with \(n\) simple modules and Loewy length \(\ell(B) = an + 1, a \geq 1\), and \(d\) an integer greater than or equal to 2. Then \(B\) is \(d\)-representation-finite if and only if

\[(a, n, d) \in \{(1, t, 2t - 1) \mid t \geq 2\} \cup \{(1, 3, 2), (1, 6, 2), (2, 3, 2)\}.

Here, an algebra is said to be \(d\)-representation-finite if it has a \(d\)-cluster-tilting module.

Understanding and classifying \(d\)-representation-finite algebras are important problems in Iyama’s higher-dimensional Auslander–Reiten theory [13, 14, 15]. While many examples exist of \(d\)-representation-finite algebras that have finite global dimension (e.g., [10, 11, 18, 21]), or are self-injective [4, 12, 16], only a few isolated examples were previously known of symmetric algebras with a \(d\)-cluster-tilting module for \(d \geq 2\). Our results give the first, to our knowledge, example of an infinite family of \(d\)-representation-finite block algebras of finite representation type.

2. PRELIMINARIES

Throughout this note, \(k\) is a field, \(A\) a finite-dimensional \(k\)-algebra, and \(d\) an integer greater than or equal to 2. We denote by \(\mod A\) the category of finite-dimensional right \(A\)-modules. The stable module category, \(\mod^Z A\), is the quotient of \(\mod A\) by the ideal of morphisms factoring through a projective module. By \(\mod^Z A\) we denote the category of \(Z\)-graded finite-dimensional \(A\)-modules, and by \(\mod^Z A\) the corresponding stable category. If \(A\) is self-injective then \(\mod A\) has

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the structure of a triangulated category, with suspension functor given by the co-syzygy functor: $\Omega^{-1} : \text{mod} A \to \text{mod} A$.

A \textit{d-cluster-tilting subcategory} of an abelian or triangulated category $\mathcal{C}$ is a functorially finite full subcategory $\mathcal{U} \subset \mathcal{C}$ satisfying

$$\mathcal{U} = \{ X \in \mathcal{C} \mid \forall U \in \mathcal{U}, \ i \in \{1, \ldots, d - 1\} : \text{Ext}^i(X, U) = 0 \} = \{ X \in \mathcal{C} \mid \forall U \in \mathcal{U}, \ i \in \{1, \ldots, d - 1\} : \text{Ext}^i(U, X) = 0 \}.$$ 

A \textit{d-cluster-tilting module} is an $A$-module $M$ for which the additive closure, $\text{add} M$, forms a $d$-cluster-tilting subcategory of $\text{mod} A$. An algebra $A$ is said to be \textit{d-representation-finite} if it has a $d$-cluster-tilting module.

We denote by $D$ the standard $k$-duality: $D = \text{Hom}_k(-, k) : \text{mod} A \to \text{mod} A^{\text{op}}$. The \textit{trivial extension} of $A$ is the algebra

$$T(A) = A \oplus DA \quad \text{with multiplication} \quad (a, f)(b, g) = (ab, ag + fb),$$

where the products $ag$ and $fb$ are given by the bimodule structure of $DA$. Trivial extension algebras are \textit{symmetric}, meaning that $T(A)$ and $DT(A)$ are isomorphic as $T(A)$-bimodules.

Let $\mathcal{C}$ be a $k$-linear triangulated category with $k$-dual $D : \mathcal{C} \to \mathcal{C}$. A \textit{Serre functor} on $\mathcal{C}$ is an auto-equivalence $S : \mathcal{C} \to \mathcal{C}$ satisfying a bi-functorial isomorphism $\mathcal{C}(X, S(Y)) \simeq \mathcal{D}(C(Y, X))$. For any symmetric algebra $A$, the syzygy functor $\Omega$ is a Serre functor on $\text{mod} A$. If $A$ is an algebra of finite global dimension, then the \textit{Nakayama functor}

$$\nu = D \circ \text{RHom}_A(-, -) \simeq - \circ \Omega^2 D : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$$

is a Serre functor on the bounded derived category $\mathcal{D}^b(A)$ of finite-dimensional $A$-modules. In this case, Happel’s theorem [27] gives a triangle equivalence

$$\mathcal{D}^b(A) \simeq \text{mod}^Z T(A),$$

under which the grade shift functor (1) on $\text{mod}^Z T(A)$ corresponds to the auto-equivalence $\nu \circ [1]$ of $\mathcal{D}^b(A)$. Note that $\nu$ is a triangle functor on $\mathcal{D}^b(A)$, and thus it commutes with $[1]$ up to natural isomorphism.

For a quiver $Q$, arrows in the path algebra $kQ$ are composed from left to right. Consequently, modules of $kQ$ correspond to covariant representations of the quiver. Let $Q$ be a quiver of simply laced Dynkin type. Then the path algebra $kQ$ [3], as well as its trivial extension $T(kQ)$ [20], is representation finite. The forgetful functor $\text{mod}^Z T(kQ) \to \text{mod} T(kQ)$ is dense (that is, every $T(kQ)$-module is gradable) [7] and there are triangle equivalences

$$\text{mod} T(kQ) \simeq (\text{mod}^Z T(kQ))/[1] \simeq \mathcal{D}^b(kQ)/[\nu \circ [1]].$$

It follows that basic $d$-cluster-tilting modules of $T(kQ)$ correspond bijectively to $d$-cluster-tilting subcategories of $\mathcal{D}^b(kQ)$ that are invariant under the functor $\nu \circ [1]$.

The full subcategory of $\mathcal{D}^b(kQ)$ formed by all indecomposable objects is equivalent to the mesh category of the stable translation quiver $\mathcal{Z}Q$, and the Auslander–Reiten translation on $\mathcal{D}^b(kQ)$ is given by $\tau = \nu \circ [-1]$. In particular, every indecomposable object in $\mathcal{D}^b(kQ)$ is isomorphic to $\tau^l(P)$ for a unique indecomposable projective $kQ$-module $P$ and integer $l \in \mathbb{Z}$. [9, Section I.5]

**Proposition 2.1 ([17 Proposition 3.4])**. Let $\mathcal{C}$ be a $k$-linear triangulated category with a Serre functor $S$. Then every $d$-cluster-tilting subcategory $\mathcal{U}$ of $\mathcal{C}$ satisfies $(S \circ [-d])(\mathcal{U}) = \mathcal{U}$.

For $A$ of finite global dimension, we write

$$\nu_d = \nu \circ [-d] : \mathcal{D}^b(A) \to \mathcal{D}^b(A).$$

Let $l, m \in \mathbb{Z}, \ l > 0$. An algebra $A$ of finite global dimension is \textit{twisted $(m/l)$-Calabi–Yau} if there exists a $k$-linear automorphism $\phi \in \text{Aut}_k(A)$ such that $\nu^l \simeq [m] \circ \phi^*$, where $\phi^*$ denotes the auto-functor $\mathcal{D}^b(A) \to \mathcal{D}^b(A)$ induced by $\phi$. If $\phi = \text{id}$, then $A$ is $(m/l)$-Calabi–Yau. The algebra $A$ is \textit{(twisted) fractionally Calabi–Yau} if it is (twisted) $(m/l)$-Calabi–Yau for some $l$ and $m$. We remark that for an $(m/l)$-Calabi–Yau algebra $A$, $m/l$ is uniquely determined by $A$ as a rational number.
Proposition 2.2 ([19, 0.3], see also [3] Theorem 8.1). Let $Q$ be a quiver of simply laced Dynkin type. The path algebra $kQ$ is $(h - 2)/h$-Calabi–Yau, where $h$ denotes the Coxeter number of the Dynkin diagram. If $Q$ is of type $A_n$, $D_{2m}$, $E_7$ or $E_8$, then $kQ$ is $(\frac{h}{2} - 1)/(\frac{h}{2})$-Calabi–Yau.

| $Q$ | $A_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|-------|-------|-------|
| $h$ | $n + 1$ | $2(n - 1)$ | 12    | 18    | 30    |

Table 1. Coxeter numbers of Dynkin quivers.

Lemma 2.3. The following isomorphisms of triangle functors $D^b(kQ) \to D^b(kQ)$ hold.

\[ [h - 2] \simeq \nu^h, \]
\[ \nu \circ [1] \simeq \tau^{1-h}, \]
\[ [2] \simeq \tau^{-h} \simeq (\nu \circ [1]) \circ \tau^{-1}. \]

Proof. The existence of an isomorphism $[h - 2] \simeq \nu^h$ of triangle functors follows from [3] Proposition 2.7(b). It implies that $[2] \simeq \nu^{-h} \circ [h] \simeq \tau^{-h}$, giving the first isomorphism in [3]. Now, $\nu \circ [1] \simeq \tau \circ [2] \simeq \tau^{1-h}$, proving (2), whence $\tau^{-h} \simeq \tau^{1-h} \circ \tau^{-1} \simeq (\nu \circ [1]) \circ \tau^{-1}$, concluding the proof of [3].

Lemma 2.4. If $T(kQ)$ is $d$-representation-finite, then $(d + 1) \mid 2(h - 1)$.

Proof. Let $\mathcal{U}$ be a $(\nu \circ [1])$-equivariant $d$-cluster-tilting subcategory of $D^b(kQ)$. By Lemma 2.3, there are isomorphisms

\[ [2(h - 1)] = [h - 2] \circ [h] \simeq \nu^h \circ [h] \simeq (\nu \circ [1])^h \]

of triangle functors $D^b(kQ) \to D^b(kQ)$ and thus $\mathcal{U}[2(h - 1)] = (\nu \circ [1])^h(\mathcal{U}) = \mathcal{U}$. On the other hand,

\[ \mathcal{U}[d + 1] = \nu_d(\mathcal{U}[d + 1]) = (\nu \circ [1])(\mathcal{U}) = \mathcal{U}, \]

and it follows that $\mathcal{U}[g] = \mathcal{U}$, where $g = \gcd(d + 1, 2(h - 1))$.

For all $X, Y \in \mathcal{U}$ and $r \in \{1, \ldots, d - 1\}$, we have $\text{Hom}_{D^b(kQ)}(X, Y[r]) = \text{Ext}^r(X, Y) = 0$. Hence $\mathcal{U}[r] = \mathcal{U}$ is impossible for all $r < d$ and, consequently, $g \in \{d, d + 1\}$. With $g \mid (d + 1)$ and $d \geq 2$, this implies that $g = d + 1$, and thus $(d + 1) \mid 2(h - 1)$.

3. Proof of Theorem 1.2

Theorem 1.2 is an application of [4] Section 5, in which a characterisation of all $d$-representation-finite self-injective Nakayama algebras was given. Throughout this section, let $B$ be a ring-indecomposable symmetric Nakayama $k$-algebra with $n$ simple modules and Loewy length $\ell(B) = an + 1$. Specifying Theorem 5.1 of [4] to this situation, we get the following result. As usual, $d$ is an integer greater than or equal to 2.

Proposition 3.1. The algebra $B$ is $d$-representation-finite if and only if at least one of the following conditions holds:

(a) $(an + 1)(d - 1) + 2 \mid 2n$;
(b) $(an + 1)(d - 1) + 2 \mid tn$, where $t = \gcd(d + 1, 2an)$.

Lemma 3.2. The condition (a) in Proposition 3.1 holds if and only if $(a, n, d) = (1, 3, 2)$.

Proof. As $(an + 1)(d - 1) + 2 > an$, the condition (a) is equivalent to $a = 1$ and $(n + 1)(d - 1) + 2 = 2n$. The last equation can be re-written as $(n + 1)(3 - d) = 4$. Since $d \geq 2$, this holds if and only if $n = 3$ and $d = 2$.

Lemma 3.3. If $B$ is $d$-representation-finite then $(d + 1) \mid 2n$. 

□
In particular, the condition (b) in Proposition 3.4 is equivalent to
\[(an + 1)(d - 1) + 2) | (d + 1)n. \tag{4}\]

**Proof.** Since \(B\) is a ring-indecomposable symmetric Nakayama algebra, the Auslander–Reiten translate \(\tau \simeq \Omega^2 \mod B \to \mod B\) cyclically permutes the simple modules \[\text{Corollary IV.2.12]. Hence \(\Omega^{2n}(S) \simeq S\) for all simple modules \(S \in \mod B\). As moreover \(\ell(\Omega^2(M)) = \ell(M)\) for all \(M \in \mod B\) \[\text{Corollary IV.2.9]}, and indecomposable \(B\)-modules are classified by their tops and Lowey lengths \[\text{Lemma IV.2.5}], it follows that \(\Omega^{2n}(M) \simeq M\) for all \(M \in \mod B\).

Let \(M\) be a \(d\)-cluster-tilting \(B\)-module. Then \(\Omega^{d+1}(M) = S_d(M) \simeq M\) in \(\mod B\) which, combined with \(\Omega^{2n}(M) \simeq M\), yields \(\Omega^{gcd(d+1,2n)}(M) \simeq M\). Since \(\text{Hom}(\Omega^r(M), M) \simeq \text{Ext}^r(M, M) = 0\) for all \(r \in \{1, \ldots, d - 1\}\) this implies that \(gcd(d + 1, 2n) \geq d\) and, as \(d \geq 2\), it follows that \((d + 1) | 2n.\)

**Lemma 3.4.** If the condition (b) in Proposition 3.4 holds, then either \((a, n, d) = (2, 3, 2)\) or \(a = 1.\)

**Proof.** If (b) holds then
\[((an + 1)(d - 1) + 2) \leq tn = (d + 1)n,\]
which is equivalent to
\[(a - 1)(d - 1)n + d + 1 \leq 2n.\]
Thus \((a - 1)(d - 1) \in \{0, 1\},\) implying that either \(a = 1\) or \(a = d = 2.\) In the latter case, it follows from (4) that \(n = d + 1 = 3.\)

Summarising the results above, we see that \(B\) is \(d\)-representation-finite if and only if either \((a, n, d) \in \{(1, 3, 2), (2, 3, 2)\},\) or \(a = 1, (d + 1) | 2n\) and
\[\left((n + 1)(d - 1) + 2\right) | (d + 1)n. \tag{5}\]
To prove Theorem 1.2 we need to show that in the latter case, either \(d = 2n - 1\) or \((d, n) = (2, 6)\) holds.

By Lemma 3.3 there exists an integer \(b \geq 1\) such that \(2n = b(d + 1).\) Multiplying each side of (5) with 2 gives
\[2((n + 1)(d - 1) + 2) | 2n(d + 1).\]
Since
\[2((n + 1)(d - 1) + 2) = 2n(d - 1) + 2(d + 1) = b(d + 1)(d - 1) + 2(d + 1) = (b(d - 1) + 2)(d + 1) \quad \text{and} \quad 2n(d + 1) = b(d + 1)^2,\]
this is equivalent to
\[(b(d - 1) + 2) | b(d + 1). \tag{6}\]
Let \(c \geq 1\) be such that \(b(d + 1) = c(b(d - 1) + 2).\) Then \((c - 1)(d - 1)b = 2(b - c) < 2b,\) whence \((c - 1)(d - 1) < 2,\) so either \(c = 1,\) or \(c = d = 2.\) Thus (6) holds if and only if either \(b(d - 1) + 2 = b(d + 1),\) or \(d = 2\) and \(3b = 2(b + 2).\) In the former case \(b = 1\) and thus \(2n = b(d + 1) = d + 1.\) In the latter, \(b = 4\) and \(2n = 4(2 + 1) = 12;\) so \((d, n) = (2, 6).\) This concludes the proof of Theorem 1.2.

4. Proof of Theorem 1.1

To start, observe that if \(Q\) is a quiver of Dynkin type \(A_n,\) then \(T(kQ)\) is a symmetric Nakayama algebra with \(n\) simple modules and Lowey length \(n + 1.\) By Theorem 1.2 such an algebra is \(d\)-representation-finite if and only if either \(d = 2n - 1,\) or \(d = 2\) and \(n \in \{3, 6\}.\) This establishes the claim in Theorem 1.1 for Dynkin quivers of type \(A.\)

**Proposition 4.1.** Let \(Q\) be an acyclic quiver. If \(T(kQ)\) is \(d\)-representation-finite for some \(d,\) then \(Q\) is a simply laced Dynkin quiver.
By Theorem 1.3, if $T(kQ)$ is $d$-representation-finite then $kQ$ is twisted fractionally Calabi–Yau. This is satisfied only if $Q$ is a Dynkin quiver.

If $\nu^l \simeq [m] \circ \phi^*$ for some $l, m \in \mathbb{Z}$, $l > 0$ and $\phi \in \text{Aut}_k(kQ)$, then $\nu^{-1}_1(kQ) \simeq (\phi^*)^{-1}(kQ)[l - m] \simeq kQ[m - l]$. On the other hand, if $Q$ is not Dynkin then $kQ$ is a representation-infinite hereditary algebra and thus, for all $l > 0$, $\nu^{-1}_1(kQ) = \tau^{-l}(kQ) \in D^b(kQ)$ is a non-projective stalk complex concentrated in degree zero [8, Proposition VIII.1.15]. So in this case, $\nu^{-1}_1(kQ)$ is not isomorphic to $kQ[l - m]$ for any $m \in \mathbb{Z}$, and thus $kQ$ is not twisted fractionally Calabi–Yau.

It remains to consider quivers of Dynkin types $D$ and $E$. For a vertex $i$ of the quiver $Q$, we denote by $P_i$ the projective cover of the simple $kQ$-module supported at $i$, and by $O_i = \text{add}(\tau^l(P_i) \mid l \in \mathbb{Z}) \subset D^b(kQ)$ the $\tau$-orbit of $P_i$ in $D^b(kQ)$. Recall that if $Q$ is a Dynkin quiver then every indecomposable object in $D^b(kQ)$ in contained in precisely one of the orbits $O_i$.

$$
\begin{array}{cccc}
1 & 2 & 3 & \cdots & m \\
\hline
4
\end{array}
$$

Figure 1. The Dynkin diagram $E_m$, $m \in \{6, 7, 8\}$.

**Proposition 4.2.** Let $Q$ be a quiver of Dynkin type $E$. Then $T(kQ)$ is not $d$-representation-finite for any $d \geq 2$.

Proof. Let $Q$ be an a quiver of Dynkin type $E_m$, $m \in \{6, 7, 8\}$, with vertices enumerated as in Figure 1. Recall that $D^b(kQ)$ is equivalent to the mesh category of the stable translation quiver $\mathbb{Z}Q$ and that, by Lemma 2.3, $[\nu \circ [1]] \simeq (\nu \circ [1]) \circ \tau^{-1}$ holds on $D^b(kQ)$. Let $X \in D^b(kQ)$ be an indecomposable object. From the shape of $\mathbb{Z}Q$, one readily reads off the following:

- $\text{Hom}_{D^b(kQ)}((\nu \circ [1])^2(X), X[4]) \simeq \text{Hom}_{D^b(kQ)}(X, \tau^{-2}(X)) \neq 0$, if $X \in O_4$;
- $\text{Hom}_{D^b(kQ)}((\nu \circ [1])^3(X), X[6]) \simeq \text{Hom}_{D^b(kQ)}(X, \tau^{-3}(X)) \neq 0$, if $X \in O_1$;
- $\text{Hom}_{D^b(kQ)}((\nu \circ [1])^{3-m}(X), X[2(3-m)]) \simeq \text{Hom}_{D^b(kQ)}(X, \tau^{3-m}(X)) \neq 0$, if $X \in O_m$;
- $\text{Hom}_{D^b(kQ)}((\nu \circ [1])(X), X[2]) \simeq \text{Hom}_{D^b(kQ)}(X, \tau^{-1}(X)) \neq 0$, otherwise.

Thus, for any $(\nu \circ [1])$-equivariant subcategory $U$ of $D^b(kQ)$, we have $\text{Hom}_{D^b(kQ)}(U, U[r]) \neq 0$ for some $r \in \{1, \ldots, 2(3-m)\}$. Observe that $2(m-3) < h - 2$ for each $m = 6, 7, 8$, where $h = 12, 18, 30$ is the Coxeter number of $E_m$, so if $D^b(kQ)$ has a $(\nu \circ [1])$-equivariant $d$-cluster-tilting subcategory then $d < h - 2$.

On the other hand, by Lemma 2.3 the existence of such a subcategory implies that $(d + 1) \mid 2(h - 1)$. Since $h - 1 \in \{11, 17, 29\}$ is a prime number and $d + 1 < h - 1$, it follows that $d + 1 = 2$ and thus $d = 1$. So $D^b(kQ)$ does not have a $(\nu \circ [1])$-equivariant $d$-cluster-tilting subcategory for any $d \geq 2$, and hence $T(kQ)$ is not $d$-representation-finite.

For the remainder of this section, let $Q$ be a quiver of Dynkin type $D_n$, with $n \geq 4$ and vertices enumerated as in Figure 2.

$$
\begin{array}{cccc}
1 & 2 & \cdots & (n-2) & n \\
\hline
(n-1)
\end{array}
$$

Figure 2. The Dynkin diagram $D_n$.

Grimeland [8] has studied invariance properties of 2-cluster-tilting subcategories of $D^b(A)$ for representation-finite hereditary algebras $A$. The following result is a direct consequence of [8, Corollary 44].

**Proposition 4.3.** The category $D^b(kQ)$ does not have a $(\nu \circ [1])$-equivariant 2-cluster-tilting subcategory; hence, $T(kQ)$ is not 2-representation-finite.
Now, assume that $\mathcal{D}_c^b(kQ)$ has a $(\nu \circ [1])$-equivariant $d$-cluster-tilting subcategory $\mathcal{U}$ for some $d \geq 3$. Since the Coxeter number of $D_n$ is $h = 2n - 2$, Lemma 4.4 tells us that
\[(d + 1) \mid 2(2n - 3) . \tag{7}\]
This immediately implies that $d \neq 3$, since $2(2n - 3) = 4n - 6$ is not divisible by 4.

**Lemma 4.4.** The $d$-cluster-tilting subcategory $\mathcal{U}$ of $\mathcal{D}_c^b(kQ)$ is contained in $\text{add} (O_1 \cup O_{n-1} \cup O_n)$, but not in $O_1$.

**Proof.** Let $X \in \mathcal{U}$ be indecomposable. Since $\mathcal{U}$ is $(\nu \circ [1])$-equivariant and $d \geq 4$, by Lemma 2.3(3) we get that
\[\text{Hom}_{\mathcal{D}_c^b(kQ)}(X, \tau^{-1}X) \simeq \text{Hom}_{\mathcal{D}_c^b(kQ)}((\nu \circ [1])(X), X[2]) = 0.\]
This implies that $X \in O_1 \cup O_{n-1} \cup O_n$.

Assume now that $U \subset O_1$, and hence that $X \in O_1$. From the shape of $\mathbb{Z}Q$, one reads off that there exists a unique (up to isomorphism) indecomposable $\tilde{X} \in O_n$ such that $\text{Hom}_{\mathcal{D}_c^b(kQ)}(X, \tilde{X}) \neq 0$. Moreover, if $f : X \to \tilde{X}$ is a non-zero morphism and $Y \in O_1$ an indecomposable object not isomorphic to $X$, then every morphism $X \to Y$ factors through $f$. Hence, $f$ induces an isomorphism $\text{Hom}_{\mathcal{D}_c^b(kQ)}(X, Y) \simeq \text{Hom}_{\mathcal{D}_c^b(kQ)}(\tilde{X}, Y)$.

Note that $O_1[1] = \hat{O}_1$. Now, for any indecomposable $Y \in \mathcal{U}$ and any $r \in \{1, \ldots, d - 1\}$, we have $\text{Hom}_{\mathcal{D}_c^b(kQ)}(X, Y[r]) = 0$, implying that $Y[r] \not\simeq X$, and thus
\[\text{Hom}_{\mathcal{D}_c^b(kQ)}(\tilde{X}, Y[r]) \simeq \text{Hom}_{\mathcal{D}_c^b(kQ)}(X, Y[r]) = 0.\]
This proves that $\tilde{X} \in \{Z \in \mathcal{D}_c^b(kQ) \mid \text{Ext}^1(Z, U) = 0, \forall i \in \{1, \ldots, d - 1\}\} = \mathcal{U}$; hence, $U \subset O_1$ is impossible. □

With the following proposition, we conclude the proof of the necessity part of Theorem 1.1.

**Proposition 4.5.** The algebra $T(kQ)$ is $d$-representation-finite only if $d = n = 4$.

**Proof.** As before, $\mathcal{U}$ denotes a $(\nu \circ [1])$-equivariant $d$-cluster-tilting subcategory of $\mathcal{D}_c^b(kQ)$. By Lemma 4.4 there exists some indecomposable object $X \in \mathcal{U}$ that is contained in either $O_{n-1}$ or $O_n$. As $[4] \simeq \tau^{-2} \circ (\nu \circ [1])^2$ by Lemma 2.3(3), we get
\[0 \neq \text{Hom}_{\mathcal{D}_c^b(kQ)}(X, \tau^{-2}(X)) \simeq \text{Hom}_{\mathcal{D}_c^b(kQ)}((\nu \circ [1])^2(X), X[4])\]
and, since $X$ and $(\nu \circ [1])^2(X)$ belong to $\mathcal{U}$, this implies that $d \leq 4$. On the other hand, from Proposition 4.3 and (7) we know that $d > 3$, hence $d = 4$.

Note that $\text{Hom}_{\mathcal{D}_c^b(kQ)}(\tau^3(X), X) \neq 0$ if and only if $0 \leq 2r \leq n - 2$. From the definitions, we get the following isomorphisms of triangle functors on $\mathcal{D}_c^b(kQ)$:
\[\nu_4 \circ \nu_4 \circ [-3] \simeq \nu_4 \circ \tau^4 \circ [3 - 4 - 3] = \tau_4.\]
As $\mathcal{U}$ is 4-cluster-tilting, it is $\nu_4$-equivariant by Proposition 2.1 and it follows that
\[0 = \text{Hom}_{\mathcal{D}_c^b(kQ)}(\nu_4 \circ [1]^3 \circ \nu_4) (X), X[3]) \simeq \text{Hom}_{\mathcal{D}_c^b(kQ)}(\tau^4(X), X)\]
implying that $4 > n - 2$, that is, $n < 6$. On the other hand, the condition (7) for $d = 4$ becomes $5 \mid 2(2n - 3)$ or, equivalently, $5 \mid (n + 1)$. Hence, $n = 4$.

To conclude the proof of Theorem 1.1, it remains only to show that $T(kQ)$ is 4-representation-finite in case $Q$ is of Dynkin type $D_4$. To this end, consider the quiver $Q$ given by Figure 3 and let
\[\mathcal{U} = \text{add}\{\tau^l(P_1 \oplus P_4) \mid l \in \mathbb{Z}\} \subset \mathcal{D}_c^b(kQ).\]
By Lemma 2.3(3), $(\nu \circ [1] \simeq \tau^{-h} = \tau^{-5}$, so $\mathcal{U}$ is $(\nu \circ [1])$-equivariant. Moreover, by Proposition 2.2
\[
\begin{array}{ccc}
1 & \xrightarrow{2} & 4 \\
\downarrow & & \downarrow \\
3
\end{array}
\]

**Figure 3. Quiver $Q$ of Dynkin type $D_4$.**
Proposition 5.1. The module \( S \) is a 4-cluster-tilting subcategory of \( \mathcal{D}^b(kQ) \). It is now straightforward to verify that \( \mathcal{U} \) is a 4-cluster-tilting subcategory of \( \mathcal{D}^b(kQ) \), thus concluding the proof of Theorem 1.1.

The module \( \mathcal{U} \) is a 4-cluster-tilting subcategory of \( \mathcal{D}^b(kQ) \), thus concluding the proof of Theorem 1.1.

Here, we give examples of \( d \)-cluster-tilting modules for \( T(kQ) \) in each of the cases listed in Theorem 1.1. The proofs of the results in this section are straightforward verifications.

For \( n \geq 2 \), let \( Q_n \) be the following quiver:

\[
Q_n : \quad 1 \rightarrow 2 \rightarrow \cdots \rightarrow n
\]

We denote by \( S_i \) the simple \( kQ \)-module supported at the vertex \( i \), and by \( P_i \) its projective cover.

**Proposition 5.1.**

(a) The module \( P_n \oplus T(kQ) \) is a \((2n-1)\)-cluster-tilting module of \( T(kQ_n) \).

(b) The module \( P_1 \oplus P_2 \oplus S_3 \oplus T(kQ_3) \) is a 2-cluster-tilting module of \( T(kQ_3) \).

(c) The module \( T(kQ_6) \oplus \bigoplus_{i=0}^{3} \Omega_{T(kQ_6)}^{3i}(P_3 \oplus P_6) \) is a 2-cluster-tilting module of \( T(kQ_6) \).

Let \( Q \) be the quiver of type \( D_4 \) given in Figure 3.

**Proposition 5.2.** The module \( P_1 \oplus P_3 \oplus T(kQ) \) is a 4-cluster-tilting module of \( T(kQ) \).

**Figure 4.** The Auslander–Reiten quiver of \( \mathcal{D}^b(kQ) \) for \( Q \) of type \( D_4 \), with the 4-cluster-tilting subcategory \( \mathcal{U} \) indicated in black.

**Figure 5.** A 2-cluster-tilting module of \( T(kQ_5) \).

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Figure 6. A 2-cluster-tilting module of $T(kQ_6)$.

Figure 7. A 2-cluster-tilting module of $T(kQ)$, with $Q$ of type $D_4$. 
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