CHARACTER CLUSTERS FOR LIE ALGEBRA MODULES OVER A FIELD OF NON-ZERO CHARACTERISTIC

DONALD W. BARNES

Abstract. For a Lie algebra $L$ over an algebraically closed field $F$ of non-zero characteristic, every finite dimensional $L$-module can be decomposed into a direct sum of submodules such that all composition factors of a summand have the same character. Using the concept of a character cluster, this result is generalised to fields which are not algebraically closed. Also, it is shown that if the soluble Lie algebra $L$ is in the saturated formation $\mathfrak{S}$ and if $V, W$ are irreducible $L$-modules with the same cluster and the $p$-operation vanishes on the centre of the $p$-envelope used, then $V, W$ are either both $\mathfrak{S}$-central or both $\mathfrak{S}$-eccentric. Clusters are used to generalise the construction of induced modules.

1. Introduction

Throughout this note, $L$ is a finite dimensional Lie algebra over the field $F$ of characteristic $p \neq 0$. Let $V$ be a finite dimensional $L$-module. To define a character for $V$, we must embed $L$ in a $p$-envelope $(L^p, [p])$. The action $\rho$ of $L$ on $V$ can be extended to $L^p$. (See Strade and Farnsteiner [4, Theorem 5.1.1].)

Definition 1.1. A character for $V$ is a linear map $c : L^p \to F$ such that for all $x \in L^p$, we have

$$\rho(x)^p - \rho(x^{[p]}) = c(x)^p1.$$

Not every module has a character, but if $F$ is algebraically closed and $V$ is irreducible, then $V$ has a character. (See Strade and Farnsteiner [4, Theorem 5.2.5].) The following is Strade and Farnsteiner, [4, Theorem 5.2.6].

Theorem 1.2. Suppose that $F$ is algebraically closed and let $(L, [p])$ be a restricted Lie algebra over $F$. Let $V$ be a finite dimensional $L$-module. Then there exist $c_i : L \to F$ and submodules $V_i$ such that $V = \oplus_i V_i$ and every composition factor of $V_i$ has character $c_i$.

This decomposition in terms of characters is functorial and is clearly useful. In this note, the concept of a character cluster is used to obtain a similar result which does not require the field to be algebraically closed. As a further application, it is shown that, if the soluble Lie algebra $L$ is in the saturated formation $\mathfrak{S}$ and $V, W$ are irreducible $L$-modules with the same cluster and the $p$-operation vanishes on the centre of the $p$-envelope used, then either both $V, W$ are $\mathfrak{S}$-central or both are $\mathfrak{S}$-eccentric. Over a perfect field, clusters are used to generalise the construction of induced modules.

2010 Mathematics Subject Classification. Primary 17B10.

Key words and phrases. Lie algebras, saturated formations, induced modules.
To simplify the exposition, we work with a restricted Lie algebra \((L, [p])\). To apply the results to a general Lie algebra, as is the case for characters, we have to embed the algebra in a \(p\)-envelope, and the clusters obtained depend on that embedding.

## 2. Preliminaries

In the following, \((L, [p])\) is a restricted Lie algebra over the field \(F\), \(\bar{F}\) is the algebraic closure of \(F\) and \(\bar{L} = \bar{F} \otimes_F L\) is the algebra obtained by extension of the field. A character of \(L\) is an \(\bar{F}\)-linear map \(c : L \to \bar{F}\). If \(\{e_1, \ldots, e_n\}\) is a basis of \(L\), then \(c\) can be expressed as a linear form \(c(x) = \sum a_i x_i\) for \(x = \sum x_i e_i\), where \(a_i \in \bar{F}\). If \(\alpha\) is an automorphism of \(\bar{F}/F\), that is, an automorphism of \(\bar{F}\) which fixes all elements of \(F\), then \(\alpha^c\) is the character \(c^\alpha(x) = \sum a_i^\alpha x_i\) and is called a conjugate of \(c\). We do not distinguish in notation between \(c : L \to \bar{F}\) and its linear extension \(\bar{L} \to \bar{F}\). We denote by \(F[c]\) the field \(F[a_1, \ldots, a_n]\) generated by the coefficients \(a_i\). It is the field generated by the \(c(x)\) for all \(x \in L\) and is independent of the choice of basis.

If \(V\) is an \(L\)-module, then \(\bar{V}\) is the \(\bar{L}\)-module \(\bar{F} \otimes_F V\). The action of \(x \in L\) on \(V\) is denoted by \(\rho(x)\). The module \(V\) has character \(c\) if \((\rho(x)^p - \rho(x^{[p]}))v = c(x)p v\) for all \(x \in L\) and all \(v \in V\).

In the universal enveloping algebra \(U(L)\), the element \(x^p - x^{[p]}\) is central. (See Strade and Farnsteiner [4], p. 203.) For the module \(V\) giving the representation \(\rho\), we put \(\phi_x = \rho(x)^p - \rho(x^{[p]})\). We then have \([\phi_x, \rho(y)] = 0\) for all \(x, y \in L\).

**Lemma 2.1.** The map \(\phi : L \to \text{End}(V)\) defined by \(\phi_x(v) = (\rho(x)^p - \rho(x^{[p]}))v\) is \(p\)-semilinear.

**Proof.** In the universal enveloping algebra \(U(L)\), we have

\[(a + b)^p = a^p + b^p + \sum_{i=1}^{p-1} s_i(a, b)\]

(see Strade and Farnsteiner [4] p. 62 equation (3),) and

\[(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b).\]

(See Strade and Farnsteiner, p. 64 Property (3).) Putting these together, we have

\[(a + b)^p - (a + b)^{[p]} = a^p + b^p - a^{[p]} - b^{[p]}\]

It follows that \(\phi_{a+b} = \phi_a + \phi_b\). Clearly, \(\phi_{\lambda a} = \lambda^p \phi_a\).

**Remark 2.2.** In the decomposition of \(\bar{V}\) given by Theorem 1.2, the summand corresponding to the character \(c\) is

\[\{v \in \bar{V} \mid (\phi_x - c(x)^p) v = 0\ \text{for some} \ r \ \text{and all} \ x \in \bar{L}\}.\]

By Lemma 2.1, we need only consider those \(x \in L\), or indeed, in some chosen basis of \(L\).
3. Clusters

**Definition 3.1.** The cluster $\mathrm{Cl}(V)$ of an $L$-module $V$ is the set of characters of the composition factors of the $L$-module $\bar{V} = \bar{F} \otimes_F V$.

**Lemma 3.2.** Suppose $c \in \mathrm{Cl}(V)$. Then the conjugates $c^\alpha$ of $c$ are in $\mathrm{Cl}(V)$.

**Proof.** Let $A/B$ be a composition factor of $\bar{V}$ and let $\{v_1, \ldots, v_k\}$ be a basis of $V$. The action $\rho(x)$ of $x \in L$ on $V$ and so also on $\bar{V}$ is given in respect to this basis by a matrix $X$ with coefficients in $F$. An automorphism $\alpha$ maps $v = \lambda_1 v_1 + \ldots + \lambda_k v_k$ to $v^\alpha = \lambda_1^\alpha v_1 + \ldots + \lambda_k^\alpha v_k$. Since $X^\alpha = X$, we have that $(xv)^\alpha = xv^\alpha$. Thus $A^\alpha, B^\alpha$ are submodules of $\bar{V}$ and $A^\alpha/B^\alpha$ is a composition factor. The linear map $\phi_x = \rho(x) - \rho(x^p)$ also commutes with $\alpha$. Thus from $\phi_x(a + B) = c(x)p a + B$, it follows that $\phi_x(a^\alpha) + B^\alpha = c^\alpha(x)p a^\alpha + B^\alpha$. Thus $c^\alpha \in \mathrm{Cl}(V)$.

The statement $(xv)^\alpha = x v^\alpha$ may suggest that $A/B$ and $A^\alpha/B^\alpha$ are isomorphic. They are not. The map $v \mapsto v^\alpha$ is not linear, as $(\lambda v)^\alpha = \lambda^\alpha v^\alpha$.

By Lemma 3.2, a cluster $\mathrm{Cl}(V)$ is a union of conjugacy classes of characters.

**Definition 3.3.** A cluster $\mathrm{Cl}(V)$ is called simple if it consists of a single conjugacy class of characters.

**Theorem 3.4.** Let $V$ be an irreducible $L$-module. Then $\mathrm{Cl}(V)$ is simple.

**Proof.** $\bar{V} = \bar{F} \otimes_F V$ has a direct decomposition $\bar{V} = \sum_i \bar{V}_c$ where the component $\bar{V}_c$ is, by Remark 2.2, the space

$$\{v \in \bar{V} \mid (\phi_x - c(x)p^r 1)^r v = 0 \text{ for all } x \in L \text{ and some } r\}.$$  

Here, we may take for $r$ the length of a composition series of $\bar{V}_c$, which is independent of $x$. Let $c \in \mathrm{Cl}(V)$. Let $\bar{V}_0 = \sum_i \bar{V}_{c^\alpha}$ where the sum is over the distinct conjugates $c^\alpha$. Let $f_x(t) = \Pi_{\alpha}(t - c^\alpha(x)p^r)$. The coefficients of $f_x(t)$ are invariant under the automorphisms of $\bar{F}/F$. Therefore for some $k$, we have that $f_x(t)p^r$ is a polynomial over $F$. As the field is not assumed to be perfect, this may require $k > 0$. Let $m_x(t)$ be the least power of $f_x(t)$ which is a polynomial over $F$. Then, with $r$ the length of a composition series of $\bar{V}_c$, we have

$$\bar{V}_0 = \{v \in \bar{V} \mid m_x(\phi_x)^r v = 0 \text{ for all } x \in L\}.$$  

The condition $m_x(\phi_x)^r v = 0$ for all $x \in L$ may be regarded as a set of linear equations over $F$ in the coordinates of $v$. These equations have a non-zero solution over $F$ since $\bar{V}_c \neq 0$. Therefore, they have a non-zero solution over $F$, that is,

$$\bar{V}_0 = \{v \in V \mid m_x(\phi_x)^r v = 0 \text{ for all } x \in L\} \neq 0.$$  

Since the $\rho_y$ commute with the $\phi_x$, $\bar{V}_0$ is a submodule of $V$. Therefore $\bar{V}_0 = V$ and it follows that the set of conjugates of $c$ is the whole of $\mathrm{Cl}(V)$.

4. The cluster decomposition

We have seen that if $c \in \mathrm{Cl}(V)$, then every conjugate $c^\alpha$ of $c$ is in $\mathrm{Cl}(V)$. It is convenient to expand our terminology and call any finite set $C$ of linear maps $c: L \rightarrow \bar{F}$ a cluster if, for each $c \in C$, all conjugates of $c$ are in $C$. With this expansion of our terminology, every cluster is a union of simple clusters.

**Theorem 4.1.** Let $(L, [p])$ be a restricted Lie algebra and let $V$ be an $L$-module. Suppose that $\mathrm{Cl}(V)$ is the union $C_1 \cup \cdots \cup C_k$ of the distinct simple clusters $C_i$. Then $V = V_1 \oplus \cdots \oplus V_k$ of submodules $V_i$ such that $\mathrm{Cl}(V_i) = C_i$. 

Proof. By Theorem 1.2, $\hat{V}$ is the direct sum over the set of characters $\nu$ of submodules $\hat{V}_i$, whose composition factors all have character $\nu$. By Remark 2.2, $\hat{V}_i$ is the space annihilated by some sufficiently high power of $(\phi_x - c(x)\nu^1)$ for all $x \in L$.

Suppose $\nu \in C_i$. Put $\hat{V}_i = \sum_{\alpha} \hat{V}_i$. Some power $m_x(t)$ of $\Pi_x(t - c^r(x)\nu^i)$ is a polynomial over $F$, and $\hat{V}_i$ is the space annihilated by $m_x(\phi_x)^r$ for all $x \in L$ and some sufficiently large $r$. Put

$$V_i = \{ v \in V \mid (m_x(\phi_x)^r v = 0 \text{ for all } x \in L) \}.$$ 

The set of conditions $m_x(\phi_x)^r v = 0$ for all $x \in L$ may be regarded as a set of linear equations over $F$ in the coordinates of $v$, so the $F$-dimension of its solution space $V_i$ in $V$ is equal to the $F$-dimension of its solution space $\hat{V}_i$ in $\hat{V}$. It follows that $V = \oplus_i V_i$. Clearly $V_i$ is a submodule of $V$ and $\text{Cl}(V_i) = C_i$.

**Theorem 4.2.** Suppose that $S \succeq L$ and let $V$ be an $L$-module. Then the components of the cluster decomposition $V = \oplus C_i V_i$ with respect to $S$ are $L$-submodules.

Proof. Although $S$ need not be a restricted algebra, it is embedded in the restricted algebra $(L, \{p\})$ and the components are defined using the operation $\{p\}$. There exists a series $S = S_0 \triangleleft S_1 \triangleleft \ldots \triangleleft S_n = L$. We use induction over $i$ to prove that $V_{C_i}$ is an $S_i$-module. Take $x \in S_i$ and consider $(x V_C + V_C)/V_C$. For $s \in S$ and $v \in V$, we have $s(xv) = x(sv) + [s, x]v$. But $[s, x] \in S_{i-1}$, so $[s, x]v \in V_C$. Thus the map $v \rightarrow xv + V_C \in (x V_C + V_C)/V_C$ is an $S$-module homomorphism. Thus the character of every composition factor of $\bar{F} \otimes_F ((x V_C + V_C)/V_C)$ is in $C$, which implies that $x V_C \subseteq V_C$.

**Remark 4.3.** The decomposition given by Theorem 4.2 depends on the $p$-operation, not merely on the algebra $S$. Changing the $p$-operation may change the decomposition as is shown by the following example. This opens the possibility that, where the minimal $p$-envelope of $S$ has non-trivial centre, judicious variation of the $p$-operation may give useful different direct decompositions.

**Example 4.4.** Let $L = \langle a_1, a_2 \mid [a_1, a_2] = 0 \rangle$ and let $V = \langle v_1, v_2 \rangle$ with $a_i v_i = v_i$ and $a_i v_j = 0$ for $i \neq j$. With $a_1^{[p]} = 0$ and $a_2^{[p]} = -a_1$, $V$ has the character $\nu$ with $c(a_1) = 0$ and $c(a_2) = 1$. The cluster decomposition with respect to $(L, \{p\})$ is simply $V = V_C$. However, with the $p$-operation $\{p\}$ with $a_i^{[p]} = 0$, the submodule $\langle v_1 \rangle$ has character $c_1$ with $c_1(a_1) = 1$ and $c_1(a_2) = 0$, while $\langle v_2 \rangle$ has character $c_2$ with $c_2(a_1) = 0$ and $c_2(a_2) = 1$. This gives the cluster decomposition $V = V_{c_1} \oplus V_{c_2}$.

5. $\mathfrak{F}$-CENTRAL AND $\mathfrak{F}$-ECCENTRIC MODULES

Let $\mathfrak{F}$ be a saturated formation of soluble Lie algebras over $F$. Comparing Theorem 4.2 with [3, Lemma 1.1] suggests a further relationship between clusters and saturated formations beyond that of [3, Theorem 6.4].

**Theorem 5.1.** Let $\mathfrak{F}$ be a saturated formation and suppose $S \in \mathfrak{F}$. Let $(L, \{p\})$ be a $p$-envelope of $S$ and suppose that $z^{[p]} = 0$ for all $z$ in the centre of $L$. Let $V, W$ be irreducible $S$-modules. Suppose that $\text{Cl}(V) = \text{Cl}(W)$. Then $V, W$ are either both $\mathfrak{F}$-central or both $\mathfrak{F}$-eccentric.

Proof. Suppose to the contrary, that $V$ is $\mathfrak{F}$-central and that $W$ is $\mathfrak{F}$-eccentric. By [2, Theorem 2.3], $\text{Hom}(V, W)$ is $\mathfrak{F}$-hypereccentric. But from [4, Theorem 5.2.7] it follows that the characters of the composition factors of $\text{Hom}(\bar{V}, \bar{W})$ are all of the
functions $c_2 - c_1$ where $c_1 \in \text{Cl}(V)$ and $c_2 \in \text{Cl}(W)$. Since $\text{Cl}(V) = \text{Cl}(W)$, we have that $0 \in \text{Cl}(\text{Hom}(V,W))$.

By assumption, we have that $z[p] = 0$ for all $z$ in the centre of $L$. As $(L,[p])$ is a $p$-envelope of $S$, we have $S \unlhd L$. By [3, Theorem 6.4], a composition factor $X$ of $\text{Hom}(V,W)$ with $\text{Cl}(X) = \{0\}$ is $\delta$-central, contrary to $\text{Hom}(V,W)$ being $\delta$-hypereccentric.

\section{6. C-induced modules}

Let $(L,[p])$ be a restricted Lie algebra over the perfect field $F$ and let $S$ be a $[p]$-subalgebra of $L$. Let $W$ be an $S$-module and let $C$ be a cluster of characters of $L$ whose restriction to $S$ is $\text{Cl}(W)$. We require that distinct members of $C$ have distinct restrictions to $S$ in which case, we say that $C$ restricts simply to $S$.

Note that, given a simple cluster $C_S$ of $S$, we can easily construct a cluster $C$ of $L$ which restricts simply to $C_S$. We take a cobasis $\{e_1, \ldots, e_n\}$ of $S$ in $L$, that is, a basis of some subspace complementary to $S$. A character $c : S \to \bar{F}$ can be extended to $L$ by assigning arbitrarily the values $c(e_i) \in \bar{F}$. If these are chosen in $F[c]$, then any automorphism which fixes the given $c$ also fixes its extension.

We want to apply the construction of $c$-induced modules (see Strade and Farnsteiner [4, Section 5.6]) to the $c$-components $\bar{W}_c$ of $W$. This construction only works for modules with character $c$. Every composition factor of $\bar{W}_c$ has character $c$, but $\bar{W}_c$ itself need not. This leads to the following definition.

\textbf{Definition 6.1.} We say that the $S$-module $W$ is amenable (for induction) if, for all $c \in \text{Cl}(W)$, $\bar{W}_c$ has character $c$.

Note that if $\bar{W}_c$ has character $c$, then for each conjugate $c^\alpha$ of $c$, $\bar{W}_{c^\alpha}$ has character $c^\alpha$. It would be nice to have a way of determining if a module $W$ is amenable which does not require analysis of $W$. The following lemmas achieve that.

\textbf{Lemma 6.2.} Let $\{s_1,\ldots,s_n\}$ be a basis of $S$ and let $W$ be an $S$-module. Let $m_i(t)$ be the minimum polynomial of $\phi_{s_i}$. Then $W$ is amenable if and only if for all $i$, $\gcd(m_i(t),m'_i(t)) = 1$.

\textbf{Proof.} The module $W$ is amenable if and only if, for all $c \in \text{Cl}(W)$ and all $i$, we have $(\phi_{s_i} - c(s_i)^p)\bar{W}_c = 0$. So $W$ is amenable if and only if for all $i$, in $\bar{F}[t]$, $m_i(t)$ has no repeated factors, that is, if and only if $\gcd(m_i(t),m'_i(t)) = 1$. As the calculation of $\gcd(m_i(t),m'_i(t))$ in $\bar{F}[t]$ is the same as in $F[t]$, the result follows.

\textbf{Lemma 6.3.} Let $W$ be an irreducible $S$-module. Then $W$ is amenable.

\textbf{Proof.} For any $s \in S$, $s^p - s^{[p]}$ is in the centre of the universal enveloping algebra of $S$ and so, for any representation $\rho$ of $S$, we have $[\rho(s_1)^p - \rho(s_1^{[p]})], \rho(s_2)] = 0$ for all $s_1, s_2 \in S$. For $c \in \text{Cl}(W)$, put $f_s(t) = \Pi(t - c^\alpha(s)^p)$ where the product is taken over the distinct conjugates of $c(s)$. Then $f_s(t)$ is a polynomial over $F$, and $\rho(s_2)$ commutes with $f_s(\phi_{s_1})$ for all $s_1, s_2 \in S$. Thus $W_0 = \{w \in W \mid f_s(\phi_s)w = 0 \text{ for all } s \in S\}$ is a submodule of $W$. The conditions $f_s(\phi_s)w = 0$ are linear equations over $F$ with non-zero solutions over $\bar{F}$ and so have non-zero solutions over $F$. Thus $W_0 \neq 0$ which implies $W_0 = W$.

As the construction being developed can be applied separately to each direct summand of $W$, we suppose that $C$ is simple. Take a basis $\{b^1,\ldots,b^k\}$ of $W$.}


Corresponding to each \( c \in C \), we have a component \( \bar{W}_c \) of \( \bar{W} = \bigoplus_c \bar{W}_c \). For each \( w \in \bar{W} \), we have \( w = \sum_c w_c \) with \( w_c \in \bar{W}_c \).

**Lemma 6.4.** Let \( w = \sum \lambda_i b^i \in \bar{W} \). Then \( w \) is invariant under the automorphisms of \( \bar{F}/\bar{F} \) if and only if the \( \lambda_i \in F \), in which case, \( (w_c)^{\alpha} = w_{c^{\alpha}} \). Further, \( sw \) is also invariant for all \( s \in S \).

**Proof.** If \( w = \sum \lambda_i b^i \) is invariant, then \( \lambda_i \) is invariant. Since \( F \) is perfect, this implies \( \lambda_i \in F \). If \( \lambda_i \in F \) for all \( i \), then clearly \( w \) is invariant. As \( W \) is an \( S \)-module, also \( sw \) is invariant. The action of \( \alpha \) permutes the \( \bar{W}_c \) and does not change the direct decomposition. That \( (w_c)^{\alpha} = w_{c^{\alpha}} \) follows. \( \square \)

Suppose that \( C \) is a simple cluster of characters of \( L \) which restricts simply to \( S \) and that \( \bar{W} \) is an amenable \( \bar{S} \)-module with \( \text{Cl}(\bar{W}) = \bar{C}\bar{S} \). For each \( c \in C \), we form the \( c \)-reduced enveloping algebras \( u(L, c) \) and \( u(S, c) \). (See Strade and Farnsteiner [4] page 226.) Since \( \bar{W} \) is amenable, we can construct the \( c \)-induced \( L \)-modules

\[
\bar{V}_c = \text{Ind}_{\bar{S}}^L(\bar{W}_c, c) = u(L, c) \otimes_{u(S, c)} \bar{W}_c
\]

and put \( \bar{V} = \bigoplus_c \bar{V}_c \). From \( \bar{V} \), we shall select an \( F \)-subspace \( V \) with \( \bar{F} \otimes_F V = \bar{V} \), which we shall show to be an \( L \)-module with \( \text{Cl}(V) = C \).

For \( x, y \in L \) in the following, we need to distinguish their product in the associative algebra \( u(L, c) \) from their product in the Lie algebra. We denote the Lie algebra product by \([x, y]\). Take a cobasis \( \{e_1, \ldots, e_n\} \) for \( S \) in \( L \). Then elements \( e_1^{r_1} e_2^{r_2} \ldots e_n^{r_n} \otimes w_c \) with \( r_i \leq p - 1 \) and \( w_c \in \bar{W}_c \) span \( \bar{V}_c \). To simplify the notation, we write \( e(r) \) for \( e_1^{r_1} e_2^{r_2} \ldots e_n^{r_n} \). For an element \( w = \sum w_c \in \bar{W} \), it is convenient to abuse notation and write \( e(r) \otimes w \) for the element \( \sum_c e(r) \otimes w_c \). It should be remembered that in this sum, the \( e(r) \) come from different algebras \( u(L, c) \) with different multiplication, and that the tensor products are over different algebras \( u(S, c) \).

Any element \( w_c \in \bar{W}_c \) is an \( \bar{F} \)-linear combination of the \( b^i \), so an element of \( \bar{V}_c \) is expressible as an \( \bar{F} \)-linear combination of the \( e(r) \otimes b^i \). It follows that the \( e(r) \otimes b^i \) form a basis of \( \bar{V} \). An automorphism \( \alpha \) maps \( e(r) \otimes w \) to \( e(r) \otimes w^{\alpha} \). Thus the invariant elements of \( \bar{V} \) are the \( F \)-linear combinations of the basis.

**Lemma 6.5.** Let \( v \in \bar{V} \) be invariant. Then \( xv \) is invariant for all \( x \in L \).

**Proof.** We use induction over \( k \) to show that \( x_1 \ldots x_k \otimes b^i \) is invariant for all \( x_1, \ldots, x_k \in L \). The result then follows trivially.

For \( s \in S \), we have \( s(1 \otimes b^i) = 1 \otimes sb^i \) which is invariant by Lemma 6.4. For \( e_j \), we have \( e_j(1 \otimes b^i) = e_j \otimes b^i \) which is invariant. Note that in this case, the multiplication is the same in all the \( u(L, c) \). Thus the result holds for \( k = 1 \).

Suppose that \( k > 1 \). We express each of the \( x_i \) as a linear combination of the \( e_j \) and an element of \( S \). Then we use the commutation rules \( xy - yx = [x, y] \) to move each factor to its correct position, giving a sum of terms of the form \( e_1^{r_1} e_2^{r_2} \ldots e_n^{r_n} s_1 \ldots s_m \otimes b^i \), but with the \( r_j \) not restricted to be less than \( p \). The terms coming from a commutator \([x, y] \) all have fewer than \( k \) factors and so are invariant. Any elements of \( S \) at the end moves past the tensor product, giving \( e_1^{r_1} e_2^{r_2} \ldots e_n^{r_n} \otimes s_1 \ldots s_m b^i \). By Lemma 6.4, \( 1 \otimes s_1 \ldots s_m b^i \) is invariant and since, in this case, \( e_1^{r_1} \ldots e_n^{r_n} \) has fewer than \( k \) factors, the term is invariant. Thus we are left to consider terms of the form \( e_1^{r_1} \ldots e_n^{r_n} \otimes b^i \). If \( r_j < p \) for all \( j \), then the term is one of our basis elements and so is invariant.
Suppose that for some \( j \), we have \( r_j \geq p \). Then we must separate the summands. The term can be written in the form \( ee_j^p e' \otimes b^i \) where \( e, e' \) are strings of cobasis elements. In the algebra \( u(\mathcal{L}, e^\alpha) \), we have \( e_j^p = e_j^{[p]} + e^\alpha(e_j)^p 1 \). Thus

\[
ee_j^p e' \otimes b^i = ee_j^{[p]} e' \otimes b^i + \sum_{\alpha} ee' \otimes e^\alpha(e_j)^p b^i_{\alpha}.
\]

But \( ee_j^p e' \otimes b^i \) has fewer than \( k \) factors and so invariant. As \( \sum_{\alpha} 1 \otimes e^\alpha(e_j)^p b^i_{\alpha} \) is invariant and \( ee' \) has fewer than \( k \) factors, it follows that \( ee_j^p e' \otimes b^i \) also is invariant. 

We define the C-induced module \( \text{Ind}_S^L(W, C) \) to be the \( F \)-subspace of invariant elements of \( V \). By Lemma 6.5, it is an \( L \)-module. Clearly \( \text{Cl}(\text{Ind}_S^L(W, C)) = C \).

To illustrate this, we calculate a simple example.

Example 6.6. Let \( F \) be the field of 3 elements and let \( L = \langle x, y \mid [x, y] = y \rangle \). Putting \( x^{[p]} = x \) and \( y^{[p]} = 0 \) makes this a restricted Lie algebra. We take \( S = \langle x \rangle \) and \( W = \langle b^1, b^2 \rangle \) with \( x b^1 = b^2 \) and \( x b^2 = -b^1 \). Over \( F \), we have \( W = W_1 \oplus W_2 \) with \( W_1 = (-b^1 - ib^2) \) and \( W_2 = (-b^1 + ib^2) \), where \( i \in F, i^2 = -1 \). Denote the action of \( S \) on \( W \) by \( p \). Then \( \rho(x)(-b^1 - ib^2) = -i(-b^1 - ib^2) \) and \( \rho(x)(-b^1 + ib^2) = i(-b^1 + ib^2) \).

We have \( \rho(x)^p - \rho(x^{[p]}) = -i(-i)^3 - (-i)(-b^1 - ib^2) = -i(-b^1 - ib^2) \).

Thus the character of \( W_1 \) must have \( c_1(x)^3 = -i \), so \( c_1(x) = i \). Similarly, we have \( c_2(x) = -i \). As distinct conjugates of a character on \( L \) in \( M \) must have distinct restrictions to \( S \), \( c_1(y) \in F[i] \). Put \( c_1(y) = \lambda = \alpha + i\beta \) where \( \alpha, \beta \in F \). Then \( c_2(y) = \bar{\lambda} \).

Note that in \( u(\mathcal{L}, c_1) \), \( y^3 = \lambda^3 = \bar{\lambda} \). In both the algebras, \( xy = y + xy \) and \( xy^2 = (y+xy)y = y^2 + y(xy) = -y^2 + y^2 x \).

In the notation used above, we have \( b_{c_1}^1 = -b^1 - ib^2 \) and \( b_{c_2}^1 = -b^1 + ib^2 \), while for \( b^2 \), we have \( b_{c_1}^2 = ib^1 - b^2 \) and \( b_{c_2}^2 = -ib^1 - b^2 \). The induced module \( V = \text{Ind}_S^L(W, C) \) has basis the six elements \( v_j^r \) for \( r = 0, 1, 2 \) and \( j = 1, 2 \). We calculate the actions of \( x, y \) on these elements.

\[
\begin{align*}
xv_0^1 &= x(1 \otimes b^1) = v_0^1 & xv_0^2 &= x(1 \otimes b^2) = -v_1^0 \\
xv_1^1 &= (y + xy) \otimes b^1 = v_1^1 + v_2^1 & xv_1^2 &= (y + xy) \otimes b^2 = v_2^1 - v_1^1 \\
xv_2^1 &= (-y^2 + y^2 x) \otimes b^1 = -v_1^2 + v_2^2 & xv_2^2 &= (-y^2 + y^2 x) \otimes b^2 = -v_2^2 - v_1^2 \\
yv_1^0 &= v_1^1 & yv_1^2 &= v_2^1 \\
yv_1^1 &= v_1^2 & yv_1^2 &= v_2^2
\end{align*}
\]

The calculations of \( yv_2^2 \) are more complicated.

\[
\begin{align*}
yv_1^1 &= y^3 \otimes (b_{c_1}^1 + b_{c_2}^1) = 1 \otimes (\lambda b_{c_1}^1 + \lambda b_{c_2}^1) \\
&= 1 \otimes ((\alpha - i\beta)(-b^1 - ib^2) + (\alpha + i\beta)(-b^1 + ib^2)) \\
&= 1 \otimes (\alpha b^1 + \beta b^2) = \alpha v_1^0 + \beta v_0^2 \\
yv_2^1 &= y^3 \otimes (b_{c_1}^2 + b_{c_2}^2) = 1 \otimes (\lambda b_{c_1}^2 + \lambda b_{c_2}^2) \\
&= 1 \otimes ((\alpha - i\beta)(ib^1 - b^2) + (\alpha + i\beta)(-ib^1 - b^2)) \\
&= 1 \otimes (-\beta b^1 + \alpha b^2) = -\beta v_0^1 + \alpha v_2^0 \\
\end{align*}
\]
Remark 6.7. As noted earlier, in the notation used above, if we are given an amenable $S$-module $W$ with simple cluster $C_S$, we can construct a simple cluster $C$ on $L$ which restricts simply to $C_S$ by choosing arbitrarily the $c_1(e_i)$ in $F[c_1]$. If we choose the $c_1(e_i)$ in $F$, then we have $c_j(e_i) = c_1(e_i)$ for all $j$. This simplifies the calculation of the action on the induced module as we then have $e^p_i = e^{|p|}_i + c(e_i)^p 1$ in all the algebras $u(L, c_i)$ and it follows that $e^p_i b^j = (e^{|p|}_i + c(e_i)^p 1)b^j$ can be calculated without using the character decomposition of $W$. That the action of $x \in L$ on a basis element $e(r) \otimes b^j$ can be calculated without using the character decomposition follows by an induction as in the proof of Lemma 6.5. It thus becomes possible to calculate the action on $\text{Ind}_S^L(W, C)$ without having to determine the eigenvalues of the $\phi_s$. In the above example, if we take $\beta = 0$, then the calculations of $yv_2^2$ and $yv_2^3$ simplify to $yv_1^2 = y^3b^i = \alpha^3b^i = \alpha v_0^i$.

Remark 6.8. Denote the category of amenable $L$-modules with cluster $C$ by $\text{AmMod}(L, C)$. The restriction functor $\text{Res}_S^L : \text{AmMod}(L, C) \to \text{AmMod}(S, C|S)$ sends an $L$-module $V$ to $V$ regarded as an $S$-module. Suppose that $C$ restricts simply to $S$. Then $\text{Ind}_S^L(V, C)$ is a functor $\text{AmMod}(S, C|S) \to \text{AmMod}(L, C)$. In the special case where $C = \{c\}$, $\text{Ind}_S^L(V, c)$ is a left adjoint to $\text{Res}_S^L$ by Strade and Farnsteiner [4, Theorem 5.6.3]. Applying this to the $\bar{W}$ in the general case gives that $\text{Ind}_S^L(V, C)$ is a left adjoint to $\text{Res}_S^L$.

References

1. D. W. Barnes, On $F$-hypercentral modules for Lie algebras, Arch. Math. 30 (1978), 1–7.
2. D. W. Barnes, On $F$-hyperexcentric modules for Lie algebras, J. Austral. Math. Soc. 74 (2003), 235–238.
3. D. W. Barnes, Ado-Iwasawa extras, J. Austral. Math. Soc. 78 (2005), 407–421.
4. H. Strade and R. Farnsteiner, Modular Lie algebras and their representations, Marcel Dekker, Inc., New York- Basel, 1988.

1 Little Wonga Rd., Cremorne NSW 2090, Australia,
E-mail address: donwb@iprimus.com.au