Abstract

We discuss the analytic continuation of the Hadamard product of two holomorphic functions under assumptions pertaining to Écalle’s Resurgence Theory, proving that if both factors are endlessly continuable with prescribed sets of singular points $A$ and $B$, then so is their Hadamard product with respect to the set $\{0\} \cup A \cdot B$. This is a generalization of the classical Hadamard Theorem in which all the branches of the multivalued analytic continuation of the Hadamard product are considered.

1 Introduction

1.1 The Hadamard product of two power series $f(\xi), g(\xi) \in \mathbb{C}[[\xi]]$ is defined by the formula

$$f(\xi) = \sum_{n \geq 0} a_n \xi^n \quad \text{and} \quad g(\xi) = \sum_{n \geq 0} b_n \xi^n \quad \implies \quad f \odot g(\xi) := \sum_{n \geq 0} a_n b_n \xi^n. \quad (1.1)$$

If the radii of convergence of $f$ and $g$ are positive, i.e. $f(\xi), g(\xi) \in \mathbb{C}\{\xi\}$, then so is the radius of convergence of $f \odot g$. From now on, we identify a convergent power series with the holomorphic germ at 0 that it defines. J. Hadamard published in 1898 the proof of his now classical theorem [Ha99], to the effect that:

Given finite subsets $A, B \subset \mathbb{C}^*$ and $K, L > 0$, if $f$ extends analytically to $\mathbb{D}_K \setminus \bigcup_{\alpha \in A} R_\alpha$ and $g$ to $\mathbb{D}_L \setminus \bigcup_{\beta \in B} R_\beta$, then $f \odot g$ extends analytically to $\mathbb{D}_{KL} \setminus \bigcup_{\gamma \in A \cdot B} R_\gamma$.

Here, we have denoted by $\mathbb{D}_M$ the open disc centred at the origin of radius $M$, by $R_\omega$ the portion of the ray $\omega \mathbb{R}_{\geq 0}$ from $\omega$ onward, and by $A \cdot B$ the “product set” of $A$ and $B$:

$$\mathbb{D}_M := D(0, M), \quad R_\omega := \omega [1, +\infty), \quad A \cdot B := \{\alpha \beta \mid (\alpha, \beta) \in A \times B\}. \quad (1.2)$$

So, this is a statement about the “principal branches” of $f$, $g$ and $f \odot g$: we are considering their analytic continuation to a part of their Mittag-Leffler stars, assuming that any singular point that appears for $f$ or $g$ when following a ray emanating from the origin is isolated (in
fact, J. Hadamard also proves a more general version, in which the various $R_\omega'$s are interpreted as portion of logarithmic spirals of the same polar slope).

In [Bo98], É. Borel studied the analytic continuation of $f \odot g$, also giving a proof of Hadamard’s theorem, with a view to giving more information on the nature of the singularities of $f \odot g$ in terms of the nature of the singularities of $f$ and $g$, at least in the case of “uniform singularities” (i.e. with trivial monodromy: poles or essential singularities). We refer the reader to [PM20a], [PM20b] for a modern study and a generalization of this result.

Interestingly, Borel also mentions something about the analytic continuation of $f$, $g$ and $f \odot g$ to other sheets than the principal one,¹ and he gives a valuable hint in a footnote which has been successfully used in the context of Écalle’s Resurgence Theory to study the convolution product $f * g$—see §1.4. It is our aim to consider other branches than the principal one for the analytic continuation of $f \odot g$, under appropriate assumptions on $f$ and $g$.

1.2 In this article, we are concerned with the case when $f$ and $g$ satisfy a property pertaining to Écalle’s Resurgence Theory [Eca81], namely a restricted form of “endless continuability”:

Definition 1.1. Given a non-empty closed discrete subset $\Omega$ of $\mathbb{C}$, what we call “$\Omega$-continuable germ” is a holomorphic germ $f$ at the origin that admits analytic continuation along any path $\gamma: [0,1] \to \mathbb{C} \setminus \Omega$ that has its initial point $\gamma(0)$ in the disc of convergence of $f$.

This is the definition that has been employed in [Eca81] with $\Omega = 2\pi i \mathbb{Z}$, or in [Sau13], [Sau15], [MS16]. It already covers a range of interesting examples, e.g. in complex dynamics. The simplest examples of $\Omega$-continuable germs are provided by meromorphic functions and algebraic functions; however, the definition puts no restriction on the nature of the singularities of the analytic continuation of an $\Omega$-continuable germ, only on their location. More general is the definition of “endless continuability” [CNP93], [KS20], and even more general that of “continuability without a cut” [Eca85].

Our main result is

Theorem A. Given non-empty closed discrete subsets $A, B \subset \mathbb{C}$, if $f$ is an $A$-continuable germ and $g$ a $B$-continuable germ, then

$$\Omega := \{0\} \cup (A \cdot B)$$

(1.3)

is closed and discrete and $f \odot g$ is an $\Omega$-continuable germ.

The proof is given in Sections 2–3.

Note that in Definition 1.1 we may have 0 belonging to $\Omega$ or not, but in all cases does an $\Omega$-continuable germ have a principal branch regular at 0. Simple examples with $\Omega = \{0,1\}$ are $f_1(\xi) := -\frac{1}{\xi} \log(1-\xi)$ and $\text{Li}_2(\xi) := \int_0^\xi f_1(\xi_1) d\xi_1$. So, Theorem A is compatible with the Hadamard Theorem, and J. Hadamard is right not to include 0 among the possibly singular points of $f \odot g$ as far as only the principal branches are concerned.

The necessity of including 0 among the possibly singular points of $f \odot g$ when we consider its analytic continuation on other sheets was already noted by Borel, who gives credit to E. Lindelöf for that point [Bo98, ftn 2]. A simple example that illustrates it is $f_0(\xi) := -\log(1-\xi)$: the germ $f_0$ is $\{1\}$-continuable, but the germ $f_0 \odot f_0$ is not; in fact, $f_0 \odot f_0 = \text{Li}_2$ is $\{0,1\}$-continuable.

¹In this article, when we speak of “principal” sheet or “principal” branch, we mean that we refer to the analytic continuation along straight line segments starting at the origin when possible, i.e. we may identify the principal sheet with the Mittag-Leffler star.
1.3 The formal Borel transform $\mathcal{B} : t \mathbb{C}[[t]] \to \mathbb{C}[[\xi]]$ is defined by:

$$\tilde{f}(t) = \sum_{n \geq 0} c_n t^{n+1} \implies \mathcal{B}\tilde{f}(\xi) = \sum_{n \geq 0} \frac{c_n}{n!} \xi^n. \quad (1.4)$$

In other words, $\mathcal{B}\tilde{f}(\xi) = e^\xi \circ \tilde{f}(\xi)$. A formal power series $\tilde{f}(t) \in t \mathbb{C}[[t]]$ is said to be an $\Omega$-resurgent series when its Borel transform $\mathcal{B}\tilde{f}$ is an $\Omega$-continuable germ [Eca81], [Sau13]. The singularities of $\mathcal{B}\tilde{f}$ then give us information on the divergence of $\tilde{f}(t)$, inasmuch as the Borel transform of a convergent power series has no singularity at all.

Now,

$$f = \mathcal{B}\tilde{f} \quad \text{and} \quad g = \mathcal{B}\tilde{g} \quad \implies \quad f \odot g = \mathcal{B}(e_1 \odot \tilde{f} \odot \tilde{g}), \quad (1.5)$$

where $e_1(t) := te^t$, hence our result can be rephrased as

**Given non-empty closed discrete subsets $A, B \subset \mathbb{C}$, if $\tilde{f}(t)$ is an $A$-resurgent series and $\tilde{g}(t)$ a $B$-resurgent series, then $e_1 \odot \tilde{f}(t) \odot \tilde{g}(t)$ is an $\Omega$-resurgent series with $\Omega$ as in (1.3).**

This is to be compared with Theorem 6.27 of [MS16], which can be rephrased as:

**Given non-empty closed discrete subsets $A, B \subset \mathbb{C}$ such that $\Omega := A \cup B \cup (A + B)$ is closed and discrete, if $\tilde{f}(t)$ is an $A$-resurgent series and $\tilde{g}(t)$ a $B$-resurgent series, then their product $\tilde{f}(t) \odot \tilde{g}(t)$ is an $\Omega$-resurgent series.**

The latter statement is a variation on a result on the multiplication of $\Omega$-resurgent series proved in [Sau13]. More generally, the stability under multiplication of the space of resurgent series (defined as Borel preimages of endlessly continuable germs or of germs continuable without a cut) is a key fact of Écalle’s Resurgence Theory.

1.4 The shift of indices in the definition of $\mathcal{B}$ (mapping $t^{n+1}$ to $\xi^n/n!$) is a convenient normalization which results in the following relation between the ordinary multiplication in $\mathbb{C}[[t]]$ and the convolution $\ast$ in $\mathbb{C}[[\xi]]$:

$$\mathcal{B}(\tilde{f} \cdot \tilde{g}) = f \ast g(\xi) := \int_0^\xi f(\xi_1)g(\xi - \xi_1) \, d\xi_1. \quad (1.6)$$

This is the usual “additive” convolution (associated with Laplace transform and Borel-Laplace summation), whereas the Hadamard product may be considered as a “multiplicative” convolution. Indeed, as was well-known already in Hadamard’s time,

$$0 < \rho < R_f \quad \text{and} \quad |\xi| < \rho R_g \implies f \odot g(\xi) = \frac{1}{2\pi i} \oint_{C_\rho} f(\zeta)g(\frac{\xi}{\zeta}) \frac{d\zeta}{\zeta}, \quad (1.7)$$

where $C_\rho$ is the parametrized circle: $s \in [0,1] \mapsto \zeta = \rho e^{2\pi is}$ (we have denoted by $R_f$ and $R_g$ the radii of convergence of $f$ and $g$).

Actually, Formula (1.7) is the basis of Hadamard’s and Borel’s arguments in [Ha99] and [Bo98], and it will be the starting point of our analysis as well. The idea, translated from Borel’s own words, is that “this expression of $f \odot g$ stays valid if one deforms the integration contour without letting it cross any singular point of the integrand” and, “the contour having
been fixed in an arbitrary manner, one obtains the analytic continuation of \( f \circ g \) by moving \( \xi \) in the plane, provided the singular points of the integrand do not cross the integration contour”.

It seems that, throughout [Bo98], Borel keeps in mind the possibility of going to the non-principal sheets and dealing with multivalued analytic continuation, yet reluctantly so, since when he explicitly mentions that possibility (in the aforementioned footnote 2 or later in Sec. III when he writes “if the functions \( f \) and \( g \) were multivalued...”) he tends to recommend to discard it (see the last paragraph of his Sec. III: “It seems to us useless to insist on the latter point... one would be led to complicated statements...”). However, he puts in a footnote (footnote 3) an idea that has been successfully adapted to study the analytic continuation of the additive convolution (1.6) of endlessly continuable germs in the context of Resurgence [Eca81], [CNP93].

We reproduce here the content of footnote 3 of [Bo98], with minor notational changes. Borel writes (1.7) as \( f \circ g(\xi) = \frac{1}{2\pi i} \oint_C f(x) g(\xi x) \frac{dx}{x} \), with \( C = C_{\rho^{-1}} \).

“Let us conceive the closed contour \( C \) as a flexible extensible thread, the singular points of \( f(\frac{1}{x}) \) as pins stuck into the plane, the singular points of \( g(\xi x) \) as pins that travel as \( \xi \) moves. It is necessary and sufficient that the thread always part the two systems of pins. Now, this will always be possible, by means of a suitable deformation, if, while travelling, the second pins never come to hit the first ones (...); the thread may acquire a very complicated form, but this is harmless.” [Bo98, ftn 3, p. 240]

In the case of the additive convolution product (1.6), instead of the closed contour \( C = C_{\rho^{-1}} \), it is the line segment \([0, \xi]\) that must be deformed. In [Sau13] and [MS16], it was shown how to construct explicitly this deformation when \( \xi \) moves in the complex plane without meeting \( A \cup B \cup (A + B) \); this is done by applying to the initial integration contour a homeomorphism that is a deformation of the identity obtained as the flow of an explicit non-autonomous vector field tailored to the situation.

A benefit of such a detailed rigorous proof with respect to the arguments given in [Eca81] or [CNP93] is that it allows for quantitative estimates. These estimates, in turn, can be used to show the stability of the space of resurgent series under nonlinear operations and not only multiplication [Sau15], [MS16], [KS20].

1.5 Our proof of Theorem A is spread over

- Section 2, which gives details on the kind of “deformation of the identity” that we need to deform the integration contour in (1.7),
- Section 3, which explains how to obtain this deformation of the identity out of the flow of an explicit non-autonomous vector field.

This proof bears some resemblance with the proof of the theorem on convolution presented in [Sau13], however a number of modifications were necessary, notably a novel method to construct a non-autonomous vector field adapted to Formula (1.7) (Section 3).

We were led to Theorem A while studying the Borel transform of the Moyal star product of two resurgent series \( \tilde{f}(t,q,p) \) and \( \tilde{g}(t,q,p) \) [LSS20]. Indeed, the Borel counterpart of the Moyal star product can be written in terms of the Borel transforms of the factors, \( f(\xi, q, p) \) and \( g(\xi, q, p) \), and the formula appears as a mixture of additive convolution with respect to \( \xi \) and Hadamard product; more specifically, it involves the Hadamard product with respect to \( \zeta \) of
the Taylor expansions $f(\xi_1, q, p + \zeta) \circ g(\xi_2, q, \zeta, p)$ and then a convolution-like integration with respect to $\xi_1$ and $\xi_2$. However, in such a many-variable context, the restrictions one needs to put on the singular locus of $f$ and $g$ are more stringent than in Definition 1.1, namely one requires “algebro-resurgence” [GGS14], [LSS20]: $f, g \in \mathbb{C}\{\xi, q, p\}$ have analytic continuation away from a proper algebraic subvariety of $\mathbb{C}^3$ (or $\mathbb{C}^{2N+1}$ if one is interested in deformation quantization of $\mathbb{C}^{2N}$ with coordinates $q_1, \ldots, q_N, p_1, \ldots, p_N$). As a result, the functions of $\zeta$ involved in the Hadamard product part of the formula have at most finitely many singularities.

Our work appears as complementary to the one of [PM20a] and [PM20b], which is devoted to the nature of the singularities of the principal branch of $f \circ g$ and contains new formulas for the monodromy at a given point $\omega = \alpha \beta$. Considering the monodromy operator $\Delta_\omega$ forces to go around $\omega$ and visit other sheets of the Riemann surface of $f \circ g$, but only the nearby sheets (the “next-to-principal” one for $\Delta_\omega(f \circ g)$, or the ones reached by circling several times around $\omega$ if one wants to iterate $\Delta_\omega$). Dealing with these nearby branches of the analytic continuation of the Hadamard product requires a deformation of the integration contour in (1.7) that is tractable by means of relatively elementary geometry; the deformation becomes much more intricate if the variable $\xi$ is allowed to travel to arbitrary sheets, but it can still be mathematically described and studied by following the method of Sections 2–3. Therefore, it would be interesting to see whether this can be used to study the nature of the singularities of the non-principal branches of $f \circ g$ in the spirit of [PM20a] and [PM20b]. Moreover, as noted in [PM20a], one might try to explore more deeply the connection with Écalle’s Resurgence Theory; the so-called “alien operators”, designed to study the singularities of endlessly continuable functions, behave particularly well in relation with the additive convolution (1.6) and they lead to contour deformations reminiscent of those employed in [PM20a]—see e.g. pp. 236–237 of [MS16].

Another avenue of research that naturally suggests itself would be to consider the Hadamard product of endlessly continuable germs $f$ and $g$ which are not $\Omega$-continuable in the sense of Definition 1.1 for any discrete closed set $\Omega$. One may conjecture that $f \circ g$ is still endlessly continuable because, during the process of analytic continuation by means of contour deformation, one “feels” the presence of only finitely many singularities of $f$ and $g$ at the same time, which is why (additive) convolution (and even iterated convolutions) could be handled in [KS20]; one might try to apply similar techniques to the Hadamard product.

## 2 Deformations of the identity adapted to a path $\gamma$

We now begin the proof of Theorem A. We thus give ourselves $A$ and $B$ non-empty closed discrete subsets of $\mathbb{C}$, and we define $\Omega$ by (1.3). We also introduce

$$A' := A \setminus \{0\}, \quad a := \min\{|\alpha| \mid \alpha \in A'\}, \quad B' := B \setminus \{0\}, \quad b := \min\{|\beta| \mid \beta \in B'\},$$

(2.1)

regardless of $0$ belonging to $A \cup B$ or not. Note that

$$\Omega = \{0\} \cup (A' \cdot B').$$

(2.2)

### 2.1 The first claim to be proved is

**Lemma 2.1.** The set $\Omega$ is closed and discrete.
Proof. Clearly, this is equivalent to saying that

\[
\text{For any } R \in \mathbb{R}_{>0}, \quad \Omega \cap \mathbb{D}_R \text{ is finite.} \quad (2.3)
\]

Let \( R \in \mathbb{R}_{>0} \). Any pair \((\alpha, \beta) \in A' \times B'\) such that \( \alpha, \beta \in \mathbb{D}_R \) must satisfy \( |\alpha| \geq a > 0, |\beta| \geq b > 0 \) and \( |\alpha\beta| \leq R \). Hence, \( \alpha \in \mathbb{D}_{R/b} \) and \( \beta \in \mathbb{D}_{R/a} \). But \( A \cap \mathbb{D}_{R/b} \) and \( B \cap \mathbb{D}_{R/a} \) are finite, by the rephrasing (2.3) of our assumption on \( A \) and \( B \), hence there only finitely many such pairs \((\alpha, \beta)\) and \( \Omega \cap \mathbb{D}_R \) is finite. \( \square \)

2.2 Let \( I := [0,1] \). We now give ourselves an \( A \)-continuable germ \( f \), a \( B \)-continuable germ \( g \), and a path \( \gamma : I \to \mathbb{C} \setminus \Omega \) that has its initial point close enough to the origin, specifically

\[
0 < |\gamma(0)| < ab. \quad (2.4)
\]

Let us choose \( \rho > 0 \) such that

\[
|\gamma(0)| \frac{b}{a} < \rho < a. \quad (2.5)
\]

Note that the radius of convergence \( R_f \) of \( f \) is at least \( a \), and the radius of convergence \( R_g \) of \( g \) at least \( b \), hence Formula (1.7) implies that

\[
f \circ g(\xi) = \frac{1}{2\pi i} \oint_{C_{\rho}} f(\zeta) g\left(\frac{\xi}{\zeta}\right) \frac{d\zeta}{\zeta} \quad \text{for } \xi \text{ close enough to } \gamma(0). \quad (2.6)
\]

To prove Theorem A, we just need to prove that \( f \circ g \) admits analytic continuation along \( \gamma \).

Without loss of generality, we may suppose that \( \gamma \) is \( C^1 \) with a derivative \( \gamma' \) that is Lipschitz.

2.3 We now make precise the notion of deformation of the identity that we have alluded to in the introduction:

Definition 2.2. We call “deformation of the identity adapted to \( \gamma \)” any family \(( \Psi_t )_{t \in I} \) of Lipschitz homeomorphisms of \( \mathbb{C} \) such that \( \Psi_0 = \text{Id}_\mathbb{C} \) and

\[
\alpha \in \{0\} \cup A' \implies \Psi_t(\alpha) = \alpha \quad \text{for all } t \in I, \quad (2.7)
\]

\[
\beta \in B' \implies \Psi_t\left(\frac{\gamma(0)}{\beta}\right) = \frac{\gamma(t)}{\beta} \quad \text{for all } t \in I. \quad (2.8)
\]

Lemma 2.3. Suppose we have found \(( \Psi_t )_{t \in I} \), a deformation of the identity adapted to \( \gamma \), then the analytic continuation of \( f \circ g \) along \( \gamma \) exists and is given, at each \( \gamma(t), t \in I \), by the formula

\[
\text{cont}_{\gamma(t)}(f \circ g)(\xi) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\rho}} f(\zeta) g\left(\frac{\xi}{\zeta}\right) \frac{d\zeta}{\zeta} \quad \text{for } \xi \text{ close enough to } \gamma(0). \quad (2.9)
\]

Proof. Since each \( \Psi_t \) is a homeomorphism of \( \mathbb{C} \), condition (2.7) entails \( \Psi_t^{-1}(\alpha) = \alpha \) for \( \alpha \in A \), whence

\[
\zeta \in \mathbb{C} \setminus A \implies \Psi_t(\zeta) \in \mathbb{C} \setminus A \quad \text{for all } t \in I. \quad (2.10)
\]

We also have \( \Psi_t^{-1}(0) = 0 \) and, by condition (2.8), \( \Psi_t^{-1}\left(\frac{\gamma(t)}{\beta}\right) = \frac{\gamma(t)}{\beta} \) for all \( \beta \in B' \), whence

\[
\zeta \neq 0 \text{ and } \frac{\gamma(t)}{\zeta} \in \mathbb{C} \setminus B' \implies \Psi_t(\zeta) \neq 0 \text{ and } \frac{\gamma(t)}{\Psi_t(\zeta)} \in \mathbb{C} \setminus B' \quad \text{for all } t \in I \quad (2.11)
\]
Indeed: \( \gamma(t) = \beta \in B' \) would require \( \zeta = \Psi_t^{-1}(\gamma(t)) = \gamma(0) \beta \). Therefore, since by (2.5) all the points of \( C_\rho \) satisfy the requirements indicated in the left-hand sides of (2.10)–(2.11), the integral in the right-hand side of (2.9) is a well-defined function \( h_t(\xi) \) of \( \xi \), which represents a holomorphic germ at \( \gamma(t) \).

The Cauchy theorem entails that, for \( t < t' \) close enough, the functions \( h_t \) and \( h_{t'} \) coincide in a neighbourhood of \( \gamma([t,t']) \), hence \( h_t \) is the analytic continuation along \( \gamma \) at \( \gamma(t) \).

3 Construction of a non-autonomous vector field

3.1 To conclude the proof of Theorem A, it is thus sufficient to find a deformation of the identity in the sense of Definition 2.2. Our tool will be the flow map induced by a non-autonomous vector field.

Lemma 3.1. Suppose \( X : I \times \mathbb{C} \to \mathbb{C} \) is locally Lipschitz (identifying \( \mathbb{C} \) and \( \mathbb{R}^2 \)) and satisfies

\[
|X(t,\zeta)| \leq K|\zeta|
\]  

with some \( K > 0 \). Then, for each \( t^* \in I \) and each initial condition \( \zeta_0 \in \mathbb{C} \), the non-autonomous vector field

\[
\frac{d\zeta}{dt} = X(t,\zeta)
\]

has a unique maximal solution \( t \mapsto \Phi^{t^*,t}(\zeta_0) \) such that \( \Phi^{t^*,t}(\zeta_0) = \zeta_0 \). This maximal solution is defined for all \( t \in I \).

Moreover, the map \((t^*,t,\zeta) \in I \times I \times \mathbb{C} \mapsto \Phi^{t^*,t}(\zeta) \in \mathbb{C} \) thus defined is locally Lipschitz, and \((\Phi^{t^*,t})_{t^*,t\in I} \) is a family of locally Lipschitz homeomorphisms of \( \mathbb{C} \) satisfying

\[
\Phi^{t^*,t*} \circ \Phi^{t^*,t} = \Phi^{t^*,t*}, \quad \Phi^{t^*,t} = \text{Id}_{\mathbb{C}}.
\]  

Proof. Apply the classical Cauchy-Lipschitz theorem about the existence and uniqueness of solutions to systems of ordinary differential equations and their regular dependence upon the initial condition. The maximal solutions are defined for all times because of the bound (3.1). □

Corollary 3.2. Suppose that we have found \( c : I \times \mathbb{C} \to \mathbb{C} \) locally Lipschitz, bounded by \( K|\zeta| \) with some \( K > 0 \), such that

\[
\alpha \in \{0\} \cup A' \implies c(t,\alpha) = 0 \quad \text{for all } t \in I,
\]

\[
\beta \in B' \implies c\left(t, \frac{\gamma(t)}{\beta}\right) = \frac{1}{\beta} \quad \text{for all } t \in I.
\]

Then the flow map \( \Psi_t := \Phi^{0,t} \) of the non-autonomous vector field

\[
X(t,z) := c(t,z) \gamma'(t)
\]

gives rise to a deformation of the identity adapted to \( \gamma \).

Proof. We have \( \Psi_t(\alpha) = \Phi^{0,t}(\alpha) = \alpha \) for all \( \alpha \in A \) by (3.4), and (3.5) shows that \( t \mapsto \frac{\gamma(t)}{\beta} \) is a particular solution of \( X \) for each \( \beta \in B' \), hence \( \frac{\gamma(t)}{\beta} = \Phi^{0,t}(\frac{\gamma(0)}{\beta}) = \Psi_t(\frac{\gamma(0)}{\beta}) \). □
3.2 Thus, we are just left with the question of finding a function $c$ meeting the requirements of Corollary 3.2. Since $\Omega$ is closed and $\gamma(I)$ is compact, we can pick $\delta, M > 0$ so that, for each $t \in I$,
\[
\delta \leq |\gamma(t)| \leq M, \quad |\gamma(t) - \alpha \beta| \geq \delta \quad \text{for all } (\alpha, \beta) \in A \times B. \quad (3.7)
\]

**Lemma 3.3.** If $\varepsilon > 0$ is chosen small enough so that

\[
\varepsilon < \frac{a}{2}, \quad \frac{a \delta}{\varepsilon} > M + \frac{2M}{a} \varepsilon, \quad (3.8)
\]

then

\[
\beta \in B' \implies \text{dist} \left( \frac{\gamma(t)}{\beta}, A' \right) \geq \varepsilon \quad \text{for all } t \in I. \quad (3.9)
\]

**Proof.** Suppose $|\frac{\gamma(t)}{\beta} - \alpha| < \varepsilon$ with $\alpha \in A'$. We would have

\[
\delta \leq |\gamma(t) - \alpha \beta| < \varepsilon |\beta| \quad (3.10)
\]

by (3.7b). We also have $|\alpha \beta| \geq a|\beta|$ by (2.1) and $|\gamma(t)| \leq M$ by (3.7a), whence

\[
a|\beta| \leq \varepsilon |\beta| + M \quad (3.11)
\]

by the triangle inequality. Now, either $|\beta| \geq \frac{2M}{a}$, but then $a|\beta| \leq \varepsilon |\beta| + M \leq (\varepsilon + \frac{\varepsilon}{a})|\beta|$ leads to a contradiction with (3.8a); or $|\beta| < \frac{2M}{a}$, but then $\frac{a \delta}{\varepsilon} < a|\beta|$ by (3.10), which is $\leq M + |\beta|\varepsilon$ by (3.11), whence the inequality $\frac{a \delta}{\varepsilon} < M + \frac{2M}{a} \varepsilon$, which contradicts (3.8b).

Let us choose a Lipschitz function $\eta = \eta_{\varepsilon, A'} : \mathbb{C} \to [0, 1]$ such that

\[
\alpha \in A' \implies \eta(\alpha) = 0, \quad \text{dist}(\zeta, A') \geq \varepsilon \implies \eta(\zeta) = 1. \quad (3.12)
\]

For instance, we may pick a Lipschitz function $\chi_{\varepsilon} : \mathbb{R}_{\geq 0} \to [0, 1]$ such that $\chi_{\varepsilon}(0) = 0$ and

\[
d \geq \varepsilon \implies \chi_{\varepsilon}(d) = 1,
\]

e.g. $\chi_{\varepsilon}(d) := \min\left\{ \frac{d}{\varepsilon}, 1 \right\}$, and define $\eta(\zeta) := \chi_{\varepsilon}\left( \text{dist}(\zeta, A') \right)$.

We conclude by checking that the function defined by

\[
c(t, \zeta) := \eta(\zeta) \frac{\zeta}{\gamma(t)} \quad (3.13)
\]

fulfils all the requirements of Corollary 3.2:

- The function $c$ is locally Lipschitz and bounded by $\frac{1}{\beta} |\zeta|$, by virtue of (3.7a), because $|\eta| \leq 1$.
- We have $c(t, 0) = 0$.
- If $\alpha \in A'$, then $c(t, \alpha) = 0$ because (3.12a) says that $\eta(\alpha) = 0$.
- If $\beta \in B'$, then (3.9) says that $\text{dist} \left( \frac{\gamma(t)}{\beta}, A' \right) \geq \varepsilon$, hence $c(t, \frac{\gamma(t)}{\beta}) = \eta \left( \frac{\gamma(t)}{\beta} \right) \frac{1}{\beta} = \frac{1}{\beta}$ by (3.12b).
Therefore, the proof of Theorem A is complete.

3.3 Remark. When $B$ is a finite set, the construction of §3.2 can be replaced by one closer to that of [Sau13], namely:

$$c(t, \zeta) := \sum_{\beta_1 \in B'} \frac{\eta_{\beta_1}(\zeta, t)}{\eta_{\beta_1}(\zeta, t) + |\zeta - \frac{\gamma(I)}{\beta_1}|} \cdot \frac{1}{\beta_1},$$

with $\eta_{\beta_1}(\zeta, t) := \text{dist} (\zeta, \{0\} \cup A' \cup B(\beta_1, t))$,

(3.14)

where $B(\beta_1, t) := \{ \gamma(I)_{\beta_2} | \beta_2 \in B' \setminus \{\beta_1\} \}$. Indeed, the denominators are all always $> 0$ precisely because $\gamma(I)$ and $\{0\} \cup (A' \cdot B')$ do not intersect.

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