On Hopf adjunctions, Hopf monads and Frobenius-type properties

Adriana Balan†

Abstract
A triple adjunction $L \dashv U \dashv R$ between monoidal categories, with $U$ strong monoidal, is called a biHopf adjunction if $L \dashv U$ is a Hopf adjunction and $U \dashv R$ is a coHopf adjunction. We show that any biHopf adjunction determines a linearly distributive functor $(R, L)$. We investigate when such a biHopf adjunction is ambidextrous ($L \cong R$), and when $L$ and $R$ become Frobenius monoidal functors.

We apply the obtained results to Hopf monads: we show that under suitable exactness assumptions, a Hopf monad $T$ on a monoidal category $C$ having as right adjoint a Hopf comonad $G$ is a Frobenius monad if $T \cong G$ and $G \cong$ are isomorphic (right) Hopf $T$-modules (in particular, $T \cong$ is a Frobenius algebra), where $\cong$ denotes the unit object of $C$; if additionally, duals exist, then a Hopf monad $T$ on an autonomous category becomes also a Frobenius monoidal functor.

Keywords (co)monoidal functor; Hopf (co)monad; Hopf adjunction; linear functor; Frobenius algebra; Frobenius monad; Frobenius monoidal functor

MSC2010 18D10; 18C15; 18A40; 16T05

1 Introduction
Hopf monads on monoidal categories, as introduced and studied in [4, 3] are categorical generalizations of Hopf algebras. Several important results on Hopf algebras have their counterpart for Hopf monads, like the fundamental theorem of Hopf modules, Maschke’s theorem on semisimplicity, etc.

It appears thus natural to ask whether Hopf monads are also Frobenius. Instead of a finite dimensional Hopf algebra (hence having a dual, which is itself a Hopf algebra), we shall now consider a Hopf monad with right adjoint, under the additional proviso that this right adjoint (comonad) is itself a Hopf comonad [7]. We call such a biHopf monad. Hopf monads on autonomous
categories automatically have right adjoints and are biHopf monads, but Hopf monads on arbitrary monoidal categories are not necessarily biHopf, not even if they have right adjoints, and we provide a simple example of such.

Before proceeding, we need to understand what do we mean by a (bi)Hopf monad $T$ on a monoidal category $C$ to be Frobenius. There are two monoidal structures that appear in the definition of a Hopf monad. Firstly, it is a monad, thus a algebra in the monoidal category of endofunctors $[C, C]$, endowed with composition as tensor product. So one may ask when a Hopf monad is also a Frobenius monad, that is, a Frobenius algebra in $[C, C]$.

Secondly, $T$ carries a comonoidal structure, thus the monoidal structure of its base category $C$ is also involved. The corresponding notion of interest is that of a Frobenius monoidal functor, which is simultaneously a monoidal and a comonoidal functor, subject to two coherence conditions which ensure that such functor preserves dual objects and Frobenius algebras.

Hopf monads induced by (tensoring with) finite dimensional Hopf algebras on the category of (finite dimensional) vector spaces are both Frobenius monads and Frobenius monoidal functors. Thus our interest is naturally motivated.

Hopf monads are completely definable in terms of Hopf adjunctions $[3]$. Bi-Hopf monads on monoidal categories are completely determined by biHopf adjunctions, that is, triple adjunctions $L \dashv U \dashv R$ with $U$ strong monoidal, such that $L \dashv U$ is a Hopf adjunction and $U \dashv R$ is a coHopf adjunction. The readers should think of $L$ as being the "induction" functor and of $R$ as being the "coinduction" functor, as these are usually called in Hopf algebra theory. If $U$ is a Frobenius functor (its left and right adjoints are isomorphic), then the associated Hopf monad $T = UL$ is a Frobenius monad, and conversely, for a Frobenius monad $T$, the forgetful functor from the associated category of algebras is Frobenius $[35]$. One step further, if $L$ is a Frobenius monoidal functor, then the associated Hopf monad $T = UL$ is again Frobenius monoidal, being a composite of such functors.

For any biHopf adjunction $L \vdash U \vdash R$ between monoidal categories, we observe that the pair $(R, L)$ constitutes a linearly distributive functor in the sense of $[10]$, this time between degenerate linearly distributive categories, that is, between monoidal categories $[4]$. A linearly distributive (shortly, linear) functor is a pair formed by a monoidal functor and a comonoidal functor (co)acting on each other, subject to several coherence conditions ensuring that these (co)actions are compatible with the (co)monoidal structure. Each Frobenius monoidal functor between arbitrary monoidal categories produces a degenerate linear functor, that is, with equal components, where the (co)strengths are provided by the (co)monoidal structure morphisms $[10]$. However, imposing an equality rather than an isomorphism is usually considered evil in category theory, thus we have

---

$^1$Our first result connecting biHopf adjunctions and linear functors should not be surprising. After all, linear functors capture much of the linear logic structures, while (representations of) Hopf algebras are known to produce models of (noncommutative or/and cyclic) linear logic $[2]$. 

---

2
provided the necessary coherence conditions which ensure that for a linear functor, the components are isomorphic if and only if one of these is a Frobenius monoidal functor, if and only if the second is so.

A triple adjunction $L \dashv U \dashv R$ between symmetric monoidal closed categories, with $U$ strong monoidal and strong closed, has been called a Wirthmüller context [20]. As $U$ is strong closed if and only if the adjunction $L \dashv U$ is Hopf, the notion of a Wirthmüller context can be considered more generally for arbitrary symmetric monoidal categories. One can go even further, by dropping the symmetry assumption, and considering separately left and right Wirthmüller contexts.

In particular, any biHopf adjunction $L \dashv U \dashv R : \mathcal{C} \to \mathcal{D}$ between monoidal categories determines a Wirthmüller context as above. In the quest of an (Wirthmüller) isomorphism between the left and the right adjoints of $U$, we show that under suitable exactness assumptions (existence and preservation of certain coreflexive equalizers), the existence of an object $C \in \mathcal{C}$ together with an isomorphism $LC \cong R\mathbb{1}$ of right $L\mathbb{1}$-comodules implies a functor isomorphism $L(C \otimes (-)) \cong R$, coherent with respect to the linear structure. One should see the object $C$ as the categorified version of the space of integrals for a Hopf algebra, while the isomorphism $LC \cong R\mathbb{1}$ is a consequence of the Fundamental Theorem of Hopf modules, applied to the dual of a finite-dimensional Hopf algebra. More details on the categorification of integrals/coinvariants, for Hopf monads on autonomous/monoidal categories, can be found in [4, 3].

The case when $C$ is isomorphic to the unit object $\mathbb{1}$ has received a special attention, as it yields isomorphism $L\mathbb{1} \cong R\mathbb{1}$ of right $L\mathbb{1}$-comodules. But for any linear functor $(R, L)$, the object $L\mathbb{1}$ is both left and right dual to $R\mathbb{1}$. Thus having an isomorphism of right $L\mathbb{1}$-comodules simply says that $L\mathbb{1}$ is a Frobenius algebra, or equivalently that $R\mathbb{1}$ is a Frobenius algebra. And under the exactness assumptions mentioned earlier, this corresponds to a natural isomorphism $R \cong L$ compatible in some precise sense with the linear structure of $(R, L)$.

If the categories involved are both autonomous, then the above results can be improved to the following: $L\mathbb{1}$ (or equivalently $R\mathbb{1}$) is a Frobenius algebra if and only if $L$ (equivalently, $R$) is a Frobenius monoidal functor, again under the above mentioned exactness assumptions.

The corresponding results for Hopf monads are now easy to obtain: first, that any biHopf monad $T$ on a monoidal category, with right adjoint (comonad) denoted by $G$, induces a linear comonad $(G, T)$; next, that an isomorphism $\overline{T} : TC \cong G\mathbb{1}$ of right Hopf $T$-modules can be extended under suitable assumptions to a functorial isomorphism $\overline{T} : T(C \otimes (-)) \cong G$. Observe that there is a natural way of obtaining such an object $C$, as the coinvariant part of the right Hopf $T$-module $G\mathbb{1}$, provided $T$ is conservative, Hopf $T$-modules have coinvariants and $T$ preserves them [3]. One step further, an isomorphism $T\mathbb{1} \cong G\mathbb{1}$ of right $T\mathbb{1}$-comodules (thus $T\mathbb{1}$ is a Frobenius algebra) induces an isomorphism $T \cong G$ of $T$-algebras, compatible in some sense with the comonoidal structure of $T$, which in turn induces a monoidal structure on $T$. In presence of duals, this
new monoidal structure on $T$ and its comonoidal structure make $T$ a Frobenius monoidal functor, such that $\chi$ is simultaneously a monoidal-comonoidal and a comonoidal-monoidal isomorphism, and consequently, $G$ is also a Frobenius monoidal functor.

The paper is organized as follows: Section $2$ briefly reviews the notions of monoidal (autonomous) categories and functors, and recalls the notion of linear functor from $[11]$, particularized to the case of monoidal categories.

The next section contains the result that biHopf adjunctions produce linear functors. Consequently, biHopf monads produce linear comonads.

Section $3$ analyzes the Wirthmüller context associated to a biHopf adjunction, yielding conditions that a biHopf adjunction $L \dashv U \dashv R$ is ambidextrous (that is, $L \cong R$), respectively that a biHopf monad is a Frobenius monad.

The last section considers Hopf adjunctions/monads on autonomous categories. In fact, the presence of duals converts any adjunction into a biHopf one, provided one of the functors is strong monoidal, and any Hopf monad into a biHopf monad. Proposition $22$, Corollary $23$ and Corollary $24$ give then the necessary conditions ensuring that in a Hopf adjunction, the left adjoint is Frobenius monoidal, respectively that a Hopf monad is a Frobenius monoidal functor. The section ends with an example of a (bi)Hopf monad on an autonomous category, more precisely on the braided autonomous category of finite-dimensional super-vector spaces, which is not a Frobenius monoidal functor, neither a Frobenius monad. This Hopf monad is obtained by tensoring with a particular Hopf algebra $A$, which as a algebra, has been showed in $[12]$ not to be a Frobenius algebra. In particular, it turns that Hopf algebras on (braided) autonomous categories are not necessarily Frobenius algebras.

2 Preliminaries

2.1 Monoidal categories and functors We shall denote by $\otimes$ the tensor product of any monoidal category encountered in this paper, while the unit object will be generically denoted by $\mathbb{1}$. We shall in the sequel omit the associativity and unit constraints, writing as the monoidal categories would be strict $[29]$. The identity morphism will be always denoted by $1$, the carrier being obvious from the context. Also, to avoid overcharge in notations, we shall omit labeling natural transformations.

Many of our proofs rely on commutative diagrams. In order to increase their readability, we shall label diagrams by $(N)$ and $(M)$ if these commute by naturality, respectively by monoidal functoriality, as in $(f \otimes 1)(1 \otimes g) = f \otimes g = (1 \otimes g)(f \otimes 1)$. Otherwise, we shall refer to previously labeled relations.

If $C$ is a monoidal category, the reversed tensor product $X \otimes^{rev} Y = Y \otimes X$ determines another monoidal structure on $C$, that we shall denote by $C^{cop}$. The opposite category of $C$ also becomes monoidal, with either the original monoidal product $\otimes$, in which case we refer to it as $C^{op}$, or with the reversed monoidal product $\otimes^{rev}$. We shall then use the notation $C^{opp,cop}$. All the above mentioned monoidal categories share the same unit object $\mathbb{1}$. 
The notion of monoidal functor between monoidal categories is well-known; a classical reference is [29]; the same goes for the notion of comonoidal functor, as well for monoidal and comonoidal natural transformations. Less familiar, but encountered in this paper, are natural transformations between functors with (different) monoidal orientation. We recall the reader the double category of monoidal categories, having as horizontal arrows the monoidal functors and as vertical arrows the comonoidal ones [23, Section 2.3]. A square is then given by a natural transformation \( \alpha : KF \to HG \), see below, subject to the coherence conditions

\[
\begin{array}{c}
\begin{array}{c}
K(X \otimes Y) \\
H(GX \otimes GY)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F(X \otimes Y) \\
G \otimes H
\end{array}
\end{array}
\]

In case both \( K \) and \( H \) are the identity functors, we call such \( \alpha : F \to G \) a monoidal-comonoidal natural transformation. By taking instead \( F \) and \( G \) to be identity, the resulting natural transformation \( \alpha : K \to H \) will be called comonoidal-monoidal.

Finally, we recall from [13] that a functor \( F : C \to D \) between monoidal categories is called Frobenius monoidal if it carries both a monoidal structure \((f_2 : F(X \otimes Y) \to F(X) \otimes F(Y), f_0 : 1 \to F 1)\) and a comonoidal structure \((F_2 : F(X \otimes Y) \to F(X) \otimes F(Y), F_0 : F 1 \to 1)\), subject to the compatibility conditions below:

\[
\begin{align}
F(X \otimes Y) \otimes FZ \xrightarrow{F_2 \otimes 1} & F(X) \otimes F(Y) \otimes FZ \\
F(X \otimes Y \otimes Z) \xrightarrow{\otimes f_2} & F(X) \otimes F(Y \otimes Z) \\
F(X \otimes F(Y \otimes Z)) \xrightarrow{1 \otimes F_2} & F(X) \otimes F(Y) \otimes FZ \\
F(X \otimes Y \otimes Z) \xrightarrow{F_2} & F(X \otimes Y) \otimes FZ
\end{align}
\]

The simplest example of a Frobenius monoidal functor is a strong monoidal one, in which case the structural monoidal/comonoidal morphisms are inverses to each other.

2.2 Autonomous categories We quickly review below the basics on autonomous categories; more details can be found in the references [25, 27, 24, 21].

\footnote{Some authors call such natural transformations lax-colax (monoidal).}
A left dual of an object $X$ in a monoidal category $\mathcal{C}$ consists of another object $SX$ together with a pair of arrows $d : 1 \to X \otimes SX$, $e : SX \otimes X \to 1$, satisfying the relations

\[
\begin{align*}
X \otimes X \otimes X &\xrightarrow{d \otimes 1} X \otimes SX \otimes X \\
\otimes e &\xrightarrow{1 \otimes e} X
\end{align*}
\]

A monoidal category is called left autonomous if each object has a left dual.

Each object $X \in \mathcal{C}$ with left dual induces adjunctions $(-) \otimes X \dashv (\_)(-) : \mathcal{C} \to \mathcal{C}$. Consequently, the left dual of an object, if exists, it is unique up to isomorphism. Assuming a choice of duals in a left autonomous category $\mathcal{C}$, the assignment $X \mapsto SX$ extends to a strong monoidal functor $S : \mathcal{C} \to \mathcal{C}$. A category is called autonomous if it is both left and right autonomous. Then $S$ and $S'$ form a contravariant pair of adjoint equivalences $S \dashv S' : \mathcal{C} \to \mathcal{C}$.

A right dual of an object $X$ in a monoidal category $\mathcal{C}$ is an object $S'X$, together with arrows $d' : 1 \to S'X \otimes X$, $e' : X \otimes S'X \to 1$, satisfying the relations

\[
\begin{align*}
X \otimes S'X \otimes X &\xrightarrow{1 \otimes d'} X \otimes S'X \otimes X \\
e' \otimes 1 &\xrightarrow{1 \otimes e'} X
\end{align*}
\]

A monoidal category is called right autonomous if each object has a right dual. In particular, once a choice of right duals is assumed, $S'$ becomes a strong monoidal functor $S' : \mathcal{C} \to \mathcal{C}$.

2.3 Frobenius algebras References on Frobenius algebras can be found in \cite{1, 22, 28, 35}. We recall below several equivalent characterizations.

Definition 1. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. A Frobenius algebra in $\mathcal{C}$ is an object $A$, simultaneously a algebra $(A, m : A \otimes A \to A, u : 1 \to A)$ and a coalgebra $(A, d : A \to A \otimes A, e : A \to 1)$, such that in the following relations hold:

\[
\begin{align*}
A \otimes A &\xrightarrow{1 \otimes d} A \otimes A \otimes A \\
d \otimes 1 &\xrightarrow{m} A \\
A \otimes A \otimes A &\xrightarrow{1 \otimes m} A \otimes A
\end{align*}
\]

\[^{3}\text{Left duals are also denoted } ^{\ast}X, ^{\perp}X, \text{ or } ^{\lor}X. \text{ Our notation is borrowed from } \cite{32}, \text{ where } S \text{ is reminiscent of the (left) star operator on a } ^{\ast}-\text{autonomous category. Also, we wanted to employ the same type of notation as for functors precisely to emphasize the (contravariant) functorial nature of the process of taking (left) duals.}\]
Frobenius monoidal functors **preserve** Frobenius algebras \([13]\); in particular, this applies to the unit object \(\mathbb{1}\) of the domain category. As an aside remark, notice that if \(A\) is a Frobenius algebra in a *braided* monoidal category \(C\), then the functors \((-) \otimes A, A \otimes (-) : C \to C\) obtained by tensoring with \(A\) are Frobenius monoidal \([13]\).

A Frobenius algebra \(A\) in a monoidal category is in particular a self-dual object, with evaluation \(A \otimes A \xrightarrow{\varepsilon} A\) and coevaluation \(\mathbb{1} \xrightarrow{\eta} A \otimes A\). Conversely, the following result holds, for which we refer to the above mentioned references.

**Lemma 2.** Let \((A, m : A \otimes A \to A, u : \mathbb{1} \to A)\) be a algebra in the monoidal category \(C\).

If the object \(A\) has left dual \(SA\), then \(A\) is a Frobenius algebra if and only if there is an isomorphism of right \(A\)-modules \(A \cong SA\), where \(A\) is a right \(A\)-module via multiplication, and the right \(A\)-module structure of \(SA\) is obtained as

\[
\begin{array}{c}
SA \otimes A \\
\downarrow \cong \\
SA \\
\end{array}
\]

Dually, if \(A\) has right dual \(S'A\), then \(A\) is a Frobenius algebra if and only if there is an isomorphism of left \(A\)-modules \(S'A \cong A\), where \(A\) is a left \(A\)-module via multiplication, and the left \(A\)-module structure of \(S'A\) is given by

\[
\begin{array}{c}
A \otimes S'A \\
\downarrow \cong \\
S'A \\
\end{array}
\]

Dual statements characterize the coalgebras which are Frobenius algebras.

A *Frobenius monad* on an arbitrary category \(C\) is a Frobenius algebra in the monoidal category of endofunctors \([C, C]\), with composition as monoidal product and identity functor as unit. In particular, it follows from Proposition 2 (see also \([30]\ Proposition 3.11\)) that a monad \((T, \mu : T^2 \to T, \eta : 1 \to T)\) having right adjoint \(G\), is a Frobenius monad if and only if there is a natural isomorphism \(\chi : T \to G\) such that

\[
\begin{array}{ccc}
T^2 & \xrightarrow{\chi} & TG \\
\downarrow \mu & & \downarrow uT G \\
T & \xrightarrow{\overline{\chi}} & GT^2 G \\
\downarrow G \mu G & & \downarrow G \mu G \\
G & \xrightarrow{G e} & G \\
\end{array}
\]

where \(u : 1 \to GT\) and \(e : TG \to 1\) are the unit, respectively the counit of the adjunction \(T \dashv G\).

For each algebra \(A\) in a monoidal category \(C\), tensoring with \(A\), say on the left, produces the monad \(T = A \otimes (-) : C \to C\). Then \(A\) is a Frobenius algebra if and only if \(T\) is a Frobenius monad.
A Frobenius functor is a functor $U$ having left adjoint $F$ which is also right adjoint to $U$. One also says that the adjunction is ambidextrous. If $U^T$ denotes the forgetful functor from the category of Eilenberg-Moore algebras $C^T$ for a monad $T$, then $T$ is a Frobenius monad if and only if $U^T$ is a Frobenius functor.}

### 2.4 Linear functors between monoidal categories

A pair of functors between monoidal categories, one of them monoidal and the other comonoidal, subject to several coherence conditions, has been called a linearly distributive functor, and it makes sense in a more general context than monoidal categories, namely linearly distributive categories. In short, a linearly distributive category is a category $C$ equipped with two monoidal structures $(C, \otimes, 1)$ and $(C, \oplus, 0)$, and two natural transformations $A \otimes (B \oplus C) \to (A \otimes B) \oplus C$, $(A \oplus B) \otimes C \to A \oplus (B \otimes C)$, subject to several naturality coherence conditions that make the two monoidal structures work well together. Any monoidal category $(C, \otimes)$ is a (degenerate) linearly distributive category, with $\otimes = \oplus$ and $1 = 0$. This is our case of interest, that we shall pursue in the sequel.

Linearly distributive functors (in short, linear functors) between linear categories were defined by Cockett and Seely in [9] and [10] as to provide a categorical settings for linear logic. To not be confounded with another notion of linear functor, namely an (enriched) functor between categories enriched over the category of $k$-vector spaces, for a commutative field $k$. These have been introduced by Cockett and Seely in [9] and [10] as to provide a categorical settings for linear logic.
Given linear functors \((R, L) : \mathcal{C} \to \mathcal{D}\) and \((R', L') : \mathcal{D} \to \mathcal{E}\) linear functors between monoidal categories, the pair \((R'R, L'L)\) becomes again a linear functor, the corresponding natural transformations being given by

\[
R'R(X \otimes Y) \xrightarrow{R'L_X Y} R'(LX \otimes RY) \xrightarrow{L'L_X Y} L'X \otimes R'RY
\]

and similar three more formulas.

Given linear functors \((R, L), (R', L') : \mathcal{C} \to \mathcal{D},\) a linear natural transformation \((R, L) \to (R' L')\) consists of a monoidal natural transformation \(\rho : R \to R'\) and a comonoidal natural transformation \(\lambda : L' \to L\) such that

\[
\begin{align*}
\lambda \circ \nu^R_L \circ (1 \otimes \rho) &= \nu^R_L \circ (\lambda \otimes 1) \\
\lambda \circ \nu^L_R \circ (\rho \otimes 1) &= \nu^L_R \circ (1 \otimes \lambda) \\
(1 \otimes \lambda) \circ \nu^R_L \circ \rho &= (\rho \otimes 1) \circ \nu^R_L \\
(\lambda \otimes 1) \circ \nu^L_R \circ \rho &= (1 \otimes \rho) \circ \nu^L_R \\
\end{align*}
\]

There are well-defined notions of vertical and horizontal compositions for linear natural transformations for which we refer again to [11], such that linear categories with linear functors and linear natural transformations organize into a 2-category, whose 2-subcategory having as objects the monoidal categories will be denoted by \textbf{LinMon}. 

9
**Example 3.** Any strong monoidal functor \((U, u_2, u_0) : C \to D\) between monoidal categories induces a linear functor by \(R = L = U\), with (co)strengths given by \(\nu_R' = \nu_R = u_2^{-1}, \nu_L' = \nu_L = u_2\). More generally, any Frobenius monoidal functor \((F, f_2, f_0, F_2, F_0) : C \to D\) provides a linear functor \((R, L)\) with equal components \(R = L = F\), such that \(\nu_R' = \nu_R = L_2 = F_2\) and \(\nu_L' = \nu_L = r_2 = f_2\) \[17\]. Conversely, the following result holds:

**Theorem 4.** Let \((R, L)\) a linear functor between monoidal categories, such that there is a natural isomorphism \(\chi : L \to R\) satisfying

\[
L(X \otimes Y) \xrightarrow{L_2} LX \otimes LY \\
\chi \downarrow \quad \quad \quad \quad \quad \downarrow \chi \otimes 1
\]

\[(4)\]

and

\[
LX \otimes RY \xrightarrow{\nu_L} L(X \otimes Y) \\
\chi \otimes 1 \quad \quad \quad \quad \quad \downarrow \chi \\
RX \otimes RY \xrightarrow{r_2} R(X \otimes Y)
\]

\[(5)\]

Then the comonoidal component functor \(L\), with the monoidal structure inherited from \(R\) via the isomorphism \(\chi\), becomes a Frobenius monoidal functor. Dually, \(R\) inherits a comonoidal structure from \(L\) and becomes a Frobenius monoidal functor. The isomorphism \(\chi : L \to R\) is then simultaneously monoidal-comonoidal and comonoidal-monoidal.

**Proof.** The monoidal structure on \(L\), namely

\[
LX \otimes LY \xrightarrow{\chi \otimes \chi} RX \otimes RY \xrightarrow{r_0} R(X \otimes Y) \xrightarrow{\chi^{-1}} L(X \otimes Y)
\]

can alternatively be described, using \[(5)\], as

\[
LX \otimes LY \xrightarrow{1 \otimes \chi} LX \otimes RY \xrightarrow{\nu_L} L(X \otimes Y)
\]

We check now the two diagrams \[(4)-(5)\] ensuring that \(L\) becomes a Frobenius
Conversely, using right duals, we obtain that
\[ (\text{5}) \implies L(X \otimes Y) \otimes LZ \xrightarrow{1 \otimes X} L(X \otimes Y) \otimes RZ \xrightarrow{\nu^L_1} L(X \otimes Y) \otimes Z \]
and
\[ L(X \otimes LY \otimes LZ) \xrightarrow{\otimes 1 \otimes X} L(X \otimes LY \otimes RZ) \xrightarrow{1 \otimes \nu^L_1} L(X \otimes LY) \]
are autonomous, the relations (4) and (5) are equivalent. We first show that (4) implies (5), using left duality:

\[ L(X \otimes Y) \xrightarrow{\lambda_1} L(X \otimes Y) \otimes SY \xrightarrow{\lambda_1} L(X \otimes Y) \otimes SY \]

\[ L(X \otimes Y) \xrightarrow{\lambda_1} L(X \otimes Y) \otimes SY \xrightarrow{\lambda_1} L(X \otimes Y) \otimes SY \]

Conversely, using right duals, we obtain that (5) implies (4):
We shall need later some facts on how a linear functor \((R, L)\) acts on the unit object \(1\). As the latter is simultaneously a algebra and a coalgebra, it is mapped by the monoidal functor \(R\) to the algebra \((R1, r_2 : R1 \otimes R1 \to R1, r_0 : 1 \to R1)\) and by the comonoidal functor \(L\) to the coalgebra \((L1, L_2 : L1 \to L1 \otimes L1, L_0 : L1 \to 1)\).

Furthermore, half of the relations \((\text{LF1}) - \text{LF2}\) ensure that \(\nu'_{\text{L}} : R1 \otimes L1 \to L1\) defines a left \(R1\)-action on \(L1\) and \(\nu'_{\text{R}} : L1 \otimes R1 \to L1\) defines a right \(R1\)-action on \(L1\), while \((\text{LF3})\) says that \(L1\) becomes an \(R1\)-bimodule, such that by \((\text{LF5})\) the comultiplication \(L_2 : L1 \to L1 \otimes L1\) respects both the left and right \(R1\)-actions. Similarly, \(R1\) becomes a \(L1\)-bicomodule using \(\nu''_{\text{R}}\) and \(\nu''_{\text{L}}\) as coactions, such that the multiplication \(r_2 : R1 \otimes R1 \to R1\) is a morphism of \(L1\)-bicomodules. Then \((\text{LF4})\) says that the left/right \(R1\)-actions on \(L1\) are morphisms of left, respectively right \(L1\)-comodules, and similarly for the \(L1\)-coactions on \(R1\).

Finally, notice that \(L1\) is both a left and right dual for \(R1\), with evaluation and coevaluation morphisms \(1 \xrightarrow{r_0} R1 \xrightarrow{\nu'_{\text{L}}} R1 \otimes L1, L1 \otimes R1 \xrightarrow{\nu'_{\text{R}}} L1 \xrightarrow{L_0} 1\), respectively \(1 \xrightarrow{r_0} R1 \xrightarrow{\nu''_{\text{L}}} L1 \otimes R1, R1 \otimes L1 \xrightarrow{\nu''_{\text{R}}} L1 \xrightarrow{L_0} 1\). Under these dualities, the multiplication of the monoid \(R1\) is transposed to the left and right \(L1\)-coactions on \(R1\), which transposed again produce the comultiplication on \(L1\).

Such a pair \((R1, L1)\) formed by a algebra and a coalgebra (co)acting on each other was called a cyclic nuclear algebra in the terminology of [17], or a linear monad in [8].

3 Hopf adjunctions and linear functors

3.1 (Co)monoidal and (co)Hopf adjunctions Let \(\mathcal{C}\) and \(\mathcal{D}\) monoidal categories. In the sequel, we shall denote objects of \(\mathcal{C}\) by \(X, Y, \ldots\) and the objects of \(\mathcal{D}\) by \(A, B, \ldots\) to make a clear distinction between them.

Recall from [26] that given an adjunction between \(\mathcal{C}\) and \(\mathcal{D}\), monoidal structures on the right adjoint are in one-to-one correspondence with comonoidal structures on the left adjoint, such that the unit and the counit of the adjunction become monoidal-comonoidal natural transformations.

We throughout consider that \((U : \mathcal{D} \to \mathcal{C}, u_2 : UA \otimes UB \to U(A \otimes B), u_0 : 1 \to U1)\) is a strong monoidal functor, having both a left adjoint \(L\) and a right adjoint \(R\).

Denote by \(\eta' : 1 \to UL\) and \(\epsilon' : LU \to 1\) the unit, respectively the counit of the adjuction \(L \dashv U : \mathcal{D} \to \mathcal{C}\). As mentioned above, \(L\) becomes a comonoidal functor, with structure morphisms

\[
\begin{align*}
L_0 : L1 &\xrightarrow{L_{u_0}} LU1 \xrightarrow{\epsilon'} 1 \\
L_2 : L(X \otimes Y) &\xrightarrow{L(u'_{2} \otimes u'_{1})} L(U(LX \otimes ULY)) \xrightarrow{L_{u_0}} LU(LX \otimes LY) \xrightarrow{\epsilon'} LX \otimes LY
\end{align*}
\]
such that the unit and the counit verify the diagrams below:

\[
\begin{array}{cccc}
X \otimes Y & \rightarrow & UL(X \otimes Y) \\
\eta' \otimes \eta & \downarrow & \downarrow U\eta \\
ULX \otimes ULY & \rightarrow & U(LX \otimes LLY)
\end{array}
\]

We state explicitly also the dual situation, to be used for later reference. Namely, in the adjunction \((U \dashv R : \mathcal{C} \rightarrow \mathcal{D}, \eta : 1 \rightarrow RU, \epsilon : UR \rightarrow 1)\), the right adjoint \(R\) inherits a monoidal structure by

\[
\eta \circ \eta'^{-1} : 1 \rightarrow RU \quad R^{-1}
\]

while the unit and the counit verify the following diagrams:

\[
\begin{array}{cccc}
A \otimes B & \rightarrow & R(UA \otimes UB) \\
\eta' \otimes \eta & \downarrow & \downarrow U\eta \\
RA \otimes RUB & \rightarrow & R(UA \otimes UB)
\end{array}
\]

For the adjunction \(L \dashv U\), the natural composites

\[
\mathbb{H}^{l} : L(- \otimes U(-)) \rightarrow L(-) \otimes LU(-) \xrightarrow{1 \otimes \epsilon'^{l}} L(-) \otimes (-)
\]

\[
\mathbb{H}^{r} : L(U(-) \otimes -) \rightarrow LU(-) \otimes L(-) \xrightarrow{\epsilon^{l} \otimes 1} (-) \otimes L(-)
\]

were called in \cite{3} the left Hopf operator, respectively the right Hopf operator. Following \textit{op. cit.}, \(L \dashv U\) is called a left Hopf adjunction if \(\mathbb{H}^{l}\) is invertible.\footnote{In this case, one also say that projection formula holds \cite{4}. Recall from \cite{13} that a functor between (left) closed monoidal categories is monoidal if and only if it is (left) closed. If additionally it is strong monoidal and right adjoint, then it is strong (left) closed if and only if all (left) Hopf operators are invertible.}

Similar terminology goes for the right-handed case: the adjunction \(L \dashv U\) is called simply a Hopf adjunction if it is both left and right Hopf adjunction.

The next lemma collect some properties of Hopf operators needed in the sequel; for convenience, these are listed in groups, illustrating an example of each by a commutative diagram, and are easy to prove by standard diagram chasing left to the reader.

**Lemma 6.** For a comonoidal adjunction \(L \dashv U\), the Hopf operators satisfy the following relations:

\[
\begin{cases}
\mathbb{H}^{l} \circ L(1 \otimes u_{0}) = 1 \\
\mathbb{H}^{r} \circ L(u_{0} \otimes 1) = 1
\end{cases}
\]
\[ \{ (\mathcal{H}' \otimes 1) \circ \mathcal{H}' = \mathcal{H}' \circ L(u_2 \otimes 1) \} \]
\[ (1 \otimes \mathcal{H}') \circ \mathcal{H}' = \mathcal{H}' \circ L(u_2 \otimes 1) \]

\[ L(X \otimes U A \otimes U B) \xrightarrow{\mathcal{H}'} L(X \otimes U A) \otimes B \]
\[ L((1 \otimes u_2) \otimes (\mathcal{H}') \circ L(u_2 \otimes 1)) \]
\[ L(X \otimes U (A \otimes B)) \xrightarrow{\mathcal{H}'} LX \otimes A \otimes B \]

\[ L(U A \otimes X \otimes U B) \xrightarrow{\mathcal{H}'} L(X \otimes U A) \otimes B \]
\[ A \otimes L(X \otimes U B) \xrightarrow{1 \otimes \mathcal{H}'} A \otimes LX \otimes B \]

\[ \{ \mathcal{H}' \circ L(1 \otimes \eta') = L_2 \} \]
\[ \{ \mathcal{H}' \circ L(\eta' \otimes 1) = L_2 \} \]

\[ X \otimes U A \xrightarrow{\eta'} UL(X \otimes U A) \]
\[ X \otimes U A \xrightarrow{\eta' \otimes 1} U L \]
\[ U L \otimes U A \xrightarrow{u_2} U(LX \otimes A) \]

\[ \{ (L_2 \otimes 1) \circ \mathcal{H}' = (1 \otimes \mathcal{H}') \circ L_2 \} \]
\[ \{ (1 \otimes L_2) \circ \mathcal{H}' = (\mathcal{H}' \otimes 1) \circ L_2 \} \]

\[ L(X \otimes Y \otimes U LZ) \xrightarrow{\mathcal{H}'} L(X \otimes Y) \otimes LZ \]
\[ L(X \otimes Y \otimes U LZ) \xrightarrow{1 \otimes \mathcal{H}'} L(X \otimes Y) \otimes LZ \]

\[ L X \otimes L (Y \otimes U LZ) \xrightarrow{1 \otimes \mathcal{H}'} LX \otimes LY \otimes LZ \]

\[ \{ (L_0 \otimes 1) \circ \mathcal{H}' = \epsilon' \} \]
\[ \{ (1 \otimes L_0) \circ \mathcal{H}' = \epsilon' \} \]

\[ LU A \xrightarrow{\mathcal{H}'} L 1 \otimes A \]
\[ LU A \xrightarrow{1 \otimes \eta'} A \]

For the purposes of this paper, we need to introduce the (straightforward) dual concept. Namely, define the left and right coHopf operators associated to the adjunction \( U \dashv R \) by

\[ \mathcal{S}_L' : R(-) \otimes (-) \xrightarrow{1 \otimes \eta''} R(-) \otimes RU(-) \xrightarrow{r_2} R((-) \otimes U(-)) \]
\[ \mathcal{S}_R' : (-) \otimes R(-) \xrightarrow{\eta' \otimes 1} RU(-) \otimes R(-) \xrightarrow{r_2} R(U(-) \otimes (-)) \]
We call the adjunction $U \dashv R$ a left coHopf adjunction if the left coHopf operator $\tilde{H}_l$ is invertible, a right coHopf adjunction if the right coHopf operator $\tilde{H}_r$ is invertible, and simply a coHopf adjunction if it is both left and right coHopf adjunction. To save space, we do not list anymore the straightforward duals of (9)-(15), with the convention that whenever we have to refer to them in the sequel, they will appear with the prefix "co".

Finally, we introduce the notion of a biHopf adjunction as being a triple adjunction $L \dashv U \dashv R$ between monoidal categories, with $U$ strong monoidal, $L \dashv U$ a Hopf adjunction and $U \dashv R$ a coHopf adjunction.

### 3.2 BiHopf adjunctions yield linear functors

Linear functors, as explained earlier, can be seen as a more general version of Frobenius monoidal functors. In our quest for the latter, we show as a first step that biHopf adjunctions produce linear functors:

**Theorem 7.** Consider a biHopf adjunction $L \dashv U \dashv R : C \to D$ between monoidal categories. Then the pair $(R, L)$ is a linear functor.

**Proof.** The four strengths and costrengths are defined as follows:

\[
\begin{align*}
\nu_R^r &: R(X \otimes Y) \xrightarrow{R(\eta_l \otimes 1)} R(U LX \otimes Y) \xrightarrow{\tilde{H}_r^{-1}} LX \otimes RY \\
\nu_R^l &: R(X \otimes Y) \xrightarrow{R(1 \otimes \eta_r)} R(X \otimes ULY) \xrightarrow{\tilde{H}_r^{-1}} RX \otimes LY \\
\nu_L^r &: RX \otimes LY \xrightarrow{H_r^{-1}} L(RX \otimes Y) \xrightarrow{L(\epsilon_r \otimes 1)} L(X \otimes Y) \\
\nu_L^l &: LX \otimes RY \xrightarrow{H_l^{-1}} L(X \otimes URY) \xrightarrow{L(1 \otimes \epsilon_r)} L(X \otimes Y)
\end{align*}
\]

Now we shall check that these form a linear functor. For each group of relations it is enough to prove only one of them, the remaining following by the passage to $C^{\text{op}}$, $C^{\text{cop}}$, $C^{\text{cop,op}}$.

**Proof of (LF1):**

\[
\begin{align*}
RX & \xrightarrow{R(\eta_l \otimes 1)} R(U \boxtimes X) \xrightarrow{\tilde{H}_r^{-1}} RX \\
R(U \boxtimes X) & \xrightarrow{R(UL \eta \otimes 1)} R(U \boxtimes RX) \xrightarrow{\tilde{H}_r^{-1}} RX \\
R(U \boxtimes RX) & \xrightarrow{R(1 \otimes \eta_r)} R(U \boxtimes ULY) \xrightarrow{\tilde{H}_r^{-1}} RX \otimes LY
\end{align*}
\]

\[\text{Recall from [7] that a functor between (left) coclosed monoidal categories is comonoidal if and only if it is (left) coclosed. If additionally it is strong monoidal and left adjoint, then it is strong (left) coclosed if and only if all (left) coHopf operators are invertible.}\]
Corollary 8. Any biHopf adjunction \( L \dashv U \dashv R \) between monoidal categories determines an adjunction \((U, U) \dashv (R, L)\) in the 2-category \( \text{LinMon} \) of monoidal categories, linear functors and linear natural transformations, with unit \((\eta^r, \epsilon^l)\) and counit \((\epsilon^r, \eta^l)\).

Proof. The only thing to verify is that the unit and the counit are indeed linear natural transformations. That is, they verify the following relations obtained from \((\text{LN})\):

\[
\begin{align*}
(\eta^r, \epsilon^l) \text{ linear:} & \\
(\eta^l \otimes 1) \circ \nu^r_R & \circ R u_2^{-1} \circ \eta^r = 1 \otimes \eta^r \\
(1 \otimes \epsilon^l) \circ \nu^l_R & \circ R u_2^{-1} \circ \eta^l = \eta^l \otimes 1 \\
\epsilon^l \circ L u_2 \circ \nu^l_R & \circ (\eta^r \otimes 1) = 1 \otimes \epsilon^l \\
\epsilon^l \circ L u_2 \circ \nu^l_R & \circ (1 \otimes \eta^r) = \epsilon^l \otimes 1 \\
(\eta^l \otimes 1) \circ \epsilon^r & = (1 \otimes \epsilon^l) \circ u_2^{-1} \circ U \nu^r_R \\
(1 \otimes \eta^l) \circ \epsilon^r & = (\epsilon^r \otimes 1) \circ u_2^{-1} \circ U \nu^r_R \\
U \nu^r_L \circ u_2 \circ (1 \otimes \eta^l) & = \eta^l \circ (\epsilon^r \otimes 1) \\
U \nu^r_L \circ u_2 \circ (\eta^l \otimes 1) & = \eta^l \circ (1 \otimes \epsilon^r)
\end{align*}
\] (18)

It is enough to check only one of the above eight relations, the rest following by symmetry. For example, the first relation in (18) is proved in the diagram below:

---

3.3 Hopf (co)monads and Hopf modules

A monad \((T, \mu : T^2 \to T, \eta : 1 \to T)\) on a monoidal category \(C\) is called a comonoidal monad if it is endowed with a comonoidal structure \((T^2 : T(X \otimes Y) \to TX \otimes TY, T_0 : T1 \to 1))\), such that the unit \(\eta\) and the multiplication \(\mu\) become comonoidal natural transformations. In other words, a comonoidal monad is a monad in the 2-category \(\text{cMon} \) of monoidal categories, comonoidal functors and comonoidal natural transformations.

[^8]
A comonoidal monad was called a left Hopf monad in [3] if the following natural transformation (called left fusion operator) is an isomorphism:

\[
\Pi_l : T(- \otimes T(-)) \xrightarrow{T_2} T(-) \otimes T^2(-) \xrightarrow{1 \otimes \mu} T(-) \otimes T(-)
\]

A right Hopf monad [3] is again a comonoidal monad as above, but this time the right fusion operator

\[
\Pi_r : T(T(-) \otimes -) \xrightarrow{T_2} T^2(-) \otimes T(-) \xrightarrow{\mu \otimes 1} T(-) \otimes T(-)
\]

is invertible.

Any adjunction \(L \dashv U : \mathcal{D} \rightarrow \mathcal{C}\) produces a monad \(T = UL\). If the adjunction is comonoidal, so is the monad. Furthermore, if the adjunction is left or right Hopf, the monad is again such [4].

Conversely, for any monad \(T\) on a category \(\mathcal{C}\) denote as earlier by \(\mathcal{C}_T\) the Eilenberg-Moore category of \(T\)-algebras, and by \(L_T \dashv U_T : \mathcal{C}_T \rightarrow \mathcal{C}\) the adjunction between the free and the forgetful functor. There is a one-to-one correspondence between comonoidal structures on the underlying functor \(T\) making it a comonoidal monad and monoidal structures on the category \(\mathcal{C}_T\), such that the forgetful functor \(U_T\) becomes strict monoidal [31], in particular exhibiting \(L_T \dashv U_T\) as a comonoidal adjunction. One step further, \(L_T \dashv U_T\) is a left/right Hopf adjunction if and only if \(T\) is a left/right Hopf monad [3, Lemma 2.18], and in such case the Hopf operators and the fusion operators are connected, say in the left-sided case, by \(\Pi_{X,Y}^l = U_T \Pi_{X,L}^l T Y\) (thus the fusion operators are \(T\)-algebra morphisms), and their inverses by

\[
\Pi_{X,(A,a)}^{-1} : TX \otimes A \xrightarrow{1 \otimes \eta} TX \otimes TA \xrightarrow{\Pi_r^{-1}} T(X \otimes TA) \xrightarrow{T(1 \otimes a)} T(X \otimes A) \quad (20)
\]

for any \(T\)-algebra \((A, TA \xrightarrow{\rho} A)\). Similar formulas hold for the right Hopf/fusion operators.

**Coinvariants** Recall from [3] that for a comonoidal monad \(T\), the free \(T\)-algebra \(T \mathbb{1}\) becomes a coalgebra in the monoidal category of \(T\)-algebras \(\mathcal{C}_T\), with comultiplication \(T_2 : T \mathbb{1} \rightarrow T \mathbb{1} \otimes T \mathbb{1}\) and counit \(T_0 : T \mathbb{1} \rightarrow \mathbb{1}\). Right \(T\)-comodules within \(\mathcal{C}_T\) were called right Hopf \(T\)-modules in [3], and the category of right Hopf \(T\)-modules denoted by \(\mathcal{H}^r(T)\), and similarly for the left-handed-case. In particular, each free \(T\)-algebra \((TX, \mu : T^2X \rightarrow TX)\) naturally becomes a right \(T\)-Hopf module with \(TX \xrightarrow{T_2} TX \otimes T \mathbb{1}\). This induces a (comparison) functor \(K : \mathcal{C} \rightarrow \mathcal{H}^r(T)\), \(KX = (TX, \mu, T_2)\).

For a right Hopf \(T\)-module \((A, TA \xrightarrow{\rho} A, A \xrightarrow{\rho} A \otimes T \mathbb{1})\), its coinvariant part \(A^{coT}\) is defined as the equalizer below, if it exists:

\[
A^{coT} \xrightarrow{i} A \xrightarrow{\rho \otimes 1} A \otimes T \mathbb{1} \quad (21)
\]

\(^{9}\)This can be rephrased by saying that the 2-category \(\text{cMon}\) admits Eilenberg-Moore objects.
We shall need in the sequel the right-handed version of [3, Theorem 6.11]:

**Theorem 9.** Let $T$ a right Hopf monad on a monoidal category $C$. If $T$ is conservative, right $T$-Hopf modules have coinvariants (the equalizer (21) exists for any right Hopf $T$-module) and $T$ preserves them, then the comparison functor $K$ is an equivalence of categories. In particular, for each right Hopf $T$-module $(A, TA \xrightarrow{\alpha} A, A \xrightarrow{\rho} A \otimes T1)$, the composite

$$TA \xrightarrow{\alpha T} TA \xrightarrow{T} A$$

is an isomorphism of right Hopf $T$-modules.

**Hopf comonads** By reversing the arrows, one obtains the notions of a monoidal comonad $(G : C \to C, g_2 : GX \otimes GY \to G(X \otimes Y), g_0 : 1 \to G1)$, respectively a left/right Hopf comonad [7] on a monoidal category, as being a monoidal comonad whose left/right cofusion operators

$$\overline{\eta} : G(-) \otimes G(-) \xrightarrow{\delta} G(-) \otimes G^2(-) \xrightarrow{g_2} G((-) \otimes G(-))$$

$$\overline{\epsilon} : G(-) \otimes G(-) \xrightarrow{\delta \otimes 1} G^2(-) \otimes G(-) \xrightarrow{g_2} G((-) \otimes (-))$$

are invertible. Same for the right case.

The comonad $G = UR$ generated by a Hopf adjunction $U \dashv R$ is a Hopf comonad and conversely, any left/right Hopf comonad $G$ on a monoidal category $C$ produces a left/right co-Hopf adjunction $U_G \dashv R_G : C \to C_G$ between the corresponding forgetful and cofree functors, where $C_G$ denotes the Eilenberg-Moore category of $G$-coalgebras. The reader can easily derive their descriptions by duality.

**3.4 BiHopf monads produce linear comonads** Recall that (co)Hopf adjunctions determine Hopf (co)monads and conversely, any Hopf (co)monad induces a (co)Hopf adjunction between the (co)free and forgetful functor. Any biHopf adjunction $L \dashv U \dashv R$ determines an adjunction between the Hopf monad $T = UL$ and the Hopf comonad $G = UR$. Consequently, the composite $(G, T)$ is a comonad in $\text{LinMon}$. Conversely, any monad $T$ on a category $C$, having a right adjoint $G$, with unit $u : 1 \to GT$ and counit $\epsilon : TG \to 1$, determines a comonad structure on $G$, with counit and comultiplication

$$\epsilon : G \xrightarrow{\eta G} TG \xrightarrow{\epsilon} 1, \quad \delta : G \xrightarrow{\mu G} GTG \xrightarrow{\alpha} GT^2G \xrightarrow{\epsilon} G2$$

The categories of $G$-coalgebras $C_G$ and of $T$-algebras $C^T$ are isomorphic [19], such that the isomorphism functor $Q : C^T \to C_G$ commutes with the forgetful functors, namely $U_G = U^TQ$. Explicitly, $Q$ maps a $T$-algebra $(A, a : TA \to A)$ to the $G$-coalgebra $(A, A \xrightarrow{\eta A} GTA \xrightarrow{\alpha} GA)$. Via the isomorphism $Q$, the
forgetful functor $U^T$ has not only a left adjoint $L^T$, but gains also a right adjoint $R^T := QR_G$ which maps an object $X \in \mathcal{C}$ to $GX$, endowed with the $T$-algebra structure $TGX \xrightarrow{\mu^T} GT^2 GX \xrightarrow{G\mu^T} GTGX \xrightarrow{G\epsilon_X} GX$.

Assume now that $T$ is a comonoidal monad. Then its adjoint $G$ becomes a monoidal comonad with structure morphisms $g_0 : 1 \xrightarrow{u} GT \xrightarrow{G\mu} G$ and $g_2 : GX \otimes GY \xrightarrow{\mu} G(TGX \otimes TGY) \xrightarrow{G(e \otimes e)} G(X \otimes Y)$, the forgetful functors $U^T$ and $U_G$ are strong monoidal and so is $Q$, thus in the triple adjunction $L^T \dashv U^T \dashv R^T$, the functor $U^T$ is strong monoidal, while its left and right adjoints are comonoidal, respectively monoidal. In particular, the monoidal structure morphisms $g_0$ and $g_2$ become $T$-algebra maps.\[10\]

One step further, suppose that $T$ is a Hopf monad and its adjoint $G$ is a Hopf comonad. We shall call such $T$ a biHopf monad.

**Remark 10.** A Hopf monad $T$ on a monoidal category which has a right adjoint is not necessarily a biHopf monad.

To see this, let $\mathcal{C}$ be the category of sets and functions $\mathbf{Set}$, with its usual cartesian monoidal structure, and let $G$ be a (finite) non-trivial group. Then the monad $T = (\_ \times G)$ is easily seen to be both left and right Hopf, while its right adjoint (comonad) $G = \mathbf{Set}(G, \_)$ is not a Hopf comonad - the cofusion operators failing to be bijective for cardinality reasons.

However, we shall later see in Remark 21 that in case the base category is autonomous, $T$ is a Hopf monad if and only if it is a biHopf monad.

For a biHopf monad $T$, with right adjoint $G$, the adjunction $L^T \dashv U^T$ is a Hopf adjunction, and $U_G \dashv R_G$ is a coHopf adjunction. As (co)Hopf adjunctions are stable under strong monoidal isomorphisms, $U^T \dashv R^T$ is a coHopf adjunction. Consequently, $L^T \dashv U^T \dashv R^T$ is a biHopf adjunction, thus Corollary 8 applies and yields the comonad $(G, T)$ in the 2-category $\mathbf{LinMon}$. We record the above discussion in the following:

**Corollary 11.** Let $T$ a biHopf monad on a monoidal category $\mathcal{C}$. Then the pair $(G, T)$ is a comonad in $\mathbf{LinMon}$, induced by the adjunction $(R^T, L^T) \dashv (U^T, U_T)$.

For further reference, we record below the formulas producing the four (co)strengths associated to the linear comonad $(G, T)$, easy consequences of relations (3), (19) and (20):

\[10\] This holds more generally: any $G$-coalgebra morphism $f : (X, X \xrightarrow{s} GX) \rightarrow (Y, Y \xrightarrow{s} GY)$ is turned by the inverse of the (isomorphism) functor $Q$ into a morphism of $T$-algebras $(X, TX \xrightarrow{\delta} TX) \rightarrow (Y, TY \xrightarrow{\delta} TGY \xrightarrow{\delta} Y)$. 

\[20\]
\[ \begin{align*}
\varpi_L : TX \otimes GY & \xrightarrow{1 \otimes \eta} TX \otimes TY \xrightarrow{T^{-1}} T(X \otimes TY) \\
\varpi_L : GX \otimes TY & \xrightarrow{\eta \otimes 1} TGX \otimes TY \xrightarrow{T^{-1}} T(TGX \otimes TY) \xrightarrow{T(1 \otimes \epsilon)} T(X \otimes Y) \\
\varpi_R : G(X \otimes Y) & \xrightarrow{\eta \otimes u} G(X \otimes GTY) \xrightarrow{T^{-1}} GX \otimes GTY \xrightarrow{1 \otimes \epsilon T} GX \otimes TY \\
\varpi_R : G(X \otimes Y) & \xrightarrow{G(1 \otimes u)} G(GTX \otimes Y) \xrightarrow{T^{-1}} GTX \otimes GY \xrightarrow{\epsilon T} TX \otimes GY
\end{align*} \] (23)

Notice additionally that all the four above (co)strengths are morphisms of \( T \)-algebras.

**Proposition 12.** If \( T \) is a biHopf monad with right adjoint comonad \( G \), then any cofree \( G \)-coalgebra becomes a right Hopf \( T \)-module using \( \varpi_R \) as in (23).

**Proof.** We have explained earlier how (cofree) \( G \)-coalgebras become \( T \)-algebras via the inverse of the isomorphism functor \( Q \). They are also right Hopf \( T \)-modules: firstly, as each \( \varpi_R : GX \rightarrow GX \otimes T \) is a morphism of \( T \)-algebras, and secondly, because by (LF2) \( \varpi_R \) induces on \( GX \) the structure of a right \( T \)-comodule. \qed

## 4 Hopf adjunctions and Frobenius-type properties

Consider a biHopf adjunction \( L \dashv U \dashv R \) between monoidal categories. Such a triple adjunction is an example of a Wirthmüller context \[20, 1\]. Although we consider only biHopf adjunctions, the reader might see that some of the results below hold with less assumptions. We are interested in finding conditions such that \( U \) is a Frobenius functor. As the left and right adjoints of \( U \) form a linear functor, it is natural to look not just for a mere isomorphism between them, but for one compatible with the linear structure. If such an isomorphism exists, then it induces \( L \cong R \) and exhibits both \( L \cong R \) as Frobenius algebras.

### 4.1 The Wirthmüller isomorphism

Motivated by the theory of (finite dimensional) Hopf algebras, but also by the development in \[20\], we first consider the existence of an object \( C \in C \), together with an isomorphism \( \chi : LC \rightarrow R \), for an object \( C \in C \).

By \((\text{LF}1)-(\text{LF}2)\), the morphism \( \nu' : R \rightarrow R \otimes L \), defined as in (27), induces a right \( L \)-comodule structure on \( R \). On the other hand, \( LC \) naturally

\[11\]More precisely, a Wirthmüller context has only been considered in the situation where the categories involved were (symmetric) monoidal closed. But the closed structure appears only in the requirement that \( U \) to be strong closed, which as said earlier, can be substituted by the more convenient formulation of Hopf adjunction.
carries a right $L\mathbb{1}$-comodule structure due to the comonoidality of $L$. We shall ask for $\chi$ to be a morphism of right $L\mathbb{1}$-comodules:

\[
\begin{array}{c}
LC \\ \chi \\ R\mathbb{1}
\end{array}
\xrightarrow{
\begin{array}{c}
L_2 \\ \chi \otimes 1 \\ \nu_R
\end{array}
}
\begin{array}{c}
LC \otimes L\mathbb{1} \\ \chi \otimes 1 \\ R\mathbb{1} \otimes L\mathbb{1}
\end{array}
\tag{24}
\]

The composite $L(C \otimes U(-)) \cong LC \otimes (-) \cong R\mathbb{1} \otimes (-) \cong RU(-)$ is an isomorphism. We want to extend it to an isomorphism $L(C \otimes (-)) \cong R$. To this purpose, we need additional assumptions.

**Theorem 13.** Let $L \dashv U \dashv R : C \rightarrow D$ a biHopf adjunction between monoidal categories, together with an isomorphism $\chi : LC \rightarrow R\mathbb{1}$ of right $L\mathbb{1}$-comodules. Assume that for each object $X \in C$, the diagram below

\[
\begin{array}{c}
X \\ \eta_l
\end{array}
\xrightarrow{
\begin{array}{c}
\eta^l \\ \eta^l \otimes 1 \\ \eta^l_\otimes
\end{array}
}
\begin{array}{c}
ULX \\ ULULX
\end{array}
\tag{25}
\]

is an equalizer, and that it is preserved by the functor $L(C \otimes (-))$. Then $\chi$ uniquely extends to a natural isomorphism $\chi : L(C \otimes (-)) \rightarrow R$ such that

\[
\begin{array}{c}
L(C \otimes X) \\ \chi
\end{array}
\xrightarrow{
\begin{array}{c}
L_2 \\ \chi \otimes 1
\end{array}
}
\begin{array}{c}
LC \otimes LX \\ R\mathbb{1} \otimes LX
\end{array}
\tag{26}
\]

**Proof.** Notice first that the diagram below commutes serially

\[
\begin{array}{c}
X \\ \eta_l
\end{array}
\xrightarrow{
\begin{array}{c}
\eta^l \\ \eta^l \otimes 1 \\ \eta^l_\otimes
\end{array}
}
\begin{array}{c}
ULX \\ ULULX
\end{array}
\xrightarrow{
\begin{array}{c}
UL\eta^l_l \\ \eta^l_\otimes
\end{array}
}
\begin{array}{c}
ULULX \\ UL\mathbb{1} \otimes ULX
\end{array}
\]

By hypothesis, the top row is an equalizer and the vertical arrow on the right is an isomorphism; consequently, the bottom row is also an equalizer, thus preserved by the right adjoint functor $R$:

\[
\begin{array}{c}
RX \\ \eta_R
\end{array}
\xrightarrow{
\begin{array}{c}
R\eta_R^l \\ R(\eta^l_\otimes
\end{array}
}
\begin{array}{c}
RULX \\ R(UL\mathbb{1} \otimes ULX)
\end{array}
\tag{27}
\]

Also by hypothesis the diagram below

\[
\begin{array}{c}
L(C \otimes X) \\ L(1 \otimes \eta^l)
\end{array}
\xrightarrow{
\begin{array}{c}
L(1 \otimes U) \\ L(\eta^l_\otimes)
\end{array}
}
\begin{array}{c}
L(C \otimes ULX) \\ L(\mathbb{1} \otimes UL)
\end{array}
\xrightarrow{
\begin{array}{c}
L(\mathbb{1} \otimes U) \\ L(\eta^l_\otimes)
\end{array}
}
\begin{array}{c}
L(C \otimes ULULX)
\end{array}
\]
is an equalizer, obtained by applying the functor \( L(C \otimes (-)) \) to \( 25 \).

We relate the above two equalizers by two vertical (iso)morphisms, as follows:

\[
\begin{array}{c}
\xymatrix{
L(C \otimes X) 
\ar[r]^-{(1 \otimes \eta^1)} & L(C \otimes ULX) 
\ar[r]^-{L(1 \otimes UL\eta^1)} & L(C \otimes ULULX) \\
\ar|\chi|^{|RULX|^1} & \ar|\chi\otimes 1|^{|RUL\chi|^1} & \ar|\chi\otimes \eta^1|^{|RUL\chi\otimes \eta^1|^1} \\
RX 
\ar[r]^-{R\eta^1} & RULX 
\ar[r]^-{R(u_2^{-1} \circ ULX)} & R(UL \otimes ULX) 
\ar|\eta^1|^{|RULX|^1} & \ar|\eta^1\otimes 1|^{|RULX|^1} & \ar|\eta^1\otimes \eta^1|^{|RULX|^1} 
}
\end{array}
\]

The diagram on the right commutes serially:

\[
\begin{array}{c}
\xymatrix{
L(C \otimes ULX) 
\ar[r]^-{(1 \otimes \eta^1)} & L(C \otimes LULX) 
\ar[r]^-{L(1 \otimes UL\eta^1)} & L(C \otimes ULULX) \\
\ar|\chi|^{|RULX|^1} & \ar|\chi\otimes 1|^{|RUL\chi|^1} & \ar|\chi\otimes \eta^1|^{|RUL\chi\otimes \eta^1|^1} \\
RX 
\ar[r]^-{R\eta^1} & RULX 
\ar[r]^-{R(u_2^{-1} \circ ULX)} & R(UL \otimes ULX) 
\ar|\eta^1|^{|RULX|^1} & \ar|\eta^1\otimes 1|^{|RULX|^1} & \ar|\eta^1\otimes \eta^1|^{|RULX|^1} 
}
\end{array}
\]

Consequently, there is a unique (natural) (iso)morphism \( \chi : L(C \otimes X) \to RX \), such that the left square in \( 28 \) commutes:

\[
R\eta^1 \circ \chi = \delta^1 \circ (\chi \otimes 1) \circ H^1 \circ L(1 \otimes \eta^1) \tag{29}
\]

By hypothesis, \( \delta^1 \) is an isomorphism. Then \( 12 \) and \( 17 \) show that \( \chi : L \to R \) verifies the relation \( 29 \) if and only if it satisfies \( 26 \), which can be seen as generalizing \( 24 \).

Finally, using \( 11 \), \( 24 \) and \text{co} \( 10 \), a quick diagram chasing will now show that for \( X = 1 \), \( \chi : LC \cong L(C \otimes 1) \to R1 \) also satisfies \( 29 \). Henceforth \( \chi \) extends \( \chi \), in the sense that \( X_1 = \chi \).\[\square\]
isomorphism \(L\) and \(R\) are such that there is an isomorphism \(\chi\) between \(L(C \otimes X \otimes Y)\) and \(L(C \otimes X) \otimes LY\).

\[
\begin{array}{ccc}
L(C \otimes X \otimes Y) & \xrightarrow{L_2} & L(C \otimes X) \otimes LY \\
\chi \downarrow & & \downarrow \chi \otimes 1 \\
R(X \otimes Y) & \xrightarrow{\nu_R} & RX \otimes LY
\end{array}
\]

In particular for \(Y = 1\), diagram (29) above implies that \(\chi\) is a morphism of right \(L\)-comodules, while taking \(X = 1\) in (30) recovers (26).

Proof. The diagram below shows that \((R\eta \otimes 1) \circ \nu_R \circ \chi = (R\eta \otimes 1) \circ (\chi \otimes 1) \circ L_2\).

By hypothesis, \((R\eta \otimes 1)\) is a monomorphism, thus (30) follows.

\[
\begin{array}{ccc}
L(C \otimes X \otimes Y) & \xrightarrow{L_2} & L(C \otimes X) \otimes LY \\
\chi \downarrow & & \downarrow \chi \otimes 1 \\
R(X \otimes Y) & \xrightarrow{\nu_R} & RX \otimes LY
\end{array}
\]

Remark 15. The hypotheses of Theorem 13 entail that \(R\eta \otimes 1\) is a regular monomorphism. It remains monic after tensoring on the right with \(LY\), for \(Y \in \mathcal{C}\), if for example the category \(\mathcal{D}\) would be right autonomous.

4.2 BiHopf adjunctions are ambidextrous We shall now turn to the special case when \(C\) is (isomorphic to) the unit object \(1\). Notice that in this case (25) is mapped by \(L\) into a split equalizer, thus an absolute one. So it is only necessary to require (27) to be an equalizer. Then Theorem 13 and Proposition 14 rephrase as follows:

Theorem 16. Let \(L \dashv U \dashv R : \mathcal{C} \to \mathcal{D}\) a biHopf adjunction between monoidal categories, such that there is an isomorphism \(\chi : L1 \to R1\) of right \(L1\)-comodules. Assume that (27) is an equalizer.
Then $\chi$ uniquely extends to a natural isomorphism $\chi : L \to R$ such that

$$
\begin{array}{c}
LX \xrightarrow{L_2} L1 \otimes LX \\
\downarrow \chi \\
RX \xleftarrow{R\eta} R1 \otimes LX
\end{array}
$$

In particular, $U$ is a Frobenius functor.

If additionally the morphism $R\eta \otimes 1 : RX \otimes LY \to RULX \otimes LY$ is monic for all $X, Y \in C$, then $\chi$ verifies the more general relation (4).

### 4.3 BiHopf monads are Frobenius monads

Having in mind the results established in the previous section, we turn now to the corresponding statements for biHopf monads. We refer to the notations introduced in Sections 3.3 and 4.1.

Let $T$ be a biHopf monad on a monoidal category $C$ having right adjoint $G$. Assume there is an object $C \in C$ and an isomorphism $\chi : TC \to G1$ of right Hopf $T$-modules, where $G1$ carries the right Hopf $T$-module structure from Proposition (12).

We shall provide conditions which will guarantee the extension of $\chi$ to a natural isomorphism $T(C \otimes X) \cong GX$ of right Hopf $T$-modules.

**Theorem 17.** Let $T$ be a biHopf monad on a monoidal category $C$. Assume there is an object $C \in C$, together with an isomorphism of right Hopf $T$-modules $\chi : TC \to G1$. If for any $X \in C$, the diagram below is an equalizer

$$
\begin{array}{c}
X \xrightarrow{\eta} TX \xrightarrow{T\eta} T^2X \\
\downarrow \chi \\
GX \xleftarrow{G\eta} G1 \otimes TX
\end{array}
$$

and it is preserved by the functor $T(C \otimes (-))$, then $\chi$ extends to a natural isomorphism $\chi : T(C \otimes X) \to GX$ of $T$-algebras such that

$$
\begin{array}{c}
T(C \otimes X) \xrightarrow{T2} T^2C \otimes TX \\
\chi \downarrow \chi \downarrow \\
GX \xleftarrow{G\eta} G1 \otimes TX
\end{array}
$$

If additionally the morphism $G\eta \otimes 1 : GX \otimes TY \to GTX \otimes TY$ is monic for all $X, Y \in C$, then $\chi$ satisfies

$$
\begin{array}{c}
T(C \otimes X \otimes Y) \xrightarrow{T2} T(C \otimes X) \otimes TY \\
\chi \downarrow \chi \downarrow \\
G(X \otimes Y) \xleftarrow{G\eta} GX \otimes TY
\end{array}
$$

In particular, $\chi$ is an isomorphism of right Hopf $T$-modules.
Proof. By hypothesis, the triple adjunction $L^T \dashv U^T \dashv R^T$ is biHopf and $\chi : L^T C \to R^T \mathbb{1}$ is a morphism of right $L^T \mathbb{1}$-comodules. As the forgetful functor $U^T$ creates limits, (25) will be an equalizer and will be preserved by $L^T(C \otimes (-))$. Consequently, theorem 13 applies to yield an isomorphism $\chi : T(C \otimes X) \to G X$ of $T$-algebras such that (32) holds.

Finally, using the additional assumptions and Proposition 14, we see that (33) also holds.

We shall now see that there is a natural way of providing an object $C \in \mathcal{C}$ together with an isomorphism $\chi : T C \to G / BD$ of Hopf $T$-modules, under some additional hypotheses.

Recall from Proposition 12 that cofree $G$-coalgebras, in particular $G \mathbb{1}$, become right Hopf $T$-modules. Assuming that the hypotheses of Theorem 9 hold, there is an isomorphism $\chi : T C \cong G \mathbb{1}$ of right Hopf modules, where $C$ is the coinvariant part of the right Hopf $T$-module $G \mathbb{1}$, given by the equalizer below:

\[
C \xrightarrow{i} G \mathbb{1} \xrightarrow{\eta \otimes 1} T \mathbb{1} \otimes G \mathbb{1}
\]

Explicitly, $\chi$ can be written as in (22):

\[
\chi : T C \xrightarrow{T i} T G \mathbb{1} \xrightarrow{\mu} GT^2 G \mathbb{1} \xrightarrow{G \mu G} GTG \mathbb{1} \xrightarrow{G \eta} G \mathbb{1}
\]

Theorem 9 implies also that for a right Hopf module which is free as a $T$-algebra, namely $(T X, \mu, T^2)$, the coinvariant part is precisely $X$. In particular, that (31) is also an equalizer. We summarize all the above into the following:

**Corollary 18.** Let $T$ a biHopf monad on a monoidal category $\mathcal{C}$, with right adjoint $G$. Assume $T$ is conservative, right $T$-Hopf modules have coinvariants (the equalizer (21) exists for any right Hopf $T$-module) and $T$ preserves coreflexive equalizers. If additionally the functor $C \otimes (-)$ preserves the equalizer (31), then $\chi$ extends to an isomorphism $\chi : T(C \otimes (-)) \to G$ of $T$-algebras satisfying (32).

Particularizing now Theorem 17 to the case $C \cong \mathbb{1}$ and using Theorem 16, we obtain:

**Theorem 19.** Let $T$ a biHopf monad on a monoidal category $\mathcal{C}$, such that there is an isomorphism $\chi : \mathbb{1} \to G \mathbb{1}$ of right Hopf $T$-modules, and (31) is an equalizer. Then $\chi$ extends to a natural isomorphism $\chi : T \to G$ of $T$-algebras such that

\[
\begin{array}{ccc}
TX & \xrightarrow{T^2} & T \mathbb{1} \otimes TX \\
\downarrow{\chi} & & \downarrow{\chi \otimes 1} \\
GX & \xrightarrow{\nu^1} & G \mathbb{1} \otimes TX
\end{array}
\]

Consequently, $T$ is a Frobenius monad.
If additionally the morphism $G\eta \otimes 1 : GX \otimes TY \to GTX \otimes TY$ is monic for all $X, Y \in C$, then \( \chi \) verifies

\[
\begin{array}{ccc}
T(X \otimes Y) & \xrightarrow{T_2} & TX \otimes TY \\
\chi & \downarrow & \downarrow \chi \otimes 1 \\
G(X \otimes Y) & \xrightarrow{\nu^r} & GX \otimes TX
\end{array}
\]

Therefore under the above exactness assumptions, a biHopf monad \( T \) on a monoidal category is a Frobenius monad if \( T \cong G \) as right \( T \)-Hopf modules.

The above results are not completely satisfactory, as they do not take into account the full strength of the (co)monoidal structure of the functors involved. We shall be able to strengthen them in the next sections, but under the additional assumptions of existence of duals.

5 Hopf adjunctions on autonomous categories

5.1 Duals create adjoints Consider a strong monoidal functor \( U : C \to D \) between autonomous categories. Then it is folklore that \( U \) preserves both left and right duals, in the sense that there are isomorphisms \( \theta : SU \to US \), respectively \( \theta' : S'U \to US' \) compatible with the (co)evaluation morphisms.

The next result has been noticed in [14, Proposition 7.6] (but see also [4, Corollary 3.12] and [5, Lemma 3.5]). As it requires the mere existence of contravariant equivalences \( S \dashv S' \) on both categories and of the natural isomorphism \( \theta : SU \to US \), without any need of monoidal or autonomous structure, we have chosen to state it in full generality.

**Lemma 20.** Let \( C, D \) categories endowed with contravariant (adjoint) equivalences \( S \dashv S' : C^{op} \to C \) and similarly for \( D \). Assume there is a natural isomorphism \( \theta : SU \to US \). Then \( U \) has a left adjoint if and only if it has a right adjoint.

**Proof.** We shall consider the case when \( U \) has a left adjoint \( L \); a similar argument can be used in case \( U \) has a right adjoint. Then \( L^{op} \) is right adjoint to \( U^{op} \), and the result follows by composing the sequence of adjunctions

\[
\begin{array}{ccc}
C & \overset{S}{\underset{S'}{\dashv}} & C^{op} & \overset{U^{op}}{\underset{L^{op}}{\dashv}} & D^{op} & \overset{S'}{\underset{S}{\dashv}} & D
\end{array}
\]

and noticing that $U \cong SU^{op} \cong S'^{-1} U S'$.

\[\text{[12] There is however an error in op. cit., in the sense that it asserts creation of an infinite chain of adjunction. In fact one can only prove that } U \text{ has left adjoint if and only if it has a right adjoint.}
\]

\[\text{[13] } S' \text{ is both left and right adjoint to } S, \text{ being part of an adjoint equivalence.}\]
As the mate of $\theta : SU \to US$ under the adjunction (contravariant equivalence) $S \dashv S'$ is again an isomorphism $\theta' : S'U \to US'$, by switching then the roles of $S$ and $S'$ in Lemma 20 we obtain that $SLS'$ is also a right adjoint for $U$, provided $U$ has left adjoint $L$. By unicity of adjoints, $S'L \cong SLS'$.

The readers might have noticed that above, the strong monoidal structure of $U$ was only used to create the isomorphism $\theta$. Fix now a left adjoint $L$ for $U$. The isomorphism of right adjoints $S'L \cong SLS'$ is in fact an isomorphism of monoidal functors, where both carry the monoidal structure inherited from $U$. As so does the comonoidal structure of $L$, there is no surprise that alternatively the monoidal structure on the "de Morgan duals" $S'L$ and $SLS'$ can equivalently be described using the comonoidal structure of $L$.

**5.2 Frobenius monoidal functors from Hopf adjunctions**

**Remark 21.** Let $U : \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor between left autonomous categories, with left adjoint $L$. Then the adjunction $L \dashv U$ is left Hopf, the inverse of the (left) Hopf operator being the composite

$$
\begin{align*}
L(-) \otimes A & \xrightarrow{L(1 \otimes \eta_{L-}) \otimes 1} L(- \otimes UA \otimes SUA) \otimes A \\
& \xrightarrow{L(1 \otimes \theta^{-1}) \otimes 1} L(U(UL(- \otimes U) \otimes SUA) \otimes A \\
& \xrightarrow{\epsilon \otimes 1} L(- \otimes UA) \otimes SA \otimes A \\
& \xrightarrow{1 \otimes e} L(- \otimes UA)
\end{align*}
$$

Similarly, an adjunction $L \dashv U$ between right autonomous categories with $U$ strong monoidal is right Hopf. By duality, if $U$ is a strong monoidal functor between (left/right) autonomous categories and has right adjoint $R$, then the adjunction $U \dashv R$ is (left/right) coHopf.

Concluding, an adjunction $L \dashv U$ between autonomous categories with $U$ strong monoidal is in fact a triple (biHopf) adjunction $L \dashv U \dashv R$, where the right adjoint satisfies $R \cong SLS' \cong S'L$ as monoidal functors.

We can now restate Theorem 16 in case of autonomous categories as follows:

**Proposition 22.** Let $L \dashv U$ be an adjunction between autonomous categories, with $U$ strong monoidal, and let $R$ denote the right adjoint of $U$, as provided by Lemma 24. Assume (23) is a coreflexive equalizer for any object $X \in \mathcal{C}$, and $L$ preserves these coreflexive equalizers.

If $L\mathbb{1}$ is a Frobenius algebra, witnessed by an isomorphism $\chi : L\mathbb{1} \cong R\mathbb{1}$ of right $L\mathbb{1}$-comodules, then $\chi$ extends to a natural isomorphism $\chi : L \cong R$ satisfying (4).

Corroborating the above with Theorem 4 and Remark 5, we obtain that

**Corollary 23.** Under the assumptions of Proposition 22, $L$ becomes a Frobenius monoidal functor with the monoidal structure inherited from $R$. 
Remark 24. Compare the above results with [33], where a very particular case of a Hopf adjunction between autonomous categories was considered.

More precisely, let $C$ denote a finite tensor category. Then the forgetful functor from the monoidal center of $C$ is strong monoidal and has left adjoint [15], in particular it induces a biHopf adjunction as above. In [33], it is shown that the left and right adjoints of $U$ are isomorphic, $L \cong R$, if and only if $LS \cong SL$ ($RS \cong SR$), if and only if $L \mathbb{1}$ (equivalently, $R \mathbb{1}$) is a self dual object, if and only if $C$ is unimodular - the latter being a specific property of finite tensor categories, corresponding to our assumption $C \cong \mathbb{1}$.

Our results in Proposition 22 and Corollary 23 are obtained in more general setting of arbitrary Hopf adjunctions between autonomous categories, but with the additional hypothesis that $L \mathbb{1}$ is not only a self-dual object, but a Frobenius algebra. The exactness conditions we impose are fulfilled in [33] because of the special properties of the category considered. Consequently, the results we obtain are more precise, in the sense that $L$ and $R$ are not just isomorphic functors, but the isomorphism between takes into account also their (co)monoidal nature; while in [33] this aspect does not seem to be considered.

5.3 Hopf monads are Frobenius monoidal functors

If instead of an adjunction $L \dashv U$ with $U$ strong monoidal between autonomous categories, we consider a Hopf monad $T$ on an autonomous category, then the associated category of $T$-algebras is autonomous [4], and the forgetful functor $U^T$ is strong monoidal. Consequently, $U^T$ has both left adjoint $L^T$ and right adjoint $R^T$ as above, resulting the biHopf adjunction $L^T \dashv U \dashv R^T$, and a structure of biHopf monad on $T$, in particular a right adjoint $G$ characterized by a monoidal isomorphism $G \cong STS' \cong STS$.

Then Theorem 19 becomes:

Corollary 25. Let $T$ a Hopf monad on an autonomous category $C$, such that [31] is an equalizer, and let $G$ denote the right adjoint of $T$, as above. Assume there is an isomorphism $\chi : T \mathbb{1} \to G \mathbb{1}$ of right Hopf $T$-modules. Then $\chi$ extends to a natural isomorphism $\chi : T \to G$ of $T$-algebras satisfying [34]. Consequently, $T$ is simultaneously a Frobenius monad and a Frobenius monoidal functor, and $\chi$ is a monoidal-comonoidal isomorphism.

We end this section by an example which shows that even in case of autonomous categories, not any Hopf monad is Frobenius monoidal, nor a Frobenius monad.

Example 26. Let $k$ be a commutative field of characteristic different from 2, and consider the $\mathbb{Z}_2$-graded Hopf algebra $A = k[x]/(x^2)$, with grading $A_0 = k$ and $A_1$ spanned by $x$. It is an algebra as usual, with the induced (polynomial) multiplication and unit 1. As for the coalgebra structure, the element 1 is group-like and $x$ is primitive. The antipode acts as $S(1) = 1$, $S(x) = -x$. This is a Hopf algebra in the braided category $C$ of (finite dimensional) $\mathbb{Z}_2$-graded vector spaces, which is not graded Frobenius (that is, it is not a Frobenius algebra in $C$), but still a Frobenius algebra in the category of vector spaces [12].
Then tensoring with $A$ yields the Hopf monad $T = A \otimes -$ on the (braided) autonomous category $C$. But $T$ fails to be a Frobenius monad, and also a Frobenius monoidal functor, because $A$ is not a Frobenius algebra.

References

[1] L. Abrams. Modules, comodules, and cotensor products over Frobenius algebras. *J. Algebra*, 219(1):201–213, 1999.

[2] R. F. Blute. Hopf algebras and linear logic. *Math. Struct. Comput. Sci.*, 6(02):189–212, 1996.

[3] A. Bruguières, S. Lack, and A. Virelizier. Hopf monads on monoidal categories. *Adv. Math.*, 227(2):745–800, 2011.

[4] A. Bruguières and A. Virelizier. Hopf monads. *Adv. Math.*, 215(2):679–733, 2007.

[5] A. Bruguières and A. Virelizier. Quantum double of Hopf monads and categorical centers. *Trans. AMS*, 364(3):1225–1279, 2012.

[6] S. Caenepeel, G. Militaru, and S. Zhu. *Frobenius and separable functors for generalized module categories and nonlinear equations*. Berlin: Springer, 2002.

[7] D. Chikhladze, S. Lack, and R. Street. Hopf monoidal comonads. *Theory Appl. Categ.*, 24(19):554–563, 2010.

[8] J. R. B. Cockett, J. Koslowski, and R. A. G. Seely. Introduction to linear bicategories. *Math. Struct. Comput. Sci.*, 10(02):165–203, 2000.

[9] J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. In M. P. Fourman, P. T. Johnstone, and A. M. Pitts, editors, *Applications of Categories in Computer Science*, Lond. Math. Soc. Lect. Note Ser. 177, pages 45–65. Cambridge Univ. Press, 1992.

[10] J. R. B. Cockett and R. A. G. Seely. Weakly distributive categories. *J. Pure Appl. Algebra*, 114(2):133 – 173, 1997.

[11] J. R. B. Cockett and R. A. G. Seely. Linearly distributive functors. *J. Pure Appl. Algebra*, 143(1-3):155 – 203, 1999.

[12] S. Dascalescu, C. Năstăsescu, and L. Năstăsescu. Frobenius algebras of corepresentations and group-graded vector spaces. *J. Algebra*, 406:226–250, 2014.

[13] B. Day and C. Pastro. Note on Frobenius monoidal functors. *New York J. Math.*, 14:733–742, 2008.
[14] B. Day and S. Ross. Quantum categories, star autonomy, and quantum groupoids. In G. Janelidze, B. Pareigis, and W. Tholen, editors, *Galois theory, Hopf algebras, and semiabelian categories*, pages 187–225. AMS, 2004.

[15] B. Day and R. Street. Centres of monoidal categories of functors. In A. Davydov, M. Batanin, M. Johnson, S. Lack, and A. Neeman, editors, *Categories in algebra, geometry and mathematical physics*, Contemp. Math. 431, pages 187–202, Providence, RI, 2007. AMS.

[16] J. M. Egger. Star-autonomous functor categories. *Theory Appl. Categ.*, 20(11):307–333, 2008.

[17] J. M. Egger. The Frobenius relations meet linear distributivity. *Theory Appl. Categ.*, 24:25–38, 2010.

[18] S. Eilenberg and G. M. Kelly. Closed categories. In S. Eilenberg, D. K. Harrison, S. MacLane, and H. Röhrl, editors, *Proceedings of the Conference on Categorical Algebra*, pages 421–562. Springer Berlin Heidelberg, 1966.

[19] S. Eilenberg and J. C. Moore. Adjoint functors and triples. *Illinois J. Math.*, 9:381–398, 1965.

[20] H. Fausk, P. Hu, and J. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11(4):107–131, 2003.

[21] P. Freyd and D. N. Yetter. Coherence theorems via knot theory. *J. Pure Appl. Algebra*, 78:49–76, 1992.

[22] J. Fuchs and C. Stignier. On Frobenius algebras in rigid monoidal categories. *Arab. J. Sci. Eng., Sect. C, Theme Issues*, 33(2):175–191, 2008.

[23] M. Grandis and R. Paré. Adjoints for double categories. *Cah. Topol. Géom. Différ. Catég.*, 45(3):193–240, 2004.

[24] A. Joyal and R. Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[25] G. M. Kelly. Many-variable functorial calculus. I. In G. M. Kelly, M. Laplaza, and S. Mac Lane, editors, *Coherence in categories*, LNM 281, pages 66–105. Springer, 1972.

[26] G. M. Kelly. Doctrinal adjunction. In G. M. Kelly, editor, *Category Seminar*, LNM 420, pages 257–280. Springer, Berlin Heidelberg, 1974.

[27] G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. *J. Pure Appl. Algebra*, 19:193–213, 1980.

[28] F. W. Lawvere. Ordinal sums and equational doctrines. In H. Appelgate, M. Barr, J. Beck, F. W. Lawvere, F. E. J. Linton, E. Manes, M. Tierney, and F. Ulmer, editors, *Seminar on Triples and Categorical Homology Theory*, LNM 80, pages 141–155. Springer, 1969.
[29] S. MacLane. *Categories for the Working Mathematician*. GTM 5. Springer-Verlag, New York, 4th edition, 1998.

[30] B. Mesablishvili and R. Wisbauer. QF functors and (co)monads. *J. Algebra*, 376:101–122, 2013.

[31] I. Moerdijk. Monads on tensor categories. *J. Pure Appl. Algebra*, 168(2-3):189–208, 2002.

[32] C. Pastro and R. Street. Closed categories, star-autonomy, and monoidal comonads. *J. Algebra*, 321(11):3494–3520, 2009.

[33] K. Shimizu. Characterizations of unimodular finite tensor categories. arXiv preprint arXiv:1402.3482, 2014.

[34] M. Shulman. Comparing composites of left and right derived functors. *New York J. Math.*, 17(75-125):12, 2011.

[35] R. Street. Frobenius monads and pseudomonoids. *J. Math. Phys.*, 45(10):3930–3948, 2004.