Full Poincaré mapping for ultra-sensitive polarimetry

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While polarisation sensing is vital in many areas of research, with applications spanning from microscopy to aerospace, traditional approaches are limited by method-related error amplification or accumulation, placing fundamental limitations on precision and accuracy in single-shot polarimetry. Here, we put forward a new measurement paradigm to circumvent this, introducing the notion of a universal full Poincaré generator to map all polarisation analyser states into a single vectorially structured light field, allowing all vector components to be analysed in a single-shot with theoretically user-defined precision. To demonstrate the advantage of our approach, we use a common GRIN optic as our mapping device and show mean errors of <1% for each vector component, enhancing the sensitivity by around three times, allowing us to sense weak polarisation aberrations not measurable by traditional single-shot techniques. Our work paves the way for next-generation polarimetry, impacting a wide variety of applications relying on weak vector measurement.

Key words: Universal full Poincaré generator; Structured light; Polarisation sensing
Introduction

Polarisation sensing methods have wide applications which range from quantum physics to clinical applications\(^1\)\(^{-15}\). They can be divided into two categories: time-resolved, where measurements are taken using a sequence of analysers in a time multiplexed manner, or single-shot, where different analysers are spatially multiplexed. Time resolved measurements can be easier to implement, but single-shot methods are crucial for applications with rapidly changing inputs or where high throughput is required. The standard measurement approaches in both methods are directly or indirectly related to a core measurement equation\(^11\)\(^{-20}\): \(S = \text{inv}(A) \cdot I\), where \(S\) is the Stokes vector to be measured, and \(I\) is the vector of intensities recorded at the detector. Matrix \(A\) is known as the instrument matrix, which is determined by the system configuration. In order to enhance precision and accuracy, numerous attempts have been made to push \(A\) towards an optimal matrix as it determines the properties of the error propagation and hence affects the performance\(^16\)\(^{-20}\). An evaluation of systematic error amplification level can be performed using the condition number (CN) of \(A\), with the theoretical minimum value\(^16\)\(^{-20}\) \(\text{CN} = \sqrt{3}\). In practice, traditional approaches require various processes to be implemented before the matrix calculation, which includes denoising, optimisation, and calibration\(^16\)\(^{-26}\). Although the detailed operating procedures are different among various polarimetry techniques, the complex error transfer and accumulation processes are fairly consistent\(^16\)\(^{-26}\), as all they essentially measure the vector property of the light beam in cumbersome and indirect ways. This places fundamental limits on what is possible in the context of ultra-sensitive single-shot polarimetry: given the error amplification factor and the separate accumulation processes, improving sensitivity is typically achieved through acquisition over longer time or through complicated and inconvenient calibration procedures\(^18\)\(^{-25}\), which means in general that either sensitivity or speed is compromised.

Here we will put forward a new approach that allows high sensitivity and speed to be realised simultaneously, for single-shot polarimetry without the error amplification factor and cumbersome processes. At the heart of our approach is the notion of Full Poincaré Beam (FPB) mapping, producing a particular form of vectorially structured light. The concept of a ‘Full Poincaré beam’ has been of great interest in optics for many years\(^27\)\(^{-32}\). This interest stems from the uniqueness of the beam property, namely that the state of polarisation (SOP) of the beam’s transverse cross section can cover the full
Poincaré sphere, hence the name – FPB. Such beams are especially important for various applications utilising structured light, such as singularity analysis and beam shaping\textsuperscript{27-32}. Most previous investigations have been concerned with the properties of the FPBs themselves, but we extend the discussion here to cover a device that maps any input to a FPB, which we term a universal full Poincaré generator (UFPG). The essential feature here is the ability to generate different FPBs given different pure uniform incident SOPs, which is possible if the element has a spatially variant birefringence that with appropriate design can provide the full range of retardances required to be a universal analyser, i.e., analyse for all states of polarisation simultaneously in a single shot rather than by a time or spatial sequence of different analysers (see Supplementary Notes 1 and 2 for more details). The resulting complex vectorially structured light field then holds all the information required to deduce the initial unknown SOP, which can easily be extracted by machine learning assisted image processing. This provides a simple intuitive route to polarisation measurement, as the recorded structured light image has a unique one-to-one link to the input SOP, which enables new prospects for direct vector sensing, rather than indirect estimation via a matrix calculation.

Figure 1 illustrates conceptually how the UFPG-based paradigm works, using a simple GRIN lens system as an example (see Discussion later for other implementation options for). We use an illumination beam with an arbitrarily chosen linear or elliptical SOP for demonstration purposes (in reality this represents the unknown input state). A FPB is generated (Fig. 1b) after this passes through the GRIN lens\textsuperscript{20, 32}. The output vector field is then filtered via a polarisation filter (PF) assembly (the right-hand circular (RC) SOP is selected as the eigenstate here) leading to a non-uniform intensity distribution (Fig. 1b) that can be recorded at the detector. Intuitively, one can understand that the brightest points within the intensity distribution must correspond to the eigenstate of the PF. The positions of these points depend upon the input state. Hence, the input SOP could – in principle – be read off directly from the positions of the brightest points, as long as the mapping of states to image position is known (see Fig. 1c). As the GRIN lens is a UFPG of order, there are two points of maximum intensity two (see Supplementary Notes 1 and 2). The nature of this UFPG sensing paradigm means that we have access, in principle, to a complete set of all possible analyser states in a single mapping. This goes beyond any other non-UFG systems, as they cannot sense the ‘complete’ set of states in the same way and hence cannot enable the same paradigm. Our approach circumvents the above-mentioned mathematical limit, as here is no minimum error propagation amplification posed by the matrix process (see Supplementary Note 3). Instead, in theory its sensitivity can
be continuously boosted, even in a single-shot, by judicious choice of detector pixel size and imaging optics, and in practice further enhanced by the use of intelligence in the imaging, such as through machine learning.

Figure. 1d provides a visualized example of the link between the incident SOPs (shown on Poincaré sphere; Fig. 1d (i)), the SOPs related locations mapped on the intensity image (Fig. 1d (ii)), and a sketch of the pixel size on the detector (Fig. 1d (iii)). In order to determine the brightest point locations properly, the essential point of the new paradigm, we adopt a pure image-based sensing process (details in next section, and **Supplementary Note 4**). Instead of dividing the pipeline into denoising, optimisation, and intensity and polarisation calibration, we have managed to combine such processes in one step by remodelling the process into a combined fitting and estimation task. It is an optimal, system-based, integrated approach that enables the notable increase in precision and accuracy.
Figure 1. Concepts and mechanisms of the UFPG paradigm. (a) Definition of a UFPG. (b) Intensity distribution generation process – including sending different incident SOPs through the GRIN lens system, obtaining corresponding output FPBs, and using a
polarisation filter (PF) to create an intensity image. (c) Intuitive explanation of the measurement process, illustrated through finding brightest points (BP) on the intensity distribution and reading the input SOP from the locations corresponding to these points on the PF determined FPB map. In practice, all points in the image feed into the estimation process. (d) Mapping of SOPs between domains. (i) Example SOPs (points $\alpha$, $\beta$, $\gamma$) on the Poincaré sphere. (ii) The corresponding points $\alpha'$, $\beta'$, $\gamma'$, mapped from points $\alpha$, $\beta$, $\gamma$ on the acquired intensity image. (iii) A sketch of the points as pixel positions recorded on the detector. The pixel size acts as a theoretical limit on the sensitivity of the new paradigm (see Supplementary Note 4).

Results

The feasibility of such a UFPG polarimeter was tested experimentally, with the results shown in Figures 2 and 3. Effective realisation of this method consists of two steps. First, fitting of a model to the intensity data is required in order to estimate the position of the brightest points. This needs to be robust against temporal or spatial noise$^{16-26}$. Secondly, the mapping between the detected positions and the related SOP should be properly identified. To implement these image processing steps, we drew upon machine-learning (ML) techniques$^{33-36}$, particularly through inspiration from the recent success of convolutional neural networks in computer vision$^{33-36}$. An overview of the ML process along with detailed mechanisms can be found in Supplementary Note 4. Importantly, rather than treating the complex polarisation measurement processes as separate steps as traditional methods have done$^{16-26}$, we implemented a single image-based ‘end-to-end’ solution$^{35,36}$ to achieve maximum efficacy. This way we avoid the need for separate procedures for denoising, optimisation and calibration (which can each introduce separate errors).
Figure 2. Measurement performance verification of the UFPG paradigm, and comparison with recent reported techniques. (a) Three example curves (curves generated using an arbitrary state generator followed by a rotating quarter wave plate; For details of the generator see Supplementary Note 2) on the Poincaré sphere as a feasibility test for our new approach. Both theoretical and experimental data are presented on the sphere. Theoretical data are shown in grey circles; experimental counterparts are indicated by star symbols with different colours for different curves. The experimental set-up is described in Supplementary Note 2. (b) Precision evaluation process. The error diagrams for $S_1$, $S_2$, $S_3$, and Euclidean distance at 200 random points are acquired from the sampled curves on the Poincaré sphere. The same measurements were conducted by a traditional point Stokes polarimeter (see Supplementary Note 5), whose measurement errors are plotted in coloured ball symbols for an intuitive comparison. (c) The errors of Stokes vector components and
parameter $DOP$ of the UFPG system are presented with calculated magnitudes. (d) The reported error performance of several modern polarimeters\textsuperscript{37}. Compared to those existing counterparts\textsuperscript{37-41}, our new approach demonstrates the exceptional measurement precision and accuracy (see more data in Supplementary Tables 1 and 2).

Different SOPs along three selected curves (a total of 900 points) were tested on the Poincaré sphere to validate the feasibility and for error analysis. Both theoretical and experimental data are demonstrated in Fig. 2a. Then 200 points were randomly sampled from the obtained data and the mean errors corresponding to elements of the Stokes vector: $S_1$, $S_2$, $S_3$, Euclidean distance, and degrees of polarisation (DOP) were calculated (Fig. 2b and 2c). We achieve exceptional measurement precision across the Poincaré sphere with stable performance and is robust compared to existing techniques – Fig. 2d, Supplementary Tables 1 and 2 also give a quantitative comparison with several established methods to demonstrate the outstanding precision and accuracy of the new paradigm. Our results also confirm the ability to differentiate small scale polarisation changes in a single-shot with a sensitivity $\sim0.8\%$ (see definition, validations and details in Supplementary Note 5), around three times better than the well-established traditional single-shot point Stokes polarimeter, and more than six times better than other modern approaches based on light-matter interactions, such as plasmonics, spin-orbit effects and metasurfaces.
Figure 3. Sensitivity validation of the UFPG system. (b) A conceptual flow chart of the mechanism of UFPG sensing paradigm. The paradigm consists of two core parts: a UFPG system, and an imaging processing method (a ML technique in this case) to determine the locations of the brightest points and thereby determination of the unknown SOP. Any small input Stokes vector change can lead to the movement of the brightest point location. Note here we focus on the relative SOP changes rather than measuring the absolute value of the SOP. (b) Schematic of the spatially variant half wave plate array. When a vertically linear SOP is normally incident upon a plate with fixed fast axis orientations shown in the figure, the output should be a horizontally linear SOP. The output vector field is given. (c) Plate with a fine tilt (δ=3°) angle. We measured the Stokes vector output in the white dotted box region along the x axis, via the UFPG method and the traditional single-shot point Stokes polarimeter. The output vector field is demonstrated as well. (d) The obtained horizontal linear component of 200 measured points using both approaches. Pink and blue dots represent the measurement points using the UFPG polarimeter and the point Stokes polarimeter. The grey line is a reference line. The additional variations are attributed to manufacturing imperfections. More details of their initial comparisons and related analysis are provided in Supplementary Note 5. A visualized ellipticity change (pseudo-magnitude) is provided as illustration of the effect of the error in the horizontal component.
While high measurement accuracy is an advantage of the new paradigm, there are notable advantages in practical sensing, such as the capability for single-shot performance and an exceptional system sensitivity. The sensitivity is vital for applications where weak polarisation variations play important roles. This includes, for example, optical super-resolution microscopes, where polarisation aberrations would detrimentally affect the system resolution\textsuperscript{42-45} or measurements of weak vector information\textsuperscript{46, 47}; as the quality of a vectorially structured light beam is destroyed. In order to compensate for such errors, having a highly sensitive sensor is crucial, since the higher the sensitivity, the more precisely the compensation system can then conduct the correction\textsuperscript{43-45}. Here, we demonstrate the sensitivity of the UFPG polarimeter (see Fig. 3a) by testing a weak polarisation aberration that was induced by a slightly tilted spatially variant half-wave plate array (see Fig. 3b and 3c). Such an array can be used, for example, to generate the vectorial light field for the depletion beam of a STED system\textsuperscript{48}, where a fine polarisation error could be catastrophic\textsuperscript{42-45}. For the demonstration, we sampled 200 points along the y axis of the UFPG system shown in Fig. 3c (white dotted boxes) by using our new paradigm and the traditional point Stokes polarimeter. Then we calculated the horizontal linear components of the output Stokes vectors recorded using two approaches and plotted them in Fig. 3d. It can be found that the performance of the UFPG polarimeter (pink data) shows higher consistency and higher stability than the traditional polarimeter (blue data) with respect to the reference line (grey). In fact, the traditional approach fails to detect any trend of statistical significance, since the variance in the data is larger than the total induced perturbation. In contrast, our approach with its high sensitivity detects the trend accurately, with the maximum perturbation around five times larger than the variance (Fig. 3d; \textit{Supplementary Note 5}). To validate our result, we used a time averaged approach\textsuperscript{20} to reduce uncertainty in the traditional approach, producing the grey curve as a reference. To elucidate our earlier discussion, the trade-off here for improved sensitivity is a complicated and difficult calibration procedure and time; yet through the use of a new physical mechanism and the intelligence in ML, our paradigm allows for the first time single-shot ultra-sensitive polarimetry.

In conclusion, we proposed a new paradigm for polarisation sensing using a novel mapping of any input SOP to a structured light field with a physical optic, which in subsequently processed by a machine learning algorithm. It featured exceptional measurement precision and accuracy in single-shot operation with robust performance. Note in this letter, we use a GRIN lens as the UFPG because it is non-pixelated, produces high beam quality, is low cost, and easily available\textsuperscript{20, 32}; however,
the UFPG can also take the form of a DMD, SLM, metasurface\textsuperscript{5,10} or an alternative device, as long as it is suitably configured.

Future directions are available for further development of this concept. The sensitivity can be enhanced further by device substitution, as the fundamental minimum scale depends upon the number of pixels in the image (see Supplementary Note 4). The use of UFPGs with higher order numbers could lead to greater robustness and precision, linked to the enhancement of the pattern complexity\textsuperscript{35,36}. The depolarisation property can be readily extracted from the contrast of the intensity distribution image to reveal more polarimetric information (see Supplementary Note 6). Furthermore, the UFPG paradigm can be adapted to single-shot full Stokes imaging (see Supplementary Note 7). There are obviously still challenges to overcome for this technique to be applied in certain future applications (such as dealing with highly scattering systems\textsuperscript{13-15}), but our work opens a unique avenue towards precise sensing of weak vectorial information, which will provide useful capability in any application that relies upon polarisation sensing.

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Competing interests

The authors declare no competing interests.

Author contributions

C.H. and M.J.B. conceived the main ideas for the new paradigm, developed the concepts, planned the modelling and experiments. J.L., J.C., J.A., B.D. and C.H. performed the ML based brightest point estimation, parameter fitting, sensitivity evaluation, and aberration analysis. J.L., J.C., J.W., J.C., J.Q., M.W., B.D. and C.H. performed the data processing. C.H. carried out the experiments, analyzed the results, and prepared the figures. P.X., D.S.E., A.F. and M.J.B.
provided supervision and scientific guidance. C.H. and M.J.B. wrote the paper. All authors participated in discussion and editing of the paper.

Additional Information

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– Supplementary Information –

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Supplementary Note

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Supplementary Note 1: Definition of universal full Poincaré generator

There are two types of existing systems that can generate a full Poincaré beam (FPB)\textsuperscript{1-3}. The first type – shown in Supplementary Figure 1a – has the functionality of transferring fixed (or limited state) state of polarisation (SOP) into a FPB. A typical system configuration is based on two liquid crystal spatial light modulators (SLMs) or a system using multiple passes from a single SLM\textsuperscript{4}. Under such a geometry, due to the SLM having a uniformly distributed slow/fast axis orientation, it is strongly polarisation dependent. It can be used to generate the FPB to some extent but cannot meet the requirements of generating a FPB with an arbitrary incident SOP. For example, if we use a uniform linear SOP that is aligned in the same direction as the fast axis orientation of the first SLM, then the modulation of such a pass would lose all functionality. Hence with only one degree of freedom introduced by the second SLM (or the second pass) an arbitrary SOP cannot be generated\textsuperscript{4,5}. Under our definition in the main article, such a system cannot be referred to as a universal full Poincaré generator (UFPG).

The second type of system is based on the functionality of transferring each arbitrary SOP into a different FPB (see Supplementary Figure 1b), which is the UFPG we defined above. It can be a linear retarder array – such as the GRIN lens we used in the main article; or a mixed diattenuator array (see Supplementary Figure 1b). The former is a combination of all effective fast axis orientations (θ from 0° to 180°) and retardance values (δ from 0° to 180°). The latter contains all possibilities of the eigenvector (determined via transmission axis orientation θ’ and eigenvector elliptical ratio b/a). The UFPG (GRIN lens) that we used in the main article, has spatially variant birefringence that provides, in effect, the full range of retardances required with order number two (see Supplementary Figure 1c). Note that the UFPG system is not limited to these two types but could also in principle be generated by other mechanisms.
Supplementary Figure 1. Two types of systems that can generate a FPB. (a) Traditional FPB generation methods – restricted by the ability of generating a FPB from only a fixed incident SOP. The device commonly consists of a mixed-layered retarder array, such as
multiple SLMs. (b) The UFPG as defined in this paper – any pure incident SOP can be used to generate a FPB. Two typical examples are given, one is a linear retarder array (with both the value of the eigenvector and eigenvalue ranging from 0° to 180°) and the second is a mixed diattenuator array (with all possible eigenvectors, θ' ranging from 0° to 180°, and b/a from 1 to +∞). (c) The order number of the UFPG system corresponding to the order of the underlying FPB. A practical implementation of a UFPG through a GRIN lens, with order number two, consisting of linear retarders. Here, the full range of retardances and angles is provided through the spatial variation of birefringence in the GRIN medium\textsuperscript{6-8}. The exact distribution of these retardances is not critical.
Supplementary Note 2: Experimental set-up and operating principles of universal full Poincaré generator polarimetry

The experiments to validate the feasibility of the PFG based paradigm were performed with the set-up that consists of a spatial light modulator (SLM, Hamamatsu, X10468–02) based polarisation state generator (PSG) and a GRIN lens-based polarisation state analyser (PSA) (Supplementary Figure 2a). We used a LED (3 W, 633 nm, $\Delta \lambda = 20$ nm) as the illumination source, then the light beam passed through a polariser (Thorlabs, LPVIS050) to generate linear polarised light at 45° with respect to the slow (modulating) axis orientation of the SLM. Here we adopted the double-path geometry using a single SLM according to the method in Ref [5], in order to generate an arbitrary SOP. This PSG (yellow dashed box) was used to generate training data for the convolutional neural network to reconstruct the polarisation filter (PF) determined vector map (see later sections). On the analysis side (blue dashed box), the UFPG system we used was a GRIN lens (Femto Technology Co. Ltd., G-B161157-S1484, NA=0.1, Pitch=2, the same as that used in Ref [8]) followed by a fixed circular polariser (CP) (Thorlabs, CP1L633), which formed the PSA. The intensity images were obtained by a camera (Thorlabs, DCC3240N). After the network was trained, the PSG was removed from the system before conducting the actual sensing experiments. As we have discussed before, the order number of the UFPG is extensible and the distribution of retardance (or diattenuation) is not confined to a particular physical arrangement. Supplementary Figure 2b shows the link between an UFPG with order number two and a GRIN lens, which was used throughout this paper. In the GRIN lens, the retardances are arranged with circular symmetry, rather than the conceptual Cartesian arrangement shown in Fig. 1 in the main article.

The UFPG paradigm is based on the concept of ‘full generation/analysis’. As a UFPG is a complete system that contains all possible SOP generating channels, conceptually through a continuous array of polarisation elements, it can create a FPB in a single shot. Those channels can also perform ‘full analysis’ of any incident SOP, in effect through the array of polarisation elements. A GRIN lens with appropriately chosen properties can provide a practical UFPG, as it behaves as a spatially variant waveplate, due to its intrinsic birefringence. For any uniform SOP at the GRIN lens input, a FPB will be
created at the GRIN lens output. The PF following the GRIN lens post-selects the chosen PF eigenstate, effectively by projecting the output state at each field point onto the eigenstate. This creates an intensity distribution that is characteristic of the state present at the GRIN lens input. The brightest points of this distribution correspond to the points at which the GRIN lens outputs the SOP equivalent to the PF eigenstate, while the same position of the brightest point on the PF-determined map (which is a certain vector field generated via the PF eigenstate incident GRIN lens under the local coordinate system) is the incident SOP. If the input state changes, then the positions of the brightest points change. There exists therefore a mapping between the input states and the output intensity distribution. For instance, when a 45° linear SOP illuminates the GRIN lens before being filtered by a PF that contains the eigenstate of right hand circular (RC) SOP, the final intensity would have two brightest points at the locations with exact RC-SOP (the number of points is determined by the order of UFPG; for the GRIN lens there are two points). Note that the PF-determined map can only be changed through using a different PF eigenstate, which would lead to a change in the locations of the brightest points on the intensity distribution for a given input SOP. It should be emphasised again that this new paradigm directly determines the SOP from the intensity pattern without recourse to conventional instrument matrix calculations.
Supplementary Figure 2. Experimental setup of the UFPG polarimeter. (a) Sketch of experiment: P, CP: fixed polariser; QWP: rotating quarter waveplate; WP: quarter waveplate (note here we illustrate using a transmission geometry the both SLM parts were actually two adjacent sections of a single SLM; in practice, the WP was transited twice in a reflection geometry to act as a half wave plate); UFPG: universal full Poincare generator, which is in this case a GRIN lens; PSG and PFG based polarimetry are shown in the dashed yellow/blue circles. (b) Sketch for the illustration of two UFPG units within a GRIN lens. In this manifestation of the UFPG, the retardance is arranged in a polar coordinate system. The black dotted line shows the symmetric line to separate two UFPG units.
Supplementary Note 3: Stokes vector measurement

In traditional approaches to Stokes vector measurement, intensities are recorded on the detector for calculation of the target SOPs$^{9-13}$. The core equations for calculation of the Stokes vector are shown in Eq. (3-1), where $S_{in}$ is the Stokes vector of the incident light field, consisting of 4 components ($S_0$, $S_1$, $S_2$, $S_3$). $A$ is an $n \times 4$ matrix known as the instrument matrix, which is determined by the optical properties of the measurement system$^{9-13}$. $I$ is the intensity information recorded by the detector.

$$I = A \cdot S_{in},$$

$$S_{in} = \text{inv}(A) \cdot I.$$  \hspace{1cm} (3-1)

In practice, we also need to consider the occurrence of errors that would affect the measurement precision and accuracy$^9$-17. These two parameters will both, however, be affected via systematic errors $\Delta A$, $\Delta I$ or random errors $\delta A$, $\delta I$. Hence three key processes – denoising, optimisation, and calibration – have been investigated and explored by numerous researchers in the quest to enhance the precision and accuracy of a polarisation sensing system by eliminating these errors$^9$-17.

Amongst the three processes, optimisation is used to deal with the instrument matrix $A$ – which determines the amplitude of the error transformation of a designed polarisation sensing system$^9$-12. Several criteria such as the condition number (CN) have been put forward to evaluate such an optimisation$^9$-12. In the context of polarisation sensing, this is defined as

$$\text{CN} (A) = \| A \| \cdot \| \text{inv}(A) \|,$$

$$\max(\mu_j) = \| A \|,$$  \hspace{1cm} (3-2)
\[ CN(A) = \frac{\mu_{\text{max}}}{\mu_{\text{min}}} \]

Here, \( \| \cdot \| \) represents the matrix L-2 norm, \( A \) is the instrument matrix during Stokes measurements; \( \mu_i \ (i = 1, 2, 3 \ldots) \) are the singular values of matrix \( A \), \( \mu_{\text{min}} \) and \( \mu_{\text{max}} \) are the minimum and maximum values of \( \mu_i \). Details can be found in Ref [9-12]. For a complete calculation of \( S_n \), at least four intensity parameters are required, which is determined by the component number of the Stokes vector. To calculate four unknown parameters, there needs to be at least four equations, hence the instrument matrix \( A \) cannot be a single column/row\( ^9-12 \). Additionally, as each row of \( A \) is related to a different analysis state, any combination of the related vectors cannot be mutually orthogonal. Hence for polarisation sensing, the minimum CN value is \( \sqrt{3} \), which is the theoretical limit with considerations of a systematic error amplification\( ^9-12 \), as opposed to the minimum CN value (CN = 1) without matrix inversion induced systematic amplification. However, in the new paradigm, we can avoid any such issues related to this matrix inversion though the use of a different computational process that is based upon the physical phenomenon of the mapping of polarisation state to an intensity pattern by the UFPG. As mentioned in main article, to deal with the additional errors such as temporal and spatial errors and noise, we introduce machine learning (ML) based ‘end to end’ method to deal all factors in one step. Details can be found in Supplementary Note 4.

Note that the ML based denoising process in polarisation imaging has been explored before with respect to neural networks specifically designed for polarisation\( ^17 \). However, in this work we place emphasis on the optimal system with specific task of estimating the location of the brightest point – we are able to increase measurement accuracy and precision with a non-polarisation specific oriented neural network, taking advantage of the ‘full analysis’ properties of UFPG based system itself.
**Supplementary Note 4: Machine learning based image processing approach for polarisation estimation**

For a chosen combination of UFPG and PF, there is a unique correspondence between the input SOP and the intensity pattern at the detector. Conceptually, we can explain intuitively that the positions of the brightest points encode the input SOP. However, practically, we wish to use the whole of the intensity pattern in order to estimate the input state, to ensure highest accuracy, particularly in the presence of noise. Furthermore, the exact mapping between the input state and intensity pattern is not perfectly known in advance, due to variations in system configuration as well as spatial and temporal noise. Rather than performing denoising or calibration step, we chose to adopt a convolutional neural network (CNN) to approach the estimation holistically\(^{18,19}\). A U-Net like network has been chosen\(^{20-22}\), due to its robustness to image noise, stability towards hyperparameter changing, and promising detection performance.

The main process of the approach is depicted in Supplementary Figure 3a and 3b with a simple flow chart. The imaging processing pipeline comprises three steps: 1) the obtained input intensity images (with unknown SOP) are fed into the CNN and the areas around the brightest points are highlighted in the output heatmap (which represent probable locations of the peak); 2) refined estimates of the brightest points are localised from the heatmap using the centrosymmetric constraint, due to the symmetry of the GRIN lens; 3) the positions are converted via a look-up table (LUT) to the SOPs. The target SOP can then be obtained as the output (see Supplementary Figure 3c for the three steps). The LUT can be built offline prior to the processing procedure and is generated from simulation images with known SOPs. This enables fast and efficient SOP prediction from the brightest points during inference.

This network follows an encoder-decoder structure (see Supplementary Figure 3b), where the encoder down-samples the input to extract deeper features, and the decoder up-samples the feature map to integrate information from the encoder at different scales. After the last convolutional layer, a heatmap with pixel values ranging from 0.0-1.0 is generated, which can be seen as a probability map for the brightest points. 57877 pairs of simulated/experimental images were used to generate the training set. The simulated images were calculated via a GRIN lens retardance model using the approaches
described in Ref [8] as the ground truth. The experimental images were acquired with known SOP input using the system shown in Supplementary Figure 2a. We generated the data by uniformly sampling on the Poincaré sphere, in order to cover as large a parameter range as possible. The locations of the brightest points were directly found from the simulated images, followed by a Gaussian distribution to indicate the local area around them to model the expected network output heatmap. Then pairs of noisy intensity images and heatmaps that have a one-to-one correspondence, were used to train the network. To increase the training set size as well as to simulate data conditions in real-world applications, data augmentation including contrast and brightness changes have been carried out to improve the robustness of the network.

During the training, the network was trained with a stochastic gradient descent (SGD) optimizer using gradients computed with backpropagation, with batch size set to 4, learning rate 0.001, momentum 0.9. A weighted L-2 loss function (Eq. (4-1)) was adopted to deal with the “imbalanced classification” problem, since the bright area only takes up a small part of the image:

\[
Loss = \frac{1}{2N} \sum_{i=0}^{N-1} w_i (v_i - v_i^*)^2 + \lambda \| A \|^2 ,
\]

where \( N \) is the total number of pixels, \( v_i \) the predicted value of the \( i \)th pixel, and \( v_i^* \) the ground truth value of the \( i \)th pixel. \( w_i \) is the weight of the \( i \)th pixel, which was set to 50 if \( v_i^* > 0 \), otherwise \( 1 \). \( \lambda = 0.0005 \) is the coefficient of the regulariser \( A \), where \( A = [a_0, a_1, a_2, \ldots, a_k] \) is the set of all parameters in the network. The network was trained over 5 epochs and converged in one hour on a PC (OS: Ubuntu 16.04; CPU: i7-4770; GPU: NVIDIA GTX 1080 Ti).

As it is shown in the main article, we demonstrate the feasibility of using the proposed ML based method to estimate the SOP. This is a highly efficient image retrieval system: the brightest points function as image signatures that are unique to SOPs in certain ranges, so the problem of predicting SOPs is simplified and converted to a search task. There are several advantages of this approach: 1) preparation of the training set is straightforward and it is easy to cover an adequate domain; 2) finding the SOP takes only 30 ms on a normal desktop GPU, enabling real-time online SOP detection; 3) the network
is invulnerable to temporal/spatial noise from the image acquisition system. Future work will centre around expanding the capability of the proposed ML based approach to enable robust and accurate real-world applications.

As the new paradigm is based on estimation the brightest point on the image, there is an intrinsic link between the image resolution (with pixel number $n \times n$) and the polarisation resolution (sensitivity $S_p$) of the system. This hardware parameter could be used to indicate the maximum sensitivity that the system can perform, assuming other noise sources are minimised, which can guide the training process of the CNN with respect to the best effective dataset. This sensitivity can be calculated as

$$S_p = K \cdot \frac{D_s}{\sqrt{R}} \cdot \sqrt{\eta}.$$  \hspace{1cm} (4.2)

where $D_s$ is the dimension of the Stokes vector, $R = \frac{\pi n^2}{4}$ represents the effective pixel number (in the GRIN lens based UFPG, we have a circular area). $K$ is a constant parameter. As the topological order $\eta$ of the GRIN lens is 2, there would in effect be half the number of pixels to determine $S_p$. Note here we assume the sampling depth is sufficient and the non-linearity of the system is low. Following the above equation, we could calculate and plot the theoretical relationship between $S_p$ and intensity image with resolution $n \times n$, if systematic and random errors are minimised (see Supplementary Figure 3d). $S_p$ can be boosted if we use a higher camera pixel resolution.
Supplementary Figure 3. Schematic of the machine learning based image processing approach to identify the unknown SOP.

(a) Simple flow chart of the UFPG based paradigm enabled by ML. (b) ML implementation process: intensity distribution images are
used as the network input, which then pass through the CNN (encoder plus decoder) then to find the brightest point locations and predict the unknown SOP. (c) One practical example of using the CNN based imaging process approach to acquire the unknown SOP from an experimental intensity distribution image. (d) The theoretical relationship between the sensitivity of the UFPG and the pixel number of the acquired image (an image with pixel number $n \times n$).
Supplementary Note 5: Statistical analysis, validation and comparison of the measurement precision/systematic sensitivity

In this section, we examine the numerical performance of the proposed UFPG method and the established method of the point Stokes polarimeter (PSP). In particular, we compare their performance based on mean square error (MSE) and mean absolute error (MAE). We denote \((Y_1, ..., Y_n)\) as the vector of \(n\) observed ground truth samples, with \((\hat{Y}_1, ..., \hat{Y}_n)\) being the predicted values. MSE and MAE are computed as,

\[
MSE = \frac{1}{n} \sum_{i=1}^{n} e_i^2, \quad MAE = \frac{1}{n} \sum_{i=1}^{n} |e_i|,
\]

where \(e_i = \hat{Y}_i - Y_i\). We first conduct an induction process (with validation cases 1, 2 and 3, details of which are explained below) via measuring a standard light field that compares to the ground truth. In terms of robustness, error plots for both methods comparing to the ground truth in all cases are demonstrated in Supplementary Figure 5 and Table 1, where a conditional mean (CM) for each point with the conditional confidence interval (CCI) related parameters are included. The definition for 90\%-CI \((P)\) throughout this work follows \(P(Y \in [F_{lower}(X), F_{upper}(X)]|X = x) = 90\%\). Throughout this work, we used \(P\) to characterize the system sensitivity of the UFPG polarimeter, as it reflects the robustness of the data variation compared with the ground truth.

The CM is estimated via least squares regression, as shown in Eq. (5-2), and CCI is estimated via quantile regression, as shown in Eq. (5-3).

\[
\hat{F} = \arg\min_{F} \frac{1}{2} \sum_{i=1}^{n} (Y_i - F(X_i))^2,
\]
\[ F = \arg \min_F \frac{1}{n} \sum_{i=1}^{n} \rho_{\tau}(Y_i - F(X_i)), \]  

(5-3)

where \( \rho_{\tau}(u) = \mu(u - H(\mu < 0)) \); \( H(\cdot) \) is an indicator function, \( \tau = (0.05, 0.95) \) corresponds to (5\%, 95\%) in quantile. \( F \) in both Eq. (5-2) and Eq. (5-3) is formulated via a Gradient Boosting algorithm\(^3\), which is also a ML method for regression and classification, presented as

\[ F(X) = \sum_{j=1}^{J_m} \gamma_{jm} H(x \in R_{jm}). \]  

(5-4)

where \( J_m \) is the number of its leaves, \( \gamma_{jm} \) is the value predicted in region \( R_{jm} \). Here the tuning parameters are set as follows: number of trees = 2, and max depth = 2 for estimating the CM for case 1 and 2. The numerical results are summarized in Supplementary Figure 4, where the upper, middle, lower lines are the 95\% quantile, the CM, and the 5\% quantile. As suggested by the results in Supplementary Figure 5 and Supplementary Table 1, the biases for the UFPG polarimeter are much smaller, and the minimal and maximal width of 90%-CI are both much narrower than that from the PSP in different cases, hence the UFPG polarimeter shows better sensitivity. The corresponding quantitative results also indicate that the UFPG polarimeter consistently provides more robust and precise measurement.

Three experiments were executed to compare the performance of the new UFPG polarimeter and the PSP: Case 1) measured a standard horizontal polarised light field from the light source (generated via a polariser, Supplementary Figure 4a); Case 2) measured the output of the horizontal middle line of the non-tilted spatial variant half-wave plate array (SVHWA) (see Supplementary Figure 4b); Case 3) measured a chosen line on the SVHWA surface. Note the measurements of case 1 and 2 are from small regions (enclosed in the white dotted boxes in Supplementary Figure 4a and 4b). As for case 3, the measurement is across a strip (see Supplementary Figure 4c). The original results in theory, using UFPG and PSP for all three cases, are plotted in Supplementary Figure 4. Then the MSE, MAE as well as sensitivity are
calculated for cases 1 and 2 (here we focus on relative error characterisation); results can be found in Supplementary Figure 5 and Table 1.

As indicated in Supplementary Figure 5 and Table 1, the UFPG method has significantly better performance than the PSP in both cases 1 and 2. Note cases 2 and 3 are measured with a SVHWA, hence additional polarisation aberration can be induced via the manufacturing processing of the plate itself, which can lead to higher error compared with the ground truth (as indicated by the results of case 2). Hence, we used case 1 as a standard sensitivity evaluation process throughout this work. We can find in Table 1 that the UFPG polarimeter shows sensitivity ranging from a minimum 0.79% to a maximum 0.81%, validating that the UFPG method provides robust performance.

Note here we did not present the equivalent results for case 3 as there is too much discrepancy in this case between the theoretical model and the real data, mostly due to differences in the expected properties and the manufactured device. Hence, there is no reliable “ground truth” for the comparison in this case. Similarly, the case in Fig. 3e in the main article also does not have such ground truth. The reference line used to aid the visualized comparison was measured by precision time-sequenced imaging Stokes vector polarimetry. Supplementary Table 2 gives a quantitative comparison of the measurement precision between the UFPG polarimeter and the PSP as well as other polarimetry counterparts (see more details in Ref [31]). The UFPG polarimeter shows superior performance.
Supplementary Figure 4. Performance evaluation of UFPG and PSP methods. (a) A test region of the horizontal linear SOP light field generated by a high precision polariser. Theoretical results together with experiment results using UFPG and PSP methods are
given. In the plots of the data in the sub-figures, the horizontal axis represents 200 points measured along the zero degree linear polarised light field (the direction within the white-dotted rectangle is shown by the “sample direction” arrow); the vertical axis shows the percentage of the vertical polarised components, which represents the error. (b) A test region of a horizontal linear SOP light field generated by the SVHWA. Theoretical results together with experimental results using UFPG and PSP methods are given. The meaning of the axes in the sub-figures are the same as in case (a). (c) A test on a strip region of a line from the SOP light field generated via the SVHWA. Theoretical results together with experiment results using UFPG and PSP methods are given. In the sub-figures we use horizontal linear components as value for comparison of different approaches. Two regions A, A’ and B, B’, that exhibit weak signal difference can be seen in UFPG and PSP sub-figures respectively. The physical locations of the regions (A, A’) and (B, B’) are the same.

Supplementary Figure 5. Statistic results of Case 1 and Case 2 from UFPG and PSP methods. Numerical data can be found in Supplementary Table 1 and Table 2. The horizontal axis represents point number and the vertical axis represents predicted value of CM (solid lines) and CCI (shaded regions) with respect to the ground truth. The blue lines in the figure are calculated via least squares regression and the shaded areas show quantiles calculated via quantile regression. Parameter definitions can be found in Supplementary Note 5.
Supplementary Table 1 and Table 2

### Table 1 Results of UFPG and PSP methods

| Method | MSE (*10⁻⁵) | MAE (*10⁻³) | System Sensitivity (min, max) |
|--------|-------------|-------------|-----------------------------|
| UFPG   | 0.72±0.65   | 2.33±1.33   | 0.79%, 0.81%                |
| PSP    | 25.31±32.54 | 13.04±9.14  | 4.65%, 4.88%                |

### Case 2

| Method | MSE (*10⁻⁵) | MAE (*10⁻³) | System Sensitivity (min, max) |
|--------|-------------|-------------|-----------------------------|
| UFPG   | 1.63±1.69   | 3.41±2.16   | 1.24%, 1.28%                |
| PSP    | 62.75±69.41 | 21.11±13.49 | 6.55%, 7.27%                |

### Table 2 Quantitative comparision data

|                        | DOP error (max) | DOP error (mean) | System Sensitivity (max; in Case 1) |
|------------------------|-----------------|------------------|-------------------------------------|
| Product 1 (UFPG)       | ±0.84%          | ±0.02%           | 0.81%                               |
| Product 2 (PSP)        | N/A             | ±1%              | 4.88%                               |
| Product 3*             | ±5%             | N/A              | N/A                                 |

**Error of Stokes parameters (mean)**

| Product 1 (UFPG)       | <1% (S₁), <1% (S₂), <1% (S₃)                       |
| Product 4 (X-shaped aperture array)* | >7.3% (S₁), >7.2% (S₂), >5.2% (S₃) |
| Product 5 (Integrated plasmonic)* | ~45% (S₁, S₂, S₃) |
| Product 6 (Dielectric metasurface)* | 7.5~15% (S₁, S₂, S₃) |

*DOP error is what the product claimed; †Refer to Ref [31] for data of other comparisons.
Supplementary Note 6: Depolarisation characterisation

Depolarisation is an important vectorial parameter in numerous techniques and applications. This parameter can also be additionally extracted via the image in the UFPG polarimeter. Supplementary Figure 6 shows the typical intensity distributions of several random SOPs selected from the Poincaré sphere under different levels of depolarisation (0% for 6a, 20% for 6b, 60% for 6c, respectively). The SOP of their polarised parts, however, remain the same. The level of contrast within the image directly represents the level of depolarisation of the target beam. Its DOP (degree of polarisation) - can be theoretically calculated via the normalised intensity value of the brightest and darkest points on the intensity image according to a simple calculation, \((I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}})\). This is another unique feature of the UFPG polarimeter that enables the calculation of depolarisation in a simple way.
Supplementary Figure 6. The relationship between depolarised beams and their corresponding intensity images. (a) Before depolarisation – point $\mu$ on the surface of the Poincare sphere, with radius $r$ from the centre $o$. Recorded intensity images of the UFPG.
polarimeter for several randomly chosen SOPs on the surface. (b) and (c) After depolarisation – points $\mu'$ and $\mu''$ in the Poincare sphere with radius $r'$ and $r''$ from the centre $o$ (the fully polarised part representing the same SOP as $\mu$). The final obtained intensity images under such depolarisation levels with previously chosen incident SOPs are also given below. The DOP of the incident beam can be calculated from the contrast level of their corresponding intensity images.
**Supplementary Note 7: Single-shot full Stokes imaging using the UFPG paradigm**

While we can implement single-shot sensing through our UFPG-based paradigm, it is also easy to expand the concept into the imaging domain. An \( n \times n \) UFPG array consisting of \( n \times n \) UFPGs is designed (see Supplementary Note 7a, 10 × 10 UFPGs are used for demonstration). Next it is integrated into the image plane of an imaging polarimeter (see Supplementary Figure 7b) – on the surface of the camera sensor combined with a polariser. The assembly can then directly perform polarisation imaging by determining the brightest point on each super-pixel (10 × 10) via the same procedure as for point sensing.
Supplementary Figure 7. Extended application for the UFPG system – single-shot full Stokes imaging. (a) A typical UFPG array with super-pixel number 10 × 10. Note that here one UFPG acts as one pixel. (b) A schematic for a new compact snapshot Stokes imaging device. It possesses the same advantages as a single UFPG, such as direct Stokes vector detection without any additional matrix calculations. PSG: polarisation state generator; P: polariser.
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