Abstract. In this paper, a generalized finite element method (GFEM) with optimal local approximation spaces for solving high-frequency heterogeneous Helmholtz problems is systematically studied. The local spaces are built from selected eigenvectors of local eigenvalue problems defined on generalized harmonic spaces. At both continuous and discrete levels, (i) wavenumber explicit and nearly exponential decay rates for the local approximation errors are obtained without any assumption on the size of subdomains; (ii) a quasi-optimal and nearly exponential global convergence of the method is established by assuming that the size of subdomains is $O(1/k)$ ($k$ is the wavenumber). A novel resonance effect between the wavenumber and the dimension of local spaces on the decay of error with respect to the oversampling size is implied by the analysis. Furthermore, for fixed dimensions of local spaces, the discrete local errors are proved to converge as $h \to 0$ ($h$ denoting the mesh size) towards the continuous local errors. The method at the continuous level extends the plane wave partition of unity method [I. Babuska and J. M. Melenk, Int. J. Numer. Methods Eng., 40 (1997), pp. 727–758] to the heterogeneous-coefficients case, and at the discrete level, it delivers an efficient non-iterative domain decomposition method for solving discrete Helmholtz problems resulting from standard FE discretizations. Numerical results are provided to confirm the theoretical analysis and to validate the proposed method.

Key words. generalized finite element method, Helmholtz equation, multiscale method, Trefftz methods, local spectral basis

AMS subject classifications. 65M60, 65N15, 65N55

1. Introduction. The Helmholtz equation models wave propagation and scattering phenomena in the frequency domain, and arises in a variety of science and engineering applications, including seismic imaging, medical ultrasound technologies, and underwater acoustics. It is well known that due to the so called pollution effect, solving the Helmholtz equation with standard finite element methods (FEMs) needs a much higher mesh resolution than typically required for a meaningful representation of the solution in the finite element spaces. Other standard numerical methods also suffer from a similar problem. In the high frequency regime, such discretizations result in very large scale and strongly indefinite linear systems of equations which are difficult to solve by classical methods. Significant research efforts have been devoted to addressing these challenging problems, which mainly focus on two directions: non-standard discretization schemes and efficient methods for solving the resulting linear systems from standard FE approximations.

In view of the drawback of standard FEMs, many attempts have been made to develop efficient non-standard FEMs for the Helmholtz equation. High-order discretization methods, especially the $hp$-FEM [31], have been shown to be effective to alleviate the difficulty arising from the pollution effect. Another class of non-standard FEMs are Trefftz-type methods which use (local) solutions of the Helmholtz equation as basis functions; see [26] for a survey. A popular choice of Trefftz-type basis functions are plane waves and associated methods include the ultraweak variational formulation [9], the plane wave partition of unity method [3], the plane wave discontinuous Galerkin method [25], the least-squares finite element method [33], and the discontinuous enrichment method [17], to cite a few. Due to the use of operator-adapted basis
functions, the Trefftz-type methods usually need much fewer degrees of freedom than conventional FEMs to achieve the same accuracy.

However, all the aforementioned non-standard FEMs are typically developed for the classical Helmholtz equation and it is not straightforward to extend these methods to Helmholtz problems in heterogeneous media, especially with multiscale features. To the best of our knowledge, there has been no theoretical justification for the $hp$-FEM for heterogeneous Helmholtz problems so far. For the Trefftz-type methods, it is difficult to explicitly construct (local) solutions of heterogeneous Helmholtz equations. In recent years, there has been an increasing interest among the numerical homogenization community in developing numerical multiscale methods for Helmholtz problems with or without heterogeneous coefficients, e.g., (generalized) multiscale FEMs [12, 18, 19, 20], LOD type methods [7, 24, 36, 37], the heterogeneous multiscale method [35], the multiscale asymptotic method [8], and the multiscale hybrid-mixed method [11]. These multiscale methods are usually based on solving some local problems numerically to get basis functions that capture the wave characteristics and local media information. The associated analysis was typically performed in the "asymptotic regime", i.e., the mesh size of the coarse grid is $O(1/k)$ ($k$ is the wave number), e.g., in [12, 36]. In fact, in this regime, the indefiniteness of the local problems largely disappears and many analysis techniques developed for positive definite problems can be applied. For multiscale methods in the preasymptotic regime, which is of more interest for practical applications, very little analysis of the error and of the effect of the wavenumber is available in the literature.

Although using non-standard FEMs for the Helmholtz equation can lead to a dramatic gain in efficiency, due to their simplicity, standard FEMs are still widely used. In this scenario, the efficient solution of the resulting large linear systems becomes a focus of research. For discrete Helmholtz problems of very large size, direct methods are in general prohibitively expensive, and classical iterative methods suffer from the problem of slow convergence [15]. Over the past two decades, robust preconditioning of the Helmholtz equation has been extensively studied and many novel preconditioners have been proposed; see [22] for a review. Here we put a special emphasis on domain decomposition methods (DDMs), as they are a natural choice for use on parallel computers. Simple extensions of state-of-the-art techniques for symmetric positive definite problems to indefinite and non-self-adjoint problems have been shown to be inefficient [15]. To obtain an efficient domain decomposition preconditioner for the Helmholtz equation, two key ingredients are needed: transmission conditions and a coarse space, suitably adapted to the characteristics of the problem. In particular, we highlight the DtN and GenEO spectral coarse spaces [5], which are constructed by selected modes of local eigenvalue problems adapted to the Helmholtz equation. Apart from practical difficulties, few domain decomposition preconditioners (and also other preconditioners) for the Helmholtz equation are amenable to rigorous analysis due to the indefinite and non-Hermitian nature of the underlying problem.

In this paper, we consider the numerical solution of high-frequency heterogeneous Helmholtz problems and deal with the two above focused topics, nonstandard FEM discretizations and efficient solvers, within the unified framework of the Multiscale Spectral Generalized Finite Element Method (MS-GFEM) [1, 2, 29, 28]. Built on the GFEM [30] which constructs the trial space by gluing local approximation spaces together with a partition of unity, the MS-GFEM achieves a high efficiency by building optimal local approximation spaces from selected eigenvectors of carefully designed local eigenvalue problems. At the continuous level, the local eigenproblems are defined on generalized harmonic spaces that consist of (local) solutions of the Helmholtz
equation under consideration with vanishing source term, and thus the method is an extension of the plane wave partition of unity method [3] to the heterogeneous coefficient case. Wavenumber-explicit and nearly exponential decay rates for the local approximation errors are derived in the preasymptotic regime and a nearly exponential error decay for the global approximation is obtained under some general assumptions on the stability of the local Helmholtz problems. The local approximation errors are a posteriori computable from the local spectra. Our analysis implies the presence of a resonance effect between the wavenumber and the dimension of the local approximation spaces, which affects the error decay with respect to the oversampling size; see Remark 3.3. In particular, it is shown that the (approximation) error of the method in the high-frequency regime does not always decay with increasing oversampling size, in contrast to the positive definite case [29]. A quasi-optimal global convergence rate for the method is established under the assumption that the size of the subdomains is $O(1/k)$. Compared to the usual Trefftz methods, a second-level discretization for solving the local problems is needed in our method. However, these local problems can be solved entirely in parallel and the local basis functions can be reused. Therefore, a significant gain in efficiency can be expected when solving problems with many different wave sources.

At the discrete level, the MS-GFEM delivers a non-iterative DDM for solving linear systems arising from standard FE discretizations of heterogeneous Helmholtz problems. The optimal local spaces for approximating the standard FE solution of the problem are constructed analogously to the continuous level by solving discrete local eigenvalue problems. Similar local and global error estimates for the discrete method are obtained under assumptions akin to those in the continuous setting. Furthermore, a proof of the convergence of the discrete eigenvalues as $h \to 0$ to those of the continuous problems is given ($h$ denoting the mesh size), thus also providing guaranteed a posteriori error estimates for the discrete problem for $h$ sufficiently small. As in the case of classical two-level DDMs for the Helmholtz equation, the local approximation spaces and the boundary conditions of the local problems in the method are suitably adapted to the Helmholtz equation, when compared with those for positive definite problems [29, 28]; see Remark 2.4. However, although both methods are based on solving some local problems and a global coarse problem, the discrete MS-GFEM can solve the problem in one shot without iteration.

Our work distinguishes itself from previous works on multiscale methods for heterogeneous Helmholtz problems in three aspects. First, a non-standard FEM discretization and an efficient method for solving discrete Helmholtz problems are unified under the same mathematical framework, which is unusual in previous studies. Second, all the local analysis of the method, including the local approximation error estimates and the convergence of the eigenvalues of the local eigenproblems, hold without the resolution condition that the size $H$ of the subdomains is $O(1/k)$. Finally, the effect of the wavenumber on the error of the method in the preasymptotic regime is systematically investigated both theoretically and numerically, especially the aforementioned error resonance phenomenon. It is worth noting that although the resolution condition is required to get a quasi-optimal global error estimate for the method, numerical results show that it is not a necessity in practice, just as the condition "$k^2 h$ is small" for the standard linear FEM for the Helmholtz equation [30] is not required either in practice. In fact, in our numerical experiments, we obtain excellent results for $kH \approx 13$, i.e., about two wavelengths per subdomain.

The remainder of this paper is structured as follows. In section 2, we introduce the model Helmholtz problem considered in this paper and describe the continuous
MS-GFEM for solving the problem. Section 3 is devoted to the local and global error estimates of the continuous method. The discrete MS-GFEM for solving the discrete Helmholtz problem is presented in the first part of section 4, followed by some technical tools used in the subsequent analysis. We focus on the convergence analysis of the discrete MS-GFEM in section 5, including the local and global error estimates and the convergence of the eigenvalues of the local eigenproblems. Numerical results are reported in section 6 to support the theoretical analysis and validate the method.

2. Problem formulation and the continuous MS-GFEM.

2.1. Model Helmholtz problem. Let \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) be a bounded Lipschitz domain with boundary \( \Gamma \). Given \( k > 0 \), we consider the following heterogeneous Helmholtz equation with mixed boundary conditions: Find \( u : \Omega \rightarrow \mathbb{C} \) such that

\[
\begin{aligned}
\begin{cases}
-\text{div}(A\nabla u) - k^2V^2u &= f, & \text{in } \Omega \\
u &= 0, & \text{on } \Gamma_D \\
A\nabla u \cdot n - ik\beta u &= g, & \text{on } \Gamma_R,
\end{cases}
\end{aligned}
\tag{2.1}
\]

where \( n \) denotes the outward unit normal to \( \Gamma \), \( \Gamma_D \cap \Gamma_R = \emptyset \), and \( \Gamma = \Gamma_D \cup \Gamma_R \). We suppose that \( |\Gamma_R| > 0 \). Throughout this paper, we make the following assumptions on the problem:

Assumption 2.1.

(i) \( A \in (L^\infty(\Omega))^{d \times d} \) is pointwise symmetric and there exists \( 0 < a_{\text{min}} < a_{\text{max}} < \infty \) such that

\[
a_{\text{min}}|\xi|^2 \leq A(x)\xi \cdot \xi \leq a_{\text{max}}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \Omega;
\]

(ii) \( V \in L^\infty(\Omega) \) and there exists \( 0 < V_{\text{min}} < V_{\text{max}} < \infty \) such that \( V_{\text{min}} \leq V(x) \leq V_{\text{max}} \) for all \( x \in \Omega \);

(iii) \( f \in L^2(\Omega), \ g \in L^2(\Gamma_R) \) and \( \beta \in L^\infty(\Gamma_R) \) is a real-valued function.

We note that the results in this paper hold for a wider class of \( f \) and \( g \) in a weaker dual space, but we omit this extension for ease of presentation.

Defining the space

\[
H_D^1(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}
\]

and the sesquilinear form \( B : H_D^1(\Omega) \times H_D^1(\Omega) \rightarrow \mathbb{C} \) by

\[
B(u, v) = \int_{\Omega} (A\nabla u \cdot \nabla v - k^2V^2uv) \, dx - ik \int_{\Gamma_R} \beta u \overline{v} \, ds, \quad \forall u, v \in H_D^1(\Omega),
\]

the weak formulation of the problem (2.1) is to find \( u^c \in H_D^1(\Omega) \) such that

\[
B(u^c, v) = F(v) := \int_{\Gamma_R} g \overline{v} \, ds + \int_{\Omega} f \overline{v} \, dx, \quad \forall v \in H_D^1(\Omega).
\]

For later use, we introduce some local sesquilinear forms. Let \( \omega \) be a subdomain of \( \Omega \) and \( u, v \in H^1(\omega) \). We define

\[
\begin{aligned}
B_\omega(u, v) &= \int_{\omega} (A\nabla u \cdot \nabla v - k^2V^2uv) \, dx - ik \int_{\Gamma_R \cap \partial \omega} \beta u \overline{v} \, ds, \\
A_\omega(u, v) &= \int_{\omega} A\nabla u \cdot \nabla v \, dx, \quad A_{\omega, k}(u, v) = \int_{\omega} (A\nabla u \cdot \nabla v + k^2V^2uv) \, dx,
\end{aligned}
\tag{2.6}
\]

\[
A_\omega(u, v) = \int_{\omega} A\nabla u \cdot \nabla v \, dx.
\]
and
\[(2.7) \quad \|u\|_{A,\omega} = \sqrt{A_\omega(u,u),} \quad \|u\|_{A,\omega,k} = \sqrt{A_{\omega,k}(u,u)}.\]

If \(\omega = \Omega\), we simply write \(\|u\|_A\) (\(\|u\|_{A,k}\)). It can be proved \cite{30} that there exists \(C_B > 0\) independent of \(k\) such that
\[(2.8) \quad |B(u,v)| \leq C_B\|u\|_{A,k}\|v\|_{A,k}, \quad \forall u, v \in H^1(\Omega).\]

We assume the well-posedness of the problem \((2.5)\) as follows.

**Assumption 2.2.** For any \(f \in L^2(\Omega)\) and \(g \in L^2(\Gamma_R)\), the problem \((2.5)\) has a unique solution \(u' \in H^1_D(\Omega)\), and there exists \(C_{\text{stab}}(k) > 0\) depending polynomially on \(k\) such that
\[(2.9) \quad \|u'\|_{A,k} \leq C_{\text{stab}}(k)(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}).\]

**Remark 2.3.** It was proved in \cite{23} that under Assumption 2.1, for \(d = 2\), the problem \((2.5)\) is uniquely solvable in \(H^1_D(\Omega)\), and for \(d = 3\), some additional assumptions on the coefficients \(A\) and \(V\) are required to obtain the unique solvability. The condition of a polynomial growth in \(k\) on the stability constants is satisfied for a wide class of problems (see, e.g., \cite{7, 16}). Some special instances \cite{10} of an exponential growth in \(k\), which makes the problem strongly unstable, are ruled out here.

**2.2. Continuous MS-GFEM.** In this subsection, we present the continuous MS-GFEM for solving the problem \((2.5)\). For completeness, we first recall the GFEM. Let \(\{\omega_i\}_{i=1}^M\) be a collection of open subsets of \(\Omega\) satisfying \(\cup_{i=1}^M \omega_i = \Omega\) and a pointwise overlap condition:
\[(2.10) \quad \exists \zeta \in \mathbb{N} \quad \forall x \in \Omega \quad \text{card}\{i \mid x \in \omega_i\} \leq \zeta.\]

Let \(\{\chi_i\}_{i=1}^M\) be a partition of unity subordinate to the open cover satisfying
\[(2.11) \quad 0 \leq \chi_i(x) \leq 1, \quad \sum_{i=1}^M \chi_i(x) = 1, \quad \forall x \in \Omega, \quad \chi_i(x) = 0, \quad \forall x \in \Omega/\omega_i, \quad i = 1, \ldots, M, \quad \chi_i \in W^{1,\infty}(\omega_i), \quad \|\nabla \chi_i\|_{L^\infty(\omega_i)} \leq \frac{C_1}{\text{diam}(\omega_i)}, \quad i = 1, \ldots, M.\]

For each \(i = 1, \ldots, M\), let \(u^p_i \in H^1(\omega_i)\) be a local particular function and \(S_n(\omega_i) \subset H^1(\omega_i)\) be a local approximation space of dimension \(n_i\) such that \(u^p_i\) and all functions in \(S_n(\omega_i)\) vanish on \(\partial \omega_i \cap \Gamma_D\). A key feature of the GFEM is to build the global particular function \(u^p\) and the trial space \(S_n(\Omega)\) by pasting the local particular functions and the local approximation spaces together using the partition of unity:
\[(2.12) \quad u^p = \sum_{i=1}^M \chi_i u^p_i, \quad S_n(\Omega) = \left\{ \sum_{i=1}^M \chi_i \phi_i : \phi_i \in S_n(\omega_i) \right\}.\]

By the assumptions on the partition of unity in \((2.11)\), we see that \(u^p \in H^1_D(\Omega)\) and \(S_n(\Omega) \subset H^1_D(\Omega)\). The final step of the GFEM is the finite-dimensional Galerkin approximation: Find \(u^G = u^p + u^s\) with \(u^s \in S_n(\Omega)\) such that
\[(2.13) \quad B(u^G, v) = F(v), \quad \forall v \in S_n(\Omega).\]
Combining (2.5) and (2.13) yields the Galerkin orthogonality:

\[ B(u^e - u^G, v) = 0 \quad \forall v \in S_n(\Omega). \]  

Obviously, the core of the GFEM is the construction of the local particular functions and of the local approximation spaces. In the MS-GFEM, the local particular functions are defined as solutions of local Helmholtz problems and the local approximation spaces are constructed by eigenvectors of local eigenvalue problems; see Theorem 2.9. We detail the construction in what follows.

For a subdomain \( \omega_i \), we introduce an oversampling domain \( \omega^*_i \) that satisfies \( \omega_i \subset \omega^*_i \subset \Omega \) as illustrated in Figure 1. On each \( \omega^*_i \), we define

\[ H^1_D(\omega^*_i) = \{ v \in H^1(\omega^*_i) : v = 0 \text{ on } \partial \omega_i \cap \Gamma_D \} \]

and the generalized harmonic space

\[ H_B(\omega^*_i) = \{ u \in H^1_D(\omega^*_i) : B_{\omega^*_i}(u, v) = 0 \quad \forall v \in H^1_D(\omega^*_i) \}. \]

\( H_B(\omega^*_i) \) consists of local solutions of the Helmholtz equation with vanishing source term. A general result on equivalent norms for Sobolev spaces (see, e.g., [34, Chapter 2]) gives that there exists \( C > 0 \), such that for any \( u \in H_B(\omega^*_i) \),

\[ \| u \|_{L^2(\omega^*_i)} \leq C \| \nabla u \|_{L^2(\omega^*_i)}. \]

Therefore, \( \| \cdot \|_{A,\omega^*_i} \) is a norm on \( H_B(\omega^*_i) \). Next we consider the following local Helmholtz problem on \( \omega^*_i \):

\[ \begin{aligned}
-\text{div}(A \nabla \psi_i) - k^2 V^2 \psi_i &= f, & \text{in } \omega^*_i \\
\psi_i &= 0, & \text{on } \partial \omega^*_i \cap \Gamma_D \\
A \nabla \psi_i \cdot n - ik \psi_i &= g, & \text{on } \partial \omega^*_i \cap \Gamma_R \\
A \nabla \psi_i \cdot n - ik V \psi_i &= 0, & \text{on } \partial \omega^*_i \cap \Omega.
\end{aligned} \]

The weak formulation of the problem (2.18) is given by: Find \( \psi_i \in H^1_D(\omega^*_i) \) such that

\[ B_{\omega^*_i}(\psi_i, v) - ik \int_{\partial \omega^*_i \cap \Omega} V \psi_i \pi ds = F_{\omega^*_i}(v), \quad \forall v \in H^1_D(\omega^*_i), \]
where
\begin{equation}
F_{\omega^*_i}(v) = \int_{\Gamma_B \cap \partial \omega_i^*} g \tau \, ds + \int_{\omega_i^*} f \tau \, dx.
\end{equation}

**Remark 2.4.** In contrast to the Dirichlet boundary conditions for local problems in the positive definite case [29], we impose impedance boundary conditions on the artificial interior boundaries for the local Helmholtz problems to guarantee their unique solvability. Such boundary conditions are commonly used as transmission conditions in DDMs for the Helmholtz equation [21].

We assume the well-posedness of the local Helmholtz problem (2.19) as follows.

**Assumption 2.5.** For each \(i = 1, \cdots, M\), the problem (2.19) has a unique solution \(\psi_i \in H^1_D(\omega_i^*)\), and there exists a constant \(C_{\text{stab}}^i(k)\) depending polynomially on \(k\) such that
\begin{equation}
\|\psi_i\|_{A,\omega_i^*,k} \leq C_{\text{stab}}^i(k) (\|f\|_{L^2(\omega^*_i)} + \|g\|_{L^2(\omega_i^* \cap \Gamma_R)}).
\end{equation}

Combining (2.5) and (2.19), we see that \(u^*|_{\omega_i^*} - \psi_i \in H_B(\omega_i^*)\), where \(u^*\) is the exact solution of the global problem. Therefore, the exact solution is locally decomposed into two parts, one being the solution of the local Helmholtz problem and another belonging to the generalized harmonic space \(H_B(\omega_i^*)\). To approximate the latter part, we follow the lines of [29] to construct a finite-dimensional space that is optimal in an appropriate sense, using the singular vectors of a compact operator involving the partition of unity function. To this end, we first give a novel identity and the resulting Caccioppoli-type inequality for functions in the generalized harmonic space, which plays a crucial role in the analysis of the continuous MS-GFEM.

**Lemma 2.6.** Assume that \(\eta \in W^{1,\infty}(\omega_i^*)\) satisfies \(\eta(x) = 0\) on \(\partial \omega_i^* \cap \Omega\). Then, for any \(u, v \in H_B(\omega_i^*)\),
\begin{equation}
A_{\omega_i^*}(\eta u, \eta v) = \int_{\omega_i^*} (A \nabla \eta \cdot \nabla \eta + k^2 V^2 \eta^2) u v \, dx.
\end{equation}
In particular,
\begin{equation}
\|\eta u\|_{A,\omega_i^*} \leq \left( a_{\text{max}}^{1/2} \|\nabla \eta\|_{L^\infty(\omega_i^*)} + k V_{\text{max}} \|\eta\|_{L^\infty(\omega_i^*)} \right) \|u\|_{L^2(\omega_i^*)},
\end{equation}
where \(a_{\text{max}}\) and \(V_{\text{max}}\) are the (spectral) upper bounds of \(A\) and \(V\) in Assumption 2.1.

The proof is given in Appendix A.

Now we are ready to construct the desired optimal space for approximating a generalized harmonic function. We start by introducing the operator
\begin{equation}
P_i : (H_B(\omega_i^*), \| \cdot, A_{\omega_i^*} \|) \to (H^1_D(\omega_i^*), \| \cdot, A_{\omega_i^*,k} \|) \text{ such that } P_i(v) = \chi_i v,
\end{equation}
where \(\chi_i\) is the partition of unity function supported on \(\omega_i\). Note that here and after, we equip the spaces \(H_B(\omega_i^*)\) and \(H^1_D(\omega_i^*)\) with the norms \(\| \cdot, A_{\omega_i^*} \|\) and \(\| \cdot, A_{\omega_i^*,k} \|\), respectively. Since \(H^1(\omega_i^*)\) is compactly embedded into \(L^2(\omega_i^*)\), Lemma 2.6 implies that \(P_i\) is a compact operator. Next we consider the following Kolmogorov \(n\)-width of the compact operator \(P_i\):
\begin{equation}
d_n(\omega_i, \omega_i^*) = \inf_{Q(n) \subset H^1_D(\omega_i)} \sup_{u \in H_B(\omega_i^*)} \inf_{v \in Q(n)} \frac{\|P_i u - v\|_{A_{\omega_i^*,k}}}{\|u\|_{A_{\omega_i^*}}}.
\end{equation}
where the first infimum is taken over all \( n \)-dimensional subspaces of \( H^{1}_{D1}(\omega_i) \). Then the optimal approximation space \( \hat{Q}(n) \) satisfies

\[
d_n(\omega_i, \omega_i^\ast) = \sup_{u \in H_B(\omega_i^\ast)} \inf_{v \in \hat{Q}(n)} \frac{\|P_i^* u - v\|_{A, \omega_i, k}}{\|u\|_{A, \omega_i^\ast}}.
\]

The following lemma gives a characterization of the \( n \)-width via the singular values and singular vectors of the compact operator \( P_i \).

**Lemma 2.7.** For each \( j \in \mathbb{N} \), let \((\lambda_j, \phi_j)\) be the \( j \)-th eigenpair (arranged in decreasing order) of the problem

\[
A_{\omega_i, k}(\chi_i \phi, \chi_i \psi) = \lambda A_{\omega_i^\ast}(\phi, \psi), \quad \forall \psi \in H_B(\omega_i^\ast).
\]

Then \( d_n(\omega_i, \omega_i^\ast) = \lambda_{n+1}^{1/2} \) and the associated optimal approximation space is given by

\[
\hat{Q}(n) = \text{span}\{\chi_i \phi_1, \cdots, \chi_i \phi_n\}.
\]

**Proof.** Let \( P_i^* : H^{1}_{D1}(\omega_i) \rightarrow H_B(\omega_i^\ast) \) denote the adjoint of the operator \( P_i \) and let \( \{\phi_i\} \) and \( \{\lambda_i\} \) denote the eigenfunctions and eigenvalues of the problem

\[
P_i^* P_i \phi = \lambda \phi.
\]

A classical result in [38, Theorem 2.5] gives that \( d_n(\omega_i, \omega_i^\ast) = \lambda_{n+1}^{1/2} \) and that the associated optimal approximation space is given by \( \hat{Q}(n) = \text{span}\{P_1 \phi_1, \cdots, P_n \phi_n\} \). We complete the proof by noting that (2.27) is the variational formulation of (2.29) \( \square \)

**Remark 2.8.** By Lemma 2.6, the eigenvalue problem (2.27) can be rewritten as

\[
(\tilde{Q}_k \phi, \psi)_{L^2(\omega_i^\ast)} = \lambda A_{\omega_i^\ast}(\phi, \psi), \quad \forall \psi \in H_B(\omega_i^\ast),
\]

where \( \tilde{Q}_k := A \nabla \chi_i \cdot \nabla \chi_i + 2k^2 V^2 \chi_i^2 \), i.e., an eigenvalue problem with weighted \( L^2 \) norm on \( \omega_i^\ast \).

From the definition and characterization of the \( n \)-width, we see that a generalized harmonic function can be well approximated by eigenvectors of the problem (2.27). This motivates the definition of the local particular function and the local approximation space for the MS-GFEM as follows.

**Theorem 2.9.** Let the local particular function and the local approximation space on \( \omega_i \) be defined as

\[
u_i := \psi_i \big|_{\omega_i}, \quad S_n(\omega_i) := \text{span}\{\phi_1 \big|_{\omega_i}, \cdots, \phi_n \big|_{\omega_i}\},
\]

where \( \psi_i \) is the solution of (2.18) and \( \phi_j \) denotes the \( j \)-th eigenfunction of the problem (2.27), and let \( u' \) be the exact solution of the problem (2.5). Then,

\[
\inf_{\varphi \in \nu_i + S_n(\omega_i)} \|\chi_i (u' - \varphi)\|_{A, \omega_i, k} \leq d_n(\omega_i, \omega_i^\ast) \|u' - \psi_i\|_{A, \omega_i^\ast}.
\]

**Proof.** Since \( u' \big|_{\omega_i} - \psi_i \in H_B(\omega_i^\ast) \), (2.32) follows from the definition and characterization of the \( n \)-width. \( \square \)
Remark 2.10. In [3], plane waves were used to construct the local approximation spaces for the homogeneous Helmholtz equation with constant coefficients in \( \mathbb{R}^2 \) as
\[
S_n = \text{span}\left\{ \exp\left(ik(x \cos \frac{2\pi q}{n_i} + y \sin \frac{2\pi q}{n_i})\right), \quad q = 0, \ldots, n_i - 1 \right\}.
\]
Note that in the constant coefficient case, plane wave solutions lie in the generalized harmonic spaces. Therefore, our method can be viewed as an extension of the method in [3] to heterogeneous-coefficients and to the inhomogeneous \((f \neq 0)\) case. A combination of the classical FEM and the plane wave partition of unity method for the Helmholtz equation can be found in [40].

3. Convergence analysis of the continuous MS-GFEM. In this section, we first derive wavenumber explicit upper bounds for the local approximation errors and then establish a quasi-optimal global convergence of the method.

3.1. Local approximation error estimates. By Theorem 2.9, the local approximation error in each subdomain is bounded by the \( n \)-width (2.25) and thus it suffices to derive upper bounds for the \( n \)-widths. For ease of notation, we omit the subscript index \( i \) in this subsection. By assuming that \( \omega \) and \( \omega^* \) are concentric (truncated) cubes with side lengths \( H \) and \( H^* \) \((H^* > H)\), respectively, we have

**Theorem 3.1.** There exist \( n_0 > 0 \) and \( b > 0 \) independent of \( k \), such that
\[
d_n(\omega, \omega^*) \leq e^n e^{-bn^{1/(d+1)}} e^{-\rho(H/H^*)bn^{1/(d+1)}}, \quad \forall n > n_0,
\]
where \( \sigma = k(H^* - H)V_{\max}/(2a_{\max}^{1/2}) \) and \( \rho(s) = 1 + s \log(s)/(1 - s) \).

**Remark 3.2.** The proof of Theorem 3.1 provides explicit values for \( n_0 \) and \( b \):
\[
n_0 = 2(4e\Theta)^d \quad \text{and} \quad b = (2e\Theta + 1/2)^{-d/(d+1)},
\]
where \( \Theta \) is defined in (2.29).

**Remark 3.3.** If \( k = 0 \), \( d_n(\omega, \omega^*) \) decays nearly exponentially with respect to \( n \) and \( H/H^* \) as shown in [29]. In the asymptotic regime, i.e., \( H \sim H^* \sim k^{-1} \), \( e^n \) is small and the decay of \( n \)-width is similar to that in the positive definite case. In the preasymptotic regime, since \( k \) only appears in the \( n \)-independent factor \( e^n \), \( d_n(\omega, \omega^*) \) still decays nearly exponentially with \( n \) and the rate is independent of \( k \). The decay of \( d_n(\omega, \omega^*) \) in the preasymptotic regime with respect to \( H/H^* \) \((H \text{ fixed})\) is nontrivial and it depends on the relation between \( k \) and \( n \). For moderate \( k \), \( d_n(\omega, \omega^*) \) decays nearly exponentially with \( H/H^* \), even for small \( n \). However, if \( k \) is large, the situation is different and we can distinguish two cases:
(i) for \( n \) sufficiently large, \( d_n(\omega, \omega^*) \) decays nearly exponentially with \( H/H^* \);
(ii) for small \( n \), \( d_n(\omega, \omega^*) \) first decreases and then stagnates as \( H/H^* \to 0 \).

Thus, there exists a resonance effect between \( k \) and \( n \) that influences the decay of \( d_n(\omega, \omega^*) \) with respect to the oversampling size \( H^* \) in the high-frequency regime. Numerical results in section 6 confirm the presence of this effect.

The key to the proof of Theorem 3.1 is to explicitly construct an \( n \)-dimensional space \( Q(n) \subset H_{D_k}^1(\omega) \) with the approximation error decaying nearly exponentially. As in [1, 29], this can be achieved by an iteration argument performed on a series of nested domains. To this end, we first construct an auxiliary \( m \)-dimensional space that approximates a generalized harmonic function with an explicit algebraic convergence rate with respect to \( m \).
LEMMA 3.4. Let $D^* \subset \Omega$ be an open connected set. There exists an $m$-dimensional space $R_m(D^*) \subset H_B(D^*)$ such that for any $u \in H_B(D^*)$,  

\begin{equation}
\inf_{v \in R_m(D^*)} \| u - v \|_{L^2(D^*)} \leq \frac{C(m)}{2\pi} \left( \gamma_d |D^*| \right) ^{1/d} a_{\min}^{-1/2} \| u \|_{A, D^*},
\end{equation}

where $|D^*|$ denotes the volume of the domain $D^*$, $\gamma_d$ is the volume of the unit ball in $\mathbb{R}^d$, and $C(m) = m^{-1/d}(1 + o(1))$.

Proof. In this proof, we use for convenience the alternative norm $\| u \|_{A, D^*, \varepsilon} := \left( \| u \|_{A, D^*}^2 + \varepsilon \| u \|_{L^2(D^*)}^2 \right)^{1/2}$ for $\varepsilon \in (0, 1)$. Consider the $n$-width of the embedding operator from $H_B(D^*)$ into $L^2(D^*)$:

\begin{equation}
\tilde{d}_{m, \varepsilon}(D^*) = \inf_{R(m) \subset L^2(D^*)} \sup_{u \in H_B(D^*)} \inf_{v \in R(m)} \frac{\| u - v \|_{L^2(D^*)}}{\| u \|_{A, D^*, \varepsilon}}.
\end{equation}

For each $j \in \mathbb{N}$, let $\phi_{j, \varepsilon}$ be the $j$-th eigenvector of the problem

\begin{equation}
(\phi, v)_{L^2(D^*)} = \lambda \left( A_{D^*} (\phi, v) + \varepsilon (\phi, v)_{L^2(D^*)} \right), \quad \forall v \in H_B(D^*)).
\end{equation}

Using the characterization of the $n$-width (see the proof of Lemma 2.7), the optimal approximation space associated with $\tilde{d}_{m, \varepsilon}(D^*)$ is given by

\begin{equation}
R_{m, \varepsilon}(D^*) = \text{span} \{ \phi_{1, \varepsilon}, \ldots, \phi_{m, \varepsilon} \}.
\end{equation}

From the definition of $\tilde{d}_{m, \varepsilon}(D^*)$, it follows that for any $u \in H_B(D^*)$,

\begin{equation}
\inf_{v \in R_{m, \varepsilon}(D^*)} \| u - v \|_{L^2(D^*)} \leq \tilde{d}_{m, \varepsilon}(D^*) \| u \|_{A, D^*, \varepsilon}.
\end{equation}

Since $H_B(D^*) \subset H^1(D^*)$, we see that

\begin{equation}
\tilde{d}_{m, \varepsilon}(D^*) \leq d_{m, \varepsilon}(D^*) := \inf_{R(m) \subset L^2(D^*)} \sup_{u \in H^1(D^*)} \inf_{v \in R(m)} \frac{\| u - v \|_{L^2(D^*)}}{\| u \|_{A, D^*, \varepsilon}}.
\end{equation}

Let $\mu_{m+1, \varepsilon}$ denote the $(m + 1)$-th eigenvalue of the problem

\begin{equation}
(\phi, v)_{L^2(D^*)} = \mu \left( A_{D^*} (\phi, v) + \varepsilon (\phi, v)_{L^2(D^*)} \right), \quad \forall v \in H^1(D^*)).
\end{equation}

It follows from the characterization of the $n$-width again that $d_{m, \varepsilon}(D^*) = \mu_{m+1, \varepsilon}^{1/2}$, which, together with (3.7) and (3.8), gives that

\begin{equation}
\inf_{v \in R_{m, \varepsilon}(D^*)} \| u - v \|_{L^2(D^*)} \leq \frac{1}{2} \mu_{m+1, \varepsilon} \| u \|_{A, D^*, \varepsilon}, \quad \forall u \in H_B(D^*).
\end{equation}

We now pass to the limit in (3.10). Without loss of generality, for each $j \in [1, m]$, we may assume that $\| \phi_{j, \varepsilon} \|_{D^*} = 1$ for any $\varepsilon \in (0, 1)$. Then, there exist a subsequence, still denoted by $\{ \phi_{j, \varepsilon} \}$, and a function $\phi_j \in H^1(D^*)$, such that $\phi_{j, \varepsilon} \rightharpoonup \phi_j$ weakly in $H^1(D^*)$ as $\varepsilon \rightarrow 0$. It is not difficult to verify that $\phi_j \in H_B(D^*)$. Now let us define

\begin{equation}
R_m(D^*) = \text{span} \{ \phi_1, \ldots, \phi_m \}.
\end{equation}

Since $H^1(D^*)$ is compactly embedded into $L^2(D^*)$, we see that for each $j \in [1, m]$, $\phi_{j, \varepsilon} \rightharpoonup \phi_j$ strongly in $L^2(D^*)$. Therefore, for each $u \in H_B(D^*)$, we have

\begin{equation}
\inf_{v \in R_{m, \varepsilon}(D^*)} \| u - v \|_{L^2(D^*)} \rightarrow \inf_{v \in R_m(D^*)} \| u - v \|_{L^2(D^*)} \quad \text{as} \quad \varepsilon \rightarrow 0.
\end{equation}
Next, we note that $\mu_{m+1,\varepsilon}/(1 - \varepsilon \mu_{m+1,\varepsilon})$ is the $(m+1)$-th eigenvalue of the problem
\begin{equation}
(\phi, v)_{L^2(D^*)} = \mu A_{D^*}(\phi, v), \quad \forall v \in H^1(D^*).
\end{equation}

Therefore, applying the classical eigenvalue asymptotics for the Laplacian [13, Chapter VI] to (3.13) yields that
\begin{equation}
\mu_{m+1,\varepsilon}^{1/2} \leq \left( \frac{\mu_{m+1,\varepsilon}}{1 - \varepsilon \mu_{m+1,\varepsilon}} \right)^{1/2} \leq \frac{C(m)}{2\pi} \left( \gamma_d |D^*|^{1/d} a_{\min}^{-1/2} \right).
\end{equation}

Taking $\varepsilon \to 0$ in (3.10) and using (3.12) and (3.14), we get the desired estimate (3.3).

Combining Lemmas 2.6 and 3.4 gives the approximation error in $R_m(D^*)$ in the energy norm restricted to a subdomain of $D^*$ as follows.

**Lemma 3.5.** Let $D \subset D^*$ be two open connected subsets of $\Omega$ such that $\delta = \text{dist}(D, \partial D^* \setminus \partial \Omega) > 0$ and let $R_m(D^*) \subset H_B(D^*)$ be the space given in Lemma 3.4. Then, for any $u \in H_B(D^*)$,
\begin{equation}
\inf_{v \in R_m(D^*)} \|u - v\|_{A,D} \leq \frac{C(m)}{2\pi} \left( \gamma_d |D^*|^{1/d} a_{\max}^{1/2} \right) \left( \frac{1}{\delta} + \frac{kV_{\max}}{a_{\max}^{1/2}} \right) \|u\|_{A,D^*}.
\end{equation}

**Proof.** Let $\eta \in C^1(D^*)$ be a cut-off function satisfying
\begin{equation}
\eta = 0 \quad \text{on } \partial D^* \cap \Omega, \quad \eta = 1 \quad \text{on } \Omega, \quad \text{and} \quad |\nabla \eta| \leq 1/\delta.
\end{equation}
Applying the Caccioppoli-type inequality (2.23) to $\eta$ and $u - v$ on $D^*$ and combining the result with Lemma 3.4, we get (3.15).

Lemma 3.5 is the starting point of the iteration argument in which the approximation result (3.15) is applied recursively on a series of nested domains between $\omega$ and $\omega^*$. Let $\delta^* = H^* - H$. For an integer $N \geq 1$, we denote $\{\omega^j\}_{j=1}^{N+1}$ as the nested family of (truncated) concentric cubes with side length $H^* - \delta^*(j - 1)/N$ such that $\omega = \omega^{N+1} \subset \omega^N \subset \cdots \subset \omega^1 = \omega^*$. Let $n = Nm$ and define
\begin{equation}
T(n, \omega, \omega^*) = R_m(\omega^1) + \cdots + R_m(\omega^N).
\end{equation}
The following lemma shows that $T(n, \omega, \omega^*)$ can deliver a sharper approximation result than (3.15).

**Lemma 3.6.** Let $u \in H_B(\omega^*)$ and $N \geq 1$ be an integer. Then,
\begin{equation}
\inf_{\varphi \in T(n, \omega, \omega^*)} \|\chi(u - \varphi)\|_{A,\omega,k} \leq 2e^\sigma \xi N \prod_{j=1}^{N-1} \left( 1 - \frac{j\delta^*}{NH^*} \right) \|u\|_{A,\omega^*},
\end{equation}
where $\sigma = k\delta^*V_{\max}/(2a_{\max}^{1/2})$ and $\xi$ is given by
\begin{equation}
\xi = \xi(N, m) = N \frac{1/d}{\pi} \left( \frac{a_{\max}}{a_{\min}} \right)^{1/2} \frac{H^*}{\delta^*} C(m).
\end{equation}

**Proof.** Applying Lemma 3.5 with $D^* = \omega^1$ and $D = \omega^2$ and using the fact that $\text{dist}(\omega^2, \partial \omega^1 \setminus \partial \Omega) = \delta^*/(2N)$ and $|\omega^1|^{1/d} = H^*$, we can find a $v^1_u \in R_m(\omega^1)$ such that
\begin{equation}
\|u - v^1_u\|_{A,\omega^2} \leq \xi(1 + \sigma/N) \|u\|_{A,\omega^1}.
\end{equation}
Since \( u - v^1_u \in H_\omega^N(\omega^2) \), we can apply Lemma 3.5 again and combine the result with (3.20) to find a \( v^2_u \in R_\omega(\omega^3) \) satisfying

\[
\|u - v^1_u - v^2_u\|_{A,\omega^3} \leq \xi^2(1 + \sigma/N)^2(1 - \frac{\delta^*}{NH^*})\|u\|_{A,\omega^1}.
\]

Repeating successively the same argument for \( N - 3 \) times, we see that there exist \( v^j_u \in R_\omega(\omega^3) \), \( j = 1, 2, \ldots, N - 1 \), such that

\[
\|u - \sum_{j=1}^{N-1} v^j_u\|_{A,\omega^N} \leq \xi^{N-1}(1 + \sigma/N)^{N-1} \prod_{j=1}^{N-2} (1 - \frac{j\delta^*}{NH^*})\|u\|_{A,\omega^1}.
\]

Finally, combining Lemma 3.4 and the Caccioppoli-type inequality (2.23) with \( \eta = \chi \), there exists a \( v^N_u \in R_\omega(\omega^N) \) such that

\[
\|\chi(u - \sum_{j=1}^N v^j_u)\|_{A,\omega^{N+1,k}} \leq C(m)\frac{\gamma^{1/d}}{2\pi} (H^* - \delta^*(N - 1)/N) c_{\min}^{-1/2}
\]

\[
\cdot (a_{\max}^{1/2} \|\nabla \chi\|_{L^\infty(\omega)} + 2kV_{\max}) \|u - \sum_{j=1}^{N-1} v^j_u\|_{A,\omega^N} \leq 2\xi \left(1 - \frac{(N - 1)\delta^*}{NH^*}\right) (\|\nabla \chi\|_{L^\infty(\omega)}\delta^*/(4N) + \sigma/N)\|u - \sum_{j=1}^{N-1} v^j_u\|_{A,\omega^N}.
\]

Without loss of generality, we assume that \( \|\nabla \chi\|_{L^\infty(\omega)} \leq 4N/\delta^* \). It follows from (3.22), (3.23), and the inequality \((1 + \sigma/N)^N \leq e^\sigma\) that

\[
\|\chi(u - \sum_{j=1}^N v^j_u)\|_{A,\omega^{N+1,k}} \leq 2(1 + \sigma/N)^N \xi^N \prod_{j=1}^{N-1} (1 - \frac{j\delta^*}{NH^*})\|u\|_{A,\omega^1} \leq 2e^{\sigma} \xi^N \prod_{j=1}^{N-1} (1 - \frac{j\delta^*}{NH^*})\|u\|_{A,\omega^1}.
\]

Since \( \sum_{j=1}^N v^j_u \in T(n, \omega, \omega^*) \), (3.19) follows immediately from (3.24).

**Proof of Theorem 3.1.** Let \( Q(n) = \{ \chi v : v \in T(n, \omega, \omega^*) \} \subset H_{T}^1(\omega) \). It follows from Lemma 3.6 that

\[
\sup_{u \in H_\omega^N(\omega)} \inf_{\varphi \in Q(n)} \frac{\|\chi u - \varphi\|_{A,\omega,k}}{\|u\|_{A,\omega^*}} \leq 2e^{\sigma} \xi^N \prod_{j=1}^{N-1} \left(1 - \frac{j\delta^*}{NH^*}\right),
\]

where \( \sigma = k\delta^*/(2a_{\max}^{1/2}) \) and \( \xi \) is given in (3.19). Denoting by

\[
\Theta = \frac{\gamma^{1/d}}{2\pi} \left(\frac{a_{\max}}{a_{\min}}\right)^{1/2} H^* - \delta^*, \quad n_0 = 2(4e\Theta)^d, \quad b = (2e\Theta + 1/2)^{-d/(d+1)},
\]

and using a similar argument as in the proof of Theorem 3.6 in [29] by specifying a particular relation between \( m \) and \( N \), we can prove that for any \( n > n_0 \),

\[
\xi^N \prod_{j=1}^{N-1} \left(1 - \frac{j\delta^*}{NH^*}\right) \leq e^{-b n^{1/(d+1)}} e^{-\rho(H/H^*)bn^{1/(d+1)}},
\]

where \( \rho(s) = 1 + s \log(s)/(1 - s) \). Inserting (3.27) into (3.25) and recalling the definition of the \( n \)-width, we get (3.1) and complete the proof.
3.2. Global error estimates. In this subsection, we first derive the global approximation error estimate and then establish a quasi-optimal convergence rate for the method under some assumptions. For convenience, we define

\[
d_{\text{max}} = \max_{i=1, \ldots, M} d_{n_i}(\omega_i, \omega_i^*), \quad C_{\text{max}}(k) = \max_{i=1, \ldots, M} C_{\text{stab}}^i(k),
\]

where \(C_{\text{stab}}^i(k)\) are the stability constants defined in Assumption 2.5. Furthermore, we assume that the oversampling domains \(\{\omega_i^*\}_{i=1}^M\) satisfy a similar pointwise overlap condition as \(\{\omega_i\}_{i=1}^M\):

\[
\exists \zeta^* \in \mathbb{N} \quad \forall x \in \Omega \quad \text{card}\{i \mid x \in \omega_i^*\} \leq \zeta^*.
\]

**Lemma 3.7.** Let \(u^p\) and \(S_n(\Omega)\) be the global particular function and the trial space for the continuous MS-GFEM, and let \(u^*\) be the solution of the problem (2.5). Then, there exists a \(\varphi \in u^p + S_n(\Omega)\) such that

\[
\|u^p - \varphi\|_{A, k} \leq \sqrt{2C_{\text{stab}}}(C_{\text{stab}} + \sqrt{2}C_{\text{max}}(k))\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial \Omega))},
\]

where \(C_{\text{stab}}(k)\) is the stability constant defined in Assumption 2.2.

**Proof.** Using (2.32) and Assumption 2.5, we see that on each subdomain \(\omega_i\), there exists a \(\varphi_i \in u^p + S_n(\omega_i)\) such that

\[
\|\varphi_i - \varphi\|_{A, \omega_i} \leq d_{n_i}(\omega_i, \omega_i^*)\|u^* - \varphi_i\|_{A, \omega_i}^* \\
\leq d_{n_i}(\omega_i, \omega_i^*)\|u^*\|_{A, \omega_i} + C_{\text{stab}}^i(k)\|f\|_{L^2(\omega_i^*)} + \|g\|_{L^2(\partial \omega_i^* \cap \partial \Omega))}.
\]

Let \(\varphi = \sum_{i=1}^M \chi_i \varphi_i \in u^p + S_n(\Omega)\). It follows from (2.10) that

\[
\|u^p - \varphi\|_{A, k}^2 = \left|\sum_{i=1}^M \chi_i (u^p - \varphi_i)\right|_{A, \omega_i,k}^2 \leq \zeta \sum_{i=1}^M \|\chi_i (u^p - \varphi_i)\|_{A, \omega_i,k}^2.
\]

Following the same lines as in the proof of Theorem 2.1 in [29], we get (3.30) by inserting (3.31) into (3.32) and using (3.29) and the stability estimate (2.9).

To derive a quasi-optimal convergence rate for the method, we introduce the solution operator \(\hat{S} : L^2(\Omega) \rightarrow H^1_D(\Omega)\) for the problem:

\[
\text{Find } \hat{u} \in H^1_D(\Omega) \text{ such that } B(\hat{u}, v) = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1_D(\Omega),
\]

i.e., \(\hat{S}(f) := \hat{u}\), and define the following quantity

\[
\eta(S_n(\Omega)) := \sup_{f \in L^2(\Omega)} \inf_{\varphi \in S_n(\Omega)} \frac{\|\hat{S}(f) - \varphi\|_{A, k}}{\|f\|_{L^2(\Omega)}}.
\]

It follows from Assumption 2.2 that the operator \(\hat{S}\) is well defined. Recalling the inequality (2.8) with the \(k\)-independent constant \(C_B\) and using a standard duality argument (see, e.g., [16]), we have

**Theorem 3.8.** Let \(u^*\) be the solution of the problem (2.5). Assuming that

\[
2C_B k V_{\text{max}} \eta(S_n(\Omega)) \leq 1,
\]

then the problem (2.13) has a unique solution \(u^G \in u^p + S_n(\Omega)\) satisfying

\[
\|u^* - u^G\|_{A, k} \leq 2C_B \inf_{\varphi \in u^p + S_n(\Omega)} \|u^* - \varphi\|_{A, k}.
\]
Proof. Since (2.13) is a finite-dimensional linear system, its unique solvability is implied by (3.36) and Assumption 2.2. Hence we restrict our attention to the proof of (3.36). Let $e_G = u^e - u^G$. We observe that
\begin{equation}
\|e_G\|^2_{A,k} = \text{Re} B(e_G, e_G) + 2k^2 \langle V^2 e_G, e_G \rangle_{L^2(\Omega)}.
\end{equation}
Using the Galerkin orthogonality (2.14) and (2.8) yields that for any $\varphi \in u^p + S_n(\Omega)$,
\begin{equation}
\|e_G\|^2_{A,k} = \text{Re} B(e_G, u^e - \varphi) + 2k^2 \langle V^2 e_G, e_G \rangle_{L^2(\Omega)} \\
\leq C_B\|e_G\|_{A,k} \|u^e - \varphi\|_{A,k} + 2k^2 V_{\text{max}}^2\|e_G\|^2_{L^2(\Omega)}.
\end{equation}
To estimate $\|e_G\|_{L^2(\Omega)}$, we consider the adjoint problem:
\begin{equation}
\text{Find } w^e \in H^1_D(\Omega) \text{ such that } B(v, w^e) = (v, e_G)_{L^2(\Omega)} \quad \forall v \in H^1_D(\Omega).
\end{equation}
Note that $\overline{w} \in H^1_D(\Omega)$ satisfies
\begin{equation}
B(\overline{w}, v) = (\overline{v}, v)_{L^2(\Omega)} \quad \forall v \in H^1_D(\Omega),
\end{equation}
i.e., $\overline{w} = \overline{S(\overline{v})}$. Choosing $v = e_G$ in (3.39) and using the Galerkin orthogonality again, we see that for any $\varphi \in S_n(\Omega)$,
\begin{equation}
\|e_G\|^2_{L^2(\Omega)} = B(e_G, w^e - \varphi) \leq C_B\|e_G\|_{A,k} \|\overline{w} - \overline{\varphi}\|_{A,k} \\
\leq C_B\|e_G\|_{A,k} \|\overline{S(\overline{v})} - \overline{\varphi}\|_{A,k},
\end{equation}
which, combining with (3.34) and the fact that $\overline{\varphi} \in S_n(\Omega)$, gives that
\begin{equation}
\|e_G\|_{L^2(\Omega)} \leq C_B \eta(S_n(\Omega)) \|e_G\|_{A,k}.
\end{equation}
Inserting (3.42) into (3.38) and using (3.35), we get
\begin{equation}
\|e_G\|^2_{A,k} \leq C_B\|e_G\|_{A,k} \|u^e - \varphi\|_{A,k} + 2(k V_{\text{max}} C_B \eta(S_n(\Omega)))^2\|e_G\|^2_{A,k} \\
\leq C_B\|e_G\|_{A,k} \|u^e - \varphi\|_{A,k} + \frac{1}{2}\|e_G\|^2_{A,k} \quad \forall \varphi \in u^p + S_n(\Omega),
\end{equation}
from which the estimate (3.36) follows. \qed

In the rest of this subsection, we derive an upper bound for $\eta(S_n(\Omega))$. To this end, we introduce the following Poincaré inequality with an explicit dependence on the diameter of a domain.

Lemma 3.9. Let $\Omega' \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $\Gamma' \subset \partial \Omega'$ with $|\Gamma'| > 0$. Then there exists a constant $C_{p}$ depending only on the shape of $\Omega'$ but not on its size, such that for any $u \in H^1(\Omega')$ with vanishing trace on $\Gamma'$,
\begin{equation}
\|u\|_{L^2(\Omega')} \leq C_{p} \text{diam}(\Omega') \|\nabla u\|_{L^2(\Omega')}.
\end{equation}
By virtue of Lemma 3.9, we define $C_p$ as the uniform Poincaré constant such that
\begin{equation}
\|u\|_{L^2(\omega_i)} \leq C_{p} \text{diam}(\omega_i) \|\nabla u\|_{L^2(\omega_i)}, \quad \forall u \in H^1_{D1}(\omega_i^*), \quad \forall i = 1, \ldots, M.
\end{equation}
Recalling Assumption 2.2 with the stability constant $C_{\text{stab}}(k)$, and combining the local approximation error estimates and the Poincaré inequality, we can prove
LEMMA 3.10. Assuming that
\[ kV_{\max} C_P H_{\max}^* \leq a_{\min}^{1/2}/\sqrt{2}, \]
then,
\[ \eta(S_n(\Omega)) \leq \sqrt{\zeta^2 (2d_{\max} (C_{\text{stab}}(k) + 2C_P H_{\max}^* a_{\min}^{-1/2}) + 3\sqrt{3}C_P H_{\max}^* a_{\min}^{-1/2})}. \]

Proof. By Assumption 2.2, for any \( f \in L^2(\Omega) \), the problem (3.33) has a unique solution \( \hat{u} \in H_{D}^1(\Omega) \) with the estimate
\[ \| \hat{u} \|_{A,k} \leq C_{\text{stab}}(k) \| f \|_{L^2(\Omega)}. \]
On each oversampling domain \( \omega^*_i \), we consider the following local Helmholtz problem:
\[ \text{Find } \tilde{\psi}_i \in H_{D}^1(\omega^*_i) \text{ such that } B_{\omega^*_i}(\tilde{\psi}_i, v) = (f, v)_{L^2(\omega^*_i)} \quad \forall v \in H_{D}^1(\omega^*_i). \]
Note that \( \tilde{\psi}_i \) satisfies the homogeneous Dirichlet boundary conditions on \( \partial \omega^*_i \cap \Omega \). Under the assumption (3.46), the local sesquilinear form in (3.49) is coercive. In fact, using (3.46) and the Poincaré inequality (3.54), we have that for any \( v \in H_{D}^1(\omega^*_i) \),
\[ \text{Re} B_{\omega^*_i}(v, v) \geq \| v \|_{A,\omega^*_i}^2 - k^2 V_{\max}^2 C_P [\text{diam}(\omega^*_i)]^2 \| \nabla v \|_{L^2(\omega^*_i)}^2 \]
\[ \geq \| v \|_{A,\omega^*_i}^2 - a_{\min} \| \nabla v \|_{L^2(\omega^*_i)}^2 / 2 \geq \| v \|_{A,\omega^*_i}^2 / 2. \]
Using the Poincaré inequality (3.54) again, it follows that
\[ \| \tilde{\psi}_i \|_{A,\omega^*_i}^2 \leq 2C_P \text{diam}(\omega^*_i) \| f \|_{L^2(\omega^*_i)} \| \nabla \tilde{\psi}_i \|_{L^2(\omega^*_i)} \]
\[ \leq 2C_P \text{diam}(\omega^*_i) a_{\min}^{-1/2} \| f \|_{L^2(\omega^*_i)} \| \tilde{\psi}_i \|_{A,\omega^*_i}. \]
Denoting by \( \Lambda = C_P H_{\max}^* a_{\min}^{-1/2} \), (3.51) leads to
\[ \| \tilde{\psi}_i \|_{A,\omega^*_i} \leq 2\Lambda \| f \|_{L^2(\omega^*_i)}, \quad \text{and} \quad \| \tilde{\psi}_i \|_{A,\omega^*_i,k} \leq 3\Lambda \| f \|_{L^2(\omega^*_i)}. \]
Furthermore, combining (3.33) and (3.49), we see that \( \hat{u} |_{\omega^*_i} - \tilde{\psi}_i \in H_{G}(\omega^*_i) \). Recalling the definition of the \( n \)-width and using (3.52), it follows that there exists \( \varphi_i \in S_n(\omega_i) \) such that
\[ \| \chi_i (\hat{u} - \tilde{\psi}_i - \varphi_i) \|_{A,\omega^*_i,k} \leq d_{n_i}(\omega_i, \omega^*_i) \| \hat{u} - \tilde{\psi}_i \|_{A,\omega^*_i} \]
\[ \leq d_{n_i}(\omega_i, \omega^*_i) \| \hat{u} \|_{A,\omega^*_i} + 2\Lambda \| f \|_{L^2(\omega^*_i)}. \]
Define \( \hat{u}^p = \sum_{i=1}^M \chi_i \tilde{\psi}_i \) and \( \varphi = \sum_{i=1}^M \chi_i \varphi_i \in S_n(\Omega) \). Using (3.48) and a similar argument as in the proof of Lemma 3.7, we get
\[ \| \hat{u} - \hat{u}^p - \varphi \|_{A,k} \leq \sqrt{2\zeta^2 d_{\max}} (\| \hat{u} \|_{A} + 2\Lambda \| f \|_{L^2(\Omega)}) \]
\[ \leq \sqrt{2\zeta^2 d_{\max}} (C_{\text{stab}}(k) + 2\Lambda) \| f \|_{L^2(\Omega)}, \]
and consequently
\[ \| \hat{u} - \varphi \|_{A,k} \leq \| \hat{u}^p \|_{A,k} + \sqrt{2\zeta^2 d_{\max}} (C_{\text{stab}}(k) + 2\Lambda) \| f \|_{L^2(\Omega)}. \]
It remains to estimate $\|\hat{u}^p\|_{A,k}$. By definition, we see that

$$
\|\hat{u}^p\|_{A,k}^2 \leq \zeta \sum_{i=1}^{M} \|\chi_i \hat{\psi}_i\|_{A,\omega_i,k}^2 \leq \zeta \sum_{i=1}^{M} \left( \|\chi_i \hat{\psi}_i\|_{A,\omega_i}^2 + k^2 \|\hat{\psi}_i\|_{L^2(\omega_i)}^2 \right).
$$

A use of the triangle inequality gives that

$$
\|\chi_i \hat{\psi}_i\|_{A,\omega_i} \leq \|\hat{\psi}_i\|_{A,\omega_i} + a_{\max}^{1/2} \|\nabla \chi_i\|_{L^\infty(\omega_i)} \|\hat{\psi}_i\|_{L^2(\omega_i)}.
$$

Without loss of generality, we assume that $k$ is sufficiently large such that $k V_{\min} \geq a_{\max}^{1/2} \|\nabla \chi_i\|_{L^\infty(\omega_i)}$. It follows that

$$
\|\chi_i \hat{\psi}_i\|_{A,\omega_i} \leq \|\hat{\psi}_i\|_{A,\omega_i} + k \|\hat{\psi}_i\|_{L^2(\omega_i)}.
$$

Combining (3.52), (3.56), and (3.58), we come to

$$
\|\hat{u}^p\|_{A,k} \leq \left( 3\zeta \sum_{i=1}^{M} \|\hat{\psi}_i\|_{A,\omega_i,k}^2 \right)^{1/2} \leq 3\sqrt{3} \zeta^{1/2} \Lambda \|f\|_{L^2(\Omega)}.
$$

Inserting (3.59) into (3.55) yields that

$$
\|\hat{u} - \varphi\|_{A,k} \leq \|f\|_{L^2(\Omega)} \leq \sqrt{\zeta^{1/2} \sqrt{2} d_{\max}(C_{\text{stab}}(k) + 2\Lambda) + 3\sqrt{3} \Lambda},
$$

and the desired estimate (3.47) follows by recalling that $\Lambda = C_P H^{*}_{\max} a_{\min}^{-1/2}$. 

Based on Theorem 3.8 and Lemma 3.10, we see that some resolution conditions on $d_{\max}$ and $H^{*}_{\max}$ need to be imposed to obtain a quasi-optimal convergence rate for the method. To do this, we define a constant:

$$
\Xi = C_B V_{\max} \sqrt{\zeta^{1/2}}.
$$

**Corollary 3.11.** Let $u^c$ be the solution of problem (2.5) and $u^G$ be the continuous MS-GFEM approximation. Suppose that

$$
d_{\max} \leq \left(8\sqrt{2} k C_{\text{stab}}(k) \Xi \right)^{-1}, \quad H^{*}_{\max} \leq \left(12\sqrt{3} k C_P a_{\min}^{-1/2} \Xi \right)^{-1},
$$

then

$$
\|u^c - u^G\|_{A,k} \leq 2C_B \inf_{\varphi \in \mathcal{U}^p + S_n(\Omega)} \|u^c - \varphi\|_{A,k}.
$$

**Proof.** The assumptions (3.35) and (3.46) are implied by (3.62) and the result follows from Lemma 3.10 and Theorem 3.8. 

**Remark 3.12.** Due to Assumption 2.2 and Theorem 3.1, the first condition in (3.62) is satisfied if the sizes of the local subspaces grow polylogarithmically in $k$. The second condition is equivalent to $H k = O(H^{*}_{\max} k) = O(1)$, where $H$ denotes the size of the subdomains. Under these conditions, Lemma 3.7, Theorem 3.1, and (3.63) imply a nearly exponential convergence rate of the method. However, the second condition is very stringent in the high frequency regime, and our numerical results in section 6 show that it is not necessary in practice to obtain near-exponential convergence.
4. Discrete MS-GFEM. In the rest of this paper, we assume that \( \Omega \) is a Lipschitz polyhedral domain for simplicity. Let \( \tau_h = \{K\} \) be a shape-regular triangulation of \( \Omega \) consisting of triangles (tetrahedrons) or rectangles if \( \Omega \) is a rectangular domain. The mesh size \( h := \max_{K \in \tau_h} \text{diam}(K) \) is assumed to be sufficiently small to resolve the high frequency features of the wave and the fine-scale details of the coefficients. Let \( U_h \subset H^1(\Omega) \) be a standard Lagrange finite element space. For simplicity, we take \( U_h \) to be the space consisting of continuous piecewise linear functions or the Q1 bilinear space for a rectangular subdivision. Let \( U_{h,D} = U_h \cap H^1_0(\Omega) \). The standard finite element method for the problem (2.5) is: Find \( u_h \in U_{h,D} \) such that

\[
B(u_h, v_h) = F(v_h) \quad \forall v_h \in U_{h,D}.
\]

In what follows, we introduce the discrete MS-GFEM for solving the problem (4.1) in parallel with the continuous MS-GFEM in section 2. Let \( \{\omega_i\}_{i=1}^M \) be an overlapping decomposition of \( \Omega \) resolved by the mesh. We extend each subdomain \( \omega_i \) by several layers of fine mesh elements to create a larger oversampling domain \( \omega_i^* \), and define

\[
\begin{align*}
U_h(\omega_i^*) &= \{v_h|_{\omega_i^*} : v_h \in U_h\}, \\
U_{h,D}(\omega_i^*) &= \{v_h \in U_h(\omega_i^*) : v_h = 0 \text{ on } \partial \omega_i^* \cap \Gamma_D \}, \\
U_{h,DI}(\omega_i^*) &= \{v_h \in U_h(\omega_i^*) : v_h = 0 \text{ on } \partial \omega_i^* \cap (\Omega \cup \Gamma_D) \}, \\
H_{h,B}(\omega_i^*) &= \{u_h \in U_{h,D}(\omega_i^*) : \mathcal{B}_{\omega_i^*}(u_h, v_h) = 0, \forall v_h \in U_{h,DI}(\omega_i^*)\},
\end{align*}
\]

where \( H_{h,B}(\omega_i^*) \) is referred to as the discrete generalized harmonic space and we note that \( H_{h,B}(\omega_i^*) \subsetneq H_B(\omega_i^*) \). Similar to (2.17), there exists \( C > 0 \) independent of \( h \), such that for any \( u_h \in H_{h,B}(\omega_i^*) \),

\[
\|u_h\|_{L^2(\omega_i^*)} \leq C\|\nabla u_h\|_{L^2(\omega_i^*)}.
\]

The proof of (4.3) is given in Appendix B. Therefore, \( \|\cdot\|_{A,\omega_i^*} \) is also a norm on \( H_{h,B}(\omega_i^*) \). Next we introduce the discrete local Helmholtz problem: Find \( \psi_{h,i} \in U_{h,D}(\omega_i^*) \) such that

\[
\mathcal{B}_{\omega_i^*}(\psi_{h,i}, v_h) - ik \int_{\partial \omega_i^* \cap \Gamma_D} V \psi_{h,i} \overline{v_h} \, ds = F_{\omega_i^*}(v_h), \quad \forall v_h \in U_{h,DI}(\omega_i^*).
\]

We proceed to construct the optimal spaces for approximating a discrete generalized harmonic function in the same spirit as before. Let \( I_{h} : C(\Omega) \rightarrow U_h \) be the standard Lagrange interpolation operator. We define the operator

\[
P_{h,i} : H_{h,B}(\omega_i^*) \rightarrow U_{h,DI}(\omega_i) \quad \text{such that} \quad P_{h,i}v_h = I_{h}(\chi_{i}v_h),
\]

where \( \chi_{i} \) is the partition of unity function supported on \( \omega_i \). For each \( n \in \mathbb{N} \), we consider the Kolmogorov \( n \)-width of \( P_{h,i} \) defined by

\[
d_{h,n}(\omega_i, \omega_i^*) = \inf_{Q(n) \subset U_{h,DI}(\omega_i)} \sup_{u_h \in H_{h,B}(\omega_i^*)} \inf_{v_h \in Q(n)} \frac{\|P_{h,i}u_h - v_h\|_{A,\omega_i,k}}{\|u_h\|_{A,\omega_i^*}}.
\]

Similar to Lemma 2.7, we have the following characterization of the \( n \)-width. We omit the proof.
Lemma 4.1. For each \( j \in \mathbb{N} \), let \((\lambda_{h,j}, \phi_{h,j})\) be the \( j \)-th eigenpair (arranged in decreasing order) of the problem
\[
A_{\omega_i^*, k}(I_h(x_i \phi_{h_j}), I_h(x_i \phi_{h_j})) = \lambda_{h_j} A_{\omega_i^*}(\phi_{h_j}, \phi_{h_j}), \quad \forall v_h \in H_{h,B}(\omega_i^*).
\]
Then \( d_{h,n}(\omega_i, \omega_i^*) = \lambda_{h,n+1}^{1/2} \) and the optimal approximation space is given by
\[
\tilde{Q}(n) = \text{span}\{I_h(x_i \phi_{h,1}), \cdots, I_h(x_i \phi_{h,n})\}.
\]

Remark 4.2. The way that we define the operator \( P_{h,i} \) in this paper also works for the positive definite case in [28] where \( P_{h,i} \) was defined in a slightly different way without involving the interpolation operator.

Before defining the local particular function and the local approximation space for the discrete MS-GFEM, we make some assumptions on the well-posedness of the discrete problems (4.1) and (4.4) analogously to the continuous level.

Assumption 4.3. There exists \( h_0 > 0 \) such that for any \( 0 < h < h_0 \),

(i) the problem (4.1) has a unique solution \( u_h^* \in U_{h,D} \), and there exists \( C_{\text{stab}}(k) \) depending polynomially on \( k \) such that
\[
\|u_h^*\|_{A,k} \leq C_{\text{stab}}(k)(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}).
\]

(ii) for each \( i = 1, \cdots, M \), the problem (4.4) is uniquely solvable in \( U_{h,D}(\omega_i^*) \), and there exists \( C_{\text{stab}}^i(k) \) depending polynomially on \( k \) such that
\[
\|\psi_{h,i}\|_{A_{\omega_i^*, k}} \leq C_{\text{stab}}^i(k)(\|f\|_{L^2(\omega_i^*)} + \|g\|_{L^2(\partial \omega_i^* \cap \Gamma_R)}).
\]

Remark 4.4. The unique solvability of the discrete problem (4.1) can be implied by that of the continuous problem (2.5) and of the continuous adjoint problem with the right-hand term in a weaker dual space; see the remark after Lemma 4.10. Similar results hold for the local problems (4.4).

Theorem 4.5. Let the local particular function and the local approximation space on \( \omega_i \) be defined as
\[
u^p_{h,i} := \psi_{h,i} |_{\omega_i} \quad \text{and} \quad S_{h,n_i}(\omega_i) := \text{span}\{\phi_{h,1}|_{\omega_i}, \cdots, \phi_{h,n_i}|_{\omega_i}\},
\]
where \( \psi_{h,i} \) is the solution of (4.4) and \( \phi_{h,j} \) denotes the \( j \)-th eigenfunction of the problem (4.7). Then,
\[
\inf_{\varphi \in \nu^p_{h,i} + S_{h,n_i}(\omega_i)} \|I_h (x_i (u_h^* - \varphi))\|_{A_{\omega_i}, k} \leq d_{h,n_i}(\omega_i, \omega_i^*) \|u_h^* - \psi_{h,i}\|_{A_{\omega_i^*}}.
\]

Proof. Noting that \( u_h^* |_{\omega_i^*} - \psi_{h,i} \in H_{h,B}(\omega_i^*) \), (4.12) follows from Lemma 4.1 and the definition of the \( n \)-width. \( \square \)

Now we proceed to define the global particular function and the trial space for the discrete MS-GFEM:
\[
u^p_h := \sum_{i=1}^M I_h (x_i \nu^p_{h,i}) \quad \text{and} \quad S_h(\Omega) := \left\{ \sum_{i=1}^M I_h (x_i \nu_{h,i}) : \nu_{h,i} \in S_{h,n_i}(\omega_i) \right\}.
\]

The last step is to solve the discrete problem on \( S_h(\Omega) \): Find \( u_h^G = u_h^p + u_h^* \), where \( u_h^* \in S_h(\Omega) \), such that
\[
\mathcal{B}(u_h^G, v_h) = F(v_h) \quad \forall v_h \in S_h(\Omega).
\]
By the definition of $S_h(\Omega)$, we see that $S_h(\Omega) \subset U_{h,D}$, and thus the discrete MS-GFEM delivers a conforming approximation of the problem (4.1). As in the continuous case, we have the Galerkin orthogonality:

$$B(u_h^* - u_h^G, v_h) = 0 \quad \forall v_h \in S_h(\Omega).$$

4.1. Technical tools. In this subsection, we present some technical tools that will be used for proving the convergence of the discrete MS-GFEM in the next section. We start with the following superapproximation estimates.

For each $u_h \in U_h$ and $K \in \tau_h$ with $h_K := \text{diam}(K) \leq \delta$, we have the Galerkin orthogonality:

$$B(u_h - u_h^G, v_h) = 0 \quad \forall v_h \in S_h(\Omega).$$

Estimate (4.18) was proved in Theorem 2.1 of [14] and (4.17) can be proved using exactly the same argument; see also [42, Chapter 3]. Next we give the multiplicative trace inequality on $K$.

$$
\begin{align*}
\|\eta^2 u_h - I_h(\eta^2 u_h)\|_{H^1(K)} &\leq C\left(\frac{h_K}{\delta} \|\nabla (\eta u_h)\|_{L^2(K)} + \frac{h_K}{\delta^2} \|u_h\|_{L^2(K)}\right), \\
\|\eta^2 u_h - I_h(\eta^2 u_h)\|_{L^2(K)} &\leq C\left(\frac{h_K^2}{\delta} \|\nabla (\eta u_h)\|_{L^2(K)} + \frac{h_K^2}{\delta^2} \|u_h\|_{L^2(K)}\right),
\end{align*}
$$

where $I_h$ is the standard Lagrange interpolation operator.

Lemma 4.6 ([14]). Assume that $\eta \in C^\infty(\Omega)$ satisfying $|\eta|_{W^{1,\infty}(\Omega)} \leq C\delta^{-j}$ for $0 \leq j \leq 2$. Then for each $u_h \in U_h$ and $K \in \tau_h$ with $h_K := \text{diam}(K) \leq \delta$,

$$\|\eta^2 u_h - I_h(\eta^2 u_h)\|_{H^1(K)} \leq C \left(\frac{h_K}{\delta} \|\nabla (\eta u_h)\|_{L^2(K)} + \frac{h_K}{\delta^2} \|u_h\|_{L^2(K)}\right),$$

$$\|\eta^2 u_h - I_h(\eta^2 u_h)\|_{L^2(K)} \leq C \left(\frac{h_K^2}{\delta} \|\nabla (\eta u_h)\|_{L^2(K)} + \frac{h_K^2}{\delta^2} \|u_h\|_{L^2(K)}\right),$$

where $I_h$ is the standard Lagrange interpolation operator.

Proof. We prove (4.18) using a standard scaling argument. Let $\hat{K}$ denote the reference element. For each $K \in \tau_h$, there exists $C > 0$ depending only on the shape regularity of the mesh such that

$$\|u\|_{L^2(\partial K)}^2 \leq C \|u\|_{L^2(K)} \|\nabla u\|_{L^2(K)} + h_K^{-1} \|u\|_{L^2(K)}^2.$$  

Using the relation between $u$ and $\hat{u}$, we have

$$\|\nabla \hat{u}\|_{L^2(\partial \hat{K})} \leq C \|\hat{u}\|_{L^2(\hat{K})} \|\nabla u\|_{L^2(K)},$$

and

$$\|\hat{u}\|_{L^2(\hat{K})} \leq C \|\det(B_K)^{-1/2} |B_K| \|u\|_{L^2(K)},$$

and

$$\|u\|_{L^2(\partial K)} \leq |\det(B_K)|^{1/2} |B_K^{-1}|^{1/2} \|\hat{u}\|_{L^2(\partial \hat{K})}.$$  

Using the relation between $u$ and $\hat{u}$, we have

$$\|\nabla \hat{u}\|_{L^2(\partial \hat{K})} \leq C \|\hat{u}\|_{L^2(\hat{K})} \|\nabla u\|_{L^2(K)},$$

and

$$\|\hat{u}\|_{L^2(\hat{K})} \leq C |\det(B_K)|^{-1/2} |B_K|^{-1/2} \|\hat{u}\|_{L^2(\partial \hat{K})}.$$  

Let $\rho_K$ denote the diameter of the largest sphere contained in $\overline{K}$. We have the following classical estimates (see, e.g., [32, Lemma 5.10])

$$\|B_K\| \leq C h_K, \quad |B_K^{-1}| \leq C h_K^{-1}, \quad C_1 \rho_K^3 \leq |\det(B_K)| \leq C_2 h_K^3.$$

Now the desired estimate (4.19) follows from (4.20)–(4.23) and the shape-regularity of the mesh.  

REMARK 4.8. Using a similar argument, it can be proved that for any \( u \in H^1(\Omega) \), there exists a constant \( C \), which depends only on the shape of \( \Omega \), such that

\[
\|u\|^2_{L^2(\partial \Omega)} \leq C(\|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} + \text{diam}(\Omega)^{-1}\|u\|^2_{L^2(\Omega)}).
\]

For each \( i = 1, \cdots, M \), we define projections \( \Pi^h_i : H^1_0(\omega^*_i) \rightarrow U_{h,D}(\omega^*_i) \) by

\[
E_{\omega^*_i}(\Pi^h_i u, v_h) = E_{\omega^*_i}(u, v_h) \quad \forall v_h \in U_{h,D}(\omega^*_i).
\]

To prove the stability and well-definedness of \( \Pi^h_i \), we need the following assumption on the continuous adjoint problem:

ASSUMPTION 4.9. For any \( \mathcal{G} \in (H^1_0(\omega^*_i))^\prime \), the adjoint problem

\[
\mathcal{B}_{\omega^*_i}(v, u) = \mathcal{G}(v) \quad \forall v \in H^1_0(\omega^*_i)
\]

has a unique solution \( u \in H^1_0(\omega^*_i) \), and there exists \( C > 0 \) which may depend on \( k \) such that

\[
\|u\|_{A,\omega^*_i,k} \leq C\|\mathcal{G}\|_{(H^1_0(\omega^*_i))'}.
\]

LEMMA 4.10. Under Assumption 4.9, there exists an \( h_0 > 0 \), such that for any \( 0 < h < h_0 \), \( \Pi^h_i \) is well-defined and satisfies

\[
\Pi^h_i u \|_{A,\omega^*_i,k} \leq C\|u\|_{A,\omega^*_i,k}, \quad \|u - \Pi^h_i u\|_{A,\omega^*_i,k} \leq C\inf_{v_h \in \mathcal{U}_{h,D}(\omega^*_i)} \|u - v_h\|_{A,\omega^*_i,k}
\]

for any \( u \in H^1_0(\omega^*_i) \), where \( C > 0 \) is independent of \( h \).

Proof. Since (4.25) is a finite-dimensional linear system, the estimate (4.28) yields the unique solvability of (4.25). Therefore, it suffices to prove (4.28). Under Assumption 4.9, it follows from [39, Theorem 2] that for any \( \varepsilon > 0 \), there exists an \( h_0 = h_0(\varepsilon) \), such that for any \( 0 < h < h_0 \),

\[
\|u - \Pi^h_i u\|_{L^2(\omega^*_i)} \leq \varepsilon\|u - \Pi^h_i u\|_{A,\omega^*_i,k},
\]

\[
\|u - \Pi^h_i u\|_{A,\omega^*_i,k} \leq C\inf_{v_h \in \mathcal{U}_{h,D}(\omega^*_i)} \|u - v_h\|_{A,\omega^*_i,k}.
\]

Therefore, the second part of (4.28) is proved. Next by choosing \( v_h = \Pi^h_i u \) in (4.25) and taking the real part of the equation, we see that

\[
\|\Pi^h_i u\|^2_{A,\omega^*_i,k} \leq C\|\Pi^h_i u\|_{A,\omega^*_i,k}\|u\|_{A,\omega^*_i,k} + 2k^2V_{\text{max}}^2\Pi^h_i u \|_{L^2(\omega^*_i)}^2.
\]

To proceed, we deduce from (4.29) that

\[
\|\Pi^h_i u\|_{L^2(\omega^*_i)} \leq \|u\|_{L^2(\omega^*_i)} + \varepsilon\left(\|\Pi^h_i u\|_{A,\omega^*_i,k} + \|u\|_{A,\omega^*_i,k}\right).
\]

Inserting (4.32) into (4.31) and taking \( \varepsilon \) sufficiently small such that \( \varepsilon < (2kV_{\text{max}})^{-1} \), we get the first part of (4.28).

REMARK 4.11. Using the same argument, we can see that if a similar condition as Assumption 4.9 is imposed on the global adjoint problem and the continuous problem (2.5) is uniquely solvable, then the discrete problem (4.1) is also uniquely solvable if \( h \) is sufficiently small.

Next we discuss the well-posedness of a saddle point problem arising from an elliptic BVP defined on the generalized harmonic space and its FE approximation.
LEMMA 4.12. Given $\mathcal{F} \in (H_D^1(\omega_1^*))'$, consider the problem of finding $u \in H_D^1(\omega_1^*)$ and $p \in H_D^1(\omega_1^*)$ such that

$$A_{\omega_1^*}(u, v) + B_{\omega_1^*}(v, p) = \mathcal{F}(v) \quad \forall v \in H_D^1(\omega_1^*),$$

(4.33)

$$B_{\omega_1^*}(u, \xi) = 0 \quad \forall \xi \in H_D^1(\omega_1^*).$$

Under Assumption 4.9, there exists a unique solution $(u, p)$ to (4.33) and

$$\|u\|_{A_{\omega_1^*}, k} + \|p\|_{A_{\omega_1^*}, k} \leq C\|\mathcal{F}\|(H_D^1(\omega_1^*))'.$$

(4.34)

Proof. By (2.17), we see that the sesquilinear form $A_{\omega_1^*}(\cdot, \cdot)$ is coercive on $H_B^1(\omega_1^*)$. For any $p \in H_D^1(\omega_1^*)$, we define $G_p \in (H_D^1(\omega_1^*))'$ such that $G_p(v) = A_{\omega_1^*}(v, p)$ and consider the problem of finding $w \in H_D^1(\omega_1^*)$ such that

$$B_{\omega_1^*}(w, v) = G_p(v) \quad \forall v \in H_D^1(\omega_1^*).$$

(4.35)

Using Assumption 4.9 and the Fredholm alternative, we see that the problem (4.35) has a unique solution $w \in H_D^1(\omega_1^*)$ with

$$\|w\|_{A_{\omega_1^*}, k} \leq C\|G_p\|(H_D^1(\omega_1^*))' \leq C\|p\|_{A_{\omega_1^*}, k}.$$

(4.36)

It follows that

$$\sup_{v \in H_D^1(\omega_1^*)} \frac{|B_{\omega_1^*}(v, p)|}{\|v\|_{A_{\omega_1^*}, k}} \geq C^{-1}\|p\|_{A_{\omega_1^*}, k}, \quad \forall p \in H_D^1(\omega_1^*).$$

(4.37)

Therefore, the inf-sup condition is verified and the result follows immediately from [4, Theorem 4.2.3].

LEMMA 4.13. Given $\mathcal{F} \in (H_D^1(\omega_1^*))'$, consider the discrete problem of finding $u_h \in U_{h,D}(\omega_1^*)$ and $p_h \in U_{h,D}(\omega_1^*)$ such that

$$A_{\omega_1^*}(u_h, v_h) + B_{\omega_1^*}(v_h, p_h) = \mathcal{F}(v_h) \quad \forall v_h \in U_{h,D}(\omega_1^*),$$

(4.38)

$$B_{\omega_1^*}(u_h, \xi_h) = 0 \quad \forall \xi_h \in U_{h,D}(\omega_1^*).$$

Under Assumption 4.9, there exists $h_0 > 0$, such that for any $0 < h < h_0$, the problem (4.38) has a unique solution $(u_h, p_h)$ with

$$\|u - u_h\|_{A_{\omega_1^*}, k} + \|p - p_h\|_{A_{\omega_1^*}, k} \leq C\left(\inf_{v_h \in U_{h,D}(\omega_1^*)} \|u - v_h\|_{A_{\omega_1^*}, k} + \inf_{q_h \in U_{h,D}(\omega_1^*)} \|p - q_h\|_{A_{\omega_1^*}, k}\right),$$

(4.39)

where $(u, p)$ denotes the solution of (4.33).

Proof. The coerciveness of $A_{\omega_1^*}(\cdot, \cdot)$ on $H_B^1(\omega_1^*)$ is implied by (4.3) and thus it suffices to prove the discrete inf-sup condition. For any $p_h \in U_{h,D}(\omega_1^*)$, we consider the discrete problem of finding $w_h \in U_{h,D}(\omega_1^*)$ such that

$$B_{\omega_1^*}(w_h, v_h) = A_{\omega_1^*}(v_h, p_h) \quad \forall v_h \in U_{h,D}(\omega_1^*).$$

(4.40)

To estimate $\|w_h\|_{A_{\omega_1^*}, k}$, we consider an auxiliary problem:

$$\text{Find } w_h \in H_D^1(\omega_1^*) \text{ such that } B_{\omega_1^*}(w_h, v) = A_{\omega_1^*}(v, p_h) \quad \forall v \in H_D^1(\omega_1^*).$$

(4.41)
By Assumption 4.9, the problem (4.41) has a unique solution \( w^h \in H^1_D(\omega^*_1) \) with
\[
\| w^h \|_{A, \omega^*_1, k} \leq C \| p_h \|_{A, \omega^*_1, k}.
\]
Note that \( B_{\omega^*_1}(w^h, v_h) = B_{\omega^*_1}(w, v) \) for any \( v_h \in U_{h,D}(\omega^*_1) \). Using Lemma 4.10 and (4.42), we see that there exists \( h_0 > 0 \), such that for any \( 0 < h < h_0 \),
\[
\| w_h \|_{A, \omega^*_1, k} \leq C \| w^h \|_{A, \omega^*_1, k} \leq C \| p_h \|_{A, \omega^*_1, k}.
\]
Now the discrete inf-sup condition follows from (4.40) and (4.43) and a similar argument as in (4.37) and the result is implied by the classical theory of mixed finite element methods (see, e.g., [4, Theorem 5.2.2]).

We conclude this section with a uniform approximation result for compact subsets of \( H^1_D(\omega) \) in \( U_{h,D}(\omega) \).

**Lemma 5.1 (Discrete Caccioppoli inequality).** Let \( S \) be a fixed compact subset of \( H^1_D(\omega) \). For any \( \varepsilon > 0 \), there exists \( h_0 = h_0(S, \varepsilon) \) such that if \( 0 < h \leq h_0 \), for each \( v \in S \), there exists \( v_h \in U_{h,D}(\omega) \) satisfying
\[
\| v - v_h \|_{H^1(\omega)} \leq \varepsilon.
\]

**5. Convergence analysis of the discrete MS-GFEM.** In this section, we first prove error estimates for the discrete MS-GFEM and then show that the eigenvalues of the discrete eigenproblems converge towards those of the continuous eigenproblems as \( h \to 0 \).

**5.1. Local and global error estimates.** The key to deriving a nearly exponential convergence rate for the local approximations of the discrete MS-GFEM is a discrete Caccioppoli inequality which is proved in detail below.

**Lemma 5.1 (Discrete Caccioppoli inequality).** Let \( \omega \subset \omega^* \) be subdomains of \( \Omega \) with \( \delta := \text{dist}(\omega, \partial \omega^* \setminus \partial \Omega) > 0 \). In addition, let \( \max_{K \cap \omega^* \neq \emptyset} h_K \leq \min\{\frac{1}{4}, \delta^{-1}\} \). Then, for each \( u_h \in H_{h,B}(\omega^*) \),
\[
\| u_h \|_{A, \omega} \leq C \delta^{-1} \| u_h \|_{L^2(\omega^*)} + \sqrt{2}k \max \| u_h \|_{L^2(\omega^*)},
\]
where \( C \) depends only on \( d \), the (spectral) bounds of the coefficients, and the shape regularity of the mesh.

**Proof.** Let \( \eta \in C^{\infty}(\omega^*) \) be a cut-off function satisfying \( 0 \leq \eta \leq 1 \) and
\[
\eta = 1 \quad \text{in} \quad \omega, \quad \eta = 0 \quad \text{on} \quad \partial \omega^*/\partial \Omega, \quad \| \eta \|_{W^{j,\infty}(\omega^*)} \leq C \delta^{-j}, \quad j = 1, 2.
\]
Using (A.3) with \( u = v = u_h \) gives that
\[
\| \eta u_h \|_{A, \omega^*} = \int_{\omega^*} (A\nabla \eta \cdot \nabla \eta) |u_h|^2 \, dx + \text{Re}[A_{\omega^*}(u_h, \eta^2 u_h)]
\]
\[
= \int_{\omega^*} (A\nabla \eta \cdot \nabla \eta + k^2 V^2 \eta^2) |u_h|^2 \, dx + \text{Re}[B_{\omega^*}(u_h, \eta^2 u_h)].
\]
Since \( u_h \in H_{h,B}(\omega^*) \), we see that \( B_{\omega^*}(u_h, I_h(\eta^2 u_h)) = 0 \) and thus
\[
B_{\omega^*}(u_h, \eta^2 u_h) = B_{\omega^*}(u_h, \eta^2 u_h - I_h(\eta^2 u_h))
\]
\[
= A_{\omega^*}(u_h, \eta^2 u_h - I_h(\eta^2 u_h)) + k^2 \int_{\omega^*} V^2 u_h(\eta^2 u_h - I_h(\eta^2 u_h)) \, dx
\]
\[- i k \int_{\partial \omega^* \cap \Gamma_R} \beta u_h(\eta^2 u_h - I_h(\eta^2 u_h)) \, ds.
\]
In what follows, we bound the RHS of (5.4) term by term. Using (4.16) and an inverse estimate (local to each $K$), we obtain

$$
\mathcal{A}_\omega(u_h, \eta^2 u_h - I_h(\eta^2 u_h))
\leq C \sum_{K \cap \omega^* \neq \emptyset} h_K \| \nabla u_h \|_{L^2(K)} \left( \delta^{-1} \| \nabla (\eta u_h) \|_{L^2(K)} + \delta^{-2} \| u_h \|_{L^2(K)} \right)
$$

(5.5)

\begin{align*}
&\leq C \delta^{-2} \sum_{K \cap \omega^* \neq \emptyset} \| u_h \|_{L^2(K)}^2 + \frac{a_{\min}}{6} \| \nabla (\eta u_h) \|_{L^2(K)}^2 \\
&\leq C \delta^{-2} \| u_h \|_{L^2(\omega^*)}^2 + \frac{1}{6} \| \eta u_h \|_{\mathcal{A}_\omega^*}^2
\end{align*}

Similarly, we can use (4.17) and the assumption that $kh_K \leq 1$ to get

$$
k^2 \int_{\omega^*} V^2 u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \, dx
\leq C k^2 \sum_{K \cap \omega^* \neq \emptyset} h_K^2 \| u_h \|_{L^2(K)} \left( \delta^{-1} \| \nabla (\eta u_h) \|_{L^2(K)} + \delta^{-2} \| u_h \|_{L^2(K)} \right)
$$

(5.6)

\begin{align*}
&\leq C \delta^{-2} \| u_h \|_{L^2(\omega^*)}^2 + \frac{1}{6} \| \eta u_h \|_{\mathcal{A}_\omega^*}^2.
\end{align*}

It remains to estimate the last term. Observe that

$$
\left| ik \int_{\partial \omega^* \cap \Gamma_R} \beta u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \, ds \right|
\leq k \| \beta \|_{L^\infty(\Gamma_R)} \sum_{K \cap \omega^* \neq \emptyset} \left| u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \right| ds.
$$

(5.7)

Using the multiplicative trace inequality (4.18) yields that

$$
\int_{\partial K} \left| u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \right| \, ds \leq \| u_h \|_{L^2(\partial K)} \| \eta^2 u_h - I_h(\eta^2 u_h) \|_{L^2(K)}
$$

(5.8)

\begin{align*}
&\leq C \left( h_K^{-1/2} \| u_h \|_{L^2(K)} + \| u_h \|_{L^2(K)}^{1/2} \| \nabla u_h \|_{L^2(K)}^{1/2} \right) \left( h_K^{-1/2} \| \eta^2 u_h - I_h(\eta^2 u_h) \|_{L^2(K)} + \| \eta^2 u_h - I_h(\eta^2 u_h) \|_{L^2(K)}^{1/2} \| \nabla (\eta^2 u_h - I_h(\eta^2 u_h)) \|_{L^2(K)}^{1/2} \right).
\end{align*}

Applying Lemma 4.6 to (5.8) and using a similar argument as in (5.5), it can be proved that

$$
\| \beta \|_{L^\infty(\Gamma_R)} \int_{\partial K} \left| u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \right| \, ds \leq h_K \left( C \delta^{-2} \| u_h \|_{L^2(K)}^2 + \frac{1}{6} \| \eta u_h \|_{\mathcal{A}_\omega^*}^2 \right).
$$

(5.9)

Inserting (5.9) into (5.7) and noting that $kh_K \leq 1$, we get

$$
\left| ik \int_{\partial \omega^* \cap \Gamma_R} V u_h (\eta^2 u_h - I_h(\eta^2 u_h)) \, ds \right| \leq C \delta^{-2} \| u_h \|_{L^2(\omega^*)}^2 + \frac{1}{6} \| \eta u_h \|_{\mathcal{A}_\omega^*}^2.
$$

(5.10)

Collecting the estimates (5.5), (5.6), and (5.10) and recalling (5.4), we arrive at

$$
| \mathcal{B}_{\omega^*}(u_h, \eta^2 u_h) | \leq C \delta^{-2} \| u_h \|_{L^2(\omega^*)}^2 + \frac{1}{2} \| \eta u_h \|_{\mathcal{A}_\omega^*}^2,
$$

(5.11)

which, combining with (5.2) and (5.3), gives (5.1).
COROLLARY 5.2. Let ω and ω* satisfy the same assumptions as in Lemma 5.1 and let h ≤ \frac{1}{4} δ, k−1. Assume that η ∈ W^{1,∞}(ω*) satisfying ||η||_{L^∞(ω*)} ≤ 1 and supp(η) ⊂ \overline{ω}. Then, for each uh ∈ H_{h,B}(ω*),

\[ ||I_h(ηuh)||_{A,ω^*,k} ≤ C(δ^{-1} + kV_{\max} + ||\nabla η||_{L^∞(ω*)})||uh||_{L^2(ω*)}, \]

where C depends on the bounds of the coefficients and the shape regularity of the mesh.

Proof. Using the stability of the interpolation operator and the assumption that ||η||_{L^∞(ω*)} ≤ 1, we have

\[ ||ηuh||_{A,ω^*,k} ≤ C||ηuh||_{A,ω^*,k} ≤ C(||ηuh||_{A,ω^*} + kV_{\max}||uh||_{L^2(ω*)}). \]

Next we use the triangle inequality, the assumptions on η, and Lemma 5.1 to obtain

\[ ||ηuh||_{A,ω^*} ≤ ||A^{1/2}uh\nabla η||_{L^2(ω^*)} + ||\eta||_{L^∞(ω^*)}||uh||_{A,ω} \]

\[ ≤ (δ^{-1} ||\nabla η||_{L^∞(ω^*)} + Cδ^{-1} + \sqrt{kV_{\max}})||uh||_{L^2(ω^*)}. \]

Inserting (5.14) into (5.13) completes the proof of (5.12).

Now we can give upper bounds for the local approximation errors. As before, we assume that ω and ω* are concentric cubes of sides length H and H*, respectively.

THEOREM 5.3. There exist n_0 > 0 and b > 0 independent of h and k, such that for any n > n_0, if h is sufficiently small, then

\[ d_{h,n}(ω, ω^*) ≤ e^{2σ}e^{-bn^{1/(d+1)}}e^{-ρ(H/H^*)bn^{1/(d+1)}}, \]

where σ = k(H^* - H)V_{\max}/(2a_{\max}^{1/2}) and ρ(s) = 1 + s \log(s)/(1 - s).

The proof of Theorem 5.3 follows the same lines as the proof of Theorem 3.1, by combining the discrete Caccioppoli inequality and a similar approximation result as Lemma 3.4 for the discrete generalized harmonic spaces, which can be proved using similar techniques and the min-max principle for eigenvalues. For the sake of space limitation, we omit the details here.

Before giving the global error estimates for the method, let us recall the stability constants \( \tilde{C}^{st}_{\text{stab}}(k) \) and \( \tilde{C}^{st}_i(k) \) defined in Assumption 4.3. Similar to (3.28), we define

\[ d_{h,max} = \max_{i=1,\cdots,M} d_{h,n_i}(ω_i, ω_i^*), \quad \tilde{C}^{st}_i(k) = \max_{i=1,\cdots,M} \tilde{C}^{st}_i(k). \]

The global approximation error of the discrete MS-GFEM is given in the following lemma, which can be proved using exactly the same technique as in Lemma 3.7.

LEMMA 5.4. Let \( u_h^* \) be the solution of the discrete problem (4.1) and let \( u_h^p \) and \( S_h(Ω) \) be the global particular function and the trial space of the discrete MS-GFEM. Then, there exists a \( ϕ_h \in u_h^p + S_h(Ω) \) such that

\[ ||u_h^* - ϕ_h||_{A,k} ≤ 2ζ\sqrt{ε}d_{h,max}(\tilde{C}^{st}_\text{stab}(k) + \sqrt{2}\tilde{C}^{st}_\text{max}(k))(||f||_{L^2(Ω)} + ||g||_{L^2(Γ_R)}). \]

Arguing as in the proof of Theorem 3.8 and Lemma 3.10, we can get a similar quasi-optimal convergence rate for the discrete method as Corollary 3.11. Before giving the result, we recall the constants \( ζ \) defined in (3.61) and \( H^*_{\text{stab}} \) defined in (3.28).
Theorem 5.5. Let \( u^G_h \) be the solution of the discrete problem (4.1) and \( u^G_i \) be the discrete MS-GFEM approximation. Suppose that
\[
d_{h,\text{max}} \leq (8\sqrt{2}kC_{\text{stab}}(k))^{-1}, \quad H_{\text{max}}^* \leq (12\sqrt{3}kC_Ia_{\text{min}}^{-1/2})^{-1},
\]
where \( C_I > 0 \) depends on the stability of the interpolation operator \( I_h \). Then,
\[
\|u^G_h - u^G_i\|_{A,k} \leq 2C_B \inf_{\varphi_h \in u^G_h + S_h(\Omega)} \|u^G_h - \varphi_h\|_{A,k}.
\]

5.2. Convergence of the eigenvalues. As we have seen in the preceding sections, the \( n \)-widths at the continuous and discrete levels are given by the square roots of the \( (n+1) \)-th eigenvalue of the continuous and discrete eigenproblems (2.27) and (4.7), respectively. In this subsection, we will prove the convergence of the discrete eigenvalues to the continuous ones as \( h \to 0 \) by proving the convergence of the discrete eigenvalues.

Let \( u^*_j \) be the solution of the discrete problem (4.1) and \( u^*_h \) be the solution of the discrete problem (4.7) such that for each \( u \in H_B(\omega^*) \), \( Tu \in H_B(\omega^*) \) satisfies
\[
A_{\omega^*}(Tu, v) = A_{\omega^*}(\chi u, \chi v), \quad \forall v \in H_B(\omega^*),
\]
and \( T_h = P_h^* P_h : H_B(\omega^*) \to H_B(\omega^*) \) such that for each \( u_h \in H_{h,B}(\omega^*) \), \( T_h u_h \in H_{h,B}(\omega^*) \) satisfies
\[
A_{\omega^*}(T_h u_h, v) = A_{\omega^*}(I_h(\chi u_h), I_h(\chi v_h)), \quad \forall v \in H_{h,B}(\omega^*).
\]

Theorem 5.6. The operators \( T \) and \( T_h \) (0 < \( h \leq 1 \)) are self-adjoint, positive, and compact operators. The continuous and discrete eigenproblems (2.27) and (4.7) can be formulated as the following spectral problems for the operators \( T \) and \( T_h \):
\[
u_j \in H_B(\omega^*), \quad Tu_j = \lambda_j u_j, \quad j = 1, 2, \ldots,
\]
\[
\lambda_1 \geq \lambda_2 \geq \cdots \lambda_j \geq \cdots, \quad \lim_{\lambda_j \to 0},
\]
and
\[
u_{h,j} \in H_{h,B}(\omega^*), \quad T_h u_{h,j} = \lambda_{h,j} u_{h,j}, \quad j = 1, 2, \ldots,
\]
\[
\lambda_{h,1} \geq \lambda_{h,2} \geq \cdots \lambda_{h,j} \geq \cdots, \quad \lim_{\lambda_{h,j} \to 0},
\]
where the eigenvalues are repeated according to their multiplicities. In order to prove convergence of the discrete eigenvalues, we use an abstract theoretical framework developed in [27] which requires some assumptions on the associated operators and function spaces.

Assumption 5.6. The operators \( T \) and \( T_h \) and the spaces \( H_B(\omega^*), H_{h,B}(\omega^*) \) satisfy the following conditions:

A1. There exist continuous linear operators \( R_h : H_B(\omega^*) \to H_{h,B}(\omega^*) \) satisfying
\[
\|R_h u\|_{A,\omega^*} \leq c_0 \|u\|_{A,\omega^*}, \quad \forall u \in H_B(\omega^*),
\]
where the constant \( c_0 \) is independent of \( h \); moreover, for any \( u, v \in H_B(\omega^*) \) and \( u_h, v_h \in H_{h,B}(\omega^*) \) satisfying
\[
\lim_{h \to 0} \|u_h - R_h u\|_{A,\omega^*} = 0, \quad \lim_{h \to 0} \|v_h - R_h v\|_{A,\omega^*} = 0,
\]
it holds that
\[
\lim_{h \to 0} A_{\omega^*}(u_h, v_h) = A_{\omega^*}(u, v).
\]
A2. The operators $T_h$ and $T$ are self-adjoint, positive, and compact, and the norms $\|T_h\| = \|T_h\|_{\mathcal{L}(H_h, B(\omega^*))}$ are uniformly bounded with respect to $h$.

A3. If $\psi_h \in H_{h,B}(\omega^*), \psi \in B(\omega^*)$ and

$$ \lim_{h \to 0} \|\psi_h - R_h \psi\|_{A,\omega^*} = 0, $$

then

$$ \lim_{h \to 0} \|T_h \psi_h - R_h T \psi\|_{A,\omega^*} = 0. $$

A4. For any sequence $\psi_h \in H_{h,B}(\omega^*)$ with $\sup_{h \in (0,1]} \|\psi_h\|_{A,\omega^*} < \infty$, there exists a subsequence $\psi_{h'}$ and a function $u \in B(\omega^*)$ such that

$$ \|T_{h'} \psi_{h'} - R_{h'} u\|_{A,\omega^*} \to 0 \quad \text{as} \quad h' \to 0. $$

By [27, Lemma 11.3 & Theorem 11.4], the following theorem holds true.

**Theorem 5.7.** Let $\{\lambda_j\}$ and $\lambda_{h,j}$ be the eigenvalues of problems (5.22) and (5.23), respectively. Assume that Assumption 5.6 holds true. Then, for each $j = 1, 2, \ldots , \lambda_{h,j} \to \lambda_j$ as $h \to 0$. Moreover, for sufficiently small $h$,

$$ |\lambda_{h,j} - \lambda_j| \leq 2 \sup_{u \in N(\lambda_j,T)} \|T_h R_h u - R_h T u\|_{A,\omega^*}, \quad j = 1, 2, \ldots , $$

where $N(\lambda_j,T)$ is the normalized eigenspace of $T$ corresponding to the eigenvalue $\lambda_j$:

$$ N(\lambda_j,T) = \{u \in B(\omega^*) : \|u\|_{A,\omega^*} = 1, \quad Tu = \lambda_j u\}. $$

By Theorem 5.7, in order to prove the convergence of the eigenvalues, it suffices to prove that conditions A1-A4 are satisfied. In fact, we have

**Theorem 5.8.** Under Assumption 4.9, conditions A1-A4 are satisfied.

**Proof.** The verification of conditions A1-A4 shares some similarities with that for positive definite problems in [28] and thus we omit some details of the proof which can be found in [28]. We start with the verification of condition A1. Let $h$ be sufficiently small. We define $R_h = \Pi_h|_{H_h(\omega^*)}$, where $\Pi_h$ is the projection defined in (4.25). It follows from the definition of $\Pi_h$ that $R_h u \in H_{h,B}(\omega^*)$ for any $u \in B(\omega^*)$. By Lemma 4.10 and (2.17), we see that (5.24) holds and that

$$ \|R_h u - u\|_{A,\omega^*} \to 0 \quad \text{as} \quad h \to 0. $$

Therefore, the first part of condition A1 is proved and the other part can be proved by using (5.32) and the same technique used in the proof of Theorem 4.2 in [28]. Condition A2 follows directly from the definition of operators $T$ and $T_h$. To verify condition A3, we first extend the definition of $T_h$ to all of $H_{h,B}^1(\omega^*)$ and observe that

$$ \|T_h \psi_h - R_h T \psi\|_{A,\omega^*} \leq \|T_h (\psi_h - R_h \psi)\|_{A,\omega^*} + \|T_h (R_h \psi - \psi)\|_{A,\omega^*}, $$

$$ + \|R_h T \psi - T \psi\|_{A,\omega^*} + \|T_h \psi - T \psi\|_{A,\omega^*}. $$

The uniform boundedness of $\|T_h\|$, (5.27), and (5.32) yield that

$$ \lim_{h \to 0} (\|T_h (\psi_h - R_h \psi)\|_{A,\omega^*} + \|T_h (R_h \psi - \psi)\|_{A,\omega^*} + \|R_h T \psi - T \psi\|_{A,\omega^*}) = 0. $$
It remains to show that \( \|T_h \psi - T \psi\|_{A,\omega^*} \to 0 \) as \( h \to 0 \). To this end, we consider an auxiliary problem of finding \( T_h \psi \in H_{h,B}(\omega^*) \) such that

\[
(5.35) \quad A_{\omega^*}(T_h \psi, v_h) = A_{\omega,k}(\chi \psi, \chi v_h), \quad \forall v_h \in H_{h,B}(\omega^*).
\]

Rewriting problems (5.20) and (5.35) as saddle point problems as in [28] and using Lemmas 4.12 and 4.13, we see that

\[
(5.36) \quad \|T_h \psi - T \psi\|_{A,\omega^*} \to 0 \quad \text{as} \quad h \to 0.
\]

Next we will prove that \( \|T_h \psi - T \psi\|_{A,\omega^*} \to 0 \). Let \( e_h = T_h \psi - T \psi \). Subtracting (5.21) from (5.35) and using (2.17), we see that \( e_h \) satisfies

\[
(5.37) \quad \|e_h\|^2_{A,\omega^*} \leq C \|\chi \psi - I_h(\chi \psi)\|^2_{A,\omega} + C\|\chi e_h - I_h(\chi e_h)\|_{A,\omega}.
\]

Moreover, there exists \( C_0 > 0 \) such that \( \|e_h\|_{H^1(\omega^*)} \leq C_0 \) for all \( h \). Given \( C_1 > 0 \), we define a subset of \( H^1_D(\omega^*) \):

\[
(5.38) \quad S = \{ \chi u : u \in H^1_D(\omega^*), \quad \|u\|_{H^1(\omega^*)} \leq C_0, \quad \|u\|_{A,\omega} \leq C_1\|u\|_{L^2(\omega^*)} \}.
\]

Since \( e_h \in H_{h,B}(\omega^*) \), the discrete Caccioppoli inequality (5.1) implies that there exists \( C_1 > 0 \) such that \( \chi e_h \in S \) for all \( h \). Furthermore, it follows from (5.14) and Rellich’s theorem that \( S \) is a compact subset in \( H^1_D(\omega) \). Hence, we can use Lemma 4.14 to assert that for any \( \varepsilon > 0 \), there exists \( h_0 > 0 \), such that if \( 0 < h < h_0 \), there exists \( v_h \in U_{h,D1}(\omega^*) \) satisfying \( \|\chi e_h - v_h\|_{H^1(\omega)} \leq C \varepsilon \), and thus

\[
(5.39) \quad \|\chi e_h - I_h(\chi e_h)\|_{A,\omega} = \|\chi e_h - v_h\|_{A,\omega} \leq C \varepsilon,
\]

where we have used that fact that \( I_h v_h = v_h \). Combining (5.37) and (5.39) shows that

\[ \|T_h \psi - T \psi\|_{A,\omega^*} \to 0 \]

which, together with (5.33), (5.34), and (5.36), give (5.28). Therefore, condition A3 is verified.

Finally, we check the validity of condition A4. Let \( \{\psi_h\} \) be a sequence satisfying \( \psi_h \in H_{h,B}(\omega^*) \) for each \( h \in (0, 1] \) and \( \sup_{h \in (0, 1]} \|\psi_h\|_{A,\omega^*} < \infty \). By (4.3), we see that \( \sup_{h \in (0, 1]} \|\psi_h\|_{A,\omega^*,k} < \infty \). Therefore, we can extract a subsequence \( \{\psi_{h'}\} \) such that \( \{\psi_{h'}\} \) converges weakly (in \( H^1_D(\omega^*) \)) to some \( \psi \in H^1_D(\omega^*) \). Applying a similar argument as in the proof of (4.3) yields that \( \psi \in H_B(\omega^*) \). Define \( u = T \psi \in H_B(\omega^*) \).

It can be proved that \( u \) satisfies (5.29) by the same argument as in the proof of Theorem 4.2 in [28] using the discrete Caccioppoli inequality. Therefore, conditions A1–A4 are verified.

6. Numerical experiments. In this section, we provide some numerical results to support the theoretical analysis and to demonstrate the effectiveness of the method.

6.1. Classical Helmholtz example. First we consider (2.1) on the unit square \( \Omega = (0, 1)^2 \) with \( \Gamma_R = \partial \Omega \), \( A(x) = I \), \( V(x) = 1 \), and \( f(x) = 0 \). The boundary data \( g \) is chosen such that the problem admits the plane-wave solution \( u^* = \exp(ikx \cdot \cdot \cdot 0.8) \).

A FE mesh of the computational domain is defined on a uniform Cartesian grid with \( h = 10^{-3} \). To implement the MS-GFEM, we first split the domain into \( M = m^2 \) non-overlapping subdomains resolved by the mesh, and then extend each subdomain by 2 layers of mesh elements to create an overlapping decomposition \( \{\omega_i\}_{i=1}^M \) of \( \Omega \).

Each overlapping subdomain \( \omega_i \) is further extended by \( \ell \) layers of mesh elements.
to create an oversampling domain $\omega^*_i$ on which the local problems are solved. Let $n_{loc}$ be the number of eigenvectors selected in each subdomain for building the local approximation space. As in [28], by introducing a Lagrange multiplier to relax the generalized harmonic constraint, the local eigenproblems (4.7) are solved in mixed formulation: Find $\lambda_h \in \mathbb{R}$, $\phi_h \in U_{h,D}(\omega^*_i)$, and $p_h \in U_{h,DI}(\omega^*_i)$ such that

$$
A_{\omega^*_i}(\phi_h, v_h) + B_{\omega^*_i}(v_h, p_h) = \lambda_h^{-1} A_{\omega^*_i,k}(I_h(\chi_i\phi_h), I_h(\chi_i v_h)) \quad \forall v_h \in U_{h,D}(\omega^*_i),
$$

$$
B_{\omega^*_i}(\phi_h, \xi_h) = 0 \quad \forall \xi_h \in U_{h,DI}(\omega^*_i).
$$

Let $u^e_h$ and $u^G_h$ be the standard FE approximation and the (discrete) MS-GFEM approximation of the problem, respectively. Denote by $\text{error}^e$ (error) the relative error between $u^G_h$ and the exact solution $u^e$ (resp, $u^e_h$), i.e.,

$$
\text{error}^e := \frac{\|u^e - u^G_h\|_{A,k}}{\|u^e\|_{A,k}}, \quad \text{error} := \frac{\|u^e_h - u^G_h\|_{A,k}}{\|u^e_h\|_{A,k}}.
$$

We test our method for two wavenumbers, $k = 100$ and $k = 200$. First, we study the decay of the errors with respect to the dimension of the local spaces for different oversampling size with $k = 100$. Figure 2 (left) displays the errors between the discrete MS-GFEM approximations and the exact solution. The horizontal asymptote arises when $n_{loc}$ is sufficiently large and the errors are dominated by the fine-scale FE approximation error. The errors between the discrete MS-GFEM approximations and the standard FE approximation are shown in Figure 2 (right) and we can clearly see that they decay nearly exponentially with respect to $n_{loc}$, agreeing well with our theoretical analysis. In addition, it is visible that the method works even without oversampling, i.e., $\ell = 0$.

Next we vary the oversampling size $\ell$ and illustrate the decay of the errors with respect to $H/H^*$ for different dimensions of local spaces and different wavenumbers in Figure 3, where $H$ and $H^*$ represent the sizes of the subdomains and the oversampling domains, respectively. We can see that for $k = 100$, the errors decay nearly exponentially with respect to $H/H^*$ for all different dimensions of local spaces and that for $k = 200$ and a large $n_{loc}$, the errors decay similarly as in the $k = 100$ case. However, for $k = 200$ and a small $n_{loc}$ (15 or 20), the errors first decrease and then stagnate with increasing $H^*$. This verifies the presence of the resonance effect described in Remark 3.3.
6.2. A scattering problem. We consider the following heterogeneous scattering problem on the unit square $\Omega = (0, 1)^2$ with $\Gamma_R = \partial \Omega$: $V(x) = 1$, $\beta(x) = 1$, $f(x) = 0$ and the coefficient $A(x)$ as illustrated in Figure 4 (left). The incident wave is taken as $u^{inc} = \exp(i \vec{k} \cdot x)$ with $\vec{k} = k(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and on $\Gamma_R$ we use first-order absorbing boundary conditions: $g = n \cdot \nabla u^{inc} - i ku^{inc}$.

The computational settings are similar to those in subsection 6.1 except that the fine-scale FE mesh size is $h = 1/1400$. The modulus of the fine-scale FE solution $u_h^e$ with $k = 130$ is displayed in Figure 4 (right). Since the exact solution of the problem is not available, $u_h^e$ is used as the reference solution for computing the errors. First we compare the decay rates of the errors with respect to $n_{loc}$ for $k = 130$ and $k = 260$ in Figure 5. It can be observed that with a fixed oversampling size, the decay rates with respect to $n_{loc}$ are roughly the same for two different wavenumbers, which confirms the theoretical analysis. In addition, as shown in Figure 6 (left), we see that for $k = 130$, the errors decay nearly exponentially with respect to $H/H^*$ and no stagnation phenomenon is observed. Finally, we illustrate the decay of the errors with respect to $m$ (the number of subdomains in one direction) for a fixed oversampling size in Figure 6 (right). One can observe that the errors generally decay dramatically with increasing $m$. We note that in this case, the quantity $H/H^*$ decreases with
increasing $m$ since the oversampling size $\ell$ is fixed.

**6.3. Marmousi problem.** In the last example, we consider the benchmark Marmousi model [41]. The problem is posed on the domain $(0, 9 \text{ km}) \times (-3 \text{ km}, 0)$ and a point source is placed near the top boundary. Homogeneous Dirichlet and impedance boundary conditions are prescribed on the top surface and on the remaining three sides, respectively. Moreover, $A(x) = I$, $\beta(x) = kV(x)$, and the velocity field $1/V(x)$ is depicted in Figure 7 (top).

The frequency for the test is taken as 20 Hz ($k = 40\pi$) and we use 10 points per (minimal) wavelength for the FE discretization, corresponding to $h = 7.5\text{m}$. In view of the geometry of the computational domain, the numbers of the subdomains in the $y$ and $x$ directions are denoted by $m$ and $3m$, respectively. Otherwise, the computational setting is the same as in the previous subsections. The real part of the discrete MS-GFEM approximation computed using about 23000 local basis functions ($m = 16$, $\ell = 9$, $n_{\text{loc}} = 30$) with a relative error less than $10^{-3}$ is plotted in Figure 7 (bottom).

First we display the decay of the eigenvalues in three subdomains in a semi-logarithmic scale in Figure 8 (left): one in the interior, one near the Dirichlet boundary and one near the impedance boundary. As predicted by the theoretical analysis, the eigenvalues in all the three subdomains decay nearly exponentially. Next we study the decay of the errors with respect to $n_{\text{loc}}$ and the results are shown in Figure 8.
Fig. 7. Marmousi problem (subsection 6.3): The velocity field of the Marmousi model (top) and the real part of $u_G^h$ (bottom) computed with $m = 16$, $\ell = 9$, and $n_{loc} = 30$.

Fig. 8. Marmousi problem (subsection 6.3): Plot of $\lambda_{h,n}$ against $n$ (left); plot of $\log_2(\text{error})$ against $n_{loc}$ (right).

(right). We see that for this realistic model with a heterogeneous velocity field, the errors of our method decay nearly exponentially with increasing $n_{loc}$ just as we have observed in the previous examples. Finally, we show the influence of the size of the subdomains by plotting the decay of the errors with respect to $H/H^*$ in Figure 9. It can be clearly seen that decreasing the size of the subdomains (increasing $m$) can significantly alleviate the stagnation of the errors with increasing oversampling size for a small $n_{loc}$. In fact, as noted in Remark 3.3, if $H \sim H^* \sim O(k^{-1})$, the method for Helmholtz problems behaves similarly to that for positive definite problems and the resonance effect disappears.

7. Summary and future work. We have performed a systematic investigation of a multiscale spectral GFEM with novel local approximation spaces for solving heterogeneous Helmholtz problems at both the continuous and the discrete level,
providing a comprehensive analysis. Wavenumber explicit error estimates for the local approximations are obtained and the influence of the wavenumber on the error of the method in the preasymptotic regime is theoretically and numerically investigated. This goes well beyond most previous studies. Furthermore, our method provides a unified mathematical framework within which Trefftz-type discretization schemes for heterogeneous Helmholtz problems and efficient solvers for discrete Helmholtz problems can be developed and analysed.

There are two important issues of the method that remain to be addressed. The first is adaptivity. The new formalism readily facilitates an adaptive choice of the mesh size in the discretization of the local problems and of the number of eigenvectors to be included in each of the local approximation spaces. The second issue we aim to address is a discontinuous formulation. In the MS-GFEM, the local basis functions are pasted together by a partition of unity to form the continuous trial and test functions. An alternative is to use a discontinuous formulation in which the continuity of the numerical solution across neighbouring subdomains is maintained in a weak sense, just as in the ultraweak variational formulation [9] or in the discontinuous enrichment method [17]. In this setting, we then expect the resulting linear system of the coarse problem to be better conditioned.

Appendix A. Proof of Lemma 2.6.

Proof. For any \( u, v \in H^1_D(\omega^*) \), a direct calculation shows that

\[
\int_{\omega^*} A \nabla (\eta u) \cdot \nabla (\eta v) \, dx = \int_{\omega^*} (A \nabla \eta \cdot \nabla \eta) u v \, dx - \int_{\omega^*} (A \nabla u \cdot \nabla \eta) \eta v \, dx
\]

\[
+ \int_{\omega^*} (A \nabla \eta \cdot \nabla v) \eta u \, dx + \int_{\omega^*} A \nabla u \cdot \nabla (\eta^2 v) \, dx.
\]

(A.1)

Exchanging \( u \) and \( v \) in (A.1), it follows that

\[
\int_{\omega^*} A \nabla (\eta v) \cdot \nabla (\eta u) \, dx = \int_{\omega^*} (A \nabla \eta \cdot \nabla \eta) u v \, dx - \int_{\omega^*} (A \nabla v \cdot \nabla \eta) \eta u \, dx
\]

\[
+ \int_{\omega^*} (A \nabla \eta \cdot \nabla u) \eta v \, dx + \int_{\omega^*} A \nabla v \cdot \nabla (\eta^2 u) \, dx.
\]

(A.2)
Adding (A.1) and (A.2) together and using the symmetry of $A$, we get

$$
\int_{\omega^*_H} A \nabla (\eta u) \cdot \nabla (\eta \bar{v}) \, dx = \int_{\omega^*_H} (A \nabla \eta \cdot \nabla \eta) u \bar{v} \, dx 
$$

(A.3)

$$
\frac{1}{2} \left( \int_{\omega^*_H} A \nabla u \cdot \nabla (\eta^2 \bar{v}) \, dx + \int_{\omega^*_H} A \nabla \bar{v} \cdot \nabla (\eta^2 u) \, dx \right).
$$

By the assumptions on $\eta$, we see that for any $u, v \in H^1_D(\omega^*_H)$, $\eta^2 u, \eta^2 v \in H^1_D(\omega^*_H)$. If, in addition, $u, v \in H_B(\omega^*_H)$, then we have

$$
B_{\omega^*_H}(u, \eta^2 v) = 0, \quad B_{\omega^*_H}(v, \eta^2 u) = 0.
$$

Therefore,

$$
\int_{\omega^*_H} A \nabla u \cdot \nabla (\eta^2 \bar{v}) \, dx = k^2 \int_{\omega^*_H} \eta^2 V^2 u \bar{v} \, dx + ik \int_{\partial \omega^*_H \cap V_R} \eta^2 \beta u \bar{v} \, ds,
$$

$$
\int_{\omega^*_H} A \nabla \bar{v} \cdot \nabla (\eta^2 u) \, dx = k^2 \int_{\omega^*_H} \eta^2 V^2 u \bar{v} \, dx - ik \int_{\partial \omega^*_H \cap V_R} \eta^2 \beta u \bar{v} \, ds.
$$

Inserting (A.5) into (A.3) yields (2.22) and the inequality (2.23) follows.

**Appendix B. Proof of (4.3).**

**Proof.** It is easy to see that for any $h > 0$, the $H^1$ seminorm is a norm on $H_{h,B}(\omega^*_H)$. Since all norms are equivalent in a finite-dimensional vector space, there exist $C_h > 0$ depending on $h$ such that

$$
\| u_h \|_{L^2(\omega^*_H)} \leq C_h \| \nabla u_h \|_{L^2(\omega^*_H)}, \quad \forall u_h \in H_{h,B}(\omega^*_H).
$$

(B.1)

It suffices to show that $\limsup_{h \to 0} C_h < +\infty$. If it doesn’t hold, then there exists a sequence $\{u_{h_n}\}_{n=1}^\infty$ with $h_n \to 0$ as $n \to \infty$, such that $\| u_{h_n} \|_{L^2(\omega^*_H)} = 1$ and $\| \nabla u_{h_n} \|_{L^2(\omega^*_H)} \leq 1/n$ hold for any $n \geq 1$. We can find a subsequence, still denoted by $\{u_{h_n}\}_{n=1}^\infty$, and a function $u_0 \in H^1_D(\omega^*_H)$, such that $u_{h_n} \to u_0$ weakly in $H^1_D(\omega^*_H)$. Since $H^1(\omega^*_H)$ is compactly embedded into $L^2(\omega^*_H)$, we see that $u_{h_n} \to u_0$ strongly in $L^2(\omega^*_H)$. It follows that $\| u_0 \|_{L^2(\omega^*_H)} = 1$ and $\nabla u_0 = 0$. Therefore, $u_0$ is a constant.

Next, we will show that $u_0 \in H_B(\omega^*_H)$. For any fixed $v \in H^1_D(\omega^*_H)$,

$$
B_{\omega^*_H}(u_0, v) = B_{\omega^*_H}(u_0 - u_{h_n}, v) + B_{\omega^*_H}(u_{h_n}, I_{h_n} v) + B_{\omega^*_H}(u_{h_n}, v - I_{h_n} v),
$$

where $I_{h_n}$ is the standard Lagrange interpolation operator. Since $\{u_{h_n}\}$ converges weakly to $u_0$ in $H^1_D(\omega^*_H)$, we have

$$
A_{\omega^*_H,k}(u_0 - u_{h_n}, v) \to 0 \quad \text{as } n \to \infty.
$$

(B.2)

Using the compact embedding of $H^1(\omega^*_H)$ into $L^2(\omega^*_H)$ and $L^2(\partial \omega^*_H)$, we see that

$$
\| u_0 - u_{h_n} \|_{L^2(\omega^*_H)} \to 0, \quad \| u_0 - u_{h_n} \|_{L^2(\partial \omega^*_H)} \to 0 \quad \text{as } n \to \infty.
$$

(B.3)

Combining (B.3) and (B.4) implies that $B_{\omega^*_H}(u_0 - u_{h_n}, v) \to 0$ as $n \to \infty$. Moreover, since $u_{h_n} \in H_{h,B}(\omega^*_H)$ and $I_{h_n} v \in \bar{U}_{h,n,D1}(\omega^*_H)$, the second term $B_{\omega^*_H}(u_{h_n}, I_{h_n} v)$ vanishes. Finally, in view of the boundedness of the sequence $\{u_{h_n}\}$, we conclude

$$
|B_{\omega^*_H}(u_{h_n}, v - I_{h_n} v)| \leq C \| v - I_{h_n} v \|_{A_{\omega^*_H,k}} \to 0 \quad \text{as } n \to \infty.
$$

(B.5)

Making $n \to \infty$ in (B.2) yields that $B_{\omega^*_H}(u_0, v) = 0$ for any $v \in H^1_D(\omega^*_H)$. Therefore, $u_0 \in H_B(\omega^*_H)$. Since $u_0$ is a constant, we see that $u_0 \equiv 0$, which contradicts with the fact that $\| u_0 \|_{L^2(\omega^*_H)} = 1$. 

\[\square\]
REFERENCES

[1] I. Babuška and R. Lipton, Optimal local approximation spaces for generalized finite element methods with application to multiscale problems, Multiscale Modeling & Simulation, 9 (2011), pp. 373–406, https://doi.org/10.1137/100791051.

[2] I. Babuška, R. Lipton, P. Sinz, and M. Stuehner, Multiscale-Spectral GFEM and optimal oversampling, Computer Methods in Applied Mechanics and Engineering, 364 (2020), p. 112960, https://doi.org/10.1016/j.cma.2020.112960.

[3] I. Babuška and J. M. Melenk, The partition of unity method, International journal for numerical methods in engineering, 40 (1997), pp. 727–758, https://doi.org/10.1002/(SICI)1097-0207(19970228)40:4<727::AID-NME66>3.0.CO;2-N.

[4] D. Boffi, F. Brezzi, and M. Fortin, Mixed finite element methods and applications, Springer-Verlag, Berlin, 2013.

[5] N. Bootland, V. Dolean, P. Jolivet, and P.-H. Tournier, A comparison of coarse spaces for Helmholtz problems in the high frequency regime, Computers & Mathematics with Applications, 98 (2021), pp. 239–253, https://doi.org/10.1016/j.camwa.2021.07.011.

[6] S. C. Brenner, L. R. Scott, and L. R. Scott, The mathematical theory of finite element methods, vol. 3, Springer, 2008.

[7] D. L. Brown, D. Gallistl, and D. Peterseim, Multiscale Petrov-Galerkin method for high-frequency heterogeneous Helmholtz equations, in Meshfree methods for partial differential equations VIII, Springer, 2017, pp. 85–115.

[8] L.-Q. Cao, J.-Z. Cui, and D.-C. Zhu, Multiscale asymptotic analysis and numerical simulation for the second order Helmholtz problem with rapidly oscillating coefficients over general convex domains, SIAM Journal on Numerical Analysis, 40 (2002), pp. 543–577, https://doi.org/10.1137/S0036142900376110.

[9] O. Cessenat and B. Despres, Application of an ultra weak variational formulation of elliptic pdes to the two-dimensional Helmholtz problem, SIAM journal on numerical analysis, 35 (1998), pp. 255–299, https://doi.org/10.1137/S0036142995285873.

[10] S. N. Chandler-Wilde, E. A. Spence, A. Gibbs, and V. P. Smyshlyaev, High-frequency bounds for the Helmholtz equation under parabolic trapping and applications in numerical analysis, SIAM Journal on Mathematical Analysis, 52 (2020), pp. 845–893, https://doi.org/10.1137/18M1234916.

[11] T. Chaumont-Frelet and F. Valentin, A multiscale hybrid-mixed method for the Helmholtz equation in heterogeneous domains, SIAM Journal on Numerical Analysis, 58 (2020), pp. 1029–1067.

[12] Y. Chen, T. Y. Hou, and Y. Wang, Exponentially convergent multiscale methods for high frequency heterogeneous Helmholtz equations, arXiv preprint arXiv:2105.04080, (2021).

[13] R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. I, Interscience Publishers, New York, 1953.

[14] A. Deslaur, J. Guzman, and A. Schatz, Local energy estimates for the finite element method on sharply varying grids, Mathematics of computation, 80 (2011), pp. 1–9, https://doi.org/10.1090/S0025-5718-2010-02353-1.

[15] O. G. Ernst and M. J. Gander, Why it is difficult to solve Helmholtz problems with classical iterative methods, Numerical analysis of multiscale problems, (2012), pp. 325–363.

[16] S. Estebanzy and J. M. Melenk, On stability of discretizations of the Helmholtz equation, in Numerical analysis of multiscale problems, Springer, 2012, pp. 285–324.

[17] C. Farhat, I. Harari, and U. Hetmaniuk, A discontinuous galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime, Computer methods in applied mechanics and engineering, 192 (2003), pp. 1389–1419, https://doi.org/10.1016/S0045-7825(02)00646-1.

[18] S. Fu, E. T. Chung, and G. Li, An Edge Multiscale Interior Penalty Discontinuous Galerkin method for heterogeneous Helmholtz problems with large varying wavenumber, Journal of Computational Physics, 441 (2021), p. 110387, https://doi.org/10.1016/j.jcp.2021.110387.

[19] S. Fu and K. Gao, A fast solver for the Helmholtz equation based on the generalized multiscale finite-element method, Geophysical Journal International, 211 (2017), pp. 797–813, https://doi.org/10.1093/gji/ggx343.

[20] S. Fu, G. Li, R. Craster, and S. Guenneau, Wavelet-based edge multiscale finite element method for helmholtz problems in perforated domains, Multiscale Modeling & Simulation, 19 (2021), pp. 1684–1709, https://doi.org/10.1137/19M1267180.

[21] M. J. Gander, F. Magoules, and F. Nataf, Optimized Schwarz methods without overlap for the Helmholtz equation, SIAM Journal on Scientific Computing, 24 (2002), pp. 38–60, https://doi.org/10.1137/S1064827501387012.
[22] M. J. Gander and H. Zhang, A class of iterative solvers for the Helmholtz equation: Factorizations, sweeping preconditioners, source transfer, single layer potentials, polarized traces, and optimized Schwarz methods, Siam Review, 61 (2019), pp. 3–76, https://doi.org/10.1137/16M109781X.

[23] I. Graham and S. Sauter, Stability and finite element error analysis for the Helmholtz equation with variable coefficients, Mathematics of Computation, 89 (2020), pp. 105–138, https://doi.org/10.1090/mcom/3457.

[24] M. Hauck and D. Peterseim, Multi-resolution localized orthogonal decomposition for Helmholtz problems, arXiv preprint arXiv:2104.11190, (2021).

[25] R. Hiptmair, A. Moiola, and I. Perugia, Plane wave discontinuous Galerkin methods for the 2d Helmholtz equation: analysis of the p-version, SIAM Journal on Numerical Analysis, 49 (2011), pp. 264–284, https://doi.org/10.1137/090761057.

[26] R. Hiptmair, A. Moiola, and I. Perugia, A survey of Trefftz methods for the Helmholtz equation, in Building bridges: connections and challenges in modern approaches to numerical partial differential equations, Springer, 2016, pp. 237–279.

[27] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, Homogenization of differential operators and integral functionals, Springer Science & Business Media, 2012.

[28] C. Ma and R. Scheichl, Error estimates for fully discrete generalized FEMs with locally optimal spectral approximations, arXiv preprint arXiv:2107.09988, (2021).

[29] C. Ma, R. Scheichl, and T. Dodwell, Novel design and analysis of generalized fe methods based on locally optimal spectral approximations, arXiv preprint arXiv:2103.09545, (2021).

[30] J. Melenk, On generalized finite element methods. Ph.D. thesis, Department of Mathematics, University of Maryland, 1995.

[31] J. M. Melenk and S. Sauter, Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation, SIAM Journal on Numerical Analysis, 49 (2011), pp. 1210–1243, https://doi.org/10.1137/090776202.

[32] P. Monk, Finite element methods for Maxwell’s equations, Oxford University Press, 2003.

[33] P. Monk and D.-Q. Wang, A least-squares method for the Helmholtz equation, Computer Methods in Applied Mechanics and Engineering, 175 (1999), pp. 121–136, https://doi.org/10.1016/S0045-7825(98)00326-0.

[34] J. Necas, Direct methods in the theory of elliptic equations, Springer Science & Business Media, Berlin, 2011.

[35] M. Ohlberger and B. Verfurth, A new heterogeneous multiscale method for the Helmholtz equation with high contrast, Multiscale Modeling & Simulation, 16 (2018), pp. 385–411, https://doi.org/10.1137/16M1108820.

[36] D. Peterseim, Eliminating the pollution effect in Helmholtz problems by local subscale correction, Mathematics of Computation, 86 (2017), pp. 1005–1036, https://doi.org/10.1090/mcom/3156.

[37] D. Peterseim and B. Verfürth, Computational high frequency scattering from high-contrast heterogeneous media, Mathematics of Computation, 89 (2020), pp. 2649–2674, https://doi.org/10.1090/mcom/3529.

[38] A. Pinkus, n-widths in Approximation Theory, Springer-Verlag, Berlin, 1985.

[39] A. Schatz and J. Wang, Some new error estimates for Ritz–Galerkin methods with minimal regularity assumptions, Mathematics of computation, 65 (1996), pp. 19–27, https://doi.org/10.1090/S0025-5718-96-00649-7.

[40] T. Strouboulis, I. Babuška, and R. Hidajat, The generalized finite element method for Helmholtz equation: theory, computation, and open problems, Computer Methods in Applied Mechanics and Engineering, 195 (2006), pp. 4711–4731, https://doi.org/10.1016/j.cma.2005.09.019.

[41] R. Versteeg, The Marmousi experience: Velocity model determination on a synthetic complex data set, The Leading Edge, 13 (1994), pp. 927–936.

[42] L. Wahlbin, Superconvergence in Galerkin finite element methods, Springer-Verlag, Berlin, 2006.