An Art and Technology Approach to Actively Engage Students in the Mathematics of the Regular Polyhedra

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Abstract
The goal of the present work is two folded: At first the author demonstrates a step-by-step problem solving approach, to prove the (convex) Regular Polyhedra Theorem using Zome tools. Then the geometric constructions of some patterns for the decoration of solids using utilities such as the Geometer’s Sketchpad will be presented.

Keywords: Regular Polyhedra, Convex Solids, Schläfli Symbols, Fractal Geometry, Zome tools.

1 Introduction

The word polyhedron comes from the Greek, meaning an object with many faces. The faces of a polyhedron are the polygons from which the polyhedron has been made. The edges of a polyhedron are the sides of its faces. The vertices of a polyhedron are the vertices of its faces. Polyhedra are involved in architecture, art, and nature. The shapes of many molecules and crystals are closely related to polyhedra, such as the regular polyhedra, pyramids, and prisms. The pyramids of Giza demonstrate an impressive level of the geometry of polyhedra which was known to the Egyptians of those times. However, our knowledge of their written mathematics is limited to two papyri, Rhind Papyrus and Moscow Papyrus. Problem number fourteen of the Moscow Papyrus, written in approximately 1890 BC, suggests a formula to compute the volume of a truncated square pyramid. Later on, in Greece, Archimedes documented that Democritus, the Greek philosopher before him who lived during the end of the fifth century BC, knew how to compute the volume of the pyramid. The best-known polyhedra that have connected numerous disciplines such as astronomy, philosophy, and art through the centuries are the regular polyhedra. They are known as the Platonic solids. There are five regular polyhedra. By a regular polyhedron we mean a polyhedron with the properties that:

(a) All its faces are congruent regular polygons.
(b) The arrangements of polygons about the vertices are all alike.

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The main goal of the present work is to first demonstrate a step-by-step method, based on Pólya’s heuristic approach [5], to prove the (convex) Regular Polyhedra Theorem using Zome tools. And then presents the geometric constructions of some patterns for the decoration of solids using utilities such as the Geometer’s Sketchpad.

2 How to Prove the Regular Polyhedra Theorem using Zome Tools

The Regular Polyhedra Theorem for convex solids can be written as follows:

**Theorem 2.1. (The Regular Polyhedra Theorem):** The number of (convex) regular polyhedra is five. Three of them have triangular faces. One has square faces. One has pentagonal faces.

The following “problem solving” activity to prove Theorem 2.1 using the exhaustion approach begins with the construction of a square and then a cube as practice for students to get familiar with Zome tools. There are many available tools in the market but we use Zome tools because students see vertices (balls) and edges (struts) more comfortably in a model using this manipulative device. Moreover, the holes on the balls are arranged symmetrically in a way that the students can understand relationship between the number of edges of a face and the number of edges emanating from a single vertex in a particular regular polyhedron. Zome tools and a companion book, *Zome Geometry* [2] provide many hands-on activities and instructions for instructors who wish to give more exposure to their students in solid constructions. Zome system consists of two types of objects: struts and balls. Zome struts come in different colors and sizes. Each color indicates a specific polygon for the cross section of a strut. The blue struts, $B$, have rectangular bases. The yellow struts, $Y$, have triangular bases. The red, $R$, green, $G$, and green-blue, $GB$, struts have pentagonal bases. Zome balls are designed with appropriate holes in specific locations for all these different struts [8].

![Figure 1: (Left) Zome struts and a ball with its symmetrical holes. (Right) Making solids](image)

Zome struts come in different sizes. We use subscripts to indicate the size of a strut. For example, $B_1$ indicates the small blue strut, $B_2$ is the middle one, and $B_3$ is the large blue strut. It is not difficult to discover that we can make a $B_3$ size edge by connecting a $B_1$ and a $B_2$ in a straight line which shows that $B_1 + B_2 = B_3$. Similar equations are true for the yellow and red struts: $Y_1 + Y_2 = Y_3$ and $R_1 + R_2 = R_3$. It also can be shown that $B_2/B_1 = B_3/B_2$ and therefore $B_2/B_1 = (B_1 + B_2)/B_2$ (similarly $Y_2/Y_1 = (Y_1 + Y_2)/Y_2$ and $R_2/R_1 = (R_1 + R_2)/R_2$). This demonstrates that the struts for these colors illustrate the Golden Proportion. As a practice, let us construct a cube. For this the students will find that because of the ball holes the easiest way of such a construction is if they use blue struts (Figure 2). After building a square one can continue the construction to build a cube (Figure 2). The cube is the first regular polyhedron that usually comes to the minds of students that satisfy the following requirements:
(a) All its faces are congruent regular polygons.
(b) The arrangements of polygons about the vertices are all alike.

The cube construction requires three-dimensional corners, each out of a ball and three equal size struts. Three struts should be connected to a common ball in such a way that each becomes perpendicular to the plane containing the other two. Students will realize that the only possible case is using three equal size blue struts. Constructing a single three-dimensional corner will show them how to complete the construction of a cube. Another activity prior to the constructive proof of Theorem 2.1 is to use green struts to make a diagonal for one of the cube faces. After that they continue making more diagonals to compose an equilateral triangle. Using this triangle as the base it is not difficult to figure out how to add more diagonals to construct a triangular pyramid, which is exhibited in Figure 2 (which is called the tetrahedron later).

![Figure 2: The construction of a cube and the inscribed tetrahedron]

To construct a regular polyhedron one may consider an order to start by using the simplest possible face that such a polyhedron can possess. By simplest we mean with the least number of sides. The simplest face then would be the equilateral triangle. To construct a three dimensional corner about a vertex out of equilateral triangles students discover that they need three, four, or five of them, nothing more and nothing less (Figure 3): Obviously, to make a 3D corner one needs at least three polygons, and also six equilateral triangles about a vertex on a plane make a flat surface, a regular triangular tessellation.

![Figure 3: The constructions of 3D corners out of three, four, and five equilateral triangles]

To build a Zome regular polyhedron out of equilateral triangles, the students should notice that what they have already constructed inside of the cube is a polyhedron with equilateral triangular faces with the property that each 3D corner is the common vertex of three triangles. Since each face of the solid is an equilateral triangle and each vertex is the meeting point of exactly three faces, this is one of the regular polyhedra, which is called the tetrahedron! The following table summarizes information for this solid. The Schläfli
Symbol is an ordered pair notation such that its first coordinate presents the number of sides of a face and the second coordinate presents the number of faces that meet in a vertex. The Schläfli symbol is named after the 19th-century mathematician Ludwig Schläfli who made many contributions in geometry. Here $F$, $V$, and $E$ show the number of faces, vertices, and edges of the solid respectively.

| Polyhedron   | Schläfli Symbol | $F$ | $V$ | $E$ | The Shape of Each Face |
|--------------|-----------------|-----|-----|-----|-------------------------|
| Tetrahedron  | (3, 3)          | 4   | 4   | 6   | Equilateral Triangle    |
| Octahedron   | (3, 4)          | 8   | 6   | 12  | Equilateral Triangle    |

Working with green struts is difficult due to the fact that they are not all the way straight. There are small bends on each end of a green strut that make it look like the letter “S”. So it is not an easy task to build a tetrahedron out of green struts independent from the blue cube. But there is a good approximation for this solid that can be obtained by using three blue struts, to make an equilateral triangle as a base, and three red struts to build the tetrahedron over this triangle (Figure 4).

The next regular polyhedron should be built with the property that four equilateral triangles meet at each vertex. To form such a vertex using Zome tools, four green struts should meet at a ball in such a way that the opposite struts have the same orientations with respect to these bends. More specifically, two opposite struts with a shorter distance between them on the ball should have a downward concave down bend near the ball and the other two struts should have an upward concave bend near the ball. If the students continue this process for each vertex they will obtain a regular solid with eight equilateral triangles faces, the octahedron! (Figure 5). The following table presents the two solids of the tetrahedron and the octahedron.

There is also an approximation for the octahedron with a triangular anti-prism (a polyhedron with a pair of identical parallel bases where the surrounding faces are all triangles). For this, we may use six blue struts, to construct two equilateral triangles for the bases and then use six red struts, to complete the structure (Figure 5).

After constructing regular solids with 3D corners that are composed from 3 and 4 triangles, the students are ready to build a new regular solid such that each of its vertices is the meeting point of 5 triangles. There are appropriate holes on the balls that are found easily to use blue struts to build five equilateral triangles about
a ball. Then completing the polyhedra, the icosahedron, with twenty faces is only a matter of time and energy (Figure 6).

![Figure 5: The octahedron and its approximation](image)

Another way of composition is possible if one uses a single ball and connects all possible red struts that can be emanated from that ball. Then adds ball and blue struts to connect the free ends of red struts (Figure 6).

| Polyhedron       | Schlafli Symbol | F  | V  | E  | The Shape of Each Face   |
|------------------|-----------------|----|----|----|----------------------------|
| Tetrahedron      | (3, 3)          | 4  | 4  | 6  | Equilateral Triangle      |
| Octahedron       | (3, 4)          | 8  | 6  | 12 | Equilateral Triangle      |
| Icosahedron      | (3, 5)          | 20 | 12 | 30 | Equilateral Triangle      |

![Figure 6: The icosahedron](image)

The three solids of the tetrahedron, octahedron, and icosahedron complete the list of regular polyhedra that one can build out of equilateral triangles.

| Polyhedron       | Schlafli Symbol | F  | V  | E  | The Shape of Each Face   |
|------------------|-----------------|----|----|----|----------------------------|
| Tetrahedron      | (3, 3)          | 4  | 4  | 6  | Equilateral Triangle      |
| Octahedron       | (3, 4)          | 8  | 6  | 12 | Equilateral Triangle      |
| Icosahedron      | (3, 5)          | 20 | 12 | 30 | Equilateral Triangle      |
| Hexahedron       | (4, 3)          | 6  | 8  | 12 | Square                    |

The students have already built the cube (hexahedron) and easily can find that making four squares about a vertex on a plane, similar to the six equilateral triangles case, will make a flat tessellation. So the conclusion is that one cannot make any more regular polyhedra with square faces. Therefore, the list of regular polyhedra with triangle and square faces includes four solids only.
To continue this mathematical investigation students need to think about using regular polygons with more sides than four for faces of a regular solid. This means they need to know the angle measures of the pentagon, hexagon, and so on, to study if they can build three dimensional corners out of some copies of each of them.

In general, students are not familiar with the formula for this so either you may use cut out pentagons and hexagons made from paper board to show them that three pentagons will make a 3D corner but three hexagons make a flat surface, or include the following theorem in your activity to establish a formula for the angle measures of regular polygons.

**Theorem 2.2.** *Any n-gon can be divided into n− 2 triangles.*

Suppose that an n-gon with vertices \(A_1\) to \(A_n\) is given. Using \(A_1\) as a common vertex for diagonals we want to divide n-gon into triangles (Figure 7). The first two sides, \(A_1A_2\) and \(A_2A_3\) with the diagonal \(A_1A_3\), identify the first triangle (here we used two sides to build one triangle). After that, each side, with the help of two adjacent diagonals identify one new triangle (Here, one side of n-gon is used to build one triangle). But the last triangle will employ the last two sides of \(A_1A_n\) and \(A_nA_{n−1}\) to be constructed (two sides for one triangle). This shows the number of constructed triangles are two less than the number of sides!

![Figure 7: An n sided polygon can be divided into n− 2 triangles](https://example.com/f7.png)

This means that the sum of angle measures of the n-gon is \((n−2)180^\circ\). Therefore, the angle measure of an angle of the regular n-gon is \((n−2)180^\circ/n\). Hence, the angle measure of each angle of the regular pentagon is \(108^\circ\). So it is possible to arrange three pentagons about a vertex and still create a 3D corner.

| Polyhedron     | Schläfi Symbol | F | V | E | The Shape of Each Face     |
|----------------|----------------|---|---|---|----------------------------|
| Tetrahedron    | (3, 3)         | 4 | 4 | 6 | Equilateral Triangle       |
| Octahedron     | (3, 4)         | 8 | 6 | 12| Equilateral Triangle       |
| Icosahedron    | (3, 5)         | 20| 12| 30| Equilateral Triangle       |
| Hexahedron     | (4, 3)         | 6 | 8 | 12| Square                     |
| Dodecahedron   | (5, 3)         | 12| 20| 30| Regular Pentagon           |

The regular solid with such vertices is called the dodecahedron. To build the dodecahedron with pentagonal faces using Zome tools one only needs to use blue struts. Since the students through the construction of the icosahedron have learned how to make a regular pentagon using blue struts they may use a pentagon as a base and build their dodecahedron on top of that (Figure 8). The students will discover that the built pentagon has two different sides for the front and the back of the solid and for only one of these sides the construction of the dodecahedron is possible.
Three regular hexagons about a vertex will make a flat surface (each angle of the regular hexagon is 120°) and arranging three faces of a regular polygon with the number of sides greater than six about a vertex on a plane is impossible. This concludes our proof of Theorem 2.1 and the set of convex regular polyhedra.

3 Patterning the Platonic Solids

Another activity that can be integrated into this topic is the decoration of the Platonic solids using patterns from Fractal geometry and also from the arts of some cultures. In this regard, a tool, such as the Geometer’s Sketchpad, is necessary. The following are only a few examples and an interested instructor can search for more patterns. The key is to analyze and discover the steps that are required to construct such designs geometrically [6].

3.1. Examples from Fractal Geometry

To make decorated tetrahedrons, octahedrons, or icosahedrons, one needs to embellish an equilateral triangle only. One design could be based on the Sierpinski Gasket. Using the Geometer’s Sketchpad construct segment AB. Use AB as the base to make the equilateral triangle ΔABC as is presented in Figure 9. Construct the interior of ΔABC. The next step is to construct the midpoints D, E, and F, of its sides AB, BC, and CA, respectively. Now iterate on A and B (point C is a dependent point to A and B so should not be used in the iteration process). Using the Geometer’s Sketchpad select A and B in order and click on “Iterate” on the Transform menu. When the dialog box appears select A as its own image (by clicking on A in the figure) and select D as the image of B. With this action, as is show in Figure 9, a self-similar triangle with the scale factor of one-half will appear on the left part of the original triangle (If the default mode is 3 iterations, then you will see three iterations on the left part). As is shown in Figures 9, one needs to add two more small replicas there, one on the right side and the other on the top part of the triangle. For this, the students should add two new maps. Click on “Structure” and choose Add New Map. For the first map D will be the image of A but B will be its own image. For the second map, after choosing “Structure” and Add New Map, choose F as the image of A and E as the image of B. Now set the iterations to only show non-point images by going to “Structure” and clicking on Non-Point Images. On “Display”, click on Final Iteration Only. A similar figure to the lower middle image in Figure 9 appears. To see the most recent iterated images and be able to increase the stages of iterations, click on the center of the interior of the original triangle and “Hide Triangle”. This will result in stage three of the Sierpinski Gasket. By selecting the entire figure and using the Plus (+) and Minus (−) keys, one experiments with different stages of iterations. Click on the Point tool, go to the Edit menu, and “Select All Points”. Then go to the Display menu and “Hide Points”. The Sierpinski Gasket is ready!
To decorate the cube one needs to make a decorated square. You may use the same process as for the Sierpinski Gasket but now divide the initiator into nine smaller squares. This will create a middle square that will able you to iterate properly. Number the squares from left to right and from top to bottom (Figure 10). Then you may obtain the following three decorated squares that are based on the application of the same idea as for Sierpinski Gasket but now on different sets of squares. The second image from the left presents the result if one applies the idea to squares numbered 1, 3, 5, 7, and 9. For the third image in Figure 10 you should use squares numbered 2, 4, 6, and 8. The last image in Figure 10 illustrates the application of the idea to the square number 5 only.

For the decoration of the dodecahedron you may use the above iteration method (the left two images in Figure 11).

### 3.2. Examples from Persian Mosaic Designs

Another interesting design can be obtained by a compass straight edge geometric construction such as the right image in Figure 11, which is a Persian traditional mosaic pattern. This design is based on the 10/3 star polygon that can be constructed using the steps in Figure 12 [7].
For this, one should notice that the radius of the circumscribed circle to the pentagon on the right image in Figure 13, $OA$, which is the distance from the center of the pentagon to a vertex, is twice the radius of the circle that is the basis for the $10/3$ star polygon at the center ($AM = MO$ in the top left image in Figure 14). The reason for this is that the two $10/3$ star polygons, one at the center $O$ and the other at the vertex $A$, are each others reflections under the tangent to the circle at point $M$ (see the middle bottom image in Figure 14 that also includes a tangent to the circle at point $N$ that is necessary to be used as the reflection line, to complete the star). By following images from the top left to the bottom right a person may complete the design properly.

4 Conclusion

The ultimate goal for a classroom should be a place for students to pursue knowledge actively. They should take responsibility for their own learning. For this, we need to shift the classroom from a lecturing-listening to a coaching-doing environment. Is this a possible procedure for all mathematics classrooms? The answer
is negative. Nevertheless, there are many undergraduate courses that at least some parts of them can be based on the student-centered problem-solving approach. The aforementioned Zome activity will let the students think, work with their hands, struggle, and stretch their minds. Art is another endeavor that can be used to open a window for students to see mathematics from a different perspective, to get involved with computer-based activities, and to understand many mathematical concepts better and deeper. In such an environment students can explore ideas from the convergence and divergence of series in calculus, using the art of Navajo [3-4] to the fundamental theorem of number theory in relation with the Pennsylvania Dutch Hex Signs [1]. Activity based instruction will bring students to the center. Art can facilitate this process.

![Figure 13: A workshop conducted by the author in the Istanbul Design Center in Turkey](image)

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