CROSS RATIO GEOMETRY
ADVANCES FOR FOUR CO-LINEAR POINTS IN THE
DESARGUES AFFINE PLANE-SKEW FIELD

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Dedicated to Girard Desargues and Karl G. C. von Staudt

ABSTRACT. This paper introduces advances in the geometry of the cross ratio of four co-linear points in in the Desargues affine plane. The cross-ratio of co-linear points of a skew field in the Desargues affine plane. The results given here have a clean rendition, based on Desargues affine plane axiomatics, skew field properties and the addition and multiplication of planar co-linear points.

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1. INTRODUCTION AND PRELIMINARIES

In the advancement of our research in the connections of axiomatic geometry and algebraic structures, we have achieved some results which we have presented in this paper. More recently, results are given about the association of algebraic structures in affine planes and in Desargues affine plane, and vice versa in [21, 22, 6, 19, 20, 18, 27, 25, 26, 25, 23]. The foundations for the study of the connections between axiomatic geometry and algebraic structures were set forth by D. Hilbert [9]. And some classic research results in this context are given, for example, by E. Artin [1], D.R. Huges and F.C. Piper [10], H. S. M Coxeter [5]. Marcel Berger in [3], Robin Hartshorne in [7].

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In this paper, we advance in study regarding the cross ratio of 4-points, in a line of the Desargues affine plane. We study and discusses the properties and results related to the cross ratio for four points, also we see the points of line as a elements of a skew field which constructed over this line on Desargues affine plane.

We use skew field properties for the proof of our results, since the cross-ratio sketch is very confusing (even with the Euclidean interpretation).

Earlier, we study the ratio of 2 and 3 points in a line on Desargues affine plane (see [23], [24], [24]), also we have shown that on each line on Desargues affine plane, we can construct a skew-field simply and constructively, using simple elements of elementary geometry, and only the basic axioms of Desargues affine plane (see [22], [6], [18], [27]).

In this paper, we utilize a method that is naive and direct, without requiring the concept of coordinates. We bases only in Desargues affine plane axiomatic and in skew field properties (the points in a line on Desargues affine plane, we think of them as elements of skew fields, which is a construct over this line).

1.1. Desargues Affine Plane. Let \( \mathcal{P} \) be a nonempty space, \( \mathcal{L} \) a nonempty subset of \( \mathcal{P} \). The elements \( p \) of \( \mathcal{P} \) are points and an element \( \ell \) of \( \mathcal{L} \) is a line.

**Definition 1.** The incidence structure \( \mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{T}) \), called affine plane, where satisfies the above axioms:

1. For each points \( \{P, Q\} \in \mathcal{P} \), there is exactly one line \( \ell \in \mathcal{L} \) such that \( \{P, Q\} \in \ell \).
2. For each point \( P \in \mathcal{P}, \ell \in \mathcal{L}, P \notin \ell \), there is exactly one line \( \ell' \in \mathcal{L} \) such that \( P \in \ell' \) and \( \ell \cap \ell' = \emptyset \) (Playfair Parallel Axiom [14]). Put another way, if the point \( P \notin \ell \), then there is a unique line \( \ell' \) on \( P \) missing \( \ell \) [15].
3. There is a 3-subset of points \( \{P, Q, R\} \in \mathcal{P} \), which is not a subset of any \( \ell \) in the plane. Put another way, there exist three non-collinear points \( \mathcal{P} \) [15].

**Desargues’ Axiom, circa 1630** [11] §3.9, pp. 60-61 [17]. Let \( A, B, C, A', B', C' \in \mathcal{P} \) and let pairwise distinct lines \( \ell^{AA'}, \ell^{BB'}, \ell^{CC'}, \ell^{AC}, \ell^{A'C'} \in \mathcal{L} \) such that

\[ \ell^{AA'} \parallel \ell^{BB'} \parallel \ell^{CC'} \] (Fig. 1(a)) or \( \ell^{AA'} \cap \ell^{BB'} \cap \ell^{CC'} = P \) (Fig. 1(b))

and \( \ell^{AB} \parallel \ell^{A'B'} \) and \( \ell^{BC} \parallel \ell^{B'C'} \).

\[ A, B \in \ell^{AB}, A'B' \in \ell^{A'B'}, \text{ and } B, C \in \ell^{BC}, B'C' \in \ell^{B'C'}. \]

Then \( \ell^{AC} \parallel \ell^{A'C'} \).

**Example 1.** In Euclidean plane, three vertexes ABC and A'B'C', are similar (in (a) are equivalent-triangle and in (b) are homothetical-triangle) the parallel lines, \( \ell^{AC}, \ell^{A'C'} \in \mathcal{L} \) in Desargues’ Axiom are represented in Fig. 1. In other words, the side AC of the triangle of \( \triangle ABC \) is parallel with the side A'C' of the triangle \( \triangle A'B'C' \), provided the restrictions on the points and lines in Desargues’ Axiom are satisfied.

A Desargues affine plane is an affine plane that satisfies Desargues’ Axiom.

**Notation 1.** Three vertexes ABC and A'B'C', which, fulfilling the conditions of the Desargues Axiom, we call 'Desarguesian'.

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1.2. Addition and Multiplication of points in a line of Desargues affine plane.

Addition of points in a line of affine plane: In an Desargues affine plane $\mathcal{A}_D = (P, L, I)$ we fix two different points $O, I \in P$, which, according to Axiom 1, determine a line $\ell^{OI} \in L$. Let $A$ and $B$ be two arbitrary points of a line $\ell^{OI}$. In plane $\mathcal{A}_D$ we choose a point $B_1$ not incident with $\ell^{OI}$: $B_1 \notin \ell^{OI}$ (we call the auxiliary point). Construct line $\ell^{B_1O}_{B_1}$, which is only according to the Axiom 2. Then construct line $\ell^{A}_{OB_1}$, which also is the only according to the Axiom 2. Marking their intersection $P_1 = \ell^{A}_{OB_1} \cap \ell^{B_1O}_{B_1}$. Finally construct line $\ell^{P_1}_{B_1}$. For as much as $\ell^{B_1}_{B_1}$ cuts the line $\ell^{OI}$ in point $B$, then this line, parallel with $\ell^{B_1}_{B_1}$, cuts the line $\ell^{OI}$ in a single point $C$, this point we called the addition of points $A$ with point $B$ (Figure 2 (a)).

Multiplication of points in a line in affine plane. Choose in the plane $\mathcal{A}_D$ one point $B_1$ not incident with lines $\ell^{OI}$, and construct the line $\ell^{B_1A}$. Construct the line $\ell^{A}_{OB_1}$, which is the only according to the Axiom 2 and cutting the line $\ell^{OB_1}$. Marking their intersection with $P_1 = \ell^{B_1A} \cap \ell^{A}_{OB_1}$. Finally, construct line $\ell^{P_1}_{B_1}$. For as much as $\ell^{B_1}_{B_1}$ cuts the line $\ell^{OI}$ in point $B$, then this line, parallel with $\ell^{B_1}_{B_1}$, cuts the line $\ell^{OI}$ in one single point $C$, this point we called the multiplication of points $A$ with point $B$ (Figure 2 (b)).

The process of construct the points $C$ for adition and multiplication of points in $\ell^{OI}$—line in affine plane, is presented in the tow algorithm form

**Addition Algorithm**

Step.1: $B_1 \notin \ell^{OI}$
Step.2: $\ell^{B_1}_{OI} \cap \ell^{A}_{OB_1} = P_1$
Step.3: $\ell^{P_1}_{B_1} \cap \ell^{OI} = C (= A + B)$

**Multiplication Algorithm**

Step.1: $B_1 \notin \ell^{OI}$
Step.2: $\ell^{A}_{B_1} \cap \ell^{OB_1} = P_1$
Step.3: $\ell^{P_1}_{B_1} \cap \ell^{OI} = C (= A \cdot B)$

In [8] and [6], we have prove that $(\ell^{OI}, +, \cdot)$ is a skew field in Desargues affine plane, and is field (commutative skew field) in the Papus affine plane.

1.3. Some algebraic properties of Skew Fields. In this section $K$ will denote a skew field $\mathbb{S}$ and $z[K]$ its center, where is the set $K$ such that

$z[K] = \{ k \in K \mid a k = k a, \ \forall a \in K \}$
Proposition 1. \( z[K] \) is a commutative subfield of a skew field \( K \).

Let now \( p \in K \) be a fixed element of the skew field \( K \). We will denote \( z_K(p) \) the centralizer in \( K \) of the element \( p \), where \( z_K(p) = \{ k \in K | pk = kp \} \).

\( z_K(p) \) is sub skew field of \( K \), but, in general, it is not commutative.

Let \( K \) be a skew field, \( p \in K \), and let us denote by \( [p_K] \) the conjugacy class of \( p \):

\[
[p_K] = \{ q^{-1}pq \mid q \in K \setminus \{0\} \}
\]

If, \( p \in z[K] \), for all \( q \in K \) we have that \( q^{-1}pq = p \).

1.4. Ratio of two and three points. In the paper \[23\], we have done a detailed study, related to the ratio of two and three points in a line of Desargues affine plane. Below we are listing some of the results for ratio of two and three points.

Definition 2. \[23\] Let there be two different points \( A, B \in \ell_{OI} \), and \( B \neq O \), in Desargues affine plane. We define as ratio of this tow points, a point \( R \in \ell_{OI} \), such that,

\[
R = B^{-1}A,
\]

we mark this, with, \( R = r(A : B) = B^{-1}A \)

For a 'ratio-point' \( R \in \ell_{OI} \), and for point \( B \neq O \) in line \( \ell_{OI} \), is a unique defined point, \( A \in \ell_{OI} \), such that \( R = B^{-1}A = r(A : B) \).

Some results for Ratio of 2-points in Desargues affine plane (see \[23\]).

- If have two different points \( A, B \in \ell_{OI} \), and \( B \neq O \), in Desargues affine plane, then, \( r^{-1}(A : B) = r(B : A) \).
- For three collinear point \( A, B, C \) and \( C \neq O \), in \( \ell_{OI} \)–line, have,

\[
r(A + B : C) = r(A : C) + r(B : C).
\]

- For three collinear point \( A, B, C \) and \( C \neq O \), in \( \ell_{OI} \)–line, have,

\[
(1) \ r(A \cdot B : C) = r(A : C) \cdot B.
(2) \ r(A : B \cdot C) = C^{-1}r(A : C).
\]

- Let’s have the points \( A, B \in \ell_{OI} \)–line where \( B \neq O \). Then have that,

\[
r(A : B) = r(B : A) \iff A = B.
\]

- This ratio-map, \( r_B : \ell_{OI} \to \ell_{OI} \) is a bijection in \( \ell_{OI} \)–line in Desargues affine plane.
The ratio-maps-set $R_2 = \{ r_B(X) | \forall X \in \ell^{O_I} \}$, for a fixed point $B$ in $\ell^{O_I}$--line, forms a skew-field with ‘addition and multiplication’ of points. This skew field $(R_2, +, \cdot)$ is sub-skew field of the skew field $(\ell^{O_I}, +, \cdot)$.

**Ratio of three points in a line on Desargues affine plane.** (see [23])

**Definition 3.** If $A, B, C$ are three points on a line $\ell^{O_I}$ (collinear) in Desargues affine plane, then we define their **ratio** to be a point $R \in \ell^{O_I}$, such that:

$$(B - C) \cdot R = A - C, \quad \text{concisely} \quad R = (B - C)^{-1}(A - C),$$

and we mark this with $r(A, B; C) = (B - C)^{-1}(A - C)$.

**Some Results for Ratio of 3-points in Desargues affine plane** ([23]).

- For 3-points $A, B, C$ in a line $\ell^{O_I}$ of Desargues affine plane, we have that,

$$r(-A, -B; -C) = r(A, B; C).$$
For 3-points $A, B, C$ in a line $\ell^{O_1}$ in the Desargues affine plane, have
\[ r^{-1}(A, B; C) = r(B, A; C). \]

- If $A, B, C$, are three different points, and different from point $O$, in a line $\ell^{O_1}$ on Desargues affine plane, then
\[ r(A^{-1}, B^{-1}; C^{-1}) = B[r(A, B; C)]A^{-1}. \]

- In the Pappus affine plane, for three point different from point $O$, in $\ell^{O_1}$-line, we have $r(A^{-1}, B^{-1}; C^{-1}) = r(A, B; C) \cdot r(B, A; O)$.

- This ratio-map, $r_{BC} : \ell^{O_1} \to \ell^{O_1}$ is a bijection in $\ell^{O_1}$-line in Desargues affine plane.

- The ratio-maps-set $R_3 = \{r_{BC}(X) \forall X \in \ell^{O_1}\}$, for a different fixed points $B, C$ in $\ell^{O_1}$-line, forms a skew-field with 'addition and multiplication' of points in $\ell^{O_1}$-line. This, skew field $(R_3, +, \cdot)$ is sub-skew field of the skew field $(\ell^{O_1}, +, \cdot)$.

2. Cross-Ratio for Four points in a line of Desargues affine plane

This section culminates in a main result in this paper. We consider the cross-ratio of co-linear points in Desargues affine planes, utilizing a method that is naive and direct without requiring planar coordinates. We define the cross-ratio of four co-linear points in a line on Desargues affine plane as a point in this line. This work carries forward earlier results that reveal the close connection between lines in the Desargues affine planes and corresponding skew fields. Skew fields properties in our proofs. Mainly, we rely on our results regarding the addition and multiplication of co-linear points in the Desargues affine plane, and the fact that a line (set of points), with addition and multiplication, forms a skew field (for more about this, see [18], [22], [6], [20], [19], [21], [27], [28], [24]).

The classical definition of the cross-ratio (see [13, 9, 2, 3]) for 4-points, is given as a product of two ratio of lengths. So, for example, for four co-linear points $A, B, C, D$,
\[ c_r(A, B; C, D) = \frac{AC}{BC} \cdot \frac{BD}{AD}, \]
where $AC, BC, BD, AD$ are the lengths of segments $[AB], [BC], [BD], [AD]$, respectively.

Since we will not use coordinates and metrics, our definitions are rely solely on the algebra and axiomatics for the Desargues affine plane.

Let us have the line $\ell^{O_1}$ in Desargues affine plane $A_D$, and four points, $A, B, C, D \in \ell^{O_1}$

**Definition 4.** If $A, B, C, D$ are four points on a line $\ell^{O_1}$ in Desargues affine plane $A_D$, no three of them equal, then we define their cross ratio to be a point:
\[ c_r(A, B; C, D) = [(A - D)^{-1}(B - D)] [(B - C)^{-1}(A - C)] \]

**Remark 1.** Similar to 'ratio', we can define it, the cross-ratio, also as
\[ c_r(A, B; C, D) = [(B - D)(A - D)^{-1}][(A - C)(B - C)^{-1}], \]
or
\[ c_r(A, B; C, D) = [(B - D)(A - C)][(A - D)^{-1}(B - C)^{-1}], \]
(or all combination of product of this 4-factors) the results would be similar, but the obtained point will always be different for each case. In \( \ell_{O1} \)-line, in Desargues affine planes, these are a different point from that of our definition, since:

\[
[(B-D)(A-D)^{-1}][(A-C)(B-C)^{-1}] \neq [(A-D)^{-1}(B-D)] [(B-C)^{-1}(A-C)].
\]

and

\[
[(B-D)(A-C)][(A-D)^{-1}(B-C)^{-1}] \neq [(A-D)^{-1}(B-D)] [(B-C)^{-1}(A-C)].
\]

also for the other cases, we would have a difference for each pair, found for the cross ratio, according to any definition we take. We are keeping our definition.

**Definition 5.** If the line \( \ell_{O1} \) in Desargues affine plane, is an infinite line (number of points in this line is \( +\infty \)), we define as follows:

\[
\begin{align*}
c_r(\infty, B; C, D) &= (B-D)(B-C)^{-1} \\
c_r(A, \infty; C, D) &= (A-D)^{-1}(A-C) \\
c_r(A, B; \infty, D) &= (A-D)^{-1}(B-D) \\
c_r(A, B; C, \infty) &= (B-C)^{-1}(A-C)
\end{align*}
\]

From this definition and from ratio definition \( \ell_{O1} \) we have that,

- \( c_r(A, B; C, D) = [(A-D)^{-1}(B-D)] [(B-C)^{-1}(A-C)] \), so
  \[
  c_r(A, B; C, D) = r(B, A; D) \cdot r(A, B; C).
  \]

- \( c_r(\infty, B; C, D) = (B-D)(B-C)^{-1} = [(D-B)^{-1}(C-B)]^{-1} \), so,
  \[
  c_r(\infty, B; C, D) = r(C, D; B).
  \]

- \( c_r(A, \infty; C, D) = (A-D)^{-1}(A-C) = (D-A)^{-1}(C-A) \), so,
  \[
  c_r(A, \infty; C, D) = r(C, D; A).
  \]

- \( c_r(A, B; \infty, D) = (A-D)^{-1}(B-D) \), so
  \[
  c_r(A, B; \infty, D) = r(A, B; D).
  \]

- \( c_r(A, B; C, \infty) = (B-C)^{-1}(A-C) \), so,
  \[
  c_r(A, B; C, \infty) = r(A, B; C).
  \]

**Some simple properties of Cross-Ratios**, which derive directly from the definition, related to the position of the points \( A, B, C, D \) in \( \ell_{O1} \)-line in Desargues affine plane.

- If \( A = B \), then
  \[
  c_r(A, B; C, D) = c_r(A, A; C, D) = [(A-D)^{-1}(A-D)][(A-C)^{-1}(A-C)] = [I][I] = I.
  \]

- If \( A = C \), then
  \[
  c_r(A, B; C, D) = c_r(A, B; A, D) = [(A-D)^{-1}(B-D)][(B-A)^{-1}(A-A)] = [(A-D)^{-1}(B-D)][(B-A)^{-1} \cdot O] = O.
  \]
Suppose that there exist two different points.

**Proof.**

Let

\[ c_r(A, B; C, D) = c_r(A, B; C, A) \]

\[ = [(A - A)^{-1}(B - A)][(B - C)^{-1}(A - C)] \]

\[ = [O^{-1}(B - A)][(B - C)^{-1}(A - C)] \]

(think that \( O^{-1} = \infty \) (point in infinity))

\[ = \infty. \]

**Theorem 1.** Let \( R \in \ell^{O_1} \), such that \( R \neq O \) and \( R \neq I \). If \( A, B, C \in \ell^{O_1} \) are three different points, then there exists a single point \( D \in \ell^{O_1} \), such that \( c_r(A, B; C, D) = R \).

**Proof.** Suppose that there exist two different points \( D \) in \( \ell^{O_1} \)-line, such that

\[ c_r(A, B; C, D) = c_r(A, B; C, D') \]

We rewrite them, cross ratios, as products of 'ratios', and we have,

\[ c_r(A, B; C, D) = [(A - D)^{-1}(B - D)][(B - C)^{-1}(A - C)] = r(B, A; D) \cdot r(A, B; C) \]

and

\[ c_r(A, B; C, D') = [(A - D')^{-1}(B - D')][(B - C)^{-1}(A - C)] = r(B, A; D') \cdot r(A, B; C) \]

So, have,

\[ r(B, A; D) \cdot r(A, B; C) = r(B, A; D') \cdot r(A, B; C) \]
we mark \( r(B, A; D) = R_1; r(A, B; C) = R_2, r(B, A; D') = R_3 \), remember that these are points of the line \( \ell^{OI} \), so they are elements of the skew-fields \((\ell^{OI}, +, \cdot)\), and have
\[
R = R_1 \cdot R_2 \quad \text{and} \quad R = R_3 \cdot R_2
\]
Thus, for it, we have
\[
R_1 \cdot R_2 = R_3 \cdot R_2 \Rightarrow R_1 \cdot R_2 - R_3 \cdot R_2 = O \Rightarrow (R_1 - R_3) \cdot R_2 = O
\]
But the points, \( R_1, R_2, R_3 \), are points of \( \ell^{OI} \)-line in Desargues affine plane, therefore, they are elements of skew-fields \( K = (\ell^{OI}, +, \cdot) \). We also know the fact that 'a skew field does not have a divisor of zero' (more on skew fields, see [4], [8], [10], [12])
\[
R_1 - R_3 = O \quad \text{or} \quad R_2 = O, \quad \text{but} \quad R_2 \neq O \Rightarrow R_1 - R_3 = O
\]
so,
\[
R_1 = R_3 \Rightarrow r(B, A; D) = r(B, A; D')
\]
and from the uniqueness of the definition for 'ratio', we have,
\[
D = D'
\]

\[ \square \]

**Theorem 2.** If \( A, B, C, D \) are distinct points in a \( \ell^{OI} \)-line, in Desargues affine plane, then
\[
c_r(-A, -B; -C, -D) = c_r(A, B; D, C)
\]

**Proof.** From cross-ratio definition [4] we have
\[
c_r(-A, -B; -C, -D) = [(-A - (-C)^{-1}(-B - (-D)^{-1}(-A - (-D))]
\]
\[
= [(-A + C)^{-1}(-B + C)][(-B + D)^{-1}(-A + D)]
\]
\[
= [[-I](A - C)^{-1}(-I)(B - C)][(-I)(B - D)^{-1}(-I)(A - D)]
\]
\[
= [(A - C)^{-1}(-I)^{-1}(-I)(B - C)][(-I)(B - D)^{-1}(-I)(A - D)]
\]
\[
= [(A - C)^{-1}(-I)(B - C)][(B - D)^{-1}(-I)(A - D)]
\]
\[
= c_r(A, B; D, C)
\]
From skew fields properties we have that \((ab)^{-1} = b^{-1}a^{-1}\) and \(ab \neq ba\), \([-I]^{-1} = -I\), and \([-I][-I] = I.\]

\[ \square \]

**Theorem 3.** If \( A, B, C, D \) are distinct points in a line, in Desargues affine plane, then
\[
c_r(A, B; C, D) = c_r(A, B; D, C)
\]

**Proof.** From cross-ratio Definition [4] have
\[
c_r(A, B; C, D) = \{[(A - D)^{-1}(B - D)]\ [B - C)^{-1}(A - C)]\}^{-1}
\]
\[
= [(B - C)^{-1}(A - C)]^{-1}[(A - D)^{-1}(B - D)]^{-1}
\]
\[
= [(A - C)^{-1}(B - C)][(B - D)^{-1}(A - D)]
\]
\[
= c_r(A, B; D, C)
\]
\[ c_r(A, B; C, D) = [(A - B)^{-1} - (A - D)^{-1}] [(A - B)^{-1} - (A - C)^{-1}]^{-1} \]

**Theorem 4.** For 4 co-linear points \( A, B, C, D \) in a line \( \ell^{01} \) in the Desargues affine plane, the cross-ratio satisfies the equation, \( c_r(A, B; C, D) \) is the ratio of the points, on this line, and the properties that satisfy a skew-field, therefore, we have association property:

\[ R = c_r(A, B; C, D) \]

In the same way as above, (always bearing in mind that the points of a line of Desargues affine planes form a skew-field related to the addition and multiplication of the points, on this line, and the properties that satisfy a skew-field) we do the following transformations.

First we have the associative property for the multiplication of points on a line, \( [R(A - C)^{-1}(B - C)](A - B)^{-1} = R[(A - C)^{-1}(B - C)(A - B)^{-1}] \)

Now we transform the expression
\[(A - C)^{-1}(B - C)(A - B)^{-1} = [(A - C)^{-1}(B + A - A - C)(A - B)^{-1}] \\
= [(A - C)^{-1}([A - C] - [A - B])(A - B)^{-1}] \\
= [I - (A - C)^{-1}[A - B]](A - B)^{-1} \\
= (A - B)^{-1} - (A - C)^{-1}[A - B](A - B)^{-1} \\
= (A - B)^{-1} - (A - C)^{-1}
\]

So, have
\[(A - B)^{-1} - (A - D)^{-1} = R\left[(A - B)^{-1} - (A - C)^{-1}\right]
\]
Hence
\[R = [(A - B)^{-1} - (A - D)^{-1}][I - (A - C)^{-1}]^{-1}
\]
so,
\[c_r(A; B; C, D) = [(A - B)^{-1} - (A - D)^{-1}]\left[(A - B)^{-1} - (A - C)^{-1}\right]^{-1}.
\]

**Theorem 5.** If \(A, B, C, D\) are distinct points in a line, in Desargues affine plane and \(I\) is unitary point for multiplications of points in same line, then
\[I - c_r(A; B; C, D) = c_r(A; C; B, D)\]

**Proof.** Let’s start the calculations, using the result of the theorem
\[I - [(A - B)^{-1} - (A - D)^{-1}][(A - B)^{-1} - (A - C)^{-1}]^{-1} = \]
\[= \{(A - C)^{-1} - (A - D)^{-1}\} \left[(A - C)^{-1} - (A - B)^{-1}\right]^{-1}\]
write,
\[I = [(A - B)^{-1} - (A - C)^{-1}]\left[(A - B)^{-1} - (A - C)^{-1}\right]^{-1}\]
so,
\[I - c_r(A; B; C, D) = I - [(A - B)^{-1} - (A - D)^{-1}][(A - B)^{-1} - (A - C)^{-1}]^{-1} \]
\[= [(A - B)^{-1} - (A - C)^{-1}]\left[(A - B)^{-1} - (A - C)^{-1}\right]^{-1} \]
\[= [\{(A - B)^{-1} - (A - C)^{-1}\} - [(A - B)^{-1} - (A - D)^{-1}]\} \cdot \left[(A - B)^{-1} - (A - C)^{-1}\right]^{-1} \]
\[= \{(A - C)^{-1} - (A - B)^{-1}\} [(A - B)^{-1} - (A - C)^{-1}]^{-1} \]
\[= (I) \left[(A - C)^{-1} - (A - B)^{-1}\right]^{-1} \left[(A - C)^{-1} - (A - D)^{-1}\right]^{-1} \]
\[= (-I) \left[(A - C)^{-1} - (A - B)^{-1}\right]^{-1} \left[(A - C)^{-1} - (A - D)^{-1}\right]^{-1} \]
\[= \left[(A - C)^{-1} - (A - D)^{-1}\right]^{-1} \left[(A - C)^{-1} - (A - B)^{-1}\right]^{-1} \]
\[= c_r(A; C; B, D)\]
from skew-field properties, we have \((-I)^{-1} = -I\) and \((-I)(-I) = I\)

**Theorem 6.** If \(A, B, C, D\) are distinct points in a line, in Desargues affine plane and \(I\) is unitary point for multiplications of points in same line, then,
Proof. (a) In theorem 3 we have proved that 
\[ c_r(A, D; B, C) = I - c_r^{-1}(A, B; C, D) \]
and from theorem 5 we have that 
\[ I - c_r(A, B; D, C) = c_r(A, D; B, C) \].

(b) From theorem 5 we have that, 
\[ c_r(A, C; D, B) = \left[I - c_r(A, B; C, D)\right]^{-1} \]
and from theorem 3 have that 
\[ I - c_r(A, B; D, C) = c_r(A, D; B, C) \].

(c) At this point we will prove that: 
\[ c_r(A, D; C, B) = \left[I - c_r(A, B; C, D)\right]^{-1}c_r(A, B; C, D) \].

Mark the cross-ratios point \( R = c_r(A, B; C, D) \), and rewrite. So we have to prove that the equation holds,
\[ [I - R^{-1}]^{-1} = [R - I]^{-1}R, \]
remember that the points are points of \( \ell^{O1} \)-line, in Desargues affine planes, and can also be thought of as elements of skew-fields \( K = (\ell^{O1}, +, \cdot) \), therefore, we can make algebraic transformations, allowed for skew-fields, and we have
\[ [I - R^{-1}]^{-1} = [R - I]^{-1}R \]
(multiply from the right with \( R^{-1} \))
\[ [I - R^{-1}]^{-1} \cdot R^{-1} = [R - I]^{-1}R \cdot R^{-1} \]
(from skew field property have that \( p^{-1}q^{-1} = (qp)^{-1} \))
\[ [R(I - R^{-1})]^{-1} = [R - I]^{-1}[R \cdot R^{-1}] \]
\[ [R \cdot I - R \cdot R^{-1}]^{-1} = [R - I]^{-1} \cdot I \]
\[ [R - I]^{-1} = [R - I]^{-1} \]
\( \square \)

**Theorem 7.** If \( A, B, C, D \) are distinct points, and different from zero-point \( O \), in a line, in Desargues affine plane and \( I \) is unitary point for multiplications of points in same line, have,
\[ c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = A \cdot c_r(A, B; C, D) \cdot A^{-1} \]

Proof. From cross-ratio definition we have,
\[ c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = [(A^{-1} - D^{-1})^{-1}(B^{-1} - D^{-1})][(B^{-1} - C^{-1})(A^{-1} - C^{-1})] \]
Points \( A, B, C, D \) and \( A^{-1}, B^{-1}, C^{-1}, D^{-1} \), are points of \( \ell^{O1} \)-line in Desargues affine plane, so are and elements of the skew field \( K = (\ell^{O1}, +, \cdot) \). First we prove
that, for two elements \(X, Y\) in a skew field \(K\), we have that \(X^{-1} - Y^{-1} = Y^{-1}(Y - X)X^{-1}\). Indeed

\[
Y^{-1}(Y - X)X^{-1} = [Y^{-1}(Y - X)]X^{-1} \\
= (Y^{-1}Y - Y^{-1}X)X^{-1} \\
= (I - Y^{-1}X)X^{-1} \\
= IX^{-1} - Y^{-1}(XX^{-1}) \\
= X^{-1} - Y^{-1}I \\
= X^{-1} - Y^{-1}.
\]

We use this result in the calculation of \(c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1})\), and have

\[
c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = [(A^{-1} - D^{-1})^{-1}(B^{-1} - D^{-1})] \\
\cdot [(B^{-1} - C^{-1})(A^{-1} - C^{-1})] \\
= [(D^{-1}(D - A)A^{-1})^{-1}(D^{-1}(D - B)B^{-1})] \\
\cdot [(C^{-1}(C - B)B^{-1})(C^{-1}(C - A)A^{-1})] \\
= [(A(D - A)^{-1}D)(D^{-1}(D - B)B^{-1})] \\
\cdot [(B(C - B)^{-1}C)(C^{-1}(C - A)A^{-1})] \\
(\text{from skew field properties } (abc)^{-1} = c^{-1}b^{-1}a^{-1}) \\
= [A(D - A)^{-1}(DD^{-1})(D - B)B^{-1}] \\
\cdot [B(C - B)^{-1}(CC^{-1})(C - A)A^{-1}] \\
(\text{from associative properties for multiplication}) \\
= [A(D - A)^{-1}(D - B)B^{-1}] [B(C - B)^{-1}(C - A)A^{-1}] \\
= A [((D - A)^{-1}(D - B)B^{-1}) [B(C - B)^{-1}(C - A)] ] A^{-1} \\
= A \cdot c_r(A, C; B, D) \cdot A^{-1}.
\]

therefore, we can say that the points, \(c_r(A, C; B, D)\) and \(c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1})\) are conjugate-points in a line of Desargues affine plane. \(\square\)

**Corollary 1.** If the point \(A \in z[K]\) (center of skew field \(K = (t^{O1}, +, \cdot)\), then,

\[c_r(A, C; B, D) = c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}).\]

**Proof.** If \(A \in z[K]\) then, \(AX = XA, \forall X \in K\), so \(AXA^{-1} = X, \forall X \in K\). So, for \(A \in z[K]\), we have that,

\[A \cdot c_r(A, C; B, D) \cdot A^{-1} = c_r(A, C; B, D).
\]

Hence

\[c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}) = c_r(A, C; B, D) \iff A \in z[K]\]

\(\square\)

**Corollary 2.** In Pappus affine plane, \(c_r(A, C; B, D) = c_r(A^{-1}, B^{-1}; C^{-1}, D^{-1}).\)
Theorem 8. If $A, B, C, D$ are distinct points in a line, in Desargues affine plane and $I$ is unitary point for multiplications of points in same line, have,

$$c_r(A; B; C; D) \neq c_r(B; A; D; C)$$

so, $c_r(A; B; C; D)$ is different point from $c_r(B; A; D; C)$.

Proof. From Definition of Cross-Ratio we have,

$$c_r(A; B; C; D) = [(A - D)^{-1}(B - D)] [(B - C)^{-1}(A - C)] = r(B; A; D) \cdot r(A; B; C)$$

and

$$c_r(B; A; D; C) = [(B - C)^{-1}(A - C)] [(A - D)^{-1}(B - D)] = r(A; B; C) \cdot r(B; A; D)$$

We mark the points, like below

$$R_1 = r(A; B; C) \quad \text{and} \quad R_2 = r(B; A; D)$$

so

$$c_r(A; B; C; D) = R_2 \cdot R_1 \quad \text{and} \quad c_r(B; A; D; C) = R_1 \cdot R_2$$

This points are in $\ell^{O1} - \text{line}$ in Desargues affine plane, so are elements of the skew fields $K = (\ell^{O1}, +, \cdot)$, which are constructet over this line, so $E, F, G, H \in K$. So we have,

$$R_2 \cdot R_1 \neq R_1 \cdot R_2 \Rightarrow c_r(A; B; C; D) \neq c_r(B; A; D; C) \quad \square$$

Corollary 3. If $A, B, C, D \in \ell^{O1}$ are distinct points in a line, in Pappus affine plane and $I$ is unital point for multiplications, then

$$c_r(A; B; C; D) = c_r(B; A; D; C)$$

Proof. If affine plane is Pappian plane, then the skew-field $(\ell^{O1}, +, \cdot)$ is commutative, then is a Field. \quad \square

We marked with $K = (\ell^{O1}, +, \cdot)$ the skew field over $\ell^{O1} - \text{line}$ in Desargues affine plane, we know that the center of the skew field $z[K]$, is a sub-skew field of $K$, moreover, $z[K]$ it is also commutative.

Theorem 9. If $A, B, C, D \in \ell^{O1}$ are distinct points in a line, in Desargues affine plane and $I$ is unital point for multiplications of points in same line, then equation

$$c_r(A; B; C; D) = c_r(B; A; D; C)$$

it’s true, if

(a): points $A, B, C, D$ are in ’center of skew-field’ $z[K]$;

(b): ratio-points $r(A; B; C)$ are in ’center of skew-field’;

(c): ratio-point $r(B; A; D)$ are in ’center of skew-field’;

(d): ratio-point $r(A; B; D)$ is in centeralizer of point $r(A; B; C)$, or vice versa.

Proof. (a) If points $A, B, C, D \in z[K]$, we have that,

$$A - D, B - D, B - C, A - C \in z[K]$$

and

$$(A - D)^{-1}, (B - D)^{-1}, (B - C)^{-1}, (A - C)^{-1} \in z[K]$$

also the production is commutative. Hence,

$$[(A - D)^{-1}(B - D)] \cdot [(B - C)^{-1}(A - C)] = [(B - C)^{-1}(A - C)] \cdot [(A - D)^{-1}(B - D)]$$
so,
\[ c_r(A, B; C, D) = c_r(B, A; D, C). \]

(b) If ratio-points \( r(A, B; C) \) are in 'center of skew-field', we have that,
\[ X \cdot r(A, B; C) = r(A, B; C) \cdot X, \quad \forall X \in K \]
so the equation is also true for the ratio-point \( r(B, A; D) \), and have,
\[ r(A, B; C) \cdot r(B, A; D) = r(B, A; D) \cdot r(A, B; C) \]
\[ [(B - C)^{-1}(A - C)] \cdot [(A - D)^{-1}(B - D)] = [(A - D)^{-1}(B - D)] \cdot [(B - C)^{-1}(A - C)] \]
\[ c_r(B, A; D, C) = c_r(A, B; C, D). \]

(c) in the same way, as in case (b).

(d) The Centralizer \( C_K(r(A, B; C)) = \{ Y \in K \mid Y \cdot r(A, B; C) = r(A, B; C) \cdot Y \} \),
and we have that, \( r(B, A; D) \in C_K(r(A, B; C)) \), so we have,
\[ r(B, A; D) \cdot r(A, B; C) = r(A, B; C) \cdot r(B, A; D) \]
so,
\[ c_r(A, B; C, D) = c_r(B, A; D, C), \]
in the same way, it is proved that if \( r(A, B; C) \in C_K(r(B, A; D)) \), then \( c_r(A, B; C, D) = c_r(B, A; D, C) \). \( \square \)

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