COUNTING THE MINIMAL NUMBER OF INFLECTIONS OF A PLANE CURVE

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Abstract. Given a plane curve $\gamma : S^1 \to \mathbb{R}^2$, we consider the problem of determining the minimal number $I(\gamma)$ of inflections which curves $\text{diff}(\gamma)$ may have, where $\text{diff}$ runs over the group of diffeomorphisms of $\mathbb{R}^2$. We show that if $\gamma$ is an immersed curve with $D(\gamma)$ double points and no other singularities, then $I(\gamma) \leq 2D(\gamma)$. In fact, we prove the latter result for the so-called plane doodles which are finite collections of closed immersed plane curves whose only singularities are double points.

1. Introduction

It is obvious that any plane curve $\gamma : S^1 \to \mathbb{R}^2$ diffeomorphic to the figure-eight must have at least two inflection points. Generalizing this observation, B. Shapiro posed in [Sh] the problem of finding/estimating the minimal number of inflection points of a given immersed plane curve having only double points under the action of the group of diffeomorphisms of the plane. He obtained a number of results for the class of the so-called tree-like curves characterized by the property that removal of any double point makes the curve disconnected.

Definition 1. A tree-like curve is a closed immersed plane curve with property: removal of any double point with its neighborhood makes the curve disconnected.

In particular, using a natural plane tree associated to any tree-like curve, he got lower and upper bounds for the number of inflections for such curves and also found a criterion when a tree-like curve can be drawn without inflections.

When we say that a curve $\gamma$ can be drawn with a certain number of inflection points we mean that there is a plane diffeomorphism $\text{diff}$ such that $\text{diff}(\gamma)$ has that many inflections. Respectively drawing is $\text{diff}(\gamma)$.

In what follows we shall work with the following natural generalization of immersed plane curves with at most double points, comp. e.g. [M1].

Definition 2. A doodle is a union of a finite number of closed immersed plane curves without triple intersections.

The main result of this note is as follows.

Theorem 1. Any doodle with $n$ double points can be drawn with at most $2n$ inflection points.

We conjecture the following stronger statement.

Conjecture 1. Any closed plane curve with $n$ double points can be drawn with at most $n+1$ inflection points.

This conjecture is true for tree-like curves.

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Theorem 2. Any tree-like curve with \( n \) double points except figure-eight can be drawn with at most \( n \) inflection points.

The bound from Theorem 2 is tight. There are examples with \( 2k \) double points, which can not be drawn with less than \( 2k \) inflections. We must take the closed curve with alternating \( 2k \) loops by turn outward and inward.

In complement to Theorems 1 and 2, we present in §3 an infinite family of topologically distinct minimal fragments forcing an inflection point which implies that the problem of defining the exact minimal number of inflection points of a given doodle is algorithmically very hard. Therefore there is no chance to obtain an explicit formula for the latter number except for some very special families of plane curves. Our results seem to support the general principle that invariants of curves and knots of geometric origin are difficult to calculate even algorithmically.

Observe that algebraic invariants of doodles similar to Vassiliev invariants of knots were introduced by V. I. Arnold in [Ar] and later considered by number of authors. See especially, [Me1], [Me2].

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2. Proofs

Proof of Theorem 1. Assume the contrary, i.e. that there exists a doodle with \( n \) double points which can not be drawn with less than \( 2n + 1 \) inflections. Let us consider a counterexample with the minimal number of double points.

Obviously our counterexample is not an embedded circle and it is connected. Consider this doodle as an (obvious) planar graph \( G \) with possible multiple edges and loops. Double points are the vertices of this graph, and the arcs connecting double points are the edges.

By faces of a doodle we mean the bounded faces of the (complement to the) planar graph. By the length of a face we mean the number of edges in its boundary.

Lemma 3. A minimal counterexample has the following properties:
• a) there are no faces of length 1.
• b) there are no faces of length 2.
• c) there are no edges of multiplicity \( \geq 3 \).

Proof. a) Assuming that there exists a face of length 1; remove temporarily its boundary and remove the resulting vertex of valency 2 by gluing two edges into one. (It might happen that there will be no vertices left.) Then we obtain a graph corresponding to a doodle with \( n - 1 \) double points.

Thus we can draw a new doodle with at most \( 2n - 2 \) inflection points. Then by returning back the removed face we add no more than 2 inflection points, see Fig. 1. Contradiction with the minimality assumption.

Figure 1. Returning the face of length 1.
b) Assuming that there exists a face of length 2, denote the vertices of this face by $A, B$, and its edges by $l_1, l_2$. Vertices $A$ and $B$ are distinct, since otherwise this common vertex would have valency 4, and therefore there exist edges joining this face with other vertices. But then our doodle has just one double point; it is easy to check that this can not be a counterexample.

Remove edges $l_1, l_2$ and contract $A$ and $B$ to one vertex called $\hat{AB}$. We obtain a new doodle with $n - 1$ double points. By the minimality of our counterexample we can draw it with no more than $2n - 2$ inflection points. Unhinging the double vertex $\hat{AB}$ and smoothing the resulting picture we add exactly two new inflection points, see Fig. 2. Contradiction with the minimality assumption.

![Figure 2. Returning pairs of double edges bounding the face of length 2.](image)

\[\text{Figure 2. Returning pairs of double edges bounding the face of length 2.}\]

c) Assume that there exists a triple edge. Consider edges $l_1, l_2, l_3$, forming this triple edge and connecting a pair of vertices called $A, B$. (Observe that $A$ and $B$ are distinct, since otherwise their valency should be 6, but the maximal valency is 4.)

Edges $l_1, l_2, l_3$ divide the plane in two finite domains and one infinite. Let us denote the finite domains by $\sigma_1, \sigma_2$.

Both vertices $A$ and $B$ have exactly one additional edge each. Either both these edges go inside $\sigma_i$ ($i = 1, 2$), or none of them goes inside $\sigma_i$. (Otherwise in the graph induced by all vertices inside $\sigma_i$ one vertex will have valency 3 and the remaining will have valency 4, but the sum of all valencies must be even!). Thus edges can not go into $\sigma_1$ and $\sigma_2$ simultaneously. Without loss of generality assume that these edges do not go into $\sigma_1$. But then either the doodle is disconnected which is impossible, or $\sigma_1$ is a domain with empty interior which is impossible by b).

\[\square\]

Notice that in our doodle there still might be double edges or loops with non-empty interior. Let us split each loop into three subedges by adding two fake vertices. Additionally in each double edges we split one of them into two subedges by adding one fake vertex.

Denote by $G'$ the obtained planar graph; it does not contain multiple edges or loops. By Fáry’s theorem it has a drawing $\zeta'$ in which all the edges are straight segments and $\zeta'$ is equivalent to the original drawing.

Denote by $\zeta''$ the drawing of the graph $G$ obtained by a smoothening of the angles between the edges at each vertex in the drawing $\zeta$ (see Fig. 3).

**Lemma 4.** In the drawing $\zeta''$ each edge of the graph $G$ contains at most one inflection.

**Proof.** If we do not split an edge, then obviously it has at most one inflection.
If an edge is split into three subedges, then it is a loop. Call it $ABC$, where $B$, $C$ are the fake vertices. Vertex $A$ is the original and hence its valency is 4. Thus it has exactly two other edges. Either both other edges go inside the triangle $\triangle ABC$ or both go outside this triangle.

If they go outside, then either this loop is a face of length 1 or our doodle is disconnected. Hence, both edges go inside $\triangle ABC$ (see Fig. 4 I) and then the loop $ABC$ has no inflections.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{It remains to consider the case when the edge splits into two subedges. Call it $ABC$ with the fake vertex $B$. Then $AC$ is an edge in the graph $G$. Consider the triangle $\triangle ABC$. If we go along the edge $ABC$ across $C$ in the doodle we go inside a triangle or along the edge $CA$. Then the part $BC$ of the edge $ABC$ has no inflection. Hence, there is at most one inflection on the edge $ABC$. In the remaining case we go outside of the triangle; then the fourth edge of $C$ also goes outside it (see Fig. 4 II).

Similarly, we need to consider the case when other edges of vertex $A$ go outside $\triangle ABC$. Then these edges do not go inside $\triangle ABC$, hence, either there is a face of length 2 or the doodle is disconnected. Both case are impossible. Since we covered all possible cases, the lemma is proved. \hfill $\square$

Lemma 3 implies that the number of inflections does not exceed the number of edges, hence it is at most $2n$. Theorem 1 is proved. \hfill $\square$

\begin{proof}[Proof of Theorem 2] We prove that the minimal number of inflections is not more than number of double points plus 1. The idea of the proof without plus 1 can be found in the remark 1.

Now assume the contrary, i.e. that there exists a tree-like curve with $n$ double points which can not be drawn with less than $n + 2$ inflections. Let us consider a counterexample with the minimal number of double points. Obviously, $n > 1$.

Consider the tree that corresponds to our curve. We split our curve into $n + 1$ closed parts of curve, these parts corresponds to vertices of the tree and points of tangency of these parts corresponds to edges of the tree (see fig. more information about appropriate tree see in [Sh]).

If the outer face corresponds to vertex of the tree, then we call this vertex bad, all other vertices are called good. Let $v_1 \ldots v_k$ be the longest path in the tree.

\begin{lemma}
For a minimal counterexample the following conditions are impossible.
\end{lemma}
a) The vertex $v_2$ ($v_{k-1}$) has degree $\deg(v_2) > 2$ and it is adjacent to $\deg(v_2) - 1$ good leaf vertices.

b) The vertex $v_2$ ($v_{k-1}$) has degree $\deg(v_2) = 2$, $v_1, v_2$ are good and $v_1 \cup v_2$ is not boundary of outer face.

Proof. a) The vertex $v_2$ is adjacent to $\deg(v_2)$ vertices and $\deg(v_2) - 1$ of them are good leaves. Consequently, there are two good leaves which are attached in a sequence. Removing these two leaves, we obtain a smaller tree-like curve, hence, it is not a counterexample. We can draw this curve with number of inflections is less than number of double points plus 1 and after that we return two deleted leaves with addition no more than two inflections (see fig. 6, left 1-3).

b) The vertex $v_2$ is adjacent only to vertices $v_1$ and $v_3$, furthermore $v_3$ is attached to outer side of $v_2$. Removing vertices $v_1$ and $v_2$, we obtain a smaller tree-like curve, hence, it is not a counterexample. We can draw this curve with number of inflections is less than number of double points plus 1 and after that we return two deleted vertices with addition no more than two inflections (see fig. 6, 4-7).

Now return to the proof of our theorem. Consider the next case, let $\deg(v_2) > 2$. The vertex $v_2$ is adjacent to at least $\deg(v_2) - 1$ leaves. Hence (by a)), one of these leaves is bad and $k > 3$ (otherwise $v_2$ is adjacent to $\deg(v_2)$ leaves). Then $v_{k-1}$, $v_k$ are good and $v_{k-1} \cup v_k$ is not a boundary of outer face, hence, $\deg(v_{k-1}) > 2$ (otherwise we have a contradiction to b)). Similarly, $v_{k-1}$ is adjacent to the bad vertex too. Then the bad vertex has degree at least two, but it is a leaf in this case. Hence, this case is not possible.

Then $v_2$ has degree 2 and, analogically, the vertex $v_{k-1}$ has degree 2. Furthermore, $v_1$, $v_2$ or $v_1 \cup v_2$ is the boundary of outer face (otherwise we have a contradiction to b)). Similarly, $v_{k-1}$, $v_k$ or $v_{k-1} \cup v_k$ is the boundary of outer face. Hence, $k = 3$ and $v_2$ is a bad vertex. Then all vertices except $v_2$ are attached to the inner side of $v_2$, but this tree-like curve can be drawn without inflections. This is a contradiction. We consider all possible cases, the theorem is proved.

Remark 1. To prove the bound without plus 1 we must prove that tree-like curves with 3 double points are not counterexamples, because our proof is based on the step from $n$ to $n - 2$. 

\[\square\]
3. On minimal fragments forcing an inflection.

Definition 3. A fragment is the union of a finite number of immersed plane curves without triple intersections (up to diffeomorphisms).

Obviously, if a doodle $\gamma$ has $k$ disjoint fragments forcing an inflection (see next definition), then any drawing of $\gamma$ contains at least $k$ inflections.

Definition 4. A fragment is called a minimal fragment forcing an inflection if the following two conditions are satisfied (see Fig. 7):

- any drawing of this fragment necessarily contains an inflection point.
- removing any double point or any curve or cutting any curve (between two double points) we obtain a fragment which can be drawn without inflection points.

![Figure 7. Fragments forcing an inflection point. $a$ – non-minimal, $b, c$ – minimal.](image)

Remark 2. Obviously, any minimal fragment forcing an inflection is connected.

In this section we construct an infinite series of minimal fragments forcing an inflection. Additionally, this construction implies the following result:

Theorem 6. There exists $c > 0$ such that the number of fragments forcing an inflection with at most $n$ double points is at least $e^{cn}$.

The above theorem is true even for fragments consisting of curves without self-intersections, but for $n$ at least some $N_0$. That fact in its turn makes it very hard not only to count the minimum number of inflections of a given doodle but also to find a criterion when a doodle can be drawn without inflections. Now let us construct a series of minimal fragments.

Definition 5. A key $b$ for the curve $z$ is a curve shown in Fig. 8.

![Figure 8. Key $b$ for a curve $z$.](image)

Lemma 7. (1) If a drawing of a curve $z$ has no inflections, then its key determines the direction of convexity of the curve $z$.

(2) If a curve $z$ is convex in the right direction, then its key can always be drawn without inflections.
(3) If a part of key is removed, then the remaining parts of the key can always be drawn without inflections.

**Proof.** Items 1 and 3 are obvious. In the right-hand of Fig. 8 it is shown how to draw a key in item 2.

Now we present an infinite series of distinct minimal fragments forcing an inflection. It consists of fragments having the following form:

- $k \geq 3$ curves bound a domain in which each curve intersects only with its neighbors and goes after crossing inside the domain (see Fig. 9 left).
- Each of these curves has either a key of type II or III or a loop close to one of its endpoints (see Fig. 9).

![Figure 9. Minimal fragments forcing inflection points.](image)

**Theorem 8.** Fragments in the above series are minimal fragments forcing inflections.

**Proof.** Consider a fragment consisting of $k$ curves (excluding keys). This fragment must contain an inflection point, because otherwise all curves are convex inwards (due to the presence of keys or loops) and the "vertices" of the $k$-gon have the same convexity, but this is impossible.

It remains to prove that this fragment is minimal.

1° If we remove something from at least one key or a loop or cut a loop or a key of type II. Then we can draw one curve convex outwards and the others convex inwards (see Fig. 10 left). After that we can draw all loops and other keys without inflections.

2° If we remove a part of a curve inside a key of type III. Then we can draw the part of a curve convex outwards, the other part of this curve and other curves convex inwards, and this key of type III without inflections (see Fig. 10 middle and right). After that we can draw all loops and keys without inflections.

3° If we remove a part of curve inside a key of type II. This case can be proved by combining the ideas of cases 1° and 2°. We draw "big" part of the curve convex outwards and other $k - 1$ curves inwards (see Fig. 10 left) and later we draw the key of type II with second part of this curve on the end of "big" part (see Fig. 10 right). After that we can draw all loops and keys without inflections.

4° If we cut a curve in the boundary of $k$-gon outside a key of type III. This case is obvious. We can do all $k$ curves with convexity in the correct direction, because we should not build a "$k$-gon".

We have considered all possible cases, so the theorem is proved.

**Proof of Theorem 2.** We will use only loops (similarly, we could use only keys of type II and III). Fixing $k > 0$, we have 2 possibilities for each loop. Hence, we have at least $2^k/k$ minimal fragments, because each fragment is considered at most $k$ times. They have exactly $2k$ double points. Now it is obvious that there exists desired $c > 0$. 

□
Figure 10.

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