QUADRATIC ALGEBRAS BASED ON $SL(NM)$ ELLIPTIC QUANTUM $R$-MATRICES

I. A. Sechin∗† and A. V. Zotov∗‡

We construct a quadratic quantum algebra based on the dynamical $RLL$-relation for the quantum $R$-matrix related to $SL(NM)$-bundles with a nontrivial characteristic class over an elliptic curve. This $R$-matrix simultaneously generalizes the elliptic nondynamical Baxter–Belavin and the dynamical Felder $R$-matrices, and the obtained quadratic relations generalize both the Sklyanin algebra and the relations in the Felder–Tarasov–Varchenko elliptic quantum group, which are reproduced in the respective particular cases $M = 1$ and $N = 1$.

Keywords: quantum quadratic algebras, elliptic integrable system, quantum dynamical $R$-matrix

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1. Sklyanin algebra

We consider the Baxter–Belavin quantum $R$-matrix [1], [2]

$$R_{12}^{BB}(h, u) = \sum_{\alpha \in \mathbb{Z}_N^2} \varphi_{\alpha}(u, h + \omega_{\alpha})T_{\alpha} \otimes T_{-\alpha}. \quad (1)$$

This definition involves the elliptic functions $\varphi_{\alpha}(u, x + \omega_{\alpha})$ and the $(N \times N)$ basis matrices $T_{\alpha}$ connected with these functions. They are defined in Appendix A. This $R$-matrix satisfies the quantum Yang–Baxter equation in $\text{Mat}(N, \mathbb{C}) \otimes^3$:

$$R_{12}^{BB}(h, z_{12})R_{13}^{BB}(h, z_{13})R_{23}^{BB}(h, z_{23}) = R_{23}^{BB}(h, z_{23})R_{13}^{BB}(h, z_{13})R_{12}^{BB}(h, z_{12}). \quad (2)$$

Here, the notation $z_{ij} = z_i - z_j$ is used, and the lower indices in $R$-matrices denote the numbers of tensor components where they act nontrivially. For instance, in (2),

$$R_{13}^{BB}(h, z_{13}) = \sum_{\alpha} \varphi_{\alpha}(z_{13}, h + \omega_{\alpha})T_{\alpha} \otimes 1_N \otimes T_{-\alpha}. \quad (3)$$

An operator $L(z)$ is called the $R$-operator for the Baxter–Belavin $R$-matrix if it satisfies the $RLL$-relation

$$R_{12}^{BB}(h, z_1 - z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R_{12}^{BB}(h, z_1 - z_2). \quad (4)$$
In [3], Sklyanin suggested a class of $L$ operators for $N = 2$. His result was later extended to arbitrary $N$, with other possible parameters of the underlying bundles over an elliptic curve also taken into account [4]–[6]. The constructed $L$ operators are connected with a quadratic algebra called the Sklyanin algebra.

We consider an $L$ operator of the form

$$L(z) = \sum_\alpha \varphi_\alpha(z, h + \omega_\alpha)S_\alpha T_\alpha.$$  

(5)

$RLL$-relation (4) for this $L$ operator is equivalent to the following quadratic relations for operators $S_\alpha$ labeled by pairs $(\alpha, \beta)$ and independent of the spectral parameters $z_1$ and $z_2$:

$$\sum_\gamma \kappa_{\gamma, \alpha-\beta}(E_1(\omega_\gamma + h) - E_1(\omega_{\alpha-\gamma} + h) + E_1(\omega_{\alpha-\gamma} + h) - E_1(\omega_{\beta+\gamma} + h))S_{\alpha-\gamma}S_{\beta+\gamma} = 0$$

(6)

for $\beta \neq 0$ and

$$\sum_\gamma \kappa_{\gamma, \alpha}(E_2(\omega_\gamma + h) - E_2(\omega_{\alpha-\gamma} + h))S_{\alpha-\gamma}S_\gamma = 0$$

(7)

for $\beta = 0$, where $E_1(z)$ and $E_2(z)$ are the functions defined in Appendix A. The collection of numbers

$$\kappa_{\alpha, \beta} = \exp\left(\frac{\pi i}{N}(\beta_1\alpha_2 - \beta_2\alpha_1)\right)$$

(8)

defines the structure constants of relations (6), (7), called the Sklyanin algebra relations. For example, the operators $S_\alpha = T_{-\alpha}$ satisfy these relations. In this case, the $RLL$-relation turns into Yang–Baxter equation (2).

Definition (5) and relations (6) and (7) can be slightly modified. The $L$ operator can be divided by a function depending on $z$ only, because this function cancels in both parts of the $RLL$-relation. We write the $\varphi_\alpha$ function explicitly:

$$\varphi_\alpha(z, h + \omega_\alpha) = \phi(z, h + \omega_\alpha)e^{(2\pi i/N)\alpha z} = \frac{\theta'(0)\theta(z + h + \omega_\alpha)}{\theta(z)\theta(h + \omega_\alpha)}e^{(2\pi i/N)\alpha z}. $$

(9)

Dividing the $L$ operator in Eq. (5) by $\theta'(0)/\theta(z)$, we obtain

$$L^h(z) = \sum_\alpha \frac{\theta(z + h + \omega_\alpha)}{\theta(h + \omega_\alpha)}e^{(2\pi i/N)\alpha z}S_\alpha T_\alpha. $$

(10)

The factor $\theta(h + \omega_\alpha)$ is independent of the spectral parameter and can therefore be eliminated by redefining $S_\alpha$. In this case, the $L$ operator takes the form

$$L^h(z) = \sum_\alpha \frac{\theta(z + h + \omega_\alpha)}{\theta(h + \omega_\alpha)}e^{(2\pi i/N)\alpha z}S_\alpha T_\alpha, \quad S_\alpha = \frac{S_\alpha}{\theta(h + \omega_\alpha)}. $$

(11)

The Sklyanin algebra relations are also modified as follows:

$$\beta \neq 0: \sum_\gamma \kappa_{\gamma, \alpha} \kappa_{\beta, \gamma}[E_1(\omega_\gamma + h) - E_1(\omega_{\beta+\gamma} + h) +
+ E_1(\omega_{\beta+\gamma} + h) - E_1(\omega_{\alpha-\gamma} + h)]\theta(h + \omega_{\beta+\gamma})\theta(h + \omega_{\alpha-\gamma})S_{\alpha-\gamma}S_{\beta+\gamma} = 0,$$

(12)

$$\beta = 0: \sum_\gamma \kappa_{\gamma, \alpha}[E_2(\omega_\gamma + h) - E_2(\omega_{\alpha-\gamma} + h)]\theta(h + \omega_{\alpha+\gamma})\theta(h + \omega_{\alpha-\gamma})S_{\alpha-\gamma}S_\gamma = 0.$$  

Moreover, we can replace the parameter $h$ in the $L$ operator with another parameter by shifting $z$ because the $R$-matrix depends on the difference $z_1 - z_2$ only. We can then define the operator

$$L^\eta(z) = L^h(z + \eta - h) = \sum_\alpha \theta(z + \eta + \omega_\alpha)e^{(2\pi i/N)\alpha z}S^\eta_\alpha T_\alpha,$$

(13)

$$S^\eta_\alpha = \tilde{S}_\alpha e^{(2\pi i/N)\alpha z(\eta - h)}.$$  

The relations for $S^\eta_\alpha$ are similar to the relations for $\tilde{S}_\alpha$ up to these exponential factors.
2. Elliptic quantum group

We consider the Felder dynamical quantum $R$-matrix [7]

$$R_{12}^F(h, u \mid q) = \sum_{i=1}^{M} \phi(u, h) E_{ii} \otimes E_{ii} + \sum_{i,j=1 \atop i \neq j}^{M} \phi(u, q_{ij}) E_{ij} \otimes E_{ji} + \sum_{i,j=1 \atop i \neq j}^{M} \phi(h, -q_{ij}) E_{ii} \otimes E_{jj},$$  \hspace{1cm} (14)

where $q_{ij} = q_i - q_j$, $E_{ij}$ are $(M \times M)$ matrices with the entries $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$, and $\phi$ is the elliptic Kronecker function defined in Appendix A. The term “dynamical” means that the $R$-matrix depends on the dynamical parameters $q_i$.

The $R$-matrix in Eq. (14) satisfies the quantum dynamical Yang–Baxter equation

$$R_{12}^F(h, z_{12} \mid q) R_{13}^F(h, z_{13} \mid q - h^{(2)}) R_{23}^F(h, z_{23} \mid q) = R_{23}^F(h, z_{23} \mid q - h^{(1)}) R_{13}^F(h, z_{13} \mid q) R_{12}^F(h, z_{12} \mid q - h^{(3)}).$$

In this equation, shifts along the Cartan subalgebra $\{E_{ii}\}$ in $\mathfrak{gl}(M)$ are used:

$$R_{12}^F(h, z_{12} \mid q - h^{(3)}) = e^{-h \hat{\partial}_3} R_{12}^F(h, z_{12} \mid q) e^{h \hat{\partial}_3}, \quad \hat{\partial}_3 = \sum_k (E_{kk})_3 \partial_{q_k}. \hspace{1cm} (15)$$

In addition to the quantum dynamical Yang–Baxter equation, the $R$-matrix also satisfies the zero-weight conditions

$$[(E_{ii})_1 + (E_{ii})_2, R_{12}^F(h, z_{12} \mid q)] = 0,$$

$$[\hat{\partial}_1 + \hat{\partial}_2, R_{12}^F(h, z_{12} \mid q)] = 0.$$

Let $h_i$, $i = 1, 2, \ldots, M$ be commuting elements. An operator $L(z \mid q)$ is called a dynamical $L$ operator with the Cartan elements $h_i$ for the Felder $R$-matrix if it satisfies the dynamical RLL-relation

$$R_{12}^F(h, z_{12} \mid q) L_1(z_1 \mid q - h^{(2)}) L_2(z_2 \mid q) = L_2(z_2 \mid q - h^{(1)}) L_1(z_1 \mid q) R_{12}^F(h, z_{12} \mid q - h \cdot h),$$

$$R_{12}^F(h, z_{12} \mid q - h \cdot h) = \exp \left[ -h \sum_k h_k \frac{\partial}{\partial q_k} \right] R_{12}^F(h, z_{12} \mid q) \exp \left[ h \sum_k h_k \frac{\partial}{\partial q_k} \right]. \hspace{1cm} (16)$$

The dynamical Yang–Baxter equation implies that the Felder $R$-matrix is a dynamical $L$ operator with the Cartan elements $h_1 = (E_{ii})_3$:

$$L_1(z \mid q) = R_{13}^F(h, z \mid q). \hspace{1cm} (17)$$

The RLL-relations can be rewritten in equivalent form if we act on both sides from the left by the operator $e^{h \hat{\partial}_1} e^{h \hat{\partial}_2}$. Using the zero-weight property $[\hat{\partial}_1 + \hat{\partial}_2, R_{12}^F(h \mid u \mid q)] = 0$, we then obtain $[e^{h \hat{\partial}_1} e^{h \hat{\partial}_2}, R_{12}^F(h \mid u \mid q)] = 0$. Therefore, we have

$$e^{h \hat{\partial}_1} e^{h \hat{\partial}_2} R_{12}^F(h, z_{12} \mid q) L_1(z_1 \mid q - h^{(2)}) L_2(z_2 \mid q) =$$

$$= e^{h \hat{\partial}_1} e^{h \hat{\partial}_2} L_2(z_2 \mid q - h^{(1)}) L_1(z_1 \mid q) R_{12}^F(h, z_{12} \mid q - h \cdot h),$$

$$e^{h \hat{\partial}_1} e^{h \hat{\partial}_2} R_{12}^F(h, z_{12} \mid q) e^{-h \hat{\partial}_2} L_1(z_1 \mid q) e^{h \hat{\partial}_2} L_2(z_2 \mid q) =$$

$$= e^{h \hat{\partial}_2} L_2(z_2 \mid q) e^{h \hat{\partial}_1} L_1(z_1 \mid q) R_{12}^F(h, z_{12} \mid q - h \cdot h).$$
and

\[ R_{12}^F(h, z_{12} | q) e^{\hbar \partial_z} L_1(z_1 | q) e^{\hbar \partial_z} L_2(z_2 | q) = e^{\hbar \partial_z} L_2(z_2 | q) e^{\hbar \partial_z} L_1(z_1 | q) R_{12}^F(h, z_{12} | q - h \cdot \hbar). \]

We define the operators \( \tilde{L}(u | q) = e^{\hbar \partial_u} L(u | q) \). Relations (16) can then be rewritten in the form

\[ R_{12}^F(h, z_{12} | q) \tilde{L}_1(z_1 | q) \tilde{L}_2(z_2 | q) = \tilde{L}_2(z_2 | q) \tilde{L}_1(z_1 | q) R_{12}^F(h, z_{12} | q - h \cdot \hbar). \quad (18) \]

In [8], Tarasov and Varchenko constructed dynamical \( L \) operators and the related quadratic algebra that is also known as the small elliptic quantum group. We consider \( q_k \) and \( q_k - \hbar h_k \) in the \( R \)-matrices in (18) as independent coordinates and use the notation \( q_k^{(2)} = q_k, q_k^{(1)} = q_k - \hbar h_k \) for these two new sets of variables. Then the \( RLL \)-relation becomes

\[ R_{12}^F(h, z_{12} | q^{(2)}) \tilde{L}_1(z_1 | q^{(1)}, q^{(2)}) \tilde{L}_2(z_2 | q^{(1)}, q^{(2)}) = \tilde{L}_2(z_2 | q^{(1)}, q^{(2)}) \tilde{L}_1(z_1 | q^{(1)}, q^{(2)}) R_{12}^F(h, z_{12} | q^{(1)}). \quad (19) \]

We consider the ansatz for the \( L \) operator

\[ \tilde{L}(z | q) = \sum_{i,j} \theta(z + q_i^{(2)} - q_j^{(1)}) t_{ij} E_{ij}, \quad (20) \]

where \( t_{ij} \) are operators that do not commute with the coordinates \( q_k^{(1)} \), but shift them by \( \hbar \) according to the rule

\[ t_{ij} f(q_1^{(1)}, \ldots, q_i^{(1)}, \ldots, q_M^{(1)}, q_1^{(2)}, \ldots, q_j^{(2)}, \ldots, q_M^{(2)}) = f(q_1^{(1)}, \ldots, q_i^{(1)} + \hbar, \ldots, q_M^{(1)}, q_1^{(2)}, \ldots, q_j^{(2)}, \ldots, q_M^{(2)}) t_{ij}, \]

where \( f \) is an arbitrary function of the \( q_k^{(1)} \). Dynamical \( RLL \)-relation (19) for this \( L \) operator is equivalent to the following quadratic relations for the \( t_{ij} \):

\[ t_{ij} t_{ik} = t_{ik} t_{ij}, \quad t_{ik} t_{jk} = \frac{\theta(q_{ij}^{(1)} - \hbar)}{\theta(q_{ij}^{(1)} + \hbar)} t_{jk} t_{ik}, \quad i \neq j, \]

\[ \frac{\theta(q_{ij}^{(2)} - \hbar)}{\theta(q_{ij}^{(2)} + \hbar)} t_{ij} t_{kl} - \frac{\theta(q_{ik}^{(1)} - \hbar)}{\theta(q_{ik}^{(1)} + \hbar)} t_{ik} t_{ij} = -\frac{\theta(\hbar)\theta(q_{ik}^{(1)} + q_{ij}^{(2)})}{\theta(q_{ik}^{(1)})\theta(q_{ij}^{(2)})} t_{il} t_{kj}, \quad i \neq k, \quad j \neq l. \]

These quadratic relations define the (small) elliptic Felder–Tarasov–Varchenko quantum group.

3. A quadratic algebra for the \( SL(NM) \) \( R \)-matrix

We consider the quantum \( R \)-matrix related to an \( SL(NM) \) bundle with a nontrivial characteristic class over the elliptic curve. This \( R \)-matrix was constructed in [9]. It simultaneously generalizes the nondynamical Baxter–Belavin quantum \( R \)-matrix and the dynamical Felder quantum \( R \)-matrix, and can be represented in the form

\[ R_{ab}^H(h, z_{12} | q) = \sum_i (E_{ii})_a (E_{ii})_b R_{12}^{BB}(h, z_{12}) + \sum_{i \neq j} (E_{ij})_a (E_{ji})_b R_{12}^{BB}(q_{ij}, z_{12}) + \sum_{i \neq j} (E_{ii})_a (E_{jj})_b \otimes 1_N \otimes 1_N \phi(h, -q_{ij}). \quad (21) \]
Here, the spaces labeled by small Latin letters are \((M \times M)\) matrix spaces in the standard basis, and the spaces labeled by numbers are \((N \times N)\) matrix spaces in basis \((A.6)\). This quantum \(R\)-matrix satisfies the dynamical quantum Yang–Baxter equation with shifts along the Cartan subalgebra corresponding to \((M \times M)\) matrices only (i.e., of the form \(h_i \otimes 1_N\)):

\[
R_{ab12}^h(z_{12} \mid q)R_{ac13}^h(z_{13} \mid q - h(b))R_{bc23}^h(z_{23} \mid q) = \\
= R_{be23}^h(z_{23} \mid q - h(a))R_{ac13}^h(z_{13} \mid q)R_{ab12}^h(z_{12} \mid q - h(c)).
\]

An operator \(L_{a1}(z \mid q^{(1)}, q^{(2)})\) is called the \(L\) operator for this quantum \(R\)-matrix if it satisfies the following \(RLL\)-relation

\[
R_{ab12}^h(z_{12} \mid q^{(2)})L_{a1}(z_1 \mid q^{(1)}, q^{(2)})L_{b2}(z_2 \mid q^{(1)}, q^{(2)}) = \\
= L_{b2}(z_2 \mid q^{(1)}, q^{(2)})L_{a1}(z_1 \mid q^{(1)}, q^{(2)})R_{ab12}^h(z_{12} \mid q^{(1)}).
\]

The main result in this paper is the description of the quadratic algebra associated with this \(RLL\)-relation. We choose an \(L\) operator in the form

\[
L_{a1}(z_1 \mid q^{(1)}, q^{(2)}) = \sum_{ij} (E_{ij})_a L_{ij}^a(z_1 \mid q^{(1)}, q^{(2)}),
\]

\[
L_{ij}(z \mid q) = \sum_\alpha \theta(z + q^{(2)}_i - q^{(1)}_j + \omega_\alpha) t^{\alpha}_{ij} T_\alpha.
\]

The operators \(t^{\alpha}_{ij}\) shift the coordinates \(q_k\) by the rule

\[
t^{\alpha}_{ij} f(q^{(1)}_1, \ldots, q^{(1)}_i, \ldots, q^{(1)}_M, q^{(2)}_1, \ldots, q^{(2)}_j, \ldots, q^{(2)}_M) = \\
= f(q^{(1)}_1, \ldots, q^{(1)}_i + h, \ldots, q^{(1)}_M, q^{(2)}_1, \ldots, q^{(2)}_j + h, \ldots, q^{(2)}_M) t^{\alpha}_{ij}.
\]

Then the \(RLL\)-relation is equivalent to the following set of quadratic relations for the generators \(t^{\alpha}_{ij}\):

1. For coincident pairs of indices \(i, j\), the elements \(\{t^{\alpha}_{ij} \mid \alpha \in \mathbb{Z}_N^2\}\) satisfy the Sklyanin algebra relations with the parameter \(\eta = q^{(2)}_i - q^{(1)}_j\).

2. For coincident second indices and distinct first indices \(i, j, k, j \neq k\),

\[
\sum_\gamma \kappa^{\alpha\beta}_\gamma \phi(h + \omega_\gamma, q^{(1)}_{jk} + \omega_{j^+ - a}) t^{\gamma^+\gamma}_{jk} t^{\beta\alpha}_{ki} = \phi(h, -q^{(1)}_{jk}) t^{\beta\alpha}_{ki} j^{\alpha}_{ji}.
\]

3. For coincident first indices and distinct second indices \(i, j, k, j \neq k\),

\[
\sum_\gamma \kappa^{\alpha\beta}_\gamma \phi(h + \omega_{-\beta - \gamma}, -q^{(2)}_{jk} - \omega_\gamma) t^{\alpha \gamma}_{ik} t^{\beta\gamma}_{ij} = \phi(h, -q^{(2)}_{jk}) t^{\beta\gamma}_{ij} i^{\alpha\gamma}_{ik}.
\]

4. For distinct first and second pairs of indices \(i, j, k, l, i \neq j, k \neq l\)

\[
\sum_\gamma \kappa^{\alpha\beta}_\gamma \phi(q^{(2)}_{ik} + \omega_\gamma, q^{(1)}_{jl} + \omega_{j^+ - a}) t^{\alpha^+\gamma}_{jk} t^{\beta\gamma}_{li} = \\
= \phi(h, -q^{(1)}_{jl}) t^{\beta\gamma}_{li} i^{\alpha\gamma}_{jk} - \phi(h, -q^{(2)}_{jk}) t^{\beta\gamma}_{ij} i^{\alpha\gamma}_{lk}.
\]

In the case \(M = 1\), there are only \(\{t^{\alpha}_{ij} \mid \alpha \in \mathbb{Z}_N^2\}\) generators satisfying the Sklyanin algebra relations, and in the case \(N = 1\) there are only the elliptic quantum groups generators \(\{t^{0}_{ij} \mid i, j \in 1, 2, \ldots, M\}\). Therefore, the constructed quadratic algebra generalizes these two quantum algebras simultaneously.

The proof of this equivalence is straightforward, it can be verified using elliptic function identities given in Appendix A. An example of the calculation for a particular tensor component is presented in Appendix B.
4. Conclusion

The quadratic algebra generalizing the elliptic quantum group and the Sklyanin algebra is constructed. On one hand, it is a classification-type result, which complements and generalizes the known structures of quadratic algebras related to bundles over elliptic curves. On the other hand, the obtained results can be applied to the description of concrete mechanical systems. It was shown in [10] that the considered quantum $R$-matrix is connected with quantum long-range spin chains and $R$-matrix-valued Lax pairs. Moreover, this particular $R$-matrix in the nonrelativistic classical limit describes a system of interacting tops. The relativistic analogue of this system was also obtained recently using a natural ansatz for the Lax pair [11]. Therefore, the result in this paper can also be regarded as the description of the operator algebra underlying the model of quantum relativistic interacting tops.

Appendix A: Elliptic functions and their properties

In the definitions of $R$-matrices in this paper, we use the Kronecker elliptic functions

$$\varphi_\alpha(u, x + \omega_\alpha) = \phi(u, x + \omega_\alpha)e^{(2\pi i/N)\alpha_2 u}, \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N},$$

which are expressed through the odd theta function

$$\theta(u) = -\sum_{k \in \mathbb{Z}} \exp\left(\pi i \tau \left( k + \frac{1}{2} \right)^2 + 2\pi i \left( k + \frac{1}{2} \right) \left( u + \frac{1}{2} \right) \right). \quad (A.2)$$

Here, $\tau$ — a complex parameter with $\text{Im} \tau > 0$ — is the modular parameter of the elliptic curve underlying all elliptic functions.

The main tool for the derivation of the quadratic relations is the addition formula (also known as the genus-one Fay identity) for the Kronecker functions

$$\phi(z, x)\phi(w, y) = \phi(z - w, x)\phi(w, x + y) + \phi(w - z, y)\phi(z, x + y) \quad (A.3)$$

and its degenerations corresponding to coincident values of variables

$$\phi(z, x)\phi(z, y) = \phi(z, x + y)(E_1(z) + E_1(x) + E_1(y) - E_1(x + y + z)), \quad (A.4)$$

where the Eisenstein functions are used:

$$E_1(z) = \frac{\theta'(z)}{\theta(z)}, \quad E_2(z) = -E_1'(z) \quad (A.5)$$

In the definition of the Baxter–Belavin quantum $R$-matrix, the basis matrices $T_\alpha$ are used. They are defined as

$$T_\alpha = T_{(\alpha_1, \alpha_2)} = e^{\pi i \alpha_1 \alpha_2 / N} Q_{\alpha_1} A_{\alpha_2},$$

$$Q_{jk} = \delta_{jk} e^{2\pi i k / N}, \quad A_{jk} = \begin{cases} 1, & j + 1 = k \mod N, \\ 0, & \text{otherwise}. \end{cases} \quad (A.6)$$
Appendix B: An example of calculation verifying the RLL-relation

We consider, for example, the \((E_{ij})_a(E_{ik})_b\)-component of the RLL-relation:

\[
R^{BB}_{12}(h, z_{12})L^i_1(z_1) L^k_2(z_2) = L^i_2(z_2) L^j_1(z_1) \phi(h, -q_{jk}^{(1)}) + L^j_2(z_2) L^k_1(z_1) R^{BB}_{12}(q_{kj}^{(1)}, z_{12}), \quad j \neq k. \tag{B.1}
\]

This relation is given in \((N \times N)\)-matrices. Decomposing it in the basis \(T_\alpha\), we obtain the following scalar relations in components \((T_\alpha)_1(T_\beta)_2\) (after canceling all exponential factors):

\[
\theta(z_2 + q_i^{(2)} - q_i^{(1)} + \omega_\beta) \theta(z_1 + q_i^{(2)} - q_j^{(1)} + h + \omega_\alpha) \phi(h, -q_{jk}^{(1)}) = \\
= \sum_\gamma \varepsilon_{\gamma \alpha} \varepsilon_{\beta \gamma} \phi(z_{12}, h + \omega_\gamma) \theta(z_1 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma) \times \\
\times t_{ji}^{\alpha - \gamma} \theta(z_2 + q_i^{(2)} - q_k^{(1)} + \omega_\beta + \gamma) - \theta(z_2 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma) \times \\
\times t_{ji}^{\alpha - \gamma} \phi(z_{12}, q_{kj}^{(1)} + \omega_\alpha - \beta - \gamma).
\]

Moving all \(t_{ab}\) to the right yields

\[
\theta(z_2 + q_i^{(2)} - q_k^{(1)} + \omega_\beta) \theta(z_1 + q_i^{(2)} - q_j^{(1)} + h + \omega_\alpha) \phi(h, -q_{jk}^{(1)}) = \\
= \sum_\gamma \varepsilon_{\gamma \alpha} \varepsilon_{\beta \gamma} \phi(z_{12}, h + \omega_\gamma) \theta(z_1 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma) \times \\
\times \theta(z_2 + q_i^{(2)} - q_k^{(1)} + h + \omega_\beta + \gamma) - \\
- \theta(z_2 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma) \theta(z_1 + q_i^{(2)} - q_k^{(1)} + h + \omega_\beta + \gamma) \times \\
\times \phi(z_{12}, q_{kj}^{(1)} + \omega_\alpha - \beta - \gamma) \theta_1^{\alpha - \gamma} \theta_2^{\beta + \gamma}.
\]

We divide both sides by \(\theta(z_2 + q_i^{(2)} - q_k^{(1)} + \omega_\beta)\) and consider the expression in the brackets in the right-hand side. It can be simplified:

\[
\phi(z_{12}, h + \omega_\gamma) \frac{\theta(z_1 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma) \theta(z_2 + q_i^{(2)} - q_k^{(1)} + h + \omega_\beta + \gamma)}{\theta(z_1 + q_i^{(2)} - q_j^{(1)} + h + \omega_\alpha) \theta(z_2 + q_i^{(2)} - q_k^{(1)} + \omega_\beta)} = \\
- \phi(z_{12}, q_{kj}^{(1)} + \omega_\alpha - \beta - \gamma) \frac{\theta(z_1 + q_i^{(2)} - q_j^{(1)} + h + \omega_\alpha) \theta(z_2 + q_i^{(2)} - q_k^{(1)} + \omega_\beta + \gamma)}{\theta(z_1 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha, q_{kj}^{(1)} + \omega_\beta + \gamma - \alpha)} = \\
\phi(z_{12}, h + \omega_\gamma) \frac{\phi(z_1 + q_i^{(2)} - q_k^{(1)} + \omega_\beta, h + \omega_\gamma)}{\phi(z_1 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma, h + \omega_\gamma)} \times \\
- \phi(z_{12}, q_{kj}^{(1)} + \omega_\alpha - \gamma) \phi(z_1 + q_i^{(2)} - q_j^{(1)} + h + \omega_\alpha, q_{kj}^{(1)} + \omega_\beta + \gamma - \alpha) \times \\
\times \phi(z_1 + q_i^{(2)} - q_k^{(1)} + \omega_\alpha - \gamma, h + \omega_\gamma) \times \\
\times \phi(z_1 + q_i^{(2)} - q_j^{(1)} + \omega_\alpha - \gamma, q_{kj}^{(1)} + \omega_\beta + \gamma - \alpha).
\]
Applying the Fay identity to $\phi$ and using the property $\phi(x, -x) = 0$, we obtain
\[
\phi(z_2 + q_1^{(1)} - q_k^{(1)} + \omega_{\beta}, h + \omega_{\gamma})\phi(z_2 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha}) = \\
\quad = \phi(q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha}, h + \omega_{\gamma})\phi(z_2 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, q_{jk}^{(1)} + h + \omega_{\beta+2\gamma-\alpha}),
\]
\[
\phi(z_1 + q_1^{(2)} - q_j^{(1)} + h + \omega_{\alpha}, q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha})\phi(z_1 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, h + \omega_{\gamma}) = \\
\quad = \phi(h + \omega_{\gamma}, q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha})\phi(z_1 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, q_{jk}^{(1)} + h + \omega_{\beta+2\gamma-\alpha}).
\]

We can pull the factor $\phi(h + \omega_{\gamma}, q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha})$ out of the numerator; by Fay’s identity, the remaining part is then exactly equal to the denominator:
\[
\phi(z_{12}, h + \omega_{\gamma})\phi(z_2 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, q_{jk}^{(1)} + h + \omega_{\beta+2\gamma-\alpha}) - \\
\quad - \phi(z_{12}, q_{jk}^{(1)} + \omega_{\alpha-\beta-\gamma})\phi(z_1 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, q_{jk}^{(1)} + h + \omega_{\beta+2\gamma-\alpha}) = \\
\quad = \phi(z_1 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, h + \omega_{\gamma})\phi(z_2 + q_1^{(2)} - q_j^{(1)} + \omega_{\alpha-\gamma}, q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha}).
\]

Using this simplification, we obtain the required relation without spectral parameters:
\[
\sum_\gamma \kappa_{\gamma,\alpha}\kappa_{\gamma,\beta}\phi(h + \omega_{\gamma}, q_{jk}^{(1)} + \omega_{\beta+\gamma-\alpha})t_{ji}^{\alpha-\gamma}t_{ki}^{\beta+\gamma} = \phi(h, -q_{jk}^{(1)})t_{ji}^{\beta}t_{ki}^{\alpha}.
\]

All other relations can be verified similarly by considering the other components of the $RLL$-relation.

**Conflicts of interest.** The authors declare no conflicts of interest.

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