Let $N^0(m,n)$ be the number of odd Durfee symbols of $n$ with odd rank $m$, and $N^0(a,M;n)$ be the number of odd Durfee symbols of $n$ with odd rank congruent to $a$ modulo $M$. We show that the odd rank can be expressed as a linear sum of ordinary ranks. We also give explicit formulas for the generating functions of $N^0(a,M;n)$ and their $\ell$-dissections where $0 \leq a \leq M-1$ and $M, \ell \in \{2,4,8\}$. From these formulas, we obtain some interesting arithmetic properties of $N^0(a,M;n)$. Furthermore, let $D^0_k(n)$ denote the number of $k$-marked odd Durfee symbols of $n$. Andrews conjectured that $D^0_2(n)$ is even if $n \equiv 4$ or $6 \pmod{8}$ and $D^0_3(n)$ is even if $n \equiv 1, 9, 11$ or $13 \pmod{16}$. Using our results on odd ranks, we prove Andrews’ conjectures.

1. Introduction and Main Results

Given a positive integer $n$, a partition $\alpha = (a_1, a_2, \cdots, a_r)$ of $n$ is a non-increasing sequence of positive integers that add up to $n$. The $a_i$’s are called the parts of $\alpha$. We denote by $l(\alpha) := r$ the number of parts in $\alpha$ and call $|\alpha| := a_1 + a_2 + \cdots + a_r$ the weight of $\alpha$. As usual, let $p(n)$ denote the number of partitions of $n$ and we agree that $p(0) = 1$ for convention. Its generating function satisfies

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}, \quad (1.1)$$

where

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad (1.2)$$

and

$$(a;q)_\infty := \lim_{n \to \infty} (a;q)_n, \quad |q| < 1. \quad (1.3)$$

Ramanujan discovered the following beautiful congruences:

$$p(5n+4) \equiv 0 \pmod{5}, \quad (1.4)$$
$$p(7n+5) \equiv 0 \pmod{7}, \quad (1.5)$$
$$p(11n+6) \equiv 0 \pmod{11}. \quad (1.6)$$

In order to explain Ramanujan’s congruences combinatorially, Dyson [11] introduced the concept of rank. The rank of a partition is defined as its largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of $n$ with rank $m$, and
let $N(a, M; n)$ denote the number of partitions of $n$ with rank $\equiv a \pmod{M}$. It is well known that
\[
R_1(z; q) := 1 + \sum_{m=\infty}^{\infty} \sum_{n=1}^{\infty} N(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^n^2}{(zq; q)_n(z^{-1}q; q)_n}.
\]

Dyson [11] conjectured that for $(r, M) \in \{(4, 5), (5, 7)\}$, $0 \leq a \leq M - 1$ and $n \geq 0$,
\[
N(a, M; Mn + r) = 1 \frac{M}{M} p(Mn + r). \tag{1.8}
\]

Atkin and Swinnerton-Dyer [8] proved (1.8) by studying the generating functions of $N(a, M; n)$ with $M = 5$ or 7. The arithmetic relations in (1.8) gave a satisfactory explanation to the congruences (1.4) and (1.5). To give a similar combinatorial interpretation to the congruence (1.6), another combinatorial quantity, the crank was guessed by Dyson [11] and finally given explicitly in the paper of Andrews and Garvan [6]. Since we are not going to consider crank in this paper, we refer the interested reader to [6].

Around 2003, Atkin and Garvan [7] considered the $k$-th moment of the rank which is defined as
\[
N_k(n) := \sum_{m=\infty}^{\infty} m^k N(m, n). \tag{1.9}
\]
Since $N(-m, n) = N(m, n)$, all the odd order moments are zero. Andrews [1] discovered that there is a rich combinatorial and enumerative structure associated with the moments of ranks. He considered a symmetrized $k$-th moment function
\[
\eta_k(n) := \sum_{m=\infty}^{\infty} \left( m + \left\lfloor \frac{k-1}{2} \right\rfloor \right) N(m, n). \tag{1.10}
\]
Again, it is easy to see that $\eta_{2k+1}(n) = 0$. By introducing the concepts of Durfee symbols and $k$-marked Durfee symbols, Andrews gave a combinatorial interpretation for $\eta_{2k}(n)$ by showing that $\eta_{2k}(n)$ equals the number of $(k + 1)$-marked Durfee symbols of $n$. For our purpose, we will not discuss Durfee symbols and $k$-marked Durfee symbols. At the end of his paper, Andrews [1] proposed several open problems. In particular, he asked the reader to prove that $\eta_4(n) \equiv 0 \pmod{5}$ if $n \equiv 24 \pmod{25}$. In this paper, we not only give a proof to this congruence but also find a companion congruence for $\eta_6(n)$.

**Theorem 1.1.** For any integer $n \geq 0$ we have
\[
\eta_4(25n + 24) \equiv 0 \pmod{5}, \tag{1.11}
\]
\[
\eta_6(49n + 47) \equiv 0 \pmod{7}. \tag{1.12}
\]

In [1] Andrews also introduced odd Durfee symbols and $k$-marked odd Durfee symbols. The motivation for Andrews to introduce odd Durfee symbols is to give a natural combinatorial explanation to an identity associated with Watson’s third order mock theta function $\omega(q)$ [20], which is defined as
\[
q \omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+1}}{(q; q^2)_n^{n+1}}. \tag{1.13}
\]
Fine [12, Eq. (26.84)] discovered that \( \omega(q) \) satisfies

\[
 q^{\omega(q)} = \sum_{n=0}^{\infty} (q; q^2)_{n+1} q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^{n+(n-1)})}. \tag{1.14}
\]

The right side of the above identity is the generating function of certain partitions. To be more specific, let \( p_\omega(n) \) be the number of partitions of \( n \) wherein at least all but one instance of the largest part is one of a pair of consecutive non-negative integers. Then (1.14) can be restated as

\[
 q^{\omega(q)} = \sum_{n=0}^{\infty} p_\omega(n) q^n. \tag{1.15}
\]

By utilizing MacMahon’s modular partitions with modulus 2, Andrews showed that each partition enumerated by \( p_\omega(n) \) has associated with it an odd Durfee symbol of \( n \).

**Definition 1.2.** An odd Durfee symbol of \( n \) is a two-rowed array with a subscript of the form

\[
\begin{pmatrix}
 a_1 & a_2 & \cdots & a_s \\
 b_1 & b_2 & \cdots & b_t
\end{pmatrix}_D
\]

wherein all the entries are odd numbers such that

1. \( 2D + 1 \geq a_1 \geq a_2 \geq \cdots \geq a_s > 0 \);
2. \( 2D + 1 \geq b_1 \geq b_2 \geq \cdots \geq b_t > 0 \); and
3. \( n = \sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j + 2D^2 + 2D + 1 \).

The odd rank of an odd Durfee symbol is the number of entries in the top row minus the number of entries in the bottom row.

The definition here is equivalent to Andrews’ original definition but in slightly different form. Actually we are using the rephrased version from the work of Ji [14].

Let \( N^0(m,n) \) denote the number of odd Durfee symbols of \( n \) with odd rank \( m \). By interchanging the rows of the symbol, it is clear that

\[
 N^0(m,n) = N^0(-m,n). \tag{1.16}
\]

Andrews [1, Sec. 8] proved that

\[
 p_\omega(n) = \sum_{m=-\infty}^{\infty} N^0(m,n). \tag{1.17}
\]

and [1, Eq. (8.3)]

\[
 R^0_1(z; q) := \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} N^0(m,n) z^n q^n \\
 = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+1}}{(zq; q^2)_{n+1} (z^{-1}q; q^2)_{n+1}}. \tag{1.18}
\]

Clearly, if we let \( z = 1 \) in (1.18), we recover Fine’s formula (1.15).

Recently, Andrews, Dixit and Yee [4] showed that \( p_\omega(n) \) also counts the number of partitions of \( n \) in which all odd parts are less than twice the smallest part. Besides its rich combinatorial meanings, the function \( p_\omega(n) \) also possess many interesting congruence
properties. Using the theory of modular forms, Waldherr \[18\] gave the first explicit example of congruences satisfied by $p_{\omega}(n)$. He proved that for any integer $n \geq 0$,

$$p_{\omega}(40n + r) \equiv 0 \pmod{5}, \quad r \in \{28, 36\}. \quad (1.19)$$

An elementary proof of these congruences was given by Andrews, Passary, Sellers and Yee \[5\]. They also established some congruences like

$$p_{\omega}(8n + 4) \equiv 0 \pmod{4}, \quad (1.20)$$

$$p_{\omega}(8n + 6) \equiv 0 \pmod{8}, \quad \text{and} \quad (1.21)$$

$$p_{\omega}(16n + 13) \equiv 0 \pmod{4}. \quad (1.22)$$

For more congruences satisfied by $p_{\omega}(n)$, we refer the reader to \[4, 5, 18, 19\]. Clearly, the roles of the odd rank $N^{0}(m,n)$ and $p_{\omega}(n)$ are similar to the roles of the rank $N(m,n)$ and $p(n)$, and knowing the properties of odd rank will be helpful to understand $p_{\omega}(n)$.

By analyzing the generating functions \[1.7\] and \[1.18\], we find that the odd rank is closely related to the ordinary rank.

**Theorem 1.3.** Let $n \geq 1$ and $m$ be integers.

1. If $n \equiv m \pmod{2}$, then

$$N^{0}(m,n) = 0. \quad (1.23)$$

2. If $n \equiv m + 1 \pmod{2}$, then we have

$$N^{0}(m,n) - N^{0}(m - 1, n - 1) = N \left( m + 1, \frac{n - m - 1}{2} \right) \quad (1.24)$$

and

$$N^{0}(m,n) = \sum_{k=0}^{n-1} N \left( m + 1 - k, \frac{n - m - 1}{2} \right). \quad (1.25)$$

An interesting consequence is

**Corollary 1.4.** If $n \leq 3m + 5$ and $n \equiv m + 1 \pmod{2}$, we have

$$N^{0}(m,n) = p \left( \frac{n - m - 1}{2} \right). \quad (1.26)$$

From Theorem \[1.3\], we know that studying the properties of odd ranks also helps us to explore the properties of the ordinary ranks.

Similar to \[1.10\], Andrews \[1\] considered a symmetrized $k$-th moment function

$$\eta_{k}^{0}(n) := \sum_{m=-\infty}^{\infty} \left( m + \left\lfloor \frac{k}{2} \right\rfloor \right) N^{0}(m,n). \quad (1.27)$$

Since $N^{0}(m,n) = N^{0}(-m,n)$, we have $\eta_{2k+1}^{0}(n) = 0$. Using Watson’s first identity on page 66 of \[20\], we have \[1 \text{ Eq. (8.5)}\]

$$R_{1}^{0}(z;q) = \frac{1}{(q^{2};q^{2})_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n}q^{3n^{2}+3n+1}}{1 - zq^{2n+1}}. \quad (1.28)$$
Using (1.28), Andrews deduced that [1, Theorem 21]
\[
\sum_{n=1}^{\infty} \eta_{2k}(n) q^n = \frac{1}{(q^2;q^2)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2 + (2k+3)n + k + 1}}{(1 - q^{2n+1})^{2k+1}}.
\] (1.29)

To give a combinatorial explanation of (1.27), Andrews introduced \( k \)-marked odd Durfee symbols.

**Definition 1.5.** A \( k \)-marked odd Durfee symbol of \( n \) is composed of \( k \) pairs of partitions into odd parts with a subscript, given by
\[
\eta^0 = \left( \alpha^{k}, \alpha^{k-1}, \ldots, \alpha^{1} \right),
\]
where \( \alpha^i \) (resp. \( \beta^i \)) are all partitions with odd parts such that
\[
\sum_{i=1}^{k} (|\alpha^i| + |\beta^i|) + 2D^2 + 2D + 1 = n,
\] (1.30)
and the following conditions are satisfied:
1. For \( 1 \leq i < k \), \( \alpha^i \) is a nonempty partition, while \( \alpha^k \) and \( \beta^i \) could be empty;
2. \( \beta_i^{i-1} \leq \alpha_i^{i-1} \leq \beta_{l(\beta^i)} \) for \( 2 \leq i \leq k \);
3. \( \beta_i^{k}, \alpha_i^{k} \leq 2D + 1 \).

Here \( \alpha_1^{i} \) (resp. \( \beta_1^{i} \)) denotes the largest part of the partition \( \alpha^i \) (resp. \( \beta^i \)), and \( \alpha_i^{l(\alpha^i)} \) (resp. \( \beta_i^{l(\beta^i)} \)) denotes the smallest part of the partition \( \alpha^i \) (resp. \( \beta^i \)).

Let \( D^0_k(n) \) denote the number of \( k \)-marked odd Durfee symbols of \( n \). Andrews [1, Corollary 29] proved that for \( k \geq 0 \),
\[
D^0_{k+1}(n) = \eta_{2k}(n).
\] (1.31)

Andrews also investigated the parity of \( D^0_k(n) \). He proved that for each \( k \geq 1 \), if \( n \equiv k - 1 \) (mod 2), then \( D^0_k(n) \) is even. Andrews [1, Conjectures A and B] then proposed the following conjectures.

**Conjecture 1.6.** \( D^0_2(n) \) is even if \( n \equiv 4 \) or 6 (mod 8).

**Conjecture 1.7.** \( D^0_3(n) \) is even if \( n \equiv 1, 9, 11 \) or 13 (mod 16).

One of the main goals of this paper is to confirm the above conjectures. To prove these results, from (1.31) it suffices to prove the corresponding congruences for \( \eta_{2k}(n) \) and \( \eta_{0}(n) \). This observation will eventually turn our attention to the arithmetic properties of \( N^0(a; M; n) \) with \( 0 \leq a \leq M - 1 \) and \( M \in \{4, 8\} \) after analyzing (1.27), where \( N^0(a; M; n) \) denotes the number of odd Durfee symbols of \( n \) with odd rank congruent to \( a \) modulo \( M \). Hence, we need to study the odd ranks modulo 4 and 8. Santa-Gadea and Lewis [15] proved a number of results on ranks and cranks modulo 4 and 8. Recently, Andrews, Berndt, Chan, Kim and Malik [3] found some results on the ranks modulo 4 and 8. For example, they showed that [3] (7.5), (7.6)
\[
N(0, 4; 2n) - N(2, 4; 2n) = (-1)^n \left( N(0, 8; 2n) - N(4, 8; 2n) \right),
\] (1.32)
\[
N(0, 4; 2n + 1) - N(2, 4; 2n + 1) = (-1)^n \left( N(0, 8; 2n + 1) + N(1, 8; 2n + 1) \right)
\]
Later Mortenson \cite{16} used different methods to prove these results and obtained generating functions for $N(a, M; n) - p(n)/M$ where $M \in \{4, 8\}$. Motivated by their works, we are able to give explicit formulas for generating functions associated with $N^0(a, M; \ell n + r)$, where $0 \leq a \leq M - 1$, $0 \leq r \leq \ell - 1$ and $M, \ell \in \{2, 4, 8\}$ (see Theorems 5.1-3.3). Using these generating functions, we prove some interesting arithmetic relations analogous to (1.8), (1.32) and (1.33).

**Theorem 1.8.** For any integer $n \geq 0$ we have
\[
N^0(0, 8; 8n + r) = N^0(4, 8; 8n + r), \quad r \in \{5, 7\},
\]
\[
N^0(1, 8; 8n + r) = N^0(3, 8; 8n + r), \quad r \in \{4, 6\}.
\]

Meanwhile, we give simple formulas for the generating functions of certain odd rank differences. To state these formulas, let $J_n := (q^m; q^m)_\infty$ and we recall a universal mock theta function
\[
g(x; q) := x^{-1} \left( 1 + \sum_{n=0}^{\infty} \frac{q^n}{(x; q)_{n+1}(q/x; q)_n} \right).
\]

**Theorem 1.9.** We have
\[
\sum_{n=0}^{\infty} \left( N^0(0, 8; 8n + 1) - N^0(4, 8; 8n + 1) \right) q^n = \frac{J_3^3}{J_1^2 J_4},
\]
\[
\sum_{n=0}^{\infty} \left( N^0(0, 8; 8n + 3) - N^0(4, 8; 8n + 3) \right) q^n = \frac{J_3^4}{J_2^3} + qg(q^2; q^4),
\]
\[
\sum_{n=0}^{\infty} \left( N^0(1, 8; 8n) - N^0(3, 8; 8n) \right) q^n = qg(q; q^4),
\]
\[
\sum_{n=0}^{\infty} \left( N^0(1, 8; 8n + 2) - N^0(3, 8; 8n + 2) \right) q^n = \frac{J_2 J_4}{J_1}.
\]

An unexpected consequence of this theorem is the following amazing relation.

**Corollary 1.10.** For any integer $n \geq 1$ we have
\[
N^0(0, 8; 16n - 5) - N^0(4, 8; 16n - 5) = p_\omega(n).
\]

As a supplementary result to Theorem 1.8 by Theorem 1.9 we prove some strict inequalities between odd ranks.

**Theorem 1.11.** For any integer $n \geq 0$ ($n \geq 1$ when $r = 0$) we have
\[
N^0(0, 8; 8n + r) > N^0(4, 8; 8n + r), \quad r \in \{1, 3\},
\]
\[
N^0(1, 8; 8n + r) > N^0(3, 8; 8n + r), \quad r \in \{0, 2\}.
\]

Finally, we present some congruences satisfied by the odd ranks.

**Theorem 1.12.** Let $k$ be a positive integer. For any integer $n \geq 0$ we have
\[
N^0(k, 2k; n) \equiv 0 \pmod{2},
\]
\[
N^0(0, 4; 8n + r) \equiv 0 \pmod{4}, \quad r \in \{5, 7\},
\]
\begin{align*}
N^0(0, 4; 16n + 13) &\equiv 0 \pmod{8}, \quad (1.46) \\
N^0(1, 4; 8n + 4) &\equiv 0 \pmod{2}, \quad (1.47) \\
N^0(1, 4; 8n + 6) &\equiv 0 \pmod{4}, \quad (1.48) \\
N^0(1, 4; 40n + r) &\equiv 0 \pmod{5}, \quad r \in \{28, 36\}, \quad (1.49) \\
N^0(2, 4; 16n + 13) &\equiv 0 \pmod{4}, \quad (1.50) \\
N^0(0, 8; 8n + r) &\equiv 0 \pmod{2}, \quad r \in \{5, 7\}, \quad (1.51) \\
N^0(0, 8; 16n + 15) &\equiv 0 \pmod{16}, \quad (1.52) \\
N^0(2, 8; 16n + 11) &\equiv 0 \pmod{8}, \quad (1.53) \\
N^0(2, 8; 16n + 13) &\equiv 0 \pmod{2}. \quad (1.54)
\end{align*}

The paper is organized as follows. In Section 2 we collect some formulas which will be used frequently in our proofs. In Section 3 we present the generating functions for $N(a, M; n)$ ($0 \leq a \leq M − 1$ and $M \in \{2, 4, 8\}$) and their $\ell$-dissections with $\ell \in \{2, 4, 8\}$. Section 4 is devoted to the proofs of all the theorems and Andrews’ conjectures. We also point out that many congruences in the literature including (1.19)–(1.22) are direct consequences of our theorems.

2. Preliminaries

From [13, Eq. (4.2)] we find
\begin{equation}
g(x; q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x; q)_{n+1}(q/x; q)_{n+1}}. \quad (2.1)
\end{equation}

It follows that
\begin{equation}
g(x; q) = g(q/x; q). \quad (2.2)
\end{equation}

For example, we have
\begin{equation}
g(iq; q^2) = g(-iq; q^2). \quad (2.3)
\end{equation}

Comparing (1.18) with (2.1) we obtain
\begin{equation}
R_0^0(z; q) = qg(zq; q^2). \quad (2.4)
\end{equation}

We define
\begin{equation}
j(x; q) := (x; q)_{\infty}(q/x; q)_{\infty}(q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} x^n, \quad (2.5)
\end{equation}

where the last equality follows from Jacobi’s triple product identity. Meanwhile, we use the following notations:
\begin{equation}
J_{a, m} := j(q^a; q^m), \quad \bar{J}_{a, m} := j(-q^a; q^m). \quad (2.6)
\end{equation}

The following product rearrangements will be used frequently:
\begin{equation}
\bar{J}_{1, 4} = \frac{J_2^2}{J_1}, \quad \bar{J}_{1, 2} = \frac{J_2^5}{J_1^2 J_4}, \quad J_{1, 2} = \frac{J_1^2}{J_2}, \quad J_{1, 4} = \frac{J_1 J_4}{J_2}. \quad (2.7)
\end{equation}
Lemma 2.1. (Cf. [17] (p. 39), [2] (12.4.4).) For generic \( x \in \mathbb{C} \) we have
\[
g(x; q) = -x^{-1} + qx^{-3}g(-qx^{-2}; q^4) - qg(-qx^2; q^4) + \frac{J_5^2}{xj(x; q)j(-qx^2; q^2)J_4^2}.
\] (2.8)

Lemma 2.2. (Cf. [17] p. 39, [2] (12.4.4).) For generic \( x \in \mathbb{C} \) we have
\[
g(x; q) + g(-x; q) = -2qg(-qx^2; q^4) + \frac{2J_5^2}{j(-qx^2; q^4)j(x^2; q^2)J_4^2},
\] (2.9)
\[
g(x; q) - g(-x; q) = -2x^{-1} + 2qx^{-3}g(-qx^{-2}; q^4) + \frac{2J_5^2}{xj(-q^3x^2; q^4)j(x^2; q^2)J_4^2}.
\] (2.10)

As some consequences to this lemma, if we replace \( q \) by \( q^2 \) and set \( x = q \) in (2.9) (resp. (2.10)), we obtain
\[
g(q; q^2) + g(-q; q^2) = -2q^{-2}g(-q^4; q^8) + 2\frac{J_4^8 J_5^2}{J_2^4 J_8^2}
\] (2.11)
\[
g(q; q^2) - g(-q; q^2) = -2q^{-1} + 2q^{-1}g(-1; q^8) + q^{-1}\frac{J_6^4 J_8}{J_2^4 J_16^2},
\] (2.12)
respectively.

Let \( \zeta_M = e^{2\pi i/M} \) throughout this paper. In the same way, we can deduce the following identities, which will be used in Section 34
\[
g(iq; q^2) + g(-iq; q^2) = -2q^2g(q^4; q^8) + 2\frac{J_3^8}{J_4^2},
\] (2.13)
\[
g(i; q) - g(-i; q) = 2i + 2iqg(q; q^4) - 2i\frac{J_5^2}{J_1^4 J_4^2},
\] (2.14)
\[
g(\zeta_M q; q^2) + g(-\zeta_M q; q^2) = -2q^2g(-q^4; q^8) + 2\frac{J_5^2 J_3^4}{J_2^2 J_8^4 J_16},
\] (2.15)
\[
g(\zeta_M q; q^2) - g(-\zeta_M q; q^2) = -2\zeta_M^{-1}q^{-1} - 2\zeta_M g(i; q^8) + (\zeta_M + \zeta_M^{-1})q^{-1}\frac{J_5^4 J_8^2}{J_2^4 J_3^8 J_16},
\] (2.16)
\[
g(i\zeta_M q; q^2) + g(-i\zeta_M q; q^2) = -2q^2g(iq^4; q^8) + 2\frac{J_4^8 J_3^2}{J_2^2 J_8^4 J_16},
\] (2.17)
\[
g(i\zeta_M q; q^2) - g(-i\zeta_M q; q^2) = 2\zeta_M q^{-1} + 2\zeta_M^{-1}q^{-1}g(-i; q^8) - (\zeta_M + \zeta_M^{-1})q^{-1}\frac{J_4^8 J_16^2}{J_2^2 J_8^4 J_3^2}.
\] (2.18)

Lemma 2.3. We have
\[
J_1^2 = \frac{J_2^4 J_5^8}{J_4^2 J_16^2} - 2g\frac{J_2^4 J_16^2}{J_8^2},
\] (2.19)
\[
J_4^4 = \frac{J_4^2 J_16^2}{J_2^2 J_8^2} - 4q\frac{J_2^4 J_16^2}{J_8^2},
\] (2.20)
\[
\frac{1}{J_2^2} = \frac{J_8^2}{J_2^2 J_16^2} + 2g\frac{J_2^4 J_16^2}{J_8^2},
\] (2.21)
\[
\frac{1}{J_1^2} = \frac{J_4^2}{J_2^2 J_16^2} + 4q\frac{J_2^4 J_16^2}{J_8^2}.
\] (2.22)
Proof. Recall two important theta functions
\[ \phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{J_5}{J_1 J_4^2}, \]
\[ \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{J_2}{J_1}. \]
From [9, Entry 25 (v), (vi)] we find
\[ \phi(q) = \phi(q^4) + 2q \psi(q^8), \] (2.23)
\[ \phi^2(q) = \phi^2(q^2) + 4q \psi^2(q^4). \] (2.24)
These two identities immediately lead to (2.21) and (2.22). Next, replacing \( q \) by \(-q\) in (2.23) and (2.24) we obtain
\[ \phi(-q) = \phi(q^4) - 2q \psi(q^8), \] (2.25)
\[ \phi^2(-q) = \phi^2(q^2) - 4q \psi^2(q^4). \] (2.26)
Since
\[ \phi(-q) = \frac{J_1^2}{J_2^2}, \] (2.27)
(2.19) and (2.20) follow immediately from (2.25) and (2.26).

**Corollary 2.4.** We have
\[ \frac{1}{J_1^6} = \left( \frac{J_8 J_4^{14}}{J_2^{19} J_1^{16}} + 8q^2 \frac{J_4^4 J_8^3 J_1^{16}}{J_2^2} \right) + 2q \left( \frac{J_4^{16} J_1^{16}}{J_2^{19} J_8} + 2 \frac{J_4^2 J_8^9}{J_1^{19} J_2^2} \right) \] (3.2)
and
\[ \frac{1}{J_1^8} = \left( \frac{J_4^{28}}{J_2^{28} J_8^8} + 16q^2 \frac{J_4^4 J_8^8}{J_2^{20}} \right) + 8q \frac{J_4^{16}}{J_2^{24}}. \] (2.29)

**Proof.** Multiplying (2.21) and (2.22), we get (2.28). Taking square on both sides of (2.22), we get (2.29). \( \square \)

3. Odd Ranks Modulo 2, 4 and 8

In this section, we obtain the generating functions of \( N^0(a, M; \ell n + r) \) with \( M, \ell \in \{2, 4, 8\}, \) \( 0 \leq a \leq M - 1 \) and \( 0 \leq r \leq \ell - 1. \) From (1.16) we know that we only need to consider \( 0 \leq a \leq \frac{M}{2} \) since \( N^0(a, M; n) = N^0(M - a; M; n). \) Moreover, when \( M, \ell \in \{2, 4, 8\}, \) from Theorem 1.3 (1) (which will be proved in Section 4) we know that \( N^0(a, M; \ell n + r) = 0 \) if \( a \) and \( r \) have the opposite parity. Thus we only need to consider the case that \( a \equiv r \pmod{2}. \)

To get the generating function for \( N^0(a, M; n), \) using (1.18) and the fact that
\[ \frac{1}{M} \sum_{j=0}^{M-1} \zeta_M^{kj} = \begin{cases} 1, & k \equiv 0 \pmod{M}, \\ 0, & k \not\equiv 0 \pmod{M}, \end{cases} \] (3.1)
we obtain the following identity:
\[ \sum_{n=0}^{\infty} N^0(a, M; n) q^n = \frac{1}{M} \sum_{j=0}^{M-1} \zeta_M^{-aj} R_1^{0j}(\zeta_M^j; q). \] (3.2)
From (2.4) we deduce that
\[
\sum_{n=0}^{\infty} N^0(a, M; n)q^n = \frac{q}{M} \sum_{j=0}^{M-1} \zeta_M^{-aj} g(\zeta_M^j q; q^2). \tag{3.3}
\]

Now we consider the odd rank modulo 2.

**Theorem 3.1.** We have
\[
\sum_{n=0}^{\infty} N^0(0, 2; 2n + 1)q^n = -qg(-q^2; q^4) + \frac{J_2^6 J_8}{J_1 J_4}, \tag{3.4}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 2; 4n + 1)q^n = \frac{J_2^6}{J_1 J_4}, \tag{3.5}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 2; 4n + 3)q^n = \frac{J_2^6}{J_1 J_4}, \tag{3.6}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 2n)q^n = -1 + g(-1; q^4) + \frac{1}{2} \frac{J_2^6 J_8}{J_1 J_4}, \tag{3.7}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 4n)q^n = -1 + g(-1; q^2) + \frac{1}{2} \frac{J_2^{15}}{J_1 J_4}, \tag{3.8}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 4n + 2)q^n = 2 \frac{J_2^6 J_8}{J_1}, \tag{3.9}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 2; 8n + 1)q^n = \frac{J_2^{12} J_8}{J_1 J_8} + 8q \frac{J_2^6 J_4^3 J_8^2}{J_1}, \tag{3.10}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 2; 8n + 3)q^n = qg(-q^2; q^4) - \frac{J_2^6 J_8}{J_1 J_4} + 4 \frac{J_2^6 J_4^5}{J_1 J_8}, \tag{3.11}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 2; 8n + 5)q^n = 2 \left( \frac{J_2^6 J_4^2}{J_1 J_4} + \frac{J_2^6}{J_1 J_8} \right), \tag{3.12}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 2; 8n + 7)q^n = -q^{-1} + q^{-1} g(-1; q^4) + \frac{1}{2} q^{-1} \frac{J_2^6 J_8}{J_1 J_4} + 8 \frac{J_2^6 J_8}{J_1 J_4}, \tag{3.13}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 8n)q^n = -1 + g(-1; q) + \frac{1}{2} \frac{J_2^{12}}{J_1 J_4} + \frac{J_2^6 J_8}{J_1 J_4}, \tag{3.14}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 8n + 2)q^n = 2 \frac{J_2^{16}}{J_1 J_4}, \tag{3.15}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 8n + 4)q^n = 4 \frac{J_2^{10}}{J_1}, \tag{3.16}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 2; 8n + 6)q^n = 8 \frac{J_2^4 J_4}{J_1}. \tag{3.17}
\]
Proof. Setting \((a, M) = (0, 2)\) in (3.3) we obtain
\[
\sum_{n=0}^{\infty} N^0(0, 2; n)q^n = \frac{q}{2} (g(q; q^2) + g(-q; q^2)).
\] (3.18)
Substituting (2.11) into (3.18), extracting the odd power terms, dividing by \(q\) and replacing \(q^2\) by \(q\), we obtain (3.4). Next, substituting (2.22) into (3.4) and extracting the even (resp. odd) power terms, we obtain (3.5) (resp. (3.6)). Now, substituting (2.28) into (3.8) and extracting the even power terms, we obtain (3.10). Finally, substituting (2.29) into (3.8) and extracting the even power terms, we obtain (3.11) (resp. (3.13)).

Replacing \(q\) by \(q^2\) and setting \(x = -q\) in Lemma 2.1, we obtain
\[
g(-q; q^2) = q^{-1} - q^{-1}g(-1; q^8) - q^2g(-q^4; q^8) + \frac{J^8_4J^2_8}{J^4_1J^8_1} - \frac{1}{2}q^{-1}J^6_4J^8_8. \] (3.19)
Substituting (2.22) and (3.19) into (3.6) and extracting the even (resp. odd) power terms, we obtain (3.11) (resp. (3.13)).

Similarly, setting \((a, M) = (1, 2)\) in (3.3) we obtain
\[
\sum_{n=0}^{\infty} N^0(1, 2; n)q^n = \frac{q}{2} (g(q; q^2) - g(-q; q^2)).
\] (3.20)
Substituting (2.12) into (3.20) and extracting the even power terms, we obtain (3.7). Next, substituting (2.22) into (3.7) and extracting the even (resp. odd) power terms, we obtain (3.8) (resp. (3.9)). Again, substituting (2.29) into (3.8) and extracting the even (resp. odd) power terms, we obtain (3.11) (resp. (3.14)). Finally, substituting (2.22) into (3.9) and extracting the even (resp. odd) power terms, we obtain (3.15) (resp. (3.17)).

Next we consider the odd rank modulo 4.

**Theorem 3.2.** We have

\[
\sum_{n=0}^{\infty} N^0(0, 4; 2n + 1)q^n = q^5g(-q^4; q^16) - q\frac{J^8_4J^2_8}{J^4_1J^8_1} + \frac{1}{2} \left( \frac{J^8_4J^2_8}{J^4_1J^8_1} + \frac{J^8_4}{J^2_8} \right),
\] (3.21)
\[
\sum_{n=0}^{\infty} N^0(0, 4; 4n + 1)q^n = \frac{J^4_2J^5_8}{J^4_1J^2_4J^8_1},
\] (3.22)
\[
\sum_{n=0}^{\infty} N^0(0, 4; 4n + 3)q^n = q^2g(-q^4; q^8) - \frac{J^8_4J^2_8}{J^4_1J^8_1} + 2\frac{J^8_4}{J^2_8},
\] (3.23)
\[
\sum_{n=0}^{\infty} N^0(0, 4; 8n + 1)q^n = \frac{J^{12}_2J^4_4}{J^{10}_1J^8_8},
\] (3.24)
\[
\sum_{n=0}^{\infty} N^0(0, 4; 8n + 3)q^n = gg(-q^2; q^4) + 2\frac{J^6_4J^5_8}{J^8_1J^2_8} - \frac{J^8_4J^2_8}{J^4_1J^8_1},
\] (3.25)
\[
\sum_{n=0}^{\infty} N^0(0, 4; 8n + 5)q^n = 4\frac{J^9_4}{J^5_1J^7_8},
\] (3.26)
\[
\sum_{n=0}^{\infty} N^0(0, 4; 8n + 7)q^n = 4\frac{J^8_4J^2_8}{J^4_1J^8_4}.
\] (3.27)
Proof. (1) Setting $(a, M) = (0, 4)$ in \((3.3)\) we obtain
\[
\sum_{n=0}^{\infty} N^0(0, 4; n)q^n = \frac{q}{4} \left( (g(q; q^2) + g(-q; q^2)) + (g(iq; q^2) + g(-iq; q^2)) \right). \tag{3.42}
\]
Substituting (2.11) and (2.13) into (3.42), we get
\[
\sum_{n=0}^{\infty} N^0(0,4;n)q^n = \frac{q}{4} (-2q^2 (g(q^4;q^8) + g(-q^4;q^8)) + 2\frac{J_4^6J_8^2}{J_4^2J_8^2} + 2J_4^3).
\] (3.43)

Replacing \( q \) by \( q^4 \) in (2.11), we obtain
\[
g(q^4;q^8) + g(-q^4;q^8) = -2q^8 g(-q^{16};q^{32}) + 2\frac{J_4^6J_8^2}{J_4^2J_8^2}.
\] (3.44)

From (3.44) we see that (3.43) reduces to
\[
\sum_{n=0}^{\infty} N^0(0,4;2n+1)q^n = q^5 g(-q^8;q^{16}) - q\frac{J_8^5J_4^2}{J_4^2J_8^2} + \frac{1}{2} \left( \frac{J_4^8J_8^2}{J_4^2J_8^2} + \frac{J_4^6}{J_4^2} \right).
\] (3.45)

This proves (3.21).

Substituting (2.22) into (3.45), we arrive at
\[
\sum_{n=0}^{\infty} N^0(0,4;2n+1)q^n = q^5 g(-q^8;q^{16}) - q\frac{J_8^5J_4^2}{J_4^2J_8^2} + \frac{1}{2} \left( \frac{J_4^8J_8^2}{J_4^2J_8^2} + \frac{J_4^6}{J_4^2} \right) + 2q\frac{J_4^6}{J_4^2J_4^2}.
\] (3.46)

Extracting the even power terms, we obtain
\[
\sum_{n=0}^{\infty} N^0(0,4;4n)q^n = \frac{1}{2} \left( \frac{J_4^9}{J_4^2J_4^2} + \frac{J_4^3}{J_4^2} \right)
\]
\[
= \frac{1}{2} \frac{J_4^3}{J_4^2J_4^2} \left( J_4^2J_4^2 + J_4^6 \right)
\]
\[
= \frac{1}{2} \frac{J_4^3}{J_4^2J_4^2} \left( J_4^2J_4^2 - 2qJ_4^2J_8^2 \right) J_8^2 + \frac{J_4^6}{J_4^2J_4^2} \left( J_4^2J_4^2 + 2qJ_4^2J_8^2 \right)
\]
\[
= \frac{J_4^6}{J_4^2J_4^2}.
\]

where in the last second line we have used (2.11) and (2.22). This proves (3.22). Now we substitute (2.22) into (3.22) and extracting the even (resp. odd) power terms, we obtain (3.24) (resp. (3.26)).

Extracting the odd power terms in (3.46), we obtain
\[
\sum_{n=0}^{\infty} N^0(0,4;4n+3)q^n = q^2 g(-q^8;q^8) - \frac{J_4^8J_8^2}{J_4^2J_8^2} + 2\frac{J_4^6}{J_4^2J_4^2}.
\] (3.47)

Substituting (2.21) into (3.47) and extracting the even (resp. odd) power terms, we get (3.25) (resp. (3.27)).

(2) These identities can be proved in the same way as (1). Alternatively, we observe that \( N^0(1,4;n) = N^0(3,4;n) \) for any integer \( n \). Hence
\[
N^0(1,2;2n) = N^0(1,4;2n) + N^0(3,4;2n) = 2N^0(1,4;2n).
\] (3.48)

Therefore, identities (3.28)–(3.31) follow directly from (3.7)–(3.9) and (3.14)–(3.17).
(3) Setting \((a, M) = (2, 4)\) in (3.3), by using (2.11) and (2.13) we obtain
\[
\sum_{n=0}^{\infty} N^0(2, 4; n)q^n = \frac{1}{4} q \left( (g(q; q^2) + g(-q; q^2)) - (g(iq; q^2) + g(-iq; q^2)) \right)
\]
\[
= \frac{1}{2} q^3 \left( g(q^4; q^8) - g(-q^4; q^8) \right) + \frac{1}{2} q^6 \frac{J^5_8 J^2_{16}}{J^4 J^0} - \frac{1}{2} q^3 \frac{J^3_8}{J^2_4}. \quad (3.49)
\]
Now replacing \(q\) by \(q^4\) in (2.12) and substituting it into (3.49), we obtain (3.35).
Substituting (2.22) into (3.35), and extracting the even power terms, we obtain \(\sum_{n=0}^{\infty} N^0(2, 4; 4n + 1)q^n\)
\[
= \frac{1}{2} \left( \frac{J^0_2}{J^0_1 J^2_4} \right) - \frac{1}{2} \left( \frac{J^3_2}{J^8_8} \right)
= \frac{1}{2} J^3_2 \left( \frac{J^5_8}{J^4_4} J^2_4 - J^1_2 \right)
= \frac{1}{2} J^3_2 \left( \frac{J^5_8}{J^4_4} (J^4_4 + 2q J^2_{16} J^8_8) - \left( \frac{J^2_{16}}{J^4_4 J^2_8} - \frac{2q J^2_{16}}{J^8_8} \right) \right)
= 2q \frac{J^3_4 J^2_{16}}{J^8_8} \quad (3.50)
\]
where in the last second line we have used (2.19) and (2.21). This proves (3.36). Other identities in (3) can be proved in a similar fashion.

Finally, we consider the odd rank modulo 8.

**Theorem 3.3.** We have

(1)
\[
\sum_{n=0}^{\infty} N^0(0, 8; 2n + 1)q^n = -q^{21} g(-q^{32}; q^{64}) + q^5 \frac{J^8_8 J^2_{16} J^0_{64}}{J^4_4 J^4_{16}} + \frac{1}{2} \frac{J^3_2}{J^1_2} + \frac{1}{2} J^3_8 \quad (3.51)
\]
\[
\sum_{n=0}^{\infty} N^0(0, 8; 4n + 1)q^n = \frac{1}{4} \left( \frac{J^0_2}{J^0_1 J^2_4} + \frac{J^3_2}{J^1_2} + 2 \frac{J^4_4}{J^8_8 J^2_2} \right), \quad (3.52)
\]
\[
\sum_{n=0}^{\infty} N^0(0, 8; 4n + 3)q^n = -q^{10} g(-q^{16}; q^{32}) + q^2 \frac{J^8_8 J^2_{16} J^0_{64}}{J^4_4 J^3_8 J^2_{32}} + \frac{1}{2} \frac{J^3_2}{J^1_2} + \frac{1}{2} J^6_8, \quad (3.53)
\]
\[
\sum_{n=0}^{\infty} N^0(0, 8; 8n + 1)q^n = \frac{1}{2} \frac{J^3_4 J^2_{16}}{J^8_8} + \frac{1}{4} \frac{J^3_8}{J^2_4} + \frac{1}{4} \frac{J^1_2 J^2_{16}}{J^4 J^0} + 2q \frac{J^2_2 J^3_8}{J^4}, \quad (3.54)
\]
\[
\sum_{n=0}^{\infty} N^0(0, 8; 8n + 1)q^n = -q^5 g(-q^8; q^{16}) + q \frac{J^8_8 J^2_{16} J^0_{64}}{J^4_4 J^3_8 J^2_{32}} + \frac{1}{2} \frac{J^3_2}{J^1_2} - \frac{1}{2} \frac{J^3_8}{J^2_4} + \frac{1}{2} J^6_8 \quad (3.55)
\]
\[
\sum_{n=0}^{\infty} N^0(0, 8; 8n + 5) q^n = 2 \frac{J_4^3 J_8^2}{J_4^6 J_8^2}, \tag{3.56}
\]
\[
\sum_{n=0}^{\infty} N^0(0, 8; 8n + 7) q^n = 2 \frac{J_2^8 J_4^2}{J_2^6 J_4^4}. \tag{3.57}
\]

(2)
\[
\sum_{n=0}^{\infty} N^0(1, 8; 2n) q^n = -\frac{1}{4} + \frac{1}{4} g(-1; q^4) + \frac{1}{2} q^4 g(q^4; q^8) + \frac{1}{8} \frac{J_0^6 J_4}{J_1^4 J_8^2} - \frac{1}{4} \frac{J_4^7}{J_4^3 J_8^4} + \frac{1}{4} \frac{J_2^8 J_2^5}{J_4^2 J_3^2 J_8^4}, \tag{3.58}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 8; 4n) q^n = -\frac{1}{4} + \frac{1}{4} g(-1; q^2) + \frac{1}{2} q^2 g(q^2; q^8) + \frac{1}{8} \frac{J_1^{15}}{J_1^4 J_4^4}, \tag{3.59}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 8; 4n + 2) q^n = -\frac{1}{2} \left( \frac{J_0^3 J_4^2}{J_1^4} + \frac{J_4 J_8}{J_2} \right), \tag{3.60}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 8; 8n) q^n = -\frac{1}{4} + \frac{1}{4} g(-1; q) + \frac{1}{2} q g(q; q^4) + \frac{1}{8} \frac{J_0^{22}}{J_1^{15} J_4^4} + \frac{2}{3} \frac{J_1^4}{J_1^4 J_4^4}, \tag{3.61}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 8; 8n + 2) q^n = \frac{1}{2} \left( \frac{J_1^{16}}{J_1^4 J_4^4} + \frac{J_2 J_4}{J_1^2} \right), \tag{3.62}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 8; 8n + 4) q^n = \frac{J_1^{10}}{J_1^4}, \tag{3.63}
\]
\[
\sum_{n=0}^{\infty} N^0(1, 8; 8n + 6) q^n = 2 \frac{J_4^4 J_4^4}{J_1^4}. \tag{3.64}
\]

(3)
\[
\sum_{n=0}^{\infty} N^0(2, 8; 2n + 1) q^n = -\frac{1}{2} q^{-1} + \frac{1}{2} g(-1; q^8) + \frac{1}{4} q^{-1} \frac{J_0^6 J_1^{16}}{J_4^6 J_8^2} + \frac{1}{4} \frac{J_5^2 J_8^2}{J_4^5 J_8^2} + \frac{1}{4} \frac{J_4^3}{J_4^2 J_8^2}, \tag{3.65}
\]
\[
\sum_{n=0}^{\infty} N^0(2, 8; 4n + 1) q^n = q \frac{J_2^4 J_8^2}{J_1^4 J_8}, \tag{3.66}
\]
\[
\sum_{n=0}^{\infty} N^0(2, 8; 4n + 3) q^n = -\frac{1}{2} q^{-1} + \frac{1}{2} g(-1; q^8) + \frac{1}{4} q^{-1} \frac{J_0^6 J_8}{J_2^6 J_4^2} + \frac{J_1^6}{J_1^4 J_3^2}, \tag{3.67}
\]
\[
\sum_{n=0}^{\infty} N^0(2, 8; 8n + 1) q^n = 4q \frac{J_2^4 J_4^2 J_8^2}{J_1^6}, \tag{3.68}
\]
\[
\sum_{n=0}^{\infty} N^0(2, 8; 8n + 3) q^n = \frac{J_2^6 J_4^5}{J_1^6 J_8^2}, \tag{3.69}
\]
\[
\sum_{n=0}^{\infty} N^0(2, 8; 8n + 5)q^n = \frac{J_2^{14} J_8^2}{J_1^8 J_4^4},
\]
\[
\sum_{n=0}^{\infty} N^0(2, 8; 8n + 7)q^n = -\frac{1}{2} q^{-1} + \frac{1}{2} q^{-1} g(-1; q^4) + \frac{1}{4} q^{-1} J_1^6 J_4^4 + 2 \frac{J_1^2 J_8^2}{J_1^4 J_4^4}.
\]

\[
\sum_{n=0}^{\infty} N^0(3, 8; 2n)q^n = -\frac{1}{4} + \frac{1}{4} g(-1; q^4) - \frac{1}{2} q^4 g(q^4; q^{16}) + \frac{1}{4} J_1^7 J_4^3 + \frac{J_1^6 J_4^2}{J_1^4 J_4^4},
\]
\[
\sum_{n=0}^{\infty} N^0(3, 8; 4n)q^n = -\frac{1}{4} + \frac{1}{4} g(-1; q^2) - \frac{1}{2} q^2 g(q^2; q^8) + \frac{1}{4} J_1^7 J_3^3 + \frac{1}{8} J_1^4 J_8^2 + \frac{J_1^5 J_4^2}{J_1^3 J_4^4},
\]
\[
\sum_{n=0}^{\infty} N^0(3, 8; 4n + 2)q^n = \frac{1}{2} \left( \frac{J_2 J_4^3 J_8^2}{J_1^3} - \frac{J_4 J_8}{J_2} \right),
\]
\[
\sum_{n=0}^{\infty} N^0(3, 8; 4n + 4)q^n = \frac{1}{8} J_1^{13} J_4^3 + 2 q \frac{J_1^7 J_8^2}{J_1^4 J_4^4},
\]
\[
\sum_{n=0}^{\infty} N^0(3, 8; 4n + 2)q^n = \frac{1}{2} \left( \frac{J_1^6 J_4^3}{J_1^4 J_4^4} - \frac{J_4 J_8}{J_1^3} \right),
\]
\[
\sum_{n=0}^{\infty} N^0(3, 8; 8n + 6)q^n = 2 \frac{J_1^4 J_4^4}{J_1^3}.
\]

\[
\sum_{n=0}^{\infty} N^0(4, 8; 2n + 1)q^n = \frac{q^{-3} - q^{-3} g(-1; q^{64}) - \frac{1}{2} q^{-3} J_3^6 J_4^4 J_8}{J_1^8 J_{128}^2} + \frac{1}{4} \left( \frac{J_8^3 J_4^2}{J_1^3 J_4^4} - 2 \frac{J_8 J_4}{J_8 J_1^4} + \frac{J_3^4 J_4^4}{J_1^4 J_4^4} \right) + \frac{1}{2} q \left( \frac{J_3^6 J_8}{J_1^4 J_4^4} - \frac{J_8 J_1^3 J_2^2}{J_1^4 J_4^4} \right),
\]
\[
\sum_{n=0}^{\infty} N^0(4, 8; 4n + 1)q^n = \frac{1}{4} \left( \frac{J_9 J_4^4}{J_1^4 J_4^4} - 2 \frac{J_4 J_8}{J_2 J_4} + \frac{J_1^3 J_4^2}{J_1^4 J_4^2} \right),
\]
\[
\sum_{n=0}^{\infty} N^0(4, 8; 4n + 3)q^n = \frac{q^{-2} - q^{-2} g(-1; q^{32}) - \frac{1}{2} q^{-2} J_8 J_1^3 J_2^2}{J_1^4 J_4^4} - \frac{1}{2} J_3^3 J_4 + \frac{J_4^8 J_1^2 J_8^2}{J_1^4 J_4^4} + \frac{J_8^6 J_1}{J_2 J_4}.
\]
Proof. We only give proofs to part (1). Parts (2)-(5) can be proved similarly.

Substituting (2.11), (2.13), (2.15), and (2.17) into the above identity, we deduce that

\[
\sum_{n=0}^{\infty} N^0(4, 8; 8n + 1)q^n = \frac{1}{4} J_1^2 J_4 + 2q \frac{J_2 J_3 J_8}{J_1^3} - \frac{1}{2} J_1^2 \frac{J_2 J_4}{J_4} + \frac{1}{4} J_1^2 J_8^2, \tag{3.82}
\]

\[
\sum_{n=0}^{\infty} N^0(4, 8; 8n + 3)q^n = q^{-1} - q^{-1} g(-1; q^{16}) - \frac{1}{2} q^{-1} \frac{J_1^6 J_{16}}{J_4^2 J_{32}} - \frac{1}{2} J_2^3 \frac{J_1^2}{J_1^2 J_4^2} - \frac{J_2^3 J_8}{J_1^2 J_4^2} J_2, \tag{3.83}
\]

\[
\sum_{n=0}^{\infty} N^0(4, 8; 8n + 5)q^n = 2 \frac{J_4^9 J_2^2}{J_1^6 J_8^2}, \tag{3.84}
\]

\[
\sum_{n=0}^{\infty} N^0(4, 8; 8n + 7)q^n = 2 \frac{J_2 J_4}{J_8 J_4}. \tag{3.85}
\]

Setting \((a, M) = (0, 8)\) in (3.3), we obtain

\[
\sum_{n=0}^{\infty} N^0(0, 8; n)q^n = \frac{1}{8} \left( (g(q; q^2) + g(-q; q^2)) + (g(\zeta q; q^2) + g(-\zeta q; q^2)) \right.
\]

\[
+ (g(i q; q^2) + g(-i q; q^2)) + (g(i \zeta q; q^2) + g(-i \zeta q; q^2)) \bigg). \tag{3.86}
\]

Substituting (2.11), (2.13), (2.15), and (2.17) into the above identity, we deduce that

\[
\sum_{n=0}^{\infty} N^0(0, 8; n)q^n = -\frac{q^3}{4} \left( (g(q^4; q^8) + g(-q^4; q^8)) + (g(i q^4; q^8) + g(-i q^4; q^8)) \right)
\]

\[
+ \frac{1}{4} q \left( \frac{J_8 J_{16}}{J_4 J_8} + \frac{J_8^2 J_{32}}{J_4^2} + 2 \frac{J_2 J_4 J_{16}}{J_2 J_8 J_{16}} \right). \tag{3.87}
\]

Replacing \(q\) by \(q^4\) in (2.11) and (2.13) and substituting them into (3.84), then using (2.11) with \(q\) replaced by \(q^{16}\), we arrive at

\[
\sum_{n=0}^{\infty} N^0(0, 8; n)q^n = -\frac{1}{4} q^3 \left( 4q^{40} g(-q^{64}; q^{128}) - 4q^{8} J_{64}^8 J_{128}^8 \right) - \frac{1}{2} q^3 \left( \frac{J_{16}^8 J_{64}^2}{J_8 J_{32}^2} + \frac{J_{32}^3}{J_{16}^3} \right)
\]

\[
+ \frac{1}{4} q \left( \frac{J_4 J_{16}}{J_8 J_{16}} + \frac{J_8^2 J_{32}}{J_4^2} + 2 J_2^3 J_{16} J_{32} J_{16} \right). \tag{3.88}
\]

Since \(N^0(0, 8; 2n) = 0\), this identity reduces to (3.51).

Substituting (2.21) and (2.22) into (3.51), then extracting the even (resp. odd) power terms, we prove (3.52) (resp. (3.53)).

Next, substituting (2.21) and (2.28) into (3.52), then extracting the odd power terms, we can prove (3.56) in the same way as (3.36). Other identities in (1) can be proved in a similar fashion. \(\square\)

We may further give generating functions for \(N^0(a, M; 16n+r)\) for some \(0 \leq a \leq M-1\), \(0 \leq r \leq 15\) and \(M \in \{2, 4, 8\}\). We give some examples in the following corollary.
Corollary 3.4. We have
\[
\sum_{n=0}^{\infty} N^0(0, 8; 16n + 13)q^n = 4 \left( \frac{J_{25}^{25} J_{8}^{2}}{J_{19}^{19} J_{7}^{7}} + 2 \frac{J_{11}^{11} J_{4}^{4}}{J_{15}^{15} J_{2}^{2}} \right), \\
\sum_{n=0}^{\infty} N^0(0, 8; 16n + 15)q^n = 16 \frac{J_{15}^{15} J_{2}^{2} J_{4}^{4}}{J_{16}^{16}}.
\] (3.89) (3.90)

Proof. Substituting (2.28) into (3.56) and extracting the odd power terms, we obtain (3.89).

Similarly, substituting (2.29) into (3.57) and extracting the odd power terms, we obtain (3.90). □

4. Proofs of the Theorems and Andrews’ Conjectures

We give proofs to the main results stated in the introduction.

Proof of Theorem 1.1. Let \( p \) be an odd prime. By (1.10) we have
\[
\eta_p - 1(n) = \sum_{m=-\infty}^{\infty} \frac{(m + \frac{p-3}{2})(m + \frac{p-5}{2}) \cdots m(m-1) \cdots (m - \frac{p-1}{2})}{(p-1)!} N(m, n)
\]
\[
\equiv \sum_{m=-\infty}^{\infty} N(m; n) \pmod{p}.
\]
Thus
\[
\eta_p - 1(n) \equiv N \left( \frac{p + 1}{2}, p; n \right) \pmod{p}. \tag{4.1}
\]

Let \( p = 5 \). By (1.8) we have \( N(3, 5; 5n + 4) = \frac{1}{5} p(5n + 4) \) and hence
\[
\eta_4(5n + 4) \equiv \frac{1}{5} p(5n + 4) \pmod{5}. \tag{4.2}
\]
Using the congruence [10, Theorem 2.3.6]
\[
p(25n + 24) \equiv 0 \pmod{25}, \tag{4.3}
\]
we complete the proof of (1.11).

Similarly, let \( p = 7 \). Using the fact that \( N(4, 7; 7n+5) = \frac{1}{7} p(7n+5) \) and the congruence
[10, Theorem 2.4.3]
\[
p(49n + 47) \equiv 0 \pmod{49}, \tag{4.4}
\]
we complete the proof of (1.12). □

Proof of Theorem 1.3. (1) By Definition 1.2 (3), since \( a_i \) and \( b_j \) are all odd, we have
\[
n \equiv s + t + 1 \pmod{2}.
\]
The rank of the corresponding Durfee symbol is \( m = s - t \). Thus we must have \( n \equiv m + 1 \pmod{2} \).

(2) Comparing (1.36) with (1.7), we deduce that
\[
R_1(z; q) = (1 - z)(1 + zg(z; q)). \tag{4.5}
\]
Replacing $q$ by $q^2$ and setting $z$ as $zq$, we obtain
\begin{align*}
R_1(zq; q^2) &= (1 - zq) (1 + zqg(zq; q^2)) \\
&= 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) z^m q^{m+2n}.
\end{align*}
(4.6)

Substituting (2.4) into (4.6), we obtain
\begin{align*}
R_1(zq; q^2) &= 1 - zq + z(1 - zq) R_0(z; q) \\
&= 1 + \sum_{n=2}^{\infty} \sum_{m=-\infty}^{\infty} (N^0(m-1, n) - N^0(m-2, n-1)) z^m q^n.
\end{align*}
(4.7)

Comparing the coefficients of $z^m q^n$ on the right sides of (4.6) and (4.7), we deduce that
\begin{align*}
N^0(m - 1, n) - N^0(m - 2, n - 1) &= N\left(m, \frac{n - m}{2}\right).
\end{align*}
(4.8)

This yields (1.24) upon replacing $m$ by $m + 1$.

Applying (1.24) repetitively, we deduce that
\begin{align*}
N^0(m,n) &= \sum_{k=0}^{m-1} \left( N^0(m - k, n - k) - N^0(m - k - 1, n - k - 1) \right) + N^0(m - n, 0) \\
&= \sum_{k=0}^{n-1} N\left(m + 1 - k, \frac{n - m - 1}{2}\right),
\end{align*}
(4.9)

where in the last equality we used the fact that $N^0(m, 0) = 0$ for any $m$, which is clear by definition.

Proof of Corollary 1.4. If $n \leq 3m + 5$, then we have $m + 1 \geq \frac{n-m-1}{2} - 1$. Note that $N(m,n) = 0$ if $|m| \geq n$. By (1.25) we have
\begin{align*}
N^0(m,n) &= \sum_{|k|<\frac{n-m-1}{2}} N\left(k, \frac{n-m-1}{2}\right) = p\left(\frac{n-m-1}{2}\right).
\end{align*}

Proof of Theorem 1.8. Comparing (3.56) with (3.84), we obtain the case $r = 5$ of (1.34). Similarly, comparing (3.57) with (3.85), we obtain the case $r = 7$ of (1.34). The equalities in (1.35) can be proved by comparing (3.63) with (3.77) and (3.64) with (3.78).

Theorem 1.8 leads to the following corollary.

Corollary 4.1. For any integer $n \geq 0$ we have
\begin{align*}
N^0(0, 4; 8n + r) &= 2N^0(0, 8; 8n + r), \quad r \in \{5, 7\},
\end{align*}
(4.10)
\begin{align*}
N^0(1, 4; 8n + r) &= 2N^0(1, 8; 8n + r), \quad r \in \{4, 6\}.
\end{align*}
(4.11)

Proof. By (1.34) we deduce that
\begin{align*}
N^0(0, 4; 8n + 5) &= N^0(0, 8; 8n + 5) + N^0(4, 8; 8n + 5) = 2N^0(0, 8; 8n + 5).
\end{align*}
(4.12)

This proves the case $r = 5$ of (4.10). The rest of the equalities can be proved in a similar fashion.
Remark 4.1. We can also prove Corollary 4.1 without using Theorem 1.8. Indeed, comparing (3.26) with (3.56), we obtain the case \( r = 5 \) of (4.10) immediately. In the same way we can prove other equalities.

Proof of Theorem 1.9. (1) Subtracting (3.82) from (3.54), we get (1.37). Similarly, subtracting (3.75) from (3.61) we get (1.39), and subtracting (3.76) from (3.62) we get (1.40).

The proof of (1.38) requires more tricks. Subtracting (3.83) from (3.55) we obtain \[
\sum_{n=0}^{\infty} (N^0(0, 8; 8n + 3) - N^0(4, 8; 8n + 3)) q^n = -q^{-1} + q^{-1}g(-1; q^{16}) - q^5g(-q^8; q^{16}) + \frac{1}{2}q^{-1} \frac{J_8^0 J_{16}}{J_4^2 J_{32}^2} + \frac{J_4^3}{J_2^2} + q \frac{J_8^0 J_{32}^2}{J_4^2 J_{16}}. \tag{4.13}
\]
To simplify the right side of the above identity, we invoke Lemma 2.1. Replacing \( q \) by \( q^2 \) and setting \( x = q \) in (2.8), after rearrangement, we obtain
\[
g(-1; q^8) - q^3g(-q^4; q^8) = 1 + qg(q; q^2) - \frac{1}{2} \frac{J_2 J_4^6}{J_4^2 J_8^4}. \tag{4.14}
\]
Replacing \( q \) by \( q^2 \) in (4.14) and then substituting it into (4.13), after rearrangement, we get
\[
\sum_{n=0}^{\infty} (N^0(0, 8; 8n + 3) - N^0(4, 8; 8n + 3)) q^n = \frac{1}{2} q^{-1} \left( \frac{J_8^0 J_{16}}{J_4^2 J_{32}^2} - \frac{J_4 J_8^6}{J_2^2 J_{16}^4} \right) + \frac{J_4^3}{J_2^2} + q \frac{J_8^0 J_{32}^2}{J_4^2 J_{16}} + qg(q^2; q^4),
\]
\[
= \frac{1}{2} q^{-1} \left( \frac{J_8^0 J_{16}}{J_4^2 J_{32}^2} - \frac{J_4 J_8^6}{J_{16}^4} \left( \frac{J_5^0 J_{32}^2}{J_4^2 J_{16}^2} + 2q^2 \frac{J_8^0 J_{32}^2}{J_4^2 J_{16}} \right) \right) + \frac{J_4^3}{J_2^2} + q \frac{J_8^0 J_{32}^2}{J_4^2 J_{16}} + qg(q^2; q^4) = \frac{J_4^3}{J_2^2} + qg(q^2; q^4). \tag{4.15}
\]
where in the last second equality we have used (2.21) with \( q \) replaced by \( q^2 \).

Proof of Corollary 1.10. Extracting the odd power terms in (1.38), we obtain
\[
\sum_{n=0}^{\infty} (N^0(0, 8; 16n + 11) - N^0(4, 8; 16n + 11)) q^n = g(q; q^2). \tag{4.16}
\]
From (1.13), (1.15) and (2.1) we know that
\[
\sum_{n=0}^{\infty} p_\omega(n) = qg(q; q^2). \tag{4.17}
\]
Comparing (4.16) with (4.17), we obtain (1.41).

Proof of Theorem 1.11. For two power series \( A_1(q) := \sum_{n=-\infty}^{\infty} a_1(n)q^n \) and \( A_2(q) := \sum_{n=-\infty}^{\infty} a_2(n)q^n \), we say that \( A_1(q) \succeq A_2(q) \) if \( a_1(n) \geq a_2(n) \) holds for any integer \( n \). For example, we have
\[
\psi^2(q) = (1 + q + q^3 + q^6 + \cdots)^2 \geq 1 + q + q^2 + q^3. \tag{4.18}
\]
We have
\[ \frac{J_2}{J_4} = \psi(q) \geq \frac{1 + q + q^2 + q^3}{1 - q^4} = \frac{1}{1 - q} = \sum_{n=0}^{\infty} q^n. \] (4.19)

Hence from (1.37) we deduce that for any \( n \geq 0 \),
\[ N^0(0, 8; 8n + 1) > N^0(4, 8; 8n + 1). \] (4.20)

Similarly, we have
\[ \frac{J_2}{J_1} = \psi(q) \geq \frac{1}{1 - q} = \sum_{n=0}^{\infty} q^n. \] (4.21)

From (2.1) we have
\[ g(q; q^2) = \sum_{n=0}^{\infty} q^{2n(n+1)} \geq \frac{1}{1 - q} = \sum_{n=0}^{\infty} q^n. \] (4.22)

Thus by (1.38) we deduce that for any \( n \geq 0 \),
\[ N^0(0, 8; 8n + 3) > N^0(4, 8; 8n + 3). \] (4.23)

Combining (4.20) with (4.23), we obtain (1.42).

Next, we observe that
\[ \frac{J_2}{J_4} = \psi(q) \geq \frac{1 + q}{1 - q} = \sum_{n=0}^{\infty} q^n. \] (4.26)

This together with (1.40) implies that for any \( n \geq 0 \),
\[ N^0(1, 8; 8n) > N^0(3, 8; 8n). \] (4.27)

Combining (4.25) with (4.27), we obtain (1.43).

**Proof of Theorem 1.12.** Since \( N(-m, n) = N(m, n) \), we deduce that
\[ N^0(k, 2k; n) = \sum_{m=-\infty}^{\infty} N^0(2km + k, n) = 2 \sum_{m=0}^{\infty} N^0(2km + k, n). \]

This proves (1.44).

Congruence (1.45) follows from (3.26) and (3.27). By (3.26) and the binomial theorem we have
\[ \sum_{n=0}^{\infty} N^0(0, 4; 8n + 5)q^n = 4 \frac{J_4^9}{J_6^2 J_8^2} \equiv 4 \frac{J_4^9}{J_2^4 J_8} \pmod{8}, \] (4.28)

which yields (1.46).

Congruences (1.47) and (1.48) follow from (3.33) and (3.34), respectively.
From (3.33) and the binomial theorem, we deduce that
\[ \sum_{n=0}^{\infty} N^0(1, 4; 8n + 4)q^n \equiv \frac{J_1^2 J_0}{J_2^2} \quad (\text{mod } 5). \tag{4.29} \]

By Euler’s pentagonal number theorem, we have
\[ J_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{4.30} \]

For any integer \( n \), the residue of \( \frac{n(3n+1)}{2} \) modulo 5 can only be 0, 1 or 2. Using this fact, (1.49) follows from (4.29).

By (3.40) and the binomial theorem we have
\[ \sum_{n=0}^{\infty} N^0(2, 4; 8n + 5)q^n \equiv 2\frac{J_8^2 J_4}{J_2^4 J_8} \quad (\text{mod } 4), \tag{4.31} \]
which yields (1.50).

Congruences in (1.51) follow from (3.50) and (3.57). Congruences (1.52) and (1.53) follow from (3.89) and (3.90), respectively. Congruence (1.54) follows from (3.68).

By the binomial theorem, from (3.69) we deduce that
\[ \sum_{n=0}^{\infty} N^0(2, 8; 8n + 3)q^n \equiv \frac{J_6^2 J_4^2 J_2^8}{J_4^8 J_6^2} \quad (\text{mod } 8), \tag{4.32} \]
from which (1.55) follows. The congruence (1.56) can be proved similarly using (3.70).

**Remark 4.2.** As mentioned in the introduction, we can prove some congruences in the literature using Theorems 1.8 and 1.12. For example, since
\[ p_\omega(2n) = N^0(1, 2; 2n) = 2N^0(1, 4; 2n), \tag{4.33} \]
from the case \( r = 4 \) of (4.11) we have
\[ p_\omega(8n + 4) = 2N^0(1, 4; 8n + 4) = 4N^0(1, 8; 8n + 4). \tag{4.34} \]
This gives a combinatorial interpretation to (1.20). Similarly, from the case \( r = 6 \) of (4.11) we have
\[ p_\omega(8n + 6) = 4N^0(1, 8; 8n + 6). \tag{4.35} \]
By (3.64) we immediately get (1.21). In the same way, using (1.46), (1.50) and the fact that
\[ p_\omega(16n + 13) = N^0(0, 4; 16n + 13) + N^0(2, 4; 16n + 13), \]
we can prove (1.22).

Waldherr’s congruence (1.19) can also be deduced from (1.49) in view of (4.33).

**Proof of Conjectures 1.6 and 1.7.** From (1.31) and (1.27) we deduce that
\[ D_2^0(n) = \eta_2^0(n) = \sum_{m=-\infty}^{\infty} \binom{m+1}{2} N^0(m, n) \equiv \sum_{m=-\infty}^{\infty} N^0(m, n) \quad (\text{mod } 2). \]
This implies
\[ \mathcal{D}_2^0(n) \equiv N^0(1,4;n) + N^0(2,4;n) \pmod{2}. \] (4.36)

Therefore, we have \( \mathcal{D}_2^0(2n) \equiv N^0(1,4;2n) \pmod{2} \). From (1.47) and (1.48) we complete the proof of Conjecture 1.6.

Similarly, we have
\[ \mathcal{D}_3^0(n) = \eta_4^0(n) = \sum_{m=-\infty}^{\infty} \left( \frac{m+2}{4} \right) N^0(m,n) \equiv \sum_{m=-\infty \atop m \equiv 2, 3, 4, 5 \pmod{8}} (N^0(m,n)). \pmod{2} \] (4.37)

Hence
\[ \mathcal{D}_3^0(n) \equiv N^0(2,8;n) + N^0(3,8;n) + N^0(4,8;n) + N^0(5,8;n) \equiv N^0(2,8;n) \pmod{2}, \] (4.38)

where we used the facts that \( N^0(3,8;n) = N^0(5,8;n) \) and \( N^0(4,8;n) \equiv 0 \pmod{2} \) by (1.41). Now by (1.51), (1.55) and (1.56) we complete the proof of Conjecture 1.7. \( \square \)

5. Open Problems

There are many interesting questions that can be investigated in the future. For example, numerical evidences suggest that the following congruences hold.

**Conjecture 5.1.** For any integer \( n \geq 0 \), we have
\[ \eta_4(125n + 99) \equiv 0 \pmod{25}, \] (5.1)
\[ \eta_4(625n + 224) \equiv 0 \pmod{125}. \] (5.2)

If we want to use the strategy in the proof of Theorem 1.1 to prove this conjecture, then one has to study the rank modulo 25 and 125, which seems to be very complicated.

We end this paper with the following question: can we prove the relation (1.41) by establishing bijections between the combinatorial quantities enumerated on both sides?

References

[1] G.E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007), 37–73.
[2] G.E. Andrews and B.C. Berndt, Ramanujan’s Lost Notebook Part I, Springer, New York, 2005.
[3] G.E. Andrews, B.C. Berndt, S.H. Chan, S. Kim and A. Malik, Four identities for third order mock theta functions, preprint.
[4] G.E. Andrews, A. Dixit and A.J. Yee, Partitions associated with Ramanujan/Watson mock theta functions \( \omega(q), \nu(q) \) and \( \phi(q) \), Res. Number Theory (2015), 1–19.
[5] G.E. Andrews, D. Passary, J. Sellers and A.J. Yee, Congruences related to the Ramanujan/Watson mock theta functions \( \omega(q) \) and \( \nu(q) \), Ramanujan J (2016), doi:10.1007/s11139-016-9812-2.
[6] G.E. Andrews and F. Garvan, Dyson’s crank of a partition, Bull. Am. Math. Soc. 18 (1988), 167–171.
[7] A.O.L. Atkin and F. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003), 343–366.
[8] A.O.L. Atkin and H.P.F. Swinnerton-Dyer, Some properties of partitions, Proc. Lond. Math. Soc. III Ser. 4 (1954), 84–106.
[9] B.C. Berndt, Ramanujan’s Notebooks, Part III, Springer-Verlag, New York, 1991.
[10] B.C. Berndt, Number Theory in the Spirit of Ramanujan, AMS, 2016.
[11] F.J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944), 10–15.
[12] N.J. Fine, Basic Hypergeometric Series and Applications, AMS, Providence, 1988.
[13] D.R. Hickerson and E.T. Mortenson, Hecke-type double sums, Appell-Lerch sums, and mock theta functions, I, Proc. Lond. Math. Soc. (3) 109 (2014), no. 2, 382–422.
[14] K.Q. Ji, The combinatorics of $k$-marked Durfee symbols, Trans. Amer. Math. Soc. 363(2) (2011), 987–1005.
[15] R. Lewis and N. Santa-Gadea, On the rank and the crank modulo 4 and 8, Trans. Amer. Math. Soc. 341 (1994), no. 1, 449–465.
[16] E.T. Mortenson, On ranks and cranks of partitions modulo 4 and 8, arXiv:1707.02674v3.
[17] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa Publishing House, New Delhi, 1988.
[18] M. Waldherr, On certain explicit congruences for mock theta functions, Proc. Amer. Math. Soc. 139 (3) (2011), 865–879.
[19] L. Wang, New congruences for partitions related to mock theta functions, 175 (2017), 51–65.
[20] G.N. Watson, The final problem, J. London Math. Soc. 11 (1936), 55–80.

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, People’s Republic of China

E-mail address: mathlqwang@163.com; wanglq@whu.edu.cn