EMBEDDINGS INTO ORLICZ SPACES VIA THE MODIFIED RIESZ POTENTIAL

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ABSTRACT. $L^1_{1}$-functions which are defined in non-smooth domains in the $n$-dimensional Euclidean space can be estimated point-wise by the modified Riesz potential of their gradients. These point-wise estimates imply embeddings into Orlicz spaces from the space $L^1_p$, $1 \leq p < n$, where the functions are defined in bounded or unbounded domains with minimum requirement of the smoothness of the boundary. The results are sharp for $L^1_{1}$-functions.

1. INTRODUCTION

It is well known that a locally Lipschitz function can be estimated point-wise by the Riesz potential of its gradient in bounded John domains, [20, Theorem], [6, Theorem 10], and hence, especially, in Lipschitz domains and in convex domains, [5, Lemma 7.16]. By modifying the Riesz potential, point-wise estimates can be generalized for functions which are defined in more irregular domains than John domains, [11, Theorem 3.4], [10, Theorem 4.4]. More precisely, for every function $u$ whose weak distributional partial derivatives are in $L^1(G)$, the pointwise estimate

$$|u(x) - u_D| \leq \int_G \frac{|
abla u(y)|}{\psi(|x - y|)^{n-1}} dy$$

holds for almost every $x \in G$. Here, $G$ is a domain in the $n$-dimensional Euclidean space and the regularity of the boundary is controlled by the function $\psi$. Hedberg’s method [13, Lemma, Theorem 1] can be extended so that this point-wise estimate leads to the Sobolev-type inequality where an Orlicz-space is the target space. Hedberg’s method has been used by A. Cianchi and B. Stroffolini for the classical Riesz potential when functions are Orlicz functions, [3, Theorem 1, Corollary 1], and by the authors for the modified Riesz potential with a special Orlicz function, [11, Theorem 1.1] and [9, Theorem 1.1], and with a general Orlicz function in [10, Corollary 3.4, Corollary 5.4]. For other papers on Orlicz embeddings of Cianchi we refer to [11, 2].

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In the present paper we show that the optimal Orlicz function for the modified Riesz potential in (1.1) can be found as a function of $\psi$ which depends on the geometry of the domain $G$. Our main theorem is the following theorem where we give the formula to the Orlicz function.

1.2. Theorem. Let $1 \leq p < n$. Let the continuous, strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ be such that $\varphi(0) = \lim_{t \to 0^+} \varphi(t) = 0$ and $\varphi$ satisfies the $\Delta_2$-condition and the inequality \[
\frac{\varphi(t_1)}{t_1} \leq \frac{\varphi(t_2)}{t_2} \text{ whenever } 0 < t_1 \leq t_2. \]

If

\[
\psi(t) = \begin{cases} 
\varphi(t) & \text{when } 0 \leq t \leq 1; \\
\varphi(1)t & \text{when } t \geq 1,
\end{cases}
\]

then there exists an $N$-function $H$ that satisfies the $\Delta_2$-condition, and

\[
H^{-1}(t) \approx \frac{t^{p-1}}{\psi(t^{-\frac{1}{p}})^n} \text{ for } t > 0.
\]

With this function we obtain the following point-wise estimate.

1.4. Theorem. Let $G$ be a domain in $\mathbb{R}^n$, $n \geq 2$. Let $1 \leq p < n$. If $H$ is the function from Theorem 1.2 and $\|f\|_{L^p(G)} \leq 1$, then there exists a constant $C$ such that the point-wise estimate

\[
H \left( \int_G \frac{|f(y)|}{\psi(|x-y|)^{n-1}} \, dy \right) \leq C(Mf(x))^p
\]

holds for every $x \in \mathbb{R}^n$. Here, $Mf$ is the Hardy-Littlewood maximal operator of $f$ and the constant $C$ depends on $n$, $p$, and the $\Delta_2$-constant of $H$ only.

By this point-wise estimate we obtain embedding results for bounded and unbounded non-smooth domains. Examples of these domains are Lipschitz domains and convex domains, but also domains with suitable outward cusps are allowed.

We define a class of domains which are controlled by the function $\psi$ from (1.3). We call these domains in Definition 2.2 as $\varphi$-cigar John domains, since our definition is a modification of [22, 2.1] where J. Väisälä has defined unbounded John domains with $\varphi(t) = t$. Hence, examples of $\varphi$-John domains are the classical bounded and unbounded John domains, but also so called $s$-John domains when $\varphi(t) = t^s$.

We have the following corollary which recovers some of the known results of the Poincaré inequality.

1.5. Corollary. If there exists $\alpha \in [1, n/(n - 1))$ such that $t^\alpha/\varphi(t)$ is increasing for $t > 0$ and if $D$ is a bounded or an unbounded $\varphi$-cigar John domain with a constant $c_J$ in $\mathbb{R}^n$, $n \geq 2$ and if $1 \leq p < n$, then with the function $H$ in Theorem 1.2 there exists a constant $C$ such that the inequality

\[
\inf_{b \in \mathbb{R}} \|u - b\|_{L^p(D)} \leq C\|\nabla u\|_{L^p(D)},
\]
holds for every $u \in L^1_{\text{loc}}(D)$ with $|\nabla u| \in L^p(D)$. Here the constant $C$ depends on $n$, $p$, $\Delta_2$-constants of $H$ and $\varphi$, and John constant $c_J$ only.

We point out that if $D$ is a bounded $s$-John domain, then $\varphi(t) = t^s$, $t \geq 0$, and this corollary yields that the \((\frac{np}{n - np + np - 1}, p)\)-Poincaré inequality holds. If $p = 1$, the result is optimal. Thus the corollary recovers some of the known results of [21, Theorem 10], [7, Corollaries 5 and 6], and [17, Theorem 2.3], but our proof is completely different from the previous proofs.

Especially, in Section 6 we construct an example of an unbounded domain which shows that the Lebesque space cannot be the target space in this corresponding embedding if $\lim_{t \to 0^+} t/\varphi(t) = \infty$.

The outline of the paper is as following: We define the domains we consider in Section 2 and we call them $\varphi$-cigar John domains. We find the suitable Orlicz function in Section 3, prove embedding theorems in Section 4, recover some Poincaré inequalities in Section 5, and in Section 6 we construct an example of an unbounded $\varphi$-cigar domain.

2. John domains

Throughout the paper we let the function $\varphi : [0, \infty) \to [0, \infty)$ satisfy the following conditions

(1) $\varphi$ is continuous,
(2) $\varphi$ is strictly increasing,
(3) $\varphi(0) = \lim_{t \to 0^+} \varphi(t) = 0$,
(4) there exists a constant $C_\varphi \geq 1$ such that

$$\frac{\varphi(t_1)}{t_1} \leq C_\varphi \frac{\varphi(t_2)}{t_2}$$

whenever $0 < t_1 \leq t_2$,
(5) $\varphi$ satisfies the $\Delta_2$-condition i.e. there exists a constant $C_{\Delta_2} \geq 1$ such that $\varphi(2t) \leq C_{\Delta_2} \varphi(t)$ for every $t > 0$.

We write

$$\psi(t) = \begin{cases} \varphi(t) & \text{if } 0 \leq t \leq 1; \\ \varphi(1)t & \text{if } t \geq 1. \end{cases}$$ \quad (2.1)

Now, if $\varphi$ satisfies the conditions (1)–(5), then $\psi$ does, too, and the constant in (4) is the same for the functions $\varphi$ and $\psi$, that is $C_\varphi = C_\psi$.

The definition of a bounded John domain goes back to F. John [16, Definition, p. 402] who defined an inner radius and an outer radius domain, and later this domain was renamed as a John domain in [18, 2.1].

We extend the definition of John domains following J. Väisälä [22, 2.1] in the classical case. Let $E$ in $\mathbb{R}^n$, $n \geq 2$, be a closed rectifiable curve with endpoints $a$ and $b$. The subcurve between $x, y \in E$ is denoted by
for all $x \in E$ we write
\[ q(x) = \min \{ \ell(E[a, x]), \ell(E[x, b]) \}, \]
where $\ell(E[a, x])$ is the length of the subcurve $E[a, x]$. 

2.2. Definition. A bounded or an unbounded domain $D$ in $\mathbb{R}^n$ is a $\varphi$-cigar John domain if there exists a constant $c_J > 0$ such that each pair of points $a, b \in D$ can be joined by a closed rectifiable curve $E$ in $D$ such that
\[ \text{Cig } E(a, b) = \bigcup \left\{ B \left( x, \frac{\psi(q(x))}{c_J} \right) : x \in E \setminus [a, b] \right\} \subset D \]
where $B(x, r)$ is an open ball centered at $x$ with a radius $r > 0$ and the function $\psi$ is defined as in (2.1).

The set $\text{Cig } E(a, b)$ is called a cigar with core $E$ joining $a$ and $b$. We point out that if $D$ is a $\varphi$-cigar John domain with $\varphi(t) = t^p$, $p \geq 1$, then it is a $\varphi$-cigar John domain with $\varphi(t) = t^q$ for every $q \geq p$. For the case $\psi(t) = \varphi(t) = t$ for all $t \geq 0$, in Definition 2.2, we refer to [22, 2.1] and [19] 2.11 and 2.13.

If $D$ is a bounded domain then the following definition from [10] Definition 4.1 for a $\psi$-John domain gives an equivalent definition to a bounded $\varphi$-cigar John domain.

2.3. Definition. A bounded domain $D$ in $\mathbb{R}^n$, $n \geq 2$, is a $\psi$-John domain if there exist a constants $0 < \alpha \leq \beta < \infty$ and a point $x_0 \in D$ such that each point $x \in D$ can be joined to $x_0$ by a rectifiable curve $\gamma : [0, \ell(\gamma)] \to D$, parametrized by its arc length, such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, $\ell(\gamma) \leq \beta$, and
\[ \psi(t) \leq \frac{\alpha}{\ell(\gamma)} \text{dist}(\gamma(t), \partial D) \quad \text{for all} \quad t \in [0, \ell(\gamma)]. \]
The point $x_0$ is called a John center of $D$ and $\gamma$ is called a John curve of $x$.

If the function $\psi$ is defined as in (2.1) with the function $\varphi$, then a bounded domain is a $\psi$-John domain if and only if it is a $\varphi$-John domain. If $\psi(t) = t$, then our definition for bounded $\psi$-John domains coincides with the definition of the classical John domains. If $\psi(t) = t^s$, $s \geq 1$, then our definition for bounded $\psi$-John domains coincides with the definition of $s$-John domains.

2.4. Theorem. Let $D$ be a bounded domain. If $D$ is a $\psi$-John domain then $D$ is a $\varphi$-cigar John domain. On the other hand, if $D$ is a $\varphi$-cigar John domain with a constant $c_J$, then $D$ is a $\psi$-John domain with constants
\[ \alpha = \frac{c_J \varphi(1) \left( \max \left\{ 2, \frac{c_J \text{diam}(D)}{\varphi(1)} \right\} \right)^2}{\psi \left( \frac{1}{2c_J} \psi \left( \frac{1}{4} \text{diam}(D) \right) \right)}, \]
Note that when $\text{diam}(D) \to \infty$, then $\alpha \to \infty$ with the same speed as $\text{diam}(D)$.

**Proof.** Assume first that $D$ is a $\psi$-John domain with a John center $x_0$. Let $a, b \in D$ and let the John curves $\gamma_1$ and $\gamma_2$ connect them to $x_0$, respectively. We may assume that $a, b \in D \setminus B(x_0, \text{dist}(x_0, \partial D))$, since inside the ball the points can be connect by two straight lines going via the center of the ball $B(x_0, \text{dist}(x_0, \partial D))$. Let $E = \gamma_1 \circ \gamma_2$. Then,

$$C_{ij} E(a, b) = \bigcup_{t \in (0, \ell(\gamma_1)]} B(\gamma_1(t), \frac{\psi(t)}{\alpha/\text{dist}(x_0, \partial D)}) \cup \bigcup_{t \in (0, \ell(\gamma_2)]} B(\gamma_2(t), \frac{\psi(t)}{\alpha/\text{dist}(x_0, \partial D)})$$

and thus $D$ is a $\varphi$-cigar John domain.

Assume then that $D$ is a $\varphi$-cigar John domain. Let us carefully choose a suitable John center so that the center is not too close to the boundary of $D$. Let $x, y \in D$ such that $|x - y| \geq \frac{1}{2} \text{diam}(D)$. Let $E$ be a core of a John cigar that connects $x$ and $y$. Then the length of $E$ is at least $\frac{1}{2} \text{diam}(D)$. Let $x_0$ be the center of $E$. Then

$$\text{dist}(x_0, \partial D) \geq \frac{\psi(\frac{1}{4} \text{diam}(D))}{\frac{c_J}{\text{diam}(D)}}$$

so we choose $r = \psi\left(\frac{1}{4} \text{diam}(D)\right)/c_J$, and hence $B(x_0, r) \subset D$. From now on this $r$ and the point $x_0$ are fixed in this proof.

For every $a \in D \setminus B(x_0, r)$ there exists a curve $E$ such that $C_{ij} E(a, x_0) \subset D$. Let $\ell(E)$ be the length of $E$, then $\ell(E) \leq 2$ or by the definition

$$\text{diam}(D) \geq 2 \frac{\psi(\ell(E)/2)}{c_J} = 2 \frac{\psi(1)\ell(E)}{2c_J}$$

i.e. $\ell(E) \leq \max\left\{2, \frac{c_J \text{diam}(D)}{\phi(1)}\right\} = \beta$.

**Figure 1.** The cigar from $a$ to $x_0$ (the solid line), the core $E$ (the dotted line) and a new carrot given by the constant $c_J M$ (the dashed line).
Note that the length of $E$ inside the ball $B(x_0, r)$ is at least $r$ and thus for the points in $E \cap \partial B(x_0, r)$ the distance to the boundary is at least $\psi(r/2)$. Let us choose that

$$M = \frac{\psi(\beta)}{\psi(\ell(E))} = \frac{\varphi(1)\beta}{\psi(\ell(E))}.$$  

Since $r \leq \ell(E) \leq \beta$ and $\psi$ is increasing, we have $M \geq 1$.

Let $z_0 \in E$ be the first point from $a$ that satisfies $z_0 \in \partial B(x_0, r)$. Let us replace $E[z_0, x_0]$ by the radius of the ball $B(x_0, r)$, if necessary. Let us denote this new arc by $E$. Let $\gamma$ be an arc $E$ parametrized by its curve length, such that $\gamma(0) = a$, $\gamma(\ell(E)) = x_0$. Since $\psi(\ell(E)) \leq \psi(\ell(E))$ we obtain that

$$\bigcup_{t \in [0, \ell(E))} B\left(\gamma(t), \frac{\psi(t)}{M c_j}\right) \setminus B(x_0, r) \subset \text{Cig}[a, x_0].$$

This yields that

$$\bigcup_{t \in [0, \ell(E))} B\left(\gamma(t), \frac{\psi(t)}{M c_j}\right) \subset D$$

and thus

$$\psi(t) \leq Mc_j \text{dist}(\gamma(t), \partial D) \leq \frac{Mc_j \beta}{\ell(E)} \text{dist}(\gamma(t), \partial D).$$

This yields that we may choose $\alpha = Mc_j \beta$. Thus, $D$ is a $\psi$-John domain with these $\alpha$ and $\beta$. □

3. Point-wise estimates

We note that by the condition (4) of $\varphi$

$$\psi(t) \leq C \varphi(1) t \quad \text{for all } t \geq 0. \quad (3.1)$$

We recall a covering lemma from [10, 4.3. Lemma] which is valid for a bounded $\varphi$-John domain. For the previous versions in classical case we refer to [8, Theorem 9.3] and in a special case to [11, Lemma 3.5].

3.2. Lemma. [10, 4.3. Lemma]. Let $\varphi$ satisfies the conditions (1)–(5). Let $\psi : [0, \infty) \to [0, \infty)$ be defined as in (2.1). Let $D$ in $\mathbb{R}^n, n \geq 2$, be a bounded $\psi$-John domain with John constants $\alpha$ and $\beta$. Let $x_0 \in D$ the John center. Then for every $x \in D \setminus B(x_0, \varphi(\text{dist}(x_0, \partial D))$ there exists a sequence of balls $(B(x_i, r_i))$ such that $B(x_i, 2r_i)$ is in $D$ for each $i = 0, 1, \ldots$, and for some constants $K = K(\alpha, \text{dist}(x_0, \partial D), \varphi(D), \varphi)$, $N = N(n)$, and $M = M(n)$

- $B_0 = B(x_0, \frac{1}{2} \text{dist}(x_0, \partial D))$;
- $\psi(\text{dist}(x, B_i)) \leq Kr_i$, and $r_i \to 0$ as $i \to \infty$;
- no point of the domain $D$ belongs to more than $N$ balls $B(x_i, r_i)$; and
3.3. **Remark.** (1) The constant $K$ in the previous lemma can be taken to be $K = \max\{2\alpha/\text{dist}(x_0, \partial D), 2\varphi(1), \varphi(\text{diam}(D))/\text{diam}(D)\}$.

(2) If $D$ is a $\varphi$-cigar John domain and the John center has been chosen as in Theorem 2.4, then

$$\frac{\alpha}{\text{dist}(x_0, \partial D)} \leq \frac{c_j^2 \varphi(1) \left( \max\left\{2, \frac{c \text{diam}(D)}{\varphi(1)}\right\} \right)^2}{\psi \left( \frac{1}{c_j^2} \varphi \left( \frac{1}{4} \text{diam}(D) \right) \right) \psi \left( \frac{5}{4} \text{diam}(D) \right)} \to \frac{32c_j^5}{\varphi(1)^4}$$

as $\text{diam}(D) \to \infty$.

We recall the following definitions. Let $G$ be an open set of $\mathbb{R}^n$. We denote the Lebesgue space by $L^p(G)$, $1 \leq p < \infty$. By $L^1_p(G)$, $1 \leq p < \infty$, we denote those locally integrable functions whose first weak distributional derivatives belongs to $L^p(G)$ i.e. $L^1_p(G) = \{u \in L^{1}_\text{loc}(G) : |\nabla u| \in L^p(G)\}$. By $W^{1,p}(G)$, $1 \leq p < \infty$, we denote those functions from $L^p(G)$ whose first weak distributional derivatives belongs to $L^p(G)$ i.e. $W^{1,p}(G) = \{u \in L^p(G) : |\nabla u| \in L^p(G)\}$.

Theorem 2.4 and Lemma 3.2 give the following point-wise estimate which we recall from [10, 4.4. Theorem].

3.4. **Theorem.** Let $\varphi$ satisfy the conditions (1)–(5). Let $\psi : [0, \infty) \to [0, \infty)$ be as defined in (2.1). Let $D$ in $\mathbb{R}^n$, $n \geq 2$, be a bounded $\varphi$-cigar John domain with a John constant $c_j$. Then there exists a finite constant $C$ and $x_0 \in D$ such that for every $u \in L^1(D)$ and for almost every $x \in D$ the inequality

$$|u(x) - u_{B(x_0, \text{dist}(x_0, \partial D))}| \leq C \int_D \frac{|\nabla u(y)|}{\psi(|x - y|)^{p-1}} dy$$

holds. Here

$$C = c \left( n, c_j, C_\varphi, C_\varphi^{\frac{1}{2}n}, \varphi(1), \min \left\{ \text{diam}(D), 1 \right\} \right).$$

We recall the definitions of $N$-functions and Orlicz spaces.

3.5. **Definition.** A function $H : [0, \infty) \to [0, \infty)$ is an $N$-function if

(N1) $H$ is continuous,

(N2) $H$ is convex,

(N3) $\lim_{t \to 0^+} \frac{H(t)}{t} = 0$ and $\lim_{t \to \infty} \frac{H(t)}{t} = \infty$.

Continuity and $\lim_{t \to 0^+} \frac{H(t)}{t} = 0$ yield that $H(0) = 0$. Let $0 < t < s$ by convexity

$$H(t) = H\left(\frac{t}{s} s + \left(1 - \frac{t}{s}\right) 0\right) \leq \frac{t}{s} H(s) + \left(1 - \frac{t}{s}\right) H(0)$$

and thus $\frac{H(t)}{t} \leq \frac{H(s)}{s}$ for $0 < t < s$.

This implies that $H$ is a strictly increasing function.
By the notation $f \lessapprox g$ we mean that there exists a constant $C > 0$ such that $f(x) \leq C g(x)$ for all $x$. The notation $f \approx g$ means that $f \lessapprox g \lessapprox f$.

Two $N$-functions $H$ and $K$ are equivalent, which is written as $H \cong K$, if there exists $m \geq 1$ such that $H(t/m) \leq K(t) \leq H(mt)$ for all $t > 0$. Equivalent $N$-functions give the same space with comparable norms. We point out that $H \cong K$ if and only if for the inverse functions $H^{-1} \approx K^{-1}$.

We assume that $H$ satisfies the $\Delta_2$-condition, that is, there exists a constant $C_H^\Delta_2$ such that
\begin{equation}
H(2t) \leq C_H^\Delta_2 H(t) \quad \text{for all } t > 0.
\end{equation}

If an $N$-function satisfies the $\Delta_2$-condition then the relations $\cong$ and $\approx$ are equivalent. The constant $C_H^\Delta_2$ is called the $\Delta_2$-constant of $H$.

Let $G$ in $\mathbb{R}^n$ be an open set. The Orlicz class is a set of all measurable functions $u$ defined on $G$ such that
\[ \int_G H(|u(x)|) \, dx < \infty. \]

We study the Orlicz space $L^H(G)$ which means the space of all measurable functions $u$ defined on $G$ such that
\[ \int_G H(\lambda |u(x)|) \, dx < \infty \]
for some $\lambda > 0$.

Whenever the function $H$ satisfies the $\Delta_2$-condition, then the space $L^H(G)$ is a vector space and it is equivalent to the corresponding Orlicz class. We study these Orlicz spaces and call their functions Orlicz functions. The Orlicz space $L^H(G)$ equipped with the Luxemburg norm
\[ \|u\|_{L^H(G)} = \inf \left\{ \lambda > 0 : \int_G \Phi \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\} \]
is a Banach space.

We recall the following theorem from [10, 1.3. Theorem].

3.7. Theorem. Let $\varphi$ satisfy the conditions (1)-(5). Let $\psi : [0, \infty) \to [0, \infty)$ be defined as in (2.1). Let $1 \leq p < n$ be given. Suppose that there exists a continuous function $h : [0, \infty) \to [0, \infty)$ such that
\begin{equation}
\sum_{k=1}^{\infty} \frac{(2^{-k}t)^n}{\psi(2^{-k}t)^{n-1}} \leq h(t) \quad \text{for all } t > 0.
\end{equation}

Let $\delta : (0, \infty) \to [0, \infty)$ be a continuous function and let $H : [0, \infty) \to [0, \infty)$ be an $N$-function satisfying the $\Delta_2$-condition. Suppose that there exists a finite constant $C_H$ such that the inequality
\begin{equation}
H \left( h(\delta(t))t + \psi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})} \right) \leq C_H t^p
\end{equation}
is satisfied for all $t > 0$. Then
\begin{equation}
\sum_{k=1}^{\infty} \frac{(2^{-k}t)^n}{\psi(2^{-k}t)^{n-1}} \leq h(t) \quad \text{for all } t > 0.
\end{equation}
holds for all $t > 0$. Let $G$ in $\mathbb{R}^n$ be an open set. If $\|f\|_{L^p(G)} \leq 1$, then there exists a constant $C$ such that the inequality

$$H \left( \int_G \frac{|f(y)|}{\psi(|x-y|)^{n-1}} dy \right) \leq C(Mf(x))^p$$

(3.10)

holds for every $x \in \mathbb{R}^n$. Here the constant $C$ depends on $n$, $p$, $C_\varphi$, $C_H$, and the $\Delta_2$-constants of $\varphi$ and $H$ only.

Our goal is to find a formula which would give all suitable functions $H$. Examples of some of these functions were given in [10, Section 6].

Here we do the preparations to find $H$. Assume that there exists $\alpha \in [1, n/(n-1))$ such that $t^\alpha/\varphi(t)$ is increasing for $t > 0$. This yields that $t^\alpha/\psi(t)$ is increasing, too. Under this condition inequality (3.8) holds: Since

$$\frac{(2^{-k}t)^n}{\varphi(t2^{-k})n-1} = \frac{(2^{-k}t)^n}{(2^{-k})n(n-1)} \frac{(2^{-k})t^{(n(n-1))}}{n-1} \leq (2^{-k}t)^{\alpha(n-1)} \frac{\varphi(t2^{-k})n-1}{\varphi(t)n-1} = 2^{-k(n-\alpha(n-1))} \frac{t^n}{\varphi(t)n-1},$$

we have

$$\sum_{k=1}^\infty \frac{(2^{-k}t)^n}{\varphi(t2^{-k})n-1} \leq C(n, \alpha) \frac{t^n}{\varphi(t)n-1}, \text{ where } C(n, \alpha) = \frac{2n-2n(n-1)}{2n-2(n-1)}.$$

Let us define the functions $h$ and $\delta$ such that

$$h(t) = C(n, \alpha) \frac{t^n}{\varphi(t)n-1} \quad \text{and} \quad \delta(t) = t^{-\frac{n}{n-1}} \text{ for all } t > 0.$$

Then,

$$h(\delta(t)t) + \varphi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})} = h \left( t^{-\frac{n}{n-1}} \right) t + \varphi \left( t^{-\frac{n}{n-1}} \right)^{1-n} \left( t^{-\frac{n}{n-1}} \right)^{n(1-\frac{1}{p})}$$

$$= \frac{C(n, \alpha) t^{-p}}{\varphi \left( t^{-\frac{n}{n-1}} \right)} \frac{t^{1-p}}{\varphi \left( t^{-\frac{n}{n-1}} \right)^{n-1}}$$

$$= \frac{(C(n, \alpha) + 1)t^{1-p}}{\varphi \left( t^{-\frac{n}{n-1}} \right)^{n-1}}.$$

If we choose

$$F^{-1}(t) = \frac{(C(n, \alpha) + 1)(t^{1/p})^{1-p}}{\varphi \left( (t^{1/p})^{-\frac{n}{n-1}} \right)^{n-1}} = \frac{(C(n, \alpha) + 1)t^{1-\frac{n}{n-1}}}{\varphi \left( t^{-\frac{n}{n-1}} \right)^{n-1}}$$

and assume that the inverse function of $F^{-1}$ exists, that is $(F^{-1})^{-1} = F$ exists, then

$$h(\delta(t)t) + \varphi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})} = F^{-1}(t^p)$$

and thus

$$F \left( h(\delta(t)t) + \varphi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})} \right) = F \left( F^{-1}(t^p) \right) = t^p.$$
Unfortunately, there is a problem with this function $F$ to be a suitable function $H$; namely, the function $F$ is not necessary convex. For example, if $n = 2$, $\varphi(t) = t^2$, and $p = 1.9$, then the function $F$ is not convex, see Figure 2. The angle at the point $(1, F^{-1}(1))$ comes from the angle of $\psi$ at the point $(1, \psi(1))$. Our main theorem, Theorem 1.2 in Introduction, corrects this point: we show that there exists an $N$-function $H$ that is equivalent with $F$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The function $F$ is not necessary convex.}
\end{figure}

Proof of Theorem 1.2 Let us write that

$$F^{-1}(t) = \frac{t^{\frac{1}{p}-1}}{\psi \left( t^{\frac{1}{p}} \right)^{n-1}}$$

for $t > 0$ and $F^{-1}(0) = 0$. Let us first show that $F^{-1}$ is strictly increasing. Assume then that $0 < s < t$. The inequality $F^{-1}(s) < F^{-1}(t)$ is equivalent to the inequality

$$\frac{\psi \left( \left( \frac{1}{s} \right)^{\frac{1}{p}} \right)^{n-1}}{\left( \frac{1}{s} \right)^{1-\frac{1}{p}}} < \frac{\psi \left( \left( \frac{1}{t} \right)^{\frac{1}{p}} \right)^{n-1}}{\left( \frac{1}{t} \right)^{1-\frac{1}{p}}}.$$ 

Recall that if $\varphi$ satisfies the condition (4), then $\psi$ does, too, and the constant is the same for both functions. Thus by the condition (4) and the inequality $p < n$ we obtain

$$\frac{\psi \left( \left( \frac{1}{s} \right)^{\frac{1}{p}} \right)^{n-1}}{\left( \frac{1}{s} \right)^{1-\frac{1}{p}}} < \frac{\psi \left( \left( \frac{1}{t} \right)^{\frac{1}{p}} \right)^{n-1}}{\left( \frac{1}{t} \right)^{1-\frac{1}{p}}} = \frac{\psi \left( \left( \frac{1}{t} \right)^{\frac{1}{p}} \right)^{n-1}}{\left( \frac{1}{t} \right)^{1-\frac{1}{p}}}.$$
Thus the function $F^{-1}$ is strictly increasing. This yields that the function $F$ exists and is strictly increasing.

Let us show that $\lim_{t \to 0^+} F^{-1}(t) = 0$. Since $p < n$ we obtain

$$\lim_{t \to 0^+} F^{-1}(t) = \lim_{t \to 0^+} \frac{t^{\frac{1}{2}-1}}{\psi \left( t^{\frac{1}{2}} \right)^{n-1}} = \lim_{t \to 0^+} \varphi(1)^{1-n} t^{\frac{1}{2}-1} = 0.$$ 

Let us show that $\lim_{t \to 0^+} F^{-1}(t) = \infty$. Since $t/\varphi(t)$ is decreasing, by the condition (4), and by $p < n$ we obtain

$$\lim_{t \to 0^+} F^{-1}(t) = \lim_{t \to 0^+} \frac{t^{\frac{1}{2}-1}}{\psi \left( t^{\frac{1}{2}} \right)^{n-1}} = \lim_{t \to 0^+} \frac{t^{\frac{1}{2}-1}}{\psi \left( t^{\frac{1}{2}} \right)^{n-1}} \geq \lim_{t \to 0^+} \varphi(1)^{n-1} = \infty.$$ 

We have shown that $F^{-1} : [0, \infty) \to [0, \infty)$ is bijective.

Let us then study the condition

$$(3.11) \quad \frac{F(s)}{s} < \frac{F(t)}{t} \quad \text{for} \quad 0 < s < t.$$ 

Since $F^{-1}$ is a strictly increasing bijection, inequality (3.11) is equivalent to

$$\frac{s}{F^{-1}(s)} < \frac{t}{F^{-1}(t)}.$$ 

Since $t^\alpha/\varphi(t)$ is increasing, then $\varphi(t)/t^\alpha$ is decreasing and $\psi(t)/t^\alpha$ is decreasing, too. We note that $1 - \frac{\alpha(n-1)}{n} > 0$, since $\alpha < \frac{n}{n-1}$. We obtain

$$\frac{s}{F^{-1}(s)} = s^{1-\frac{\alpha}{p} + 1-\frac{\alpha(n-1)}{n}} \frac{\psi \left( s^{\frac{1}{n}} \right)^{n-1}}{\left( s^{\frac{1}{n}} \right)^{\alpha}} = s^{1-\frac{\alpha}{p} + 1-\frac{\alpha(n-1)}{n}} \frac{\psi \left( s^{\frac{1}{n}} \right)^{n-1}}{\left( s^{\frac{1}{n}} \right)^{\alpha}} < t^{1-\frac{\alpha}{p} + 1-\frac{\alpha(n-1)}{n}} \frac{\psi \left( t^{\frac{1}{n}} \right)^{n-1}}{\left( t^{\frac{1}{n}} \right)^{\alpha}} = \frac{t}{F^{-1}(t)}$$

and thus inequality (3.11) holds.

Let us then show that $F^{-1}(cs) \geq 2F^{-1}(s)$ for all $s \geq 0$ with $c = 2^{\frac{\alpha}{p}}$. The inequality $F^{-1}(cs) \geq 2F^{-1}(s)$ is equivalent to

$$\frac{\psi \left( \left( \frac{1}{c} \right)^{\frac{1}{n}} \right)^{n-1}}{\left( \frac{1}{c} \right)^{1-\frac{\alpha}{p}}} \leq \frac{\psi \left( \left( \frac{1}{c} \right)^{\frac{1}{n}} \right)^{n-1}}{\left( \frac{1}{c} \right)^{1-\frac{\alpha}{p}}}.$$
By the condition (4) of $\varphi$ and the inequality $p < n$, we obtain
\[
2 \frac{\psi \left( \left( \frac{1}{c_s} \right)^\frac{1}{n-1} \right)}{\left( \frac{1}{c_s} \right)^{\frac{1}{n-1}}} = 2 \left( \psi \left( \left( \frac{1}{c_s} \right)^\frac{1}{n-1} \right) \right)^{n-1} = \left( \psi \left( \left( \frac{1}{c_s} \right)^\frac{1}{n-1} \right) \right)^{n-1} \left( \frac{1}{s} \right) \leq \left( \psi \left( \left( \frac{1}{c_s} \right)^\frac{1}{n-1} \right) \right)^{n-1} \left( \frac{1}{s} \right) = \frac{\psi \left( \left( \frac{1}{c_s} \right)^\frac{1}{n-1} \right)}{\left( \frac{1}{c_s} \right)^{\frac{1}{n-1}}}.
\]
The inequality $F^{-1}(cs) \geq 2F^{-1}(s)$ yields that $F$ satisfies the $\Delta_2$-condition.

Let us write $F(t) = s$. Then $F^{-1}(s) = t$. Since $F$ is increasing, we have
\[
F(2t) = F(2F^{-1}(s)) \leq F(F^{-1}(cs)) = cs = cF(t).
\]

P. H"ast"o has shown in [15, Proposition 5.1] that if $f : [0, \infty) \to [0, \infty)$ satisfies the $\Delta_2$-condition and $x \mapsto f(x)/x$ is increasing, then $f$ is equivalent to a convex function. Since $F$ satisfies inequality (3.11) and the $\Delta_2$-condition, we obtain that $F$ is equivalent to a convex function $H$.

Using $\lim_{t \to 0^+} F^{-1}(t) = 0$ and the bijectivity, we obtain
\[
\lim_{t \to 0^+} \frac{F(t)}{t} = \lim_{t \to 0^+} \frac{t}{F^{-1}(t)} = \lim_{t \to 0^+} \frac{t \psi \left( \left( \frac{1}{c_s} \right)^\frac{1}{n-1} \right)}{\left( \frac{1}{c_s} \right)^{\frac{1}{n-1}}} = \lim_{t \to 0^+} \varphi(1)^{n-1} t^{1-\frac{1}{p}+\frac{1}{n-1}} = 0
\]
and thus also $\lim_{t \to 0^+} \frac{H(t)}{t} = 0$. This gives that $H$ is right continuous at the origin. Thus by convexity the function $H$ is continuous on $[0, \infty)$.

Since $\varphi(t)/t^\alpha$ is decreasing and $\alpha < \frac{n}{n-1}$, we obtain
\[
\lim_{t \to \infty} \frac{F(t)}{t} = \lim_{t \to \infty} \frac{t}{F^{-1}(t)} = \lim_{t \to \infty} t^{1-\frac{1}{p}} \varphi \left( \left( \frac{1}{c_s} \right)^n \right)^{n-1} \geq \lim_{t \to \infty} t^{1-\frac{1}{p}+\frac{1}{n-1}} \left( \frac{\varphi(1)}{1^{\alpha}} \right)^{n-1} = \infty.
\]
Since the functions $F$ and $H$ are equivalent, this yields that
\[
\lim_{t \to \infty} \frac{H(t)}{t} = \infty.
\]
Thus we have shown that the function $H$ satisfies the conditions (N1) – (N3).

\[\square\]

3.12. Remark. Later it is crucial to us that
\[
H^{-1}(t) \approx \frac{t^{\frac{1}{p}-1}}{\psi \left( \left( \frac{1}{c_s} \right)^{\frac{1}{p-1}} \right)} = \frac{t^{\frac{1}{p}-1}}{\varphi(1)^{\frac{1}{p-1}} \left( \frac{1}{c_s} \right)^{\frac{1}{p-1}}} = \varphi(1)^{1-\alpha} t^{\frac{1}{p-1}}
\]
for $0 < t \leq 1$. Namely, then for every $\varphi$ the function $H$ satisfies $H(t) \approx t^{\frac{1}{p-1}}$ whenever $0 < t \leq 1$. 

3.13. Example. Functions \( \varphi(t) = t^\alpha / \log^\beta(e + 1/t) \), \( \alpha \in [1, \frac{n}{n-1}) \) and \( \beta \geq 0 \), satisfy the assumptions of Theorem 1.2.

Now, the proof for our second main theorem, Theorem 1.4 in Introduction, follows easily:

Proof of Theorem 1.4. Theorem 3.7 and Theorem 1.2. □

As a corollary we obtain from Theorem 1.4 and Theorem 3.4:

3.14. Corollary. Let \( 1 \leq p < n \). Let the function \( H \) be as in Theorem 1.2. If \( D \) is a bounded \( \varphi \)-cigar John domain with a constant \( c_J \), then there exist a constant \( C \) and a point \( x_0 \in D \) such that the point-wise estimate

\[
H \left( |u(x) - u_{B(x_0, \text{dist}(x_0, \partial D))}| \right) \leq C(M|\nabla u(x)|)^p
\]

holds for all \( u \in L^1_p(D) \) with \( \|\nabla u\|_{L^p(D)} \leq 1 \) and for almost every \( x \in D \). Here the constant \( C \) depends on \( n, p, C_H, C^\Delta_2, C^\Delta_2, c_J, \varphi(1) \) and \( \min\{\text{diam}(D), 1\} \) only.

4. On embeddings

Corollary 3.14 is essential in the proofs of the following Theorem 4.1 and Theorem 4.3

4.1. Theorem (Bounded domain, \( 1 < p < n \)). Assume that \( \varphi \) satisfies the conditions \((1)-(5)\), \( C^\varphi = 1 \) in the condition \((4)\), and there exists \( \alpha \in [1, \frac{n}{n(n-1)}) \) such that \( t^\alpha/\varphi(t) \) is increasing for \( t > 0 \). Let \( \psi \) be defined as in \((2.1)\). Let \( D \subset \mathbb{R}^n, n \geq 2 \), be a bounded \( \varphi \)-cigar John domain with a constant \( c_J \). Let \( 1 < p < n \). Then there exists an \( N \)-function \( H \), that satisfies \( \Delta_2 \)-condition and

\[
H^{-1}(t) \approx \frac{t^{1/\alpha-1}}{\psi\left(t^{-\frac{1}{\alpha}}\right)^{n-1}} \text{ for all } t > 0,
\]

and there exists a constant \( C < \infty \) such that the inequality

\[
\|u - u_D\|_{L^p(D)} \leq C\|\nabla u\|_{L^p(D)},
\]

holds for every \( u \in L^1_p(D) \). Here the constant \( C \) depends on \( n, p, C_H, C^\Delta_2, C^\Delta_2, c_J, \varphi(1) \) and \( \min\{\text{diam}(D), 1\} \) only.

Proof. Assume that \( \|\nabla u\|_{L^p(D)} \leq 1 \). Corollary 3.14 yields that

\[
H \left( |u(x) - u_{B(x_0, \text{dist}(x_0, \partial D))}| \right) \leq C(M|\nabla u(x)|)^p,
\]
where the constant $C$ depends on $n$, $p$, $C_{H}^{\Delta_2}$, $C_{\varphi}^{\Delta_2}$, $c_J$, and $\min\{\text{diam}(D), 1\}$ only. By integrating over $D$ and using the fact that the maximal operator is bounded whenever $1 < p < n$, we obtain that

$$
\int_{D} H \left( \left| \frac{u(x) - u_{B(x_0, \text{dist}(x_0, \partial D))}}{\text{dist}(x_0, \partial D)} \right|^p \right) \, dx \leq C \int_{D} (M|\nabla u(x)|)^p \, dx
$$

$$
\leq C \int_{D} |\nabla u(x)|^p \, dx \leq C.
$$

This yields that the inequality

$$
\|u - u_{B(x_0, \text{dist}(x_0, \partial D))}\|_{L^\mu(D)} \leq C
$$

holds for every $u \in L^p_{\mu}(D)$ with $\|\nabla u\|_{L^p(D)} \leq 1$. By applying this inequality to the function $u/\|\nabla u\|_{L^p(D)}$ we obtain that

$$
\|u - u_{B(x_0, \text{dist}(x_0, \partial D))}\|_{L^\mu(D)} \leq C \|\nabla u\|_{L^p(D)}.
$$

We may assume w.l.o.g. that $\|\nabla u\|_{L^p(D)} \neq 0$. Let denote $B = B(x_0, \text{dist}(x_0, \partial D))$. By the triangle inequality

$$
\|u - u_B\|_{L^\mu(D)} \leq \|u - u_D\|_{L^\mu(D)} + \|u_B - u_D\|_{L^\mu(D)}.
$$

Here,

$$
\|u_B - u_D\|_{L^\mu(D)} = \|u_B - u_D\|_{L^\mu(D)} \leq \frac{\|1\|_{L^\mu(D)} \|u - u_B\|_{L^\mu(D)}}{|D|}
$$

$$
\leq C \frac{\|1\|_{L^\mu(D)} \|1\|_{L^{\mu'}(D)} \|u - u_B\|_{L^\mu(D)}}{|D|}
$$

where $H^*$ is the conjugate function of $H$ and $C$ is the constant in Hölder’s inequality.

Next we show that $\|1\|_{L^\mu(D)} \|1\|_{L^{\mu'}(D)} \approx |D|$. Since the function $H$ is continuous and strictly increasing, there exists a unique $\lambda > 0$ such that

$$
H \left( \frac{1}{\lambda} \right) |D| = \int_{D} H \left( \frac{1}{\lambda} \right) \, dx = 1
$$

i.e. $\lambda = \|1\|_{L^\mu(D)}$. By solving $\lambda$ we obtain

$$
\|1\|_{L^\mu(D)} = \frac{1}{H^{-1} \left( \frac{1}{\lambda} \right)}.
$$

Similarly, we obtain

$$
\|1\|_{L^{\mu'}(D)} = \frac{1}{(H^*)^{-1} \left( \frac{1}{\lambda} \right)}.
$$

Since

$$
t \leq H^{-1}(t)(H^*)^{-1}(t) \leq 2t
$$

for all $t \geq 0$, see for example [4] Lemma 2.6, p. 56] , we obtain that

$$
\|1\|_{L^\mu(D)} \|1\|_{L^{\mu'}(D)} = \frac{1}{H^{-1} \left( \frac{1}{|D|} \right)(H^*)^{-1} \left( \frac{1}{|D|} \right)} \leq |D|.
$$
Hence, we have shown that
\[ \|u - u_D\|_{L^p(D)} \leq C \|\nabla u\|_{L^p(D)} \]
for every \( u \in L^1_p(D) \).

\[ \square \]

4.2. Example. Let us choose that \( \varphi(t) = t^s, \ s \in (1, \frac{n}{n-1}) \). We have calculated in Remark 3.12 that for every \( \varphi \) the function \( H \) satisfies \( H(t) \approx t^{\frac{np}{n-1}} \) whenever \( 0 < t \leq 1 \). If \( t > 1 \), then
\[ H^{-1}(t) \approx \frac{t^{\frac{1}{p}}}{\psi\left(t^{\frac{1}{n}}\right)^{n-1}} = \frac{t^{\frac{1}{p}}}{\varphi\left(t^{\frac{1}{n}}\right)^{n-1}} = t^{\frac{1}{p} - 1 \frac{n-1}{n}} = t^{\frac{1}{p} - 1 \frac{n-1}{n}} \]
and thus we have that \( H(t) \approx t^{\frac{np}{n-1} - 1} \) for \( t > 1 \).

4.3. Theorem (Bounded domain, \( p = 1 \)). Assume that the function \( \varphi \) satisfies the conditions (1)–(5), \( C_\varphi = 1 \) in the condition (4), and there exists \( \alpha \in [1, \frac{n}{n-1}) \) such that \( t^\alpha / \varphi(t) \) is increasing for \( t > 0 \). Let \( \psi \) be defined as in (2.7). Let \( D \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded \( \varphi \)-cigar John domain with a constant \( c_J \). Then there exists an \( N \)-function \( H \), that satisfies \( \Delta_2 \)-condition and
\[ H^{-1}(t) \approx \frac{1}{\psi\left(t^{\frac{1}{n}}\right)^{n-1}} \text{ for all } t > 0, \]
such that the inequality
\[ \|u - u_D\|_{L^1(D)} \leq C \|\nabla u\|_{L^1(D)}, \]
holds for some constant \( C \) and for every \( u \in L^1_p(D) \). Here the constant \( C \) depends on \( n \), \( C_\varphi^{\Delta_2} \), \( C_\varphi^{\Delta_2} \), and \( c_J \) only.

Proof. Let us consider functions \( u \in L^1_p(D) \) such that \( \|\nabla u\|_{L^1(D)} \leq 1 \). The center ball \( B(x_0, \text{dist}(x_0, \partial D)) \) is written as \( B \). In the proof of Theorem 2.4 we had chosen \( x_0 \) so that \( \text{dist}(x_0, \partial D) \geq \psi\left(\frac{1}{2} \text{diam}(D)\right)/c_J \). We show that there exists a constant \( C < \infty \) such that the inequality
\[ \int_D H(|u(x) - u_D|) \, dx \leq C \]
holds whenever \( \|\nabla u\|_{L^1(D)} \leq 1 \). This yields the claim as in the proof of Theorem 4.1.

Since \( H \) is increasing, we first estimate
\[ \int_D H(|u(x) - u_D|) \, dx \leq \sum_{j \in \mathbb{Z}} \int_{|x| \in D : 2^j \|u(x) - u_D| \leq 2^{j+1}} H(2^{j+1}) \, dx. \]
Let us define
\[ v_j(x) = \max\left\{ 0, \min\{ |u(x) - u_D| - 2^j, 2^j \} \right\} \]
Thus for all $x \in D$. If $x \in \{x \in D : 2^j < |u(x) - u_B| \leq 2^{j+1}\}$, then $v_{j-1}(x) \geq 2^{j-1}$. We obtain

$$
\int_D H(|u(x) - u_B|) \, dx \leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : |v_j(x)| \geq 2^j\}} H(2^{j+2}) \, dx.
$$

(4.5)

By the triangle inequality we have

$$
v_j(x) = |v_j(x) - (v_j)_B + (v_j)_B| \leq |v_j(x) - (v_j)_B| + |(v_j)_B|.
$$

By the $(1, 1)$-Poincaré inequality in a ball $B$, [5, Section 7.8], there exists a constant $C(n)$ such that

$$
|(v_j)_B| = (v_j)_B = \int_B v_j(x) \, dx \leq \int_B |u(x) - u_B| \, dx \leq C(n)|B| \frac{1}{2^j} \int_B |\nabla u(x)| \, dx \leq C(n)|B| \frac{1}{2^j}.
$$

We continue to estimate the right hand side of inequality (4.5)

$$
\int_D H(|u(x) - u_B|) \, dx \leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : |v_j(x)| \geq 2^j\}} H(2^{j+2}) \, dx
$$

$$
\leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : |v_j(x)| \geq 2^j\}} H(2^{j+2}) \, dx + \sum_{2^j \leq C(n)|B| \frac{1}{2^j}} \int_D H(2^{j+2}) \, dx
$$

$$
\leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : |v_j(x)| \geq 2^j\}} H(2^{j+2}) \, dx + \sum_{j=-\infty}^{j_0} \int_D H(2^{j+2}) \, dx,
$$

where $j_0 = \lceil \log(C(n)|B| \frac{1}{2^j}) \rceil$.

Assume first that $\text{diam}(D)$ is so large that $j_0 \leq -2$. When $t < 1$, then $\psi(t^{-1/n}) = \varphi(1)t^{-1/n}$ by (2.1) and thus

$$
H^{-1}(t) = \frac{1}{\psi(t^{-1/n})} = \varphi(1)^{1-n}t^{1/n}.
$$

Thus for $t < 1$ we obtain that $H(t) \approx t^{\frac{1}{n-1}}$. This yields that

$$
\sum_{j=-\infty}^{j_0} \int_D H(2^{j+2}) \, dx \approx |D| \sum_{j=-\infty}^{\lceil \log(C)|B| \frac{1}{2^j} \rceil} 2^{n(j+2)} \leq C|D|2^{\frac{1}{n-1}} \frac{\log C|B| \frac{1}{2^j}}{n-1}
$$

$$
\leq C|D||B|^{\frac{1}{n-1}} = C|D||B|^{-1}
$$

$$
\leq C \frac{\text{diam}(D)^n}{\psi(\frac{1}{d} \text{diam}(D))/c_j^n}.
$$

This constant does not blow up when $\text{diam}(D) \to \infty$:

$$
\frac{\text{diam}(D)^n}{(\psi(\frac{1}{d} \text{diam}(D))/c_j^n) \to 4^ne_j^n \varphi(1)^n} \text{ as } \text{diam}(D) \to \infty.
$$
Assume then that \( \text{diam}(D) \) is small. This yields that for every \( j_0 \in \mathbb{Z} \) the sum \( \sum_{j=-2}^{j_0} H(2^{j+2}) \) is finite and depends on \( j_0 \). We obtain

\[
(4.8) \quad \sum_{j=-\infty}^{j_0} \int_D H(2^{j+2}) \, dx \leq \sum_{j=-\infty}^{-2} \int_D H(2^{j+2}) + \sum_{j=-2}^{j_0} H(2^{j+2}) < \infty.
\]

Then, we will find an upper bound for the sum

\[
\sum_{j \in \mathbb{Z}} \int_{\{x \in D : |\nabla v_j(x)| \geq 2^{j+1}\}} H(2^{j+2}) \, dx.
\]

Since \( \|\nabla v\|_{L^1(D)} \leq \|\nabla u\|_{L^1(D)} \leq 1 \), Corollary [3.14] yields that

\[
\sum_{j \in \mathbb{Z}} \int_{\{x \in D : |\nabla v_j(x)| \geq 2^{j+1}\}} H(2^{j+2}) \, dx = \sum_{j \in \mathbb{Z}} \int_{\{x \in D : H(2^{j+2}) \geq 2^{j+1}\}} H(2^{j+2}) \, dx
\]

\[
\leq \sum_{j \in \mathbb{Z}} \int_{\{x \in D : CM|\nabla v_j| \geq 2^{j+1}\}} H(2^{j+2}) \, dx.
\]

We choose for every \( x \in \{x \in D : \text{CM}|\nabla v_j| \geq 2^{j+1}\} \) a ball \( B(x, r_x) \), centered at \( x \) and with radius \( r_x \) depending on \( x \), such that

\[
C \int_{B(x, r_x)} |\nabla v_j(y)| \, dy \geq \frac{1}{2} H(2^{j-1})
\]

with the understanding that \( |\nabla v| \) is zero outside \( D \). By the Besicovitch covering theorem (or the \( \delta \)-covering theorem) we obtain a subcovering \( \{B_k\}_{k=1}^{\infty} \) so that we may estimate by the \( \Delta_2 \)-condition of \( H \)

\[
\sum_{j \in \mathbb{Z}} \int_{\{x \in D : |\nabla v_j(x)| \geq 2^{j+1}\}} H(2^{j+2}) \, dx \leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} \int_{B_k} H(2^{j+2}) \, dx
\]

\[
\leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} |B_k| H(2^{j+2}) \leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^{\infty} C|B_k| H(2^{j+2}) \int_{B_k} |\nabla v_j(y)| \, dy
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \int_D |\nabla v_j(y)| \, dy.
\]

Let \( E_j = \{x \in D : 2^j < |u(x) - u_{|\partial D}| \leq 2^{j+1}\} \). Since \( |\nabla v_j| \) is zero almost everywhere in \( D \setminus E_j \) and \( |\nabla u(x)| = \sum_j |\nabla v_j(x)| \chi_{E_j}(x) \) for almost every \( x \in D \), we obtain

\[
(4.9) \quad \sum_{j \in \mathbb{Z}} \int_{\{x \in D : |\nabla v_j(x)| \geq 2^{j+1}\}} H(2^{j+2}) \, dx \leq C \int_D |\nabla u(y)| \, dy \leq C.
\]

Estimates \( (4.6), (4.7), (4.8) \) and \( (4.9) \) imply inequality \( (4.4) \). \( \square \)

4.10. Remark. Corollary [1.5] in Introduction follows from Theorem [4.1] and Theorem [4.3]
4.11. *Remark.* In Theorem 4.3 the $N$-function $H$ is the best possible in a sense that it cannot be replaced by any $N$-function $K$ that satisfies the $\Delta_2$-condition and

$$
\lim_{t \to \infty} \frac{K(t)}{H(t)} = \infty.
$$

In [10, Theorem 7.2] we have shown that the corresponding embedding in Theorem 4.3 does not hold if

$$
\lim_{t \to 0^+} t^n K \left( \frac{1}{\varphi(t)^{n-1}} \right) = \infty.
$$

This is valid for this function $K$. By the definitions of $H^{-1}$ and $\psi$ we obtain that

$$
\lim_{t \to 0^+} t^n K \left( \frac{1}{\varphi(t)^{n-1}} \right) = \lim_{s \to \infty} \frac{1}{s} K \left( \frac{1}{\varphi \left( s^{\frac{n}{n-1}} \right)^{n-1}} \right) = \lim_{s \to \infty} \frac{K \left( H^{-1}(s) \right)}{H \left( H^{-1}(s) \right)} = \infty,
$$

and thus there does not exist a constant $c$ such that

$$
\|u - u_D\|_{L^p(D)} \leq c \|\nabla u\|_{L^1(D)},
$$

for every $u \in L^1_p(D)$.

4.12. *Theorem* (Unbounded domain, $1 \leq p < n$). Assume that the function $\varphi$ satisfies the conditions (1)–(5), $C_\varphi = 1$ in the condition (4), and there exists $\alpha \in [1, n/(n - 1))$ such that $t^\alpha / \varphi(t)$ is increasing for $t > 0$. Let the function $\psi$ be defined as in (2.1). Let $D$ in $\mathbb{R}^n$, $n \geq 2$, be an unbounded domain that satisfies the following conditions:

- (a) $D = \bigcup_{i=1}^\infty D_i$, where $|D_i| > 0$;
- (b) $D_i \subset D_{i+1}$ for each $i$;
- (c) each $D_i$ is a bounded $\varphi$-cigar John domain with a constant $c_J$.

Let $1 \leq p < n$. Then, there exists an $N$-function $H$, that satisfies $\Delta_2$-condition and

$$
H^{-1}(t) \approx \frac{t^{\frac{1}{p-1}}}{\psi \left( t^{\frac{n}{n-1}} \right)^{n-1}} \text{ for all } t > 0,
$$

and there exists a constant $C$ such that the inequality

$$
\inf_{b \in \mathbb{R}} \|u - b\|_{L^p(D)} \leq C \|\nabla u\|_{L^p(D)},
$$

holds for every $u \in L^1_p(D)$. Here the constant $C$ depends on $n$, $p$, $C_\Delta$, $C_\Delta^\circ$, and $c_J$ only.

The proof follows the proof of [14, Theorem 4.1].

*Proof.* By Theorems 4.1 and 4.3 there exists a constant $C$ such that the inequality

$$
\|u - u_D\|_{L^p(D)} \leq C \|\nabla u\|_{L^p(D)},
$$

(4.13)
holds for each \( D_i \) and all \( u \in L^1_p(D) \). The constant \( C \) does not blow up when the diameter of \( D_i \) tends to infinity. In the case \( 1 < p < n \) this is clear. In the case \( p = 1 \), we refer to the discussion after (4.7) in the proof of Theorem 4.3. The constant depends on \( D_1 \) but this does not cause a problem.

When \( \|\nabla u\|_{L^p(D)} \leq 1 \) inequality (4.13) yields that there exists a constant \( C < \infty \) such that the inequality

\[
\int_{D_i} H(|u(x) - u_{D_i}|) \, dx \leq C,
\]

holds; here the constant \( C \) is independent of \( i \).

Let us write

\[
u_i = \int_{D_i} u(x) \, dx = \frac{1}{|D_i|} \int_{D_i} u(x) \, dx.
\]

The triangle inequality yields that

\[
|\nu_i| \leq \int_{D_i} |u(x) - \nu_i| \, dx + \int_{D_i} |u(x)| \, dx.
\]

Since \( D_i \) satisfies inequality (4.13), we have \( u \in L^p(D_1) \subset L^1(D_1) \) and thus the second term is finite. Again, by inequality (4.13) we obtain that

\[
\int_{D_i} |u(x) - \nu_i| \, dx \leq \frac{C\|1\|_{L^p(D_1)}}{|D_i|} \|u - u_{D_i}\|_{L^p(D_1)} \leq \frac{C\|1\|_{L^p(D_1)}}{|D_i|} \|\nabla u\|_{L^p(D_1)} \leq \frac{C\|1\|_{L^p(D_1)}}{|D_1|} \|\nabla u\|_{L^p(D)} < \infty.
\]

Thus the real number sequence \( (u_i) \) is bounded and hence there exists a convergent subsequence \( (u_{i_j}) \) and \( b \in \mathbb{R} \) such that \( u_{i_j} \to b \).

Since \( H \) is continuous,

\[
\lim_{j \to \infty} \chi_{D_{i_j}} H(|u(x) - u_{i_j}|) = \chi_D H(|u(x) - b|).
\]

Fatou’s lemma and the modular form of (4.13) yield that

\[
\int_D H(|u(x) - b|) \, dx \leq \liminf_{j \to \infty} \int_D \chi_{D_{i_j}} H(|u(x) - u_{i_j}|) \, dx
\]

\[
= \liminf_{j \to \infty} \int_{D_{i_j}} H(|u(x) - u_{i_j}|) \leq \liminf_{j \to \infty} C = C
\]

for every \( u \in L^1_{\text{loc}}(D) \) with \( \|\nabla u\|_{L^p(D)} \leq 1 \). This yields that there exists a constant \( C \) such that the inequality

\[
\|u - b\|_{L^1(D)} \leq C
\]
holds for every \( u \in L^1_p(D) \) with \( \|\nabla u\|_{L^p(D)} \leq 1 \). The claim follows by applying this inequality to the function \( u/\|\nabla u\|_{L^p(D)}. \)

4.14. Example. Let the function \( \varphi \) be defined as in Theorem 4.12. The following unbounded domains satisfy the assumptions of Theorem 4.1:

(a) \( \{(x', x_n) \in \mathbb{R}^n : x_n \geq 0 \text{ and } |x'| < \psi(x_n)\}. \)

(b) \( \mathbb{R}^2 \setminus \{(x, \varphi(x)) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(x, -\varphi(x)) \in \mathbb{R}^2 : 0 \leq x \leq 1\}. \)

5. On Poincaré inequalities

As a special case we recover results for Poincaré domains. We recall that a bounded domain \( D \) is called a \((q, p)\)-Poincaré domain, where \( q, p \in [1, \infty) \), if there is a constant \( C < \infty \) such that the inequality

\[
(5.1) \quad \|u - u_D\|_{L^q(D)} \leq C\|\nabla u\|_{L^p(D)}
\]

holds for all \( u \in W^{1,p}(D) \). Inequality (5.1) is the \((q, p)\)-Poincaré inequality. We note that for a bounded domain \( D \) inequality (5.1) holds if and only the inequality

\[
\inf_{b \in \mathbb{R}} \|u - b\|_{L^q(D)} \leq C_1\|\nabla u\|_{L^p(D)}
\]

holds, the constants \( C \) and \( C_1 \) depend on each other and \(|D|\) only. Let us recall results for bounded \( \varphi \)-John domains in the case \( \varphi(t) = t^s \), for a fixed \( s \geq 1 \). A bounded \( t^s \)-John domain is usually called \( s \)-John domain. A bounded \( s \)-John domain is a \((p, p)\)-Poincaré domain whenever \( s \in \left[1, \frac{n}{n+1}\right) \). [24 Theorem 10]. So, a bounded \( s \)-John domain is a \((p, p)\)-Poincaré domain for all \( p \geq 1 \) if \( s \in \left[1, \frac{n}{n+1}\right) \). A bounded \( s \)-John domain is a \((1, p)\)-Poincaré domain if \( s \in \left(1, \frac{n}{n-1}\right) \), where \( \lambda \in [n-1, n] \) is the Minkowski dimension of the boundary of the domain, [22 Theorem 1.3]. A bounded \( s \)-John domain is a \((\frac{np}{n-1} - p, p)\)-Poincaré domain for every \( 1 \leq p < s(n-1) + 1 \) if \( s \in \left[1, \frac{n}{n-1}\right] \). [7] Corollaries 5 and 6], [17 Theorem 2.3]. The exponent \( \frac{np}{s(n-1)-p+1} \) is the best possible in the class of bounded \( s \)-John domains, we refer to [7] p. 442. Our Theorems 4.1 and 4.3 with \( \varphi(t) = t^s \) give that a bounded \( s \)-John domain is a \((\frac{np}{n-1} - p, p)\)-Poincaré domain if \( 1 \leq p < n \) and \( s \in \left[1, \frac{n}{n-1}\right) \). Thus our result is optimal in the case \( p = 1 \). On the other hand, our method does not cover the case \( s = \frac{n}{n-1} \). Note that our proof totally differs from the previous proofs in [21 Theorem 10], [7] Corollaries 5 and 6], and [17 Theorem 2.3].

6. Lebesgue space cannot be a target space

In this section we give an example which shows that for certain unbounded \( \varphi \)-cigar John domains the target space cannot be a Lebesgue space. The idea is that at near the infinity the target space should be \( L^{np/(n-p)} \) but local structure of the domain may not allow so good
integrability. We assume a priori that the function $\varphi$ has the properties (1)–(5). Later on we give extra conditions to the function $\varphi$.

We construct a mushrooms-type domain. Let $(r_m)$ be a decreasing sequence of positive real numbers converging to zero. Let $Q_m$, $m = 1, 2, \ldots$, be a closed cube in $\mathbb{R}^n$ with side length $2r_m$. Let $P_m$, $m = 1, 2, \ldots$, be a closed rectangle in $\mathbb{R}^n$ which has side length $r_m$ for one side and $2\varphi(r_m)$ for the remaining $n-1$ sides. Let $Q$ be the first quarter of the space i.e. all coordinates of the points in $Q$ are positive. We attach $Q_m$ and $P_m$ together creating 'mushrooms' which we then attach, as pairwise disjoint sets, to the side $\{(0, x_2, \ldots, x_n) : x_2, \ldots, x_n > 0\}$ of $Q$ so that the distance from the mushroom to the origin is at least 1 and at most 4, see Figure 3. We assume that a priori the function $\varphi$ has the properties (1)–(5), but we have to assume here also that $\varphi(r_m) \leq r_m$. We need copies of the mushrooms. By an isometric mapping we transform these mushrooms onto the side $\{(x_1, 0, \ldots, x_n) : x_1, x_3, \ldots, x_n > 0\}$ of $Q$ and denote them by $Q_m^*$ and $P_m^*$. So again the distance from the mushroom to the origin is at least 1 and at most 4. We define

$$ G = \text{int} \left( Q \cup \bigcup_{m=1}^{\infty} \left( Q_m \cup P_m \cup Q_m^* \cup P_m^* \right) \right). $$

See Figure 3. We omit a short calculation which shows that $G$ is a $\varphi$-cigar John domain.

Let us define a sequence of piecewise linear continuous functions $(u_k)_{k=1}^{\infty}$ by setting

$$ u_k(x) := \begin{cases} F(r_k) & \text{in } Q_k, \\ -F(r_k) & \text{in } Q_k^*, \\ 0 & \text{in } Q_0, \end{cases} $$

Figure 3. Unbounded $\varphi$-cigar John domain.
where the function $F$ will be given in (6.2). Then the integral average of $u_k$ over $G$ is zero for each $k$.

The gradient of $u_k$ differs from zero in $P_m \cup P_m^*$ only and

$$|\nabla u_k(x)| = \frac{F(r_m)}{r_m}, \text{ when } x \in P_m \cup P_m^*.$$ 

Note that

$$\int_G |\nabla u_k(x)|^p \, dx = 2 \int_{P_m} \left( \frac{F(r_m)}{r_m} \right)^p = 2r_m (\varphi(r_m))^{p-1} \frac{F(r_m)^p}{r_m^p}.$$ 

We require that

$$\int_G |\nabla u_k(x)|^p \, dx = 1.$$ 

Hence, we define

$$F(r_m) = \left( \frac{r_m^{p-1}}{2\varphi(r_m)^{p-1}} \right)^{1/p}.$$ 

Let $H$ be an $N$-function. Then,

$$\inf_{b \in \mathbb{R}} \int_G H(|u_k(x) - b|) \, dx \geq \inf_{b \in \mathbb{R}} \left( |Q_m| \cdot |H(F(r_m) - b)| + |Q_m^*| \cdot |H(-F(r_m) - b)| \right) \geq r_m^p H(F(r_m)).$$ 

Hence, we have

$$r_m^p H(F(r_m)) = r_m^p H\left( \left( \frac{r_m^{p-1}}{2\varphi(r_m)^{p-1}} \right)^{1/p} \right) \geq r_m^p H\left( \frac{1}{2} \left( \frac{r_m^{p-1}}{\varphi(r_m^{p-1})} \right)^{1/p} \right).$$ 

Thus, there does not exist a positive constant $C$ such that the inequality

$$\inf_{b \in \mathbb{R}} \|u - b\|_{L^p(G)} \leq C \|\nabla u\|_{L^p(G)}$$

could hold for all $u$ from the appropriate space if

$$\lim_{t \to 0^+} t^p H\left( \frac{1}{2} \left( \frac{t^{p-1}}{\varphi(t)^{p-1}} \right)^{1/p} \right) = \infty.$$ 

Assume that $\lim_{t \to 0^+} t/\varphi(t) = \infty$. If $H(t) = t^q$, then we obtain that the inequality does not hold if

(6.3) $$q \geq \frac{np}{n - p}.$$ 

Assume then that we have a sequence $(s_j)$ of positive numbers going to infinity. For each $s_j$ we may choose points $x(j)$ and $y(j)$ such that the balls $B(x(j), s_j)$ and $B(y(j), s_j)$ are subsets of the first quadrant and $B(x(j), 3s_j) \cap B(y(j), 3s_j) = \emptyset$. We define a sequence of piecewise linear continuous functions $(v_j)_{j=1}^\infty$ by setting

$$v_j(x) := \begin{cases} 
  s_j^{\frac{np}{n-p}} & \text{in } B(x_j^1, s_j), \\
  -s_j^{\frac{np}{n-p}} & \text{in } B(x_j^2, s_j), \\
  0 & \text{in } G \setminus \left( B(x_j^1, 2s_j) \cup B(x_j^2, 2s_j) \right). 
\end{cases}$$
Now we have

$$\int_G |\nabla u|^p \, dx \leq C s_j \left( \frac{s_j - \frac{m}{p}}{s_j} \right)^p \leq C$$

for some constant $C$. On the other hand, for any $b \in \mathbb{R}$

$$\int_G H(|u_j(x) - b|) \, dx \geq C s_j H(|s_j - \frac{m}{p}| - b) + C s_j H(|s_j - \frac{m}{p}| - b)$$

$$\geq C s_j H(|s_j - \frac{m}{p}|) . $$

Thus, there does not exist a positive constant $C_1$ such that the inequality $\inf_b \|u - b\|_{L^q(G)} \leq C_1 \|\nabla u\|_{L^p(G)}$ could hold for all $u$ from the appropriate space if

$$\lim_{s \to \infty} s^m H(s^{- \frac{m}{p}}) = \lim_{s \to \infty} s^m \frac{n}{p} H \left( \frac{1}{s} \right) = \infty .$$

By choosing $H(t) = t^q$, we obtain that the inequality does not hold if

$$(6.4) \quad q < \frac{np}{n - p} .$$

If $\lim_{t \to 0^+} t/\varphi(t) = \infty$ and if there were an embedding with the Lebesgue space $L^q$ as a target space, then by (6.3) we would have $q < \frac{np}{n - p}$ and by (6.4) we would have $q \geq \frac{np}{n - p}$. Thus the target space cannot be a Lebesgue space. The target space can be $L^q$ only if $\lim_{t \to 0^+} t/\varphi(t) < \infty$. And in this case $q = \frac{np}{n - p}$. Note that the limit $\lim_{t \to 0^+} t/\varphi(t)$ exists since $\varphi$ is increasing and $\varphi \geq 0$. If $\lim_{t \to 0^+} t/\varphi(t) = m > 0$, then there exists $t_0 > 0$ such that $\frac{1}{2m} \varphi(t) \leq t \leq 2m \varphi(t)$. Thus, we have proved the following theorems.

6.5. Theorem. Let $\varphi$ satisfy (1)–(5), and assume that $\lim_{t \to 0^+} t/\varphi(t) = \infty$. Let $G$ be the unbounded $\varphi$-cigar John domain constructed in (6.1). Let $1 \leq p < n$. Then there do not exist numbers $q \in \mathbb{R}$ and $C \in \mathbb{R}$ such that the inequality

$$\inf_{b \in \mathbb{R}} \|u - b\|_{L^q(G)} \leq C \|\nabla u\|_{L^p(G)}$$

could hold for all $u \in L^1_p(G)$.

6.6. Theorem. Let $\varphi$ satisfy (1)–(5), and assume that $\lim_{t \to 0^+} t/\varphi(t) = m < \infty$. Let $G$ be the unbounded $\varphi$-cigar John domain constructed in (6.1). Assume that there exist numbers $q \in \mathbb{R}$ and $C \in \mathbb{R}$ such that the inequality

$$\inf_{b \in \mathbb{R}} \|u - b\|_{L^q(G)} \leq C \|\nabla u\|_{L^p(G)}$$

holds for all $u \in L^1_p(G)$. Then $q = \frac{np}{n - p}$ and there exists $t_0 > 0$ such that $\varphi(t) \approx t$ for all $t \in (0, t_0)$. 

\[ \]
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