Bayesian Rotation Inversion of KIC 11145123

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Abstract

A scheme of Bayesian rotation inversion, which allows us to compute the probability of a model of a stellar rotational profile, is developed. The validation of the scheme with simple rotational profiles and the corresponding sets of artificially generated rotational shifts has been successfully carried out, and we can correctly distinguish the (right) rotational model, prepared beforehand for generating the artificial rotational shifts, from the other (wrong) rotational model. The Bayesian scheme is applied to a γ Dor–δ Sct-type hybrid star, KIC 11145123, leading to a result that the convective core of the star might be rotating much faster (~10 times faster) than the other regions of the star. The result is consistent with that previously suggested by Hatta et al. based on a three-zone modeling, further strengthening their argument from a Bayesian point of view.

Unified Astronomy Thesaurus concepts: Asteroseismology (73); Delta Scuti variable stars (370); Stellar interiors (1606); Stellar rotation (1629); Bayesian statistics (1900); Model selection (1912)

1. Introduction

Stellar internal rotation plays an essential role in a variety of stellar internal dynamics (e.g., Maeder 2009), such as generation of magnetic fields via the dynamo mechanism and transportation of chemical elements caused by rotationally induced mixing. It is therefore of great importance for us to study stellar internal rotation theoretically and observationally, the latter of which, in particular, has recently become feasible thanks to the establishment of asteroseismology (e.g., Aerts et al. 2010) brought about by high-precision photometric observations by spacecraft such as Kepler (Koch et al. 2010) and TESS (Ricker et al. 2014), leading to numerous asteroseismic inferences on the internal rotation of various types of stars, including solar-like stars (e.g., Benomar et al. 2015, 2018; Schunker et al. 2016a, 2016b), early-type main-sequence stars (e.g., Kurtz et al. 2014; Saio et al. 2015; Schmid & Aerts 2016; Pápics et al. 2017; Christophe et al. 2018; Ouazzani et al. 2019; Li et al. 2020), and evolved stars (e.g., Beck et al. 2012; Mosser et al. 2012; Di Mauro et al. 2018; Deheuvels et al. 2020).

The current understanding of the internal rotation of main-sequence stars is summarized by Aerts et al. (2019); that almost all of the main-sequence stars investigated so far are exhibiting nearly rigid rotation throughout, which was not expected based on previous hydrodynamical numerical simulations of angular momentum transfer inside stars (e.g., Tayar & Pinsonneault 2013; Eggenberger et al. 2017). To fill the gap between the observation and the theory, several mechanisms of angular momentum transfer by, for instance, internal gravity waves or magnetic fields have been proposed (e.g., Cantiello et al. 2014; Rogers 2015; Fuller et al. 2019). Thus, asteroseismic studies have definitely contributed to propelling understanding of the stellar rotation.

Interestingly, there are also a few asteroseismic studies suggesting the existence of a rotational velocity gradient inside stars (e.g., Benomar et al. 2018); stars are rotating rigidly throughout most of the interiors, but not completely. One such star for which inside rotational velocity shear has been suggested is KIC 11145123 (Hatta et al. 2019), which is a γ Dor–δ Sct-type hybrid star (Bradley et al. 2015) and has been actively studied based on its well-resolved frequency splitting for pressure (p), gravity (g), and mixed modes, revealing the evolutionary stage (Kurtz et al. 2014; Hatta et al. 2021), asphericity (Gizon et al. 2016), and internal rotation (Kurtz et al. 2014; Hatta et al. 2019).

Hatta et al.’s (2019) primary focus was on investigating the latitudinally differential rotation of the star, but, by carefully checking the behaviors of estimates obtained via rotation inversion, they found a hint that the convective core of the star might be rotating five to six times faster than the other regions of the star. Although the suggestion of the fast core rotation might appear to be incompatible with the current understanding that main-sequence stars are rotating almost rigidly, it is actually not the case. Previous asteroseismic studies, especially those focusing on early-type main-sequence stars with the convective core and broad radiative region above, have utilized high-order g modes, which are not established in the convective core, to infer rotation rates in the deep region; what they have estimated are rotation rates in the deep radiative regions. In contrast, Hatta et al. (2019) used mixed modes, which have finite sensitivity, though small, inside the convective core, enabling them to obtain a hint of the rotation rate of the convective core located beneath the radiative region. They actually confirmed that the star is rotating almost rigidly throughout the radiative region and that the convective core is the exception.

Saio et al. (2021) provided another asteroseismic study that inferred convective-core rotation of fast-rotating γ Dor stars by fitting characteristic dips in observed g-mode period spacings caused by the coupling between inertia modes and high-order g modes. Though they found no hint of rotational velocity shears...
(between the convective core and the radiative region above) among their targets, comparison between the two studies should be helpful for putting further constraints on theoretical calculations of angular momentum transfer inside stars.

In this paper, we would like to take a further step in terms of investigating convective-core rotation of stars based on a newly developed scheme of Bayesian rotation inversion that enables us to compute the probabilities of models of rotational profiles based on the so-called global likelihood (e.g., Gregory 2005). The first goal is to present the scheme of Bayesian rotation inversion, and the second goal is to apply the scheme for KIC 11145123 and examine whether the fast convective-core rotation is obtained or not.

The structure of the paper is as follows. In Section 2, the mathematical formulations of the Bayesian rotation inversion are presented after a brief introduction to asteroseismic rotation inversion, which is based on the perturbative approach. Note that the perturbative approach is justified for inferring the internal rotation of KIC 11145123 because of the fact that the star is a slow rotator with a rotation period of about 100 days, which is much longer than the dynamical timescale, as well as the oscillation period of the star of a few hours. In Section 3, validation of the developed scheme is carried out with simple artificial data sets. We apply the Bayesian scheme to KIC 11145123 in Section 4, where the basic properties of the star, the comparison of modeled rotational profiles, and the results finally obtained are featured. We lastly give a summary in Section 5.

Finally, we have a note on the reference stellar models used in this paper. As reference stellar models for rotation inversion, we first chose two models; one is a model constructed by Kurtz et al. (2014) assuming single-star evolution, which was used in Hatta et al. (2019), and the other is a nonstandard model of the star constructed by Hatta et al. (2021), taking the effects of some interactions with other stars into account. We, however, did not see any significant differences in the results of rotation inversion no matter which model was taken as a reference model; thus, we will present the results obtained based on the newer model computed by Hatta et al. (2021) in this study. Some of the global stellar parameters of the reference model are as follows: \( M = 1.36 M_\odot \), \( Y_{\text{ini}} = 0.26 \), \( Z_{\text{ini}} = 0.002 \), age = \( 2.160 \times 10^9 \) yr. The star is represented by a relatively low-mass stellar model around the terminal-age main-sequence stage, exhausting most of the hydrogen at the central hydrogen-burning region. For more information on the reference model, see Hatta et al. (2021).

2. Method

2.1. Rotation Inversion

It is relatively simple to mathematically describe the frequency splitting caused by rotation (the rotational splitting) when the internal rotation is slow compared with the dynamical timescale and the oscillation periods of the system. In this case, we can treat rotation as a small perturbation to the system; then, we can relate the internal rotation to the rotational shifts in frequencies based on the first-order perturbative approach (see more details in, e.g., Unno et al. 1989; Aerts et al. 2010). The explicit form for the rotational shift \( \delta \omega / \mu \), where \( i \) represents a particular set of mode indices, namely, the radial order \( n \), the spherical degree \( l \), and the azimuthal order \( m \) thus derived is as follows:

\[
d_i = \int K_i(x, \mu) \Omega(x, \mu) dx d\mu + e_i,
\]

in which the internal rotation \( \Omega(x, \mu) \) is expressed as a function of a position inside the star, represented by the fractional radius \( (x = r/R) \) and the cosine of the colatitude \( \theta (\mu = \cos \theta) \). The observational uncertainty for the rotational shift is given by \( e_i \).

The rotational splitting kernel \( K_i(x, \mu) \) can be obtained by calculating the linear adiabatic oscillation of a certain reference model, and the explicit form can be found in, e.g., Aerts et al. (2010).

Then, what we have to do to estimate the internal rotation \( \Omega(x, \mu) \) is invert the set of Equation (1) (rotation inversion), where the number of equations is identical to that of the observed rotational shifts. Techniques such as the regularized least-squares method (e.g., Tikhonov & Arsenin 1977) and the optimally localized averaging method (Backus & Gilbert 1967), both of which are well established, have been frequently used in helioseismology, which is also the case in asteroseismology.

Note that, in asteroseismology, we do not have a large number of rotational shifts (around a few dozen at most) compared with the case in helioseismology (of the order of \( 10^5 \)); thus, it is sometimes difficult to draw definitive conclusions based on just one method (even when it is one of the standard methods), and comparisons of results obtained via different methods can help us to better understand the inversion results, which is another reason why we attempt to develop a new scheme of Bayesian rotation inversion in this study. In other words, there is no all-around method that enables us to completely solve any inverse problems; i.e., each inversion technique provides us with the corresponding estimate based on a particular criterion adopted for the technique. Therefore, we should be cautious not to jump to seemingly satisfactory conclusions, which is especially the case in asteroseismology, where the relative scarcity of observed rotational shifts easily leads to the ill-posedness of the inverse problems.

2.2. Basic Points in Bayesian Statistics

In this section, we would like to give a few basic points in Bayesian statistics. Bayesian statistics allows us to, for example, investigate global properties of probabilities of parameters or compute probabilities of models, based on the latter of which we can conduct model comparison among possible models. In particular, such capability of the model comparison strongly motivates us to construct an inversion scheme based on Bayesian statistics, and the application can be found in Section 4 with which the possibility of the fast convective-core rotation of KIC 11145123 is tested. For more thorough introductions and discussions on Bayesian statistics in astronomical contexts, readers should refer to, e.g., Gregory (2005). Applications in global and local helioseismology can also be found in Kashyap et al. (2021) and Jackiewicz (2020), respectively.

Then, let us introduce one of the most fundamental equations in Bayesian statistics, Bayes’ theorem, which has the following form:

\[
p(\theta|d) = \frac{p(d|\theta)p(\theta)}{p(d)},
\]

where
which can be derived based on the definition of the conditional probability. Data sets (obtained by observation) and parameters (to be estimated) are represented by \( d \) and \( \theta \) following the notation in Benomar et al. (2009). The probabilities \( p(\theta|d) \), \( p(d|\theta) \), \( p(\theta) \), and \( p(d) \) should read the posterior probability of the parameters given the data set, the probability of the data set given the parameters (or the likelihood of the parameters), the prior probability of the parameters, and the probability of the data set marginalized by all of the parameters (the so-called global likelihood), respectively.

What Bayes’ theorem (Equation (2)) indicates is actually not complex to interpret; though we have to begin with uninformative states (represented by the prior probability \( p(\theta) \)), once we conduct observations, we can estimate the probability of obtaining the resultant data sets assuming a set of parameters (represented by the likelihood of the parameters \( p(d|\theta) \)). Finally, combining the prior probability and the likelihood enables us to update our understanding of the parameters (represented by the posterior probability \( p(\theta|d) \)).

2.3. Formulation

One of the goals in the Bayesian rotation inversion is to compute the posterior probability of the parameters describing a rotational profile given the observed rotational shifts \( p(\Omega(x, \mu)|d) \). To this end, we need to determine functional forms of the prior probability \( p(\Omega(x, \mu)) \) and the likelihood \( p(d|\Omega(x, \mu)) \) beforehand, as shown in Section 2.3.1. We also describe how to carry out model comparisons in the Bayesian framework in Section 2.3.2. In this subsection, we basically follow the formulations in Benomar et al. (2009).

2.3.1. Priors and Likelihoods

For the prior probability, we adopt a uniform distribution whose range is reasonably broad, as we, a priori, do not have strong constraints on the parameters to be estimated. Although it is generally recognized that the choice of functional forms of prior probabilities could affect Bayesian inferences (e.g., Benomar et al. 2009), this seems not to be the case in our study, as will be seen in Section 4, which is another reason for adopting simple uninformative prior probabilities for the parameters. The specific range for the prior probability is later shown in Sections 3.2 and 4.2.

For the likelihood, we assume that an observed rotational shift \( d_i \) is a realization from a Gaussian distribution whose mean is given by the first term on the right-hand side in Equation (1) with its standard deviation identical to that of the corresponding observational uncertainty \( e_i \). Based on the assumption, the explicit form for the likelihood of the parameters given the rotational shift is as follows:

\[
p(d_i|\Omega(x, \mu)) = \frac{1}{\sqrt{2\pi e_i}} \exp \left[ -\frac{1}{2} \left( d_i - \int K_i(x, \mu)\Omega(x, \mu)dx d\mu \right)^2 \right]. \tag{3}
\]

By further assuming that the observed rotational shifts are statistically independent of each other, the explicit form for the likelihood of the parameters given the set of observed rotational shifts is the product of Equation (3) as follows:

\[
p(d|\Omega(x, \mu)) = \prod_{i=1}^{N} p(d_i|\Omega(x, \mu)), \tag{4}
\]

where the number of observed rotational shifts is denoted by \( N \).

Based on the determined prior probability and the likelihood of the parameters given the set of observed rotational shifts, we can calculate the posterior probability of the parameters following Equation (2), and we can subsequently obtain estimates for the parameters by, for instance, choosing a set of parameters for which the posterior probability is maximum (called maximum a posteriori estimation).

Note that we have not determined an explicit functional form for the rotational profile \( \Omega(x, \mu) \) yet, which is a necessary step for us to compute the likelihood (we have to compute the integration inside Equation (3)). Several specific rotational profiles (and the corresponding results of the Bayesian rotation inversion) can be found in Sections 3 and 4. It should also be noted that calculation of the posterior probability requires us to carry out numerical integrations via, for instance, the Markov Chain Monte Carlo (MCMC) method if the number of parameters used to describe a rotational profile is so large that it is computationally impossible to directly evaluate the posterior probability.

2.3.2. Model Comparison Based on Global Likelihoods

In this subsection, we would like to mention a model comparison based on Bayesian statistics. The important quantity is the global likelihood \( p(d) \), which is a normalization constant in Equation (2). We can confirm the importance of the global likelihood by reconsidering Bayes’ theorem (Equation (2)), which can be rewritten as

\[
p(\theta|d, M_j) = \frac{p(d|\theta, M_j)p(\theta, M_j)}{p(d|M_j)}, \tag{5}
\]

where a model \( M_j \) representing a certain set of parameters is explicitly expressed, and the global likelihood can read the likelihood of the model \( M_j \) given the data set.

Then, let us consider the posterior probability of the model \( M_j \) as follows:

\[
p(M_j|d) = \frac{p(d|M_j)p(M_j)}{p(d)} \tag{6},
\]

and let us compare the posterior probability of the model \( M_j \) and that of another model, \( M_k \). Taking the ratio between the two posterior probabilities (called the odds ratio, \( O_{M_j,M_k} \); Gregory 2005) leads to

\[
O_{M_j,M_k} = \frac{p(M_j|d)}{p(M_k|d)} = \frac{p(d|M_j)}{p(d|M_k)}, \tag{7}
\]

in which the ratio of the posterior probabilities of the models is expressed by the ratio of the global likelihoods of the models (note that it is assumed that \( p(M_j) = p(M_k) \) here).

It is conventionally thought that the model \( M_j \) is substantially favored compared with the model \( M_k \) when \( O_{M_j,M_k} > 3 \) (e.g., Jeffreys 1998). This threshold can be explained with a simple example where there are just two models, \( M_j \) and \( M_k \). In that case, the posterior probability of the model \( M_j \) can be expressed as \([1 + O_{M_j,M_k}]^{-1}\). Thus, the threshold 3 corresponds to the probability 0.75. In this way, the global likelihoods \( p(d|M) \) are
such essential quantities that we can select the most probable model given the data set, which is practically demonstrated in Sections 3 and 4.

3. Simple Tests

The Bayesian scheme explained in Section 2 is tested in this section. We first introduce two models of the rotational profile, from which two specific rotational profiles, one with a rotational velocity shear and the other without it, are artificially generated. The corresponding sets of rotational shifts are given as well (Section 3.1). We carry out Bayesian rotation inversion with the artificial data sets to compute the posterior probabilities and global likelihoods, based on which whether we can choose the correct model of rotational profile or not is checked (Section 3.2).

3.1. Artificial Rotational Shifts

A rotational shift of a certain mode \(d_i\) can be computed following Equation (1) once we specify a “true” rotational profile \(\Omega(x, \mu)\) and calculate the rotational splitting kernel \(K(x, \mu)\) of the model. We present two models of rotational profile \(\Omega(x)\) as below (note that we do not consider the latitudinal dependence of the internal rotation for simplicity in this section). The first one is described with two parameters, namely, a rotation rate of the core \(\Omega_1\) and that of the envelope \(\Omega_2\) (the red lines in Figure 1). The other one has a linear profile in terms of the fractional radius, which is parameterized with a rotation rate of the center of the star \(\Omega_c\) and the surface rotation rate \(\Omega_s\) (the blue line in Figure 1). Let us call the former model \(M_{sh}\) and the latter one \(M_{lin}\). Specific parameters for two particular rotational profiles are as follows:

\[
(\Omega_{1, \text{true}}, \Omega_{2, \text{true}}) = (1.50, 0.980)
\]  

and

\[
(\Omega_{c, \text{true}}, \Omega_{s, \text{true}}) = (0.950, 0.980)
\]

in units of \(2\pi \times 0.01\) day\(^{-1}\). Let us call the rotational profiles thus specified \(\Omega_{sh}\) and \(\Omega_{lin}\), respectively.

Note that, in this study, we concentrate on main-sequence stars with a convective core, as is the case for KIC 11145123. Accordingly, we assume that the boundary in \(\Omega_{sh}\) is fixed to be the convective boundary; \(\Omega_1\) and \(\Omega_s\) represent the rotation rates of the convective core and radiative envelope.

For computing the splitting kernels \(K(x, \mu)\), the linear adiabatic oscillation of the reference model, which is the nonstandard model of KIC 11145123 constructed by Hatta et al. (2021); see Section 1), has been calculated via the linear adiabatic oscillation code GYRE (Townsend & Teitler 2013). Based on the eigenfunctions and eigenfrequencies thus obtained, we calculate the splitting kernels \(K(x, \mu)\) (the explicit form can be found in, e.g., Aerts et al. 2010). We have computed splitting kernels for 33 eigenmodes, namely, 20 high-order \(g\) modes with \((l, m) = (1, 1)\), three low-order \(p\) modes with \((l, m) = (1, 1)\), five low-order mixed modes with \((l, m) = (2, 1)\), and five low-order mixed modes with \((l, m) = (2, 2)\). The eigenmodes are thus chosen, since these types of modes (high-order \(g\) modes and low-order \(p/mixed\) modes) are frequently observed for \(\gamma\) Dor-\(\delta\) Sct-type hybrid stars such as KIC 11145123, though the relatively larger number of modes compared with the actual observation is prepared to render rotation inversion as robust as possible. It should be noted that the results of the simple test here are qualitatively the same even if we use a smaller set of splitting kernels, which is identical to that in the case of KIC 11145123 (see Section 4.1).

Then, based on the rotational profiles, namely, \(\Omega_{sh}\) and \(\Omega_{lin}\), and the splitting kernels \(K(x, \mu)\), we have computed the corresponding sets of rotational shifts following Equation (1), in which the observational uncertainties \(e_i\) are assumed to be realizations from a Gaussian distribution whose mean and standard deviation are zero and \(10^{-3}\) (in units of \(2\pi \times 0.01\) day\(^{-1}\)). The standard deviation of the Gaussian distribution is chosen so that it is around a typical observational uncertainty of the rotational shifts for KIC 11145123. It is also assumed that the observational uncertainties \(e_i\) are statistically independent of each other. We thus have two sets of 33 artificially generated rotational shifts with which Bayesian rotation inversion is to be carried out in the following sections. We denote the artificial data set for the rotational profile with a velocity shear as \(\delta\omega_{sh}\) and that for the linear rotational profile as \(\delta\omega_{lin}\).

3.2. Bayesian Rotation Inversion with the Artificial Rotational Shifts

In this section, the Bayesian scheme has been utilized with the set of rotational shifts \((\delta\omega_{sh}, \delta\omega_{lin})\) in order to test whether we can correctly choose the right model of the rotational profile \((M_{sh} \text{ or } M_{lin})\) via the Bayesian scheme or not.

As mentioned in Section 2.3.1, the first thing we have to do is specify the parameters to describe \(\Omega(x, \mu)\), for which the posterior probabilities are to be computed. For simplicity, we have used the models of rotational profile \(M_{sh}\) described by \((\Omega_1, \Omega_2)\) and \(M_{lin}\) described by \((\Omega_c, \Omega_s)\) (Figure 1), the former of which contains \(\Omega_{sh}\), and the latter of which contains \(\Omega_{lin}\).

Second, prior probabilities for the prepared parameters have to be specified. We assume that the prior probabilities are uniform, as follows:

\[
\Omega_1, \Omega_2, \Omega_c, \Omega_s \sim U[0, 50],
\]

where \(q \sim U[a, b]\) means that \(q\) is a random variable uniformly distributed in a range from \(a\) to \(b\). The rotation rates are in units of \(2\pi \times 0.01\) day\(^{-1}\). The joint prior probability is computed by taking the products of prior probabilities, assuming that the parameters are statistically independent of each other.
Based on the priors and likelihoods, which can be computed with Equations (3) and (4), we calculate the numerator of the right-hand side in Equation (2), integrate it over the parameter space to obtain the normalization constant $p(d)$ (or the global likelihood), and, finally, compute the posterior probability of the parameters given the data set. Since the number of parameters is just two for both of the models and it is not computationally expensive to numerically carry out such two-dimensional computations, we have directly calculated the posterior probabilities.

Figure 2 shows two posterior probability density functions given $\delta\omega_{\text{sh}}$. One is computed based on $M_{\text{sh}}$, and the other is computed based on $M_{\text{lin}}$ ($P_{\text{sh}}$ and $P_{\text{lin}}$, respectively, in Figure 2). Since both of the posterior probabilities are unimodal, we can estimate the parameters by, for instance, taking a parameter set that maximizes the corresponding posterior probability (the maximum a posteriori estimate).

It is apparent in Figure 2 that the maximum a posteriori estimate for $P_{\text{sh}}$ is almost identical to the prepared parameters (Equation (8); red dotted lines in Figure 2), while those for $P_{\text{lin}}$ are biased. The important point, however, is that we cannot determine which model describes the data set $\delta\omega_{\text{sh}}$ more appropriately by just comparing the posterior probabilities without knowing the prepared parameters (Equation (8)) beforehand. As described in Section 2.3.2, such a model comparison should be achieved by comparing the global likelihood, which is the normalization constant in Equation (2).

In the case of this simple test with $\delta\omega_{\text{sh}}$, we have obtained the following odds ratio (defined by Equation (7)): $\log O_{M_{\text{lin}}, M_{\text{sh}}} = 23.10$ (Table 1), which is much larger than the conventionally adopted criterion $\log 3 \sim 0.5$ (e.g., Jeffreys 1998). It is therefore correctly inferred that, in the light of the Bayesian scheme, the model of a rotational profile with a velocity shear $M_{\text{sh}}$ is more favorable to describe the artificially generated data set $\delta\omega_{\text{sh}}$.

The same is true when $\delta\omega_{\text{lin}}$ is used for the test (Figure 3). The obtained odds ratio is $\log O_{M_{\text{lin}}, M_{\text{sh}}} = 135.6$ (Table 1). It is clear that model $M_{\text{lin}}$ is preferred, which is the right one we have used to generate the data set $\delta\omega_{\text{lin}}$. It should also be instructive to mention that the estimates are biased unless we have chosen the correct model (see $P_{\text{sh}}$ in Figure 3), highlighting the importance of the model comparison in the Bayesian context achieved by computing global likelihoods.

4. Applying the Method to KIC 11145123

The Bayesian scheme demonstrated in the previous sections is applied to one of the Kepler targets, KIC 11145123, to infer its internal rotation profile, especially focusing on the convective-core rotation. After we present the rotational shifts and corresponding splitting kernels for KIC 11145123 (Section 4.1), basic setups for the Bayesian rotation inversion are given (Section 4.2), based on which the posterior probabilities and global likelihoods are computed, and the model comparison is conducted as well (Section 4.3). Then, the validation of the results obtained is carried out in Section 4.4. We finally provide a brief discussion of the results in Section 4.5.

4.1. Data and Splitting Kernels

We have used a set of rotational shifts and observational uncertainties that were measured and determined by Kurtz et al. (2014; see Tables 1 and 2 in the paper). The set is composed of 23 eigenmodes, namely, 15 high-order $g$ modes with $(l, m) = (1, 1)$, two low-order $p$ modes with $(l, m) = (1, 1)$, three low-order mixed modes with $(l, m) = (2, 2)$, and three low-order mixed modes with $(l, m) = (2, 2)$. The mode identification...
for the mode set has been conducted based on the nonstandard model of the star constructed by Hatta et al. (2021), according to which the star is a low-mass star at the terminal-age main-sequence stage. Some of basic global parameters of the nonstandard model can also be found in the last paragraph of Section 1.

The corresponding splitting kernels are calculated using the nonstandard model as a reference model. We have three types of splitting kernels. The first type corresponds to high-order $g$ modes and has sensitivity in the deep radiative region just above the convective core (Figure 4). The second and third types correspond to low-order $p$ and low-order mixed modes, respectively (Figures 5 and 6). Both of them have sensitivity in the outer envelope, but the low-order mixed modes with $(l, m) = (2, 1)$ have sensitivity in the high-latitude region as well, which is not the case for other modes with $(l, m) = (1, 1)$ or $(l, m) = (2, 2)$. It should be emphasized that only the mixed-mode splitting kernels have finite sensitivity inside the convective core, which renders the detection of the convective-core rotation possible, as has been shown in Hatta et al. (2019). A close look at the splitting kernels around the convective boundary is presented and discussed in Section 4.5.

### 4.2. Models of Rotational Profile and Priors

For carrying out Bayesian rotation inversion in the case of KIC 11145123, we parameterize the rotational profile as follows:

$$\Omega(x, \mu) = \Omega_0(x) + \mu^2 \Omega_1(x),$$  \hspace{1cm} (11)

where $\Omega_0(x)$ can have a linear profile described with two parameters, namely, the rotation rates at the center $\Omega_c$ and surface $\Omega_s$ (see the blue line in Figure 1), or a velocity shear with four parameters, namely, the rotation rate below a velocity shear boundary $x_{sh}$, which is assumed to be uniform at $\Omega_{\text{core}}$, the rotation rate $\Omega_{\text{rad}}$ at another side of the shear boundary fixed to be the convective-core boundary ($x_{czb} \sim 0.045$), and that at the surface $\Omega_s$ (see Figure 7).

Another function, $\Omega_1(x)$, which is related to the latitudinal dependence of the internal rotation, can be zero everywhere (with no parameters) or have a linear profile (with two parameters, in almost the same way as a linear profile of $\Omega_0(x)$ but with additional indices as $\Omega_{1c}$ and $\Omega_{1s}$). We have taken
such latitudinal dependence of the internal rotation into account in order to evaluate the effect on the global likelihood, though it is not a primary subject to be investigated in this study. We note that the final inference on the convective-core rotation is clearly indicating that the rotational profile of the model is substantially favored compared with model $M_j$ when $\log O_{M,M_j} > 0.5$.

With the definitions for $\Omega_{\ell}(x)$ and $\Omega_{\ell_1}(x)$, there are four ways of parameterization of the rotational profile in total. Let us denote the models as follows: $M_1d2p$ for linear $\Omega_0$ and zero $\Omega_1$, $M_{1d4p}$ for shear $\Omega_0$ and zero $\Omega_1$, $M_{2d4p}$ for linear $\Omega_0$ and linear $\Omega_1$, and $M_{2d6p}$ for shear $\Omega_0$ and linear $\Omega_1$, where the subscripts $d$ and $p$ stand for the dimension of a modeled rotational profile and the number of parameters in the model, respectively (Table 2).

Then, what we have to specify is the corresponding priors for the parameters. As described in Section 3.2, each prior probability for a certain parameter is assumed to be uniform, as follows:

$$\Omega_c, \Omega_s, \Omega_{core}, \Omega_{rad} \sim U[0.1, 30],$$
$$\Omega_{cc}, \Omega_{is} \sim U[-10, 10],$$

and

$$x_{sh} \sim U[0.010, 0.055].$$

The parameters representing rotation rates are in units of $2\pi \times 0.01$ day$^{-1}$, and $q \sim U[a, b]$ means that $q$ is a random variable uniformly distributed in a range from $a$ to $b$, as described in Section 3.2. The joint prior probability is computed by taking products of the prior probabilities based on the assumption that the parameters are statistically independent of each other.

### 4.3. Results

For each way of parameterization, the posterior probability of the parameters is computed based on the likelihood of the parameters and the joint prior probability. The likelihood is calculated with Equations (3) and (4). We carry out the so-called Metropolis method (Metropolis et al. 1953), which is one of the standard algorithms to carry out MCMC, to evaluate the posterior probability. The convergence of samples generated via the Metropolis method has been checked by visual inspection (see Figure 8) with several different sets of initial values for the samples. A typical number of iterations required is of the order of $10^5$, and small fractions of samples are discarded from the final samples, as they are considered to be samples in the burn-in periods (a period during which obtained samples are not thought to be realizations from the posterior probability distribution we would like to sample). More information on the principles of MCMC, how to manage the outcomes of MCMC, and so on can be found, for example, in Gregory (2005).

We consider that the posterior probabilities have been successfully sampled via MCMC based on the convergence of MCMC samples (see Figure 8). Then, we compute the global likelihood for each way of parameterization of the rotational profile to carry out the Bayesian model comparison. Note that the global likelihood $p(d)$, which is the normalization constant in Equation (2), cannot be determined with the MCMC samples alone. We therefore need some tools to compute the global likelihoods. This can be accomplished by following the procedures proposed by Chib & Jeliazkov (2001), in which an exact value of the posterior probability of a particular set of parameters is directly evaluated, and then, the global likelihood is calculated based on Equation (2). We have confirmed that the method works correctly for simple cases where we can analytically compute the posterior probability and global likelihood.

The decimal logarithms of the odds ratios thus obtained are $\log O_{M_{1d2p},M_{1d4p}} = 4710$, $\log O_{M_{1d4p},M_{2d4p}} = 4090$, $\log O_{M_{2d4p},M_{2d6p}} = 5390$, and $\log O_{M_{1d2p},M_{2d6p}} = 4770$ (Table 2), clearly indicating that the rotational profiles with the velocity shear ($M_{1d4p}$ and $M_{2d6p}$) are more favored than those without the shear ($M_{1d2p}$ and $M_{2d4p}$). The validity of the computation of the global likelihoods has been checked with several different settings in MCMC, and the global likelihoods have been rounded to no more than four significant figures (Table 2).

### Table 2

Comparison of Odds Ratios among the Prepared Models

| Model Name | $M_{1d2p}$ | $M_{1d4p}$ | $M_{1d4p}$ | $M_{2d4p}$ |
|-----------|------------|------------|------------|------------|
| Dimension | 1D         | 2D         | 1D         | 2D         |
| Radial Profile | Linear | Linear | Shear | Shear |
| $M_i/M_f$ | $M_{1d2p}$ | $M_{1d4p}$ | $M_{1d4p}$ | $M_{2d4p}$ |
| 0         | -620.0     | -4710      | -5390      |
| 620.0     | 0          | -4090      | -4770      |
| 4710      | 4090       | 0          | -680.0     |
| 5390      | 4770       | 680.0      | 0          |

Note. Properties of the prepared models are described by two factors, namely, the dimension of the modeled rotational profile (dimension) and the kind of radial rotational profile of the model (radial profile; the top table). The middle and bottom tables show odds ratios $O_{M_i,M_f}$ (expressed in the decimal logarithm) computed given the observed rotational shifts with or without the mixed modes, respectively. Based on Equation (7), it is apparent that $O_{M_i,M_f} = (O_{M_i,M_f})^{-1}$. It is generally thought that model $M_i$ is substantially favored compared with model $M_f$ when $\log O_{M_i,M_f} > 0.5$.
We can also confirm that based on the posterior probability for the models with the velocity shear, the fast core rotation is again inferred. Figure 9 shows the two-dimensional contour map of the (marginalized) posterior probability density function as a function of \((\Omega_{\text{conv}}, x_{\text{sh}})\), which is computed for the model \(M_{2d6p}\). Note that \(\Omega_{\text{conv}}\) is used instead of \(\Omega_{\text{core}}\) to describe the latitudinally averaged rotation rate of the convective core. When we adopt the maximum a posteriori estimates for the parameters, they are \((\Omega_{\text{conv}}, x_{\text{sh}}) \sim (13.4, 0.0493)\); thus, the fast convective-core rotation has been inferred as in the case of Hatta et al. (2019; see Figure 8 for \(\Omega_{\text{sh}}\) and \(\Omega_{\text{rad}}\), which are of the order of 1 in units of \(2\pi \times 0.01 \text{ day}^{-1}\) and much slower than \(\Omega_{\text{conv}}\).

4.4. Validity of the Results

Since the models used for Bayesian model comparison in the last section are much more complex than those in Section 3, we would like to check whether we can conduct Bayesian model comparison even with such complex models, which eventually enables us to validate the results obtained in the last section. To this end, we present the same test as described in Section 3, except for the following two points. First, the maximum a posteriori estimates for the parameters, which are determined with the posterior probability given model \(M_{2d6p}\) (computed in the last section), are used for constructing an artificial rotational profile and the corresponding rotational shifts. Let us call the artificial rotational shifts \(\delta \omega_{2d6p}\). Second, to compute the posterior probabilities given the artificial rotational shifts, we have used the four models, namely, \(M_{1d2p}\), \(M_{1d4p}\), \(M_{2d4p}\), and \(M_{2d6p}\) (see the definitions of the models in Section 4.2). What we would like to check is whether we can correctly choose the right model \(M_{2d6p}\) (in this case) among the models based on the global likelihoods or not.

The global likelihoods have been computed via the method of Chib & Jeliazkov (2001). The decimal logarithms of the resultant odds ratios are as follows: \(\log O_{M_{2d6p},M_{2d4p}} = 20540\), \(\log O_{M_{2d4p},M_{2d6p}} = 14390\), and \(\log O_{M_{2d6p},M_{1d4p}} = 668.4\) (Table 3). It is therefore evident that the correct model, \(M_{2d6p}\), is preferred to the other models from the Bayesian perspective. This can also be confirmed when we see Figure 10, where the prepared parameters (represented by red dotted lines in the figure) can be estimated with little biases only if we choose the correct model \(M_{2d6p}\) (see \(P_{2d6p}\) in the figure).

4.5. Discussion

Based on the results obtained in Section 4.3, it has been inferred via the Bayesian scheme that the convective core of KIC 11145123 is rotating approximately 10 times faster than the other regions of the star. We have also confirmed the validity of the results in Section 4.4. In this section, we have a brief discussion about the inferred position of the velocity shear.

Figure 11 shows the Brunt–Väisälä frequency of the reference model (black curve) and the inferred position of the velocity shear (blue dashed line). What we would like to point out is that the inferred position of the velocity shear \(x_{\text{sh}} \sim 0.05\) is slightly above the convective boundary \(x_{\text{sh}} \sim 0.045\), at which the square of the Brunt–Väisälä frequency is zero, and that the shear is located in the overshoot zone (shown as the blue shaded area in Figure 11). In the overshoot zone, where overshooting has been modeled as a diffusive process...
following Herwig (2000) in the case of the reference model, the square of the Brunt–Väisälä frequency is positive so that the overshoot zone is included in the g-mode cavity. This is fairly relevant to the inversion analysis, since not only the mixed modes but also the high-order g modes should also have sensitivity for the fast core rotation (compare the black and orange curves in Figure 12).

The relevance described in the last paragraph can actually be confirmed by a test in which the Bayesian rotation inversion is carried out without the mixed-mode rotational shifts. We then have the following odds ratio in the decimal logarithmic scale: log \( O_{M_{\text{damp}},M_{\text{damp}}} = 20 \) (Table 2). Note that the latitudinal dependence is not considered here for simplicity. Although the differences in the global likelihoods are much smaller than those computed with the mixed modes included in the analysis, we can still claim that the model with a velocity shear \( M_{1,4p} \) is more favorable in terms of the global likelihood; the inference of the velocity shear has just become more marginal.

The same analysis with the artificial rotational shifts \( \delta \omega_{1h} \) in Section 3.2 has shown that it becomes rather marginal to distinguish the two models of rotational profile (one with a velocity shear \( M_{1h} \) and the other without it \( M_{\text{lin}} \)) if we exclude the mixed modes. The odds ratio expressed in the decimal logarithm is log \( O_{M_{1h},M_{\text{lin}}} = 1.25 \). This is mainly because we artificially locate the position of the shear identical to the convective-core boundary, and there are no modes that are sensitive to the artificially generated fast convective-core rotation other than the mixed modes. However, in the case of KIC 11145123, high-order g modes can be sensitive to the fast core rotation that slightly invades the radiative region, possibly leading to the marginal preference for the model with a velocity shear \( M_{1,4p} \) despite the absence of the mixed-mode rotational shifts in the analysis.

5. Summary

With the increasing number of stars for which asteroseismic rotation inversion can be carried out, developing and improving inversion techniques is of great importance for us to render our inferences on the internal rotation of stars as robust and reliable as possible. In this study, we first present a scheme of Bayesian rotation inversion that enables us to compute the probability of a model of the rotational profile and thus select the most reliable model among multiple models prepared by us beforehand (model comparison via the global likelihood). We then conduct a simple test for the scheme using two models of the rotational profile, based on which two specific sets of rotational profiles and the corresponding rotational shifts are artificially generated. It has been shown that we can successfully choose the correct model of the rotational profile among the prepared models.

Then, we have applied the Bayesian scheme to one of the Kepler targets, KIC 11145123, for which fast convective-core rotation has been suggested by Hatta et al. (2019). Focusing on the convective-core rotation of the star, four models of rotational profile are constructed. The global likelihoods of the four models thus computed clearly indicate that the models with fast convective-core rotation are favored, supporting the previous suggestion by Hatta et al. (2019) from a Bayesian perspective. The estimated parameters have been used to construct an artificial rotational profile and corresponding rotational shifts, based on which the validity of the obtained results has been checked.

In addition to the inference on the convective-core rotation, it has been suggested that the position of the rotational velocity shear is not at the convective boundary but located within the overshoot zone. Since it is generally thought that there are still numerous uncertainties in the physics around the boundary between the convective core and the radiative region above for early-type main-sequence stars (e.g., the position of the convective boundary, the extent of the overshoot zone, the possible rotational velocity shear, the possible dynamo mechanisms, and so on), the results of the study could pose a unique challenge to, for instance, numerical simulations of the dynamo mechanisms inside the convective core of early-type main-sequence stars.

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