In recent theoretical and experimental investigations, researchers have linked the low-energy field theory of a Weyl semimetal gapped with a charge-density wave (CDW) to high-energy theories with axion electrodynamics. However, it remains an open question whether a lattice regularization of the dynamical Weyl-CDW is in fact a single-particle axion insulator (AXI). In this Letter, we use analytic and numerical methods to study both lattice-commensurate and incommensurate minimal (magnetic) Weyl-CDW phases in the mean-field state. We observe that, as previously predicted from field theory, the two inversion- (I-) symmetric Weyl-CDWs with \( \phi = 0, \pi \) differ by a topological axion angle \( \delta \theta = \pi \). However, we crucially discover that neither of the minimal Weyl-CDW phases at \( \phi = 0, \pi \) is individually an AXI; they are instead quantum anomalous Hall (QAH) and “obstructed” QAH insulators that differ by a fractional translation in the modulated cell, analogous to the two phases of the Su-Schrieffer-Heeger model of polyacetylene. Using symmetry indicators of band topology and non-abelian Berry phase, we demonstrate that our results generalize to multiband systems with only two Weyl fermions, establishing that minimal Weyl-CDWs unavoidably carry nontrivial Chern numbers that prevent the observation of a static magnetoelectric response.

We discuss the experimental implications of our findings, and briefly outline generalizations to nonmagnetic semimetals with additional Weyl points.

In condensed matter physics, one of the most important tools is low-energy field theory. From the \( \mathbf{k} \cdot \mathbf{p} \) Hamiltonian of a solid-state material, an effective action can be developed to characterize the robust, and frequently topological, long-wavelength response effects [1–9]. However, to extrapolate from a low-energy field theory to an experimentally observable response effect, one must carefully complete the theory to short (UV) wavelengths – specifically, two field theories that are identical at the \( \mathbf{k} \cdot \mathbf{p} \) level may differ at large momenta, leading to distinct physical interpretations. For example, the \( \mathbf{k} \cdot \mathbf{p} \) Hamiltonian \( \mathcal{H}(\mathbf{q}) = \sigma^x q_x + \sigma^y q_y \) can characterize one of the twofold Dirac points in a graphene-like 2D semimetal [10–12], or the isolated Dirac point on the surface of a time-reversal- (T-) symmetric 3D topological insulator (TI) [13–16]. In both cases, \( \mathcal{H}(\mathbf{q}) \) carries the response of a half-level 2 + 1-D Chern-Simons theory [4, 5, 13, 14, 17]. For a 2D semimetal, the UV completion must account for additional Dirac fermions at higher momenta, which are required be present by the parity anomaly [17–23], leading to an overall integer-valued Hall conductivity [4, 21, 24–26]. However, in the case of a 3D TI, the Dirac fermion is unpaired, and a T-preserving UV completion is impossible without accounting for the bulk of the system, which conversely contributes an anomalous half-integer Hall conductivity to a magnetically gapped TI surface [4, 5, 13, 14, 17, 22, 23, 27–31]. Some of the most intriguing low-energy field theories involve condensed matter realizations of high-energy electrodynamics [6]. Specifically, in 3D insulators, the long-wavelength linear response can be formulated as an effective action[32]:

\[
S[A_\mu] = \frac{1}{4\pi^2} \int d^4x \epsilon^{\mu\nu\lambda\sigma} (\theta \partial_\mu + v_\mu) A_\nu \partial_\lambda A_\sigma , \tag{1}
\]

in which \( A_\mu \) is the electromagnetic gauge potential [2, 4, 5]. In Eq. (1), \( v_\mu = (v, 0) \) is a constant vector:

\[
\sigma^{ij}_H = \frac{e^2 v k}{h} \epsilon_{k\ell j} \tag{2}
\]

where \( \sigma^{ij}_H \) is the Hall conductivity tensor. For gapped periodic systems:

\[
v = C(\mathbf{k} \cdot \mathbf{R}_i) \mathbf{G}_i , \tag{3}
\]

where \( \mathbf{R}_i \) is a primitive lattice vector, \( \mathbf{G}_i \) is a primitive reciprocal lattice vector, and \( C(\mathbf{k} \cdot \mathbf{R}_i) = v_i/|\mathbf{G}_i| = v_i \) is the weak Chern number [Fig.1(b)], which can be evaluated as the integral of the abelian Berry curvature over a plane in the Brillouin zone (BZ) normal to \( \mathbf{R}_i \) [33] [4, 34–36]. In Eq. (1), \( \theta \) is known as the axion angle, and governs magnetoelectric response [4, 5]: in terms of the non-abelian Berry connection \( \mathcal{A} \):

\[
\theta[\mathcal{A}] = \frac{1}{4\pi} \int d^3k \epsilon^{ijk} \text{tr} \left( \mathcal{A}_i \partial_j \mathcal{A}_k - \frac{2i}{3} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_k \right) . \tag{4}
\]

Because \( \theta \) is the coefficient of \( \mathbf{E} \cdot \mathbf{B} \) in Eq. (1), then spatial inversion (I) and T symmetries (as well as other, more complicated crystal and magnetic symmetries) [28] act to quantize \( \theta \) as a \( \mathbb{Z}_2 \) topological invariant for which \( \theta = 0 \) (\( \theta = \pi \)) is the trivial (topological) value [4, 5, 31][37]. In
FIG. 1. (a) An inversion- (I-) symmetric axion insulator (AXI) [4, 5, 27, 28, 38, 39] with \( \nu_{x,y,z} = 0, \theta = \pi \) in Eq. (1). (b) A weak, \( x \)-directed, \( I \)-symmetric stack of Chern insulators with a nontrivial quantum anomalous Hall (QAH) effect [34, 35] \( \nu_{x} = -1, \nu_{y} = \nu_{y} = \theta = 0 \). (c) A superposition of the insulators in (a) and (b) is can be deformed into an "obstructed" QAH (oQAH) insulator \( \nu_{x} = -1, \nu_{y} = \nu_{y} = 0, \theta = \pi \) equivalent to shifting the QAH insulator in (b) by a half-lattice translation. A finite-sized oQAH insulator either exhibits coexisting surface and hinge states or exactly one fewer (or one more) QAH surface state, depending on whether \( I \) symmetry is weakly broken.

In several recent proposals [1, 2, 66, 67], it was also recognized that the low-energy field theory of a bulk-gapless topological material could realize a gapped \( \theta = \pi \) phase in the presence of a correlated charge-density wave (CDW). Specifically, in Weyl semimetals (WSMs) – whose bulk Fermi pockets [Weyl points (WPs)] are sources and sinks of Berry curvature characterized by integer-valued topological (chiral) charges [68–70] [Fig. 2(a)] – it was shown at the \( k \cdot p \) level that:

\[
\theta(x, t) = \theta_{0} + \phi(x, t),
\]

where \( \theta_{0} = Q \cdot x \) is the contribution to \( \theta(x, t) \) from two WPs separated by a momentum \( Q \), and \( \phi(x, t) \) is the (dynamical) phase of the CDW order parameter [1, 2, 66, 67]. The appearance of an axionic response was attributed to the chiral anomaly in quantum field theory. In the mean-field approximation, \( \phi(x, t) \) is a Goldstone mode, and hence can be tuned freely; uniform shifts of \( \phi(x, t) \) are the current-carrying sliding mode of the CDW. In disordered or incommensurate CDWs, however, \( \phi(x, t) \) is typically pinned to a non-universal value [71–73]. The proposals in [1, 2, 66, 67] have recently been revisited in light of the discovery of high-temperature WPs in the archetypal CDW compound (TaSe\(_{2}\))\(_{2}\)I [74–77], where nonlinear magnetoresistance consistent with a gapped dynamical Weyl-CDW phase was observed in (TaSe\(_{2}\))\(_{2}\)I below the CDW transition temperature [78].

Confusingly, the gapped dynamical Weyl-CDW phase is frequently labeled an AXI in the literature. However, because Eq. (5) was derived from a \( k \cdot p \) approximation, the relationship between the field-theory analysis and mean-field band topology should not be taken for granted. There have remained open and urgent questions whether there exists a UV completion in which \( \delta\theta_{\phi} = \theta_{\phi = \pi} - \theta_{\phi = 0} \) mod \( 2\pi = \pi \) emerges due to the topology of band electrons, and whether the bulk at \( \phi = 0, \pi \) is a single-particle electronic AXI. In this Letter, we use analytic and numerical methods to demonstrate that the UV completion of the simplest dynamical Weyl-CDW phase is in fact not an AXI, but is simply an \( I \)-symmetric QAH insulator. However, we discover that, depending on the CDW angle \( \phi \), the Weyl-CDW phase can be decomposed into two topologically distinct QAH phases – a QAH insulator or an “obstructed” QAH (oQAH) insulator [Fig. 1(b,c)] – that differ by fractional lattice translation. Crucially, because \( \theta \) is origin-dependent in the presence of a background QAH [4, 30, 53], then we find that the QAH and oQAH phases still differ by an origin-(gauge-) independent, topological axion angle \( \delta\theta_{\phi} = \pi \) that reflects a difference in \( I \)-quantized “Chern number polarization” [53, 79]. This provides a direct analogy between axionic CDWs and the Su-Schrieffer-Heeger model of polyacetylene [80], in which both phases are trivial atomic limits that differ by a fractional lattice translation corresponding to an \( I \)-quantized topological polarization. We demonstrate that the relative axionic response of two
\( T \)-symmetric Weyl-CDWs can be attributed to the single-particle quasiparticle band structure. Additionally, we generalize our findings to multi-band systems with two WPs and to incommensurate CDWs, establishing that QAH insulators and topological phase shifts \( \delta \theta_\phi = \pi \) are generic in minimal \( T \)-symmetric Weyl-CDWs. We conclude by discussing the experimental implications of our findings in the larger search for AXIs, and briefly outline generalizations to \( T \)-symmetric WSMs with additional WPs.

Model: We begin by introducing a simple model of a minimal \( T \)-broken, \( T \)-symmetric magnetic WSM with two WPs and with orthorhombic lattice vectors of length \( a, b, c \) in the \( \hat{x}, \hat{y}, \hat{z} \) directions, respectively [81]:

\[
H_0 = \left( \sum_R \left[ it_x c_R^\dagger \sigma^x c_{R+\hat{x}} + it_y c_R^\dagger \sigma^y c_{R+\hat{y}} + t_z c_R^\dagger \sigma^z c_{R+\hat{z}} \right] + \sum_R m/2 \left( c_R^\dagger \sigma^z c_{R+\hat{x}} + c_R^\dagger \sigma^z c_{R+\hat{y}} - 2 c_R^\dagger \sigma^z c_R \right) \right) + \sum_R t_z \cos\left( \frac{Qc}{2} \right) + h.c.,
\]

where \( m/2 > t_x, t_y, t_z > 0 \). While opaque in position space, Eq. (6) assumes a simpler form in \( k \)-space:

\[
H(k) = -2t_x \sigma^x \sin(k_x a) + t_y \sigma^y \sin(k_y b) + 2t_z \sigma^z \cos(k_c c) - m \sigma^z \left[ 2 - \cos(k_x a) - \cos(k_y b) \right].
\]

Eq. (7) is gapped at half filling at all \( k \) points away from two WPs at \( k = (0, 0, \pm Q/2) \) with chiral charges \( |C| = 1 \) [Fig. 2(a)]. The WPs in Eq. (7) are related by \( \mathcal{I} \), which is represented by:

\[
\mathcal{I}H(k)\mathcal{I}^{-1} = \sigma^z H(-k)\sigma^z.
\]

As shown in [82, 83] and in the Supplementary Material (SM), the occupied parity (\( \mathcal{P} \)) eigenvalues imply the \( k \)-space Chern numbers \( C(ck_z = 0) \) mod 2 = -1, \( C(ck_z = \pi) \) mod 2 = 0, mandating the appearance of the \( |C| = 1 \) WPs.

We next construct a \( \mathbf{k} \cdot \mathbf{p} \) expansion of Eq. (7) about the two WPs:

\[
H(q) \approx - 2t_x a q_x \sigma^x + 2t_y b q_y \sigma^y + 2t_z c q_z \sin\left( \frac{Qc}{2} \right) \sigma^z \tau^z,
\]

where the Pauli matrices \( \vec{\tau} \) act in the space of \( c_{1/2,k} \) in:

\[
c_{R} \approx \sum_k C_{1k} e^{iR\cdot[(Q/2)\vec{x}+\vec{k}]} + C_{2k} e^{-iR\cdot[(Q/2)\vec{x}-\vec{k}]}.
\]

Eq. (9) can be gapped by a CDW distortion:

\[
H_{CDW} = 2 \sum_R |\Delta| \cos(QR_z + \phi) c_R^\dagger \sigma^z c_R + \Delta \sum_k (c_{k-\frac{\pi}{2}\vec{z}}^\dagger \sigma^z \tilde{c}_{k+\frac{\pi}{2}\vec{z}} e^{-i\phi} + h.c.),
\]

which breaks the translation symmetry of Eqs. (6) and (7), coupling the two WPs. At the \( \mathbf{k} \cdot \mathbf{p} \) level, Eq. (12) induces a mass in Eq. (9):

\[
V_\phi = |\Delta| \sigma^z (\tau^x \cos \phi + \tau^y \sin \phi).
\]

Crucially, \( \mathcal{I} \) symmetry is now represented in Eqs. (9) and (13) by:

\[
\mathcal{I}H(q)\mathcal{I}^{-1} = \sigma^z \tau^z H(-q)\sigma^z \tau^z,
\]

such that Eq. (13) only preserves \( \mathcal{I} \) (centered at the origin) for \( \phi = 0, \pi \). For \( Q \neq \pi/c \) and \( |\Delta| > 0 \), this model is gapped for all \( \phi \).

Consistent with previous works [1, 2, 66, 67], we recognize that a domain wall between \( \phi = 0, \pi \) is equivalent to the critical point between a trivial insulator and an AXI [4, 28]. Correspondingly, because \( \{H(q), V_\phi\} = 0 \) for all \( \mathcal{I} \)-breaking defects in the space \( (\Delta, \phi) \) will bind chiral modes [84] (the axion strings of Ref. 1). In [1, 2, 4, 6, 85], the authors used the chiral anomaly to motivate the appearance of chiral modes, specifically identifying the relationship \( \theta = (\pi/2)(1 - \text{sgn}(|\cos \phi|)) \) mod 2\( \pi \) for \( \phi = 0, \pi \). The change in axion angle \( \delta \theta_\phi = \delta \phi \) is also consistent with magnetic symmetry-based indicators \( \{\hat{z}_4, \hat{z}_2, \hat{z}_{2i}\} \) for 3D crystals with only \( \mathcal{I} \) and translation symmetries [28, 43, 44, 47, 49–51, 56–58]:

\[
\hat{z}_4 = \frac{1}{2} \sum_{k_x, k_y, k_z} (n^\uparrow_+ - n^\downarrow_-) \text{mod } 4,
\]

\[
\hat{z}_{2i} = \frac{1}{2} \sum_{k_x, k_y, k_z \equiv \pi/c} (n^\uparrow_+ - n^\downarrow_-) \text{mod } 2,
\]

where \( n^\uparrow_\pm \) are the number of states with \( \pm 1 \) parity eigenvalues at \( k_x \). Specifically, \( \delta \phi = \pi \) in Eq. (13) implies that \( \delta \hat{z}_4 = 2 \). However, because weak Chern numbers are only \( \mathcal{I} \)-symmetry-indicated modulo 2 [82, 83], \( \delta \hat{z}_4 = 2 \) does not in itself indicate an AXI transition. Furthermore, when \( \nu_z \neq 0 \), defining \( \theta \) uniquely requires the specification of a reference state and \( \mathcal{I} \) center (i.e., an origin) [53]. Correspondingly, if there are other bulk or surface contributions to the topological response (e.g., other massive Dirac fermions at larger momenta, or a background QAH), then defects in \( V_\phi \) will host additional states that coexist with and obscure the AXI bound states. Therefore, in order to fully determine the topology of the Weyl-CDW, we must examine the lattice-regularized UV completion [Eqs. (7) and (12)] in detail.

Commensurate UV Completion: When \( Q = 2\pi/Nc, N \in \mathbb{Z}^+ \) in Eq. (7), the CDW is lattice-commensurate, and Eq. (7) remains periodic with respect to the reciprocal lattice vector \( \mathbf{G} = Q\hat{z} \) of the folded (reduced) BZ (rBZ). Folding into the rBZ translates bands with \( |k_z| > \pi/(Nc) \) into the rBZ [depicted in Fig. 2(b) for \( N = 3 \)]. For all values of \( N \), the two WPs fold into a linear fourfold (Dirac) [86] degeneracy at the rBZ boundary.
FIG. 3. Bulk ($\nu_z$) and slab ($G_z$) Chern numbers for an $N = 3$ ($Q = 2\pi/3c$) commensurate CDW. (a) The $y$-surface spectral function at $E = 0$ as a function of $k'_x$ and $k'_z$ in the surface rBZ exhibits a flat band. (b) The $y$-surface spectral function at $k'_x = 0$ as a function of $E$ and $k'_z$ exhibits $C(3k'_z, c) = -1$ spectral flow. The spectral functions in (a,b) are the same for $\phi = 0, \pi$, and together imply that $\nu_z(\phi = 0, \pi) = -1$. To measure the bulk axion angle $\theta$, we construct an $I$-symmetric, $z$-directed slab of Eq. (7) with 15 layers (5 unit cells) and $Q = 2\pi/3c$. (c) The $y$-directed slab Berry phase [28] exhibits $G_z = -5 \left(G_z = -4\right)$ spectral flow for $\phi = 0 \left(\phi = \pi\right)$. Along with $\nu_z(\phi = 0, \pi) = -1$ obtained from (a,b), the spectral flow in (c) implies through Eq. (16) that $\theta_{\phi=0,\pi} = 0, \pi$, such that the Weyl-CDW at $\phi = 0 \left(\phi = \pi\right)$ is a QAH (oQAH) insulator [Fig. 1(b,c)], and that $\delta\theta_{\phi} = \pi$. Further calculation details are provided in the SM.

From the bulk parity eigenvalues, we deduce that, for all $N \in \mathbb{Z}^+$, $C_z(\{k_z c\} < \pi/N) = -1, C_z(\{k_z c\} > \pi/N) = 0$ [87]. This implies that $\nu_z = -1$ in the rBZ [Fig. 2(b)], independent of whether $\phi = 0, \pi$. Combining $\nu_z = -1$ with the $k \cdot p$ analysis in the text preceding Eq. (15) and fixing the origin to $z = 0$ in the modulated cell, we find that $\phi = 0 \left[\phi = \pi\right]$ corresponds to a $\{2001\}$ $z$-directed weak Chern (i.e. QAH) insulator (see Fig. 1) with $\nu_{x,y} = 0$, $\nu_z = -1$ and $\theta = 0 \left[\theta = \pi\right]$. Despite the QAH and oQAH insulators differing by a translation $(Nc/2) \hat{z}$, $\delta\theta_{\phi} = \pi$, independent of the choice of origin. In the SM, we provide further details, and analytically compute $\nu_{x,y,z}$ and $\delta\theta_{\phi}$ for $N = 2$.

To explicitly determine the bulk topology, we will employ model-agnostic numerical methods [28, 30, 53] to extract $\nu_z(\phi)$ and $\delta\theta_{\phi}$. To begin, we fix the origin to the $I$ center at $(x,y,z) = (0,0,0)$ in the modulated cell, and form an $R_y$-directed, $I$-symmetric slab. The Hall conductance $G_{H,i}$ of the slab consists of an extensive contribution from the bulk QAH and an intensive contribution from $\theta$ that either reflects the bulk magnetoelectric polarizability or a QAH effect offset from the origin [4, 30, 53]:

$$G_{H,i} = \sigma_{H,i}L_i + (e^2/\hbar)c^z(i\delta\theta_{\phi})$$

where $\sigma_{H,i} = e^2/2\pi$ is the Hall conductivity (given by a weak Chern number $\nu_z$), where $L_i$ is the (lattice-regularized) thickness of the slab. Because a slab is a quasi-2D system, it carries a quantized Chern number $\nu_i$ that is related to Eq. (16) by $\nu_i = e^2/\hbar = G_{H,i}$. For $I$-symmetric slab geometries, $\theta$ remains quantized to the bulk value, and provides an odd-integer contribution to $\delta\theta_{\phi}$.

Next, we cut the $N = 3$ model in Fig. 3(a,b) into an $I$-symmetric, $z$-directed slab geometry with $L_z/3c = 5$ unit cells, and calculate the $y$-directed non-abelian Berry phase (Wilson loop) $W$ [28, 92-94] of the slab [Fig. 3(c)], whose winding indicates that $G_z(0) = -5 = \nu_z L_z/3c$. Along with $\nu_z(\phi = 0, \pi) = -1$, the difference $|\delta G_z| = 1$ indicates through Eq. (16) that $\theta_{\phi=0,\pi} = 0, \pi$, such that the insulating Weyl-CDW is a QAH (oQAH) insulator for $\phi = 0 \left(\phi = \pi\right)$.

**Incommensurate UV completion:** Having demonstrated that $I$-symmetric Weyl-CDWs are either QAH or oQAH insulators in the commensurate case, we next explore the case of incommensurate modulation. Although an incommensurate CDW is not translationally-
invariant, neither QAH nor oQAH phases require translation symmetry [4, 5, 27, 28, 38, 39]. Consequently, Eq. (16) still applies, without modification, to \( \mathbf{z} \)-directed slabs of Eq. (7) with incommensurate values of \( Q \).

To confirm this result, we cut \( H_0 + H_{\text{CDW}} \) [Eqs. (7) and (12)] with \( Q = \varphi \pi / 2c \) (where \( \varphi \) is the golden ratio) into an \( I \)-symmetric rod geometry. First, we observe an extensive number of QAH surface states along the rod for \( \phi = 0, \pi; [\phi = 0 \text{ is shown in Fig. 4(a)]; we additionally calculate that there is exactly one fewer chiral mode per surface at \( \phi = \pi \). Next, to measure \( \theta \), we cut the incommensurate Weyl-CDW into an \( I \)-symmetric, \( \mathbf{z} \)-directed slab geometry and calculate the \( \hat{y} \)-directed Berry phase, as we did in the commensurate case [Fig. 3]. In the slab geometry, \( |\delta G_z| = 1 \) between \( \phi = 0, \pi \) [Fig. 4(b)]. Furthermore, in incommensurate CDWs, a tuning cycle in \( \phi \) changes the bulk wavefunctions, but not the bulk energy spectrum [95, 96], such that, if an incommensurate Weyl-CDW with \( \phi = 0 \) is bulk-insulating, then an incommensurate Weyl-CDW with arbitrary \( \phi \) is also bulk insulating. Additionally, the weak QAH invariant cannot change without a bulk gap closure, whereas the value of \( \theta \) is free to wind between 0 and \( \pi \) at \( I \)-breaking CDW angles away from \( \phi = 0, \pi \) [4, 5, 28]. As shown in the SM, \( |\delta G_z| = 1 \) in Fig. 4(b), along with \( G_{H,z} \) calculated for successive rational approximants of an irrational \( Q \), imply that, as in the commensurate case (Fig. 3), the incommensurate Weyl-CDW carries the relative axis angle \( \delta \theta_0 = \pi \).

**Experimental Implications:** Our results have several implications for the search for axionic responses in Weyl-CDW materials. First, we have demonstrated that a large QAH effect is unavoidable and guaranteed by band topology for both commensurate and incommensurate Weyl-CDWs, independent of \( \phi \). Second, the interplay between lattice and phase-angle defects, which both bind 1D chiral modes, is a fruitful area for future study, though one must cautiously separate contributions from \( \theta \) and those from a background QAH effect.

Next, we emphasize that the quantized axionic response \( \delta \theta_0 = \pi \) in Weyl-CDWs is measurable through the dynamical dependence of the quasi-2D QAH effect on \( \phi \), rather than through the static magnetoelectric polarizability at fixed \( \phi \) [97, 98]. Furthermore, soliton-like defects in \( \phi \) can in principle be manipulated by exciting the CDW sliding mode. Finally, in \( I \)-symmetric, magnetic Weyl-CDWs, our results highlight the experimental and theoretical difficulty of distinguishing QAH, oQAH, and AXI phases. However, our results do imply that, by carefully computing \( k \)-space Chern numbers and then zone-folding, it is possible to predict the topology of Weyl-CDWs in real materials without performing intensive quasiperiodic calculations.

**Outlook:** Going forward, our analysis can also be extended to CDWs and spin-density waves [99–102] in \( T \)-symmetric semimetals, including Dirac [86, 103, 104], Weyl, and nodal-line semimetals [105–108], which have recently been shown to exhibit signatures of higher-order topology [54, 89, 109, 110]. Most interestingly, because rotation- and \( T \)-symmetric HOTIs [28, 55] can be formed from weak stacks of 2D TIs [47, 50], then it is likely that rotation-symmetric CDWs in \( T \)-symmetric WSMs, such as the CDW in (TaSe\(_4\))\(_2\)I [74], may also exhibit non-trivial response effects. Specifically, a CDW can fold the four WPs in a minimal rotation- and \( T \)-symmetric WSM into an eightfold double Dirac point for which \( T \)-symmetric, line-like defects bind helical modes equivalent to HOTI hinge states [111, 112]. Additionally, recent experiments have demonstrated hinge-state-like step-edge helical modes and robust edge supercurrents in rotation-and \( T \)-symmetric WSMs [113–117]. This provides further motivation for the elucidation of quantized response effects in \( T \)-symmetric HOTIs and Weyl-CDWs with trivial \( \theta \) angles, which are currently unknown.

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[1] Z. Wang and S.-C. Zhang, Physical Review B 87, 161107 (2013).
[2] Y. You, G. Y. Cho, and T. L. Hughes, Physical Review B 94, 085102 (2016).
[3] M. Laubach, C. Platt, R. Thomale, T. Neupert, and S. Rachel, Phys. Rev. B 94, 241102 (2016).
[4] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Physical Review B 78, 195424 (2008).
[5] A. M. Essin, J. E. Moore, and D. Vanderbilt, Physical review letters 102, 146805 (2009).
[6] F. Wilczek, Phys. Rev. Lett. 58, 1799 (1987).
[7] S. C. Zhang, T. H. Hansson, and S. Kivelson, Physical review letters 62, 82 (1989).
[8] B. I. Halperin, P. A. Lee, and N. Read, Physical Review B 47, 7312 (1993).
[9] D. T. Son, Physical Review X 5, 031027 (2015).
[10] D. P. DiVincenzo and E. J. Mele, Phys. Rev. B 29, 1685 (1984).
[11] G. W. Semenoff, Phys. Rev. Lett. 53, 2449 (1984).
[12] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009).
[13] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).
[14] L. Fu and C. L. Kane, Phys. Rev. B 76, 045302 (2007).
[15] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature 452, 970 (2008).
[16] Y. Xia, D. Qian, D. Hsieh, L. Wray, A. Pal, H. Lin, A. Bansil, D. Grauer, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature physics 5, 398 (2009).
[17] M. F. Lapa, Phys. Rev. B 99, 235144 (2019).
Note that for gapless systems, such as Weyl semimetals, in this Letter we will use Greek indices as spacetime indices, and Roman indices as purely spatial indices.

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C(\mathbf{k} \cdot \mathbf{R}_B) will depend on \mathbf{k}.
Throughout this Letter, we choose parameters for which $\nu = -1$.

This convention relies on $N$ being odd to avoid slabs with fractional modulated cells. In the SM, we consider a Weyl-CDW with $N = 2$, which can only be cut into an $\mathbb{I}$-symmetric slab with a half-integer number of modulated cells.

B. J. Wieder, Z. Wang, H. Weng, Q. Wu, X. Dai, and Z. Fang, Phys. Rev. B 88, 125427 (2013).

Z. Wang, Y. Sun, X.-Q. Chen, C. Franchini, G. Xu, H. Weng, X. Dai, and Z. Fang, Phys. Rev. B 85, 195320 (2012).

Y. Kim, B. J. Wieder, C. L. Kane, and A. M. Rappe, Phys. Rev. Lett. 115, 036806 (2015).

C. Fang, Y. Chen, H.-Y. Kee, and L. Fu, Physical Review B 92, 081201 (2015).

B. J. Wieder and C. L. Kane, Phys. Rev. B 94, 155108 (2016).

L. M. Schoop, M. N. Ali, C. StràÀyer, A. Topp, A. Varykhalov, D. Marchenko, V. Duppel, S. S. P. Parkin, B. V. Lotsch, and C. R. Ast, Nature Communications 7, 11696 EP (2016), article.

M. Lin and T. L. Hughes, Phys. Rev. B 98, 241103 (2018).

C.-Z. Li, A.-Q. Wang, C. Li, W.-Z. Zheng, A. Brinkman, D.-P. Yu, and Z.-M. Liao, Phys. Rev. Lett. 124, 156601 (2020).

B. J. Wieder, Y. Kim, A. M. Rappe, and C. L. Kane, Phys. Rev. Lett. 116, 186402 (2016).

B. Bradlyn, J. Cano, Z. Wang, M. G. Vergniory, C. Felser, R. J. Cava, and B. A. Bernevig, Science , 10.1126/science.aaf5037 (2016).

F.-T. Huang, S. Joon Lim, S. Singh, J. Kim, L. Zhang, J.-W. Kim, M.-W. Chu, K. M. Rabe, D. Vanderbilt, and S.-W. Cheong, Nature Communications 10, 4211 (2019).

Y.-B. Choi, Y. Xie, C.-Z. Chen, J.-H. Park, S.-B. Song, J. Yoon, B. J. Kim, T. Taniguchi, K. Watanabe, H.-J. Lee, J.-H. Kim, K. Chung Fong, M. N. Ali, K. Tuen Law, and G.-H. Lee, arXiv e-prints , arXiv:1909.02537 (2019), arXiv:1909.02537 [cond-mat.mes-hall].

A. Kononov, G. Abulizi, K. Qu, J. Yan, D. Mandrus, K. Watanabe, T. Taniguchi, and C. Schönenberger, arXiv e-prints , arXiv:1911.02414 (2019), arXiv:1911.02414 [cond-mat.mes-hall].

W. Wang, S. Kim, M. Liu, F. A. Cevallos, R. J. Cava, and N. P. Ong, arXiv e-prints , arXiv:2011.08598 (2019), arXiv:2011.08598 [cond-mat.supr-con].

I.-L. Liu, C. Heikes, T. Yildirim, C. Eckberg, T. Metz, S. Ran, I. Ratcliff, William, J. Paglione, and N. P. Butch, arXiv e-prints , arXiv:2005.02277 (2019), arXiv:2005.02277 [cond-mat.mtrl-sci].
I. PARITY (INVERSION) EIGENVALUE ANALYSIS OF THE MINIMAL MAGNETIC WEYL SEMIMETAL TIGHT-BINDING MODEL WITHOUT MODULATION

Here we compute the parity [inversion (I)] eigenvalues of the occupied bands of the unperturbed tight-binding model with two Weyl points (WPs), which is described by the Bloch Hamiltonian:

\[ H_0(k) = -2(t_x \sigma^x \sin k_x a + t_y \sigma^y \sin k_y b) + 2t_z \sigma^z (\cos k_z c - \cos \frac{Q_c}{2}) - m \sigma^z (2 - \cos k_x a - \cos k_y b). \]  

Eq. (1) respects \( I \) symmetry, which is represented by:

\[ \mathcal{I} H_0(k) \mathcal{I}^{-1} = \sigma^z H_0(-k) \sigma^z. \]  

| \((k_x a, k_y b, k_z c)\) | \(n_x^+\) | \(n_y^+\) |
|-----------------|--------|--------|
| \((0, 0, 0)\)   | 0      | 1      |
| \((\pi, 0, 0)\) | 1      | 0      |
| \((0, \pi, 0)\) | 1      | 0      |
| \((\pi, \pi, 0)\) | 1      | 0      |
| \((\pi, 0, \pi)\) | 1      | 0      |
| \((0, \pi, \pi)\) | 1      | 0      |
| \((\pi, \pi, \pi)\) | 1      | 0      |

TABLE I. Valence parity [inversion (I)] eigenvalue multiplicities \( (n_x^+\) for the unmodulated Hamiltonian \( H_0 \) of a minimal \( I \)-symmetric Weyl semimetal \[ Eq. (1) \].

We will now deduce the parity eigenvalues in the \( ck_z = 0, \pi \) planes. First, at \( k_z c = 0 \), Eq. (1) at the \( I \)-invariant momenta (TRIM points) is given by:

\[ H_0(0, 0, 0) = 2t_z (1 - \cos \frac{Q_c}{2}) \sigma^z, \]  

\[ H_0(\pi/a, \pi/b, 0) = -\left[ 4m - 2t_z (1 - \cos \frac{Q_c}{2}) \right] \sigma^z, \]  

\[ H_0(\pi/a, 0, 0) = H_0(0, \pi/b, 0) = -\left[ 2m - 2t_z (1 - \cos \frac{Q_c}{2}) \right] \sigma^z. \]  

We observe that \( H_0(0, 0, 0), H_0(\pi/a, 0, 0), \) and \( H_0(0, \pi/b, 0) \) are proportional to the matrix \( I = \sigma^z \) \[ Eq. (2) \], implying that, for \( m \geq 2t_z \), the occupied parity eigenvalues at \( k = (\pi/a, 0, 0), (0, \pi/b, 0) \), and \( (\pi/a, \pi/b, 0) \) are positive, but the occupied parity eigenvalue at \( k = (0, 0, 0) \) is negative. Using the \( \mathbb{Z}_2 \) parity index for an \( I \)-symmetric, time-reversal-(\( T^- \)) broken 2D insulator \[ 1–4 \], we then compute the Chern number \( C(k_z c) \) of the occupied bands in the \( k_z c = 0 \) plane, which we find to be \( C(k_z c = 0) \mod 2 = -1 \). Conversely, in the \( k_z c = \pi \) plane:

\[ H_0(0, 0, \pi/c) = -2t_z (1 + \cos \frac{Q_c}{2}) \sigma^z, \]  

\[ H_0(\pi/a, \pi/b, \pi/c) = -\left[ 4m + 2t_z (1 + \cos \frac{Q_c}{2}) \right] \sigma^z, \]  

\[ H_0(\pi/a, 0, \pi/c) = H_0(0, \pi/b, \pi/c) = -\left[ 2m + 2t_z (1 + \cos \frac{Q_c}{2}) \right] \sigma^z. \]
implying that all of the parity eigenvalues of the occupied bands are positive, such that \( C(k_z = \pi) \mod 2 = 0 \). Because \( C(k_z = \pi) = C(k_z = 0) \mod 2 = 1 \), then there must be a set of WPs with net chiral charge \( |C| = 1 \) in each half of the Brillouin zone (BZ) [5], taking BZ halves to be indexed by positive and negative values of \( k_z \). We have verified that Eq. (1) only features two total WPs with \( |C| = 1 \) chiral charges [6].

We summarize the parity eigenvalues in Table I; we will find this information useful for future calculations (see Sec. II).

### II. Symmetry Indicators for N-fold Commensurate Modulation

We will now extend the previous analysis in Sec. I to the more general case of commensurate, \( \mathcal{I} \)-symmetric CDWs characterized by \( Q = 2\pi/Nc \) in Eq. (1). This is accomplished by folding the BZ into the reduced BZ (rBZ) – in which we label crystal momenta \( \mathbf{k}' \) – and then counting the parity eigenvalues that have been folded onto each TRIM point in the rBZ. In the \( Nk_{z,c} = 0 \) plane in the rBZ, the occupied Bloch states have been folded from all of the BZ planes at \( k_{z,c} = 2\pi m/N, m \in \mathbb{Z} \) in the larger BZ of the unmodulated structure. Crucially, we recognize that pairs of Bloch states folded from values of \( k_z \) away from \( k_{z,c} = 0, \pi \) will carry net-zero parity eigenvalues, because generic \( \mathbf{k} \) points away from \( k_{z,c} = 0, \pi \) are not \( \mathcal{I} \)-symmetric [7]. Conversely, Bloch states folded from TRIM points in the original (unmodulated) cell will contribute the parity eigenvalues listed in Table I.

| \((k_{\alpha}', a, k_{\beta}', b, k_{\gamma}', Nc)\) | \(n_+^a\) | \(n_-^a\) |
|------------------|--------|--------|
| \((0, 0, 0)\) | \(N/2\) | \(N/2\) |
| \((\pi, 0, 0)\) | \((N/2) + 1\) | \((N/2) - 1\) |
| \((0, \pi, 0)\) | \((N/2) + 1\) | \((N/2) - 1\) |
| \((\pi, \pi, 0)\) | \((N/2) + 1\) | \((N/2) - 1\) |
| \((0, 0, \pi)\) | \(\left[(N/2) - 1\right] + \text{Dirac}\) | \(\left[(N/2) - 1\right] + \text{Dirac}\) |
| \((\pi, 0, \pi)\) | \((N/2)\) | \((N/2)\) |
| \((0, \pi, \pi)\) | \((N/2)\) | \((N/2)\) |
| \((\pi, \pi, \pi)\) | \((N/2)\) | \((N/2)\) |

**Table II.** Valence parity eigenvalue multiplicities \((n_+^a)\) for the Hamiltonian of an \( \mathcal{I} \)-symmetric Weyl-CDW with \( N \)-fold modulation \( Q = 2\pi/Nc \) in Eq. (1), in the case in which \( N \) is even. The symbol Dirac represents the parity eigenvalue contribution from gapping the Dirac degeneracy that forms at the rBZ boundary (see the discussion in the main text).

For \( Q = 2\pi/Nc \) modulation with an even value of \( N \), the \( k_{z,c} = 0, \pi \) planes and \( N - 2 \) planes at generic values of \( k_z \) from the original BZ are folded into the \( k_{z,c}' = 0 \) plane in the rBZ. Conversely, the occupied bands at \( k_{z,c}' = \pi \) in the rBZ are folded from the \( k_{z,c}' \)-indexed planes containing the two WPs in the original BZ, as well as \( N - 2 \) planes at generic values of \( k_z \). It is important to note that, at the rBZ TRIM point \( \mathbf{k}' = (0, 0, \pi/Nc) \), the two WPs from the BZ have become folded into a fourfold Dirac degeneracy [8] in the rBZ. As discussed in the main text, there is only one \( \mathcal{I} \)-symmetric mass for the Dirac degeneracy if it directly gaps, such that the Dirac point either contributes two negative or two positive parity eigenvalues to the set of valence parity eigenvalues. The resulting distribution of valence parity eigenvalues is listed in Table II.

| \((k_{\alpha}', a, k_{\beta}', b, k_{\gamma}', Nc)\) | \(n_+^a\) | \(n_-^a\) |
|------------------|--------|--------|
| \((0, 0, 0)\) | \((N - 1)/2\) | \((N + 1)/2\) |
| \((\pi, 0, 0)\) | \((N + 1)/2\) | \((N - 1)/2\) |
| \((0, \pi, 0)\) | \((N + 1)/2\) | \((N - 1)/2\) |
| \((\pi, \pi, 0)\) | \((N + 1)/2\) | \((N - 1)/2\) |
| \((0, 0, \pi)\) | \(\left[(N - 1)/2\right] + \text{Dirac}\) | \(\left[(N - 3)/2\right] + \text{Dirac}\) |
| \((\pi, 0, \pi)\) | \((N + 1)/2\) | \((N - 1)/2\) |
| \((0, \pi, \pi)\) | \((N + 1)/2\) | \((N - 1)/2\) |
| \((\pi, \pi, \pi)\) | \((N + 1)/2\) | \((N - 1)/2\) |

**Table III.** Valence parity eigenvalue multiplicities \((n_+^a)\) for the Hamiltonian of an \( \mathcal{I} \)-symmetric Weyl-CDW with \( N \)-fold modulation \( Q = 2\pi/Nc \) in Eq. (1), in the case in which \( N \) is odd. The symbol Dirac represents the parity eigenvalue contribution from gapping the Dirac degeneracy that forms at the rBZ boundary (see the discussion in the main text).

For \( Q = 2\pi/Nc \) modulation with an odd value of \( N \), the situation is similar to the even case. In the \( k_{z,c}' = 0 \) plane of the rBZ, there are folded Bloch states originating from the \( k_{z,c}' = 0 \) plane in the BZ of the unmodulated structure, as well as Bloch states from \( N - 1 \) additional planes at generic values of \( k_z \). In the \( Nk_{z,c}' = \pi \) plane of the rBZ,
there are states originating from the $k_z = \pi$ plane of the original BZ, Bloch states from the two $k_z$-indexed planes containing WPs in the original BZ, and Bloch states originating from $N - 3$ additional planes at generic values of $k_z$. As in the even case, at the rBZ TRIM point $k' = (0, 0, \pi/Nc)$, the two WPs from the BZ have become folded into a fourfold Dirac degeneracy [8] in the rBZ. The resulting distribution of valence parity eigenvalues is listed in Table III.

For all $N \in \mathbb{Z}^+$, when the Dirac degeneracy at $k'_z = \pi$ is directly gapped, the resulting occupied parity eigenvalues are determined by the CDW phase $\phi$. As determined through the $k \cdot p$ analysis in the main text, when $\phi = 0$ ($\phi = \pi$), the Dirac degeneracy is split to contribute two additional negative (positive) parity eigenvalues. Because the Weyl-CDW with commensurate $N$ has 3D translation and $I$ symmetries, then it respects the symmetries of magnetic space group $\text{MSG} 2.4$ ($P\bar{1}$), numbered in the convention of Belov, Nerenova, and Smirnova (BNS) [7, 9] (the Hamiltonian also respects additional symmetries that do not affect the analysis performed in this Letter). Previous works [10–18] have determined the symmetry-based indicators in MSG $2.4$ ($P\bar{1}$) to be given by:

$$\hat{z}_4 \equiv \frac{1}{2} \sum_{k_x, k_y \in \text{TRIMS}} (n_x^a - n_x^c) \mod 4,$$ (9)

and:

$$\hat{z}_{2i} \equiv \frac{1}{2} \sum_{k_x, k_y \in \text{TRIMS}} (n_x^b - n_x^c) \mod 2,$$ (10)

where in particular, odd values of weak $\hat{z}_{2i}$, along with the presence of a bulk gap at all $k$ points (i.e., the absence of WPs), indicate that the Chern number $C_i(k_i)$ mod 2 = 1 at all values of $k_i$. Using the parity eigenvalues listed in Tables II and III, we deduce that:

$$\hat{z}_4(\phi = n\pi) = 1 + (-1)^n, \quad \hat{z}_{2x} = \hat{z}_{2y} = 0, \quad \hat{z}_{2z} = 1. \quad \text{(11)}$$

In particular, the nontrivial, $\phi$-independent weak Chern index $\hat{z}_{2z} = 1$ directly indicates the presence of an unavoidable QAH background with an odd Chern number, as discussed throughout the main text.

### III. EXTENSION TO INCOMMENSURATE MODULATION

In this section, we will show that the previous zone-folding arguments in Sec. II can be extended beyond integer values of $N$ in $Q = 2\pi/Nc$. To begin, in the case in which $Q = 2\pi M/Nc$, where $M$ and $N$ are coprime positive integers, then the rBZ continues to have a well-defined primitive reciprocal lattice vector $G_z = (2\pi/Nc)\hat{z}$. However, when $Q = 2\pi M/Nc$, the two WPs are separated by a momentum $N\delta k'_z c = 2\pi M$. This implies that $M$ planes indexed by $k_z$ with $|k_z| < \delta k'_z/2$ in the original BZ — whose occupied bands each carry a Chern number $\nu_z = -1$ — will be folded onto the same value of $k'_z$ in the rBZ. Thus, in the gapped Weyl-CDW phase with $Q = 2\pi M/Nc$, there is a weak Chern number $\nu_z = -M$. When $M$ is odd, the two WPs in the original BZ continue to fold onto the point in the rBZ, and the resulting parity eigenvalue analysis in Sec. II applies. However, when $M$ is even, the two WPs fold onto the $\mathbf{k}' = 0$ point in the rBZ. Nevertheless, the $k \cdot p$ Dirac Hamiltonian that we analyzed in the main text is agnostic to the TRIM point around which it is formulated, and therefore also characterizes the fourfold Dirac point that forms at $\mathbf{k}' = 0$ when $|\Delta| = 0$ and $M$ is even. Therefore, the dependence of the strong ($\hat{z}_4$) and weak ($\hat{z}_{2i}$) indices on the CDW phase $\phi$ [Eq. (11)] continues to imply that, for a $\hat{z}$-directed slab with thickness $L_z$ of a Weyl-CDW with modulation $Q = 2\pi M/Nc$, the anomalous Hall conductance $G_{H,z}$ is given by:

$$G_{H,z} = \frac{e^2}{h} \left( -\frac{ML_z}{Nc} + \frac{\theta_\phi}{\pi} \right),$$ (12)

where $\delta \theta_\phi = \theta_{\phi = \pi} - \theta_{\phi = 0} \mod 2\pi = \pi$ relative to a fixed choice of origin (see the main text for a more general expression for slab Hall conductance).

Crucially, because any irrational modulation can be expressed as the limit of a sequence of rational approximants [19], this implies that our results also extend to Weyl-CDWs with incommensurate modulation. Taking the limit in which $N/M$ becomes irrational while keeping the slab Chern number $ML_z/Nc$ fixed, we conclude that, for both commensurate and incommensurate Weyl-CDWs formed from minimal $I$-symmetric Weyl semimetals, the anomalous Hall conductance of a $\hat{z}$-directed slab with thickness $L_z$ is given by:

$$G_{H,z} = \frac{e^2}{2\pi h} (QL_z + 2\theta_\phi).$$ (13)
Because $\delta \theta_{\phi} = \pi$ for every rational $Q$, then we conclude that $\delta \theta_{\phi} = \pi$ remains true in the limit of irrational $Q$. To leading order in the thermodynamic limit, Eq. (13) is consistent with the statement that the CDW preserves the anomalous Hall conductance obtained by integrating the $k$-space Chern numbers between the WPs of the parent (unmodulated) Weyl semimetal.

IV. EXPLICIT MODEL OF A COMMENSURATE WEYL-CDW WITH $N = 2$ MODULATION

In the case of $Q = 2\pi/Nc = \pi/c$ ($N = 2$) modulation (i.e., a Peierls distortion), the unmodulated Hamiltonian $H_0$ in Eq. (1) can simply be re-expressed in a larger unit that is doubled in the $z$- (c-axis-) direction. The position-space embedding of the $N = 2$ modulated Hamiltonian then arises from four total orbitals, which are distributed into pairs located at $(x, y, z) = (0, 0, 0)$ and $(0, 0, c)$ within the doubled cell. Introducing a vector of Pauli matrices $\vec{\mu}$ to index the orbitals at $z = 0, c$, we obtain the unmodulated Hamiltonian:

$$H_0(k') = 2(-t_x \sigma^x \mu^z \sin k'_z a - t_y \sigma^y \mu^0 \sin k'_y b + t_z \cos k'_z \sigma^z \mu^z)
- m(2 - \cos k'_x a - \cos k'_y b) \sigma^z \mu^z,$$

where $k'$ indexes momentum in the reduced BZ (rBZ). In the rBZ, the reciprocal lattice vectors are given by $G'_x = (2\pi/a)x$, $G'_y = (2\pi/b)y$, and $G'_z = (\pi/c)z$, such that $H(2ck'_z + 2n)$ is now related to $H(k'z')$ by the reciprocal lattice vector $G'_z$. Eq. (14) is $I$ symmetric; using induction from the real-space data [20] (i.e., the positions of the four orbitals), we deduce that the matrix representative of $I$ at each of the eight rBZ TRIM points $k'_a$ is given by:

$$I(k'_a) = \sigma^z \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{-i2k'_z} \end{pmatrix}.$$

We next consider the modulation induced by the CDW distortion, which is given by $H_{CDW}$ in the main text. For the case in which $N = 2$, $QR_z \in \pi Z$, such that:

$$\cos(QR_z + \phi) = (-1)^{QR_z/\pi} \cos \phi.$$

Eq. (16) implies that the on-site modulation $H_{CDW}$ assumes a form:

$$H_{CDW} = |\Delta| \cos \phi \sigma^z \mu^z.$$

Surprisingly, for $N = 2$ (Peierls) modulation, $H_{CDW}$ is $I$-symmetric at all values of $\phi$, distinctly unlike the more generic CDWs discussed throughout this Letter. Because $I = \sigma^z \mu^z$ at the four TRIM points in the $2ck'_z = \pi$ plane [Eq. (15)], then the $k \cdot p$ expansion of $H_0 + H_{CDW}$ around $k' = (0, 0, \pi/c)$ acquires a Dirac mass $V_\phi = I \cos \phi$. Consequently, for $|\Delta| > 0$, the Dirac degeneracy at $k' = (0, 0, \pi/c)$ splits, leading at half filling to two occupied bands at $k' = (0, 0, \pi/c)$ with negative [positive] parity ($I$) eigenvalues when $\text{sgn} [\cos \phi]$ is positive [negative].

For values of $|\Delta|$ small enough not to invert bands elsewhere in the rBZ, the occupied parity eigenvalues at the four TRIM points in the $2ck'_z = 0$ plane are given by $\{\pm,+,++;++;++\}$ for all values of $\phi$. Conversely, the valence parity eigenvalues in the $2ck'_z = \pi$ plane are $\phi$-dependent, and are given by $\{-,+-,++;+;++\}$ when $\text{sgn} [\cos \phi]$ is positive and by $\{++;+-,++;+;++\}$ when $\text{sgn} [\cos \phi]$ is negative (unlike in the case of general modulation, for $N = 2$ modulation $Q = \pi/c$, the bulk is gapless at $\phi = \pm \pi/2$). We observe that the occupied parity eigenvalues computed for $Q = \pi/c$ ($N = 2$) modulation are consistent with the valence parity eigenvalue formulas for even-$N$ modulation listed in Table II. Using the symmetry-based indicators $\{\hat{z}_1|\hat{z}_2;\hat{z}_2;\hat{z}_2\}$ for 3D crystals with $I$ and translation symmetry (see Sec. II and the main text) [10–18], we determine that the occupied bands of the $N = 2$ minimal Weyl-CDW are characterized by the indices $\{2001\}$ [$\{0001\}$] when $\text{sgn} [\cos \phi]$ is positive [negative]. From the slab Berry phase analysis performed in the main text, we conclude [using the origin choice specified by the embedding in Eq. (14)] that the Weyl-CDW with $N = 2$ modulation is a weak Chern (QAH) insulator with $\nu_z = -1$, $\theta = \nu_x = \nu_y = 0$ when $\text{sgn} [\cos \phi]$ is positive, and consequently is an “obstructed” QAH (oQAH) insulator with $\theta = \pi$, $\nu_z = -1$, $\nu_x = \nu_y = 0$ when $\text{sgn} [\cos \phi]$ is negative.

In the four-band model $H_0 + H_{CDW}$, we find that $\delta \theta_{\phi} = \theta_{\phi=\pi} - \theta_{\phi=0} \bmod 2\pi$ can in fact be directly computed. At nonzero values of $k'_z$ and $k'_y$, the Hamiltonian $H_0 + H_{CDW}$ is generically gapped for all values of $\phi$, allowing us to specialize momentarily to the line $k'_z = k'_y = 0$ along which the bulk gap is smallest (or vanishing for $\phi = \pm \pi/2$):

$$H(0, 0, k'_z, \phi) = \sigma^z(2t_z \mu^z \cos k'_z c + \mu^z |\Delta| \cos \phi).$$

First, we fix $\phi$ to a value away from $\phi = \pm \pi/2$; this opens a bulk gap at all $k'$ points in the rBZ. Next, we tune $t_z$ to zero, which does not close a bulk gap. With $t_z \to 0$, $[H_0 + H_{CDW}, \mu^z] = 0$, allowing us to divide the two occupied
bands into $\mu^z = \pm 1$ sectors. When $\phi$ is fixed to a value $\phi \neq \pm \pi/2$ and then $t_z$ is set to zero without closing the bulk gap, $H_0 + H_{CDW}$ assumes the $k'_z$-independent form of a 2D Chern insulator [21], but only within one of the $\mu^z = \pm 1$ sectors [Eqs. (14) and (17)]; in the other ($\mu^z = \mp 1$) sector, the occupied band carries a trivial Chern number.

In the case in which $\cos \phi > 0$, the band with $\mu^z = 1$ [$\mu^z = -1$] in $H_0 + H_{CDW}$ carries the Chern number $C_z(k'_z) = -1$ [$C_z(k'_z) = 0$] for all values of $k'_z$ in the rBZ. In the definition of $I$ symmetry in Eq. (15), the $\mu^z = 1$ ($\mu^z = -1$) sector is $k'_z$-independent ($k'_z$-dependent). First, we note that because the $\mu^z = 1$ subspace of $H_0 + H_{CDW}$ is $k'_z$-independent, and has a $k'_z$-independent embedding [from the definition of $I$ in Eq. (15)], then we can inverse-Fourier-transform the $k'_z$ component of the $\mu^z = 1$ subspace of $H_0 + H_{CDW}$ to realize a $C_z = -1$ Chern insulator at $z = 0$. Next, we recognize that, while the $\mu^z = -1$ subspace in the $I$ embedding is $k'_z$-dependent [Eq. (15)], the occupied band in the $\mu^z = -1$ subspace is topologically trivial. We have confirmed that the trivial occupied band in the $\mu^z = -1$ subspace of $H_0 + H_{CDW}$ with $\cos \phi > 0$ can be deformed into the limit of a weak stack [10, 11, 17, 18, 23] of one $C_z = -1$ Chern insulator in the $z = 0$ plane plus one Wannier orbital at $(x, y, z) = (0, 0, c)$ per doubled cell.

In the case in which $\cos \phi < 0$, we find that the resulting orbitals and Chern insulators are related to the previous case of $\cos \phi > 0$ by a (now-fractional) lattice translation $t = c\hat{z}$, similar to the relationship between the two phases of the Su-Schrieffer-Heeger (SSH) model of polyacetylene [24]. To see this, we first note that when $\cos \phi < 0$, the band with $\mu^z = 1$ [$\mu^z = -1$] in $H_0 + H_{CDW}$ carries the Chern number $C_z(k'_z) = 0$ [$C_z(k'_z) = -1$] for all values of $k'_z$ in the rBZ. We then again note that, in the definition of $I$ symmetry in Eq. (15), the $\mu^z = 1$ ($\mu^z = -1$) sector is $k'_z$-independent ($k'_z$-dependent). This implies that a trivial band in the $\mu^z = 1$ sector can be inverse-Fourier-transformed into a Wannier orbital at $(x, y, z) = (0, 0, 0)$, but also implies a more complicated dependence on $k'_z$ for the $C_z(k'_z) = -1$ band in the $\mu^z = -1$ sector. However, if we shift our origin by an amount $t = c\hat{z}$, we restore the previous relationship between the embedding in $I$ and the $\mu^z = \pm 1$ sectors of the $k'$-space Hamiltonian $H_0 + H_{CDW}$; specifically, we again realize a situation in which the band with nontrivial Chern number lies in the same $\mu^z$ sector as the sector of $I$ with no $k'_z$ dependence. This allows us to once again inverse-Fourier-transform the $k'_z$ components of the two occupied bands of $H_0 + H_{CDW}$ with $\cos \phi < 0$ to realize a weak stack of Wannier orbitals and Chern insulators. However, in the case of $\cos \phi < 0$, the weak stack consists of one Wannier orbital at $(x, y, z) = (0, 0, 0)$ plus one $C_z = -1$ Chern insulator in the $z = c$ plane, which we recognize as an oQAH insulator with the topological indices $\theta = \pi, \nu_z = -1, \nu_x = \nu_y = 0$. The case of $N = 2$ modulation thus emphasizes the origin-dependence of $\theta$ when $\nu_{x,y,z} \neq 0$. However, in the presence of $I$ symmetry, the difference $\delta \theta_\phi = \pi$ between $\cos \phi > 0$ and $\cos \phi < 0$ is origin- and gauge-independent and topological, analogous to the difference in the positions of the Wannier centers between the two phases of the SSH model [24–26].

We can also analytically determine that $\delta \theta_\phi = \pi$ for an $I$-symmetric Weyl-CDW with $N = 2$ modulation by appealing to the field-theoretic discussion in the main text and expressing $\theta$ in terms of the non-abelian Berry connection $A$. While the Chern-Simons form $CS_3[A]$ appearing in the integrand of Eq. (2) of the main text can be a bit cumbersome to evaluate due to gauge-fixing in the presence of nonzero weak Chern numbers, we can simplify the analysis by computing the difference $\delta \theta_\phi$ [27, 28]:

$$\delta \theta_\phi = \theta_{\phi = \pi} - \theta_{\phi = 0} \mod 2\pi = \frac{1}{4\pi} \int CS_3[A(\phi = \pi)] - CS_3[A(\phi = 0)]$$

$$= \frac{1}{16\pi} \int d^3k'dk_4 e^{i\mu\lambda\rho \nu} \text{tr}(\Omega_{\mu
u} \Omega_{\lambda\rho}),$$

where $\Omega$ is the non-abelian Berry curvature for the occupied states of the Hamiltonian:

$$H(k', k_4) = H_0(k') + k_4\sigma^z \mu^z,$$

and where the integral in Eq. (20) is taken within the range $k_4 = -|\Delta|$ to $k_4 = |\Delta|$. $H(k', k_4)$ is $I$-symmetric, where we define $I$-symmetry as leaving $k_4$ invariant (since $H_{CDW}$ is $I$-even). The presence of $I$ symmetry causes the contribution to the integral in Eq. (20) to be zero from every $I$-symmetric region that does not enclose a gapless point of $H(k', k_4)$ at half filling. Thus, we can deform the region of integration to an infinitesimal ball surrounding the degeneracy at $(k', k_4) = (0, 0, \pi/2c, 0)$. We next expand the Hamiltonian $H(k', k_4)$ about the degenerate point in $(k', k_4) = (0, 0, \pi/2c, 0) + (q', q_4)$, and obtain the $k \cdot p$ Hamiltonian of a four-dimensional Dirac fermion:

$$H(q', q_4, \Delta) \approx t_x q' \sigma^x \mu^0 + t_y b(q') \sigma^y \mu^0 + t_z c(q') \sigma^z \mu^0 + q_4 \sigma^y \mu^z,$$

For the Dirac Hamiltonian in Eq. (22), the integral $e^{i\mu\lambda\rho \nu} \text{tr}(\Omega_{\mu
u} \Omega_{\lambda\rho}) = 16\pi^2 \delta(q, q_4)$ in Eq. (20) takes the form of a point source [27, 29], allowing the integral to be computed directly:

$$\delta \theta_\phi = \theta_{\phi = \pi} - \theta_{\phi = 0} \mod 2\pi = \pi.$$
Having previously established that $\nu_z = -1$, $\nu_x = \nu_y = 0$ for $H_0 + H_{CDW}$ when $\cos \phi < 1$, we conclude that the $N = 2$ Weyl-CDW with phase $\cos \phi < 1$ is an oQAH insulator with the symmetry-based indicators $\{000\}$ and the topological indices $\theta = \pi$, $\nu_z = -1$, $\nu_x = \nu_y = 0$, in agreement with more general formulation in Sec. II, and consistent with our analysis of the other Weyl-CDW models in this Letter [though we again note that the relative assignment of $\theta$ depends on the choice of origin, which we have here chosen to be $(x, y, z) = (0, 0, 0)$].

### V. WEYL-CDWS WITH NONTRIVIAL CHERN NUMBERS AT $k_z = \pi$

In this section, we consider a slightly distinct Weyl semimetal (WSM) from the model studied in the main text. Here, we again consider a WSM with $\mathcal{I}$ and 3D translation symmetries, but one in which the $k_z = \pi/c$ plane now carries a nontrivial Chern number $C(k_zc = \pi) = 1$, instead of the $k_z = 0$ plane. To realize this configuration of Chern numbers [$C(k_z = 0) = 0$, $C(k_zc = \pi) = 1$], we reverse the signs of $t_x$, $t_y$, and $m$ in $H_0$ in the main text. We again introduce the on-site CDW mass $H_{CDW}$ with a commensurate modulation in the $\hat{z}$-direction by $Q = 2\pi/Nc$, and then again fold bands into the rBZ. For the WSM in this section, each band outside of the rBZ now contributes $+1$ to the weak Chern number, because each plane outside of the rBZ either has the $k$-space Chern number $C(|k_zc| > \pi/N) = 1$ or $C(|k_zc| < \pi/N) = 0$. In this case, independent of the CDW phase $\phi$, we find that the gapped Weyl-CDW phase still unavoidably carries the weak Chern numbers $\nu_x = \nu_y = 0$, $\nu_z = N - 1$. We next determine the parity eigenvalues and symmetry-based indicators [10–18] of the WSM in the presence of a CDW with commensurate modulation $Q = 2\pi/Nc$. For even values of $N$, the resulting band topology is the same as in the previous analysis performed in Sec. II. However, when $N$ is odd, the weak symmetry index $z_{2z} = N - 1 \mod 2 = 0$, implying that $\nu_z$ is even (though we have shown in the previous paragraph that $\nu_z$ cannot be zero). Next, we determine the strong index $z_{4z}$ by repeating the analysis used to construct Table III. Through this analysis, we find that there are $2N + 2$ positive and $2N - 2$ negative valence parity eigenvalues in the $Nk_z'c = 0$ plane, and $2N - 2$ negative and $2N$ positive parity eigenvalues and one fourfold Dirac fermion in the $Nk_z'c = \pi$ plane. Computing the strong index $z_{4z}$, we find that:

$$z_{4z}(\phi = n\pi) = 1 - (-1)^n.$$  \hspace{1cm} (24)

Thus, when $\phi = 0$ the Weyl-CDW, in this section is a $\nu_z = N - 1$ weak Chern insulator with the completely trivial parity indices $\{000\}$, and when $\phi = \pi$, the Weyl-CDW is an oQAH insulator with the parity indices $\{200\}$. Crucially, from the analysis performed in this section and throughout this Letter, we determine that the oQAH phase of the Weyl-CDW carries the topological indices $\nu_x = N - 1$, $\nu_z = \nu_y = 0$, and that the Weyl-CDWs at $\phi = 0, \pi$ differ by a topological axion angle $\delta \theta_0 = \pi$.

To confirm the presence of a nontrivial axion angle at $\theta = \pi$ (in the origin-dependent convention employed throughout this Letter), we perform numerical calculations for the case of commensurate modulation with $N = 3$. In Fig. 1,
we have calculated the $\hat{y}$-directed Berry phase of an $I$-symmetric, $\hat{z}$-directed slab with 15 atomic layers, corresponding to $L_z/3c = 5$ unit cells of the modulated Weyl-CDW phase. When $\phi = 0$, the Berry phase implies a slab Chern number $G_z(\phi = 0) = 10$, which is consistent with a weak Chern number $\nu_z = N - 1 = 2$ through $G_z = L_z\nu_z/3c$. Conversely, when $\phi = \pi$, the Berry phase in Fig. 1 implies a slab Chern number of $G_z(\phi = \pi) = 11$. This is consistent with the observation that $|\delta \tilde{z}_4| = 2$ indicates a difference in the parity of the Chern number of an $I$-symmetric slab [2, 4, 10, 17, 18]. As discussed in the main text, for a fixed, $I$-symmetric slab geometry with more than one layer, $|\delta G_z| = 1$ implies that $\delta \theta = \pi$. From the zone-folding analysis performed earlier in this section and the discussion in the main text, Fig. 1 implies that, given our choice of origin, the Weyl-CDW is an oQAH insulator with $\theta = \pi$, $\nu_z = 2$, $\nu_x = \nu_y = 0$. Crucially, the difference $\delta \theta = \theta - \theta = 0 \mod 2\pi = \pi$ is origin-independent and topological.

1. X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B 74, 085308 (2006).
2. T. L. Hughes, E. Prodan, and B. A. Bernevig, Phys. Rev. B 83, 245132 (2011).
3. L. Fu and C. L. Kane, Phys. Rev. B 76, 045302 (2007).
4. A. M. Turner, Y. Zhang, R. S. K. Mong, and A. Vishwanath, Phys. Rev. B 85, 165120 (2012).
5. X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
6. T. M. McCormick, I. Kimchi, and N. Trivedi, Physical Review B 95, 075133 (2017).
7. C. J. Bradley and A. P. Cracknell, *The Mathematical Theory of Symmetry in Solids* (Clarendon Press, Oxford, 1972).
8. S. M. Young, S. Zaheer, J. C. Y. Teo, C. L. Kane, E. J. Mele, and A. M. Rappe, Phys. Rev. Lett. 108, 140405 (2012).
9. N. Belov and T. Neronova, N.N. and Smirnova, Sov. Phys. Crystallogr. 2, 311 (1957).
10. Y. Xu, L. Elcoro, Z. Song, B. J. Wieder, M. G. Vergniory, N. Regnault, Y. Chen, C. Felser, and B. A. Bernevig, arXiv e-prints, arXiv:2003.00012 (2020), arXiv:2003.00012 [cond-mat.mtrl-sci].
11. L. Elcoro, B. J. Wieder, Z. Song, Y. Xu, N. Regnault, B. Bradlyn, and B. A. Bernevig, To be submitted.
12. H. C. Po, A. Vishwanath, and H. Watanabe, Nat. Comm. 8, 50 (2017).
13. H. Watanabe, H. C. Po, and A. Vishwanath, Science advances 4, eaat8685 (2018).
14. E. Khalaf, Physical Review B 97, 205136 (2018).
15. B. J. Wieder and B. A. Bernevig, arXiv preprint arXiv:1810.02373 (2018).
16. J. Yu, Z.-D. Song, and C.-X. Liu, arXiv e-prints, arXiv:2003.01275 (2020), arXiv:2003.01275 [cond-mat.mes-hall].
17. H. Kim, K. Shiozaki, and S. Murakami, Phys. Rev. B 100, 165202 (2019).
18. R. Takahashi, Y. Tanaka, and S. Murakami, Phys. Rev. Research 2, 013300 (2020).
19. W. Rudin et al., *Principles of mathematical analysis*, Vol. 3 (McGraw-hill New York, 1964).
20. J. Cano, B. Bradlyn, Z. Wang, L. Elcoro, M. G. Vergniory, C. Felser, M. I. Aroyo, and B. A. Bernevig, Phys. Rev. B 97, 035139 (2018).
21. B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).
22. N. Marzari, A. A. Mostofi, J. R. Yates, I. Souza, and D. Vanderbilt, Rev. Mod. Phys. 84, 1419 (2012).
23. Z. Song, T. Zhang, Z. Fang, and C. Fang, Nat. Commun. 9, 3530 (2018).
24. W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
25. N. Varnava and D. Vanderbilt, Phys. Rev. B 98, 245117 (2018).
26. O. Zilberberg, S. Huang, J. Guglielmon, M. Wang, K. P. Chen, Y. E. Kraus, and M. C. Rechtsman, Nature 553, 59 (2018).
27. X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Physical Review B 78, 195424 (2008).
28. A. M. Essin, J. E. Moore, and D. Vanderbilt, Physical review letters 102, 146805 (2009).
29. S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. Ludwig, New Journal of Physics 12, 065010 (2010).