Complexified Path Integrals and the Phases of Quantum field Theory

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Abstract
The path integral by which quantum field theories are defined is a particular solution of a set of functional differential equations arising from the Schwinger action principle. In fact these equations have a multitude of additional solutions which are described by integrals over a complexified path. We discuss properties of the additional solutions which, although generally disregarded, may be physical with known examples including spontaneous symmetry breaking and theta vacua. We show that a consideration of the full set of solutions yields a description of phase transitions in quantum field theories which complements the usual description in terms of the accumulation of Lee-Yang zeroes. In particular we argue that non-analyticity due to the accumulation of Lee-Yang zeros is related to Stokes phenomena and the collapse of the solution set in various limits including but not restricted to, the thermodynamic limit. A precise demonstration of this relation is given in terms of a zero dimensional model. Finally, for zero dimensional polynomial actions, we prove that Borel resummation of perturbative expansions, with several choices of singularity avoiding contours in the complex Borel plane, yield inequivalent solutions of the action principle equations.

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1 Introduction

Many of the ideas presented in this paper have also appeared previously in [1]. The intent of this article is to assemble what we view as the important points in a short and coherent summary, and to add some results concerning a relation between Lee-Yang zeroes and Stokes phenomena.

Quantum field theories may be defined either by a path integral or by a set of functional differential equations which follow from the Schwinger action principle [2],

$$\delta \langle t_1 | t_2 \rangle = \langle t_1 | \delta S | t_2 \rangle .$$  \hfill (1)

For the sake of illustration, consider a zero dimensional “quantum field theory” defined by the action $S(\phi)$. The generating functional for correlation functions of $\phi$ is

$$Z(J) = \int_{-\infty}^{+\infty} d\phi \exp(-S(\phi) + J \phi) ;$$ \hfill (2)

where it is assumed that the integral is convergent. This is a solution of the action principle equations

$$\left( S'(\partial J) + J \right) Z(J) = 0$$ \hfill (3)

$$\left( \partial g_i - \frac{\partial S(\partial J)}{\partial g_i} \right) Z(g, J) = 0 ;$$ \hfill (4)

where $g_i$ are the parameters of the theory. For the specific action

$$S(\phi) = \frac{1}{2} \mu \phi^2 + \frac{g}{4} \phi^4 ;$$ \hfill (5)

equations (3) and (4) become

$$(g \partial^3 J + \mu \partial J + J) Z(J) = 0$$ \hfill (6)

$$(\partial g - \frac{1}{4} \partial^3 J) Z(J) = (\partial &mu; - \frac{1}{2} \partial^2 J) Z(J) = 0 .$$ \hfill (7)

Although all these equations are included in the Schwinger action principle, it will be convenient to refer to equations involving variations of the fields (e.g., (3)) as Schwinger-Dyson equations, and those arising from variation of parameters (e.g., (4)) as action principle equations.

In general the equations (3) and (4) have a several parameter class of solutions. For the action (5), the corresponding equations (6) and (7) have a three parameter class of solutions, which includes (2). The Schwinger-Dyson equations are solved by

$$Z(J) = \sum_I c_I (g, \mu) \int_{\Gamma_I} d\phi \exp(-S(\phi) + J \phi) ;$$ \hfill (8)

where $\Gamma_I$ are inequivalent integration paths in the complex $\phi$ plane over which the integral converges and $c_I$ are arbitrary functions of the coupling constants $g$ and $\mu$. The number of such contours matches the order of the differential equation (3). Figure 1 shows the domains of convergence, $\cos(4 \text{arg}(\phi)) > 0$, and a basis set of contours for real positive $g$. 
Roughly speaking, the action principle requires the coefficients $c_I$ to be independent of the parameters $\mu$ and $g$. This statement is imprecise since, if $g$ is taken to be complex and the argument of $g$ is varied sufficiently, the contours of integration $\Gamma_I$ must be changed to maintain convergence. For a general polynomial action, this statement holds for the “top coupling” associated with the highest power of $\phi$ in the action. For a given top coupling $g$, each contour $\Gamma_I$ belongs to an equivalence class of contours for which integration gives the same result. The equivalence classes (see figure 2) consist of contours for which $|\phi| \to \infty$ within the same domains of convergence; the action is analytic in $\phi$ except at infinity. There is always a choice of contour $\Gamma_I$ within an equivalence class which can be held fixed while making infinitesimal variations of the top coupling. The action principle requires that the coefficients associated with these contours do not vary as one makes infinitesimal changes in the couplings.

Figure 2. For $\text{arg}(g_4) = 0$, (a) the contours $B$ and $B'$ are equivalent. However for $\text{arg}(g_4) = 2\pi/3$, (b) the integral over $B'$ remains convergent while the integral over $B$ diverges.

To dispel doubt that the “exotic” solutions with complex integration contours have physical relevance, consider the effective action for the theory defined by (5), with real $\mu \leq 0$ and real $g > 0$. The expectation value for $\phi$ is a solution of the equation

$$J = \frac{d\Gamma}{d\phi};$$

(9)
where $\Gamma$ is the one particle irreducible effective potential. The tree level effective potential (in this case the action) has minima at $\phi = 0, \pm \sqrt{\mu/g}$. The minimum $\langle \phi \rangle = \sqrt{\mu/g} + \cdots$ corresponds to a solution of the Schwinger-Dyson equations with an integral representation involving a sum of the contours $A$ and $C$ drawn in figure 1. These integrals have contributions only from the saddle point at $\phi = \sqrt{-\mu/g}$. Hence symmetry breaking vacua, whose existence is normally attributed to a thermodynamic limit, exist even in zero dimensions when the full set of solutions of the Schwinger-Dyson and action principle equations are considered. Note that no symmetry breaking term has been added to the action; the symmetry is broken by the choice of integration path.

Note that the solutions associated with the contours $A$ and $C$ in figure 1 have complex parts which do not appear at any order in a perturbative expansion about the saddle point. However, due to the linearity of the Schwinger-Dyson equations, one can sum the contours to get a solution for which the non-perturbative contribution is real. The reader might be concerned that the “exotic” solutions which are not integrals over real $\phi$ do not satisfy the axioms of Euclidean quantum field theory. Although one can easily arrange for the Greens functions to be real, one might still worry that even Greens functions are not manifestly positive. In fact one should postpone these questions for the higher dimensional case as perversities of the exotic solutions could vanish in the thermodynamic/continuum limit. In fact, we expect this to be the case, as symmetry breaking vacua and theta vacua are examples of exotic solutions.

In the subsequent section, we argue that the appearance of phase boundaries in quantum field theories is intimately related to a collapse of the solution set in the thermodynamic limit. This proposal can be made concrete in a zero dimensional analogue, for which we demonstrate a correspondence between the collapse of the solution set as a top coupling is set to zero and the accumulation of Lee-Yang zeros, leading to a non-analyticity in a coupling constant. In this context, the limit of a vanishing top coupling is analogous to the thermodynamic limit. The complementary descriptions of phase boundaries in terms of the accumulation of Lee-Yang zeroes and the collapse of the solution set share a common origin in Stokes phenomena.

In section 3, we prove the equivalence of Borel resummation of the perturbative expansion about saddle points and exotic solutions of the Schwinger-Dyson equations, for various singularity avoiding contours in the Borel plane. Finally, in section 4, we argue that the action principle may emerge from the Schwinger-Dyson equations under suitable conditions, due to the collapse of the solution set in the thermodynamic limit, rather than being an independent set of equations.

2 Collapse of the Solution Set and the Accumulation of Lee-Yang Zeros

When complex values of the parameters of a quantum field theory are considered, phase boundaries which appear in the thermodynamic limit can be understood in terms of the accumulation of zeroes of the partition function which pinch the real axis, known as Lee-Yang zeroes [3]. As we shall explicitly demonstrate below, the accumulation of Lee-Yang zeroes is not confined to the thermodynamic limit. In zero dimensional polynomial theories, Lee-Yang zeroes also accumulate in the limit that the top coupling $g_T$ goes to zero. In this limit, a non-analyticity in the coupling $g_T-1$ appears. In this context, we will propose an alternative (and complementary) description of the appearance of phase boundaries in terms of the collapse of the solution set of the Schwinger-Dyson

\footnote{The standard way to non-perturbatively describe symmetry breaking vacua is to introduce a small symmetry breaking term which is taken to zero only after taking an infinite volume limit.}
and action principle equations.

Consider the zero dimensional action $S = \sum_{l=1}^{T} g_l \phi^l$. There is a branch point at $g_T = 0$, due to the necessity of rotating the contours of integration in the integral representation, in order to maintain convergence as the argument of $g_T$ is varied. It is always possible to find a contour in an equivalence class which can be held fixed under infinitesimal variations of the top coupling (see figure 2), so as to satisfy the action principle. However large variations in the argument of the top coupling require changes in the contour. In particular, under a rotation by $2\pi T$ the solutions transists among $T$ Riemann sheets. The solutions are analytic in the couplings $g_l$ for $l < T$, except at infinity, since these couplings do not effect the domains of convergence. In the limit $g_T \to 0$, the solution develops a branch point in the new top coupling, $g_{T-1}$. The limit $g_T \to 0$ is analogous to a thermodynamic limit. For the case $T = 3$ we will explicitly show that this limit is accompanied by the accumulation of Lee-Yang zeroes in the complex $g_{T-1}$ plane.

The appearance of this branch point can be understood in terms of the collapse of the solution set of the Schwinger-Dyson and action principle equations. In the limit $g_T \to 0$, with fixed $\arg(g_T)$ solutions of the Schwinger-Dyson and action principle equations either diverge, vanish or have a finite limit. It is easy to see which by considering the overlap of the domains of convergence in the complex $\phi$ plane for the case in which $g_T$ is the top coupling or for which $g_{T-1}$ is the top coupling ($g_T = 0$). If a convergent contour for $g_T \neq 0$ is equivalent to one which lies within a single wedge of convergence for the case $g_T = 0$, then the integral will vanish in the $g_T \to 0$ limit with fixed $\arg(g_T)$. On the other hand if a convergent contour for $g_T \neq 0$ is not equivalent to any contour lying within the wedges of convergence for $g_T = 0$, then the $g_T \to 0$ limit is divergent. A finite $g_T \to 0$ limit exists only if an equivalent contour lies within two different wedges of convergence for the $g_T = 0$ case. Figure 3 illustrates the various possibilities for the case $T = 4$.

![Diagram](image)

Figure 3. For $\arg(g_4) = 0$ and $\arg(g_3) = 3\pi/4$, the integral over the contour $\Gamma_A$ or the equivalent contour $\Gamma'_A$ vanishes in the $g_4 \to 0$ limit, while the integral over $\Gamma_B$ is finite. For $\arg(g_4) = 0$ and $\arg(g_3) = \pi$ the integral over $\Gamma_C$ diverges.

Suppose that the argument of $g_T$ is kept fixed as $g_T \to 0$. In this limit the behavior of the partition function, defined by a particular contour of integration in the complex $\phi$ plane, will change discontinuously as the argument of $g_{T-1}$ is varied across certain critical values. For example the $g_T \to 0$ limit may go from finite to divergent upon crossing a Stokes line. However it is possible to keep this limit finite by considering variations in the contour of integration which violate the action principle by terms which vanish in the $g_T \to 0$ limit. For $g_T = 0$, these variations are equivalent to analytic continuation in $g_{T-1}$. 

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To illustrate how this works, consider the case $T = 4$. As one varies the argument of $g_3$ keeping that of $g_4$ fixed, a generic solution of the Schwinger-Dyson and action principle equations will become alternately divergent, finite, or vanishing in the $g_4 \rightarrow 0$ limit. One can keep the limit finite by adding contours such as $\Gamma_A$ in figure 3 when the argument of $g_3$ enters a wedge in which the contribution from this contour vanishes as $g_4 \rightarrow 0$.

This process is illustrated in figure 4. The domains of convergence for $g_4 \neq 0$ are indicated in light gray, while the domains of convergence for $g_4 = 0$, $g_3 \neq 0$ are indicated in dark gray. Starting with a solution of the Schwinger-Dyson and action principle equations corresponding the contour $A$, $\arg(g_3)$ is varied from $\pi$ to $\pi/2$ keeping $\arg(g_4) = 0$. Initially the solution is finite as $g_4 \rightarrow 0$, since the contour $A$ lies asymptotically within two domains of convergence for $g_4 = 0$. At $\arg(g_3) = 3\pi/4$ the contour is modified to $C \equiv A + B$. Note that the integration over $B$ vanishes in the $g_4 \rightarrow 0$ limit for a neighborhood of $\arg(g_3) = 3\pi/4$, since $B$ is equivalent to a contour lying within a single domain of convergence when $g_4 = 0$. One can continue varying $\arg(g_3)$ to $\pi/2$ without a change in the asymptotic behavior as $g_4 \rightarrow 0$; the partition function remains finite in this limit. On the other hand, had the contour been fixed as $A$, the integral would be divergent as $g_4 \rightarrow 0$ for $\arg(g_3)$ on the other side of the Stokes line $\arg(g_3) = 5\pi/8$. For $\arg(g_3) = \pi/2$ and $g_4 = 0$, the contour $A$ does not lie within the domains of convergence asymptotically, unlike the the contour $C$.

![Figure 4](image_url)

Figure 4. As the argument of $g_3$ is varied, the contour is changed to keep the integral finite in the $g_4 \rightarrow 0$ limit.

Considering larger variations of $\arg(g_3)$ one can repeat this process to give a solution of the Schwinger-Dyson equations which is finite and satisfies the action principle in the $g_4 \rightarrow 0$ limit. For $g_4 = 0$ this process corresponds to analytic continuation in $g_3$. The changes in contour necessary to maintain a finite $g_4 \rightarrow 0$ limit (and avoid Stokes phenomena) give rise to the third order branch point at $g_3 = 0$.

We thus arrive at the picture that non-analyticity in a coupling constant arises due to the collapse of the solution set of the Schwinger-Dyson and action principle equations. In the example we have given, non-analyticity in $g_{T-1}$ arises as the top coupling $g_T \rightarrow 0$. In this limit, some solutions are finite, while others diverge or vanish. Because the equations are linear and certain solutions vanish in this limit, various classes of solutions with inequivalent integral representations also coalesce as $g_T \rightarrow 0$. This permits larger variations of the contour which satisfy the action principle. At the same time larger variations of the contour may become necessary to obtain a finite partition function as coupling constants are varied. While we have explicitly demonstrated this mechanism for the $g_T \rightarrow 0$ limit of a zero dimensional polynomial theory, we propose that the same phenomena hold true for the thermodynamic limit, $N \rightarrow \infty$ where $N$ is the number of degrees of freedom, in a multi-dimensional field theory. In other words, non-analyticity in a parameter of a quantum field theory should be related to the collapse of the solution set of the Schwinger-Dyson and action principle equations in the $N \rightarrow \infty$ limit.
The analogy we have proposed between a non-analyticity arising in the $g_T \to 0$ limit of a zero dimensional theory and non-analyticities arising in the thermodynamic limit is strengthened by noting that the $g_T \to 0$ limit is accompanied by the accumulation of Lee-Yang zeroes along Stokes lines in the complex $g_{T-1}$ plane. One can explicitly see how this occurs for $T = 3$. Consider the partition function

$$Z = \int_C d\phi e^{-\left(\frac{g}{2} \phi^3 + \frac{\mu}{2} \phi^2\right)};$$

(10)

for positive real $g$ and a contour $C$ which goes to infinity with $\arg(\phi) = \pm 2\pi/3$, as in figure 5. The integral (10) can be evaluated in terms of an Airy function;

$$Z = 2\pi i e^{-\frac{1}{12} \mu^3 g^{-1/3} Ai\left(\frac{\mu^2}{4 g^{4/3}}\right)}.$$

(11)

Figure 5.

The zeroes of $Ai(z)$ lie along the negative real $z$ axis. Since the argument of the Airy function in (11) is $z = \mu^2/4 g^{4/3}$, zeroes of the partition function accumulate on the imaginary axis in the complex $\mu$ plane as $g \to 0$, pinching the real axis at $\mu = 0$. In fact, $\arg(\mu) = \pm \pi/2$ are Stokes lines. The $g \to 0$ behavior of $Z$ in the neighborhood of $\arg(\mu) = -\pi/2$ can be seen by inspecting figure 6, in which the domains of convergence for $g \neq 0$ and $g = 0$ are indicated by the lighter and darker shaded regions respectively. For $\arg(\mu) = -\pi/2 + \epsilon$, $Z$ diverges in the $g \to 0$ limit, with the leading term in a saddle point expansion given by

$$Z \sim \sqrt{\frac{2\pi}{\mu}} \exp\left(-\frac{1}{6} \frac{\mu^3}{g^2}\right).$$

(12)

On the other side of the Stokes line, $\arg(\mu) = -\pi/2 - \epsilon$, $Z$ converges in the $g \to 0$ limit, with the leading term in a saddle point expansion given by

$$Z \sim \sqrt{\frac{2\pi}{\mu}} \exp\left(-\frac{1}{6} \frac{\mu^3}{g^2}\right).$$

(13)
Figure 6. On either side of the Stokes line at \( \arg(\mu) = -\pi/2 \), the integral is either finite (c) or divergent (a) in the \( g \to 0 \) limit. Zeroes of the partition function accumulate as \( g \to 0 \) for values of \( \mu \) on the Stokes line (b).

On the Stokes line, \( \arg(\mu) = -\pi/2 \), the two saddle point expansions (12) and (13) become comparable in magnitude as \( g \to 0 \), since the real part of the exponential in (12) vanishes. The integral over the contour in figure 5 is equivalent to the sum of the integrals over two constant phase (steepest descent) contours, \( C_1 \) and \( C_2 \) in figure 7, which pass through classical solutions for which the real part of the action is degenerate. The accumulation of zeroes on the Stokes line is related to the fact that the relative phase of the two integrals oscillates wildly with variations of \( \mu \) in the \( g \to 0 \) limit, due to the factor of \( \exp(\mu^3/g^2) = \exp(i|\mu|^3/g^2) \).

Figure 7. Steepest descent contours passing through the classical solutions at \( \phi = 0, -\mu/g \), on the Stokes line \( \arg(\mu) = -\pi/2, \arg(g) = 0 \).

Thus, the accumulation of Lee-Yang zeroes and the collapse of the solution set of the Schwinger-Dyson and action principle equations are complementary descriptions of the appearance of non-analyticity in a parameter of a zero dimensional theory as the top coupling \( g_T \to 0 \). The two descriptions have a common origin in Stokes phenomena. We conjecture that these are also complementary descriptions of non-analyticity in parameters of quantum field theories which arise in the thermodynamic limit \( N \to \infty \). Upon completion of this work, we discovered [4], in which Lee-Yang zeroes in a 1D model with a wetting transition were shown to lie along Stokes lines associated with the asymptotic expansion in \( N \). The correspondence between Stokes lines and Lee-Yang zeroes has also been suggested in [5], in the context of Ising and gauge models.

3 Borel Resummation and Complexified Path Integrals

It is well known that the loop expansion in quantum field theory is an asymptotic rather than a convergent series. One approach to obtain non-perturbative information from the loop expansion is the Borel resummation, in which a convergent series (the Borel transform) in a variable \( t \) is obtained from the asymptotic series in \( \hbar \). The Borel transform is then inverted to give a function having the correct asymptotic expansion, but which contains non-perturbative information. Since there are actually an infinite number of such functions, differing by essential singularities at \( \hbar \to 0 \), it is a non-trivial statement that the Borel transformation corresponds to the path integral. Moreover, there are frequently singularities of the Borel transform on the positive real \( t \) axis, due to instantons and renormalons, which prevents an unambiguous Borel resummation.
Under generic conditions, the number of classical solutions is equal to the number of independent solutions of the Schwinger-Dyson equations. For an arbitrary polynomial action in a zero dimensions,

\[ S(\phi) = \sum_{n=1}^{T} \frac{g_n}{n} \phi^n ; \]  

we will prove that various Borel resummations of perturbative expansions about classical solutions satisfy both the Schwinger-Dyson equations and the action principle equations, and therefore correspond to various complexified path integrals.

Consider the partition function

\[ Z = \int_{\Gamma} d\phi e^{\frac{1}{\hbar} S(\phi)} ; \]  

where the path \( \Gamma \) is equivalent to a steepest descent path passing through a dominant saddle point (classical solution) \( \phi = \bar{\phi}_{\alpha} \). The loop expansion yields a contribution to the generating function of the form;

\[ Z_{\alpha} \approx \sqrt{\frac{\pi \hbar}{S''(\bar{\phi}_{\alpha})}} \sum_{n=0}^{\infty} c_n \hbar^n . \]  

This series is asymptotic, but its Borel transform defined by

\[ B_{\alpha}(t) = \sqrt{\frac{\pi}{S''(\bar{\phi}_{\alpha})}} \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n + \frac{1}{2})} t^n ; \]  

converges to

\[ B_{\alpha}(t) = \frac{\sqrt{t}}{2\pi i} \oint_{C} d\phi \frac{1}{t - (S(\phi) - S(\bar{\phi}_{\alpha}))} ; \]  

where in the vicinity of \( t = 0 \) the contour \( C \) encloses, in the opposite sense (see figure 8), the two poles \( \phi_{\alpha,j}(t) \), for \( j = 1, 2 \), which coalesce to \( \bar{\phi}_{\alpha} \) as \( t \to 0 \). All the other poles are taken to lie outside the contour. The Borel transform has a singularity when one of the exterior poles coalesces with one of the interior poles, which occurs when \( t = S(\bar{\phi}_{\alpha'}) - S(\bar{\phi}_{\alpha}) \) where \( \bar{\phi}_{\alpha'} \) is a neighboring classical solution. Doing the \( \phi \) integral gives

\[ B_{\alpha}(t) = \sqrt{t} \sum_{j=1,2} (-1)^j \frac{1}{S'(\phi_{\alpha,j}(t))} . \]  

\[ \text{Figure 8.} \]
Thus far everything we have said is well known [6]. We now exhibit an exact relation between the Borel resummation and the exotic solutions of the Schwinger-Dyson and action principle equations. We invert the Borel transform by writing

\[ Z_\alpha = e^{-\frac{1}{\hbar}S(\bar{\phi}_\alpha)} \int_\Sigma dt e^{-\frac{1}{\sqrt{t}} B_\alpha(t)} \]  

(20)

with an as yet unspecified integration contour \( \Sigma \) in the complex \( t \) plane. Equivalence of (20) with a solution of the Schwinger-Dyson equations requires

\[ e^{-\frac{1}{\hbar}S(\bar{\phi}_\alpha)} \int_\Sigma dt e^{-\frac{1}{\sqrt{t}} \frac{1}{\hbar} B_\alpha(t)} \]

(21)

where \( \Gamma_I \) indicate open paths in the complex \( \phi \) plane which asymptotically lie within the domains of convergence determined by the the top coupling \( g_T \). Since \( C \) and \( c_I \Gamma_I \) are inequivalent, (21) involves a non-trivial exchange in the order of integration under which contours are not preserved. Instead of directly proving (21), we will show that (20) solves the Schwinger-Dyson equations and satisfies the action principle.

We set \( \hbar = 1 \) in what follows. If \( Z_\alpha \) satisfies both the Schwinger-Dyson equations and the action principle then it is annihilated by the operators;

\[ \hat{L} = \sum_{n=2}^{T} (n-1)g_n \partial_{g_{n-1}} - g_1 ; \]

(22)

and

\[ \hat{H}_n = \partial_{g_n} - \frac{1}{n} \partial^n_{g_1} . \]

(23)

It is convenient to define the quantity

\[ F_\alpha \equiv \int_\Sigma dt e^{-t} B_\alpha(t) \int_\Sigma \frac{1}{\sqrt{t}} \Big( \frac{1}{\hbar} B_\alpha(t) \Big) . \]

(24)

To show that \( \hat{L}Z_\alpha = 0 \), for \( Z_\alpha \) defined in (20), it suffices to show that \( \hat{L}F_\alpha = 0 \) where

\[ \hat{L} \equiv \sum_{n=2}^{T} (n-1)g_n D_{g_{n-1}} - g_1 ; \quad D_{g_n} \equiv \partial_{g_n} - \partial_{g_n} S(\bar{\phi}_\alpha) . \]

(25)

Before proceeding, we list several simple but useful identities. Due to the equation of motion, \( S'(\bar{\phi}_\alpha) = 0 \), one has

\[ \frac{\partial}{\partial g_{n}} S(\bar{\phi}_\alpha) = \frac{1}{n} \bar{\phi}_n \]

(26)

Identities obtained by differentiating the relation \( t - S(\phi_{\alpha,j}(t)) + S(\bar{\phi}_\alpha) = 0 \), which defines \( \phi_{\alpha,j}(t) \), are

\[ \frac{\partial}{\partial t} \phi_{\alpha,j} = \frac{1}{S'(\phi_{\alpha,j})} \]

(27)

\[ \frac{\partial}{\partial g_{n}} \phi_{\alpha,j} = \frac{1}{n} \left( \phi_{\alpha,j} - \bar{\phi}_n \right) S'(\phi_{\alpha,j}) \]

(28)
where the equations of motion have been used again in deriving the last equation.

Using these identities and the equations of motion, one can show that

$$
\sum_{n=2}^{T} [(n - 1)g_n D_{g_{n-1}} - g_1] \frac{1}{S'(\phi_{\alpha,j})} = 0;
$$

implying $\hat{L}_F = 0$, or $\hat{L}_Z = 0$, for any integration path in $t$.

We are not done yet, since the equation $\hat{L}_Z = 0$ is a combination of the Schwinger-Dyson and the action principle equations. The integration path $\Sigma$ will be constrained further by requiring the action principle to be separately satisfied. To this end, consider the quantity

$$
\mathcal{A} \equiv [klD_{g_l}D_{g_k} + (k + l)D_{g_{k+l}}]F
$$

which will vanish if the action principle is also satisfied. Using the same identities discussed above, $\mathcal{A}$ may be rewritten as

$$
\mathcal{A} = \int_{\Sigma} dt e^{-t} \frac{\partial}{\partial t} \left[ e^{-t} \sum_{j} (-1)^j[klD_{g_l}D_{g_k} + (k + l)D_{g_{k+l}}]\phi_{\alpha,j}(t) \right] = \sum_{j} (-1)^j e^{-t} kl \frac{\partial}{\partial g_k} \frac{\partial}{\partial g_l} \phi_{\alpha,j}(t) \bigg|_{\partial \Gamma}.
$$

Clearly $\mathcal{A}$ vanishes if the contour $\Sigma$ begins at $t = 0$ and ends at $Re(t) = +\infty$, avoiding singularities. The contribution from the boundary at $t = 0$ vanishes because $\phi_{\alpha,1}(t)$ and $\phi_{\alpha,2}(t)$ coalesce at $t = 0$, so that the factor of $\sum_{j} (-1)^j$ in (31) leads to a cancellation. Closed contours encircling branch cuts in the complex $t$ plane also give $\mathcal{A} = 0$. So long as a contour $\Sigma$ for which $\mathcal{A} = 0$ is used, the Borel resummation gives a solution of the Schwinger-Dyson and action principle equations.

It would be very interesting to see to what extent this analysis extends to theories with non-zero dimension. The analysis is surely more difficult since, among other possible complications, there are singularities in the Borel plane due to renormalons as well as finite action classical solutions (instantons).

The “exotic” solutions of the Schwinger-Dyson equations given by $Z = c_I Z_I$, where the $Z_I$ are generated from classical solutions by Borel resummation, may in some sense be thought of as a generalized form of theta vacua, in which the $c_I$ play the role of theta parameters. In the usual approach a particular theta vacuum is selected by adjusting a surface term in the action and integrating over real fields. The surface term term effects the Schwinger-Dyson equations only at the space-time boundary, so in the infinite volume limit its role is only to set a boundary condition. It does this by putting a different weight on the contributions to $Z$ coming from perturbative expansions about different classical solutions. We conjecture that when appropriately resummed, this is equivalent to a weighted sums over complexified path integrals.

4 Emergence of Action Principle in the Thermodynamic Limit

So far we have treated the Schwinger-Dyson and action principle equations as independent. This is certainly true for a finite number of degrees of freedom. If the action principle is not imposed,
the manner in which a solution of the Schwinger-Dyson equations changes as one varies a coupling is un-determined, as the there is a continuous set of solutions to the Schwinger-Dyson equations for any value of the coupling. However, with certain assumptions, the action principle arises due to the collapse of the solution set of the Schwinger-Dyson equations in the thermodynamic limit.

Suppose that the solution set collapses in the thermodynamic limit, i.e. some solutions coalesce while others diverge, such that are solutions of the form

$$Z \sim \exp(-NF)$$

as \(N \to \infty\) where there are only discrete possibilities for the free energy \(F\). Discreteness of the solutions automatically fixes the dependence on the couplings. Let us assume that this dependence is described by a differential equation of the form

$$\hat{O}_\alpha Z \equiv (\frac{\partial}{\partial g_\alpha} - \hat{K}_\alpha) Z[g_\alpha, J_i] = 0 ;$$

where \(\hat{K}\) is a linear operator\(^3\). We will show below that these equations are equivalent to the action principle equations. The argument follows from the commutation relations of operators associated with the action principle and the Schwinger-Dyson equations.

Writing the action in the form

$$S\{\phi_i\} = g_\alpha f_\alpha \{\phi_i\},$$

the operators associated with the Schwinger-Dyson equations are

$$\hat{L}_i \equiv g_\alpha \frac{\partial f_\alpha}{\partial \phi_i} \mid_{\{\phi_j\} \to \{\partial J_j\}} - J_i$$

while those associated with the action principle are

$$\hat{H}_\alpha \equiv \partial_{g_\alpha} - f_\alpha \mid_{\{\phi_j\} \to \{\partial J_j\}} .$$

Thus one obtains the commutation relations

$$[\hat{L}_i, \hat{H}_\alpha] = 0 , \quad [\hat{H}_\alpha, \hat{H}_\beta] = 0 .$$

Let us now write \(\hat{O}_\alpha = \hat{H}_\alpha + \Delta_\alpha\). If \(\hat{L}_i Z = 0\) and \(\hat{O}_\alpha Z = 0\), then \([\hat{O}_\alpha, \hat{L}_i] Z = 0\), or

$$\hat{L}_i \hat{\Delta}_\alpha Z = 0 .$$

If \(\hat{\Delta}_\alpha Z\) is non-zero, it is a solution of the Schwinger Dyson equations. Moreover there is only one discrete possibility: \(\Delta_\alpha Z = c_\alpha \{g_\beta\} Z\), where \(c_\alpha\) is some function of the couplings, so that \(\hat{H}_\alpha Z = -c_\alpha \{g_\beta\} Z\). The coefficients \(c_\alpha\) can be absorbed by a coupling constant dependent re-scaling \(Z \to Z' = e^{\Omega(g_\beta)} Z\), where

$$\hat{H}_\alpha Z' = [\hat{H}_\alpha, \Omega] Z - e^\Omega c_\alpha Z = ((\partial_{g_\alpha} - c_\alpha) e^\Omega) Z = 0 .$$

The existence of a solution to the equations \((\partial_{g_\alpha} - c_\alpha) e^\Omega = 0\) requires \(\partial_{g_\alpha} c_\beta - \partial_{g_\beta} c_\alpha = 0\), which follows from \((\partial_{g_\alpha} c_\beta - \partial_{g_\beta} c_\alpha) Z = -[\hat{H}_\alpha, \hat{H}_\beta] Z = 0\). The re-scaled partition function satisfies both the Schwinger-Dyson and Schwinger action principle equations, \(\hat{L}_i Z' = \hat{H}_\alpha Z' = 0\), even though we started with just the Schwinger-Dyson equations.

\(^3\)The operator \(\hat{K}_\alpha\) is necessarily linear for this equation to make sense in the thermodynamic limit.
5 Non-Polynomial Actions

Although we have focused on polynomial actions, our discussion of the solution set of the Schwinger action principle equations can be readily extended to non-polynomial actions. A simple non-polynomial action is that one plaquette QED, with action \[ S = \beta \cos \theta. \] The generating functional

\[
Z(J, \tilde{J}) = \int_{-\pi}^{\pi} d\theta e^{\beta \cos \theta + J e^{i\theta} + \tilde{J} e^{-i\theta}}; \quad (38)
\]

is a solution of the differential equations

\[
\left[ \frac{\beta}{2} (\partial_J - \partial_{\tilde{J}}) + (J \partial_J - \tilde{J} \partial_{\tilde{J}}) \right] Z(J, \tilde{J}) = 0 \quad (39)
\]
\[
\left[ \partial_{\tilde{J}} - \frac{1}{2} (\partial_J + \partial_{\tilde{J}}) \right] Z(J, \tilde{J}) = 0 \quad (40)
\]
\[
\partial_J \partial_{\tilde{J}} Z(J, \tilde{J}) = Z(J, \tilde{J}); \quad (41)
\]

where (39) and (40) are the Schwinger-Dyson and action principle equations, while (41) is a constraint equation. In fact these equations have a two parameter (one if you neglect the normalization) class of solutions given by linear combinations of basis solutions

\[
Z(J, \tilde{J}) = \int_{\Sigma} d\theta e^{\beta \cos \theta + J e^{i\theta} + \tilde{J} e^{-i\theta}}; \quad (42)
\]

for contours \( \Sigma \) equivalent to either \( \Sigma_1 \) and \( \Sigma_2 \) in figure 9 (assuming real positive \( \beta \)).

\[ \begin{array}{c}
\begin{array}{c}
\Sigma_1 \\
-\pi \\
\end{array}
\end{array} \quad | \theta \\
\begin{array}{c}
\Sigma_2 \\
-\pi \\
\end{array} \quad | \theta \]

Figure 9.

Note that integration over \( \Sigma_1 - \Sigma_2 \) is equivalent to the integral over the usual compact path \( \theta = [-\pi, \pi] \). The possible physical relevance of the exotic solutions, upon generalizing to a theory in a finite number of dimensions, is not manifest as it was for polynomial theories. In the former case, symmetry breaking vacua were clearly related to exotic solutions. Our experience with the polynomial theories leads us to speculate that the exotic solutions for lattice gauge theories are related to physically realizable phases of gauge theory. Certainly we expect that the appearance of phase boundaries in gauge theories, via the accumulation of Lee-Yang zeros, is closely related to the collapse of the solution set in the thermodynamic limit.
6 Conclusions and Outlook

We have examined the complete set of solutions of the differential equations which follow from
the Schwinger action principle. While only one of these solutions corresponds to the usual path
integral, the other solutions, which involve complexified path integrals have physical relevance.
On the one hand the manner in which the full solution set collapses in the thermodynamic (or
analogous $g_T \to 0$) limit is related, via Stokes phenomena, to the accumulation of Lee-Yang
zeroes at phase boundaries. On the other hand, exotic solutions may themselves be physical, with
theta vacua and symmetry breaking vacua as known examples. In the zero dimensional case, we
have proven that Borel resummations of perturbation series, having various singularity avoiding
contours of integration in the complex Borel variable, solve the action principle equations and
therefore correspond to various complexified “path” integrals.

While we have explicitly discussed the solution set of the Schwinger Dyson and action prin-
iple equations for zero dimensional models, one can readily generalize to lattice models in multi-
dimensions, in which case one finds a huge solution set. The basis set of solutions to the Schwinger-
Dyson equations for a scalar field theory on a lattice can be written in terms of the zero dimensional
solutions $Z^{(0)}(J)$ as follows;

$$Z = \exp(K_{ij} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j}) \prod_{k} Z_{k}^{(0)}(J_k)$$

where $K_{ij}$ is the lattice kinetic term, and the zero dimensional solution $Z_{k}^{(0)}$ may be different
at each lattice site $k$. For a polynomial potential, the number of independent solutions grows
exponentially with the number of lattice sites. Like internal symmetries, space-time symmetries
may be broken by the choice of integration paths. Determining the collapse of the solution set in
the thermodynamic and continuum limits is a difficult problem. It would be very interesting if
exotic solutions with different integration paths at different sites have physical relevance.

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