On R-duals of Type III in Hilbert Spaces

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ABSTRACT. Following work by Casazza, Kutyniok, and Lammers and its development by Stoeva and Christensen, we provide some novel characterizations of R-dual sequences of type III in Hilbert spaces. We systematically extend the construction procedure by basing it on a choice of an antiunitary involution. For certain classes of R-duals of type III, we derive a representation of the associated frame operator in terms of spectral measures.

KEY WORDS: frames, Riesz sequence, Riesz basis, spectral representation, R-dual of type I, R-dual of type III.

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1. Introduction

In this paper we consider frame/Riesz sequence properties for a sequence \( F = \{f_i\}_{i \in I} \) in a Hilbert space \( \mathcal{H} \) and the corresponding sequence depending on two orthonormal bases, the so-called Riesz dual, or R-dual, sequences \( \Omega = \{\omega_j\}_{j \in I} \) generated by a combined action of certain operators on one of these orthonormal bases. Sequences of this form were introduced in [1] by Casazza, Kutyniok, and Lammers with the purpose of deriving duality principles for frames in an arbitrary separable Hilbert space \( \mathcal{H} \). For each sequence in \( \mathcal{H} \), they constructed a corresponding sequence depending on the choice of two orthonormal bases with a kind of duality relation between them, and then used this construction to derive duality principles for frames. They called the constructed sequence R-dual sequence. R-dual sequences have since been considered by several authors; see [5]–[7]. One of the extensions most important for the present paper appears in the work of Stoeva and Christensen, which introduces various classes of R-duals, for example, aimed at obtaining general versions of the duality principle in Gabor frames [8]. The authors of [7] gave an equivalent condition of two sequences to be R-duals.

In this paper we present variations of these constructions, which are based on choices of certain isometric operators on \( \ell^2(I) \) that help implement the duality principles. Each choice of operator gives rise to a construction of R-dual sequences that have the same desirable properties as the original construction in [1], relating frame/Riesz sequence properties of the sequence \( F = \{f_i\}_{i \in I} \) to properties of its R-dual \( \Omega = \{\omega_j\}_{j \in I} \), as studied by Casazza in [1] and also by Christensen and Kim in [5] and by Stoeva and Christensen in [8]. Finally, given an R-dual of a frame of type III, we derive a representation for its frame operator (see Theorem 4.4) via the spectral theorem. The main results appear in Section 3 and Section 4. Section 2 contains some basic definitions and results.

2. Preliminaries

In what follows, we will review basic definitions of frame and Riesz basis and present types I and III of R-duals; for more details, we refer the interested reader to [4], [3], and [1]. Throughout this paper, \( \mathcal{H} \) is a separable Hilbert space, \( I_{\mathcal{H}} \) is the identity operator on \( \mathcal{H} \), and \( I \) is a countable index set.

A collection of vectors \( F = \{f_i\}_{i \in I} \) in \( \mathcal{H} \) is a Bessel sequence if there exists a constant \( B > 0 \) such that
\[
\sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2
\]
for all \( f \in \mathcal{H} \). If, in addition, there is a constant \( A > 0 \) such that
\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2
\]
for all \( f \in \mathcal{H} \), then \( F = \{ f_i \}_{i \in I} \) is a frame for \( \mathcal{H} \). The constants \( A \) and \( B \) are called frame bounds. A frame \( F = \{ f_i \}_{i \in I} \) is \( A \)-tight if \( A = B \). The synthesis operator of \( F = \{ f_i \}_{i \in I} \) is defined by
\[
T_F : \ell^2(I) \rightarrow \mathcal{H}, \quad T_F \{ t_i \}_{i \in I} = \sum_{i \in I} c_i f_i.
\]
Given a frame \( F = \{ f_i \}_{i \in I} \) in \( \mathcal{H} \), its frame operator is
\[
S_F := T_F T_F^* : \mathcal{H} \rightarrow \mathcal{H}, \quad S_F f = \sum_{i \in I} \langle f, f_i \rangle f_i,
\]
where \( T_F^* \) is the adjoint of \( T_F \), which is given by \( T_F^* f = \{ \langle f, f_i \rangle \}_{i \in I} \). In this case, the operator \( S_F \) is bounded, invertible, self-adjoint, and positive. Moreover, the sequence \( \tilde{F} = \{ S_F^{-1} f_i \}_{i \in I} \) is also a frame for \( \mathcal{H} \) and satisfies the reconstruction formula \( f = \sum_{i \in I} \langle f, f_i \rangle S_F^{-1} f_i \) for every \( f \in \mathcal{H} \). The sequence \( \tilde{F} = \{ S_F^{-1} f_i \}_{i \in I} \) is called the canonical dual frame of \( F = \{ f_i \}_{i \in I} \). Also, any sequence \( G = \{ g_i \}_{i \in I} \) in \( \mathcal{H} \) which is not the canonical dual and satisfies \( f = \sum_{i \in I} \langle f, g_i \rangle f_i \) is called an alternate dual frame of \( F = \{ f_i \}_{i \in I} \).

A collection of vectors \( \Omega = \{ \omega_j \}_{j \in I} \) in \( \mathcal{H} \) is a Riesz sequence if there exist constants \( C, D > 0 \) such that
\[
C \sum_{j \in I} |c_j|^2 \leq \| \sum_{j \in I} c_j \omega_j \|^2 \leq D \sum_{j \in I} |c_j|^2
\]
for all finite sequences \( \{c_j\}_{j \in I} \). The numbers \( C \) and \( D \) are called Riesz bounds. A Riesz sequence \( \Omega = \{ \omega_j \}_{j \in I} \) is a Riesz basis for \( \mathcal{H} \) if \( \overline{\operatorname{span}} \{ \omega_j \}_{j \in I} = \mathcal{H} \).

We are now ready to introduce the main definitions used in this paper. We begin with the following well-known properties:

- \( F = \{ f_i \}_{i \in I} \) is a Bessel sequence in \( \mathcal{H} \) if and only if \( T_F^* \) is a well-defined bounded operator on \( \mathcal{H} \);
- \( F = \{ f_i \}_{i \in I} \) is a frame for \( \mathcal{H} \) if and only if \( T_F T_F^* : \mathcal{H} \rightarrow \mathcal{H} \) is a bounded invertible operator;
- \( F = \{ f_i \}_{i \in I} \) is a Riesz sequence in \( \mathcal{H} \) if and only if \( T_F^* T_F : \ell^2(I) \rightarrow \ell^2(I) \) is a bounded invertible operator.

**Definition 2.1** [1]. Let \( \{ e_i \}_{i \in I} \) and \( \{ h_i \}_{i \in I} \) be orthonormal bases for \( \mathcal{H} \). Let \( F = \{ f_i \}_{i \in I} \subseteq \mathcal{H} \) be such that \( \sum_{i \in I} |\langle f_i, e_j \rangle|^2 < \infty \) for any \( j \in I \). The R-dual of \( F = \{ f_i \}_{i \in I} \) with respect to the orthonormal bases \( \{ e_i \}_{i \in I} \) and \( \{ h_i \}_{i \in I} \) is defined as the sequence given by
\[
\omega_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i, \quad j \in I.
\]
This R-dual is said to be of type I in [8]. Below we give the definition of R-duals of type III, which are essential to our main result in the next section.

**Definition 2.2** [8]. Let \( F = \{ f_i \}_{i \in I} \) be a frame for \( \mathcal{H} \) with frame operator \( S_F \). Let \( \{ e_i \}_{i \in I} \) and \( \{ h_i \}_{i \in I} \) denote orthonormal bases for \( \mathcal{H} \), and let \( Q : \mathcal{H} \rightarrow \mathcal{H} \) be a bounded bijective operator with \( \| Q \| \leq \sqrt{|S_F|} \) and \( \| Q^{-1} \| \leq \sqrt{|S_F^{-1}|} \). The R-dual of type III of \( F = \{ f_i \}_{i \in I} \) with respect to the triple \( (\{ e_i \}_{i \in I}, \{ h_i \}_{i \in I}, Q) \) is the sequence \( \Omega = \{ \omega_j \}_{j \in I} \) defined by
\[
\omega_j = \sum_{i \in I} \langle S_F^{-1/2} f_i, e_j \rangle Q h_i, \quad j \in I.
\]
In this case, \( F = \{f_i\}_{i \in I} \) obtained as
\[
 f_i = \sum_{j \in I} \langle \omega_j, (Q^*)^{-1}h_i \rangle S_F^{1/2} e_j, \quad i \in I. \tag{2}
\]

Relation (2) does not imply, in general, that \( F = \{f_i\}_{i \in I} \) is an R-dual of type III of \( \Omega = \{\omega_j\}_{j \in I} \); that is, this definition of R-duality is not symmetric. However, an appropriate choice of \( Q \) ensures the following symmetry property of the sequences \( F = \{f_i\}_{i \in I} \) and \( \Omega = \{\omega_j\}_{j \in I} \) (see [8; Theorem 4.4]).

**Theorem 2.3.** Let \( F = \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \), and let \( \Omega \) be a Riesz sequence with the same optimal bounds as \( F = \{f_i\}_{i \in I} \). Denote the synthesis operator for \( F = \{f_i\}_{i \in I} \) by \( T_F \) and the frame operators for \( F = \{f_i\}_{i \in I} \) and \( \Omega \) by \( S_F \) and \( S_{\Omega} \), respectively. If \( \dim(\ker T_F) = \dim(\text{span}\{\omega_j\}_{j \in I} \}^\perp) \), then there exist orthonormal bases \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) for \( \mathcal{H} \) such that
\[
 \omega_j = \sum_{i \in I} \langle f_i, S_F^{-1/2} e_j \rangle \widetilde{S}_{\Omega}^{1/2} h_i, \quad j \in I,
\]
where \( S_{\Omega}^{1/2} \) is an extension of \( S_{\Omega}^{1/2} \) to an operator on \( \mathcal{H} \).

The sequence \( \Omega = \{\omega_j\}_{j \in I} \) defined in (3) is called the symmetrical R-dual of type III of \( F = \{f_i\}_{i \in I} \) with respect to the triple \( (\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, S_{\Omega}^{1/2}) \). In this case, for all \( i \in I \),
\[
 f_i = \sum_{j \in I} \langle \omega_j, S_{\Omega}^{-1/2} h_i \rangle \tilde{S}_{\Omega}^{1/2} e_j, \tag{4}
\]
that is, \( F = \{f_i\}_{i \in I} \) is an R-dual of type III of \( \Omega = \{\omega_j\}_{j \in I} \) with respect to the triple \( (\{h_i\}_{i \in I}, \{e_i\}_{i \in I}, S_{\Omega}^{1/2}) \).

### 3. Characterizing R-Duality in \( \mathcal{H} \)

We first consider symmetrical R-duals of type III. In [8; Theorem 4.3] it was proved that if \( F = \{f_i\}_{i \in I} \) is a frame sequence and \( \Omega = \{\omega_j\}_{j \in I} \) is an R-dual of \( F = \{f_i\}_{i \in I} \) of type III, then the following assertions hold:

(i) \( F = \{f_i\}_{i \in I} \) is a frame for \( \mathcal{H} \) if and only if \( \Omega = \{\omega_j\}_{j \in I} \) is a Riesz sequence; in the affirmative case, the bounds for \( F = \{f_i\}_{i \in I} \) are also bounds for \( \Omega = \{\omega_j\}_{j \in I} \);

(ii) \( F = \{f_i\}_{i \in I} \) is a Riesz sequence if and only if \( \Omega \) is a frame for \( \mathcal{H} \); in the affirmative case, the bounds for \( F = \{f_i\}_{i \in I} \) are also bounds for \( \Omega = \{\omega_j\}_{j \in I} \);

(iii) \( \Omega = \{\omega_j\}_{j \in I} \) is a Riesz basis if and only if \( F = \{f_i\}_{i \in I} \) is a Riesz basis.

The following section presents another viewpoint of the construction of R-duals of type III. In order to develop this viewpoint, we introduce some additional terminology.

**Definition 3.1.** A map \( J: \mathcal{H} \to \mathcal{H} \) is called conjugate-linear if
\[
 J(\lambda_1 u + \lambda_2 v) = \overline{\lambda_1} J(u) + \overline{\lambda_2} J(v)
\]
for all \( \lambda_1, \lambda_2 \in \mathbb{C} \) and all \( u, v \in \mathcal{H} \). If, in addition, \( J \) is bijective and isometric, then it is called an antiunitary operator on \( \mathcal{H} \). It is called an involution if \( J^2 = I_\mathcal{H} \).

The following lemma characterizes antiunitary involutions.

**Lemma 3.2.** Let \( \mathcal{H} \) denote a Hilbert space, and let \( J_1: \mathcal{H} \to \mathcal{H} \) be an antiunitary involution. Let \( J: \mathcal{H} \to \mathcal{H} \) denote a bijection. Then the following conditions are equivalent:

(i) for all \( u, v \in \mathcal{H} \), \( \langle J u, v \rangle = \overline{\langle u, J v \rangle} \) and \( J^2 = I_\mathcal{H} \);

(ii) \( J \) is conjugate-linear and satisfies \( J^2 = I_\mathcal{H} \);

(iii) there exists a unitary map \( U \) such that \( J = U J_1 \) and \( J_1 U J_1 = U^* \).
Proof. (i) To prove the conjugate linearity of \( J \), we observe that, by (i), for all \( \lambda \in \mathbb{C} \) and all \( u, v \in \mathcal{H} \), we have
\[
\langle J\lambda u, v \rangle = \overline{\langle \lambda u, Jv \rangle} = \overline{\lambda} \langle Ju, v \rangle,
\]
which gives \( J\lambda u = \overline{\lambda} Ju \). By a similar reasoning, we obtain \( J(u + v) = Ju + Jv \), which proves (ii).

To prove that \( (ii) \implies (iii) \), we set \( U = JJ_1 \). Then \( U \) is isometric, bijective, and linear, and hence unitary. Furthermore, \( UJ_1 = JJ_1^2 = J \), and \( J^2 = JJ_1^2 = I_\mathcal{H} \) implies
\[
UJ_1UJ_1 = JJ_1J_1JJ_1J_1 = I_\mathcal{H},
\]
and thus \( U^* = J_1UJ_1 \).

Let us prove \( (iii) \implies (i) \). If \( J = UJ_1 \) is as in \( (iii) \), then \( J^2 = UJ_1UJ_1UU^* = I_\mathcal{H} \). Since \( J \) is bijective, conjugate-linear, and isometric, polarization yields
\[
\langle Ju, Ju \rangle = \langle u, u \rangle
\]
for all \( u \) and \( w \), so that setting \( w = Ju \) and using \( J^2 = I_\mathcal{H} \), we obtain
\[
\langle Ju, v \rangle = \overline{\langle u, Jv \rangle},
\]
which is (i). \( \square \)

We will explain below that any choice of an antiunitary involution gives rise to a construction of R-dual sequences that have all the desirable properties of the original construction. The original construction in [1] is based on the following concrete choice of \( J_1 \): \( \ell^2(I) \to \ell^2(I) \):
\[
J_1((x_i)_{i \in I}) = (\overline{x}_i)_{i \in I}.
\]
Since each Hilbert space is isometrically isomorphic to some \( \ell^2(I) \), this example also establishes that antiunitary involutions exist on every Hilbert space \( \mathcal{H} \).

The next lemma shows how \( J_1 \), as defined above, enters the definition of R-duals.

Lemma 3.3. Under the assumptions of Theorem 2.3, for any \( i, j \in I \),
\[
\omega_j = T_{\overline{\mathcal{H}}}J_1T_{\overline{F}}^*(e_j),
\]
where \( \overline{\mathcal{H}} := \{ S_{\overline{h}} \}_{h \in I} \) and \( \overline{F} := \{ S_{\overline{f}} \}_{f \in I} \). Also,
\[
f_i = T_{\overline{F}}J_1T_{\overline{\Omega}}^*(h_i),
\]
where \( \overline{F} := \{ S_{\overline{e}} \}_{e \in I} \) and \( \overline{\Omega} := \{ S_{\overline{\omega}} \}_{\omega \in I} \).

Proof. By a direct and simple calculation we obtain (5), which is an operator representation of the symmetrical type-III R-duality of \( \Omega = \{ \omega_j \}_{j \in I} \) and \( F = \{ f_i \}_{i \in I} \) expressed by (3) and (4). \( \square \)

Following this example, we define the symmetrical R-dual associated to an antiunitary involution \( J \), a frame \( F = \{ f_i \}_{i \in I} \), and a Riesz sequence \( \Omega = \{ \omega_j \}_{j \in I} \) by replacing \( J_1 \) with \( J \), which leads to
\[
\omega_j = T_{\overline{\mathcal{H}}}JT_{\overline{F}}^*(e_j).
\]
Our aim is to prove that, for any choice of \( J \), this construction yields a notion of an R-dual that has all the desired properties of the original construction. We will use the following simple Lemma.

Lemma 3.4. Let \( U : \ell^2(I) \to \ell^2(I) \) be a linear operator, and let \( \tilde{U} = JUJ \), where \( J \) is an antiunitary involution. Then \( \tilde{U} \) is linear, \( \| U \| = \| \tilde{U} \| \), and \( \tilde{U}^* = JU^*J \).

Proof. \( \langle \tilde{U}f, g \rangle = \langle JUJf, g \rangle = \langle UJf, Jg \rangle = \langle Jf, U^*Jg \rangle = \langle f, JU^*Jg \rangle \). \( \square \)
The next result is as follows.

**Proposition 3.5.** Given a Bessel sequence \( F = \{ f_i \}_{i \in I} \) in \( \mathcal{H} \), let \( \mathcal{H}_F = \text{span} \{ f_i \}_{i \in I} \), and let \( P_F : \mathcal{H} \to \mathcal{H}_F \) be an orthogonal projection. Assume that \( F = \{ f_i \}_{i \in I} \) is a Riesz sequence. Then the following statements hold:

(i) \( \tilde{F} = \{ Pf_i \}_{i \in I} \) is a Riesz sequence if and only if \( P_F P^* P F : \mathcal{H} \to \mathcal{H}_F \) is an invertible operator, where \( P = S_F^{-1/2} \); 
(ii) \( T^*_\Omega = JT^*_E T^*_F J T^*_H \) and \( T^*_F = JT^*_H T^*_\Omega J T^*_E \), where \( E = \{ e_i \}_{i \in I} \) and \( H = \{ h_i \}_{i \in I} \). In particular, \( \Omega = \{ \omega_j \}_{j \in I} \) is a Bessel sequence in \( \mathcal{H} \) if \( F = \{ f_i \}_{i \in I} \) is a Bessel sequence in \( \mathcal{H} \), and vice versa.

**Proof.** We first note that \( \tilde{F} \) is a Bessel sequence in \( \mathcal{H} \) with \( T^*_F = T^*_F P^* \), because, for \( f \in \mathcal{H} \),

\[
\sum_{i \in I} |\langle f, Pf_i \rangle|^2 \leq B\|P^* f\|^2 = B\|P\|^2 \|f\|^2
\]

and \( T^*_F f = \{ \langle f, Pf_i \rangle \}_{i \in I} \). In particular, \( \tilde{F} \) is a Bessel sequence if \( T^*_F \) is a bounded operator on \( \mathcal{H} \). To prove (i), note that, on the other hand, we have

\[
T^*_F T^*_F = T^*_F P^* P T_F = T^*_F P_F P^* P P_F T_F, 
\]

and \( T_F : \ell^2(I) \to H_F \) is invertible. Hence \( T^*_F : H_F \to \ell^2(I) \) is invertible; thus, \( \tilde{F} \) is a Riesz sequence if and only if \( T^*_F T^*_F \) is an invertible operator on \( \ell^2(I) \), and this is so if and only if \( P_F P^* P F \) is an invertible operator on \( \mathcal{H}_F \), as desired.

(ii) We have

\[
T^*_\Omega f = \{ \langle f, \omega_j \rangle \}_{j \in I} = \{ \langle f, T^*_H (T^*_F e_j) \rangle \}_{j \in I} = \{ \langle T^*_F f, (T^*_F e_j) \rangle \}_{j \in I} = \{ \langle T^*_F f, T^*_F e_j \rangle \}_{j \in I} \equiv \{ T^*_F (T^*_F e_j) \}_{j \in I} = \{ T^*_F (T^*_H J T^*_E) f \}.
\]

This means that \( T^*_\Omega = JT^*_E T^*_F J T^*_H \). The argument for \( T^*_F \) is similar.

The following theorem shows that any antiunitary involution can be employed to define a notion of an R-dual of type III.

**Theorem 3.6.** Let \( F = \{ f_i \}_{i \in I} \) be a frame sequence, and let \( \Omega = \{ \omega_j \}_{j \in I} \) be an R-dual of type III of \( F = \{ f_i \}_{i \in I} \) associated to an antiunitary involution \( J \). Then

\[
T^*_\Omega T^*_\Omega = (T^*_H J T^*_E) (T^*_E T^*_F J T^*_H) = T^*_H J T^*_F J T^*_H,
\]

and

\[
T^*_\Omega T^*_\Omega = (J T^*_E T^*_F J T^*_H) (T^*_H J T^*_E) T^*_F J T^*_H.
\]

Furthermore,

(i) \( \Omega = \{ \omega_j \}_{j \in I} \) is a Riesz sequence in \( \mathcal{H} \) if and only if \( F = \{ f_i \}_{i \in I} \) is a frame for \( \mathcal{H} \);

(ii) \( F = \{ f_i \}_{i \in I} \) is a Riesz sequence in \( \mathcal{H} \) if and only if \( \Omega = \{ \omega_j \}_{j \in I} \) is a frame for \( \mathcal{H} \).

**Proof.** A straightforward computation using the relations \( J^2 = I_{\mathcal{H}} \) and \( T_E T_E^* = I_{\mathcal{H}} \), as well as Lemma 3.4 and Proposition 3.5, yields formulas (8) and (9).

(i) \( \Omega = \{ \omega_j \}_{j \in I} \) is a Riesz sequence if and only if \( T^*_\Omega T^*_\Omega \) is a bounded invertible operator on \( \ell^2(I) \), and this is so if and only if

\[
T^*_F J T^*_H J T^*_F := U
\]
is invertible (because $S_F^{-1/2}$ is bounded and invertible). Let now $U := JT_F^*T_H^*J$. Then $U : \ell^2(I) \to \ell^2(I)$ is a positive invertible operator. Finally, to prove the proposition, it is enough to show that $F$ is a frame for $\mathcal{H}$ if and only if $T_FU_T^* : \mathcal{H} \to \mathcal{H}$ is invertible.

Let $V := U^{1/2}$; then $V$ is a positive invertible operator on $\ell^2(I)$, and we have

$$T_FU_T^* = (VT_F^*)^*(VT_F^*).$$

In particular, $F$ is a frame for $\mathcal{H}$ if and only if $T_FT_F^* : \mathcal{H} \to \mathcal{H}$ is a bounded invertible operator, which is so if and only if $T_F^* : \mathcal{H} \to \ell^2(I)$ is an embedding, that is, if and only if $VT_F^* : \mathcal{H} \to \ell^2(I)$ is an embedding, or, equivalently, $(VT_F^*)^*(VT_F^*) = T_FU_T^*$ is an invertible operator, as desired.

(ii) $F = \{f_i\}_{i \in I}$ is a Riesz sequence in $\mathcal{H}$ if and only if $T_F^*T_F$ is a bounded invertible operator on $\ell^2(I)$, that is,

$$T_O J T^*_E T^*_G J T^*_O := W$$

is invertible (because $S_{T_O}^{-1/2}$ is bounded and invertible). Let $W := J T^*_E T^*_G J$. An argument similar to that used in (i) proves that $\Omega = \{\omega_j\}_{j \in I}$ is a frame for $\mathcal{H}$. \qed

The generalization using antunitary involutions highlights the algebraic and geometric properties of the map $J$ that lead to the desirable properties of the R-dual. Further benefits of this additional freedom of choice in the design of R-duals remain to be explored. For example, it is conceivable that a clever choice of $J$ may yield R-duals with additional symmetry properties.

4. Representation of $S_O$

In [7] Chuang and Zhao characterized R-duals $\Omega = \{\omega_j\}_{j \in I}$ of type I of a given frame $F$ by conditions formulated without explicit reference to the construction procedure of such duals. In principle, this result shows that the frame operator $S_O$ is computable from $S_F$, although the proof of the theorem is not explicit. The rest of this section is devoted to a more explicit description of $S_O$ in terms of $S_F$.

**Theorem 4.1** [7]. Let $F = \{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$, and let $\Omega = \{\omega_j\}_{j \in I}$ be a Riesz sequence in $\mathcal{H}$. Denote the synthesis operator for $F = \{f_i\}_{i \in I}$ by $T_F$, the frame operator of $F$ by $S_F$, and the frame operator of $\Omega = \{\omega_j\}_{j \in I}$ by $S_O$. Then $\Omega = \{\omega_j\}_{j \in I}$ is an R-dual of type I of $F = \{f_i\}_{i \in I}$ if and only if the following two conditions hold:

(i) there exists an antunitary operator $\Lambda : \mathcal{H} \to \overline{\text{span}} \Omega$ such that $S_O = \Lambda S_F \Lambda^{-1}$,

(ii) $\dim(\ker T_F) = \dim((\text{span}\{\omega_j\}_{j \in I})^\perp)$.

In the following theorem we present a representation for the operator $S_O$ associated with R-duals of types I and III. Let now the countable index set $I$ be the set $\mathbb{Z}$ of integers. We use some ideas of [2].

**Theorem 4.2.** Let $F = \{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$, and let $\Omega = \{\omega_j\}_{j \in I}$ be an R-dual of type I of $F = \{f_i\}_{i \in I}$ with respect to orthonormal bases $\{e_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$. Denote the frame operator of $F = \{f_i\}_{i \in I}$ and $\Omega = \{\omega_j\}_{j \in I}$ by $S_F$ and $S_O$, respectively, and suppose that

$$\sum_{j \in I} |\langle S_O h_0, h_j \rangle| < \infty. \quad (10)$$

Then there exist operators $\{V_j\}_{j \in I}$ on $\mathcal{H}$ such that

(i) $V_j(S_O h_0) = S_O h_j$.
(ii) if, for each \( k \in I \), \( \{ \mathcal{V}_j(h_k) \}_{j \in I} \) is a Bessel sequence with Bessel bound \( B \), then there exist bounded operators \( \{ \Lambda_k \}_{k \in \mathbb{Z}} \), which satisfy \( \sup_{k \in \mathbb{Z}} \| \Lambda_k \| < \infty \), so that \( S_\Omega \) associated with \( F = \{ f_i \}_{i \in I} \) has the representation
\[
S_\Omega f = \sum_{i \in \mathbb{Z}} \langle f_i, f_0 \rangle \Lambda_i(f)
\]  
with unconditional convergence in the operator norm.

Proof. Let \( \mathcal{U} : \mathcal{H} \to \mathcal{H} \) be unitary shift operator \( \mathcal{U}h_i := h_{i+1} \), and let \( \mathcal{V} : \mathcal{H} \to \mathcal{H} \) be the operator defined by \( \mathcal{V} = S_\Omega \mathcal{U} S_\Omega^{-1} \). We define \( \{ \mathcal{V}_j \}_{j \in I} \) on \( \mathcal{H} \) by
\[
\mathcal{V}_j := \mathcal{V}^j = S_\Omega \mathcal{U}^j S_\Omega^{-1}.
\]
Then (i) is obviously fulfilled, since
\[
\mathcal{V}_j(S_\Omega h_0) = \mathcal{V}^j(S_\Omega h_0) = S_\Omega \mathcal{U}^j h_0 = S_\Omega h_j \quad \text{for all } j \in I.
\]
Assertion (ii) is proved by calculations similar to those in the proof of Theorem 4.5 in [2].

For type III, the above procedure for computing \( \langle S_\Omega h_0, h_j \rangle \) does not work, and we have to change our strategy. First, recall (see, e.g., [9]) that a representation of \( S_\Omega \) in terms of simple operators (projections) is called a spectral representation of the operator \( S_\Omega \).

Let \( T : \mathcal{H} \to \mathcal{H} \) be a bounded self-adjoint linear operator. For \( \lambda \in \mathbb{R} \), let \( T_\lambda = T - \lambda I \mathcal{H} \). We denote the positive square root of \( T_\lambda^2 \) by \( |T_\lambda| \); the operator \( T_\lambda^+ = \frac{1}{2}(|T_\lambda| + T_\lambda) \) is called the positive part of \( T_\lambda \), and the operator \( T_\lambda^- = \frac{1}{2}(|T_\lambda| - T_\lambda) \) is called the negative part of \( T_\lambda \). We have \( T_\lambda = T_\lambda^+ - T_\lambda^- \), and the spectral family \( \mathcal{E} \) of \( T \) is defined by \( \mathcal{E} = \{ E_\lambda \}_{\lambda \in \mathbb{R}} \), where \( E_\lambda \) is the projection of \( \mathcal{H} \) onto the null space \( \mathcal{N}(T_\lambda^+) \) of \( T_\lambda^+ \). Let
\[
m = \inf \frac{\langle Tx, x \rangle}{\| x \| = 1} \quad \text{and} \quad M = \sup \frac{\langle Tx, x \rangle}{\| x \| = 1}.
\]
The following theorem holds [9; Theorem (9.2-1)].

**Theorem 4.3.** Let \( T : \mathcal{H} \to \mathcal{H} \) be a bounded self-adjoint linear operator on a complex Hilbert space \( \mathcal{H} \). Then \( T \) has the spectral representation
\[
T = mE_m + \int_m^M \lambda dE_\lambda,
\]
where the integral is to be understood in the sense of uniform operator convergence. Also, for all \( x, y \in \mathcal{H} \),
\[
\langle Tx, y \rangle = mW(m) + \int_m^M \lambda dW(\lambda),
\]
where \( W(\lambda) = \langle E_\lambda x, y \rangle \), and the integral is an ordinary Riemann–Stieltjes integral.

Now we are ready to present a representation for the operator \( S_\Omega \) associated with R-duals of type III.

**Theorem 4.4.** Let \( F = \{ f_i \}_{i \in I} \) be a frame for \( \mathcal{H} \) with frame operator \( S_F \), and let \( \Omega = \{ \omega_j \}_{j \in I} \) be a symmetrical R-dual of type III of \( F = \{ f_i \}_{i \in I} \) with respect to the triple \( \{ e_i \}_{i \in I}, \{ h_i \}_{i \in I}, S_\Omega^{1/2} \).

Suppose that
\[
\sum_{j \in I} |\langle S_\Omega h_0, h_j \rangle| < \infty.
\]
Assume there exist operators \( \{ \mathcal{V}_j \}_{j \in I} \) on \( \mathcal{H} \) such that

(i) \( \mathcal{V}_j(S_\Omega h_0) = S_\Omega h_j \) for all \( j \in I \);
(ii) there is a constant $B > 0$ such that, for each $k \in I$, the set $\{V_j(h_k)\}_{j \in I}$ is a Bessel sequence with Bessel bound $B$.

Then there exist bounded operators $\{\Lambda_i\}_{i \in I}$, which satisfy $\sup_{i \in I} \|\Lambda_i\| < \infty$, and constants $C_i$ such that $S_\Omega$ associated with $F = \{f_i\}_{i \in I}$ has the representation

$$S_\Omega f = \sum_{i \in I} (M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i) \Lambda_i(f)$$

(15)

with unconditional convergence in the operator norm.

**Proof.** Since $S_\Omega$ is a bounded self-adjoint linear operator on $H$, by Eq. (13) we have

$$\langle S_\Omega h_0, h_i \rangle = MW(m) + \int_m^M \lambda dW(\lambda)$$

$$= MW(m) + MW(M) - MW(m) - \int_m^M W(\lambda) d\lambda$$

(the integral is the ordinary Riemann–Stieltjes integral), where $W(\lambda) = \langle E_\lambda h_0, h_i \rangle$ and $E_\lambda$ is the projection of $H$ onto the null space $N(S_\Omega^+)$ of $S_\Omega^+$ for $S_\Omega = S_\Omega - \lambda I$. Therefore,

$$\langle S_\Omega h_0, h_i \rangle = MW(M) - \int_m^M W(\lambda) d\lambda. \quad (16)$$

On the other hand, $\Omega = \{\omega_j\}_{j \in I}$ is a Riesz sequence in $H$, and hence it is a Riesz basis for $V := \text{span} \Omega$. Thus, the sequence $\{S_\Omega^{-1/2} \omega_j\}_{j \in I}$ is an orthonormal basis for $V$ [8; Lemma (1.1)] (note that $N(S_\Omega^+)$ is a Riesz sequence in $H$, and hence it is a Riesz basis for $V$). Consider the extension $S_\Omega^{-1/2}$ of $S_\Omega^{-1/2}$ to an operator on $H$ as in [8; Lemma (1.3)]; $\{S_\Omega^{-1/2} \omega_j\}_{j \in I}$ is an orthonormal basis for $V$, too. Therefore, the orthogonal projection $E_M$ of $H$ onto $N(S_\Omega^+)$ is given by

$$E_M f = \sum_{j \in I} \langle f, S_\Omega^{-1/2} \omega_j \rangle S_\Omega^{-1/2} \omega_j, \quad f \in H. \quad (17)$$

It is enough to prove that if we define $E_M$ by (17), then $E_M f = f$ for $f \in N(S_\Omega^+)$ and $E_M f = 0$ for $f \in (N(S_\Omega^+))^\perp$; the first equation follows by the orthonormality of $\{S_\Omega^{-1/2} \omega_j\}_{j \in I}$, and the second, by the fact that the range of $S_\Omega^{-1/2}$ equals $N(S_\Omega^+)$, because $S_\Omega^{-1/2}$ is a bijection on $N(S_\Omega^+)$. Therefore,

$$\langle E_M h_0, h_i \rangle = \sum_{j \in I} \langle h_0, S_\Omega^{-1/2} \omega_j \rangle \langle S_\Omega^{-1/2} \omega_j, h_i \rangle = \langle S_F^{-1/2} f_1, S_F^{-1/2} f_0 \rangle$$

(17)

(the second equality follows from (4)). We now have

$$S_\Omega f = \sum_{j \in I} \langle f, h_j \rangle S_\Omega(h_j) = \sum_{j \in I} \langle f, h_j \rangle V_j(S_\Omega(h_0)) = \sum_{j \in I} \langle f, h_j \rangle V_j \left( \sum_{i \in I} \langle S_\Omega(h_0), h_i \rangle h_i \right)$$

$$= \sum_{i \in I} \langle S_\Omega h_0, h_i \rangle \sum_{j \in I} \langle f, h_j \rangle V_j(h_i) = \sum_{i \in I} (M(E_M h_0, h_i) - C_i) \sum_{j \in I} \langle f, h_j \rangle V_j(h_i)$$

$$= \sum_{i \in I} (M(S_F^{-1/2} f_1, S_F^{-1/2} f_0) - C_i) \sum_{j \in I} \langle f, h_j \rangle V_j(h_i),$$
where $C_i := \int M W(\lambda) \, d\lambda$. Defining $\Lambda_i$ by $\Lambda_i(f) = \sum_{j \in I}(f, h_j)\mathcal{V}_j(h_i)$, we obtain

$$S_\Omega f = \sum_{i \in I}(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)\Lambda_i(f),$$

as desired. The proof of convergence is almost the same as in Theorem 4.5 of [2], but we review it. The sequence $\{\mathcal{V}_j(h_i)\}_{j \in I}$ is a Bessel sequence with bound $B$, and hence, for any finite set $J \subset I$,

$$\left\|\sum_{j \in J}(f, h_j)\mathcal{V}_j(h_i)\right\|^2 \leq B \sum_{j \in J}|\langle f, h_j \rangle|^2 = B\|f\|^2,$$

that is, the series defining $\Lambda_i$ are unconditionally convergent and $\|\Lambda_i\| \leq \sqrt{B}$ for each $k \in \mathbb{Z}$.

On the other hand, for finite subsets $I_1$ and $I_2$ of $I$, it follows from the above calculations that

$$\left\|\sum_{i \in I_1}(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)\sum_{j \in I_2}(f, h_j)\mathcal{V}_j(h_i)\right\|$$

$$= \left\|\sum_{i \in I_1}(S_\Omega h_0, h_i)\sum_{j \in I_2}(f, h_j)\mathcal{V}_j(h_i)\right\|$$

$$\leq \left\{\sum_{i \in I_1}|\langle S_\Omega h_0, h_i \rangle|\right\}\left\|\sum_{j \in I_2}(f, h_j)\mathcal{V}_j(h_i)\right\|$$

$$\leq \left\{\sum_{i \in I_1}|\langle S_\Omega h_0, h_i \rangle|\right\}\sqrt{B}\|f\|.$$

The convergence of the series (10) implies that the series in the construction of $\Lambda_i$ are unconditionally convergent. Finally, for a finite subset $J \subset I$, we have

$$\left\|S_\Omega - \sum_{i \in J}(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)\Lambda_i\right\|$$

$$= \sup_{\|f\|=1}\left\|S_\Omega(f) - \sum_{i \in J}(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)\Lambda_i(f)\right\|$$

$$= \sup_{\|f\|=1}\left\|\sum_{i \in J}(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)\Lambda_i(f)\right\|$$

$$\leq \left\{\sum_{i \in J}|(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)|\right\}\sup_{\|f\|=1}\sup_{i \in J}\|\Lambda_i(f)\|$$

$$\leq \left\{\sum_{i \in J}|(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)|\right\}\sqrt{B}.$$

Now, using relations (10) and $|\langle S_\Omega h_0, h_i \rangle| = |(M(S_F^{-1/2} f_i, S_F^{-1/2} f_0) - C_i)|$, we see that the operators converge to $S_\Omega$ unconditionally in the operator norm.

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