ABOUT THE POLIGNAC’S CONJECTURE

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Abstract. In this note we present a method to bound gaps between infinitely many consecutive prime numbers via the divergence of the series of reciprocals of the prime numbers and a consequence of a test for convergence of series of positive numbers. Using this alternative method we indicate how our results are related to the Polignac’s conjecture.

1. Introduction

The study of the gap between consecutive prime numbers, that is, the investigation of the behavior of the sequence \( g_n = p_{n+1} - p_n \), has attracted attention of the most prominent mathematicians in the area. Here, \( p_n \) denotes the \( n \)th prime number. Two of the most important problems related to this subject are the twin-prime conjecture and the Polignac’s conjecture. The first one states that there exist infinitely many primes for which \( g_n = 2 \) and the second conjecture generalizes the first one, by stating that for any positive even integer \( k \), there exist infinitely many primes for which \( g_n = k \). As far as we know, until this moment both conjectures were still not proved or disproved. However, several works have contributed to the progress of this subject. Let us indicate some of them.

In (4) Dan Goldston, János Pintz, and Cem Yıldırım (also indicated as GPY) presented a solution for a long-standing open problem. They proved that there are infinitely many primes for which the gap to the next prime is as small as we want compared to the average gap between consecutive primes. They showed that

\[
\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\ln(p_n)} = 0.
\]

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There, the approach adopted is usually referred as the level of distribution of primes in arithmetic progressions and, with an additional assumption on this level of distribution the showed that

\[(1.2) \lim \inf_{n \to \infty} p_{n+1} - p_n \leq 16.\]

Latter, in [5] the same authors considerably improved (1.1) proving that

\[(1.3) \lim \inf_{n \to \infty} \frac{p_{n+1} - p_n}{\sqrt{\ln(p_n) \ln(\ln(p_n))}} < \infty.\]

This result shows that there exist pairs of primes nearly within the square root of the average spacing.

In 2013, Yitang Zhang [11] published his celebrated paper providing the first proof of finite gaps between prime numbers. There it is shown that

\[(1.4) \lim \inf_{n \to \infty} p_{n+1} - p_n \leq 7 \cdot 10^7.\]

His approach was a refinement of the work of Goldston, Pintz and Yildirim on the small gaps between consecutive primes [4] and a major ingredient of the proof is a stronger version of the Bombieri-Vinogradov theorem [11]. A nice exposition of the Zhang’s proof can be found in [6].

The improvement of the Zhang’s numerical bound on the gaps was obtained right after. For instance, the work of Polymath8 [10] and Maynard [8] presented a reduction of Zhang’s bound to 4680 and 600, respectively. In particular, in Maynards work the proofs involved quite different methods to Zhang and brought the upper bound down to 600 using the Bombieri - Vinogradov Theorem (not Zhangs stronger alternative) and an improvement on GPY results. We refer to [9] and references there in for more information about the developments of the investigation about gaps of prime numbers.

The main feature of this note is a new approach to deal with gaps of infinitely many consecutive prime numbers. Our method involves the divergence of the series
of the reciprocals of the prime numbers and a consequence of a version of a test for convergence of series of positive terms.

As it is presented at the final of this note, our method may be used as an alternative to deal with the Polignac’s conjecture.

Let us start with the following classical result.

**Lemma 1.1.** The series $\sum \frac{1}{p_n}$ diverges.

The next result is a consequence of the prime number theorem (see [1, p. 79]).

We will make use of the asymptotic notation $u(x) \sim v(x)$, meaning that

$$\lim_{x \to \infty} \frac{u(x)}{v(x)} = 1.$$

**Lemma 1.2.**

$$p_n \sim n \ln(n).$$

See, for instance, [1, p.16], for a proof.

Next we will use $f(x) = o(g(x))$, as $x \to \infty$ to denote

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

Let us define a class of summable positive sequences that will play an importante role in this note. We will say that a pair $(\{b_n\}, r)$ satisfies the $*$-condition, in which $\{b_n\}$ is a summable sequence of positive terms and $r > 0$, if there exists $n_0 > 0$ for which

$$\frac{b_n}{b_{n+1}} = 1 + \frac{r}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right),$$

for all $n > n_0$.

The next result is a test for convergence of series of positive terms.

**Lemma 1.3.** Let $(\{b_n\}, r)$ be a pair satisfying the $*$-condition, for some $r > 0$. If $\{a_n\}$ is a sequence of positive terms such that there exists $n_1 > 0$ for which

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{r}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right),$$
for all \( n > n_1 \), then \( \{a_n\} \) is summable.

**Proof.** Since \( \sum_{n=1}^{\infty} b_n \) converges and

\[
\frac{b_n}{b_{n+1}} = 1 + \frac{r}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right) \leq \frac{a_n}{a_{n+1}},
\]

for all \( n > \max\{n_0, n_1\} \), the comparison test of second kind (see [2, p. 114]) implies that \( \sum a_n \) converges.

\[\square\]

An immediate consequence of the previous lemma is the following key result for the main result of this note. The idea is to use the contrapositive argument in Lemma 1.3.

**Corollary 1.3.1.** Let \( \{a_n\} \) be a sequence of positive terms. If \( \sum a_n \) diverges then for every pair \( \{b_n\}, r \) satisfying the \(*\)-condition and every \( n_0 > 0 \), there exists a \( n > n_0 \) for which

\[
\frac{a_n}{a_{n+1}} < 1 + \frac{r}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right).
\]

**Proof.** It suffices to use the contrapositive of Lemma 1.3. \[\square\]

The main result of this note is as follows.

**Theorem 1.4.** If there exists a pair \( \{b_n\}, r \) satisfying the \(*\)-condition for some \( r \geq 2 \) then

\[
\liminf_{n \to \infty} g_n \leq r.
\]

**Proof.** Suppose that there exists a pair \( \{b_n\}, r \) satisfying the \(*\)-condition with \( r \geq 2 \). From Lemma 1.3 and Corollary 1.3.1 we have that

\[
\frac{p_{n+1}}{p_n} < 1 + \frac{r}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right),
\]

for infinitely many values of \( n \). That is,

\[
g_n < p_n \left(\frac{r}{n \ln(n)} + o\left(\frac{1}{n \ln(n)}\right)\right),
\]
for infinitely many values of \( n \).

By Lemma 1.2, \( n \sim n \ln(n) \), hence

\[
g_n < n \ln(n) \left( \frac{r}{n \ln(n)} + o \left( \frac{1}{n \ln(n)} \right) \right),
\]

for infinitely many values of \( n \). Therefore

\[
\liminf_{n \to \infty} g_n \leq r.
\]

\[ \square \]

2. Final remarks

In this note we have presented an alternative and simple method to deal with the Polignac’s conjecture. In particular, Theorem 1.4 indicates that if one find a pair \( (\{b_n\}, r) \) satisfying the \( * \)-condition with \( r = 2 \) then the twin-prime conjecture would be true. More generally if, for each \( r \geq 2 \) even, one finds a pair \( (\{b_n\}, r) \) satisfying the \( * \)-condition then the Polignac’s conjecture would be true.

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