Abstract

An open superstring field theory action has been proposed which does not suffer from contact term divergences. In this paper, we compute the on-shell four-point tree amplitude from this action using the Giddings map. After including contributions from the quartic term in the action, the resulting amplitude agrees with the first-quantized prescription.
1 Introduction

To study non-perturbative features of superstring theory, a field theory action for the superstring could be useful. As shown by Wendt in 1989 [1], Witten’s cubic action [2] for open superstring field theory has divergent contact term problems due to colliding interaction-point operators. If the Neveu-Schwarz string field $A$ carries picture $-1$, the cubic action is $\int (AQ^3 + ZA^3)$ where $Z = \{Q, \xi\}$ is the picture-raising operator which is inserted at the interaction-point [1]. Since $Z$ has a singularity with itself, the four-point amplitude computed using the above action has contact term divergences which break gauge invariance and make the action inconsistent. Similar contact term problems exist with other choices of picture [1] and with other open superstring field theory actions such as the light-cone and covariantized light-cone actions [4].

In 1995, an action for open superstring field theory was proposed which does not suffer from contact term divergences [1]. This action was constructed by embedding the $N=1$ superstring into the $N=2$ superstring and resembles a Wess-Zumino-Witten action. Using modified Green-Schwarz variables, this open superstring field theory action can be written in a manifestly $SO(3,1)$ super-Poincaré invariant manner [1]. This spacetime-supersymmetric version of the action is easily generalized to any compactification to four dimensions which preserves at least $N=1$ $d=4$ supersymmetry. It is also possible to write this field theory action using Ramond-Neveu-Schwarz variables and, at least in the Neveu-Schwarz sector, this can be done in a manifestly $SO(9,1)$ Lorentz invariant manner.

In this paper, we shall use this action to explicitly compute the four-point open superstring tree amplitude. When the external states are on-shell, this amplitude will be shown to agree with the first-quantized result. There are no contact term divergences, and the quartic vertex plays a crucial role in cancelling a finite contact term contribution coming from two cubic vertices.

In section 2, we shall review the Wess-Zumino-Witten-like action for open superstring field theory. In section 3, we shall discuss gauge fixing and the Giddings map for computing four-point amplitudes using open string field theory. In section 4, we shall explicitly compute the four-point on-shell tree amplitude and will prove equivalence with the first-quantized result.
2 Review of Superstring Field Theory Action

For any critical ($\hat{c} = 2$) N=2 superconformal representation with N=2 generators $[T, G^+, G^-, J]$, one can construct the following open string field theory action:

\[
S = \frac{1}{2} \int [(e^{-\Phi} G_0^+ e^{\Phi})(e^{-\Phi} \tilde{G}_0^+ e^{\Phi}) - \int_0^1 dt (e^{-t\Phi} \partial_t e^{t\Phi}) \{e^{-t\Phi} G_0^+ e^{t\Phi}, e^{-t\Phi} \tilde{G}_0^+ e^{t\Phi}\}].
\]

(1)

which resembles the Wess-Zumino-Witten action $\int [(g^{-1} dg)^2 + \int dt (g^{-1} dg)^3]$. The open string fields $\Phi$ are glued together using Witten’s midpoint interaction and are restricted to be U(1)-neutral with respect to $J$. The open string fields carry Chan-Paton factors which will be suppressed throughout this paper.

The fermionic operators $G_0^+$ and $\tilde{G}_0^+$ are constructed from the N=2 superconformal generators in the following manner: After twisting, the fermionic generator $G^+$ carries conformal weight +1 and the fermionic generator $G^-$ carries conformal weight +2. Furthermore, after bosonizing the U(1) generator $J = \partial H$, one can construct SU(2) currents $[J^{++} = e^{iH}, J = \partial H, J^{--} = e^{-iH}]$ of conformal weight [0, 1, 2] after twisting. Commuting the zero modes of these SU(2) currents with the fermionic generators, one obtains two new fermionic generators, $\tilde{G}^+ = [G^-, J^{++}]$ and $\tilde{G}^- = [G^+, J^{--}]$, of conformal weight +1 and +2 respectively. $G_0^+$ is defined to be the zero mode of $G^+$ and $\tilde{G}_0^+$ is defined to be the zero mode of $\tilde{G}^+$. Note that $\{G_0^+, \tilde{G}_0^+\} = 0$ and after twisting, the U(1) anomaly implies that non-vanishing correlation functions on a sphere must carry +2 U(1) charge.

The action of (1) can be justified by considering the following three examples of critical N=2 superconformal representations. The first example is the open self-dual string where $\Phi$ depends on the string modes of the variables $[x^{\alpha\bar{\alpha}}, \psi^\alpha, \bar{\psi}^\alpha]$ for $\alpha, \bar{\alpha} = 1$ to 2. For this N=2 superconformal representation, $G^+ = \psi_\alpha \partial x^{\alpha+}, \tilde{G}^+ = \bar{\psi}_\alpha \partial x^{\alpha-},$ and $J = \psi_\alpha \bar{\psi}_\alpha$. Since there are no massive physical fields, the dependence of $\Phi$ on the non-zero modes of $[x^{\alpha\bar{\alpha}}, \psi^\alpha, \bar{\psi}^\alpha]$ gives rise to auxiliary and gauge fields. The massless physical field comes from dependence on the zero mode of $x^{\alpha\bar{\alpha}}, \Phi(x^{\alpha\bar{\alpha}})$, which when plugged into (1) reproduces the Donaldson-Nair-Schiff action for D=4 self-dual Yang-Mills.

The second example is the N=2 embedding of the superstring using the standard RNS worldsheet variables $[x^\mu, \psi^\mu, b, c, \xi, \eta, \phi]$ where $\mu = 0$ to 9 and
the super-reparameterization ghosts have been fermionized as $\beta = e^{-\phi} \partial \xi$ and $\gamma = e^{\phi} \eta$ [7]. In this example, the Neveu-Schwarz contribution to the field theory action is obtained by defining $\Phi = \xi_0 A$ where $A$ is the standard Neveu-Schwarz string field in the $-1$ picture. (The Ramond contribution to the superstring field theory action is not yet known.) In the $N=2$ embedding of the superstring, $G_0^+$ is the BRST charge $Q$ and $\tilde{G}_0^+$ is the zero mode of $\eta$. So the linearized equation of motion coming from the action of (1) is

$$0 = G_0^+ \tilde{G}_0^+ \Phi = Q \eta_0 (\xi_0 A) = QA$$

as desired. Furthermore, since $J = bc + \xi \eta$, the $U(1)$-neutrality condition implies that $\Phi$ has zero RNS ghost-number.

The third example is the ‘hybrid’ description of the superstring using the four-dimensional superspace variables $[x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}]$ where $m = 0$ to 3 and $\alpha, \dot{\alpha} = 1$ to 2, combined with a chiral boson $\rho$ and the variables of a $c = 9$ $N=2$ superconformal field theory which describes a six-dimensional compactification manifold [9]. In this example, the compactification-independent part of the field theory action is obtained by defining $\Phi$ to be a string field depending only on the four-dimensional superspace variables. (The compactification-dependent contribution to the action was worked out in [8] and involves two other string fields in addition to $\Phi$.) The massless and first massive level of this action have been explicitly computed in [8] and [10] and describe a super-Yang-Mills and massive spin-2 multiplet in $N=1 \ d = 4$ superspace.

3 Gauge Fixing and the Giddings Map

Similar to the Wess-Zumino-Witten action, the action of (1) is invariant under the gauge transformation

$$\delta e^\Phi = (G_0^+ \Omega) e^\Phi + e^\Phi (\tilde{G}_0^+ \tilde{\Omega})$$

where $\Omega$ and $\tilde{\Omega}$ are arbitrary string fields of $-1$ $U(1)$-charge. At linearized level, this transformation reduces to $\delta \Phi = G_0^+ \Omega + \tilde{G}_0^+ \tilde{\Omega}$ which allows the gauge-fixing conditions

$$G_0^- \Phi = \tilde{G}_0^- \Phi = 0$$

(2)

where $G_0^-$ and $\tilde{G}_0^-$ are the zero modes of $G^-$ and $\tilde{G}^- = [G^+, J_0^{-}]$. The gauge-fixing conditions of (1) can be obtained, for example, by choosing

$$\Omega = -(T_0)^{-1} G_0^- \Phi, \quad \tilde{\Omega} = -(T_0)^{-2} \tilde{G}_0^- G_0^+ \Phi,$$
so that \( G_0^{-} \delta \Phi = -G_0^{-} \Phi \) and \( \tilde{G}_0^{-} \delta \Phi = -\tilde{G}_0^{-} \Phi \) where we have used that \( \{G_0^{-}, G_0^{+}\} = \{\tilde{G}_0^{-}, \tilde{G}_0^{+}\} = T_0 \).

In this gauge, the linearized propagator is \( \mathcal{P} = (T_0)^{-2} \tilde{G}_0^{-} \tilde{G}_0^{-} \) since \( \tilde{G}_0^{+} G_0^{+} \Phi = 0 \) is the linearized equation of motion and \( \mathcal{P} \tilde{G}_0^{+} G_0^{+} \Phi = \Phi \). Although this propagator looks complicated, we shall show in section 4 that it can be simplified to \( \mathcal{P} = (T_0)^{-1} J_0^{-}\) when computing on-shell tree amplitudes.

Although the action of (1) contains vertices with arbitrary numbers of string fields, only the cubic and quartic vertices will be necessary for computing four-point tree amplitudes. Expanding the action of (1), one obtains

\[
S = \int \left( \frac{1}{2} G_0^{+} \Phi \tilde{G}_0^{+} \Phi - \frac{1}{6} \Phi \{G_0^{+} \Phi, \tilde{G}_0^{+} \Phi\} - \frac{1}{24} [\Phi, G_0^{+} \Phi][\Phi, \tilde{G}_0^{+} \Phi] + \cdots \right). \tag{3}
\]

Using Witten’s gluing prescription \cite{6} for string fields, the cubic and quartic vertex from (3) are described by the diagrams

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram1}
\caption{}
\end{figure}

For four-point tree amplitudes, contributions can come from two cubic vertices or from one quartic vertex. The Witten diagram for two cubic vertices connected by a propagator of length \( \tau \) is given by

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram2}
\caption{}
\end{figure}
where the propagator is integrated along the contour $c$. The Witten diagram for a quartic vertex is given by Figure 2 in the limit that $\tau \to 0$.

As was shown by Giddings [11], it is convenient to perform a Schwarz-Christoffel transformation from the diagram of figure 2 to the upper half plane such that the four external strings are mapped to the points $\pm \alpha(\tau)$ and $\pm (\alpha(\tau))^{-1}$. If strings [(1), (2), (3), (4)] are mapped to the points $[-\alpha, \alpha, \alpha^{-1}, -\alpha^{-1}]$ in this order, then $0 < \alpha(\tau) \leq \delta$ where $\delta \equiv \sqrt{2} - 1$, $\alpha(\tau = 0) = \delta$ and $\alpha(\tau = \infty) = 0$. On the other hand, if strings [(1), (2), (3), (4)] are mapped to the points $[-\alpha^{-1}, -\alpha, \alpha, \alpha^{-1}]$ in this order, then $\delta \leq \alpha(\tau) < 1$ where $\alpha(\tau = 0) = \delta$ and $\alpha(\tau = \infty) = 1$.

4 Computation of Four-Point Amplitude

The tree-level scattering amplitude for four external string fields labeled by the letters $[A, B, C, D]$ gets contributions either from the diagram of Figure 2 or from the second diagram of Figure 1. There are 24 different ways to match the string fields $[A, B, C, D]$ with the external legs [(1), (2), (3), (4)] and we shall restrict our attention to those 4 combinations where $[A, B, C, D]$ are cyclically ordered. For contributions coming from the diagram of Figure 2, these 4 combinations split into two ‘s-channel’ contributions, $A_s$, where string $A$ is associated with leg (2) or leg (4) and two ‘t-channel’ contributions, $A_t$, where string $A$ is associated with leg (1) or leg (3). The four cyclically related combinations coming from the quartic vertex of Figure 1 will be called $A_q$.

In this section, it will be shown that the sum of these three contributions, $A = A_s + A_t + A_q$, reproduces the first-quantized result for the on-shell scattering amplitude which can be written as [7]

$$A = \int_0^1 d\alpha \langle (\int d^2z \mu_\alpha(z, \bar{z}) G^- (z)) \rangle$$

$$\Phi_A(-\alpha^{-1}) G^+_0 \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \tilde{G}^+_0 \Phi_D(\alpha^{-1}) \rangle.$$

where $\mu_\alpha(z, \bar{z})$ is the appropriate Beltrami differential for an $\alpha$-dependent parameterization of the modulus and $\langle \rangle$ signifies the two-dimensional correlation function in the upper half-plane. To relate (4) to the standard expression for the scattering amplitude of four Neveu-Schwarz strings, recall from section 2 that $\Phi = \xi_0 V(-1)$ where $V(-1)$ is the Neveu-Schwarz vertex operator in the $-1$ picture. Furthermore, $G^- = b$, $\tilde{G}^+_0 \Phi = \eta_0 \xi_0 V(-1) = V(-1)$ and
\[ G_0^+ \Phi = Q \xi_0 V^{(-1)} = V^{(0)} \] where \( V^{(0)} \) is the Neveu-Schwarz vertex operator in the zero picture. So (4) implies that

\[
\mathcal{A} = \int_0^1 d\alpha \langle (\int d^2 z \mu_\alpha (z, \bar{z}) b(z)) \xi_0 V_A^{(-1)} (-\alpha^{-1}) V_B^{(0)} (-\alpha) V_C^{(0)} (\alpha) V_D^{(-1)} (\alpha^{-1}) \rangle,
\]

which agrees with the first-quantized prescription of [12] in the large Hilbert space, i.e. in the Hilbert space including the \( \xi \) zero mode.

### 4.1 Computation of s-channel contribution

We shall first compute the contribution \( \mathcal{A}_s \) from s-channel diagrams. Using the cubic vertices and propagator \( \mathcal{P} \) from section 3 and including the various combinatorial factors,

\[
\mathcal{A}_s = 2(3)_2 \frac{1}{2 \pi i} \int \frac{dw}{2\pi i} J^- (w) \langle (G_0^+ \Phi_A (4) G_0^+ \Phi_B (1) + G_0^+ \Phi_A (4) G_0^+ \Phi_B (1)) \rangle,
\]

where \( \langle \rangle \rangle \) signifies the two-dimensional correlation function in the Witten diagram of Figure 2. Note that one can always choose the \( G_0^+ \) and \( \tilde{G}_0^+ \) operators of the cubic vertices to act on external legs since

\[
\langle (G_0^+ \Phi_A (1) \tilde{G}_0^+ \Phi_B (2) + \tilde{G}_0^+ \Phi_A (1) G_0^+ \Phi_B (2)) \Phi_C (3) \rangle =
\]

\[
\langle \Phi_A (1) (\tilde{G}_0^+ \Phi_B (2) G_0^+ \Phi_C (3) + G_0^+ \Phi_B (2) \tilde{G}_0^+ \Phi_C (3)) \rangle
\]

by deforming the contour integral of \( G^+ \) and \( \tilde{G}^+ \) off of \( \Phi_A (1) \).

The propagator \( \mathcal{P} = (T_0)^{-2} G_0^- \tilde{G}_0^- \) can be simplified by writing \( \tilde{G}_0^- = [G_0^+, J_0^-] \) and deforming the contour integral of \( G^+ \) off of \( J_0^- \). When the external string fields are on-shell, i.e. \( G_0^+ \tilde{G}_0^+ \Phi = 0 \), the contour integral of \( G^+ \) only contributes by hitting the remaining \( G_0^- \) of the propagator. Since \( \{G_0^+, G_0^-\} = T_0 \), this means that the propagator can be simplified to \( \mathcal{P} = (T_0)^{-1} J_0^- \) when all external string fields are on-shell. As usual, it is convenient to rewrite \( (T_0)^{-1} = \int_0^\infty d\tau e^{-\tau T_0} \) so the s-channel contribution is

\[
\mathcal{A}_s = \frac{1}{4} \int_0^\infty d\tau (\int \frac{dw}{2\pi i} J^- (w) \langle (G_0^+ \Phi_A (4) \tilde{G}_0^+ \Phi_B (1) + G_0^+ \Phi_A (4) G_0^+ \Phi_B (1)) \rangle) \langle (G_0^+ \Phi_C (2) \tilde{G}_0^+ \Phi_D (3) + \tilde{G}_0^+ \Phi_C (2) G_0^+ \Phi_D (3)) \rangle \rangle_W
\]
where the contour $c$ is that of Figure 2.

After performing the Giddings map of Figure 2 to the upper half plane using the ordering $[(1), (2), (3), (4)] \to [-\alpha, \alpha, \alpha^{-1}, -\alpha^{-1}]$,

$$A_s = -\frac{1}{4} \int_0^{\delta} d\alpha \left( \frac{d\tau}{d\alpha} \right) \int_c \frac{dz}{2\pi i} \frac{dz}{dw} J^{--}(z)$$

$$(G_0^+ \Phi_A(-\alpha^{-1}) \tilde{G}_0^+ \Phi_B(-\alpha) + \tilde{G}_0^+ \Phi_A(-\alpha^{-1}) G_0^+ \Phi_B(-\alpha))$$

$$(G_0^+ \Phi_C(\alpha) \tilde{G}_0^+ \Phi_D(\alpha^{-1}) + \tilde{G}_0^+ \Phi_C(\alpha) G_0^+ \Phi_D(\alpha^{-1}))$$

where $\tilde{c}$ is the Giddings map of the contour $c$ and we have used that $J^{--}$ has conformal weight 2 so $J^{--}(z) = (\frac{dz}{dw})^2 J^{--}(w)$. The overall minus sign comes from the fact that $\alpha$ decreases as $\tau$ increases.

4.2 Computation of t-channel contribution

Performing a re-identification of the external strings with the external legs, one finds that the t-channel contribution to the scattering amplitude is

$$A_t = \frac{1}{4} \int_0^{\infty} d\tau \int_c \frac{dw}{2\pi i} \frac{dz}{dw} J^{--}(w)$$

$$(G_0^+ \Phi_D(\alpha^{-1}) \tilde{G}_0^+ \Phi_A(\alpha) + \tilde{G}_0^+ \Phi_D(\alpha^{-1}) G_0^+ \Phi_A(\alpha))$$

$$(G_0^+ \Phi_B(\alpha) \tilde{G}_0^+ \Phi_C(\alpha^{-1}) + \tilde{G}_0^+ \Phi_B(\alpha) G_0^+ \Phi_C(\alpha))$$

where we have now used the Giddings map with the ordering $[(1), (2), (3), (4)] \to [-\alpha^{-1}, -\alpha, \alpha, \alpha^{-1}]$.

If there were no $G_0^+$ and $\tilde{G}_0^+$ operators, one could sum $A_s$ and $A_t$ to get an integral $\int_0^{1} d\alpha f(\alpha)$. This is what happens in the open bosonic string four-point amplitude where $\int_0^{1} d\alpha f(\alpha)$ can be related to the Veneziano amplitude using the fact that

$$\frac{d\tau}{d\alpha} \int_c \frac{dz}{2\pi i} \frac{dz}{dw} b(z) = \int d^2 z \mu_\alpha(z, \bar{z}) b(z)$$
where \( \mu_\alpha \) is the Beltrami differential corresponding to the \( \alpha \) modulus \([11]\). However, for the open superstring four-point amplitude, one first has to perform contour deformations of the \( G^+_0 \) and \( \tilde{G}^+_0 \) operators in \( \mathcal{A}_t \) until they appear in the same manner as in the expression for \( \mathcal{A}_s \). As will now be shown, these contour deformations produce a finite contact term which is cancelled by the contribution from the quartic vertex.

Consider the expression

\[
Y = \frac{d\tau}{d\alpha} \langle \int \frac{dz}{2\pi i} (\frac{dz}{dw}) J^{--}(z) G^+_0 \Phi_D(\alpha^{-1}) \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) \tilde{G}^+_0 \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \rangle
\]

\[
= -\langle (\int d^2z \mu_\alpha(z, \bar{z}) J^{--}(z)) \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) \tilde{G}^+_0 \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) G^+_0 \Phi_D(\alpha^{-1}) \rangle
\]

which is one of the terms in \( \mathcal{A}_t \). Deforming the \( G^+ \) contour off of \( \Phi_D(\alpha^{-1}) \), one gets

\[
Y = -\langle (\int d^2z \mu_\alpha(z, \bar{z}) \tilde{G}^-(z)) \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) \tilde{G}^+_0 \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \Phi_D(\alpha^{-1}) \rangle.
\]

Now deform the contour of \( \tilde{G}^+ \) off of \( \Phi_B(-\alpha) \) to get

\[
Y = -\langle (\int d^2z \mu_\alpha(z, \bar{z}) \tilde{G}^-(z)) \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \tilde{G}^+_0 \Phi_D(\alpha^{-1}) \rangle
\]

\[
+\langle (\int d^2z \mu_\alpha(z, \bar{z}) T(z)) \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \Phi_D(\alpha^{-1}) \rangle
\]

where \( T \) is the stress-tensor. Finally, writing \( \tilde{G}^-(z) = [G^+_0, J^{--}(z)] \) and deforming the \( G^+ \) contour off of \( J^{--}(z) \), one gets

\[
Y = -\langle (\int d^2z \mu_\alpha(z, \bar{z}) J^{--}(z)) \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) G^+_0 \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \tilde{G}^+_0 \Phi_D(\alpha^{-1}) \rangle
\]

\[
+\frac{\partial}{\partial \alpha} \langle \tilde{G}^+_0 \Phi_A(-\alpha^{-1}) \Phi_B(-\alpha) G^+_0 \Phi_C(\alpha) \Phi_D(\alpha^{-1}) \rangle
\]

where we have used that \( \int d^2z \mu_\alpha(z, \bar{z}) T(z) \) has the effect of taking the variation with respect to the modulus \( \alpha \).

Similarly,

\[
\frac{d\tau}{d\alpha} \langle \int \frac{dz}{2\pi i} (\frac{dz}{dw}) J^{--}(z) \tilde{G}^+_0 \Phi_D(\alpha^{-1}) G^+_0 \Phi_A(-\alpha^{-1}) G^+_0 \Phi_B(-\alpha) \tilde{G}^+_0 \Phi_C(\alpha) \rangle
\]

(8)
\[
\langle - \left( \int d^2 z \mu_\alpha (z, \bar{z}) J^- (z) \right) \rangle \\
G_0^+ \Phi_A (-\alpha^{-1}) \tilde{G}_0^+ \Phi_B (-\alpha) \tilde{G}_0^+ \Phi_C (\alpha) \ G_0^+ \Phi_D (\alpha^{-1}) \\
- \frac{\partial}{\partial \alpha} \langle G_0^+ \Phi_A (-\alpha^{-1}) \Phi_B (-\alpha) \tilde{G}_0^+ \Phi_C (\alpha) \Phi_D (\alpha^{-1}) \rangle.
\]

Plugging (7) and (8) into (6), one finds

\[
A_t = - \frac{1}{4} \int_0^1 d\alpha \langle \left( \int d^2 z \mu_\alpha (z, \bar{z}) J^-(z) \right) \rangle 
\]

\[
(G_0^+ \Phi_A (-\alpha^{-1}) \tilde{G}_0^+ \Phi_B (-\alpha) + \tilde{G}_0^+ \Phi_A (-\alpha^{-1}) \ G_0^+ \Phi_B (-\alpha)) \\
(G_0^+ \Phi_C (\alpha) \tilde{G}_0^+ \Phi_D (\alpha^{-1}) + \tilde{G}_0^+ \Phi_C (\alpha) \ G_0^+ \Phi_D (\alpha^{-1})) \\
+ \frac{1}{4} \int_\delta^1 d\alpha \frac{\partial}{\partial \alpha} \langle \tilde{G}_0^+ \Phi_A (-\alpha^{-1}) \Phi_B (-\alpha) \tilde{G}_0^+ \Phi_C (\alpha) \Phi_D (\alpha^{-1}) \rangle \\
- G_0^+ \Phi_A (-\alpha^{-1}) \Phi_B (-\alpha) \tilde{G}_0^+ \Phi_C (\alpha) \Phi_D (\alpha^{-1}) \rangle.
\]

Comparing with (5), one finds

\[
A_s + A_t = - \frac{1}{4} \int_0^1 d\alpha \langle \left( \int d^2 z \mu_\alpha (z, \bar{z}) J^-(z) \right) \rangle 
\]

\[
(G_0^+ \Phi_A (-\alpha^{-1}) \tilde{G}_0^+ \Phi_B (-\alpha) + \tilde{G}_0^+ \Phi_A (-\alpha^{-1}) \ G_0^+ \Phi_B (-\alpha)) \\
(G_0^+ \Phi_C (\alpha) \tilde{G}_0^+ \Phi_D (\alpha^{-1}) + \tilde{G}_0^+ \Phi_C (\alpha) \ G_0^+ \Phi_D (\alpha^{-1})) \\
- \frac{1}{4} \langle \tilde{G}_0^+ \Phi_A (-\delta^{-1}) \Phi_B (-\delta) \tilde{G}_0^+ \Phi_C (\delta) \Phi_D (\delta^{-1}) \\
- G_0^+ \Phi_A (-\delta^{-1}) \Phi_B (-\delta) \tilde{G}_0^+ \Phi_C (\delta) \Phi_D (\delta^{-1}) \rangle.
\]

Note that we have used the “cancelled propagator” argument of [3] to ignore the surface term coming from \( \alpha = 1 \), but this argument cannot be used to ignore the surface term coming from \( \alpha = \delta \).
4.3 Computation of quartic vertex contribution

It will now be shown that this surface term from $\alpha = \delta$ is precisely cancelled by the contribution $A_q$ from the quartic vertex. Using the action of (3) and including only contributions with the cyclic order $[A,B,C,D]$,

$$A_q = - \sum_{cyclic} \frac{1}{24} \left( \Phi_A(-\delta^{-1}) G_0^+ \Phi_B(-\delta) - G_0^+ \Phi_A(-\delta^{-1}) \Phi_B(-\delta) \right) \tag{11}$$

$$\left( \Phi_C(\delta) \tilde{G}_0^+ \Phi_D(\delta^{-1}) - \tilde{G}_0^+ \Phi_C(\delta) \Phi_D(\delta^{-1}) \right)$$

where $\sum_{cyclic}$ denotes a sum over the four cyclic permutations of $[A,B,C,D]$. Using properties of the cyclic sum, the sixteen terms in $A_q$ can be reordered as

$$A_q = - \sum_{cyclic} \frac{1}{24} \left( 2\Phi_A G_0^+ \Phi_B \Phi_C \tilde{G}_0^+ \Phi_D \right.$$

$$- \Phi_A G_0^+ \Phi_B \tilde{G}_0^+ \Phi_C \Phi_D - G_0^+ \Phi_A \Phi_B \Phi_C \tilde{G}_0^+ \Phi_D \left. \right)$$

where we have suppressed the $z$ location of the four external strings. Note that the quartic vertex is invariant under cyclic rotations of these $z$ locations. Deforming the contour of $\tilde{G}^+$ off of $\Phi_C$ in the second term, one gets

$$A_q = - \sum_{cyclic} \frac{1}{24} \left( 3\Phi_A G_0^+ \Phi_B \Phi_C \tilde{G}_0^+ \Phi_D \right.$$  

$$- \tilde{G}_0^+ \Phi_A G_0^+ \Phi_B \Phi_C \Phi_D - G_0^+ \Phi_A \Phi_B \Phi_C \tilde{G}_0^+ \Phi_D \right) \tag{12}$$

But the last two terms of (12) cancel each other after summing over cyclic permutations, so

$$A_q = - \sum_{cyclic} \frac{1}{8} \left( \Phi_A G_0^+ \Phi_B \Phi_C \tilde{G}_0^+ \Phi_D \right) \tag{13}$$

$$= - \frac{1}{8} \left( \Phi_A G_0^+ \Phi_B \Phi_C \tilde{G}_0^+ \Phi_D - \Phi_A \tilde{G}_0^+ \Phi_B \Phi_C G_0^+ \Phi_D \right) + \frac{X}{8}$$

where

$$X = \left( \tilde{G}_0^+ \Phi_A \Phi_B G_0^+ \Phi_C \Phi_D - G_0^+ \Phi_A \Phi_B \tilde{G}_0^+ \Phi_C \Phi_D \right).$$
Deforming the $G^+$ contour off of $\Phi_B$ in the first term and then deforming the $\tilde{G}^+$ contour off of $\Phi_D$, one finds

$$A_q = -\frac{1}{8} \langle - G^+_0 \Phi_A \Phi_B \Phi_C \tilde{G}^+_0 \Phi_D$$

$$= -\Phi_A \Phi_B G^+_0 \Phi_C \tilde{G}^+_0 \Phi_D - \Phi_A \tilde{G}^+_0 \Phi_B \Phi_C G^+_0 \Phi_D \rangle + \frac{X}{8}$$

Finally, deforming the $G^+_0$ contour off of $\Phi_D$ in the last term, one finds

$$A_q = \frac{X}{4}. \quad (14)$$

So combining (14) with (10),

$$A = A_s + A_t + A_q = -\frac{1}{4} \int_0^1 d\alpha \langle ( \int d^2z \mu_\alpha (z, \bar{z}) J^{--} (z)) \rangle \quad (15)$$

where we have used contour deformations of $G^+$ and $\tilde{G}^+$ to write all four terms in the same form. Note that there are no surface term contributions coming from these deformations due to the “cancelled propagator” argument which can be used when $\alpha = 0$ or $\alpha = 1$.

So we have proven our claim that, after including the contribution from the quartic vertex of the action of (1), the second-quantized prescription agrees with the first-quantized prescription for on-shell tree-level four-point open superstring scattering amplitudes.

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