A MIXED METHOD FOR TIME-TRANSIENT ACOUSTIC WAVE PROPAGATION IN METAMATERIALS

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Abstract. In this paper we develop a finite element method for acoustic wave propagation in Drude-type metamaterials. The governing equation is written as a symmetrizable hyperbolic system with auxiliary variables. The standard mixed finite elements and discontinuous finite elements are used for spatial discretization, and the Crank–Nicolson scheme is used for time discretization. The a priori error analysis of fully discrete scheme is carried out in details. Numerical experiments illustrating the theoretical results and metamaterial wave propagation, are included.

1. Introduction

Metamaterials usually mean the materials with artificial micro/nano-scale structures which show unconventional macro-scale material properties which are not observed in natural materials. The unconventional material properties of metamaterials have many potential applications in wave propagation. For example, cloaking devices, which hide internal objects from external detection using wave refraction, can be made by an appropriate design of metamaterial device. Therefore devising metamaterials and its numerical simulations are research topics of great interest nowadays.

There are three major classes of metamaterials, which are for acoustic, electromagnetic, and elastodynamic wave propagation. In this paper we only consider acoustic wave propagation in metamaterials. In time-harmonic cases some of these wave propagation equations coincide under special circumstances but they are all different in time transient wave propagation. For the theory of electromagnetic metamaterials and time-domain finite element methods we refer to, e.g., [3] [11] [12] [19] and the references in [10] for more comprehensive list of previous studies. There are also previous studies on elastodynamic metamaterials in, e.g., [13] [14] [17].

To the best of our knowledge there are very limited number of previous studies on numerical methods for time transient acoustic wave propagation in metamaterials. In [3] some acoustic metamaterial models, the acoustic counterpart of doubly negative index materials in electromagnetics [12] [19], are studied. In the paper the authors proposed a form of symmetrizable hyperbolic system as the governing equations of acoustic wave propagation in metamaterials. In addition, they proved existence of weak solutions and showed numerical experiments with the finite difference method.

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In this paper we develop a finite element method for the system proposed in [3] and prove the a priori error analysis. For spatial discretization we use the mixed finite element for the Poisson equation and some discontinuous finite element spaces. To circumvent lower convergence rate of the pressure variable in some mixed finite element pairs, we propose a novel local post-processing which gives numerical pressure with higher order approximation properties (See Subsection 3.3).

The paper is organized as follows. In Section 2 we first introduce symbols and notation in the paper, and then present the governing equations for the acoustic wave propagation in metamaterials as well as the energy estimate. In Section 3 we introduce finite element discretization for the system and prove the a priori error analysis. In particular, we show that a local post-processing for the pressure variable can be used to obtain a numerical pressure which has better approximation property than the original numerical pressure. In Section 4 we present the results of numerical experiments which illustrate our theoretical results and exotic wave propagation in metamaterials.

2. Preliminaries

2.1. Notation. Let Ω ⊂ R^d, d = 2 or 3, be a polygonal/polyhedral domain with Lipschitz boundary. Throughout this paper we assume that T_h is a triangulation of Ω without hanging nodes.

We use L^r(Ω) to denote the Lebesgue space with the norm

\[ \|v\|_{L^r} = \begin{cases} \left( \int_Ω |v|^r \, dx \right)^{1/r}, & \text{if } 1 \leq r < \infty, \\ \text{esssup}_{x \in Ω} \{v(x)\}, & \text{if } r = \infty. \end{cases} \]

For a domain D ⊂ Ω, L^2(D) and L^2(D; R^d) be the sets of R- and R^d-valued square integrable functions with inner products \( (v, v')_D := \int_D vv' \, dx \) and \( (v, v')_D := \int_D v \cdot v' \, dx \). We will use \( (\cdot, \cdot)_D \) instead of \( (\cdot, \cdot)_D \) if \( D = \Omega \). For an integer \( t \geq 0 \) \( P_t(D) \) and \( P_t(D; R^d) \) are the spaces of R- and R^d-valued polynomials of degree \( \leq t \) on \( D \).

In the paper \( H^s(D), s \geq 0, \) denotes the Sobolev space based on the \( L^2 \)-norm with s-differentiability on the domain \( D \). We refer to [7] for a rigorous definition of this space. The norm on \( H^s(D) \) is denoted by \( \|\cdot\|_{s,D} \) and \( D \) is omitted if \( D = \Omega \). If \( \rho \) is a nonnegative function in \( L^∞(Ω) \), then \( \|v\|_\rho \) and \( \|v\|_\rho \) denotes the \( \rho \)-weighted \( L^2 \)-norms \( (\int_Ω \rho|v|^2 \, dx)^{1/2} \) and \( (\int_Ω \rho v \cdot v \, dx)^{1/2} \).

For \( T > 0 \) and a separable Hilbert space \( X \), let \( C^m([0,T]; X) \) denote the set of functions \( f : [0, T] \rightarrow X \) that are continuous in \( t \in [0, T] \). For an integer \( m \geq 1 \), we define

\[ C^m([0,T]; X) = \{ f | \partial^i f / \partial t^i \in C^0([0, T]; X) \}, \quad 0 \leq i \leq m \}, \]

where \( \partial^i f / \partial t^i \) is the i-th time derivative in the sense of the Fréchet derivative in \( X \) (cf. [20]). For a function \( f : [0, T] \rightarrow X \), the Bochner norm is defined as

\[ \|f\|_{L^r(0, T; X)} = \begin{cases} \left( \int_0^T \|f(s)\|_X^r \, ds \right)^{1/r}, & 1 \leq r < \infty, \\ \text{esssup}_{t \in (0, T)} \|f(t)\|_X, & r = \infty. \end{cases} \]

We define \( W^{k,r}(0, T; X) \) for a non-negative integer \( k \) and \( 1 \leq r \leq \infty \) as the closure of \( C^k([0, T]; X) \) with the norm \( \|f\|_{W^{k,r}(0, T; X)} = \sum_{i=0}^k \|\partial^i f / \partial t^i\|_{L^r(0, T; X)} \). The semi-norm \( \|f\|_{W^{k,r}(0, T; X)} \) is defined by \( \|f\|_{W^{k,r}(0, T; X)} = \|\partial^k f / \partial t^k\|_{L^r(0, T; X)}. \)
Finally, for a normed space $X$ with its norm $\| \cdot \|_X$ and functions $f_1, f_2 \in X$, $\| f_1, f_2 \|_X$ will be used to denote $\| f_1 \|_X + \| f_2 \|_X$, and $\| f_1, f_2, f_3 \|_X$ is defined similarly.

2.2. A metamaterial model of acoustic wave propagation. A system of equations for the acoustic wave propagation with velocity and pressure unknowns is

$$
\rho \frac{\partial v}{\partial t} + \text{grad} p = f,
$$

$$
\kappa^{-1} \frac{\partial p}{\partial t} + \text{div} v = g
$$

with the density $\rho$ and the bulk modulus $\kappa$. In conventional material models the coefficients $\rho$ and $\kappa$ are fixed uniformly positive functions in $\Omega$. In this paper we are interested in metamaterial models such that $\rho$ and $\kappa^{-1}$ are frequency-dependent, more precisely, the temporal Fourier transform of the equations with frequency $\omega$ satisfy

$$
-i\omega \hat{\rho}(\omega) \hat{v}(\omega) + \text{grad} \hat{p}(\omega) = \hat{f}(\omega),
$$

$$
-i\omega \hat{\kappa}^{-1}(\omega) \hat{p}(\omega) + \text{div} \hat{v}(\omega) = \hat{g}(\omega)
$$

with

$$
\hat{\rho}(\omega) = \rho_a \left(1 - \frac{\Omega_\rho^2}{\omega^2 - \omega_\rho^2}\right),
$$

$$
\hat{\kappa}^{-1}(\omega) = \kappa_a^{-1} \left(1 - \frac{\Omega_\kappa^2}{\omega^2 - \omega_\kappa^2 + i\gamma\omega}\right), \quad \gamma \geq 0
$$

where $\rho_a, \kappa_a > 0$ are functions in $\Omega$ with uniform positive lower bounds, $\Omega_\rho \geq 0$, $\Omega_\kappa \geq 0$ are functions in $\Omega$, $\omega_\rho > 0$, $\omega_\kappa > 0$, $\gamma \geq 0$ are constants in $\Omega$, and $\hat{v}, \hat{p}, \hat{f}, \hat{g}$ are the temporal Fourier transforms of $v, p, f, g$, respectively.

To obtain a system of time-dependent equations we introduce new variables $u, w, q, r$ satisfying

$$
i\omega \hat{v}(\omega) = (\omega^2 - \omega_\rho^2)\hat{u}(\omega), \quad -i\omega \hat{u}(\omega) = \hat{v}(\omega),
$$

$$
i\omega \hat{p}(\omega) = (\omega^2 + i\gamma\omega - \omega_\kappa^2)\hat{q}(\omega), \quad -i\omega \hat{q}(\omega) = \hat{p}(\omega).
$$

The system of time-dependent equations are

$$
\rho_a \frac{\partial v}{\partial t} + \text{grad} p + \rho_a \Omega_\rho^2 u = f,
$$

$$
\kappa_a^{-1} \frac{\partial p}{\partial t} + \text{div} v + \kappa_a^{-1} \Omega_\kappa^2 q = g,
$$

$$
\frac{\partial u}{\partial t} - v + \omega_\rho^2 w = 0,
$$

$$
\frac{\partial w}{\partial t} - u = 0,
$$

$$
\frac{\partial q}{\partial t} - p + \gamma q + \omega_\kappa^2 r = 0,
$$

$$
\frac{\partial r}{\partial t} - q = 0.
$$

We assume that the material of wave propagation in $\Omega$ consists of a conventional positive index material (PIM) on a subdomain $\Omega_P \subset \Omega$ and a negative index material (NIM) on $\Omega \setminus \Omega_P$. The PIM and NIM materials are mathematically modeled by the values of $\Omega_\rho$ and $\Omega_\kappa$, i.e., $\Omega_\rho = \Omega_\kappa = 0$ on $\Omega_P$ and $\Omega_\rho, \Omega_\kappa > 0$ on $\Omega \setminus \Omega_P$. Note that the first two equations in (2.1) are decoupled from the other equations on the
domain $\Omega_P$ because $\Omega_P = \Omega_\kappa = 0$ on $\Omega_P$. From this observation we may develop numerical methods which solve different sets of equations on the PIM and NIM domains. However, we will focus on a monolithic numerical method for the system because monolithic approaches can cover problems with varying interfaces between PIM and NIM in a unified manner. They can be used for shape optimization problems for metamaterial device design, which is our future research interest.

For boundary conditions of (2.1) let $\Gamma_D, \Gamma_N$ be the subsets of $\partial \Omega$ such that $\Gamma_D \cap \Gamma_N = \emptyset, \Gamma_D \cup \Gamma_N = \partial \Omega$. Then imposed boundary conditions are

$$p(t) = p_D(t) \quad \text{on} \quad \Gamma_D, \quad \mathbf{v}(t) \cdot \mathbf{n} = v_N(t) \quad \text{on} \quad \Gamma_N$$

with given functions $p_D$ on $(0, T] \times \Gamma_D$ and $v_N$ on $(0, T] \times \Gamma_N$, where $\mathbf{n}$ is the unit outward normal vector field on $\Gamma_N$.

To write a variational form of the system let us define function spaces

$$W = L^2(\Omega; \mathbb{R}^4), \quad Q = L^2(\Omega), \quad V = \{ \mathbf{v} \in W : \text{div} \mathbf{v} \in L^2(\Omega) \}$$

where $\text{div} \mathbf{v}$ is defined in the sense of distributions. For future reference we define $\mathcal{X} := V \times Q \times W \times Q \times Q$ with the norm induced by the $L^2$ norms of the function spaces. We also define $\rho_u, \rho_w, \rho_q, \rho_r$ to denote (nonnegative) weights

$$\rho_u = \rho_u \Omega_\kappa^2, \quad \rho_w = \rho_w \omega_p^2 \Omega_\kappa^2, \quad \rho_q = \kappa_q^{-1} \Omega_\kappa^2, \quad \rho_r = \kappa_r^{-1} \omega_r^2 \Omega_\kappa^2$$

in the rest of this paper.

For well-posedness of (2.1) we recall the following result from [3, Theorem 3.1].

**Theorem 2.1.** For (2.1) suppose that initial data $(\mathbf{v}(0), p(0), u(0), w(0), q(0), r(0)) \in \mathcal{X}$ satisfy $\mathbf{v}(0), u(0), w(0) \in H^1(\Omega; \mathbb{R}^4), \quad p(0), q(0), r(0) \in H^1(\Omega)$. In addition, suppose that $f \in C^1([0, T]; W)$, $g \in C^1([0, T]; Q)$ hold. Then there exists a unique solution

$$(v, p, u, w, q, r) \in C^1([0, T]; \mathcal{X}) \cap C^0([0, T]; \mathcal{X})$$

for the given initial data and $f$, $g$.

For finite element discretization we need to consider a variational form of (2.1). For simplicity of presentation we assume the homogeneous boundary condition

$$p(t) = 0 \quad \text{on} \quad \partial \Omega$$

for all $t \in (0, T]$.

For simplicity of presentation we will use $\dot{\mathbf{v}}$ instead of $\partial \mathbf{v}/\partial t$ in the rest of this paper.

**Definition 2.2.** For $f \in L^1((0, T]; W)$, $g \in L^1((0, T); Q)$, we say $(\mathbf{v}, p, u, w, q, r) \in H^1((0, T], \mathcal{X}) \cap L^2((0, T]; \mathcal{X})$ a weak solution of (2.1) if it satisfies

$$(\rho_u \dot{\mathbf{v}}, \mathbf{v}') - (p, \text{div} \mathbf{v}') + (\rho_u \mathbf{u}, \mathbf{v}') = (f, \mathbf{v}'), \quad (\kappa_a^{-1} \dot{p}, p') + (\text{div} \mathbf{v}, p') + (\rho_q \mathbf{q}, p') = (g, p'), \quad (\dot{u}, \mathbf{u}') - (v, \mathbf{u}') + (\omega_u^2 \mathbf{w}, \mathbf{u}') = 0, \quad (\mathbf{w}, \mathbf{w}') - (\mathbf{u}, \mathbf{u}') = 0, \quad (\dot{q}, q') - (p, q') + (\gamma q, q') + (\omega_q^2 r, q') = 0, \quad (\dot{r}, r') - (q, r') = 0$$

for $(\mathbf{v}', p', u', w', q', r') \in \mathcal{X}$ and for almost every $t \in (0, T)$. 

One can easily check by the integration by parts that the solution in Theorem 2.1 with the boundary condition (2.4) is a weak solution satisfying (2.5).

We remark that the above variational form can cover general boundary conditions with some necessary modifications. For the boundary condition (2.2) we replace $V$ by

$$V_N = \{ v \in W : \text{div} v \in L^2(\Omega), v \cdot n = v_N \text{ on } \Gamma_N \}$$

and replace (2.5a) by

$$(\rho_\alpha v, v') - (p, \text{div} v') + (\rho_\alpha u, v') = (f, v') - \int_{\Gamma_D} p_D v' \cdot n \, ds$$

for the test function $v'$ in $V_N^0 := \{ v \in W : \text{div} v \in L^2(\Omega), v \cdot n = 0 \text{ on } \Gamma_N \}$.

**Theorem 2.3.** If $(v, p, u, w, q, r)$ is a solution of (2.1) satisfying (2.3), then

$$(2.6) \quad \|v, p, u, w, q, r\|_{L^\infty((0,T);X)} \leq C_1 \|v(0), p(0), u(0), w(0), q(0), r(0)\|_X + C_2 \|f, g\|_{L^1((0,T);W \times Q)}$$

holds with $C_2$ which may depend on $T$. Moreover, if we define

$$(2.7) \quad E_0(t)^2 = \|u(t)\|_{\rho_\alpha}^2 + \|v(t)\|_{\rho_\alpha}^2 + \|w(t)\|_{\rho_\alpha}^2 + \|p(t)\|_{\kappa^{-1}}^2 + \|q(t)\|_{\rho_\alpha}^2 + \|r(t)\|_{\rho_\alpha}^2$$

with the weighted (semi)-norm $\|\cdot\|_{\rho_\alpha} = \rho_u, \rho_\alpha, \rho_w, \kappa^{-1}, \rho_q, \rho_r$, then

$$(2.8) \quad E_0(t) \leq E_0(0) + C \int_0^t (\|f(s)\|_0 + \|g(s)\|_0) \, ds$$

with $C > 0$ depending only on $\|\rho_\alpha^{-1}\|_{L^\infty}$ and $\|\kappa\|_{L^\infty}$.

**Proof.** Recall that $(v, p, u, w, q, r)$ satisfies (2.5). If we choose $(v', p', u', w', q', r') = (v, p, u, w, q, r)$ in (2.5) and add all the equations, then we get

$$(2.9) \quad \frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{\rho_\alpha}^2 + \|v(t)\|_{\rho_\alpha}^2 + \|w(t)\|_{\rho_\alpha}^2 + \|p(t)\|_{\kappa^{-1}}^2 + \|q(t)\|_{\rho_\alpha}^2 + \|r(t)\|_{\rho_\alpha}^2 \right) + (\gamma q, q)$$

$$+ \left( (\rho_\alpha - 1) u, v \right) + \left( (\rho_q - 1) q, p \right)$$

$$+ \left( (\omega^2 - 1) w, u \right) + \left( (\omega^2 - 1) - 1) r, q \right) = (f, v) + (g, p).$$

Let $E_1(t)^2 = \|u(t)\|_{\rho_\alpha}^2 + \|v(t)\|_{\rho_\alpha}^2 + \|w(t)\|_{\rho_\alpha}^2 + \|p(t)\|_{\kappa^{-1}}^2 + \|q(t)\|_{\rho_\alpha}^2 + \|r(t)\|_{\rho_\alpha}^2$. The Cauchy–Schwarz inequality with the above identity gives

$$\frac{d}{dt} E_1(t)^2 \leq CE_1(t)^2 + \|f(t)\|_0 + \|g(t)\|_0 E_1(t)$$

with $C$ depending on $\rho_u, \rho_q, \omega_\rho, \omega_\kappa$. By Gronwall lemma one can obtain

$$E_1(t) \leq E_1(0) + C(t) \int_0^t (\|f(s)\|_0 + \|g(s)\|_0) \, ds.$$

Then (2.6) follows from the equivalence of

$$(\sqrt{E_1(t)} \text{ and } \|u(t), v(t), w(t), p(t), q(t), r(t)\|_{X}).$$

To prove (2.8) we choose $(v', p', u', w', q', r') = (v, p, u, w, q, r)$ in (2.5) and add all the equations. Then we get

$$\frac{1}{2} \frac{d}{dt} E_0(t)^2 + (\gamma q, q) = (f, v) + (g, p).$$
If \( E_0(t) \leq E_0(0) \), then there is nothing to prove, so we assume \( E_0(t) > E_0(0) \) and will prove (2.8) in the rest of the proof.

First, we prove (2.8) assuming that \( E_0(t) = \text{esssup}_{s \in [0,t]} E_0(s) \). Since \((\gamma \rho q, q) \geq 0\), integration of the above identity from 0 to \( t \) gives

\[
E_0(t)^2 - E_0(0)^2 \leq 2 \max\{\|\rho_a^{-1}\|_{L^\infty}, \|\kappa_a\|_{L^\infty}\} \int_0^t (\|f(s)\|_0 + \|g(s)\|_0) E_0(s) \, ds
\]

Then

\[
E_0(t) \leq \frac{E_0(0)^2}{E_0(t)} + 2 \max\{\|\rho_a^{-1}\|_{L^\infty}, \|\kappa_a\|_{L^\infty}\} \int_0^t (\|f(s)\|_0 + \|g\|_0) \, ds
\]

(2.10)

\[
\leq E_0(0) + 2 \max\{\|\rho_a^{-1}\|_{L^\infty}, \|\kappa_a\|_{L^\infty}\} \int_0^t (\|f(s)\|_0 + \|g(s)\|_0) \, ds
\]

which proves (2.8).

If \( 0 < E_0(t) < \text{esssup}_{s \in [0,t]} E_0(s) \), then there exists \( 0 \leq t_0 < t \) such that \( E_0(t_0) = \text{esssup}_{s \in [0,t_0]} E_0(s) \) and \( E_0(t) < E_0(t_0) \). By the same argument as above, we can obtain the inequality (2.10) for \( E_0(t_0) \). Then

\[
E_0(t) < E_0(t_0) \leq E_0(0) + 2 \max\{\|\rho_a^{-1}\|_{L^\infty}, \|\kappa_a\|_{L^\infty}\} \int_0^{t_0} (\|f(s)\|_0 + \|g(s)\|_0) \, ds
\]

\[
\leq E_0(0) + 2 \max\{\|\rho_a^{-1}\|_{L^\infty}, \|\kappa_a\|_{L^\infty}\} \int_0^{t_0} (\|f(s)\|_0 + \|g(s)\|_0) \, ds,
\]

so (2.8) is proved.

By observing (2.1a) and (2.1b), the auxiliary variables \((u, w, q, r)\) interact with \((v, p)\) only on the NIM domain on which \(\Omega_p\) and \(\Omega_\kappa\) are strictly positive. In fact, the physical meaning of \((u, w, q, r)\) on \(\Omega_P\) is not clear, so there is no natural way to determine the initial data of \((u, w, q, r)\) on \(\Omega_P\). In the following theorem we show that \((v, p)\) in (2.5) is independent on the initial data of \((u, w, q, r)\) on \(\Omega_P\). As a consequence, any choice of initial data \((u, w, q, r)\) on \(\Omega_P\) is allowed to obtain a unique \((v, p)\). This argument can be extended to our numerical scheme, so there is no concern in the choice of numerical initial data of \((u, w, q, r)\) on \(\Omega_P\).

**Theorem 2.4.** Given \(f \in L^1((0,T);W)\), \(g \in L^1((0,T);Q)\), and initial data \(U(0) \in \mathcal{X}\), (2.1) has a unique weak solution. In addition, suppose that \(U_i \in H^1((0,T);X) \cap L^2((0,T);X), i = 1, 2\) are the weak solutions for the two sets of initial data \(U_i(0) \in \mathcal{X}, i = 1, 2\). For \(U_i(t) := (v_i(t), q_i(t), w_i(t), r_i(t))\) with \(i = 1, 2\), if

\[
(2.11) \quad v_1(0) = v_2(0), \quad p_1(0) = p_2(0) \quad \text{on } \Omega,
\]

\[
(2.12) \quad u_1(0) = u_2(0), \quad w_1(0) = w_2(0), \quad q_1(0) = q_2(0), \quad r_1(0) = r_2(0) \quad \text{on } \Omega \setminus \overline{\Omega_P},
\]

then the same identities hold for the weak solutions \(U_1(t), U_2(t)\) for \(t \in (0,T]\).

**Proof.** If \(U_i, i = 1, 2\) are weak solutions for given \(f, g\), and initial data \(U(0) \in \mathcal{X}\). Then \(U_1 - U_2\) is a weak solution for \(f = 0, g = 0\), and zero initial data. Then \(U_1 = U_2\) follows by (2.6).
To prove the second part of the assertion, let \( U_i, i = 1, 2 \) be the weak solutions for the initial data
\[
(v_i(0), p_i(0), u_i(0), w_i(0), q_i(0), r_i(0)), \quad i = 1, 2
\]
satisfying (2.11) and (2.12). Let \((v, p, u, w, q, r)\) be the difference \( U_1 - U_2 \) and define \( E_0(t) \) as in (2.7). Then \( E_0(t) \) satisfies (2.8) with \( f = 0 \) and \( g = 0 \). Moreover, \( E_0(0) = 0 \) because of (2.11), (2.12), so \( E_0(t) = 0 \) holds for all \( t \in (0, T] \) and the assertion follows. \( \square \)

3. Finite elements discretization and error analysis

3.1. Finite elements for spatial discretization. Recall that \( \mathcal{T}_h \) is a triangulation of \( \Omega \) without hanging nodes. In the rest of this paper we assume that the density functions \( \rho_{\sigma} \) with \( \sigma = u, w, q, r \) are in \( W_h^{1,\infty}(\mathcal{T}_h) \) where
\[
W_h^{1,\infty}(\mathcal{T}_h) := \{ \rho \in L^2(\Omega) : \rho|_K \in L^\infty(K), \text{ grad}(\rho|_K) \in L^\infty(K; \mathbb{R}^d) \quad \forall K \in \mathcal{T}_h \}
\]
with the norm \( \| \rho \|_{W_h^{1,\infty}} := \sup_{K \in \mathcal{T}_h}(\| \rho|_K \|_{L^\infty(K)} + \| \text{grad}(\rho|_K) \|_{L^\infty(K)}) \).

Finite element discretization of the first order differential equation form of acoustic wave equations, is studied in \([8]\). We extend the approach in \([8]\) to include the BDM\(_1\) element of the first kind if \( k \) is a fixed integer. For given \( k \geq 0 \) we set \( S_k(K) \) as either BDM\(_{k+1}(K)\) or RTN\(_k(K)\), and define \( V_h \) as the finite element space
\[
(3.1) \quad V_h = \{ v \in V : v|_K \in S_k(K), \quad K \in \mathcal{T}_h \}.
\]
By this definition \( V_h \) is the Brezzi–Douglas–Marini or the Nédélec element of the second kind if \( S_k(K) = \text{BDM}_{k+1}(K) \) \([6, 16]\), and is the Raviart–Thomas or the Nédélec element of the first kind if \( S_k(K) = \text{RTN}_k(K) \) \([15, 18]\). For the details on the definition of \( V_h \) with \( S_k(K) = \text{BDM}_{k+1}(K) \) or \( S_k(K) = \text{RTN}_k(K) \), we refer to \([5, 14]\) and the original articles \([6, 15, 16, 18]\). For \( W_h, \) let \( W_h(K) \) be
\[
W_h(K) = \begin{cases} 
\mathcal{P}_{k+1}(K; \mathbb{R}^d) & \text{if } S_k(K) = \text{BDM}_{k+1}(K), \\
\mathcal{P}_k(K; \mathbb{R}^d) & \text{if } S_k(K) = \text{RTN}_k(K),
\end{cases}
\]
and define \( W_h, Q_h \) as
\[
(3.2) \quad W_h = \{ v \in L^2(\Omega; \mathbb{R}^d) : v|_K \in W_h(K) \},
\]
\[
(3.3) \quad Q_h = \{ q \in L^2(\Omega) : q|_K \in \mathcal{P}_k(K) \}.
\]
Let \( X_h \) be \( V_h \times Q_h \times W_h \times W_h \times Q_h \times Q_h \). For \( f \in C^0([0, T]; W_h) \), \( g \in C^0([0, T]; Q) \) the semidiscrete problem is to find \( (v_h, p_h, u_h, w_h, q_h, r_h) \in C^1((0, T], X_h) \)
which satisfies

\begin{align}
(3.4a) \quad & (\rho_a \dot{v}_h, v') - (p_h, \text{div } v') + (\rho_a u_h, v') = (f, v') , \\
(3.4b) \quad & (\kappa_a^{-1} p_h, p') + (\text{div } u_h, p') + (\rho_q q_h, p') = (g, p') , \\
(3.4c) \quad & (\dot{u}_h, u') - (v_h, u') + (\omega_p^2 w_h, u') = 0 , \\
(3.4d) \quad & (\dot{w}_h, w') - (u_h, w') = 0 , \\
(3.4e) \quad & (\dot{q}_h, q') - (p_h, q') + (\gamma q_h, q') + (\omega_p^2 r_h, q') = 0 , \\
(3.4f) \quad & (\dot{r}_h, r') - (q_h, r') = 0
\end{align}

for \((v', p', u', w', q', r') \in X_h\) and for all \(t \in (0, T]\).

We will not discuss an error analysis for semidiscrete solutions in the paper. Instead, we will show a detailed error analysis of fully discrete solutions in the subsection below.

### 3.2. Error analysis of fully discrete solutions

In this subsection we consider fully discrete solutions of (3.4) with the Crank-Nicolson scheme and show the a priori error estimates.

For \(T > 0\) let \(\Delta t = T/N\) for a natural number \(N\) and define \(\{t_n\}_{n=0}^N\) by \(t_n = n \Delta t\). For a variable \(\sigma : [0, T] \to X\) for a Hilbert space \(X\), we will use \(\sigma^n_h\) and \(\sigma^n\) for the numerical solution of \(\sigma\) at \(t_n\) and \(\sigma(t_n)\), respectively. The variable \(\sigma\) can be \(u, v, w, p, q, r\) in the problem. As such, \(f^n\) and \(g^n\) will denote \(f(t_n)\) and \(g(t_n)\) for a time-dependent functions \(f \in C^0([0, T]; W)\) and \(g \in C^0([0, T]; Q)\). For simplicity we will also use the definitions

\[
\dot{v}^{n+\frac{1}{2}} := \frac{1}{\Delta t} (v^{n+1} - v^n), \quad v^{n+\frac{1}{2}} := \frac{1}{2} (v^n + v^{n+1})
\]

for any sequence \(\{v^n\}_{n=0}^N\) with the upper index \(n\).

The Crank-Nicolson scheme of (2.5) is the following: For given \(U^n := (v^n, p^n_h, u^n_h, w^n_h, q^n_h, r^n_h), \quad f^n, \quad f^{n+1}, \quad g^n, \quad g^{n+1}\), we find \(U^{n+1}_h := (v^{n+1}_h, p^{n+1}_h, u^{n+1}_h, w^{n+1}_h, q^{n+1}_h, r^{n+1}_h) \in X_h\) such that

\begin{align}
(3.5a) \quad & (\rho_a \dot{v}_h^{n+\frac{1}{2}}, v') - (p_h^{n+\frac{1}{2}}, \text{div } v') + (\rho_a u_h^{n+\frac{1}{2}}, v') = (f^{n+\frac{1}{2}} + \frac{\Delta t}{2}, v') , \\
(3.5b) \quad & (\kappa_a^{-1} \partial_t p_h^{n+\frac{1}{2}}, p') + (\text{div } u_h^{n+\frac{1}{2}}, p') + (\rho_q q_h^{n+\frac{1}{2}}, p') = (g^{n+\frac{1}{2}} + \frac{\Delta t}{2}, p') , \\
(3.5c) \quad & (\partial_t u_h^{n+\frac{1}{2}}, u') - (v_h^{n+\frac{1}{2}}, u') + (\omega_p^2 w_h^{n+\frac{1}{2}}, u') = 0 , \\
(3.5d) \quad & (\partial_t w_h^{n+\frac{1}{2}}, w') - (u_h^{n+\frac{1}{2}}, w') = 0 , \\
(3.5e) \quad & (\partial_t q_h^{n+\frac{1}{2}}, q') - (p_h^{n+\frac{1}{2}}, q') + (\gamma q_h^{n+\frac{1}{2}}, q') + (\omega_p^2 r_h^{n+\frac{1}{2}}, q') = 0 , \\
(3.5f) \quad & (\partial_t r_h^{n+\frac{1}{2}}, r') - (q_h^{n+\frac{1}{2}}, r') = 0
\end{align}

for all \((v', p', u', w', q', r') \in X_h\).

For the well-definedness of this fully discrete scheme, we show that \(U^{n+1}_h = 0\) if

\begin{align}
U^n_h = 0, \quad f^n = f^{n+1} = 0, \quad g^n = g^{n+1} = 0.
\end{align}
To show it, assume that (3.6) is true. Then (3.5) becomes

\begin{align*}
(3.7a) & \quad \frac{1}{\Delta t} (\rho_n v_{n+1}^l, v') - \frac{1}{2} (p_{h+1}^n, \text{div } v') + \frac{1}{2} (\rho_n w_{h+1}^n, v') = 0, \\
(3.7b) & \quad \frac{1}{\Delta t} (\kappa_n^{-1} p_{h+1}^n, p') + \frac{1}{2} (\text{div } v_{h+1}^n, p') + \frac{1}{2} (\rho_q q_{h+1}^n, p') = 0, \\
(3.7c) & \quad \frac{1}{\Delta t} (u_{h+1}^n, u') - \frac{1}{2} (v_{h+1}^n, u') + \frac{1}{2} (\omega_n^2 w_{h+1}^n, u') = 0, \\
(3.7d) & \quad \frac{1}{\Delta t} (w_{h+1}^n, w') - \frac{1}{2} (u_{h+1}^n, w') = 0, \\
(3.7e) & \quad \frac{1}{\Delta t} (q_{h+1}^n, q') - \frac{1}{2} (p_{h+1}^n, q') + \frac{1}{2} (\gamma q_{h+1}^n, q') + \frac{1}{2} (\omega_n^2 r_{h+1}^n, q') = 0, \\
(3.7f) & \quad \frac{1}{\Delta t} (r_{h+1}^n, r') - \frac{1}{2} (q_{h+1}^n, r') = 0.
\end{align*}

Let $P_h$ and $P_h$ be the standard $L^2$ projections into $Q_h$ and $W_h$. If we take

$$
v' = v_{h+1}^n, \quad p' = p_{h+1}^n, \quad u' = P_h(\rho_u u_{h+1}^n),
$$

$$
u' = \omega_n^2 P_h(\rho_u w_{h+1}^n), \quad q' = P_h(\rho_q q_{h+1}^n), \quad r' = \omega_n^2 P_h(\rho_q q_{h+1}^n),
$$
in (3.7a) and add all the equations, then we get

$$
\frac{1}{\Delta t} \left( \|v_{h+1}^n\|^2_{\rho_v} + \|p_{h+1}^n\|^2_{\rho_p} + \|u_{h+1}^n\|^2_{\rho_u} + \|w_{h+1}^n\|^2_{\rho_w} + \|q_{h+1}^n\|^2_{\rho_q} + \|r_{h+1}^n\|^2_{\rho_r} \right) + \left( \gamma \omega_n^2 \rho_q r_{h+1}^n, r_{h+1}^n \right) = 0,
$$

so $v_{h+1}^n = 0, p_{h+1}^n = 0$. From these, $u_{h+1}^n = w_{h+1}^n = 0$ follows by taking $u' = u_{h+1}^n$ and $w' = \omega_n^2 w_{h+1}^n$ in (3.7c) and (3.7d), and then by adding them. Finally, $q_{h+1}^n = r_{h+1}^n = 0$ follows by taking $q' = q_{h+1}^n$ and $r' = \omega_n^2 r_{h+1}^n$ in (3.7e) and (3.7f), and then by adding them. Therefore, $U_{h+1}^n = 0$.

For the error analysis we use $e_{n}^\sigma = \sigma_n - \sigma_{h}^n$ for the error of variable $\sigma$ ($\sigma = v, u, p, q, r$) at $t = t_n$. For error equations we consider the difference of the average of (2.9) at $t_n$ and $t_{n+1}$, and the fully discrete scheme (3.5). Then the error equations are

\begin{align*}
\rho_n (\dot{v}_{h+1}^n - \partial_t v_{h+1}^n, v') - (c_p^n, \text{div } v') + \frac{1}{2} (\rho_{u} c_{u}^{n+\frac{1}{2}}, v') = 0, \\
\kappa_n (p_{h+1}^n - \partial_t p_{h+1}^n, p') + (\text{div } c_v^n, p') + \frac{1}{2} (\rho_{q} c_{q}^{n+\frac{1}{2}}, p') = 0, \\
(u_{h+1}^n - \partial_t u_{h+1}^n, u') - (c_v^n, u') + \frac{1}{2} (\omega_n^2 c_{w}^{n+\frac{1}{2}}, u') = 0, \\
(w_{h+1}^n - \partial_t w_{h+1}^n, w') - (c_u^n, w') = 0, \\
(q_{h+1}^n - \partial_t q_{h+1}^n, q') - (c_p^n, q') + \frac{1}{2} (\gamma q_{h+1}^n, q') + \frac{1}{2} (\omega_n^2 c_{r}^{n+\frac{1}{2}}, q') = 0, \\
(r_{h+1}^n - \partial_t r_{h+1}^n, r') - (c_q^n, r') = 0.
\end{align*}

For $v' \in H^s(\Omega; \mathbb{R}^d), s > \frac{1}{2}$, we define $\Pi_h$ as the canonical interpolation operators of RTN or BDM element which satisfy

$$
\text{div } \Pi_h v' = P_h \text{div } v', \quad \|v' - \Pi_h v'\|_0 \leq Ch^m \|v'\|_m
$$

with $m := \max\{s, k + 1 + \delta\}$ where $\delta = 1$ if $V_h$ is a BDM element and $\delta = 0$ if $V_h$ is an RTN element.
Using $\Pi_h, P_h, P_h$, we can define the decomposition of errors

\begin{align}
(3.10) & \quad e^n_v = e^{I,n}_v + e^{h,n}_v := (v^n - \Pi_h v^n) + (\Pi_h v^n - v^n_h), \\
(3.11) & \quad e^n_u = e^{I,n}_u + e^{h,n}_u := (u^n - P_h u^n) + (P_h u^n - u^n_h), \\
(3.12) & \quad e^n_w = e^{I,n}_w + e^{h,n}_w := (w^n - P_h w^n) + (P_h w^n - w^n_h), \\
(3.13) & \quad e^n_p = e^{I,n}_p + e^{h,n}_p := (p^n - P_h p^n) + (P_h p^n - p^n_h), \\
(3.14) & \quad e^n_q = e^{I,n}_q + e^{h,n}_q := (q^n - P_h q^n) + (P_h q^n - q^n_h), \\
(3.15) & \quad e^n_r = e^{I,n}_r + e^{h,n}_r := (r^n - P_h r^n) + (P_h r^n - r^n_h).
\end{align}

For estimates of the interpolation errors denoted by $e^{I,n}_\sigma$ for a variable $\sigma$, let us use a generic symbol $I_h \sigma^n$ to denote the interpolation of the exact solution $\sigma^n$ into the corresponding finite element space. More specifically, $I_h = \Pi_h$ if $\sigma = v$, $I_h = P_h$ if $\sigma = u, w$, and $I_h = P_h$ if $\sigma = p, q, r$. Then it holds that

\begin{equation}
\|e^{I,n}_\sigma\|_0 = \|\sigma^n - I_h \sigma^n\|_0 \leq Ch^s \|\sigma^n\|_s,
\end{equation}

with

\begin{align}
(3.17) & \quad \frac{1}{2} < s \leq k + 1 + \delta \quad \text{if } \sigma = v, \\
(3.18) & \quad 0 \leq s \leq k + 1 + \delta \quad \text{if } \sigma = u, w, \\
(3.19) & \quad 0 \leq s \leq k + 1 \quad \text{if } \sigma = p, q, r.
\end{align}

By (3.9) we can obtain

\begin{align*}
(\epsilon^{h,n}_p, \text{div } v') = 0 \quad \forall v' \in V_h, \\
(\text{div } e^{h,n}_w, q') = 0 \quad \forall q' \in Q_h.
\end{align*}

By these identities, the orthogonality of $L^2$ projections, and some algebraic manipulations, the previous error equations are reduced to

\begin{align}
(3.20a) & \quad \left( \rho \partial_t e^{h,n+\frac{1}{2}}_v, v' \right) - \left( e^{h,n+\frac{1}{2}}_p, \text{div } v' \right) + \left( \rho u^{h,n+\frac{1}{2}}_v, v' \right) = F^n_v(v'), \\
(3.20b) & \quad \left( \kappa^{-1} \partial_t e^{h,n+\frac{1}{2}}_p, p' \right) + \left( \text{div } e^{h,n+\frac{1}{2}}_w, p' \right) + \left( \rho q^{h,n+\frac{1}{2}}_p, p' \right) = F^n_p(p'), \\
(3.20c) & \quad \left( \partial_t e^{h,n+\frac{1}{2}}_u, u' \right) - \left( e^{h,n+\frac{1}{2}}_v, u' \right) + \left( \omega^{\gamma}_{\rho} e^{h,n+\frac{1}{2}}_w, u' \right) = F^n_u(u'), \\
(3.20d) & \quad \left( \partial_t e^{h,n+\frac{1}{2}}_w, w' \right) - \left( e^{h,n+\frac{1}{2}}_u, w' \right) = F^n_w(w'), \\
(3.20e) & \quad \left( \partial_t e^{h,n+\frac{1}{2}}_q, q' \right) - \left( e^{h,n+\frac{1}{2}}_p, q' \right) + \left( \gamma e^{h,n+\frac{1}{2}}_w, q' \right) + \left( \omega^{\gamma}_{\rho} e^{h,n+\frac{1}{2}}_r, q' \right) = F^n_q(q'), \\
(3.20f) & \quad \left( \partial_t e^{h,n+\frac{1}{2}}_r, r' \right) - \left( e^{h,n+\frac{1}{2}}_q, r' \right) = F^n_r(r').
\end{align}
where
\begin{align}
F^p_n(v') &= -\left(\rho_a \left(\partial_t v^{n+\frac{1}{2}} - \dot{v}^{n+\frac{1}{2}}\right), v'\right) - \left(\rho_a \epsilon^{n+\frac{1}{2}}_u, v'\right), \\
F^p_n(p') &= -\left(\kappa_a^{-1} \left(\partial_t P_h p^{n+\frac{1}{2}} - \dot{p}^{n+\frac{1}{2}}\right), p'\right) - \left(\rho_q \epsilon^{n+\frac{1}{2}}_e, p'\right), \\
F^n_{u}(u') &= -\left(\partial_t P_h u^{n+\frac{1}{2}} - \dot{u}^{n+\frac{1}{2}}, u'\right), \\
F^n_{w}(w') &= -\left(\partial_t P_h w^{n+\frac{1}{2}} - \dot{w}^{n+\frac{1}{2}}, w'\right), \\
F^n_{q}(q') &= -\left(\partial_t P_h q^{n+\frac{1}{2}} - \dot{q}^{n+\frac{1}{2}}, q'\right), \\
F^n_{r}(r') &= -\left(\partial_t P_h r^{n+\frac{1}{2}} - \dot{r}^{n+\frac{1}{2}}, r'\right).
\end{align}

In the discussions below we will use $\mathcal{E}^n$ defined as
\begin{equation}
(\mathcal{E}^n)^2 = ||\epsilon^h_{u,n}||^2_{\rho_u} + ||\epsilon^h_{v,n}||^2_{\rho_v} + ||\epsilon^h_{w,n}||^2_{\rho_w} + ||\epsilon^h_{p,n}||^2_{\rho_p} + ||\epsilon^h_{q,n}||^2_{\rho_q} + ||\epsilon^h_{r,n}||^2_{\rho_r}.
\end{equation}

**Proposition 3.1.** For given $f \in C^0([0,T];W)$, $g \in C^0([0,T];Q)$ and initial data $(\mathbf{v}(0), \mathbf{u}(0), \mathbf{w}(0), p(0), q(0), r(0)) \in X$ suppose that $(\mathbf{v}, \mathbf{u}, \mathbf{w}, p, q, r)$ is a weak solution of \((3.21)\). Assume that numerical initial data
\begin{align}
(\mathbf{v}(0), p_h(0), \mathbf{u}(0), \mathbf{w}(0), q_h(0), r_h(0)) \in \mathcal{X}_h
\end{align}
satisfy
\begin{equation}
\begin{aligned}
&\|u(0) - u_h(0)\|_{\rho_u} + \|v(0) - v_h(0)\|_{\rho_v} + \|w(0) - w_h(0)\|_{\rho_w} \\
&+ \|p_h(0) - p_h(0)\|_{\rho_p} + \|P_h q(0) - q_h(0)\|_{\rho_q} + \|P_h r(0) - r_h(0)\|_{\rho_r} \\
&\leq C_0' h^s, \quad \frac{1}{2} < s \leq k + 1 + \delta
\end{aligned}
\end{equation}
with $C_0'$ independent of $h$. We also assume that the exact solution $(\mathbf{v}, \mathbf{u}, \mathbf{w}, p, q, r)$ and $\rho_u$, $\kappa_a^{-1}$, $\rho_v$, $\rho_w$, $\rho_q$, $\rho_r$ satisfy the regularity assumptions \((3.25), (3.26), (3.27)\) below.

If \(\{(\mathbf{v}_n^h, \mathbf{u}_n^h, \mathbf{w}_n^h, p_h^n, q_h^n, r_h^n)\}_{n=1}^N\) is a numerical solution obtained by the fully discrete scheme \((3.5)\), then
\begin{equation}
(\mathcal{E}^n)^2 \leq C_0 h^s + C_1 (\Delta t)^2 + C_2 h^s, \quad \frac{1}{2} < s \leq k + 1 + \delta
\end{equation}
with constants $C_0$, $C_1$, $C_2$ such that $C_0$ depends on $C_0'$ in \((3.23)\) and
\begin{equation}
\|\rho_u, \rho_v, \rho_w\|_{L^\infty} < \infty,
\end{equation}
$C_1$ depends on
\begin{equation}
\|\mathbf{v}, \mathbf{u}, \mathbf{w}, p, q, r\|_{W^{3,1}(0,T;L^2)}, \quad \|\rho_u, \rho_v, \rho_w, \rho_q, \rho_r\|_{L^\infty},
\end{equation}
and $C_2$ depends on
\begin{equation}
\|\dot{\mathbf{v}}\|_{L^1(0,T;H^s)}, \quad \|\dot{\mathbf{p}}\|_{L^1(0,T;H^{s-1})}, \quad \|\rho_u, \kappa_a^{-1}\|_{W^{1,\infty}}, \quad \|\kappa_a, \rho_v, \rho_w, \rho_q, \rho_r\|_{L^\infty}
\end{equation}
with $s_0 = \max\{0, s - 1\}$.

**Proof.** Let us take $(v', p', u', w', q', r') \in \mathcal{X}_h$ as
\begin{align}
(\epsilon^{h,n+\frac{1}{2}}_v, \epsilon^{h,n+\frac{1}{2}}_p, P_h(\rho_u \epsilon^{h,n+\frac{1}{2}}_u), P_h(\rho_v \epsilon^{h,n+\frac{1}{2}}_w), P_h(\rho_q \epsilon^{h,n+\frac{1}{2}}_q), P_h(\rho_r \epsilon^{h,n+\frac{1}{2}}_r))
\end{align}
in (3.20) and add all the equations, then we can get

\[
\frac{1}{2} (E^{n+1})^2 - (E^n)^2 + \Delta t \left( (\gamma \rho_q \varepsilon^q_{h,n+\frac{1}{2}}) + \varepsilon^h_{n+\frac{1}{2}} \right) \\
= \Delta t \left( F^{n}_w (\varepsilon^h_{n+\frac{1}{2}}) + F^{n}_p (\varepsilon^p_{h,n+\frac{1}{2}}) + F^{n}_q (P_h (\rho_u \varepsilon^u_{h,n+\frac{1}{2}})) \right) \\
+ \Delta t \left( F^{n}_w (P_h (\rho_w \varepsilon^w_{h,n+\frac{1}{2}})) + F^{n}_q (P_h (\rho_q \varepsilon^q_{h,n+\frac{1}{2}})) + F^{n}_r (P_h (\rho_r \varepsilon^r_{h,n+\frac{1}{2}})) \right) \\
=: R^n.
\]

The proof of (3.24) has three steps. In the first step, we shall prove

\[
\|R^n\|_{0} \leq (C_{1,n} (\Delta t)^2 + C_{2,n} h^s) (E^n + E^{n+1}), \quad \frac{1}{2} < s \leq k + 1 + \delta
\]

with \(C_{1,n}\) depending on

\[
\|v, u, w, p, q, r\|_{W^{3,1}(t_n, t_{n+1}; L^2)}, \quad \|\rho_u, \rho_a, \rho_w, \rho_q, \rho_r\|_{L^\infty},
\]

and with \(C_{2,n}\) depending on

\[
\|\dot{v}\|_{L^1(t_n, t_{n+1}; H^s)}, \quad \|\dot{p}\|_{L^1(t_n, t_{n+1}; H^s)}, \quad \|\rho_u, \rho_a, \rho_w, \rho_q, \rho_r\|_{L^\infty}
\]

with \(s_0 = \max\{0, s - 1\}\). The more detailed dependence of \(C_{1,n}\), \(C_{2,n}\) will be clarified in the proof of (3.29). In the second step, we prove

\[
E^n \leq E^0 + 2 \sum_{i=0}^{n-1} (C_{1,i} (\Delta t)^2 + C_{2,i} h^s), \quad \frac{1}{2} < s \leq k + 1 + \delta.
\]

In the third step, we prove

\[
E^0 \leq C_0 h^s, \quad \frac{1}{2} < s \leq k + 1 + \delta
\]

with \(C_0\) depending on \(C_0^s\), the shape regularity of \(T_h\), and \(\|\rho_u, \rho_a, \rho_w\|_{L^\infty} < \infty\).

Note that the conclusion (3.24) follows from (3.28), (3.29), (3.32), (3.33) by taking \(C_1 = \sum_{0 \leq i \leq n} C_{1,i}, C_2 = \sum_{0 \leq i \leq n} C_{2,i}\). Therefore we will devote the rest of proof to prove (3.29), (3.32), and (3.33).

Since the proof of (3.29) is long, we show (3.32) and (3.33) first assuming that (3.29) is proved. For (3.32), note that

\[
E^{n+1} - E^n \leq 2(C_{1,n} (\Delta t)^2 + C_{2,n} h^s)
\]

is obtained by (3.28) and (3.29). Then (3.32) follows by induction. For (3.33), the triangle inequality and \(\|\rho_u, \rho_a, \rho_w\|_{L^\infty} < \infty\) give

\[
E^0 \leq \|\varepsilon^0 w\|_{\rho_u} + \|\dot{\varepsilon}^0 w\|_{\rho_a} + \|\varepsilon^0 w\|_{\rho_w} + C_0 h^s, \quad \frac{1}{2} < s \leq k + 1 + \delta
\]

with \(C > 0\) depending on the shape regularity of \(T_h\) and \(\|\rho_u, \rho_a, \rho_w\|_{L^\infty}\).

Before we prove (3.29), let us review some interpolation error estimates from time discretization schemes. Since the interpolation operator \(I_h(= \Pi_h, P_h, \bar{P}_h)\) is independent in time, the time derivative of \(I_h \sigma\) is same as \(I_h \dot{\sigma}\) as long as they
are well-defined pointwisely in time. Then, assuming that a general variable \( \sigma \in L^2(\Omega, t_{n+1}; L^2) \) is sufficiently regular, we can obtain

\[
\Delta t \| \sigma^{n+\frac{1}{2}} - \tilde{\sigma} t \sigma^{n+\frac{1}{2}} \|_0 = \| \Delta t \sigma^{n+\frac{1}{2}} - (\sigma^{n+1} - \sigma^n) \|_0 \\
\leq C \Delta t^2 \| \sigma \|_{W^{3,1}(\Omega, t_{n+1}; L^2)},
\]

(3.35)

\[
\Delta t \| \tilde{\sigma} t \sigma^{n+\frac{1}{2}} - \tilde{\sigma} I_h \sigma^{n+\frac{1}{2}} \|_0 = \| \sigma^{n+1} - I_h \sigma^{n+1} - (\sigma^n - I_h \sigma^n) \|_0 \\
= \left\| \int_{t_n}^{t_{n+1}} (\tilde{\sigma}(s) - I_h \tilde{\sigma}(s)) \, ds \right\|_0 \\
\leq C h^s \| \tilde{\sigma} \|_{L^1(\Omega, t_{n+1}; H^s)}
\]

(3.36)

with \( s \) satisfying the range conditions in \([3.17], [3.18], [3.19]\).

For the proof of (3.35), it suffices to estimate the terms in (3.24) by the definition of \( R^n \) in \([3.28]\).

By \([3.16], [3.35], [3.36]\), and the triangle inequality, and by assuming that the exact solution \( v \) is sufficiently regular, one can show

\[
\Delta t |F_v^n(e_v^{h_n+\frac{1}{2}})| \\
\leq C \left( (\Delta t)^2 \| v \|_{W^{3,1}(\Omega, t_{n+1}; L^2)} + h^s \| \dot{v} \|_{L^1(\Omega, t_{n+1}; H^s)} \right) \| e_v^{h_n+\frac{1}{2}} \|_{\rho_u}
\]

(3.37)

with \( C > 0 \) depending on \( \| \rho_u, \rho_a \|_{L^\infty}, \| \rho_u \|_{L^\infty}^{\frac{1}{2}} \). Noting the identity

\[
F_v^n(P_h(\rho_u e_u^{h_n+\frac{1}{2}})) = - (\tilde{\sigma} I_h u^{n+\frac{1}{2}} - \tilde{\sigma} I_h u^{n+\frac{1}{2}}, P_h(\rho_a e_u^{h_n+\frac{1}{2}}))
\]

(3.38)

\[
= - (\tilde{\sigma} u^{n+\frac{1}{2}} - \tilde{\sigma} u^{n+\frac{1}{2}}, P_h(\rho_a e_u^{h_n+\frac{1}{2}}))
\]

and the inequality \( \| P_h(\rho_a e_u^{h_n+\frac{1}{2}}) \|_0 \leq \| \rho_u e_u^{h_n+\frac{1}{2}} \|_0 \leq \| \sqrt{\rho_u} \|_{L^\infty} \| e_u^{h_n+\frac{1}{2}} \|_{\rho_u} \), we obtain

\[
\Delta t |F_v^n(P_h(\rho_u e_u^{h_n+\frac{1}{2}}))| \leq C \| v \|_{W^{3,1}(\Omega, t_{n+1}; L^2)} \| e_v^{h_n+\frac{1}{2}} \|_{\rho_u}
\]

(3.39)

\[
\text{with } C > 0 \text{ depending on } \| \rho_u \|_{L^\infty} \text{ by (3.35). A completely similar argument gives}
\]

\[
\Delta t |F_v^q(P_h(\rho_q e_q^{h_n+\frac{1}{2}}))| \leq C \| v \|_{W^{3,1}(\Omega, t_{n+1}; L^2)} \| e_v^{h_n+\frac{1}{2}} \|_{\rho_u},
\]

(3.40)

\[
\Delta t |F_q(P_h(\rho_q e_q^{h_n+\frac{1}{2}}))| \leq C \| v \|_{W^{3,1}(\Omega, t_{n+1}; L^2)} \| e_q^{h_n+\frac{1}{2}} \|_{\rho_u},
\]

(3.41)

\[
\text{with } C > 0 \text{ depending on } \| \rho_u \|_{L^\infty}, \| \rho_q \|_{L^\infty}, \| \rho_q \|_{L^\infty}, \text{ respectively.}
\]

Now we only need to estimate the \( F_v^n(e_p^{h_n+\frac{1}{2}}) \)-involved term but it needs an additional discussion because the standard approximation theory with \( Q_h \) gives only a bound of \( O(h^{k+1}) \). To obtain an estimate of \( \| e_p^{h_n+\frac{1}{2}} \|_{\hat{\kappa}_n^{-1}} \) with a bound of \( O(h^s) \), \( \frac{1}{2} \leq s \leq k + 1 + \delta \), we will use

\[
\left| \left( \kappa_a^{-1} \tilde{\sigma} e_p^{h_n+\frac{1}{2}}, p' \right) \right| = \left| \left( (\kappa_a^{-1} - P_0 \kappa_a^{-1}) \tilde{\sigma} e_p^{h_n+\frac{1}{2}}, p' \right) \right| \\
\leq C h \| \kappa_a^{-1} \|_{L^\infty} \| \tilde{\sigma} e_p^{h_n+\frac{1}{2}} \|_0 \| p' \|_{\kappa_a^{-1}},
\]

(3.42)

\[
\left| \left( \rho_a e_p^{h_n+\frac{1}{2}}, p' \right) \right| = \left| \left( (\rho_a - P_0 \rho_a) e_p^{h_n+\frac{1}{2}}, p' \right) \right| \\
\leq C h \| \rho_a \|_{L^\infty} \| \tilde{\sigma} e_p^{h_n+\frac{1}{2}} \|_0 \| p' \|_{\kappa_a^{-1}}.
\]

(3.43)
By (3.16), (3.42), (3.43), (3.35), (3.36), assuming that the exact solutions are sufficiently regular, one can obtain

\[(3.44) \quad \Delta t F_p^n [e_p^{h,n+\frac{1}{2}}] \leq C \left( (\Delta t)^2 \| \hat{p} \|_{W^{2,1}(t_n,t_{n+1};L^2)} + h^s \| \hat{p} \|_{L^1(t_n,t_{n+1};H^{s})} \right) \| e_p^{h,n+\frac{1}{2}} \|_{\kappa^s_n^{-1}} \]

for \(0 \leq s \leq k + 1 + \delta\) with \(C > 0\) depending on \(\|\kappa^s_n\|_{L^\infty}, \|\rho_a\|_{W^{1,\infty}_n}, \|\kappa^s_n^{-1}\|_{W^{1,\infty}_n}\). Combining (3.37), (3.38), (3.39), (3.40), (3.41), (3.44), and the triangle inequality with the definition of \(R^n\), we can obtain (3.29) with \(C_{1,n}, C_{2,n}\) with the dependence described in (3.30), (3.31).

**Theorem 3.2.** Suppose that the assumptions of Proposition 3.1 hold and \(\delta\) is defined in the same way. Then

\[(3.45) \quad \| u^n - u^n_h \|_{\rho_a} + \| v^n - v^n_h \|_{\rho_a} + \| w^n - w^n_h \|_{\rho_a} \leq E^n + Ch^s \| u, v, w \|_{C^0([0,T];H^s)} \]

with \(\frac{1}{2} < s \leq k + 1 + \delta\) where \(C\) depends on the shape regularity of \(T_h\) and the degree \(k\). Similarly,

\[(3.46) \quad \| p^n - p^n_h \|_{\kappa^s_n^{-1}} + \| q^n - q^n_h \|_{\rho_q} + \| r^n - r^n_h \|_{\rho_r} \leq E^n + Ch^s \| p, q, r \|_{C^0([0,T];H^s)} \]

with \(0 \leq s \leq k + 1\).

**Proof.** By the triangle inequality and the definition of \(E^n\),

\[(3.47) \quad \| u^n - u^n_h \|_{\rho_a} + \| v^n - v^n_h \|_{\rho_a} + \| w^n - w^n_h \|_{\rho_a} \leq E^n + \| e^{I,n}_u \|_{\rho_a} + \| e^{I,n}_v \|_{\rho_a} + \| e^{I,n}_w \|_{\rho_a}. \]

By the approximation properties of \(\Pi_h\) and \(P_h\),

\[(3.48) \quad \| e^{I,n}_u \|_{\rho_a} + \| e^{I,n}_v \|_{\rho_a} + \| e^{I,n}_w \|_{\rho_a} \leq Ch^s \| u, v, w \|_{C^0([0,T];H^s)}, \quad \frac{1}{2} < s \leq k + 1 + \delta \]

holds with \(C > 0\) depending on the shape regularity of \(T_h\) and \(\|\rho_a, \rho_q, \rho_r\|_{L^\infty}\).

Then (3.45) follows.

Similarly, the triangle inequality gives

\[(3.46) \quad \| p^n - p^n_h \|_{\kappa^s_n^{-1}} + \| q^n - q^n_h \|_{\rho_q} + \| r^n - r^n_h \|_{\rho_r} \leq E^n + \| e^{I,n}_p \|_{\kappa^s_n^{-1}} + \| e^{I,n}_q \|_{\rho_q} + \| e^{I,n}_r \|_{\rho_r}. \]

Then

\[(3.49) \quad \| e^{I,n}_p \|_{\kappa^s_n^{-1}} + \| e^{I,n}_q \|_{\rho_q} + \| e^{I,n}_r \|_{\rho_r} \leq Ch^s \| p, q, r \|_{C^0([0,T];H^s)}, \quad 1 \leq s \leq k + 1 \]

holds with \(C > 0\) depending on the shape regularity of \(T_h\) and \(\|\kappa^s_n^{-1}, \rho_q, \rho_r\|_{L^\infty}\), so (3.46) follows.

### 3.3 Error analysis for post-processed solutions.

If \(V_h\) is a BDM element, then \(\delta = 1\) and the optimal convergence rate in (3.46) is one order lower than the one in (3.45). This lower convergence rate can be circumvented by a local post-processing which we introduce below.

Throughout this subsection we assume that the exact solutions are sufficiently regular and we will not concern about low regularity of exact solutions. In our local post-processing, we first find \(\{p^{n,s}_h\}_{n=0}^{N-1}\), a new numerical solution approximating
p(t_n + \Delta t/2), not p(t_n). The goal is to show that \( \| p_h^{n,*} - p(t_n + \Delta t/2) \|_0 \) can have \( O(h^{k+2}) \) convergence rate.

To define the local post-processing let us define

\[
Q_h^* = \{ q \in L^2(\Omega) : q|_K \in P_{k+2}(K), \quad K \in T_h \}.
\]

We define \( p_h^{n,*} \in Q_h^* \) as

\[
(3.49) \quad \int_K p_h^{n,*} \, dx = \int_K p_h^{n+\frac{1}{2}} \, dx,
\]

\[
(3.50) \quad (\nabla p_h^{n,*}, \nabla p')_K = - (\rho_0 \partial_t \nu_h^{n+\frac{1}{2}}, \nabla p')_K - (\rho_0 \nu_h^{n+\frac{1}{2}}, \nabla p')_K + (f^{n+\frac{1}{2}}, \nabla p')_K, \quad \forall p' \in Q_h^*
\]

for all \( K \in T_h \). Note that this post-processing is solving a system with a block diagonal matrix such that the size of each matrix block is the number of DOFs of \( Q_h^* \) on one simplex \( K \). Therefore, the computational costs are negligibly small compared to the computational costs of the original linear system.

**Lemma 3.3.** Suppose that the assumptions of Proposition 3.1 hold and \( V_h \) is a BDM element. If \( \{ p_h^{n,*} \}_{n=0}^{N-1} \) is defined as in (3.49), (3.50), and the exact solution \((v, p, u, w, q, r)\) is sufficiently regular, then

\[
\| p(t_n + \Delta t/2) - p_h^{n,*} \|_0 \leq C((\Delta t)^2 + h^{k+2})
\]

with a constant \( C > 0 \) which depends on the constants \( C_0, C_1, C_2 \) in Proposition 3.1. \( \| u \|_{C_0([0,T];H^{k+1})}, \| p \|_{C_0([0,T];H^{k+2})}, \) and \( \| p \|_{W^2,\infty(t_n,t_{n+1};L^2)} \).

**Proof.** Recall that \( p^{n+\frac{1}{2}} = \frac{1}{2}(p^n + p^{n+1}) = \frac{1}{2}(p(t_n) + p(t_{n+1})) \). We first note that

\[
\| p(t_n + \Delta t/2) - p^{n+\frac{1}{2}} \|_0 \leq C(\Delta t)^2 \| p \|_{W^2,\infty(t_n,t_{n+1};L^2)}
\]

by an argument with the Taylor expansion. By the triangle inequality it suffices to estimate \( \| p^{n+\frac{1}{2}} - p_h^{n,*} \|_0 \).

To show an error estimate of \( \| p^{n+\frac{1}{2}} - p_h^{n,*} \|_0 \) consider the error equation

\[
\left( \nabla (p^{n+\frac{1}{2}} - p_h^{n,*}), \nabla p' \right) = - \left( \rho_0 (\dot{\nu}^{n+\frac{1}{2}} - \partial_t \nu_h^{n+\frac{1}{2}}), \nabla p' \right) - \left( \rho_0 (\nu_h^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}), \nabla p' \right) \quad \forall p' \in Q_h^*
\]

from the definition of \( p_h^{n,*} \) and (2.1a).

For \( P_h^* \), the \( L^2 \) projection to \( Q_h^* \), we can rewrite this equation as

\[
\left( \nabla (P_h^* p^{n+\frac{1}{2}} - p_h^{n,*}), \nabla p' \right) = - \left( \nabla (p^{n+\frac{1}{2}} - P_h^* p^{n+\frac{1}{2}}), \nabla p' \right) - \left( \rho_0 (\dot{\nu}^{n+\frac{1}{2}} - \partial_t \nu_h^{n+\frac{1}{2}}), \nabla p' \right) - \left( \rho_0 (\nu_h^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}), \nabla p' \right).
\]
If we take \( p' = P_h^* p^{n+\frac{1}{2}} - P_h^{n+\frac{1}{2}} \) and use the Cauchy–Schwarz inequality, then we get
\[
\| \text{grad}(P_h^* p^{n+\frac{1}{2}} - P_h^{n+\frac{1}{2}}) \|_0 
\]
(3.51) \[
\leq \| \text{grad}(p^{n+\frac{1}{2}} - P_h^* p^{n+\frac{1}{2}}) \|_{0,K} + C \left( \| \hat{v}^{n+\frac{1}{2}} - \hat{\partial}_t u_h^{n+\frac{1}{2}} \|_0 + \| u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}} \|_{\rho_a} \right) 
\]
=: I_1 + I_2 + I_3
with \( C > 0 \) depending on \( \rho_a \) and \( \Omega_p \).

To estimate \( I_1 \), assuming that \( p \) is sufficiently regular, we use the Bramble–Hilbert lemma and get
\[
\| \text{grad}(p^{n+\frac{1}{2}} - P_h^* p^{n+\frac{1}{2}}) \|_{0,K} \leq Ch^{k+1} \| p^n, p^{n+1} \|_{k+2,K}, \quad \forall K \in \mathcal{Th}.
\]
An estimate of \( I_3 \) is obtained by Theorem 3.2 as
\[
\| u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}} \|_{\rho_a} \leq C((\Delta t)^2 + h^{k+2})
\]
under the assumption that \( u \) is sufficiently regular.

We estimate \( I_2 \) by estimating
\[
I_2^a := \| \hat{v}^{n+\frac{1}{2}} - \hat{\partial}_t u^{n+\frac{1}{2}} \|_0, \quad I_2^b := \| \hat{\partial}_t e_v^{n+\frac{1}{2}} \|_0, \quad I_2^c := \| \hat{\partial}_t e_u^{n+\frac{1}{2}} \|_0.
\]
By (3.35) and (3.36),
\[
I_2^a \leq C((\Delta t)^2 \| v \|_{W^{3,\infty}(t_n,t_{n+1};L^2)},
\]
(3.55)
\[
I_2^b \leq C(h^{k+1} \| v \|_{L^\infty(t_n,t_{n+1};H^{k+1})}),
\]
(3.56)
\[
I_2^c \leq C(h^{k+1} \| v \|_{L^2(t_n,t_{n+1};H^{k+1})}).
\]

The estimate of \( I_2^c \) is more technical. First, note that it is enough to estimate \( \| \hat{\partial}_t e_v^{n+\frac{1}{2}} \|_{\rho_a} \) since \( \rho_a > 0 \) is uniformly positive. It is known (cf. [1, 2]) that there is a decomposition \( \tilde{\partial}_t e_v^{n+\frac{1}{2}} = v_0 + v_1 \) with \( v_0, v_1 \in V_h \) such that
\[
\text{div} v_0 = 0, \quad (\rho_a v_0, v_1) = 0, \quad \| \text{div} v_1 \|_0 \leq C \| \text{div} v_1 \|_0
\]
with \( C > 0 \) independent of \( h \). Using this decomposition, we can rewrite (3.20a) as
\[
(\rho_a v_0 + v_1), v' - \left( e_p^{h,n+\frac{1}{2}}, \text{div} v' \right) + \left( \rho_a e_u^{h,n+\frac{1}{2}}, v' \right) = F^n(v').
\]
If \( v' = v_1 \), then
\[
\| v_1 \|_0^2 \leq (C(e_p^{h,n+\frac{1}{2}} + e_u^{h,n+\frac{1}{2}}) \| v_1 \|_0 + |F^n(v_1)|).
\]
Recall that \( e_p^{h,n+\frac{1}{2}} \|_0, e_u^{h,n+\frac{1}{2}} \|_{\rho_a} \) are estimated in Proposition 3.1 and \( |F^n(v_1)| \) is estimated by (3.37). As a consequence,
\[
\| v_1 \|_0 \leq C(h^{k+1} + (\Delta t)^2)
\]
holds with \( C > 0 \) depending on \( \| v \|_{W^{3,\infty}(t_n,t_{n+1};L^2)}, \| \text{div} v \|_{L^\infty(t_n,t_{n+1};H^{k+1})} \), and the constants \( C_0, C_1, C_2 \) in Proposition 3.1. If \( v' = v_0 \), then we get
\[
\| v_0 \|_0^2 \leq \| e_u^{h,n+\frac{1}{2}} \|_0 \| v_0 \|_0 + |F^n(v_0)|.
\]
An estimate of \( \| v_0 \|_0 \) can be obtained by a completely similar argument for the estimate of \( \| v_1 \|_0 \). Therefore, by combining the estimates of \( \| v_0 \|_0 \) and \( \| v_1 \|_0 \), we have
\[
I_2 \leq C((\Delta t)^2 + h^{k+1})
\]
with \( C > 0 \) depending on \( \| v \|_{W^{3,\infty}(t_n,t_{n+1};L^2)}, \| \text{div} v \|_{L^\infty(t_n,t_{n+1};H^{k+1})} \), and the constants \( C_0, C_1, C_2 \) in Proposition 3.1.
By combining (3.51), (3.52), (3.53), (3.55), (3.56), (3.57), we obtained
\[ \| \nabla (p_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}}) \|_0 \leq C(\Delta t)^2 + h^{k+1}. \]
An element-wise Poincaré inequality and the above estimate give
\[ \| p_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \|_0 \leq Ch \| \nabla (p_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}}) \|_0 \leq C(\Delta t)^2 + h^{k+1}. \]
From this we can obtain
\[ \| p_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \|_0 \leq \| p_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \|_0 + \| p_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \|_0 \]
\[ \leq Ch^{k+2} \| p_h^{n+\frac{1}{2}} \|_{k+2} + C(\Delta t)^2 + h^{k+1}, \]
which is the desired estimate.

4. Numerical results

In this section we present the results of numerical experiments to illustrate the validity of our theoretical analysis.

In the first set of experiments we use a manufactured solution and show the convergence rates of errors with various finite element discretizations. Specifically, let \( \Omega = [0,1] \times [0,1] \) with the subdomain \( \Omega_0 = [3/8, 5/8] \times [0,1] \). We set \( \rho_a = \kappa_a = \omega = \omega_\gamma = 1, \gamma = 0 \) on \( \Omega \) whereas \( \Omega_\rho, \Omega_\kappa \) are defined as
\[
\Omega_\rho = \Omega_\kappa = \begin{cases} 
1 & \text{on } \Omega_0 \\
0 & \text{on } \Omega \setminus \Omega_0 
\end{cases}
\]
A manufactured solution is constructed with
\[ w(x,y) = \frac{(1 + \sin t)(x^2y + xy^2)}{\cos(2t)(x + y + \cos x)}, \quad r(x,y) = \cos(3t)xy, \]
and the other functions \( u(x,y), v(x,y), q(x,y), p(x,y), f(x,y), g(x,y) \) are defined by (2.1).

Table 1. Errors and convergence rates with \( V_h(K) \times Q_h(K) \times W_h(K) = BDM_1(K) \times P_0(K) \times P_1(K; \mathbb{R}^d) \) for the exact solution in (4.1).

| \( h \) | \( \| v - v_h \|_0 \) | \( \| u - u_h \|_0 \) | \( \| w - w_h \|_0 \) | \( \| p - p_h \|_0 \) |
|------|-----------------|-----------------|-----------------|-----------------|
| 8    | 5.53e-03        | 9.51e-03        | 3.65e-03        | 1.65e-01        |
| 16   | 1.34e-03        | 2.04            | 2.39e-03        | 1.99            | 9.15e-04        | 2.00            | 8.26e-02        | 1.00            |
| 32   | 3.49e-04        | 1.94            | 5.98e-04        | 2.00            | 2.29e-04        | 2.00            | 4.13e-02        | 1.00            |
| 64   | 8.42e-05        | 2.05            | 1.49e-04        | 2.00            | 5.72e-05        | 2.00            | 2.06e-02        | 1.00            |

| \( h \) | \( \| p - p_h \|_0 \) | \( \| q - q_h \|_0 \) | \( \| r - r_h \|_0 \) |
|------|-----------------|-----------------|-----------------|
| 8    | 5.53e-03        | 9.51e-03        | 3.65e-03        |
| 16   | 1.34e-03        | 2.04            | 2.39e-03        | 1.99            | 9.15e-04        | 2.00            |
| 32   | 3.49e-04        | 1.94            | 5.98e-04        | 2.00            | 2.29e-04        | 2.00            |
| 64   | 8.42e-05        | 2.05            | 1.49e-04        | 2.00            | 5.72e-05        | 2.00            |
Table 2. Errors and convergence rates with $V_h(K) \times Q_h(K) \times W_h(K) = \text{RTN}_0(K) \times P_0(K) \times P_0(K; \mathbb{R}^d)$ for the exact solution in (4.1).

| $\frac{1}{h}$ | $\|v - v_h\|_0$ | $\|u - u_h\|_0$ | $\|w - w_h\|_0$ | $\|p - p_h\|_0$ |
|--------------|----------------|----------------|----------------|----------------|
|              | error rate     | error rate     | error rate     | error rate     |
| 8            | 1.78e-01       | 7.50e-02       | 7.61e-02       | 1.65e-01       |
| 16           | 8.86e-02       | 1.01           | 3.74e-02       | 1.00           | 3.81e-02       | 1.00         |
| 32           | 4.43e-02       | 1.00           | 1.87e-02       | 1.00           | 1.90e-02       | 1.00         |
| 64           | 2.21e-02       | 1.00           | 9.34e-03       | 1.00           | 9.52e-03       | 1.00         |

Table 3. Errors and convergence rates with $V_h(K) \times Q_h(K) \times W_h(K) = \text{BDM}_2(K) \times P_1(K) \times P_2(K; \mathbb{R}^d)$ for the exact solution in (4.1).

| $\frac{1}{h}$ | $\|v - v_h\|_0$ | $\|u - u_h\|_0$ | $\|w - w_h\|_0$ | $\|p - p_h\|_0$ |
|--------------|----------------|----------------|----------------|----------------|
|              | error rate     | error rate     | error rate     | error rate     |
| 8            | 1.78e-01       | 7.50e-02       | 7.61e-02       | 1.65e-01       |
| 16           | 8.86e-02       | 1.01           | 3.74e-02       | 1.00           | 3.81e-02       | 1.00         |
| 32           | 4.43e-02       | 1.00           | 1.87e-02       | 1.00           | 1.90e-02       | 1.00         |
| 64           | 2.21e-02       | 1.00           | 9.34e-03       | 1.00           | 9.52e-03       | 1.00         |

We consider 4 different spatial discretizations such that the local finite element spaces $V_h(K) \times Q_h(K) \times W_h(K)$ are

(4.2) $\text{BDM}_1(K) \times P_0(K) \times P_1(K; \mathbb{R}^d)$, $\text{RTN}_0(K) \times P_0(K) \times P_0(K; \mathbb{R}^d)$,
(4.3) $\text{BDM}_2(K) \times P_1(K) \times P_2(K; \mathbb{R}^d)$, $\text{RTN}_1(K) \times P_1(K) \times P_1(K; \mathbb{R}^d)$.

For triangulation we use the structured meshes obtained by the bisection of uniform $N \times N$ squares of $\Omega$ for $N = 8, 16, 32, 64$. Then the maximum mesh size is a multiple of $h = 1/N$ with a uniform constant independent of $N$. We use the Crank–Nicolson scheme for time discretization with $\Delta t = h$ for (4.2) and with $\Delta t = h^2$ for (4.3) in order to see optimal convergence rates of spatial discretization.

1The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.
Table 4. Errors and convergence rates with $V_h(K) \times Q_h(K) \times W_h(K) = RTN_1(K) \times P_1(K) \times P_1(K; \mathbb{R}^d)$ for the exact solution in (4.1).

| $\frac{1}{h}$ | $\|v - v_h\|_0$ error | $\|u - u_h\|_0$ error | $\|w - w_h\|_0$ error | $\|p - p_h\|_0$ error |
|----------------|------------------------|------------------------|------------------------|------------------------|
| 8              | 2.97e-03                | 2.23e-03                | 2.75e-03                | 4.10e-03                |
| 16             | 6.93e-04                | 5.58e-04                | 6.88e-04                | 1.01e-03                |
| 32             | 2.01e-04                | 1.79e-04                | 1.72e-04                | 2.55e-04                |
| 64             | 4.56e-05                | 3.49e-05                | 4.30e-05                | 6.40e-05                |

| $\frac{1}{h}$ | $\|p - p_h\|_0$ error | $\|q - q_h\|_0$ error | $\|r - r_h\|_0$ error |
|----------------|------------------------|------------------------|------------------------|
| 8              | 2.97e-03                | 2.23e-03                | 2.75e-03                |
| 16             | 6.93e-04                | 5.58e-04                | 6.88e-04                |
| 32             | 2.01e-04                | 1.79e-04                | 1.72e-04                |
| 64             | 4.56e-05                | 3.49e-05                | 4.30e-05                |

Errors. For these four different discretization schemes the errors and convergence rates are presented in Tables 1 - 4.

In these experiments the errors of $\|u - u_h\|_0$, $\|v - v_h\|_0$, $\|w - w_h\|_0$, $\|p - p_h\|_0$, $\|q - q_h\|_0$, $\|r - r_h\|_0$ are computed at $T = 0.25$ whereas the error $\|p - p_h\|_0$ is computed at $T = \frac{1}{2} \Delta t$. Although we used the weighted norms $\|\cdot\|_{\rho_u}$, $\|\cdot\|_{\rho_w}$, $\|\cdot\|_{\rho_q}$, $\|\cdot\|_{\rho_r}$ in our error analysis for the errors of $u$, $w$, $q$, $r$, here we compute the standard $L^2$ norms for those errors. Since the $L^2$ norms are the upper bounds of the weighted norms, the optimal convergence rates of the $L^2$ norm errors are the results stronger than the optimal convergence rates of the weighted norm errors.

In all of these experiments we can see the convergence rates which are expected in our error analysis.

Figure 1. Wave propagation with $p_D$ in (4.4), $\mu_f = 18$, at $t = 0.2, 0.4$.
In the second set of experiments we set $\Omega = [0, 2] \times [0, 2]$ with $\Omega_0 = [3/5, 4/5] \times [0, 2]$. We assume that the parameters are given as

$$
\Omega_\rho = \Omega_\kappa = \begin{cases}
80 & \text{on } \Omega_0 \\
0 & \text{on } \Omega \setminus \Omega_0
\end{cases}, \quad \omega_\rho = \omega_\kappa = 40 \quad \text{on } \Omega,
$$

so the medium is a metamaterial on $\Omega_0$ but is a conventional material on $\Omega \setminus \Omega_0$.

We remark that the choices of these parameters are made without consideration of physical ranges of parameter values. We also set

$$
p_D(t, x, y) = \begin{cases}
10 \sin(\mu_f \pi (x + y - 10t)) & \text{if } t > x + y \text{ and } x < 3/5, \\
0 & \text{otherwise}
\end{cases}
$$

(4.4)

where $\mu_f$ is a constant. In the following experiments we impose the boundary condition (2.2) with $p_D$ in the above and $\Gamma_D = \partial \Omega$. Speaking more intuitively,
this condition gives an incoming wave propagation from the bottom-left corner of \( \Omega \) with frequency \( 5\mu_f \).

\[ \text{Figure 4. Wave propagation with } p_D \text{ in } \{4.5\} \text{ at } t = 0.06, 0.16, 0.36, 0.48 \]

We present the results of three experiments for \( \mu_f = 18, 19, 20 \). The finite elements with \( V_h(K) \times Q_h(K) \times W_h(K) = RTN_1(K) \times P_1(K) \times P_1(K; \mathbb{R}^d) \) are used for spatial discretization and \( \mathcal{T}_h \) is the structured mesh obtained by bisecting 50 \( \times \) 50 uniform squares of \( \Omega \). The Crank–Nicolson scheme is used for time discretization with \( \Delta t = 0.002 \).

The wave propagation patterns are presented in Figure 1, Figure 2, and Figure 3 for \( \mu_f = 18 \), \( \mu_f = 19 \), and \( \mu_f = 20 \), respectively. We call the three regions the left, the middle, and the right subdomains. In all of the figures in Figures 1–3 the wave propagation patterns look standard plane waves in the lower part of the left subdomain whereas they are more complicated due to the waves reflected by the interface of the left and the middle subdomains. We can clearly see reversed wave propagation patterns on the metamaterial layer \( \Omega_0 \) in the three figures at \( t = 0.4 \).
In addition, wave propagation patterns on the right subdomain in the figures at $t = 0.4$, are nearly plane waves with propagation directions similar to the patterns on the left subdomain. One can see that the details of wave propagation patterns, particularly the shapes and directions of the reversed patterns on the metamaterial layer, depend on the frequency of waves.

In the last experiment we set $\Omega = [0, 2] \times [0, 2]$ with $\Omega_0 = [3/5, 1] \times [0, 2]$, and the parameters are

$$\Omega_\rho = \Omega_\kappa = \begin{cases} 80 & \text{on } \Omega_0 \\ 0 & \text{on } \Omega \setminus \Omega_0 \end{cases}, \quad \omega_\rho = \omega_\kappa = 80 \quad \text{on } \Omega.$$

We also set

$$p_D(t, x, y) = \begin{cases} 10 \exp(-(1 + \sin(20\pi(x^2 + (y - 1)^2 - 10t)))) & \text{if } y - 1 < 0.1, \\ 0 & \text{otherwise.} \end{cases} \quad (4.5)$$

We impose the boundary condition (2.2) with the above $p_D$ on the left-side $\{0\} \times [0, 2]$ and with 0 on the other sides of $\Omega$. The finite elements are $V_h(K) \times Q_h(K) \times W_h(K)$ and $T_h$ is the structured mesh same as the second set of experiments. $\Delta t = 0.002$ with the Crank–Nicolson scheme. The wave propagation patterns are presented in Figure 4. One can see that wave propagation patterns are not conventional on the metamaterial layer $\Omega_0$.

5. Conclusion

In this paper we developed finite element methods for acoustic wave propagation in the Drude-type metamaterials. We combined the mixed finite elements for the Poisson equations and piecewise discontinuous finite element spaces for spatial discretization. For time discretization we use the Crank–Nicolson scheme. We carried out the a priori error analysis and proposed a local post-processing scheme to overcome low approximation property of the pressure for the BDM type finite elements. The numerical experiments show the validity of our theoretical analysis as well as atypical wave propagation patterns in metamaterials.

References

[1] Douglas N. Arnold, Richard S. Falk, and R. Winther, Preconditioning in $H(\text{div})$ and applications, Math. Comp. 66 (1997), no. 219, 957–984. MR 1401938 (97i:65177)
[2] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther, Multigrid in $H(\text{div})$ and $H(\text{curl})$, Numer. Math. 85 (2000), no. 2, 197–217. MR 1754719 (2001d:65161)
[3] C. Bellis and B. Lombard, Simulating transient wave phenomena in acoustic metamaterials using auxiliary fields, Wave Motion 86 (2019), 175–194. MR 3904336
[4] Daniele Boffi, Franco Brezzi, and Michel Fortin, Mixed finite element methods and applications, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013. MR 3097958
[5] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods, Springer Series in computational Mathematics, vol. 15, Springer, 1992. MR MR2233925 (2008i:35211)
[6] Franco Brezzi, Jr. Jim Douglas, and L. D. Marini, Two families of mixed finite elements for second order elliptic problems, Numer. Math. 47 (1985), no. 2, 217–235. MR 799685 (87g:65133)
[7] Lawrence C. Evans, Partial differential equations, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR 1625845 (99e:35001)
[8] Tunc Geveci, On the application of mixed finite element methods to the wave equations, RAIRO Modél. Math. Anal. Numér. 22 (1988), no. 2, 243–250. MR 945124 (89h:65116)
[9] Yunqing Huang, Jichun Li, and Wei Yang, *Modeling backward wave propagation in metamaterials by the finite element time-domain method*, SIAM J. Sci. Comput. **35** (2013), no. 1, B248–B274. MR 3033069

[10] Jichun Li, *A literature survey of mathematical study of metamaterials*, Int. J. Numer. Anal. Model. **13** (2016), no. 2, 230–243. MR 3421776

[11] Jichun Li and Yunqing Huang, *Time-domain finite element methods for Maxwell’s equations in metamaterials*, Springer Series in Computational Mathematics, vol. 43, Springer, Heidelberg, 2013. MR 3013583

[12] Jichun Li and Aihua Wood, *Finite element analysis for wave propagation in double negative metamaterials*, J. Sci. Comput. **32** (2007), no. 2, 263–286. MR 2320572

[13] Graeme W. Milton and Pierre Seppecher, *Realizable response matrices of multi-terminal electrical, acoustic and elastodynamic networks at a given frequency*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **464** (2008), no. 2092, 967–986. MR 2379501

[14] Graeme W. Milton and John R. Willis, *On modifications of Newton’s second law and linear continuum elastodynamics*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **463** (2007), no. 2079, 855–880. MR 2293080

[15] J.-C. Nédélec, *Mixed finite elements in $\mathbb{R}^3$*, Numer. Math. **35** (1980), no. 3, 315–341. MR 592160 (81k:65125)

[16] ———, *A new family of mixed finite elements in $\mathbb{R}^3$*, Numer. Math. **50** (1986), no. 1, 57–81. MR 864305 (88e:65145)

[17] A. N. Norris and A. L. Shuvalov, *Elastic cloaking theory*, Wave Motion **48** (2011), no. 6, 525–538. MR 2811881

[18] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606. MR 0483555 (58 #3547)

[19] Wei Yang, Yunqing Huang, and Jichun Li, *Developing a time-domain finite element method for the Lorentz metamaterial model and applications*, J. Sci. Comput. **68** (2016), no. 2, 438–463. MR 3519188

[20] Kosaku Yosida, *Functional analysis*, 6th ed., Springer Classics in Mathematics, Springer-Verlag, 1980.