Abstract. For nonpositively curved perturbations of parabolic cylinders we establish the existence of a logarithmically large resonance free region. We use an escape function construction in a compact part of the manifold and near infinity we use the method of complex scaling. To the author’s knowledge this is the first proof of a large resonance free region for manifolds with cusps.

1. Introduction

Resonance free regions near the essential spectrum have been extensively studied since the foundational work of Lax-Phillips [LaPh] and Vainberg [Vai]. They have been applied to resonant wave expansions and local wave decay, and this remains an active area [BoHä08, GuNa09] with many open questions. Recent applications include resolvent estimates, local smoothing estimates, and Strichartz estimates [Bur04, NoZw, BuGuHa]. The distribution of resonances is also of geometric interest as it is closely related to the dynamical structure of the set of trapped classical trajectories. Typically, more trapping results in more resonances near the essential spectrum. In particular, the largest resonance free regions exist when there is no trapping.

In this paper we study logarithmic resonance free regions for nontrapping manifolds with two ends, a cusp and a funnel. General hypotheses are given in §2, but consider the following

Example. Let \( \mathbb{H}^2 \) be the hyperbolic upper half plane. Let \((X, g)\) be a compactly supported metric perturbation of the \textit{parabolic cylinder}, \( \langle z \mapsto z + 1 \rangle \setminus \mathbb{H}^2 \) for which the curvature remains nonpositive. As we will show in §3.2, this implies that \((X, g)\) has no trapped geodesics.

Theorem. Let \( X \) be a manifold of dimension \( n \), either as in the example above or as in §3. Then there exist \( M_0, M_1 > 0 \) such that for all \( \chi \in C_0^\infty(X) \), the cutoff resolvent \( \chi(\Delta_g - s(n - 1 - s))^{-1} \chi \) continues meromorphically from \( \text{Re } s > (n - 1)/2 \) to the region

\[
\{ s \in \mathbb{C} : |\text{Im } s| > M_0, \text{Re } s > -M_1 |\text{Im } s| \},
\]

with only finite rank poles, which are independent of \( \chi \). Moreover, for any \( M_2 > 0 \) there exists \( M_3 > 0 \) such that the continuation is holomorphic in the region

\[
\{ s \in \mathbb{C} : |\text{Im } s| > M_3, \text{Re } s > -M_2 \log |\text{Im } s| \}.
\]

Here \( \Delta_g \) is the unique self-adjoint extension of the nonnegative Laplace-Beltrami operator on \( C_0^\infty(X) \). We will see in §3.4 that the essential spectrum of \( \Delta_g \) is given by \( [(n - 1)^2/4, \infty) \).
Initially the resolvent \((\Delta_g - s(n - 1 - s))^{-1}\) is defined only on \(\{s: \Re s > (n - 1)/2, s(n - 1 - s) \not\in \sigma_{pp}\}\). Light gray shading indicates the region to which \(\chi(\Delta_g - s(n - 1 - s))^{-1}\chi\) can be continued meromorphically, and dark gray shading indicates the region to which it can be continued holomorphically. Black dots represent poles of the resolvent, that is to say finite multiplicity eigenvalues, and crosses represent poles of the meromorphic continuation, that is to say finite multiplicity resonances.

Hence, on \(\{s: \Re s > (n - 1)/2, s(n - 1 - s) \not\in \sigma_{pp}\}\) where \(\sigma_{pp}\) is the pure point spectrum of \(\Delta_g\),

\[(\Delta_g - s(n - 1 - s))^{-1}: L^2(X) \to L^2(X).\]

However, \(\|(\Delta_g - s(n - 1 - s))^{-1}\|_{L^2(X) \to L^2(X)} \to \infty\) as \(\Re s \to (n - 1)/2\). Nonetheless, we show that for any \(\chi \in C_0^\infty(X)\) we can meromorphically continue the cutoff resolvent

\[\chi(\Delta_g - s(n - 1 - s))^{-1}\chi: L^2(X) \to L^2(X),\]

across the essential spectrum \(\{s \in \mathbb{C}: \Re(s) = (n - 1)/2\}\), and that there is a logarithmic high-energy resonance-free region (see Figure 1).

In particular, in the region where the continuation of the cutoff resolvent is holomorphic, the spectrum of \(\Delta_g\) is absolutely continuous (see for example [ReSi, Theorem XIII.20]).

The poles of the meromorphic continuation in (1.3) are called resonances, and in the case of asymptotically hyperbolic manifolds work of Mazzeo and Melrose [MaMe87] and Guillarmou [Gui05] shows that the continuation extends to \(\mathbb{C}\setminus\{n/2 - k, k \in \mathbb{N}\}\), and [Gui05, Theorem 1.4] gives a necessary and sufficient condition for the continuation to extend to \(\mathbb{C}\).

Such logarithmic resonance free regions go back to work of Lax and Phillips [LaPh] and Vainberg [Vai] on obstacle scattering in Euclidean space. When \(X\) is Euclidean outside of a compact set, the result of the Theorem is proven for very general nontrapping perturbations.
of the Laplacian by Sjöstrand and Zworski in [SjZw07, Theorem 1], which extends earlier work of Martinez [Mar02] and Sjöstrand [Sjö90].

In [CaVo02, Corollary 1.2], Cardoso and Vodev, extending work of Burq [Bur98], prove resolvent estimates for very general infinite volume manifolds which imply an exponentially small resonance free region of the form \( \{ s \in \mathbb{C} : |\text{Im } s| \geq M_2, \text{Re } s - (n-1)/2 \geq -e^{-M_1|s|} \} \). To the author’s knowledge the result in the present paper is the first proof of a large resonance free region for manifolds with cusps.

In many situations we also have information about the distribution of resonances away from the essential spectrum. For example, when \( (X, g) \) is the quotient of \( \mathbb{H}^2 \) by \( z \mapsto z + 1 \), Borthwick [Bor, §5.3] explicitly calculates the integral kernel of \( (\Delta_g - s(1-s))^{-1} \) in terms of Bessel functions and shows that the meromorphic continuation in (1.3) is holomorphic in \( \mathbb{C} \setminus \{1/2\} \), and has only a simple pole at \( s = 1/2 \). In [GuZw97, Theorem 1.3], Guillopé and Zworski study more general Riemann surfaces and, under a condition on the 0-volume of the surface, prove the existence of infinitely many resonances and give optimal lower and upper bounds on their number in disks. We give an application of their result to our setting in §3.3.

The above Theorem also provides a first step in support of the following

**Conjecture** (Fractal Weyl upper bound). Let \( \Gamma \) be a geometrically finite discrete group of isometries of \( \mathbb{H}^2 \) such that \( X = \Gamma \backslash \mathbb{H}^2 \) is a smooth surface of infinite area. Let \( R(X) \) denote the set of resonances of \( X \) counted with multiplicity, let \( K \subset T^*X \) be the set of maximally extended geodesics which are precompact, and let \( m \) be the Hausdorff dimension of \( K \). Then for any \( C_0 > 0 \) there is \( C_1 > 0 \) such that

\[
\# \{ s \in R(X) : r \leq \text{Im } s \leq r + 1, \text{Re } s \geq -C_0 \} \leq C_1 r^{m/2-1}.
\]

The terminology comes from the fact that this is a partial generalization to the case of resonances of the Weyl asymptotic for eigenvalues of a compact manifold. If \( \Gamma \backslash \mathbb{H}^2 \) has funnels but no cusps, this is follows from work of Zworski [Zwo99] and Guillopé-Lin-Zworski [GLZ04], and if it has cusps but no funnels, this follows from work of Selberg [Sel]. The remaining case is the one where \( \Gamma \backslash \mathbb{H}^2 \) has both cusps and funnels. The methods of the present paper, combined with those of [SjZw07], provide a possible program toward this conjecture. The crucial missing ingredient is the adaptation of the escape function in the cusp (constructed in §8 below) to a trapping situation.

We will take a semiclassical approach to the Theorem, and accordingly introduce the operator

\[
P \overset{\text{def}}{=} h^2 \left( e^{-\varphi} \Delta_g e^\varphi - \frac{(n-1)^2}{4} \right) - 1, \quad (1.4)
\]

with \( \varphi \in C^\infty(X) \) satisfying (3.7) and (3.10) below. It is sufficient to proceed as follows. We will prove that there exists a family of operators \( R(\zeta) : L^2_{\text{comp}}(X) \to L^2_{\text{loc}}(X) \) such that \( R(\zeta) = (P - \zeta)^{-1} \) for \( \text{Im } \zeta > 0 \), and there exists \( C > 0 \) and \( h_0 > 0 \) such that for \( 0 < h \leq h_0 \),

\[
R(\zeta) : L^2_{\text{comp}}(X) \to L^2_{\text{loc}}(X)
\]

is meromorphic with finite rank poles in \( \{ \zeta \in \mathbb{C} : |\zeta| \leq 1/C \} \).
Moreover, for any \( C > 0 \), there exists \( h_0 > 0 \) such that for \( 0 < h \leq h_0 \),

\[
R(\zeta) : L^2_{\text{comp}}(X) \to L^2_{\text{loc}}(X) \quad \text{is holomorphic in} \quad \{ \zeta \in \mathbb{C}; |\zeta| \leq Ch \log(1/h) \}. \tag{1.6}
\]

Observe that (1.5) implies (1.1), and that (1.6) implies (1.2).

To prove (1.5) and (1.6), in §7 we will use the results of §5 and §6 to holomorphically deform \( P \) into a family of nonselfadjoint operators \( P_R \), parametrized by \( R \in [1, \infty) \), such that for every \( \chi \in C_0^\infty(X) \) there exists \( R \) such that \( \chi P_R = \chi P \). We will show that (1.5) and (1.6) follow respectively from the claims that

For all \( R \), the spectrum of \( P_R \) in \( \{ \zeta \in \mathbb{C}; |\zeta| \leq 1/C \} \) consists of a discrete set of eigenvalues of finite multiplicity, and this spectrum is independent of \( R \). \tag{1.7}

and

\( P_5 \) has no spectrum in \( \{ \zeta \in \mathbb{C}; |\zeta| \leq Ch \log(1/h) \} \). \tag{1.8}

The choice of \( R = 5 \) here is for convenience: as a result of (1.7) we see that if \( P_R \) has no spectrum in \( \{ |\zeta| \leq Ch \log(1/h) \} \) for one value of \( R \), the same is true for all \( R \).

To obtain the \( P_R \) we use the method of complex scaling of Aguilar and Combes \[AgCo71\] and Simon \[Sim72\], following the geometric approach of Sjöstrand and Zworski \[SjZw91\]. Our assumptions on \( X \) allow us to separate variables near infinity, and the conjugation by \( e^\varphi \) allows us to write \( P \) near infinity as a direct sum of one-dimensional Schrödinger operators \( Q_{jm} \). We will define \( P_R \) by deforming these \( Q_{jm} \) holomorphically. For this procedure we follow Zworski \[Zwo99\], who defines a complex scaling based on separation of variables in the case where \( X \) is a convex co-compact quotient of \( \mathbb{H}^2 \). The main new difficulties in our case are that our manifold has a cusp, and that it is only analytic outside of a compact set. This requires holomorphic deformations which are substantially different from those of \[Zwo99\].

To prove (1.8) we will microlocally deform the operator \( P_5 \) in a compact region of the phase space \( T^*X \) to an operator \( P_{5,\varepsilon} \), and we will prove that

\[
\|u\|_{L^2_\varphi(X)} \leq \frac{C}{h \log(1/h)} \|P_{5,\varepsilon}u\|_{L^2_\varphi(X)}, \tag{1.9}
\]

where \( \|u\|_{L^2_\varphi(X)} \) is defined by \( \|e^\varphi u\|_{L^2(X)} \). We deform by conjugating by an exponential weight constructed using a nontrapping escape function, that is to say a function which is uniformly increasing along geodesic trajectories in \( T^*X \). This conjugation preserves the spectrum, and will allow us to apply a positive commutator estimate to obtain (1.9).

An outline of the paper is as follows.

- In §2 we give our assumptions on \( X \).
- In §3 we present basic consequences of our assumptions for \( \Delta_g \) and give examples of manifolds satisfying these assumptions.
- In §4 we review the results for pseudodifferential operators needed in §7 and §9.
- In §5 we define the complex scaled operators in the cusp.
- In §6 we define the complex scaled operators in the funnel.
• In §7 we define the full complex scaled operators and prove (1.7). We further prove that (1.7) ⇒ (1.5) and (1.8) ⇒ (1.6).
• In §8 we define a nontrapping escape function using the assumptions from §2.
• In §9 we combine the complex scaling and the escape function to prove (1.9) and as a consequence (1.8).
• In the Appendix we give computations of sectional curvature which are needed for the examples in §3.2.

2. Assumptions on $X$.

Our assumptions are divided into two classes. The first class, given in §2.1, concerns only the structure of the manifold near infinity, and under those assumptions we will prove the meromorphic continuation. More precisely we will prove (1.7), and as a consequence (1.5) and (1.1). In particular the construction of the $P_R$ will depend only on the assumptions in §2.1. The second class of assumptions are global dynamical assumptions, and they will be needed to prove (1.8), and as a consequence (1.6) and (1.2). We present them in §2.2 but will use them only in §8 and §9.

2.1. General assumptions on $X$. These assumptions are needed throughout the paper.

Let $X$ be a smooth Riemannian manifold with infinity given by a cusp and a funnel. More precisely, we have

$$X = X_0 \cup X_C \cup X_F,$$

Figure 2. The manifold $X$ and its decomposition into $X_0$, $X_C$, and $X_F$. 
where $X_0$ is a compact manifold with boundary such that $\partial X_0 = \partial X_C \cup \partial X_F$, $X_0 \cap X_C = \partial X_C$, $X_0 \cap X_F = \partial X_F$, $X_C \cap X_F = \emptyset$. We assume that

$$X_C \cong [0, \infty)_r \times S_C, \quad g|_{X_C} = dr^2 + \frac{e^{-2r}}{\beta_C^2(r)} \sigma_C, \quad (2.1)$$

$$X_F \cong [0, \infty)_r \times S_F, \quad g|_{X_F} = dr^2 + \frac{e^{2r}}{\beta_F^2(r)} \sigma_F, \quad (2.2)$$

where $(S_j, \sigma_j)$ are compact Riemannian manifolds of dimension $n - 1$, and each $\beta_j : [0, \infty) \to (0, \infty)$ is an analytic function which extends to a holomorphic function on a conic neighborhood of the positive real axis. We require further that

$$|\beta_j(z) - 1| \leq 1/3 \quad (2.3)$$

in this conic neighborhood. Using Cauchy estimates, we find that

$$|\beta_j^{(k)}(z)| \leq C_k |z|^{-k}, \quad (2.4)$$

uniformly in a slightly smaller conic region. Fix $\theta > 0$ such that $\{z \in \mathbb{C} : |\arg z| < \theta\}$. For convenience of exposition, take $\theta \in (0, \pi/2)$ small enough that

$$\tan \theta \leq 1/2. \quad (2.5)$$

A larger $\theta$ would allow a better constant $M_1$ in (1.1), but we do not pursue this here. Under a stronger assumption on $\beta_F$ we would be able to apply the results of [MaMe87] and [Gui05] to obtain a meromorphic continuation to $\mathbb{C}$. We instead prove our meromorphic continuation directly, which allows us to treat a slightly more general case and also makes the presentation more self-contained.

In the case of the parabolic cylinder $\beta_F \equiv \beta_C \equiv 1$, and in the case of a hyperbolic funnel $\beta_F(r) = e^{r+R} \text{sech}(r + R)$ for some $R > 0$. By taking $X_0$ larger if necessary, we may assume without loss of generality that

$$|\beta_j''(r)| + |\beta_j'(r)| \leq \beta_j(r)/2, \quad j \in \{C,F\}, r \in [0, \infty). \quad (2.6)$$

We also introduce the shorthand notation

$$X_{Ca} \overset{\text{def}}{=} X_C \cap \{r \geq a\}, \quad X_{Fa} \overset{\text{def}}{=} X_F \cap \{r \geq a\}.$$
2.2. Global dynamical assumptions on $X$. These assumptions are needed only in §8 and §9 where we prove the holomorphic continuation.

Assume that $X$ is nontrapping. This means that no maximally extended geodesic is precompact. Assume also that if $\gamma \subset X$ is a maximally extended geodesic, then $\gamma \cap X_C$ has at most one connected component.

We will see in §3.2 that these assumptions are satisfied in the case of nonpositive curvature.

3. Preliminaries

This section is organized as follows.

- In §3.1 we give consequences of the assumptions in 2.1 for geodesic trajectories on $X$.
- In §3.2 we show that a nonpositive curvature assumption implies the dynamical assumptions in 2.2 giving us a class of manifolds to which our results are applicable.
- In §3.3 we give some examples of manifolds satisfying our assumptions which have infinitely many resonances.
- In §3.4 we compute $\Delta_g$ near infinity, and specify the conjugation used to define $P$ in (1.4). We then deduce the essential spectrum, and give the separation of variables which we use in §7 to define the complex scaled operators $P_R$.

3.1. Dynamics near infinity. Geodesics in the funnel escape to infinity either as $t \to \infty$ or as $t \to -\infty$. Indeed from (2.2) we see that $p$, the geodesic Hamiltonian, is given by

$$p = \rho^2 + \beta_F(r)^2 e^{-2r \alpha},$$

in the funnel $X_F$, where $\rho$ is dual to $r$, and $\alpha$ is the geodesic hamiltonian of $S_F$. From this we conclude that, along geodesic flowlines, we have

$$\dot{r}(t) = H_p \rho = 2\rho(t),$$

$$\dot{\rho}(t) = -H_p r = 2 \left[ 1 - \frac{\dot{\beta}_F(r(t))}{\beta_F(r(t))} \right] (p - \rho(t)^2),$$

so long as the trajectory remains within $X_F$. Dividing the second equation by $p - \rho(t)^2$ and integrating both sides we find that

$$\rho(t) = \sqrt{p} \tanh \left[ 2\sqrt{p} \left( t + \int_0^t \frac{\beta_F(r(s))}{\beta_F(r(s))} ds \right) + \tanh^{-1} \left( \frac{\rho(0)}{\sqrt{p}} \right) \right], \quad (3.1)$$

Now fix $r(0) \in (0, \infty)$, observe that by (2.6) we have $|\dot{\beta}_F(r)/\beta_F(r)| \leq 1/2$ for $r \geq r(0)$, and take $\rho(0) \geq 0$. We claim that

$$\lim_{t \to -\infty} r(t) = \infty.$$
If \( \rho(0) = \sqrt{p} \) then \( \dot{\rho}(t) \equiv 0 \) and \( r(t) = r(0) + 2\sqrt{pt} \), so we have the claim. Otherwise observe that we always have \( \dot{\rho} > 0 \) and \( \dot{r} \geq 0 \), so by (3.1) we have \( \lim_{t \to \infty} \rho = \sqrt{p} \). Consequently \( \dot{r} \) is uniformly bounded below and we again have the claim.

By contrast geodesics in the cusp can escape to infinity only if \( \rho \equiv 0 \), and otherwise must enter \( X_0 \) in finite time. Indeed, in this case we have

\[
p = \rho^2 + \beta_C(r)^2 e^{2r\alpha},
\]

where again \( \rho \) is dual to \( r \), and this time \( \alpha \) is the geodesic hamiltonian of \( S_C \). Now

\[
\dot{r}(t) = H_p \rho = 2\rho(t),
\]

\[
\dot{\rho}(t) = -H_p r = -2 \left[ 1 + \frac{\dot{\beta}_C(r(t))}{\beta_C(r(t))} \right] \left( \rho - \rho(t)^2 \right),
\]

and consequently

\[
\rho(t) = \sqrt{p} \tanh \left[ -2\sqrt{p} \left( t + \int_0^t \frac{\dot{\beta}_C(r(s))}{\beta_C(r(s))} ds \right) + \tanh^{-1} \left( \frac{\rho(0)}{\sqrt{p}} \right) \right],
\]

while the trajectory remains in \( X_C \). Once again, if \( \rho(0) = \sqrt{p} \) we have \( r(t) = r(0) + 2\sqrt{pt} \) and hence \( \lim_{t \to \infty} r(t) = \infty \). Otherwise observe that if the trajectory remained in \( X_C \) always, then (3.2) would be always valid. But according to (3.2) we would have \( \rho \to \sqrt{p} \) which would force \( r(t) \to -\infty \), which is impossible.

### 3.2. A family of examples

In this section \( d_g(p,q) \) denotes the distance between \( p \) and \( q \) with respect to the Riemannian metric \( g \), and \( L_g(c) \) denotes the length of a curve \( c \) with respect to \( g \).

Generalizing slightly the example at the beginning of the paper, let \( n \geq 2 \), let \((\mathbb{H}^n, g_h)\) be the hyperbolic upper half space,

\[
(\mathbb{H}^n, g_h) = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 > 0 \right\}, x_1^{-2} (dx_1^2 + \cdots + dx_n^2),
\]

and \((X, g_h)\) be the parabolic cylinder obtained by taking the quotient by \( \{ x \mapsto x + c_1, \ldots, x \mapsto x + c_{n-1} \} \), where the \( c_j \) are linearly independent vectors in \( \mathbb{R}^{n-1} = \{0, x_2, \ldots, x_n\} \). Define \((X, g)\) by taking \( g \) as follows. Take real numbers \( 0 < x^m < x^M \), and let \( X_C = X \cap \{x_1 \geq x^M\} \) and \( X_F = X \cap \{x_1 \leq x^m\} \). The \( r \) coordinate in the case of \( X_C \) is \( r = \log x_1 - \log x^M \), and in the case of \( X_F \) it is \( r = \log x^m - \log x_1 \). The metrics \( \sigma_C \) and \( \sigma_F \) are taken such that \( dr^2 + e^{2r} \sigma_C \) and \( dr^2 + e^{-2r} \sigma_F \) agree with \( g_h \) in \( X_C \) and \( X_F \) respectively. The functions \( \beta_C \) and \( \beta_F \) can be taken to be any functions satisfying the hypotheses in the first paragraph of (2.1). Now let \( g \) be any metric with all sectional curvatures nonpositive obeying (2.1) and (2.2). The calculation in the Appendix shows that the curvature in \( X_C \) and \( X_F \) is nonpositive so long as (2.6) holds.

The assumptions in (2.2) will follow from the following classical theorem, (see for example [BrHa99 Theorem III.H.1.7]).
Proposition 3.1 (Stability of quasi-geodesics). Let \((\mathbb{H}^n, g_h)\) be hyperbolic \(n\)-space, let \(p, q \in \mathbb{H}^n\), and let \(\gamma_h : [t_1, t_2] \to \mathbb{H}^n\) be the geodesic from \(p\) to \(q\). Suppose \(c : [t_1, t_2] \to \mathbb{H}^n\) satisfies
\[
\frac{1}{C_1} |t - t'| \leq d_{g_h}(c(t), c(t')) \leq C_1 |t - t'|, 
\]for all \(t, t' \in [t_1, t_2]\). Then
\[
\max_{t \in [t_1, t_2]} d_{g_h}(\gamma_h(t), c(t)) \leq C_2, 
\]where \(C_2\) depends only on \(C_1\).

To apply this theorem, observe first that just as \(g_h\) descends to a metric on \(X\), so \(g\) lifts to a metric on \(\mathbb{H}^n\). By abuse of notation we call the lifted metric \(g\) as well. Observe that by construction we have a constant \(C_g\) such that
\[
\frac{1}{C_g} g \leq g_h \leq C_g g. 
\]
Indeed on a compact set this is true for any pair of metrics, and an explicit calculation shows it to be true in \(X_C\) and \(X_F\). We will show that if \(c\) is a unit speed \(g\)-geodesic in \(\mathbb{H}^n\), then \(\frac{1}{C_1} d_g(p, q) \leq d_{g_h}(p, q) \leq C_1 d_g(p, q)\), for all \(p, q \in \mathbb{H}^n\), with a constant \(C_1\) which depends only on \(C_g\). For this last we compute as follows: let \(\gamma\) be a unit speed \(g\)-geodesic from \(p\) to \(q\). Then
\[
d_{g_h}(p, q) \leq L_{g_h}(\gamma) = \int_{t_1}^{t_2} \sqrt{g_h(\dot{\gamma}, \dot{\gamma})} dt \leq \sqrt{C_g} \int_{t_1}^{t_2} \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt 
= \sqrt{C_g} L_{g}(\gamma) = \sqrt{C_g} d_g(p, q).
\]
Notice that for the last equality we have used the nonpositive curvature assumption; more specifically we used the fact that in this setting geodesics are always distance-minimizing. This proves the second inequality of (3.6), and the first follows from the same calculation once we observe that the hypothesis (3.5) is unchanged if we switch \(g\) and \(g_h\).

Now (3.4) implies the nontrapping assumption immediately because if \(\gamma_h\) is a \(g_h\)-geodesic, then for any \(x \in X\) we have
\[
\lim_{t \to \infty} d_{g_h}(\gamma_h, x) = \lim_{t \to \infty} d_g(\gamma_h, x) = \infty,
\]and consequently the same must be true if \(\gamma_h\) is replaced by a \(g\)-geodesic \(\gamma\).

We must check finally that if \(\gamma\) is a maximally extended \(g\)-geodesic in \(X\), then \(\gamma \cap X_C\) has at most one connected component. We will actually check that for \(N\) sufficiently large, if \(\gamma\) is instead a maximally extended \(g\)-geodesic in \(\mathbb{H}^n\), then \(\gamma \cap \{x_1 \geq N\}\) has at most one connected component. The conclusion will then follow provided we redefine our decomposition of \(X\) so that \(X_C = \{x_1 \geq N\}\).
We argue by way of contradiction. Recall from the dynamics computations in §3.1 that $\rho$ is nonincreasing in $X_C$. As a result, if $\gamma \cap \{ x_1 \geq N \}$ is to have two connected components, then $\gamma$ must pass through $\{ x_1 \leq x^M \}$ in the intervening time, and there must exist times $t_1 < t_2 < t_3$ such that $x_1(\gamma(t_1)) = x_1(\gamma(t_3)) = N$, $x_1(\gamma(t_2)) = x^M$. Now the $g_h$-geodesic $\gamma_h: [t_1, t_3] \to \mathbb{H}^n$ joining $\gamma(t_1)$ to $\gamma(t_3)$ has $x_1(\gamma(t)) \geq N$ for all $t \in [t_1, t_3]$. It follows that $d_{g_h}(\gamma_h(t_2), \gamma(t_2)) \geq \log N - \log x^M$, and if $N$ is large enough this violates (3.4).

3.3. Examples with infinitely many resonances. In this section we specialize the examples of §3.2 to the case $n = 2$, $\beta_C \equiv 1$, $\beta_F = e^{r \sech(r + R)}$ for some $R > 0$.

Let $R(s)$ denote the meromorphic continuation of $\chi(\Delta_g - s(1 - s))^{-1}\chi$. In this case, $R(s)$ is meromorphic in $\mathbb{C}$, and near each pole $s_0$ we have

$$R(s) = \chi \left( \sum_{j=1}^{k} \frac{A_j}{(s - s_0)^j} + A_0(s) \right) \chi,$$

where the $A_j$ for $1 \leq j \leq k$ are finite rank operators $L^2_{\text{comp}}(X) \to L^2_{\text{loc}}(X)$ and $A_0(s)$ is holomorphic near $s_0$. The multiplicity of a pole, $m(s_0)$ is given by

$$m(s) \overset{\text{def}}{=} \text{rank} \left( \sum_{j=1}^{k} A_j \right).$$

We put

$$(X^0_F, g_h) = ([0, \infty) \times (\mathbb{R}/\ell\mathbb{Z}), dr^2 + \cosh^2 rdt^2),$$

with $\ell$ chosen as a function $c_1, x^m$ and $R$ so that

$$(X_F, g) \cong ([1, \infty) \times (\mathbb{R}/\ell\mathbb{Z}), dr^2 + \cosh^2 rdt^2).$$

We define the 0-volume of $X$ by

$$0^-\text{vol}(X) \overset{\text{def}}{=} \text{vol}_g(X_0) + \text{vol}_g(X_C) - \text{vol}_{g_h}(X^0_F \setminus X_F).$$

Now [GuZw97, Theorem 1.3] says that

**Proposition 3.2** (Bounds on the number of resonances). If $0^-\text{vol}(X) \neq 0$, then there exists a constant $C$ such that

$$r^2/C \leq \sum_{|s| \leq r} m(s) \leq Cr^2, \quad r > C.$$

Hence provided we ensure that $0^-\text{vol}(X) \neq 0$ by adjusting the metric in $X_0$, the meromorphic continuation in (1.3) will have order $r^2$ resonances in a disk of radius $r$, but none of them will be in the logarithmic region (1.2).
3.4. The essential spectrum and separation of variables. In the funnel the nonnegative Laplace-Beltrami operator, $\Delta_g$, is given by

$$\Delta_g|_{X_F} = \frac{1}{\sqrt{|g|}} \sum_{k,l=1}^{n} D_k \left( \sqrt{|g|} g^{kl} D_l \right)$$

$$= g^{rr} D_r^2 + \frac{1}{\sqrt{|g|}} \left( D_r \sqrt{|g|} \right) g^{rr} D_r + (D_r g^{rr}) D_r + (e^{-\tau} \beta_F(r))^2 \Delta_{S_F}$$

$$= D_r^2 + i(n-1) \left( \frac{\beta'_F(r)}{\beta_F(r)} - 1 \right) D_r + (e^{-\tau} \beta_F(r))^2 \Delta_{S_F},$$

where $|g| = \det g = (e^{-\tau} \beta_F(r))^{-2(n-1)} \det \sigma_j$, $D_k = -i \partial_k$, and $\Delta_{S_F}$ is the Laplacian on $S_F$. Suppose $\varphi \in C^\infty(X)$ obeys

$$\varphi|_{X_F} = \frac{n-1}{2} (-r + \log \beta_F(r)). \quad (3.7)$$

Then conjugating by the unitary operator:

$$e^\varphi : L^2([0, \infty) \times S_F, drdV_{S_F}) \rightarrow L^2 \left( X_F, e^{-2\varphi} drdV_{S_F} \right)$$

$$u(r,y) \mapsto e^\varphi u(r,y),$$

where $dV_{S_F}$ is the volume form on $S_F$ (note that $e^{-2\varphi} drdV_{S_F}$ is the volume form on $X_F$) gives

$$e^{-\varphi}(\Delta_g|_{X_F})e^\varphi = D_r^2 + (e^{-\tau} \beta_F(r))^2 \Delta_{S_F} + \frac{(n-1)^2}{4} + V_F(r), \quad (3.9)$$

with

$$V_F(r) \overset{\text{def}}{=} -\frac{(n-1)^2}{2} \frac{\beta'_F(r)}{\beta_F(r)} - \frac{n-1}{2} \frac{\beta''_F(r)}{\beta_F(r)} + \frac{n^2-1}{4} \frac{(\beta'_F(r))^2}{(\beta_F(r))^2}$$

obeying $|V_F^{(k)}(r)| \leq C_k |r|^{-k-1}$.

Similarly in the cusp we must impose

$$\varphi|_{X_C} = \frac{n-1}{2} (r + \log \beta_C(r)). \quad (3.10)$$

to obtain

$$e^\varphi : L^2([0, \infty) \times S_C, drdV_{S_C}) \rightarrow L^2 \left( X_C, e^{-2\varphi} drdV_{S_C} \right)$$

$$u(r,y) \mapsto e^\varphi u(r,y),$$

and

$$e^{-\varphi}(\Delta_g|_{X_C})e^\varphi = D_r^2 + (e^{\tau} \beta_C(r))^2 \Delta_{S_C} + \frac{(n-1)^2}{4} + V_C(r), \quad (3.12)$$

with

$$V_C(r) \overset{\text{def}}{=} \frac{(n-1)^2}{2} \frac{\beta'_C(r)}{\beta_C(r)} - \frac{n-1}{2} \frac{\beta''_C(r)}{\beta_C(r)} + \frac{n^2-1}{4} \frac{(\beta'_C(r))^2}{(\beta_C(r))^2}$$

obeying $|V_C^{(k)}(r)| \leq C_k |r|^{-k-1}$. 
This shows that the essential spectrum of $\Delta_g$ is $\{(n-1)^2/4, \infty\}$: the compactness of $X_0 \subset X$ prevents its geometric properties from affecting this part of the spectrum (see for example [ReSi, Theorem XIII.14, Corollary 3]).

This gives rise to the following separation of variables. Let $j \in \{C, F\}$, and let $(\lambda_{j1}, \lambda_{j2}, \ldots)$ be the sequence of eigenvalues of $S_j$, in increasing order and with multiplicity. Now if $u$ is a function supported in $X_j$ we may write

$$u(r, y) = \sum_{m=1}^{\infty} u_{jm}(r) \phi_{jm}(y),$$

where $\phi_{jm}$ is the eigenfunction corresponding to $\lambda_{jm}$. If $u \in L^2_{\text{loc}}(X_j)$, we have $u_{jm} \in L^2_{\text{loc}}([0, \infty))$, and

$$u_{jm}(r) = \int_{S_j} u(r, y) \phi_{jm}(y) dS_y,$$

for almost every $r$. We may now write

$$Pu(r, y) = \left( \sum_{m=1}^{\infty} [Q_j(h^2\lambda_{jm})u_{jm}(r)] \phi_{jm}(y) \right),$$

with the $Q_j(\alpha)$ a family of ordinary differential operators parametrized by $\alpha \in [0, \infty)$, given by

$$Q_F(\alpha) = h^2D_r^2 + e^{-2r}\alpha\beta_F(r) + h^2V_F(r) - 1,$$

and

$$Q_C(\alpha) = h^2D_r^2 + e^{2r}\alpha\beta_C(r) + h^2V_C(r) - 1.$$ (3.14)

We will treat the $V_j$ as lower order terms, and use only the fact that they are holomorphic in $\{z \in \mathbb{C}: |\arg z| \leq \theta\}$, and satisfy

$$|V_j^{(k)}(z)| \leq C|z|^{-k-1}$$

uniformly there. We will not use the explicit formulas for them derived in §3.4, which is important for applications to general relativity, more precisely to the Schwarzschild and De Sitter-Schwarzschild models for static black holes: see for example [SáZw97, BoHä08, MeSáVa].

Finally we discuss the domain of $P$. The domain of $\Delta_g$ is given by

$$H^2(X) \overset{\text{def}}{=} \{ u \in L^2(X): \Delta_g u \in L^2(X) \}.$$ 

By definition $P$ is an unbounded operator on $L^2_\varphi(X) \overset{\text{def}}{=} \{ e^\varphi u : u \in L^2(X) \}$, and its domain is

$$H^2_\varphi(X) \overset{\text{def}}{=} \{ u \in L^2_\varphi(X): e^{-\varphi}\Delta_g e^\varphi u \in L^2_\varphi(X) \} = \{ e^\varphi u : u \in H^2(X) \}.$$ 

4. PSEUDODIFFERENTIAL OPERATORS

We review here standard material from microlocal analysis, following the presentation of [EvZw] (see also [DiSj99]).
4.1. Pseudodifferential operators on \( \mathbb{R}^n \). We use the notation \( S^{m,k}_\delta(\mathbb{R}^n) \) to denote the symbol class of functions \( a = a(h,x,\xi), a \in C^\infty((0,1) \times T^*\mathbb{R}^n) \) which obey

\[
\partial_x^\alpha \partial_\xi^\beta a \leq C_{\alpha,\beta} h^{-k-\delta(|\alpha|+|\beta|)}(1+|\xi|^2)^{(m-|\beta|)/2},
\]

(4.1)

uniformly in \( T^*\mathbb{R}^n \). The \textit{semiclassical principal symbol} of \( a \) is the equivalence class of \( a \) in \( S^{m,k}_\delta(\mathbb{R}^n)/S^{m-1,k-1}_\delta(\mathbb{R}^n) \).

We quantize \( a \in S^{m,k}_\delta(\mathbb{R}^n) \) to an operator \( \text{Op}(a) \) using the formula

\[
(\text{Op}_\hbar(a)u)(x) = \frac{1}{(2\pi \hbar)^n} \int e^{i(x-y)\cdot\xi/\hbar} a(h,x,\xi) u(y) dy d\xi,
\]

(4.2)

and say that \( \text{Op}_\hbar(a) \in \Psi^{m,k}_\delta(\mathbb{R}^n) \). Observe that \( \sup \text{Op}_\hbar(a)u \subset \sup \text{a}(h,\cdot,\xi) \). From [EvZw] Theorem 4.22 we know that for \( a \in S^{0,0}_\delta(\mathbb{R}^n) \) with \( \delta \leq 1/2 \), the operator \( \text{Op}(a) \) is bounded \( L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) uniformly in \( h \). Using the composition property [EvZw] Theorem 8.2] we see that if \( a \in S^{m,k}_\delta(\mathbb{R}^n) \) with \( \delta \leq 1/2 \), then \( h^k \text{Op}(a) \) is uniformly bounded \( H^s_\hbar(\mathbb{R}^n) \to H^{s-m}_\hbar(\mathbb{R}^n) \) for all \( s \), where

\[
\|u\|_{H^s_\hbar(\mathbb{R}^n)} \overset{\text{def}}{=} \|(1+h^2\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}.
\]

Moreover if \( A \in \Psi^{m,k}_\delta(\mathbb{R}^n) \) and \( B \in \Psi^{m',k'}_\delta(\mathbb{R}^n) \), then \( AB \in \Psi^{m+m',k+k'}_\delta(\mathbb{R}^n) \) and commutator \([A,B] \in \Psi^{m+m'-1,k+k'-1}_\delta(\mathbb{R}^n) \). If \( a \) and \( b \) are respectively the semiclassical principal symbols of \( A \) and \( B \), then the semiclassical principal symbol of \([A,B]\) is \( \text{i} \hbar^{-1}\{a,b\} \), where \( \{\cdot,\cdot\} \) denotes the Poisson bracket.

For \( K \subset T^*\mathbb{R}^n \) such that either \( K \) or its complement is precompact in the \( \xi \) variable, we say that \( a \in S^{m,k}_\delta(\mathbb{R}^n) \) is \textit{elliptic} on \( K \) if

\[
|a| \geq c h^{-k}(1+|\xi|^2)^{m/2},
\]

(4.3)

uniformly for \((x,\xi) \in K \). We say that \( A \in \Psi^{m,k}_\delta(\mathbb{R}^n) \) is elliptic on \( K \) if its principal symbol \( a \) is.

We define the semiclassical wavefront set of a pseudodifferential operator \( A \in \Psi^{m,k}_\delta(\mathbb{R}^n) \), which we denote by \( \text{WF}_\hbar A \), as follows. We say that a point \((x_0,\xi_0) \in T^*\mathbb{R}^n \) is not in \( \text{WF}_\hbar A \) if \(|\partial^\alpha a| = O(h^\infty) \) near \((x_0,\xi_0) \) for any multiindex \( \alpha \), where \( a \) is the full symbol of \( A \) (that is to say, \( A = \text{Op}_\hbar(a) \)). What will be important for us is that

\[
\text{WF}_\hbar A \cap \text{WF}_\hbar B = \emptyset \implies \|AB\|_{H^{-N}_\hbar(\mathbb{R}^n) \to H^N_\hbar(\mathbb{R}^n)} = O(h^N), \quad \forall N \in \mathbb{N},
\]

(4.4)

provided at least one of \( A \) and \( B \) is the quantization of a function in \( C^\infty_0(T^*\mathbb{R}^n) \). This again follows from the composition formula [EvZw] Theorem 8.2]..

The wavefront set allows us to define a notion of local invertibility in a region where an operator is elliptic. If \( A \in \Psi^{m,k}_\delta(\mathbb{R}^n) \) with \( \delta < 1/2 \) is elliptic on \( K \subset T^*\mathbb{R}^n \), we have the following estimate

\[
\|Bu\|_{H^s_\hbar(\mathbb{R}^n)} \leq C h^k \|ABu\|_{H^{s+m}_\hbar(\mathbb{R}^n)},
\]

(4.5)
for $h \in (0, h_0]$, provided $WF_h B \subset K^\circ$ and either $K$ or its complement is precompact in the $\xi$ variable. The constants $C$ and $h_0$ depend on finitely many derivatives of the principal symbol of $A$.

We will also need the sharp Gårding inequality [EvZw. Theorem 4.2]. This says that if $A \in \Psi_0^m(\mathbb{R}^n)$ has principal symbol $a$ obeying $a \geq 0$ on $K \subset T^*\mathbb{R}^n$, then

$$
\langle ABu, Bu \rangle_{L^2(\mathbb{R}^n)} \geq -C h \|Bu\|_{L^2(\mathbb{R}^n)}^2,
$$

provided $WF_h B \subset K^\circ$ and either $K$ or its complement is precompact in the $\xi$ variable.

4.2. Pseudodifferential operators on a manifold. The results in the previous section can be directly extended to the case of a noncompact manifold $X$, provided we require our estimates to be uniform only on compact subsets of $X$. For this reason the distinction between $L^s(\mathbb{R}^n)$ and $L^s_h(\mathbb{R}^n)$ will not be relevant for this section, as these two spaces are the same on compact subsets of $X$.

Write $S^m_{\delta}(X)$ for the symbol class of functions $a \in C^\infty((0, 1) \times T^*X)$ satisfying (4.1) on coordinate patches (note that this condition is invariant under change of coordinates). The principal symbol corresponding to $a$ is the equivalence class of $a$ in $S^m_{\delta}(X)/S^{m-1,k-1}_{\delta}(X)$.

We quantize $a \in S^m_{\delta}(X)$ to an operator $a^\ell \in \Psi^m_{\delta}(X)$ by using a partition of unity and the formula (4.2) in coordinate patches. Our quantization depends on the choice of partition of unity and on the choice of coordinates, but the class $\Psi^m_{\delta}(X)$ does not. Observe that $supp a^\ell u \subset supp a(h_\ast, \xi)$. In this case we have boundedness of the operator only on compact subsets of $X$, that is if $A \in \Psi^m_{\delta}(X)$ and $K_0 \subset X$ is compact, then $h^k A$ is uniformly bounded $H^s_h(K_0) \to H^{s-m}(K_0)$, where

$$
\|u\|_{H^s_h(K_0)} \overset{\text{def}}{=} \| (1 + h^2 \Delta_g)^{s/2} u \|_{L^2(K_0)}.
$$

Just as in the case $X = \mathbb{R}^n$, if $A \in \Psi^m_{\delta}(X)$ and $B \in \Psi^{m', k'}_{\delta}(X)$, then $AB \in \Psi^{m+m', k+k'}_{\delta}(X)$ and commutator $[A, B] \in \Psi^{m+m'-1,k+k'-1}_{\delta}(X)$. If $a$ and $b$ are respectively the semiclassical principal symbols of $A$ and $B$ (the principal symbol remains invariantly defined for manifolds, although the total symbol does not), then the semiclassical principal symbol of $[A, B]$ is $ih^{-1}\{a, b\}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket arising from the natural symplectic form on $T^*X$.

We say that $a \in S^m_{\delta}(X)$ is elliptic on a set $K \subset T^*X$ if it satisfies (4.3) in local coordinates on $K$ and if either $K$ or its complement is precompact in the fibers. We say that $A \in \Psi^m_{\delta}(X)$ is elliptic on $K$ if its principal symbol $a$ is elliptic on $K$.

We define the semiclassical wavefront set of a pseudodifferential operator $A \in \Psi^m_{\delta}(X)$, which we denote by $WF_h A$, by using, in local coordinates, the criterion for a pseudodifferential operator on $\mathbb{R}^n$. That this notion is invariant under change of coordinates follows from, for example [EvZw (8.43)]. We again have

$$
WF_h A \cap WF_h B = \emptyset \implies \|AB\|_{H^{-N}(X) \to H^N(X)} = O(h^N), \ \forall N \in \mathbb{N}, \quad (4.7)
$$
provided at least one of \( A \) and \( B \) is the quantization of a function in \( C^\infty_0(T^*X) \).

The wavefront set allows us to define a notion of local invertibility in a region where an operator is elliptic. If \( A \in \Psi^{n,k}_\delta(X) \) is elliptic on a set \( K \subset T^*X \), we have the following estimate

\[
\|Bu\|_{H^s_h(K_0)} \leq Ch^k\|ABu\|_{H^{s+m}_h(K_0)},
\]

provided \( K_0 \subset X \) is compact, WF\(_h\) \( B \subset K^c \), and either \( K \) or its complement is precompact in the fibers.

Finally we again have a sharp Gårding inequality. This says that if \( A \in \Psi^{0,0}_\delta(X) \) has principal symbol \( a \) obeying \( a \geq 0 \) on \( K \subset T^*X \),

\[
\langle ABu, Bu \rangle_{L^2(K_0)} \geq -Ch\|Bu\|_{L^2(K_0)}^2,
\]

provided \( K_0 \subset X \) is compact, WF\(_h\) \( B \subset K^c \), and either \( K \) or its complement is precompact in the fibers.

### 4.3. Exponentiation of operators

For \( G \in C^\infty_0(T^*X) \) and \( \varepsilon \in [0, C_0h \log(1/h)] \), we will be interested in operators of the form \( e^{\varepsilon G^\varepsilon/h} \). Recall that \( G^\varepsilon \) is compactly supported in space and hence bounded \( L^2(X) \to L^2(X) \) (and also \( L^2_{\varepsilon}(X) \to L^2_{\varepsilon}(X) \)) uniformly in \( h \). We write

\[
e^{\varepsilon G^\varepsilon/h} = \sum_{j=0}^{\infty} \frac{(\varepsilon/h)^j}{j!}(G^\varepsilon)^j,
\]

with the sum converging in the \( L^2 \) norm operator topology, although this convergence is not uniform as \( h \to 0 \). To show that \( e^{\varepsilon G^\varepsilon/h} \) is a pseudodifferential operator, we use Beals’s characterization of pseudodifferential operators [EvZw, Theorem 8.13]. This says that a bounded operator \( A \) on \( L^2(X) \) is a pseudodifferential operator in class \( \Psi^{0,k}_\delta(X) \) if and only if we have

\[
\|\text{ad}_{L_1} \cdots \text{ad}_{L_N} A\|_{L^2(X) \to L^2(X)} = \mathcal{O}(h^{N-k-2\delta}),
\]

for all \( N = 0, 1, 2, \ldots \) and any \( L_1, \ldots, L_N \) compactly supported differential operators of order 1. Here \( \text{ad}_B A \) is defined \( [A,B] \). We use this to show that \( e^{\varepsilon G^\varepsilon/h} \in \Psi^{0,k}_\delta(X) \) for \( k = C_0||G|| \) and for any \( \delta > 0 \). Indeed, for \( N = 0 \) we have (using \( \| \cdot \| \) to denote the \( L^2 \) operator norm),

\[
\| e^{\varepsilon G^\varepsilon/h} \| \leq \sum_{j=0}^{\infty} \frac{(C_0 \log(1/h))^j}{j!}\| G \|^j = e^{C_0 \log(1/h)||G||} = h^{-C_0||G||}.
\]

For \( N = 1 \) we have

\[
\| \text{ad}_L e^{\varepsilon G^\varepsilon/h} \| \leq \sum_{j=1}^{\infty} \frac{(C_0 \log(1/h))^j}{(j-1)!}\| \text{ad}_L G \|\| G \|^{j-1}
\]

\[
= C_0 \log(1/h)\| \text{ad}_L G \| h^{-C_0||G||} \leq Ch^{1-C_0||G||-2\delta}.
\]

Similarly, for \( N \geq 2 \) we have

\[
\| \text{ad}_{L_1} \cdots \text{ad}_{L_N} e^{G^\varepsilon/h} \| \leq C_N(\log(1/h))^Nh^{1-C_0||G||} \leq \tilde{C}_Nh^{1-C_0||G||-2\delta}.
\]
To understand $e^{\varepsilon G^\ell/h}$ better we will use the following Taylor expansion:

$$e^{\varepsilon G^\ell/h} = \text{Id} + \varepsilon h^{-1} G^\ell + \varepsilon h^{-1} G^\ell \left( \int_0^1 e^{\varepsilon s G^\ell/h} (1 - s) ds \right) \varepsilon h^{-1} G^\ell. \quad (4.11)$$

Observe that (4.11) and (4.7) imply that

$$\text{WF}_h(e^{\varepsilon G^\ell/h} - \text{Id}) \subset \text{WF}_h(G^\ell) \quad (4.12)$$

Furthermore, if $A \in \Psi^{m,k}(X)$, then $e^{-\varepsilon \text{ad}_{G^\ell/h}} A e^{\varepsilon G^\ell/h} = e^{-\varepsilon \text{ad}_{G^\ell/h}} A$. We again write a Taylor series:

$$e^{-\varepsilon \text{ad}_{G^\ell/h}} A = \sum_{j=0}^\infty \varepsilon^j \left( \frac{\text{ad} G^\ell}{h} \right)^j A,$$

with convergence in the $L^2$ norm operator topology. Together with Beals’s characterization (4.10), this shows that $e^{-\varepsilon \text{ad}_{G^\ell/h}} A e^{\varepsilon G^\ell/h} = A - \varepsilon [A, G^\ell/h] + \varepsilon^2 R,$

where $R \in \Psi^{-\infty,k}(X)$.

5. Complex scaling in the cusp

We deform the operator in the cusp using the method of complex scaling for one dimensional Schrödinger operators.

5.1. Outline. We use the separation of variables derived in §3.4. We will deform the operators $Q_C(\alpha)$ by replacing them with $Q_{R,C}(\alpha)$, where

$$Q_{R,C}(\alpha) \overset{\text{def}}{=} \frac{h^2 D_r^2}{(1 + if''_C)^2} + e^{2(r + if''_C)} \alpha \beta_C(r + if''_C) + h B_C(r + if''_C) - 1. \quad (5.1)$$

The function $f_C(r)$, which depends also on $R$ and $\alpha$, will be discontinuous in $R$ and $\alpha$ but smooth in $r$, and $'$ here and below denotes differentiation with respect to $r$. The function $f''_C$ will be constructed below in §5.2, but for now let us only specify that it will satisfy $|\arg(r + if''_C(r))| < \theta$ for $r > 0$, and $f''_C = 0$ for $r \leq R$. The operator $B_C$ is given by

$$B_C(r + if''_C) \overset{\text{def}}{=} \frac{f''_C}{(1 + if''_C)^3} h D_r + h V_C(r + if''_C).$$

This deformation is motivated by the complex change of variables $r \mapsto r + if''_C$. In other words, we have

$$u' \mapsto \frac{u'}{1 + if''_C}, \quad u'' \mapsto \frac{u''}{(1 + if''_C)^2} - \frac{if''_C u'}{(1 + if''_C)^3}. \quad (5.2)$$

The first term of (5.2) gives us the first term of (5.1). The second term is smaller by a factor of $h$, and becomes a part of $B$. This idea will be explained further in §7 when we use the $Q_{R,C}$ to define $P_R$. 
The semiclassical principal symbol of $Q_{R,C}$ is
\[
q_{R,C}(\alpha) \overset{\text{def}}{=} \rho^2 \left(1 + if_C^2\right)^2 + e^{2(r + if_C)}\alpha\beta_C(r + if_C) - 1
\]
We will choose $f_C$ such that we have $f_C = 0$ for $r \leq R$, and such that for a fixed $R_C > 0$, $\delta > 0$,
\[
|\Re q_{R,C}| \leq \delta \implies |\Im q_{R,C}| \leq -\delta,
\]
for all $r \geq R + R_C$ and $\alpha \geq 0$. Also we will have
\[
|\Re q_{R,C}| \leq \delta \implies |\Im q_{R,C}| \leq 0,
\]
for all $r \geq 0$ and $\alpha \geq 0$. Crucially, $R_C$ and $\delta$ are independent of $R$. In fact, they will depend only on $\theta$. Next for every $\varepsilon' > 0$ there is a $\delta' > 0$ such that
\[
|q_{R,C}| \leq \delta' \implies |f_C(r)| + |f'_C(r)| + |f''_C(r)| \leq \varepsilon'
\]
for all $r \geq 0$ and $\alpha \geq 0$. Finally, for each $\alpha \geq 0$ we will have $R^\alpha_C > 0$ such that
\[
|\Re q_{R,C}| \leq \frac{2}{3}e^{2r},
\]
for all $r \geq R^\alpha_C$.

5.2. Construction of $f_C$. We construct here the function $f_C$ used in the definition of the deformed operators (5.1). The exponential term provides some ellipticity whenever $\alpha \neq 0$, but this ellipticity is not uniform for $\alpha$ small. Consequently we take $f_C \equiv 0$ unless $\alpha$ is very small.

5.2.1. The case $\alpha$ bounded below. Unless $\alpha$ is extremely small, the $e^{2r}$ term will provide the ellipticity needed in (5.3) – (5.6). We accordingly put
\[
f_C(r) \overset{\text{def}}{=} 0, \quad \text{when } \alpha > \frac{\tan \theta}{4} e^{-2(R + \pi/\tan \theta)}.
\]
Observe that in this case, we have $\Im q_{R,C} \equiv 0$. Moreover, using (2.3), we have
\[
\Re q_{R,C} \geq e^{2r} \alpha \beta_C(r) - 1 \geq \frac{2}{3}e^{2r} \alpha - 1 \geq 1,
\]
provided $e^{2r} \geq 3\alpha^{-1}$, and hence we have shown that
\[
R_C \geq \frac{1}{2} \log \frac{12}{\tan \theta} + \frac{\pi}{\tan \theta} \implies (5.3) - (5.6) \text{ hold for } \alpha > \frac{\tan \theta}{4} e^{-2(R + \pi/\tan \theta)}.
\]

5.2.2. The case $\alpha$ small. For $0 \leq \alpha \leq \frac{\tan \theta}{4} e^{-2(R + \pi/\tan \theta)}$, we define $f_C$ as follows.

We describe the construction in each of the three regions depicted in Figure 3. For ease of exposition we define $f_C$ so that it is continuous and piecewise smooth. To obtain a smooth $f_C$ one convolves with an approximate identity.
Figure 3. The contour $f_C$ for $0 < \alpha \leq \frac{\tan \theta}{4} e^{-2(R+\pi/\tan \theta)}$. We use the standard complex scaling while $e^{2r} \alpha$ is small enough to be negligible. When this is no longer the case, we “level off” $f_C$ at a place such that $\sin(2f_C)$ is negative, so that $\text{Im} \ q_{fC}$ stays uniformly negative thanks to the $\sin(2f_C)$ term.

Observe that

$$\text{Re} \ q_{R,C} = (1 - f_C^2) \frac{\rho^2}{|1 + if_C'|^4} + e^{2r} \alpha (\cos 2f_C \ \text{Re} \ \beta_C - \sin 2f_C \ \text{Im} \ \beta_C) - 1,$$

$$\text{Im} \ q_{R,C} = 2f_C \frac{\rho^2}{|1 + if_C'|^4} + e^{2r} \alpha (\cos 2f_C \ \text{Im} \ \beta_C + \sin 2f_C \ \text{Re} \ \beta_C).$$

1. In region I we construct $f_C$ such that (5.4) holds, and such that $f_C'(R+1, \alpha) = \tan \theta$. Put

$$f_C = C_1 e^{C_2/(R-r)}.$$

For $|1 + if_C'|^{-4} \rho^2 \leq 1/2$, observe that, using (2.3), we have

$$\text{Re} \ q_{R,C} \leq \frac{1}{2} + \frac{4}{3} e^{2(R+1)} \alpha - 1 \leq -\frac{1}{4}.$$

Meanwhile for $|1 + if_C'|^{-4} \rho^2 \geq 1/2,$

$$\text{Im} \ q_{R,C} \leq -f_C' + e^{2(R+1)} \alpha (\cos (2f_C) \ \text{Im} \ \beta_C + \sin (2f_C) \ \text{Re} \ \beta_C).$$

But observe that $\sin(2f_C) \leq 2f_C$, and $| \text{Im} \ \beta_C | \leq C_3 f_C$ uniformly on compact sets. Hence we have

$$e^{2(R+1)} \alpha | \cos (2f_C) \ \text{Im} \ \beta_C + \sin (2f_C) \ \text{Re} \ \beta_C| \leq C_4 f_C,$$

and for $C_2$ large we have (5.4). Finally take $C_1$ such that $f_C'(R+1) = \tan \theta.$
(2) In region II we take \( f_C' = \tan \theta \). The point \( R_C^\alpha \) is chosen such that
\[
\frac{\tan \theta}{8} \leq e^{2R_C^\alpha \alpha} \leq \frac{3 \tan \theta}{8}, \tag{5.8}
\]
and such that \( f_C(R_C^\alpha) = 3\pi/4 + \pi k \) for some \( k \in \mathbb{N}_0 \). To see that these two requirements are consistent, we must check that \( f_C \) climbs by enough over the interval \( (5.8) \).

The amount by which \( f_C \) climbs over the interval \( (5.8) \) is
\[
\tan \theta \left( \log \frac{3(\tan \theta)/8 - (\tan \theta)/8}{\alpha} \right) = \tan \theta \left( \log \frac{\tan \theta}{4\alpha} \right).
\]
To make this greater than \( \pi \), we need \( \alpha \leq e^{-2\pi/\tan \theta \tan \theta/4} \), which we have. In the case \( \alpha = 0 \), we take \( R_C^\alpha = \infty \), and there is no region III.

We further check that if \(|1 + if_C'|^{-4} \rho^2 \leq 1/2\), then, using \( (5.8) \) and \( (2.5) \), we have
\[
\Re q_{R,C} \leq \frac{1}{2} + e^{2R_C^\alpha \alpha} |\beta_C| - 1 \leq \frac{\tan \theta}{2} - \frac{1}{2} \leq -\frac{1}{4},
\]
while if \(|1 + if_C'|^{-4} \rho^2 \geq 1/2\), then
\[
\Im q_{R,C} \leq -\tan \theta + e^{2R_C^\alpha \alpha} |\beta_C| \leq -\frac{\tan \theta}{2}.
\]

(3) In region III we take \( f_C \equiv f_C(R_C^\alpha) \). As a result we have \( \sin(f_C) \equiv -1 \). In this region we have, again using \( (5.8) \) and \( (2.5) \),
\[
\Im q_{R,C} \leq -\frac{2}{3} e^{2\rho^2} \alpha \leq -\frac{2}{3} e^{2R_C^\alpha \alpha} \leq -\frac{\tan \theta}{12}.
\]

As a result of this construction we have
\[
R_C \geq 1 \implies (5.3)-(5.6) \text{ hold for } \alpha \leq \frac{\tan \theta}{4} e^{-2(R+\pi/\tan \theta)}.
\]

6. Complex scaling in the funnel

As we did in the cusp, we deform the operator \( P \) in the funnel using the method of complex scaling. We follow [Zwo99 §4], extending and simplifying those methods.

6.1. Outline. We again use the separation of variables derived in §3.4. We will deform the operators \( Q_F(\alpha) \) by replacing them with \( Q_{R,F}(\alpha) \), where
\[
Q_{R,F}(\alpha) \overset{\text{def}}{=} \frac{h^2 D_r^2}{(1 + if_F)^2} + e^{-2(r+if_F)\alpha} \beta_F(r + if_F) + hB_F(r + if_F) - 1. \tag{6.1}
\]
Here once again \( f_F \) will be specified below in §6.2, the symbol \( ' \) denotes differentiation with respect to \( r \), and \( B_F \) is a first order differential operator given by
\[
B_F(r + if_F) \overset{\text{def}}{=} \frac{f''_F}{(1 + if_F)^2} hD_r + hV_F(r + if_F).
\]
The semiclassical principal symbol of $Q_{R,F}$ is given by

$$q_{R,F}(\alpha) \overset{\text{def}}{=} \frac{\rho^2}{(1 + i f'_F)^2} + e^{-2(r + i f_F)} \alpha \beta_F (r + i f_F) - 1$$

We will choose $f_F$ such that we have $f_F = 0$ for $r \leq R$, and such that for a fixed $R_F > 0$, $\delta > 0$

$$|\Re q_{R,F}| \leq \delta \implies |\Im q_{R,F}| \leq -\delta,$$  \hfill (6.2)

for all $r \geq R + R_F$ and $\alpha \geq 0$. Meanwhile we will also have

$$|\Re q_{R,F}| \leq \delta \implies |\Im q_{R,F}| \leq 0$$  \hfill (6.3)

for all $r \geq 0$ and $\alpha \geq 0$. Once again $R_F$ and $\delta$ depend only on $\theta$. Finally for every $\varepsilon' > 0$ there is a $\delta' > 0$ such that

$$|q_{R,F}| \leq \delta' \implies |f_F(r)| + |f'_F(r)| + |f''_F(r)| \geq \varepsilon'$$  \hfill (6.4)

for all $r \geq 0$ and $\alpha \geq 0$.

6.2. **Construction of $f_F$.** This construction is complicated by the fact that the exponential term from (3.14) causes oscillations in $\Im q_{R,F}$ as $f$ varies. The exponential damping makes these oscillations small as $r \to \infty$, but this damping is not uniform for $\alpha$ large.

Observe that

$$\Re q_{R,F} = (1 - f'_F)^2 \frac{\rho^2}{1 + i f'_F} + e^{-2r} \alpha \cos 2f_F \Re \beta_F + \sin 2f_F \Im \beta_F - 1,$$

$$\Im q_{R,F} = 2f'_F \frac{\rho^2}{1 + i f'_F} + e^{-2r} \alpha \cos 2f_F \Im \beta_F - \sin 2f_F \Re \beta_F.$$

6.2.1. **The case $\alpha$ bounded above.** For $0 \leq \alpha \leq 6e^{2R}$, the exponential damping term is uniform in $\alpha \theta$ and we use the standard complex scaling. More precisely, we use the same scaling we used for the case $\alpha = 0$ in §5.2.2 above, but with $R_C$ replaced by $R_F^0$ taken such that $6e^{2(r - R_F^0)} \leq \tan \theta/2$. From this, using (2.3) and (2.5), we obtain that for $r \geq R_F^0$, we have

$$|e^{-2(r + i f_F)} \alpha \beta_F (r + i f_F)| \leq e^{-2R_F^0} \cdot 6e^{2R} \cdot \frac{4}{3} \leq \frac{2 \tan \theta}{3},$$

and hence, for $|1 + i f'_F|^{-4} \rho^2 \leq 1/2$, we have

$$\Re q_{R,F} \leq \frac{1}{2} + \frac{2 \tan \theta}{3} - 1 \leq -\frac{1}{6},$$

and for $|1 + i f'_F|^{-4} \rho^2 \geq 1/2$, we have, for $r \geq R_F^0 + 1$

$$\Im q_{R,F} \leq -\tan \theta + \frac{2 \tan \theta}{3} \leq -\frac{\tan \theta}{3},$$

because $f'_F(r) = \tan \theta$ when $r \geq R_F^0 + 1$.

$$R_F \geq R_F^0 - R + 1 \implies (6.2) \text{ and } (6.4) \text{ hold for } 0 \leq \alpha \leq 6e^{2R}.$$
Figure 4. The contour $f_F$ for $\alpha > 6e^{2R}$. In region I we have $e^{-2r}\alpha$ large enough that it makes $\text{Re}\ q_{R,F}$ is uniformly positive. In region II we arrange $f_F$ so that $\sin 2f$ is positive, so that $\text{Im}\ q_{R,F}$ is uniformly negative. In region III we use the fact that $e^{-2r}\alpha$ is small enough to be negligible, and we may revert to the standard complex scaling.

6.2.2. The case $\alpha$ large. For $\alpha > 6e^{2R}$, we define $f_F$ as follows.

We describe the construction in each of the three regions depicted in Figure 4. Once again for ease of exposition we define $f_F$ so that it is piecewise smooth, and to obtain a smooth $f_F$ one convolves with an approximate identity.

1. In region I we use $f_F = C_1 e^{C_2/(R-r)}$ as we did in region I of §5.2.2. We adjust $C_1$ and $C_2$ so that $f_F = 0$ for $r \leq R$, such that $f_F(r) = \pi/8$ when $e^{-2r}\alpha = 5$, and such that $\text{Im}\ q_{R,F} \geq 0$ everywhere (for more details see the discussion of region I in §5.2.2). In this case we have, using (2.3),

$$\text{Re}\ q_{R,F} \geq e^{-2r}\alpha(\cos(2f_F)\text{Re}\ \beta_F + \sin(2f_F)\text{Im}\ \beta_F) - 1 \geq \frac{1}{3\sqrt{2}} e^{-2r}\alpha - 1 > \frac{1}{6}.$$

2. In region II we take $f_F \equiv \pi/8$, and $R_F^\alpha$ is chosen such that $e^{-2R_F^\alpha}\alpha = \tan \theta/2$. In this region we have, using (2.3),

$$\text{Im}\ q_{R,F} \leq e^{-2r}\alpha(\cos(2f_F)\text{Im}\ \beta_F - \sin(2f_F)\text{Re}\ \beta_F) \leq -\frac{\tan \theta}{2} \cdot \frac{1}{3\sqrt{2}}.$$

3. In region III we take $f_F(r, \alpha) = (r - R_F^\alpha) \tan \theta + \pi/8$. Now if $|1 + if_F|^{-4} \rho^2 \leq 1/2$ we have, using $e^{-2r}\alpha = \tan \theta/2$, (2.3) and (2.5),

$$\text{Re}\ q_{R,F} \leq e^{-2r}\alpha(\cos(2f_F)\text{Re}\ \beta_F + \sin(2f_F)\text{Im}\ \beta_F) - \frac{1}{2} \leq 2\tan \theta - \frac{1}{2} \leq -\frac{1}{4},$$

while if $|1 + if_F|^{-4} \rho^2 \geq 1/2$ we have

$$\text{Im}\ q_{R,F} \leq -\tan \theta + e^{-2r}\alpha(\cos(2f_F)\text{Im}\ \beta_F - \sin(2f_F)\text{Re}\ \beta_F) \leq -\tan \theta + \frac{2\tan \theta}{3}.$$

1 It may be that the point $z = \log(\alpha/5)/2 + i\pi/8$ does not have $|\text{arg}\ z| < \theta$ when $\alpha = 6e^2$, so that $\beta_F$ is not holomorphic there. This can be remedied by taking $X_0$ larger in the decomposition in §2.1.
This construction gives us \((6.2) - (6.4)\) for \(\alpha > 6e^{2R}\) without any additional restriction on \(R_F\).

7. The complex scaled operator and a characterization of resonances.

This section follows in part [SjZw91, §3] and [Zwo99, §§4-5]. Using the notation of [§3.4, (5.1), and (6.1)] we now define the complex scaled operator as follows:

1. When \(\text{supp} u \subset X_0\), put \(P_Ru \overset{\text{def}}{=} Pu\).
2. When \(\text{supp} u \subset X_C\), put
   \[
P_Ru \overset{\text{def}}{=} \sum_{m=1}^{\infty} \left[ Q_{R,C}(h^2\lambda_{Cm})u_{Cm}(r) \right] \phi_{Cm}(y).
   \]
3. When \(\text{supp} u \subset X_F\), put
   \[
P_Ru \overset{\text{def}}{=} \sum_{m=1}^{\infty} \left[ Q_{R,F}(h^2\lambda_{Fm})u_{Fm}(r) \right] \phi_{Fm}(y).
   \]

By linearity this definition can be applied to all \(u \in C^\infty_0(X)\). We extend it to \(H^2_\varphi(X)\), the domain of \(P\), using the following

**Lemma 7.1.** The operator \(P_R\) is bounded \(H^2_\varphi(X) \to L^2_\varphi(X)\).

We use the notation
\[
\|u\|_{H^m_\varphi(X)} \overset{\text{def}}{=} \|(1 + P)^{m/2}u\|_{L^2_\varphi(X)} = \|e^\varphi(1 + P)^{m/2}u\|_{L^2(X)}.
\]

**Proof.** We clearly have
\[
\|P_Ru\|_{L^2_\varphi(X)} \leq \|u\|_{H^2_\varphi(X)}, \quad \text{when } \text{supp} u \subset \{P_f \equiv P\}.
\]

Meanwhile, for \(j \in \{C, F\}\),
\[
\|Pu\|_{L^2_\varphi(X_j)} = \sum_{m=1}^{\infty} \|Q_j(h^2\lambda_{jm})u_{jm}\|_{L^2((0,\infty))},
\]
\[
\|P_Ru\|_{L^2_\varphi(X_j)} = \sum_{m=1}^{\infty} \|Q_{R,j}(h^2\lambda_{jm})u_{jm}\|_{L^2((0,\infty))},
\]
and hence it is enough to prove that
\[
\|Q_{R,j}(h^2\lambda_{jm})u_{jm}\|_{L^2((0,\infty))} \leq C \left( \|Q_j(h^2\lambda_{jm})u_{jm}\|_{L^2((0,\infty))} + \|u_{jm}\|_{L^2((0,\infty))} \right) + \|h^2D^2_{\varphi}u_{jm}\|_{L^2((0,\infty))}
\]
(7.2)
with a constant uniform in \( m \). This is because
\[
\sum_{m=1}^{\infty} \| Q_j(h^2 \lambda jm) u_{jm} \|^2_{L^2([0,\infty))} = \| Pu \|^2_{L^2(x_j)},
\]
\[
\| u_{jm} \|^2_{L^2([0,\infty))} + \| h^2 D_r^2 u_{jm} \|^2_{L^2([0,\infty))} \leq \| Q_j(h^2 \lambda jm) u_{jm} \|^2_{H_{h,\infty}(x_j)}.
\]

Now to prove (7.2), observe that
\[
\| Q_{R,j}(h^2 \lambda jm) u_{jm} \|_{L^2} \leq C \left( \| h^2 D_r^2 u_{jm} \|_{L^2} + \| e^{\pm 2r} h^2 \lambda jm u_{jm} \|_{L^2} + \| h B_j u_{jm} \|_{L^2} + \| u_{jm} \|_{L^2} \right),
\]
with ‘+’ in the case \( j = C \) and ‘-’ in the case \( j = F \). Each of these terms is bounded by
\[
C(\| u_{jm} \|_{L^2([0,\infty))} + \| h^2 D_r^2 u_{jm} \|_{L^2([0,\infty))}),
\]
with the exception of the second. For that one we observe that
\[
\| e^{\pm 2r} h^2 \lambda jm u_{jm} \|_{L^2} = \| \beta_j(r)^{-1}(Q_j(h^2 \lambda jm) - (h D_r)^2 - h^2 V_j + 1) u_{jm} \|_{L^2}
\]
\[
\leq C(\| Q_j(h^2 \lambda jm) u_{jm} \|_{L^2} + \| u_{jm} \|_{L^2} + \| h^2 D_r^2 u_{jm} \|_{L^2}).
\]

We now establish the existence of a domain of invertibility for the \( Q_{jR} \).

**Lemma 7.2.** There exist \( h_0 > 0 \) and \( c_0 > 0 \) such that for all \( R \geq 1 \), and \( h \in (0, h_0] \) the operator \((P_R - \zeta)^{-1}\) is meromorphic with finite rank poles in
\[
\{ \zeta \in \mathbb{C}: |\zeta| < c_0 \} \cup \{ \zeta \in \mathbb{C}: |\text{Re} \, \zeta| < c_0, \text{Im} \, \zeta > 0 \}.
\]

To prove this, we first use the results of \[5\] and \[6\] to construct inverses near infinity of the \( Q_{jf} \).

**Lemma 7.3.** Let \( W \in C^\infty(\mathbb{R}) \) have \( \text{supp} \, W \subset (-\infty, 1] \) and \( W(r) \geq 1 \) for \( r \leq 0 \). For \( j \in \{C, F\} \), put \( W_{R,j}(r) \equiv W(r - R - R_j) \), with \( R_C \) as in the end of \[5\] and \( R_F \) as in the end of \[6\]. Define
\[
Q_{R,j}^W(\alpha) = Q_{R,j}(\alpha) - i W_{R,j}
\]
Then there exist \( h_0 > 0 \) and \( c_0 > 0 \) such that for all \( R \geq 1 \), and \( h \in (0, h_0] \), \( j \in \{C, F\} \), \( m \in \mathbb{N} \) and \( \alpha \geq 0 \), the operator \((Q_{R,j}^W(\alpha) - \zeta)^{-1}\) is invertible on \( L^2(\mathbb{R}) \) with
\[
\| (Q_{R,j}^W(\alpha) - \zeta)^{-1} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq C(1 + \text{Im} \, \zeta)^{-1}
\]
when \( \zeta \) is in
\[
\{ \zeta \in \mathbb{C}: |\zeta| < c_0 \} \cup \{ \zeta \in \mathbb{C}: |\text{Re} \, \zeta| < c_0, \text{Im} \, \zeta > 0 \}.
\]

**Proof of Lemma 7.3** Let \( q_{R,j}^W \) denote the semiclassical principal symbol of \( Q_{R,j}^W \). Then
\[
q_{R,j}^W = q_{R,j} - i W = \frac{\rho^2}{(1 + i f_j)^2} + e^{\pm 2(r + i f_j)} \alpha \beta_j(r + i f_j) - 1 - i W_{R,j}(r),
\]
with ‘+’ for \( j = C \) and ‘-’ for \( j = F \).

In the case \( j = F \) there exists a constant \( C_\delta \) depending on the \( \delta \) in (6.2) and (6.3),
\[
(\rho)^2/C_\delta \leq \frac{|q_{R,j}^W - \zeta|}{1 + \text{Im} \, \zeta} \leq C_\delta (\rho)^2.
\]
uniformly for \( r, \rho \in \mathbb{R}, \alpha \geq 0, R \geq 1, \) and \( \zeta \in \{7.4\}. \) Hence \((1 + \text{Im } \zeta)^{-1}(q_{R,j}^W - \zeta) \in S^2(\mathbb{R})\) is elliptic. By \((4.5)\), there exists \( h_0 > 0 \) depending on \( C_\delta \) and on finitely many derivatives of \( q_{R,j}^W \) such that \((q_{R,j}^W(\alpha) - \zeta)(1 + \text{Im } \zeta)^{-1}\) is invertible with uniformly bounded inverse.

In the case \( j = C \) more care is needed because the exponentially growing potential function prevents us from applying the results of \( 4.1 \). We instead take a more “bare hands” approach, and construct separately a left and a right inverse satisfying \((7.3)\). As a left approximate inverse, take

\[
\text{Op}_r \left( \frac{1 + e^{2r}}{q_{R,C}^W - \zeta} \right) \circ \frac{1}{1 + e^{2r}},
\]

where

\[
\text{Op}_r \left( \frac{1 + e^{2r}}{q_{R,C}^W - \zeta} \right) u(r) \overset{\text{def}}{=} \frac{1}{2\pi h} \int \int e^{i\rho(r-r')/h} \frac{1 + e^{2r'}}{q_{R,C}^W(\rho, r') - \zeta} u(r') dr' d\rho.
\]

Observe that the estimates \((5.3)\) and \((5.4)\) ensure that \( q_{R,C}^W - \zeta \) is never zero. We wish to write

\[
\text{Op}_r \left( \frac{1 + e^{2r}}{q_{R,C}^W - \zeta} \right) \circ \frac{1}{1 + e^{2r}} \circ (Q_{R,j}^W - \zeta) = \text{Id} + \mathcal{O}_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}(h),
\]

where the estimate on the error term is uniform in \( R \geq 1, \alpha \geq 0, \) and \( \zeta \in \{7.4\}, \) and then remove the remainder by Neumann series. Notice that \((1 + e^{2r})^{-1} \circ Q_{R,j}^W \in \Psi^2(\mathbb{R})\) and \( \text{Op}_r \left( (1 + e^{2r})/(q_{R,C}^W - \zeta) \right) \in \Psi^0(\mathbb{R}), \) and hence the following computation makes sense when \( u \) is in Schwartz class (we use here \( \hat{u}(\rho) \overset{\text{def}}{=} \int e^{-ir\rho/h} u(r) dr \)):

\[
\text{Op}_r \left( \frac{1 + e^{2r}}{q_{R,C}^W - \zeta} \right) \circ \frac{1}{1 + e^{2r}} \circ (Q_{R,j}^W - \zeta) u(r) =
\]

\[
= \frac{1}{(2\pi h)^2} \int \int e^{i\rho(r-r')/h} e^{i\rho'/h} q_{R,C}^W(\rho, r') - \zeta \hat{u}(\rho') d\rho' dr' d\rho
\]

\[
= \frac{1}{(2\pi h)^2} \int \int e^{i\rho(r-r')/h} e^{i\rho'/h} \frac{q_{R,C}^W(\rho', r') - q_{R,C}^W(\rho, r')}{q_{R,C}^W(\rho, r') - \zeta} \hat{u}(\rho') d\rho' dr' d\rho.
\]

To study the remainder, we observe that

\[
q_{R,C}^W(\rho', r') - q_{R,C}^W(\rho, r') = \frac{\rho'^2}{(1 + if_C')^2} - \frac{\rho^2}{(1 + if_C)^2} = (\rho' - \rho)(\rho' + \rho)(1 + if_C')^{-2},
\]

and hence an integration by parts in \( r' \) shows that

\[
\frac{1}{(2\pi h)^2} \int \int e^{i\rho(r-r')/h} e^{i\rho'/h} \frac{q_{R,C}^W(\rho', r') - q_{R,C}^W(\rho, r')}{q_{R,C}^W(\rho, r') - \zeta} \hat{u}(\rho') d\rho' dr' d\rho
\]

\[
= \frac{i}{(2\pi h)^2} \int \int e^{i\rho(r-r')/h} e^{i\rho'/h} \partial_r \left[ \frac{(\rho' + \rho)(1 + if_C')^{-2}}{q_{R,C}^W(\rho, r') - \zeta} \right] \hat{u}(\rho') d\rho' dr' d\rho
\]

\[
= i h \left[ \text{Op}_r(\tilde{g}) \circ (hD_r) + (hD_r) \circ \text{Op}_r(\tilde{g}) \right] u,
\]

where

\[
\tilde{g}(\rho, r') \overset{\text{def}}{=} \partial_r \left[ (1 + if_C'(r'))^{-2}(q_{R,C}^W(\rho, r') - \zeta)^{-1} \right].
\]
Now thanks to the estimates (5.3), (5.4) and (5.6), we have \((1 + \text{Im } \zeta) \rho g(\rho, r') \in S^{-1}(\mathbb{R})\), uniformly in \(\alpha \geq 0, R \geq 1\) and \(\zeta \in (7.4)\). This implies (7.5), and a Neumann series argument then constructs a full left inverse which obeys (7.3).

To obtain a right inverse, we show in the same way that

\[
(Q^W_{R,j} - \zeta) \circ \frac{1}{1 + e^{2r}} \circ \text{Op}_t \left( \frac{1 + e^{2r}}{q^W_{R,C} - \zeta} \right) = \text{Id} + O_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})}(h),
\]

where

\[
\text{Op}_t \left( \frac{1 + e^{2r}}{q^W_{R,C} - \zeta} \right) u(r) = \frac{1}{2\pi i} \int \int e^{i\rho(r - r')/h} \frac{1 + e^{2r}}{q^W_{R,C}(\rho, r) - \zeta} u(r') dr'd\rho.
\]

Proof of Lemma 7.2. We first define a partition of unity and a family of smooth cutoffs as follows. Let \(\theta_0 \in C^\infty_0(X)\) and \(\theta_C, \theta_F \in C^\infty(X)\) have \(\text{supp } \theta_C \subset X_C, \text{supp } \theta_F \subset X_F\), and \(\theta_0 + \theta_C + \theta_F = 1\). Let \(\chi_0 \in C^\infty_0(X)\) and \(\chi_C, \chi_F \in C^\infty(X)\) have \(\text{supp } \chi_C \subset X_C, \text{supp } \chi_F \subset X_F\), \(\text{supp } \chi_0 \subset \{P = R\}\), and \(\chi_0 \theta_0 = \theta_0, \chi_C \theta_C = \theta_C, \chi_F \theta_F = \theta_F\).

Fix \(\zeta_0\) with \(\text{Im } \zeta_0 > 0\) and put \(G_0(\zeta_0) = (P - \zeta_0)^{-1}\). For \(j \in \{C, F\}\) define the operator \(G_j(\zeta)\) on \(X_j\) according to the separation of variables by putting \(G_j(\zeta) u_{jm} \overset{\text{def}}{=} (Q^W_{R,j}(\hbar^2 \lambda_{jm}) - \zeta)^{-1} u_{jm}\).

Put \(E(\zeta) \overset{\text{def}}{=} \chi_0 G_0(\zeta_0) \theta_0 + \chi_F G_F(\zeta) \theta_F + \chi_C G_C(\zeta) \theta_C\). Then

\[
(P_R - \zeta) E(\zeta) = \text{Id} + K(\zeta),
\]

where

\[
K(\zeta) \overset{\text{def}}{=} (P_R - \zeta) (\chi_0 G_0(\zeta_0) \theta_0 + \chi_F G_F(\zeta) \theta_F + \chi_C G_C(\zeta) \theta_C) - \text{Id}
\]

\[
= [P_R, \chi_0] G_0(\zeta_0) \theta_0 + \chi_0(\zeta_0 - \zeta) G_0(\zeta_0) \theta_0 + \sum_{j \in \{C, F\}} [P_R, \chi_j] G_j(\zeta) \theta_j + \chi_j K_j(\zeta) \theta_j,
\]

and, for \(j \in \{C, F\}\),

\[
K_j u_{jm} \overset{\text{def}}{=} i(Q^W_{R,j}(\alpha) - \zeta)^{-1} W_{R,j} u_{jm}.
\]

Each of the terms of \(K(\zeta)\) is a bounded map from \(L^2 \rightarrow H^1_{\text{comp}}\), and hence compact by Rellich’s theorem. Now for each fixed \(\zeta_0\), the operator \(K(\zeta)\) is holomorphic in \(\zeta\) and compact. If we take \(\zeta_0 \in (7.4)\) with \(\text{Im } \zeta_0\) sufficiently large, then at \(\zeta = \zeta_0\) the operator \(\text{Id} + K(\zeta)\) is invertible because \(G_0(\zeta_0), G_C(\zeta)\) and \(G_F(\zeta)\) have small norm. Consequently by analytic Fredholm theory the family of operators \((\text{Id} + K(\zeta))^{-1}\) is meromorphic in \((7.4)\) with finite rank poles.

To obtain a left inverse we use \(F(\zeta) \overset{\text{def}}{=} \theta_0 G_0(\zeta_0) \chi_0 + \theta_F G_F(\zeta) \chi_F + \theta_C G_C(\zeta) \chi_C\) and apply the analytic Fredholm theory to \(F(\zeta)(P_R - \zeta)\).

The following lemma about cutoff resolvents provides the link between \(P\) and \(P_R\).
Lemma 7.4. If $\chi \in C_0^\infty(X)$ satisfies $\chi P = \chi P_R$, then
\[
(\chi(P - \zeta)^{-1}\chi = \chi(P_R - \zeta)^{-1}\chi
\] (7.6)
for any $\zeta$ with $\text{Im} \zeta \gg 1$ sufficiently large which is not in the spectrum of $P$ or of $P_R$.

Proof. Let $v \in L^2_{\text{comp}}(X)$ be supported in the set $\{P_R \equiv P\}$. By assumption there exists a unique $u \in H^2(\phi)(X)$ such that
\[
(P - \zeta)u = v.
\]
To prove the lemma we will use this $u$ to construct $u_R \in H^2_{h,\phi}(X)$ which solves
\[
(P_R - \zeta)u_R = v
\]
and obeys $\chi u = \chi u_R$. Indeed, on the support of $\chi$ we may just take $u_R = u$. For $j \in \{C,F\}$, observe that by (3.13) in $X_{jR}$ we have
\[
(Q_j(h^2\lambda_{jm}) - \zeta)u_{jm} = 0,
\]
for all $m$. The ordinary differential operator
\[
Q_j(h^2\lambda_{jm}) = h^2(D_r^2 + e^{\pm 2\pi} \lambda_{jm}\beta_j(r) + V_j(r)) - 1
\]
extends holomorphically to
\[
\tilde{Q}_j(h^2\lambda_{jm}) \equiv h^2(D_r^2 + e^{\pm 2\pi} \lambda_{jm}\beta_j(z) + V_j(z)) - 1,
\]
for $z$ satisfying $|\arg z| < \theta$. Moreover there exists $\tilde{u}_{jm}(z)$ holomorphic in $\{|\arg z| < \theta\}$ with $\tilde{u}_{jm}(z)\{z \in [0,\infty)\} = u_{jm}$ such that
\[
(\tilde{Q}_j(h^2\lambda_{jm}) - \zeta)\tilde{u}_{jm} = 0.
\]
Let $\Gamma_j(\alpha) = \{z \in \mathbb{C} : z = r + if_j(r)\}$. Using (7.1) and (7.2), we see that it remains to show that $\tilde{u}_{jm}|_{\Gamma_j(h^2\lambda_{jm})} \in L^2([0,\infty))$, after which we will be able to take $u_{R,jm} \equiv \tilde{u}_{jm}|_{\Gamma_j(h^2\lambda_{jm})}$.

For that we introduce a new contour, given by $\Gamma_{j,T}(\alpha) \equiv \{z \in \mathbb{C} : z = r + i\chi_T(r)f_j(r)\}$, where $\chi_T(r) \equiv \chi_0(r/T)$ and $\chi_0 \in C_0^\infty((1/2,3/2);[0,1])$ is identically 1 near 1. Now let
\[
\tilde{V}(r) = h^2(e^{\pm 2\pi} \lambda_{jm}\beta_j(z) + V_j(z))|_{z=r+i\chi_T(r)f_j(r)} - 1 - \zeta.
\]
We apply
\[
\sum_{k=0}^2 \|(\tilde{h}D_r)^k u\|_{L^2(I_2)} \leq C \left(\|(\tilde{h}D_r)^2 u + \tilde{V}(r)u\|_{L^2(I_1)} + \|u\|_{L^2(I_1\setminus I_2)}\right), \quad \tilde{h} \in (0,\tilde{h}_0],
\] (7.7)
with $I_2 \subset (0,\infty)$ a bounded open interval containing supp $\chi_T(r)$, and $I_1 \subset (0,\infty)$ a bounded open interval containing $I_2$, and with $\tilde{h} = \tilde{h}(1 + T^2)^{-1/2}$ and $T \gg 1$. Letting $T \to \infty$, we have
\[
\|\tilde{u}_{jm}|_{\Gamma_j(h^2\lambda_{jm})}\|_{L^2([0,\infty))} \leq C \left(\|u_{jm}\|_{L^2([0,\infty))} + \|u_{jm}\|_{L^2([0,\infty))}\right).
\]
The proof of (7.7) is very similar to that of Lemma 7.3 and we omit it. \hfill \Box

The next lemma removes the dependence on $\chi$. To state it we put
\[
\pi_{\zeta_0,R} : L^2_{\phi}(X) \to L^2_{\phi}(X), \quad \pi_{\zeta_0,R} \equiv \int_{\zeta_0} (P_R - \zeta)^{-1}d\zeta,
\]
where $\gamma_0$ is a circle around $\zeta_0$ taken small enough that it encloses no poles of $(P_R - \zeta)^{-1}$ except for $\zeta_0$ (if $\zeta_0$ is not a pole, then $\pi_{\zeta_0,R}$ is the zero operator).

**Lemma 7.5.** If $\chi \in C^\infty_0(X)$ has $\chi \equiv 1$ on $X_0$, then for all $R > 1$ and $\zeta_0 \in \{|\zeta_0| < c_0\} \cup \{|\text{Re } \zeta_0| < c_0, \text{Im } \zeta_0 > 0\}$, we have
\[
\pi_{\zeta_0,R}(L_\varphi^2(X)) = \pi_{\zeta_0,R}\chi(L_\varphi^2(X)).
\]

**Proof.** If $\zeta_0$ is not a pole of $(P_R - \zeta)^{-1}$ there is nothing to prove. Otherwise, we must show that
\[
\pi_{\zeta_0,R}(L_\varphi^2(X)) \subset \pi_{\zeta_0,R}\chi(L_\varphi^2(X)),
\]
or equivalently, that if $u \in L_\varphi^2(X)$ is orthogonal to the image of $\pi_{\zeta_0,R}\chi$, then $\pi_{\zeta_0,R}^*u = 0$. Suppose now that $v \in L_\varphi^2(X)$ has $\chi v = v$. Then
\[
\langle \pi_{\zeta_0,R}^*u, v \rangle_{L_\varphi^2} = \langle u, \pi_{\zeta_0,R}v \rangle_{L_\varphi^2} = \langle u, \pi_{\zeta_0,R}\chi v \rangle_{L_\varphi^2} = 0,
\]
and consequently $\pi_{\zeta_0,R}^*u$ vanishes in the set where $\chi \equiv 1$. But we have $(P_R - \zeta_0)^k\pi_{\zeta_0,R}^*u = 0$ for some $k \in \mathbb{N}$, and we will use this to show that $\pi_{\zeta_0,R}^*u$ vanishes identically.

To do this observe that it remains to show that the restrictions $(\pi_{\zeta_0,R}^*u)|_{X_C}$, $(\pi_{\zeta_0,R}^*u)|_{X_F}$ vanish identically. Separating variables implies that for each $j \in \{C, F\}$ and $m \in \mathbb{N}$, the functions
\[
(\pi_{\zeta_0,R}^*u|_{X_j})_{jm}(r) \overset{\text{def}}{=} \int_{S_j} \pi_{\zeta_0,R}^*u|_{X_j}(r,y)\phi_{jm}(y)dS_y
\]
solve an ordinary differential equation in $r$, namely
\[
Q_{R,j}(h^2\lambda_{jm})^k(\pi_{\zeta_0,R}^*u|_{X_j})_{jm} = 0,
\]
and hence if any one of them vanishes in an open set it vanishes identically. $\square$

Now because $\chi(P_R - \zeta)^{-1}\chi$ continues meromorphically to [7.4] for each $R$ (with poles independent of $\chi$), and because the equation [7.6] is holomorphic, $\chi(P - \zeta)^{-1}\chi$ continues meromorphically to the same region. Because the left hand side of [7.6] is independent of $R$, the right hand side is as well, and we have [1.7], and consequently [1.5] as well.

### 8. Escape function constructions

In this section we construct the escape functions which will provide ellipticity where the complex scaling does not. We construct three escape functions: $G_0$ is supported in a neighborhood of $T^*X_0$, $G_C$ is supported in $T^*X_C$ and $G_F$ is supported in $T^*X_F$. The nontrapping assumption of §2.2 allows us to use a construction based on that in [VaZw00, §4] to obtain $G_0$. However, the operator $G_0^e$ does not respect the separation of variables which we use to define $P_5$, and hence we must have $\{G_0 \neq 0\} \subset \{P_5 = P\}$. The escape functions $G_F$ and $G_C$ do respect the separation of variables, and bridge the gap between $G_0$ and the region where $P_5 \neq P$. 
Lemma 8.1. Fix $R = 5$, and let $f_C$ and $f_F$ be as in §5 and §6. There exist $\delta_0, \delta_p, \delta_f > 0$ and functions $G_0, G_C, G_F \in C_0^\infty (T^*X)$ with the following properties:

$$\text{supp } G_0 \cap T^*X_C \subset \{ r \leq 3 \}, \quad \text{supp } G_0 \cap T^*X_F \subset \{ r \leq 3 \},$$
$$\text{supp } G_C \subset T^*X_C, \quad \text{supp } G_F \subset T^*X_F.$$ 

Moreover, if $G = G_0 + G_C + G_F$, then, in the notation of §3.1,

$$H_p G \geq \delta_0, \quad \text{on } \left[ T^*X \setminus (T^*X_{C5} \cup T^*X_{F5}) \right] \cap \{ |p - 1| \leq \delta_p \}, \quad (8.1)$$

and for $j \in \{ C, F \},

$$H_{Re(q_j,\alpha)} G_j \geq \delta_0, \quad \text{on } \left[ T^*X_j \setminus T^*X_j(5+R_j) \right] \cap \{ |p - 1| \leq \delta_p \} \cap \left\{ \sum_{k \in \{0,1,2\}} | f_j^{(k)} (r) | \leq \delta_f \right\}. \quad (8.2)$$

Finally $G_C$ and $G_F$ are functions only of $r, \rho$ and $p$.

**Proof.** We will construct $G_0, G_C,$ and $G_F$ such that the estimates (8.1) and (8.2) hold only on $p^{-1}(1)$ rather than on $\{ |p - 1| \leq \delta_p \}$. Then (8.1) and (8.2) will follow on the larger set for a slightly smaller $\delta_0$, provided $\delta_p$ is sufficiently small.

Let $C_F$ be a positive constant to be specified later, and let $\psi \in C_0^\infty ((0, \infty); [0, 1])$ have $\psi \equiv 1$ near 1.

1) We put

$$G_F \overset{\text{def}}{=} C_F \rho \chi_F(r) \psi(p),$$

where $\chi_F \in C_0^\infty (\mathbb{R}; [0, \infty))$ is supported in $[1, R_F + 6]$, and obeys $\chi'_F \geq 0$ on $[1, R_F + 5]$, $\chi_F(2) = 1$, and $\chi'_F = 1$ on $[2, R_F + 5]$. We then have

![Figure 5](image_url)

**Figure 5.** The escape functions $G_F$ and $G_C$ in the funnel and in the cusp respectively. Gray shading indicates the region where $H_p G_F < 0$.

$$\frac{1}{C_F \psi(p)} H_p G_F = - (\partial_r p) \partial_\rho (\rho \chi_F(r)) + (\partial_\rho p) \partial_r (\rho \chi_F(r))$$

$$= 2 \left[ 1 - \frac{\beta'_F(r)}{\beta_F(r)} \right] (p - \rho^2) \chi_F(r) + 2 \rho^2 \chi'_F(r)$$

$$\geq (p - \rho^2) \chi_F(r) + \rho^2 \chi'_F(r),$$
by \((2.6)\). From this it follows that
\[
H_p G_F \geq 0, \quad \text{on } T^* X \setminus T^* X_{F(R_F+5)}
\]
and
\[
H_p G_F \geq C_F, \quad \text{for } 2 \leq r \leq R_F + 5, \quad p = 1.
\]

Next
\[
\frac{1}{C_F \psi(p)} H_{Re q_H F(\alpha)} G_F = -\text{Re } \partial_r^2 \left( \frac{\rho^2}{(1 + i f_f)^2} + e^{-2(r + i f_f)} \beta_F(r + i f_f) \alpha \right) \chi_F(r) + \text{Re } \partial_p^2 \left( \frac{\rho^2}{(1 + i f_f)^2} + e^{-2(r + i f_f)} \beta_F(r + i f_f) \alpha \right) \rho \chi_F'(r).
\]
\[
(8.3)
\]

We observe first that
\[
\text{Re } \partial_p^2 \left( \frac{\rho^2}{(1 + i f_f)^2} + e^{-2(r + i f_f)} \beta_F(r + i f_f) \alpha \right) \rho = 2 \text{Re } \frac{\rho^2}{(1 + i f_f)^2} \geq \rho^2,
\]
provided \(\delta_f\) (and consequently \(f'_F\)) is sufficiently small.

Second, we have
\[
\partial_r^2 \left( \frac{\rho^2}{(1 + i f_f)^2} + e^{-2(r + i f_f)} \beta_F(r + i f_f) \alpha \right) = \mathcal{O}(\delta_f) \rho^2 + (-2 + \mathcal{O}(\delta_f)) \left( e^{-2(r + i f_f)} \beta_F(r + i f_f) \alpha \right),
\]
where \(\mathcal{O}(\delta_f)\) indicates a function of \(r\) only (which can be written explicitly in terms of \(f_F\) and \(\beta_F\)) which is bounded by a constant times \(\delta_f\) uniformly for \(r \in [3, R_F + 5]\).

Substituting \((8.4)\) and \((8.5)\) into \((8.3)\) we find that
\[
\frac{1}{C_F \psi(p)} H_{Re q_H F(\alpha)} G_F \geq \rho^2 \chi_F'(r) + \mathcal{O}(\delta_f) \rho^2 \chi_F(r) + 2 \text{Re } \left( e^{-2(r + i f_f)} \beta_F(r + i f_f) \alpha \right) \chi_F(r),
\]
for \(r \in [3, R_F + 5]\). Provided \(\delta_f\) is sufficiently small this implies
\[
\frac{1}{C_F \psi(p)} H_{Re q_H F(\alpha)} G_F \geq \frac{1}{2} \rho^2 \chi_F'(r) + e^{-2r} \alpha \chi_F(r)
\]
\[
= \frac{1}{2} \rho^2 \chi_F'(r) + (p - \rho^2) \chi_F(r),
\]
for \(r \in [3, R_F + 5]\). From this we obtain
\[
H_{Re q_H F(\alpha)} G_F \geq \frac{1}{2} C_F, \quad \text{for } 3 \leq r \leq R_F + 5, \quad p = 1.
\]

2) We put
\[
\tilde{G}_C \overset{\text{def}}{=} -\rho \chi_C(r) \psi(p),
\]
where \(\chi_C \in C^\infty([0, \infty); [0, \infty))\) obeys \(\chi_C(0) = 1\), \(\chi'_C \equiv -(2(R_C + 5))^{-1}\) on \([0, R_C + 5]\), and \(\chi_C \equiv 0\) for \(r\) sufficiently large.
We then have
\[ \frac{1}{\psi(p)} H_p \tilde{G}_C = 2 \left[ 1 - \frac{\beta'_C(r)}{\beta_C(r)} \right] (p - \rho^2) \chi_C(r) - 2\rho^2 \chi'_C(r) \geq (p - \rho^2) \chi_C(r) - \rho^2 \chi'_C(r), \]
by (2.6). From this it follows that
\[ H_p \tilde{G}_C \geq 0, \quad \text{everywhere} \]
and
\[ H_p \tilde{G}_C \geq C_C, \quad \text{for } 0 \leq r \leq R_C + 5, \quad p = 1, \]
for some constant $C_C$. A calculation just like that in 1) above gives
\[ H_{\text{Req}, C(\alpha)} \tilde{G}_C \geq \frac{1}{2}, \quad \text{for } 3 \leq r \leq R_C + 5, \quad p = 1. \]
The problem with this function is that it is not smooth past $R_C$, and to fix this we must combine it with a function defined on $X_0$.

![Figure 6](image.png)

**Figure 6.** In this figure the gray lines represent level curves of $\alpha$ at $p = 1$, the horizontal line corresponding to $\alpha = 0$. The regions of support for $\tilde{G}_C$, $G_{\text{00}}$, and $\tilde{G}_0$ are delineated by black lines. This figure assumes that the form of the geodesic hamiltonian $p = \rho^2 + e^{2\gamma} \beta_C(r)^2 \alpha$ extends to a neighborhood of $X_C$.

3) For $\xi$ in the closure of $\left[ T^* X \setminus (T^* X_C \cup T^* X_{F_2}) \right] \cap p^{-1}(1)$, let
\[ T_{\xi}^- \overset{\text{def}}{=} \sup \{ t \in (-\infty, 0] : \exp(t H_p) \xi \in T^* X_C \cup T^* X_{F_2} \}, \]
\[ T_{\xi}^+ \overset{\text{def}}{=} \inf \{ t \in [0, \infty) : \exp(t H_p) \xi \in T^* X_C \cup T^* X_{F_2} \}. \]
By the nontrapping assumption in \( \S 2.2 \), we have \( T^-_\xi > -\infty \) and \( T^+_\xi < \infty \). Moreover, the assumptions in \( \S 2.2 \) imply that one of the following three possibilities holds:

\[
\begin{align*}
\exp(T^-_\xi H_p)\xi &\in T^*X_{F2} \quad \text{and} \quad \exp(T^+_\xi H_p)\xi \in T^*X_{F2}, \quad (8.6) \\
\exp(T^-_\xi H_p)\xi &\in T^*X_C \quad \text{and} \quad \exp(T^+_\xi H_p)\xi \in T^*X_{F2}, \quad (8.7) \\
\exp(T^-_\xi H_p)\xi &\in T^*X_{F2} \quad \text{and} \quad \exp(T^+_\xi H_p)\xi \in T^*X_C, \quad (8.8)
\end{align*}
\]

That is to say, it is not the case that

\[
\exp(T^-_\xi H_p)\xi \in T^*X_C \quad \text{and} \quad \exp(T^+_\xi H_p)\xi \in T^*X_C.
\]

We now construct a escape functions supported near \( \exp([T^-_\xi, T^+_\xi] H_p)\xi \) according to which of (8.6), (8.7) and (8.8) holds. In the case (8.6) we construct one escape function, which we call \( G_{\xi 0} \). In the cases (8.7) and (8.8) if the geodesic through \( \xi \) is transversal to \( \partial T^*X_C \) we construct one escape function, which we call \( G_{\xi C} \). In the cases (8.7) and (8.8) if the geodesic through \( \xi \) is not transversal to \( \partial T^*X_C \) the situation is still more delicate and we construct two escape functions, \( G_{\xi 0} \) and \( G_{\xi C} \).

Before proceeding to a precise description of the constructions, we give a rough outline (see Figure 6): We will sum the \( G_{\xi 0} \) (after passing to a finite subfamily) to obtain \( G_{00} \), an escape function whose support overlaps that \( \tilde{\xi} \). For \( \xi \in \Sigma \) if \( \partial T^*X_C \) is a sufficiently small neighborhood of \( \xi \), the set

\[
V_{\xi 0} \overset{\text{def}}{=} \{ \exp(t H_p)\left[ U_\xi \cap \Sigma_\xi \right] : T^-_\xi - 1 < t < T^+_\xi + 1 \}
\]

is disjoint from \( T^*X_C \), and it is diffeomorphic to \( (\Sigma_\xi \cap U_\xi) \times (T^-_\xi - 1, T^+_\xi + 1) \). We use this diffeomorphism to define product coordinates on \( V_{\xi 0} \). For \( \varphi_\xi \in C^\infty_0(\Sigma_\xi \cap U_\xi; [0, 1]) \) and \( \chi_\xi \in C^\infty_0((-1, T^-_\xi + 1]) \). Then put

\[
G_{\xi 0} \overset{\text{def}}{=} C_{00} \chi_\xi \varphi_\xi \psi(p), \quad H_p G_{\xi 0} = C_{00} \chi'_\xi \varphi_\xi \psi(p).
\]

Note that \( G_\xi \in C^\infty_0(V_{\xi 0}) \). We define \( \chi_\xi \) and \( \varphi_\xi \) as follows. First take \( \chi_\xi \) such that

\[
t \in \text{supp} \chi_\xi \implies \exp(t H_p)\xi \notin T^*X_{F3},
\]

and

\[
\exp(t H_p)\xi \notin T^*X_{F2} \implies \chi'_\xi(t) = 1.
\]

Next take \( \varphi_\xi \) to be identically 1 near \( \xi \) but with support small enough that \( \text{supp} G_{\xi 0} \cap T^*X_{F3} = \text{supp} G_{\xi 0} \cap T^*X_C = \emptyset \). We will necessarily have \( H_p G_{\xi 0} < 0 \) in some parts of \( T^*X_{F2} \setminus T^*X_{F3} \), but we will choose \( C_F \gg C_00 \) so that the \( G_F \) term overcomes this.

Let \( V'_{\xi 0} \subset V_{\xi 0} \) be an open set containing \( \{ \exp(t H_p)\xi : T^-_\xi \leq t \leq T^+_\xi \} \) such that \( \varphi_\xi \equiv \chi'_\xi \equiv 1 \) on \( V'_{\xi 0} \). This set will be important when we pass to a subfamily of the \( G_{\xi 0} \).
• In the case (8.7), we distinguish two subcases. Because our construction will depend only on the geodesic through ξ and not on ξ itself, we may assume that ξ ∈ ∂T∗X_0, and hence that T_ξ^- = 0.
  
  If ∂T∗X_0 is transversal to the geodesic through ξ, the construction is very similar to the case (8.6) above. Namely, let U_ξ be a neighborhood of ξ small enough that the set
  
  \[ V_\xi \overset{\text{def}}{=} \{ \exp(tH_\rho)[U_\xi \cap \partial T^*X_0] : -1 < t < T_\xi^+ + 1 \} \]
  
  is diffeomorphic to (∂T∗X_0 ∩ U_ξ) × (−1, T_ξ^+ + 1). Using this diffeomorphism to define product coordinates on V_ξ, put
  
  \[ G_\xi \overset{\text{def}}{=} C_0C_0'(\exp(tH_\rho)[U_\xi \cap \partial T^*X_0]) : -1 < t < T_\xi^+ + 1 \}
  
  where \( \varphi_\xi \in C_0^\infty(\partial T^*X_0 ∩ U_ξ; [0, 1]) \) and \( \chi_\xi \in C_0^\infty([0, T_\xi^+ + 1]) \) are as follows. Take \( \chi_\xi(0) = 1 \), and require that
  
  \[ t \in \text{supp} \chi_\xi \implies \exp(tH_\rho)\xi \not\in T^*X_0, \]
  
  and
  
  \[ \exp(tH_\rho)\xi \not\in T^*X_0 \implies \chi_\xi'(t) = 1. \]

  Next take \( \varphi_\xi \) such that \( \varphi_\xi = \rho \) near ξ and has support small enough that \( \text{supp}G_\xi ∩ T^*X_0 \not\in \varnothing \). We will necessarily have \( H_\rho G_\xi < 0 \) in some parts of \( T^*X_0 \not\in T^*X_0 \), but we will choose \( C_F \gg C_0C_0' \) so that the \( G_\rho \) term overcomes this.

  In this case, let \( V'_\xi \subset V_\xi \) be an open set containing \( \{ \exp(tH_\rho)\xi : 0 \leq t \leq T_\xi^+ \} \) such that \( \varphi_\xi \equiv 1 \) on \( V'_\xi \), and such that \( V'_\xi \subset \{ \chi_\xi = 1 \} \cup T^*X_0 \).

  If ∂T∗X_0 is not transversal to the geodesic through ξ, we proceed in two steps. First we put
  
  \[ G_0 \overset{\text{def}}{=} C_0C_0'\chi_\xi\varphi_\xi\psi(p) \]
  
  with \( \chi_\xi \) and \( \varphi_\xi \) just as in the case (8.6) above, and also take \( V'_0 \) as in the case (8.6). For \( G_\xi \) we cannot repeat the procedure of the transversal case, and so we take any function in \( C_0^\infty(T^*X_0 \not\in T^*X_0) \) with \( \text{supp}G_\xi \subset V'_0 \) such that \( G_\xi + G_\xi' \) is continuous in a neighborhood of ξ. This will produce a region where \( H_\rho G_\xi < 0 \), but if we take \( C_00 \gg 1 \) this will be overcome by the \( G_00 \) term.

  Let \( V'_\xi \) be neighborhood of ξ with \( V'_\xi \) contained in the set where \( G_\xi + G_\xi' \) is continuous.

  • The case (8.8) is very similar to the case (8.7) and we only outline it briefly. In this case we may assume that \( T_\xi^+ = 0 \). In the transversal subcase we put
  
  \[ V_\xi \overset{\text{def}}{=} \{ \exp(tH_\rho)[U_\xi \cap \partial T^*X_0] : -1 + T_\xi^- < t < +1 \} \]
  
  and
  
  \[ G_\xi \overset{\text{def}}{=} C_0C_0'\chi_\xi\varphi_\xi\psi(p), \quad H_\rho G_\xi = C_0C_0'\chi_\xi\varphi_\xi\psi(p), \]
  
  where \( \varphi_\xi \in C_0^\infty(\partial T^*X_0 ∩ U_ξ; [0, 1]) \) as before and \( \chi_\xi \in C_0^\infty([-1 - T_\xi^- , 0]) \) as before with but the difference that \( \chi_\xi(0) = -1 \). In the nontransversal subcase there is no difference in the construction.
Let $K_{00}$ be the closure of the intersection of $\left[ T^*X \setminus (T^*X_{F2} \cup T^*X_{C}) \right] \cap p^{-1}(1)$ with the set of $\xi$ for which (8.6) holds. Take $\{\xi_1, \ldots, \xi_N\}$ such that the $V_{\xi_j}^0$ cover the set $K_{00}$, and put

$$G_{00} \overset{\text{def}}{=} \sum_{j=1}^N G_{\xi_j}^0.$$ 

Next let $K_{0C}$ be the closure of the intersection of $\left[ T^*X \setminus (T^*X_{F2} \cup T^*X_{C}) \right] \cap p^{-1}(1)$ with the set of $\xi$ for which (8.6) does not hold (and hence one of (8.7) and (8.8) does). Take $\{\xi_1, \ldots, \xi_N\}$ such that the $V_{\xi_j}^C$ cover the set $K_{0C}$, and put

$$\tilde{G}_0 \overset{\text{def}}{=} \sum_{j=1}^N G_{\xi_j}^C.$$ 

4) We now observe that $\tilde{G}_0 + \tilde{G}_C$ is continuous by construction, and there is a constant $c$ such that

$$H_p(\tilde{G}_0 + \tilde{G}_C) \geq c > 0 \quad \text{on} \quad \left[ T^*X \setminus (T^*X_{F2} \cup T^*X_{C(R_C+5)}) \right] \cap \{p = 1\},$$

where $\tilde{G}_0 + \tilde{G}_C$ is differentiable (namely everywhere off of the set $\partial T^*X_C$). Hence there exists a regularization $\tilde{G}$ of $\tilde{G}_0 + \tilde{G}_C$, such that

$$H_p(\tilde{G}_0 + \tilde{G}) \geq c/2 > 0 \quad \text{on} \quad \left[ T^*X \setminus (T^*X_{F2} \cup T^*X_{C(R_C+5)}) \right] \cap \{p = 1\}.$$ 

Let $\tilde{\chi} \in C_0^\infty(T^*X, [0, 1])$ have $\tilde{\chi} \equiv 1$ on $T^*X_{C4}$ and $\tilde{\chi} \equiv 0$ on $T^*X \setminus T^*X_{C3}$. Now put

$$G_C \overset{\text{def}}{=} \tilde{\chi} \tilde{G}, \quad G_0 \overset{\text{def}}{=} (1 - \tilde{\chi}) \tilde{G} + G_{00}.$$ 

Provided $C_F$ is sufficiently large, we now have the lemma. \qed

9. Microlocal deformation of $P_5$ and proof of (1.9)

In this section $C$ denotes a finite constant which may change from line to line, and which asserted to be uniform only in $0 < h \ll 1$, and in some cases also uniform in $m \in \mathbb{N}$.

Fix $\varepsilon \in [M_1h, M_2h \log(1/h)]$, let $G_0, G_C, G_F \in C_0^\infty(T^*X)$ be as in §8, and define

$$e_G \overset{\text{def}}{=} e^{\varepsilon G_C^0/h} e^{G_C^0/h} e^{G_C^0/h}, \quad P_\varepsilon \overset{\text{def}}{=} e^{-1}_G P e_G, \quad P_{5,\varepsilon} \overset{\text{def}}{=} e^{-1}_G P_5 e_G.$$ 

Recall that from the discussion in §4.3 we know that for $j \in \{0, C, F\}$, the $e^{G_j^0/h}$ are pseudodifferential operators, and that $e^{G_j^0/h} - \text{Id}$ maps $L^2$ functions to compactly supported smooth functions with a loss of $h^{C}$ for $C > 0$ which depends on $M_2$.

The following lemma allows us to reduce (1.9) to three estimates, one on a compact part of the manifold, one on the funnel, and one on the cusp. Recall the Sobolev space notation of (7.1).
Lemma 9.1. The estimate [1.9] is a consequence of the following three estimates for $u \in C_0^\infty(X)$:

\[
\|u\|_{H^2_{\varphi,h}(X)} \leq \frac{C}{h \log(1/h)} \|P_{5,\varepsilon} u\|_{L^2_\varphi(X)}, \quad \text{supp } u \cap X_{C4} = \text{supp } u \cap X_{F4} = \emptyset. \tag{9.1}
\]

\[
\|u\|_{H^2_{\varphi,h}(X)} \leq \frac{C}{h \log(1/h)} \|P_{5,\varepsilon} u\|_{L^2_\varphi(X)}, \quad \text{supp } u \subset X_{C3}. \tag{9.2}
\]

\[
\|u\|_{H^2_{\varphi,h}(X)} \leq \frac{C}{h \log(1/h)} \|P_{5,\varepsilon} u\|_{L^2_\varphi(X)}, \quad \text{supp } u \subset X_{F3}. \tag{9.3}
\]

To prove it, we will need the following fact about commutators. This would follow from the discussion in §4.2 if $P_5$ were a pseudodifferential operator.

Lemma 9.2. For $\chi \in C_0^\infty(X)$ such that $\chi P \equiv \chi P_5$, we have

\[
\|[P_{5,\varepsilon}, \chi] u\|_{L^2_\varphi(X)} \leq C h \|u\|_{H^1_{\varphi,h}(X)}.
\]

Proof of Lemma 9.2. We take $\tilde{\chi} \in C_0^\infty(X)$ with $\tilde{\chi} \chi \equiv \chi$ and $\tilde{\chi} P \equiv \tilde{\chi} P_5$, and write

\[
\chi e^{-1}_G P_5 e_G = \chi e^{-1}_G P e_G - \chi e^{-1}_G (1 - \tilde{\chi}) P e_G + \chi e^{-1}_G (1 - \tilde{\chi}) P_5 e_G.
\]

By (4.7) and Lemma 7.1, the last two terms are of size $O_{L^2_\varphi(X) \to L^2_\varphi(X)}(h^{\infty})$ (as well as smoothing and compactly supported in space), and hence negligible. A similar calculation for $e^{-1}_G P_5 e_G \chi$ then shows that

\[
[P_{5,\varepsilon}, \chi] = [P_\varepsilon, \chi] + O_{L^2_\varphi(X) \to L^2_\varphi(X)}(h^{\infty}),
\]

from which the lemma follows. \hfill \Box

Proof of Lemma 9.1. We take $\chi_0, \chi_C, \chi_F \in C^\infty(X; [0, 1])$ such that

1. $\chi_0 + \chi_C + \chi_F \equiv 1$,
2. $\chi_0 \equiv 1$ on $X \setminus (X_{C3} \cup X_{F3})$,
3. $\chi_0 \equiv 0$ on $X_{C4} \cup X_{F4}$,
4. $\text{supp } \chi_F \subset X_F$, $\text{supp } \chi_C \subset X_C$.

First,

\[
\|u\|^2_{H^2_{\varphi,h}(X)} \leq C \left( \|\chi_0 u\|^2_{H^2_{\varphi,h}(X)} + \|\chi_C u\|^2_{H^2_{\varphi,h}(X)} + \|\chi_F u\|^2_{H^2_{\varphi,h}(X)} \right) \tag{9.4}
\]

\[
\leq \frac{C}{h^2 \log^2(1/h)} \left( \|P_{5,\varepsilon} \chi_0 u\|^2_{L^2_\varphi(X)} + \|P_{5,\varepsilon} C u\|^2_{L^2_\varphi(X)} + \|P_{5,\varepsilon} \chi_F u\|^2_{L^2_\varphi(X)} \right).
\]

We now observe that, for $j \in \{0, C, F\}$,

\[
\|P_{5,\varepsilon} \chi_j u\|^2_{L^2_\varphi(X)} \leq C \left( \|\chi_j P_{5,\varepsilon} u\|^2_{L^2_\varphi(X)} + \|P_{5,\varepsilon} \chi_j u\|^2_{L^2_\varphi(X)} \right),
\]

and hence

\[
\|P_{5,\varepsilon} \chi_j u\|^2_{L^2_\varphi(X)} \leq C \left( \|\chi_j P_{5,\varepsilon} u\|^2_{L^2_\varphi(X)} + \|P_{5,\varepsilon} \chi_j u\|^2_{L^2_\varphi(X)} \right).
\]
Now
\[
\|[(P_{5,\varepsilon},X_j)u]\|_{L^2_h(X)}^2 \leq C h^2 \|u\|_{H^2_{\text{osc}}(X)}^2,
\]
(9.5) gives (1.9), provided \( h \) is sufficiently small that the \( \|u\|_{H^2_{\text{osc}}(X)}^2 \) term of (9.5) can be absorbed into the left hand side of (9.4). Note that (9.5) follows from Lemma 9.2. \( \square \)

Next we use the following lemma to reduce (9.2) and (9.3) to a family of one-dimensional estimates. We use the notational shorthands
\[
Q_{5,jm} \overset{\text{def}}{=} Q_{5,j}(h^2 \lambda_{jm}) \quad \text{and} \quad Q_{5,jme} \overset{\text{def}}{=} e^{-\varepsilon \Op_{\ell}(G_j)/h} Q_{5,j}(h^2 \lambda_{jm}) e^{\varepsilon \Op_{\ell}(G_j)/h}.
\]

**Lemma 9.3.** The estimates (9.2) and (9.3) follow from
\[
\|u_{Cm}\|_{H^2_h([0,\infty))} \leq \frac{C}{h \log 1/h} \|Q_{5,Cme} u_{Cm}\|_{L^2_h([0,\infty))}, \quad u_{Cm} \in C_0^\infty((3,\infty)),
\]
and
\[
\|u_{Fm}\|_{H^2_h([0,\infty))} \leq \frac{C}{h \log 1/h} \|Q_{5,Fme} u_{Fm}\|_{L^2_h([0,\infty))}, \quad u_{Fm} \in C_0^\infty((3,\infty)),
\]
respectively, provided the constant \( C \) is uniform in \( m \).

**Proof.** Using the notation of (3.4) and (7.1), we have
\[
\|u\|_{L^2_{\text{osc}}(X)}^2 \leq 2 \|u\|_{L^2(X)}^2 + \|Pu\|_{L^2(X)}^2 = \sum_{m=1}^\infty 2 \|u_{jm}\|_{L^2([0,\infty))}^2 + \|Q_j(h^2 \lambda_{jm}) u_{jm}\|_{L^2([0,\infty))}^2.
\]
Then
\[
\|Q_j(h^2 \lambda_{jm}) u_{jm}\|_{L^2([0,\infty))} \leq C \left( \|Q_{5,jm} u_{jm}\|_{L^2([0,\infty))} + \|u_{jm}\|_{H^2_h([0,\infty))} \right),
\]
and
\[
\|Q_{5,jm} u_{jm}\|_{L^2} \leq \|Q_{5,jme} u_{jm}\|_{L^2} + \|Q_{5,jm} - Q_{5,jme}\|_{L^2} u_{jm}\|_{L^2}.
\]
We observe that because of (4.13), we have
\[
\|(Q_{5,jm} - Q_{5,jme}) u_{jm}\|_{L^2} \leq \varepsilon \|u_{jm}\|_{H^1_h}.
\]
Now
\[
\sum_{m=1}^\infty \|Q_{5,jme} u_{jm}\|_{L^2([0,\infty))}^2 = \|e^{-\varepsilon G_{j}^{\ell}/h} P_5 e^{\varepsilon G_{j}^{\ell}/h} u\|_{L^2(X)}^2,
\]
and
\[
\|e^{-\varepsilon G^{\ell}/h} P_5 e^{\varepsilon G^{\ell}/h} u\|_{L^2(X)} = \|e^{-1} P_5 e_G u\|_{L^2(X)} + \mathcal{O}(h^\infty) \|u\|_{L^2(X)},
\]
(9.8) give the conclusion.

To prove (9.8), we take \( \chi \in C^\infty(X) \) such that \( \chi \equiv 1 \) on \( X_{j(9/2)} \) and \( \chi \equiv 0 \) on \( X \setminus X_{j(7/2)} \), and then \( \tilde{\chi} \) with the same properties but such that \( \chi \tilde{\chi} \equiv \chi \), and write
\[
e^{-\varepsilon G^{\ell}/h} P_5 e^{\varepsilon G^{\ell}/h} u = e^{-\varepsilon G^{\ell}/h} \tilde{\chi} P_5 e^{\varepsilon G^{\ell}/h} \chi u + e^{-\varepsilon G^{\ell}/h} (1 - \tilde{\chi}) P_5 e^{\varepsilon G^{\ell}/h} \chi u.
\]
We have
\[
e^{-\varepsilon G^{\ell}/h} (1 - \tilde{\chi}) P_5 e^{\varepsilon G^{\ell}/h} \chi u = e^{-\varepsilon G^{\ell}/h} (1 - \tilde{\chi}) P_5 e^{\varepsilon G^{\ell}/h} \chi u = \mathcal{O}(h^\infty) u
\]
by (4.7), and we have
\[
e^{-\varepsilon G^{\ell}/h} \tilde{\chi} P_5 e^{\varepsilon G^{\ell}/h} \chi u = e^{-1} \tilde{\chi} P_5 e_G \chi u + \mathcal{O}(h^\infty) u
\]
by (4.7), (4.12), and Lemma 7.1. Finally we observe that
\[ e_G^{-1} \tilde{\chi} P_5 e_G \chi u = e_G^{-1} P_5 e_G u + O(h^{\infty})u. \]

To obtain (1.9), and as a consequence (1.8) and the Theorem, it remains to prove (9.1), (9.6) and (9.7). These proofs are based on the arguments in [SjZw07] §4.1 (and see that paper for references to earlier works which use a similar method, including [Mar02]), where similar estimates are derived under different assumptions near infinity.

Let \( \tilde{\psi} \in C_0^\infty(\mathbb{R}) \) have \( \tilde{\psi} \equiv 1 \) near 1, and supp \( \tilde{\psi} \subset \{ p: |p - 1| < \delta_p \} \), where \( \delta_p \) is as in Lemma 8.1.

**Proof of (9.1).** Take \( u \in C_0^\infty(X) \), supp \( u \cap X_{C4} = \supp u \cap X_{F4} = \emptyset \). We have \( P_{5,\varepsilon} u = P_{\varepsilon} u \), and hence (9.1) reduces to
\[ \| u \|_{H^{2,h}_u(X)} \leq \frac{C}{h \log(1/h)} \| P_{\varepsilon} u \|_{L^{2}_u(X)}. \]

But we have
\[ \| (\mathrm{Id} - \tilde{\psi}(P))u \|_{H^{2,h}_u(X)} \leq C \| P_{\varepsilon} (\mathrm{Id} - \tilde{\psi}(P))u \|_{L^{2}_u(X)}, \]
by (4.8). Meanwhile
\[ \| P_{\varepsilon} \tilde{\psi}(P)u \|_{L^{2}_u(X)} \| \tilde{\psi}(P)u \|_{L^{2}_u(X)} \geq \left| \langle P_{\varepsilon} \tilde{\psi}(P)u, \tilde{\psi}(P)u \rangle_{L^{2}_u(X)} \right| \geq \left| \text{Im} \langle P_{\varepsilon} \tilde{\psi}(P)u, \tilde{\psi}(P)u \rangle_{L^{2}_u(X)} \right| \geq \frac{\varepsilon}{C} \| \tilde{\psi}(P)u \|^2_{L^{2}_u(X)}, \]
where for the last step we have used (4.13) and (4.9), observing that the principal symbol of \( (P_{\varepsilon} - P_{\varepsilon}^*)/2i \) is given by \(-\varepsilon \{ p, G \}\), so that Lemma 8.1 gives us the necessary nonnegativity.

**Proof of (9.6) and (9.7).** Fix \( j \in \{ C, F \} \) and \( m \in \mathbb{N} \), and take \( u_{jm} \in C_0^\infty([R_j - 3/2, \infty)) \). We have
\[ \| (\mathrm{Id} - \tilde{\psi}(P))u_{jm} \|_{H^{2}_{(0,\infty)}} \leq C \| Q_{5,jm\varepsilon} (\mathrm{Id} - \tilde{\psi}(P))u_{jm} \|_{L^2([0,\infty))}, \]
by (4.5). Meanwhile
\[ \| Q_{5,jm\varepsilon} \tilde{\psi}(P)u_{jm} \|_{L^2} \| \tilde{\psi}(P)u_{jm} \|_{L^2} \geq \left| \langle Q_{5,jm\varepsilon} \tilde{\psi}(P)u_{jm}, \tilde{\psi}(P)u_{jm} \rangle_{L^2} \right| \geq \frac{\varepsilon}{C} \| \tilde{\psi}(P)u_{jm} \|^2_{L^2}, \]
where for the last step we have used (4.13) and (4.6). This time the principal symbol of \( (Q_{5,jm\varepsilon} - Q_{5,jm\varepsilon}^*)/2i \) is given by \( \text{Im} q_{ij} - \varepsilon \text{Re} q_{ij}, G \). In this case the necessary nonnegativity is given by Lemma 8.1 together with (5.3) and (5.4) in the case \( j = C \), and by Lemma 8.1 together with (6.2) and (6.3) in the case \( j = F \).
Here we give a computation for the curvature of a metric of the form (2.1) or (2.2). This is important for the examples in §3.2. For this section only, let \((S, \tilde{g})\) be a compact Riemannian manifold, and let \(X = \mathbb{R} \times S\) be equipped with the metric
\[
g = dr^2 + f(r)^2 \tilde{g},
\]
where \(f\) is any smooth function satisfying \(f > 0\) everywhere. Let \(p \in X\), let \(P\) be a two-dimensional subspace of \(T_pX\), and let \(K(P)\) be the sectional curvature of \(P\) with respect to \(g\). We will show that if \(\partial_r \in T_pX\), then
\[
K(P) = -\frac{f''(r)}{f(r)},
\]
while if \(P \subset T_pS\), then
\[
K(P) = \frac{\tilde{K}(P)}{f(r)^2} - \frac{f'(r)^2}{f(r)^2},
\]
where \(\tilde{K}(P)\) is the sectional curvature of \(P\) with respect to \(\tilde{g}\).

To perform these computations, we will work in coordinates \((x^0, \ldots, x^n) = (r, x^1, \ldots, x^n)\), and write
\[
g = g_{\alpha\beta} dx^\alpha dx^\beta = dr^2 + g_{ij} dx^i dx^j = dr^2 + f(r)^2 \tilde{g}_{ij} dx^i dx^j,
\]
using the Einstein summation convention. We use Greek letters for indices which include 0, that is indices which include the direction \(r\), and Latin letters for indices which do not. For brevity we write \(\partial_\alpha = \partial_{x^\alpha}\) and \(\partial_j = \partial_{x^j}\). We then have
\[
\partial_\alpha g_{\rho\sigma} = 0, \quad \partial_r g_{jk} = 2f^{-1} f' g_{jk}, \quad \partial_\hbar g_{jk} = f^2 \partial_\hbar \tilde{g}_{jk}.
\]
We write \(\Gamma\) for the Christoffel symbols of \(g\), and \(\tilde{\Gamma}\) for those of \(\tilde{g}\). These are given by
\[
\Gamma^r_{\rho\sigma} = \Gamma^\alpha_{\rho\sigma} = 0, \quad \Gamma^r_{jk} = -f^{-1} f' g_{jk}, \quad \Gamma^i_{jr} = f^{-1} f' \tilde{\Gamma}^i_{jr}, \quad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}.
\]
We define the Riemann curvature tensor by
\[
R_{\alpha\beta\gamma\delta} = \partial_\alpha \Gamma^\delta_{\beta\gamma} + \Gamma^\epsilon_{\beta\gamma} \Gamma^\delta_{\alpha\epsilon} - \partial_\beta \Gamma^\delta_{\alpha\gamma} - \Gamma^\epsilon_{\alpha\gamma} \Gamma^\delta_{\beta\epsilon}.
\]
Now if \(P \subset T_pX\) is spanned by a pair of orthogonal unit vectors \(V^\alpha \partial_\alpha\) and \(W^\alpha \partial_\alpha\), then \(K(P) = R_{\alpha\beta\gamma\delta} V^\alpha W^\beta W^\gamma V^\delta\), and similarly for \(\tilde{R}\) and \(\tilde{K}\). In our case we have
\[
R_{ijk}^\ell = \tilde{R}_{ijk}^\ell + \Gamma^r_{jk} \Gamma^\ell_{ir} - \Gamma^r_{ik} \Gamma^\ell_{jr} = \tilde{R}_{ijk}^\ell + (f^{-1} f')^2 (f' g_{jk} + \delta_j^\ell g_{ik}),
\]
\[
R_{rjk}^r = \partial_r \Gamma^r_{jk} - \Gamma^m_{rk} \Gamma^r_{jm} = -(f^{-1} f' g_{jk})' + (f^{-1} f')^2 g_{jk} = -f^{-1} f'' g_{jk}.
\]
In the case where \(\partial_r \in P\) we take \(V = \partial_r\) and \(W = W^j \partial_j\) any unit vector in \(T_pX\) orthogonal to \(V\). Then we have
\[
K(P) = R_{rjk} W^j W^k = -f^{-1} f'' g_{jk} W^j W^k = -f^{-1} f''.
\]
Meanwhile if $\partial_r \perp P$ we may write $V = V^j \partial_j$ and $W = W^j \partial_j$. Then
\[
K(P) = R^m_{ijk\ell} V^i W^j W^k V^\ell \\
= \left( \tilde{R}^m_{ijk\ell} + (f^{-1})^2 (f')^2 (-\delta^m_i g_{jk} + \delta^m_j g_{ik}) \right) g_m \delta V^i W^j W^k V^\ell \\
= \left( f^2 \tilde{R}^m_{ijk\ell} + (f^{-1})^2 (f')^2 (-g_i g_{jk} + g_j g_{ik}) \right) V^i W^j W^k V^\ell.
\]
using the fact that $fV$ and $fW$ are orthogonal unit vectors for $\tilde{g}$, we see that
\[
K(P) = f^{-2} \tilde{K}(P) - (f^{-1})^2 (f')^2.
\]

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