COMPLEX MONGE-AMPERE OPERATORS VIA
PSEUDO-ISOMORPHISMS: THE WELL-DEFINED CASES

TUYEN TRUNG TRUONG

ABSTRACT. Let $X$ and $Y$ be compact Kähler manifolds of dimension 3. A bimeromorphic map $f : X \to Y$ is pseudo-isomorphic if $f : X - I(f) \to Y - I(f^{-1})$ is an isomorphism.

Let $T = T^+ - T^-$ be a current on $Y$, where $T^\pm$ are positive closed $(1,1)$ currents which are smooth outside a finite number of points. We assume that the following condition is satisfied:

Condition 1. For every curve $C$ in $I(f^{-1})$, then in cohomology $\{T\}\{C\} = 0$.

Then, we define a natural push-forward $f_*(\varphi dd^c u \wedge f^*(T))$ for a quasi-pls function $u$ and a smooth function $\varphi$ on $Y$. We show that this pushforward satisfies a Bedford-Taylor’s monotone convergence type.

Assume moreover that the following two conditions are satisfied

Condition 2. The signed measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$.

Condition 3. For every curve $C$ in $I(f^{-1})$, the measure $T \wedge [C]$ has no Dirac mass.

Then, we define a Monge-Ampere operator $MA(f^*(T)) = f^*(T) \wedge f^*(T) \wedge f^*(T)$ for $f^*(T)$. We show that this Monge-Ampere operator satisfies several continuous properties, including a Bedford-Taylor’s monotone convergence type when $T$ is positive. The measures $MA(f^*(T))$ are in general quite singular. Also, note that it may be not possible to define $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$.

1. INTRODUCTION

Let $X$ and $Y$ be compact Kähler manifolds of dimension 3. A bimeromorphic map $f : X \to Y$ is pseudo-isomorphic if the map $g = f|_{X - I(f)} : X - I(f) \to Y - I(f^{-1})$ is an isomorphism. Here $I(f)$ and $I(f^{-1})$ are the indeterminate sets of $f$ and $f^{-1}$, both have dimensions at most 1. (In fact, Bedford-Kim [3] showed that if $I(f)$, and hence $I(f^{-1})$, is non-empty then it must be of pure dimension 1). We let $\Gamma_g \subset (X - I(f)) \times (Y - I(f^{-1}))$ be the graph of $g$, and $\Gamma_f = \text{the closure of } \Gamma_g \text{ in } X \times Y \text{ the graph of } f$. Let $\pi_1, \pi_2 : X \times Y \to X, Y$ be the natural projections, and occasionally we use the same notations for the restrictions to $\Gamma_g, \Gamma_f$.

Given a meromorphic map $f : X \to Y$ and a smooth closed $(1,1)$ form $\theta$ on $Y$, the pullback $f^*(\theta)$ is well-defined as a $(1,1)$ current, however, in general is singular on $I(f)$. To see an explicit example, consider the simple map $J : \mathbb{P}^3 \to \mathbb{P}^3$ given by the formula $J[x_0 : x_1 : x_2 : x_3] = [1/x_0 : 1/x_1 : 1/x_2 : 1/x_3]$. If $u$ is a smooth function then $J^* dd^c u$ will have singularities of the form $1/(x_i^2 x_j^2)$ near the curves of indeterminacy $x_i = x_j = 0$. Therefore, a priori it is not clear whether we can define the Monge-Ampere operator $MA(J^* dd^c u) = J^* dd^c u \wedge J^* dd^c u \wedge J^* dd^c u$ in a reasonable manner.
In [13], we show that if \( f: X \to Y \) is a pseudo-isomorphism in dimension 3, and \( \theta \) is a smooth closed \((1,1)\) form on \( Y \) such that in cohomology
\[
\{\theta\}, \{C\} = 0,
\]
for every curve \( C \) in \( I(f^{-1}) \), then we have a well-defined Monge-Ampere operator \( MA(f^*(\theta)) \).

For example, the map \( J \) above is not yet pseudo-isomorphic, but if we let \( X \to \mathbb{P}^3 \) be the blowup at the 4 points \( e_0 = [1:0:0:0], e_1 = [0:1:0:0], e_2 = [0:0:1:0], e_3 = [0:0:0:1] \), then the induced map \( J_X \) is a pseudo-isomorphism. Hence, for any smooth function \( u \) on \( Y = X \), we can define the Monge-Ampere operator \( J_X^*dd^c u \) in a reasonable and consistent way, even though as we saw above this \((1,1)\) current is quite singular. Note that if we write \( dd^c u = \alpha^+ - \alpha^- \), where \( \alpha^\pm \) are positive closed smooth \((1,1)\) forms, there may be no reasonable and consistent way to define the wedge products \( J_X^*(\alpha^\pm) \land J_X^*(\alpha^\pm) \land J_X^*(\alpha^\pm) \). For such an intersection to be well-defined, we may want to show that \( J_X^*(\alpha^\pm) \) have locally bounded potentials near \( I(J_X) \). However, we have the following result

**Lemma 1.1.** Let \( C \) be an irreducible curve in \( I(J_X) \). Then \( J_X(C) = D \) is another irreducible curve in \( I(J_X) \). If \( \omega \) is a positive closed smooth \((1,1)\) form on \( X \) such that \( \{\omega\}, \{D\} > 0 \), then the local potentials of \( J_X^*(\omega) \) are unbounded near \( C \).

**Proof.** That \( J_X(C) = D \) is an irreducible curve in \( J_X(C) \) can be checked directly (see the last section in [13]). Now we prove the claim about the unboundedness of the local potentials of \( J_X^*(\omega) \) near \( C \). Assume otherwise. Then by Bedford-Taylor’s results [4], the wedge intersection of currents \( J_X^*(\omega) \land [C] \) is well-defined as a positive measure on \( X \). In particular, in cohomology
\[
0 \leq \{J_X^*(\omega) \land [C]\} = \{J_X^*(\omega)\}, \{C\} = \{\omega\}, \{J_X\}, \{C\}.
\]
However, we can check that in cohomology \( \{J_X\}, \{C\} = -\{D\} \) (see the last section in [13]). Hence we obtain
\[
\{\omega\}, \{J_X\}, \{C\} = -\{\omega\}, \{D\} < 0.
\]
by assumption. This is a contradiction. \( \square \)

The purpose of this short note is to extend the Monge-Ampere operator \( MA(f^*(T)) = f^*(T) \land f^*(T) \land f^*(T) \) in [13] to currents \( T \) which can be singular on a finite number of points. The points are allowed to be in \( I(f^{-1}) \). The main motivation for this is that given a psef cohomology class \( \eta \in H^{1,1}(X) \), it may not be able to find a positive closed smooth form \( \theta \) in that class, while if we allow a mild singularity there may be a positive closed \((1,1)\) current in the class of \( \eta \) with that singularity. Moreover, if we allow more singularity for \( T \), then the current \( f^*(T) \) may be more singular and hence it makes it more difficult to define \( MA(f^*(T)) \).

We show that the Monge-Ampere operator so defined satisfies various continuous properties, see in particular Theorems [2,8] and [2,13], Lemmas [2,7] and [2,9] and the last subsection of the paper. In the proof of the continuous properties, we will use the following approximation of positive closed smooth \((1,1)\) currents, due to Demailly [6].

**Definition 1.2.** Let \( Y \) be a compact Kähler manifold with a Kähler form \( \omega_Y \). Let \( T = \alpha + dd^c u \) be a positive closed \((1,1)\) current on \( Y \), where \( \alpha \) is a smooth closed \((1,1)\) form
and \( u \) is a quasi-psh function. Let \( u_j \) be a sequence of smooth quasi-psh functions on \( Y \) decreasing to \( u \) such that \( \alpha + dd^c u_j + \epsilon \omega_Y \geq 0 \) for all \( j \), here \( \epsilon > 0 \) is a positive constant. Then we say that \( \alpha + dd^c u_j \) is a good approximation of \( T = \alpha + dd^c u \).

Here we summarize the main results.

**Theorem 1.3.** Let \( f: X \to Y \) be a pseudo-isomorphism in dimension 3. Let \( T = T^+ - T^- \) be a difference of two positive closed currents \((1,1)\) currents on \( Y \), both are smooth outside a finite number of points. These points are allowed to be in \( I(f^{-1}) \).

Assume that for every curve \( C \) in \( I(f^{-1}) \) we have in cohomology \( \{T\}, \{C\} = 0 \).

We write \( f^*(T) = \Omega + dd^c u \), where \( \Omega \) is a smooth closed \((1,1)\) form and \( u = u^+ - u^- \) is a difference of two quasi-psh functions.

1) (Bedford-Taylor’s monotone convergence.) Let \( u_j^\pm \) be smooth quasi-psh functions decreasing to \( u^\pm \). Then for any smooth function \( \varphi \) on \( X \), the following limit exists

\[
\lim_{j \to \infty} f_*(\varphi(\Omega + dd^c u_j^+ - dd^c u_j^-)) \wedge f^*(T).
\]

We denote the limit by \( f_* (\varphi f^*(T) \wedge f^*(T)) \).

2) Let \( S \) be a smooth closed \((1,1)\) form on \( Y \). Let \( u_j^\pm \) be as in 1). Then

\[
\lim_{j \to \infty} \int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge f^*(T) = \int_Y S \wedge f_*(\varphi f^*(T) \wedge f^*(T)).
\]

3) Assume further that \( T \) satisfies the following two conditions:

i) The measure \( T \wedge T \wedge T \) has no mass on \( I(f^{-1}) \).

ii) For each curve \( C \) in \( I(f^{-1}) \) then the measure \( T \wedge [C] \) has no Dirac mass.

Then there is a natural and well-defined wedge intersection of currents \( T \wedge f_*(\varphi f^*(T) \wedge f^*(T)) \). In view of 2) above, we define the Monge-Ampere operator \( MA(f^*(T)) \) by the formula

\[
< MA(f^*(T)), \varphi > := \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).
\]

4) Assumptions are as in 3). Assume further that \( T \) is a positive current. We write \( T = \alpha + dd^c v \), where \( \alpha \) is a smooth closed \((1,1)\) form and \( v \) is a quasi-psh function. Let \( \alpha + dd^c v_n \) be a good approximation of \( T = \alpha + dd^c v \), in the sense of Definition 1.2. Then for any smooth function \( \varphi \) we have

\[
\lim_{n \to \infty} \int_Y (\alpha + dd^c v_n) \wedge f_*(\varphi f^*(T) \wedge f^*(T)) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).
\]

In other words, we have a double Bedford-Taylor’s monotone type convergence

\[
\lim_{n \to \infty} \lim_{j \to \infty} \int_X \varphi f^*(\alpha + dd^c v_n) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).
\]

5) Assumptions are as in 4). Assume moreover that \( f^*(T \wedge T) \) has no mass on \( I(f) \) (for example if \( T \) is smooth near \( I(f^{-1}) \)). If \( \Omega + dd^c u_j \) is a good approximation of \( f^*(T) = \Omega + dd^c u \) in the sense of Definition 1.2. then

\[
\lim_{j \to \infty} \int_X \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)).
\]
6) Assumptions are as in 3). If $T$ is smooth or positive then $MA(f^*(T)) = f^*(T \wedge T \wedge T)$. Here, since the measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$, the pullback $f^*(T \wedge T \wedge T)$ is well-defined.

**Remark 1.4.** In 6) of Theorem 3, a priori the measure $f^*(T \wedge T \wedge T)$ is quite singular near $I(f)$, even if $T$ is smooth. Also, note that there may be no reasonable and consistent manner to define the terms $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$, so we need to define $f^*(T) \wedge f^*(T) \wedge f^*(T)$ directly. See Lemma 1.1 and the discussion before it.

### 2. Definition of the Monge-Ampere Operator

We will consider the following class of currents

**Definition 2.1.** Class $(A)$. A closed $(1,1)$ current $T$ is in class $(A)$ if $T = T^+ - T^-$ where $T^\pm$ are positive closed $(1,1)$ currents which are smooth outside a finite number of points.

**Remark 2.2.** The essential property that we need in the above definition is that in $W - A$, here $W$ is an open neighborhood of $I(f^{-1})$ and $A$ is a finite set, the currents $T^\pm$ are smooth (in fact, continuous is enough). Outside $W - A$, $T^\pm$ may have mild singularity such that $T \wedge T \wedge T$ is well-defined. For example, following Bedford-Taylor [4], we need only to require that $T^\pm$ have locally bounded potentials.

**Remark 2.3.** That $T^\pm$ may have singular points on $I(f^{-1})$ makes it difficult to define the individual wedge products of currents $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$. This is because the preimage of a point on $I(f^{-1})$ may be a whole curve on $I(f)$. So a priori $f^*(T^\pm)$ may be singular on a whole curve contained in $I(f)$, see for example Lemma 1.1. Hence, in the below, we will define $f^*(T) \wedge f^*(T) \wedge f^*(T)$ directly, not via the wedge products $f^*(T^\pm) \wedge f^*(T^\pm) \wedge f^*(T^\pm)$.

We will consider the following three conditions

**Condition 1.** For every curve $C$ in $I(f^{-1})$, then in cohomology $\{T\}, \{C\} = 0$.

**Condition 2.** The signed measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$.

**Condition 3.** For every curve $C$ in $I(f^{-1})$, the measure $T \wedge [C]$ has no Dirac mass.

**Remark 2.4.** If $T$ is in Class $(A)$, then the measure $T \wedge T \wedge T$ has no mass on $I(f^{-1})$, except possibly a finite number of points on $I(f^{-1})$ where $T$ is not smooth. Hence Condition 2 is equivalent to that $T \wedge T \wedge T$ has no Dirac masses at these points.

**Remark 2.5.** If $T$ is smooth then $T$ satisfies both Conditions 2 and 3.

If $T$ is a positive closed $(1,1)$ current in Class $(A)$ and satisfies Condition 1, then it automatically satisfies Condition 3. Because in this case the wedge product of currents $T \wedge [C]$ is well-defined as a positive measure, and the total mass is $\{T\}, \{C\}$. However, if $T$ is not positive then this implication is not automatic.

Assume that the Monge-Ampere operator $MA(f^*(T)) = f^*(T) \wedge f^*(T) \wedge f^*(T)$ is well-defined. Then, formally, for a smooth function $\varphi$ on $X$ we have

$$\int_X \varphi f^*(T) \wedge f^*(T) \wedge f^*(T) = \int_Y T \wedge f_*(\varphi f^*(T) \wedge f^*(T)), \tag{2.1}$$
provided that both wedge intersections of currents \( f_*(\varphi f^*(T) \wedge f^*(T)) \) and \( T \wedge f_*(\varphi f^*(T) \wedge f^*(T)) \) are well-defined. The remaining of this note is to define these under the assumption that \( T \) is in Class \( \mathcal{A} \) and satisfies Conditions 1, 2 and 3.

**Remark 2.6** (Justification for the approach.). Under Condition 1, we showed in \cite{13} that \( f^*(T) \wedge f^*(T) = f^*(T \wedge T) \), so one may attempt to define \( MA(f^*(T)) \) in a different way

\[
(2.2) \quad \int_X \varphi f^*(T) \wedge f^*(T) \wedge f^*(T) = \int_X f_*(\varphi f^*(T)) \wedge T \wedge T.
\]

At first look, this approach seems to have equal footing with our approach in Equation (2.1). To justify what approach is more reasonable, let us consider a more general problem. Assume that \( S \) is another \((1,1)\) current which is smooth, and we want to define \( f^*(S) \wedge f^*(T) \wedge f^*(T) \).

Our approach in Equation (2.1) is to define

\[
\int_X \varphi f^*(S) \wedge f^*(T) \wedge f^*(T) := \int_Y S \wedge f_*(\varphi f^*(T) \wedge f^*(T)).
\]

The right hand side of the above expression is well-defined, since \( S \) is smooth, provided that the current \( f_*(\varphi f^*(T) \wedge f^*(T)) \) is defined. Moreover, the equality is justified by proving a continuity property, see Lemma 2.7 below.

The approach in Equation (2.2) is to define either

\[
\int_X \varphi f^*(S) \wedge f^*(T) \wedge f^*(T) := \int_Y f_*(\varphi f^*(S)) \wedge T \wedge T,
\]

or

\[
\int_X \varphi f^*(S) \wedge f^*(T) \wedge f^*(T) := \int_Y f_*(\varphi f^*(T)) \wedge S \wedge T.
\]

Since \( T \) may not be smooth, the equalities between the two sides of the above two expressions are not justified, if \( \varphi \) is not a constant.

From this simple consideration, we see that the definition in Equation (2.1) is more reasonable. Moreover, we will show later that if either \( T \) is smooth or positive, then the definitions in Equations (2.1) and (2.2) are the same.

Now we state and prove the continuous property referred to in the above remark.

**Lemma 2.7.** (Bedford-Taylor’s monotone convergence.) Assume \( S \) is a smooth closed \((1,1)\) form and \( T \) is a current in the class \((\mathcal{A})\) and satisfies Condition 1. We write \( f^*(T) = \Omega + dd^c u \), where \( \Omega \) is a smooth closed \((1,1)\) form and \( u = u^+ - u^- \) is the difference of two quasi-psh functions. Let \( u^+_j \) be a sequence of smooth quasi-psh functions decreasing to \( u^+ \). We denote \( u_j = u^+_j - u^-_j \). Then

\[
\lim_{j \to \infty} \int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y S \wedge f_*(\varphi (\Omega + dd^c u) \wedge f^*(T)).
\]

Here the current \( f_*(\varphi (\Omega + dd^c u) \wedge f^*(T)) \) is defined in Equation (2.3) below.

**Proof.** First, we show that for each \( j \)

\[
\int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y S \wedge f_*(\varphi (\Omega + dd^c u_j) \wedge f^*(T)).
\]
Here both sides are well-defined, since \( \varphi, S, \Omega \) and \( u_j \) are smooth. The term \( f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) \) is defined as follows, by Meo’s results:

\[
f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = (\pi_1)_*(\pi_1^*(\varphi(\Omega + dd^c u_j)) \wedge \pi_1^*(f^*(T)) \wedge [\Gamma_f]).
\]

Now we can approximate \( f^*(T) \) by smooth closed \((1,1)\) forms \( \gamma_n = \gamma_n^+ - \gamma_n^- \). Here \( \gamma_n^\pm \) positive closed smooth \((1,1)\) forms with uniformly bounded masses, and converges locally uniformly on \( X - \text{supp}(f) \) to \( f^*(T) \).

Then it can be seen, by dimension reason (see for example the proof of Lemma 5 in [12]), that

\[
\int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \lim_{n \to \infty} \int_Y \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-).
\]

Since all \( \varphi, S, \Omega, u_j \) and \( \gamma_n^\pm \) are all smooth, we have

\[
\int_X \varphi f^*(S) \wedge (\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-) = \int_Y S \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-)).
\]

Now

\[
f_*(\varphi(\Omega + dd^c u_j) \wedge (\gamma_n^+ - \gamma_n^-)) = (\pi_2)_*(\pi_1^*(\varphi(\Omega + dd^c u_j) \wedge \pi_1^*(\gamma_n^+ - \gamma_n^-) \wedge [\Gamma_f]).
\]

The limit when \( n \to \infty \) of the right hand side is \((\pi_2)_*(\pi_1^*(\varphi(\Omega + dd^c u_j) \wedge \pi_1^*(f^*(T)) \wedge [\Gamma_f]). \) This is because the limit of \( \pi_1^*(\varphi) \pi_1^*(\Omega + dd^c u_j) \wedge \pi_1^*(\gamma_n^+ - \gamma_n^-) \wedge [\Gamma_f] \) is \( \pi_1^*(\varphi) \pi_1^*(\Omega + dd^c u_j) \wedge \pi_1^*(f^*(T)) \wedge [\Gamma_f]. \)

Therefore the claim is proved. Using this claim and part 2) of Theorem 2.8 below, the lemma follows.

### 2.1. Definition of the current \( f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) \)

The extension by zero of the current \( \pi_1^*(f^*(T)) \wedge [\Gamma_g] \) (the latter has bounded mass by Meo’s result [10]). Let \( u \) be a quasi-psh function on \( X \). Theorem 1.2 in [13] shows that the current \( \pi_1^*(uf^*(T)) \wedge [\Gamma_g] \) has bounded mass, and we let \( (\pi_1^*(uf^*(T)) \wedge [\Gamma_g])^\circ \) denote its extension by zero. In [13], we defined

\[
(2.3) \quad f_*(\varphi dd^c u \wedge f^*(T)) := (\pi_2)_*(\pi_1^*(\varphi) \wedge dd^c(\pi_1^*(uf^*(T)) \wedge [\Gamma_g])^\circ)
\]

We now prove a Bedford-Taylor’s monotone convergence theorem for this operator.

**Theorem 2.8.** Assume that \( T \) is in Class \((A)\) and satisfies the Condition 1. Then

1) If \( u \) is a smooth quasi-psh function on \( X \), we have

\[
f_*(\varphi dd^c u \wedge f^*(T)) = (\pi_2)_*(\pi_1^*(\varphi) \wedge dd^c(\pi_1^*(uf^*(T)) \wedge [\Gamma_f]).
\]

The right hand side above is the (correct) usual definition in the case \( u \) is smooth.

2) Let \( u \) be a quasi-psh function on \( X \), and let \( u_j \) be a sequence of smooth quasi-psh functions decreasing to \( u \). Then

\[
\lim_{j \to \infty} f_*(\varphi dd^c u_j \wedge f^*(T)) = f_*(\varphi dd^c u \wedge f^*(T)).
\]
Proof. 1) A modification of the proof of Theorem 1.3 in [13] shows that
\[ \pi_1^*(f^*(T)) \cap [\Gamma_f] = (\pi_1^*(f^*(T)) \cap [\Gamma_g])^0 + \sum_j \lambda_j [V_j]. \]

Here \( \lambda_j \geq 0 \) is a constant, and \( V_j \) are varieties of dimension 2 contained in \( \Gamma_f - \Gamma_g \). Moreover, \( \pi_2(V_j) \) are contained in the finite set of singular points of \( T \).

Since \( u \) is smooth, it is not difficult to check that
\[ (2.4) \quad \pi_1^*(u) (\pi_1^*(f^*(T)) \cap [\Gamma_g])^0 = (\pi_1^*(u) \pi_1^*(f^*(T)) \cap [\Gamma_g])^0. \]

Since we will use similar arguments later on, we give here a detailed proof. Using \( T = T^+ - T^- \), we may assume that \( T \) is positive. We may also assume that \( 0 \geq u \geq -M \). Then \( \pi_1^*(u) (\pi_1^*(f^*(T)) \cap [\Gamma_g])^0 \) is bounded between the two negative currents \( 0 \) and \( -M (\pi_1^*(f^*(T)) \cap [\Gamma_g])^0 \). Both these currents have no mass on \( \Gamma_f - \Gamma_g \), so is \( \pi_1^*(u) (\pi_1^*(f^*(T)) \cap [\Gamma_g])^0 \). On \( \Gamma_g \), \( \pi_1^*(u) (\pi_1^*(f^*(T)) \cap [\Gamma_g])^0 \) equals \( (\pi_1^*(u) \pi_1^*(f^*(T)) \cap [\Gamma_g])^0 \), and the current \( (\pi_1^*(u) \pi_2^*(f^*(T)) \cap [\Gamma_g])^0 \) has no mass on \( \Gamma_f - \Gamma_g \) by definition. Therefore, the two currents in Equation (2.4) are the same on \( Y \).

For any \( j \), since \( \pi_2(V_j) \) is a point, by the dimension reason we see immediately that
\[ (\pi_2)_*(\pi_1^*(f) dd^c \pi_1^*(u) \cap [V_j]) = 0. \]

Therefore we obtain
\[ (\pi_2)_*(\pi_1^*(f) \cap dd^c \pi_1^*(uf^*(T)) \cap [\Gamma_f]) = (\pi_2)_*(\pi_1^*(f) \cap dd^c (\pi_1^*(uf^*(T)) \cap [\Gamma_g])^0), \]

and the latter was defined to be \( f_*(\varphi dd^c u \cap f^*(T)) \) in Equation (2.3).

2) From Equation (2.4), it suffices to show that
\[ \lim_{j \to \infty} (\pi_1^*(uf_j^*(T)) \cap [\Gamma_g])^0 = (\pi_1^*(uf^*(T)) \cap [\Gamma_g])^0. \]

The proof of this is similar to that used to prove Equation (2.4). We can assume that \( T \) is positive, all \( u_j \) and \( u \) are negative. Let \( R \) be one cluster point of the left hand side. Then \( R \) is negative, \( R \geq 0 \) the right hand side, and on \( \Gamma_g \) then \( R \) is the right hand side. Since the right hand side has no mass on \( \Gamma_f - \Gamma_g \) by definition, we conclude that \( R = \) the right hand side.

We write \( f^*(T) = \Omega + dd^c u \), where \( \Omega \) is a smooth closed \((1,1)\) form, and \( u = u^+ - u^- \) is a difference of two quasi-psh functions. By Theorem [2.3] the pushforward
\[ f_*(\varphi f^*(T) \cap f^*(T)) := f_*(\varphi \Omega \cap f^*(T)) + f_*(\varphi dd^c u^+ \cap f^*(T)) - f_*(\varphi dd^c u^- \cap f^*(T)) \]
is well-defined. Moreover, if \( u^+_j \) is a sequence of smooth quasi-psh functions decreasing to \( u^+ \) then
\[ \lim_{j \to \infty} f_*(\varphi (\Omega + dd^c (u^+_j - u^-_j)) \cap f^*(T)) = f_*(\varphi f^*(T) \cap f^*(T)). \]

This Bedford-Taylor’s monotone convergence type implies the following

Lemma 2.9. The definition in Equation (2.5) is independent of the choice of \( \Omega \) and \( u \) in \( f^*(T) = \Omega + dd^c u \).
2.2. **Definition of the current** \( T \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) \). Let \( f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) \) be the current defined in the previous subsection. We now define the intersection \( T \wedge f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) \). Without loss of generality we may assume that \( 0 \leq \varphi \leq 1 \).

We recall that from Theorem 2.8, if \( u = u^+ - u^- \) where \( u^\pm \) are quasi-psh functions, and \( u_j^\pm \) are smooth quasi-psh functions decreasing to \( u^\pm \) then

\[
  f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = \lim_{j \to \infty} f_*(\varphi(\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge f^*(T))
\]

\[
  = \lim_{j \to \infty} (\pi)^*_2(\pi^*_1(\varphi)R^+ + \pi^*_1(\varphi)R^-).
\]

While the sequence \( \pi^*_1(\Omega + dd^c u_j^+ - dd^c u_j^-) \wedge f^*(T) \wedge [\Gamma_f] \) may not have a limit, it is a compact sequence and we let \( R^+ - R^- \) be a cluster point. Here \( R^\pm \) are positive closed currents of bidimension \((1,1)\) supported in \( \Gamma_f \). By the result discussed in the previous paragraph, we have

\[
  f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = (\pi)^*_2(\pi^*_1(\varphi)R^+ - \pi^*_1(\varphi)R^-).
\]

Since we assumed that \( 0 \leq \varphi \leq 1 \), we have

\[
  0 \leq (\pi)^*_2(\pi^*_1(\varphi) \wedge R^\pm) \leq (\pi)^*_2(R^\pm)
\]

**Remark 2.10.** Note that, under Condition 1 and the assumption that \( T \) is in class \((A)\), then

\[
  (\pi)^*_2(R^+ - R^-) = f_*(f^*(T) \wedge f^*(T)) = f_*(f^*(T \wedge T)) = T \wedge T
\]

has no mass on \( I(f^{-1}) \). Here we used that \( f_*(f^*) = Id \) on positive closed \((1,1)\) and \((2,2)\) currents, see Theorem 1 in [11]. However, each \( (\pi)^*_2(R^\pm) \) may have mass on \( I(f^{-1}) \). Therefore, if \( \varphi \) is not a constant, \( (\pi)^*_2(\pi^*_1(\varphi)R^+ - \pi^*_1(\varphi)R^-) \) may have mass on \( I(f^{-1}) \).

Since the currents \( (\pi)^*_2(\pi^*_1(\varphi) \wedge R^\pm) \) are positive DSH currents in the sense in Dinh-Sibony \([7, 8]\), they are \( C \)-flat in the sense of Bassanelli \([2]\). By Federer-type \( C \)-flatness theorem (Theorem 1.24 in \([2]\)), the restrictions of \( (\pi)^*_2(\pi^*_1(\varphi) \wedge R^\pm) \) to \( I(f^{-1}) \) are well-defined as a current on \( I(f^{-1}) \).

Note that on \( Y - I(f^{-1}) \), then \( f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = f_*(\varphi)T \wedge T \). Let \( (f_*(\varphi)T \wedge T)^o \) be the extension by zero of this current from \( Y - I(f^{-1}) \) to \( Y \). From the discussion above, and taking the bidimension of the various currents into consideration, we obtain the following result

**Lemma 2.11.**

\[
  f_*(\varphi(\Omega + dd^c u) \wedge f^*(T)) = (f_*(\varphi)T \wedge T)^o + \sum_j (\chi_j^+ - \chi_j^-)[C_j].
\]

Here \( C_j \) are irreducible components of dimension 1 of \( I(f^{-1}) \), and \( \chi_j^\pm \) are bounded positive measurable functions on \( C_j \).

Let \( A \) be the finite set where \( T \) is not smooth. Since \( f_*(\varphi) \) is a difference of two quasi-psh functions and \( T^\pm \) are continuous on \( Y - A \), by results in Fornaess-Sibony \([9]\) and Demailly \([5]\) (Section 4, Chapter 3), the current \( f_*(\varphi)T \wedge T \) is well-defined on \( Y - A \). Moreover, a monotone convergence property holds. Therefore, since \( T^\pm \wedge T^\pm \) are positive closed currents with no mass on \( I(f^{-1}) \), an argument similar to that in the proof of Equation (2.4) concludes that \( f_*(\varphi)T \wedge T \) has no mass on \( I(f^{-1}) - A \). By dimension reason, we
see that $f_*(\varphi)T \land T$ extends as a current on $Y$. The extension current is the same as the current $(f_*(\varphi)T \land T)^\circ$ defined before Lemma 2.11.

By Lemma 2.11 to define $T \land f_*(\varphi(\Omega + dd^c u) \land f^*(T))$, it is enough to define $T \land (f_*(\varphi)T \land T)^\circ$ and $T \land \chi^\pm_j[C_j]$ for each $j$. We note that $T \land T = \mu^+ - \mu^-$, where $\mu^\pm$ are positive measures which are smooth on $Y - A$. If Condition 2 is satisfied, then we can choose $\mu^\pm$ to have no mass on $A$. Similarly, $T \land [C_j]$ is a difference of two positive measures, which we can take to have no Dirac mass if Condition 3 is satisfied.

The following continuous property is a simple result in measure theory. For completeness, we include a proof of it here.

**Lemma 2.12.** Assume that $T$ is in Class (A) and Condition 2) is satisfied.

Let $\gamma_n$ be a sequence of uniformly continuous functions on $Y$ which converges to $f_*(\varphi)$ as currents. Moreover, assume that $\gamma_n = f_*(\varphi)$ on an open set $W$ with $W \cap I(f^{-1}) = \emptyset$, such that $T$ is smooth on $X - W - I(f^{-1})$. Then the sequence $T \land (\gamma_nT \land T)^\circ = \gamma_nT \land T$ converges to $f_*(\varphi)(\mu^+ - \mu^-)$. Here the measure $f_*(\varphi)(\mu^+ - \mu^-)$ is well-defined on $Y - A$, and is defined to be 0 on the finite set $A$.

A similar result holds when we consider the measures $T \land [C_j]$ and the functions $\chi^\pm_j$.

**Proof.** Since $\gamma_n$ is smooth, and $T \land T$ has no mass on $I(f^{-1})$, we have $(\gamma_nT \land T)^\circ = \gamma_nT \land T$.

Since $\mu^\pm$ are positive smooth measures on $Y - A$, we have

$$\lim_{n \to \infty} \gamma_nT \land T \land T = f_*(\varphi)(\mu^+ - \mu^-),$$

on $Y - A$.

Since $\mu^\pm$ are positive measures with no mass on $A$, any cluster point of $\gamma_n\mu^\pm$, which is bounded by $\mu^\pm$, also has no mass on $A$. Therefore we obtain

$$\lim_{n \to \infty} \gamma_nT \land T \land T = f_*(\varphi)(\mu^+ - \mu^-),$$

on all of $A$. \hfill \Box

By Lemma 2.12 the wedge intersection $T \land f_*(\varphi(\Omega + dd^c u) \land f^*(T))$ is well-defined using a continuous property.

### 2.3. The case $T$ is smooth or positive

We now show that in case $T$ is smooth or positive then the Monge-Ampere in our approach Equation (2.1) and the approach in Equation (2.2) are the same.

We first consider the case where $T$ is smooth. Then, by Theorem 2.8 the Monge-Ampere operator $MA(f^*(T)) = f^*(T) \land f^*(T) \land f^*(T)$ defined in Equation (2.1) is

$$< MA(f^*(T)), \varphi > = \lim_{j \to \infty} \varphi f^*(T) \land (\Omega + dd^c u_j) \land f^*(T),$$

where $u_j$ is an appropriate sequence of smooth functions converging to $u$. Since $T$ satisfies Condition 1, we have $f^*(T) \land f^*(T) = f^*(T \land T)$. Then

$$\lim_{j \to \infty} \int_X \varphi f^*(T) \land (\Omega + dd^c u_j) \land f^*(T) = \lim_{j \to \infty} \int_X \varphi(\Omega + dd^c u_j) \land f^*(T \land T)$$

$$= \lim_{j \to \infty} \int_X f_*(\varphi(\Omega + dd^c u_j)) \land T \land T.$$
Since $T$ is smooth on $Y$ and $\Omega + dd^c u_j \to f^*(T)$, the limit of the sequence of measures $f_\ast((\varphi + dd^c u_j)) \cap T \cap T$ is exactly $f_\ast((\varphi f^*(T)) \cap T \cap T$. It is also the same as $f_\ast((\varphi f_\ast(f^*(T))) \cap T \cap T = f_\ast(\varphi T \cap T \cap T$. Here we use that $f_\ast f^* = Id$ on positive closed (1,1) and (2,2) currents, by Theorem 1 in [11]. Thus the proof for the case $T$ is smooth is completed.

Now we consider the case $T$ is positive. We have the following

**Theorem 2.13.** Assume $T$ is a positive closed (1,1) current in Class $(A)$ and satisfies Condition 1. Then

1) \begin{equation}
(2.7) \quad f_\ast(\varphi f^*(T) \wedge f^*(T)) = f_\ast(\varphi T \wedge T.
\end{equation}

2) Assume moreover that $T$ satisfies Conditions 2), 3). We write $T = \alpha + dd^c v$, where $\alpha$ is a smooth closed (1,1) form and $v$ is a quasi-psh function. Let $\alpha + dd^c v_n$ be a good approximation of $T$ in the sense of Definition [12]. Then, for any smooth function $\varphi$ on $X$ we have

$$\lim_{n \to \infty} \int_X (\alpha + dd^c v_n) \wedge f_\ast(\varphi f^*(T) \wedge f^*(T)) = \int_X T \wedge f_\ast(\varphi f^*(T) \wedge f^*(T)).$$

3) Assumptions are as in 2). Assume moreover that $f^*(T \cap T)$ has no mass on $I(f)$. We write $f^*(T) = \Omega + dd^c u$, where $\Omega$ is a smooth closed (1,1) form and $u$ is a quasi-psh function. Let $\Omega + dd^c u_j$ be a good approximation of $f^*(T) = \Omega + dd^c u$ in the sense of Definition [12]. Then

$$\lim_{n \to \infty} \int_X \varphi f^*(T) \wedge (\Omega + dd^c u_j) \wedge f^*(T) = \int_Y T \wedge f_\ast(\varphi f^*(T) \wedge f^*(T)).$$

**Proof.** 1) In this case, $f^*(T) = \Omega + dd^c u$, where $\Omega$ is a smooth closed (1,1) form and $u$ is a quasi-psh function. Let $u_j$ be a sequence of smooth quasi-psh functions decreasing to $u$.

By the monotone convergence in Equation (2.6), we have

$$f_\ast(\varphi f^*(T) \wedge f^*(T)) = \lim_{n \to \infty} f_\ast(\varphi(\Omega + dd^c u_j) \wedge f^*(T)).$$

By Theorem 1 in [11], $f_\ast f^* = Id$ for positive closed (1,1) and (2,2) currents. Since $T$ satisfies Condition 1 and is in Class $(A)$, for every smooth closed (1,1) form $\alpha$ we can apply Theorem 1.1 in [13] to obtain

$$f_\ast(\alpha \wedge f^*(T)) = f_\ast(f^*(f_\ast(\alpha) \wedge f^*(T)) = f_\ast(f^*(f_\ast(\alpha) \wedge T) = f_\ast(\alpha) \wedge T.$$

We now claim that

$$f_\ast(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = f_\ast(\varphi) f_\ast((\Omega + dd^c u_j) \wedge f^*(T))$$

for every $j$. We choose $\Omega + dd^c u_j$ a good approximation for $f^*(T)$, in the sense of Definition [12]. Therefore $\Omega + dd^c u_j + \epsilon \omega_X$ is positive for every $j$, here $\epsilon > 0$ is a constant. Since $\varphi$ is bounded, $f_\ast(\varphi(\Omega + dd^c u_j + \epsilon \omega_X) \wedge f^*(T))$ is bounded by $f_\ast((\Omega + dd^c u_j + \epsilon \omega_X) \wedge f^*(T))$. The latter, as seen in the last paragraph, is the same as $f_\ast((\Omega + dd^c u_j + \epsilon \omega_X) \wedge T$. It has no mass on $I(f^{-1})$. Therefore, $f_\ast(\varphi(\Omega + dd^c u_j) \wedge f^*(T))$ also has no mass on $I(f^{-1})$. Since $f_\ast(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = f_\ast(\varphi) f_\ast((\Omega + dd^c u_j) \wedge T$ on $Y - I(f^{-1})$, we conclude that the equality holds on all of $Y$. 

Recall that $A$ is the finite set of points where $T$ is not smooth. Since 
\[ \lim_{j \to \infty} f_*(\varphi) f_* (\Omega + dd^c u_j) = f_*(\varphi) f_* (T) = f_*(\varphi) T \] on $Y$, we conclude that on $Y - A$
\[ \lim_{j \to \infty} f_*(\varphi) f_* (\Omega + dd^c u_j) \land T = f_*(\varphi) T \land T. \]

By dimension reason, the above limit also holds on all of $Y$.

2) We need to show that
\[ \lim_{n \to \infty} (\alpha + dd^c v_n) \land (f_*(\varphi) T \land T)^0 = (f_*(\varphi) T \land T)^0. \]

First, since $\alpha + dd^c v_n$ is smooth, we have
\[ (\alpha + dd^c v_n) \land (f_*(\varphi) T \land T)^0 = (f_*(\varphi)(\alpha + dd^c v_n) \land T \land T)^0. \]

Therefore, it suffices to show that any cluster point of $(f_*(\varphi)(\alpha + dd^c v_n) \land T \land T)^0$ has no mass on $I(f^{-1})$.

Since $\alpha + dd^c v_n$ is a good approximation of $\alpha + dd^c v$, there is a constant $\epsilon > 0$ such that $\alpha + dd^c v_n + \epsilon \omega_Y$ is positive for every $n$. We write
\[ (f_*(\varphi)(\alpha + dd^c v_n) \land T \land T)^0 = \mu_{1,n} - \mu_2, \]

Here
\[ \mu_{1,n} = (f_*(\varphi)(\alpha + dd^c v_n + \epsilon \omega_Y) \land T \land T)^0, \]
\[ \mu_2 = (f_*(\varphi) \epsilon \omega_Y \land T \land T)^0, \]

are positive measures.

Since $\mu_{1,n}, \mu_2$ are bounded by the positive measures
\[ \nu_{1,n} = (\alpha + dd^c v_n + \epsilon \omega_Y) \land T \land T, \]
\[ \nu_2 = \epsilon \omega_Y \land T \land T, \]

it suffices to show that $\nu_2$ and any cluster point of $\nu_{1,n}$ have no mass on $I(f^{-1})$.

Since $T$ is smooth outside a finite number of points and $\omega_Y$ is smooth, it is easy to see that $\nu_2$ has no mass on $I(f^{-1})$.

The limit of $\nu_{1,n}$ is $(T + \epsilon)T \land T$ also has no mass on $I(f^{-1})$, since $T \land T \land T$ has no mass on $I(f^{-1})$ by Condition 3). Here, we use that monotone convergence holds, since $T \land T$ is smooth outside a finite number of points.

3) The proof is similar to the proof of 2). We write $T = \alpha + dd^c v$, where $\alpha$ is a smooth closed $(1,1)$ form, and $v$ is a quasi-psh function. Let $\alpha + dd^c v_n$ be a good approximation of $T$ in the sense of Definition 1.2. Hence we can assume that $\alpha + dd^c v_n + \epsilon \omega_Y \geq 0$ for all $n$, here $\epsilon$ is a positive constant.

Then it is easy to see that
\[ \lim_{j \to \infty} \int_X \varphi f^*(T) \land (\Omega + dd^c u_j) \land f^*(T) \land T = \lim_{j \to \infty} \int_X \varphi f^*(\alpha + dd^c v_n) \land (\Omega + dd^c u_j) \land f^*(T). \]

For each $n, j$ then as in 1) and previous sections, we can show that
\[ \int_X \varphi f^*(\alpha + dd^c v_n) \land (\Omega + dd^c u_j) \land f^*(T) = \int_Y (\alpha + dd^c v_n) \land f_*(\varphi(\Omega + dd^c u_j) \land f^*(T)). \]
Therefore, to prove 2), it suffices to show that
\[
\lim_{j \to \infty} \lim_{n \to \infty} (\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = (f_*(\varphi)T \wedge T \wedge T)^0.
\]
Since \( f : X - I(f) \to Y - I(f^{-1}) \) is a pseudo-isomorphism, the above equality holds on \( Y - I(f^{-1}) \). Therefore, we only need to show that any cluster point of
\[
\lim_{j \to \infty} \lim_{n \to \infty} (\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T))
\]
has no mass on \( I(f^{-1}) \).
As in the proof of 1), we have that
\[
(\alpha + dd^c v_n) \wedge f_*(\varphi(\Omega + dd^c u_j) \wedge f^*(T)) = (f_*(\varphi)(\alpha + dd^c v_n) \wedge f_*(\Omega + dd^c u_j) \wedge T)^0.
\]
We write
\[
(f_*(\varphi)(\alpha + dd^c v_n) \wedge f_*(\Omega + dd^c u_j) \wedge T)^0 = \mu_{j,n} - \mu_{1,j,n} - \mu_{2,j,n},
\]
where
\[
\mu_{j,n} = (f_*(\varphi)(\alpha + dd^c v_n + \omega_Y) \wedge f_*(\Omega + dd^c u_j + \omega_X) \wedge T)^0,
\]
\[
\mu_{1,j,n} = (f_*(\varphi)\omega_Y \wedge f_*(\Omega + dd^c u_j + \omega_X) \wedge T)^0,
\]
\[
\mu_{2,j,n} = (f_*(\varphi)(\alpha + dd^c v_n + \omega_Y) \wedge \epsilon f_*(\omega_X) \wedge T)^0.
\]
Note that \( \mu_{j,n}, \mu_{1,j,n}, \mu_{2,j,n} \) are positive measures and are bounded by the following positive measures
\[
\nu_{j,n} = (\alpha + dd^c v_n + \omega_Y) \wedge f_*(\Omega + dd^c u_j + \omega_X) \wedge T,
\]
\[
\nu_{1,j,n} = \omega_Y \wedge f_*(\Omega + dd^c u_j + \omega_X) \wedge T,
\]
\[
\nu_{2,j,n} = (\alpha + dd^c v_n + \omega_Y) \wedge f_*(\omega_X) \wedge T.
\]
Hence, it suffices to show that the following limits exist and have no mass on \( I(f^{-1}) \)
\[
\lim_{j \to \infty} \lim_{n \to \infty} \nu_{j,n},
\]
\[
\lim_{j \to \infty} \lim_{n \to \infty} \nu_{1,j,n},
\]
\[
\lim_{j \to \infty} \lim_{n \to \infty} \nu_{2,j,n}.
\]
a) The first limit is
\[
\lim_{j \to \infty} \lim_{n \to \infty} (\alpha + dd^c v_n + \omega_Y) \wedge f_*(\Omega + dd^c u_j + \omega_X) \wedge T
\]
\[
= \lim_{j \to \infty} f_*(\Omega + dd^c u_j + \omega_X) \wedge (T + \omega_Y) \wedge T
\]
\[
= (T + f_*(\omega_X)) \wedge (T + \omega_Y) \wedge T.
\]
Here we used that \( T \) is smooth outside a finite number of points, hence monotone convergence holds. In the resulting limit:
- The term \( T \wedge T \wedge T \) has no mass on \( I(f^{-1}) \) by Condition 3).
- The term \( T \wedge \omega_Y \wedge T \) has no mass on \( I(f^{-1}) \) since \( T \) is smooth outside a point and \( \omega_Y \) is smooth.
- The term \( f_*(\omega_X) \wedge \omega_Y \wedge T \) has no mass on \( I(f^{-1}) \) since \( T \) is smooth outside a finite number of points, \( f_*(\omega_X) \) has no mass on proper analytic subvarieties, and \( \omega_Y \) is smooth.
- Now we show that the last term \( f^*(\omega) \wedge T \wedge T \) has no mass on \( I(f^{-1}) \). By assumption, \( f^*(T \wedge T) \) has no mass on \( I(f) \), hence it is a positive current, and the positive measure \( \omega_X \wedge f^*(T \wedge T) \) has no mass on \( I(f) \). Therefore, the pushforward \( f^*(\omega_X \wedge f^*(T \wedge T)) \) is well-defined as a positive measure with no mass on \( I(f^{-1}) \). On \( Y - I(f^{-1}) \), then \( f^*(\omega) \wedge T \wedge T = f^*(\omega_X \wedge f^*(T \wedge T)) \). Therefore, \( f^*(\omega) \wedge T \wedge T \geq f^*(\omega_X \wedge f^*(T \wedge T)) \), which can be computed cohomologically, are the same. We conclude that \( f^*(\omega) \wedge T \wedge T = f^*(\omega_X \wedge f^*(T \wedge T)) \) on \( Y \).

Here we use the following properties of pseudo-isomorphisms in dimension 3: \( f^*(\zeta \wedge \eta) = f^*(\zeta) \wedge f^*(\eta) \) (see [3]) for \( \zeta \in H^{1,1} \) and \( \eta \in H^{2,2} \).

Hence we conclude that the first limit has no mass on \( I(f^{-1}) \).

b) Using a similar argument, we have that the second and third limits also have no mass on \( I(f^{-1}) \), as wanted.

\[ \square \]

**References**

[1] **Note.** More relevant references will be added later.

[2] G. Bassanelli, *A cut-off theorem for plurisubharmonic currents*, Forum Math. 6 (1994), no. 5, 567–595.

[3] E. Bedford and K.-H. Kim, *Dynamics of pseudo-automorphisms of 3 spaces: periodicity versus positive entropy*, arXiv: 1101.1614.

[4] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampere equation*, Invent. Math. 37 (1976), no. 1, 1–44.

[5] J.-P. Demailly, *Complex analytic and differential geometry*, Online book, version of Thursday 10 September 2009.

[6] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geometry 1. (1992), 361–409.

[7] T.-C Dinh and N. Sibony, *Regularization of currents and entropy*, Ann. Sci. Ecole Norm. Sup. (4), 37 (2004), no 6, 959–971.

[8] T.-C Dinh and N. Sibony, *Pull-back of currents by holomorphic maps*, Manuscripta Math. 123 (2007), no 3, 357–371.

[9] J. E. Fornaess and N. Sibony, *Oka’s inequality for currents and applications*, Math. Ann. 301 (1995), 399–419.

[10] Michel Meo, *Inverse image of a closed positive current by a surjective analytic map*, (in French), C. Acad. Sci. Paris Ser. I Math. 322 (1996), no 12, 1141–1144.

[11] T.T. Truong, : *Some dynamical properties of pseudo-automorphisms in dimension 3*, accepted in Transactions of the American Mathematical Society. [ArXiv:1304.4100]

[12] T.T. Truong, : *Pullback of currents by meromorphic maps*, preprint. To appear in Bulletin de Societe Mathematiques de France.

[13] T.T. Truong, : *Pseudo-isomorphisms in dimension 3 and applications to complex Monge-Ampere operators*, arXiv: 1403.5325.

**Department of Mathematics, Syracuse University, Syracuse NY 13244**

*E-mail address: tutruong@syr.edu*