A NOTE ON SPECIAL ORTHOGONAL GROUPS FOLLOWING WALDSPURGER

BINYONG SUN AND CHEN-BO ZHU

ABSTRACT. The purpose of this note is to verify that the archimedean multiplicity one theorems shown for orthogonal groups (as well as general linear and unitary groups) in a previous paper of the authors remain valid for special orthogonal groups. The necessary ingredients to establish this variant are due to Waldspurger.

Theorem 0.1. Let $G$ be a special orthogonal group $SO(p, q)$ or $SO_n(\mathbb{C})$, $p, n \geq 1$, $q \geq 0$. Let $G'$ be $SO(p-1, q)$ or $SO_{n-1}(\mathbb{C})$, viewed as a subgroup of $G$ as usual. Then for every irreducible Casselman-Wallach smooth representation $V$ of $G$, and $V'$ of $G'$, one has that
\[
\dim \text{Hom}_{G'}(\hat{V} \hat{\otimes} V', \mathbb{C}) \leq 1.
\]
Here $\hat{\otimes}$ stands for the completed projective tensor product of Hausdorff locally convex topological vector spaces.

We follow the general set-up of [SZ, Section 3].

Let $(A, \tau)$ be a (finite-dimensional) commutative involutive algebra over $\mathbb{R}$, and let $E$ be a (non-degenerate finitely generated) Hermitian $A$-module, with a Hermitian form $\langle \cdot, \cdot \rangle_E : E \times E \to A$. Denote by $U(E)$ the group of $A$-linear automorphisms of $E$ preserving the form $\langle \cdot, \cdot \rangle_E$. Write $E_\mathbb{R} := E$, viewed as a real vector space. Denote by $\hat{U}(E)$ the subgroup of $GL(E_\mathbb{R}) \times \{\pm 1\}$ consisting of pairs $(g, \delta)$ such that either
\[
\delta = 1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle u, v \rangle_E, \quad u, v \in E,
\]
or
\[
\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.
\]
This contains $U(E)$ as a subgroup of index two.

First assume that $(A, \tau)$ is simple. If $\tau$ is nontrivial, we put
\[
U_s(E) := U(E) \quad \text{(This is a general linear group or a unitary group.)}
\]

First version on May 14, 2010.
and
\[ \hat{U}_s'(E) := \hat{U}_s(E) := \hat{U}(E). \]
Otherwise, \( \tau \) is trivial and \( A = \mathbb{R} \) or \( \mathbb{C} \). Then we have \( U(E) = O(E) \), and \( \hat{U}(E) = O(E) \times \{ \pm 1 \} \), in the usual notations. We shall put
\[ U_s(E) := SO(E) \subset O(E), \]
and following Waldspurger [Wa]
\[ \hat{U}_s(E) := \left\{ (g, \delta) \in \hat{U}(E) = O(E) \times \{ \pm 1 \} \mid \det(g) = \delta^{\frac{\dim A_E + 1}{2}} \right\} \]
and
\[ \hat{U}_s'(E) := \left\{ (g, \delta) \in \hat{U}(E) = O(E) \times \{ \pm 1 \} \mid \det(g) = \delta^{\frac{\dim A_E}{2}} \right\}. \]
In general, write
\[(A, \tau) = (A_1, \tau_1) \times (A_2, \tau_2) \times \cdots \times (A_r, \tau_r)\]
as a product of simple commutative involutive algebras over \( \mathbb{R} \). Then
\[ E = E_1 \times E_2 \times \cdots \times E_r, \]
where
\[ E_i := A_i \otimes_A E \]
is naturally a Hermitian \( A_i \)-module. We put
\[ U_s(E) := U_s(E_1) \times U_s(E_2) \times \cdots \times U_s(E_r) \subset U(E), \]
and
\[ \hat{U}_s(E) := \hat{U}_s(E_1) \times_{\{ \pm 1 \}} \hat{U}_s(E_2) \times_{\{ \pm 1 \}} \cdots \times_{\{ \pm 1 \}} \hat{U}_s(E_r) \]
\[ := \left\{ (g_1, g_2, \cdots, g_r, \delta) \mid (g_i, \delta) \in \hat{U}_s(E_i), i = 1, 2, \cdots, r \right\} \subset \hat{U}(E). \]
The latter \( (\hat{U}_s(E)) \) contains the former as a subgroup of index two. Denote by \( \chi_{s,E} \) the quadratic character on \( \hat{U}_s(E) \) with kernel \( U_s(E) \). Likewise we define a group \( \hat{U}_s'(E) \) which contains \( U_s(E) \) as a subgroup of index two. Denote by \( \chi_{s,E}' \) the quadratic character on \( \hat{U}_s'(E) \) with kernel \( U_s(E) \).

Write
\[ u_s(E) := \{ x \in \text{End}_A(E) \mid \langle xu, v \rangle_E + \langle u, xv \rangle_E = 0, u, v \in E \} \]
SPECIAL ORTHOGONAL GROUPS

for the Lie algebra of \( U_s(E) \) (which is also the Lie algebra of \( U(E) \)). Let the groups \( \bar{U}_s(E) \) and \( \bar{U}'_s(E) \) act on \( U_s(E) \) and \( u_s(E) \) by

\[
\begin{cases}
(g, \delta) \cdot x := gx\delta g^{-1}, & x \in U_s(E), \\
(g, \delta) \cdot x := \delta gxg^{-1}, & x \in u_s(E).
\end{cases}
\]  

(2)

Also they act on \( E \) by

\[
\begin{cases}
(g, \delta) \cdot u := \delta gu, & (g, \delta) \in \bar{U}_s(E), u \in E, \\
(g, \delta) \cdot u := gu, & (g, \delta) \in \bar{U}'_s(E), u \in E.
\end{cases}
\]  

(3)

It is by now standard (see for example [SZ, Section 7]) that Theorem 0.1 is implied by the first assertion of the following theorem in the case of \( A = \mathbb{R} \) or \( \mathbb{C} \), \( \tau \) trivial.

**Theorem 0.2.** One has that

\[
C_{\chi_s,E}^{-\infty}(U_s(E) \times E) = 0
\]

and

\[
C_{\chi_s,E}^{-\infty}(u_s(E) \times E) = 0.
\]

(4)

(5)

Note that \((g, \delta) \mapsto (\delta g, \delta)\) is a group isomorphism from \( \bar{U}_s(E) \) onto \( \bar{U}'_s(E) \) fixing \( U_s(E) \). Thus Theorem 0.2 is equivalent to

**Theorem 0.3.** One has that

\[
C_{\chi'_s,E}^{-\infty}(U_s(E) \times E) = 0
\]

and

\[
C_{\chi'_s,E}^{-\infty}(u_s(E) \times E) = 0.
\]

For \( E \) as in (1), put

\[
\text{sdim}(E) := \sum_{i=1}^{r} \max\{\text{rank}_{A_i}E_i - 1, 0\} + \dim_{\mathbb{R}} E_{\mathbb{R}}.
\]

We argue by induction on \( \text{sdim}(E) \) and so will assume that Theorem 0.2 (and hence Theorem 0.3) holds whenever \( \text{sdim}(E) \) is smaller.

Without loss of generality, in the remaining part of this note, assume that \((A, \tau)\) is simple and \( E \) is faithful as an \( A \)-module. Let \( x \) be a semisimple element of \( U_s(E) \) or \( u_s(E) \). Denote by \( A_x \) the subalgebra of \( \text{End}_A(E) \) generated by \( A \), \( x \) and \( x^\tau \). Here \( \tau \) is the involution of \( \text{End}_A(E) \) given by

\[
\langle xu, v \rangle_E = \langle u, x^\tau v \rangle_E, \quad u, v \in E.
\]
Proposition 0.5. imply the following

\[ x \in U_s(E_x) \text{ if } x \in U(E) \text{ and } x \in u_s(E_x) \text{ if } x \in u(E). \]

As in [SZ, Section 5], Harish-Chandra’s method of descent and the above lemma imply the following

**Proposition 0.5.** Every element of \( C_{X_s}^{-\infty}(U_s(E) \times E) \) is supported in \( (Z_E \times U_E) \times E \), and every element of \( C_{X_s}^{-\infty}(u_s(E) \times E) \) is supported in \( (\mathfrak{z}_E \oplus N_E) \times E \), where \( Z_E \) is the scalar multiplications (by \( A \)) in \( U_s(E) \), \( \mathfrak{z}_E \) is the scalar multiplications (by \( A \)) in \( u_s(E) \), \( U_E \) is the set of unipotent elements of \( U_s(E) \), and \( N_E \) is the set of nilpotent elements of \( u_s(E) \).

By the first assertion of the above proposition, (3) will imply (4). So we only need to prove (5).

Let \( v \) be a non-degenerate element of \( E \) (i.e., \( \langle v, v \rangle_E \) is invertible in \( A \)), and denote by \( E_v \) the orthogonal complement of \( v \) in \( E \). The second key observation of Waldspurger [Wa] is the following

**Lemma 0.6.** The map \( (g, \delta) \mapsto (g|_{(E_v)_w}, \delta) \) identifies the stabilizer of \( v \) in \( \tilde{U}_s(E_v) \) with the group \( \tilde{U}_s'(E_v) \). Furthermore, the restriction to \( \tilde{U}_s'(E_v) \) of the module \( u_s(E) \) is isomorphic to \( u_s(E_v) \times E_v \times u_s(A v) \). Here \( u_s(A v) \) carries the trivial \( \tilde{U}_s'(E_v) \)-action.

Again Harish-Chandra’s method of descent and the above lemma imply the following

**Proposition 0.7.** Every element of \( C_{X_s}^{-\infty}(u_s(E) \times E) \) is supported in \( u_s(E) \times \Gamma_E \), where \( \Gamma_E := \{ u \in E \mid \langle u, u \rangle_E = 0 \} \) is the null cone of \( E \).

The (same and key) argument of [SZ, Section 4] (reduction within the null cone) works in the setting of this note and we have

**Proposition 0.8.** Assume that every element of \( C_{X_s}^{-\xi}(u_s(E) \times E) \) is supported in \( (\mathfrak{z}_E \oplus N_E) \times \Gamma_E \), then

\[ C_{X_s}^{-\xi}(u_s(E) \times E) = 0. \]

Here \( C_{X_s}^{-\xi}(u_s(E) \times E) \) denotes the subspace of tempered generalized functions in \( C_{X_s}^{-\infty}(u_s(E) \times E) \).
Now Propositions 0.5 and 0.7 imply that the hypothesis of Proposition 0.8 is satisfied. (This completes the step of reduction to the null cone.) Together with Proposition 0.8, they imply that (6) always holds. Then a general principle due to Aizenbud and Gourevitch ([AGS, Theorem 4.0.2]) implies that (5) also holds. □

References

[AGS] A. Aizenbud, D. Gourevitch and E. Sayag, Generalized Harish-Chandra descent, Gelfand pairs, and an Archimedean analog of Jacquet-Rallis’s theorem, Duke Math. Jour. 149, (2009), 509-567.

[SZ] B. Sun and C.-B. Zhu, Multiplicity one theorems: the Archimedean case, preprint, http://www.math.nus.edu.sg/~matzhucb/publist.html, 2008.

[Wa] J.-L. Waldspurger, Une variante d’un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann, arXiv:0911.1618

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P.R. China
E-mail address: sun@math.ac.cn

Department of Mathematics, National University of Singapore, 2 Science drive 2, Singapore 117543
E-mail address: matzhucb@nus.edu.sg