Stars in M theory

(made up of intersecting branes)

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ABSTRACT

We study stars in M theory. First, we obtain the analog of Oppenheimer – Volkoff equations in a suitably general set up. We obtain analytically the asymptotic solutions to these equations when the equations of state are linear. We study perturbations around such solutions in several examples and, following a standard method, use their behaviour to determine whether an instability is present or not. In this way, we obtain a generalisation of the corresponding results of Chavanis. We also find that stars in M theory have instabilities. Therefore, if sufficiently massive, such stars will collapse. We discuss the significance of these (in)stabilities within the context of Mathur’s fuzz ball proposal.
1. Introduction

Stars in our 3+1 dimensional universe are expected to collapse if they are sufficiently massive. Depending on their masses, they may collapse to form compact objects like white dwarfs or neutron stars, or collapse to form black holes. The compact objects may collapse further if they gain sufficient mass, for example, by accretion. Thus, ultimately, all sufficiently massive objects are expected to collapse to form black holes.

Stars and other compact objects may be taken to be static and spherically symmetric, with their constituents obeying appropriate equations of state. The equilibrium configurations are then determined by Oppenheimer – Volkoff (OV) equations. By studying perturbations around such equilibrium configurations, one determines the onset of instability which signals the onset of collapse [1] – [4]. Studying the collapse further and determining the end state of collapse is complicated, and is a field of ongoing research [5]. Nevertheless, a sufficiently massive object is generally assumed to ultimately collapse to form a black hole.

It is clearly of interest to study stars and their collapse dynamics in string/M theory. In general, for the same entropic reasons which are explained in [6, 7] where early universe was studied in string/M theory, we may assume that stars in string/M theory are made up of $N$ stacks of intersecting branes. The equations of state for intersecting branes given in these studies are applicable here also. Or, they can be derived using similar methods. In the following, we will restrict ourselves to M theory. The corresponding string theory results are straightforward to obtain.

In this paper, we consider a $D = n_c + m + 2$ dimensional spacetime where $n_c$ dimensional space is assumed to be compact and toroidal, and the $(m+1)$ dimensional transverse space, with $m \geq 2$, is assumed to be non compact. We assume suitable isometries along the compact space and, further, that the stars are made up of $\mathcal{N}$ number of non interacting multicomponent fluids. This set up, for suitable choices of $(D, n_c, \mathcal{N})$ and suitable equations of state, is applicable to the stars made up of intersecting branes in M theory.

We then obtain the analog of OV equations in this general set up. Solving these OV type equations generally and analytically is not possible. The standard procedure is to resort to numerical methods.

However, following the methods of [8, 9, 10, 11], it is possible to determine analytically whether an instability is present or not. In this method, one
first obtains the asymptotic solutions of the OV equations. Such solutions, referred to in [8] as singular solutions, can be obtained analytically when the equations of state are linear. One then studies the behaviour of the perturbations in the mass of the star around these asymptotic solutions. This behaviour is sufficient to determine the presence or absence of the instability: if the perturbations exhibit damped oscillations then an instability is present; and if they exhibit a monotonous behaviour with no oscillations then no instability is present. Technically, determining which behaviour is exhibited by the perturbations involves determining the sign of a certain discriminant. See [9, 12, 13, 14] for detailed explanations of a similar instability, called gravothermal catastrophe, which occur in isothermal spheres.

We follow this method. Taking the equations of state to be linear, we obtain analytically the asymptotic solutions to the OV type equations. We then study the perturbations around the asymptotic solutions. A general answer regarding the presence or absence of an instability is algebraically quite involved. Hence, and in order to illustrate the nature of the results, we consider a few select examples only and study the perturbations. These examples are: (1) Stars which are effectively same as those in [11]. (2) Stars for which $N = 1$ and $n_c$ is arbitrary. (3) Stars made up of $M2$ branes, or $M5$ branes. And, (4) stars made up of four stacks of intersecting branes, denoted as $22'55'$ configuration. Similar results hold for stars made up of three stacks of $22'2''$ intersecting brane configuration.

Our results are a generalisation of those of [11] and are qualitatively similar. Thus, e.g. in Example (2) with the equation of state being given by $p = w\rho$ where $0 \leq w \leq 1$, we find that there is no instability if $m \geq 9$. Thus in these dimensions, there will not be any collapse, and a stable equilibrium configuration is possible for any value of mass. In the other examples of stars in M theory, $n_c$ and $m$ are fixed and $D = 11$, and instabilities are present. Thus, a sufficiently massive stars in these examples will collapse.

We then discuss the significance of these (in)stability results, particularly within the context of Mathur’s fuzzball proposal [15].

This paper is organised as follows. In section 2, we present the general set up and write down the analog of OV equations. In section 3, we incorporate the linear equations of state. In section 4, we obtain the asymptotic solutions and study the perturbations. In section 5, we specialise to the stars in M theory, obtain asymptotic solutions, and study the perturbations. In section 6, we discuss the significance of the (in)stability results. In section 7, we
conclude with a brief summary and mention a few topics for further studies. In three appendices, we provide some useful formulas.

2. General Set Up

We are interested in studying the static solutions describing stars made up of \( \mathcal{N} \) stacks of M theory branes, intersecting according to the BPS rules whereby two stacks of five branes intersect along three common spatial directions; a stack each of two branes and five branes intersect along one common spatial direction; and two stacks of two branes intersect along zero common spatial direction. We model these intersecting M theory branes by \( \mathcal{N} \) separately conserved energy momentum tensors \( T_{MN(I)} \) and appropriate equations of state among their components. We write the equations of motion and an appropriate ansatz for the metric which can describe such static configurations of M theory stars.

We take the spatial directions of the brane worldvolumes to be toroidal and assume necessary isometries. Let the spacetime coordinates be given by \( x^M = (t, x^i, r, \theta^a) \) where \( i = 1, 2, \cdots, n_c \) and \( a = 1, 2, \cdots, m \). The total spacetime dimension \( D = n_c + m + 2 \) which = 11 for stars in M theory. The coordinates \( x^i \) describe the compact, \( n_c \) dimensional, toroidal space; and the radial coordinate \( r \) and the spherical coordinates \( \theta^a \) describe the non compact, \( (m + 1) \) dimensional, transverse space. In the following, we will assume that \( m \geq 2 \). In standard notation and with \( \kappa^2 = 8\pi G_D = 1 \), the equations of motion may be written as

\[
\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} = T_{MN} = \sum_{I} T_{MN(I)} \tag{1}
\]

\[
\sum_{M} \nabla_M T^{M}_{N(I)} = 0 \tag{2}
\]

where \( T_{MN} \) is the total energy momentum tensor of non interacting multicomponent fluids and \( T_{MN(I)} \) the energy momentum tensor for the \( I^{th} \) component fluid, \( I = 1, 2, \cdots, \mathcal{N} \). For stars in M theory, \( T_{MN} \) is the total energy momentum tensor for intersecting branes and \( T_{MN(I)} \) is the energy momentum tensor for the \( I^{th} \) stack of branes, \( I = 1, 2, \cdots, \mathcal{N} \).

In the following, we study the static solutions which are spherically symmetric in the \( (m + 1) \) dimensional transverse space. The line element \( ds \)
which can describe such static intersecting branes may now be written as
\[ ds^2 = g_{MN} \, dx^M \, dx^N = -e^{2\lambda_0} \, dt^2 + \sum_i e^{2\lambda_i} (dx^i)^2 + e^{2\lambda} \, dr^2 + e^{2\sigma} \, d\Omega_m^2 \tag{3} \]
where \( d\Omega_m \) is the standard line element on an \( m \) dimensional unit sphere.

The energy momentum tensors \( T_{MN(I)} \) may all be assumed to be diagonal. We denote these diagonal elements as
\[ (T_0^0(I), T_i^i(I), T_r^r(I), T_a^a(I)) = (p_0I, p_iI, \Pi_I, p_aI) \tag{4} \]
where \( p_0I = -\rho_I \) and \( p_aI = p\Omega_I \) for all \( a \). The total energy momentum tensor is now given by
\[ T^M_N = \text{diag} \, (p_0, p_i, \Pi, p_a) \] where \( p_0 = \sum_I p_0I = -\rho \) and
\[ \rho = \sum_I \rho_I \ , \ p_i = \sum_I p_iI \ , \ \Pi = \sum_I \Pi_I \ , \ p_a = p_\Omega = \sum_I p\Omega_I \ . \tag{5} \]

In the case of the stars being studied here, one also has \( \Pi_I = p_aI = p\Omega_I \).

**Equations of motion**

Let
\[ \alpha = (0, i, a) \ , \ \lambda^\alpha = (\lambda^0, \lambda^i, \lambda^a) \ , \ p_{aI} = (p_0I, p_iI, p_aI) \tag{6} \]
where \( p_0I = -\rho_I \) and
\[ \lambda^a = \sigma \ , \ p_{aI} = p\Omega_I = p_I \]
for all \( a \). For the static solutions which are spherically symmetric in the \((m + 1)\) dimensional transverse space, the fields \((\lambda^\alpha, \lambda)\) and \((p_{aI}, \Pi_I)\) are all assumed to depend only on the coordinate \( r \). Define
\[ \Lambda = \sum_{\alpha} \lambda^\alpha = \lambda^0 + \sum_i \lambda^i + m\sigma \tag{7} \]
\[ T_I = \sum_M T_{M(I)}^M = \Pi_I + \sum_{aI} p_{aI} \ . \tag{8} \]

It is straightforward to calculate the Riemann tensor components corresponding to the metric given by equation (3), and obtain the equations of motion. Using the above definitions and after some algebra, it follows that
\[ (\Pi_I)_r = -\Pi_I \Lambda_r + \sum_{aI} p_{aI} \lambda^\alpha_r \tag{9} \]
\[ \Lambda_r^2 - \sum_{\alpha} (\lambda^\alpha_r)^2 = 2 \sum_I \Pi_I \, e^{2\lambda} + m(m - 1) \, e^{2\lambda - 2\sigma} \tag{10} \]
\[ \lambda_\alpha + (A_r - \lambda_r) \lambda_\rho = \sum_I \left( -p_{\alpha I} + \frac{T_I}{D - 2} \right) e^{2\lambda} + \delta^{\alpha\rho} (m - 1) e^{2\lambda - 2\sigma} \]  

(11)

where the subscripts \( r \) denote \( r \)-derivatives. We define a function \( f(r) \) by

\[ e^{2\lambda - 2\sigma} = \frac{1}{r^2 f} \]  

(12)

and a mass function \( M(r) \) by

\[ M(r) = r^{m-1} (1 - f) \quad \leftrightarrow \quad f = 1 - \frac{M}{r^{m-1}} \]  

(13)

Note that either of the functions \( f \) and \( M \) may be traded for the function \( \lambda(r) \), and that the line element for the \((m+1)\) dimensional transverse space given in equation (3), written in terms of \( f \), becomes

\[ e^{2\lambda} dr^2 + e^{2\sigma} d\Omega_m^2 = e^{2\sigma} \left( \frac{dr^2}{r^2 f} + d\Omega_m^2 \right) \]  

(14)

Also, by a suitable change of variable, the equations of motion (9) – (11) can be written more compactly as shown in Appendix A.

**Reduction to \( d = m + 2 \) dimensions**

We will now dimensionally reduce on the compact \( n_c \) dimensional toroidal space from \( D = n_c + m + 2 \) dimensions to \( d = m + 2 \) dimensions. Let \( x^M = (t, x^i, r, \theta^a) \) be the \( D \) dimensional spacetime coordinates as before and \( x^\mu = (t, r, \theta^a) \) be the \( d \) dimensional spacetime coordinates. Consider the \( D \) dimensional line element \( ds \) given by equation (3), and denote its \( d \) dimensional part as follows:

\[ ds_d^2 = g_{\mu\nu(d)} \, dx^\mu \, dx^\nu = -e^{2\lambda_0} dt^2 + e^{2\lambda} dr^2 + e^{2\sigma} d\Omega_m^2 \]  

(15)

Upon dimensional reduction, symbolically, we have the following:

\[ S \sim \int d^D x \sqrt{-g} \, R \]

\[ = \int d^d x \sqrt{-g_{(d)}} \, e^{\Lambda^c} \left( R_{(d)} + \cdots \right) , \quad \Lambda^c = \sum_i \lambda^i \]

\[ = \int d^d x \sqrt{-\bar{g}} \left( \bar{R} + \cdots \right) , \quad \bar{g}_{\mu\nu} = e^{2\Lambda_0} g_{\mu\nu(d)} \]
where $\tilde{g}_{\mu\nu}$ is the $d$ dimensional Einstein frame metric. Hence the line element $\tilde{ds}_d$ in the $d$ dimensional Einstein frame is given by

$$\tilde{ds}_d^2 = \tilde{g}_{\mu\nu} \, dx^\mu \, dx^\nu = -e^{2\tilde{\lambda}_0} \, dt^2 + e^{2\tilde{\lambda}_r} \, dr^2 + e^{2\tilde{\sigma}} \, d\Omega_m^2$$

(16)

where

$$\tilde{\lambda}^a = \lambda^a + \frac{\Lambda}{m}, \quad \tilde{\lambda} = \lambda + \frac{\Lambda}{m},$$

(17)

and $\tilde{\lambda}^a = \tilde{\sigma}$ for all $a$. Also, let

$$\tilde{\Lambda} = \tilde{\lambda}_0 + \frac{m}{2} \tilde{\sigma}, \quad \tilde{\chi} = \tilde{\lambda} - \tilde{\sigma} = \tilde{\lambda}_0 + (m-1)\tilde{\sigma}.$$  

(18)

We then have, with the functions $\chi$ and $\tau$ as defined in Appendix A,

$$\tilde{\Lambda} = \Lambda + \frac{\Lambda}{m}, \quad \tilde{\chi} = \chi = \Lambda - \sigma$$

and

$$e^{2\lambda - 2\sigma} = e^{2\tilde{\lambda} - 2\tilde{\sigma}} = \frac{1}{r^2 f}, \quad r_r = \frac{dr}{d\tau} = e^{\Lambda - \lambda} = e^{\tilde{\Lambda} - \tilde{\lambda}} = r \sqrt{f} \, e^\chi.$$  

Note that it is equally convenient to use $\tilde{\lambda}_i$ or $\lambda^i$; that $\Lambda^c$ is related to $\tilde{\lambda}^i$ as follows: $\tilde{\Lambda}^c = \sum_i \tilde{\lambda}^i = \frac{n+2m}{m} \Lambda^c$; that $\sum_i \tilde{a}^i b^i = \sum_i a^i \tilde{b}^i = \sum_i a^i \tilde{b}^i + a^c \tilde{b}^c$ where $\tilde{a}^i = a^i + \frac{2}{m}$; $\tilde{a}^c = \sum_i a^i$, and similarly for $\tilde{b}^i$ and $\tilde{b}^c$; and, lastly, that $\sum_i \tilde{a}^i a^i > 0$ if $a^i$ do not all vanish.

It now follows from equations (9) – (11) that

$$(\Pi_I)_r = -\Pi_I \, \tilde{\Lambda}_r + p_{0I} \, \tilde{\lambda}_0^0 + m \, p_{0I} \, \tilde{\sigma}_r + \sum_i \left(p_{iI} - \frac{T_{iI}}{m}\right) \tilde{\lambda}_r^i + \sum_I \left(p_{0I} - \frac{T_{0I}}{m}\right) \left(\frac{2\Lambda^c_I}{m}\right) \tilde{\lambda}_r$$

(19)

$$2\tilde{\lambda}_0^0 \tilde{\sigma}_r + (m-1) (\tilde{\sigma}_r)^2 = \frac{2}{m} \sum_I \Pi_I \, e^{2\lambda} + (m-1) \, e^{2\tilde{\lambda} - 2\tilde{\sigma}} + \frac{B}{m}$$

(20)

$$\tilde{\sigma}_r + (\tilde{\Lambda}_r - \tilde{\lambda}_0) \, \tilde{\sigma}_r = \sum_I \left(-p_{0I} + \frac{T_{0I}}{m}\right) \, e^{2\lambda} + (m-1) \, e^{2\tilde{\lambda} - 2\tilde{\sigma}}$$

(21)

$$\tilde{\lambda}_r^0 + (\tilde{\Lambda}_r - \tilde{\lambda}_0) \, \tilde{\lambda}_r^0 = \sum_I \left(-p_{0I} + \frac{T_{0I}}{m}\right) \, e^{2\lambda}$$

(22)

$$\tilde{\chi}_r^0 + (\tilde{\Lambda}_r - \tilde{\lambda}_0) \, \tilde{\chi}_r^0 = \sum_I \left(-p_{0I} + \frac{T_{0I}}{m}\right) \, e^{2\lambda}$$

(23)
where $\mathcal{T}_I = \Pi_I + p_{0I} + m p_{0I}$ and $\mathcal{B} = \sum_i (\mathcal{A}_I)^2 + \frac{(\mathcal{A}_I)^2}{m} = \sum_i \mathcal{A}_I^2 \mathcal{A}_I^2$. Using the diffeomorphic freedom in defining the radial coordinate, we now set $e^\sigma = r$. Equations (20) and (21) become

$$r \mathcal{A}_I^0 = \sum I \frac{\Pi_I}{m} r^2 e^{2\mathcal{A}} + \frac{m-1}{2} (e^{2\mathcal{A}} - 1) + \frac{r^2 \mathcal{B}}{2m} \tag{24}$$

$$r (\mathcal{A}_I^0 - \mathcal{A}_I) = \sum I \left( \frac{\Pi_I + p_{0I}}{m} \right) r^2 e^{2\mathcal{A}} + (m-1) (e^{2\mathcal{A}} - 1) . \tag{25}$$

**Stars**

For the stars, we have $\Pi_I = p_{0I} = p_I$. Then $\mathcal{T}_I = (m + 1)p_I - \rho_I$, and equations (19) and (22) – (25) become

$$(p_I)_r = -(\rho_I + p_I) \mathcal{A}_I^0 + \sum I \left( p_I - \frac{(m+1)p_I - \rho_I}{m} \right) \mathcal{A}_I^r + p_I \left( \frac{2\mathcal{A}_I^c}{m} \right) \tag{26}$$

$$\mathcal{A}_I^r + (\mathcal{A}_r - \mathcal{A}_I) \mathcal{A}_I^0 = \sum I \left( \rho_I + \frac{(m+1)p_I - \rho_I}{m} \right) e^{2\mathcal{A}} \tag{27}$$

$$\mathcal{A}_I^i + (\mathcal{A}_r - \mathcal{A}_I) \mathcal{A}_I^j = \sum I \left( -p_I + \frac{(m+1)p_I - \rho_I}{m} \right) e^{2\mathcal{A}} \tag{28}$$

$$r \mathcal{A}_I^0 = \sum I \frac{p_I}{m} r^2 e^{2\mathcal{A}} + \frac{m-1}{2} (e^{2\mathcal{A}} - 1) + \frac{r^2 \mathcal{B}}{2m} \tag{29}$$

$$r (\mathcal{A}_I^0 - \mathcal{A}_I) = \sum I \left( p_I - \rho_I \right) r^2 e^{2\mathcal{A}} + (m-1) (e^{2\mathcal{A}} - 1) . \tag{30}$$

Since $e^\sigma = r$, we also have

$$\tilde{d}s_d^2 = -e^{2\mathcal{A}} dt^2 + \frac{dr^2}{f} + r^2 d\Omega_m^2 , \quad f = e^{-2\mathcal{A}} = 1 - \frac{M}{r^{m-1}} .$$

Thus, the line element $\tilde{d}s_d$ in the $d = m + 2$ dimensional Einstein frame takes the standard form. Therefore the functions appearing in it, e.g. the mass
function $M$, may be interpreted in the standard way. Note that equations (24) and (25), equivalently equations (29) and (30), give

$$r \tilde{\lambda}_r = \sum_I \frac{\rho_I}{m} r^2 e^{2\tilde{\lambda}} - \frac{m-1}{2} (e^{2\tilde{\lambda}} - 1) + \frac{r^2 B}{2m}.$$ 

The definition $M(r) = r^{m-1} (1 - f) = r^{m-1} (1 - e^{-2\tilde{\lambda}})$ then gives

$$M_r = \frac{r^m}{m} \left( 2 \sum_I \rho_I e^{-2\lambda} + B e^{-2\tilde{\lambda}} \right)$$

which leads to the familiar expression $M = \frac{2}{m} \int_0^r dr \left( r^{m} \rho \right)$ when $\lambda^i = 0$.

Equations (26) – (31) are the analog of OV equations. They describe the static equilibrium configurations of stars in $D = n_c + m + 2$ dimensional space-time. These configurations are independent of the $n_c$ dimensional compact toroidal coordinates and are spherically symmetric in the $(m+1)$ dimensional transverse space. The corresponding equations when $n_c = 0$, and hence $D = m + 2$, are given in [11] and they follow from the above ones by formally setting $\lambda^i = 0$, taking $\mathcal{N} = 1$, and ignoring the $\tilde{\lambda}^i_{rr}$ equation. The standard four dimensional OV equations follow upon further setting $m = 2$.

### 3. Linear equations of state

To solve equations (26) – (30) and obtain the solutions for the fields $(\lambda^a, p_{aI}, \Pi_I)$, one further requires equations of state which give $p_{aI}$ and $\Pi_I$ as functions of $\rho_I$. In this paper, we will take the equations of state to be linear and write them as

$$p_{aI} = w^I_a \rho_I, \quad \Pi_I = w^I_{\Pi} \rho_I$$

where $(w^I_a, w^I_{\Pi})$ are constants, $w^I_0 = -1$ since $p_{0I} = -\rho_I$, and $w^I_a = w^I_{\Pi} = w^I$ since $p_{aI} = p_{\Pi I} = p_I$ for all $a$. We show in Appendix B that when the equations of state are linear, the fields $(\lambda^a, \rho_I)$ can be expressed in terms of $\mathcal{N} + 1$ independent fields, denoted as $(l^I, l^* )$. One then has to solve the equations for $l^I$ and $l^*$ only.

For the stars we study here, $w^I_{\Pi} = w^I_a = w^I$ since $\Pi_I = p_{aI} = p_I$. Define

$$c^{aI} = -w^I_a + \frac{(m+1)w^I + \sum_j w^I_j - 1}{n_c + m}$$

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\[ \tilde{c}^{\alpha I} = c^{\alpha I} + \sum_j c^{jI} = -w_\alpha^I + \frac{(m+1)w^I - 1}{m}. \]  

(34)

Thus \( \tilde{c}^{\alpha I} = \tilde{c}^I \) and

\[ \tilde{c}^0I = 1 + w^I + \tilde{c}^I, \quad \tilde{c}^{iI} = -w_i^I + w^I + \tilde{c}^I, \quad \tilde{c}^I = \frac{w^I - 1}{m}. \]

Now consider equation (26). Upon using \( p_{\alpha I} = w_\alpha^I \rho_I \), it gives

\[ w^I (\ln \rho_I)_r = - (1 + w^I) \tilde{\lambda}_r^0 - \sum_i \tilde{c}^{iI} \lambda_r^i + w^I \left( \frac{2\Lambda_c}{m} \right) \]  

(35)

from which it follows that

\[ \rho_I = \rho_{I0} e^{\phi_I} e^{\frac{2\Lambda_c}{m}}, \quad w^I \phi^I = - (1 + w^I) \tilde{\lambda}_0^0 - \sum_i \tilde{c}^{iI} \lambda^i \]  

(36)

where \( \rho_{I0} \) is a constant. Therefore

\[ \rho_I e^{2\lambda} = \rho_{I0} e^{\phi^I + 2\lambda}, \quad r^2 \rho_I e^{2\lambda} = \rho_{I0} e^{\phi^I + 2\lambda + 2\tilde{\sigma}}. \]

For any function \( X(r(\tilde{\sigma})) \), using \( r = e^{\tilde{\sigma}} \), we have

\[ X_{\tilde{\sigma}} = r X_r, \quad X_{\tilde{\sigma}\tilde{\sigma}} = r^2 X_{rr} + r X_r \]

where the subscripts \( \tilde{\sigma} \) denote \( \tilde{\sigma} \)-derivatives. Then

\[ r^2 \left( X_{rr} + (\tilde{\lambda}_r - \tilde{\lambda}^0_r) X_r \right) = X_{\tilde{\sigma}\tilde{\sigma}} + (\tilde{\chi}_{\tilde{\sigma}} - \tilde{\lambda}_{\tilde{\sigma}}) X_{\tilde{\sigma}} \]

where \( \tilde{\chi} = \tilde{\Lambda} - \tilde{\sigma} = \tilde{\lambda}_0^0 + (m - 1) \tilde{\sigma} \). Equations (27) – (30), written in terms of \( \tilde{\sigma} \), now become

\[ \tilde{\lambda}_{\tilde{\sigma}\tilde{\sigma}} + (\tilde{\chi}_{\tilde{\sigma}} - \tilde{\lambda}_{\tilde{\sigma}}) \tilde{\lambda}_0^0_{\tilde{\sigma}} = \sum_I \tilde{c}^{0I} \rho_{I0} e^{\phi^I + 2\lambda + 2\tilde{\sigma}} \]  

(37)

\[ \tilde{\lambda}_{i\tilde{\sigma}\tilde{\sigma}} + (\tilde{\chi}_{\tilde{\sigma}} - \tilde{\lambda}_{\tilde{\sigma}}) \tilde{\lambda}_i^0_{\tilde{\sigma}} = \sum_I \tilde{c}^{iI} \rho_{I0} e^{\phi^I + 2\lambda + 2\tilde{\sigma}} \]  

(38)

\[ \tilde{\lambda}_0^0_{\tilde{\sigma}} = \sum_I \frac{w^I}{m} \rho_{I0} e^{\phi^I + 2\lambda + 2\tilde{\sigma}} + \frac{m - 1}{2} \left( e^{2\tilde{\lambda}} - 1 \right) + \frac{r^2 B}{2m} \]  

(39)
\[(\tilde{\lambda}_0^\sigma - \tilde{\lambda}_\sigma) = \sum_I \left(\frac{w_I^I - 1}{m}\right) \rho_{I0} e^{\phi_I + 2\tilde{\lambda}_0 + 2\tilde{\sigma}} + (m - 1) \left( e^{2\tilde{\lambda}} - 1 \right) \] (40)

where \( r^2 B = \sum_i \tilde{\lambda}_i^\sigma \lambda_i^\sigma \). Thus, equations (37) – (40) describe the stars when the equations of state are linear as given in equation (32) with \( w^n_I = w^I \).

4. Asymptotic solutions and perturbations

We now study the solutions to equations (37) – (40) in the limit \( r \to \infty \). We consider the ansatz

\[ \tilde{\lambda}_0^0 = \tilde{\lambda}_0^0 + \tilde{s}_0 \tilde{\sigma} + \tilde{u}_0, \quad \lambda^i = \lambda_0^i + s^i \tilde{\sigma} + u^i, \quad \tilde{\lambda} = \tilde{\lambda}_0 + \tilde{s} \tilde{\sigma} + \tilde{u} \]

where \((\tilde{\lambda}_0^0, \lambda_0^i, \tilde{\lambda}_0)\) and \((\tilde{s}_0, s^i, \tilde{s})\) are constants, \( \tilde{\lambda}_0^0 \) and \( \lambda_0^i \) can be set to zero with no loss of generality, and \((\tilde{u}_0, u^i, \tilde{u})\) are functions of \( r \). In the limit \( r \to \infty \), the \( \tilde{\lambda}_0 \) and the \( \tilde{\sigma} \) terms give the leading order asymptotic solutions, which are referred to in [8] as singular solutions. The functions \( \tilde{u} \)'s give the next order corrections, and thus are the perturbations around the asymptotic solutions. In this paper, we analyse the equations of motion to zeroth order in the \( \tilde{u} \)'s to obtain the asymptotic solutions; and analyse to the first order in the \( \tilde{u} \)'s to obtain perturbation equations.

Set \( \tilde{\lambda}_0^0 = \lambda_0^i = 0 \) with no loss of generality. Then

\[ \Lambda^c = S^c \tilde{\sigma} + U^c, \quad \tilde{\lambda}^i = \tilde{s}_i \tilde{\sigma} + \tilde{u}_i, \quad \phi^I = q^I \tilde{\sigma} + y^I \]

where

\[ S^c = \sum_i s^i, \quad U^c = \sum_i u^i; \quad \tilde{s}_i = s^i + \frac{S^c}{m}, \quad \tilde{u}_i = u^i + \frac{U^c}{m} \]

and, as follows from equation (36),

\[ w^I q^I = - (1 + w^I) \tilde{s}_0 - \sum_i \tilde{c}_{iI} s^i \] (41)

\[ w^I y^I = - (1 + w^I) \tilde{u}_0 - \sum_i \tilde{c}_{iI} u^i \] (42)

Also, write \( r^2 B = \sum_i \tilde{\lambda}_i^\sigma \lambda_i^\sigma = B_0 + 2B_1 + B_2 \) where

\[ B_0 = \sum_i \tilde{s}_i s^i, \quad B_1 = \sum_i s^i \tilde{u}_\sigma = \sum_i \tilde{s}_i u^i, \quad B_2 = \sum_i \tilde{u}_\sigma u^i. \]
Using the above expressions, we now expand equations (37) – (40) to zeroth and first order in the \( \tilde{u} \)'s. At zeroth order, equating the powers of \( r \) gives immediately

\[
2 + q^I + 2\tilde{s} = \tilde{s} = 0 .
\] (43)

Hence,

\[
r^2(\rho_I e^{2\lambda}) = \rho_{I0} e^{2\tilde{\lambda}_0 + y^I + 2\tilde{u}} = R_I (1 + y^I + 2\tilde{u} + \cdots)
\]

\[
e^{2\tilde{\lambda} - 1} = e^{2\tilde{\lambda}_0 + 2\tilde{u}} - 1 = (e^{2\tilde{\lambda}_0} - 1) + e^{2\tilde{\lambda}_0} (2\tilde{u} + \cdots)
\]

\[
\tilde{\chi}_\sigma - \tilde{\lambda}_\sigma = \alpha + (\tilde{u}^0_\sigma - \tilde{u}_\sigma) , \quad \alpha = m - 1 + \tilde{s}^0 - \tilde{s}
\]

where \( R_I = \rho_{I0} e^{2\tilde{\lambda}_0} \). Then equations (37) – (42) give

\[
\alpha \tilde{s}^0 = \sum_I \tilde{c}^{0l} R_I , \quad \alpha \tilde{s}^i = \sum_I \tilde{c}^{il} R_I
\] (44)

\[
\tilde{s}^0 = \sum_I \frac{w^I}{m} R_I + m - \frac{1}{2} \left( e^{2\tilde{\lambda}_0} - 1 \right) + \frac{B_0}{2m}
\] (45)

\[
\tilde{s}^0 - \tilde{s} = \sum_I \left( \frac{w^I - 1}{m} \right) R_I + (m - 1) \left( e^{2\tilde{\lambda}_0} - 1 \right)
\] (46)

\[
2w^I = (1 + w^I) \tilde{s}^0 + \sum_i \tilde{c}^{ij} \tilde{s}^j
\] (47)

at zeroth order. And, they give

\[
\tilde{u}^0_\sigma + \alpha \tilde{u}^0_\sigma + \tilde{s}^0 (\tilde{u}^0_\sigma - \tilde{u}_\sigma) = \sum_I \tilde{c}^{0l} R_I (y^I + 2\tilde{u})
\] (48)

\[
\tilde{u}^i_\sigma + \alpha \tilde{u}^i_\sigma + \tilde{s}^i (\tilde{u}^0_\sigma - \tilde{u}_\sigma) = \sum_I \tilde{c}^{il} R_I (y^I + 2\tilde{u})
\] (49)

\[
\tilde{u}^0_\sigma = \sum_I \frac{w^I}{m} R_I (y^I + 2\tilde{u}) + (m - 1) e^{2\tilde{\lambda}_0} \tilde{u} + \frac{B_1}{m}
\] (50)

\[
\tilde{u}^0_\sigma - \tilde{u}_\sigma = \sum_I \left( \frac{w^I - 1}{m} \right) R_I (y^I + 2\tilde{u}) + (m - 1) e^{2\tilde{\lambda}_0} (2\tilde{u})
\] (51)

\[
w^I y^I = -(1 + w^I) \tilde{u}^0 - \sum_i \tilde{c}^{ij} u^i
\] (52)
at first order. Note from equation (46) that

\[ \alpha = m - 1 + \tilde{s}^0 - \tilde{s} = \sum_I \left( \frac{w_I - 1}{m} \right) R_I + (m - 1) e^{2\tilde{\lambda}_0}. \]

We now make a few general remarks, thereby also summarising very briefly several results in the studies of stars which will be used here. See [8] – [14] for more details.

(1) Given the equations of state, the constants \((w^\alpha, \tilde{c}^\alpha_I)\) are known. The zeroth order equations are simply algebraic equations for the constants \((\tilde{s}^\alpha, R_I, e^{2\tilde{\lambda}_0})\). The first order equations are linear differential equations for the perturbations \(\tilde{u}^\alpha\). They are of the type encountered in standard ‘small oscillations’ problems in mechanics, and may be solved by standard techniques. Nevertheless, solving these equations and obtaining a general answer is algebraically quite involved. Hence, to proceed further and to illustrate the nature of the results, we will consider a few particular examples.

(2) It follows from \(\tilde{\lambda} = \tilde{\lambda}_0 + \tilde{u}\) and from the definition \(\frac{M(r)}{r^{m-1}} = 1 - e^{-2\tilde{\lambda}}\) that, in the limit of large \(r\),

\[ \frac{M(r)}{r^{m-1}} \to 1 - e^{-2\tilde{\lambda}_0} \]

and that \(\tilde{u}\) is the perturbation in the mass of the star.

(3) The size of the spherically symmetric star is defined as the radius \(r_*\) at which the pressure vanishes. The criteria for finiteness of \(r_*\) are given in [16]. The radius \(r_*\) is infinite for stars made up of perfect fluids with equation of state \(p = w \rho\) where \(0 \leq w \leq 1\). It is a standard procedure then to construct composite configurations consisting of perfect fluid cores and constant density crusts to render \(r_*\) finite, see [8] for example. Or, to confine the system within a spherical box of radius \(r_*\), see [11] for example.

OV equations are then solved numerically. The solutions contain one free parameter, taken in [11] to be \(x_{ch} \propto \sqrt{\rho_0} r_*\) which is a measure of central density. Let \(y_{ch} = \frac{M}{r_*^m}\) which measures the mass of the star. It is found from the numerical analyses that: (i) As \(x_{ch}\) increases from zero to \(\infty\), \(y_{ch}\) increases from zero to a (first) maximum \(y_1\) at \(x_1\), thereafter exhibits damped oscillations, asymptoting to a value \(y_s\). The asymptotic value \(y_s\) is typically several percent smaller than \(y_1\) in magnitude. (ii) The behaviour for large
values of $x_{ch}$ is described well by the analytical asymptotic solutions and the perturbations around it. (iii) The solutions are unstable beyond the first maximum which is at $(x_{ch}, y_{ch}) = (x_1, y_1)$.

As a consequence, one has the following. For a given value of central density, the radius $r_\ast$ must be $< r_1$ where $r_1$ corresponds to $x_1$. The mass of the star must then be less than $(y_1 r_{1\ast}^m)$. A more massive star will be unstable and will collapse.

(4) In [11], Chavanis studied static spherically symmetric equilibrium configurations of stars in $D = m + 2$ dimensional spacetime, $n_c = 0$ and $\mathcal{N} = 1$ in our notation. He found that, for $m \geq m_{cr}(w) \sim 9$, $y_{ch}$ increases monotonously from zero to $y_s$, effectively making $x_1$ infinite and $y_1 = y_s$, see Figures 20 – 23 in [11]. The perturbations around the corresponding analytical asymptotic solutions exhibit a monotonous behaviour with no oscillations.

As a consequence, one has the following. For $m \geq 9$, $x_1$ is effectively infinite. Then $r_{1\ast}$ is infinite which makes the upper limit $(y_s r_{1\ast}^m)$ on the mass of the star also infinite. Hence, a star can be arbitrarily massive and stable when $m \geq 9$.

**Example (1):** $\tilde{c}^I = 0 \implies s^I = 0$

We first consider an example where the corresponding stars are effectively same as those in [11]. Consider the case where $w^I = \frac{(m+1)w^I - 1}{m}$. Then $\tilde{c}^I = 0$ from which it follows that $s^I = 0 = B_0 = B_1$, so we omit the tilde’s on $s$’s. We then get $s^0 = \frac{2w}{1+w}$ which implies that $w^I = w$ for all $I$ [17]. Then

$$\alpha = m - 1 + \frac{2w}{1+w}, \quad c^0 = c^0 = \frac{m - 1 + (m + 1)w}{m}.$$ 

Note that $(1 + w)\alpha = m\tilde{c}^0$. The zeroth order equations (44) – (46) give

$$\alpha s^0 = c^0 \sum I R_I \implies R = \sum I R_I = \frac{2mw}{(1 + w)^2}$$

$$s^0 = \frac{w}{m} R + \frac{m - 1}{2} \left(e^{2\lambda_0} - 1\right) \implies e^{2\lambda_0} = \frac{D}{(m - 1)(1 + w)^2}$$
where \( D = (m - 1)(1 + w)^2 + 4w \). The remaining relation
\[
s^0 = \frac{w - 1}{m} R + (m - 1) \left( e^{2\lambda_0} - 1 \right)
\]
is now satisfied identically.

At first order, since \( \tilde{c}^i I = 0 \) and \( w^I = w \), we have
\[
w y^I = - (1 + w) \tilde{u}^0, \quad \tilde{u}^i + \alpha \tilde{u}_\sigma^i = 0 .
\]
Thus, \( y^I = y \) is independent of \( I \) and \( \tilde{u}^i \). Equations (48), (50), and (51) now give
\[
\tilde{u}^0 + \alpha \tilde{u}^0 + s^0 (\tilde{u}^0 - \tilde{u}_\sigma^0) = \tilde{c}^0 R (y + 2\tilde{u}) ,
\]
\[
\tilde{u}^0 = \frac{w}{m} R (y + 2\tilde{u}) + (m - 1) e^{2\lambda_0} \tilde{u}
\]
\[
\tilde{u}^0 - \tilde{u}_\sigma^0 = \frac{w - 1}{m} R (y + 2\tilde{u}) + (m - 1) e^{2\lambda_0} (2\tilde{u}) ,
\]
from which it follows that
\[
\tilde{u}^0 = - \frac{2w}{1 + w} \tilde{u}^0 + \left( m - 1 + \frac{4w}{1 + w} \right) \tilde{u}
\]
\[
\tilde{u}_\sigma^0 = - \frac{2}{1 + w} \tilde{u}^0 - (m - 1) \tilde{u}
\]
\[
\tilde{u}_\sigma^0 = w \tilde{u}_\sigma + \frac{D}{1 + w} \tilde{u} .
\]

The \( \tilde{u}^0_\sigma \) and \( \tilde{u}_\sigma \) equations above are of the type
\[
A x^I + B y^I = ax + by , \quad P x^I + Q y^I = px + qy \tag{53}
\]
in an obvious notation. Solving, for example, for \( (x', x) \) in terms of \( (y', y) \) gives \( x' = a_1 y' + a_2 y \) and \( x = a_3 y' + a_4 y \); one then gets \( a_3 y'' + a_4 y' = x' = a_1 y' + a_2 y \) which gives an equation for \( y'' \). An equation for \( x'' \) also follows similarly. One can show in this way that both \( x'' \) and \( y'' \) obey the same equation, namely
\[
(PB - QA) \left( \ast \right)'' - (Pb + pB - QA - qA) \left( \ast \right)' + (pb - qa) \left( \ast \right) = 0 \tag{54}
\]

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where \((*) = x, y\). Applying this formula, it follows straightforwardly that \(\tilde{u}^0\) and \(\tilde{u}\) obey the same equation, namely

\[
\tilde{u}^0_{\tilde{\sigma}} + \alpha \tilde{u}^0_{\tilde{\sigma}} + \frac{2D}{(1+w)^2} \tilde{u}^0 = 0
\]

\[
\tilde{u}_{\tilde{\sigma}} + \alpha \tilde{u}_{\tilde{\sigma}} + \frac{2D}{(1+w)^2} \tilde{u} = 0 .
\]

Except for the presence of \(\tilde{u}\)'s obeying the equation \(\tilde{u}_{\tilde{\sigma}} + \alpha \tilde{u}_{\tilde{\sigma}} = 0\), the results in this example are same as the corresponding ones of [11] for which \(n_c = 0\). This is because the choice of \(w_i\), leading to \(s^i = 0\), decouples the effects of compact space.

**Example (2) : \(\tilde{c}^{i I} \neq 0\), \(N = 1\)**

Consider an example where \(\tilde{c}^{i I} \neq 0\) but \(N = 1\). Using \(s = 0\), \(q' = -2\), and omitting the \(I\)-scripts, the zeroth order equations (41), (44) – (46) give

\[
2w = (1+w) s^0 + \sum_i \tilde{c}^i s^i
\]

\[
\alpha s^0 = \tilde{s}^0 R , \quad \alpha s^i = \tilde{s}^i R
\]

\[
\tilde{s}^0 = \frac{w}{m} R + \frac{m-1}{2} \left( e^{2\tilde{\lambda}_0} - 1 \right) + \frac{B_0}{2m}
\]

\[
\tilde{s}^0 = \frac{w-1}{m} R + (m-1) \left( e^{2\tilde{\lambda}_0} - 1 \right)
\]

\[
\Rightarrow \quad \alpha = m - 1 + \tilde{s}^0 = \frac{w-1}{m} R + (m-1) e^{2\tilde{\lambda}_0}
\]

where \(B_0 = \sum_i \tilde{s}^i s^i\). We now get

\[
\left( \tilde{s}^i, s^i, R \right) = \left( \tilde{c}^i, c^i, \alpha \right) \frac{\tilde{s}^0}{\tilde{\sigma}^0},
\]

using which we obtain

\[
s^0 = \frac{2w}{(1+w)(1+\gamma)} , \quad \gamma = \frac{\sum_i \tilde{c}^i c^i}{(1+w)\tilde{\sigma}^0}.
\]
Then $\tilde{s}^i$ and $\alpha = m - 1 + \tilde{s}^0$ follow, then $R$, then $e^{2\lambda_0}$ from, for example, the last of the zeroth order equations:

$$(m - 1) e^{2\lambda_0} = \alpha + \frac{1 - w}{m} R = \frac{\alpha}{1 + \gamma} \left( \frac{\mathcal{D}}{mc^0(1 + w)} + \gamma \right).$$

Also, $\mathcal{B}_0 = \sum_i \tilde{s}^i s^i = \gamma(1 + w) \frac{(s^0)^2}{2\tilde{\sigma}}$. All the results of Example (1) will follow from the present ones upon setting $\gamma = 0$.

The first order equations (42), (48) – (51) give

$$w y + (1 + w) \tilde{u}^0 + \sum_i \tilde{c}^i u^i = 0$$

$$\tilde{u}^0_{\tilde{\sigma}} + \alpha \tilde{u}^0_{\tilde{\sigma}} + s^0 (\tilde{u}^0_{\tilde{\sigma}} - \tilde{u}^0_{\tilde{\sigma}}) = \tilde{c}^0 R (y + 2\tilde{u})$$

$$\tilde{u}^i_{\tilde{\sigma}} + \alpha \tilde{u}^i_{\tilde{\sigma}} + s^0 (\tilde{u}^0_{\tilde{\sigma}} - \tilde{u}^0_{\tilde{\sigma}}) = \tilde{c}^i R (y + 2\tilde{u})$$

$$\tilde{u}^0_{\tilde{\sigma}} = \frac{w}{m} R (y + 2\tilde{u}) + (m - 1) e^{2\lambda_0} \tilde{u} + \frac{\mathcal{B}_1}{m}$$

$$\tilde{u}^0_{\tilde{\sigma}} - \tilde{u}_{\tilde{\sigma}} = \frac{w - 1}{m} R (y + 2\tilde{u}) + (m - 1) e^{2\lambda_0} (2\tilde{u})$$

where $\mathcal{B}_1 = \sum_i s^i u^i$.

It can now be seen from the $\tilde{u}^0_{\tilde{\sigma}}$ and $\tilde{u}^i_{\tilde{\sigma}}$ equations, and from the zeroth order results for $\tilde{s}^i$, that

$$F^i_{\tilde{\sigma}} + \alpha F^i_{\tilde{\sigma}} = 0 \quad F^i = \tilde{u}^i - \tilde{c}^i \tilde{u}^0_{\tilde{\sigma}}.$$

Although $F^i$ admit the general solutions $F^i = F^i_0 + F^i_1 e^{-\alpha \tilde{\sigma}}$, we will set the integration constants $F^i_{0,1}$ to zero and thus take $\tilde{u}^i = \tilde{c}^i \tilde{u}^0_{\tilde{\sigma}}$. It is now straightforward to obtain $y$ and $\mathcal{B}_1$. They are given by

$$\tilde{s}^0 y = -2 \tilde{u}^0 \quad \mathcal{B}_1 = \gamma(1 + w) \tilde{s}^0 \tilde{u}^0_{\tilde{\sigma}}.$$

Using the above expression for $y$, and after some algebra, we now get

$$\tilde{u}^0_{\tilde{\sigma}} + \alpha \tilde{u}^0_{\tilde{\sigma}} + \frac{2 \alpha}{1 + \gamma} \left( \frac{\mathcal{D}}{mc^0(1 + w)} + \gamma \right) \tilde{u}^0 = 0$$

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\[
\left(1 - \frac{\gamma(1 + w)s^0}{mc_0^2}\right) \ddot{\tilde{u}}_0 = -\frac{2\alpha w}{mc_0^2} \tilde{u}^0 + \alpha \left(1 + \frac{(1 + w)s^0}{mc_0^2}\right) \ddot{u}
\]
\[
\ddot{\tilde{u}}_0 - \ddot{\tilde{u}} = \frac{2\alpha(1 - w)}{mc_0^2} \tilde{u}^0 + 2\alpha \ddot{u}.
\]

The \(\ddot{u}_0\) and \(\ddot{u}\) equations above are of the type given in equation (53). Hence, applying the consequent formula (54) and after some algebra, one can show that both \(\ddot{u}_0\) and \(\ddot{u}\) obey the same equation, given above for \(\ddot{u}_0\).

The results of this example provide a generalisation of the corresponding ones of [11]. In this example, \(N = 1\) and the effects of compact space appear through a single parameter \(\gamma\).

**Analysis of perturbation equation**

Consider the equation for the perturbations

\[
(\ast)'' + \alpha (\ast)' + 2\alpha \left(1 + \frac{(1 - w)s^0}{mc_0^2}\right) (\ast) = 0
\]

where \((\ast) = \ddot{u}^0\) or \(\ddot{u}\), and we have used

\[
\frac{\alpha}{1 + \gamma} \left(\frac{D}{mc_0^2(1 + w)} + \gamma\right) = \alpha + \frac{1 - w}{m} R, \quad R = \frac{\alpha s^0}{c_0}.
\]

Then the solutions are given by \((\ast) \sim e^{k}\tilde{\sigma}\) where

\[
k = \frac{-\alpha \pm \sqrt{\Delta}}{2}, \quad \Delta = \alpha^2 - 8\alpha \left(1 + \frac{(1 - w)s^0}{mc_0^2}\right).
\]

If \(\Delta\) is negative then \(k\) is complex and the solutions exhibit damped oscillations. Such oscillations imply an instability towards a collapse of a sufficiently massive star. If \(\Delta\) is zero or positive then \(k\) is real and the solutions are exponentially damped but with no oscillations. Such non oscillating, monotonous behaviour implies stability [11]. The fluctuations of \(\ddot{u}\) are related to the fluctuations of the mass function \(M(r)\) and, hence, a negative \(\Delta\) denotes an instability towards a collapse of a sufficiently massive star whereas a zero or a positive \(\Delta\) denotes stability.
In order to analyse the sign of $\Delta$, we write it as

$$\Delta = \kappa_1 \left\{ \alpha mc^0 - 8 (mc^0 + (1-w)s^0) \right\}$$

$$= \kappa_1 \left\{ (m-9) mc^0 + s^0 (mc^0 - 8(1-w)) \right\}$$

$$= \kappa_2 \left\{ (m-9)(1+w)(1+\gamma) mc^0 + 2w (mc^0 - 8(1-w)) \right\}$$

$$= \kappa_2 \left\{ b_2 w^2 + b_1 w + b_0 \right\}$$

where $\kappa_1 = \frac{\alpha}{mc^0}$, $\kappa_2 = \frac{s^0}{(1+w)(1+\gamma)}$, and

$$b_2 = (m-3)^2 + \gamma(m+1)(m-9)$$

$$b_1 = 2(m-9)(m+1+\gamma m)$$

$$b_0 = (m-1)(m-9)(1+\gamma) .$$

We take $c_i$ to not all vanish, then the sum $\sum_i \tilde{c}_i c_i > 0$ . Assume that $w > \frac{1}{1+m}$ . It is, indeed, physically natural that $0 \leq w \leq 1$ . It then follows that $(1+w, \tilde{c}_0, \gamma, \tilde{s}_0, \alpha, \kappa_1, \kappa_2)$ are all positive and, hence,

$$Sgn \Delta = Sgn (b_2 w^2 + b_1 w + b_0) .$$

We now analyse the sign of $\Delta$ . Write $b_2 w^2 + b_1 w + b_0 = Q_0 + Q_1$ where

$$Q_0 = (m-3)^2 w^2 + 2(m-9)(m+1)w + (m-1)(m-9)$$

and

$$Q_1 = \gamma(m-9)(1+w) mc^0 = m(m-9) \sum_i \tilde{c}_i c_i .$$

Note that the polynomial $Q_0$ is same as that analysed in [11]; that the discriminant of $Q_0$ is $128 m (9-m)$ ; that $\sum_i \tilde{c}_i c_i > 0$ ; and that the polynomial $Q_1$ may be thought of as arising due to the effects of compact space. Now consider $Q_0 + Q_1$ for different ranges of $m$ .

$m > 9$ : In this case, $Q_1$ is positive. The discriminant of $Q_0$ is negative, hence $Q_0$ is positive for all $w$ . Therefore, $\Delta$ is positive.

$m = 9$ : In this case, $Q_0 = 36 w^2$ and $Q_1 = 0$ . Hence, $\Delta$ is zero or positive.
m < 9: In this case, $Q_1 < 0$. Consider $Q_0$. It is simple to show \cite{11} that $Q_0$ is negative for $0 \leq w \leq 1$. Hence, $\Delta$ is negative for $0 \leq w \leq 1$.

Thus, it follows that the perturbations exhibit monotonous and non-oscillating behaviour for $m \geq 9$ for any value of $w$. Hence, a star can be arbitrarily massive and stable in these cases. For $m < 9$, and for the physically relevant range $0 \leq w \leq 1$, the perturbations exhibit damped oscillations. Therefore there will be instability and, hence, sufficiently massive stars will collapse.

5. M theory branes: U duality relation among $(p_{\alpha I}, \Pi_I)$

M theory has U duality symmetries. Incorporating these symmetries leads to a relation among the components $(p_{\alpha I}, \Pi_I)$ of the energy momentum tensor for the $I^{th}$ stack of branes. We will explain this relation below. Then, we apply the general results of sections 3 and 4 to M theory branes.

We use the U duality symmetries here the same way as in our earlier works \cite{7}. We note that, for a given $\mathcal{N}$, different intersecting brane configurations of M theory can be related to each other by suitable U duality operations – namely, by suitable dimensional reduction and uplifting to and from type IIA string theory, and T and S dualities in type IIA/B string theories. For a metric of the form given in equation (3), such an operation leads to relations among $\lambda^a$ which in turn, through their equations of motion, imply relations among $(p_{\alpha}, \Pi)$. Although only time dependent cases in early universe were studied in \cite{7}, the U duality details of these works carry over and are applicable to the static cases also with only a few minor changes. We present only the main results here, see \cite{7} for details.

It can be shown that the relations among $(p_{\alpha}, \Pi)$ for intersecting branes of M theory, obtained by applying U duality operations as described above, are all satisfied if the individual $(p_{\alpha I}, \Pi_I)$ obey the relation

$$p_{\parallel I} = \Pi_I + p_0 I + p_{\perp I} + m \left(p_{\Omega I} - p_{\perp I}\right) \quad (55)$$

where $p_{\parallel I}$ and $p_{\perp I}$ are the pressures along the directions that are parallel and transverse to the worldvolume of the $I^{th}$ stack of branes. The above relation is a consequence of U duality symmetries and, therefore, must always be valid independent of the details of the equations of state. We further take $p_{\Omega I} = p_{\perp I}$ which is natural since the sphere directions are transverse to the
branes. Furthermore, in the case of early universe studied previously as well as in the case of stars that are being studied here, the constituent matter components satisfy $T_{r(l)}^r = T_{a(l)}^a$, thus $\Pi_l = p_{aI}$. Using $p_{0l} = -\rho_l$ and $p_{aI} = p_{aI} = p_{lI} = p_l$, equation (55) then becomes $p_{lI} = -\rho_l + 2p_l$ [6, 7].

Consider the case where the equations of state are linear, as given in equations (32). For M theory branes, let $p_{lI} = \rho_l$ and $p_{lI} = \rho_l$. Taking $w_{lI} = \rho_{lI} = \rho_I$, and then $w_{lI} = \rho_I$ for stars, the U duality relation in equation (55) then gives

$$w_{lI} = w_{lI} - 1 + w = -1 + 2w .$$

For intersecting branes in M theory, these relations further lead to an elegant structure which is shown in Appendix C.

Consider the constants $\tilde{c}^{lI}$ defined in equation (34). For intersecting branes in M theory, $w'_{lI} = w'_{lI}$ if $i \in ||I$, namely if $x^i$ is a worldvolume coordinate of the $I^{th}$ stack of branes, and $w'_{lI} = w'_{lI}$ otherwise. Similarly, let $\tilde{c}^{lI} = \tilde{c}^{lI}$ if $i \in ||I$ and $\tilde{c}^{lI} = \tilde{c}^{lI}$ otherwise. Then, using the U duality relation $w'_{lI} = -1 + 2w'$, it follows that

$$\tilde{c}^{lI} = \frac{w' - 1}{m} , \quad \tilde{c}^{lI} = \tilde{c}^{lI} + 1 - w = (1 - m) \tilde{c}^{lI} . \quad (56)$$

Hence, for any $a^i$ with $a^c = \sum_i a^i$, it follows that

$$\sum_i \tilde{c}^{lI} a^i = (1 - w) \left( \sum_i a^i - \frac{a^c}{m} \right) . \quad (57)$$

With $w'_{aI}$, and thus $\tilde{c}^{aI}$, specified for intersecting branes, it is now a straightforward exercise to apply the general results of sections 3 and 4 to stars made up of intersecting branes in M theory. In the following, we present two examples: the $N = 1$ case of $M2$ and $M5$ brane stars, and the $N = 4$ case of $22'55'$ intersecting brane configuration.

Other intersecting brane configurations can be analysed by same method as shown for the $N = 4$ case. The results for the $N = 3$ case of $22'2''$ intersecting brane configuration are almost identical to those of the $N = 4$ case.

**Example (3) : $N = 1$ case – M2 and M5 branes**

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Consider stars made up of M2 branes or M5 branes. Then \( \mathcal{N} = 1 \), \( D = 11 \), \( n_c = 2 \) or 5, \( m = 7 \) or 4, and \( \bar{c}^i = \bar{c}^\parallel = \frac{(m-1)(1-w)}{m} \). Hence,

\[
\sum_i \bar{c}^i c^i = \frac{m n_c (\bar{c}^\parallel)^2}{n_c + m} = \frac{n_c (m-1)^2 (1-w)^2}{m (n_c + m)},
\]

which \( = \frac{8}{7} (1-w)^2 \) for M2 branes and \( = \frac{5}{4} (1-w)^2 \) for M5 branes.

This is a special case of Example (2) described in section 4. Therefore, the analysis and the results of Example (2) apply straightforwardly. Since \( m < 9 \), the perturbations exhibit damped oscillations for \( 0 \leq w \leq 1 \). Therefore there will be instability and, hence, sufficiently massive M2 or M5 brane stars will collapse.

**Example (4) : \( \mathcal{N} = 4 \) case – 22'55' configuration**

For the BPS intersecting brane configurations of the M theory, we have

\[ n_c + m = 9, \quad w^I_\perp = w^I_\parallel = w^I, \quad w^I_\parallel = -1 + 2w^I. \]

Also \( \bar{c}^I = \bar{c}^\parallel I \) if \( i \in \parallel \) and \( \bar{c}^I = \bar{c}^\perp I \) otherwise. From the relation between \( w^I_\parallel \) and \( w^I_\perp \), it follows that

\[
\bar{c}^\perp I = \frac{w^I - 1}{m}, \quad \bar{c}^\parallel I = 1 - w^I + \bar{c}^\perp I = (1 - m) \bar{c}^\perp I.
\]

In the 22'55' : (12, 34, 13567, 24567) configuration of the M theory, there are two stacks each of two branes and five branes whose spatial worldvolume directions are indicated. For this configuration, we have

\[ \mathcal{N} = 4, \quad n_c = 7, \quad m = 2, \quad \bar{c}^\parallel I = - \bar{c}^\perp I = \frac{1 - w^I}{2}. \]

Using \( \bar{c}^\parallel I = 1 - w^I + \bar{c}^\perp I \) and then \( \bar{c}^\perp I = \frac{w^I - 1}{m} \), we have

\[
\sum_i \bar{c}^I s^i = (1 - w^I) \sum_{i \in \parallel} s^i + \bar{c}^\perp I S^c = (1 - w^I) \left( \sum_{i \in \parallel} s^i - \frac{S^c}{m} \right),
\]

\[
\sum_i \bar{c}^I u^i = (1 - w^I) \sum_{i \in \parallel} u^i + \bar{c}^\perp I U^c = (1 - w^I) \left( \sum_{i \in \parallel} u^i - \frac{U^c}{m} \right).
\]
Consider sums of the type $\sum \tilde{c}^I X_I$. Denoting the $X_I$'s for the 22'55' configuration as $(X_2, X_2', X_5, X_5')$ and using $\tilde{c}^\parallel = - \tilde{c}^\perp$, we get

$$
\begin{align*}
\sum_I \tilde{c}^1 X_I &= \tilde{c}^\parallel (X_2 - X_2' + X_5 - X_5') \\
\sum_I \tilde{c}^2 X_I &= \tilde{c}^\parallel (X_2 - X_2' - X_5 + X_5') \\
\sum_I \tilde{c}^3 X_I &= \tilde{c}^\parallel (-X_2 + X_2' + X_5 - X_5') \\
\sum_I \tilde{c}^4 X_I &= \tilde{c}^\parallel (-X_2 + X_2' - X_5 + X_5') \\
\sum_I \tilde{c}^{5,6,7} X_I &= \tilde{c}^\parallel (-X_2 - X_2' + X_5 + X_5') .
\end{align*}
$$

(58)

**Zeroth order:** At zeroth order, we have $\tilde{s} = 0$, $q^I = -2$ and, using the above expression for $\sum_i \tilde{c}^I s^i$, we get from equation (41) that

$$
\sum_{i \in \parallel} s^i = \frac{2w^I - (1 + w^I) \tilde{s}^0}{1 - w^I} + \frac{S^c}{m} .
$$

We now consider the case where $w^I = w$ for all $I$, and omit the $I$ superscripts on $\tilde{c}^0I$ and $w$’s. Note that $\tilde{c}^I$ will depend on $I$ since $\tilde{c}^I = \tilde{c}^\parallel I$ if $i \in \parallel$ and $\tilde{c}^I = \tilde{c}^\perp I$. Since $w^I = w$ for all $I$, the above expression implies that the sum $\sum_{i \in \parallel} s^i$ must be same for all $I$. Thus, for the 22'55' : (12, 34, 13567, 24567) configuration, it follows that

$$
s^1 + s^2 = s^3 + s^4 = s^1 + s^3 + s^5 + s^6 + s^7 = s^2 + s^4 + s^5 + s^6 + s^7
$$

$$
\implies s^3 = s^2 , \quad s^4 = s^1 , \quad s^5 + s^6 + s^7 = 0 , \quad S^c = 2 (s^1 + s^2) .
$$

It can then be shown that $\tilde{s}^i = s^i + \frac{S^c}{m}$ satisfy the relations

$$
\tilde{s}^3 = \tilde{s}^2 , \quad \tilde{s}^4 = \tilde{s}^1 , \quad \tilde{s}^5 + \tilde{s}^6 + \tilde{s}^7 = \tilde{s}^1 + \tilde{s}^2 .
$$

Consider now equation (44) for $\tilde{s}^i : \alpha \tilde{s}^i = \sum_I \tilde{c}^I R_I$. Denoting the $R_I$’s for the 22'55' configuration as $(R_2, R_2', R_5, R_5')$ and using $\tilde{c}^\parallel = - \tilde{c}^\perp$, we get

$$
\alpha \tilde{s}^1 = \sum_I \tilde{c}^I R_I = \tilde{c}^\parallel (R_2 - R_2' + R_5 - R_5')
$$

23
\[
\alpha \tilde{s}^2 = \sum_I \tilde{c}^2 I R_I = c^\parallel (R_2 - R_2' - R_5 + R_5') \\
\alpha \tilde{s}^3 = \sum_I \tilde{c}^3 I R_I = c^\parallel (-R_2 + R_2' + R_5 - R_5') \\
\alpha \tilde{s}^4 = \sum_I \tilde{c}^4 I R_I = c^\parallel (-R_2 + R_2' - R_5 + R_5') \\
\alpha \tilde{s}^{5,6,7} = \sum_I \tilde{c}^{5,6,7} I R_I = c^\parallel (-R_2 - R_2' + R_5 + R_5') .
\]

The three relations on the \(\tilde{s}^i\) can now be seen to imply that

\[
R_2 = R_2' = R_5 = R_5' = \frac{R}{4}, \quad \sum_I R_I ,
\]

\[
\Longrightarrow \quad \tilde{s}^i = 0 \quad \Longrightarrow \quad s^i = S^c = B_0 = B_1 = 0.
\]

Since \(S^c = 0\), we omit the tilde’s on \(s^i\)’s. Using equations (44) – (46), we then get the same zeroth order results as in Example (1), namely

\[
s^0 = \frac{2w}{1 + w}, \quad c^0 = \frac{m - 1 + (m + 1)w}{m},
\]

\[
R = \sum_I R_I = \frac{2mw}{(1 + w)^2}, \quad (m - 1) e^{2\lambda w} = \frac{D}{(1 + w)^2}
\]

where \(D = (m - 1)(1 + w)^2 + 4w\) and \(m = 2\).

**First order**: Consider now the equations at the first order in the \(\tilde{u}^i\)’s for the case where \(w^I = w\) for all \(I\). Then \((w^I, c^0 I, R_I)\) do not depend on \(I\) and we omit the \(I\)–scripts on them. Noting that \(\tilde{s}^i = B_1 = 0\) and \(R_I = R\) now, the first order equations (48) – (51) for \(y^I\) and the \(\tilde{u}^i\)’s are given by

\[
w y^I = -(1 + w) \tilde{u}^0 - (1 - w) \left( \sum_{i \in I} u^i - \frac{U^c}{m} \right) \quad (59)
\]

\[
\tilde{u}_\tilde{\sigma}^0 + \alpha \tilde{u}_\tilde{\sigma}^0 + \tilde{s}^0 (\tilde{u}_\tilde{\sigma}^0 - \tilde{u}_\tilde{\sigma}) = \sum_I \tilde{c}^0 \frac{R}{4} (y^I + 2\tilde{u}) \quad (60)
\]

\[
\tilde{u}_\tilde{\sigma}^i + \alpha \tilde{u}_\tilde{\sigma}^i = \sum_I \tilde{c}^i \frac{R}{4} (y^I + 2\tilde{u}) \quad (61)
\]
\[
\tilde{u}_\sigma^0 = \sum_I \frac{w}{m} \frac{R}{4} (y^I + 2\tilde{u}) + (m - 1) e^{2\lambda_0} \tilde{u}
\]  \hspace{1cm} (62)

\[
\tilde{u}_\sigma^0 - \tilde{u}_\sigma = \sum_I \left( \frac{w - 1}{2} \right) \frac{R}{4} (y^I + 2\tilde{u}) + (m - 1) e^{2\lambda_0} (2\tilde{u}) .
\]  \hspace{1cm} (63)

Note that \( y^I \) depends on \( u^i \), and also on \( I \) through the sum \( \sum_{i\in I} u^i \). However, with \( w^I = w \) for all \( I \), the equations for \( \tilde{u}_0 \) and \( \tilde{u} \) depend not on the individual \( y^I \) but only on their sum \( \sum_I y^I \). Now, for the \( N = 4, 22'55' : (12, 34, 13567, 24567) \) configuration, we have \( m = 2 \) and

\[
\sum_I \left( \sum_{j\in I} u^i - \frac{U^c}{m} \right) = 2 \frac{U^c - N \frac{U^c}{m}}{m} = 0 .
\]

Therefore, for the \( 22'55' \) configuration, the sum \( \sum_I y^I \) is independent of \( u^i \). Then

\[
\frac{R}{4} \sum_I y^I = -\frac{1+w}{w} R \tilde{u}^0
\]

and the equations for \( \tilde{u}_0 \) and \( \tilde{u} \) become the same as those in Example (1). In particular, it follows that the perturbations exhibit damped oscillations and, hence, there is instability. Therefore, sufficiently massive stars made up of \( 22'55' \) intersecting branes will collapse.

Equation for \( \tilde{u}^i \) depend on \( I \) through the term \( \sum_I \tilde{c}^I (y^I + 2\tilde{u}) \), and hence through the sums \( \sum_I \tilde{c}^I \) and \( \sum_I \tilde{c}^I y^I \). Such sums for the \( 22'55' \) configuration are given by the expressions given in equation (58), from which it can be seen that

\[
\sum_I \tilde{c}^I = 0 \quad \Rightarrow \quad \sum_I \tilde{c}^I y^I = \sum_I \tilde{c}^I \left( \sum_{j\in I} u^j \right) .
\]

For \( I : (2, 2', 5, 5') \), the sums \( \sum_{j\in I} u^j \) are given by \( (u^1 + u^2) \), \( (u^3 + u^4) \), \( (u^1 + u^3 + u^5 + u^6 + u^7) \), and \( (u^2 + u^4 + u^5 + u^6 + u^7) \). We then get

\[
\tilde{u}_{\sigma\sigma}^i + \alpha \tilde{u}_\sigma^i = \frac{R}{4} \sum_I \tilde{c}^I y^I
\]

where

\[
\sum_I \tilde{c}^I y^I = 2 \tilde{c}^I (u^1 - u^4)
\]

25
\[
\begin{align*}
\sum_I \tilde{c}^{2I} y^I &= 2 \parallel (u^2 - u^3) \\
\sum_I \tilde{c}^{3I} y^I &= 2 \parallel (u^3 - u^2) \\
\sum_I \tilde{c}^{4I} y^I &= 2 \parallel (u^4 - u^1) \\
\sum_I \tilde{c}^{5,6,7} y^I &= 2 \parallel (u^5 + u^6 + u^7) .
\end{align*}
\]

(64)

It then follows easily that

\[(*_{1})_{\tilde{\sigma} \tilde{\sigma}} + \alpha(*_{1})_{\tilde{\sigma}} - R \parallel (*_{1}) = 0\]

where \((*_{1}) = (\tilde{u}^1 - \tilde{u}^4), (\tilde{u}^2 - \tilde{u}^3), \text{ and } (u^5 + u^6 + u^7) ; \text{ and }\]

\[(*_{2})_{\tilde{\sigma} \tilde{\sigma}} + \alpha(*_{2})_{\tilde{\sigma}} = 0\]

where \((*_{2}) = (\tilde{u}^1 + \tilde{u}^4), (\tilde{u}^2 + \tilde{u}^3), (\tilde{u}^5 + \tilde{u}^6 - 2\tilde{u}^7), \text{ and } (\tilde{u}^5 - 2\tilde{u}^6 + \tilde{u}^7) .\]

\[\mathcal{N} = 3 \text{ case – } 22'2'' \text{ configuration}\]

We now consider the \(22'2'' : (12, 34, 56)\) configuration of the M theory where there are three stacks of two branes whose spatial worldvolume directions are indicated. For this configuration, we have \(\mathcal{N} = 3, n_c = 6, \text{ and } m = 3 . \) Further analysis proceeds in the same way as for the \(\mathcal{N} = 4 \) case. The steps differ in details, but the final results for the \(\mathcal{N} = 3 \) case of \(22'2''\) intersecting brane configuration are almost identical to those of the \(\mathcal{N} = 4 \) case, but now with \(m = 3 . \)

Thus: \(s^i = 0; \) the same zeroth order results as in Example (1) are obtained; and, the equations for \(\tilde{u}^0 \) and \(\tilde{u} \) become the same as those in Example (1). In particular, it follows that the perturbations exhibit damped oscillations and, hence, there is instability. Therefore, sufficiently massive stars made up of \(22'2''\) intersecting branes will collapse.

6. Significance of (in)stabilities – Mathur’s fuzzball proposal

We now discuss the significance of the (in)stability results for the stars in M theory, within the context of Mathur’s fuzzball proposal [15]. We draw a
corollary of this proposal, which is perhaps obvious to many readers; discuss its implication for the equations considered here; and make a few comments.

We now have that stars in M theory made up of $M_2$ or $M_5$ branes, or the $\mathcal{N} = 4$  22'55' intersecting branes, or the $\mathcal{N} = 3$  22'2' intersecting branes all have instabilities. Therefore, if sufficiently massive, such stars in M theory will collapse presumably to form black brane configurations. The results of [11] that stable, arbitrarily massive, equilibrium configurations of stars may exist if $m \geq 9$ are not applicable to the stars in M theory made up of branes and intersecting branes because $D = 11$ and $n_c + m = 9$ in M theory, and $n_c > 1$ for branes and intersecting branes.

From many points of view, such a result is only to be expected. However, consider Mathur’s fuzz ball proposal [15]. According to this proposal, in essence, there is no horizon. $^1$ At distances greater than of the order of Schwarzschild radius, the spacetime geometry is indistinguishable from a black hole / black brane geometry but, at shorter distances, the spacetime is filled with ‘fuzz’.

It then follows as a corollary of the fuzz ball proposal that

- The equilibrium configurations of stars of arbitrarily high mass must exist which must be stable and not collapse.

This must be true irrespective of whether the stars are neutral or charged, rotating or non rotating. In the following, we proceed with the assumption that the fuzz ball proposal and, consequently, its corollary above are valid.

We now make a remark regarding fuzz ball configurations. A generic fuzz ball state is highly quantum and has sizeable fluctuations of metric and other fields. Its connection to classical geometry is very much more involved than the standard quantum to classical correspondence. Here, we envision the fuzzball configurations as describing intersecting branes with all the high energy, highly interacting excitations included. Our analysis requires only the energy momentum tensor and the equations of state describing such configurations. They may be derived in certain approximations as in [6] or by using symmetries as in [7]. Or they may be postulated based on underlying microphysics if known, or based simply on educated guesswork.

$^1$Several horizonless pictures have been advocated from several different points of view over time. For a sample of such works, see [18] – [23].
The fuzz ball situation here may be likened to studying stars made up of photons, or nuclear matter, or Bose–Einstein condensates. These constituents are highly quantum. Nevertheless, to study the properties of such stars, the equations of state for the constituents are sufficient. For photons, one has $(\text{pressure}) = (\text{density})/3$; for nuclear matter, one postulates different equations of states; and for Bose–Einstein condensates, they may be derived from statistical mechanics.

It is important to find the equilibrium horizonless configurations implied by the fuzz ball proposal, and understand the reasons for their stability. One may therefore seek stationary configurations since equilibrium ones are in general of this type. If one further assumes no rotation then one may restrict to static and spherically symmetric configuration. Then one may proceed, for example, by following the methods of [8, 9, 10, 11] as done in this paper: First obtain OV type equations corresponding to the equilibrium configurations of the ‘fuzz’; derive or postulate appropriate equations of state for the ‘fuzz’, which may at least capture its stabilising aspects; then obtain the asymptotic solutions to these equations; study the behaviour of perturbations in the mass of the star around the asymptotic solutions; and, determine whether an instability is present or not depending on whether perturbations are oscillatory or not.

The OV type equations, and the equations of state, given in this paper for stars in M theory predict instabilities. Therefore, the assumed validity of the corollary above implies that although these equations may describe well the stars in M theory, they are not sufficiently general to describe the ‘fuzz’ which must be able to prevent the collapse. We suspect that, at the least, the isometries along the compact space needs to be removed or modified. Note that the absence of such isometries invalidates our use of U duality symmetries in [7] to obtain their effects on the equations of state. One then has to postulate, or derive by other means, equations of state for the ‘fuzz’ which must be able to prevent the collapse. This, however, is a tall order and it is presently not clear to us how to proceed.

7. Conclusions

We summarise briefly the results presented in this paper. Our aim is to study the stars in M theory and determine whether they are stable or not.
We start with appropriate higher dimensional spacetime, applicable to stars in M theory. We obtain the analog of OV equations. The equations of state are taken to be linear. We then follow the standard methods to determine the presence or absence of instabilities: we obtain analytically the asymptotic solutions and study perturbations around it.

A general answer is algebraically quite involved. Hence, we consider a few examples. In the process, we obtain a generalisation of the results of Chavanis given in [11]. We also obtained the results for stars in M theory made up of $M_2$ and $M_5$ branes; and the $\mathcal{N} = 4$ $22'55'$ intersecting branes. The results for $\mathcal{N} = 3$ $22'2''$ intersecting branes are similar to the $\mathcal{N} = 4$ case. All these stars in M theory have instabilities. Therefore, if sufficiently massive, they will collapse to form black brane configurations. We then discussed the significance of these (in)stabilities within the context of Mathur’s fuzz ball proposal.

We now conclude by mentioning a few topics for further studies. With the OV type equations given here, one can study stars in M theory, and also stars in more general Kaluza–Klein spacetimes. The equations of state for the star’s constituents in the later spacetimes may need to be postulated. However, one can use this freedom and explore the consequences of various types of equations of state.

One can also study the collapse of such stars, and obtain the analog of Oppenheimer–Snyder collapse scenario. This is likely to be an interesting area of research because of its similarities to the time dependent cosmological studies. It will be interesting to explore the similarities between big bang / big crunch scenarios and the collapse of a star down the singularity to form a black hole. And, in particular, explore whether a bounce in the scale factor during the early universe evolution implies a bounce for the collapsing star and, if so, whether it leads to stability.

At a technical level, we have not analysed all the intersecting brane configurations in full generality. For example, in all the cases here, we end up with $w' = w$. This needs generalisation. Also, one can consider more general equations of state and explore the possible consequences. This will require numerical analysis and is likely to be challenging. However, this may yet be the most practical way to model the ‘fuzz’ and to find the stable equilibrium configurations.
Appendix A : Equations of motion in a more compact form

In this Appendix, we write the equations of motion (9) – (11) in a more compact form. Note that equation (11) suggests a change of variable from \( r \) to \( \tau \) given by

\[
e^\lambda \, dr = e^\Lambda \, d\tau .
\]  

(65)

The line element for the \((m + 1)\) dimensional transverse space given in equation (3), written in terms of \( \tau \), becomes

\[
e^{2\lambda} dr^2 + e^{2\sigma} d\Omega_m^2 = e^{2\chi} \left( e^{2\lambda} d\tau^2 + d\Omega_m^2 \right)
\]  

(66)

where the field \( \chi \) is defined by

\[
\chi = \Lambda - \sigma = \lambda^0 + \sum_i \lambda^i + (m - 1)\sigma .
\]  

(67)

It follows from the above definitions that

\[
r_\tau = \frac{dr}{d\tau} = e^{\Lambda - \lambda} = r \sqrt{f} \, e^\chi
\]  

(68)

and that, for any function \( X(r(\tau)) \),

\[
X_\tau = e^{\Lambda - \lambda} X_r , \quad X_{\tau\tau} = e^{2(\Lambda - \lambda)} \left( X_{rr} + (\Lambda_r - \lambda_r) X_r \right)
\]  

(69)

where the subscripts \( \tau \) denote \( \tau \)-derivatives. Equations (9) – (11), expressed in terms of \( \tau \), become

\[
\left( \Pi_I \right)_\tau = -\Pi_I \Lambda_\tau + \sum_a p_{aI} \lambda^a_\tau
\]  

(70)

\[
\Lambda^2_\tau - \sum_a (\lambda^a_\tau)^2 = 2 \sum_I \Pi_I e^{2\lambda} + m(m - 1) e^{2\chi}
\]  

(71)

\[
\lambda^a_{\tau\tau} = \sum_I \left( -p_{aI} + \frac{T_I}{D - 2} \right) e^{2\lambda} + \delta^{a\alpha} (m - 1) e^{2\chi} .
\]  

(72)

Equation for the field \( \chi \) defined in equation (67) is then given by

\[
\chi_{\tau\tau} = \sum_I \left( \Pi_I + p_{aI} \right) e^{2\lambda} + (m - 1)^2 e^{2\chi} .
\]  

(73)
Defining $G_{\alpha\beta}$, $G^{\alpha\beta}$, and $h_{\alpha I}$ by

$$G_{\alpha\beta} = 1 - \delta_{\alpha\beta}, \quad G^{\alpha\beta} = \frac{1}{D-2} - \delta^{\alpha\beta},$$

$$h_{\alpha I} = \sum_{\beta} G_{\alpha\beta} \left( -p_{\beta I} + \frac{T_I}{D-2} \right) = \Pi_I + p_{\alpha I}, \quad (74)$$

the above equations may be written compactly as

$$(\Pi_I)_\tau = -2\Pi_I \Lambda_\tau + \sum_\alpha h_{\alpha I} \lambda^\alpha_\tau \quad (75)$$

$$\sum_{\alpha\beta} G_{\alpha\beta} \lambda^\alpha_\tau \lambda^\beta_\tau = 2 \sum_I \Pi_I e^{2\Lambda} + m(m-1) e^{2\chi} \quad (76)$$

$$\lambda^\alpha_\tau = \sum_{\beta I} G^{\alpha\beta} h_{\beta I} e^{2\Lambda} + \delta^{\alpha\beta} (m-1) e^{2\chi}. \quad (77)$$

$$\chi_\tau = \sum_I h_{\Omega I} e^{2\Lambda} + (m-1)^2 e^{2\chi}. \quad (78)$$

### Appendix B : Linear equations of state

When the equations of state are linear, the fields $(\lambda^\alpha, \rho_I)$ can be expressed in terms $N + 1$ independent fields, denoted as $(l^I, l^*)$. Consider the linear equations of state given by equations (32), namely

$$p_{\alpha I} = w^I_\alpha \rho_I, \quad \Pi_I = w^\Pi_I \rho_I \quad (79)$$

where $(w^I_\alpha, w^\Pi_I)$ are constants, $w^I_0 = -1$ since $p_{0I} = -\rho_I$, and $w^I_\alpha = w^I_\Omega = w^I$ since $p_{\alpha I} = p_{\Omega I} = p_I$ for all $a$. Then

$$h_{\alpha I} = \Pi_I + p_{\alpha I} = z^I_\alpha \Pi_I \quad (80)$$

where $z^I_\alpha$ are constants and $z^I_\alpha = z^I_\Omega$ for all $a$. The sets of constants $z^I_\alpha$ and $(w^I_\alpha, w^\Pi_I)$ where $w^I_0 = -1$ are related to each other as follows:

$$z^I_\alpha = 1 + \frac{w^I_\alpha}{w^\Pi_I}; \quad w^I_\alpha = \frac{z^I_\alpha - 1}{1 - z^I_0}, \quad w^\Pi_I = \frac{1}{1 - z^I_0}. \quad (80)$$

Equation (75) can now be solved using equation (80) to give

$$\Pi_I = \Pi_{I0} e^{-2\Lambda + l^I} \quad \rightarrow \quad h_{\alpha I} e^{2\Lambda} = z^I_\alpha \left( \Pi_{I0} e^{l^I} \right) \quad (81)$$
where $\Pi_{I0}$ are constants and $l^I$ are defined by

$$l^I = \sum_\alpha z^I_\alpha \lambda^\alpha .$$  \hspace{1cm} (82)

An expression for $\rho_I$ follows easily since $\Pi_I = w^I_\Pi \rho_I$. Using the above expressions now, equations for $\lambda^\alpha$, $l^I$, and $\chi$ become

\begin{align*}
\lambda^\alpha_{\tau\tau} &= \sum_I z^{\alpha I} \left( \Pi_{I0} e^{l^I} \right) + \delta^{\alpha a} (m-1) e^{2\chi} \hspace{1cm} \text{(83)} \\
l^I_{\tau\tau} &= \sum_J G^{I J} \left( \Pi_{J0} e^{l^J} \right) + z^I_{\Omega} m(m-1) e^{2\chi} \hspace{1cm} \text{(84)} \\
\chi_{\tau\tau} &= \sum_I z^I_\Omega \left( \Pi_{I0} e^{l^I} \right) + (m-1)^2 e^{2\chi} \hspace{1cm} \text{(85)}
\end{align*}

where

$$z^{\alpha I} = \sum_\beta G^{\alpha\beta} z^I_\beta , \quad G^{I J} = \sum_{\alpha\beta} G^{\alpha\beta} z^I_\alpha z^J_\beta .$$  \hspace{1cm} (86)

The above set of equations may be written even more compactly, and $\lambda^\alpha$ may be expressed in terms of $(l^I, \chi)$, by defining the following:

$$\hat{I} = (I, *) , \quad \hat{l}^I = (l^I, l^*), \quad \hat{z}^I_\alpha = (z^I_\alpha, z^*_\alpha) , \quad \Pi_{I0} = (\Pi_{I0}, \Pi_\Omega)$$

where $z^I_\alpha = z^I_\Omega = (z^I_\alpha, z^*_\alpha)$ for all $\alpha$, and

$$l^* = 2\chi = \sum_\alpha z^*_\alpha \lambda^\alpha , \quad z^*_\alpha = 2 \left( 1 - \frac{\delta_{aa}}{m} \right) , \quad \Pi_{*0} = \frac{m(m-1)}{2} .$$

Equations (83), (84) and (85) may then be written as

\begin{align*}
\lambda^\alpha_{\tau\tau} &= \sum_J z^{\alpha J} \left( \Pi_{J0} e^{l^J} \right) \hspace{1cm} \text{(87)} \\
l^{\hat{I}}_{\tau\tau} &= \sum_J \hat{G}^{\hat{I} \hat{J}} \left( \Pi_{J0} e^{l^J} \right) \hspace{1cm} \text{(88)}
\end{align*}

where

$$z^{\hat{I}} = \sum_\beta G^{\alpha\beta} z^I_\beta , \quad \hat{G}^{\hat{I} \hat{J}} = \sum_{\alpha\beta} G^{\alpha\beta} z^{\hat{I}}_\alpha z^\hat{J}_\beta .$$  \hspace{1cm} (89)
It follows that $z^{\alpha*} = \frac{2}{m} \delta^{\alpha \alpha}$, and that $\hat{G}^{ij}$ are given by

$$\hat{G}^{ij} = G^{ij}, \quad \hat{G}^{i*} = \hat{G}^{*i} = n^i = (n^i, n^*) = 2 \left( z^i, z^*_\Omega \right).$$

Thus

$$\hat{G}^{i*} = \hat{G}^{*i} = 2 z^i, \quad \hat{G}^{**} = 2 = \frac{4(m-1)}{m}.$$

When $l^I$ are all independent, equation (88) can be inverted to give

$$\Pi_{j_0} e^{l^I} = \sum_j \hat{G}_{ij} l^j_{\tau\tau}$$

where $\hat{G}_{ij}$ are the elements of the inverse of the matrix formed by $\hat{G}^{ij}$, and are given by

$$\hat{G}_{ij} = G_{ij} + \frac{n_i n_j}{N}, \quad \hat{G}_{is} = \hat{G}_{si} = -\frac{n_i}{N}, \quad \hat{G}_{ss} = \frac{1}{N}.$$

Here $G_{ij}$ are the elements of the inverse of the matrix formed by $G^{ij}$ and

$$n_i = \sum_j G_{ij} n^j, \quad N = n^* - \sum_j n_j n^j.$$

From equations (87) and (90), it now follows that

$$\lambda^\alpha = \sum_{ij} z^{\alpha i} \hat{G}_{ij} l^j + L^\alpha (\tau - \tau_0) + c^\alpha$$

$$= \sum_j z^\alpha_j l^j + z^\alpha_* l^* + L^\alpha (\tau - \tau_0) + c^\alpha$$

(91)

where $(L^\alpha, c^\alpha) = (L^\Omega, c^\Omega)$ for all $a$,

$$z^\alpha_j = (z^\alpha_j, z^\alpha_*) = \sum_i z^{\alpha i} \hat{G}_{ij} = \sum_{\beta I} C^{\alpha \beta} z^i \hat{G}_{ij},$$

and $(L^\alpha, c^\alpha)$ are constants obeying the following constraints:

$$l^I = \sum_{\alpha} z^I_\alpha \lambda^\alpha \implies \sum_{\alpha} z^I_\alpha L^\alpha = \sum_{\alpha} z^I_\alpha c^\alpha = 0.$$

(92)
Substituting the expression for $\lambda^\alpha$ from equation (91) into equation (76) and after some algebra, it follows that

$$\sum_{i,j} \hat{G}_{ij} \tau^i l^j = 2 \sum_i \Pi_{i0} e^i - \sum_{\alpha\beta} G_{\alpha\beta} L^\alpha L^\beta .$$  \hspace{1cm} (93)

Note that

$$\sum_{\alpha} \hat{z}_\alpha^* L^\alpha = 0 \implies \sum_{\alpha} L^\alpha = L^\Omega$$

and, hence, that

$$- \sum_{\alpha\beta} G_{\alpha\beta} L^\alpha L^\beta = \sum_{\alpha} (L^\alpha)^2 - (\sum_{\alpha} L^\alpha)^2$$

$$= (L^\Omega)^2 + \sum_i (L^i)^2 + (m-1)(L^\Omega)^2 \geq 0 .$$

One may now solve equations (88) and (93) and obtain $\hat{t}(\tau)$ . Equation (91) then gives $\lambda^\alpha(\tau)$ . One now has to make a choice of $r$ using the diffeomorphic freedom in defining it. In this paper, $r$ is chosen to be given by $r = e^{\tilde{\beta}} = e^{\sigma + \frac{\Lambda c_m}{m}}$ . Equation (68) then gives $\sqrt{f} = \tilde{\sigma} e^{-\chi}$, thereby completing the solutions.

It is easy to see that, when written in terms of the variable $r$, $(\tau - \tau_0)$ in equation (91) for $\lambda^\alpha$ is to be replaced by a function $F(r)$ which obeys

$$F_{rr} + (\Lambda - \lambda_r) F_r = 0 \iff F_{\tau\tau} = 0 .$$

It is also easy to see that $\tilde{\lambda}^\alpha = \lambda^\alpha + \frac{\Lambda c}{m}$ are given by

$$\tilde{\lambda}^\alpha = \sum_{i,j} \tilde{\hat{z}}_{ij}^\alpha \hat{G}_{ij} \tau^i l^j + \tilde{L}^\alpha (\tau - \tau_0) + \tilde{c}^\alpha$$

$$= \sum_i \tilde{z}_i^\alpha l^i + \tilde{z}_s^\alpha l^s + \tilde{L}^\alpha (\tau - \tau_0) + \tilde{c}^\alpha$$  \hspace{1cm} (94)

where $\tilde{z}_j^\alpha = (\tilde{z}_j^\alpha, \tilde{z}_s^\alpha) = \sum_{i,j} \tilde{z}_{ij}^\alpha \hat{G}_{ij}$ and

$$\tilde{X}^\alpha = X^\alpha + \frac{\sum_i X^i}{m} , \quad X^\alpha = (z^\alpha, L^\alpha, c^\alpha) .$$
For example, using $z^\alpha_I = \sum_\beta G^{\alpha \beta} z_\beta^I$ and $\sum_\alpha L^\alpha = L^\Omega$, it follows that

$$z^\alpha_I = -z^I_\alpha + \frac{z^I_0 + mz^I_\Omega}{m}, \quad \tilde{L}^\alpha = L^\alpha - \frac{L^0 + (m-1)L^\Omega}{m}.$$ 

**Appendix C : M theory branes**

For intersecting branes in M theory, the U duality relations further lead to an elegant structure. Firstly, it can be argued [7] that the equations of state may be written in terms of one single function only. For example, they may be written as

$$h_{\alpha I} = \Pi_I + p_{\alpha I} = z^I_\alpha \mathcal{F}(\{\ast\}_I), \quad \alpha = \{0, \|, \bot, \Omega\}$$

where $\{\ast\}_I$ denote brane quantities such as the number and the nett charge of the branes in $I^{th}$ stack, the constant coefficients $z^I_\alpha$ and the functional dependence of $\mathcal{F}$ on the brane quantities $\{\ast\}$ are same for all $I$, and

$$z_\Omega = z_\bot, \quad z_\| = z_0 + z_\bot$$

as follows from $p_{\Pi I} = p_{\bot I}$ and from equation (55). Note that equations (95) may still lead to equations of state of the type considered in equation (80) if $\mathcal{F}$ depends on $\Pi$ only and if the function $\mathcal{F}(\Pi)$ is such that, for example, $\mathcal{F}(\Pi) = u^{(lo)}(\Pi)$ or $u^{(hi)}(\Pi)$ for low or high magnitudes of $\Pi$. Equation (80) then follows, with $z^I_\alpha = z_\alpha u^I$ where $u^I = u^{(lo)}$ or $u^{(hi)}$ according to whether the magnitude of $\Pi_I$ is low or high.

Consider the case where $z^I_\alpha = z_\alpha u^I$ and $z_\alpha$ obey equations (96). It then follows from equations (86) and from BPS intersection rules that

$$z^\alpha_I = \left(\frac{(n_I + 1)}{9} z_0 + z_\bot - z_\alpha \right) u^I$$

$$G^{IJ} = 2z_0 \left( z_\bot - z_0 \delta^{IJ} \right) u^I u^J.$$ 

The form of $z^\alpha_I$ and the relations among them are consequences of U duality symmetries of M theory. The relations among $z^\alpha_I$ will be reflected in the solutions for $\lambda^\alpha$ when written in terms of $l^I$, see equation (91). Note the elegant structure of $G^{IJ}$ and its independence on $n_I$. These features of $G^{IJ}$
are consequences of both the U duality symmetries and the BPS intersection rules. Similar $G^{IJ}$ was also present in our time dependent early universe studies.

Note that if $G^{IJ}$ are of the form

$$G^{IJ} = a \left( b + \delta^{IJ} \right) u^I u^J$$  \hspace{1cm} (99)

then $G_{IJ}$ are given by

$$G_{IJ} = \frac{\beta + \delta_{IJ}}{a u^I u^J}, \quad \beta = -\frac{b}{N b + 1} . \hspace{1cm} (100)$$

Thus, for the BPS intersecting branes in M theory, it follows from equation (98) that

$$G_{IJ} = \frac{1}{2z_0^2} u^I u^J \left( \frac{z_\perp}{N z_\perp - z_0} - \delta_{IJ} \right) . \hspace{1cm} (101)$$

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