ORTHOGONALITY AND BOUNDARY CONDITIONS IN QUANTUM MECHANICS

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One-dimensional particle states are constructed according to orthogonality conditions, without requiring boundary conditions. Free particle states are constructed using Dirac’s delta function orthogonality conditions. The states (doublets) depend on two quantum numbers: energy and parity ("+" or ";-"). With the aid of projection operators the particles are confined to a constrained region, in a way similar to the action of an infinite well potential. From the resulting overcomplete basis only the mutually orthogonal states are selected. Four solutions are found, corresponding to different non-commuting Hamiltonians. Their energy eigenstates are labeled with the main quantum number n and parity "+" or ";-". The energy eigenvalues are functions of n only. The four cases correspond to different boundary conditions: (I) the wave function vanishes on the boundary (energy levels: 1\(^+\), 2\(^+\), 3\(^+\), 4\(^+\), ...), (II) the derivative of the wavefunction vanishes on the boundary (energy levels 0\(^+\), 1\(^-\), 2\(^+\), 3\(^-\), ...), (III) periodic boundary conditions (energy levels: 0\(^+\), 2\(^+\), 2\(^-\), 4\(^+\), 4\(^-\), 6\(^+\), 6\(^-\), ...), (IV) periodic boundary conditions (energy levels: 1\(^+\), 1\(^-\), 3\(^+\), 3\(^-\), 5\(^+\), 5\(^-\), ...). Among the four cases, only solution (III) forms a complete basis in the sense that any function in the constrained region, can be expanded with it. By extending the boundaries of the constrained region to infinity, only solution (III) converges uniformly to the free particle states. Orthogonality seems to be a more basic requirement than boundary conditions. By using projection operators, confinement of the particle to a definite region can be achieved in a conceptually simple and unambiguous way, and physical operators can be written so that they act only in the confined region.

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1. INTRODUCTION

In quantum mechanics the physical states corresponding to different eigenvalues are orthogonal [1-3]. Therefore orthogonality is much more than a convenient tool for expanding functions in terms of a suitable orthogonal basis. On the other hand the space of quantum states is a normed space and normalization of states is required. Usually, when the coordinates are extending to infinity, in order to secure normalizability, boundary conditions have to be imposed. But this procedure in quantum mechanics becomes too rigid. We show in the case of an infinite square well (Sec. 3), for which normalizability is always secured, that by imposing boundary conditions, part of the solutions can be missing. In this case, in order to get more solutions, we impose orthogonality alone. In this way new solutions are obtained. The energy levels are not changed. The difference is that with the boundary conditions the energy states are singlets, with the orthogonality conditions the energy states may be also doublets. This brings also changes in the bases of the vector space. This is extensively discussed in Sec. 3.

In Sec. 2 we obtain the doublet energy states of a free particle by imposing the Dirac’s delta function conditions of orthogonality only, without boundary conditions. Thus working with Dirac’s representation theory of quantum mechanics (1), one can get a larger basis of states and new possibilities compared to quantum mechanics which is defined in Hilbert space. Working in the Dirac’s abstract space of vector states does not require boundary conditions [4], as long as they are not projected onto the coordinate states.

In Sec. 3 we find solutions for the infinite square well problem by redefining it so that by using a proper projection operator, the free particle solution is projected into the confined subspace inside the infinite square well. The projected free particle solution forms an overcomplete nonorthogonal basis. We find solutions to the problem after sifting from the overcomplete solutions the orthogonal ones. In this way we find additional solutions to the ones usually quoted in textbooks [2,3]. The new solutions satisfy new boundary conditions. But the solutions so obtained are not resulting only from the possibility to reformulate boundary conditions. We prove that there are no solutions to mixed boundary conditions, namely vanishing of the wavefunction at one end and vanishing of the derivative at the other end and vice versa.

2. ENERGY EIGENVECTORS OF A FREE PARTICLE

In this section we derive the energy eigenvectors of a free particle with mass m and momentum p, in one dimension, on the infinite x axis. We shall do it first in an abstract way, without projecting them into coordinates. The free particle energy (Hamiltonian) operator is

\[ \hat{H} = \frac{\hat{p}^2}{2m}. \]  

(2.1)
with the energy eigenvalue $E$ satisfying the equation

$$
\hat{H}|E\rangle = \frac{\hat{p}^2}{2m}|E\rangle = E|E\rangle,
$$

(2.1a)

where $\hat{p}$ is the momentum operator and $p$ its eigenvalue satisfying the eigenvalue equation

$$
\hat{p}|p\rangle = p|p\rangle.
$$

(2.2)

The energy eigenvectors are obviously also the eigenvectors of $\hat{p}^2$. We shall find them in the following way. Let us first note that

$$
\hat{p}^2|p\rangle = p^2|p\rangle; \quad \hat{p}^2|-p\rangle = p^2|-p\rangle;
$$

(2.3)

therefore any combination of these two states will be also an eigenvector of the momentum squared operator:

$$
|p^2\rangle = A|p\rangle + B|-p\rangle,
$$

(2.4)

where $A$ and $B$ are constants. Following Dirac [1], we shall require the following normalization:

$$
\langle p|p'\rangle = \delta (p - p'),
$$

(2.5)

$$
\langle p^2|p'^2\rangle = \delta (p^2 - p'^2) = \frac{1}{2|p|}[\delta (p - p') + \delta (p + p')].
$$

(2.6)

One can prove that the only combination of (2.4) that satisfies Eq. (2.6) (up to a phase factor) is

$$
|p^2_+\rangle = \frac{1}{\sqrt{2|p|}}(|p\rangle + |-p\rangle).
$$

(2.7)

We can find a second possibility (again up to a phase factor) in the form

$$
|p^2_-\rangle = \frac{\text{sign}(p)}{2i|p|}(|p\rangle - |-p\rangle) = \frac{1}{2i\sqrt{|p|}}(|p\rangle - |-p\rangle),
$$

(2.8)

where

$$
\text{sign}(p) = \begin{cases} 
1 & , \text{for } p > 0, \\
0 & , \text{for } p = 0, \\
-1 & , \text{for } p < 0.
\end{cases}
$$

(2.9)

One can easily verify that the vectors (2.7) and (2.8) are mutually orthogonal. Moreover, they form a complete orthonormal basis, which can be proved in the following way:

$$
\int_0^\infty \left(|p^2_+\rangle\langle p^2_+| + |p^2_-\rangle\langle p^2_-|\right) dp^2 = \int_0^\infty (|p\rangle\langle p| + |-p\rangle\langle -p|) dp = \int_{-\infty}^\infty |p\rangle\langle p|dp = \hat{I},
$$

(2.10)

where $\hat{I}$ is the identity operator. The above results allow us to conclude that for each energy eigenvalue there are two degenerate states having the same energy eigenvalue $E_+ = E_-$ for which the following relations are satisfied:

$$
E_+ = E_- = \frac{p^2}{2m} \equiv E, \quad \langle E_+|E_-\rangle = 0,
$$

(2.11)
\[ \langle E_+ | E'_+ \rangle = \langle E_- | E'_- \rangle = \delta (E - E') = \delta \left( \frac{p^2}{2m} - \frac{p'^2}{2m} \right) = \frac{m}{|p|} \left[ \delta (p - p') + \delta (p + p') \right], \quad (2.11) \]

\[ |E+\rangle = \sqrt{\frac{m}{2|p|}} \left( |p| + |-p| \right), \quad |E_-\rangle = -i \sqrt{\frac{m}{2|p|}} \left( |p| - |-p| \right). \quad (2.12) \]

The projection into x space proceeds via the momentum wavefunctions

\[ \langle x | p \rangle = \frac{1}{\sqrt{2\pi \hbar}} \exp \left( \frac{ipx}{\hbar} \right), \quad (2.13) \]

(normalized according to (2.5)) and leads to twofold energy degenerate wavefunctions

\[ \langle x | E_+ \rangle = \sqrt{\frac{m}{\pi \hbar |p|}} \cos \left( \frac{|p|x}{\hbar} \right), \quad (2.14a) \]

\[ \langle x | E_- \rangle = \sqrt{\frac{m}{\pi \hbar |p|}} \sin \left( \frac{|p|x}{\hbar} \right), \quad (2.14b) \]

According to their transformation properties with respect to x we will call the solution (2.14a) a positive parity wavefunction and the solution (2.14b) the negative parity wavefunction. Accordingly we can define the energy states with a definite parity:

\[ |E, \Pi \rangle = \begin{cases} 
|E_+\rangle, & \text{for } \Pi = 1, \\
|E_-\rangle, & \text{for } \Pi = -1,
\end{cases} \quad (2.15) \]

and write the completeness relation in the form

\[ \hat{I} = \int_0^\infty (|E, 1\rangle \langle E, 1| + |E, -1\rangle \langle E, -1|) \, dE. \quad (2.16) \]

### 3. THE INFINITE POTENTIAL WELL

Let us consider the case of a particle in an infinite potential well. The potential is defined as:

\[ V(x) = \begin{cases} 
0, & -a \leq x \leq a, \\
\infty, & |x| > a,
\end{cases} \quad (3.1) \]

i.e. the particle is confined in a "bandwidth" of width a [4]. The time independent Schrödinger equation (in the stationary state) has the form

\[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \langle x | E \rangle = E \langle x | E \rangle, \quad \langle a | E \rangle = \langle -a | E \rangle = 0, \quad (3.2) \]

where \( \hbar \) is the Planck constant, \( m \) the mass of the particle, and \( E \) the energy eigenvalue.

By using orthogonality considerations rather than boundary conditions we will find additional solutions, such that the energy states will be twofold degenerate. Let \( \hat{H} \) be the
Hamiltonian (the energy operator). As an observable it is required to be Hermitian; therefore \( \langle E | \hat{H} | E' \rangle = E \langle E | E' \rangle = E' \langle E | E' \rangle \); and after subtracting both sides, we find that physical states have to satisfy the orthogonality condition

\[
(E - E')' \langle E | E' \rangle = 0. \tag{3.3}
\]

These conditions are quite general and do not explicitly have boundary conditions. Boundary conditions are required when the scalar product has to be finite. But in the case of an infinite square well there is no problem that the scalar products and normalizations will be finite; therefore the condition (3.3) seems to be more adequate than boundary conditions.

The eigenvalues and orthogonal eigensolutions of Eq. (3.2), as they usually appear in textbooks [2,3], are

\[
E_n = \frac{p_n^2}{2m} = \frac{\hbar^2 k_n^2}{2m}, \quad k_n = \frac{n\pi}{2a}, \quad n = 1, 2, 3, ..., \tag{3.4}
\]

\[
\langle x | E_{n-} \rangle = \frac{1}{\sqrt{a}} \sin (k_n x), \quad \text{for } n = 2, 4, 6, ..., \tag{3.5a}
\]

\[
\langle x | E_{n+} \rangle = \frac{1}{\sqrt{a}} \cos (k_n x), \quad \text{for } n = 1, 3, 5, ..., \tag{3.5b}
\]

Let us use a different approach for finding the energy solutions, namely that of Eq. (3.3). In Sec. 2 we found twofold degenerate energy solutions for the free particle case:

\[
\langle x | E_+ \rangle = \sqrt{\frac{m}{\pi \hbar |p|}} \cos \left( \frac{|p| x}{\hbar} \right), \quad \langle x | E_- \rangle = \sqrt{\frac{m}{\pi \hbar |p|}} \sin \left( \frac{|p| x}{\hbar} \right). \tag{3.6}
\]

Let us project the whole free particle vector space into a subspace given by the projection operator:

\[
\hat{X}_a = \int_{-a}^{a} |x\rangle \langle x| dx, \tag{3.7}
\]

i.e. we now confine the free particle to exist only in \(-a \leq x \leq a\). In section 2 the normalizations of the free particle states were

\[
E_+ = E_- = \frac{p^2}{2m} = E, \quad \langle E_+ | E'_+ \rangle = 0, \quad \langle E_- | E'_- \rangle = \delta (E - E'), \quad \delta \left( \frac{p^2}{2m} - \frac{p'^2}{2m} \right) = \frac{m}{\sqrt{|pp'|}} \left[ \delta (p - p') + \delta (p + p') \right]. \tag{3.8}
\]

Now, in the constrained vector subspace, the delta functions will be replaced by (the incomplete delta functions [5]):

\[
\langle E_+ | \hat{X}_a | E'_+ \rangle = \langle E_- | \hat{X}_a | E'_- \rangle = \frac{m}{\sqrt{|pp'|}} \left( \frac{\sin \left[ a (p - p') / \hbar \right]}{\pi (p - p')} + \frac{\sin \left[ a (p + p') / \hbar \right]}{\pi (p + p')} \right). \tag{3.9}
\]

For \( p = p' \), we have

\[
\langle E_+ | \hat{X}_a | E_+ \rangle = \langle E_- | \hat{X}_a | E_- \rangle = \frac{ma}{\pi \hbar |p|}, \tag{3.9a}
\]
According to Eq. (3.5), physical states, which are eigenstates of Hermitian operators (observables), should be orthogonal for different energies. In Eq. (3.9) this will happen if

\[ p - p' = \frac{n_1 \pi \hbar}{a}, \quad \text{and} \quad p + p' = \frac{n_2 \pi \hbar}{a}, \quad n_1 \text{ and } n_2 \text{ integers} \quad (3.10) \]

There are two solutions of Eqs. (3.10) (characterized by the positive integer \( n \)):

\[(I) : \quad p_{2n} = \frac{n \pi \hbar}{a}, \quad (3.11a)\]

\[(II) : \quad p_{2n+1} = \frac{n \pi \hbar}{a} + \frac{\pi \hbar}{2a}. \quad (3.11b)\]

Let us describe the wavefunctions, the solutions to (3.1), in the following way:

\[ \langle x|E(|p|), \Pi \rangle = \begin{cases} \frac{1}{\sqrt{a}} \cos \left( \frac{|p| \pi x}{\hbar} \right), & \text{for } \Pi = 1, \\ \frac{1}{\sqrt{a}} \sin \left( \frac{|p| \pi x}{\hbar} \right), & \text{for } \Pi = -1, \end{cases} \quad (3.12) \]

where \( \Pi \) is the parity eigenvalue. One should note that the solutions (3.11a) and (3.11b) are valid for both parities (see Eq. (3.8)). Therefore we can get four different orthonormal solutions:

\[(I) : \quad \langle x|E(p_{2n-1}), 1 \rangle = \frac{1}{\sqrt{a}} \cos \left( \frac{(2n-1) \pi x}{2a} \right), \quad \langle x|E(p_{2n}), -1 \rangle = \frac{1}{\sqrt{a}} \sin \left( \frac{n \pi x}{a} \right); \quad (3.13a)\]

\[(II) : \quad \langle x|E(p_{2n}), 1 \rangle = \frac{1}{\sqrt{a}} \cos \left( \frac{n \pi x}{a} \right), \quad \langle x|E(p_{2n-1}), -1 \rangle = \frac{1}{\sqrt{a}} \sin \left( \frac{(2n-1) \pi x}{2a} \right); \quad (3.13b)\]

\[(III) : \quad \langle x|E(p_{2n}), 1 \rangle, \quad \langle x|E(p_{2n}) , -1 \rangle; \quad (3.13c)\]

\[(IV) : \quad \langle x|E(p_{2n-1}), 1 \rangle, \quad \langle x|E(p_{2n-1}), -1 \rangle, \quad (3.13d)\]

where solution (I) corresponds to the standard solution (3.6) for which the wavefunctions vanish on the boundary. Solution (II) corresponds to boundary conditions for which the derivatives of the wavefunctions vanish on the boundary. Solution (III) is the (complete) basis of Fourier series, with periodic (symmetric) boundary conditions. Solution (IV) corresponds to periodic (antisymmetric) boundary conditions. The sum of solutions (III) and (IV) is equal to the sum of solutions (I) and (II); therefore only three solutions are linearly independent. The main problem with solutions (I-IV) is that although each solution separately is an orthonormal basis, the solutions are not always orthogonal to each other. This means that they must correspond to different Hamiltonians. Therefore the projection into a confined subspace generated four Hamiltonians (three linearly independent) for which physically acceptable solutions represent particles confined in the coordinate subspace. We shall discuss in more detail these Hamiltonians in Sec. 4.

Before that, let us consider mixed boundary conditions, i.e. vanishing of the wavefunction at \( x=a \) and vanishing of the derivative at \( x=-a \) or vice versa. We will prove by contradiction...
that there are no solutions for these problems. Let us first assume that there are solutions.
Let us call such a solution \( \psi(x) \), which satisfies
\[
\psi(a) = \psi'(-a) = 0,
\]
(3.14)

The most general form of the solution is
\[
\psi(x) = A \cos(hx) + B \sin(hx),
\]
(3.15)
where \( A, B \) and \( h \) are constants. After substituting the boundary conditions (3.14) into Eq. (3.15) we obtain
\[
A \cos(ha) + B \sin(ha) = 0, \quad (3.16a)
\]
\[
hA \sin(ha) + hB \cos(ha) = 0. \quad (3.16b)
\]
Eqs. (3.16a) and (3.16b) can be rewritten as:
\[
A \cos(ha) = -B \sin(ha), \quad (3.17a)
\]
\[
A \sin(ha) = -B \cos(ha). \quad (3.17b)
\]
Dividing Eq. (3.17a) by Eq. (3.17b), we obtain:
\[
\frac{\cos(ha)}{\sin(ha)} = -\frac{\sin(ha)}{\cos(ha)},
\]
(3.18)
which can be rewritten as
\[
\cos^2(ha) + \sin^2(ha) = 0, \quad (3.19)
\]
which is a contradiction as the right-hand-side of Eq. (3.19) should be equal to one. In this way we arrive at twofold degenerate energy states. We see that in the case of the infinite well the orthogonality requirement leads to all solutions, while by imposing boundary conditions on the wavefunction some of the solutions are missing. In the next section we demonstrate that we could have guessed this result by considering the completeness relation in the \( x \) subspace constrained by the well.

4. THE HAMILTONIANS AND COMPLETENESS OF THE BASES

The four Hamiltonians, corresponding to the solutions (I-IV), can be represented in the following way:
\[
H_I = \sum_{n=1}^{\infty} E(p_{2n-1}) \hat{X}_a |E(p_{2n-1}), 1\rangle \langle E(p_{2n-1}), 1| \hat{X}_a + \sum_{n=1}^{\infty} E(p_{2n}) \hat{X}_a |E(p_{2n}), -1\rangle \langle E(p_{2n}), -1| \hat{X}_a,
\]
(4.1)
\[
H_{II} = \sum_{n=1}^{\infty} E(p_{2n-1}) \hat{X}_a |E(p_{2n-1}), -1\rangle \langle E(p_{2n-1}), -1| \hat{X}_a + \sum_{n=1}^{\infty} E(p_{2n}) \hat{X}_a |E(p_{2n}), 1\rangle \langle E(p_{2n}), 1| \hat{X}_a,
\]
(4.2)
The new basis (4.11) is overcomplete and nonorthogonal. In the quantization process the i.e., in the subspace the projection operator becomes the identity operator (for the subspace).

Let us now consider the problem of completeness of the bases (3.13a-3.13d). First we should constrain the coordinate space with the projection operator (3.7), all states, including the states of Eq. (3.13a) (which vanish at the ends of the well) are obtained. Note that the presence of the projection operator assures that no particle can be found outside the well.

The normalizations were chosen to agree with the normalizations of Eqs. (3.13a-3.13d). The Hamiltonians (4.1-4.4) are linearly dependent:

\[ H_I = H_{III} + H_{IV}, \]

and each one of them, separately, represents a particle confined to a well. For instance, in the example

\[ H_I \left( \hat{X}_a | E \left( p_{2n-1} \right), 1 \right) = E \left( p_{2n-1} \right) \left( \hat{X}_a | E \left( p_{2n-1} \right), 1 \right) \]
\[ H_I \left( \hat{X}_a | E \left( p_{2n-1} \right), -1 \right) = E \left( p_{2n-1} \right) \left( \hat{X}_a | E \left( p_{2n-1} \right), -1 \right) \]

the states of Eq. (3.13a) (which vanish at the ends of the well) are obtained. Note that the presence of the projection operator assures that no particle can be found outside the well.

Let us now consider the problem of completeness of the bases (3.13a-3.13d). First we should specify what is the meaning of completeness in the framework of the Dirac formalism. If we start with the unconstrained vector space, then the completeness is expressed through the identity operator \( \hat{I} \):

\[ \hat{I} = \int_0^\infty (|E, 1\rangle \langle E, 1| + |E, -1\rangle \langle E, -1|) \, dE = \int_{-\infty}^{\infty} |p\rangle \langle p| \, dp = \int_{-\infty}^{\infty} |x\rangle \langle x| \, dx. \]

If we constrain the coordinate space with the projection operator (3.7), all states, including any basis states, has to be projected into the subspace. For example the momentum basis has to be projected onto

\[ |p\rangle \Rightarrow \hat{X}_a |p\rangle, \]

and completeness means:

\[ \int_{-\infty}^{\infty} \hat{X}_a |p\rangle \langle p| \hat{X}_a \, dp = \hat{X}_a \hat{I} \hat{X}_a = \hat{X}_a, \]

i.e., in the subspace the projection operator becomes the identity operator (for the subspace). The new basis (4.11) is overcomplete and nonorthogonal. In the quantization process the
eigenstates of the Hamiltonian can form an orthogonal basis. Are the bases (3.13a-3.13d) complete? In order to answer this question, let us construct the projection operators generating the bases (3.13a-3.13d):

\[ P_I = \sum_{n=1}^{\infty} \hat{X}_a (|E(p_{2n-1}), 1\rangle \langle E(p_{2n-1}), 1| + |E(p_{2n}), -1\rangle \langle E(p_{2n}), -1|) \hat{X}_a, \]  
\[ P_{II} = \sum_{n=1}^{\infty} \hat{X}_a (|E(p_{2n-1}), -1\rangle \langle E(p_{2n-1}), -1| + |E(p_{2n}), 1\rangle \langle E(p_{2n}), 1|) \hat{X}_a, \]  
\[ P_{III} = \sum_{n=1}^{\infty} \hat{X}_a (|E(p_{2n}), -1\rangle \langle E(p_{2n}), -1| + |E(p_{2n-1}), 1\rangle \langle E(p_{2n-1}), 1|) \hat{X}_a, \]  
\[ P_{IV} = \sum_{n=1}^{\infty} \hat{X}_a (|E(p_{2n-1}), -1\rangle \langle E(p_{2n-1}), -1| + |E(p_{2n-1}), 1\rangle \langle E(p_{2n-1}), 1|) \hat{X}_a, \]

with the linear dependence:

\[ P_I + P_{II} = P_{III} + P_{IV}. \]

Is the projection operator \( P_I \) equal to \( \hat{X}_a \)? The answer is negative. If we apply the above projection operators to an arbitrary function \( f(x) \) we obtain functions with support in the subspace only, with the properties

\[ \langle x|P_I|f \rangle \implies \langle \pm a|P_I|f \rangle = 0, \]  
\[ \langle x|\hat{X}_a|f \rangle \implies \lim_{x \to a} \langle \pm a|\hat{X}_a|f \rangle = \lim_{x \to a} f(\pm a), \]

i.e., \( P_I \) projects only to functions which are vanishing at the boundary; therefore the basis (3.13a) can be complete only with respect to functions vanishing on the boundary. The basis (3.13b) can be complete only with respect to functions whose derivative vanishes on the boundary. The basis (3.13c) is the basis for Fourier series expansion and is known to be complete in the subspace. The basis (3.13d) is related to the basis (3.13c) via

\[ \langle x|E(p_{2n-1}), 1\rangle = \cos \left( \frac{\pi x}{2a} \right) \langle x|E(p_{2n}), 1\rangle + \sin \left( \frac{\pi x}{2a} \right) \langle x|E(p_{2n}), -1\rangle, \]  
\[ \langle x|E(p_{2n-1}), -1\rangle = \cos \left( \frac{\pi x}{2a} \right) \langle x|E(p_{2n}), -1\rangle - \sin \left( \frac{\pi x}{2a} \right) \langle x|E(p_{2n}), 1\rangle, \]

but any expansion in terms of the basis (3.13d) will miss a constant term (which is present in the Fourier series), and for this reason the basis (3.13d) is not complete. Up to that constant term, any expansion with the basis (3.13d) can be brought to a non vanishing combination of Fourier series; therefore it is complete for functions not having a constant term in their Fourier expansion.

5. SUMMARY AND CONCLUSIONS

In this work one dimensional particle states were constructed according to orthogonality conditions, without requiring boundary conditions. Free particle states were constructed using Dirac’s delta function orthogonality conditions Eq. (2.11). The states, given in Eq. (2.12), depended on two quantum numbers: energy and parity ("+" or "-"'). The eigenvalues
depended only on the energy, therefore the states were doublets in energy (with the exception of a singlet at zero energy). With the aid of the projection operator of Eq. (3.7), the particles were confined to a constrained region, in a way similar to the action of an infinite-well potential. From the resulting overcomplete basis, only the mutually orthogonal states are selected from Eq. (3.9). Four solutions were found: Eqs. (3.13a-3.13d), corresponding to different non-commuting Hamiltonians of Eqs. (4.1-4.4). Their energy eigenstates were labeled with the main quantum number \( n \) and parity "+" or ",-". The energy eigenvalues were functions of \( n \) only. The four cases corresponded to different boundary conditions:

(I) The wave function vanishes on the boundary (energy levels: \( 1^+, 2^-, 3^+, 4^- \),...). This is the standard model of the infinite square well.

(II) The derivative of the wavefunction vanishes on the boundary (energy levels \( 0^+, 1^-, 2^+, 3^- \),...).

(II) Periodic boundary conditions (energy levels: \( 0^+, 2^+, 2^-, 4^+, 4^-, 6^+, 6^- \),...). The eigenfunctions of Eq. (3.13.c) coincide with the basis of the Fourier series.

(IV) Periodic boundary conditions (energy levels: \( 1^+, 1^-, 3^+, 3^- \),...).

Among the four cases, only solution (III) forms a complete basis, in the sense that any function in the constrained region, can be expanded with it. By extending the boundaries of the constrained region to infinity, it seems that only the doublets of solution (III) converge to the free particle doublet states.

By confining a particle to a constrained subspace and requiring, as the basic physical condition, the orthogonality of states, we obtained four exclusive solutions, corresponding to different boundary conditions. Therefore orthogonality seems to be a more basic requirement than boundary conditions. By using projection operators, confinement of the particle to a definite region can be achieved in a simple and unambiguous way. All physical operators can be written so that they act only in the confined region. This method seems to be superior to the boundary condition models.

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