On optimal heuristic randomized semidecision procedures, with application to proof complexity

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Acceptors and proof systems

- \(A\) is an acceptor for language \(L\) if
  - \(\forall x \in L \, A(x) = 1\),
  - \(\forall x \notin L \, A(x)\) does not stop.

- [Cook, Reckhow, 70s] A \textbf{proof system} for language \(L\) is a polynomial-time surjective mapping \(\Pi: \{0, 1\}^* \to L\).
  - \(w\) is called a \(\Pi\)-proof of \(f(w)\).

- Proof system from acceptor:
  - \(\Pi_A : [\text{protocol of } A(x)] \mapsto x\).

- An \textbf{automatizable proof system} is a pair \((\Pi, B)\):
  - \(\forall x \in L \, B(x)\) outputs a \(\Pi\)-proof of \(x\) in time \(\leq \text{poly}(\text{size of the shortest } \Pi\text{-proof of } x)\).
  - \(\forall x \notin L \, B(x)\) does not stop.

- \((\Pi_A, \tilde{A})\) is an automatizable proof system.
Propositional proof systems

- **Propositional** proof systems: proof systems for the language of Boolean tautologies $\text{TAUT}$.
- Every algorithm for $\text{TAUT}$ yields a proof system, but not vice versa.
- $\text{NP} = \text{coNP}$ iff there is a proof system that has a polynomial-size proof for every tautology.
- $\text{P} = \text{NP}$ iff there is an automatizable proof system that has a polynomial-size proof for every tautology.
Simulation and Optimality

- A proof system $\Pi_1$ \textit{p-simulates} a proof system $\Pi_2$ if there exists a polynomial-time computable function $f$ that maps $\Pi_2$-proofs to $\Pi_1$-proofs.
- An acceptor $A_1$ \textit{simulates} an acceptor $A_2$ if for all $x \in L$, the running time of $A_1(x)$ is polynomial in the running time of $A_2(x)$.
- [Krajíček, Pudlák, 1989] \exists \ p-optimal proof system for TAUT $\iff$ \exists optimal acceptor for TAUT.
- [Messner, 1999] For every paddable language $L$, \exists \ p-optimal proof system for $L$ $\iff$ \exists optimal acceptor for $L$.
- [Cook, Krajíček, 2007] \exists \ p-optimal proof system with 1 bit of nonuniform advice.
Heuristic proof systems and acceptors

- Distributional proving problem: \((L, D)\) where \(D\) is polynomial-time samplable distribution on \(\overline{L}\).
- Heuristic proof system for \((L, D)\) is a randomized algorithm \(\Pi(x, w, d)\):
  - Running time of \(\Pi(x, w, d)\) is \(\text{poly}(|x|, |w|, d)\).
  - (Completeness) \(\forall x \in L \forall d \in \mathbb{N} \exists w \Pr\{\Pi(x, w, d) = 1\} > \frac{1}{2}\).
  - (Soundness) \(\Pr_{x \leftarrow D_n}\{\exists w \Pr\{\Pi(x, w, d) = 1\} > \frac{1}{4}\} < \frac{1}{d}\).
- Heuristic acceptor for \((L, D)\) is a randomized algorithm \(A(x, d)\):
  - \(\forall x \in L \forall d \in \mathbb{N} A(x, d) = 1\).
  - \(\Pr_{x \leftarrow D_n}\{\Pr\{A(x, d) \text{ stops}\} > \frac{1}{4}\} < \frac{1}{d}\}.
- Median running time:
  - \(\min\{t \mid \Pr\{A(x, d) \text{ runs in } \leq t \text{ steps}\} \geq \frac{1}{2}\}\).
Optimal heuristic acceptor

**Theorem.** For every r.e. language $L$ and p-samplable $D$ with support in $\overline{L}$ there exists an acceptor $U(x, d)$ for $(L, D)$ that has optimal (up to $\text{poly}(|x|, d)$) median time.

**Construction (sketch):**

Optimal heuristic acceptor $U(x, d)$:

- In parallel for all $1 \leq i \leq \log^* n$:
  1. Execute $A_i(x, d')$; Let $T_i$ be its running time.
  2. Verify the correctness of $A_i$:
     - Repeat for many times:
       - $r \leftarrow D_n$,
       - If $A_i^{\leq T_i}(r, d') = 1$ too often,
       - then put a black point;
     - and verify that the number of black points is small.
  3. Return "1".

- Execute the semidecision procedure for r.e. language $L$. 
Further research

- Some recent observations (unpublished)
  - An optimal heuristic automatizable proof system (under weak enough notion of automatizability).
  - \( \exists \) one-way functions \( \implies \) \( \exists \) p-samplable distribution \( D \) on \( \text{Taut} \) such that every heuristic acceptor for \((\text{Taut}, D)\) is not polynomial bounded.
  - A universal distribution on \( \text{Taut} \) that dominates distributions on \( \text{Taut} \) that are provably correct or certify their results.

- Open questions
  - Construct an optimal heuristic proof system.
  - Extend the equivalence between p-optimal proof systems and optimal acceptors to the heuristic case.