Transverse QCD Dynamics
Near the Light Cone

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Abstract

Starting from the QCD Hamiltonian in near-light cone coordinates, we study the dynamics of the gluonic zero modes. Euclidean 2+1 dimensional lattice simulations show that the gap at strong coupling vanishes at intermediate coupling. This result opens the possibility to synchronize the continuum limit with the approach to the light cone.

1 Introduction

The solution of Quantum Chromodynamics on the light cone is still an unsolved theoretical task the present status of which is reviewed in ref. \cite{1}. In a recent paper \cite{2} a formulation of QCD in coordinates near the light cone has been proposed which has the advantage of keeping a direct link \cite{3–5} to equal time theories. The Cauchy problem is well defined in near light cone coordinates, since the initial data are given on a space like surface. This formulation avoids the solution of constraint equations which, on the quantum field theoretic level, may be very complicated. The problem of a nontrivial vacuum appears in a solvable form related to the transverse dynamics. This is physically very appealing, since in high energy reactions the incoming particles propagate near the light cone and interact mainly exchanging particles with transverse momenta. We would like to connect successful models for the soft transverse nonperturbative dynamics of high energy reactions (cf. refs. \cite{6, 7}) to the underlying QCD Hamiltonian. In diffractive reactions, fast Lorentz contracted hadrons experience the confining forces of QCD in their transverse extensions and interact with other fast moving hadrons via soft interactions. In a theory of total cross sections the nonperturbative infrared dynamics in the transverse plane is essential. The objective of this

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paper is to investigate the effective transverse Hamiltonian in QCD near the light cone for small light cone momenta.

Near light cone QCD has a nontrivial vacuum which we claim cannot be neglected even in the light cone limit. We will demonstrate the existence of massless excitations in the zero mode theory. These excitations do not decouple in the light cone limit. Genuine nonperturbative techniques must be used to investigate the behavior of this limit. In principle the additional parameter which labels the coordinate system can be chosen arbitrarily. We will show that the zero mode Hamiltonian depends on an effective coupling constant containing this parameter and evolves towards an infrared fixed point. Therefore, we propose a sophisticated choice of the frame dependence which facilitates calculations in the near light cone frame. We use the infrared fixed point of the zero mode Hamiltonian to follow a trajectory in the space of couplings, where high resolution is synchronized with the light cone limit. Note, we consider the parameter associated with the frame dependence as yet another coupling constant. As shown in Ref. [8] the zero mode sector is also relevant to the definition of M-theory in light cone coordinates.

We choose the following near light cone coordinates which smoothly interpolate between the Lorentz and light front coordinates:

\[
x^t = x^+ = \frac{1}{\sqrt{2}} \left\{ \left( 1 + \frac{\eta^2}{2} \right) x^0 + \left( 1 - \frac{\eta^2}{2} \right) x^3 \right\},
\]

\[
x^- = \frac{1}{\sqrt{2}} (x^0 - x^3).
\] (1)

The transverse coordinates \(x^1, x^2\) are unchanged; \(x^t = x^+\) is the new time coordinate, \(x^-\) is a spatial coordinate. As finite quantization volume we will take a torus and its extension in \(\cdot\), as well as in \(1, 2\) direction is \(L\). The scalar product of two 4-vectors \(x\) and \(y\) is given with \(\vec{x}_\perp \vec{y}_\perp = x^1 y^1 + x^2 y^2\) as

\[
x_\mu y^\mu = x^- y^+ + x^+ y^- - \eta^2 x^- y^- - \vec{x}_\perp \vec{y}_\perp
\]

\[
= x^- y^+ + x^+ y^- + \eta^2 x_+ y_+ - \vec{x}_\perp \vec{y}_\perp.
\] (2)

Obviously, the light-cone is approached as the parameter \(\eta\) goes to zero. For non-zero \(\eta\), the transition to the coordinates introduced above can be formally identified as a Lorentz-boost combined with a linear transformation, which avoids time dependent boundary conditions [4]. The boost parameter \(\beta = v_3\) is given by

\[
\beta = \frac{1 - \eta^2/2}{1 + \eta^2/2},
\] (3)

indicating that for \(\eta^2 \to 0\) the relative velocity \(v_3 \to c (\equiv 1)\). Of course, this is connected to the well-known interpretation of the ’tilted’ light-cone frame in terms of the infinite momentum frame.

The choice of near light cone coordinates allows to quantize the theory on a space-like finite interval of length \(L\) in \(x^-\) at equal times, i.e. \(\Delta x^+ = 0\). The invariant length squared of this interval for \(\Delta x^2_\perp = 0\) is related to the length of the compact \(x^-\) dimension.

\[
\Delta s^2 = \Delta x^- \Delta x^+ + \Delta x^+ \Delta x^- - \eta^2 (\Delta x^-)^2 - \Delta x^2_\perp
\]

\[
= -\eta^2 L^2.
\] (4)
For simplicity we consider also the transverse dimensions periodic in $L$. Previous work of the St. Petersburg and Erlangen groups [3, 4] assumed a fixed tilted coordinate system with fixed transverse ultraviolet cut off. Our main purpose in this paper is to consider the zero mode fields on a transverse lattice with *varying* transverse lattice spacing $a$. We propose to approach light-cone dynamics by synchronizing the continuum limit $\Lambda = \pi/a \to \infty$ with the light-cone limit $x^+ \to \frac{1}{\sqrt{2}}(x^0 + x^3)$. The vacuum fluctuations in the transverse directions induce a second order phase transition which allows to have a continuum limit of the lattice theory. Thereby we can eliminate the cut off in a controlled way, preserving the nontrivial vacuum structure.

The gauge fixing procedure in the modified light-cone gauge $\partial^- A^- = 0$ involves zero modes dependent on the transverse coordinates. These zero mode fields carry zero linear momentum $p^-$ in near light cone coordinates, but finite amount of $p^0 + p_3$. They correspond to ”wee” partons in the language of the original parton model of Feynman. In $SU(2)$ the zero mode fields $a_-(x_\perp)$ are proportional to $\tau^3$, i.e. they can be chosen color diagonal. The use of an axial gauge is very natural for the light-cone Hamiltonian even more so than in the equal-time Hamiltonian. The asymmetry of the background zero mode naturally coincides with the asymmetry of the space coordinates on the light cone. The zero mode fields describe disorder fields. Depending on the effective coupling the zero mode transverse system will be in the massive or massless phase. Its second order phase transition allows to perform the continuum limit in the zero mode Hamiltonian. Our main conjecture is that the continuum limit and light cone limit can be realized simultaneously at this critical point. The evolution of the coupling determines the approach to the light cone with transverse resolution approaching zero. The resulting relation is reminiscent of the simple behavior one gets from considering naive scaling relations and dynamics in the infinite momentum frame. We will also calculate the contribution of the two dimensional zero modes to the ground state energy and show that it scales with the three dimensional volume. This estimate demonstrates very simply the relevance of modes with lower dimensionality to the full problem in the case of critical second order behavior.

The naive considerations for a simultaneous light cone and continuum limit go as follows.

We characterize the 'infinite' momentum frame by giving the momenta of the proton and photon in usual coordinates. The fast moving proton carries $(P, 0_\perp, P)$ and the photon $q = (\nu M^2, \sqrt{Q^2}, -\nu M^2)$, where $\nu$ is the energy transfer in the laboratory. The dominant contribution comes from energy conserving transitions, where the energy of the quark in the final state equals the sum of initial quark and photon energies. For collinear quarks with momentum $p = (x_B P, 0_\perp, x_B P)$, where $x_B$ denotes the momentum fraction, one explicitly finds
\[
\sqrt{Q^2 + \left(x_B P - \frac{\nu M}{2P}\right)^2} = x_B P + \frac{\nu M}{2P},
\]
which yields the "scaling variable" $x_B = \frac{Q^2}{2M\nu}$ as long as $P \geq Q/x_B$, i.e. the "infinite" momentum has to be large enough:
\[
\gamma M = P \geq Q = 1/a.
\]
At the same time, the photon has a wavelength $\propto 1/Q$ which resolves transverse details of size $a$ in the hadronic wavefunction. By eq. (6), the $\gamma$ factor of the "infinite" momentum is related to $\eta$. Therefore we expect that the parameter $\eta$ approaches zero with the transverse
One of the objectives of this paper is to derive the precise relation between $\eta$ and $a$. We will simulate the zero mode dynamics on a $(2 + 1)$ dimensional lattice to find the fixed point of the effective coupling. The light-like limit is governed by an infrared fixed point (cf. ref. [8]). Most previous discussions of the zero mode problem have been on the classical equations of motion level, here we will work on the fully nonperturbative quantum level. Our work is not directly related to M-theory, but some of the consequences may be relevant for other studies with light like compactification.

2 Near Light Cone QCD Hamiltonian

In Ref. [2] the near light cone Hamiltonian has been derived. A similar Hamiltonian has been obtained in equal time coordinates [9,10]. Here we will sketch the derivation. We restrict ourselves to the color gauge group $SU(2)$ and dynamical gluons; only an external (fermionic) charge density $\rho_m$ is considered here.

Since the $A^a_\perp$ coordinates have no momenta conjugate to them, the Weyl gauge $A^a_\perp = 0$ is the starting point for a canonical formulation. The canonical momenta of the dynamical fields $A^a_- , A^a_i$ are given by

$$\Pi^a_- = \frac{\partial L}{\partial F^a_+} = F^a_+ ,$$

$$\Pi^a_i = \frac{\partial L}{\partial F^a_+} = F^a_- + \eta^2 F^a_+ .$$

From this, we get the Weyl gauge Hamiltonian density

$$\mathcal{H}_W = \frac{1}{2} \Pi^a_+ \Pi^a_- + \frac{1}{2} F^a_2 F^a_2 + \frac{1}{2\eta^2} \sum_{i=1,2} \left( \Pi^a_i - F^a_ - \right)^2 .$$

The Hamiltonian has to be supplemented by the original Euler–Lagrange equation for $A_+$ as constraints on the physical states (Gauss’ Law constraints)

$$G^a_+(x_\perp, x^-) |\Phi\rangle = \left( D^{ab}_+ \Pi^b_- + D^{ab}_+ \Pi^b_+ + g \rho^a_m \right) |\Phi\rangle$$

$$= \left( D^{ab}_- \Pi^b_- + G^a_\perp \right) |\Phi\rangle = 0 .$$

In order to obtain a Hamiltonian formulated in terms of unconstrained variables, one has to resolve the Gauss’ Law constraint. Via unitary gauge fixing transformations [9,10] a solution of Gauss’ Law with respect to components of the chromo–electric field $\Pi_-$ can be accomplished. This gives a Hamiltonian independent of the conjugate gauge fields $A_-$, i.e. the latter become cyclic variables. Classically this would correspond to the light front gauge $A_- = 0$. However, this choice is not legitimate if we want to consider the theory in a finite box. Instead, the (classical) Coulomb light front gauge $\partial_- A_- = 0$ is compatible with gauge invariance and periodic boundary conditions. The reason is that $A_-$ carries information on the (gauge invariant) eigenvalues of the spatial Polyakov line matrix

$$\hat{P}(x_\perp) = P \exp \left[ ig \int dx^- A_-(x_\perp, x^-) \right] ,$$

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where $P$ is the path ordering at infinity.
which can be written in terms of a diagonal matrix with the zero mode field $a^3_\perp(x_\perp) = a_\perp(x_\perp)$

\[ \hat{P}(x_\perp) = V \exp [igLa_\perp(x_\perp)] V^\dagger. \]  

(12)

Obviously we have to keep these ‘zero modes’ $a_\perp(x_\perp)$ as dynamical variables, while the other components of $A_\perp$ are eliminated. The zero mode degrees of freedom are independent of $x^-$ and, therefore, correspond to quantities with zero longitudinal momentum $p_-$. 

In order to eliminate the momentum $\Pi_-$, conjugate to $A_\perp$, by means of Gauss’ Law, one needs to ‘invert’ the covariant derivative $D_\perp$ which simplifies to $d_\perp = \partial_\perp - ig[a_\perp,...]$. On the space of physical states one can simply make the replacement¹

\[ \Pi_-(x_\perp,x_-) \rightarrow p_-(x_\perp) - \left( d_-^{-1} \right) G_\perp(x_\perp,y^-). \]

(13)

The operator $p_-(x_\perp)$ is also diagonal and $p^3_\perp(x_\perp)$ is the momentum conjugate to the zero mode $a^3_\perp(x_\perp)$. It has eigenvalue zero with respect to $d_-$, i.e. $d_- p_- = 0$, and is therefore not constrained.

The appearance of the zero modes implies a residual Gauss’ Law, which arises from the $x_-$ integration over eq. (10) for the $a = 3$ component using periodic boundary conditions in the $x_-$ direction. This constraint on two dimensional fields can be handled in full analogy to QED, since it concerns the diagonal part of color space. A further Coulomb gauge fixing in the $SU(2)$ 3–direction eliminates the color neutral, $x_-$–independent, two–dimensional longitudinal gauge field in favor of a neutral chromo–electric field

\[ e_\perp(x_\perp) = g\nabla_\perp \int dy^-dy_\perp d(x_\perp - y_\perp) \left\{ f^{3ab}A^a_\perp(y_\perp,y^-)\Pi^b_\perp(y_\perp,y^-) + \rho^3_m(y_\perp,y^-) \right\} \frac{\tau^3}{2}. \]

(14)

Here we use the periodic Greens function of the two dimensional Laplace operator

\[ d(z_\perp) = -\frac{1}{L^2} \sum_{\vec{n} \neq 0} \frac{1}{p^2_\vec{n}} e^{i\vec{n}z_\perp}, \quad p_n = \frac{2\pi}{L} \vec{n}, \]

(15)

where $\vec{n} = (n_1, n_2)$ and $n_1, n_2$ are integers.

As a remnant of the local Gauss’ Law constraints, a global condition

\[ Q^3|\Phi'\rangle = \int dy^- dy_\perp \left\{ f^{3ab}A^a_\perp(y_\perp,y^-)\Pi^b_\perp(y_\perp,y^-) + \rho^3_m(y_\perp,y^-) \right\} |\Phi'\rangle = 0 \]

(16)

emerges. The physical meaning of this equation is that the neutral component of the total color charge, including external matter as well as gluonic contributions, must vanish in the sector of physical states. It can be shown (see [2]) that there must be $\hat{Q}_{12}(x_\perp) = 0$ everywhere in transverse space in order to avoid an infinite Coulomb energy. The two conditions together suggest that physical states have to be color singlets.

The final Hamiltonian density in the physical sector explicitly reads (in terms of unconstrained $A_\perp$ and $\Pi_\perp$ obtained after a shift is made by subtracting averages) [2]

\[ H = \text{tr} \left[ \partial_\perp A_2 - \partial_2 A_1 - ig[A_1, A_2] \right]^2 + \frac{1}{\eta^2} \text{tr} \left[ \Pi_\perp - (\partial_\perp A_1 - ig[a_\perp, A_\perp]) \right]^2 \]

\[ + \frac{1}{\eta^2} \text{tr} \left[ \frac{1}{L} e_\perp - \nabla_\perp a_\perp \right]^2 + \frac{1}{2L^2} p^3_\perp(x_\perp)p^2_\perp(x_\perp) \]

\[ + \frac{1}{L^2} \int_0^L \! dz^- \int_0^L \! dy^- \sum_{p,q,n} \left[ G_{\perp pq}(x_\perp, z^-)G_{\perp pq}(x_\perp, y^-) \right] e^{i2\pi n(z^- - y^-)/L}, \]

(17)

¹The inversion of $d_-$ can be explicitly performed in terms of its eigenfunctions, cf. [1].
where $p$ and $q$ are matrix labels for rows and columns, $a_{\pm q} = (a_{\pm})_{qq}$ and the prime indicates that the summation is restricted to $n \neq 0$ if $p = q$. The operator $G_\perp (x_\perp, x^-)$ is defined as

$$G_\perp = \nabla_\perp \Pi_\perp + gf^{abc} \tau^a A^b_\perp \left( \Pi_\perp^c - \frac{1}{L_\perp} e_\perp^c \right) + g \rho_m .$$ (18)

The last two terms of the Weyl gauge Hamiltonian come from the original term $\Pi_\perp^2$ with squared electric field strengths in $x^-$ direction. After elimination of $\Pi_\perp$ the zero mode part and the light cone Coulomb energy in the axial gauge remain. In the Coulomb term one sees the role of the zero mode fields as infrared regulators of the spatial momenta $p_\perp = 2\pi n/L$ which are quantized due to the compact interval $L$. Since the two dimensional theory has been Coulomb gauge fixed, the electric field $\vec{e}_\perp$ replaces the canonical momentum of the longitudinal, neutral gauge field which has been eliminated.

We note that the terms containing $\vec{e}_\perp$ and the momentum $\vec{\Pi}_\perp$, have the pre-factor $1/\eta^2$. Physically this pre-factor signals the increase of transverse electric energies with the boost factor $\gamma = (\sqrt{2}\eta)^{-1}$. The boost also couples transverse electric fields with transverse magnetic fields. In the light-cone limit the pre-factor diverges and the adjacent brackets become constraint equations. This reflects the corresponding reduction of the number of degrees of freedom if one goes exactly on the light cone. We do not follow this procedure, but keep a finite $\eta$ as a kind of Lagrange parameter.

A characteristic feature of an exact light-cone formulation is the triviality of the ground state. This may simplify explicit calculations, e.g. of the hadron spectrum. However, the light cone vacuum is definitely not trivial in the zero mode sector. In fact, already in ref [2] we have shown strong and weak coupling solutions of the zero mode Hamiltonian. Massless modes influence the dynamics in the light cone limit, whereas massive modes decouple when $\eta \to 0$. In contrast to earlier work, we solve the zero mode Hamiltonian in this paper numerically showing the transition from the massive phase to the massless phase.

The zero mode degrees of freedom $a_\perp$ couple to the transverse three dimensional gluon fields $A_\perp$ via the second magnetic term in $\mathcal{H}$, the Coulomb term and directly via the electric field $\vec{e}_\perp$. We remark that the transverse electric fields $\Pi_\perp$ and $\vec{e}_\perp$ are dual to the magnetic fields $\partial_\perp A_\perp$ and $\nabla_\perp a_\perp$. This duality is typical for the light cone and is absent in the equal time case. Since duality plays an important role in supersymmetric QCD its role in light cone theories should deserves to be investigated in greater detail.

In the following we neglect the couplings between the three and two dimensional fields and consider the pure zero mode Hamiltonian

$$h = \int d^2x \left[ \frac{1}{2L} p^3_\perp(\vec{x}_\perp) p^3_\perp(\vec{x}_\perp) + \frac{L}{2\eta^2}(\nabla_\perp a^3_\perp)^2 \right].$$ (19)

The global constraint, eq. (18), does not contain $a_\perp$ and $p_\perp$ and, consequently, is irrelevant for the time being. Even at this level of severe approximations the zero mode Hamiltonian differs from the corresponding one in QED. The reason is the hermiticity defect of the canonical momentum $p^\perp$. In the Schrödinger representation eq. (19)

$$h = \int d^2x \left[ -\frac{1}{2L} J(a^3_\perp(\vec{x}_\perp)) \frac{\partial}{\partial a^3_\perp(\vec{x}_\perp)} J(a^3_\perp(\vec{x}_\perp)) \frac{\partial}{\partial a^3_\perp(\vec{x}_\perp)} + \frac{L^2}{2\eta^2}(\nabla_\perp a^3_\perp)^2 \right],$$ (20)

contains the Jacobian $J(a_\perp)$ which equals the Haar measure of $SU(2)$

$$J(a^3_\perp(\vec{x}_\perp)) = \sin^2 \left( \frac{gL}{2} a^3_\perp(\vec{x}_\perp) \right).$$ (21)
It stems from the gauge fixing procedure, effectively introducing curvilinear coordinates. It also appears in the functional integration volume element for calculating matrix elements. It is convenient to introduce dimensionless variables
\[
\varphi(\vec{x}_\perp) = \frac{g L a^3}{2} (\vec{x}_\perp),
\]
which vary in a compact domain \(0 \leq \varphi \leq \pi\). We regularize the above Hamiltonian \(h\) by introducing a lattice spacing \(a\) between transversal lattice points \(\vec{b}\). Next we appeal to the physics of the infinite momentum frame and factorize the reduced true energy from the Lorentz boost factor \(\gamma = \sqrt{2/\eta}\) and the cut off by defining \(h_{\text{red}}\)
\[
h = \frac{1}{2 \eta a} h_{\text{red}}
\]
In the continuum limit of the transverse lattice theory we let \(a\) go to zero. For small lattice spacing we obtain the reduced Hamiltonian
\[
h_{\text{red}} = \sum_{\vec{b}} \left\{ -g_{\text{eff}}^2 \frac{1}{J} \frac{\partial}{\partial \varphi(\vec{b})} J \frac{\partial}{\partial \varphi(\vec{b})} + \frac{1}{g_{\text{eff}}^2} \sum_{i=1,2} \left( \varphi(\vec{b}) - \varphi(\vec{b} + \vec{e}_i) \right)^2 \right\}
\]
with the effective coupling constant
\[
g_{\text{eff}}^2 = \frac{g^2 L \eta}{4a}
\]
The first part of the Hamiltonian contains the kinetic (electric) energy of the \(SU(2)\) rotators on a half circle at each lattice point and the second part gives the potential (magnetic) energy of these rotators due to the differences of angles at nearest neighbor sites. For further discussion we define these electric and magnetic parts as:
\[
h_{\text{red}} = h_e + h_m
\]
\[
h_e = \sum_{\vec{b}} h_{\vec{b}}^e
\]
\[
h_m = \sum_{\vec{b}} \sum_{i=1,2} h_{\vec{b},\vec{b}+\vec{e}_i}^m
\]
In the strong coupling domain where we have analytical solutions of the zero mode Hamiltonian, the numerical Hamiltonian lattice theory agrees with the analytical solutions for the mass gap. The real question concerns the continuum limit of the zero mode system which occurs outside of the strong coupling region. For this purpose we have to find the region of vanishing mass gap for the lattice Hamiltonian.

3 Lattice Calculation of the Zero Mode Hamiltonian

We solve the equivalent lattice theory in a Euclidean formulation. The zero mode Hamiltonian represents a 2 + 1 dimensional theory in two spatial and one time direction. The lattice has \(N_x a = N_y a\) extensions in transverse space and \(N_T \Delta \tau = T\) extension in near light cone time. To set up the density matrix one has to write down the Trotter formula for the given Hamiltonian. Using \(h_{\text{red}} = h_e + h_m\) we have
\[
\exp(-T h_{\text{red}}) = \lim_{N_T \to \infty} \left[ \exp(-\Delta \tau h_m/2) \exp(-\Delta \tau h_e) \exp(-\Delta \tau h_m/2) \right]^{N_T},
\]
where each time evolution step $\Delta \tau$ can be separately done for the electric and magnetic part of the Hamiltonian. For definiteness, we choose $\Delta \tau = a/2$ in all the following. In the Appendix we show that this choice of $\Delta \tau$ optimizes the updating procedure since it generates approximately equal widths for the weight functions resulting from the kinetic and potential energies. The different time slices will labeled with the index $l$. The electric Hamiltonian can be evaluated by inserting products of complete set single site eigenfunctions $C_n$ (cf. [2]). Practically a maximal number $N_{\text{max}} = 100$ of eigenfunctions is fully sufficient to reach convergence in the interval of couplings we need.

$$
\langle \varphi_{l+1}|h^e|\varphi_l \rangle = \sum_{n_{l+1},n_l} \langle \varphi_{l+1}|n_{l+1}\rangle \langle n_{l+1}|h^e|n_l \rangle \langle n_l|\varphi_l \rangle \approx \sum_{n_l=0}^{N_{\text{max}}} C_{n_l}(\varphi_{l+1}) g_{\text{eff}}^2 n_l(n_l + 2) C_{n_l}(\varphi_l) \quad (28)
$$

with the (single site) eigenfunctions and eigenvalues given as:

$$
C_{n_l}(\varphi_l) = \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin((n_l + 1)\varphi_l)}{\sin \varphi_l} \right\}, \quad (29)
$$

$$
h^e C_{n_l}(\varphi_l) = g_{\text{eff}}^2 n_l(n_l + 2) C_{n_l}(\varphi_l). \quad (30)
$$

The eigenfunctions form an orthonormal set with respect to a scalar product which contains the Jacobian in the measure. The Jacobian $J(\varphi(\vec{b}))$ has been defined above (21). The magnetic part is diagonal in $\{\varphi(\vec{b})\}$ i.e. local in time:

$$
\langle \{\varphi_l(\vec{b})\}|h_m|\{\varphi_l(\vec{b})\} \rangle = h_m(\{\varphi_l(\vec{b})\}) = \sum_{\vec{b},\vec{b}'} \sum_{i=1,2} h_{\vec{b},\vec{b}+\vec{e}_i}^m (\varphi_l(\vec{b}) - \varphi_l(\vec{b}+\vec{e}_i)) \quad (31)
$$

The full partition function is given by an integral over all time slices

$$
Z = \text{tr} \exp[-T h_{\text{red}}] = \int \prod_{\vec{b}} \left( J(\varphi(\vec{b})) d\varphi(\vec{b}) \right) \langle \{\varphi(\vec{b})\}| \exp[-T h_{\text{red}}]|\{\varphi(\vec{b})\} \rangle \quad (32)
$$

$$
= \int \prod_{\vec{b},l} \left( J(\varphi(\vec{b})) d\varphi(\vec{b}) \sum_{n_l=0}^{N_{\text{max}}} C_{n_l}(\varphi_{l+1}(\vec{b})) \exp \left[ -g_{\text{eff}}^2 n_l(n_l + 2) \Delta \tau \right] C_{n_l}(\varphi_l(\vec{b})) \right) \times \prod_l \exp \left[ -h_m(\{\varphi_l(\vec{b})\}) \Delta \tau \right].
$$

Because of the Jacobian the dominant contributions to the partition function come from $\varphi_l$ values around $\frac{\pi}{2}$. The Hamiltonian is invariant under reflections $\varphi_l - \frac{\pi}{2} \rightarrow \frac{\pi}{2} - \varphi_l$. In the strong coupling limit, where $g_{\text{eff}}^2$ is large, the half rotators act almost independently on each lattice site producing a large mass gap. The system is disordered. For decreasing coupling constant $g_{\text{eff}}^2$ the movement of the individual rotators becomes locked from one site to the next. Long range correlations develop. The order parameter (or “magnetization”) of the system is the expectation value of the trace of the Polyakov line (11), which on the lattice has the form

$$
P = \frac{1}{2} \text{tr} \vec{P} = \frac{1}{N^2} \sum_{\vec{b},\tau} \cos \varphi(\vec{b},\tau) \quad (33)
$$

The order parameter is odd under the above symmetry operation. Operator expectation values are evaluated with the density matrix defined by $h_{\text{red}}$ and $T$:

$$
\langle O \rangle = \frac{1}{Z} \text{tr} \langle O \exp[-T h_{\text{red}}] \rangle. \quad (34)
$$
At each particular $\beta_g$

$$\beta_g \equiv 1/g_{\text{eff}}^2 \quad (35)$$

we simulate lattice sizes with equal number of sites in all directions $N = N_x = N_y = N_T$ and $N = 4, 6, 8, 12, 16$. A Metropolis algorithm is used for updating with the steps size and the number of hits adapted to $\beta_g$. In order to tabulate the Boltzmann weights related to the timelike links we also discretized the continuous angle variables to a system of $N_\varphi = 60$ orientations: $\varphi_j = j\pi/N_\varphi$, $j = [0, N_\varphi]$.

In this explorative investigation we generated between 5000 and 50 000 uncorrelated configurations depending on the $\beta_g$ and lattice size. All our calculations were done on a cluster of AlphaStations with the single site Metropolis updating algorithm sketched above, so the accuracy can be improved using a more powerful algorithms and/or more computing resources.

We calculate the $\beta_g$ dependence of the following quantities: average electric energy $\varepsilon_e$ and magnetic energy $\varepsilon_m$ per site, the average of the absolute value of the order parameter $\langle |P| \rangle$, the susceptibility $\chi$ and the normalized fourth cumulant $g_r$ (which gives the deviation of the moments of the Polyakov expectation value from a pure Gaussian behavior):

$$\varepsilon_{e,m} = \frac{1}{N^2} \langle h_{e,m} \rangle \quad (36)$$

$$\chi = N^3 \left( \langle P^2 \rangle - \langle P \rangle^2 \right) \quad (37)$$

$$g_r = \frac{\langle P^4 \rangle}{\langle P^2 \rangle^2} - 3. \quad (38)$$

Firstly we calculated the ground state energy for strong couplings $g_{\text{eff}}^2$ and compared with the exact calculation [2]. The obtained results from the lattice agree with the analytical result. For higher values of $\beta_g$ the lattice calculation agrees within $10-20\%$ with the previous effective double site calculation for the energy per site [2]. The ultraviolet regularization with $\Delta \tau$ influences the final lattice result. The energies are measured with high accuracy and practically do not depend on the infrared cutoff, i.e. on the total lattice size for $N \geq 8$. The situation for other variables is different.

In order to see the emergence of a massless phase, we investigated the (connected) time correlation functions of the (trace of the) Polyakov line operators:

$$K(0, \tau) = \frac{1}{N^4} \left\langle \sum_{\vec{b},\vec{b}'} \cos(\varphi(\vec{b}, 0)) \cos(\varphi(\vec{b}', \tau)) \right\rangle - \langle |P| \rangle^2 \quad (39)$$

The correlation masses are obtained from a fit of the time correlation functions $K(0, \tau)|\tau = n\Delta \tau$ to a parameterization taking the periodicity in time into account

$$K(0, n\Delta \tau) = c \cosh \left[ m \Delta \tau \left( n - \frac{N_T}{2} \right) \right] \quad (40)$$

The obtained masses for different lattice sizes are shown in Fig.1a. For each fixed $\beta_g$ we extrapolated the mass to the limit $N \to \infty$ using a linear $1/N$ parameterization

$$m(\beta_g, N) = m_0(\beta_g) + \frac{C(\beta_g)}{N} \quad (41)$$
Our data for $m(\beta_g, N)$ can be fitted with this linear form (41) for $\beta_g > \sim 3$. The resulting dependence of $m_0$ on $\beta_g$ is presented in Fig.1b. The mass vanishes in the infinite volume limit near $\beta \approx 5$, which is a first guess for the critical coupling $\beta_g^\ast$.

In order to extract more exact information on the critical behavior

$$m(\beta_g) \propto \left( \frac{1}{\beta_g} - \frac{1}{\beta_g^\ast} \right)^\nu .$$  \hspace{1cm} (42)

from our data on small systems we have done a Finite Size Scaling (FSS) [12,13] analysis of the variables $\langle |P| \rangle$, $\chi$ and $g_r$, searching for the critical coupling $\beta_g^\ast$ in the interval $3 \leq \beta_g \leq 9$. We could determine the critical indices $\beta$, $\gamma$ and $\nu$ using an approach employed for $SU(2)$ gauge theory at the finite temperature transition by Engels et al. [14]. The general form of the scaling relations for a variable $V$ ($V = \langle |P| \rangle, \chi, g_r$) is

$$V(t, N) = N^{\rho/\nu} F_V \left( tN^{1/\nu}, g_iN^\gamma \right) ,$$  \hspace{1cm} (43)

where $\rho$ is the corresponding critical index ($\beta, \gamma, 0$) for the respective quantities and $t$ is the reduced inverse coupling $\beta_g$: \[ t = \frac{\beta_g^\ast - \beta_g}{\beta_g} . \]  \hspace{1cm} (44)

In practice eq. (43) is computed near $t = 0$. Expanding up to first order in $t$ and taking into account only the largest irrelevant exponent $y_1 \equiv -\omega$ one obtains

$$V(t, N) = N^{\rho/\nu} \left[ c_0 + \left( c_1 + c_2 N^{-\omega} \right) tN^{1/\nu} + c_3 N^{-\omega} \right] .$$  \hspace{1cm} (45)
With this parameterization we analyze our data for lattice sizes $N = 4, 6, 8, 12, 16$. The $\beta_g$ dependence in eq. (45) is valid near the critical point. The quality of the fit to the data in different intervals on $\beta_g$ can serve as a guide to localize the critical point $\beta_g^*$. We find that in our case this parameterization can be applied for $\langle |P| \rangle$ and $\chi$ only for $\beta_g > \sim 5$: for smaller $\beta_g$ the $\chi^2/D.F.$ becomes too large. Finally we use the parameterization of eq. (45) in the interval $5 \leq \beta_g \leq 9$ (see Fig 2) where $|t| \lesssim 0.3$ and an expansion of first order in $t$ still is applicable. In this $\beta_g$ interval the fits for $\langle |P| \rangle$ and $\chi$ yield a $\chi^2/D.F. \sim 2$. To have a reliable error estimate for the extracted parameters we use the jackknife method. The results for $\beta_c^*$ and critical indices $\beta$, $\gamma$ and $\nu$ are presented in Table 1. We use $\omega = 1$, but in fact the results do not depend on $\omega$ in the interval $0.8 \leq \omega \leq 1.2$.

It should be noticed that the system frequently flips between the two ordered states on a finite lattice. Therefore the expectation value $\langle P \rangle$ in equation (37) vanishes with good accuracy. Analogously to the treatment of the magnetization in the 3-dimensional Ising model, we should - instead of the expression for $\chi$ in eq. (37) - define the susceptibility in the broken phase as $\chi_{\text{broken}} = N^3 (\langle P^2 \rangle - \langle |P| \rangle^2)$. For the Ising case this susceptibility is supposed to converge towards the correct infinite volume limit. Due to lack of statistics, however, we did not use separate expressions for $\chi$ in the two phases. Our data for $g_r$ are even much less accurate which excludes the possibility to use them in the FSS analysis.

The values found for the critical indices $\beta$, $\gamma$ and $\nu$ are not far from those of the Ising model ($D=3$) (see Table 1). They are much less accurate but also in agreement with a high statistics analysis [14] of the $SU(2)$ deconfinement transition at finite temperature. We interpret the behavior of the light cone zero mode theory as a consequence of the underlying $Z(2)$ symmetry of the Hamiltonian. $Z(2)$ transformations correspond to reflections of the

Figure 2: Comparison of the theoretical parameterizations given in eqs. (44-45) for $\langle |P| \rangle$ and $\chi$ with the lattice data results as a function of $\beta_g$ for different lattice sizes $N$. 

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\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
$V$ & Transverse QCD (2 + 1) & Ising ($D = 3$) \\
\hline
$< |P| >$ & $\beta$ & $0.21 \pm 0.06$ & $0.3267(10)$ \\
& $\nu$ & $0.57 \pm 0.06$ & $0.6289(8)$ \\
& $\beta_g^*$ & $6.3 \pm 1.1$ & \\
\hline
$\chi$ & $\gamma$ & $1.13 \pm 0.36$ & $1.239(7)$ \\
& $\nu$ & $0.52 \pm 0.10$ & $0.6289(8)$ \\
& $\beta_g^*$ & $5.7 \pm 1.2$ & \\
\hline
\end{tabular}
\caption{Critical parameters for the transverse QCD near the light cone. The Ising indices are from ref. \cite{15}.}
\end{table}

angle variables $\varphi$ around $\frac{\pi}{2}$. The Hamiltonian and the measure of eq. (17) are invariant under these transformations. The resulting critical behavior is common to Ising like models. The critical coupling itself is a nonuniversal quantity and we have found a rough value. More extended work with the lattice Hamiltonian near the light cone is needed to clarify the continuous transition further.

4 Conclusions

The scaling analysis gives indications that there is a second order transition between a phase with massive excitations at strong coupling and a phase with massless excitations in weak coupling. To reach a higher accuracy large scale simulations of the Hamiltonian zero mode system are needed. In this context it is advisable to treat the coupled system of three and two dimensional modes together. A calculation in the epsilon expansion \cite{11} gives the zero of the $\beta$-function as an infrared stable fixed point. This shows that the limit of large longitudinal dimensions $L$ is well defined. Using the running coupling constant $g_{\text{eff}}^2$ of the zero mode system we have (cf. (42))

$$ m a = \frac{1}{\zeta_0 g_{\text{eff}}^2} (g_{\text{eff}}^2 - g_{\text{eff}}^{*2})^\nu $$

with $g_{\text{eff}}^{*2} = 0.17 \pm 0.03$ and $\nu = 0.56 \pm 0.05$ from our lattice calculations (cf. Table 1). The mass can be interpreted as an inverse correlation length $ma = a/\xi$. When the correlation length $\xi$ approaches the temporal size of the lattice $\xi \approx N_T a/2$ in the infrared limit we can identify $ma$ with $a/L$, where $L$ is our infrared length scale. Therefore the effective coupling $g_{\text{eff}}^2 = g^2 \eta \frac{L^2}{4a}$ (cf. (23)) runs with $a/L$ in the following way

$$ g_{\text{eff}}^2 = g_{\text{eff}}^{*2} + (\zeta_0 g_{\text{eff}}^{*2})^{1/\nu} \left( \frac{a}{L} \right)^{1/\nu}. $$

The coupling to the three dimensional modes produces the usual evolution of $g^2$ in $SU(2)$ QCD, where the coupling $g^2(a)$ at the lattice scale $a$ is related to the coupling $g_0^2$ defined at
the infrared scale $\approx L$ as follows

$$g^2(a) = \frac{g_0^2}{1 + \frac{g_0^2}{4\pi^2} \frac{22}{3} \log \frac{L}{a}}.$$  \hfill (48)

Combining the eqs. (47) and (48) we can synchronize the approach to the light cone, i.e. the limit $\eta \to 0$ with the continuum limit $a \to 0$. The condition that the three-dimensional evolution of $g^2$ has to be compatible with the two-dimensional evolution of $g_{\text{eff}}^2$ towards $g_{\text{eff}}^2$ yields that for $a \to 0$ the light cone parameter $\eta$ approaches zero as

$$\eta(a) \sim \frac{g_{\text{eff}}^2}{\pi^2} \frac{22}{3} \frac{a}{L} \log \frac{L}{a}.$$  \hfill (49)

This relation is similar to the naive scaling result deduced in section 1 from physics arguments besides logarithmic modification. Although at the outset we had as parameters $N_L = L/a$ and $\eta$ in the Hamiltonian we reduce this multiparameter problem to a problem with one single coupling constant which can be chosen as $g$.

We can use the energy and the critical behavior of the Euclidean 2+1 dimensional system to obtain its contribution to the ground state energy of the system in 3 + 1 dimensions. In our calculations the energy of the zero mode Hamiltonian $\langle h \rangle$ near the critical coupling is proportional to $N_L^2/\eta$. Using the scaling of $\eta$ determined from the compatibility of 2-dimensional and 3-dimensional dynamics, we can eliminate $\eta$ in favor of $N_L = L/a$ and get a zero mode energy which grows linearly with the 3-dimensional volume. That means the zero mode dynamics becomes relevant for the full problem, i.e.

$$\langle ha \rangle \sim N_L N_L^2.$$  \hfill (50)

In reality the zero mode system is coupled to the 3 + 1 dimensional system the extension of which is large. So we expect that the couplings of the 2 + 1 dimensional system can get modified. Because of the universal dynamics near the infrared fixed point this will not change the qualitative form of the volume dependence obtained in eq. (50). The behavior of the zero mode Hamiltonian has to be taken into account when we try to solve for the infinite momentum frame solutions of the complete Hamiltonian. The discussion \cite{1,16} until now often centers on the order of limits. In order to guarantee a meaningful limit of the near light cone theory, one first has to take $L \to \infty$ and then one can take $\eta \to 0$. This is a workable procedure in solvable 1+1 dimensional models. In QCD in 3 + 1 dimensions it hardly can serve as a prescription. Numerical solutions have to be obtained with a finite cutoff and the continuum limit can only be reached approximately via scaling relations. It is in this case that the role of zero mass excitations will crucially enter the physics. As one sees above the synchronization of the continuum limit with the light cone limit leads to results which are independent of the near light cone parameter $\eta$. The independence of the ground state energy near the critical coupling gives us back Lorentz invariance which states that the physics should not depend on the reference frame. The vacuum energy is not allowed to depend on $\eta$. There are dynamical zero modes in near light cone QCD which contain the physics of the nonperturbative QCD vacuum, the physics of the gluon and quark condensates. The nontrivial structure of the nonabelian gauge theory enters in our calculation through the kinetic operator of the rotators which at finite resolution have a finite mass gap vanishing only in the continuum limit.
The near light cone description of QCD based on a modified axial gauge \( \partial_+ A_+ = 0 \) yields a very natural formulation of high energy scattering. Polyakov variables near the light cone have already entered the calculations of high energy cross sections in two different ways: For the geometrical size of cross sections in ref. [6, 7] the correlations between Polyakov lines along the projectile and target directions are important. For this situation an interpolating gauge may be useful. For the energy dependence of cross sections an evolution equation of Polyakov line correlators approaching the light cone has been discussed in ref. [17] which generalizes the BFKL equation. This situation is close to the treatment in this paper. Now the decisive step is to back up our theoretical work on effective transverse QCD dynamics near the light cone with phenomenological consequences for high energy scattering.

Appendix

A choice of \( \Delta \tau \) is proposed by comparing Monte-Carlo weights \( W_t \) and \( W_s \) which link nearest neighbors in time and space respectively

\[
W_t(\varphi_1, \varphi_2) = \langle \varphi_1(x, y, \tau + 1) | e^{-\Delta r h_t} | \varphi_2(x, y, \tau) \rangle \sqrt{J(\varphi_1)J(\varphi_2)}, \quad (51)
\]

\[
W_s(\varphi_1, \varphi_2) = \langle \varphi_1(x + 1, y, \tau) | e^{-\Delta r h_m} | \varphi_2(x, y, \tau) \rangle. \quad (52)
\]

Figure 3: Weights \( W_t(\varphi_1, \varphi_2) \) of time-like and \( W_t(\varphi_1, \varphi_2) \) of space-like links for different \( \Delta \tau \). Figures (a,b,c) correspond to \( \Delta \tau = a/10, a/2, a \).

The probability distribution for the update of the field \( \varphi(x, y, \tau) \) is given by the product of weights for all links connecting it to neighbors in time and space. So the updating procedure for \( \varphi \) is more effective when \( W_t \) and \( W_s \) have similar distributions near the angle \( \varphi = \pi/2 \) where the Jacobian has its maximum. For \( \Delta \tau = a/10 \) (cf. Fig. 3a) the spatial weights \( W_s \) have a much wider distribution than the timelike weights \( W_t \), whereas for \( \Delta \tau = a \) (cf. Fig. 3c) the situation is reverse. For the intermediate case \( \Delta \tau = a/2 \) (cf. Fig. 3b) the widths of the distributions are very similar, therefore the updating procedure is optimal.

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