SATURATED CONFIGURATION AND NEW LARGE CONSTRUCTION OF EQUIANGULAR LINES

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Abstract. A set of lines through the origin in Euclidean space is called equiangular when any pair of lines from the set intersects with each other at a common angle. We study the maximum size of equiangular lines in Euclidean space and use graph theoretic approach to prove that all the currently known construction for maximum equiangular lines in $\mathbb{R}^d$ cannot add another line to form a larger equiangular set of lines if $14 \leq d \leq 20$ and $d \neq 15$. We give new constructions of large equiangular lines which are 248 equiangular lines in $\mathbb{R}^{42}$, 200 equiangular lines in $\mathbb{R}^{41}$, 168 equiangular lines in $\mathbb{R}^{40}$, 152 equiangular lines in $\mathbb{R}^{39}$ with angle $1/7$, and 56 equiangular lines in $\mathbb{R}^{18}$ with angle $1/5$.

1. Introduction

A set of lines through the origin in Euclidean space is called equiangular when any pair of lines from the set intersects with each other at a common angle. The study of equiangular lines often refers to equiangular tight frames (ETFs) which are the maximum separation codes on sphere attaining Welch bounds $[24, 9, 10]$. ETFs are highly related to strongly regular graphs $[3, 23]$. The problem of determining the maximum cardinality $N(d)$ of equiangular lines in $\mathbb{R}^d$ was extensively studied for the last 70 years. It is not hard to see that $N(2) = 3$ which is realized by the diagonal of a regular hexagon. The problem becomes not trivial for the higher dimension. Haantjes $[14]$ in 1948 who first showed that $N(3) = N(4) = 6$. The maximum equiangular line in $\mathbb{R}^3$ is the six diagonals of icosahedron with angle $\arccos(\frac{1}{\sqrt{5}})$. In 1966, van Lint and Seidel $[21]$ determined $N(d)$ for $5 \leq d \leq 7$. In 1973, Lemmens and Seidel $[15]$ extended the knowledge up to dimension 23. The methods to give upper bounds for equiangular lines are diverse. Barg and Yu $[4]$ used the semidefinite programming method. Greaves et al. $[12]$ studied it by the analysis of Seidel matrix. Balla et al. $[2]$ relied on probability and Ramsey theory to achieve asymptotic bounds. Glazyrin et al. $[11]$ suggested zonal spherical function on derived sets to obtain better bounds for infinitely many dimensions and so many other methods. Gerzon $[15]$ proved that $N(d) \leq \frac{d(d+1)}{2}$. However, most of the cases are far away from this bound. Currently only dimensions $d = 2, 3, 7$ and 23 attain this bound and the equality holds if and only if tight spherical 5-designs exist $[8]$. Notice that the maximum equiangular lines in $\mathbb{R}^7$ and $\mathbb{R}^{23}$ are universal optimal codes $[6]$ and they are subsets of $E_8$ root system and Leech lattice respectively. $E_8$ root system was recently shown to be the solution of sphere packing in $\mathbb{R}^8$ $[22]$, so as Leech lattice for $\mathbb{R}^{24}$ $[7]$.

To the best of our knowledge, the ranges of $N(d)$ for $2 \leq d \leq 43$ are listed in Table 1 (see $[1, 4, 13, 18, 26]$).

| $d$ | 2 | 3–4 | 5 | 6 | 7–13 | 14 | 15 | 16 | 17 |
|-----|----|-----|----|----|------|----|----|----|----|
| $N(d)$ | 3 | 6 | 10 | 16 | 28 | 28–29 | 36 | 40–41 | 48–49 |
| $d$ | 18 | 19 | 20 | 21 | 22 | 23–41 | 42 | 43 |
| $N(d)$ | 56–60 | 72–75 | 90–95 | 126 | 176 | 276 | 276–288 | 344 |

Notice that the lower bound of $\mathbb{R}^{18}$ is 56, which is a new construction shown in this paper (see Section 3). We are interested in the open cases of $N(d)$ for all $d \leq 43$. Though we cannot determine the value of $N(d)$, but our results give the hint of that all the current best known construction of equiangular lines should be maximal in that dimension (see Conjecture 3).
A line through the origin in Euclidean space can be represented by any one of the opposite unit vectors that are parallel to the line. Whenever we talk about such a line in this article, we will sometimes refer to it as a unit vector that represents the line.

2. Determination of equiangular lines being saturated

A set $X$ of lines through the origin in $\mathbb{R}^d$ is called equiangular with the angle $\alpha$ if each pair of lines in $X$ forms the angle $\arccos \alpha$. Given such a set $X$, is it possible to find another line $\ell$ in the span of $X$ such that $\ell$ intersects each line in $X$ at the same common angle $\alpha$? If not, then the equiangular line set $X$ is called saturated in $\mathbb{R}^d$. The answer to this question is negative if $|X|$ has reached its known upper bound, for example the 28 lines in $\mathbb{R}^7$, consisting of all permutations of the vector $(1, 1, 1, 1, 1, 0, 0)$ in $\mathbb{R}^8$ (all of which lie in the 7-dimensional subspace $\sum_{i=1}^{8} x_i = 0$ of $\mathbb{R}^8$). In the cases where $N(d)$ has not been determined (for instance $d = 14, 16, 20, \text{ and } 42$), it is not an easy task to determine whether another line could be added to the current known construction of equiangular sets of lines.

We propose the following algorithm to answer this question in small dimensions. First we choose a subset $X' = \{v_1, v_2, \ldots, v_d\}$ of $X$ that forms a basis of the span of $X$. Then we find the set $V_0$ of all unit vectors in the span of $X$ whose inner products with each of vectors in $X'$ are $\pm \alpha$. Since we are looking for lines, we only need one vector from each pair of opposite vectors $w$ and $-w$ in $V_0$; call this set of representatives $V$. Note that the difference set $X \setminus X'$ must be a subset of $V$. Now we construct a simple graph $G$ whose vertex set is $V$, and two vertices $v$ and $v'$ in $G$ are adjacent if and only if $\langle v, v' \rangle = \pm \alpha$. Then a saturated equiangular lines that contains $X'$ has the cardinality $d + \omega(G)$, where $\omega(G)$ is the clique number of $G$. Although this does not directly answer the question whether or not $X$ is saturated in that dimension, the number $d + \omega(G)$ is still an upper bound for the number of equiangular lines that contains $X$. If $d + \omega(G) = |X|$, then we can conclude that $X$ is saturated.

1. Find a basis $B = \{b_i : 1 \leq i \leq r\}$ in $X$ that spans $\mathbb{R}^r$
2. Solves for all unit vectors $v_i$ in $\mathbb{R}^8$ which intersect with every vectors in $B$ with angle $\pm \alpha$.
3. Make a graph $G = (V, E)$, $V = \{v_i\}$; $(v_i, v_j) \in E$ if and only if $\langle v_i, v_j \rangle = \pm \alpha$
4. Compute $N = |B| + \omega(G)$ (the clique number of $G$)
5. if $|X| = N$ then
6. return $X$ is saturated.
7. end if

Using this algorithm, we are able to establish the following result, which does not seem to appear anywhere in the literature.

**Theorem 1.** The following sets are saturated equiangular lines:

- The 28 lines in $\mathbb{R}^{14}$, in Tremain [20], p. 24 (or below).
- The 40 lines in $\mathbb{R}^{16}$, in Tremain [20], p. 25.
- The 48 lines in $\mathbb{R}^{17}$, in Lemmens and Seidel [15], section 2.
- The 54 lines in $\mathbb{R}^{18}$, in Szöllősi [18].
- The 72 lines in $\mathbb{R}^{19}$, constructed by Asche (see Taylor [19], Theorem 8.1, or below).
- The 90 lines in $\mathbb{R}^{20}$, constructed by Taylor [19], Theorem 8.2 (or below).

**Example 1** (from [20], p. 24). A construction of 28 equiangular lines in $\mathbb{R}^{14}$ using (7, 3, 1)-designs was given by Tremain.

Figure 1 is a graphic representation on 28 column vectors in $\mathbb{R}^{14}$: each $\circ$, $\bullet$, and $\ast$ shall be replaced by $\sqrt{1/5}$, $-\sqrt{1/5}$, and $\sqrt{2/5}$, respectively; empty squares shall be filled with 0. One checks immediately that this indeed gives 28 equiangular lines in $\mathbb{R}^{14}$ with angle 1/5. The following computation is executed in Sage [16]. We label these 28 vectors by $w_1, w_2, \ldots, w_{28}$ from left to right. The vectors $\{w_{2k}: k = 1, 2, \ldots, 14\}$ form a basis for $\mathbb{R}^{14}$. Let $C$ be the following set of vectors in $\mathbb{R}^{14}$:

$$C := \{v \in \mathbb{R}^{14} : \langle v, w_{2k} \rangle = \pm \frac{1}{5}, \forall 1 \leq k \leq 14; \langle v, w_2 \rangle = \frac{1}{5}\}.$$  

Among those $2^{13}$ vectors in $C$, there are 378 unit vectors; call the collection of these unit vectors $V$. Using these vectors in $V$ as vertices, a simple graph $G$ is constructed by connecting $v, v' \in G$ whose inner product is $\pm \frac{1}{5}$. Finally we verify that the clique number $\omega(G) = 14$, which means that
the maximum cardinality of equiangular lines in $\mathbb{R}^{14}$ that contains $X'$ is $|X'| + \omega(G) = 28$. This implies that the 28 equiangular lines above in $\mathbb{R}^{14}$ is saturated. This computation in effect reduces the number of combinations of $\{\frac{1}{3}, -\frac{1}{3}\}$-inner products to be checked from $2^{27}$ to $2^{13}$.

Example 2 ([19, Theorems 8.1 and 8.2]). We hereby give the construction of 90 equiangular lines in $\mathbb{R}^{20}$ and 72 equiangular lines in $\mathbb{R}^{19}$ by Asche, and verify that these sets are saturated.

The constructions come from the Witt design (also known as the Steiner triple system $S(24, 8, 5)$, see [25]). We first list all the 8-subsets of $\{1, 2, \ldots, 24\}$ in lexicographical order, and any such subset which differs from the some subset already found in fewer than 4 elements is discarded. This procedure picks out 759 8-subsets of $\{1, 2, \ldots, 24\}$, with the first subset being $\{1, 2, 3, 4, 5, 6, 7, 8\}$; let $C$ be the collection of these 759 subsets. Let $\{e_i : i = 1, \ldots, 24\}$ denote the standard basis for $\mathbb{R}^{24}$, and let $e_\Sigma := \sum_{i=1}^{24} e_i$. For any $E \in C$ with $1 \in E$, define $f(E) := (4 \sum_{i \in E} e_i - 4e_1 - e_\Sigma)/\sqrt{80}$. Define

$$
\begin{align*}
c &:= 4e_1 + e_\Sigma; \\
c_1 &:= e_2 + e_3 + e_{10} + e_{12} + e_{13} + e_{14} + e_{21} + e_{24}; \\
c_2 &:= e_2 + e_3 + e_6 + e_7 + e_{18} + e_{19} + e_{22} + e_{23}.
\end{align*}
$$

There are 90 sets $E_1, \ldots, E_{90}$ in $C$ such that $1 \in E$ and $f(E_i)$ is perpendicular to each of $e_1 - e_2$, $c$, $c_1$, and $c_2$, for all $1 \leq i \leq 90$. Moreover, it is readily checked that the unit vectors $f(E_i)$, $1 \leq i \leq 90$, form equiangular lines with common angle $\frac{\pi}{2}$ and live in a 20-dimensional subspace $W$ of $\mathbb{R}^{24}$. The list of these 8-sets $E_i$ is given in Table 2 by the lexicographic order.

It can be checked that $X' := \{f(E_j) : j \in J\}$ forms a basis for $W$, where $J = \{6, 7, 13, 19, 21, 24, 27, 34, 43, 45, 48, 52, 57, 61, 66, 70, 74, 80, 82, 89\}$. Let $C$ be the following set of vectors in $W$:

$$
C := \{v \in W : \langle v, f(E_j) \rangle = \pm \frac{1}{5}, \ \forall \ j \in J; \langle v, f(E_0) \rangle = \frac{1}{5}\}.
$$

Among those $2^{19}$ vectors in $C$, there are only 70 vectors of unit length; in fact they are the remaining 70 unit vectors $f(E_k)$ (or their opposites) for $k \in \{1, 2, \ldots, 90\} \setminus J$. From here we can conclude that these 90 lines form a saturated equiangular line set in $\mathbb{R}^{20}$. Note that our procedure again reduces the number of vectors to be checked from $2^{89}$ to $2^{19}$.

Inside the above 90 lines, we can pick out 72 lines by discarding those 18 8-sets that contains 3 from Table 2. The resulting vectors $f(E_i)$, $i = 19, \ldots, 90$, are also perpendicular to $e_1 - e_3$ in $\mathbb{R}^{24}$. Therefore those 72 equiangular lines live in a 19-dimensional subspace of $\mathbb{R}^{24}$.
3. Construction of a large equiangular subset of lower rank

A large equiangular set can be found from a larger set from higher dimensional spaces. For example, 48 lines in \( \mathbb{R}^7 \), 54 lines in \( \mathbb{R}^8 \), 72 lines in \( \mathbb{R}^9 \), 90 lines in \( \mathbb{R}^{10} \), 126 lines in \( \mathbb{R}^{11} \), and 176 lines in \( \mathbb{R}^{12} \) can all be found among the 276 equiangular lines in \( \mathbb{R}^{23} \), sitting inside various lower dimensional subspaces. Similar stories also happen to \( \mathbb{R}^7 \). The 28 equiangular lines in \( \mathbb{R}^7 \) contain maximum size of equiangular lines in \( \mathbb{R}^6 \) (16 lines) and \( \mathbb{R}^5 \) (10 lines). A linearly independent subset \( T \) of vectors inside an equiangular set \( X \) generates the maximal subset of \( X \) that contains in the span of \( T \), usually of lower rank. In this section we mention two such constructions.

3.1. 248 equiangular lines in \( \mathbb{R}^{42} \) with angle 1/7. The existence of 344 equiangular lines in \( \mathbb{R}^{43} \) with angle 1/7 follows from Taylor’s result on the doubly transitive group \( \text{PGU}(3,7^2) \) [19], and can also be constructed from the strongly regular graph \( \text{SRG}(344, 168, 92, 72) \) [5], which induces the Gram matrix \( G = [(v_i, v_j)]_{i,j=1}^{344} \) of these lines. We randomly select 42 columns from \( G \) and verify that they form a linearly independent set of vectors. Then we collect the column vectors of \( G \) which belong to the span of these vectors; call this collection \( X \). By picking out the corresponding rows and columns of \( X \) from \( G \), the resulting matrix is the Gram matrix of equiangular unit vectors of rank 42 and angle 1/7. The best result among a few thousand runs of this experiment gave 248 equiangular lines in \( \mathbb{R}^{42} \) with angle 1/7.

For sake of comparison, we recall the inequality (2), which is the so-called relative bound for equiangular lines.

**Theorem 2** ([21], p.342). Let \( X \) be an equiangular set with angle \( \alpha \) in \( \mathbb{R}^r \). If \( r < \frac{1}{\alpha^2} \), then

\[
|X| \leq R(r, \alpha) := \frac{r(1-\alpha^2)}{1-r\alpha^2}.
\]
Table 3. Relative bounds and the sizes of known constructions of equiangular set with angle $1/7 \in \mathbb{R}^d$

| $d$ | $R(d, 1/7)$ | Found |
|-----|--------------|--------|
| 14  | 288          | 248    |
| 16  | 246          | 200    |
| 17  | 213          | 168    |
| 18  | 187          | 152    |
| 19  | 156          |        |

Proceeding in a similar fashion, we look for large subsets in $\mathbb{R}^d$ of the 344 equiangular lines in $\mathbb{R}^{43}$. The best results\(^1\) are listed in Table 3, with a comparison with the relative bounds (which are also the best upper bounds so far). Although our constructions do not reach the relative bounds, we doubt if there are more equiangular lines with angle $1/7$ in these dimensions. Notice that if our construction are the maximum in that dimension and angle, then there will be new results for nonexistence of two associated strongly regular graphs (with 288, 246 vertices, respectively).

3.2. 56 equiangular lines in $\mathbb{R}^{18}$ with angle $1/5$. Among the 72 equiangular lines in $\mathbb{R}^{19}$ with angle $1/5$ given in Example 2 of Section 2, we randomly select 18 of them and collect all the vectors from those 72 vectors that fall into the span of these 18 vectors. After a few hundred runs of the experiments, the best result we find is a collection of 56 equiangular lines of rank 18 with angle $1/5$. Specifically, there are two kinds of such 56 vectors. Let

\[
    u_1 := e_4 + e_5 + 6e_6 - 3e_7 + e_8 + e_9 + 6e_{10} + e_{11} - 3e_{12} + 6e_{13} - 3e_{14} + e_{15} + e_{16} + e_{17},
\]

(3)

\[
    u_2 := 5e_4 + 5e_5 - 3e_6 - 3e_7 - 4e_8 + 5e_9 - 4e_{11} - 4e_{15} + 5e_{16} - 4e_{17} - 3e_{18} + 6e_{19} - 4e_{20} + 6e_{22} - 3e_{23}.
\]

The collections we find are perpendicular to either $\{c, c_1, c_2, e_1 - e_2, e_1 - e_3, u_1\}$ or $\{c, c_1, c_2, e_3 - e_2, c_1 - c_3, u_2\}$ (the vectors $c, c_1, c_2$ are listed in (1); the coordinates of $u_1$ and $u_2$ in (3) may be permuted in specific ways.) This finding raises the lower bound of $N(18)$ from 54 to 56, see Table 1. Also from the method described in Section 2, we confirm that both configurations of 56 equiangular lines in $\mathbb{R}^{19}$ are saturated.

4. Discussions

All other equiangular sets of lines in Theorem 1 are checked to be saturated using the similar procedures to the Examples above. But it took too long when we tried to examine 248 equiangular lines in $\mathbb{R}^{42}$. Our algorithm would require to pick out unit vectors among $2^{41}$ (roughly 2.2 trillion) possible candidates.

In light of Theorem 1, we propose this following conjecture.

**Conjecture 3.** The following table gives on the maximal equiangular lines on the specified dimensions:

| $d$ | $N(d)$ |
|-----|--------|
| 14  | 28     |
| 16  | 40     |
| 17  | 48     |
| 18  | 56     |
| 19  | 72     |
| 20  | 90     |

Conjecture 3 is coherent to Peter Casazza’s conjecture which states that all the maximal sizes of equiangular lines are even numbers except for $\mathbb{R}^2$. We know that the crucial steps are to do the classification of equiangular lines. For instance, the 36 equiangular lines in $\mathbb{R}^{15}$ with angle $\frac{1}{7}$ has been classified to be 227 different classes [17]. We believe that there should be less than 227 different classes of 28 equiangular lines in $\mathbb{R}^{14}$. If there are not so many different classes, then in conjunction of our methods, we might be able to prove $N(14) = 28$.

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\(^1\)Examples on these Gram matrices can be downloaded at [http://math.ntnu.edu.tw/~yclin/Gram1-7/](http://math.ntnu.edu.tw/~yclin/Gram1-7/)
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