COSMOLOGY IN A TEST TUBE:
THEORY OF DOMAIN WALLS FORMATION IN BINARY FLUIDS

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Formation of domain walls during a rapid phase transition in a quasi one dimensional Cahn-Hilliard equation describing binary fluids in a thin tube is studied. Density of kinks scales as a sixth root of quench rate for equal concentrations and like a square root of quench rate for unequal concentrations of fluids. For a slow inhomogeneous transition the density is linear in velocity of temperature front. This paper is first theoretical study of topological defects formation in a system with conserved order parameter.

It has been pointed out some time ago that topological defects which formed during subsequent symmetry breaking phase transitions can provide seeds for structure formation in the early universe [1]. Kibble [2] gave a detailed theory of defect formation in I order phase transitions which proceed by bubble nucleation. Bubbles are born with random orientation of order parameter; when they coalesce they can give rise to a nontrivial topological winding number. This prediction was verified in relatively simple beautiful experiments in liquid crystals [3]. Disclinations were observed and theoretical relation between bubble density and disclination density was verified. All this at nearly room temperature and with at most the aid of an optical microscope.

A scenario for II order transitions was put forward by Zurek [4]. He observed that the order parameter goes out of equilibrium some time before the critical point is crossed. In this way a unique finite correlation length \( \xi \) proportional to the forth root of the quench rate is frozen-in. \( \xi \) is the scale which determines density of topological defects after the transition. Experimental verification is not as clear as for I order transitions. Experiments in superfluid \( ^4\text{He} \) were done [5] just to be falsified later on. There are spectacular experiments in superfluid \( ^3\text{He} \) [6]. Density of detected vortices is consistent with the order of magnitude Zurek’s prediction. However the tricky detection is somewhat indirect, there is no way to control the quench rate and verify the scaling of \( \xi \) with the rate, finally the precise origin of detected vortices was also recently put in question [7]. There is urgent need for an experiment as simple as that in liquid crystals.

A good candidate for such an experiment are binary fluids described by Cahn-Hilliard plus Navier-Stokes equations. Order parameter is conserved and proportional to the difference in fluid concentrations. An example are aniline and cyclohexan. They do not mix for temperatures less than \( T_c = 30.9^\circ\text{C} \). The transition is second order with a standard Ginzburg-Landau free energy. The fluids differ in optical density and thus domain walls between them can be detected by optical means. A rapid quench should result in a quench rate dependent density of domain walls. The best experimental approach is to quench the fluids in a thin tube, which makes the system effectively one dimensional. In one dimension the quench-made domain walls cannot be eradicated by mutual annihilation thanks to order parameter conservation - they are a permanent record of the nonequilibrium transition. The transition in one dimension is, strictly speaking, a crossover; the correlation length does not diverge at \( T_c \). This deviation from a mean-field is important only in a very narrow critical regime which is irrelevant for our nonequilibrium quenches. In fact we do not see any deviation from a mean-field in our numerical simulations.

Dynamics of binary fluid separation in one spatial dimension is described by model H dynamics which consists of the Cahn-Hilliard (CH) equation for conservative real order parameter \( \phi \)

\[
\dot{\phi} = \left[ -\epsilon(t,x)\phi + \phi^3 - \phi'\right]' + \xi',
\]

where \( \epsilon = \frac{\partial_t}{\partial_x} \) and \( \phi' = \frac{\partial_x}{\partial_t} \). CH equation is usually supplemented by Navier-Stokes equation but in one dimension the incompressibility condition makes this perturbation trivial. We allow for variation of the symmetry breaking parameter \( \epsilon \) (temperature/pressure) both in space and in time. \( \xi \) is assumed to be a white Gaussian noise with nonvanishing cumulants

\[
\langle \xi(t,x) \rangle = 0,
\]

\[
\langle \xi(t_1,x_1)\xi(t_2,x_2) \rangle = 2T \delta(t_1 - t_2) \delta(x_1 - x_2).
\]

TRANSITION WITH EQUAL CONCENTRATIONS. To begin with let us consider a uniform linear phase transition with

\[
\epsilon(t,x) = \frac{t}{\tau}
\]

with \( \phi = 0 \) on average at the initial \( t = -\infty \), which is preserved by CH evolution. Any uniform transition close

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to the critical point at $\epsilon = 0$ can be described by such a linearized $\epsilon$.

For $t < 0$ the system is in a symmetric phase: $\phi$ is subject to small fluctuations around 0. For this stage of the quench and also for the onset of spinodal instability, just after $\epsilon$ crossed 0, Eq.[4] can be linearized in $\phi$,

$$\dot{\phi} = \left[ -\frac{t}{\tau} + 3\langle \phi^2 \rangle \right] \phi'' - \phi''' + \xi'''. \quad (4)$$

The mean-field ($\langle \phi^2 \rangle$) is kept just to control validity of the linearization. We solve Eq.(4) by neglecting the mean-field term and performing a Fourier transformation

$$\phi(t, x) = \int_{-\infty}^{+\infty} dk \, e^{ikx} \phi(t, k) . \quad (5)$$

Eq.(4) can be solved for any $k$ with a help of its Green function and the correlations [3]. The power spectrum $P(t, k)$ of the fluctuations is

$$\langle \phi^* (t, k) \phi(t, p) \rangle = P(t, k) \delta(k - p) ,$$

$$P(t, k) = \frac{T \sqrt{\tau |k|}}{\pi} \, e^{\frac{k^2 (t^2 + k^2)}{\tau^2}} \int_{-\infty}^{+\infty} ds \, e^{-s^2} . \quad (6)$$

The fluctuations, measured by $\langle \phi^2 \rangle = \int dk \, P(t, k)$, are small for $t < 0$. At some $t > 0$ and the linearized approximation involved in solving Eq.(4) breaks down. Assuming that $t/\tau^{2/3} > 0$ is sufficiently large the error function integral in (4) is constant for small $k^2$ (but it is still essential to suppress the divergence for large $k^2$). The remaining exponent is peaked at $k^2 \approx t/3\tau$ with a maximum of $\exp(4t^3/27\tau^2)$. This maximum begins to blow up at $t \approx \tau^{2/3}$. It is around this time that $\langle \phi^2 \rangle$ passes through $t/\tau$. This is the moment when kinks of width given by the corresponding healing length of $\tau^{1/6}$ begin to form. Fluctuations with $|k| > \hat{k} = \tau^{-1/6}$ are irrelevant for kink formation at $\hat{t}$. Density of kinks which are going to form can be identified with the density of zeros of $\phi(\hat{t}, x)$ smoothed over $|k| > \hat{k}$ which is, according to a formula from [3],

$$n = \frac{\pi}{2} \sqrt{\frac{\int_{-\hat{k}}^{+\hat{k}} dk \, k^2 \, P(\hat{t}, k)}{\int_{-\hat{k}}^{+\hat{k}} dk \, P(\hat{t}, k)}} . \quad (8)$$

Introducing an integration variable $k/\hat{k}$ one can see that any $\tau$-dependence can be factorized in front of the integrals so that

$$n \sim \frac{1}{\tau^{1/6}} \quad (9)$$

for any $\tau$. Results from numerical simulations consistent with this prediction are shown in Fig.1. In the limit of adiabatic transition $\tau \to \infty$ the system stays close to the critical point for a time long enough to order at long distance. For a fast quench substantial amount of initial disorder is frozen into the ordered phase in a form of kinks. In one dimension, because of $\phi$-conservation, these domain walls have no freedom to diffuse around and mutually annihilate. There is no “phase ordering kinetics” to erase this trace of disorder. The kinks are permanent record of the phase transition.

**TRANSITION WITH UNEQUAL CONCENTRATIONS.** The two fluids may differ by average concentration. In that case the conserved average $\phi$ is $\langle \phi \rangle = 0$. We take $\langle \phi \rangle = M > 0$ for definiteness. An uniform $\phi = M$ configuration is stable against small perturbations if $\epsilon < 3M^2$ when $M$ is outside the interval between the two inflection points of the double-well potential. For $\epsilon < M^2 \phi = M$ is not a minimum of the effective potential but it cannot decay because of $\phi$-conservation; $\phi = M$ is bigger than the positive minimum at $\sqrt{\tau}$ for $\epsilon > 0$ or the minimum at $0$ for $\epsilon < 0$.

At $\epsilon = M^2 \phi = M$ coincides with the minimum of the potential. When $M^2 < \epsilon < 3M^2 \phi = M$ is stable against small perturbations but again it is not a minimum of the double well potential. This time, however, $\phi$-conservation does not forbid its decay to the minima at $\pm \sqrt{\tau}$. The decay can proceed by thermal nucleation of antikink-kink pairs (AKP) provided that $T$ is large as compared to a barrier. To estimate minimal $T$ necessary for nucleation let us introduce $\epsilon = M^2 + \varepsilon$ with $\varepsilon < 0$ and expand $\phi = M + \phi$. The effective potential can be approximated by $\frac{1}{2} \varepsilon^2 \langle \phi^2 \rangle$. Fluctuations around $M$ are $\langle \phi^2 \rangle \sim T/\sqrt{|\varepsilon|}$. They can result in AKP nucleation if they reach beyond the positive inflection point at $\sqrt{(3M^2 - \varepsilon)/2}$ or in other words ($M - \sqrt{(3M^2 - \varepsilon)(3M^2 - \varepsilon)/2} \approx \langle \phi^2 \rangle$. To first order in $|\varepsilon|/M^2$ nucleation takes place at $|\varepsilon| < (TM^2)^{2/5} = \varepsilon_n$. Nucleation time is $\sim \varepsilon^{-2}$. If the transition proceeds at a finite rate, $\varepsilon = t/\tau$, there may be not enough time for thermal nucleation. Nucleation can only happen if at $\varepsilon = -\varepsilon_n$ the nucleation time is shorter than the time left till $\varepsilon$ reaches 0. This condition is satisfied if

$$\tau (TM^2)^{6/5} \gg 1 . \quad (10)$$

If the transition is fast enough or $T$ is sufficiently small, when the opposite condition holds, no AKP are nucleated for $M^2 < \epsilon < 3M^2; \phi$ remains fluctuating around $M$ until $\epsilon$ crosses $3M^2$ and spinodal decomposition due to instability of $\phi = M$ begins. This case can be analyzed as follows. As a first step we define $\epsilon(t) = 3M^2 + t/\tau$ and expand $\phi = M + \phi$. Equation (3) when linearized in $\phi$ gives Eq.(1) but with $\phi$ replaced by $\phi$. By the same token as for equal concentrations, at $t = \tau^{2/3}$ fluctuations blow up with momentum peaked at $\pm k \sim \pm \tau^{-1/6}$. In contrast to the $M = 0$ case, the growth of this instability is halted very quickly much before any kinks are formed. To see this let us pick a wave $\phi = a(t) \cos(\hat{k}x)$. At around
\( t = \tau^{2/3} \) maxima of this wave enter the area beyond the inflection point of the actual double-well potential, \( \phi > \sqrt{(3M^2 + \tau^{-1/3})/3} \) - their growth is slowed down. At the same time minima get further and further into the unstable regime between the inflection points so it seems that their growth rate should be accelerated. If it were so they would quickly hop to the neighborhood of the negative inflection point and the initial \( \cos(kx) \) would distort into a periodic array of kinks-antikinks with density \( \sim k \). It cannot be so because it would obviously violate \( \phi \)-conservation. The growth of \( a(t) \) stops as soon as the maxima cross the inflection point. At this stage the cosine wave is still an almost negligible fluctuation as the maxima cross the inflection point. At this stage the landscape of correlated \( \tilde{\phi} \), \( \bar{\phi} \), \( \hat{\phi} \) of the "cosine" are more negative than the other. 

\[ \frac{\hat{\phi}}{M} \]\n
\[ \frac{\hat{\phi}}{M} \]

\[ \sim \frac{1}{4\pi \epsilon(t)} \] in the limit of very slow \( v \). We will argue \textit{a posteriori} that the sharp step is a good approximation of any generic front for \( v \to 0 \).

We solve the problem of kink generation behind the moving front by perturbative expansion around \( v = 0 \). At \( v = 0 \) there is a static \( \phi \)-front, \( \phi(x) = H(x) \) - a step in \( \phi \) at \( x \approx 0 \) interpolating between \( -\sqrt{-\epsilon} \) at \( x = -\infty \) and \( 0 \) at \( x = +\infty \). Its width is given by \( \approx \frac{1}{\epsilon^{1/2}} + \frac{1}{\epsilon^{1/2}} \).

Let us now switch on small \( v > 0 \). The \( \epsilon \)-front \((v \to 0)\) is slowly moving on. If \( \phi \) were not conserved, the \( \phi \)-front would follow moving in step with \( \epsilon \)-front and leaving no kinks behind \( \hat{k} \). For our conserved \( \phi \) it is not possible, kinks must inevitably appear. To see in some detail how it happens let us substitute \( \phi(t, x) = H(x - vt) + \psi(t, x) \) with \( \psi = O(v) \) to Eq.(10) and keep only \( O(v) \) terms. We are interested in length scales large as compared to the width of the step \( H(x) \) and that of the \( \epsilon \)-front. That is why we keep only up to the second \( x \)-derivative which is responsible for diffusion. Far from \( x \approx vt \) we obtain

\[ \dot{\psi}(t, y) = \epsilon_+ \psi''(t, y) + v \psi'(t, y) + v \sqrt{\epsilon_2/2} \theta(t) \delta(y) , \] (14)

where \( y = x - vt \). We take into account a \( \delta \)-like source term at \( y = 0 \) which is a long-wavelength approximation to \( vH' \). We also set \( \epsilon_- = \epsilon_+ / 2 \) for simplicity. The source term is switched on at \( t = 0 \) when the \( \epsilon \)-front starts to move, hence the Heaviside function. The solution of Eq.(14) is straightforward,

\[ \psi(t, y) = v \sqrt{\epsilon_2/2} \int_0^t dt' e^{-\frac{(y + v(t-t'))^2}{4\pi \epsilon_+ (t-t')}} . \] (15)

The source term produces \( \psi \) at \( y = 0 \) at constant rate. It spreads around by diffusion but at the same time it is carried to negative \( y \) with velocity \( -v \). For \( y > 0 \) the penetration by diffusion dominates at first but at \( v^2 t^2 \sim \epsilon_+ t \) the two processes balance one another and \( \psi(t, y > 0) \) saturates. From this time on all the \( \psi \) is carried directly from the source to \( y < 0 \) with velocity \( +v \). This effectively means that the \( \phi \)-front halted, while the \( \epsilon \)-front keeps moving on. A supercooled phase with a slightly positive \( \phi \) is growing in between them with velocity \( v \). When its width exceeds \( \sqrt{2/\epsilon_+} \), \( \phi \) decays towards positive ground state. From this time on we have a negative \( \phi \)-step moving together with \( \epsilon \)-step and the whole story repeats itself at spatial intervals of \( \epsilon_+/v \). Density of kinks is

\[ n \sim \frac{v}{\epsilon_+} \] for \( v \to 0 \).

(16)

It should be stressed that the whole process is deterministic, kinks are made at regular intervals. Noise is required to begin the process; it also adds some irregularity on top of the regular pattern.

Note that for small \( v \) the relevant length scale is \( \epsilon_+/v \). For small enough \( v \), it far exceeds the \( \epsilon \)-front width and
the width of $H(x)$. This justifies the sharp step in Eq. (13) and the long-wavelength approximations involved in our derivation of Eq. (14).

Let us now turn to the opposite large-\(v\) limit where we anticipate the transition to be effectively homogeneous. Any generic \(\epsilon(t, x)\) can be linearized around \(\epsilon = 0\),

\[\epsilon(t, x) = \frac{vt - x}{vt} \equiv \alpha (vt - x) . \quad (17)\]

At any fixed \(x\) the transition proceeds at the rate of \(1/\tau\) just like in Eq. (4). If it were homogeneous it would enhance the momentum \(k = \tau^{-1/6}\). For

\[v \gg \alpha^5 \quad \text{or} \quad v \gg \tau^{-5/6}\]

this momentum scale is much bigger than the slope \(\alpha\) and the relevant field fluctuations do not feel the inhomogeneity. This is where the transition is effectively homogeneous and Eq. (14) applies.

**CONCLUSION.** We studied dynamics of domain walls formation during a quench in effectively one dimensional binary fluids. We believe that our predictions can be verified by experiments in thin test tubes. In fact first steps in this direction has been already made [10]. One can test the 1/6 scaling for equal concentrations, which is the most direct analogue of Kibble-Zurek scenario for second order transitions. A more intriguing case are unequal concentrations where we also get scaling with the quench rate in a range of parameters. Its 1/2 exponent should be much easier to measure than the 1/6. Finally, unlike for nonconserved order parameter [3], density of kinks is linear in front velocity for a slow inhomogeneous quench. We believe that binary fluids are a unique opportunity of an almost bare eye detection of topological defects. We also believe that one can not only detect topological defects but also proceed and measure scaling of their density with quench rate. In this paper we lay theoretical foundations for the experimental work in progress [10].

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[1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (C.U.P., Cambridge, 1994).
[2] T.W.B. Kibble, J. Phys. A 9, 1387 (1976); Phys.Rep. 67, 183 (1980).
[3] I. Chuang, R. Dürrer, N. Turok and B. Yurke, Science 251, 1336 (1991); M.J. Bowick, L. Chandar, E.A. Schiff and A.M. Srivastava, Science 263 (1994).
[4] W.H. Żurek, Nature 317, 505 (1985); Acta Phys.Polon. B24, 1301 (1993); Nature 368, 292 (1994); Phys. Rep. 276 177 (1996).
[5] P.C. Hendry et al., Nature 368, 315 (1994); M.E. Dodd et al., Phys. Rev. Lett., 81, 3703 (1998).
[6] C. Baierle et al., Nature 382, 332 (1996); V.M.H. Ruutu et al., Nature 382, 334 (1996); V.M.H. Ruutu et al., Phys. Rev. Lett. 80, 1465 (1998).
[7] I.S. Aranson, N.B. Kopnin and V.M. Vinokur, Phys.Rev.Lett. 83, 2600 (1999).
[8] F. Liu and G. F. Mazenko, Phys. Rev. B46, 5963 (1992); B.I. Halperin, “Physics of Defects”, proceedings of Les Houches, Session XXXV 1980 NATO ASI (North Holland Press, 1981) p.816.
[9] J. Dziarmaga, P. Laguna and W.H. Żurek, Phys.Rev.Lett. 82, 4749 (1999); N.B. Kopnin and E.V. Thuneberg, Phys.Rev.Lett. 83, 116 (1999).
[10] J. Dziarmaga, J. Mayer, M. Sadzikowski, in preparation.

![FIG. 1. log(n) as a function of log(τ) for M = 0 (the top plot) and for M = 2 (the bottom plot) according to numerical simulations. For M = 0 the slope is 0.18 ± 0.01 as compared to the theoretical 1/6 ≈ 0.17. For M = 2 the slope saturates for log(τ) > 2 at 0.48 ± 0.05 as compared to the theoretical 0.50. At low τ the M = 0, 2 results tend to be the same. Vertical size of a point is its statistical error. Simulations were done at T = 10^{-5} on a ∆x = 1, ∆t = 0.01 lattice of size 1024 with periodic boundary conditions. \(\epsilon(t)\) was swept from 3M^2 − 10^\epsilon^{-1/3} to 3M^2 + 10^\epsilon^{-1/3}. Kinks were counted at final time. Density n is an average over many runs.](image-url)
FIG. 2. Two snapshots of $\phi$ as a function of $x$ for $M = 2, \tau = 128, T = 10^{-5}$ taken from numerical simulations. The thin line is $\phi$ at $t = 0.8\hat{t}$ when spinodal decomposition begins. The amplitude of fluctuations around $M = 2$ is magnified 100 times. The thick line is $\phi$ at $t = 1.6\hat{t}$ when kinks are already well defined.