EFFECTIVE MULTIPLE MIXING IN SEMIDIRECT PRODUCT ACTIONS

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Abstract. We prove effective decay of certain multiple correlation coefficients for measure preserving, mixing Weyl chamber actions of semidirect products of semisimple groups with $G$-vector spaces. These estimates provide decay for some semisimple groups of higher rank.

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1. Introduction

Ergodicity and various notions of mixing constitute the measure theoretic incarnation of statistical uniformity in measurable dynamics. The relationships among those notions are not completely understood; it is not known, for example, if every mixing map $T$ on a measure space $X$ is mixing of all orders. For $\mathbb{Z}^d$-actions, $d \geq 2$, Ledrappier gave a counterexample in [10] later generalized by Schmidt (see [15]) through the introduction of certain algebraic dynamical systems. General $\mathbb{R}^d$-actions can also have a wide range of
behaviors depending on the nature of the action and properties of the object acted upon.

In the context of actions of a highly noncommutative group (such as a semisimple group) the situation changes drastically. Many topological and measurable distribution phenomena follow not from the peculiarities of the action, but rather on the nature of the acting group. This is manifest in the following two basic results: the Howe - Moore vanishing theorem [5] which implies that every ergodic action of such a group is mixing and Mozes’s theorem [12] stating that for well behaved semisimple Lie groups, mixing implies multiple mixing of all orders. It is the rich geometry and non trivial interplay between certain subgroups of semisimple groups that lies at the heart of both these results and accounts for this exceptional behavior of semisimple groups. For example, Mautner’s phenomenon (see [2]) in various guises plays a major role in both cited results. A comprehensive survey emphasizing this interplay can be found in [16].

Mozes’s theorem relies on geometric considerations concerning how the group acts on a certain space of measures. It uses compactness in an essential way, preventing an immediate quantitative refinement even with specific hypotheses on the action and the target space. The Howe - Moore theorem, on the other hand, is essentially a representation theoretic result; for this reason, it is possible to make quantitative by utilizing the $L^2$-theory and a clever application of tensor products.

The next question that arises is whether we can give a quantitative form of Mozes’s theorem whose asymptotics agree with Howe - Moore for 2-mixing. More explicitly, let $G$ be a suitable group with a measure preserving action on a probability measure space $(X,\mu)$ denoted $g \cdot x$. For any $k + 1$-tuple $f_i \in L^\infty_0(X)$ (bounded, zero mean functions) form the correlation integral

$$\int_X f_0(x)f_1(g_1^{-1} \cdot x) \cdots f_k(g_k^{-1} \cdot x) d\mu(x)$$

The task is to bound this integral in terms of data (as explicit as possible) intrinsic to the $f_i$, to the acting group $G$ and to a given notion of growth of the acting tuple $(g_i)$. This paper is mainly concerned with the Weyl chamber action of groups $G$ where $G$ is the semidirect product of a semisimple group $G$ with a vector space by means of a representation of one on the other. Such groups are naturally found as subgroups of semisimple groups (but are not restricted to such a role) and we exploit this inclusion to get quantitative decay for some semisimple groups of higher rank. Since the precise statement of our result is too technical to give in the introduction, we will give a simplified description here and refer to Section 4 for the full version.

Note that this paper relies heavily on the work of Wang in [18] and cannot possibly be made self contained without copying that work verbatim. References to proofs in [18] will abound and the reader is advised to consult it for verifying several statements made here.
Following Wang we let $G$ be a connected semisimple almost algebraic (see section 2 for precise definitions) $K$-group where $K$ is a local field of characteristic zero together with a $K$-rational representation $\rho : G \to GL(V)$ satisfying certain conditions. Let the group $G \ltimes \rho V$ act on a probability space $(X, \mu)$ via measure-preserving transformations so that the action is mixing; thus we obtain a unitary representation on $L^2(X)$ that distributes over pointwise (a.e.) products of functions. Restricting it to $G \ltimes \rho 0$ we get a (still mixing) action of the semisimple group $G$; relative to a Cartan decomposition $G = KD^+FK$ we let $k + 1$ elements

$$a^i \in D^+$$

of $G$ with $a^0 = I$ act on $k + 1$ bounded, zero mean, $K$-finite functions $f_i$ of bounded spectral support. Define

$$\mathcal{R}(a) = \left| \sum_{i=1}^{k-1} \lambda\left(\frac{a^k}{a^i}\right) \right| \left| \sum_{i=1}^{k} \varrho(a^i)^{-1} \right|$$

where $\varrho$ and $\lambda$ are lowest and highest weights respectively for the representation $\rho$. Our main result (3.5) will imply the following estimate:

**Theorem 1.1.** Let $a^i$ be as above; there exists an $L^2$-dense subspace $\mathcal{D}$ of bounded, zero-mean, $K$-finite functions and constants $C, C'$ depending only on the action so that if

$$\min\left( \min_{i=0,\ldots,k-1} |\lambda\left(\frac{a^k}{a^i}\right)|, \min_{i=1,\ldots,k} |\varrho(a^i)^{-1}| \right) > C',$$

$f_i$ are in $\mathcal{D}$ and $d_i = \dim(K \cdot f_i)$, we have the bound

$$\left| \int_X f_0(x) f_1((a^1)^{-1} \cdot x) \cdots f_k((a^k)^{-1} \cdot x) d\mu(x) \right| \leq s^{2qC} \prod_i ||f_i||_{\infty} d_i \mathcal{R}(a)^{-\frac{q}{2}}$$

where $s$ depends on the function $f_i$ and the action; the exponent $q$ only depends on the action.

For the action on smooth vectors we will get bounds for a wider class of functions defined by the finiteness of certain Sobolev norm (i.e. a norm involving derivatives of various orders in certain directions in the group) for the functions $\sigma(g) \cdot f : G \to L^\infty(X)$.

Note that if a semisimple group of higher rank has an adequate number of subgroups that are semidirect products of the form above (for example, covering the entire Cartan action), we can extend the bounds to those groups.

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1. See section 2 for definitions.
2. By 'action' we will refer to all data involved in it, including the group structure. The independence is on the functions in the correlation integral.
3. Explicitly, $s$ is such that all the $f_i$ lie in the image $P_s(\mathcal{L}_{0,K})$ for the approximate projection $P_s$ defined in section 2.
We will give the extension briefly for the group $SL(n, \mathbb{K})$ in Section 5.2, following the demonstration given in [6], and discuss how to get such bounds in some generality in Section 5.

Our method was inspired by a proof of quantitative decay given in [6]. Initially we had worked out the case $SL(n, \mathbb{R})$ only, when the paper of Zhenqi Jenny Wang [18] was brought to our attention which generalized computations in [6]; this led us to expand the setting and try to isolate parts of the computation that can work in more general settings. Apart from methods, we borrow many notations and notions from these expositions, so the reader is advised to consult them for further reference. For complete proofs of unproven statements found here and in Howe-Tan’s book, see [9], chapters 1, 2 and Appendices 1-3.

While this paper was being written, I was notified that similar results were obtained by Bjorklund, Einsiedler and Gorodnik in [3]; that work treats the full action of semisimple groups over local fields and adeles acting on suitable homogeneous spaces. It covers actions of rank 1 groups with spectral gap as well as nonsplit groups, two situations that that we cannot treat here, and provides uniform estimates for Sobolev vectors. Their method is dynamical in nature and examines the quantitative properties of the orbit of the correlation measures by the acting elements; in that sense, it is close to the spirit of Mozes’ original method. In contrast, our method is spectral and builds on consecutive approximations by nicely behaved functions. Since semidirect products are our main focus, for the application of our estimates to semisimple groups we need these groups to contain sufficiently many semidirect products. This excludes groups of split rank at most 1 and almost direct products of such groups.

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2. **Set up and central notions**

In the following sections, we lay down notation, the central objects of study and the tools we will use in the proofs.

2.1. **Semidirect products and excellent representations.** Let $\mathcal{K}$ be a local field of characteristic zero, i.e. the real or complex numbers or a finite extension of a $p$-adic field. Having fixed $\mathcal{K}$, whenever we talk about an algebraic group as a functor we will indicate it by a tilde, e.g. $\tilde{G}$. Then $G$ will denote the group of $\mathcal{K}$-rational points of $\tilde{G}$. 
Let $G$ be the group of $\mathcal{K}$-rational points of a connected semisimple algebraic group $\tilde{G}$. Let $\tilde{D}$ be a maximal $\mathcal{K}$-split torus and $\tilde{B}$ a minimal parabolic containing $\tilde{D}$. Write $\mathbf{X}(\tilde{D})$ for the characters of $\tilde{D}$ defined over $\mathcal{K}$. The choice of a parabolic group $B$ determines an ordering of the characters; let $X^+$ be the set of positive characters with respect to the given ordering.

As in [18] let $\mathcal{K}^0 = \{x \in \mathbb{R} | x \geq 0\}$ and $\overline{\mathcal{K}} = \{x \in \mathbb{R} | x \geq 1\}$ when $\mathcal{K}$ is Archimedean. When $\mathcal{K}$ is non-Archimedean, we fix a uniformizer $q$ with $|q|^{q-1}$ the cardinality of the residue field of $\mathcal{K}$. Then correspondingly $\mathcal{K}^0 = \{q^n | n \in \mathbb{Z}\}$ and $\overline{\mathcal{K}} = \{q^{-n} | n \in \mathbb{N}\}$.

Define subgroups $D^0$ and $D^+$ of $D$ by

$$D^0 = \{d \in D | \chi(d) \in \mathcal{K}^0 \text{ for each } \chi \in \mathbf{X}(\tilde{D})\},$$

$$D^+ = \{d \in D | \chi(d) \in \overline{\mathcal{K}} \text{ for each } \chi \in \mathbf{X}^+\}.$$

We call $D^+$ the positive Weyl chamber in $D$ (relative to the prescribed data).

Next, we denote the centralizer of $\tilde{D}$ in $\tilde{G}$ by $\tilde{Z}$ and transfer the ordering of $\mathbf{X}(\tilde{D})$ to $\mathbf{X}(\tilde{Z})$ by inclusion. Let

$$Z_+ = \{z \in Z | ||\chi(z)|| \geq 1 \text{ for each } \chi \in \mathbf{X}(Z)^+\}$$

and

$$Z_0 = \{z \in Z | ||\chi(z)|| = 1 \text{ for each } \chi \in \mathbf{X}(Z)^+\}.$$

Note that these are $\mathcal{K}$-subgroups of the $\mathcal{K}$-group $Z$. We then have the following decomposition (see [4] and the discussion in [18]):

**Lemma 2.1.** There exists a good maximal compact subgroup $K$ of $G$ such that

1. $N_G(D) \subset KD$.
2. We have the decomposition $G = K(Z_+/Z_0)K$ such that for each $g \in G$, there exists a unique element $z$ of $Z_+$ modulo $Z_0$ so that $g \in KzK$.
3. There exists a finite subset $F \subset C_G(D)$ so that $G = K(D^+F)K$ and for each $g \in G$ there exist unique $d \in D^+$ and $f \in F$ so that $g \in KdfK$.

Recall that any semisimple group is the almost direct product of its almost simple factors, $G = \prod G_i$. We will assume that no factor is compact, although this can be avoided at the expense of making the statements of the theorems more complicated. We opt for simplicity.

Now let $\rho : G \to \text{GL}(V)$ be a representation on a finite dimensional $\mathcal{K}$-vector space $V$ with the following properties:

1. $\rho$ is continuous when $\mathcal{K} = \mathbb{R}$ and $\mathcal{K}$-rational in all other cases;

\[\text{In the case } \mathcal{K} = \mathbb{R} \text{ one can take } G \text{ to be any semisimple Lie group and our results are true in that generality. We will restrict to algebraic groups for simplicity of notation. See [18] for the modifications that need to be made to accommodate this case.}\]
(2) for each almost simple factor $G_i$, the only $\rho(G_i)$-fixed point in $V$ is 0.

Such representations are called excellent in [18]. We will also assume once and for all that $\ker(\rho) < Z(G)$, i.e. that the representation is not far from faithful. By means of this representation we define the main acting object of this work: let

$$\mathfrak{G} = G \rtimes \rho V$$

be the semidirect product of $G$ with $V$ by means of $\rho$. This is a $\mathcal{K}$-group whose unipotent radical over $\mathcal{K}$ coincides with $V$. Since $G$ is semisimple and $\text{char}(\mathcal{K}) = 0$, the representation $\rho$ is completely reducible and thus $V$ breaks into irreducible components

$$V = \bigoplus_{i=1}^N V_i.$$

Now we introduce notation involving $V$ and $\rho$:

1. $\| \cdot \|$ denotes a $K$-invariant norm on $V$.
2. The restriction of $\rho$ on $V_i$ is denoted by $\rho_i$.
3. $\Phi_i$ is the set of weights of $\rho_i$ with respect to $D$ on $V_i$.
4. $\lambda_i$ resp. $\gamma_i$ are the highest resp. lowest weights of $\rho_i$.
5. For each $i$ and each weight $w$ of $\rho_i$, $V_w$ is the corresponding weight subspace of $V_i$ (there will be no problem distinguishing irreducible components).
6. $\Phi$ is the set of roots of $G$ with respect to $D$.
7. For each $\omega \in \Phi$, denote by $g_\omega$ the root space corresponding to the root.
8. $\delta_B$ is the modular function of the Borel subgroup $B$ that determines the ordering on $\Phi$.
9. $\{\omega_1, \cdots, \omega_n\} \subset \Phi^+$ is the set of simple roots in $\Phi^+$.
10. $q_i := \left(\frac{4}{9}\right)^{\#\Phi_i - 1}$ if $\dim V_{\lambda_i} > 1$, otherwise $q_i := \left(\frac{4}{9}\right)^{\#\Phi_i - 2}$.

More details about the aspects of root systems and weights we will use can be found in [18, Section 3] and the references therein. The concepts listed above will play a role in the explicit bounds we will give for the correlations that we will now introduce.

2.2. Actions and unitary representations. Let $(X, \mu)$ be a probability space, $\mathcal{H} = L_0^2(X)$ the Hilbert space of square integrable functions on $X$ orthogonal to the constants, $\langle \rangle$ the inner product, $\mathcal{L} = L_0^\infty(X) \subset \mathcal{H}$ and $\sigma$ a measure-preserving, mixing action of $\mathfrak{G}$ on $X$; we always use the notation $g \cdot x$ for $\sigma(g)(x)$. The action on $\mathcal{H}$ defined by

$$(g \cdot f)(x) := f(g^{-1} \cdot x)$$

is a unitary representation of $G$; we call $g \cdot f$ a translate of $f$ by $g$, suppressing mention of the action. Note that the representation is multiplicative, i.e. it distributes over pointwise (and a.e. pointwise) products of functions:

$$g \cdot (f \cdot h) = (g \cdot f)(g \cdot h).$$
Furthermore, for each irreducible component \( V_i \) of \( V \), the representation \( \sigma|_{V_i} \) has no fixed vectors in \( \mathcal{H} \). A fixed vector would give an invariant matrix coefficient on \( \mathcal{H} \), but mixing implies the decay of all such coefficient.

For any linear space of functions on \( X \), a subscript \( K \) denotes \( K \)-finite functions in that space. A function \( f \) is called \( K \)-finite if the space \( \langle K \cdot f \rangle \subset \mathcal{H} \) is finite dimensional. We will deal especially with the algebra of \( K \)-finite bounded functions \( L^K \) and \( L^2 \)-dense subspaces in it. As the representation induced on \( L^2(X) \) by the action is unitary, by the Peter-Weyl theorem \( K \)-finite vectors are dense in \( L^2(X) \).

2.3. Form of the bounds. After this setup, we can now describe the kind of correlations we will bound and in what way. Recall the notation in the previous section. Let \( f_i \) be bounded, \( K \)-finite functions on \( X \). For simplicity, assume that \( V = V_1 \) so \( \rho \) is irreducible. Abbreviate \( \rho_1 \) and \( \lambda_1 \) by \( \rho \) and \( \lambda \).

Of course, in the general case we can restrict to each irreducible component, obtain bounds there and choose the best. The mixing property provides us this luxury.

Let \( a_i \in D^+ \) be the acting elements; for uniformity, define \( a^0 = 1 \). Order the elements according to their \( \lambda \) values, i.e. \( |\lambda(a^k)| > |\lambda(a^i)| \) for \( i > j \).

**Theorem 2.2.** With notations as above, there exists an \( L^2 \)-dense subspace \( \mathcal{D} \) of bounded, zero mean, \( K \)-finite vectors such that, for

\[
\min \left( \min_{i=0,\ldots,k-1} |\lambda\left(\frac{a^k}{a^i}\right)|, \min_{i=1,\ldots,k} |\rho\left(a^i\right)^{-1}| \right) > C',
\]

\( C' \) independent of the \( f_i \),

\[
\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) d\mu(x)
\]

\[
\leq s^2 C \prod_i ||f_i||_{\infty} \dim(K \cdot f_i) \left( \sum_{i=1}^{k-1} \lambda\left(\frac{a^k}{a^i}\right) \right)^{-\frac{3}{2}} \left( \sum_{i=1}^{k} \rho\left(a^i\right)^{-1} \right)^{-\frac{3}{2}}
\]

where \( s \) is a parameter depending only on the \( f_i \) and \( C \) depends only on \( G \) and the action, but not the \( f_i \).

In fact, this is weaker than what we will prove in Section 4, but this version is given here for simplicity.

Notice that the \( a^i \) are in a positive Weyl chamber (in \( D \)) and a given weight is evaluated on them in both terms on the right hand side, so the summands do not cancel each other. The space \( D \) and the parameter \( s \) will become explicit in the next sections; in later sections we will examine actions for which \( s \) can be eliminated and \( D \) is replaced by all bounded, zero mean \( K \)-finite vectors satisfying appropriate smoothness assumptions.

\[5\] So we see that all we need in fact is that there are no invariant vectors in \( \mathcal{H} \) for any \( V_i \).
With this uniformity, we will then be able to pass to arbitrary Sobolev vectors (of sufficiently high order) by using an observation from \cite{8} based on a computation in \cite[Lemma 4.4.2.3]{19}.

2.4. **Approximate Projections.** Given a finite dimensional normed vector space \((V, \|\cdot\|)\) over \(\mathbb{K}\) with \(\mathbb{K}\)-invariant norm, we denote by \(\hat{V}\) the unitary dual, i.e. the topological group of all additive unitary characters of \(V\). For \(x = (x_i), y = (y_i) \in V\) let

\[(x, y) = \sum x_i y_i\]

be the standard bilinear form on \(V\). Choosing a fixed non trivial unitary character \(\zeta\) of \(\mathbb{K}\), define the map \(V \rightarrow \hat{V}\) by

\[v \mapsto \zeta((v, \cdot)) =: \zeta_v\]

This correspondence is a topological group isomorphism between \(V\) and \(\hat{V}\) through which we will usually identify the two. In this situation, given \(v, w \in V\), we denote \([v, w] = \zeta_v(w)\).

Under \((\cdot, \cdot)\) we naturally define the transpose of a linear operator; define \(\rho^* : G \rightarrow GL(V)\) to be the inverse transpose of \(\rho\),

\[\rho^*(v) := (\rho^{-1})^T(v)\]

This provides an identification of the dual action of \(G\) on \(V^*\) with the action \(\rho^*\) on \(V\), given the topological isomorphism above. Furthermore, if \(\rho\) is irreducible and excellent on \(V\), so is \(\rho^*\); finally, \(\| \cdot \|\) is \(\rho^*(\mathbb{K})\)-invariant as well. See Section 6.1 of \cite{18} for these facts.

For \(f \in L^1(V)\) and \(\chi \in \hat{V}\), define the Fourier transform

\[\hat{f}(\chi) = \int_V \overline{\chi(v)} f(v) \, dm(v)\]

Where \(dm(v)\) is a Haar measure on \(V\). Using the topological identification of \(V\) and \(\hat{V}\), we can view the Fourier transform as a function on \(V\) by the formula

\[\hat{f}(w) = \int_V \zeta_{-w}(v) f(v) \, dm(v) = \int_V [-w, v] f(v) \, dm(v)\]

in the bracket notation of the pairing.

We will use repeatedly the following theorems (Plancherel, inversion and duality):

**Theorem 2.3.** There is a normalization of the dual Haar measure \(dm(\chi)\) on \(\hat{V}\) so that:

1. The Fourier transform extends to an isometry \(L^2(V) \rightarrow L^2(\hat{V})\).
2. If both \(f\) and \(\hat{f}\) are integrable, then for almost every \(v \in V\)

\[f(v) = \int_{\hat{V}} \chi(v) \hat{f}(\chi) \, dm(\chi)\]
Every $v \in V$ defines a unitary character of $\hat{V}$ through the pairing $(v, \chi) \rightarrow \chi(v)$ which furnishes a canonical topological isomorphism between $V$ and $\hat{V}$.

The Schwartz-Bruhat space $\mathcal{S}(V)$ is just the usual Schwartz space when $\mathcal{K}$ is Archimedean; in the non-Archimedean case, it consists of compactly supported, locally constant functions on $V$. The main properties of $\mathcal{S}(V)$ are that its functions are dense in $L^2(V)$ and the Fourier transform furnishes a topological isomorphism $\mathcal{S}(V) \simeq \mathcal{S}(\hat{V})$. For more details about the Fourier analysis facts we will use [17].

Given a Schwartz function $\phi$ on $\hat{V}$ and $f \in L^1_K$ define

\begin{equation}
(2.7) \quad P_\phi(f) := \int_V \hat{\phi}(x) (x \cdot f) \, dm(x)
\end{equation}

Here we use the formulation of [9], Chapter 11 for Banach-space valued integrals. Because of the rapid decay of the Fourier transform $\hat{\phi}$, $P_\phi(f)$ retains differentiability properties of $f$ and the inequality

\begin{equation}
(2.8) \quad ||P_\phi(f)||_p \leq ||\hat{\phi}||_{L^1(V)} ||f||_p, \quad 1 \leq p \leq \infty
\end{equation}

shows that it is bounded on all the spaces we will consider. Some structural properties of this operator (the case $\mathcal{K} = \mathbb{R}$ is worked out in [6], chapter I; the general case has no new features regarding these properties) include

- $P$ is self-adjoint (with respect to the inner product of $\mathcal{H}$) for real $\phi$; more generally, $P_\phi^* = P_{\hat{\phi}}^*$.
- Operator multiplication transforms to pointwise multiplication of functions:

\begin{equation}
(2.9) \quad P_{\phi \psi} = P_\phi \circ P_\psi
\end{equation}

This property plus linearity in the subscript shows that $P$ is a homomorphism from the pointwise algebra of Schwartz functions to self adjoint operators on $\mathcal{H}$.
- Given the context of Section 2, for $g \in G$,

\begin{equation}
(2.10) \quad \sigma(g) P_\phi \sigma(g^{-1}) = P_{\phi \rho(g^{-1})}.
\end{equation}

When $\phi$ is $K$-invariant, this equation implies that $P_\phi$ commutes with the $K$-action and thus $K$-finite vectors are $L^2$-dense in the range of $P_\phi$.

Note that we will usually identify $\phi \in \mathcal{S}(\hat{V})$ with $\phi(\zeta) \in \mathcal{S}(V)$. With that identification, the action of $G$ in (2.10) corresponds to the representation $\rho^*$ on $\mathcal{S}(V)$, i.e. when we think of $\phi$ as a function on $V$, we have

\begin{equation}
(2.11) \quad \sigma(g) P_\phi \sigma(g^{-1}) = P_{\phi \rho^*(g^{-1})}.
\end{equation}

Convergence and limits involving $P$ are obtained using positivity: for $\phi \geq 0$, $P_\phi$ is a positive semidefinite operator. To see this, simply use (2.9)
and self-adjointness:
\[
\langle P\phi(f), f \rangle = \langle P\sqrt{P\phi(f)}, f \rangle = \langle P\sqrt{P\phi(f)}, P\sqrt{P\phi(f)} \rangle \geq 0
\]
This way we see that \( P\phi \geq P\psi \) and thus \( ||P\phi||_2 \geq ||P\psi||_2 \) when \( \phi \geq \psi \). Thus, if \( \phi_j \) increase or decrease monotonically to a bounded function on \( V \), the \( P\phi_j \) converge strongly to a bounded, self-adjoint operator on \( \mathcal{H} \). Although we will mostly deal directly with the \( P\phi \), since we cannot guarantee control on the \( L^\infty \) norm for the limits in general, we will use them as a tool to simplify calculations. Of course, under additional assumptions about the smoothness of the vectors, integration by parts in (2.7) transforms the sequence into one which is \( L^p \)-convergent for any \( p \geq 1 \), but since we want to treat \( K \)-finite vectors that will not be necessarily smooth, we avoid the use of the limit operators as such.

Now let \( S \) be a subset of \( V \) with the property that its characteristic function \( \chi_S \) can be pointwise approximated by a sequence of decreasing compactly supported Schwartz functions; we call such sets admissible and all sets we will deal with will be admissible. In particular, we will be interested in the annuli
\[
\text{Ann}(s) := \{ x \in V | s^{-1} < ||x|| < s \}.
\]
Recall that the norm on \( V \) is assumed \( K \)-invariant. The characteristic function of each annulus \( \chi_{\text{Ann}(s)} \) can be approximated by a sequence of smooth functions with the properties
\[
\phi^k_s \equiv 1 \text{ on } \text{Ann}(s)
\]
\[
\text{supp}(\phi^k_s) \subset \text{Ann}\left(s + \frac{1}{k}\right)
\]
\[
\phi^k_s \leq \phi^l_s \text{ for } l \leq k
\]
From this definition, the sequence \( P_{s,k} := P_{\phi^k_s} \) consists of positive, decreasing, self-adjoint (see [6] for the easy computation) bounded operators on \( L^2(X) \) and thus has a strong limit for fixed \( s \) as \( k \) tends to infinity which by (2.9) is idempotent, since \( \phi^k_s \to \chi_{\text{Ann}(s)} \). Note that the image under \( P_s = \lim P_{s,k} \) of \( L^\infty_0(X) \) is \( L^\infty_0 \)-dense in \( L^\infty_0 \) since the \( P_s \) form a system of projections that tends to the identity operator in \( L^2(X) \) as \( s \) goes to infinity.

The properties of \( P\phi \) listed above imply trivially some important facts:
- If \( \text{supp}(\phi) \subset S \), then
  \[
  P_S(P\phi) = P\phi
  \]
  (2.12)
- If \( S \) is invariant under rotations, then for any \( g \in K \),
  \[
  \sigma(g)P_S\sigma(g^{-1}) = P_S
  \]
  (2.13)
- By the previous property, when \( S \) or \( \phi \) are \( K \)-invariant, \( P_S \) or \( P\phi \) commutes with the action of \( K \) and thus \( K \)-finite vectors are dense in the range of \( P_S \) or \( P\phi \).
The following ad hoc notation will be convenient: if \( P_S(f) = f \) for some set \( S \), we say that the 'spectral support' of \( f \) lies in \( S \); when \( S \) is replaced in the subscript by a Schwartz function, the notion will refer to its support. Intuitively, \( P_S \) restricts the 'spectrum' of \( f \) to lie in \( S \), so a function which is unaffected by this application is justified in being called 'spectrally supported in \( S \)'. Note that \( P_S(L^2) \) is a closed vector subspace of \( L^2 \) since the \( P_S \) are norm bounded (for fixed \( S \)).

With this notation in hand, we can define explicitly the dense subspace of \( L_K \) where we will bound the coefficients effectively.

**Definition 2.4.** We define \( D \) to be the union

\[
D = \bigcup_{s>0, k>s^2} P_{\phi_k}(L_K)
\]

and call it the space of spectrally bounded functions in \( L_K \).

It is easy to see that this space is \( L^2 \)-dense in \( L_K \). Of course the specific choice \( k > s^2 \) is not important, we just need some leeway for approximations and we do not want \( k \) to be too small as to cause problems with stretching annuli.

### 3. Main Results

**3.1. Behavior and bounds on projection operators.** First, we need a lemma on how pointwise multiplication of functions behaves with respect to the operators \( P_{\phi} \). Below we identify \( \hat{V} \) with \( V \) and the two dual Haar measures by the isomorphism in 2.3 (compatibility in the computations below is guaranteed by (2.6)). Recall the notation \([u, z] = \zeta_u(z)\) for \( u, z \in V \) (keep in mind the usual case \([u, z] = e^{i\langle u, z \rangle}\)). The result is

**Lemma 3.1.** Let \( \phi, \psi \in \mathcal{G}(V) \) and \( f, g \in L^2(X) \) be such that the pointwise (a.e.) product \( P_{\phi}(f)P_{\psi}(g) \) is in \( L^2(X) \). Suppose \( \omega \in \mathcal{G}(V) \) is identically equal to one on \( \text{supp}(\phi) + \text{supp}(\psi) \); then \( P_\omega(P_{\phi}(f)P_{\psi}(g)) = P_{\phi}(f)P_{\psi}(g) \).

**Proof.** Compute:

\[
P_\omega(P_{\phi}(f)P_{\psi}(g)) = \int \hat{\omega}(z) \int \hat{\phi}(x) \rho(z + x)f \ dm(x) \int \hat{\psi}(y) \rho(z + y)g \ dm(y) \ dm(z)
\]

\[
= \int \hat{\omega}(z) \int \hat{\phi}(x) \rho(x) \ dm(x) \int \hat{\psi}(y - z) \rho(y)g \ dm(y) \ dm(z)
\]

\[
= \int \int \rho(x)f \rho(y)g \int \hat{\omega}(z) \hat{\phi}(x - z) \hat{\psi}(y - z) \ dm(z) \ dm(x) \ dm(y)
\]
Now expand the inner integral using the definition of the Fourier transform, valid for $L^1$ functions:

\[
\int \hat{\omega}(z) \hat{\phi}(x - z) \hat{\psi}(y - z) \, dm(z)
= \int \int \int \omega(u_3)[-z, u_3] \phi(u_1)[- (x - z), u_1] \\
\cdot \psi(u_2)[- (y - z), u_2] \, dm(u_2) \, dm(u_1) \, dm(u_3) \, dm(z)
= \int \phi(u_1)[-x, u_1] \psi(u_2)[-y, u_2] \\
\cdot \left( \int \int \omega(u_3)[-z, u_3] [z, u_1] [z, u_2] \, dm(u_3) \, dm(z) \right) \, dm(u_1) \, dm(u_2)
= \int \phi(u_1)[-x, u_1] \psi(u_2)[-y, u_2] \\
\cdot \left( \int [z, u_1 + u_2] \int \omega(u_3)[-z, u_3] \, dm(u_3) \, dm(z) \right) \, dm(u_1) \, dm(u_2)
\]

The integral in the parentheses is simply

\[
\int [z, u_1 + u_2] \hat{\omega}(z) \, dm(z) = \omega(u_1 + u_2) = 1
\]

by Fourier inversion and the fact that $u_1 \in \text{supp}(\phi)$, $u_2 \in \text{supp}(\psi)$. Untangling the remaining integrals we get the required result. \qed

**Corollary 3.2.** Let $\text{supp}(\phi) \subset S$ and $\text{supp}(\psi) \subset T$ for admissible sets $S$ and $T$. Then

(3.1) \[ P_{S+T}(P_\phi P_\psi) = P_\phi P_\psi \]

The relations (3.2) and (2.12) form the core of the main computation.

In the sequel, we will examine how restricting a unit (in the $L^2$-norm) $K$-finite vector $f$ to the image of an approximate projection $P_\phi$ for suitable $\phi$ affects its norm. This was accomplished in greater generality in [18] from which we will draw notation and results, noting the places in that paper where they are treated. The idea of estimating matrix coefficients (non-uniformly) by looking at the effect the representation has on their 'spectral support' (i.e. the smallest set with an approximate characteristic function $\phi$ such that $P_\phi(f) = f$) and then estimating norms of functions with restricted spectral support is a major theme in chapter 5 of [6]; the non-commutativity of $K$ in our setting increases the complexity of this method considerably. However, the detailed analysis in [18] allows one to carry it out effectively.

In order to state the second main lemma and principal ingredient for bounding norms of projected vectors, we need some additional concepts from [18]. Recall the list of notations from Section 2 and assume that $\rho = \rho_1$ is irreducible with highest / lowest weights $\lambda, \varrho$. For $\psi \in \Phi_1$, Let $\pi_\psi(v)$ be the projection of $v$ on the weight space $V_\psi$. 
Define the 'cones'

\[ \text{Cone}_1(c, s) = \{ v \in V : \|\pi_\lambda(v)\| \leq c \text{ and } \|v\| \geq s \}, \]
\[ \text{Cone}_2(c, s) = \{ v \in V : \|\pi_\phi(v)\| \leq c \text{ and } \|v\| \geq s \}. \]

See [13] Proposition 6.1 for the fundamental properties of these sets. We will not use Proposition 6.1 itself here, but we will follow verbatim the computations in Proposition 7.1 which uses Proposition 6.1 in a crucial way.

Observe that the norm \( \| \cdot \|_{\infty} \) on \( V \) defined by

\[ \|v\|_{\infty} = \max_{\phi \in \Phi_1} \|\pi_\phi(v)\| \]

is equivalent to the given norm since \( \dim(V) < \infty \), so in particular \( \|v\|_{\infty} \leq C \|v\| \); we will use this observation below.

**Lemma 3.3.** Let \( f \) be \( K \)-finite with \( \|f\|_2 = 1 \), \( \dim(K \cdot f) = d_f \), \( a^i \in D^+ \) for \( i = 1, \cdots, k \) ordered in increasing \( \pi_\lambda(a^i) \) with sufficiently high norms\(^6\) depending only on the action, \( \text{Ann}(s) \) the annulus defined in Section 2.4 and \( F_s \in \mathcal{S}(V) \) with compact support inside the set

\[ X_1(a, s) = \text{Ann}(s) \cap \left( \sum_i \rho^*(a^i) \left( \text{Ann}(s^{-1}, s) \right) \right). \]

Then for some positive \( C \) independent of \( a \), \( s \) and \( f \) we have the bounds

\[ \|P_{F_s}(f)\|_2 \leq Cs^q d_f^{\frac{q}{2}} \sum_i \phi(a^i)^{-1} |^{-\frac{q}{2}}. \]

Similarly, if the support of \( F_s \) is in the set

\[ X_2(a, s) = \text{Ann}(s) \cap \left( \sum_i \rho^*(\frac{a^i}{a^k}) \left( \text{Ann}(s^{-1}, s) \right) \right), \]

then as above

\[ \|P_{F_s}(f)\|_2 \leq Cs^q d_f^{\frac{1}{2}} \lambda \left( \frac{a^k}{a^i} \right)^{-\frac{1}{2}}. \]

**Proof.** The proof is essentially contained in the proof of Proposition 7.1 of [13]. We indicate how to extract the relevant parts for our lemma and explain the correspondences. All references to numbered sections will belong to [13]. We will only examine the first situation, since the second one is identical.

In the course of proving that proposition, the author examines (pages 31-38) a Schwartz function \( \hat{\Pi}(F_s) \eta \) and the projection of a unit vector with respect to that function, bounding the Hilbert space norm of

\[ \hat{\Pi}(F_s^{\frac{1}{2}}) \eta \]

\(^6\)In the sense of [18].

\(^7\)The reason we use \( F_s \) for the function rather than the usual Greek letters is to facilitate the comparison with the computation in [13].
in the notation of that paper, where
\[ F_s = (a\omega)^{-1}(h_s \cdot g_s)^{1/2}; \]
in our notation \( \eta \) is \( f \) and \( \hat{\Pi}(F_s^2) = P_{F_s^+}, \omega = 1 \) because we are only considering the positive Weyl chamber and the precise definition of \( F_s \) in [18] is irrelevant; in order to carry out the computations, we only need \( F_s \) to be Schwartz and its support contained in one of the two cones defined above, for specific \( 0 < c \ll 1 \) and \( s > 0 \). There, it is claimed that how small \( c \) needs to be depends on \( s \); however, the only dependence of \( c \) on \( s \) that is necessary there is that \( cs^{-1} < C \) with \( C \) depending only on the action. Since our \( c \) here is going to be of the form \( c = sA \) where \( A \) does not involve \( s \), we see that in order to ensure \( cs^{-1} < C \) all we need is to bound \( A \) by a constant depending only on the action; this translates to the norm bounds on the \( a^i \) in the statement. See p.27 of [18] for the specific requirements on \( c \).

By the discussion above, the reductions from page 31 to page 34 carry over to our \( F_s \), resulting in the situation where we want to bound
\[ \| \hat{P}_s f \|_2 \]
where \( \hat{P} \) is the approximate projection operator for the regular representation of \( K \times V \) on \( \mathcal{H} \) and \( f \) is a \( K \)-invariant unit norm vector. At that point we use the containment of the supports. In our case, observe that our \( X_1(a, s) \) is contained in the set
\[ E_1 = \{ v \in \text{Ann}(s) : s^{-1} \leq \| \sum_{i=1}^k \rho^s(a^i)v \| \leq s \} \]
which becomes, after writing \( v \) in terms of the weights, applying \( \rho^s \) to each coordinate in \( V_\psi \) and switching summations
\[ E_1 = \{ v \in \text{Ann}(s) : s^{-1} \leq \| \sum_{\psi \in \Phi_1} (\sum_{i=1}^k \psi(a^i)^{-1})\pi_\psi(v) \| \leq s \}. \]
Note the inverses because we are decomposing with respect to the \( \Phi_1 \) of \( \rho \). Using the equivalence of norms \( \| \cdot \| \) and \( \| \cdot \|_\infty \) we see that this set is contained in
\[ S_1 = \{ v \in \text{Ann}(s) : \| \pi_\rho(v) \| \leq C|s| \sum_{i=1}^k \rho(a^i)^{-1}|^{-1} \}. \]
Since \( a^i \in D^+ \), for large \( \min_{1 \leq i \leq k} |a^i| \) (in any norm on \( G \)) the coefficient on the right hand side of the definition on \( S_1 \) is going to be small. In particular, \( S_1 \) will be contained in the cone
\[ \text{Cone}_1(C|s| \sum_{i=1}^k \rho(a^i)^{-1}|^{-1}, s^{-1}). \]
Having this containment, the argument from pages 34-37 goes through without change leading to the desired conclusion analogous to (7.22), (7.23) there.

□

Since admissible sets can be approximated from above by Schwartz functions and the operators $P_s,k$ are monotone, we get the corollary

**Corollary 3.4.** With notation as in Lemma 3.3 and $S$ an admissible set contained in one of the $X_i(a_i,s)$, we have the corresponding bound from that lemma for $\|P_S(f)\|_2$.

### 3.2. Main theorem

Now consider $k \geq 2$ distinct elements $a^i \in D^+, i = 1 \cdots k$ as above and $k + 1$ functions $f_i \in D, i = 0 \cdots k$ of $L^2$ norm 1. Order the $a^i$ in increasing highest weight valuations and define $a^0 = I$. We want to bound the correlation integral

$$\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x).$$

Since $P_{s,k}(f) \to f$ for any $f \in L^2$ and we only have finitely many $f_i$, we can assume that all $f_i$ are in the image of $P_{s,l}$ for some $s,l$ (and thus certainly in the image of $P_{s'}$ where $s' = s + \frac{2}{3}$).

For notational convenience, abbreviate $\text{Ann}(s)$ by $(s)$ and denote its image under $\rho^*(a^i)$ simply by $a^i(s)$. We will also denote the action of $\rho^*(a^i)$ on the $\phi_k^i$ defined above simply by $a^i(s,k)$.

**Theorem 3.5.** Let $a^i, f_i, s$ be as above. Let

$$d_i = \dim \langle K \cdot f_i \rangle.$$

There exists a positive constant $C'$ independent of the $f_i$ such that if

$$\min_i(\min_{i=0,\ldots,k-1} |\lambda(a^k_{a^i})|, \min_{i=1,\ldots,k} |\rho(a^i)^{-1}|) > C',$$

we have the bound

$$\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x)$$

$$\leq C s^{2a_d} d_0^{\frac{1}{d}} d_k^{\frac{1}{d}} \left| \sum_{i=1}^{k-1} \lambda(a^k_{a^i}) \right|^{-\frac{3}{2}} \left| \sum_{i=1}^{k} \rho(a^i)^{-1} \right|^{-\frac{3}{2}}$$

where $C$ only depends on the $L^\infty$ norms of the $f_i$.

**Proof.** The correlation can be written as

$$\int_X P_{s,l}(f_0) a^1 \cdot P_{s,l}(f_1) \cdots a^k \cdot P_{s,l}(f_k)$$

which by (2.10) becomes

$$\int_X P_{s,l}(f_0) P_{a^i(s,l)}(a^1 \cdot f_1) \cdots P_{a^k(s,l)}(a^k \cdot f_k).$$
We now use Lemma 3.1 repeatedly to conclude that
\[ P_{\alpha_1(s,l)}(a^1 \cdot f_1) \cdots P_{\alpha_k(s,l)}(a^k \cdot f_k) \in \mathcal{P}(\mathcal{L}) \]
where \( \Sigma \) is the iterated sum set \( \sum_{i=1}^k a^i(s') \); thus in particular if
\[ z := P_{\alpha_1(s,l)}(a^1 \cdot f_1) \cdots P_{\alpha_k(s,l)}(a^k \cdot f_k) \]
then \( P_\Sigma(z) = z \). Thus the integral above becomes
\[ (3.9) \int_X P_{s,l}(f_0) P_\Sigma(z) \]
Now \( P_\Sigma \) is an orthogonal projection so we can transfer \( P_\Sigma \) from \( z \) to \( P_{s,l}(f_0) \), getting
\[ P_\Sigma(P_{s,l}(f_0)) = P_{\chi_\Sigma \phi_s^l}(f_0). \]
Here we are abusing notation a little bit, since the last expression need not be a bounded function; we will take this shortcut to mean that we have an arbitrary Schwartz function \( \phi \) dominating the function \( \chi_\Sigma \) and we are applying \( P_{\phi} \) to both terms of the 'inner product'; the rightmost term is unaffected, while the leftmost has spectral support approximately equal to that of \( \chi_\Sigma \phi_s^l \) since \( \phi \) is arbitrary and the support of \( \chi_\Sigma \phi_s^l \) is easily seen to be an admissible set (also see Corollary 3.3). Thus the integral becomes
\[ \int_X P_{\chi_\Sigma \phi_s^l}(f_0) \]
Write \( U_0 := \chi_\Sigma \phi_s^l \) and apply \( (a^k)^{-1} \) to all terms of the integral, giving
\[ (3.10) \int_X (a^k)^{-1} \cdot P_{U_0}(f_0) P_{s,l}(f_k) \prod_{i=1}^{k-1} (a^k)^{-1} \cdot a^i \cdot P_{s,l}(f_i) \]
By unitarity, the value of the integral is not affected. So, now we can repeat the reasoning above, summing the indices for all factors except for \( P_{s,l}(f_k) \), and conclude that this integral is equal to
\[ (3.11) \int_X (a^k)^{-1} \cdot P_{U_0}(f_0) P_{s,l}(f_k) \prod_{i=1}^{k-1} (a^k)^{-1} \cdot a^i \cdot P_{s,l}(f_i) \]
\[ = \int_X P_{s,l}(f_k) P_{\Sigma_k}(z_k) \]
\[ = \int_X P_{\chi_\Sigma \phi_s^l}(f_k) z_k \]
\[ (3.12) = \int_X P_{U_0}(f_0) a^k \cdot P_{U_k}(f_k) \prod_{i=1}^{k-1} a^i \cdot P_{s,l}(f_i) \]
where
\[ \Sigma_k = \sum_{j=0}^{k-1} (a^k)^{-1} \cdot a^j(s'), \]
\[ z_k = \prod_{i=0}^{k-1} (a^k)^{-1} \cdot a^i \cdot P_{s,l}(f_i) \]
and \( U_k = \chi_{\Sigma_k} \phi^f_s \).

Denote by \( U_0 \) and \( U_k \) respectively also the supports of the corresponding functions (which, note, are bounded above by 1 and thus by the characteristic functions of the supports). We can now immediately apply Lemma 3.3 with \( U_0 \) and \( U_k \) in the place of the two situations for \( F_s \) considered there, bounding
\[ \| P_{U_0}(f_0) \|_2 \text{ and } \| a^k \cdot P_{U_k}(f_k) \|_2 = \| P_{U_k}(f_k) \|_2. \]

In order to finish the proof, we simply bound (3.12) by the \( L^\infty \) norms of the functions \( f_i \) for \( i \neq 0, k \) and then use Cauchy’s inequality on the two remaining terms to finish the proof. \( \square \)

3.3. Examples. In this section we see what the bounds obtained above mean for two particular cases of semidirect products. The choice of these examples is not arbitrary: these groups will occur in Section 5 as subgroups (locally) of split simple groups of higher rank.

First consider the case of \( G = SL(2, K) \ltimes K^2 \) where the action is the standard matrix action on 2-vectors. The action is irreducible and there are only two weights; the roots of \( SL(2, K) \) are
\[ \text{diag}(a, a^{-1}) \rightarrow a^2 \]
and the weights of the standard representation on \( K^2 \) are
\[ \text{diag}(a, a^{-1}) \rightarrow a \]
with the obvious weight spaces \( V_1 = \{(v_1, 0) \in K^2\} \) and \( V_2 = \{(0, v_2) \in K^2\} \).

Take the weight
\[ \text{diag}(a, a^{-1}) \rightarrow a \]
to be positive, so this is the highest weight. The highest weight space is one dimensional, so the exponent \( q \) defined in Section 2 is in our case 1. Write
\[ a^i = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix} \]
for \( k + 1 \) elements of the positive Weyl chamber with \( 1 = a_0 < a_1 < \cdots < a_k \)
and if \( i > j \), \( \frac{a_i}{a_j} > C_0 \) for some \( C_0 \) depending on the action of \( G \) on \( X \).

Applying the preceding discussion to the hypotheses of Theorem 3.5, we get
Corollary 3.6. In the setting of Theorem 3.5 and $G = SL(2, K) \rtimes K^2$ with the standard action, we get the bound

$$\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) d\mu(x) \leq C s^2 d_0^\frac{k}{dk} \left( \sum_{i=0}^{k-1} \frac{d_k}{a_i} \right)^{-\frac{1}{2}} \left( \sum_{i=1}^{k} \sum_{i=1}^{a_i} \right)^{-\frac{1}{2}}.$$

Note how in the case $k = 1$ we recover the bound from Chapter 5 of [6].

For the second example, consider the action of $SL(2, K)$ on its Lie algebra over $K$, denoted simply by $\mathfrak{g}$ and being equivalent to $S^2(K^2)$, the second symmetric power of $K^2$ (in the case $\text{char}(K) = 0$ that we are considering). The weights and weight spaces in this case coincide with the roots and the highest weight space (pick $\text{diag}(a, a^{-1}) \to a^2$ as positive) is again one dimensional. Therefore, by the same procedure as above, we have:

Corollary 3.7. In the setting of Theorem 3.5 and $G = SL(2, K) \rtimes \mathfrak{g}$ with the adjoint action on the Lie algebra, we get the bound

$$\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) d\mu(x) \leq C s^2 d_0^\frac{k}{dk} \left( \sum_{i=0}^{k-1} \frac{d_k}{a_i} \right)^{-\frac{1}{2}} \left( \sum_{i=1}^{k} \sum_{i=1}^{a_i} \right)^{-\frac{1}{2}}.$$

4. Extending the bound

In this section we examine the limits of our method and describe the broadest class of subspaces of $L^2_0(X)$ where we can get an effective estimate depending only on the action and various norms of the functions in the correlation integral.

4.1. Beyond spectral restriction. Observe that whenever we have an estimate of the form

$$(4.1) \quad \|P(s)(f) - f\|_2 \leq C\|f\|' s^{-A}$$

for all $s > 0$, $C$ and $A$ independent of $f$ and $\| \cdot \|'$ an appropriate norm, we can use a $2\epsilon$ argument plus the uniform Hölder inequality to eliminate $s$:

$$\left| \int_X f_0 \cdots a^k \cdot f_k dx \right| \leq \sum_{i=0}^{k} \prod_{j \neq i} \|f_j\|_\infty \|P(s)(f_i) - f_i\|_2$$

$$+ \left| \int_X P(s)(f_0) \cdots a^i \cdot a^i \cdot P(s)(f_k) dx \right|$$

$$\leq C \left( \sum_{i=0}^{k} \|f_i\|' \prod_{j \neq i} \|f_j\|_\infty \right) s^{-A} + C' s^{2d_0^\frac{k}{dk} d_0^\frac{k}{dk} \Re(a)}.$$
where $R(a)$ is the factor in (3.6) depending on $a$. Choosing $s = R(a)^\epsilon$ and optimizing for $\epsilon$ to get the best overall exponent, we can get a uniform bound for all $K$-finite vectors with finite $\| \cdot \|'$-norm.

In order to axiomatize this estimate, we introduce, for each $A > 0$, the norms

\begin{align}
\| f \|_{-A} &= \sup_{0 < s < \infty} s^{-A} \| P_{B((0,s))} f \|_2, \\
\| f \|_{+A} &= \sup_{0 < s < \infty} s^A \| P_{B((0,s))} f \|_2, \\
\| f \|_{\pm A} &= \| f \|_{-A} + \| f \|_{+A},
\end{align}

where $B((0,s))$ is the ball centered at the origin of radius $s$ with respect to the given norm on $V$. Define $L^A_K$ to be the subspace of $L^2_K$ where $\| f \|_{\pm A}$ is finite. Note that for each $A > 0$,

$$ P(s)(L^A_K) \subset L^A_K $$

for all $s > 0$ and thus

$$ \mathcal{D} \subset L^A_K. $$

Furthermore, note that $\| \cdot \|_{\pm A}$ is comparable to $\| \cdot \|_2$ on each $P(s)(L^A_K)$, but not on $\mathcal{D}$ or $L^A_K$. However, taking radial functions $\psi_{A,\epsilon}(r)$ that equal $r^A$ on $B((0,1-\epsilon))$, $r^{-A}$ on $B((0,1+\epsilon))$, are smooth and equal to 1 on $B((0,1+\epsilon/2)) \setminus B((0,1-\epsilon/2))$ we see that

$$ \bigcup_{A > 0} L^A_K $$

is $L^2$-dense in $L^A_K$ (because $P_{\psi_{A,\epsilon}}(f) \to f$ as $\epsilon, A \to 0$).

The spaces $L^A_K$ form the broadest category of spaces where our method extends to give effective bounds. This should be understood in the sense that if we use as inputs only Theorem 3.5 and the basic structure of the projection operators $P$, we definitely need an estimate like (4.1) to remove the dependence on $s$. In this broadest class, our main result takes the form

**Theorem 4.1.** Let $f_i \in L^A_K$ of $L^2$-norm 1, $a_i$ and $d_i$ as in Theorem 3.5. Under the assumption (3.5), we have the bound

$$ \int_X f_0(x) a_1 \cdot f_1(x) \cdots a_k \cdot f_k(x) d\mu(x) $$

$$ \leq d_0^{1/2} d_k^{1/2} \left( \prod_{i=0}^k \| f_i \|_\infty \| f_i \|_{\pm A} \right) \left( \sum_{i=1}^k \lambda_{a_i}^{k} \left| \sum_{i=1}^k g(a_i)^{-1} \right| \right)^{-\frac{d_0}{2(k+1)}}. $$

**4.2. Beyond $K$-finiteness.** For smooth actions in the non-Archimedean case, $K$-finiteness is automatic since the stabilizer of any smooth vector is open (and thus of finite index in the maximal compact (open) subgroup of $G$). In the Archimedean case, however, smooth vectors can be far from $K$-finite. In order to pass from $K$-finite vectors to a larger class we can use the argument given in [8, Theorem 3.1]; we consider Sobolev vectors $f$, i.e.
functions on which the \( m \)-fold action of the Casimir element \( \Omega \) of the Lie algebra of \( G \) is defined and the following norm is finite (Theorem 4.4.2.1 in [19] does not require any restrictions on the norm):

\[
\| f \|_{\infty, A, m} := \max(\| \Omega^m(f) \|_{\infty}, \| \Omega^{m}(f) \|_{\pm, A}) < \infty.
\]

For such \( f \) with \( \| f \|_{\infty, A, m} < \infty \) for sufficiently large \( m \), we mimic the computation in [8] line for line: for each projection of each \( f \) onto \( K \)-isotypic components we get the estimate (4.5) which only requires \( \| a^i \| \) to be large enough depending only on the action. Note that each \( K \)-type of a smooth function satisfies the conditions of Theorem 4.1 because of the form of the projection (convolution with a character of a compact Lie group). In place of the Cauchy-Schwartz inequality in [8] we use Hölder’s inequality and the different powers of the \( \dim(\mu) \) that occur do not cause the sum over \( \hat{K} \) to diverge because they are still beaten by a polynomial of greater degree for large enough \( m \) ([19, Lemma 4.4.2.3]).

Write \( \mathcal{L}^A \) for the subspace of \( \mathcal{L} \) defined by the finiteness of (4.4). The result becomes

**Corollary 4.2.** For \( f_i \in \mathcal{L}^A \) with \( \| f \|_{\infty, A, m} < \infty \) for \( m > m_0 \) and \( \| f_i \|_2 = 1 \), \( \| a^i \| > C' \) with \( C' \) and \( m_0 \) depending only on the action, we have the bound

\[
\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x)
\]

\[
\leq \left( \prod_{i=0}^{k} \| f \|_{\infty, A, m}^2 \left( \sum_{0=1}^{k-1} \lambda \left( \frac{a^k}{a^i} \right) \left( \sum_{i=1}^{k} \| g(a^i) \|^{-1} \right) \right) \right)^{-\frac{Aq}{2(A+1q)}}
\]

(4.6)

It is possible to replace the \( L^\infty \) norm in the definition of \( \| f \|_{\infty, A, m} \) with an \( L^2 \) norm by increasing \( m \) (depending on the space \( X \)). Furthermore, one can replace the \( L^\infty \) norms in all the arguments from Theorem 3.5 and beyond by various \( L^p \) norms depending on the number of functions in the correlation, but we chose the simpler and more uniform presentation using \( L^\infty \).

The spaces \( \mathcal{L}^A \) and \( \mathcal{L}^A_K \) contain natural classes of functions. Informally, if we think of \( D \) as the space of functions with finite Fourier series supported away from zero, we should think of \( \mathcal{L}^A \) as a space of functions with rapidly convergent Fourier series (or equivalently, in the Archimedean case, sufficiently smooth functions). We require this rapid decay at zero as well as infinity. The finiteness of the norm \( \| f \|_{-A} \) is a quantitative version of the

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8Actually, even if the magnitude of the acting elements depended on \( s \), the argument still goes through for the following reason: the operators \( P(s) \) commute with the action of \( K \) and \( K \)-isotypic components are given by a convolution over \( K \); therefore, the \( K \)-isotypic components of \( P(s) f \) are the \( P(s) \)-images of \( K \)-isotypic components of \( f \). Since \( s \) does not change throughout the summation, any possible requirement on the magnitude of the acting elements in terms of \( s \) is preserved.
assertion that there are no \( \sigma(V) \)-invariant functions. Note that this finiteness holds at least for functions in the range of \( P_\phi \) for any \( \phi \in C^1(V) \) for Archimedean \( V \) and holds always for smooth vectors in the non-Archimedean case. Furthermore, it is not hard to see that in the Archimedean case, if the matrix coefficient \( \langle \sigma(v) \cdot f, f \rangle \) is smooth with sufficiently many derivatives bounded on \( V \), then \( \|f\|_{+,A} \) is finite; the permissible \( A \) depend on the degree of smoothness.

5. Simple groups of higher rank

5.1. Semidirect products inside simple groups. In this section we describe how to use the results obtained so far to get effective multiple mixing in simple split groups of higher rank (and by extension for semisimple groups with finite center all of whose almost simple factors have higher rank). We do this by locating sufficiently many semidirect products inside the simple group to which we can apply the main results. We keep notation from previous sections when referring to the functions \( f_i \) in the definition of the multiple correlation, the Cartan elements \( a_i \) etc. A detailed description of the reduction process outlined below can be found in [13] using a slightly different language.

In our setting, we are given a simple algebraic group split over \( K \) of rank greater than or equal to 2, a maximal \( K \)-split torus \( D \), root system \( \Phi = \Phi(G, D) \) and ordering \( \Phi^+ \); consider a mixing action \( \sigma \) of \( G \) on a standard probability space \( (X, \mu) \). We want to apply the results above to bound correlation coefficients for the Weyl chamber action \( \sigma(D^+) \) on \( X \). In order to achieve this, following the proof of Proposition (1.6.2) in [11] we do the following: given a positive simple root \( \omega \in \Phi^+ \), we choose another root \( \omega' \) that is not orthogonal to \( \omega \). Then from the Dynkin diagram this pair of roots corresponds to either an \( A_2 \) system or a \( C_2 \) system, so we get a surjective morphism \( SL(3) \to \langle U_{\pm \omega}, U_{\pm \omega'} \rangle =: G_\omega \) (respectively \( Sp(4) \to \langle U_{\pm \omega}, U_{\pm \omega'} \rangle =: G_\omega \) ) with finite central kernel. Furthermore, if \( \omega \) corresponds to the root \( \bar{\omega} \) (in either group) then its kernel in \( D \cap G_\omega \) corresponds to the kernel in the diagonal \( A \subset SL(3) \) (resp. \( A \subset Sp(4) \) ) of \( \bar{\omega} \).

So far, we have associated to each root of \( G \) a corresponding torus in the uppermost copy of \( SL(2) \) inside one of our rank 2 groups, along with an isogeny to \( G_\omega < G \) that carries that torus to the image of the one parameter group associated to \( \omega \). In both groups, this \( SL(2) \) copy comes with an action on a vector space \( V \) (\( K^2 \) in the \( SL(3) \) case, \( K^3 \) in the \( Sp(4) \) case; see [1], Sections I.1.4 - I.1.5 for details), i.e. we have an isogeny from \( SL(2, K) \times V \) to its image in \( G_\omega \) with the positive diagonal in \( SL(2) \) mapping to a one parameter semigroup in the positive Weyl chamber of \( G \). Decompose \( D = \ker(\omega)D_\omega \) so that \( D_\omega \) corresponds to the diagonal of \( SL(2) \). The image of the \( SL(2) \) in \( G_\omega \) (i.e. the group generated by \( U_{\pm \omega}, \) see [7] Chapter XI) commutes with \( \ker(\omega) \) by observing that their Lie algebras commute. The greatest significance of this observation is that any maximal compact subgroup \( K_\omega \)
of that image commutes with \( \ker(\omega) \). This fact plus the \( K \)-finiteness of the \( f_i \) imply the \( K_\omega \)-finiteness of the translates of the \( f_i \) by elements in \( \ker(\omega) \); note that these translates are no longer necessarily \( K \)-finite.

The second consideration we need involves the divergence of the acting elements \( A \in D^+ \) to infinity in the group \( G \). Our theorems, being about semidirect products, were formulated so that the divergence hypothesis takes the form given in inequality (2.2). In the group \( G \), the quickest way to link divergence to (2.2) is form a norm \( \| \cdot \| \) on \( D \) by transporting the norm from the Lie algebra (normed as a \( K \)-vector space) via a complete collection of one parameter subgroups spanning \( D \) defined over \( K \). Then the correspondence between the norms in \( V \) that appear in the bound (3.6) and norms in \( G \) is given by a positive constant in the exponent, using the fact that all norms in a finite dimensional \( K \)-space are equivalent.

Now suppose \( A \in D^+ \) acts on a function \( f \) in the action defined above; we can write \( A = aC \) with \( a \) being (the image in \( D^+ \) of) an element of the diagonal group of \( \text{SL}(2) \), \( C \) centralized by the maximal compact of that \( \text{SL}(2) \), \( f \) affords an action of \( \text{SL}(2) \ltimes V \) with no invariant vectors for \( V \) on \( L^2(X) \) (mixing descends to subgroups and an isogeny has finite kernel, so we get no invariant vectors for the \( V \) factor). Doing this for all terms in a correlation \( A^i \cdot f_i \) we get \( K \)-finite vectors \( f \) for an action of \( \text{SL}(2, K) \ltimes V \) and we can thus apply the results obtained so far to that context. In order to ensure that at least some tuple \( (a) \) among the various choices goes to infinity (by the remarks of the previous paragraph, we can talk about distance either in terms of norm on \( G \) or in terms of the weights of ratios of the \( a \)), we use the assumption that the original tuple \( A \) goes to infinity and for each root there will be at least one other root (perhaps the same) so that the corresponding one parameter subgroups have a large ratio; if the original tuple goes to infinity, this process can be carried out for all \( f_i \) and at this point we can apply our results for \( \text{SL}(2, K) \ltimes V \) mentioned in Corollaries 3.6 and 3.7 to bound the correlation.

If \( G \) is a semisimple split group without compact factors and all its simple factors satisfy the properties posited above, we can extend the discussion above to this setting. Note that our methods cannot extend to groups of rank 1 and their products. To see this, observe that for the 2-correlations our results essentially reduce to those of [18] with some modifications or [6], Chapter V, and prove effective mixing for mixing actions of each group they apply to. Since groups of rank 1 by themselves do not need be effectively mixing (take for example the decay for the complementary series in [6] Chapter V, Proposition 3.1.5), the method is inapplicable to that case.

5.2. From \( SL(n, K) \) to \( SL(2, K) \ltimes K^2 \). Let’s illustrate the procedure above in the simple context of \( SL(n) \). The first step in the derivation of our bound is to drop from \( SL(n, K) \) to the semi-direct product \( SL(2, K) \ltimes K^2 \). In this section, denote by \( K(n) \) the maximal compact of \( SL(n) \). We will not describe the reduction in detail here, since it can be found in detail in pages 227 -
228 of [6] in the archimedean case, and the non-archimedean case is similar. We simply give a pictorial sketch of the reduction:

- $\text{SL}(2, \mathcal{K}) \rtimes \mathcal{K}^2$ embeds in $\text{SL}(n, \mathcal{K})$ in the following ways (all elements not depicted are zero, 1 on the diagonal).

$$\text{SL}(2, \mathcal{K}) \rtimes \mathcal{K}^2 \ni \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$j, l \rightarrow \begin{pmatrix} 1_j & 1_l \\ \vdots & \vdots \\ j & a & \cdots & b & x \\ \vdots & \vdots & \vdots \\ l & c & \cdots & d & y \\ \vdots & \vdots & 1 \end{pmatrix}$$

- In our bound, we have diagonal elements acting. We can extract a single element in the diagonal group of $\text{SL}(2)$ as follows:

$$a = \begin{pmatrix} \cdots & a_j & \cdots \\ & \cdots & a_l & \cdots \\ & & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a \\ b & \cdots \\ b^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_1 & a' \\ c & \cdots \\ c & a_n \end{pmatrix}$$

where $b$ and $c$ are defined in the archimedean case by $c = \sqrt{a_j a_l}$ and $b = \frac{a_j}{c}$; in the non-archimedean case, we have the same definition if the difference of exponents of $q$ is even, otherwise we will compensate by adding and subtracting a

$$\text{diag}(\cdots, q^{\frac{1}{2}}, \cdots, q^{-\frac{1}{2}}, \cdots)$$

to bring the matrix entries back in $\mathcal{K}$; obviously, the decay is not affected by this tweak.

Note that $a'$ commutes with the specific copy of $K(2)$ inside $\text{SL}_{jl}$ so if $f$ is a $K(n)$-finite function, the new function $a' \cdot f$ is $K(2)$-finite for the action of that $\text{SL}_{jl}$ copy.
Therefore, in order to derive estimates on the initial correlation integral, we are led to consider \((X, \mu)\) with a mixing action of \(G := \text{SL}(2, \mathbb{K}) \ltimes \mathbb{K}^2\) and bound integrals of the form

\[
\int_X f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x)
\]

where

\[
a^i = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix}
\]

and the \(f_i\) are bounded, zero mean functions satisfying \(K\)-finiteness properties inherited from the original \(f_i\). From this point on, all we need to do is apply Corollary 3.6 paying attention to the remarks about divergence in Section 5.1.

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