Separators for Planar Graphs that are Almost Trees

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Abstract

We prove that a connected planar graph with \( n \) vertices and \( n + \mu \) edges has a vertex separator of size \( O(\sqrt{\mu} + 1) \), and this separator can be computed in linear time.

1. Result

We first provide a relatively self-contained proof of the claim. A shorter proof using known tools is described in Section 1.1.

Theorem 1.1. Let \( G \) be a vertex connected planar graph with \( n \) vertices and \( n + \mu \) edges, and weights \( w : V(G) \to \mathbb{R}^+ \) on the vertices. Then, one can compute, in linear time, a vertex separator for \( G \) of size \( O(\sqrt{\mu} + 1) \)

Proof: The proof in depicted in Figure 1. Assume \( \mu > 0 \), as otherwise the result is immediate. Let \( T \) be a spanning tree of \( G \), and let \( M \) be the remaining \( \mu + 1 \) edges; i.e., \( M = E(G) \setminus E(T) \). Let \( S \) be the minimal subtree of \( T \) that contains all the vertices of \( V(M) \). Let \( U \) be the set of all vertices of \( S \) that are either of degree three (or higher), or belong to \( V(M) \). Since the only leafs of \( S \) are vertices of \( V(M) \), it follows that \( |U| \leq 2|V(M)| \leq 4(\mu + 1) \).

Decompose \( S \) into maximal set of paths, such that their endpoints are in \( U \) (and no vertex of \( U \) is contained in the interior of such a path), and let \( \Pi \) be this collection of paths. Observe that \( |\Pi| \leq |U| = O(\mu) \).

Consider assigning the weight of every vertex of \( T \setminus S \) to its nearest vertex in \( S \). As such, under the new weights \( w' \), we have \( w'(S) = w(T) \), where a weight of a path is the total weight of the vertices in its interior.

Consider the planar graph \((U, \Pi \cup M)\) with weights on the edges and vertices. It has \( O(\mu) \) vertices and \( O(\mu) \) edges. As such, by Lipton and Tarjan planar separator theorem [LT79], it has a balanced separator \( Z \subseteq V(G) \) of size \( O(\sqrt{|U|}) = O(\sqrt{\mu}) \), as desired.

A vertex \( u \in Z \), with weight \( w'(u) \) might see its weight decrease to \( w(u) \) in the original graph, because of various trees attached to \( u \) with total weight \( w'(u) - w(u) \). Since \( u \) is in the separator, all these trees get separated when \( Z \) is removed. Namely, the separator set \( Z \) is still a balanced separator in \( G \).

The only case where the above argument fails, is if the computed separator has a single vertex \( z \) with majority of the weight (i.e., \( w'(z) > (2/3)w'(U) \)). Furthermore, the vertex \( z \) might have a tree \( T' \) attached to it with weight exceeding \( (2/3)w(G) \). But this can be fixed by just adding the vertex separator \( z' \) of \( T' \) to the separator set, thus implying the claim (in particular, this case, the separator is made out of two vertices \( z \) and \( z' \)).

It is easy to verify that the above algorithm can be implemented in linear time.

The above proof works for any family of graphs that is minor closed and has a sublinear sized separator. In particular, if a graph in this family with \( \mu \) vertices has separator of size \( g(\mu) \), then the above proof implies that a connected graph in this family with \( n \) vertices and \( n + \mu \) edges, has a separator of size \( O(g(\mu)) \).

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(A) A tree with $\mu + 1$ additional edges.

(B) The endpoints of the additional edges (i.e., $M$).

(C) The spanning tree $S$ of these endpoints.

(D) The additional vertices of degree 3 in $S$ (i.e., $U$).

(E) The planar graph $H$ induced by all the vertices of interest by the tree $S$, and the $\mu + 1$ additional edges. This graph has $O(\mu)$ vertices and edges.

(F) The weighted vertex separator

(G) There might be one collapsed tree that has the majority of the mass of the original graph.

(H) Add the vertex separator of this tree to the separator set.

(I) The resulting separator in the original graph.

Figure 1: A proof in pictures of Theorem 1.1.
1.1. A proof using known tools

We next provide a shorter proof using known tools – the proof was pointed out to us by Chandra Chekuri.

Proof: Let $T$ be a spanning tree of $G$, and let $M = E(G) \setminus E(T)$ be the remaining set of $\mu + 1$ edges.

Let $g$ be the treewidth of $G$. By Robertson et al. [RST94, Theorem 6.2], the graph $G$ has a grid minor of size $g' \times g'$, where $u = \lfloor (g - 5)/6 \rfloor$. But then, $G$ must contain $\lfloor u/2 \rfloor^2$ vertex disjoint cycles. However $G$ contains at most $\mu + 1$ disjoint cycles, since every cycle must contain an edge of $M$. It follows that $g^2 = O(\mu)$. Namely, $g = O(\sqrt{\mu})$. A graph with tree width $g$, has a separator of size $O(g)$, thus implying the graph has a separator of size $O(\sqrt{\mu} + 1)$.

The above argument is well known. see Demaine et al. [DFHT05, Theorem 4.4],

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