Schemes of transmission of classical information via quantum channels with many senders: discrete and continuous variables cases

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Superadditivity effects in the classical capacity of discrete multi-access channels (MACs) and continuous variable (CV) Gaussian MACs are analysed. New examples of the manifestation of superadditivity in the discrete case are provided including, in particular, a channel which is fully symmetric with respect to all senders. Furthermore, we consider a class of channels for which input entanglement across more than two copies of the channels is necessary to saturate the asymptotic rate of transmission from one of the senders to the receiver. The 5-input entanglement of Shor error correction codewords surpass the capacity attainable by using arbitrary two-input entanglement for these channels. In the CV case, we consider the properties of the two channels (a beam-splitter channel and a “non-demolition” XP gate channel) analyzed in [Czekaj et al., Phys. Rev. A 82, 020302 (R) (2010)] in greater detail and also consider the sensitivity of capacity superadditivity effects to thermal noise. We observe that the estimates of amount of two-mode squeezing required to achieve capacity superadditivity are more optimistic than previously reported.

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I. INTRODUCTION

Quantum communication is a dynamically developing branch of quantum information theory [4]. One of its central notions is that of a quantum communication channel [4,5,6], which models information transfer from senders to receivers using quantum resources. The amount of information which can be encoded in quantum states and reliably sent through a quantum channel is measured, depending on the communication scenario, by various channel capacities: (i) classical capacity \( C \), defined as the maximal rate at which classical information can be transmitted through the channel; (ii) classical private capacity \( P \), which is the classical capacity pertaining to the case when the transmitted bits are hidden from an environment; (iii) quantum capacity \( Q \) characterizing the size of the Hilbert space of states which can be transmitted through the channel. Quantum effects, associated with quantum channels, that have recently attracted much attention are so-called “activations” and “superadditivities”. For the quantum capacity \( Q \), various activations were based on bound entanglement, but the most spectacular result was recently obtained in Ref. [6], where an activation of the type \( 0 \otimes 0 > 0 \) was shown. In the case of private capacity \( P \), the corresponding superadditivity was found in Ref. [7] (see also Ref. [8]). Quantum superadditivity of the classical capacity \( C \) in the case of Multiple Access Channels (MAC’s) was shown in the Ref. [9] for discrete variables and in Ref. [10] for continuous variables. The question of additivity of \( C \) is still open for the one-sender one-receiver scenario, although a substantial breakthrough on the superadditivity of the Holevo function has recently been achieved in [11].

In the present paper we study a variety of quantum multiple access channels exhibiting superadditivity effects for classical capacity. We do this for both discrete and continuous variable (CV) systems. In particular, for the discrete variable case, we provide a new symmetric scenario where both senders can benefit from capacity superadditivity. This is in contrast to earlier examples studied in Ref. [9] where one of the senders only played a role of an assistant with respect to the other fixed sender. We furthermore go beyond the standard dense coding protocol, which is based on two particle entanglement and present examples of channels where multipartite entanglement is required to achieve the optimum channel capacity. The use of multipartite entanglement can be seen as the next step in the direction of optimization of the classical capacity of quantum channels. In particular it is shown that the 5-qubit error correction codeword [4] entangled across 5 inputs beats any 2-input based entanglement encodings for these channels.

In the CV context, we study the examples of Gaussian channels, introduced in Ref. [10] in greater detail. We extend the analysis of non-additive capacity regions and also study the dependence of the classical capacity of the channels on the choice of the set of input states. We show that for low energies, protocols using two-mode entanglement surpass both coherent state and standard single mode squeezed state encodings. Furthermore, we analyze the sensitivity of the superadditivity effects to thermal noise and show that the protocols are relatively
sensitive to thermal noise or losses in that 15 percent of power loss is sufficient to destroy the effect.

The work is organized as follows. All necessary definitions are introduced in Sec. II. Sections IIIA, IID, and IIE are devoted to the discrete variable case where we provide: a proof of the classical additivity of capacity regions (Sec. IIIA), an example of a symmetric MAC, exhibiting superadditivity of the classical capacity (Sec. IIII), an analysis of the influence of multipartite entanglement on the capacity regularizations (Sec. IIIII) and an example of the superadditivity of regularized capacity regions (Sec. IIIE). Continuous variable MAC’s are studied in Sections IV A, IVB, and IV C wherein: the locality rule for continuous variable MAC’s is presented Sec. IV A the dependence of the classical channel capacities on the choice of input states is studied in Sec. IVB, while the influence of thermal noise is analyzed in Sec. IV C.

II. BASIC DEFINITIONS

The transmission of classical information through a quantum channel corresponds to the following communication sequence [4]:

\[ x \mapsto \rho_x \mapsto \Phi(\rho_x) \mapsto \text{tr}[\Phi(\rho_x)E_y] \mapsto y. \] (1)

The sender maps the message \( x \) taken from some alphabet, into a state \( \rho_x \) of a quantum system, which in turn is sent through a quantum channel \( \Phi \) to the receiver. The quantum channel models the interaction of \( \rho_x \) with the environment. It is assumed that none of the users have access to the environment. The receiver obtains the state \( \Phi(\rho_x) \) and performs a measurement \( \{E_y\} \) yielding some output result \( y \) from which he tries to infer the message sent by the sender. The receiver knows the set of states \( \{\rho_x\} \) as well as the respective probabilities \( p_x \) with which they are input to the channel. We distinguish two cases: (i) the states \( \{\rho_x\} \) belong to a finite dimensional quantum space and \( x \) is a discrete variable (DV); (ii) the states \( \{\rho_x\} \) are states of a bosonic system and \( x \) is a continuous variable (CV). In the latter situation, a restriction on the average energy sent through the channel must be imposed to obtain a meaningful concept of channel capacity, since cranking up the power of transmission indefinitely allows perfect transfer of information. The restriction usually takes the form of a constraint on the average photon number of the input ensemble \( \{p_x, \rho_x\} \):

\[ \text{tr}[\hat{N} \int p_x \rho_x dx] \leq N, \] where \( \hat{N} \) is the photon number operator.

The sender may perform an encoding of his messages into code states to reduce the probability that a message deciphered from the measurement outcome disagrees with the one sent through the channel. Code states belong to the Hilbert space \( \mathcal{H}^\otimes n \), describing the input of \( n \) copies of the channel \( \Phi \), i.e. \( \Phi^\otimes n \). As \( n \to \infty \) the probability of a decoding error can be made arbitrary small.

The maximal rate at which information can be reliably transmitted through a quantum channel is defined as its classical capacity \( C \). By the well known result [2], the "single shot" classical capacity \( C^{(1)}(\Phi) \) is bounded by the Holevo quantity:

\[ C^{(1)}(\Phi) \leq \chi(\Phi) = \max_{(p_x, \rho_x)} \left( S(\rho(\hat{\rho})) - \sum_x p_x S(\Phi(\rho_x)) \right). \] (2)

where \( \hat{\rho} = \sum_x p_x \rho_x \) is the mean input state and \( S(\rho) = -\text{tr}[\rho \log \rho] \) is the von Neuman entropy. It can be shown that the above capacity can be achieved by product code states over the copies of \( \mathcal{H} \) (Holevo-Schumacher-Westmoreland coding theorem [3]).

However, the input Hilbert space \( \mathcal{H}^\otimes n \) allows also for entangled states, which may be useful for overcoming the above bound. This possibility is quantitatively taken into account by considering the so-called regularized classical capacity:

\[ C^{(n)}(\Phi) = \lim_{n \to \infty} \frac{1}{n} \chi(\Phi^\otimes n). \] (3)

The importance of considering entangled encodings is highlighted by Hastings’ recent work [11], who showed that there do exist channels for which \( C^{(n)}(\Phi) > \chi(\Phi) \).

In this paper we consider multiple access channels (MAC’s), where there are at least two senders (we will denote them as \( A, B, \ldots \)), transmitting to one receiver \( R \). Each sender sends his message independently of the other senders, i.e. their inputs are completely uncorrelated. They know only the input ensembles and agree upon a set of rules governing the use of the channel; the first \( n_1 \) uses of the channel consists of sending states from a fixed first ensemble, next \( n_2 \) uses of the channel consist of states chosen from a second ensemble and so on. This procedure is called time sharing [12].

For the case of two senders, a MAC acts as a mapping:

\[ \rho_{xA} \otimes \rho_{xB} \mapsto \Phi(\rho_{xA} \otimes \rho_{xB}). \] (4)

Here \( x_A \) and \( x_B \) are messages pertaining to senders \( A \) and \( B \) respectively.

The capacity region \( \mathcal{R}(\Phi) \) of the classical MAC \( \Phi \) characterized by the conditional probability distribution \( p(y_R|x_A, x_B) \) is defined as a set of vectors \( \mathcal{R} = \{R_A, R_B\} \) of rates, simultaneously achievable by adequate coding and time sharing. The capacity region \( \mathcal{R}(\Phi) \) of a classical two-sender MAC is given by the convex hull of the rates \( \{R_A, R_B\} \) for which there exist probability distributions \( p_{x_A, x_B} \) of transmitted symbols and a joint probability distribution \( p_{x_A, x_B | y_R} = p(y_R|x_A, x_B) p_{x_A, x_B} \) such that [12]:

\[ R_A \leq I(X_A : Y | X_B) \] (5)
\[ R_B \leq I(X_B : Y | X_A) \] (6)
\[ R_A + R_B \leq I(X_A, X_B : Y). \] (7)

where \( I(X_A, X_B : Y) \) denotes the mutual information and \( I(X_A : Y | X_B), I(X_B : Y | X_A) \) are conditional mutual information quantities. These quantities are related
to the Shannon entropy \( H(X) = - \sum_x p_x \log p_x \) and conditional entropy \( H(Y|X) = H(X,Y) - H(X) \) as follows: \( I(X:Y) = H(X,Y) - H(Y|X), \; I(X:Y|Z) = H(X,Y|Z) - H(Y|X,Z) \). In the opposite way, for each vector of rates \( R \in \mathcal{R}(\Phi) \) there exist input symbols probability distribution \( p(x_A, x_B, Q) = p(x_A|Q)p(x_B|Q)p(Q) \) that following set of inequalities is fulfilled:

\[
R_A \leq I(X_A:Y|X_B, Q) \tag{8}
\]

\[
R_B \leq I(X_B:Y|X_A, Q) \tag{9}
\]

\[
R_A + R_B \leq I(X_A, X_B:Y|Q). \tag{10}
\]

Random variable \( Q \) refers to time sharing procedure.

For the case of a quantum MAC \( \Phi \) with two senders, a useful notion is that of a “classical-quantum” state: 

\[
\rho = \sum_{x_A, x_B} p_{x_A} p_{x_B} e_{x_A} \otimes e_{x_B} \otimes \Phi(p_{x_A}, \rho_{x_B}) \tag{11}
\]

where \( \{e_{x_A}\}, \{e_{x_B}\} \) are projectors onto the standard basis of the Hilbert space controlled by sender \( A \) (B) and \( \{p_{x_A}, \rho_{x_B}\} \) is the ensemble of code states of \( A \) (B).

The single-shot capacity region \( \mathcal{R}^{(1)}(\Phi) \) is obtained as a convex closure of all rates \( (R_A, R_B) \), for which there exist classical-quantum states \( \rho \) fulfilling the following set of inequalities:

\[
R_A \leq I(X_A:Y|X_B) \tag{12}
\]

\[
R_B \leq I(X_B:Y|X_A) \tag{13}
\]

\[
R_T = R_A + R_B \leq I(X_A, X_B:Y). \tag{14}
\]

In distinction to the case of classical channels, the mutual information is now given in terms of the von Neumann entropy \( I(X_A, X_B:Y) = S(\rho_{AB}) + S(\rho_I) - S(\rho_{ABR}) \) and \( I(A:B) = \sum_{x_B} p_{x_B} I(A:B = \rho_{xB}) \). The von Neumann entropy is defined as \( S(\rho) = - \text{tr}[\rho \log \rho] \). \( R_T \) denotes the total capacity and is defined as \( R_T = \sum_i R_i \). In the following, we will often refer to the notion of the regularized capacity region \( \mathcal{R}^{(\infty)}(\Phi) = \lim_{n \to \infty} \mathcal{R}(\Phi^{\otimes n})/n \).

Finally, we shall use the notion of parallel composition of MAC’s, which we illustrate here by an example of two classical channels (denoted by \( \Phi_I \) and \( \Phi_{II} \)) and two senders (\( A \) and \( B \)). In parallel composition sender \( A \) has access to input ports \( X^I_A, X^I_B \) of the first (second) channel. \( X^I_B, X^I_B \) denote input ports controlled by sender \( B \). For each input port \( X^I_2 \) there is a set of possible signals which can be sent through the channel. The channels operate synchronously, which means that communication process can be divided into steps. In each step, user \( A \) sends the vector of symbols \( x_A = \{x_{A}^I, x_{A}^{II}\} \) while sender \( B \) sends symbols \( x_B = \{x_{B}^I, x_{B}^{II}\} \). In each step a given channel is used by every user exactly once. At the end of the communication step the receiver obtains the output \( y = \{y^I, y^{II}\} \).

Let \( p(y^I|x^I_A, x^I_B), (p(y^{II}|x^{II}_A, x^{II}_B)) \) be the transition probabilities for the MAC’s \( \Phi_I (\Phi_{II}) \), then the transition probability for the parallel composition is given by:

\[
p(y|x_A, x_B) = p(y^I|x_A^I, x_B^I) p(y^{II}|x_A^{II}, x_B^{II}) \tag{15}
\]

The parallel composition of quantum MAC’s is defined as the straightforward generalization of the above concept.

**III. QUANTUM MACS IN FINITE DIMENSIONAL SPACES**

**A. Additivity theorem for classical discrete multi-access channels**

We shall state the additivity theorem for capacity regions of classical discrete MACs in full generality. First recall that the capacity region \( \mathcal{R}(\Phi) \) for a classical MAC with arbitrary number of senders is given by the convex hull of the \( \{R_i\} \) which fulfill:

\[
R_S \leq I(X_S:Y|R|S^C) \tag{16}
\]

where \( S \) enumerates all subsets of senders and \( R_S = \sum_{S \subseteq E} R_i \), while \( S^C \) is the complement of the set \( S \). For the 2-to-1 channels this reduce to the simple form of Eqs. (9,10). The capacity region evaluated for fixed probability distribution of input symbols \( \tilde{p} = p(Q^I, Q^{II}) \prod_i p(X^I_i, X^{II}_i|Q^I_i, Q^{II}) \) has the form (cf. Eq. (10)):

\[
\tilde{R} = \{R \in \mathbb{R}^n : \forall_{S \subseteq E} R_S \leq I(X_S:Y|X_{SC},Q), \forall_{i} R_i \geq 0 \}. \tag{17}
\]

The **additivity theorem** states that the achievable capacity region \( \mathcal{R} \) of a channel being the parallel composition of MACs is the geometrical sum of capacity regions of the constituting channels. More formally, suppose \( n \) MACs are used parallelly, with each channel having \( m \) senders. Let \( \tilde{R} = \{R_1, \ldots, R_m\} \) be the vector of achievable rates for the composite channel, then the capacity additivity theorem states that \( \tilde{R} \) can be written as a sum of vectors \( \tilde{R}^{(j)} \) describing the capacity region of the \( j \)-th MAC:

\[
\tilde{R} = \{\bigotimes_i \Phi_i \} = \sum_i \mathcal{R}(\Phi_i). \tag{18}
\]

The additivity theorem for the case of channels with two senders is graphically depicted in FIG. [a].

Here we prove only simple 2-to-1 scenario \( \mathcal{R}(\Phi_I \otimes \Phi_{II}) = \mathcal{R}(\Phi_I) + \mathcal{R}(\Phi_{II}) \), complete prove will be postponed to appendix. We start with \( \langle \rangle \). The outline is as follows: for arbitrary chosen vector of rates \( \tilde{R} = (R_A, R_B) \in \mathcal{R}(\Phi_I \otimes \Phi_{II}) \), complete prove will be postponed to appendix. We start with \( \langle \rangle \).
\begin{align}
H(Y|Q) &= \sum_{q} p(q)H(Y|Q = q) \quad (23) \\
&\leq \sum_{\{q^I,q^{II}\}} p(\{q^I,q^{II}\})(H(Y^I|Q^I = q^I) \\
&\quad + H(Y^{II}|Q^{II} = q^{II})) \\
&= \sum_{q^I} p(q^I)H(Y^I|Q^I = q^I) \\
&\quad + \sum_{q^{II}} p(q^{II})H(Y^{II}|Q^{II} = q^{II})) \\
&= H(Y^I|Q^I) + H(Y^{II}|Q^{II}), \quad (26)
\end{align}

where in Eq. (24) we again make use of entropy subadditivity. In similar way one can show Eq. (20). To prove Eq. (22) it is enough to observe, that conditional transition probability describing setup \(\Phi_I \otimes \Phi_{II}\) factorizes (see Eq. (15)), hence we can write:

\begin{align}
H(Y|X_A, X_B, Q) &= - \sum_{x_A, x_B, y, q} p(y|x_A, x_B, q) \\
&= - \sum_{x_A^I, x_B^I, y^I, q^I} p_I(y^I|x_A^I, x_B^I) \\
&\quad - \sum_{x_A^{II}, x_B^{II}, y^{II}, q^{II}} p_{II}(y^{II}|x_A^{II}, x_B^{II}) \\
&= H(Y^I|X_A^I, X_B^I, Q^I) + H(Y^{II}|X_A^{II}, X_B^{II}, Q^{II}). \quad (29)
\end{align}

Now we are going to show that \(\hat{R}(\Phi_I \otimes \Phi_{II}) \subseteq \hat{R}(\Phi_I \otimes \Phi_{II})\). By the definition of capacity region, there exist input symbol probability distribution \(\hat{p}\) that the rates vector \(\hat{R}\) obeys Eqs. (5-7). Using Eqs. (19-22) we can bound RHS of Eqs. (6-7) in following way:

\begin{align}
R_A &\leq I(X_A : Y|X_B, Q) \quad (33) \\
&= H(Y|X_B, Q) - H(Y|X_A, X_B, Q) \quad (34) \\
&\leq H(Y^I|X_B^I, Q^I) + H(Y^{II}|X_B^{II}, Q^{II}) \\
&\quad - H(Y|X_A, X_B, Q) \\
&= H(Y^I|X_B^I, Q^I) + H(Y^{II}|X_B^{II}, Q^{II}) \quad (36) \\
&\quad - H(Y^I|X_A^I, X_B^I, Q^I) - H(Y^{II}|X_A^{II}, X_B^{II}, Q^{II}) \\
&= I(X_A^I : Y^I|X_B^I, Q^I) + I(X_A^{II} : Y^{II}|X_B^{II}, Q^{II}). \quad (37)
\end{align}
Analogical expression can be write for $R_B$.

$$
R_A + R_B \leq I(X_A, X_B : Y|Q) \tag{38}
$$

$$
= H(Y|Q) - H(Y|X_A, X_B, Q) \tag{39}
$$

$$
\leq H(Y'|Q^I) + H(Y'|Q^{II}) \tag{40}
$$

$$
- H(Y|X_A, X_B, Q)
= H(Y'|Q^I) + H(Y'|Q^{II}) \tag{41}
$$

$$
- H(Y|X_A^I, X_B^I, Q^I)
- H(Y|X_A^{II}, X_B^{II}, Q^{II})
= I(X_A^I, X_B^I : Y|Q^I) \tag{42}
$$

$$
I(X_A^I, X_B^I : Y|Q^I, Q^{II}) + I(X_A^{II}, X_B^{II} : Y|Q^{II}). \tag{43}
$$

These inequalities define region $\hat{R}(\Phi_I \otimes \Phi_{II})$ which is fixed probability capacity region for $\hat{p}$. We have shown capacity region inclusion.

We shall move to $\hat{R}(\Phi_I \otimes \Phi_{II}) = \hat{R}(\Phi_I) + \hat{R}(\Phi_{II})$. Fixed probability capacity region $\hat{R}_I$ obtained for input symbol probability distribution $\hat{p}_I$ is given by:

$$
R_A \leq I(X_A^I : Y'|X_B^I, Q^I) \tag{44}
$$

$$
R_B \leq I(X_B^I : Y'|X_A^I, Q^I) \tag{45}
$$

$$
R_A + R_B \leq I(X_A^I, X_B^I : Y'|Q^I) \tag{46}
$$

$$
+ I(X_A^{II}, X_B^{II} : Y|Q^{II}). \tag{47}
$$

Geometrical hull $\hat{R}_I + \hat{R}_{II}$ can be easy obtained as a convex hull of sums of vertices of the fixed probability capacity regions $\hat{R}_I, \hat{R}_{II}$ and is equal to the region $\hat{R}$. Because $\hat{R}$ was chosen arbitrary, we have proven that $\hat{R}(\Phi_I \otimes \Phi_{II}) \subseteq \hat{R}(\Phi_I) + \hat{R}(\Phi_{II})$.

(2) Let $\hat{R}_I \in \hat{R}(\Phi_I)$ belong to fixed probability capacity region with associated with input symbols probability $\hat{p}_I$. Similar we have for $\hat{R}_{II}$. It is easy to check by direct evaluation of Eq. (8) and Eq. (10) that rates vector $\hat{R}_I + \hat{R}_{II}$ belongs to fixed probability capacity region of $\Phi_I \otimes \Phi_{II}$ obtained for input symbols probability distribution $\hat{p} = \hat{p}_I \hat{p}_{II}$. That proofs $\hat{R}_I + \hat{R}_{II} \in \hat{R}(\Phi_I \otimes \Phi_{II})$.

### B. Superadditivity

Superadditivity is defined as the situation when for a certain type of capacity $\tilde{C}$ and two channels $\Phi_I, \Phi_{II}$, the following holds:

$$
\tilde{C}(\Phi_I \otimes \Phi_{II}) > \tilde{C}(\Phi_I) + \tilde{C}(\Phi_{II}). \tag{49}
$$

One may distinguish the following types of superadditivity: (a) superadditivity of channel capacity, when $\tilde{C} = C^{(\infty)}$ (see Eq. (3)), (b) superadditivity of Holevo capacity, when $\tilde{C} = C$, (c) self superadditivity, if $\tilde{C} = \chi$ and $\Phi_I = \Phi_{II}$. For self superadditivity, $C^{(\infty)} > C^{(1)}$. Note that the RHS of (19) expresses the capacity achieved with product inputs on $\Phi_I$ and $\Phi_{II}$. Superadditivity means that using encoded states that are correlated (entangled) across uses of channels is advantageous.

In the context of MACs superadditivity effects are identified in terms of the capacity regions: $R(\Phi_I \otimes \Phi_{II}) \supseteq R(\Phi_I) + R(\Phi_{II})$ where $+$ denotes the geometrical sum of two regions. Superadditivity occurs if there exists a vector in the region $R(\Phi_I \otimes \Phi_{II})$ which cannot be expressed as a sum of two vectors from $R(\Phi_I), R(\Phi_{II})$ respectively. To prove superadditivity effects in terms of the capacity regions it is enough to show that the maximal rate achieved by one of the senders (say sender $A$) exhibits superadditivity. This means that we may concentrate only on the rate of a single sender or, in other words, show the effect only by analysis of its ,,coordinate” (or ,,dimension”) in the multidimensional geometric regions $C(\Phi_I \otimes \Phi_{II}), C(\Phi_I)$ and $C(\Phi_{II})$.

### C. Superadditivity effect in symmetric channels

Examples of channels presented in [9, 10], which exhibit superadditivity effects, are highly unsymmetrical. One of the senders performs there a “remote” dense coding on the part of an entangled state transmitted by the other. In the described communication schemes one sender is a true sender who transmits messages while the role of the others is only to help in the communication process since their transmission rates is equal 0. It might suggest that in the channels based on the dense coding scheme there is only a single super sender who takes advantage of the entangled state transmission. This is not the case as shown here. A channel can be constructed that is symmetric with respect to the exchange of senders facilitating a superadditivity effect for all of them.

Here we consider a channel $\Phi$ (see 2) with two senders: $A$ and $B$. Each of the senders controls two 1-qbit lines. The channel operates in two modes: $F$ and $S$. Each occurs with probability 1/2. In the first mode, the operation of the channel is depicted in FIG. (2b). In the second mode, $A$ and $B$ are swapped, i.e. lines $A_1$ and $A_2$ now belong to $B$ while $B_1$ and $B_2$ to $A$. The channel is explicitly symmetric w.r.t. the senders. Information that the first (second) case occurred is sent to the receiver as a label $|F\rangle (|S\rangle)$. The cross at the end of lines denotes replacement of the transmitted state by a completely mixed state. The action of the controlled $\sigma_z$ gate is: $|00\rangle \otimes I + |01\rangle \otimes |\sigma_z + 10\rangle \otimes |\sigma_z + 11\rangle \otimes |\sigma_y \rangle$.

The capacity region $R(\Phi)$ is upper bounded by the following inequalities: $R_A \leq 1, R_B \leq 1, R_A + R_B \leq 1$, as a direct consequence of the dimensionality of the output space (one-qubit space).
Each user is supplied with an additional one qubit identity connection with receiver. These two channels will be jointly referred to as the channel $\mathcal{I}$. Note that its capacity region $\mathcal{R}(\mathcal{I})$ is given by $R_A \leq 1, R_B \leq 1, R_A + R_B \leq 2$.

The upper bound for $\mathcal{R}(\Phi + \mathcal{I})$ thus becomes $R_A \leq 2, R_B \leq 2, R_A + R_B \leq 3$. On the other hand, the lower bound for the achievable capacity region of the composite channel $\mathcal{R}(\Phi \otimes \mathcal{I})$ can be seen in FIG. 3. To see this, we present a protocol which achieves the capacity $(2.5, 0)$. Due to symmetry of the channel, it follows that the rates $(0, 2.5)$ are also achievable. Notice immediately that the rates $(1, 2)$ and $(2, 1)$ can be obtained by product code states. All the other rates presented in FIG. 3 are obtained by time sharing.

Consider the following protocol: sender $A$ sends the states $|i⟩|i'⟩$ with probability $1/8$ where $|i⟩ \in \{|0⟩, \ldots, |1⟩\}$ are all possible standard basis states of two qubits, while $|i'⟩ \in \{|0⟩, |1⟩\}$. The two-qubit states $|i⟩$ are input to $\Phi$ while $|i'⟩$ input to the supporting identity channel $\mathcal{I}$. $B$ sends the fixed state $1/\sqrt{2}(|0⟩ + |1⟩)$ with one qubit of the Bell state sent through line $B_2$ and the other through the supporting channel.

For given $\{i, i'\}$ the receiver gets
\[
\rho_{ii'} = \frac{1}{2} |F⟩⟨F| \otimes |1⟩⟨1|_{A_0} (i_{A_2}) \otimes \frac{1}{8} |0⟩⊗ |i'⟩⟨i'|,
\]
\[
+ \frac{1}{2} |S⟩⟨S| \otimes \frac{1}{8} |0⟩⊗ |i⟩’⟨i’|,\]

$|F⟩, |S⟩$ denotes the mode of operation of channel $\Phi$. The output state consists of the mode label and 6 qubits. The first 4 qubits are output by the channel $\Phi$, while the 5-th and 6-th qubits are outputs pertaining to $\mathcal{I}$. If channel $\Phi$ works in mode 1, either the identity operation $I$ or $\sigma_x$ is performed on the line $A_2$. However states sent by sender $A$ $|0⟩$ and $|1⟩$ are invariant under the mentioned operations since the receiver obtains an unchanged state from the line $A_2$. If the channel $\Phi$ operates in mode 2, the controlled $\sigma_x$ gate fired by the state $|i⟩$ from sender $A$, is performed on half of the Bell state input by sender $B$. The result of this operation is denoted by $|\psi_i⟩$. The entropy of the conditional output state $\rho_{ii'}$ is equal to 4.5. Note that entropy has the same value for each input state $|i⟩|i'⟩$.

The mean output state is $\bar{\rho} = \frac{1}{8} \sum_{i,i'} \rho_{ii'}$ and can be written as:
\[
\rho = \frac{1}{2} \left( |F⟩⟨F| \frac{1}{64} ^T \otimes + |S⟩⟨S| \frac{1}{64} ^T \otimes \right) \quad (50)
\]
\[
= \frac{1}{128} ^T . \quad (51)
\]

It has entropy $S(\rho) = 7$. In presented scheme, sender $B$ transmits all the time the same state and attains a rate of 0. Since the setup $\Phi \otimes \mathcal{I}$ can be viewed as a channel with single sender $A$ while the state from the helper-sender $B$ is formally included to the environment. By Holevo’s theorem (see [2], we obtain that the rate that sender $A$ can attain is thus 2.5 bits.

Although rates $(2.5, 0)$ and $(0, 2.5)$ are achieved in the protocol where there is still one true sender while the other is helper-sender and there is no superadditivity of total rates $R_T = R_A + R_B$, potentially both of the senders can take advantage of entangled state transmission.

D. Necessity of multi particle entanglement to approach regularized capacity region

In this section we give an example of a channel where senders must use multiparticle entanglement states to achieve the regularized capacity region.

We start by describing the class of channels $\Phi_{n,n'}$ that will be used in the search for superadditivity effects. The channels have one distinguished sender $A$ and $n$ helper-senders $B_i$. Sender $A$ controls $n'$ of 2-qubit lines which are measured in the standard basis by the channel (alternatively it can be seen as he controls $n'$ of 2-bit lines), senders $B_i$ control only 1-qubit lines. Each time the channel is used, one of the helper-senders is attributed

FIG. 2: Channel $\Phi$ from Sec. III C a) channel $\Phi$ working in parallel with identity channel $\mathcal{I}$, waved line denotes entangled state, b) and c) two modes of work of channel $\Phi$.

FIG. 3: Lower bound for achieved capacity region for the channel $\Phi$ from Sec. III C working in parallel with identity channel. Thick lines refers to the upper bound for geometrical sum of capacity regions of component channels.
to each 2-bit line of sender $A$. One helper-sender can be attributed only to one line of sender $A$. Selected helper senders become active helper-senders. It means that they participate in transmission of messages from sender $A$. States from the active helper-sender is modified by the unitary operation from the set $I, \sigma_x, \sigma_y, \sigma_z$ which is triggered by the state of the appropriate line of sender $A$. States of the others helper-senders become unchanged. Described selection of active helper-senders is performed in a random way. Each selection can be chosen with equal probability. States transmitted by $A$ are absorbed (i.e. the output degrees of freedom of $A$ which is triggered by the state of the appropriate line of sender $A$). The receiver obtains only the states coming from senders $B_i$ and a label $w$ with information about attribution of active helper-senders to lines of $A$. For example if $n = 3, n' = 2$, the label $w = \{2, 3\}$ tells receiver that states from senders $B_2$ and $B_3$ were chosen as the targets of the unitaries controlled by first and second 2-qbit line of sender $A$ respectively. This channel is schematically depicted in FIG. 4. Note that the message included in the label $w$ may be represented as a $n' \lfloor \log_2 n \rfloor$-qubit state $|w\rangle = |i_1\rangle_1 ... |i_{n'}\rangle_n$, where $i_k$ is the number of the helper-sender chosen to be the target of the unitary operation controlled by $k$-th line of sender $A$. $(.)_k$ denotes binary representation of the value $i_k$. For example in the above mentioned case of $n = 3$ the label $w = 2, 3$ corresponds to $|w\rangle = |10, 11\rangle$. We shall use this notation in the analysis of a specific example.

Here we study the parallel setup of $m$ copies of the channel $\Phi_{n,n'}^{\otimes m}$ from the class described above. For simplicity we will denote the channel by $\Phi$. Please note that in following part of this section we choose $n' = 1$. In the setup $\Phi^{\otimes m}$, senders $B_i$ can send at most $m$-particle entangled states through their inputs. Entanglement cannot be transmitted through inputs of two different senders (see Fig. 5).

![FIG. 5: The parallel setup of channels described in Sec. III D. The inputs of the channels used for transmission of entangled states are shown. The present case consists of channels with two helper-senders $n = 2$, each of which can send three particle entangled states $m = 3$.](image)

We focus on the upper bound for the achievable rate for sender $A$. We restrict ourselves to the scheme where helper-senders $B_i$ send one state at all times. Vectors of rates for such schemes take the form $(R_A, 0)$. Formally we can consider the channel $\Phi_{w,w'}^{\otimes m}$ in the setup as a 1-to-1 channel and determine the capacity $C_A(\Phi_{w,w'}^{\otimes m})$ of the sender $A$. Now we prove that the upper bound for the capacity $C_A(\Phi_{w,w'}^{\otimes m})$ has the form:

$$C_A(\Phi_{w,w'}^{\otimes m}) \leq n \sum_{i=0}^{m} p^i (1-p)^{m-i} \binom{m}{i} \min(2i, m)$$  \hspace{1cm} (52)

where $n$ is the number of helper-senders, $m$ is the number of channels used for transmission that is equivalent to the number of parties in the entangled state pertaining to $B_i$, and $p = 1/n$.

**Proof:** First we find an upper bound for the Holevo capacity of the setup $\Phi_{w,w'}^{\otimes m}$ in the case when the helper-sender $B_i$ was active $l_i$ times. Then we use these results to calculate the upper bound for the capacity of $\Phi_{w,w'}^{\otimes m}$.

The orthogonal label $|w_j\rangle$ describes which sender $B_i$ was active in the $j$-th copy of $\Phi$. Label $|w\rangle = |w_1, \ldots, w_m\rangle = |w_1\rangle \otimes \ldots \otimes |w_m\rangle$ is the complete list of the active helper-senders in the setup. Given the label we know that sender $B_i$ was active $l_i$ times. The probability of occurrence of the situation described in $|w\rangle$ is given by $p_w = p^m$.

Suppose that $|w\rangle$ is obtained as the result of $\Phi_{w,w'}^{\otimes m}$. This fixes the attribution of senders $B_i$ to the lines of $A$. We denote this case as $\Phi_{w,w'}^{\otimes m}$. Now, the $m$ uses of the channel $\Phi$ can be thought as $n$ separate channels $\Gamma_i, m$. The input of each channel $\Gamma_i$ consists of the subset of lines from $A$ and all lines from $B_i$. None of the $\Gamma_i$ share input lines with any other $\Gamma_j$. Each channel $\Gamma_i$ has $2i$ qbits input from sender $A$, $m$ qbits input from sender $B_i$.

![FIG. 4: The channel described in Sec. III D with $n = 5$ helper-senders and $n' = 3$ lines belonging to sender $A$. The message represented by the label $w = 1, 2, 4$ is additionally sent to the receiver which may be represented as a „flag” state $|w\rangle = |001, 010, 100\rangle$.](image)
and a $m$ qbit output. The equivalence $\Phi_w^{\otimes m} = \otimes_i \Gamma_i$ is depicted in FIG. 6.

Taking into account dimensionality, one can infer that $A$ can transmit at most $\min(2l_i, m)$ classical bits of information through $\Gamma_i$. Given $|w\rangle$, the channels $\Gamma_i$ work independently. There cannot be entanglement shared between $\Gamma_i$ and $\Gamma_j$ because sender $A$ transmits only classical states and users $B_i$ and $B_j$ cannot share entanglement due to definition of MAC. This leads to the total conditional capacity:

$$C_A(\Phi_w^{\otimes m}) = \sum_{i=1}^{n} C(\Gamma_i) = \sum_i \min(2l_i, m)$$ (53)

The following observation is helpful for the calculation of $C_A(\Phi_w^{\otimes m})$. Consider channel $\Delta(\rho) = \sum_{w} p_w \Delta_w(\rho)|w\rangle\langle w|$ which acts with probability $p_w$ as channel $\Delta_w$. Assume again that the label $|w\rangle$ is sent to the receiver which identifies the case that occurs. For this channel we have:

$$C(\Delta) = \max_{\{p_x, \rho_x\}} S(\Delta(\sum x p_x \rho_x)) - \sum_x p_x S(\Delta(\rho_x))$$ (54)

$$= \max_{\{p_x, \rho_x\}} S\left(\sum_{w} p_w \Delta_w(\sum x p_x \rho_x)|w\rangle\langle w|\right) - \sum_x p_x S\left(\sum_{w} p_w \Delta_w(\rho_x)|w\rangle\langle w|\right)$$ (55)

$$= \max_{\{p_x, \rho_x\}} \sum_w p_w \left\{ S\left(\Delta_w(\sum x p_x \rho_x)\right) + H\left(\{p_w\}\right) \right\} - \sum_x p_x S\left(\Delta_w(\rho_x)\right) - H\left(\{p_w\}\right)$$ (56)

$$\leq \sum_w p_w \max_{\{p^w_x, \rho^w_x\}} \left\{ S\left(\Delta_w(\sum x p^w_x \rho^w_x)\right) - \sum_x p^w_x S\left(\Delta_w(\rho^w_x)\right)\right\}$$ (57)

$$= \sum_w p_w C(\Delta_w),$$ (58)

where equality occurs if the same ensemble achieves the capacity of each channel $\Delta_w$. Similar argumentation can be use to show that rates achieved for channel $\Delta$ in certain protocol obey:

$$R(\Delta) = \sum_w p_w R(\Delta_w),$$ (59)

where $R(\Delta_w)$ are the rates achieved by this protocol in case of $\Delta_w$.

Using the above observation, and substituting $\Delta = \Phi_w^{\otimes m}, p_w = p^m$ and bound $C(\Delta_w)$ by $C_A(\Phi_w^{\otimes m})$ (53), we obtain:

$$C_A(\Phi_w^{\otimes m}) \leq p^m \sum_w \left(\min(2l_1(w), m) + \ldots + \min(2l_n(w), m)\right),$$ (60)

$$= p^m \sum_{l_1 + \ldots + l_m = m} \frac{m!}{l_1! \ldots l_m!} \left(\min(2l_1, m) + \ldots + \min(2l_n, m)\right)$$ (63)

$$= np^m \sum_{l=0}^{m} \binom{m}{l} \min(2l, m) \alpha_l$$ (64)

$$= n \sum_{l=0}^{m} \binom{m}{l} \min(2l, m) p^m (n-1)^{m-l}$$ (65)

$$= n \sum_{l=0}^{m} p^l (1-p)^{m-l} \binom{m}{l} \min(2l, m).$$ (66)
In Eq. (63) we collected in the common factor all \( w \) with the same \( \{l_1, \ldots, l_n\} \). Because formulas with \( \min(l_2, m), \ldots, \min(l_n, m) \) in Eq. (65) have the same form as the one with \( \min(l_1, m) \), we omitted them and introduced in Eq. (65) factor \( \eta \). Moreover we introduced \( \alpha_1 = \sum_{l_2 + \ldots + l_n = m-l} (\frac{m-1}{l_2, \ldots, l_n}) \). In Eq. (66) we used the relation

\[
\sum_{k_1 + \ldots + k_n = m} \binom{m}{k_1, \ldots, k_n} = n^m. \tag{67}
\]

Recalling that \( p = 1/n \) leads the relation \( (n - 1)p = (n - 1)/n = 1 - p \) which was used in Eq. (66).

![Diagram](image.png)

**FIG. 7:** Upper bound for reguralized capacity \( C_A^{(m)} = \frac{1}{m}C_A(\Phi^{\otimes m}) \) as a function of \( m \) - number of channel copies the capacity is evaluated on - and number of helper sender \( m \) in the channel \( \Phi \).

The upper bound given by (62) is achieved in the case of \( m \in \{1, 2, 5\} \) by the protocols which runs as follows: \( A \) transmits with equal probability all states from the standard basis of his \( 2m \)-qubit input space of \( \Phi^{\otimes m} \) while all \( B_i \)'s transmit either state \( |0\rangle \) from the standard base, one of the Bell states \( |\phi^+\rangle \) or \( |0_L\rangle \) - the 5 qubit correction code word (see [4]) for \( m = 1, m = 2 \) or \( m = 5 \) case respectively.

\[
|0_L\rangle = \frac{1}{\sqrt{6}}[|00000\rangle + |10101\rangle + |01001\rangle + |10100\rangle](68)
\]

\[
|01010\rangle - |11011\rangle - |00110\rangle - |11100\rangle
\]

\[
-|11110\rangle - |00011\rangle - |11110\rangle - |00111\rangle
\]

\[
-|10001\rangle - |01100\rangle - |10111\rangle + |00011\rangle]
\]

To prove this, we will show that for each \( w \), an ensemble used in the protocol gives (62). Equality in Eq. (57) then becomes a simple consequence of relation Eq. (59).

Assuming knowledge of \( w \), the output entropy of the channel is equal to 0 for each state transmitted in described protocol. Hence we have to check if, under condition of \( w \), the mean output state entropy \( S(\rho_w) \) reaches \( \sum \min(2l_i(w), m) \). Since senders \( B_i \) are uncorrelated, we can focus only on sender \( B_i \) and consider only value \( l_i \). Let the set \( \{e\} \) contain all positions where the state coming from sender \( B_i \) was affected by the channel \( \Phi \) (\(| e | = l_i \)). We denote by \( \mathcal{E}^k(\rho) \) the completely depolarizing channel acting on \( k \)-th qbit of the state \( \rho \). For given \( e, \) part of the mean output state coming from \( B_i \) has the form \( \rho_e = (\bigotimes_{j=1}^k \mathcal{E}^{(e_j)}) [|\phi\rangle \langle \phi|] \) where \( e_j \) denotes the \( j \)-th element of \( e \) and \( |\phi\rangle \) is \(|0\rangle \), \(|\Phi^+\rangle \) or \(|0_L\rangle \). The condition of whether \( S(\rho_e) = \min(2l_i, m) \) occurs for all \( e \) was checked numerically. The program enumerated all \( e \), then for each \( e \) it computed state \( \rho_e \) and its entropy \( S(\rho_e) \). Obtained results confirmed that \( S(\rho_e) = \min(2l_i, m) \) for \(|0\rangle \), \(|\Phi^+\rangle \) and \(|0_L\rangle \).

In the presented protocol, entanglement increases diversity of the mean output state. An important feature of the code state from the 5 qubit correction code is that the increase of entropy of the output state depends only on the number of qbits affected by the unitary. It does not depend on localization of affected qbits. We cannot exceed \( m \) bits of entropy per state hence the closer \( l_i \) is to \( m \) the smaller is the entropy increase. Due to the asymptotical equipartition property, for \( n > 1 \) the larger the entanglement in the state, the smaller is the chance that \( l_i \) will be close to the \( m \).

The above analysis opens the possibility of further analysis concerning type of entanglement is the best in case of various channels. The possible classification of noise with respect to classes of entanglement seems especially interesting, for instance one can ask whether there are any channels for which cluster type entanglement is the best in saturating the asymptotic rates of the channel. We leave these type of questions for further research.

E. Supperadditivity of regularized capacity.

We now turn to the study of the supperadditivity effect for regularized capacity. We will investigate a setup which consists of two channels of the type already described in Sec. III. For the channel \( \Phi_I \) we choose \( n = 10, n' = 9 \) and for the channel \( \Phi_{II} \) we choose \( n = 10, n' = 1 \). We are interested in maximal transmission rate from sender \( A \), that is the case when all senders \( B_i \) help sender \( A \) by transmitting the same states all the time. Their rates are equal 0. Formally we can include senders \( B \) to the environment and view channels \( \Phi_I \) and \( \Phi_{II} \) as 1-to-1 channels.

First we show that for channel \( \Phi_I \) and \( \Phi_{II} \), upper bounds \( C_A \) for rates achievable by sender \( A \) fulfill \( C_A^{(\infty)} > C_A^{(1)} \). For this, we consider a protocol where senders \( B_i \) transmits one of the Bell states and show that this protocol achieves a regularized rate strictly greater than \( C_A^{(1)} \). Calculation will be performed for general sizes of the set of selected helper-senders equal \( n' \). The single shot capacity is given by the joint dimensionality of states of the selected helper-senders and it reads: \( C_A^{(1)} = n' \). In case of two uses of the channel and Bell states transmission, the probability that the same set of selected helper-senders was chosen twice is \( p = 1/\binom{m}{n} \). With probability \( 1-p \) sets
of the selected helper-senders in the first and second uses of the channel differ in at least one sender $B_i$. It means that input Bell state of two sender, lets say $B_1$ and $B_2$, was affected by the channel only once and, due to dense coding, the states carry full information from appropriate 2-qubit input lines of $A$ (line 1 and 2). In this case sender $A$ take advantage of transmission of additional 2 bits of information. Under condition of output label $w$, output entropy of the channel is 0 therefore the rate achievable by the protocol for given $w$ is equal to entropy of the mean output state (strictly speaking entropy of the part coming from senders $B$). As usual, sender $A$ transmits with equal probability all states from the standard basis. Recalling to Eq. 59 the rate achievable by sender $A$ is at least $R_A = p2n' + (1 - p)(2n' + 2) = 2n'(1 - p)$. This leads to $C_A^0 \geq (1/2)R_A = n' + (1 - p) > n' = C_A^{(1)}$.

Now we pass to the supperadditivity of the regularized capcities. We again refer to [18]. We will show that $C_A^{(1)} (\Phi_I \otimes \Phi_{II}) > C_A^{(\infty)} (\Phi_I) + C_A^{(\infty)} (\Phi_{II})$. We first provide upper bounds for $C_A^{(\infty)} (\Phi_I)$ and $C_A^{(\infty)} (\Phi_{II})$.

Recall that in this situation, entangled states can be transmitted only through the inputs controlled by the same users. Channel capacity is upper bounded by the minimum value of logarithm of its input and output spaces. Therefore for the channel $\Phi_{m,n}$, we have $C_A^{(m)} \leq 1/m \min(2m', mn) = \min(2n', n)$ and it leads to $C_A^{(\infty)} (\Phi_I) \leq \min(2 \times 9, 10) = 10$ and $C_A^{(\infty)} (\Phi_{II}) \leq \min(2 \times 1, 10) = 2$.

Now we move to the case $\Phi_I \otimes \Phi_{II}$, i.e. the case where entanglement between inputs of channels $\Phi_I$ and $\Phi_{II}$ controlled by the same user is allowed. One can use the following protocol to provide a lower bound for $C_A^{(1)} (\Phi_I \otimes \Phi_{II})$: sender $A$ only uses inputs of $\Phi_I$, and sends each state from the standard basis with the same probability; through channel $\Phi_{II}$, he sends only one chosen state $\{|00\}$ all the time. It is easy to see that channel $\Phi_{II}$ does not change the states coming from senders $B_i$ and in fact it can be seen as an identity channel. Senders $B_i$ send one chosen Bell state $\{\Phi^+\}$. The first qubit of the Bell state goes through the channel $\Phi_I$ while the second through the channel $\Phi_{II}$. If the qubit is affected by the channel $\Phi_I$, the dense coding scheme is reproduced. Each time the setup $\Phi_I \otimes \Phi_{II}$ is used, all the lines controlled by $A$ find as a target different Bell states. Therefore rate achieved by protocol is given by the dimensionality of input space of channel $\Phi_I$ controlled by sender $A$ and reads 18 bits. It is lower bound for $C_A^{(1)} (\Phi_I \otimes \Phi_{II})$ and shows that $C_A^{(\infty)} (\Phi_I \otimes \Phi_{II}) \geq C_A^{(1)} (\Phi_I \otimes \Phi_{II}) \geq 18 > 12 \geq C_A^{(\infty)} (\Phi_I) + C_A^{(\infty)} (\Phi_{II})$ and proofs that supperadditivity effect indeed occurs.

IV. QUANTUM GAUSSIAN MACS

We shall now consider the capacity properties of Gaussian Multi-Access channels. Before going further, we first collect certain basic notions and definitions that will be subsequently useful.

Recall first the concept of classical Gaussian multiple access channels [12]. Inputs and outputs of classical CV gaussian MACs are real numbers. The gaussian MAC models the influence of additive gaussian noise $Z$ (with variance $S$) on the total input signal, i.e. the output is

$$Y = \sum_i X_i + Z$$  \hspace{1cm} (69)

To prevent unphysical infinite capacities, the power constraints are imposed on the input signals $(X_i^2) \leq P_i$. Under these constraints, the capacity region for the classical gaussian MAC channel is given by [12]:

$$\sum_i R_i \leq C(\sum_i P_i / S)$$  \hspace{1cm} (70)

where $C(x) = 1/2 \log(1 + x)$.

For a quantum Gaussian MAC, the input and the output spaces are described by infinite dimensional Hilbert spaces, isomorphic to those describing a finite number of bosonic modes [18]. The latter are equipped with the “position” and “momentum” canonical observables $\{\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n\}$ fulfilling the commutation rules $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{i,j}$, where $i, j$ enumerate modes of the system. States of a bosonic system can be expressed in terms of characteristic functions $\chi(\xi) = \text{Tr}[\rho W_\xi]$ where $W_\xi = \exp(-i\xi^T R)$ is the so-called Weyl operator and $\hat{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)^T$ is the vector of canonical observables [18, 19]. Gaussian states are the states whose characteristic functions are gaussian:

$$\chi(\xi) = \exp\left[-\frac{1}{4} \xi^T \gamma + i d^T \xi\right]$$  \hspace{1cm} (71)

where $d$ is the displacement vector (with $d_j = \text{tr}(\rho \hat{R}_j)$) and $\gamma$ is the covariance matrix with entries $\gamma_{jk} = 2\text{tr}[\rho(\hat{R}_j - d_j)(\hat{R}_k - d_k)] - iJ^{(n)}_{jk}$ that completely define the Gaussian state. $J^{(n)}$ is the symplectic form for the multimode system:

$$J^{(n)} = \bigoplus_{i=1}^n J, \quad J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right).$$  \hspace{1cm} (72)

Gaussian channels are defined as mappings that transform gaussian states into gaussian states. They can be expressed as transformations of $\gamma$ and $d$:

$$\gamma \rightarrow X \gamma X^T + Y$$  \hspace{1cm} (73)
$$d \rightarrow Xd$$  \hspace{1cm} (74)

Complete positivity of the channel is guaranteed by the condition:

$$Y + iJ - iX^T JX \succeq 0.$$  \hspace{1cm} (75)
We now show how to determine $X,Y$ for an arbitrary gaussian channel $\Phi$. Recall that the action of any general channel is given by: $\Phi(\rho_s) = \text{tr}_e \left[ \hat{U}(\rho_s \otimes \rho_e)\hat{U}^\dagger \right]$, where $\hat{U} = \exp(-i\hat{H})$ is a unitary operation generated by a Hamiltonian $\hat{H}$. Gaussian channels are generated by Hamiltonians $\hat{H}$ that are quadratic in the canonical operators: $\hat{H} = i\hat{R}^T \hat{b}\hat{R}$, where $\hat{b}$ is a $2n \times 2n$ hermitian matrix. Here $\rho_s$ is the input state and $\rho_e$ is the state of environment. Now, for gaussian channels, both $\rho_s$ and $\rho_e$ are gaussian states with covariance matrices $\gamma_s, \gamma_e$ and displacement vectors $d_s, d_e$ respectively. The displacement of the output state depends linearly on $d_e$. As any displacement of output states by a constant vector is a unitary operation and as such it does not influence the channel capacity, we assume that $d_e = 0$. The action of $\hat{U}$ on the canonical observables can be identified with the linear transformation $\hat{U}^\dagger \hat{R}^T \hat{U} = M \hat{R}^T$. Now, we express $M$ in block form with respect to a system/environment partition: $M = \begin{pmatrix} M_{ss} & M_{se} \\ M_{es} & M_{ee} \end{pmatrix}$. From the latter one obtains $X = M_{ss}$ and $Y = M_{ee} \gamma_e^T M_{se}^T$.

Finally, note that in the context of quantum gaussian channels, power constraints are usually expressed as a limitation on a mean number of photons transmitted per channel use.

Squeezed states represent an important class of gaussian states for communication tasks. A one–mode squeezed state saturates the Heisenberg uncertainty principle, with lower quantum noise (variance) in one of the quadratures as compared with a coherent state. In the photon number basis a one mode vacuum squeezed state has the following form:

$$|\zeta, 0\rangle = \sqrt{\text{sech} r} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left[ -\frac{1}{2} e^{i\phi} \tanh r \right]^n |2n\rangle \tag{76}$$

where $r$ is the squeezing parameter. In terms of the covariance matrix formalism, the $\phi = 0$ squeezed vacuum state is described by

$$\gamma = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix} \tag{77}$$

and displacement vector $d = 0$. Displacing a squeezed vacuum state using the displacement operator $\hat{D}_d$ leads to a state with unchanged covariance matrix but with the displacement vector $d = \tilde{d}$. In the two-mode case, we shall utilize the two-mode squeezed vacuum state, with squeezing of the relative position $x_1 - x_2$ and total momentum $p_1 + p_2$. The covariance matrix of this state takes the form \[20\]:

$$\gamma = H^T \text{diag}(e^{-2r}, e^{2r}, e^{2r}, e^{-2r}) H \tag{78}$$

where:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{79}$$

while the displacement vector $d = 0$.

Lastly, for calculation of channel capacities we shall require the entropy of $n$-mode gaussian states $\rho$. This is given by the formula \[20\]:

$$S(\rho) = \sum_{j=1}^{n} g(\frac{v_j - 1}{2}) \tag{80}$$

in terms of normal modes of the system. Here $g(x) = (x + 1) \ln(x + 1) - x \ln x$ is the entropy of a normal mode with average occupation number $x$. The $v_j$’s are the symplectic eigenvalues of the covariance matrix $\gamma$ corresponding to the state $\rho$, i.e. the square roots of the eigenvalues of the matrix $-J^{(n)} \gamma J^{(n)}$. (Note that the symplectic spectrum for each mode is doubly degenerate and that in the entropy formula each value is taken only once).

### A. Locality rule for classical gaussian MAC

The analysis of capacity regions is more intricate in the CV case than in the discrete variable case, already for classical channels. This is intimately related to the fact that the capacities are dependent on power constraints which may lead to various scenarios. To see this, consider an example of two 1-to-1 classical channels channels $\Phi_1$ and $\Phi_2$ with noise levels $N_1$ and $N_2$ and the same power constraints $P$. We assume that $N_1 < N_2$. Suppose each channel works separately, then $\Phi_1$ achieves the capacity $\tilde{C}_1 = \frac{1}{2} \log \left[ 1 + \tilde{P}/N_1 \right]$ \[12\]. Now suppose the channels work in a parallel setup. The sender aims to maximize the total capacity $C_T = C_1 + C_2 = \frac{1}{2} \log \left[ 1 + P_1/N_1 \right] + \log \left[ 1 + P_2/N_2 \right]$ where $P_i$ is the power allocated to channel $\Phi_i$. One demands that the total power available to the user in this case is identical to the total power used when the channels were utilized separately, i.e. $P_1$ and $P_2$ obey the constraint $P_1 + P_2 \leq 2\tilde{P}$. Now since the noise levels $N_1, N_2$ are different, the senders can increase the total capacity by allocating more power in the transmission through the channel with the lower level of noise. When $N_1 + 2\tilde{P} < N_2$, the optimal choice is to put $P_1 = 2\tilde{P}$. In the other case, the optimal allocation is determined from the relation $N_1 + P_1 = N_2 + P_2$. Using this power redistribution, the sender can achieve capacity $\tilde{C} > C_1 + C_2$. This process of optimisation is the so–called waterfilling scheme(see e.g. \[12\]).

Thus, we see that for Gaussian channels the additivity theorem of Sec.\[11\]A cannot be stated as such. However, observe that the local rates depend only on the local power constraints (cf. Eq. \[73\]). This means that, in a multuser scenario, adding a resource (channel or energy) to one sender never helps the others beat their maximal achievable rates (power constraints pertaining to different users are not allowed to be combined, hence no inter-user waterfilling effect can take place). We call this observation the locality rule for classical Gaussian...
We shall focus our attention on the following protocols: We consider only transmission of Gaussian states. We will also point out cases where the locality rule is broken. Thus characterized by a loss \( \cos \theta \) of the vacuum state while the probability of displacement is chosen, as is standard, to be a Gaussian distribution \( p(x, p) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + p^2}{2\sigma^2}\right) \) with \( \sigma^2 = 2N_1 \). Sender \( B \) transmits a fixed chosen coherent state all the time. The receiver performs homodyne detection on both quadratures to decode the message. This is a typical setup for transmission of information through optical fibers. The achievable rate depends only on the output power corresponding to user \( A \) and reads:

\[
R_{\text{coherent}}^A \leq \log(1 + \sin^2 \theta N_A) \tag{83}
\]

This rate refers to the case of a lossy channel with transmittivity \( T = \sin^2 \theta \) in case when sender performs encoding in coherent states and receiver performs homodyne detection on the output. It depends only on power constraints for \( A \) and manifestly obeys the locality rule.

2. Senders \( A \) and \( B \) use single-mode squeezed vacuum states. Both users transmit states which are squeezed in the same canonical variable, say \( x \). A encodes his message in the displacement of the variable \( x \) of his state, whose value is Gaussian distributed with variance \( \sigma_x^2 \). The receiver performs homodyne detection only on \( x \). This setup was studied in [11]. The rate is given by:

\[
R_{A}^{\text{squeezed}} = \frac{1}{2} \log \left[ 1 + \frac{\sigma_x^2 \sin^2 \theta}{\sin^2 \theta - 2R + \cos^2 \theta - 2r} \right] \tag{84}
\]

where \( R \) and \( r \) denote the squeezing parameters of the \( x \) quadrature for senders \( A \) and \( B \) respectively. The energy constraints can be written as: \( \sigma_x^2 \leq 4(N_A - \sinh^2 R), \sinh^2 r \leq N_B \). User \( A \) performs optimization of the parameter \( R \), that is he optimizes the power allocation between squeezing and mean square displacement. For fixed \( \theta \), in the limit \( N_A \to \infty, N_B \to \infty \) we get asymptotically

\[
R_{A}^{\text{squeezed}} \to \log[1 + N_A] \tag{85}
\]

3. Sender \( A \) again sends coherent states, encoding his message in the displacement of both canonical variables. The displacement has probability density distribution as in case [11]. Sender \( B \) transmits a two-mode squeezed state, one mode through \( \Phi_\theta \) and the second one through the extra resource \( I \). The receiver has access to the output of \( \Phi_\theta \) and \( I \). The decoding consists of a joint measurement of the canonical variables \( x_0, x_1 \) and \( p_0, p_1 \) on the output modes of the setup. To achieve this, the output modes of \( \Phi_\theta \) and \( I \) are mixed on a 50 : 50 beam splitter followed by homodyne measurements of \( x_1 \) and \( p_2 \) on the output modes of the 50 : 50 beam splitter. In this setup, the sender \( B \) is assumed to make use of entangled states.
The formula for the transmission rate is now given by:
\[ R_{A}^{opt} = \log \left[ 1 + \frac{\sigma^2 \sin^2 \theta}{2(\cosh r - \cos \theta \sinh r)^2} \right]. \] (86)

Here \( \sigma^2 = 2N_A \) is the variance in the displacement of canonical variables in \( A \)'s mode. Here, \( r \) denotes the squeezing parameter of the two mode squeezed state sent by \( B \). The imposed power constraints imply that \( \sinh^2 \theta = N_B/2 \). For given \( N_A \), the optimal values of \( [\theta, r] \) lie on curve:
\[ \cos \theta = \tanh r \] (87)
which leads to the following maximal rate formula:
\[ R_{A}^{ent-opt} = \log [1 + N_A]. \] (88)

\( R_{A}^{ent-opt} \) is in fact equal to the rate achievable by a one mode ideal channel case when sender performs encoding in coherent states and receiver performs homodyne detection on the output. Thus entanglement can be used to completely overcome power loss in the case of coherent state encoding. The same effect is also obtained in the second case, without entanglement, as described above, but only in the asymptotic regime of infinite power (see Eq. (85)).

It is indeed worth noting, for comparison, that the limit \( N_A \to \infty, N_B \to \infty \) of Eq. (87) under the constraint Eq. (86) leads to:
\[ R_{A}^{squeezed} \leq \frac{1}{2} \log [1 + 16N_A] \approx \frac{1}{2} \log [1 + N_A] = \frac{1}{2} R_{A}^{ent-opt}. \] (89)

Comparison of this result with Eq. (85) shows that one mode squeezed states transmission requires much higher squeezing to reach the rates achievable by two mode squeezed state transmission.

We can also calculate two upper bounds \( R_{A}(\Phi_\theta) \) for transmission rates only through channel \( \Phi_\theta \):

1. Bound based on maximal entropy of a state with mean number of photons equal to the mean number of photons in the output mode of the channel \( \Phi_\theta \). We shall refer to it as to the output entropy bound. This tells us how large a rate is achievable if no entanglement is allowed in the communication protocol and is given by:
\[ R_{A}^{prod-bound} = g(N_{out}) \] (90)
where:
\[ g(N_{out}) = g\left(\sqrt{N_A \sin^2 \theta + N_B \cos^2 \theta} \right) \] (91)

2. Bound based on maximal entropy of a state with mean number of photons equal to the mean number of photons in the input mode \( A \) of the channel \( \Phi_\theta \). This may be referred to as an input entropy bound. This bound cannot be violated by any type of communication protocol, entanglement-free or entanglement-aided, and it tells us how much information can be transmitted with given energy constraints if sender \( A \) is connected to the receiver by a one mode ideal line. We will check how close the protocols described above approach this bound, which is given by:
\[ R_{A}^{max} \leq g(N_A). \] (92)

The bounds \( R_{A}^{prod-bound} \) and \( R_{A}^{max} \) allow us to express the theoretical maximum rate for sender \( A \) in the form \( \min(R_{A}^{prod-bound}, R_{A}^{max}) \).

FIG. 9 shows the behaviour of rates achievable by different encoding schemes for the beam splitter channel \( \Phi_\theta \) as a function of the energy constraint \( N_B \) for sender \( B \). Rates are evaluated for the following setups of channel parameters \( \theta \) and energy constraints \( N_A \) for sender \( A \): (A) \( \theta = \pi/4, N_A = 10^6 \); (B) \( \theta = 0.2, N_A = 10^6 \); (C) \( \theta = 0.5, N_A = 10^3 \). We will check how close the protocols described above approach this bound, which is given by
\[ R_{A}^{max} \leq g(N_A). \] (92)

FIG. 9 shows the behaviour of rates achievable by different encoding schemes and parameter regimes as a function of the energy constraint \( N_B \) for sender \( B \). The bounds and bound correspond directly to the points (1-5) in the main text. Figure a) presents the situation where using entangled states quickly becomes more efficient than using any product state encoding, while on the other hand Fig. b) presents a situation where entanglement cannot beat the upper bound for rates achievable for product states encoding. In Fig. c) we consider the behaviour of rates for large range of values of \( N_B \). We can observe that in the low \( N_B \) range, the strategy using entanglement states is the best among the three considered approaches. However increasing \( N_B \) leads to a maximal value of \( R_{A}^{opt} \) after which further growth leads to a diminishing rate. This can be explained by increasing of the entanglement of the output state with the erased mode. In case of \( R_{A}^{squeezed} \) the situation looks different. It was shown in the paper [10] that in the limit \( N_A \to \infty, N_B \to \infty \) rate \( R_{A}^{squeezed} \) asymptotically approaches the upper bound for the transmission rate for sender \( A \) expressed by \( R_{A}^{max} \). It has to be reiterated...
FIG. 10: a) Lines present bounds of the areas where \( R^{\text{ent}}_A > R^{\text{prod–bound}}_A \). They are given by the condition \( R^{\text{ent}}_A / R^{\text{prod–bound}}_A = 1 \). Lines refer to the cases of \( N_A = 10^3, 10^6, 10^9 \). For given \( N_A \) the superadditive region lies above the corresponding line. b) The dotted line delimits the area where \( R^{\text{ent}}_A > R^{\text{prod–bound}}_A \) for \( N_A = 10^3 \). Notice that the superadditive region is considered in a large scale of \( N_B \). The figure shows that for there is large window of parameters \( N_B, \theta \) for which \( R^{\text{ent}}_A > R^{\text{prod–bound}}_A \) can be observed.

here that this strategy requires extremely high squeezing to approach the maximal rate achieved by the protocol using a two mode squeezing scenario.

In case of the protocol using two mode squeezed states it is interesting to ask about the lower limit of squeezing for which the rate achieved starts to be higher than the rate achieved by any protocol based only on product states encoding. FIG. 10(a) presents demarcation curves \( R^{\text{ent}}_A / R^{\text{prod–bound}}_A = 1 \) in the \( \theta – N_B \) parameter plane for three different values of the parameter \( N_A \). For fixed \( N_A \), with increasing \( N_B \), we move above the demarcation curve and fall into the area where \( R^{\text{ent}}_A > R^{\text{prod–bound}}_A \). The minimal mean photon number in the entangled state, required to approach this area, amounts to around \( N_B = 1, 0.6, 0.55 \) for \( N_A = 10^3, 10^6, 10^9 \). These values of \( N_B \) refer to the following squeezing levels which are experimentally realistic: 5.72dB, 4.55dB, 4.37dB. The demarcation curve is crossed as \( \theta \) equals 0.28, 0.1, 0.02 or transmissivity 0.077, 0.01, 0.0004. For large \( N_A \), the locality rule is broken for \( \theta \approx 0 \). In this regime the setup reproduces the continuous variable dense coding scheme. \( N_B = 1 \) means that we use two mode squeezed state with squeezing equal 5.72dB which is reasonable value for experimental setup. In FIG. 10(b), we change the scale of observation and show that breaking of the locality rule occurs for quite a large range of the parameter \( N_B \) and \( \theta \).

C. Realization of XP gate by linear optics and one mode squeezed states. Influence of noise on superadditivity effect

In this section, we shall start with details of realization of the three input quantum non demolition channel \( \Phi \) presented in FIG. 8(b). This will lead naturally to a discussion of the interplay between and noise (or imperfections) and superadditivity.

The channel \( \Phi : A X A P B \rightarrow R \) acts as follows \( \Phi(\rho_{A X A P} \otimes \rho_B) = \text{tr}_{A X A P} \left[ \hat{U}(\rho_{A X A P} \otimes \rho_B)\hat{U}^\dagger \right] \). Sender A holds lines \( A X \) and \( A P \), while sender B holds line \( B \). \( \hat{U} \) is an unitary operator of the form \( \hat{U} = \exp[-i(\hat{x}_X\hat{p}_B - \hat{p}_X\hat{x}_B)] \), which can be factorized as follows:

\[
\begin{align*}
\hat{U} &= \exp[-i(\hat{x}_X\hat{p}_B - \hat{p}_X\hat{x}_B)] \\
&= \exp\left[\frac{i}{2}\hat{x}_X\hat{p}_B\right]\exp[-i\hat{x}_B\hat{p}_B]\exp[i\hat{p}_X\hat{x}_B].
\end{align*}
\]

The XP interaction, appearing here, can be obtained by measurement-induced continuous-variable quantum interactions as described in Ref. [13]. An experimental proof–of–concept has been presented in Ref. [14].

The superadditive effect of single user capacity (breaking of the locality rule) for this channel was considered by us in [10] for this channel, where the XP interaction was assumed to be ideally implemented. However, the method of measurement-induced continuous-variable quantum interactions is, in practice, imperfect and introduces errors in the output states. To study such errors it is useful to write down how canonical observables are transformed by the realization of the XP gate [13, 14]:

\[
\begin{align*}
\hat{x}_1^{\text{out}} &= \hat{\tilde{x}}_1^{\text{in}} - \sqrt{\alpha} \hat{x}_0 - \sqrt{\beta} \hat{x}_{S_1}, \\
\hat{p}_1^{\text{out}} &= \hat{\tilde{p}}_1^{\text{in}} - \frac{1-T}{\sqrt{T}} \hat{p}_2^{\text{in}} + \sqrt{T} \beta \hat{p}_{S_2}, \\
\hat{x}_2^{\text{out}} &= \hat{\tilde{x}}_2^{\text{in}} + \frac{1-T}{\sqrt{T}} \hat{\tilde{x}}_1^{\text{in}} - \sqrt{\alpha} \hat{p}_0 + \sqrt{T} \beta \hat{x}_{S_1}, \\
\hat{p}_2^{\text{out}} &= \hat{\tilde{p}}_2^{\text{in}} - \sqrt{\alpha} \hat{p}_0 + \sqrt{T} \beta \hat{p}_{S_2}
\end{align*}
\]

where: \( \alpha = (1-T)/(1-\eta)/(1+T)\eta, \beta = (1-T)/(1+T) \), \( \hat{x}_{S_1}, \hat{p}_{S_2} \) are canonical observables of two different modes in squeezed states, \( \eta \) is efficiency of the homodyne detectors inside the XP gate realisation and \( \hat{x}_0, \hat{p}_0 \) are canonical observables of two different modes in the coherent states used that homodyne detectors. Parameer \( T \) depends on the configuration of the XP gate realisation and can be manipulated. Choosing \( T = \frac{1}{2} (3 - \sqrt{3}) \), the coefficients of \( \hat{\tilde{p}}_2^{\text{in}} \) and \( \hat{\tilde{x}}_2^{\text{in}} \) in Eqs. (95)–(77) become –1 and 1. We will hereby represent this XP gate realization as a quantum noisy channel described by transformation matrices \( X, Y \), using the method outlined in Sec. IV

Here we assume that errors introduced by linear optical elements can be neglected in comparison with natural noise due to the physical generation of highly squeezed
states. Then
\[
X = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad Y = \begin{pmatrix}
\sigma_1^2 & 0 & 0 & 0 \\
0 & \sigma_2^2 & 0 & 0 \\
0 & 0 & \sigma_3^2 & 0 \\
0 & 0 & 0 & \sigma_4^2
\end{pmatrix}
\]
(99)
with \(\sigma_1^2 = \alpha + \beta e^{-2s}, \sigma_2^2 = \alpha / T + \beta T e^{-2s}\). We assumed that squeezed states in both modes have the same squeezing level. This gate reproduces the ideal XP gate in the limit of infinite squeezing \(s \to \infty\) and ideal homodyne detection \(\eta \to 1\). If all XP gates used in the implementation of the considered channel have the same parameters, we can collect all noise components under the common factor \(\sigma_{\text{noise}}^2 = \sigma_1^2 + \sigma_2^2\).

Now suppose that sender B also has access to the input \(B'\) of a one mode ideal channel \(I\) (see Fig. 8(b)). We are interested in the maximal rate \(R_A^{(1)}(\Phi \otimes I)\) for sender A under the following protocol \(10\) when sender A transmits displaced squeezed one mode vacuum states. States transmitted through line \(A_X\) (\(AP\)) are squeezed in the canonical observable \(\hat{x}(\hat{p})\), where the squeezing parameter is \(R\). Sender A encodes his message in the displacement during encoding. The displacement has Gaussian distribution with variance \(\sigma^2\). Squeezed B continuously transmits a constant fixed two mode squeezed vacuum state, with squeezing parameter \(r\). One mode is transmitted through line \(B\) and the other through line \(B'\). The receiver performs joint homodyne detection of \(x_B - x_{B'}\) and \(p_B + p_{B'}\) on the output of channels \(\Phi\) and \(I\) to decode the message.

The rate in this case is now calculated to be:
\[
R_A^{(1)} = \log \left(1 + \frac{\sigma^2}{\omega^2 \cos^2 \omega} \right)
\]
(100)

The imperfections in the implementation of the desired unitary evolution appear in the form of the extra noise term \(\sigma_{\text{noise}}^2 / 2\) in the expression for maximal rate, as compared to the ideal case described in \(10\).

For a more realistic description, we shall model the influence of various unavoidable imperfections, associated \(e.g.\) with the implementation of displacement during encoding, the measurement process realized by the receiver for decoding, and the interaction with environment at finite temperature, by thermal noise channels. This type of a channel is a 1-to-1 lossy channel mixing an input state with a thermal state \(\rho_{\text{NT}}\) containing an average of \(N_T\) photons, at a beam splitter with transmissivity \(T = \cos^2 \omega\). The receiver receives only one of the output modes of this mixing beam splitter. The covariance matrix \(\gamma\) of an input state is transformed by this channel as follows
\[
\gamma \rightarrow T \gamma + (1 - T) \gamma_{\text{NT}},
\]
(101)
where \(\gamma_{\text{NT}} = N_{T}I\). Below we assume that XP gate is perfect, however we put the thermal noise channels parameterized by \(\omega\) and \(N_{Th}\) at two places: between the output of the non-demolition channel \(\Phi\) and receiver and between the output of supporting channel \(I\) and receiver. Now the transmission rate for the upper sender, using the same protocol as described earlier in the noiseless case, is modified and is calculated here to be
\[
R_A^{(1)} = \log \left(1 + \frac{\sigma^2 \cos^2 \omega}{\left(e^{-2R} + e^{-2r} \right) \cos^2 \omega + (1 + N_{TTh}) \sin^2 \omega} \right)
\]
(102)

In a similar way, we now also model effects of noise on the beamsplitter MAC channel \(\Phi_\theta\) discussed earlier (Fig. 8(a)) in point 3 of Sec. IV.B We again place the thermal noise channel (parametrised by \(\omega, N_{TTh}\)) between output of the \(\Phi_\theta\) channel and receiver and between output of the supporting channel \(I\) and receiver. In this case, we obtain now the following rate of the upper sender:
\[
R_A^{(1)} = \log \left(1 + \frac{\sigma^2 \sin^2 \theta T}{\left(\cosh r - \cos \theta \sinh r \right)^2 T + (1 + 2N_{TTh}) \left(1 - T \right)} \right)
\]
(103)
Here \(\theta\) is the parameter of the BS channel \(\Phi_\theta\), \(T = \cos^2 \omega\) is the transmissivity of the thermal noise channel and \(N_{TTh}\) is the mean photon number of the environment. In Fig. 11 we use this result to illustrate how the capacity \(R_A^{(1)}(\Phi_\theta \otimes I)\) changes with the parameters of the thermal noise channel. Even if the effect of thermal noise channel is small - its transmissivity is large and \(N_{TTh} = 0\) - the capacity gain becomes negligible and from \(T = 0.85\) no enhancement over the upper bound for rates obtained using product codes is observed. This scenario corresponds to the case where there are only losses in the thermal channel.

Now we use the presented results to discuss the possibility of experimental verification of the considered superadditivity effects in the context of available technological resources 28, 29. For homodyne detection we will assume quantum efficiency at the level \(\eta = 99\%\) as in 30 and the dark noise level at 20dB below the shot noise of the local oscillator. For power constraints of sender A: \(N_A = 1000\) we remain in the regime of linear approximation of homodyne detection (mean number of photons of local oscillator is on the level \(4 \times 10^6\)). Note finally, that the highest observed value of single mode squeezing 28 is at the level of 10 dB which corresponds to mean number of photon 2.025.

We start with a discussion of the setup \(\Phi \otimes I\) in the context of the implementation of the XP gate presented in 14. In that experiment, squeezing of 5.6 dB and detectors with quantum efficiency \(\eta = 98\%\) were used. A quadrature transfer coefficient of \(T_P = SNR_{\text{out}} / SNR_{\text{in}} = 0.4\) was reported, where \(SNR_{\text{in(out)}}\) is the signal-to-noise ratio for the signal input (output) of the gate. This coefficient \(T_P\) can be translated to \(\sigma_{\text{noise}}^2 = 5\) in the model of the \(\Phi\) gate. Assuming this value in Eq. (109) and comparing the rate with the output entropy bound leads to the conclusion that the superadditivity effect described here cannot be achieved for these parameters. On the other hand, implementation of
the $XP$ gate with the use of 10 dB squeezing leads to $\sigma^2_{\text{noise}} = 0.098$ and moves the transmission into the superadditivity effect regime.

For the setup $\Phi_\theta \otimes I$ the situation is more optimistic. With a realistic loss level of 5% on the optical elements and homodyne detection efficiency as described above one gets $\cos^2 \omega = 0.94, N_Y = 0.09$. Our results show that the superadditivity effect can be observed for $\theta = 0.25$ for squeezing upwards of 7.8 dB (mean photon number 2.1). Thus one can draw the conclusion that a loop-hole free verification of the superadditivity effect can be done with the present state of art quantum optical experimental techniques.

Formulas (100, 102, and 103) can be understood in a generic signal-to-noise "fenomenological" scheme as $R_A = \log(1 + \sigma^2_{\text{signal}}/\sigma^2_{\text{noise}})$. The variance $\sigma^2_{\text{signal}}$ describes how spread out are the input states of the sender in phase space and $\sigma^2_{\text{noise}}$ describes the effective noise level associated with measurement of displacement which here is the carrier of classical information. Senders can manipulate $\sigma^2_{\text{signal}}$ and $\sigma^2_{\text{noise}}$ by changing energy allocation used for displacement and squeezing. In this way one can bring nearer to the bound for channel capacity. Noise introduced by imperfection of elements of the communication setup plays the role of a lower bound for $\sigma^2_{\text{noise}}$ and the user can not decrease measurement error below its level.

V. CONCLUSIONS

The superadditivity of classical capacity regions have been reported previously in case of discrete \cite{11} and continuous variables (Gaussian) \cite{10} cases. Here we have analysed these problems in more detail. We have been able to show that asymmetry of the channel is not crucial for the superadditivity effect. Even more interestingly, we have proven explicitly that two-input entanglement is not enough in some cases - in our case we have shown analytically that at least 5-input entanglement is needed. It is interesting that here the 5-qubit code error correction code has been used (see \cite{4}) to beat the C-type (classical) multi-access capacity which so far was a tool related to Q-type (quantum) bipartite capacity. Moreover we do not know of any example in bipartite classical capacity where more than two-input entanglement is needed to achieve the asymptotic bound. In fact the celebrated effect of breaking of additivity of Holevo function \cite{11} needs two copies of the channel. We believe that our result will inspire the search for requirement of multipartite entanglement for achieving the asymptotic Holevo capacity in the bipartite case. In both bipartite and multiuser cases, this opens the intriguing question concerning which types of multipartite entanglement (bipartite quantum code-words, cluster, Dicke-type etc.) are the best for achieving asymptotic classical capacities. We leave this type of questions for further research.

In case of continuous variables, we carefully compared different scenarios including the one described by Yen and Shapiro Ref. \cite{10}. We incorporated explicitly imperfection of the schemes into calculations. The success of superadditivity depends on power of the light used and may be destroyed by thermal noise or even by loses if they are large enough. On the other hand we found that the condition for two mode squeezing used for the effect may not be very demanding (4.55$dB$). This opens the possibility of experimental confirmation of the effect in near future. Again the question of channels that require multipartite CV-type entanglement is quite natural and may be not easy in case of Gaussian channels.

VI. APPENDIX

Here we prove that capacity regions of discrete classical $n$-to-1 channels are additive:

$$R(\Phi_1 \otimes \Phi_{II}) = R(\Phi_1) + R(\Phi_{II}), \quad (104)$$

where $R(\Phi_1) + R(\Phi_{II}) = \{u_I + u_{II} : u_I \in R(\Phi_1), u_{II} \in R(\Phi_{II})\}$. For simplicity it is assumed that both channels, $\Phi_1, \Phi_{II}$ have the same number of senders. This situation
is easy to obtain by formal extension of the set of senders for one of the channels. The messages transmitted by these additional senders are then always lost. We will use short-hand notation $R_s = \sum_{i \in S} R_i$ for vector of rates $R \in \mathbb{R}^n$ where $R_i$ is $i$-th element of $R$ and $S \subseteq E$ is a subset of senders $E$. The proof for the $n$-to-1 case follows the same principles as for 2-to-1 channels. Here we provide only the parts that are distinct from that latter case.

We start with \[ \tilde{R}(\Phi_I \otimes \Phi_{II}) \subseteq \tilde{R}(\Phi_I \otimes \Phi_{II}) \subseteq \tilde{R}(\Phi_I \otimes \Phi_{II}). \] Let us assume that senders transmit with rates given by vector $\tilde{R}$. As in 2-to-1 case this vectors has to belong to the fixed probability capacity region for input symbols probability distribution $\tilde{p} = p(Q^I, Q^{II}) \prod_i p(X^I_i, X^{II}_i | Q^I, Q^{II})$. Below we provide upper bound for this region:

\[ \tilde{R}_S \leq I(X_S : Y | X_{SC}, Q) \]

\[ = H(Y | X_{SC}, Q) - H(Y | X_S, X_{SC}, Q) \]

\[ = H(Y | X_{SC}, Q) - H(Y^I | X^I_S, X^I_{SC}, Q^I) + H(Y^{II} | X^{II}_S, X^{II}_{SC}, Q^{II}) \]

\[ \leq H(Y | X^I_S, X^I_{SC}, Q^I) + H(Y^{II} | X^{II}_S, X^{II}_{SC}, Q^{II}) \]

\[ = I(X^I_S : Y^I | X^I_{SC}, Q^I) + I(X^{II}_S : Y^{II} | X^{II}_{SC}, Q^{II}). \]

where Eq. (109) is based on the factorisation of conditional probabilities defining the channel action for the product channel, while in Eq. (109) we use entropy subadditivity. On the other hand, evaluation of Eq. (10) for input symbols probability distribution $\tilde{p} = \tilde{p}_I \otimes \tilde{p}_{II} = (p(Q^I) \prod_i p(X^I_i | Q^I)) (p(Q^{II}) \prod_i p(X^{II}_i | Q^{II}))$ leads to the region:

\[ \tilde{R} = \{ R \in \mathbb{R}^n : \forall_{i \in E} R_i \geq 0 \}

\[ \forall_{S \subseteq E} \tilde{R}_S \leq I(X^I_S : Y^I | X^I_{SC}, Q^I) + I(X^{II}_S : Y^{II} | X^{II}_{SC}, Q^{II}). \]

Combining this result with the bound Eq. (108), it is easy to see that $\tilde{R}(\Phi_I \otimes \Phi_{II}) \subseteq \tilde{R}(\Phi_I \otimes \Phi_{II})$ holds.

It remains to be shown now that

\[ \tilde{R}(\Phi_I \otimes \Phi_{II}) = \tilde{R}(\Phi_I) + \tilde{R}(\Phi_{II}), \]

where again $\tilde{R}(\Phi_I) (\tilde{R}(\Phi_{II}))$ is evaluated for margin probability distribution $\tilde{p}_I (\tilde{p}_{II})$. The following argument is based on the fact that fixed probability capacity region is a polymatroid [20].

**Definition 1.** Let $E = \{1, \ldots, n\}$ and $f : 2^E \rightarrow \mathbb{R}_+$ be a set functions (i.e. function that maps subsets of $E$ into $\mathbb{R}_+$). The polyhedron:

\[ B(f) = \{ x \in \mathbb{R}^n : \forall_{S \subseteq E} S \leq f(s), \forall_{i \in S} x_i \geq 0 \} \]

is a polymatroid if the set function $f$ satisfies: (i) $f(\emptyset) = 0$, (ii) $S \subseteq T \Rightarrow f(S) \leq f(T)$, (iii) $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$.

**Lemma 1.** The fixed probability capacity region (cf. Eq. (17)) is a polymatroid.

**Proof.** Observe that conditional mutual information $I(X_S : Y | X_{SC}, Q)$ plays the role of a set function $f(S)$, in the above equation. All we have to do now is to check conditions (i) – (iii) defining the polymatroid. By definition if there is no sender, mutual information is equal to 0, which proves (i). Now, let us write

\[ f(T) = I(X_T : Y | X_{TC}, Q) \]

\[ = H(Y | X_{TC}, Q) - H(Y | X_T, X_{TC}, Q) \]

\[ \geq H(Y | X_{SC}, Q) - H(Y | X_S, X_{SC}, Q) \]

\[ = I(X_S : Y | X_{SC}, Q) \]

\[ = f(S) \]
condition (iii):

\[ f(S) + f(T) = I(X_S : Y | X_{SC}, Q) + I(X_T : Y | X_{TC}, Q) \]
\[ = H(Y | X_{SC}, Q) - H(Y | X_S, X_{SC}, Q) + H(Y | X_{TC}, Q) - H(Y | X_T, X_{TC}, Q) \]
\[ = H(Y, X_{SC}, Q) - H(X_{SC}, Q) + H(Y, X_{TC}, Q) - H(X_{TC}, Q) \]
\[ = H(Y, X_{SC}, Q) + H(Y, X_{TC}, Q) - 2H(Y | X_E, Q) \]
\[ \geq H(Y, X_{SC} \cup T, Q) \]

where \( \bar{R}^I_S \in \bar{\mathcal{R}}(\Phi_I), \bar{R}^{II}_S \in \bar{\mathcal{R}}(\Phi_{II}) \) and by definition obey Eq. (17).

Finally, we will show \( \bar{\mathcal{R}}(\Phi_I \otimes \Phi_{II}) \subseteq \bar{\mathcal{R}}(\Phi_I) + \bar{\mathcal{R}}(\Phi_{II}) \). Since there is equivalence between the vertex and the half space representation of a convex polyhedron, we only have to show that each vertex \( v \in \bar{\mathcal{R}}(\Phi_I \otimes \Phi_{II}) \) can be expressed as \( v = u + w \) where \( u \) and \( w \) are suitable vertices of \( \bar{\mathcal{R}}_I \) and \( \bar{\mathcal{R}}_{II} \) respectively.

As we have seen, the fixed probability capacity region is polymatroid. This leads to a key property of the set of its vertices. Let \( \pi \) be an ordered choice from the set of senders \( E \). For each ordered choice \( \pi \), there is a vertex \( v \) with entries: \( v_{\pi_1} = f(\pi_1), v_{\pi_i} = f(\{\pi_1, \ldots, \pi_i\}) - f(1, \ldots, \pi_i - 1) \) \( \forall i \in \pi \) \( v_i = 0 \). On the other hand, we can always find an ordered choice \( \pi \) which defines given vertex. It may happen that more than one ordered choice gives the vertex with the same entries. E.g. in the 2-to-1 case, fixed probability capacity region is given by the vertices:

\[ \pi = \{1\} : \left( \begin{array}{c} I(X_1^I : Y^I | X_2^I, Q^I) + I(X_1^{II} | X_2^{II}, Q^{II}) \\ 0 \end{array} \right) \]
\[ \pi = \{1, 2\} : \left( \begin{array}{c} I(X_1^I : Y^I | X_2^I, Q^I) + I(X_2^{II} | X_1^{II}, Q^{II}) \\ I(X_2^I : Y^I | X_1^I, Q^I) + I(X_2^{II} | X_1^{II}, Q^{II}) \end{array} \right) \]
\[ \pi = \{2\} : \left( \begin{array}{c} I(X_1^I : Y^I | X_2^I, Q^I) + I(X_1^{II} | X_2^{II}, Q^{II}) \\ I(X_1^I : Y^I | X_2^I, Q^I) + I(X_2^{II} | X_1^{II}, Q^{II}) \end{array} \right) \]

Using the chain rule, we obtain that for a given ordered choice \( \pi \), rates achieved in vertex \( v(\pi) \) are:

\[ R_{\pi_i} = I(X_{\pi_i}^I : Y^I | X_{\pi_{i+1}}^I, \ldots, X_{\pi_n}^I, Q^I) + I(X_{\pi_i}^{II} | X_{\pi_{i+1}}^{II}, \ldots, X_{\pi_n}^{II}, Q^{II}) \]

and can be viewed as a sum of the vectors of rates \( u(\pi) \) and \( v(\pi) \) with entries:

\[ R_{\pi_i}^{I} = I(X_{\pi_i}^I : Y^I | X_{\pi_{i+1}}^I, \ldots, X_{\pi_n}^I, Q^I) \]
\[ R_{\pi_i}^{II} = I(X_{\pi_i}^{II} | X_{\pi_{i+1}}^{II}, \ldots, X_{\pi_n}^{II}, Q^{II}) \]

which in an obvious way belong to fixed probability capacity regions \( R_I \) and \( R_{II} \) respectively. This completes the proof.

The proof for \( n \) MACs can be obtained through induction of the above proof. Indeed it suffices to divide the set of \( n \) MACs into two MACs – one composite MAC consisting of \( n-1 \) MACs and a second channel consisting of the remaining MAC, and apply the above solution to them.

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[1] Note that the assumption that each channel has the same number of inputs does not in any way influence the generality of the theorem, since $m$ can be chosen to be the maximal number of senders allowed among all considered channels and the definition of the other channels can be extended simply in the following manner $p(y|x_A, x_B, \ldots, x_m) = p(y|x_A, x_B, \ldots, x_w)$ if $w < m$.

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[30] Note that we are working in the bright states regime. We do not need single photon resolution. In those conditions quantum efficiency on the level $\eta = 99\%$ is available e.g. PIN photodiodes.