Frame-independent formulation of Newtonian mechanics

Katarzyna Grabowska
konieczn@fuw.edu.pl
Paweł Urbański
urbanski@fuw.edu.pl
Physics Department, University of Warsaw
Hoża 69, 00-681 Warszawa

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Abstract

Based on ideas of W. M. Tulczyjew, a frame independent formulation of analytical mechanics in the Newtonian space-time is presented. The differential geometry of affine values i.e., the differential geometry in which affine bundles replace vector bundles and sections of one dimensional affine bundles replace functions on manifolds, is used. Lagrangian and hamiltonian generating objects, together with the Legendre transformation independent on inertial frame are constructed.

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1 Introduction

Mathematical formulation of analytical mechanics is usually based on objects that have vector character. So is the case of the most of mathematical physics. We use tangent vectors as infinitesimal configurations, cotangent vectors as momenta, we describe dynamics using forms (symplectic form)
and multivectors (Poisson bracket) and finally we use an algebra of smooth functions. However, there are cases where we find difficulties while working with vector objects. For example, in the analytical mechanics of charged particles we have a problem of gauge dependence of lagrangians. In Newtonian mechanics there is a strong dependence on inertial frame, both in lagrangian and hamiltonian formulation. In the mechanics of non-autonomous system we are forced to chose a reference vector field on the space-time that fulfills certain conditions or we cannot write the dynamics at all. In all those cases the traditional language of differential geometry seems to introduce too much mathematical structure. In other words, there is too much structure with comparison to what is really needed to define and describe the behavior of the system. As a consequence we have to put in an additional information to the system such as gauge or reference frame.

Gauge independence of the Lagrangian formulation of Newtonian dynamics can be achieved by increasing the dimension of the configuration space of the particle. The four dimensional space-time of general relativity is replaced by the five dimensional manifold (as in the Kaluza theory) ([6], [3]). This approach encounters serious conceptual difficulties and cannot be considered satisfactory. An alternate approach is proposed in the present note. The four dimensional space-time is used as the configuration space. The phase space is no longer a cotangent bundle and not even a vector bundle. It is an affine bundle modelled on the cotangent bundle of the space-time manifold. The Lagrangian is a section of an affine line bundle over the tangent bundle of the space-time manifold. The proper geometric tools are provided by the geometry of affine values. We call the geometry of affine values the differential geometry that is built using sections of one-dimensional affine bundle over the manifold instead of functions on the manifold. The affine bundle we use is equipped with the fiber action of the group \((\mathbb{R}, +)\), so we can add reals to elements of fibres and real functions to sections, but there is no distinguished ”zero section”. Those elements of the geometry of affine values that are needed in the Newtonian mechanics are described in section 4.1, the complete presentation of the theory can be found in [1]. In fact the geometry of affine values appeared much earlier in works of W.M. Tulczyjew and his collaborators (see e.g. [12]) and has been successfully applied to the description of the dynamics of charged particles ([3]).

The aim of this paper is to show how the geometry of affine values works in the simplest case of analytical mechanics in the Newtonian space-time. The idea of applying affine geometry to this problem comes from Prof
Wlodzimierz Tulczyjew and his group ([5], [6], [2]). Another proposition of the frame independent formulation of the Newtonian mechanics by increasing the dimension of the space-time can be found in [4].

2 Newtonian space-time

The Newtonian space-time is a system \((N, \tau, g)\) where \(N\) is a four-dimensional affine space with the model vector space \(V\), \(\tau\) is a non-zero element of \(V^*\) and \(g: E_0 \rightarrow E_0^*\) represents an Euclidean metric on \(E_0 = \ker \tau\). The elements of the space \(N\) represent events. The time elapsed between two events is measured by \(\tau\):

\[
\Delta t(x, x') = \langle \tau, x - x' \rangle.
\]

The distance between two simultaneous events is measured by \(g\):

\[
d(x, x') = \sqrt{\langle g(x - x'), x - x' \rangle}.
\]

The space-time \(N\) is fibrated over the time \(T = N/E_0\) which is one-dimensional affine space modelled on \(\mathbb{R}\). By \(\eta\) we will denote the canonical projection

\[
\eta: N \rightarrow T,
\]

by \(i\) the canonical embedding

\[
i: E_0 \rightarrow V,
\]

and by \(i^*\) the dual projection

\[
i^*: V^* \rightarrow E_0^*.
\]

By means of \(i\) and \(i^*\) we can define a contravariant tensor \(g'\) on \(V^*\):

\[
g' = i \circ g^{-1} \circ i^*.
\]

The kernel of \(g'\) is a one-dimensional subspace of \(V^*\) spanned by \(\tau\).

Let \(E_1\) be an affine subspace of \(V\) defined by the equation \(\langle \tau, v \rangle = 1\). The model vector space for this subspace is \(E_0\). An element of \(E_1\) can represent velocity of a particle. The affine structure of \(N\) allows us to associate to an element \(u\) of \(E_1\) the family of inertial observers that move in the space-time.
with the constant velocity \( u \). This way we can interpret an element of \( E_1 \) as an inertial reference frame. For a fixed inertial frame \( u \), we define the space \( Q \) of world lines of all inertial observers. It is the quotient affine space \( N/\{u\} \). The space-time \( N \) becomes the product of affine spaces

\[
N = Q \times T.
\]

The model vector space for \( Q \) is the quotient vector space \( V/\{u\} \) that can be identified with \( E_0 \). The corresponding canonical projection is

\[
\iota_u : V \to E_0 : v \mapsto \iota_u(v) = v - \langle \tau, v \rangle u
\]

and the splitting \( V = E_0 \times \mathbb{R} \) is given by

\[
V \ni v \mapsto (\iota_u(v), \langle \tau, v \rangle) \in E_0 \times \mathbb{R}.
\]

The dual splitting is given by

\[
V^* \ni p \mapsto (\iota^*(p), \langle p, \tau \rangle) \in E_0^* \times \mathbb{R}.
\]

The tangent bundle \( T N \) we identify with the product \( N \times V \) and the subbundle \( VN \) of vectors vertical with respect to the projection on time, with \( N \times E_0 \). Consequently, the bundle \( V^1N \) of infinitesimal configurations (positions and velocities) of particles moving in the space-time \( N \) is identified with \( N \times E_1 \). When the inertial frame \( u \) is chosen, \( E_1 \) is identified with \( E_0 \) and \( V^1N \) is identified with \( VN \).

The vector dual \( V^*N \) for \( VN \) is a quotient bundle of \( N \times V^* \) by the one-dimensional subbundle \( N \times \langle \tau \rangle \). We can identify it with \( N \times E_0^* \). Using the inertial frame we can make it a subbundle of \( T^*N \).

3 Analytical mechanics in the fixed inertial frame

In the following section we will present the analytical mechanics of one particle in the Newtonian space-time in the fixed inertial frame \( u \). First we concentrate on the inhomogeneous formulation, suitable for trajectories parameterized by the time, then we pass to the homogeneous one. The homogeneous formulation accepts all parameterizations.
3.1 Inhomogeneous dynamics described in the fixed inertial frame

Let \( u \in E_1 \) represent an inertial frame. For a fixed time \( t \in T \), the phase space for a particle with mass \( m \) with respect to the inertial frame \( u \) is \( T^* N_t \simeq N_t \times E_0^* \), where \( N_t = \eta^{-1}(t) \). The collection of phase spaces form a phase bundle \( \mathcal{V}^* N \simeq N \times E_0^* \). Phase space trajectories of the system are solutions of the well-known equations of motion:

\[
\dot{p} = -d_s \varphi(x) \quad \dot{x} = g^{-1}\left(\frac{p}{m}\right) + u.
\]

where \((x, p, \dot{x}, \dot{p}) \in \mathcal{V}^1 \mathcal{V}^* N \subset \mathcal{T} \mathcal{V}^* N \simeq N \times E_0^* \times V \times E_0^* \) and \( \varphi : N \to \mathbb{R} \) is a potential. Subscript \( s \) in \( d_s \) means that we differentiate only in spatial directions i.e. the directions vertical with respect to the projection on time, therefore \( d_s \varphi(x) \in E_0^* \). The equations define a vector field on \( \mathcal{V}^* N \) with values in \( \mathcal{V}^1 \mathcal{V}^* N \), i.e. a section of the bundle \( \mathcal{V}^1 \mathcal{V}^* N \to \mathcal{V}^* N \). The image of the vector field \( \square \) we will call the inhomogeneous dynamics and denote by \( D_{i,u} \). It can be generated directly by the lagrangian

\[
\ell_{i,u} : \mathcal{V}^1 N \to \mathbb{R} : (x, w) \mapsto \frac{m}{2}(g(w-u), w-u) - \varphi(x).
\]

The procedure of generating the inhomogeneous dynamics from the lagrangian \( \square \) is as follows. The image of the vertical derivative \( d_s \ell_{i,u} \) is a submanifold of

\[
\mathcal{V}^* \mathcal{V}^1 N \simeq N \times E_1 \times E_0^* \times E_0^*
\]

which is canonically isomorphic (as an affine space) to

\[
\mathcal{V}^1 \mathcal{V}^* N \simeq N \times E_0^* \times E_1 \times E_0^*.
\]

This isomorphism we obtain by a reduction of the canonical isomorphism

\[
\alpha_N : \mathcal{T} \mathcal{T}^* N \to \mathcal{T}^* \mathcal{T} N
\]

(for the definition of \( \alpha_N \) see \( \square \)), which for affine spaces assumes the form

\[
\alpha_N(x, a, v, b) = (x, v, b, a).
\]

After the reduction, we get

\[
\alpha^1_N : N \times E_0^* \times E_1 \times E_0^* \to N \times E_1 \times E_0^* \times E_0^*.
\]
Using $\alpha^1_N$ we can obtain $D_{i,u}$ from $d_s \ell_{i,u}$ by taking an inverse-image:

$$D_{i,u} = (\alpha^1_N)^{-1}(d_s \ell_{i,u}(V^1 N)).$$

The dynamics $D_{i,u}$ cannot be generated directly from a hamiltonian by means of the canonical Poisson structure on $V^* N$, which is the reduced canonical Poisson (symplectic) structure of $T^* N$. In the coordinates adapted to the structure of the bundle $(t, x^i, p_i)$ the Poisson bi-vector is given by

$$\Lambda = \partial p_i \wedge \partial x^i.$$  

Symplectic leaves for this Poisson structure are cotangent bundles $T^* N_t$, where $N_t = \eta^{-1}(t)$. It follows that every hamiltonian vector field is vertical with respect to the projection on time. However, using reference frame $u$, we can generate first the vertical part of the dynamics (1), i.e the equations

$$\dot{p} = -d_s \varphi(x) \quad \dot{x} = g^{-1}(\frac{p}{m}),$$

and add the reference vector field $u$. The hamiltonian function for the problem reads

$$h_{i,u}(x, p) = \frac{1}{2m} \langle p, g^{-1}(p) \rangle + \varphi(x),$$

where $(x, p) \in V^* N$.

The system (3) can be generated also from lagrangian function defined on $VN$ by the formula:

$$\ell_{i,u}(x, w) = \frac{m}{2} \langle g(w), w \rangle - \varphi(x).$$

We identify the fiber over $t$ of $VN$ with $TN_t$ and use the standard procedure to generate a submanifold $D_{u,t}$ in $T^* N_t$. The collection of these submanifolds give us the system (3).

The dynamics (1) and the generating procedures depend strongly on the choice of the reference frame. In particular, the relation velocity-momenta is frame-dependent which means that we have to redefine the phase manifold for the particle to obtain frame-independent dynamics. Also the hamiltonian formulation will be possible if we replace the canonical Poisson tensor by a more adapted object.
3.2 Homogeneous dynamics described in the fixed inertial frame

In the homogeneous formulation of the dynamics infinitesimal configurations are pairs \((x, v)\) \(\in N \times V^+ \subset N \times V \simeq TN\) where \(V^+\) is an open set of vectors such that \(\langle \tau, v \rangle > 0\).

The homogeneous lagrangian is an extension by homogeneity of the \(\ell_{i,u}\) from (3.1) and is given by the formula:

\[
\ell_{h,u}(x, v) = \frac{m}{2\langle \tau, v \rangle} \langle g(i_u(v)), i_u(v) \rangle - \langle \tau, v \rangle \varphi(x).
\]  

This choice guaranties that the action calculated for a piece of the world line, which is one dimensional oriented submanifold of the space-time, does not depend on its parametrization. However, we still have to use the fixed inertial frame \(u\).

The image of the differential of \(\ell_{h,u}\) is a lagrangian submanifold of \(T^*TN \simeq N \times V \times V^* \times V^*\). An element \((x, v, \alpha_x, \alpha_v)\) is in the image of \(d\ell_{h,u}\) if it satisfies the following equations

\[
\begin{align*}
\alpha_x &= -(\tau, v) d\varphi(x), \\
\alpha_v &= \frac{m}{\langle \tau, v \rangle} i_u^* g \circ i_u(v) - \frac{m}{2\langle \tau, v \rangle^2} \langle g(i_u(v)), i_u(v) \rangle \tau - \varphi(x) \tau.
\end{align*}
\]  

The image of \(d\ell_{h,u}(N \times V^+)\) by the mapping \(\alpha^{-1}_N\) is a lagrangian submanifold of \(TT^*N\). This submanifold we will call the homogeneous dynamics and denote by \(D_{h,u}\). An element \((x, p, \dot{x}, \dot{p})\) of \(TT^*N \simeq N \times V^* \times V \times V^*\) is in \(D_{h,u}\) if

\[
\begin{align*}
\dot{x} &= v, \\
\dot{p} &= -(\tau, v) d\varphi(x), \\
p &= \frac{m}{\langle \tau, v \rangle} i_u^* g \circ i_u(v) - \frac{m}{2\langle \tau, v \rangle^2} \langle g(i_u(v)), i_u(v) \rangle \tau - \varphi(x) \tau
\end{align*}
\]  

for some \(v \in V^+\), i.e. \(\langle \tau, v \rangle > 0\). We observe that \(D_{h,u}\) does not project on the whole \(T^*N\), but \((x, p)\) must satisfy the following equation:

\[
\frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle + \varphi(x) = 0.
\]  

The equation (8) is the analog of the mass-shell equation \(p^2 = m^2\) in the relativistic mechanics. Since there is the difference in signature of \(g'\) between
the Newtonian and the relativistic case, we obtain here paraboloid of constant mass instead of relativistic hyperboloid. The mass-shell will be denoted by $K_{m,u}$.

It is possible to generate the dynamics $D_{h,u}$ directly by a generalized hamiltonian system. The hamiltonian generating object (see [9]) is the family

$$
N \times V^* \times V^+ \xrightarrow{-H_{h,u}} \mathbb{R},
$$

(9)

where

$$
H_{h,u}(x, p, v) = \langle p, v \rangle - \ell_{h,u}(x, v) \in \mathbb{R}.
$$

(10)

This family can be simplified. The fibration $\zeta$ can be represented as a composition $\zeta'' \circ \zeta'$, where

$$
\zeta': N \times V^* \times V^+ \rightarrow N \times V^* \times \mathbb{R}_+: (x, p, v) \mapsto (x, p, < \tau, v >),
$$

and

$$
\zeta'': N \times V^* \times \mathbb{R}_+ \rightarrow N \times V^*: (x, p, r) \mapsto (x, p).
$$

Equating to zero the derivative of $H_{h,u}$ along the fibres of $\zeta'$ we obtain the relation

$$
v = \frac{< \tau, v >}{m} g^{-1} \circ \iota^*(p) + < \tau, v > u.
$$

(11)

It follows that the family (9) is equivalent (generates the same object) to the reduced family

$$
N \times V^* \times \mathbb{R}_+ \xrightarrow{-\tilde{H}_{h,u}} \mathbb{R},
$$

(12)

where

$$
\tilde{H}_{h,u}(x, p, r) = r\left(\frac{1}{2m}\langle p, g'(p) \rangle + \langle p, u \rangle + \varphi(x)\right).
$$

(13)

No further simplification is possible.
The critical set $S(\tilde{H}_{h,u}, \zeta'')$ is the submanifold

$$\left\{ (x, p, r) \in N \times V^* \times \mathbb{R}_+; \frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle + \varphi(x) = 0 \right\}$$

and its image $\zeta''(S(\tilde{H}_{h,u}, \zeta''))$ is the mass shell $K_{m,u}$.

The function $H_{h,u}$ is zero on $S(\tilde{H}_{h,u}, \zeta'')$ and projects to the zero function on $K_{m,u}$. However, a Dirac system with the zero function on the constraints $K_{m,u}$ does not generate $D_{h,u}$. The lagrangian submanifold $\bar{D}_{h,u} \subset T^*\mathbb{T}N$ generated by this system is exactly the characteristic distribution of $K_{m,u}$, i.e.

$$\bar{D}_{h,u} = (TK_{m,u})^\perp$$

and does not respect the condition $\langle \tau, v \rangle > 0$. We have only $D_{h,u} \subset \bar{D}_{h,u}$.

4 The dynamics independent on inertial frame

4.1 The geometry of affine values

First, let us give some definitions. A vector space $V$ with distinguished non-zero element $v$ we will call a special vector space. A canonical example of a special vector space is $(\mathbb{R}, 1)$. It will be denoted by $I$. If $A$ is an affine space then $\text{Aff}(A, \mathbb{R})$ – the vector space of all affine functions with real values on $A$ – is a special vector space with distinguished element $1_A$ being a constant function on $A$ equal to 1. The space $\text{Aff}(A, \mathbb{R})$ will be denoted by $A^\dagger$ and called a vector dual for $A$. Having a special vector space $(V, v)$ we can define its affine dual by choosing a subspace in $V^*$ of those linear functions that take the value 1 on $v$:

$$V^\dagger = \{ \varphi \in V^* : \varphi(v) = 1 \}.$$  

We have that

**Theorem 1** ([1]) For $(V, v)$ and $A$ such that $\dim V < \infty$ and $\dim A < \infty$

$$(V^\dagger)^\dagger = V, \quad (A^\dagger)^\dagger = A.$$
An affine space modelled on a special vector space will be called a *special affine space*. Similar definitions we can introduce for bundles: a *special vector bundle* is a vector bundle with distinguished non-vanishing section and a *special affine bundle* is an affine bundle modelled on a special vector bundle.

The geometry of affine values is, roughly speaking, the differential geometry built on the set of sections of one-dimensional special affine bundle $Z$ over $M$ modelled on $M \times \mathbb{I}$, instead of just functions on $M$. The bundle $Z$ will be called a *bundle of affine values*. Since $Z$ is modelled on $M \times \mathbb{I}$ we can add reals in each fiber of $Z$, i.e $Z$ is an $\mathbb{R}$-principal bundle. The vertical vector field on $Z$ which is the fundamental vector field for the action of $\mathbb{R}$ will be denoted by $X_Z$. Let us now consider an example of a bundle of affine values: If $(V, v)$ is a special vector space, then we have the quotient vector space $V_0 = V / \langle v \rangle$. The vector spaces $V$ and $V_0$ together with the canonical projection form an example of a bundle $V$ of affine values. The appropriate action of $\mathbb{R}$ in the fibers is given by

$$V \times \mathbb{R} \ni (w, r) \mapsto (w - rv) \in V$$

and the fundamental vector field $X_V$ is a constant vector field equal to $v$ on $V$.

The affine analog of the cotangent bundle $T^*M$ in the geometry of affine values is called a *phase bundle* and denoted by $PZ$. We define an equivalence relation in the set of pairs of $(m, \sigma)$, where $m \in M$ and $\sigma$ is a section of $Z$. We say that $(m, \sigma), (m', \sigma')$ are equivalent if $m = m'$ and $d(\sigma - \sigma')(m) = 0$, where we have identified the difference of sections of $Z$ with a function on $M$. The equivalence class of $(m, \sigma)$ is denoted by $d\sigma(m)$. The set of equivalence classes is denoted by $PZ$ and called the *phase bundle* for $Z$. It is, of course, the bundle over $M$ with the projection $d\sigma(m) \mapsto m$. It is obvious that $PZ \to M$ is an affine bundle modelled on the cotangent bundle $T^*M \to M$.

As an example we construct a phase bundle for the bundle of affine values built out of a special vector space. In the set of all sections of the bundle $V$ there is a distinguished set of affine sections, since $V$ and $V_0$ as vector spaces are also affine spaces. We observe that there are affine representatives in every equivalence class $d\sigma(m)$ that differ by a constant function. There is also one linear representative, i.e. such an affine section that takes value 0 at the point $0 \in V_0$. The set of elements of a phase bundle can be therefore identified with a set of pairs: point in $m$ and a linear injection from $V_0$ to $V$. Moreover, we observe that such linear injections are in one-to-one
correspondence with linear functions on $V$ such that they take value 1 on $v$ (or the canonical vector field $X_V$ evaluated on the function gives 1). The image of a linear section is a level-0 set of the corresponding function. The functions that correspond to linear sections form the affine dual $V^\dagger$, therefore we have
\[ \mathcal{P}V \cong V_0 \times V^\dagger. \] (14)

### 4.2 Frame independent lagrangian

Now, we will collect all the homogeneous lagrangians for all inertial frames and construct for them a universal object which does not depend on an inertial frame. It is convenient to treat a lagrangian as a section of the trivial bundle $N \times V \times \mathbb{R} \rightarrow N \times V$ rather than as a function.

For two reference frames $u$ and $u'$, we have the difference
\[ \ell_{h,u}(x,v) - \ell_{h,u'}(x,v) = m\langle g(u' - u), \frac{\partial}{\partial u'}(v) \rangle. \]

Let us denote $\tau_{u'\mapsto u}^* g(u' - u)$ by $\sigma(u',u)$. With this notation
\[ \ell_{h,u}(x,v) - \ell_{h,u'}(x,v) = m\langle \sigma(u',u), v \rangle. \]

For $\sigma$ we have the following equalities
\[ \sigma(u',u) = -\sigma(u,u'), \] (15)
\[ \sigma(u'',u') + \sigma(u',u) = \sigma(u'',u). \] (16)

In the $E_1 \times N \times V \times \mathbb{R}$, we introduce the following relation:
\[ (u,x,v,r) \sim (u',x',v',r') \iff \left\{ \begin{array}{l} x = x', \\ v = v', \\ r = r' + m\langle \sigma(u',u), v \rangle. \end{array} \right. \] (17)

From (15) we obtain that $\sim$ is symmetric and reflexive, from (16) that it is transitive, therefore it is an equivalence relation. Since the relation does not affect $N$ at all, it is obvious that in the set of equivalence classes we have a cartesian product structure $N \times W$. In $W$ we distinguish two elements: $w_0 = [u, 0, 0]$ and $w_1 = [u, 0, -1]$,
\[ w_0 = \{(u,0,0) : u \in E_1\}, \quad w_1 = \{(u,0,-1) : u \in E_1\}, \]
and two natural operations:

\[ + : W \times W \to W \quad \circ : \mathbb{R} \times W \to W \]

\[
[u, v, r] + [u', v', r'] = [\frac{u+u'}{2}, v + v', r + r' + m(\langle \sigma(u, \frac{u+u'}{2}), v \rangle + \langle \sigma(u', \frac{u+u'}{2}), v' \rangle)],
\]

(18)

\[
\alpha \circ [u, v, r] = [u, \alpha v, \alpha r].
\]

The above operations are well defined that can be checked by direct calculation. Some more calculation one needs to show that

**Proposition 1** \((W, +, \circ)\) is a vector space with \(w_0\) as the zero-vector. Moreover \((W, w_1)\) is a special vector space such that \(W/<w_1> \simeq V\)

The canonical projection \(W \to V\) will be denoted by \(\zeta\). It follows from \textbf{4.2} that quadruples \((u, x, v, \ell_{h,u}(x, v))\) and \((u', x, v, \ell_{h,u'}(x, v))\) are equivalent. Consequently, frame dependent lagrangian defines a section \(\ell_h\) over \(N \times V^+\) of the one-dimensional special affine bundle (a bundle of affine values) \(N \times W \to N \times V\) which does not depend on the inertial frame. The section \(\ell_h\) will be called an **affine lagrangian** for the homogeneous mechanics independent on the choice of inertial frame. In the following we show that the bundle \(N \times W \to N \times V\) carries a structure, which can be used for generating the frame-independent dynamics. We begin with the construction of the phase space.

**4.3 Phase space**

In the frame dependent formulation of the dynamics, the phase space for the massive particle is \(T^*N \simeq N \times V^*\). For each frame \(u\) we have the Legendre mapping

\[
\mathcal{L}_u : TN \supset N \times V^+ \to T^*N
\]

\[
(x, v) \mapsto \frac{m}{2\langle \tau, v \rangle} g \circ \iota_u(v) - \frac{m}{2\langle \tau, v \rangle^2} (g(\iota_u(v)), \iota_u(v)) \tau - \varphi(x) \tau,
\]

(19)

i.e. the vertical derivative of \(\ell_{h,u}\) with respect to the projection \(T^*N \to N\).

Since \(\ell_{h,u}(x, v) - \ell_{h,u'}(x, v) = m(\sigma(u', u), v)\), we have also

\[
\mathcal{L}_u(v) - \mathcal{L}_{u'}(v) = m\sigma(u', u).
\]

(20)
Proposition 2  A mapping $\Phi_{u',u}: T^*N \rightarrow T^*N$ defined by

$$\Phi_{u',u}(x, p) = (x, p + m\sigma(u', u))$$

has the following properties

1. $\Phi_{u',u}(K_{m,u'}) = K_{m,u}$,

2. it is a symplectomorphism of the canonical symplectic structure on $T^*N$,

3. $T\Phi_{u',u}(D_{h,u'}) = D_{h,u}$.

Proof. The image of $L_u$ is $K_{m,u}$, so the first property is an immediate consequence of (20) and the definition of $\Phi_{u',u}$. The mapping $\Phi_{u',u}$ is a translation by a constant vector. It follows that it is a symplectomorphism. Consequently,

$$T\Phi_{u',u}((TK_{m,u'})^\sharp) = (TK_{m,u})^\sharp$$

and

$$T\Phi_{u',u}(\bar{D}_{h,u'}) = \bar{D}_{h,u}.$$  

Since $\Phi_{u',u}$ respects the time orientation, we have also

$$T\Phi_{u',u}(D_{h,u'}) = D_{h,u}.$$  

The above observation suggests the following equivalence relation in $E_1 \times N \times V^*$:

$$(u, x, p) \sim (u', x', p') \iff \begin{cases} x = x' , \\
p = p' + m\sigma(u', u) . \end{cases} \quad (21)$$

Again, we have the obvious structure of the cartesian product in the set of equivalence classes: $N \times P$. The set $N \times P$ will be called an affine phase space. The set $P$ is an affine space modelled on $V^*$:

$$[u, p] + \pi = [u, p + \pi] \text{ for } \pi \in V^*.$$  

An element of $P$ will be denoted by $p$.  

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It follows from Proposition 4.3 that \( N \times P \) is a symplectic manifold and the isomorphism of tangent and cotangent bundles assumes the form
\[
\beta : T(N \times P) \simeq N \times P \times V \times V^* \longrightarrow T^*(N \times P) \simeq N \times P \times V^* \times V
\]
\[
\beta : (x, p, v, a) \mapsto (x, p, a, -v)
\]
\[\text{(22)}\]
Moreover, the equivalence classes of the elements of mass-shells form the universal mass shell \( K_m \) and the elements of frame dependent dynamics form the universal dynamics \( D_h \) which is contained in \( (T K_m)^\beta \).

A straightforward calculation shows that the function
\[
E_1 \times N \times V^* \ni (u, x, p) \mapsto \frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle
\]
is constant on equivalence classes and projects to a function on \( N \times P \). We denote this function by \( \Psi_m \). It follows that the generating object \( \Psi_m \) of the dynamics \( D_{h,u} \) defines a generating object
\[
\begin{array}{c}
N \times P \times \mathbb{R}_+ \xrightarrow{-\tilde{H}_h} \mathbb{R}, \\
\zeta'' \downarrow \\
N \times V^*
\end{array}
\]
\[\text{(23)}\]
of the dynamics \( D_h \), where
\[
\tilde{H}_h(x, p, r) = r(\Psi_m + \varphi(x)).
\]
\[\text{(24)}\]

### 4.4 Lagrangian as a generating object

In the previous section we have constructed the frame independent dynamics \( D_h \) and a hamiltonian generating object. Now, we show that the frame independent affine lagrangian \( \ell_h \) is also a generating object of \( D_h \). \( \ell_h \) is a section of a bundle of affine values \( N \times W \rightarrow N \times V \) and its differential is a section of \( \mathcal{P}(N \times W) \rightarrow N \times V^+ \). For a given frame \( u \), we identify \( N \times W \) with \( N \times V \times \mathbb{R} \) and a section of \( \zeta \) with a function on \( N \times V \). Consequently, an affine covector \( a \in \mathcal{P}(N \times W) \) is represented by a covector \( a_u \in T^*(N \times V) = N \times V \times V^* \times V^* \). It follows from \( \text{(12)} \) that \( a_u = (x, v, a, b) \) and \( a_u' = (x, v, a, b + m\sigma(u, u')) \) represent the same element of \( \mathcal{P}(N \times W) \). In the process of generation of frame dependent dynamics we use the canonical isomorphism \( \alpha_N : TT^*N \rightarrow T^*TN \) \[\text{(3.1)}\]. We observe that
\[
\alpha_N(x, a + m\sigma(u, u'), v, b) = (x, v, b, a + m\sigma(u, u')), 
\]
hence \(\alpha_N\) defines an isomorphism
\[
\alpha : T(N \times P) \to P(N \times W)
\]
and the image of \((D_h)\) is the image of \(dl_h\).

Now, we can summarize our constructions. We have canonical symplectic structure on \(N \times P\) with the corresponding mapping
\[
\beta : T(N \times P) \longrightarrow T^*(N \times P),
\]
which forms the basis for the hamiltonian formulation of the dynamics. Together with \(\alpha\) it gives rise to the following diagram (Tulczyjew triple):

\[
\begin{array}{ccc}
(T^*(N \times P), \omega_{N \times P}) & \xleftarrow{\beta} & (T(N \times P), d_T \omega_P) & \xrightarrow{\alpha} & (P(N \times W), \omega_{N \times W}) \\
N \times P & \searrow & N \times V
\end{array}
\]

### 4.5 The Legendre transformation

The Legendre transformation is the passage from lagrangian to hamiltonian generating object. In previous sections it was done with the knowledge of the Legendre transformation for the frame dependent dynamics. In that case we make use of the canonical symplectomorphism \(\gamma_M : T^*T^*M \to T^*TM\) generated by \(\langle \,, \rangle : TM \times_M T^*M \to \mathbb{R}\), where \(M\) is a manifold and \(\langle \,, \rangle\) is the canonical pairing between vectors and covectors. It follows that the image \(\gamma(L)\) of a lagrangian submanifold \(L\) generated by a lagrangian \(\ell\) is generated by a Morse family
\[
\ell - \langle \,, \rangle : TM \times_M T^*M \to \mathbb{R},
\]
where \(TM \times_M T^*M\) is considered a fibration over \(T^*M\) (see [2] for details).

Now, we show that analogous procedure can be applied in the case of the affine framework. First, we observe that every element \(w \in W\) defines, in natural way, an affine function on \(P\):
\[
f_w(p) = \langle p, v \rangle - r, \quad \text{where} \quad w = [u, v, r], \ p = [u, p]. \tag{25}
\]
Indeed, when we take another representative of \(w\) and \(p\), e.g. \((u', v, r')\) and \((u', p')\) respectively, then we obtain
\[
\langle p', v \rangle - r' = \langle p - m\sigma(u', u), v \rangle - r + m\langle \sigma(u', u), v \rangle = \langle p, v \rangle - r.
\]
The element \( w_1 \) defines the constant function equal to 1 on \( P \):

\[
f_{w_1}(p) = \langle p, 0 \rangle - (-1) = 1.
\]

This implies the following:

**Proposition 3** There is a natural isomorphism between \( P^\dagger \) and \( (W, w_1) \) given by

\[
f_{[u,v,r]}([u,p]) = \langle p, v \rangle - r.
\]

It means that also \( W^\dagger \simeq P \).

With this isomorphism we have (see (14)) \( P(N \times W) \simeq N \times V \times V^* \times P \) and \( \alpha: T(N \times P) \rightarrow P(N \times W) \) assumes the form

\[
\alpha: (x, p, v, a) \mapsto (x, v, a, p).
\]

\((W, w_1)\), being a special vector space, has a structure of a one-dimensional affine bundle modelled on \( V \times I \). The action of the group \((\mathbb{R}, +)\) in the fiber over \( V \) comes from the natural action in the fiber of \( E_1 \times V \times \mathbb{R} \rightarrow E_1 \times V \). The fundamental vector field \( X_W \) for this action is a constant vector field with value \( w_1 \) at every point.

Now, we need a pairing between \( P \) and \( V \), which reduces to \( \langle \cdot, \cdot \rangle \) (as a section of the trivial bundle \( V \times V^* \times \mathbb{R} \)) in the vector case. The pairing is a section of \( P \times W \) over \( P \times V \) defined by

\[
P \times V \ni (p, v) \mapsto \langle p, v \rangle = [u, v, \langle p, v \rangle] \in W, \quad \text{where } p = [u, p].
\]

The above definition is correct, i.e. does not depend on the choice of representatives:

\[
[u, v, \langle p, v \rangle] = [u', v, \langle p, v \rangle - \langle m\sigma(u'), u \rangle, v] = [u', v, \langle p', v \rangle].
\]

It remains to show that the pairing \((28)\) generates an isomorphism between \( T(N \times P) \) and \( P(V \times W) \).

**Proposition 4** There is a natural symplectomorphism between \( P((N \times W) \times (N \times P)) \) and \( P(N \times W) \ominus T^*(N \times P) \).
Proof. It is enough to check that any section of \((N \times W) \times (N \times P)\) over \((N \times V) \times (N \times P)\) is equivalent to a section \(\sigma\) of the form

\[ \sigma(x, v, y, p) = \sigma_0(v) + f_1(x) - f_2(p) - f_3(y), \]

where \(\sigma_0\) is a linear section of \(W \to V\) and functions \(f_i\) are affine.

Similar arguments show that \(P((N \times W) \times (N \times P)) \cong N \times N \times V\times V^\ast \times P\).

The canonical diagonal inclusion \(N \subset N \times N\) implies the projection

\[ V^* \times V^* \to V^*: (a, b) \mapsto a + b \]

and consequently a relation between \(P((N \times W) \times (N \times P))\) and \(P((N \times W) \times (N \times P))\).

With this relation a section of \(P(N \times W \times P)\) over \(N \times V \times P\) defines a submanifold of

\[ P((N \times W) \times (N \times P)) \supset P(N \times W) \ominus T^*(N \times P) \]

i.e., a symplectic relation

\[ T^*(N \times P) \to P(N \times W). \]

In particular, the differential of the pairing \(\langle , \rangle\) generates a relation

\[ \gamma: T^*(N \times P) \cong N \times P \times V^* \times V \to P(N \times W) \cong N \times V \times V^* \times P \] (29)

It easy task to verify that this relation has the following representation

\[ \gamma: (x, v, p, a, v) \mapsto (x, -v, a, p) \in N \times V \times V^* \times P. \]

We see from (26) and (22) that \(\gamma = \alpha \circ \beta^{-1}\), and consequently \(\gamma \circ \alpha(D_h) = \beta(D_h)\). Following the general rule for composing of generating objects, we conclude that \(\beta(D_h)\) is generated by the Morse family

\[ N \times P \times V^+ \overset{-H_h}{\longrightarrow} \mathbb{R}, \]

where

\[ H_h(x, v, p) = \langle p, v \rangle - \ell_h(x, v). \] (31)

As in the frame-dependent case, this family can be reduced to the family.
4.6 Comments

Frame independent inhomogeneous formulation of the dynamics can be obtained by the reduction of the homogeneous lagrangian with respect to the embedding $N \times V^1 \hookrightarrow N \times V$. The Legendre transformation leads to a hamiltonian which is no longer a function, but a section of certain bundle over $N \times P$. It requires also extension of the notion of a Poisson bracket to sections of an affine values bundle (see [1], [11]).

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