Generalized Inheritance

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(Dated: August 18, 2015)

Generalized inheritance is used with the almost-conformal Killing equation. Examples are the flat FRW, Kasner, and deSitter metrics. The volume changes in FRW while transitioning to a stiff fluid are discussed. An inheritance current is implicit in the generalized condition.

Keywords: inheritance, ACKV, FRW, deSitter

I. INTRODUCTION

Solutions and models in general relativity are increasingly being generalized to include more physically realistic matter content and symmetry descriptions. The symmetries associated with Killing vectors and conformal Killing vectors are defined through the Lie derivative of the metric

$$\mathcal{L}_\xi g_{ab} = \xi_a;_b + \xi_b;_a = \kappa (\nabla \cdot \xi) g_{ab}$$

(1)

with zero divergence defining the Killing vector (KV) and $$\kappa = 1/2$$ the conformal Killing vector (CKV). Other types of generators are classified by their divergence behavior:

$$(\nabla \cdot \xi)_a = 0, \homothetic \text{ vector} \quad (2a)$$

$$(\nabla \cdot \xi)_{ab} = 0, (\nabla \cdot \xi)_a \neq 0, \special \text{ conformal vector} \quad (2b)$$

$$(\nabla \cdot \xi)_{ab} \neq 0, \proper \text{ conformal vector} \quad (2c)$$

The metric symmetries do not automatically extend to other curvature functions or to matter sources in the spacetime. For a fluid with unit flow vector $$\hat{U}^a$$ and a CKV satisfying Eq.(1), Maartens et al. established the kinematic result

$$\mathcal{L}_\xi \hat{U}^a = -\frac{\kappa}{2}(\nabla \cdot \xi)\hat{U}^a + V^a$$

(3)
where \( V^a \dot{U}_a = 0 \) and \( \dot{U}_a \dot{U}^a = -1 \). It is a kinematic constraint which compares velocity parameters in generalized matter content across general relativity solution sets \([2, 16, 17]\). Coley and Tupper \([7, 8, 18]\) used the \( V^a = 0 \) case to define *inheritance* as the condition which conformally maps flow lines onto flow lines, i.e.

\[
\mathcal{L}_\xi \dot{U}^a = -\frac{\kappa}{2} (\nabla \cdot \xi) \dot{U}^a
\]  

(4)

Several generalizations of the conformal Killing equation have been suggested. Matzner \([19]\) suggested

\[
\nabla_b [\xi^{(a;b)}] \sim \text{const} \cdot \xi^a.
\]  

(5)

Later work \([20, 21]\) considered generalizations of the \( \kappa = 0 \) Yano-Bochner equation \([22, 23]\), defining an almost CKV (ACKV).

\[
\nabla_a [\tilde{\xi}^{a;b} + \tilde{\xi}^{b:a} - \kappa (\nabla \cdot \tilde{\xi}) g^{a b}] = 0
\]  

(6)

with \( \kappa \) generalizing the CKV divergence contribution (\( \xi^a \) is a CKV, \( \tilde{\xi}^a \) is an ACKV).

Using the Riemann tensor to write the double derivatives, the ACKV equation can also be written as

\[
\nabla_b \nabla^b \tilde{\xi}^a + R^a_b \tilde{\xi}^b + (1 - \kappa) \nabla^a (\nabla \cdot \tilde{\xi}) = 0
\]  

(7)

The ”almost” idea has been used in several contexts. A common use is to describe an ”almost symmetry” as one that asymptotically becomes a metric symmetry \([20]\). Another is to consider symmetry generators on surfaces that are distorted from spherical symmetry \([24, 25]\), or generators in perturbed manifolds with an exact symmetry \([26, 27]\). Just as the conformal Killing equation was generalized with a divergence, the *inheritance* condition can be extended with a divergence.

Here we suggest a generalization of the *inheritance* condition by taking a divergence of the mapping equation for the four-velocity

\[
\nabla_a [\mathcal{L}_\xi \dot{U}^a + \frac{\kappa}{2} \dot{U}^a (\nabla \cdot \tilde{\xi})] = 0.
\]  

(8)

The Lie derivative has been scaled to match the scaling in the ACKV equation, Eq.\((6)\). *Inheritance* is implicit in the conformal Killing equation for proper, timelike CKV’s \([18]\). Coupling ”almost-inheritance” with ”almost-symmetry”, examines the role of inheritance for vectors obeying the almost-conformal-Killing equation and allows for scalings beyond
the conformal $\kappa = 1/2$. We discuss the effects which follow from a generalized inheritance condition for a timelike vector, $\xi^a = \chi \hat{U}^a$. For this case, one difference will be in the value of the scaling parameter $\kappa$. Another is the use of $\kappa$ as a modeling parameter for homothetic vectors. For $\nabla \cdot \xi = \text{const}$, $\kappa$ does not appear in the ACKV equation, Eq.(6), but does remain in the almost-inheritance condition. This allows some tracking of symmetry transitions. In the next section we consider the ACKV parameters. The ACKV equation and almost inheritance are discussed in the third part of the paper with some metric examples.

II. ALMOST INHERITANCE

Conformal Parameters

A timelike CKV is $\xi^a = \chi \hat{U}^a$ with parameters $\xi^a \xi_a = -\chi^2$, $\kappa = 1/2$, and velocity parameters of the CKV observer. One of the differences between a KV, a CKV, and an ACKV is the scaling along the flow of the vector trajectory. For Killing and conformal Killing vectors, Eq.(1) projected by $\hat{U}^a$ and $\hat{U}^b$ relates the divergence to the change in the norm

$$\kappa(\nabla \cdot \xi) = 2 \dot{\chi}$$  \hspace{1cm} (9)

KV's with zero divergence have constant norm along their trajectory. The CKV norm change is related to the divergence with $\kappa = 1/2$. For KVs and CKVs, the expansion of the observer velocity is also related to $\dot{\chi}$. Using Eq.(9), the expansion $\Theta := \nabla_a \hat{U}^a$ is

$$\Theta = 3(\dot{\chi}/\chi).$$  \hspace{1cm} (10)

A direct velocity projection is not possible with the ACKV equation and there is no simple relation between the ACKV observer expansion and derivatives of $\chi$. Next we consider possible solutions to the almost inheritance condition and possible relations between $\Theta$, $\chi$, and its derivatives for ACKV $\bar{\xi}^a$.

General Parameters

For a timelike ACKV, $\bar{\xi}^a = \chi \hat{U}^a$, the generalized inheritance condition is

$$\nabla_a \left[ \hat{U}^a \left( \frac{\kappa - 2}{2} \dot{\chi} + \frac{\kappa}{2} \Theta \right) \right] = 0$$  \hspace{1cm} (11)
or, in expanded form.
\[
\left(\frac{\kappa-2}{2}\right)\ddot{\chi} + \Theta(\kappa-1)\dot{\chi} + \frac{\kappa}{2}(\dot{\Theta} + \Theta^2)\chi = 0
\]  
(12)
with \(\dot{\chi} = \chi_\alpha \tilde{U}^\alpha\). There are several solutions. An obvious one, which follows from Eq.(11) by inspection, is
\[
\Theta = \left(\frac{2-\kappa}{\kappa}\right)\dot{\chi}
\]
a generalization of the conformal Eq.(10). A further generalization of this choice is
\[
\Theta = \Theta_1\left(\frac{\dot{\chi}}{\chi}\right)
\]  
(13)
With these choices for \(\Theta\), the generalized inheritance condition can be written as a product.
\[
(\dot{\chi} + \Theta_1\frac{\dot{\chi}^2}{\chi}) \left[\left(\frac{\kappa-2}{2}\right) + \frac{\kappa}{2}\Theta_1\right] = 0
\]
There are two solutions.
\[
(1) \quad \Theta_1 = \frac{2-\kappa}{\kappa}
\]  
(14)
\[
(2) \quad \Theta_1 = -\left(\frac{\dot{\chi}}{\chi}\right)\left(\frac{\dot{\chi}}{\chi}\right)
\]  
(15)
with the second solution implying the relation \(\Theta = -\dot{\chi}/\chi\). When the ACKV has constant divergence, the modeling relation
\[
\nabla \cdot \tilde{\xi} = \dot{\chi} + \chi \Theta = \dot{\chi}(1 + \Theta_1) = \delta
\]  
(16)
implies \(\dot{\chi} = \text{const}\) and almost inheritance eliminates solution (2) used by itself. When both solutions are simultaneously true, the almost inheritance condition implies no expansion.
\[
(\frac{\dot{\chi}^2}{\chi^2})\Theta_1 = 0
\]  
(17)
There are other solution limits when used with the complete general inheritance condition. For example, the second solution implies
\[
\frac{\kappa}{2}\left(\frac{-\dot{\chi}}{\chi} + \dot{\Theta} + \Theta^2\right) = 0
\]
For metrics with \(\dot{\Theta} + \Theta^2 = 0\), either \(\kappa = 0\) or \(\dot{\chi} = 0\), with \(\dot{\chi} = \text{const}\). For these metrics
\[
\Theta(\kappa-1)\dot{\chi} + \left(\frac{\kappa-2}{2}\right)\dot{\chi} = 0
\]  
(18)
Observers with a non-zero expansion and constant \(\dot{\chi}\) would have \(\kappa = 1\). There are several interesting solutions that satisfy this condition and will be discussed in the next section.
III. ACKV EXAMPLES

The examples we consider are flat FRW, Kasner, and deSitter metrics. For geodesic and irrotational metrics the ACKV equation, Eq. (7), is

$$\chi_{ba} \dot{U}^a + \chi^b \Theta - \dot{U}_b \nabla_a \chi^a - \chi^a \dot{U}_{ba} - (\kappa - 2) \nabla_b (\nabla \cdot \xi) = -2R_{ba} \xi^a$$  \hspace{1cm} (19)

Projecting with $\dot{U}^b$ and spatial vector $S^b$, the equations are

$$\chi_{ba} \dot{U}^a \dot{U}^b + \chi \Theta + \nabla_a \chi^a - (\kappa - 2) \dot{U}^b \nabla_b (\nabla \cdot \xi) = -2\chi R_{ba} \dot{U}^a \dot{U}^b$$  \hspace{1cm} (20)

$$\chi_{ba} \dot{U}^a S^b + \chi^b \Theta S^b - \chi^a S^b \dot{U}_{ba} - (\kappa - 2) S^b \nabla_b (\nabla \cdot \xi) = -2\chi R_{ba} S^b \dot{U}^a$$  \hspace{1cm} (21)

FRW

Flat FRW is a simple example of solution (2) metric.

$$ds^2 = -dt^2 + A^2(t)(dx^2 + dy^2 + dz^2)$$  \hspace{1cm} (22)

with $\dot{U}^a = [1, 0, 0, 0]$ and $\Theta = 3\dot{A}/A$. The timelike ACKV equation is

$$\ddot{\chi} + \dot{\chi} \Theta + \nabla_a \chi^a - (\kappa - 2) \dot{U}^b \nabla_b (\nabla \cdot \xi) = -2\chi R_{ba} \dot{U}^a \dot{U}^b$$  \hspace{1cm} (23)

$$\frac{(\kappa - 2)}{\chi} \dot{U}^t \nabla_t [\chi, t + 3 \frac{\dot{A}}{A}] = -6 \frac{\ddot{A}}{A}$$  \hspace{1cm} (24)

Let $A = A_0 t^n$, with for example, $n = 2/3$ FRW dust and $n = 1/2$. This is the Tolman radiation solution \[28\]. The second almost inheritance solution, Eq. (12), for the comoving expansion and $\chi = \chi(t)$ is

$$\chi = [(\Theta_1 + 1)(c_1 t + c_2)]^{\frac{1}{\Theta_1 + 1}}$$  \hspace{1cm} (25)

$$\Theta = \frac{\Theta_1}{1 + \Theta_1} \left( \frac{c_1}{c_1 t + c_2} \right)$$  \hspace{1cm} (26)

where $(c_1, c_2)$ are assumed constant. Taking $c_2 = 0$, and comparing to the FRW expansion $\Theta = 3(\dot{A}/A)$, $\Theta_1$ is determined from the known solutions.

$$\Theta_1 = \frac{3n}{1 - 3n}$$  \hspace{1cm} (27)

Using these the ACKV equation relates $\kappa$, $\Theta_1$ and the FRW power, $n$.

$$-(\kappa - 2) \left( \frac{1}{\Theta_1 + 1} - 1 \right) \left( \frac{1}{\Theta_1 + 1} + 3n \right) = 6n(n - 1)$$  \hspace{1cm} (28)

$$\kappa = 2n$$
The parameter set is determined in general by the power law behavior of the metric, \( \Theta_1 = 3n/(1-3n) \), \( \kappa = 2n \). For dust, the values are \( \Theta_1 = -2, \kappa = 4/3 \) and for radiation \( \Theta_1 = -3, \kappa = 1 \). The spatial ACKV equation is identically zero. The stress-energy components for this FRW metric are

\[
8\pi\rho = 3n^2/t^2 \tag{29a}
\]
\[
8\pi P = n(2 - 3n)/t^2 \tag{29b}
\]

\( 1/3 \leq n \leq 2/3 \) is required for physical stress and acoustic speed \( \leq 1 \). The stress-energy is well behaved for the stiff fluid value, \( n = 1/3 \), but \( \Theta_1 \) is not. The stiff fluid with \( n = 1/3 \) has expansion \( \Theta = 1/t \) and satisfies the relation \( \dot{\Theta} + \Theta^2 = 0 \). This is the special solution discussed in Section 2 where the ACKV has a constant divergence together with the stiff fluid case \( \Theta_1 = 1, \kappa = 1 \). The discontinuity in \( \Theta_1 \) is related to the behavior of the FRW spatial volume, \( V \).

\[
dV = t^{3n}dV_3
\]

with \( dV_3 = dx dy dz \). The way the volume changes along the FRW flow is

\[
\dot{V} = 3nt^{3n-1}dV_3
\]

and for the stiff perfect fluid, \( \dot{V} = \text{const} \). The transition from a non-stiff to a stiff equation of state reflects a change in the volume structure of the fluid. This is like the transition from a body centered cubic structure to a face centered cubic structure in a heated iron wire. Here the symmetry change is the appearance of the homothetic ACKV [29, 30] for the stiff fluid and the transition from solution (2) to solution (1) with the associated discontinuities in the parameter sets.

**Kasner**

Kasner vacuum, with four-velocity \( \bar{U}^a = [1, 0, 0, 0] \), is an example of a solution (1) metric.

\[
ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2
\]

with \( p_1 + p_2 + p_3 = 1 \) and \( p_1^2 + p_2^2 + p_3^2 = 1 \). The metric is geodesic and irrotational with expansion \( \Theta = 1/t \) such that \( \dot{\Theta} + \Theta^2 = 0 \). The timelike ACKV equation is

\[
\chi_{\mu,\mu} + \chi_{\mu}/t - \chi_{\mu} - \chi_{t}/t = 0 \tag{30a}
\]
\[
(k - 2)\bar{U}^b\nabla_b(\nabla \cdot \bar{\xi}) = 0 \tag{30b}
\]
and requires that either $\bar{\xi}^a$ has a constant divergence or $\kappa = 2$ with the divergence not determined. If the divergence is not constant, the first almost inheritance condition enters and, with $\kappa = 2$, requires zero expansion. The constant divergence determines $\chi$

$$\dot{\chi} + \frac{\chi}{t} = \delta \quad \text{or} \quad \frac{(t\chi)_t}{t} = \delta,$$

thus

$$\chi = \frac{\delta t}{2} + \frac{c_1}{t}$$

The almost inheritance condition determines $\kappa$

$$\frac{2c_1}{t^3} + \left(\frac{\delta}{2} - \frac{c_1}{t^2} - \frac{\kappa}{2}\right)\frac{1}{t} = 0, \quad c_1 = 0, \quad \kappa = 1$$

The spatial ACKV equation is an identity. Kasner has a set of homothetic vectors with the ACKV form part of the known homothetic CKV. The relation between the expansion and $\chi$ is $\Theta = \dot{\chi}/\chi$ with $\Theta_1 = 1$. The Kasner parameter set is $\Theta_1 = 1, \kappa = 1$. There is no phase behavior in Kasner since, even with anisotropic volume expansion, the volume has the same form over the parameter range $dV = t^{p_1 + p_2 + p_3}dV_3 = tdV_3$.

### deSitter

The third example is the deSitter form of the FRW metric with $A(t) = A_0 e^{\alpha t}$. The expansion is constant, $\Theta = 3\alpha$ and $\chi$ follows as

$$\chi = \chi_0 \exp(3\alpha t/\Theta_1).$$

The almost inheritance condition is

$$\left(\frac{\kappa - 2}{2}\right) \frac{9\alpha^2}{\Theta_1^2} + 3\alpha(\kappa - 1)\frac{3\alpha}{\Theta_1} + \frac{\kappa}{2}(9\alpha^2) = 0$$

$$\kappa - 2 + 2(\kappa - 1)\Theta_1 + \kappa \Theta_1^2 = 0$$

with solutions $\Theta_1 = (2 - \kappa)/\kappa$ and $\Theta_1 = -1$. The ACKV equation is

$$\frac{(2 - \kappa)(\frac{3\alpha}{\Theta_1} + 3\alpha)}{\Theta_1} \frac{3\alpha}{\Theta_1} = 6\alpha^2$$

$$\frac{(2 - \kappa)(\Theta_1 + 1)}{3} \frac{\Theta_1^2}{\Theta_1}$$

The deSitter parameters are the original conformal set $\kappa = 1/2, \Theta_1 = 3$. 
IV. INHERITANCE CURRENT

Currents can be associated with Killing structures. Ruiz, Palenzuela and Bona [32] used a Killing vector with a stress-energy to define a conserved current

\[ J^a_{RPB} = T^{ab} \xi_b \] (35)

The generalized inheritance condition, Eq. (8), was constructed from the inheritance condition, Eq. (4). The zero divergence can also be used to define a conserved inheritance current

\[ J^a = \mathcal{L}_\xi U^a + \frac{\kappa}{2} U^a (\nabla \cdot \xi) \] (36)

For the timelike ACKV considered here, the inheritance current is

\[ J^a = \left[ \left( \frac{\kappa - 2 + \kappa \Theta_1}{2} \right) \dot{\chi} \right] \dot{U}^a = \left[ \Theta \left( \frac{\kappa - 2 + \kappa \Theta_1}{2 \Theta_1} \right) \chi \right] \dot{U}^a \] (37)

and, with a dependence on expansion, describes the rate of change of a geodesic cross section [33]. The current is zero for solution (1). For non-stiff FRW, using the relation between parameters, \( \kappa = 2 \Theta_1 / [3(1 + \Theta_1)] \), the inheritance current reflects the CKV parameter structure and is also zero for \( \Theta_1 = \Theta \).

\[ J^a = \left[ \left( \frac{\Theta_1 - 3}{3} \right) \dot{\chi} \right] \dot{U}^a \] (38)

The FRW solution has volume transition behavior in moving to the stiff fluid. The FRW inheritance current is non-zero for the non-stiff fluid with its changing cross section.

\[ J^a = \left[ \dot{\chi} \frac{4n - 1}{1 - 3n} \right] \dot{U}^a, \quad n > 1/3. \] (39)

and reflects the same volume discontinuity at \( n = 1/3 \).

Consider a small region of the FRW space interior to a hypersurface, \( S^a \) with normal \( \dot{U}^a \). For zero divergence, Gauss’ theorem provides the integral of the current over the hypersurface boundary

\[ \oint J^a dS_a = 0 \] (40)

This implies that the integral is hypersurface independent. Substituting we have

\[ \oint \frac{1 - 4n}{1 - 3n} \dot{\chi} t^{3n} dV_3 = 0 \] (41)

Using \( \Theta = \Theta_1 (\dot{\chi}/\chi) \) and the solutions, Eqs. (25,26), the integral can be written as

\[ \oint \left( 1 - 4n \right) \left( 1 - 3n \right)^{3n - 1} \frac{1}{t^{3n}} c_1^{1/1 + 1} t^{3n} dV_3 = 0 \] (42)

The current can be interpreted as maintaining the size of the base coordinate 3-volume.
V. DISCUSSION

We have suggested an almost inheritance condition to be used with the ACKV equation. It is a two parameter, \([\theta_1, \kappa]\), model generalizing the conformal relation between the observer expansion and the ACKV norm with parameters \([\theta = 3, \kappa = 1/2]\). With this generalization, two solutions for generalized inheritance were presented and FRW, Kasner and de Sitter were used as examples. The three metrics were chosen because they span the class of the almost-inheritance solutions and because they all appear frequently in current models. Almost inheritance, applied to these spacetimes, can extend the range of the many current applications. The FRW metric is frequently used in general relativity extensions. The extended parameter set allowed by the generalized inheritance condition extends the range of perfect fluid spaces, linking the equation of state to the almost conformal parameter. A new behavior is the possible phase transition in transfer to a stiff fluid. The Kasner metric is used in models of the very early universe and is useful in considering anisotropic volume expansion. As a background metric it is used to explore solutions of differential equations and in developing extended classifications. The deSitter universe, with accelerated expansion, is an important model of both our early and our asymptotic universe. Like the Kasner metric, it is used as a background space to explore equation solutions and to support deSitter process descriptions.

By themselves the solutions presented here are useful in discussing divergence behaviors for the ACKV. In the almost-inheritance model, Kasner has a constant divergence ACKV with the solution (1) parameter set \([1, 1]\). The power law FRW example, \(A(t) = A_0 t^n\), used both solutions, with the non-stiff perfect fluids having parameter set \([3n/(1-3n), 2n]\). The stiff fluid solution (1) has set \([1, 1]\). deSitter has the original conformal parameters.

The four-velocities in the examples used were all simple comoving identities. Outgoing, marginally bound geodesic Schwarzschild with four-velocity

\[
\dot{U}^a = \left[(1-2m_0/r)^{-1}, \sqrt{2m_0/r}, 0, 0\right]
\]

has expansion \(\Theta = (3/2)\sqrt{2m_0/r^3}\) and also obeys the condition \(\dot{\Theta} + \Theta^2 = 0\). Studying the relation between ACKVs and generalized inheritance for velocities with more structure...
would be of interest.

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