POINT TO POINT TRAVELING WAVE AND PERIODIC TRAVELING WAVE INDUCED BY HOPF BIFURCATION FOR A DIFFUSIVE PREDATOR-PREY SYSTEM

HONGYONG ZHAO*
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China

DAIYONG WU
1Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China
2Department of Mathematics
Anqing Normal University
Anqing 246133, China

ABSTRACT. In this paper, we consider a diffusive Leslie-Gower predator-prey system with prey subject to Allee effect. First, taking into account the diffusion of both species, we obtain the existence of traveling wave solution connecting predator-free constant steady state and coexistence steady state by using the upper and lower solutions method. However, due to the singularity in the predator equation, we need construct a positive suitable lower solution for the prey density. Such a traveling wave solution can model the spatial-temporal process where the predator invades the territory of the prey and they eventually coexist. Second, taking into account two cases: the diffusion of both species and the diffusion of prey-only, we prove the existence of small amplitude periodic traveling wave train solutions by using the Hopf bifurcation theory. Such traveling wave solutions show that the predator invasion leads to the periodic population densities in the coexistence domain.

1. Introduction. The predator-prey models played a very important role in the theoretical studies of invasive species, resource management and environment protection. The Leslie-Gower [2, 15] type models are that the density of predators follows a logistic dynamics with a varying carrying capacity proportional to the density of prey. The importance of the Leslie-Gower model is highlighted by Collings [5], who argued that ratio-dependent Leslie model provides a way to avoid the biological control paradox wherein classical prey-dependent exploitation models generally do not allow for a prey equilibrium density that is both low and stable.

2010 Mathematics Subject Classification. Primary: 35K57, 35C07; Secondary: 34C23.
Key words and phrases. Traveling wave solution, predator-prey, Allee effect, Hopf bifurcation.

The work is partially supported by the National Natural Science Foundation of China (Nos 11571170, 31570417); the Natural Science Foundation of Anhui Province of China (No 1608085 MA14); the Key Project of Natural Science Research of Anhui Higher Education Institutions of China (No KJ2018A0365).

* Corresponding author: Hongyong Zhao.
As it was said in [3], spatial features are now solidly established as essential considerations in ecology both in terms of theory and practice, and the mathematical challenges in advancing understanding of the role of space in ecology are substantial and mathematically seductive. Taking into account the spatial diffusion, system (2) in [21] can become as follows (also see system (1.1) in [22]):

\[
\begin{cases}
    u_t = d_1 \Delta u + u(1-u) \left( \frac{u}{b} - 1 \right) - \beta uv, & x \in \mathbb{R}, t > 0, \\
    v_t = d_2 \Delta v + \mu v \left( 1 - \frac{v}{u} \right), & x \in \mathbb{R}, t > 0, \\
    u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) \geq 0, & x \in \mathbb{R},
\end{cases}
\]

(1)

where 0 < b < 1 represents Allee effect threshold. \( \beta \) and \( \mu \) are positive constants. \( d_1 \) and \( d_2 \) denote the diffusion coefficients corresponding to the prey \( u \) and the predator \( v \) with \( d_1 > 0 \) and \( d_2 \geq 0 \). In system (1), the habitat is the whole real line, the predator follows a logistic dynamics with a varying carrying capacity proportional to the density of the prey, and the prey is subject to Allee effect.

In recent years, traveling wave solutions of reaction-diffusion equations have attracted increasing interest (see e.g., [6, 7, 28, 18, 27, 9, 10, 8]). It is well known that traveling wave is one kind of special solutions to the evolutionary systems, which has a fixed shape and translates at a constant speed \( c \) as time evolves. In addition, the minimum wave speed has a possible link to the population spreading speed. In fact, it is conjectured that the minimum wave speed is identical to the population spreading speed, which is proved to be true for some ecological models (see e.g., [16, 17, 11]). For the existence of the traveling wave solutions, a standard approach, such as the monotone iteration or comparison argument, has been established that is very efficient to deal with the traveling wave solutions for the monotone systems. However, for predator-prey systems, it is very difficult to prove the existence of traveling wave solutions since predator-prey systems do not generate monotone semiflows. Recently, the analysis of the phase plane can be useful for the existence of traveling wave solutions for predator-prey systems, where original system becomes a four-dimensional ODE system, but the geometric structure in \( \mathbb{R}^4 \) will be very complex (see e.g., [10, 11]). The application of Schauder’s fixed point theorem with the help of upper and lower solutions has also been proved to be quite successful. Although this method is very standard, the construction of suitable upper and lower solutions can be very challenge.

For system (1), Ni and Wang [22] analyzed the nonnegative constant steady state solutions and their stabilities, and investigated the stationary patterns induced by diffusions. To the best of our knowledge, few works have been done for the existence of traveling wave for system (1).

In this paper, we mainly establish the existence and nonexistence of traveling waves connecting predator-free steady state and coexistence steady state by constructing suitable upper and lower solutions. This kind of traveling wave can be called as wave of invasion (see [25]), which is of ecological interest since it corresponds to a situation where an environment is initially inhabited only by the prey species at its carrying capacity, and a small invasion of the predator drives the system to a stable coexistence steady state displacing an unstable predator-free steady state [12].

On the other hand, we will study the existence of small amplitude periodic traveling wave train solutions for our system based on the Hopf bifurcation theory. It is noted that these solutions are periodic in \( \xi = x + ct \) given by (3). As pointed out
in [29], periodic traveling wave train solutions are spatio-temporal patterns which have periodic profile, maintain their shape and move at a constant speed.

Our work is organized as follows. Section 2 is devoted to the existence of traveling wave solution connecting predator-free constant steady state and coexistence steady state by constructing the upper and lower solution method. By constructing the sequence of iterations, we obtain the asymptotic boundary conditions at $+\infty$ (see (5)). Section 3 is concerned with the existence of small amplitude wave train solutions, taking into account two cases: the diffusion of both species and the diffusion of prey-only, by Hopf bifurcation theory.

2. Existence of traveling wave solution. In this section, we will discuss the existence of traveling wave solution for system (1) with the diffusion of two species, i.e., $d_1 > 0$ and $d_2 > 0$.

For the convenience of discussion, we introduce transformations $\tilde{x} = \frac{x}{\sqrt{d_1}}$. Then, dropping the tildes, system (1) can be rewritten as

\[
\begin{align*}
    u_t &= \Delta u + u(1 - u) \left( \frac{u}{b} - 1 \right) - \beta uv, x \in \mathbb{R}, t > 0, \\
    v_t &= d \Delta v + \mu v \left( 1 - \frac{u}{u} \right), x \in \mathbb{R}, t > 0, \\
    u(x, 0) &= u_0(x) > 0, v(x, 0) = v_0(x) \geq 0, x \in \mathbb{R}.
\end{align*}
\]

(2)

It is clear that system (2) has the predator-free constant steady states $(b, 0)$ and $(1, 0)$. For the sake of discussion, denote $\bar{\beta} = \frac{(1 - \sqrt{b})^2}{b}$. As pointed out in [22], for positive constant steady states, the following results are obtained.

(i) If $\beta > \bar{\beta}$, then system (2) has no positive constant steady state.
(ii) If $\beta < \bar{\beta}$, then system (2) has two positive constant steady states denoted as $(u_*, u_*)$ and $(u^*, u^*)$ with $u_* < u^* < 1$, where

\[
    u_* = \frac{1}{2} \left( 1 + b - \beta b - \sqrt{(1 + b - \beta b)^2 - 4b} \right),
\]
\[
    u^* = \frac{1}{2} \left( 1 + b - \beta b + \sqrt{(1 + b - \beta b)^2 - 4b} \right).
\]

(iii) If $\beta = \bar{\beta}$, then system (2) has a unique positive constant steady state $(\sqrt{b}, \sqrt{b})$.

Remark 1. We can see that if $\beta > \bar{\beta}$, then system (2) has no coexistence steady state. This implies that when the capturing rate $\beta$ is large, system (2) cannot achieve a state of coexistence of both species.

In the following, we always assume $\beta < \bar{\beta}$. According to [22], constant steady state $(u_*, u_*)$ are unstable for all $d, \mu > 0$. So we only focus on the existence of traveling wave solutions

\[
    (u(x, t), v(x, t)) = (u(x + ct), v(x + ct)) = (u(\xi), v(\xi))
\]

for system (2) connecting the predator-free constant steady state $(1, 0)$ and the coexistence steady state $(u^*, u^*)$. Here $c > 0$ is the wave speed. Then the traveling wave solution satisfies the following form

\[
\begin{align*}
    u_{\xi \xi} - cu_{\xi} + u(1 - u) \left( \frac{u}{b} - 1 \right) - \beta uv &= 0, \xi \in \mathbb{R}, \\
    dv_{\xi \xi} - cv_{\xi} + \mu v \left( 1 - \frac{u}{u} \right) &= 0, \xi \in \mathbb{R}
\end{align*}
\]

(4)
with the boundary conditions
\[
\lim_{\xi \to -\infty} (u(\xi), v(\xi)) = (1, 0), \quad \lim_{\xi \to \infty} (u(\xi), v(\xi)) = (u^*, u^*).
\]

(5)

Here, subscripts denote differentiation with respect to the corresponding variable.

First, we introduce the following function spaces
\[
X = \{ \Phi = (u, v) | \Phi \in C(\mathbb{R}, \mathbb{R}^2) \},
\]
\[
X_0 = \left\{ (u, v) \in X \bigg| \frac{1+b}{2} \leq u(\xi) \leq 1, 0 \leq v(\xi) \leq 1, \xi \in \mathbb{R} \right\},
\]

and give the definition of the upper and lower solutions.

**Definition 2.1.** A pair of functions \((\hat{u}, \hat{v})\) and \((\check{u}, \check{v})\) in \(X_0\) are coupled upper and lower solutions of system (4) if \(\hat{u} \geq \check{u}, \hat{v} \geq \check{v}\), and there exists a finite set \(D = \{ D_j \in \mathbb{R} : j = 1, 2, \ldots, n \}\) such that \(\hat{u}, \check{u}, \hat{v}, \check{v} \in C^2(\mathbb{R}\setminus D)\), and \((\hat{u}, \hat{v}), (\check{u}, \check{v})\) satisfy the following inequalities
\[
\hat{u}_{\xi\xi} - c\hat{u}_{\xi} + \hat{u}(1 - \hat{u}) \left( \frac{\hat{u}}{b} - 1 \right) - \beta \hat{u} \hat{v} \leq 0, \hat{u}(\xi) \geq 1, \hat{u}(\xi) \geq u^*,\]
\[
\hat{u}_{\xi\xi} - c\hat{u}_{\xi} + \hat{u}(1 - \hat{u}) \left( \frac{\hat{u}}{b} - 1 \right) - \beta \hat{u} \hat{v} \geq 0, \hat{u}(\xi) \leq 1, \hat{u}(\xi) \leq u^*,\]
\[
\check{v}_{\xi\xi} - c\check{v}_{\xi} + \mu \check{v} \left( 1 - \frac{\check{v}}{u} \right) \leq 0, \check{v}(\xi) \leq 0, \check{v}(\xi) \geq u^*,\]
\[
\check{v}_{\xi\xi} - c\check{v}_{\xi} + \mu \check{v} \left( 1 - \frac{\check{v}}{u} \right) \geq 0, \check{v}(\xi) \geq 0, \check{v}(\xi) \leq u^*.
\]

(6)

Define the functions
\[
H_1(u, v) = \beta_0 u + u(1 - u) \left( \frac{u}{b} - 1 \right) - \beta uv,
\]
\[
H_2(u, v) = \beta_0 v + \mu v \left( 1 - \frac{v}{u} \right),
\]

for some constant \(\beta_0\). Taking \(\beta_0 > \max \left\{ \frac{3}{b} + 1 + \beta, \frac{\mu(3-b)}{b+1} \right\}\), we easily obtain that for \(\frac{b+1}{2} \leq u \leq 1\) and \(0 \leq v \leq 1\), \(H_1(u, v)\) is nondecreasing in \(u\) and nonincreasing in \(v\), and \(H_2(u, v)\) is nondecreasing in \(u\) and \(v\). So system (4) is equivalent to the following equation
\[
\begin{align*}
\left\{ & u_{\xi\xi} - cu_{\xi} - \beta_0 u + H_1(u, v) = 0, \\
& dv_{\xi\xi} - cv_{\xi} - \beta_0 v + H_2(u, v) = 0.
\right.
\end{align*}
\]

Denote
\[
\lambda_1 = \frac{c - \sqrt{c^2 + 4\beta_0}}{2}, \quad \lambda_2 = \frac{c + \sqrt{c^2 + 4\beta_0}}{2},
\]
\[
\lambda_3 = \frac{c - \sqrt{c^2 + 4d\beta_0}}{2d}, \quad \lambda_4 = \frac{c + \sqrt{c^2 + 4d\beta_0}}{2d}.
\]

For \((u(\xi), v(\xi)) \in X_0\), we consider an operator \(P = (P_1, P_2) : X_0 \to X\) defined as follows
\[
P_1(u, v)(\xi) = \frac{1}{\lambda_2 - \lambda_1} \left( \int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} + \int_{\xi}^{+\infty} e^{\lambda_2(\xi - s)} \right) H_1(u, v)(s) ds,
\]
\[
P_2(u, v)(\xi) = \frac{1}{d(\lambda_4 - \lambda_3)} \left( \int_{-\infty}^{\xi} e^{\lambda_3(\xi - s)} + \int_{\xi}^{+\infty} e^{\lambda_4(\xi - s)} \right) H_2(u, v)(s) ds.
\]
Clearly, a fixed point of \( P \) is a solution of system (4). Similar to Lemma 3.2 of [13] and Lemma 3.2 of [19], we can prove the existence of the fixed point by using the Schauder’s fixed point theorem and obtain the following result (also see [29, 4]).

**Lemma 2.2.** Let \( c > 0 \). Suppose that system (4) has a pair of upper and lower solutions \( (\hat{u}, \hat{v}) \) and \( (\tilde{u}, \tilde{v}) \) in \( X_0 \) satisfying

\[
\tilde{u}'(\xi-) \geq \tilde{u}'(\xi+), \hat{v}'(\xi-) \geq \hat{v}'(\xi+), \tilde{u}'(\xi-) \leq \tilde{u}'(\xi+), \hat{v}'(\xi-) \leq \hat{v}'(\xi+), \xi \in D.
\]

Then system (4) has a positive solution \((u, v)\) such that \( \hat{u}(\xi) \leq u(\xi) \leq \tilde{u}(\xi) \) and \( \hat{v}(\xi) \leq v(\xi) \leq \tilde{v}(\xi) \) for all \( \xi \in \mathbb{R} \).

In what follows, we focus on constructing the upper and lower solutions \((\hat{u}(\xi), \hat{v}(\xi))\) and \((\tilde{u}(\xi), \tilde{v}(\xi))\). However, there is a negative power nonlinearity in the predator equation of system (4). Thus, we need a positive lower solution for the prey density.

For constructing the proper upper and lower solutions, we first assume that \( \beta < \hat{\beta} \triangleq \frac{(1-b)^2}{4b} \). Note that \( \hat{\beta} < \beta \). It is easy to obtain that if \( \beta < \hat{\beta} \) holds, then \( u_* \) and \( u^* \) exist, and

\[
u_* < \frac{b+1}{2} < u^*.
\]

Assume that \( c > 2\sqrt{d\mu} \). Define

\[
\hat{u}(\xi) = 1,
\]

\[
\hat{u}(\xi) = \begin{cases} \frac{1+b}{2}, \xi \geq 0, \\
\frac{1 - \frac{b}{2} e^{r_1 \xi}}, \xi < 0,
\end{cases}
\]

\[
\hat{v}(\xi) = \begin{cases} 1, \xi \geq 0, \\
e^{r_2 \xi}, \xi < 0,
\end{cases}
\]

\[
\hat{v}(\xi) = \begin{cases} 0, \xi \geq \frac{1}{\eta r_1 - r_2} \ln \frac{1}{q}, \\
e^{r_2 \xi} - q e^{r_1 \xi}, \xi < \frac{1}{\eta r_1 - r_2} \ln \frac{1}{q},
\end{cases}
\]

where

\[
r_2 = \frac{c - \sqrt{c^2 - 4d\mu}}{2}, 0 < r_1 \leq r_2, \frac{r_2}{r_1} < \eta < \min \left\{ \frac{c}{2r_1}, \frac{2r_2}{r_1} \right\}
\]

and

\[
q = -\frac{2\mu}{(b+1)(d\eta^2 r_1^2 - c\eta r_1 + \mu)} + 1.
\]

Note that \( r_2 < \eta r_1 < \frac{c}{2} \). It is clear that \( d\eta^2 r_1^2 - c\eta r_1 + \mu < 0 \). It shows that \( q > 1 \).

First, note that \( \hat{u}(\xi) = 1 \). It is easy to obtain that

\[
\hat{u}_{\xi\xi} - c\hat{u}_\xi + \hat{u}(1-\hat{u}) \left( \frac{\hat{u}}{b} - 1 \right) - \beta \hat{u}\hat{v} = -\beta \hat{u}\hat{v} \leq 0.
\]

Next, for \( \xi \geq 0 \), we have

\[
\hat{u}_{\xi\xi} - c\hat{u}_\xi + \hat{u}(1-\hat{u}) \left( \frac{\hat{u}}{b} - 1 \right) - \beta \hat{u}\hat{v} = \frac{1+b}{2} \left( \frac{(1-b)^2}{4b} - \beta \right) \geq 0.
\]
For $\xi < 0$, we obtain that $\hat{u}(\xi) = 1 - \frac{1-b}{2}e^{r_1\xi}$ and $\hat{v}(\xi) = e^{r_2\xi}$. From $r_1 \leq r_2 < \xi$, it follows that

$$
\hat{u}_{\xi\xi} - c\hat{u}_{\xi} + \hat{u}(1 - \hat{u}) \left( \frac{\hat{u}}{b} - 1 \right) - \beta \hat{u} \hat{v} \\
= -\frac{1-b}{2} r_1 e^{r_1\xi}(r_1 - c) + \left( 1 - \frac{1-b}{2} e^{r_1\xi} \right) \left( \frac{(1-b)^2}{4b} e^{r_1\xi}(2 - e^{r_1\xi}) - \beta e^{r_2\xi} \right) \\
\geq \left( 1 - \frac{1-b}{2} e^{r_1\xi} \right) e^{r_1\xi} \left( \frac{(1-b)^2}{4b} - \beta \right) \geq 0.
$$

Now, we show that

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) \leq 0.
$$

For $\xi \geq 0$, $\hat{u}(\xi) = \hat{v}(\xi) = 1$. Then,

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) = \mu(1 - 1) = 0.
$$

For $\xi < 0$, $\hat{v}(\xi) = e^{r_2\xi}$. It is obtained that

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) = -\mu \frac{e^{2r_2\xi}}{\hat{u}(\xi)} \leq 0.
$$

Thus, it holds that for all $\xi$,

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) \leq 0.
$$

Finally, we will prove that

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) \geq 0.
$$

For $\xi \geq \frac{1}{\eta r_1 - r_2} \ln \frac{1}{q}$, $\hat{v}(\xi) = 0$. It is clear that

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) = 0.
$$

For $\xi < \frac{1}{\eta r_1 - r_2} \ln \frac{1}{q}$, $\hat{v}(\xi) = e^{r_2\xi} - qe^{\eta r_1\xi}$. Note that $\frac{1}{\eta r_1 - r_2} \ln \frac{1}{q} < 0$, $\hat{u}(\xi) \geq \frac{b+1}{2}$ and $\eta r_1 \leq 2r_2$. It is obtained that

$$
d\hat{v}_{\xi\xi} - c\hat{v}_{\xi} + \mu \hat{v} \left( 1 - \frac{\hat{v}}{u} \right) = e^{r_2\xi}(dr_2^2 - cr_2 + \mu) - qe^{\eta r_1\xi}(d\eta r_1^2 - c\eta r_1 + \mu) - \frac{\mu(e^{r_2\xi} - qe^{\eta r_1\xi})^2}{b+1} \\
\geq -qe^{\eta r_1\xi}(d\eta r_1^2 - c\eta r_1 + \mu) - 2\mu e^{2r_2\xi} \\
\geq e^{\eta r_1\xi} \left( -q(d\eta r_1^2 - c\eta r_1 + \mu) - \frac{2\mu}{b+1} \right) \geq 0.
$$

Therefore, $(\hat{u}, \hat{v})$ and $(\hat{u}, \hat{v})$ are a pair of upper and lower solutions of system (2). Note that the case of $c = 2\sqrt{d\mu}$ can be proved by letting $c$ tend to $2\sqrt{d\mu}$ from the right (see Theorem 4.1 in [14], Theorem 4.2 in [26]). By Lemma 2.2, system (4) has
a positive solution \((u, v)\) satisfying \(\hat{u}(\xi) \leq u(\xi) \leq \tilde{u}(\xi)\) and \(\hat{v}(\xi) \leq v(\xi) \leq \tilde{v}(\xi)\) for all \(\xi \in \mathbb{R}\). In what follows, we will prove that

\[
\lim_{\xi \to -\infty} (u(\xi), v(\xi)) = (1, 0), \quad \lim_{\xi \to +\infty} (u(\xi), v(\xi)) = (u^*, u^*).
\]

It is clear that

\[
1 = \lim_{\xi \to -\infty} \hat{u}(\xi) \leq \liminf_{\xi \to -\infty} u(\xi) \leq \limsup_{\xi \to -\infty} u(\xi) \leq \lim_{\xi \to -\infty} \tilde{u}(\xi) = 1.
\]

\[
0 = \lim_{\xi \to -\infty} \hat{v}(\xi) \leq \liminf_{\xi \to -\infty} v(\xi) \leq \limsup_{\xi \to -\infty} v(\xi) \leq \lim_{\xi \to -\infty} \tilde{v}(\xi) = 0.
\]

Thus, \(\lim_{\xi \to -\infty} (u(\xi), v(\xi)) = (1, 0)\). Next, we will show that

\[
\lim_{\xi \to +\infty} (u(\xi), v(\xi)) = (u^*, u^*).
\]

Denote

\[
u^+ = \limsup_{\xi \to +\infty} u(\xi), \quad u^- = \liminf_{\xi \to +\infty} u(\xi),
\]

\[
v^+ = \limsup_{\xi \to +\infty} v(\xi), \quad v^- = \liminf_{\xi \to +\infty} v(\xi).
\]

Obviously, it is obtained that

\[
u(\xi) \geq \hat{u}(\xi) \geq 1 + \frac{b}{2} > 0.
\]

So,

\[
u^- = \liminf_{\xi \to +\infty} u(\xi) \geq 1 + \frac{b}{2} > 0.
\]

By the second equation of system (2), we obtain that

\[
\begin{cases}
 v_1 \geq d\Delta v + \mu v \left(1 - \frac{2v}{1+b}\right), & x \in \mathbb{R}, t > 0, \\
v(x, 0) = v(x), & x \in \mathbb{R}.
\end{cases}
\] (7)

By Lemma 2.4 and Lemma 2.5 in [29] (also see Corollary 1 and Proposition 2.1 in [1]), it holds that

\[
v^- = \liminf_{\xi \to +\infty} v(\xi) \geq 1 + \frac{b}{2} > 0.
\]

In the following, similar to the methods in [29], we will obtain the relation of inequality between \(\hat{u}, \hat{v}, \tilde{u}\) and \(\tilde{v}\). Note that \(\lambda_1 < 0 < \lambda_2\). For sufficiently small \(\varepsilon > 0\), we can choose

\[
N = \frac{1}{\lambda_1} \ln \left(\frac{-\varepsilon(\lambda_2 - \lambda_1)\lambda_1}{\beta_0}\right) > 0,
\]

and there exists \(M_1\) and \(\xi_1 > M_1 + N\) such that \(u(\xi_1) > u^* - \varepsilon\) and for \(\xi > M_1\),

\[
u(\xi) < u^* + \varepsilon, \quad v(\xi) > v^- - \varepsilon.
\]
Thus, we can obtain that
\[ u^+ - \varepsilon < u(\xi_1) = \frac{1}{\lambda_2 - \lambda_1} \left( \int_{-\infty}^{\xi_1-N} e^{\lambda_1(s)} H_1(u,v)(s) ds \right. \\
\quad \left. + \int_{\xi_1-N}^{\xi_1} e^{\lambda_1(s)} H_1(u,v)(s) ds + \int_{\xi_1}^{+\infty} e^{\lambda_2(s)} H_1(u,v)(s) ds \right) \]
\[ \leq \frac{H_1(u^+ + \varepsilon, v^- - \varepsilon)}{\lambda_2 - \lambda_1} \left( \int_{-\infty}^{\xi_1-N} e^{\lambda_1(s)} ds + \int_{\xi_1}^{+\infty} e^{\lambda_2(s)} ds \right) \]
\[ + \frac{H_1(1,0)}{\lambda_2 - \lambda_1} \int_{-\infty}^{\xi_1-N} e^{\lambda_1(s)} ds \]
\[ \leq \frac{H_1(u^+ + \varepsilon, v^- - \varepsilon)}{\beta_0} - \frac{\beta_0 e^{\lambda_1 N}}{\lambda_2 - \lambda_1} \]
\[ = (u^+ + \varepsilon) + \frac{1}{\beta_0} (u^+ + \varepsilon)(1 - u^+ - \varepsilon) \left( \frac{u^+ + \varepsilon}{b} - 1 \right) \]
\[ - \frac{\beta}{\beta_0} (u^+ + \varepsilon)(v^- - \varepsilon) + \varepsilon. \]

By the arbitrariness of \( \varepsilon \), we can obtain that
\[ u^+(1 - u^+) \left( \frac{u^+}{b} - 1 \right) - \beta u^+ v^- \geq 0. \] (8)

Similarly, we have that
\[ u^-(1 - u^-) \left( \frac{u^-}{b} - 1 \right) - \beta u^- v^+ \leq 0. \] (9)

Moreover
\[ v^+ \left( 1 - \frac{v^+}{u^+} \right) \geq 0, v^- \left( 1 - \frac{v^-}{u^-} \right) \leq 0. \] (10)

Note that \( v^+ \geq v^- > 0 \). From (10), it follows that
\[ v^+ \leq u^+, v^- \geq u^- . \]

Furthermore, according to \( \frac{b+1}{2} < u^- \leq u^+ < 1 \), it is obtained that
\[ (1 - u^-) \left( \frac{u^-}{b} - 1 \right) \geq (1 - u^+) \left( \frac{u^+}{b} - 1 \right) \]
and
\[ (1 - u^+) \left( \frac{u^+}{b} - 1 \right) \geq \beta v^+ \geq \beta u^-, (1 - u^-) \left( \frac{u^-}{b} - 1 \right) \leq \beta v^- \leq \beta u^- . \]

Denote \( u^+_1 = 1 \) and \( u^-_1 = \frac{b+1}{2} \). Construct sequences \( \{u^+_k\}_{k=1,2,\ldots} \) and \( \{u^-_k\}_{k=1,2,\ldots} \), where
\[ u^+_{k+1} = \frac{b + 1 + \sqrt{(b - 1)^2 - 4b\beta u^-_k}}{2}, \quad u^-_{k+1} = \frac{b + 1 + \sqrt{(b - 1)^2 - 4b\beta u^+_k}}{2}. \] (11)

Obviously, \( u^+_k \) is decreasing in \( k \) and \( u^-_k \) is increasing in \( k \). For all integer \( k \), it holds that \( u^+ \leq u^+_k \) and \( u^- \geq u^-_k \). Moreover, it is easy to obtain that for all integer \( k \),
\[ u^-_k \leq u^* \leq u^+_k . \] (12)
In fact, if it is false, then for some integer \( k_0 \), at least one of the following inequalities holds
\[
 u_{k_0}^+ < u^* < u_{k_0}^-.
\]

First, assume that for some integer \( k_0 \),
\[
u_{k_0}^+ < u^*.
\]
By the first equation of (11), we can obtain that
\[
u_{k_0 + 1}^- > \frac{b + 1 + \sqrt{(1 - b)^2 - 4b\beta u^*}}{2} = u^*.
\]
Then, by the second equation of (11), we have that
\[
u_{k_0 + 2}^+ < \frac{b + 1 + \sqrt{(1 - b)^2 - 4b\beta u^*}}{2} = u^*.
\]
So,
\[
u^+ \leq u_{k_0 + 2}^+ < u^* < u_{k_0 + 1}^- \leq u^-.
\]
It contradicts with \( u^+ \geq u^- \). Similarly, the case for \( u_{k_0}^- > u^* \) can be treated.

Next, for all integer \( k \), it is seen that
\[
u^+ - \nu^- = \frac{b + 1 - \sqrt{(b - 1)^2 - 4b\beta u^-}}{2} - \frac{b + 1 - \sqrt{(b - 1)^2 - 4b\beta u^+}}{2}
\]
\[
= \frac{2b\beta}{\sqrt{(b - 1)^2 - 4b\beta u^-} + \sqrt{(b - 1)^2 - 4b\beta u^+}}(u^+ - u^-)
\]
\[
\leq \frac{2b\beta}{\sqrt{(b + 1 - b\beta)^2 - 4b - b\beta}}(u^+ - u^-)
\]
\[
= \rho(u^+ - u^-),
\]

where \( \rho = \frac{2b\beta}{\sqrt{(b + 1 - b\beta)^2 - 4b - b\beta}} \). Under the condition \( \rho < 1 \), i.e.,
\[
\beta < \tilde{\beta} \triangleq \frac{(1 - b)^2}{4b} \frac{1}{\sqrt{\frac{(1 - b)^2}{2} + \frac{(b + 1)^2}{2} + \frac{b + 1}{2}}},
\]
we can obtain that
\[
\lim_{k \to \infty} (u^+_k - u^-_k) = 0.
\]

It follows from (12) and (13) that
\[
u^+ \leq \lim_{k \to \infty} u^+_k = u^* = \lim_{k \to \infty} u^-_k \leq u^-.
\]

Note that \( u^+ \geq u^- \). Thus, \( u^+ = u^- = u^* \). Therefore, this completes the proof of existence of a traveling wave solution connecting \((1, 0)\) and \((u^*, u^*)\).

Similar to the proof of Theorem 2.6 in [4] (also see [29]), it is obtained that for \( 0 < c < 2\sqrt{d\mu} \), system (2) has no positive traveling wave solution connecting \((1, 0)\) and \((u^*, u^*)\).

Note that \( \tilde{\beta} < \hat{\beta} \). Summarizing the above discussion, we obtain the following result.

**Theorem 2.3.** Assume that \( \beta < \tilde{\beta} \). For \( c \geq 2\sqrt{d\mu} \), system (2) has a positive positive traveling wave solution connecting \((1, 0)\) and \((u^*, u^*)\). For \( 0 < c < 2\sqrt{d\mu} \), system (2) has no nonnegative traveling wave solution connecting \((1, 0)\) and \((u^*, u^*)\).
3. Wave train solutions induced by Hopf bifurcation. In this section, we will study the existence of small amplitude periodic solutions of system (4) based on Hopf bifurcation theory, which is equivalent to wave train solutions of system (2).

3.1. Predator and prey diffuse. Denote

\[ \alpha = -\frac{3}{b}(u^*)^2 + \left(\frac{2(b+1)}{b} - \beta \right)u^* - 1. \]

It is obvious that \( \alpha = \frac{2u^*}{b}(-u^* + \frac{b+1}{2}) \). Note that \( \frac{du^*}{d\beta} < 0 \) and \( u^* = \frac{b+1}{2} \) if and only if

\[ \beta = \tilde{\beta} \triangleq \frac{(1-b)^2}{2(1+b)b}. \]

Thus, we can obtain that \( \alpha < 0 \) if \( 0 < \beta < \tilde{\beta} \), \( \alpha = 0 \) if \( \beta = \tilde{\beta} \), and \( \alpha > 0 \) if \( \tilde{\beta} < \beta < \bar{\beta} \). Furthermore, denote \( f(z) = z f_1(z) \), where

\[ f_1(z) = -(1-z) \left( \frac{z}{b} - 1 \right) + \beta z. \]

It is clear that \( f_1(u^*) = 0 \) and \( f'_1(u^*) > 0 \). Thus,

\[ \beta u^* - \alpha = f'(u^*) = f_1(u^*) + u^* f'_1(u^*) > 0. \]

Under the condition \( \alpha > 0 \), we can obtain that \( \frac{\alpha^2}{\beta u^*} < \mu < \alpha \) holds.

First, we transform system (4) into first order ordinary differential equations. Let \( u'(\xi) = U(\xi) \) and \( v'(\xi) = V(\xi) \). Then system (4) becomes as follows

\[
\begin{align*}
    u'(\xi) &= U(\xi), \\
    U'(\xi) &= cU(\xi) - u(\xi)(1 - u(\xi)) \left( \frac{u(\xi)}{b} - 1 \right) + \beta u(\xi)v(\xi), \\
    v'(\xi) &= V(\xi), \\
    V'(\xi) &= \frac{c}{d} V(\xi) - \frac{\mu}{d} v(\xi) \left( 1 - \frac{v(\xi)}{u(\xi)} \right).
\end{align*}
\]

It is clear that \((u^*, 0, 0, u^*, 0)\) is an equilibrium point of (14). The Jacobian matrix of the linearization of (14) at \((u^*, 0, 0, u^*, 0)\) is given by

\[
\begin{pmatrix}
    0 & 1 & 0 & 0 \\
    -\alpha & c & \beta u^* & 0 \\
    0 & 0 & 0 & 1 \\
    -\frac{\mu}{d} & 0 & \frac{\mu}{d} & \frac{c}{d}
\end{pmatrix}.
\]

(15)

Thus, the corresponding characteristic equation is

\[ F_1(\lambda, d) \triangleq \lambda^4 - c \left( 1 + \frac{1}{d} \right) \lambda^3 + \left( \frac{c^2}{d} - \frac{\mu}{d} + \alpha \right) \lambda^2 + \left( \frac{\mu - \alpha}{d} + \beta u^* - \alpha \right) \mu = 0. \]

(16)

To obtain the existence of periodic solutions induced by Hopf bifurcation, we seek for a pair of purely imaginary roots of (16). First, we assume that \( \frac{\alpha^2}{\beta u^*} < \mu < \alpha \). Let
\[ \lambda = i\omega \ (\omega > 0) \] be a root of (16). Next, substitute \( i\omega \) into (16). Then, separating the real and imaginary parts, we can obtain that

\[
\begin{align*}
\omega^2 &= \frac{\alpha - \mu}{1 + d} \\
\omega^4 &= \left( \frac{\mu}{d} - \alpha - \frac{c^2}{d} \right) \omega^2 + \frac{(\beta u^* - \alpha)\mu}{d} = 0. 
\end{align*}
\] (17)

Note that \( \alpha - \mu > 0 \). Thus, it follows from (17) that

\[ G(d) \triangleq (\beta u^* \mu - \alpha^2)(1 + d)^2 + (\mu - \alpha)(c^2 - 2\alpha)(1 + d) - (\mu - \alpha)^2 = 0. \] (18)

By \( \mu > \frac{\alpha^2}{\beta u^*} \), one can obtain that (18) has a unique root \( d = d_H > -1 \). Note that

\[ d_H = \frac{(\alpha - \mu)}{2(\beta u^* \mu - \alpha^2)} \left( c^2 - 2\alpha + \sqrt{(c^2 - 2\alpha)^2 + 4(\beta u^* \mu - \alpha^2)} \right) - 1. \]

It is easy to see that \( d_H \) is strictly increasing on \( c \), and there exists a unique

\[ c_* = \sqrt{\frac{\mu(\beta u^* - \mu)}{(\alpha - \mu)}} \]

such that \( d_{\text{Hopf}}(c_*) = 0 \), and \( d_H(c) > 0 \) for \( c > c_* \).

In the following, we will analyze the transversity conditions with \( d = d_H(c) \) for \( c > c_* \). From (16), it follows that

\[
\text{Re} \left( \frac{d\lambda}{dd} \right)^{-1} \bigg|_{d = d_H(c)} = \frac{2(\mu - \alpha)d_H(c)h(d_H(c))}{(1 + d_H(c))^2g(d_H(c))},
\]

where

\[
g(d_H(c)) = \left( \frac{(\beta u^* - \alpha)^2\mu^2}{(d_H(c))^4} + \frac{(\mu - \alpha)(\mu - c^2)}{(d_H(c))^2(1 + d_H(c))} \right)^2 + \frac{(\alpha - \mu)^3c^2}{(d_H(c))^2(1 + d_H(c))^3} > 0
\]

and

\[
h(d_H(c)) = \left. \frac{d(G(d_H(c)))}{dd} \right|_{d = d_H(c)} = -\frac{3G(d_H(c))}{(d_H(c))^4} + \frac{G'(d_H(c))}{(d_H(c))^3} = \frac{G'(d_H(c))}{(d_H(c))^3} > 0.
\]

Then, \( \text{Re} \left( \frac{d\lambda}{dd} \right)^{-1} \bigg|_{d = d_H(c)} < 0 \). Hence, based on the Hopf bifurcation theorem [20], it is obtained that system (4) possesses the small amplitude periodic solutions with period

\[ T(d) = 2\pi \sqrt{\frac{1 + d}{\alpha - \mu} + O(d - d_H(c))}. \]

This periodic solution corresponds to a small amplitude traveling wave train solution of system (2).

**Theorem 3.1.** Assume that \( \beta < \bar{\beta} \) and \( \frac{\alpha^2}{\beta u^*} < \mu < \alpha \). Then there exists a unique \( c_* > 0 \) such that for \( c > c_* \), as the parameter \( d \) crosses the Hopf bifurcation points \( d = d_H(c) \), system (2) has a family of wave train solutions with period

\[ T(d) = 2\pi \sqrt{\frac{1 + d}{\alpha - \mu} + O(d - d_H(c))}, \]
where

\[ c_* = \sqrt{\frac{\mu (\beta u^* - \mu)}{\alpha - \mu}} \]

and

\[ d_H(c) = \frac{(\alpha - \mu)}{2(\beta u^* \mu - \alpha^2)} \left( c^2 - 2\alpha + \sqrt{(c^2 - 2\alpha)^2 + 4(\beta u^* \mu - \alpha^2)} \right) - 1. \]

3.2. Sedentary predator and diffusing prey. As we know, in the animal world, most predators are fast-moving for food, e.g., tigers have to run faster to catch a running deer. However, there are other cases in ecosystems where predators move slower than the prey (see e.g., [24, 23]). For example, the pacman frog which has a wide mouth that enables it to swallow prey that crosses its path uses a ‘sit-and-wait’ strategy. The web-builders spider also applies the same strategy to prey upon small insects. Thus it is a relevant case to consider. In this subsection, we will assume that the predator moves very slowly relative to the prey. This is reasonable that we take \( d = 0 \).

Let \( u'(\xi) = U(\xi) \). Then system (4) becomes as follows

\[
\begin{align*}
  u'(\xi) &= U(\xi), \\
  U'(\xi) &= c U(\xi) - u(\xi)(1 - u(\xi)) \left( \frac{u(\xi)}{b} - 1 \right) + \beta u(\xi)v(\xi), \\
  v'(\xi) &= \frac{\mu}{c} v(\xi) \left( 1 - \frac{v(\xi)}{u(\xi)} \right).
\end{align*}
\]

(19)

It is clear that \((u^*, 0, u^*)\) is an equilibrium point of (19). The Jacobian matrix of the linearization of (19) at \((u^*, 0, u^*)\) is given by

\[
\begin{pmatrix}
  0 & 1 & 0 \\
  -\alpha & c & \beta u^* \\
  \frac{\mu}{c} & 0 & -\frac{\mu}{c}
\end{pmatrix}.
\]

(20)

Thus, the characteristic equation of (19) is

\[
F_2(\lambda, c) \triangleq \lambda^3 + \left( \frac{\mu}{c} - c \right) \lambda^2 + (\alpha - \mu) \lambda + \frac{\mu}{c}(\alpha - \beta u^*) = 0.
\]

(21)

We will search for a pair of pure imaginary roots of (21). Substituting \( \lambda = i\omega (\omega > 0) \) into (21) and gathering terms, we can obtain \( \omega^2 = \alpha - \mu \) and \( \omega^2 = \frac{\mu(\beta u^* - \alpha)}{c^2 - \mu} \). Thus, under the condition \( \alpha > \mu \), if

\[ c = c_H \triangleq \sqrt{\frac{\mu(\beta u^* - \alpha)}{\alpha - \mu} + \mu}, \]

then (21) has a pair of pure imaginary roots. Regarding \( \lambda = \lambda(c) \) as a function of \( c \) and differentiating (21), we obtain

\[
\frac{d\lambda}{dc} = -\frac{\partial F_2/\partial c}{\partial F_2/\partial \lambda} = \frac{(\mu + c^2)\lambda^2 + \mu(\alpha - \beta u^*)}{3c^2\lambda^2 + 2c(\mu - c^2)\lambda + c^2(\alpha - \mu)}.
\]

It is clear that

\[
\frac{d\lambda}{dc} \bigg|_{c=c_H} = \frac{(\mu + c_H^2)(\mu - \alpha) + \mu(\alpha - \beta u^*)}{2c_H^2(\mu - \alpha) + 2c_H(\mu - c_H^2)\omega^2}.
\]
Thus,

\[
\text{Re}\left( \frac{d\lambda}{dc} \bigg|_{c=c_H} \right) = \frac{(\mu + c_H^2)(\alpha - \mu) + \mu(\beta u^* - \alpha)}{2c_H^2(\alpha - \mu) + 2(c_H^2 - \mu)} > 0.
\]

**Theorem 3.2.** Assume that \( \alpha > \mu \). Then as the parameter \( c \) crosses the Hopf bifurcation points \( c = c_H \), system (2) with \( d = 0 \) has a family of wave train solutions with period

\[
T(c) = \frac{2\pi}{\sqrt{\alpha - \mu}} + O(c - c_H).
\]

4. **Conclusion.** In this paper, we study the traveling wave solutions of a reaction-diffusion predator-prey system with prey subject to Allee effect. Here, the growth rate of the predator does not depend on predation rate explicitly, but rather obeys the logistic growth with a varying carrying capacity proportional to the density of the prey. By constructing the upper and lower solutions and the sequence of iterations, we prove the existence of traveling wave solution connecting predator-free steady state \((1, 0)\) and coexistence steady state \((u^*, u^*)\) for wave of speed \( c \geq 2\sqrt{d\mu} \). We also find that system (2) has no traveling wave solution connecting predator-free steady state \((1, 0)\) and coexistence steady state \((u^*, u^*)\) for wave of speed \( 0 < c < 2\sqrt{d\mu} \). Especially, for overcoming the singularity in the predator equation, we construct a positive lower solution for the prey density. Subsequently, take into account two cases: the diffusion of both species and the diffusion of prey-only, we obtain the existences of small amplitude periodic traveling wave train. It shows that the predator invasion leads to the periodic population densities in the coexistence domain.

**Acknowledgments.** We would like to thank the editor and reviewers for their valuable comments and suggestions, which significantly improved the quality of our paper indeed.

**REFERENCES**

[1] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.*, 30 (1978), 33–76.

[2] M. A. Aziz-Alaoui and M. Daher Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, *Appl. Math. Lett.*, 16 (2003), 1069–1075.

[3] S. Cantrell, C. Cosner and S. G. Ruan, *Spatial Ecology, Mathematical and Computational Biology Series*, Chapman & Hall/CRC Mathematical and Computational Biology Series, CRC Press, Boca Raton, FL, 2010.

[4] Y.-Y. Chen, J.-S. Guo and C.-H. Yao, Traveling wave solutions for a continuous and discrete diffusive predator-prey model, *J. Math. Anal. Appl.*, 445 (2017), 212–239.

[5] J. B. Collings, The effect of the functional response on the bifurcation behavior of a mite predator-prey interaction model, *J. Math. Biol.*, 36 (1997), 149–168.

[6] S. R. Dunbar, Traveling wave solutions of diffusive lotka-volterra equations, *J. Math. Biol.*, 17 (1983), 11–32.

[7] S. R. Dunbar, Traveling wave solutions of diffusive lotka-volterra equations: a heteroclinic connection in \( \mathbb{R}^4 \), *Trans. Amer. Math. Soc.*, 286 (1984), 557–594.

[8] C.-H. Hsu, C.-R. Yang, T.-H. Yang and T.-S. Yang, Existence of traveling wave solutions for diffusive predator-prey type systems, *J. Differ. Equations*, 252 (2012), 3040–3075.

[9] W. Z. Huang, Traveling wave solutions for a class of predator-prey systems, *J. Dyn. Differ. Equ.*, 24 (2012), 633–644.

[10] J.-H. Huang, G. Lu and S. G. Ruan, Existence of traveling wave solutions in a diffusive predator-prey model, *J. Math. Biol.*, 46 (2003), 132–152.
[11] W. Z. Huang, A geometric approach in the study of traveling waves for some classes of non-monotone reaction-diffusion systems, *J. Differ. Equations*, **260** (2016), 2190–2224.

[12] Y. H. Huang and P. X. Weng, Traveling waves for a diffusive predator-prey system with general functional response, *Nonlinear Anal. Real World Appl.*, **14** (2013), 940–959.

[13] Y.-L. Huang and G. Lin, Traveling wave solutions in a diffusion system with two preys and one predator, *J. Math. Anal. Appl.*, **418** (2014), 163–184.

[14] W. Z. Huang and M. A. Han, Non-linear determinacy of minimum wave speed for a Lotka-Volterra competition model, *J. Differ. Equations*, **251** (2011), 1549–1561.

[15] A. Korobeinikov, A Lyapunov function for Leslie-Gower predator-prey models, *Appl. Math. Lett.*, **14** (2001), 697–699.

[16] M. A. Lewis, B. T. Li and H. F. Weinberger, Spreading speed and linear determinacy for two-species competition models, *J. Math. Biol.*, **45** (2002), 219–233.

[17] B. T. Li, H. F. Weinberger and M. A. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, *Math. Biosci.*, **196** (2005), 82–98.

[18] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, **60** (2007), 1–40.

[19] G. Lin, Invasion traveling wave solutions of a predator-prey system, *Nonlinear Anal.*, **96** (2014), 47–58.

[20] J. E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications*, Applied Mathematical Sciences, Vol. 19. Springer-Verlag, New York, 1976.

[21] W. J. Ni and M. X. Wang, Dynamical properties of a Leslie-Gower prey-predator model with strong Allee effect in prey, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 3409–3420.

[22] W. J. Ni and M. X. Wang, Dynamics and patterns of a diffusive Leslie-Gower prey-predator model with strong Allee effect in prey, *J. Differ. Equations*, **261** (2016), 4244–4274.

[23] S. E. Riechert, *Spiders as Representative ‘Sit-and-Wait’ Predators*, Natural Enemies: The Population Biology of Predators, Parasites and Diseases, John Wiley & Sons, 2009.

[24] H. M. Safuan, I. N. Towers, Z. Jovanoski and H. S. Sidhu, On travelling wave solutions of the diffusive Leslie-Gower model, *Appl. Math. Comput.*, **274** (2016), 362–371.

[25] N. Shigesada and K. Kawasaki, *Biology Invasions: Theory and Practice*, Oxford University Press, Oxford, 1997.

[26] A. I. Volpert, V. A. Volpert and V. A. Volpert, *Traveling Wave Solutions of Parabolic Systems*, Translations of Mathematical Monographs, 140. American Mathematical Society, Providence, RI, 1994.

[27] J. H. Wu and X. F. Zou, Traveling wave fronts of reaction-diffusion systems with delay, *J. Dyn. Differ. Equ.*, **13** (2001), 651–687.

[28] X. B. Zhang and H. L. Zhu, Dynamics and pattern formation in homogeneous diffusive predator-prey systems with predator interference or foraging facilitation, *Nonlinear Anal. Real World Appl.*, **48** (2019), 267–287.

[29] W. Zuo and J. Shi, Traveling wave solutions of a diffusive ratio-dependent Holling-tanner system with distributed delay, *Comm. Pure Appl. Math.*, **17** (2018), 1179–1200.

Received November 2018; revised March 2019.

E-mail address: Hyzho1967@126.com

E-mail address: wudy99018163.com