Influence of localization transition on dynamical properties for an extended Aubry–André–Harper model

X L Zhao¹,², Z C Shi³,⁴, C S Yu¹ and X X Yi²

¹School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, People’s Republic of China
²Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, People’s Republic of China
³Department of Physics, Fuzhou University, Fuzhou 350002, People’s Republic of China
⁴Fujian Key Laboratory of Quantum Information and Quantum Optics (Fuzhou University), Fuzhou 350116, People’s Republic of China

E-mail: yixx@nenu.edu.cn

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Abstract
We show the localization transition and its influence on two dynamical processes of an extended Aubry–André–Harper model with incommensurate on-site and hopping potentials. After specifying an extended Aubry–André–Harper model, we check the localization transition for all the eigenstates and eigenenergy band splitting behavior versus a system parameter. To check the influence of localization transition on dynamical processes, firstly, the slowly pumping of edge states are examined. The adiabatic pumping is suppressed in the localized region. Then by quantum Lyapunov control method in terms of different control Hamiltonians, we prepare an edge-localized state in the nonlocal region. The control effect is suppressed in the localized region compared to that in the nonlocal region. In the dynamical processes, the system acts as conductor for the excitation in the nonlocal region and insulator in the localized region. Then we adopt the occupation imbalance between even and odd sites and entropy to indicate the localization transition further. Finally, the experimental schemes based on cold atoms trapped in quasiperiodic optical lattice and coupled optical waveguide arrays are suggested.

Keywords: localization transition, adiabatic pumping, Lyapunov control

(Some figures may appear in colour only in the online journal)

1. Introduction
Localization is an intriguing phenomenon in many-body systems in the presence of disorder [1]. For example, Anderson localization explains the transport mechanism for metal–insulator phase transition in solids [2]. This localization is interpreted as the interference effect of the electron wave functions in the presence of disorder.

Analogous but different to Anderson localization in one dimension (1D) version [3–10], localization transition occurs in the Aubry–André–Harper (AAH) model with on-site incommensurate modulations [3, 4]. This is a model intermediate between random and periodic systems with quasiperiodic modulated on-site potentials. It can reduce from the description of a two-dimensional (2D) quantum Hall system by using Landau gauge for the magnetic field [5]. The critical point for the localization transition is related to the ratio of the on-site and the hopping amplitudes. In a subsequent of variants of this model, extensive arguments about localization properties have been discussed [10–22].

AAH model has attracted wide attention since it can be used to study topological properties for the 2D counterparts [16–18]. But the intensity of the magnetic field sets the obstacle for this model to be explored in solid systems currently. Fortunately, the developing quantum simulation systems [23, 24] offer platforms to implement such models such
as by non-interacting $^{87}$K Bose–Einstein condensate in quasi-random optical lattice [10] or interacting fermions in a one-dimensional quasi-random optical lattice [13, 14]. Besides, coupled optical waveguide arrays can also be used to implement such a model with tunable on-site and hopping parameters [15, 16, 25–30].

In this work, we consider an extended AAH model with hopping and on-site potentials both being modulated incommensurately. We use the inverse participation ratio (IPR): $\text{IPR} = \sum_n |\psi(n)|^4$ [31], to indicate the degree of localization. Here $\psi(n)$ are the amplitudes for a state on site $n$. Average localization diagrams versus two modulation phases are exhibited. Then we specify a system to show the localization transition in terms of all the eigenstates and the energy spectrum. The splitting behavior of the energy spectrum coincides with the localization transition of the eigenstates. In addition, this transition may have significant effects on dynamical properties of the system. Thus we show the influence of the localization on slowly pumping of an edge state; preparing the edge state by Lyapunov control methods. It is shown that this model acts as a conductor for the excitation on the chain in the nonlocal region, however as insulator in the localized region which blocks the transportation processes. In order to reveal the distribution patterns of the eigenstates on the chain which IPR may not reflect accurately, we use occupation imbalance between even and odd sites and the entropy of the states to check the localization transition [13, 14]. It can be seen that the localization transition behavior coincides with the one IPR indicates. Finally we propose the discussions may be checked in the system consist of cold atoms trapped in quasiperiodic optical lattice [10, 13, 14] or coupled optical waveguide arrays [15, 16, 25–30].

Since the Lyapunov control method would be employed to prepare an edge state, we give a brief introduction to this method here. It is a ‘close-loop design, open-loop apply’ strategy which has been applied to finish control goals in various systems effectively [32–36]. In this method, after choosing suitable control Hamiltonians based on controlled systems, control fields are designed based on Lyapunov functions. Then the system is driven to the goal state. According to LaSalle’s invariant principle [37], the control fields tend to vanish when the system is asymptotically steered to a target. In this work, we adopt two different control Hamiltonians in terms of one Lyapunov function to design the control fields to prepare an intriguing edge state.

This work is organized as follows. In section 2, we show the average localization for all the eigenstates versus system parameters and specify the extended AAH model which will be discussed further. Then we show that the splitting behavior for the energy spectrum coincides with the localization transition of the eigenstates. In section 3, we consider the influence of the localization on adiabatic pumping of the edge states; the procedure to prepare the edge-localized state by state-distance Lyapunov control method in terms of two different control Hamiltonians. Then in section 4, to further reveal the localization patterns on the chain, besides IPR, we employ occupation imbalance between even and odd sites and entropy of the eigenstates to show the localization behavior. Then we discuss the experimental schemes to realize this model based on cold atoms in quasiperiodic optical lattice and coupled optical waveguide arrays. Finally, we conclude in section 5.

2. The extended AAH model

In this work, we consider an extended 1D AAH model with both incommensurate on-site potentials and nearest-neighbor hopping interactions which is described by the following tight-binding Hamiltonian ($\hbar = 1$ is assumed in this work):

$$
\hat{H}_0 = \sum_{n=1}^{N} v \cos(2\pi bn + \theta_n) \hat{c}_n^{\dagger} \hat{c}_n + \sum_{n=1}^{N-1} [\lambda_0 + \lambda \cos(2\pi bn + \theta)] \hat{c}_n^{\dagger} \hat{c}_{n+1} + \text{h.c.},
$$

where $\hat{c}_n^{\dagger}$ and $\hat{c}_n$ are the polarized Fermionic creation and annihilation operators for site $n$. $N$ is the total number of sites on the 1D AAH chain. The terms in the first line in (1) represent the incommensurate on-site potentials where $v$ is the amplitude. The last terms describe the kinetic energy from the nearest-neighbor hopping. $\lambda_0$ is the amplitude which is taken as the energy unit throughout this work and $\lambda$ indicates the modulation amplitude. Even this Hamiltonian is a mutation of the quasiperiodic model derived from the tight-binding square lattice with next-nearest couplings [19–22]; it can be implemented by cold atoms trapped in quasiperiodic optical lattice [10, 13, 14] or coupled optical waveguide arrays [15, 16, 25–30].

Note that whether $b$ is rational or not influences the localization properties of such kinds of systems [19–22]. When $b$ is a rational number, the adjustable on-site potentials have the periodicity of $1/b$ which is determined by the magnetic field penetrating the 2D counterpart of the 1D chain [3, 5]. Due to Bloch theorem, no localization transition occurs in this case. Whereas in this work, we will focus on the case that $b$ is the irrational number $(\sqrt{5} - 1)/2$, namely, the inverse of golden mean. Mathematically, localization transition occurs with $b$ being an irrational Diophantine number [38]. We will exhibit the localization transition and its influence on dynamical processes in the following.

2.1. Localization transition and fractal energy band

We review the localization transition to draw forth the system we will discuss further. To indicate the degree of localization for a state, the $\text{IPR} = \sum_n |\psi(n)|^4$ is employed [31], where $\psi(n)$ are amplitudes for the state projecting on site $n$. It can be seen that when $|\psi(n)|$ distributes homogeneously over $N$ sites on a chain, namely $|\psi(n)| \sim 1/\sqrt{N}$, then $\text{IPR} \sim 1/N$. Whereas, if $|\psi(n)|$ locates mainly over a range $\zeta$, namely $|\psi(n)| \sim 1/\sqrt{\zeta}$, then $\text{IPR} \sim 1/\zeta$. Thus IPR tends to vanishing for nonlocal states but finite values for localized states while $N$ is large. Considering there may be various kinds of localization patterns on the chain, to reveal the localization further, we exhibit that also by occupation imbalance between
the even and odd sites [13, 14] and entropy for the eigenstates to check the localization transition IPR reflects.

First we check the average localization over all the eigenstates as a function of $\lambda/\lambda_0$ and $v/\lambda_0$ in Hamiltonian (1) for $\theta_0 = 0$ and $\pi/2$ in figure 1. It can be seen that when $v = 0$, only hopping interactions remain which leads to the eigenstate being nonlocal for $\lambda$. For finite $v/\lambda_0$, with increasing of $\lambda/\lambda_0$, the nonlocal region tends to expand as the shaded area shows in figure 1. This may result from that the hopping leads to diffusion of the distributions. When $\theta_0 = 0$, the boundary for localization transition as shown in figure 1(a) fulfills the condition $(v/2)^2 + \lambda^2 = \lambda_0^2$. However when $\theta_0 = \pi/2$, the boundary tends to be a rectangle as in figure 1(b). Later we will focus on the extended AAH model with $v/\lambda_0 = 1$.

Then we show the localization transition diagram for all the eigenstates versus $\lambda/\lambda_0$ by two examples with different lengths in figure 2 when $v/\lambda_0 = 1$, $\theta = 0$ and $\theta_0 = \pi/2$. It can be seen that the localization transition occurs not only in the ground state but also in the excited states which hints that this transition can occur in relatively high temperature. According to figure 1(b), $|\lambda/\lambda_0| \approx 1$ is the transition point for localization when $|v/\lambda_0| \lesssim 2$. The transition of average localization is clear and yet there are localized states in the nonlocal region. In particular the location for the most obvious localized state (the states are ordered according to the corresponding eigenenergies from small to large) is related to the inverse of golden mean $b = (\sqrt{5} - 1)/2$, independent on the length of the AAH chain. We find that the location of the most obvious localized states in the nonlocal region can be expressed as GI ($N \times b$). Here GI ($x$) denotes the function which returns the minimum integer among those larger than $x$ and $N$ is the number of sites on the chain. For example, for the chain with 1000 sites: GI (1000 × $b$) = 619 and 40 sites: GI (40 × $b$) = 25, and so on. Yet there are other eigenstates related to $b$ in the Hamiltonian which has similar localization character but not obviously localized in the nonlocal region. We show the IPR for the eigenstates in the nonlocal region ($\lambda/\lambda_0 = 0.5$) of different lengths in figure 3. It can be seen that the GI ($N \times b$) th states locate obviously in the nonlocal region for the range of lengths. However, explanations for these elusive localized states may need further work.

The localization properties and self-similar structures of distributions for eigenstates at the localization boundary in similar systems have been investigated widely [39–41]. In this work, we find that the energy band versus $\lambda/\lambda_0$ splits in a fractal way in the nonlocal region. And the splitting behavior in the nonlocal region is different to that in the local region. Then we check the energy band versus $\lambda/\lambda_0$ in figure 4. It can be seen that the energy band splits in the nonlocal region with a fractal structure related to the inverse of golden mean. For example, for the chain of $N = 1000$ as in figure 4, a simple examine is: $90/146 \simeq (146)/(90 + 146) \simeq (90 + 146)/(146 + 90 + 146) \ldots \simeq 0.618$ which approximates $b = (\sqrt{5} - 1)/2$. With increasing of $N$, the

Figure 1. The average localization transition diagrams indicated by IPR over all the eigenstates for $\theta_0 = 0$ and $\theta_0 = \pi/2$ when $\theta = 0$. When $v = 0$, only hopping potential exists which leads to nonlocal region. We have used a chain composed of 40 sites. The other parameter are $b = (\sqrt{5} - 1)/2$ and $v/\lambda_0 = 1$ in the Hamiltonian (1). We would check the localization transition for a range of $\lambda/\lambda_0$ when $v/\lambda_0 = 1$ as the red dotted line shows in (b).

Figure 2. The IPR for the eigenstates of two chains for a range of $\lambda/\lambda_0$ when $b = (\sqrt{5} - 1)/2 \approx 0.618$. We show (a) $N = 1000$ and (b) $N = 40$ sites on the chain for comparing when $v/\lambda_0 = 1$, $\theta = 0$ and $\theta_0 = \pi/2$. With increasing of $\lambda/\lambda_0$, the IPR for some eigenstates decrease due to the hopping interaction dominates. The eigenstates are denoted by index $\mu$. 

Figure 3. It can be seen that the IPR for some eigenstates are finite for $v/\lambda_0 = 1$ and $\theta = 0$ as in figure 3. This may result from that the hopping leads to diffusion of the distributions. When $\theta_0 = 0$, the IPR for the eigenstates increase with increasing of $v/\lambda_0$.
which is elusive. But the transition in $\lambda/\lambda_0$ coincides with the localization transition in this system, we check the dependence of the IPR for the eigenstates $\mu$ on site $n$ which are adiabatic pumping for an edge state and preparing the edge state by Lyapunov control method.

3. Influence of the localization on dynamical processes

3.1. Adiabatic pumping for the edge-localized state

The appearance of edge-localized states whose eigenenergies span the band gap connecting different energy bands is usually treated as a signature of nontrivial topological phase. Topological phase with edge states has been discussed in quasicrystals which is attributed to higher dimensional systems [16]. The AAH model in this work is an extension for such a quasicrystal chain which can be implemented by cold atom system trapped in quasi-periodic potentials [10, 13, 14] or coupled optical waveguide array systems [15, 16]. Some topological properties can be checked by adiabatic pumping of the edge states. Since there is localization transition in this system, we check the influence of this transition on the pumping processes for the edge states by slowly varying $\theta$ and $\theta_e$ in figures 6 and 7 respectively. It may approximate to an adiabatic process when the pumping is slow enough. Such pumping processes can be checked by solving the Schrödinger equations:

$$i\partial_t \psi_n = \left[ \lambda_0 + \lambda \cos(2\pi n (n - 1) + \theta(t)) \right] \psi_{n-1} + \left[ \lambda_0 + \lambda \cos(2\pi b(n + \theta(t))) \right] \psi_{n+1} + v \cos(2\pi bn + \theta_e(t)) \psi_n,$$

where $\psi_n (n = 1, 2, \ldots, N)$ are the amplitudes for the wavefunction on site $n$. To approximate the adiabatic pumping process, we assume that: $\theta(t) = 1.2 + t(2\pi - 1.2)/\tau (\theta_e$ fixed) extended except for the obvious edge-localized GI $(N \times b)$ th state. When $\lambda/\lambda_0$ is larger than 1, all the eigenstates are localized. Later, one can see that the localization transition influences dynamical processes based on the transport properties of the excitations on the chain. The dynamical processes are adiabatic pumping for an edge state and preparing the edge state by Lyapunov control method.

Figure 3. The IPR for the eigenstates (denoted by $\mu$) in a range of length (denoted by $N$) in the nonlocal region with $\lambda/\lambda_0 = 0.5$. It can be seen that the most obviously localized at GI $(N \times b)$, for example, GI$(200 \times b) = 124$. The position of the other localized eigenstates are related to inverse of the golden mean $b = (\sqrt{5} - 1)/2$. The other parameters are $\nu/\lambda_0 = 1$, $\theta = 0$ and $\theta_e = \pi/2$.

Figure 4. The energy spectrum as a function of $\lambda/\lambda_0$ when $\nu/\lambda_0 = 1$, $\theta = 0$ and $\theta_e = \pi/2$. We have set $N = 1000$ for the chain. The energy band is zoom in to show that the energy is splitting in a way of fractal related to the inverse of the golden mean $b = (\sqrt{5} - 1)/2$. The numbers denote the amount of energy levels in the bands. The change of the splitting behavior coincides with the localization transition in figure 2(a).

Figure 5. The population of the eigenvectors for $\lambda/\lambda_0 = 0.5$, 1, 2 and 5 for chains with $N = 40$ sites. The degree of localization for $\lambda/\lambda_0 = 5$ seems less than those of $\lambda/\lambda_0 = 2$ and 1 which coincides with that as figures 1 and 2 show. This is because with increasing of $\lambda/\lambda_0$, the hopping dominates, $n$ and $\mu$ denote the position of the sites on the chain and index for the eigenstates respectively. The other parameters are same to those in figure 2(b). The GI$(40 \times b) = 25$ states are denoted by red-thick lines.

ratios would approach to $(\sqrt{5} - 1)/2$ which is elusive. But the energy levels in the bands tend to be bunching and crossing when $\lambda/\lambda_0$ strides over 1. The bunching and crossing of the energy levels correspond to the localization of the eigenstates which can be confirmed in figure 5. This change of the splitting behavior coincides with the localization transition in figure 2.

To show the localization in detail, the distributions for different eigenstates are plotted in figure 5 by examples. It can be seen that in the nonlocal region, most of the eigenstates are...
where \( \lambda/\lambda_0 \) is 0.8 when the phase \( \theta = 0 \). \( (a_2) \) The pumping of the 25th eigenstate by slow scan of \( \theta(t) = 1.5 + n(2\pi - 1.5)/\tau \). \( (b_1) \) and \( (b_2) \) are those when \( \lambda/\lambda_0 = 2 \) for the slowly varying \( \theta(t) = 1.2 + n(2\pi - 1.2)/\tau \). \( (a_1) \) and \( (b_1) \) are those when \( \lambda/\lambda_0 = 2 \) for the slowly varying \( \theta(t) = 3 + t(2\pi - 3)/\tau \). \( (a_2) \) and \( (b_2) \) are those when \( \lambda/\lambda_0 = 2 \) for the slowly varying \( \theta(t) = 3 + t(2\pi - 3)/\tau \). \( (a_3) \) and \( (b_3) \) are those when \( \lambda/\lambda_0 = 2 \) for the slowly varying \( \theta(t) = 3 + t(2\pi - 3)/\tau \). \( (a_4) \) and \( (b_4) \) are those when \( \lambda/\lambda_0 = 2 \) for the slowly varying \( \theta(t) = 3 + t(2\pi - 3)/\tau \).

\( \theta(t) = 0 \) denotes evolving time in units of \( 1/\lambda_0 \) and \( \tau \) denotes the rate for the pumping processes. The larger \( \tau \) is, the more slowly the pumping. The larger \( \tau \) is, the more slowly the pumping. The larger \( \tau \) is, the more slowly the pumping.

or \( \theta(t) = 3 + t(2\pi - 3)/\tau \) (\( \theta \) fixed). Here \( t \) denotes the evolution time. The time is in units of \( 1/\lambda_0 \) here and after. \( \tau \) determines the speed of evolution. The larger \( \tau \) is, the more slowly the evolution will be, namely, the more degree of the process verges on an adiabatic one.

In terms of slowly varying \( \theta \) or \( \theta_s \), the energy spectrum are shown in \( (a_1) \)s and \( (b_1) \)s in figures 6 and 7 respectively. In terms of \( \theta \), both in the extended and localized region, the signature of topological phase transition can be observed. For \( \theta_s \), topological phase transition occurs in the nonlocal region but in the localized region since the edge states are pinned to one end of the chain. Even though we do not focus on topological property of this model, we exhibit nontrivial topological hints of this model in the appendix.

\( (a_2) \)s and \( (b_2) \)s in figures 6 and 7 show the slowly pumping for the edge states in the nonlocal and local regions as \( \theta(t) \) or \( \theta_s(t) \) varies slowly. The influence of localization on the pumping processes is obvious. In the nonlocal region, as \( \theta(t) \) or \( \theta_s(t) \) varies slowly, the edge state can be adiabatically pumped into a bulk band and then become an edge state localized at the opposite end of the chain. However in the localized region, the bulk states become localized and the adiabatic pumping processes are suppressed.

According to quantum adiabatic theorem [42, 43], it is necessary that the system evolves slowly enough to remain the state as the instantaneous eigenstate of the time dependent Hamiltonian and the gap between the eigenvalues also plays a critical role. Whether the state is pumped to the goal state successfully can be indicated by defining the fidelity as \( F_f = |\langle \varphi(\tau) | \varphi_0 \rangle|^2 \), where \( |\varphi(\tau) \rangle \) is the state when the pumping process finishes. We check the fidelity \( F_f \) as a function of \( \tau \) in the nonlocal and localized region in figure 8 which correspond to the pumping processes in figures 6(a2) and (b2). It confirms that \( \theta(t) \) evolves slowly enough is necessary for the state being adiabatically pumped to the goal state successfully. And in the localized region, the failure of the pumping process may result from that the gaps between the related energy levels are to narrow (even cross) as shown in figure 4. This follows the statement of adiabatic theorem [42, 43].

The pumping processes are based on transportation of the excitations on the chain. Thus in figure 9, we show the transportation for excitations on the chain in the two different localization regions respectively. It shows that in the nonlocal region, the excitations spread on the chain as they propagate. But in the localized region, the spreading is suppressed and obviously localized oscillation appears. From this point of view, one may conclude that the system acts as conductor in the nonlocal region and insulator in the localized region for the excitations. This reflects the origin for the failure of adiabatic pumping of the edge states in the localized region. This also signifies that the localization transition can be reflected by dynamical process of the excitations. Next we
would examine the suppression of the control effect in the localized region compare to that in the nonlocal region when we use Lyapunov control method to prepare the edge state.

3.2. Prepare edge-localized state by Lyapunov control

One may want to positively influence the dynamics of a quantum system to achieve an object by control methods. Usually a specified state can be chosen as the control goal. In this extended AAH model, the localized edge state (GI (N x b) th state) in the nonlocal region is an intriguing target which is shown in figure 10. Its corresponding eigenenergy locates in the gap apart from others. Quantum Lyapunov control method may be an effective tool to prepare such states since this method has been employed to prepare eigenstates effectively for many quantum systems [32–36]. Considering this control method is also based on transport properties of the excitations, we would check the influence of localization on the control effect.

In quantum Lyapunov control, in order to achieve a goal, control fields need to be designed based on Schrödinger equation: (i\hbar/\partial t)\psi(t) = (\hat{H}_0 + \sum_n f_{2n}(t) \cdot \hat{H}_{Cn})\psi(t), where \hat{H}_0 and \hat{H}_{Cn} denote the free Hamiltonian of the controlled system and the control Hamiltonians, respectively. f_{2n}(t) are the control fields need to be designed. Generally, there are three kinds of design schemes in this method [32–36]. Here we consider the Hilbert–Schmidt state-distance scheme to prepare the edge-localized eigenstate as shown in figure 10. We denote the target state as |ϕ⟩ here and the instantaneous state at time t during the control process as |ϕ(t)⟩. According to the Hilbert–Schmidt state-distance scheme in quantum Lyapunov control method, the Lyapunov function can be written as bellow:

\[ V_L = \frac{1}{2}(1 - |\langle ϕ | ϕ(t)⟩|^2), \]  \tag{3}

where |⟨ϕ | ϕ(t)⟩|^2 denotes the transition probability from |ϕ⟩ to |ϕ(t)⟩. It reflects the distance between the states |ϕ⟩ and |ϕ(t)⟩ in Hilbert space. According to the design process in Lyapunov control method [32–36], the first-order time derivative for V_L need to be calculated as:

\[ \partial_t V_L = -\sum_n f_{2n}(t) \cdot |⟨ϕ | ϕ⟩| \times \text{Im}[e^{i \arg⟨ϕ | ϕ⟩}⟨ϕ | \hat{H}_{Cn} | ϕ(t)⟩], \]  \tag{4}

where we employ Im [•] to denote the imaginary part of •. Next f_{2n}(t) need to be designed in order to achieve a control goal. According to the physical meaning of V_L in (3), the control fields f_{2n}(t) act in a way to make the distance between |ϕ⟩ and |ϕ(t)⟩ shrink. Thus a succinct and valid choice is

\[ f_{2n}(t) = T_n \cdot \text{Im}[e^{i \arg⟨ϕ | ϕ⟩}⟨ϕ | \hat{H}_{Cn} | ϕ(t)⟩], \]  \tag{5}

where T_n > 0 are the free constants. When ⟨ϕ | ϕ⟩ = 0, the angle arg⟨ϕ | ϕ⟩ is uncertain. Without loss of generality, we set arg⟨ϕ(t) | ϕ⟩ = 0 in this work.

Until now we have not specified the exact forms for the control Hamiltonians \hat{H}_{Cn}. Their forms are based on the controlled systems. For such an AAH chain, there may be various control Hamiltonians which can be adopted to achieve the goal. Here we consider time dependent hopping-control Hamiltonian (Lyapunov control-A) with the elements:

\[ \hat{H}_{ch}(m, n) = \delta_{m,n+1} V_c \cos[2\pi Kbm + X \cos(2\pi Zb\omega_c t)] + \text{h.c.}, \]  \tag{6}

and the time dependent on-site-control Hamiltonian (Lyapunov control-B) with the elements:

\[ \hat{H}_{co}(m, n) = \delta_{m,n} V_c \cos[2\pi Kbm + X \cos(2\pi Zb\omega_c t)], \]  \tag{7}

where V_c, K, X and Z are free control Hamiltonian parameters. The control Hamiltonians are all Hermitian to insure the evolution unitary and control fields real. Such kinds of control fields can be realized by electro-optic region modulator

Figure 9. (b) The evolution of IPR for an excitation when the excitation initially locates at the middle site on the chain versus \( \lambda/\lambda_0 \). (a) The evolution for the excitation when \( \lambda/\lambda_0 = 2 \) and (c) the evolution for the excitation when \( \lambda/\lambda_0 = 0.5 \). The other parameters are \( N = 40, \theta = 0 \) and \( \theta_0 = \pi/2 \).

Figure 10. The eigenenergies for the chain when \( \theta = 0 \) and \( \theta_0 = \pi/2 \) for the chain with \( N = 40 \) when \( \lambda/\lambda_0 = 0.5 \). In this case, the 25th eigenstate is the edge-localized state even in the nonlocal region with gap-located eigenenergy. We choose it as the control target to prepare by Lyapunov control method.
where we have set the designed control excitations in the waveguides on-site and hopping potentials along the propagation of the guide systems, the parameters can be tuned by changing the

When this model is implemented in coupled optical wave-trapped in quasiperiodic optical lattice systems exhibit the control performance in the same parameters, i.e., the identical

In detail, we show the evolution for the excitations on the chain during the Lyapunov control processes in figure 12. Compared to the ergodic oscillation without control when the initial excitation locates on one site on the chain in figure 9(a), the state is steered to the goal state in both control procedures.

Even the fidelity can reach high values in both control processes, however the oscillations of the control fields \( f_A \) and \( f_B \) in figure 11 increase the complexity in operation. Thus we may adopt optimal strategies to optimize the control fields to reduce this complexity. In Lyapunov control, the control fields are designed to make the state converge to the objective state. So we redesign the control fields to optimize the control processes. And compared to the amplitudes of control fields, the sign of them ensure the state evolves monotonously to the control target. Thus we adjust the amplitudes of the control fields but maintain the sign of them. Here we should confirm that the control fields are redesigned but simply change the amplitudes specified by the Lyapunov control method. Since the state approaches the target, the amplitudes of the control fields become small. The alternating frequency of the sign for the control fields would be sensitive to the amplitudes of them. However there are other factors influence the control performance, for example, the form of the control Hamiltonians and the amplitudes and combination of the parameters in the control Hamiltonians. Optimal control strategies may be employed to improve the control performance, for example, choosing appropriate combinations of \( V_x, K, X \) and \( Z \) in the control Hamiltonians which is beyond the scope of this work.

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Even the fidelity can reach high values in both control processes, however the oscillations of the control fields \( f_A \) and \( f_B \) in figure 11 increase the complexity in operation. Thus we may adopt optimal strategies to optimize the control fields to reduce this complexity. In Lyapunov control, the control fields are designed to make the state converge to the objective state. So we redesign the control fields to optimize the control processes. And compared to the amplitudes of control fields, the sign of them ensure the state evolves monotonously to the control target. Thus we adjust the amplitudes of the control fields but maintain the sign of them. Here we should confirm that the control fields are redesigned but simply change the amplitudes specified by the Lyapunov control method. Since the state approaches the target, the amplitudes of the control fields become small. The alternating frequency of the sign for the control fields would be sensitive to the amplitudes of them. However there are other factors influence the control performance, for example, the form of the control Hamiltonians and the amplitudes and combination of the parameters in the control Hamiltonians. Optimal control strategies may be employed to improve the control performance, for example, choosing appropriate combinations of \( V_x, K, X \) and \( Z \) in the control Hamiltonians which is beyond the scope of this work.
where $\xi$ is a positive constant. So the modulated control field would be

$$f_M(t) = \begin{cases} M(t), & f_o(t) > 0, \\ -M(t), & f_o(t) < 0, \end{cases}$$

(9)

where $f_M(t)$ denote the modulated control fields. $f_o(t)$ is the not-modulated control field designed by Lyapunov control method at time $t$. In this situation, the sign of $f_o(t)$ and the modulated ones $f_M(t)$ are identical to make the state converge to the target monotonously. $f_{AM}$ and $f_{BM}$ will be used to denote the redesigned control fields corresponding to Lyapunov control-A and -B cases. By this modulation, the control procedure mainly becomes controlling the time interval between sequential pulses since the envelope function is given. This reduces the control complexity in practice. From figure 11, we can see that the fidelity can reach similar high values by the optimized control fields at the same terminated time compared to the not-modulated cases.

For generality, we choose 30 different initial states to check the control results in figure 13. The initial states are normalized and projecting to the sites with randomly amplitudes. It can be seen that the fidelity can reach high values by both control Hamiltonians and Lyapunov control-A performs slightly better than Lyapunov control-B in average.

As mentioned above, the localization condition influences the transport properties of this AAH model. Since Lyapunov control based on transportation of the excitations, it is natural to conjecture the Lyapunov control performance would be suppressed in the localized region. Thus we choose the same control parameters and initial state as mentioned above to examine the influence of localization. The control goal state is still the 25th edge eigenstate for the chain with $N = 40$. In the parameter range $\lambda/\lambda_0 \in [0.1, 3.1]$, we show the average control results for Lyapunov control-A and -B processes at time $t = 2000$ in figure 14. It is obvious that in the localized region, namely $\lambda/\lambda_0 > 1$, the control effect have been suppressed in both Lyapunov control-A and -B processes. The suppression may be interpreted as that the transportation of the excitation has been blocked in the localized region. So not only in the edge state pumping processes but also in the Lyapunov control processes, the localization constrain the transportation of the excitations on the chain which results to obviously different performance of the dynamical processes in nonlocal and localized regions.
4. Discussions

4.1. Occupation imbalance and entropy as the localization indicator

Even IPR can be employed to indicate the degree of localization, it may not be a perfect indicator to show various kinds of localization patterns. Thus besides IPR, we also use the quantity called occupation imbalance to reveal the localization transition. It is defined as: 

$$\mathcal{I} = |N_e - N_o|,$$

where $N_e$ and $N_o$ are the summation for the particle density projecting on even and odd lattice sites. This quantity is similar to the one employed in experiments [13, 14] to measure ergodicity for a quantum system. Since the difference of occupation $N_e - N_o$ tends to vanish for even-distributed occupation patterns but finite for states tend to projecting mainly on even or odd sites. Thus $\mathcal{I}$ reflects the distribution characters for the states in terms of the occupations on even and odd sites. Besides, the entropy defined as $\mathcal{E} = \text{Tr}[\rho \log \rho]$ is also adopted to reflect the distribution characters, where $\rho$ is the density matrix and $\text{Tr}[\star]$ denotes the trace of $\star$. Considering entropy reflects the degree of confusion for the distribution of a state, $\mathcal{E}$ would be small when the normalized distribution tends to flock together. Conversely, it would be large. We show $\mathcal{I}$ and $\mathcal{E}$ as a function of $\lambda/\lambda_0$ for all eigenstates of a chain with $N = 1000$ in figure 15. It can be seen that localization transitions occur in terms of $\mathcal{I}$ and $\mathcal{E}$ which are consistent with that in terms of IPR. A feature reflected in figure 2 is that with increasing of $\lambda/\lambda_0$ ($\nu/\lambda_0$ is fixed), the degree of localization for the eigenstates tends to decrease. Correspondingly, the occupation imbalance between even and odd sites tends to decrease and the degree of confusion tends to increase with increasing of $\lambda/\lambda_0$. These behaviors may result from that the hopping interaction has the effect of dispersing the localization.

4.2. Realization of the model by cold atoms trapped in quasiperiodic optical lattice

Even the extended AAH model originally generates from solid physics [3, 4], however the necessary intense magnetic field is unavailable in such systems currently. Fortunately the developing quantum simulation based on cold atoms trapped in optical lattice provides platforms to study such many-body systems [10, 23, 24]. The 1D optical lattice for the extended AAH model can be realized by superimposing an optical lattice with lattice constant $\pi/k_2$ onto a primary one with incommensurate lattice constant $\pi/k_1$. The potentials for such a superimposed lattice can be described by:

$$V(x) = s_1\sin^2(k_1x + \theta') + s_2\sin^2(k_2x + \theta'_c),$$

where $x$ denotes position along the chain, $\theta'$ and $\theta'_c$ are free phases introduced to describe the shift between the lattices, $s_1$ and $s_2$ ($s_1 \gg s_2$) are the amplitudes in units of the recoil energy $E_r = k_1^2/(2m)$, $m$ is the effective mass for atoms. In this case, the Hamiltonian reads:

$$H = \frac{-\nabla_x^2}{2m} + V(x).$$

After second quantization [23, 46] in terms of Wannier functions, the on-site incommensurate modulation in Hamiltonian (1) is obtained when $\nu \sim s_2$, $\theta_e = \nu \pi/k_1$, and $b = k_2/k_1$. Here we focus on the refinement for the hopping modulations to some extent. We expect the hopping modulation is amended by the displacement of the extremum for the lattice potential $V(x)$ in (10) around the sites: $x_n = n\pi/k_1$ where $n$ denote the positions for the sites. Since tunneling between sites depends on distance minus-exponentially, such a relation can be assumed as: $J_{n,n+1} \approx e^{-(\xi_1 - x_n)}$, where $\xi = b_1/(2\sqrt{V})$ when $s_1 \gg E_r$ [23]. Then the deviation of the position for extremum of $V(x)$ in (10), to the first order in $s_2/s_1$ reads:

$$\delta x_n = -\frac{s_2b\sin(2n\pi b) + 2\theta'_c}{2\cos(2\theta')s_1 k_1} - \frac{\tan(2\theta')}{{k_1\cos(2\theta')}},$$

In this case, one gains:

$$x_{n+1} - x_n = \frac{\pi}{k_1} - \frac{s_2b\sin(\nu \pi b)}{2\cos(2\theta')s_1 k_1}\cos(2n\pi b + \theta_e),$$

where $\theta_e = \nu \pi/k_1$. Thus the hopping terms in the lowest order of $s_2/s_1$ are

$$J_{n,n+1} \approx 1 + \frac{s_2b\sin(\nu \pi b)}{2\cos(2\theta')s_1 E_r}\cos(2n\pi b + \theta_e).$$

This coincides with the incommensurate modulations of hopping in Hamiltonian (1) when $\lambda \sim s_2b\sin(\nu \pi b)/(2\cos(2\theta')s_1 E_r)$. Thus the extended AAH model can be implemented by cold atoms trapped in quasiperiodic optical lattice.
4.3. Realization of the model by coupled optical waveguide array system

Besides the cold atom system in optical lattice, this extended AAH model can also be simulated in coupled optical waveguide arrays [15, 16, 25–30, 47]. The dynamical processes are mapped to the propagation of the probe light along the waveguides. Such arrays of waveguides can be manufactured on optical bulk materials by using femtosecond laser pulses [25, 26] or by applying high resolution, large field electron-beam lithography combine with reactive ion etching technique on AlGaAs substrate [15]. The hopping can be tuned by varying the spacing between the waveguides which determines $\lambda$, $\theta$ in Hamiltonian (1). The on-site potentials can be modulated by changing the widths of the waveguides which determines $\nu$ and $\theta_i$ [15, 25–27]. In this apparatus, the fluorescence microscopy technique can be employed to observe the light intensity in the waveguides and the distribution for the wave functions can be examined by measuring the intensity of differential light at the output interface. Disorder exists but small and can be factored out since the localization length associated with the disorder is much larger than the spacing between the waveguides and widths of them [15]. Since the intrinsic loss of the waveguides can be turned weak and identical for all sites, it can also be factored out in this model.

The above discussions about transport properties are in linear scope. However when one utilizes optical waveguide systems, the nonlinearity is usually considered when the intensity of the probe light injected into the system is large enough. The influence of nonlinearity on some dynamical processes have been explored theoretically and experimentally [27–30, 47]. The Kerr-type nonlinearity is usually considered. The transportation of the excitation versus localization may change in the presence of nonlinearity. The dynamical equations in the presence of Kerr-type nonlinearity can be obtained by adding the term:

$$\kappa |\psi_n|^2 \psi_n$$ (15)

to right hand side of equations (2) with time-invariant $\theta$ and $\theta_i$. Here $\kappa$ is the Kerr nonlinear coefficient indicating the strength of the third-order nonlinearity [27–30, 47]. It can be interpreted that the on-site potentials add terms proportional to $\kappa |\psi_n|^2$ in the presence of this kind of nonlinearity. We use the performance of transportation for excitations on the chain for a range of $\kappa/\lambda_0$ to check the influence on localization. Thus we show the average IPR at time $t = 100$ over the cases that the initial excitations locate only on one site in each simulation in figure 16. Namely, each site on the chain acts as a position for each initial excitation in the simulations. This is a statistical and dynamical viewpoint to check the localization property in the presence of nonlinearity. From figure 16, it can be seen that the localization condition have changed obviously in the presence of Kerr-type nonlinearity. For example, in the original nonlocal region ($\lambda/\lambda_0 \lesssim 1$), the Kerr-type nonlinearity leads to localized behavior for the excitations. However, with increasing of $\kappa$, higher-order nonlinearity need to be considered. The nonlinear terms $\kappa |\psi_n|^2 \psi_n$ break down to describe the dynamics [48]. Since the sensitive dependence on initial conditions in nonlinear dynamical processes, more rigorous and general influence of nonlinearity on the dynamical process may be needed to study the elusive relation between nonlinearity and localization.

5. Conclusion

In conclusion, we have studied the influence of localization transition for an extended AAH model on adiabatic pumping for an edge state and preparing the edge state by Lyapunov control method. The on-site and hopping potentials are both modulated incommensurately in this model. For a specified model, not only the ground state but also most of the excite eigenstates exhibit localization transition. And the change of the energy spectrum splitting coincides with the localization transition. The influence of localization transition on dynamical processes are explored by two examples: edge state adiabatic pumping and preparing the edge states by Lyapunov control method. But the adiabatic pumping fails and Lyapunov control effect are suppressed in the localized region. These can be interpreted as that in the nonlocal region, the extended AAH chain acts as conductor for the excitation but insulator in the localized region. Besides IPR, the occupation imbalance between even and odd sites and the entropy for the eigenstates can reveal the localization transition and coincide with IPR reveals. Finally, we propose that such a model can be implemented by cold atoms trapped in quasiperiodic optical lattice. Besides, the system composed of coupled optical waveguides can also be used to check the discussions. In the optical waveguide system, we show the influence of Kerr-type nonlinearity on the localization transition in the dynamical viewpoint by means of statistical method.
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Appendix

The edge states traversing the bulk gaps is a signature for nontrivial topological phase [49]. The topological indexes which reflect the bulk property are usually defined in the momentum space in the periodic boundary condition. However the periodic boundary condition is broken since the periodic parameter is irrational number $(\sqrt{5} - 1)/2$ in Hamiltonian (1). However the edge states as a function of $\theta$ ($\theta_i$) behave like the edge states in general topological systems [49]. One can see that after the crossing or immersing into the bands, the GI $(N \times b)$-states would locate in the opposite ends of the chain. For example, in figures A1(a) and (b) the edge states in the panels locates at opposite ends of the chains but in (c) they locate at the same end. This is because the crossing of the edge states or immersing of them divides the system into different phases. This is the hints of nontrivial topological property and chiral signature which can be seen by comparing distributions of the edge states in figures 6 and 7. Similar but different discussions about topological properties can be found in [16]. The behavior of the edge states in this model is related to lengths of the chains which can be confirmed by figure A1. When $b$ is a rational number, the behavior of the edge states in the real space is independent of length. Thus this behavior results from that the irrational periodic parameter $(\sqrt{5} - 1)/2$ breaks the space translation symmetry.

In this work, we focus on the localization transition and its influence on adiabatic pumping and Lyapunov control processes. The crossover of localization transition and topological phase transition is intriguing and needs further investigations.

ORCID iDs

X L Zhao https://orcid.org/0000-0003-3738-2957

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