Notes on the Kinematic Structure of the Three–Flavor Neutrino Oscillation Framework

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Abstract

These notes present a critique of the standard three–flavor neutrino oscillation framework. The design proposal of the MINOS at Fermilab based on a two mass eigenstate framework may require serious reconsideration if there is strong mixing between all three flavors of neutrinos. For the LSND and KARMEN neutrino oscillation experiments, the amplitude of neutrino oscillation of the “one mass scale dominance” framework vanishes for certain values of mixing angles as a result of opposite signs of two equal and opposite contributions. Recent astronomical observations leave open the possibility that one of the neutrino mass eigenstates may be non–relativistic in some instances. Neutrino oscillation phenomenology with a superposition of two relativistic, and one non–relativistic, mass eigenstates is constructed. It is concluded that if the transition from the non-relativistic to the relativistic regime happens for energies relevant to the Reactor and the LSND neutrino oscillation experiments then one must consider an ab initio analysis of the existing data.

1 Introduction

There are at least two aspects to neutrino oscillation phenomenology. The first is the kinematic aspect where neutrino oscillations are modeled in terms of mass squared differences associated with the underlying neutrino mass eigenstates and mixing angles. The exact form of the standard kinematic analysis depends on whether the weak neutrino eigenstates are of Dirac type, of Majorana type, or of the type recently proposed in Ref. [1], and further on whether or not one allows for CP violation in the neutrino sector. The second aspect is dynamical in nature. This includes the well–known MSW modifications to neutrino oscillations [2], the gravitationally induced neutrino oscillation phases [3], and neutrino spin flips in strong gravitational and magnetic fields [4].

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In this paper I shall confine my attention to the kinematical aspect of the phenomenon. The neutrino–oscillation parameters determined by kinematic analysis may then be used to determine which dynamical aspects may be important. It must be noted that even the kinematical aspects of neutrino phenomenology \[5\]-\[10\] are only now beginning to be understood properly \[8\]-\[10\].

As soon as the preprint from Kamioka reporting the zenith-angle dependence of the atmospheric neutrino anomaly \[11\] arrived, it became clear that the phenomenon of neutrino oscillations could no longer be studied in terms of the two–flavor analysis and that a great amount of confusion had arisen when the intuition and arguments from the two–flavor studies were carried over, without due reflection, to a realistic three–flavor analysis. Similar observations were made by Learned, Pakvasa, and Weiler \[12\] (and few others) when they first learned of the Kamiokande collaboration’s initial results on the depletion in the flux of atmospheric muon neutrinos relative to the flux of atmospheric electron neutrinos. Due to conceptual and experimental complexities associated with the neutrino–oscillation physics there are, at present, a large number of papers devoted to the three–flavor analysis. Still, some fundamental aspects of the three–flavor neutrino–oscillation analysis remain to be discussed. Some of the remarks that I present are seemingly trivial, but in view of their possible relevance, I take the liberty of presenting them in this paper in the hope of establishing a better conceptual understanding.

In the next section I briefly review the standard three–flavor neutrino oscillation framework. Section 3 is devoted to MINOS at Fermilab and other long–baseline experiments. Section 4 concentrates on the LSND neutrino oscillation experiment and KARMEN. Section 5 is confined to a brief discussion of the Reactor \(\nu_e\) experiments. Section 6 presents modification to neutrino–oscillation phenomenology with a superposition of relativistic and non-relativistic mass eigenstates. Section 7 presents the conclusion. In addition to these seven sections, the paper contains three appendices. Appendix A provides further details on standard neutrino oscillation phenomenology. Appendix B contains a simple theorem on the inverted mass hierarchy. Appendix C presents a pedagogic discussion of neutrino oscillations and energy conservation.

2 Three–Flavor Neutrino Oscillations: A Brief Review

Because of its simplicity, the two–flavor neutrino oscillation framework has dominated the design and understanding of neutrino oscillation experiments. However, as the observational and experimental indications for actual neutrino

\[2\] Section 6 is essentially based on a recent work of the author with Goldman \[13\].
oscillations mount, newer and conceptually more sophisticated neutrino oscillation experiments must be considered. In these experiments, in my opinion, the insights gained from the two–flavor neutrino oscillation frameworks may prove to be inadequate. It is, therefore, important to have a fresh ab initio look at the three–flavor neutrino oscillations. I begin with a brief review of the three–flavor neutrino oscillation scenario and soon find myself embedded in insights and results that would be either counterintuitive from a two–flavor neutrino oscillation point of view, or entirely absent unless the three–flavor neutrino oscillations are invoked. Many of these results arise from a larger number of relative phases — relative phases that can lead, for example, to important cancellations.

Assuming, (a) neutrinos to be Dirac particles, that (b) no CP violation occurs in the neutrino sector, that (c) both the flavor eigenstates and the mass eigenstates are relativistic in the laboratory frame, and that (d) all three mass eigenstates are stable, the kinematically induced neutrino oscillation probability, in a three–flavor neutrino oscillation scenario, reads: 

\[ P_{\ell\ell}' (L, \{\eta_k\}) = \int_{E_{\text{min}}}^{E_{\text{max}}} dE f_\ell(E) P_{\ell\ell} (E, L, \{\eta_k\}) , \] (1)

where

\[ P_{\ell\ell} (E, L, \{\eta_k\}) = \delta_{\ell\ell} - 4 U_{\ell 1} U_{\ell 1}^* U_{\ell 2} U_{\ell 2}^* \sin^2 \left( \frac{\varphi_{0}}{2} \right) - 4 U_{\ell 1} U_{\ell 1}^* U_{\ell 3} U_{\ell 3}^* \sin^2 \left( \frac{\varphi_{0}}{2} \right) - 4 U_{\ell 2} U_{\ell 2}^* U_{\ell 3} U_{\ell 3}^* \sin^2 \left( \frac{\varphi_{0}}{2} \right) . \] (2)

The kinematic phase that appears in the above equation is defined as

\[ \varphi_{0} = \frac{2 \pi}{\lambda_{\text{osc}}} \frac{L}{E} . \] (3)

Equations (1) and (2) require several additional comments and observations. These remarks are enumerated in Appendix A. However, the following definitions need to be noted immediately:

(i) The oscillations length, \( \lambda_{\text{osc}}^\nu \), is defined as

\[ \lambda_{\text{osc}}^\nu = \frac{2 \pi}{\alpha} \frac{E}{\Delta m_{\nu}^2} . \] (4)

The kinematic phase may also be written as: \( \varphi_{0} = 1.27 \Delta m_{\nu}^2 \times (L/E) \). These kinematic phases may be modified for dynamical reasons. Here, \( \alpha = \beta/2; \beta = 2.54 \) is the usual factor that arises from expressing

When no ambiguity is likely to arise I write \( P_{\ell\ell} (L, \{\eta_k\}) = P (\nu_\ell \to \nu_\ell) \).
$E$ in MeV, $L$ in meters, and $\Delta m^2_{jk}$ in eV$^2$. $E$ refers to neutrino kinetic energy, $\sqrt{p^2 + m^2}$, and $L$ is the distance between the creation region and the detection region for the neutrino oscillation event. The five neutrino oscillation parameters $\{\eta_k\}$ in equation (1) are the two mass squared differences and the three mixing angles: $\eta_1 = \Delta m^2_{21}$, $\eta_2 = \Delta m^2_{32}$, $\eta_3 = \theta$, $\eta_4 = \beta$, and $\eta_5 = \psi$. The third mass squared difference is then given by $\Delta m^2_{31} = \Delta m^2_{21} + \Delta m^2_{32}$.

(ii) $f_\ell(E)$ is the neutrino flux, of flavor $\ell$, at energy $E$. The $f_\ell(E)$ is normalized to unity:

$$\int_{E_{\text{min}}}^{E_{\text{max}}} dE \ f_\ell(E) = 1.$$  \hfill (5)

$E_{\text{min}}$ and $E_{\text{max}}$ refer, respectively, to the minimum and maximum energy relevant to the neutrino beam and the neutrino detector.

(iii) An element of the $3 \times 3$ unitary neutrino–mixing–matrix, $U(\theta, \beta, \psi)$, is labeled as $U_{\ell j}$ with $\ell$ standing for any of the three neutrino flavors, $\ell = e, \mu, \tau$; and $j$ representing any of the three mass eigenstates, $j = 1, 2, 3$. The explicit expression for the mixing matrix that I use is (Maiani representation [14], with CP phase $\delta$ set equal to zero)

$$U(\theta, \beta, \psi) = \begin{pmatrix} c_\theta c_\beta & s_\theta c_\beta & s_\beta \\ -c_\theta s_\beta s_\psi - s_\theta c_\psi & c_\theta c_\phi - s_\theta s_\beta s_\phi & c_\beta s_\psi \\ -c_\theta s_\beta c_\phi + s_\theta s_\psi & -s_\theta s_\beta c_\phi - c_\theta s_\phi & c_\beta c_\psi \end{pmatrix}, \hfill (6)$$

where $c_\theta = \cos(\theta)$, $s_\theta = \sin(\theta)$, etc.

3 The $\nu_\mu \to \nu_\tau$ MINOS at Fermilab, and Other Long–Baseline Experiments

For the sake of concreteness let us choose a generic long–baseline $\nu_\mu \to \nu_\tau$ experiment and consider the mass hierarchy such that

$$\lambda_{3(2, 1)}^{\text{osc}} \ll L, \quad \lambda_{21}^{\text{osc}} \sim L.$$  \hfill (7)

\footnote{Or perhaps more precisely, $E$ represents the expectation value of the kinematic Hamiltonian in the appropriate neutrino–flavor eigenstate.}
Using Eq. (1) and averaging $\sin^2(\cdot \cdot)$ associated with the $\lambda_3^{osc}(2,1)$ terms to 0.5 I obtain

$$\mathcal{P}(\nu_\mu \rightarrow \nu_\tau) = A_C(\beta, \psi) + A_O(\theta, \beta, \psi) \int_{E_{\text{min}}}^{E_{\text{max}}} dE \, f_{\nu_\mu}(E) \sin^2 \left( \frac{C_{21}}{E} \right), \quad (8)$$

where the $\theta$–independent constant contribution, $A_C(\beta, \psi)$, and the amplitude, $A_O(\theta, \beta, \psi)$, of the oscillatory term, in $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau)$ are:

$$A_C(\beta, \psi) = 2 \cos^4(\beta) \cos^2(\psi) \sin^2(\psi) \quad , \quad (9)$$

$$A_O(\theta, \beta, \psi) = 4 \left[ \cos(\psi) \sin(\theta) + \sin(\beta) \sin(\psi) \cos(\theta) \right] \times \left[ \sin(\psi) \sin(\theta) - \sin(\beta) \cos(\psi) \cos(\theta) \right] \times \left[ \sin(\beta) \cos(\psi) \sin(\theta) + \sin(\psi) \cos(\theta) \right] \times \left[ \sin(\beta) \sin(\psi) \sin(\theta) - \cos(\psi) \cos(\theta) \right]. \quad (10)$$

Oftentimes one attempts to understand this experiment, and similar experiments, within the framework of a two–flavor neutrino oscillation. Such an analysis can be misleading. In the physically required three–flavor neutrino oscillations I see that there is a constant piece in $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau)$ and an oscillatory term. Depending on the mixing angles the constant piece may make a significant contribution to the $\mathcal{P}(\nu_\mu \rightarrow \nu_\tau)$.

The counterpart of Eq. (8) for the $\nu_e \rightarrow \nu_\tau$ Long–Baseline Experiments reads:

$$\mathcal{P}(\nu_e \rightarrow \nu_\tau) = B_C(\beta, \psi) + B_O(\theta, \beta, \psi) \int_{E_{\text{min}}}^{E_{\text{max}}} dE \, f_{\nu_e}(E) \sin^2 \left( \frac{C_{21}}{E} \right), \quad (11)$$

where

$$B_C(\beta, \psi) = 2 \cos^2(\beta) \sin^2(\beta) \cos^2(\psi) \quad , \quad (12)$$

$$B_O(\theta, \beta, \psi) = 4 \cos^2(\beta) \cos(\theta) \sin(\theta) \times \left[ \sin(\psi) \cos(\theta) + \sin(\beta) \cos(\psi) \sin(\theta) \right] \times \left[ \sin(\psi) \sin(\theta) - \sin(\beta) \cos(\psi) \cos(\theta) \right]. \quad (13)$$

Similarly, the counterpart of Eq. (8) for the $\nu_e \rightarrow \nu_\mu$ Long–Baseline Experiments reads:

$$\mathcal{P}(\nu_e \rightarrow \nu_\mu) = C_C(\beta, \psi) + C_O(\theta, \beta, \psi) \int_{E_{\text{min}}}^{E_{\text{max}}} dE \, f_{\nu_e}(E) \sin^2 \left( \frac{C_{21}}{E} \right), \quad (14)$$

where
\[
C_c(\beta, \psi) = 2 \cos^2(\beta) \sin^2(\beta) \sin^2(\psi), \quad (15)
\]
\[
C_o(\theta, \beta, \psi) = 4 \cos^2(\beta) \cos(\theta) \sin(\theta) 
\times [\cos(\psi) \sin(\theta) + \sin(\beta) \sin(\psi) \cos(\theta)] 
\times [\cos(\psi) \cos(\theta) - \sin(\beta) \sin(\psi) \sin(\theta)]. \quad (16)
\]

It may be worthwhile to observe how, with respect to the mixing angle \(\psi\), a “large” (“small”) \(B_c(\beta, \psi)\) will complement a “small” (“large”) \(C_c(\beta, \psi)\). In particular, by measuring (in the same beam) the constant pieces in \(P(\nu_e \to \nu_\tau)\) and \(P(\nu_e \to \nu_\mu)\) I can directly measure the mixing angles \(\beta\) and \(\psi\):

\[
\psi = \tan^{-1} \left( \frac{C_c(\beta, \psi)}{B_c(\beta, \psi)} \right), \quad (17)
\]

\[
\beta = \frac{1}{2} \sin^{-1} \left( \frac{\sqrt{2} B_c(\beta, \psi)}{\cos(\psi)} \right) = \frac{1}{2} \sin^{-1} \left( \frac{\sqrt{2} C_c(\beta, \psi)}{\sin(\psi)} \right). \quad (18)
\]

In Eq. (18) the angle \(\psi\) is to be substituted from Eq. (17).

Hence, the angle \(\theta\), the remaining of the three mixing angles, can be obtained by measuring the amplitude of the oscillatory term in \(P(\nu_\mu \to \nu_\mu)\), or equivalently any one of the \(A_o(\theta, \beta, \psi), B_o(\theta, \beta, \psi), C_o(\theta, \beta, \psi)\). Within the scenario represented by Eq. (7), the three probabilities (see comments below that require an additional measurement) for \(\nu_\mu \to \nu_\tau, \nu_e \to \nu_\tau, \text{and } \nu_e \to \nu_\mu\), measure four parameters — the three mixing angles \((\theta, \beta, \psi)\) and one of the two independent mass square differences \(\Delta m^2_{21}\). The long–baseline experiments, by their very nature (within the scenario under consideration), determine only \(\Delta m^2_{21}\). The \(\Delta m^2_{32}\) must then be determined either in a reactor experiment or an experiment like the LSND neutrino oscillation experiment.

In principle, the three \(P(\nu_\ell \to \nu_\ell)\) considered above can be measured at a very similar \(f_{\nu_\ell}(E)\). However, an additional measurement of one of the three \(P(\nu_\ell \to \nu_\ell)\) must be done at a sufficiently different \(f_{\nu_\ell}(E)\). This becomes obvious if I note that there are nine oscillation probabilities,

\[
P(\nu_e \to \nu_e), \quad P(\nu_e \to \nu_\mu), \quad P(\nu_e \to \nu_\tau),
\]

\[
P(\nu_\mu \to \nu_\mu), \quad P(\nu_\mu \to \nu_\mu), \quad P(\nu_\mu \to \nu_\tau),\quad (19)
\]

\[
P(\nu_\tau \to \nu_\mu), \quad P(\nu_\tau \to \nu_\mu), \quad P(\nu_\tau \to \nu_\tau),
\]

that can be measured at a given energy \(E\). Because of the three unitarity–imposed conditions,

\[
P(\nu_\ell \to \nu_\ell) + P(\nu_\ell \to \nu_\mu) + P(\nu_\ell \to \nu_\tau) = 1, \quad \ell = e, \mu, \tau, \quad (20)
\]
and the three conditions

\[
\mathcal{P}(\nu_e \rightarrow \nu_\mu) = \mathcal{P}(\nu_\mu \rightarrow \nu_e), \\
\mathcal{P}(\nu_e \rightarrow \nu_\tau) = \mathcal{P}(\nu_\tau \rightarrow \nu_e), \\
\mathcal{P}(\nu_\mu \rightarrow \nu_\tau) = \mathcal{P}(\nu_\tau \rightarrow \nu_\mu),
\]

arising from the tentative assumption that there is no CP violation in the neutrino sector, only three of the \(\mathcal{P}(\nu_\ell \rightarrow \nu_\ell)[f(E)]\) are independent. Therefore, to determine the four parameters \((\theta, \beta, \psi, \Delta m^2_{21})\) one needs one more measurement \(\mathcal{P}(\nu_\ell \rightarrow \nu_\ell')[f(E') \neq f(E)]\).

4 Experiments Dedicated to \(\bar{\nu}_\mu \rightarrow \bar{\nu}_e\) in the Appearance Mode: LSND and KARMEN

The published results of the LSND Neutrino Oscillation Experiment (NOE) and Karlsruhe–Rutherford Medium Energy Neutrino Experiment (KARMEN) both present results on \(\bar{\nu}_\mu \rightarrow \bar{\nu}_e\) in the appearance (of \(\nu_e\)) mode \([15,16]\). For LSND NOE \(L \simeq 30\) m and for KARMEN \(L \simeq 17\) m. KARMEN, which is less sensitive than LSND NOE, sees no neutrino–oscillation signal. The latest LSND NOE results, if interpreted within neutrino oscillation framework, yield

\[
\mathcal{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e)|_{LSND} = \left[0.31^{+0.11}_{-0.10} \pm 0.05\right] \times 10^{-2}.
\]

For the \(\mu^+\) decay at rest, the normalized–to–unity Michel energy spectrum of the \(\bar{\nu}_\mu\) is given by

\[
f_{\bar{\nu}_\mu}(y) = - \frac{2y_3^2}{y^4} \left(3 - \frac{2y_2}{y}\right), \tag{22}
\]

where \(y \equiv 1/E\), and

\[
y_2 = \frac{1}{E_{max}}; \quad E_{max} = 52.8\) MeV . \tag{23}
\]

Substituting Eq. (22) into Eq. (1) I obtain:

\[
\mathcal{P}(\bar{\nu}_\mu \rightarrow \bar{\nu}_e) = -4 U_{e1} U_{\mu1} U_{e2} U_{\mu2} \times \mathcal{O}(L, \Delta m^2_{21}) -4 U_{e1} U_{\mu1} U_{e3} U_{\mu3} \times \mathcal{O}(L, \Delta m^2_{31}) -4 U_{e2} U_{\mu2} U_{e3} U_{\mu3} \times \mathcal{O}(L, \Delta m^2_{32}). \tag{24}
\]

\[5\] In Eqs. (14) to (21) all \(\mathcal{P}(\nu_\ell \rightarrow \nu_\ell')\) refer to same energy spectrum \(f(E)\).
In the above expression I have introduced a new function $\mathcal{O}(L, \Delta m^2_{\nu})$. It is defined as:

$$\mathcal{O}(L, \Delta m^2_{\nu}) = 2y^3 \left[ \left( \frac{C_1}{3y^2} + \frac{C_2}{3} \right) \sin (2y C_3) + \left( \frac{5C_2^5}{6y^2} - \frac{1}{4y^2} \right) \cos (2y C_3) \right. \left. + 2C_3 \sin (2y C_3) - \frac{2y C_4^3}{3} \cos (2y C_3) + \frac{1}{4y^2} - \pi C_5^3 \right], \quad (25)$$

where $C_3 = \alpha L \Delta m^2_{\nu}$ and

$$\text{Si}(x) \equiv \int_0^x \frac{\sin(t)}{t} dt, \quad \text{Ci}(x) \equiv \int_0^x \frac{\cos(t)}{t} dt. \quad (26)$$

Given the mixing angles and detector efficiencies (that can depend not only on $E$ but also on the location of an event within the detector), the function $\mathcal{O}(L, \Delta m^2)$ is a measure of the neutrino oscillation probability. I shall call $\mathcal{O}(L, \Delta m^2)$ the probability function. The probability function, corresponding to LSND’s $L \simeq 30$ m, is graphed in Fig. 1.

As it should, $\mathcal{O}(L, \Delta m^2)$ approaches 0.5 as $\Delta m^2 \rightarrow \infty$.

In an experiment using the neutrinos from the decay at rest of $\mu^+$ (and in the absence of any directed magnetic fields), the event density within the detector is measured by the raw event density function, $\mathcal{E}(L, \Delta m^2)$,

$$\mathcal{E}(L, \Delta m^2) \equiv \frac{\epsilon(L)}{L^2} \times \mathcal{P}(\varphi_\mu \rightarrow \varphi_\nu) \quad ,$$

$$\equiv \epsilon(L) \mathcal{E}'(L, \Delta m^2) \quad , \quad (27)$$

where $\Delta m^2 = \{\Delta m^2_{31}, \Delta m^2_{32}\}$ and $\epsilon(L)$ is the energy-integrated average efficiency at $L$ inside the detector, and the definition of $\mathcal{E}'(L, \Delta m^2)$, the event density function, is obvious from Eq. (27). In actual experimental situations, such as LSND’s NOE and KARMEN, the gain in event rate with decreasing $L$ is inevitably shadowed by an increased background [17]. By measuring the $L$–dependence of the event density function within a detector one may rule out one set of solutions over the other.

To explore the physical content of Eq. (24) for $\mathcal{P}(\varphi_\mu \rightarrow \varphi_\nu)$ let us consider an often–invoked scenario in which mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ are almost degenerate [18]. Setting $\Delta m^2_{32} \simeq \Delta m^2_{31} \equiv \Delta m^2_{3(2,1)}$, and using Eq. (1), Eq. (24) becomes:

$$\mathcal{P}(\varphi_\mu \rightarrow \varphi_\nu) = A_{21} \mathcal{O}(L, \Delta m^2_{21}) + A_{3(2,1)} \mathcal{O}(L, \Delta m^2_{3(2,1)}) \quad , \quad (28)$$

with the two oscillation amplitudes defined as
\[ A_{21} = 4 \cos^2(\beta) \cos(\theta) \sin(\theta) \left[ \cos(\psi) \sin(\theta) + \sin(\beta) \sin(\psi) \cos(\theta) \right] \]
\[ \times \left[ \cos(\psi) \cos(\theta) - \sin(\beta) \sin(\psi) \sin(\theta) \right], \]
\[ A_{3(2,1)} = 4 \cos^2(\beta) \sin^2(\beta) \sin^2(\psi). \]

(29)

In the scenario indicated above one usually assumes \( m_3 \gg m_2 \simeq m_1 \). For \( \Delta m^2_{21} \ll \Delta m^2_{3(2,1)} \), and based on the \( \Delta m^2 \)-dependence of \( \sin^2(y C_{\mu}) \), one neglects the term associated with \( \Delta m^2_{21} \) and assumes that the dominant contribution to \( P(\nu_{\mu} \rightarrow \nu_e) \) comes from the \( \Delta m^2_{3(2,1)} \) term. This is the basic scenario that the one–mass scale dominance framework considers.

I now make two observations in this context.

**Observation I** For \( \beta = n \pi/2 \) and/or \( \psi = n \pi, n = 0, 1, 2, \ldots \), the oscillation amplitude \( A_{3(2,1)} \) associated with the \( \mathcal{O}(L, \Delta m^2_{3(2,1)}) \) term, the dominant term of the so called “one mass–scale dominance” identically vanishes.

**Observation II** Refer to Fig. 1, and consider a set of mass–squared differences such that \( \Delta m^2_{21} \) lies roughly between 1 and 2 eV\(^2\) and \( \Delta m^2_{3(2,1)} \) is greater than 2 eV\(^2\). Then

\[ \mathcal{O} \left( L, \Delta m^2_{21} \right) \geq \mathcal{O} \left( L, \Delta m^2_{3(2,1)} \right). \]

(30)

For the above configuration of mass–squared differences, Eq. (30) violates one of the assumptions of the “one mass–scale dominance.” That is, the \( \geq \) in the above expression is in contradiction to the expected \( \ll \) of the the one mass–scale dominance analysis.

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\(^6\) For “and” in “and/or” above the three–flavor analysis reduces to a two–flavor framework.

\(^7\) Note should be taken that specific values of \( \Delta m^2 \) considered here refer to \( L = 30 \) meters for LSND NOE. In addition, as the detector has a width of about 8 meters centered at \( L = 30 \) meters, depending on the oscillation length under consideration, appropriate \( L \)-integration of Eq. (30) may need to be considered. Similar considerations apply to KARMEN.
Observations I and II suggest that the $\mathcal{O}(L, \Delta m^2_{21})$ term may make a dominant contribution to $\mathcal{P}(\overline{\nu}_\mu \to \overline{\nu}_e)$ for certain values of $\beta$, $\psi$, and $\Delta m^2_{21}$.

For the indicated–example configuration of mass–squared differences, the canonical wisdom of the one–mass scale dominance fails, in part, because $\mathcal{P}(\overline{\nu}_\mu \to \overline{\nu}_e)$ evaluated at

$$E_{\text{average}} \equiv \int_{\infty}^{y_2} dy \frac{f_{\overline{\nu}_\mu}(y)}{y} ,$$

(31)

does not equal Michel–spectrum averaged $\mathcal{P}(\overline{\nu}_\mu \to \overline{\nu}_e)$. This is particularly true for low $\Delta m^2$, that is, before the $\Delta m^2$–region where $\mathcal{O}(L, \Delta m^2)$ reaches its ($\Delta m^2 \to \infty$)–value of one half [which in turn coincides with the average value of $\sin^2(\cdots)$].

The above discussion should not be interpreted to mean that the existing neutrino–oscillation data necessarily imply the specific values of parameters for which the one–mass scale dominance requires revision. It remains possible that actual neutrino–oscillation parameters satisfy the canonical one–mass scale analysis. In fact, the LSND NOE relevant $\Delta m^2$ of Ref. [18] lies within the applicability of one–mass scale dominance analysis as will be shown elsewhere.
However, it is important to define the unexpected boundaries where the one-mass scale dominance analysis can, and does, break down.

Figure 2. The functions $O'(L, \Delta m^2_{\nu}, E_{\text{min}})$ for LSND NOE.

For various reasons, such as detector efficiency, neutrino background, etc. [15,16], one may wish to introduce an energy cutoff in $E_\nu$. For LSND’s decay–at–rest data this energy cutoff corresponds to $E_{\text{min}} \simeq 20 \text{ MeV}$. Under such circumstances, assuming that there is no cut on $E_{\text{max}}$ except that dictated by the Michel spectrum itself, the $O(L, \Delta m^2_{\nu})$ in the above discussion should be replaced by

$$O'(L, \Delta m^2_{\nu}, E_{\text{min}}) = \left( \frac{1}{4 y_2} - \frac{4 y_2^2 - 12 y_1 y_2 + 9 y_1^2}{4 y_1^2 y_2} \right)^{-1} \left[ \frac{3}{2 y_1} - \frac{y_2}{2 y_1^2} - \frac{1}{y_2} + C_{\nu} \sin (2 y_2 C_{\nu}) - \frac{C_{\nu} y_2}{y_1} \sin (2 y_1 C_{\nu}) + \frac{1}{y_2} \cos (2 y_2 C_{\nu}) + \frac{y_2}{2 y_1^2} \cos (2 y_1 C_{\nu}) - \frac{3}{2 y_1} \cos (2 y_1 C_{\nu}) + 3 C_{\nu} \sin (2 y_2 C_{\nu}) - 3 C_{\nu} \sin (2 y_1 C_{\nu}) + 2 y_2 C_{\nu}^2 \cos (2 y_1 C_{\nu}) - 2 y_2 C_{\nu}^2 \cos (2 y_2 C_{\nu}) \right], \quad (32)$$
where $y_1 = 1/E_{\text{min}}$. The new function $\mathcal{O}'(L, \Delta m^2_{\nu}, E_{\text{min}})$ is plotted in Fig. 2 for LSND. The differences and similarities between the functions $\mathcal{O}(L, \Delta m^2_{\nu}, E_{\text{min}})$ and $\mathcal{O}'(L, \Delta m^2_{\nu})$ are apparent in Fig. 2.

5 Experiments Dedicated to $\nu_e \rightarrow \nu_e$ in the Disappearance Mode: Reactor $\nu_e$ Experiments

These experiments are performed with reactor $\nu_e$, which have a typical energy of about 5 MeV. For this section I shall assume $E = 5$ MeV. In extracting the five neutrino–oscillations parameters from the existing data we must use the exact known spectra and detector efficiencies.

At the present time there exist at least four results on $\mathcal{P}(\nu_e \rightarrow \nu_e)$ from reactor experiments [19–22]. These experiments work in the $\nu_e$ disappearance mode. The core and the reactor sizes place a lower limit on the distance between the core and the detector. For the latest experiment from Savannah River we have $L$ of about 18 m, for one position of the detector, and roughly 24 m for another. The overall best limit on $\mathcal{P}(\nu_e \rightarrow \nu_e)$ comes from the Bugey experiment [21]. Results from all reactor experiments are consistent with the null–oscillation hypothesis. The oscillation hypothesis, within the framework of a two–flavor neutrino oscillation, for the Savannah River experiment [22] yields $\Delta m^2 = 3.84 \text{ eV}^2$. The existing cosmological arguments suggest that $\sum m_i \leq 30 \text{ eV}$. Thus, the range $0 \text{ eV}^2 \leq \Delta m^2_{\nu} \leq 10^3 \text{ eV}^2$ holds particular interest.

There are two independent $\Delta m^2_{\nu}$. Let these be $\Delta m^2_{21}$ and $\Delta m^2_{32}$. Then, the third $\Delta m^2_{ij}$ is simply given by: $\Delta m^2_{21} = \Delta m^2_{31} + \Delta m^2_{32}$. Associated with these $\Delta m^2_{\nu}$ are three oscillation lengths (of which only two are independent): $\lambda_{21}^{\text{osc}}$, $\lambda_{32}^{\text{osc}}$, and

$$\lambda_{31}^{\text{osc}} = \frac{\lambda_{21}^{\text{osc}} \lambda_{32}^{\text{osc}}}{\lambda_{21}^{\text{osc}} + \lambda_{32}^{\text{osc}}}.$$  \hspace{1cm} (33)

The initial three independent length scales associated with the masses $m_i$ of the mass eigenstates $|\nu_i\rangle$ reduce to two independent length scales. The physical origin of this fact lies in the assumption that all three mass eigenstates are relativistic.

To gain some understanding of the neutrino–oscillation parameter space $(\Delta m^2_{21}, \Delta m^2_{32}, \theta, \beta, \psi)$ suppose that $\Delta m^2_{21} \simeq \Delta m^2_{32} \simeq 10^3 \text{ eV}^2$. Then, $\lambda_{21}^{\text{osc}} \simeq \lambda_{32}^{\text{osc}} \simeq 3 \text{ cm}$ and a detector placed at $L \gg 3 \text{ cm}$ sees only an overall $\nu_e$ deficit:

$$\mathcal{P}(\nu_e \rightarrow \nu_e) = 1 - 2 \cos^2(\beta) \left( \sin^2(\beta) + \cos^2(\beta) \cos^2(\theta) \sin^2(\theta) \right).$$  \hspace{1cm} (34)
This expression follows by setting all $\sin^2(\cdots)$ in Eq. (11) equal to 0.5, as is appropriate when the corresponding $\lambda^{\text{osc}} \gg$ the relevant $L$, setting $\ell = \ell' = e$, and exploiting certain trigonometric identities.

For this scenario, the reactor experiment is unable to determine any of the $\Delta m^2_{ji}$. It is only sensitive to the two of the three mixing angles. The $\nu_e$ deficit, if $\Delta m^2_{21} \simeq \Delta m^2_{32} \simeq 10^3$ eV$^2$, in reactor experiments is independent of the mixing angle $\psi$. The Savannah River experiment yields (in the “integrated rate test”) $\mathcal{P}(\nu_e \rightarrow \nu_e) = [98.7 \pm 0.6 \, \text{(stat.)} \pm 3.7 \, \text{(syst.)}] \times 10^{-2}$ for “position 1” of the detector and $\mathcal{P}(\nu_e \rightarrow \nu_e) = [105.5 \pm 1.0 \, \text{(stat.)} \pm 3.7 \, \text{(syst.)}] \times 10^{-2}$ for “position 2” of the detector. These rates are in good agreement with the no–oscillation hypothesis.

Since, in the above defined scenario, the condition $L \gg 3$ cm for the Sun is satisfied by thirteen orders of magnitude, and if CP is not violated in the neutrino sector, and a solar neutrino deficit arises solely from a kinematically induced neutrino oscillation, then Eq. (34) is valid for Sun as well. That is, in the above defined scenario, $\mathcal{P}(\nu_e \rightarrow \nu_e)|_{\text{Reactor}} = \mathcal{P}(\nu_e \rightarrow \nu_e)|_{\text{Solar}}$. But since $\mathcal{P}(\nu_e \rightarrow \nu_e)|_{\text{Solar}} \sim 0.5$, the scenario $\Delta m^2_{21} \simeq \Delta m^2_{32} \simeq 10^3$ eV$^2$ is ruled out. So, at least one of the $\Delta m^2_{ji} \ll 10^3$ eV$^2$.

This is a generally known conclusion but, to the best of our knowledge, here it has been reached via an entirely new chain of arguments.

6 Neutrino Oscillations with a Superposition of Relativistic and Non-Relativistic Mass Eigenstates

The existing indications of neutrino oscillations [23,13,11] arise from the data that contains neutrino energies in sub-MeV to GeV range. The neutrino oscillation phenomenology within the standard three–flavor framework contains the fundamental assumption that neutrino mass eigenstates that superimpose to yield the neutrino flavor eigenstates are relativistic. This assumption is supported by the cosmological argument [24] that the sum of the neutrino masses have an upper bound of about 30 eV

$$\sum_{\ell} m(\nu_{\ell}) \leq 30 \text{ eV} \quad . \quad (35)$$

However, very recent astronomical observations [25,27] raise potentially serious questions on the validity of the standard cosmological model. First [25,26], the UC Berkeley’s Extreme Ultraviolet Explorer satellite’s observations of the Coma cluster of galaxies indicates that this cluster of galaxies may contain a submegakelvin cloud of $\sim 10^{13} M_\odot$ baryonic gas. Second [27], the discovery by the German x-ray satellite Rosat in which a sample of 24 Seyfert galaxies con-
tained 12 that were accompanied by a pairs of x-ray sources, almost certain to be high redshift quasars, aligned on either side of the galaxy. These observations and the associated interpretations, if correct, may place severe questions to cosmological models that depend on the ratio of photon to baryonic density in the universe and its size.

Tentatively, therefore, I relax the cosmological constraint (35) completely and explore the resulting consequences for three-flavor neutrino oscillation framework. The latest kinematic limits on neutrino masses are much less severe [28, 29]:

\begin{align}
    m(\nu_\tau) &< 23 \text{ MeV} \\
    m(\nu_\mu) &< 0.17 \text{ MeV} \\
    m(\nu_\tau) &< 10 - 15 \text{ eV}
\end{align}

The existing and the proposed neutrino oscillation experiments involve neutrino energies from a fraction of a MeV, if not less, to several hundred GeV for the upper-energy end of the neutrino beams. It remains possible that for some of the experiments (or a certain sector of an experiment) the mass eigenstates |\nu_3\rangle and |\nu_2\rangle are non-relativistic. Towards the end of understanding such a possible situation I now consider the interplay of non-relativistic and relativistic mass eigenstates in three-flavor neutrino oscillation framework.

Consider a physical situation where we have two relativistic, |\nu_1\rangle and |\nu_2\rangle, and one non-relativistic, |\nu_3\rangle, neutrino mass eigenstates.

At \( t = 0, x = 0 \), assume that a source creates a \( \nu_\ell \)

\[ |\nu_\ell\rangle = U_{\ell 1} |\nu_1\rangle + U_{\ell 2} |\nu_2\rangle + U_{\ell 3} |\nu_3\rangle \]

The spatial envelope, which is assumed to be “sufficiently” narrow, associated with \{ |\nu_1\rangle, |\nu_2\rangle \} evolves towards the detector as \( x \approx t \), while the spatial envelope of |\nu_3\rangle evolves towards the detector as

\[ x \simeq (p/m_3) t = \left( \frac{\sqrt{2m_3(E - m_3)}}{m_3} \right) t \]

Therefore, the \{ |\nu_1\rangle, |\nu_2\rangle \} arrives at the detector at time \( t_I \simeq L \), while the |\nu_3\rangle arrives at the detector at a time \( t_{II} = m_3L/\sqrt{2m_3(E - m_3)} \). For the above considered energies and for a source–detector distance of a few tens (or greater) of meters the detected |\nu_\ell'\rangle has no overlap with |\nu_3\rangle at the registered event at \( t_I \), and similarly the detected |\nu_\ell'\rangle has no overlap with \{ |\nu_1\rangle, |\nu_2\rangle \} at the registered event at \( t_{II} \). With these observations at hand one can easily evaluate the modification to the neutrino oscillation probability [2]. The modified expression reads:
\[ P(E, L, \{\xi_k\}) = (U_{\nu3} U_{\ell 3})^2 + \left[ (U_{\nu1} U_{\ell 1} + U_{\nu2} U_{\ell 2})^2 - 4 U_{\nu1} U_{\ell 1} U_{\nu2} U_{\ell 2} \sin^2 \left( \varphi_{21}^0 \right) \right] \]. \quad (41)

The first term on the rhs of the above equation is the contribution from the non–relativistic mass eigenstate to the \( P(E, L, \{\xi_k\}) \) at the \( \nu_\ell \) event at time \( t_{II} \) while the second term is the contribution from the relativistic mass eigenstates to the \( \nu_\ell \) event at the earlier time \( t_I \). Contained in the above expression is the fundamental “collapse of the wave packet” postulate of the orthodox interpretation of the quantum mechanics. That is, given a single \( \nu_\ell \) emitted at the source, if the event occurs at \( t_I \) no event occurs at \( t_{II} \), and vice versa.

There are several observations that one may make about the result (41). These observations follow.

First:

The neutrino oscillation probability now contains only one length scale, \( \xi_1 = \Delta m^2_{21} \), \( \xi_2 = \theta \), \( \xi_3 = \beta \), and \( \xi_4 = \psi \). However, this loss of length scale is related to a manifestly different expression for neutrino oscillation probabilities.

Second:

If \( m_3 \) is in the range of a fraction of an MeV to a few tens of MeV, one cannot base the analysis of existing neutrino oscillation data in terms of Eq. (2), or Eq. (41), alone. For the zenith–angle dependence of the atmospheric neutrino anomaly data [11] neutrino energies in the GeV range are involved. This meets the requirements under which Eq. (4) is operative. On the other hand, the physical situation for part of the LSND events (energy range between 20 MeV and Michel spectrum cutoff of 52.8 Mev [15]), and the reactor experiments (average \( \overline{\nu}_e \) energy about 5 MeV [19–22]) [11], the physical conditions for Eq. (41) may be satisfied.

Third:

For the solar–neutrino deficit one may speculate that \( m_3 \) may be such that an energy–dependent transition takes place from the non-relativistic regime to relativistic regime, thus accounting for the apparent energy dependence of the solar–neutrino deficit. So, consider a situation where the length scales are such that all \( \sin^2 \left( \varphi_{21}^0 \right) \) average to \( 1/2 \). Then

\[ P_{ee} \text{ [Eq. (2)]} = 1 - 2 U_{e1}^2 U_{e2}^2 - U_{e1}^2 U_{e3}^2 - 2 U_{e2}^2 U_{e3}^2 \],

\[ P_{ee} \text{ [Eq. (41)]} = U_{e1}^4 + U_{e2}^4 + U_{e3}^4 \].
Exploiting unitarity of the mixing matrix I immediately see that above speculation, within the defined context, has no consequence because

\[ \text{Unitarity } \Rightarrow P_{ee} [\text{Eq. (2)}] = P_{ee} [\text{Eq. (11)}] \ . \]

Fourth:

In astrophysical environments, such as for neutrinos observed in the supernova 1987a \cite{30,31}, it may happen that \( m_3 \) is such that the evolution of \(|\nu_3\rangle\) envelope does not escape the astrophysical environment. That is, \( p/m \) associated with the envelope of \(|\nu_3\rangle\) is less than the escape velocity

\[ \text{Non Relativistic: } E < m + \frac{r_g}{2r} \ , \] (42)

where \( r_g \equiv 2GM \) is the gravitational radius of the astrophysical object. Under these circumstances Eq. (11), for a detector at Earth, reduces to:

\[
P_{\ell\ell'} (E, L, \{\xi_k\}) = (U_{\ell 3} U_{\ell' 3})^2 \Theta \left( E - m + \frac{r_g}{2r} \right) 
+ (U_{\ell 1} U_{\ell 1} + U_{\ell 2} U_{\ell 2})^2 - 4U_{\ell 1} U_{\ell 1} U_{\ell 2} U_{\ell 2} \sin^2 (\varphi_{21}^0) \ . \] (43)

In the above expression \( \Theta(\cdot\cdot\cdot) \) is the usual step function, vanishing for its argument less than zero and equal to unity for its argument greater or equal to unity.

Now I rewrite Eq. (11) in a form that makes the deviations of result (11) explicit from the corresponding two–flavor scenario with two relativistic mass eigenstates. This new form of Eq. (11) reads:

\[
P_{\ell\ell'} (E, L, \{\xi_k\}) = \Delta_{\ell\ell'} - A_{\ell\ell'} \sin^2 (\varphi_{21}^0) \ , \] (44)

with \( \Delta_{\ell\ell'} = (U_{\ell 3} U_{\ell 3})^2 + (U_{\ell 1} U_{\ell 1} + U_{\ell 2} U_{\ell 2})^2 \), or more explicitly

\[
\Delta = \delta + \begin{pmatrix} s_\beta^4 + c_\beta^4 - 1 & 2c_\beta^2 s_\beta^2 s_\psi & 2c_\beta^2 s_\beta^2 c_\psi \\ 2c_\beta^2 s_\beta^2 s_\psi & 2c_\beta^2 s_\beta^2 (c_\beta^2 s_\psi^2 - 1) & 2c_\beta^2 c_\beta^2 s_\psi^2 \\ 2c_\beta^2 s_\beta^2 c_\psi^2 & 2c_\beta^2 c_\beta^2 s_\psi^2 & 2c_\beta^2 c_\beta^2 (c_\beta^2 s_\psi^2 - 1) \end{pmatrix} \ , \] (45)

and

\[
A_{\ell\ell'} \equiv 4U_{\ell 1} U_{\ell 1} U_{\ell 2} U_{\ell 2} \ . \] (46)

In Eq. (45) \( \delta \) is a \( 3 \times 3 \) identity matrix.

Only when both \( \beta \) and \( \psi \) vanish does Eq. (44) reduce to the expression for the two flavor scenario with two relativistic mass eigenstates — for then \( U(\theta, \beta, \psi) \)
becomes block diagonal with no mixing with $|\nu_3\rangle$, as

$$\beta = \psi = 0 : \quad \Delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} s_{2\theta}^2 & -s_{2\theta} & 0 \\ -s_{2\theta} & s_{2\theta}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is, therefore, concluded that there is enough richness in the three flavor neutrino oscillation phenomenology that unless an argument can be made that this structure is incapable of accommodating the existing neutrino oscillation data there is no necessity to put forward the sterile neutrino hypothesis. I hasten to add that the purpose of this paper was not to show that the uncovered structure and length scales can necessarily accommodate the existing and the forthcoming data, but only to reveal the often ignored structure and the hidden complexity of the three–flavor neutrino oscillation phenomenology.

7 Conclusions

In these notes I presented a critique of the standard three–flavor neutrino oscillation framework. I have argued that the standard three–flavor neutrino oscillation framework has rich structure that must not be ignored in making various approximations and designing experiments. Specifically, the design proposal of the MINOS at Fermilab based on a two–mass eigenstate framework may require serious revision if there is strong mixing between all three flavors of neutrinos. The general lesson to learn here is that:

Even if one is searching for neutrino oscillations between two flavors it may not be advisable to work in terms of the well known formula (cf, [32, p. 17]):

$$P(\nu_a \rightarrow \nu_b) = \sin^2(2\theta) \sin^2 \left(1.27 \frac{\Delta m^2 L}{E}\right),$$

where $\Delta m^2$ is measured in eV$^2$, $L$ in meters (kilometers), and $E$ in MeV (GeV).

Detailed analytical and graphical analysis of LSND NOE revealed the interesting roles that various relative phases play in neutrino oscillation probabilities. For example, in the context of LSND NOE and KARMEN, the amplitude $A_{3(2,1)}$, defined in Eq. (29), vanishes for certain values of mixing angles as a result of opposite signs of two equal and opposite contributions. \footnote{Such cancellations are a generic feature of three, or more, state systems and range from neutrino oscillation phenomenology to laser oscillation without population inversion [33].} I further
argued, for instance, why the canonical wisdom of “one mass scale dominance” may fail in certain unexpected sectors of the neutrino–oscillation parameter space. By comparing the solar neutrino deficit and the results on $\nu_e$ disappearance in the reactor experiments I reached the conclusion that the standard three–flavor neutrino oscillation framework requires one of the mass–squared differences to be $\ll 10^3$ ev$^2$. This, when coupled with the very recent astronomical observations, left open the possibility that one of the mass eigenstates may be non–relativistic and hence introduce a fundamental modification to the standard neutrino oscillation phenomenology without invoking a sterile neutrino. If the transition from the non-relativistic to the relativistic regime happens for energies relevant to the Reactor and LSND NOE then one must consider an ab intio analysis of the existing data.

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A Comments and Observations on Eqs. (1) and (2) and A Brief Outline of the Standard Neutrino–Oscillation Phenomenology

Equations (1) and (2) require several comments and observations. These remarks follow.

While analyzing experimental situations, the integral appearing in Eq. (2) is evaluated by summing over approximately–constant bins of \( f_\ell(E) \). In this context I note that the integral

\[
I_\beta = \int_{E_{\text{min}}}^{E_{\text{max}}} dE \sin^2 \left( 2 \pi \frac{L}{\lambda_{\text{osc}}} \right) = \int_{E_{\text{min}}}^{E_{\text{max}}} dE \sin^2 \left( \frac{C_\beta}{E} \right), \tag{A.1}
\]

with \( C_\beta = \alpha L \Delta m^2_\beta \) may be evaluated by introducing \( y \equiv 1/E \), and using the result

\[
\int_{E_{\text{min}}}^{E_{\text{max}}} dE \sin^2 \left( \frac{C_\beta}{E} \right) = \int_{y_1}^{y_2} \left( - \frac{dy}{y^2} \right) \sin^2 (y C_\beta) = - \frac{\cos(2 y_2 C_\beta)}{2 y_2} - \frac{\cos(2 y_1 C_\beta)}{2 y_1} + \frac{1}{2 y_2} + \frac{1}{2 y_1}. \tag{A.2}
\]

Referring to Eq. (2), it is important to note how it arises. Apart from the already indicated assumptions, the fundamental assumption of the neutrino–oscillation phenomenology, contained in Eq. (2), is that weak flavor eigenstates for \( \nu_e, \nu_\mu, \nu_\tau \), and \( \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau \), are not mass eigenstates. Why this is so, no one knows at the present time and the question remains a deep mystery. To appreciate the nature of this mystery, note that from the 1939 classic paper of Wigner [34] we have learned that every physical state is described by two Casimir invariants associated with the Poincaré group of space–time symmetries. These Casimir invariants are directly connected to mass and spin. Why, therefore, do weak interactions and space–time symmetries intermingle in such a manner as to prefer linear superpositions of mass eigenstates? I shall return to this subject elsewhere building on some recent work on the subject [1,35–37].

Here is a brief outline of the standard neutrino–oscillation phenomenology. Assume that in the “creation region,” \( R_c \), located at \( \vec{r}_c \), a weak eigenstate with energy \( E \) (with appropriate uncertainty dictated not only by the Heisenberg Principle but also determined by the masses associated with mass eigenstates) denoted by \( | \nu_\ell, R_c \rangle \), is produced at \( \vec{r}_c \), with the clock set to \( t = 0 \). Each of the
three neutrino mass eigenstates shall be represented by $|\nu_i\rangle$; $i = 1, 2, 3$. So I have the linear superposition:

$$|\nu_\ell, \mathcal{R}_c\rangle = \sum_{i=1,2,3} U_{\ell i} |\nu_i\rangle \quad ,$$

(A.3)

where $\ell = e, \mu, \tau$ represent the weak flavor eigenstates (corresponding to electron, muon, and tau neutrinos, respectively). The $|\nu_1\rangle$, $|\nu_2\rangle$, and $|\nu_3\rangle$ correspond to the three mass eigenstates of masses $m_1$, $m_2$, and $m_3$, respectively.

Under the already–indicated assumptions the unitary mixing matrix $U_{\ell i}$ may be parameterized by three angles and is given by Eq. (6).

At a time “$t$”= $t > 0$, I wish to study the weak flavor eigenstate in the “detector region,” $\mathcal{R}_d$, located at $\vec{r}_d$. Under the above indicated assumptions the neutrino evolution is given by the expression

$$|\mathcal{R}_d\rangle = \exp \left( -\frac{i}{\hbar} \int_{t_c}^{t_d} H dt + \frac{i}{\hbar} \int_{\vec{r}_c}^{\vec{r}_d} \vec{P} \cdot d\vec{x} \right) |\nu_\ell, \mathcal{R}_c\rangle \quad .$$

(A.4)

Here $H$ is the time translation operator, the Hamiltonian, associated with the system; $\vec{P}$ is the operator for spatial translations, the momentum operator, and $[H(t, \vec{x}), \vec{P}(t, \vec{x})] = 0$. It shall be noted that $|\nu_\ell, \mathcal{R}_c\rangle$ has evolved to a state $|\mathcal{R}_d\rangle$ that, in general, is not an eigenstate associated with $\nu_e, \nu_\mu, \text{or } \nu_\tau$. Instead, it is a linear superposition of states associated with $\nu_e, \nu_\mu, \text{or } \nu_\tau$. The “neutrino oscillation probability” from a state $|\nu_\ell, \mathcal{R}_c\rangle$ to another state $|\nu_{\ell'\prime}, \mathcal{R}_d\rangle$ is now obtained by calculating the projection $\langle \nu_{\ell'\prime}, \mathcal{R}_d | \mathcal{R}_d \rangle$, i.e., the amplitude for $|\nu_\ell, \mathcal{R}_c\rangle \rightarrow |\nu_{\ell'\prime}, \mathcal{R}_d\rangle$, and then multiplying it by its complex conjugate. An algebraic exercise that exploits (i) the unitarity of the neutrino mixing matrix $U(\theta, \beta, \psi)$, (ii) orthonormality of the mass eigenstates, (iii) certain trigonometric identities, makes (iv) the standard observation (for neutrino oscillations) that $(E_{\ell'\prime} - E_\ell) = \Delta m^2_{\ell'\prime} c^4/(2E)$, where $\Delta m^2_{\ell'\prime} \equiv m^2_{\ell'\prime} - m^2_\ell$ and $E_{\ell'\prime} \simeq E_\ell \equiv E$, and (v) sets $|\vec{r}_d - \vec{r}_c| = L$, yields Eq. (2) for $P_{\ell'\ell}(E, L, \{\eta_k\})$.

### B A Simple Theorem on the Inverted Mass Hierarchy

Under

$$\Delta m^2_{21} \longleftrightarrow \Delta m^2_{32} \quad ,$$

(B.1)

and simultaneous change in $U(\theta, \beta, \psi)$ of the form

$$U_{\ell'1} U_{\ell 1} \longleftrightarrow \pm U_{\ell'3} U_{\ell 3} \quad ,$$

(B.2)

$$U_{\ell'2} U_{\ell 2} \longleftrightarrow \pm U_{\ell'2} U_{\ell 2} \quad ,$$

(B.3)
the \( P(\nu_\ell \rightarrow \nu_\ell') \), given by Eq. (4), remains unchanged.  

The invariance of various neutrino oscillation probabilities \( P(\nu_\ell \rightarrow \nu_\ell') \), under the above defined change in neutrino–oscillation parameters, does not imply that other physical phenomena too remain unaffected under the change in the neutrino–oscillation parameters defined above.

For considerations on the evolution of the universe, and formation of the large–scale structures in the universe, it may be noted that given a mass eigenstate configuration \( m_3 \gg m_2 \geq m_1 \) (by definition, the “standard mass hierarchy”), the corresponding “inverted mass hierarchy” (by definition, \( m_3 \geq m_2 \gg m_1 \)) defined by the above symmetry, necessarily has a larger \( \sum_i m_i \) than the \( \sum_i m_i \) associated with the (corresponding) standard mass hierarchy.  

Within the above framework (i.e., three flavor neutrino oscillation phenomenology with Dirac neutrinos without CP violation), the neutrino–oscillation experiments cannot distinguish between the standard mass hierarchy and the inverted mass hierarchy. Additional physical phenomenon must be considered to distinguish the standard and inverted mass hierarchies.

C Energy Conservation and Neutrino Oscillations

This section is included here purely for pedagogic reasons, and may be skipped by the “experts.”

Let us begin with posing a pedagogic question. To keep the algebraic details simple, I will confine to a situation where the mixing angles \( \beta \) and \( \psi \) vanish. Then, consider an electron neutrino in the creation region:

\[
|\nu_e, R_c\rangle = c_\theta |\nu_1\rangle + s_\theta |\nu_2\rangle ,
\]

and, again for the sake of argument, I will confine to a situation where the neutrino mass eigenstates have same momenta, and hence different energies. The expectation of value of energy in the state given by (C.1) is

\[
\langle E_{\nu_e} \rangle = c_\theta^2 E_1 + s_\theta^2 E_2 .
\]

\(^{9}\) Either the upper + sign, or the lower − sign, in both of the above equations is meant to be taken in any given transformation.

\(^{10}\) Similar comments apply in the context of gravitational effects on physical phenomenon involving neutrinos in astrophysical environments. An exception is the gravitationally induced modification to the neutrino–oscillation probabilities, where a similar symmetry as mentioned above exists.  

21
The state $|\nu_e, \mathcal{R}_c \rangle$ evolves with time, and at any given time it has a definite probability of being found (on measurement) as an electron neutrino, and one minus that probability of being found as a muon neutrino. The energy expectation value of the energy remains constant in time. But now suppose I make a measurement and detect a muon neutrino. The linear superposition of the electron and muon neutrino state now collapses to muon neutrino

$$|\nu_\mu, \mathcal{R}_d \rangle = -s_\theta |\nu_1 \rangle + c_\theta |\nu_2 \rangle \quad .$$

(C.3)

The expectation value of the energy is now

$$\langle E_{\nu_\mu} \rangle = s_\theta^2 E_1 + c_\theta^2 E_2 \quad .$$

(C.4)

The posed question is: Where did the excess energy,

$$\Delta E_{\text{excess}} \equiv \langle E_{\nu_\mu} \rangle - \langle E_e \rangle = c_{2\theta} (E_2 - E_1) \quad ,$$

(C.5)

come from?

To answer this question it should be realized that in order for the detector to detect a muon neutrino it must be able to make a measurement over a period

$$\Delta t \approx \frac{c}{\lambda_{\text{osc}}/4} = \frac{\hbar E}{\Delta m_{21}^2 c^4}; \quad E = c_{2\theta}^2 E_1 + s_\theta^2 E_2 \quad .$$

(C.6)

As such, the act of measurement must impart a minimum energy (via the scattering process in the detector) $\Delta E_{\text{Heisenberg}} = \hbar/(2\Delta t)$ to the detected muon neutrino. It is readily found that

$$\Delta E_{\text{Heisenberg}} = \frac{\Delta m_{21}^2 c^4}{2E} \quad .$$

(C.7)

Referring to Eq. (C.3), now note that $E_2 - E_1 \approx \Delta m_{21}^2 c^4/(2E)$. Thus

$$\Delta E_{\text{excess}} = c_{2\theta} \Delta E_{\text{Heisenberg}} \quad .$$

(C.8)

So what at first might have appeared as a violation of energy conservation is precisely the energy, or more accurately the $c_{2\theta}$ fraction of $\Delta E_{\text{Heisenberg}}$, that the act of measurement imparts to the detected muon neutrino.

For a more formal discussion of this point see Ref. [9].

\[11\] In this section I shall not set $\hbar$ and $c$ equal to unity.
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