DENSITY OF VALUES OF LINEAR MAPS ON QUADRATIC SURFACES.

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ABSTRACT. In this paper we investigate the distribution of the set of values of a linear map at integer points on a quadratic surface. In particular we show that this set is dense in the range of the linear map subject to certain algebraic conditions on the linear map and the quadratic form that defines the surface. The proof uses Ratner’s Theorem on orbit closures of unipotent subgroups acting on homogeneous spaces.

1. INTRODUCTION

We are motivated by the following general problem.

Problem 1.1. If $X$ is some rational surface in $\mathbb{R}^d$ and $F : X \to \mathbb{R}^s$ is a polynomial map, then what can one say about the distribution of the set $\{F(x) : x \in X \cap \mathbb{Z}^d\}$ in $\mathbb{R}^s$?

In full generality problem 1.1 is unapproachable via available techniques, however if $X$ has a large group of symmetries then the problem can be studied from a dynamical systems point of view. The main result of this paper deals with a special case of the above problem and is stated in the following Theorem.

Theorem 1.2. Suppose $Q$ is a quadratic form on $\mathbb{R}^d$ such that $Q$ is non-degenerate, indefinite with rational coefficients and signature $(p, q)$. For $a \in \mathbb{Q} \setminus \{0\}$ define $X_\mathbb{R} = \{x \in \mathbb{R}^d : Q(x) = a\}$ and $X_\mathbb{Z} = \{x \in \mathbb{Z}^d : Q(x) = a\}$, suppose that $X_\mathbb{Z}$ is non empty. Let $M : \mathbb{R}^d \to \mathbb{R}^s$ be a linear map such that

1. The following inequalities hold, $d > 2s$ and $\text{rank}(Q|_{M=0}) > 2$.
2. The quadratic form $Q|_{M=0}$ is indefinite.
3. For all $\alpha \in \mathbb{R}^s \setminus \{(0, \ldots, 0)\}$, $\alpha M$ is non rational.

Then $M(X_\mathbb{Z}) = \mathbb{R}^s$.

Density of the set in problem 1.1 has been established in some special cases, Theorem 1.2 gives conditions sufficient for density in the situation described in the Theorem. The simplest instance of problem 1.1 is when $X_\mathbb{Z} = \mathbb{Z}^d$ and $F : \mathbb{R}^d \to \mathbb{R}^s$ is a linear map, in this case we have $F(X_\mathbb{Z}) = \mathbb{R}^s$ provided that $s < d$ and condition 3 of Theorem 1.2 is satisfied for the map $F$. This result can be proved using elementary methods, see [Ca57], Theorem 1, page 64, for example.

The case where $X_\mathbb{Z} = \mathbb{Z}^d$ and $F$ is a quadratic form is dealt with by the so called Oppenheim conjecture which says, provided that $Q$ is a non degenerate, indefinite quadratic form in $d \geq 3$ variables such that $Q$ is not a multiple of a rational quadratic form, then $Q(\mathbb{Z}^d) = \mathbb{R}$. After 50 years of unsuccessful attempts to prove the conjecture using methods from analytic number theory (see [Ma97] for a survey...
of partial results obtained using these methods), it was finally proved in the late 80’s by G.A. Margulis in [Ma89]. His method of proof used a surprising connection, first noted by Ranghunathan, between the orbits of $SO(Q)$ in the space $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ and the Oppenheim conjecture. In particular, what Ranghunathan noticed was that if you could show that $SO(Q)SL_3(\mathbb{Z})$ was dense in $SL_3(\mathbb{R})$, then this would be enough to prove the Oppenheim conjecture.

Since the successful proof of the Oppenheim conjecture similar lines of reasoning have been used by Dani and Margulis in [DM90] to show that, for a pair $(Q, L)$, consisting of a non degenerate quadratic form, $Q$, and a non-zero linear form, $L$, the set $\{((Q(x), L(x)) : x \in \mathbb{Z}^3)\}$ is dense in $\mathbb{R}^2$ provided for all $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, $\alpha Q + \beta L^2$ is not rational and the plane given by $\{x \in \mathbb{R}^3 : L(x) = 0\}$ is tangent to the surface given by $\{x \in \mathbb{R}^3 : Q(x) = 0\}$. This result was later extended by A. Gorodnik in [Go04] who showed that if $(Q, L)$ is a pair as before, in dimension $d \geq 4$, such that $Q|_{L=0}$ is indefinite and for all $\alpha, \beta \in \mathbb{R} \setminus \{0\}$, $\alpha Q + \beta L^2$ is not rational, then $\{(Q(x), L(x)) : x \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}^2$. Recently in [DS08] Dani and Shrikrishna prove what is in some sense an extension of the above work of Gorodnik. They prove density for a system consisting of $d-2$ linear forms and a quadratic form under similar algebraic conditions as in [Go04]. The major difference is that the result of [DS08] applies only to almost all linear forms.

The key fact that enables us to study the above examples is that the group of isometries of the maps in question are large. Moreover, except in the latter example they are generated by one parameter unipotent subgroups and it is the absence of this property that is responsible for the result of [DS08] only being applicable almost everywhere.

We now make some remarks concerning the conditions of Theorem 1.2.

**Remark 1.3.** There is no reason to expect that the inequalities of condition 1 are optimal, however they are necessary for the proof to work. It is reasonable to expect some inequalities of this type to be necessary. The condition that $\text{rank}((Q|_{M=0}) > 2$ is analogous to the condition that $d > 2$ in the Oppenheim conjecture and as such we believe this condition to be necessary, although no counter examples have been found to indicate this.

**Remark 1.4.** Condition 2 is possibly stronger than is strictly necessary, however it is a natural condition and comparable with conditions imposed in [Go04]. It implies the necessary condition that the set $X_\mathbb{R} \cap \{x \in \mathbb{R}^d : M(x) = b\}$, for some $b \in \mathbb{R}^s$ is non compact. To see that this condition is necessary, suppose $X_\mathbb{R} \cap \{x \in \mathbb{R}^d : M(x) = b\}$ is compact. Hence $X_\mathbb{R} \cap \{x \in \mathbb{R}^d : |M(x) - b| \leq \epsilon\}$ is also compact and therefore contains only finitely many integer points. Hence if $b \notin \mathbb{Z}^s$, we can make $\epsilon$ small enough so that $X_\mathbb{R} \cap \{x \in \mathbb{R}^d : |M(x) - b| \leq \epsilon\}$ contains no integer points, but then there exists an open set $B_\epsilon(b) \subset \mathbb{R}^s$ such that there is no $x \in X_\mathbb{R}$ with $M(x) \in B_\epsilon(b)$.

**Remark 1.5.** Condition 3 is necessary since otherwise $M(\mathbb{Z}^d)$ would not even be dense in $\mathbb{R}^s$.

2. Set up

2.1. **Construction of a dynamical system.** The strategy we will use to prove Theorem 1.2 is analogous to that used by Margulis to prove the Oppenheim conjecture. We will use the following Theorem of M. Ratner found in [Ra94].
Theorem 2.1. (Ranghunathan’s topological conjecture) Let $G$ be a connected Lie group and $U$ a subgroup of $G$ generated by one parameter unipotent subgroups, given a lattice $\Gamma$ of $G$ and any $x \in G/\Gamma$, the closure of the orbit $Ux$ is equal to the orbit of a closed connected subgroup $F$ such that $U \leq F \leq G$.

In order to make use of Theorem 2.1 we need to construct a dynamical system, so define $G_R = \{ g \in SL_d(\mathbb{R}) : Q(gx) = Q(x) \}$, and let $G = G^*_R$ be the connected component containing the identity of $G_R$. Let $\Gamma = G \cap SL_d(\mathbb{Z})$ and $H = \{ g \in G : M(gx) = M(x) \}$.

It is a standard fact that $G \cong SO(p,q)^o$ is a connected Lie group. Since a priori, $H$ may not be generated by one parameter unipotent subgroups, our first aim is to define $H^* < H$ such that $H^*$ is generated by unipotent subgroups, we will then consider the dynamical system that arises from $H^*$ acting on $G/\Gamma$. Note that condition 2 of Theorem 1.2 implies that $H$ will be non compact, and so there is hope that such an $H^*$ exists, in section 2.2 an explicit description of $H^*$ is given.

It is known that $\Gamma$ is a lattice in $G$ so long as $G$ is non-compact and defined over the rationals, which happens when $p, q \geq 1$ and $Q$ is a rational form. In particular both of these conditions follow from assumptions of Theorem 1.2 so in our case $\Gamma$ is a lattice in $G$.

2.2. A canonical form for the system. We would like to use linear transformations to transform our system $(Q, M)$ into something more manageable. For two pairs $(Q_1, M_1)$ and $(Q_2, M_2)$ we say $(Q_1, M_1) \sim (Q_2, M_2)$ if and only if there exists $g_d \in GL_d(\mathbb{R})$ and $g_s \in GL_s(\mathbb{R})$ such that $(Q_1(x), M_1(x)) = (Q_2(g_d x), g_s M_2(g_s x))$.

The following result, adapted from [Go04], is reproduced below for completeness and will be used to establish a more general form.

Lemma 2.2. Every pair $(Q, L)$, where $Q$ is a non-degenerate quadratic form on $\mathbb{R}^d$ with signature $(p, q)$, and $L$ is a non-zero linear form on $\mathbb{R}^d$, is equivalent to one and only one of the following pairs:

1. If $\text{rank}(Q|_{L=0}) = d - 1$ then either,
   a. $(Q, L) \sim (\sum_{i=1}^p x_i^2 - \sum_{i=p+1}^d x_i^2, x_1)$
   b. $(Q, L) \sim (\sum_{i=1}^p x_i^2 - \sum_{i=p+1}^d x_i^2, x_d)$

2. If $\text{rank}(Q|_{L=0}) = d - 2$ then $(Q, L) \sim (2x_1x_d + \sum_{i=2}^p x_i^2 - \sum_{i=p+1}^d x_i^2, x_1)$.

Proof. By Sylvester’s Law we can always transform $Q \sim \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^d x_i^2$. Next by applying an element of $SO(p, q)$ to the system it is possible to ensure that the coefficient of $x_1$ in $L(x)$ is non zero. Now use the transformation $x_1 \rightarrow x_1 + L'(x)$ for $L'$ a linear form in the remaining variables, to get that

$$(Q, L) \sim (x_1^2 + x_1 L''(x) + Q'(x), x_1)$$

for $Q'$ and $L''$ a quadratic form and linear form respectively, in the variables not including $x_1$. Note that $Q'$ has signature $(p - 1, q)$ or $(p, q - 1)$ if $\text{rank}(Q|_{L=0}) = d - 1$ and signature $(p - 1, q - 1)$ if $\text{rank}(Q|_{L=0}) = d - 2$. Suppose we are in the first case, apply a transformation in the variables not including $x_1$, to get that

$$(Q, L) \sim \left( x_1^2 + 2 \sum_{i=2}^d \alpha_i x_i x_1 + \sum_{i=2}^p x_i^2 - \sum_{i=p+1}^d x_i^2, x_1 \right),$$
where \( \hat{p} = p \) or \( \hat{p} = p + 1 \). Next, we use transformations of the form

\[
x_k \to \begin{cases} 
x_k - \alpha_k x_1 & \text{for } 2 \leq k \leq \hat{p} \\
x_k + \alpha_k x_1 & \text{for } \hat{p} + 1 \leq k \leq d 
\end{cases}
\]

to get

\[
(Q, L) \sim \left( x_1^2 \left( 1 - \sum_{i=2}^{\hat{p}} \alpha_i^2 + \sum_{i=\hat{p}+1}^{d} \alpha_i^2 \right) \right) + \sum_{i=2}^{\hat{p}} x_i^2 - \sum_{i=\hat{p}+1}^{d} x_i^2, x_1 \right).
\]

If the coefficient of \( x_1^2 \) is positive, then \( \hat{p} = p \) and we see that we are in case 1a of the Lemma, similarly if the coefficient of \( x_1^2 \) is negative, then \( \hat{p} = p + 1 \) and we see that we are in case 1b of the Lemma, after relabeling.

Suppose that \( \operatorname{rank}(Q|_{L=0}) = d - 2 \), apply a transformation in the variables not including \( x_1 \) to get that

\[
(Q, L) \sim \left( x_1^2 + 2 \sum_{i=2}^{d} \alpha_i x_i x_1 + \sum_{i=p+1}^{d} x_i^2 - \sum_{i=\hat{p}+1}^{d} x_i^2, x_1 \right).
\]

Next, for \( 2 \leq k \leq d - 1 \) we use transformations of the form \( x_k \to x_k \pm \alpha_k x_1 \) to make \( \alpha_i \) zero for all \( i \neq d \). Note that \( \alpha_d \neq 0 \), otherwise \( Q \) would be degenerate, so to finish off we use the transformation \( x_d \to \frac{2}{\alpha_d} (x_d - x_1) \) and we see that we are in the second case of the Lemma.

We can now prove the main Lemma of this section.

**Lemma 2.3.** For any pair \((Q, M)\), where \( Q \) is a non-degenerate quadratic form on \( \mathbb{R}^d \) with signature \((p, q)\), and \( M : \mathbb{R}^d \to \mathbb{R}^s \) is a linear map, if \( Q|_{M=0} \) is indefinite then \((Q, M) \sim (Q_0, M_0)\), where

\[
Q_0(x) = Q_{m+1,\ldots,s}(x) + 2 \sum_{i=1}^{m} x_i x_{d-i+1} + \sum_{i=s+1}^{s+r} x_i^2 - \sum_{i=s+r+1}^{s+r+n} x_i^2
\]

\[
M_0(x) = (x_1, \ldots, x_s)
\]

and \( m = d - s - \operatorname{rank}(Q|_{M=0}) \) and \( Q_{m+1,\ldots,s}(x) \) is a non degenerate quadratic form in variables \( x_{m+1}, \ldots, x_s \) with signature \((p', q')\), such that the following relations hold, \( r = p - m - p' \geq 1 \) and \( n = q - m - q' \geq 1 \).

**Proof.** We proceed by induction on \( s \). For \( s = 1 \) we know from Lemma 2.2 that the conclusion of the Lemma holds, so suppose the Lemma holds for \( s \leq k - 1 \). Let \( s = k \), and suppose that \( M = (L_1, \ldots, L_k) \) for \( L_1, \ldots, L_k \) non zero linear forms on \( \mathbb{R}^d \).

If \( \operatorname{rank}(Q|_{L_1=0}) = d - 1 \), using Lemma 2.2 it is clear that we can transform our system into

\[
\begin{pmatrix} Q \\ M \end{pmatrix} \sim \begin{pmatrix} \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{d} x_i^2 \\ (x_1, L'_2, \ldots, L'_k) \end{pmatrix},
\]

where \( L'_2, \ldots, L'_k \) are linear forms on \( \mathbb{R}^d \) and \( l \in \{1, d\} \). Next we can eliminate the coefficient of \( x_l \) in \( L'_2, \ldots, L'_k \) by subtracting some multiple of \( x_l \), then relabel \( x_l \to x_1 \) and \( x_1 \to x_l \) and apply the inductive hypothesis to see that the conclusion of the Lemma holds.
If rank \((Q|_{L_1=0}) = d - 2\), using Lemma 2.2 we get
\[
\begin{pmatrix}
Q
M
\end{pmatrix}
\sim
\begin{pmatrix}
2x_1x_d + \sum_{i=2}^p x_i^2 - \sum_{i=p+1}^{d-1} x_i^2 \\
(x_1, L'_2, \ldots, L'_k)
\end{pmatrix}.
\]
Again we can eliminate the coefficient of \(x_1\) in \(L'_2, \ldots, L'_k\) by subtracting some multiple of \(x_1\). Suppose that the coefficient of \(x_d\) is zero for each \(L'_2, \ldots, L'_k\), in this case we are in position to apply the inductive hypothesis and get to the conclusion of the lemma. Suppose the coefficient of \(x_d\) in \(L'_i\) is non zero for some \(2 \leq i \leq k\), without loss of generality suppose that \(i = 2\), in particular suppose \(L'_2(x) = L''(x) + \alpha dx_d\) for some linear form \(L''\) in variables \(x_2, \ldots, x_{d-1}\). Use a transformation of the form \(x_d \to \frac{1}{\alpha_d} (x_d - L''(x))\) to get that
\[
\begin{pmatrix}
Q
M
\end{pmatrix}
\sim
\begin{pmatrix}
\frac{2}{\alpha_d} x_1x_d + \sum_{i=2}^{d-1} x_i \beta_i x_1 + \sum_{i=2}^p x_i^2 - \sum_{i=p+1}^{d-1} x_i^2 \\
(x_1, x_d, L'_3, \ldots, L'_k)
\end{pmatrix}
\]
Next, for \(2 \leq k \leq d - 1\) we use transformations of the form \(x_k \to x_k \pm \beta_k x_1\) to make \(\beta_i\) zero for all \(i\). After we have done this we end up with
\[
\begin{pmatrix}
Q
M
\end{pmatrix}
\sim
\begin{pmatrix}
Q_{1,d}(x) + \sum_{i=2}^p x_i^2 - \sum_{i=p+1}^{d-1} x_i^2 \\
(x_1, x_d, L'_3, \ldots, L'_k)
\end{pmatrix}
\]
so if we relabel \(x_2 \to x_d\) and \(x_d \to x_2\) and eliminate these co-ordinates from \(L'_3, \ldots, L'_k\) we see we can apply the inductive hypothesis and get the desired conclusion. The assertion that \(Q_{m+1, \ldots, s}(x)\) is a non degenerate quadratic form in variables \(x_{m+1}, \ldots, x_s\) follows from the fact that \(Q\) is non degenerate. We see that \(Q_{m+1, \ldots, s}(x)\) has signature \((p', q')\) where \(r = p - m - p'\) and \(n = q - m - q'\) because the signature of \(2 \sum_{i=1}^m x_i x_d + \sum_{i=s+1}^{s+r} x_i^2 - \sum_{i=s+r+1}^{s+m} x_i^2\) is \((r + m, n + m)\). Finally, the assumption that \(Q|_{M=0}\) is indefinite means that \(r \geq 1\) and \(n \geq 1\). 

Define \(g_d \in GL_d(\mathbb{R})\) and \(g_s \in GL_s(\mathbb{R})\) such that we have \(Q(x) = Q_0(g_d x)\) and \(M(x) = g_s M_0(g_d x)\). The conjugate dynamical system consisting of \(\hat{H}\) acting on \(\hat{G}/\Gamma\) is defined by relation \(g_d \hat{G} g_d^{-1} = G\), and the corresponding relation for subgroups of \(G\). Note that for all \(g \in \hat{G}\) we have \(Q_0(gx) = Q_0(x)\) and for all \(h \in \hat{H}\) we have \(Q_0(hx) = Q_0(x)\) and \(M_0(hx) = M_0(x)\).

2.3. Definition of \(H^+\). We will use the notation \(I_p\) to denote the \(p \times p\) identity matrix and \(I_{p,q}\) to denote the indefinite identity matrix with signature \((p, q)\). For \(1 \leq i \leq m\) and \(t_i \in \mathbb{R}^{d-s-m}\) we define the linear transformations
\[
U_{t_i} : \begin{cases}
x_j \to x_j & \text{for } j \leq s \text{ and } j > d - m, j \neq d - 1 + i \\
x_j \to x_j + t_{ij} x_i & \text{for } s + 1 \leq j \leq d - m \\
x_{d-1+i} \to x_{d-1+i} - I_{r,n} t_i x' - \frac{1}{2} \frac{\partial}{\partial x'} I_{r,n} t_i x_i & \text{for } x' = (x_{s+1}, \ldots, x_{d-m})
\end{cases}
\]
One can check that \(\hat{H}\) contains the subgroups \(U_i = \{U_{t_i} : t_i \in \mathbb{R}^{d-s-m}\}\) and the subgroup \(D_{0,0,0}\) defined as follows
\[
D_{s_1, s_2, s_3} = \begin{pmatrix}
I_{s_1-i_1-i_2-i_3} & SO(r+i_1+i_3, n+i_2+i_3) & I_{m-i_3}
\end{pmatrix},
\]
for convenience we denote $D_{0,0,0} = D$, so we see that

$$\hat{H}^* = U_1 \ldots U_mD \subseteq \hat{H}.$$ 

One sees that $\hat{H}^*$ defined in this way is connected and generated by one parameter unipotent subgroups since the $U_i$’s are themselves unipotent subgroups and conditions 1 and 2 of Theorem 1.2 imply that $D$ is generated by one parameter unipotent subgroups. To see this note that rank $(Q|_{M=0}) > 2$ implies that $r + n \geq 3$. Moreover, as noted in Lemma 2.3 the fact that $Q|_{M=0}$ is indefinite implies that $r \geq 1$ and $n \geq 1$.

3. Lemmas concerning subspaces invariant under the action of subgroups of $SO(p,q)$.

We wish to obtain information about the subgroup $F$ such that $H^* \subseteq F \subseteq G$ and for any $x \in G/\Gamma$ the orbit $\overline{H^*x} = Fx$. First we introduce some notation, let $L$ be the space of $d$ dimensional linear forms defined over $\mathbb{R}$ and $M = (L_1, \ldots, L_s)$ for $L_i \in \mathcal{L}$. For any group $G$ let $\mathcal{L}^G = \{L \in L : L(gx) = L(x) \text{ for all } g \in G\}$ be the ‘fixed vectors of $G$’.

In particular we are interested in subspaces of $\mathcal{L}$ invariant under $F$, the main work of this section is to classify such subspaces. The first step is to show that there are no invariant rational subspaces in the fixed vectors of $H^*$, this is then extended to algebraic subspaces and finally to any subspace defined over $\mathbb{C}$.

First we prove that the set of vectors fixed by $H^*$ is exactly the set of linear forms that make up $M$.

We will use the notation $\mathcal{L}_m = \langle x_{s+1}, \ldots, x_{d-m} \rangle$.

**Lemma 3.1.** $\mathcal{L}^{H^*} = \langle L_1, \ldots, L_s \rangle$.

**Proof.** It is clear that $\langle L_1, \ldots, L_s \rangle \subseteq \mathcal{L}^{H^*}$. So suppose $\langle L_1, \ldots, L_s \rangle \subsetneq \mathcal{L}^{H^*}$. This means there exists $T \in \mathcal{L}^{H^*}$ such that $T \notin \langle L_1, \ldots, L_s \rangle$, this is equivalent to saying, there exists $T \in \mathcal{L}^{H^*}$ such that $T \notin \langle x_1, \ldots, x_s \rangle$. Since $T \notin \langle x_1, \ldots, x_s \rangle$, if for $t \in \mathbb{R}^d$ we write $T(x) = tx$, we can suppose $t_1, \ldots, t_s = 0$. Furthermore since $D_m < \hat{H}^*$, we may suppose that $t_{s+1}, \ldots, t_{d-m} = 0$ since $SO(r,n)^0$ has no fixed vectors. But we also have $U_1 \ldots U_m < \hat{H}^*$ and the space $\langle x_{d-m+1}, \ldots, x_d \rangle$ is clearly not $U_1 \ldots U_m$ invariant. This implies that $t_{d-m+1}, \ldots, t_d = 0$, and thus $T = 0$ and we have a contradiction. □

Recall, condition 3 of Theorem 1.2 is that for all $\alpha \in \mathbb{R}^* \setminus \{(0, \ldots, 0)\}$, $\alpha M$ is non rational. From this and Lemma 3.1 we can deduce the following.

**Corollary 3.2.** There exists no non trivial $F$ invariant subspaces defined over $\mathbb{Q}$ and contained in $\mathcal{L}^{H^*}$.

**Proof.** Suppose there exists $U \subseteq \mathcal{L}^{H^*}$ such that $U$ is defined over $\mathbb{Q}$, or in other words, $U$ has a rational basis. By Lemma 3.1 it is clear $U \subseteq \langle L_1, \ldots, L_s \rangle$. Now let $u_1, \ldots, u_a$ be a basis for $U$, then by the preceding remark, we can write $u_i$ as a linear combination of the $L_i$’s. Since we are supposing $u_i$ is rational, this is in contradiction with condition 3 of Theorem 1.2 since this condition implies that no linear combination of $L_i$ is rational. □
We continue the process of classifying $F$ invariant subspaces of $\mathcal{L}$. Since $D \leq F$ any $F$ invariant subspace will be $D$ invariant, it is for this reason the next Lemma, which classifies two distinct possibilities for any $D$ invariant subspace, will be useful.

**Lemma 3.3.** If $V \subseteq \mathcal{L}$ is a $D$ invariant subspace then either

$(1) \ V \subseteq \mathcal{L}^D$

$(2) \ V = \mathcal{L}_m \oplus U$ where $U \subseteq \mathcal{L}^D$.

**Proof.** First note that if $V \subseteq \mathcal{L}^D$, then it is clear that $V$ will be $D$ invariant. Suppose that $V \subseteq \mathcal{L}$ is a $D$ invariant linear subspace such that $V \nsubseteq \mathcal{L}^D$, since $\mathcal{L} = \mathcal{L}^D \oplus \mathcal{L}_m$ there exists $v \in V$ such that $v = v_1 + v_2$ for $v_1 \in \mathcal{L}^D$ and $v_2 \in \mathcal{L}_m$, with $v_2 \neq 0$. Now for any $d \in D$ we have $d(v - v) = v_1 + dv_2 - (v_1 + v_2) = dv_2 - v_2 \in V$, since $\mathcal{L}^D$ consists of forms fixed by $D$. We can choose $d \in D$ so that $w = dv_2 - v_2 \neq 0$, but $w \in V \cap \mathcal{L}_m$, this means that $\langle Dw \rangle$ is a $D$ invariant subspace such that $\langle Dw \rangle \subseteq \mathcal{L}_m$, but this implies $\langle Dw \rangle = \mathcal{L}_m$ because $D$ acts irreducibly on $\mathcal{L}_m$. Since $\langle Dw \rangle = \mathcal{L}_m$, we have that $\mathcal{L}_m \subseteq V$ and as $v = v_1 + v_2 \in V$ for $v_1 \in \mathcal{L}^D$ and $v_2 \in \mathcal{L}_m$ we have $v_1 = v - v_2 \in V$ and so we see that $V = \mathcal{L}_m \oplus (V \cap \mathcal{L}^D)$, which implies that we are in the second case. \qed

For the same reasons as before it will be useful to classify distinct possibilities for any $\widehat{H}^*$ invariant subspace of $\mathcal{L}$. We will use the following notation $J_m = \begin{pmatrix} 0 & 0 & B_m \\ 0 & I_{d-2m} & 0 \\ B_m & 0 & 0 \end{pmatrix}$ where $B_m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \text{Mat}_m(\mathbb{R})$.

**Lemma 3.4.** If $V \subseteq \mathcal{L}$ is a $\widehat{H}^*$ invariant subspace then either

$(1) \ V \subseteq \mathcal{L}^{\widehat{H}^*}$

$(2) \ V = J_m \mathcal{L}_0 \oplus U$ where $U \subseteq J_m \mathcal{L}^{\widehat{H}^*}$.

**Proof.** First note that if $V \subseteq \mathcal{L}^{\widehat{H}^*}$ then it is clear that $V$ will be $\widehat{H}^*$ invariant. Suppose that $V \subseteq \mathcal{L}$ is an $\widehat{H}^*$ invariant linear subspace such that $V \nsubseteq \mathcal{L}^{\widehat{H}^*}$, since $D \leq \widehat{H}^*$ it is clear $V$ will be $D$ invariant, thus by Lemma 3.3 either

$(1) \ V \subseteq \mathcal{L}^D$

$(2) \ V = \mathcal{L}_m \oplus U$ where $U \subseteq \mathcal{L}^D$.

If we are in case (1), then since $\mathcal{L}^D = \mathcal{L}^{\widehat{H}^*} \oplus \langle x_{d-m+1}, \ldots, x_d \rangle$ we can suppose there exists $v \in V$ such that $v = v_1 + v_2$ for $v_1 \in \mathcal{L}^{\widehat{H}^*}$ and $v_2 \in \langle x_{d-m+1}, \ldots, x_d \rangle$. Suppose that $v_2 \neq 0$, then there exists $u \in U_1 \ldots U_m < \widehat{H}^*$ such that $uv_2 \notin \mathcal{L}^D$ and hence $uv_2 \notin V$, but is a contradiction since $V$ is supposed to be $\widehat{H}^*$ invariant.

If we are in case (2), from the definitions of $U_1 \ldots U_m$, $\mathcal{L}_m$ and $J_m$ we see that $\langle U_1 \ldots U_m \mathcal{L}_m \rangle = \langle J_m \mathcal{L}_0 \rangle$, then since $U_1 \ldots U_m < \widehat{H}^*$, the fact that $\mathcal{L}_m \subseteq V$ and $V$ is $\widehat{H}^*$ invariant implies that $V = J_m \mathcal{L}_0 + U$ for some $U \subseteq \mathcal{L}$. Moreover, since $\mathcal{L} = J_m \mathcal{L}^{\widehat{H}^*} \oplus J_m \mathcal{L}_0$ we see that $V = J_m \mathcal{L}_0 \oplus (V \cap J_m \mathcal{L}^{\widehat{H}^*})$ which implies we are in the second case of the Lemma. \qed
We can reformulate the above as follows.

**Corollary 3.5.** If $V \subseteq \mathcal{L}$ is a $H^*$ invariant subspace then either

1. $V \subseteq \mathcal{L}^{H^*}$
2. $V = g_d^T J_m \mathcal{L}_0 \oplus U$ where $U \subseteq g_d^T J_m \hat{\mathcal{L}}^{H^*}$.

**Proof.** Note that if $V \subseteq \mathcal{L}$ is a $H^*$ invariant subspace, then $g_d^{-T} V$ is a $\hat{H}^*$ invariant subspace, and that $g_d^T \mathcal{L}^{H^*} = \hat{\mathcal{L}}^{H^*}$. □

The next Lemmas are central to the arguments in the next section. We use $\overline{\mathbb{Q}}$ to denote the algebraic closure of $\mathbb{Q}$.

**Remark 3.6.** A key fact that will be used in the following will be Proposition 3.2 in [Sh91] which says that $F = F(\mathbb{R})^\circ$, where $F$ is an algebraic group, defined over $\mathbb{Q}$ and that the radical of $F$ is a unipotent algebraic group defined over $\mathbb{Q}$. In particular this means $F \cap SL_d(\mathbb{Q}) = F$. This will be used in conjunction with the Levi decomposition $F = F_r F_u$ where $F_r$ is reductive and $F_u$ is the unipotent radical, see [OV90], chapter 6, page 282, for details. In particular we see that $F_r$ is in fact semisimple and $F$, can be chosen to be defined over $\mathbb{Q}$.

The following Lemma and its Corollary can be seen as refinements of Corollary 3.2.

**Lemma 3.7.** There exists no non trivial $F$ invariant subspaces defined over $\overline{\mathbb{Q}}$ and contained in $\mathcal{L}^{H^*}$.

**Proof.** Suppose for a contradiction there exists at least one non trivial $F$ invariant subspace defined over $\overline{\mathbb{Q}}$ and contained in $\mathcal{L}^{H^*}$. Define $V$ to be the unique maximal $F$ invariant subspace such that $V \subseteq \mathcal{L}^{H^*}$ and $V$ is defined over $\overline{\mathbb{Q}}$. For $\sigma \in \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$, we write $\sigma (V)$ to mean the vector space spanned by the vectors that arise from applying $\sigma$ to all components of all basis vectors of $V$. It follows that for any $\sigma \in \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$ we have that $\sigma (V)$ is $F \cap SL_d(\mathbb{Q})$ invariant. By remark 3.6 we have $F \cap SL_d(\mathbb{Q}) = F$, and therefore $\sigma (V)$ is $F$ invariant. Additionally, this means that $\sigma (V)$ is $H^*$ invariant, hence Corollary 3.5 implies, either $\sigma (V) \subseteq \mathcal{L}^{H^*}$, or $\sigma (V) = g_d^T J_m \mathcal{L}_0 \oplus U$ where $U \subseteq g_d^T J_m \hat{\mathcal{L}}^{H^*}$. But dim $(\sigma (V)) \leq s < d - s \leq \text{dim} (g_d^T J_m \mathcal{L}_0 \oplus U)$ and so we must have that $\sigma (V) \subseteq \mathcal{L}^{H^*}$. By definition of $V$, we see that $\sigma (V) \subseteq V$, and thus $\sigma (V) = V$ by considering dimensions. By a Proposition on page 30 of [Sh91] this means that $V$ has a rational basis, contradicting Corollary 3.2. □

**Corollary 3.8.** There exists no non trivial $F$ invariant subspaces defined over $\overline{\mathbb{Q}}$ and contained in $g_d^T \mathcal{L}^D$.

**Proof.** Lemma 3.7 implies that if $V$ is an $F$ invariant subspace defined over $\overline{\mathbb{Q}}$ and contained in $g_d^T \mathcal{L}^D$ then $V$ is not contained in $\mathcal{L}^{H^*}$. But since $V$ should also be $H^*$ invariant Corollary 3.5 implies the only other option is that $V = g_d^T J_m \mathcal{L}_0 \oplus U$ where $U \subseteq g_d^T J_m \hat{\mathcal{L}}^{H^*}$. This is impossible since in this case $V$ would not be contained in $g_d^T \mathcal{L}^D = \mathcal{L}^{H^*} \oplus g_d^T \langle x_{d-m+1}, \ldots, x_d \rangle$. □

We can now use Corollary 3.8 together with results of Shah detailed in remark 3.6 to establish that $F$ is semisimple and to further classify $F$ invariant subspaces of $\mathcal{L}$.

**Lemma 3.9.** $F$ is semisimple.
Proof. We prove that $\mathcal{L}^{F_u} = \mathcal{L}$, this implies that the unipotent radical is trivial and thus by remark 3.6 we will be done. Since $\mathcal{L}^{F_u}$ is $F_u$ invariant and $F_u$ is reductive there exists an $F_u$ invariant complement, $W$, to $\mathcal{L}^{F_u}$ such that $\mathcal{L} = \mathcal{L}^{F_u} \oplus W$. Let $S = W$ or $\mathcal{L}^{F_u}$, note that since $F_u$ can be defined over $\mathbb{Q}$ in either case $S$ can be defined over $\overline{\mathbb{Q}}$. Now since $D \subseteq \overline{F}$ it’s clear that $g_d D g_d^{-1} \subseteq F$. Moreover since $D$ is reductive, by Malcev’s Theorem, found in [OV90], Theorem 5, page 287, there exists $u \in F_u$ such that $u^{-1} g_d D g_d^{-1} u \subseteq F_u$. This means that $S$ is $u^{-1} g_d D g_d^{-1} u$ invariant and hence $g_d^{-1} u^T S$ is $D$ invariant, and thus by Lemma 3.3 either $u^T S \subseteq g_d^T \mathcal{L}^D$, or $u^T S = g_d^T \mathcal{L}_m \oplus U$ where $U \subseteq g_d^T \mathcal{L}^D$. Moreover, by comments on page 10 of [Bo66] we see that $u \in F_u \cap SL_d(\mathbb{Q})$ and so $u^T S$ is defined over $\overline{\mathbb{Q}}$, thus, Corollary 3.8 implies that $u^T S = g_d^T \mathcal{L}_m \oplus U$ where $U \subseteq g_d^T \mathcal{L}^D$. This is a contradiction since $u^T \mathcal{L}^{F_u}$ and $u^T W$ should have an empty intersection. This shows us either $\mathcal{L}^{F_u} = 0$ or $W = 0$. Finally we note that since $F_u$ is unipotent $\mathcal{L}^{F_u} \neq 0$, this proves the Lemma.

Lemma 3.10. There exists no non trivial $F$ invariant subspaces defined over $\overline{\mathbb{Q}}$ and contained in $\mathcal{L}$.

Proof. By Lemma 3.9 $F$ is semisimple and hence completely reducible. Suppose for a contradiction that $\mathcal{L} = V_1 \oplus \ldots \oplus V_k$ where each $V_i$ is a non trivial, irreducible $F$ invariant subspace. By remark 3.6 $F$ can be defined over $\mathbb{Q}$, this implies that each of the $V_i$ can be chosen to be defined over $\overline{\mathbb{Q}}$. Because $H^* < F$ each $V_i$ is $H^*$ invariant. Thus by Corollary 3.8 all but one of the spaces $V_i$ are contained in $\mathcal{L}^{H^*}$. Without loss of generality let $W = V_1 \oplus \ldots \oplus V_{k-1} \subseteq \mathcal{L}^{H^*}$, but since each $V_i$ is $F$ invariant so is $W$. It follows that we have found an $F$ invariant subspace defined over $\overline{\mathbb{Q}}$ contained in $\mathcal{L}^{H^*}$ in contradiction with Lemma 3.7. Thus we have a contradiction as required and our original decomposition of $\mathcal{L}$ must have been trivial.

The following is the final Lemma of the section and completes our classification of $F$ invariant subspaces of $\mathcal{L}$. In the course of the proof the notations of weights and weight spaces are used, the reader is directed to numerous books on Lie groups for details of this subject, for instance [OV90].

Lemma 3.11. There exists no non trivial $F$ invariant subspaces defined over $\mathbb{C}$ and contained in $\mathcal{L}$.

Proof. Let $\mathfrak{f}$ denote the Lie algebra of $F$, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{f}$ and $\mathfrak{t}^*$ its dual. By Lemma 3.9 $\mathfrak{f}$ is semisimple. We use the language and notation of Serre, [Se87]. Lemma 3.11 says that, if $\mathcal{L}_\mathbb{Q}$ is the space of linear forms defined over $\overline{\mathbb{Q}}$ then $\mathcal{L}_\mathbb{Q}$ is an irreducible $\mathfrak{f}$-module. Thus the Theorem of the highest weight, which can be found on page 60 of [Se87], implies that $\mathcal{L}_\mathbb{Q}$ has a highest weight. If $\mathcal{L}_\mathbb{C}$ is the space of linear forms defined over $\mathbb{C}$, we claim that the weights of $\mathcal{L}_\mathbb{C}$ are the same as the weights of $\mathcal{L}_\mathbb{Q}$. It is easy to see that the weights of $\mathcal{L}_\mathbb{Q}$ form a subset of the weights of $\mathcal{L}_\mathbb{C}$. The situation when there is a weight of $\mathcal{L}_\mathbb{C}$, that is not a weight of $\mathcal{L}_\mathbb{Q}$, is the one we should rule out. Since $\mathbb{Q}$ and $\mathbb{C}$ are algebraically closed we can decompose $\mathcal{L}_\mathbb{Q} = \bigoplus_{\omega \in \mathfrak{t}^*} \mathcal{L}_\mathbb{Q}^\omega$ and $\mathcal{L}_\mathbb{C} = \bigoplus_{\omega \in \mathfrak{t}^*} \mathcal{L}_\mathbb{C}^\omega$. We can write $\mathcal{L}_\mathbb{C} = \mathcal{L}_\mathbb{Q} \otimes \mathbb{C}$, and thus combined with our decomposition for $\mathcal{L}_\mathbb{Q}$ we get that $\mathcal{L}_\mathbb{C} = \bigoplus_{\omega \in \mathfrak{t}^*} \mathcal{L}_\mathbb{Q}^\omega \otimes \mathbb{C}$. Comparing this with the earlier decomposition of $\mathcal{L}_\mathbb{C}$ shows us for any weight, $\omega$, we have $\mathcal{L}_\mathbb{Q}^\omega \otimes \mathbb{C} = \mathcal{L}_\mathbb{C}^\omega$. Thus, if $\mathcal{L}_\mathbb{Q}^\omega = 0$, then $\mathcal{L}_\mathbb{C}^\omega = 0$, this implies the claim. Therefore $\mathcal{L}_\mathbb{C}$ has a highest weight and thus $\mathcal{L}_\mathbb{C}$ is an irreducible $\mathfrak{f}$-module, this implies the Lemma.
4. Proof of the main Theorem.

To prove Theorem 1.2 we will proceed to show that the group $F$ obtained from the application of Ratner’s Theorem is $G = SO(p, q)^o$, once this is established Theorem 1.2 follows by standard arguments. Since, in the course of proving this fact some cumbersome notation is used, possibly obscuring the underlying idea, an outline of the proof is presented as follows.

1. The Lie algebra of $\hat{F}$ is decomposed into subspaces defined in terms of 4 by 4 block matrices.
2. Then it is shown that, if the intersection of these subspaces with the Lie algebra of $\hat{F}$ is trivial in certain cases, then $F$ will have non trivial invariant subspaces contained in $L^{H^*}$, contradicting Lemma 3.11.
3. Therefore the intersection of these subspaces with the Lie algebra of $\hat{F}$ is non trivial, this is used to show that $\hat{F}$ contains an enlarged copy of $H^*$.
4. This process can then be repeated until it is shown that $SO(p, q)^o \leq F$.

We now proceed with the actual proof.

**Lemma 4.1.** $F = SO(p, q)^o$.

**Proof.** We look at the Lie algebra of $\hat{F}$, denoted $\mathfrak{f}$. Since $\hat{H}^* = U_1 \ldots U_mD \leq \hat{F}$ the Lie algebra of $\hat{H}^*$, denoted $\mathfrak{h}^*$, is a subalgebra of $\mathfrak{f}$. The aim is to show that this implies one of the following,

1. $U_1 \ldots U_mD_{1,0,0} \leq \hat{F}$,
2. $U_1 \ldots U_mD_{0,1,0} \leq \hat{F}$,
3. $U_1 \ldots U_{m-1}D_{0,0,1} \leq \hat{F}$.

The first two cases occur when $r + n < d - 2m$ and the third case occurs when $r + n = d - 2m$. Once we have established the above claim, then we can replace $(r, n, m)$ by $(r + 1, n, m)$ in the first case, by $(r, n + 1, m)$ in the second case and by $(r + 1, n + 1, m - 1)$ in the last case. Then it is possible to repeat the entire argument until the claim of the Lemma has been obtained.

Let

$$f = \begin{pmatrix} f_{11} & f_{21} & f_{31} & f_{41} \\ f_{12} & f_{22} & f_{32} & f_{42} \\ f_{13} & f_{23} & f_{33} & f_{43} \\ f_{14} & f_{24} & f_{34} & f_{44} \end{pmatrix}$$

where $f_{11}, f_{14}, f_{41}, f_{44} \in \text{Mat}_m(\mathbb{R})$, $f_{12}^T, f_{21}, f_{12}^T, f_{21} \in \text{Mat}_{s-m,m}(\mathbb{R})$, $f_{22} \in \text{Mat}_{s-m,m}(\mathbb{R})$, $f_{32}, f_{23}^T \in \text{Mat}_{d-s-m,m}(\mathbb{R})$ and $f_{31}, f_{43}, f_{34}, f_{43}^T \in \text{Mat}_{d-s-m,m}(\mathbb{R})$. Since $\mathfrak{f}$ is a subalgebra of $\mathfrak{so}(Q_0)$ we have that any $f \in \mathfrak{f}$ must satisfy the relation $f^TQ' + Q'f = 0$, where $Q'$ is the matrix that defines the quadratic form $Q_0$. So, suppose that $f \in \mathfrak{f}$, and compute as follows,

$$f^T \begin{pmatrix} 0 & 0 & 0 & B_m \\ 0 & Q'' & 0 & 0 \\ 0 & 0 & I_{r,n} & 0 \\ B_m & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & B_m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{r,n} & 0 \\ B_m & 0 & 0 & 0 \end{pmatrix} = 0,$n

where $Q''$ is the matrix that defines $Q_{m+1, \ldots, s}$. The above computation yields the following, $f_{14}, f_{44} \in \mathfrak{so}(B_m)$, $f_{22} \in \mathfrak{so}(Q'')$, $f_{33} \in \mathfrak{so}(r, n)$, $f_{12}Q'' = -B_m f_{34}^T$, $f_{13}I_{r,n} = -B_m f_{44}^T$, $f_{11}B_m = -B_m f_{44}^T$. 


\( f_{23} I_{r,n} = -Q'' f_{12}^T, \) \( f_{21} B_m = -Q'' f_{12}^T \) and \( f_{31} B_m = -I_{r,n} f_{13}^T. \) Considering these relations, define the following subspaces,

\[
\begin{align*}
\mathbf{v}^+ &= \{ \begin{pmatrix} 0 & s & 0 & 0 \\ 0 & 0 & 0 & -B_m s^T Q'' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : s \in \text{Mat}_{s-m, m}(\mathbb{R}) \} \\
\mathbf{v}^- &= \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & -Q'' s^T B_m & 0 & 0 \end{pmatrix} : s \in \text{Mat}_{m, s-m}(\mathbb{R}) \} \\
\mathbf{v} &= \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{r,n} s^T Q' & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : s \in \text{Mat}_{s-m, d-s-m}(\mathbb{R}) \} \\
\mathbf{a} &= \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \} \\
\mathbf{m} &= \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \} \\
\mathbf{d} &= \{ \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_m d B_m \end{pmatrix} : d \in \text{Mat}_m(\mathbb{R}) \} \\
\mathbf{u}^- &= \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -B_m t^T I_{p,q} & 0 & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix} : t \in \text{Mat}_{d-s-m, m}(\mathbb{R}) \} \\
\mathbf{u}^+ &= \{ \begin{pmatrix} 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{p,q} t^T B_m \\ 0 & 0 & 0 & 0 \end{pmatrix} : t \in \text{Mat}_{d-s-m, m}(\mathbb{R}) \} \\
\mathbf{b}^+ &= \begin{pmatrix} 0 & 0 & 0 & \mathfrak{so}(B_m) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Since \( D \leq \hat{F}, \) the Lie algebra \( m \) is a subalgebra of \( f. \) Therefore, considering \( f, m, v \) and \( a \) as vector spaces, we have that \( m \oplus ((v \oplus a) \cap f) \leq f. \)
Let \( v_k = \{ t \in v \text{ such that } s_{ij} = 0 \text{ for all } j \neq k \} \), in other words, \( v_k \) is the subspace of \( v \) that consists of the \( k \)th row of \( v \). Note that \( v_k \) is a Lie algebra but \( v \) is not unless \( s - m = 1 \). Again by considering the \( v_k \)'s and \( v \) as vector spaces we can write \( v = v_1 \oplus \ldots \oplus v_{s-m} \) where we cannot split the \( v_k \)'s into further \( m \) invariant subalgebras since \([h,v] \in v_k \) for all \( h \in m \) and \( v \in v_k \). So to summarise the above,

\[
\begin{align*}
m \oplus ((v_1 \oplus \ldots \oplus v_{s-m} \oplus a) \cap f) & \subseteq f.
\end{align*}
\]

Since \( f \) is a Lie algebra and hence closed under the Lie bracket, we have the following implication, if \( f \cap v^\perp \neq 0 \), then \( f \cap v \neq 0 \), this follows since \( h^\ast \) is a subalgebra of \( f \) and \( h^\ast = u^- \oplus m \), then choosing \( h \in h^\ast \) and \( v \in v^\perp \) we see that \([h,v] \cap v \) contains non trivial elements. Additionally, we note that if \( r + n < d - 2m \), then the above observations imply that \( f \cap v \neq 0 \), since otherwise \( f \) would be contained in a subspace of the form

\[
\begin{align*}
\begin{pmatrix}
* & 0 & * & * \\
0 & * & 0 & 0 \\
* & 0 & * & * \\
* & 0 & * & *
\end{pmatrix}
\end{align*}
\]

This would imply that there exists an \( F \) invariant subspace \( V \subseteq \mathcal{L}^{H^+} \) contradicting Lemma 3.11. Therefore at least one of the \( v_k \neq 0 \), so without loss of generality suppose that \( v_{s-m} \neq 0 \). This means that

\[
\begin{align*}
f & \supseteq m \oplus ((v_1 \oplus \ldots \oplus v_{s-m} \oplus a) \cap f) \\
& = m \oplus v_{s-m} \oplus ((v_1 \oplus \ldots \oplus v_{s-m-1} \oplus a) \cap f) \\
& \cong \begin{cases} 
\mathfrak{so} (r, n+1) \oplus ((v_1 \oplus \ldots \oplus v_{s-m-1} \oplus a) \cap f) & \text{if } q' \geq 1 \\
\mathfrak{so} (r-1, n) \oplus ((v_1 \oplus \ldots \oplus v_{s-m-1} \oplus a) \cap f) & \text{otherwise},
\end{cases}
\end{align*}
\]

where in the last step we use that \( v_{s-m} \neq 0 \) and the following relation

\[
v_k \oplus \mathfrak{so} (r,n) \cong \begin{cases} 
\mathfrak{so} (r+1, n) & \text{for } 1 \leq k \leq p' \\
\mathfrak{so} (r,n+1) & \text{for } q' < k \leq s-m
\end{cases}
\]

which holds provided that \( v_k \neq 0 \). So in the case that \( r + n < d - 2m \), we have shown that either, \( U_1 \ldots U_mD_{1,0,0} \leq \tilde{F} \) or \( U_1 \ldots U_mD_{0,1,0} \leq \tilde{F} \).

We are left with the case when \( r + n = d - 2m \). In this case the subspaces, \( a, b^\perp \) and \( v \) become trivial and we are really dealing with \( 3 \) by \( 3 \) matrices. For this case we introduce the additional notation. Let \( \tilde{\mathfrak{d}} = \{ d \in \mathfrak{d} \text{ such that } d_{lk} = 0 \text{ for } lk \neq ii \} \) be the subspace of \( \mathfrak{d} \) consisting of elements of \( \mathfrak{d} \) such that every entry is zero except for the entry in the \( jj \)th matrix position. Moreover, let \( u_k^\pm = \{ u \in u^\pm \text{ such that } t_{kl} = 0 \text{ for } l \neq k \} \) be the subspace of \( u^\pm \) which consists of the \( k \)th row of \( u^\pm \).

Considering \( u^\pm \) and the \( u_k^\pm \)'s as vector spaces, we can write \( u^\pm = u_1^\pm \oplus \ldots \oplus u_m^\pm \). Note that the \( u_k^\pm \)'s are Lie algebras and that we cannot split the \( u_k^\pm \)'s into further \( m \) invariant subalgebras since, as the action of \( m \) on \( u_k^\pm \) is irreducible we have that \([s,u] \in u_k^\pm \) for all \( s \in m \) and \( u \in u_k^\pm \). Since \( U_1 \ldots U_mD \leq F \) we have \( u^- \oplus m \leq f \) and that

\[
f \supseteq u^- \oplus m \oplus \left( (\mathfrak{d} \oplus u^+ \oplus b^+) \cap f \right),
\]

where again we consider the decomposition as a decomposition of vector spaces.
If \( f \cap b^+ \neq 0 \), then \( f \cap u^+ \neq 0 \), since \( f \) is closed under the Lie bracket. Explicitly, \( f \) contains \( h^* = u^+ \oplus m \), and so taking \( b \in b^+ \) and \( h \in h^* \) and checking that \( [u, b] \cap u^+ \) contains non-trivial elements verifies the assertion.

This observation and Lemma 4.11 means at least one of the \( u^+_k \) is not zero since otherwise \( f \) would be contained in a subspace of the form

\[
\begin{pmatrix}
* & 0 & 0 \\
* & * & 0 \\
* & * & *
\end{pmatrix},
\]

which would mean there was an \( F \) invariant subspace \( V \) such that \( V \subseteq L^{H^*} \). Suppose without loss of generality that \( u^+_m \neq 0 \), since \( f \) is closed under the Lie bracket this implies that \( f \cap \widehat{\delta} \neq 0 \), where we can check this claim as before. Now we can compute

\[
f \supseteq u^+ \oplus m \oplus (\langle d \oplus u^+ \rangle \cap f)
\]

\[
\supseteq \bigoplus_{i=1}^{m} u^{-}_i \oplus m \oplus \widehat{\delta} \oplus u^+_m \oplus \left( \bigoplus_{i=1}^{m-1} u^+_i \right) \cap f
\]

\[
\supseteq \bigoplus_{i=1}^{m} u^{-}_i \oplus so (r + 1, n + 1).
\]

In the second step we use that \( d \supseteq \widehat{\delta} \neq 0 \) and in the last step we use that \( u^+_m \oplus m \oplus \widehat{\delta} \oplus u^+_m \cong so (r + 1, n + 1) \), provided that \( u^+_m \), \( u^+_m \) and \( \widehat{\delta} \) are not zero.

This means that \( U_1 \ldots U_{m-1} D_{0,0,1} \leq F \) and we are done by the initial remarks. \( \square \)

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Rewrite \( \overline{M (X_Z)} = \{ M (x) : x \in X_Z \} \) and by using the definition of \( H^* \) we see \( \{ M (x) : x \in X_Z \} = \{ M (H^* x) : x \in X_Z \} \). Now we can restrict our attention to a \( \Gamma \) orbit in \( X_Z \) to get \( \{ M (H^* x) : x \in X_Z \} \supseteq \{ M (H^* G x) : x \in X_Z \} \). By Lemma 4.1 and Ratner’s Theorem we have \( \{ M (H^* G x) : x \in X_Z \} \supseteq \{ M (G x) : x \in X_Z \} \) and since \( G \), being the identity component of \( SO (p, q) \), acts transitively on connected components of \( X_R \) we have \( \{ M (G x) : x \in X_Z \} = \{ M (x) : x \in X_R \} \) since if \( X_R \) is not connected, then if \( x \in X_Z \) we have \( -x \in X_Z \) and \( x \) and \( -x \) lie in the two separate components of \( X_R \). The fact that \( Q \) is non-degenerate and indefinite means that \( X_R \cap \{ x \in R^d : M (x) = b \} \) is non-empty for every \( b \in R^d \) or, in other words, that \( \{ M (x) : x \in X_R \} = R^d \) and so we are done. \( \square \)

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