PTSC: a New Definition for Structural Controllability under Numerical Perturbations

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Abstract—This paper proposes a novel notion for structural controllability under structured numerical perturbations, namely the perturbation-tolerant structural controllability (PTSC), on a single-input structured system whose entries can be classified into three categories: fixed zero entries, unknown generic entries whose values are fixed but unknown, and perturbed entries that can take arbitrary complex values. Such a system is PTSC if, for almost all values of the unknown generic entries in the parameter space, the corresponding controllable system realizations can preserve controllability under arbitrary complex-valued perturbations with their structure prescribed by the perturbed entries. This new notion can characterize the generic property in controllability preservation under structured numerical perturbations. We give a necessary and sufficient condition for a single-input system to be PTSC, whose verification has polynomial time complexity. Our results can serve as some feasibility conditions for the conventional structured controllability radius problems from a generic view.

Index Terms—Structural controllability, strong structural controllability, structured numerical perturbations, generic property

I. INTRODUCTION

In recent years, security has been becoming an attractive issue in the control and estimation of cyber-physical systems, such as chemical processes, power grids and transportation networks [14]. The robustness of various system properties have been investigated under internal faults (like disconnections of links/nodes [2, 5]) or external attacks (like adversarial sensor/actuator attacks [3]), including stability [6], stabilization [7], controllability and observability [8–12]. Particularly, as a fundamental system property, controllability/observability under structural perturbations (such as link/node/actuator/sensor removals or deletions) has been extensively explored on its robust performance. To name a few, [8] considered observability preservation under sensor removals, [9] investigated controllability preservation under simultaneous link and node failures, while [10–12] systematically studied the involved optimization problems with respect to link/node/actuator/sensor removals or deletions from a computational perspective. Since controllability/observability is a generic property that depends mainly on the system structure [13], its robustness is mainly dominated by the robustness property of the corresponding graphs.

Note that structural perturbation is a kind of numerical perturbation that makes the corresponding link has a zero weight. In the more general case where the perturbed links do not necessarily result in zero weights, controllability robustness has also been studied by computing the distance (in terms of the 2-norm or the Frobenius norm) from a controllable system to the set of uncontrollable systems [14–17]. Such a notion, also named controllability radius, was first proposed by [14], and then developed by several other researchers on its efficient computations [15–17]. Recently, by restricting the perturbation matrices to a prescribed structure, the so-called structured controllability radius problem (SCRP) has also attracted researchers’ interest, i.e., the problem of determining the smallest (Frobenius or 2-) norm additive perturbation with a prescribed structure for which controllability fails to hold [18]. Towards this problem, various numerical algorithms have been proposed [19–21]. However, due to the nonconvexity of this problem, all these algorithms are suboptimal [20]. Moreover, since most of these algorithms adopted some relaxation and iterative techniques and owing to the involved rounding errors, there is usually no guarantee that the returned numerical perturbations can make the original system uncontrollable.

On the other hand, to avoid the potential numerical issues, strong structural controllability (SSC), a notion proposed by Mayeda and Yamada [22], could also be used to measure the controllability robustness of a system against numerical perturbations. In the SSC theory, the system parameters are divided into two categories, indeterminate parameters and fixed zero parameters. A system is SSC, if whatever values (other than zero) the indeterminate parameters of the system may take, the system is controllable. Criteria for SSC in the single-input case was given in [22], and then extended to the multi-input cases in [23–24], as well as allowing the existence of parameters that can take arbitrary values, including zero and nonzero [25]. Note for SSC to measure controllability robustness, the numerical perturbations should have the same zero/nonzero structure as the original systems. While in practice, perturbations could be imposed to only partial system components (such as a subset of links of a network) and do not necessarily have the same structure as the original systems.

In this paper, we propose a new definition for structural controllability under structured numerical perturbations, namely, the perturbation-tolerant structural controllability (PTSC). The entries for system matrices are classified into three categories: fixed zero entries, unknown generic entries whose values are fixed but unknown (they can be seen as randomly selected values), and perturbed entries that can take arbitrary complex values. The notion of PTSC is defined in the following way: a system is PTSC if, for almost all values of the unknown generic entries in the parameter space, the corresponding controllable system realizations can preserve controllability under arbitrary complex-valued perturbations with structure prescribed by the
perturbed entries. The main contributions of this paper are as follows:

1) We propose a novel notion PTSC to study controllability preservation for a single-input structured system under structured numerical perturbations. In PTSC, the perturbation structure can be arbitrary relative to the structure of the original system. This notion provides a new view in studying the robustness of structural controllability other than structural perturbations.

2) We have shown PTSC can characterize the generic property that, depending on the structure of the original system and the perturbations, for almost all of the controllable system realizations, either they can preserve controllability under arbitrary complex-valued perturbations with the given structure, or there is a perturbation with the given structure that can make the corresponding system fail to be controllable. This is beneficial in studying the SCRPs from a generic view.

3) We give a necessary and sufficient condition for a single-input system to be PTSC, whose verification has polynomial time complexity. The derivation is based on the one-edge preservation principle and a series of nontrivial results on the roots of determinants of generic matrix pencils.

The rest is organized as follows. Section II introduces the PTSC notion and proves the involved genericity for single-input systems. Section III presents some preliminaries required for our further derivations. Section IV gives a necessary and sufficient condition for a single-input system to be PTSC. Section V discusses the application of PTSC on the SCRPs. The last section concludes this paper.

Notations: Given an integer \( p \geq 0 \), define \( J_p = \{1, \ldots, p\} \). For a \( p \times q \) matrix \( M \), \( M[I_1, I_2] \) denotes the submatrix of \( M \) whose rows are indexed by \( I_1 \) and columns by \( I_2 \), \( I_1 \subseteq J_p, I_2 \subseteq J_q \). For a vector \( b, b_i \) denotes the \( i \)th entry of \( b \).

II. THE NOTION OF PTSC

A. Structured Matrix

Before presenting the notion of PTSC, we first introduce the so-called structured matrix. A structured matrix is a matrix whose entries are either fixed zero (denoted by 0) or indeterminate parameters (denoted by \( * \)). For description simplicity, we may simply say the entry represented by \( * \) is a nonzero entry. Let \( \{0, *\}^{p \times q} \) be the set of all \( p \times q \) dimensional structured matrices. For \( M \in \{0, *\}^{p \times q} \), the following two sets of matrices are defined:

\[ S_M = \{ M \in \mathbb{C}^{p \times q} : M_{ij} = 0 \text{ if } \hat{M}_{ij} = 0 \}, \]
\[ \overline{S}_M = \{ M \in \mathbb{C}^{p \times q} : M_{ij} = 0 \text{ if } \hat{M}_{ij} = 0, M_{ij} \neq 0 \text{ if } \hat{M}_{ij} = * \}. \]

Any \( M \in S_M \) is called a realization of \( \hat{M} \). For two structured matrices \( \hat{M}, \hat{N} \in \{0, *\}^{p \times q}, \triangledown \) is the entry-wise OR operation, i.e., \( (\hat{M} \triangledown \hat{N})_{ij} = * \text{ if } \hat{M}_{ij} = * \) or \( \hat{N}_{ij} = * \); otherwise \( (\hat{M} \triangledown \hat{N})_{ij} = 0 \).

A structured matrix could also be seen as a matrix whose entries are parameterized by the free parameters of its \( * \) entries, and therefore is sometimes called a generic matrix \(^{[24]}\). For a generic matrix \( M \) and a constant matrix \( N \) with the same dimension, \( M + \lambda N \) defines a generic matrix pencil, which can be seen as a matrix-valued polynomial of free parameters in \( M \) and the variable \( \lambda \).

B. Notion of PTSC

Consider the linear time invariant (LTI) system
\[ \dot{x}(t) = Ax(t) + bu(t), \]
where \( A \in \mathbb{C}^{n \times n}, b \in \mathbb{C}^{n \times 1} \). It is known that \((A, b)\) is controllable, if and only if the controllability matrix \( C(A, b) \) defined as follows has full row rank.

\[ C(A, b) = [b, Ab, \ldots, A^{n-1}b]. \]

Let \( \bar{F} = \{0, *\}^{n \times (n+1)} \) be a structured matrix that specifies the structure of the perturbation (matrix) \([\Delta A, \Delta b]\), that is, \( \bar{F}_{ij} = 0 \) implies \([\Delta A, \Delta b]_{ij} = 0 \). In other words, \([\Delta A, \Delta b]\) is in \( \bar{S}_F \). It is emphasized that throughout this paper, the perturbations are allowed to be complex-valued. We will also call the system \((A + \Delta A, b + \Delta b)\) the perturbed system.

**Definition 1** (PTC): System \((A, b)\) in \( \bar{F} \) is said to be perturbation-tolerantly controllable (PTC) with respect to \( \bar{F} \), if for all \((\Delta A, \Delta b)\in S_F\), \((A + \Delta A, b + \Delta b)\) is controllable.

If \((A, b)\) is controllable but not PTC w.r.t. \( \bar{F} \) (i.e., there exists a \((\Delta A, \Delta b)\in S_F\) making \((A + \Delta A, b + \Delta b)\) uncontrollable), \((A, b)\) is said to be perturbation-sensitively controllable (PSC) w.r.t. \( \bar{F} \).

Let \( \bar{A} \in \{0, *\}^{n \times n}, \bar{b} \in \{0, *\}^{n \times 1} \) be the structured matrices specifying the sparsity pattern of \( A, b \), respectively. That is, \( A \in S_{\bar{A}} \) and \( b \in S_{\bar{b}} \).

**Definition 2** (Structural controllability): \((\bar{A}, \bar{b})\) is said to be structurally controllable, if there exists a realization \((A, b)\in S_{[\bar{A}, \bar{b}]}\) so that \((A, b)\) is controllable.

A property is called generic for a set of systems if, depending on the (common) structure of parameterized systems in this set, either this property holds for almost all of the system parameters in the corresponding parameter space, or this property does not hold for almost all of the system parameters \(^{[13]}\). It is well-known that controllability is a generic property in the sense that, if \((A, b)\) is structurally controllable, then all realizations of \((\bar{A}, \bar{b})\) are controllable except for a set with zero Lebesgue measure in the corresponding parameter space. For a structurally controllable pair \((\bar{A}, \bar{b})\), let \( CS(\bar{A}, \bar{b}) \) denote the set of all controllable complex-valued realizations of \((\bar{A}, \bar{b})\).

**Proposition 1**: With \((\bar{A}, \bar{b})\) and \( \bar{F} \) defined above, suppose that \((\bar{A}, \bar{b})\) is structurally controllable. Then, either for all \((A, b)\in CS(\bar{A}, \bar{b})\), \((A, b)\) is PTC w.r.t. \( \bar{F} \), or for almost all \((A, b)\in CS(\bar{A}, \bar{b})\) except for a set with zero Lebesgue measure in the corresponding parameter space, \((A, b)\) is PSC w.r.t. \( \bar{F} \).

**Proof**: Let \( p_1, \ldots, p_r \) be variables that the \( r \) nonzero entries of \([A, b]\) take, and \( \bar{p}_1, \ldots, \bar{p}_l \) be variables that the \( l \) perturbed entries of \([\Delta A, \Delta b]\) take. Denote by \( p \doteq (p_1, \ldots, p_r) \) and \( \bar{p} \doteq (\bar{p}_1, \ldots, \bar{p}_l) \). It turns out det \( C(A + \Delta A, b + \Delta b) \) can be expressed as

\[ \det C(A + \Delta A, b + \Delta b) = f(p)g(\bar{p})h(p, \bar{p}), \]

where \( f(p) \) (resp. \( g(\bar{p}) \)) denotes the polynomial of \( p \) (resp. \( \bar{p} \)) with real coefficients, and \( h(p, \bar{p}) \) denotes the polynomial of \( p \) and \( \bar{p} \) where at least one \( p_i \) (\( i \in \{1, \ldots, r\} \)) and one \( \bar{p}_j \) (\( j \in \{1, \ldots, l\} \)) both have a term with degree no less than one.

It can be seen that, if neither \( g(\bar{p}) \) nor \( h(p, \bar{p}) \) exists in \( \bar{F} \), then for all \((A, b)\in CS(\bar{A}, \bar{b})\), \((A, b)\) is PTC w.r.t. \( \bar{F} \), as in
plex values. PTSC of the perturbed structured system requires but unknown values (they can be seen as randomly generated, the system matrices can be divided into three categories, namely, the fixed zero entries, the unknown generic entries which take fixed but unknown values (they can be seen as randomly generated values), and the perturbed entries which can take arbitrarily complex values. PTSC of the perturbed structured system requires that for almost all values of the unknown generic entries making the original system controllable, the corresponding perturbed systems are controllable for arbitrary values of the perturbed entries.

Recall that SSC is defined as follows.

**Definition 4 (SSC):** \((\bar{A}, \bar{b})\) is said to be SSC, if every \((A, b)\) \in \(S_{[\bar{A}, \bar{b}]}\) is controllable.

As mentioned earlier, SSC could be seen as the ability of a system to preserve controllability under perturbations that have the same structure as the system itself, with the constraint that the perturbed entries of the resulting system cannot be zero. It is thus clear that the essential difference between PTSC and SSC lies in two aspects: First, the perturbed entries can take arbitrary values including zero in PTSC, while they must take nonzero values in SSC. Second, in SSC, all nonzero entries can be perturbed, while in PTSC, an arbitrary subset of entries (prescribed by the perturbation structure) can be perturbed, and the remaining entries remain unchanged. Because of them, neither criteria for SSC can be converted to those for PTSC, nor the reverse.

### III. Preliminaries and Terminologies

In this section, we introduce some preliminaries as well as terminologies in graph theory and structural controllability.

#### A. Graph Theory

If not specified, all graphs in this paper refer to directed graphs. A graph is denoted by \(G = (V, E)\), where \(V\) is the vertex set, and \(E \subseteq V \times V\) is the edge set. For a graph \(G = (V, E)\) with \(N\) vertices, a path from vertex \(v_i\) to vertex \(v_j\) is a sequence of edges \((v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)\) where each edge belongs to \(E\). For a set \(E_s \subseteq E\), \(G-E_s\) denotes the graph obtained from \(G\) after deleting the edges in \(E_s\); similarly, for \(V_s \subseteq V\), \(G-V_s\) denotes the graph after deleting vertices in \(V_s\) and all edges incident to vertices in \(V_s\). For two graphs \(G_1 = (V_1, E_1)\), \(i = 1, 2\), \(G_1 \cup G_2\) denotes the graph \((V, E \cup V_2)\).

A graph \(G = (V, E)\) is said to be bipartite if its vertex set can be divided into two disjoint parts \(V_1\) and \(V_2\) such that no edge has two ends within \(V_1\) or \(V_2\). The bipartite graph \(G\) is also denoted by \((V_1, V_2, E)\). A matching of a bipartite graph is a subset of its edges among which any two do not share a common vertex. The maximum matching is the matching with the largest number of edges among all possible matchings. The number of edges contained in a maximum matching of a bipartite graph \(G\) is denoted by \(\nu(G)\). For a weighted bipartite graph \(G\), where each edge is assigned a non-negative weight, the weight of a matching is the sum of all edges contained in this matching. The minimum weight maximum matching (resp. maximum weight maximum matching) is the minimal weight (resp. maximal weight) over all maximum matchings of \(G\).

The generic rank of a structured matrix \(\bar{M}\), given by \(\text{rank}(\bar{M})\), is the maximum rank \(\bar{M}\) can achieve as the function of its free parameters. The bipartite graph associated with a structured matrix \(\bar{M}\) is given by \(B(\bar{M}) = (R, C, E)\), where the left (resp. right) vertex set \(R (C)\) corresponds to the row index (column index) set of \(\bar{M}\), and the edge set corresponds to the set of non-zero entries of \(\bar{M}\), i.e., \(E = \{(i, j) : i \in R, j \in C, M_{ij} \neq 0\}\). It is known that, \(\text{rank}(\bar{M})\) equals the cardinality of the maximum matching of \(B(\bar{M})\).

#### C. Relations with SSC

We may also revisit PTSC from the standpoint of the perturbed structured system \([\bar{A}, \bar{b}] \in \bar{F}\). In this system, entries of system matrices can be divided into three categories, namely, the fixed zero entries, the unknown generic entries which take fixed but unknown values (they can be seen as randomly generated values), and the perturbed entries which can take arbitrarily complex values. PTSC of the perturbed structured system requires
B. **DM-Decomposition**

Dulmage-Mendelsohn decomposition (DM-decomposition for short) is a unique decomposition of a bipartite graph w.r.t. maximum matchings. Let $G = (V^+, V^-, E)$ be a bipartite graph. For $M \subseteq E$, we denote by $V^+(M)$ (resp. $V^-(M)$) the set of vertices in $V^+$ (resp. $V^-$) incident to edges in $M$. An edge of $G$ is said to be admissible, if it can be contained in some maximum matching of $G$.

**Definition 5** (DM-decomposition): The DM-decomposition of a bipartite graph $G = (V^+, V^-, E)$ is to decompose $G$ into subgraphs $G_k = (V_k^+, V_k^-, E_k)$ ($k = 0, 1, \ldots, d, \infty$) (called DM-components of $G$) satisfying:

1. $V^* = \bigcup_{i=0}^{\infty} V_k^*$, $V_i^+ \cap V_j^+ = \emptyset$ for $i \neq j$, with $* = +$ and $-$; $E_k = \{(v^+, v^-) \in E : v^+ \in V_k^+, v^- \in V_k^\}$. 
2. For $0 \leq k \leq d$ (consistent components): $\text{nt}(G_k) = |V_k^+| = |V_k^-|$, and each $e \in E_k$ is admissible in $G_k$: for $k = 0$ (horizontal tail): $\text{nt}(G_0) = |V_0^+| = |V_0^-|$, if $V_0^+ \neq \emptyset$, and each $e \in E_0$ is admissible in $G_0$; for $k = \infty$ (vertical tail): $\text{nt}(G_\infty) = |V_\infty^+| = |V_\infty^-|$, if $V_\infty^+ \neq \emptyset$, and each $e \in E_\infty$ is admissible in $G_\infty$;
3. $E_{kl} = \emptyset$ unless $1 \leq k \leq l \leq d$, and $E_{kl} \neq \emptyset$ only if $1 \leq k \leq l \leq d$, where $E_{kl} = \{e \in E : V^+\{e\} \subseteq V_k, V^-\{e\} \subseteq V_l\}$;
4. $G$ cannot be decomposed into more consistent components $1)-3)$.

For an $m \times l$ matrix $M$, the DM-decomposition of $B(M)$ into digraphs $G_k = (V_k^+, V_k^-, E_k)$ ($k = 0, 1, \ldots, d, \infty$) corresponds to the fact that there exist two permutation matrices $P \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{l \times l}$ satisfying

$$PMQ = \begin{bmatrix}
M_0 & M_{01} & \cdots & M_{0l} & M_{0\infty} \\
0 & M_1 & \cdots & M_{1l} & M_{1\infty} \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & M_d & M_{d\infty} \\
0 & 0 & \cdots & \cdots & \cdots
\end{bmatrix},$$

(3)

where the submatrix $M_k = M[V_k^+, V_k^-]$ corresponds to $G_k$ ($k = 0, 1, \ldots, d, \infty$). Matrix $M$ is also called the DM-decomposition of $M$.

A bipartite graph is said to be DM-irreducible if it cannot be decomposed into more than one nonempty component in the DM-decomposition. DM-decomposition is closely related to the irreducibility of the determinant of a generic matrix. Recall that a multivariable polynomial $f$ is irreducible if it cannot be factored as $f = f_1 f_2$ with $f_1, f_2$ being polynomials with smaller degrees than $f$.

**Lemma 1** [26, Theorems 2.2.24, 2.2.28]: For a bipartite graph $G(M) = (V^+, V^-, E(M))$ associated with a generic square matrix $M$, the following conditions are equivalent:

1. $G(M)$ is DM-irreducible;
2. $\text{nt}(G(M) - \{v_1, v_2\}) = \text{nt}(G(M)) - 1$ for any $v_1 \in V^+$ and $v_2 \in V^-;
3. \det M$ is irreducible.

C. **Structural Controllability**

Given $(\bar{A}, \bar{b})$, let $X, U$ denote the sets of state vertices and input vertices respectively, i.e., $X = \{x_1, \ldots, x_n\}$, $U = \{x_{n+1}\}$. Denote the edges by $\mathcal{E}_X.X(\bar{A}) = \{(x_i, x_j) : Aij \neq 0\}$, $\mathcal{E}_U.X(\bar{b}) = \{(x_{n+1}, x_i) : \bar{b}_i \neq 0\}$. Let $G(\bar{A}, \bar{b}) = (X \cup U, \mathcal{E}_X.X(\bar{A}) \cup \mathcal{E}_U.X(\bar{b}))$ be the system graph associated with $(\bar{A}, \bar{b})$. A state vertex $x \in X$ is said to be input-reachable, if there exists a path from an input vertex $x_{n+1}$ to $x$ in $G(\bar{A}, \bar{b})$. Similarly, $G(F) = (X \cup U, \mathcal{E}_F)$ denotes the graph associated with the perturbation structure $\bar{F}$, where $\mathcal{E}_F = \{(x_i, x_j) : F_{ij} \neq 0, 1 \leq j \leq n, 1 \leq i \leq n + 1\}$.

**Lemma 2** [26]: System $(\bar{A}, \bar{b})$ in (1) is structurally controllable, if and only if 1) every state vertex $x \in X$ is input-reachable; 2) $\text{rank}(\bar{A}, \bar{b}) = n$.

IV. **NECESSARY AND SUFFICIENT CONDITION**

In this section, we present a necessary and sufficient condition for the PTSC in the single-input case.

A. **One-edge Preservation Principle**

At first, a one-edge preservation principle is given as follows, which is fundamental to our subsequent derivations.

**Proposition 2** (One-edge preservation principle): Suppose $(\bar{A}, \bar{b})$ is structurally controllable. $(\bar{A}, \bar{b})$ is PSSC w.r.t. $\bar{F}$, if and only if there exists one edge $e \in E(\bar{F})$, such that $[\bar{A}, \bar{b}] \cup F(e)$ is PSSC w.r.t. $\bar{F}(e)$, where $F(e)$ denotes the structured matrix associated with the graph $\bar{G}(F) - \{e\}$, and $\bar{F}(e)$ the structured matrix obtained from $\bar{F}$ by preserving only the entry corresponding to $e$.

**Proof**:

Let $p = (p_1, \ldots, p_r)$ and $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_s)$ be defined similarly in the the proof of Proposition 1. From the analysis in that proof, $(\bar{A}, \bar{b})$ is PSSC w.r.t. $\bar{F}$, if and only if there exists a $\bar{p}_j, j \in \{1, \ldots, l\}$, that has a degree no less than one in $\det(C(\bar{A} + \Delta \bar{A}, \bar{b} + \Delta \bar{b}))$ (expressed in (2)), where $(\bar{A}, \bar{b})$ and $(\Delta \bar{A}, \Delta \bar{b})$ are realizations of $(\bar{A}, \bar{b})$ and $(\Delta \bar{A}, \Delta \bar{b})$ respectively, with the corresponding parameters being $p$ and $\bar{p}$. Let $e$ be the edge corresponding to $\bar{p}_j$. Suppose that the coefficient of $\bar{p}_j^k$ is nonzero for some degree $k \geq 1$. Since the coefficient of $\bar{p}_j^k$ is a polynomial of $p$ and $\bar{p}_j \{\bar{p}_j\}$ in $\det(C(\bar{A} + \Delta \bar{A}, \bar{b} + \Delta \bar{b}))$, it always equals that in $\det(C(\bar{A} + \Delta \bar{A}, \bar{b} + \Delta \bar{b}))$, where $(\bar{A}, \bar{b})$ has the system graph $\bar{G}(\bar{A}, \bar{b}) \cup \bar{G}(\bar{F}) - \{e\}$, and $(\Delta \bar{A}, \Delta \bar{b})$ corresponds to the perturbation $\bar{F}(e)$, noting that $[\bar{A} + \Delta \bar{A}, \bar{b} + \Delta \bar{b}] = [\bar{A} + \Delta \bar{A}, \bar{b} + \Delta \bar{b}]$ in the symbolic operation sense. Upon observing this, the proposed statement follows immediately.

It is remarkable that the one-edge preservation principle does not mean the perturbation of only one entry is enough to destroy controllability; Instead, we means we can regard $|\mathcal{E}_F| - 1$ entries of $\bar{F}$ as unknown generic entries (in other words, their values can be chosen randomly; but not fixed zero) and find suitable values for the last entry. This principle indicates that verifying the PTSC w.r.t. an arbitrary perturbation structure can be reduced to an equivalent problem with a single-edge perturbation structure. Having observed this, in the following, we will give the conditions for the absence of zero uncontrollable modes and nonzero uncontrollable modes, respectively, in the single-edge perturbation scenario. Recall that an uncontrollable mode for $(\bar{A}, \bar{b})$ is a $\lambda \in \mathbb{C}$ making $\text{rank}([\bar{A} - \lambda I, \bar{M}, \bar{B}]) < n$. Then, based on Proposition 1 conditions for PTSC with a general perturbation structure will be obtained.

B. **Condition for Zero Mode**

Let $\bar{H} = [\bar{A}, \bar{b}]$. For $j \in \{1, \ldots, n + 1\}$, let $r_j = \text{rank}(\bar{H}|J_n, J_{n+1}\{j\})$. Define sets $\mathcal{I}_j$ and $\mathcal{I}_j^+$ as

$$\mathcal{I}_j = \{i \in J_n : \text{rank}(\bar{H}|J_j, J_{n+1}\{j\}) = r_j, |\mathcal{I}| = r_j\},$$

$$\mathcal{I}_j^+ = \{j_n|J_n \setminus w \in \mathcal{I}_j\}.$$
Based on these definitions, the following proposition gives a necessary and sufficient condition for the absence of zero uncontrollable modes in the single-edge perturbation scenario.

**Proposition 3:** Suppose that $(A, b)$ is structurally controllable, and there is only one nonzero entry in $F$ with its position being $(i, j)$. Then, for almost all $(A, b) \in CS(A, b)$, there is no $(\Delta A, \Delta b) \in S_F$ such that a nonzero $n$-vector $q$ exists making $q^T[A + \Delta A, b + \Delta b] = 0$, if and only if $i \notin I^*_c$.

To prove Proposition 3, we need the following lemma.

**Lemma 3:** Given a matrix $H \in \mathbb{C}^{p \times q}$ of rank $r - 1$, let $x \in \mathbb{C}^p$ be a nonzero vector in the left null space of $H$. Then, for any $i \in J_n$, $x_i \neq 0$, and if only if $H[J_n \setminus \{i\}, J_q]$ is of full row rank.

**Proof:** The proof is quite standard, thus omitted here.

**Proof of Proposition 3** Sufficiency: Since $(A, b)$ is structurally controllable, $\text{rank}(H) = n$, which means that no $r_j = n$ or $n - 1$. If $r_j = n$, then $I^*_c = \emptyset$, which immediately indicates that no $q(\neq 0)$ exists making $q^T[A + \Delta A, b + \Delta b] = 0$ for almost all $(A, b) \in CS(A, b)$. Now suppose $r_j = n - 1$. A vector $q(\neq 0)$ making $q^T[A + \Delta A, b + \Delta b]$ must lie in the left null space of $H[J_n, J_{n+1} \setminus \{j\}]$, for almost all $H \in S_F$. As $r_j = n - 1$, for almost all $(A, b) \in CS(A, b)$, $J^*_c$ consists of all the nonzero positions of $q$ according to Lemma 3. As a result, if $i \notin J^*_c$,

$$q^T[A + \Delta A, b + \Delta b][J_n, \{j\}] = \sum_{k \in I^*_c} q_k [A, b]_{kj} \neq 0,$$

where the inequality is due to the fact that $[A, b]$ has full row rank.

Necessity: Assume that $i \notin J^*_c$. As $I^*_c \neq \emptyset$ and $(A, b)$ is structurally controllable, $[A, b][J_n, J_{n+1} \setminus \{j\}]$ has rank $n - 1$ for all $(A, b) \in CS(A, b)$. Let $q$ be a nonzero vector in the left null space of $[A, b][J_n, J_{n+1} \setminus \{j\}]$. According to Lemma 3 as $i \in I^*_c$, we have $q_i \neq 0$ for almost all $(A, b) \in CS(A, b)$. By setting $[\Delta A, \Delta b]_{ij} = -1/q_i \sum_{k \in I^*_c \setminus \{i\}} q_k [A, b]_{kj}$, we get

$$q^T[A + \Delta A, b + \Delta b][J_n, \{j\}] = \sum_{k \in I^*_c} q_k [A, b]_{kj} = 0,$$

which makes $q^T[A + \Delta A, b + \Delta b] = 0$. \hfill \square

**Remark 2:** From the proof of Proposition 3 provided $(A, b)$ is structurally controllable, $i \notin I^*_c$ is equivalent to that, $\text{rank}(H[J_n, J_{n+1} \setminus \{j\}]) = n$ (corresponding to $r_j = n$) or $\text{rank}(H[J_n \setminus \{i\}, J_{n+1} \setminus \{j\}]) = n - 2$ (corresponding to $r_j = n - 1$) but $\text{rank}(H[J_n \setminus \{i\}, J_{n+1} \setminus \{j\}]) < n - 1$). Moreover, since adding a column to a matrix can increase its rank by at most one, the latter two conditions are mutually exclusive.

**C. Condition for Nonzero Mode**

In the following, we present a necessary and sufficient condition for the absence of nonzero uncontrollable modes using the DM-decomposition.

For $j \in J_{n+1}$, let $j_c = J_{n+1} \setminus \{j\}$. Moreover, define a generic matrix pencil as $H_\lambda = [A - \lambda I, b], H_\lambda^c = H_{\lambda}[J_n, \{j\}]$, and $H_\lambda^c = H_{\lambda}[J_n, j_c]$. Here, the subscript $\lambda$ indicates a matrix-valued function of $\lambda$. Let $B(H_\lambda) = (V^+, V^-) \in \mathbb{C}^{p \times q}$ be the bipartite graph associated with $H_\lambda$, where $V^+ = \{x_1, \ldots, x_n\}$, $V^- = \{v_1, \ldots, v_{n+1}\}$, and $E = \{(x_i, v_k) : E_i \cup E_{A, b}\}$ with $E_i = \{(x_i, v_k) : i = 1, \ldots, n\}$, $E_{A, b} = \{(x_i, v_k) : (A, b)_{ik} \neq 0\}$. No parallel edges are included even if $E_i \cap E_{A, b} \neq 0$. An edge is called a $\lambda$-edge if it belongs to $E_i$, and a self-loop if it belongs to $E_i \cap E_{A, b}$. Note by definition, a self-loop is also a $\lambda$-edge. Let $B(H_\lambda^c)$ be the bipartite graph associated with $H_\lambda^c$, that is, $B(H_\lambda^c) = B(H_\lambda) - \{v_j\}$.

**Lemma 4:** Suppose $(A, b)$ is structurally controllable. Then $\text{mt}(B(H_\lambda^c)) = n$ for all $j \in J_{n+1}$.

**Proof:** If $j = n + 1$, it is obvious that $\text{mt}(B(H_\lambda^c)) = n$ as $E_i$ is a matching with size $n$. Now consider $j \in \{1, \ldots, n\}$. As $(A, b)$ is structurally controllable, from Lemma 2 there is a path from $x_{j+1}$ to $x_j$ in the system graph $G(A, b)$. Denote such a path by $(x_{j+1}, x_j, \ldots, (x_{j-1}, x_j))$ with $(j_1, \ldots, j_r) \subseteq J_n$ and $j_r = j$. Since each $(x_j, x_{j+1})$ in $G(A, b)$ corresponds to $(x_{j+1}, v_{k+1})$ in $B(H_\lambda)$, $(x_{j+1}, v_{k+1}), (x_j, v_j, \ldots, (x_{j-1}, v_{j-1})) \cup \{(x_i, v_i) : i \notin J_n \cup \{j_1, \ldots, j_r\}\}$ forms a matching with size $n$ in $B(H_\lambda^c)$. \hfill \square

Let $G_k = (V_k^+, V_k^-, E_k)$ $(k = 0, 1, \ldots, d)$ be the DM-components of $B(H_\lambda^c)$. From Lemma 2, we know that both the horizontal tail and the vertical one are empty. Accordingly, let $M_k^I$ be the DM-decomposition of $H_\lambda^c$ with the corresponding permutation matrices $P$ and $Q$, i.e.,

$$PH_k^I Q = \begin{pmatrix} M_1^I(\gamma) & \cdots & M_d^I(\gamma) \\ 0 & \ddots & 0 \\ 0 & \cdots & M_d^I(\gamma) \end{pmatrix} = M_k^I(\gamma). \quad (4)$$

Moreover, define $M_0^I = PH_0^I$. Suppose that $x_i$ is the $\tilde{i}$th vertex in $V_k^+$ $(1 \leq i \leq |V_k^+|, 1 \leq \tilde{i} \leq d)$.

For $k \in \{1, \ldots, d\}$, let $\gamma_{\text{min}}(G_k^c)$ and $\gamma_{\text{max}}(G_k^c)$ be respectively the minimum number of $\lambda$-edges and maximum number of $\lambda$-edges contained in a matching among all maximum matchings of $G_k^c$. Afterwards, define a boolean function $\gamma_{\text{nz}}(\cdot)$ for $G_k^c$ as

$$\gamma_{\text{nz}}(G_k^c) = \begin{cases} 1 & \text{if } \gamma_{\text{max}}(G_k^c) - \gamma_{\text{min}}(G_k^c) > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

From Lemma 2 in the appendix, $\gamma_{\text{nz}}(G_k^c) = 1$ means $\det M_k^I(\gamma)$ has at least one nonzero root for $\lambda$, while $\gamma_{\text{nz}}(G_k^c) = 0$ means the contrary. The following lemma shows that $\gamma_{\text{nz}}(G_k^c)$ can be determined in polynomial time via the maximum/minimum weight maximum matching algorithms.

**Lemma 5:** Assign the following weight function $W_k : \mathcal{E}_k \to \{0, 1\}$ for $G_k^c$ as

$$W_k(e) = \begin{cases} 1 & \text{if } e \in \text{a } \lambda\text{-edge} \\ 0 & \text{if } e \in \mathcal{E}_k \setminus \mathcal{E}_I. \end{cases}$$

Then, it is true that

$$\gamma_{\text{max}}(G_k^c) = \text{the maximum weight maximum matching of } G_k^c,$$

$$\gamma_{\text{min}}(G_k^c) = \text{the minimum weight maximum matching of } G_k^c.$$
\( \bar{V}_{k^*} = V_k^- \cup V_{k+1}^- \cup \cdots \cup V_i^- \), \( \bar{e}_{k^*} = \{(x_i,v_i) \in \mathcal{E} : x_i \in \bar{V}_{k^*}, v_i \in \bar{V}_{k^*}\} \), and the weight \( W(e) : \bar{e}_{k^*} \rightarrow \{0,1\} \)
\[
W(e) = \begin{cases} 
1 & \text{if } e \in \mathcal{E}_k \\
0 & \text{otherwise}.
\end{cases}
\]

In other words, \( \bar{G}_{k^*}^c \) is the subgraph of \( B(H_K^c) \) induced by vertices \( \bar{V}_k^c \) and \( \bar{V}_{k^*} \).

**Proposition 4:** Suppose \((\bar{A}, \bar{b})\) is structurally controllable, and there is only one nonzero entry in \( F \) with its position being \((i, j)\). Then, for almost all \((\bar{A}, \bar{b}) \in \text{CS}(\bar{A}, \bar{b})\), there is a \( (\Delta \bar{A}, \Delta \bar{b}) \in \mathcal{S}_F \) such that a nonzero \( n \)-vector \( q \) exists making \( q^T [\bar{A} + \Delta \bar{A} - \lambda I, \bar{b} + \Delta \bar{b}] = 0 \) for some nonzero \( \lambda \in \mathbb{C} \), if and only if there exists a \( k \in \Omega_j \) associated with which the minimum weight maximum matching of \( \bar{G}_k^c \), defined above is less than \(|\bar{V}_k^c|\).

The proof relies on a series of nontrivial results on the roots of determinants of generic matrix pencils, which is postponed to the appendix.

**D. Necessary and Sufficient Condition**

We are now giving a necessary and sufficient condition for PTSC with general perturbation structures.

**Theorem 1:** Consider a structurally controllable pair \((\bar{A}, \bar{b})\) and the perturbation structure \( F \). For each edge \( e = (x_i, x_j) \in \mathcal{E}_F \), let \([\bar{A}^c, \bar{b}^c] = [\bar{A}, \bar{b}] \setminus F(e) \), with \( F(e) \) defined in Proposition 2. Moreover, let \( \Omega_j \) and \( \bar{G}_k^c \) be defined in the same way as in Proposition 4, in which \((\bar{A}, \bar{b})\) shall be replaced with \((\bar{A}^c, \bar{b}^c)\).

Then, \((\bar{A}, \bar{b})\) is PTSC w.r.t. \( F \), if and only if for each edge \( e = (x_i, x_j) \in \mathcal{E}_F \), it holds simultaneously:

1. \( \text{grank}(H[\mathcal{J}_n, \mathcal{J}_{n+1}\setminus \{j\}]) = n \) or \( \text{grank}(H[\mathcal{J}_n\setminus \{i\}, \mathcal{J}_{n+1}\setminus \{j\}]) = n - 2 \), with \( H = [\bar{A}^c, \bar{b}^c] \);
2. \( \Omega_j = \emptyset \), or otherwise for each \( k \in \Omega_j \), the minimum weight maximum matching of the bipartite \( \bar{G}_k^c \), is \(|\bar{V}_k^c|\).

**Proof:** Follows immediately from Propositions 2-4.

Since each step in Theorem 1 can be implemented in polynomial time, its verification has polynomial complexity. To be specific, for each edge \( e \in \mathcal{E}_F \), to verify Condition 1), we can invoke the Hopcroft-Karp algorithm twice, which incurs time complexity \( O(n^{0.5}|\mathcal{E}_F| \cup \mathcal{E}_F|) \rightarrow O(n^{2.5}) \). As for Condition 2), the DM-decomposition incurs \( O(n^{2.5}) \), and computing the minimum weight maximum matching of \( \bar{G}_k^c \), costs \( O(n^3) \) [29]. Since \(|\Omega_j| \leq n \), for each \( e \in \mathcal{E}_F \), verifying Condition 2) takes at most \( O(n^{2.5} + n^2 n^3) \). To sum up, verifying Theorem 1 incurs time complexity at most \( O(\mathcal{E}_F(n^{2.5} + n^4)) \), i.e., \( O(\mathcal{E}_F(n^4)) \). The procedure for verifying PTSC can be summarized as follows.

**Algorithm for verifying PTSC for \((\bar{A}, \bar{b})\) w.r.t. \( F \):**

1. Check structural controllability of \((\bar{A}, \bar{b})\). If yes, continue; otherwise, break and return false.
2. For each \( e = (x_i, x_j) \in \mathcal{E}_F \), construct \([\bar{A}^c, \bar{b}^c] \), and implement the following steps:
   2.1 Check whether \( \text{grank}(H[\mathcal{J}_n, \mathcal{J}_{n+1}\setminus \{j\}]) = n \) or \( \text{grank}(H[\mathcal{J}_n\setminus \{i\}, \mathcal{J}_{n+1}\setminus \{j\}]) = n - 2 \), with \( H = [\bar{A}^c, \bar{b}^c] \).
   If yes, continue; otherwise, return false.
   2.2 Construct \( \Omega_j \) and \( \bar{G}_k^c \), associated with \([\bar{A}^c, \bar{b}^c] \).
   2.3 For each \( k \in \Omega_j \), check whether the minimum weight maximum matching of \( \bar{G}_k^c \), is equal to \(|\bar{V}_k^c|\). If yes, continue; otherwise, break and return false.
3. If not break, return true.

**Example 2 (Example 7 continuing):** Let us revisit Example 1. Consider the perturbation \([\Delta A_2, \Delta b_2]\). For edge \( e = (x_3, x_4) \), the DM-decomposition of \( H_5^c (j = 5) \) associated with \([\bar{A}^c, \bar{b}^c] \) and the corresponding \( M_5^c \) are respectively

\[
M_5^c = \begin{bmatrix} 
-\lambda & 0 & c & 0 \\
0 & -\lambda & a & f \\
-\lambda & a & 0 & 0 \\
f & a & 0 & 0 \\
c & a & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

from which, \( i^* = 1 \) and \( \Omega_j = \emptyset \). It means Condition 2) of Theorem 1 is satisfied. Similar analysis could be applied to the edge \( e = (x_3, x_1) \), and it turns out that both conditions in Theorem 1 hold. Therefore, \((\bar{A}, \bar{b})\) is PTSC w.r.t. \([\Delta A_1, \Delta b_1]\), which is also consistent with Example 1.

**V. Implications to SCRPs**

PTC reflects the ability of a numerical system to preserve controllability under structured perturbations. This notion is closely related to the SCRCP studied in [13, 20, 21], where the problem is formulated as searching the smallest perturbations (in terms of the Frobenius norm or 2-norm) with a prescribed structure that result in an uncontrollable system.

It turns out that the SCRCP is feasible, and if only if the original system is PSC w.r.t. the corresponding perturbation structure (considering complex-valued perturbations). Hence, before implementing any numerical algorithms on the (single-input) SCRCP, we can check whether the corresponding structured system is PTSC w.r.t. the perturbation structure. If the answer is yes and the original numerical system is controllable, then there cannot exist numerical perturbations with the prescribed structure for which the perturbed system is uncontrollable; otherwise, with probability 1 (before looking at the exact parameters of the original system), such a structured numerical perturbation exists.

**VI. Conclusion**

This paper proposes a novel notion of PTSC to study controllability preservation for a structured system under structured numerical perturbations. It is shown this notion can characterize the generic property in controllability preservation for structured systems under structured numerical perturbations. A necessary and sufficient condition is given for a single-input system to be PTSC w.r.t. a prescribed perturbation structure. Readers can refer to [28] for extensions of this work to the multi-input case.
Next, we prove 2). Consider a self-loop with the entry being $t_l - \lambda$ (1 ≤ l ≤ r). As $B(P_\lambda)$ is DM-reducible, every nonzero entry must be contained in $\det(P_\lambda)$ by Definition[5] which means $t_l - \lambda$ is contained in some term $(t_l - \lambda)f$ of $\det(P_\lambda)$, where $f$ denotes a polynomial over variables $\{t_1, ..., t_r\} \setminus \{t_l\}$ and $\lambda$. This term can be written as the sum of two terms $t_l f$ and $t_l \lambda f$, which indicates $\det(P_\lambda)$ contains at least two monomials whose degrees for $\lambda$ differ from each other. Then, following the similar reasoning to the proof of 1), $\det(P_\lambda)$ contains at least one nonzero root.

We are now proving 3). Suppose such a nonzero root exists that is independent of $T_i$ for some $i \in \{1, ..., n\}$, and denote it by $z$. Let $T[I_1, I_2]$ be the set of variables in $t_1, ..., t_r$ that appear in $M[I_1, I_2]$ for $I_1, I_2 \subseteq J_n$, and let $R(T_i)$ (resp. $C(T_i)$) be the set of rows (resp. column) indices of variables $T_i \subseteq \{t_1, ..., t_r\}$. Suppose $[P_\lambda]_{k_0, i} = \lambda$ for some $k_0 \in J_n \setminus R(T_i)$ ($k_0$ can be empty). Upon letting all $t_k \in T_i$ be zero, we obtain ($P_\lambda = M - zE$)

$$\det(P_\lambda) = \sum_{j=1}^{n} (-1)^{j+1} |P_{ij}| \det(P_\lambda[J_n \setminus \{j\}, J_n \setminus \{i\}]) = z \cdot \det(P_\lambda[J_n \setminus \{k_0\}, J_n \setminus \{i\}]) = 0,$$

which indicates

$$\det(P_\lambda[J_n \setminus \{k_0\}, J_n \setminus \{i\}]) = 0,$$

as $z \neq 0$. Since $[P_\lambda]_{j_0, j} \neq 0$ has full generic rank from Lemma[1] it concludes that $z$ depends solely on the variables $T[J_n \setminus \{k_0\}, J_n \setminus \{i\}]$. Similarly, because of (7), for each $t_k \in T_i$, fixing all $t_k \in T_i \setminus \{t_k\}$ to be zero yields

$$\det(P_\lambda[J_n \setminus R(\{t_k\}), J_n \setminus \{i\}]) = 0,$$

which indicates that $z$ depends on the variables $T[J_n \setminus R(\{t_k\}), J_n \setminus \{i\}]$, being independent of the remaining variables. Taking the intersection of $T[J_n \setminus \{j\}, J_n \setminus \{i\}$ over all $j \in R(T_i) \cup \{k_0\}$, we obtain $T[\Theta, J_n \setminus \{i\}]$, where $\Theta \equiv J_n \setminus R(\{t_k\}) \cup \{k_0\}$. That is, $z$ depends on variables $T[\Theta, J_n \setminus \{i\}]$, and makes $P_\lambda[\Theta, J_n \setminus \{i\}]$ rank row deficient. However, for each pair $(j, l), j \in J_n \setminus \{i\}, l \in R(T_i)$, it holds

$$\text{grank}(P_\lambda[\Theta, J_n \setminus \{i\}]) \geq \text{grank}((M - \lambda E)[J_n \setminus \{i\}, J_n \setminus \{j\}]) - |R(T_i) \cup \{k_0\} - 1| \equiv n - |R(T_i) \cup \{k_0\}| = |\Theta|,$$

where (a) is due to that $P_\lambda[\Theta, J_n \setminus \{i\}]$ is obtained by deleting $|R(T_i)\cup\{k_0\}$ - 1 rows from $P_\lambda[J_n \setminus \{l\}, J_n \setminus \{j\}]$, and (b) comes from $\text{grank}(P_\lambda[J_n \setminus \{l\}, J_n \setminus \{j\}]) = n - 1 \cdot 2$ by Lemma[1]. That is, after deleting any column from $P_\lambda[\Theta, J_n \setminus \{i\}]$, the resulting matrix remains of full row generic rank, which induces at least one nonzero polynomial equation constraint on $z$ and $T[\Theta, J_n \setminus \{i, j\}]$. This indicates $z$ depends on $T[\Theta, J_n \setminus \{i, j\}]$, or equivalently, being independent of $T[\Theta, \{j\}]$, for each $j \in J_n \setminus \{i\}$. It finally concludes that $z$ is independent of the variables $T[\Theta, J_n \setminus \{i\}]$, causing a contraction. Therefore, the assumed $z$ cannot exist.

\[\square\]

Lemma 8: Let $M_\chi^J$ and $\Omega_j$ be defined in (4) and (6). For
each \( k \in \Omega_j \), let \( \tilde{M}_{ki}\) \((\lambda) \) \( = M_{ki}\) \( \Delta \) \( \tilde{V}_{ki} \), \( \tilde{V}_{ki} \), \( \lambda \) \( \Delta \) \( \tilde{M}_{ki}(\lambda) \) \( = \begin{bmatrix} M_{ki}(\lambda) & \cdots & M_{ki}(\lambda) \\ 0 & \ddots & 0 \\ \vdots & \ddots & \ddots \end{bmatrix} \). Then, \( \tilde{M}_{ki}(\lambda) \) generically has full row rank when \( \lambda \in \{ z \in \mathbb{C} \setminus \{ 0 \} : \det M_{ki}(z) = 0 \} \), if and only if the minimum weight maximum matching of the bipartite \( G_{ki}^{\tilde{M}_{ki}(\lambda)} \) is less than \( |V_{ki}^{\tilde{M}_{ki}(\lambda)}| \).

**Proof:** Letting structural controllability of \( \det \) \( \Delta \) \( \tilde{G}_{ki}^{\tilde{M}_{ki}(\lambda)} \) be less than \( |V_{ki}^{\tilde{M}_{ki}(\lambda)}| \). Then, \( \tilde{G}_{ki}^{\tilde{M}_{ki}(\lambda)} \) has a nonzero root assumption. Maximum matching with size \( n_1 - 1 \) from its structure. Suppose that \( G_{ki}^{\tilde{M}_{ki}(\lambda)} \) has a matching weight less than \( |V_{ki}^{\tilde{M}_{ki}(\lambda)}| \). Each edge not incident to \( V_{ki}^{\tilde{M}_{ki}(\lambda)} \) must have a matching with size \( n_1 - 1 \). Indeed, if this is not true, then any maximum matching of \( G_{ki}^{\tilde{M}_{ki}(\lambda)} \) must match \( V_{ki}^{\tilde{M}_{ki}(\lambda)} \), which certainly leads to a weight equaling \( |V_{ki}^{\tilde{M}_{ki}(\lambda)}| \), noting that each edge not incident to \( V_{ki}^{\tilde{M}_{ki}(\lambda)} \) has a zero weight. Furthermore, due to the DM-irreducibility of \( G_{ki}^{\tilde{M}_{ki}(\lambda)} \), from Lemma 7 any nonzero root of \( \det M_{ki}(\lambda) \) cannot be independent of the variables in the \( k \)th column of \( M_{ki}(\lambda) \). Therefore, \( \det M_{ki}(\lambda) \) has a nonzero root and the fact that \( \det M_{ki}(\lambda) \) is mathematically multirank allows \( M_{ki}(\lambda) \) to be a zero rank. The second equality is due to \( (P[\Delta A, \Delta b])_{ij} = [\Delta A, \Delta b]_{ij} \). Upon defining \( q^i = \hat{q} P \), we have

\[
q^i (\Delta A, \Delta b)_{ij} = -1/\hat{q} \sum_{l=1}^{n} \hat{q} (P[\Delta A - z I, b])_{ij} = 0,
\]

which comes from the fact \( q^i H^j = 0 \) and \( Q \) is invertible.

**Necessity:** For the existence of \( q \) making the condition in Proposition 4 satisfied, it is necessary \( H^j \) should be of rank deficient at some nonzero value for \( \lambda \) (generically). Denote such a value by \( z \) for the sake of distinguishing it from the variable \( \lambda \). Since DM-decomposition does not alter the rank, \( M_{ki}^j \) should be of row rank deficient too. From the block-triangular structure of \( M_{ki}^j \) (see (3)), there must exist some \( k \in \{ 1, ..., n \} \), such that \( M_{ki}^j(z) \) is singular generically. From Lemma 7 such an integer \( k \) must correspond to a \( G_{ki}^{\tilde{M}_{ki}(\lambda)} \) satisfying \( \gamma_{\bar{m}}(G_{ki}^{\tilde{M}_{ki}(\lambda)}) = 1 \). We consider two cases: i) \( k > i^* \), and ii) \( k \leq i^* \).

In case i), since \( k > i^* \), from the upper block-triangular structure of \( M_{ki}^j \), it can be seen that \( M_{ki}^j[\mathcal{J}_{n_1 \setminus \{ i \}}, \mathcal{J}_{n_1}] \) is of row rank deficient when \( \det M_{ki}^j(z) = 0 \). Note that \( \gamma_{\bar{m}}(M_{ki}^j) \geq n - 1 \) as otherwise \( \gamma_{\bar{m}}(H_z) < n \), which is contradictory to the structural controllability of \( (A, b) \). Consequently, \( M_{ki}^j \) has a left null space with dimension one. Denote by \( \hat{q} \) the vector spanning that space. From Lemma 3 \( \hat{q}_i = 0 \). As a result, for any \( \Delta A, \Delta b \in S_{[\Delta A, \Delta b]} \),

\[
q^i \{ M_{ki}^j + (P[\Delta A, \Delta b])[\mathcal{J}_{n_1 \setminus \{ i \}}, \mathcal{J}_{n_1}] \} = \hat{q} P M_{ki}^j \neq 0,
\]

where (a) results from \( (P[\Delta A, \Delta b])_{ij} = [\Delta A, \Delta b]_{ij} \), and the inequality from \( \hat{q} P H_z = 0 \) meaning that \( z \) will be an uncontrollable mode (noting that \( \hat{q}^i (M_{ki}^j, M_{ki}^j) = \hat{q}^i P H_z = 0 \), and \( Q \) is invertible). Consequently, case i) cannot lead to the required results.

Therefore, \( k \) must fall into case ii). Now suppose that the minimum weight maximum matching of \( G_{ki}^{\tilde{M}_{ki}(\lambda)} \) is equal to \( |V_{ki}^{\tilde{M}_{ki}(\lambda)}| \). Then, from Lemma 8 and by the block-triangular structure of \( M_{ki}^j \), we obtain that \( M_{ki}^j[\mathcal{J}_{n_1 \setminus \{ i \}}, \mathcal{J}_{n_1}] \) is generically of row rank deficient. Following the similar reasoning to case i), it turns out that the requirement in Proposition 4 cannot be satisfied. This proves the necessity.

\[
\begin{align*}
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