Evolution of the D3-brane for dynamical embeddings

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Abstract

The Dirac-Born-Infeld (DBI) action of the 3-dimensional brane for its dynamical embeddings and the gauge fields has been studied. The evolution of both the D3-brane and the ambient space has been obtained. For the special constraint put on the transverse coordinates a family of the contracting and expanding spaces has been found.

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1 Introduction

In the presence of D-branes supersymmetry of the background is broken. The survived supersymmetries are given by the Killing spinors. If these spinors are projected onto the D-brane, then one obtains as a result an equation for the survived supersymmetry generators [1]. The number of the unbroken supersymmetries is equal to the dimension of the solution space of this equation. The supersymmetry charges are minimal in the case of the BPS states. It means that masses are equal to the gauge charges. Thus the Hamiltonian (the energy of the system) takes the special form. For the D_p-branes described by the DBI action, relations between the embeddings, gauge fields and geometric and topological properties of the ambient space are discussed in [2]. This is related to the concept of a calibrated submanifold [3]. The calibrated submanifold minimizes the DBI action. Thus the calibration bound gives the BPS bound. In the present paper evolution of both a D3-brane and the ambient space has been investigated. This evolution is induced by the deformed metric which is related to the vanishing of the DBI Lagrangian. In Section 2 we recall the form of the DBI action and constraints which appear in this action in the case when both the gauge fields and embeddings are dynamic. Since the Lagrangian for DBI action is invariant under diffeomorphism the Hamiltonian for the DBI action is just equal to the sum of constraints. In Section 3 we present the case of a
dynamic embedding, which means that both the coordinates transverse to the brane and the gauge fields depend on time. We put an isotropic constraint on the transverse velocities. This constraint depends on two parameters. In this case the deformed metric leads to the evolution both of the D-brane and the ambient space. For the different parameters we are going to obtain different evolutions of the D-brane and the ambient space. For the special values of these parameters the de Sitter space is obtained.

2 DBI Lagrangian and the constraints

The low energetic action in the flat ambient space-time for a $Dp$-brane is given by the expression:

$$ S = -T_p \int_{\mathcal{M}_{p+1}} e^{-\phi} \left( - \det \left( \gamma_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta} + B_{\alpha\beta} \right) \right)^{1/2} d^{p+1}\xi + T_p \int_{\mathcal{M}_{p+1}} \sum_i C_i \wedge \exp \left( 2\pi \alpha' F + B \right), \quad (2.1) $$

where: $\phi$ is a dilaton field, $\alpha, \beta = 0, 1, ..., p$ and $p$ is a spatial dimension of a $Dp$-brane. The metric $\gamma_{\alpha\beta}$ on the worldvolume is induced by the background metric $g_{MN}$:

$$ \gamma_{\alpha\beta} = g_{MN} \partial_\alpha X^M \partial_\beta X^N, $$

$X^M$ is the embedding of $\mathcal{M}_{p+1}$ into the ambient spacetime: $X^M = (X^\alpha, X^a)$, $a = 1, ..., 9-p$. The RR fields are denoted as $C_i$, $F_{\alpha\beta}$ is the abelian gauge field strength on the brane and $B_{\alpha\beta}$ is the pullback of the background NS 2-form $B$. The symbols $F$ and $B$ denote:

$$ F = F_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta, \quad B = B_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta. $$

In the last integral in Eq. (2.1) are picked up only those forms with degree which is equal to the dimension of the $Dp$-brane. In the case of the non-flat backgrounds the action (2.1) is corrected by the non-linear terms in the curvature forms, both of $M$ and the ambient spacetime [4].

It is well-known that a gauge field $A$ with the strength $F$ produces $D(p-2)$-brane by the WZ action and the fundamental string by coupling to the background NS antisymmetric field $B$. In this paper the purely geometrical backgrounds have been taken into account only. Thus the WZ action vanishes. The case with $p = 3$ only has been considered. In [5, 6] the problem of the stability of supertubes and branes has been discussed.

Thus the action (2.1) is reduced to the DBI action and takes the form:

$$ S = T_3 \int_{\mathcal{M}_4} d^4\xi e^{-\phi} \sqrt{- \left( 1 - \frac{1}{2} \gamma^{\alpha\gamma} \gamma^{\beta\delta} F_{\alpha\delta} F_{\beta\gamma} \right) \det \gamma - P f f^2 (F), \quad (2.2) $$
where $\mathcal{F}_{\alpha\beta} = 2\pi\alpha' F_{\alpha\beta}$ (in the case considered $B = 0$) and $P f f^2 (\mathcal{F}) = \det (\mathcal{F})$.

In the non-flat background cases one should add to the action (2.2) non-linear terms in the curvature. The Lagrangian in the Eq.(2.2) is expressed by the electric field $E_m = -F_{0m}$ and the magnetic field $B_m = \frac{1}{2} \varepsilon_{mnp} F^{np}$ and assumes the following form:

$$L = T_3 e^{-\phi} \sqrt{1 + (2\pi\alpha')^2 \gamma^{00} E^2 + (2\pi\alpha')^2 B^2} \det \gamma - (2\pi\alpha')^4 (E \cdot B)^2,$$

where $E^2 = \gamma^{mn} E_m E_n$ and $B^2 = \gamma^{mn} B_m B_n$. We also redefine the tension $T_3$ by the dilaton field $\phi$ in the following way:

$$T_3 e^{-\phi} \rightarrow T_3.$$

In this redefinition is hidden an assumption that $\phi$ is constant on the worldvolume. The metric $\gamma$ is induced by the backgrounds given by the supergravity solutions, e.g. [7]. One can notice that the Lagrangian can be rewritten in the form:

$$L = T_3 \sqrt{1 + (2\pi\alpha')^2 B^2} \det \gamma - E^T M E,$$

where the entries of the matrix $M$ are given by:

$$M^{mn} = (2\pi\alpha')^2 \gamma^{00} \gamma^{mn} \det \gamma + (2\pi\alpha')^4 B^m B^n \quad (2.4)$$

and $B^n = \gamma^{mn} B_m$.

The canonical coordinates for the embedding $X$ and the gauge field $A$ are:

$$(X^M, P_M), (\Pi^\alpha, A_\alpha).$$

The canonical momenta are given by:

$$P_M = \frac{\partial L}{\partial (\partial_0 X^M)} = - \frac{T_3}{2} \sqrt{-\det (\gamma + F)} (G^{00} + G^{0\alpha}) \partial_0 X^N g_{MN},$$

$$\Pi^m = \frac{\partial L}{\partial (\partial_0 A_m)} = - \frac{2\pi\alpha' T_3}{2} \sqrt{-\det (\gamma + F)} (G^{m0} - G^{0m}),$$

$$\Pi^0 = \frac{\partial L}{\partial (A_0)} = 0,$$

where:

$$G^{\alpha\beta} = (G^{-1})^{\alpha\beta} = \left[ (\gamma + F)^{-1} \right]^{\alpha\beta}.$$

We define the following matrices:

$$\mathcal{P}_M = (\mathcal{P}_M^\alpha) = G^{-T} e_M + G^{-1} e_M,$$

$$\mathcal{E} = (\mathcal{E}^{\alpha\beta}) = G^{-1} - G^{-T},$$
where:

\[ e_M = g_{MN} e^N, \]
\[ e^N = (e^N_\alpha) = (\partial_\alpha X^N). \]

One can observe that the matrices \( P \) and \( E \) obey the relation:

\[ P_M e^M + E F = 2I, \] (2.10)

in the worldvolume coordinates this relation has the form:

\[ P^\alpha_M e^M + \mathcal{E}^{\alpha\gamma} \mathcal{F}_{\gamma\beta} = 2\delta^\alpha_\beta. \] (2.10a)

The square of \( P_M \) is:

\[ g^{MN} P_M P_N = \mathcal{E}\gamma\mathcal{E} + 4G^{-1} + 2 \left( G^{-1} \mathcal{F}G^{-T} + G^{-T} \mathcal{F}G^{-1} \right). \] (2.11)

From the relation (2.10a) the following formulas for \( \alpha = 0 \) and \( \beta = 0 \) have been obtained, respectively:

\[ 2\pi\alpha' P_M e^M_\beta + \Pi^m \mathcal{F}_{\beta m} = 2\pi\alpha' T_3 \sqrt{-\det (\mathcal{G} + \mathcal{F})} \delta^0_\beta, \] (2.12)
\[ 2\pi\alpha' P_M \partial_0 X^M + \Pi^m \mathcal{F}_{0m} = 2\pi\alpha' T_3 \sqrt{-\det (\mathcal{G} + \mathcal{F})}, \] (2.13)

where \( P_M \) and \( \Pi^m \) are related to \( P_M \) and \( E \) as follows:

\[ P_M = T_3 \sqrt{-\det (\mathcal{G} + \mathcal{F})} P_M^0, \]
\[ \Pi^m = \frac{2\pi\alpha' T_3}{2} \sqrt{-\det (\mathcal{G} + \mathcal{F})} E^0 m. \]

For \( \beta = m \) one obtains the worldspace diffeomorphism constraint [8]:

\[ 2\pi\alpha' P_M \partial_m X^M + \Pi^m \mathcal{F}_{mn} = 0. \] (2.14)

There are also two other constraints [8]:

- the Hamiltonian constraint (which follows from (2.11)):

\[ P_M P_N g^{MN} + \Pi^m \gamma_{mn} + T_3^2 \det [(\mathcal{G} + \mathcal{F})_{mn}] = 0, \]

- the Gauss law:

\[ \partial_m \Pi^m = 0. \]

The Hamiltonian constraint was considered in the static embedding \( X \) for different configurations in [9].

Let us assume that the embedding \( X \) is not static and has the form:

\[ X(\xi) = (\xi^0, \xi^m, X^a (\xi^0, \xi^m)) \] (2.15)
and the metric $g_{MN}$ is "diagonal":

$$(g_{MN}) = \begin{pmatrix} g_{00} & (g_{mn}) \\ (g_{mn}) & (g_{ab}) \end{pmatrix}$$

with the signature $(-1, +1, ..., +1)$. Thus $-g_{00} \geq 0$. For the embedding $X$ the induced metric $\gamma_{\alpha\beta}$ takes the form:

$$\begin{align*}
\gamma_{00} &= g_{00} + g_{ab}X^aX^b, \\
\gamma_{0m} &= g_{ab}X^a\partial_m X^b, \\
\gamma_{m0} &= g_{ab}\partial_m X^a X^b, \\
\gamma_{mn} &= g_{mn} + g_{ab}\partial_m X^a \partial_n X^b.
\end{align*}$$

(2.16)

We restrict ourselves to a homogenous case: $\partial_m X^a = 0$. Thus:

$$\det (\gamma_{\alpha\beta}) = \gamma_{00} \det (\gamma_{mn})$$

and the matrix $M$ takes the form:

$$M^{mn} = (2\pi\alpha')^2 \gamma^{mn} \det (\gamma_{rp}) + (2\pi\alpha')^4 B^m B^n.$$  

The Lagrangian in this case takes the form:

$$L = T_3 \sqrt{- \left( g_{00} + g_{ab}X^a X^b \right) \left( 1 + (2\pi\alpha')^2 B^2 \right) \det (\gamma_{mn}) - EM \bar{E}}. $$

Note that $-\gamma_{00} > 0$. The momenta $P_a$ transverse to the worldvolume and the momenta $\Pi^m$ have the form:

$$P_a = -\frac{T_3^2 \det (\gamma_{mn})}{L} \left( 1 + (2\pi\alpha')^2 B^2 \right) X^b g_{ab}$$

and

$$\Pi^m = -\frac{T_3^2}{L} M^{mn} E_n. $$

respectively. The tangent momentum to the worldvolume $P_m$ is obtained from the diffeomorphism constraint (2.14):

$$P_m = -\Pi^a \mathcal{F}_{mn}. $$

(2.19)

and is expressed by the the Poynting vector $S_m = \varepsilon_{mnp} E^n B^p$ on the worldvolume:

$$P_m = -\frac{T_3^2 (2\pi\alpha')^3 \det (\gamma_{rs})}{L} S_m,$$

where $E^n = \gamma^{nm} E_m$. The momentum $P_M$ has the form:

$$P_M = (\mathcal{H}, -\Pi^a \mathcal{F}_{mn}, P_a),$$

(2.18)
where $\mathcal{H}$ is the energy density and $P_n$ is given by (2.17). The square of $P_M$ is:

$$P_M P^M = g^{00} \mathcal{H}^2 + g^{mn} P_m P_n + g^{ab} P_a P_b,$$

where:

$$g^{mn} P_m P_n = \frac{(2\pi\alpha')^4}{T_3^4} \frac{T^4}{L^2} [E \times B]^2 \det^2 (\gamma_{mn}),$$  \hspace{1cm} (2.19)

$$g^{ab} P_a P_b = \frac{T_3^4}{L^2} \left( 1 + (2\pi\alpha')^2 B^2 \right) X^2 \det^2 (\gamma_{mn})$$  \hspace{1cm} (2.20)

and $X^2 = g_{ab} \dot{X}^a X^b$, the vector product is defined as:

$$(E \times B)_m = \varepsilon_{mnp} E^n B^p.$$  

The Hamiltonian constraint takes the form:

$$P_M P^M + \Pi^m \Pi^a \dot{\gamma}_{mn} + T_3^4 \left( 1 - \frac{1}{2} \mathcal{F}_{mn} \mathcal{F}^{mn} \right) \det (\gamma_{pr}) = 0,$$

where:

$$\Pi^m \Pi^a \dot{\gamma}_{mn} = \frac{(2\pi\alpha')^4}{T_3^4} \frac{T^4}{L^2} \left[ E^2 \det^2 (\gamma_{mn}) + 2 (2\pi\alpha')^2 (E \cdot B)^2 \det (\gamma_{mn}) + (2\pi\alpha')^4 B^2 (E \cdot B)^2 \right] =$$

$$= \frac{(2\pi\alpha')^4}{T_3^4} \frac{T^4}{L^2} \left[ E \det (\gamma_{mn}) + (2\pi\alpha')^2 (E \cdot B) B \right]^2,$$

$$\frac{1}{2} \mathcal{F}_{mn} \mathcal{F}^{mn} = - (2\pi\alpha')^2 B^2.$$  \hspace{1cm} (2.21)

Thus the square of the energy density is:

$$-g^{00} \mathcal{H}^2 L^2 = (2\pi\alpha')^2 T_3^4 [E \times B]^2 \det^2 (\gamma_{mn}) + (2\pi\alpha')^2 \frac{T_3^4}{T_3^3} \left[ E \det (\gamma_{mn}) + (2\pi\alpha')^2 B (E \cdot B) \right]^2 +$$

$$T_3^4 \left( 1 + (2\pi\alpha')^2 B^2 \right) \cdot \dot{X}^2 \det^2 (\gamma_{mn}) + T_3^4 \left( 1 + (2\pi\alpha')^2 B^2 \right) \det (\gamma_{mn}).$$

For $B = E = 0$ the Lagrangian (2.3) has the form:

$$L = T_3 \sqrt{- \left( g_{00} + g_{ab} \dot{X}^a X^b \right) \det (\gamma_{mn})}.$$  

Hence one can obtain:

$$\mathcal{H}^2 = T_3^2 \frac{g_{00} \det (\gamma_{mn})}{1 + g^{00} X^2}$$  \hspace{1cm} (2.22)

and:

$$P_a = T_3 \frac{\dot{X}^b g_{ab} \sqrt{\det (\gamma_{mn})}}{\sqrt{1 + g^{00} X^2} / -g_{00}}.$$  \hspace{1cm} (2.23)
Note that $|g_{00}| \geq X^2$. The Hamiltonian constraint has the form:

$$-g^{00}\mathcal{H}^2 - \mathbf{P}^2 = T_3^2 \det (\gamma_{mn}).$$

where $\mathbf{P}^2 = g^{ab} P_a P_b$. This last equation has the same structure as the equation of the motion of a relativistic particle with the mass $m_0 = T_3 \sqrt{\det (\gamma_{mn})}$. In the case of D0-brane (D-particle) moving in the background with the metric $g_{MN}$ the formulas (2.22) and (2.23) give the energy and momentum of the D-particle:

$$\mathcal{H} = \frac{T_3 \sqrt{-g_{00}}}{1 + g^{00} X^2},$$

$$P_a = -\frac{T_3 \sqrt{-g_{00} X^b g_{ab}}}{\sqrt{1 + g^{00} X^2}}.$$  \hspace{1cm} (2.24)

The mass of this D-particle is $T_0$.

The energy density as the function of the momenta has the form:

$$\mathcal{H} = \sqrt{-g_{00} \left( T_3^2 \left( 1 + (2\pi \alpha')^2 B^2 \right) \det (\gamma_{mn}) + g^{ab} P_a P_b + g^{mn} P_m P_n + \Pi^m \Pi^n \gamma_{mn} \right) \left[ E \times B \det (\gamma_{mn}) + E \det (\gamma_{mn}) + B (E \cdot B) \right] \left[ E \times B \det (\gamma_{mn}) + L \right]^2 \det (\gamma_{mn})},$$

where:

$$W_{mn} = \left( 1 + (2\pi \alpha')^2 B^2 \right) \gamma_{mn} - (2\pi \alpha')^2 B_m B_n.$$  \hspace{1cm} (2.26)

The energy density is the monotonically increasing function of the momenta. So it is bounded from bottom by:

$$\mathcal{H} \geq T_3 \sqrt{-g_{00} \left( 1 + (2\pi \alpha')^2 B^2 \right) \det (\gamma_{mn})}.$$

The equation (2.21) can also be expressed as a sum of the squares:

$$-g^{00}\mathcal{H}^2 = \frac{(2\pi \alpha')^2 T_3^4}{L^2} \left[ E \times B \det (\gamma_{mn}) + E \det (\gamma_{mn}) + B (E \cdot B) \right]^2 +$$

$$+ \frac{T_3^2 \left( 1 + (2\pi \alpha')^2 B^2 \right)}{2L^2} \left[ T_3 \sqrt{1 + (2\pi \alpha')^2 B^2} \left| X \right| \sqrt{\det (\gamma_{mn})} \right] \det (\gamma_{mn}) +$$

$$+ \frac{T_3^2 \left( 1 + (2\pi \alpha')^2 B^2 \right)}{2L^2} \left[ T_3 \sqrt{1 + (2\pi \alpha')^2 B^2} \left| X \right| \sqrt{\det (\gamma_{mn})} - L \right] \det (\gamma_{mn}).$$  \hspace{1cm} (2.27)
One can deduce that the energy square is bounded by the following configurations:

\[-g^{00} \mathcal{H}^2 \geq \frac{(2\pi\alpha')^2}{L^2} T_3^4 \left[ E \times B \det (\gamma_{mn}) + E \det (\gamma_{mn}) + B (E \cdot B) \right]^2 + \frac{T_3^2 (1 + (2\pi\alpha')^2 B^2)}{2 L^2} \left[ T_3 \sqrt{1 + (2\pi\alpha')^2 B^2} \left| X \right| \sqrt{\det (\gamma_{mn}) + L} \right]^2 \det (\gamma_{mn}) \cdot (2.28)\]

One obtains the equality when:

\[T_3 \sqrt{1 + (2\pi\alpha')^2 B^2} \left| X \right| \sqrt{\det (\gamma_{mn})} = L. \quad (2.29)\]

The inequality (2.28) is the BPS bound [9]. In the case when \(E = B = 0\) the condition (2.29) gives:

\[2X^2 \det (\gamma_{mn}) = -g_{00}.\]

For the static configuration \(X^a = 0\) one obtains:

\[-g_{00} \left( 1 + (2\pi\alpha')^2 B^2 \right) \det (\gamma_{mn}) = E \cdot B.\]

The BPS configuration for \(X^a \neq 0\) has the energy:

\[-g^{00} \mathcal{H}_{BPS}^2 = T_3^4 \left( 2\pi\alpha' \right)^2 \left[ E \times B \det (\gamma_{mn}) + E \det (\gamma_{mn}) + B (E \cdot B) \right]^2 \left[ 1 + (2\pi\alpha')^2 B^2 \right] \left| X \right|^2 \det (\gamma_{mn}) + 2T_3^2 \left[ 1 + (2\pi\alpha')^2 B^2 \right]^2 \det (\gamma_{mn}) \cdot (2.30)\]

The expression under the square root in (2.3) has to be positive in order to get the real Lagrangian. This condition puts the constraint on the allowed configurations:

\[- \left( 1 + (2\pi\alpha')^2 \gamma^{00} E^2 + (2\pi\alpha')^2 B^2 \right) \det \gamma \geq (2\pi\alpha')^2 \left( E \cdot B \right)^2, \quad (2.31)\]

since \((E \cdot B)^2 > 0\). Integrating the square root of (2.31) over the D3-brane \(M\) one obtains:

\[\int_M \sqrt{- \left( 1 + (2\pi\alpha')^2 B^2 + (2\pi\alpha')^2 \gamma^{00} E^2 \right) \det \gamma d^4x} \geq (2\pi\alpha')^2 \int_M E \cdot B d^4x.\]

The r.h.s. of the above inequality is expressed by the second Chern character \(ch_2(L)\) of the line bundle \(L\) over \(M\). This second Chern character is expressed by the curvature form \(F\) of the line bundle as follows:

\[ch_2(L) = -\frac{1}{8\pi^2} F \wedge F = \frac{1}{2\pi^2} E \cdot B d^4x.\]
Thus one obtains:

\[
\int_M \sqrt{-\left(1 + (2\pi\alpha')^2 B^2 + (2\pi\alpha')^2 \gamma^{00} E^2\right)} \det \gamma \geq 2\pi^2 (2\pi\alpha')^2 \int_M ch_2(L) .
\]

(2.32)

For the embedding under consideration the formula (2.31) takes the form:

\[
\left(-\gamma^{00} - (2\pi\alpha')^2 E^2 - (2\pi\alpha')^2 \gamma^{00} B^2\right) \det (\gamma_{mn}) \geq (2\pi\alpha')^4 (E \cdot B)^2 .
\]

(2.33)

One can notice that:

\[
-\gamma^{00} - (2\pi\alpha')^2 E^2 - (2\pi\alpha')^2 \gamma^{00} B^2 \geq 0 .
\]

(2.34)

This condition leads to the relation:

\[
-\gamma^{00} \geq (2\pi\alpha')^2 E^2 + (2\pi\alpha')^2 \gamma^{00} B^2
\]

\((-\gamma^{00} > 0)\). Thus in the induced metric \(\gamma\) with the signature \((- ++ +)\) we get a bound on the allowed magnetic and electric fields:

\[
|\gamma^{00}| \geq (2\pi\alpha')^2 E^2 - (2\pi\alpha')^2 |\gamma^{00}| B^2 .
\]

This relation is in agreement with the result of [10] which says that the electric field has the maximal value. In the case when the last inequality is saturated, it means that fields \(E\) and \(B\) are maximal. Thus one obtains that \(E \cdot B = 0\), so the DBI Lagrangian vanishes.

### 3 Evolution of the D3-brane

In this section we consider a D3-brane embedded in the non-static and homogeneous way in the background given by the supergravity solutions. Let us deform the time component of the metric \(\gamma\) in such a way that it becomes original for \(E = B = 0\). The simplest deformation which fulfils the above conditions is obtained from (2.34).

The condition (2.34) can be expressed by the function \(V\) as follows:

\[
V \geq 0 ,
\]

(3.1)

where:

\[
V = -\gamma^{00} \left(1 + (2\pi\alpha')^2 B^2\right) - (2\pi\alpha')^2 E^2 .
\]

(3.2)

The condition (3.1) agrees with the signature of the induced metric \((- ++ +)\). Thus the deformed metric \(dl'^{2}\) on the brane looks like:

\[
dl'^{2} = -V \left(d\xi^0\right)^2 + \gamma_{mn} d\xi^m d\xi^n .
\]

(3.3)

The condition that \(V = 0\) determines a certain region on which the metric \(dl'^{2}\) is degenerated. These configurations, for which \(V = 0\), correspond to the vanishing of the Lagrangian \(L\).
Let us consider the background metric given by:

$$ds^2 = -\lambda_0 dt^2 + \lambda_1 \sum_{i=1}^{\tilde{d}-1} dX_i^2 + \lambda_2 dr^2 + r^2 \lambda_3 d\Omega_{d+1}. \quad (3.4)$$

This metric describes a \((d + 2)\)-brane which is wrapped on \(S^{d+1}\). For the embedding

$$X^M (\xi^0, \xi^m) = (\xi^0, \xi^m, X^a (\xi^0)) \quad (3.5)$$

(where \(a = 4, ..., 9\)) the induced \(dl^2\) metric on \(M\) has the form:

$$\gamma_{00} = -\lambda_0 + \lambda_1 \sum_{i=4}^{\tilde{d}-1} X_i^2 + \lambda_2 r^2 + r^2 \lambda_3 \varphi^2, \quad (3.6)$$

$$\gamma_{mn} = \lambda_1 \delta_{mn}, \text{for } 3 \leq \tilde{d} - 1. \quad (3.7)$$

In ten dimensions \(\tilde{d} - 1 = 9 - d\) and \(\gamma_{m0} = 0\), where:

$$\varphi^2 = h_{rs} \varphi^r \varphi^s, \quad (3.8)$$

and \(h_{rs} = h_{rs} (\varphi) (r, s = 1, ..., d + 1)\) is the metric on \(S^{d+1}\). Thus the deformed metric is:

$$dl'^2 = - \left( \lambda_0 - \lambda_1 \sum_{i=4}^{\tilde{d}-1} X_i^2 - \lambda_2 r^2 - r^2 \lambda_3 \varphi^2 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 E^2 \right) (d\xi^0)^2 + \lambda_1 d\xi_a d\xi^a. \quad (3.9)$$

Using spherical coordinates \((\rho, \theta, \psi)\) on \(M\) the above metric assumes the form:

$$dl'^2 = - \left( \lambda_0 - \lambda_1 \sum_{i=4}^{\tilde{d}-1} X_i^2 - \lambda_2 r^2 - r^2 \lambda_3 \varphi^2 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 E^2 \right) (d\xi^0)^2 + \lambda_1 d\rho^2 + \rho^2 \lambda_1 d\Omega_2. \quad (3.10)$$

Let us now compare this metric with the Reissner-Nordström-like metric which describes a charged black hole. The standard form of the metric describing a Reissner-Nordström-like black hole in four dimensions is the following:

$$ds^2 = -f (r) dt^2 + f^{-1} (r) dr^2 + r^2 d\Omega_2 \quad (3.11)$$

where \(f\) is equal to zero for two values of \(r\). These zeros are ordered in such a way: \(r_+ > r_-\), and \(r_+\) defines an event horizon. In the case when \(r_+ = r_-\) the black hole is extremal. In order to obtain the Reissner-Nordström-like black
hole for the metric $dl'$ the following constraint should be put on the metric components:

$$
\left( \lambda_0 - \lambda_1 \sum_{i=4}^{d-1} X_i^2 - \lambda_2 r^2 - r^2 \lambda_3 \varphi^2 \right) \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 \mathbf{E}^2 \right) \lambda_1 = 1. 
$$

(3.12)

In the case when $r = \varphi = 0$ the above condition assumes the form:

$$
\lambda_1^2 \left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2 - \lambda_1 \left[ \lambda_0 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 \mathbf{E}^2 \right] + 1 = 0.
$$

The solutions for this equations with respect to $\lambda_1$ are as follows:

$$
\lambda_1(\pm) = \frac{\lambda_0 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 \mathbf{E}^2}{2 \left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2} \\
\pm \frac{\sqrt{\left[ \lambda_0 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 \mathbf{E}^2 \right]^2 - 4 \left( 1 + (2\pi\alpha')^2 B^2 \right)^2 \sum_{i=4}^{d-1} X_i^2}}{2 \left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2}.
$$

(3.13)

$\lambda_1$ is real if:

$$
D = \left[ \lambda_0 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 \mathbf{E}^2 \right]^2 - 4 \left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2 \geq 0.
$$

(3.14)

The only one positive solution of (3.13) exists when $D = 0$. This condition relates $\lambda_0$ to $X_i$, $\mathbf{E}$ and $\mathbf{B}$:

$$
\left[ \lambda_0 \left( 1 + (2\pi\alpha')^2 B^2 \right) - (2\pi\alpha')^2 \mathbf{E}^2 \right]^2 = 4 \left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2. 
$$

(3.15)

In this case the solution (3.13) is:

$$
\lambda_1 = \frac{1}{\sqrt{\left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2}}
$$

(3.16a)

and:

$$
\lambda_0 = \frac{(2\pi\alpha')^2 \mathbf{E}^2 \pm 2 \sqrt{\left( 1 + (2\pi\alpha')^2 B^2 \right) \sum_{i=4}^{d-1} X_i^2}}{1 + (2\pi\alpha')^2 B^2}.
$$

(3.16b)
Note that $\lambda_0 > 0$ for all configurations if one chooses the sign $+$ in (3.16b). In the case when the sign $-$ is chosen the allowed configurations are restricted by the following condition:

$$\frac{(2\pi\alpha')^2 E^2}{2 \sqrt{1 + (2\pi\alpha')^2 B^2}} \geq \sqrt{\sum_{i=4}^{d-1} X_i}.$$ 

Thus the metric $dl''$ has the form:

$$dl'' = -\sqrt{\left(1 + (2\pi\alpha')^2 B^2\right)} \sqrt{\sum_{i=4}^{d-1} X_i (d\xi^0)^2} + \frac{1}{\sqrt{\left(1 + (2\pi\alpha')^2 B^2\right)} \sqrt{\sum_{i=4}^{d-1} X_i}} (d\rho^2 + \rho^2 d\Omega_2).$$

(3.17)

In the static case, i.e. $\dot{X}_i = \dot{r} = \dot{\phi} = 0$, the condition (3.12) gives:

$$\left(\lambda_0 \left(1 + (2\pi\alpha')^2 B^2\right) - (2\pi\alpha')^2 E^2\right) \lambda_1 = 1.$$  

(3.18)

Thus:

$$dl'' = -\left(\lambda_0 \left(1 + (2\pi\alpha')^2 B^2\right) - (2\pi\alpha')^2 E^2\right) (d\xi^0)^2 + \frac{1}{\lambda_0 \left(1 + (2\pi\alpha')^2 B^2\right) - (2\pi\alpha')^2 E^2} (d\rho^2 + \rho^2 d\Omega_2).$$  

(3.19)

This metric describes the spacetime with the magnetic and electric fields. Moreover this spacetime has the event horizon which is given by the vanishing of the Lagrangian since $|\gamma_{00}| = |\lambda_0|$.

In the metric (3.17) we make the change of the variable $\xi_0$ assuming that the magnetic field $B$ is constant on $M$:

$$\tau (t) = \int^t F (t') dt',$$ 

(3.20)

where $t = \xi_0$ and:

$$F (t) = \left[\left(1 + (2\pi\alpha')^2 B^2\right) \sum_{i=4}^{d-1} X_i (t)\right]^{1/4}.$$ 

(3.21)

In this new coordinate $\tau$ the metric (3.17) takes the form:

$$dl'' = -d\tau^2 + \left[\frac{1}{F (\tau)}\right]^2 (d\rho^2 + \rho^2 d\Omega_2),$$

(3.22)

where the function $f (\tau)$ is the inverse function to the function (3.20):

$$t = f (\tau).$$
In order to predict, how does the deformed metric (3.17) behave, let us assume the following form of the function $F(t)$:

$$F(t) = \Lambda^{1/2} t^{\alpha/2},$$

where $\alpha$ and $\Lambda$ are constants. This function is related to the Kaster-Traschen dynamic solutions [11] in the case when $\alpha = 1$. These solutions have been generalized to the branes in [12]. The metric (3.17) for this function takes the form:

$$dl'^2 = -\Lambda t^{\alpha/2} dt^2 + \Lambda^{-1} t^{-\alpha/2} d\mathbf{x}^2.$$  

(3.24)

In the coordinate $\tau$ related to $t$ by

$$t(\tau) = \left[(1 + \alpha/4) \sqrt{\Lambda} \tau\right]^{4/(4+\alpha)},$$  

(3.25)

(for $\alpha \neq -4$) the metric (3.24) becomes:

$$dl'^2 = -d\tau^2 + \Lambda^{-1} \left[(1 + \alpha/4) \sqrt{\Lambda}\right]^{-2\alpha/(4+\alpha)} \tau^{-2\alpha/(4+\alpha)} d\mathbf{x}^2.$$  

(3.26)

In the case when $\alpha = -4$ the variables $t$ and $\tau$ are related with each other as follows:

$$t(\tau) = \exp\left(\frac{\tau}{\sqrt{\Lambda}}\right).$$  

(3.27)

The metric (3.24) for $\alpha = -4$ is:

$$dl'^2 = -d\tau^2 + \Lambda^{-1} \exp\left(-\tau/\sqrt{\Lambda}\right) d\mathbf{x}^2.$$  

(3.28)

If $\alpha \in (-\infty, -4) \cup (0, +\infty)$, then $2\alpha/(4 + \alpha) > 0$ and the metric (3.28) represents the four-dimensional space-time being contracted from the phase with the finite space intervals at $\tau = 0$ to the phase with these intervals going to zero. For $\alpha = 0$ the metric is static. On the other hand the expanding space-time is obtained for $\alpha \in (-4, 0)$ (because $2\alpha/(4 + \alpha) < 0$) starting from an initial singularity. The Kasner metric is obtained for $\alpha = 4/3$ ([12]). The special case corresponds to $\alpha = -4$ with the metric (3.28). It is the de Sitter metric being contracted from the maximal space distance $\Lambda^{-1/2}$ for $\tau = 0$. In this way one obtains a family of contracting and expanding directions tangent to the D3-brane in the case when the constraint (3.23) holds.

The distance $l$ in the transverse directions $X_i$ with respect to the ambient metric (3.4) is given by:

$$l = \int \sqrt{\lambda_1} ds,$$

where $s$ is a parameter on a curve in the $X_i$ directions. From (3.16) and (3.23) one obtains:

$$l = \left\{ \begin{array}{ll} \frac{\sqrt{\lambda_1(4-\alpha)}}{\ln t} & \text{for } \alpha \neq 4 \\ \frac{\ln t}{\sqrt{\Lambda}} & \text{for } \alpha = 4. \end{array} \right.$$  

(3.29)

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The distance $l$ expressed by $\tau$ has the form:

$$l = \begin{cases} 
\frac{4}{\sqrt{\Lambda(4-\alpha)}} \left[ (1 + \alpha/4) \sqrt{\Lambda} \right]^{4-\alpha} \text{ for } \alpha \neq -4, +4 \\
\frac{1}{2\sqrt{\Lambda}} \ln \left( \frac{2\sqrt{\Lambda} \tau}{\sqrt{\Lambda}} \right) \text{ for } \alpha = +4 \\
\frac{1}{2\sqrt{\Lambda}} \exp \left( \frac{2\tau}{\sqrt{\Lambda}} \right) \text{ for } \alpha = -4
\end{cases}$$

(3.30)

The distance $l$ should be positive, so $4 > \alpha$. The transverse directions do expand for $\alpha \in (-4, +4)$ and contract for $\alpha \in (-\infty, -4)$. The cases when $\alpha = -4$ and $\alpha = +4$ correspond to the expanding transverse directions. To summarize, for some values of $\alpha$ the tangent directions contract while the transverse directions expand:

- For $\alpha \in (-\infty, -4)$ the tangent and transverse directions contract.
- For $\alpha \in (-4, 0)$ the tangent and transverse directions expand from the initial singularity.
- For $\alpha \in (0, +4]$ the tangent directions contract while the transverse directions expand.
- For $\alpha = -4$ the tangent directions are described by the contracting de Sitter metric with the maximal size $\Lambda^{-1/2}$ whereas the transverse directions expand. The minimal size of the transverse space is $2^{-1}\Lambda^{-1/2}$.
- For $\alpha = 0$ the tangent directions are static (they do not depend on $\tau$), the transverse directions expand from an initial point which is not singular.

## 4 Conclusions

The metric induced on the D3-brane has been deformed by adding the electric and magnetic fields to $\gamma_{00}$. This new metric has been compared to the Reissner-Nordström-like metric. This comparison has been made because we have expected the appearance of the singularities on the brane in the case when the DBI Lagrangian vanishes. Vanishing the DBI Lagrangian can be interpreted as a result of the strong coupling ($T_3 \to 0$) [8]. In this case description of the brane by the DBI action is unvalid since the DBI Lagrangian is obtained in the low energetic approximation.

A family of expanding and contracting branes has been obtained for the embedding $X$ restricted by the Eq. (3.23). The DBI Lagrangian vanishes for $t = 0$. This corresponds to a special state of the D3-brane. This state has been interpreted as an initial singularity (in the case of expansion) or a final singularity (in the case of contraction). Other solutions with the vanishing DBI Lagrangian have been obtained which do not possess singularities on the brane worldvolume. These solutions have been expressed by the de Sitter metrics (Eq. (3.28)).
The behavior of the directions transverse to the D3-brane is given as a function of time by Eq.(3.30). In this way a dynamical model of the space-time with a D3-brane embedded in it has been obtained.

5 References

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