GREENBERG’S CONJECTURE AND CYCLOTOMIC TOWERS

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Abstract. We describe Greenberg’s pseudo-null conjecture, and prove a result describing conditions under which the pseudo-null conjecture for a number field $K$ implies the conjecture for finite extensions of $K$. We then apply the result to the cyclotomic $\mathbb{Z}_p$-tower above a cyclotomic field of prime roots of unity, verifying the conjecture for a large class of cyclotomic fields.

1. Greenberg’s conjecture

In the late 1950’s Iwasawa introduced a powerful technique for studying class groups and unit groups of number fields. Motivated by the theory of curves over finite fields, Iwasawa’s theory of $\mathbb{Z}_p$-extensions has since become a widely used tool in algebraic number theory, Galois theory, and arithmetic geometry. We describe in this section a conjecture of Greenberg concerning the structure of a classical Iwasawa module, and we mention a Galois theoretic consequence concerning free pro-$p$-extensions of number fields.

Let $K$ be an algebraic number field and $p$ an odd prime. By a multiple $\mathbb{Z}_p$-extension $K_\infty/K$ we mean a Galois extension with Galois group $\Gamma \simeq \mathbb{Z}_p^d$ for
some positive integer $d$. In what follows we will be particularly interested in two such extensions of $K$ for which we reserve the following notation:

- $K^{cyc}/K$ denotes the *cyclotomic* $\mathbb{Z}_p$-extension of $K$.
- $\tilde{K}/K$ denotes the compositum of all $\mathbb{Z}_p$-extensions of $K$.

Let $F$ be a finite extension of $K$ contained in $K_{\infty}$, and denote by $A(F)$ the Sylow $p$-subgroup of the ideal class group of $F$. The Galois group of $F/K$ acts on $A(F)$ in the natural way, making $A(F)$ into a $\mathbb{Z}_p[\text{Gal}(F/K)]$-module. As $F$ varies over all finite subextensions the $A(F)$ form an inverse system (under norm maps) and we denote by $A$ the inverse limit. The group $A$ then carries a natural structure as a module over the Iwasawa algebra

$$\mathbb{Z}_p[[\Gamma]] := \varprojlim_F \mathbb{Z}_p[\text{Gal}(F/K)].$$

It is common to study $A$ by identifying the $A(F)$ with Galois groups as follows. By class field theory, the group $A(F)$ is isomorphic to the Galois group, $X_F$, of the maximal abelian unramified $p$-extension of $F$ (the *$p$-Hilbert class field of $F$*). The isomorphism respects the Galois module structure, the action of $\text{Gal}(F/K)$ on $X_F$ being inner automorphism. The $X_F$ form an inverse system (the maps being given by restriction of automorphisms) and the limit $X$ is the Galois group of the maximal abelian unramified pro-$p$-extension of $K_{\infty}$. So $X \cong A$. 
The Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$ is non-canonically isomorphic to the power series ring

$$\Lambda := \mathbb{Z}_p[[T_1, T_2, \ldots, T_d]],$$

where topological generators $\gamma_i$ of $\Gamma$ are sent to $1 + T_i$. So the $\mathbb{Z}_p[[\Gamma]]$-module structure of $A$ is studied via the $\Lambda$-module structure of $X$ (noting that $T_i x = x^{\gamma_i - 1}$).

For $K_{\infty}/K$ any multiple $\mathbb{Z}_p$-extension Greenberg ([3], Theorem 1) has shown $X$ to be a finitely generated torsion $\Lambda$-module. In particular, the annihilator of $X$, $\text{Ann}_\Lambda(X)$, is non-trivial. Traditionally, annihilators of classical Iwasawa modules have been of much interest. The Main conjecture of Iwasawa theory gives the factors of the annihilator of $X$ for the cyclotomic $\mathbb{Z}_p$-extension of a number field $K$ as essentially the $p$-adic $L$-functions attached to $K$. There is also a two variable Main conjecture for certain $\mathbb{Z}_p^2$-extensions arising from the theory of elliptic curves.

Greenberg ([3], Conjecture 3.4) has conjectured that for the cyclotomic $\mathbb{Z}_p$-extension $K^{cyc}/K$ of a totally real field $K$, the module $X$ is finite. If a totally real field $K$ satisfies Leopoldt’s conjecture the extensions $K^{cyc}$ and $\tilde{K}$ coincide (i.e. $K$ has only one $\mathbb{Z}_p$-extension). Furthermore, when $\Lambda = \mathbb{Z}_p[[T]]$ it can be shown that a module being finite is equivalent to having an annihilator of height
at least 2. With this in mind the above conjecture is a special case of the more general conjecture ([5], Conjecture 3.5):

**Conjecture 1.** Let $K$ be any number field and $	ilde{K}$ the compositum of all $\mathbb{Z}_p$-extensions of $K$. Then $\text{Ann}_\Lambda(X)$ has height at least 2.

A $\Lambda$-module whose annihilator has height at least 2 is said to be pseudo-null, and we will refer to Conjecture 1 above as Greenberg’s conjecture, or just the pseudo-null conjecture.

The point of this note is two-fold. First, we prove a “going-up” theorem for the pseudo-null conjecture. Namely, if $K$ is a number field, and $F$ is a finite extension of $K$ in $\tilde{K}$, we give conditions under which Greenberg’s conjecture for $K$ implies Greenberg’s conjecture for $F$ (Theorem 6). The result is an exercise in utilizing several equivalent formulations of the conjecture. Versions of these formulations have appeared in Lannuzel and Nguyen-Quang-Do ([9], Theorem 4.4) as well as work of McCallum ([11] and this author [10]). Secondly, as an application of the result, we consider the example $K = \mathbb{Q}(\zeta_p)$ and $F = \mathbb{Q}(\zeta_{p^n})$. We verify the conjecture for a certain class of such $K$’s, implying the conjecture for each field in the corresponding $\mathbb{Z}_p$-tower.

The key argument in both results is reduced to a capitulation problem, namely the need for a set of ideals, or ideal classes, to become principal when extended to an appropriate field. For the “going-up” result, the resolution of this problem is provided by an equivalent form of the conjecture, stating that all ideal classes
capitulate in $\tilde{K}$. In verifying the conjecture for $\mathbb{Q}(\zeta_p)$ capitulation is obtained by more direct means. We state our second result here.

Let $K = \mathbb{Q}(\zeta_p)$, $E = \mathcal{O}_K^\times$ and $U = \mathcal{O}_{K_{\pi}}^\times$, where $\pi$ is the unique prime of $K$ above $p$. Denote by $\mathcal{E}$ the closure of $E$ in $U$. We denote by $\lambda_p$ the Iwasawa lambda invariant of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\zeta_p)$. Let $v_p$ denote the $p$-adic valuation. In Section 4 we prove

**Theorem 1.** Suppose $K = \mathbb{Q}(\zeta_p)$ satisfies the following conditions:

1. Vandiver’s conjecture
2. $\lambda_p = 1$.
3. $v_p(|(U/\mathcal{E})[p^\infty]|) \leq v_p(|A(K)|)$.

Then for all $n \geq 1$ the pseudo-null conjecture holds for $\mathbb{Q}(\zeta_{pn})$.

We mention here one Galois theoretic consequence of the pseudo-null conjecture for cyclotomic fields. The existence of free pro-$p$-extensions (Galois extensions with Galois group a free pro-$p$-group) has been the subject of much study. See for example the list of known results in [15]. Let $K = \mathbb{Q}(\zeta_{pn})$ for some $n > 0$, and let $\Omega_K$ denote the maximal pro-$p$ extension of $K$ which is unramified at all primes not dividing $p$. Let $\mathcal{G}_K$ denote the Galois group.

Since free pro-$p$-extensions are unramified outside $p$, such extensions of $K$ are contained in $\Omega_K$. We will see that $\mathcal{G}_K$ is a free pro-$p$ group exactly when $p$ is a regular prime (since the number of relations defining $\mathcal{G}_K$ is equal to the $p$-rank
of the class group of $K$). When $p$ is an irregular prime the group $\mathcal{G}_K$ is not free, but we may look for free pro-$p$ quotients. Let $r_2$ denote the number of complex places of $K$. Then Leopoldt’s conjecture predicts $r_2 + 1$ independent $\mathbb{Z}_p$-extensions of $K$, and so the maximal rank of a free pro-$p$-extension of $K$ is bounded above by $r_2 + 1$. The following is proved in [9], as well as [11]:

**Theorem 2.** Suppose that $K = \mathbb{Q}(\zeta_{p^n})$ satisfies Greenberg’s conjecture. Then $\mathcal{G}_K$ has a free pro-$p$-quotient of rank $r_2 + 1$ if and only if $p$ is regular.

We give here a brief outline of the paper. In Section 2, we introduce several auxiliary $Λ$-modules and Galois groups needed for the later study. Theorem 3 and Lemma 1 are the key results of this section, implying a sufficient condition for a standard Iwasawa module to be torsion free (Corollary 1). In Section 3 we recall and provide several equivalent formulations of Greenberg’s pseudo-null conjecture, and we state and prove one of our main results (the “going-up” theorem). Finally, in Section 4 we turn to the example furnished by cyclotomic fields, proving Theorem 1 above.

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2. Auxiliary modules

For a number field $K$ and a prime number $p$, we call a field extension of $K$ $p$-ramified if it is unramified at all primes of $K$ not dividing $p$. We fix the following notation:

The fields:

- $\Omega_K$ the maximal pro-$p$, $p$-ramified extension of $K$
- $\tilde{K}$ the compositum of all $\mathbb{Z}_p$-extensions of $K$
- $L_\infty$ the maximal abelian unramified pro-$p$-extension of $\tilde{K}$
- $M_\infty$ the maximal abelian $p$-ramified pro-$p$-extension of $\tilde{K}$
- $N_\infty$ the extension of $\tilde{K}$ generated by $p$-power roots of $p$-units of $\tilde{K}$

The Galois groups:

- $G_K$ the Galois group of $\Omega_K/K$
- $\Gamma$ the Galois group of $\tilde{K}/K$
- $X$ the Galois group of $L_\infty/\tilde{K}$
- $Y$ the Galois group of $M_\infty/\tilde{K}$
- $Y'$ the Galois group of $N_\infty/\tilde{K}$

The Galois groups $Y$ and $Y'$ carry an action of $\Gamma$ via conjugation, just as $X$, making them into $\Lambda$-modules. We shall see that for certain base fields $K$, the pseudo-null conjecture may be formulated in terms of the $\Lambda$-module structure of $Y$ (in particular, that $Y$ is $\Lambda$-torsion free). The module $Y$ is known to be finitely generated, and, for $K/\mathbb{Q}$ abelian, have $\Lambda$-rank equal to $r_2$, where $r_2$ denotes the
number of complex places of \( K \) ([1]). For a \( \Lambda \)-module \( M \) we write \( \text{Tor}_\Lambda(M) \) for the \( \Lambda \)-torsion submodule. The following result is due to McCallum.

**Theorem 3** ([1], Theorem 3). Suppose there is only one prime of \( K \) above \( p \), and \( \tilde{K} \) contains all \( p \)-power roots of unity. Then \( \text{Tor}_\Lambda(Y') = 0 \).

*Remark 1*: The proof of this result involves a detailed analysis of the filtration

\[ E_F^u \subset E_F^n \subset E_F^{\text{loc}} \subset E_F, \]

where \( E_F \) denotes the units \( \mathcal{O}_F[1/p]^\times \) of a finite extension \( F \) of \( K \) in \( \tilde{K} \), and the superscripts denote certain classes of universal norms (see Section 4 of [1] for the precise definitions). The torsion submodule of \( Y' \) is contained in the kernel of a surjective map of Galois groups. The Pontryagin dual of this kernel is \( \lim_{\rightarrow} F(E_F/E_F^p) \otimes \mathbb{Q}_p/\mathbb{Z}_p \), and is shown to be zero by considering each graded factor from the filtration.

*Remark 2*: In particular, the result tells us \( \text{Tor}_\Lambda(Y) \) fixes the field \( N_{\infty} \). This observation, combined with Lemma 1 below, gives our approach to verifying the pseudo-null conjecture.

The group \( \mathcal{G}_K \) has a minimal free presentation

\[ 1 \rightarrow R \rightarrow F_g \rightarrow \mathcal{G}_K \rightarrow 1, \]
where $F_g$ is the free pro-$p$-group on $g$ generators and $R$ is the normal closure of a finitely generated subgroup (the group of relations for $G_K$). Denote by $s$ the minimal number of (topological) generators of $R$. The numbers $g$ and $s$ are equal to the $\mathbb{F}_p$-dimensions of $H^i(G_K, \mathbb{Z}/p\mathbb{Z})$, $i = 1, 2$ respectively (see Chapter 4 of [13]).

Let $G_K^{ab}$ denote the maximal abelian quotient of $G_K$, and $M_K$ the maximal abelian $p$-ramified pro-$p$-extension of $K$ (so $G_K^{ab} = \text{Gal}(M_K/K)$). The field $M_K$ is an abelian, $p$-ramified extension of $\tilde{K}$ (the Galois group of $M_K/\tilde{K}$ is just the torsion subgroup of $G_K^{ab}$), and so is contained in the field $M_\infty$. Hence we have a natural map from $Y$ to $G_K^{ab}$ given by restriction of automorphisms. We refer the reader to [11] for a proof of the following.

**Lemma 1** ([11], Lemma 24). Suppose $K$ satisfies Leopoldt’s conjecture. If $G_K$ is a one-relator group (i.e. $s = 1$), then the map

$$\text{Tor}_A(Y) \rightarrow G_K^{ab}$$

is the zero map if and only if $\text{Tor}_A(Y) = 0$.

The following is an immediate consequence of Theorem 3 and Lemma 1:
Corollary 1. If $K$ is a number field satisfying the hypotheses of Theorem 3 and Lemma 1, then

$$M_K \subset N_\infty \text{ implies } \text{Tor}_\Lambda(Y) = 0.$$ (1)

3. Equivalent formulations

We have introduced the natural Iwasawa modules $X$ and $Y$ in the last section. The Galois action on each of the $X_F$ is also compatible with regard to extensions of ideal classes, so we may form the $\Lambda$-module $\varprojlim F X_F$ as well. Recall the groups $\text{Ext}^i_\Lambda(\cdot, \Lambda)$ are the right derived functors of $\text{Hom}_\Lambda(\cdot, \Lambda)$.

Theorem 4. Let $p$ be an odd prime and let $K$ be a number field with a unique prime above $p$. Then $\text{Ext}^1_\Lambda(X, \Lambda)$ is the Pontryagin dual of $\varprojlim F X_F$, where the $F$ vary over the finite extensions of $K$ in $\bar{K}$.

Proof: Let $m$ denote the unique maximal ideal of $\Lambda = \mathbb{Z}_p[[T_1, \ldots, T_r]]$, and define

$$\omega_n(T_i) = (1 + T_i)^{p^n} - 1.$$ (2)

The result is obtained by establishing the isomorphism

$$H^r_m(X) \simeq \varprojlim F X_F.$$
where $H^i_{\mathfrak{m}}(X)$ denotes Grothendieck’s local cohomology relative to the $\mathfrak{m}$-primary sequences
\[ x_n = (p^n, \omega_n(T_1), \ldots, \omega_n(T_r)). \]

The desired result is then a consequence of (a version of) Grothendieck’s local duality; namely
\[ \text{Ext}^N_{\Lambda}(X, \Lambda) \simeq \text{Hom}_{\mathbb{Z}/p}(H^i_{\mathfrak{m}}(X), \mathbb{Q}/\mathbb{Z}), \]

where $N$ denotes the length of the $\mathfrak{m}$-primary sequence. A good reference for this material is Chapter 3 of [1].

The details establishing (2) can be found in Theorem 8 of [11], where McCallum proves a similar result for the Galois group $X'$ of the maximal abelian unramified pro-$p$-extension of $\tilde{K}$ in which all primes dividing $p$ are completely decomposed. The proof translates easily to this case, simply replacing the decomposition group with inertia. □

Let $\mu_n$ denote the group of $n$-th roots of unity. As above, we let $X'_F$ denote the Galois group of the maximal abelian unramified extension of $F$ in which all primes dividing $p$ are completely decomposed. We write $X'$ for $X'_{\tilde{K}}$.

**Theorem 5.** Let $p > 5$ be a prime and suppose $\mu_p$ is in $K$. If $K$ has a unique prime ideal $\mathfrak{p}$ dividing $p$, then the following are equivalent:
(a) $X$ is pseudo-null  
(b) $X'$ is pseudo-null  
(c) $\text{Tor}_\Lambda(Y) = 0$  
(d) $\varinjlim F X'_F = 0$  
(e) $\varinjlim F X_F = 0$,  

where the fields $F$ vary over all finite extensions of $K$ in $\bar{K}$.

**Proof:** $(a) \iff (b)$. Recall $\Gamma = \text{Gal}(\bar{K}/K)$. We let $\Gamma_\wp$ denote the decomposition group of $\wp$ in $\Gamma$, and let $\Lambda_\wp = \mathbb{Z}_p[\Gamma/\Gamma_\wp]$. There is a natural surjection $X \to X'$ whose kernel is generated as a $\mathbb{Z}_p$-module by the Frobenius automorphisms corresponding to the primes above $p$, and therefore is finitely generated as a module over $\Lambda_\wp$. As a $\Lambda$-module, the annihilator of $\Lambda_\wp$ has height equal to the $\mathbb{Z}_p$-rank of $\Gamma_\wp$ (this is just the augmentation ideal in $\mathbb{Z}_p[\Gamma_\wp]$). Since there is only one prime of $K$ above $p$, its decomposition group has finite index in $\Gamma$, and therefore our assumption on $p$ makes $\Lambda_\wp$ pseudo-null. Therefore the kernel of the surjection $X \to X'$ is pseudo-null, and $X$ and $X'$ are pseudo-isomorphic.

$(a) \iff (c)$. This follows from a duality due to Jannsen ([8], Theorem 5.4) relating the $\Lambda$-modules $X'$ and $Y$, together with a structure theorem for $Y$ due to Nguyen-Quang-Do (Corollary 14 of [11] or Theorem 4.4 of [9]).
(c) ⇔ (d). In proving the results cited in the previous case, one shows, in particular, that

\[ \text{Tor}_\Lambda(Y) \cong \text{Ext}^1_\Lambda(X', \Lambda) \]

([9], Theorem 9). But \( \text{Ext}^1_\Lambda(X', \Lambda) \) is known to be the Pontryagin dual of \( \varinjlim_{F} X'_F \) (\([9]\), Theorem 8). The result then follows.

(c) ⇔ (e). Grothendieck’s local duality can be used to show that a torsion \( \Lambda \)-module is pseudo-null if and only if \( \text{Ext}^1_\Lambda \) vanishes (\([9]\), Lemma 6). This implies, in particular, that \( \text{Ext}^1_\Lambda(X, \Lambda) \) and \( \text{Ext}^1_\Lambda(X', \Lambda) \) are isomorphic, yielding

\[ \text{Tor}_\Lambda(Y) \cong \text{Ext}^1_\Lambda(X, \Lambda) \]

as well. Theorem 4 then finishes the proof. □

Remark: Various forms of these equivalences have certainly appeared elsewhere. In \([9]\), Lannuzel and Nguyen-Quang-Do prove the equivalence of (a), (c), and (e) under slightly different hypotheses. Namely, no restriction is made on the number of primes of \( K \) dividing \( p \), but rather it is assumed that all finite extensions of \( K \) in \( \tilde{K} \) satisfy Leopoldt’s conjecture. Formulation (c) has been used by McCallum \([10]\) and this author \([10]\) to verify Greenberg’s conjecture for certain classes of cyclotomic fields.
The following theorem provides sufficient conditions for when the pseudo-null conjecture for a number field $K$ implies the conjecture for a finite extension of $K$ in $\tilde{K}$. We apply this to the cyclotomic tower in Section 4.

**Theorem 6.** Let $p \geq 5$ be a prime and suppose $\mu_p$ is contained in $K$. Suppose $K$ has a unique prime $\wp$ dividing $p$. Then, if $F \subset \tilde{K}$ is a finite extension of $K$ satisfying

1. $\wp$ is non-split in $F/K$
2. $\dim_{\mathbb{F}_p} H^2(\mathcal{G}_F, \mathbb{Z}/p\mathbb{Z}) \leq 1$
3. Leopoldt’s conjecture,

then Greenberg’s conjecture for $K$ implies Greenberg’s conjecture for $F$.

**Proof:** Let $K$ and $F$ be number fields satisfying the above hypotheses, and assume the pseudo-null conjecture holds for $K$. We apply the notation introduced in Section 2 to the field $F$ (so we have $\Omega_F$, $\mathcal{G}_F$, $M_F$, etc.) If the $\mathbb{F}_p$-dimension of $H^2(\mathcal{G}_F, \mathbb{Z}/p\mathbb{Z})$ is 0, then $\mathcal{G}_F$ is a free pro-$p$-group. A structure theorem for $Y$ due to Nguyen Quang Do ([12], Proposition 1.7) then implies $\text{Tor}_A(Y) = 0$. Hence by formulation (c) of Theorem 5 Greenberg’s conjecture holds for $F$.

If the $\mathbb{F}_p$-dimension of $H^2(\mathcal{G}_F, \mathbb{Z}/p\mathbb{Z})$ is 1, then such an $F$ satisfies the hypotheses of Theorem 3 and Lemma 1, and so Corollary 1 applies. Namely, Greenberg’s
pseudo-null conjecture will hold for $F$ provided $M_F \subset N_\infty$, and hence it will suffice to show the extension $M_F/\tilde{F}$ is generated by $p$-power roots of $p$-units of $\tilde{F}$.

We consider the field $F^{cyc} = FK^{cyc}$, the cyclotomic $\mathbb{Z}_p$-extension of $F$. By assumption, this field contains all $p$-power roots of unity. Recall the group $\mathcal{G}^{ab}_F = \text{Gal}(M_F/F)$. The subgroup $\text{Gal}(M_F/F^{cyc})$ has the same torsion subgroup (which is just $\text{Gal}(M_F/\tilde{F})$) and $\mathbb{Z}_p$-rank 1 less. In particular, we have a non-canonical isomorphism

$$\text{Gal}(M_F/F^{cyc}) \simeq \text{Gal}(\tilde{F}/F^{cyc}) \times \text{Gal}(M_F/\tilde{F}).$$

We let $L$ denote the fixed field of the first factor (so $M_F = \tilde{F}L$.)

The Galois group $\text{Gal}(L/F^{cyc})$ is isomorphic to the torsion subgroup of $\mathcal{G}^{ab}_F$, and hence is a finite $p$-group. Since $F^{cyc}$ contains all $p$-power roots of unity, the extension $L/F^{cyc}$ is just a Kummer extension, generated by $p$-power roots of elements of $F^{cyc}$,

$$L = F^{cyc}(x_1^{1/p^{m_1}}, x_2^{1/p^{m_2}}, \ldots, x_n^{1/p^{m_n}}).$$

Further, the ideals $(x_i)$ are $p^{m_i}$-th powers of ideals of $F^{cyc}$, say $(x_i) = \mathfrak{J}_i^{p^{m_i}}$.

The extension $M_F/\tilde{F}$ is also generated by the $x_i^{1/p^{m_i}}$, and the ideals $(x_i)$ are the $p^{m_i}$-th powers of the ideals $\mathfrak{J}_i$ extended to $\tilde{F}$. But here is the key: the ideal
classes $[\mathfrak{a}]$ become principal classes when extended to $\tilde{F}$. This follows from the fact that $F^\text{cyc} \subset \tilde{K}$ and, having assumed the pseudo-null conjecture holds for $K$ (using formulation (e) of Theorem 5), the fact that all ideal classes become principal in $\tilde{K}$.

For a generator $x_i^{1/p^m_i}$ of $M_F/\tilde{F}$ we now know the ideal $(x_i)$ is the $p^m_i$-th power of a principal ideal, say

$$(x_i) = (y_i)^{p^m_i}.$$ 

The elements $x_i$ and $y_i^{p^m_i}$ must differ by a unit, say $x_i = uy_i^{p^m_i}$. But clearly, an extension generated by a $p^m_i$-th root of $x_i$ is also generated by a $p^m_i$-th root of $x_i/(y_i^{p^m_i}) = u$, and so the extension $M_F/\tilde{F}$ is generated by $p$-power roots of units on $\tilde{F}$. This implies $M_F \subset N_\infty$ which, by Corollary 1 and Theorem 5, implies Greenberg's conjecture for $F$. \Box

4. Cyclotomic Fields

We fix $p$ a prime number and consider more closely the case of the cyclotomic fields $K = \mathbb{Q}(\zeta_{p^n})$. Recall the group $\mathcal{G}_K$ has a minimal presentation as a pro-$p$-group with $g$ generators and $s$ relations, where $g$ and $s$ are equal to the $\mathbb{F}_p$-dimensions of $H^1(\mathcal{G}_K, \mathbb{Z}/p\mathbb{Z})$ and $H^2(\mathcal{G}_K, \mathbb{Z}/p\mathbb{Z})$ respectively.
Lemma 2. Let $p$ be a prime and let $K = \mathbb{Q}(\zeta_{p^n})$ for some natural number $n$. Let $\alpha$ denote the $\mathbb{Z}/p\mathbb{Z}$-rank of the $p$-class group of $K$. Then

$$g = \frac{p^n + p^{n-1} + 2}{2} + \alpha$$

$$s = \alpha.$$

Proof: These computations are not new, and we give here just a sketch. Let $\Omega'_K$ be the maximal $p$-ramified extension of $K$ with Galois group $G'_K$. Since $K$ contains the group $\mu_p$, and $G_K$ is the maximal pro-$p$ quotient of $G'_K$, we have

$$H^i(G_K, \mathbb{Z}/p\mathbb{Z}) \simeq H^i(G'_K, \mu_p).$$

The $\mathbb{Z}/p\mathbb{Z}$-dimensions of the latter groups can be obtained by considering the sequence

$$1 \to \mu_p \to \mathcal{O}_{\Omega'_K}[1/p]^\times \xrightarrow{p} \mathcal{O}_{\Omega'_K}[1/p]^\times \to 1.$$

The $p$-power map on $\mathcal{O}_{\Omega'_K}[1/p]^\times$ is surjective by the maximality of $\Omega'_K$ over $K$ (since $p$-th roots of $p$-units generate $p$-ramified extensions). Taking cohomology of the sequence with respect to the Galois group $G'_K$ yields a long exact sequence which may be broken into the following pair of short exact sequences.

$$0 \to \frac{\mathcal{O}_K[1/p]^\times}{(\mathcal{O}_K[1/p]^\times)^p} \to H^1(G'_K, \mu_p) \to C(K)[p] \to 0.$$
\[
0 \to \frac{C(K)}{pC(K)} \to H^2(G'_K, \mu_p) \to H^2(G'_K, O_{\Omega'_K}[1/p]^\times)[p] \to 0,
\]

where \(C(K)\) denotes the ideal class group of \(K\). The group \(H^2(G'_K, O_{\Omega'_K}[1/p]^\times)\) injects into the Brauer group \(B(K)\), and can be shown to be 0 by considering its behavior in the exact sequence

\[
0 \to B(K) \to \bigoplus_v B(K_v) \xrightarrow{\sum \text{inv}} \mathbb{Q}/\mathbb{Z} \to 0.
\]

A simple dimension count then gives

\[
g = r_2 + 1 + \alpha
\]

\[
s = \alpha
\]

where \(r_2 = (p^n + p^{n-1})/2\), as desired. \(\Box\)

If \(p\) is a regular prime, \(\alpha = 0\) for \(\mathbb{Q}(\zeta_p^n)\), \(n \geq 0\). Hence \(s = 0\), implying \(\text{Tor}_\Lambda(Y) = 0\), establishing Greenberg’s conjecture for each field in the cyclotomic tower.

The following corollary is an immediate consequence of Theorem 6 and Lemma 2.
Corollary 2. Let $p$ be an irregular prime. Let $n > 0$ be such that $\mathbb{Q}(\zeta_p^n)$ has a cyclic $p$-class group. Then Greenberg’s conjecture for $\mathbb{Q}(\zeta_p)$ implies Greenberg’s conjecture for $\mathbb{Q}(\zeta_p^n)$.

Proof: In the notation of Theorem 6, with $K$ as above, let $F = \mathbb{Q}(\zeta_p^n)$ for some positive integer $n$ satisfying the hypothesis. The field $K$ has a unique prime $\pi$ above $p$, and $\pi$ is totally ramified in $F/K$, and hence non-split. The dimension of $H^2(G_F, \mathbb{Z}/p\mathbb{Z})$ is less than or equal to 1 by our assumption of cyclic $p$-class groups. Since $F/\mathbb{Q}$ is abelian, implying Leopoldt’s conjecture for $F$, the hypotheses of Theorem 6 are satisfied, as desired. □

Finally, we prove Theorem 1 by providing a class of cyclotomic fields $\mathbb{Q}(\zeta_p)$, satisfying the hypotheses of Corollary 2, for which the pseudo-null conjecture is true. A similar class was first given by McCallum ([11], Theorem 1). He considered such fields with $p$-class group isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We provide here a slight generalization of that class, allowing for cyclic $p$-class groups of arbitrary $p$-power order, as well as apply Corollary 2 to extend the conjecture to all fields in the cyclotomic $\mathbb{Z}_p$-tower. We restate Theorem 1 here.

Theorem 7. Suppose $K = \mathbb{Q}(\zeta_p)$ satisfies the following conditions:

1. Vandiver’s conjecture
2. $\lambda_p = 1$.
3. $v_p(|(U/E)[p^\infty]|) \leq v_p(|A(K)|)$. 
Then for all $n \geq 1$ the pseudo-null conjecture holds for $\mathbb{Q}(\zeta_{p^n})$.

Remark 1: Condition (2) is heuristically true for approximately 75% of all irregular primes and experimentally true for 75% of the irregular primes up to 12 million, according to [2] (for these primes, $\lambda_p$ is just the index of irregularity of $p$).

Remark 2: Letting $K_n = \mathbb{Q}(\zeta_{p^n+1})$ and $A_n = A(K_n)$, the hypotheses of Vandiver’s conjecture and $\lambda_p = 1$ imply

$$A_n \simeq X/((1 + T)^{p^n} - 1)X,$$

where $X = \mathbb{Z}_p[[T]]/(T + p^a)$ (see Theorem 10.16 and Proposition 13.22 of [14]). In particular this yields isomorphisms

$$A_n \simeq \mathbb{Z}/p^{a+n}\mathbb{Z}$$

for all $n \geq 0$, and so (3) is a condition on cyclic groups of $p$-power order.

Remark 3: Since $A(K)$ is cyclic, there is only one Bernoulli number $B_i$, $2 \leq i \leq p - 3$, divisible by $p$. If $B_{p-j}$ denotes this term (so $\varepsilon_j A(K)$ is the non-trivial term of the idempotent decomposition of $A(K)$), then $L_p(s, \omega^{1-j})$ is the only non-trivial $p$-adic $L$-function attached to $K$. It follows from Theorem 8.25
of [14] that

\[(U/E)[p^\infty] \simeq \mathbb{Z}/p^m\mathbb{Z},\]

where \(m = v_p(L_p(1, \omega^{1-j}))\). This valuation may be computed in terms of the characteristic power series \(f(T)\) of \(\lim_{\leftarrow n} A(K_n)\). Under the assumption \(\lambda_p = 1\) this power series has the form \(f(T) = (T + cp^a)u\), where \(u\) is a unit, \(p^a\) is the order of the cyclic group \(A(K)\), and

\[f((1 + p)^s - 1) = L_p(s, \omega^{1-j}).\]

So the valuation of \(L_p\) at \(s = 1\) equals the valuation of \(f(p) = (p + cp^a)u\).

If \(a > 1\), \(v_p(f(p)) = 1\), and condition (3) is satisfied. If, on the other hand, \(a = 1\), \(v_p(f(p))\) depends on the value of \(c \pmod{p}\). The valuation will again be 1 provided \(c \not\equiv -1 \pmod{p}\). This congruence has been checked for \(p < 4000\) in [6], although tables are only given for \(p < 400\) and \(3600 < p < 4000\). For these values the congruence condition is satisfied.

Suppose \(K = \mathbb{Q}(\zeta_p)\) satisfies (1)-(3) above. Since \(A(K)\) is cyclic, say of order \(p^a\), the group \(G\) is a one-relator group and Lemma 1 applies. We will utilize this lemma to show \(\text{Tor}_A(Y) = 0\). In light of Corollary 1, it suffices to show \(M_K \subset N_\infty\), and so we consider the structure of \(G_K^{ab}\) in more detail.
Lemma 3. Suppose $K$ satisfies hypothesis (1) and (2) of Theorem 7. Then the torsion subgroup of $G_{ab}^K$ is cyclic.

Proof: Let $J_K$ denote the idele group of $K$, with $K^\times$ embedded diagonally. Let $U$ be the subgroup of ideles which are units at $\pi$ (the prime of $K$ above $p$) and 1 elsewhere, and let $U'$ be the subgroup of ideles which are 1 at $\pi$ and units elsewhere. Class field theory gives an isomorphism

$$G_{ab}^K \simeq \text{pro-}p\text{-completion of } J_K/(\overline{K^\times U'}),$$

where the overline denotes the closure.

If we let $\overline{E}$ denote the closure of the embedding of the units of $K$ in $U$, then in fact we have an exact sequence

$$0 \longrightarrow U_1/\overline{E_1} \longrightarrow G_{ab}^K \longrightarrow A(K) \longrightarrow 0,$$

where the subscript 1 indicates we are taking units congruent to 1 modulo $\pi$.

Since $U_1$ has $\mathbb{Z}_p$-rank $[K : \mathbb{Q}] = p - 1$ and $\overline{E_1}$ has $\mathbb{Z}_p$-rank $(p-3)/2$ (by Leopoldt’s conjecture, which holds for $K$), the $\mathbb{Z}_p$-rank of $G_{ab}^K$ is $(p+1)/2$ ($p \neq 2$ by the assumption $\lambda_p = 1$).
We claim the torsion in $G_{K}^{ab}$ comes from $U_{1}/E_{1}$, and show this by considering an idele $(a_{v})$ whose image in $G_{K}^{ab}$ is a torsion element, say of order $p^{m}$. So

$$(a_{v})^{p^{m}} \in K^{\times}U^{\prime},$$

say $(a_{v})^{p^{m}} = \alpha(u_{v})$ (where we abuse notation writing $\alpha$ for both the element of $K^{\times}$ as well as its diagonal image in $J_{K}$). This implies $\alpha$ is a $p^{m}$-th power in $K_{\pi}$, the $\pi$-adic completion of $K$. Let $\mathfrak{a}$ then be the ideal of $K$ such that $\mathfrak{a}^{p^{m}} = (\alpha)$.

We want to show the class of $\mathfrak{a}$ is principal.

Let $K_{m-1} = \mathbb{Q}(\zeta_{p^{m}})$, so $K_{m-1}(\alpha^{1/p^{m}})$ is an unramified extension. Since the class of $\mathfrak{a}$ lies in $A(K)^{-}$ (by Vandiver’s conjecture), the Kummer pairing implies the Galois group of $K_{m-1}(\alpha^{1/p^{m}})/K_{m-1}$ is trivial. Hence $\alpha$ must be a $p^{m}$-th power in $K_{m-1}$ as well, which means the ideal class of $\mathfrak{a}$ is principal when extended to $K_{m-1}$ (represented by a principal ideal generated by a $p^{m}$-th root of $\alpha$). But the map from $A(K)$ to $A(K_{m-1})$ is injective ([14], Proposition 13.26), and so $\mathfrak{a}$ must have represented a principal class in $A(K)$ as well. Hence the torsion in $G_{K}^{ab}$ maps to 0 in $A(K)$.

We now just need to determine the torsion subgroup of $U_{1}/E_{1}$. We may consider each factor of the idempotent decomposition separately. Since $\varepsilon_{i}E_{1} = 0$ for $i = 0$ and for $i$ odd, and each $\varepsilon_{i}U_{1} \simeq \mathbb{Z}_{p}$, we obtain

$$U_{1}/E_{1} \simeq (\mathbb{Z}_{p})^{(p+1)/2} \oplus \bigoplus_{i \text{ even}} \varepsilon_{i}U_{1}/\varepsilon_{i}E_{1}.$$
For even $i$ the terms $\varepsilon_i U_1 / \varepsilon_i E_1$ are equal to $\varepsilon_i U_1^+ / \varepsilon_i E_1^+$, where the superscript $+$ indicates we are looking at units in the local subfield fixed by the automorphism of order 2. Vandiver’s conjecture implies the cyclotomic units $C_1^+$ have index prime to $p$ in $E_1^+$ ([(14), Theorem 8.2]), and so it suffices to consider the quotients $\varepsilon_i U_1^+ / \varepsilon_i C_1^+$. But Theorem 8.25 of [(14)] states

$$[\varepsilon_i U_1^+ : \varepsilon_i C_1^+] = p^{v_p(L_p(1,\omega^i))}.$$

Since $A(K)$ is cyclic there is only one non-trivial $L_p(s,\omega^i)$, and hence only one cyclic factor, say of order $p^m$, in the torsion subgroup of $U_1/E_1$. \hfill $\square$

**Proof of Theorem 7:** The field $\widetilde{K}$ is in fact the fixed field of the torsion subgroup of $G_K^{ab}$, and so the extension $M_K/\widetilde{K}$ is a Kummer extension with $\text{Gal}(M_K/\widetilde{K}) \simeq \mathbb{Z}/p^m\mathbb{Z}$. With $A(K) \simeq \mathbb{Z}/p^a\mathbb{Z}$, condition (3) of the Theorem just states $m \leq a$.

To show that $M_K$ is contained in $N_{\infty}$, we need to show that $M_K/\widetilde{K}$ is generated by a $p$-th power root of a unit of $\widetilde{K}$. The argument, as in the proof of Theorem 6, is reduced to a capitulation problem.

Consider the extension $M_K/K_{m-1}$. There is a non-canonical isomorphism

$$\text{Gal}(M_K/K_{m-1}) \simeq \text{Gal}(\widetilde{K}/K_{m-1}) \times \text{Gal}(M_K/\widetilde{K}).$$
We let $L$ denote the fixed field of the first factor. The extension $L/K_{m-1}$ is a Kummer extension, and we may write

$$L = K_{m-1}(x^{1/p^m})$$

for some $x$ in $K_{m-1}$ where the ideal $(x)$ is of the form $(x) = \mathfrak{J}^{p^m}P$, where $P$ is the principal ideal of $K_{m-1}$ lying above $p$.

Since, in particular, $\mathfrak{J}$ represents a class of order dividing $p^m$ in $A(K_{m-1})$, condition (3) implies the class of $\mathfrak{J}$ is an extension of a class from $A(K)$ (recall the map $A(K) \to A(K_{m-1})$ is just an injection $\mathbb{Z}/p^a\mathbb{Z} \hookrightarrow \mathbb{Z}/p^{a+m-1}\mathbb{Z}$). We let $\mathfrak{A}$ be a representative ideal of the class that extends to the class of $\mathfrak{J}$.

Since the $p$-Hilbert class field of $K$ is contained in $\bar{K}$, and the class of $\mathfrak{A}$, and therefore $\mathfrak{J}$, becomes principal in $\bar{K}$. The extension $M_K/\bar{K}$ is also generated by a $p^m$-th root of $x$, and the ideal $(x)$ in $\bar{K}$ is now the $p^m$-th power of a principal ideal,

$$(x) = (y)^{p^m}.$$ 

The elements $x$ and $y^{p^m}$ then differ by a unit, i.e. $x = uy^{p^m}$. But clearly the extension $M_K$ is also generated by the $p^m$-th root of $x/y^{p^m} = u$, and so the field $M_K$ is contained in $N_\infty$. □

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