VARIATIONAL AND OPERATOR METHODS FOR MAXWELL-STOKES SYSTEM

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Dedicated to Professor Wei-Ming Ni on the occasion of his 70th birthday

Abstract. In this paper we revisit the nonlinear Maxwell system and Maxwell-Stokes system. One of the main feature of these systems is that existence of solutions depends not only on the natural of nonlinearity of the equations, but also on the type of the boundary conditions and the topology of the domain. We review and improve our recent results on existence of solutions by using the variational methods together with modified De Rham lemmas, and the operator methods. Regularity results by the reduction method are also discussed and improved.

1. Introduction. In this paper we revisit the questions of solvability and regularity of solutions to the nonlinear differential systems involving curl, including the linear, semilinear and quasilinear Maxwell system and Maxwell-Stokes system. One of the main feature of these systems is that existence of solutions depends not only on the natural of nonlinearity of the equations, but also on the type of the boundary conditions, and the topology of the domains. We shall show existence of solutions by using the variational methods, the monotone operator and compact operator method, and the reduction method. The results obtained in [48, 49] will be represented, generalized and improved.

This paper is organized as follows. Section 2 provides preliminary results on the operator curl, including the elementary facts about the operator curl, various spaces of vector fields, and div-curl-gradient inequalities. We shall also explain the reduction method.

In section 3 we introduce the question of effects of domain topology on the Maxwell equations and Maxwell-Stokes equations. In subsection 3.1 we show that the type of the boundary condition for the unknown potential function should be determined suitably according to the domain topology, in order for the boundary value problem to be solvable. In subsection 3.2 we briefly derive a general magneto-static model with topology effect.

2010 Mathematics Subject Classification. Primary: 35Q61; Secondary: 35A15, 35J20, 35J47, 35J50, 35J57, 35J61, 35J62, 35Q60, 47J30, 78A25.

Key words and phrases. Curl equation, Maxwell equations, Maxwell-Stokes system, magneto-static problem, modified de Rham lemma, variational method, reduction method, domain topology.

This work was partially supported by the National Natural Science Foundation of China grant no. 11671143, and 11431005.

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In section 4 we state the modified de Rham lemmas for functionals vanishing on $W^{1,p}_0(\Omega, \text{div } 0)$, $W^{1,p}_0(\Omega, \text{div } 0)$ and on $W^{1,p}_n(\Omega, \text{div } 0)$ respectively, and describe the potential functions in several examples.

In section 5 we consider quasilinear Maxwell-Stokes systems that have variational structure. We use the minimization method together with the modified de Rham lemmas to show existence of weak solutions.

In sections 6 and 7 we study linear and semilinear Maxwell-Stokes system, and prove existence of weak solutions, then show regularity and derive estimates of the solutions.

2. Preliminaries of the operator $\text{curl }^2$.

2.1. Some facts about $\text{curl }^2$. In this subsection we briefly describe properties of the operators $\text{curl}$ and $\text{curl }^2$. The precise statements of these facts will be given in the next subsections.

1. Degenerate ellipticity of the operator $\text{curl }^2$. In the three dimensions, for a vector field $u = (u_1(x), u_2(x), u_3(x))^T$, $\text{curl } u \equiv \nabla \times u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T$.

$\text{curl }^2$ is a degenerately elliptic operator: if we write $\text{curl }^2 u = \left( -\partial_{33}u_1 - \partial_{32}u_2 + \partial_{31}u_3, \partial_{23}u_1 - \partial_{21}u_2 - \partial_{22}u_3 + \partial_{23}u_3, \partial_{13}u_1 - \partial_{12}u_2 - \partial_{11}u_3 - \partial_{13}u_3 \right)$, then

$$\sum_{\alpha,\beta,i,j} A_{ij}^\alpha \xi^i_{\alpha} \xi^j_{\beta} = \frac{1}{2} \left\{ |\xi|^2 - (\xi_1^1 + \xi_2^2 + \xi_3^3)^2 \right\}$$

$$+ \frac{1}{2} \left\{ (\xi_2^2 - \xi_1^1)^2 + (\xi_3^3 - \xi_1^1)^2 + (\xi_3^3 - \xi_2^2)^2 \right\},$$

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha \beta} \eta^i_{\alpha} \eta^j_{\beta} = |\eta \times \xi|^2, \quad \forall \eta, \xi \in \mathbb{R}^3.$$

Hence for a boundary value problem (BVP for short) of the equations involving $\text{curl}$ (curl equation for short), in order to have well-posedness, type of boundary conditions cannot be arbitrarily prescribed. On the other hand, if $\Omega$ has no holes, $\text{curl }^2$ is globally elliptic in the sense that the quadratic form $a(u, v) = \int_{\Omega} \text{curl } u \cdot \text{curl } v dx$ is coercive in the space $H^1_{\text{div}}(\Omega, \text{div } 0)$.

2. Curl and the divergence-free condition $\text{div } u = 0$. The operators $\text{curl}$, $\text{div}$ and $\Delta$ are connected by the equality $\Delta = -\text{curl }^2 + \nabla \text{div } u$.

If $\text{div } u = 0$, then $\text{curl }^2 u = -\Delta u$, which could simplify the analysis, and BVP of curl equation coupled with the divergence-free condition and with a suitable boundary condition (such as prescribing the tangential component $u_\tau$) is elliptic and satisfies the complementing condition of Agmon-Douglis-Nirenberg [1]. Unfortunately, the divergence-free condition is not always available (for instance in the Meissner model), and the requirement brings extra unknown potential term into the equation.
3. curl and domain topology. Description of the kernel and image of the operator curl is useful in the study of existence and uniqueness of BVPs of curl-equations. In a simply-connected domain a smooth vector field with its curl vanishes is a gradient of a scalar function, and in a domain without holes a smooth and divergence-free vector field is a curl of some vector field. These conclusions are not true if the domain is multiply-connected and with holes, and description of the kernel and image of curl involves the spaces $H^1(\Omega)$ and $H^2(\Omega)$ associated with domain topology.

4. curl and compactness. For a vector field $u \in L^2(\Omega, \mathbb{R}^3)$, control of both curl $u$ and div $u$ yields interior control of $\nabla u$, but not globally on $\Omega$, hence we have difficulty of lack of compactness. On a simply-connected domain without holes and with a $C^2$ boundary, control of both curl $u$ and div $u$ together with control of either normal component or tangential component of $u$ on the boundary yields a global control of $\nabla u$ on $\Omega$, and certain type of Poincaré inequality is available.

5. Hodge decomposition and nonlinearities. For linear systems, one often uses Hodge decomposition $u = v + \nabla \phi$, div $v = 0$, where $\phi$ and $v$ satisfy certain boundary conditions according to one’s needs. For nonlinear systems, Hodge decomposition does not help as much as for linear systems, but it may yield some compactness for special nonlinearities.

2.2. Spaces of functions and vector fields. Most of materials in this subsection can be found in [25, 29]. Let us consider a domain $\Omega$ satisfying the following conditions:

(O1) $\Omega$ is a bounded $C^r$ domain in $\mathbb{R}^3$, $r \geq 1$, and is locally situated on one side of its boundary $\partial \Omega$, and $\partial \Omega$ has $m+1$ connected components $\Gamma_j$, $j = 1, \cdots, m+1$, where $\Gamma_{m+1}$ is the boundary of the unbounded component of $\Omega^c = \mathbb{R}^3 \setminus \Omega$.

(O2) There exist $N$ 2-dimensional $C^r$ manifolds $\Sigma_i$, non-tangential to $\partial \Omega$, such that $\Sigma_i \cap \Sigma_k = \emptyset$ for $i \neq k$, and $\hat{\Omega} = \Omega \setminus \left( \sum_{i=1}^{N} \Sigma_i \right)$ is simply-connected and Lipschitz.

The number $N$ in (O2) is called the first Betti number of $\Omega$, which is equal to the number of handles of $\Omega$; and the number $m$ in (O1) is called the second Betti number of $\Omega$, which is equal to the number of holes in $\Omega$. We say $\Omega$ is simply-connected if $N = 0$. We say $\Omega$ has no holes if $m = 1$, namely if $\partial \Omega$ is connected. We use $\nu$ to denote the unit outer normal vector of $\partial \Omega$ which points to the outside of $\Omega$.

We use $C^{k+a}(\Omega)$, $L^p(\Omega)$ and $W^{k,p}(\Omega)$ to denote the Hölder spaces, Lebesgue spaces and Sobolev spaces for real valued functions, use $C^{k+a}(\Omega, \mathbb{C})$, $L^p(\Omega, \mathbb{C})$ and $W^{k,p}(\Omega, \mathbb{C})$ to denote the corresponding spaces of complex-valued functions, and use $C^{k+a}(\Omega, \mathbb{R}^3)$, $L^p(\Omega, \mathbb{R}^3)$ and $W^{k,p}(\Omega, \mathbb{R}^3)$ to denote the spaces of vector fields. However the norms both for scalar functions and for vector fields will be denoted by $\| \cdot \|_{C^{k+a}(\Omega)}$, $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_{W^{k,p}(\Omega)}$. For $1 < p < \infty$, $p'$ denotes the conjugate number, and $p_*$ denotes the index of the Sobolev imbedding $W^{1,p}(\Omega) \subset L^{p_*}(\Omega)$ in $\mathbb{R}^3$, namely

$$p' = \frac{p}{p-1}, \quad p_* = \begin{cases} \frac{3p}{3-p} & \text{if } 1 \leq p < 3, \\ +\infty & \text{if } p \geq 3. \end{cases}$$
We denote by \((p_*)'\) the conjugate number of the Sobolev index \(p_*\) in \(\mathbb{R}^3\), and \((p')_*\) the Sobolev index in \(\mathbb{R}^3\) of the conjugate number \(p'\):

\[
(p_*)' = \begin{cases} 
\frac{p_*}{p_* - 1} & \text{if } 1 \leq p_* < +\infty \\
1 & \text{if } p_* = +\infty 
\end{cases}
\]

\[
(p')_* = \begin{cases} 
\frac{3p'}{4p' - 3} & \text{if } 1 \leq p' < 3 \\
+\infty & \text{if } p' \geq 3 
\end{cases}
\]

\[
(1)
\]

Remark 1. Let \(1 < p < \infty\).

(i) If \(r \geq (p_*)'\), then \(r_* \geq p'\).

(ii) If \(1 \leq s \leq (p')_*\), then \((s')_* \geq p\).

Denote

\[
\langle \zeta, \eta \rangle_{\partial \Omega, 1/p} = \langle \zeta, \eta \rangle_{W^{-1/p, p}(\partial \Omega), W^{1/p, p'}(\partial \Omega)}.
\]

If \(F(\Omega)\) and \(G(\partial \Omega)\) denote space of scalar functions on \(\Omega\) or on \(\partial \Omega\), we set

\[
\hat{F}(\Omega) = \{ \phi \in F(\Omega) : \int_{\Omega} \phi(x) dx = 0 \},
\]

\[
\hat{G}(\partial \Omega) = \{ \psi \in G(\partial \Omega) : \int_{\partial \Omega} \psi dS = 0 \}.
\]

Let \(\Delta\) denote the Laplacian operator. For \(1 \leq p \leq \infty\) we denote

\[
L^p(\Omega, \Delta) = \{ u \in L^p(\Omega) : \Delta u \in L^p(\Omega) \},
\]

\[
L^p(\Omega, \Delta) = \{ u \in L^p(\Omega, \Delta) : \Delta u = 0 \},
\]

\[
W^{2,p}_{00}(\Omega) = \{ u \in W^{2,p}(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \}.
\]

(4)

We use \(u_T\) to denote the tangential component on \(\partial \Omega\) of \(u\), namely

\[
u \times u \times \nu.
\]

Very often we use notation with the subscript \(T\), for instance \(u^0_T\), to denote a given vector field on \(\partial \Omega\) which is tangential to \(\partial \Omega\). We denote

\[
\mathbb{H}_1(\Omega) = \{ u \in L^2(\Omega, \mathbb{R}^3) : \text{curl } u = 0 \text{ and div } u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial \Omega \},
\]

\[
\mathbb{H}_2(\Omega) = \{ u \in L^2(\Omega, \mathbb{R}^3) : \text{curl } u = 0 \text{ and div } u = 0 \text{ in } \Omega, u_T = 0 \text{ on } \partial \Omega \}.
\]

(5)

It is well-known that \(\dim \mathbb{H}_1(\Omega) = N\) and \(\dim \mathbb{H}_2(\Omega) = m\), where \(m\) and \(N\) are given in \((O_1)\) and \((O_2)\). We denote by \(\mathbb{H}_j(\Omega)_{L^2(\Omega)}^\perp\) the orthogonal complementary of \(\mathbb{H}_j(\Omega)\) in \(L^2(\Omega, \mathbb{R}^3)\). When \(\partial \Omega\) is of \(C^2\), using regularity of div-curl system we have

\[
\| h_j \|_{W^{1,p}(\Omega)} \leq C_j(\Omega, p) \| h_j \|_{L^p(\Omega)}, \quad \forall h_j \in \mathbb{H}_j(\Omega), \ j = 1, 2.
\]

(6)
For $1 \leq p \leq \infty$, we define the following Sobolev type spaces of vector fields

$$
W^p(\Omega, \text{div}) = \{ u \in L^p(\Omega, \mathbb{R}^3) : \text{div} u \in L^p(\Omega) \},
$$

$$
W^p(\Omega, \text{curl}) = \{ u \in L^p(\Omega, \mathbb{R}^3) : \text{curl} u \in L^p(\Omega, \mathbb{R}^3) \},
$$

$$
\mathcal{H}(\Omega, \text{curl}) \equiv \mathcal{W}^2(\Omega, \text{curl}), \quad \mathcal{H}(\Omega, \text{div}) \equiv \mathcal{W}^2(\Omega, \text{div}).
$$

**Remark 2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $1 < p < \infty$.

(i) There exists a continuous map

$$
\gamma : L^p(\Omega, \Delta) \to W^{-1/p, p}(\partial \Omega),
$$

such that for any $\psi \in L^p(\Omega, \Delta)$,

$$
\langle \gamma(\psi) \rangle_{\partial \Omega, 1/p} = \int_{\Omega} \{ \psi \Delta \xi_\eta - \xi_\eta \Delta \psi \} \, dx, \quad \forall \eta \in W^{1/p, p'}(\partial \Omega),
$$

where $\xi_\eta \in W^{2,p'}(\Omega)$ is the lifting of $(0, \eta) \in W^{1+1/p, p'}(\partial \Omega) \times W^{1/p,p'}(\partial \Omega)$, such that

$$
\xi_\eta = 0 \quad \text{and} \quad \frac{\partial \xi_\eta}{\partial \nu} = \eta \quad \text{on} \ \partial \Omega,
$$

$$
\| \xi_\eta \|_{W^{2,p'}(\Omega)} \leq C(\Omega, p') \| \eta \|_{W^{1/p, p'}(\partial \Omega)}.
$$

If $\psi \in C^1(\bar{\Omega})$ then $\gamma(\psi) = \psi$ on $\partial \Omega$.

(ii) There exist continuous trace maps

$$
\gamma_\nu : W^p(\Omega, \text{div}) \to W^{-1/p, p}(\partial \Omega),
$$

$$
\gamma_\tau : W^p(\Omega, \text{curl}) \to T W^{-1/p, p}(\partial \Omega, \mathbb{R}^3),
$$

such that, for any $u \in W^p(\Omega, \text{div})$ and $v \in W^p(\Omega, \text{curl})$,

$$
\langle \gamma_\nu(u), \eta \rangle_{\partial \Omega, 1/p} = \int_{\Omega} \{ \phi_\eta \text{div} u + \nabla \phi_\eta \cdot u \} \, dx, \quad \forall \eta \in W^{1/p, p'}(\partial \Omega),
$$

$$
\langle \gamma_\tau(v), z \rangle_{\partial \Omega, 1/p} = \int_{\Omega} \{ w_z \cdot \text{curl} v - v \cdot \text{curl} w_z \} \, dx, \quad \forall z \in T W^{1/p, p'}(\partial \Omega, \mathbb{R}^3),
$$

where $\phi_\eta \in W^{1,p'}(\Omega)$ is a lift of $\eta$, and $w_z \in W^{1,p'}(\Omega, \mathbb{R}^3)$ is a lifting of $z$ such that

$$
\| \phi_\eta \|_{W^{1,p'}(\Omega)} \leq C(\Omega, p') \| \eta \|_{W^{1/p, p'}(\partial \Omega)},
$$

$$
\| w_z \|_{W^{1,p'}(\Omega)} \leq C(\Omega, p') \| z \|_{W^{1/p, p'}(\partial \Omega)}.
$$

If $u \in C^1(\bar{\Omega}, \mathbb{R}^3)$, then $\gamma_\nu(u) = \nu \cdot u$ and $\gamma_\tau(u) = \nu \times u$.

If $X(\Omega)$ denotes a space of vector fields, then we set

$$
X(\Omega, \text{div} 0) = \{ u \in X(\Omega) : \text{div} u = 0 \text{ in } \Omega \},
$$

$$
X(\Omega, \text{curl} 0) = \{ u \in X(\Omega) : \text{curl} u = 0 \text{ in } \Omega \},
$$

$$
X_{\nu 0}(\Omega) = \{ u \in X(\Omega) : u_\nu = 0 \text{ on } \partial \Omega \},
$$

$$
X_{\tau 0}(\Omega) = \{ u \in X(\Omega) : \nu \cdot u = 0 \text{ on } \partial \Omega \}. \quad (8)
$$
In particular,
\[ W_{t_0}^{k,p}(\Omega, \mathbb{R}^3) = \{ u \in W^{k,p}(\Omega, \mathbb{R}^3) : u_T = 0 \}, \]
\[ W_{n_0}^{k,p}(\Omega, \mathbb{R}^3) = \{ u \in W^{k,p}(\Omega, \mathbb{R}^3) : \nu \cdot u = 0 \}, \]
\[ W_1^{1,p}(\Omega, \mathbb{R}^3, u_T) = \{ u \in W^{1,2}(\Omega, \mathbb{R}^3) : u_T = u_T^0 \text{ on } \partial \Omega \}, \]
\[ W_{t_0}^{k,p,\ast}(\Omega, \mathbb{R}^3) : \text{ the dual space of } W_{t_0}^{k,p}(\Omega, \mathbb{R}^3), \]
\[ W_{n_0}^{k,p,\ast}(\Omega, \mathbb{R}^3) : \text{ the dual space of } W_{n_0}^{k,p}(\Omega, \mathbb{R}^3). \]

Denote the paring between \( W_{t_0}^{k,p,\ast}(\Omega, \mathbb{R}^3) \) and \( W_{t_0}^{k,p}(\Omega, \mathbb{R}^3) \) by
\[ (\cdot, \cdot)_{W_{t_0}^{k,p,\ast}(\Omega, \mathbb{R}^3), W_{t_0}^{k,p}(\Omega)}. \]

Denote the spaces of tangential vector fields on \( \partial \Omega \) by
\[ TC^{k+\alpha}(\partial \Omega, \mathbb{R}^3) = \{ w \in C^{k+\alpha}(\partial \Omega, \mathbb{R}^3) : \nu \cdot w = 0 \text{ on } \partial \Omega \}, \]
\[ TW^{s,p}(\partial \Omega, \mathbb{R}^3) = \{ w \in W^{s,p}(\partial \Omega, \mathbb{R}^3) : \nu \cdot w = 0 \text{ on } \partial \Omega \text{ in the sense of trace} \}, \]
\[ TW^{-s,p'}(\partial \Omega, \mathbb{R}^3) : \text{ the dual space of } TW^{s,p}(\partial \Omega, \mathbb{R}^3). \]

2.3. **Decomposition of \( L^2(\Omega, \mathbb{R}^3) \).** The kernel of curl and div can be represented by
\[ \mathcal{H}(\Omega, \text{curl} 0) = \text{grad} H^1(\Omega) \oplus_{L^2(\Omega)} H_1(\Omega), \]
\[ \mathcal{H}(\Omega, \text{div} 0) = \mathcal{H}^T(\Omega, \text{div} 0) \oplus_{L^2(\Omega)} H_2(\Omega), \]
\[ \mathcal{H}_{n_0}(\Omega, \text{div} 0) = \mathcal{H}_{n_0}^T(\Omega, \text{div} 0) \oplus_{L^2(\Omega)} H_1(\Omega), \]
and the images of curl are given by
\[ \text{curl } H^1(\Omega, \mathbb{R}^3) = \text{curl } H_{n_0}^1(\Omega, \text{div} 0) = \mathcal{H}_T(\Omega, \text{div} 0), \]
\[ \text{curl } H_{n_0}^1(\Omega, \text{div} 0) = \mathcal{H}_{n_0}^T(\Omega, \text{div} 0), \]
where
\[ \mathcal{H}_T(\Omega, \text{div} 0) = \{ u \in \mathcal{H}(\Omega, \text{div} 0) : \langle u \cdot \nu, 1 \rangle_{\Sigma_{1/2}} = 0, \ j = 1, \ldots, m + 1 \}, \]
\[ \mathcal{H}_{n_0}^T(\Omega, \text{div} 0) = \{ u \in L^2(\Omega, \mathbb{R}^3) : \text{div } u = 0, \ u \cdot \nu|_{\partial \Omega} = 0, \ (u \cdot \nu, 1)_{\Sigma_{1/2}} = 0, \ i = 1, \ldots, N \}, \]
see for instance [25, p.222, Proposition 3; p.226, Remark 5]). We have the following decomposition of \( L^2(\Omega, \mathbb{R}^3) \) (see [25, section 4.1]):
\[ L^2(\Omega, \mathbb{R}^n) = \mathcal{H}^T(\Omega, \text{div} 0) \oplus_{L^2(\Omega)} H_2(\Omega) \oplus_{L^2(\Omega)} \text{grad} H_0^1(\Omega), \]
\[ L^2(\Omega, \mathbb{R}^n) = \mathcal{H}_{n_0}^T(\Omega, \text{div} 0) \oplus_{L^2(\Omega)} H_1(\Omega) \oplus_{L^2(\Omega)} \text{grad} H_0^1(\Omega). \]

2.4. **The div-curl-gradient inequalities.**

**Lemma 2.1.** Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^{k+2} \) boundary, \( k \) is a non-negative integer, and \( 1 < p < \infty \).

(i)
\[ ||u||_{W^{k+1,p}(\Omega)} \leq C(\Omega, k, p) \{ ||\text{div } u||_{W^{k,p}(\Omega)} + ||\text{curl } u||_{W^{k,p}(\Omega)} \]
\[ + ||u||_{L^p(\Omega)} + ||\nu \cdot u||_{W^{k+1-1/p, p}(\partial \Omega)} \}. \]

If \( u \in H_1(\Omega)_{L^p(\Omega)} \), then the term \( ||u||_{L^p(\Omega)} \) can be dropped from the right side.
Lemma 2.4. For any \( p \in H(\Omega) \), \( h \), \( h \in H(\Omega) \), respectively we need the following Lemma 2.3. For \( 1 < p < \infty \), there exist unique \( \frac{\| u \|_{L^p(\Omega)}}{\| u \|_{L^p(\Omega)}} \mid \leq C(\Omega, k, \alpha) \{ \| \text{div} u \|_{C^{k+\alpha}(\Omega)} + \| \text{curl} u \|_{C^{k+\alpha}(\Omega)} \}
\end{equation}

If \( u \in H_1(\Omega) \), then the term \( \| u \|_{L^2(\Omega)} \) can be dropped from the right side.

**Lemma 2.2.** Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^{k+2+\alpha} \) boundary, \( k \) is a non-negative integer, and \( 0 < \alpha < 1 \).

(i) 
\[
\| u \|_{C^{k+1+\alpha}(\Omega)} \leq C(\Omega, k, \alpha) \{ \| \text{div} u \|_{C^{k+\alpha}(\Omega)} + \| \text{curl} u \|_{C^{k+\alpha}(\Omega)} \}
\]

(ii) 
\[
\| u \|_{C^{k+1+\alpha}(\Omega)} \leq C(\Omega, k, \alpha) \{ \| \text{div} u \|_{C^{k+\alpha}(\Omega)} + \| \text{curl} u \|_{C^{k+\alpha}(\Omega)} \}
\]

If \( u \in H_1(\Omega) \), then the term \( \| u \|_{L^2(\Omega)} \) can be dropped from the right side.

These inequalities and more general versions can be found in literature. For instance, (15) and (16) with \( p = 2 \) can be found in Theorem 3 on page 205, and Proposition 6 on page 237 in [25], also see [16, 29, 56, 39]. (15) and (16) with \( 1 < p < \infty \) can be found in [65, 6, 31]. (17) and (18) can be found in [12]. For a domain with Lipschitz boundary, see for instance [42, 55, 22, 4, 23] and the references therein.

To show that the term \( \| u \|_{L^p(\Omega)} \) can be dropped when \( u \) is orthogonal to \( H_1(\Omega) \) or \( H_2(\Omega) \) respectively we need the following Lemma 2.3. For \( 1 \leq p < \infty \) we define

\[
H^p_1(\Omega) = \{ u \in L^p(\Omega, \mathbb{R}^3) : \text{curl} u = 0 \text{ and } \text{div} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \partial \Omega \},
\]

\[
H^p_2(\Omega) = \{ u \in L^p(\Omega, \mathbb{R}^3) : \text{curl} u = 0 \text{ and } \text{div} u = 0 \text{ in } \Omega, \text{ curl} \nabla \phi = 0 \text{ on } \partial \Omega \}.
\]

**Lemma 2.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary. Then \( H^p_1(\Omega) \) and \( H^p_2(\Omega) \) are independent of \( p \) for any \( 1 < p < \infty \), and for any \( 0 < \alpha < 1 \)

\[
H^p_1(\Omega) = H^1(\Omega) \subset C^\alpha(\Omega, \mathbb{R}^3), \quad H^p_2(\Omega) = H^2(\Omega) \subset C^\alpha(\Omega, \mathbb{R}^3).
\]

**Proof.** It is well-known that \( H^1(\Omega), H^2(\Omega) \subset C^\alpha(\Omega, \mathbb{R}^3) \) for any \( 0 < \alpha < 1 \), see [25]. So we only need to consider the case when \( 1 < p < 2 \). We prove the conclusion for \( H^p_1(\Omega) \). Let us fix \( 1 < p < 2 \) and \( h \in H^p_1(\Omega) \). Applying (15) with \( k = 0 \) we see that \( h \in W^{1,p}(\Omega, \mathbb{R}^3) \subset L^{p_1}(\Omega, \mathbb{R}^3) \) with \( p_1 = 3p/(3-p) \). By iteration we get \( h \in L^{p_2}(\Omega, \mathbb{R}^3) \) with \( p_2 = 3p_{k-1}/(3-p_{k-1}) \), and \( p_2 - p_{k-1} \geq \delta = p^2/(3-p) \). After a finite time of iterations we find \( h \in L^{p_k}(\Omega, \mathbb{R}^3) \) with \( p_k > 3 \), then from (15) \( h \in W^{1,p_k}(\Omega, \mathbb{R}^3) \subset C^\alpha(\Omega, \mathbb{R}^3) \). Using (15) we get \( h \in W^{1,s}(\Omega, \mathbb{R}^3) \) for any \( s > 1 \), so \( h \in C^\alpha(\Omega, \mathbb{R}^3) \) for any \( 0 < \alpha < 1 \).

The following lemma gives a description of the kernel of curl in \( L^p(\Omega, \mathbb{R}^3) \).

**Lemma 2.4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary and \( 1 < p < \infty \). For any \( u \in L^p(\Omega, \mathbb{R}^3) \) with \( \text{curl} u = 0 \) in \( \Omega \), there exist unique \( \phi \in W^{1,p}(\Omega) \) and \( h_1 \in H_1(\Omega) \) such that

\[
u = \nabla \phi + h_1, \quad \| \phi \|_{W^{1,p}(\Omega)} + \| h_1 \|_{L^p(\Omega)} \leq C(\Omega, p) \| u \|_{L^p(\Omega)}.
\]
Proof. The result when \( p = 2 \) is well-known, see [25]. Let \( 1 < p < \infty \) and \( \mathbf{u} \in L^p(\Omega, \text{curl} 0) \). From [31] there exist unique \( \mathbf{v} \in W_{n0}^{1,p}(\Omega, \text{div} 0) \), \( \psi \in W^{1,p}(\Omega) \) and \( \mathbf{h} \in H^1(\Omega) \) such that

\[
\mathbf{u} = \text{curl} \mathbf{v} + \nabla \psi + \mathbf{h},
\]

\[
\|\mathbf{v}\|_{W^{1,p}(\Omega)} + \|\psi\|_{W^{1,p}(\Omega)} + \|\mathbf{h}\|_{L^p(\Omega)} \leq C_1(\Omega, p)\|\mathbf{u}\|_{L^p(\Omega)}.
\]

Denote \( \mathbf{u}_1 = \text{curl} \mathbf{v} \). By [57, Theorem 5.4], or [32, p.648-650], the following problem

\[
\Delta \xi = 0 \quad \text{in} \ \Omega, \quad \frac{\partial \xi}{\partial \nu} = \nu \cdot \mathbf{u}_1 \quad \text{on} \ \partial \Omega,
\]

has a unique weak solution \( \xi \in \dot{W}^{1,p}(\Omega) \), and

\[
\|\xi\|_{L^p(\Omega)} + \|\nabla \xi\|_{L^p(\Omega)} \leq C_2(\Omega, p)\|\mathbf{u}_1\|_{L^p(\Omega)}.
\]

Let \( \mathbf{w} = \mathbf{u}_1 - \nabla \xi \). Then \( \mathbf{w} \in W^p(\Omega, \text{curl} 0) \) and

\[
\int_{\Omega} \mathbf{w} \cdot \nabla \eta dx = 0, \quad \forall \eta \in W^{1,p'}(\Omega),
\]

which implies \( \text{div} \mathbf{w} = 0 \) in \( \Omega \) and \( \nu \cdot \mathbf{w} = 0 \) on \( \partial \Omega \) in the weak sense. So \( \mathbf{w} \in H^1_0(\Omega) \), and

\[
\|\mathbf{w}\|_{L^p(\Omega)} \leq C_3(\Omega, p)\|\text{curl} \mathbf{v}\|_{L^p(\Omega)}.
\]

Now \( \mathbf{u} = \nabla \phi + \mathbf{h}_1 \), where \( \phi = \psi + \xi \in \dot{W}^{1,p}(\Omega) \), \( \mathbf{h}_1 = \mathbf{w} + \mathbf{h} \in H^1_0(\Omega) = H^1(\Omega) \). \( \square \)

The following lemma gives a description of the image of \( \text{curl} \) in \( L^p(\Omega, \mathbb{R}^3) \). Define

\[
\mathcal{H}^{\nu,j,p}(\Omega, \text{div} 0) = \{ \mathbf{u} \in W^p(\Omega, \text{div} 0) : \langle \nu \cdot \mathbf{u}, 1 \rangle_{\Gamma_j} = 0, \ j = 1, \cdots, m + 1 \},
\]

\[
\mathcal{H}^{\Sigma_i,p}(\Omega, \text{div} 0) = \{ \mathbf{u} \in W^p_{n0}(\Omega, \text{div} 0) : \langle \nu \cdot \mathbf{u}, 1 \rangle_{\Sigma_i} = 0, \ i = 1, \cdots, N \}.
\]

\[\text{(20)}\]

**Lemma 2.5.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary and \( 1 < p < \infty \). Then

\[
\text{curl} W^{1,p}(\Omega, \mathbb{R}^3) = \text{curl} W_{n0}^{1,p}(\Omega, \text{div} 0) = \mathcal{H}^{\nu,p}(\Omega, \text{div} 0),
\]

\[
\text{curl} W_{n0}^{1,p}(\Omega, \mathbb{R}^3) \subseteq \mathcal{H}^{\nu,p}(\Omega, \text{div} 0).
\]

**Proof.** When \( p = 2 \) the conclusion is well-known, see (12).

(i) Prove \( \text{curl} W^{1,p}(\Omega, \mathbb{R}^3) \subseteq \mathcal{H}^{\nu,p}(\Omega, \text{div} 0) \). Assume \( \mathbf{u} = \text{curl} \mathbf{w} \) for some \( \mathbf{w} \in W^{1,p}(\Omega, \mathbb{R}^3) \). For any \( \mathbf{h}_2 \in H^2(\Omega) \), since \( \mathbf{h}_2|_{\partial \Omega} = 0 \) on \( \partial \Omega \), we have

\[
\int_{\Omega} \mathbf{u} \cdot \mathbf{h}_2 dx = \int_{\Omega} \text{curl} \mathbf{w} \cdot \mathbf{h}_2 dx = - \int_{\partial \Omega} \mathbf{w} \cdot (\nu \times \mathbf{h}_2) dS = 0.
\]

Taking \( \mathbf{h}_2 = \nabla \xi_j \), where \( \xi_j \) is a harmonic function in \( \Omega \) and \( \xi_j = \delta_{jk} \) on \( \Gamma_k \), the above equality gives

\[
0 = \int_{\Omega} \mathbf{u} \cdot \nabla \xi_j dx = \int_{\Omega} \text{div} (\xi_j u) dx = \langle \gamma_\nu(\mathbf{u}), \xi_j \rangle_{\partial \Omega, 1/p} = \langle \gamma_\nu(\mathbf{u}), 1 \rangle_{\Gamma_j, 1/p}.
\]

(ii) Prove \( \mathcal{H}^{\nu,p}(\Omega, \text{div} 0) \subseteq \text{curl} W^{1,p}(\Omega, \mathbb{R}^3) \). Assume \( \mathbf{u} \in \mathcal{H}^{\nu,p}(\Omega, \text{div} 0) \). Since \( C^1(\Omega, \mathbb{R}^3) \) is dense in \( W^p(\Omega, \text{div} 0) \) (see for instance [25, p.204, Theorem 1]1), we can take \( \mathbf{u}_k \in C^1(\Omega, \mathbb{R}^3) \) such that \( \mathbf{u}_k \to \mathbf{u} \) in \( W^p(\Omega, \text{div} 0) \). Then \( \text{div} \mathbf{u}_k \to \text{div} \mathbf{u} = 0 \) in \( L^p(\Omega) \) and \( \gamma_\nu(\mathbf{u}_k) \to \gamma_\nu(\mathbf{u}) \) in \( W^{-1,p'}(\partial \Omega) \). Let \( \phi_k \in H^1(\Omega) \) be the unique weak solution of

\[
\Delta \phi_k = \text{div} \mathbf{u}_k \quad \text{in} \ \Omega, \quad \frac{\partial \phi_k}{\partial \nu} = c_k = \frac{1}{|\partial \Omega|} \int_{\Omega} \text{div} \mathbf{u}_k dx \quad \text{on} \ \partial \Omega.
\]

\[1\] If we use the fact that \( C^1(\Omega, \text{div} 0) \) is dense in \( W^p(\Omega, \text{div} 0) \) [59, p.87-89, Lemma II.2.5.5], we make our proof shorter.
Then \( \phi_k \in H^2(\Omega) \cap W^{2,p}(\Omega) \) and \( \phi_k \to 0 \) in \( W^{2,p}(\Omega) \). Set \( v_k = u_k - \nabla \phi_k \). Then \( v_k \in H(\Omega, \text{div} \, 0) \cap H^1(\Omega, \mathbb{R}^3) \), \( v_k \to u \) in \( L^p(\Omega, \mathbb{R}^3) \), and \( \gamma_\nu(v_k) = \gamma_\nu(u_k) - c_k \to \gamma_\nu(u) \) in \( W^{-1/p,p}(\partial \Omega) \). Since \( \text{div} \, v_k = 0 \), we use the second equality in (11) to find
\[
\eta_k \in H^2(\Omega, \text{div} \, 0) \text{ and } h_{2k} \in H^2(\Omega) \text{ such that }
\]
\[
v_k = z_k + h_{2k}, \quad \|z_k\|_{L^2(\Omega)}^2 + \|h_{2k}\|_{L^2(\Omega)}^2 = \|v_k\|_{L^2(\Omega)}^2.
\]
We can write \( h_{2k} = \nabla \xi_k \), where \( \xi_k \) is a harmonic function in \( \Omega \) and \( \xi_k = c_{k,j} \) is a constant on \( \Gamma_j \) for \( j = 1, \cdots, m+1 \), and by (6),
\[
|c_{k,j}| \leq C(\Omega)\|h_{2k}\|_{H^1(\Omega)} \leq C(\Omega)\|h_{2k}\|_{L^2(\Omega)}, \quad j = 1, \cdots, m+1,
\]
\[
\int_{\Omega} h_{2k}^2 dx = \int_{\Omega} v_k \cdot h_{2k} dx = \int_{\Omega} v_k \cdot \nabla \xi_k dx = \int_{\partial \Omega} \xi_k \nu \cdot v_k dS
\]
\[
= \sum_{j=1}^{m+1} \int_{\Gamma_j} \nu \cdot v_k dS \leq (m+1)C(\Omega)\|h_{2k}\|_{L^2(\Omega)}\|\gamma_\nu(v_k)\|_{W^{-1/p,p}(\partial \Omega)}.
\]
Hence
\[
\|h_{2k}\|_{L^2(\Omega)} \leq (m+1)C(\Omega)\|\gamma_\nu(v_k)\|_{W^{-1/p,p}(\partial \Omega)} \to 0 \quad \text{as } k \to \infty.
\]
By this and (6), \( h_{2k} \to 0 \) in \( L^p(\Omega, \mathbb{R}^3) \), hence \( z_k \to u \) in \( L^p(\Omega, \mathbb{R}^3) \). By the first line of (12), we can find
\[
w_k \in H^1(\Omega, \text{div} \, 0) \cap H^2(\Omega)
\]
such that \( \text{curl} \, w_k = z_k \). By (15) \( w_k \) is uniformly bounded in \( W^{1,p}(\Omega, \mathbb{R}^3) \). After passing to a subsequence, we may assume that \( w_k \to w \) weakly in \( W^{1,p}(\Omega, \mathbb{R}^3) \) and strongly in \( L^p(\Omega, \mathbb{R}^3) \), \( w \in W^{1,p}_{00}(\Omega, \text{div} \, 0) \) and \( \text{curl} \, w = u \) in \( \Omega \). Therefore \( u \in \text{curl} \, W^{1,p}_{00}(\Omega, \text{div} \, 0) \).

(iii) Prove \( \text{curl} \, W^{1,p}_{00}(\Omega, \mathbb{R}^3) \subseteq \mathcal{H}^{\Sigma,p}(\Omega, \text{div} \, 0) \). Assume \( u = \text{curl} \, w \) for \( w \in W^{1,p}_{00}(\Omega, \mathbb{R}^3) \). For any \( y \in W^{1,q}(\Omega, \text{curl} \, 0) \), since \( \nu \times w = 0 \) on \( \partial \Omega \), we have
\[
\int_{\Omega} u \cdot y dx = \int_{\partial \Omega} \text{curl} \, w \cdot y dS = \int_{\partial \Omega} (\nu \times w) \cdot y dS = 0.
\]
We first take \( y = \nabla \xi \) for any \( \xi \in W^{2,q}(\Omega) \) in the above equality to get
\[
0 = \int_{\Omega} u \cdot \nabla \xi dx = \int_{\Omega} \text{div} \, (\xi u) dx = \langle \gamma_\nu(u), \xi \rangle_{\partial \Omega, 1/p}.
\]
hence \( \gamma_\nu(u) = 0 \) on \( \partial \Omega \). We then take \( y = h_1 \in H^1_0(\Omega) \), where \( h_1 = \nabla \eta \) on \( \tilde{\Omega} = \Omega \setminus \Sigma \), and the jump of \( \eta \) on \( \Sigma_i \) is a constant \( c_i \), and we get
\[
0 = \int_{\Omega} u \cdot \nabla \eta dx = \int_{\Omega} u \cdot \nabla \eta dx = \int_{\Omega} \text{div} \, (\eta u) dx
\]
\[
= \sum_{i=1}^{N} \langle \nu \cdot u, [\eta] \rangle_{\Sigma_i, 1/p} = \sum_{i=1}^{N} c_i \langle \nu \cdot u, 1 \rangle_{\Sigma_i, 1/p}.
\]
Since \( c_i \)'s can be any numbers, we find \( \langle \nu \cdot u, 1 \rangle_{\Sigma_i, 1/p} = 0 \) for \( i = 1, \cdots, N \).

(iv) Prove \( \mathcal{H}^{\Sigma,p}_{00}(\Omega, \text{div} \, 0) \subseteq \text{curl} \, W^{1,p}_{00}(\Omega, \text{div} \, 0) \). Assume \( u \in \mathcal{H}^{\Sigma,p}_{00}(\Omega, \text{div} \, 0) \). As in (ii) we find
\[
v_k \in H(\Omega, \text{div} \, 0) \cap H^1(\Omega, \mathbb{R}^3),
\]
\( \mathbf{v}_k \to \mathbf{u} \) in \( \mathcal{W}^p(\Omega, \text{div} 0) \); and find \( \psi_k \in \dot{W}^{1,p}(\Omega) \) such that

\[
\int_\Omega \nabla \psi_k, \nabla \eta dx = \int_\Omega \mathbf{v}_k \cdot \nabla \eta dx, \quad \forall \eta \in W^{1,p}(\Omega),
\]

\[
\|\psi_k\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)\|\mathbf{v}_k\|_{L^p(\Omega)}.
\]

Moreover, \( (21) \) holds with \( \psi_k \) replaced by \( \psi_k - \psi_l \) and \( \mathbf{v}_k \) replaced by \( \mathbf{v}_k - \mathbf{v}_l \), so

\[
\|\psi_k - \psi_l\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)\|\mathbf{v}_k - \mathbf{v}_l\|_{L^p(\Omega)},
\]

hence \( \psi_k \) converges strongly in \( W^{1,p}(\Omega) \), and the limit \( \psi \in \dot{W}^{1,p}(\Omega) \) satisfies \( (21) \) with \( \psi_k \) replaced by \( \psi \) and \( \mathbf{v}_k \) replaced by \( \mathbf{u} \). Since \( \text{div} \mathbf{u} = 0 \) in \( \Omega \) and \( \nu \cdot \mathbf{u} = 0 \) on \( \partial \Omega \), we see that \( \psi = 0 \). Thus \( \psi_k \to 0 \) strongly in \( W^{1,p}(\Omega) \).

Let \( \mathbf{y}_k = \mathbf{v}_k - \nabla \psi_k \). Then \( \mathbf{y}_k \in H_{00}(\Omega, \text{div} 0) \) and \( \mathbf{y}_k \to \mathbf{u} \) strongly in \( L^p(\Omega, \mathbb{R}^3) \). By the third equality in \( (11) \) we can find \( \mathbf{z}_k \in H_{00}^1(\Omega, \text{div} 0) \) and \( \mathbf{h}_{1k} \in \mathbb{H}_1(\Omega) \) such that

\[
\mathbf{y}_k = \mathbf{z}_k + \mathbf{h}_{1k}, \quad \|\mathbf{z}_k\|_{L^2(\Omega)} + \|\mathbf{h}_{1k}\|_{L^2(\Omega)} = \|\mathbf{y}_k\|_{L^2(\Omega)}.
\]

After passing to a subsequence we may assume that \( \mathbf{h}_{1k} \to \mathbf{h}_1 \in \mathbb{H}_1(\Omega) \). Thus

\[
\int_\Omega |\mathbf{h}_{1k}|^2 dx = \int_\Omega \mathbf{y}_k \cdot \mathbf{h}_{1k} dx \to \int_\Omega \mathbf{u} \cdot \mathbf{h}_1 dx = 0,
\]

so \( \mathbf{z}_k \to \mathbf{u} \) in \( L^p(\Omega, \mathbb{R}^3) \) as \( k \to \infty \). From the second line of \( (12) \) we can find

\[
\mathbf{w}_k \in H_{00}^1(\Omega, \text{div} 0) \cap \mathbb{H}_1(\Omega) \in H_{00}^1(\Omega, \text{div} 0),
\]

such that \( \text{curl} \mathbf{w}_k = \mathbf{z}_k \). By \( (16) \) \( \mathbf{w}_k \) is uniformly bounded in \( W^{1,p}(\Omega, \mathbb{R}^3) \). Since \( \mathbf{z}_k \to \mathbf{u} \) in \( L^p(\Omega, \mathbb{R}^3) \), after passing to a subsequence, we may assume that \( \mathbf{w}_k \to \mathbf{w} \) weakly in \( W^{1,p}(\Omega, \mathbb{R}^3) \) and strongly in \( L^p(\Omega, \mathbb{R}^3) \), \( \mathbf{w} \in W_{00}^1(\Omega, \text{div} 0) \) and \( \text{curl} \mathbf{w} = \mathbf{u} \) in \( \Omega \). Therefore \( \mathbf{u} \in \text{curl} W_{00}^{1,p}(\Omega, \text{div} 0) \).

2.5. **Reduction method.** The reduction method is useful to study BVPs of curl systems (see [48, 49]). Let us explain this method by the magneto-static model

\[
\begin{aligned}
&\text{curl} [\mathcal{H}(x, \text{curl} \mathbf{u})] = \mathbf{J}(x), \quad \text{div} \mathbf{u} = 0 & &\text{in} \, \Omega, \\
&\mathbf{u} \cdot \nu = 0 & &\text{on} \, \partial \Omega.
\end{aligned}
\]

Assume \( \mathcal{H} \) satisfies condition \( (H) \) in section 3, in particular, \( \mathcal{H}(x, \mathbf{z}) \) is invertible in \( \mathbf{z} \). So there exists \( \mathcal{B}(x, \mathbf{w}) \) such that \( \mathbf{w} = \mathcal{H}(x, \mathbf{z}) \) if and only if \( \mathbf{z} = \mathcal{B}(x, \mathbf{w}) \).

If \( (22) \) has a solution \( \mathbf{u} \in H^1(\Omega, \mathbb{R}^3) \), then \( \mathbf{J} \in H^1(\Omega, \text{div} 0) \) (see Lemma 2.5 or [48, Lemma 2.1]), hence there exists \( \mathbf{v} \in H_{00}^1(\Omega, \text{div} 0) \) such that

\[
\text{curl} \mathbf{v} = \mathbf{J},
\]

so \( \text{curl} [\mathcal{H}(x, \text{curl} \mathbf{u}(x)) - \mathbf{v}(x)] = \mathbf{0} \). From the first line in \( (11) \), there exist \( \mathbf{h}_1 \in \mathbb{H}_1(\Omega) \) and \( \phi \in H^1(\Omega) \), such that

\[
\mathcal{H}(x, \text{curl} \mathbf{u}(x)) - \mathbf{v}(x) = \mathbf{h}_1 + \nabla \phi.
\]

So \( (22) \) is reduced to

\[
\begin{aligned}
&\text{curl} \mathbf{u} = \mathcal{B}(x, \mathbf{v} + \mathbf{h}_1 + \nabla \phi), \quad \text{div} \mathbf{u} = 0 & &\text{in} \, \Omega, \\
&\mathbf{u} \cdot \nu = 0 & &\text{on} \, \partial \Omega.
\end{aligned}
\]
Taking divergence in the first equality in (24), and using the fact $\nu \cdot \text{curl} \mathbf{u} = \nu \cdot \text{curl} \mathbf{u}_T = 0$, we see that $\phi$ satisfies
\[
\begin{cases}
\text{div} \left[ \mathcal{B}(x, \mathbf{v} + \mathbf{h}_1 + \nabla \phi) \right] = 0 & \text{in } \Omega, \\
\nu \cdot \mathcal{B}(x, \mathbf{v} + \mathbf{h}_1 + \nabla \phi) = 0 & \text{on } \partial \Omega.
\end{cases}
\]  
(25)

We may get regularity and estimates of $\mathbf{u}$ as follows. With the $\mathbf{v}$ given, and taking $\mathbf{h}_1$ as a parameter, we view (25) as a co-normal problem for $\phi$ and derive estimates. Then we view (24) as a div-curl system and derive estimates of $\mathbf{u}$.

We may get existence of $\mathbf{u}$ as follows. We first fix $\mathbf{v}$ and $\mathbf{h}_1$, and find a solution $\phi = \phi[\mathbf{v}]$ of (25) by using, say, the monotonicity operator method. Then we find an $\mathbf{h}_1 = \mathbf{h}_1[\mathbf{v}] \in H_0^1(\Omega)$ such that
\[
\int_{\Omega} \mathcal{B}(x, \mathbf{v} + \mathbf{h}_1[\mathbf{v}] + \nabla \phi[\mathbf{v}]) \cdot \mathbf{h} \, dx = 0, \quad \forall \mathbf{h} \in H_1(\Omega).
\]
Then
\[
\mathcal{B}(x, \mathbf{v} + \mathbf{h}_1[\mathbf{v}] + \nabla \phi[\mathbf{v}]) \in H^*_0(\Omega, \text{div } 0).
\]  
(26)

So (24) with $\mathbf{h}_1 = \mathbf{h}_1[\mathbf{v}]$ and $\phi = \phi[\mathbf{v}]$ has a solution $\mathbf{u}$, which is a solution of (22).

3. Maxwell equations and domain topology. The Maxwell Equations (1865) are the following
\[
\begin{cases}
\text{div} (\varepsilon \mathbf{E}) = \rho, & \text{(Gauss’s law)} \\
\text{div} \mathbf{B} = 0, & \text{(Gauss’s law of induction)} \\
\text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, & \text{(Faraday’s law of induction)} \\
\text{curl} \mathbf{H} = \mathbf{j} + \frac{\partial}{\partial t}(\varepsilon \mathbf{E}), & \text{(Ampère’s law)}
\end{cases}
\]  
(27)

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field, $\mathbf{j}$ is the electric current density, $\rho$ is the charge density, $\varepsilon$ is the permittivity. Consider (27) in $\mathbb{R}^3$, and from the second equation in (27), we can write
\[
\mathbf{B} = \text{curl} \mathbf{u},
\]  
(28)

where $\mathbf{u}$ is the magnetic potential. Since two magnetic potentials differ by a gradient describes the same magnetic field, it has been well accepted that the field strength ($\mathbf{E}, \mathbf{B}$) (or ($\mathbf{E}, \mathbf{H}$)), instead of the electromagnetic potentials, is fundamental to describe an electromagnetic field, until Y. Aharonov and D. Bohm discovered in 1959 the observable effect of magnetic potentials [2]. As mentioned in [68] that, the work [2] raised the question of what constitutes an intrinsic and complete description of electromagnetism. Wu and Yang [68] showed that, on a multiply-connected domain, the field strength under-describes electromagnetism and the phases (contour integrals of electromagnetic potentials) over-describes electromagnetism, but the phase factor of the integral of electromagnetic potentials
\[
\exp\left(\frac{ie}{\hbar c} \oint A_\mu dx^\mu\right)
\]
provides a complete description that is neither too much nor too little.

The original form of the Maxwell equations was a system of 20 equations and 20 unknowns, involving magnetic potential. Oliver Heaviside (1884) re-wrote the equations in a vector form without potential.
We would like to examine the problem of complete description of electromagnetism from the PDE point of view. Since Aharonov-Bohm effect is due to non-trivial topology of the domain, our question will be the following:

**Question 3.1.** Is it possible to find a complete description of electromagnetism by PDEs of electric field $E$ and magnetic potential $u$, together with consideration of domain topology?

This question motivated our study on the following problems:

(i) Examine effects of domain topology on the magneto-static problem.

(ii) Study PDEs of electric field $E$ and magnetic potential $u$, with domain topology as parameters.

### 3.1. Boundary conditions and domain topology

We tried in [49] to understand effects of domain topology on the Maxwell system (27) by considering a quasi-static approximation. Neglecting the term $\frac{\partial}{\partial t}(\varepsilon E)$ in the fourth equation of (27), we get the equality

$$\text{curl} \, H = j.$$  

For linear material the $B$-$H$ relation is given by $B = \mu H$, where $\mu$ is the permeability. We consider nonlinear material and assume that the $B$-$H$ relation is given by

$$H = H(x, B),$$  

where $H(x, z)$ is a nonlinear vector-valued function. So the static magnetic problem is

$$\text{curl} \left[ H(x, \text{curl} \, u) \right] = j(x) \quad \text{in } \Omega.$$  

Let $j \in L^2(\Omega, \mathbb{R}^3)$. From (12), a necessary condition for a BVP of (30) to be solvable is

$$j \in \begin{cases} H(\Omega, \text{div} 0) & \text{if } \Omega \text{ has no holes,} \\ H^\Gamma(\Omega, \text{div} 0) & \text{if } \Omega \text{ has holes.} \end{cases}$$

If $j$ does not satisfy this condition, then $j$ should be replaced by $j + \nabla \phi$ such that

$$j + \nabla \phi \in \begin{cases} H(\Omega, \text{div} 0) & \text{if } \Omega \text{ has no holes,} \\ H^\Gamma(\Omega, \text{div} 0) & \text{if } \Omega \text{ has holes.} \end{cases}$$

Hence we consider a BVP of the Maxwell-Stokes system instead of the Maxwell system, and the boundary condition for $\phi$ should be suitably determined according to the domain topology. For a more general system, with $j$ replaced by $f(x, u)$, if the domain has no holes, we can formulate the problem

\begin{align*}
\text{curl} \left[ H(x, \text{curl} \, u) \right] &= f(x, u) + \nabla \phi, \quad \text{div} \, u = 0 \quad \text{in } \Omega, \\
u^T = u_0^T, \quad \phi = 0 & \quad \text{on } \partial \Omega; 
\end{align*}

while for a domain with holes, the BVP is

\begin{align*}
\text{curl} \left[ H(x, \text{curl} \, u) \right] &= f(x, u) + \nabla \phi, \quad \text{div} \, u = 0 \quad \text{in } \Omega, \\
u^T = u_0^T, \quad \frac{\partial \phi}{\partial \nu} = -c(f, u) & \quad \text{on } \partial \Omega, 
\end{align*}

where

$$c(f, u)(x) = \frac{1}{|\Gamma_i|} \langle 1, \nu \cdot f(x, u(x)) \rangle_{H^{1/2}(\Gamma_i), H^{-1/2}(\Gamma_i)} \quad \text{on } \Gamma_i, \quad i = 1, \ldots, m + 1.$$  

The physical meaning of the term $\nabla \phi$ will be clear in Remark 3 below.
bounded set \( K \) and (31) and (32). We need the following

\[ w \in \text{loc}\ (\bar{\Omega}), \]

there exists \( C \) such that

\[ |\nabla_x H(x, z)| \leq C_0. \]

\[ f \in C^1_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3, \mathbb{R}^3), \]

and there exists a constant \( K_0 \) such that

\[ |\nabla_x f(x, u)| \leq K_0(|u| + 1), \quad |\nabla_u f(x, u)| \leq K_0. \]

\[ \lim_{|u| \to \infty} \frac{|f(x, u)|}{|u|} = 0 \quad \text{uniformly for } x \in \bar{\Omega}. \]

Condition (H) is used in [49], which implies the conditions (H1), (H2), (H3) in [49, subsection 2.3], while condition (H2) implies condition (H4) in [49]:

\[ \langle H(x, z_2) - H(x, z_1), z_2 - z_1 \rangle \geq \mu |z_2 - z_1|^2, \quad \mu > 0. \]

Condition (f) here is same as \((f_1) + (f_2)\) in [49, subsection 4.2].

Let us start with (31) on a domain without holes. Given \( u_0^\Omega \in TC^{2+\alpha}(\partial \Omega, \mathbb{R}^3) \), there exists \( U \in C^{2+\alpha}(\partial \Omega, \text{div} 0) \) such that \( U_T = u_0^\Omega \) on \( \partial \Omega \) (see [45]). Let \( v = u - U \). Then (31) is equivalent to

\[
\begin{cases}
\text{curl } [H_1(x, \text{curl } v)] = \mathcal{P}_0 f_1(x, v), & \text{div } v = 0 \quad \text{in } \Omega, \\
\nu_T = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where

\[ H_1(x, z) = H(x, z + \text{curl } U), \quad f_1(x, u) = f(x, u + U), \]

and

\[ \mathcal{P}_0 : L^2(\Omega, \mathbb{R}^3) \to H(\Omega, \text{div} 0) \]

is the Leray projection, namely, for any \( w \in L^2(\Omega, \mathbb{R}^3) \), \( \mathcal{P}_0 w = w + \nabla \xi_w \), where \( \xi_w \in H^1(\Omega) \) satisfies

\[-\Delta \xi_w = \text{div } w \quad \text{in } \Omega, \quad \xi_w = 0 \quad \text{on } \partial \Omega.\]

Define

\[ G(v) = \text{curl } H_1(x, \text{curl } v), \quad F(v) = \mathcal{P}_0 f_1(x, v) : C^2^{2+\alpha}(\bar{\Omega}, \text{div} 0) \to C^1(\bar{\Omega}, \text{div} 0).\]

When \( \Omega \) has no holes, \( G \) is homeomorphic, and \( F \) is compact. Using the Schauder fixed point theorem we have

**Proposition 1** ([49], Theorem 5.9). Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) without holes and with a \( C^{3+\alpha} \) boundary, \( 0 < \alpha < 1 \), \( H \) and \( f \) satisfy (H) and (f) respectively, and \( u_0^\Omega \in TC^{2+\alpha}(\partial \Omega, \mathbb{R}^3) \). Then (31) has a solution

\[ (u, \phi) \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3) \times C^1(\bar{\Omega}), \]

and it satisfies the estimate

\[ \|u\|_{C^{2+\alpha}(\bar{\Omega})} + \|\phi\|_{C^1(\bar{\Omega})} \leq C(\Omega, \alpha, \|u_0^\Omega\|_{C^{2+\alpha}(\partial \Omega)}), \quad (34)\]

\(^3\text{When } \Omega \text{ is a bounded domain, we say } H \in C^{k+\alpha}(\Omega \times \mathbb{R}^3) \text{ if } H \in C^{k+\alpha}(\Omega \times K, \mathbb{R}^3) \text{ for any bounded set } K \subset \mathbb{R}^3.\)
where the constant \( C \) depends also on the constants in \((H)\) and \((f)\).

If \( \Omega \) has holes, we consider BVP (32). With \( v = u - U \) as above, (32) is equivalent to
\[
\begin{aligned}
curl[H_1(x, \nabla v)] &= \mathcal{P}_1 f_1(x, v), \quad \text{div} v = 0 \quad &\text{in} \, \Omega, \\
v_T &= 0 \quad &\text{on} \, \partial \Omega,
\end{aligned}
\]
here
\[
\mathcal{P}_1 : \mathcal{H}(\Omega, \text{div}) \to \mathcal{H}^T(\Omega, \text{div})
\]
is a projection, \( \mathcal{P}_1(w) = w + \nabla \zeta_w \), where \( \zeta_w \) is a weak solution of
\[
-\Delta \zeta_w = \text{div} w \quad \text{in} \, \Omega, \quad \frac{\partial \zeta_w}{\partial \nu} = -c(w) \quad \text{on} \, \partial \Omega, \quad \int_{\Omega} \zeta_w \, dx = 0,
\]
\[
c(w)(x) = c_i(w) = \frac{1}{|F_i|} \int_{\Gamma_i} \nu \cdot w(x) \, dS \quad \text{on} \, \Gamma_i, \quad i = 1, \ldots, m + 1.
\]
We need the following condition on \( f(x,u) \):
\[
(f') \quad f \in C^{1+\alpha}_\text{loc}(\Omega \times \mathbb{R}^3, \mathbb{R}^3), \quad \text{and the following holds uniformly for } x \in \bar{\Omega}:
\]
\[
\lim_{|u| \to \infty} \left\{ \frac{|f(x,u)| + |\nabla_x f(x,u)|}{|u|} + |\nabla_u f(x,u)| \right\} = 0.
\]
This condition implies conditions \((f_1)\) and \((f_2')\) in [49].

**Proposition 2** ([49], Theorem 6.4). Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^{3+\alpha} \) boundary, \( 0 < \alpha < 1 \), \( \mathcal{H} \) and \( f \) satisfy \((H)\) and \((f')\) respectively, and \( u_0^T \in \mathcal{C}^{2+\alpha}(\partial \Omega, \mathbb{R}^3) \). Then (32) has a solution \((u, \phi) \in C^{2+\alpha}(\Omega, \mathbb{R}^3) \times C^{1+\alpha}(\Omega) \). If \( u \) is chosen to satisfy the \( \mathbb{H}_2(\Omega) \)-orthogonality condition (113) in \( C^{2+\alpha}(\Omega, \mathbb{R}^3) \), then \((u, \phi)\) satisfies the estimate (34), with the constant \( C \) depends also on the constants in \((H)\), \((f')\) and (113).

### 3.2. A magneto-static model with domain topological effect. A general magneto-static model which includes the effect of domain topology was derived in [51] under the following assumptions:

(i) The vector-valued function \( \mathcal{H}(x, \cdot) \) in the \( H-B \) relation (29) has an inverse \( B(x, \cdot) \).

(ii) The current density \( j \) satisfies the Ohm’s law
\[
\begin{aligned}
j &= j_a + \sigma E, \\
\end{aligned}
\]
where \( \sigma \geq 0 \) is the electric conductivity, and \( j_a \) is the applied current density.

We first derive a quasi-static approximation of the Maxwell system (27). Assume that
\[
B, \partial_t B \in L^2(0, T; L^2(\Omega, \mathbb{R}^3)), \quad E \in L^2(0, T; H^1(\Omega, \mathbb{R}^3)), \quad \forall T > 0.
\]
From (27) \( \text{div} B = 0 \). By the second line of (11) we write
\[
B = \text{curl} w + h_2,
\]
where \( w(t, \cdot) \in H^1_{\text{ad}}(\Omega, \text{div} 0), h_2(t, \cdot) \in \mathbb{H}_2(\Omega) \). Then from the third line of (27)
\[
0 = \partial_t B + \text{curl} E = \text{curl} (E + \partial_t w) + \partial_t h_2,
\]
then
\[
-\partial_t h_2 = \text{curl} (E + \partial_t w) \in \text{curl} H^1(\Omega, \mathbb{R}^3) = \mathcal{H}^T(\Omega, \text{div} 0) \subseteq [\mathbb{H}_2(\Omega)]^T_{L^2(\Omega)}.
\]

\[\text{See for instance [33, p.199-120].}\]
So $\partial_t h_2 = 0$ and $\text{curl}(E + \partial_t w) = 0$. By the first line of (11)

$$E + \partial_t w = -\nabla \xi + \tilde{h}_1,$$

where $\xi(t, \cdot) \in H^1(\Omega)$ and $\tilde{h}_1(t, \cdot) \in \mathbb{H}_1(\Omega)$. Introduce

$$A = w + \nabla \zeta,$$

where

$$\zeta(t, x) = \int_0^t \xi(s, x) ds.$$

Then

$$E = -\partial_t A + \tilde{h}_1, \quad B = \text{curl} A + h_2, \quad H(t, x) = \mathcal{H}(x, \text{curl} A + h_2).$$

Using the fourth line of (27), (29) and (36), we have

$$\sigma \partial_t A - \partial_t (\varepsilon E) + \text{curl} [\mathcal{H}(x, \text{curl} A + h_2(x))] = j_a + \sigma \tilde{h}_1. \quad (37)$$

Set

$$u = A - \nabla \Phi, \quad q = -\partial_t \Phi,$$

where

$$\Delta \Phi(t, x) = \text{div} A(t, x)$$

in $\Omega$ for all $t \geq 0$. Neglecting the displacement current $\partial_t (\varepsilon E)$ in (37), we get the following

**Problem.** *Quasilinear parabolic Maxwell-Stokes system*

$$\begin{cases} 
\sigma \partial_t u + \text{curl} [\mathcal{H}(x, \text{curl} u + h_2(x))] = j_a + \sigma \tilde{h}_1 + \sigma \nabla q, & t > 0, \quad x \in \Omega, \\
\text{div} u = 0, & t > 0, \quad x \in \Omega. 
\end{cases} \quad (38)$$

Now we derive time-independent model. Assume $u, \tilde{h}_1, j_a, q, \varepsilon$ and $\rho$ are independent of $t$, $\sigma$ is a positive constant, and denote

$$\phi(x) = \sigma q(x), \quad h_1(x) = \sigma \tilde{h}_1(x), \quad \Phi(x, t) = \Phi_0(x) - \frac{t}{\sigma} \phi(x).$$

We have

$$E(x) = \frac{1}{\sigma} \nabla \phi(x) + \frac{1}{\sigma} h_1(x), \quad B(x) = \text{curl} u(x) + h_2(x),$$

$$H(x) = \mathcal{H}(x, \text{curl} u(x) + h_2(x)). \quad (39)$$

**Problem.** *Quasilinear elliptic Maxwell-Stokes system*

$$\begin{cases} 
\text{curl} [\mathcal{H}(x, \text{curl} u + h_2(x))] = j_a + h_1(x) + \nabla \phi & \text{in } \Omega, \\
\text{div} u = 0 & \text{in } \Omega. \quad (40)
\end{cases}$$

**Remark 3.** About the physical meaning of $\nabla \phi$ in (40): Mathematically, the gradient term $\nabla \phi$ is necessarily introduced into the equation in (40) to balance the equality, which is necessary for solvability of the BVP. On the other hand, from (39)

$$\nabla \phi = \sigma E - h_1,$$

so $\nabla \phi$ represents both electric field $E$ and domain topology through $h_1$. 
Now we consider solvability of (40), with boundary condition $\phi = 0$ on $\partial \Omega$, and $u$ satisfies either the tangential trace boundary condition
\[ u_T = u_T^0 \quad \text{on } \partial \Omega, \tag{41} \]
or the tangential curl boundary condition
\[ \nu \times \mathcal{H}(x, \text{curl } u + h_2(x)) = \nu \times H^0 \quad \text{on } \partial \Omega. \tag{42} \]
Assume $j_a \in H^2_0(\Omega)^\perp_{L^2(\Omega)}$, which is a necessary condition for (40) to have a solution (see [48, Lemma 2.1]). Then by the first equality in (14) we write
\[ j_a = J + \nabla \zeta, \quad J \in H^0(\Omega, \text{div } 0), \quad \zeta \in H^1_0(\Omega). \]
Taking $\phi = -\zeta$, we can write the equation as
\[
\begin{cases}
\text{curl } [\mathcal{H}(x, \text{curl } u + h_2(x))] = h_1(x) + J(x) & \text{in } \Omega, \\
\text{div } u = 0 & \text{in } \Omega.
\end{cases} \tag{43}
\]
In the following we assume that
\[
\Omega \text{ is a bounded domain in } \mathbb{R}^3 \text{ with a } C^2 \text{ boundary, } \mathcal{H} \text{ satisfies condition } (H),
\]
\[
J \in H^0(\Omega, \text{div } 0), \quad h_1 \in H^1_1(\Omega), \quad h_2 \in H^2_2(\Omega),
\tag{44}
\]
First we consider problem (43)-(41).

**Proposition 3** ([51]). Assume (44) and $u_T^0 \in TH^{1/2}(\partial \Omega, \mathbb{R}^3)$. Then (43)-(41) has a weak solution $u \in H^1(\Omega, \mathbb{R}^3)$, and it is unique in $H^1(\Omega, \mathbb{R}^3) \cap H^2_2(\Omega)^\perp_{L^2(\Omega)}$.

**Proof.** Since $J + h_1 \in H^0(\Omega, \text{div } 0)$, there exists a unique
\[ j_0 \in H^1_{10}(\Omega, \mathbb{R}^3) \cap H^2_2(\Omega)^\perp_{L^2(\Omega)} \]
such that $\text{curl } j_0 = J + h_1$. Take $U \in H^1(\Omega, \text{div } 0)$ such that $U_T = u_T^0$ on $\partial \Omega$ and write $v = u - U$. By condition $(H)$, (43)-(41) is solvable if and only if there exist $\phi \in H^1(\Omega)$, $h'_1 \in H^1_1(\Omega)$, such that the following has a solution $v$:
\[
\begin{cases}
\text{curl } v = B(x, j_0 + h'_1 + \nabla \phi) - h_2 - \text{curl } U & \text{in } \Omega, \\
\text{div } v = 0 & \text{in } \Omega, \\
\nu_T = 0 & \text{on } \partial \Omega.
\end{cases} \tag{45}
\]
From the second equality in (12) we know that, for the given $j_0$ and $h'_1$, (45) is solvable if and only if the following co-normal problem
\[
\begin{cases}
\text{div } [B(x, j_0 + h'_1 + \nabla \phi)] = 0 & \text{in } \Omega, \\
\nu \cdot B(x, j_0 + h'_1 + \nabla \phi) = \nu \cdot (h_2 + \text{curl } u_T^0) & \text{on } \partial \Omega,
\end{cases} \tag{46}
\]
has a solution $\phi$ which satisfies the orthogonality condition
\[
\int_\Omega \langle B(x, j_0 + h'_1 + \nabla \phi) - \text{curl } U, h \rangle dx = 0, \quad \forall h \in H^1_1(\Omega). \tag{47}
\]
Choose an orthonormal basis $\{e_i\}_{i=1}^N$ of $H^1_1(\Omega)$, write
\[ \xi = (\xi_1, \ldots, \xi_N)^t, \quad h'_1 = h'_{1, \xi} = \sum_{i=1}^N \xi_i e_i. \]
By the monotone operator method we can show that, (46) has a unique solution $\phi = \phi_\xi \in H^1(\Omega)$. Set
\[
f(\xi) = (f_1(\xi), \ldots, f_N(\xi))^t, \quad c = (c_1, \ldots, c_N)^t,
\]
where
\[
f_i(\xi) = \int_{\Omega} \langle B(x, j_0 + h^1_{i, \xi} + \nabla \phi), e_i \rangle dx, \quad c_i = \int_{\Omega} e_i \cdot \text{curl} \nu dx, \quad i = 1, \ldots, N.
\]
Then (47) can be written as $f(\xi) = c$. As in [48, proof of Theorem 4.9], we can show that $f(\xi)$ is a continuous map in $\mathbb{R}^N$, and for large $R$ it holds that
\[
\langle f(\xi) - c, \xi \rangle \geq 0 \quad \forall |\xi| = R.
\]
By the accurate angle theorem for continuous maps in $\mathbb{R}^N$ (see [72, Proposition 2.8]), there exists $\xi_0 \in B_R$ such that
\[
f(\xi_0) - c = 0.
\]
Let $\phi_{\xi_0}$ be the solution of (46) associated with $h^1_{\epsilon, \xi_0}$. Then $(h^1_{\epsilon, \xi_0}, \phi_{\xi_0})$ solves problem (46)-(47).

Next we consider (43)-(42). In addition to (44) we need the following
\[
\nu \times \mathbf{H}^0 \in TH^{1/2}(\partial \Omega, \mathbb{R}^3), \quad \nu \cdot \text{curl} \mathbf{H}^0_T = \nu \cdot \mathbf{J} \quad \text{on} \ \partial \Omega,
\]
\[
\int_{\Omega} (\mathbf{h}_1 + \mathbf{J}) \cdot \mathbf{h} dx = \langle \pi, \mathbf{h}, \nu \times \mathbf{H}^0 \rangle_{\partial \Omega, 1/2}, \quad \forall \mathbf{h} \in \mathbf{H}^1(\Omega).
\]
Here $\mathbf{H}^0_T = (\nu \times \mathbf{H}^0) \times \nu$, and $\pi$ is the tangential trace map $\mathbf{w} \mapsto \mathbf{w}_\tau$ from $\mathbf{H}(\Omega, \text{curl})$ to $H^{-1/2}(\partial \Omega, \mathbb{R}^3)$.

**Proposition 4** ([51]). Assume (44) and (48). Then (43)-(42) has a weak solution, and it is unique in $H^1_{n_0}(\Omega, \text{div} 0) \cap \mathbf{H}^1(\Omega)_{\perp L^2(\Omega)}$.

**Proof.** Let
\[
X = \mathbf{H}^1_{n_0}(\Omega, \text{div} 0) \cap \mathbf{H}^1(\Omega)_{\perp L^2(\Omega)}.
\]
By (15), both $\|\mathbf{w}\|_{H^1(\Omega)}$ and $\|\text{curl} \mathbf{w}\|_{L^2(\Omega)}$ are equivalent norms in $X$. By Lax-Milgram lemma, there is a continuous and strongly monotone operator $\mathbf{A}_{h_2} : X \to X^*$ such that
\[
\langle \mathbf{A}_{h_2}(\mathbf{w}), \mathbf{v} \rangle_{X^*, X} = \int_{\Omega} \langle \mathbf{H}(\mathbf{w}, \text{curl} \mathbf{w} + \mathbf{h}_2), \text{curl} \mathbf{v} \rangle dx, \quad \forall \mathbf{v} \in X.
\]
By Browder-Minty theorem, $\mathbf{A}_{h_2}$ is surjective, so there exists a unique $\mathbf{u} \in X$ such that
\[
\int_{\Omega} \langle \mathbf{H}(\mathbf{x}, \text{curl} \mathbf{u} + \mathbf{h}_2), \text{curl} \mathbf{v} \rangle dx = \int_{\Omega} (\mathbf{h}_1 + \mathbf{J}) \cdot \mathbf{v} dx - \langle \pi, \mathbf{v}, \nu \times \mathbf{H}^0 \rangle_{\partial \Omega, 1/2},
\]
\[
\forall \mathbf{v} \in X.
\]
For any $\mathbf{w} \in H^1(\Omega, \mathbb{R}^3)$ we can write
\[
\mathbf{w} = \mathbf{v} + \mathbf{h} + \nabla \phi, \quad \mathbf{v} \in H^1_{n_0}(\Omega, \text{div} 0) \cap \mathbf{H}^1(\Omega)_{\perp L^2(\Omega)}, \quad \mathbf{h} \in \mathbf{H}^1(\Omega), \quad \phi \in H^1(\Omega).
\]
Since
\[
\langle \mathbf{A}_{h_2}(\mathbf{u}), \mathbf{h} + \nabla \phi \rangle_{X^*, X} = 0,
\]
and using the last equality in (48), we see that (49) holds also for $\mathbf{h} + \nabla \phi$, hence it holds for $\mathbf{w}$. \hfill \Box
4. Modified de Rham lemmas.

4.1. A modified de Rham lemma in $W^{1,p}_0(\Omega, \mathbb{R}^3)$. The classical de Rham lemma stated in the form of Proposition I.1.1 in [64, p.11] or Lemma II.1.1 in [27, p.144] says that a bounded linear functional vanishing on $H^1_0(\Omega, \text{div} \, 0)$ can be represented by a gradient $\nabla \phi$ with the potential $\phi \in L^2(\Omega)$. Note that no information on trace of $\phi$ on $\partial \Omega$ is given in the lemma. A modified de Rham’s lemma for functionals vanishing on $H^1_0(\Omega, \text{div} \, 0)$ was given in [49], where the trace on $\partial \Omega$ of the potential $\phi$ is identified. As mentioned in [49, p.10] that, the statement of the lemma and its proof given there remain valid for functionals vanishing in $W^{1,p}_0(\Omega, \text{div} \, 0)$ for $1 < p < +\infty$ Here, we give a precise statement and complete proof of them.

**Lemma 4.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $1 < p < +\infty$.

(i) Given $\phi \in \dot{L}^p(\Omega)$ and $\zeta \in W^{-1/p',p'}(\partial\Omega)$, we can define a linear and bounded functional $T_{\phi,\zeta}$ on $W^{1,p}_0(\Omega, \mathbb{R}^3)$, such that for $w \in W^{1,p}_0(\Omega, \mathbb{R}^3)$,

$$
(T_{\phi,\zeta}, w)_{W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)} = -\int_{\Omega} \phi \text{div } w \, dx + \langle \zeta, \nu \cdot w \rangle_{\partial\Omega, 1/p'}.
$$

(ii) The set

$$\pi_{t_0}^{p',-1/p'}(\Omega) = \{ T_{\phi,\zeta} \in W^{1,p}_0(\Omega, \mathbb{R}^3) : \phi \in \dot{L}^p(\Omega), \zeta \in W^{-1/p',p'}(\partial\Omega) \},$$

is homeomorphic to $\dot{L}^p(\Omega) \times W^{-1/p',p'}(\partial\Omega)$.

(iii) For $\phi \in \dot{L}^p(\Omega)$ we denote $T_{\phi,0}$ by $T_\phi$. Then for $w \in W^{1,p}_0(\Omega, \mathbb{R}^3)$,

$$
(T_{\phi}, w)_{W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)} = -\int_{\Omega} \phi \text{div } w \, dx.
$$

$\pi_{t_0}^p(\Omega) = \{ T_\phi : \phi \in \dot{L}^p(\Omega) \}$ is homeomorphic to $\dot{L}^p(\Omega)$.

Proof. We prove (ii). Consider $T_{\phi,\zeta} \in \pi_{t_0}^{p',-1/p'}(\Omega)$. For any $\eta \in W^{1/p',p'}(\partial\Omega)$, take $u_\eta \in W^{1,p}_0(\Omega, \mathbb{R}^3)$ such that

$$\text{div } u_\eta = h_\eta \equiv \frac{1}{|\Gamma|} \int_{\partial\Omega} \eta dS \quad \text{in } \Omega, \quad \nu \cdot u_\eta = \eta \quad \text{on } \partial\Omega,$$

$$\|u_\eta\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)\|\eta\|_{W^{1/p',p}(\partial\Omega)}.$$

Since $\phi \in \dot{L}^p(\Omega)$,

$$\langle \zeta, \eta \rangle_{\partial\Omega, 1/p'} = T_{\phi,\zeta}[u_\eta] \leq C(\Omega, p)\|T_{\phi,\zeta}\|_{W^{1,p}_0(\Omega)} \|\zeta\|_{W^{1/p',p}(\partial\Omega)},$$

hence

$$\|\zeta\|_{W^{-1/p',p}(\partial\Omega)} \leq C(\Omega, p)\|T_{\phi,\zeta}\|_{W^{1,p}_0(\Omega)}.$$

---

5This lemma, which is often called “classical de Rham lemma”, is actually a result being stated in several forms, and proved using various methods by many mathematicians, among them was J. L. Lions [36, p.67-69] whose proof uses de Rham’s theorem on currents in [26, p95, Theorem 17]. See the comments in [64, p.11] and [27, P226]. A constructive proof can be found in [66, 58], [57, Theorem 1.1] and [27, Lemma III.1.1], and an abstractive proof can be found in [13, Theorem IV.2.3].

6We can take $(u_\eta, p_\eta)$ to be the solution of the Stokes problem

$$-\Delta u_\eta + \nabla p_\eta = 0 \quad \text{and} \quad \text{div } u_\eta = h_\eta \quad \text{in } \Omega, \quad u_\eta = \eta \nu \quad \text{on } \partial\Omega.$$

The solution exists and is unique, and

$$\|u_\eta\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)(\|h_\eta\|_{L^p(\Omega)} + \|\eta\|_{W^{1/p',p}(\partial\Omega)}) \leq C(\Omega, p)\|\eta\|_{W^{1/p',p}(\partial\Omega)},$$

see [15, 5], also see [27, Theorem IV.6.1], [13, Theorem IV.6.6].
Set
\[ \psi = |\phi|^{p'/p-1}\phi - c, \quad \text{where} \quad c = \frac{1}{|\Omega|} \int_{\Omega} |\phi|^{p'/p-1}\phi dx. \]

Then \( \psi \in L^p(\Omega) \),
\[ \|\psi\|_{L^p(\Omega)} \leq 2^{p+1} \|\phi\|_{L^{p'}(\Omega)}', \quad \int_{\Omega} \phi \psi dx = \int_{\Omega} |\phi|^{p'} dx. \]

Since \( \int_{\Omega} \psi dx = 0 \), we can find \( \mathbf{v}_\psi \in W_0^{1,p}(\Omega, \mathbb{R}^3) \) such that \(^7\)
\[ \text{div} \mathbf{v}_\psi = \psi \quad \text{in} \ \Omega, \quad \|\mathbf{v}_\psi\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)\|\psi\|_{L^p(\Omega)}. \]

Then
\[ \int_{\Omega} |\phi|^{p'} dx = \int \phi \text{div} \mathbf{v}_\psi dx = -T_{\phi,\zeta}[\mathbf{v}_\psi] \leq C(\Omega, p)\|T_{\phi,\zeta}\|_{W_0^{1,p',*}(\Omega)}\|\psi\|_{L^p(\Omega)}, \]
\[ \leq C(\Omega, p)\|T_{\phi,\zeta}\|_{W_0^{1,p',*}(\Omega)}\left(\|\phi\|_{L^{p'}(\Omega)}\right)^{p'/p}, \]
from which and since \( p' = p' - 1/p = 1 \), we get
\[ \|\phi\|_{L^{p'}(\Omega)} \leq C(\Omega, p)\|T_{\phi,\zeta}\|_{W_0^{1,p',*}(\Omega)}. \]  \( \text{(54)} \)

Combining (i), (53) and (54) we have
\[ \|T_{\phi,\zeta}\|_{W_0^{1,p',*}(\Omega)} \leq \|\phi\|_{L^{p'}(\Omega)} + \|\zeta\|_{W^{-1,p',p'}(\partial\Omega)} \leq C(\Omega, p)\|T_{\phi,\zeta}\|_{W_0^{1,p',*}(\Omega)}. \]

So \( \mathcal{T}^{p',-1/p'}_{t_0}(\Omega) \) is homeomorphic to \( \mathcal{L}^{p'}(\Omega) \times W^{-1/p',p'}(\partial\Omega). \)

For \( \phi \in L^{p'}(\Omega) \) we can also define a functional \( T_{\phi} \) on \( W_0^{1,p}(\Omega, \mathbb{R}^3) \) by
\[ T_{\phi}[\mathbf{w}] = -\int_{\Omega} \phi \text{div} \mathbf{w} dx. \]

In the same manner we define \( T_{\phi,c} \) for \( \phi \in L^{p'}(\Omega) \) and \( c \in \mathbb{R} \) (see (50)). If furthermore \( \phi \) has zero trace on \( \partial\Omega \) then we may write \( T_{\phi} \) as a "gradient", \( T_{\phi} = \nabla \phi \), which is understood as a functional on \( W_0^{1,p}(\Omega, \mathbb{R}^3) \).

If \( \phi \) does not have zero trace, when considered as a functional on \( W_0^{1,p}(\Omega, \mathbb{R}^3) \), one may write \( T_{\phi} \) as gradient \( \nabla \phi \),
\[ \langle \nabla \phi, \mathbf{w} \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = -\int_{\Omega} \phi \text{div} \mathbf{w} dx. \]

However, when considered as a functional on \( W_0^{1,p}(\Omega, \mathbb{R}^3) \), writing \( T_{\phi} \) as gradient \( \nabla \phi \) may cause some confusion.

**Lemma 4.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary and \( 1 < p < +\infty \). Assume that \( T \in W_0^{1,p',*}(\Omega, \mathbb{R}^3) \) and
\[ \langle T, \mathbf{w} \rangle_{W_0^{1,p',*}(\Omega), W_0^{1,p}(\Omega)} = 0, \quad \forall \mathbf{w} \in W_0^{1,p}(\Omega, \text{div} 0). \]  \( \text{(55)} \)

Then there exist \( \phi \in \mathcal{L}^{p'}(\Omega) \) and \( c \in \mathbb{R} \) such that \( T = T_{\phi,c} \in \mathcal{T}^{p',-1/p'}_{t_0}(\Omega) \).

---

\(^7\)Let \( (\mathbf{v}_\psi, q_\psi) \) be the solution of
\[ -\Delta \mathbf{v}_\psi + \nabla q_\psi = 0 \quad \text{and} \quad \text{div} \mathbf{v}_\psi = \psi \quad \text{in} \ \Omega, \quad \mathbf{v}_\psi = 0 \quad \text{on} \ \partial\Omega. \]

Then \( \mathbf{v}_\psi \in W_0^{1,p}(\Omega, \mathbb{R}^3) \), and satisfies the \( W^{1,p} \) estimate. See [15, 5].
Proof. Step 1. We first show that if \( T \in W^{1,p,\ast}_0(\Omega, \mathbb{R}^3) \) is such that (55) holds, then there exist \( \phi \in L^p(\Omega) \) and \( \zeta \in W^{-1/p',p}(\partial\Omega) \) such that \( T = T_{\phi,\zeta} \). The proof is a modification of the proof for the classical de Rham lemma (see for instance [13, p.243]). If \( X \) is a Banach space and \( X^\ast \) is its dual space, for \( A \subset X \) we denote
\[
A^\perp = \{ f \in X^\ast : \langle f, x \rangle_{X^\ast, X} = 0, \ \forall x \in A \}.
\]
Let
\[
X = W^{1,p}_0(\Omega, \mathbb{R}^3), \quad Z = W^{1,p}_0(\Omega, \text{div} 0), \quad \mathfrak{T} = \mathfrak{T}^{p',-1/p'}_{0}((\Omega).
\]
We need prove \( Z^\perp \subseteq \mathfrak{T} \). Since \( X \) is reflexive, we know that
\[
\mathfrak{T}^\perp = \{ w \in W^{1,p}_0(\Omega, \mathbb{R}^3) : \langle T, w \rangle_{W^{1,p,\ast}_0(\Omega), W^{1,p}_0(\Omega)} = 0, \ \forall T \in \mathfrak{T} \},
\]
and we only need to show that \( \mathfrak{T}^\perp \subseteq Z \). Let \( u \in \mathfrak{T}^\perp \). From (55), for any \( \psi \in L^p(\Omega) \) and \( \xi \in W^{-1/p',p'}(\partial\Omega) \) we have
\[
\langle T_{\psi,\xi}, u \rangle_{W^{1,p,\ast}_0(\Omega), W^{1,p}_0(\Omega)} = -\int_\Omega \psi \text{div} u + \langle \xi, \nu \cdot u \rangle_{\partial\Omega, 1/p'} = 0.
\]
Taking \( \psi \in W^{1,p}_0(\Omega) \) and \( \xi = 0 \) in the above equality we see that \( \text{div} u = 0 \) in the weak sense, hence \( u \in W^{1,p}_0(\Omega, \text{div} 0) \). This verifies that \( \mathfrak{T}^\perp \subseteq Z \).

Step 2. Let \( T = T_{\phi,\zeta} \) satisfy (55). We show that \( \zeta \) is equal to a constant number. From (50) and (55), for any \( w \in W^{1,p}_0(\Omega, \text{div} 0) \) we have
\[
0 = \langle T_{\phi,\zeta}, w \rangle_{W^{1,p,\ast}_0(\Omega), W^{1,p}_0(\Omega)} = -\int_\Omega \phi \text{div} w + \langle \zeta, \nu \cdot w \rangle_{\partial\Omega, 1/p'}.
\]
(56)

Denote
\[
\zeta_0 = \frac{1}{|\partial\Omega|} \langle \zeta, 1 \rangle_{\partial\Omega, 1/p'}.
\]
For any \( \eta \in W^{1/p',p}(\partial\Omega) \), we choose \( w_\eta \in W^{1,p}_0(\Omega, \text{div} 0) \) such that \(^8\)
\[
\nu \cdot w_\eta = (\eta - \eta_0) \text{ on } \partial\Omega, \quad \text{where } \eta_0 = \frac{1}{|\partial\Omega|} \langle 1, \eta \rangle_{\partial\Omega, 1/p'}.
\]
Then
\[
\langle \zeta, \eta_0 \rangle_{\partial\Omega, 1/p'} = \eta_0 \langle \zeta, 1 \rangle_{\partial\Omega, 1/p'} = \frac{1}{|\partial\Omega|} \langle 1, \eta \rangle_{\partial\Omega, 1/p'} \langle \zeta, 1 \rangle_{\partial\Omega, 1/p'} = \langle \zeta_0, \eta \rangle_{\partial\Omega, 1/p'},
\]
\[
\langle \zeta - \zeta_0, \eta \rangle_{\partial\Omega, 1/p'} = \langle \zeta, \eta - \eta_0 \rangle_{\partial\Omega, 1/p'} = \langle \zeta, \nu \cdot w_\eta \rangle_{\partial\Omega, 1/p'} = 0.
\]
Here we have used (56). Hence \( \zeta = \zeta_0 \) is a constant. \( \square \)

\(^8\)Let \((w_\eta, q_\eta)\) be the solution of
\[
-\Delta w_\eta + \nabla q_\eta = 0 \text{ and } \text{div } w_\eta = 0 \text{ in } \Omega, \quad w_\phi = (\eta - \eta_0) \nu \text{ on } \partial\Omega.
\]
Then \( w_\eta \in W^{1,p}_0(\Omega, \text{div} 0) \) is needed. Existence and regularity of \( w_\eta \) for \( C^2 \) domains see [15, 5].
Lemma 4.3. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $1 < p < \infty$.

(i) Given $\phi \in \dot{L}^{p'}(\Omega)$ and $z \in TW^{-1/p',p'}(\partial\Omega, \mathbb{R}^3)$, we can define a linear and bounded functional on $W_{n0}^{1,p}(\Omega, \mathbb{R}^3)$, which is denoted by $T_{\phi,z}$, such that for $w \in W_{n0}^{1,p}(\Omega, \mathbb{R}^3)$,

$$
\langle T_{\phi,z}, w \rangle_{W_{n0}^{1,p'}(\Omega), W_{n0}^{1,p}(\Omega)} = -\int_{\Omega} \phi \div w \, dx + \langle z, w_{\partial\Omega} \rangle_{\partial\Omega, 1/p'}.
$$

(ii) The set

$$
T_{n0}^{p',-1/p'}(\Omega) = \{ T_{\phi,z} \in W_{n0}^{1,p'}(\Omega, \mathbb{R}^3) : \phi \in \dot{L}^{p'}(\Omega), z \in TW^{-1/p',p'}(\partial\Omega, \mathbb{R}^3) \}
$$

is homeomorphic to $\dot{L}^{p'}(\Omega) \times TW^{-1/p',p'}(\partial\Omega, \mathbb{R}^3)$.

(iii) For $\phi \in \dot{L}^{p'}(\Omega)$ we denote $T_{\phi,0}$ by $T_{\phi}$. Then for $w \in W_{n0}^{1,p}(\Omega, \mathbb{R}^3)$,

$$
\langle T_{\phi}, w \rangle_{W_{n0}^{1,p'}(\Omega), W_{n0}^{1,p}(\Omega)} = -\int_{\Omega} \phi \div w \, dx.
$$

4.2. A modified de Rham lemma in $W_{n0}^{1,p}(\Omega, \mathbb{R}^3)$.

Lemma 4.4. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $1 < p < \infty$.

Assume that $T \in W_{n0}^{1,p,\ast}(\Omega, \mathbb{R}^3)$ and

$$
\langle T, w \rangle_{W_{n0}^{1,p'}(\Omega), W_{n0}^{1,p}(\Omega)} = 0, \quad \forall w \in W_{n0}^{1,p}(\Omega, \div 0).
$$

Then there exists $\phi \in \dot{L}^{p'}(\Omega)$ such that $T = T_{\phi} \in T_{n0}^{p'}(\Omega)$.

\[9\text{We can take } (u_y, p_y) \text{ to be the solution of the Stokes problem}
\]

$$
-\Delta u_y + \nabla p_y = 0 \quad \text{and} \quad \div u_y = 0 \quad \text{in } \Omega, \quad u_y = y \quad \text{on } \partial\Omega.
$$

The solution exists and is unique, and

$$
\|u_y\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)\|y\|_{W^{1/p',p}(\partial\Omega)}.
$$

see [15], [5], also see [27, Theorem IV.6.1], [13, Theorem IV.6.6].
Proof. The proof is similar to the proof of Lemma 4.2 but simpler. Let

\[ X = W^{1,p}(\Omega, \mathbb{R}^3), \quad Z = W^{1,p}(\Omega, \text{div} 0), \quad \mathcal{F} = \mathcal{F}_0(\Omega). \]

Let \( u \in \mathcal{F}^\perp \). Then for any \( \phi \in W_0^{1,p}(\Omega) \subset \mathcal{F} \) we have

\[ 0 = \langle T_\phi, u \rangle_{W_0^{1,p}(\Omega), W_0^{1,p}(\Omega)} = -\int_\Omega \phi \text{div} u \, dx = \int_\Omega \nabla \phi \cdot u \, dx. \]

It implies that \( \text{div} u = 0 \), hence \( u \in W_0^{1,p}(\Omega, \text{div} 0) = Z \). Thus \( \mathcal{F}^\perp \subset Z \), hence \( Z^\perp \subset \mathcal{F} \).

4.3. **Representation of a curl-functional vanishing in** \( W^{1,p}_0(\Omega, \text{div} 0) \). We use \( M(3), S^0(3), S^+(3) \) to denote the set of all \( 3 \times 3 \) matrices, non-negative symmetric matrices, and positive definite matrices, respectively. The following condition for a matrix-valued function \( A(x) \) is used frequently:

\[ (A_0) \quad A \in L^\infty(\Omega, S^+(3)) \text{ and there exist positive constants } \lambda(A) < \Lambda(A) \text{ such that } \]

\[ \lambda(A)\|\xi\|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda(A)\|\xi\|^2, \quad \forall x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3. \]

Let \( 1 < p < \infty \). For any \( w \in L^p(\Omega, \mathbb{R}^3) \), we can define \( T_{\text{curl} w} \in W^{1,p}_0(\Omega, \mathbb{R}^3) \) by

\[ (T_{\text{curl} w}, v)_{W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)} = \int_\Omega w \cdot \text{curl} v \, dx, \quad \forall v \in W^{1,p}_0(\Omega, \mathbb{R}^3). \]

For simplicity of notation we shall write \( T_{\text{curl} w} \) by \( \text{curl} w \), so we have

\[ (\text{curl} w, v)_{W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)} = \int_\Omega w \cdot \text{curl} v \, dx, \quad \forall v \in W^{1,p}_0(\Omega, \mathbb{R}^3). \]

As an example assume \( A \) satisfies \( A_0 \), \( u \in W^{p'}_0(\Omega, \mathbb{R}^3) \) and \( g \in L^p(\Omega, \mathbb{R}^3) \). Then \( w = A \text{curl} u - g \in L^p(\Omega, \mathbb{R}^3) \), and we have

\[ (\text{curl} (A \text{curl} u - g), v)_{W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)} = \int_\Omega (A \text{curl} u - g) \cdot \text{curl} v \, dx, \quad \forall v \in W^{1,p}_0(\Omega, \mathbb{R}^3). \]

In particular we have

\[ (\text{curl} (A \text{curl} u - g), \nabla \phi)_{W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)} = 0, \quad \forall \phi \in W^{1,p}_0(\Omega). \]

In the following we assume that

\[ 1 < p < \infty, \quad A \text{curl} u, g \in L^p(\Omega, \mathbb{R}^3), \quad U \in L^r(\Omega, \mathbb{R}^3) \text{ for some } r \geq (p^*)'. \]

**Lemma 4.5.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a Lipschitz boundary, \( p, r, \) \( A \text{curl} u, g, U \) satisfy \( (61) \), such that

\[ \int_\Omega \{ (A \text{curl} u + g, \text{curl} z) + (U, z) \} \, dx = 0 \quad \forall z \in W^{1,p}_0(\Omega, \text{div} 0). \]

Then there exists \( \phi = \phi_0 + h \in L^p(\Omega) \), where \( \phi_0 \in W^{1,r}_0(\Omega) \) is the unique solution of

\[ \Delta \phi_0^0 = \text{div} U \quad \text{in } \Omega, \quad \phi_0^0 = 0 \quad \text{on } \partial \Omega, \]

and \( h \in L^p(\Omega, \Delta 0) \), such that

\[ \text{curl} [A(x)\text{curl} u + g] + U = \nabla \phi, \]

(64)
which holds in the sense of $W^{-1,p'}(\Omega, \mathbb{R}^3)$, namely, for all $z \in W^{1,p}_0(\Omega, \mathbb{R}^3)$,
\[
\int_\Omega \{ \langle A \text{curl} \ u + g, \text{curl} \ z \rangle + \langle U, z \rangle \} \, dx = \int_\Omega \nabla \phi^0_U \cdot z \, dx - \int_\Omega h \text{div} \ z \, dx,
\]
\[
\forall z \in W^{1,p}_0(\Omega, \mathbb{R}^3).
\]

The function $\phi$ is unique up to an additive constant.

Proof. Since $U \in L^r(\Omega, \mathbb{R}^3)$ for some $r \geq p'$, the integral $\int_\Omega U \cdot z \, dx$ is a continuous functional on $W^{1,p}_0(\Omega, \mathbb{R}^3)$. Denote
\[
f = \text{curl} \{ A(x) \text{curl} \ u + g \} + U \in W^{-1,p'}(\Omega, \mathbb{R}^3).
\]

Then
\[
\langle f, z \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)} = \int_\Omega \{ \langle A \text{curl} \ u + g, \text{curl} \ z \rangle + \langle U, z \rangle \} \, dx,
\]
\[
\forall z \in W^{1,p}_0(\Omega, \mathbb{R}^3).
\]

$\nabla \phi^0_U \in L^r(\Omega, \mathbb{R}^3) \subseteq L^{p'}(\Omega, \mathbb{R}^3)$ defines a functional on $W^{1,p}_0(\Omega, \mathbb{R}^3)$ by
\[
\langle \nabla \phi^0_U, w \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)} = \int_\Omega \nabla \phi^0_U \cdot w \, dx = -\int_\Omega \phi^0_U \text{div} \ w \, dx,
\]
\[
\forall w \in W^{1,p}_0(\Omega, \mathbb{R}^3),
\]
and it vanishes on $W^{1,p}_0(\Omega, \text{div} 0)$. From this and (62) we see that $f - \nabla \phi^0_U$ vanishes on $W^{1,p}_0(\Omega, \text{div} 0)$. Applying the classical de Rham Lemma (see for instance [13, Theorem IV.2.4]) to $f - \nabla \phi^0_U$ we get a function $h \in \dot{L}^{p'}(\Omega)$ such that
\[
f - \nabla \phi^0_U = \nabla h,
\]

which holds as an equality for elements in $W^{-1,p'}(\Omega, \mathbb{R}^3)$, and this $h$ is unique up to an additive constant. Using (59) we have, for any $w \in W^{1,p}_0(\Omega, \mathbb{R}^3)$,
\[
\langle \nabla h, w \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)} = \langle f - \nabla \phi^0_U, w \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)}
\]
\[
= \int_\Omega \{ \langle A \text{curl} \ u - g, \text{curl} \ w \rangle + \langle U - \nabla \phi^0_U, w \rangle \} \, dx.
\]

In particular, using (60), for any $\psi \in C^2_c(\Omega)$ we have
\[
\langle \nabla h, \nabla \psi \rangle_{W^{-1,p'}(\Omega), W^{1,p}_0(\Omega)}
\]
\[
= \int_\Omega \{ \langle A \text{curl} \ u + g, \text{curl} \nabla \psi \rangle + \langle U - \nabla \phi^0_U, \nabla \psi \rangle \} \, dx = 0.
\]

Hence $\Delta h = 0$ in the sense of distribution, so $h \in \dot{L}^{p'}(\Omega, \Delta 0)$. Since $r_* \geq p'$, so
\[
\phi^0_U \in W^{1,r_*}_0(\Omega) \subset L^{r_*}(\Omega) \subset L^{p'}(\Omega),
\]

hence $\phi \in L^{p'}(\Omega)$. \qed

4.4. Representation of a curl-functional vanishing in $W^{1,p}_0(\Omega, \text{div} 0)$.

Lemma 4.6. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, $p, r, A \text{curl} \ u, g, U$ satisfy (61), such that
\[
\int_\Omega \{ \langle A \text{curl} \ u + g, \text{curl} \ z \rangle + \langle U, z \rangle \} \, dx = 0 \quad \forall z \in W^{1,p}_0(\Omega, \text{div} 0).
\]

(65)
Let \( \phi^0_U \in W^{1,r}_0(\Omega) \) be the unique weak solution of (63). Then for all \( z \in W^{1,p}_{t_0}(\Omega, \mathbb{R}^3) \),
\[
\int_{\Omega} \{ \langle A \cdot \nabla u + g, \nabla z \rangle + \langle U, z \rangle \} \, dx = \int_{\Omega} \nabla \phi^0_U \cdot z \, dx, \quad \forall z \in W^{1,p}_{t_0}(\Omega, \mathbb{R}^3).
\] (66)

**Proof.** Denote
\[
f = \text{curl} [A(x) \nabla u + g] + U \in W^{1,p,*}_{t_0}(\Omega, \mathbb{R}^3).
\]
Then
\[
\langle f, w \rangle_{W^{1,p,*}_{t_0}(\Omega), W^{1,p}_{t_0}(\Omega)} = \int_{\Omega} \{ \langle A \cdot \nabla u + g, \nabla w \rangle + \langle U, w \rangle \} \, dx,
\]
\[
\forall w \in W^{1,p}_{t_0}(\Omega, \mathbb{R}^3).
\]

From Remark 1 \( \phi^0_U \in L^{r^*}(\Omega) \subseteq L^{p^*}(\Omega) \). Since
\[
\nabla \phi^0_U \in L^r(\Omega, \mathbb{R}^3) \subseteq L^{p^*}(\Omega, \mathbb{R}^3),
\]
and \( \phi^0_U \) has trace zero on \( \partial \Omega \), \( \nabla \phi^0_U \) defines a continuous functional on \( W^{1,p}_{t_0}(\Omega, \mathbb{R}^3) \) by
\[
\langle \nabla \phi^0_U, w \rangle_{W^{1,p,*}_{t_0}(\Omega), W^{1,p}_{t_0}(\Omega)} = - \int_{\Omega} \phi^0_U \text{div} \, w \, dx,
\]
and it vanishes on \( W^{1,p}_{t_0}(\Omega, \text{div} 0) \). From this and (65) we see that
\[
\langle f - \nabla \phi^0_U, w \rangle_{W^{1,p,*}_{t_0}(\Omega), W^{1,p}_{t_0}(\Omega)} = 0, \quad \forall w \in W^{1,p}_{t_0}(\Omega, \text{div} 0).
\]

We apply Lemma 4.2 (also see [45, Lemma 2.2]) and conclude that, there exists a function \( \psi \in \hat{L}^{p^*}(\Omega) \) and a constant \( c \), such that
\[
\langle f - \nabla \phi^0_U, w \rangle_{W^{1,p,*}_{t_0}(\Omega), W^{1,p}_{t_0}(\Omega)} = \langle T\psi, w \rangle_{W^{1,p,*}_{t_0}(\Omega), W^{1,p}_{t_0}(\Omega)}
\]
\[
= - \int_{\Omega} \psi \text{div} \, w \, dx + c \int_{\partial \Omega} \nu \cdot \text{curl} \, w \, dS = - \int_{\Omega} (\psi - c) \text{div} \, w \, dx, \quad \forall w \in W^{1,p}_{t_0}(\Omega, \mathbb{R}^3).
\] (67)

Let \( \xi \in W^{2,p}_{t_0}(\Omega) \) satisfy \( \Delta \xi = \psi - c \) in \( \Omega \) and \( \xi = 0 \) on \( \partial \Omega \). Then \( \nabla \xi \in W^{1,p}_{t_0}(\Omega, \mathbb{R}^3) \). Taking \( w = \nabla \xi \) in (67) and using (66) we get
\[
\int_{\Omega} (\nabla \phi^0_U) \cdot \nabla \xi \, dx = - \int_{\Omega} (\psi - c)^2 \, dx.
\]

Since \( \text{div} \, (U - \nabla \phi^0_U) = 0 \) in \( \Omega \) and \( \xi = 0 \) on \( \partial \Omega \), we have
\[
\int_{\Omega} (U - \nabla \phi^0_U) \cdot \nabla \xi \, dx = 0,
\]
so \( \psi - c \equiv 0 \). Since \( \psi \in \hat{L}^{p^*}(\Omega) \), we have \( c = 0 \), so \( \psi = 0 \), \( f = \nabla \phi^0_U \), which holds as elements in \( W^{1,p,*}_{t_0}(\Omega, \mathbb{R}^3) \). So (66) holds.

**Remark 4.** Under the conditions of Lemma 4.6, the classical de Rham lemma gives a function \( \phi \in L^{p^*}(\Omega) \) such that \( f = \nabla \phi \) holds in \( W^{-1,p'}(\Omega, \mathbb{R}^3) \), but it does not give information whether \( \phi \) has trace on \( \partial \Omega \) and whether the trace is constant or zero.

On the other hand, by using [45, Lemma 2.2] (or Lemma 4.2) we get \( \phi \in L^{p^*}(\Omega) \) and \( c \in \mathbb{R} \) such that the equality \( f = T_{\phi,c} \) holds in \( W^{1,p,*}_{t_0}(\Omega, \mathbb{R}^3) \), and in Lemma 4.6 we obtain more information of \( \phi \) by showing that \( \phi \in W^{1,p'}_{t_0}(\Omega) \), hence \( f = T_{\phi} = \nabla \phi \).
Corollary 1. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, $1 < p < \infty$, and $\mathcal{A}$ satisfies $(A_0)$. Let $u \in W^{1,p}_0(\Omega, \text{div} 0)$ be such that
\[
\int_{\Omega} \langle \mathcal{A} \text{curl} u, \text{curl} v \rangle dx = 0, \quad \forall v \in W^{1,p'}_0(\Omega, \text{div} 0). \tag{68}
\]
Then $u \in \mathbb{H}_2(\Omega)$. In other words, the only $W^{1,p}$-weak solutions of the following
\[
\begin{cases}
\text{curl} (\mathcal{A} \text{curl} u) = 0, & \text{in } \Omega, \\
\text{div } u = 0 & \text{on } \partial \Omega
\end{cases}
\tag{69}
\]
are all $h_2 \in \mathbb{H}^p_2(\Omega) = \mathbb{H}_2(\Omega)$.

Proof. When $p \geq 2$, we take $v = u$ in (68) to get $\text{curl} u = 0$, so $u \in \mathbb{H}_2(\Omega)$. Now assume $1 < p < 2$. From (68) and using Lemma 4.6 with $g = U = 0$, we see that (68) holds for all $v \in W^{1,p'}_0(\Omega, \mathbb{R}^3)$, so $u$ is a $W^{1,p}$-weak solution of (69). Then $\mathcal{A} \text{curl} u \in L^p(\Omega, \text{curl} 0)$. By Lemma 2.4, there exist $\phi \in W^{1,p}(\Omega)$ and $h_1 \in \mathbb{H}^p_2(\Omega)$ such that
\[
\mathcal{A} \text{curl} u = \nabla \phi + h_1.
\]
So $\phi$ is a weak solution of the following co-normal problem
\[
\text{div } (\mathcal{A}^{-1} \nabla \phi) = -\text{div } f \quad \text{in } \Omega, \quad \nu \cdot (\mathcal{A}^{-1} \nabla \phi) = -\nu \cdot f \quad \text{on } \partial \Omega,
\]
where $f = \mathcal{A}^{-1} h_1 \in L^2(\Omega, \mathbb{R}^3)$. Since this problem has a unique solution in $\dot{H}^1(\Omega)$, so $\phi \in \dot{H}^1(\Omega)$, $\text{curl } u = \mathcal{A}^{-1} (\nabla \phi + h_1) \in L^2(\Omega, \mathbb{R}^3)$, and we can take $v = u$ in (68) to get $\text{curl } u = 0$. Thus $u \in \mathbb{H}_2(\Omega)$.

4.5. Representation of a curl-functional vanishing on $W^{1,p}_0(\Omega, \text{div} 0)$. We need the following facts. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $1 < p < \infty$.

(i) If $\phi \in L^{p'}(\Omega, \Delta)$, then $\phi$ has trace $\gamma(\phi) \in W^{-1/p', p'}(\partial \Omega)$ (see Remark 2). Therefore it induces a functional on $W^{1,p}(\Omega, \mathbb{R}^3)$, which is denoted by $\nabla \phi$, such that
\[
\langle \nabla \phi, w \rangle_{W^{1,p}, \cdot, W^{1,p}(\Omega)} = -\int_{\Omega} \phi \text{div } wd\xi + \langle \gamma(\phi), \nu \cdot w \rangle_{\partial \Omega, 1/p'}, \quad w \in W^{1,p}(\Omega, \mathbb{R}^3),
\tag{70}
\]
\[
\forall w \in W^{1,p}(\Omega, \mathbb{R}^3).
\]
(ii) For any $U \in L^r(\Omega, \mathbb{R}^3)$, there exists a unique $\phi_U \in \dot{W}^{1,r}(\Omega)$ such that
\[
\int_{\Omega} (\nabla \phi_U - U) \cdot \nabla \xi dx = 0, \quad \forall \xi \in W^{1,s}(\Omega),
\tag{71}
\]
where $1/r + 1/s = 1$, see [57, Theorem 5.4]. So $\phi_U$ is a weak solution of
\[
\Delta \phi_U = \text{div } U \quad \text{in } \Omega, \quad \frac{\partial \phi_U}{\partial \nu} = \nu \cdot U \quad \text{on } \partial \Omega.
\]

Lemma 4.7. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, $1 < p < \infty$, $\mathcal{A} \text{curl } u, g, U$ satisfy (61) with $\nu \times g \in TW^{-1/p, p'}(\partial \Omega, \mathbb{R}^3)$, $W_T \in TW^{-1/p, p'}(\partial \Omega, \mathbb{R}^3)$, and the following equality holds for all $z \in W^{1,p}_0(\Omega, \text{div} 0)$:
\[
\int_{\Omega} \{ \langle \mathcal{A} \text{curl } u + g, \text{curl } z \rangle + \langle U, z \rangle \} dx + \langle W_T, z \rangle_{\partial \Omega, 1/p'} = 0. \tag{72}
\]
Then there exists a unique \( \phi \in \dot{L}^{p'}(\Omega) \) such that, for all \( w \in W^{1,p}(\Omega, \mathbb{R}^3) \),
\[
\int_{\Omega} \left\{ \langle A \text{curl} \ u + g, \text{curl} \ w \rangle + \langle U, w \rangle \right\} dx + \langle W_T, w \rangle_{\partial\Omega,1/p'} = (\nabla \phi, w)_{W^{1,p}(\Omega),W^{1,p}(\Omega)}. \tag{73}
\]
Moreover, \( \phi = \phi_U + h \), where \( \phi_U \in \dot{L}^{r^*}(\Omega) \) satisfies (71), and \( h \in \dot{L}^{p'}(\Omega, \Delta 0) \).

**Proof.**

**Step 1.** From Remark 1, \( \phi_U \in L^{r^*}(\Omega) \subseteq L^{p'}(\Omega) \), \( \nabla \phi_U \in L^{r}(\Omega, \mathbb{R}^3) \subseteq L^{p'}(\Omega, \mathbb{R}^3) \),
\[
\int_{\Omega} \nabla \phi_U : \xi dx = 0, \quad \forall \xi \in W^{1,p}_{0}\Omega(\Omega, \text{div} 0).
\]

We define a functional \( T \) on \( W^{1,p}_{0}\Omega(\Omega, \mathbb{R}^3) \) by
\[
(T, v)_{W^{1,p}_{0}\Omega(\Omega, \mathbb{R}^3)} = \int_{\Omega} \left\{ \langle A \text{curl} \ u + g, \text{curl} \ v \rangle + \langle U - \nabla \phi_U, v \rangle \right\} dx + \langle W_T, v \rangle_{\partial\Omega,1/p'}.
\]

From (73) \( T \) vanishes on \( W^{1,p}_{0}\Omega(\Omega, \text{div} 0) \). By Lemma 4.4, there exists \( h \in \dot{L}^{p'}(\Omega) \) such that \( T = T_h \), so for all \( v \in W^{1,p}_{0}\Omega(\Omega, \mathbb{R}^3) \) we have
\[
\int_{\Omega} \left\{ \langle A \text{curl} \ u + g, \text{curl} \ v \rangle + \langle U - \nabla \phi_U, v \rangle + h \text{div} \ v \right\} dx + \langle W_T, v \rangle_{\partial\Omega,1/p'} = 0. \tag{74}
\]

Set \( \phi = \phi_U + h \). Since \( r^* \geq p' \), \( \phi_U \in L^{p'}(\Omega) \), hence \( \phi \in L^{p'}(\Omega) \), it induces a functional \( T_\phi \) on \( W^{1,p}_{0}\Omega(\Omega, \mathbb{R}^3) \), and we can write (74) as
\[
\int_{\Omega} \left\{ \langle A \text{curl} \ u + g, \text{curl} \ v \rangle + \langle U, v \rangle \right\} dx + \langle W_T, v \rangle_{\partial\Omega,1/p'} = (T_\phi, v)_{W^{1,p}_{0}\Omega(\Omega, \mathbb{R}^3)}.
\]

**Step 2.** For any \( w \in W^{1,p}(\Omega, \mathbb{R}^3) \), let \( \psi_w \in W^{2,p}(\Omega) \) be such that
\[
\psi_w = 0, \quad \frac{\partial \psi_w}{\partial \nu} = \nu \cdot w \quad \text{on } \partial\Omega. \tag{75}
\]

Taking \( v = w - \nabla \psi_w \in W^{1,p}_{0}\Omega(\Omega, \mathbb{R}^3) \) as a test field in (74), and using the facts
\[
\int_{\Omega} (U - \nabla \phi_U) \cdot \nabla \psi_w dx = 0, \quad (\nabla \psi_w)_{|\partial\Omega} = 0 \quad \text{on } \partial\Omega,
\]
we get
\[
\int_{\Omega} \left\{ \langle A \text{curl} \ u + g, \text{curl} \ w \rangle + \langle U - \nabla \phi_U, w \rangle + h (\text{div} w - \Delta \psi_w) \right\} dx + \langle W_T, w \rangle_{\partial\Omega,1/p'} = 0, \quad \forall w \in W^{1,p}(\Omega, \mathbb{R}^3). \tag{76}
\]

Although \( \psi_w \) is not uniquely determined by \( w \), for any two functions \( \psi_w^{(1)}, \psi_w^{(2)} \in W^{2,p}(\Omega) \) satisfying condition (75) for the same \( w \), we have \( \psi_w^{(1)} = \psi_w^{(2)} \in W^{2,p}_{0}\Omega(\Omega) \), hence
\[
\int_{\Omega} h \Delta \psi_w^{(1)} dx = \int_{\Omega} h \Delta \psi_w^{(2)} dx.
\]

So the integral \( \int_{\Omega} h \Delta \psi_w dx \) depends only on \( \nu \cdot w \).

For any \( \xi \in W^{1,p}_{0}\Omega(\Omega), \eta \in W^{2,p}_{0}\Omega(\Omega) \), taking \( w = \nabla \xi \) and \( \psi_w = \xi + \eta \) in (76) we get
\[
\int_{\Omega} h \Delta \eta dx = 0, \quad \forall \eta \in W^{2,p}_{0}\Omega(\Omega).
\]
Hence $h \in L^p(\Omega, \Delta 0)$, so $h$ has trace $\gamma(h) \in W^{-1/p', p'}(\partial \Omega)$. Therefore it induces a functional $\nabla h$ on $W^{-1/p}(\Omega, \mathbb{R}^3)$ given by (70).

**Step 3.** Now we show that for any $\zeta \in \dot{L}^p(\Omega, \Delta 0)$, $w \in W^{1,p}(\Omega, \mathbb{R}^3)$, it holds that

$$\int_{\Omega} \zeta (\text{div } w - \Delta \psi_w) dx = -\langle \nabla \zeta, w \rangle_{W^{1,p}(\Omega), W^{1,p}(\Omega)}. \quad (77)$$

To prove, let us first assume $\zeta \in \dot{W}^{1,p'}(\Omega, \Delta 0)$. Then since $\Delta \zeta = 0$ and $\psi_w$ satisfies (75), we have

$$\int_{\Omega} \zeta \Delta \psi_w dx = \int_{\partial \Omega} \zeta \frac{\partial \psi_w}{\partial \nu} dS = \int_{\partial \Omega} \zeta \nu \cdot w dS,$$

$$\int_{\Omega} \zeta (\text{div } w - \Delta \psi_w) dx = -\int_{\Omega} \nabla \zeta \cdot w dx = -\langle \nabla \zeta, w \rangle_{W^{1,p}(\Omega), W^{1,p}(\Omega)}.$$

So in this case (77) is true.

For a general $\zeta \in L^p(\Omega, \Delta 0)$, we take $\zeta_n \in W^{1,p}(\Omega, \Delta 0)$ such that $\zeta_n \to \zeta$ in $L^p(\Omega)$. Then for any $w \in W^{1,p}(\Omega, \mathbb{R}^3)$ we have

$$\int_{\Omega} \zeta_n (\text{div } w - \Delta \psi_w) dx \to \int_{\Omega} \zeta (\text{div } w - \Delta \psi_w) dx.$$

By the trace theorem in $L^p(\Omega, \Delta)$ (see Remark 2) $\gamma(\zeta_n) \to \gamma(\zeta)$ in $W^{-1/p', p'}(\partial \Omega)$. From (70) $\nabla \zeta_n \to \nabla \zeta$ in $W^{1,p}(\Omega)$. Hence for any $w \in W^{1,p}(\Omega, \mathbb{R}^3)$ we have

$$\langle \nabla \zeta_n, w \rangle_{W^{1,p}(\Omega), W^{1,p}(\Omega)} \to \langle \nabla \zeta, w \rangle_{W^{1,p}(\Omega), W^{1,p}(\Omega)}.$$

Now we apply (77) to $\zeta_n$ and then let $n \to \infty$ to get (77) for $\zeta$.

**Step 4.** Using (77) we can write (76) as follows: for all $w \in W^{1,p}(\Omega, \mathbb{R}^3)$,

$$\int_{\Omega} \{ \langle A \text{curl } u + g, \text{curl } w \rangle + \langle U, w \rangle \} dx + \langle W_T, w \rangle_{\partial \Omega, 1/p'} = \langle \nabla \phi, w \rangle_{W^{1,p}(\Omega), W^{1,p}(\Omega)},$$

where $\phi = \phi_U + h$. So we get (73) for all $w \in W^{1,p}(\Omega, \mathbb{R}^3)$.

The above lemma (when $p = 2$) makes it possible to solve the following BVP

$$\begin{cases}
\text{curl } (A \text{curl } u + g) + U = \nabla \phi & \text{in } \Omega, \\
\nu \times (A \text{curl } u + g) = W_T & \text{on } \partial \Omega,
\end{cases} \quad (78)$$

by minimizing the associated energy functional on $H^1_{\text{div}}(\Omega, \text{div } 0)$.

**Remark 5.** In Lemma 4.7 the function $\phi$ is unique up to an additive constant. It suggests that in (78) we should not pose any extra boundary condition on $\phi$.

5. **Quasilinear Maxwell-Stokes system.** In this section we study existence of solutions to the following

$$\begin{cases}
\text{curl} [\nabla_x P(x, \text{curl } u)] + c(x)|u|^{p-2} u = \nabla_u F(x, u) + \nabla \phi & \text{in } \Omega, \\
\text{div } u = 0 & \text{in } \Omega, \\
u_T = u_T^0, \quad \phi = 0 & \text{on } \partial \Omega,
\end{cases} \quad (79)$$
where \(1 < p < +\infty\). An important example of (79) is the quasilinear magneto-static Maxwell equations
\[
\begin{align*}
\text{curl}[\mathbf{H}(x, \text{curl} \mathbf{u})] &= \mathbf{f}(x, \mathbf{u}), \quad \text{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\
\mathbf{u}_T &= \mathbf{u}_T^0, \quad \text{on } \partial \Omega,
\end{align*}
\] (80)
and the corresponding quasilinear Maxwell-Stokes system ([49])
\[
\begin{align*}
\text{curl}[\mathbf{H}(x, \text{curl} \mathbf{u})] &= \mathbf{f}(x, \mathbf{u}) + \nabla \phi, \quad \text{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\
\mathbf{u}_T &= \mathbf{u}_T^0, \quad \phi = 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (81)
Other examples include the model of Meissner states of superconductivity [17, 41, 10, 46, 47, 35], nonlinear Maxwell equations ([71, 8, 30, 48]), the extended magneto-static Born-Infeld model [19, 20].

**Example.** Consider equations (81) with
\[
\mathbf{H}(\mathbf{z}) = h(|\mathbf{z}|^q) \mathbf{z},
\]
where \(h(t)\) is a continuous positive function on \((0, +\infty)\). We can write \(\mathbf{H}(\mathbf{z}) = \nabla P(\mathbf{z})\), where
\[
G(t) = - \int_t^c \frac{h(s)}{|s|^{1-2/q}} ds, \quad P(\mathbf{z}) = \frac{1}{q} G(|\mathbf{z}|^q).
\]
This example includes the \(p\)-curl \(\text{curl} \) system
\[
\text{curl} ([\text{curl}]^{p-2} \text{curl} \mathbf{u}) = \mathbf{f}(x, \mathbf{u}),
\]
with \(q = p - 2\) and \(h(s) = s\).

**Definition 5.1.** We say that \((\mathbf{u}, \phi)\) is a weak solution of (79) if \(\mathbf{u} \in \mathcal{W}^{1,p}_t(\Omega, \text{div} 0, \mathbf{u}_T^0), \phi \in \mathcal{L}^p(\Omega)\) with trace zero on \(\partial \Omega\), such that
\[
\int_\Omega \left( (\nabla_x P(\mathbf{z}) \text{curl} \mathbf{u}, \text{curl} \mathbf{w}) + c(x) |\mathbf{u}|^{p-2} \mathbf{u} \cdot \mathbf{w} - (\nabla_x F(x, \mathbf{u}), \mathbf{w}) \right) dx = \langle T_0, \mathbf{w} \rangle_{\mathcal{W}_h^{1,p}((\Omega)), \mathcal{W}_h^{1,p}((\Omega))}, \quad \forall \mathbf{w} \in \mathcal{W}_h^{1,p}(\Omega, \mathbb{R}^3),
\]
(82)
where \(T_0\) is the functional defined in (52).

We shall also consider Maxwell-Stokes problem with the natural boundary condition
\[
\begin{align*}
\text{curl}[\nabla_x P(\mathbf{z}) \text{curl} \mathbf{u}] + c(x) |\mathbf{u}|^{p-2} \mathbf{u} &= \nabla_x F(x, \mathbf{u}) + \nabla \phi \quad \text{in } \Omega, \\
\text{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\nu \times [\nabla_x P(\mathbf{z}) \text{curl} \mathbf{u}] &= \mathbf{W}_T, \quad \nu \cdot \mathbf{u} = 0 \quad \text{on } \partial \Omega,
\end{align*}
\] (83)
where \(1 < p < +\infty\). In (83) no boundary condition on \(\phi\) is prescribed, see Remark 5.

**Definition 5.2.** We say \((\mathbf{u}, \phi)\) is a weak solution of (83) if \(\mathbf{u} \in \mathcal{W}^{1,p}_{n_0}((\Omega, \text{div} 0), \phi \in \mathcal{L}^p(\Omega)\) and
\[
\int_\Omega \left( (\nabla_x P(\mathbf{z}) \text{curl} \mathbf{u}, \text{curl} \mathbf{w}) + c(x) |\mathbf{u}|^{p-2} \mathbf{u} \cdot \mathbf{w} - (\nabla_x F(x, \mathbf{u}), \mathbf{w}) \right) dx + \langle \mathbf{W}_T, \mathbf{w} \rangle_{\partial \Omega, 1/p'} = \langle T_0, \mathbf{w} \rangle_{\mathcal{W}^{1,p}_{n_0}((\Omega), \mathcal{W}^{1,p}_{n_0}((\Omega))}, \quad \forall \mathbf{w} \in \mathcal{W}^{1,p}_{n_0}(\Omega, \mathbb{R}^3),
\]
(84)
where \(T_0\) is the functional defined in (57).
Definitions 5.1 and 5.2 do not require \( \phi \) to lie in some Sobolev space, hence in (82) we prefer writing \( T_\phi \) instead of \( \nabla \phi \), see (32).

5.1. *Existence by minimization.* To avoid technical complication, we assume the following conditions:

\( P \in C^{0}_{loc}(\bar{\Omega} \times \mathbb{R}^3), P(x,z) \) is differentiable in \( z \) and \( \nabla_z P(x,z) \in C^{0}_{loc}(\bar{\Omega} \times \mathbb{R}^3, \mathbb{R}^3) \), \( P(x,z) \) is strictly convex in \( z \), and there exist positive constants \( C_1, C_2, C_3 \) such that

\[
C_1(|z|^p - 1) \leq P(x,z) \leq C_2(|z|^p + 1),
\]

\[
|\nabla_z P(x,z)| \leq C_2(|z|^{p-1} + 1).
\]

\( F \in C^{0}_{loc}(\bar{\Omega} \times \mathbb{R}^3), F(x,z) \) is differentiable in \( z \) and

\[
\nabla_u F(x,u) \in C^{0}_{loc}(\bar{\Omega} \times \mathbb{R}, \mathbb{R}^3),
\]

and there exist \( 1 < p_1 < p_* \) and a positive constant \( C_3 \) such that

\[
|\nabla_u F(x,u)| \leq C_3(|u|^{p_1-1} + 1),
\]

\[
\lim_{|u| \to \infty} \frac{|F(x,u)|}{|u|^p} = 0 \quad \text{uniformly for } x \in \Omega.
\]

\[
(c) \quad c \in C(\bar{\Omega}), c(x) \geq 0 \text{ on } \Omega, \text{ and if } \Omega \text{ has holes then assume } \min_{\bar{\Omega}} c(x) > 0. \quad (85)
\]

(\( U_0 \)) \( u_j^0 \in TW^{1,p-1,p}(\partial\Omega, \mathbb{R}^3) \).

**Theorem 5.3.** Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary, \( 1 < p < \infty \), \( \Omega, F, c \) and \( u_j^0 \) satisfy (\( P^p \)), (\( F^p \)), (c) and (\( U_0 \)) respectively. Then (79) has a weak solution

\[
(u, \phi) \in W^{1,p}_0(\Omega, \text{div } 0, u_j^0) \times W^{1,r}_0(\Omega),
\]

where \( r = p_*/(p_1 - 1) \).

**Proof.** Weak solutions of (79) can be obtained by variational methods. The case with \( p = 2 \) is given in [49, Proposition 3.1]. Now consider \( 1 < p < +\infty \). Set

\[
E[u] = \int_{\Omega} \{ P(x, \text{curl } u) + \frac{c(x)}{p} |u|^p - F(x, u) \} \, dx,
\]

\[
c_p = \inf_{u \in W^{1,p}_0(\Omega, \text{div } 0, u_j^0)} E[u].
\]

From (\( F^p \)), for any \( \varepsilon > 0 \) we have \( C(\varepsilon) \) such that

\[
F(x,u) \leq \varepsilon |u|^p + C(\varepsilon). \quad (86)
\]

**Case 1.** \( \Omega \) has no holes and \( c(x) \geq 0 \) on \( \Omega \). Using (16) for the case \( H_2(\Omega) = \{ 0 \} \) and Sobolev imbedding we have

\[
\|u\|_{L^p(\Omega)} \leq C(\Omega,p) \{ \|\text{curl } u\|_{L^p(\Omega)} + \|\text{div } u\|_{L^p(\Omega)} + \|u_T\|_{W^{1/4,p}(\partial\Omega)} \}. \quad (87)
\]

Let \( \{ u_j \} \subset W^{1,p}_0(\Omega, \text{div } 0, u_j^0) \) be a minimizing sequence, \( E[u_j] = c_p + \varepsilon_j \to c_p \). From (\( P^p \), (86) and (87) we have

\[
c_p \|\text{curl } u_j\|_{L^p(\Omega)}^p - c|\Omega|
\]

\[
\leq c_p + \varepsilon_j + \varepsilon C(\Omega,p) \{ \|\text{curl } u_j\|_{L^p(\Omega)}^p + \|u_T\|_{W^{1/4,p}(\partial\Omega)}^p \} + C(\varepsilon)|\Omega|.
\]
Taking $\varepsilon > 0$ small such that $\varepsilon C(\Omega, p) < c$, we see that $||\text{curl}\ u_j||_{L^p(\Omega)}$ is bounded. By (16) and (87) $\{u_j\}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^3)$, hence it has a subsequence, still denoted by $\{u_j\}$, and $u \in W^{1,p}_0(\Omega, \text{div }0, u_j^0)$, such that $u_j \to u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$ and strongly in $L^p(\Omega, \mathbb{R}^3)$ for $1 < p_1 < p^*$. Condition $(P^p)$ implies that the functional

$$w \mapsto \int_{\Omega} P(x, w)dx$$

is weakly lower semi-continuous on $L^p(\Omega, \mathbb{R}^3)$, and $(F^p)$ implies that the functional

$$u \mapsto \int_{\Omega} F(x, u)dx$$

is continuous on $L^p(\Omega, \mathbb{R}^3)$. Therefore

$$\mathcal{E}[u] \leq \liminf_{j \to \infty} \mathcal{E}[u_j],$$

so $u$ is a minimizer of $\mathcal{E}$ on $W^{1,p}_0(\Omega, \text{div }0)$, and it satisfies, for all $w \in W^{1,p}_0(\Omega, \text{div }0)$,

$$\int_{\Omega} \{\langle \nabla w, P(x, \text{curl } u) \rangle + c(x)\|u\|^{p-2}u \cdot w - \langle \nabla u , F(x, u) \rangle \}dx = 0. \quad (88)$$

By conditions $(P^p)$ and $(F^p)$ we can verify that

$$g \equiv \nabla_x F(x, \text{curl } u) \in L^r(\Omega, \mathbb{R}^3),$$

$$U \equiv c(x)\|u\|^{p-2}u - \nabla u F(x, u) \in L^r(\Omega, \mathbb{R}^3),$$

where

$$r = p_*/(p_* - 1) > p_*/(p_1 - 1) = (p_*)'.$$

Applying Lemma 4.6 without the term $\mathcal{A}(x)\text{curl } u$, we conclude that, there exists $\phi \in W^{1,p}_0(\Omega)$ such that the equality

$$\text{curl } \{\nabla_x P(x, \text{curl } u)\} + c(x)\|u\|^{p-2}u - \nabla u F(x, u) = \nabla \phi$$

holds as elements in $W^{1,p_0}_0(\Omega, \mathbb{R}^3)$, so $(u, \phi)$ is a weak solution of (79).

**Case 2.** $\Omega$ has holes and $c(x) \geq c_0 > 0$ on $\Omega$. Take $0 < \varepsilon < c_0/(2p)$ We have

$$c\|\text{curl } u_j\|_{L^p(\Omega)} - c|\Omega| + \frac{c_0}{p}\|u_j\|_{L^p(\Omega)}$$

$$\leq \int_{\Omega} \{P(x, \text{curl } u_j) + c(x)\|u_j\|^p\}dx$$

$$= \mathcal{E}[u_j] + \int_{\Omega} F(x, u)dx \leq c_p + \varepsilon_j + \varepsilon\|u_j\|_{L^p(\Omega)} + C(\varepsilon)|\Omega|.$$}

So $\|\text{curl } u_j\|_{L^p(\Omega)} + \|u_j\|_{L^p(\Omega)}$ is bounded. Using (16) we see that $\{u_j\}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^3)$. The rest of the proof is same as in case 1. \Box

**Theorem 5.4.** Let $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, $1 < p < \infty$, $P$, $F$ and $u_j^0$ satisfy $(P^p)$, $(F^p)$ and $(U_0)$ respectively, and $\mathcal{W}_T \in TW^{-1/p'}(\partial \Omega, \mathbb{R}^3)$. Assume $c(x)$ is a non-negative continuous function on $\Omega$, and if $\Omega$ is multiply-connected then assume $c(x) \geq c_0$ for some $c_0 > 0$. Then (83) has a weak solution

$$(u, \phi) \in W^{1,p}_0(\Omega, \text{div }0) \times \hat{L}^p(\Omega).$$
Proof. The functional associated with (83) is

\[ G[u] = \int_{\Omega} \left\{ P(x, \text{curl } u) + \frac{c(x)}{p} |u|^p - F(x, u, u) \right\} dx + \langle W_T, u \rangle_{\partial \Omega, 1/p'} . \]

Similar to the proof of Theorem 5.3, we can show that the functional \( G \) has a minimizer \( u \) on \( W^{1,p}_{\text{ad}}(\Omega, \text{div } 0) \). Hence for all \( w \in W^{1,p}_{\text{ad}}(\Omega, \text{div } 0) \) we have

\[
\int_{\Omega} \left\{ \langle \nabla_x P(x, \text{curl } u), \text{curl } w \rangle + c(x)|u|^{p-2}u \cdot w - \langle \nabla_u F(x, u, w) \rangle dx \right.
\]
\[ + \langle W_T, w \rangle_{\partial \Omega, 1/p'} = 0. \]

Then we apply Lemma 4.7 without the term \( \mathcal{A} \text{curl } u \) and with

\[
g \equiv \nabla_x P(x, \text{curl } u) \in L^q(\Omega, \mathbb{R}^3),
\]
\[
U \equiv c(x)|u|^{p-2}u - \nabla_u F(x, u) \in L^r(\Omega, \mathbb{R}^3),
\]

where \( r = p_*/(p_1 - 1) \), to conclude that, there exists \( \phi = \phi_U + h \in L^p(\Omega) \), where \( \phi_U \in \dot{W}^{1,p}(\Omega) \subset L^p(\Omega) \), hence the functional \( T_\phi \) is well-defined on the space of test fields \( W^{1,p}_{\text{ad}}(\Omega, \mathbb{R}^3) \).

In Theorem 5.3, \( r = p_*/(p_1 - 1) > (p_*)' \), by Remark 1 we have \( r_* > p' \). So the weak solution \( (u, \phi) \) satisfies \( \phi \in W^{1,p}_0(\Omega) \subset L^p(\Omega) \), hence the functional \( T_\phi \) is well-defined on the space of test fields \( W^{1,p}_{\text{ad}}(\Omega, \mathbb{R}^3) \).

Now we make comments regarding regularity of the weak solutions of (79) and (83).

(i) When \( p = 2 \), the weak solutions of the quasilinear Maxwell-Stokes system (79) have Schauder regularity if the data are suitably smooth (see [49]):

\[
u \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3), \quad \phi \in C^{1+\alpha}(\bar{\Omega}).
\]

For the Maxwell type equations, Schauder regularity of weak solutions has been established for many quasilinear models, see [10, 46, 47, 35, 50] for the model of Meissner states of superconductivity, [48, 49] for the nonlinear Maxwell equations.

If the divergence-free condition \( \text{div } u = 0 \) is removed from (79) or (83), then regularity of the weak solutions may be lower. For instance, the weak solutions of the extended magneto-static Born-Infeld model have only \( C^{1+\alpha} \) regularity (see [19, 20]):

\[
u \in C^{2+\alpha}(\bar{\Omega}, \text{div } 0) + \text{grad} C^{2+\alpha}(\bar{\Omega}).
\]

(ii) When \( p \neq 2 \), equation in (79) is degenerately elliptic, we expect that the weak solutions have lower regularity. For the weak solutions of the \( p \)-curl \( p \)-curl system, the interior regularity has been obtained in [21, Theorem 4.1],

\[
u \in C^{1+\alpha}_{\text{loc}}(\Omega, \text{div } 0) + \text{grad} C^{1+\alpha}_{\text{loc}}(\Omega).
\]

5.2. Examples. As an application of Theorem 5.3 we consider solvability of the following system

\[
\begin{cases}
\text{curl } [a(x, |\text{curl } u|^p)] \text{curl } u + c(x)|u|^{p-2}u = g(x, |u|^p)u + \nabla \phi & \text{in } \Omega, \\
\text{div } u = 0 & \text{in } \Omega, \\
u_T = u_T^p, \quad \phi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\[10^\text{It is interesting that the relation between the } p \text{-curl curl system with the } p \text{-Laplace equation was observed there.}]}
The case where \( p = 2 \) has been studied in [49]. For \( 1 < p < \infty \) we need the following conditions. Denote \( \mathbb{R}_+ = [0, +\infty) \).

(a) \( a \in C^0_\text{loc}(\bar{\Omega} \times \mathbb{R}_+) \), \( \frac{\partial a}{\partial t} \in C^0_\text{loc}(\bar{\Omega} \times \mathbb{R}_+) \). There exist positive constants \( \lambda, \Lambda \) such that
\[
\lambda t^{1-2/p} \leq a(x,t) \leq \Lambda t^{1-2/p}, \quad a(x,t) + p \frac{\partial a}{\partial t}(x,t)t > 0, \quad \forall x \in \Omega, \ t > 0.
\]

(g) \( g \in C^0_\text{loc}(\bar{\Omega} \times \mathbb{R}_+) \), and there exists a positive constant \( C \) such that
\[
|g(x,t)| \leq Ct^{1-2/p}, \quad \forall x \in \Omega, \ t \geq 0,
\]
\[
\lim_{t \to +\infty} \frac{g(x,t)}{t^{1-2/p}} = 0 \quad \text{uniformly for} \ x \in \Omega.
\]

**Corollary 2.** Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary, \( 1 < p < +\infty \), \( a, c, g \) and \( u_T^0 \) satisfy (a), (c), (g) and \((U_0)\) respectively. Then (89) has a weak solution
\[
(u, p) \in W_t^{1,p}(\Omega, \text{div} 0,u_0^0) \times W_0^{1,r}(\Omega)
\]
with \( r = p_+(p-1) \).

**Proof.** Extend \( a(x,t) \) and \( g(x,t) \) evenly for \( t < 0 \). Set
\[
c(x,t) = \int_0^t a(x,s) \frac{ds}{s^{1-2/p}}, \quad h(x,t) = \int_0^t g(x,s) \frac{ds}{s^{1-2/p}},
\]
\[
P(x,z) = \frac{1}{p} c(x,|z|^p), \quad F(x,u) = \frac{1}{p} h(x,|u|^p).
\]
Then
\[
\frac{\partial P}{\partial z_i}(x,z) = a(x,|z|^p) z_i, \quad \frac{\partial F}{\partial u_i}(x,u) = g(x,|u|^p) u_i,
\]
\[
\frac{\partial^2 P}{\partial z_i \partial z_j}(x,z) = a(x,|z|^p) \delta_{ij} + p \frac{\partial a}{\partial t}(x,|z|^p) |z|^{(p-2)} z_i z_j.
\]
From (a) and (g) we see that \( P(x,z) \) and \( F(x,u) \) satisfy \((P^p)\) and \((F^p)\) with \( p_1 = p \), and for \( \xi \neq 0 \),
\[
I = \sum_{i,j=1}^3 \frac{\partial^2 P}{\partial x_i \partial x_j}(x,z) \xi_i \xi_j = a(x,|z|^p) |\xi|^2 + p \frac{\partial a}{\partial t}(x,|z|^p) |z|^{(p-2)} (z \cdot \xi)^2 > 0.
\]
So \( P(x,z) \) is strictly convex in \( z \), and existence follows from Theorem 5.3. \( \square \)

If a Maxwell-Stokes system has a weak solution \((u, \phi)\), and if we can use the special structure of the equation to show that \( \nabla \phi = 0 \), then \( u \) is a solution of the corresponding Maxwell system. As an example, we examine solvability of the following Maxwell system
\[
\begin{align*}
\text{curl} [\nabla P(x, \text{curl } u)] + c(x)|u|^{p-2} u &= J(x), \quad \text{div } u = 0 \quad \text{in } \Omega, \\
u_T &= u_T^0, \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( J \) is given.

**Corollary 3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a \( C^2 \) boundary, \( 1 < p < \infty \), \( P \) and \( u_T^0 \) satisfy the conditions in Theorem 5.3, \( J \in W^r(\Omega, \text{div} 0) \) with \( r \geq (p_+)' \).

If either
\[
\begin{align*}
\text{(i)} & \quad 1 < p < +\infty, \ c(x) \equiv 0, \ \Omega \text{ has no holes}; \text{ or} \\
\text{(ii)} & \quad p = 2 \text{ and } c(x) = c_0 \text{ is a positive constant};
\end{align*}
\]
then (90) has a weak solution
\[ \mathbf{u} \in W^{1,p}_t(\Omega, \text{div} \mathbf{0}, \mathbf{u}_T^0). \]

**Proof.** By Theorem 5.3 the following problem has a weak solution \((\mathbf{u}, \phi)\) with \(\phi \in W^{1,r}_0(\Omega)\):
\[
\begin{aligned}
\begin{cases}
\text{curl} [\nabla z P(x, \text{curl} \mathbf{u})] + c(x)|\mathbf{u}|^{p-2}\mathbf{u} = \mathbf{J}(x) + \nabla \phi, & \text{in } \Omega, \\
\text{div} \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u}_T = \mathbf{u}_T^0, & \phi = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
If either (i) or (ii) holds, from the equation we see that \(\Delta \phi = 0\) in the sense of distribution. Since \(\phi = 0\) on \(\partial \Omega\), we see that \(\phi \equiv 0\) on \(\Omega\). Hence \(\mathbf{u}\) is a weak solution of (90).

6. **Linear Maxwell-Stokes system.** We review and generalize some existence and regularity results in [49, Appendix B] for the linear Maxwell-Stokes system
\[
\begin{aligned}
\begin{cases}
\text{curl} (\mathcal{A} \text{curl} \mathbf{u}) = \mathbf{J} + \nabla \phi, & \text{in } \Omega, \\
\text{div} \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u}_T = \mathbf{u}_T^0, & \phi = 0 & \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]
and the linear Maxwell type system
\[
\begin{aligned}
\begin{cases}
\text{curl} (\mathcal{A} \text{curl} \mathbf{u}) = \mathbf{J}, & \text{in } \Omega, \\
\text{div} \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u}_T = \mathbf{u}_T^0, & \phi = 0 & \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]
Note that in the case where the domain \(\Omega\) has holes, then the operator \(\mathbf{u} \mapsto \text{curl} (\mathcal{A} \text{curl} \mathbf{u})\) is degenerate, which causes difficulties in analysis. One could add a positive term \(\mathcal{B} \mathbf{u}\) to the left side of the equation in (91) and consider the equation of the form
\[
\text{curl} (\mathcal{A} \text{curl} \mathbf{u}) + \mathcal{B} \mathbf{u} = \mathbf{J} + \nabla \phi,
\]
where \(\mathcal{B} \in C^0(\overline{\Omega}, S_+^+(3))\), then the corresponding BVP is non-degenerate and existence problem becomes much easier. However in this section we are interested in the degenerate case, and we wish to examine new phenomena for (91) and (92).

BVPs of the type (92) have been studied by many authors, see the classical works [25, 16, 40] and the references therein. For the recent research works, see for instance [54, 38, 4, 71, 7, 6], just name a few.

In [49], for \(\mathbf{J} \in L^2(\Omega, \mathbb{R}^3)\) or \(\mathbf{J} \in H^{1*}_0(\Omega, \mathbb{R}^3)\), we got existence and regularity of weak solutions \((\mathbf{u}, \phi) \in H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega)\) of (91) and (92). Here we study weak solutions \((\mathbf{u}, \phi) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega)\), with \(\mathbf{J} \in L^r(\Omega, \mathbb{R}^3)\) or \(\mathbf{J} \in W^{1,p,r}_0(\Omega, \mathbb{R}^3)\). Assume that
\[ \Omega \text{ is a bounded domain in } \mathbb{R}^3 \text{ with a } C^2 \text{ boundary, } \quad 1 < p < \infty, \]
\(\mathcal{A}\) satisfies \((A_0)\), \(\mathbf{u}_T^0 \in W^{1/p'}(\partial \Omega, \mathbb{R}^3)\).

**Definition 6.1.** Let \(\mathbf{J} \in L^r(\Omega, \mathbb{R}^3)\) with \(r \geq (p_*)'\). We say that \((\mathbf{u}, \phi)\) is a weak solution of (91) if \(\mathbf{u} \in W^{1,p}_t(\Omega, \text{div} \mathbf{0}, \mathbf{u}_T^0)\), \(\phi \in L^p(\Omega)\) with trace zero on \(\partial \Omega\), such that
\[
\int_\Omega \{\langle \mathcal{A} \text{curl} \mathbf{u}, \text{curl} \mathbf{w} \rangle - \mathbf{J} \cdot \mathbf{w} \} \, dx = \langle T_\phi, \mathbf{w} \rangle_{W^{1,p}_0(\Omega, \mathbb{R}^3, W^{1,p}_0(\Omega, \mathbb{R}^3))},
\]
\[ \forall \mathbf{w} \in W^{1,p}_0(\Omega, \mathbb{R}^3). \]
If $J \in W^{1,p',s}_0(\Omega, \mathbb{R}^3)$, then (91) should be written in the following way:

$$
\begin{aligned}
\begin{cases}
\text{curl} \left( A \text{curl} u \right) = J + T_{\phi,c}, & \text{in } \Omega, \\
u_\Gamma = u_\Gamma^0, & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
$$

(95)

**Definition 6.2.** Let $J \in W^{1,p',s}_0(\Omega, \mathbb{R}^3)$. We say that $(u, \phi)$ is a weak solution of (95) if $u \in W^{1,p}_1(\Omega, \text{div } 0, u_\Gamma^0)$, $\phi \in L^p(\Omega)$ and there exists $c \in \mathbb{R}$ such that

$$
\int_{\Omega} \langle A \text{curl } u, \text{curl } w \rangle dx = \langle J + T_{\phi,c}, w \rangle_{W^{1,p',s}_0(\Omega), W^{1,p'}_0(\Omega)},
$$

$$
\forall w \in W^{1,p'}_0(\Omega, \mathbb{R}^3).
$$

(96)

**Remark 6.** In (82) the space of the test fields $w$ is $W^{1,p'}_0(\Omega, \mathbb{R}^3)$, in which the $u$ component of the solutions also lie; while in (94) the space of the test fields is $W^{1,p'}_0(\Omega, \mathbb{R}^3)$. This difference is due to the difference in the structure of the equations.

### 6.1. Existence of solutions

Given $J \in L^r(\Omega, \mathbb{R}^3)$, we decompose it into a divergence-free field and a gradient field:

$$
J = J_0 - \nabla \phi_J,
$$

(97)

where $J_0 \in W^r(\Omega, \text{div } 0)$, and $\phi_J \in W^{1,r}_0(\Omega)$ is a weak solution of

$$
- \Delta \phi_J = \text{div } J \quad \text{in } \Omega, \quad \phi_J = 0 \quad \text{on } \partial \Omega,
$$

(98)

$$
\| \nabla \phi_J \|_{L^r(\Omega)} \leq C(\Omega, r) \| J \|_{L^r(\Omega)}.
$$

Note that $\nabla \phi_J \in H^s_2(\Omega)$, hence if $J$ satisfies the following orthogonality condition

$$
\int_{\Omega} J \cdot h_2 dx = 0, \quad \forall h_2 \in H^s_2(\Omega),
$$

(99)

then $J_0$ also satisfies this condition.

For $J \in W^{1,p',s}_0(\Omega, \mathbb{R}^3)$, the meaning of being divergence-free is not clear, and existence of decomposition similar to (97) is more complicated. Denote the connected components of $\partial \Omega$ by $\Gamma_j$, $j = 1, \cdots, m + 1$, and $\Gamma = \bigcup_{j=1}^{m+1} \Gamma_j$. Denote, for $1 < s < \infty$,

$$
W^{k,s}_c(\Omega) = \{ \xi \in W^{k,s}(\Omega) : \xi = \text{constant } \text{on } \Gamma_j, \ j = 1, \cdots, m + 1 \},
$$

$$
\mathcal{W}^{k,s,s}_0(\Omega, \Gamma) = \{ f \in W^{k,s,s}_0(\Omega, \mathbb{R}^3) : \langle f, \nabla \xi \rangle_{W^{k,s,s}_0(\Omega, \mathbb{R}^3), W^{k,s,s}_0(\Omega)} = 0, \forall \xi \in W^{k+1,s}_c(\Omega) \}.
$$

(100)

**Lemma 6.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary and $1 < p < \infty$. Assume that $J \in W^{1,p',s}_0(\Omega, \mathbb{R}^3)$ is such that

$$
\langle J, h_2 \rangle_{W^{1,p',s}_0(\Omega), W^{1,p'}_0(\Omega)} = 0, \quad \forall h_2 \in H^s_2(\Omega).
$$

(101)

In addition, assume there exists a number $s$ with

$$
1 < s \leq (p')_s,
$$

(102)

such that $J$ can be extended to a bounded functional on $W^{1,s}_c(\Omega)$ in the following sense:

...
Proposition 5. Let \( \Omega, p, A, u_0^p \) satisfy (93), \( J \in W^{1,p'}_0(\Omega, \mathbb{R}^3) \).

(i) The functional
\[
\xi \mapsto \langle J, \nabla \xi \rangle_s
\]

is bounded on \( W^{1,s}_c(\Omega) \), where \( \langle \cdot, \cdot \rangle_s \) denotes the paring between \( W^{1,s}_c(\Omega) \) and its dual space;

(ii) If \( \xi \in W^{1,s}_c(\Omega) \) and \( \nabla \xi \in W^{1,p'}_{10}(\Omega, \mathbb{R}^3) \), then
\[
\langle J, \nabla \xi \rangle = \langle J, \nabla \xi \rangle_{W^{1,p'}_{10}(\Omega), W^{1,p'}_{10}(\Omega)}.
\]

Then there exists a unique \( \phi_J \in W^{1,s'}_{0}(\Omega) \) such that
\[
\int_\Omega \langle \nabla \phi_J, \nabla \xi \rangle dx = -\langle J, \nabla \xi \rangle_s, \quad \forall \xi \in W^{1,s}_c(\Omega).
\]

Let \( J_0 = J + \nabla \phi_J \). Then \( J_0 \in W^{1,s'}_{10}(\Omega, \Gamma) \).

Proof. Denote, for \( r = s' = s/(s-1) \),
\[
X = W^{1,r}_0(\Omega), \quad Y = W^{1,s}_c(\Omega),
\]
\[
a(\phi, \psi) = \int_\Omega \langle \nabla \phi, \nabla \psi \rangle dx, \quad F[\psi] = -\langle J, \nabla \psi \rangle_s,
\]
\[
N_X = \{ \phi \in X : a(\phi, \psi) = 0, \forall \psi \in Y \},
\]
\[
N_Y = \{ \psi \in Y : a(\phi, \psi) = 0, \forall \phi \in X \}.
\]

Obviously \( N_X = \{ 0 \} \) and \( \text{grad} N_Y = H^1_2(\Omega) = H^2_2(\Omega) \). For any \( \xi \in N_Y \), since \( h_2 = \nabla \xi \in H^2_2(\Omega) \), from (101) and (103) we have
\[
F[\xi] = -\langle J, \nabla \xi \rangle_s = -\langle J, h_2 \rangle_{W^{1,p'}_{10}(\Omega), W^{1,p'}_{10}(\Omega)} = 0.
\]

So by [32, Theorem 1.1] there exists a unique \( \phi_J \in W^{1,r}_0(\Omega) \) such that (104) holds. Set \( J_0 = J + \nabla \phi_J \). Then for any \( w \in W^{1,p'}_{10}(\Omega, \mathbb{R}^3) \) we have
\[
\langle J_0, w \rangle_{W^{1,p'}_{10}(\Omega), W^{1,p'}_{10}(\Omega)} = \langle J, w \rangle_{W^{1,p'}_{10}(\Omega), W^{1,p'}_{10}(\Omega)} + \int_\Omega \nabla \phi_J \cdot w dx.
\]

Since \( s \leq (p')_s \), we have
\[
W^{2,p'}_c(\Omega) \subseteq W^{(p')_s}(\Omega) \subseteq W^{1,s}_c(\Omega), \quad \text{grad} W^{2,p'}_c(\Omega) \subseteq W^{1,p'}_{10}(\Omega, \mathbb{R}^3).
\]

So for any \( \xi \in W^{2,p'}_c(\Omega) \) we have
\[
\langle J_0, \nabla \xi \rangle_{W^{1,p'}_{10}(\Omega), W^{1,p'}_{10}(\Omega)} = \langle J, \nabla \xi \rangle_s + \int_\Omega \nabla \phi_J \cdot \nabla \xi dx = 0.
\]

Hence \( J_0 \in W^{1,s,s}(\Omega, \Gamma) \). \( \square \)

**Proposition 5.** Let \( \Omega, p, A, u_0^p \) satisfy (93), \( J \in W^{1,p'}_{10}(\Omega, \mathbb{R}^3) \).

(i) Eq.(95) has weak solutions \( (u, \phi) \in W^{1,p}_t(\Omega, \text{div} 0, u_0^p) \times L^p(\Omega) \) in the sense of Definition 6.2 if and only if (101) holds.

(ii) If \( J \in W^{1,s,s}_c(\Omega, \Gamma) \) with \( s \) satisfying (102), then (92) has a weak solution \( u \in W^{1,p}_t(\Omega, \text{div} 0, u_0^p) \).

(iii) If \( J \) satisfies (101) and can be extended to a bounded functional on \( W^{1,s}_c(\Omega) \) with \( s \) satisfying (102), then (91) has a weak solution
\[
(u, \phi) \in W^{1,p}_t(\Omega, \text{div} 0, u_0^p) \times W^{1,s}_0(\Omega).
\]
Proof. The proof has been given in [49] for the case when \( p = 2 \), and the general case when \( 1 < p < \infty \) can be proved by a similar argument.

Given \( u_0^0 \in TW^{1/p, p}(\partial \Omega, \mathbb{R}^3) \), we can find \( U \in W^{1,p}(\Omega, \mathbb{R}^3) \) such that
\[
\text{div } U = 0 \quad \text{in } \Omega, \quad U_T = u_0^0 \quad \text{on } \partial \Omega, \tag{105}
\]
\[
\|U\|_{W^{1,p}(\Omega)} \leq C(\Omega, p)\|u_0^0\|_{W^{1/p, p}(\partial \Omega)}.
\]

Set \( v = u - U \), \( g = A(x) \text{ curl } U(x) \). We transfer (95) and (92) to the following
\[
\left\{
\begin{aligned}
\text{curl } (A \text{ curl } v + g) &= J + T_{\phi, c}, & \text{div } v = 0 & \quad \text{in } \Omega, \\
\n\end{aligned}
\right.
\]
\[
\nu_T = 0, \quad \text{on } \partial \Omega, \tag{106}
\]
and
\[
\left\{
\begin{aligned}
\text{curl } (A \text{ curl } v + g) &= J, & \text{div } v = 0 & \quad \text{in } \Omega, \\
\nu_T &= 0, \quad \text{on } \partial \Omega. \tag{107}
\end{aligned}
\right.
\]

**Step 1.** If \( (u, \phi) \in W^{1,p}_t(\Omega, \text{div } 0, u_0^0) \times L^p(\Omega) \) is a weak solution of (95), then \((v, \phi)\) is a weak solution of (106) in the following sense: there exists \( c \in \mathbb{R} \) such that
\[
\int_{\Omega} (A \text{ curl } v + g, \text{ curl } w) \, dx = (J + T_{\phi, c}, w)_{W^{1,p'}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)}, \quad \forall w \in W^{1,p'}_{t0}(\Omega, \mathbb{R}^3). \tag{108}
\]

Hence \( v \in W^{1,p}_0(\Omega, \text{div } 0) \) satisfies
\[
\int_{\Omega} (A \text{ curl } v, \text{ curl } w) \, dx = (f, w)_{W^{1,p'}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)}, \quad \forall w \in W^{1,p'}_{t0}(\Omega, \text{div } 0), \tag{109}
\]
where \( f \in W^{1,p'}_{t0}(\Omega, \mathbb{R}^3) \) and
\[
(f, w)_{W^{1,p', *}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)} = - \int_{\Omega} g \cdot \text{ curl } w \, dx + (J, w)_{W^{1,p', *}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)}.
\]

Note that (101) is necessary for solvability of (95). Now assume (101) holds. Denote \( V = W^{1,p}_0(\Omega, \text{div } 0), \quad W = W^{1,p'}_{t0}(\Omega, \text{div } 0), \)
\[
a(v, w) = \int_{\Omega} (A \text{ curl } v, \text{ curl } w) \, dx,
\]
\[
N_V = \{ v \in V : a(v, w) = 0, \quad \forall w \in W \},
\]
\[
N_W = \{ w \in W : a(v, w) = 0, \quad \forall v \in V \}.
\]

By Corollary 1 \( N_V = N_W = \mathbb{H}_2(\Omega) \), and by condition (101) we have
\[
(f, w)_{W^{1,p', *}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)} = 0, \quad \forall w \in N_W.
\]

Using Lemma 2.1 (ii), we can apply [32, Theorem 1.1] to conclude that there exists \( v \in V \) such that
\[
a(v, w) = (f, w)_{W^{1,p', *}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)}, \quad \forall w \in W.
\]

So (109) holds. Then we apply Lemma 4.2 to the functional
\[
w \mapsto a(v, w) - (f, w)_{W^{1,p', *}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)}
\]
on \( W^{1,p'}_{t0}(\Omega, \mathbb{R}^3) \), and conclude that there exist \( \phi \in \hat{L}^p(\Omega) \) and \( c \in \mathbb{R} \) such that
\[
a(v, w) = (f + T_{\phi, c}, w)_{W^{1,p', *}_{t0}(\Omega), W^{1,q}_{t0}(\Omega)}, \quad \forall w \in W^{1,p'}_{t0}(\Omega, \mathbb{R}^3).
\]
So \((v, \phi)\) is a weak solution of \((106)\) in the sense of \((108)\). Conclusion (i) is proved.

**Step 2.** Assume \(J \in \mathcal{W}^{1,s}_0(\Omega, \Gamma)\). Since \(H_2(\Omega) \subset \text{grad} \mathcal{W}^{2,s}_0(\Omega)\), so \(J\) satisfies (101). From step 1, Eq. (106) has weak solutions \((v, \phi) \in \mathcal{W}^{1,p}_0(\Omega, \text{div} 0) \times L^p(\Omega)\) in the sense of (108). From (108) we have, for all \(w \in \mathcal{W}^{1,p'}_0(\Omega, \mathbb{R}^3)\),

\[
\int_\Omega (A \text{curl} v + g, \text{curl} w) dx = \langle J, w \rangle_{W^{1,p'}_0(\Omega), W^{1,p'}_0(\Omega)}
\]

By a similar but easier proof, for (91) with \(J = J_0\), we have

\[
\int_\Omega (\phi, \Delta x_n) dx + c \int_\Omega (1, \nabla \phi)_{\partial \Omega, 1/p} \int_\Omega (\phi) \Delta x_n dx = \int_\Omega (\phi) \Delta x_n dx = \int_{E_n} (\phi - c)^2 dx.
\]

Hence \(\phi = c\) on \(\bigcup_{n=1}^\infty E_n = \Omega\). Since \(\phi \in \mathcal{L}^p(\Omega)\), so \(c = 0\). Thus \(\phi = c = 0\). Therefore \(v\) is a weak solution of (107). Conclusion (ii) is proved.

**Step 3.** Assume \(J\) can be extended to a bounded functional on \(\mathcal{W}^{1,s}_0(\Omega)\) with \(1 < s < (p')_s\). From Lemma 6.3 there exists a unique \(\phi_J \in \mathcal{W}^{1,s'}_0(\Omega)\) such that (104) holds, and \(J_0 = J + \nabla \phi_J \in \mathcal{W}^{1,s}_0(\Omega, \Gamma)\). We repeat the argument in step 2 with \(J\) replaced by \(J_0\), and get a weak solution \(v \in \mathcal{W}^{1,p}_0(\Omega, \text{div} 0)\) of (107) with \(J\) replaced by \(J_0\). Then \((v, \phi_J)\) is a weak solution of (106) with \(\phi_J \in \mathcal{W}^{1,s}_0(\Omega)\).

By a similar but easier proof, for (91) with \(J \in \mathcal{L}^r(\Omega, \mathbb{R}^3)\), we have

**Lemma 6.4.** Let \(\Omega, p, A, u_0^1\) satisfy (93), and \(J \in \mathcal{L}^r(\Omega, \mathbb{R}^3)\) with \(r \geq 3p/(3+p)\).

(i) Eq. (91) has weak solutions \((u, \phi) \in \mathcal{W}^{1,p}_0(\Omega, \text{div} 0, u_0^0) \times L^p(\Omega)\) in the sense of (94) if and only if (99) holds. If \(J\) satisfies (99), then (91) has a weak solution

\[
(u, \phi_J) \in \mathcal{W}^{1,p}_0(\Omega, \text{div} 0, u_0^0) \times \mathcal{W}^{1,r}_0(\Omega).
\]

(ii) If \(J\) satisfies (99) and \(\text{div} J = 0\) in \(\Omega\), then (92) has a weak solution

\[
u \in \mathcal{W}^{1,p}_t(\Omega, \text{div} 0, u_0^0).
\]

In Proposition 5 (iii), the index \(s\) satisfies (102), by Remark 1 \((s')_s \geq p\), so the weak solution \((u, \phi)\) satisfies \(\phi \in \mathcal{W}^{1,s}_0(\Omega) \subset L^{(s')_s}(\Omega) \subset L^p(\Omega)\). In Lemma 6.4 (i), \(r \geq 3p/(3+p)\), so \((s')_s \geq p\), and the weak solution \((u, \phi)\) satisfies \(\phi \in \mathcal{W}^{1,s'}_0(\Omega) \subset L^{r_*}(\Omega) \subset L^p(\Omega)\). Therefore in both cases the functional \(T_{\phi}\) is well-defined on \(\mathcal{W}^{1,p'}_0(\Omega, \mathbb{R}^3)\).

**6.2. Regularity and a priori estimates.** From Corollary 1 we know that if (91) has a weak solution \((u, \phi)\), then all the solution are given by \((u + h_2, \phi)\) with \(h_2 \in \mathbb{H}_2(\Omega)\). The following concept of \(\mathbb{H}_2(\Omega)\)-orthogonality was introduced in [49, Definition 2.7, Proposition 2.9] for the purpose of having uniqueness modulo \(\mathbb{H}_2(\Omega)\), and a priori estimates.
Definition 6.5. (i) Let \( u \in W^{k,p}(\Omega, \mathbb{R}^3) \). We say \( u \) satisfies \( H_2(\Omega) \)-orthogonality condition in \( W^{k,p}(\Omega, \mathbb{R}^3) \) if there exists \( U \in W^{k,p}(\Omega, \mathbb{R}^3) \) such that
\[
\begin{align*}
\text{div} U &= 0 \quad \text{in} \quad \Omega, \\
U_T &= \mathbf{u}_T \quad \text{on} \quad \partial \Omega, \quad u - U \in H_2(\Omega)_{L^2(\Omega)}, \\
\|U\|_{W^{k,p}(\Omega)} &= C(\Omega, k, p)\|u_T\|_{W^{k-1/p,p}(\partial \Omega)}.
\end{align*}
\]
(111)
\[
\begin{align*}
\|u\|_{W^{k,p}(\Omega)} &= \|u\|_{H_2(\Omega)}, \\
\|\phi\|_{L^2(\Omega)} &= 0 \quad \text{in} \quad \Omega, \quad \text{curl} \mathbf{u} = \mathbf{u}_T \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
(ii) Let \( u \in C^{k+\alpha}(\Omega, \mathbb{R}^3) \). We say \( u \) satisfies \( H_2(\Omega) \)-orthogonality condition in \( C^{k+\alpha}(\Omega, \mathbb{R}^3) \) if there exists \( U \in C^{k+\alpha}(\Omega, \mathbb{R}^3) \) such that (111) holds and
\[
\|U\|_{C^{k+\alpha}(\Omega)} \leq C(\Omega, k, \alpha)\|u_T\|_{C^{k+\alpha}(\partial \Omega)}.
\]
(113)

Proposition 6. Assume \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with a \( C^3 \) boundary, \( \mathcal{A} \in C^1(\Omega, S^*(\mathbb{R}^3)) \) satisfies (A0), \( J \in L^r(\Omega, \mathbb{R}^3) \), \( u^0_T \in TW^{2-1/q,q}(\partial \Omega, \mathbb{R}^3) \), and \((u, \phi) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega)\) is a weak solution of (91), where \( 1 < p < \infty \), \( r \geq 3p/(3+p) \), \( q = \min\{r, p\} \). Then \( u \in W^{2,q}(\Omega, \mathbb{R}^3) \), \( \phi \in W^{1,3}(\Omega) \), and
\[
\begin{align*}
\|u\|_{W^{2,q}(\Omega)} &\leq C_1\{\|J\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}\}, \\
\|\nabla \phi\|_{L^r(\Omega)} &\leq C(\Omega, r)\|J\|_{L^r(\Omega)},
\end{align*}
\]
(114)
where \( C_1 \) depends on \( \Omega, p, r, \|\mathcal{A}\|_{C^0(\Omega)}, \|\mathcal{A}\|_{C^0(\Omega)}, \|\mathcal{A}\|_{C^1(\Omega)}, \|\mathcal{A}\|_{C^1(\Omega)} \), and the constants in (A0).

If furthermore \( u \) satisfies the \( H_2(\Omega) \) orthogonality condition (112) in \( W^{2,q}(\Omega, \mathbb{R}^3) \), then
\[
\|u\|_{W^{2,q}(\Omega)} \leq C_2\{\|J\|_{L^p(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}\},
\]
(115)
where \( C_2 \) depends on \( \Omega, p, r, \|\mathcal{A}\|_{C^0(\Omega)}, \|\mathcal{A}\|_{C^0(\Omega)}, \|\mathcal{A}\|_{C^1(\Omega)}, \|\mathcal{A}\|_{C^1(\Omega)} \), the constants in (A0) and (112).

Proof. The case where \( p = 2 \) has been proved in [49, Appendix B]. Here we consider the general case where \( 1 < p < \infty \).

Step 1. Let \( J_0 \) and \( \phi_J \) be given in the decomposition of \( J \) in (97). Then
\[
\|\phi_J\|_{W^{1,r}(\Omega)} + \|J_0\|_{L^r(\Omega)} \leq C(\Omega, r)\|J\|_{L^r(\Omega)}.
\]
(116)
(97) is also the decomposition of \( J \) in \( L^q(\Omega, \mathbb{R}^3) \), so the above inequalities remain true if \( r \) is replaced by \( q \). Let \( (u, \phi) \in W^{1,p}(\Omega, \mathbb{R}^3), u^0_T \in L^3(\Omega) \) be a weak solution of (91). From Lemma 6.4, \( \phi = \phi_J \), and \( u \) is a weak solution of (91) with \( J \) replaced by \( J_0 \). Denote \( w = \mathcal{A} \text{curl} u \). Then \( w \in L^p(\Omega, \mathbb{R}^3) \) and
\[
\begin{align*}
\text{curl} w &= J_0, \\
\text{div} (A^{-1}w) &= 0 \quad \text{in} \quad \Omega, \\
\nu \cdot (A^{-1}w) &= \nu \cdot \text{curl} u^0_T \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
By Theorem 2.2 (3) in [32], \( J_0 \) has decomposition
\[
J_0 = \text{curl} w_0 + \nabla \zeta + h_2, \quad w_0 \in W^{1,r}_0(\Omega, \text{div} 0), \quad \zeta \in W^{1,r}_0(\Omega), \quad h_2 \in H_2(\Omega).
\]
\( \Delta \zeta = \text{div} J_0 = 0 \), so \( \zeta = 0 \). \( J_0 \) satisfies the orthogonality condition (99), so \( h_2 = 0 \). Hence
\[
\text{curl} w_0 = J_0, \quad \text{div} w_0 = 0 \quad \text{in} \quad \Omega, \quad \nu \cdot w_0 = 0 \quad \text{on} \quad \partial \Omega.
\]
(117)
Since \( w - w_0 \in W^{1,q}(\Omega, \mathbb{R}^3) \), and \( \text{curl}(w - w_0) = 0 \), by Lemma 2.4 we can write
\[
\begin{align*}
w - w_0 &= \nabla \psi + h_1, \quad \psi \in W^{1,q}(\Omega), \quad h_1 \in H_2(\Omega), \\
\|\nabla \psi\|_{L^q(\Omega)} + \|h_1\|_{L^q(\Omega)} &\leq C(\Omega, q)\|w - w_0\|_{L^q(\Omega)}.
\end{align*}
\]
(118)
Plugging (118) into (116) we have
\[
\begin{align*}
\frac{\text{div} [A^{-1}(\nabla \psi)]}{\frac{\text{div} [A^{-1}(w_0 + h_1)]}{\text{in} \quad \Omega,} \\
\nu \cdot [A^{-1}(\nabla \psi)] &= \nu \cdot \text{curl} u^0_T - \nu \cdot [A^{-1}(w_0 + h_1)] \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
(119)
Since \( \nu \cdot \text{curl } u^0_T \in W^{1-1/q,q}(\partial \Omega) \), we apply the \( L^p \) estimates for oblique derivative problem (see [34, Chapter 6]) to get
\[
\psi \in W^{2,q}(\Omega), \text{ so } w = w_0 + \nabla \psi + h_1 \in W^{1,q}(\Omega, \mathbb{R}^3).
\] (120)

Now curl \( u = A^{-1}w \in W^{1,q}(\Omega, \mathbb{R}^3) \). Using (16) with \( k = 1 \) we have \( u \in W^{2,q}(\Omega, \mathbb{R}^3) \),
\[
\|u\|_{W^{2,q}(\Omega)} \leq C(\Omega,q)\{\|A^{-1}w\|_{W^{1,q}(\Omega)} + \|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\partial \Omega)}\}. \tag{121}
\]

Next we derive more precise estimate on \( u \). Applying the \( L^p \) estimates of oblique derivative problems to (119) we have
\[
\|\nabla \psi\|_{W^{1,q}(\Omega)} \leq C(\Omega,q)\{\|\nabla \psi\|_{L^q(\Omega)} + \|\text{div } [A^{-1}(w_0 + h_1)]\|_{L^q(\Omega)} + \|\nu \cdot \text{curl } u^0_T\|_{W^{1,q}(\partial \Omega)}\}.
\]

Using (122) and by proof by contradiction, we can show that, for the weak derivative problems to (119), we have
\[
\|u\|_{W^{2,q}(\Omega)} \leq C(\Omega,q)\{\|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\partial \Omega)}\}.
\]

From these inequalities, (6) and (118) we get
\[
\|\nabla \psi\|_{W^{1,q}(\Omega)} \leq C\{\|w - w_0\|_{L^q(\Omega)} + \|J_0\|_{L^q(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}\}.
\]

Then using (120) we get
\[
\|u\|_{W^{2,q}(\Omega)} \leq C\{\|w - w_0\|_{L^q(\Omega)} + \|J_0\|_{L^q(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}\}.
\]

Plugging this back to (121), we get
\[
\|u\|_{W^{2,q}(\Omega)} \leq C\|w\|_{L^q(\Omega)} + \|J_0\|_{L^q(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}.
\]

Then using \( \|w\|_{L^q(\Omega)} \leq \|A\|_{C^0(\Omega)} \|u\|_{W^{1,q}(\Omega)} \) and using interpolation, we get
\[
\|u\|_{W^{2,q}(\Omega)} \leq C_1\{\|J_0\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}\},
\]
where \( C_1 \) depends on \( \Omega, p, r, \|A\|_{C^0(\Omega)}, \|A^{-1}\|_{C^1(\Omega)} \), and the constants in (A0). So we get (114).

**Step 2.** If \( (u, \phi) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega) \) is a weak solution of (91) with \( u \) satisfying the \( H^2(\Omega) \) orthogonality condition (112) in \( W^{2,q}(\Omega, \mathbb{R}^3) \), letting \( \mathcal{U} \in W^{2,q}(\Omega, \mathbb{R}^3) \) satisfy (111) and (112), and letting \( v = u - \mathcal{U} \), then \( (v, \phi) \in W^{2,q}_0(\Omega, \mathbb{R}^3) \times L^q(\Omega) \) is a weak solution of (106) with \( g = \mathcal{A} \text{curl } \mathcal{U} \in W^{1,q}(\Omega, \mathbb{R}^3) \), and \( v \) is orthogonal to \( H^2(\Omega) \). Applying (114) we have
\[
\|v\|_{W^{2,q}(\Omega)} \leq C_1\{\|J - \text{curl } g\|_{L^q(\Omega)} + \|v\|_{L^q(\Omega)}\}. \tag{122}
\]

Using (122) and by proof by contradiction, we can show that, for the weak solutions \( (v, \phi) \) of (106) with \( v \) orthogonal to \( H^2(\Omega) \),
\[
\|v\|_{L^q(\Omega)} \leq C_3\{\|J\|_{L^q(\Omega)} + \|g\|_{W^{1,q}(\Omega)}\}, \tag{123}
\]
where \( C_3 \) depends on \( \Omega, p, r, \mathcal{A} \) and the constants in (A0) and in (112), but is independent of \( J, g \) and the solutions.

From (122) and (123) we find
\[
\|v\|_{W^{2,q}(\Omega)} \leq C\{\|J\|_{L^q(\Omega)} + \|g\|_{W^{1,q}(\Omega)}\} \leq C_2\{\|J\|_{L^q(\Omega)} + \|u^0_T\|_{W^{2-1/q,q}(\partial \Omega)}\},
\]
where \( C_2 \) depends only on \( C_1, C_3 \) and the constant in (112). \qed
The proof of Proposition 6 actually gives the following $L^p$ estimate for (91):

**Corollary 4.** Assume $\Omega, A$ satisfy the conditions in Proposition 6, $J \in L^p(\Omega, \mathbb{R}^3)$, $u^0_\tau \in \mathcal{W}^{2-1/r, r}(\partial \Omega, \mathbb{R}^3)$, and $(u, \phi) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega)$ is a weak solution of (91) with $1 < p < r$. Then $u \in W^{2,r}(\Omega, \mathbb{R}^3)$, $\phi \in W^{0,1}_0(\Omega)$, and the estimates (114) holds with $q$ replaced by $r$.

**Proposition 7.** Assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^{3+\alpha}$ boundary, $0 < \alpha < 1$, $A \in C^{1+\alpha}(\Omega, S^+(3))$ satisfies (A0), $J \in C^\alpha(\Omega, \mathbb{R}^3)$ and $u^0_\tau \in TC^{2+\alpha}(\partial \Omega, \mathbb{R}^3)$. Let $(u, \phi) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega)$ be a weak solution of (91) with $1 < p < \infty$. Then $u \in C^{2+\alpha}(\Omega, \mathbb{R}^3)$, $\phi \in C^{1+\alpha}(\Omega)$,

\[
\|u\|_{C^{2+\alpha}(\Omega)} \leq C_2 \{\|J\|_{C^\alpha(\Omega)} + \|u\|_{L^p(\Omega)} + \|u^0_\tau\|_{C^{2+\alpha}(\partial \Omega)}\}, \\
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Corollary 5. Assume $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a $C^2$ boundary, $A$ and $B$ satisfy $(A_0)$ and $(B_0)$ respectively, and $f(x, u) = \nabla u F(x, u)$, where $F$ satisfies $(F^p)$ with $p = 2$, and $u_0^p \in TH^{1/2}(\partial \Omega, \mathbb{R}^3)$. Then (127) has a weak solution

$$(u, \phi) \in H^1_0(\Omega, \text{div} \ 0) \times H^1_0(\Omega)$$

As mentioned in [49, p.13, footnote 8], using variational methods, existence of solutions to (127) can be obtained for the Maxwell-Stokes system with more general nonlinearities. Let us consider

$$F \in C^0_{\text{loc}}(\Omega \times \mathbb{R}^3), \quad f(x, u) = \nabla u F(x, u) \in C^0_{\text{loc}}(\Omega \times \mathbb{R}, \mathbb{R}^3)$$

with $f(0, 0) = 0$. We apply Lemma 4.6 with $E$ having structure. By the Mountain-pass lemma [3] (also see [60, p.109, Theoren II.6.1]) we know that $E$ has a non-zero critical point $u \in X$. Since $u \in H^1(\Omega, \mathbb{R}^3) \subset L^6(\Omega, \mathbb{R}^3)$, by $(F)$ we have $F(x, u(x)) \in L^r(\Omega, \mathbb{R}^3)$ with $r = 6/(p-1) \geq 6/5 = (2r)'$. We apply Lemma 4.6 with

$$U(x) = B(x)u(x) - f(x, u(x)) \in L^r(\Omega, \mathbb{R}^3)$$

to get $\phi \in W^{1,r}_0(\Omega)$ such that $(u, \phi)$ is a weak solution of (127).
By conditions \((A_0)\) and \((B_0)\), the following eigenvalue problem
\[
\begin{cases}
\text{curl}(A \text{curl } v) + B v = \lambda v, & \text{div } v = 0 \quad \text{in } \Omega, \\
v_T = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a sequence of eigenvalues \(\{\lambda_n\}_{n=1}^{\infty}\) with 0 < \(\lambda_1 \leq \lambda_2 \leq \cdots\), each has finite multiplicity, and \(\lambda_n \to \infty\). So when \(F(x, u)\) is even in \(u\), we can use the method in the proof of [60, p.112, Theorem II.6.6] and use Lemma 4.6 to get the conclusion. \(\square\)

Regularity of the weak solutions to \((127)\) with \(|f(x, u)| \leq C(1 + |u|^p)\) with 1 < \(p \leq 3\) has been proved in [49, Proposition3.4]. Now we prove a regularity result for all \(1 < p < 6\).

**Proposition 8.** Assume \(\Omega\) is a bounded domain in \(\mathbb{R}^3\) with a \(C^3\) boundary, \(A \in C^1(\Omega, S^+(3))\) satisfies \((A_0)\), \(B\) satisfies \((B_0)\),
\[f(x, u) = \nabla u F(x, u),\]
where \(F\) satisfies \((F)\). Let \((u, \phi) \in H^2(\Omega, \text{div } 0) \times W^{1, r}(\Omega), r = 6/(p - 1)\), be the weak solution of \((128)\) obtained in Theorem 7.1. Then
\[u \in W^{2, q}(\Omega, \mathbb{R}^3) \cap C^{1+\beta}(\bar{\Omega}, \mathbb{R}^3), \quad \phi \in W^{1, q}_0(\Omega) \cap C^\beta(\bar{\Omega})\]
for any \(1 < q < \infty\) and \(0 < \beta < 1\).

If furthermore \(\Omega\) has \(C^{3+\alpha}\) boundary, \(A \in C^{1+\alpha}(\bar{\Omega}, S^+(3))\) and \(f \in C^\alpha_{\text{loc}}(\Omega \times \mathbb{R}^3, \mathbb{R}^3)\), where \(0 < \alpha < 1\), then \(u \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)\), \(\phi \in C^{1+\alpha}(\bar{\Omega})\).

**Proof.** Set \(J(x) = f(x, u(x))\). As in the proof of Theorem 7.1 we know that \(J \in L^r(\Omega, \mathbb{R}^3)\) with \(r = 6/(p - 1)\). Applying Corollary 4 we know that \(u \in W^{2, r}(\Omega, \mathbb{R}^3)\). We repeat this process \(k\) times to find that \(u \in W^{2, r_k}(\Omega, \mathbb{R}^3)\) and \(\phi \in W^{1, r_k}_0(\Omega)\), with \(r_k \geq 3/2\). Then we have \(u \in W^{1, q}(\Omega, \mathbb{R}^3)\) and \(\phi \in W^{1, q}_0(\Omega)\) for any \(1 < q < \infty\), hence \(u \in C^{1+\beta}(\bar{\Omega}, \mathbb{R}^3)\) and \(\phi \in C^\beta(\bar{\Omega})\) for any \(0 < \beta < 1\).

If \(f \in C^\alpha_{\text{loc}}\) and \(u \in C^{1+\beta}\), then \(J \in C^\alpha(\bar{\Omega}, \mathbb{R}^3)\). By Proposition 7 we have \(u \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)\) and \(\phi \in C^{1+\alpha}(\bar{\Omega})\). \(\square\)

Among semilinear Maxwell type systems, the model of Meissner states of superconductivity is an interesting system
\[
\begin{cases}
-\lambda^2 \text{curl } A = (1 - |A|^2)A & \text{in } \Omega, \\
\lambda (\text{curl } A)_T = H_T^c & \text{on } \partial \Omega,
\end{cases}
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^3\). Note that \((129)\) does not have the divergence-free requirement. In fact the solution \(A\) of \((129)\) is unique if \(H_T^c\) is small [10], so we can not require \(\text{div } A = 0\). On the other hand, for small solutions \((129)\) can be transferred to a system with divergence-free condition. In fact, if \(A\) is a solution of \((129)\) and
\[\|A\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{2}},\]
then \(H = \lambda \text{curl } A\) is a divergence-free solution of the following
\[
\begin{cases}
-\lambda^2 \text{curl } [F(\lambda^2 |\text{curl } H|^2)\text{curl } H] = H & \text{in } \Omega, \\
H_T = H_T^c & \text{on } \partial \Omega, \\
\lambda |\text{curl } H|_{L^\infty(\Omega)} < \frac{2}{\sqrt{27}},
\end{cases}
\]
where \( F(u) \) is a positive valued function defined on \([0, 2/\sqrt{27}]\). When \( \Omega \) is simply-connected and without holes, existence of the solution to (130) with small boundary data is proved in [41]. Existence and regularity of the solution for boundary data under the optimal bound

\[
\mathcal{H}^r_T \in C^{2+\alpha}(\partial \Omega, \mathbb{R}^3), \quad \|\mathcal{H}^r_T\|_{C^{\alpha}(\partial \Omega)} < \sqrt{\frac{5}{18}},
\]

is obtained in [10], and the concentration behavior of the solution as \( \lambda \to 0 \) is also established. The restriction for the domain to be simply-connected and without holes is removed in [35], hence the results in [10] remain true for a general bounded domain. More precise results of the location of concentration points is proved in [70], and the full model of Meissner states are studies in [44, 50]. These results for (130) give the corresponding results for (129).

However (129) and (130) are not equivalent for large solutions with \( \|A\|_{L^\infty(\Omega)} > 1/\sqrt{3} \).

**Question 7.2.** Let \( \Omega \) be a bounded domain \( \Omega \subset \mathbb{R}^3 \).

(i) Study existence and regularity of solutions to (129) such that

\[
\frac{1}{\sqrt{3}} < \max |A(x)| < 1.
\]

(ii) Study existence and regularity of solutions to (129) satisfying \( \max |A(x)| = 1 \).

Regarding question (i), existence of solutions on radially symmetric domains has been obtained in [69], while the problem for a general domain remains open. Question (ii) is related to the vortex solutions of Ginzburg-Landau equations for superconductivity [17], and we guess that there may exist solutions which satisfy \( \max |A(x)| = 1 \) and have singularities. This could happens in the 2-dimensional problem. In fact, the vector field in [52, p.5037, (1.12)] satisfies \( |v(x)| = 1 \) and \( \text{curl} \ v = 2 \sin \phi_0 \), a constant, hence \( v \) is a solution of the corresponding 2-dimensional problem on the disc, but \( v \) does not lie in \( H^1(\Omega, \mathbb{R}^2) \).

It is interesting to compare (129) with the following equation

\[
\lambda^2 \text{curl}^2 u = (1 - |u|^2) u \quad \text{in} \ \Omega, \quad u_T = u_0^T \quad \text{on} \ \partial \Omega,
\]

which can be viewed as an approximation of the relaxed curl-minimization problem

\[
\inf_{u \in \mathcal{H}(\Omega, S^2, u_0^T)} \int_\Omega |\text{curl} u|^2 dx,
\]

\[
\mathcal{H}(\Omega, S^2, u_0^T) = \{ u \in \mathcal{H}(\Omega, \text{curl}) : |u(x)| = 1 \ \text{a.e.,} \ u_T = u_0^T \ \text{on} \ \partial \Omega \}.
\]

The original curl-minimizing problem

\[
\inf_{u \in \mathcal{W}(\Omega, S^2, u_0)} \int_\Omega |\text{curl} u|^2 dx,
\]

\[
\mathcal{W}(\Omega, S^2, u_0) = \{ u \in \mathcal{H}(\Omega, \text{curl}) : |u(x)| = 1 \ \text{a.e.,} \ u = u_0 \ \text{on} \ \partial \Omega \},
\]

was proposed in the study of the asymptotic limit of nematic liquid crystals [52, p.5037] and [43, p.380, Problem 4], and very little is known.

In recent years, various semilinear curl systems have been studied, see for instance [61, 62, 11, 37, 9, 63, 73] and the references therein.

**Remark 7.** In this paper we require the domain to have a \( C^2 \) or \( C^{2+\alpha} \) boundary. It will be interesting to study these nonlinear problems on Lipschitz domains, which is interesting both in mathematics and in engineering.
Appendix A. List of notations.

\[ F(\Omega), \ G(\partial \Omega) \]  
\[ \mathcal{H}(\Omega, \text{div}), \ \mathcal{H}(\Omega, \text{curl}) \]  
\[ \mathbb{H}_1(\Omega), \ \mathbb{H}_2(\Omega), \ \mathbb{H}_1^p(\Omega), \ \mathbb{H}_2^p(\Omega) \]  
\[ H(\Omega, \text{div}), \ H(\Omega, \text{curl}) \]  
\[ H^1(\Omega), \ H^2(\Omega), \ H^p_1(\Omega), \ H^p_2(\Omega) \]  
\[ H^\Gamma(\Omega, \text{div} 0), \ H^\Sigma_0(\Omega, \text{div} 0) \]  
\[ H^\Gamma(\Omega, \text{div} 0), \ H^\Sigma_0(\Omega, \text{div} 0), \ L^p(\Omega, \Delta), \ L^p(\Omega, \Delta 0) \]  
\[ M(3), \ S^0(3), \ S^+(3) \]  
\[ TC^{k+\alpha}(\partial\Omega, \mathbb{R}^3), \ TW^{s,p}(\partial\Omega, \mathbb{R}^3), \ TW^{-s,p'}(\partial\Omega, \mathbb{R}^3) \]  
\[ W^{2,p}_{00}(\Omega), \ W^{k,p}_{00}(\Omega, \mathbb{R}^3), \ W^{1,p}(\Omega, \mathbb{R}^3, \mathbf{n}_\Gamma^0) \]  
\[ W^{k,p,\star}_{00}(\Omega, \mathbb{R}^3), \ W^{k,p,\star}(\Omega, \mathbb{R}^3) \]  
\[ W^{k,s}(\Omega, \text{curl}), \ W^p(\Omega, \text{div}) \]  
\[ W^{k,s}(\Omega, \Gamma), \ W^{k,s}(\Omega, \Gamma) \]  
\[ X(\Omega, \text{curl} 0), \ X(\Omega, \text{div}), \ X_{\partial\Omega}(\Omega), \ X_{n_0}(\Omega) \]  
\[ \langle \zeta, \eta \rangle_{\partial\Omega, 1/p} \]  
conditions \((A_0)\)  
conditions \((f)\), \((H)\)  
conditions \((c)\), \((F^p)\), \((P^p)\), \((U_0)\)  
conditions \((a)\), \((g)\)  
conditions \((B_0)\), \((F)\)

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Received for publication February 2019.

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