Boundary behaviour of RW’s on planar graphs and convergence of LERW to chordal $\text{SLE}_2$

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Abstract

This paper concerns a random walk on a planar graph and presents certain estimates concerning the harmonic measures for the walk in a grid domain which estimates are useful for showing the convergence of a LERW (loop-erased random walk) to an SLE (stochastic Loewner evolution). We assume that the walk started at a fixed vertex of the graph satisfies the invariance principle as in Yadin and Yehudayoff [16] in which the convergence of LERW to a radial SLE is established in this setting. Our main concern is chordal case, where a random walk is started at a boundary vertex of a simply connected grid domain and conditioned to exit it through another boundary vertex specified in advance. The primary contribution of the present paper is an estimate, which states that the excursion of the conditioned walk leaves an intrinsic neighborhood of its initial point not ‘along’ the boundary but through an intrinsic interior of the domain with high probability. Based on this result we give a proof for the convergence to the chordal SLE, a result that has recently been proved by Suzuki [12] under an analyticity assumption on the boundary of the domain arising in the limit.
1 Introduction

This paper concerns a random walk on a planar graph imbedded in the plane and provides certain estimates concerning the harmonic measures for the walk in a domain. The estimates obtained are used for showing the convergence of a loop-erased random walk (LERW) to a stochastic Loewner evolution (SLE). Our essential hypothesis is that the walk started at one fixed vertex of the planar graph satisfies the invariance principle (as in [16]): properly scaled trajectory of it weakly converges to that of the planar Brownian motion with respect to a metric which disregards the difference of time parametrization. We do not assume the symmetry of the random walk, while the planarity of the graph plays an essential role as in [16].

The loop-erased random walk is a process obtained by erasing loops one by one from a random walk on a graph in chronological order. It was introduced by Lawler [3] as a version of self-avoiding random walk focusing the central limit behavior (functional limit theorem) in dimensions $\geq 4$, and there have appeared many works studying various aspects of it ([2], [7], [5], [8], [9], [10], etc.).

The stochastic—or Schramm—Loewner evolutions (SLE$_\kappa$) are a family of random trajectories obtained as a solution of the Loewner differential equation (in the complex plane) driven by the process $\sqrt{\kappa}W(t)$, where $W$ is a one-dimensional standard Brownian motion and $\kappa$ is a...
positive parameter. The SLE’s are introduced by Schramm in [13], in which it is conjectured, based on ample evidence provided by foregoing works and his paper itself, that the scaling limit of LERW on \( \mathbb{Z}^2 \) must be SLE\(_2\).

This conjecture by Schramm is proved by Lawler, Schramm and Werner [6] where a scaled LERW on some regular lattices is shown to converge to a radial SLE\(_2\), a version of SLE path that starts from an interior point of the domain and ends up at a boundary point of it. Dapeng Zhan [17] have studied LERW’s on the square lattice in a multiply connected domain and proved the existence of their scaling limit; in the case of a simply connected domain in particular, he has proved the convergence to a chordal SLE\(_2\), another version of SLE path that travels from boundary to boundary. Yadin and Yehudayoff [16] extend the result of [6], the convergence of LERW to a radial SLE to that for the natural random walks on planar graphs under a natural setting (the same as ours) of the problem. Recently Suzuki [12] have obtained a chordal version of their result: the LERW in a simply connected domain conditioned to connect two boundary vertices converges to a chordal SLE\(_2\) curve in a setting similar to [16] under the assumptions (1) the invariance principle holds uniformly for starting points of the walk and more seriously (2) the boundary of the domain is locally analytic at the starting boundary point of the random walk from whose trajectory (or rather its time reversal) the LERW is derived.

In this paper we are concerned with the chordal case of LERW on a planar graph as in [12]. For the chordal case of LERW we need unlike the radial case to deal with an excursion of random walk (a random walk path in a domain connecting a boundary point with another one), and we are forced to estimate the harmonic measure of the random walk started at a vertex on (or near) the boundary and conditioned to exit the domain through another boundary vertex that is specified in advance. The SLE\(_2\) curve is conformally invariant, to which we intend to show the LERW’s obtained from such excursions scaled by the sizes of the domains converge. The approximation must accordingly be effected uniformly for the domain as far as its sizes (measured by the inner radius with respect to an appropriately chosen point) is large enough and in order to obtain such uniformity we wish to find a certain estimate concerning the distribution of ‘entrance’ of the excursion into the substantial interior of the domain. The primary contribution of the present paper is such an estimate (Proposition 4.6) that states that the excursion leaves an intrinsic neighborhood of its initial point not ‘along’ the boundary but through an intrinsic interior of the domain with high probability. The result plays a key role in our verification of the convergence of LERW to the chordal SLE: it refines the result of [12] by removing the second extra assumption mentioned above.

The rest of the paper is organized as follows. In Section 2 we introduce our planar graph and the random walk on it, and then state Hypothesis (H), our basic assumption in this paper, and derive a lemma from it by using a key result of [16]. In Section 3 we discuss some geometric properties of a conformal map from a simply connected domain onto the unit disc and derive preliminary facts used in the next section. Section 4 consists of 5 subsections and provides various estimates concerning the hitting distribution of the random walk; Proposition 4.6, the primary result of the paper, is proved in the third subsection of it. Section 5 concerns the convergence of a LERW on the planar graph to a chordal SLE\(_2\). We break this section into
four subsections. A brief review of the chordal SLE in $\mathbb{H}$ is provided in Section 5.1. In Section 5.2 we present an account of the chordal SLE in a simply connected domain together with some facts concerning it. The statement of the convergence result together with its abridged proof is given in the last two subsections.

## 2 Planar graph and Hypothesis (H)

Here we introduce our random walk on a planar graph as well as Hypothesis (H), and discuss on the fundamental facts on the harmonic measures of the scaled walk. The setting is the same as in [16]. We formulate a key idea used in it as a lemma that is convenient to apply in the rest of the present paper.

### 2.1 Planar graph and random walks on it.

For any $x, y \in \mathbb{C}$, we write $[x, y]$ for $\{(1 - t)x + ty : 0 \leq t \leq 1\}$, the directed line segment emanating from $x$ and ending at $y$. (The same square brackets is used to designate a closed interval of $\mathbb{R}$, but this will cause no confusion.) Let $V$ be a countable subset of $\mathbb{C}$. Let $p : V \times V \mapsto [0, 1]$ be such that $\sum_{v \in V} p(u, v) = 1$ and put $E = \{(u, v) : p(u, v) > 0\}$. A pair $(u, v) \in E$ is identified with the directed segment $[u, v]$ and is called an edge. The pair $G = (V, E)$ may be considered to be a graph of directed edges $[u, v] \in E$ with weight $p(u, v)$. We are concerned with the Markov chain whose transition probability is $p(u, v)$.

In this paper, we assume that the graph $G$ satisfies the following properties.

1. $G$ is a planar graph, namely any two edges are disjoint unless they have at least one common endpoint.

2. For any compact set $K \subset \mathbb{C}$, $\#(K \cap V) < \infty$. ($\#$ designates the cardinality of a set.)

3. The Markov chain $(S_k)_{k=0}^{\infty}$ on $V$ with transition probability $p(u, v)$ is irreducible.

For simplicity we further suppose that $0 \in V$.

We suppose that for each $v \in V$ there is given a random sequence $(S^v_k)_{k=0}^{\infty} \subset V$ defined on a probability space $(\Omega, \mathcal{F}, P)$ that constitutes a Markov chain with transition probability $p(u, v)$ such that $S^v_0 = v$ and that any two sequences with distinct initial vertices are independent. We denote the linear interpolation of $(S^v_k)$ by $(\tilde{S}^v_t)_{t \geq 0}$: $\tilde{S}^v_t$ travels along the edge $[S^v_n, S^v_{n+1}]$ with unit speed for $t \in [n, n + 1]$. For any subset $U$ of $\mathbb{C}$ and for $v \in V$ we define the first exit time of $S^v$ from $U$, denoted by $\tau_U$, as the least positive integer $k$ such that the segment $[S^v_{k-1}, S^v_k]$ contains a point of $\mathbb{C} \setminus U$:

$$\tau_U = \inf\{k \geq 1 : [S^v_{k-1}, S^v_k] \setminus U \neq \emptyset\}.$$ 

If a set $K$ intersects the segment $[S^v_{k-1}, S^v_k]$ for $k = \tau_U$, then $S^v$ is said to exit $U$ through $K$. ($K$ will be contained in the complement of $U$ in our use.) (In the case $v \in \partial U$ we shall
modify the definition of $\tau_U$ in Section 4.3, until that time we shall not encounter the situation where the modification is needed.) Obviously $\tau_U$ depends on $v$ and we sometimes indicate this dependence by writing $\tau^v_U$ but usually do not when it is clear from the context. We also suppose that the standard Brownian motion on $\mathbb{C}$ is defined on $(\Omega, \mathcal{F}, P)$ and denote it by $W_t$ and write $W_t^z = z + W_t$, $z \in \mathbb{C}$ (so that $W_0^z = z$). The first exit time of $W_t^z$ from $U$ will be denoted by $\tau_U^W$: $\tau_U^W = \inf\{t > 0 : W_t^z \notin U\}$.

For a set $A \subset \mathbb{C}$ we write $V(A)$ for $V \cap A$ and denote by $G(A)$ the subgraph of $G$ of which the vertex set is $V(A)$ and the edges are those $(u, v) \in E$ such that $[u, v] \subset A$. A path in $G(A)$ is a finite sequence $u_0, \ldots, u_n$ such that $(u_{k-1}, u_k)$ is an edge of $G(A)$ for each $k = 1, \ldots, n$. If $v \in V(U)$, $\tau_U$ may agrees with the first time when $S^v_n$, considered to be a walk on $G$, exits the subgraph $G(U)$.

2.2. The metric of the path space and hypothesis (H).

Denote by $\mathcal{C}^*$ the space of finite continuous plane curves. Here it is understood that a curve (also called path) is oriented and represented by $\gamma \in C[0, \tau]$, a continuous map of a finite interval $[0, \tau]$ into $\mathbb{C}$, but two such maps are identified if they are transformed to each other by some changes of parametrization that preserve orientation. We consider $\mathcal{C}^*$ as a metric space with the metric $d^*_U$ defined as follows: for $\gamma_j \in C[0, \tau_j]$ ($j = 1, 2$),

$$d^*_U(\gamma_1, \gamma_2) = \inf \sup_{0 \leq t \leq \tau_1} |\gamma_1(t) - \gamma_2 \circ \chi(t)|,$$

where the infimum is taken over all homeomorphisms $\chi : [0, \tau_1] \mapsto [0, \tau_2]$ with $\chi(0) = 0$. The metric space $\mathcal{C}^*$ is separable and complete.

Under this metric we shall consider the convergence of probability measures on $\mathcal{C}^*$ induced by $\tilde{S}^v$ that is stopped on exiting a domain $D$ (i.e., at the time $\tau_D$). Given a map $\gamma \in C[0, \tau]$ with $\gamma(0) \in D$, that represents an element $\mathcal{C}^*$, let $\gamma^D$ denote the restriction of $\gamma$ on $[0, \tau_D \wedge \tau]$ where $\tau_D = \inf\{t \in [0, \tau] : \gamma(t) \notin D\}$ and put

$$d_D(\gamma_1, \gamma_2) = d^*_U(\gamma^D_1, \gamma^D_2).$$

For $h > 0$ we scale our random walk by $h > 0$. The scaling is plainly given by simply multiplying $S_n^v$ by $h$. Thus the scaled walk $S_n^{h,v} := hS_n^v$ ($v \in V$) is the random walk on $hV = \{hv : v \in V\}$ started at $hv$; also write $\tilde{S}^{h,v}$ for the linear interpolation of $S^{h,v}$. We denote the open unit disk by $\mathbb{D}$ and suppose that the walk $S^0$ satisfies invariance principle in the following sense:

$$(H) \left\{ \begin{array}{l}
\text{the law of the scaled walk } \tilde{S}^{h,0} \text{ stopped on exiting } \mathbb{D} \text{ weakly converges to the law of Brownian motion } W^0 \text{ stopped on exiting } \mathbb{D} \text{ as } h \downarrow 0, \text{ where the weak convergence is relative to the metric } d_\mathbb{D}.
\end{array} \right.$$  

In [16] it is deduced from this hypothesis that for any $v \in V$ with $hv \in \mathbb{D}$, the Markov chain $hS^v$ killed on exiting $\mathbb{D}$ behaves like a Brownian motion killed on $\partial \mathbb{D}$ as $h \downarrow 0$ as far as the hitting distributions are concerned. We formulate a consequence of this in the next subsection as Lemma 2.2. All applications of (H) in this paper will be via it or its corollaries.

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2.3. A consequence of Hypothesis (H)

Denote by \( \Gamma_r(a) \) the disc of radius \( r > 0 \) centered at \( a \in \mathbb{C} \):

\[
\Gamma_r(a) = \{ u \in \mathbb{C} : |u - a| < r \}.
\] (2.1)

Let \( r > 0 \) and \( U \) be a domain that contains \( \Gamma_{r'} \setminus \Gamma_r \) for some \( r' > r \). A (continuous) curve \( \gamma \in C([0, \tau]) \) is said to encompass the disc \( \Gamma_r(a) \) in \( U \) if there is a pair \( s < s' \leq \tau \gamma \) such that

1. \( \gamma[0, s'] \subset U, \gamma[s, s'] \subset U \setminus \Gamma_r(a), \gamma(s) = \gamma(s') \) and
2. the argument \( \arg(\gamma(t) - a) \) continuously varies from \( \arg(\gamma(s) - a) \) either to \( \arg(\gamma(s) - a) + 2\pi \) or to \( \arg(\gamma(s) - a) - 2\pi \) as \( t \) increases from \( s \) to \( s' \).

Here \( \gamma[s, s'] \) designates the restriction of \( \gamma \) to \( [s, s'] \). Because of the condition (1) the term ‘encompass’ entails that encompassing is made before exiting \( U \). A random walk encompasses \( \Gamma_r(a) \) in \( U \) if its linear interpolation does. The following result is essentially Proposition 4.1 of [16] that we adapt and modify to the present need and notation.

**Proposition 2.1.** For any \( \varepsilon > 0 \) and \( M \geq 1 \) there exists \( \eta = \eta(\varepsilon, M) > 0 \) such that for any positive number \( r \) one can choose \( h_0 = h_0(\varepsilon, \lambda, M) > 0 \) so that if \( 0 < h < h_0 \) and \( a \in M \mathbb{D} \), then for all \( u \in V(h^{-1} \eta_{\lambda}(a)) \),

\[
P \left[ S^{h,u} \text{ encompasses } \Gamma_{\eta r}(a) \text{ in } \Gamma_r(a) \right] > 1 - \varepsilon.
\]

**Proof.** In our proposition the statement in [16] Proposition 4.1 is modified in two ways. Firstly it is stated for a simply connected domain \( D \) by means of a conformal map \( \varphi_D \). Our proposition is specialized to the case \( D = (M + 1) \mathbb{D} \). Secondly the choice of \( h_0 \) may depend on \( a \) in [16], while it does not in ours. This independence of \( h_0 \) from \( a \) is verified by examining the proof in [16]. (We provide more details in Appendix for the latter.) \( \square \)

In (H) the domain \( \mathbb{D} \) plays no intrinsic role: it may be replaced by any bounded domain containing the origin because of the scaling property of Brownian motion, hence the assertion for \( M > 1 \) follows from that for \( M = 1 \), so we usually state results only for the case \( M = 1 \) in the sequel.

We formulate Lemma 2.2 mentioned previously, in terms not of the scaled walk \( S^{h,u} \) but of \( S^u \) itself for convenience of later applications. Let \( U \) be a domain of \( \mathbb{C} \) and \( K \) a compact set that is contained in the complement of \( U \). For \( u \in V(U) \) and \( 0 < r < \rho \), define

\[
q_{K,U}(u) = P[S^u \text{ exits } U \text{ through } K]
\]

and

\[
q^{(0)}_{K,U}(u; r) = P[S^0 \circ \theta_{\sigma(\Gamma_r(u))} \text{ exits } U \text{ through } K | \sigma_{\Gamma_r(u)} < \tau_{\partial U}],
\]

where \( \sigma_B \) (or \( \sigma(B) \)) is the first hitting time of a set \( B \) by \( S^u \): \( \sigma_B = \sigma_{\mathbb{C} \setminus B} \) and \( \theta_{\sigma} \) denotes the usual shift operator so that \( S^0 \circ \theta_{\sigma} \) is the the Markov chain \( (S^0_{n+\sigma})_{n=0,1,2,...} \).
Lemma 2.2. Let $\eta = \eta(\varepsilon,1)$ be as in Proposition 2.1. Then for any $\varepsilon > 0$ and $\lambda > 1$ one can choose $R > 1$ independently of $U$ and $K$ so that if $\rho \geq R$, then for all $u \in V(\mathbb{D})$ with $\text{dist}(u, \partial U) > r$, 
\[(1 - \varepsilon)q^{(0)}_{K,U}(u; \eta \lambda) < q_{K,U}(u) < (1 - \varepsilon)^{-1}q^{(0)}_{K,U}(u; \eta \lambda).\]

Proof. We omit $K, U$ from $q^{(0)}_{K,U}(v; r)$ and $q_{K,U}(u)$. Since by strong Markov property $q^{(0)}(u; \eta \lambda)$ is a convex sum of $q(v)$ over $v \in \Gamma_{\eta \lambda}(u)$ with $P[S^0_{\sigma(\Gamma_{\eta \lambda}(u))} = v] > 0$, there exist two sites $u^*$ and $u_*$ in $V(\Gamma_{\eta \lambda}(u))$ such that

$q(u_*) \leq q^{(0)}(u; \eta \lambda) \leq q(u^*)$. \hspace{1cm} (2.2)

By the maximal principle applied to the stopped chain $(S^w_{n \wedge \tau(U)})_{n=0,1,...}$ there exists a path $\gamma$ in $G(U)$ such that $q(v) \geq q(u^*)$ for $v \in \gamma$ and $\gamma$ connects $u^*$ with a site outside $\Gamma_{\lambda}(u)$. Then by Proposition 2.1 the walk $S^u$ intersects $\gamma$ before exiting $\Gamma_{\eta \lambda}(u)$ with a probability larger than $1 - \varepsilon$, so that

$q(u) > (1 - \varepsilon)q(u^*)$,

which combined with (2.2) shows the first inequality of the lemma. Repeating the same argument with $u$ and $u_*$ in place of $u^*$ and $u$, respectively, we obtain $q(u_*) > (1 - \varepsilon)q(u)$, hence the second inequality of the lemma.

We may analogously define

$q^W_{K,U}(x) = P[W^x \text{ exits from } U \text{ through } K]$.

Corollary 2.3. Let $\eta = \eta(\varepsilon,1)$ be as in Proposition 2.1 and $u \in V(U)$ be such that $q^W_{K,U}(u) > 0$. Suppose that for any $\alpha > 0$, there exists positive constants $R_0 > 1$ and $1 < \lambda \leq \text{dist}(u, \partial U)$ such that if $\rho \geq R_0$, then $q^W_{K,U}(u) > 0$ and

$1 - \alpha \leq q^{(0)}_{K,U}(u; \eta \lambda)/q^W_{K,U}(u) \leq 1 + \alpha$. \hspace{1cm} (2.3)

Then for any $\alpha > 1$, there exists $R'$ such that for $\rho \geq R'$,

$1 - \alpha \leq q_{K,U}(u)/q^W_{K,U}(u) \leq 1 + \alpha$. \hspace{1cm} (2.4)

Here if $R_0$ is independent of $u, U$ and $K$ (when these vary in any fashion), then so is $R$. If for some constant $c$

$q^{(0)}_{K,U}(u; \eta \lambda)/q^W_{K,U}(u) < c$ \hspace{1cm} (2.5)

in place of (2.3), then $q_{K,U}(u)/q^W_{K,U}(u) < 2c$ in place of (2.4).

Remark 1. The shifted walk $S^0 \circ \theta_{\sigma(\Gamma_{\eta \lambda}(u))}$ behaves like Brownian motion $W^u$ (under scaling by $1/\rho$) as long as they are kept away from the boundary of $U$ with some sufficient distance. In order to ensure condition (2.3) or something like that we need some condition for the pair $K$ and $U$. If $U$ is nice, such a condition will be satisfied for any $K$. As is discussed in the next section we are concerned with a conformal map $\varphi_D$ from a simply connected domain
Lemma 2.4. for simplicity we verify the following

Proof. A

Lemma 2.4. For any \( \varepsilon > 0 \) there exists a positive number \( \eta = \eta(\varepsilon) < 1/2 \) such that for any positive number \( \lambda < 1/2 \) one can choose \( h_0 = h_0(\varepsilon, \lambda) > 0 \) so that if \( h < h_0 \) and \( z \in (1 - \lambda) \mathbb{D} \), then

\[
P \left[ S^{h,0} \circ \theta_{\sigma(\Gamma_{\eta\lambda}(a))} \text{ encompasses } \Gamma_{\eta\lambda}(a) \text{ in } \Gamma_{\lambda}(a) \mid \sigma_{\eta\lambda}(a) < \tau_{D} \right] > 1 - \varepsilon.
\]

Proof. For each \( s \in (1/2, 1) \) let \( A_s \) and \( B_s \) denote the events defined by

\[
A_s = \text{[Brownian motion } W^0 \text{ hits } \Gamma_{s\eta\lambda}(a) \text{ before exiting } s \mathbb{D}],
\]

\[
B_s = \text{[Brownian motion } W^0 \text{ hits } \Gamma_{s\eta\lambda}(a) \text{ before exiting } s \mathbb{D}].
\]

where \( \sigma_W^\Gamma \) denotes the first hitting time of \( \Gamma \) by \( W^0 \), and put \( p(s) = P[A_s] \) and \( q(s) = P[B_s | A_s] \). Then \( q(1) \geq 1 - \varepsilon/3 \) if \( \eta = \eta(\varepsilon) \) is chosen small enough, and \( p(s)/p(1/s) \uparrow 1 \) and \( q(s) \rightarrow q(1) \) as \( s \uparrow 1 \). Fix \( s < 1 \) so that \( p(s)/p(1/s) \geq 1 - \varepsilon/4 \) and \( q(s) \geq 1 - \varepsilon/4 \), Noting that the boundaries of both events \( A_s \) and \( B_s \) are null we apply the assumed invariance principle (H) to see that our random walk \( S^0 \) and the Brownian motion \( W^0 \) can be both defined on the same probability space so that if the event \( C_h \) is defined by

\[
C_h = [d_\mathbb{D}(\tilde{S}^{h,0}, W^0) < (1 - s)\eta \lambda],
\]

then for some \( h_0 = h_0(\varepsilon, \lambda) \)

\[
P[C_h] \geq 1 - \frac{1}{4}\varepsilon P[A_s \cap B_s] \quad 0 < h < h_0. \tag{2.6}
\]

In the definitions of \( A_1 \) and \( B_1 \) replace \( W^0 \) by \( S^{h,0} \) and let \( A_1^{RW} \) and \( B_1^{RW} \) be the corresponding events. Then, from (2.6) it follows that

\[
P[B_1^{RW} \cap A_1^{RW}] \geq P[B_s \cap A_s \cap C_h] \geq (1 - \varepsilon/4)P[B_s \cap A_s] = (1 - \varepsilon/4)q(s)p(s)
\]

and

\[
P[A_1^{RW}] \leq P[A_1^{RW} \cap C_h] + (1 - P[C_h]) \leq P[A_1/s] + 4^{-1}\varepsilon P[A_s] \leq (1 + \varepsilon/4)p(1/s);
\]

hence the probability in question is bounded from below by \( (1 + \varepsilon/4)^{-1}(1 + \varepsilon/4)^3 \geq 1 - \varepsilon \) as required.
3 Preliminary lemmas of geometric nature

Let $D$ be a simply connected domain of the complex plane and $\hat{o}$ and $v_0$ be fixed points of $D$ and $V(\partial D)$, respectively. By the Riemann mapping theorem there exists a (unique) conformal map of $\varphi_D$ onto $\mathbb{D}$ such that

$$\varphi_D(\hat{o}) = 0 \quad \text{and} \quad \varphi_D(v_0) = 1. \quad (3.1)$$

To be precise $v_0$ must be understood to be a prime end of $D$: otherwise $v_0$ may correspond multiple points of the unit circle $\partial U$, hence multiple $\varphi_D$’s, and in such a case it is understood that any one of them is selected. (For instance if $D$ is an upper half plane with a slit $[0, i]$, every point $s \in [0, 1)$, should be counted twice. It is known that any conformal map of $D$ onto $\mathbb{D}$ naturally induces one to one correspondence between the set of prime ends of $D$ and $\partial \mathbb{D}$ [11], in particular if $D$ is a Jordan domain, the prime ends are identified with the boundary points. For more details see Section 5.4; until there we shall not encounter any serious problem that necessitates to use the concept of prime ends.)

In this section we collect certain simple geometric relations between the subsets of $D$ and their images $\subset \mathbb{D}$ by $\varphi_D$. Although the planar graph $G$ is irrelevant to the analysis of this section, the results obtained have consequences on the random walk on it which are included in this section as Corollaries 3.3 and 3.5.

For any non-empty set $A \subset \mathbb{C}$, denote by $\text{dist}(A, B)$ the distance between $A$ and another $B \neq \emptyset$, by $\text{diam} A$ the diameter of $A$ and by $\text{in-rad}_x A$ the inner radius of $A$ with respect to $x \in A$: $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$, $\text{diam} A = \sup\{|x - y| : x, y \in A\}$ and $\text{in-rad}_x A = \text{dist}(x, \mathbb{C} \setminus A)$. We continue to denote by $\Gamma_r(a)$ the open disc of radius $r$ centered at $a$ (as defined in (2.1)).

3.1 Elementary bounds on distortion under $\varphi_D$

Put

$$\rho_D = \text{in-rad}_\hat{o} D = \text{dist}(\hat{o}, \mathbb{C} \setminus D).$$

A version of the Koebe distortion theorem [11 Corollary 1.4] says that if $x \in D$ and $\delta = 1 - |\varphi_D(x)|$,

$$\delta(2 - \delta)/4 \leq |\varphi_D'(x)|\text{dist}(x, \partial D) \leq \delta(2 - \delta). \quad (3.2)$$

Taking $x = 0$, this gives $\rho_D \leq 1/|\varphi_D'(\hat{o})| \leq 4\rho_D$ and, employing another form of the Koebe distortion theorem [11 Theorem 1.3], we obtain that for $0 < r \leq 1$,

$$\varphi_D^{-1}(r\mathbb{D}) - \hat{o} \supset \frac{r}{(1 + r)^2\varphi_D'(\hat{o})} \mathbb{D} \supset \frac{1}{4r\rho_D}\mathbb{D};$$

similarly for $0 < r < 1$, $\varphi_D^{-1}(r\mathbb{D}) - \hat{o} \subset [4r\rho_D/(1 - r)^2]\mathbb{D}$.

**Lemma 3.1.** There exists a positive increasing function $\kappa(\ell)$ on the interval $0 \leq \ell < 1$ such that if $K \subset D$ is a compact connected set and $\text{diam} \varphi_D(K) \geq \ell$, then $\text{diam} K \geq \kappa(\ell)\rho_D$, entailing that if a line segment $[x, y]$ is contained in $\overline{D}$ and $|x - y| < \kappa(\ell)\rho_D$, then $|\varphi_D(x) - \varphi_D(y)| \leq \ell$. It in particular follows that if $z \in D$ and $1 - |\varphi_D(z)| \geq \delta$, then

$$\text{dist}(z, \partial D) \geq \kappa(\delta)\rho_D \quad \text{and} \quad |\varphi_D'(z)| \leq [(2 - \delta)\delta/\kappa(\delta)]/\rho_D.$$
Proof. The distortion theorem says that \(|(\varphi_D^{-1})'(z)\varphi_D'(\hat{o})| \geq (1 - |z|)/(1 + |z|)^3 \geq \delta/8\) for \(|z| < 1 - \delta\), which combined with the inequality \(\rho_D \leq 1/|\varphi'_D(\hat{o})|\) shows the first bound of the lemma (with \(\kappa(\ell) = \ell\delta/8\)) if \(\varphi_D(K) \subset (1 - \delta)\mathbb{D}\). Consider the case \(\varphi_D(K) \not\subset (1 - \delta)\mathbb{D}\). We suppose \(\varphi_D(K) \not\subset \frac{1}{2}\mathbb{D}\) for definiteness. Let \(r_*\) be the infimum of \(r\) such that \(\varphi_D(K) \subset r\mathbb{D}\) and put \(D_* = \varphi_D^{-1}(r_*\mathbb{D})\). Then \(r_* \geq 1/2\), hence in-rad\(\delta)(D_*) \geq \frac{1}{4}r_*\rho_D \geq \frac{1}{8}\rho_D\) and from the Beurling estimate it follows that the harmonic measure of \(K\) in \(D_* \setminus K\) from \(\hat{o}\) is bounded above by \(c'\sqrt{\rho_D^{-1}\text{diam}\ K}\) with a universal constant \(c' > 0\) ([4, Corollary 3.78 (the second formula)]), whereas the same harmonic measure is bounded below by a positive multiple of \(\text{diam}\ \varphi_D(K)\) owing to the conformal invariance of the harmonic measure, since \(\varphi_D(K)/r_*\) is a connected subset of the closed unit disc \(\mathbb{D}\). This entails that

\[
\text{diam}\ \varphi_D(K) \leq c''\sqrt{\rho_D^{-1}\text{diam}\ K},
\]

which gives the required inequality with \(\kappa(\ell) = (\ell/c'')^2\).

Remark 2. We may take \(\kappa(\delta) = c\delta^2\) with a universal constant \(c > 0\) as is indicated in the proof, but we do not need it in this paper.

3.2. Domains \(B_r(x), U_\delta\) and \(Q(x)\).

For \(a \in D\) and \(r > 0\) let \(B_r(a) = \varphi_D^{-1}(\Gamma_r(a)), \) namely

\[
B_r(a) = \{u \in D : |\varphi_D(u) - \varphi_D(a)| < r\}.
\]

For \(\delta > 0\) put

\[
U_\delta = U_\delta^{D,v_0} = \{x \in D : \text{dist}(\varphi_D(x), \partial\mathbb{D}) < \delta\}
\]

and

\[
U_{r,\delta} = U_\delta \cap B_{r+\delta}(v_0).
\]

By Lemma 3.1

\[
\text{dist}(\partial U_\delta, \partial D) \geq \kappa(\delta)\rho_D. \tag{3.3}
\]

We adapt a method found in [4]. Given \(\delta > 0\) and \(x \in D \cap \partial U_\delta\) and let \(z^* = z^*(x, \delta)\) be a point of \(\partial D\) closest to \(x\) and set \(r = \text{dist}(x, \partial D) = |z^* - x|\). For \(0 < \kappa < 2\), let

\[
Q_\kappa = Q_\kappa(x, D) = \text{the connected component of } \Gamma_{\kappa r}(z^*) \cap D
\]

which contains a point of the segment \([z^*, x]\).

We write \(Q(x)\) for \(Q_1 = Q_1(x, D)\). The following result (as well as its proof) is a simple modification of that found in the proof of Lemma 5.4 in [6]. (The modification, although not substantial at all, make simpler and clearer the arguments developed later.)

Lemma 3.2. Let \(Q(x) = Q_1(x, D)\) be defined as above, \(\omega\) the component of \(\partial Q(x) \cap D\) containing \(x\) and \(D(x)\) the component of \(D \setminus \omega\) that does not contain \(\hat{o}\). There exists a universal constant \(m > 1\) such that if \(0 < \delta < 1/m\) and \(x \in D \cap \partial U_\delta\), then \(\omega \cup \Gamma_{r/4}(x)\) and \(\partial B_{m\delta}(x) \cap D\) are disjoint; in particular only the following two alternatives are possible:

1. \(Q(x) \subset D(x);\)
2. \(Q(x) \subset D \setminus D(x);\)

in either case

\[
D(x) \cup \Gamma_{r/4}(x) \subset B_{m\delta}(x). \tag{3.4}
\]
Proof. By the Koebe 1/4 theorem we have $i\cdot \text{rad}_xB_\delta(x) \geq \frac{1}{4}(2-\delta)/|\varphi_D(x)|$, which combined with \([3.2]\) gives the inequality $r \leq \frac{1}{4}\text{in-rad}_xB_\delta(x)$ so that for all $m > 1$,

$$\Gamma_{r/4}(x) \subset B_{m\delta}(x).$$ \hfill (3.5)

Let $A_+$ denote the annulus $\Gamma_{\frac{r}{4}}(z^*) \setminus \Gamma_r(z^*) = \{ z : r \leq |z^* - z| < \frac{5}{4}r \}$. Observe the following two simple facts, the former is obvious from the definitions of $A_+$ and $Q(x)$; the latter follows from the conformal invariance of harmonic measure.

(a) With a probability greater than a positive universal constant the Brownian path $W_t$ started at $x$ moves around $Q(x)$ counterclockwise in such a way that for some epoch $t_0 > 0$ the path $W[0, t_0]$ is contained in $A_+ \cup \Gamma_{r/4}(x)$ and as $t$ ranges over $[0, t_0]$ its argument about $z^*$, $\arg(W_t - z^*)$, extends through at least an interval of length $11\pi/6$ while confined in $(\theta - \pi/6, \theta + 11\pi/6)$ where $\theta = \arg(x - z^*)$; by symmetry the same thing but in the opposite direction of rotation holds.

(b) The probability that the Brownian motion started at $x$ leaves $B_{m\delta}(x)$ before hitting $\partial D$ may be made arbitrarily small for $\delta < 1/m$ by choosing $m$ large.

Taking \((3.5)\) into account it follows from (a) and (b) above that if $m$ is large so that both the probabilities in (a) are larger than that in (b), then $\partial B_{m\delta}(x) \cap D$ cannot disconnect $x$ from $\partial D$ in either of the two components of $Q_{5/4}(x \cup \Gamma_{r/4}(x))$ since otherwise any encompassing path in (a) must hits $\partial B_{m\delta}(x)$ earlier than $\partial D$ (entailing that the event in (b) is contained in one of two events in (a)).

In the same way, but considering the annulus $A_- = \Gamma_r(x) \setminus \Gamma_{\frac{r}{4}}(x)$ in place of $A_+$, we see that for $m$ large enough $\partial B_{m\delta}(x) \setminus \partial D$ cannot disconnect $x$ from $\partial D$ in either of the two components of $Q(x) \setminus (Q_{3/4} \cup \Gamma_{r/4}(x))$.

Simple topological arguments then verify that $\partial B_{m\delta}(x) \cap (\omega \cup \Gamma_{r/4}(x)) = \emptyset$ and either (1) or (2) holds.

\[ \square \]

Corollary 3.3. For any $\delta_0$ and $M$ there exists a constant $R$ (depending only on $\delta_0$ and $M$) such that if $D \subset M\rho_D \mathbb{D}$ and $\rho_D \geq R$, then for some universal constant $c > 0$,

$$P[S^u \text{ exits } B_{m\delta}(u) \text{ through } \partial D] > c \quad \text{if } u \in V(D \setminus U_{\delta_0}),$$

where $\delta = 1 - |\varphi_D(u)|$ and $m$ is the universal constant appearing in Lemma 3.2.

Proof. We apply Lemma 3.2 with $u$ in place of $x$. Let $A_-$ be the annulus defined in its proof. Then in the case (1) of Lemma 3.2 the connected component of $(A_- \cap D) \cup \Gamma_{r/4}(u)$ is contained in $B_{m\delta}(u)$, so that

$$P[S^u \text{ exits } B_{m\delta}(u) \text{ through } \partial D] \geq P[S^u \text{ encompasses } \Gamma_{3r/4}(z^*) \text{ in } A_- \cup \Gamma_{r/4}].$$

Owing to Corollary 2.3 (and the remark after it) the right side is bounded below by half the corresponding probability for Brownian motion, which is a universal constant. In the case (2), a similar reasoning shows the result. \[ \square \]

3.3. Construction of a domain $U$. 

We give a consequence of Lemma 3.2 in a form that is actually used in our substantial application of Lemma 3.2. Let $m$ be the universal constant described in Lemma 3.2

**Lemma 3.4.** For each $0 < \delta < 1/m$, there exists a simply connected subdomain $U$ of $U_{m\delta, \delta}$ such that both of the sets

$$C := \partial U \cap \partial D, \quad C^* := \partial U \cap (D \cap \partial U)$$

are not empty and that if

$$\Omega = \mathbb{C} \setminus (\partial U \setminus C),$$

then for some $x^* \in C$ and some universal constant $c_* > 0$,

$$h^{bm}(y, C_{in}^*; \Omega) \geq c_* \text{ for } y \in \Gamma_{\kappa(\delta)}(x^*),$$

where $\kappa$ is the function given in Lemma 3.4. $C_{in}^*$ means the inner side of $C^*$ relative to $U$, and $h^{bm}(y, A; \Omega)$ denotes the harmonic measure of $A$ in $\Omega$ from $y$ (so that

$$h^{bm}(y, C_{in}^*; \Omega) = P[W^y(\tau_\Omega) \in C^* \text{ and } \exists \varepsilon > 0, \forall s \in (0, \varepsilon), W^y(\tau_\Omega - s) \in U].$$

**Proof.** Take $\varphi^1_D(1 - \delta)$ for $x$ in Lemma 3.2 which we are to apply. Let $Q = Q(D, x)$, $z^* = z^*(x, \delta)$ and $r (= |x - z^*|)$ be as in Section 3.2 and $D(x)$ and $\omega$ be as defined in Lemma 3.2. Our construction of $U$ apply for either of two alternatives in Lemma 3.2 while $x^*$ is chosen in a different fashion.

Put $U^0 = U_\delta \setminus [D(x) \cup \Gamma_{r/4}(x)]$ (recall $D(x)$ is the component of $D \setminus \omega$ that does not contain $\partial$) and define $U$ by

$$U = \text{the component of } U_\delta \setminus U^0 \text{ that contacts } \partial D.$$  

$U_\delta \cap \partial U$ consists of two disjoint arcs, $C_\pm$, say, of $[D(x) \cup \Gamma_{r/4}(x)]$; and $C$ (given by (3.6)) is the part of $\partial D$ (in contact with $U$) cut by the endpoints of $C_\pm$ that agree those of $\omega$. (To be precise either of the endpoints of $\omega$ do not always cut $\partial D$ if the cluster set to the prime end associated with it is not point, but this causes no problem in below.) Similarly $C^*$ is an arc cut from the inner boundary of $U_\delta$ by the other endpoints of $C_\pm$. Clearly $\varphi_D(U) \subset \Gamma_{m\delta}(1)$, and we see that $U \subset U_{m\delta, \delta}$.

Now consider the case (1) and we take $x^* = z^*$, which is certainly in $C$. If the intersection of $\Gamma_{r/2}(z^*) \setminus Q(x)$ with $U$ is empty, then $\text{dist}(\partial \Omega, z^*) \geq r/2$, and it is plain to see (3.7). If this intersection is not empty, we modify $U$ by subtracting the set $\Gamma_r(z^*) \setminus Q(x)$ from it and taking the component of the resulting set that intersects $\Gamma_{r/4}(x)$ and define $C^*$ by (3.6). Then we also have (3.7).

In the case (2) we take as $x^*$ the point where the ray that issues from $z^*$ and passes through $x$ first falls on $\partial D$. Then, on observing that any continuous curve from $x^*$ that reaches $\partial \Omega_{out}$ (the $z^*$ side of $\partial \Omega$) within $\Omega$ must travel a distance more than $\pi r/3$, we see (3.7) holds true. The proof of Lemma 3.4 is finished.

Let $H^{RW}(u, C_{in}^*; \Omega)$ denote the harmonic measure of $C_{in}^*$ for the walk $S^u$ in $\Omega$. (For the present purpose the detailed definition is irrelevant and any reasonable one may be adopted.) Then we have
Corollary 3.5. For any \( \delta_0 > 0 \) and \( M \geq 1 \) there exists a constant \( R \) (depending only on \( \delta_0 \) and \( M \)) such that if \( D \subset M \rho_D \mathbb{D} \) and \( \rho_D \geq R \), then for some universal constant \( c > 0 \),

\[
H^{RW}(u, C^{*\nu}_\text{in}; \Omega) > c \quad \text{for} \quad u \in \Gamma_{\kappa(\delta)\rho_D/8}(x^*) \cap \Omega,
\]

where \( m \) is the universal constant appearing in Lemma 3.2.

Proof. Comparing the harmonic measure on the left side of (3.7) with the corresponding harmonic measure for the random walk \( S^0 \circ \theta_{\sigma(\Gamma_M(\kappa(\delta)\rho_D/8)(y)))} \) we infer that the latter, which may be written as \( q^{(0)}_{\text{in}, \Omega}(u; \kappa(\delta)\rho_D/8) \), is larger than \( 2c \) if \( \rho_D \) is large enough, and an application of Lemma 2.2 concludes the proof.

4 Estimates of hitting distributions of random walks

Let \( G = (V, E) \) be the planar graph and \( (S^v_n)_{n=0}^\infty \) the random walk on \( V \) starting at \( v \) described in Section 2. A bounded domain \( D \) is called a grid domain if its boundary consists of edges of the graph \( G \). Define

\[
\mathcal{D} = \{ D : D \text{ is a simply connected and bounded grid domain} \}.
\]

In this section we shall give several estimates of harmonic measures of the walks \( S^v_n, v \in D \) for various subdomains of \( D \in \mathcal{D} \) that are defined by means of a conformal map \( \varphi_D : D \mapsto \mathbb{D} \). What causes the problem is that the map \( \varphi_D \) may distort the metric property of \( D \) unrestrictedly when the points approach the boundary of \( D \), so that the direct application of the invariance principle—such as that stated in Corollary 2.3—would be impossible since we must verify condition (2.3) or (2.5).

In what follows, as in the preceding section, a point \( \hat{o} \) of \( D \in \mathcal{D} \) is suitably chosen as a ‘reference point’ which together with \( v_0 \in \partial D \) uniquely determines the conformal map \( \varphi_D : D \mapsto \mathbb{D} \) via the condition (3.1). With this being taken into account define

\[
\mathcal{D}_{R,M} = \{ (\hat{o}, D) : \hat{o} \in D \in \mathcal{D}, v_0 \in V(\partial D), \rho_D > R, D \subset M \rho_D \mathbb{D} \},
\]

where \( \rho_D = \text{in-rad}_D \) as in Section 3. For sake of brevity we write \( D \in \mathcal{D}_{R,M} \) instead of \((\hat{o}, D) \in \mathcal{D}_{R,M} \) with the understanding that \( \hat{o} \) is assigned to \( D \) in some way (cf. the beginning of Section 5.2 and (5.7) for the choice of \( \hat{o} \)). We shall be interested in a lower bound of \( \rho_D \) and apart from it no significance will be attached to a particular choice of \( \hat{o} \) for the discussion made in the present section. A boundary point \( v_0 \in V(\partial D) \) is also supposed to be assigned to \( D \in \mathcal{D} \) to determine \( \varphi_D \) uniquely, but our analysis will be carried out so as to be independent of it. The dependence on \( M \) is needed because the bounds in Proposition 2.1 or in (2.3) (with a nice \( U \)) are not ensured by (H) if \( u \) is indefinitely far from the origin. If these estimates are valid without the restriction \( |u| < M \), then it is unnecessary to impose the boundedness of \( D \) indicated by \( M \). In any case we shall do not take much care of the restriction imposed by \( M \).

Throughout this and the next section we continue to use the notations \( U_{\delta}, U_{r,\delta}, B_r(x) \) and \( \Gamma_r(x) \) (the first three are defined in Section 3.2 and the last by (2.1)); also suppose the
condition (H) introduced in Section 2 to be valid. The subscript $D$ is dropped from $\varphi_D$ in the proofs, if $D$ is clear from the context. In addition we bring in

$$\tau_U = \sup\{t > \tau_U - 1 : \bar{S}^u_t \notin U\}.$$ (4.1)

For instance the statement that $S^u$ exits $U_{r,\delta}$ through $\partial U_{\delta} \cap U_{\delta}$ is expressed as $\bar{S}^u(\tau_{U_{r,\delta}}) \in U_{\delta}$, while $S^u(\tau_{U_{r,\delta}})$ is possibly in $U_{r,\delta}$. Note that the expressions $\bar{S}^u(\tau_{U_{r,\delta}}) \notin D$, $\bar{S}^u(\tau_{U_{r,\delta}}) \in \partial D$ and $\bar{S}^u(\tau_{U_{r,\delta}}) \in \partial D$ all mean the thing if $D$ is a grid domain.

4.1. Simple properties of the planar graph

Here we state some elementary results that follow from Hypothesis (H);

1. $\max\{|u - v| : (u, v) \in E, u, v \in V(r\mathbb{D})\} = o(r)$ as $r \to \infty$.
2. $\sup\{\text{dist}(z, V) : z \in r\mathbb{D}\} = o(r)$ as $r \to \infty$.
3. $\sup_{D \in \mathcal{D}_{R,M}} \sup_{(u,v) \in E,u \in \partial U,v \in D} |\varphi_D(u) - \varphi_D(v)| \to 0$ as $R \to \infty$ (for each $M$).
4. For any $\delta > 0, \varepsilon > 0$ and $M > 1$, there exists $R > 1$ such that if $D \in \mathcal{D}_{R,M}$, $|\varphi_D(x)| \leq 1 - \delta$, then there is a path of $G$ of diameter less than $\varepsilon \kappa(\delta) \rho_D$ that encircles $x$ in $D$ and $B_\varepsilon(x)$ contains a vertex of $V(D)$. ($\kappa$ is the function specified in Lemma 3.1)

(1) and (2) are readily verified directly from (H). For (3) use Lemma 3.1 on noting that if $(u, v) \in E$, $u \in \partial U$ and $v \in D$, then $[u, v] \subset \partial U$ in view of planarity of $G$ and the definition of grid domain. (4) also follows from Lemma 3.1 together with (H).

The facts listed above, which are easy to grasp, will be applied without explicitly mentioning of their use in most cases in the later discussions. From them it follows that given $M > 1$, $\eta \in (0, 1/10)$ and $\delta > 0$, we can choose $R > 1$ large enough that for every $y \in \partial D$ the subgraph $G(B_\delta(y) \setminus U_{\eta\delta})$ is so spatial as to contain a fine network of paths: e.g., the annulus $B_\delta(y) \setminus B_{\delta/2}(y)$ contains a path of $G$ that connects the two disconnected parts of $B_\delta(y) \setminus B_{\delta/2}(y) \cap U_{\eta\delta}$. It will be tacitly supposed that $R$ is large enough according to the arguments developed in below.

4.2. Starting near the boundary (unconditional case)

The walk starting near the boundary is relevant to our analysis. In this subsection we verify that the probability of such a walk escaping immediate absorption to the boundary is small and use this fact to derive a result on the harmonic measure of the walk in $D \in \mathcal{D}_{R,M}$.

In the next subsection we consider the behavior of the walk conditioned to escape immediate absorption.

The next lemma is a slight improvement of Proposition 4.5 of [16] (for the present setting) and the corresponding one verified in the proof of Lemma 5.4 of [6] (for the simple random walk on the square lattice). The proof is similar except for our use of Lemma 3.2. (An extended form is found in the proof of Corollary 4.9)

**Lemma 4.1.** For any $\delta_0 > 0$ and $M > 1$ there exists $R$ such that for $\delta \geq \delta_0$, $D \in \mathcal{D}_{R,M}$ and $u \in V(U_{\delta})$,

$$P[S^u(\tau_{B_{\delta_0}(u)}) \in \partial D] > c,$$ (4.2)
where $m$ and $c$ are the same universal constants specified in Lemma 3.2 and Corollary 3.3 respectively.

**Proof.** Owing to Corollary 3.3 there exists a constant $R$ (depending only on $\delta_0$ and $M$) such that for $\delta \geq \delta_0/3$,

$$P[S^u(\tau_{B_{m\delta}(u)}) \in \partial D] > c, \quad \text{if} \quad u \in U_\delta \setminus U_{\delta/5}. \quad (4.3)$$

Let $u \in U_{\delta/5}$ and put $p(v) = P[S^v(\tau_{B_{m\delta}(u)}) \in \partial D]$ for $v \in B_{m\delta}(u)$. Since $p(v)$ is harmonic in $B_{m\delta}(u)$, a maximum principle shows that there exists a path $\gamma$ of $G(B_{m\delta}(u))$ that connects $u$ with $\partial B_{m\delta}(v) \setminus \partial D$ such that $p(v) \leq p(u)$ for $v \in \gamma$. (Here the phrase that $\gamma = (\gamma_k)_{k=0}^{\infty}$ ‘connects’ $u \in V$ with a set $A \subset \mathbb{C}$ means that $\gamma_0 = u$ and $(\gamma_{n-1}, \gamma_n) \cap A \neq \emptyset$.) If $\gamma$ intersects the curve $\partial U_{\delta/4} \cap D$ (in $B_{m\delta}(u)$), then $p(u) > c$ according to what we have shown in the preceding paragraph.

Consider the case when $\gamma$ does not, and suppose $\varphi_D(u)$ is real and positive for convenience of description. Put $U_\delta^+ = \varphi^{-1}(\mathbb{D}^+) \cap U_\delta$, where $\mathbb{D}^+ = \{z \in \mathbb{D} : \Im z > 0\}$. By replacing $R$ by a larger one if necessary we can then find two vertices $u_+$ and $u_-$ such that

$$u_+ \in U_{\delta/3}^+ \setminus B_{m\delta/3}(u) \cup U_{\delta/4} \quad \text{and} \quad B_{m\delta/3}(u_+) \subset B_{m\delta}(u).$$

Now let $\delta \geq \delta_0$ so that we may apply (4.3) with $u_+$ and $\delta/3$ in place of $u$ and $\delta$, respectively. Then, we infer in the same way as above that there are paths $\gamma_{\pm}$ such that $\gamma_\pm$ connects a vertex $u_\pm$ with $\partial D$ in $B_{\delta/3}(u_\pm)$ (respectively) and $p(v) > c$ for $v \in \gamma_+ \cup \gamma_-$. If $\gamma$ does not intersect the curve $\partial U_{\delta/4} \cap D$, then $\gamma$ must cross either $\gamma_+$ or $\gamma_-$, hence $p(u) > c$. The proof of Lemma 4.1 is complete. □

**Corollary 4.2.** There exists a universal constant $c_1 < 1$ such that for any $\delta_0 > 0$ and $M > 1$ there exists a constant $R$ such that for $\delta_0 \leq \delta < 1/4$, $D \in \mathcal{D}_{R,M}$ and $u \in V(U_\delta)$ with $|\Im \varphi_D(u)| < \delta/4$,

$$P[S^u(\tau_{U_{\delta,\delta}})] \in U_\delta] < c_1.$$  

(See (4.1) for $\tau_U$.)

**Proof.** Given $\delta_0 > 0$, take $\delta \geq \delta_0$ and $u \in U_\delta$. By Lemma 4.1 $P[S^u(\tau_{U_{\delta,\delta}}) \notin U_\delta] > c$ whenever $u \in U_{\delta/2m}$ and $|\Im \varphi(u)| < \delta/2$. The general case of $|\Im \varphi(u)| < \delta/4$ is reduced to what is just verified. Indeed, on putting $U = U_{\delta/2,\delta} \setminus U_{\delta/2m}$, with the help of Lemma 3.1 an application of Corollary 2.3 shows that if $|\Im \varphi(u)| < \delta/4$, then $P[S^u(\tau_U) \notin U_\delta \setminus U_{\delta/2m}] > c'$ with some universal constant $c' > 0$, provided that $R$ is large enough, so that the required inequality ensues at least with $c_1 = 1 - cc'$. □

**Corollary 4.3.** For any $\varepsilon > 0$, $\alpha > 0$ and $M > 1$ there exists $R$ and $\delta > 0$ such that for all $D \in \mathcal{D}_{R,M}$ and $u \in V(U_\delta)$,

$$P[S^u(\tau_{B_{n}(u)}) \in D] < \varepsilon.$$  

(4.4)

**Proof.** Taking $N$ so large that $\varepsilon < (1 - c)^N$, we have only to apply Lemma 4.1 repeatedly at most $N$ times by starting with $\delta = \delta_0 = m^{-N}\alpha$ to arrive at the inequality of the lemma. □

By virtue of the bound (4.4) we can control the probability of the random walk badly behaving near the boundary, and an application of Corollary 2.3 leads to the following corollary.
By the conformal invariance of Brownian hitting probability the constants \( q_B \) and \( \tau^W \) can choose the following probability for Brownian motion. Among them we shall need the following one.

**Corollary 4.4.** For any \( \varepsilon > 0, r > 0 \) and \( M > 1 \) there exists \( R > 1 \) such that if \( I \subset \partial \mathbb{D} \) is an arc of length \( 2r \) centered at 1 and \( J = \varphi_D^{-1}(I) \), then for \( u \in V(D \setminus U_\varepsilon) \),

\[
(1 - \varepsilon) P[W^u(\tau^W_D) \in J] \leq P[S^u(\tau_D) \in J] \leq (1 + \varepsilon) P[W^u(\tau^W_D) \in J],
\]

where \( W^u \) denotes a planar Brownian motion started at \( u \) and \( \tau^W_D \) its first exit time from \( B \).

**Proof.** Let \( 0 < \eta < 1 \) and for \( u \in V(D \setminus U_\varepsilon) \) and \( 0 < \delta < (r \wedge \varepsilon)/2 \), denote by \( A = A_{\delta, \eta, u} \) the event that the walk enters into \( U_\delta \) substantially through \( \partial U_\delta \cap \partial U_{(1+\eta)r,\delta} \):

\[
A = \{ S^u(\tau_D \setminus U_\delta) \in U_{(1+\eta)r,2\delta} \}
\]

and make decomposition

\[
P[S^u(\tau_D) \in J] = P(\{ S^u(\tau_D) \in J \} \cap A) + P(\{ S^u(\tau_D) \in J \} \setminus A).
\]

According to Corollary 4.4 for any \( \varepsilon_1 > 0 \) there exists \( \delta_1 = \delta_1(\eta r, \varepsilon_1) > 0 \) and \( R = R(\eta r, \varepsilon_1) \) such that \( P[S^u(\tau_{B_{\delta_1}(w)}) \in D] < \varepsilon_1 \) for \( w \in U_\delta, \delta \leq \delta_1 \), implying that the second probability of the decomposition is less than \( \varepsilon_1 \). On taking \( \varepsilon_1 = \frac{1}{2} \varepsilon \inf_{w \in V(D \setminus U_\varepsilon)} P[W^u(\tau^W_D) \in J] \) this yields that

\[
P(\{ S^u(\tau_D) \in J \} \setminus A) \leq \frac{1}{2} \varepsilon P[W^u(\tau^W_D) \in J].
\]

On the other hand, applying Corollary 2.3 we deduce that for each \( \eta \) and \( \delta \leq \delta_1 \), we can choose \( R \) large so that

\[
P(A) \leq (1 + \frac{1}{2} \varepsilon) P[W^u(\tau^W_D) \in \partial U_{(1+\eta)r,\delta}].
\]

By the conformal invariance of Brownian hitting probability the constants \( \delta \) and \( \eta \) may have been chosen small (independently of \( R \)) so that the right side above is at most \( (1 + \frac{1}{2} \varepsilon) P[W^u(\tau^W_D) \in J] \). As a consequence, we may assert that for \( R \) large enough,

\[
P(A) \leq (1 + \frac{1}{2} \varepsilon) P[W^u(\tau^W_D) \in J].
\]

This together with (4.7) concludes the upper bound of (4.5).

In a similar way, putting \( q_\eta(u) = P[S^u(\tau_D \setminus U_\delta) \in U_{(1-\eta)r,2\delta}] \), we infer on the one hand \( q_\eta(u) \leq P[S^u(\tau_D) \in J] + \frac{1}{2} \varepsilon q_\eta(u) \), and on the other hand \( q_\eta(u) \geq P[W^u(\tau^W_D) \in J](1 - \frac{1}{2} \varepsilon) \), yielding the lower bound in (4.5). The proof is complete.

A simple modification of the proof above shows extensions of (4.5) that provide for a certain class of events \( B \) the upper and lower bounds of \( P(B \cap \{ S^u(\tau_D) \in J \}) \) in terms of the corresponding probability for Brownian motion. Among them we shall need the following one.

**Corollary 4.5.** Let \( J \) be as in Corollary 4.4. Let \( u \in D \setminus U_\delta \). Let \( \Gamma \) be a set of paths of \( G(D) \) and \( \Gamma' \) a measurable set of continuous curves in \( D \) such that for any \( \delta > 0 \) small enough, one can choose \( R \) so that for some constant \( c > 0 \),

\[
P[S^u[0, \tau_D \setminus U_\delta] \in \Gamma \text{ and } S^u(\tau_D \setminus U_\delta) \in U_{\frac{1}{2}r,2\delta}] > c P[W^u[0, \tau_D \setminus U_\delta] \in \Gamma' \text{ and } W^u(\tau^W_D) \in J].
\]

(4.8)
Then, for any $\delta > 0$ small enough one can choose $R$ so that

$$
P \left[ S^u[0, \tau_{D \setminus U_\delta}] \in \Gamma \mid S^u(\tau_D) \in J \right] > \frac{3}{4} P \left[ W^u[0, \tau_{D \setminus U_\delta}] \in \Gamma' \mid W^u(\tau_D^W) \in J \right].$$

If $\delta$ and $R$ are chosen independently of $u$ and $D$ in (4.8), then so are they in (4.9). (Here $S^u[k, n] = (S^n_j)_{k \leq j \leq n}$ and analogously for $W^u[s, t]$.)

Proof. In view of the preceding corollary it suffices to show

$$
P \left[ S^u[0, \tau_{D \setminus U_\delta}] \in \Gamma \text{ and } S^u(\tau_D) \in J \right] > \frac{3}{4} P \left[ S^u[0, \tau_{D \setminus U_\delta}] \in \Gamma \text{ and } S^u(\tau_{D \setminus U_\delta}) \in U_{2r, 2\delta} \right].$$

(4.10)

By Corollary 4.4 it follows that for all $\delta$ small enough, we can choose $R$ so that if $w \in U_{2r, 2\delta}$, then $P[S^w(\tau_D) \in J] > 3/4$. Hence the right side of (4.10) is less than $P[S^u[0, \tau_{D \setminus U_\delta}] \in \Gamma, S^u(\tau_{D \setminus U_\delta}) \in U_{2r, 2\delta}$ and $S^u(\tau_D) \in J]$, which plainly entails (4.10).

4.3. Starting near the boundary (conditional case)

The main result of this subsection (Proposition 4.6) concerns the walk conditioned to escape immediate absorption into the complement of $D$. It provides an estimate of a conditional probability, given that the walk started at $v_0$ immediately enters into $D$ and hits $\partial U_{\Lambda \delta, \delta} \cap D$ before leaving $D$. Proposition 4.6, while playing a crucial role in the next section, is not used in the succeeding subsections of the present section.

Let $U$ be a domain of $\mathbb{C}$ whose boundary contains a vertex $v \in V$. The definition given in Section 2 of the first exit time $\tau_U$ for the walk starting from a boundary point of $U$ may be written as

$$
\tau_U = 1 + \tau_U \circ \theta_1 \quad \text{if } S^v_1 \in U, \\
= 1 \quad \text{if } S^v_1 \notin U. 
$$

(4.11)

Here $\theta_n$ denotes the usual shift operator acting on random walk paths. When there are more than two prime ends which are associated with $v$, the condition $S^v_1 \notin U$ in (4.11) must be replaced by another one for the present purpose. Let exactly $j$ prime ends, $v_k, \ldots, v_j$, say, correspond to $v$. Then any continuous curve in $U$ approaching $v$ may be considered to approach one of these prime ends and not any other. We say $u \in V$ is a neighbor site of $v_k$ in $U$ if $[v, u]$ is an edge of $G(U)$ and the segment $[u, v)$ approaches $v_k$ ($k = 1, \ldots, j$). Let $\text{nbd}_U(v_k)$ denote the set of neighbor sites of $v_k$ in $U$. Then the latter condition in (4.11) is replaced by

$$
\tau_U = 1 \quad \text{if } S^v_1 \notin \text{nbd}_U(v_k) 
$$

(4.12)

($k = 1, \ldots, j$) in order to distinguish $v_k$ from the others. In the sequel we adopt the latter definition (4.12) with convention that

If we consider $S^v$ with $v \in V(\partial D)$, the same letter $v$ is understood to designate a prime end of $\partial D$ that is associated with $v$.

For typographical reason we often write $U(\Lambda \delta, \delta)$ for $U_{\Lambda \delta, \delta}$. 

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Proposition 4.6. For any $\varepsilon > 0$ there exists $\Lambda > 8$ such that for any $\delta_0 > 0$ and any $M > 1$, one can find $R > 1$ such that if $\delta_0 \leq \delta < 2/\Lambda$, $D \in \mathcal{D}_{R,M}$, $v \in \overline{U}_{\delta,\delta}$ and $P[S^v(\tau_{U(\Lambda,\delta,\delta)})] > 0$, then
\[ P[\overline{S}^v(\tau_{U(\Lambda,\delta,\delta)}) \in U_\delta | S^v(\tau_{U(\Lambda,\delta,\delta)}) \in D] < \varepsilon. \quad (4.13) \]
(The event under $P$ says that the walk exits $U_{\Lambda,\delta}$ from either one of its two narrow edges; see [4.3] for the notation $\tau_U$.)

Proof. Let $U^+_{\delta}$ denote the upper half of the annulus $U_\delta$ which is defined to be the $\varphi^{-1}$-image of $\{\Re z > 0\} \cap \varphi(U_\delta)$. By symmetry it suffices to show
\[ P[\overline{S}^v(\tau_{U(\Lambda,\delta,\delta)}) \in U^+_{\delta} | S^v(\tau_{U(\Lambda,\delta,\delta)}) \in D] < \varepsilon. \quad (4.14) \]

The rest of the proof is broken into five steps. First we prove the proposition when $v = v_0$ in the steps 1 through 4. The general case readily follows from this special case and is dealt with in the step 5.

Step 1. Put
\[ T = \tau_{U(\Lambda,\delta,\delta)} \quad \text{and} \quad \bar{T} = \bar{\tau}_{U(\Lambda,\delta,\delta)}, \]
and let $p(v)$ denote the conditional probability on the left-hand side of (4.14):
\[ p(v) = P[\overline{S}^v_T \in U^+_{\delta} | S^v_T \in D], \quad v \in V(\overline{U}_{\Lambda,\delta}). \quad (4.15) \]

We claim that for $0 < r < (\Lambda - 1)\delta$,
\[ p(y) \geq p(v_0) \quad \text{if} \quad P[S^v_T \in D] > 0 \quad \text{and} \quad y \in V(\overline{U}_r \cap \partial D), \quad (4.16) \]
where $U^+_{r,\delta} = U^+_{\delta} \cap U_{r,\delta}$. For the proof we use the fact that if $X = (X_n)$ is our walk $(S_n)$ killed at $T$ and conditioned on exiting through $\partial U_{\Lambda,\delta,\delta} \setminus \partial D$ and if $h(v) = P[S^v_T \in U^+_{\delta}]$, then $X$ is the $h$-transform of our walk $(S_n)$; in particular the process $X$ is Markovian and $p(v)$ is a harmonic function of it. Thus by Maximum principle applied to $X$ there exists a path in $U_{\Lambda,\delta,\delta}$ connecting $v_0$ to $J := U^+_{\delta} \cap \partial U_{\Lambda,\delta,\delta}$ in $U_{\Lambda,\delta,\delta}$ such that $p(y) \geq p(v_0)$ for $y$ on the path. If $\gamma^{v_0}$ denotes such a path, then the linear interpolation $S$ of the conditioned walk $X$ starting at $y \in \overline{U}_{r,\delta} \cap \partial D$ must hit $\gamma^{v_0} \cup J$ before exiting $U_{\Lambda,\delta,\delta}$ (recall $v_0$ is in a corner of $U^+_{\delta}$), and the strong Markov property of $X$ concludes the claim (4.16).

Step 2. Let $U$ be a simply connected subdomain of $U_{\Lambda,\delta,\delta}$ such that if
\[ C = \partial U \cap \partial D, \quad C^* = \partial U \cap \partial U_{\delta} \cap \partial D, \quad (4.17) \]
then
\[ C \quad \text{and} \quad C^* \quad \text{are both non-empty,} \quad C \quad \text{is connected} \quad \text{and} \quad C \subset \partial U^+_{\delta}. \]

Let $v^*$ be a vertex in $C$. In the next step $U$ and $v^*$ will be specified more explicitly (by means of $D(x)$ defined in the preceding section), whereas in the present step they may be rather arbitrary except for the restriction just mentioned.
Put
\[ \Omega = C \setminus \partial U \setminus C. \]
It is convenient to bring in
\[ C^\circ = C \setminus \partial U \setminus C. \]
Plainly \( \partial \Omega = \partial U \setminus C = \partial U \setminus C^\circ \) and \( C^\circ \) disconnects \( U \) from \( \Omega \setminus U \) in the graph \( G(\Omega) \). Although the probability in question depends on our random walk \( S^v \) restricted to \( D \), we are to extend it to \( \Omega \) through \( C^\circ \). While how to choose the walk outside \( U \) is at our disposal, we use the walk \( S^v \) itself but with understanding that the extended walk distinguishes the same vertex of \( D \) according as it is reached by the walk from a vertex of \( C^\circ \) within \( \Omega \setminus U \) or within \( D \).

It is to distinguish two sides of \( \partial \Omega = \partial U \setminus C^\circ \) what we actually need in the sequel, and the following notation may allow us to dispense with the formal definition. Thus we write
\[ \langle C^* \rangle_{\text{out}} = \{ v \in V(D) : [u, v] \in E \text{ and } [u, v] \cap C^* \neq \emptyset \text{ for some } u \in V(U \cup C^\circ) \} \]
(the subscript ‘out’ reflects the fact that the directed segment \([u, v]\) in the definition above is an ‘outward’ boundary edge relative to \( U \) although \( U \) does not appear in the notation). By means of this notation the event that the walk exits from \( \Omega \) by crossing \( C^* \) with an edge directed outward from \( U \cup C^\circ \) is expressed in the formula \( S^v_{\tau(\Omega)} \in \langle C^* \rangle_{\text{out}} \). We shall use \( \langle \partial U \rangle_{\text{out}} \) in the analogous sense.

**Step 3.** Let \( v^* \in V(C^\circ) \) and \( L \) denote the last exit time from \( \Omega \setminus U \) of the process \( S^v_n, n < \tau_\Omega \):
\[
L = \begin{cases} 
\max\{0 \leq n < \tau_\Omega : S^v_n \notin U \} & \text{if } S^v_1 \notin U, \\
0 & \text{if } S^v_1 \in U,
\end{cases}
\]
so that \( S^v_L \in C^\circ \) unless \( L + 1 = \tau_\Omega \) since \( C \) is a section cut from the boundary of the grid domain \( D \). We are to compute
\[
q := P[S^v_L \in C^\circ, \bar{S}^v_T \circ \theta_L \in U^+_\delta, S^v_T \circ \theta_L \in D],
\]
the probability that the walk is found in \( C^\circ \) at the epoch \( L \) and continued thereafter and then exits \( U_{M, \delta} \) through \( U^+_\delta \cap \partial U_{M, \delta} \) without landing on \( \partial D \). (Here the shift operator \( \theta_L \) acts on \( T \) as well as on \( S^v \) as usual.) Noting that if \( n < \tau_\Omega, S^v_n \in C^\circ \) and \( \bar{S}^v_T \circ \theta_n \in U^+_\delta \), then \( L = n \), we deduce that
\[
q = \sum_{n=0}^{\infty} \sum_{y \in C^\circ} P[L = n < \tau_\Omega, S^v_n = y; \bar{S}^v_T \circ \theta_n \in U^+_\delta, S^v_T \circ \theta_n \in D]
= \sum_{n=0}^{\infty} \sum_{y \in C^\circ} P[S^v_n = y, n < \tau_\Omega; \bar{S}^v_T \circ \theta_n \in U^+_\delta, S^v_T \circ \theta_n \in D]
= \sum_{n=0}^{\infty} \sum_{y \in C^\circ} P[S^v_n = y, n < \tau_\Omega]P[\bar{S}^v_T \in U^+_\delta, S^v_T \in D]
= \sum_{y \in C^\circ} G_\Omega(v^*, y)P[S^v_T \in D]P\left[\bar{S}^v_T \in U^+_\delta \mid S^v_T \in D\right], \tag{4.18}
\]
where \( G_\Omega(v^*, y) = \sum_{n=0}^{\infty} P[S_n^y = y, n < \tau_\Omega] \). Recalling the notation \( \langle C^* \rangle_{\text{out}} \) (introduced after \( \Omega \) is) we have

\[
P[S_T^y \in D] \geq P[S_T^y(\Upsilon) \in \langle C^* \rangle_{\text{out}}].
\]

Hence

\[
q \geq \sum_{y \in \mathcal{C}^o} G_\Omega(v^*, y) P[S_T^y(\Upsilon) \in \langle C^* \rangle_{\text{out}}] p(y),
\]

where \( p(y) \) is the conditional probability defined in Step 1.

Noting that if \( S_T^y(\Upsilon) \circ \theta_L \in \langle C^* \rangle_{\text{out}} \) and \( S_T^y \in \mathcal{C}^o \), then \( S_T^y(\Upsilon) \in \langle C^* \rangle_{\text{out}} \) and vice versa, we have

\[
\sum_{y \in \mathcal{C}^o} G_\Omega(v^*, y) P[S_T^y(\Upsilon) \in \langle C^* \rangle_{\text{out}}] = \sum_{n=0}^{\infty} \sum_{y \in \mathcal{C}^o} P[n < \tau_\Omega, S_n^y = y; S_T^y(\Upsilon) \circ \theta_n \in \langle C^* \rangle_{\text{out}}]
\]

\[
= \sum_{n=0}^{\infty} P[L = n, S_n^y \in \mathcal{C}^o; S_T^y(\Upsilon) \circ \theta_n \in \langle C^* \rangle_{\text{out}}]
\]

\[
= P[S_T^{\circ \circ}(\Upsilon) \in \langle C^* \rangle_{\text{out}}].
\]

Thus in view of (4.16) and (4.19)

\[
q \geq p(v_0) P[S_T^{\circ \circ}(\Upsilon) \in \langle C^* \rangle_{\text{out}}].
\]

**Step 4.** In this step we verify the inequality of the proposition for \( v = v_0 \). First we claim that one can choose the pair of \( U \) and \( v^* \) so that for some universal constant \( c^0 > 0 \)

\[
P[S_T^{\circ \circ}(\Upsilon) \in \langle C^* \rangle_{\text{out}}] \geq c^0 \quad \text{if } \Lambda \geq m
\]

and that \( U \) satisfies (along with those stated at (4.17))

\[
U \subset U_{2m, \delta, \delta}^+.
\]

Here \( m \) is the universal constant specified in Lemma 3.2. To this end we take up \( U \) as given by Lemma 3.3 but with (4.22) in place of the inclusion \( U \subset U_{m, \delta, \delta} \) (note that \( \varphi(U_{2m, \delta, \delta}) \) is simply a rotation of \( \varphi(U_{m, \delta, \delta}) \) and the special choice of \( x \) as \( \varphi^{-1}(1-\delta) \) in the proof carries no significance except for this inclusion). As \( v^* \) we take a vertex in \( \partial D \) closest to \( x^* \in \partial U \) (described in Lemma 3.3). Then an easy application of Corollary 2.3 verifies (4.21) with \( c^0 = c^* / 2 \), provided \( R \) is large enough and \( D \in \mathbb{D}_{R, M} \). Plainly we have (4.22). Thus the claim has been proved.

From the last expression of \( q \) in (4.18) and by using the last exit decomposition as in Step 3 (but in reverse direction) we have

\[
q \leq \sum_{y \in \mathcal{C}^o} G_\Omega(v^*, y) P[S_T^y \in U_\delta^+]
\]

\[
\leq \sum_{y \in \mathcal{C}^o} G_\Omega(v^*, y) \sum_{u \in (\partial U)_{\text{out}} \cap U_\delta} P[S_T^y(\Upsilon) = u] P[S_T^u \in U_\delta^+]
\]

\[
\leq P[S_T^{\circ \circ}(\Upsilon) \in (\partial U)_{\text{out}} \cap U_\delta] \sup_{u \in (\partial U)_{\text{out}} \cap U_\delta} P[S_T^u \in U_\delta^+]
\]

\[
\leq \sup_{u \in (\partial U)_{\text{out}} \cap U_\delta} P[S_T^u \in U_\delta^+],
\]

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and by repeated applications of Corollary 4.2 we conclude \( q \leq c_1^{\Lambda - 2m} \) with a universal constant \( c_1 < 1 \) (\( [a] \) denotes the largest integer that does not exceed a real number \( a \)). From (4.20) and (4.21) it therefore follows that \( p(v_0) \leq c_1^{\Lambda - 2m}/c^\circ \), showing (1.14) for \( v = v_0 \).

**Step 5.** Consider the general case \( v \in \overline{U}_{\delta,\delta} \). We must prove that \( p(v) \), the probability defined by (4.15), can be made arbitrarily small by taking \( \Lambda \) large enough. If \( v \in \partial D \), the same proof as above apply since \( 1 \in \partial \mathbb{D} \) may be replaced by any point of \( \partial \mathbb{D} \). Let \( v \in D \cap \overline{U}_{\delta,\delta} \).

We suppose \( v \in U_{\delta}^- = \varphi^{-1}(\{\exists z < 0\}) \cap U_{\delta} \) for simplifying the description. Let \( \gamma \) be a path from \( v \) to \( J \) in \( U_{\Lambda,\delta} \) on which \( p(v) \leq p(u) \), where \( J = U_{\delta}^+ \cap \partial U_{\Lambda,\delta} \) as before. Let \( w \) be a vertex on \( \partial D \) closest to \( \varphi^{-1}(e^{i\delta/2}) \) and \( A \) the event that the walk \( S^w \) exits \( U_{\Lambda,\delta}^+ := U_{\Lambda,\delta} \cap U_{\delta}^+ \) without hitting \( \gamma \). Then, noting that the event \( S_T^w \in D \) (with \( T = \tau_{U_{\Lambda,\delta}} \) as before) entails \( S_T^w(\tau_{U_{\Lambda,\delta}}) \in D \) and writing \( B \) for the latter event, we infer that

\[
P[A \mid S_T^w \in D] = \frac{P[A, S_T^w \in D \mid B]}{P[S_T^w \in D \mid B]} \leq \frac{P[A \mid B]}{P[S_T^w \in D \mid B]}.
\]

In view of what is noted at the beginning of this step the last ratio as well as \( p(w) \) may be made arbitrarily small by taking \( \Lambda \) (and \( R \)) large enough. On the other hand on using the strong Markov property of the walk conditioned on \( S_T^w \in D \) (as in Step 1) we deduce the inequality

\[
p(w) \geq P[\gamma \text{ is hit before exiting } U_{\Lambda,\delta}^+, S_T^w \in U_{\delta}^+ \mid S_T^w \in D] \geq (1 - P[A \mid S_T^w \in D])p(v).
\]

Hence \( p(v) \) may be made arbitrarily small. The proof of Proposition 4.6 is complete. \( \square \)

Taking \( \varepsilon = 1/2 \) in Proposition 4.6 we plainly obtain the following

**Corollary 4.7.** For any \( \delta_0 > 0 \) and \( M > 1 \), one can find \( R > 1 \) such that if \( \delta_0 \leq \delta < 1/4 \), \( (\hat{\delta}, D) \in \mathbb{D}_{R,M} \), \( v \in V(\overline{U}_{\delta}) \), \( P[S^v(\tau_{U(\delta,\delta)}) \in D] > 0 \) and \( |\Im \varphi_D(v)| < \delta \), then for some universal integer \( m^* \geq 4 \),

\[
P[S_T^v(\tau_{U(m^*,\delta)}) \in \partial U_{\delta} \cap D \mid S_T^v(\tau_{U(m^*,\delta)}) \in D] > 1/2. \tag{4.23}
\]

### 4.4. Hitting distribution of \( \partial D \)

This and the next subsections, in which we do not use Proposition 4.6, primarily concern the hitting distribution

\[
H_D(u, b) := P[S_T^u = b], \quad u \in V(D), \ b \in \partial D,
\]

and provide some estimates of it. In later applications we need extend it to \( b \in V(D) \) by

\[
H_D(u, b) := P[S^u \text{ visits } b \text{ before exiting } D], \quad u \in V(D) \setminus \{b\}.
\]

If \( u \in V(\partial D), u \neq b \) and \( \hat{u} \) is a prime end that is associated with \( u \), we set

\[
H_D(\hat{u}, b) = \sum_{v \in \text{nbd}_D(\hat{u})} P[S_T^u = v]H_D(y, b)
\]

and for \( \delta > 0 \) we define the hitting function\( H_D(u, \partial D) \) by

\[
H_D(u, \partial D) := \lim_{\varepsilon \to 0} H_D(u, \hat{u}), \quad u \in V(D).
\]
to be consistent to the definition of $\tau_U$ in (4.11). In the sequel, however, we write simply $v$ for $\hat{v}$ according to the convention advanced right before Proposition 4.6.

The most results presented below are essentially the same as what are found in [16]. We give proofs to some of them, which are simpler than those in [16] mainly owing to Corollary 2.3 although the idea of the proofs are the same as in [16]. In the proofs we shall often drop the subscript $D$ from $H_D$ as well as from $\varphi_D$.

Given $r > 0$ and $b \in V(\overline{U_{r/4}})$, put $I = \{ e^{i\theta} \varphi_D(b) : |\theta| \leq r \}$, $J = \varphi_D^{-1}(I)$. The next result is essentially the same as Lemma 5.8 of [16].

**Lemma 4.8.** For any $0 < r < \frac{1}{10}$ and $M > 1$ there exists $R$ such that if $D \in \mathbb{D}_{R,M}$ and $b \in V(\overline{U_{r/4}})$, then for all $v \in V(D \setminus U_r)$ and $w \in V(D \setminus B_{4r}(b))$ with $P[S^w(\tau_D) \in J] > 0$,

$$P[S^w(\tau_D) = b \mid S^w(\tau_D) \in J] \leq c_2 P[S^v(\tau_D) = b \mid S^v(\tau_D) \in J]$$

for some universal constant $c_2$.

**Proof.** For simplicity we suppose $b \in \partial D$, the arguments below being readily adapted to the case $b \in V(\overline{U_{r/4}})$. Put $q_J(y) = P[S^y(\tau_D) = b \mid S^y(\tau_D) \in J]$, $y \in V(D)$. It suffices to prove that if $v \in V(D \setminus U_r)$, then for some universal constant $c^0 > 0$,

$$q_J(y) \leq c^0 q_J(v) \quad \text{for} \quad y \in V(B_{4r}(b) \setminus \overline{B_{3r}(b)}). \quad (4.24)$$

For $q_J(w)$ ($w \notin B_{4r}(b)$) is a convex combination of $q_J(y)$, $y \in V(B_{4r}(b) \setminus \overline{B_{3r}(b)})$, provided that $R$ is so large that no edge in $G(D)$ joins $B_{3r}(b)$ and $D \setminus B_{4r}(b)$.

Since $q_J$ is harmonic for the walk conditioned to exit $D$ through $J$, there exists a path $\gamma^y$ of $G(D)$ joining $y$ and $b$ such that $q_J(y) \leq q_J(z)$ for all $z \in V(\gamma^y)$. Now let $v \in V(D \setminus U_r)$ and consider the event, denoted by $B$, that the path $(S^v_n)_{0 \leq n < \tau_D}$ enters into $B_{2r}(b)$ avoiding $U_{r/2}$ as the landing place:

$$B = \{ S^v(\tau_D \setminus B_{2r}(b)) \in D \setminus U_{r/2} \}.$$

We apply Corollary 4.5 and Corollary 2.3 to see that given the event $B$ as well as the event $S^v(\tau_D) \in J$ occurs, the conditional probability that the walk $S^v$ crosses $\gamma^y$ before exiting $D$ is bounded below by a universal positive constant. (To this end one may first verify that under this conditioning the conditional probability that for any $\eta > 0$ small enough, the walk starting at a vertex in $B_{(2+\eta)r}(b) \setminus U_{r/2}$ exits $B_{3r}(b)$ through the upper half of $J$ is bounded below by a universal constant, and by symmetry the same is true for the exiting through the lower half.) On the other hand, applying Corollaries 4.5 and 2.3 again we infer that $P[B \mid S^v(\tau_D) \in J] \geq c_1$ with a universal constant $c_1 > 0$. Hence, the conditional probability of the walk $S^v$ crossing $\gamma^y$ before exiting $D$ given $S^v(\tau_D) \in J$ is bounded below by a positive universal constant, $c_4$ say, from which, on using the strong Markov property of the conditioned walk, we infer

$$q_J(v) \geq P[S^v(\tau_D) = b, S^v[0, \tau_D] \cap \gamma^y \neq \emptyset \mid S^v(\tau_D) \in J] \geq c_4 q_J(y),$$

showing (4.24) with $c^0 = 1/c_4$. This finishes the proof of Lemma 4.8. \qed
Corollary 4.9. For any $0 < \delta < \frac{1}{10}$ and $M > 1$ there exists $R = R(\delta)$ such that if $D \subseteq \mathbb{D}_{R,M}$ and \( b \in V(U_{\delta/4}) \), then for all \( v \in V(D \setminus U_\delta) \) and \( w \in V(U_{2\delta} \setminus B_{1/2}(b)) \),

\[
H_D(w, b) \leq c_3 H_D(v, b) \quad (4.25)
\]

with a universal constant \( c_3 > 0 \).

Proof. Let \( I \) and \( J \) be as in Lemma 4.8 with \( r = \delta \) and \( R \) be chosen large enough. Then according to Corollary 4.4 uniformly for \( D \subseteq \mathcal{D}_{R,M} \) and \( v \in D \setminus U_\delta \),

\[
H(v; J) > P[S^w(\tau_D) \in J] \geq \frac{1}{2} P[W^v(\tau_\delta) \in I] \geq h(\delta),
\]

(4.26)

with \( h(\delta) = c \delta^2 \) for some \( c > 0 \) (one may take \( c = 1/8\pi \)). According to Corollary 4.3 we can find a number \( \eta = \eta(\delta) \) such that \( P[S^w(\tau_{B_\delta(w)}) \in D] \leq h(\delta) \) for \( w \in U_{\eta\delta} \). Hence if \( w \in U_{\eta\delta} \setminus B_{3\delta}(b) \), then

\[
H(w; J) \leq h(\delta). \quad (4.27)
\]

On the other hand for \( w \in (U_{2\delta} \setminus U_{\eta\delta}) \setminus B_{1/2}(b) \), we see \( P[W^v(\tau_\delta) \in I] \leq 2c^\circ h(\delta) \) where \( c^\circ \) is a universal constant. Thus the bound (4.27) with \( h(\delta) \) replaced by \( c^\circ h(\delta) \) is valid also for such \( w \). Combined with (4.26) we then conclude that if \( w \in U_{2\delta} \setminus B_{1/2}(b) \) and \( v \in D \setminus U_\delta \),

\[
\frac{H(w, b)}{H(v, b)} = \frac{P[S^w(\tau_D) = b \mid S^v(\tau_D) \in J]}{P[S^v(\tau_D) = b \mid S^v(\tau_D) \in J]} \frac{H(w; J)}{H(v; J)} \leq c_2 c^\circ,
\]

where \( c_2 \) is the constant in Lemma 4.8. The proof of Corollary 4.9 is complete.

Remark 3. The estimate of Corollary 4.9 holds also for any pair \( w \in U_\delta \) and \( v \in D \setminus U_\delta \) such that \( \varphi_D \) sends both \( w \) and \( v \) into \( \{ z \in \mathbb{D} : k\delta < |z - \varphi(b)| \leq (k+1)\delta \} \) for some \( k = 3, 4, \ldots \).

4.5. Poisson kernel approximation

Let \( \lambda \) be the usual Poisson kernel for \( \mathbb{D} \):

\[
\lambda(z, w) = \frac{1 - |z|^2}{|z - w|^2} \quad (z \in \mathbb{D}, w \in \partial\mathbb{D}).
\]

The following proposition is proved in [6] when \( G \) is the square lattice and in [16] in the same setting as ours.

Proposition 4.10. For any \( \varepsilon > 0 \) and \( M > 1 \) there exists \( R_0 = R_0(\varepsilon, M) \) and \( \eta > 0 \) such that if \( D \subset \mathcal{D}_{R,M}, b \in V(\partial D) \) and \( a \in V(D \setminus U_\varepsilon) \) (i.e., \( a \in V(D) \) and \( |\varphi_D(a)| < 1 - \varepsilon \)), then for \( b' \in \overline{B}_\eta(\varphi_D(b)) \) with \( H_D(\delta, b') > 0 \),

\[
\left| \frac{H_D(a, b')}{H_D(\delta, b')} - \lambda(\varphi_D(a), \varphi_D(b)) \right| < \varepsilon.
\]

For our application in the next section we work with the upper half plane \( \mathbb{H} = \{ z : \Re z > 0 \} \) and it is convenient to translate the formula of Proposition 4.10 as in the next corollary.

Corollary 4.11. For any constants \( \epsilon > 0 \) and \( M > 1 \), and any compact set \( K \) of \( \mathbb{H} \), there exists \( R > 0 \) such that if \( D \subset \mathcal{D}_{R,M}, \psi : D \rightarrow \mathbb{H} \) is a conformal map with \( \psi(\delta) = i \) and \( b \in V(\partial D) \) with \( H_D(\delta, b) > 0 \), then for all \( y \) and \( w \) from \( V(D) \) such that \( \psi(y), \psi(w) \in K \),

\[
\left| \frac{H_D(w, b)}{H_D(y, b)} - \frac{|\psi(y) - \psi(b)|^2 \Re \psi(w)}{|\psi(w) - \psi(b)|^2 \Re \psi(y)} \right| < \epsilon.
\]
5 Convergence of LERW to chordal SLE₂

This section concerns the convergence to a chordal SLE₂ of the loop erasure of the random walk on the planar graph $G$ started at a boundary vertex of a grid domain $D$ and conditioned to exit at another boundary vertex. After giving a brief exposition of the chordal Loewner chain together with a few preliminary lemmas in Subsections 1 and 2 we state our result on the convergence of LERW (Theorem [5.3]) and advance an abridged proof of it in Section 5.3.

5.1 Chordal Loewner Chain for a simple curve in $\mathbb{H}$

A chordal Loewner chain is the solution of a type of Loewner equation that describes the evolution of a continuum growing from the boundary to the boundary of a simply connected domain of $\mathbb{C}$. For our present purpose we have only to consider the case when the continuum is a simple curve. In this subsection we consider the special case when the domain is $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$, the upper half plane and the curve grows from the origin to the infinity in $\mathbb{H}$, general case will be considered in the next subsection.

Suppose that $\gamma : [0, \infty) \to \mathbb{H}$ is a simple curve with $\gamma(0) \in \mathbb{R}, \gamma(0, \infty) \subset \mathbb{H}$. Then, for each $t \geq 0$, there exists a unique conformal map $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$. It is noted that $g_t$ can be continuously extended to the (two sided) boundary of $\mathbb{H} \setminus \gamma(0, t]$ along $\gamma(0, t]$. For each $t$, there exists the limit

$$\lim_{z \to \infty} z(g_t(z) - z),$$

called the half-plane capacity of $\gamma[0, t]$; $\text{hcap}(\gamma[0, t])$ is real, and increasing and continuous in $t$. If $\gamma$ is parametrized by $\frac{1}{2}$ times the half-plane capacity (so that $\text{hcap}(\gamma[0, t]) = 2t$), then according to Loewner’s theorem $g_t$ satisfies his differential equation

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z, \quad (5.1)$$

where $U(t) = g_t(\gamma(t))$ and $U(t)$ is a $\mathbb{R}$-valued continuous function (see [1]). The equation (5.1) is called the chordal Loewner equation and $U(t)$ the driving function. The family $g_t, t \geq 0$ is called the chordal Loewner chain generated by a curve $\gamma$ and/or driven by a function $U(t)$.

Conversely, given a continuous function $U(\cdot) : [0, \infty) \to \mathbb{R}$, one can solve the ordinary differential equation (5.1) for each $z \in \mathbb{H}$ to obtain the solution $g_t(z)$ up to the time $T_z := \sup\{t > 0 : |g_t(z) - U(t)| > 0\}$. A function $\gamma$ may be defined by $\gamma(t) = \lim_{z \to U(t), z \in \mathbb{H}} g_y^{-1}(z)$, provided the limit exists. If $U(t)$ has a sufficient regularity, this gives a simple curve $\gamma[0, t] = \{z \in \mathbb{H} : T_z \leq t\}$ and then for $t > 0$, $g_t(z)$ is a conformal map from $\mathbb{H} \setminus \gamma[0, t]$ onto $\mathbb{H}$. If $U(\cdot)$ is the driving function of a simple curve $\gamma$ in particular, we can recover $\gamma$ from $U(\cdot)$. If this is the case the curve $\gamma$ may be said to be driven by $U$ (via the Loewner chain).

It is known that if we take a linear Brownian path $\sqrt{\kappa} W(t)$ as $U(t)$, a simple (random) curve $\gamma$ is driven by $U$ for $0 < \kappa \leq 4$ with probability one; in the case $\kappa > 4$, the procedure above still produces a curve which is, however, no longer a simple curve. In either case the random curve driven by $\sqrt{\kappa} W$ is called a chordal $\text{SLE}_\kappa$ curve in $\mathbb{H}$.

In the rest of this subsection let $U(t)$ be a real continuous function of $t \geq 0$. We shall need the following lemma, which is Lemma 2.1 in [1] but restricted to the case when a simple curve is driven by $U$. 

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Lemma 5.1. Suppose that a simple curve $\gamma(t)$ is driven by $U(t)$. Put

$$k(t) = \sqrt{t} + \max_{0 \leq s \leq t} |U(s) - U(0)|.$$  \hspace{1cm} (5.2)

Then, for any $t > 0$, $c^{-1}k(t) \leq \text{diam}(\gamma[0,t]) \leq ck(t)$ for a universal constant $c > 0$.

From the Loewner equation (5.1) we have

$$|g_t(z) - z| < t \cdot \sup_{0 < s \leq t} \frac{2}{|g_s(z) - U(s)|},$$  \hspace{1cm} (5.3)

which the upper bound (easier half) in Lemma 5.1 is deduced from. While the next lemma improves this upper bound, our application of it concerns its another aspect.

Lemma 5.2. Put for $t > 0$ and $z \in \mathbb{H}$

$$\lambda_t = \min_{0 \leq s \leq t} |z - U(s)| \quad \text{and} \quad \mu_t = \min_{0 \leq s \leq t} |g_s(z) - U(s)|.$$

Then $\mu_t > \frac{1}{2} \left( \lambda_t + \sqrt{\lambda_t^2 - 8t} \right)$ as long as $\lambda_t \geq \sqrt{8t}$.

Proof. The proof rests only on (5.3). We deduce from it that $\lambda_t - \mu_t < 2t/\mu_t$. Indeed this inequality certainly holds true for $t_*$ at which the minimum $\mu_t$ is attained so that $\mu_t = \mu_{t_*} = |g_{t_*}(z) - U(t_*)|$; hence it does also for $t$. We may rewrite it as $\mu_t^2 - \lambda_t \mu_t + 2t > 0$. If $\lambda_t \geq \sqrt{8t}$, then, putting $\xi_\pm(t) = \frac{1}{2} \left( \lambda_t \pm \sqrt{\lambda_t^2 - 8t} \right)$, we have either $\mu_t < \xi_-(t)$ or $\mu_t > \xi_+(t)$. Clearly $\xi_-(t) \leq \xi_+(t)$ and $\mu_t - \xi_+(t)$ is continuous and positive for $t$ small enough, hence we must have $\xi_+(t) < \mu_t$ for all $t \leq t_0$, where $t_0 = t_0(z)$ is determined by $\lambda_{t_0} = \sqrt{8t_0}$.

The expression for the logarithmic derivative of the imaginary part of $g_t$ derived from (5.1) we deduce that for $z \in \mathbb{H}$, $t > 0$,

$$\frac{3 \text{Re} g_t(z)}{3 z} = \exp \left( - \int_0^t \frac{2}{|g_s(z) - U(s)|^2} ds \right).$$  \hspace{1cm} (5.4)

Lemma 5.2 combined with (5.3) and (5.4) entails the following corollary.

Corollary 5.3. If $\lambda_t \geq \sqrt{8t}$, then

$$|g_t(z) - z| < \frac{4t}{\lambda_t + \sqrt{\lambda_t^2 - 8t}} \quad \text{and} \quad 1 \geq \frac{\text{Re} g_t(z)}{3 z} > \exp \left( - \frac{8t}{(\lambda_t + \sqrt{\lambda_t^2 - 8t})^2} \right).$$

5.2. Chordal Loewner chains in simply connected domains

We adapt the formulation of [14]. Let $D$ be a simply connected domain with two distinct boundary points $v_0, v_e \in \partial D$ (to be precise these should be prime ends). Let $\psi : D \to \mathbb{H}$ be a conformal map with $\psi(v_e) = 0, \psi(v_0) = \infty$. Although $\psi$ is not unique, any other such map can be written as $y \psi$ for some $y > 0$, so that $v_0, v_e$ being given, the map $\psi$ is in one-to-one correspondence to a point $\hat{o}$ on the curve $(\psi^{-1}(iy))_{0 < y < \infty}$ (running from $v_e$ to $v_0$ in $D$) via the equation $\psi(\hat{o}) = i$. We shall take this $\hat{o}$ as a reference point attached to $D$. 

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For a simple curve $\gamma : (0, T) \to D$ connecting $v_e$ and $v_0$ so that $\gamma(0+) = v_e$ and $\gamma(T-) = v_0$, let $g_t$ be the Loewner chain generated by the curve $\psi \circ \gamma : (0, T) \to \mathbb{H}$. Put $D(t) := D \setminus \gamma(0, t]$ and define $\psi_t : D(t) \mapsto \mathbb{H}$ by

$$\psi_t = g_t \circ \psi|_{D(t)}, \quad t \in [0, \infty),$$

where $\psi|_{D(t)}$ designates the restriction of $\psi$ to $D(t)$. Now we reparametrize the curve $\psi \circ \gamma$ in $\mathbb{H}$ by half plane capacity so that $2t = \text{hcap}(\psi \circ \gamma[0, t]), 0 \leq t < \infty$. The driving function $U(t)$ of the chain $g_t$ is then given by

$$U(t) = \psi_t(\gamma(t)).$$

The family of conformal maps $\psi_t, t \geq 0$ may also be called a chordal Loewner chain (in $D$) with driving function $U(t)$. For each $s > 0$, the curve $\gamma^{(s)}(t) := \gamma(s + t)$ connects $\gamma(s)$ and $v_0$ in $D(s)$, and $\psi_s$ conformally maps $D(s)$ onto $\mathbb{H}$ with $\psi_s(\gamma(s)) = U(s), \psi_s(v_e) = \infty$. On putting

$$g^{(s)}_t = g^{(s)}_{s+t} \circ g^{-1}_s \quad \text{and} \quad \psi^{(s)}_t = \psi_{s+t}|_{D(s)},$$

substitution into $U(s + t) = \psi_{s+t}(\gamma(s + t))$ yields

$$U(s + t) = \psi^{(s)}_t(\gamma^{(s)}(t)).$$

It follows that $\psi^{(s)}_t = g^{(s)}_t \circ \psi_s, g^{(s)}_t$ (and $\psi^{(s)}_t$) is the Loewner chain generated by the curve $\gamma^{(s)}$ and $U^{(s)}(t) := U(s + t)$ is the driving function of the chain $\psi^{(s)}_t$ in $D(s)$.

Define $\hat{o}(t) \in D(t)$ by

$$\psi_t(\hat{o}(t)) = U(t) + i.$$ (5.7)

Then $\hat{o} = \hat{o}(0)$, and $\hat{o}(t)$ will serve as an appropriate reference point of $D(t)$ for our purpose.

The following lemma, proved in [12], plays a significant role to show Theorem 5.5 below.

**Lemma 5.4.** ([12, Lemma 2.4]) Let $\hat{o}(t)$ and $\rho_D(t)$ be as above. Given $T > 1$ and $\epsilon > 0$, put $\hat{T} := \sup\{t \in [0, T] : |U(t)| < 1/\epsilon\}$. There then exists a constant $c(T, \epsilon) > 0$, which does not depend on $(\gamma(t), t \geq 0)$ nor on $D$, such that $\text{in-rad}_{\hat{o}(t)} D(t) \geq c(T, \epsilon) \rho_D$ for $t < \hat{T}$.

**Proof.** In [12] the constant $c(T, \epsilon)$ is allowed to depend on $(D, \gamma(0))$, which is not important therein since $D$ is fixed in the setting of its main theorem. In the first half of the proof of Lemma 2.4 of [12], its substantial part, it is verified that

$$|\phi(\hat{o}(t)) - \phi(\gamma(t'))| \geq 2^{-1} e^{-4\hat{T}} \quad \text{if} \quad t' \leq t < \hat{T}. \quad (5.8)$$

We write down the other half (with slight modification of wording) to ensure that $c(T, \epsilon)$ can be taken independently of $(D, \gamma(0))$. In [14] (the proof of Corollary 4.3) it is proved that

$$\text{in-rad}_{\hat{o}(t)}(D) \geq c_0(T, \epsilon) \text{in-rad}_{\hat{o}(0)}(D) \quad \text{and} \quad \text{dist}(\phi(\hat{o}(t)), \mathbb{R}) \leq M(T, \epsilon).$$

According to the first inequality it suffices for the proof of the lemma to show that

$$|\hat{o}(t) - \gamma(t')| \geq c_1(T, \epsilon) \text{dist}(\hat{o}(t), \partial D) \quad t' \leq t < \hat{T}.$$
If $|\dot{\gamma}(t) - \gamma(t')| < 2^{-1}\text{dist}(\dot{\gamma}(t), \partial D)$, we may take $1/2$ for $c_1(T, \varepsilon)$; otherwise, applying \cite{5,8} and the distortion theorem in turn yields

$$2^{-1}e^{-4T} \leq |\phi(\dot{\gamma}(t)) - \phi(\gamma(t'))| \leq 16|\dot{\gamma}(t) - \gamma(t')|\frac{\text{dist}(\phi(\dot{\gamma}(t)), \mathbb{R})}{\text{dist}(\dot{\gamma}(t), \partial D)}.$$ 

Hence $|\dot{\gamma}(t) - \gamma(t')| \geq [e^{-4T}/32M(T, \varepsilon)]\text{dist}(\dot{\gamma}(t), \partial D)$ as desired. \hfill \Box

5.3. Convergence of driving function

Let $G = (V, E)$ be a planar irreducible graph whose edges are directed and weighted as described in Section 2. For any finite sequence $\omega = (\omega_0, \ldots, \omega_N)$ in $V$, the loop erasure of $\omega$, denoted by $\text{LE}(\omega)$, is the sequence obtained by erasing the loops in it in chronological order.

To be precise $\text{LE}(\omega) = (\hat{\omega}_0, \ldots, \hat{\omega}_L)$ is defined as follows: on putting $\hat{\omega}_0 = \omega_0$, for $k = 0, 1, \ldots$, inductively define $s_k = \max\{n \leq N : \omega_n = \hat{\omega}_k\}$, $\hat{\omega}_{k+1} = \omega_{s(k)+1}$. If $\omega$ is a path in $G$, namely $(\omega_k, \omega_{k+1}) \in E$ for $k = 1, \ldots, N$, then $\text{LE}(\omega)$ is a self-avoiding path in $G$. For our present purpose we consider the loop erasure $\text{LE}(\omega^-)$, where $\omega^- = (\omega_N, \ldots, \omega_0)$, the time-reversal of $\omega$: $\text{LE}(\omega^-)$ is obtained from $\omega$ by erasing the loops in anti-chronological order and tracing the resulting path in that order.

Let $D \in \mathcal{D}$, a simply connected grid domain (see Section 4), $v_0, v_e \in V(\partial D)$ be two distinct boundary vertices and $\psi : D \rightarrow \mathbb{H}$ be as in the preceding subsection, so that $\psi(v_0) = \infty$, $\psi(v_e) = 0$. Define $\hat{\psi}$ by $\hat{\psi} = \psi^{-1}(i)$ and put $\rho_D = \text{in-rad}_\partial(D)$ as before. Let $S^x$ be a natural random walk on $G$ started at $x$ (see Section 2 for detailed description) and suppose that $H_D(v_0, v_e) > 0$ (see Section 4.4 for $H_D$). Let $\Gamma_{v_0,v_e}$ denote an excursion, a natural random walk path, in $D$ started at $v_0$ and conditioned to hit $\partial D$ at $v_e$, where it is stopped.

We identify a path in $G$ with the curve obtained by linearly interpolating it, and accordingly use the same expressions $\Gamma_{v_0,v_e}$, $\text{LE}(\Gamma_{v_0,v_e})$ or the like to denote the (polygonal) curves corresponding to the random walk path in $G$ the expressions originally designate, which abuse of notation will not give rise to any confusion. It is recalled that a chordal $\text{SLE}_2$ curve in $\mathbb{H}$ is the random curve that generates the Loewner chain in $\mathbb{H}$ whose driving function is $\sqrt{2}W(t)$, in particular the curve starts at $0$ a.s. Note that $\text{LE}(\Gamma_{v_0,v_e}^-)$ starts at $v_e$, which $\psi$ sends to $0 \in \partial \mathbb{H}$.

**Theorem 5.5.** If the grid domain $D \in \mathcal{D}$ expands to the whole complex plane $\mathbb{C}$ in such a way that $\rho_D \rightarrow \infty$ and $D/\rho_D$ is confined in a compact set, then the simple curve $\psi \circ \text{LE}(\Gamma_{v_0,v_e}^-)$ converges to the chordal $\text{SLE}_2$ curve in $\mathbb{H}$ with respect to driving function.

The convergence “with respect to driving function” in Theorem 5.5 is paraphrased in more precise terms as follows: the driving function of the random curve $\psi \circ \text{LE}(\Gamma_{v_0,v_e}^-)$ under the conditional probability given that the walk $S^x_{v_0}$ exits $D$ through $v_e$ converges weakly (in the usual sense) to $(\sqrt{2}W^0(t) : t \geq 0)$, the linear Brownian motion path started at $0$ and scaled by $\sqrt{2}$. It is noted that the law of the inverse image by $\psi$ of $\text{SLE}_2$ in $\mathbb{H}$ is independent of the choice of $\psi$ apart from the time-change by scaling with a constant factor.

Suzuki \cite{12} proves the theorem above under the additional assumption that $\partial D$ is locally analytic at $v_0$ and $S^x$ satisfy invariance principle uniformly for the initial point $x \in V$. As is remarked in \cite{12} the first assumption of the local analiticity at $v_0$ is imposed to assure the
conclusion of Proposition 4.6, while the uniformity of invariance principle with respect to initial points of the random walk may be replaced by our hypothesis (H) owing to the results of [16] that are cited as Propositions 2.1 and 4.10 in the present paper.

In what follows we present main steps of the proof of Theorem 5.5, which is similar to those found in [14] or [12], focusing the description on the new ingredients that are special to the present setting.

We designate the self-avoiding path LE((Γ₀,γ₀),v) in G by γ = (γ₀,γ₁,...,γℓ) where γ₀ = vₑ and γℓ = v₀. By the same symbols γ and γ₀,j we also denote the corresponding simple curves according to the convention mentioned previously.

Let gₜ and U(t) be the Loewner chain generated by the simple curve ψ(γ₀,j)⊂ H ∪ {0} and its driving function, respectively. On recalling that gₜ is defined for the curve ˜ψ(γ₀,j) whose time parameter is changed by a function χ so that 2t = hcap ˜ψ([0,χ(t)]), it is appropriate to define for j = 0,1,2,..., tⱼ = χ−¹(j), namely

\[ tⱼ = \frac{1}{2} hcap ψ(γ₀,j). \]

We put Uⱼ := U(tⱼ) and Dⱼ := D \ γ₀,j and often write t(j) for tⱼ. Following [14] we bring in the moving reference point ˆoⱼ := ψ⁻¹(tⱼ)i + Uⱼ. (Here ψᵢ is defined by (5.5), and ˆoⱼ = ˆo(tⱼ) in the notation of Section 5.2.) In radial case, such a point is fixed at the ‘origin’ of D toward which the loop erasure evolves, while in chordal case, ˆoⱼ must be appropriately moved along with j so that there remains a sufficient space around ˆoⱼ in each Dⱼ, j = 0,1,2...,k (for a suitable k << ℓ), which allows us to apply the invariance principle and its consequences obtained in Section 4.

For any ε > 0, let

\[ m := \min\{j ≥ 1 : tⱼ ≥ ε² or |Uⱼ − U₀| ≥ ε}\].

**Lemma 5.6.** There exists a constant c > 0 such that for each ε > 0 and M > 1, there exists R > 0 such that if D ∈ Dᵣ, then

\[ |U_m − U₀| < 2ε, \quad |E[U_m − U₀]| ≤ ce³ \quad and \quad |E[(U_m − U₀)^² − 2t_m]| ≤ ce³. \]

(Although U₀ = 0, we enter U₀ in the formulae above to indicate how they show when the starting position γ₀ of the curve γ is not mapped to the origin by ψ.)

Proving this lemma is an essential step for Theorem 5.5. We give a proof, omitting some details, and indicate where we need Proposition 4.6. We make use of a martingale which is suggested in [6] and adopted in [10, 12] as an observable for the same purpose as ours. We define

\[ Mⱼ := \frac{1}{Z₀} H_D(w,γⱼ)/H_D(v₀,γⱼ) \],

(5.9)

(for any δ > 0 and w ∈ V(D)), where

\[ Z₀ = Z(D,δ,vₑ) = H_D(δ,vₑ)/H_D(v₀,vₑ). \]
Then \( M_j \) is a martingale with respect to the filtration generated by \( \gamma[0, j], j = 0, 1, 2, \ldots \), of which fact an abridged proof is given in \([6]\) (see \([16, 12]\) for a detailed proof).

**Remark 4.** \( Z_0 \) is unbounded and acts as a normalizing constant. In \([6]\) and \([16]\), the radial SLE being concerned, \( v_0 \) is replaced by \( \hat{o} \) and the normalization is not needed (one may take \( Z_0 = 1 \)). Suzuki \([12]\), dealing with the chordal case, adopts a different normalization: \( Z_0 = H_D(v_0; A) \), where \( A = \psi^{-1}([-1, 1]) \). The difference is of only technical matter, although with our choice the proof is simpler (owing to Corollary 4.9).

**Proof of Lemma 5.6.** First we notice a fact which underlies the arguments given below. It is shown in \([14]\) (the first paragraph of the proof of Proposition 4.1 of it) that there exists a universal constants \( c \) and \( R \) such that if \( \rho_D \geq R \) and \( \varepsilon < 1/R \),

\[
\text{in-rad}_{\delta}(D_m) \geq c \rho_D. \tag{5.10}
\]

(In \([14]\) the boundedness of edge length over \( G \) is used, which can be plainly replaced by the property (1) of Section 4.1.)

We are to derive a neat expression of \( 1/Z_0 M_{m} \) as well as \( 1/Z_0 \) (see \([5,12]\) and \([5,13]\) below) by applying Proposition 4.10 (Poisson kernel approximation). Let \( 0 < \varepsilon < 1/2 \) and let \( B_{\varepsilon}(a) \) and \( U_\delta \) be defined as in Section 3.2 (with \( (D, \hat{o}) \)). Then, on writing \( B(\varepsilon^2, v_0) \) for \( B_{\varepsilon^2}(v_0) \)

\[
H_{D_j}(v_0, \gamma_j) = \sum_{y \in C_\varepsilon} P[S^m(\tau_B(\varepsilon^2, v_0)) = y | H_{D_j}(y, \gamma_j)]. \tag{5.11}
\]

where \( C_\varepsilon \) is the set of all vertices \( y \) for which the summand is positive (hence located along \( D \cap \partial B_{\varepsilon^2}(v_0) \)). We split the sum on the right-hand side into two parts according as \( y \in U_\delta \) or not. It is here that we apply Proposition 4.6. Owing to it we can choose \( \delta = \delta(\varepsilon) > 0 \) (independent of \( D_j \)) so that for \( D \) with \( \rho_D \) large enough,

\[
P\left( |\varphi_D(S^m(\tau_B(\varepsilon^2, v_0)))| > 1 - \delta \big| S^m(\tau_B(\varepsilon^2, v_0)) \in D \right) = O(\varepsilon^3).
\]

It is easy to see that \( \gamma[0, t_m] \) is confined in a small neighborhood of \( v_0 \) (cf. the argument given for \((5.14)\) below if necessary) and on using Corollary 4.9 we observe that the conditioning on the event \( S^m(\tau_B(\varepsilon^2, v_0)) \in D \) can be replaced by conditioning on \( S^m(\tau_{D_j}) = \gamma_j, \) implying that on the right-hand side of \((5.11)\) the proportion of the contribution of the sum on \( y \in U_\delta \) to the whole sum is \( O(\varepsilon^3) \).

Once \( \delta \) is determined we apply Proposition 4.10 or rather Corollary 4.11 for computation of the sum over \( V(D) \setminus U_\delta \). It is immediate to see

\[
\frac{1}{Z_0} = \sum_{y \in C_\varepsilon \setminus U_\delta} \frac{\Im \psi(y)}{|\psi(y) - U_0|^2} \left( 1 + O(\varepsilon^3) \right) \tag{5.12}
\]

(provided that \( \rho_D \) is large enough depending only on \( \varepsilon \)). We wish to apply Corollary 4.11 with \( D_m \) in place of \( D \), which is to result in the formula

\[
\frac{1}{Z_0 M_m} = \frac{H_{D_m}(v_0, \gamma_m)}{H_{D_m}(w, \gamma_m)} = \frac{|\psi_{\tau_m}(w) - U_m|^2}{\Im \psi_{\tau_m}(w)} \sum_{y \in C_\varepsilon \setminus U_\delta} \frac{\Im \psi_{\tau_m}(y)}{|\psi_{\tau_m}(y) - U_m|^2} \left( 1 + O(\varepsilon^3) \right). \tag{5.13}
\]
Since $\psi(\partial B_{e_2}(v_0) \setminus U_3)$ is distorted by the mapping $g_{t_m}$, this application requires justification (specifically to ensure the condition that $\psi_{t_m}(y)$ and $\psi_{t_m}(w)$ remain in a compact set of $\mathbb{H}$), which however is given by Corollary 5.3 (see (5.15) below).

In view of (3) of Section 4.1 we have
\[
\text{diam}(\psi_{t(m-1)}(\gamma[t_{m-1}, t_m])) \leq \epsilon^2
\]
whenever $\rho_{D_3}$ is large enough, which fact together with Lemma 5.1 (its harder half) applied to the Loewner chain $\tilde{g}_t := g_{t+t(m-1)} \circ g_{t(m-1)}^{-1}$ implies $t_m - t_{m-1} = O(\epsilon^4)$ and $U(t_m) - U(t_{m-1}) = O(\epsilon^2)$; hence, by the definition of $m$, $t_m < 2\varepsilon^2$ and $\sup\{|U(s) - U(0)| : s \in [0, t_m]\} < 2\varepsilon$. (5.14)

For $z \in \mathbb{D}$ we have
\[
\psi \circ \varphi_D^{-1}(z) = \frac{1 - z}{1 + z},
\]
in particular for $y \in C_{\varepsilon}$, $|\psi(y)| \sim 2/\varepsilon^2$. Hence, according to Corollary 5.3 we obtain for $y \in C_{\varepsilon}$
\[
|\psi_{t(m)}(y) - \psi(y)| = O(\epsilon^4) \quad \text{and} \quad \frac{\Im \psi_{t(m)}(y)}{\Im \psi(y)} = 1 + O(\varepsilon^6);
\]
moreover, using the Loewner equation (5.11) in addition, we also infer that if $w \in V(D)$ is subject to
\[
\Im \psi(w) \geq \frac{1}{2}, \quad |\psi(w)| \leq 3,
\]
then
\[
\psi_s(w) - \psi(w) = \frac{2s}{\psi(w) - U_j} + O(\varepsilon^3) \quad \text{for} \quad s \in [0, t_m],
\]
entailing $\Im \psi_{t(m)}(w) \geq \frac{1}{3}, \quad |\psi_{t(m)}(w)| \leq 4$. Noting $U_m/\psi(y) = O(\varepsilon^3)$ we apply the relation (5.15) to find that the ratio of each term of the sum in (5.12) and the corresponding one of the sum appearing in (5.13) is $1 + O(\varepsilon^3)$, hence so is the ratio of the two sums. Hence by (5.13) and (5.12)
\[
M_m = \frac{\Im \psi_{t(m)}(w)}{|\psi_{t(m)}(w) - U_m|^2} \left(1 + O(\varepsilon^3)\right)
= \Im \left(\frac{-1}{\psi_{t(m)}(w) - U_m}\right) + O(\varepsilon^3).
\]

Let $F(z, u) = \Im[-1/(z - u)], z \in \mathbb{C}, u \in \mathbb{R}$ and compute the difference $F(\psi_{t(m)}(w), U_m) - F(\psi(w), U_0)$ by means of Taylor expansion. With the help of (5.17) as well as $0 = E[M_m - M_0] = E[F(\psi_{t(m)}(w), U_m) - F(\psi(w), U_0)]$, this leads to
\[
\Im \left\{\frac{1}{(\psi_{t(m)}(w) - U_0)^2} E[U_m - U_0] + \frac{1}{(\psi_{t(m)}(w) - U_0)^3} E[(U_m - U_0)^2 - 2t_m]\right\} = O(\varepsilon^3).
\]
(5.18)

Take two distinct vertices $w_1$ and $w_2$ in place of $w$ so that $\psi(w_1) = i + O(\varepsilon^3)$ and $\psi(w_2) = e^{i\pi/3} + O(\varepsilon^3)$, and you find out the required relation of the lemma. □
Proof of Theorem 5.5. Having proved Lemma 5.6 it is easy to adapt the arguments given in [6] for our proof of Theorem 5.5. Let $T > 1$ and $\epsilon_1 > 0$ and put $\hat{T} = \sup\{t \in [0, T] : |U(t)| < 1/\epsilon_1\}$. Let $\epsilon > 0$ be small enough. Let $m_0 = 0$ and define $m_n$ inductively by

$$m_n := \min\{j > m_{n-1} : t_j - t_{m_{n-1}} \geq \epsilon^2 \text{ or } |U_j - U_{m_{n-1}}| \geq \epsilon\}.$$  

Let $N := \max\{n \in \mathbb{N} : t_{m_n} < \hat{T}\}$. By a Markovian nature of the walk $S^{v_0}$ conditioned on $\gamma[0, j]$ (Lemma 3.2 of [6]) together with the Huygens property (5.6) of the Loewner chain, we apply Lemma 5.6 with $(D_{m_n}, v_0, \gamma_{m_n}, \hat{\delta}_{m_n}, U(\cdot + t_{m_n}))$ in place of $(D, v_0, v_\epsilon, \hat{\delta}, U)$. The application is secured owing to Lemma 5.4 which shows that in-rad$_\delta(D_j)/\rho_D$ is bounded below by a positive constant for $n \leq N$. It then follows that there exists $R = R(\epsilon, \epsilon_1, T, M) > 0$ such that if $D \in \mathcal{D}_{R, M}$, then for any $n \leq N$

$$|U_{m_{n+1}} - U_{m_n}| < 2\epsilon, \quad \mathbf{E}[U_{m_{n+1}} - U_{m_n} \mid \gamma[0, m_n]] = O(\epsilon^3),$$  

and

$$\mathbf{E}[(U_{m_{n+1}} - U_{m_n})^2 \mid \gamma[0, m_n]] = \mathbf{E}[2(t_{m_{n+1}} - t_{m_n}) \mid \gamma[0, m_n]] + O(\epsilon^3).$$

It is a more or less standard issue of the probability theory to deduce from these three relations that the law of $U(t)$ weakly converges to that of a scaled Brownian motion $\sqrt{2}W(t)$ as $\rho_D \to \infty$ and is carried out in [6] (Section 3.3: especially the arguments following Eqs.(3.16-17)).

Remark 5. Another observable. In [6] the random variables $h^+_j =$the number of visits to $w$ by $\Gamma^{v_0, \gamma_j}[0, n_j]$ is adopted as a martingale observable for deriving the estimate corresponding to Lemma 5.6 (see Proposition 3.4 of [6]). In the present setting we have

$$E[h^+_j \mid \gamma[0, j]] = \frac{\hat{G}_j(v_0, w)\hat{H}_j(w, \gamma_j)}{\hat{H}_j(v_0, \gamma_j)},$$

where $\hat{\cdot}$ indicates the corresponding objects for the walk conditioned to exits $D$ through $v_\epsilon$. We can adapt the proof in [6], provided that in our assumption (H) the weak convergence is understood relative to not the metric $d_\mathbb{D}$ but the usual metric for uniform convergence of functions of $t \geq 0$ on each finite interval. This modification of (H) is needed since for the observable we need to approximate the Green function $G_j(y, w)$ by the corresponding one for Brownian motion.

Remark 6. From the convergence of the driving process it follows that the image of loop erasure $\psi(\text{LE}((\Gamma^{v_0, v_\epsilon})^-))$ converges relative to Hausdorff metric if the time (parametrized by the h-capacity) is restricted to a finite interval (cf. [6], [14]).

5.4. Uniform convergence.

Suzuki [12] points out that if the transition probability $p(u, v)$ has an invariant measure $\pi$ and the dual walk with respect to it as well as the original walk satisfies invariance principle (as given in [12]), then the convergence of $\psi(\text{LE}(\Gamma^{v_0, v_\epsilon}))$ with respect to the metric of ‘uniform’ convergence of path functions follows from their convergence with respect to driving function (i.e., convergence of their driving functions). His reasoning is based on a result by Sheffield
and Sun [15, Corollary 1.7], which asserts that if a sequence of random simple curves in \( \mathbb{D} \) and that of their time-reversals both converge to SLE\(_{\kappa} \) curves with \( \kappa \leq 4 \) with respect to driving function, then the weak convergence with respect to the metric \( d^*_{\mathbb{D}} \) holds true. It is shown by Lawler [5] that for symmetric random walks (in fact Markov chains), the law of \( \text{LE}((\Gamma^{\nu_0,\nu_0})^-) \) agrees with that of \( (\text{LE}(\Gamma^{\nu_0,\nu_0})^-) \). Suzuki observes that the symmetry assumption is dispensable so that the result extends to general Markov chains. Note that on using this result, \( \text{LE}((\Gamma^{\nu_0,\nu_0})^-) \) has the same distribution as \( L((\Gamma^*\nu_0,\nu_0)^-) \) under the existence of \( \pi \), where \( \Gamma^* \) designates the excursion of the dual walk. (Cf. Section 5.2 of [12] for more details).

In below we formulate the corresponding result in the usual setting in which we consider the loop erasure of the scaled random walk that is confined in a fixed domain. For simplicity we shall assume \( p(u, v) = p(v, u) \) so that the condition on the dual walk mentioned above is plainly true.

Let \( D \) be a bounded and simply connected domain of \( \mathbb{C} \) (not necessarily a grid domain). We are concerned with the correspondence between points of \( \partial D \) and those of the unit circle \( \partial \mathbb{D} \), which is nicely given by means of prime ends of \( D \), ideal boundary points. We do not give any definition of prime ends for which the readers may be referred to [11] or [1]. What is needed in this paper are the facts mentioned below. There is a natural one-one correspondence between the set of prime ends and \( \partial U \). We write \( \partial_{pmd} D \) for the set of all prime ends of \( D \) and denote the correspondence by \( \iota \). Let \( \varphi : D \mapsto \mathbb{D} \) be any conformal map and denote by \( \varphi \cup \iota \) its extension to \( D \cup \partial_{pmd} D \) by means of \( \iota \). A topology of \( D \cup \partial_{pmd} D \) is given so that \( \varphi \cup \iota \) is a homeomorphism from \( D \cup \partial_{pmd} D \) onto \( \overline{\mathbb{D}} \). For \( \zeta \in \partial \mathbb{D} \) let \( C(\varphi, \zeta) \) be the set of all limit points of \( \varphi^{-1}z \) as \( z \in \mathbb{D} \to \zeta \):

\[
C(\varphi, \zeta) = \bigcap_{r>0} \{ \varphi^{-1}(z) : z \in \mathbb{D}, |z - \zeta| < r \}.
\]

The correspondence \( \varphi \) is natural in the sense that the topology induced is independent of the choice of \( \varphi \) and if a sequence \( x_n \in D \) converges to \( x \in \partial_{pmd} D \), then \( \text{dist}(x_n, C(\varphi, \iota(x))) \) tends to zero (and under an additional condition on \( (x_n) \) the converse holds). (Cf. Section 2.5 of [11].)

**Theorem 5.7.** Suppose, in addition to the basic hypothesis (H), that \( p \) is symmetric: \( p(u, v) = p(v, u) \). Let \( D \) be a simply connected domain, \( \varphi \) be a conformal map of \( D \) onto \( \mathbb{D} \) (as above) and \( a, b \in \partial_{pmd} D \) be two distinct prime ends of \( D \). Let \( (\rho_n) \) be a sequence of positive numbers and \( (u(n)) \) and \( (v(n)) \) be two sequences in \( V(D) \) such that both \( u(n)/\rho_n \) and \( v(n)/\rho_n \) are in \( \hat{D} \) and \( \rho_n \to \infty \), \( u(n)/\rho_n \to a \) and \( v(n)/\rho_n \to b \) as \( n \to \infty \) and that \( S^{u(n)} \), the random walk started at \( u(n) \), arrives at \( v(n) \) before exiting \( \rho_n D = \{ \rho_n x : x \in D \} \) with positive probability. Let \( \mu_n \) be the conditional probability law of the path \( \Gamma_n := (S^{u(n)}(k) : k = 0, \ldots, \sigma_{v(n)}) \), given that \( S^{u(n)} \) visits \( v(n) \) before exiting \( \rho_n D \), and \( \gamma_n \) be the linear interpolation of the loop erasure \( \text{LE}(\Gamma_n) \). Then the law of \( \varphi \circ \gamma_n \) induced from \( \mu_n \) weakly converges relative to the metric \( d_{\mathbb{D}} \) to that of the chordal SLE\(_2 \) curve from \( \varphi(a) \) to \( \varphi(b) \) in \( \mathbb{D} \).

**Proof.** We reduce the problem to Theorem 5.5. To this end let \( V_n \) be the set of all vertices that the walk \( S^{u(n)} \) arrives before exits \( \rho_n D \) with positive probability:

\[
V_n = \{ w \in V : P[S^{u(n)}_k = w \text{ for some } k < \tau_{\rho_n D}] > 0 \}
\]
and define a grid domain $D_n \in \mathcal{D}$ as the smallest one among those that contains $V_n$ (in their interior). Then $S_{\Omega(n)}$ exits $D_n$ and $\rho_n D$ at the same time and $D_n/\rho_n$ converges to $D$ in the Carathéodory sense (see [10], Proposition 3.63). Noting that in Theorem 5.5 $v_0$ and/or $v_e$ have not to be taken from boundary; they may be vertices near the boundary (see Proposition 3.10), the premise in Corollary 1.7 of [15] mentioned above is readily verified.

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