COINTEGRATION IN CONTINUOUS TIME FOR FACTOR MODELS

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ABSTRACT. We develop cointegration for multivariate continuous-time stochastic processes, both in finite and infinite dimension. Our definition and analysis are based on factor processes and operators mapping to the space of prices and cointegration. The focus is on commodity markets, where both spot and forward prices are analysed in the context of cointegration. We provide many examples which include the most used continuous-time pricing models, including forward curve models in the Heath-Jarrow-Morton paradigm in Hilbert space.

1. INTRODUCTION

We aim at developing a formalism to the concept of cointegration in continuous time. Cointegration has since the seminal paper of Engle and Granger [19] become a very popular concept for stochastic modelling of dependent time series of data, in particular in economics. For example, the price series of two financial assets can be non-stationary, while one may find that a linear combination of these is stationary. Cointegration provides a framework for analysing and modelling time series that explains such observable features in data.

Although there has been a huge development in continuous-time financial models over the last decades, the literature on cointegration for continuous-time stochastic processes and its application to finance is relatively scarce. A non-exhaustive list of papers in this stream of research include Comte [15], Duan and Pliska [17], Duan and Theriault [18], Nakajima and Ohashi [31], Paschke and Prokopczuk [33], Benth and Koekebakker [10], and recently Farkas et al. [23]. In the present paper we formalise ideas on cointegration in continuous time for factor processes, and extend these to cointegration for stochastic processes with infinite dimensional state space. The latter will provide a theoretical framework for studying cointegration in forward and futures markets, say.

Comte [15] presents an in-depth analysis on classical cointegration and its extension to continuous-time models, where continuous-time autoregressive moving average processes (CARMA) play a central role. Duan and Pliska [17] analyse a specific cointegrated asset price model, and show that pricing options will not be influenced by cointegration. Their paper has triggered many theoretical and empirical
studies, including Nakajima and Ohashi [31], Paschke and Prokopczuk [33], Benth and Koekebakker [10] and Farkas et al. [23]. Duan and Theriault [18] extend cointegration to continuous-time forward price models. Benth and Koekebakker [10], and more recently Benth [6], focus on the relationship between cointegration in spot and forward markets, and propose cointegration models for forward markets. Contrary to the conclusions of Duan and Pliska [17], these two papers argue that in commodity markets the pricing measure may preserve cointegration. We refer to Back and Prokopczuk [4] for a review of modelling and pricing in commodity markets.

Starting with spot price models, we discuss a framework for cointegration based on factor models. Our concept makes use of a set of stochastic processes, which we call factors, which explains the dynamics of prices via a linear transformation. This yields a vector-valued price dynamics, for which one can introduce the concept of cointegration. The following example is frequently referred to in the text, and explains our ideas in a simple setting.

**Example 1.** Consider the classical spot price model for two commodity markets, given by the two-factor model,

\[
S_i(t) = X_i(t) + X_3(t), i = 1, 2.
\]

We assume \( X_3(t) = \mu t + \sigma B_3(t) \) being a drifted Brownian motion and \( X_i(t) \) being two Ornstein-Uhlenbeck processes,

\[
dX_i(t) = -\alpha_i Y_i(t) dt + \eta_i dB_i(t),
\]

with constants \( \alpha_i > 0, \eta_i > 0, i = 1, 2 \). Here, \((B_1, B_2, B_3)\) is a trivariate Brownian motion, possibly correlated. This model was proposed in the univariate case by Lucia and Schwartz [29] for electricity spot prices and extended to cross-commodity markets by Paschke and Prokopczuk [32] for oil markets (see also Duan and Pliska [17] and Benth and Koekebakker [10] for general analysis). For example, \((S_1, S_2)\) can model the joint spot price dynamics in the coal and electricity market, or in two different electricity markets. Since the bivariate Ornstein-Uhlenbeck process \((X_1, X_2)\) admits a limiting Gaussian distribution, the price difference process \(S_1(t) - S_2(t)\) will have a limiting distribution. On the other hand, each marginal price process \(S_i\) is non-stationary since the drifted Brownian motion \(X_3\) is (unless \( \mu = \sigma = 0 \)). According to Duan and Pliska [17], the processes \(S_1\) and \(S_2\) are cointegrated. We notice that the definition of the bivariate price process \((S_1, S_2)\) involves three factor processes \(X_1, X_2, X_3\), and a linear combination of these. Introducing the matrix

\[
\mathcal{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},
\]

we represent the vector \( \mathbf{S} := (S_1, S_2)^\top \) as \( \mathbf{S}(t) = \mathcal{P}(X_1(t), X_2(t), X_3(t))^\top \). We assume that all elements \( \mathbf{x} \in \mathbb{R}^n, n \in \mathbb{N} \), are column vectors, and \( \mathbf{x}^\top \) is the transpose of \( \mathbf{x} \). Cointegration is achieved since there exists a vector \( \mathbf{c} = (1, -1)^\top \in \mathbb{R}^2 \) such that the process \( \mathbf{c}^\top \mathbf{S} = X_1 - X_2 \) admits a limiting distribution.

Based on multivariate spot price models of the form introduced in this example, we analyse forward prices derived from processes with certain affinity properties. In this context, polynomial processes (see Filipovic and Larsson [25]) constitute an important case, along with the more specific CARMA processes. We present results on the cointegration relationship between spot and forward markets, with a particular attention to pricing measures and the application to commodity markets.
Our analysis of spot and forward markets motivates the definition of cointegration for stochastic processes in infinite dimensions. We introduce a concept for modelling cointegrated forward curves, following the HJM-paradigm (see Heath, Jarrow and Morton [26]) of modelling forward prices directly rather than explaining these via spot models (we refer to Benth, Šaltytė Benth and Koekebakker [7] for an extensive analysis of forward modelling in energy markets). Since forward curves can be modelled as stochastic processes in Hilbert space of real-valued functions on \( \mathbb{R}_+ \) (see Benth and Krühner [11, 12]), we concentrate our analysis on formulating cointegration via linear operators on Hilbert spaces. We show how cointegration in Hilbert space can be related to the finite dimensional case. It turns out that product Hilbert spaces provide a natural framework for modelling, and we give several examples including infinite dimensional factor processes capturing stationary and non-stationary effects as well as non-Gaussianity. We also include a discussion of some recent empirical studies on forward gas markets by Geman and Liu [22] viewed in our cointegration context.

The results of this paper is presented as follows: in Section 2 we define and analyse cointegration for multivariate spot price models based on factor processes. The question of forward pricing in cointegrated spot markets is analysed in Section 3, where we give a description of cointegration of forwards. Finally, in Section 4 we introduce cointegration for Hilbert-space valued stochastic processes, and apply this to cross-commodity forward prices modelled within the HJM-approach.

2. Cointegration for factor models

Suppose that \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a complete probability space equipped with a right-continuous filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) where \( \mathcal{F}_t \) contains all sets of \( \mathcal{F} \) of probability zero (i.e., satisfying the usual conditions). Let \( \{S(t)\}_{t \geq 0} \in \mathbb{R}^d \) be \( d \) asset prices in a given market, where we define

\[
S(t) = P X(t), \quad t \geq 0
\]

for an adapted stochastic process \( \{X(t)\}_{t \geq 0} \in \mathbb{R}^n \) and \( P \in \mathbb{R}^{d \times n} \). The matrix \( P \) is hereafter referred to as the pricing matrix, and \( X \) the factor process of the market. For convenience, we assume that the number of factors \( n \) is at least equal to the number of assets \( d \), i.e., \( n \geq d \). We reserve the notation \( \{e_i\}_{i=1}^k \) for the canonical basis vectors in \( \mathbb{R}^k \), where the dimension \( k \in \mathbb{N} \) will be clear from the context.

**Definition 2.** The pricing matrix \( P \) is called minimal if all factors \( X_i(t), i = 1, \ldots, n \) are represented in \( S(t) \).

The definition simply says that we have all necessary factors to define the price dynamics \( S \). It does not say that all factors are present in each price coordinate \( S_j(t), j = 1, \ldots, d \).

**Lemma 3.** \( P \) is minimal if and only if \( e_i \notin \ker(P) \) for all \( i = 1, \ldots, n \).

**Proof.** Obviously,

\[
X(t) = \sum_{j=1}^n X_j(t)e_j,
\]

where \( X_j(t) = X^\top(t)e_j \). Hence,

\[
S(t) = \sum_{j=1}^n X_j(t)Pe_j.
\]
If \( e_i \in \ker(\mathcal{P}) \) for given \( i \in \{1, \ldots, n\} \), then \( S \) will not depend on \( X_i \). Opposite, if \( S \) is not depending on \( X_i \), then \( \mathcal{P}e_i = 0 \).

We restrict our considerations to minimal pricing matrices \( \mathcal{P} \). We also confine ourselves to non-degenerate markets, that is, markets where we have \( d \) distinct price processes for the \( d \) assets. Hence, we assume that \( \mathbb{P}(S_k(t) = S_j(t), t \geq 0) = 0 \) for any \( k, j = 1, \ldots, d \) with \( k \neq j \). A sufficient condition for this to hold is when the \( d \) row vectors of \( \mathcal{P} \) are linearly independent, that is, when \( \text{rank}(\mathcal{P}) = d \). We restrict our analysis to this case.

Assuming the price process \( S \) is defined by \( \Psi \) with \( \mathcal{P} \) being a minimal pricing matrix having full rank, we define cointegration as follows: Let \( P_X(t, \cdot) \) denote the probability distribution of \( X(t) \) defined on \( \mathcal{B}(\mathbb{R}^n) \), the Borel \( \sigma \)-algebra on \( \mathbb{R}^n \). For any \( x \in \mathbb{R}^n \), we denote by \( P_{c^\top X}(t, \cdot) \) the probability distribution of the real-valued random variable \( x^\top X(t) \). Furthermore, denote by \( \Psi_X(t, z) \) the characteristic function of \( X(t) \), defined for \( z \in \mathbb{R}^n \) as

\[
\Psi_X(t, z) = \mathbb{E} \left[ e^{iz^\top X(t)} \right].
\]

Sometimes, one is using the cumulant function instead of the characteristic function, where the cumulant \( \kappa_X(t, z) := \log \Psi_X(t, z) \) with \( \log \) denoting the distinguished logarithm (see e.g. Sato [35]). We see that \( \Psi_X(t, z) = \hat{P}_X(t, z) \), where \( \hat{P}_X(t, z) \) is the Fourier transform of the distribution \( P_X(t, \cdot) \).

**Definition 4** (Definition of cointegration). We say that \( S \) is cointegrated if there exists \( c \in \mathbb{R}^d \) and a probability distribution \( \mu_c \) on \( \mathcal{B}(\mathbb{R}) \) such that \( P_{c^\top \mathcal{P}X}(t, \cdot) \) converges to \( \mu_c \) when \( t \to \infty \). We call \( c \) a cointegration vector for \( S \).

In the definition of cointegration, the convergence is in the sense of probability measures (see Def. 2.2 in Sato [35]). The definition of cointegration means that there exists a linear combination of the price process vector \( \{S(t)\}_{t \geq 0} \) which admits a limiting probability distribution, where the linear combination is represented by the cointegration vector \( c \). We recall that \( c^\top S(t) = c^\top \mathcal{P}X(t) \), and thus \( P_{c^\top \mathcal{P}X}(t, \cdot) \) is the probability distribution of \( c^\top S(t) \), which must converge to a probability distribution when time tends to infinity in order to achieve cointegration.

Definition 4 includes trivially the case \( c = 0 \in \mathbb{R}^d \), since \( P_{0^\top \mathcal{P}X}(t, \cdot) = \delta_0(\cdot) \) with \( \delta_0 \) being the Dirac measure at zero. From a practical viewpoint, we are obviously not interested in this degenerate case of a cointegration vector, but include \( c = 0 \) in any case for completeness. If we choose \( n = d \) and \( \mathcal{P} = I \), the \( d \times d \) identity matrix, we have \( S = X \). Thus the definition of cointegration can also be directly applied for the factor process.

As the next result shows, cointegration can be characterized by convergence of characteristic functions when time tends to infinity:

**Proposition 5** (Cumulant characterisation of cointegration). If \( S \) is cointegrated with cointegration vector \( c \in \mathbb{R}^d \), then \( \lim_{t \to \infty} \Psi_X(t, z^\top c) = \Psi_{\mu_c}(z) \) uniformly (in \( z \in \mathbb{R} \)) on any compact set, with \( \Psi_{\mu_c} \) being the characteristic function of the distribution \( \mu_c \). Opposite, if there exists a \( c \in \mathbb{R}^d \) and a complex-valued function \( z \mapsto \Psi_c(z) \) on \( \mathbb{R} \) which is continuous at \( z = 0 \) such that \( \lim_{t \to \infty} \Psi_X(t, z^\top c) = \Psi_c(z) \) for every \( z \in \mathbb{R} \), then \( S \) is cointegrated with cointegration vector \( c \).

**Proof.** By Definition 4 of cointegration we have that \( P_{c^\top \mathcal{P}X}(t, \cdot) \to \mu_c \) for a probability distribution \( \mu_c \). It is well-known (see e.g., Sato [35] Prop. 2.5 (vi)) that this
P is a pricing matrix satisfying the assumptions of minimality and full rank. In this model, \((X, c)\) the existence of a unique \(c\) ker\((c)\) of \((X\) system for the factor process \(c\) then for any \(d < n\) that \(a\) system of prices \(c\) particular, if \(C\) rank, \(\text{rank}(P)\) follows from the fact that the \(P\) may not give rise to a cointegration vector for \(X\) may fail to exist, so even if \(X\) admits a cointegration vector, it may not give rise to a cointegration vector for \(S\). If \(d = n\), then \(c := P^{-1}a\) since \(P\) is invertible due to the full rank assumption. However, the typical situation is that \(d < n\), and then the linear system \(P^\top c = a\) is over-determined and in general will not possess a solution.

We have the following convenient definition:

**Definition 6.** Denote by \(C_X\) the set of all cointegration vectors for \(X\) and \(C_S\) the set of all cointegration vectors for \(S\).

We note from the discussion above that if \(c \in C_S\), then \(P^\top c \in C_X\). Hence, \(P^\top C_S \subset C_X\). For many specifications of \(P\), this inclusion is strict, telling that the set of cointegration vectors for \(S\) is restricted compared to the range of cointegration possibilities given by the vector \(X\). But if \(a \in C_X\) is in the image of \(P^\top\), then we have the existence of a unique \(c \in \mathbb{R}^d\) such that \(P^\top c = a\), that is, \(c \in C_S\). Uniqueness of \(c\) follows from the fact that the \(n \times d\) matrix \(P^\top\) has full column rank \(d\). In particular, if \(C_X \subset \text{Range}(P^\top)\), then \(P^\top C_S = C_X\).

We define a cointegrated pricing system:

**Definition 7.** If \(P \in \mathbb{R}^{d \times n}\) is a pricing matrix, i.e., minimal and with \(\text{rank}(P) = d\), and \(c \in \mathbb{R}^d\) is such that \(P^\top c \in S_X\), we say that \((P, c)\) is a cointegrated pricing system for the factor process \(X\).

If \((P, c)\) is a cointegrated pricing system for the factor process \(X\), we can define a system of prices \(S(t) = PX(t)\) which becomes cointegrated for the vector \(c\), according to Definition 4. To a given \(a \in S_X\), there may exist many cointegrated pricing systems \((P, c)\); indeed all possible combinations of pricing matrices \(P\) and vectors \(c\) such that \(P^\top c = a\).

Let us return to Example 1, where we considered two spot price dynamics given by 1. We recall the pricing matrix \(P\) in 2, and the factor process \(X(t) = (X_1(t), X_2(t), X_3(t))\) for \(X_3\) a drifted Brownian motion and \((X_1, X_2)\) a bivariate Ornstein-Uhlenbeck process. As \(P\) has two independent row vectors, it has full rank, \(\text{rank}(P) = 2\). Moreover, we easily see that \(Pe_i \neq 0\) for \(i = 1, 2, 3\). Indeed, \(\ker(P)\) has dimension 1 and is spanned by the vector \((1, 1, -1)\). We conclude that \(P\) is a pricing matrix satisfying the assumptions of minimality and full rank. In this model, \((X_1, X_2)\) is a 2-dimensional Ornstein-Uhlenbeck process,

\[
X_i(t) = X_i(0)e^{-\alpha_i t} + \int_0^t \eta_i e^{-\alpha_i(t-s)} dB_i(s) \quad i = 1, 2.
\]

We find \((X_1(t), X_2(t)) \overset{d}{\rightarrow} \mathcal{N}(0, C)\).
where the covariance matrix $C \in \mathbb{R}^{2 \times 2}$ is

$$C = \begin{bmatrix} \frac{\eta^2}{2 \alpha_1} & \rho_{\eta \eta} \frac{m \alpha_2}{\alpha_1 + \alpha_2} \\ \rho_{\eta \eta} \frac{m \alpha_2}{\alpha_1 + \alpha_2} & \frac{\eta^2}{2 \alpha_2} \end{bmatrix}.$$ 

Here, $\rho$ is the correlation between $B_1$ and $B_2$, and $\rightarrow$ denotes limit in distribution. Hence, since $(X_1, X_2)$ has a limiting distribution, we find from the non-stationarity of $X_3$ that $\mathcal{C}_X = \{ a \in \mathbb{R}^3 \mid a_3 = 0 \}$. In particular, $\mathcal{C}_X$ is a vector space with basis vectors $e_1$ and $e_2$. We remark that in general, $\mathcal{C}_X$ does not need to be a vector space. If we for example substitute $X_1$ and $X_2$ with two stationary stochastic processes which are not jointly stationary, we have that a linear combination of the two may fail to be stationary even though they are marginally stationary. We find further that $\mathcal{C}_S$ is the vector space spanned by the vector $(1,-1)^T$. Finally, the range of $\mathcal{P}^T$ is spanned by the two vectors $(1,0,1)^T$ and $(0,1,1)^T$, i.e., the row vectors of $\mathcal{P}$. Thus, if $a \in \mathcal{C}_X$, then $a$ is in the range of $\mathcal{P}^T$ only when $a = k(1,-1,0)^T$ for $k \in \mathbb{R}$. Therefore, $\mathcal{P}^T \mathcal{C}_S \subset \mathcal{C}_X$, with a strict inclusion in this case. From these considerations, we also see that there exists many pricing systems $(\mathcal{P},c)$, indeed, for a fixed $\mathcal{P}$ we have a continuum of $c \in \mathcal{C}_S$. But we may also choose different pricing matrices. For example, if

$$\mathcal{P} = \begin{bmatrix} a & b & w \\ u & v & 1 \end{bmatrix},$$

for any $a,b,u,v,w \in \mathbb{R}$ such that $\mathcal{P}$ is minimal and non-degenerate, we can use $c = (1,-w)^T$ to define a pricing system, where $c^T \mathcal{P} X(t) = (a - uw)X_1(t) + (b - vw)X_2(t)$. Such a pricing matrix $\mathcal{P}$ is relevant when modelling two commodities that do not share the same denominator. For example, gas and coal typically have different energy units than power, and we will have a conversion factor (heat rate) between them modelled by $w$ in the present context.

The particular example discussed above motivates some further analysis of the set $\mathcal{C}_X$. In many situations, as in the example, we can single out a subset of factors from $X$ which has a limit in distribution, i.e., $X^m(t) := (X_1(t),\ldots,X_m(t))^T$ with $m \leq n$ for which $P X^m(t,\cdot) \to \mu^m$ for a probability distribution $\mu^m$ on $\mathbb{R}^m$ as $t \to \infty$. Then $\mathcal{C}_{X_1} \subset \mathcal{C}_X$, where

$$\mathcal{C}_{X_1}^m := \{ a \in \mathbb{R}^n \mid a_{m+1} = \ldots = a_n = 0 \}.$$ 

Remark that we do not in general have equality between $\mathcal{C}_{X_1}^m$ and $\mathcal{C}_X$ as there may be cointegration between some of the factors $X_{m+1},\ldots,X_n$ that may not hold jointly with the first $m$ factors. For convenience, we have assumed that the subset of factors which has a limiting distribution consists of the first $m$. Since we may re-label the factors, this assumption is of course without loss of generality. We observe that $\mathcal{C}_{X_1}^m$ is a vector space, and that in the case $m = n$, we trivially have $\mathcal{C}_{X_1}^n = \mathcal{C}_X = \mathbb{R}^n$. When $m < n$, any $(\mathcal{P},c)$ such that $\mathcal{P}^T c \in \mathcal{C}_{X_1}^m$ will be a cointegrated pricing system for $X$. We observe that these considerations are in line with the example above, where $\mathcal{C}_{X_1}^2 = \mathcal{C}_X$ since the two first factors have jointly a limiting distribution, while the last factor is non-stationary. We have the following general result:

**Lemma 8.** Suppose $P_{X_{n-1}}(t,\cdot)$ has a limiting distribution, while $P_{X_n}(t,\cdot)$ does not have a limiting distribution. If $X_n$ is independent of $X^{n-1}$, then $\mathcal{C}_X = \mathcal{C}_{X_1}^{n-1}$. 
**Proof.** Let $c \in \mathcal{C}_X$ with $c_n \neq 0$. By independence, we find for $z \in \mathbb{R}$

$$
\Psi_{c \uparrow X}(t, z) = \mathbb{E}[e^{iz(t-t_0)}X(t)]
$$

$$
= \mathbb{E}[e^{iz(c_1X_1(t)+\cdots+c_{n-1}X_{n-1}(t))}]\mathbb{E}[e^{izc_nX_n(t)}]
$$

$$
= \Psi_{X_{n-1}}(t, z(c_1, \ldots, c_{n-1})^\top)\Psi_{X_n}(t, zc_n).
$$

For every $z$, $\Psi_{X_{n-1}}(t, z(c_1, \ldots, c_{n-1})^\top)$ will have a limit, while there exists a Borel set $A_0$ with positive Lebesgue measure such that $\Psi_{X_n}(t, z) = 0$ does not have a limit for every $x \in A_0$ (this could be the whole of the real line, or some subset with infinite Lebesgue measure). But then for all $z \in A_0/c_n$ we have that $\Psi_{X_n}(t, zc_n)$ does not have a limit, and in conclusion $\Psi_{c \uparrow X}(t, z)$ does not have a limit for every $z \in \mathbb{R}$ as $t \to \infty$. This violates the assumption that $c \in \mathcal{C}_X$ with $c_n \neq 0$. Thus, $c_n = 0$, showing the claim. \hfill \Box

Remark that in Example 1 the non-stationary drifted Brownian motion is not necessarily independent of the two other factors, showing that the assumption of independence is sufficient, but not necessary.

Notice that if $X^m$ admits a stationary limit, and $(X_{m+1}, \ldots, X_n)$ is dependent on $X^m$, we may have non-trivial $c \in \mathcal{C}_X \setminus \mathcal{C}_X^m$. Indeed, consider $n = 3$ and the processes

$$
X_1(t) = \int_0^t \exp(-\alpha_1(t-s)) dB_1(s),
$$

and

$$
X_i(t) = \mu t + \int_0^t \exp(-\alpha_i(t-s)) dB_i(s), \ i = 2, 3,
$$

for constants $\mu, \alpha_i > 0, i = 2, 3$ and a trivariate Brownian motion $(B_1, B_2, B_3)$ being correlated. Then $X = (X_1, X_2, X_3)^\top$ have dependent coordinates, and for any vector $c = (a, b, -b)^\top \in \mathbb{R}^3, a, b \in \mathbb{R}$, we find that

$$
c^\top X(t) = a \int_0^t e^{-\alpha_1(t-s)} dB_1(s) + b \left( \int_0^t e^{-\alpha_2(t-s)} dB_2(s) - \int_0^t e^{-\alpha_3(t-s)} dB_3(s) \right)
$$

which will converge in distribution to a normally distributed random variable with zero mean as $t \to \infty$. Hence, $c \notin \mathcal{C}_X^m$. Here, $X_1$ has a limit in distribution, while $X_i, i = 2, 3$ both will have a mean $\mu t$ and thus there does not exist any limiting distribution. This is an example with $m = 1$ and $n = 3$. We remark that the example is slightly pathological, as we could have assumed $n = 4$ with $X_4(t) = \mu t$, and defined $\tilde{X}_4(t) = \int_0^t \exp(-\alpha_4(t-s)) dB_4(s), i = 2, 3$. Then, with $\mathbf{X} := (X_1, \tilde{X}_2, \tilde{X}_3, X_4)^\top$ we are back to the situation with $m = n - 1 = 3$ and $X_4$ being (trivially) independent of $\mathbf{X}^3 = (X_1, \tilde{X}_2, \tilde{X}_3)^\top$.

We have the following remark, which gives a practical consequence of our considerations so far:

**Remark 9.** In a practical application we can model a system of $d$ commodity price dynamics with cointegration as follows: first, we assume that we have $m$ factor processes which jointly admit a limiting distribution, and $n - m$ non-stationary processes, with $n \geq d$. Then we know that any $(\mathcal{P}, c)$ such that $\mathcal{P}^\top c \in \mathcal{C}_X^m$ will be a cointegrated pricing system. This provides us with a constraint on the possible specifications of $(\mathcal{P}, c)$ which can be used in the next step on specifying parametric models for the factor processes and estimating on data. As long as we know that
\(X^m\) admits a limiting distribution, we can characterize a set of admissible pricing systems \((\mathcal{P},c)\) before any further specification and estimation on data.

In the analysis so far we have exclusively thought of the price dynamics \(S\) in (3) as being on an arithmetic form. However, commonly one models cointegration on the logarithm of prices, \(\ln S := (\ln S_1, \ldots, \ln S_d)^T\). If we suppose that \(\ln S\) satisfies

\[
\ln S(t) = \mathcal{P}X(t), \quad t \geq 0,
\]

we can repeat the analysis above for a geometric price dynamics. Energy markets like gas and power have frequently experienced negative prices, and hence an arithmetic price dynamics may be attractive.

2.1. Particular model specifications. Recalling Example 1, we may for the cross-commodity spot price dynamics (1) assume a general (non-stationary) dynamics \(X_t\) and a bivariate process \(Y := (X_1, X_2)^T\) with the property that \(P_X(t, \cdot) \rightarrow P_\infty(\cdot)\) for some probability distribution \(P_\infty\) on \(\mathbb{R}^2\). Then, we find for any \(c = (k, -k)^T \in \mathcal{C}_S\) that the characteristic function of the random variable \(c^T \mathcal{P}X(t)\) is

\[
\Psi_{c^T \mathcal{P}X}(t, z) = \mathbb{E}\left[e^{i(k(X_1(t) - X_2(t)))}\right] = \Psi_X(t, z \mathcal{P}^T c) = \hat{P}_X(t, z c^T) \rightarrow \hat{P}_\infty(z c^T),
\]

for every \(z \in \mathbb{R}\) as \(t \rightarrow \infty\). But then by Prop. 5, \(c^T \mathcal{P}X(t)\) has a limiting distribution \(\mu_c(z) = \hat{P}_\infty(z c^T)\). This shows that we may significantly go beyond the dynamics discussed in (1) that preserves cointegration, and has a marginal structure with a (long term) non-stationary factor and a (short term) factor modelling the ”stationary” variations. In this Subsection we discuss various other particular specifications of these factors beyond classical Ornstein-Uhlenbeck models.

The so-called Lévy stationary (LS) processes provides us with a flexible class of stationary models which can be applied as dynamics for \(X^m = (X_1, \ldots, X_m^T)\), \(m \in \mathbb{N}\). To this end, assume

\[
X^m(t) = \int_{-\infty}^t G(t - s) dL(s),
\]

where \(L = (L_1, \ldots, L_k)^T\) is a two-sided square integrable \(k\)-dimensional Lévy process with zero mean and \(u \mapsto G(u)\) is a measurable mapping from \(\mathbb{R}_+\) into the space of \(m \times k\) matrices with elements \(g_{ij} \in L^2(\mathbb{R}_+)\), \(i = 1, \ldots, m, j = 1, \ldots, k\). We remark that the assumption on the \(g_{ij}\)'s ensures that \(X^m\) is a well-defined mean zero square integrable stochastic process with values in \(\mathbb{R}^m\). LS processes form a subclass of the more general Lévy semistationary processes considered in e.g. Barndorff-Nielsen, Benth and Veraart [5].

As the following Lemma shows, \(X^m = (X_1, \ldots, X_m)^T\) is strictly stationary:

**Lemma 10.** The process \(X^m = (X_1, \ldots, X_m)^T\) defined in (4) is a strictly stationary process, that is, for any \(\tau \geq 0\) and \(r \in \mathbb{N}\), the \(m \times r\)-dimensional random matrices \((X^m(t_1 + \tau), \ldots, X^m(t_r + \tau))\) and \((X^m(t_1), \ldots, X^m(t_r))\) have the same probability distribution.

**Proof.** Let \(t_\ell, \ell = 1, \ldots, r\) be an increasing sequence of times on \(\mathbb{R}\), and notice that \((X^m(t_1), X^m(t_2), \ldots, X^m(t_r))^T \in \mathbb{R}^{mr}\). By the independent increment property of
Lévy processes, it follows that with \( \psi \) denoting the cumulant of \( L(1) \) and \( z^\top = ((z_1^1, \ldots, z_m^1), (z_1^2, \ldots, z_m^2), \ldots, (z_1^n, \ldots, z_m^n)) \in \mathbb{R}^{mn} \),

\[
E \left[ e^{iz^\top(X_1^m(t_1), \ldots, X_1^m(t_r))} \right] = E \left[ e^{iz^\top \int_{-\infty}^{t_1} G(t_1-s) \, dL(s) + \cdots + iz^\top \int_{-\infty}^{t_r} G(t_r-s) \, dL(s)} \right]
\]

\[
\times \cdots \times E \left[ e^{iz^\top \int_{t_{r-1}}^{t_r} G(t_r-s) \, dL(s)} \right] = \exp \left( \int_{-\infty}^{t_1} \psi \left( (G(t_1-s)^\top z_1 + \cdots + G(t_r-s)^\top z_r) \right) \, ds \right)
\]

\[
\times \exp \left( \int_{t_1}^{t_2} \psi \left( (G(t_2-s)^\top z_2 + \cdots + G(t_r-s)^\top z_r) \right) \, ds \right)
\]

\[
\times \cdots \times \exp \left( \int_{t_{r-1}}^{t_r} \psi \left( (G(t_r-s)^\top z_r) \right) \, ds \right).
\]

Here we have used the notation \( z_i := (z_1^1, \ldots, z_m^i)^\top \). Thus, after a change of variables, we see that the characteristic function of \( (X_1^m(t_1), \ldots, X_1^m(t_r)) \) depends on \( (t_2 - t_1, t_3 - t_2, \ldots, t_r - t_{r-1}) \) only, and we can conclude that the probability distribution of \( (X_1^m(t_1 + \tau), \ldots, X_1^m(t_r + \tau)) \) equals that of \( (X_1^m(t_1), \ldots, X_1^m(t_r)) \) for any \( \tau > 0 \). Strict stationarity follows.

Remark that any linear combination of \( X_1, \ldots, X_m \) is strictly stationary whenever \( (X_1, \ldots, X_m) \) is strictly stationary. If the real-valued process \( \{U(t)\}_{t \geq 0} \) is a strictly stationary process, we have that its probability distribution \( P_U(t, \cdot) \) satisfies \( P_U(t + \tau, \cdot) = P_U(t, \cdot) \) for all \( t \geq 0 \), for any given \( \tau \geq 0 \). Hence, \( P_U(t, \cdot) \equiv P_U(\cdot) \), that is, it is independent of time \( t \). This implies trivially that \( P_U(t, \cdot) \rightarrow P_U(\cdot) \) when \( t \rightarrow \infty \), and moreover, the characteristic function of \( U(t) \) is also independent of \( t \). Hence, for any pricing system \( (\mathcal{P}, c) \), where \( \mathcal{P}^\top \cdot c \in \mathbb{C}_m^\infty \), we have that \( \mathcal{P}^\top X(t) = (e_1^\top \mathcal{P}^\top c)X_1(t) + \cdots + (e_m^\top \mathcal{P}^\top c)X_m(t) \), i.e., a linear combination of \( X_1, \ldots, X_m \), which is a strictly stationary process. We note in passing that if the characteristic function of a stochastic process \( V(t) \) is independent of \( t \), it holds that \( P_V(t, \cdot) = P_V(\cdot) \). This implies stationarity in the sense that the probability distribution of \( V(t) \) is invariant of time, however, it does not necessarily imply strict stationarity. In Benth [9], cointegration models based on LS processes were proposed and analysed.

An example of an LS process is given by

\[
X_i(t) = \eta_i \int_{-\infty}^{t} e^{-\alpha_i(t-s)} \, dB_i(s),
\]

for \( i = 1, \ldots, m \) with \( B = (B_1, \ldots, B_m)^\top \) being a two-sided \( m \)-dimensional Brownian motion (possibly correlated) and \( \alpha_1, \ldots, \alpha_m, \eta_1, \ldots, \eta_m \) positive constants. Then it holds that the distribution function of \( (X_1, \ldots, X_m) \) is time invariant and equal to \( \mathcal{N}(0, C) \), with covariance matrix \( C \in \mathbb{R}^{m \times m} \) having diagonal elements \( \eta_i^2/(2\alpha_i) \), \( i = 1, \ldots, m \) and off-diagonal elements \( \rho_{ij} \eta_i \eta_j/\alpha_i + \alpha_j \) for \( \rho_{ij} \) being the correlation coefficient between \( B_i \) and \( B_j \), \( i \neq j \). In fact, this example is a particular case of so-called continuous-time autoregressive moving average (CARMA) processes, as we discuss next.
For $p \in \mathbb{N}$, define the matrix $A \in \mathbb{R}^{p \times p}$ as

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & -\alpha_{p-3} & \ldots & -\alpha_1 \\
\end{bmatrix},$$

for positive constants $\alpha_k$, $k = 1, \ldots, p$. Consider the $p$-dimensional Ornstein-Uhlenbeck process

$$dY(t) = AY(t) \, dt + e_p \, dL(t).$$

Here we recall that $\{e_i\}_{i=1}^p$ are the $p$ canonical basis vectors in $\mathbb{R}^p$ and $L$ is a (two-sided) real-valued square-integrable Lévy process with zero mean. Following Brockwell [13], we define a CARMA($p$, $q$) process $Z$ for $q < p, p, q \in \mathbb{N}$ by

$$Z(t) = b^\top Y(t),$$

for $b \in \mathbb{R}^p$, where $b = (b_0, b_1, \ldots, b_q, 0, \ldots, 0)^\top$ and $b_q = 1$. We observe that for $q = 0$, $b = e_1$ and we say in this case that $Z$ is a continuous-time autoregressive process of order $p$ (a CAR($p$)-process in short). We suppose that the $p$ eigenvalues of $A$ have negative real part, which yields that $Z$ is strictly stationary with

$$Z(t) = \int_{-\infty}^t b^\top e^{A(t-s)} e_p \, dL(s).$$

Thus, with $G(s) := b^\top \exp(A(s)) e_p$, a CARMA($p$, $q$)-process is an example of a real-valued LS-process.

We want to apply CARMA-processes as factors in a cointegration model (see Comte [15] for an extensive analysis of cointegration based on CARMA processes). To this end, let $X$ be an $n$-dimensional process, and $X^m = (X_1, \ldots, X_n)$ for $m < n$ be an $m$-dimensional CARMA-process. A simple way to define such a process is as follows: given an $m$-dimensional two-sided square integrable Lévy process $L = (L_1, \ldots, L_m)^\top$ with zero mean. For $i = 1, \ldots, m$, let $X_i$ be as in (5), that is, a CARMA($p_i$, $q_i$)-process driven by $L_i$ and with matrix $A_i \in \mathbb{R}^{p_i \times p_i}$ having eigenvalues with negative real part. In the notation of LS-processes in (5), this means that the $m \times m$-matrix-valued function $G(u)$ has diagonal elements $g_{ii}(u) := b_i^\top \exp(A_i(s)) e_p$, and off-diagonal elements being zero. By Lemma [10] $X^m$ is an $m$-dimensional strictly stationary process. Benth and Koekkebakker [10] consider such models in the context of cointegration. We remark in passing that CARMA-processes has been applied to model commodities like oil and power (see e.g. Paschke and Prokopczuk [32] and Benth, Klüppelberg, Müller and Vos [31]). Multivariate CARMA-processes going beyond the simple specification we consider here have been proposed and analysed by Marquardt and Stelzer [39]. Their definition will yield an LS-process [5] with the matrix-valued function $G$ having non-zero off-diagonal elements. Thus, we do not only have dependency through the Lévy processes, but also functional dependencies between the coordinates in vector-valued CARMA process. Such multivariate CARMA processes is further studied by Schlemm and Stelzer [35] and Kevei [28]. Taking these extensions into account, we have a rich class of stationary processes available for cointegration modelling.
It is well-known (see e.g. Benth and Šaltytė Benth [8] and Benth, Šaltytė Benth and Koekebakker [7]) that a CARMA($p,q$)-process on a discrete time scale will define an ARMA($p,q$) time series. Furthermore, as is demonstrated in Aadland, Benth and Koekebakker [1], the process $X(t) = \int_0^t Z(s) \, ds$, where $Z$ is a CAR($p$)-process, becomes a non-stationary process. Hence, it may serve as a non-stationary factor process in modelling the price dynamics $S$. Indeed, we can use a set of $n - m$ dependent CAR-processes to define non-stationary processes $X_{m+1}, \ldots, X_n$ in this way. As Aadland, Benth and Koekebakker [1] show, these processes will become integrated autoregressive times series on a discrete time scale. Aadland, Benth and Koekebakker [1] model cointegration in a freight rate market using CAR-processes, both for the stationary and the non-stationary processes.

3. Forwards Pricing under Cointegration

Denote the forward prices at time $t \geq 0$ of contracts delivering the underlying assets $S = (S_1, \ldots, S_d)^\top$ at time $T \geq t$ by $F(t,T) := (F_1(t,T), \ldots, F_d(t,T))^\top \in \mathbb{R}^d$. The price vector of the $d$ assets are defined by $S(t) \in \mathbb{R}^d$ in [3] with $\mathcal{P}$ being minimal and of full rank. Thus, we suppose an arithmetic model for the spot market. Assume $\mathbb{Q} \sim \mathbb{P}$ is a pricing measure such that $X(t) \in \mathbb{R}^n$ is $\mathbb{Q}$-integrable for all $t > 0$. Then, the forward price vector $F(t,T)$ is defined as (see Benth, Šaltytė Benth and Koekebakker [2]),

$$F(t,T) = \mathbb{E}_\mathbb{Q}[S(T) | \mathcal{F}_t].$$  

Hence, by the definition of $S$ we find that

$$F(t,T) = \mathcal{P}\mathbb{E}_\mathbb{Q}[X(T) | \mathcal{F}_t].$$

To proceed our analysis, the following definition of affinity is convenient:

**Definition 11.** The stochastic process $\{X(t)\}_{t \geq 0}$ is said to be affine with respect to $\mathbb{Q}$, or $\mathbb{Q}$-affine for short, if there exist measurable deterministic functions $(t,T) \mapsto A(t,T) \in \mathbb{R}^{n \times n}$ and $(t,T) \mapsto a(t,T) \in \mathbb{R}^d$ such that

$$\mathbb{E}_\mathbb{Q}[X(T) | \mathcal{F}_t] = A(t,T)X(t) + a(t,T)$$

for $0 \leq t \leq T < \infty$.

A trivial example of a $\mathbb{Q}$-affine process $X$ is the case when $X$ is an $n$-dimensional $\mathbb{Q}$-Brownian motion $B = (B_1, \ldots, B_n)^\top$. Then $a = 0$, and $A$ is the covariance matrix with elements $\rho_{ij}t$ for $\rho_{ii} = 1$ and $\rho_{ij}$ being the correlation between $B_i$ and $B_j$, $i \neq j$. A less trivial example is provided by Ornstein-Uhlenbeck processes. We show next the affinity property for $\mathbb{Q}$-semimartingales which are polynomial processes (see e.g. Cuchiero, Keller-Ressel and Teichmann [16] and Filipovic and Larsson [24] for a definition and analysis of polynomial processes).

**Proposition 12.** Assume the $\mathbb{Q}$-dynamics of $X$ is a polynomial process in $\mathbb{R}^n$. Then $X$ is $\mathbb{Q}$-integrable and $\mathbb{Q}$-affine, with the functions $(t,T) \mapsto a(t,T)$ and $(t,T) \mapsto A(t,T)$ being homogeneous, i.e., $A(t,T) = A(T-t)$ and $a(t,T) = a(T-t)$ (with a slight abuse of notation).

**Proof.** A polynomial process has finite moments (Lemma 2.17 in Cuchiero, Keller-Ressel and Teichmann [16]), and thus $X$ is $\mathbb{Q}$-integrable. Following the definition of a polynomial process (see e.g. Cuchiero, Keller-Ressel and Teichmann [16] and Filipovic and Larsson [24]), we know that for the generator $\mathcal{G}$ of $X$, there exists a
matrix $G \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$ such that $Gx = Gx + b$. This holds true since $x$ is a first order polynomial and the generator is preserving the order when applied to polynomials. Therefore, from the martingale problem of polynomial processes,

$$
\mathbb{E}_Q[X(T) \mid \mathcal{F}_t] = X(t) + \int_t^T (G\mathbb{E}_Q[X(s) \mid \mathcal{F}_t] + b) \, ds.
$$

and thus,

$$
\mathbb{E}_Q[X(T) \mid \mathcal{F}_t] = e^{G(t-T)}X(t) + \int_t^T e^{G(t-s)}b \, ds.
$$

The result follows.

We remark in passing that the class of polynomial processes has a much richer structure than really needed for the $Q$-affinity. The generator of a polynomial process preserves the order of any polynomial, while $Q$-affinity only requires that the generator preserves the first order polynomials.

As an example, consider an Ornstein-Uhlenbeck process in $\mathbb{R}^n$ with $Q$-dynamics

$$
dX(t) = (\mu + CX(t)) \, dt + \Sigma dW(t).
$$

Here, $\mu \in \mathbb{R}^n$, $C \in \mathbb{R}^{n \times n}$, $\Sigma \in \mathbb{R}^{n \times m}$ and $W$ is an $m$-dimensional Brownian motion. A direct calculation reveals that for $t \leq T$,

$$
X(T) = e^{C(T-t)}X(t) + \int_t^T e^{C(T-s)}\mu \, ds + \int_t^T e^{C(T-s)}\Sigma dW(s),
$$

and thus

$$
\mathbb{E}_Q[X(T) \mid \mathcal{F}_t] = e^{C(T-t)}X(t) + \int_0^{T-t} e^{Cs} \mu \, ds.
$$

In conclusion, $Q$-affinity holds with $A(T-t) = \exp(C(T-t))$ and $a(T-t) = \int_0^{T-t} \exp(Cs) \mu \, ds$. Note that both $A$ and $a$ are homogeneous in time. Whenever $C$ is an invertible matrix, we find

$$
a(T-t) = C^{-1}(e^{C(T-t)} - I)\mu
$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

We have the following simple result:

**Corollary 13.** If $\{X(t)\}_{t \geq 0}$ is $Q$-integrable and $Q$-affine process in $\mathbb{R}^n$, then

$$
F(t, T) = PA(t, T)X(t) + PA(t, T).
$$

Moreover, if $PA(t, T) = \tilde{A}(t, T)P$ for some $\tilde{A}(t, T) \in \mathbb{R}^{d \times d}$, $0 \leq t \leq T < \infty$, then

$$
F(t, T) = \tilde{A}(t, T)S(t) + PA(t, T) \text{ (i.e., the forward price vector is affine in the asset price $S$.)}
$$

**Proof.** This is trivial from the definition of affinity.

We note that in the case $d = n$, we have forward prices which are affine in the underlying spot when $P$ and $A(t, T)$ commutes for all $t \leq T$.

Let us next turn to the question of cointegration in the forward market. As $t \leq T < \infty$, it is natural to switch to the Musiela parametrization, and express forward prices in terms of time to maturity $x := T - t$ rather than time of maturity $T$. I.e., introduce the random fields $f(t, x)$ for $x \geq 0$ by

$$
f(t, x) := F(t, t + x) .
$$
Hence, we find in the case of \( \{X(t)\}_{t \geq 0} \) being \( \mathbb{Q} \)-affine and \( \mathbb{Q} \)-integrable that
\[
f(t, x) = \mathcal{P}\mathcal{A}(t, t + x)X(t) + a(t, t + x).
\]

The following Proposition holds:

**Proposition 14.** Fix \( x \geq 0 \), and suppose that \( X \) is \( \mathbb{Q} \)-integrable and \( \mathbb{Q} \)-affine, with \( \mathcal{A} \) and \( a \) homogeneous (e.g., \( \mathcal{A}(t, T) = \mathcal{A}(T - t) \) and \( a(t, T) = a(T - t) \)). Then \( t \mapsto f(t, x) \) is cointegrated if there exists a vector \( c \in \mathbb{R}^d \) such that \( c^\top \mathcal{P}\mathcal{A}(x) \in \mathcal{C}_X \), or, equivalently, \((\mathcal{P}\mathcal{A}(x), c)\) is a cointegrated pricing system.

**Proof.** This follows readily from the definitions and the fact that for homogeneous \( \mathcal{A} \) and \( a \), \( \mathcal{A}(t, t + x) = \mathcal{A}(x) \) and \( a(t, t + x) = a(x) \).

**Remark 15.** We emphasise that \( x \geq 0 \) is fixed in Proposition 14. This means that it is the dynamics of the forward contracts with fixed time to maturity that is cointegrated. This can be viewed as a roll-over contract, where one fixes the time to maturity and ”rolls over” the position when time progresses. The actual forward price dynamics will in general not be cointegrated as it will depend on \( \mathcal{A}(t, T) \) and \( a(t, T) \), which varies with time \( t \). Benth and Koekkebakker (10) make a similar observation for a more particular HJM-type cointegrated forward price model. If \( x = 0 \), or equivalently \( t = T \), we are back to the spot price case. Propositions 14 and 17 show that polynomial processes can be used to build cointegrated forward price models.

Consider the case when the \( \mathbb{Q} \)-dynamics of \( X(t) \in \mathbb{R}^3 \) is such that \( X_3(t) \) is a non-stationary process and \( (X_1, X_2) \) admits a limiting distribution. From previous considerations we then have that \( \mathcal{C}_X = \{a \in \mathbb{R}^3 | a_3 = 0\} \). In the context of Example 1 for any pricing matrix \( \mathcal{P} \in \mathbb{R}^{2 \times 3} \) and \( c \in \mathbb{R}^2 \), we find that \( c^\top \mathcal{P} \in \mathcal{C}_X \) if and only if \( c^\top \mathcal{P}e_3 = 0 \) (e.g., the third coordinate of \( c^\top \mathcal{P} \) is equal to zero). With \( p_{ij} \) denoting the \( ij \)th element of \( \mathcal{P} \), we find that \( c^\top \mathcal{P} \in \mathcal{C}_X \) if and only if \( c_1p_{13} + c_2p_{23} = 0 \). Let us analyse this for non-trivial \( c \) (e.g., \( c \neq 0 \)) and \( \mathcal{P} \) being minimal. Minimality of \( \mathcal{P} \) means that \( \mathcal{P}e_i \neq (0, 0)^\top \) for \( i = 1, 2, 3 \), and in particular for \( i = 3 \) we find \( (p_{13}, p_{23}) \neq (0, 0) \). Thus, we find that \((\mathcal{P}, c)\) is a cointegrated pricing system if and only if either \( c_2, p_{13} \neq 0 \) and \( c_1/c_2 = -p_{23}/p_{13} \) or \( c_1, p_{23} \neq 0 \) and \( c_2/c_1 = -p_{13}/p_{23} \). If \( X \) is \( \mathbb{Q} \)-affine with a matrix \( \mathcal{A}(t, T) = \mathcal{A}(T - t) \in \mathbb{R}^{3 \times 3} \) satisfying \( e_1^\top \mathcal{A}(x)e_3 = e_2^\top \mathcal{A}(x)e_3 = 0 \) yields that \( c^\top \mathcal{P}\mathcal{A}(x) \in \mathcal{C}_X \) for any cointegrated pricing system \((\mathcal{P}, c)\).

As a particular case of the above, consider the factor process
\[
dX(t) = \left(\mu + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} X(t)\right) dt + \Sigma dW(t)
\]
with \( \Sigma, C \in \mathbb{R}^{2 \times 2} \), \( \mu \in \mathbb{R}^3 \) and \( 0 = (0, 0)^\top \). Further, \( W \) is assumed to be a trivariate \( \mathbb{Q} \)-Brownian motion. Here, \( (X_1, X_2) \) will be a bivariate OU process with mean-reversion matrix \( C \) and noise vector \((e_1^\top \Sigma dW(t), e_2^\top \Sigma dW(t))^\top \), which admits a limiting distribution whenever \( C \) has eigenvalues with negative real part. The process \( X_3 \) is a drifted Brownian motion. Then,
\[
\mathbb{E}_Q[X(T)|\mathcal{F}_t] = \mathcal{A}(T - t)X(t) + a(T - t),
\]
where
\[
\mathcal{A}(T - t) = \begin{bmatrix} e^{C(T-t)} & 0 \\ 0' & 1 \end{bmatrix},
\]
and \(a(T-t) = \int_0^{T-t} A(y) \mu \, dy\). Thus, for \((\mathcal{P}, c)\) being a cointegrated pricing system, the forward prices will also be cointegrated. We see that we obtain cointegration both for the spot (under \(Q\)) and the forward prices for rather general models of \(X\), including a full correlation structure between the three noises \(W\) and flexible mean reversion matrix \(C\).

Note that if \(A\) is not homogeneous, that is, \(X\) is \(Q\)-affine for a non-homogeneous \(A\), then we may lose cointegration in the forward process. In the case \(a\) is non-homogeneous, we may recover cointegration as long as \(A\) is homogeneous by considering the "de-trended" forward price vector \(\tilde{f}(t, x) := f(t, x) - \mathcal{P}a(t, x)\). In that case, \(\tilde{f}(t, x)\) is cointegrated whenever \(c^\top \mathcal{P} A(x) \in C_X\). Indeed, this is a relevant case for commodity markets with seasonally varying prices. For example, in power markets, where prices are highly influenced by weather conditions, it may appear that \(a\) is not homogeneous. Indeed, the factor model \(\{\mathcal{P}, \mathcal{C}\}\) can be used as a model for spot prices with \(\mu\) being time dependent, i.e. \(t \mapsto \mu(t)\) for some measurable real-valued function being bounded on compacts. Then \(a(t, T) = \int_0^T A(T-s) \mu(s) \, ds\) is not in general homogeneous. Typically, \(\mathbf{\mu}(t)\) models a seasonal mean price, towards which the stationary part of \(\mathbf{X}\) mean reverts (see e.g. Benth, Šaltyte Benth and Koekebakker [7] for models of this type with seasonality).

3.1. General LS-processes. In general, LS-process will not be \(Q\)-affine. In this subsection we analyse forward pricing involving LS-processes.

For \(x \geq 0\), we define the \(\mathbb{R}^m\)-valued random field \(\tilde{X}^m(t, x)\) by

\[
\tilde{X}^m(t, x) := \int_{-\infty}^t G(t-s+x) \, dL(s),
\]

with \(G\) and \(L\) being as in the definition of the LS-process in \(\{\mathcal{P}\}\). We assume that this is the \(Q\)-dynamics of \(\tilde{X}^m(t, x)\). In particular, for \(x = 0\), we are back to \(\tilde{X}^m(t)\) as in \(\{\mathcal{P}\}\) (but now considered as a dynamics with respect to \(Q\)). Moreover, following the proof of Lemma \(\{\mathcal{P}\}\), the stochastic process \(t \mapsto \tilde{X}^m(t, x)\) is strictly stationary for every \(x \geq 0\). It is simple to see that

\[
E_Q[\tilde{X}^m(T) \mid \mathcal{F}_t] = E_Q[\tilde{X}^m(0) \mid \mathcal{F}_t] = \tilde{X}^m(t, T-t),
\]

by appealing to the independent increment property of Lévy processes. Hence, assuming a factor process \(\mathbf{X}\) which is \(Q\)-integrable, where \(\mathbf{X}^m\) is given by an LS-process as in \(\{\mathcal{P}\}\) with respect to the probability \(Q\), we find that

\[
f(t, x) = \mathcal{P} \left( \tilde{X}_1(t, x), \ldots, \tilde{X}_m(t, x), E_Q[X_{m+1}(T) \mid \mathcal{F}_t], \ldots, E_Q[X_n(T) \mid \mathcal{F}_t] \right)^\top.
\]

We see that any \(c \in \mathbb{R}^d\) such that \(c^\top \mathcal{P} \in C_X^m\) implies that \(c^\top f(t, x)\) becomes a linear combination of \(\tilde{X}_1(t, x), \ldots, \tilde{X}_m(t, x)\), and therefore strictly stationary. Hence, \(c\) will be a cointegration vector for \(f(t, x)\).

The classes of CARMA-processes and their multivariate extensions discussed in the previous section provide a rich class of LS-processes that can be used for modelling cointegrated forward prices under the Musiela parametrization.

3.2. Factor models of geometric type. Classically, pricing models in finance have been geometric. In our context, we recall from \(\{\mathcal{P}\}\) that this means a spot
price dynamics \( S \) of the form \( \ln S(t) = \mathcal{P}X(t) \). The forward price vector \( F(t, T) = (F_1(t, T), \ldots, F_d(t, T))^{\top} \) will be given by
\[
F_i(t, T) = \mathbb{E}_Q [\exp(e_i^{\top} \mathcal{P}X(T)) | \mathcal{F}_t]
\]
for \( t \leq T \) and \( i = 1, \ldots, d \). We recall that \( \{e_i\}_{i=1}^d \) are the canonical basis vectors in \( \mathbb{R}^d \), thus \( e_i^{\top} \mathcal{P}X(T) \) is the \( i \)th coordinate of the vector \( \mathcal{P}X(t) \), i.e., \( \ln S_i(t) \). We are naturally led to define the following class of factor processes:

**Definition 16.** A process \( X \) is called exponentially \( Q \)-affine if for every \( z \in \mathbb{R}^n \), \( z^{\top}X(T) \) has finite exponential moment under \( Q \) and there exist measurable mappings \( (t, T) \mapsto \alpha(t, T; z) \in \mathbb{R}^n \) and \( (t, T) \mapsto a(t, T; z) \in \mathbb{R} \) such that
\[
\mathbb{E}_Q[\exp(z^{\top} X(T)) | \mathcal{F}_t] = \exp \left( \alpha(t, T; z)^{\top} X(t) + a(t, T; z) \right)
\]
for all \( t \leq T \).

For exponential \( Q \)-affine factor processes, we have:

**Proposition 17.** If \( X \) is an \( n \)-dimensional exponential \( Q \)-affine factor process, then \( F(t, T), t \leq T \) has coordinates
\[
F_i(t, T) = \exp \left( \alpha(t, T; \mathcal{P}^{\top} e_i)^{\top} X(t) + a(t, T; \mathcal{P}^{\top} e_i) \right)
\]
for \( i = 1, \ldots, d \).

**Proof.** This follows immediately from the definition of exponential affinity and (17).

For example, if \( X \) is given by (12) (under \( Q \)), it will be exponentially \( Q \)-affine, as the following Lemma shows:

**Lemma 18.** Suppose that \( X \) is the factor process in \( \mathbb{R}^3 \) defined in (12). Then \( X \) is exponential \( Q \)-affine, with \( \alpha(t, T; z) = \mathcal{A}(T - t)^{\top} z \) and
\[
a(T - t; z) = \int_0^{T-t} z^{\top} \mathcal{A}(s) \mu + z^{\top} \mathcal{A}(s) \Sigma C \Sigma^{\top} \mathcal{A}(s)^{\top} z \, ds.
\]
Here, \( \mathcal{A} \) is defined in (13) and \( C \) is the \( 3 \times 3 \) covariance matrix of \( \mathcal{W} \).

**Proof.** It holds that
\[
X(T) = \mathcal{A}(T - t)X(t) + \int_0^{T-t} \mathcal{A}(s) \mu \, ds + \int_t^T \mathcal{A}(T - s) \Sigma d\mathcal{W}(s),
\]
where \( \mathcal{A}(s) \) is defined in (13). As the stochastic integral on the right hand side is a Wiener integral, it is a Gaussian random variable and hence \( z^{\top} X(t) \) has finite exponential moment for every \( z \in \mathbb{R}^n \). By the independent increment property of Brownian motion and the \( X(t) \) being \( \mathcal{F}_t \)-measurable, we find
\[
\mathbb{E}_Q \left[ \exp(z^{\top} X(T)) | \mathcal{F}_t \right] = \exp \left( z^{\top} \mathcal{A}(T - t)X(t) + \int_0^{T-t} z^{\top} \mathcal{A}(s) \mu \, ds \right)
\]
\[
\times \mathbb{E}_Q \left[ \exp \left( \int_t^T z^{\top} \mathcal{A}(T - s) \Sigma d\mathcal{W}(s) \right) \right]
\]
\[
= \exp \left( z^{\top} \mathcal{A}(T - t)X(t) + \int_0^{T-t} z^{\top} \mathcal{A}(s) \mu \, ds \right)
\]
\[ \times \exp \left( \int_t^T z^\top A(T - s) \Sigma C \Sigma^\top A(T - s)^\top z \, ds \right). \]

The result follows. \(\square\)

We remark in passing that one can easily extend the above Lemma to higher dimensions than 3. Observe that both \(\alpha\) and \(a\) are homogeneous, i.e., depending only on the time to maturity \(T - t\). Let \((\mathcal{P}, \mathcal{C})\) for \(c \in \mathbb{R}^2\) and \(\mathcal{P} \in \mathcal{R}^{2 \times 3}\) be a cointegrated pricing system (under \(\mathbb{Q}\)), i.e., \(c^\top \mathcal{P} \in \mathcal{C}_X\). We have then

\[ \ln f_i(t, x) = e_i^\top \mathcal{P} A(x) X(t) + a(x; \mathcal{P}^\top e_i) \]

for \(i = 1, 2\). Moreover, as we have seen earlier, \(c^\top \mathcal{P} A(x) \in \mathcal{C}_X\), and therefore the logarithmic forward prices \(f_1(t, x)\) and \(f_2(t, x)\) are cointegrated for the cointegration vector \(c\).

Next, let us focus on general LS-processes as factors in a geometric model. Suppose that \(X^m\) has \(\mathbb{Q}\)-dynamics defined as in (5). We find the following:

**Proposition 19.** Assume \(X^m\) is an \(m\)-dimensional process with \(\mathbb{Q}\)-dynamics as in (5). Let \(t \leq T\). If \(z^\top X^m(t)\) has finite exponential moment under \(\mathbb{Q}\) for \(z \in \mathbb{R}^m\), then

\[ \mathbb{E}_\mathbb{Q} \left[ \exp \left( z^\top \int_{-\infty}^T G(T - s) \, dL(s) \right) \mid \mathcal{F}_t \right] \]

\[ = \exp \left( z^\top \int_t^T G(T - s) \, dL(s) + \int_0^{T-t} \psi_Q (G(s)^\top z) \, ds \right) \]

where \(\psi_Q\) is the cumulant of \(L(1)\) under \(\mathbb{Q}\).

**Proof.** Since \(\int_{-\infty}^t G(T - s) \, dL(s)\) is \(\mathcal{F}_t\)-measurable, and the Lévy process \(L\) has independent increments, it follows that

\[ \mathbb{E}_\mathbb{Q} \left[ \exp \left( z^\top \int_{-\infty}^T G(T - s) \, dL(s) \right) \mid \mathcal{F}_t \right] \]

\[ = \exp \left( z^\top \int_t^T G(T - s) \, dL(s) \right) \mathbb{E}_\mathbb{Q} \left[ \exp \left( \int_t^T z^\top G(T - s) \, dL(s) \right) \right] \]

\[ = \exp \left( z^\top \int_t^T G(T - s) \, dL(s) + \int_t^T \psi_Q (G(s)^\top z) \, ds \right). \]

and the result follows. \(\square\)

Express the factor process as \(X = (X^m, \hat{X}) \in \mathbb{R}^n\), for \(\hat{X}\) being a process in \(\mathbb{R}^{n-m}\), \(n > m \in \mathbb{N}\). We further suppose that \(X^m\) is an LS-process under \(\mathbb{Q}\), as in (5). Consider a cointegrated pricing system \((\mathcal{P}, \mathcal{C})\), that is, \(\mathcal{P}^\top c \in \mathcal{C}_X\), and introduce the following representation of the \(d \times n\)-matrix \(\mathcal{P}\): let \(\mathcal{P}^m \in \mathbb{R}^{d \times m}\) and \(\hat{\mathcal{P}} \in \mathbb{R}^{d \times (n-m)}\) be such that \(\mathcal{P} = [\mathcal{P}^m \hat{\mathcal{P}}]\). Then for \(i = 1, \ldots, d\),

\[ e_i^\top \mathcal{P} X(T) = e_i^\top \mathcal{P}^m X^m(T) + e_i^\top \hat{\mathcal{P}} \hat{X}(T). \]

Assume that \(e_i^\top \mathcal{P}^m X^m(T)\) and \(e_i^\top \hat{\mathcal{P}} \hat{X}(T)\) have finite exponential moment under \(\mathbb{Q}\), and that they are conditionally independent with respect to \(\mathcal{F}_t\) for all \(t \leq T\).
Then it holds for \( i = 1, \ldots, d \) and \( t \leq T \) that
\[
F_i(t, T) = \mathbb{E}_Q \left[ \exp \left( e_i^\top \mathcal{P} X(T) \right) \mid \mathcal{F}_t \right]
\]
\[
= \mathbb{E}_Q \left[ \exp \left( e_i^\top \mathcal{P}^m X^m(T) \right) \mid \mathcal{F}_t \right] \mathbb{E}_Q \left[ \exp \left( e_i^\top \hat{\mathcal{P}} \hat{X}(T) \right) \mid \mathcal{F}_t \right]
\]
\[
= \exp \left( e_i^\top \mathcal{P}^m \int_{-\infty}^{t} G(T - s) \, dL(s) + \int_{0}^{T-t} \psi_Q \left( G(s)^\top \mathcal{P}^{m,\top} e_i \right) \, ds \right)
\]
\[
\times \mathbb{E}_Q \left[ \exp \left( e_i^\top \hat{\mathcal{P}} \hat{X}(T) \right) \mid \mathcal{F}_t \right]
\]
where we used Prop. 19 with \( z = \mathcal{P}^{m,\top} e_i \) in the last equality. We find that
\[
\ln f_i(t, x) = e_i^\top \mathcal{P}^m \int_{-\infty}^{t} G(t - s + x) \, dL(s) + \int_{0}^{x} \psi_Q \left( G(s)^\top \mathcal{P}^{m,\top} e_i \right) \, ds
\]
\[
+ \ln \mathbb{E}_Q \left[ \exp \left( e_i^\top \hat{\mathcal{P}} \hat{X}(t + x) \right) \mid \mathcal{F}_t \right],
\]
for \( i = 1, \ldots, d \) and \( x \geq 0 \). The last term is nonlinear in the vector \( \hat{X} \), and \( c \) may fail to be a cointegration vector for \( \ln f(t, x) \), even in the case when \( \mathcal{P}^\top c \in \mathcal{C}^m_\mathbb{Q} \).

However, typically in applications, \( \hat{X} = U \) for a Lévy process \( U \) in \( \mathbb{R}^{n-m} \). In the simplest case, \( U(t) = \mu t + \Sigma dW(t) \) for \( \mu \in \mathbb{R}^{n-m}, \Sigma \) an \( (n-m) \times (n-m) \) volatility matrix and \( W \) a \( \mathbb{Q} \)-Brownian motion in \( \mathbb{R}^{n-m} \). Suppose that \( U \) is independent of \( L \). Then it follows that \( e_i^\top \mathcal{P}^m X^m(T) \) and \( e_i^\top \hat{\mathcal{P}} \hat{X}(T) \) are conditionally independent with respect to \( \mathcal{F}_t \) for all \( t \leq T \). Moreover, \( e_i^\top \hat{\mathcal{P}} \hat{X}(T) \) have finite exponential moment under \( \mathbb{Q} \) when \( U \) is a drifted Brownian motion as exemplified above. Without any loss of generality, we assume that the coordinates \( W_i, i = 1, \ldots, n - m \), of \( W \) are independent. Denoting \( \kappa_Q \) the cumulant function of \( U \), we find by resorting to the independent increment property of Lévy processes that
\[
\mathbb{E}_Q \left[ \exp \left( e_i^\top \hat{\mathcal{P}} \hat{X}(t + x) \right) \mid \mathcal{F}_t \right] = \exp \left( e_i^\top \hat{\mathcal{P}} \hat{X}(t) + x \kappa_Q \left( \hat{\mathcal{P}}^\top e_i \right) \right).
\]
In this case we have that
\[
\ln f(t, x) = c^\top \mathcal{P} \left( \frac{X^m(t, x)}{X(t)} \right) + h(x)
\]
with \( h(x) \in \mathbb{R}^d \) with coordinates
\[
h_i(x) = \int_{0}^{x} \psi_Q \left( G(s)^\top \mathcal{P}^{m,\top} e_i \right) \, ds + x \kappa_Q \left( \hat{\mathcal{P}}^\top e_i \right).
\]
After a simple modification of Lemma 19, we know that \( X^m(t, x) \) is a strictly stationary process in \( \mathbb{R}^m \). In this case, any \( c \in \mathbb{R}^d \) such that \( \mathcal{P}^\top c \in \mathcal{C}^m_\mathbb{Q} \) is a cointegration vector for \( \ln f(t, x) \). Thus, the cointegration vector for the spot yields cointegration of the forwards as well.

### 3.3. Market probability \( \mathbb{P} \) vs. pricing measure \( \mathbb{Q} \)

Throughout this Section we have assumed a factor process specified directly under the pricing measure \( \mathbb{Q} \) in our analysis of cointegration for forward markets. Indeed, we have supposed a cointegrated spot model under the pricing measure \( \mathbb{Q} \) rather than under the market probability \( \mathbb{P} \). In practice, the situation is more likely that one has a cointegrated spot model under the market probability \( \mathbb{P} \), and introduces a pricing measure \( \mathbb{Q} \) to
price forwards on the spot prices. The next step is to analyse possible cointegration of the forward prices.

A common approach in commodity and energy markets for introducing a pricing measure \( Q \) is to consider structure preserving equivalent probabilities (see Benth, Šaltytė Benth and Koekebakker [7], Benth et al. [9], Benth and Koekebakker [10], Eydeland and Wolyniec [20], Geman [21], Lucia and Schwartz [29], to mention just a few). By this we mean a probability \( Q \sim P \) that preserves the probabilistic structure of the factor process \( X \). In commodity markets, one typically chooses the Esscher and Girsanov transforms as the approach to construct pricing measures, with constant market price of risk (see Benth, Šaltytė Benth and Koekebakker [7] for an introduction of the Esscher transform in commodity markets and, e.g., Karatzas and Shreve [27] for a general analysis of the Girsanov transform). Roughly speaking, any cointegrated pricing system \((P, c)\) for \( P \) will also become a cointegrated pricing system for \( Q \) when we use the Esscher and Girsanov transform with constant market price of risk to the factor process. We emphasise that to apply these transforms, we need to have a factor process driven by Lévy processes with finite exponential moments of some order.

For CARMA processes driven by Brownian motion one can introduce pricing measures that are structure preserving, where the coefficients \( \alpha_i, i = 1, \ldots, p \) in the CARMA matrix \( A \) in (6) is changed (see Benth and Šaltytė Benth [8]). Restricting to Ornstein-Uhlenbeck processes, one can find a similar structure preserving pricing measure which slows down the speed of mean reversion, even for processes driven by positive Lévy processes (see Benth and Ortiz-Latorre [13]). Thus, we see that for a rich class of CARMA and Ornstein-Uhlenbeck processes, we can introduce pricing measures \( Q \) that preserves stationarity of the factor process \( X^m \). Combined with an Esscher transform for the remaining \( \tilde{X} \), where \( X = (X^m, \tilde{X}) \), we see that the essential probabilistic characteristics for cointegration under \( P \) is transferred to \( Q \). In conclusion, we can obtain cointegration in the spot market under \( P \) which is transferred to \( Q \). In this way, our analysis of cointegration in the forward market in this Section can be linked to practice.

In general, there is no equivalence of cointegrated pricing systems under the market probability \( P \) and the chosen pricing measure \( Q \). For example, considering the measure change in Benth and Ortiz-Latorre [13] which is reducing the speed of mean reversion of an Ornstein-Uhlenbeck process, we can in fact ”kill” the mean reversion, and turn the stationary Ornstein-Uhlenbeck dynamics under \( P \) into a non-stationary dynamics under \( Q \). Such a situation may also occur in the case of CARMA processes using the measure change suggested in Benth and Šaltytė Benth [8]. We see that we may alter the space of possible cointegration pricing systems when going from \( P \) to \( Q \). For example, considering the simple three-factor model by Lucia and Schwartz in Example [1] by introducing a measure change as in Benth and Ortiz-Latorre [13] which kills the mean reversion of \( X_1 \) and \( X_2 \), we end up with three non-stationary (indeed, drifted Brownian motions) processes under \( Q \). In this case, the only cointegrated pricing systems under \( Q \) are those \((P, c)\) for which \( P^c = 0 \in \mathbb{R}^3 \). Interestingly, for this example, we can price forwards and recover cointegration for the forward prices, as these will be given in terms linear combination of the factor processes, which are cointegrated with respect to \( P \).
4. Cointegration for Hilbert-valued stochastic processes

We want to generalize the concept of cointegration to Hilbert-valued stochastic processes. Recall from above that we defined cointegration in finite dimensions by the following scheme of linear mappings for a cointegration pricing system \((P, c)\) and a factor process \(X \in \mathbb{R}^n\):

\[
X \in \mathbb{R}^n \xrightarrow{P} S(t) \in \mathbb{R}^d \xrightarrow{c^*} c^T S(t) \in \mathbb{R}.
\]

I.e., for a cointegration pricing system \((P, c)\), we use a linear operator \(P\) to map the factor vector from the factor space \(\mathbb{R}^n\) to the price space \(\mathbb{R}^d\), and next the linear operator \(c\) to map the price vector from the price space to the real line, which we can think of as the cointegration space. We lift this to Hilbert-valued stochastic processes:

Let \(F, P\) and \(C\) be three separable Hilbert spaces, denoting the factor, price and cointegration space, resp. We denote \(\langle \cdot, \cdot \rangle_i\), the inner product with associated norm \(|\cdot|_i\), for \(i = F, P, C\). Assume \(P \in L(F, P)\), which we call the price operator and \(C \in L(P, C)\) the cointegration operator. For \(\{X(t)\}_{t \geq 0}\) being a \(F\)-valued predictable process, we define the price process

\[
Y(t) = PX(t), t \geq 0
\]

which becomes a \(P\)-valued predictable process.

**Definition 20.** We say that \((P, C)\) is a cointegration pricing system if the \(C\)-valued stochastic process \(\{C^P X(t)\}_{t \geq 0}\) admits a limiting distribution. We say that the price process \(Y(t) = PX(t)\) for given \(P \in L(F, P)\) is cointegrated if there exists a \(C \in L(P, C)\) such that \(CY(t)\) admits a limiting distribution.

Obviously, if \(Y\) in (20) is cointegrated, then \((C, P)\) is a cointegration pricing system for the given cointegration operator \(C\).

We recall from infinite dimensional stochastic analysis (see e.g. Peszat and Zabczyk [34]) that the distribution of a \(C\)-valued random variable \(Z\) is defined as the image measure \(P_Z\) on the Borel sets \(\mathcal{B}(C)\) of \(C\), that is \(P_Z(A) = \mathbb{P}(Z \in A)\) for \(A \in \mathcal{B}(C)\). The definition of cointegration demands the existence of a probability measure \(P_\infty\) on \(\mathcal{B}(C)\) such that \(P_\infty \circ P_X(t) \rightarrow P_\infty\) when \(t \rightarrow \infty\), where the limit is in the sense of probability measures, e.g., for every bounded measurable function \(g : C \rightarrow \mathbb{R}\), it holds for \(t \rightarrow \infty\)

\[
\int_C g(u) P_\infty(du) \rightarrow \int_C g(u) P_\infty(du).
\]

Denote the cumulant functional of \(X\) by \(\Psi_X(t, v), v \in F\), defined as

\[
\Psi_X(t, v) := \log \mathbb{E} \left[ e^{i(v, X(t))_F} \right],
\]

where \(\log\) is the distinguished logarithm (see e.g. Sato [35]). We have the following equivalent characterization of cointegration:

**Proposition 21.** \(Y\) is cointegrated if and only of there exists a \(C \in L(P, C)\) and a cumulant function \(\Psi_C\) such that

\[
\lim_{t \rightarrow \infty} \Psi_X(t, P^* C^* u) = \Psi_C(u)
\]

for all \(u \in C\).
Proof. It holds that $⟨u, CPX(t)⟩_C = ⟨PC*u, X(t)⟩_F$ for every $u ∈ C$. Thus,

$$\log E \left[ e^{i⟨u, CPX(t)⟩_C} \right] = Ψ_X(t, PC*u).$$

If $(P, C)$ is a cointegrated pricing system, then $Ψ_C$ is the cumulant function of $P_∞$.

We next connect cointegration in Hilbert space to cointegration in finite dimensions, as considered in the previous sections:

**Proposition 22.** Let $(C, P)$ be a cointegration pricing system (for the factor process $X$ in $F$). Assume that $\dim(\ker C^⊥) := d < ∞$ and $\dim(\ker P^⊥) := n < ∞$ for $n, d ∈ \mathbb{N}$. Then for every $T ∈ C^*$, there exist $c_f ∈ \mathbb{R}^d, P ∈ \mathbb{R}^{d×n}$ and an $\mathbb{R}^n$-valued factor process $X(t)$ such that

$$TCPX(t) = c_f^T P X(t),$$

and where the real-valued process $t → c_f^T P X(t)$ admits a limiting distribution.

Proof. For any $u ∈ P$, $u^⊥ := u - \text{Proj}_{\ker C} u ∈ \ker C^⊥$ and for an ONB $\{h_i\}_{i=1}^d$ in $\ker C^⊥$ we find,

$$u^⊥ = \sum_{i=1}^d ⟨u^⊥, h_i⟩_P h_i = \sum_{i=1}^d ⟨u, h_i⟩_P h_i - \sum_{i=1}^d \langle \text{Proj}_{\ker C} u, h_i⟩_P h_i = \sum_{i=1}^d ⟨u, h_i⟩_P h_i.$$

But then, since $Cu = C u^⊥$ for every $u ∈ P$, it follows

$$CPX(t) = \sum_{i=1}^d ⟨PX(t), h_i⟩_P C h_i.$$

Next, for any $v ∈ F$, we have that $v^⊥ := v - \text{Proj}_{\ker P} v ∈ \ker P^⊥$, and for an ONB $\{f_j\}_{j=1}^n$ in $\ker P^⊥$ it holds that

$$v^⊥ = \sum_{j=1}^n ⟨v, f_j⟩_F f_j.$$

Since $P v = P v^⊥$ for any $v ∈ F$, we derive

$$PX(t) = \sum_{j=1}^n ⟨X(t), f_j⟩_F P f_j.$$

From this we find

$$CPX(t) = \sum_{i=1}^d \sum_{j=1}^n ⟨X(t), f_j⟩_F P f_j h_i.$$  \hfill (21)

Define the $d × n$-matrix $P := \{(P f_j, h_i)_P\}_{i=1,\ldots,d,j=1,\ldots,n}$ and the $\mathbb{R}^n$-valued factor process $X(t) := (⟨X(t), f_1⟩_F, \ldots, ⟨X(t), f_n⟩_F)^T$. Finally, we introduce $c_f := (TCh_1, \ldots, TCh_d)^T ∈ \mathbb{R}^d$, and the representation of $TCPX(t)$ follows.

Note that for any $θ ∈ \mathbb{R}$,

$$θ TCPX(t) = ⟨X(t), P^* C^* T^* θ⟩_F.$$
Therefore, by the assumption that $CPX(t)$ admits a limiting distribution in combination with Prop. [21] there exists a function $\mathbb{R} \ni \theta \mapsto \Psi_{TC}(\theta) \in \mathbb{C}$ given by

$$
\Psi_{TC}(\theta) = \lim_{t \to \infty} \Psi_X(t, \mathcal{P}^*C^*T^* \theta) = \Psi_{c}(T^* \theta).
$$

The function $\Psi_{TC}$ is a cumulant function, since $\mathcal{T}$ is a continuous linear operator (see e.g. Sato [35, Prop. 2.5 (viii)]). The Proposition follows. \hfill $\Box$

Notice that for any $\mathcal{T} \in \mathcal{C}^*$, $\mathcal{TC} \in \mathcal{P}^*$, and we can interpret $(\mathcal{TC}, \mathcal{P})$ as a cointegration pricing system with cointegration space $\mathcal{C}$, which explains the subscript. Given the factor process $\mathcal{X}$ (see e.g. Sato [35, Prop. 2.5 (viii)]). The Proposition follows.

Definition 23. A cointegration pricing system $(\mathcal{C}, \mathcal{P})$ has a finite dimensional realization (FDR) if, for $n, d \in \mathbb{N}$, there exists an $\mathbb{R}^n$-valued factor process $\mathcal{X}$, a $d \times n$ pricing matrix $\overline{\mathcal{P}}$ and a $c \in \mathbb{C}^{\times d}$ such that $CPX(t) = c^\top \overline{\mathcal{P}} \mathcal{X}(t)$.

In view of Proposition [22] we have an FDR when $\ker \mathcal{C}^\perp$ and $\ker \mathcal{P}^\perp$ are finite dimensional. In this case, $c = (Ch_1, \ldots, Ch_d)^\top$ with $\{h_i\}_{i=1}^d$ being the ONB of $\ker \mathcal{C}^\perp$. If $(\mathcal{C}, \mathcal{P})$ is a general cointegration pricing system which has an FDR, then for any $\mathcal{T} \in \mathcal{C}$ we find that

$$
\mathcal{TC} \mathcal{P} \mathcal{X}(t) = c_T^\top \overline{\mathcal{P}} \mathcal{X}(t),
$$

for $c_T := \mathcal{T} c = (\mathcal{T} c_1, \ldots, \mathcal{T} c_d)^\top \in \mathbb{R}^d$. Hence, $(\mathcal{T} c, \overline{\mathcal{P}})$ will be a finite dimensional cointegrated pricing system for the factor process $\mathcal{X}$. We remark that Definition [23] does not really rest on the fact that there exist any limiting distribution, but as we work with cointegration in this paper, we focus on cointegration pricing systems, that is, pricing systems $(\mathcal{C}, \mathcal{P})$ for which $CPX(t)$ admits a limiting distribution.

We remark that we have not assumed any minimality of the pricing matrix $\overline{\mathcal{P}}$ in the above considerations. We recall from the proof of Prop. [22] that the $d \times n$-matrix $\overline{\mathcal{P}}$ has elements $\langle \mathcal{P} f_j, h_i \rangle_p$, and minimality is achieved as long as this matrix has full rank. However, the next proposition shows that we must take into account a possible finite-dimensionality of the factor process $X$ as well. Indeed, another situation where we may have an FDR is when the factor process has a finite-dimensional state space:

Proposition 24. Assume that the factor process $\{X(t)\}_{t \geq 0}$ takes values in $\mathbb{F}_n \subset \mathbb{F}$, where $\dim(\mathbb{F}_n) := n < \infty$ for $n \in \mathbb{N}$. Then any cointegration pricing system $(\mathcal{C}, \mathcal{P})$ has a finite dimensional realization, with $\overline{\mathcal{P}} = \text{Id}$ (the identity matrix on $\mathbb{R}^n$). $X(t) := ((X(t), f_1)^\top, \ldots, (X(t), f_n)^\top) \in \mathbb{R}^n$ for an ONB $\{f_j\}_{j=1}^n$ of $\mathbb{F}_n$ and $c = (\mathcal{CP} f_1, \ldots, \mathcal{CP} f_n)^\top \in \mathbb{C}^{\times n}$. 


Proof. If \(X(t) \in F_n\), then \(X(t) = \sum_{j=1}^{n} (X(t), f_j)_F f_j\) and therefore

\[
CPX(t) = \sum_{j=1}^{n} (X(t), f_j)_F CP f_j = c^T \text{Id} X(t).
\]

The result follows. \(\square\)

This result indicates strongly the possible non-uniqueness of the FDR, since depending on the pricing operator \(P\), one may specify a different \(\overline{P}\) than the identity matrix, and thus also different \(c\). It also shows that the question of minimality depends on \(C, P\) and the possible finite dimensionality of \(X\).

Let us now focus on the case where \(F\) and \(P\) can be represented as product spaces, e.g., when \(F = H^x_n\) and \(P = K \times d\) for two separable Hilbert spaces \(H\) and \(K\). We denote the inner product as usual by \((\cdot, \cdot)_i\) with corresponding norms \(| \cdot |_i\), where the subscript indicates the space, here \(i = H, K\). The inner product on the product space \(F\) is then given by \((u, v)_F = \sum_{j=1}^{n} (u_j, v_j)_H\) for \(u = (u_1, \ldots, u_n) \in F\) and \(v = (v_1, \ldots, v_n) \in F\) (and likewise for \(P\)).

To make an example, suppose we have given a factor process \(X \in H^x_n\) and a pricing operator \(P\) given as an \(d \times n\)-matrix of operators \(P = \{P_{ij}\}_{i=1,\ldots,d;j=1,\ldots,n}\) with \(P_{ij} \in L(H,K)\). Then the pricing vector will be \(Y(t) = PX(t)\), which is a \(K \times d\)-valued stochastic process. Indeed, we have that \(Y = (Y_1,\ldots,Y_d)^T\) with

\[
Y_i(t) = \sum_{j=1}^{n} P_{ij} X_j(t)
\]

for \(i = 1,\ldots,d\). In analogy with Example 4, we assume that \((X_1,\ldots,X_{n-1})^T \in H^{x(n-1)}\) admits a limiting distribution, while \(X_n\) may be non-stationary. We observe that any \(C = (C_1,\ldots,C_d)^T\) with \(C_i \in L(K,C)\) will be such that \(C \in L(K \times d, C)\).

Under the condition \(\sum_{i=1}^{d} C_i P_{in} = 0\) we find \(CY(t) = \sum_{i=1}^{d} \sum_{j=1}^{n-1} C_i P_{ij} X_j(t)\), that is, a \(C\)-valued stochastic process not depending on \(X_n\) but only on \(X_j\) for \(j = 1,\ldots,n-1\). This provides us with a simple example of a cointegration pricing system.

A way to generate a system of factor processes \(X \in H^x_n\) can be as follows: consider an \(\mathbb{R}^m\)-valued stochastic process \(\{Z(t)\}_{t \geq 0}\) and \(A \in L(\mathbb{R}^m, H^x_n)\). For \(b \in H^x_n\), define the factor process

\[
X(t) = AZ(t) + b.
\]

We remark that \(A\) can be represented as an \(n \times m\)-matrix with elements in \(H\). Indeed, the columns of this matrix will be given by the action of \(A\) on the canonical basis vectors in \(\mathbb{R}^m\). If \(H\) is some space of functions on \(\mathbb{R}_+\), we may relate the factor process \(X\) to the affine models of forward prices from the previous section, i.e., the affine forward models provide a class of factors in an infinite dimensional framework. The existence of a limiting distribution of one or more of the factors \(X_j, j = 1,\ldots,n\) can be traced back to the process \(Z\). Indeed, this simplified case relates us back to the models consider in Section 3, for example the polynomial processes in Proposition 12.

4.1. A discussion of cross-commodity forward markets. Let us now focus specifically on commodity forward markets, and start with a discussion on cross-commodity models. Suppose we have \(d\) forward markets, with forward price dynamics denoted by \(f_i(t,x), i = 1,\ldots,d\) and \(x \in \mathbb{R}_+\) being time to maturity. We
are aiming at a $d$-dimensional model of the forward curve dynamics $t \mapsto f(t, \cdot) = (f_1(t, \cdot), \ldots, f_d(t, \cdot))^\top$. We choose $H$ to be a Hilbert space of real-valued measurable functions on $\mathbb{R}_+$. Following the analysis in Benth and Krühner [12], a convenient choice of such a space could be the so-called Filipovic space of absolutely continuous functions (see Appendix A for a definition).

Based on the analysis in Benth and Krühner [12] (see also Benth and Krühner [11]), the forward price dynamics $\{f(t, \cdot)\}_{t \geq 0}$ can be expressed as a $H^{d \times d}$-valued stochastic process

\begin{equation}
    df(t, \cdot) = \partial f(t, \cdot) \, dt + \beta(t, f(t, \cdot)) \, dt + \sigma(t, f(t, \cdot)) \, dL(t)
\end{equation}

where $L$ is a $V$-valued square-integrable Lévy process with zero mean and $V$ being a separable Hilbert space. We use the notation $\partial$ for the $d \times d$ matrix-operator

\begin{equation}
    \partial = \begin{bmatrix}
        \frac{\partial}{\partial x} & 0 & \cdots & 0 \\
        0 & \frac{\partial}{\partial x} & \cdots & 0 \\
        \vdots & \vdots & \ddots & \vdots \\
        0 & 0 & \cdots & \frac{\partial}{\partial x}
    \end{bmatrix},
\end{equation}

with $\partial/\partial x$ being the derivative operator on the functions in $H$. We assume that this operator is a densely defined unbounded operator on $H$ which is the generator of a $C_0$-semigroup (the shift semigroup). This holds if we choose $H$ to be the Filipovic space, say. Further, the measurable mappings $\sigma : \mathbb{R}_+ \times H^{d \times d} \to L(V, H^{d \times d})$ and $\beta : \mathbb{R}_+ \times H^{d \times d} \to H^{d \times d}$ are assumed to satisfy the Lipschitz conditions stated in Peszat and Zabczyk [34], Section 9.2] such that there exists a unique mild predictable càdlàg solution to (22).

The function $\beta$ models the risk premium in this cross-commodity model of forward curves. We note in passing that (22) is formulated under $\mathbb{P}$, and to ensure an arbitrage-free dynamics there must exist a probability $\mathbb{Q} \sim \mathbb{P}$ such that the $\mathbb{Q}$-dynamics of $f$ is

\begin{equation}
    df(t, \cdot) = \partial f(t, \cdot) \, dt + dM(t)
\end{equation}

where $M$ is a $H^{d \times d}$-valued (local) $\mathbb{Q}$-martingale (see Benth & Krühner [11]). We will not pursue the existence of such a $\mathbb{Q}$ in further detail here.

We may view the cross-commodity forward model (22) in our cointegration context by choosing the factor process $X$ to be equal to the price vector process $f$. Thus, we have $H = K$ and $n = d$, with a pricing matrix $\mathcal{P}$ simply being the identity operator in $H^{d \times d}$. In particular, we let $\mathcal{P} = H^{d \times d}$, i.e., the pricing space is the product space. In many markets, prices are naturally varying over seasons. For example in power markets, prices are typically higher in heating and cooling seasons. Such a behaviour may be modelled into $\beta$. Further, many commodities are based on distinguishable resources, with oil and gas as prime examples. For such commodities, one may expect non-stationarity effects in prices. Other sources of non-stationarity are technological changes and inflation. Such non-stationarity could possibly be modelled in the $\beta$, as well, for example by adding dependency on additional (non-stationary) stochastic factors $Z$, e.g., assuming a drift of the form $\beta(t, f(t, \cdot), Z(t))$.

Cointegration in this context could be formulated as follows: There is an operator $C \in L(H^{d \times d}, H)$ such that the $H$-valued stochastic process $t \mapsto g(t, \cdot) := Cf(t, \cdot)$ admits a limiting distribution. In many applications one is interested in the spread between two or more forward markets, and it is natural to consider linear combinations of the forward curves, which again will be an element in the space of
existence of an invariant measure of Filipovic space, we may resort to Tehranchi \[37\] for sufficient conditions for the time-homogeneity and Lipschitzianity of \(\sigma\).

No-arbitrage condition with the volatility situation where this drift condition is not needed. Tehranchi \[37\] treats HJM models, which has a nonlinearity in the drift satisfying a one has that sunshine over the day is varying with season, and so is the average wind speed, stochastic process measuring the total generation of such. Since the amount of power generation from photovoltaic and wind, and we let \(Z\) still in place, the process will likely show an increasing trend, at least on a short term horizon. Hence, \(Z\) may be thought of as a non-stationary stochastic process. Assume now that \(\beta(t, f(t, \cdot), Z(t)) = (\beta_1 Z(t), \beta_2 Z(t))^\top\) for \(\beta_1, \beta_2\) two constants, which is an \(\mathbb{R}^2\)-valued stochastic process, and thus trivially in \(H^{2,2}\). Further, we let the volatility be constant, in the sense that \(\sigma(s, f(s, \cdot)) = \Sigma \in L(V, H^{2,2})\). Under this specification, we choose \(\mathcal{C} := (\beta_2, -\beta_1)\), which will commute with \(\partial/\partial x\), and we find for \(g(t, \cdot) := \beta_2 f_1(t, \cdot) - \beta_1 f_2(t, \cdot)\) the cointegration process \(g\) will be an Ornstein-Uhlenbeck process with unbounded operator \(\partial/\partial x\) and volatility \(\Sigma\). Invariant measures for Lévy-driven Ornstein-Uhlenbeck processes are thoroughly discussed in Applebaum \[3\] (see also references therein). Although Tehranchi \[37\] considers more general HJM-models with Gaussian noise, one can apply his methods to conclude that \(g\) in \[24\] admits a limiting distribution if we choose \(H\) to be the Filipovic space (see Appendix \[A\]). We remark in passing that Tehranchi \[37\] makes use of the fact that the shift semigroup \(S(t)\) is a strict contraction on a convenient subspace of the Filipovic space.

So far we have only considered arithmetic forward models. To introduce a geometric model, of the form \(F(t, x) := \exp(f(t, x))\), with \(f\) defined by the dynamics \[22\] and \(\exp(f) := (\exp(f_1), \ldots, \exp(f_d))\), we must impose additional structure on the Hilbert space \(H\). Indeed, it has to be closed under exponentiating, that is, for any \(h \in H\), it must hold that \(\exp h \in H\). If \(H\) is a Banach algebra under pointwise multiplication, this holds true, since in that case we have \(|h^n|_H \leq |h|^n_H\) and thus...
| exp h|_H ≤ exp |h|_H < ∞. We remark that after an appropriate scaling of the norm in the Filipovic space, it becomes a Banach algebra (see Benth and Krühner [11]).

### 4.2. A three-factor example.

We end this Section with a concrete example adopted from Benth [6]. Let H be a Hilbert space of real-valued measurable functions on \( \mathbb{R}_+ \). Consider a three factor processes \( X = (X_1, X_2, X_3)' \in H^{3 \times 3} \) given by \( X_3(t) = L(t) \) where \( L \) is an \( \mathbb{R} \)-valued Lévy process and for \( x \in \mathbb{R}_+ \),

\[
X_k(t, x) = h_k(t, x) + \int_0^t g_k(t + x - s) \, dU_k(s), \ k = 1, 2.
\]

Here, for \( k = 1, 2 \), \( U_k \) are \( \mathbb{R} \)-valued Lévy processes with zero mean and finite variance, and \( h_k(t, \cdot), g_k \in H \). In the next lemma, we state conditions such that \( \{X_k(t)\}_{t \geq 0} \) becomes an \( H \)-valued stochastic process.

**Lemma 25.** Suppose that the shift semigroup \( \{S(t)\}_{t \geq 0} \) is bounded on \( H \), i.e., \( S(t) \in L(H) \) for all \( t \geq 0 \). If \( \int_0^t |g_k(s + \cdot)|_H^2 \, ds < \infty \) for every \( t \geq 0 \), then \( \{X_k(t)\}_{t \geq 0} \) defined in (25) is an \( H \)-valued stochastic process. Its cumulant is

\[
\log \mathbb{E}[\exp (i(h, X_k(t)))_H] = i(h, h_k(t))_H + \int_0^t \psi_{\psi_{U_k}}( (h, g_k(s + \cdot))_H) \, ds,
\]

for \( h \in H \) and \( \psi_{\psi_{U_k}} \) the cumulant of \( U_k(1) \).

**Proof.** Fix \( t \geq 0 \). By assumption, it holds that \( g_k(t-s+\cdot) = S(t-s)g_k(\cdot) \in H \) for all \( s \in [0, t] \). From Peszat and Zabczyk [34], the stochastic integral \( \int_0^t g_k(t-s+\cdot) \, dU_k(s) \) is well-defined and defines an element in \( H \) if \( \int_0^t |g_k(t-s+\cdot)|_H^2 \, ds < \infty \), which holds by assumption. Thus, \( \{X_k(t)\}_{t \geq 0} \) is an \( H \)-valued stochastic process.

We have that the operator \( G(t-s)(h) = (h, g_k(t-s + \cdot))_H \) is a linear functional on \( H \). Moreover, by the Cauchy-Schwartz inequality,

\[
\int_0^t G^2(t-s)(h) \, ds = \int_0^t (h, g_k(t-s+\cdot))_H^2 \, ds 
\leq |h|_H^2 \int_0^t |g_k(s+\cdot)|_H^2 \, ds.
\]

Hence, by the integrability assumption on the norm of \( g_k \), \( s \to G(t-s)(h) \) is \( U_k \)-integrable on \([0, t]\), and by linearity we find

\[
(h, \int_0^t g_k(t-s+\cdot) \, dU_k(s))_H = \int_0^t (h, g_k(t-s+\cdot))_H \, dU_k(s) = \int_0^t G_k(t-s)(h) \, dU_k(s).
\]

Hence,

\[
\log \mathbb{E} \left[ \exp \left( i(h, \int_0^t g_k(t-s+\cdot) \, dU_k(s))_H \right) \right] = \log \mathbb{E} \left[ i \int_0^t G(t-s)(h) \, dU_k(s) \right] = \int_0^t \psi_{\psi_{U_k}}( G(s)(h) ) \, ds.
\]

Since \( U_k \) is a zero mean square integrable Lévy process, its cumulant becomes

\[
\psi_{\psi_{U_k}}(z) = -\frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} (e^{izy} - 1 - izy) \ell(dy)
\]

for \( \sigma \geq 0 \) a constant and \( \ell \) the Lévy measure (see Applebaum [2]). We have

\[
|e^{izy} - 1 - izy| = |iz|^2 \int_0^y e^{izu} \, du \, dx \leq \frac{1}{2} z^2 y^2.
\]
and therefore \( \psi_{U_k}(G(s)(h)) \) is integrable on \([0, t]\) whenever \( G(s)(h) \in L^2([0, t]) \), which holds by assumption after appealing to the Cauchy-Schwartz inequality, as argued above.

In the Lemma 25 above we assumed that the function \([0, t] \ni s \mapsto |g_k(s + \cdot)|_H \in \mathbb{R}_+ \) is in \( L^2([0, t]) \). As the shift operator \( S(t) \) is assumed continuous, a sufficient condition for this to hold is that \( s \mapsto \|S(s)\|_{op} \in L^2([0, t]) \). Whenever the family of shift operators defines a strongly continuous semigroup, say, this holds true. If in addition \( \{S(t)\}_{t \geq 0} \) is exponentially stable, we have that \( s \mapsto \|S(s)\|_{op} \in L^2(\mathbb{R}_+) \). If \( s \mapsto |g_k(s + \cdot)|_H \in L^2(\mathbb{R}_+) \) and \( h_k(t) \) has a limit in \( H \) as \( t \to \infty \), it follows from Lemma 25 that \( \{X_k\}_{t \geq 0} \) admits a limiting distribution in \( H \).

Introduce next the pricing operator \( \mathcal{P} \in L(H^{\times 3}, H^{\times 2}) \) simply as

\[
(26) \quad \mathcal{P} = \begin{bmatrix} \text{Id} & 0 & \text{Id} \\ 0 & \text{Id} & \text{Id} \end{bmatrix},
\]

where \( \text{Id} \) is the identity operator on \( H \). If we assume \( H \) to be a Banach algebra, we can define the exponential forward price dynamics for a bivariate commodity market by

\[
(27) \quad F(t) := \exp(\mathcal{P}X(t)).
\]

Following the analysis in Benth \([6]\), we can choose \( h_k(t) \) to ensure an arbitrage-free dynamics (see Prop. 2 in \([6]\)). One can also think of \( h_k \) as a model for the market price of risk/risk premium in the forward market.

In this bivariate cross commodity forward price model, we see that \( \ln F(t) = \mathcal{P}X(t) \), and thus for any \( \mathcal{C} \in L(H^{\times 2}, C) \), we have

\[
(28) \quad \mathcal{C} \ln F(t) = \mathcal{C}_1X_1(t) + \mathcal{C}_2X_2(t) + (\mathcal{C}_1 + \mathcal{C}_2)X_3(t).
\]

Here we have represented the operator \( \mathcal{C} \) in matrix form, i.e.,

\[
\mathcal{C} = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix}
\]

for \( \mathcal{C}_i \in L(H, C), i = 1, 2 \). Letting \( \mathcal{C}_2 = -\mathcal{C}_1 \), we find \( \mathcal{C} \ln F(t) = \mathcal{C}_1(X_1(t) - X_2(t)) \).

In the next lemma, we state sufficient conditions for \( \mathcal{C}_1(X_1(t) - X_2(t)) \) to admit a limiting distribution in \( H \), which thus yield sufficient conditions for having a cointegrated model.

**Lemma 26.** Assume that the shift operator \( S(t) \) is bounded in \( H \) for all \( t \geq 0 \) and \( |g_k(s + \cdot)|_H \in L^2(\mathbb{R}_+) \) for \( k = 1, 2 \). If \( h_\infty := \lim_{t \to \infty} (h_1(t) - h_2(t)) \) exists in \( H \), then \( \mathcal{C}_1(X_1(t) - X_2(t)) \) admits a limiting distribution in \( H \). This limiting distribution has cumulant

\[
\lim_{t \to \infty} \log E[\exp(\mathcal{i}(h, \mathcal{C}_1(X_1(t) - X_2(t))))] = \mathcal{C}_1h_\infty + \int_0^\infty \psi_U((\mathcal{C}_1^*h, g_1(s + \cdot))_H, -(\mathcal{C}_1^*h, g_2(s + \cdot))_H) \, ds
\]

where \( \psi_U \) is the cumulant of the bivariate Lévy process \( U = (U_1, U_2) \) and \( h \in C \).

**Proof.** We find, following Lemma 25, that the processes \( \int_0^t g_k(t - s + \cdot) \, dU_k(s) \) in \( H \) both admit a limiting distribution. Moreover, by using the same argument for marginal integrability as in the proof of Lemma 25, we find that \( s \mapsto \psi_U((\mathcal{C}_1^*h, g_1(s + \cdot))_H, -(\mathcal{C}_1^*h, g_2(s + \cdot))_H) \) is integrable on \( \mathbb{R}_+ \) for any \( h \in C \). The result follows. \( \Box \)
A special case is to choose $C = H$ and $C_1 = \text{Id}$. Thus, $X_1(t) - X_2(t)$ admits in particular a limiting distribution when the conditions in Lemma 26 are fulfilled.

Geman and Liu [22] perform an empirical analysis of cointegration between the gas forward markets at Henry Hub (US) and National Balancing Point (UK). They introduce various measures on the forward curves to study how integrated the markets are. More specifically, it is proposed to measure the distance between the average of the respective forward curves, or simply the distance between the implied spot prices (closest maturity forwards), or the distance between some geometric weighted average of forward prices. In our context, the latter two distance measures can be expressed as $|C_1 X_1(t) - C_1 X_2(t)|$ with $C = \mathbb{R}$ and $C_1 \in H^*$. For example, in the case of closest forwards (or spot), we choose $C_1 = \delta_0$, the evaluation operator at zero, assuming that this is continuous on $H$.

The average of the forward curve is not possible to represent via a linear operator $C_1$ in a geometric model. However, if we choose to work with an arithmetic model, this would simply become an integral operator on the curves in $H$.

In view of the results in Section 3, one can find spot models that lead to cointegration of forward prices with given time to maturity. In the context of Geman and Liu [22], measuring the difference of the average of the forward curves at given maturity-times could lead to stationarity and thus the conclusion that the markets are cointegrated. However, Geman and Liu [22] do not find evidence for cointegration of the two gas forward markets in Henry Hub and National Balancing Point. This could be explained by a possible term structure of the risk premium (which can be traced back in the $\beta$ function above) and thus the need for more sophisticated choices of operators $C$ to reveal a potential cointegration.

**Appendix A. The Filipovic space**

We present the Filipovic space following Filipovic [24]: Let $w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a monotonely increasing function with $w(0) = 1$ and $\int_0^\infty w^{-1}(x) \, dx < \infty$. Introduce the Filipovic space, denoted $H_w$, as the space of absolutely continuous functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ for which

$$|f|_w^2 := f^2(0) + \int_0^\infty w(x)(f'(x))^2 \, dx < \infty,$$

where $f'$ is the weak derivative of $f$. With the inner product

$$(f,g)_w = f(0)g(0) + \int_0^\infty w(x)f'(x)g'(x) \, dx$$

for $f, g \in H_w$, $H_w$ becomes a separable Hilbert space. The shift operator $S(t) : f \mapsto f(t + \cdot)$ for $t \geq 0$ defines a $C_0$-semigroup on $H_w$ which is quasi-contractive and uniformly bounded. The generator of $S(t)$ is the derivative operator. The evaluation map $\delta_x : f \mapsto f(x)$ is a linear functional on $H_w$. Finally, from Benth and Krühner [11], $H_w$ becomes a Banach algebra after appropriate rescaling of the norm $| \cdot |_w$, that is, if $f, g \in H_w$, then $fg \in H_w$ and $\|fg\|_w \leq \|f\|_w \|g\|_w$ with $\| \cdot \|_w := c| \cdot |_w$ for a suitable constant $c > 0$ depending on $\int_0^\infty w^{-1}(x) \, dx$. 


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