A NOTE ABOUT THE $\ell^p$-IMPROVING PROPERTY OF THE AVERAGE OPERATOR

JOSÉ MADRID

Abstract. In this paper we give a short proof of the $\ell^p$-improving property of the average operator along the square integers and more general quadratic polynomials. Moreover we obtain a similar result for some higher degree polynomials. We also show an elementary proof of the $\ell^p$-improving property of the average operator along primes.

1. Introduction

For $f \in \ell^2(\mathbb{Z})$. Define the average of $f$ along the polynomial $P$ mapping the integers to the integers, by

$$A_P^N f(x) := \frac{1}{N} \sum_{k=1}^{N} f(x + P(k)),$$

(1.1)

In the case $P(x) = x^d$ we will denote this average by $A_d^N f$ for $d > 2$ and $A_N f$ for $d = 2$. Along this paper $\|f\|_p$ denotes the $\ell^p$-norm of the function $f: \mathbb{Z} \to \mathbb{R}$ and $p' = \frac{p}{p-1}$.

We prove that if $3/2 < p \leq 2$ and $P(x) = ax^2 + bx + c \in \mathbb{Z}[X]$ is a quadratic polynomial with no negative coefficients, then $A_P^N f$ satisfies an $\ell^p$-improving estimate:

$$N^{-2/p'} \|A_P^N f\|_{p'} \lesssim N^{-2/p} \|f\|_p,$$

for every $f: \mathbb{Z} \to \mathbb{R}$. The range $(3/2, 2]$ is optimal. We are able to extend this result for the higher degree polynomials $P(x) = x^d$, however, so far we can do that only in a (probably) not optimal range. We also obtain a similar improving result for averages along primes through our method.

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The regularity on $\ell^p$-spaces of the averages operator along the square integers was originally studied by Bourgain in [1]. The $\ell^p$-improving estimates for compact supported functions were recently studied in [4], analyzing a good approximation for the corresponding multiplier, coming from the Hardy Littlewood circle method. The argument in that paper use strongly the fact that $P(x) = x^2$ and it does not extend even for the case $p(x) = x^2 + x$ as pointed out by the authors, due to some difficulties coming from the minor arcs, in that paper they asked about whether or not we continue having the $\ell^p$-improving estimate for other quadratic polynomials like $x^2 + x$, and for higher degree polynomials (see conjecture 6.3 in [4]). In this paper we give a positive answer to that question for $d = 2$ and in the higher degree case for $P(x) = x^d$ in the the range $(2 - 4/(2 + d(2^d + 2)), 2]$ through a complete different argument, in particular we recover the Theorem 1.1 in [4]. The main ingredient is to find a relation between the average operator and the discrete fractional integral operator in order to use the results from [12] and [10]. This method also works for the average operator along primes as discussed below. Variants of this were studied in [8], [11] and [13], for some other interesting related results see [3], [5], [8], [7], [9].

**Theorem 1.1.** For every $3/2 < p \leq 2$ there is a constant $C_p > 0$, such that for all $N \in \mathbb{N}$ we have that
\[
\|A_N f\|_{p'} \leq C_p N^{2/p' - 2/p} \|f\|_p,
\]
for every function $f : \mathbb{Z} \to \mathbb{R}$.

The next results is an extension of this theorem.

**Theorem 1.2.** Let $P(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ be a quadratic polynomial with no negative coefficients. For every $3/2 < p \leq 2$ and $N \in \mathbb{N}$, there is a constant $C_p > 0$, such that for all $N \in \mathbb{N}$ we have that
\[
\|A_N^P f\|_{p'} \leq C_p \left(2a + \frac{b}{N}\right) (2aN + b)^{2/p' - 2/p} \|f\|_p,
\]
for every function $f : \mathbb{Z} \to \mathbb{R}$.

In particular, if $P(x) = x^2$, we have that $a = 1$, $b = 0$ and $c = 0$, we recover the previous theorem.

For all $d > 2$ we define
\[
\tilde{p}_d = 2 - \frac{4}{2 + d(2^d + 2)}.
\]

The next theorem establish the desired result when $p$ is close to 2, more precisely when $p > \tilde{p}_d$. 
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**Theorem 1.3.** For every $\tilde{p}_d < p \leq 2$, there is a constant $C = C(p, d) > 0$, such that for all $N \in \mathbb{N}$ we have that

$$\|A_N^d f\|_{p'} \leq CN^{d/p'-d/p} \|f\|_p,$$

for every function $f : \mathbb{Z} \to \mathbb{R}$.

**Remark 1.4.** Using Theorem 1 in [10] we can go slightly lower than $\tilde{p}_d$ for $d > 11$.

Our final result discuss the improving property of the average operator along primes. This result was recently established in [5] in this paper we present an elementary proof of this fact.

For every $f : \mathbb{Z} \to \mathbb{R}$ we define the average operator along primes to be

$$A_N f(x) = \frac{1}{N} \sum_{p \leq N} f(x - p) \log p$$

where the sum is taken over all primes with size at most $N$.

**Theorem 1.5.** Let $1 < p \leq 2$, then there exists a constant $C_p > 0$ such that for all $N \in \mathbb{N}$ we have that

$$\|A_N f\|_{p'} \leq C_p N^{1/p} \|f\|_p, \quad (1.2)$$

for every $f : \mathbb{Z} \to \mathbb{R}$.

**Remark 1.6.** In the case $p > 2$, if $f$ is supported in $[0, N]$ the inequality (1.2) follows immediately as a consequence of the $\ell^p$-boundedness of the operator $A_N$ established by Bourgain in [2] and Hölder inequality.

2. Preliminaries

We write $A \lesssim B$ if there exists an absolute constant $C$ such that $A \leq CB$. If the constant depends on parameter $\lambda$ we denote that with a subscript, such as $A \lesssim_\lambda B$. We write $A \sim B$ if both $A \lesssim B$ and $B \lesssim A$. $\delta_k$ denotes the classical Dirac delta function supported at the point $k$, more precisely $\delta_k(k) = 1$ and $\delta_k(n) = 0$ for all $n \neq k$.

Let us focus in the case $P(x) = x^d$. Let $d \geq 2$ be an integer, we start observing that an inequality like

$$N^{-d/p'} \|A_N^d f\|_{p'} \lesssim N^{-d/p} \|f\|_p, \quad (2.3)$$

for functions $f : \mathbb{Z} \to \mathbb{R}$ supported in $[-N^d, N^d]$ along the polynomial $P(x) = x^d$ would be optimal in terms of magnitude, in fact, it is enough to consider $f = \chi_{[-N^d, N^d]}$, in this case $\|f\|_p = 2^{1/p} N^{d/p}$ while $\|A_N^d f\|_{p'} \geq \|A_N^d f\|_{\ell^{p'}[-N^d,0]} = N^{d/p}.$
Moreover, we observe, that in order to have the $\ell^p$-improving property (2.3), the condition $p \geq 2 - \frac{1}{d} =: p_d$ is necessary. In fact, if $f = \delta_0$, then $\|f\|_p = 1$ and $\|A_N^d f\|_{p'} = \left( N \frac{1}{N^{p'}} \right)^{1/p'} = \frac{1}{N^{1/p'}}$, then (2.3) holds for $f = \delta_0$ only if $N^{-d/p'} N^{-1/p} \lesssim N^{-d/p}$ which is equivalent to have $\frac{d-1}{p} \leq \frac{d}{p'}$, i.e. $p > p_d$. In particular if $d = 2$ then $p_2 = 3/2$ is the endpoint, and $p'_2 = 3$.

2.1. **Structure of the paper.** Section 3 contains the proof of our main theorems for averages operators along polynomials. In section 4 we present the proof of our Theorem 1.5 for averages operators along primes.

3. **PROOF OF MAIN RESULTS**

Now we are in position to start proving our main theorems, a key idea will be to relate the average operator with the discrete fractional integral operator. For $f : \mathbb{Z} \to \mathbb{R}$ we define

$$I_{d, \lambda} f(n) = \sum_{m=1}^{\infty} \frac{f(n - m^d)}{m^\lambda}$$

where $0 < \lambda < 1$ and $d \geq 1$ is an integer. We denote by $I_{\lambda} f = I_{2, \lambda} f$. This is the well known discrete fractional integral operator.

3.1. **Case $d=2$, $P(x) = x^2$.**

**Proof of Theorem 1.1.** We start observing that by Young inequality

$$\|A_N f\|_2 = \|K_N \ast f\|_2 \leq \|K_N\|_1 \|f\|_2 = \|f\|_2 = N^{2/2 - 2/2} \|f\|_2 \quad (3.4)$$

where $K_N = \frac{1}{N} \sum_{k \leq N} \delta_{-k^2}$. Then we can focus in the case when $p \in (3/2, 2)$, we define

$$\lambda = 1 - \left( \frac{2}{p} - \frac{2}{p'} \right).$$

Thus: $0 < \lambda < 1$ and $\frac{1-\lambda}{2} = \frac{1}{p} - \frac{1}{p'}$. Moreover, since $p > \frac{3}{2}$ we have that $2 > \frac{1}{p-1} = \frac{p'}{p}$. Then

$$\frac{1}{p} > \frac{2}{p} - \frac{2}{p'} = 1 - \lambda \quad \text{and} \quad \frac{1}{p'} < \lambda.$$

Then, as a consequence of Theorem A and Theorem 1 in [12] we know that there exists a constant $C = C(p)$ such that

$$\|I_{\lambda} h\|_{p'} \leq C\|h\|_p,$$
for every $h \in \ell^p$. Then we observe that

$$A_N f(x) := \frac{1}{N} \sum_{k \leq N} f(x + k^2)$$

$$\leq N^{\lambda-1} \sum_{k \leq N} \frac{f(x + k^2)}{k^\lambda}$$

$$\leq N^{\lambda-1} I_\lambda g(-x)$$

for all $x \in \mathbb{Z}$, where $g$ is given by $g(y) := f(-y)$ for every $y \in \mathbb{Z}$. Using this we obtain

$$\|A_N f\|_{p'} \leq N^{\lambda-1} \|I_\lambda g\|_{p'} \leq C N^{\lambda-1} \|g\|_p = C N^{\lambda-1} \|f\|_p.$$

Therefore

$$\|A_N f\|_{p'} \leq C N^{2/p'-2/p} \|f\|_p.$$

□

3.2. General case $d = 2$. Now we are in position to extend this $\ell^p$-improving property to any quadratic polynomial.

Proof of Theorem 1.2. We define $g : \mathbb{Z} \to \mathbb{R}$ by

$$g(4am) = f(m) \text{ for all } m \in \mathbb{Z} \text{ and } g(n) = 0 \text{ if } 4a \nmid n.$$

We observe that since $f$ is supported in $[-(aN^2 + bN + \frac{b^2}{4a}), aN^2 + bN + \frac{b^2}{4a}]$ then we have that $g$ is supported in $[-(2aN + b)^2, (2aN + b)^2]$. Therefore

$$A_{N}^p f(x) = \frac{1}{N} \sum_{n \leq N} f(x + an^2 + bn + c)$$

$$= \frac{1}{N} \sum_{n \leq N} g(4ax + 4a^2n^2 + 4abn + 4ac)$$

$$= \frac{1}{N} \sum_{n \leq N} g(4a(x + c) - b^2 + (2an + b)^2)$$

$$\leq \frac{(2aN + b)}{N} \frac{1}{2aN + b} \sum_{k \leq 2aN + b} g(4a(x + c) - b^2 + k^2)$$

$$= \left(2a + \frac{b}{N}\right) A_{2aN+b} g(4a(x + c) - b^2) .$$
Using the previous calculus, as a consequence of Theorem 1.1 we obtain
\[ \|A_N f\|_{p'} \leq \left( 2a + \frac{b}{N} \right) \|A_{2aN+b} g\|_{p'} \]
\[ \leq \left( 2a + \frac{b}{N} \right) (2aN + b)^{2/p'-2/p} C_p \|g\|_p \]
\[ \leq \left( 2a + \frac{b}{N} \right) (2aN + b)^{2/p'-2/p} C_p \|f\|_p, \]
for every \( p \in (3/2, 2] \).

3.3. Case \( d > 1 \). We will adapt the idea used in the proof of Theorem 1.

Proof of Theorem 1.3. The case \( p = 2 \) follows from Young inequality similarly to (3.4). Let \( p \in (\tilde{p}_d, 2) \), we define \( \lambda = 1 - \left( \frac{d}{p} - \frac{d}{p'} \right) \). Thus:
\[ \lambda > 0, \quad \lambda < 1 \quad \text{and} \quad 1 - \lambda = \frac{1}{p} - \frac{1}{p'} \).
Moreover, using that \( p > p_d \) we see \( \frac{1}{p} > 1 - \lambda \) and \( \frac{1}{p'} < \lambda \). As a consequence of Theorem 1 in [10] we know that if \( \lambda > \lambda_d \) (this condition is equivalent to \( p > \tilde{p}_d \)) then there exists a constant \( C = C(p, d) \) such that
\[ \|I_{d,\lambda} h\|_{p'} \leq C \|h\|_p, \]
for every \( h \in l^p \), where \( \lambda_d := 1 - \frac{1}{2^{p-1}+1} \).

Then we observe that
\[ A_N^d f(x) \leq N^{d-1} I_{k,\lambda} g(-x) \]
for all \( x \in \mathbb{Z} \), where \( g \) is given by \( g(y) := f(-y) \) for every \( y \in \mathbb{Z} \). Using this we obtain
\[ \|A_N^d f\|_{p'} \leq N^{d-1} \|I_{\lambda} g\|_{p'} \leq C N^{d-1} \|f\|_p, \]
which is the desired result.

4. Averages along primes

For \( N \in \mathbb{N} \) and \( \lambda \in (0, 1) \) we define the truncated fractional integral operator along primes to be
\[ J_{\lambda,N} f(x) = \sum_{p \leq N} \frac{f(x-p)}{p^\lambda} \log p. \]
for every \( f : \mathbb{Z} \rightarrow \mathbb{R} \), where the sum is taken over all the primes \( p \) with size at most \( N \).
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**Lemma 4.1.** Let $\lambda \in (0, 1)$ and $C > 0$ be a constant. Assume that $1 < p < \frac{1}{1-\lambda}$ and $\frac{1}{q} = \frac{1}{p} - (1 - \lambda)$. Then

$$\sup_{\alpha \leq CN^{-1/q}\|f\|_p} \alpha^q \{x; J_{\lambda,N}f(x) > \alpha\} \lesssim \|f\|_p^q.$$

**Proof.** We assume with loss of generality that $f \geq 0$. We start enumerating the primes with size at most $N$.

$$\{p \leq N; p \text{ prime}\} := \{p_1, p_2, \ldots, p_{r_N}\}.$$ We recall that

$$p_n \sim n \log n \text{ for all } n,$$

more precisely $n \log n + n \log \log n - n \leq p_n \leq n \log n + n \log \log n$.

We observe that

$$\|J_{\lambda,N}f\|_p \leq \left( \sum_{n \leq r_N} \frac{\log p_n}{p_n^{\lambda}} \right) \|f\|_p$$

and

$$\sum_{n \leq r_N} \frac{\log p_n}{p_n^{\lambda}} \leq \sum_{n \leq r_N} \frac{\log n}{n^{\lambda}(\log n)^\lambda} \leq \frac{(\log r_N)^{1-\lambda}}{n^{\lambda}} \lesssim (r_N \log r_N)^{1-\lambda} \lesssim N^{1-\lambda}.$$ Therefore

$$\{|x; J_{\lambda,N}f(x) > \alpha/2\} \lesssim \frac{1}{\alpha^p} \|J_{\lambda,N}f\|_p^p \leq \frac{1}{\alpha^p} \left( \sum_{n \leq r_N} \frac{\log p_n}{p_n^{\lambda}} \right)^p \|f\|_p^p \lesssim \frac{1}{\alpha^p} N^{p(1-\lambda)} \|f\|_p^p \lesssim \frac{1}{\alpha^p} \|f\|_p^q \lesssim \frac{1}{\alpha^q} \|f\|_p^q$$

for every $\alpha \leq CN^{-1/q}\|f\|_p$. □
**Proof of Theorem 1.5.** Let $p \in (1, 2)$. We start observing that by Hölder inequality and the prime number theorem

$$\mathcal{A}_N f(x) \leq \left(\frac{\log N}{N}\right)^{\frac{1}{p}} \|f\|_p \leq \frac{C}{N^{\frac{1}{p'}}} \|f\|_{p'},$$

for some constant $C = C_p$. Moreover

$$\mathcal{A}_N f(x) \leq N^{\lambda-1} J_{\lambda,N} f(x) \text{ for all } x \in \mathbb{Z}.$$  

where $\lambda := 1 - \left(\frac{1}{p} - \frac{1}{p'}\right)$. Then, using the previous lemma we obtain

$$\sup_{\alpha > 0} \alpha^{p'} \left|\left\{x; \mathcal{A}_N f(x) > \alpha\right\}\right| = \sup_{\alpha \leq CN^{-1/p'}} \alpha^{p'} \left|\left\{x; \mathcal{A}_N f(x) > \alpha\right\}\right| \leq \sup_{\alpha \leq CN^{-1/p'}} \alpha^{p'} \left|\left\{x; J_{\lambda,N} f(x) > \alpha/N^{\lambda-1}\right\}\right| \lesssim N^{p'\left(\lambda-1\right)} \|f\|_{p'}.$$

Therefore

$$\sup_{\alpha > 0} \alpha \left|\left\{x; \mathcal{A}_N f(x) > \alpha\right\}\right|^{\frac{1}{p'}} \lesssim N^{\frac{1}{p'} - \frac{1}{p}} \|f\|_{p'}.$$  

This means that $\mathcal{A}_N$ is of weak type $(p, p')$ for every $p \in (1, 2)$, then as a consequence of the Marcinkiewicz interpolation theorem we conclude that

$$\|\mathcal{A}_N f\|_{p'} \lesssim N^{\frac{1}{p'} - \frac{1}{p}} \|f\|_{p'},$$

for all $p \in (1, 2)$. The case $p = p' = 2$ is easier, this follows as consequence of Young inequality

$$\|\mathcal{A}_N f\|_2 = \|K_N * f\|_2 \leq \|K_N\|_1 \|f\|_2 = \|f\|_2 = N^{1/2 - 1/2} \|f\|_2$$

where $K_N(x) := \frac{1}{N} \sum_{\delta \leq N} \delta_n(x) \log p$ for every $x \in \mathbb{Z}$. \hfill \Box

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Department of Mathematics, UCLA, 520 Portola Plaza, Los Angeles, CA 90095, USA

E-mail address: jmadrid@math.ucla.edu