Distributed generalized Nash equilibrium seeking in aggregative games on time-varying networks

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Abstract—We design the first fully-distributed algorithm for generalized Nash equilibrium seeking in aggregative games on a time-varying communication network, under partial-decision information, i.e., the agents have no direct access to the aggregate decision. The algorithm is derived by integrating dynamic tracking into a projected pseudo-gradient algorithm. The convergence analysis relies on the framework of monotone operator splitting and the Krasnosel’skii–Mann fixed-point iteration with errors.

I. INTRODUCTION

An aggregative game is a collection of inter-dependent optimization problems associated with noncooperative decision makers, or agents, where each agent is affected by some aggregate effect of all the agents [1]. Remarkably, aggregative games arise in several applications, such as demand side management in the smart grid [2], e.g. for charging/discharging electric vehicles [3], demand-response regulation in competitive markets [4], congestion control in traffic and communication networks [5]. The common denominator is the presence of a large number of selfish agents, whose aggregate actions may disrupt the shared infrastructure, e.g. the power grid or the transportation network, if left uncontrolled.

Designing solution methods for multi-agent equilibrium problems in noncooperative games has recently gained high research interest. Several authors have developed semi-decentralized and distributed equilibrium seeking algorithms for games without coupling constraints [6], and for games with coupling constraints [7], [8], [9], [10].

With focus on the generalized Nash equilibrium (GNE) problem, the formulations in [9], [10] have introduced an elegant approach based on monotone operator theory [11] to characterize the equilibrium solutions as the zeros of a monotone operator. Not only is the monotone-operator-theoretic approach general — e.g., unlike variational inequalities, smoothness of the cost functions is not required — but also computationally viable, since several algorithmic methods to solve monotone inclusions are already well established, e.g. operator-splitting methods [11, §26].

However, in the aforementioned literature on noncooperative equilibrium computation, it is assumed that the agents have direct access to the decisions of all their competitors, allowing every agent to evaluate its cost function without the need of extra communication. This game setup is known as full-decision information.

Recently, in the broader context of network games, the authors in [12], [13] propose fully-distributed algorithms for equilibrium seeking under partial-decision information. In [12], to deal with the lack of information, the agents are endowed with auxiliary variables, namely, the estimates of the decisions of the other agents. Then, a consensus protocol is combined with accelerated projected-pseudo-gradient dynamics to steer the estimates towards their real value and, consequently, the decisions to a Nash equilibrium, in the same time-scale. In [13], similar ideas are developed in the general framework of monotone operator theory to design an algorithm for games with coupling constraints. The algorithms proposed in [12], [13] require a number of auxiliary variables (namely the estimates of the decisions of the other agents) which is proportional to the number of agents in the game. From a practical perspective, this can be regarded as a drawback in terms of memory storage and communication requirements, especially in games with very large number of agents.

Scalability with respect to the population size indeed motivates us to focus on aggregative games. In this context, the authors in [14] propose an algorithm that relies on dynamic tracking, a technique extensively used in distributed optimization [15]. Specifically, the authors embed dynamic tracking of the aggregate decision in a projected-pseudo-gradient update to compute a Nash equilibrium in aggregative games without coupling constraints, in a fully-distributed fashion. Unfortunately, the extension of the methodology in [14] to generalized Nash equilibrium problems currently is missing, since it presents several technical issues. In the context of aggregative games with coupling constraints, an algorithm is proposed in [16], however with important limitations: it requires a very large number of distributed communication rounds before each strategy update; convergence is guaranteed to approximate solutions (i.e., ε-Nash equilibria) only; the communication network must be time-invariant.

Contribution: In this paper, we propose the first discrete-time, fully-distributed algorithm to compute a generalized Nash equilibrium in aggregative games with coupling constraints, over a time-varying communication network, under partial-decision information. The algorithm is obtained by combining dynamic tracking, projected-pseudo-gradient and Krasnosel’skii–Mann dynamics. The key approach to prove convergence of our proposed algorithm relies on applying and
Organized the framework of operator splitting methods [11] and fixed-point iteration with errors [17].

Organization of the paper: In Section II, we formalize the generalized Nash equilibrium seeking problem for aggregative dynamics by a time-varying communication network. In Section III, we present a fully-distributed algorithm and discuss its interpretation from an operator theoretic and fixed-point perspective. In Section IV, we establish global convergence of the proposed method. To corroborate the theory, in Section V, we study the performance of the proposed method on a Nash–Cournot game. Concluding remarks and future research directions are discussed in Section VI.

Basic notation: \( \mathbb{R} \) denotes the set of real numbers, and \( \mathbb{R} := \mathbb{R} \cup \{ \infty \} \) the set of extended real numbers. \( \mathbf{0} \) (1) denotes a matrix/vector with all elements equal to 0 (1); to improve clarity, we may add the dimension of these matrices/vectors as subscript. \( A \otimes B \) denotes the Kronecker product between the matrices \( A \) and \( B \). For a square matrix \( A \in \mathbb{R}^{n \times n} \), its transpose is \( A^\top \); \( A > 0 \) (\( \succeq 0 \)) stands for positive definite (semidefinite) matrix. Given \( A > 0 \), \( \| \cdot \|_A \) denotes the A-induced norm, such that \( \| x \|_A = \sqrt{x^\top A x} \). \( \| A \| \) denotes the largest singular value of \( A \). Given \( N \) matrices \( A_1, \ldots, A_N \), blockdiag \( (A_1, \ldots, A_N) \) denotes a block diagonal matrix with \( A_1, \ldots, A_N \) as diagonal blocks. Given \( N \) vectors \( x_1, \ldots, x_N \), \( x := \text{col}(x_1, \ldots, x_N) = [x_1, \ldots, x_N]^{\top} \), \( \tilde{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \), \( x_{-i} := \text{col}(x_1, \ldots, x_i, \ldots, x_N) \); given a vector \( z \), \( (z, x_{-i}) := \text{col}(x_1, \ldots, x_i, z, x_{i+1}, \ldots, x_N) \).

Operator theoretic definitions: \( \text{Id}() \) denotes the identity operator. The mapping \( I_{S} : \mathbb{R}^{n} \rightarrow \{0, \infty\} \) denotes the indicator function for the set \( S \subseteq \mathbb{R}^{n} \), i.e., \( I_{S}(x) = 0 \) if \( x \in S \), and \( \infty \) otherwise. For a closed set \( S \subseteq \mathbb{R}^{n} \), the mapping \( \text{proj}_{S} : \mathbb{R}^{n} \rightarrow S \) denotes the projection onto \( S \), i.e., \( \text{proj}_{S}(x) = \text{argmin}_{\psi \in S} \| y - x \| \). The set-valued mapping \( N_{S} : \mathbb{R}^{n} \rightarrow 2^{\mathbb{R}^{n}} \) denotes the normal cone operator for the set \( S \subseteq \mathbb{R}^{n} \), i.e., \( N_{S}(x) = \emptyset \) if \( x \notin S \), \( \{ v \in \mathbb{R}^{n} : v \perp_{S} u \leq 0 \} \) otherwise. For a function \( \psi : \mathbb{R}^{n} \rightarrow \mathbb{R} \), \( \text{dom}(\psi) := \{ x \in \mathbb{R}^{n} : \psi(x) < \infty \} \); \( \partial \psi : \text{dom}(\psi) \rightarrow \mathbb{R}^{n} \) denotes its subdifferential set-valued mapping, defined as \( \partial \psi(x) := \{ v \in \mathbb{R}^{n} : \psi(z) \geq \psi(x) + \langle v, z - x \rangle \} \) for all \( z \in \text{dom}(\psi) \). A set-valued mapping \( F : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) is (strictly) monotone if \( (u - v)^{\top}(y - x) \geq 0 \) (\( > 0 \)) for all \( u \neq y \in \mathbb{R}^{n} \), \( u \in F(x) \), \( v \in F(y) \); \( F \) is restricted (strictly) monotone on \( Y \subseteq \mathbb{R}^{n} \) if \( z^{*} - z \in [s^{*} - s] \geq 0 \) (\( > 0 \)) for all \( z, s \in [s^{*} - s] \). A set-valued mapping \( F : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) is L-Lipschitz continuous, with \( L > 0 \), if \( \| F(x) - F(y) \| \leq L \| x - y \| \) for all \( x, y \in \mathbb{R}^{n} \); \( F \) is nonexpansive if it is 1-Lipschitz continuous; \( F \) is \( \eta \)-averaged, with \( \eta \in (0, 1) \), if \( \| F(x) - F(y) \|^2 \leq \| x - y \|^2 - \frac{\eta}{1-\eta} \| (I - F)(x) - (I - F)(y) \|^2 \) for all \( x, y \in \mathbb{R}^{n} \); \( F \) is \( \beta \)-cocoercive, with \( \beta > 0 \), if \( \beta F \) is \( \frac{1}{\beta} \)-averaged.

II. Problem statement

Consider a set of \( N \) agents indexed by \( \mathcal{I} = \{1, \ldots, N\} \). The \( i \)-th agent is characterized by a local strategy set \( \Omega_i \subseteq \mathbb{R}^{n} \) and a cost function \( J_i(x_i, \bar{x}) \), which depends on the decision of agent \( i \), \( x_i \), and on the aggregate of all agent decisions, i.e.,

\[
\bar{x} := \frac{1}{N} \sum_{i=1}^{N} x_i.
\]

Moreover, we assume that the collective strategy profile \( \bar{x} := \text{col}(x_1, \ldots, x_N) \in \mathbb{R}^{Nn} \) must satisfy a coupling constraint, described by the affine function \( \bar{x} \mapsto C \bar{x} - c \), where \( C = [C_1 \ldots C_N] \in \mathbb{R}^{m \times Nn} \), \( c = \sum_{i=1}^{N} c_i \in \mathbb{R}^{m} \), and \( C_i, c_i \) are local parameters known to agent \( i \) only. In summary, the aim of each agent \( i \), given the decision variables of the other agents, i.e., \( x_{-i} := \text{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \), is to choose a strategy \( x_i \) that solves its local optimization problem, according to the game setup above, i.e., \( \forall i \in \mathcal{I} \):

\[
\begin{aligned}
\arg\min_{x_i \in \Omega_i} & \quad J_i(x_i, \frac{1}{N} \sum_{j \neq i} x_j) \\
\text{s.t.} & \quad x_i \in \Omega_i \\
& \quad C_i x_i - c_i \leq \sum_{j \neq i} (c_j - C_j x_j)
\end{aligned}
\]

where the last constraint is equivalent to \( C \bar{x} - c \leq 0 \).

Assumption 1: For all \( i \in \mathcal{I} \) and any fixed \( u \in \frac{1}{N} \sum_{j \neq i} \Omega_j \), the function \( J_i(\cdot, \frac{1}{N} \sum_{j \neq i} x_j + u) \) is convex and continuously differentiable, \( \Omega_i \subseteq \mathbb{R}^{n} \) is non-empty, compact and convex. The global feasible set \( K := \{ \bar{x} \in \bigcap_{i=1}^{N} \Omega_i | C \bar{x} - c \leq 0 \} \) is non-empty and satisfies Slater’s constraint qualification.

From a game-theoretic perspective, our goal is to distributively compute a generalized Nash equilibrium of the aggregative game described by the \( N \) inter-dependent optimization problems in (1).

Definition 1 (Generalized Nash equilibrium): A collective strategy \( x^* \) is a generalized Nash equilibrium (GNE) of the game in (1) if \( x^* \in K \) and, for all \( i \in \mathcal{I} \) and for all \( z \) such that \( (z, x^*_i) \in K \),

\[
J_i(x^*_i, \bar{x}) \leq J_i\left( z, \frac{1}{N} \sum_{j \neq i} x_j \right) \quad \square
\]

A. Communication networks

We consider a time-varying network to model the communications among agents over time. At each stage \( k \), the communication is described by an undirected graph \( \mathcal{G}_k = (\mathcal{I}, \mathcal{E}_k) \), where \( \mathcal{I} \) is the set of vertices (agents) and \( \mathcal{E}_k \subseteq \mathcal{I} \times \mathcal{I} \) is the set of edges. An unordered pair of vertices \((i, j)\) belongs to \( \mathcal{E}_k \) if and only if agents \( j \) and \( i \) can exchange information. The set of neighbors of agent \( i \) at stage \( k \) is defined as \( \mathcal{N}_i(k) = \{ j | (i, j) \in \mathcal{E}_k \} \). Next, we assume the graphs sequence \( \{ \mathcal{G}_k \}_{k \in \mathbb{N}} \) to be \( Q \)-connected.

Assumption 2: There exists an integer \( Q \geq 1 \) such that the graph \( (\mathcal{I}, \bigcup_{k=1}^{Q} \mathcal{E}_{k+k}) \) is connected, for all \( k \geq 0 \). \( \square \)

This assumption ensures that the intercommunication intervals are bounded for agents that communicate directly. In other words, every agent sends information to each of its neighboring agents at least once every \( Q \) time intervals.

We consider a mixing matrix \( W(k) = [w_{ij}(k)] \) associated with \( \mathcal{G}_k \), whose elements satisfy the following assumption.
Assumption 3: For all $k \in \mathbb{N}$, the matrix $W(k) = [w_{i,j}(k)]$ satisfies the following conditions:

(i) (Edge utilization) Let $i, j \in \mathcal{I}$, $i \neq j$. If $(i, j) \in \mathcal{E}_k$, $w_{i,j}(k) \geq \epsilon$, for some $\epsilon > 0$; $w_{i,j}(k) = 0$ otherwise;

(ii) (Positive diagonal) For all $i \in \mathcal{I}$, $w_{i,i}(k) \geq \epsilon$;

(iii) (Double-stochasticity) $W(k)1 = 1$, $1^\top W(k) = 1^\top$. \hfill \Box

Assumption 3 is strong but typical for multilateral coordination and optimization [13]. For an undirected graph it can be fulfilled, for example, by using Metropolis weights:

$$w_{i,j}(k) = \begin{cases} \left(\max\{|N_i'(k), |N_j'(k)|\}\right)^{-1} & \text{if } (i, j) \in \mathcal{E}_k, \\ 1 - \sum_{\ell \in \mathcal{N}_i} w_{i,\ell}(k) & \text{if } i = j. \end{cases}$$

(2)

Finally, let us introduce the transition matrices $\Psi(k, s)$ from time $s$ to $k$:

$$\Psi(k, s) = W(k)W(k-1)\cdots W(s+1)W(s),$$

(3)

for $0 \leq s < k$, where $\Psi(k, k) = W(k)$, for all $k$. The following statement shows the convergence properties of the transition matrix $\Psi(k, s)$.

Lemma 1 ([13] Lemma 5.3.1): Let Assumptions 2 hold true. Then, the following statements hold:

(i) $\lim_{k \to \infty} \Psi(k, s) = (1/N)11^\top$, for all $s \geq 0$.

(ii) The convergence rate of $\Psi(k, s)$ is geometric, i.e.,

$$\|\Psi(k, s) - (1/N)11^\top\| \leq \theta^k$$

for all $k \geq s \geq 0$, where $\theta := (1 - \epsilon/(4N^2))^{-2}$ and

$$\rho := (1 - \epsilon/(4N^2))^{1/Q} \in (0, 1),$$

(4)

with $Q$ as in Assumption 2 and $\epsilon$ as in Assumption 3. \hfill \Box

B. GNE as zeros of a monotone operator

As first step, we characterize a GNE of the game in terms of the KKT conditions of the coupled optimization problems in (1). For each agent $i \in \mathcal{I}$, let us introduce the Lagrangian function $L_i$, defined as

$$L_i(x, \lambda_i) := J_i(x_i, \bar{x}) + \lambda_{\Omega_i}(x_i) + C_i^\top \lambda_i,$$

where $\lambda_i \in \mathbb{R}^{m_0}$ is the dual variable of agent $i$ associated with the coupling constraints, and $\lambda_{\Omega_i}$ is the indicator function. It follows from [20, §12.2.3] that the set of strategies $x^*$ is a GNE of the game in (1) if and only if the following coupled KKT conditions are satisfied for some $\lambda_1, \ldots, \lambda_N \in \mathbb{R}_{\geq 0}^m$:

$$\forall i \in \mathcal{I}: \begin{cases} 0 \in \nabla x_i J_i(x^*_i, \cdot) + \nabla \Omega_i(x^*_i) + C_i^\top \lambda_i, \\ 0 \leq \lambda_i^\top \perp - (C x^* - c) \geq 0. \end{cases}$$

(5)

The next proposition characterizes the subclass of variational generalized Nash equilibria (v-GNE) as the solution set of the KKT conditions in (5) with equal dual variables, i.e., $\lambda^*_1 = \ldots = \lambda^*_N$, or equivalently as the solution to a specific variational inequality [11] or equivalently as the zero set of the set-valued mapping

$$U: \mathbb{R}^m \times \mathbb{R}^m \ni (x, \lambda) \mapsto \left[ \begin{array}{c} N_{\Omega}(x) + F(x) + C^\top \lambda \\
\mathbb{R}^{m_0}_{\geq 0}(\lambda) - (C x - c) \end{array} \right],$$

(6)

where $\lambda \in \mathbb{R}^m$, $\Omega := \prod_{i=1}^N \Omega_i$, $N_{\Omega} = \partial \Omega$ is the normal cone operator associated with a set $\mathcal{S}$ and $F$ is the so-called pseudo-gradient (PG) mapping defined as

$$F(x) = \text{col}(\nabla_{x_1} J_1(x_1, \bar{x}), \ldots, \nabla_{x_N} J_N(x_N, \bar{x})).$$

(7)

To emphasize the structure of $F$, we define

$$F_i(x, \bar{x}) := \nabla_{x_i} J_i(x_i, \bar{x}), \quad \forall i \in \mathcal{I},$$

and the extended PG mapping

$$\tilde{F}(x, u) := \text{col}(F_1(x_1, u_1), \ldots, F_N(x_N, u_N)),$$

(9)

where each component mapping $F_i$ is given by (8). With this notation, we have $\tilde{F}(x, 1 \otimes \bar{x}) = F(x)$. Next, we assume Lipschitz continuity of the mapping $\tilde{F}$ in (9), which is usual in the context of games under partial-decision information, see e.g. [13] Assumption 3, [14] Assumption 3.

Assumption 4: The mapping $\tilde{F}$ in (9) is uniformly Lipschitz continuous over $\Omega \times \Omega$, where $\Omega := \prod_{i=1}^N (1/\sum_{i=1}^N \Omega_i)$, i.e., there exists some $L_{\tilde{F}} > 0$ such that, for all $x, y \in \Omega$ and $u, w \in \Omega$,

$$\|\tilde{F}(x, u) - \tilde{F}(y, w)\| \leq L_{\tilde{F}} \|x - y\|.$$

(10)

Proposition 1: Let Assumption 1 hold. Then, the following statements are equivalent:

(i) $x^*$ is a variational GNE of the game in (1);

(ii) $\exists \lambda^* \in \mathbb{R}_{\geq 0}^m$ such that the pair $(x^*, \lambda^*)$ is a solution to the KKT in (5), for all $i \in \mathcal{I}$;

(iii) $x^*$ is a solution to VI$(F, K)$;

(iv) $\exists \lambda^* \in \mathbb{R}_{\geq 0}^m$ such that $\text{col}(x^*, \lambda^*) \in \text{zer}(U)$.

(11)

Proof: The equivalences $\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)}$ are proven in [22] Th. 3.1 while (iii)$\Leftrightarrow$(iv) follows by [23] Th. 3.1. \hfill \Box

The following assumptions on the PG mapping are standard (e.g. [8] Th. 3. [10] Assumption 2), [24] Assumption 3) and sufficient to ensure the convergence of standard GNE seeking algorithms.

Assumption 5 (Cocoercive pseudogradient): The mapping $F$ in (7) is $\chi$-cocoercive over $\Omega$. \hfill \Box

When $F$ is $\xi$-strongly monotone and $L_F$-Lipschitz over $\Omega$, then $F$ is also $(\xi/L_F^2)$-cocoercive over $\Omega$. However, in general, cocoercive mappings are not necessarily strongly monotone, e.g. the gradient of a (non-strictly) convex function.

Remark 1 (Existence and uniqueness of a v-GNE): It follows by [25] Cor. 2.2.5 that $\text{VI}(F, K)$ has a non-empty and compact solution set, since $K$ is non-empty, compact and convex and $F$ is continuous, by Assumption 1. Furthermore, when $F$ is strictly monotone, then the solution to $\text{VI}(F, K)$, (i.e., the v-GNE of the game), is unique [25] Th. 2.3.3. \hfill \Box

C. Boundedness of the dual variables

In this section, we formally establish the boundedness of the dual solution set of $\text{VI}(F, K)$ or, equivalently, of the dual part of the monotone inclusion $\text{col}(x^*, \lambda^*) \in \text{zer}(U)$.

Lemma 2: Let Assumptions 1 hold true. If $\text{col}(x^*, \lambda^*) \in \text{zer}(U)$, then $\lambda^* \in D^*$, where $D^* \subset \mathbb{R}_{\geq 0}^m$ is bounded. \hfill \Box
Proof: The boundedness of the dual solution set $D^*$ follows by [23 Prop. 3.3] since $\mathcal{V}(F, K)$ has a non-empty solution set by Remark 1 there exists a vector $x \in \text{dom}(F)$ satisfying Slater’s constraint qualification by Assumption I and the primal solution set of $\mathcal{V}(F, K)$ is bounded since $\Omega$ is bounded by Assumption I.

In the remainder of the paper, we assume that each agent can estimate a superset of $D^*$ and define $D := \{\lambda \in \mathbb{R}^m_0 | \lambda_j \leq B_D \text{ for all } j \in \{1, \ldots, m\}\}$. We have that $D$ is bounded and $D^* \subseteq D$. Furthermore, it follows that the set of zeros of the operator $\text{zer}(D)$ corresponds to the set of zeros of the following operator with bounded domain:

$$U_D : [x] \mapsto \left[ \begin{array}{c} N_{\Omega}(x) + F(x) + C^T \lambda \\ N_D(\lambda) - (C^T \lambda) \end{array} \right].$$

Remark 2: In the context of constrained distributed optimization, an estimate of the optimal dual solution set can be construct based on a Slater’s vector, see [26, §4.2], [27 §3.A (2)]. The extension of these estimation methods to generalized noncooperative games would rely on Lagrangian duality theory for variational inequalities [23]. In practice, the agents do not need an accurate estimate of the optimal dual solution set $D^*$ and can simply construct a local superset $D$ by taking the upper bound $B_D$ large enough.

D. A standard semi-decentralized algorithm

It follows by Proposition I that the original GNE seeking problem corresponds to the following monotone inclusion problem:

$$\text{find } \omega^* = \text{col}(x^*, \lambda^*) \text{ s.t. } 0 \in U_D(\omega^*).$$

Next, we recall a standard semi-decentralized GNE seeking algorithm obtained by solving the monotone inclusion problem in (10) by means of a preconditioned forward-backward (pFB) splitting [24 Alg. 1].

**Algorithm 1** Semi-decentralized v-GNE seeking

In parallel, for all $i \in I$:

$$x_i^{k+1} = \text{proj}_{\Omega}(x_i^k - \alpha_i(F_i(x_i^k, \hat{x}^k) + C_i^T \hat{\lambda}^k))$$

Central coordinator:

$$\lambda^{k+1} = \text{proj}_D(\lambda^k + \beta(Cx^{k+1} - Cx^k - c))$$

If the step sizes $\{\alpha_i\}_{i \in I}$ and $\beta$ are chosen small enough, then the sequence $(\text{col}(x_i^{k}, \lambda^{k}))_{k \in \mathbb{N}}$ generated by Algorithm 1 converges to some $\text{col}(x^*, \lambda^*) \in \text{zer}(U)$, where $x^*$ is a v-GNE, see [24 Th. 1] for a formal proof of convergence.

Remark 3: Algorithm 1 is not distributed. In fact, at each iteration $k$, a central coordinator is needed to:

(i) gather and broadcast the average strategy $\bar{x}^k$;
(ii) update and broadcast the dual variable $\lambda^k$.

III. A DISTRIBUTED GNE SEEKING ALGORITHM

A. Towards a fully distributed algorithm

A first step towards a fully-distributed algorithm consists of endowing each agent with a copy, $\lambda_i$, of the dual variable and enforcing consensus on the local copies. Consider the set-valued mapping $T$, obtained by augmenting $U_D$ with the local copies of the dual variable:

$$T : [x] \mapsto \left[ \begin{array}{c} N_{\Omega}(x) + F(x) + \frac{1}{N} C_i^T \lambda \\ N_D(\lambda) + L m \lambda - \frac{1}{N} C_i(Cx - c_i) \end{array} \right],$$

where $\lambda = \text{col}(\lambda_1, \ldots, \lambda_N)$, $D = \prod_{i=1}^N D_i$, $C_i = I_N \otimes C_i$, $c_i = 1 \otimes c$, $L m = L \otimes I_m$, and $L := I_N - \frac{1}{N} 1_1 1_1^T$ represents the projection onto the disagreement space.

**Remark 4:** When the local copies of the dual variable are equal, i.e., $\lambda \in D := \{1_N \otimes \lambda, | \lambda \in D \}$, the first row block of $T$ corresponds to that of $U_D$, while each of the $N$ components of the second row block of $T$ describes the same slack complementarity condition, namely, the second row block of $U_D$.

We note that the mapping $T$ in (11) can be written as the sum of two operators, i.e.,

$$T_1 : \text{col}(x, \lambda) \mapsto \text{col}(F(x), L m \lambda + \frac{1}{N} C_i),$$

$$T_2 : \text{col}(x, \lambda) \mapsto N_{\Omega}(x) \times N_D(\lambda) + S \text{col}(x, \lambda),$$

where $S$ is a skew-symmetric linear mapping defined as

$$S := \frac{1}{N} \left[ \begin{array}{cc} 0 & C_i^T \\ -C_i & 0 \end{array} \right].$$

The formulation $T = T_1 + T_2$ is called splitting of $T$, and will be exploited in different ways later on. The next lemma shows that $T_2$ is maximally monotone and that $T_1$ is cocoercive and strictly monotone with respect to the consensus subspace of the dual variables, i.e., $\Omega \times D$.

**Lemma 3:** Let Assumptions I-5 hold true. The following statements hold:

(i) $T_2$ in (13) is maximally monotone on $\Omega \times D$;
(ii) $T_1$ in (12) is $\delta$-cocoercive, with $0 < \delta \leq \min\{1, \chi\}$ and restricted-strictly-monotone over $\Theta := \Omega \times D$, i.e., for all $\omega \in \Theta$, $\omega \in (\Omega \times D) \setminus \Theta$, it holds that $(T_1(\omega) - T_1(\omega))^T (\omega - \omega) > 0$;
(iii) $T$ is maximally monotone on $\Omega \times D$ and restricted-strictly-monotone over $\Theta$.

**Proof:** See Appendix A.

The next proposition exploits the restricted strict monotonicity of $T$ to shows that the v-GNE of the original game are fully characterized by the zeros of $T$.

**Proposition 2:** Let Assumption I hold true. The following statements hold:

(i) $\text{zer}(T) \neq \emptyset$;
(ii) If $\text{col}(x^*, \lambda^*) \in \text{zer}(T)$, then $x^*$ is a v-GNE and $\lambda^* = \text{col}(\lambda^*, \ldots, \lambda^*)$, with $\lambda^* \in \mathbb{R}^m_{\geq 0}$.

**Proof:** See Appendix B.

To find a zero of $T$, we exploit a preconditioned version of the forward-backward method [11 §25.6] on the splitting (12)-(13), similarly to [10, 24], thus obtaining Algorithm 2.

Proof: See Appendix C.
Then, we show that, if the step sizes in the main diagonal are chosen according to Algorithm 2 to a v-GNE if the step-sizes are chosen according to the following choices.

Local projected pseudo-gradient update:
\[ \hat{x}_i^k = \text{proj}_{\Omega_i}(x_i^k - \alpha_i(F_i(x_i^k, \hat{x}^k) + C_i^T \lambda_i^k)), \]
\[ d_i^k = 2C_i \hat{x}_i^k - C_i x_i^k - c_i, \]
\[ \lambda_i^k = \text{proj}_D(\lambda_i^k + \beta_i(d_i^k - \lambda_i^k + \hat{\lambda}_i^k)), \]

Local Krasnosel’skii–Mann process:
\[ x_i^{k+1} = x_i^k + \gamma_i^k(\hat{x}_i^k - x_i^k), \]
\[ \lambda_i^{k+1} = \lambda_i^k + \gamma_i^k(\hat{\lambda}_i^k - \lambda_i^k), \]

Remark 5: The local auxiliary variables \(d_i\)'s are introduced to cast Algorithm 2 in a more compact form.

Remark 6 (Algorithm 2 as a fixed-point iteration): Our convergence analysis is based on recasting the dynamics generated by Algorithm 2 as the fixed-point iteration
\[ \omega^{k+1} = \omega^k + \gamma^k R(\omega^k - \omega^k), \quad (k \in \mathbb{N}) \]
where \( \omega^k = \text{col}(x^k, \lambda^k) \) is the stacked vector of the iterates and \( R \) is the so-called pFB operator, defined as
\[ R := (\text{Id} + \Phi^{-1}T_2) - \text{Id} - \Phi^{-1}T_1, \]
where \( T_1, T_2 \) in (12)-(13) characterize the splitting of \( T \), and \( \Phi \) is the so-called preconditioning matrix, here chosen as
\[ \Phi := \left[ \begin{array}{c} \alpha^{-1}_d - \frac{1}{\beta_d^2} C_T \frac{1}{\beta_d^2} C_T \end{array} \right], \]

To conclude this section, we note that the projected-pseudo-gradient updates in Algorithm 2 can be cast compactly as
\[ \hat{x}^k = \text{proj}_\Omega(x^k - \alpha_d(F(x^k, \hat{x}^k) + C_d^T \hat{\lambda}_d^k)), \]
\[ \hat{\lambda}_d^k = \text{proj}_D(\lambda_d^k + \beta_d(d_d^k - \lambda_d^k + \hat{\lambda}_d^k)), \]

where
\[ \hat{x}_i^k = 1 \otimes \hat{x}_i^k, \quad \hat{\lambda}_i^k = 1 \otimes \hat{\lambda}_i^k, \quad d_i^k = 1 \otimes d_i^k \]
and \( C_d := \text{blkdiag}(C_1, \ldots, C_N) \).

We note that Algorithm 2 is not distributed, since the local updating rule of each agent requires the knowledge of
(i) the average strategy, i.e., \( \bar{x}_i = \frac{1}{N} \sum_{j=1}^{N} x_j^k \),
(ii) the average dual variable, i.e., \( \bar{\lambda}_i^k = \frac{1}{N} \sum_{j=1}^{N} \lambda_j^k \),
(iii) the aggregate value \( \bar{d}_i^k = \frac{1}{N} \sum_{j=1}^{N} d_j^k \), which characterizes the violation of the coupling constraints, i.e.,
\[ \frac{\lambda_d}{N} \sum_{j=1}^{N} d_i^k = \frac{\lambda_d}{N} \sum_{j=1}^{N} (2C_j \bar{x}_j^k - C_j x_j^k - c_j). \]

B. A fully-distributed algorithm via dynamic tracking

To implement Algorithm 2 fully-distributively over a time-varying network, we approximate its updates by endowing each agent \( i \) with some surrogate variables (or estimates), i.e., \( \sigma_i, \tilde{y}_i, \text{and } z_i \), that dynamically track the true aggregates \( \bar{x}_i, \tilde{d}_i \) and \( \bar{\lambda}_i \), respectively. Then, to mitigate the errors due to the inexactness of the surrogate variables, we relax the projected-pseudo-gradient iterations by means of a Krasnosel’skii–Mann (KM) process [11] eq.(5.12), whose step-sizes are set according to the following design choice.

Assumption 7: The sequence \( (\gamma^k)_{k \in \mathbb{N}} \) satisfies the following conditions:
(i) (non-increasing) \( 0 \leq \gamma^{k+1} \leq \gamma^k \leq 1 \), for all \( k \geq 0; \)
(ii) (summable) \( \sum_{k=0}^{\infty} \gamma^k = \infty; \)
(iii) (square-summable) \( \sum_{k=0}^{\infty} (\gamma^k)^2 < \infty. \)

For example, Assumption 7 is satisfied for step sizes of the form \( \gamma^k = \frac{k}{k+1} \), where \( \frac{1}{2} < b \leq 1 \).

The proposed algorithm relies on agents constructing an estimate of the aggregates by mixing information drawn from local neighbors and making a subsequent relaxed projected-pseudo-gradient step, as in Algorithm 2. To build the estimates \( \sigma_i, \tilde{y}_i, z_i \), at every iteration \( k \), agent \( i \) receives \( \sigma_j^k, \tilde{y}_j^k, z_j^k \)’s from its neighbors, \( j \in N_i(k) \), and aligns its intermediate estimates according to the following rules:
\[ \tilde{z}_i^k := \sum_{j=1}^{N} w_{i,j}(k) z_j^k, \]
\[ \tilde{y}_i^k := \sum_{j=1}^{N} w_{i,j}(k) \tilde{y}_j^k, \]
\[ \tilde{\sigma}_i^k := \sum_{j=1}^{N} w_{i,j}(k) \sigma_j^k, \]

Then, on the basis of \( \tilde{\sigma}_i^k, \tilde{y}_i^k \) and \( \tilde{z}_i^k \), agent \( i \) updates its strategy \( x_i^{k+1} \), its dual variable \( \lambda_i^{k+1} \), and the new estimates \( \sigma_i^{k+1}, \tilde{y}_i^{k+1}, z_i^{k+1} \) as formalized in the next table that summarizes the proposed algorithm.

\[ x_i^{k+1} = \text{proj}_\Omega(x_i^k - \alpha_d(F(x_i^k, \hat{x}_i^k) + C_d^T \hat{\lambda}_d^k)), \]
\[ \hat{\lambda}_d^k = \text{proj}_D(\lambda_d^k + \beta_d(d_d^k - \lambda_d^k + \hat{\lambda}_d^k)), \]

where
\[ \hat{x}_i^k = 1 \otimes \hat{x}_i^k, \quad \hat{\lambda}_i^k = 1 \otimes \hat{\lambda}_i^k, \quad d_i^k = 1 \otimes d_i^k \]
Algorithm 3. Fully-distributed v-GNE seeking

Initializaton: For all $i \in I$: set $\bar{x}_i^{-1}, x_i^0, \tilde{x}_i^{-1} \in \Omega_i$, $\lambda_i^0 \in \mathbb{R}_{\geq 0}$, $\sigma_i^0 = x_i^0$, $\tilde{z}_i^0 = \lambda_i^0$, $y_i^0 = 2C_i \bar{x}_i^{-1} - C_i x_i^{-1} - C_i; \alpha_i, \beta_i$ as in Assumption 6 and $(\gamma_k)_{k \in \mathbb{N}}$ as in Assumption 7.

For all $i \in I$:

Communication and distributed averaging:
\[
\tilde{x}_i^k = \frac{1}{N} \sum_{j=1}^{N} w_{i,j}(k) x_j^k, \quad y_i^k = \frac{1}{N} \sum_{j=1}^{N} w_{i,j}(k) y_j^k, \quad \tilde{z}_i^k = \frac{1}{N} \sum_{j=1}^{N} w_{i,j}(k) z_j^k,
\]

Local strategy update and dynamic tracking of $d^k$:
\[
\hat{x}_i^k = C_i (2 \bar{x}_i - x_i^k) - C_i (2 \bar{x}_i - x_i^k) - C_i (2 \bar{x}_i - x_i^k) - C_i (2 \bar{x}_i - x_i^k),
\]
\[
\lambda_i^k = \rho_i (\lambda_i^k + \beta_i (y_i^{k+1} - \lambda_i^k + \tilde{z}_i^k)),
\]

Local Krasnosel’skii–Mann process:
\[
x_i^{k+1} = x_i^k + \gamma (\tilde{x}_i^k - x_i^k), \quad \lambda_i^{k+1} = \lambda_i^k + \gamma (\tilde{z}_i^k - \lambda_i^k),
\]

Local dynamic tracking of $\tilde{x}^{k+1}$ and $\tilde{\lambda}^{k+1}$:
\[
\tilde{x}_i^{k+1} = \tilde{x}_i^k + \tilde{\lambda}_i^{k+1} - \tilde{x}_i^k, \quad \tilde{z}_i^{k+1} = \tilde{z}_i^k + \tilde{\lambda}_i^{k+1} - \tilde{z}_i^k.
\]

Note that the projected-pseudo-gradient updates in Algorithm 3 can be recast in a compact form as
\[
\tilde{x}_i^k = \text{proj}_{\Omega_i} (x_i^k - \sigma_i (F_i(x_i^k, \tilde{x}_i^k) + C_i \tilde{z}_i^k)), \quad \tilde{\lambda}_i^k = \text{proj}_{\Delta_i} (\lambda_i^k + \beta (y_i^{k+1} - \lambda_i^k + \tilde{z}_i^k)),
\]

where
\[
\tilde{x}_i^k = W_i (x_i^k - \sigma_i x_i^k) + C_i \tilde{z}_i^k, \quad \tilde{\lambda}_i^k = \lambda_i^k + \beta_i (y_i^{k+1} - \lambda_i^k + \tilde{z}_i^k).
\]

The main technical challenge to invoke [17] Th. 5.5] and, in turn, prove the convergence of Algorithm 3 is to find a step-size sequence $(\gamma_k)_{k \in \mathbb{N}}$ that complies with (C.1), such that the relaxed error sequence $(\gamma_k \| e(k) \|)_{k \in \mathbb{N}}$ satisfies (C.2). We immediately note that if $(\gamma_k)_{k \in \mathbb{N}}$ is chosen as in Assumptions 7 then it already satisfies (C.1). In the following subsection, we show that (C.2) is also satisfied.

A. Analysis of the relaxed error sequence

In the next lemma, we show a fundamental invariance property of Algorithm 3, namely, at each stage $k$, the averages among the estimates $\sigma_i^k$, $y_i^k$, and $z_i^k$ are equivalent to the correspondent aggregate true values we aim to track.

Lemma 4: Let Assumption 3 hold true and set the initial conditions $\sigma_i^0$, $y_i^0$, $z_i^0$ as in Algorithm 3, for all $i \in I$. Then, the following equations hold for all $k \geq 0$:

(i) $\tilde{x}_i^k = \frac{1}{N} \sum_{j=1}^{N} \sigma_j^k = e^k$;
(ii) $\tilde{y}_i^k = \frac{1}{N} \sum_{j=1}^{N} y_j^k = d^k$;
(iii) $\tilde{z}_i^k = \frac{1}{N} \sum_{j=1}^{N} z_j^k = \tilde{d}^k$.

Proof: See Appendix E.

The next lemma provides upper bounds for the estimation errors at each stage $k$ of Algorithm 3.

Lemma 5: Let Assumptions 1[13] hold true. Then, there exist some positive constants $B_0$, $B_1$, and $\delta$ and a diminishing scalar sequence $(\phi_k)_{k \in \mathbb{N}}$ defined as
\[
\phi_k = \delta_1 \rho_k + \delta_2 \sum_{l=1}^{k} \rho_k^{l-1} \rho_k^{l-1},
\]

with such that the following upper bounds hold for all $k \in \mathbb{N}$:

(i) $\| \sigma_i^k - 1 \otimes \tilde{x}_i^k \| \leq 2B_0 \rho_k + B_1 \sum_{s=1}^{k} \rho_k^{s-1}$;
(ii) $\| \tilde{z}_i^k - 1 \otimes \tilde{\lambda}_i^k \| \leq 2B_0 \rho_k + \rho_k B_0 \sum_{s=1}^{k} \rho_k^{s-1}$;
(iii) $\| y_i^k - 1 \otimes \tilde{d}_i^k \| \leq 2B_0 \rho_k + \sum_{s=1}^{k} \rho_k^{s-1} \phi_k + \phi_k$.

Proof: See Appendix E.

By exploiting the upper bounds in Lemma 5 and a result on the convergence of scalar sequences, which is recalled next, we can show that the estimates asymptotically converge to their correspondent aggregate true values.

Lemma 6 (22) Lemma 3.1): Let $(\delta_k)_{k \in \mathbb{N}}$ be a sequence.

(i) If $\lim_{k \to \infty} \delta_k = 0$ and $0 < \tau < 1$, then $\lim_{k \to \infty} \sum_{l=0}^{k} \tau - \delta l = \delta/(1 - \tau)$.
(ii) If $\delta_k \geq 0$ for all $k$, $\sum_{k=0}^{\infty} \delta_k < \infty$ and $0 < \tau < 1$, then $\sum_{k=0}^{\infty} \tau - \delta l < \infty$.

Proposition 3: Let Assumptions 1[13] hold true. Then, the following statements hold:

(i) $\lim_{k \to \infty} \| \sigma_i^k - 1 \otimes \tilde{x}_i^k \| = 0$;
(ii) $\lim_{k \to \infty} \| \tilde{z}_i^k - 1 \otimes \tilde{\lambda}_i^k \| = 0$;
(iii) $\lim_{k \to \infty} \| y_i^k - 1 \otimes \tilde{d}_i^k \| = 0$.

□
Proof: (i) From the upper bound in Lemma 5(i), we have
\[
\limsup_{k \to \infty} \left\| (W(k) \otimes I_n)\sigma^k - 1 \otimes \bar{x}^k \right\| \leq \limsup_{k \to \infty} \left( \theta B\rho^{k} + \theta B_0 \sum_{s=1}^{k} \rho^{k-s} \gamma^{s-1} \right) \leq 0,
\]
where \( \lim_{k \to \infty} \rho^k = 0 \), since \( 0 < \rho < 1 \) by Lemma 11 and \( \lim_{k \to \infty} \sum_{s=1}^{k} \rho^{k-s} \gamma^{s-1} = 0 \) by Lemma 6(a), since \( 0 < \rho < 1 \) and \( \lim_{k \to \infty} \gamma^k = 0 \) by Assumption 7. Hence, \( \lim_{k \to \infty} \left\| (W(k) \otimes I_n)\sigma^k - 1 \otimes \bar{x}^k \right\| = 0 \). The proofs of (ii) and (iii) are analogous. \hfill \blacksquare

Next, we derive an upper bound for the error \( e^k \) that directly depends on the estimation errors in Lemma 5.

Lemma 7: Let Assumptions 13 hold true. Then, the following bound holds for all \( k \in \mathbb{N} \):
\[
\| e^k \| \leq \| a_d \| \| L_F \| \| \sigma^k - 1 \otimes \bar{x}^k \| + \| \beta_d \| \| y^{k+1} - 1 \otimes \bar{d}^k \| + \| \alpha \| \| \| C_d \| \| \| \beta_1 \| \| z^k - 1 \otimes \bar{\lambda}^k \|.
\]
Proof: See Appendix C. \hfill \blacksquare

Finally, by combining the upper bounds in Lemmas 5 and 7 and exploiting a result on the convergence of scalar sequences, i.e., Lemma 6(b), we show that condition (C.2) holds, namely, the relaxed error sequence \( (\gamma^k \| e^k \|)_{k \in \mathbb{N}} \) is summable.

Lemma 8: Let Assumptions 13 hold true. The sequence \( (\gamma^k \| e^k \|)_{k \in \mathbb{N}} \), with \( e^k \) as in (23), is summable, i.e.,
\[
\sum_{k=0}^{\infty} \gamma^k \| e^k \| < \infty.
\]
Proof: See Appendix D. \hfill \blacksquare

Now, we can prove the convergence of Algorithm 3.

Theorem 2: Let Assumptions 13 hold true, the step sizes \( \{\alpha, \beta\} \in \mathbb{R} \) be set as in Assumption 6 and \( (\gamma^k)_{k \in \mathbb{N}} \) as in Assumption 7. Then, the sequence \( (\text{col}(x^k, \lambda^k))_{k \in \mathbb{N}} \) generated by Algorithm 3 globally converges to some \( \text{col}(x^*, \lambda^*) \in \text{zer}(T), \) where \( x^* \) is a v-GNE of the game in (1). \hfill \blacksquare

For all \( k \in \mathbb{N} \), the iterations of Algorithm 3 can be cast as the Krasnosel’ski–Mann process with errors \( \omega^{k+1} = \omega^k + \gamma^k (R(\omega^k) + e^k - \omega^k), \) where \( \omega^k = \text{col}(x^k, \lambda^k), \) \( R \) as in (16) and \( e^k \) as in (23). By (17) Th. 5.5, the sequence \( (\omega^k)_{k \in \mathbb{N}} \) converges to some \( \omega^* \in \text{fix}(R), \) since \( R \) is averaged, thus nonexpansive, with \( \text{Assumption 7} \) and (C.1)–(C.2) hold, by Assumption 6 and Lemma 8, respectively. To conclude, we note that \( \omega^* \in \text{fix}(R) = \text{zer}(\Phi^{-1}T_1 + \Phi^{-1}T_2), \) by (11) Prop. 25.1 (iv), and that \( \text{zer}(\Phi^{-1}T_1 + \Phi^{-1}T_2) = \text{zer}(T) \neq \varnothing, \) with \( T \) as in (11), since \( \Phi > 0, \) by Lemma 9 and \( T_1 + T_2 = T. \) Since \( \omega^* \in \text{zer}(T), \) then \( x^* \) is a v-GNE of the game in (1), by Proposition 2(ii). \hfill \blacksquare

V. NUMERICAL SIMULATIONS

In this section, we study the performance of the proposed algorithm on a class of network Nash–Cournot games with market capacity constraints. Such games represent an instance of generalized aggregative Nash games. In Section V.A, we describe the player cost functions and strategy sets and verify that the necessary assumptions are satisfied. In Section V.B we compare the performance of our algorithm against a standard semi-decentralized method (Algorithm 1).

A. Generalized network Nash–Cournot game

We extend the network Nash–Cournot game model proposed in [14][14][14] with additional market capacity constraints. Specifically, consider \( N \) firms that compete over \( m \) markets. Let firm \( i \)'s production and sales at location \( l \) be denoted by \( g_{i,l} \) and \( s_{i,l}, \) respectively, while its cost of production at location \( l \) is denoted by \( f_{i,l}(g_{i,l}) \) and defined as follows:
\[
f_{i,l}(g_{i,l}) = a_{i,l}g_{i,l}^2 + g_{i,l}b_{i,l},
\]
where \( a_{i,l} \) and \( b_{i,l} \) are scaling parameters for agent \( i. \)

The goods sold by firm \( i \) at location \( l \) fetch a revenue \( p(s_i) s_{i,l}, \) where \( p(s_i) \) denote the sales price at location \( l \) and \( s_i = \sum_{i=1}^{N} s_{i,l} \) represents the aggregate sales at location \( l. \) The market price is set according to an inverse demand function which depends on the aggregate of the network, i.e.,
\[
p_i(s_i) = d_l - s_l,
\]
where \( d_l \) is the overall demand for location \( l. \) Each firm \( i \) has a production limitation at location \( l, \) described by \( u_{i,l}. \) Moreover, the overall production in each market \( l \) must meet the correspondent demand \( d_l \) and do not exceed a maximum capacity \( r_l. \) Hence, the coupling constraints \( g_{i,l} \leq \sum_{i=1}^{N} g_{i,l} \leq r_l, \) for all \( l = 1, 2, \ldots, m, \) have to be satisfied.

Overall, each firm \( i, \) given the strategies of the other firms, aims at solving the following optimization problem:
\[
\begin{aligned}
\text{argmin}_{\{g_{i,l}, s_{i,l}\}_{i=1}^{m}} & \sum_{i=1}^{m} (f_{i,l}(g_{i,l}) - p_i(s_i) s_{i,l}) \\
\text{s.t.} \quad & \sum_{i=1}^{m} g_{i,l} \geq \sum_{i=1}^{m} s_{i,l}, \\
& g_{i,l}, s_{i,l} \geq 0, \quad g_{i,l} \leq u_{i,l}, \quad l = 1, \ldots, m, \\
& d_l \leq \sum_{i=1}^{m} g_{i,l} \leq r_l, \quad l = 1, \ldots, m.
\end{aligned}
\]
Effectively, the payoff function of firm \( i \) is parametrized by nodal aggregate sales and its constraints depend on the other firms’ strategies, thus leading to a generalized aggregative game. In this example, we assume that the firms communicate over a dynamic network to cope with the lack of aggregate information, which is necessary to compute their optimal production and sale strategies.

Next, we show that the proposed network Nash–Cournot game does satisfy our technical setup. Let \( x_i = \text{col}(g_{i,1}, \ldots, g_{i,n}, s_{i,1}, \ldots, s_{i,n}) \in \mathbb{R}^{2m} \) denote the strategy vector of agent \( i \) and \( x = \text{col}(x_1, \ldots, x_N) \) denote the collective strategy profile. The cost function of agent \( i \) is quadratic, convex in \( x_i, \) continuously differentiable and can be cast in a compact form as
\[
J_i(x_i, \bar{x}) = x_i^\top A_i x_i + b_i^\top x_i + (\Delta \bar{x})^\top x_i,
\]
where \( A_i := \text{diag}(a_{i,1}, \ldots, a_{i,m}, 0, \ldots, 0), \) \( \Delta = \text{diag}(0, I_m) \) and \( b_i := \text{col}(b_{i,1}, \ldots, b_{i,m}, -d_{i,1}, \ldots, -d_{i,m}). \) The local feasible set of firm \( i \) is non-empty (for an adequate choice of \( u_{i,l}'s \) ), convex, compact and reads as \( \Omega_i := \{ x_i \in \mathbb{R}^{2m} | \sum_{i=1}^{m} g_{i,l} \geq \sum_{l=1}^{m} s_{i,l}, g_{i,l}, s_{i,l} \geq 0, g_{i,l} \leq u_{i,l}, l = 1, \ldots, m, \}. \)
The coupling constraints are affine and can be written in compact form as in (1), with \( C_i = \begin{bmatrix} 0 & I_m \end{bmatrix} \) and \( c_i = \frac{1}{N} \text{col}(r_1, \ldots, r_m, -d_1, \ldots, -d_m) \), for all \( i \in \mathcal{I} \). Thus, Assumption 1 is satisfied.

The pseudo gradient mapping \( F \) is affine and reads as
\[
F(x) = P x + b,
\]
with
\[
P = 2A + \frac{1}{N} I \otimes \Delta + \frac{1}{N} (11^T \otimes \Delta),
\]
\( A = \text{blkdiag}(A_1, \ldots, A_N) \) and \( b = \text{col}(b_1, \ldots, b_N) \). By a direct inspection of the eigenvalues of \( P \), we can show that \( F \) is strongly monotone and Lipschitz continuous, when the coefficients \( a_{i,j} \)'s are positive. Hence, Assumption 5 is satisfied. In particular, it follows by [23] p.79 that \( F \) is \( \chi \)-cocoercive with \( \chi := \|P\|^{-1} \). Moreover, since \( F \) is strongly monotone and the sets \( \Omega_i \) are compact, it follows by Remark 1 that there exists a unique v-GNE. The mapping \( \hat{F} \) is affine and reads as
\[
\hat{F}(x, \sigma) = (2A + \frac{1}{N} I \otimes \Delta)x + (I \otimes \Delta)\sigma + b.
\]
Similarly, it can be shown that \( \hat{F} \) is \( L_{\hat{F}} \)-Lipschitz continuous with \( L_{\hat{F}} := \max_{i,j} \{a_{i,j}, 1\} \). Thus, Assumption 4 is satisfied.

B. Simulations studies

In our numerical study we consider a network Nash-Cournot game played by 20 firms, i.e., \( N = 20 \), over 10 markets, i.e., \( m = 10 \). All the parameters of the game are drawn from uniform distributions and fixed over the course of the entire simulations. Specifically, for all \( i \in \mathcal{I} \) and \( l \in \{1, \ldots, m\} \), we set the parameters of production cost in (25) as \( a_{i,l} \in \mathcal{U}(2,3) \) and \( b_{i,l} \in \mathcal{U}(2,12) \), where \( \mathcal{U}(t, \tau) \) denotes the uniform distribution over an interval \( [t, \tau] \) with \( t < \tau \). We set the production capacities of firm \( i \) as \( u_{i,l} \in \mathcal{U}(50,100) \) for all \( l \in \{1, \ldots, n\} \) and for all \( i \in \mathcal{I} \). Moreover, the demand at market \( l \) is set as \( d_l \in \mathcal{U}(90,100) \), while the market capacity as \( r_l \in \mathcal{U}(d_l,2d_l) \) for all \( l \in \{1, \ldots, m\} \).

At each iteration \( k \), the firms communicate according to a randomly generated and connected small world, where each node has 4 neighbors. To create a doubly stochastic mixing matrix \( W(k) \), we exploit the Metropolis weighting rules in (2). Thus, Assumptions 2 and 3 are satisfied. The agents update their decisions and their estimates as in Algorithm 3. The step-sizes \( \{\alpha_i, \beta_i\}_{i \in \mathcal{I}} \) are chosen according to Assumption 6 where the global parameter \( \tau \) is set 5% larger than the theoretical lower bound \( \frac{1}{2 \delta_2} \), where \( \delta = \min\{1, \|P\|\} \) and \( P \) as in (28).

In Figure 1 we show the trajectories of the sequences of normalized residuals \( \|x_k - x^*\|/\|x^0 - x^*\| \) for different choices of the step-size sequence \( \gamma^k \). Moreover, we compare the trajectories of Algorithm 3 with those obtained with Algorithm 1 [24] Alg. 1, which is a semi-decentralized algorithm and works under the assumption of full-decision information, i.e., the firms have access to the real aggregate information at each stage \( k \) of the algorithm. As expected, the semi-decentralized algorithm converges faster than the fully-distributed counterpart. Interestingly, we notice that convergence is achieved also in the case of fixed relaxation step in the KM process, e.g. \( \gamma^k = 1 \) for all \( k \geq 0 \), which is not supported by our theoretical analysis.

In Figure 2 we compare the trajectories of the consensus disagreement \( \|L \otimes I_m \lambda^k\| \) for two choices of the step-size sequence \( \gamma^k \). Moreover, we compare the trajectories of Algorithm 3 with those obtained with Algorithm 1 [24] Alg. 1, which is a semi-decentralized algorithm and works under the assumption of full-decision information, i.e., the firms have access to the real aggregate information at each stage \( k \) of the algorithm. As expected, the semi-decentralized algorithm converges faster than the fully-distributed counterpart. Interestingly, we notice that convergence is achieved also in the case of fixed relaxation step in the KM process, e.g. \( \gamma^k = 1 \) for all \( k \geq 0 \), which is not supported by our theoretical analysis.

VI. Conclusion

For a general class of aggregative games with linear coupling constraints over time-varying communication networks, we have designed the first single-layer, fully-distributed algorithm to compute a variational generalized Nash equilibrium. Global convergence can be established via monotone-operator-theoretic and fixed-point arguments, integrated with a dynamic tracking methodology.

The analysis approach in this paper is genuinely novel, hence opens up a number of new research directions. Motivated by the numerical results of Section V, it would be valuable to explore the computational aspects of the proposed method, e.g. how the connectivity of the communication networks influences the convergence speed. Whether or not the proposed algorithm converges with fixed step sizes in the Krasnosel’skii-Mann process is currently an open question. Finally, it would be highly valuable to relax the assumption of double-stochasticity of the mixing matrices.
A. Proof of Lemma [3]

(i) $T_2$ is the sum of two terms: $S$ in (17) which is a linear, skew symmetric mapping, thus maximally monotone [11] Ex. 20.30]; and $N_O \times N_D$ which is maximally monotone since is the direct sum of maximally monotone operators [11] Prop. 20.23] (i.e., the normal cones of the closed convex sets $\Omega$ and $D$). Hence, the maximal monotonicity of $S + N_O \times N_D = T_2$ follows by [11] Cor. 24.4 (i) since dom($S$) $= \mathbb{R}^{(n+m)N}$.

(ii) $F$ is $\chi-$cocoercive, by Assumption [5] and $L_m$ is 1-cocoercive by (21) p.79, since $L_m$ is a linear, positive semi-definite mapping with $\|L_m\| = 1$. It follows that the direct sum $T_1(\cdot) = F(\cdot) \times (L_m \cdot 1_+ c_1)$ is $\delta-$cocoercive, for all $\delta$ such that $0 < \delta \leq \min\{1, \chi\}$. Now, we show that $T_1$ is restricted strictly monotone w.r.t. $\Theta$. Let $\omega = \text{col}(x, \lambda) \notin \Theta$, hence $\lambda = \lambda + \lambda_1$, with $\lambda_1 \in D^*$ and $0 \neq \lambda_1 \in D \setminus D^*$. Let $\omega' = \text{col}(x', \lambda') \notin \Theta$, hence $\lambda' = \lambda' \in D^*$ and $\lambda_1 = 0$. The following inequalities show that $T_1$ in (13) is restricted strictly monotone w.r.t. $\Theta$:

$$
(T_2(\omega) - T_2(\omega'))^T(\omega - \omega') = (F(x) - F(x'))^T(x - x') + (\lambda - \lambda')^T L_m (\lambda - \lambda') \\
\geq \chi\|F(x) - F(x')\|^2 + (\lambda_1)^T L_m \lambda_1 \\
\geq e_{\gamma_2}(\lambda_1) L_m > 0,
$$

where $L_m = (L \otimes I_m)$ and $e_{\gamma_2}(L) = 1$ is the second smallest eigenvalue of $L = I - \frac{1}{\lambda} 1 1^T$. The first inequality follows by the cocoercivity of $F$ (Assumption [5]) and since $\lambda_1 \parallel L_m = \lambda_1^T L_m = 0$ such that $L_m \lambda_1 = 0, \lambda_1 = 0$. (iii): The maximal monotonicity of $T = T_1 + T_2$ follows by [11] Cor. 24.4 (i), since $T_1$ is cocoercive (thus maximally monotone [11] Example 20.31]), $T_2$ is maximally monotone and dom($T_1$) $= \mathbb{R}^{(n+m)N}$. Moreover, since $T_1$ is also restricted-strictly monotone with respect to $\Theta$, then $T$ enjoys the same property. \hfill \blacksquare

B. Proof of Proposition [2]

(i) By Proposition [1] there exists $\lambda^* \in \mathbb{R}^{\geq 0}$ such that $\text{col}(x^*, \lambda^*) \in \text{zer}(U)$, where $x^*$ is a v-GNE. Define $\omega^* = \text{col}(x^*, \lambda^*)$, with $\lambda^* = 1_N \otimes \lambda^*$, then we have $T(\omega^*) \notin 0$. In fact, each component of the first row block of $T(\omega^*)$ reads as $N_{O^*}(x^+_1) + \nabla_x J(x^+_1, x^*) + C_1^T \lambda^* \ni 0$ while each component of the second row block of $T(\omega^*)$ reads as $N_{R^*}(\lambda^*) - \frac{1}{2}(C \omega^* - \omega^*) \ni 0$, since $N_{R^*}(\lambda^*) - \frac{1}{2}(C \omega^* - \omega^*) \ni 0$ and $\frac{1}{2}N_{R^*} = N_{R^*}$. Hence, $\text{zer}(T) \neq \emptyset$. (ii) From the first part of the proof, we know that there exists $\omega^* \in \Theta$ such that $\omega^* \in \text{zer}(T)$. Now, we show that all the zeros of $T$ lie in $\Theta$. By contradiction, let $\omega' \in \text{zer}(T)$ and assume $\omega' \notin \Theta$. Then, $0 \in T(\omega') \ni 0 \in T(\omega')$ and Lemma [3] (iii) yields $0 = (0 - 0)^T (\omega^* - \omega') > 0$, which is impossible. Therefore, $\omega^* \in \Theta$, namely $\omega' \ni \omega^* = \text{col}(x^*, 1 \otimes \lambda^*)$. Now, by substituting $\omega'$ into $T$ (since $(L \otimes I_m)(1 \otimes \lambda^*) = 0$) we recover that $\omega' \in \text{zer}(T) \ni \text{col}(x^*, \lambda^*) \in \text{zer}(U)$, which, by Proposition [1] holds if and only if $x^*$ is a v-GNE.

C. Proof of Theorem [1]

To prove convergence of Algorithm 2 we follow the same technical reasoning of the proof in [10] Alg. 1. Specifically, the proof is divided in two parts to show that:

(1) Algorithm 2 corresponds to the fixed-point iteration in (15), i.e., $\omega^{k+1} = \omega^k + \beta_1 (R(\omega^k) - \omega^k)$, where $R := (I + \Phi^{-1}T_2)^{-1} \circ (I - \Phi^{-1}T_1)$ is the so-called pFB operator.

(2) If the step sizes are set as in Assumption [6] then $R$ is an averaged operator. Hence, (15) globally converges to some $\omega^* := \text{col}(x^*, \lambda^*) \in \text{fix}(R)$. Since $\text{fix}(R) = \text{zer}(T)$, with $T$ as in (11), then $x^*$ is a v-GNE, by Proposition [2].

(1): Let us recast Algorithm 2 in a compact form as

$$\begin{align*}
\hat{x}^k &= \text{proj}_{\Omega}(x^k - \alpha_d(\hat{F}(x^k, \hat{x}^k) + C_1^T \hat{\lambda}^k)), \\
\hat{\lambda}^k &= \text{proj}_{D}(x^k + \beta_d(\hat{x}^k - \lambda^k + \hat{\lambda}^k)), \\
\lambda^{k+1} &= \lambda^k + \gamma_k(\hat{x}^k - x^k), \\
x^{k+1} &= \hat{x}^k + \gamma_k(\hat{x}^k - x^k),
\end{align*}$$

Since $\text{proj}_{\Omega}(I + N_{\Omega})^{-1} \hat{F}(x^k, 1 \otimes \hat{x}^k) = F(x^k)$ and $C_1^T \hat{\lambda}^k = C_1^T (1 \otimes \lambda^k) = \frac{1}{\lambda} C_1 \lambda^k$, it follows from (29) that $\text{proj} \{Id + N_{\Omega}(\hat{x}^k) \in \lambda^k - \alpha_d F(x^k) + \gamma C_1^T \lambda^k\}$, which leads to

$$-F(x^k) \in N_{\Omega}(\hat{x}^k) + \frac{1}{\lambda} C_1^T \hat{\lambda}^k + \alpha_d^{-1}(\hat{x}^k - x^k) - \frac{1}{\lambda} C_1^T (\hat{\lambda}^k - \lambda^k),$$

where we used $\alpha_d^{-1} N_{\Omega}(\hat{x}^k) = N_{\Omega}(\hat{x}^k)$. Similarly, since $1 \otimes \hat{x}^k = \frac{1}{\lambda} (2C_1 \hat{x}^k - C_1 x^k - c_1)$ and $1 \otimes \hat{\lambda}^k = ((I - \frac{1}{\lambda} 1 1^T) \otimes I_m) \lambda^k = (L \otimes I_m) \lambda^k$, it follows from (30) that $\text{proj} \{Id + N_{R_{\mathbb{R}^n}N}(\hat{\lambda}^k) \in \lambda^k - \beta_d (\frac{1}{\lambda} (2C_1 \hat{x}^k - C_1 x^k - c_1) - \lambda^k, which leads to

$$\begin{align*}
-L_m \lambda^k - \frac{1}{\lambda} C_1 \hat{x}^k &\in N_{R_{\mathbb{R}^n}N}(\hat{\lambda}^k) - \frac{1}{\lambda} C_1 \hat{\lambda}^k \\
\frac{1}{\lambda} C_1 \hat{x}^k &\notin \frac{1}{\lambda} C_1 \hat{\lambda}^k
\end{align*}$$

Let $\omega^* := \text{col}(x^*, \lambda^*)$, then the inclusions in (33)–(34) can be cast in compact form as $-T_1(\omega^*) \in T_2(\omega^*) + \Phi(\omega^* - \omega^*)$, where $T_1, T_2$ and $\Phi$ as in (12), (13) and (17), respectively. By making $\omega^* \equiv \text{explicit in (33)}$, we obtain

$$\omega^* = \text{proj} \{Id + \Phi^{-1}T_2(\omega^*) \in T_2(\omega^*) + \Phi(\omega^* - \omega^*)\},$$

which corresponds to $\omega^* = \Phi(\omega^*)$, where $\Phi$ is the pFB operator in (16). Finally, it follows by (31)–(32) that $\omega^{k+1} = \omega^k + \gamma_k(\Phi(\omega^k) - \omega^k)$, which concludes the proof.

(2): Next, we introduce some technical statements that we exploit later on in this proof.

Lemma 9: Let the step-sizes $\{\alpha_d, \beta_d\}_{d \in I} \satisfy Assumption [6] Then the following statements hold:

(i) $\Phi - \tau I \geq 0$, with $\tau$ as in Assumption [6]

(ii) $\|\Phi^{-1}\| \leq \tau^{-1}$.
Proof: (i): By the generalized Gershgorin circular theorem [29, Th. 2], each eigenvalue $\mu$ of the matrix $\Phi$ in (17) satisfies at least one of the following inequalities:

\begin{align}
\mu &\geq \alpha_i - 1 - \|C_i^T\|, \quad \forall i \in I, \tag{37} \\
\mu &\geq \beta_i - \frac{1}{f} \sum_{j=1}^{N} \|C_j^T\|, \quad \forall i \in I. \tag{38}
\end{align}

Hence, if we set the step-sizes $\alpha_i, \beta_i$ as in Assumption 6 the inequalities (37)-(38) yield to $\mu \geq \tau$. It follows that the smallest eigenvalue of $\Phi$, i.e., $\mu_{\min}(\Phi)$, satisfies $\mu_{\min}(\Phi) \geq \tau > 0$. Thus, $\Phi - \tau I$ is positive semi-definite.

(ii): Let $\mu_{\max}(\Phi)$ be the largest eigenvalue of $\Phi$. We have that $\mu_{\max}(\Phi) \geq \mu_{\min}(\Phi) \geq \tau$. Moreover, $\|\Phi\| = \mu_{\max}(\Phi) \geq \mu_{\min}(\Phi) = \frac{\tau}{1-\tau}$, hence $\|\Phi\| \leq \tau^{-1}$.

Lemma 10: Let Assumptions 1 and 5 hold and the step-sizes $\{\alpha_i, \beta_i\}_{i \in I}$ satisfy Assumption 6. The following properties hold in the $\Phi$-induced norm (i.e., $\|\cdot\|_{\Phi}$):

(i) $\Phi^{-1}T_1$ is $\delta\tau$-coercive and $(\text{Id} - \Phi^{-1}T_1)$ is $\frac{1}{\delta\tau}$-averaged;

(ii) $\Phi^{-1}T_2$ is maximally monotone and $(\text{Id} - \Phi^{-1}T_2)^{-1}$ is $\frac{1}{\delta\tau}$-averaged.

Proof: (i): Since $T_1$ is single-valued and $\Phi^{-1}$ nonsingular, by Lemma 2 (i), for each $\omega, \omega' \in \Omega \times \mathbb{R}^{N \times N}$

\begin{align}
\|\Phi^{-1}T_1(\omega) - \Phi^{-1}T_1(\omega')\|_{\Phi}^2 &= \|T_1(\omega) - T_1(\omega')\|^2_{\Phi^{-1}} \\
\|\Phi^{-1}T_1(\omega) - \Phi^{-1}T_1(\omega')\|_{\Phi}^2 &\leq \frac{\delta\tau}{\|\Phi\|} \|T_1(\omega) - T_1(\omega')\|^2_{\Phi^{-1}} \\
&= \frac{\delta\tau}{\|\Phi\|} \|\Phi^{-1}T_1(\omega) - \Phi^{-1}T_1(\omega')\|^2_{\Phi}. \tag{39}
\end{align}

In other words, $\Phi^{-1}T_1$ is $\delta\tau$-coercive in the $\Phi$-induced norm. It follows from [11, Prop. 4.33] that $(\text{Id} - \Phi^{-1}T_1)$ is $\frac{1}{\delta\tau}$-averaged in the $\Phi$-induced norm.

(ii): By Lemma 3.7 (ii), $\Phi^{-1}T_2$ is maximally monotone in the $\Phi$-induced norm, since $T_2$ is maximally monotone by Lemma 5 (i). By [11, Prop. 23.7], the resolvent mapping $(\text{Id} + \Phi^{-1}T_2)$ is $\frac{1}{\delta\tau}$-averaged (or firmly-nonexpansive, see [11, Remark 4.24]) in the $\Phi$-induced norm, since $\Phi^{-1}T_2$ is maximally monotone in the same norm.

Lemma 11: Let Assumptions 1, 5 hold and the step-sizes $\{\alpha_i, \beta_i\}_{i \in I}$ satisfy Assumption 6. Then, the pFB operator $R = (\text{Id} + \Phi^{-1}T_2)^{-1} \circ (\text{Id} - \Phi^{-1}T_1)$ is $\nu$-averaged in the $\Phi$-induced norm (i.e., $\|\cdot\|_{\Phi}$, with $\nu = \frac{\delta\tau}{\delta\tau + 1}(\frac{1}{\delta\tau + 1})$.

Proof: By [11, Proposition 2.4], the mapping $R$ is $(\frac{2\delta\tau}{\delta\tau + 1})$-averaged with respect to $\|\cdot\|_{\Phi}$, since composition of $(\text{Id} + \Phi^{-1}T_2)^{-1}$ and $(\text{Id} - \Phi^{-1}T_1)$ which are $\frac{1}{\delta\tau}$- and $\frac{1}{\delta\tau}$-averaged in $\|\cdot\|_{\Phi}$, respectively, by Lemma 10. Moreover, $\frac{2\delta\tau}{\delta\tau + 1} \in (\frac{1}{2}, 1)$, since $\tau > \frac{\delta\tau}{\delta\tau + 1}$, by Assumption 6.

The fixed-point iteration (15), that corresponds to Algorithm 2 by the first part of this proof, is the Krasnosel’skiĭ-Mann iteration on the mapping $R$, which is $\nu$-averaged, with $\nu \in (\frac{1}{2}, 1)$, by Lemma 11. The convergence of (15) to some $\omega^* := \text{col}(x^*, \lambda^*) \in \text{fix}(R)$ follows by [11, Prop. 5.15]. To conclude, we note that $\omega^* \in \text{fix}(R) = \text{zer}(\Phi^{-1}T_1 + \Phi^{-1}T_2)$, by [11, Prop. 25.1 (iv)], and that $\text{zer}(\Phi^{-1}T_1 + \Phi^{-1}T_2) = \text{zer}(T)$, with $T$ as in [11], since $\Phi > 0$, by Lemma 2 (i), and $T_1 + T_2 = T$. Since the limit point $\omega^* \in \text{zer}(T) \neq \partial$, by Proposition 2 (i), then $x^*$ is a v-GNE of the game in (11), by Proposition 2 (ii), thus concluding the proof.

D. Proof of Lemma 2

We prove equation (i) by induction. At step zero, $\bar{x}^0 = \bar{x}^0$ holds if the estimates are initialized as $\sigma^0 = \sigma^0$, for all $i \in I$. At step $k$, we assume that $\bar{x}^k = \bar{x}^k$. To conclude the proof, we show that relation (i) holds at step $k + 1$:

\begin{align}
\sigma^{k+1} &= \frac{1}{N}(1^T \otimes I_n)((W(k) \otimes I_n)\sigma^k + x^{k+1} - x^k), \\
&= \frac{1}{N}(1^T \otimes I_n)(W(k) \otimes I_n)\sigma^k + x^{k+1} - x^k, \\
&= \sigma^k + x^{k+1} - x^k = x^{k+1}.
\end{align}

The first equality follows from the updating rule of the $\sigma_i$’s in Algorithm 3, the second follows by definition of $\bar{x}^k$, i.e., $\bar{x}^k = \frac{1}{N}(1^T \otimes I_n)x^k$, the third follows since the mixing matrix $W(k)$ is column stochastic, i.e., $1^TW(k) = 1^T$, by Assumption 3 while the last equality follows from the induction step $k$, i.e., $\sigma^k = \bar{x}^k$. The proof of equations (ii) and (iii) are analogous.

E. Proof of Lemma 5

For ease of notation, this proof is developed for the scalar case, i.e., $n = m = 1$. In this case, we can write $\|\sigma^0 - 1 \otimes \bar{x}^k\| = \|W(k) \otimes I_n)\sigma^0 - 1 \otimes \bar{x}^k\| = \|W(k)\sigma^0 - \bar{x}^k\|.$

(i): The update of the estimates $\sigma_i$’s in Algorithm 3 can be written in a compact form as

\begin{align}
\sigma^{k+1} = W(k)\sigma^k + x^{k+1} - x^k. \tag{41}
\end{align}

By telescoping [41], we obtain

\begin{align}
\sigma^{k+1} &= W(k)(W(k-1)\sigma^{k-1} + x^k - x^{k-1}) \\
&= x^{k+1} - x^k \\
&= \Psi(k, k-1)\sigma^{k-1} + \Psi(k, k)(x^k - x^{k-1}) \\
&= x^{k+1} - x^k, \\
&= \cdots \\
&= W(k, 0)\sigma^0 + \sum_{s=1}^{k} \Psi(k, s)(x^s - x^{s-1}) \\
&= x^{k+1} - x^k. \tag{42}
\end{align}

Since $\sigma^{k+1} - x^{k+1} + x^k = W(k)\sigma^k$, by (41), it follows by (42) that

\begin{align}
W(k)\sigma^k = \Psi(k, 0)\sigma^0 + \sum_{s=1}^{k} \Psi(k, s)(x^s - x^{s-1}).
\end{align}

Now, consider $\bar{x}^k$, which may be written as follows:

\begin{align}
\bar{x}^k = \bar{x}^{k-1} + (\bar{x}^k - \bar{x}^{k-1}) = \bar{x}^0 + \sum_{s=1}^{k} (\bar{x}^s - \bar{x}^{s-1}).
\end{align}

By Lemma 4 we have that $\bar{x}^k = \bar{x}^0 + \sum_{s=1}^{k} (\bar{x}^s - \bar{x}^{s-1}) = \frac{1}{N}1^T\sigma^0 + \sum_{s=1}^{k} \frac{1}{N}1^T(x^s - x^{s-1}). \tag{43}$
From equations (42) and (43), we have the following:

\[
||W(k)\sigma^k - \bar{x}^k||
= ||\left(\Psi(k, 0) - \frac{1}{N}11^T\right)\sigma^0 \\
+ \sum_{s=1}^k \left(\Psi(k, s) - \frac{1}{N}11^T\right)(x^s - x^{s-1})|| \\
\leq ||\Psi(k, 0) - \frac{1}{N}11^T||\|\sigma^0\| \\
+ \sum_{s=1}^k ||\Psi(k, s) - \frac{1}{N}11^T||\|x^s - x^{s-1}\| \\
\leq \theta \rho^k \|\sigma^0\| + \sum_{s=1}^k \theta \rho^{k-s} \|x^s - x^{s-1}\| \tag{44}
\]

The last inequality follows since \(||\Psi(k, s) - \frac{1}{N}11^T|| \leq \theta \rho^{k-s}\) for all \(k \geq s \geq 0\), by Lemma [1]. Next, we find an upper bound for \(\|x^s - x^{s-1}\|\). The update of the decisions \(x_i\)'s can be written in a compact form as \(x^{k+1} = x^k + \gamma_k (\hat{x}^k - x^k)\).

We note that \(\hat{x}_i^k, x_i^k \in \Omega_i\), for all \(k \geq 0\) since \(\hat{x}_i^k\) is obtained by projecting onto \(\Omega_i\) and \(x_i^k = (1 - \gamma_k) x_i^{k-1} + \gamma_k \hat{x}_i^k\) is a convex combination of elements of the convex set \(\Omega_i\). Since all the sets \(\Omega_i\) are compact, by Assumption [1] it follows that for some constant \(B_0\), we have:

\[
\|x^s - x^{s-1}\| = \gamma^{s-1} \|\hat{x}^s - x^{s-1}\| \leq \gamma^{s-1} B_0. \tag{45}
\]

By combining (45) and (44), we obtain

\[
||W(k)\sigma^k - \bar{x}^k|| \leq \theta \rho^k B_0 + \sum_{s=1}^k \theta \rho^{k-s} \|x^s - x^{s-1}\| B_0,
\]

where we exploited the initialization step \(\sigma^0 = x^0 \in \Omega\), from which \(\|\sigma^0\| \leq B_0\).

(ii): The update of the estimates \(z_i\)'s in Algorithm 3 can be written in a compact form as

\[
z^{k+1} = W(k)z^k + \lambda^{k+1} - \lambda^k. \tag{46}
\]

By telescoping (46), we obtain

\[
||W(k)z^k - \bar{\lambda}^k|| \leq \theta \rho^k \|z^0\| + \sum_{s=1}^k \theta \rho^{k-s} \|\lambda^s - \lambda^{s-1}\|. \tag{47}
\]

Now, we estimate \(\|\lambda^s - \lambda^{s-1}\|\). The update of the decisions \(\lambda_i\)'s can be written in a compact form as \(\lambda^{k+1} = \lambda^k + \gamma_k (\hat{\lambda}^k - \lambda^k\).\)

We note that \(\hat{\lambda}_i^k, \lambda_i^k \in D\), for all \(k \geq 0\) since \(\hat{\lambda}_i^k\) is obtained by projecting onto \(D\) and \(\lambda_i^k = (1 - \gamma_k) \lambda_i^{k-1} + \gamma_k \hat{\lambda}_i^k\) is a convex combination of elements of the convex set \(D\). Since \(D\) is compact, it follows that for some constant \(B_D\), we have

\[
||W(k)z^k - \bar{\lambda}^k|| \leq \theta \rho^k B_D + \sum_{s=1}^k \theta \rho^{k-s} \|\lambda^s - \lambda^{s-1}\| B_D.
\]

(iii): The update of the estimates \(y_i\)'s in Algorithm 3 can be written in a compact form as

\[
y^{k+1} = W(k)y^k + C_D(2\bar{x}^k - x^k) \\
- C_D(2\bar{x}^{k-1} - x^{k-1}). \tag{48}
\]

By telescoping (48), we obtain

\[
y^{k+1} = \Psi(k, 0)y^0 + \sum_{s=1}^k \Psi(k, s) \\
\cdot \left( C_D(2x^s - x^{s-1}) - C_D(2x^{s-2} - x^{s-2}) \right) \\
+ C_D(2\bar{x}^k - x^k) - C_D(2\bar{x}^{k-1} - x^{k-1}). \tag{49}
\]

Now, consider \(\tilde{y}_i^k\), which may be written as:

\[
\tilde{y}_i^k = y_i^0 + \sum_{s=1}^k (\tilde{y}_i^{s-1} - \tilde{y}_i^{s-2}) + \tilde{y}_i^k - \tilde{y}_i^{k-1}.
\]

By Lemma [4], we have that \(\tilde{y}_i^k = d_i^0 = \frac{1}{N}1^T C_D(2\bar{x}^k - x^k) - c\), for all \(s \geq 0\), which leads to

\[
\tilde{y}_i^k = \frac{1}{N}1^T y_i^0 + \sum_{s=1}^k \frac{1}{N}1^T \\
\cdot \left( C_D(2x^s - x^{s-1}) - C_D(2x^{s-2} - x^{s-2}) \right) \\
+ \frac{1}{N}1^T (C_D(2\bar{x}^k - x^k) - C_D(2\bar{x}^{k-1} - x^{k-1})). \tag{50}
\]

From the relations (49) and (50), we have the following:

\[
||y^{k+1} - \tilde{y}_i^k|| \\
= ||\left(\Psi(k, 0) - \frac{1}{N}11^T\right)y^0 + \sum_{s=1}^k (\Psi(k, s) - \frac{1}{N}11^T) \\
\cdot \left( C_D(2x^s - x^{s-1}) - C_D(2x^{s-2} - x^{s-2}) \right) \\
+ \frac{1}{N}1^T (C_D(2\bar{x}^k - x^k) - C_D(2\bar{x}^{k-1} - x^{k-1})). || \tag{51}
\]

where the last inequality follows since \(||\Psi(k, s) - \frac{1}{N}11^T|| \leq \theta \rho^{k-s}\) for all \(k \geq s \geq 0\), by Lemma [1].

Now we upper bound \(||2\bar{x}^k - x^k|| - (2\bar{x}^{k-1} - x^{k-1}||)

\[
||2\bar{x}^k - x^k|| - (2\bar{x}^{k-1} - x^{k-1}| | \\
\leq 2\|\bar{x}^k - \bar{x}^{k-1}\| + \|x^k - x^{k-1}\| || \\
\leq 2\|\bar{x}^k - \bar{x}^{k-1}\| + \gamma^{k-1} B_0, \tag{52}
\]

where the second inequality follows from (45). Consider the term \(2\|\tilde{\bar{x}}^k - \tilde{\bar{x}}^{k-1}\|\). By exploiting the compact update (20) and the nonexpansiveness of the projection operator, we have

\[
||\tilde{\bar{x}}^k - \tilde{\bar{x}}^{k-1}|| \\
\leq \|\tilde{\bar{x}}^k - \tilde{\bar{x}}^{k-2}\ - \alpha \| \\
\cdot \left( \tilde{F}(x^{s-1}, W(s-1)\sigma^{s-1}) - \tilde{F}(x^{s-2}, W(s-2)\sigma^{s-2}) \right) \\
+ C_D^T W(s-1)z^{s-1} - C_D^T W(s-2)z^{s-2} || \\
\leq \|\tilde{\bar{x}}^k - \tilde{\bar{x}}^{s-2}|| \\
+ L_F ||\alpha_d|| \left[ \||W(s-1)\sigma^{s-1} - W(s-2)\sigma^{s-2}|| \\
+ ||\alpha_d|| \||W(s-1)z^{s-1} - W(s-2)z^{s-2}|| \\
\leq \gamma^{s-2} B_0 + L_F ||\alpha_d|| \left[ \||\tilde{\bar{x}}^k - \tilde{\bar{x}}^{s-2}|| \\
+ ||\alpha_d|| \||W(s-1)z^{s-1} - W(s-2)z^{s-2}||, \tag{53}
\right.
\]

\]

where the second inequality follows from the Lipschitz continuity of \(\tilde{F}\), while the last inequality follows from the relation \(||\tilde{y}_i^k|| = \sqrt{||a||^2 + ||b||^2} \leq ||a|| + ||b||\). Now, we upper bound the last two terms in (53). Since \(W(s-2)\sigma^{s-2} = \)
\[\sigma^{s-1} - x^{s-1} + x^{s-2} \text{ by (41)}, \text{ then we can write} \]
\[
\|W(s-1)\sigma^{s-1} - W(s-2)\sigma^{s-2}\|
\leq \|W(s-1)\sigma^{s-1} - \sigma^{s-1} + x^{s-1} - x^{s-2}\|
\leq \|W(s-1)\sigma^{s-1} - \sigma^{s-1}\| + \|x^{s-1} - x^{s-2}\|
\leq \|W(s-1)\sigma^{s-1} - \sigma^{s-1}\| + \|x^{s-1} - x^{s-2}\|
\leq \|W(s-1)\sigma^{s-1} - 1\overline{x}^{s-1}\| + \|\sigma^{s-1} - 1\overline{x}^{s-1}\| + \gamma^{s-2}B_\Omega
\leq \theta B_\Omega^{s-1} + \theta B_\Omega \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1}
+ \theta B_\Omega \rho^{s-2} + \theta B_\Omega \sum_{\ell=1}^{s-2} \rho^{(s-2)-\ell} \gamma^{\ell-1} + \gamma^{s-2}B_\Omega
\leq 2\theta B_\Omega \rho^{s-2} + 4\theta B_\Omega \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1}. \tag{54}
\]

The third inequality follows by substituting to \(|W(s-1)\sigma^{s-1} - 1\overline{x}^{s-1}|\) and \(|\sigma^{s-1} - 1\overline{x}^{s-1}|\) the upper bound derived in Lemma 5 (i). Equivalently, for the last addend is (53), we can write
\[
\|W(s-1)z^{s-1} - W(s-2)z^{s-2}\|
\leq 2\theta B_D \rho^{s-2} + 4\theta B_D \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1}. \tag{55}
\]

By combining (53) with (54) and (55), we obtain
\[
\|
\hat{x}^s - \bar{x}^{s-1}\|
\leq \|\alpha_d\|2\theta(L_F B_\Omega + \|C_d\|B_D) \rho^{s-2}
+ \|\alpha_d\|4\theta(L_F B_\Omega + \|C_d\|B_D) \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1}
+ (B_\Omega + \|\alpha_d\|L_F B_\Omega) \gamma^{s-2}
\leq \epsilon_1 \rho^{s-2} + \epsilon_2 \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1}.
\tag{56}
\]

By substituting (56) into (52), we obtain
\[
\|(2\hat{x}^s - x^s) - (2\overline{x}^{s-1} - x^{s-1})\|
\leq 2\epsilon_1 \rho^{s-2} + 2\epsilon_2 + \epsilon_3 \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1} + \gamma^{s-2}B_\Omega
\leq 2\epsilon_1 \rho^{s-2} + 2\epsilon_2 + \epsilon_3 + B_\Omega \sum_{\ell=1}^{s-1} \rho^{(s-1)-\ell} \gamma^{\ell-1}
\leq \phi^{s-1}, \tag{57}
\]
with \((\phi^k)_{k \in \mathbb{N}}\) as in (24). The second inequality follows since \(\gamma^{s-1} \leq \gamma^{s-2}\), by Assumption 8. Finally, by combining (57) and (51), we obtain the upper bound in Lemma 5 (iii).\]

\[\text{F. Proof of Lemma 7}\]

From \(\|\hat{z}\| = \sqrt{\|a\|^2 + \|b\|^2} \leq \|a\| + \|b\|\), it follows that
\[
\|e^k\| = \|\text{col}(\hat{x}^k, \hat{\lambda}^k) - \text{col}(\bar{x}^k, \bar{\lambda}^k)\|
\leq \|\hat{x}^k - \bar{x}^k\| + \|\hat{\lambda}^k - \bar{\lambda}^k\|. \tag{58}
\]

Consider \(\|\hat{x}^k - \bar{x}^k\|\), where \(\bar{x}^k\) and \(\bar{x}^k\) are defined in (20) and (18), respectively. By exploiting the nonexpansiveness of the projection, we can write
\[
\|\hat{x}^k - \bar{x}^k\|
\leq \|\alpha_d\|\|\tilde{F}(x^k, \sigma^k) - \tilde{F}(x^k, \hat{x}^k) + C_d^T (\hat{x}^k - \lambda^k)\|
\leq L_F \|\alpha_d\|\|(W(k) \cap I_m)\sigma^k - 1 \otimes x^k\|
+ \|\alpha_d\|\|C_d\|\|(W(k) \cap I_m)z^k - 1 \otimes \lambda^k\|, \tag{59}
\]

where the second inequality follows by the triangular inequality and the Lipschitz continuity of \(\tilde{F}\) (Assumption 4). Now, consider \(\|\tilde{x}^k - \lambda^k\|\), where \(\tilde{x}^k\) and \(\lambda^k\) are defined in (21) and (19), respectively. By exploiting the nonexpansiveness of the projection, we have
\[
\|\tilde{x}^k - \lambda^k\| \leq \|\beta_d\|\|(W(k) \cap I_m)z^k - 1 \otimes \tilde{x}^k\|
+ \|\beta_d\|\|y^{k+1} - 1 \otimes d^{k}\|. \tag{60}
\]

Finally, by combining (60) and (59) with (58) we obtain the upper bound in Lemma 7.\]

\[\text{G. Proof of Lemma 8}\]

By substituting the bounds on the estimation errors of Lemma 5 into the error bound in Lemma 7 we obtain
\[
\gamma^k\|e^k\| \leq \alpha_1 \gamma^k \rho^k + \alpha_2 \gamma^k \sum_{s=1}^{k} \rho^{-s} \gamma^{s-1}
\leq \alpha_1 \gamma^k \rho^k + \alpha_2 \gamma^k \sum_{s=1}^{k} \rho^{-s} \gamma^{s-1}, \tag{61}
\]
where \(\alpha_1, \alpha_2, \alpha_3\) and \(\alpha_4\) are positive constants defined as \(\alpha_1 := \theta B_\Omega \|\alpha_d\|L_F + \|C_d\|B_D\|\beta\|\) and \(\alpha_2 := \theta B_\Omega \|\alpha_d\|L_F + \|\alpha_d\|\|C_d\|\|C_d\|\|\beta\|\), \(\alpha_3 := \theta B_\Omega \|\alpha_d\|L_F + \|\alpha_d\|\|C_d\|\|\beta\|\), \(\alpha_4 := \theta B_\Omega \|\alpha_d\|L_F + \|\alpha_d\|\|C_d\|\|\beta\|\). Now, we show that each term on the right-hand side of (61) is summable, hence also the sequence \((\gamma^k\|e^k\|)_{k \in \mathbb{N}}\) is such, i.e., \(\sum_{k=0}^{\infty} \gamma^k\|e^k\| < \infty\).

Term 1: To establish the convergence of \(\sum_{k=0}^{\infty} \gamma^k \rho^k\), we note that \(\gamma^k \leq \gamma^0\) for all \(k \in \mathbb{N}\) by Assumption 7 implying that \(\sum_{k=0}^{\infty} \gamma^k \rho^k \leq \gamma^0 \sum_{k=0}^{\infty} \rho^k < \infty\), since \(0 < \rho < 1\) by Lemma 1.

Term 2: Since \(\gamma^k \leq \gamma^{s-1}\), for all \(k \geq s-1\) (Assumption 7), the following relations hold for the second term in the right-hand side of (61):
\[
\sum_{k=0}^{\infty} \gamma^k \left( \sum_{s=1}^{k} \rho^{-s} \gamma^{s-1}\right) = \sum_{k=0}^{\infty} \sum_{s=1}^{k} \rho^{-s} \gamma^{s-1}\gamma^s
\leq \sum_{k=0}^{\infty} \sum_{s=1}^{k} \rho^{-s} (\gamma^{s-1})^2. \tag{62}
\]
It follows by Lemma 5 (b) that \(\sum_{k=0}^{\infty} \sum_{s=1}^{k} \rho^{-s} (\gamma^{s-1})^2 < \infty, \text{ since } \sum_{k=0}^{\infty} (\gamma^k)^2 < \infty, \gamma^k \geq 0\text{ for all } k \text{ (Assumption 7)}\) and \(0 < \rho < 1\).
Term 3: By exploiting the definition of the sequence $(\phi^k)_{k \in \mathbb{N}}$ in Lemma [5] we can write
\[\sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \gamma^k \phi^k = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \gamma^k \left( \delta_1 \rho^{-k-1} + \delta_2 \sum_{\ell=1}^{k} \rho^{\ell-k-1} \right) = \delta_1 \sum_{k=0}^{\infty} \gamma^k \rho^{-k-1} + \delta_2 \sum_{k=0}^{\infty} \gamma^k \rho^{-k-1} \leq \delta_1 \rho^{-1} \sum_{k=0}^{\infty} \gamma^k \rho^{-k-1} + \delta_2 \sum_{k=0}^{\infty} \gamma^k \rho^{-k-1} \leq \delta_1 \gamma^0 \sum_{k=0}^{\infty} \rho^{-k-1} + \delta_2 \sum_{k=0}^{\infty} \rho^{-k} (\gamma^k)^2\]
By exploiting the same technical reasoning in (i) and (ii), we can show that each term on the right-hand side of the previous inequality globally converges. Therefore, we conclude that \(\sum_{k=0}^{\infty} \gamma^k \phi^k < \infty\).

Term 4: Since \(\gamma^k \leq \gamma^s\), for all \(k \geq s\) (Assumption [7]), the following relations hold for the last term in the right-hand side of (61):
\[\sum_{k=0}^{\infty} \sum_{s=1}^{k} \rho^{k-s} \phi^{s-1} = \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \rho^{k-s} \phi^{s-1} \leq \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \rho^{k-s} (\gamma^s)^{-1} \phi^{s-1} \]
It follows by Lemma [6](b) that \(\sum_{k=0}^{\infty} \sum_{s=1}^{k} \rho^{k-s} \phi^{s-1} < \infty\), since \(\sum_{k=0}^{\infty} \gamma^k \phi^k < \infty\) by (iii), and \(0 < \rho < 1\)
To conclude, since all the terms in the right-hand side of (61) are summable, then we have \(\sum_{k=0}^{\infty} \gamma^k \phi^k < \infty\). 

REFERENCES

[1] M. Jensen, “Aggregative games and best-reply potentials,” Economic Theory, Springer, vol. 43, pp. 45–66, 2010.
[2] W. Saad, Z. Han, H. Poor, and T. Başar, “Game theoretic methods for the smart grid,” IEEE Signal Processing Magazine, pp. 86–105, 2012.
[3] Z. Ma, D. Callaway, and I. Hiskens, “Decentralized charging control of large populations of plug-in electric vehicles,” IEEE Trans. on Control Systems Technology, vol. 21, no. 1, pp. 67–78, 2013.
[4] N. Li, L. Chen, and M. A. Dahleh, “Demand response using linear supply function bidding,” IEEE Transactions on Smart Grid, vol. 6, no. 4, pp. 1827–1838, 2015.
[5] J. Barrera and A. Garcia, “Dynamic incentives for congestion control,” IEEE Trans. on Automatic Control, vol. 60, no. 2, pp. 299–310, 2015.
[6] S. Grammatico, F. Parise, M. Colombino, and J. Lygeros, “Decentralized convergence to Nash equilibria in constrained deterministic mean field control,” IEEE Trans. on Automatic Control, vol. 61, no. 11, pp. 3315–3329, 2016.
[7] S. Grammatico, “Dynamic control of agents playing aggregative games with coupling constraints,” IEEE Trans. on Automatic Control, vol. 62, no. 9, pp. 4537–4548, 2017.
[8] D. Pacagnan, B. Gentile, F. Parise, M. Kamgarpour, and J. Lygeros, “Nash and Wardrop equilibria in aggregative games with coupling constraints,” IEEE Transactions on Automatic Control, available online at https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=8395077, 2018.
[9] G. Belgioioso and S. Grammatico, “Semi-decentralized Nash equilibrium seeking in aggregative games with coupling constraints and non-differentiable cost functions,” IEEE Control Systems Letters, vol. 1, no. 2, pp. 400–405, 2017.
[10] P. Yi and L. Pavel, “An operator splitting approach for distributed generalized nash equilibrium computation,” Algorithmica, vol. 102, pp. 111–121, 2019.
[11] H. H. Bauschke, P. L. Combettes et al., Convex analysis and monotone operator theory in Hilbert spaces. Springer, 2017, vol. 2011. 
[12] T. Tatarenko, W. Shi, and A. Nedić, “Accelerated gradient play algorithm for distributed nash equilibrium seeking,” in 2018 IEEE Conference on Decision and Control (CDC). IEEE, 2018, pp. 3561–3566.
[13] L. Pavel, “A doubly-augmented operator splitting approach for distributed GNE seeking over networks,” in 2018 IEEE Conference on Decision and Control (CDC). IEEE, 2018, pp. 3529–3534.
[14] J. Koshal, A. Nedić, and U. Shanbhag, “Distributed algorithms for aggregative games on graphs,” Operations Research, vol. 64, no. 3, pp. 680–704, 2016.
[15] A. Nedić, A. Olshevsky, and W. Shi, “Achieving geometric convergence for distributed optimization over time-varying graphs,” SIAM Journal on Optimization, vol. 27, no. 4, pp. 2597–2633, 2017.
[16] F. Parise, B. Gentile, and J. Lygeros, “A distributed algorithm for average aggregative games with coupling constraints,” arXiv preprint arXiv:1706.04634, 2017.
[17] P. L. Combettes, “Quasi-fejérian analysis of some optimization algorithms,” in Studies in Computational Mathematics. Elsevier, 2001, vol. 8, pp. 115–152.
[18] K. Margellos and S. Grammatico, “Projected-gradient algorithms for finite-dimensional variational inequalities and complementarity problems,” Springer Verlag, 2003.
[19] F. Facchinei and J. Pang, Finite-dimensional variational inequalities and complementarity problems. Springer Science & Business Media, 2007.
[20] A. Auslender and M. Teboulle, “Lagrangian duality and related multiplier methods for variational inequality problems,” SIAM Journal on Optimization, vol. 10, no. 4, pp. 1097–1115, 2000.
[21] G. Belgioioso and S. Grammatico, “Projected-gradient algorithms for generalized equilibrium seeking in aggregative games are preconditioned forward-backward methods,” in 2018 European Control Conference (ECC). IEEE, 2018, pp. 2188–2193.
[22] F. Facchinei and J.-S. Pang, Finite-dimensional variational inequalities and complementarity problems. Springer Science & Business Media, 2007.
[23] A. Nedić and A. Ozdaglar, “Subgradient methods for saddle-point problems,” Journal of optimization theory and applications, vol. 142, no. 1, pp. 205–228, 2009.
[24] M. Zhu and S. Martínez, “On distributed convex optimization under inequality and equality constraints,” IEEE Transactions on Automatic Control, vol. 57, no. 1, pp. 151–164, 2012.
[25] S. S. Ram, A. Nedić, and V. V. Veeravalli, “Distributed stochastic subgradient projection algorithms for convex optimization,” Journal of optimization theory and applications, vol. 147, no. 3, pp. 516–545, 2010.
[26] D. G. Feingold, R. S. Varga et al., “Block diagonally dominant matrices and generalizations of the gerschgorin circle theorem.” Pacific Journal of Mathematics, vol. 12, no. 4, pp. 1241–1250, 1962.
[27] P. L. Combettes and B. C. Vu, “Variable metric forward–backward splitting with applications to monotone inclusions in duality.” Optimization, vol. 63, no. 9, pp. 1289–1318, 2014.
[28] P. L. Combettes and I. Yamada, “Compositions and convex combinations of averaged nonexpansive operators,” Journal of Mathematical Analysis and Applications, vol. 425, no. 1, pp. 55–70, 2015.

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