Global Solution of the 3D Relativistic Vlasov–Poisson System for a Class of Large Data

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Abstract

For a class of arbitrary large initial data with radial symmetry or cylindrical symmetry, we prove the existence of global solutions for the 3D relativistic Vlasov–Poisson system for the plasma physics case. The compact support assumption is not imposed for both cases. The essential lower bound assumption of the angular momentum in the previous work of Glassey and Schaeffer (Math Methods Appl Sci 24:143–157, 2001) is not imposed on the initial data for the cylindrical symmetry case.

Keywords Large data global solution · Relativistic Vlasov–Poisson · Moments method

1 Introduction

We are interested in the large data Cauchy problem for the 3D relativistic Vlasov–Poisson system in the plasma physics case (RVP-PP), which reads as follows.

\[
\begin{align*}
\frac{\partial_t f + \hat{v} \cdot \nabla_x f + E \cdot \nabla_v f}{\Delta \phi} &= 0, \\
\Delta \phi &= \rho(t), \quad \rho(t) := \int_{\mathbb{R}^3} f(t, x, v) dv, \\
\int_{\mathbb{R}^3} f(0, x, v) dv &= f_0(x, v),
\end{align*}
\]

(1.1)

where \( \hat{v} := v/\sqrt{1+|v|^2} \), \( f(t, x, v) \geq 0 \) denotes the distribution of particles and \( \rho(t, x) \) denotes the density of particles.

From (1.1), the backward characteristics associated with the RVP-PP system read as follow,

\[
\begin{align*}
\frac{d}{ds} X(s; t, x, v) &= \hat{V}(s; t, x, v), \\
\frac{d}{ds} V(s; t, x, v) &= \nabla_x \phi(s, X(s; t, x, v))
\end{align*}
\]

(1.2)
For convenience in notation, if without causing confusion, we usually drop the dependence of characteristics with respect to \((t, x, v)\), which is fixed for most of time, and abbreviate characteristics as \((\mathcal{X}(s), \mathcal{V}(s))\).

For the RVP-PP system (1.1), the following conservation laws hold,

\[
E(t) := \int_{\mathbb{R}^n} |\nabla_x \phi(t)|^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle v \rangle f(t, x, v)dxdv = E(0), \quad \|f(t, x, v)\|_{L^p_{x,v}} = \|f(0, x, v)\|_{L^p_{x,v}},
\]

(1.3)

where \(p \in [1, \infty]\).

It’s worth to mention that the gravitational case, with \(+E \cdot \nabla_v f\) replaced by \(-E \cdot \nabla_v f\) in (1.1), is also very interesting. Correspondingly, the + sign in the conservation law of \(E(t)\) in (1.3) is replaced by – for the gravitational case. For large initial data, blow up can occur for the relativistic gravitational case, see Glassy-Schaeffer [5]. Moreover, there is a large literature devoted to the study of the non-relativistic Vlasov–Poisson system for both the plasma physics case and the gravitational case, e.g., with \(\hat{v}\) replaced by \(v\) in (1.1) is the non-relativistic Vlasov–Poisson system in the plasma physics case. We do not try to elaborate it here but refer readers to Lions-Perthame [15], Pfaffelmoser [18], Schaeffer [20], Lione-Ben-Artzi-Calogero-Pankavich [1], Pankavich [17], Illner-Rein [7, 8], Illner-Rein [9] and references therein for more detailed introduction.

For the rest of discussion, we restrict ourselves to the plasma physics case. A remarkable result by Lions-Perthame [15] (see also Pfaffelmoser [18] and Schaeffer [20]) showed that the non-relativistic Vlasov–Poisson system in the plasma physics case admits global classic solution for very general initial data. Whether the analogue of Lions-Perthame [15] holds for 3D RVP-PP system (1.1) remains an open problem. Indeed, the conservation law for RVP-PP system, see (1.3), is much weaker than the corresponding conservation law for the non-relativistic case, in which the second order momentum is bounded over time.

If the initial data is smooth and small, then the system (1.1) admits global solution. Moreover, the regularity of initial data can be propagated, and the density and its derivatives decay sharply over time, see [13, 21].

However, the picture of the large data problem of the RVP-PP system seems far from complete. A well-known result by Glassy-Schaeffer [5] says that the RVP-PP system (1.1) admits global classical solution if the initial data has radial symmetry and also has compact support in both \(x\) and \(v\).

With another assumption on the angular momentum, the spherical symmetry assumption imposed on the initial data \(f_0(x, v)\) of the RVP-PP system (1.1) can be relaxed to the cylindrical symmetry. The result of Glassy-Schaeffer [6] says that the RVP-PP system (1.1) admits global solution if the cylindrically symmetric initial data has compact support in both \(x\) and \(v\) and its angular momentum is bounded away from zero, c.f., (1.5). Without loss of generality, we assume that \(f_0(x, v)\) is cylindrically symmetric in \(x_1x_2\)-plane, which will be called the horizontal plane in later context. More precisely, the following equality holds for any \(x, v \in \mathbb{R}^3\),

\[
f_0(x_1, x_2, x_3, v_1, v_2, v_3) = f_0(|(x_1, x_2)|, 0, x_3, |(v_1, v_2)|, 0, v_3).
\]

(1.4)

The lower bound assumption on the angular momentum of particles can be understood in the following sense,

(Angular momentum assumption) \(f_0(x, v) = 0, \text{ if } (x, v) \in \{(x, v) : x, v \in \mathbb{R}^3, |x_1v_2 - x_2v_1| \leq C\}\),

(1.5)
where $C$ is some absolute constant.

Due to the gradient structure of the electric field and the cylindrical symmetry of solution, a crucial advantage of imposing the lower bound angular momentum assumption is that the space characteristics are far away from the $z$-axis because the angular momentum $X_1(s; t, x, v) V_2(s; t, x, v) - X_2(s; t, x, v) V_1(s; t, x, v)$ is conserved over time. More precisely, from the cylindrical symmetry of solution, for simplicity of notation, we define $\phi : \mathbb{R}^t \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ as follows,

$$\tilde{\phi}(t, r, x_3) := \phi(t, r, 0, x_3), \quad \nabla_x \phi = \frac{(x_1, x_2, 0)}{r} \partial_r \tilde{\phi}(t, r, x_3)$$

$$+ (0, 0, 1) \partial_{x_3} \tilde{\phi}(t, r, x_3), \quad r := |(x_1, x_2)|. \quad (1.6)$$

Hence, from (1.6) and (1.2), we have

$$\frac{d}{ds} \left( X_1(s; t, x, v) V_2(s; t, x, v) - X_2(s; t, x, v) V_1(s; t, x, v) \right)$$

$$= \left( X_1(s; t, x, v) X_2(s; t, x, v) - X_2(s; t, x, v) X_1(s; t, x, v) \right)$$

$$\times \partial_r \tilde{\phi}(s, |(X_1(s; t, x, v), X_2(s; t, x, v))|, X_3(s; t, x, v)) = 0.$$

That is to say, the angular momentum $X_1(s; t, x, v) V_2(s; t, x, v) - X_2(s; t, x, v) V_1(s; t, x, v)$ is conserved over time.

In this paper, we are interested in improve previous results of Glassey-Schaeffer for the RVP-PP system (1.1) for both the radial symmetry case in [5] and the cylindrical symmetry case in [6]. More precisely, for the radial symmetry case, our main result is stated as follows,

**Theorem 1.1** Assume that the initial data $f_0(x, v) \in H^s(\mathbb{R}_x^3 \times \mathbb{R}_v^3), s \in \mathbb{Z}^+, s \geq 6$ is radial in the sense that, $\forall R \in SO(3), f_0(Rx, Rv) = f_0(x, v)$. Moreover, we assume that the initial data decays polynomially as $(x, v) \rightarrow \infty$ in the following sense,

$$\sum_{\alpha \in \mathbb{Z}_+^3, |\alpha| \leq s} \| (1 + |x| + |v|)^N \nabla_{x,v}^\alpha f_0(x, v) \|_{L^2_{x,v}} < +\infty, \quad N_r := 100. \quad (1.7)$$

Then the RVP-PP system (1.1) admits global solution in $H^s(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$.

For the cylindrical symmetry case, our main result is stated as follows,

**Theorem 1.2** If the cylindrically symmetric initial data (in the sense of (1.4)) $f_0(x, v) \in H^s(\mathbb{R}_x^3 \times \mathbb{R}_v^3), s \in \mathbb{Z}^+, s \geq 6$ satisfies the following estimate,

$$\sum_{\alpha \in \mathbb{Z}^3_+, |\alpha| \leq s} \| (1 + |x| + |v|)^N \nabla_{x,v}^\alpha f_0(x, v) \|_{L^2_{x,v}}$$

$$+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x_1 v_2 - x_2 v_1|^{13}} f_0(x, v) dx dv < +\infty, \quad N_{c} := 10^8. \quad (1.8)$$

Then the RVP-PP system (1.1) admits global solution in $H^s(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$.

A few remarks are in order.

**Remark 1.1** The main merit of this paper are two new observations in the cylindrical symmetry case.

The first observation is a new weighted space-time estimate with a choice of singular weighted function, see the estimate (3.9) in Lemma 3.1. This weighted space-time estimate
helps to understand the close to the $z$-axis scenario. It tells us that particles are not very localized around the $z$-axis.

The second observation is that a smoothing effect is available for the action of electric field along characteristics after carefully analyzing the time resonance set, which is the set of frequencies such that the phase of oscillation in time is zero. The method of studying time (as well as space) resonance set was introduced by Germain-Masmoudi-Shatah [2] in the study of NLS. Now, it has very wide applications in the study of nonlinear dispersive equations, see e.g., [3, 4, 10–12] and the nonlinear wave equations, see [19].

Now, we give a conceptual description about this smoothing effect. Roughly speaking, \( \int_{t_1}^{t_2} P \phi (\varphi (X(s))) \, ds \) is smoother than \( P \phi (\varphi (X(s))) \) itself if the frequency is localized away from the time resonance set. Here \( P \) is a Fourier multiplier operator. This type of smoothing effect was pointed out by Klainerman-Staffilani [14] for the relativistic Vlasov–Maxwell system. Due to the different speeds of the electromagnetic field and massive particles, the smoothing effect is more intuitive and transparent for the relativistic Vlasov–Maxwell system. However, for the RVP-PP system (1.1), the smoothing effect is rather obscure and technical. We exploit the smoothing effect by doing normal form transformation after localizing away from the time resonance set, e.g., see the localization of frequencies in (4.20) and the normal form transformation in (4.31).

**Remark 1.2** The compact support assumptions in previous results of Glassey-Schaffer [5, 6] are removed for both the radial case and the cylindrical symmetry case. Despite that the lower bound assumption of the planar momentum is removed, as stated in (1.5), we still need an vanishing order condition at zero planar momentum for the initial data. The number of order “13” in (1.8) can definitely be improved. However, there are additional difficulties in removing completely the vanishing order condition. Roughly speaking, the additional difficulty is caused by the case \( |(v_1, v_2)|^2 \sim |v| \sim |\xi| \sim |(x_1, x_2)|^{-1}, |v| \gg 1 \), where \( |\xi| \) measures the size of frequency variable. In this case, the aforementioned two observations don’t provide stronger control than the conservation law (1.3). In some sense, the vanishing order condition, which is conserved over time, says that this worst scenario rarely occurs. However, for generic initial data, it’s not apriori clear or expected that the worst scenario rarely occurs. See also [22] for additional remarks and properties of the RVP-PP system that might shine some light on the future study.

**Remark 1.3** Last and also the least, the plausible goal of optimizing \( s \), and \( N_r, N_c \) is not pursued here.

### 1.1 Notation

For any two numbers \( A \) and \( B \), we use \( A \lesssim B \) and \( B \gtrsim A \) to denote \( A \leq CB \), where \( C \) is an absolute constant. We use the convention that all constants which only depend on the initial data, e.g., the conserved quantities in (1.3), will be treated as absolute constants. For any \( u \in \mathbb{R}^3 \setminus \{0\} \), we use \( \tilde{u} \) to denote the direction of \( u \), i.e., \( \tilde{u} := u/|u| \).

We fix an even smooth function \( \tilde{\psi} : \mathbb{R} \to [0, 1] \), which is supported in \([-3/2, 3/2]\) and equals to “1” in \([-5/4, 5/4]\). For any \( k, j \in \mathbb{Z} \), \( j > 0 \), we define the cutoff functions \( \psi_k, \psi_{\leq k}, \psi_{\geq k} : \bigcup_{n \in \mathbb{Z}^+} \mathbb{R}^n \to \mathbb{R} \) as follows,

\[
\psi_k(x) := \tilde{\psi}(|x|/2^k) - \tilde{\psi}(|x|/2^{k-1}), \quad \psi_0(x) := \tilde{\psi}(|x|),
\]

\[
\forall j \in (0, \infty) \cap \mathbb{Z}, \varphi_j(x) := \psi_j(x).
\]
1.2 Local Theory and the Reduction of the Proof

Because our assumption on the initial data is stronger than the assumption imposed on the distribution function in [16], by using the same argument used by Luk-Strain [16] for the relativistic Vlasov–Maxwell system, which is more difficult, we can reduce the proof of global existence to the $L^\infty_x$-estimate of the electric field $\nabla_x \phi$, which corresponds to the acceleration of the speed of particles.

More precisely, assume that $T^*$ is the maximal time of existence of solution. If we can show that $\nabla_x \phi \in L^\infty ([0, T^*) \times \mathbb{R}_3^4)$, then the lifespan of the solution can be extended to $[0, T^* + \epsilon]$ for some positive number $\epsilon$. That is to say, the solution of RVP-PP (1.1) exists globally in time.

Plan of this Paper

- In Sect. 2, we use the observation that particles will travel away from the source of acceleration, i.e., the origin, to control the majority set in the radial case.
- In Sect. 3, we control the electric field for a fixed time and prove a weighted space-time estimate for the distribution function, which provides a good control for the electric field near the $z$-axis, for the cylindrical symmetry case.
- In Sect. 4, we show that a smoothing effect is available for the time integral of electric field along characteristics, which enables us to control strongly the majority set, in the cylindrical symmetry case.
- In Sect. 5, thanks to the strong control of the majority set, we show that the high order moment grows at most polynomially for both the radial case and the cylindrical symmetry case. The boundedness of the high order moment implies the boundedness of electric field in any finite time. Hence finishing the proof of theorem 1.1 and theorem 1.2.

2 Control of the Majority Set in the Radial Case

In this section, we mainly control the velocity characteristics for the radial case, which will ultimately provide a control for the high order moment, see Sect. 5. Throughout this section, $f(t, x, v)$ is assumed to be radial even without explicit statement. We recall that $f(t, x, v)$ is radial means that $\forall R \in SO(3), f(t, Rx, Rv) = f(t, x, v)$.

To control the $L^\infty_t$-norm of the electric field, we use the classic moment method. Since our goal is to show that the electric field doesn’t blow up in finite time, any polynomial growth rate or even exponential growth rate in time is acceptable. To account for the growth in $t$ and to measure the maximum of moments of solution within the time interval $[0, t]$, we define

$$M^r(t) := (1 + t)^{2n_r} + \sup_{s \in [0, t]} \int \int (1 + |v|)^{n_r} f(s, x, v) dvdx, \quad n_r := N_r/10 = 10,$$

(2.1)

where $N_r$ is defined in (1.7). As long as we can show that $M^r(t) \in L^\infty ([0, T^*])$, we know that $\nabla_x \phi \in L^\infty ([0, T^*) \times \mathbb{R}_3^4)$, see Lemma 2.2.

Moreover, we define a set of majorities of particles, which initially localize around zero, at time $s$ as follows,

$$R^r(t, s) := \{ (X(s; t, x, v), V(s; t, x, v)) : |X(0; t, x, v)| + |V(0; t, x, v)| \leq (M^r(t))^{1/(2n_r)} \}. $$

(2.2)
Thanks to the fast polynomial decay assumption on the initial data, see (1.7), we know that \( f(s, X(s; t, x, v), V(s; t, x, v)) \) is very small outside \( R^r(t, s) \), \( \forall s \in [0, t] \). Hence, it would be sufficient to control the contribution from the majority set \( R^r(t, s) \). To control the majority set \( R^r(t, s) \), it’s crucial to control the electric field.

We have two basic estimates for the electric field. The first estimate is stated in the following Lemma. Roughly speaking, the first estimate says that, outside a small ball of the origin, the electric field is not so strong. We remark that this estimate is available mainly because of the radial symmetry.

**Lemma 2.1** Under the radial assumption, for any \( t \in [0, T^*) \), \( x \in \mathbb{R}^3 / \{0\} \), the following point-wise estimate holds for the electric field,

\[
|\nabla_x \phi(t, x)| \lesssim \frac{1}{|x|^2}. \tag{2.3}
\]

**Proof** Recall that the distribution \( f \) is radial, which implies that the density function \( \rho(t, x) \) is radial,

\[
\forall x \in \mathbb{R}^3 / \{0\}, \quad \rho(t, Rx) = \int_{\mathbb{R}^3} f(t, Rx, v) dv = \int_{\mathbb{R}^3} f(t, Rx, R\omega) |\det(R)| d\omega = \int_{\mathbb{R}^3} f(t, x, \omega) d\omega = \rho(t, x).
\]

Hence \( \phi(t, x) \) is also radial. Define

\[
\tilde{\phi}(t, r) := \phi(t, r, 0, 0), \quad \tilde{\rho}(t, r) := \rho(t, r, 0, 0),
\]

\[
\implies \phi(t, x) = \tilde{\phi}(t, |x|), \quad \rho(t, x) = \tilde{\rho}(t, |x|).
\]

The Poisson equation for \( \phi \) in (1.1) is reduced as follows,

\[
(\partial_r^2 + \frac{2}{r} \partial_r) \tilde{\phi}(t, r) = \tilde{\rho}(t, r), \quad \partial_r^2 (r \tilde{\phi})(t, r) = r \tilde{\rho}(t, r), \quad \implies \partial_r (r \tilde{\phi}) = \int_0^r s \tilde{\rho}(t, s) ds + c
\]

\[
\implies r \tilde{\phi} = \int_0^r \int_0^r s \tilde{\rho}(t, s) ds d\tau + cr = \int_0^r s(r - s) \tilde{\rho}(t, s) ds + cr
\]

\[
\implies \tilde{\phi}(t, r) = \int_0^r s \tilde{\rho}(t, s) ds - \frac{1}{r} \int_0^r s^2 \tilde{\rho}(t, s) ds + c,
\]

\[
\implies \partial_r \tilde{\phi}(t, r) = \frac{1}{r^2} \int_0^r s^2 \tilde{\rho}(t, s) ds
\]

Hence, from the above equality and the conservation law in (1.3), the following estimate holds point-wisely,

\[
\implies \nabla_x \phi = \frac{x}{|x|} \partial_r \tilde{\phi}(t, |x|) = \frac{x}{|x|^3} \int_0^{|x|} s^2 \tilde{\rho}(t, s) ds, \quad \implies \nabla_x \phi
\]

\[
= \frac{x}{|x|} |\nabla_x \phi|, \quad |\nabla_x \phi| \lesssim \frac{1}{|x|^2}. \tag{2.4}
\]

Moreover, as summarized in the following Lemma, we have another estimate for the electric field. Roughly speaking, the second estimate says that the electric field is not too strong even if it’s very close to the origin. This estimate mainly follows from the definition of the high order moment in (2.1) and the fact that the first moment is always bounded from the above because of the conservation law (1.3).
Lemma 2.2 Let $\delta \in (0, 10^{-10})$ be some fixed sufficiently small constant. Under the radial assumption, the following estimate holds for any $t \in [0, T^*)$,

$$\|\nabla_x \phi(t, x)\|_{L^\infty} \lesssim 1 + (M^r(t))^{(5+\delta)/(3-\delta)(n_r-1)}. \quad (2.5)$$

Proof Note that

$$\begin{align*}
\nabla_x \phi(t, x) &= \int_{\mathbb{R}^3} \frac{\rho(t, y)(y-x)}{|y-x|^3} dy = \int_{|y-x| \leq \kappa} \frac{\rho(t, y)(y-x)}{|y-x|^3} dy \\
&\quad + \int_{|y-x| \geq \kappa} \frac{\rho(t, y)(y-x)}{|y-x|^3} dy.
\end{align*} \quad (2.6)$$

From the conservation law in (1.3), the second part of the above equation is controlled as follows,

$$\left| \int_{|y-x| \geq \kappa} \frac{\rho(t, y)(y-x)}{|y-x|^3} dy \right| \lesssim \frac{1}{\kappa^2}. \quad (2.7)$$

For the first part, we use the Hölder inequality by choosing $p = (3-\delta)/2$ and $q = (3-\delta)/(1-\delta)$. As a result, we have

$$\left| \int_{|y-x| \leq \kappa} \frac{\rho(t, y)(y-x)}{|y-x|^3} dy \right| \lesssim \left( \int_{|y-x| \leq \kappa} \frac{1}{|y-x|^{2p}} dy \right)^{1/p} \left( \int_{\mathbb{R}^3} (\rho(t, y))^q dy \right)^{1/q} \lesssim \kappa^{2\delta/(3-\delta)} \|\rho(t, x)\|_{L^q}. \quad (2.8)$$

Since $\|f(t, x, v)\|_{L^q_{x,v}}$ is bounded from (1.3), the following estimate hold,

$$\rho(t, x) = \int_{|v| \leq R} f(t, x, v) dv + \int_{|v| > R} f(t, x, v) dv$$

$$\lesssim R^3 + R^{-6/(1-\delta)} \left( \int_{\mathbb{R}^3} (1 + |v|)^{6/(1-\delta)} f(t, x, v) dv \right)$$

$$\lesssim \left( \int_{\mathbb{R}^3} (1 + |v|)^{6/(1-\delta)} f(t, x, v) dv \right)^{(1-\delta)/(3-\delta)}. \quad (2.9)$$

Since first moment is bounded all time due to the conservation law in (1.3), for any $m, n \in \mathbb{R}_+$, s.t., $m < n$, we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^m f(t, x, v) dv dx = \int_{|v| \leq R} \int_{\mathbb{R}^3} (1 + |v|)^m f(t, x, v) dv dx$$

$$+ \int_{|v| \geq R} \int_{\mathbb{R}^3} (1 + |v|)^m f(t, x, v) dv dx$$

$$\lesssim R^{m-1} + R^{m-n} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^n f(t, x, v) dv dx \right)$$

$$\lesssim \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^n f(t, x, v) dv dx \right)^{(m-1)/(n-1)}. \quad (2.9)$$

From the above two estimates, we have

$$\|\rho(t, x)\|_{L^q} \lesssim \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_r} f(t, x, v) dv dx \right)^{(5+\delta)/(3-\delta)(n_r-1)}. \quad (2.9)$$
Hence, from the estimates (2.6–2.9) and the definition of $M^r(t)$ in (2.1), after letting $\kappa = 1$, we have
\[
\|\nabla_x \phi(t, x)\|_{L^\infty_t} \lesssim 1 + (M^r(t))^{(5+\delta)/(3-\delta)(n_r-1)}.
\] (2.10)
Hence finishing the proof of the desired estimate (2.5).

Remark 2.1 We remark that there is no evidence that the obtained estimate (2.5) is sharp because here we only used the conservation laws in (1.3). Despite the index of power in (2.5) looks arbitrary, here we provide an example to explain why it is so. In view of definition of $M^r(t)$ in (2.1). If $f(t, x, v)$ is supported in the following set
\[
\{(x, v) : x, v \in \mathbb{R}^3, |v| \sim 2^j\}, 
\]
then, roughly speaking, $M^r(t)$ provides no better control than the first moment $M_1(t)$. Moreover, if we assume that on the Fourier side, $\hat{f}(t, \xi, v)$ is supported in the following set,
\[
\{(|\xi|, v) : |\xi|, v \in \mathbb{R}^3, |\xi| \sim 2^k\},
\]
then by using either the conservation law of the first momentum or the conservation law of $L^\infty_x$ of $f(t, x, v)$, the following estimate holds after optimizing two strategies,
\[
(\text{example case}) \quad |\nabla_x \phi| \lesssim \min\{2^{2k-j}, 2^{-k+3j}\} \lesssim 2^{5j/3} \sim (M^r(t))^{5/(3(n_r-1))}.
\]
From the above estimate, we can see the appearance of the index $5/(3(n_r-1))$ modulo the loss of $\delta$.

With the above two estimates of the electric field, in the following Proposition 2.1, we control the final state $R^r(t, t)$ of characteristics that start from the majority set $R^r(t, 0)$, see (2.2). Intuitively speaking, if using the initial data with compact support as an example, the following proposition provides an estimate on the radii of support of solution.

**Proposition 2.1** Let $\delta$ be the small constant used in Lemma 2.2. Under the radial assumption, for any $t \in [0, T^*)$, the following relation holds for some sufficiently large absolute constant $C$,
\[
R^r(t, t) \subset B(0, C(M^r(t))^{1/(2n_r)}) \times B(0, C(M^r(t))^{(5+2\delta)/(6-2\delta)(n_r-1)}),
\] (2.11)
where $R^r(t, t)$ is defined in (2.2).

**Proof** Let $t \in [0, T^*)$ be fixed. Since we can let the absolute constant $C$ to be sufficiently large, it would be sufficient to consider the case $M^r(t) \gg 1$.

As preliminaries, we first study the evolution of the associated characteristics of the RVP-PP system (1.1). From (1.1) and (1.2), as a result of direct computations, we have
\[
\frac{d}{ds} |X(s)| = \frac{X(s)}{|X(s)|} \cdot \nabla \phi(s), \quad \frac{d}{ds} |V(s)| = \frac{V(s) \cdot X(s)}{|V(s)||X(s)|} |\nabla_x \phi(X(s))|.
\] (2.12)
From the above two equations, we can see that the quantity $V(s) \cdot X(s)$ plays an essential role. As a result of direct computations, we have
\[
\frac{d}{ds} \left( \frac{V(s) \cdot X(s)}{|V(s)|} \right) = |\nabla \phi(s)| + \frac{|X(s)|}{|V(s)|} \left( 1 - \frac{(V(s) \cdot X(s))^2}{|V(s)|^2 |X(s)|^2} \right) |\nabla_x \phi(X(s))| \geq 0.
\] (2.13)
From the above equation, we know that the quantity $V(s) \cdot X(s)$ is an increasing function with respect to time “$s$”.

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Note that, from the Eq. (2.12), the following rough estimate holds for the length of $X(s; t, x, v)$,

$$ |X(s; t, x, v)| \leq |X(0; t, x, v)| + |s| \leq 2(M^r(t))^{1/(2n_r)}, \quad s \in [0, t]. \quad (2.14) $$

We define the maximal time such that the velocity characteristic doesn’t exceed the threshold as follows,

$$ \tau := \sup \{ s : s \in [0, t], \forall \kappa \in [0, s], \ |V(\kappa; t, x, v)| \leq (M^r(t))^{(5+2\delta)/(6-2\delta)(n_r-1)} \}. $$

From the continuity of characteristics, we know that $\tau > 0$. If $\tau = t$, then there is nothing left to be proved. It remains to consider the case when $0 < \tau < t$.

Note that, from the Eq. (2.12), we know that $|V(s; t, x, v)|$ is decreasing if $V(s; t, x, v) \cdot X(s; t, x, v) < 0$. Hence, at the time $\tau$, we have $V(\tau; t, x, v) \cdot X(\tau; t, x, v) \geq 0$. Otherwise, it contradicts the definition of the maximal time. From the monotonicity of $V(s; t, x, v) \cdot X(s; t, x, v)$, see the Eq. (2.13), we know that $V(s; t, x, v) \cdot X(s; t, x, v) \geq 0$ for $s \in [\tau, t]$, which implies that $|V(s; t, x, v)| \geq |V(\tau; t, x, v)|$ for all $s \in [\tau, t]$ from the Eq. (2.12). To sum up, for all $s \in [\tau, t]$, we have

$$ |V(s; t, x, v)| \geq |V(\tau; t, x, v)| = (M^r(t))^{(5+2\delta)/(6-2\delta)(n_r-1)}, \quad V(s; t, x, v) \cdot X(s; t, x, v) \geq 0. \quad (2.15) $$

Starting from the time $\tau$, from the above estimate and the Eq. (2.13), we have the following estimate for any $s \in [\tau, t]$,

$$ \frac{V(s; t, x, v) \cdot X(s; t, x, v)}{|V(s; t, x, v)|} = \frac{V(\tau; t, x, v) \cdot X(\tau; t, x, v)}{|V(\tau; t, x, v)|} + \int_\tau^s \frac{d}{ds} \left( \frac{V(\kappa; t, x, v) \cdot X(\kappa; t, x, v)}{|V(\kappa; t, x, v)|} \right) d\kappa \gtrsim s - \tau. \quad (2.16) $$

From the Eq. (2.12) and the estimates (2.15), which tells us that $\forall s \in [\tau, t], |V(s)| \gg 1$, and (2.16), we have

$$ |X(s; t, x, v)|^2 - |X(\tau; t, x, v)|^2 = \int_\tau^s \frac{d}{ds} |X(\kappa; t, x, v)|^2 d\kappa \gtrsim \int_\tau^s (\kappa - \tau) d\kappa \gtrsim (s - \tau)^2. $$

From the above estimate, the estimate (2.10), and the equation in (1.2), the following estimate holds for any $s \in [\tau, t]$,

$$ |V(s; t, x, v)| \leq |V(\tau; t, x, v)| + \int_\tau^s \left| \frac{d}{ds} V(\kappa; t, x, v) \right| d\kappa \lesssim (M^r(t))^{(5+2\delta)/(6-2\delta)(n_r-1)} + \int_\tau^{\tau+\gamma} (M^r(t))^{(5+2\delta)/(6-2\delta)(n_r-1)} d\kappa \left| \int_\tau^{\tau+\gamma} \frac{1}{(\kappa - \tau)^\gamma} d\kappa \right| \lesssim (M^r(t))^{(5+2\delta)/(6-2\delta)(n_r-1)}, \quad \text{by letting } \gamma = (M^r(t))^{-(5+2\delta)/(6-2\delta)(n_r-1)}. \quad (2.17) $$
To sum up, our desired conclusion (2.11) holds from (2.14) and (2.17).

3 A Space-Time Estimate for the Distribution Function in the Cylindrical Symmetry Case

Throughout this section, \( f(t, x, v) \) is assumed to be cylindrical symmetric even without explicit statement. Our main goal in this section is to show a singular weighted space-time estimate for the distribution function in the cylindrical symmetry case, which is summarized in Proposition 3.1. This is also the first observation mentioned in Remark 1.1. Thanks to this estimate, we know that particles are not too concentrated near \( z \)-axis.

Firstly, as preparation for proving Proposition 3.1, we introduce the set-up for the cylindrical symmetric case and obtain some basic tools, which are point-wise estimates of the localized electric field and are stated in Lemma 3.1.

As in the radial case, we also use the classic moment method, to control the electric field over time. We define

\[
M^c(t) := (1 + t)^{n_c^2} + \sup_{s \in [0, t]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_c} f(s, x, v)dvdx, \quad n_c := N_c/10,
\]

where \( N_c \) is defined in (1.8).

Comparing with the definition of the \( M'(t) \) for the radial setting, see (2.1), the definition of \( M^c(t) \) is conceptually same but technically different in terms of the growth rate and the order of moment. The reason is that, since the cylindrical symmetric case is much harder, we don’t have much room left to close the argument, we choose \( n_c \) very large such that the absolute constant \( \epsilon := 10/n_c \) is very small.

For the sake of making easier comparison at the dyadic level because we will do dyadic decomposition for the velocity variable and the frequency variable later, for any fixed \( t \in [0, T^\ast) \), where \( T^\ast \) denotes the maximal time of existence, we define

\[
M_t := \inf \{ k : k \in \mathbb{Z}_+, 2^k \geq (M^c(t))^{1/(n_c-1)} \}.
\]

To better identifying bad parts of the electric field, for any \( k \in \mathbb{Z}, j_2 \in \mathbb{Z}_+, j_1 \in [0, j_2 + 1] \cap \mathbb{Z}, \) we define the localized electric field as follows,

\[
E_{k; j_1, j_2}(t, x) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{K}_k(y) f(t, x - y, v)\varphi_{j_1}(v_1, v_2)\varphi_{j_2}(v)dydv,
\]

\[
\tilde{K}_k(y) := \int_{\mathbb{R}^3} e^{iy \cdot i\xi} i^{2k} |\xi|^{-2} \varphi_k(\xi)d\xi,
\]

\[
E(t, x) = \sum_{k, j_2 \in \mathbb{Z}_+, j_1 \in [0, j_2 + 1] \cap \mathbb{Z}} E_{k; j_1, j_2}(t, x),
\]

where, by doing integration by parts in \( \xi \) many times, the following estimate holds for the kernel \( \tilde{K}_k(y) \),

\[
|\tilde{K}_k(y)| \lesssim 2^{2k} (1 + 2^k |y|)^{-N_5^c}.
\]

As summarized in the following Lemma, we first prove several rough estimates for the electric field, in which we only exploit the conservation laws in (1.3) and the benefit of the cylindrical symmetry of solution in a straightforward way. The following Lemma plays the same role as Lemmas 2.1 and 2.2 for the radial setting. As in the radial setting, we don’t expect them to be sharp and only use them to have a general sense of the electric field.
Lemma 3.1 Let $\epsilon := 10/n_c$. For any fixed $t \in [0, T^*)$, the following rough estimate holds for the localized electric field,

$$\|E_{k; j_1, j_2}(t, x)\|_{L^\infty_t L^2_x} \lesssim \min\{2^{-k+2j_1+j_2}, 2^{2k-j_2}, 2^{2k-n_c}M^c(t)\}, \quad (3.5)$$

$$\|E(t, x)\|_{L^\infty_t L^2_x} \lesssim 2^{5\mathcal{M}_t/3+2e\mathcal{M}_t}. \quad (3.6)$$

Moreover, under the cylindrical symmetry assumption, for any $x \in \mathbb{R}^3$ s.t., $|(x_1, x_2)| \neq 0$, we have

$$|E_{k; j_1, j_2}(t, x)| \lesssim 1 + \min \left\{ \frac{2j_1+e\mathcal{M}_t}{|(x_1, x_2)|^{1/2}}, \frac{2^{k-j_2}+e\mathcal{M}_t}{|(x_1, x_2)|} \right\}. \quad (3.7)$$

Proof Note that the desired estimate (3.5) holds straightforwardly from the estimate of kernel in (3.4), the volume of support of $\nu$, and the conservation law (1.3). After summing up the obtained estimate (3.5) with respect to $k, j_1, j_2$, our desired rough estimate (3.6) for the electric field holds.

Now, we focus on the proof of the desired estimate (3.7). Note that, (3.7) holds directly from (3.5) if $|x| \leq 2^{-k+e\mathcal{M}_t/2}$. It would be sufficient to consider the case $|x| \geq 2^{-k+e\mathcal{M}_t/2}$. From the estimate of kernel in (3.4) and the cylindrical symmetry of solution, we have

$$|E_{k; j_1, j_2}(t, x)| \lesssim 1 + 2^k \int_{\mathbb{R}^3} \int_{|x| \leq 2^{-k+e\mathcal{M}_t/10}} f(t, x - y, v)\varphi_{j_1}(v_1, v_2)\varphi_{j_2}(v)dydv$$

$$\lesssim 1 + 2^{k+e\mathcal{M}_t} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|(x_1, x_2)|} f(t, x - y, v)\varphi_{j_1}(v_1, v_2)\varphi_{j_2}(v)dydv \lesssim 1 + \frac{2^{k+e\mathcal{M}_t-j_2}}{|(x_1, x_2)|}. \quad (3.8)$$

After optimizing the above estimate with the obtained estimate (3.5), our desired estimate (3.7) holds.

With the above preparation, we are ready to state and prove the main result of this section, which is summarized in the following proposition.

Proposition 3.1 Let $\epsilon^* := \epsilon/100, \epsilon := 10/n_c$. Under the cylindrical symmetry assumption, for any $t \in [0, T^*)$, s.t., $\mathcal{M}_t \gg 1$, the following weighted space-time estimate holds,

$$A(t) := \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|(v_1, v_2)|^{2+2e^*}}{|(x_1, x_2)|^{1-2e^*}} f(s, x, v)dx dv ds \lesssim 2^{5e\mathcal{M}_t}. \quad (3.9)$$

As a by-product, the following $L^1_t L^\infty_x$-type estimate holds for any $t_1, t_2 \in [0, t]$.

$$\int_{t_1}^{t_2} \|E_{k; j_1, j_2}(t, \cdot)\|_{L^\infty_x} dt \lesssim 2^{e\mathcal{M}_t} + 2^{k-j_1+j_2+6e\mathcal{M}_t}. \quad (3.10)$$

Proof Define

$$\phi(x) := \begin{cases} 
2 & x \in [2, \infty) \\
2 + (x - 2)^3 & x \in [1, 2] \\
x^3 & x \in [0, 1) \\
0 & x \in (-\infty, 0]
\end{cases}, \quad \phi_1(x) := \phi(2^{-l}x). \quad (3.11)$$

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From the above explicit formula of cutoff function, we have
\[ \phi'(x) \geq 0, \quad \forall x \in (0, \infty), \quad \left| \frac{x\phi'(x)}{\phi(x)} \right| \lesssim 1, \quad \phi'_\ell(x) := 2^{-\ell} \phi(2^{-\ell} x), \quad \left| \frac{x\phi'_\ell(x)}{\phi(x)} \right| \lesssim 1. \] (3.12)

For \( \mu \in \{+, -\} \), we choose a weight function as follows,
\[
\omega_\mu(x, v) := \left( \mu| (v_1, v_2) || (x_1, x_2) || (v_1 x_1 + v_2 x_2) \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + (x_1 v_2 - x_2 v_1)^2 \right)^{\epsilon^*} \\
\times \phi\left( \mu \left( \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + \frac{1}{2} \right) \right)^{-1}.
\] (3.13)

As a result of direct computation, we have
\[
(v_1 \partial_{x_1} + v_2 \partial_{x_2}) \omega_\mu(x, v) = \epsilon^* \left( \mu| (v_1, v_2) || (x_1, x_2) || (v_1 x_1 + v_2 x_2) \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + (x_1 v_2 - x_2 v_1)^2 \right)^{-1} \\
\times \left[ \mu| (v_1, v_2) || (x_1, x_2) || (v_1 x_1 + v_2 x_2) \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + (x_1 v_2 - x_2 v_1)^2 \right] \\
+ (v_1 v_2) || (x_1, x_2) || (v_1 x_1 + v_2 x_2) \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + (x_1 v_2 - x_2 v_1)^2 \right)^{\epsilon^*} \\
\times \phi\left( \mu \left( \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + \frac{1}{2} \right) \right)^{-1} \mu(x_1 v_2 - x_2 v_1)^2.
\] (3.14)

From the above equality, we have
\[
\mu(v_1 \partial_{x_1} + v_2 \partial_{x_2}) \omega_\mu(x, v) \gtrsim \left( \mu| (v_1, v_2) || (x_1, x_2) || (v_1 x_1 + v_2 x_2) \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + (x_1 v_2 - x_2 v_1)^2 \right)^{-\epsilon^*} \\
\gtrsim \left( \mu| (v_1, v_2) || (x_1, x_2) || (v_1 x_1 + v_2 x_2) \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) + (x_1 v_2 - x_2 v_1)^2 \right)^{1-\epsilon^*} \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) \gtrsim 0.
\] (3.15)

Let
\[ I_{\mu}(t) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |(v_1, v_2)| \omega_\mu(x, v) f(t, x, v) dx dv. \] (3.16)

As a result of direct computation, we have
\[ \frac{d}{dt} I_{\mu}(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |(v_1, v_2)| \hat{\nu} \cdot \nabla_x ( \omega_\mu(x, v) ) f(t, x, v) dx dv \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} E(t, x) \cdot \nabla_v \left( |(v_1, v_2)| \omega_\mu(x, v) \right) f(t, x, v) dx dv.
\]

From the above equality and the estimate (3.15), we have
\[ \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |(v_1, v_2)|^{2+2\epsilon^*} \phi_{-10 \Omega R} \left( \mu \frac{x_1 v_1 + x_2 v_2}{(v_1, v_2) || (x_1, x_2)} \right) f(s, x, v) dx dv ds \]
\begin{equation}
|I_\mu(t)| + |I_\mu(0)| + \left| \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} E(s, x) \cdot \nabla_v ((v_1, v_2)|\omega_\mu(x, v)|) f(s, x, v) dx dv ds \right|.
\tag{3.17}
\end{equation}

Note that, from the estimate (3.13) and the conversation law (1.3), we have

\begin{equation}
\forall s \in [0, t], \quad |I_\mu(s)| \lesssim 1
+ \int_{|x| \leq 2^{2s}/5} \int_{|v| \leq 2^{2s}} |(v_1, v_2)|\omega_\mu(x, v) f(t, x, v) dx dv \lesssim 2^{eM_t}.
\tag{3.18}
\end{equation}

From the volume of support, we have

\begin{equation}
\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|(v_1, v_2)|^{2+2e^*}}{|(x_1, x_2)|^{1-2e^*}(v)} 
\left( 1 - \phi_{-10M_t} \left( \frac{x_1 v_1 + x_2 v_2}{|(v_1, v_2)|| (x_1, x_2)|} \right) \right)
\lesssim 1 + \int_0^t \int_{|x| \leq 2^{2s}/5} \int_{|v| \leq 2^{2s}} |(v_1, v_2)|^{2+2e^*} 
\left( 1 - \phi_{-10M_t} \left( \frac{x_1 v_1 + x_2 v_2}{|(v_1, v_2)|| (x_1, x_2)|} \right) \right)
\lesssim 1 + 2^{-10M_t + 10eM_t} \lesssim 1.
\tag{3.19}
\end{equation}

Lastly, we estimate the contribution from the nonlinear effect in (3.17). Recall (3.13). As a result of direct computation, we have

\begin{equation}
|\nabla_v \omega_\mu(x, v)| \lesssim \frac{|(x_1, x_2)|^{2e^*}}{|(v_1, v_2)|^{1-2e^*}}.
\tag{3.20}
\end{equation}

After localizing the sizes of \(|(v_1, v_2)|\) and \(|v|\) and localizing the electric field, from the estimate (3.20), we have

\begin{equation}
\left| \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} E(s, x) \cdot \nabla_v ((v_1, v_2)|\omega_\mu(x, v)|) f(s, x, v) dx dv ds \right|
\lesssim \sum_{k \in \mathbb{Z}, j_1, j_2 \in \mathbb{Z}^+, j_1 \in [0, j_2 + 2]^c \mathbb{Z}} H_{k; j_1', j_2'}(t),
\tag{3.21}
\end{equation}

where

\begin{equation}
H_{k; j_1', j_2'}(t) := \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |(x_1, x_2)|^{2e^*} |(v_1, v_2)|^{2e^*} |E_{k; j_1', j_2'}(s, x)| f(s, x, v)
\psi_{j_1'}(v_1, v_2) \psi_{j_2'}(v) dx dv ds.
\tag{3.22}
\end{equation}

By using the volume of support of \(u\) and the estimate of the kernel \(\tilde{K}_k(y)\) in (3.4), we have

\begin{equation}
|E_{k; j_1', j_2'}(s, x)| \lesssim \min\{2^{-k+2j_1'+j_2'}, 2^{2k-j_2'}\}.
\tag{3.23}
\end{equation}

From the above estimate, we can rule out the case when \(k \geq 4M_t\) and the case \(j_2' \geq 2M_t\) or \(j_2 \geq 2M_t\). Since there are at most \(M_t^3\) cases left, it would be sufficient to let \(k, j_1, j_2, j_1', j_2'\) all be fixed.
Based on the relative size of \( j_2 \) and \( j_2' \), we separate into two cases as follows.

- If \( j_2' \leq j_2 \).

  Note that, from the cylindrical symmetry and the estimate of the kernel \( \tilde{K}_k(y) \) in (3.4), we have

  \[
  |E_k; j_1, j_2'(s, x)| \lesssim 2^{eM_r} + \int_{\mathbb{R}^3} \int_{|y| \leq 2^{-k+eM_r}/10} 2^{k+eM_r/2} f(s, x - y, u) \psi_{j'_1}(u_1, u_2) \psi_{j'_2}(u) dy du.
  \]

  Therefore, from the above estimate and the definition of \( A(t) \) in (3.9), we have

  \[
  \int_0^t \|E_k; j_1, j_2'(s, x)\|_{L^\infty} ds \lesssim 2^{2eM_r} + 2^{k+eM_r} \epsilon^{-2j_1+j_2} A(t) \tag{3.23}
  \]

  From the above \( L^1 \) \( L^\infty \)-type estimate and the \( L^1 \) \( L^\infty \)-\( L^1 \) type bilinear estimate, the following estimate holds if \( k \leq 2j_1' - 2eM_r \),

  \[
  |H_{k; j_1', j_2'}(t)| \lesssim 2^{2eM_r} + 2^{k+eM_r} - 2j_1' + j_2 A(t) \lesssim 2^{3eM_r} + 2^{-eM_r} A(t) \tag{3.24}
  \]

  If \( k \geq 2j_1' - 2eM_r \), then from the rough estimate of the localized electric field in (3.22), we have

  \[
  |H_{k; j_1', j_2'}(t)| \lesssim 2^{-k+2j_1' + j_2 + 3eM_r} - 2^{-j_2} \lesssim 2^{5eM_r} \tag{3.25}
  \]

  To sum up, in whichever case, from the estimates (3.24) and (3.25), we have

  \[
  |H_{k; j_1', j_2'}(t)| \lesssim 2^{3eM_r} + 2^{-eM_r} A(t). \tag{3.26}
  \]

- If \( j_2' \geq j_2 \).

  Recall (3.21). Note that, after changing coordinates \( x \rightarrow x + y \), we have

  \[
  |H_{k; j_1, j_2'}(t)| \lesssim 2^{5eM_r} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_k(y) f(s, x - y, u) f(s, x, v) \psi_{j_1'}(u_1, u_2) \psi_{j_2'}(u) dy dx du ds
  \]

  \[
  \lesssim 2^{5eM_r} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(s, x, u) \tilde{E}_{k; j_2, j_2'}(s, x) \psi_{j_1'}(u_1, u_2) \psi_{j_2'}(u) dx du ds,
  \]

  where

  \[
  \tilde{E}_{k; j_2, j_2'}(s, x) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_k(y) f(s, x + y, u) \psi_{j_1'}(u_1, u_2) \psi_{j_2'}(v) dy dv.
  \]

  Similar to the obtained estimates (3.23) and (3.24), the following estimate holds if \( k \leq 2j_1 - 2eM_r \),

  \[
  |H_{k; j_1, j_2'}(t)| \lesssim 2^{3eM_r} + 2^{k+eM_r} - 2j_1 + j_2' A(t) \lesssim 2^{3eM_r} + 2^{-eM_r} A(t). \tag{3.27}
  \]

  Note that, from the estimate of the kernel \( \tilde{K}_k(y) \) in (3.4) and the conservation law (1.3), we have

  \[
  |\tilde{E}_{k; j_2, j_2'}(s, x)| \lesssim 2^{-k+2j_1+j_2}.
  \]
From the above estimate, we have the following estimate holds if \( k \geq 2j_1 - 2eM_t \),

\[
|H_{k; j_1, j_2}^{j_1, j_2}(t)| \lesssim 2^{-k+2j_1+j_2+3eM_t-j_2'} \lesssim 2^{5eM_t}.
\]

To sum up, we have

\[
\left| \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} E(s, x) \cdot \nabla_v \left( (v_1, v_2) |\omega_{\mu}(x, v)| f(s, x, v) dxdvd\sigma \right) \right| \lesssim 2^{5eM_t} + 2^{-eM_t} A(t).
\]

Combining the above estimate with the estimates (3.17–3.19), we have

\[
A(t) \lesssim 2^{5eM_t} + 2^{-eM_t} A(t), \quad \implies \quad A(t) \lesssim 2^{5eM_t}.
\]

Hence finishing the proof of our desired estimate (3.9). The desired estimate (3.10) holds directly from the obtained estimates (3.9) and (3.23).

We end this section with some remarks about the above Proposition.

**Remark 3.1** The idea of exploiting spacetime type estimate is not new. From [6][Lemma 2], the following type of spacetime estimate was obtained via different approach,

\[
\int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(x_1v_2 - x_2v_1)^2}{|(x_1, x_2)|^3(|v)|} f(s, x, v) dxdvd\sigma \lesssim 1.
\]

Modulo the loss of \( \epsilon \) order, the obtained estimate (3.9) in Proposition 3.1 is an improvement in the sense that it provides strong control even when \((x_1, x_2)\) is parallel to \((v_1, v_2)\).

**Remark 3.2** An essential improvement upon the spacetime estimate, e.g., a stronger weight function in (3.9), helps to remove the vanishing order condition of initial data in (1.8). Despite many unsuccessful attempts, it’s not clear to the author whether the obtained estimate (3.9) is optimal or how to improve the spacetime estimate (3.9).

### 4 Control of the Majority Set in the Cylindrical Symmetry Case

As the obtained Proposition 2.1 for the radial symmetry setting, in this section, we mainly control the corresponding majority set for the cylindrical symmetric case by using the estimates for the electric field and the spacetime estimate of distribution function obtained in the previous section. Once again, throughout this section, \( f(t, x, v) \) is assumed to be cylindrical symmetric even without explicit statement.

First of all, as the definition of majority set in (2.2) for the radial case, we define a majority set of particles in the cylindrical symmetry case at time \( s \) as follows,

\[
R^{cyl}(t, s) := \{ (X(s; t, x, v), V(s; t, x, v)) : |X(0; t, x, v)| + |V(0; t, x, v)| \leq 2^{2M_t/2} \},
\]

where \( M_t \) is defined in (3.2).

Moreover, to measure the size of velocity characteristics \(|V(s; t, x, v)|\) with respect to the dyadic level \( \mathbb{M}_t \), we define

\[
\beta_t(x, v) := \sup_{s \in [0, t]} \inf_{k \in \mathbb{R}_+} \{ k \in \mathbb{R}_+ : |V(s; t, x, v)| \leq 2^{kM_t} \}, \quad \beta_t
\]

\[
:= \sup \{ \beta_t(x, v) : (x, v) \in R^{cyl}(t, t) \}.
\]

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As summarized in the following proposition, we estimate the upper bound of $\beta_t$ defined above by using a bootstrap argument with a divided plan. The core part of the proof, which is to control the accumulation effect of the electric field along the characteristics time, is divided into three cases, which are studied case by case from Lemmas 4.1 to 4.3.

**Proposition 4.1** Let $\epsilon := 10/n_c$. Under the cylindrical symmetry assumption, for any $t \in [0, T^*)$, s.t., $\mathcal{M}_t \gg 1$, we have

$$\beta_t \leq 1 - \epsilon/2.$$

**Proof** We aim to prove the following estimate

$$\sup_{s \in [0, t]} |V(s)| \leq 2(1-\epsilon)\beta^1 \mathcal{M}_t + 1, \quad \beta^1 := \max\{\beta_t, 1\}.$$  (4.4)

Define $\zeta$ be the first time that the speed of particle reaches a threshold as follows

$$\zeta := \sup\{s \in [0, t] : \forall \kappa \in [0, s], |V(\kappa)| \leq 2(1-\epsilon)\beta^1 \mathcal{M}_t\}.$$  (4.5)

From the continuity of the speed characteristics, we know that $\zeta \in (0, t)$. Since our desired estimate (4.4) holds straightforwardly if $\zeta = t$, we only have to consider the case $\zeta \in (0, t)$. Let

$$\zeta^* := \sup\{s : s \in [\zeta, t], \forall \kappa \in [\zeta, s], |V(\kappa)| \in 2(1-\epsilon)\beta^1 \mathcal{M}_t [99/100, 101/100]\}.$$  (4.6)

Note that, from (1.2), we have

$$\frac{d}{ds} |V(s)| = \tilde{V}(s) \cdot E(s, X(s)).$$

Recall the decomposition of the electric field in (3.3). From the estimate (4.7) in Lemma 4.1, the estimate (4.15) in Lemma 4.2, and the estimate (4.16) in Lemma 4.3, we know that for any $s \in [\zeta, t]$, we have

$$|V(s)| \leq |V(\zeta)| + 2(1-2\epsilon)\mathcal{M}_t \leq 2(1-\epsilon)\beta^1 \mathcal{M}_t + 2(1-2\epsilon)\mathcal{M}_t \leq (1 + 2^{-1000})(1-\epsilon)\beta^1 \mathcal{M}_t.$$  

Hence improving the bootstrap assumption, which implies that our desired estimate (4.4) is true. Recall the definition of $\beta_t$ in (4.2). We have

$$2\beta^1 \mathcal{M}_t \leq 2(1-\epsilon)\beta^1 \mathcal{M}_t + 1, \quad \Rightarrow \beta_t \mathcal{M}_t \leq (1 - \epsilon)\mathcal{M}_t + 10, \quad \Rightarrow \beta_t \leq (1 - \epsilon/2).$$

Hence finishing the proof of our desired estimate (4.3). \hfill \Box

After using the dyadic decomposition for the electric field defined in (3.3), in the following Lemma, we first rule out some trivial cases by using the rough estimates obtained for the localized electric field in Lemma 3.1 and the conservation of angular momentum, c.f., (4.13).

**Lemma 4.1** Let $\epsilon := 10/n_c$. Under the cylindrical symmetry assumption and with the notation introduced in Proposition 4.1, for any $s_1, s_2 \in [\zeta, \zeta^*] \subset [0, t]$, the following estimate holds,

$$\sum_{(k, j_1, j_2) \in \mathcal{B}^c} \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) ds \lesssim 2^{\epsilon(1-2\epsilon)} \mathcal{M}_t,$$  (4.7)

where the index set $\mathcal{B}$ is defined as follows,

$$\mathcal{B} := \{(k, j_1, j_2) : k, j_1, j_2 \in \mathbb{Z}_+, k \in [2j_1 - 4\epsilon \mathcal{M}_t, 2j_1 + 4\epsilon \mathcal{M}_t], j_2 \in [(1 - 5.5\epsilon)\mathcal{M}_t, (1 + \epsilon)\mathcal{M}_t], j_1 \geq 5\mathcal{M}_t/8 + 10\epsilon \mathcal{M}_t\}.$$  (4.8)
Proof. We split possible scenarios of \( (k, j_1, j_2) \in B^c \) into four cases as follows.

- If \( j_2 \geq (1 + \epsilon) M t \).

Recall (3.3). From the rough estimate of the localized electric field (3.5) in Lemma 3.1, we have

\[
\sum_{k \in Z_+, j_2 \geq (1 + \epsilon) M t} \left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) \, ds \right| \lesssim \sum_{k \in Z_+, j_2 \geq (1 + \epsilon) M t} \min\{2^{-k + 3j_2}, 2^{2k - nj_2} M^c(t)\}
\]

\[
\lesssim \sum_{k \in Z_+, j_2 \geq (1 + \epsilon) M t} \min\{2^{-k + 3j_2}, 2^{2k - 10j_2}\} \lesssim \sum_{k \in Z_+, j_2 \geq (1 + \epsilon) M t} 2^{-k/100} \lesssim 1. \tag{4.9}
\]

- If \( k \not\in [2j_1 - 4\epsilon M t, 2j_1 + 4\epsilon M t] \) and \( j_2 \leq (1 + \epsilon) M t \).

From the obtained estimate (3.23) and the rough estimate of the localized electric field (3.5) in Lemma 3.1, we have

\[
\sum_{k \in Z_+, k \not\in [2j_1 - 4\epsilon M t, 2j_1 + 4\epsilon M t]} \left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) \, ds \right| \lesssim 2^{\epsilon M t} \min\{2^{k - 2j_1 + j_2}, 2^{-k + 2j_1 + j_2}\} \lesssim 2^{j_2 - 3 \epsilon M t} \lesssim 2^{(1 - 2 \epsilon) M t}. \tag{4.10}
\]

- If \( k \in [2j_1 - 4\epsilon M t, 2j_1 + 4\epsilon M t], j_2 \leq (1 - 5.5 \epsilon) M t \).

From the estimate (3.10) in Proposition 3.1 and the rough estimate of the localized electric field (3.5) in Lemma 3.1, we have

\[
\left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) \, ds \right| \lesssim \int_{s_1}^{s_2} \| E_{k, j_1, j_2}(s, \cdot) \|_{L^2} \, ds \lesssim 2^{5\epsilon M t} + \min\{2^{k - 2j_1 + j_2 + 6\epsilon M t}, 2^{-k + 2j_1 + j_2 + \epsilon M t}\} \lesssim 2^{j_2 + 3.5 \epsilon M t} \lesssim 2^{(1 - 2 \epsilon) M t}. \tag{4.11}
\]

- If \( k \in [2j_1 - 4\epsilon M t, 2j_1 + 4\epsilon M t], j_2 \in [(1 - 5.5 \epsilon) M t, (1 + \epsilon) M t], j_1 \leq 5 M t / 8 + 10 \epsilon M t \).

We first rule out the case \(|X(s)|\) is relatively large. From the rough estimate (3.7) in Lemma 3.1, we have

\[
\left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) \, ds \right| \lesssim 2^{10 \epsilon M t} + 2^{-k - j_2 + 2\epsilon M t} 2^{2j_1 - 2j_1 - 15 \epsilon M t} \lesssim 2^{(1 - 3 \epsilon) M t}. \tag{4.12}
\]

Now, it remains to consider the case when \(|(X_1(s), X_2(s))|\) is relatively small. Let

\[
J(s) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x_1 v_2 - x_2 v_1|^{-13} \psi_{\geq -100 \epsilon M t} (x_1 v_2 - x_2 v_1) f(s, x, v) \, dx \, dv.
\]
From our assumption on the initial data, see (1.7), we know that $0 \leq J(0) \lesssim 1$. As a result of direct computation, we have

$$\frac{d}{ds}J(s) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x_1 v_2 - x_2 v_1|^{-13} \psi_{\geq -100 \mathfrak{M}_t} (x_1 v_2 - x_2 v_1) \partial_s f(s, x, v) dx dv = 0,$$

$$\implies 0 \leq J(s) = J(0) \lesssim 1. \quad (4.13)$$

From the above estimate of $J(s)$, we have

$$|E_{k, j_1, j_2}(s, X(s))| \psi_{< 2j_1 - 2\mathfrak{M}_t, +15e \mathfrak{M}_t} (X_1(s), X_2(s)) \lesssim 1,$$

$$\leq 1 + 2^{2k} \int_{\mathbb{R}} \int_{|v| \leq 2^{k+4} \mathfrak{M}_t /10} f(s, X(s) - y, v) \psi_{< 2j_1 - 2\mathfrak{M}_t, +15e \mathfrak{M}_t} (X_1(s), X_2(s)) \psi_{j_1} (v_1, v_2) \psi_{j_2} (v) dy dv$$

$$\leq 1 + 2^{2k} \int_{\mathbb{R}} \int_{|v| \leq 2^{k+4} \mathfrak{M}_t /10} f(s, X(s) - y, v) \psi_{< -100 \mathfrak{M}_t} (X_1(s) - y_1) v_2 - (X_2(s) - y_2) v_1$$

$$\times \psi_{< 2j_1 - 2\mathfrak{M}_t, +15e \mathfrak{M}_t} (X_1(s), X_2(s)) \psi_{j_1} (v_1, v_2) \psi_{j_2} (v) dy dv + 2^{2k} \int_{\mathbb{R}} \int_{|v| \leq 2^{k+4} \mathfrak{M}_t /10} f(s, X(s) - y, v)$$

$$\leq 1 + 2^{2k} \int_{\mathbb{R}} \int_{|v| \leq 2^{k+4} \mathfrak{M}_t /10} (X_1(s) - y_1) v_2 - (X_2(s) - y_2) v_1 f(s, X(s) - y, v) dy dv + 2^{2k+3j_2} 100 \mathfrak{M}_t + 10e \mathfrak{M}_t,$$

$$\leq 2^{2k+3j_2} - 100 \mathfrak{M}_t + 10e \mathfrak{M}_t \lesssim 2^{10} (10e) \mathfrak{M}_t. \quad (4.14)$$

Hence our desired estimate (4.7) holds from the above obtained estimates (4.9), (4.10), (4.12), and (4.14).

In the following Lemma, we rule out further the case when $|(X_1(s), X_2(s))|$ is relatively large and the case when $|(X_1(s), X_2(s))|$ is very small by using again the conservation law of angular momentum, c.f., (4.13).

**Lemma 4.2** Let $\epsilon := 10/n_c$. Under the cylindrical symmetry assumption and with the notation introduced in Proposition 4.1, for any $(k, j_1, j_2) \in B$, see (4.8), the following estimate holds for any $s_1, s_2 \in [\zeta, \zeta^*] \subset [0, t]$.

$$\left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) \left( \psi_{\geq 2j_1 - 2\mathfrak{M}_t + 15e \mathfrak{M}_t} (X_1(s), X_2(s)) \right) \right.$$

$$+ \psi_{\leq (17j_1) /13 - 2e \mathfrak{M}_t} (X_1(s), X_2(s))) ds \left| \lesssim 2^{1 - 3e} \mathfrak{M}_t. \quad (4.15)$$

**Proof** By using the same argument as in the obtained estimate (4.12), we have

$$\left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k, j_1, j_2}(s, X(s)) \psi_{\geq 2j_1 - 2\mathfrak{M}_t + 15e \mathfrak{M}_t} (X_1(s), X_2(s))) ds \right|$$

$$\lesssim 1 + 2^{k - j_2 + 2e \mathfrak{M}_t} 2^{2\mathfrak{M}_t} - 2j_1 - 15e \mathfrak{M}_t \lesssim 2^{1 - 3e} \mathfrak{M}_t.$$
It remains to consider the case \(|(X_1(s), X_2(s))| \lesssim 2^{(\sigma_1-17j_1)/13-2\epsilon}\). From the estimate (4.13), we have

\[
|E_{k,j_1,j_2}(s, X(s))| \psi_{\leq (\sigma_1-17j_1)/13-2\epsilon}\psi(X_1(s), X_2(s)) \\
\lesssim 1 + 2^{2k} \int_{\mathbb{R}^3} \int_{|y| \leq 2-k+\epsilon} f(s, X(s) - y, v) \psi_{\leq (\sigma_1-17j_1)/13-2\epsilon}\psi(X_1(s), X_2(s))dydv \\
\lesssim 1 + 2^{2k} (2^{-k+\epsilon})^{10} + 2^{(\sigma_1-17j_1)/13-2\epsilon})^{13} z^{13j_1} \\
\int_{\mathbb{R}^3} \int_{|y| \leq 2-k+\epsilon} |(X_1(s) - y_1)v_2 - (X_2(s) - y_2)v_1|^{13} \\
\times \psi_{\geq 100\epsilon}\psi((X_1(s) - y_1)v_2 - (X_2(s) - y_2)v_1)f(s, X(s) - y, v)dydv \\
+ 2^{2k+3j_2-100\epsilon+10\epsilon}\lesssim 2^{(1-\epsilon)\sigma_1}. 
\]

Hence finishing the proof of our desired estimate (4.15).

With the above preparation, in the following Lemma, we focus on the main case of interest, in which we know appropriate localization of frequency, \(|(v_1, v_2)|, |v|, \) and the distance to z-axis, i.e., \(|(X_1(s), X_2(s))|\). This is also the case related to the second observation we mentioned in Remark 1.1 about the smoothing effect. After further localization on the Fourier side, see the decomposition (4.17), for the piece in which the oscillation phase in time has a lower bound, see (4.30), we exploit the smoothing effect by doing integration by parts in time.

**Lemma 4.3** Let \(\epsilon := 10/n_c\). Under the cylindrical symmetry assumption and with the notation introduced in Proposition 4.1, for any \((k, j_1, j_2) \in B,\) see (4.8), the following estimate holds for any \(s_1, s_2 \in [\xi, \xi^*] \subseteq [0, t],\)

\[
\left| \int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k,j_1,j_2}(s, X(s)) \psi_{(\sigma_1-17j_1)/13-2\epsilon}\psi(X_1(s), X_2(s))ds \right| \\
\lesssim 2^{(1-\epsilon)\sigma_1}. 
\]  

**Proof** Let

\[
l_1 := j_1 - \sigma_1 - 5\epsilon, \quad \alpha = \frac{2}{3}l_1, \quad l_2 := \alpha - 20\epsilon, \\
\theta_{V(s)}(v) := (\tilde{V}(s) - \hat{v})/|\tilde{V}(s) - \hat{v}|.
\]

Based on the possible size of \(\theta_{V(s)}(v) \cdot \xi/|\xi|, |\tilde{V}(s) - \hat{v}|, \) and the angle between \((-X_2(s), X_1(s), 0)/|(X_1(s), X_2(s))|\) and \(V(s, v),\) we decompose the integral in (4.16) into three parts as follows,

\[
\int_{s_1}^{s_2} \tilde{V}(s) \cdot E_{k,j_1,j_2}(s, X(s)) \psi_{(\sigma_1-17j_1)/13-2\epsilon}\psi(X_1(s), X_2(s))ds \\
= \sum_{i=1,2,3} \int_{s_1}^{s_2} H_{k,j_1,j_2}^i(s)ds, 
\]  

where

\[
H_{k,j_1,j_2}^i(s) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix(s,\xi - iv\hat{v} \cdot \xi)} i\tilde{V}(s) \cdot \xi |\xi|^{-2} \tilde{g}(s, \xi, v) \\
\varphi_{j_1}(\xi) \varphi_{j_2}(v) \varphi_{k}(\xi) \varphi_{l_1,\alpha}(v, X(s), \tilde{V}(s)) \\
\times \psi_{(\sigma_1-17j_1)/13-2\epsilon}\psi_{2j_2-10\epsilon}(X_1(s), X_2(s))dv d\xi, 
\]  

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where the orthonormal frame $\psi_{\alpha}(v, \tilde{V}(t))$ is defined as follows,

$$
\psi_{\alpha}(v, X(s), \tilde{V}(s)) := \psi_{l_1}(|\tilde{v} - \tilde{V}(s)|) + \psi_{\geq l_1}(|\tilde{v} - \tilde{V}(s)|) \psi_{\leq \alpha}(\theta V(s)(v) \times (-X_2(s), X_1(s), 0) / ((X_1(s), X_2(s)))).
$$

(4.21)

The estimate of $H_{k,j_1,j_2}^1(s)$. Note that, in terms of kernel, we have

$$
H_{k,j_1,j_2}^1(s) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{V}(s) \cdot \tilde{K}_k(y) f(s, x - y, v) \varphi_{j_1}(\psi_2(v))(\varphi_{j_2}(v) \psi_{\alpha}(v, X(s), \tilde{V}(s)))dudy,
$$

(4.22)

where the kernel $\tilde{K}_k(y)$ was defined in (3.3). Note that, for any fixed $a, b \in \mathbb{S}^2$, $a \neq b$, $0 < \epsilon \ll 1$, we have

$$
S_{a,b} := \{c : c \in \mathbb{S}^2, |(a - c) \times b| \leq \epsilon\}, \quad |S_{a,b}| \lesssim \epsilon^{3/2}.
$$

The above claim follows from the following argument. Let $e_1 := b$, $e_2 := (a - (b \cdot a)b) / |a - (b \cdot a)b|$, $e_3 = e_1 \times e_2$. Then $\{e_1, e_2, e_3\}$ is an orthonormal frame. In terms of the orthonormal frame, we have

$$
a = a_1 e_1 + a_2 e_2, \quad c = \cos \theta \cos \phi e_1 + \cos \theta \sin \phi e_2 + \sin \theta e_3,
$$

$$
\theta \in [-\pi/2, \pi/2], \phi \in [0, 2\pi],
$$

$$
(a - c) \times b = -(a_2 - \cos \theta \sin \phi)e_3 + \sin \theta e_2,
$$

$$
|(a - c) \times b| \leq \epsilon \implies |\sin \theta| \leq \epsilon, \quad |a_2 - \sin \phi| \lesssim \epsilon,
$$

$$
|a - c| \lesssim |a_1 - \cos \phi| + |a_2 - \sin \phi| + \epsilon \lesssim \epsilon^{1/2},
$$

$$
\Rightarrow \left|\{(\theta, \phi) : \theta \in [-\pi/2, \pi/2], \phi \in [0, 2\pi], |\sin \theta| \leq \epsilon, |a_2 - \sin \phi| \lesssim \epsilon\}\right| \lesssim \epsilon^{3/2}.
$$

(4.23)

If $v \in supp_\nu(\psi_{\leq \alpha}(\theta V(s)(v) \times (-X_2(s), X_1(s), 0) / ((X_1(s), X_2(s)))) \psi_n(|\tilde{v} - \tilde{V}(s)|))$, $n \in [1, 2] \cap \mathbb{Z}$, after letting $\epsilon = 2^{n+\alpha}$, the following estimate holds from estimates in (4.23),

$$
Vol(supp_\nu(\psi_{\leq \alpha}(\theta V(s)(v) \times (-X_2(s), X_1(s), 0) / ((X_1(s), X_2(s)))) \psi_n(|\tilde{v} - \tilde{V}(s)|)) \varphi_{j_2}(v))) \lesssim 2^{3j_2} 2^{(n+\alpha)/2}, |\tilde{v} - \tilde{v}| \leq |v|^{-2}, \quad \Rightarrow 2^{n-2} \leq |	ilde{V}(s) - \tilde{v}| + 2^{-2M_1 + 3eM_1} \lesssim 2^{(n+\alpha)/2} + 2^{-2M_1 + 3eM_1},
$$

$$
\Rightarrow n \leq \max\{\alpha, -2M_1 + 2eM_1\} + C = \alpha + C,
$$

(4.24)

where $C$ is some absolute constant.

From the above estimate, the estimate of the kernel $\tilde{K}_k(y)$ in (3.4), and the volume of support of $v$, we have

$$
|H_{k,j_1,j_2}^1(s)| \lesssim 2^{eM_1}(2^{-k+3j_2+2j_1} + 2^{-k+3j_2+3\alpha}) \lesssim 2^{(1-3e)M_1}.
$$

(4.25)
The estimate of $H^2_{k,j_1,j_2}(s)$. Note that, in terms of kernel, we have

$$H^2_{k,j_1,j_2}(s) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(s, X(s) - y, v) \tilde{K}_{k,j_2}(y, v, V(s))(1 - \varphi_{l,\alpha}(v, X(s), \tilde{V}(s)))$$

\[
\times \psi_{[(\mathcal{M}_t - 17j_1)/13 - 2\mathcal{M}_t, 2j_1 - (2 - 15\epsilon)\mathcal{M}_t]}(X_1(s), X_2(s))\varphi_{j_1}(\theta)\varphi_{j_2}(v)dv dy, \quad (4.26)
\]

where

$$\tilde{K}_{k,j_2}(y, v, V(s)) = \int_{\mathbb{R}^3} e^{i\gamma \xi} i \xi |\xi|^{-2} \varphi_k(\xi) \psi_{\leq l_2}(\theta V(s)(v) \cdot \xi) d\xi. $$

By doing integration by parts in $\theta V(s)(v)$ direction and $(\theta V(s)(v))^\perp$ directions, the following estimate holds for the kernel $\tilde{K}_{k,j_2}(y, v, V(s))$,

$$|\tilde{K}_{k,j_2}(y, v, V(s))| \lesssim 2^{k+l_2}(1 + 2^{k+l_2}|y \cdot \theta V(s)(v)|)^{-N_0^3} (1 + 2^k|y \times \theta V(s)(v)|)^{-N_0^3}. $$

(4.27)

From the above estimate, we know that “$y$” is localized inside a cylinder with base in the plane perpendicular to $\theta V(s)(v)$. Due to the cutoff function $(1 - \varphi_{l,\alpha}(v, X(s), \tilde{V}(s)))$ in (4.26), the angle between $-(X_2(s), X_1(s), 0)/|\mathcal{X}(X_1(s), X_2(s))|$ and $\theta V(s)(v)$ is greater than $2^\alpha$, which means that the intersection of the cylinder with any $x_1x_2$ plane is less than $(2^{-k-\alpha})^2$.

Moreover, note that, for the case we are considering, we have

$$|(X_1(s), X_2(s))| \geq 2^{(\mathcal{M}_t - 17j_1)/13 - 2\mathcal{M}_t} \geq 2^{-2j_1 - l_2 + 10\epsilon\mathcal{M}_t} \geq 2^{-k-l_2 + 5\epsilon\mathcal{M}_t}.$$  

(4.28)

Therefore, from the cylindrical symmetry of solution and the estimate (3.9) in Proposition 3.1, we have

\[
\int_{j_1}^{t_2} \left|H^2_{k,j_1,j_2}(s)\right|ds \lesssim 2^{k+l_2} \int_{j_1}^{t_2} \left|\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + 2^{k+l_2}|y \cdot \theta V(s)(v)|)^{-N_0^3} (1 + 2^k|y \times \theta V(s)(v)|)^{-N_0^3} f(s, X(s) - y, v)\varphi_{j_1}(v_1, v_2)\varphi_{j_2}(v)(1 - \varphi_{l,\alpha}(v, X(s), \tilde{V}(s)))
\psi_{[(\mathcal{M}_t - 17j_1)/13 - 2\mathcal{M}_t, 2j_1 - (2 - 15\epsilon)\mathcal{M}_t]}(X_1(s), X_2(s))dv dy ds \lesssim 1 + \int_{j_1}^{t_2} \int_{\mathbb{R}^3} \int_{|y| \leq 2^{-k-l_2 + \epsilon\mathcal{M}_t}/2} 2^{k+l_2-\delta \alpha + 6\epsilon\mathcal{M}_t/10} |X_1(s) - y_1, X_2(s) - y_2|^{2} \psi_{[(\mathcal{M}_t - 17j_1)/13 - \mathcal{M}_t, 2j_1 - (2 - 15\epsilon)\mathcal{M}_t]}(X_1(s), X_2(s))\varphi_{j_1}(u_1, u_2)\varphi_{j_2}(v)f(s, X(s) - y, v)dv dy ds \lesssim 1 + 2^{2k+l_2-\delta \alpha + 6\mathcal{M}_t/2.2}j_2 + j_1 \lesssim 2^{1-\mathcal{M}_t/2.2}j_2. (4.29)
\]

The estimate of $H^3_{k,j_1,j_2}(s)$. Recall (4.20). Note that, for any $(v, \xi) \in \text{supp}(\psi_{\leq l_2}(\theta V(s)(v) \cdot \xi)\varphi_{j_1}(v, X(s), \tilde{V}(s)))$, we have

\[
|\tilde{V}(s) - \tilde{v}| \geq |\tilde{V}(s) - \tilde{v}| - 2^{-2\mathcal{M}_t + 4\epsilon\mathcal{M}_t} \geq 2^j, \quad \Rightarrow |\tilde{V}(t) \cdot \xi - \tilde{v} \cdot \xi| = |\tilde{V}(t) - \tilde{v}|\theta_{V(t)}(v) \cdot \xi | \geq 2^{j+l_2}. \quad (4.30)
\]
Let $g(s, x, v) := f(s, x + s\hat{v}, v)$. For $H^3_{k,j_1,j_2}(s)$, we do integration by parts in time once. As a result, we have

\[
\int_{s_1}^{s_2} H^3_{k,j_1,j_2}(s)\,ds = \text{End}_{k,j_1,j_2}(s_1, s_2) + \tilde{H}^1_{k,j_1,j_2}(s_1, s_2) + \tilde{H}^2_{k,j_1,j_2}(s_1, s_2),
\]

(4.31)

where

\[
\text{End}_{k,j_1,j_2}(s_1, s_2) := \sum_{a=1,2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{iX(s_a)\cdot\xi - is_a\hat{v} \cdot \xi} i\hat{V}(s_a) \cdot \xi |\xi|^{-2} \tilde{g}(s_a, \xi, v)\psi_k(\xi)
\]

\[
\times [\varphi_{j_1}(\hat{\psi})\varphi_{j_2}(v)(\tilde{V}(s_a) \cdot \xi - \hat{v} \cdot \xi)^{-1}\psi_{>l_2}(\theta_{V(s_a)}(v) \cdot \tilde{\xi})(1 - \varphi_{l_1,a}(v, X(s_a), \tilde{V}(s_a)))
\]

\[
\times \psi_{|1\mathfrak{m}_l - 17j_1|/13 - 2e\mathfrak{m}_l,2j_1,(2 - 15e)\mathfrak{m}_l}](X_1(s_a), X_2(s_a))dud\xi
\]

\[
\times \sum_{a=1,2} \sum_{n\in[1,2]\cap Z} \sum_{l\in[1,2]\cap Z} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(s, x, X(s_a) - y, v)\tilde{K}^0_{k,n,l}(y, X(s_a), V(s_a), v)|\varphi_{j_1}(v_1, v_2)\varphi_{j_2}(v)dydv,
\]

(4.32)

\[
\tilde{H}^1_{k,j_1,j_2}(s_1, s_2) := \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{iX(s)\cdot\xi - is\hat{v} \cdot \xi} i\hat{V}(s) \cdot \xi |\xi|^{-2} \tilde{g}(s, \xi, v)
\]

\[
\times [\varphi_{j_1}(\hat{\psi})\varphi_{j_2}(v)\partial_s \tilde{g}(s, \xi, v)(1 - \varphi_{l_1,a}(v, X(s), \tilde{V}(s)))]
\]

\[
\times \psi_{|1\mathfrak{m}_l - 17j_1|/13 - 2e\mathfrak{m}_l,2j_1,(2 - 15e)\mathfrak{m}_l}](X_1(t), X_2(t))\tilde{V}(s) \cdot \xi - \hat{v} \cdot \xi)^{-1}\psi_{>l_2}(\theta_{V(s)}(v) \cdot \tilde{\xi})dud\xi ds,
\]

(4.33)

\[
\tilde{H}^2_{k,j_1,j_2}(s_1, s_2) := \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{iX(s)\cdot\xi} \partial_s (i\hat{V}(s) \cdot \xi |\xi|^{-2} \psi_{>l_2}(\theta_{V(s)}(v) \cdot \tilde{\xi})
\]

\[
\times \psi_{|1\mathfrak{m}_l - 17j_1|/13 - 2e\mathfrak{m}_l,2j_1,(2 - 15e)\mathfrak{m}_l}](X_1(t), X_2(t))\tilde{V}(s) \cdot \xi - \hat{v} \cdot \xi)^{-1}(1 - \varphi_{l_1,a}(v, X(s), \tilde{V}(s)))\varphi_k(\xi)|\varphi_{j_1}
\]

\[
\times (v_1, v_2)\varphi_{j_2}(v)\tilde{f}(s, \xi, v)dydvds,
\]

(4.34)

where the kernels $\tilde{K}^i_{k,n,l}(y, X(s_a), V(s_a), v), i \in \{0, 1\}$, are defined as follow,

\[
\tilde{K}^0_{k,n,l}(y, X(s_a), V(s_a), v) := \int_{\mathbb{R}^3} e^{iy \cdot \xi} i\hat{V}(s_a) \cdot \xi |\xi|^{-2} \varphi_k(\xi)(\tilde{V}(s_a) \cdot \xi - \hat{v} \cdot \xi)^{-1}
\]

\[
\times \varphi_{l_1,a}(v, X(s_a), \tilde{V}(s_a))\psi_{n}(|\tilde{v} - \tilde{V}(s_a)|)
\]

\[
\times \psi_{|1\mathfrak{m}_l - 17j_1|/13 - 2e\mathfrak{m}_l,2j_1,(2 - 15e)\mathfrak{m}_l}](X_1(s_a), X_2(s_a))(1 - \varphi_{l_1,a}(v, X(s_a), \tilde{V}(s_a)))d\xi,
\]

(4.35)

\[
\tilde{K}^1_{k,n,l}(y, X(s), V(s), v)
\]

\[
:= \int_{\mathbb{R}^3} e^{iy \cdot \xi} \partial_s (i\hat{V}(s) \cdot \xi |\xi|^{-2} \psi_{>l_2}(\theta_{V(s)}(v) \cdot \tilde{\xi}))(\tilde{V}(s) \cdot \xi - \hat{v} \cdot \xi)^{-1}
\]

\[
\times \psi_{l_1}(\theta_{V(s)}(v) \cdot \tilde{\xi})\psi_n(|\tilde{v} - \tilde{V}(s)|)(1 - \varphi_{l_1,a}(v, X(s), \tilde{V}(s))))\varphi_k(\xi)d\xi.
\]

(4.36)
• The estimate of $End_{k,j_1,j_2}(t_1,t_2)$.

Recall (4.32). Note that, by using the estimate (4.30) and doing integration by parts in $\theta V(t_0)(v)$ direction and $(\theta V(s_a)(v))^{\perp}$ directions, the following estimate holds for the kernel $\tilde{K}^0_{k,l_1,l_2}(y, V(s_a), v)$:

$$ |\tilde{K}^0_{k,n,l}(y, X(s_a), V(s_a), v)| \lesssim 2^{k+l-n} \psi(|(2\omega_l - 17j_1)/13 - 2\omega_l, 2j_1 - (2 - 15\epsilon)\omega_l|)(X_1(s_a), X_2(s_a)) $$

$$ \times (1 + 2^{k+l}|y \cdot \theta V(s_a)(v)|)^{-N_3^3}(1 + 2^k|y \times \theta V(s_a)(v)|)^{-N_3^3}. $$

From the above estimate, the estimate (4.28), and the cylindrical symmetry of the distribution function, we have

$$ |End_{k,j_1,j_2}(s_1, s_2)| \lesssim \sum_{a=1,2} \sum_{n \in [1,2]} \sum_{l \in [l_1,2]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(s_a, X(s_a) - y, v) 2^{k-n} $$

$$ (1 + 2^{k+l}|y \cdot \theta V(s_a)(v)|)^{-N_3^3}(1 + 2^k|y \times \theta V(s_a)(v)|)^{-N_3^3} \varphi_{j_1}(v) \varphi_{j_2}(v) $$

$$ \psi(|(2\omega_l - 17j_1)/13 - 2\omega_l, 2j_1 - (2 - 15\epsilon)\omega_l|)(X_1(s_a), X_2(s_a)) dy dv $$

$$ \lesssim 2^{2\epsilon \omega_l} + 2^{k-l_1-k-\alpha+4\epsilon \omega_l} 2^{-j_2 - 2^{-(2j_1 - 17j_1)/13}} $$

$$ \lesssim 2(1-10\epsilon)\omega_l. $$

(4.38)

• The estimate of $\tilde{H}^1_{k,j_1,j_2}(s_1, s_2)$.

Recall (4.33) and (1.1). Note that,

$$ \partial_s g(s, x, v) = \nabla_x \phi(s, x + s\hat{v}) \cdot \nabla_v f(s, x + s\hat{v}), v). $$

Hence, after doing integration by parts in $v$, we have

$$ \tilde{H}^1_{k,j_1,j_2}(s_1, s_2) := \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{iX(s)v} \psi(|(2\omega_l - 17j_1)/13 - 2\omega_l, 2j_1 - (2 - 15\epsilon)\omega_l|) $$

$$ \times (X_1(s), X_2(s)) \hat{V}(s) \cdot \xi |\xi|^2 \varphi_{j_1}(\xi) \hat{f}(s, \xi - \eta, v) \nabla_x \phi(s, \eta) $$

$$ \cdot \nabla_v (\varphi_{j_1}(v_1, v_2) \varphi_{j_2}(v)(1 - \varphi_{j_1,\alpha}(v, X(s), \hat{V}(s))) $$

$$ \times \varphi_{j_2}(\theta V(s)(v) \cdot \hat{\xi}) \varphi_{j_1}(|\xi - \hat{V}(s)|) \hat{V}(s) \cdot \xi - \hat{v} \cdot \xi)^{-1} dy dv d\eta d\xi ds $$

$$ = \sum_{n \in [1,2]} \sum_{l \in [l_1,2]} \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{K}^2_{k,n,l}(y, V(s), v) \cdot \nabla_x \phi(s, X(s) - y) $$

$$ \times f(s, X(s) - y, v) dy dv ds, $$

(4.39)

where the kernel $\tilde{K}^2_{k,n,l}(y, V(s), v)$ is defined as follows,

$$ \tilde{K}^2_{k,n,l}(y, V(s), v) := \int_{\mathbb{R}^3} e^{iX(s)v} \nabla_v (\varphi_{j_1}(v_1, v_2) \varphi_{j_2}(v)(1 - \varphi_{j_1,\alpha}(v, X(s), \hat{V}(s))) $$

$$ \hat{V}(s) \cdot \xi - \hat{v} \cdot \xi)^{-1} \psi(|\xi - \hat{V}(s)|) \hat{V}(s) \cdot \xi |\xi|^2 \varphi_{j_1}(\xi) $$

$$ \psi(|(2\omega_l - 17j_1)/13 - 2\omega_l, 2j_1 - (2 - 15\epsilon)\omega_l|)(X_1(s), X_2(s)) d\xi. $$

(4.40)

By doing integration by parts in $\theta V(s)(v)$ direction and $(\theta V(s)(v))^{\perp}$ directions, as a result of direct computation, we have

$$ |\tilde{K}^2_{k,n,l}(y, V(s), v)| \lesssim 2^{k+l} \left(2^{-2l-2n-j_2} + 2^{-l-n-j_2}\right)(1 + 2^{k+l}|y \cdot \theta V(s)(v)|)^{-N_3^3} $$

$$ \times (1 + 2^k|y \times \theta V(s)(v)|)^{-N_3^3}. $$

(4.41)
Moreover, from the estimate (3.10) in Proposition 3.1, and the rough estimate (3.5) in Lemma 3.1, we have

\[
\int_{s_1}^{s_2} \| \nabla_x \phi(s, \cdot) \|_{L^\infty_x} \, ds \\
\lesssim \sum_{k, j_1, j_2 \in \mathbb{Z}, j_1, j_2 \in \{0, j_2 + 2\} \cap \mathbb{Z}} \min\{2^{-k + 2j_1 + j_2}, 2^{2k - j_2}, 2^{2k - n_j} M^c(t), 2^{2e M_j} + 2^{k - 2j_1 + j_2 + 6e M_j} \} \\
\lesssim 2^{(1 + 4e) M_j}.
\]  

(4.42)

From the above estimate, the estimate of kernel in (4.41), the estimate (4.28), and the cylindrical symmetry of the distribution function, we have

\[
\bigg| \tilde{H}^1_{k, j_1, j_2}(s_1, s_2) \bigg| \\
\lesssim \sum_{n \in \{1, 2\} \cap \mathbb{Z}, l \in \{1, 2\} \cap \mathbb{Z}} \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \| \nabla_x \phi(s, \cdot) \|_{L^\infty_x} \\
\frac{2^{k+l-k-\alpha+e M_j/10}}{|(X_1(t) - y_1, X_2(t) - y_2)|} (2^{-2l-2n-j_2} + 2^{-l-n-j_1}) \\
\times (1 + 2^{k+1} |y \cdot \theta(V(s))|)^{-N_3} (1 + 2^k |y \times \theta(V(s))|)^{-N_3} \psi_{j_1}(v_1, v_2) \psi_{j_2}(v) \\
\times f(s, X(s) - y, v) \psi_{l}(\mathbb{M}_j - 17 j_1)/13 - 2e M_j, 2j_1 - (2 - 15e) M_j)(X_1(s), X_2(s)) dy dv ds \\
\lesssim \int_{s_1}^{s_2} 2^{-j_2 + 4e M_j} (2^{-l_1 - \alpha - j_1} + 2^{-l_2 - 2l_1 - \alpha - j_2}) 2^{-\mathbb{M}_j - 17 j_1)/13} \| \nabla_x \phi(s, \cdot) \|_{L^\infty_x} \, ds \\
\lesssim 2^{15e M_j} (2^{-l_1 - \alpha - j_1} + 2^{-l_2 - 2l_1 - \alpha - j_2}) 2^{-\mathbb{M}_j - 17 j_1})/13 \lesssim 2^{(1 - 3e) M_j}.
\]  

(4.43)

- The estimate of \( \tilde{H}^2_{k, j_1, j_2}(s_1, s_2) \).

Recall (4.34) and the definition of kernel \( \tilde{K}^1_{k,n,l}(y, V(s), v) \) in (4.36). As a result of direct computations, the following estimate holds for the kernel after doing integration by parts in \( \xi \) in \( \theta(V(s)) \) direction and \( (\theta(V(s)) \) direction,

\[
\bigg| \tilde{K}^1_{k,n,l}(y, V(s), v) \bigg| \\
\lesssim 2^{k+l-n} \frac{2^{-l-n} (1 - e) M_j}{\nabla_x \phi(s, X(s))} 2^{-a-(1-e) M_j} \psi_{n}(\tilde{V} - V(s))(1 + 2^{k+1} |y \cdot \theta(V(s))|)^{-N_3} (1 + 2^k |y \times \theta(V(s))|)^{-N_3} \\
\psi_{l}(\mathbb{M}_j - 17 j_1)/13 - 2e M_j, 2j_1 - (2 - 15e) M_j)(X_1(s), X_2(s)) \\
\lesssim 2^{k-n+6e M_j} (2^{-l-n} M_j \nabla_x \phi(s, X(s)) + 2^{-a+n} (1 - 17 j_1 - \mathbb{M}_j)) 2^{-\mathbb{M}_j - 17 j_1)/13} (1 + 2^{k+1} |y \cdot \theta(V(s))|)^{-N_3} \\
\times (1 + 2^k |y \times \theta(V(s))|)^{-N_3} \psi_{l}(\mathbb{M}_j - 17 j_1)/13 - 2e M_j, 2j_1 - (2 - 15e) M_j)(X_1(s), X_2(s)).
\]

From the above estimate of kernel, the estimate (4.28), the cylindrical symmetry of the distribution function, the obtained estimate (4.42), and the estimate (3.9) in Proposition 3.1,
we have

\[
\left| \tilde{H}^2_{k,j_1,j_2}(s_1, s_2) \right| \\ \lesssim \sum_{n \in [1,2]} \sum_{l \in [1,2]} 2^{k-n+13\epsilon} M_l 2^{-k-\alpha} 2^{(17j_1-M_l)/13} \times (2^{(1+6\epsilon)M_l} 2^{-l-n-M_l} 2^{-j_2} + 2^{-\alpha+n} 2^{-2j_1+j_2}) \\ \lesssim 2^{-2l_1-l_2-\alpha+30\epsilon} M_l 2^{-j_2+(17j_1-M_l)/13} + 2^{-2\alpha} 2^{-2j_1+j_2+(17j_1-M_l)/13+30\epsilon} M_l \\ \lesssim 2^{(1-3\epsilon)M_l},
\]

(4.44)

Recall the decomposition (4.31), from the estimates (4.38), (4.38), and (4.38), we have

\[
\left| \int_{s_1}^{s_2} H^3_{k,j_1,j_2}(s) ds \right| \lesssim 2^{(1-3\epsilon)M_l}.
\]

Recall the decomposition (4.17). Our desired estimate (4.16) holds from the above estimate and the estimates (4.25) and (4.29). \(\square\)

5 Proof of Main Theorems

As summarized in Proposition 2.1 for the radial symmetry case and Proposition 4.1 for the cylindrical symmetry case, now we have good control of the majority set. It enables us to show the sub-linearity of the high momentum. The proof of Theorem 1.1 and Theorem 1.2 will follow in the exactly same sprite. For the sake of readers, we still provide detailed proof for both cases here.

5.1 Proof of Theorem 1.1

Let \(\delta\) be the small constant used in Lemma 2.2. Based on the possible size of \(|v|\), we decompose the \(n_r\)-th moment into two parts as follows,

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_r} |f(t, x, v)| dx dv \\
= \int_{\mathbb{R}^3} \int_{|x| \geq (M' (t))^{(5+3\delta)/(6-2\delta) (n_r-1)}} (1 + |v|)^{n_r} |f(t, x, v)| dx dv, \\
+ \int_{\mathbb{R}^3} \int_{|x| \leq (M' (t))^{(5+3\delta)/(6-2\delta) (n_r-1)}} (1 + |v|)^{n_r} |f(t, x, v)| dx dv.
\]

Note that, from the relation (2.11) in Proposition 2.1, we have \(|X(0; t, x, v)| + |V(0; t, x, v)| \gtrsim (M' (t))^{1/(2n_r)}\) if either \(|x| \gtrsim (M' (t))^{(1+\delta)/(2n_r)}\) or \(|v| \gtrsim (M' (t))^{(5+3\delta)/(6-2\delta) (n_r-1)}\). If \(|x| \gtrsim (M' (t))^{(1+\delta)/(2n_r)}\), then we have

\[
|X(0; t, x, v)| \geq |x| - t \geq |x| - (M' (t))^{1/(2n_r)} \gtrsim (1 + |x|).
\]

Therefore, from the above estimate and the assumption on the initial data in (1.7), the following estimate holds if \(|v| \gtrsim (M' (t))^{(5+3\delta)/(6-2\delta) (n_r-1)}\) regardless the size of \(|x|\),

\[
|f(t, x, v)| = |f_0(X(0; t, x, v), V(0; t, x, v))| \leq (M' (t))^{-4} (1 + |x|)^{-4}. \tag{5.1}
\]
Moreover, if $|v| \gtrsim (M^r(t))^{5/(2n_r)}$, from the Eq. (2.12) and the estimate (2.5), we have

$$|V(0; t, x, v)| \gtrsim |v| - \int_0^t \left(1 + (M^r(t))^{(5+\delta)/(3-\delta)(n_r-1)}\right) ds \gtrsim (1 + |v|).$$

From the above two estimates and the assumption on the initial data in (1.7), the following estimate holds if $|v| \gtrsim (M^r(t))^{(5+\delta)/(6-2\delta)(n_r-1)}$ regardless the size of $|x|$,

$$|f(t, x, v)| = |f_0(X(0; t, x, v), V(0; t, x, v))| \lesssim (1 + |x|)^{-\delta}(1 + |v|)^{-n_r-\delta}.$$ 

Therefore, from the above estimate, we have

$$\int_{\mathbb{R}^3} \int_{|v| \geq (M^r(t))^{(5+\delta)/(6-2\delta)(n_r-1)}} (1 + |v|)^{n_r} |f(t, x, v)| dx dv \lesssim (M^r(t))^{(5+\delta)n_r/((6-2\delta)(n_r-1))}.$$ 

From the conservation law (1.3), we have

$$\int_{\mathbb{R}^3} \int_{|v| \leq (M^r(t))^{(5+\delta)/(6-2\delta)(n_r-1)}} (1 + |v|)^{n_r} |f(t, x, v)| dx dv \lesssim (M^r(t))^{(5+\delta)n_r/((6-2\delta)(n_r-1))}.$$ 

To sum up, we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_r} |f(t, x, v)| dx dv \lesssim (M^r(t))^{(5+\delta)n_r/((6-2\delta)(n_r-1))}.$$ 

Since the above estimate holds for any $t \in [0, T^*)$ and $M^r(t)$ is an increasing function with respect to $t$, the following estimate holds for any $s \in [0, t]$,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_r} |f(s, x, v)| dx dv \lesssim (M^r(s))^{(5+\delta)n_r/((6-2\delta)(n_r-1))}$$

$$\leq (M^r(t))^{(5+\delta)n_r/((6-2\delta)(n_r-1))}.$$ 

Hence

$$M^r(t) = \sup_{s \in [0, t]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_r} |f(s, x, v)| dx dv + (1 + t)^{2n_r}$$

$$\lesssim (M^r(t))^{(5+\delta)n_r/((6-2\delta)(n_r-1))} + (1 + t)^{2n_r}, \quad \implies M^r(t) \lesssim (1 + t)^{2n_r}. \quad (5.2)$$

Therefore, from the estimate (2.5) in Lemma 2.2, we have

$$\|\nabla_x \phi(t, x)\|_{L^\infty_{x}} \lesssim (1 + t)^{(2+\delta)n_r/((3-\delta)(n_r-1))}.$$ 

We have shown the desired fact that $\nabla_x \phi \in L^\infty([0, T^*) \times \mathbb{R}^3)$. Hence finishing the proof of Theorem 1.1. \(\square\)

### 5.2 Proof of Theorem 1.2

From the conservation law (1.3), we have

$$\int_{\mathbb{R}^3} \int_{|v| \leq 2^{(1-\epsilon/3)n_r}} (1 + |v|)^{n_c} f(t, x, v) dx dv \lesssim 2^{(n_c-1)(1-\epsilon/3)n_r} \lesssim (M^c(t))^{1-\epsilon/3}, \quad (5.3)$$

where $\epsilon := 10/n_c$.

Recall the definition of the majority set $R^\text{cvf}(t, t)$ in (4.1) and the definition of $\beta_t$ in (1.1). From the estimate (4.3) in Proposition 4.1, we know that $|X(0; t, x, v)| + |V(0; t, x, v)| \geq \ldots$
Moreover, recall (1.2), from the $L_\infty$-type estimate of electric field in (3.6), we have
\[
|v| - |V(0; t, x, v)| \lesssim 2^{5M/r_i/3+2\epsilon M_i}, \quad \Rightarrow |v| \sim |V(0; t, x, v)|, \quad \text{if } |v| \gtrsim 2^{5M/r_i},
\]
(5.5)
\[
|x| - |X(0; t, x, v)| \lesssim 2^{\epsilon M_i}, \quad \Rightarrow |x| \sim |X(0; t, x, v)|, \quad \text{if } |x| \gtrsim 2^{2\epsilon M_i}.
\]
(5.6)

To sum up, after combining the above estimates (5.4–5.6), the following estimate holds if $|v| \gtrsim 2^{(1-\epsilon/3)M_i}$,
\[
|f(t, x, v)| = |f_0(X(0; t, x, v), V(0; t, x, v))| \lesssim (1 + |x|)^{-4}(1 + |v|)^{-n_c-4}.
\]
From the above estimate, we have
\[
\left| \int_{\mathbb{R}^3} \int_{|v| \gtrsim 2^{(1-\epsilon/3)M_i}} (1 + |v|)^{n_c} f(t, x, v) dx dv \right| \lesssim 1.
\]
(5.7)

Therefore, recall (3.1), from the estimates (5.3) and (5.7), we know that the following estimate holds for any $t \in [0, T)$,
\[
\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_c} f(t, x, v) dx dv \right| \lesssim \left( M^c(t) \right)^{1-\epsilon/3}.
\]
(5.8)

From the above estimate and the fact that $M^c(t)$ is an increasing function with respect to $t$, the following estimate holds for any $s \in [0, t]$,
\[
M^c(t) = \sup_{s \in [0, t]} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |v|)^{n_c} f(s, x, v) dx dv \right| + (1 + t)^{n_c^2} \lesssim \left( M^c(t) \right)^{1-\epsilon/3}
\]
\[
+ (1 + t)^{n_c^2}, \quad \Rightarrow M^c(t) \lesssim (1 + t)^{n_c^2}.
\]
(5.9)

From the $L_\infty$-type estimate of the electric field in (3.6), we know that the desired fact that $\nabla_x \phi \in L_\infty([0, T^*) \times \mathbb{R}^3)$ holds. Hence finishing the proof of Theorem 1.2.

We end the final section by giving a remark on the growth rate of the electric field $\nabla \phi$.

**Remark 5.1** Again, since our goal is to show global existence, it only requires us to show that $\|\nabla \phi(t, \cdot)\|_{L_\infty}$ doesn’t blow up in finite time. From the estimate of momentun, see (5.2) for the radial symmetry case and see (5.9) for the cylindrical symmetry case, and the rough estimate of electric field (2.5) for the radial symmetry case and (3.6) for the cylindrical symmetry case, as a byproduct, we know that $\|\nabla \phi(t, \cdot)\|_{L_\infty}$ grows at most polynomial rate in time. There is no chance that they are optimal because of the large order of moment we choose for simplicity.

A very interesting result by Pankavich [17] shows that $\|\nabla \phi(t, \cdot)\|_{L_\infty}$ actually decays sharply over time for compactly supported radial initial data, we refer readers to [17] and reference therein for more details in the line of research about the optimal decay (growth) rate.

Lastly, we discuss one of the essential difficulties of proving decay estimate of the electric field for the cylindrical symmetry case. To prove decay estimates for the large data case, it’s very important to have a good control of characteristics, see [17][Lemma 2.1]. Unlike the radial case, in which the second time derivative of the radius of space characteristics
is non-negative definite, e.g., see (2.13) and also [17][(2.1)], this sign-definite property is not always true in the cylindrical symmetric case. Without this sign-definite property, it’s mysterious about the precise distribution of characteristics for large time, which is vital to the decay estimate of the electric field.

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**Data/Code Availability**  The manuscript has no associated data or code.

**Declarations**

**Conflict of interest**  The author declares that he has no conflict of interest.

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