Spin squeezing by tensor twisting and Lipkin-Meshkov-Glick dynamics in a toroidal Bose-Einstein condensate with spatially modulated nonlinearity

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We propose a scheme for spin-squeezing in the orbital motion of a Bose-Einstein condensate (BEC) in a toroidal trap. A circular lattice couples two counter-rotating modes and squeezing is generated by the nonlinear interaction spatially modulated at double the lattice period. By varying the amplitude and phase of the modulation, various cases of the twisting tensor can be directly realized, including one-axis twisting and the two-axis counter-twisting. Our scheme naturally realizes the Lipkin-Meshkov-Glick model with the freedom to vary all its parameters simultaneously.

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Introduction: Squeezing in an ensemble of two-level systems [1, 2] is a quantum phenomenon with applications ranging from quantum information [3] to precision metrology [2, 4]. Since the canonical two-level system is a spin-½ particle, this is often referred to as “spin squeezing”, but physical realizations include a broad range of systems defined by the same Lie algebra, such as two-component BECs [3, 6] or polarized light [7]. A seminal paper by Kitagawa and Ueda [1] established nonlinear dynamics as a natural way to generate spin-squeezing, via two distinct mechanisms: by one-axis twisting (OAT) on the Bloch sphere with Hamiltonian $\hat{H} \sim \hat{T}_z^2$, and two-axis counter-twisting (TACT) with Hamiltonian $\hat{H} \sim \hat{T}_x^2 - \hat{T}_y^2$, with $\hat{T}_{x,y,z}$ being components of the collective spin operator. The latter scenario is more efficient in generating strong spin squeezing, however no experiment has achieved it directly, but schemes have been proposed to convert OAT into effective TACT Hamiltonians [8].

The spin-squeezing Hamiltonian quadratic in $\hat{T}_{x,y,z}$ is part of the more general Lipkin-Meshkov-Glick (LMG) model [9], an important exactly solvable model introduced to test various many-body approximation methods. The model is relevant in several fields, including nuclear physics [9, 10], quantum criticality and phase transitions [11], molecular magnetism [12], multi-particle entanglement [13] and classical bifurcations [14, 15]. The broad applicability of the LMG Hamiltonian, $\hat{H}_{\text{LMG}} = \hbar \chi [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2] + V (\hat{J}_x^2 - \hat{J}_y^2)$, arises from the independent variation of the three parameters $\omega, W, V$. Therefore, ultracold atomic systems, with their extreme tunability and control, seem a natural choice for a LMG simulator. But as yet only a limited case with one varied parameter (keeping $W = V$ fixed) has been demonstrated with BEC [15].

In this paper, we propose a way to implement general LMG with independent multi-parameter control as well as direct spin squeezing by both TACT and OAT, with a BEC in a ring trap with periodic angular modulation of the external potential and of the inter-particle interactions [16, 17]. Two counter-propagating rotational modes act as pseudo-spin coupled by the periodic potential; and the spatially modulated nonlinearity controls and varies the various components of the twisting tensor $\chi$ that covers OAT, TACT, and LMG as special cases [18].

Our study is also motivated by the rapid advances in recent years in toroidal trapping of ultracold atoms by a variety of ways: by intersecting a “sheet” beam with a Laguerre-Gaussian beam [19] or a beam passed through a ring-shaped mask [20, 21], with painted potentials [22], and with magnetic waveguides [23]. Notably, the study of LMG and spin-squeezing provides a distinct new line of study with such traps which have so far been primarily used to examine superfluid flow [19, 21] and interferometric effects [22].

FIG. 1: (Color online) Schematic of our model of a BEC in a toroidal trap with a circular lattice (black line) and spatially modulated nonlinearity controls and varies the various components of the twisting tensor $\chi$ that covers OAT, TACT, and LMG as special cases [18].
Derivation of the Hamiltonian: Consider a BEC of $N$ atoms in a toroidal trap with a superimposed potential $U$ and spatially varying nonlinearity, with the latter having twice the periodicity of the potential around the ring as shown in Fig. 1. For example, in a Laguerre-Gaussian implementation this could be achieved by modulating the amplitudes and phases of the component beams. The spatially varying nonlinearity can be produced by optically induced Feshbach resonance [24], as demonstrated for linear geometry in a recent experiment [16].

The dynamics is driven by the Hamiltonian [25]

$$
\hat{H} = \int dr \hat{\Psi}^\dagger(r) \left[ \frac{-\hbar^2}{2m} \nabla^2 + U(r, t) + \frac{g(r, t)}{2} \hat{\Psi}^\dagger \hat{\Psi} \right] \hat{\Psi}(r). \tag{1}
$$

Here $g(r, t) = 4\pi \hbar^2 a(r, t)/m$ is the nonlinearity parameter, with $a$ being the scattering length, $m$ the particle mass, and $\hat{\Psi}(r), \hat{\Psi}^\dagger(r')$ are, respectively the annihilation and creation bosonic field operators satisfying the commutation relations $[\hat{\Psi}(r), \hat{\Psi}^\dagger(r')] = \delta(r-r')$.

Using cylindrical co-ordinates $r = (r, z, \varphi)$, with $z$ along ring axis, we assume tight harmonic confinement in radial and axial directions $U(r, t) = \frac{1}{2}m \omega_r^2 r^2 + \frac{1}{2}m \omega_z^2 (r - R)^2 + U(\varphi, t)$ with $R$ being mean ring radius. The azimuthal potential is taken to be weakly sinusoidal rotating along the ring with frequency $\omega$, creating a circular Bragg grating, $U(\varphi, t) = \hbar u_x \cos[2q(\varphi + \omega t)] + \hbar u_y \sin[2q(\varphi + \omega t)]$, with $q$ an integer and amplitudes assumed small, $u_{x,y} \ll \omega_z r$. The nonlinear parameter is given a periodic spatio-temporal dependence, $g(r, t) = g(\varphi, t) = g_0 + g_1 \cos[4q(\varphi + \omega t) - \alpha]$, where $g_0, g_1$ and $\alpha$ are constants.

We expand the field operators $\hat{\Psi}(r) = \sum_n \hat{a}_n \psi_n(r)$ in terms of the mode functions of the ring trap,

$$
\psi_n(r) = \frac{1}{\sqrt{2\pi R \sigma_z \sigma_R}} e^{in\varphi} \times e^{(r-R)^2/(4\sigma_z^2)} \times e^{-z^2/(4\sigma_R^2)} \tag{2}
$$

with $\sigma_{z,R} = \sqrt{\hbar/(2m \omega_{z,R}^2)}$ being the half-widths of the mode functions in the respective directions. Assuming that the tight trapping allows only the ground state to be populated in the axial and radial directions, the Hamiltonian can be written in terms of the annihilation and creation operators $\hat{a}_n$ and $\hat{a}_n^\dagger$ as

$$
\hat{H} = \frac{\hbar^2}{2m R^2} \sum_n n^2 \hat{a}_n^\dagger \hat{a}_n + \frac{\hbar u_x}{2} \sum_n (\hat{a}_n^\dagger \hat{a}_{n-2q} e^{i2q\omega t} + \hat{a}_n \hat{a}_{n+2q} e^{-i2q\omega t}) + \frac{\hbar u_y}{2i} \sum_n (\hat{a}_n^\dagger \hat{a}_{n-2q} e^{i2q\omega t} - \hat{a}_n \hat{a}_{n+2q} e^{-i2q\omega t})
$$

$$
+ \frac{1}{16\pi^2 \sigma_z \sigma_R R} \sum_{n,k,l,s} \hat{a}_n^\dagger \hat{a}_k^\dagger \hat{a}_l \hat{a}_s \left[ g_0 \delta_{n+k,l+s} + \frac{g_1}{2} \left( \delta_{n+k,l+s+4q} e^{i(4q\omega t - \alpha)} + \delta_{n+k,l+s-4q} e^{-i(4q\omega t - \alpha)} \right) \right]. \tag{3}
$$

The Hamiltonian simplifies further if only two counter-propagating modes with $n = \pm q$ are initially populated, and the energy gap manifest in the diagonal terms inhibit transitions to other modes. Absorbing the time dependence of the fields into the definition of the mode operators, $\hat{a}_q \equiv \hat{a} e^{iq\omega t}, \hat{a}_{-q} \equiv \hat{b} e^{-i q\omega t}$, and introducing spin operators $\hat{J}_{x,y,z}$ as $\hat{J}_x = \frac{1}{2} (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger)$, $\hat{J}_y = \frac{1}{2i} (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger)$, and $\hat{J}_z = \frac{1}{2} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$, the Hamiltonian reduces to

$$
\hat{H} = \hbar \left[ u_x \hat{J}_x + u_y \hat{J}_y + 2q \omega \hat{J}_z - \chi_0 \hat{J}_z^2 + \frac{\chi_1 \cos \alpha}{2} \left( \hat{J}_x^2 - \hat{J}_y^2 \right) + \frac{\chi_1 \sin \alpha}{2} \left( \hat{J}_x \hat{J}_y + \hat{J}_y \hat{J}_x \right) \right], \tag{4}
$$

where we have defined effective one-dimensional nonlinear coefficients $\chi_{0,1} \equiv g_{0,1}/(8\pi^2 \hbar \sigma_z \sigma_R R) = a_{0,1} \times \sqrt{\omega_{z,R}^2/(\pi R)}$, and we dropped a scalar term $\chi_0 \left( N^2 - \frac{1}{2} N \right)$ that creates an irrelevant global phase.

It is worth noting that although the sign of the unmodulated nonlinear term $-\hbar \chi_0 \hat{J}_z^2$ suggests attractive inter-particle interaction, it actually stems from repulsive interaction for positive $g_0$. This apparent contradiction stems from the fact that for $J_z \approx 0$ the counter-rotating modes $\hat{a}$ and $\hat{b}$ are almost equally populated and the condensate forms a standing wave with pronounced interference fringes (see Fig. 1). Therefore, the particles are effectively compressed to half of the volume that would be occupied if they were all circulating in the same direction ($J_z \approx \pm N/2$) with no interference fringe. Thus, states with larger $|J_z|$ correspond to lower interaction energy.

Spin-squeezing by OAT and TACT: The terms linear in $\hat{J}_{x,y,z}$ generate rotation on the Bloch sphere with angular velocity vector $(u_x, u_y, 2q \omega)$. Spin squeezing is generated by nonlinear terms the coefficients of which can be writ-
ten as a twisting tensor \[18\]:

\[
\chi = \begin{pmatrix}
\frac{N}{2} \cos \alpha & \frac{N}{2} \sin \alpha & 0 \\
0 & -\frac{N}{2} \cos \alpha & 0 \\
-1 & 0 & -\chi_0
\end{pmatrix}.
\]

(5)

Since \(J_x J_z = \frac{N}{2} (\frac{N}{2} + 1)\), adding multiples of a unit matrix to \(\chi\) only leads to a global phase shift. For \(\chi_1 = 0\) (no spatial modulation of the nonlinearity) the position of the Bloch interference fringes plays no role. The Hamiltonian is invariant with respect to rotations around \(J_z\), and the nonlinear dynamics corresponds to OAT. When \(\chi_1 \neq 0\) the nonlinearity is modulated, and the interaction energy is affected by the relative positions of the interference fringes and the maxima of the nonlinearity (see Fig. 1). In this case the twisting tensor has eigenvalues \(-\chi_0\) and \(\pm |\chi_1|/2\). The maximum squeezing rate is determined by the difference between the largest and the smallest eigenvalues of the twisting tensor \[18\]. For positive \(\chi_0\) and states that are nearly Gaussian and optimally positioned on the Bloch sphere, on the axis of the middle eigenvalue, the maximum squeezing rate is

\[
d\xi^2/ dt = -N (\chi_0 + |\chi_1|/2) \xi^2,
\]

where the squeezing parameter \(\xi^2\) is defined as the ratio of the minimum variance of the uncertainty ellipse of the state and the variance of the spin coherent state. To achieve this maximum rate, the state should be centered on the Bloch sphere along the axis corresponding to the middle eigenvalue, and kept optimally oriented by continuous rotation of the system at frequency \(N (\frac{\chi_0 + |\chi_1|}{2})\) around this axis, where \(\lambda_{a,b,c}\) are the eigenvalues in ascending order; here, if \(|\chi_1| \leq \chi_0\), this frequency is \(N (\frac{1}{2} |\chi_1| - \frac{1}{2} \chi_0)\).

In the special case when the middle eigenvalue is exactly between the other two eigenvalues, \(\chi\) can be transformed to a diagonal form \(\chi = (\chi_a, -\chi_b, 0)\) corresponding to TACT (see \[18\] for details); here \(2 \chi\) is the difference between the largest and smallest eigenvalues. In our model it corresponds to the condition \(|\chi_1| = \frac{1}{3} \chi_0\). Since the modulation maxima and minima are proportional to \(\chi_0 \pm |\chi_1|\), the maximum and minimum are \(\frac{2}{3} \chi_0\) and \(\frac{1}{3} \chi_0\), so that TACT occurs if the maximum value of the modulated nonlinearity is five times larger than the minimum (see Fig. 1). In this case, the state naturally remains optimally oriented, no rotation is needed.

**The LMG Model and Phase transitions:** Special cases of our Hamiltonian \[4\] correspond to the LMG model: Setting \(u_x = \alpha = 0\) leads to LMG model with \(\Omega = 2\omega\), \(W = \chi_0\), and \(V = \chi_1/2\), and setting \(u_y = \omega = \alpha = 0\) leads to LMG model with \(\Omega = u_x\), \(W = -\frac{1}{2} \chi_0 - \frac{3}{2} \chi_1\), and \(V = \frac{1}{4} \chi_0 - \frac{1}{4} \chi_1\) with the roles of the coordinates in \(H_{LMG}\) permutated \(J_x \rightarrow J_y \rightarrow J_z \rightarrow J_x\).

To illustrate the applicability of our Hamiltonian to the LMG model, we show a possible LMG phase diagram in Fig. 2. In this example, we assume \(u_x = \omega = \alpha = 0\) and \(\chi_0 > 0\), varying \(\chi_1 \in (-\chi_0, 0)\) and \(u_x\). Such a choice may be natural, for instance, if one needs to avoid rotation of the fields. The zones denoted I—IV correspond to those studied in \[20\], reflecting different qualitative behavior of the energy surfaces and different properties of the density of eigenstates of the Hamiltonian. Crossing the boundary between various zones corresponds to various phase transitions; for example crossing from I to II by decreasing \(|u_x|\) with \(\chi_0 = 0\) corresponds to the classical bifurcation at the transition from Rabi to Josephson dynamics observed in \[15\].

**Fastest squeezing within the LMG model:** The conditions for fastest squeezing can be achieved with the parameters used for the LMG phase diagram above. The state is positioned along \(J_z\) axis on the Bloch sphere and rotation \(u_x\) should be applied at the optimum frequency defined earlier. These conditions are fulfilled by parameters of the red broken line between points a—d in Fig. 2. We show the corresponding time evolution of the squeezing parameter \(\xi^2\) in Fig. 3. The best squeezing properties correspond to TACT (line d) and as can be seen, even a relatively small deviation of the TACT condition leads to a strong deterioration of squeezing in the last stages (line c). For the respective parameter values we also show the density of states \(\rho(\epsilon)\) as a function of the Hamiltonian eigenvalues \(\epsilon\) with qualitatively different shapes in zone II (one peak) and III (two peaks that merge in the limit of TACT, point d of Fig. 2).

**Conditions for the physical parameters:** The evolution is confined to the tangential degree of freedom if the radial and axial confinement is sufficiently tight so that occupation of higher modes of these degrees of freedom is energetically forbidden. This is achieved if \(\hbar q^2/(2mR^2) \ll \omega_{R,z}\), or \(\sigma_{R,z} \ll R/q = \lambda_0/(2\pi)\) where \(\lambda_0\) is the atomic de Broglie wavelength along the ring.

To avoid transitions to other circular modes, the corresponding couplings should be sufficiently small. In the Hamiltonian \[3\] the terms with \(u_{xy}\) couple modes \(n = \pm q\) also to unwanted modes \(n = \pm 3q\). The probability of such transitions would oscillate with small mag-
FIG. 3: (Color online) Time evolution of the squeezing parameter $\xi$ with $N = 300$ and optimum rotation $u_z$, with different values of $\chi_1$ marked in Fig. 2: $a \rightarrow$ OAT, $\chi_1 = 0$; $b \rightarrow$ $\chi_1 = -0.2\frac{\pi}{\sigma_0}$; $c \rightarrow$ $\chi_1 = -0.9\frac{\pi}{\sigma_0}$; $d \rightarrow$ TACT, $\chi_1 = -\frac{3}{2}\chi_0$. Insets show the corresponding density of states $\rho$ as function of the energy spectrum $\epsilon$ of the Hamiltonian; red bars indicate energy spread of states used for the curves.

Squeezing Parameter $\xi$ (dB)

Scaled time, $N\tau$

0 1 2 3 4 5

$\xi$ = $-20$ $-15$ $-10$ $-5$ $0$ $5$ $10$ $15$ $20$

State preparation and detection: A spin coherent state can be prepared by rotating a BEC with the chosen winding number $q$. One way to do that is via Raman transition to transfer the angular momentum from LG beams to a non-rotating BEC [19]. The resulting state is an eigenstate of $\hat{J}_z$ with eigenvalue $N/2$. Any other spin coherent state can be prepared from this state by application of operators $\hat{J}_{x,y,z}$ by varying the parameters $u_{x,y}$ and $\omega$ to rotate the state on the Bloch sphere.

For detection $\hat{J}_{x,y,z}$, the procedure can be reversed: first the state is rotated so that the operator of interest corresponds to $\hat{J}_z$, then a Raman process converts the $\hat{J}_z = N/2$ eigenstate to a non-rotating BEC. Finally the proportion of the non-rotating atoms is measured; for the setup in Ref. [19] this can be done by releasing the condensate and observing the size of the central hole in the interference pattern produced by the free-falling atoms. Alternately, one can measure $\hat{J}_z$ by coupling the ring resonator to a linear atomic waveguide (formed, e.g., by a red-detuned horizontal laser beam) positioned tangentially near the ring. Atoms circulating in the opposite orientations would leak to the waveguide and propagate in opposite directions towards the waveguide ends where they can be detected.

Outlook: We have shown that a BEC in a toroidal trap with a circular lattice and a spatially modulated nonlinearity can be used to realize a general LMG model and comprehensive spin-squeezing procedures, including direct two-axes twisting; with independent control over all the relevant parameters. Our model should be useful for creating strongly squeezed states for precision quantum measurements and testing critical phenomena in a range of physical systems described by the LMG model. It also provides a different realm of physics to be explored with the rapidly advancing technology of toroidal trapping of ultracold atoms. We conclude with some comments about our model: Unlike most schemes for bosonic pseudospin squeezing [8, 9], it can directly generate rotation about an arbitrary Bloch axis by varying $u_{x,y}$ and $\omega$. Any relevant ratio of the LMG parameters can be achieved by co-ordinate rotation, if $\chi_1$ can access $[0,\pi/2\chi_0]$ or $[0,-\pi/2\chi_0]$. Although our examples did not use it, our model allows additional control via temporal modulation ($\omega \neq 0$) of the nonlinearity.

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