Nonsingular spacetime defect

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Abstract

A nonsingular localized static classical solution is constructed for standard Einstein gravity coupled to an $SO(3) \times SO(3)$ chiral model of scalars [Skyrme model]. This solution corresponds to a spacetime defect and its construction proceeds in three steps. First, an Ansatz is presented for a solution with nonsimply-connected topology of the spacetime manifold. Second, an exact vacuum solution of the reduced field equations is obtained. Third, matter fields are included and a particular exact solution of the reduced field equations is found. The latter solution has a diverging total energy, but its existence at least demonstrates that a regular defect-type solution having nonsimply-connected topology is possible with nontrivial matter fields.

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I. INTRODUCTION

It can be argued [1, 2] on general grounds that the small-length-scale structure of quantum spacetime is nontrivial. This structure has been called a quantum spacetime foam. Over larger length-scales (lower energy-scales), an effective classical spacetime manifold emerges and the crucial question is if that effective spacetime is perfectly smooth or not. Particle/wave propagation over Swiss-cheese-type manifolds has been studied and the problem can, in principle, be solved exactly [3, 4]. The simplest Swiss-cheese-type manifold has identical static “defects” (alternatively called “holes” or “knots”), where each defect provides nontrivial topology. The particular defect considered in Ref. [4] has, however, a divergent (delta-function-type) Ricci curvature scalar and does not solve the vacuum Einstein equations.

The goal, now, is to construct a nonsingular defect solution by use of appropriate coordinates and matter fields. For the defect topology at hand (holes with antipodal points identified; see below), it has been suggested [5] to use the gravitating $SO(3)$ Skyrme model [6–10] with an additional interaction term [11] allowing for negative energy-density contributions. The motivation for using the $SO(3)$ Skyrme field is to allow for the possibility of having a topologically stable solution consistent with the boundary conditions at the defect core. Negative energy-density contributions may turn out to be essential for a satisfactory defect solution over a nonsimply connected space-time [12, 13]. For this reason, our analysis allows for the possibility of having negative energy-density contributions, even though, at this stage, we can do without.

The present article uses a new Ansatz for these fields and gives special attention to the behavior of the reduced field equations at the defect core. We are, then, able to obtain a nonsingular defect solution in the gravitating $SO(3)$ Skyrme model.

The main results of this article are, first, an exact vacuum solution and, second, an exact non-vacuum solution. Physically, the vacuum solution will be relevant far away from localized energy-momentum distributions. The importance of the particular non-vacuum solution found is that it shows how the matter is distributed, given that the topology is the same as that of the vacuum solution.

II. MANIFOLD

The 4-dimensional spacetime manifold considered in this article is static for appropriate coordinates and has topology

$$M_4 = \mathbb{R} \times M_3.$$  (2.1)
The nontrivial topology appears in the 3-space $M_3$, which is, in fact, a noncompact, orientable, nonsimply-connected manifold without boundary. Up to a point, $M_3$ is homeomorphic to the 3-dimensional real-projective space,

$$M_3 \simeq \mathbb{R}P^3 - \{\text{point}\}. \quad (2.2)$$

Further details can be found in Refs. [4, 5]. Here, only the absolutely necessary information will be given.

For the explicit construction of $M_3$, we perform local surgery on the 3-dimensional Euclidean space $E_3 = (\mathbb{R}^3, \delta_{mn})$. We use the standard Cartesian and spherical coordinates on $\mathbb{R}^3$,

$$\vec{x} \equiv |\vec{x}| \hat{x} = (x^1, x^2, x^3) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta), \quad (2.3)$$

with $x^m \in (-\infty, +\infty)$, $r \geq 0$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$. Now, $M_3$ is obtained from $\mathbb{R}^3$ by removing the interior of the ball $B_b$ with radius $b$ and identifying antipodal points on the boundary $S_b \equiv \partial B_b$. With point reflection denoted by $P(\vec{x}) = -\vec{x}$, the 3-space $M_3$ is given by

$$M_3 = \left\{ \vec{x} \in \mathbb{R}^3 : (|\vec{x}| \geq b > 0) \land (P(\vec{x}) \cong \vec{x} \text{ for } |\vec{x}| = b) \right\}, \quad (2.4)$$

where $\cong$ stands for point-wise identification (Fig. 1).

The single set of coordinates (2.3) does not suffice for an appropriate description of $M_3$. The reason is simply that two different values of these coordinates may correspond to a single point. For example, $\vec{x} = (b, 0, 0)$ and $\vec{x} = (-b, 0, 0)$ correspond to the same point of $M_3$. A relatively simple covering of $M_3$ uses three sets of coordinates (also called charts or patches), labeled by $n = 1, 2, 3$. Each coordinate chart surrounds one of the three Cartesian coordinate axes. These coordinates are denoted

$$(X_n, Y_n, Z_n), \quad \text{for } n = 1, 2, 3, \quad (2.5)$$

Figure 1. Three-space $M_3$ obtained by surgery from $\mathbb{R}^3$: interior of the ball with radius $b$ removed and antipodal points on the boundary of the ball identified (as indicated by open and filled circles).
and are, despite appearances, triples of non-Cartesian coordinates. Specifically, the set of coordinates surrounding the $x^2$-axis segment with $|x^2| \geq b$ is given by

$$X_2 = \begin{cases} \phi & \text{for } 0 < \phi < \pi, \\ \phi - \pi & \text{for } \pi < \phi < 2\pi, \end{cases} \quad (2.6a)$$

$$Y_2 = \begin{cases} r - b & \text{for } 0 < \phi < \pi, \\ b - r & \text{for } \pi < \phi < 2\pi, \end{cases} \quad (2.6b)$$

$$Z_2 = \begin{cases} \theta & \text{for } 0 < \phi < \pi, \\ \pi - \theta & \text{for } \pi < \phi < 2\pi, \end{cases} \quad (2.6c)$$

with ranges

$$X_2 \in (0, \pi), \quad (2.7a)$$

$$Y_2 \in (-\infty, \infty), \quad (2.7b)$$

$$Z_2 \in (0, \pi). \quad (2.7c)$$

The other two sets, $(X_1, Y_1, Z_1)$ and $(X_3, Y_3, Z_3)$, are defined similarly.

In the following, we consider spherically symmetric fields and it suffices to consider one coordinate chart, which we take to be $(2.6)$. The notation is, furthermore, simplified as follows:

$$(X, Y, Z, T) \equiv (X_2, Y_2, Z_2, T), \quad (2.8)$$

where the time coordinate has been added in order to describe the spacetime manifold $M_4$.

### III. FIELDS AND ACTION

The spacetime manifold $(2.1)$ of the previous section is now equipped with a metric $g_{\mu\nu}(X)$, whose dynamics is taken to be governed by the standard Einstein–Hilbert action $[14]$. In addition, there is a scalar field $\Omega(X) \in SO(3)$, with self-interactions governed by a Skyrme term in the action $[6]$ and by another term $[11]$ whose coupling constant $\gamma$ is taken to be nonnegative, allowing for negative energy-density contributions.

Specifically, the combined action of the pure-gravity sector, labeled ‘grav,’ and the matter sector, labeled ‘mat,’ is given by $(c = \hbar = 1)$

$$S = \int_{M_4} d^4X \sqrt{-g} \left( \mathcal{L}_{\text{grav}, \text{EH}} + \mathcal{L}_{\text{mat}, \text{kin}} + \mathcal{L}_{\text{mat, Skyrme}} + \mathcal{L}_{\text{mat, metastab}} \right), \quad (3.1a)$$
with Lagrange densities

\[ \mathcal{L}_{\text{grav, EH}} = \frac{1}{16\pi G_N} R, \quad (3.1b) \]

\[ \mathcal{L}_{\text{mat, kin}} = \frac{f^2}{4} \text{tr}(\omega_\mu \omega^\mu), \quad (3.1c) \]

\[ \mathcal{L}_{\text{mat, Skyrme}} = \frac{1}{16e^2} \text{tr} \left( [\omega_\mu, \omega_\nu] [\omega^\mu, \omega^\nu] \right), \quad (3.1d) \]

\[ \mathcal{L}_{\text{mat, metastab}} = \gamma \frac{1}{48e^2} \left( \text{tr}(\omega_\mu \omega^\mu) \right)^2, \quad (3.1e) \]

in terms of the Ricci curvature scalar \( R \) and

\[ \omega_\mu \equiv \Omega^{-1} \partial_\mu \Omega. \quad (3.1f) \]

The \( SO(3) \times SO(3) \) global symmetry of the matter sector is realized on the scalar field by the following transformation with constant parameters \( S_L, S_R \in SO(3) \):

\[ \Omega(X) \rightarrow S_L \cdot \Omega(X) \cdot S_R^{-1}, \quad (3.2) \]

where the central dot denotes matrix multiplication. As mentioned at the end of Sec. III, the generic argument \( X \) of the fields and the measure \( d^4X \) in the integral (3.1a) correspond to only one of the three coordinate charts needed to cover \( M_4 \).

IV. ANSATZ AND FIELD EQUATIONS

A. Ansatz

A spherically symmetric Ansatz for the metric is given by the following line element:

\[ ds^2 = -\exp \left[ 2 \tilde{\kappa}(W) \right] dT^2 + \exp \left[ 2 \tilde{\lambda}(W) \right] dY^2 + W \left( dZ^2 + \sin^2 Z dX^2 \right), \quad (4.1a) \]

\[ W \equiv b^2 + Y^2, \quad (4.1b) \]

where, as explained at the end of Sec. III, we only show the coordinates of one chart with \( Y \in (-\infty, \infty) \). For later convenience, two further functions are introduced:

\[ \tilde{\kappa}(W) \equiv \exp \left[ \tilde{\lambda}(W) \right], \quad (4.2a) \]

\[ \tilde{\mu}(W) \equiv \exp \left[ \tilde{\nu}(W) \right]. \quad (4.2b) \]
The scalar field is given by the hedgehog-type Ansatz \[5, 6\],

\[
\Omega = \cos (\tilde{F}(r^2)) \mathbb{1} - \sin (\tilde{F}(r^2)) \hat{x} \cdot \vec{S} + (1 - \cos (\tilde{F}(r^2))) \hat{x} \otimes \hat{x}, \tag{4.3a}
\]

\[
\tilde{F}(b^2) = \pi, \tag{4.3b}
\]

\[
S_1 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_2 \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.3c}
\]

with \((\hat{x} \otimes \hat{x})^{ab} = \hat{x}^a \hat{x}^b\) in components. Note that, because of the boundary condition \(4.3\) \(b\) at \(|\vec{x}| = b\), it is possible to use the single coordinate chart \(2.3\) with the further identification \(r^2 = b^2 + Y^2 = W\) for the coordinates used in the metric \(4.1\). In other words, the topology of \(M_3\) (see Fig. 1) is trivially consistent with the hedgehog field having boundary value \(\tilde{F}(b^2) = \pi\).

The arguments of the tilde-functions in the above Ansätze have dimension length-square. Related functions with lengths as arguments can be defined as follows:

\[
\nu \left( \sqrt{b^2 + Y^2} \right) \equiv \tilde{\nu} \left( b^2 + Y^2 \right), \tag{4.4}
\]

and similarly for \(\lambda\) and \(F\). The functions without tilde resemble those of the previous literature \[6–11\], but we prefer to work with the tilde-functions.

The relevant nonvanishing components of the Riemann tensor for the Ansatz \(4.1\) are:

\[
R_{YTY}^T = -2 \left[ \tilde{\nu}' + 2Y^2 \left( -\tilde{\nu}'/\tilde{\kappa} + \tilde{\nu}'' + \tilde{\nu}''/\tilde{\kappa} \right) \right], \tag{4.5a}
\]

\[
R_{ZTZ}^T = -2Y^2 \tilde{\nu}'/\tilde{\kappa}^2, \tag{4.5b}
\]

\[
R_{XTX}^T = \sin^2 Z R_{ZTZ}^T, \tag{4.5c}
\]

\[
R_{ZYZ}^Y = \frac{2Y^2 (b^2 + Y^2) \tilde{\kappa}'/\tilde{\kappa} - b^2}{(b^2 + Y^2) \tilde{\kappa}^2}, \tag{4.5d}
\]

\[
R_{XYX}^Y = \sin^2 Z R_{ZYZ}^Y, \tag{4.5e}
\]

\[
R_{XZX}^Z = \sin^2 Z \left( 1 - \frac{Y^2}{(b^2 + Y^2) \tilde{\kappa}^2} \right), \tag{4.5f}
\]

where the prime stands for differentiation with respect to \(W\). The components not shown in \(4.5\) either vanish or can be computed from the ones above by the usual symmetry properties.
B. Reduced field equations

The derivation of the variational equations is straightforward \[15\]. Henceforth, we will use the following dimensionless model parameters and dimensionless variables:

\[ \tilde{\eta} \equiv 8\pi\eta \equiv 8\pi G_N f^2, \quad (4.6a) \]
\[ w \equiv (e f)^2 W = (y_0)^2 + y^2, \quad (4.6b) \]
\[ y \equiv e f Y, \quad (4.6c) \]
\[ y_0 \equiv e f b. \quad (4.6d) \]

The reduced Einstein equations and matter field equation will be given in Appendix A. From these equations, one obtains three ordinary differential equations (ODEs):

\[ \tilde{\kappa}'(w) = \tilde{\kappa} \left( \frac{w(1 - \tilde{\kappa}^2) + y_0^2}{4w(w - y_0^2)} \right) + \tilde{\eta} Q_1[\tilde{F}(w), \tilde{\kappa}(w), \tilde{\nu}(w)], \quad (4.7a) \]
\[ \tilde{\nu}'(w) = \frac{\tilde{\kappa}(w)^2}{4(w - y_0^2)} - \frac{1}{4w} + \tilde{\eta} Q_2[\tilde{F}(w), \tilde{\kappa}(w), \tilde{\nu}(w)], \quad (4.7b) \]
\[ \tilde{F}''(w) = Q_3[\tilde{F}(w), \tilde{\kappa}(w), \tilde{\nu}(w)], \quad (4.7c) \]

where the prime now stands for differentiation with respect to \( w \) and the three \( Q_n \) are certain functionals given in Appendix A. Specifically, the functionals \( Q_1 \) and \( Q_2 \) are given by the parts proportional to \( \tilde{\eta} \) in (A4a) and (A4b), while \( Q_3 \) is given by the right-hand side of (A4c).

The ODEs (4.7) are to be solved with the following boundary conditions for the non-vacuum solution:

\[ \tilde{F}(y_0^2) = \tilde{F}(\infty) = \pi, \quad (4.8a) \]
\[ \tilde{\kappa}(y_0^2) = 0, \quad \tilde{\nu}(\infty) = 0. \quad (4.8b) \]

Physically, the reason for choosing these boundary conditions is to obtain asymptotic flatness and to have a non-vanishing matter contribution which allows for an analytic solution. For the vacuum solution, (4.8a) is to be replaced by \( \tilde{F}(y_0^2) = \tilde{F}(\infty) = 0 \).

The field configuration (4.3) with boundary conditions (4.8) has vanishing winding number [recall (2.2) and the further homeomorphism \( SO(3) \simeq \mathbb{R}P^3 \)]. It is, therefore, not a genuine Skyrmion [which has \( F(0) = \pi \) and \( F(\infty) = 0 \), for unit winding number]. Our matter solution may very well turn out to be unstable, but that is not important for the main goal of this article, i.e., finding a non-trivial solution with matter fields (see Sec. VII for further discussion).
V. VACUUM SOLUTION

For vacuum matter fields,
\[ \tilde{F}(w) = 0, \]  
the ODEs \((4.7a)\) and \((4.7b)\) have \(Q_1 = Q_2 = 0\) and can be solved exactly for boundary conditions \((4.8b)\),
\[ \tilde{\kappa}(w) \equiv \exp[\tilde{\lambda}(w)] = \sqrt{1 - \frac{(y_0)^2}{w} - \frac{\ell}{\sqrt{w}}}, \]  
\[ \tilde{\mu}(w) \equiv \exp[\tilde{\nu}(w)] = \sqrt{1 - \frac{\ell}{\sqrt{w}}}, \]
with a dimensionless constant \(\ell\) and the definition \(w \equiv y^2 + (y_0)^2\) in terms of the ‘radial’ coordinate \(y \in (\infty, +\infty)\) and the defect parameter \(y_0 > 0\) (the corresponding dimensional parameter \(b\) is also nonzero and positive). Incidentally, the same \(\tilde{\kappa}(w)\) and \(\tilde{\mu}(w)\) solutions are obtained for the case \(\tilde{F}(w) = \pi\) and \(\tilde{\eta} = 0\), which corresponds to the setup of the boundary conditions \((4.8a)\) and vanishing Newton’s constant, \(G_N = 0\) (see Sec. VI for details).

At this point, we can make four general remarks. First, the vanishing of the metric component \(\tilde{\kappa}^2\) at \(Y = 0\) implies that the defect of Fig. 1 at that point \(Y = 0\) and fixed time \(T\) has zero physical extent along the radial direction (i.e., \(ds = 0\) at \(Y = 0\) for \(dT = dX = dZ = 0\)).

Second, the real \(\tilde{\kappa}\) and \(\tilde{\mu}\) functions from \((5.1)\) cover the whole of the manifold \(M_4\) for the following parameters:
\[ \ell < y_0, \]  
where \(y_0\) has been assumed positive. The topology of this solution is \(\mathbb{R} \times M_3\) with \(M_3\) given by \((2.2)\).

Third, the solution behaves asymptotically \((w \sim y^2 \rightarrow \infty)\) as follows:
\[ \tilde{\kappa}^2 \sim \frac{1}{(1 - \ell/|y|)}, \]  
\[ \tilde{\mu}^2 \sim (1 - \ell/|y|), \]
which is to be compared to the standard Schwarzschild metric \([14]\) with mass \(M\), having the respective components \(1/(1 - 2G_NM/r)\) and \((1 - 2G_NM/r)\). Note that the vacuum solution \((5.1)\) with \(\ell < 0\) would produce “antigravity,” namely, a point mass far away from the defect core would not be attracted towards it but repulsed.

Fourth, with the effective radial coordinate \(\zeta \equiv \sqrt{b^2 + Y^2}\) for \(b > 0\), the metric of Sec. VI takes precisely the standard Schwarzschild form for all values \(\zeta \in [b, \infty)\), in line with
Birkhoff’s theorem as noted in Ref. [16]. The crucial point, however, is that the proper description of the topology of the manifold requires the coordinate $Y \in (-\infty, +\infty)$.

For $\ell < y_0$, all Riemann-curvature-tensor components (4.5) are finite over the whole manifold, also at the defect core, $y = 0$ or $w = (y_0)^2$. Specifically, we find for the relevant nonvanishing Riemann tensor components:

\begin{align}
R^T_{TYT} &= (e^2 f^2) \frac{\ell (w - y_0^2)}{w^2 (\sqrt{w} - \ell)}, \\
R^T_{ZTZ} &= -\frac{\ell}{2\sqrt{w}}, \\
R^Y_{ZYZ} &= -\frac{\ell}{2\sqrt{w}}, \\
R^Z_{XZX} &= \sin^2 Z \frac{\ell}{\sqrt{w}}.
\end{align}

More importantly, the Kretschmann scalar obtained by contraction of the Riemann tensor with itself,

$$K \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 12 e^4 f^4 \frac{\ell^2}{w^3},$$

remains finite over the whole of the manifold $M_4$, because $w \geq (y_0)^2 > 0$. This behavior contrasts with that of the Schwarzschild metric over $\mathbb{R}^4$, for which $K$ diverges at the point $r = 0$.

A brief discussion of the geodesics from the metric (4.1) with functions (5.1b) and (5.1c) is given in Appendix B. The main result of this appendix is the existence of radial geodesics passing through the center, which illustrates the difference between our spacetime and that of the standard Schwarzschild solution (cf. the fourth general remark above). Three follow-up papers [16–18] describe the global structure of the new vacuum solution, in particular for the black-hole case $0 < y_0 < \ell$. These follow-up papers also give further details of the exact solution (5.1) at the defect core [19].

Note, finally, that the regular flat-spacetime metric is given by (5.1b) and (5.1c) with $\ell = 0$. (Electromagnetic wave propagation over this flat spacetime can be calculated with the methods of Refs. [3, 4].) The actual value of the free parameter $\ell$ in the metric from (5.1) will have to be determined by adding matter fields (the same applies for the determination of $M$ in the standard Schwarzschild solution).
VI. NON-VACUUM SOLUTION

A. General solution

For the case of the constant Skyrme function \( \tilde{F}(w) = \pi \), it is possible to solve the reduced field equations (A4) exactly. The vacuum solution of Sec. V also had a constant Skyrme function, \( \tilde{F}(w) = 0 \), and we can give a combined discussion by introducing the constant

\[ F = \pm 1, \tag{6.1} \]

so that the constant Skyrme function is given by \( \tilde{F}(w) = \arccos F \). A further definition introduces the re-scaled function

\[ \tilde{\sigma}(w) \equiv \frac{1}{\sqrt{1 - y_0/w}} \tilde{\kappa}(w), \tag{6.2} \]

which will simplify certain expressions below.

With these constant Skyrme functions for \( F = \pm 1 \), the differential equation for \( \tilde{\kappa}(w) \) reduces to a differential equation of Bernoulli type which can be solved with standard methods, while the differential equation for \( \tilde{\mu}(w) \) can be solved directly after insertion of the solution for \( \tilde{\kappa}(w) \). For \( \tilde{\kappa}(w) \) vanishing identically, one obtains \( \tilde{\mu}(w) = (w_0/w)^{1/4} \) with constant \( w_0 \).

For the case of nonvanishing \( \tilde{\kappa}(w) \), one obtains the following solutions for \( F = \pm 1 \):

\[ \tilde{F}(w) = \arccos F, \tag{6.3a} \]
\[ \tilde{\sigma}(w) = \left( 1 + C_1/\sqrt{w} + \tilde{\eta}(1 - F) \left[ (1 - 4\gamma/3)(1 - F)/w - 2 \right] \right)^{-1/2}, \tag{6.3b} \]
\[ \tilde{\mu}(w) = \sqrt{3} C_2 \frac{1}{\tilde{\sigma}(w)}, \tag{6.3c} \]

having fixed the overall signs of \( \tilde{\kappa} \) and \( \tilde{\mu} \). There exists a finite positive value of \( \gamma \) above which the non-vacuum solution does not exist.

In preparation of the subsequent discussion, we already give the expressions of two curvature invariants. The Ricci scalars of the two solutions given in (6.3) read

\[ R \big|_{F=\pm1} = 4 e^2 f^2 \tilde{\eta} (1 - F)/w. \tag{6.4} \]

Similarly, the Kretschmann scalar of the vacuum solution (6.3) is given by

\[ K \big|_{F=+1} = 12 e^4 f^4 C_1^2/w^3 \tag{6.5a} \]

and the one of the non-vacuum solution (6.3) by

\[ K \big|_{F=-1} = (4/9) e^4 f^4 \left( 27 C_1^2 w - 72 C_1 \tilde{\eta} \sqrt{w} [w + 8\gamma - 6] \right. \]
\[ \left. + 16 \tilde{\eta}^2 [9 w^2 + 6 w (4\gamma - 3) + 14 (3 - 4\gamma)^2] \right)/w^4, \tag{6.5b} \]

where the dependence on \( C_2 \) has canceled out (see the remark in the next subsection).
B. Boundary conditions and asymptotics

The solution (6.3) of the ODEs (A4) is fixed by the boundary conditions on $\tilde{F}(w)$, $\tilde{\kappa}(w)$, and $\tilde{\mu}(w)$ at the defect position $w = y_0^2$. For $\tilde{F}(w)$, the boundary conditions are $\tilde{F}(y_0^2) = \pi$ and $\tilde{F}'(y_0^2) = 0$. The value of $\tilde{\sigma}(w)$ at $w = y_0^2$ fixes the constant $C_1$ of the solution (6.3). Subsequently, the value of $\tilde{\mu}(w)$ at $w = y_0^2$ fixes the constant $C_2$. Remark that the actual value of $C_2$ has no direct physical significance, as it can be changed by a re-scaling of the coordinate $T$, according to (4.1a) and (4.2b).

As an example, consider this particular set of boundary conditions

$$\tilde{F}(y_0^2) = \pi, \quad \tilde{F}'(y_0^2) = 0,$$  \hspace{0.5cm} (6.6a)

$$\tilde{\sigma}(y_0^2) = \left(1 + 4 \tilde{\eta} \left[(1 - 4\gamma/3) / y_0^2 - 1\right]\right)^{-1/2},$$  \hspace{0.5cm} (6.6b)

$$\tilde{\mu}(y_0^2) = \frac{1}{1 - 4 \tilde{\eta}} \frac{1}{\tilde{\sigma}(y_0^2)},$$  \hspace{0.5cm} (6.6c)

which gives the following constants in the solution (6.3):

$$\mathcal{F} = -1,$$  \hspace{0.5cm} (6.7a)

$$C_1 = 0,$$  \hspace{0.5cm} (6.7b)

$$C_2 = \frac{1}{\sqrt{3}} \frac{1}{1 - 4 \tilde{\eta}}.$$  \hspace{0.5cm} (6.7c)

The solution with $C_1 \neq 0$ is obtained by shifting the boundary value $\tilde{\sigma}(y_0^2)$ away from the value on the right-hand side of (6.6b).

Turning towards the asymptotics at spatial infinity, the solution (6.3) has the following behavior for $w \to \infty$:

$$\tilde{F}(w) \sim \pi,$$  \hspace{0.5cm} (6.8a)

$$\tilde{\kappa}(w) \sim (1 + C_\infty + C_1 / \sqrt{w})^{-1/2},$$  \hspace{0.5cm} (6.8b)

$$\tilde{\mu}(w) \sim \sqrt{3} C_2 \left[1 + C_\infty + C_1 / \sqrt{w}\right]^{1/2},$$  \hspace{0.5cm} (6.8c)

with

$$C_\infty \equiv -2 \tilde{\eta} (1 - \mathcal{F}) = \begin{cases} 0 & \text{for } \mathcal{F} = +1, \\ -4 \tilde{\eta} & \text{for } \mathcal{F} = -1. \end{cases}$$  \hspace{0.5cm} (6.8d)

With the choice

$$C_2 = \frac{1}{\sqrt{3}} \frac{1}{1 + C_\infty},$$  \hspace{0.5cm} (6.9)
one has the asymptotic values

\[ \tilde{F}_\infty = \pi, \] (6.10a)
\[ \tilde{\kappa}_\infty = 1/\sqrt{1 + C_\infty}, \] (6.10b)
\[ \tilde{\mu}_\infty = 1/\sqrt{1 + C_\infty}. \] (6.10c)

Re-scaling the coordinates \( T \) and \( Y \) of the Ansatz (4.1) by the same factor,

\[ \hat{T} = T/\sqrt{1 + C_\infty}, \] (6.11a)
\[ \hat{Y} = Y/\sqrt{1 + C_\infty}, \] (6.11b)

then reproduces asymptotically the vacuum-solution functions (5.1b) and (5.1c) with

\[ \ell = -C_1/(1 + C_\infty)^{3/2}, \] (6.12a)

if \( y_0 \) is also re-scaled,

\[ \hat{y}_0 = y_0/\sqrt{1 + C_\infty}. \] (6.12b)

Two remarks are in order. First, the particular \( C_2 \) value (6.9) and coordinate re-scaling (6.11) make the solution (6.3) consistent with the previous boundary condition on \( \tilde{\nu}(\infty) \) from (4.8b). Second, for this \( C_2 \) value and coordinate re-scaling, the functional behavior of (6.3) in the theory with coupling constant \( \gamma = 3/4 \) is identical to that of the vacuum-solution functions (5.1), but the spacetimes are different [the re-scaling of \( Y \) and \( b \) changes the angular part of the line element (4.1a)].

Expanding on the last remark, consider the curvature invariants given in the final paragraph of Sec. VI A. The curvature invariants (6.4) and (6.5) vanish for \( w \to \infty \), as do the other 12 independent curvature invariants [20–23] which have been calculated explicitly but shall not be displayed here [24]. Hence, both solutions from (6.3) locally approach flat Minkowski spacetime. But, asymptotically, the non-vacuum solution (6.3) for \( \mathcal{F} = -1 \) does not follow the vacuum solution (6.3) for \( \mathcal{F} = +1 \), as the non-vacuum solution has \( K \propto 1/w^2 \) for \( \tilde{\eta} \neq 0 \), according to (6.5b), and the vacuum solution has \( K \propto 1/w^3 \), according to (6.5a). This unusual behavior of the non-vacuum solution is due to the relatively slow decrease of the energy-momentum tensor with increasing values of \( w \) as will be discussed in the next subsection.
C. Energy-momentum tensor

The energy-momentum tensor $T_{\mu\nu}$ is diagonal for the Ansatz fields and only the diagonal components need to be discussed. The dimensionless energy density takes the form

$$t_{tt} \equiv \frac{T_{TT}}{(e^2 f^4)} = \frac{6}{w} C_2^2 \left[1 + \mathcal{A}(w)\right] \left(1 - \mathcal{F}\right) \left[1 + C_1/\sqrt{w} - 2\tilde{\eta}[1 - \mathcal{A}(w)(1 - \mathcal{F})]\right],$$

(6.13)

with

$$\mathcal{A}(w) \equiv (1 - 4\gamma/3)\frac{(1 - \mathcal{F})}{2w}.$$

(6.14)

For the non-vacuum case $\mathcal{F} = -1$, this gives a diverging total energy when integrated over the whole space (the volume integral includes $w$-integration over terms which asymptotically go as $w^{-1/2}$). One has $t_{tt} < 0$ if one of the square brackets in (6.13) becomes negative, but not both.

The dimensionless radial pressure component is given by

$$t_{yy} \equiv \frac{T_{YY}}{(e^2 f^4)} = -\frac{2(w - y_0^2)}{w^2[1 + C_1/\sqrt{w} - 2\tilde{\eta}[1 - \mathcal{A}(w)(1 - \mathcal{F})]]}.$$ 

(6.15)

For $\mathcal{F} = -1$, the component $t_{yy}$ vanishes if $w = -1 + 4\gamma/3$. The component $t_{tt}$ vanishes and the component $t_{yy}$ develops a pole in $w$ if

$$\sqrt{w} = -\frac{C_1}{2(1 - 4\gamma)} \left(1 \mp \sqrt{1 - \frac{16\tilde{\eta}(1 - 4\tilde{\eta})}{C_1^2}(1 - 4\gamma/3)}\right),$$

(6.16)

or, for the special case of $\tilde{\eta} = 1/4$, if

$$\sqrt{w} = -\frac{(1 - 4\gamma/3)}{C_1}.$$ 

(6.17)

As $\sqrt{w} \geq y_0$, these poles can be avoided by taking the size of the defect, $y_0$, sufficiently large.

The dimensionless angular pressure components are given by

$$t_{zz} \equiv \frac{T_{ZZ}}{(e^2 f^4)} = \frac{(1 - \mathcal{F})^2(1 - 4\gamma/3)}{w},$$

(6.18a)

$$t_{xx} \equiv \frac{T_{XX}}{(e^2 f^4)} = (\sin Z)^2 t_{zz}.$$ 

(6.18b)

Three remarks can be made. First, only the $t_{yy}$ component depends explicitly on $y_0$ (and goes to zero for $w \to y_0^2$), because, for constant $\tilde{F}(w)$, the other components do not depend on $\tilde{\kappa}$. Second, the angular pressure components vanish identically for the particular value $\gamma = 3/4$, whereas they become negative for $\gamma > 3/4$. Third, in terms of the energy density
Figure 2. Top row: functions \( \tilde{F}(w) \), \( \tilde{\kappa}(w) \), and \( \tilde{\mu}(w) \) of the non-vacuum solution (6.3). Bottom row: corresponding dimensionless energy density \( t_{tt} \) and pressure components \( t_{yy} \) and \( t_{zz} \). The model parameters are \( y_0 = 1 \), \( \tilde{\eta} = 1/8 \), and \( \gamma = 0 \). The solution parameters (from the boundary conditions at the defect core and the model parameters) are \( F = -1 \), \( C_1 = 0 \), and \( C_2 = 2/\sqrt{3} \).

Figure 3. Same as Fig. 2 but with \( \gamma = 3/8 \).

Figure 4. Same as Fig. 2 but with \( \gamma = 3/4 \).
$\rho$ and the pressures $p_n$ of an imperfect fluid, the asymptotic behavior of the Skyrme field is $\rho \sim +4/w$, $p_y \sim -4/w$, and $p_x \sim p_z = O(1/w^2)$, so that $\rho + p_y$ vanishes to leading order (recall that, for a perfect fluid, precisely the combination $\rho + p$ enters the hydrostatic equilibrium equation [25]).

As a concrete example, consider the energy-momentum densities for the $C_1 = 0$ case discussed in the second paragraph of Sec. VI B. Figure 2 shows the components of the energy-momentum tensor for a particular choice of parameters, including $\gamma = 0$. Figures 3 and 4 use the same parameters but now with $\gamma = 3/8$ and $\gamma = 3/4$, respectively. Observe that the value of the central energy density $t_{tt}(y_0^2)$ decreases with increasing $\gamma$.

VII. DISCUSSION

The exact solution from Sec. VI has a constant nonvanishing Skyrme function and diverging total energy. Most likely, this particular non-vacuum solution is unstable and decays to a Skyrmion-type defect solution by emitting out-going waves of scalars. It remains to obtain this final stable Skyrmion-type defect solution with boundary conditions $\tilde{F}(y_0^2) = \pi$ and $\lim_{w \to \infty} \tilde{F}(w) = 0$.

Note that, strictly speaking, the $SO(3)$ scalar field is absent in the standard model of elementary-particle physics. But the nonlinear sigma model resurfaces if the gauge fields are eliminated and the Higgs-field modulus is frozen. Hence, the toy model of matter fields considered in the present article is not completely removed from realistic physics. As such, there may even be a connection with ideas linking the quantum structure of spacetime (having an energy scale $E_{\text{Planck}} \equiv \sqrt{\hbar c^5/G_N}$ and a length scale $hc/E_{\text{Planck}}$) to the Higgs-boson and top-quark masses [13]. We refrain from making even wilder speculations as to phenomenology. In fact, the main focus of this article remains purely theoretical, namely, finding a nonsingular spacetime-defect solution with nontrivial matter fields.

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Appendix A: ODEs

In this appendix, the relevant ordinary differential equations (ODEs) are given in detail, whereas they were only given in abbreviated form in (4.7). It turns out to be helpful to
define the following dimensionless auxiliaries:

\[
\begin{align*}
  u_1 &\equiv \left(1 - \frac{4}{3} \gamma\right) \sin^2 \left(\tilde{F}(w)/2\right), \\
  u_2 &\equiv \left(1 - \frac{2}{3} \gamma\right) \sin^2 \left(\tilde{F}(w)/2\right) + \frac{w}{4},
\end{align*}
\]  

dropping the argument \( w \) of \( u_1 \) and \( u_2 \).

The reduced Einstein equations can now be written in the form \( G^Y_Y = 8\pi G_N T^Y_Y \), \( G^Y_Y - G^T_T = 8\pi G_N (T^Y_Y - T^T_T) \), and \( G^Z_Z = 8\pi G_N T^Z_Z \) [it is also found that \( G^X_X = G^Z_Z, T^X_X = T^Z_Z \)]. In terms of dimensionless quantities, these reduced Einstein equations are

\[
\begin{align*}
  e^{-2\lambda} \left(1 + 2 \sqrt{w - y_0^2 \tilde{\nu}'}\right) - 1 &= \\
  \tilde{\eta} \left(4 \sin^2 \left(\tilde{F}/2\right) \left(\frac{u_1}{w - y_0^2} + 1\right) - 2 e^{-2\lambda} \left(u_2 - \frac{y_0^2}{4}\right) \tilde{F}'' + \frac{\gamma}{4} e^{-4\lambda} \left(w - y_0^2\right) \tilde{F}''\right), \\
  2\sqrt{w - y_0^2} \left(\tilde{\lambda}' + \tilde{\nu}'\right) &= \\
  \tilde{\eta} \left(- (4u_2 - y_0^2) \tilde{F}'' + \frac{\gamma}{3} e^{-2\lambda} \left(w - y_0^2\right) \tilde{F}''\right), \\
  e^{-2\lambda} \sqrt{w - y_0^2} \left(\tilde{\nu}' - \tilde{\lambda}'\right) \left(1 + \sqrt{w - y_0^2 \tilde{\nu}'}\right) + \sqrt{w - y_0^2 \tilde{\nu}''} &= \\
  \tilde{\eta} \left(- \frac{4u_1 \sin^2 \left(\tilde{F}/2\right)}{w - y_0^2} + \frac{1}{2} e^{-2\lambda} \left(w - y_0^2\right) \tilde{F}'' \left(1 - \frac{\gamma}{6} e^{-2\lambda} \tilde{F}''\right)\right).
\end{align*}
\]

The reduced matter field equation is

\[
\begin{align*}
  0 &= \frac{e^{2\lambda}}{w} \left(\sin \tilde{F} \left(u_1 + \frac{w}{2}\right) - \left(w - y_0^2\right) \left(1 + 4u_2 \left(\tilde{\nu}' - \tilde{\lambda}'\right)\right) + 2u_2\right) \tilde{F}' \\
  &- \left(1 - \frac{2}{3} \gamma\right) \left(w - y_0^2\right) (\sin \tilde{F}) \tilde{F}'' \\
  &- \frac{2}{3} \gamma e^{-2\lambda} \left(w - y_0^2\right) \left(2w \left(w - y_0^2\right) \left(3\tilde{\lambda}' - \tilde{\nu}'\right) - 5w + 2y_0^2\right) \tilde{F}'' \\
  &- 4 \left(w - y_0^2\right) \left(u_2 - \gamma e^{-2\lambda} w \left(w - y_0^2\right) \tilde{F}''\right) \tilde{F}''.
\end{align*}
\]

Using \( \tilde{\kappa}(w) \) and \( \tilde{\mu}(w) \) as defined in (4.2) and solving for \( \tilde{\kappa}', \tilde{\mu}', \) and \( \tilde{F}' \), one obtains the
The last two Christoffel symbols are seen to diverge at the defect core \((w = y_0^2)\), but the particle motion can still be regular, as we will now show by a simple example [using dimensionless Christoffel symbols \(\gamma_{bc}^a \equiv (ef)^{-1} \Gamma_{BC}^A\)].

**Appendix B: Vacuum-solution geodesics**

For the vacuum solution of Sec. \(\Box\) the nontrivial Christoffel symbols are

\[
\begin{align*}
\frac{1}{ef} \Gamma_{TY}^T &= \frac{1}{w^3 - \ell w}, \\
\frac{1}{ef} \Gamma_{TT}^Y &= \frac{(\sqrt{w^3 - \ell w}) \ell/2}{w^3 - \ell w}, \\
\frac{1}{ef} \Gamma_{YY}^Y &= \frac{\sqrt{w} \ y_0^2 - (w + y_0^2) \ell/2}{w \sqrt{w^3 - \ell w}}.
\end{align*}
\]

The last two Christoffel symbols are seen to diverge at the defect core \((w = y_0^2)\), but the particle motion can still be regular, as we will now show by a simple example [using dimensionless Christoffel symbols \(\gamma_{bc}^a \equiv (ef)^{-1} \Gamma_{BC}^A\)].
Consider, in fact, the geodesic equation for a particle moving solely in the $y$-direction and denote the nonvanishing dimensionless velocity components $u^t = dt/d\lambda$ and $u^y = dy/d\lambda$, where $\lambda$ is a dimensionless parameter and $c = 1$. The particle motion is, then, given by the following two equations:

$$0 = \frac{du^t}{d\lambda} + 2 \gamma^t_{ty} u^t u^y, \quad (B2a)$$

$$0 = \frac{du^y}{d\lambda} + \gamma^y_{tt} u^t u^t + \gamma^y_{yy} u^y u^y. \quad (B2b)$$

The first equation can be solved for $u^t$:

$$u^t = \left(1 - \frac{\ell}{\sqrt{w}}\right)^{-1}. \quad (B3)$$

Inserting this $u^t$ into the second equation gives

$$\frac{d^2 y}{d\lambda^2} + \frac{-y^2 \ell + 2 y^2_0}{2 y (y^2 + y^2_0)} \left(-\ell + \sqrt{y^2_0 + y^2} \right) \left(\frac{dy}{d\lambda}\right)^2 = \frac{\ell}{2 y \left(\ell - \sqrt{y^2_0 + y^2}\right)}. \quad (B4)$$

Finally, replace in (B4)

$$\frac{dy}{d\lambda} = \frac{1}{2y} \frac{dy^2}{d\lambda} , \quad \frac{d^2 y}{d\lambda^2} = \frac{1}{2y} \frac{d^2 y^2}{d\lambda^2} - \frac{1}{4y^3} \left(\frac{dy^2}{d\lambda}\right)^2 \quad (B5)$$

to obtain

$$\frac{d^2 y^2}{d\lambda^2} - \frac{1}{4 (y^2_0 + y^2)} \left(1 + \frac{\sqrt{y^2_0 + y^2}}{\sqrt{y^2_0 + y^2 - \ell}} \right) \left(\frac{dy^2}{d\lambda}\right)^2 = -\frac{\ell}{\sqrt{y^2_0 + y^2 - \ell}}, \quad (B6)$$

which remains finite along the trajectory as long as $\ell < y_0$.

For the special case $\ell = 0$ (flat spacetime with a “hole”), it is possible to obtain explicit solutions of (B2). Up to arbitrary time shifts, the radial geodesics are given in terms of two real constants $A$ and $B$, with $B$ taken positive:

$$y(t) = Ay_0, \quad (B7a)$$

$$y(t) = \begin{cases} 
\pm y_0 \sqrt{(B t)^2 + 2B t} & \text{for } t \geq 0, \\
\mp y_0 \sqrt{(B t)^2 - 2B t} & \text{for } t < 0,
\end{cases} \quad (B7b)$$

where the upper entries before $y_0$ on the right-hand side of (B7b) correspond to a positive asymptotic velocity and lower entries to a negative asymptotic velocity. Observe that $y^2$ from the second solution has a cusp at $t = 0$. 

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