Embedding Diagrams for the Reissner-Nordström spacetime

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We consider embedding diagrams for the Reissner-Nordström spacetime. We embed the \((r-t)\) and \((r-\phi)\) planes into 3-Minkowski/Euclidean space and discuss the relation between the diagrams and the corresponding curvature scalar of the 2-metrics.

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Embedding of a curved surface in the Euclidean space visually demonstrates the “curvedness” or curvature of the surface. Embedding diagrams therefore become excellent tools for visualizing the curvature of spacetime, i.e. the gravitational field. The particle trajectories could easily be seen and they would be analogues of the field lines in electromagnetic theory - they are indeed field lines of gravitational field. Of the most interesting objects in this context is black hole causing critical warping of space around it so that no null ray can propagate out of it. The embedding diagram would make this phenomenon transparent and visually illuminating.

In \cite{1}, a departure is made, from the conventional embedding process, in that the \((r-t)\) plane of spherically symmetric spacetimes is embedded into \((2+1)\)-Minkowski spacetime. Conventional embeddings are generally restricted to constant-time slices of the geometry and hence convey information only regarding the spatial curvature. Intuitively speaking the kinematic part of the field, gravitational potential sits in \((\phi_r, \phi_t, \phi_\theta)\) plane, \(\phi_r\) being embedded with the character of the embedding diagram. The \((r)\) has to be carried out over patches for \(r\) we follow the same method (which was first employed in \cite{4}) for the spatial embedding. Of course the embedding would make this phenomenon transparent and visually illuminating.

In \cite{5}, we consider embedding diagrams for the Reissner-Nordström spacetime. We embed the \((r-t)\) plane embedding has been studied \cite{6} and we follow the same method (which was first employed in \cite{3}) for the spatial embedding. Of course the embedding has to be carried out over patches for \(r_+ \leq r \leq r_- \leq r \leq r_+\), \(Q^2/2M \leq r \leq r_-\) where \(r_{\pm} = M \pm \sqrt{M^2 - Q^2}\) are the two horizons. In the former case, it cannot go as far in as \(r_-\) and can even stop above \(r_+\) if \(Q^2/M^2 > 8/9\), while the spatial embedding can go down to \(r = Q^2/2M\), the hard core radius, covering all the three patches. Of particular interest is the patch below \(r_-\).

We begin with RN spacetime metric described by

\[
\begin{align*}
&ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)}dr^2 + r^2(\theta^2 + \sin^2 \theta d\phi^2) \\
&\text{(1)}
\end{align*}
\]

We first consider the embedding of the \((r-t)\) plane in \((2+1)\)-Minkowski spacetime. Following \cite{6}, let us reduce the metric to the \((r-t)\) plane,

\[
\begin{align*}
&ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)}dr^2 \\
&\text{(2)}
\end{align*}
\]

which we wish to embed into the flat \((T, X, Y)\) spacetime. There are two horizons located at

\[
\begin{align*}
r_+ &= M + \sqrt{M^2 - Q^2}, \quad r_- = M - \sqrt{M^2 - Q^2} \\
\end{align*}
\]

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FIG. 1: Embedding diagram for RN spacetime \((r > r_+)\). The flanges represent the asymptotically flat regions. The cones represent their respective interiors.

By considering the intermediate coordinates

\[ \rho = \sqrt{X^2 - T^2}, \quad \psi = \tanh^{-1}(T/X) \] (4)

in the region \(r > r_+\) and

\[ \rho = \sqrt{T^2 - X^2}, \quad \psi = \tanh^{-1}(X/T) \] (5)

in the region \(r_- < r < r_+\), it is possible to show that the embedding is completely specified by taking

\[ Y(r) = \int_{r_+}^{r} \left[ \frac{1 - \frac{1}{4r^2} \left( \frac{d\mu}{dr} \right)^2}{\mu} \right] dr \] (6)

where

\[ \mu = 1 + 2\Phi = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad \kappa = \frac{1}{2} \frac{d\mu}{dr} |_{r=r_+}. \]

As shown in [1], note that the radical in the integrand near the horizon would go as \(-\frac{d^2\mu/dr^2}{2\kappa}\) which implies that embedding cannot proceed when \(\frac{d^2\mu}{dr^2}\) turns positive. That would happen at \(r = 3Q^2/2M\) which would lie between the two horizon unless \(Q^2/M^2 > 8/9\). Fig. 1 shows the embedding diagram. The two protruding flanges indicate the two asymptotically flat regions while the two half cones represent the corresponding region enclosed by the two horizons.

The scalar curvature of the \((r - t)\) plane is given by

\[ R = \frac{2 (2Mr - 3Q^2)}{r^4} \] (7)

That is embedding can proceed only until \(R = 0\) and not beyond it.

We now turn to the embedding of a constant time slice, the \((r - \phi)\) plane of RN geometry. Following [1] as before, we have to consider embeddings both in the Euclidean as well as Minkowski flat space depending upon the signature of the \((r - \phi)\) plane. The constant time equatorial plane slice would have the reduced metric,

\[ ds^2 = \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} dr^2 + r^2 d\phi^2 \] (8)
FIG. 2: Embedding of \((r - \phi)\) plane of RN spacetime for \(r > r_+\). One asymptotically flat region is shown. The lower end of the 'funnel' corresponds to the outer horizon at \(r_+\).

For \(r > r_+\), \(r\) it is spacelike and we embed this metric into the Euclidean space. To embed it, we employ the cylindrical polar coordinates in the Euclidean space, with the angle \(\phi\) being common for both. Thus the Euclidean metric is

\[
ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2
\]

For constant \(z\), a translation in \(\phi\), \((r, \phi) \rightarrow (r, \phi + \delta\phi)\) and \((\rho, \phi) \rightarrow (\rho, \phi + \delta\phi)\) must produce the same proper displacement in both metrics. Hence we must set

\[
\rho = r
\]

Now the embedding can be completed by specifying \(z\) as a function of \(r\). This is done by requiring agreement of the metrics (8) and (9) at constant \(\phi\). Thus

\[
d\rho^2 + dz^2 = \frac{1}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} dr^2,
\]

which can be solved to give (setting \(\rho = r\))

\[
z(r) = \int_{r_+}^r \sqrt{\frac{2Mr - Q^2}{r^2 - 2Mr + Q^2}} dr
\]

where we have set \(z(r_+) = 0\) and have chosen the appropriate sign for \(z\). That means there is no problem until \(r = Q^2/2M < r_-\). Here we have to consider all the three patches for embedding. For \(r > r_+\), the embedding diagram is shown in Fig. 2, which is quite similar to the Schwarzschild (see [3]).

For \(r_- < r < r_+\), \((1 - 2M/r + Q^2/r^2) < 0\) and the coordinate \(r\) is timelike. We therefore embed the metric (8), rewritten for convenience as

\[
ds^2 = \frac{-1}{2M/r - 1 - \frac{Q^2}{r^2}} dr^2 + r^2 d\phi^2
\]

into \((2 + 1)\)-Minkowski spacetime given by the metric,

\[
ds^2 = -dz^2 + d\rho^2 + \rho^2 d\phi^2.
\]

Once again we set \(\rho = r\). This reflects the fact that though \(r\) is timelike, it still functions as the area radius of the spacetime. Proceeding as before, we require that

\[-dz^2 + d\rho^2 = \frac{-1}{2M/r - 1 - \frac{Q^2}{r^2}} dr^2\]

from which we get

\[
z(r) = \int_{r}^{r_+} \sqrt{\frac{2Mr - Q^2}{2Mr - r^2 - Q^2}} dr
\]
where we have again set \( z(r_+) = 0 \). The embedding diagram is shown in Fig. 3. The top of Fig. 3 would join smoothly to the bottom of the funnel of Fig. 2.

For \( r < r_- \), \( r \) is once again spacelike and the embedding is carried out as before (Eqn. (8) through Eqn. (11)), except that the limits of integration in the expression for \( z \) would now range from \( r \) to \( r_- \). This time, however, the expression under the radical in eqn. (11) changes sign at \( r = \frac{3Q^2}{2M} \) and that is where the embedding terminates. The diagram (blown up to see the detail) is shown in Fig. 4. Again, on the same scale, the diagrams in Figs 2 and 3 would smoothly join at the ends.

The scalar curvature for this metric is

\[
R = \frac{-2 \left( Mr - Q^2 \right)}{r^4}
\]  

which changes sign at \( r = \frac{Q^2}{M} \), lying between the two horizons. Unlike the \((r - t)\) plane case, where embedding terminated when curvature turned zero, here the embedding can proceed beyond \( r = \frac{Q^2}{2M} \) as far in as \( r = \frac{Q^2}{2M} \).

In this case we have embeddings for both positive as well as negative curvature regions. It begins with negative curvature in Fig. 2, changing sign in Fig. 3, and it is all positive in Fig. 4.

Fig. 1 refers to the whole field incorporating the contributions of both potential and field energy while the spatial embedding diagrams as shown in Figs 2-4 refer to the contribution of the field energy alone. The embedding diagrams do reflect when the curvature changes sign from negative to positive. This would mean the contribution of field energy changing its sense from attractive to repulsive and it happens at \( r = \frac{Q^2}{M} \), where interestingly \( d\Phi/dr \) also changes sign; i.e., gravitational field changes sign [6]. As mentioned earlier, the contributions from the potential as well as the field energy act in unison and hence they should change their sense at the same radius. The point is that for attractive gravity, the field energy must be negative and the vice-versa [4]. That is why the two have to change sign at the same radius.

However, the difference is that the field energy only links to the spatial curvature [2] while the other to the potential gradient. This happens because curvature in the \((r - \phi)\) plane is proportional to \( d\Phi/dr \) while the curvature in the \((r - t)\) plane measures the tidal acceleration between two neighbouring geodesics which is given by \( d^2\Phi/dr^2 \), and the embedding in this case terminates where it changes sign at \( r = \frac{3Q^2}{2M} \). On the other hand the embedding for the \((r - \phi)\) case terminates where the gravitational potential \( \Phi \) changes sign at \( r = \frac{Q^2}{2M} \). It is interesting that embedding of the \((r - t)\) plane (i.e. the whole field) hinges on the tidal acceleration while that of the \((r - \phi)\) plane on the potential. This feature we do not quite understand and would like to probe it further in future.

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[FIG. 3: Embedding diagram for the \((r - \phi)\) plane of RN spacetime for \( r_- < r < r_+ \). On the same scale as Fig. 2, the top of this diagram would join smoothly to the bottom of the one in Fig. 2.]
FIG. 4: Embedding of the $(r - \phi)$ plane of RN spacetime for $r < r_-$. The diagram is enlarged in order to see the detail.

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