Uncertainty equalities and uncertainty relation in weak measurement

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I. INTRODUCTION

Uncertainty principle is one of the basic tenets of quantum mechanics. The initial spirit of uncertainty principle was postulated by Heisenberg \cite{1}. Kennard first mathematically derived the Heisenberg uncertainty relation \cite{2}. The most famous and popular form is the Heisenberg-Robertson uncertainty relation \cite{3}.

\[ \Delta A^2 \Delta B^2 \geq \frac{1}{4} \langle [A,B] \rangle^2, \quad (1) \]

for any observables $A$, $B$, and any state $| \psi \rangle$, where the variance of an observable $X$ in state $| \psi \rangle$ is defined as $\Delta X^2 = \langle \psi | X^2 | \psi \rangle - \langle \psi | X | \psi \rangle^2$ and the commutator is defined as $[A,B] = AB - BA$. A stronger extension of the Heisenberg-Robertson uncertainty relation \cite{4} was made by Schrödinger \cite{4}, which is generally formulated as

\[ \Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle [A,B] \rangle \right|^2 + \left| \frac{1}{2} \langle \{A,B\} \rangle - \langle A \rangle \langle B \rangle \right|^2, \quad (2) \]

where the anticommutator is defined as $\{A,B\} = AB + BA$, and $\langle X \rangle$ is defined as the expectation value $\langle \psi | X | \psi \rangle$ for any operator $X$ with respect to the normalized state $| \psi \rangle$.

However, the above two uncertainty relations have the problem that they may be trivial even when $A$ and $B$ are incompatible on the state $| \psi \rangle$. In order to correct this problem, Maccone and Pati \cite{5} presented two stronger uncertainty relations based on the sum of variances. The first one reads

\[ \Delta A^2 + \Delta B^2 \geq \frac{\pi}{2} \left| \langle [A,B] \rangle \right| + \left| \langle \psi | A + i B | \psi \rangle \right|^2, \quad (3) \]

which is valid for arbitrary states $| \psi \rangle$ orthogonal to the state of the system $| \psi \rangle$, where the sign should be chosen so that $\pm i \langle [A,B] \rangle$ (a real quantity) is positive. The second uncertainty relation is

\[ \Delta A^2 + \Delta B^2 \geq \frac{1}{2} \left| \langle \psi_A^\perp B | A + B | \psi \rangle \right|^2. \quad (4) \]

Here $| \psi_A^\perp B \rangle \propto (A + B - \langle A + B \rangle | \psi \rangle)$ is a state orthogonal to $| \psi \rangle$. Maccone and Pati also derived an amended Heisenberg-Robertson uncertainty relation

\[ \Delta A \Delta B \geq \frac{\pm i \frac{\pi}{4} \langle [A,B] \rangle}{1 - \frac{1}{2} \langle \psi | A | \psi \rangle^2 \pm \frac{e^{i \alpha} B | \psi \rangle}{2}}, \quad (5) \]

which is stronger than the Heisenberg-Robertson uncertainty relation \cite{4}.

Recently, two stronger Schrödinger-like uncertainty relations have been proved which go beyond the Maccone and Pati’s uncertainty relation \cite{6}. The new relations provide stronger bounds whenever the observables are incompatible on the state $| \psi \rangle$. The first uncertainty relation is

\[ \Delta A^2 + \Delta B^2 \geq \left| \langle [A,B] \rangle \right| + \left| \langle \{A,B\} \rangle - 2 \langle A \rangle \langle B \rangle \right| + \left| \langle \psi | A - e^{i \alpha} B | \psi \rangle \right|^2, \quad (6) \]

which is valid for arbitrary states $| \psi \rangle$ orthogonal to the state of the system $| \psi \rangle$ and stronger than the Maccone and Pati’s uncertainty relation \cite{5}. Here $\alpha$ is a real constant, if $\langle \{A,B\} \rangle - 2 \langle A \rangle \langle B \rangle > 0$, then $\alpha = \arctan \left( -\frac{\langle [A,B] \rangle}{1 - \langle \psi | A | \psi \rangle^2} \right)$; if $\langle \{A,B\} \rangle - 2 \langle A \rangle \langle B \rangle < 0$, then $\alpha = \pi + \arctan \left( -\frac{\langle [A,B] \rangle}{1 - \langle \psi | A | \psi \rangle^2} \right)$, and while $\langle \{A,B\} \rangle - 2 \langle A \rangle \langle B \rangle = 0$, the relation \textsuperscript{(6)} will reduce to \textsuperscript{(4)}. The second uncertainty relation is

\[ \Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle [A,B] \rangle \right|^2 + \left| \frac{1}{2} \langle \{A,B\} \rangle - \langle A \rangle \langle B \rangle \right|^2 \quad (7) \]

which is stronger than the Schrödinger uncertainty relation \cite{4}.

However, these new state dependent uncertainty relations have some problem \cite{7}, but some state independent uncertainty relations \cite{8,9} immune from the drawback. Maccone and Pati’s uncertainty relations \cite{5} still are very important and have some generalizations. Two variance-based uncertainty equalities have been proved recently by Yao et al. \cite{10} on the trend of stronger uncertainty relations \cite{5}, for all pairs of incompatible observables $A$ and $B$. Meanwhile, two uncertainty relations in weak measurement were introduced by Pati et al. \cite{11} for variances of two non-Hermitian operators corresponding to two non-commuting observables.

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In this work we derive and proof two uncertainty equalities, which hold for all pairs of incompatible observables \( A \) and \( B \). We also give an uncertainty relation in weak measurement for two non-Hermitian operators corresponding to two non-commuting observables.

II. UNCERTAINTY EQUALITIES

In this section, we construct and prove two uncertainty equalities which imply the uncertainty inequalities \([6]\) and \([7]\).

Uncertainty relation 1

\[
\Delta A^2 + \Delta B^2 = |\langle [A, B] \rangle + \langle A, B \rangle - 2\langle A \rangle \langle B \rangle| + \sum_{n=1}^{d-1} |\langle \psi |A - e^{i\alpha} B|\psi_n^\perp \rangle|^2,
\]

(8)

where \(|\psi\rangle, |\psi_n^\perp\rangle_{n=1}^{d-1}\) comprise an orthonormal complete basis in the \( d \)-dimensional Hilbert space.

Proof: To prove our uncertainty relation, let us define the operator \( \Pi = I - |\psi\rangle \langle \psi | \), \( A = A - \langle A \rangle I \), \( B = B - \langle B \rangle I \) and the state \(|\phi\rangle = (A - e^{i\alpha} B)|\psi\rangle\), we have

\[
\langle \psi | \Pi | \psi \rangle = \langle \psi | (A - e^{-i\alpha} B)(I - |\psi\rangle \langle \psi |)(A - e^{i\alpha} B)|\psi \rangle
\]

\[
= \langle \psi | (\bar{A} - e^{i\alpha} \bar{B})(\bar{A} - e^{-i\alpha} \bar{B})|\psi \rangle
\]

\[
= \Delta A^2 + \Delta B^2 - 2\text{Re}(e^{i\alpha} \langle \psi | \bar{A} \bar{B} |\psi \rangle).
\]

(9)

There exists \( \tau = -\alpha \), so that \( e^{i\tau} \langle \psi | \bar{A} \bar{B} |\psi \rangle \) is real, and it can be written as \(|\langle \psi | \bar{A} \bar{B} |\psi \rangle|\), we obtain

\[
\langle \psi | (\bar{A} - e^{i\alpha} \bar{B})(\bar{A} - e^{-i\alpha} \bar{B})|\psi \rangle
\]

\[
= \Delta A^2 + \Delta B^2 - |\langle [A, B] \rangle + \langle A, B \rangle - 2\langle A \rangle \langle B \rangle|,
\]

(10)

Since \( \Pi \) is the orthogonal complement to \(|\psi\rangle\), \( |\psi\rangle\), we can choose an arbitrary orthogonal decomposition of the projector \( \Pi \)

\[
\Pi = \sum_{n=1}^{d-1} |\psi_n^\perp\rangle \langle \psi_n^\perp |,
\]

(11)

where \(|\psi\rangle, |\psi_n^\perp\rangle_{n=1}^{d-1}\) comprise an orthonormal complete basis in the \( d \)-dimensional Hilbert space. Whence, Eq. \([11]\) can be rewritten as

\[
\sum_{n=1}^{d-1} |\langle \psi | (\bar{A} - e^{i\alpha} \bar{B})|\psi_n^\perp \rangle|^2
\]

\[
= \sum_{n=1}^{d-1} |\langle \psi | A - e^{i\alpha} B|\psi_n^\perp \rangle|^2
\]

\[
= \Delta A^2 + \Delta B^2 - |\langle [A, B] \rangle + \langle A, B \rangle - 2\langle A \rangle \langle B \rangle|,
\]

(12)

which is equivalent to \([8]\).

Uncertainty relation 2

\[
\Delta A^2 \Delta B^2 = \left[ \frac{1}{2} \langle [A, B] \rangle + \frac{1}{2} \langle A, B \rangle - \langle A \rangle \langle B \rangle \right]^2 
\]

\[
(1 - \frac{1}{2} \sum_{n=1}^{d-1} |\langle \psi | A - e^{i\alpha} B|\psi_n^\perp \rangle|^2)^2
\]

(13)

where \(|\psi\rangle, |\psi_n^\perp\rangle_{n=1}^{d-1}\) comprise an orthonormal complete basis in the \( d \)-dimensional Hilbert space.

Proof: To prove our uncertainty equality, let us define the operator \( \Pi = I - |\psi\rangle \langle \psi | \), \( A = A - \langle A \rangle I \), \( B = B - \langle B \rangle I \) and the unnormalized state \(|\phi\rangle = (\bar{A} - e^{i\tau} \bar{B})|\psi\rangle\), we have

\[
\langle \psi | \Pi | \psi \rangle
\]

\[
= \langle \psi | (\bar{A} - e^{-i\tau} \bar{B})(I - |\psi\rangle \langle \psi |)(\bar{A} - e^{i\tau} \bar{B})|\psi \rangle
\]

\[
= \langle \psi | (\bar{A} - e^{-i\tau} \bar{B})(\bar{A} - e^{i\tau} \bar{B})|\psi \rangle
\]

\[
= \Delta A^2 + \Delta B^2 - 2\text{Re}(e^{i\tau} \langle \psi | \bar{A} \bar{B} |\psi \rangle),
\]

(14)

There exists \( \tau = -\alpha \), so that \( e^{i\tau} \langle \psi | \bar{A} \bar{B} |\psi \rangle \) is real, and it can be written as \(|\langle \psi | \bar{A} \bar{B} |\psi \rangle|\), we obtain

\[
\langle \psi | (\bar{A} - e^{i\alpha} \bar{B})(\bar{A} - e^{-i\alpha} \bar{B})|\psi \rangle
\]

\[
= \Delta A^2 + \Delta B^2 - 2\text{Re}(e^{i\alpha} \langle \psi | \bar{A} \bar{B} |\psi \rangle)
\]

\[
= \Delta A^2 + \Delta B^2 - 2 - 2\frac{\text{Re}(e^{i\alpha} \langle \psi | \bar{A} \bar{B} |\psi \rangle)}{\Delta A \Delta B},
\]

(15)

Similarly, we choose the projector \( \Pi \)

\[
\Pi = \sum_{n=1}^{d-1} |\psi_n^\perp\rangle \langle \psi_n^\perp |
\]

(16)

Then Eq. \([15]\) can be rewritten as

\[
\sum_{n=1}^{d-1} |\langle \psi | (\bar{A} - e^{i\alpha} \bar{B})|\psi_n^\perp \rangle|^2
\]

\[
= \sum_{n=1}^{d-1} |\langle \psi | \bar{A} - e^{-i\alpha} \bar{B}|\psi_n^\perp \rangle|^2
\]

\[
= 2 - 2\frac{\text{Re}(e^{i\alpha} \langle \psi | \bar{A} \bar{B} |\psi \rangle)}{\Delta A \Delta B},
\]

(17)

which is equivalent to \([13]\).

The two uncertainty equalities \([8]\) and \([13]\) are valid for all pairs of incompatible observables. If we retain only one term associated with \(|\psi_n^\perp\rangle_{n=1}^{d-1}\) in the summation and discard the others, the uncertainty equalities \([8]\) and \([13]\) reduce to the uncertainty relations \([6]\) and \([7]\), respectively.

III. UNCERTAINTY RELATION IN WEAK MEASUREMENT

First introduced by Aharonov, Albert, and Vaidman \([12]\), weak values are complex numbers that one can define the weak value of \( A \) using two states: an initial state
\[ |\psi\rangle, \text{called the preselection, and a final state } |\varphi\rangle, \text{called the postselection. the weak value of } A \text{ has the form} \]
\[ \langle A \rangle_w = \frac{\langle \varphi | A | \psi \rangle}{\langle \varphi | \psi \rangle}. \tag{18} \]

For a given preselected and post-selected ensemble, define the operator \( A_w \) as
\[ A_w = \frac{\Pi_{\varphi} A}{p}, \tag{19} \]
where \( \Pi_{\varphi} = |\varphi\rangle\langle \varphi | \) and \( p = |\langle \varphi | \psi \rangle|^2 \). This has many properties please reference \[11\].

Here, we construct an uncertainty relation in weak measurement for variances of two non-Hermitian operators \( A_w \) and \( B_w \), corresponding to two non-commuting observables \( A \) and \( B \). The uncertainty relation quantitatively express the impossibility of jointly sharp preparation of pre- and post-selected (PPS) quantum states \( |\psi\rangle \) and \( |\varphi\rangle \) for the weak measurement of incompatible observables.

**Uncertainty relation 3**

\[ \Delta A_w^2 + \Delta B_w^2 \geq \frac{1}{p} \langle \varphi | [A,B] | \varphi \rangle + \frac{1}{p} \langle \varphi | [A,B] | \varphi \rangle - 2 \langle A_w \rangle (B_w)^* + \left| \langle \psi | A_w - e^{i\alpha} B_w | \psi^\perp \rangle \right|^2. \tag{20} \]

which is valid for two non-Hermitian operators \( A_w \) and \( B_w \), where \( p \) is equivalent to \( |\langle \varphi | \psi \rangle|^2 \).

**Proof:** To prove this relation we define the variance for any general (non-Hermitian) operator \( X \) in a state \( |\psi\rangle \) which can be defined as \[13\ [14]\]
\[ \Delta X^2 = \langle \psi | (X - \langle X \rangle) (X^\dagger - \langle X^\dagger \rangle) | \psi \rangle. \tag{21} \]

The variance of the non-Hermitian operation \( A_w \) in the quantum \( |\psi\rangle \) can be defined as
\[ \Delta A_w^2 = \langle \psi | (A_w - \langle A_w \rangle) (A_w^\dagger - \langle A_w^\dagger \rangle) | \psi \rangle, \tag{22} \]
where \( \langle A_w \rangle = \langle \psi | A_w | \psi \rangle \) and \( \langle A_w^\dagger \rangle = \langle \psi | A_w^\dagger | \psi \rangle = \langle A_w \rangle^* \). \( \Delta A_w^2 \) can also be expressed as
\[ \Delta A_w^2 = \langle \psi | A_w A_w^\dagger | \psi \rangle - \langle \psi | A_w | \psi \rangle \langle \psi | A_w^\dagger | \psi \rangle. \tag{23} \]

Similarly, for Hermitian operator \( B \), we can define the operator
\[ B_w = \frac{\Pi_{\varphi} B}{p}. \tag{24} \]

Then, the uncertainty for \( B_w \) can also be defined as
\[ \Delta B_w^2 = \langle \psi | B_w B_w^\dagger | \psi \rangle - \langle \psi | B_w | \psi \rangle \langle \psi | B_w^\dagger | \psi \rangle. \tag{25} \]
To prove our uncertainty relation in weak measurement, we introduce a general inequality
\[ ||C^\dagger | \psi \rangle - e^{i\tau} D^\dagger | \psi \rangle + k (|\psi\rangle - |\psi\rangle)||^2 \geq 0, \tag{26} \]
where \( C^\dagger \equiv A_w^\dagger - \langle A_w^\dagger \rangle \) and \( D^\dagger \equiv B_w^\dagger - \langle B_w^\dagger \rangle \).

By expanding the square modulus, we have
\[ \Delta A_w^2 + \Delta B_w^2 \geq -\lambda k^2 - \beta k + \pi, \tag{27} \]
where \( \lambda \equiv 2(1 - \text{Re}(\langle \psi | \bar{\psi} \rangle)), \pi \equiv 2\text{Re}(e^{i\tau} \langle \psi | CD^\dagger | \psi \rangle), \) and \( \beta \equiv 2\text{Re}(\langle \psi | (C - e^{-i\tau} D) | \psi \rangle) \). We choose the value of \( k \) that maximizes the right-hand-side of (27), namely \( k = -\beta/2\lambda \), we get
\[ \Delta A_w^2 + \Delta B_w^2 \geq \frac{\beta^2}{4\lambda} + \pi. \tag{28} \]

The above inequality can be rewritten as
\[ \Delta A_w^2 + \Delta B_w^2 \geq \frac{\text{Re}(\langle \psi | (C - e^{-i\tau} D) | \psi \rangle)}{2(1 - \text{Re}(\langle \psi | \bar{\psi} \rangle))} + 2\text{Re}(e^{i\tau} \langle \psi | CD^\dagger | \psi \rangle) \tag{29} \]

Suppose \( |\bar{\psi}\rangle = \cos \theta |\psi\rangle + e^{i\phi} \sin \theta |\psi^\perp \rangle \), where \( |\psi^\perp \rangle \) is orthogonal to \( |\psi\rangle \), by taking the limit \( \theta \to 0 \), the state \( |\bar{\psi}\rangle \) reduces to \( |\psi\rangle \) and then the above inequality can be reexpressed as
\[ \Delta A_w^2 + \Delta B_w^2 \geq \text{Re}(e^{i\phi} \langle \psi | (A_w - e^{-i\tau} B_w) | \psi^\perp \rangle)^2 + 2\text{Re}(e^{i\tau} \langle \psi | CD^\dagger | \psi \rangle), \tag{30} \]

there exists \( \tau = -\alpha \) so that \( e^{i\tau} \langle \psi | CD^\dagger | \psi \rangle \) is real, and it can be written as \( \langle \psi | CD^\dagger | \psi \rangle \), and then the second term becomes \( \text{Re}(e^{i\phi} \langle \psi | (A_w - e^{i\alpha} B_w | \psi^\perp \rangle)^2 \), we can choose \( \phi \) so that this term in square brackets is real, so that this term can be expressed as \( \langle \psi | A_w - e^{i\alpha} B_w | \psi^\perp \rangle^2 \). Whence, inequality \[30\] becomes
\[ \Delta A_w^2 + \Delta B_w^2 \geq \langle \psi | A_w - e^{i\alpha} B_w | \psi^\perp \rangle^2 + 2\langle \psi | CD^\dagger | \psi \rangle. \tag{31} \]

The last term can be rewritten as
\[ 2\langle \psi | CD^\dagger | \psi \rangle = |\langle CD^\dagger \rangle| \geq |\langle CD^\dagger + DC^\dagger \rangle|, \tag{32} \]
where
\[ \langle CD^\dagger + DC^\dagger \rangle = \frac{1}{p} \langle \varphi | [A,B] | \varphi \rangle - \langle A_w | B_w \rangle - \langle A_w | B_w \rangle^* \tag{33} \]

and
\[ \langle CD^\dagger - DC^\dagger \rangle = \frac{1}{p} \langle \varphi | [A,B] | \varphi \rangle - \langle A_w | B_w \rangle + \langle A_w | B_w \rangle^*. \tag{34} \]

We combine Eqs. \[33\] and \[34\], Eq. \[32\] becomes
\[ 2|\langle CD^\dagger \rangle| \geq \frac{1}{p} \langle \varphi | [A,B] | \varphi \rangle + \frac{1}{p} \langle \varphi | [A,B] | \varphi \rangle - 2\langle A_w | B_w \rangle^*. \tag{35} \]

Combining Eqs. \[32\] and \[35\], we obtain the uncertainty relation \[20\].
IV. CONCLUSIONS

In this work, we derived two new uncertainty equalities for sum and product of variances of a pair of incompatible observables, which hold for all pairs of incompatible observables $A$ and $B$. In fact, one can obtain a series of inequalities by retaining 1 to $d - 2$ terms within the set $\{|\psi_n^+\rangle_{n=1}^{d-1}\}$. We also derived an uncertainty relation in weak measurement for two non-Hermitian operators $A_w$ and $B_w$ corresponding to two non-commuting observables $A$ and $B$. The uncertainty relation quantitatively expresses the impossibility of jointly sharp preparation of PPS quantum states $|\psi\rangle$ and $|\phi\rangle$ for measuring incompatible observables during the weak measurement.

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