A Description and Proof of a Generalised and Optimised Variant of Wikström’s Mixnet

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1 Introduction

In this paper, we describe an optimised variant of Wikström’s mixnet which shuffles vectors of ElGamal ciphertexts in parallel. We then show in detail that this construction is secure.

A verifiable shuffle takes a packet of ciphertexts, which it re-encrypts and shuffles to produce an output packet. More specifically, a cryptographic shuffle of ElGamal encryptions \( e = (e_1, \ldots, e_N) \) is another list of ElGamal encryptions \( e' = (e'_1, \ldots, e'_N) \), which contains the same plaintexts \( m_i \) in permuted order. Given \( e \) and \( e' \) we may wish to prove that they have this relationship, this called a proof of shuffle.

Wikström’s verifiable mixnet as we refer to it here was first presented in “Proofs of Restricted Shuffles” by Terelius and Wikström[2], building on Wikström’s previous work in [3]. Specifically we take the optimised variant for ElGamal which appears to be in common use; for instance, it is presented in Haenni et al’s pseudo-code algorithms for implementing Wikström’s verifiable mixnet [1]. We extend the mixnet to support parallel shuffles, where each \( e_i \) and \( e'_i \) are themselves vectors of related ciphertexts. (The possibility of doing this is proven by the Wikström’s result but we wish to show that this particular instance with its optimisations is secure.)

2 Notation

- \( G_q \) is a cyclic group of prime order \( q \) in which both the decisional and computational Diffie-Hellman problems are hard. We will use the multiplicative notation for the group operation. As usually, by \( \mathbb{Z}_q \) we denote the field of integers modulo \( q \).
- \( A^N \) is the set of vectors of length \( N \) containing elements of \( A \). We will denote vectors in bold, for instance \( \mathbf{a} \). We will denote the \( i \)th element using subscript; for instance as \( a_i \).
- Similarly, \( A^{N \times N} \) is the set of square matrices of order \( N \) containing elements of \( A \). We will denote matrices using upper case letters, for instance \( M \). We will denote the \( i \)th column of \( M \) as \( M_i \) and the element \( i \)th row and \( j \)th column as \( M_{i,j} \).

A matrix \( M \), containing only 0 and 1 values, is a permutation matrix, if every column and every row contains exactly one 1.
- \(PC_{h,h_1}(m,r)\), for \(m,r \in \mathbb{Z}_q\) and \(h,h_1 \in G_q\), is defined as \(h' h_1^m\) (note that \(h\) and \(h_1\) are group elements and hence the multiplication here denotes the group multiplication).
- \(PC_{h,h_1}(m,r)\) is known as a Pedersen commitment.
- \(EPC_{h_1,h_2}(m,r)\), for \(m \in \mathbb{Z}_q^N\) and \(r \in \mathbb{Z}_q\), is defined as \(h' \prod_{i=1}^N h_i^m\) (otherwise known as an extended Pedersen commitment).
- \(PC_{h_1,h_2}(M,r)\), for \(M \in \mathbb{Z}_q^{N \times N}\) and \(r \in \mathbb{Z}_q^N\), is \((c_1, \ldots, c_n)\) where \(c_i = h_i'^N \prod_{j=1}^N h_j^{M_{ij}}\), which means that \(c_i\) is the extended Pedersen commitment to the \(i\)th column of \(M\).
- \(\text{Enc}_{g, pk}(m,r)\) for \(m \in G_q\) and \(r \in \mathbb{Z}_q\) is \((g^r,pk'm)\) (the ElGamal encryption of the group element \(m\)).
- \(\text{ReEnc}_{g, pk}(e,r)\), for \(e \in G_q^2\) and \(r \in \mathbb{Z}_q\) is \((e_1 g^r, e_2 pk')\).
- \(\text{Enc}_{g, pk}(m,r)\), for \(m \in G_q^w\) and \(r \in \mathbb{Z}_q^w\), is \(\text{Enc}_{g, pk}(m_1, r_1), \ldots, \text{Enc}_{g, pk}(m_w, r_w)\).
- \(\text{ReEnc}_{g, pk}(e,r)\), for \(e \in (G_q^2)^w\) and \(r \in \mathbb{Z}_q^w\), is \(\text{ReEnc}_{g, pk}(e_1, r_1), \ldots, \text{ReEnc}_{g, pk}(e_w, r_w)\).
- \(\langle a, b \rangle\), for \(a \in \mathbb{Z}_q^N\) and \(b \in \mathbb{Z}_q^N\) is \(\sum_{i=1}^N a_i b_i \mod q\).
- \(\text{AB}, \) for \(A \in \mathbb{Z}_q^{N \times M}\) and \(B \in \mathbb{Z}_q^{N \times w}\), is a matrix in \(\mathbb{Z}_q^{N \times w}\) where the value in the \(i\)th row and \(j\)th column is equal to \(\sum_{k=1}^M A_{i,k} B_{k,j}\).
- \(\text{MX}, \) for \(M \in \mathbb{Z}_q^{M \times N}\) and \(x \in \mathbb{Z}_q^N\), is a vector of length \(M\) where \(i\)th position is equal to \(\sum_{j=1}^N M_{i,j} x_j\). (Note that both this and the next definition are consistent with treating \(x\) as a column, and row vector respectively, and applying the definition of matrix multiplication definition from above.)
- \(\text{xM}, \) for \(M \in \mathbb{Z}_q^{N \times M}\) and \(x \in \mathbb{Z}_q^N\), is a vector of length \(M\) where the \(i\)th position is equal to \(\sum_{j=1}^N x_j M_{i,j}\).
- \(\alpha x, \) for \(a \in \mathbb{Z}_q\) and \(x \in \mathbb{Z}_q^N\), is a vector of length \(N\) where \(i\)th position is equal to \(\alpha x_i\).
- \(x^\alpha, \) for \(a \in \mathbb{Z}_q\) and \(x \in \mathbb{Z}_q^N\), is a vector of length \(N\) where \(i\)th position is equal to \(x_i^\alpha\).
- For two vectors \(x,y \in \mathbb{Z}_q^N\) we sometimes abuse notation by writing \(x + y,\) \(x \times y,\) and \(x^y\) to denote the pairwise addition, multiplication, and exponentiation of the vectors respectively.
- For a matrix \(M,\) by \(\pi_M\) we denote the permutation of the set \(\{1, \ldots, N\}\) defined by \(M,\) that is such a permutation that for each vector \(x\) we have \(x = (y_{\pi(1)}, \ldots, y_{\pi(N)})\), where \(y = \text{MX}\).
- A binary relation \(R\) for a set statements of \(S\) and witnesses \(W\) is a subset of the cartesian product of \(S\) and \(W\).
- For two binary relations \(R\) and \(R'\), we denote by \(R \wedge R'\) a relation between \((S \times S')\) and \((W \times W')\) the cartesian product of the statements and witness of \(R\) and \(R'\). The relation is said to hold when both the subrelations hold.
- For two binary relations \(R\) and \(R'\), we denote by \(R \lor R'\) a relation between \((S \times S')\) and \((W \times W')\). The relation is said to hold when either subrelations holds.
- For two binary relations \(R\) and \(R'\) where \(W = W'\), we denote by \(R \cap R'\) a relation between \((S \times S')\) and \((W\). The relation is said to hold when both the subrelations hold. We will abuse notation by writing \(R \cap R'\) when \(W \neq W'\) but are both cartesian products with subgroups in common.
3 Shuffle Proof - Description and Proof

**Algorithm 1: Interactive ZK-Proof of Extended Shuffle**

**Common Input**: A group generator $g \in G_q$, public key $pk \in G_q$, matrix commitment $c \in G_q^N$, commitment parameters $h, h_1, \ldots, h_N \in G_q$, ciphertext vectors $e_1, \ldots, e_N \in (G_q^2)^w$ and $e'_1, \ldots, e'_N \in (G_q^2)^w$.

**Private Input**: Permutation matrix $M \in \mathbb{Z}_q^{N \times N}$, randomness $r \in \mathbb{Z}_q^w$ and randomness $R \in \mathbb{Z}_q^{w \times N}$, such that $c = C_{h,h_1,\ldots,h_N} (M, r)$ and $e'_i = \text{ReEnc}_{g, pk}(e_{a_i(i)}, R_{a_i(i)})$.

1. $V$ chooses $u \in \mathbb{Z}_q^N$ randomly and hands it to $P$.
2. $P$ computes $u' = Mu$. Then $P$ chooses $\hat{r} \in \mathbb{Z}_q^N$ at random and computes

$$
\begin{align*}
\tilde{r} &= r_1 + \cdots + r_N, \\
\tilde{r}^* &= \hat{r} + \sum_{i=1}^{N-1} \left( \tilde{r}_i \prod_{j=i+1}^{N} u'_j \right), \\
\tilde{r}^* &= Ru
\end{align*}
$$

$P$ randomly chooses $\omega, \omega' \in \mathbb{Z}_q^N$, $\omega_1, \omega_2, \omega_3 \in \mathbb{Z}_q$, and $\omega_4 \in \mathbb{Z}_q^N$, and hands the following values to $V$:

$$
\begin{align*}
\tilde{c}_0 &= h_1, \\
\tilde{c}_i &= h_\tilde{r} \tilde{c}_{i-1}^N \quad (i \in \{1, \ldots, N\}) \\
t_1 &= h^{\omega_1} \\
t_2 &= h^{\omega_2} \\
t_3 &= h^{\omega_3} \prod_{i=1}^{N} h_i^\omega \\
t_4 &= \text{ReEnc}_{g, pk}(\prod_{i=1}^{N} e_i^{\omega'}, -\omega_4) \\
\hat{t}_i &= h^{\omega_4} \tilde{c}_{i-1}^N \quad (i \in \{1, \ldots, N\})
\end{align*}
$$

3. $V$ chooses a challenge $c \in \mathbb{Z}_q$ at random and sends it to $P$.
4. $P$ then responds,

$$
\begin{align*}
s_1 &= \omega_1 + c \cdot \tilde{r}, \\
s_2 &= \omega_2 + c \cdot \tilde{r}^*, \\
s_3 &= \omega_3 + c \cdot \tilde{r}, \\
s_4 &= \omega_4 + c \cdot \tilde{r}^*
\end{align*}
$$

$$
\begin{align*}
\hat{s} &= \omega + c \cdot \tilde{r} \\
\hat{s}' &= \omega' + c \cdot u'
\end{align*}
$$

5. $V$ accepts if and only if

$$
\begin{align*}
t_1 &= (\prod_{i=1}^{N} e_i / \prod_{i=1}^{N} h_i)^{-c} h_{\tilde{r}_i} \\
t_2 &= (\tilde{c}_N / h_1 \prod_{i=1}^{N} u_i)^{-c} h_{\tilde{r}_i} \\
t_3 &= (\prod_{i=1}^{N} e_i^{u_i})^{-c} h_{\tilde{r}_i} \prod_{i=1}^{N} h_i^\omega \\
t_4 &= \text{ReEnc}_{g, pk}(\prod_{i=1}^{N} e_i^{u_i})^{-c} \prod_{i=1}^{N} e_i^{s_i} \\
\hat{t}_i &= \hat{c}_i^{-c} h_{\tilde{c}_{i-1}}
\end{align*}
$$
Formal Security Statement In the security statement for the presented shuffle algorithm, we will use the following notation.

- $\mathcal{R}_{com}(h, h_1, \ldots, h_N)(m, r, m', r')$ is a relationship between the commitment parameters $(h, h_1, \ldots, h_N)$ and $(m, m' \in \mathbb{Z}_q^N, r, r' \in \mathbb{Z}_q)$ which holds if and only if $\mathsf{EPC}(m, r) = \mathsf{EPC}(m', r')$ and $m \neq m'$.
- $\mathcal{R}_π(h, h_1, \ldots, h_N, c)(M, r)$ is the relationship between the commitment parameters $(h, h_1, \ldots, h_N)$, a commitment $c \in G_q$, a permutation matrix $M \in \mathbb{Z}_q^{N \times N}$, and $r \in \mathbb{Z}_q^N$ which holds if $C_{h, h_1, \ldots, h_N}(M, r) = c$.
- $\mathcal{R}_{\text{ReEnc}, pk}(g, pk, (e_1, \ldots, e_N), (e'_1, \ldots, e'_N))(\pi_M, (r'_1, \ldots, r'_N))$, where $\pi_M$ is a permutation of the set $\{1, \ldots, N\}$, is the relation which holds if an only if $e'_i = \mathsf{ReEnc}_{g, pk}(e_{\pi_M(i)}), r'_{\pi_M(i)}$.

**Proposition 1.** Algorithm 2 is a perfectly complete, sound, and statistical honest verifier zero-knowledge 4-message proof of the relationship $\mathcal{R}_{com} \lor (\mathcal{R}_π \lor \mathcal{R}_{\text{ReEnc}, pk})$.

Since it is infeasible under the discrete logarithm assumption to find a pair satisfying $\mathcal{R}_{com}$. Thus, the proposition computationally implies a proof of knowledge of $\mathcal{R}_π \lor \mathcal{R}_{\text{ReEnc}, pk}$.

That is for a statement $(h, h_1, \ldots, h_N, c, g, pk, (e_1, \ldots, e_N), (e'_1, \ldots, e'_N))$ we can extract a witness $(M, r, (r'_1, \ldots, r'_N)))$ such that $\mathcal{R}_π(h, h_1, \ldots, h_N, c)(M, r)$ and $\mathcal{R}_{\text{ReEnc}, pk}(g, pk, (e_1, \ldots, e_N), (e'_1, \ldots, e'_N))$ ($\pi_M, (r'_1, \ldots, r'_N))$, unless we find a discrete log.

To prove the proposition, one needs to show the correctness, the zero-knowledge, and the soundness properties. For completeness of the presentation, we demonstrate those properties in the following subsections.

**Zero-knowledge** The honest-verifier zero-knowledge simulator chooses $\hat{c}_1, \ldots, \hat{c}_N \in G_q$, $\hat{s}, \hat{s}', \mathbf{u} \in \mathbb{Z}_q^N$, $\mathbf{s}, \mathbf{s}', \mathbf{u} \in \mathbb{Z}_q^N$, and $t_1, t_2, t_3, t_4, \hat{t}$, by the equations in step five.

We can observe that the statistical distance between a real and a simulated transcript is negligible in $q$.

- $\mathbf{u}$ are distributed uniformly in $\mathbb{Z}_q^N$ in both.
- $\hat{c}_1, \ldots, \hat{c}_N$ are distributed uniformly in both transcripts. In the simulated one, it is easily seen by construction. In the real transcript $\hat{c}_i = g^{\hat{t}_i} \hat{c}_{i-1}^{\mathbf{w}_i}$, where $\hat{t}_i \in_R \mathbb{Z}_q$, which randomly distributes them in $G_q$ as well.
- The challenge $c$ is uniformly distributed in both.
- In both transcripts, $S = s_1, s_2, s_3, s_4, \hat{s}, \hat{s}'$ are distributed uniformly in their domains by their definitions (in the simulated transcript it is readily visible; in the real transcript, it is because $\omega$’s are distributed uniformly).
- In both transcripts, the above values determine the values of $t_1, t_2, t_3, t_4, \hat{t}$ by the equations of Step 5.
**Correctness** We will now show the above protocol is correct, which means that in an honest run, the verifier accepts the proof.

We first show the shape of honest $\hat{c}_i$.

\[
\hat{c}_1 = h^{\hat{r}_1} h_1^{u_1'} \\
\hat{c}_2 = h^{\hat{r}_2} u_1' \\
\hat{c}_2 = h^{\hat{r}_2 (h^{\hat{r}_1} h_1^{u_1'}) u_2'} \\
\hat{c}_2 = h^{\hat{r}_2 + \hat{r}_1} u_1' h_1^{u_1'} \\
\hat{c}_2 = h^{\hat{r}_2 + \sum_{i=1}^{\alpha-1} \hat{r}_i} \prod_{j=1}^{\alpha-1} u_j' h_1^{u_1'} \\
\hat{c}_2 = h^{\hat{r}_2 + \sum_{i=1}^{\alpha-1} \hat{r}_i} \prod_{j=1}^{\alpha-1} u_j' h_1^{u_1'}
\]

by definition of $\hat{c}_1$ and $\hat{c}_0$

by definition of $\hat{c}_2$

by definition of $\hat{c}_1$

by algebraic manipulation

by algebraic manipulation

Now we will continue by induction:

\[
\hat{c}_\alpha = h^{\hat{r}_\alpha} e^{u_1'} \\
\hat{c}_\alpha = h^{\hat{r}_\alpha} (h^{\hat{r}_{\alpha-1}} + \sum_{i=1}^{\alpha-1} \hat{r}_i) \prod_{j=1}^{\alpha-1} u_j' h_1^{u_1'} \\
\hat{c}_\alpha = h^{\hat{r}_\alpha} \prod_{j=1}^{\alpha-1} u_j' h_1^{u_1'} \\
\hat{c}_\alpha = h^{\hat{r}_\alpha + \sum_{i=1}^{\alpha-1} \hat{r}_i} \prod_{j=1}^{\alpha-1} u_j' h_1^{u_1'}
\]

by definition of $\hat{c}_\alpha$

by definition of $\hat{c}_{\alpha-1}$ (ind. hypothesis)

by algebraic manipulation

by algebraic manipulation

Now on to the main thing. Note that in the following, we use the fact that $c_i$ is a commitment to a permutation matrix $M$ (and we will use the definition of a permutation matrix).

\[
t_1 = \prod_{i=1}^{N} c_i / \prod_{i=1}^{N} h_i \overset{?}{=} h^{c \cdot \sum_{i=1}^{N} r_i} \overset{\text{verification definition (Step 5)}}{=}\]

\[
h^{c \cdot \sum_{i=1}^{N} r_i} = \prod_{i=1}^{N} c_i / \prod_{i=1}^{N} h_i \overset{?}{=} h^{c \cdot \sum_{i=1}^{N} r_i} \overset{\text{by definition of } t_1}{=}\]

\[
h^{c \cdot \sum_{i=1}^{N} r_i} = \prod_{i=1}^{N} c_i / \prod_{i=1}^{N} h_i \overset{?}{=} h^{c \cdot \sum_{i=1}^{N} r_i} \overset{\text{by definition of } s_1}{=}\]

\[
\prod_{i=1}^{N} c_i / \prod_{i=1}^{N} h_i \overset{?}{=} h^{c \cdot \sum_{i=1}^{N} r_i} \overset{\text{by algebraic manipulation}}{=}\]

\[
(h^{\sum_{i=1}^{N} r_i} \prod_{i=1}^{N} h_i / \prod_{i=1}^{N} h_i) \overset{?}{=} h^{c \cdot \sum_{i=1}^{N} r_i} \overset{\text{by definition of } c_i \text{ and } \hat{r}}{=}\]

\[
h^{c \cdot \sum_{i=1}^{N} r_i} = h^{c \cdot \sum_{i=1}^{N} r_i} \overset{\text{by algebraic manipulation}}{=}\]
\[
\begin{align*}
  t_2 &= (\hat{c}_N / h_1)^{-c} h^{s_2} \\
  h^{o_2} &= (\hat{c}_N / h_1)^{-c} h^{s_2} \\
  h^{o_2} &= (\hat{c}_N / h_1)^{-c} h^{o_2 + c r^o} \\
  (\hat{c}_N / h_1)^{e} &= h^{c r^o} \\
  (h^N \cdot \sum_{i=1}^{N} e_i, u_i) &= h^{c r^o} \\
  h^c (\hat{c}_N + \sum_{i=1}^{N} e_i) &= h^c (\hat{c}_N + \sum_{i=1}^{N} e_i) \\
  h^c (\hat{c}_N + \sum_{i=1}^{N} e_i) &= h^c (\hat{c}_N + \sum_{i=1}^{N} e_i) \\
  \prod_{i=1}^{N} c_i^{u_i} &= h^{c r^o} \\
  (\prod_{i=1}^{N} EPC(M_i, r_i)^{u_i}) &= EPC(c \cdot u', c \cdot \bar{r}) \\
  (\prod_{i=1}^{N} EPC(M_i, r_i)^{u_i}) &= EPC(c \cdot u', c \cdot \langle r, u \rangle) \\
  (\prod_{i=1}^{N} EPC(u_i M_i, r_i)^{u_i}) &= EPC(c \cdot u', c \cdot \langle r, u \rangle) \\
  (EPC(M u, \langle r, u \rangle)) &= EPC(c \cdot u', c \cdot \langle r, u \rangle) \\
  (EPC(c \cdot M u, c \cdot \langle r, u \rangle)) &= EPC(c \cdot u', c \cdot \langle r, u \rangle) \\
  (EPC(c \cdot M u, c \cdot \langle r, u \rangle)) &= EPC(c \cdot M u, c \cdot \langle r, u \rangle) \\
  \end{align*}
\]

Verification definition

By definition of \( t_2 \)

By definition of \( s_2 \)

By algebraic manipulation

By the properties of \( \hat{c}_N \)

By algebraic manipulation and definition of \( u' \)

By definition \( r^o \)

Verification definition

By definition of \( t_3 \)

By definition of \( s_3 \) and \( s'_i \)

By algebraic manipulation

By definition of \( c_i \)

By definition of \( \bar{r} \)

By algebraic manipulation
\[ t_4 \equiv \text{ReEnc}(\prod_{i=1}^{N} e_i^{u_i}, -s_4) \quad \text{Verification definition} \]

\[ \text{ReEnc}(\prod_{i=1}^{N} e_i^{\omega_i}, -\omega_4) \equiv \text{ReEnc}(\prod_{i=1}^{N} e_i^{s_i'}, -s_4) \quad \text{By definition of } t_4 \]

\[ \prod_{i=1}^{N} e_i^{\omega_i} \cdot \text{Enc}(1, -\omega_4) \equiv (\prod_{i=1}^{N} e_i^{u_i})^c \prod_{i=1}^{N} (e_i^{s_i'}) \cdot \text{Enc}(1, -s_4) \quad \text{By definition of } \text{ReEnc} \]

\[ \prod_{i=1}^{N} e_i^{\omega_i} \cdot \text{Enc}(1, -\omega_4) \equiv (\prod_{i=1}^{N} e_i^{u_i})^c \prod_{i=1}^{N} (e_i^{s_i'}) \cdot \text{Enc}(1, -\omega_4 - c \cdot r^*) \quad \text{By definition of } s_4 \]

\[ (\prod_{i=1}^{N} e_i^{\omega_i})^c \equiv (\prod_{i=1}^{N} e_i^{s_i'}) \cdot \text{Enc}(1, -c \cdot r^*) \quad \text{By algebraic manipulation} \]

\[ (\prod_{i=1}^{N} e_i^{u_i})^c \equiv (\prod_{i=1}^{N} e_i^{s_i'}) \cdot \text{Enc}(1, -c \cdot r^*) \quad \text{By definition of } e' \text{ and } u' \]

\[ \text{Enc}(1, c \cdot r^*) \equiv (\prod_{i=1}^{N} \text{Enc}_{pk}(1, R_i)^{u_i})^c \quad \text{By algebraic manipulation} \]

\[ \text{Enc}(1, c \cdot r^*) \equiv \text{Enc}(1, c \cdot Ru) \quad \text{By algebraic manipulation} \]

\[ \text{Enc}(1, c \cdot r^*) \equiv \text{Enc}(1, c \cdot r^*) \quad \text{By definition of } r^* \]

\[ \hat{t}_i \equiv \hat{c}_i^c \cdot h^i \cdot \hat{s}_i' \quad \text{Verification definition} \]

\[ h^i \cdot \hat{\omega}_i \cdot \hat{c}_i' \equiv \hat{c}_i^c \cdot h^i \cdot \hat{s}_i' \quad \text{By definition of } \hat{t}_i \]

\[ h^i \cdot \hat{\omega}_i \cdot \hat{c}_i' \equiv \hat{c}_i^c \cdot h^i \cdot \hat{\omega}_i + c \cdot \hat{c}_i' \quad \text{By definition of } \hat{s}_i \text{ and } \hat{s}_i' \]

\[ \hat{c}_i^c \equiv h^i \cdot \hat{c}_i' \cdot c \cdot u_i' \quad \text{By algebraic manipulation} \]

\[ (h^i \cdot \hat{c}_i')^c \equiv h^i \cdot \hat{c}_i' \cdot c \cdot u_i' \quad \text{By definition of } \hat{c}_i \]

\[ h^i \cdot \hat{c}_i' \cdot c \cdot u_i' \equiv h^i \cdot \hat{c}_i' \cdot c \cdot u_i' \quad \text{By algebraic manipulation} \]

**Soundness** We follow the structure of the original proof, as presented in [2], and present the extractor in two parts. First, we show that, for two different transcripts with the same \( u \) but different \( c \), we can extract witness for certain sub-statements. In the extended extractor
we show that, given witnesses for these sub-statements which hold for $n$ different $u$, we can extract witness to the main statements.

**Basic extractor.** Given two accepting transcripts

$$(u, \hat{c}, t_1, t_2, t_3, t_4, c, s_1, s_2, s_3, \hat{s}, s')$$

$$(u, \hat{c}, t_1, t_2, t_3, t_4, c^*, s_1^*, s_2^*, s_3^*, \hat{s}^*, s'^*)$$

with $c \neq c^*$, the basic extractor computes

$$\bar{r} = (s_1 - s_1^*)/(c - c^*)$$
$$\bar{r} = (s_3 - s_3^*)/(c - c^*)$$
$$\hat{f} = (s - s^*)/(c - c^*)$$

$$r^\circ = (s_2 - s_2^*)/(c - c^*)$$
$$r^* = (s_4 - s_4^*)/(c - c^*)$$
$$u' = (s' - s'^*)/(c - c^*)$$

Note that we reuse symbols from the Algorithm 1. While they denote analogous entities, they are not necessarily identical (if the transcripts have not been obtained in the honest way).

We will prove that

$$\prod_{j=1}^{N} c_j = EPC(1, \bar{r})$$
$$\prod_{j=1}^{N} c'_j = EPC(u', \bar{r})$$
$$\prod_{i=1}^{N} e_{i}^{u'} = Enc_{pk}(1, r^\circ) \cdot \prod_{i=1}^{N} e_{i}^{u}$$

$$\hat{c}_i = PC_{h, \hat{c}_{i-1}}(u'_i, \hat{r}_i)$$
$$\hat{c}_N = PC_{h, h_1}(\prod_{i=1}^{N} u_i, r^\circ)$$

The proof consists of simple algebraic transformations:

$$\left(\frac{\prod_{j=1}^{N} c_j}{\prod_{j=1}^{N} c_j^{c_t_1}}\right)^{\frac{1}{c - c^*}} = \prod_{j=1}^{N} c_j$$

(tautology)

$$\left(\frac{h^s / (\prod_{j=1}^{N} h_j)^{-c}}{h^s / (\prod_{j=1}^{N} h_j)^{-c}}\right)^{\frac{1}{c - c^*}} = \prod_{j=1}^{N} c_j$$

(by the verification definition)

$$h^{\frac{s - s^*}{c - c^*}} \prod_{j=1}^{N} h_j = \prod_{j=1}^{N} c_j$$

(by algebraic manipulation)

$$EPC(1, s_1 - s_1^*/c - c^*) = \prod_{j=1}^{N} c_j$$

(by definition of EPC)

$$EPC(1, \bar{r}) = \prod_{j=1}^{N} c_j$$

(by definition of \(\bar{r}\))
\[
\left(\frac{\prod_{j=1}^{N} e_j^{u_j} t_3}{\prod_{j=1}^{N} e_j^{u_j} c^* t_3}\right)^{\frac{1}{c^*}} \rightarrow \prod_{j=1}^{N} e_j^{u_j} \quad \text{Tautology}
\]

\[
\left(\frac{h_j^{s_j} \prod_{j=1}^{N} h_j^{s_j}}{h_j^{s_j} \prod_{j=1}^{N} h_j^{s_j}}\right)^{\frac{1}{c^*}} \rightarrow \prod_{j=1}^{N} e_j^{u_j} \quad \text{By verification definition}
\]

\[
h_i^{s_i - s_i^*} \prod_{i=1}^{N} h_i^{s_i - s_i^*} = \prod_{j=1}^{N} e_j^{u_j} \quad \text{By algebraic manipulation}
\]

\[EPC\left(\frac{s_i - s_i^*}{c - c^*}, s_3 - s_3^*\right) = \prod_{j=1}^{N} e_j^{u_j} \quad \text{By definition of EPC}
\]

\[EPC(u', \tilde{r}) = \prod_{j=1}^{N} e_j^{u_j} \quad \text{By definition of } u' \text{ and } \tilde{r}
\]

\[
\left(\frac{\prod_{i=1}^{N} (e_i^{u_i})^{c_t} t_4}{\prod_{i=1}^{N} (e_i^{u_i})^{c_t} c^* t_4}\right)^{\frac{1}{c^*}} \rightarrow \prod_{i=1}^{N} e_i^{u_i} \quad \text{Tautology}
\]

\[
\left(\frac{\prod_{i=1}^{N} (e_i^{u_i})^{s_t} \text{Enc}(1, -s_4)}{\prod_{i=1}^{N} (e_i^{u_i})^{s_t} \text{Enc}(1, -s_4)}\right)^{\frac{1}{c^*}} \rightarrow \prod_{i=1}^{N} e_i^{u_i} \quad \text{By verification definition}
\]

\[
\prod_{i=1}^{N} e_i^{s_i^j - s_i^{j*}} \text{Enc}(1, \frac{s_4 - s_3^*}{c - c^*}) = \prod_{i=1}^{N} e_i^{u_i} \quad \text{By algebraic manipulation}
\]

\[
\prod_{i=1}^{N} e_i^{r_j^i - r_j^{i*}} = \text{Enc}_p k(1, r^*) \prod_{i=1}^{N} e_i^{u_i} \quad \text{By algebraic manipulation}
\]

\[
\prod_{i=1}^{N} e_i^{u_i'} = \text{Enc}_{p k}(1, r') \prod_{i=1}^{N} e_i^{u_i} \quad \text{By definition of } r_j^i \text{ and } u_i'
\]
Now, for each $i \in \{1, \ldots, N\}$

$$
\left( \frac{\hat{c}_i^t \hat{t}_i}{\hat{c}_i^{*t} \hat{t}_i} \right)^{-1} = \hat{c}_i \\
\left( \frac{h_i^{s_i^t} \hat{c}_i^{s_i^t}}{h_i^{s_i^{*t}} \hat{c}_i^{s_i^{*t}}} \right)^{-1} = \hat{c}_i \\
\frac{s_i - s_i^*}{c - c^*} \frac{s_i - s_i^*}{c - c^*} = \hat{c}_i
$$

Tautology

By verification definition

By algebraic manipulations

Extended Extractor We now sketch the extended extractor which, for a given statement
(see the common input in Algorithm 1), for $n$ different witnesses extracted by the basic extractor, produces the witnesses to the main statement. Let the collective output of the basic extractors be denoted as $\hat{\mathbf{r}}$, $\hat{\mathbf{r}}^\diamond$, $\hat{\mathbf{r}} \in \mathbb{Z}_q^n$, $\mathbf{R}^*, \hat{\mathbf{R}}^* \in \mathbb{Z}_q^{W \times N}$, and $\hat{\mathbf{R}}, U' \in \mathbb{Z}_q^{N \times N}$ extracted from the primary challenges $U \in \mathbb{Z}_q^{N \times N}$. We denote by $U_i$ the $i$th column of $U$ which is the challenge vector from the $i$th run of the basic extractor, and by $U_{j,i}$ the $j$th element of the challenge vector from the $i$th run of the basic extractor.

First note with overwhelming probability the set of $U_i$s is linearly independent, concretely the probability is bounded by $q^{-2}$. From linear independence, it follows that there exists $A \in \mathbb{Z}_q^{N \times N}$ such that $UA_i$ is the $i$th standard unit vector in $\mathbb{Z}_q$ which we will denote by $\mathbb{1}_i$. Clearly,
\[ c_l = \prod_{i=1}^{N} (c^{UA_l})_i \] since \( UA_l \) is \( I_l \)

\[ c_l = \prod_{i=1}^{N} c_{\sum_{j=1}^{N} U_{i,j} A_{j,l}} \] by definition of \( UA_l \)

\[ c_l = \prod_{j=1}^{N} \left( \prod_{i=1}^{N} c_{U_{i,j} A_{j,l}} \right) \] by algebraic manipulation

\[ c_l = \prod_{j=1}^{N} EPC(U_{j,l}', \tilde{r}_{j,l})^{A_{j,l}} \] by some algebraic manipulation and \( \prod_{i=1}^{N} c_{U_{i,j}} = EPC(U_{j,l}', \tilde{r}_{j,l}) \)

\[ c_l = \prod_{j=1}^{N} EPC(U_{j,l}' A_{j,l}, \langle \tilde{r}_{j,l}, A_{j,l} \rangle) \] by algebraic manipulation

\[ c_l = EPC\left( \sum_{j=1}^{N} U_{j,l}' A_{j,l}, \langle \tilde{r}_{j}, A_{l} \rangle \right) \] by algebraic manipulation

\[ c_l = EPC(U_{l}' A_{l}, \langle \tilde{r}_{l}, A_{l} \rangle) \] by algebraic manipulation

Therefore, we can open \( c \) to the matrix \( M \), where the \( l \)th column of \( M \) is \( U_{l}' A_{l} \), with randomness \( \langle \tilde{r}, A_{l} \rangle \). In other words we open \( c = U'A \) using randomness \( \tilde{r}A \).

We expect \( M \) to be a permutation matrix, but if it is not, then one can find a witness to \( R_{com} \) (which, as has been mentioned, can only happen with negligible probability, under our security assumptions). We extract in two different ways depending on whether \( M1 \neq 1 \).

**Option one** If \( M1 \neq 1 \), then let \( u'' = M1 \) and note that

\[ u'' \neq 1 \text{ and } EPC(1, \tilde{r}_j) = \prod_{i=1}^{N} c_i = \prod_{i=1}^{N} c_i^{1_i} = EPC(u'', \tilde{r}A) \]

in which case we found a witness breaking the commitment scheme.

**Option two** If \( M1 = 1 \), then recall Theorem 1 from “Proofs of Restricted Shuffles”, which states that \( M \) is a permutation matrix if and only if \( M1 = 1 \) and \( \prod_{i=1}^{N} \langle m_i, x \rangle - \prod_{i=1}^{N} x_i = 0 \). Since \( M1 = 1 \) and \( M \) is not a permutation matrix, then \( \prod_{i=1}^{N} \langle m_i, x \rangle - \prod_{i=1}^{N} x_i \neq 0 \). The Schwartz–Zippel says that if you sample, a non-zero polynomial, at a random point the chance that it equals zero is negligible in the order of the underlying field; hence, with overwhelming probability there exists \( j \in \{1, \ldots, N\} \) such \( \prod_{i=1}^{N} \langle m_i, U_{j,i} \rangle - \prod_{i=1}^{N} U_{i,j} \neq 0 \). Since this is true with overwhelming probability, we require it to be true and rewind if this is not
the case. (Strictly speaking we should take $N+1$ extractions from the basic extractor, if we recover a different $M$ we win, if we get the same $M$ then $U_{t+1}$ is actually independent of $M$ and the lemma can be applied.)

Let $u'' = MU_j$ and note that

$$u'' \neq U_j$$

Which must be true since

$$\prod_{i=1}^{N} U'_{i,j} = \prod_{i=1}^{N} U_{i,j} \neq \prod_{i=1}^{N} u''$$

The correctness of $U'$ We now show that $U'_l = MU_l$ for all $l \in [1, N]$ or we can find a witnesses to $R_{\text{com}}$. Let $u'' = MU_l$ and by assumption $u'' \neq U'_l$.

$$EPC(U'_l, \tilde{r}_l) = \prod_{i=1}^{N} c_{i}^{U'_{l,i}} = EPC(u'', (\tilde{r}A, U_j))$$

Extracting the randomness We having shown that if $M$ is not a permutation matrix we can extract a witness to $R_{\text{com}}$. We now show that we can extract $R \in \mathbb{Z}_q^{w \times N}$ such that $e'_i = \ldots$
\textbf{ReEnc}_{\text{pk}}(\mathbf{e}_{\pi(i)}^{}, R_{\pi(i)}^{}).

\begin{align*}
\mathbf{e}_l &= \prod_{i=1}^{N} (e^{U_A}_i), & \text{since } U_A_i \text{ is } \mathbb{I}_l & \quad (1) \\
\mathbf{e}_l &= \prod_{i=1}^{N} e^{U_i^R A_j}_i, & \text{by definition of } U_A_i & \quad (2) \\
\mathbf{e}_l &= \prod_{i=1}^{N} \left( \prod_{j=1}^{N} e^{U_j^A A_j}_i \right) & \text{by algebraic manipulation} & \quad (3) \\
\mathbf{e}_l &= \prod_{j=1}^{N} \left( \prod_{i=1}^{N} e^{U_j A_j}_i \right) & \text{by algebraic manipulation} & \quad (4) \\
\mathbf{e}_l &= \prod_{j=1}^{N} \left( \prod_{i=1}^{N} e^{\sum_{j=1}^{N} U_j^R A_j} \text{Enc}_{\text{pk}}(1, -R_j^*) \right) & \text{since } \prod_{i=1}^{N} e^{U_j}_i = \prod_{i=1}^{N} e^{U_j^R A_j}_i & \quad (5) \\
\mathbf{e}_l &= \prod_{j=1}^{N} \left( \prod_{i=1}^{N} e^{U_j^A A_j} \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) \right) & \text{by algebraic manipulation} & \quad (6) \\
\mathbf{e}_l &= \prod_{i=1}^{N} (e^{U_j^R A_j}) \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) & \text{by algebraic manipulation} & \quad (7) \\
\mathbf{e}_l &= \prod_{i=1}^{N} (e^{M U_A}_i) \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) & \text{since } U^* = MU & \quad (8) \\
\mathbf{e}_l &= \prod_{i=1}^{N} (e^{M U}_i) \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) & \text{since } U_A_i = \mathbb{I}_l & \quad (9) \\
\mathbf{e}_l &= \prod_{i=1}^{N} (e^{M U}_i) \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) & \text{since } M \mathbb{I}_l = M_l & \quad (10) \\
\mathbf{e}_l &= \prod_{i=1}^{N} (e^{M J}_i) \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) & \text{since } M \mathbb{I}_l = M_l & \quad (11) \\
\mathbf{e}_l &= e^{\pi_{M^l}(i)}_{\pi_M} \text{Enc}_{\text{pk}}(1, -\langle R_j^*, A_j \rangle) & \text{by definition of } \pi_M & \quad (12) \\
\end{align*}

We have now shown that \( \text{ReEnc}_{\text{pk}}(\mathbf{e}_l, \langle R_j^*, A_j \rangle) = e^{\pi_{M^l}(i)}_{\pi_{M}} \); hence, \( R_l = \langle R_j^*, A_j \rangle \) which concludes the proof.

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