A Stern-Gerlach Experiment in Time

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Abstract

In non-relativistic quantum mechanics, path integrals are normally derived from the Schrödinger equation. This assumes the two formalisms are equivalent. Since time plays a very different role in the Schrödinger equation and in path integrals, this may not be the case.

We here derive path integrals directly by imposing two requirements: correct behavior in the classical limit and the most complete practicable symmetry between time and space.

With these requirements, the path integral formalism predicts quantum fluctuations over the time dimension analogous to the quantum fluctuations seen over the three space dimensions. For constant potentials there is no effect. But the coupling between rapidly varying electromagnetic fields and the quantum fluctuations in time should be detectable.

We consider a variation on the Stern-Gerlach experiment in which a particle with a non-zero electric dipole moment is sent through a rapidly varying electric field, oriented parallel to the particle’s trajectory. The Schrödinger equation predicts changes to the precession frequency of the wave function about the trajectory but no physical splitting of the beam. With the approach here, path integrals predict the changes to the precession frequency and in addition that the beam will be split in velocity and time.
INTRODUCTION

Problem

In the context of non-relativistic quantum mechanics, path integrals are normally derived directly from the Schrödinger equation or, at least, validated against it [1, 2, 3, 4]. This approach makes the implicit assumption that the two formalisms are equivalent. Since the views of time most naturally associated with the Schrödinger equation and with path integrals are very different, this implicit assumption is not entirely unproblematic.

Different views of time in Schrödinger equation and path integrals

To lay a foundation, we give a brief review of a typical derivation of path integrals from the Schrödinger equation. We start with the time independent Schrödinger equation for one particle [72]

\[ i \frac{\partial}{\partial t} \psi(t, \vec{x}) = -\frac{1}{2m} \nabla^2 \psi(t, \vec{x}) + V(\vec{x}) \psi(t, \vec{x}). \]  

(1)

For infinitesimal times we can use this to write \( \psi(t + \Delta t) \) in terms of \( \psi(t) \)

\[ \psi(t + \Delta t, \vec{x}) \approx \psi(t, \vec{x}) + \frac{i\Delta t}{2m} \nabla^2 \psi(t, \vec{x}) - i\Delta t V(\vec{x}) \psi(t, \vec{x}). \]  

(2)

If a function \( \psi(x') \) is sufficiently smooth, we can expand it around a point \( x \) as

\[ \psi(x') \approx \psi(x) + (x' - x) \frac{d\psi(x)}{dx} + \frac{1}{2} (x' - x)^2 \frac{d^2\psi(x)}{dx^2}. \]  

(3)

which lets us write

\[ \psi(x) - \frac{1}{4ia} \frac{d^2\psi(x)}{dx^2} \approx \sqrt{-\frac{-ia}{\pi}} \int dx' \exp \left( ia (x' - x)^2 \right) \psi(x'). \]  

(4)

By taking \( a \) as \( \frac{m}{2\Delta t} \) and using the identity \( 1 + \delta \approx e^\delta \) for small \( \delta \) we may write [2] as

\[ \psi(t + \Delta t, \vec{x}) \approx \sqrt{\frac{m}{2\pi i\Delta t}} \int d\vec{x}' \exp \left( i\frac{m (\vec{x} - \vec{x}')^2}{2\Delta t} - i\Delta t V(\vec{x}') \right) \psi(t, \vec{x}'). \]  

(5)

If we do this repeatedly, we can push \( \psi(t) \) forwards in time, one \( \Delta t \) at a time, to get the value of \( \psi(t') \) at arbitrary times

\[ \psi(t'', \vec{x}'') = \int d\vec{x}' K(t'', \vec{x}''; t', \vec{x}) \psi(t', \vec{x}') \]  

(6)
where the kernel $K(t'', \vec{x}''; t', \vec{x}')$ represents the repeated application of (5). The kernel is then given by the Trotter product formula

$$
\lim_{N \to \infty} \int d\vec{x}_1 \ldots d\vec{x}_{N-1} \sqrt{\frac{m}{2\pi i \varepsilon}}^{4N} \exp \left( i \varepsilon \sum_{j=0}^{N-1} \left( \frac{m}{2} \left( \frac{\vec{x}_{j+1} - \vec{x}_j}{\varepsilon} \right)^2 - V(\vec{x}_j) \right) \right)
$$

with

$$
\varepsilon \equiv \frac{t'' - t'}{N}.
$$

The summand is the discrete form of the non-relativistic time-independent Lagrangian

$$
\frac{m}{2} \left( \frac{\vec{x}_{j+1} - \vec{x}_j}{\varepsilon} \right)^2 - V(\vec{x}_j) \xrightarrow{\varepsilon \to 0} \frac{m}{2} \vec{v}^2 - V(\vec{x}) = L(\vec{x}, \vec{v})
$$

so we may identify the argument of the exponential as $i$ times the classical action $S$

$$
S[\pi] \equiv \int dt L[\pi]
$$

and identify the product of integrations as a sum over paths $D[\pi]$

$$
K(t'', \vec{x}'', t', \vec{x}') = \int D[\pi] e^{iS[\pi]}
$$

where the paths $\pi$ are defined in terms of their coordinates at a series of discrete times. We have used a very intuitive view of time: we defined the wave function at one time, then pushed it forward step by step till we arrived at the wave function at any later time.

But when we look at the final expression, it is just as natural to see it as defined over time. The paths are naturally defined as trajectories over the domain $t'$ to $t''$ and the action as a functional over such trajectories. We are free to see the final product from the “block universe perspective,” in which we see all time as existing at once (even if we normally experience it sequentially).

The question then is what happens if we develop path integrals by taking (11) as the starting point?

**Normalization**

When we replaced the $\nabla^2$ in (11) with a Gaussian integral we chose the normalization constant $a = \frac{m}{2\Delta t}$ using the quietly popular “whatever-works” methodology. In general, a certain arbitrariness about normalization seems to be a common feature of path integrals...
The one unavoidable normalization requirement is that the sum of the probabilities of all possibilities be one, expressed in quantum mechanical terms as unitarity

\[ \forall a, 1 = \sum_{\{b\}} P(b | a) = \sum_{\{b\}} |K(b; a)|^2 \]  

(12)

where \( a \) represents the starting state(s) and \( \{b\} \) the set of all possible outcomes. In the case of the Schrödinger equation, (12) is enforced at every time \( t \). Assuming a wave function is properly normalized and obeys the Schrödinger equation, it will satisfy

\[ \frac{dp(t)}{dt} = 0 \]  

(13)

where \( p \) is the probability

\[ p(t) \equiv \int d\vec{x} \psi^*(t, \vec{x}) \psi(t, \vec{x}). \]  

(14)

We are requiring that \( p(t) = 1 \) at all times from \( t' \) to \( t'' \), inclusive. Given that we only have direct knowledge of the probabilities at the endpoints \( t' \) and \( t'' \), insisting that \( p(t) = 1 \) at all times in between is a stronger requirement than is strictly necessary. And there are two specific problems with normalizing on these intermediate hypersurfaces.

The first problem is the implicit selection of a specific set of spacelike hypersurfaces on which to define the probability density. Such a selection is not manifestly invariant under all possible Lorentz transformations. This is not in itself provably wrong – for one thing the Schrödinger equation is only supposed to be valid for non-relativistic quantum mechanics – but it is troubling.

One of the troubled is Suarez, who raised the possibility that if we assume standard quantum mechanics is correct, with sufficient ingenuity we could demonstrate retrotemporal causal influences. He proposed a specific alternative to standard quantum mechanics, relativistic nonlocality (RNL), to avoid this difficulty (and some others). RNL has been refuted experimentally by Stefanov, et al. But the experimental refutation of RNL does not reduce the force of Suarez’s original objections; it merely indicates that one possible resolution of them does not work.

The second problem is that in the Schrödinger equation, we are overlooking the possibility of quantum jumps in time. As we know, there is nothing quantum mechanical systems enjoy more than tunneling through barriers in space. Given that time and space are to a large extent interchangeable, we might expect that if quantum particles tunnel through barriers
in space freely, they might “tunnel in time” as well. That is to say, they might not be completely well-defined by their wave functions on any specific spacelike hypersurface.

ASSUMPTIONS

Before developing our approach we need to explicate the assumptions on which it will be based. While this is always sound practice, in any discussion involving the nature of time it is imperative. As Schulman noted in his reply to Casati, Chirikov, and Zhirov’s response to his “Opposite Thermodynamic Arrows of Time”: “They and I find ourselves in a situation that is common in discussions of the ‘arrow of time,’ namely no disagreement on technical issues and no agreement on basic assumptions.”

We will be playing a game of “as if” based on three assumptions:

1. time is a kind of space (time/space symmetry),
2. all time is to be seen at once (block universe perspective), and
3. there is no fundamental direction to time (time reversal symmetry).

Because of the importance of these points to our argument we briefly review the evidence for them.

**Time/space symmetry**

As Minkowski famously put it, “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.” This approach is fundamental to special and general relativity; for an extended review see Barbour’s *The End of Time*. While the situation in quantum mechanics is less clear, there is nothing to refute this principle here either; for an extended review see Stenger’s *Timeless Reality*. In general, phenomena seen for space (or space and momentum) are seen for time (or time and energy), e.g., diffraction and interference effects and the uncertainty principle.

Diffraction and interference effects reinforce the view that time and space are interchangeable. Just as we see diffraction and interference in space, we see them in time. The first
article was Moshinsky’s appropriately named “Diffraction in time” [20]. Experimental confirmations include [21, 22, 23, 24, 25, 26]. “Quantum beats” – self-interference in time – have also been predicted and seen [27, 28, 29].

And just as we have uncertainty relations between space and momentum, we have uncertainty relations between time and energy. An uncertainty principle between time and energy was proposed by Heisenberg ([30] as translated in [31]). This played a critical role in establishing the principles of the quantum. Einstein’s attempt to refute the uncertainty principle using his celebrated “clock-in-a-box” experiment was in turn refuted by Bohr using the time/energy uncertainty relation (and also general relativity!: see for instance Bohr’s account in [32]). If there is not some kind of an uncertainty principle between time and energy, a serious attack on the self-consistency of quantum mechanics could easily be mounted.

A particularly cogent review of the uncertainty principle between time and energy is given by Busch [33]. He argues that the time/energy uncertainty relationship is valid but does not stand on quite the same footing at the position/momentum one. (We will be arguing this is because the standard quantum mechanics formalism is asymmetric in its handling of time and space. We will be proposing a more symmetric approach.)

Hilgevoord [34, 35] argues that one must keep the definitions of time and space parallel to those for energy and momentum if there is to be the same kind of uncertainty principle for both. In particular, Hilgevoord notes the need for care in distinguishing between the time coordinate associated with the implicit space/time grid against which motion is being measured and the time coordinate associated with any specific particle trajectory. (This is an idea we take advantage of below.)

While the literature is not completely unambiguous, we will assume that the time/energy uncertainty relationship stands on as firm a basis as the space/momentum relationship, and therefore that any deviation represents either an asymmetry in the experimental setup or an asymmetry in the formalism, the first of which should be discounted, the second eschewed.

**Block universe perspective**

Implicit in time/space symmetry is the block universe perspective. If time is a space dimension, then since we may consider space “all-at-once” we may also consider time “all-at-once.”
We will need to invoke the block universe perspective when it comes time to normalize our paths. We will be normalizing them across time, not at each time; this makes sense only if it makes sense to see time all at once.

The block universe perspective has dramatic experimental support in the form of Wheeler’s delayed choice experiment [36]. This is a double-slit experiment with a twist. In a standard double slit experiment, if one checks which slit the particle went through, the interference pattern is lost. In a delayed choice experiment, the decision as to whether or not to check which slit the particle went through is made after the particle has (nominally) already gone through one or both slits. As Wheeler [37] put it: “In the new ‘delayed-choice’ version of the experiment one decides whether to put in the half-silvered mirror or take it out at the very last minute. Thus one decides whether the photon ‘shall have come by one route, or by both routes’ after it has ‘already done its travel’.” The predictions of quantum mechanics have been confirmed experimentally: recent experiments include [38, 39].

Wheeler has further pointed out (in his “Great Smoky Dragon” experiment [40]) that one may even perform a version of the delayed choice experiment across cosmological distances and times, by taking advantage of gravitational lensing effects.

It is in our view difficult to make sense of these results except by taking the block universe perspective. If we imagine the particle in question having a definite position in time, then to reproduce the experimental results, the particle would have to scoot ahead (in time) to see what experimental setup was waiting for it, then dart back (in time) to take both doors or just one, depending. We are not saying it is beyond imagination to come up with some way to make this plausible. But we find it simpler to take the experimental evidence as given.

If the delayed choice experiment is insufficiently persuasive, then we may consider the “quantum eraser” experiments, first proposed by Scully [41, 42]. We have the same experimental setup as with the delayed choice experiment, but now, after collecting the “welcher weg” or “which path” information we deliberately erase the information, while maintaining quantum coherence. The erasure of the “welcher weg” information restores the original interference pattern. Now our rather harried particle has not only to scoot ahead in time to see whether its path is being observed, it has to scoot still further ahead to see if that information is being kept or discarded. The quantum eraser has also been confirmed, see Herzog, et al [43].

There are reviews of the literature for the delayed choice and quantum eraser experiments.
in Ghose and in Auletta.

**Time reversal symmetry**

Both time/space symmetry and the block universe perspective could be brought into question by the detection of any hard evidence for an underlying direction to time. But – provided we include $CPT$ in our definition of reversal in time – there does not appear to be any evidence for such an asymmetry. Per Rosner “The discrete symmetries $C$ (charge inversion), $P$ (parity, or space reflection), and $T$ (time reversal) are preserved by strong and electromagnetic processes, but violated by weak decays. For a brief period of several years, it was thought that the products $CP$ and $T$ were preserved by all processes, but that belief was shattered with the discovery of $CP$ violation in neutral kaon decays in 1964. The product $CPT$ seems to be preserved, as is expected in local Lorentz-invariant quantum field theories.” If we look at the motion of a set of particles and look at the same motion after the $CPT$ operation is applied, we cannot tell which was the original and which the reversed, assuming the set of particles is not large enough that entropic considerations come into play.

We will assume for the rest of this investigation, that the directional character associated with time is *entirely* a product of entropy. A strong entropic gradient was established by the Big Bang, and we have been going downhill since. As Lebowitz put it “Laboratory systems are prepared in states of low Boltzmann entropy by experimentalists who are themselves in low-entropy states. Like other living beings, they are born in such states and maintained there by eating nutritious low-entropy foods, which in turn are produced by plants using low-entropy radiation coming from the Sun.”

Or to put it another way, at the level of propagators and particles there is no causal direction; that is to be found only in collections of particles, i.e. beams and other impulsive things. “For a transmitting aerial, the effective temperature of the source is made much larger than that of the surroundings, for a receiving aerial, the effective temperature of the load is made much less than that of the surroundings. There is no ‘one-sidedness’ in nature relating to the way in which oscillating electrons radiate energy, as some authors imply.”

If all of the directionality of time is given by entropy, then if somehow a push could be
administered from the opposite direction in time, then we might see arrows of cause and effect going both ways. This disconcerting possibility is the subject of Schulman’s “Opposite Thermodynamic Arrows of Time”[16]. He finds no irresolvable contradictions.

**Transactional Interpretation**

We will rely on Cramer’s [50, 51] Transactional Interpretation of quantum mechanics. Unlike most interpretations, the Transactional Interpretation is manifestly consistent with our assumptions: it treats time as a space dimension, takes the block universe perspective, and treats the forward and backward directions in time symmetrically.

For example, Cramer gives as the interpretation of a particle that is emitted and then absorbed: “But an equally valid interpretation of the process is that a four-vector standing wave has been established between emitter and absorber. As a familiar 3-space standing wave is a superposition of waves traveling to the right and left, this four-vector standing wave is the superposition of advanced and retarded components. It has been established between the terminating boundaries of the emitter, which blocks passage of the advanced wave further down the time stream, and the absorber, which blocks passage of the retarded wave further up the time stream. This space-time standing wave is the transaction...”

As an additional benefit, the Transactional Interpretation works well with path integrals. We may take the offer as the sum $K(b; a)$ of all paths from emitter to absorber; the acceptance as the sum $K^\dagger(b; a)$ of all paths from absorber back to emitter.

**FOUR DIMENSIONAL PATH INTEGRALS**

**Abstract definition of the path integral**

All path integral calculations begin by forming the sum of all paths $\pi(b; a)$ from a state (or set of states) $a$ to a state (or set of states) $b$

$$K(b; a) = \sum_{\{\pi(b; a)\}} e^{iS[\pi(b; a)]}.$$  \hspace{1cm} \hspace{1cm} (15)

$a$ defines the set of paths consistent with whatever the preparation procedure is; $b$ is the set consistent with a specific measurement procedure.
The probability of \( b \) given \( a \) is

\[
P (b | a) = K^\dagger (b; a) K (b; a) .
\] (16)

There is nothing in this approach to insist that the time of \( a \) be less than the time of \( b \), although in all cases we consider it will be 80.

Our problem is to define the Lagrangian, paths, and normalization appropriate to (15). We focus on a single charged particle in an electromagnetic field. In most cases, we will be starting at a specific point \( x' \) and ending at \( x'' \), both to be understood as four-vectors \( (\pi (b; a) \equiv \pi (x''; x')) \).

**Lagrangian**

For any given problem a wide range of Lagrangians will give the same classical trajectories. Since we are insisting on time/space symmetry, we will only consider Lorentz invariant Lagrangians. For the motion of a charged particle in an electromagnetic field, we may, per Goldstein52, pick any of the form

\[
L = -mf \left( u^2 \right) - qu^\mu A_\mu
\] (17)

with

\[
u^\mu \equiv \left( \frac{dt}{d\tau}, \frac{d\vec{x}}{d\tau} \right) = \left( \dot{t}, \dot{\vec{x}} \right)
\]

\[A^\mu \equiv \left( \phi, \vec{A} \right)\] (18)

where \( \tau \) is the proper time along the path, provided

\[
\frac{\partial f (y)}{\partial y} \bigg|_{y=1} = \frac{1}{2} .
\] (19)

Since with the choice

\[
f \left( u^2 \right) = \frac{1}{2} u^2
\] (20)

the Lagrangian is quadratic in the four-velocity \( u \) – and therefore particularly manageable – the obvious choice is

\[
L = -\frac{1}{2} mu^\mu u_\mu - qu^\mu A_\mu
\] (21)

or

\[
L = -\frac{1}{2} m\dot{t}^2 + \frac{1}{2} m\dot{x}^2 - q\dot{t}\phi + q\dot{x} \cdot \vec{A}.
\] (22)
This gives the classical equations of motion

\[ m\ddot{t} = -q\dot{\phi} + q\dot{t}\phi,0 - q\dot{x}_j A_{j,0} \]  

\[ m\ddot{x}_i = -q\dot{A}_i - q\dot{t}\phi, j + q\dot{x}_j A_{j,i} \]  

(23) (24)

While we appear to be dealing with four variables, we can use the equations of motion to show that

\[ \frac{d}{d\tau} (i^2 - \vec{x}^2) = 0 \Rightarrow i^2 - \vec{x}^2 = 1 \]  

(25)

and thereby eliminate \( \dot{t} \) and \( t \) in favor of \( \dot{x} \) and \( x \). If all we were interested in was the classical trajectories, this would be a logical next step. However, since it is quantum fluctuations in time we wish to model, the dependence on \( t \) and the terms

\[ -\frac{1}{2}mt^2 - qt\phi \]  

are of the essence. For future reference, (23,24) imply

\[ \frac{d}{d\tau} (m\dot{t}) = \vec{v} \cdot \vec{E} \sqrt{1 - \vec{v}^2} \]  

(27)

While the Lagrangian (21) satisfies our requirements, there is no guarantee it is the “correct” Lagrangian from a quantum mechanical point of view. We may think of the classical trajectory as being like the river running through the center of a valley; the quantum fluctuations as corresponding to the topography of the surrounding valley. Many different topologies of the valley are consistent with the same course for the river. However, as (21) does produce the correct classical trajectories, is symmetric between time and space, and is, of the choices given by (17), the easiest to work with, it is the obvious one to try first.

Paths

When computing a path integral we normally do the sum over all paths in space

\[ \pi = (\vec{x}(t)) \]  

(28)

with \( \vec{x}' \equiv \vec{x}(t') \) and \( \vec{x}'' \equiv \vec{x}(t'') \). To achieve the greatest practicable symmetry between time and space we have to include all paths that vary in time as well, i.e. something more like

\[ \pi \equiv (t(\tau), \vec{x}(\tau)) \]  

(29)
To achieve symmetry between $t$ and $\vec{x}$ we have to let the time coordinate of a path vary just as we let the space coordinate vary.

In the normal case, when we wish to describe all the paths from $x'$ at $t'$ to $x''$ at $t''$ (taking just one space dimension for simplicity) we may model them in terms of their offset from a straight line
\[ x(t) = x' + \frac{x'' - x'}{t'' - t'} (t - t') + \sum_{n=1}^{\infty} a_n \sin(\omega_n (t - t')) \] (30)
where $\omega_n$ is given by
\[ \omega_n \equiv \frac{n\pi}{t'' - t'} \] (31)
and the $a_n$ are real. This guarantees that all paths that begin at $t', x'$ and end at $t'', x''$ are included: there is a one-to-one mapping between the set of all paths $\pi (x''; x')$ and the set of all $\{a_n\}$.

We would like to do the same thing for paths defined in time. There are, however, two problems:

1. What are we going to use for the time dimension? If time is now a dependent variable, what should we use for the independent variable? [81]

2. And what do we mean by using time as a dependent variable?

We address the first problem by defining the time $t$ associated with a particle trajectory in terms of $T$, the time in the lab frame [82]. The idea is that the time associated with a particle trajectory has, like Vonnegut’s Billy Pilgrim [53], become slightly “unstuck” from time as defined in the laboratory frame. $T$ takes the role that $\tau$ has had in the Lagrangian [83]. In the non-relativistic case, $t \approx T$.

By using $T$ as the reference index we can now define paths that fluctuate in time exactly as we defined paths that fluctuated in space. We may describe them in terms of their offset from a straight line
\[ t(T) = t' + \frac{t'' - t'}{T'' - T'} (T - T') + \sum_{n=1}^{\infty} a_n \sin(\omega_n (T - T')) \] (32)
where $\omega_n$ is given by
\[ \omega_n \equiv \frac{n\pi}{T'' - T'} \] (33)
and the $a_n$ are real. This will guarantee that all paths begin at $T', t'$ and end at $T'', t''$ and that all are included. Again, there is a one-to-one mapping between the set of all paths $\pi(t'', t')$ and the set of all $\{a_n\}$.

We still have the second problem, what do we mean by using time as a dependent variable? In particular, there is nothing to keep the $t(T)$ in (29) from being less than $T'$ or more than $T''$ at various points in its trajectory (see fig. 1). This is unavoidable, given we are treating time as a space dimension. Paths in space are allowed to zig left before zagging right and therefore paths in time must by the assumption of time/space symmetry have the same right. But this means that our trajectories can sample the electromagnetic fields before the trajectory starts and after it ends. This does not in itself create an immediate problem for causality – we are insisting $T'' \geq T'$ – but it is perhaps a bit unnerving.

The simplest solution is to treat this as a formal device for generating experiments; this is legitimate if unhelpful.

A second solution is to observe that these fluctuations will be of order $\hbar$ and therefore small. Most paths will spend little time before $t = T'$ and little time after $t = T''$. This helps to explain why these have not been seen, but does not address the point of principle.

A third solution is to note that the starting point of a path should be regarded as just
as subject to quantum uncertainty as anything else. We do not know, except to within $\delta t \sim \hbar/\delta \omega$, where a path starts or ends. The variation in $t(T)$ merely reflects our unavoidable uncertainty on this point.

This in turn leads us to an interesting if perhaps outré view of a wave function as an object extended in time as well as in space, where the extension in time represents our uncertainty about the particle’s position in time, just as the extension in space represents our uncertainty about its position in space. Then the usual wave function $\psi$ at a crisply defined lab time $T$ is a three-dimensional average over the “true” four dimensional wave function

$$\psi^{(3)}(T, \vec{x}(T)) = \int_{-\infty}^{\infty} dt_T \psi^{(4)}(t_T, \vec{x}_T)$$

$$= \langle \psi^{(4)} \rangle_{t_T}$$

(34)

The Schrödinger equation is then describing not the full four dimensional wave function but a three dimensional average of it. Assuming for the sake of argument that this is the case, then we may test this approach by looking for time-time correlations too subtle to be captured by the time-averaged $\psi^{(3)}$.

As an aside, perhaps the simplest – and certainly the most Machian – approach we could take to the lab time is to treat the lab coordinates, $T, \vec{X}$, as representing averages over the wave function of the rest of the universe, loosely

$$T \equiv \langle \tilde{\psi} | t | \tilde{\psi} \rangle$$

$$\vec{X} \equiv \langle \tilde{\psi} | \vec{x} | \tilde{\psi} \rangle$$

(35)

where $\tilde{\psi}$ is the rest of the wave function of the universe, the part complementary to the $\psi$ under examination. Then we have no absolute time, instead time is defined by comparing the expectation of the time operator over a (usually) small part of the wave function of the universe to the expectation of the time operator over the rest of the wave function of the universe. The difference between $t$ and $T$ is that $t$ is the time for $\psi$, $T$ the time for $\tilde{\psi}$, $\psi$’s complement. The assumption that this is an acceptable approximation is essentially the assumption that quantum interference terms between $\psi$ and $\tilde{\psi}$ may be ignored, in other words that we may ignore quantum interference terms between experiment and the observing apparatus.
Fortunately we need only the assumption that the “lab time” is sufficiently well-defined to use as a background grid, a matrix of imagined marks created by averaging over the times associated with an Avogadro’s numbers’ worth of particles, and corresponding in function to those faint penciled lines and pinpricks artists inscribe on a canvas to mark out lines of perspective and vanishing points, essential for the construction, but not themselves part of the final work.

Normalization

If there are quantum fluctuations in time, then we expect that the integral of the probability density may vary in time. We can define a probability density at each laboratory time $T$ by

$$\rho(T, \vec{x}_T) \equiv \int dt_T \psi^*(t_T, \vec{x}_T) \psi(t_T, \vec{x}_T)$$

(36)

with a probability at each time of

$$p(T) \equiv \int d\vec{x}_T \rho(T, \vec{x}_T)$$

(37)

We need to allow for the possibility that

$$\frac{dp(T)}{dT} \neq 0$$

(38)

while still keeping the probability normalized in some sense [88].

We only actually measure probabilities at start and finish, at $T'$ and $T''$. Hence having

$$1 = \int d\vec{x}_{T'} \rho(T', \vec{x}_{T'})$$

$$= \int d\vec{x}_{T''} \rho(T'', \vec{x}_{T''})$$

(39)

at times $T'$ and $T''$ is mandatory. But requiring

$$1 = \int d\vec{x}_T \rho(t_T, \vec{x})$$

(40)

for arbitrary $T$ is not.

To force [89] we try normalizing the probability amplitude from $a$ to $b$ with respect to the probability amplitude from $a$ to $\{b\}$, the set of all possible outcomes

$$\tilde{K}(b; a) \equiv \frac{1}{\sqrt{N(a)}} K(b; a)$$

(41)
where the normalization is
\[ N(a) \equiv \sum_{\{b\}} K^\dagger(b; a) K(b; a) \] (42)

Therefore
\[
\sum_{\{b\}} P(b | a) = \sum_{\{b\}} \tilde{K}^\dagger(b; a) \tilde{K}(b; a) \\
= \sum_{\{b\}} \frac{K^\dagger(b; a) K(b; a)}{\sqrt{N(a)} \sqrt{N(a)}} \\
= \frac{1}{\sqrt{N(a)}} \sum_{\{b\}} K^\dagger(b; a) K(b; a) \\
= 1.
\] (43)

However, this does not quite work. For instance, if we replace sums over \( b \) with integrals over \( \vec{x}' \), and take the usual free kernel as a test case
\[
\sqrt{m^2 \pi i} \left( T' - T \right) e^{im(\vec{x}' - \vec{x})^2} \\
\] (44)
we get
\[
N = \left( \frac{m}{2\pi i(T'' - T')} \right)^3 \int d\vec{x}'' e^{-im(\vec{x}'' - \vec{x})^2} e^{im(\vec{x}'' - \vec{x})^2} \\
= \left( \frac{m}{2\pi i(T'' - T')} \right)^3 \int d\vec{x}'' \\
= \infty.
\] (45)

This is an unfortunate side effect of dealing with what is really a distribution.

Since the kernel is a distribution, we may hope to get control by selecting appropriate test functions, e.g. the Gaussians [89].

\[
\varphi_a(T', \vec{x}) \equiv \frac{1}{(\pi \sigma^2)^{3/4}} \exp \left( -\frac{(\vec{x} - \vec{x}_a)^2}{2\sigma^2} + ik_i \cdot (\vec{x} - \vec{x}_a) \right). \\
\] (46)

These are centered at \( \vec{x}_a \), have momentum \( \vec{k}_a \), and are normalized to one at \( T = T' \).

We can now compute the normalization for a specific \( \varphi_a \)
\[
N_a = \int d\vec{x}'' \varphi_a^*(T'', \vec{x}'') \varphi_a(T'', \vec{x}'') \\
\] (47)
\[
\varphi_a(T'', \vec{x}'') \equiv \int d\vec{x} K(T'', \vec{x}''; T', \vec{x}') \varphi_a(T', \vec{x}) \\
\] (48)
We will assume we have discovered the kernel is proportional to

$$\sqrt{\frac{m}{T'' - T'}} e^{i \frac{m}{2} (x'' - x')^2}$$

(49)

(the semi-classical approximation will give this, for instance) so

$$\tilde{K} (x''; x') = \frac{1}{\sqrt{N_a}} \sqrt{\frac{m}{T'' - T'}} e^{i \frac{m}{2} (x'' - x')^2}.$$  (50)

By (48)

$$\varphi_a (T''', \vec{x}'', \vec{x}''') = \frac{1}{\sqrt{N_a}} \frac{1}{(\pi \sigma^2)^{3/4}} \sqrt{\frac{2\pi i}{f (T'' - T')}}$$

$$\times \exp \left( \frac{-(\vec{x}''' - \vec{x}'')^2}{2\sigma^2 f (T'' - T')} + i \frac{\vec{k} a \cdot (\vec{x}''' - \vec{x}'')}{f (T'' - T')} - i \frac{\vec{k}^2 a (T''' - T')}{2mf (T'' - T')} \right)$$

(51)

with

$$f (T) \equiv 1 + i \frac{T}{m\sigma^2}.$$  (52)

This gives for the probability density at $T'''$

$$\rho (T''', \vec{x}''') = \frac{1}{N_a} \frac{(2\pi)^3}{\sqrt{\pi \sigma^2 |f (T'' - T')|^2}} \exp \left( -\frac{(\vec{x}''' - (\vec{x}_a + \frac{\vec{k} a}{m} (T'' - T')))^2}{\sigma^2 |f (T'' - T')|^2} \right).$$

(53)

The requirement that this be normalized to one gives

$$N_a = (2\pi)^3.$$  (54)

The normalized kernel is

$$\tilde{K} (x''; x') = \frac{1}{\sqrt{2\pi}} K (x''; x')$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{m}{T'' - T'}} e^{i \frac{m}{2} (x'' - x')^2}.$$  (55)

This is the standard free kernel up to an inessential factor of $\frac{1}{\sqrt{i}}$.

In principle, this approach to normalization is clumsier than the standard: we have to normalize for each specific starting wave function $\varphi (x')$. In other words we have

$$K (x''; x') \rightarrow \tilde{K} [\varphi] (x''; x').$$

(56)

The expression for the normalization constant for a given $\varphi$ is

$$N [\varphi] = \int d\vec{x}'' \left| \int d\vec{x}' \left( K (x''; x') \varphi (x') \right) \right|^2.$$  (57)
This gives us a formal method to determine the normalization constant for path integrals, letting us avoid the ad hoc method employed to get (4).

We will be making only a limited use of this normalization procedure; we are primarily interested in establishing the self-consistency of the approach we are taking here. But the basic idea – normalize the amplitude to get to a specific outcome by the sum of the amplitudes to get to any outcome – is sufficiently simple that a number of variations on the theme are possible. For instance, the initial wave function could be defined with respect to an observer \( a \) but then measured by an observer \( b \) moving relativistically with respect to \( a \). (It can be useful to keep in mind that \( K (b; a) \) represents correlation not causality.)

**Four dimensional kernel**

The path integral for a charged particle of mass \( m \) and charge \( q \) going from point \( x' \) to \( x'' \) in the presence of fields \( A_\mu (x) \) is therefore

\[
K (x''; x') = \int D [\pi] e^{-i \int_{T'}^{T''} dT \mathbb{R} u^\mu A_\mu}
\]

using \( u = (u_0, \vec{u}) \) for the four velocity (We will usually suppress explicit notation of the dependence of \( K \) on \( T' \) and \( T'' \)).:

\[
K (x''; x') = \lim_{N \to \infty} \int \prod_{i=1}^{N-1} dt_i d\vec{x}_i \exp \left( -i \varepsilon \sum_{j=1}^{N} \left( m u_j^2 + q u_j^\mu A_\mu (x_j) \right) \right)
\]

\[
x_j \equiv \left( \frac{t_{j+1} + t_j}{2}, \frac{x_{j+1} + x_j}{2} \right)
\]

\[
u_j^\mu \equiv \left( \frac{t_{j+1} - t_j, x_{j+1} - x_j, y_{j+1} - y_j, z_{j+1} - z_j}{\varepsilon} \right)
\]

\[
u_j^2 = \left( u_0^2 + \vec{u}_j^2 \right)
\]

\[
\varepsilon \equiv \frac{T'' - T'}{N}
\]

\[
(t_0, \vec{x}_0) = (t', \vec{x}')
\]

\[
(t_N, \vec{x}_N) = (t'', \vec{x}'')
\]
Comparison to three dimensional kernel

We compare (59) to the standard three dimensional kernel

\[ K(x'', x') \equiv \lim_{N \to \infty} \int_{N-1}^{N} \prod_{i=1}^{N-1} \exp \left( \frac{i \varepsilon}{N} \sum_{j=1}^{N-1} \left( \frac{m \vec{v}_j^2}{2} - q \phi (t_j, \vec{x}_j) + qv_i A_i (t_j, \vec{x}_j) \right) \right) \]

\[ \vec{v}_j \equiv \left( \frac{x_{j+1} - x_j}{\varepsilon}, \frac{y_{j+1} - y_j}{\varepsilon}, \frac{z_{j+1} - z_j}{\varepsilon} \right) \]

\[ \varepsilon \equiv \frac{t'' - t'}{N}. \]

From (59) we can get (60) by

1. replacing \( u_j^0 A_0 \to \phi \) and \( \vec{u}_j \cdot \vec{A} \to \vec{v}_j \cdot \vec{A} \), i.e. taking the non-relativistic limit, and

2. eliminating the \( \int_{N-1}^{N} \prod_{i=1}^{N-1} dt_i \exp \left( -i \varepsilon \sum_{j=1}^{N-1} u_j^3 \right) \), i.e. getting rid of the integrals over \( dt \), the quantum fluctuations in time.

The first change merely indicates that (59) is a possible extension to the relativistic regime of (60); the second is the interesting one. In general, as we will see below, normalization keeps the \( dt \) integrations from having any effect on the resulting kernel unless the potentials mix the time and space coordinates.

While the inclusion of quantum fluctuations in time is a novelty, it is difficult to see how one could get a correct relativistic generalization of the Schrödinger equation without them.

Consider a frame \( A \) in which they are not present. Only \( d\vec{x}_A \) integrations are used. Now consider a frame \( B \) going by the first at, say, \( 1/2c \) in the \( x \) direction. In the \( B \) frame, the integrals over \( dx_A \) will look like a combination of integrals in \( dx_B \) and \( dt_B \), in other words to \( B \), they will look as if the integrals include integrals over quantum fluctuations in time.

SEMI-CLASSICAL APPROXIMATION

We may define the semi-classical approximation as the kernel which results when we expand the Lagrangian around the classical trajectory. We define

\[ t = \bar{t} + \delta t \]
\[ x_i = \bar{x}_i + \delta x_i \]

where \( \bar{t}(T) \) and \( \bar{x}_i(T) \) represent the classical trajectory. We keep the first three terms of the expansion in \( \delta x \). This is of course exact for potentials quadratic in \( \delta x \), is a good first
approximation in many other cases, and in general can help in understanding the qualitative characteristics of the behavior.

We start with the Lagrangian \( (21) \) and write out the 0\(^{th} \), 1\(^{st} \), and 2\(^{nd} \) terms, using integration by paths to eliminate terms linear in \( \delta u \)

\[
L \approx -\frac{1}{2} m \bar{u} \dot{u} - q u A^\mu (\bar{x}) \]

\[
+ \left( m \bar{u} + q \dot{\bar{A}}_\mu (\bar{x}) - q \bar{u}^\kappa A_{\kappa,\mu} (\bar{x}) \right) \delta x^\mu
\]

\[
- \frac{1}{2} m \delta u \delta u^\mu - q \left( A_{\mu,\nu} (\bar{x}) + A_{\mu,\nu} (\bar{x}) \right) \delta u^\mu \delta x^\nu - \frac{1}{2} q \bar{u}^\kappa A_{\kappa,\mu\nu} (\bar{x}) \delta x^\mu \delta x^\nu. \quad (62)
\]

The first line integrated over \( dT \) gives the classical action \( \bar{S} \). The coefficient of \( \delta x \) in the second line is the classical equation of motion. It is identically zero per \( (23, 24) \). We know that only the antisymmetric part of \( A, A_{\mu,\nu} (\bar{x}) - A_{\nu,\mu} (\bar{x}) \), has physical meaning. Therefore we eliminate the \( A_{\mu,\nu} (\bar{x}) + A_{\mu,\nu} (\bar{x}) \) term with a gauge transformation \( A_\mu \to A_\mu + \lambda_\mu \) such that

\[
\partial^\mu \partial_\mu \lambda = -\frac{1}{2} \left( A_{\mu,\nu} (\bar{x}) + A_{\mu,\nu} (\bar{x}) \right) \]

reducing the Lagrangian to

\[
- \frac{1}{2} m \bar{u} \dot{u} - q u A^\mu (\bar{x}) - \frac{1}{2} m \delta u \delta u^\mu - \frac{1}{2} q \bar{u}^\kappa A_{\kappa,\mu\nu} (\bar{x}) \delta x^\mu \delta x^\nu \quad (64)
\]

We break out the time and space parts

\[
L = -\frac{1}{2} m \ddot{x}^2 + \frac{1}{2} m \dot{x}_i \dot{x}_i - q \dot{q} \phi (\bar{x}) + q \ddot{x}_k A_k (\bar{x})
\]

\[
- \frac{1}{2} m \delta t^2 - \frac{1}{2} \left( q \dot{\phi}_{00} (\bar{x}) - q \ddot{x}_k A_{k,00} (\bar{x}) \right) \delta t^2
\]

\[
+ \left( q \dot{\phi}_{0i} (\bar{x}) - q \ddot{x}_k A_{k,0i} (\bar{x}) \right) \delta t \delta x_i
\]

\[
+ \frac{1}{2} m \delta x_i \delta x_i - \frac{1}{2} \left( q \dot{\phi}_{ij} (\bar{x}) - q \ddot{x}_k A_{k,ij} (\bar{x}) \right) \delta x_i \delta x_j. \quad (65)
\]

The critical term is \( q \dot{\phi}_{0k} (\bar{x}) - q \ddot{x}_k A_{k,0i} (\bar{x}) \). If this is zero, then we can separate the problem into its time and space parts. The effects of the time part will then drop out during normalization, and we will be left with just the last line, the usual space-space path integral.

As usual, we can get an explicit formula for the kernel in terms of the action for the corresponding classical problem – at the expense of some slightly formal manipulations.

We start by Wick rotating, replacing \( t \) by \( i x_4 \) (so sums over the corresponding indices will run from 1 to 4). This gives

\[
K (x''; x') \equiv i^{N-1} \lim_{N \to \infty} \int \prod_{j=1}^{N-1} dx_j dy_j dz_j dA dx_j \exp \left( i \varepsilon \sum_{j=1}^{N-1} \left( m u_j^2 + q u_j^\mu A_\mu (x_j) \right) \right) \quad (66)
\]
Since the Lagrangian is quadratic in the integration variables, the integrals telescope. Per (2, 4), these give
\[ K(x''; x') = i^{N-1} A^{4N} \sqrt{\frac{\partial^2 \bar{S}(x''; x')}{\partial x'_\mu \partial x'_\nu}} e^{i\bar{S}(x''; x')} . \] (67)

The \( A^{4N} \) is the result of the fact that we did not include the per-step normalization factor, \( \frac{1}{\lambda} = \sqrt{\frac{m}{2\pi\hbar}} \) in each integration, as we are normalizing at the end of the calculation. \( \frac{\partial^2 \bar{S}(x''; x')}{\partial x''^\mu \partial x'^\nu} \) is the van Vleck Pauli determinant of \( \bar{S} \)

\[
\begin{vmatrix}
\frac{\partial^2 \bar{S}}{\partial x''^\mu \partial x''^\nu} & \frac{\partial^2 \bar{S}}{\partial x''^\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x''^\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x''^\mu \partial x'\nu} \\
\frac{\partial^2 \bar{S}}{\partial x'\mu \partial x''^\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} \\
\frac{\partial^2 \bar{S}}{\partial x'\mu \partial x''^\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} \\
\frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu} & \frac{\partial^2 \bar{S}}{\partial x'\mu \partial x'\nu}
\end{vmatrix} . \] (68)

Now we undo the Wick rotation, replacing \( x_4 \) by \( -it \). The only effect is to replace the van Vleck Pauli determinant by its negative

\[ K(x''; x') = -i^{N-1} A^{4N} \sqrt{\frac{\partial^2 \bar{S}(x''; x')}{\partial x''^\mu \partial x'\nu}} e^{i\bar{S}(x''; x')} . \] (69)

This is the unnormalized kernel, with the normalization to be computed using (57). At that time the factor of \( -i^{N-1} A^{4N} \), being independent of \( x' \) and \( x'' \), will cancel out. Knowing this factor is doomed in any case we take as our final result

\[ K(x''; x') = \sqrt{\frac{\partial^2 \bar{S}(x''; x')}{\partial x''^\mu \partial x'\nu}} e^{i\bar{S}(x''; x')} . \] (70)

There are three differences between this and the standard result

1. The van Vleck Pauli determinant is four by four rather than three by three, as it includes time-time and time-space terms.

2. The classical action \( \bar{S} \) is in general different, even though the classical trajectories are the same. The differences represent in general the difference between a relativistic and a non-relativistic approach.

3. The major difference is that the resulting kernel is used in a different way: it is applied to four rather than three dimensional wave functions, opening up possibilities for interference in time not present in the three dimensional case.
We recall we are taking the standard three dimensional wave functions as being the average in time of the four dimensional wave functions, with the time dimension being understood as representing our uncertainty as to when the particle is located, just as the first three dimensions of a wave function may be taken as representing our uncertainty as to where it is located.

**Free kernel**

In the case of the free particle (70) is exact. Since the free action is

$$\frac{m (t'' - t')^2 - (\vec{x}'' - \vec{x}')^2}{\Delta T}$$

(71)

the free kernel is

$$K (x''; x') = \frac{1}{N_a^{1/2}} \left( \frac{m}{T'' - T'} \right)^2 e^{-i \frac{m}{2} (x'' - x')^2} e^{-i \frac{m}{2} (t'' - t')^2}$$

(72)

which we may factor into time and space pieces as

$$K^{(t)} (t''; t') = \frac{1}{N_a^{1/8}} \sqrt{\frac{m}{i (T'' - T')}} e^{-i \frac{m}{2} (t'' - t')^2}$$

$$K^{(x)} (\vec{x}''; \vec{x}') = \frac{1}{N_a^{3/8}} \sqrt{\frac{m}{i (T'' - T')}} e^{i \frac{m}{2} (\vec{x}'' - \vec{x}')^2}$$

(73)

where we have chosen the individual and overall phases so that we have the useful symmetry property

$$K^{(t)}_T (t''; t') = K^{(x)}_T (x''; x') = K^{(t)}_T (x''; x') = K^{(x)}_T (x''; x') .$$

(74)

If preparation and measurement are relative to the same reference frame –the usual case – then $K^{(t)}$ will cancel out during normalization. If $t \approx T$ we will be left with a constant factor times the usual free kernel in three dimensions

$$K^{(3)} (t''; \vec{x}''; t'; \vec{x}') = \sqrt{\frac{m}{2 \pi i (t'' - t')}} e^{i \frac{m}{2} (\vec{x}' - \vec{x}')^2} .$$

(75)
Free wave function

We still need to normalize the kernel appropriately. Per discussion above, this can in general only be done relative to a specific set of test functions. We select for \( \varphi_a \) the test functions of (46) appropriately extended to include the time dimension

\[
\varphi_a (t', \vec{x}') \equiv \frac{1}{\sqrt{2\pi \sigma_t^2}} \exp \left( -\frac{(t' - t_a)^2}{2\sigma_t^2} - i\omega_a (t' - t_a) \right) \times \frac{1}{\sqrt{2\pi \sigma_x^2}} \exp \left( -\frac{(\vec{x}' - \vec{x}_a)^2}{2\sigma_x^2} + i\vec{k}_a \cdot (\vec{x}' - \vec{x}_a) \right).
\](76)

We do not require that \( \omega_a = \sqrt{k_a^2 + m^2} \), although this is certainly the most obvious choice. Given our principle of time/space symmetry it would make sense to use the same value of the standard deviation for both time and space, to set \( \sigma_t = \sigma_x \). But keeping \( \sigma_t \) and \( \sigma_x \) distinct will let us compare the three and four dimensional approaches by letting \( \sigma_t \to 0 \).

We will normally start with \( t_a = T' \). We define the “lab time” associated with a specific wave function by

\[
T [\varphi] \equiv \langle t \rangle = \int dt \varphi^* (x) t \varphi (x)
\]

which for (76) gives \( T' \) as expected. And we have for the classical position associated with a particle

\[
\vec{X} \equiv \langle \vec{x} \rangle = \int dt d\vec{x} \varphi^* (x) \vec{x} \varphi (x) = \vec{x}_a
\]

again as expected.

Given \( \varphi \) defined at one lab time \( T' \) we get \( \varphi \) at lab time \( T'' \) by applying the kernel (72) to it

\[
\varphi_{T''} (x'') = \int dx' K_{T'' - T'} (x''; x') \varphi_{T'} (x')
\]

A straightforward calculation gives

\[
\varphi_a (x'') = \varphi_a^{(t)} (t'') \varphi_a^{(\vec{x})} (\vec{x}'')
\]

\[
\varphi_a^{(t)} (t'') = \frac{1}{Na^{1/8} \sqrt{2\pi \sigma_t^2}} \frac{1}{f_t (T'' - T'')} \sqrt{\frac{2\pi}{2\pi \sigma_t^2}} \exp \left( -\frac{(t'' - t_a)^2}{2\sigma_t^2} - i\omega_a (t'' - t_a) - i\omega_a (T'' - T') \frac{2m f_t (T'' - T'')}{} \right)
\]

\[
\varphi_a^{(\vec{x})} (\vec{x}'') = \frac{1}{Na^{3/8} \sqrt{2\pi \sigma_x^2}} \frac{1}{f_x (T'' - T')} \sqrt{\frac{2\pi}{2\pi \sigma_x^2}} \exp \left( -\frac{(\vec{x}' - \vec{x}_a)^2}{2\sigma_x^2} + i\vec{k}_a \cdot (\vec{x}' - \vec{x}_a) \right)
\]
\[
\times \exp \left( -\frac{(\vec{x}' - \vec{x}_a)^2}{2\sigma_t^2f_x(T'' - T')} + i \frac{\vec{k}_a \cdot (\vec{x}' - \vec{x}_a)}{f_x(T'' - T')} - i \frac{\vec{k}_a^2(T'' - T')}{2mf_x(T'' - T')} \right)
\]

\[f_{t,x}(T) \equiv 1 + i \frac{T}{m\sigma_{t,x}^2}. \quad (80)\]

Since

\[
\frac{(2\pi)^4}{N_a} = \int dx'' \varphi_a^* (x'') \varphi_a (x'') \quad (81)
\]

we have

\[N_a = (2\pi)^4 \quad (82)\]

so the free kernels are

\[
K_{(\text{free})}^{(t)} (t'', t') = \sqrt{-\frac{m}{2\pi i (T'' - T')}} e^{-\frac{im}{2} (t'' - t')^2}
\]

\[
K_{(\text{free})}^{(\vec{x})} (\vec{x}'', \vec{x}) = \sqrt{-\frac{m}{2\pi i (T'' - T')}} e^{\frac{im}{2} (\vec{x}'' - \vec{x})^2}. \quad (83)
\]

and the normalized probability distributions are

\[
p_a (x'') = p_a^{(t)} (t'') p_a^{(\vec{x})} (\vec{x}'') \sqrt{\frac{1}{\pi \sigma_t^2 |f_t(T'' - T')|^2}} \exp \left( -\frac{(t'' - (t_a + u_0(T'' - T')))^2}{\sigma_t^2 |f_t(T'' - T')|^2} \right)
\]

\[
p_a^{(t)} (t'') = \sqrt{\frac{1}{\pi \sigma_x^2 |f_x(T'' - T')|^2}} \exp \left( -\frac{(\vec{x}'' - (\vec{x}_a + \vec{u}(T'' - T')))^2}{\sigma_x^2 |f_x(T'' - T')|^2} \right)
\]

\[
u \equiv \frac{\omega_a \vec{k}_a}{m} \quad (84)
\]

From this it is apparent that the “center-of-probability” of the particle, \(\langle t, \vec{x} \rangle\), is moving with four-velocity \(u\). The three velocity is \(\omega/k\), which is independent of \(T\).

If we let \(\sigma_t \to 0\), then the wave functions goes from being four to three dimensional as does its the probability distribution. But if we let \(\sigma_t\) start finite, then it will get still more finite (as it were) with time \(|f_t(T'' - T')|^2 \sim \frac{T'' - T'}{m^2 \sigma_t^4}\). In other words, three dimensional objects stay three dimensional and four dimensional, four. Both assumptions are self-consistent. It might be difficult to test either without reference to the other.

**Non-relativistic case**

Since we are primarily interested in the non-relativistic case (we are using relativistically invariant Lagrangians and the like only to guarantee time/space symmetry) we ask what \((70)\) looks like in the non-relativistic limit.
Of course, first we need to define what we mean by the non-relativistic limit. We take advantage of (27) and define the non-relativistic limit as being given by cases where the integral of $\vec{v} \cdot \vec{E}$ over a typical path is small. In other words the acceleration of the time variable is small, i.e. $d\ddot{t}/dT \approx 0$. The classical time may be approximately given by the linear part of (32)

$$\bar{t}(T) \approx t' + \frac{t'' - t'}{T'' - T'} (T - T')$$

$$= t' + \gamma (T - T'), \quad \gamma \equiv \frac{t'' - t'}{T'' - T'}$$

and

$$\dot{\bar{t}} \approx \gamma$$

(85)

This formula for the Lagrangian considerably simplifies our path integrals. In particular, unless $\phi,\phi_{,0i}(\bar{x}) - \bar{v}_kA_{k,0i}(\bar{x})$ is non-zero, the time and space parts will decouple. If they decouple, the time part will cancel out during normalization, just as it did in the free case [91]. Therefore, if there is to be an effect of the $dt$ integrations, $\phi(\bar{x}) - \bar{v}_kA_k(\bar{x})$ must depend on both time and space. If it does not, the four and three dimensional calculations will give the same results – at least for those cases where the semi-classical approximation is valid.

The Lagrangian (65) becomes

$$L = -\frac{1}{2}m\gamma^2 + \frac{1}{2}m\gamma^2\bar{v}_i\bar{v}_i - q\gamma\phi(\bar{x}) + q\gamma\bar{v}_kA_k(\bar{x})$$

$$-\frac{1}{2}m\delta t^2 - \frac{1}{2}q\gamma (\phi_{,00}(-\bar{x}) - \bar{v}_kA_{k,00}(-\bar{x})) \delta t^2$$

$$+q\gamma (\phi_{,0i}(\bar{x}) - \bar{v}_kA_{k,0i}(\bar{x})) \delta t\delta x_i$$

$$+\frac{1}{2}m\delta \bar{x}_i\delta \bar{x}_i - \frac{1}{2}q\gamma (\phi_{,ij}(\bar{x}) - \bar{v}_kA_{k,ij}(\bar{x})) \delta x_i\delta x_j$$

(87)

or discarding constant terms and letting $\gamma \to 1$

$$L = \frac{1}{2}m\bar{v}_i\bar{v}_i - q\gamma\phi(\bar{x}) + q\bar{v}_kA_k(\bar{x})$$

$$-\frac{1}{2}m\delta t^2 - \frac{1}{2}q (\phi_{,00}(\bar{x}) - \bar{v}_kA_{k,00}(\bar{x})) \delta t^2$$

$$+q (\phi_{,0i}(\bar{x}) - \bar{v}_kA_{k,0i}(\bar{x})) \delta t\delta x_i$$

$$+\frac{1}{2}m\delta \bar{x}_i\delta \bar{x}_i - \frac{1}{2}q (\phi_{,ij}(\bar{x}) - \bar{v}_kA_{k,ij}(\bar{x})) \delta x_i\delta x_j.$$ (88)

This is the usual non-relativistic Lagrangian (as in 60) plus terms in $\delta t^2$ and $\delta t\delta \bar{x}$.

This formula for the Lagrangian considerably simplifies our path integrals. In particular, unless $\phi_{,0i}(\bar{x}) - \bar{v}_kA_{k,0i}(\bar{x})$ is non-zero, the time and space parts will decouple. If they decouple, the time part will cancel out during normalization, just as it did in the free case [91]. Therefore, if there is to be an effect of the $dt$ integrations, $\phi(\bar{x}) - \bar{v}_kA_k(\bar{x})$ must depend on both time and space. If it does not, the four and three dimensional calculations will give the same results – at least for those cases where the semi-classical approximation is valid.
A STERN-GERLACH EXPERIMENT IN TIME

Experimental arrangement

*Mutatis mutandis*, if we are to see a difference between the four and three dimensional calculations we need a field which mixes space and time, e.g.

$$V(t, \vec{x}) \sim f(t) g(\vec{x}).$$  \hspace{1cm} (89)

Consider a particle with an electric dipole moment traveling through a potential given by

$$\phi(t, \vec{x}) = -x(E_0 + E_1 t).$$  \hspace{1cm} (90)

If the electric dipole is $\vec{p}$, the interaction energy is

$$V(t, \vec{x}) = -\vec{p} \cdot \vec{E}(t, \vec{x})$$
$$= \vec{p} \cdot \nabla \phi(t, \vec{x})$$
$$= -p_x (E_0 + E_1 t).$$  \hspace{1cm} (91)

Such a potential might be generated by a capacitor perpendicular to the $x$ axis with the two plates at $x_1$ and $x_2$ (see fig. 2). A voltage applied at $x_2$ with value $-(E_0 + E_1 T)$, will create a time-varying electric field between $x_1$ and $x_2$ parallel to the $x$ axis and of size $E_0 + E_1 T$.

For a quantum mechanical particle, the value of $\vec{p}$ will be given by the electric dipole operator $\hat{p}$. We will assume this has a set of eigenvalues $\{p\}$. We will work in the basis
in which \( \hat{p} \) is diagonal along the \( x \) axis. We consider the wave function p-component by p-component

\[
\hat{p}\varphi_{p}(x') = p\varphi_{p}(x')
\] (92)

or

\[
\psi(x') = \sum_{\{p\}} c_{p}\varphi_{p}(x')
\] (93)

where the \( \varphi_{p}(x') \) are the eigenfunctions of \( \hat{p} \). Then

\[
\psi(x'') = \sum_{\{p\}} c_{p}\int dx' K_{p}(x''; x')\varphi_{p}(x').
\] (94)

There is a formal resemblance between the Stern-Gerlach experiment[54, 55, 56] and this one. Consider a Stern-Gerlach experiment with the beam of magnetic dipoles \( \vec{\mu} \) going in the \(+x\) direction and with the magnetic field \( \vec{B} = (0, 0, B_{z}) \) varying along the \( z \) axis, \( B_{z,z} \neq 0 \). We may go from this to ours by making the replacements

\[
\begin{align*}
z & \rightarrow t \\
t & \rightarrow T \\
\vec{\mu} & \rightarrow \vec{p} \\
B_{z} & \rightarrow E_{x} \\
B_{z,z} & \rightarrow E_{x,t}.
\end{align*}
\] (95)

In most treatments of the Stern-Gerlach experiment, the “collapse of the wave function” is assigned responsibility for the observed space quantization. It would appear however that if the finite extent of the wave function along the \( z \) axis is modeled explicitly, e.g. as a Gaussian, there is in fact no need to invoke the collapse; coherent self-interference within the wave function suffices to produce the space quantization [57]. We will see a similar result here: coherent self-interference in time will produce time quantization.

**Electric dipole potential**

We first need to derive the correct form of the electric dipole interaction. For two particles we have

\[
L = -\frac{1}{2}m_{1}\dot{r}_{1}^{2} + \frac{1}{2}m_{1}\dot{x}_{1}^{2} - \frac{1}{2}m_{2}\dot{r}_{2}^{2} + \frac{1}{2}m_{2}\dot{x}_{2}^{2} - V(\vec{x}_{1}, \vec{x}_{2})
\]
\[-q \dot{t}_1 \phi (x_1) + q \dot{t}_2 \phi (x_2) \quad (96)\]

where \(V\) is the potential that holds the dipole together. We define the center of mass \(x\) and the relative \(\tilde{x}\) coordinates by

\[
x \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad x_1 = x + \frac{m_2}{m_1 + m_2} \tilde{x} \\
\tilde{x} \equiv x_1 - x_2 \quad x_2 = x - \frac{m_1}{m_1 + m_2} \tilde{x}
\]

and rewrite (96) as

\[
L = -\frac{1}{2} m \dot{t}^2 + \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \dot{\tilde{m}}^2 + V (\tilde{x}) \\
- q \left( \dot{t} + \frac{m_2}{m_1 + m_2} \dot{\tilde{x}} \right) \phi \left( x + \frac{m_2}{m_1 + m_2} \tilde{x} \right) \\
+ q \left( \dot{t} - \frac{m_1}{m_1 + m_2} \dot{\tilde{x}} \right) \phi \left( x - \frac{m_1}{m_1 + m_2} \tilde{x} \right)
\]

with

\[
m \equiv m_1 + m_2, \quad \tilde{m} \equiv \frac{m_1 m_2}{m_1 + m_2} \quad (99)
\]

Now we assume \(\tilde{x}\) small relative to \(x\) and expand \(\phi (x + \alpha \tilde{x})\) around \(\phi (x)\) in powers of \(\alpha \tilde{x}\)

\[
\phi (x + \alpha \tilde{x}) = \phi (x) + \alpha \tilde{x} \partial_\mu \phi (x) \quad (100)
\]

letting us rewrite the last two terms of (98) in terms of \(V_{\text{dipole}} (x, \tilde{x}) = q \dot{t} \phi (x) + q \dot{t} \phi_0 (x) + q \dot{t} \tilde{x} \cdot \nabla \phi (x) \quad (101)\)

The full path integral Lagrangian is given by

\[
- i \varepsilon \sum_{j=1}^{N} \left( \frac{m}{2} u_j^2 + \frac{\tilde{m}}{2} \tilde{u}_j^2 + V (\tilde{x}) + V_{\text{dipole}} (x, \tilde{x}) \right)
\]

The \(\tilde{x}\) system is not directly visible to us. Using the cumulant approximation \[4\] to lowest order, we may replace the values of the relative variables in \(V_{\text{dipole}}\) by their averages

\[
V_{\text{dipole}} (x, \tilde{x}) \approx q \langle \dot{t} \rangle \phi (x) + q \langle \dot{t} \phi_0 (x) + q \langle \dot{t} \tilde{x} \cdot \nabla \phi (x) \quad (103)\]

where the average of a relative quantity \(\tilde{Q}\) is defined by

\[
\langle \tilde{Q} (x) \rangle = \lim_{N \to \infty} \frac{\langle x_N | \prod_{i=1}^{N} d \tilde{x}_i \tilde{Q} (\{ \tilde{x}_k \}) \exp \left( -i \varepsilon \sum_{j=1}^{N} \left( \frac{m}{2} u_j^2 + V (\tilde{x}) \right) \right) | x_0 \rangle}{\langle x_N | \prod_{i=1}^{N-1} d \tilde{x}_i \exp \left( -i \varepsilon \sum_{j=1}^{N} \left( \frac{m}{2} u_j^2 + V (\tilde{x}) \right) \right) | x_0 \rangle} \quad (104)\]
We assume the wave functions in $|x_0\rangle$ and $|x_N\rangle$ are represented by some suitable time-average. Taking (103) term by term, the first term is

$$\langle \dot{t} \rangle = \langle \dot{t}_1 - \dot{t}_2 \rangle = \frac{\langle \dot{t}^{\prime \prime}_1 - \dot{t}^{\prime}_1 \rangle - \langle \dot{t}^{\prime \prime}_2 - \dot{t}^{\prime}_2 \rangle}{T^{\prime \prime} - T^{\prime}} \approx \frac{(T^{\prime \prime} - T^{\prime}) - (T^{\prime \prime} - T^{\prime})}{T^{\prime \prime} - T^{\prime}} = 0$$

(105)

where we are assuming that $\langle t^{\prime \prime} \rangle \approx T^{\prime \prime}$ and so on.

The second involves a dipole moment along the time dimension

$$p_0 \equiv q \langle \dot{t} \rangle .$$

(106)

To show a dipole in time, a system has to have an asymmetry under $T$ (just as to show a permanent spatial dipole, a system has to have an asymmetry under $P$). We will assume our electric dipole is coming from a system symmetric under $T$, so the second term is also zero.

We are left with the third term. The electric dipole moment is defined as

$$\vec{p} \equiv q \langle \vec{x} \rangle$$

(107)

so we have

$$V_{dipole} \approx -\dot{t} \vec{p} \cdot \vec{E}(\vec{x})$$

(108)

which is the same interaction as before, (91), times a factor of $\dot{t}$. The contribution to the action for a specific eigenfunction of $\hat{p}$ is

$$S_{dipole} = -\int_{T_1}^{T_2} dT V_{dipole} = \int_{T_1}^{T_2} dT \frac{dt}{dT} p E(T) = p \int_{t_1}^{t_2} dt E(t) = p \langle E \rangle (t_2 - t_1)$$

(109)

We define the impulsive approximation as letting $\Delta T \equiv T_2 - T_1 \to 0$ while holding $E_0 \Delta T$ and $E_1 \Delta T$ small but finite

$$E_0 \Delta T \to E_0$$

$$E_1 \Delta T \to E_1.$$

(110)

This approximation corresponds to letting the two plates of our capacitor get closer and closer while keeping the potential across them unchanged.
Kernel

Since there is time but no space dependence in the interaction term, we may focus our attention on time. The kernel in the three space dimensions will be given by the free kernel $K^{(x)}$ as in (73).

We start with the total action in time

$$ S(3; 0) = S_{\text{free}}(3; 2) + S_{\text{free}}(2; 1) + S_{\text{free}}(1; 0) + S_{\text{dipole}}(2; 1) $$  \hspace{1cm} (111)

The interesting part is $S_{\text{dipole}}$. To lowest order this will be given by $p\langle E(t) \rangle \Delta T$ where the average of $E$ is taken over the unperturbed classical trajectory, a straight line from $(T_1, t_1)$ to $(T_2, t_2)$ (see fig. 3). We define

$$ \bar{T} \equiv \frac{T_2 + T_1}{2}, \quad \bar{t} \equiv \frac{t_2 + t_1}{2} $$  \hspace{1cm} (112)

so

$$ S_{\text{dipole}}(2; 1) \approx p (E_0 + E_1 \bar{t}) \Delta T = p (E_0 + E_1 \bar{t}) $$  \hspace{1cm} (113)

The total action $S(3; 0)$ is now given by

$$ S(3; 0) = -\frac{m (t_3 - t_2)^2}{2 T_3 - T_2} - \frac{m (t_2 - t_1)^2}{2 T_2 - T_1} - \frac{m (t_1 - t_0)^2}{2 T_1 - T_0} + pE_0 + pE_1 \frac{t_2 + t_1}{2} $$  \hspace{1cm} (114)
and the corresponding kernel by

\[
K^{(t)}_p (t_3, t_0) = \sqrt{\frac{m}{-2\pi i (T_3 - T_2)}} \sqrt{\frac{m}{-2\pi i (T_2 - T_1)}} \sqrt{\frac{m}{-2\pi i (T_1 - T_0)}} \int dt_2 dt_1 \times \exp \left( -i \frac{m}{2} (t_3 - t_2)^2 - i \frac{m}{2} (t_2 - t_1)^2 - i \frac{m}{2} (t_1 - t_0)^2 \right) \times \exp \left( ip E_0 + ip E_1 \frac{t_2 + t_1}{2} \right).
\]

Rewriting \( T_1 \) and \( T_2 \) in terms of \( \bar{T} \) and \( \Delta T \) and discarding terms of order \( \Delta T \) and higher and of order \( E^2 \) we get

\[
K^{(t)}_p (t_3, t_0) = K^{(t)}_{(free)} (t_3; t_0) \exp (ip E_0) \exp \left( ip E_1 \frac{T_3 - T_0}{T_3 - T_0} \right). \tag{116}
\]

Wave function

We start by assuming our initial wave function is a Gaussian in time

\[
\varphi_{T_0} (t_0) = \frac{1}{\sqrt[4]{\pi \sigma^2_t}} \exp \left( -\frac{(t_0 - T_0)^2}{2\sigma^2_t} - i \omega_0 (t_0 - T_0) \right). \tag{117}
\]

We compute \( \varphi(t_3)_{T_3} \) by applying the kernel \( \tag{116} \) to this

\[
\varphi_{T_3} (t_3) = \int dt_0 K^{(t)}_p (t_3; t_0) \varphi_{T_0} (t_0)
\]

\[
\varphi_{T_3}^{(t)} (t_3) = \frac{1}{\sqrt[4]{\pi \sigma^2_t}} \sqrt{\frac{1}{f_1 (T_0 - T_3)}} \times \exp \left( -\frac{(t_3 - T_0)^2}{2\sigma^2_t f_1 (T_0 - T_3)} - i \frac{\omega_3 (t_3 - T_0)}{f_1 (T_0 - T_3)} - i \frac{\omega^2_3 (T_0 - T_3)}{2mf_1 (T_0 - T_3)} \right) \times \exp \left( ip E_0 + ip E_1 \frac{T_3 - T_0}{T_3 - T_0} \right)
\]

\[
\omega_3 \equiv \omega_0 - \frac{p E_1}{f_1 (T_3 - T_0)} \frac{T_3 - T_0}{T_3 - T_0}
\]

\[
f_1 (T_0 - T_3) = 1 - i \frac{T_3 - T_0}{m \sigma^2_t} \tag{119}
\]

The corresponding probability distribution for a single component is given by

\[
p^{(t)}_{T_3} (t_3) = \frac{1}{\pi \sigma^2_t |f_1 (T_3 - T_0)|^2} \exp \left( -\frac{(t_3 - \left(T_0 + \frac{\omega_3}{m} (T_3 - T_0)\right))^2}{\sigma^2_t |f_1 (T_3 - T_0)|^2} \right). \tag{120}
\]

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The velocity in the $x$ direction is

\[ v_x = \frac{\omega_3}{k_x}, \tag{121} \]

and

\[
\frac{\omega_3}{m} (T_3 - T_0) = \frac{\omega_0}{m} (T_3 - T_0) - p E_1 (T_3 - \bar{T}) = \frac{\omega_0}{m} (T_3 - T_0) + \Delta \omega (T_3 - \bar{T}) \\
\Delta \omega \equiv -pE_1 \\
\Rightarrow \\
\Delta v_x = \frac{\Delta \omega}{k_x} = -\frac{pE_1}{k_x}. \tag{122}
\]

The physical picture is clear. Up to time $\bar{T}$ the particle is going with velocity $v_x = \omega_0/k_x$. At time $\bar{T}$ the particle gets a kick $\Delta v$ and moves with increased or decreased velocity thereafter. When $E_1$ is not zero, we get a velocity-splitting term of order $|pE_1|$ and sense given by the sign of $-\vec{p} \cdot \vec{E}_1$.

As a double-check, we ask if this is physically reasonable. We return to the Stern-Gerlach experiment. As noted, we may interpret the observed space quantization as due to the extension of the wave function in the $z$ direction interacting with a magnetic field that varies in the $z$ direction. Take, for definiteness, the case of a magnetic dipole pointing in the $+z$ direction with the magnetic field increasing in the $+z$ direction, $B_{z,z} > 0$. The part of the wave function on the $+z$ side of the trajectory experiences a negative potential energy $\propto -\mu B_{z,z}$ and that on the $-z$ side a positive. This creates an overall torque in the $+z$ direction. To anthropomorphize slightly, the wave function attempts to maximize its time on the lower energy side of the axis by turning towards the region with lower potential energy.

We see a similar effect here. Take, for definiteness, the case where the electric dipole points in the $+x$ direction and the electric field is increasing with time, $E_1 > 0$. The potential energy $\sim -pE_1$. Anthropomorphizing again, this particle would rather slow down, to take advantage of the increased electric field to come. And this is what we see in $v_x$; it will be reduced by $|pE_1|$. A particle with its electric dipole aligned in the opposite sense would prefer to speed up, to get out of the interaction region before the electric field gets still stronger. It is the finite extension of the wave function in time which opens up this kind of possibility.
For this effect to be observable, we need the value of $\Delta \omega$ to be greater than the width of the wave function in energy, $\delta \omega$. We expect from the uncertainty principle that $\delta \omega \sim 1/\sigma_t$. If $\sigma_t \to 0$ then $\delta \omega \to \infty$ and the effect will be unobservable. This is just what we said before: $\sigma_t \to 0$ implies the wave function is in fact three-dimensional, so in this case no splitting in velocity should be seen.

We need an estimate of $\delta \omega$, if we are to put the four-dimensional wave function to the test. If we assume that the four-dimensional wave function is composed of waves with $\omega^2 = \vec{k}^2 + m^2$ then a reasonable first estimate of $\delta \omega$ is given by

$$\delta \omega \sim \frac{k}{\omega} \delta k \quad (123)$$

and our condition becomes

$$|pE_1| >> \left| \frac{k}{\omega} \delta k \right| \quad (124)$$

We note analogous concerns apply to the standard Stern-Gerlach effect: the beam must be sufficiently well localized in $k_z$ for the impetus $\Delta k_z$ from the magnetic field to be detectible.

There is one other interesting term in (119), the factor of

$$\exp \left( ipE_0 + ipE_1 \frac{T - T_0}{T_3 - T_0} t_3 \right) \quad (125)$$

(This does not contribute to the probability distribution because it is purely oscillatory.) If we hold $T_0$ and $\bar{T}$ fixed, while letting $T_3 \to \infty$, we get a change in frequency of

$$- \left( pE_0 + pE_1 T \right) \quad (126)$$

which is just the precession predicted by the Schrödinger equation (see (131) below).

**Results using Schrödinger equation**

To complete the analysis we now solve the same problem using the Schrödinger equation.

$$i \frac{\partial}{\partial T} \psi(T, \vec{x}) = -\frac{1}{2m} \nabla^2 \psi(T, \vec{x}) - \vec{p} \cdot \vec{E} \psi(T, \vec{x}) \quad (127)$$

or if we write $\psi$ in terms of the $p$ components of the $\hat{p}$ operator

$$i \frac{\partial}{\partial T} \varphi_p(T, \vec{x}) = -\frac{1}{2m} \nabla^2 \varphi_p(T, \vec{x}) - p (E_0 + E_1 T) \varphi_p(T, \vec{x}) \quad (128)$$
We may solve using separation of variables.

\[ \psi_p(T, \vec{x}) = \xi_p(T) \chi(\vec{x}) \]  

(129)

giving

\[ \begin{pmatrix} i \frac{\partial}{\partial T} + \frac{\vec{k}^2}{2m} \end{pmatrix} \xi_p(T) = -p (E_0 + E_1 T) \xi_p(T) \]

\[ \frac{\vec{k}^2}{2m} \chi(\vec{x}) = -\frac{1}{2m} \nabla^2 \chi(\vec{x}). \]  

(130)

To lowest order, the change in frequency per component from the dipole interaction will be given by

\[ \Delta \omega = \frac{1}{\Delta T} \int_{T_1}^{T_2} dT' \left( -p (E_0 + E_1 T') \right) = -pE \left( \bar{T} \right) \]  

(131)

which is the same as the earlier result(126). We see no \( \Delta v \) term and therefore no splitting of the beam in time.

This implies that a time sensitive detector will see one hump if the Schrödinger equation is correct, but two (or more) humps if four dimensional path integrals should be used. If the eigenfunctions of the electric dipole are \( \{ p_i \} \) then the humps will be spaced in velocity by \( -(p_{i+1} - p_i) E_1 \), and if this spacing is greater than \( \delta \omega \sim (\omega/k)\delta k \), they should be observable.

**Comparison of four dimensional path integrals to Schrödinger equation**

So there is a clear difference between the results with four dimensional path integrals and the Schrödinger equation. With the Schrödinger equation we predict the electric dipole will precess around the x axis, but we do not predict a split of the beam in velocity or time. With four dimensional path integrals we predict the change in precession and in addition that the beam will be split in velocity and time.

This is a non-relativistic effect; we used a Lorentz invariant Lagrangian solely to ensure time/space symmetry.

The splitting in velocity is not induced by the collapse of the wave function; we “used up” the collapse when we broke the incoming wave function up into eigenfunctions of the electric dipole operator.
The proposed experiment is clearly only a very crude test of these ideas. A subtler approach would be required to see the effect of the second order corrections to the action.

The principle advantage of the approach given here is the much greater symmetry between time and space. The principle disadvantage is that it is not entirely clear what is meant by this greater symmetry between time and space. The Machian approach to time suggested above may provide a useful line of attack.

DISCUSSION

We have observed that the assumptions about time implicit in the Schrödinger equation and in path integrals are very different. The Schrödinger equation takes a classical view of time, seeing time instant by instant and using the present to define the future. Path integrals most naturally see time all at once, from the ‘block universe perspective’. This suggests that the two formalisms may not be completely equivalent. And that therefore it could be useful to quantize path integrals independently, working from first principles.

We have done this by starting with an abstract formula for path integrals and imposing two requirements: correct behavior in the classical limit and the most complete practicable symmetry between time and space. We refer to this as “path integral quantization” to emphasize that we are taking path integrals as our starting point.

Path integral quantization predicts quantum fluctuations over the time dimension analogous to the quantum fluctuations seen over the three space dimensions. For constant potentials there is no effect. But in the presence of rapidly varying electromagnetic fields, the coupling between these quantum fluctuations in time and the fields should be detectable.

We considered in particular the case where a particle with a non-zero electric dipole moment is sent along the \( x \) axis though a rapidly varying electric field, also along the \( x \) axis. The Schrödinger equation predicts precession around the \( x \) axis but no physical splitting of the beam. Path integral quantization predicts the precession, \( and \) that the beam will be split in velocity and time.

Path integral quantization may be generalized to include QED: sums over \( \vec{k} \) become sums over \( \omega \) and \( \vec{k} \), the condition \( \omega^2 = \vec{k}^2 + m^2 \) becomes \( \omega^2 \approx \vec{k}^2 + m^2 \) (quantum fluctuations of \( \omega \) around \( \sqrt{\vec{k}^2 + m^2} \) are permitted), and normalization and renormalization have to be handled from the block universe perspective. But this is beyond the scope of this work.
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[72] We use natural units ($\hbar = c = 1$) and take the metric as having signature $(1, -1, -1, -1)$.

Except as noted, the summation convention is in effect: repeated Greek indices are summed from 0 to 3; repeated Roman from 1 to 3. Overdots, i.e. $\dot{f}$, are used to indicate differentiation.
with respect to the current path parameter (either the proper time along the path or else time as defined in the lab frame). Overbars, i.e. $\bar{x}_i$, are used to indicate classical trajectories.

[73] QED uses a similar approach: the canonical commutation relations are used to define a Hamiltonian that we then use to march the fields forward in time.

[74] For a particularly clear review of the “block universe,” Nahin.

[75] These difficulties are present in field theory as well. As Weinberg puts it “…although the path-integral formalism provides us with manifestly Lorentz-invariant rules, it does not make clear why the S-matrix calculated in this way is unitary. As far as I know, the only way to show that the path-integral formalism yields a unitary S-matrix is to use it to reconstruct the canonical formalism, in which unitarity is obvious.”

[76] Not completely unavoidable: see Calderon’s Life is a Dream.

[77] Since we are only looking at electrodynamics, we do not need the qualification about CPT.

[78] This is the familiar Einstein-Ritz debate: “While Einstein believes that one may restrict oneself to this case without essentially restricting the generality of the consideration, Ritz regards this restriction as not allowed in principle. If one accepts the latter point of view, experience requires one to regard the representation by means of the retarded potentials as the only possible one, provided one is inclined to assume that the fact of the irreversibility of radiation processes has to be present in the laws of nature. Ritz considers the restriction to the form of the retarded potentials as one of the roots of the Second Law, while Einstein believes that the irreversibility is exclusively based on reasons of probability.” as translated in Zeh’s comprehensive survey.

[79] Of course, when calculating with path integrals, it is easy to eliminate contradictions: 1) construct the set of all paths, 2) throw out all self-contradictory ones, 3) sum over the rest, 4) normalize appropriately, 5) declare the result your answer. And if there are no self-consistent paths, 6) declare the problem ill-posed.

[80] No paradox or backwards-in-time travel is implicit in selecting $t_a > t_b$. For instance, if we were a detective attempting to retrodict the path of a quantum bullet, we might easily wish to take the state $a$ as corresponding to the impact of the bullet and then attempt to infer the probability of all states $b$ corresponding to its firing.

[81] Defining time along a quantum mechanical path is a non-trivial problem: for an exploration of the difficulties, see and other articles in.
[82] Per Goldstein, we have considerable freedom in the choice of the parameter. Proper time is the most obvious. However, if we used proper time we would have $t$ and $\vec{x}$ parameterized in terms of $\tau$, $t = t(\tau)$ and $\vec{x} = \vec{x}(\tau)$. But $\tau$ is in turn defined as an integral over $t$ and $\vec{x}$, $d\tau = d\sqrt{t^2 + \vec{x}^2}$. This would leave us with circular definitions.

[83] From here forward, $\dot{f}$ means $df/dT$.

[84] And we must change $\tau$ to $T$ in (30) giving $x(T) = x' + \frac{x''-x'}{2!}(T-T') + \sum_{n=1}^{\infty} a_n \sin (\omega_n T)$.

[85] These under- and overshoots may show up as small violations of the optical theorem. We are not exploring this line of attack here, but Bennett [65, 66, 67] has argued that small violations of the optical theorem may already have been seen. To be sure, Bennett’s conclusions are not unquestioned: see Valentini [68].

[86] We now switch to treating the lab time $T$ as an index, writing $t_T$ rather than $t(T)$.

[87] The conceptual difficulties are compounded by the fact that such expectation values are almost always computed using the assumption that there is a well-defined space-time over which the relevant integrations may be done. How to integrate over a space-time that is not defined until we have done the integrations? It is like trying to make a bed while standing on it.

[88] “Keeping an open mind is a virtue – but as the space engineer James Oberg once said, not so open that your brains fall out.” – Sagan [69]

[89] Kaiser [70] discusses how an arbitrary wave form may be analyzed as a sum of Gaussian wavelets.

[90] From the point of view of the three-dimensional kernel, these quantum fluctuations in time look like a separate quantum system. They could be handled by the method of influence functionals [1, 4].

[91] Unless the observer at $x''$ is moving relative to the observer at $x'$.

[92] There is a particularly good discussion of this in Baym [71].