Cross-intersecting families and primitivity of symmetric systems

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Abstract

Let $X$ be a finite set and $\mathcal{P} \subseteq 2^X$, the power set of $X$, satisfying three conditions: (a) $\mathcal{P}$ is an ideal in $2^X$, that is, if $A \in \mathcal{P}$ and $B \subset A$, then $B \in \mathcal{P}$; (b) For $A \in 2^X$ with $|A| \geq 2$, $A \in \mathcal{P}$ if $\{x, y\} \in \mathcal{P}$ for any $x, y \in A$ with $x \neq y$; (c) $\{x\} \in \mathcal{P}$ for every $x \in X$.

The pair $(X, \mathcal{P})$ is called a symmetric system if there is a group $\Gamma$ transitively acting on $X$ and preserving the ideal $\mathcal{P}$. A family $\{A_1, A_2, \ldots, A_m\} \subseteq 2^X$ is said to be a cross-$\mathcal{P}$-family of $X$ if $\{a, b\} \in \mathcal{P}$ for any $a \in A_i$ and $b \in A_j$ with $i \neq j$. We prove that if $(X, \mathcal{P})$ is a symmetric system and $\{A_1, A_2, \ldots, A_m\} \subseteq 2^X$ is a cross-$\mathcal{P}$-family of $X$, then

$$\sum_{i=1}^{m} |A_i| \leq \begin{cases} |X| & \text{if } m \leq \frac{|X|}{\alpha(X, \mathcal{P})}, \\
m \alpha(X, \mathcal{P}) & \text{if } m \geq \frac{|X|}{\alpha(X, \mathcal{P})}, \end{cases}$$

where $\alpha(X, \mathcal{P}) = \max\{|A| : A \in \mathcal{P}\}$. This generalizes Hilton’s theorem on cross-intersecting families of finite sets, and provides analogs for cross-$t$-intersecting families of finite sets, finite vector spaces and permutations, etc. Moreover, the primitivity of symmetric systems is introduced to characterize the optimal families.

Key words: intersecting family, cross-intersecting family, symmetric system, Erdős-Ko-Rado theorem

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1 Introduction

A family $\mathcal{A}$ of sets is said to be intersecting if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. A classical result on intersecting families is due to Erdős, Ko and Rado, which says that if $\mathcal{A}$ is an intersecting family consisting of $k$-element subsets of an $n$-element set with $n \geq 2k$, then $|A| \leq \binom{n-1}{k-1}$, and if $n > 2k$, equality holds if and only if every subset in $\mathcal{A}$ contains a fixed element.

The Erdős-Ko-Rado theorem has many generalizations, analogs and variations. First, the notion of intersection is generalized to $t$-intersection, and finite sets are analogous to finite vector spaces, permutations and other mathematical objects. Second, intersecting families are generalized to cross-intersecting families: $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are said to be cross-intersecting if $A \cap B \neq \emptyset$ for any $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, $i \neq j$. Clearly, if $\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_m = \mathcal{A}$, then $\mathcal{A}$ is an intersecting family. Combining the two points of view, we may consider the cross-$t$-intersecting families over finite vector spaces, permutations, etc.

A nice result on cross-intersecting families is given by Hilton [19] as follows.

**Theorem 1.1** (Hilton [19]) Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be cross-intersecting families of $k$-element subsets of an $n$-element set $X$ with $\mathcal{A}_1 \neq \emptyset$. If $k \leq n/2$, then

$$\sum_{i=1}^{m} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{k}, & \text{if } m \leq \frac{n}{k}; \\ m \binom{n-1}{k-1}, & \text{if } m \geq \frac{n}{k}. \end{cases} \quad (1)$$

Unless $m = 2 = n/k$, the bound is attained if and only if one of the following holds:

(i) $m < n/k$ and $\mathcal{A}_1 = \{A \subset X : |A| = k\}$, and $\mathcal{A}_2 = \ldots = \mathcal{A}_m = \emptyset$;
(ii) $m > n/k$ and $|\mathcal{A}_1| = |\mathcal{A}_2| = \ldots = |\mathcal{A}_m| = \binom{n-1}{k-1}$;
(iii) $m = n/k$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are as in (i) or (ii).

Recently, Borg gives a simple proof of the above theorem [7], and generalizes it to labeled sets [4] and permutations [8]. Inspired by his proofs we shall present a general result on cross-intersecting, or cross-$t$-intersecting families of finite sets, finite vector spaces, permutations, etc. To do this, we introduce a general definition.
Let $X$ be a finite set and $\mathcal{P} \subseteq 2^X$, the power set of $X$, satisfying three conditions as follows:

(a) $\mathcal{P}$ is an ideal in $2^X$, that is, if $A \in \mathcal{P}$ and $B \subset A$, then $B \in \mathcal{P}$;
(b) For $A \in 2^X$ with $|A| \geq 2$, $A \in \mathcal{P}$ if $\{x, y\} \in \mathcal{P}$ for any $x, y \in A$ with $x \neq y$;
(c) $\{x\} \in \mathcal{P}$ for every $x \in X$.

Note that condition (a) is essential and (c) is to avoid trivial cases. If ignore conditions (b) and (c), the pair $(X, \mathcal{P})$ is an (abstract) simplicial complex in topology, or a hereditary family in extremal set theory (see e.g. [12, p.86] or [6]). If ignore (b), $\mathcal{P}$ is called a full hereditary family in [12, p.86]. Condition (b) is not redundant in most discussions on extremal combinatorics, and is necessary in our argument.

Clearly, $\mathcal{P}$ defines a binary relation "$\sim_{\mathcal{P}}$" on $X$: $x \sim_{\mathcal{P}} y$ if and only if $\{x, y\} \in \mathcal{P}$ for any $x, y \in X$. This relation is reflexive and symmetric, i.e., $x \sim_{\mathcal{P}} x$ for every $x \in X$, and $x \sim_{\mathcal{P}} y$ implies $y \sim_{\mathcal{P}} x$. Conversely, given a reflexive and symmetric binary relation "$\sim$" on $X$, we can get an ideal $\mathcal{P}$ in $2^X$: $A \subset X$ is in $\mathcal{P}$ if $a \sim b$ for any $a, b \in A$. Moreover, $\mathcal{P}$ also defines a property on $2^X$: a subset $A$ of $X$ has the property $\mathcal{P}$ if $A \in \mathcal{P}$. Therefore, we call the pair $(X, \mathcal{P})$ a $\mathcal{P}$-system, or a system, for short.

An element of $\mathcal{P}$ is also called a $\mathcal{P}$-subset of $X$. A family $\{A_1, A_2, \ldots, A_m\} \subseteq 2^X$ is said to be a cross-$\mathcal{P}$-family of $X$ if $\{a, b\} \in \mathcal{P}$ for any $a \in A_i$ and $b \in A_j$ with $i \neq j$. By definition we see that if $\{A_1, A_2, \ldots, A_m\}$ is a cross-$\mathcal{P}$-family and $A_1 = A_2 = \cdots = A_m = A$, then $A$ is a $\mathcal{P}$-subset. Write

$$\alpha(X, \mathcal{P}) := \max\{|A| : A \in \mathcal{P}\}$$

and

$$\alpha_m(X, \mathcal{P}) := \max\left\{ \sum_{i=1}^{m} |A_i| : \{A_1, A_2, \ldots, A_m\} \text{ is a cross-$\mathcal{P}$-family} \right\}.$$

A cross-$\mathcal{P}$-family $\{A_1, A_2, \ldots, A_m\}$ is said to be optimal if $\sum_{i=1}^{m} |A_i| = \alpha_m(X, \mathcal{P})$.

We call a system $(X, \mathcal{P})$ symmetric if there is a group $\Gamma$ transitively acting on $X$ and preserving the property $\mathcal{P}$, i.e., for every pair $a, b \in X$ there is a $\gamma \in \Gamma$ such that $b = \gamma(a)$, and $A \in \mathcal{P}$ implies $\delta(A) \in \mathcal{P}$ for every $\delta \in \Gamma$. In this case we say that the group $\Gamma$ transitively acts on $(X, \mathcal{P})$. 

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Two typical examples of symmetric systems are as follows.

**Example 1.2** For a positive integer \( n \), let \([n]\) denote the set \( \{1, 2, \ldots, n\} \). By \( \mathcal{C}^k_n \) we denote the set of all \( k \)-element subsets of \([n]\), as known for \( \binom{n}{k} \) in many literatures. Then \( |\mathcal{C}^k_n| = \binom{n}{k} \). A subset \( \mathcal{A} \) of \( \mathcal{C}^k_n \) is said to be a \( t \)-intersecting family if \( |A \cap B| \geq t \) for any \( A, B \in \mathcal{A} \), where \( 1 \leq t \leq k \). For convenience, we regard the empty set as a \( t \)-intersecting family. Let \( i_t \) be the collection of all \( t \)-intersecting families in \( \mathcal{C}^k_n \). Then, it is clear that \( i_t \) is an ideal of the power set of \( \mathcal{C}^k_n \), and satisfies condition (b). When \( t = 1 \), \( i_t \) is abbreviated as \( i \). The Erdős-Ko-Rado theorem and Theorem 1.1 say that \( \alpha(C_n, i) = \binom{n-1}{k-1} \) and \( \alpha_m(C_n, i) = \max \{ \binom{n}{k}, m \binom{n-1}{k-1} \} \) for \( n \geq 2k \), respectively. In fact, Erdős, Ko and Rado [13] also proved \( \alpha(C^k_n, i_t) = \binom{n-t}{k-t} \) for \( t > 1 \) and \( n \geq n_0(k, t) \), a sufficiently large positive integer depending on \( k \) and \( t \). The smallest \( n_0(k, t) = (k - t + 1)(t + 1) \) was determined by Frankl [14] for \( t \geq 15 \) and subsequently determined by Wilson [27] for all \( t \). It is well known that the symmetric group \( S_n \) transitively acts on \( \mathcal{C}^k_n \) in a natural way, and preserves \( i_t \). Therefore, \( (\mathcal{C}^k_n, i_t) \) is symmetric.

**Example 1.3** Let \( \mathcal{L}_{n,k}(q) \) denote the set of all \( k \)-dimensional subspaces of an \( n \)-dimensional vector space over a \( q \)-element field. Then \( |\mathcal{L}_{n,k}(q)| = \binom{n}{k} = \frac{\{n\}!}{\{k\}! \{n-k\}!} \) where \( \{k\} = 1 + q + \cdots + q^{k-1} \) and \( \{n\}! = \{k\} \{k-1\} \cdots \{1\} \). A subset \( \mathcal{A} \) of \( \mathcal{L}_{n,k}(q) \) is said to be a \( t \)-intersecting family if \( \dim(A \cap B) \geq t \) for any \( A, B \in \mathcal{A} \), where \( 1 \leq t \leq k \). We still use \( i_t \) to denote the collection of all \( t \)-intersecting families in \( \mathcal{L}_{n,k}(q) \), and abbreviate \( i_t \) as \( i \). That \( \alpha(\mathcal{L}_{n,k}(q), i) = \binom{n-1}{k-1} \) was first established by Hsieh [18] for \( k < n/2 \), and by Greene and Kleitman [16] for \( k \mid n \). For \( t \geq 2 \), Frankl and Wilson [15] proved that \( \alpha(\mathcal{L}_{n,k}(q), i_t) = \max \{ \binom{n-t}{k-t}, \binom{2k-t}{k} \} \) for \( n \geq 2k-t \). Analogously to \( (\mathcal{C}^k_n, i_t) \), the general linear group \( GL(n, q) \) transitively acts on \( \mathcal{L}_{n,k}(q) \) and preserves \( i_t \). Therefore, \( (\mathcal{L}_{n,k}(q), i_t) \) is also symmetric.

To our knowledge, there is no information on \( \alpha_m(\mathcal{C}^k_n, i_t) \) for \( t > 1 \) and \( \alpha_m(\mathcal{L}_{n,k}(q), i_t) \) for \( t \geq 1 \).

In this paper we shall generalize Theorem 1.1 to all symmetric systems \((X, p)\) up to \( \alpha(X, p) \). The main result will be presented in the next section. To characterize the optimal cross-\( p \)-families we introduce the primitivity of the symmetric systems, and give its main characters in Section 3. As applications of results in Section 3, we prove in Section 4 that the symmetric systems defined on finite sets, finite vector spaces and symmetric groups are all primitive.
except a few trivial cases.

2 Cross-intersecting families of symmetric systems

Given a system \((X, \mathcal{P})\), we can construct a simple graph, written as \(G(X, \mathcal{P})\), whose vertex set is \(X\), and \(\{a, b\}\) is an edge if \(\{a, b\} \notin \mathcal{P}\). Then every subset of \(X\) in \(\mathcal{P}\) corresponds to an independent set of \(G(X, \mathcal{P})\). Conversely, given a simple graph \(G\), we obtain a system \((X(G), \mathcal{P}(G))\), where \(X(G)\) is the vertex set \(V(G)\) of \(G\) and \(\mathcal{P}(G)\) consists of all independent sets of \(G\). It is clear that \(\alpha(X(G), \mathcal{P}(G)) = \alpha(G)\), the independence number of \(G\).

By \(I(X, \mathcal{P})\) we denote the set of all maximal-sized \(\mathcal{P}\)-subsets of \(X\). Similarly, for a graph \(G\), let \(I(G)\) denote the set of all maximal-sized independent sets of \(G\). For \(B \subseteq V(G)\), let \(G[B]\) denote the induced subgraph of \(G\) by \(B\).

The notations introduced below have graph-theoretic intuition.

Let \((X, \mathcal{P})\) be a \(\mathcal{P}\)-system. For \(B \subseteq X\), we abbreviation \(\alpha(B, \mathcal{P} \cap 2^B)\) as \(\alpha(B, \mathcal{P})\). Clearly, \(\alpha(B, \mathcal{P})\) equals \(\alpha(G[B])\), where \(G = G(X, \mathcal{P})\). For \(A \subseteq X\), set

\[
N_{X, \mathcal{P}}[A] = A \cup \{b \in X : \{a, b\} \notin \mathcal{P} \text{ for some } a \in A\}
\]

and

\[
\bar{N}_{X, \mathcal{P}}[A] = X - N_{X, \mathcal{P}}[A].
\]

If there is no possibility of confusion, we abbreviate \(N_{X, \mathcal{P}}[A]\) as \(N[A]\). From definition we see that \(N[\emptyset] = \emptyset; N[A] = X\) if \(A \in I(X, \mathcal{P})\); if both \(B \subseteq A\) and \(C \subseteq \bar{N}[A]\) are in \(\mathcal{P}\), then \(B \cup C \in \mathcal{P}\).

We call \((X, \mathcal{P})\) connected (disconnected) if the graph \(G(X, \mathcal{P})\) is connected (disconnected). By definition we see that \((X, \mathcal{P})\) is disconnected if and only if there is a proper subset \(A \subset X\) such that \(\bar{N}[A] = X - A\), and, \((X, \mathcal{P})\) is symmetric if and only if \(G(X, \mathcal{P})\) is vertex-transitive.

In the context of vertex-transitive graphs, the “No-Homomorphism” lemma is useful to get bounds on the size of independent sets.

**Lemma 2.1** (Albertson and Collins [1]) Let \(G\) and \(H\) be two graphs such that \(G\) is vertex-transitive and there exists a homomorphism \(\phi : H \rightarrow G\).
Then \( \frac{\alpha(G)}{|V(G)|} \leq \frac{\alpha(H)}{|V(H)|} \), and equality holds if and only if for each \( I \in I(G), \phi^{-1}(I) \in I(H) \).

In the above lemma, by taking \( H \) as an induced subgraph of \( G \) and \( \phi \) as the embedding mapping, we obtain the following theorem, which is more convenient in our argument.

**Theorem 2.2** (*Cameron and Ku [11]*) Let \( G \) be a vertex-transitive graph and \( B \) a subset of \( V(G) \). Then any independent set \( S \) in \( G \) satisfies that \( \frac{|S|}{|V(G)|} \leq \frac{\alpha(G[B])}{|B|} \), equality implies that \( |S \cap B| = \alpha(G[B]) \).

In [28], the second author of this paper proved Lemma 2.3 and Theorem 3.2 below in terms of graph theory. He also introduced the concept of imprimitive independent sets of a vertex-transitive graph. For completeness we restate them in terms of symmetric systems and provide proofs for them.

**Lemma 2.3** Let \((X,p)\) be a symmetric system. Then \( \frac{|A|}{|N(A)|} \leq \frac{\alpha(X,p)}{|X|} \) for an arbitrary \( p \)-subset \( A \) of \( X \). Equality implies that \( |S \cap N[A]| = |A| \) for every \( S \in I(X,p) \), and \( \frac{\alpha(N[A],p)}{|N[A]|} = \frac{\alpha(X,p)}{|X|} \).

**Proof.** Let \( C \) be a maximal-sized \( p \)-subset of \( N[A] \). Clearly, \( A \cup C \) is a \( p \)-subset of \( X \) and
\[
\frac{|A \cup C|}{|X|} = \frac{|A| + \alpha(N[A],p)}{|N[A]| + |N[A]|} \leq \alpha(X,p).
\]
Since \( \frac{\alpha(N[A],p)}{|N[A]|} \geq \frac{\alpha(X,p)}{|X|} \) by Theorem 2.2, \( \frac{|A|}{|N[A]|} \leq \frac{\alpha(X,p)}{|X|} \). Equality implies that \( \frac{\alpha(N[A],p)}{|N[A]|} = \frac{\alpha(X,p)}{|X|} \) and \( \alpha(X,p) = \alpha(N[A],p) + |A| \). Again by Theorem 2.2 we have that \( |S \cap N[A]| = |\alpha(N[A],p)| \) and \( |S| = |S \cap N[A]| + |S \cap N[A]| \) for every \( S \in I(X,p) \). Therefore, \( |S \cap N[A]| = |A| \) for every \( S \in I(X,p) \), completing the proof. \( \square \)

In [28], a graph \( G \) is called **IS-imprimitive** (independent-set-imprimitive) if there is an independent set \( A \) of \( G \) such that \( |A| < \alpha(G) \) and \( \frac{|A|}{|N[A]|} = \frac{\alpha(G)}{|V(G)|} \), and \( A \) is called an **imprimitive independent set** of \( G \). In any other case, \( G \) is called **IS-primitive**. In this paper, we say a system \((X,p)\) is **\( p \)**-imprimitive (\( p \)-primitive) if the graph \( G(X,p) \) is IS-imprimitive (IS-primitive); a \( p \)-subset \( A \) is called imprimitive if \( A \) is an imprimitive independent set of \( G(X,p) \). From definition we see that a disconnected symmetric system \((X,p)\) is \( p \)-imprimitive and hence a \( p \)-primitive symmetric system \((X,p)\) is connected.

We now contribute to \( \alpha_m(X,p) \). Note that in a series of papers [4789]...
Borg determined this value for various cross-intersecting families. An important step in his proofs was inequality (2) below he established for some special intersecting families. We find that the inequality for $p$-subsets in symmetric systems is a consequence of Theorem 2.2 stated as follows.

**Corollary 2.4** Let $(X, p)$ be a symmetric system, and let $A$ be a $p$-subset of $X$. Then

$$|A| + \frac{\alpha(X, p)}{|X|}|\bar{N}[A]| \leq \alpha(X, p).$$

Equality holds if and only if $A = \emptyset$ or $|A| = \alpha(X, p)$ or $A$ is an imprimitive $p$-subset.

**Proof.** If $A = \emptyset$ or $|A| = \alpha(X, p)$, equality trivially holds. Suppose that $0 < |A| < \alpha(X, p)$ and $B$ is a maximal-sized $p$-subset in $\bar{N}[A]$, that is, $|B| = \alpha(\bar{N}[A], p)$. Then $A \cup B$ is also a $p$-subset of $X$, so $|A| + |B| \leq \alpha(X, p)$, and Theorem 2.2 implies that $\frac{|B|}{|\bar{N}[A]|} \geq \frac{\alpha(X, p)}{|X|}$. Therefore,

$$|A| + \frac{\alpha(X, p)}{|X|}|\bar{N}[A]| \leq |A| + |B| \leq \alpha(X, p).$$

If $\alpha(X, p) = |A| + \frac{\alpha(X, p)}{|X|}|\bar{N}[A]| = |A| + \frac{\alpha(X, p)}{|X|}(|X| - |\bar{N}[A]|)$, then $\frac{|A|}{|\bar{N}[A]|} = \frac{\alpha(X, p)}{|X|}$, i.e., $A$ is an imprimitive $p$-subset. □

The following theorem is the main result of this paper.

**Theorem 2.5** Let $(X, p)$ be a connected symmetric system, and let $\{A_1, A_2, \ldots, A_m\}$ be a cross-$p$-family over $X$ with $A_1 \neq \emptyset$. Then

$$\sum_{i=1}^{m} |A_i| \leq \begin{cases} |X| & \text{if } m \leq \frac{|X|}{\alpha(X, p)}, \\
\alpha(X, p) & \text{if } m \geq \frac{|X|}{\alpha(X, p)}, \end{cases}$$

and the bound is attained if and only if one of the following holds:

(i) $m < \frac{|X|}{\alpha(X, p)}$ and $A_1 = X$, $A_2 = \ldots = A_m = \emptyset$,

(ii) $m > \frac{|X|}{\alpha(X, p)}$ and $A_1 = \ldots = A_m = I \in I(X, p)$,

(iii) $m = \frac{|X|}{\alpha(X, p)}$ and either $A_1, A_2, \ldots, A_m$ are as in (i) or (ii), or there is an imprimitive $p$-subset $A$ such that $A \subseteq A_i, i = 1, 2, \ldots, m$, and $\{A'_1, A'_2, \ldots, A'_m\}$ is a cross-$p$-family and a partition of $\bar{N}[A]$, where $A'_i = A_i - A, i = 1, 2 \ldots, m$.

**Proof.** Following Borg’s notation in [7,8,9], write $A_i^* = \{a \in A_i : \{a, b\} \in p$ for every $b \in A_i\}$, $A'_i = A_i - A_i^*$, $A^* = \bigcup_{i=1}^{m} A_i^*$ and $A' = \bigcup_{i=1}^{m} A'_i$. It is
clear that $A^*$ is a $p$-subset and $A' \subseteq \bar{N}[A^*]$. From definition it follows that $A_i \cap A_j \subseteq A_i^* \cap A_j^*$, therefore $A_i' \cap A_j' = \emptyset$ for $i \neq j$, thus $|A'| = \sum_{i=1}^m |A'_i|$. By Corollary 2.4 we have that

\[
\sum_{i=1}^m |A_i| = \sum_{i=1}^m |A'_i| + \sum_{i=1}^m |A^*_i| \leq |A'| + m|A^*| \leq |\bar{N}[A^*]| + m|A^*|
\]

\[
= \left| X \right| \frac{\alpha(X, p)}{\alpha(X^* \setminus \{X\})} \left( |\bar{N}[A^*]| + |A^*| \right) + \left( m - \frac{|X|}{\alpha(X, p)} \right) |A^*|
\]

\[
\leq |X| + \left( m - \frac{|X|}{\alpha(X, p)} \right) |A^*|.
\]

If $m < \frac{|X|}{\alpha(X, p)}$, then $\sum_{i=1}^m |A_i| \leq |X|$, and equality implies $A^* = \emptyset$, hence $A_i = A_i'$ for every $i \in [m]$, and we thus have that the corresponding graph $G(X, p)$ is a union of the induced subgraphs $G(X, p)[A_i]$’s. Then, the connectivity of $(X, p)$ yields that one of them is $X$ and the others are empty, as (i).

If $m > \frac{|X|}{\alpha(X, p)}$, then $\sum_{i=1}^m |A_i| \leq m \alpha(X, p)$ and equality implies that $A_1^* = \cdots = A_m^* = A^*$ and $|A^*| = \alpha(X, p)$, as (ii).

If $m = \frac{|X|}{\alpha(X, p)}$, then $\sum_{i=1}^m |A_i| \leq |X|$, and equality implies that $A_1^* = \cdots = A_m^* = A^*$ and $\frac{\alpha(X, p)}{|X|} |\bar{N}[A^*]| + |A^*| = \alpha(X, p)$. Then Corollary 2.4 implies that $|A^*| = 0$ or $|A| = \alpha(X, p)$ or $A^*$ is an imprimitive $p$-subset. In the last case, $\{A_1', A_2', \ldots, A_m'\}$ is a cross-$p$-family, and a partition of $\bar{N}[A^*]$. \(\square\)

From the above theorem we see that if $(X, p)$ is symmetric and $p$-primitive (hence connected), then $\alpha_m(X, p)$ is uniquely determined by $\alpha(X, p)$, i.e.,

\[
\alpha_m(X, p) = \max \{ |X|, m \alpha(X, p) \},
\]

and an optimal cross-$p$-family is one of the forms $\{X, \emptyset, \ldots, \emptyset\}$ and $\{A, A, \ldots, A\}$ where $A \in p$ with $|A| = \alpha(X, p)$.

For the $(X, p)$ dealt with in this field, however, $\alpha(X, p)$ is usually well known, and the symmetric property of $(X, p)$ is easy to verify. So we concentrate on the primitivity of symmetric systems in the next two sections.
Primitivity of symmetric systems

This concept comes from permutation groups. Let \( X \) be a set, and \( \Gamma \) a group transitively acting on \( X \). Then \( \Gamma \) is said to be imprimitive on \( X \) if it preserves a nontrivial partition of \( X \), called a block system, each element of which is called a block. In any other case \( \Gamma \) is primitive on \( X \). More precisely, \( \Gamma \) is imprimitive on \( X \) if there is nontrivial partition \( X = \bigcup_{i=1}^{k} X_i \) such that \( \gamma(X_i) \) is a block of the partition for every \( \gamma \in \Gamma \) and \( i = 1, 2, \ldots, k \). Here \( \gamma(X_i) \) denotes the set \( \{ \gamma(x) : x \in X_i \} \).

A classical result on the primitivity of group actions is the following theorem (cf. [20, Theorem 1.12]).

**Theorem 3.1** Suppose that a group \( \Gamma \) transitively acts on \( X \). Then \( \Gamma \) is primitive on \( X \) if and only if for each \( a \in X \), \( \Gamma_a \) is a maximal subgroup of \( \Gamma \). Here \( \Gamma_a = \{ \gamma \in \Gamma : \gamma(a) = a \} \), the stabilizer of \( a \in X \).

The following theorem explains why a symmetric system is called primitive or imprimitive.

**Theorem 3.2** Let \((X, p)\) be an imprimitive symmetric system, \( A \) a maximal-sized imprimitive \( p \)-subset of \( X \), \( D = X - N[A] \), and let \( \Gamma \) be the group transitively acting on \((X, p)\). Then \( \frac{\alpha(D, p)}{|D|} = \frac{\alpha(X, p)}{|X|} \) and \( \{ \sigma(D) : \sigma \in \Gamma \} \) forms a partition of \( X \).

**Proof.** First, suppose that \( A \) and \( B \) are two imprimitive \( p \)-subsets of \( X \), and write \( C = A \cup (B - N[A]) \). We claim that \( C \) is a \( p \)-subset satisfying \( N[C] = N[A] \cup N[B] \) and \( \frac{|C|}{|N[C]|} = \frac{\alpha(X, p)}{|X|} \).

To prove this claim we write \( N[A] \cup N[B] = M \). From definition it is easily seen that \( C \) is also a \( p \)-subset and \( N[C] \subseteq M \). Since \( \frac{|B|}{|N[B]|} = \frac{\alpha(X, p)}{|X|} \), by Lemma 2.3 we have that \( |S \cap N[B]| = |B| \) for all \( S \in I(X, p) \). So, \( B \cup (S - N[B]) \) is also a maximal-sized \( p \)-subset of \( X \) for every \( S \in I(X, p) \). By repeating this process for the maximal-sized \( p \)-subset \( B \cup (S - N[B]) \) and the imprimitive \( p \)-subset \( A \) we have that

\[
A \cup ((B \cup (S - N[B])) - N[A]) = (B - N[A]) \cup ((S - N[B]) - N[A]) = C \cup (S - M)
\]

is also a maximal-sized \( p \)-subset of \( X \), which implies that \( |S \cap M| = |C| \)

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for every $S \in I(X,p)$. Given a $u \in X$, suppose there are $r$ maximal-sized $p$-subsets containing $u$. Since $(X,p)$ is symmetric, it is easily seen that the number $r$ is independent on the choice of $u$. Let us count pairs $(x,S)$ with $x \in M \cap S$, $S \in I(X,p)$, in two ways. Since $|M \cap S| = |C|$ for every $S \in I(X,p)$, the number of the pairs is clearly equal to $|C||I(X,p)|$. On the other hand, for each $x \in M$ there are $r$ $S$’s in $I(X,p)$ with $x \in S$. So the number is also equal to $r|M|$, proving $r|M| = |C||I(X,p)|$. Similarly, by counting pairs $(x,S)$ with $x \in S \in I(X,p)$ in two ways we obtain $r|X| = \alpha(X,p)|I(X,p)|$. Combining the above two equalities gives

$$\frac{|C|}{|M|} = \frac{\alpha(X,p)}{|X|}.$$ 

Hence $N[C] = M$ and $\frac{|C|}{|N[C]|} = \frac{\alpha(X,p)}{|X|}$, proving our claim.

We now close the proof of the theorem. Let $A$ be a maximal-sized imprimitive $p$-subset of $X$. From definition it follows that $N[\sigma(A)] = \sigma(N[A])$ for all $\sigma \in \Gamma$. Suppose that there exists a $\sigma \in \Gamma$ such that $\sigma(D) \neq D$ and $\sigma(D) \cap D \neq \emptyset$. Then $\sigma(N[A]) \neq N[A]$, hence $|N[A] \cup \sigma(N[A])| > |N[A]|$. Set $A' = A \cup (\sigma(A) - N[A])$. Then $A'$ is also a $p$-subset of $X$. By the above claim we have that $N[A'] = N[A] \cup \sigma(N[A])$ and $\frac{|A'|}{|N[A']|} = \frac{\alpha(X,p)}{|X|} = \frac{|A|}{|N[A]|}$, which implies $|A'| > |A|$. On the other hand, from definition it follows that each element of $\sigma(D) \cap D$ does not belong to $N[A] \cup \sigma(N[A])$, so $N[A'] \neq X$, yielding $|A'| < \alpha(X,p)$. It contradicts the maximality of $A$, thus proving that $\sigma(D) = D$ or $\sigma(D) \cap D = \emptyset$ for each $\sigma \in \Gamma$. The transitivity of $\Gamma$ on $X$ implies that $X = \cup_{\sigma \in \Gamma} \sigma(D)$. Furthermore, for any $\sigma, \gamma \in \Gamma$, if $\sigma(D) \cap \gamma(D) \neq \emptyset$, then $(\gamma^{-1}\sigma)(D) \cap D \neq \emptyset$, implying $(\gamma^{-1}\sigma)(D) = D$, i.e., $\sigma(D) = \gamma(D)$. Therefore, $\{\sigma(D) : \sigma \in \Gamma\}$ is a partition of $X$. □

By Theorem 3.2 and Theorem 3.1 we obtain the following consequences.

**Corollary 3.3** Suppose that a group $\Gamma$ transitively acts on $(X,p)$. Then $(X,p)$ is $p$-primitve if one of the following conditions holds.

(i) $\Gamma$ is primitive on $X$, or equivalently, $\Gamma_a$ is a maximal subgroup of $\Gamma$ for each $a \in X$.

(ii) $\Gamma$ is imprimitive on $X$, but each block $D$ satisfies $\frac{\alpha(D,p)}{|D|} > \frac{\alpha(X,p)}{|X|}$.
4 Primitivity of some classical symmetric systems

Finite sets, finite vector spaces and permutations are among the most important finite structures in combinatorics, especially in extremal combinatorics. In what follows we prove the primitivity of three symmetric systems defined on them.

Proposition 4.1 \((C^k_n, i_t)\) is \(i_t\)-primitive for \(n \geq (k - t + 1)(t + 1)\) unless \(n = 2k \geq 4\) and \(t = 1\).

Proof. Since the case \(n \leq 3\) is trivial, we assume that \(n \geq 4\). From Example 1.2 we know that \((C^k_n, i_t)\) is symmetric and \(\alpha(C^k_n, i_t) = \binom{n-t}{k-t}\) for \(n \geq (k-t+1)(t+1)\). Consider the action of the symmetric group \(S_n\) on \(C^k_n\). It is well known that for each \(A \in C^k_n\), the stabilizer \(S_{n,A}\) of \(A\) is isomorphic to \(S_k \times S_{n-k}\), which is a maximal subgroup of \(S_n\) if \(n \neq 2k\) (See e.g. [3]). Therefore, \((C^k_n, i_t)\) is \(i_t\)-primitive when \(n \neq 2k\). It is easily seen that \(\{A, [2k] - A\}\) is a block in \(C^k_{2k}\) under the action of \(S_{2k}\), and every block is of this form. On the other hand, 
\[
\frac{\alpha((A,A), i_t)}{2} = \frac{\binom{2k-t}{k}}{2} = \frac{\alpha(C^k_{2k}, i_t)}{|C^k_{2k}|}\text{ for all } 1 \leq t \leq k, \text{ and equality holds if and only if } t = 1.
\]
By Corollary 3.3 \((C^k_{2k}, i_t)\) is \(i_t\)-primitive for \(t > 1\). It is clear that \((C^k_{2k}, i)\) is disconnected, hence \(i_t\)-imprimitive. \(\square\)

Proposition 4.2 \((\mathcal{L}_{n,k}(q), i_t)\) is \(i_t\)-primitive for all \(n \geq 2k - t\).

Proof. It is well known [2] that for each \(A \in \mathcal{L}_{n,k}(q)\), the stabilizer of \(A\) is a maximal subgroup of \(GL(n, q)\). By Corollary 3.3 \((\mathcal{L}_{n,k}(q), i_t)\) is \(i_t\)-primitive. \(\square\)

In the foregoing two examples, the primitivity of systems follows directly from the primitivity of groups acting on them. However, it is not always the case, as we shall see.

Let us consider the set \(S_n\). A subset \(A\) of \(S_n\) is said to be \(t\)-intersecting if any two permutations in \(A\) agree in at least \(t\) points, i.e. for any \(\sigma, \tau \in A\), \(|\{i \in [n] : \sigma(i) = \tau(i)\}| \geq t\). We still denote this property by \(i_t\). When \(t = 1\), Deza and Frankl [11] showed that a 1-intersecting subset \(A \subseteq S_n\) has size at most \((n-1)!\) and conjectured that for \(t\) fixed, and \(n\) sufficiently large depending on \(t\), a \(t\)-intersecting subset \(A \subseteq S_n\) has size at most \((n-t)!\). Cameron and Ku [10] proved a 1-intersecting subset of size \((n-1)!\) is a coset of the stabilizer of a point. A few alternative proofs of Cameron and Ku’s result are given in [23], [17] and [26]. To show the transitivity of \((S_n, i_t)\) we consider the action
of $S_n$ on itself by the multiplication on the left. It is evident that the action is transitive, but is far from primitive because the stabilizer of a point is the identity.

**Proposition 4.3** $(S_n, i_t)$ is $i_t$-primitive unless $n = 3$ and $t = 1$.

**Proof.** The case $n = 2$ is trivial. If $n = 3$, it is easy to verify that the graph $G(S_3, i)$ is disconnected and hence $i$-imprimitive, while $(S_3, i_t)$ for $t = 2, 3$ is $i_r$-primitive. We now assume that $n \geq 4$.

We first prove that $(S_n, i_t)$ is connected, i.e., the corresponding graph $G(S_n, i_t)$ is connected. Since $i_t \subseteq i_1$ for $t \geq 2$, it suffices to prove that $G(S_n, i)$ is connected. For any pair $\gamma, \eta \in S_n$, let $A_j = \{ i \in [n] : \eta(j) \neq i \neq \gamma(j) \}$ for $1 \leq j \leq n$. Clearly, $|A_j| \geq n - 2$. For every $J \subseteq [n]$, if $|J| = 2$, then $|\cup_{j \in J} A_j| = |A_j| = n - 2 \geq 2$. Suppose that $|J| \geq 3$. Then, for each $k \in [n]$, since there are at most two points $i_1, i_2 \in [n]$ such that $\gamma(i_1) = \eta(i_2) = k$, we can find a $j \in J$ such that $k \in A_j$, so $\cup_{j \in J} A_j = [n]$. Therefore $|\cup_{j \in J} A_j| \geq |J|$ for all $J \subseteq [n]$. By the well-known Hall theorem [24] on distinct representatives of subsets, there is a system of distinct representatives $i_1, i_2, \ldots, i_n$ for $A_1, A_2, \ldots, A_n$. Define a permutation $\tau$ by $\tau(j) = i_j$ for $1 \leq j \leq n$. It is clear that both $\{\eta, \tau\}$ and $\{\tau, \gamma\}$ belong to $E(G(S_n, i))$, proving that $G(S_n, i)$ is connected.

Suppose that $(S_n, i_t)$ is $i_r$-imprimitive for some $n \geq 4$ and $t \geq 1$. Let $A$ be a maximal-sized imprimitive $i$-subset of $S_n$, and $D = N[A] = S_n - N[A]$. From Theorem 3.2 it follows that $\frac{\alpha(D, i_t)}{|D|} = \frac{\alpha(S_n, i_t)}{|S_n|}$, and $\tau D \cap D = \emptyset$ or $D$ for all $\tau \in S_n$, and Theorem 2.2 implies that $|S \cap D | = \alpha(D, i_t)$ for every $S \in I(S_n, i_t)$. Let $\sigma$ be a fixed $n$-cycle permutation in $S_n$, and $H = \{ \sigma, \sigma^2, \ldots, \sigma^n = 1 \}$, the cyclic group generated by $\sigma$. Then any two distinct elements of a right coset of $H$ disagree at every point. Therefore $H \rho \subseteq N[\{\rho\}]$ for every $\rho \in S_n$, so $HA \subseteq N[A]$. Set $B = \{ \rho \in S_n : H \rho \subset D \}$ and $C = \{ \rho \in S_n : H \rho \cap N[A] \neq \emptyset \}$. We now complete the proof by two cases.

Case 1: $t \geq 2$. For any $\tau, \rho \in S_n$, set $F_i = F_i(\tau, \rho) = \{ j : \tau(j) = \sigma^i \rho(j) \}$, $i = 1, 2, \ldots, n$. It is easily seen that for every $j \in [n]$ there is a unique $i \in [n]$ such that $j \in F_i$, which yields $\sum_{i=1}^n |F_i| = n$. From this we see that there are at least half $F_i$’s with at most one point, meaning that there are at least $\lceil n/2 \rceil$ i’s such that $\tau$ and $\sigma^i \rho$ do not agree on $t$ points. In other words, $|H \rho \cap N[\{\tau\}]| \geq \lceil \frac{n}{2} \rceil \geq 2$, which implies that $B = \emptyset$ and $D \subset \cup_{i \in C} H \rho$. If $\sigma D \cap D \neq \emptyset$, then $\sigma D = D$, hence $HD = D$, contradicting $B = \emptyset$. We therefore obtain that
\(\sigma D \cap D = \emptyset\). Moreover, since \(C_{|\sigma D|} = \frac{\alpha(D)}{|D|} = \frac{\alpha(S)}{|S|}\), from Theorem 2.2, it follows that \(|S \cap \sigma D| = \alpha(S)\), for every \(S \in I(S_n, i_t)\). Note that for each \(S_D \in I(D, i_t)\), we have \((A \cup S_D) \cap \sigma D = \emptyset\) for every \(S \in I(S_n, i_t)\). Recalling that \(HA \subseteq N[A]\), we have \((A \cup S_D) \cap \sigma D = A \cap \sigma D \subseteq HA \cap \sigma D = \sigma(HA \cap D) \subseteq \sigma(N[A] \cap D) = \emptyset\), yielding a contradiction. Thus \((S_n, i_t)\) is \(i_t\)-primitive for \(t \geq 2\).

Case 2: \(t = 1\). By definition we see that \(|A \cap H| \leq 1\). On the other hand, from \(HA \subseteq N[A]\) and \(|A| = \frac{\alpha(S)}{|S_n|} = \frac{1}{n}\) it follows that \(N[A] = HA\), that is, \(N[A]\) is a union of some right cosets of \(H\), so \(D\) is a union of other right cosets of \(H\), i.e., \(D = HB\). By definition we also have that \(A \subseteq \bar{N}[D] \subseteq \bar{N}[H\rho]\) for every \(\rho \in B\). However, if \(\tau \in \bar{N}[H\rho]\), i.e. \(F_i(\tau, \rho) = \{j : \tau(j) = \sigma^i\rho(j)\} \neq \emptyset\) for every \(i \in [n]\), then

\[F_i(\sigma^k\tau, \rho) = \{j : \sigma^k\tau(j) = \sigma^i\rho(j)\} = \{j : \tau(j) = \sigma^{i-k}\rho(j)\} = F_{i-k}(\tau, \rho) \neq \emptyset\]

for all \(i, k \in [n]\) (here \(i - k\) is taken to be the least positive residue modulo \(n\), therefore \(H\tau \subseteq \bar{N}[H\rho]\). From this it follows that \(N[A] = HA \subseteq \cap_{\rho \in B} \bar{N}[H\rho] = \bar{N}[D]\), which implies that \((S_n, i)\) is disconnected, yielding a contradiction. Thus \((S_n, i)\) is \(i\)-primitive for \(n \geq 4\). \(\square\)

Analogously, we may consider the primitivity of symmetric systems defined on labeled sets [4] (or signed sets [5], colored sets [25] etc) and some other permutations (see [21], [22] and [26]).

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