ON $p$-ADIC $q$-$l$-FUNCTIONS AND SUMS OF POWERS

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**Abstract.** In this paper, we give an explicit $p$-adic expansion of

$$
\sum_{j=1}^{np} \frac{(-1)^j q^j}{[j]_q}
$$

as a power series in $n$. The coefficients are values of $p$-adic $q$-$L$-function for $q$-Euler numbers.

§1. Introduction

Let $p$ be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, cf.[1, 4, 6, 10]. Let $u_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q - 1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Kubota and

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Leopoldt proved the existence of meromorphic functions, $L_p(s, \chi)$, defined over the $p$-adic number field, that serve as $p$-adic equivalents of the Dirichlet $L$-series, cf.[10, 11]. These $p$-adic $L$-functions interpolate the values
\[ L_p(1-n, \chi) = -\frac{1}{n} (1 - \chi_n(p)p^{n-1}) B_n, \chi_n, \text{ for } n \in \mathbb{N} = \{1, 2, \ldots, \}, \]
where $B_n, \chi_n$ denote the $n$th generalized Bernoulli numbers associated with the primitive Dirichlet character $\chi$, and $\chi_n = \chi w^{-n}$, with $w$ the Teichmüller character, cf.[8, 10].

In [10], L. C. Washington have proved the below interesting formula:
\[ \sum_{j=1}^{np} \frac{1}{j^r} = - \sum_{k=1}^{\infty} \binom{-r}{k} (pn)^k L_p(r + k, w^{1-k-r}), \text{ where } \binom{-r}{k} \text{ is binomial coefficient.} \]

To give the $q$-extension of the above Washington result, author derived the sums of powers of consecutive $q$-integers as follows:

\[ \sum_{j=1}^{n} q^l = - \sum_{k=1}^{\infty} \binom{-r}{k} (pn)^k L_{p,q}(r + k, w^{1-k-r}) - (q-1) \sum_{k=1}^{p-1} B_p^{(n)}(r, a : F), \]

where $L_{p,q}(s, \chi)$ is $p$-adic $q$-L-function (see [7]). Indeed, this is a $q$-extension result due to Washington, corresponding to the case $q = 1$, see [10]. For a fixed positive integer $d$ with $(p, d) = 1$, set
\[ X = X_d = \operatorname{lim}_N \mathbb{Z}/dp^N, \]
\[ X_1 = \mathbb{Z}_p, X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p, \]
\[ a + dp^N\mathbb{Z}_p = \{ x \in X | x \equiv a \mod p^N \}, \]
where \( a \in \mathbb{Z} \) satisfies the condition \( 0 \leq a < dp^N \), (cf.[3, 4, 9]). We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), and write \( f \in UD(\mathbb{Z}_p) \), if the difference quotients 
\[
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
\]
have a limit \( f'(a) \) as \( (x, y) \to (a, a) \), cf.[3]. For \( f \in UD(\mathbb{Z}_p) \), let us begin with the expression

\[
\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \quad \text{cf.}[1, 3, 4, 7, 8, 9],
\]

which represents a \( q \)-analogue of Riemann sums for \( f \). The integral of \( f \) on \( \mathbb{Z}_p \) is defined as the limit of those sums (as \( n \to \infty \)) if this limit exists. The \( q \)-Volkenborn integral of a function \( f \in UD(\mathbb{Z}_p) \) is defined by

\[
I_q(f) = \int_X f(x)d\mu_q(x) = \int_{X_d} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x)q^x, \quad \text{cf.} \ [2, 3].
\]

It is well known that the familiar Euler polynomials \( E_n(z) \) are defined by means of the following generating function:

\[
F(z, t) = \frac{2}{e^t + 1} e^{zt} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}, \quad \text{cf.}[1, 5].
\]

We note that, by substituting \( z = 0 \), \( E_n(0) = E_n \) are the familiar \( n \)-th Euler numbers. Over five decades ago, Carlitz defined \( q \)-extension of Euler numbers and polynomials, cf.[1, 4, 5]. Recently, author gave another construction of \( q \)-Euler numbers and polynomials (see [1, 5, 9]). By using author’s \( q \)-Euler numbers and polynomials, we gave the alternating sums of powers of consecutive \( q \)-integers as follows:

\[
[2]q \sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m = (-1)^{n+1} q^n \sum_{l=0}^{m-1} \binom{m}{l} q^{nl} E_{l, q}[n]_q^{m-l} + \left((-1)^{n+1} q^{n(m+1)} + 1\right) E_{m, q},
\]

where \( E_{l, q} \) are \( q \)-Euler numbers (see [5]). From this result, we can study the \( p \)-adic interpolating function for \( q \)-Euler numbers and sums of powers due to author [7]. Throughout this paper, we use the below notation:

\[
[x]_q = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1},
\]

\[
[x]_{-q} = \frac{1 - (-q)^x}{1 - q} = 1 - q + q^2 - q^3 + \cdots + (-q)^{x-1}, \quad \text{cf.}[5, 9].
\]
Note that when $p$ is prime $[p]_q$ is an irreducible polynomial in $Q(q)$. Furthermore, this means that $Q(q)/[p]_q$ is a field and consequently rational functions $r(q)/s(q)$ are well defined mod $[p]_q$ if $(r(q), s(q)) = 1$. In a recent paper [5] the author constructed the new $q$-extensions of Euler numbers and polynomials. In Section 2, we introduce the $q$-extension of Euler numbers and polynomials. In Section 3 we construct a new $q$-extension of Dirichlet’s type $l$-function which interpolates the $q$-extension of generalized Euler numbers attached to $\chi$ at negative integers. The values of this function at negative integers are algebraic, hence may be regarded as lying in an extension of $Q_p$. We therefore look for a $p$-adic function which agrees with at negative integers. The purpose of this paper is to construct the new $q$-extension of generalized Euler numbers attached to $\chi$ due to author and prove the existence of a specific $p$-adic interpolating function which interpolate the $q$-extension of generalized Bernoulli polynomials at negative integer. Finally, we give an explicit $p$-adic expansion

$$\sum_{j=1}^{np} \frac{(-1)^j q^j}{[j]_q^r},$$

as a power series in $n$. The coefficients are values of $p$-adic $q$-$l$-function for $q$-Euler numbers.

### 2. Preliminaries

For any non-negative integer $m$, the $q$-Euler numbers, $E_{m,q}$, were represented by

$$\int_{Z_p} [x]^m d\mu_q(x) = E_{m,q} = [2]_q \left( \frac{1}{1 - q} \right)^m \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{1}{1 + q^{i+1}}, \text{ see [9].}$$

Note that $\lim_{q \to 1} E_{m,q} = E_m$. From Eq.(2), we can derive the below generating function:

$$F_q(t) = [2]_q e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{1}{1 + q^{j+1}} (-1)^j \left( \frac{1}{1-q} \right)^j \frac{t^j}{j!} = \sum_{j=0}^{\infty} E_{n,q} \frac{t^n}{n!}.$$

By using $p$-adic $q$-integral, we can also consider the $q$-Euler polynomials, $E_{n,q}(x)$, as follows:

$$E_{n,q}(x) = \int_{Z_p} [x + t]^n d\mu_q(t) = [2]_q \left( \frac{1}{1 - q} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{(-q^x)^k}{1 + q^{k+1}} \right), \text{ see [5, 9].}$$
Note that
\[ E_{n,q}(x) = \int_{\mathbb{Z}_p} ([x]_q + q^x [t]_q)^n d\mu_{-q}(x) = \sum_{j=0}^{n} \binom{n}{j} q^{jx} E_{j,q}[x]^{n-j}. \]

By (4), we easily see that
\[ \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = F_q(x, t) = \left[2\right]q e_1 - q \sum_{j=0}^{\infty} \frac{(-1)^j}{1 + q^{j+1}} q^{jx} \left(\frac{1}{1 - q}\right)^j \frac{t^j}{j!}. \]

From (6), we derive
\[ F_q(x, t) = \left[2\right]q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]} q^t = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \]

3. On the \(q\)-analogue of Hurwitz’s type \(\zeta\)-function associated with \(q\)-Euler numbers

In this section, we assume that \(q \in \mathbb{C}\) with \(|q| < 1\). It is easy to see that
\[ E_{n,q}(x) = \frac{[2]_q}{[2]_q^n} [m]_q^n \sum_{a=0}^{m-1} (-1)^a q^a E_{n,q^a}(a + x/m), \text{ see \cite{1}}, \]
where \(m\) is odd positive integer. From (7), we can easily derive the below formula:
\[ E_{k,q}(x) = \frac{d^k}{dt^k} F_q(x, t)|_{t=0} = \left[2\right]q \sum_{n=0}^{\infty} (-1)^n q^n [n + x]_q^k. \]

Thus, we can consider a \(q\)-\(\zeta\)-function which interpolates \(q\)-Euler numbers at negative integer as follows:

**Definition 1.** For \(s \in \mathbb{C}\), define
\[ \zeta_{E,q}(s, x) = [2]q \sum_{m=1}^{\infty} \frac{(-1)^n q^n}{[n + x]_q^s}. \]

Note that \(\zeta_{E,q}(s, x)\) is meromorphic function in whole complex plane.

By using Definition 1 and Eq.(8), we obtain the following:
Proposition 2. For any positive integer \( k \), we have
\[
\zeta_E,q(-k, x) = E_{k,q}(x).
\]

Let \( \chi \) be the Dirichlet character with conductor \( f \in \mathbb{N} \). Then we define the generalized \( q \)-Euler numbers attached to \( \chi \) as
\[
F_{q,\chi}(t) = [2]_q \sum_{n=0}^{\infty} e^{[n]_q t} \chi(n)(-1)^n q^n = \sum_{n=0}^{\infty} E_{n,\chi,q} t^n.
\]

Note that
\[
E_{n,\chi,q} = \frac{2}{[2]_q f} \sum_{a=0}^f \chi(a)(-1)^a q^a E_{n,q,f}(\frac{a}{f}), \text{ where } f (= \text{odd}) \in \mathbb{N}.
\]

By (9), we easily see that
\[
\frac{d^k}{dt^k} F_{q,\chi}(t)|_{t=0} = E_{k,\chi,q} = [2]_q \sum_{n=1}^{\infty} \chi(n)(-1)^n q^n [n]_q^k
\]

Definition 3. For \( s \in \mathbb{C} \), we define Dirichlet’s type \( l \)-function as follows:
\[
l_{q}(s, \chi) = [2]_q \sum_{n=1}^{\infty} \frac{\chi(n)(-1)^n q^n}{[n]_q^s}.
\]

From (11) and Definition 3, we can derive the below theorem.

Theorem 4. For \( k \geq 1 \), we have
\[
l_{q}(-k, \chi) = E_{k,\chi,q}.
\]

In [5], it was known that
\[
[2]_q \sum_{l=0}^{n-1} (-1)^l q^l \left[ l \right]_q^m = ((-1)^{n+1} q^n E_{m,q}(n) + E_{m,q}), \text{ where } m, n \in \mathbb{N}.
\]
From (4) and (12), we derive

\[
[2]_q \sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m
\]

(13)

\[
= (-1)^{n+1} q^n \sum_{l=0}^{m-1} \left( \begin{array}{c} m \\ l \end{array} \right) q^n E_{l,q}[n]_q^{m-l} + \left( (-1)^{n+1} q^{n(m+1)} + 1 \right) E_{m,q}.
\]

Let \(s\) be a complex variable, and let \(a\) and \(F (= \text{odd})\) be the integers with \(0 < a < F\). We now consider the partial \(q\)-zeta function as follows:

\[
H_q(s, a : F) = \frac{[2]_q}{[2]_{q^F}} \sum_{m \equiv a(F)} \frac{q^m (-1)_m^m}{[m]_q^s} = (-1)^a q^a \frac{[F]^{-s}}{[2]_{q^F}} \zeta_{E,q^F}(s, \frac{a}{F}).
\]

(14)

For \(n \in \mathbb{N}\), we note that \(H_q(-n, a : F) = (-1)^a q^a \frac{[F]^{-n}}{[2]_{q^F}} E_{n,q^F}(\frac{a}{F})\). Let \(\chi\) be the Dirichlet's character with conductor \(F (= \text{odd})\). Then we have

\[
l_q(s, \chi) = [2]_q \sum_{a=1}^{F} \chi(a) H_q(s, a : F).
\]

(15)

The function \(H_q(s, a : F)\) will be called the \(q\)-extension of partial zeta function which interpolates \(q\)-Euler polynomials at negative integers. The values of \(l_q(s, \chi)\) at negative integers are algebraic, hence may be regarded as lying in an extension of \(\mathbb{Q}_p\). We therefore look for a \(p\)-adic function which agrees with \(l_q(s, \chi)\) at the negative integers in Section 4.

§4. \(p\)-ADIC \(q\)-FUNCTIONS AND SUMS OF POWERS

We define \(< x > = \frac{[x]_q}{w(x)}\), where \(w(x)\) is the \(\text{Teichmüller}\) character. When \(F (= \text{odd})\) is multiple of \(p\) and \((a, p) = 1\), we define a \(p\)-adic analogue of (14) as follows:

\[
H_{p,q}(s, a : F) = \frac{(-1)^a q^a}{[2]_{q^F}} < a >^{-s} \sum_{j=0}^{\infty} \left( \begin{array}{c} -s \\ j \end{array} \right) q^{ja} \left( \frac{[F]_q}{[a]_q} \right)^j E_{j,q^F}, \text{ for } s \in \mathbb{Z}_p.
\]

(16)
Thus, we note that
\begin{equation}
H_{p,q}(-n, a : F) = \frac{(-1)^a q^a}{[2]_q^F} < a >^n \sum_{j=0}^n \binom{n}{j} q^ja \left( \frac{[F]_q}{[a]_q} \right)^j E_{j,q}
\end{equation}
\begin{equation}
= \frac{(-1)^a q^a}{[2]_q^F} w^{-n}(a)[F]^n q E_{n,q} \left( \frac{a}{F} \right) = w^{-n}(a)H_q(-n, a : F), \text{ for } n \in \mathbb{N}.
\end{equation}

We now construct the $p$-adic analytic function which interpolates $q$-Euler number at negative integer as follows:

\begin{equation}
l_{p,q}(s, \chi) = [2]_q \sum_{a=1 \atop (a,p)=1}^F \chi(a)H_{p,q}(s, a : F).
\end{equation}

In [9], it was known that
\begin{equation}E_{k,\chi,q} = \int_X \chi(x)[x]^k q d\mu_{-q}(x), \text{ for } k \in \mathbb{N}.
\end{equation}
For $f(= \text{odd}) \in \mathbb{N}$, we note that
\begin{equation}E_{k,\chi,q} = \frac{[2]_q}{[2]_q^F} \frac{[f]_q^n}{[f]_q^F} \sum_{a=0}^{f-1} \chi(a)(-1)^a q^a E_{n,q} \left( \frac{a}{f} \right).
\end{equation}

Thus, we have
\begin{equation}l_{p,q}(-n, \chi) = [2]_q \sum_{a=1 \atop (p,a)=1}^F \chi(a)H_{p,q}(-n, a : F) = \int_X \chi w^{-n}(x)[x]^n q d\mu_{-q}(x)
\end{equation}
\begin{equation}
= E_{n,\chi w^{-n},q} - [p]_q^n \chi w^{-n}(p)E_{n,\chi w^{-n},q^p}.
\end{equation}
In fact,
\begin{equation}l_{p,q}(s, \chi) = \frac{[2]_q}{[2]_q^F} \sum_{a=1}^F (-1)^a q^a < a >^{-s} \chi(a) \sum_{j=0}^\infty \binom{-s}{j} q^ja \left( \frac{[F]_q}{[a]_q} \right)^j E_{j,q}, \text{ for } s \in \mathbb{Z}_p.
\end{equation}
This is a $p$-adic analytic function and has the following properties for $\chi = w^t$:

\[ l_{p,q}(-n, w^t) = E_{n,q} - [p]_q^n E_{n,q}, \quad \text{where } n \equiv t \pmod{p - 1}, \]

\[ l_{p,q}(s, t) \in \mathbb{Z}_p \text{ for all } s \in \mathbb{Z}_p \text{ when } t \equiv 0 \pmod{p - 1}. \]

If $t \equiv 0 \pmod{p - 1}$, then $l_{p,q}(s_1, w^t) \equiv l_{p,q}(s_2, w^t) \pmod{p}$ for all $s_1, s_2 \in \mathbb{Z}_p$, $l_{p,q}(k, w^t) \equiv l_{p,q}(k + p, w^t) \pmod{p}$. It is easy to see that

\[ l_{p,q}(s, t) \in \mathbb{Z}_p \text{ for all } s \in \mathbb{Z}_p \text{ when } t \equiv 0 \pmod{p - 1}. \]

From (22) and (22-1), we derive

\[ \frac{1}{r + k - 1} \binom{-r}{k} \binom{1 - r - k}{j} = \frac{-1}{j + k} \binom{-r}{k + j - 1} \binom{k + j}{j}. \]

By using (13), we see that

\[ \frac{r}{r + k} \binom{-r - 1}{k} \binom{1 - r - k}{j} = \binom{-r}{k + j} \binom{k + j}{j}. \]

For $s \in \mathbb{Z}_p$, we define the below $T$-Euler polynomials:

\[ T_{n,q}(s, q : F) = (-1)^a q^a \sum_{k=0}^{\infty} \binom{-s}{k} \frac{[a]_q}{F^k q^k} q^{ak} \left((-1)^n q^{n F(k + 1)} - 1\right) E_{k,q}. \]
From (23) and (24), we derive

\[
\sum_{l=0}^{n-1} (-1)^F l + a q^{F l + a} [F l + a]_q^r
\]

\[
= -\sum_{s=0}^{\infty} \left(-\frac{r}{s}\right) [a]_q^{-r} \left(\frac{[F]_q}{[a]_q}\right)^s (-q^{s+1})^a (-q^F)^n \sum_{l=0}^{s-1} \left(\frac{s}{l}\right) q^n F l E_{l, q^F} [n]_{q^F}^{s-l}
\]

\[
= \frac{w^{-r}(a)}{[2]_q^F} T_{n, q}(r, a : F).
\]

First, we evaluate the right side of Eq.(26) as follows:

\[
\sum_{s=0}^{\infty} \left(-\frac{r}{s}\right) [a]_q^{-r} \left(\frac{[F]_q}{[a]_q}\right)^s (-q^{s+1})^a (-q^F)^n \sum_{l=0}^{s-1} \left(\frac{s}{l}\right) q^n F l E_{l, q^F} [n]_{q^F}^{s-l}
\]

\[
= \sum_{s=0}^{\infty} \sum_{k=0}^{r} \frac{r}{r+k} \left(-\frac{r-1}{k}\right) [a]_q^{-r} q^{ak + F n} (-1)^n [F n]_{q^F}^k \left(-\frac{q}{2}\right)^a \sum_{l=0}^{\infty} \left(-\frac{r-k}{l}\right) q^a l
\]

\[
\left(\frac{[F]_q}{[a]_q}\right)^l E_{l, q^F} q^n F l.
\]

It is easy to check that

\[
q^n F l = \sum_{j=0}^{l} \left(\frac{l}{j}\right) [n F]_q^j (q-1)^j = 1 + \sum_{j=1}^{l} \left(\frac{l}{j}\right) [n F]_q^j (q-1)^j.
\]

Let

\[
J_{p, q}^{(k)}(s, a : F) = \sum_{j=1}^{k} w^j(a) \left(\frac{k}{j}\right) (q-1)^j < a >^j H_{p, q}(s, a : F),
\]

and

\[
K_{p, q}(s, a : F) = \frac{(-1)^a q^a}{[2]_q^F} < a >^{-s} \sum_{l=0}^{\infty} \left(-\frac{s}{l}\right) q^a l \left(\frac{[F]_q}{[a]_q}\right) E_{l, q^F} \sum_{j=1}^{l} \left(\frac{l}{j}\right) [n F]_q^j (q-1)^j.
\]

For \(F = p, r \in \mathbb{N}\), we see that

\[
[2]_q \sum_{a=1}^{p-1} \sum_{l=0}^{n-1} \frac{(-1)^a q^a l}{[a + pl]_q^r} = [2]_q \sum_{j=1}^{np} \frac{q^j (-1)^j}{[j]_q^p}.
\]
For \( s \in \mathbb{Z}_p \), we define \( p \)-adic analytically continued function on \( \mathbb{Z}_p \) as

\[
K_{p,q}(s, \chi) = [2]_q \sum_{a=1}^{p-1} \chi(a) \left( J_{p,q}^{(k)}(s, a : F) + q^{ak}K_{p,q}(s, a : F) \right),
\]

(31)

\[
T_{p,q}(s, \chi) = [2]_q \sum_{a=1}^{p-1} \chi(a)T_{n,q}(s, a : F), \quad \text{where } k, n \geq 1.
\]

From (24)-(31), we derive

\[
[2]_q \sum_{j=1}^{np} \frac{q^j (-1)^j}{[j]_q^r} = - \sum_{k=0}^{\infty} \frac{r}{r+k} \left( \begin{array}{c} -r - 1 \\ k \end{array} \right) (-1)^n q^{pn} \left[ \begin{array}{c} k \\ q \end{array} \right]_{p,q} (r+k, w^{-r-k})
\]

\[
- \sum_{k=0}^{\infty} \frac{r}{r+k} \left( \begin{array}{c} -r - 1 \\ k \end{array} \right) (-1)^n q^{pn} \left[ \begin{array}{c} k \\ q \end{array} \right]_{p,q} (r+k, w^{-r-k}) - T_{p,q}(r, w^{-r}).
\]

Therefore we obtain the following theorem:

**Theorem 5.** Let \( p \) be an odd prime and let \( n \geq 1 \), and \( r \geq 1 \) be integers. Then we have

\[
[2]_q \sum_{j=1}^{np} \frac{q^j (-1)^j}{[j]_q^r} = - \sum_{k=0}^{\infty} \frac{r}{r+k} \left( \begin{array}{c} -r - 1 \\ k \end{array} \right) (-1)^n q^{pn} \left[ \begin{array}{c} k \\ q \end{array} \right]_{p,q} (r+k, w^{-r-k})
\]

(32)

\[
- \sum_{k=0}^{\infty} \frac{r}{r+k} \left( \begin{array}{c} -r - 1 \\ k \end{array} \right) (-1)^n q^{pn} \left[ \begin{array}{c} k \\ q \end{array} \right]_{p,q} (r+k, w^{-r-k}) - T_{p,q}(r, w^{-r}).
\]

For \( q = 1 \) in (32), we have

\[
2 \sum_{j=1}^{np} \frac{(-1)^j}{[j]_q^r} = - \sum_{k=0}^{\infty} \frac{r}{k+r} \left( \begin{array}{c} -r - 1 \\ k \end{array} \right) (-1)^n (pn)^k l_p(r+k, w^{-r-k}),
\]

where \( n \) is positive integer.
Remark A. Let $p$ be an odd prime. Then we have
\[
\sum_{j=1}^{p-1} \frac{(-1)^j q^j}{[j]_q} = \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q}.
\]

Proof. To prove Remark A, it is sufficient to show that
\[
\left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right)^2 = \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} - (1-q) \sum_{j=1}^{p-1} (-1)^j \right)
\]
\[
= \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} (-1)^j \frac{1}{[j]_q} - (1-q) \right)
\]
\[
= \left( \sum_{j=1}^{p-1} \frac{(-1)^j}{[j]_q} \right) \left( \sum_{j=1}^{p-1} \frac{(-1)^j q^j}{[j]_q} \right).
\]

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