Infrared fractal propagator for a charged fermion

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Abstract. The infinite range of the Coulomb forces implies that the electron can interact with softer and softer photons. This motivates us to look for an electron propagator with a fractal structure. Using Schwinger’s proper time formulation, some work by Bloch and Nordsieck and a scheme of differential regularization, we are able to obtain an expression for the electron propagator with an anomalous dimension $\gamma$. In contrast to previous discussions on the subject we maintain gauge invariance throughout and obtain an unambiguous number for the anomalous dimension $\gamma$ of the electron. This leads us to a discussion of the fractal dimension of electron paths where we are able to obtain a number based on an analogy with Brownian motion. Since our result is gauge invariant we discuss the possibility of measuring $\gamma$ experimentally and finally we briefly touch on the case of quantum gravity.

1. Introduction

There are ample reasons to look for a fractal propagator for the electron of the form

$$S(k) = \left( \frac{\kappa}{i\Lambda} \right)^\gamma \Gamma(1 + \gamma) \left\{ \frac{\kappa - k}{[k^2 + \kappa^2 - i0^+]^{1+\gamma}} \right\},$$

(1)

wherein $\hbar \kappa = mc$, $\Lambda$ is a short distance length scale, the fractional exponent $\gamma$ is a function of the coupling strength $\alpha = (e^2/\hbar c)$ and the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds$$

with $\Re(z) > 0$.

This expression has a branch cut instead of a pole and thus according to the spectral decomposition, it represents a range of masses instead of a particle with a single mass. On an intuitive level, the electron will never truly be free due to the zero mass of the photon. Hence, what truly propagates is the dressed electron constantly interacting with its cloud of photons which should be represented by a particle with a range of masses instead of a definite sharp mass. More formally, truly free particles [1] are labeled by irreducible representations of the Poincaré group in terms of mass and spin. However, charged particles cannot consistently be assigned precise masses [2–4] since they continually interact with massless photons. The resulting electromagnetic fields are too long ranged to rigorously define “in” and “out” fixed mass states and one must introduce concepts such as infraparticles (For a review see [5]).

One may also expect a degree of self-similarity at wavelengths below the electron mass since the only scale-breaking term in QED is the mass of the electron. Intuitively, one might think of stepping back farther and farther from an electron world line and seeing contributions to its
dressed structure from longer and longer wavelengths, i.e. softer and softer virtual photons. This would suggest a fractal [6] structure, as in the expression above which will be made more precise below. Such notions of scaling and fractality are not new in QED and in quantum field theory in general, but are often considered in the high energy, ultraviolet limit [7,8]. In this case, additional complications arise since more and more charged excitations must be included, but again, one sees fractional exponents in the form of anomalous dimensions and the renormalization group – another reflection of a non-trivially realized scale invariance in the theory, but now at short distances.

Another reason to look for an expression of the form Eq.(1) is that the fractional exponent makes the above propagator for the electron non-local [9] which would be expected due to the infinite range of the electromagnetic forces. This non-locality which here appears in a natural way has been previously introduced ad-hoc [10,11] as a regularization tool.

Finally we note that the above expression is non-analytic and more specifically it can not be obtained through a perturbative expansion in \( \alpha \). This would be in agreement with the results of Dyson [12] who showed that the results of a power series in \( \alpha \) can not be convergent so that a fermion can not properly be represented by a propagator which is analytic in \( \alpha \).

There is a rich literature on the subject with many different methods used for arriving at a result similar to Eq.(1) without any universal agreement as to the value of \( \gamma \). Appelquist and Carazzone [13] take \( \gamma = -(\alpha/\pi) + \ldots \) to leading order. This differs from older work based on summing logarithms [14]. The fourth volume of the Landau and Lifschitz course of theoretical physics [15] at first appears to be in agreement with \( \gamma \neq 0 \) but ultimately chooses \( \gamma = 0 \) by assigning a small mass to the photon. The photon mass implies a broken gauge symmetry, as well as the broken conformal invariance of the free Maxwell field. The null result \( \gamma = 0 \) is also in conflict with our physical understanding that no sharp mass can be assigned to a charged particle. The effect of assigning the photon a small mass is clear from the physical picture of photon processes. As one backs away from the world line of an electron, thereby going to larger and larger wavelengths, a photon mass prevents the production of more and more photons without limit and the scaling behavior of the electromagnetic fields is thereby broken [16]. The broken gauge symmetry is physically unacceptable. It has been argued [17] that the change from a pole into a branch point has measurable physical implications in measurements of “1/\( \omega \)” noise in the Schrödinger non-relativistic limit of the relativistic Dirac equation. A path integral [18] approach using the Schwinger [19] proper time representation of the propagator and some work by Bloch and Nordsieck [20] on soft photon emission gives the same sort of result, but with the final answer given only for charged scalar fields, and generally considered to be gauge invariant in such a way that the singularity structure can be returned to a simple pole by a choice of gauge. Fried also discusses this problem [21] as do Johnson and Zumino [22], and Stefanis and collaborators [23]. Batalin, Fradkin and Schvartsman have made a similar gauge dependent calculation for scalar particles [24].

The major defects of the existing calculations are that they do not directly give a Dirac propagator raised to a fractional power. Existing calculations are ambiguous and not gauge invariant.

2. Gauge Invariant Calculation

The following calculation which is included for the sake of completeness was presented in a previous paper cited [25]. For an electron moving through an external electromagnetic field, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), the Dirac propagator

\[
(-i\gamma^\mu + \kappa) G(x, y; A) = \delta(x - y) ,
\]

\[
d_\mu = \partial_\mu - i \left( \frac{e A_\mu}{\hbar c} \right) ,
\]

(3)
may be solved employing the function $\Delta(x, y; A)$:

$$
\Delta(x, y; A) = \int \gamma_5 G(x, z; A) \gamma_5 G(z, y; A) d^4 z ,
$$

$$
G(x, y; A) = (i \delta + \kappa) \Delta(x, y; A) ,
$$

$$
\left( \frac{\partial^2}{\partial t^2} + \kappa^2 \right) \Delta(x, y; A) = \delta(x - y) ,
$$

$$
\delta^2 = -d^\mu d_\mu - \frac{e}{2\hbar c} \sigma^{\mu\nu} F_{\mu\nu} .
$$

Formally introducing the Hamiltonian of the electron as

$$
\mathcal{H}_{tot} = \mathcal{H} + \mathcal{H}_{spin} ,
$$

$$
\mathcal{H} = \frac{1}{2m} \left\{ \left( \frac{p - e A}{\gamma_5 c} \right)^2 + m c^2 \right\} ,
$$

$$
\mathcal{H}_{spin} = - \left( \frac{e \hbar}{4mc} \right) \sigma^{\mu\nu} F_{\mu\nu}
$$

and employing the operator representation $p_\mu = -i\hbar \partial_\mu$, one may define the amplitude for the electron to go from $y$ to $x$ in a proper time $\tau$ as the matrix element

$$
\mathcal{G}(x, y, \tau; A) = \langle x | e^{-i\mathcal{H}_{tot} \tau/\hbar} | y \rangle .
$$

From Eqs.(4), (5) and (6) follows the electron propagator expression

$$
\Delta(x, y; A) = \frac{i\hbar}{2m} \int_0^\infty \mathcal{G}(x, y, \tau; A) d\tau ,
$$

$$
h \mathcal{G}(x, y; A) = \left( mc - \frac{e}{c} A(x) \right) \Delta(x, y; A) .
$$

The physical significance of $\mathcal{H}(p, x)$ can be made manifest in the formal classical limit $\hbar \to 0$. Hamilton’s equations in proper time,

$$
u^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial \mathcal{H}}{\partial p_\mu} \quad \text{and} \quad f^\mu = \frac{dp_\mu}{d\tau} = -\frac{\partial \mathcal{H}}{\partial x^\mu} ,
$$

directly yield the the Lorentz force on a charge equation of motion

$$
m \frac{dv^\mu}{d\tau} = \frac{e}{c} F_{\mu\nu} v_\nu .
$$

Alternatively one may employ the Lagrangian formalism,

$$
\mathcal{L}(v, x; A) = \frac{1}{2} m \left( v^\mu v_\mu - c^2 \right) + \frac{e}{c} v^\mu A_\mu(x) ,
$$

$$
\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{v}^\mu} \right) = \left( \frac{\partial \mathcal{L}}{\partial v^\mu} \right) ;
$$

i.e. Eqs.(10) also imply Eq.(9).

For long wavelength modes of the electromagnetic field, one may with a sufficient degree of accuracy neglect the spin flip term in the Hamiltonian Eq.(5). In such a case we may approximate Eq.(6) by the Lagrangian path integral formulation

$$
\mathcal{G}(x, y, \tau; A) \approx \int_{X(0)=y}^{X(\tau)=x} e^{i S[X; A]/\hbar} \prod_\sigma dX(\sigma) ,
$$

$$
S[X; A] = \int_0^\tau \mathcal{L}(\dot{X}(\sigma), X(\sigma); A) d\sigma .
$$
Now, let us consider an electron following a world line path $P$ from $x$ to $y$ in a proper time $\tau$. Since the general path $P$ contributing to the functional integral Eq.(11) represents the virtual motion of an electron, one finds, in general, that $c^2 \tau^2 \neq -(x - y)^2$. It is only in the classical limit $\hbar \to 0$ that $c^2 \tau^2 = -dX^\mu dX_\mu$. In quantum mechanics the amplitude for a process is the coherent sum of all amplitudes for all the different ways in which that process could happen. Thus, one integrates through all possible proper times the electron could accumulate while going from $x$ to $y$. Consider two different paths, $P_1$ and $P_2$, contributing to the functional integral in Eqs.(11). Although the endpoints $y$ and $x$ are the same, the proper time of the two paths are different. The relativistic “toy model” analogy is to consider two twins starting at the same age at $y$ taking two different paths, $P_1$ and $P_2$, and meeting again at $x$ when their ages are in general different.

The interaction between the electron and the electromagnetic vector potential is described by the action

$$S_{\text{int}}(P; A) = \int_0^\tau \mathcal{L}_{\text{int}}(\dot{X}(\sigma), X(\sigma); A)d\sigma,$$

$$S_{\text{int}}(P; A) = \frac{e}{c} \int_0^\tau A_\mu(X(\sigma))\dot{X}^\mu(\sigma)d\sigma,$$

$$S_{\text{int}}(P; A) = \frac{e}{c} \int_P A_\mu(X)dX^\mu, \quad (12)$$

where the integral is along the world line $P$. To describe an electron moving through a vacuum region with zero point electromagnetic fields, one averages over the vacuum field fluctuations according to the rule

$$e^{iS_{\text{int}}(P; A)/\hbar} \to \langle 0 | e^{iS_{\text{int}}(P; A)/\hbar} | 0 \rangle_+,$$

$$e^{iS_{\text{int}}(P; A)/\hbar} \to e^{iS_{\text{self}}(P)/\hbar},$$

$$S_{\text{self}}(P) = \frac{\hbar c}{2} \int_P \int_P D_{\mu\nu}(x_1 - x_2)dx_1^\mu dx_2^\nu. \quad (13)$$

In the above Eq.(13), the subscript “$+$” denotes time ordering, $\hat{A}_\mu(x)$ denotes the operator vector potential field and the photon propagator is given by

$$D_{\mu\nu}(x_1 - x_2) = \frac{i}{\hbar c} \langle 0 | \hat{A}_\mu(x_1)\hat{A}_\nu(x_2) | 0 \rangle_+. \quad (14)$$

The action form in Eq.(13) is of a well-known form [26], and we have bypassed the usual Bloch-Nordsieck replacement of $c^2 \gamma^\mu$ by four velocity $v^\mu$ by simply evaluating a phase in the soft photon infrared limit that we are considering. The propagator may be written

$$D_{\mu\nu}(x - y) = \left(\eta_{\mu\nu} - (1 - \xi)\frac{\partial_\mu\partial_\nu}{\partial x^2}\right)D(x - y),$$

$$D(x - y) = \int \frac{4\pi}{k^2 - i0^+} e^{ik\cdot(x - y)} \frac{d^4k}{(2\pi)^4},$$

$$D(x - y) = \frac{i}{\pi} \left\{ \frac{1}{(x - y)^2 + i0^+} \right\}. \quad (15)$$

where the parameter $\xi$ fixes a gauge. Because the world line of the electron never begins nor ends (charge conservation), the partial derivative terms in Eq.(15) do not contribute to the self-action in Eq.(13): Independently of the gauge parameter $\xi$ we have

$$S_{\text{self}}(P) = \frac{\hbar c}{2} \int_P \int_P D(x_1 - x_2)dx_1^\mu dx_2^\mu. \quad (16)$$
In the absence of any external field (above and beyond the vacuum fluctuation operator \( \hat{A} \)) we have now derived expressions for the renormalized vacuum electron propagator

\[
\tilde{G}(x - y) = \int S(k) e^{ik \cdot (x - y)} \frac{d^4 k}{(2\pi)^4}
\]

\[
\tilde{G}(x - y) = (0| G(x, y; \hat{A}) |0)_{\perp}
\]

\[
\tilde{G}(x - y) = (i\partial + \kappa) \tilde{\Delta}(x - y),
\]

\[
\tilde{\Delta}(x - y) = \frac{i\hbar}{2m} \int_0^\infty \tilde{G}(x - y, \tau) d\tau.
\] (17)

The functional integral expression for \( \tilde{G}(x - y, \tau) \) is given by

\[
\tilde{G}(x - y, \tau) = \int_{X(0) = y}^{X(\tau) = x} e^{i\tilde{S}[X; A]/\hbar} \prod_\sigma dX(\sigma),
\]

\[
\tilde{S}[X] = \int_0^\tau \mathcal{L}_0(\dot{X}(\sigma)) d\sigma + S_{self}[X],
\] (18)

wherein the free electron Lagrangian is

\[
\mathcal{L}_0(\dot{X}) = \frac{1}{2} m_0(\dot{X}^\mu \dot{X}_\mu - c^2),
\] (19)

and the self-action is given by Eqs.(15) and (16) as

\[
S_{self}[X] = \frac{i\hbar}{2\pi} \int_0^\tau \int_0^\tau \frac{\dot{X}^\mu(\sigma_1)\dot{X}_\mu(\sigma_2) d\sigma_1 d\sigma_2}{(X(\sigma_1) - X(\sigma_2))^2 + i0^+}.
\] (20)

The divergent piece of the self-action

\[
\Re e S_{self}[X] = \frac{\Delta m}{2} \int_0^\tau (\dot{X}^\mu(\sigma) \dot{X}_\mu(\sigma) - c^2) d\sigma,
\]

\[
|\Delta m| = \infty.
\] (21)

The formally infinite self-mass can be described by a finite physical mass \( 0 < m = (m_0 + \Delta m) < \infty \). Thus, Eq.(18) is renormalized to

\[
\tilde{G}(x - y, \tau) = \int_{X(0) = y}^{X(\tau) = x} e^{i\tilde{S}[X; A]/\hbar} \prod_\sigma dX(\sigma),
\]

\[
\tilde{S}[X] = \int_0^\tau \mathcal{L}_m(\dot{X}(\sigma)) d\sigma + iW[X]
\]

\[
\mathcal{L}_m(\dot{X}) = \frac{1}{2} m(\dot{X}^\mu \dot{X}_\mu - c^2)
\]

\[
W[X; \tau] = \Re m S_{self}[X],
\]

\[
W[X; \tau] = \frac{\hbar}{2\pi} \int_0^\tau \int_0^\tau \frac{\dot{X}^\mu(\sigma_1)\dot{X}_\mu(\sigma_2) d\sigma_1 d\sigma_2}{(X(\sigma_1) - X(\sigma_2))^2}.
\] (22)

For a straight-line path \( X^\mu(\sigma) = V^\mu \sigma \) with \( V^\mu V_\mu = -c^2 \), one finds

\[
W(\tau) = \frac{\hbar}{2\pi} \int_0^\tau \int_0^\tau \frac{d\sigma_1 d\sigma_2}{(\sigma_1 - \sigma_2)^2},
\] (23)
which can be made finite with the formal differential regularization [27]

$$\frac{d^2W(\tau)}{d\tau^2} = \frac{\hbar \alpha}{\pi \tau^2}.$$  \hfill (24)

The solution to Eq.(24) with a logarithmic cut-off \(\Lambda\) is

$$W(\tau) = -\frac{\hbar \alpha}{\pi} \ln \left( \frac{c\tau}{2\Lambda} \right).$$  \hfill (25)

From Eqs.(18) and (25), one finds

$$\tilde{G}(x - y, \tau) \approx e^{-W(\tau)/\hbar} \tilde{G}_m(x - y, \tau)$$  \hfill (26)

wherein \(\tilde{G}_m(x - y, \tau)\) is the proper time Green’s function for a particle of mass \(m\) with the corresponding free Lagrangian \(\mathcal{L}_m(\dot{X})\). In detail and to exponentially lowest order in \(\alpha\),

$$\tilde{G}_m(x - y, \tau) = \int \left\{ e^{-i\hbar (k^2 + \kappa^2) \tau/2m} e^{ik \cdot (x - y)} \right\} \frac{d^4k}{(2\pi)^4},$$  \hfill (27)

Eqs.(17) and (27) imply

$$\tilde{\Delta}(x - y) = \int D(k) e^{ik \cdot (x - y)} \frac{d^4k}{(2\pi)^4},$$

$$D(k) = \left( \frac{\kappa}{i\Lambda} \right)^{\alpha/\pi} \left\{ \frac{\Gamma(1 + (\alpha/\pi))}{[k^2 + \kappa^2 - i0]^\left(1+(\alpha/\pi)\right)} \right\},$$

$$\tilde{G}(x - y) = \int S(k) e^{ik \cdot (x - y)} \frac{d^4k}{(2\pi)^4},$$

$$S(k) = (\kappa - k)D(k).$$  \hfill (28)

Eq.(28) is equivalent to the central Eq.(1) of this work with

$$\gamma = \frac{\alpha}{\pi} + \ldots$$  \hfill (29)

in agreement with the magnitude, but not the sign, of \(\gamma\) in Appelquist and Carazzone [13].

3. Fractal Dimension

The path integral expressions for the QED charged Dirac propagator employed here allow one to assign a fractal dimension to the paths taken by charged particles in the infrared limit. The fractal nature of particle paths has been discussed in the literature (see, for example [28, 29]). Abbott and Wise [30], working from nonrelativistic quantum mechanics, found 2 as the dimension of a quantum mechanical path, as opposed to 1 for a classical path. Cannata and Ferrari [31] extended this work for spin-1/2 particles and find different results not only in the classical and quantum mechanical limits, but also in the non-relativistic and relativistic limits. Intuitively one can understand the dimension 2 result for the nonrelativistic quantum mechanical case by thinking of the Schrödinger equation as a diffusion equation in imaginary time [33]. For diffusion one has the distance \(r\) a particle covers in time \(t\) satisfying a relationship
of the form $t \propto r^d$ wherein $d$ is the fractal dimension of the “path”. For example, $t \propto r^2$ in the diffusion limit, and $t \propto r$ in the ballistic (simple path) limit [34]. Here we have a closely analogous situation but with a 4-dimensional Hamiltonian $\mathcal{H}$ and with fractal diffusion in proper time.

To avoid some spin algebraic ceremony, let us here consider a spin zero charged particle with mass $m = \hbar K/c$ moving in the vacuum. The propagator will have the LSZ spectral representation [32]

$$D(k) = \int_{K^2}^{\infty} \left[ \frac{\rho(k^2)}{k^2 + \kappa^2 - i0^+} \right] d(k^2). \quad (30)$$

Eq.(30) may equivalently be written in the form

$$D(k) = i \int_0^{\infty} e^{-i(K^2+k^2)s} U(s) ds, \quad (31)$$

wherein

$$e^{-iK^2s} U(s) = \int_{K^2}^{\infty} \rho(k^2) e^{-in^2} d(k^2). \quad (32)$$

Eqs.(1) and (31) imply

$$U(s) = \left( \frac{Ks}{\Lambda} \right)^\gamma \quad (33)$$

so that

$$D(k) = i \left( \frac{K}{\Lambda} \right)^\gamma \int_0^{\infty} s^\gamma e^{-i(K^2+k^2)s} ds. \quad (34)$$

Let $\sigma = (K/\Lambda)^\gamma s^{1+\gamma}$ so that

$$D(k) = \frac{i}{1+\gamma} \int_0^{\infty} \exp \left[ -i(K^2+k^2)(\Lambda/K)^{\gamma/(1+\gamma)} \sigma^{2/d} \right] d\sigma, \quad (35)$$

with the fractal dimension

$$d = 2(1+\gamma) \approx 2 + \frac{2\alpha}{\pi} + \ldots; \quad (36)$$

i.e. the “path” through space-time as a function of an “internal proper time” parameter $\sigma$ has a dimension slightly higher than that of a two-dimensional surface. This excess over dimension 2 can be thought of as due to an additional roughening of the path of a charged particle due to interactions with vacuum fluctuations. All previous discussions have ignored the effects of self-interaction via long-range fields.

Since the fractal dimension of Brownian motion has a physical meaning and as we have obtained a result which does not depend on any gauge and thus is a candidate to be experimentally measurable we would like to explore the possibility of measuring the anomalous dimension of $\gamma$ or at the very least thinking of a situation where it would yield measurable consequences. We need a context where the only interactions of the electrons are with itself. It became known after the discovery of the wave nature of the electrons that on the whole light optics could be carried to electron optics although with some caveats [35]. The special case of electron diffraction from multiple slits was published in 1961 by Jönsön [36] and later by Tonomura et al [37] where the latter collaboration used one electron at a time to produce the diffraction pattern thus showing that the pattern is effectively produced by freely propagating electrons. There is the distinct possibility that the anomalous dimension, as a number characterizing infrared effects will have measurable consequences for these experiments which are directly comparable to theory.
4. Extensions
For quantum gravity one can do the analysis in much the same way as for quantum electrodynamics. The ultraviolet divergences need not worry us since we are dealing with a strictly infrared problem. There should be graviton-graviton interactions, but if we neglect these as small compared to graviton-electron interactions we can simply repeat what was done for electrostatics but now with the Newtonian limit of gravity. Recognizing that we need to replace the repulsive electrostatic self-interaction, say \(+\frac{e^2}{r}\), with the attractive gravitational self-interaction, say \(-\frac{Gm^2}{r}\), suggests an asymptotic form of the Dirac propagator exponent
\[
\gamma \approx \frac{1}{\pi \hbar c} (e^2 - Gm^2) + \ldots
\]  \ (37)
If \(m = \frac{|e|}{\sqrt{G}}\), which is the ADM [38, 39] mass of charged shell of charge \(e\), regularized by its own gravity, then one recovers an effectively free propagator. Given that one generally makes measurements using the electromagnetic interaction, and the suggestion that quantum mechanics might be linked to self-interaction [40], it is interesting to consider what this might imply for the role of gravity in the quantum measurement problem [41]. In particular, since one has a connection between mass and charge which is non-perturbative in Newton’s \(G\), and implies a mass near the Planck mass \(\sim 10^{-5}\) gm, which may be thought to be in the neighborhood of a putative classical-quantum boundary.

5. Conclusions
We have provided a simple and intuitive path integral description of how the propagator for a charged Dirac particle is modified by soft self-energy radiative corrections. The result is a self-similar (fractal) object with the non-locality one would expect for a particle carrying an infinite range field. The extension of this treatment to other interactions was briefly explored, and the special case of soft graviton corrections quantitatively discussed.

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