THE VALUATION DIFFERENCE RANK OF AN ORDERED DIFFERENCE FIELD

SALMA KUHLMANN, MICKAËL MATUSINSKI, AND FRANÇOISE POINT

Abstract. There are several equivalent characterizations of the valuation rank of an ordered field (endowed with its natural valuation). In this paper, we extend the theory to the case of an ordered difference field and introduce the notion of difference rank. We characterize the difference rank as the quotient modulo the equivalence relation naturally induced by the automorphism (which encodes its growth rate). In analogy to the theory of convex valuations, we prove that any linearly ordered set can be realized as the difference rank of an ordered difference field.

1. Introduction

The theory of real places and convex valuations is a special chapter in valuation theory; it is a basic tool in real algebraic geometry. Surveys can be found in [5], [6] and [7]. An important isomorphism invariant of an ordered field is its rank as a valued field, which has several equivalent characterizations: via the ideals of the valuation ring, the value group, or the residue field. This can be extended to ordered fields with extra structure, giving a characterization completely analogous to the above, but taking into account the corresponding induced structure on the ideals, value group, or residue field (see [2] for ordered exponential fields).

In this paper, we push this analogy to the case of an (ordered) difference field. The leading idea is to identify the difference rank of a non-archimedean ordered field as the quotient by the equivalence relation that the automorphism induces on the set of infinitely large field elements. In Section 2 we start by a key remark regarding equivalence relations defined by monotone maps on chains, and briefly review the theory of convex valuations and rank. In Section 3 we represent this invariant via equivalence relations induced by addition and multiplication on the field. This approach allows us to develop in Section 4 the notion of difference compatible valuations, introduce the difference rank, and consider in particular isometries, weak isometries and $\omega$-increasing automorphisms. The main results of the paper are in the last Section 5: Theorem 5.2 and its Corollaries 5.3 and 5.4.

Finally, we note that some of our notions and results generalize to the context of an arbitrary (not necessarily ordered) valued field, see [3].

2. Preliminaries on the rank and principal rank of an ordered field

We begin by the following key observation:
For every convex valuation \( w \) a valuation compatible with the order is thus also said to be a convex valuation.

Let \( K \) be a totally ordered set, and assume that \( w \) is order preserving, i.e. \( a \leq a' \) implies \( \varphi(a) \leq \varphi(a') \), for all \( a, a' \in S \). We assume further that \( \varphi \) has an orientation, i.e. \( \varphi(a) \geq a \) for all \( a \in S \) (\( \varphi \) is a right shift) or \( \varphi(a) \leq a \) for all \( a \in S \) (\( \varphi \) is a left shift). We set \( \varphi^0(a) := a \) and \( \varphi^{n+1}(a) := \varphi(\varphi^n(a)) \) for \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

It is then straightforward that the following defines an equivalence relation on \( S \):
1. If \( \varphi \) is a right shift, set \( a \sim \varphi a' \) if and only if there is some \( n \in \mathbb{N}_0 \) such that \( \varphi^n(a) \geq a' \) and \( \varphi^n(a') \geq a \).
2. If \( \varphi \) is a left shift, set \( a \sim \varphi a' \) if and only if there is some \( n \in \mathbb{N}_0 \) such that \( \varphi^n(a) \leq a' \) and \( \varphi^n(a') \leq a \).

The equivalence classes \([a]_\varphi\) of \( \sim_\varphi \) are convex and closed under application of \( \varphi \). By the convexity, the order on \( S/\sim_\varphi \) induces an order on \( S \) such that \([a]_\varphi < [b]_\varphi\) if and only if \( a' < b' \) for all \( a' \in [a]_\varphi \) and \( b' \in [b]_\varphi \).

On the negative cone \( G^{<0} \) of an ordered abelian group \( G \), the archimedean equivalence relation \( \sim_\varphi \) is obtained by setting \( \varphi(a) := 2a \), and \( v_G \) is the map \( a \mapsto [a]_\varphi \). The order on \( G := G^{<0}/\sim_\varphi \) is the one induced by the order of \( G^{<0} \) as above. We call \( v_G(G^{<0}) := \Gamma \) the value set of \( G \). By convention we also write \( v_G(G) := \Gamma \) extending the archimedean equivalence relation to the positive cone of \( G \) by setting \( v_G(g) := v_G(-g) \). The natural valuation \( v_G \) on \( G \) satisfies the ultrametric triangle inequality, and in particular we have: \( v_G(x + y) = \min\{v_G(x), v_G(y)\} \) if \( \text{sign}(x) = \text{sign}(y) \).

We gather some basic facts about valuations compatible with the order of an ordered field. Throughout this paper, \( K \) will be a non-archimedean ordered field, and \( v \) will denote its non-trivial natural valuation, that is, its valuation ring \( R_v \) is the convex hull of \( \mathbb{Q} \) in \( K \). We set: \( P_K := K^{>0} \setminus R_v \), \( G = v(K) \) and \( \Gamma = v_G(G^{<0}) \).

The natural valuation on \( K \) satisfies \( v(x + y) = \min\{v(x), v(y)\} \) if \( \text{sign}(x) = \text{sign}(y) \) and for all \( a, b \in K : a \geq b > 0 \ \Rightarrow \ v(a) \leq v(b) \).

Let \( w \) be a valuation of \( K \), with valuation ring \( R_w \), valuation ideal \( I_w \), value group \( w(K) \) and residue field \( Kw \). Then \( w \) is called compatible with the order if and only if it satisfies, for all \( a, b \in K : a \geq b > 0 \ \Rightarrow \ w(a) \leq w(b) \).

For the following characterizations of compatible valuations, see [4] Proposition 5.1, or [5] Theorem 2.3 and Proposition 2.9, or [7] Lemma 3.2.1, or [8] Lemma 7.2:

**Lemma 2.2.** The following assertions are equivalent:
1) \( w \) is a valuation compatible with the order of \((K,<)\),
2) \( R_w \) is a convex subset of \((K,<)\),
3) \( I_w \) is a convex subset of \((K,<)\),
4) \( I_w < 1 \),
5) the image of the positive cone \( P \) of \((K,<)\) under the residue map \( K \ni a \mapsto aw \in Kw \) is a positive cone \( Pw \) in \( Kw \).

A valuation compatible with the order is thus also said to be a convex valuation. For every convex valuation \( w \), we denote by \( U_w^{>0} := \{a \in K \mid w(a) = 0 \land a > 0\} \) the group of positive units of \( R_w \). It is a convex subgroup of the ordered multiplicative group \((K^{>0},,1,<)\) of positive elements of \( K \).
Let \( w \) and \( w' \) be valuations on \( K \). We say that \( w' \) is finer than \( w \), or that \( w \) is coarser than \( w' \) if \( R_w \subsetneq R_{w'} \). This is equivalent to \( I_w \subsetneq I_{w'} \). By Lemma 2.2 4), if \( w' \) is convex, then \( w \) also is convex. Conversely, a convex subring containing 1 is a valuation ring, see [1] Section 2.2.2. The natural valuation \( v \) of \( K \) is the finest convex valuation (and is characterized by the fact that its residue field is archimedean). The set \( \mathcal{R} \) of all valuation rings \( R_w \) of convex valuations \( w \neq v \) (i.e. all corings of \( v \)) is totally ordered by inclusion. Its order type is called the rank of \((K, +, \cdot, 0, 1, <)\). For convenience, we will identify it with \( \mathcal{R} \). For example, the rank of an archimedean ordered field is empty since its natural valuation is trivial.

The rank of the rational function field \( K, \cdot \) is (isomorphic to) the rank of \( R \). The correspondence \( \gamma \mapsto \gamma_0 \) is order preserving (and is characterized by the fact that its residue field is archimedean). The set \( \mathcal{R} \) of all valuation rings \( R_w \) of convex valuations \( w \neq v \) is totally ordered by inclusion. Its order type is called the rank of \( G \) (respectively of an ordered group) is an isomorphism invariant. To every convex valuation ring \( R_w \), we associate a convex subgroup \( G_w := \{ v(a) \mid a \in K \land w(a) = 0 \} = v(U_w^{>0}) \). We call \( G_w \) the convex subgroup associated to \( w \). Note that \( G_v = \{ 0 \} \). Conversely, given a convex subgroup \( G_w \) of \( v(K) \), we define \( w : K \to v(K)/G_w \) by \( w(a) = v(a) + G_w \). Then \( w \) is a convex valuation with \( v(U_w^{>0}) = G_w \) (and \( v \) is finer than \( w \) if and only if \( G_w \neq \{ 0 \} \)). We call \( w \) the convex valuation associated to \( G_w \). We summarize:

**Lemma 2.3.** The correspondence \( R_w \mapsto G_w \) is an order preserving bijection, thus \( \mathcal{R} \) is (isomorphic to) the rank of \( G \).

We want to analyze the rank of \( G \) further and relate it to the value set \( \Gamma \) of \( G \). For \( G_w \neq \{ 0 \} \) a convex subgroup, we associate \( \Gamma_w := v_G(G_w^{>0}) \) a non-empty final segment of \( \Gamma \). Conversely, if \( \Gamma_w \) is a non-empty final segment of \( \Gamma \), then \( G_w = \{ g \mid g \in G, v_G(g) \in \Gamma_w \} \cup \{ 0 \} \) is a convex subgroup, with \( \Gamma_w = v_G(G_w) \).

Let us denote by \( \Gamma^{fs} \) the set of non-empty final segments of \( \Gamma \), totally ordered by inclusion. We summarize:

**Lemma 2.4.** The correspondence \( G_w \mapsto \Gamma_w \) is an order preserving bijection, thus the rank of \( G \) is (isomorphic to) \( \Gamma^{fs} \).

**Theorem 2.5.** The correspondence \( R_w \mapsto \Gamma_w \) is an order preserving bijection, thus \( \mathcal{R} \) is (isomorphic to) \( \Gamma^{fs} \).

A final segment which has a smallest element is a principal final segment. Let \( \Gamma^* \) denote the set \( \Gamma \) with its reversed ordering, then \( \Gamma^* \) is (isomorphic to) the totally ordered set of principal final segments:

**Lemma 2.6.** The map from \( \Gamma \) to \( \Gamma^{fs} \) defined by \( \gamma \mapsto \{ \gamma' \mid \gamma' \in \Gamma, \gamma' \geq \gamma \} \) is an order reversing embedding. Its image is the set of principal final segments.

Recall that a convex subgroup \( G_w \) of \( G \) is called principal generated by \( g, g \in G \), if \( G_w \) is the minimal convex subgroup containing \( g \). The principal rank of \( G \) is the subset of the rank of \( G \) consisting of all principal \( G_w \neq \{ 0 \} \).

**Lemma 2.7.** Let \( G_w \neq \{ 0 \} \) be a convex subgroup. Then \( G_w \) is principal if and only if \( v_G(G_w) = \Gamma_w \) is a principal final segment.
Lemma 2.8. The map \( G_w \mapsto \min v_G(G_w) \) is an order reversing bijection. Thus the principal rank of \( G \) is (isomorphic to) \( \Gamma^* \).

A convex subring \( R_w \neq R_v \) is a **principal convex subring** generated by \( a \) for \( a \in P_K \) if \( R_w \) is the smallest convex subring containing \( a \). The **principal rank** of \( K \) is the subset \( R_{pr} \) of \( R \) consisting of all principal \( R_w \in R \).

**Theorem 2.9.** The correspondence \( R_w \mapsto \Gamma_w \) is an order preserving bijection between \( R_{pr} \) and the principal rank of \( G \), thus \( R_{pr} \) is (isomorphic to) \( \Gamma^* \).

### 3. The Rank and Principal Rank via Equivalence Relations

In this section, we exploit Remark 2.1 to give an interpretation of the rank and principal rank as quotients via an appropriate equivalence relation, thereby providing alternative proofs for Theorem 2.5 and Theorem 2.9. It is precisely this approach that we will generalize to the difference rank in the next sections.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
P_K & \xrightarrow{\varphi} & P_K \\
\mid & v \mid & v \\
G^<0 & \xrightarrow{\varphi_G} & G^<0 \\
\mid & \mid & \mid \\
v_G & \xrightarrow{\varphi_G} & v_G \\
v_G(G) & \xrightarrow{\varphi_G} & v_G(G)
\end{array}
\]

with \( \varphi(a) := a^2 \) for all \( a \in P_K \),

\[ \varphi_G(v(a)) := v(\varphi(a)) \text{ for all } a \in P_K, \]

that is \( \varphi_G(g) = 2g \) for all \( g \in G^<0 \), and

\[ \varphi_G(v_G(g)) := v_G(\varphi_G(g)) \text{ for all } g \in G^{<0}, \]

that is \( \varphi_G(\gamma) = \gamma \) for all \( \gamma \in \Gamma \), so that \( \varphi_G \) is just the identity map.

We consider the equivalence relations associated to the monotone maps \( \varphi \), \( \varphi_G \) and \( \varphi_G \) as in Remark 2.1 and note that \( \sim_{\varphi} \) is just multiplicative equivalence \( \sim^* \) on \( P_K \), \( \sim_{\varphi} \) just archimedean equivalence on \( G \) and \( \sim_{\varphi} \) just equality on \( \Gamma \). Suppose that \( \sim_1 \) and \( \sim_2 \) are two equivalence relations defined on the same set. Recall that \( \sim_1 \) is said to be **coarser** than \( \sim_2 \) if \( \sim_2 \)-equivalence implies \( \sim_1 \)-equivalence. The following straightforward observation will be useful for the proof of Theorem 3.2 below:

**Lemma 3.1.** The equivalence relation \( \sim_{\varphi} \) is coarser than the archimedean equivalence relation with respect to addition on \( P_K \). The equivalence classes of \( \sim_{\varphi} \) are closed under addition and multiplication.

We further note that

\[ \varphi^n_G(v(a)) = v(\varphi^n(a)) \text{ and } \varphi^n_G(v_G(g)) = v_G(\varphi^n_G(g)) \]

thus

\[ a \sim_{\varphi} a' \text{ if and only if } v(a) \sim_{\varphi_G} v(a') \text{ if and only if } v_G(v(a)) \sim_{\varphi_G} v_G(v(a')). \]
Thus we have an order reversing bijection from $P_K/\sim_{\varphi}$ onto $\Gamma/\sim_{\varphi} = \Gamma$. Thus the chain $[P_K/\sim_{\varphi}]_{\mathfrak{is}}$ of initial segments of $P_K/\sim_{\varphi}$ ordered by inclusion is isomorphic to $\Gamma_{\mathfrak{is}}$. Theorems 2.3 and 2.9 will therefore immediately follow from the following result

**Theorem 3.2.** The rank $R$ is isomorphic to $[P_K/\sim_{\varphi}]_{\mathfrak{is}}$ and the principal rank $R^{pr}$ is isomorphic to the subset of $[P_K/\sim_{\varphi}]_{\mathfrak{is}}$ of initial segments which have a last element.

**Proof.** First we note that if $R_w$ is a convex valuation ring, then clearly $R_w^{>0} \setminus R_v^{>0}$ is an initial segment of $P_K$, and moreover $[a]_{\sim_{\varphi}}$ is the last class in case $R_w$ is principal generated by $a$. Furthermore, if $R_w$ intersects an equivalence class $[a]_{\sim_{\varphi}}$ then it must contain it, since the sequence $a^n; n \in \mathbb{N}_0$ is cofinal in $[a]_{\sim_{\varphi}}$ and $R_w$ is a convex subring. We conclude that $(R_w^{>0} \setminus R_v^{>0})/\sim_{\varphi}$ is an initial segment of $P_K/\sim_{\varphi}$. Conversely set $I_w = \{[a]_{\varphi} | a \in R_w^{>0} \setminus R_v^{>0}\}$. Given $I \in [P_K/\sim_{\varphi}]_{\mathfrak{is}}$, we show that there is a convex valuation ring $R_w$ such that $I_w = I$. Given $I$, let $(\bigcup I)$ denote the set theoretic union of the elements of $I$ and $-(\bigcup I)$ the set of additive inverses. Set $R_w = -(\bigcup I) \cup R_v \cup (\bigcup I)$. We claim that $R_w$ is the required ring. Clearly, $I_w = I$. Further $R_w$ is convex (by its construction), and strictly contains $R_v$. We leave it to the reader, using Lemma 3.1, to verify that $R_w$ is a ring, and that $R_w$ is principal generated by $a$ if $[a]_{\sim_{\varphi}}$ is the last element of $I$. \hfill $\square$

Note that the principal rank determines the rank, that is if ordered fields have (isomorphic) principal ranks, then they have (isomorphic) ranks.

4. **The difference analogue of the rank**

In this section, we develop a difference analogue of what has been reviewed above. That is, we develop a theory of difference compatible valuations, in analogy to the theory of convex valuations. The automorphism will play the role that multiplication plays in the previous case.

Let $\sigma$ be an order preserving automorphism of $K$. We assume $\sigma \neq$ identity, i.e. $(K, <, \sigma)$ is a non-trivial ordered difference field. Note that $\sigma$ satisfies for all $a, b \in K : v(a) \leq v(b)$ if and only if $v(\sigma(a)) \leq v(\sigma(b))$ and thus induces an order preserving automorphism $\sigma_G$ and $\sigma_T$ such that the following diagram commutes:

$$
\begin{array}{ccc}
P_K & \xrightarrow{\sigma} & P_K \\
v & \xrightarrow{\sim} & v \\
G^{<0} & \xrightarrow{\sigma_G} & G^{<0} \\
v_G & \xrightarrow{\sim} & v_G \\
v_G(G) & \xrightarrow{\sigma_T} & v_G(G)
\end{array}
$$

with $\sigma_G(v(a)) := v(\sigma(a))$ for all $a \in P_K$, and

$\sigma_T(v_G(g)) := v_G(\sigma_G(g))$ for all $g \in G^{<0}$. 

Now let $w$ be a convex valuation on $K$. Say $w$ is \emph{$\sigma$-compatible} if for all $a, b \in K$:

$$w(a) \leq w(b) \quad \text{if and only if} \quad w(\sigma(a)) \leq w(\sigma(b)).$$

The subset $R_\sigma := \{ R_w \in R : R_w \text{ is } \sigma\text{-compatible} \}$ is the \emph{$\sigma$-rank} of $(K, \prec, \sigma)$. Similarly, the subset of all convex subgroups $G_w \neq \{0\}$ such that $\sigma_G(G_w) = G_w$ ($\sigma_G$-invariant) is the \emph{$\sigma$-rank} of $G$. Finally, we denote by $\sigma_{T}\Gamma^{fs}$ the subset of final segments $\Gamma_w$ such that $\sigma_T(\Gamma_w) = \Gamma_w$.

The following analogues of Lemmas 2.2, 2.3 and 2.4 are verified by straightforward computations, using basic properties of valuations rings on the one hand and of automorphisms on the other (e.g. $\sigma(A) \subseteq B$ if and only if $A \subseteq \sigma^{-1}(B)$ and $\sigma(A) \subseteq B$ if and only if $\sigma(-A) \subseteq -B$):

\textbf{Lemma 4.1.} The following assertions are equivalent for a convex valuation $w$:

1) $w$ is $\sigma$-compatible
2) $w$ is $\sigma^{-1}$-compatible
3) $\sigma(R_w) = R_w$
4) $\sigma(I_w) = I_w$
5) $\sigma(R_{\sigma}^{-0} \setminus R_{\sigma}^{<0}) = R_{\sigma}^{<0} \setminus R_{\sigma}^{\geq0}$
6) the map $Kw \to Kw$ defined by $aw \mapsto \sigma(a)w$ is well-defined and is an order preserving automorphism of $Kw$ (with the ordering induced by $Pw$).

We shall call $R_w$ $\sigma$-compatible if any of the above equivalent conditions holds.

\textbf{Lemma 4.2.} The correspondence $R_w \mapsto G_w$ is an order preserving bijection from $R_\sigma$ onto the $\sigma_G$-rank of $G$.

\textbf{Lemma 4.3.} The correspondence $G_w \mapsto \Gamma_w$ is an order preserving bijection from the $\sigma_G$-rank of $G$ onto $\sigma_T\Gamma^{fs}$.

We deduce from Lemmas 4.2 and 4.3 the following information. An automorphism $\sigma$ is an \emph{isometry} if $v(\sigma(a)) = v(a)$ for all $a \in K$, equivalently $\sigma_G$ is the identity automorphism, and a \emph{weak isometry} if $\sigma_T$ is the identity automorphism. Every isometry is a weak isometry. Note that if $\Gamma$ is a rigid chain, then $\sigma$ is necessarily a weak isometry. If $\sigma$ is a weak isometry, then $\sigma_T(v_G(g)) = v_G(\sigma_G(g)) = v_G(g)$, thus $g$ is archimedean equivalent to $\sigma_G(g)$ for all $g$, and so every convex subgroup is $\sigma_G$-invariant.

\textbf{Corollary 4.4.} If $\sigma$ is a weak isometry, then $R_\sigma = R$.

\textbf{Corollary 4.5.} The correspondence $R_w \mapsto \min \Gamma_w$ is an order (reversing) isomorphism from $R_\sigma \cap R^{pr}$ onto the chain $\{ \gamma : \sigma_T(\gamma) = \gamma \}$ of fixed points of $\sigma_T$.

At the other extreme $\sigma$ is said to be $\omega$-\emph{increasing} if $\sigma(a) > a^n$ for all $n \in N_0$ and all $a \in P_K$.

\textbf{Remark 4.6.} Note that $\sigma$ is $\omega$-increasing if and only if $\sigma_T$ is a \emph{strict left shift}, that is, $\sigma_T(\gamma) < \gamma$ for all $\gamma \in \Gamma$. Thus if $\sigma$ $\omega$-increasing then $\sigma_T$ has no fixed points.

\textbf{Corollary 4.7.} If $\sigma$ is $\omega$-increasing then $R_\sigma \cap R^{pr}$ is empty.

Recall that the Hahn group over the chain $\gamma$ and components $R$, denoted $H_T.R$, is the totally ordered abelian group whose elements are formal sums $g = \sum g_\gamma 1_\gamma$, \ldots
with well-ordered support \( g := \{ \gamma : g_\gamma \neq 0 \} \). Addition is pointwise and the order lexicographic. Similarly, the the field of generalized power series over the ordered abelian group \( G \) (or Hahn field over \( G \)), denoted \( \mathbb{R}((G)) \), is the totally ordered field whose elements are formal series \( s := \sum g_t t^\gamma \), with well-ordered support \( s := \{ g : s_g \neq 0 \} \). Addition is pointwise, multiplication is given by the usual convolution formula, and the order is lexicographic.

**Lemma 4.8.** Any order preserving automorphism \( \sigma_G \) of the chain \( \Gamma \) lifts to an order preserving automorphism \( \sigma_G \) of the Hahn group \( G \) over \( \Gamma \), and \( \sigma_G \) lifts in turn to an order preserving automorphism \( \sigma \) of the Hahn field over \( G \).

**Proof.** Set \( \sigma_G(\sum g_\gamma 1_\gamma) := \sum g_\gamma 1_{\sigma_G(\gamma)} \) and \( \sigma(\sum s_g t^\gamma) := \sum s_t t^{\sigma_G(g)} \).

**Corollary 4.9.** Given any order type \( \tau \) there exists a maximally valued non-trivial ordered difference field \( (K, <, \sigma) \), such that the order type of \( R_{\sigma} \cap R_{\sigma}^{pr} \) is \( \tau \).

**Proof.** Set \( \mu := \tau^* \), and consider e.g. the linear ordering \( \Gamma := \sum_\mu \mathbb{Q}^\geq 0 \), that is, the concatenation of \( \mu \) copies of the non-negative rationals. Fix a non-trivial order automorphism \( \eta \) of \( \mathbb{Q}^{> 0} \). Define \( \tau_\Gamma \) to be the uniquely defined order automorphism of \( \Gamma \) fixing every \( 0 \in \mathbb{Q}^{\geq 0} \) in every copy and extending \( \eta \) otherwise on every copy. Set e.g. \( G := H_{\Gamma, \mathbb{R}} \). Then \( \sigma_G \) lifts canonically to \( \sigma_G \) on \( G \). Now set e.g. \( K := \mathbb{R}((G)) \). Again \( \sigma_G \) lifts canonically to an order automorphism of \( K \), our required \( \sigma \).

In the next section, we will exploit appropriate equivalence relations to define the principal difference rank and construct difference fields of arbitrary difference rank.

5. The principal \( \sigma \)-rank

Our aim now is to state and prove the analogues to Theorems 3.2, 2.6 and 2.9. However, scrutinizing the proof of Theorem 3.2, we quickly realize that in order to obtain an analogue of Lemma 3.1 (which is essential for the arguments), we need further condition on \( \sigma \). Thus from now on we will assume that \( \sigma(a) \geq a^2 \) for all \( a \in P_K \). It follows by induction that \( \sigma^n(a) \geq a^{2^n} \). Thus given \( n \in \mathbb{N}_0 \), there exists \( l \in \mathbb{N}_0 \) such that \( \sigma^l(a) \geq a^n \). Note that our condition on \( \sigma \) is fulfilled for \( \omega \)-increasing automorphism.

A convex subring \( R_\omega \neq R_v \) is \( \sigma \)-principal generated by \( a \) for \( a \in P_K \) if \( R_\omega \) is the smallest convex \( \sigma \)-compatible subring containing \( a \). The \( \sigma \)-principal rank of \( K \) is the subset \( R_\omega^{pr} \) of \( R_\sigma \) consisting of all \( \sigma \)-principal \( R_\omega \in R \). The maps \( \sigma, \sigma_G \) and \( \sigma_\Gamma \) are order preserving and we can define the corresponding equivalence relations \( \sim_\sigma, \sim_{\sigma_G}, \sim_{\sigma_\Gamma} \). As before we have

\[ a \sim_\sigma a' \] if and only if \( v(a) \sim_{\sigma_G} v(a') \) if and only if \( v_G(v(a)) \sim_{\sigma_\Gamma} v_G(v(a')) \).

Thus we have an order reversing bijection from \( P_K/\sim_\sigma \) onto \( \Gamma/\sim_{\sigma_\Gamma} \). Thus the chain \( [P_K/\sim_\sigma]^{\omega \rightarrow \omega} \) of initial segments of \( P_K/\sim_\sigma \) ordered by inclusion is isomorphic to \( [\Gamma/\sim_{\sigma_\Gamma}]^{\omega \rightarrow \omega} \).

**Lemma 5.1.** (i) The equivalence relation \( \sim_\sigma \) is coarser than the archimedean equivalence relations with respect to addition and multiplication on \( P_K \).

(ii) The equivalence classes of \( \sim_\sigma \) are thus closed under addition, multiplication and \( \sigma \).
8. SALMA KUHLMANN, MICKAËL MATUSINSKI, AND FRANÇOISE POINT

Proof. If $a$ is archimedean equivalent to $b$ then $v(a) = v(b)$ so $v(a) \sim_{G} v(b)$ certainly and therefore $a \sim_{F} b$. If $a$ is multiplicatively equivalent to $b$ so that $a^{n} \geq b$ and $b^{n} \geq a$ for some $n \in \mathbb{N}$, then choose $l$ large enough so that $\sigma^{l}(a) \geq a^{n}$ and $\sigma^{l}(b) \geq b^{n}$. Clearly, the condition on $\sigma$ implies that $a \sim_{F} \sigma(a)$. Recall that the natural valuation on $K$ satisfies $v(x + y) = \min\{v(x), v(y)\}$ if $\text{sign}(x) = \text{sign}(y)$. One easily deduces from this fact and (i) that the equivalence classes of $\sigma$ are closed under addition. Similarly, the natural valuation $v_{G}$ on $G$ satisfies $v_{G}(x + y) = \min\{v_{G}(x), v_{G}(y)\}$ if $\text{sign}(x) = \text{sign}(y)$. Again one easily deduces from this fact and (i) that the equivalence classes of $\sigma$ are closed under multiplication. □

We can now prove:

**Theorem 5.2.** The $\sigma$-rank $\mathcal{R}_{\sigma}$ is isomorphic to $[\mathbf{P}_{K} / \sim_{\sigma}]^{\text{is}}$ and the principal $\sigma$-rank $\mathcal{R}_{\sigma}^{pr}$ is isomorphic to the subset of $[\mathbf{P}_{K} / \sim_{\sigma}]^{\text{is}}$ of initial segments which have a last element.

Proof. First we note that if $R_{w}$ is a convex $\sigma$-compatible valuation ring, then clearly $R_{w}^{>0} \cap \mathbf{P}^{>0}$ is an initial segment of $\mathbf{P}_{K}$. Furthermore, if $R_{w}$ intersects a $\sigma$-equivalence class $[a]_{\sim_{\sigma}}$ then it must contain it, since the sequence $\sigma(a)^{n}; n \in \mathbb{N}$ is cofinal in $[a]_{\sim_{\sigma}}$ and $R_{w}$ is a convex subring. We conclude that $(R_{w}^{>0} \cap \mathbf{P}^{>0})/\sim_{\sigma}$ is an initial segment of $\mathbf{P}_{K}$ and moreover $[a]_{\sim_{\sigma}}$ is the last class in case $R_{w}$ is $\sigma$-principal generated by $a$. Conversely set $\mathcal{I}_{w} = \{[a]_{\sigma} | a \in R_{w}^{>0} \cap \mathbf{P}^{>0}\}$. Given $\mathcal{I} \in [\mathbf{P}_{K} / \sim_{\sigma}]^{\text{is}}$, we show that there is a $\sigma$-compatible convex valuation ring $R_{w}$ such that $\mathcal{I}_{w} = \mathcal{I}$. Given $\mathcal{I}$, let $(\bigcup \mathcal{I})$ denote the set theoretic union of the elements of $\mathcal{I}$ and $-(\bigcup \mathcal{I})$ the set of additive inverses. Set $R_{w} = - (\bigcup \mathcal{I}) \cup R_{w} \cup (\bigcup \mathcal{I})$. We claim that $R_{w}$ is the required ring. Clearly, $\mathcal{I}_{w} = \mathcal{I}$. Further $R_{w}$ is convex (by its construction), and strictly contains $R_{w}$. We leave it to the reader, using Lemma [1.1] to verify that $R_{w}$ is a $\sigma$-compatible subring, and that $R_{w}$ is $\sigma$-principal generated by $a$ if $[a]_{\sim_{\sigma}}$ is the last element of $\mathcal{I}$. □

**Corollary 5.3.** $\mathcal{R}_{\sigma}$ is (isomorphic to) $(\Gamma / \sim_{\sigma})^{\text{is}}$.

**Corollary 5.4.** $\mathcal{R}_{\sigma}^{pr}$ is (isomorphic to) $(\Gamma / \sim_{\sigma})^{*}$.

We now can $\omega$-increasing automorphisms of arbitrary principal difference rank:

**Corollary 5.5.** Given any order type $\tau$ there exists a maximally valued ordered field endowed with an $\omega$-increasing automorphism of principal difference rank $\tau$.

Proof. Set $\mu := \tau^{*}$, and consider e.g. the linear ordering $\Gamma := \sum_{\mu} \mathbb{Q}$, that is, the concatenation of $\mu$ copies of the non-negative. Let $\ell$ be e.g. translation by $-1$ on $\mathbb{Q}$. Define $\sigma_{\tau}$ to be the uniquely defined order automorphism of $\Gamma$ extending $\ell$ on every copy. It is a left shift. Set e.g. $G := H_{\mathbb{Q}}\mathbb{R}$. Then $\sigma_{\tau}$ lifts canonically to $\sigma_{G}$ on $G$. Now set e.g. $K := \mathbb{R}((G))$. Again $\sigma_{G}$ lifts canonically to an order automorphism of $K$, our required $\sigma$. □

**References**

[1] Engler, A. J. and Prestel, A.: *Valued Fields*, Springer Monographs in Math. (2005)
[2] Kuhlmann, S.: *Ordered exponential fields*, Fields Institute Monographs 12 (2000)
[3] Kuhlmann, S., Matusinski, M. and Point, F.: *The automorphism group of a valued field*, preprint (2013)
[4] Lam, T. Y.: *The theory of ordered fields*, in: Ring Theory and Algebra III (ed. B. McDonald), Lecture Notes in Pure and Applied Math. 55 Dekker, New York (1980), 1–152
THE VALUATION DIFFERENCE RANK OF AN ORDERED DIFFERENCE FIELD

[5] Lam, T. Y.: *Orderings, valuations and quadratic forms*, Amer. Math. Soc. Regional Conference Series in Math. 52, Providence (1983)

[6] Lang, S.: *The theory of real places*, Ann. Math. 57 (1953), 378–391

[7] Prieß-Crampe, S.: *Angeordnete Strukturen. Gruppen, Körper, projektive Ebenen*, Ergebnisse der Mathematik und ihrer Grenzgebiete 98, Springer (1983)

[8] Prestel, A.: *Lectures on Formally Real Fields*, Springer Lecture Notes in Math. 1093 (1984)

Universität Konstanz, FB Mathematik und Statistik, 78457 Konstanz, Germany

E-mail address: salma.kuhlmann@uni-konstanz.de

IMB, Université Bordeaux 1, 33405 Talence, France

E-mail address: mmatusin@math.u-bordeaux1.fr

Institut de mathématique, Le Pentagone, Université de Mons, B-7000 Mons, Belgium

E-mail address: Francoise.Point@umons.ac.be