ISOPARAMETRIC POLYNOMIALS AND SUMS OF SQUARES

JIANQUAN GE AND ZIZHOU TANG∗

Abstract. Hilbert’s 17th problem asks that whether every nonnegative polynomial can be a sum of squares of rational functions. It has been answered affirmatively by Artin. However, the question as to whether a given nonnegative polynomial is a sum of squares of polynomials is still a central question in real algebraic geometry.

In this paper, we solve this question completely for the nonnegative polynomials associated with isoparametric polynomials, initiated by E. Cartan, which define the focal submanifolds of the corresponding isoparametric hypersurfaces.

1. Introduction

A real polynomial \( p(x) \) in \( n \) variables is called positive semidefinite (psd for short) or nonnegative if \( p(x) \geq 0 \) for all \( x \in \mathbb{R}^n \); it is called a sum of squares (sos) if there exist real polynomials \( h_j \) such that \( p = \sum h_j^2 \). It is a central question in real algebraic geometry whether a given psd polynomial is sos (cf. \[] and references therein). As any psd or sos polynomial can be made homogeneous by adding an extra variable which preserves psd or sos, it is convenient to work with homogeneous polynomials (forms). Let \( P_{n,d} \) and \( \Sigma_{n,d} \) denote the sets of psd and sos forms in \( n \) variables of even degree \( d \), respectively. In this terminology, it is clear that \( P_{n,d} \supseteq \Sigma_{n,d} \) and the question above asks whether or when a psd form \( p(x) \in P_{n,d} \) belongs to \( \Sigma_{n,d} \).

The question above goes back to Minkowski’s thesis defence in 1885. It was Hilbert \[28\] who showed that the equality \( P_{n,d} = \Sigma_{n,d} \) holds only in the following four cases:

\[
P_{n,2} = \Sigma_{n,2}, \quad P_{1,d} = \Sigma_{1,d}, \quad P_{2,d} = \Sigma_{2,d}, \quad P_{3,4} = \Sigma_{3,4}.
\]

It follows that there exists a psd but non-sos form \( p(x) \in P_{n,d} \setminus \Sigma_{n,d} \) if \( n \geq 3 \) and \( d \geq 6 \), or \( n \geq 4 \) and \( d \geq 4 \). Hilbert’s proof used complex algebraic curves, and had no explicit example of a psd polynomial that is non-sos. 77 years later, such an example was first constructed by Motzkin \[35\]. Since then many scattered examples were constructed by Robinson, Choi-Lam, Lax-Lax, Schm"udgen, and Reznick, etc.
Algorithms were also studied extensively and applied to many aspects like optimization theory, robotics and even self-driving cars (cf. [2]). In the smallest cases: \((n, d) = (3, 6)\) and \((4, 4)\), Blekherman [5] first gave a complete unified geometric description of the difference between \(psd\) and \(sos\) forms.

After the above remarkable theorem, Hilbert [29] showed that any \(psd\) form in \(P_{3,d}\) \((d \geq 6)\) is a sum of squares of rational functions instead of polynomials. He then posed his famous Hilbert’s 17th Problem in 1900 ICM: Must every \(psd\) form be a sum of squares of rational functions (\(sosr\) for short)? This was answered affirmatively by Artin [3] using orderings of fields. However, Artin’s proof gives no specific \(sosr\) representation of a \(psd\) form. Uniform denominators \(|x|^{2r}\) with sufficiently large \(r\) were proved to exist for positive definite forms by Pólya and Reznick, i.e., if \(p(x) \in \mathbb{P}_{a,d}\) and \(p(x) > 0\) whenever \(x \neq 0\), then \(|x|^{2r}p(x) \in \Sigma_{n,d}\). There are also many nonnegative, non-sos polynomials with zeroes are known to become sos, after multiplying by \(|x|^{2r}\). This holds for instance for the Motzkin and Robinson polynomials (see Reznick [12]). On the other hand, there exist nonnegative polynomials \(f\) that will never become sum of squares after multiplying by \(|x|^{2r}\) (and in fact by any positive polynomial). This is due to existence of so-called “bad points” (see Reznick [12], page 16). Such a polynomial \(f\) with bad points was given by Delzell

\[
D(w,x,y,z) := w^2(x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2) + z^8 \in \mathbb{P}_{4,8},
\]

which has no \(sosr\) representation with a uniform denominator \(|x|^{2r}\) for any \(r\). One way to establish that a polynomial \(f\) of degree \(2d\) is not sos is to show that no polynomial of degree \(d\) vanishes on the zero-set of \(f\). This will be used in the proof of Theorems [12] [34] [35] [42] [43] for some classes of isoparametric polynomials. However, this technique no longer applies after multiplication by \(|x|^{2r}\). For more history and developments we refer to the wonderful surveys [12] [43] by Reznick and [7] by Bochnak-Coste-Roy.

In this paper we mainly consider the problem on a series of specific \(psd\) forms with significant geometric background, namely, isoparametric polynomials. It originated from the study of isoparametric hypersurfaces in unit spheres by E. Cartan [8] in the 1930s. Through a long history of efforts (e.g., [36] [37] [20] [1] [17] [16] [18] [45] [9] [30] [33] [34] [11] [12] [13], etc.), isoparametric hypersurfaces in unit spheres have been completely classified up to isometry. Equivalently, the isoparametric polynomials on Euclidean spaces have been completely classified up to orthogonal transformations.

A hypersurface of a Riemannian manifold is called isoparametric if its nearby parallel hypersurfaces have constant mean curvature, or equivalently, it is locally a regular level set of an isoparametric function \(f\) (i.e., \(|\nabla f|^2\) and \(\Delta f\) are functions of \(f\), cf. [21] [41]), or a regular leaf of an isoparametric foliation (i.e., a singular Riemannian foliation of codimension 1 with constant mean curvature regular leaves, cf. [21] [23].
In unit spheres (or real space forms), Cartan showed that a hypersurface is isoparametric if and only if it has constant principal curvatures.

A fundamental result of Münzner [36] states that an isoparametric hypersurface \( M \) in a unit sphere \( S^{n-1} \) is an open part of a level hypersurface of an isoparametric function \( f = F|_{S^{n-1}} \). Here \( F \), called a Cartan-Münzner polynomial (or isoparametric polynomial), is a homogeneous polynomial of degree \( g \) on \( \mathbb{R}^n \) satisfying the Cartan-Münzner equation:

\[
\begin{align*}
|\nabla F|^2 &= g^2|x|^{2g-2}, \\
\Delta F &= \frac{g^2}{2}(m_- - m_+)|x|^{g-2},
\end{align*}
\]  

where \( \nabla F, \Delta F \) denote the gradient and Laplacian of \( F \) on \( \mathbb{R}^n \), respectively, \( m_\pm \) denotes the multiplicities of the maximal and minimal principal curvatures of \( M \) with respect to the normal direction \( \nabla f / |\nabla f| \), and \( g = \deg(F) \) is equal to the number of distinct principal curvatures of \( M \).

It is easy to see that \( |\nabla f|^2 = g^2(1 - f^2) \) on the unit sphere \( S^{n-1} \), and thus \( \text{Image}(f) = [-1, 1] \), \( f^{-1}(t), t \in (-1, 1) \), is a regular level set (thus an isoparametric hypersurface) and \( f^{-1}(\pm 1) =: M_\pm \) are smooth submanifolds, called focal submanifolds, of codimension \( m_\pm + 1 \) in \( S^{n-1} \). In fact, given an isoparametric hypersurface \( M \) in \( S^{n-1} \), it is clear that \( M \) has exactly two focal submanifolds, say \( M_\pm \). One then defines the corresponding isoparametric function \( f \) on \( S^{n-1} \) by \( f(x) := \cos(g \text{dist}(x, M_\pm)) \), where \( \text{dist}(x, M_\pm) \) is the spherically oriented distance from \( x \) to the focal submanifold \( M_\pm \) of \( M \). We remark that if one takes \( M_- \) instead of \( M_+ \), the corresponding function becomes \(-f\). It turns out that the function

\[
F(x) := |x|^g f(x/|x|) = |x|^g \cos(g \text{dist}(x/|x|, M_+))
\]

is well-defined, and is exactly the corresponding Cartan-Münzner polynomial on \( \mathbb{R}^n \).

For a systematic introduction of isoparametric theory, we refer to the excellent book by Cecil and Ryan [10] and to a more updated survey by Chi [14].

Using an elegant topological method, Münzner [36] proved the remarkable result that the number \( g \) must be 1, 2, 3, 4, or 6 (see a new simplified proof by Fang [19]). Now since \(-1 \leq f(x) \leq 1 \), we have \(-|x|^g \leq F(x) \leq |x|^g \) on \( \mathbb{R}^n \). Thus we have infinitely many psd forms \( G_\pm \) and \( H_\pm \) defined by

\[
\begin{align*}
G_\pm^g(x) &:= |x|^g \pm F(x) \in P_{n,g} & g \text{ is even}, g = 2, 4, 6; \\
H_\pm^g(x) &:= |x|^{2g} - F(x)^2 \in P_{n,2g} & g = 1, 2, 3, 4, 6.
\end{align*}
\]

It is then natural to ask whether these explicit psd forms (known to be nonnegative from their geometric background) are sos or not. In this paper we solve this problem completely in accordance with the classification of isoparametric hypersurfaces in unit spheres. The proof will use representation theory of Clifford algebra for the cases when
$g = 4$, and deep geometric property of isoparametric hypersurfaces for the cases when $g = 6$.

For $g = 1$, isoparametric hypersurfaces are just hyperspheres $S^{n-2} \subset S^{n-1}$ and, up to a congruence, $F(x) = x_1$ is a coordinate function and thus $H_F$ is trivially sos.

Similarly, for $g = 2$, they are the Clifford torus $S^{k-1} \times S^{n-k-1} \subset S^{n-1}$ and, up to a congruence, $F(x) = \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2$ and thus $G_F^\pm$ are trivially sos.

For $g = 3$, Cartan showed that they are tubes around one of the four Veronese projective planes $FP^2 \subset S^{3m+1}$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ with $m = 1, 2, 4, 8$. We show in Section 2 that $H_F$ is always sos with an explicit expression, not only for these four isoparametric polynomials with $g = 3$ but also for $g = 1, 2, 4, 6$, simply by using the Cartan-M"unzner equation (1.1), Euler’s formula and Lagrange’s identity.

For $g = 6$, there are only two classes of homogeneous isoparametric hypersurfaces in $S^7$ and $S^{13}$ with $m_+ = m_- = m = 1, 2$ respectively. We show in Section 5 that for both isoparametric polynomials $F(x)$, neither of $G_F^\pm$ is sos.

The case $g = 4$ is the most difficult case as in the classification process, because it is the only case in which there are infinitely many homogeneous and nonhomogeneous isoparametric hypersurfaces. Fortunately, due to the classification, we only need to consider the isoparametric polynomials of OT-FKM type and the exceptional two homogeneous cases with $(m_+, m_-) = (2, 2), (4, 5)$, which will be solved in Sections 4 and 3 respectively. For both of these two types, we can always write $F(x)$ as $|x|^4$ minus some given sos form (see (3.1), (4.1), (5.1)), thus $G_F^-$ is automatically sos. However, it turns out that only in a few (though still infinitely many) classes $G_F^\pm$ is sos. For the sake of clarity, we list the classification of these sos forms in the following tables, where $k \in \mathbb{N}$, $(4,3)^I$ denotes the unique OT-FKM type with $(m_+, m_-) = (4,3)$ of the indefinite class, $G_F^\pm, H_F$ are psd forms in (1.2) with $F$ expressed in (2.1), (3.1), (4.1), (5.1) for $g = 3, 4, 6$ respectively.

| $g$ | 1 | 2 | 3 | 6 |
|-----|---|---|---|---|
| $G_F^+$ | sos | non-sos |
| $G_F^-$ | sos | non-sos |
| $H_F$ | sos | sos | sos |

For the non-sos psd forms $G_F^\pm$, we give them a simple and explicit sosr expression with a uniform denominator $|x|^2$ for $g = 4$ and $|x|^4$ for $g = 6$ in Section 2. Note

\[ F' := -F \]

is an isoparametric polynomial with multiplicities $(m_+, m_-)$ determining the same class of isoparametric hypersurfaces with converse focal submanifolds $M'_\pm = M_F$. Hence we regard them as equivalent. The class $(4,5)$ would be replaced by $(5,4)$ for the sake of consistency (see Section 5).
Table 2. Classification of sos forms $G^\pm_F, H_F$ for $g = 4$

| $(m_+,m_-)$ | $(2,2)$ | $(5,4)$ | $(1,k)$ | $(2,2k-1)$ | $(3,4)$ | $(4,3)^I$ | $(5,2)$ | $(6,1)$ | others |
|-------------|---------|---------|---------|-----------|--------|--------|--------|--------|--------|
| $G^+_F$     | non-sos | non-sos | sos     | sos       | sos    | sos    | sos    | sos    | non-sos |
| $G^-_F$     | sos     | sos     | sos     | sos       | sos    | sos    | sos    | sos    | sos    |
| $H_F$       | sos     | sos     | sos     | sos       | sos    | sos    | sos    | sos    | sos    |

that these forms are not positive definite, as they have non-trivial zero sets ($\mathbb{R}M^\pm$ for $G^\pm_F$). These examples are the supplement to Artin’s theorem on Hilbert’s 17th problem, which is beyond the scope of Pólya and Reznick’s theorem. Note also that $G^\pm_F$ have infinitely many zeroes and these non-sos psd forms have at least 8 variables. This can be compared with a low dimensional rigidity result of Choi-Lam-Reznick [16] which shows that a psd form in $P_{4,4}$ or $P_{3,6}$ with more than 11 or 10 projective zeroes must be sos.

It needs to be emphasized, that the zeroes of $G^\pm_F$ are also of special importance because of their rich geometric properties as the focal submanifolds of isoparametric hypersurfaces in $S^{n-1}$. For example, they are austere submanifolds (thus minimal) with constant principal curvatures independent of the choice of normal directions (cf. [27, 25]).

For the cases $g = 4$ (resp. $g = 6, m_+ = m_- = 1$) with $(m_+,m_-) = (2,2)$, $(5,4)$, $(4k,l - 4k - 1)^D$ (OT-FKM type with $m_+ \equiv 0 \pmod{4}$ of the definite class), we have shown a stronger result that, any quadratic form (resp. cubic form) vanishing on $(G^+_F)^{-1}(0) \cap S^{n-1} = M_-$ (resp. either of $M_\pm$) is identically zero, which implies the non-sos property of $G^+_F$. In particular, the focal submanifold $M_-$ is not quadratic (resp. cubic). This answers partially an important question of Solomon [44]. In fact, Solomon [44] had gotten the sos cases of $G^+_F$ of Table 2 (with $(3,4)$ and $(4,3)^I$ cases missing). He remarked that, the question as to whether both focal varieties might be quadratic seems difficult in general. This is important for estimates of eigenvalues and eigenfunctions of the Laplacian on isoparametric hypersurfaces, as Solomon showed that each quadratic form vanishing on one focal submanifold is an eigenfunction on every isoparametric hypersurface and the other focal submanifold in that family.

In Section 6, besides further discussion on the zeroes of $G^+_F$ and the Solomon question, we provide some clearer formulae of the psd forms $G^+_F$ for the isoparametric polynomials of OT-FKM type. For example, we get the interesting psd forms $G_{km}$ for $m = 1,2,3,4$ (see [6.1, 6.5, 6.6, 6.7]), including an elementary non-sos psd form:

$$G_{k4}(X,Y) := |X|^2|Y|^2 - |(X,Y)|^2 \in P_{8k,4} \setminus \Sigma_{8k,4}, \text{ for } X,Y \in \mathbb{H}^k, \ k \geq 2.$$ 

This immediately shows that the Cauchy-Schwarz inequality holds but Lagrange’s identity does not hold for quaternions. By the sos expression of $H_F$, we will also give an explicit sosr expression of $G_{k4}$ with a uniform denominator (see the identity (6.3)).
which generalizes Lagrange’s identity for quaternions. Moreover, we will discuss some applications to orthogonal multiplications, and to the sos problem on the Grassmanian $Gr_2(\mathbb{R}^l)$ that relates closely to the celebrated result of Blekherman-Smith-Velasco [6] and to the sos problem of Harvey-Lawson [27].

Though classified completely via many efforts, isoparametric hypersurfaces in unit spheres deserve to be even more attractive research objects. Because round spheres and Clifford tori are $g = 1$ and $g = 2$ isoparametric hypersurfaces with appropriate convexity, they are technically easier to be treated on many rigidity problems in geometric analysis than those with $g \geq 3$. From this point of view, our study of the sos problem on all isoparametric polynomials provides such an example of attempt. In particular, this algebraic study has various applications to geometry, e.g., as mentioned, to the Solomon question on eigenvalue’s estimates.

2. General results from Cartan-Münzner equation

In this section, we present some general results that can be easily deduced from the Cartan-Münzner equation (1.1), including (i) that the psd forms $H_F$ in (1.2) are always sos, and (ii) that the psd forms $G_{F}^\pm$ in (1.2) can be expressed as a sum of squares of rational functions with a uniform denominator $|x|^2$ for $g = 4$ and $|x|^4$ for $g = 6$. We also provide explicit formulae for the first nontrivial case when isoparametric polynomials are of degree $g = 3$. Explicit formulae for $g = 4, 6$ will be provided in sections later.

Let $F(x)$ be an isoparametric polynomial of degree $g \in \{1, 2, 3, 4, 6\}$ and $H_F(x) := |x|^{2g} - F(x)^2$ be the psd form as in (1.2). We first show

**Proposition 2.1.** $H_F$ is sos, i.e., a sum of squares of forms of degree $g$.

**Proof.** As $F(x)$ is homogeneous of degree $g$, $\langle \nabla F(x), x \rangle = gF(x)$ by Euler’s formula. Then the conclusion follows directly from Lagrange’s identity and the Cartan-Münzner equation (1.1):

$$|\nabla F(x) \wedge x|^2 = |\nabla F(x)|^2|x|^2 - \langle \nabla F(x), x \rangle^2 = g^2(|x|^{2g} - F(x)^2) = g^2 H_F(x),$$

where $\wedge$ is the exterior product. \(\square\)

For $g = 2, 4, 6$, let $G_{F}^\pm(x) := |x|^g \pm F(x)$ be the psd forms as in (1.2). We have

**Proposition 2.2.** For even $g$, $|x|^{g-2}G_{F}^\pm(x)$ is sos, i.e., a sum of squares of forms of degree $g - 1$.

**Proof.** Taking the gradient of $G_{F}^\pm(x)$, we have

$$\nabla G_{F}^\pm(x) = g|x|^{g-2}x \pm \nabla F(x).$$

Using Euler’s formula and the Cartan-Münzner equation (1.1), we get

$$|\nabla G_{F}^\pm(x)|^2 = g^2|x|^{2g-2} + |\nabla F(x)|^2 \pm 2g|x|^{g-2}\langle x, \nabla F(x) \rangle = 2g^2|x|^{g-2}(|x|^g \pm F(x)).$$
Thus $|x|^{g-2}G_F^\pm(x) = |\nabla G_F^\pm(x)|^2/2g^2$ is sos. \hfill \Box

As introduced in Section 1, Cartan classified isoparametric hypersurfaces in unit spheres with $g = 3$, showing that they are tubes around one of the four Veronese projective planes $M_\pm \cong \mathbb{P}^2 \subset S^{3m+1}$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ with $m_+ = m_- = m = 1, 2, 4, 8$. Cartan’s isoparametric polynomial on $\mathbb{R}^5$ with $g = 3, m = 1$ is defined by

$$F_C(x) = x_0^3 + \frac{3}{2}x_0(x_2^2 + x_3^2 - 2x_1^2) + \frac{3\sqrt{3}}{2}x_1(x_2^2 - x_3^2) + 3\sqrt{3}x_2x_3x_4,$$

for $x = (x_0, \cdots, x_4) \in \mathbb{R}^5$, which will be used in Section 5 for the case $g = 6, m = 1$. The other three Cartan polynomials on $\mathbb{R}^8, \mathbb{R}^{14}, \mathbb{R}^{26}$ with $g = 3, m = 2, 4, 8$ can be defined similarly as

$$F_C(x_0, x_1, X_2, X_3, X_4) = x_0^3 + \frac{3}{2}x_0(\sqrt{2}|X_2|^2 + |X_3|^2 - 2|X_4|^2 - 2x_1^2)
+ \frac{3\sqrt{3}}{2}x_1(|X_2|^2 - |X_3|^2) + 3\sqrt{3}\text{Re}(X_2X_3X_4),$$

where $x_0, x_1 \in \mathbb{R}, X_2, X_3, X_4 \in \mathbb{C}, \mathbb{H}, \mathbb{O}$ for $m = 2, 4, 8$, respectively, and $\text{Re}$ denotes the real part. Note that the two focal submanifolds $M_\pm = F_C^{-1}(\pm 1) \cap S^{3m+1} \cong \mathbb{P}^2$ are antipodal to each other in the sphere and their union $M_+ \cup M_-$ is exactly the set of spherical zeroes of the sos form $H_{FC} \in P_{3m+2,6}$ in (1.2). Hence $M_+ \cup M_-$ is a cubic variety but separately neither of $M_\pm$ is cubic. This is different from the case of $g = 4$, where $M_+ \cup M_-$ is a quartic variety as zeroes of $H_F$ but always non-quadratic as shown by Solomon [44]. Moreover, $M_+$ is always quadratic as zeroes of the sos quartic form $G_F$ while $M_-$ is often non-quadratic as we will show in the following sections.

3. On isoparametric with $g = 4, (m_+, m_-) = (2, 2), (5, 4)$

In this section, for the two exceptional homogeneous isoparametric hypersurfaces with $g = 4, (m_+, m_-) = (2, 2), (5, 4)$ in $S^9$ and $S^{10}$, respectively, we prove that any quadratic form vanishing on $M_-$ is identically zero, which implies that $M_-$ is non-quadratic and the psd form $G_F^+$ of (1.2) is non-sos. According to Solomon [44], the corresponding isoparametric polynomials $F(x)$ are given by (3.1) and (3.2) below, which immediately shows that $G_F^-$ is sos and $M_+$ is quadratic in both cases.

Before the proof, we first prepare two lemmas. This part treats with polynomials in $\mathbb{R}^5$. Consider the Horn form $h$ and the Choi-Lam quartic non-sos psd form $H$ [15]:

$$h(x_1, \cdots, x_5) = (x_1 + \cdots + x_5)^2 - 4(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1),$$
$$H(x_1, \cdots, x_5) = h(x_1^2, \cdots, x_5^2).$$

Denote by $Z$ the spherical zeroes of $H$, that is, $Z = \{x \in S^4 \mid H(x) = 0\}$. 
Lemma 3.1. \( Z \) is a union of ten circles. More precisely, 
\[
Z = S_1^+ \cup S_2^+ \cup S_3^+ \cup S_4^+ \cup S_5^+.
\]
where \( S_1^+ = \{ (\pm \frac{1}{\sqrt{2}}, a, 0, 0, b) \mid a^2 + b^2 = \frac{1}{2} \} \), 
\[
S_2^+ = \{ (b, \pm \frac{1}{\sqrt{2}}, a, 0, 0) \mid a^2 + b^2 = \frac{1}{2} \},
\]
\[
S_3^+ = \{ (0, b, \pm \frac{1}{\sqrt{2}}, a, 0) \mid a^2 + b^2 = \frac{1}{2} \},
\]
\[
S_4^+ = \{ (0, 0, b, \pm \frac{1}{\sqrt{2}}, a) \mid a^2 + b^2 = \frac{1}{2} \},
\]
\[
S_5^+ = \{ (a, 0, 0, b, \pm \frac{1}{\sqrt{2}}) \mid a^2 + b^2 = \frac{1}{2} \}.
\]

Proof. Observe that there are two equalities
\[
h = (x_1 - x_2 + x_3 + x_4 - x_5)^2 + 4x_2x_4 + 4x_3(x_5 - x_4),
\]
\[
h = (x_1 - x_2 + x_3 - x_4 + x_5)^2 + 4x_2x_5 + 4x_1(x_4 - x_5).
\]
It follows that \( H(x) \geq 0 \) for \( x_2^2 \geq x_3^2 \) by the first equality, and \( H(x) \geq 0 \) for \( x_2^2 \leq x_3^2 \)
by the second equality. Thus, \( H \) is indeed a \( psd \) form. To determine the spherical zeroes, we consider two cases when \( x_2^2 \geq x_3^2 \) and when \( x_2^2 \leq x_3^2 \), and make use of the two equalities mentioned above respectively. The determination of \( Z \) is complicated but elementary and will be omitted. \( \square \)

Lemma 3.2. Any quadratic form \( P \) vanishing on \( Z \) is identically zero. In particular, 
\( Z \) is not quadratic.

Proof. Suppose that \( P = P(x_1, \cdots, x_5) \) is a quadratic form vanishing on \( Z \). We observe that \( P(x_1, x_2, x_3, 0, 0) \) vanishing on \( S_2^+ \) is of the form \( \lambda(x_1^2 - x_2^2 + x_3^2) \) with \( \lambda \) being a real number. Similarly, the corresponding conclusions hold for \( P(0, x_2, x_3, x_4, 0) \), 
\( P(0, 0, x_3, x_4, x_5) \), \( P(x_1, 0, 0, x_4, x_5) \) and \( P(x_1, x_2, 0, 0, x_5) \) on \( S_3^+ \), \( S_4^+ \), \( S_5^+ \) and \( S_1^+ \), respectively. Clearly, these imply that \( P \) is identically zero. \( \square \)

Let us now consider the isoparametric polynomial \( F \) with \( g = 4, (m_+, m_-) = (2, 2) \).
According to Solomon [44], \( F \) comes from the map
\[
| (X \wedge X)^* |^2 \quad \text{for} \ X \in \Lambda^2(\mathbb{R}^5) \cong \mathbb{R}^{10},
\]
where \( \wedge \) is the exterior product, and \( * \) is the Hodge star operator \( * : \Lambda^4(\mathbb{R}^5) \to \Lambda^1(\mathbb{R}^5) \cong \mathbb{R}^5 \). Choose an oriented orthonormal basis \( \{ e_1, \cdots, e_5 \} \) in \( \mathbb{R}^5 \). Represent
\[
X = x_1e_1 \wedge e_2 + x_2e_1 \wedge e_3 + x_3e_1 \wedge e_4 + x_4e_1 \wedge e_5 + x_5e_2 \wedge e_3
\]
\[
+ x_6e_2 \wedge e_4 + x_7e_2 \wedge e_5 + x_8e_3 \wedge e_4 + x_9e_3 \wedge e_5 + x_{10}e_4 \wedge e_5.
\]
It is clear that
\[
\frac{1}{2} (X \wedge X)^* = (x_5x_{10} - x_6x_9 + x_7x_8)e_1 + (-x_2x_{10} + x_3x_9 - x_4x_8)e_2
\]
\[
+ (x_1x_{10} - x_3x_7 + x_4x_6)e_3 + (-x_1x_9 + x_2x_7 - x_4x_5)e_4 + (x_1x_8 - x_2x_6 + x_3x_5)e_5.
\]
The corresponding isoparametric polynomial \( F(X) = |X|^4 - 2| (X \wedge X)^* |^2 \) is
\[
F(x_1, \cdots, x_{10}) = (x_1^2 + \cdots + x_{10}^2)^2 - 8 \left\{ (x_5x_{10} - x_6x_9 + x_7x_8)^2
\right.
\]
\[
\left. + (-x_2x_{10} + x_3x_9 - x_4x_8)^2 + (x_1x_{10} - x_3x_7 + x_4x_6)^2
\right.
\]
\[
\left. + (-x_1x_9 + x_2x_7 - x_4x_5)^2 + (x_1x_8 - x_2x_6 + x_3x_5)^2 \right\}
\]
(3.1)
On the other hand, restricting Lemma 3.2, we see that $Q$ is convenient, denote by $R$. It is clear that $G$ is not quadratic and $G(+1)$ is non-sos.

**Theorem 3.3.** Any quadratic form $Q = Q(x_1, \ldots, x_{10})$ vanishing on $M_-$ is identically zero. In particular, $M_-$ is not quadratic and $G(+1)$ on $\mathbb{R}^{10}$ is non-sos.

**Proof.** For $x = (x_1, \ldots, x_{10})$, let $G(x) = G(+1)/2 = (|x|^4 + F(x))/2$, namely,

$$G(x) = |x|^4 - 4\left((x_5x_{10} - x_6x_9 + x_7x_8)^2 + (-x_2x_{10} + x_3x_9 - x_4x_8)^2 + (x_1x_{10} - x_3x_7 + x_4x_6)^2 + (-x_1x_9 + x_2x_7 - x_4x_5)^2 + (x_1x_8 - x_2x_6 + x_3x_5)^2\right).$$

It is clear that $M_- = \{x \in \mathbb{R}^9 \mid G(x) = 0\}$. We prove in the following steps. For convenience, denote by $\mathbb{R}^{5}_{ijklm}$ the 5-space with coordinates $x_i, x_j, x_k, x_l, x_m$.

(I) Restricting $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{23679}_{25}$, one gets

$$G(0, x_2, x_3, 0, 0, x_6, x_7, 0, x_9, 0) = (x_2^2 + x_6^2 + x_9^2 + x_3^2 + x_7^2) - 4(x_2x_6^2 + x_6x_9 + x_9x_3^2 + x_3x_7^2 + x_7x_2^2)$$

$$= H(x_2, x_6, x_9, x_3, x_7).$$

On the other hand, restricting $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{1458,10}_{5}$, one gets

$$G(x_1, 0, 0, x_4, x_5, 0, 0, x_8, 0, x_{10}) = (x_1^2 + x_5^2 + x_4^2 + x_3^2 - 4(x_2x_5^2 + x_5x_3^2 + x_3x_4^2 + x_4x_8^2 + x_8x_1^2)$$

$$= H(x_1, x_5, x_4, x_3, x_8).$$

Suppose now that a quadratic form $Q = Q(x_1, \ldots, x_{10})$ vanishes on $M_-$. Applying Lemma [Lemma 3.2] we see that $Q = Q(x)$ is a bilinear form on $\mathbb{R}^{23679}_{25} \times \mathbb{R}^{1458,10}_{5}$.

(II) Let us restrict $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{5}_{2456,10}$. Then

$$G(0, x_2, 0, x_4, x_5, x_6, 0, 0, 0, x_{10}) = (x_2^2 + x_4^2 + x_5^2 + x_6^2 + x_{10}^2) - 4(x_2x_4^2 + x_4x_5^2 + x_5x_6^2 + x_6x_{10}^2)$$

$$= H(x_2, x_4, x_5, x_6, x_{10}).$$

On the other hand, restricting $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{5}_{13789}$, one gets

$$G(x_1, 0, x_3, 0, 0, x_7, x_8, 0, x_9, 0) = (x_1^2 + x_3^2 + x_7^2 + x_8^2 + x_9^2 - 4(x_1x_3^2 + x_3x_7^2 + x_7x_8^2 + x_8x_9^2) + x_9^2)$$

$$= H(x_1, x_3, x_7, x_8, x_9).$$

Applying Lemma [Lemma 3.2] and summarizing the arguments above, we can write $Q$ as

$$Q(x_1, \ldots, x_{10}) = a_1x_1x_2 + a_2x_1x_6 + a_3x_2x_8 + a_4x_3x_4 + a_5x_3x_5 + a_6x_3x_{10} + a_7x_4x_7 + a_8x_4x_9 + a_9x_5x_7 + a_{10}x_5x_9 + a_{11}x_6x_8 + a_{12}x_7x_{10} + a_{13}x_9x_{10},$$

Clearly, the focal submanifold $M_+ = F^{-1}(1) \cap S^9 \cong Gr_2(\mathbb{R}^5)$ is quadratic and $G_F$ of (1.2) is sos. For the other focal submanifold $M_- = F^{-1}(-1) \cap S^9 \cong CP^3$, we have...
with real numbers $a_1, \cdots, a_{13}$.

(III). Let us restrict $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{34569}_5$. Then

$$G(0, 0, x_3, x_4, x_5, x_6, 0, 0, x_9, 0)$$

$$= (x_3^2 + x_5^2 + x_4^2 + x_6^2 + x_9^2)^2 - 4(x_3^2 x_5^2 + x_4^2 x_6^2 + x_4^2 x_9^2 + x_5^2 x_6^2 + x_5^2 x_9^2)$$

$$= H(x_3, x_5, x_4, x_6, x_9).$$

On the other hand, restricting $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{1278,10}_5$, one gets

$$G(x_1, x_2, 0, 0, 0, 0, x_7, x_8, 0, x_{10})$$

$$= (x_1^2 + x_2^2 + x_4^2 + x_7^2 + x_{10}^2)^2 - 4(x_1^2 x_2^2 + x_2^2 x_7^2 + x_1^2 x_7^2 + x_2^2 x_{10}^2 + x_{10}^2 x_2^2)$$

$$= H(x_1, x_7, x_2, x_{10}).$$

Applying Lemma 3.2 and summarizing the arguments above, we deduce

$$Q(x_1, \cdots, x_{10})$$

$$= a_2 x_1 x_6 + a_6 x_3 x_{10} + a_7 x_4 x_7 + a_9 x_5 x_7 + a_{11} x_6 x_8 + a_{13} x_9 x_{10}.$$

(IV). Let us restrict $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{24678}_5$. Then

$$G(0, x_2, 0, x_4, 0, x_6, x_7, x_8, 0, 0)$$

$$= (x_2^2 + x_4^2 + x_6^2 + x_7^2 + x_8^2)^2 - 4(x_2^2 x_4^2 + x_4^2 x_6^2 + x_4^2 x_7^2 + x_6^2 x_7^2 + x_6^2 x_8^2)$$

$$= H(x_2, x_6, x_4, x_7).$$

On the other hand, restricting $x \in \mathbb{R}^{10}$ to $\mathbb{R}^{1359,10}_5$, one gets

$$G(x_1, 0, x_3, 0, x_5, 0, 0, 0, x_9, x_{10})$$

$$= (x_1^2 + x_2^2 + x_5^2 + x_9^2 + x_{10}^2)^2 - 4(x_1^2 x_2^2 + x_5^2 x_9^2 + x_2^2 x_5^2 + x_2^2 x_9^2 + x_2^2 x_{10}^2 + x_{10}^2 x_9^2)$$

$$= H(x_1, x_9, x_3, x_5, x_{10}).$$

Applying Lemma 3.2 and summarizing the arguments above, we deduce

$$Q(x_1, \cdots, x_{10}) = a_2 x_1 x_6 + a_9 x_5 x_7.$$

(V). We observe that $M_-$ contains the set

$$\{(0, 0, x_3, 0, x_5, 0, x_7, 0, 0, 0) | x_3^2 = x_5^2 + x_7^2 = \frac{1}{2}\},$$

and thus the assumption that $Q(x)$ vanishes on $M_-$ implies $Q(x) = a_2 x_1 x_6$. At last, we observe that $M_-$ contains the set

$$\{(x_1, 0, 0, 0, 0, x_6, 0, 0, x_9, 0) | x_3^2 = x_1^2 + x_6^2 = \frac{1}{2}\},$$

and thus the assumption that $Q(x)$ vanishes on $M_-$ implies $Q(x) \equiv 0$. □
Now we turn to the isoparametric polynomial with \( g = 4 \), \((m_+, m_-) = (4, 5)\). In fact, we consider the equivalent version (see the footnote \[1\]) with \((m_+, m_-) = (5, 4)\) for the sake of consistency. According to Solomon [44], it comes from the map

\[
|Z \wedge Z|^2 \text{ for } Z \in \Lambda^2(C^5) \cong C^{10} \cong \mathbb{R}^{20}.
\]

Choose an oriented orthonormal basis \(\{e_1, \ldots, e_5\}\) in \(C^5\). Represent

\[
Z = z_1e_1 \wedge e_2 + z_2e_1 \wedge e_3 + z_3e_1 \wedge e_4 + z_4e_1 \wedge e_5 + z_5e_2 \wedge e_3 + z_6e_2 \wedge e_4 + z_7e_2 \wedge e_5 + z_8e_3 \wedge e_4 + z_9e_3 \wedge e_5 + z_{10}e_4 \wedge e_5.
\]

It is clear that the isoparametric polynomial \(F'\) defined by

\[
F'(Z) = |Z|^4 - 2|Z \wedge Z|^2
\]

is equal to

\[
F'(z_1, \cdots, z_{10}) = \left(\sum |z_i|^2 \right)^2 - 8\left\{|(z_5z_{10} - z_6z_9 + z_7z_8)|^2 + |(z_2z_{10} + z_3z_9 - z_4z_8)|^2 + |(z_1z_{10} - z_3z_7 + z_4z_6)|^2 + |(z_1z_9 - z_2z_7 - z_4z_5)|^2 + |(z_1z_8 - z_2z_6 + z_3z_5)|^2\right\},
\]

with \(z_j = x_j + \sqrt{-1}y_j, j = 1, \cdots, 10\). It is clear that the focal submanifold \(M^ {13}_+ = (F')^{-1}(+1) \cap S^{19}\) is quadratic and \(G_{F'}^-\) of (1.2) is sos. As before, we define \(G'\) by \(G'(z) = G^+_{F'}/2 = (F'(z) + |z|^4)/2\), namely,

\[
G'(z_1, \cdots, z_{10}) = \left(\sum |z_i|^2 \right)^2 - 4\left\{|(z_5z_{10} - z_6z_9 + z_7z_8)|^2 + |(-z_2z_{10} + z_3z_9 - z_4z_8)|^2 + |(z_1z_{10} - z_3z_7 + z_4z_6)|^2 + |(z_1z_9 - z_2z_7 - z_4z_5)|^2 + |(z_1z_8 - z_2z_6 + z_3z_5)|^2\right\}.
\]

It is clear that \(G'(z) \geq 0\), and \(M^ {14}_- = \{z \in S^{19} | G'(z) = 0\}\). Taking \(y_1 = \cdots = y_{10} = 0\), this isoparametric polynomial \(F'\) with multiplicities \((5, 4)\) becomes \(F\) in (3.1), replacing \(z_j\) by \(x_j\). As a consequence, we get from Theorem 3.3

**Corollary 3.4.** The psd form \(G^+_{F'} = 2G'\) of (1.2) on \(\mathbb{R}^{20}\) is non-sos.

Furthermore, we can show

**Theorem 3.5.** Any quadratic form \(Q\) on \(\mathbb{R}^{20}\) vanishing on \(M^ {14}_-\) is identically zero. In particular \(M^ {14}_-\) is not quadratic.

**Proof.** Suppose that a quadratic form \(Q\) vanishes on \(M_-\). At first, let us take \(y_1 = \cdots = y_{10} = 0\), or take \(x_1 = \cdots = x_{10} = 0\). Applying Theorem 3.3 we see that \(Q\) is a bilinear form on \(\{x_1, \cdots, x_{10}\}\) and \(\{y_1, \cdots, y_{10}\}\). Namely, \(Q = \sum a_{ij}x_iy_j\), with \(a_{ij} \in \mathbb{R}\) and \(i, j = 1, \cdots, 10\).
Next, let us take $x_i = y_i$ for $i = 1, \cdots, 10$. Then,
\[
\frac{1}{4}G'(x_1, \cdots, x_{10}, x_1, \cdots, x_{10}) = G(x_1, \cdots, x_{10}).
\]
By the assumption, $\sum a_{ij}x_ix_j$ vanishes on the spherical zeroes of $G$. Applying Theorem 3.3 again, we see that $a_{ij} = -a_{ji}$, for $i, j = 1, \cdots, 10$.

Now for $i < j$, considering the value of $Q$ at the zero point $z$ of $G'$ with
\[
z_i = x_i = \frac{1}{\sqrt{2}}, \quad z_j = \sqrt{-1}y_j = \frac{\sqrt{-1}}{\sqrt{2}}
\]
and all other $z_k = 0$ where $z_iz_j$ appears in $|(Z \wedge Z)^r|^2$ of $G'$, we see that $a_{ij} = 0$ for $(i, j) = (5, 10), (6, 9), (7, 8), (2, 10), (3, 9), (4, 8), (1, 10), (3, 7), (4, 6), (1, 9), (2, 7), (4, 5), (1, 8), (2, 6), \text{ or } (3, 5)$.

For any other pair $(i, j)$ with $i < j$, we can also show that $a_{ij} = 0$. For simplicity, without loss of generality we take $(i, j) = (1, 2)$ for example. Since there are items $z_1z_{10}$ and $z_2z_{10}$ in $|(Z \wedge Z)^r|^2$, we have one zero point $Z$ of $G'$ with $z_1 = x_1 = \frac{1}{2}$, $z_2 = \sqrt{-1}y_2 = \frac{\sqrt{-1}}{\sqrt{2}}$, $z_{10} = \sqrt{-1}y_{10} = \frac{\sqrt{-1}}{\sqrt{2}}$ and all other $z_k = 0$. Thus
\[
0 = Q(Z) = \frac{1}{4}a_{12} + \frac{1}{2\sqrt{2}}a_{1,10} = \frac{1}{4}a_{12},
\]
as we have shown $a_{1,10} = 0$. \hfill $\square$

4. On isoparametric of OT-FKM type with $g = 4$

In this section, we classify the classes of isoparametric polynomials of hypersurfaces of OT-FKM type with $g = 4$ such that $G_F^\pm = |x|^g + F(x)$ of $(1.2)$ is sos.

Recall that an OT-FKM type isoparametric polynomial is defined as (cf. [37, 20])

\[
F(x) = |x|^4 - 2m \sum_{\alpha=0}^{m} \langle P_\alpha x, x \rangle^2, \quad x \in \mathbb{R}^{2l},
\]
where $\{P_0, \cdots, P_m\}$ is a symmetric Clifford system on $\mathbb{R}^{2l}$, i.e., $P_\alpha$‘s are symmetric matrices satisfying $P_\alpha P_\beta + P_\beta P_\alpha = 2\delta_{\alpha\beta}I_{2l}$. Then the multiplicity pair is $(m_+, m_-) = (m, l - m - 1)$. Two Clifford systems $\{P_0, \cdots, P_m\}$ and $\{Q_0, \cdots, Q_m\}$ on $\mathbb{R}^{2l}$ are called algebraically equivalent if there exists $A \in O(\mathbb{R}^{2l})$ such that $Q_\alpha = AP_\alpha A^t$ for all $\alpha \in \{0, \cdots, m\}$. They are called geometrically equivalent when there exists $B \in O(\text{Span}\{P_0, \cdots, P_m\})$ such that $\{Q_0, \cdots, Q_m\}$ and $\{B(P_0), \cdots, B(P_m)\}$ are algebraically equivalent, which give two isoparametric polynomials that are congruent under an orthogonal transformation of $\mathbb{R}^{2l}$. We will apply representation theory of Clifford algebra to prove Theorem 4.1 below.

As introduced in Section 1, we can define the psd forms $G_F^\pm \in P_{2l,4}$ as $(1.2)$. Clearly, $G_F^- = |x|^g - F(x)$ is sos and the focal submanifold $M_+ = F^{-1}(1) \cap \mathbb{S}^{2l-1}$ is
quadratic and defined as

\[(4.2) \quad M_{m,+}^{m,+2m} = \{ x \in S^{2l-1} \mid \langle P_{\alpha} x, x \rangle = 0, \alpha = 0, \cdots, m \} \]

From now on, we write \( G_F = G_F^+ / 2 \) for simplicity. Then

\[(4.3) \quad G_F(x) = (F(x) + |x|^4)/2 = |x|^4 - \sum_{\alpha=0}^{m} \langle P_{\alpha} x, x \rangle^2. \]

We are concerned with whether the other focal submanifold \( M_- = F^{-1}(-1) \cap S^{2l-1} \) is quadratic or not. Note that \( M_- \) is just the set of spherical zeroes of \( G_F \) and can be expressed as

\[(4.4) \quad M_{2m,+}^{2m,+} = G_F^{-1}(0) \cap S^{2l-1} = \{ x \in S^{2l-1} \mid Px = x, \text{ for some } P \in \Sigma \}, \]

where \( \Sigma = \{ P \in \text{Span}\{P_0, \cdots, P_m\} \mid |P|^2 = \text{tr}(PP^t) = 2l \} \) is the Clifford sphere (see [20]). Therefore, if the psd form \( G_F \) is a sum of squares of quadratic forms, then \( M_- \) is obviously quadratic. It turns out that for almost all cases \( G_F \) is non-sos.

**Theorem 4.1.** The psd form \( G_F \) in (4.3) on \( \mathbb{R}^{2l} \), associated with the OT-FKM type, is sos if and only if \( m = 1, 2 \), or \((m_+, m_-) = (m, l - m - 1) = (5, 2), (6, 1), (3, 4) \) or (4, 3) of the indefinite class.

**Proof.** We first show the sufficiency. For \( m = 1, 2 \), Solomon [44] had proven that \( G_F(x) \) in (4.3) is a sum of squares of quadratic forms. We repeat the proof for the sake of completeness. In these cases, \( l = km, (m_+, m_-) = (1, k - 2), \) for any integer \( k \geq 3 \) for \( m = 1 \); or \((m_+, m_-) = (2, 2k - 3), \) for any integer \( k \geq 2 \) for \( m = 2 \). The coordinate \( x \in \mathbb{R}^{2l} \) can be written as \( x = (X, Y) \in \mathbb{F}^k \oplus \mathbb{F}^k \) where \( F = \mathbb{R} \) for \( m = 1 \) and \( F = \mathbb{C} \) for \( m = 2 \), respectively. Without loss of generality, we can write the Clifford system \( \{P_0, \cdots, P_m\} \) in matrix form as (4.7) below where \( E_1 \) corresponds to the complex structure on \( \mathbb{R}^l \cong \mathbb{C}^k \) in the case of \( m = 2 \). Then the isoparametric polynomial \( F(x) \) can be written as

\[F(X, Y) = (|X|^2 + |Y|^2)^2 - 2(|X|^2 - |Y|^2)^2 + 4|\langle X, Y \rangle|^2,\]

where \( \langle \cdot, \cdot \rangle \) denotes the Hermitian inner product if \( m = 2 \). Using Lagrange’s identity

\[|X|^2|Y|^2 = |X \wedge Y|^2 + |\langle X, Y \rangle|^2,\]

where \( X \wedge Y \in \Lambda^2(\mathbb{F}^k) \) is the exterior product, we obtain the sos-expression of Solomon:

\[(4.5) \quad G_F(X, Y) = 4(|X|^2|Y|^2 - |\langle X, Y \rangle|^2) = 4|X \wedge Y|^2 = 4 \sum_{1 \leq i < j \leq k} |X_i Y_j - X_j Y_i|^2,\]

for any \( X = (X_1, \cdots, X_k), \ Y = (Y_1, \cdots, Y_k) \in \mathbb{F}^k \).

For the cases of \((m_+, m_-) = (5, 2), (6, 1), (3, 4) \) or (4, 3) of the indefinite class (for the definition see Subsection 4.2 below), the isoparametric foliations are just those.
OT-FKM type isoparametric foliations with converse multiplicities \((m_-, m_+)\). Correspondingly the psd forms \(G_F\) can be expressed as a sum of squares of quadratic forms (see [20]). We repeat the proof for the sake of completeness. The isoparametric polynomials \(F(x)\) in these cases can be defined in the following unified way. Let \(\{P_0, \cdots, P_8\}\) be the Clifford system on \(\mathbb{R}^{16}\) corresponding to the Clifford algebra \(\{E_1, \cdots, E_7\}\) on the Octonions \(\mathbb{R}^8\) (see also (4.26) below). This Clifford system has the following property
\[
\sum_{\alpha=0}^{8} \langle P_{\alpha} x, x \rangle^2 = |x|^4, \quad x \in \mathbb{R}^{16}.
\]
Therefore, taking \(m = 5, 6, 3, 4\) respectively, the corresponding isoparametric polynomials \(F(x)\) can be defined as
\[
F(x) = |x|^4 - 2 \sum_{\alpha=0}^{m} \langle P_{\alpha} x, x \rangle^2.
\]
Thus
\[
G_F(x) = |x|^4 - \sum_{\alpha=0}^{m} \langle P_{\alpha} x, x \rangle^2 = \sum_{\alpha=m+1}^{8} \langle P_{\alpha} x, x \rangle^2, \quad x \in \mathbb{R}^{16}.
\]
We remark that for \(m = 1, 2\), the formula (4.6) gives another sos expression of \(G_F\) different from those in (4.5) (remarked also by Solomon in [44]).

To prove the necessity, we only need to show for all other cases that the nonnegative polynomial \(G_F(x)\) is not a sum of squares of quadratic forms. We will show this case-by-case in the following subsections. \(\square\)

4.1. \(m \equiv 0 \pmod{4}\), definite case. A Clifford system \(\{P_0, \cdots, P_m\}\) is called definite if \(P_0 \cdots P_m = \pm I_{2l}\) (for indefinite case see Subsection 4.2 below). In fact, we obtain the following stronger result.

**Theorem 4.2.** For the case of \(m \equiv 0 \pmod{4}\), when the Clifford system \(\{P_0, \cdots, P_m\}\) is definite, any quadratic form \(Q\) vanishing on \(M_-\) is identically zero. In particular, \(M_-\) is not quadratic, and the psd form \(G_F\) in (4.3) is non-sos.

**Proof.** Without loss of generality, we can write the Clifford system \(\{P_0, \cdots, P_m\}\) in matrix form under the decomposition \(\mathbb{R}^{2l} = E_+(P_0) \oplus E_-(P_0) \cong \mathbb{R}^l \oplus \mathbb{R}^l\), where \(E_{\pm}(P_0)\) are the eigenspaces of the eigenvalues \(\pm 1\) of \(P_0\), by
\[
(4.7) \quad P_0 = \begin{pmatrix} I_l & 0 \\ 0 & -I_l \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}, \quad P_{\alpha+1} = \begin{pmatrix} 0 & E_\alpha \\ -E_\alpha & 0 \end{pmatrix}, \quad 1 \leq \alpha \leq m - 1,
\]
where \(\{E_1, \cdots, E_{m-1}\}\) generates a Clifford algebra on \(\mathbb{R}^l\), i.e., \(E_\alpha\)’s are skew-symmetric matrices satisfying \(E_\alpha E_\beta + E_\beta E_\alpha = -2\delta_{\alpha\beta} I_l.\)
Recall (4.4) that \( E_+ (P_0) \cap S^{2l-1} \subset M_- \). Thus any quadratic form \( Q(x) = \sum_{i,j=1}^{2l} q_{ij} x_i x_j \) vanishing on \( M_- \) can be expressed in matrix form \( Q = (q_{ij}) \) by

\[
Q = \begin{pmatrix}
0 & B \\
B^t & 0
\end{pmatrix}.
\]

Moreover, since for any \( u \in \mathbb{R}^l \), \( x = (u, u) \in E_+ (P_1) \), and by (4.4), \( E_+ (P_1) \cap S^{2l-1} \subset M_- \), we have

\[
0 = Q(x) = 2\langle Bu, u \rangle,
\]

which implies that \( B \) is skew-symmetric.

Similarly, for each \( \alpha = 1, \ldots, m-1 \), for any \( u \in \mathbb{R}^l \), \( x = (u, -E_{\alpha} u) \in E_+(P_{\alpha+1}) \), and by (4.4), \( E_+(P_{\alpha+1}) \cap S^{2l-1} \subset M_- \), the equalities

\[
0 = Q(x) = 2\langle Bu, E_{\alpha} u \rangle
\]

imply

\[
BE_{\alpha} = -E_{\alpha} B, \quad \alpha = 1, \ldots, m-1.
\]

Now for the case of \( m \equiv 0 \pmod{4} \), when \( \{P_0, \ldots, P_m\} \) is definite, i.e.,

\[
P_0 \cdots P_m = \begin{pmatrix}
E_1 \cdots E_{m-1} & 0 \\
0 & E_1 \cdots E_{m-1}
\end{pmatrix} = \pm I_{2l},
\]

it follows from (4.9) that

\[
BE_1 \cdots E_{m-1} = (-1)^{m-1} E_1 \cdots E_{m-1} B,
\]

which implies that \( B = -B \) as \( m \) is even and \( E_1 \cdots E_{m-1} = \pm I_l \), and thus \( B = 0 \). \( \square \)

4.2. \( m \equiv 0 \pmod{4} \), \((m_+, m_-) \neq (4, 3)\), indefinite cases. With the same notations as in the last subsection, a Clifford system \( \{P_0, \ldots, P_m\} \) is called indefinite if

\[
P_0 \cdots P_m = \begin{pmatrix}
E_1 \cdots E_{m-1} & 0 \\
0 & E_1 \cdots E_{m-1}
\end{pmatrix} \neq \pm I_{2l}.
\]

Recall from [20] that each Clifford system is algebraically equivalent to a direct sum of irreducible Clifford systems. An irreducible Clifford system \( \{P_0, \ldots, P_m\} \) on \( \mathbb{R}^{2l} \) exists precisely for the following values of \( m \) and \( l = \delta(m) \) in Table 3.

**Table 3. Dimension \( \delta(m) \) of irreducible representation of Clifford algebra**

| m   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \( \cdots \) | \( m + 8 \) |
|-----|---|---|---|---|---|---|---|---|-------------|-------------|
| \( \delta(m) \) | 1 | 2 | 4 | 4 | 8 | 8 | 8 | 8 | \( \cdots \) | \( 16\delta(m) \) |

For \( m \equiv 0 \pmod{4} \), there are exactly two algebraic equivalence classes of irreducible Clifford systems \( \{P_0, \ldots, P_m\} \) on \( \mathbb{R}^{2\delta(m)} \) (resp. Clifford algebras \( \{E_1, \ldots, E_{m-1}\} \) on
R^{δ(m)}), distinguished from each other by the choice of sign in \( P_0 \cdots P_m = \pm I_2 \delta(m) \) (resp. \( E_1 \cdots E_{m-1} = \pm I_{δ(m)} \)). They are geometrically equivalent because replacing \( P_m \) by \(-P_m\) swaps them. If one constructs all direct sums of both of the irreducible algebraic classes with altogether \( k \) summands, then the invariant \(|\text{tr}(P_0 \cdots P_m)|\) (invariant under geometric equivalence) takes on \( 1 + \lceil \frac{k}{2} \rceil \) different values. Thus there are exactly one definite and \( \lceil \frac{k}{2} \rceil \) indefinite geometric equivalence classes of Clifford systems on \( \mathbb{R}^{2l} \) with \( l = k\delta(m) \). For indefinite case, \( k \geq 2 \) is necessary and the system is reducible.

With these investigations prepared, we are ready to show

**Theorem 4.3.** For the case of \( m \equiv 0 \) (mod 4), \( (m+, m-) \neq (4, 3) \), when the Clifford system \((P_0, \cdots, P_m)\) is indefinite, the psd form \( G_F \) in (4.3) is non-sos.

**Proof.** We prove it by contradiction. Assume there are quadratic forms \( Q_1, \cdots, Q_N \) such that

\[
\sum_{i=1}^{N} Q_i(x)^2 = G_F(x) = |x|^4 - \sum_{α=0}^{m} \langle P_α x, x \rangle^2.
\]

Then each quadratic form \( Q_i \) vanishes on \( M_- \), which implies that \( Q_i \)'s are in the same form as in (4.8) and (4.9), i.e.,

\[
Q_i = \begin{pmatrix} 0 & B_i \\ B_i^t & 0 \end{pmatrix}, \quad B_iE_α = -E_αB_i, \quad α = 1, \cdots, m-1,
\]

where each \( B_i \) is skew-symmetric. Recalling the representation (4.7) of the Clifford system, the equation (4.11) is now equivalent to the identity

\[
\sum_{i=1}^{N} \langle B_i u, v \rangle^2 = \frac{1}{4} G_F(x)
\]

\[
= \frac{1}{4} \left( |(u, v)|^4 - ((u, -v), (u, v))^2 - ((v, u), (u, v))^2 - 4 \sum_{α=1}^{m-1} \langle E_α u, v \rangle^2 \right)
\]

\[
= |u|^2 |v|^2 - \langle u, v \rangle^2 - \sum_{α=1}^{m-1} \langle E_α u, v \rangle^2, \quad x = (u, v) \in \mathbb{R}^l \oplus \mathbb{R}^l.
\]

This identity involves much information. For example, it follows

\[
\sum_{i=1}^{N} |B_i|^2 = l^2 - lm,
\]

where

\[
\sum_{i=1}^{N} |B_i|^2 = \sum_{p,q=1}^{l} \sum_{i=1}^{N} \langle B_i e_p, e_q \rangle^2,
\]

for an orthonormal basis \( \{e_p\} \) of \( \mathbb{R}^l \). This formula holds independent of reducibility.
On the other hand, we consider the decomposition of \( \{P_0, \ldots, P_m\} \) on \( \mathbb{R}^{2l} \) with \( l = k\delta(m) \) into a direct sum of \( k \geq 2 \) irreducible Clifford systems on \( \mathbb{R}^{2\delta(m)} \) (denoted with a superscript \( r = 1, \ldots, k \)) so that
\[
\mathbb{R}^{2l} = \mathbb{R}^{2\delta(m)} \oplus \cdots \oplus \mathbb{R}^{2\delta(m)} \quad (P_0, \ldots, P_m) = (P_0^1, \ldots, P_m^1) \oplus \cdots \oplus (P_0^k, \ldots, P_m^k).
\]
Here the irreducible Clifford systems \( \{P_0^r, \ldots, P_m^r\} \) on \( \mathbb{R}^{2\delta(m)} \) can be expressed in the form as (4.7) so that
\[
P_0^r = \begin{pmatrix} I_{\delta(m)} & 0 \\ 0 & -I_{\delta(m)} \end{pmatrix}, \quad P_1^r = \begin{pmatrix} 0 & I_{\delta(m)} \\ I_{\delta(m)} & 0 \end{pmatrix}, \quad P_{r+1}^r = \begin{pmatrix} 0 & E_r \\ -E_r & 0 \end{pmatrix},
\]
\( \alpha = 1, \ldots, m-1 \), where \( \{E_1^r, \ldots, E_{m-1}^r\} \) generates an irreducible Clifford algebra on each \( \mathbb{R}^{\delta(m)} \) of the decomposition of \( \{E_1, \ldots, E_{m-1}\} \) on \( \mathbb{R}^l = \mathbb{R}^{\delta(m)} \oplus \cdots \oplus \mathbb{R}^{\delta(m)} \).

Now for the case of \( m \equiv 0 (\bmod 4) \), when \( \{P_0, \ldots, P_m\} \) is indefinite (\([k/2]\) classes in total), according to the statements at the beginning of this subsection, without loss of generality, there is some \([k+1/2] \leq r_0 < k \) such that
\[
P_0^r \cdots P_m^r = \begin{pmatrix} E_1^r \cdots E_{m-1}^r \\ 0 \\ E_1^r \cdots E_{m-1}^r \end{pmatrix} = \begin{cases} I_{2\delta(m)} & r \leq r_0 \\ -I_{2\delta(m)} & r > r_0. \end{cases}
\]
Furthermore, we can set
\[
E_1^1 = \cdots = E_k^1, \quad E_{m-1}^1 = \cdots = E_{m-1}^k = \cdots = -E_1^0 = \cdots = -E_k^0.
\]
Therefore, as proved in the last subsection, any quadratic form \( Q \) vanishing on \( M_\ast \) is identically zero when restricted to each irreducible component \( \mathbb{R}^{2\delta(m)} \) due to (4.9), (4.10) and (4.17). Regarding \( B_i \) as skew-symmetric operator on \( \mathbb{R}^l \), we can rewrite \( B_i \) with respect to the irreducible decomposition (4.15) as
\[
B_i: \mathbb{R}^l = \mathbb{R}^{\delta(m)} \oplus \cdots \oplus \mathbb{R}^{\delta(m)} \to \mathbb{R}^{\delta(m)} \oplus \cdots \oplus \mathbb{R}^{\delta(m)} = \mathbb{R}^l
\]
\[
B_i = \begin{pmatrix} B_{i1} & \cdots & B_{ik} \\ \vdots & \ddots & \vdots \\ B_{ik} & \cdots & B_{kk} \end{pmatrix},
\]
where each \( B_{ir}^r = 0 \) since it acts as the restriction of \( Q_i \) to the \( r \)-th component \( \mathbb{R}^{2\delta(m)} \subset \mathbb{R}^{2l} \), and \( B_{ir}^r \) acts on \( \mathbb{R}^{\delta(m)} \) as \( B_i \) is skew-symmetric. Let \( \{e_a^r\}_{a=1}^{\delta(m)} \) be an orthonormal basis of the \( r \)-th component \( \mathbb{R}^{\delta(m)} \subset \mathbb{R}^l \). Since \( E_\alpha = E_\alpha^1 \oplus \cdots \oplus E_\alpha^k \) acts orthogonally, it follows from (4.13) that for \( r \neq s \),
\[
\sum_{i=1}^N |B_{ir}^r|^2 = \sum_{a,b=1}^{\delta(m)} |e_a^r|^2 |e_b^s|^2 - \langle e_a^r, e_b^s \rangle^2 - \sum_{a=1}^{m-1} \langle e_a^r, e_a^s \rangle^2 = \delta(m)^2,
\]
and thus

\[
(4.21) \quad \sum_{i=1}^{N} |B_i|^2 = \sum_{r,s=1}^{k} \sum_{i=1}^{N} |B_{r,s}^i|^2 = \sum_{r \neq s} \sum_{i=1}^{N} |B_{r,s}^i|^2 = (k^2 - k)\delta(m)^2 = l^2 - l\delta(m).
\]

Then comparing (4.14) and (4.21) we obtain \( m = \delta(m) \), which implies immediately that \( m = 4 \) or \( m = 8 \).

Let us consider now the case when \( k \geq 3 \). Noting that

\[
E_1 \cdots E_{m-1} = E_1^1 \cdots E_{m-1}^1 \oplus \cdots \oplus E_1^k \cdots E_{m-1}^k = \begin{pmatrix}
I_{r_0 \delta(m)} & 0 \\
0 & -I_{(k-r_0)\delta(m)}
\end{pmatrix},
\]

we deduce from (4.10) in the same way as before that for each \( i \),

\[
(4.22) \quad B_{r,s}^i = 0 \quad \text{for either} \quad 1 \leq r, s \leq r_0, \quad \text{or} \quad r_0 + 1 \leq r, s \leq k.
\]

This contradicts (4.20) if \( k \geq 3 \). In fact, similar arguments as in (4.21) imply

\[
(4.23) \quad \sum_{i=1}^{N} |B_i|^2 = 2 \sum_{r \leq r_0} \sum_{s=1}^{N} |B_{r,s}^i|^2 = (k^2 - r_0^2 - (k - r_0)^2)\delta(m)^2.
\]

This contradicts (4.14) and (4.21), since \( \delta(m) \geq m, \ r_0^2 \geq r_0, \ (k - r_0)^2 \geq k - r_0, \) and \( r_0^2 + (k - r_0)^2 > r_0 + (k - r_0) = k \) if \( k \geq 3 \). Hence we are only left with considering the case when \( k = 2 \), namely, the indefinite classes \((m_+, m_-) = (4, 3)\) and \((8, 7)\).

The indefinite class \((m_+, m_-) = (4, 3)\) has been excluded in the assumption. We deduce a contradiction for the last class \((m_+, m_-) = (8, 7)\) as follows. Firstly it follows from (4.22) that in this case (4.19) becomes

\[
B_i : \quad \mathbb{R}^l = \mathbb{R}^8 \oplus \mathbb{R}^8 \to \mathbb{R}^8 \oplus \mathbb{R}^8 = \mathbb{R}^l
\]

\[
B_i = \begin{pmatrix} 0 & C_i \\ -C_i^t & 0 \end{pmatrix},
\]

where \( C_i = B_i^{12} : \mathbb{R}^8 \to \mathbb{R}^8 \). The equation (4.13) becomes

\[
(4.25) \quad \sum_{i=1}^{N} \left( (C_i u^2, u^1) - (C_i u^2, v^1) \right)^2 = (|u^1|^2 + |u^2|^2)(|v^1|^2 + |v^2|^2)
\]

\[
- \left( (u^1, v^1) + (u^2, v^2) \right)^2 - \sum_{\alpha=1}^{7} \left( \langle E_{\alpha}^1 u^1, v^1 \rangle + \langle E_{\alpha}^2 u^2, v^2 \rangle \right)^2,
\]

for any \( u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^8 \oplus \mathbb{R}^8 \). Restricting to \( u^2 = v^2 = 0 \) (or \( u^1 = v^1 = 0 \)), we have

\[
(4.26) \quad \langle u^1, v^1 \rangle^2 + \sum_{\alpha=1}^{7} \langle E_{\alpha}^r u^1, v^1 \rangle^2 = |u^1|^2 |v^1|^2, \quad (u^1, v^1) \in \mathbb{R}^8 \oplus \mathbb{R}^8, \quad r = 1, 2,
\]
which is also trivially implied by the Clifford algebra on the Octonions $\mathbb{R}^8$. Then the equation (4.25) becomes

\begin{equation}
\sum_{i=1}^{N} \left( (C_i v^2, u^1) - (C_i u^2, v^1) \right)^2 = |u^1|^2|v^2|^2 + |u^2|^2|v^1|^2 - 2 \langle u^1, v^1 \rangle \langle u^2, v^2 \rangle - 2 \sum_{\alpha=1}^{7} \langle E_{\alpha}^1 u^1, v^1 \rangle \langle E_{\alpha}^2 u^2, v^2 \rangle.
\end{equation}

Taking $(v^1, v^2) = (-u^1, u^2)$ in (4.27), we obtain

\begin{equation}
\sum_{i=1}^{N} (C_i u^2, u^1)^2 = |u^1|^2|u^2|^2, \quad u = (u^1, u^2) \in \mathbb{R}^8 \oplus \mathbb{R}^8.
\end{equation}

Thus by canceling the equalities in the form (4.28), the equation (4.27) becomes

\begin{equation}
\sum_{i=1}^{N} (C_i u^2, u^1) \langle C_i u^2, v^1 \rangle = \langle u^1, v^1 \rangle \langle u^2, v^2 \rangle + \sum_{\alpha=1}^{7} \langle E_{\alpha}^1 u^1, v^1 \rangle \langle E_{\alpha}^2 u^2, v^2 \rangle,
\end{equation}

for any $u = (u^1, u^2), \ v = (v^1, v^2) \in \mathbb{R}^8 \oplus \mathbb{R}^8$. Taking $(v^1, v^2) = (u^2, u^1)$ in (4.29) and using (4.18), we find

\begin{equation}
\sum_{i=1}^{N} (C_i u^1, u^2) \langle C_i u^2, v^2 \rangle = \langle u^1, u^2 \rangle^2 - \sum_{\alpha=1}^{6} \langle E_{\alpha}^1 u^1, u^2 \rangle^2.
\end{equation}

For any fixed unit vector $u^1 \in \mathbb{R}^8$, it follows from (4.23) that \{ $u^1, E_1^1 u^1, \cdots, E_7^1 u^1$ \} constitutes an orthonormal basis of $\mathbb{R}^8$. Taking contraction of (4.30) with respect to this basis for $u^2$, we obtain

\begin{equation}
\sum_{i=1}^{N} (C_i u^1, u^1) \text{tr}(C_i) = -4|u^1|^2.
\end{equation}

Lastly, taking contraction of (4.31) for $u^1$, we obtain the following contradiction

\begin{equation}
\sum_{i=1}^{N} \left( \text{tr}(C_i) \right)^2 = -32.
\end{equation}

4.3. $m \equiv 3 \pmod{4}, \ (m_+, m_-) \neq (3,4)$. Note that for all cases of $m \neq 0 \pmod{4}$, there exists exactly one geometric equivalence class of Clifford systems \{ $P_0, \cdots, P_m$ \} on $\mathbb{R}^{2l}$ with $l = k\delta(m)$ and thus exactly one congruence class of isoparametric polynomials (cf. [20]). Using the techniques in previous subsections and the representation theory of Clifford algebra for this case, we can show

**Theorem 4.4.** For the case of $m \equiv 3 \pmod{4}, \ (m_+, m_-) \neq (3,4)$, the psd form $G_F$ in (4.3) is non-sos.
Proof. As in the last subsection, we prove it by contradiction. Assume there are quadratic forms \(Q_1, \ldots, Q_N\) such that (4.11) holds, i.e.,

\[
\sum_{i=1}^{N} Q_i(x)^2 = G_F(x) = |x|^4 - \sum_{\alpha=0}^{m} \langle P_\alpha x, x \rangle^2.
\]

We still have the formulae (4.7-4.9) and (4.11-4.14). For this case, there always exists a skew-symmetric operator \(E_m \in O(l)\) such that \(\{E_1, \ldots, E_{m-1}, E_m\}\) generates a Clifford algebra on \(\mathbb{R}^l\) of definite class, i.e., \(E_1 \cdots E_m = I_l\), corresponding to a Clifford system \(\{P_0, \ldots, P_m, P_{m+1}\}\) on \(\mathbb{R}^{2l}\). It follows from (4.12) that \(E_m = -E_1 \cdots E_{m-1}\) commutes with each \(B_i\) as \(m-1\) is even, i.e., \(A_i := B_i E_m = E_m B_i\) is symmetric. Taking \(v = E_m u\), the equation (4.13) gives

\[
\sum_{i=1}^{N} \langle B_i u, E_m u \rangle^2 = \sum_{i=1}^{N} \langle A_i u, u \rangle^2 = |u|^4, \quad u \in \mathbb{R}^l.
\]

Taking Hessian of both sides of the equation (4.32), it follows

\[
\sum_{i=1}^{N} \left( 2 A_i u^t A_i + \langle A_i u, u \rangle A_i \right) = 2 uu^t + |u|^2 I_l, \quad u \in \mathbb{R}^l,
\]

where \(u\) is regarded as a column vector in \(\mathbb{R}^l\). Taking trace of (4.33), we see

\[
\sum_{i=1}^{N} \left( 2A_i^2 + \langle A_i u, u \rangle \text{tr}(A_i) \right) = (2 + l)|u|^2, \quad u \in \mathbb{R}^l,
\]

which is equivalent to

\[
\sum_{i=1}^{N} \left( 2A_i^2 + \text{tr}(A_i) A_i \right) = (2 + l)I_l.
\]

Noting that by (4.14),

\[
\text{tr}(\sum_{i=1}^{N} 2A_i^2) = 2 \sum_{i=1}^{N} |B_i|^2 = 2(l^2 - lm),
\]

we obtain from (4.34) that

\[
\sum_{i=1}^{N} \left( \text{tr}(A_i) \right)^2 = l(2 + 2m - l) \geq 0,
\]

which holds if and only if each \(\text{tr}(A_i) = 0\) and \(m = 3\) or \(7\) (when \(m \equiv 3 \pmod{4}\)), as \(l = k\delta(m)\) increases much more quickly than \(m\). Hence we are only left with considering the cases \((3, 4)\) and \((7, 8)\), where the \((3, 4)\) case has been excluded in the assumption.

We deduce a contradiction for the last case \((m_+, m_-) = (7, 8)\) as follows. In this case, \(l = 16\), \(k = 2\), \(\delta(m) = 8\). According to the representation theory of Clifford algebra, we know there also exists a skew-symmetric operator \(\tilde{E}_m \in O(l)\) such that \(\{E_1, \ldots, E_{m-1}, \tilde{E}_m\}\) generates a Clifford algebra on \(\mathbb{R}^l\) of indefinite class, i.e.,
$E_1 \cdots E_{m-1} \tilde{E}_m \neq I_l$. The difference between $E_m$ and $\tilde{E}_m$ can be shown by their irreducible decompositions as (4.18), i.e.,

$$
\mathbb{R}^l = \mathbb{R}^8 \oplus \mathbb{R}^8 \rightarrow \mathbb{R}^8 \oplus \mathbb{R}^8 = \mathbb{R}^l
$$

(4.35)

$$
E_m = \begin{pmatrix} E_1^m & 0 \\ 0 & E_1^m \end{pmatrix},
\tilde{E}_m = \begin{pmatrix} E_1^m & 0 \\ 0 & -E_1^m \end{pmatrix},
$$

where $E_1^m = -E_1^1 \cdots E_1^{m-1} \in O(8)$. With respect to this decomposition, we rewrite the skew-symmetric operator $B_i$ as (4.19), i.e.,

$$
\mathbb{R}^l = \mathbb{R}^8 \oplus \mathbb{R}^8 \rightarrow \mathbb{R}^8 \oplus \mathbb{R}^8 = \mathbb{R}^l
$$

$$
B_i = \begin{pmatrix} B_{11}^i E_1^m & B_{12}^i \\ -(B_{12}^i)^t & B_{22}^i \end{pmatrix},
$$

where $B_{11}^i$ and $B_{22}^i$ are skew-symmetric. Since $B_i E_m = E_m B_i$, we have

$$
B_{11}^i E_1^m = E_1^m B_{11}^i, \quad B_{22}^i E_1^m = E_1^m B_{22}^i, \quad B_{12}^i E_1^m = E_1^m B_{12}^i.
$$

Setting $\tilde{A}_i := (B_i \tilde{E}_m + \tilde{E}_m B_i)/2$, we derive from the identities above and (4.35) that

$$
(4.36)
$$

$$
\tilde{A}_i = \begin{pmatrix} B_{11}^i E_1^m & 0 \\ 0 & -B_{22}^i E_1^m \end{pmatrix} =: \begin{pmatrix} \tilde{A}_{1i} & 0 \\ 0 & \tilde{A}_{2i} \end{pmatrix}.
$$

Taking $v = \tilde{E}_m u$ in (4.13), we obtain the following formula similar to (4.32)

$$
(4.37)
$$

$$
\sum_{i=1}^N \langle B_i u, \tilde{E}_m u \rangle^2 = \sum_{i=1}^N \langle \tilde{A}_i u, u \rangle^2 = |u|^4, \quad u \in \mathbb{R}^l.
$$

Analogously, from (1.37) we can derive formulae (4.33-4.34) for $\tilde{A}_r$ in place of $A_r$. In particular, the following equality holds

$$
(4.38)
$$

$$
\sum_{i=1}^N \left( 2(\tilde{A}_{ri})^2 + \text{tr}(\tilde{A}_{ri}^1 + \tilde{A}_{ri}^2) \tilde{A}_{ri}^r \right) = 18I_8, \quad r = 1, 2.
$$

In the same way, by restricting to $u = (u^1, 0) \in \mathbb{R}^8 \oplus \mathbb{R}^8$ or $u = (0, u^2) \in \mathbb{R}^8 \oplus \mathbb{R}^8$, we can derive similar formulae (4.33-4.34) for $\tilde{A}_r^r$ ($r = 1, 2$) in place of $A_l$ with $l$ replaced by 8, i.e.,

$$
(4.39)
$$

$$
\sum_{i=1}^N \left( 2(\tilde{A}_{ri})^2 + \text{tr}(\tilde{A}_{ri}^1 + \tilde{A}_{ri}^2) \tilde{A}_{ri}^r \right) = 10I_8, \quad r = 1, 2.
$$

Since

$$
0 = \text{tr}(A_i) = \text{tr}(B_{11}^i E_1^m) + \text{tr}(B_{22}^i E_1^m) = \text{tr}(\tilde{A}_{1i}) - \text{tr}(\tilde{A}_{2i}),
$$
it follows from (4.38) and (4.39) that
\[ \sum_{i=1}^{N} (\hat{A}_i^r)^2 = -\sum_{i=1}^{N} (B_i^{rr})^2 = I_8, \quad \sum_{i=1}^{N} \text{tr}(\hat{A}_i^r)\hat{A}_i^r = 8I_8, \quad r = 1, 2. \]

It follows from (4.40) and the Cauchy-Schwartz inequality
\[ 64 = \sum_{i=1}^{N} (\text{tr}(\hat{A}_i^r))^2 \leq \sum_{i=1}^{N} |B_i^{rr}|^2 |E_m^1|^2 = 64, \quad r = 1, 2, \]
that there exist \( b_i \in \mathbb{R} \) such that \( \sum_{i=1}^{N} (b_i)^2 = 1 \) and
\[ B_i^{11} = b_i E_m^1 = -B_i^{22}, \quad i = 1, \ldots, N. \]
Here the second equality holds because of the relation \( \text{tr}(B_i^{11} E_m^1) = -\text{tr}(B_i^{22} E_m^1) \).
Combining (4.40) and (4.34), we also have
\[ \sum_{i=1}^{N} B_i^{12}(B_i^{12})^t = \sum_{i=1}^{N} (B_i^{12})^t B_i^{12} = 8I_8, \]
which is, however, useless in deducing the contradiction.

Now we go back to analyze the equation (4.13) with respect to the irreducible decomposition (4.35). In this case, we rewrite (4.13) as:
\[ \sum_{i=1}^{N} \left( (B_i^{11} u^1, v^1) + (B_i^{22} u^2, v^2) + (B_i^{12} u^2, v^1) - (B_i^{12} u^1, v^1) \right)^2 \]
\[ = (|u^1|^2 + |u^2|^2)(|v^1|^2 + |v^2|^2) - \left( \langle u^1, v^1 \rangle + \langle u^2, v^2 \rangle \right)^2 \]
\[ - \sum_{\alpha=1}^{6} \left( \langle E_m^1 u^1, v^1 \rangle + \langle E_m^2 u^2, v^2 \rangle \right)^2, \quad u^1, u^2, v^1, v^2 \in \mathbb{R}^8. \]
In the same way as (4.26)-(4.30), by restricting to \( u^2 = v^2 = 0 \) (or \( u^1 = v^1 = 0 \)) in (4.42) and using (4.34), firstly we see
\[ \sum_{i=1}^{N} \langle B_i^{rr} u^1, v^1 \rangle^2 = \langle E_m^1 u^1, v^1 \rangle^2 = |u^1|^2 |v^1|^2 - \langle u^1, v^1 \rangle^2 - \sum_{\alpha=1}^{6} \langle E_m^\alpha u^1, v^1 \rangle^2, \]
for any \( u^1, v^1 \in \mathbb{R}^8 \) and \( r = 1, 2 \). Next by restricting to \( (v^1, v^2) = (-u^1, u^2) \) in (4.42), it follows
\[ \sum_{i=1}^{N} \langle B_i^{12} u^2, u^1 \rangle^2 = |u^1|^2 |u^2|^2, \quad (u^1, u^2) \in \mathbb{R}^8 \oplus \mathbb{R}^8. \]

Substituting these identities into (4.42) and finally restricting to \( (v^1, v^2) = (u^2, u^1) \) in (4.42) and using (4.18), analogous to (4.30) we have the following
\[ \sum_{i=1}^{N} \langle B_i^{12} u^1, u^1 \rangle \langle B_i^{12} u^2, u^2 \rangle = \langle u^1, u^2 \rangle^2 + \langle E_m^1 u^1, u^2 \rangle^2 - \sum_{\alpha=1}^{6} \langle E_m^\alpha u^1, u^2 \rangle^2 \]
\[ + 2 \sum_{i=1}^{N} \langle B_i^{11} u^1, u^2 \rangle \left( \langle B_i^{12} u^2, u^2 \rangle - \langle B_i^{12} u^1, u^1 \rangle \right), \]
where (4.41) has been used in the calculation
\[
\sum_{i=1}^{N} (B_{i}^{11}u, u) (B_{i}^{22}u, u) = (E_{1}^{1}u, u)^2.
\]
Then as in the case of (8, 7) of indefinite class, for any fixed unit vector \(u \in \mathbb{R}^8\), \(\{u, E_{1}^{1}u, \ldots, E_{1}^{7}u\}\) constitutes an orthonormal basis of \(\mathbb{R}^8\). Taking sum of (4.43) for \(u^2 = u, E_{1}^{1}u, \ldots, E_{1}^{7}u\), it gives
\[
\sum_{i=1}^{N} (B_{i}^{12}u, u) \text{tr}(B_{i}^{12}) = -4|u|^2,
\]
since \((B_{i}^{11}u, u) = (B_{i}^{11}u, E_{1}^{1}u) = \cdots = (B_{i}^{11}u, E_{6}^{1}u) = 0\) as \(B_{i}^{11}\) and \(B_{i}^{11}E_{1}^{a}\) (\(a = 1, \ldots, 6\)) are skew-symmetric, and
\[
(B_{i}^{12}E_{1}^{1}u, E_{1}^{1}u) - (B_{i}^{12}u, u) = 0
\]
as \(B_{i}^{12}E_{1}^{1} = E_{1}^{2}B_{i}^{12}\). At last, taking contraction of (4.44) on \(u\), we arrive at the following contradiction
\[
\sum_{i=1}^{N} \left( \text{tr}(B_{i}^{12}) \right)^2 = -32. \tag{4.44}
\]

4.4. \(m \equiv 1, 2 \mod{4}, m \geq 5, (m_+, m_-) \neq (5, 2), (6, 1)\). Using the conclusions in the last subsection for the case of \(m \equiv 3 \mod{4}\), we can show

**Theorem 4.5.** For the cases of \(m \equiv 1, 2 \mod{4}, m \geq 5, \) and \((m_+, m_-) \neq (5, 2), (6, 1)\), the psd form \(G_F\) in (4.3) is non-sos.

**Proof.** As in the last subsection, we prove it by contradiction. Assume there are quadratic forms \(Q_1, \ldots, Q_N\) such that (4.11) holds, i.e.,
\[
\sum_{i=1}^{N} Q_i(x)^2 = G_F(x) = |x|^4 - \sum_{\alpha=0}^{m} \langle P_\alpha x, x \rangle^2.
\]
Firstly it follows from Table 3 that \(l > 8\) as \(m \geq 5\) and the cases of (5, 2), (6, 1) have been excluded. For the Clifford system \(\{P_0, \cdots, P_m\}\) on \(\mathbb{R}^{2l}\), \(\{P_0, \cdots, P_3\}\) is also a Clifford system on \(\mathbb{R}^{2l}\) corresponding to an isoparametric polynomial \(F'(x) = |x|^4 - 2 \sum_{\alpha=0}^{3} \langle P_\alpha x, x \rangle^2\) of OT-FKM type with multiplicities \((3, l-4)\) (which is not equal to \((3, 4)\)). Then the assumption above expresses the nonnegative polynomial \(G_{F'}\) as a sum of squares of quadratic forms:
\[
G_{F'} = |x|^4 - \sum_{\alpha=0}^{3} \langle P_\alpha x, x \rangle^2 = \sum_{i=1}^{N} Q_i(x)^2 + \sum_{\alpha=4}^{m} \langle P_\alpha x, x \rangle^2,
\]
which contradicts Theorem 4.4. \(\square\)
5. On isoparametric with $g = 6$

In this section, we aim to prove that both psd polynomials $G_F^\pm(x)$ in (1.2) are not sos, for isoparametric hypersurfaces with $g = 6$ in $S^7$ ($m_+ = m_- =: m = 1$) and in $S^{13}$ ($m_+ = m_- =: m = 2$) respectively.

**Theorem 5.1.** For $g = 6$, both $G_F^\pm = |x|^6 \pm F(x) \in P_{6m+2,6}$ in (1.2) are non-sos.

In fact, for the case of $m = 1$, we can establish the following stronger result.

**Theorem 5.2.** Any cubic form on $\mathbb{R}^8$ vanishing on the focal submanifold $M_+$ or $M_-$ of dimension 5 is identically zero. In particular, $M_+$ and $M_-$ are not cubic, i.e., not intersections of zeroes of cubic forms, and thus $G_F^\pm$ in (1.2) are non-sos.

**Proof.** According to the classification of isoparametric hypersurfaces with $(g,m) = (6,1)$ by [17], the isoparametric polynomial $F(x)$ is uniquely determined up to congruences. A beautiful observation of Miyaoka [32] states that the isoparametric hypersurfaces are exactly the pull-back of the isoparametric hypersurfaces with $(g,m) = (3,1)$ through the Hopf fiberation. Then we can write the isoparametric polynomial of degree 6 as the composition $F = F_C \circ \pi$, where $F_C$ is Cartan’s isoparametric polynomial of degree 3 as in (2.1), and $\pi : \mathbb{R}^8 \to \mathbb{R}^5$ is the Hopf fiberation

$$\pi(u,v) = (|v|^2 - |u|^2, 2uv), \quad x = (u,v) \in \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8.$$

Let $V : S^2 \times S^3 \to S^7 \subset \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8$ be the map

$$V(t,q) = \left(\frac{\sqrt{3}}{2} (t_1 \mathbf{i} + t_2 \mathbf{j})q, (t_0 + \frac{1}{2} t_1 \mathbf{i} - \frac{1}{2} t_2 \mathbf{j})q\right), \quad t = (t_0, t_1, t_2) \in S^2, \quad q \in S^3.$$

We claim that the image $V(S^2 \times S^3)$ is exactly the focal submanifold $M_+ = F^{-1}(1) \cap S^7 = \pi^{-1}(M_C^\pm)$ (diffeomorphic to $\mathbb{R}P^2 \times S^3$), where $M_C^\pm := F_C^{-1}(1) \cap S^4$ is the focal submanifold of $F_C$, i.e., the Veronese surface $\mathbb{R}P^2$ in $S^4$. In fact, the Cartan polynomial on $\mathbb{R}^5$ of degree 3 in (2.1) can be rewritten as

$$F_C(y) = \frac{3\sqrt{3}}{2} \det \begin{pmatrix} -\frac{1}{\sqrt{3}} y_0 + y_1 & y_2 & y_4 \\ y_2 & \frac{2}{\sqrt{3}} y_0 & y_3 \\ y_4 & y_3 & -\frac{1}{\sqrt{3}} y_0 - y_1 \end{pmatrix}, \quad y = (y_0, \cdots, y_4) \in \mathbb{R}^5.$$

Then let $y$ be a point in the image of $\pi \circ V$, i.e.,

$$y = \pi \circ V(t,q) = \left(\frac{1}{2} (2t_0^2 - t_1^2 - t_2^2), \frac{\sqrt{3}}{2} (t_1^2 - t_2^2), \sqrt{3} t_0 t_1, \sqrt{3} t_0 t_2, \sqrt{3} t_1 t_2\right).$$

It follows that

$$F \circ V(t,q) = F_C(y) = \frac{3\sqrt{3}}{2} \det \left( -\frac{1}{\sqrt{3}} t_3 + \sqrt{3} \begin{pmatrix} t_1 \\ t_0 \\ t_2 \end{pmatrix} \begin{pmatrix} t_1 & t_0 & t_2 \end{pmatrix} \right) = 1.$$
Now let $\Phi(x)$ be a cubic form on $\mathbb{R}^8$ vanishing on the focal submanifold $M_+$. Decompose it as

$$\Phi(u,v) = \psi(v) + P(u,v) + Q(u,v) + \varphi(u), \quad x = (u,v) \in \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8,$$

where $\psi, P, Q, \varphi$ are cubic forms with degree 0, 1, 2, 3 on $u \in \mathbb{H} \cong \mathbb{R}^4$ (and thus with degree 3, 2, 1, 0 on $v \in \mathbb{H}$) respectively. For example, we can set $P(u,v) = P_u(v)$ where $P_u := \sum_{i=1}^{4} u_i P_i$ for certain real symmetric $(4 \times 4)$ matrices $P_i$'s, $u = (u_1, \cdots, u_4) \in \mathbb{H} \cong \mathbb{R}^4$, and $P_u = P_u(v) = (P_u v, v)$ is the quadratic form associated to $P_u$.

As $M_+$ is parameterized by $(u,v) = \mathcal{V}(t,q)$ in (5.1), we investigate firstly the evaluation of $\Phi$ on the points with $t_1 = t_2 = 0$. It follows that $\Phi(0,v) = \psi(v) \equiv 0$ for any $v \in \mathbb{H}$, and thus

$$\Phi(u,v) = P(u,v) + Q(u,v) + \varphi(u).$$

Then we consider the evaluation of $\Phi$ on the points with $r := \sqrt{t_1^2 + t_2^2} > 0$. Setting $t_1/r =: \cos \theta$, $t_2/r =: \sin \theta$ and $w := (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) q \in S^3$, we calculate

$$\Phi\left(\frac{\sqrt{3}}{2} (t_1 \mathbf{i} + t_2 \mathbf{j}) q, (t_0 + \frac{1}{2} t_1 \mathbf{i} - \frac{1}{2} t_2 \mathbf{j}) q\right)
= \Phi\left(\frac{\sqrt{3}}{2} r w, -t_0 (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) w + \frac{1}{2} r e^{-2\theta} \mathbf{k} w\right)
= \frac{\sqrt{3}}{2} r P\left(w, -t_0 (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) w + \frac{1}{2} r e^{-2\theta} \mathbf{k} w\right)
+ \frac{3}{4} r^2 \left(-t_0 Q\left(w, (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) w\right) + \frac{1}{2} r Q\left(w, e^{-2\theta} \mathbf{k} w\right)\right)
+ \frac{3\sqrt{3}}{8} r^3 \varphi(w)
$$

$$= \frac{\sqrt{3}}{2} r^2 t_0^2 P\left(w, (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) w\right)
- \frac{\sqrt{3}}{2} r^2 t_0 \left(P\left(\cos \theta \mathbf{i} + \sin \theta \mathbf{j} w, e^{-2\theta} \mathbf{k} w\right) + \frac{\sqrt{3}}{2} Q\left(\cos \theta \mathbf{i} + \sin \theta \mathbf{j} w\right)\right)
+ \frac{\sqrt{3}}{8} r^3\left(P\left(w, e^{-2\theta} \mathbf{k} w\right) + \sqrt{3} Q\left(w, e^{-2\theta} \mathbf{k} w\right) + 3 \varphi(w)\right)
$$

$$= 0,$$

for any $0 < r < 1$ with $r^2 + t_0^2 = 1$, $\theta \in \mathbb{R}$ and for any $w \in \mathbb{H}$. By comparing the degree of $r$ or using a coordinate translation ($r = \cos \phi, t_0 = \sin \phi$), one can easily deduce from the preceding identity that

$$P\left(w, (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) w\right) \equiv 0,$$

$$P\left(\cos \theta \mathbf{i} + \sin \theta \mathbf{j} w, e^{-2\theta} \mathbf{k} w\right) + \frac{\sqrt{3}}{2} Q\left(\cos \theta \mathbf{i} + \sin \theta \mathbf{j} w\right) \equiv 0,$$

$$P\left(w, e^{-2\theta} \mathbf{k} w\right) + \sqrt{3} Q\left(w, e^{-2\theta} \mathbf{k} w\right) + 3 \varphi(w) \equiv 0, \quad \text{for any } \theta \in \mathbb{R}, w \in \mathbb{H}.$$
The identity (5.4) will lead to
\[
\cos^2 \theta P(w, i w) + \sin^2 \theta P(w, j w) + \sin 2 \theta P_w(i w, j w) \equiv 0, \quad \text{for any } \theta \in \mathbb{R}, w \in \mathbb{H}.
\]
This implies
\[
(5.7) \quad P(w, i w) = P(w, j w) = P_w(i w, j w) \equiv 0, \quad \text{for any } w \in \mathbb{H}.
\]
Computing the identity (5.5), we obtain
\[
\cos \theta \cos 2 \theta P_w(i w, w) - \sin \theta \sin 2 \theta P_w(j w, k w) + \frac{\sqrt{3}}{2} \cos \theta Q(w, i w)
\]
\[
+ \sin \theta \cos 2 \theta P_w(j w, w) - \cos \theta \sin 2 \theta P_w(i w, k w) + \frac{\sqrt{3}}{2} \sin \theta Q(w, j w) \equiv 0,
\]
for any \( \theta \in \mathbb{R} \) and \( w \in \mathbb{H} \). This implies
\[
(5.8) \quad P_w(i w, w) = P_w(j w, k w) = \frac{\sqrt{3}}{2} Q(w, i w) =: \beta(w) =: \beta,
\]
\[
P_w(j w, w) = P_w(i w, k w) = \frac{\sqrt{3}}{2} Q(w, j w) =: \gamma(w) =: \gamma,
\]
for any \( w \in \mathbb{H} \), where \( \beta, \gamma \) are denoted to be the corresponding cubic forms.

Computing the identity (5.6), we deduce
\[
\cos^2 2 \theta P(w, w) + \sin^2 2 \theta P(w, k w) + 3 \varphi(w)
\]
\[
- \sin 4 \theta P_w(w, k w) + \sqrt{3} \cos 2 \theta Q(w, w) - \sqrt{3} \sin 2 \theta Q(w, k w) \equiv 0,
\]
for any \( \theta \in \mathbb{R} \) and \( w \in \mathbb{H} \). This implies
\[
(5.9) \quad P(w, w) = P(w, k w) = -3 \varphi(w) =: \alpha(w) =: \alpha,
\]
P\(_w(w, k w) = Q(w, w) = Q(w, k w) \equiv 0, \quad \text{for any } w \in \mathbb{H}
\]
where \( \alpha \) is denoted to be the corresponding cubic form.

As \( \{w, i w, j w, k w\} \) form an orthonormal basis of \( \mathbb{H} \) for any \( w \in S^3 \), under this basis we deduce the matrix \( P_w \) from (5.7, 5.8, 5.9) as
\[
|w|^2 P_w(w, i w, j w, k w) = (w, i w, j w, k w) \begin{pmatrix}
\alpha & -\beta & \gamma & 0 \\
-\beta & 0 & 0 & \gamma \\
\gamma & 0 & 0 & \beta \\
0 & \gamma & \beta & \alpha
\end{pmatrix}.
\]
Or alternatively,

\begin{equation}
|w|^4 P_w = (w, i w, j w, k w) \begin{pmatrix}
\alpha & -\beta & \gamma & 0 \\
-\beta & 0 & 0 & \gamma \\
\gamma & 0 & 0 & \beta \\
0 & \gamma & \beta & \alpha
\end{pmatrix} (w, i w, j w, k w)^t.
\end{equation}

Let \( w = 1, i, j, k \) respectively, and let \( \alpha_i, \beta_i, \gamma_i, P_i \) \( (i = 1, 2, 3, 4) \) denote the corresponding values of \( \alpha(w), \beta(w), \gamma(w) \) and \( P_w \). In this way,

\begin{equation}
P_1 = \begin{pmatrix}
\alpha_1 & -\beta_1 & \gamma_1 & 0 \\
-\beta_1 & 0 & 0 & \gamma_1 \\
\gamma_1 & 0 & 0 & \beta_1 \\
0 & \gamma_1 & \beta_1 & \alpha_1
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
0 & \beta_2 & -\gamma_2 & 0 \\
\beta_2 & \alpha_2 & 0 & -\gamma_2 \\
-\gamma_2 & 0 & \alpha_2 & -\beta_2 \\
0 & -\gamma_2 & -\beta_2 & 0
\end{pmatrix},
\end{equation}

\begin{equation}
P_3 = \begin{pmatrix}
0 & \beta_3 & -\gamma_3 & 0 \\
\beta_3 & \alpha_3 & 0 & -\gamma_3 \\
-\gamma_3 & 0 & \alpha_3 & -\beta_3 \\
0 & -\gamma_3 & -\beta_3 & 0
\end{pmatrix}, \quad P_4 = \begin{pmatrix}
\alpha_4 & -\beta_4 & \gamma_4 & 0 \\
-\beta_4 & \alpha_4 & 0 & \gamma_4 \\
\gamma_4 & 0 & \beta_4 & \alpha_4 \\
0 & \gamma_4 & \beta_4 & \alpha_4
\end{pmatrix}.
\end{equation}

Noting that \( P_w = \sum_{i=1}^4 w_i P_i \) is linear about \( w \), we see that the \((1, 4)\) entry and \((2, 3)\) entry of \( P_w \) vanish for any \( w \in \mathbb{H} \). Calculating the \((1, 4)\) entry and \((2, 3)\) entry of \(|w|^4 P_w\) from the preceding formula (5.10), we obtain the following two equations:

\[-2\gamma(w_1 w_2 + w_3 w_4) - 2\beta(w_1 w_3 - w_2 w_4) \equiv 0,\]

\[2\gamma(w_1 w_2 + w_3 w_4) - 2\beta(w_1 w_3 - w_2 w_4) \equiv 0,\]

which implies that \( \beta(w) = \gamma(w) \equiv 0 \). Now since \( Q(u, v) \) is linear about \( v \), we conclude from (5.8), (5.9) that \( Q(u, v) \equiv 0 \). Moreover, \( P_w \) can be given explicitly

\begin{equation}
|w|^4 P_w = (w, i w, j w, k w) \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha
\end{pmatrix} (w, i w, j w, k w)^t
\end{equation}

\[
= \alpha \begin{pmatrix}
w_1^2 + w_4^2 & w_1 w_2 + w_3 w_4 & w_1 w_3 - w_2 w_4 & 0 \\
w_1 w_2 + w_3 w_4 & w_2^2 + w_3^2 & 0 & w_2 w_4 - w_1 w_3 \\
w_1 w_3 - w_2 w_4 & w_2^2 + w_3^2 & w_1 w_2 + w_3 w_4 & 0 \\
0 & w_2 w_4 - w_1 w_3 & w_1 w_2 + w_3 w_4 & w_1^2 + w_2^2
\end{pmatrix}.
\]

On the other hand, a calculation by using formula (5.11) implies

\[P_w = \sum_{i=1}^4 w_i P_i = \text{diag}(\alpha_1 w_1 + \alpha_4 w_4, \alpha_2 w_2 + \alpha_3 w_3, \alpha_2 w_2 + \alpha_3 w_3, \alpha_1 w_1 + \alpha_4 w_4).\]
Combining these two expressions of $P_w$ will lead us to
\[ |w|^4(\alpha_1w_1 + \alpha_4w_4) = \alpha(w_1^4 + w_4^4), \quad |w|^4(\alpha_2w_2 + \alpha_3w_3) = \alpha(w_2^4 + w_3^4), \]
\[ \alpha(w_1w_2 + w_3w_4) = \alpha(w_1w_3 - w_2w_4) \equiv 0, \quad \text{for any } w \in \mathbb{H}. \]
This implies that $\alpha \equiv 0$, $P_w \equiv 0$ and thus $P(u, v) = P_u(v) \equiv 0$, and $\varphi(u) \equiv 0$ by \([5.9]\).
In conclusion, $\Phi(u, v) = P(u, v) + Q(u, v) + \varphi(u) \equiv 0$, any cubic form $\Phi$ vanishing on $M_+$ is identically zero as desired.

Now we turn to consider the question on $M_- := F^{-1}(-1) \cap S^7$ which is diffeomorphic but not antipodal (nor isometric) to $M_+$. A cubic form vanishing on $M_-\ldots$ may not vanish on $M_+$. However, the images of $M_\pm$ under the Hopf fibration $\pi$
\[ M_-^C := \pi(M_-) = F_C^{-1}(-1) \cap S^4 = -F_C^{-1}(1) \cap S^4 = -\pi(M_+) = -M_+^C \]
are antipodal to each other. Observing the identity \((5.3)\), we parameterize points of $M_+$ alternatively by
\[ M_+ = \left\{ \left( \frac{\sqrt{3}}{2} \cos \phi \ w, -\sin \phi \ (\cos \theta \ i + \sin \theta \ j)w + \frac{1}{2} \cos \phi \ e^{-2\theta \ k}w \right) \mid \theta, \phi \in \mathbb{R}, \ w \in S^3 \right\}. \]
A long but straightforward calculation shows that $M_-$ can also be parameterized as $M_+^C$ above by
\[ M_- = \left\{ \left( \sin \phi \ (\cos \theta \ i + \sin \theta \ j)w + \frac{1}{2} \cos \phi \ e^{2\theta \ k}w, -\frac{\sqrt{3}}{2} \cos \phi \ w \right) \mid \theta, \phi \in \mathbb{R}, \ w \in S^3 \right\}. \]
In fact, let $t_0 := \sin \phi, \ t_1 := \cos \phi \cos \theta, \ t_2 := \cos \phi \sin \theta$ as before, then it can be easily verified from $M_- = \pi^{-1}(-M_+^C)$ by the parametrization of $M_+^C$ in \((5.2)\). Then by the same argument as on $M_+$, we can show that any cubic form $\Phi$ vanishing on $M_-$ is identically zero. \[ Q.E.D. \]

To prove Theorem \(5.1\) the last case we are left with considering is the isoparametric polynomial with $(g, m) = (6, 2)$. Fortunately, isoparametric hypersurfaces in this case have been classified by Miyaoka \((33, 34)\) to be the unique homogeneous class. Moreover, the explicit formulæ of the isoparametric polynomials representing the homogeneous isoparametric hypersurfaces with $g = 6, m = 1, 2$, denoted by $F_1(x)$ ($x \in \mathbb{R}^8$) and $F_2(X)$ ($X \in \mathbb{R}^{14}$) respectively, were given by Ozeki and Takeuchi \((37)\), and then were simplified by Peng and Hou \((40)\). The points $X \in \mathbb{R}^{14}$ are written in terms of the skew-Hermitian matrix representation of the exceptional simple Lie algebra $\mathfrak{g}_2$, while the points $x \in \mathbb{R}^8$ are identified with the real symmetric matrices in $\mathfrak{p}$ of the Cartan decomposition $\mathfrak{g}_2 = \mathfrak{k} + \sqrt{-1} \mathfrak{p}$ with $\mathfrak{k}$ being the real skew-symmetric part. Explicitly, we can write $X = K + \sqrt{-1}x$ with the real symmetric part $x$ and the real skew-symmetric part $K$ in the following form \((41)\):
\[ x = \begin{pmatrix} 0 & Y & -Y \\ Y^t & T & S \\ -Y^t & -S & -T \end{pmatrix}, \quad K = \begin{pmatrix} 0 & u & u \\ -u^t & U & V \\ -u^t & V & U \end{pmatrix}, \]
\[ \]
where

\[ Y = \frac{1}{\sqrt{3}}(y_1, y_2, y_3), \quad u = \frac{1}{\sqrt{3}}(u_1, u_2, u_3), \]

\[ T = \begin{pmatrix} t_1 & \frac{1}{\sqrt{2}}y_4 & \frac{1}{\sqrt{2}}y_5 \\ \frac{1}{\sqrt{2}}y_4 & t_2 & \frac{1}{\sqrt{2}}y_6 \\ \frac{1}{\sqrt{2}}y_5 & \frac{1}{\sqrt{2}}y_6 & t_3 \end{pmatrix}, \quad S = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}, \]

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_4 & u_5 \\ -u_4 & 0 & u_6 \\ -u_5 & -u_6 & 0 \end{pmatrix}, \quad V = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, \]

where \( t_1 + t_2 + t_3 = 0, y_i, u_i \) (\( i = 1, \ldots, 6 \)) are real numbers. Then the isoparametric polynomials with \( g = 6, m = 1, 2 \), can be given as

\[ (5.13) \quad F_1(x) := 18\text{tr}(x^6) - \frac{5}{4}(\text{tr}(x^2))^3, \quad F_2(X) := 18\text{tr}(X^6) - \frac{5}{4}(\text{tr}(X^2))^3. \]

Clearly we have \( F_1(x) = -F_2(X) \) for \( X = \sqrt{-1}x \) with \( K = 0 \).

Now we are ready to prove Theorem 5.1 for the last case \( (g, m) = (6, 2) \). Assume that \( G_{F_2}(X) := |X|^6 \pm F_2(X) = \sum_\alpha \Phi_{\alpha}(X)^2 \) is a sum of squares of some cubic forms \( \Phi_{\alpha}(X) := \Phi_{\alpha}(K, x) \) on \( X = K + \sqrt{-1}x := (K, x) \in \mathbb{R}^{14} \). Then restricting to \( \sqrt{-1}p \) \( (X = \sqrt{-1}x = (0, x) \in \mathbb{R}^8 \text{ with } K = 0) \), we deduce

\[ G_{F_1}^+(x) := |x|^6 \mp F_1(x) = |X|^6 \pm F_2(X) = \sum_\alpha \Phi_{\alpha}(0, x)^2 \]

is a sum of squares of cubic forms. This contradicts Theorem 5.2. \( \square \)

We conclude this section with a remark. It can be conjectured that a statement similar to Theorem 5.2 holds in the case of \( (g, m) = (6, 2) \). More precisely, we conjecture that any cubic form on \( \mathbb{R}^{14} \) vanishing on the focal submanifold \( M_+ \text{ or } M_- \) of dimension 10 is identically zero. Unfortunately, due to extremely complicated computations, we failed to give a proof.

6. Further remarks and applications

In this section we present further discussions on the \( \text{psd} \) forms \( G_{F}^+ \) of (1.2) and their zeroes.

6.1. Relations with Lagrange’s identity. Let us start with the \( \text{psd} \) forms \( G_{F}^+ \), denoted by \( G_{km}(x) := \frac{1}{4}G_{F}^+(x) \), for isoparametric polynomials on \( \mathbb{R}^{2l} \) of OT-FKM type with \( g = 4, l = k\delta(m), k \geq 2, m = 1, 2, 3 \) and the definite class of \( m = 4 \), respectively.

The nonnegativity of \( G_{km}(x) \) shows Cauchy-Schwarz’s inequality for real, complex and quaternionic vectors in \( \mathbb{F}^k \) with \( \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) for \( m = 1, 2, 4, \) respectively, namely,

\[ G_{km}(X, Y) = |X|^2|Y|^2 - |\langle X, Y \rangle_H|^2 \geq 0, \quad \text{for } x = (X, Y) \in \mathbb{F}^k \oplus \mathbb{F}^k, \quad k \geq 2, \]
where $\langle X, Y \rangle_H = \sum_{i=1}^k X_i Y_i$ is the Hermitian product. For $m = 1, 2$, the nonnegativity also follows from the Lagrange identity

$$|X|^2 |Y|^2 - |\langle X, Y \rangle_H|^2 = |X \wedge Y|^2, \quad \text{for } X, Y \in \mathbb{R}^k \text{ or } \mathbb{C}^k, \quad k \geq 2,$$

which is exactly Solomon’s sos expression of $G_{km}(x)$ in (4.3). However for $m = 4$, usually one needs another approach to prove the nonnegativity (see [22] for example), since there no longer holds Lagrange’s identity for quaternions; in fact, there do not even exist sos expressions for $G_{km}(x)$ (proved in Theorem 4.12). As for the expression in (6.1), we recall (4.13) where $G_{km}(x) = \frac{1}{4} G_F^+(x) \in P_{2l,4} \setminus \Sigma_{2l,4}$ can be rewritten as

$$G_{km}(u, v) = |u|^2 |v|^2 - \langle u, v \rangle^2 - \sum_{\alpha=1}^{m-1} \langle E_{\alpha} u, v \rangle^2, \quad x = (u, v) \in \mathbb{R}^l \oplus \mathbb{R}^l.$$  

Now $m = 4$, $l = 4k$, there is a natural isomorphism $\mathbb{R}^l \cong \mathbb{H}^k$ which identifies $(u, v) = (X, Y) \in \mathbb{H}^k \oplus \mathbb{H}^k$. Because the Clifford system is definite, the Clifford algebra $\{E_1, E_2, E_3\}$ on $\mathbb{R}^{4k}$ corresponds to the left quaternionic product by $\{i, j, k\}$ on $\mathbb{H}^k$ and thus

$$|\langle X, Y \rangle_H|^2 = \langle u, v \rangle^2 + \langle E_1 u, v \rangle^2 + \langle E_2 u, v \rangle^2 + \langle E_3 u, v \rangle^2.$$  

On the other hand, Proposition 2.2 tells us the following sosr expression for $G_{k4}(x)$ with a uniform denominator $|x|^2$:

$$4|x|^2(|X|^2 |Y|^2 - |\langle X, Y \rangle_H|^2) = |\nabla G_{k4}(x)|^2, \quad \text{for } x = (X, Y) \in \mathbb{H}^k \oplus \mathbb{H}^k, \quad k \geq 2.$$  

However, this does not generalize Lagrange’s identity for quaternions. Alternatively, we use the sos expression for $H_F(x)$ in Proposition 2.1

$$|\nabla F(x) \wedge x|^2 = 16 H_F(x) = 16 G_F^+(x) G_F^-(x).$$

This indeed generalizes Lagrange’s identity for quaternions:

$$4 \left( \sum_{\alpha=0}^4 \langle P_{\alpha} x, x \rangle^2 \right) \left( |X|^2 |Y|^2 - |\langle X, Y \rangle_H|^2 \right) = \left| \sum_{\alpha=0}^4 \langle P_{\alpha} x, x \rangle P_{\alpha} x \wedge x \right|^2,$$

where for $x = (X, Y)$,

$$P_0 x = (X, -Y), \quad P_1 x = (Y, X),$$

$$P_{\alpha+1} x = (E_\alpha X, -E_\alpha Y) = (i Y, -i X), \quad (j Y, -j X), \quad (k Y, -k X),$$

for $\alpha = 1, 2, 3$, respectively. Note also that when $X, Y \in \mathbb{R}^k$ or $\mathbb{C}^k$, (6.3) reduces to the classical Lagrange identity. Summarizing the arguments above, we have shown

**Proposition 6.1.** For $X, Y \in \mathbb{H}^k$, $k \geq 2$, the psd polynomial $|X|^2 |Y|^2 - |\langle X, Y \rangle_H|^2$ is not sos. However, the polynomial $(|X|^2 + |Y|^2)(|X|^2 |Y|^2 - |\langle X, Y \rangle_H|^2)$ is sos with a concrete representation. Furthermore, a generalized Lagrange identity (6.3) holds for the quaternionic case.
Now we turn to the case of $m = 3$. The Clifford system $\{P_0, \cdots, P_3\}$ is just that of $m = 4$ by deleting $P_4$ from (6.3). In this way, we can rewrite the psd form $G_{k3}(x) \in P_{8k,4}$, $k \geq 2$, by: for $x = (X, Y) \in \mathbb{H}^k \oplus \mathbb{H}^k$,

$$G_{k3}(X, Y) = G_{k4}(X, Y) + \left( \text{Re}(k\langle X, Y \rangle_H) \right)^2$$

(6.5)

$$= |X|^2|Y|^2 - |\langle X, Y \rangle_H|^2 + \left( \text{Re}(k\langle X, Y \rangle_H) \right)^2.$$ 

Similarly, we can rewrite the cases of $m = 1, 2$ for quaternions by:

$$G_{2k,2}(X, Y) = G_{k3}(X, Y) + \left( \text{Re}(j\langle X, Y \rangle_H) \right)^2$$

(6.6)

$$= |X|^2|Y|^2 - \left( \text{Re}(\langle X, Y \rangle_H) \right)^2 - \left( \text{Re}(i\langle X, Y \rangle_H) \right)^2;$$

$$G_{4k,1}(X, Y) = G_{2k,2}(X, Y) + \left( \text{Re}(i\langle X, Y \rangle_H) \right)^2$$

(6.7)

$$= |X|^2|Y|^2 - \left( \text{Re}(\langle X, Y \rangle_H) \right)^2,$$

for $X, Y \in \mathbb{H}^k \cong \mathbb{C}^{2k} \cong \mathbb{H}^{4k}$, $k \geq 2$.

Noting that $G_{k3}$ corresponds to $\frac{1}{8}G^+_F$ for isoparametric polynomials of OT-FKM type with $(m_+, m_-) = (3, 4k - 4)$, we conclude from Theorem 4.1 the following

**Corollary 6.2.** The psd form $G_{k3} \in P_{8k,4}$ is sos if $k = 2$, and non-sos if $k \geq 3$.

**Remark 6.3.** From the corollary above and (6.6)(6.7), one can see that the non-sos psd form $G_{k4} \in P_{8k,4}$ (resp. $G_{k3} \in P_{8k,4}$) would turn to be sos if an additional square of a quadratic form is added in the case of $k = 2$ (resp. of $k \geq 3$).

### 6.2. Applications to orthogonal multiplication.

Recall that an orthogonal multiplication of type $[p, q, r]$, $p \leq q$, is a bilinear map

$$T : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^r$$

such that $|T(u, v)| = |u||v|$ for all $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. The existence problem of a given type $[p, q, r]$ has been studied many times but is still open (see [31] and references therein). The case when $p = q$ is of particular interest for its important applications in geometry, e.g., harmonic maps from $S^{2p-1}$ to $S^r$ by Hopf construction. For example, this will produce the classical harmonic maps (i.e., Hopf fibrations) from $S^{2m-1}$ to $S^m$ with $m = 1, 2, 4, 8$. Furthermore, by deforming the Hopf construction into a harmonic map, one can establish many harmonic representations in the classes of homotopy groups of spheres (see [38, 39]).

Now for the infinitely many classes of isoparametric polynomials $F(x)$ of OT-FKM type, we have shown in (4.13) that

$$\frac{1}{8}G^+_F(x) = |u|^2|v|^2 - \langle u, v \rangle^2 - \sum_{\alpha=1}^{m-1} \langle E_\alpha u, v \rangle^2 \in P_{2l,4}, \quad x = (u, v) \in \mathbb{R}^l \times \mathbb{R}^l.$$  

(6.8)
This result shows immediately:

**Corollary 6.4.** Orthogonal multiplications of type \([l, l, m, r]\) \((r \geq 1)\) with the form

\[ T(u, v) = \left( \langle u, v \rangle, \langle E_1 u, v \rangle, \cdots, \langle E_{m-1} u, v \rangle, T_1(u, v), \cdots, T_r(u, v) \right), \]

where \(T_i(u, v)\)'s are nonzero bilinear functions, exist if and only if the Clifford algebra \( \{E_1, \cdots, E_{m-1}\} \) on \( \mathbb{R}^{2d} \) occurs in the sos cases of Theorem 4.1. In this case,

\[ (m, l) = (1, k\mathbb{R}, (2, 2k), (3, 8), (4, 8), (5, 8), (6, 8), k \geq 2. \]

### 6.3. Applications to the sos problem on Grassmannian \(Gr_2(\mathbb{R}^l)\).

The non-sos psd forms \(G_F^+ \in P_{2,4}\) will provide examples of psd quadratic forms that are non-sos on the oriented Grassmannian \(Gr_2(\mathbb{R}^l)\). For our purpose, we regard \(Gr_p(\mathbb{R}^l)\) as a quadratic variety in \(S_{(l)}^{l-1} \subset \Lambda^p(\mathbb{R}^l)\) by the Plücker relations, though \(Gr_2(\mathbb{R}^l) \cong Q^{l-2}(\mathbb{C}) = \{[z] \in \mathbb{CP}^{l-1} | z_1^2 + \cdots + z_l^2 = 0\}\) is more understandable, via \(x \wedge y \rightarrow x + iy\) where \(x, y \in \mathbb{R}^l\) is an orthonormal basis of an oriented plane.

Recall that a real homogeneous polynomial is called nonnegative (psd) on a variety \(X \subset \mathbb{CP}^n\) if its evaluation at each real point is nonnegative, and is called sos on \(X\) if it is a polynomial sum of squares modulo the defining polynomials, ideal \(I(X)\) consisting of polynomials vanishing on \(X\). For example, if \(X\) is the Grassmannian, the ideal \(I(X)\) is generated by the quadratic forms of the Plücker relations (cf. \[24\]). In particular, an exterior 2-form \(\omega \in \Lambda^2(\mathbb{R}^l)\) (\(F = \mathbb{R}\) or \(\mathbb{C}\)) is decomposable (namely, belongs to \(Gr_2(\mathbb{R}^l)\)) if and only if \(\omega \land \omega = 0\), which represents \(\binom{l}{2}\) independent quadratic relations.

A celebrated result of Blekherman-Smith-Velasco \[12\] identifies all of the real projective varieties \(X \subset \mathbb{CP}^n\), on which every nonnegative quadratic function is a polynomial sum of squares modulo the defining ideal of \(X\), to be those varieties of minimal degree \(\text{deg}(X) = \text{codim}(X) + 1\). This generalizes Hilbert’s theorem from \(X = \mathbb{CP}^1 \subset \mathbb{CP}^2\) by Veronese maps (i.e., \(P_{2,d} = \Sigma_{2,d}\)), or \(X = \{[x^2 : xy : xz : y^2 : yz : z^2] \in \mathbb{CP}^5 \mid [x : y : z] \in \mathbb{CP}^2\}\) (i.e., \(P_{3,4} = \Sigma_{3,4}\)) to all varieties of minimal degree (which have been classified in algebraic geometry). It is well known that the Grassmannian \(Gr_2(\mathbb{C}^l) \subset \mathbb{CP}^{l-1}\) \((l > 4)\) is not of minimal degree. Hence, there exist non-sos nonnegative quadratic forms on \(Gr_2(\mathbb{C}^l)\) \((l > 4)\). Illustrating this result of Blekherman-Smith-Velasco, our examples (Corollary 6.5 below) give explicit examples of such non-sos nonnegative quadratic forms on \(Gr_2(\mathbb{C}^l)\). Recently, Bettiol-Kummer-Mendes \[4\] showed, among several important results, that the closed convex cone \(P_{Gr_k(\mathbb{C}^l)}\) of nonnegative quadratic forms on \(Gr_k(\mathbb{C}^l)\) \((2 \leq k \leq l - 2\) and \(l \geq 5)\) is not a spectrahedral shadow. In their proof the existence of a non-sos nonnegative quadratic form on \(Gr_k(\mathbb{C}^l)\) plays a key role.

\[2\]Rigorously, the case of \((m, l) = (1, 2)\) does not come from isoparametric polynomials with \(g = 4\) because now \(m_+ = l - m - 1 = 0.\)
Now let \( \{E_1, \ldots, E_{m-1}\} \) be a Clifford algebra on \( \mathbb{R}^l \), \( (l = k\delta(m) \geq 8, m \geq 3) \), whose associated psd form \( G_F^{-} \in P_{2l,4} \) is non-sos in Theorem 4.1. Let \( \varphi_\alpha \in \Lambda^2(\mathbb{R}^l) \) be the exterior 2-forms defined by \( \varphi_\alpha(u,v) := (E_\alpha u, v), \alpha = 1, \ldots, m-1 \). Note that we can also regard \( \varphi_\alpha \) as linear functions on \( \Lambda^2(\mathbb{R}^l) \) by setting \( \varphi_\alpha(u \wedge v) := \varphi_\alpha(u, v) \). Then using Lagrange’s identity we can rewrite (6.8) as
\[
\Phi(\omega) := |\omega|^2 - \sum_{\alpha=1}^{m-1} (\varphi_\alpha(\omega))^2, \quad \omega \in \Lambda^2(\mathbb{R}^l),
\]
is a non-sos psd quadratic form on \( Gr_2(\mathbb{C}^l) \subset \mathbb{CP}(l)^-1 \).

**Corollary 6.5.** Associated to each non-sos psd form \( G_F^{-} \in P_{2l,4} \) in Theorem 4.1
\[
\Phi_F(\omega) := |\omega|^2 - \sum_{\alpha=1}^{m-1} (\varphi_\alpha(\omega))^2, \quad \omega \in \Lambda^2(\mathbb{R}^l),
\]
is a non-sos psd quadratic form on \( Gr_2(\mathbb{C}^l) \subset \mathbb{CP}(l)^-1 \).

**Proof.** Otherwise, suppose there were linear functions \( \psi_1, \ldots, \psi_r \) (exterior 2-forms in \( \Lambda^2(\mathbb{R}^l) \)) such that
\[
\Phi_F(\omega) = \sum_{i=1}^{r} (\psi_i(\omega))^2 + P(\omega),
\]
for some quadratic form \( P \) in the span of the Plücker relations. Restricting \( \Phi \) to \( u \wedge v \in Gr_2(\mathbb{R}^l) \), we would get
\[
|u \wedge v|^2 - \sum_{\alpha=1}^{m-1} (\varphi_\alpha(u \wedge v))^2 = \sum_{i=1}^{r} (\psi_i(u \wedge v))^2 \in \Sigma_{2l,4},
\]
a contradiction to (6.9). \( \square \)

Furthermore, these non-sos forms also provide counterexamples to a generalized version of the Harvey-Lawson sos problem ([27, Question 6.5]).

**Problem 6.6 (Harvey-Lawson sos Problem).** Given a p-form \( \varphi \in \Lambda^p(\mathbb{R}^l) \) such that
\[
\Phi(u_1, \ldots, u_p) := |u_1 \wedge \cdots \wedge u_p|^2 - \varphi(u_1, \ldots, u_p)^2, \quad u_1, \ldots, u_p \in \mathbb{R}^l,
\]
is nonnegative with nontrivial zeroes (i.e., \( \varphi \) has comass one), are there exterior p-forms \( \psi_1, \ldots, \psi_r \) such that the following sos expression holds
\[
\Phi(u_1, \ldots, u_p) = \sum_{i=1}^{r} \psi_i(u_1, \ldots, u_p)^2 ?
\]

In other words, it asks whether all the psd quadratic forms \( \Phi(\omega) = |\omega|^2 - \varphi(\omega)^2 \) are sos on \( Gr_p(\mathbb{C}^l) \subset \mathbb{CP}(l)^-1 \). Thus Corollary 6.5 shows that for \( p = 2 \), this sos problem cannot be generalized to ask for a sos expression of psd quadratic forms with the form \( \Phi(\omega) = |\omega|^2 - \sum_{\alpha=1}^{m} (\varphi_\alpha(\omega))^2 \) when \( m \geq 2 \).
6.4. **Zeros of the non-sos psd forms** \(G^+_F\). To conclude this section, we would like to discuss further on the question of Solomon ([44]) as to whether both focal submanifolds of an isoparametric hypersurface with \(g = 4\) in a unit sphere

\[ M_\pm = (G^+_F)^{-1}(0) \cap S^{n-1} \]

are quadratic.

The focal submanifolds \(M_+\) are already quadratic (see (3.1, 3.2, 4.2)). Still this question is important because of his result: A quadratic form vanishing on one focal submanifold is an eigenfunction of the minimal isoparametric hypersurface \(M\) corresponding to the second known eigenvalue \(2 \dim M\) of the Laplacian on \(M\). A well known conjecture of Yau asserts that the first eigenvalue is \(\dim M\) for a closed embedded minimal hypersurface \(M\) in a unit sphere, which was proved in the isoparametric case by Tang and Yan [47]. As we have shown in Sections 3 and 4, \(M_-\) in the exceptional two classes (2, 2), (5, 4) and the OT-FKM type with \(m \equiv 0 (mod 4)\) of the definite class are not quadratic, by proving that there exist no nonzero quadratic forms vanishing on them. In other classes \(M_-\) may admit non-trivial quadratic forms vanishing on them. For example, for those OT-FKM type whose Clifford system \(\{P_0, \cdots, P_m\}\) can be extended to a Clifford system \(\{P_0, \cdots, P_m, P_{m+1}\}\) (there are many such classes, e.g., when \(m \equiv 3, 5, 6, 7 (mod 8)\) or \(m \equiv 0 (mod 4)\) of the indefinite class), it is not difficult to show that the extended quadratic form

\[ P_{m+1}(x) := \langle P_{m+1}, x \rangle \]

vanishes on \(M_-\). However, it seems probable that for all the classes with non-sos \(G^+_F\) in Table 2 \(M_-\) is not a quadratic variety. If so, it would give a complete answer to the question of Solomon.

**Acknowledgements.** We thank sincerely Q.S.Chi, R.Miyaoka, B.Reznick, B.Solomon and G.Thorbergsson for their interest. We also want to thank anonymous referees for reading the manuscript very carefully and making a number of valuable comments.

**References**

[1] U. Abresch, *Isoparametric hypersurfaces with four or six distinct principal curvatures*, Math. Ann. 264 (1983), 283–302.

[2] A. A. Ahmadi and A. Majumdar, *DSOS and SDSOS Optimization: More Tractable Alternatives to Sum of Squares and Semidefinite Optimization*, SIAM J. Appl. Algebra Geom. 3 (2019), no. 2, 193–230.

[3] E. Artin, *Über die Zerlegung definiter Funktionen in Quadrate*, Hamb. Abh. 5 (1927), 100–115; see Collected Papers (S. Lang, J. Tate, eds.), Addison-Wesley 1965, reprinted by Springer-Verlag, New York, et. al., pp. 273–288.

[4] R. G. Bettiol, M. Kummer and R. Mendes, *Convex Algebraic Geometry of Curvature Operators*, SIAM J. Appl. Algebra Geom., 5 (2021), no. 2, 200–228.
ISOPARAMETRIC POLYNOMIALS AND SUMS OF SQUARES

[5] G. Blekherman, *Nonnegative polynomials and sums of squares*, J. Amer. Math. Soc. **25** (2012), no. 3, 617–635.

[6] G. Blekherman, G. G. Smith and M. Velasco, *Sums of squares and varieties of minimal degree*, J. Amer. Math. Soc. **29** (2016), no. 3, 893–913.

[7] J. Bochnak, M. Coste and M-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin, 1998.

[8] E. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, Ann. Mat. Pura Appl. **17** (1938), no. 1, 177–191.

[9] T. E. Cecil, Q. S. Chi, and G. R. Jensen, *Isoparametric hypersurfaces with four principal curvatures*, Ann. of Math. **166** (2007), no. 1, 1–76.

[10] T. E. Cecil and P. J. Ryan, *Geometry of hypersurfaces*, Springer Monographs in Mathematics. Springer, New York, 2015. xi+596 pp.

[11] Q. S. Chi, *Isoparametric hypersurfaces with four principal curvatures, II*, Nagoya Math. J. **204** (2011), 1–18.

[12] Q. S. Chi, *Isoparametric hypersurfaces with four principal curvatures, III*, J. Differential Geom. **94** (2013), 487–522.

[13] Q. S. Chi, *Isoparametric hypersurfaces with four principal curvatures, IV*, J. Differential Geom. **115**(2) (2020), 225–301. DOI:10.4310/jdg/1589853626

[14] Q. S. Chi, *The isoparametric story, a heritage of Élie Cartan*, Proceedings of the International Consortium of Chinese Mathematicians, 2018, International Press of Boston (2020), 197–260.

[15] M. D. Choi and T. Y. Lam, *Extremal positive semidefinite forms*, Math. Ann. **231** (1977), 1–18.

[16] Choi, M. D., T. Y. Lam and B. Reznick, *Real zeros of positive semidefinite forms, I*, Math. Z. **171** (1980), 1–25.

[17] J. Dorfmeister and E. Neher, *Isoparametric hypersurfaces, case g = 6, m = 1*, Comm. Algebra **13** (1985), 2299–2368.

[18] F. Q. Fang, *On the topology of isoparametric hypersurfaces with four distinct principal curvatures*, Proc. Amer. Math. Soc. **127** (1999), 259–264.

[19] F. Q. Fang, *Dual submanifolds in rational homology spheres*, Sci. China Math. **60** (2017), no. 9, 1549–1560.

[20] D. Ferus, H. Karcher and H. F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502.

[21] J. Q. Ge, *Isoparametric foliations, diffeomorphism groups and exotic smooth structures*, Adv. Math. **302** (2016), 851–868.

[22] J. Q. Ge, F. G. Li and Y. Zhou, *Some generalizations of the DDVV and BW inequalities*, Trans. Amer. Math. Soc. **374** (2021), 5331–5348.

[23] J. Q. Ge and M. Radeschi, *Differentiable classification of 4-manifolds with singular Riemannian foliations*, Math. Ann. **363** (2015), 525–548.

[24] J. Q. Ge and Z. Z. Tang, *Isoparametric functions and exotic spheres*, J. Reine Angew. Math. **683** (2013), 161–180.

[25] J. Q. Ge, Z. Z. Tang and W. J. Yan, *Normal scalar curvature inequality on the focal submanifolds of isoparametric hypersurfaces*, Int. Math. Res. Not. IMRN **2020**, no. 2, 422–465.

[26] J. Harris, *Algebraic geometry. A first course*. Graduate Texts in Mathematics, **133**. Springer-Verlag, New York, 1995. xx+328 pp.

[27] R. Harvey and H.B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[28] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formenquadraten*, Math. Ann. **32** (1888), 342–350; see Ges. Abh. 2, 154–161, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.
[29] D. Hilbert, Über ternäre definite Formen, Acta Math. 17 (1893), 169–197; see Ges. Abh. 2, 345–366, Springer, Berlin, 1933, reprinted by Chelsea, New York, 1981.
[30] S. Immervoll, On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres, Ann. of Math. 168 (2008), No. 3, 1011–1024.
[31] K. Y. Lam, Some new results on composition of quadratic forms, Invent. Math. 79 (1985), 467–474.
[32] R. Miyaoka, The linear isotropy group of $G_2/\text{SO}(4)$, the hopf fibering and isoparametric hypersurfaces, Osaka J. Math. 30 (1993), 179–202.
[33] R. Miyaoka, Isoparametric hypersurfaces with $(g, m) = (6, 2)$, Ann. of Math. 177 (2013), 53–110.
[34] R. Miyaoka, Errata of “isoparametric hypersurfaces with $(g, m) = (6, 2)$ ”, Ann. of Math. 183 (2016), 1057–1071.
[35] T. S. Motzkin, The arithmetic-geometric inequality, pp. 205–224 in Inequalities (O. Shisha, ed.) Proc. of Sympos. at Wright-Patterson AFB, August 19–27, 1965, Academic Press, New York, 1967; also in Theodore S. Motzkin: Selected Papers, Birkhäuser, Boston (D. Cantor, B. Gordon and B. Rothschild, eds.).
[36] H. F. Münzner, Isoparametrische Hyperflächen in Sphären, I and II, Math. Ann. 251 (1980), 57–71 and 256 (1981), 215–232.
[37] H. Ozeki and M. Takeuchi, On some types of isoparametric hypersurfaces in spheres, I and II, Tohoku Math. J. 27 (1975), 515–559 and 28 (1976), 7–55.
[38] C. K. Peng and Z. Z. Tang, On representing homotopy classes of spheres by harmonic maps, Topology 36 (1997), 867–879.
[39] C. K. Peng and Z. Z. Tang, Harmonic maps from spheres to spheres, Topology 37 (1998), 39–44.
[40] C. K. Peng and Z. X. Hou, A remark on the isoparametric polynomials of degree 6, in Differential Geometry and Topology, Proceedings Tianjin 1986-1987, ed. by Jiang, B. et al., Lecture Notes in Math., vol. 1369 (Springer, Berlin/New York, 1989), pp. 222–224.
[41] C. Qian and Z. Z. Tang, Isoparametric functions on exotic spheres, Adv. Math. 272 (2015), 611–629.
[42] B. Reznick, Some concrete aspects of Hilbert’s 17th problem, Real algebraic geometry and ordered structures (Baton Rouge, LA, 1996), 251–272, Contemp. Math. 253, Amer. Math. Soc., Providence, RI, 2000.
[43] B. Reznick, On Hilbert’s construction of positive polynomials, 2007, arXiv:0707.2156
[44] B. Solomon, Quartic isoparametric hypersurfaces and quadratic forms, Math. Ann. 293 (1992), 387–398.
[45] S. Stolz, Multiplicities of Dupin hypersurfaces, Invent. Math. 138 (1999), 253–279.
[46] Z. Z. Tang, Isoparametric hypersurfaces with four distinct principal curvatures, Chinese Sci. Bull. 36 (1991), 1237–1240.
[47] Z. Z. Tang and W. J. Yan, Isoparametric foliation and Yau conjecture on the first eigenvalue, J. Differential Geom. 94 (2013), 521–540.
[48] G. Thorbergsson, Singular Riemannian foliations and isoparametric submanifolds, Milan J. Math. 78 (2010), 355–370.
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P. R. China

Email address: jqge@bnu.edu.cn

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P. R. China

Email address: zztang@nankai.edu.cn