On the number of ordinary circles

Hossein Nassajian Mojarrad*          Frank de Zeeuw†

Abstract

We prove that any \( n \) points in \( \mathbb{R}^2 \), not all on a line or circle, determine at least \( \frac{1}{4}n^2 - O(n) \) ordinary circles (circles containing exactly three of the \( n \) points). The main term of this bound is best possible for even \( n \). Our proof relies on a recent result of Green and Tao on ordinary lines.

1 Introduction

The classical Sylvester-Gallai theorem states that any finite non-collinear point set in \( \mathbb{R}^2 \) spans at least one ordinary line (a line containing exactly two of the points). A more sophisticated statement is the Dirac-Motzkin conjecture, according to which for \( n \neq 7, 13 \), every non-collinear set of \( n \) points in \( \mathbb{R}^2 \) determines at least \( n/2 \) ordinary lines. This conjecture was proved in 2013 by Green and Tao \cite{7} for all \( n \) larger than some fixed threshold \( N_{CT} \) (see Section 2.1).

It is natural to ask the corresponding question for ordinary circles (circles that contain exactly three of the given points); see for instance Section 7.2 in \cite{3} or Chapter 6 of \cite{8}. Elliott \cite{5} introduced this question in 1967, and proved that any \( n \) points, not all on a line or circle, determine at least \( \frac{2}{63}n^2 - O(n) \) ordinary circles. He suggested, cautiously, that the optimal bound is \( \frac{1}{5}n^2 - O(n) \). Elliott’s bound was improved by Bálintová and Bálint \cite{2} to \( \frac{5}{133}n^2 - O(n) \) in 1994, and Zhang \cite{10} obtained \( \frac{1}{18}n^2 - O(n) \) in 2011.

We use the result of Green and Tao \cite{7} to prove the following theorem, providing a tight lower bound for the number of ordinary circles determined by a large point set. It solves Problem 7.2.6 in \cite{3} (and disproves Elliott’s suggested bound).

Theorem 1.1. If \( n \) points in \( \mathbb{R}^2 \) are not all on a line or circle, then they determine at least \( \frac{1}{4}n^2 - O(n) \) ordinary circles.

If \( n \) is even, the main term \( \frac{1}{4}n^2 \) is best possible, as shown by Construction 1.2 below (which is based on a construction in Zhang \cite{10}). But for odd \( n \), the best upper bound that we know for the main term is \( \frac{3}{8}n^2 \), provided by Construction 1.3. It remains an open problem to determine the right coefficient for odd \( n \).

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1When we say that \( f(n) \) is at least \( g(n) - O(h(n)) \), we mean that there is a constant \( C \) such that for all \( n \) we have \( f(n) \geq g(n) - C \cdot h(n) \).
We have not attempted to determine the linear term in the bound more precisely, as it
depends on the threshold $N_{GT}$ in the Green-Tao theorem, which is currently quite large.
The steps in our proof would be easy to quantify, given a better threshold.

**Construction 1.2.** Let $n$ be even. Take any two concentric circles $C_1$ and $C_2$. Let $S_1$ and
$S_2$ be the vertex sets of regular polygons of size $n/2$ on $C_1$ and $C_2$ that are “aligned” in the
sense that their points lie at the same set of angles from the common center (see Figure 1).

Any ordinary circle determined by $S_1 \cup S_2$ contains two points from one circle and one
from the other. Because of the symmetry of the configuration, given $p, q \in S_1$ and a point
of $S_2$ that is not equidistant from $p$ and $q$, the circle that these three points determine must
pass through another point of $S_2$. On the other hand, if $r \in S_2$ is equidistant from $p$ and $q$,
then $p, q, r$ determine an ordinary circle.

First suppose $n/2$ is odd. Then all $\binom{n/2}{2}$ pairs of points from $S_1$ have exactly one
equidistant point in $S_2$, which gives $\binom{n/2}{2}$ ordinary circles with two points from $S_1$. Similarly, there are $\binom{n/2}{2}$ ordinary circles with two points from $S_2$. Thus this set determines
$2\binom{n/2}{2} = \frac{1}{4}n^2 - O(n)$ ordinary circles.

Now suppose $n/2$ is even. Then there are $\frac{1}{2}\binom{n/2}{2}$ pairs of points from $S_1$ that have no
equidistant point in $S_2$, so these give no ordinary circles. There are $\frac{1}{2}\binom{n/2}{2}$ pairs of points
from $S_1$ that have exactly two equidistant points in $S_2$, so these give $\binom{n/2}{2}$ ordinary circles. Again we get $\frac{1}{4}n^2 - O(n)$ ordinary circles.

Figure 1: Construction 1.2 and 1.3 each with two of its ordinary circles.

**Construction 1.3.** Let $n$ be odd. Take Construction 1.2 with $n + 1$ points and remove
an arbitrary point. The removal breaks $O(n)$ ordinary circles, and it creates $\frac{1}{8}n^2 - O(n)$
new ordinary circles. Indeed, this is the number of four-point circles through a fixed point,
which become ordinary circles after the fixed point is removed. Thus this set determines
$\frac{3}{8}n^2 - O(n)$ ordinary circles.
2 Tools

In this section we introduce the three main ingredients of our arguments: the results of Green and Tao [7]; a basic property of tangent lines to algebraic curves; and circle inversion.

2.1 The results of Green and Tao

The following point sets are crucial in the proof of Green and Tao [7], and will also be central to our proof. In [4], these constructions are ascribed to Böröczky. Further information as well as wonderful pictures can be found in [7].

We use the notation \([x : y : z]\) for points in the projective plane \(\mathbb{P}^2\), so points of the form \([x : y : 0]\) are points on the line at infinity. We say that a set \(A\) in \(\mathbb{R}^2\) is projectively equivalent to a set \(B\) in \(\mathbb{P}^2\) if, after embedding \(\mathbb{R}^2\) in \(\mathbb{P}^2\) by \((x, y) \mapsto [x : y : 1]\), there is a bijective projective transformation that maps \(A\) to \(B\).

Construction 2.1 (Böröczky Examples). For every \(m \in \mathbb{N}\), we define

\[
X_{2m} = \{ \cos \frac{2\pi j}{m} : \sin \frac{2\pi j}{m} : 1 : 0 \leq j < m \} \cup \{ -\sin \frac{\pi j}{m} : \cos \frac{\pi j}{m} : 0 : 0 \leq j < m \},
\]

which consists of \(m\) points on a unit circle and \(m\) points on the line at infinity in \(\mathbb{P}^2\). The Böröczky examples are those sets in \(\mathbb{R}^2\) that are projectively equivalent to one of the following:

- The set \(X_{2m}\);
- The set \(X_{2m}\) plus or minus one point.

Note that after projective equivalence, the circle in a Böröczky example can be transformed to any type of conic, but the line stays a line, and the line and conic stay disjoint. It is shown in [7] that \(X_{2m}\) determines exactly \(m\) ordinary lines, and that for certain points, adding or removing that point gives a set of \(n\) points with at least \(\frac{3}{4}n - 3\) ordinary lines (see Remark 2.5 below).

We state two theorems from Green and Tao [7], in forms that are convenient for our proof. A cubic curve is an algebraic curve of degree three.

**Theorem 2.2 (Green-Tao I).** For every \(K \in \mathbb{R}\) there exists an \(N_K \in \mathbb{N}\), such that the following holds for any non-collinear set \(P\) of \(n \geq N_K\) points in \(\mathbb{R}^2\). If \(P\) determines fewer than \(Kn\) ordinary lines, then all but \(O(K)\) points of \(P\) lie on a cubic curve.

We actually need a more detailed conclusion that gives more information on the nature of the ordinary lines. In the next lemma, we deduce this information from the proof in [7].

**Lemma 2.3.** In the conclusion of Theorem 2.2, any ordinary line of \(P\) is either a tangent line to the cubic curve, or it passes through one of \(O(K)\) fixed points.

**Proof.** It is proved in [7] that up to \(O(K)\) added or removed points, \(P\) has one of the forms:

- \(n \pm O(K)\) points on a line;
- The Böröczky example \(X_{2m}\), for some \(m = \frac{n}{2} \pm O(K)\);
• A subgroup $H$ of $n \pm O(K)$ nonsingular points of an irreducible cubic.

In the first case, the line can be trivially extended to a cubic by, say, adding two lines that are disjoint from $P$. Then any ordinary line must pass through one of the $O(K)$ points of $P$ not on this cubic.

Consider the second case. As described in [7, Proposition 2.1], every ordinary line of $X_{2m}$ is a tangent line to the unit circle. If any new ordinary line is created when adding or removing $O(K)$ points, then it must pass through one of these points.

For the last case, we refer to [7] for the definition and properties of the group structure on an irreducible cubic. It is explained in the proof of Proposition 2.6 of [7] that if $a \in H$ has $a \neq -2 \cdot a$ (in this group), then the line through $a$ and $-2 \cdot a$ is an ordinary line of $H$. The fact that $a + a + (-2 \cdot a) = 0$ means that this line is tangent to the cubic at $a$. As in the second case, adding or removing $O(K)$ points only creates ordinary lines passing through these points.

We now come to the second theorem from Green-Tao [7], which gives more precise information when $K = 1$, and thereby proves the Dirac-Motzkin conjecture for large $n$.

**Theorem 2.4** (Green-Tao II). There exists $N_{GT} \in \mathbb{N}$ such that the following holds for every set $P$ of $n \geq N_{GT}$ non-collinear points in $\mathbb{R}^2$. If $n$ is even, then $P$ determines at least $\frac{1}{2}n$ ordinary lines; if $n$ is odd, then $P$ determines at least $\frac{3}{4}n - 3$ ordinary lines. Moreover, there exists $C_{GT} \in \mathbb{N}$ such that, if a set $P$ of $n \geq N_{GT}$ non-collinear points in $\mathbb{R}^2$ determines less than $n - C_{GT}$ ordinary lines, then $P$ is a Böröczky example.

**Remark 2.5.** For the sake of brevity we have used a somewhat broader definition of Böröczky example than that in [7]; in the second case of Construction 2.1, Green and Tao actually only remove a specific point from the line at infinity, or they add the center of the unit circle to the set (see [7, Proposition 2.1, 2.3]). Therefore, Böröczky examples in our sense are not all extremal in the sense of Theorem 2.4, but this does not matter in our proof. We only use the converse statement, that every extremal set is a Böröczky example.

### 2.2 Tangent lines

We need a few simple notions from the theory of algebraic curves. We use the terminology from Fischer [6], and we temporarily use the projective plane $\mathbb{P}^2$ because the definition of the polar curve is more natural there. For a homogeneous polynomial $F \in \mathbb{R}[x, y, z]$, which we assume to be square-free, we write

$$V(F) = \{[x : y : z] \in \mathbb{P}^2 : F(x, y, z) = 0\}$$

for the algebraic curve that it defines in $\mathbb{P}^2$.

**Definition 2.6** (Polar curve). Let $C = V(F)$ be an algebraic curve in $\mathbb{P}^2$ not containing a line, and $p = [p_1 : p_2 : p_3] \in \mathbb{P}^2$ a fixed point. We define the homogeneous polynomial

$$D_p(F) = p_1 \frac{\partial F}{\partial x_1} + p_2 \frac{\partial F}{\partial x_2} + p_3 \frac{\partial F}{\partial x_3},$$

and we call $D_p(C) = V(D_p(F))$ the polar curve of the $C$ with respect to $p$. 

The proof of the following lemma can be found in [6, Chapter 4].

**Lemma 2.7.** Let \( C \subset \mathbb{P} \mathbb{R}^2 \) be an algebraic curve not containing a line and \( p \in \mathbb{P} \mathbb{R}^2 \) a fixed point. The tangency points on \( C \) of the tangent lines from \( p \) to \( C \) are contained in \( C \cap D_p(C) \).

**Corollary 2.8.** Let \( C \) be an algebraic curve of degree \( d \) in \( \mathbb{R}^2 \) and \( p \in \mathbb{R}^2 \) a fixed point. At most \( d(d-1) \) lines through \( p \) are tangent to \( C \) (or contained in \( C \)).

**Proof.** Since \( C \) has degree \( d \), the polynomial defining \( C \) has at most \( d \) linear factors, corresponding to the lines contained in \( C \). Remove these linear factors and let \( \tilde{C} \) be the curve defined by the remaining polynomial; it does not contain a line. Let \( e \) be the degree of \( \tilde{C} \).

Let \( \overline{C} \) be the projective closure (see [6, Chapter 2]) of \( \tilde{C} \), which is a curve in \( \mathbb{P} \mathbb{R}^2 \), also of degree \( e \). Applying Bézout’s theorem (see [6, Chapter 2]) to \( \overline{C} \) and \( D_p(\overline{C}) \) gives

\[
| \overline{C} \cap D_p(\overline{C}) | \leq \deg \overline{C} \cdot \deg D_p(\overline{C}) = e(e-1).
\]

It follows by Lemma 2.7 that there are at most \( e(e-1) \) tangency points on \( \overline{C} \) from the tangent lines passing through \( p \), so there are at most \( e(e-1) \) tangent lines from \( p \) to \( \overline{C} \).

The same bound then holds in \( \mathbb{R}^2 \) for tangent lines from \( p \) to \( C \) that are not contained in \( C \). Together with the \( d-e \) lines contained in \( C \), this gives the bound \( d(d-1) \).

\[ \square \]

### 2.3 Circle inversion

A key tool in our proof is circle inversion. We quickly introduce it here so that we can fix notation, and we summarize the properties that we use. See for instance [1] for more details and background.

It seems that inversion was first used in this context by Motzkin [9]. He used it to prove that for any point set that is not on a line or circle, there is a line or circle containing exactly three of the points. Inversion was also the main tool in each of the papers [5, 2, 10] that established lower bounds on the number of ordinary circles.

**Definition 2.9.** Let \( p \in \mathbb{R}^2 \) be a fixed point. The circle inversion of radius 1 with center \( p = (x_p, y_p) \) is the mapping \( I_p : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
I_p(x, y) = \left( \frac{x - x_p}{(x - x_p)^2 + (y - y_p)^2}, \frac{y - y_p}{(x - x_p)^2 + (y - y_p)^2} \right)
\]

for \( (x, y) \neq p \), and \( I_p(p) = p \).

Our choice to set \( I_p(p) = p \) is somewhat awkward as it makes the map discontinuous. The natural setting for inversion is the Riemann sphere, which can be viewed as \( \mathbb{R}^2 \) together with a point at infinity. Inversion then exchanges \( p \) with the point at infinity. However, because in our proof we also deal with the projective plane, which is \( \mathbb{R}^2 \) together with a line at infinity, we keep the Riemann sphere out of sight to avoid confusion.

We use the following properties of inversion.

- A line containing \( p \) is sent to itself, and a line not containing \( p \) is sent to a circle containing \( p \);
• A circle containing \( p \) is sent to a line not containing \( p \), and a circle not containing \( p \) is sent to a circle not containing \( p \);

• An algebraic curve of degree \( d \) is sent to an algebraic curve of degree at most \( 2d \).

The last property follows easily from Definition 2.9.

3 Even point sets

In this section we prove Theorem 1.1 for point sets of even size; point sets of odd size are treated in the next section, with a somewhat different proof.

In both proofs we apply a circle inversion in a point of the point set, find an ordinary line in the inverted set, and then observe that this line is inverted back to a circle that also contains the center of inversion, which is therefore an ordinary circle. However, this only works if the ordinary line in the inverted set does not pass through the center of inversion. This leads to the following notion.

Definition 3.1. Let \( P \) be a set of \( n \) points in \( \mathbb{R}^2 \) and \( q \in \mathbb{R}^2 \) a fixed point. We call an ordinary line of \( P \) a non-\( q \) ordinary line if it does not pass through \( q \).

We now prove a lower bound on the number of non-\( q \) ordinary lines. Similar bounds were proven in \([5, 2, 10]\), but they were considerably weaker, due to \([7]\) not being available.

Lemma 3.2. The following holds for every odd \( n \). If \( P \) is a set of \( n \) non-collinear points in \( \mathbb{R}^2 \) and \( q \notin P \) is a fixed point, then \( P \) determines at least \( \frac{3}{4}n - O(1) \) non-\( q \) ordinary lines.

Proof. First suppose that \( P \) determines at least \( 2n \) ordinary lines. Then the fact that there are less than \( n \) ordinary lines containing \( q \) implies that there are at least \( n \) non-\( q \) ordinary lines. Thus we can assume that \( P \) has less than \( 2n \) ordinary lines. We ensure that \( n \) is large enough to let us apply Theorems 2.2 and 2.4 by choosing the constant in the term \( O(1) \) large enough.

We apply Theorem 2.2 with \( K = 2 \) and deduce that all but \( O(1) \) points of \( P \) lie on a cubic curve. According to Lemma 2.3, an ordinary line of \( P \) is either a tangent line to the curve, or it passes through one of \( O(1) \) fixed points. By Corollary 2.8, at most \( 3 \cdot 2 \) tangent lines of the cubic pass through \( q \). On the other hand, for any fixed point, at most one of the ordinary lines through that point hits \( q \). Thus altogether \( O(1) \) of the ordinary lines of \( P \) pass through \( q \). Therefore, if \( P \) determines at least \( n - C_{\text{GT}} \) ordinary lines, then at least \( n - O(1) \) of these are non-\( q \) ordinary lines.

On the other hand, suppose that \( P \) determines less than \( n - C_{\text{GT}} \) ordinary lines. Theorem 2.4 then states that \( P \) must be an odd Böröczky example, so it has at least \( \frac{3}{4}n - 3 \) ordinary lines. Each of these ordinary lines is either a tangent line to a fixed conic, or it passes through one of \( O(1) \) fixed points. By the same argument as above, at least \( \frac{3n}{4} - O(1) \) of these ordinary lines are non-\( q \) ordinary lines. \( \square \)

Now we can state our main theorem in the even case.

Theorem 3.3. The following holds for every even \( n \). If a set \( P \) of \( n \) points in \( \mathbb{R}^2 \) is not contained in a line or circle, then \( P \) determines at least \( \frac{1}{4}n^2 - O(n) \) ordinary circles.
Proof. We first prove that for every $p \in P$ there are at least $\frac{3}{4} n - O(1)$ ordinary circles passing through $p$. Consider the set $I_p(P - p)$, and note that $|I_p(P - p)| = n - 1$ is odd. Since $P$ does not lie on a line or circle, the points of $I_p(P - p)$ do not lie on a line. By Lemma 3.2, $I_p(P - p)$ has at least $\frac{3}{4} (n - 1) - O(1)$ non-$p$ ordinary lines. After applying $I_p$ to $I_p(P - p)$, each of these lines is inverted back to an ordinary circle of $P$ containing $p$.

Let $oc(P)$ be the total number of ordinary circles of $P$ and $oc_p(P)$ the number of ordinary circles of $P$ passing through $p$. Since each of the ordinary circles passes through exactly three points of $P$, by double counting we have:

$$3oc(P) = \sum_{p \in P} oc_p(P) \geq n \cdot \left( \frac{3}{4} n - O(1) \right) = \frac{3}{4} n^2 - O(n).$$

This completes the proof. \qed

4 Odd point sets

We state a lemma similar to Lemma 3.2 for point sets with an even number of points.

Lemma 4.1. The following holds for every even $n$. If $P$ is a set of $n$ non-collinear points in $\mathbb{R}^2$ and $q \notin P$ is fixed, then $P$ determines at least $\frac{1}{2} n - O(1)$ non-$q$ ordinary lines.

Proof. The proof is very similar to the proof of Lemma 3.2 and we omit the details. The only difference is that in this even case, the Böröczky example is projectively equivalent to $X_n$ with $n$ even, and thus the lower bound for the number of ordinary lines is $\frac{1}{2} n$. \qed

Note that we could now proceed as in Theorem 3.3. However, the result would be that $P$ determines at least $\frac{1}{4} n^2 - O(n)$ ordinary circles, which is weaker than the bound in Theorem 3.3. Therefore we use a more involved argument to obtain the same bound as in the even case. The key observation is that in an even Böröczky example, the points are exactly evenly distributed over two disjoint algebraic curves. Thus, if an even point set is distributed over two (low-degree) algebraic curves, but not evenly, then it is not a Böröczky example.

Theorem 4.2. The following holds for every odd $n$. If a set $P$ of $n$ points in $\mathbb{R}^2$ is not contained in a line or circle, then $P$ determines at least $\frac{1}{4} n^2 - O(n)$ ordinary circles.

Proof. We can assume that $P$ has less than $\frac{1}{3} n(n - 1 - C_{GT})$ ordinary circles (with the constant $C_{GT}$ from Theorem 2.4); otherwise we are already done. Then by the pigeonhole principle there exists a point $p \in P$ such that $I_p(P - p)$ has less than $n - 1 - C_{GT}$ ordinary lines.

By Theorem 2.4, $I_p(P - p)$ must be a Böröczky example, i.e., it is projectively equivalent to $X_{n-1}$. Since $|I_p(P - p)| = n - 1$ is even, $I_p(P - p)$ consists of $\frac{1}{2} (n - 1)$ points on a line and $\frac{1}{2} (n - 1)$ on a disjoint circle. If we apply $I_p$ to $I_p(P - p)$, then we obtain two algebraic curves $C_1$ and $C_2$ (of degree at most 4), such that all the points of $P$ lie on these two curves, except possibly $p$. Since the line and the circle in $X_{n-1}$ are disjoint, $p$ is the only point that could be common to $C_1$ and $C_2$. Because the line and the circle in $X_{n-1}$ each contain $\frac{1}{2} (n - 1)$ points, both $C_1$ and $C_2$ contain either $\frac{1}{2} (n - 1)$ or $\frac{1}{2} (n + 1)$ points of $P$, depending on whether each contains $p$ or not.

We split the rest of the proof into the following two cases:
• For every point $q \in P - p$, $I_q(P - q)$ is not a Böröczky example;

• There exists a point $r \in P - p$ such that $I_r(P - r)$ is a Böröczky example.

In the first case, for every $q \in P - p$, $I_q(P - q)$ has at least $n - O(1)$ non-$q$ ordinary lines by Theorem 2.4, which implies $oc_q(P) \geq n - O(1)$. By double-counting we get

$$3oc(P) \geq \sum_{q \in P - p} oc_q(P) \geq (n - 1)(n - O(1)) = n^2 - O(n),$$

which is even better than the desired bound.

Consider the second case. Since the circle and the line in $X_{n-1}$ are disjoint, we have

$$I_r(C_1 - r) \cap I_r(C_2 - r) = \emptyset,$$

and as a result, $p$ cannot be common to $C_1$ and $C_2$, so $C_1 \cap C_2 = \emptyset$. Moreover, $p$ must belong to one of $C_1$ or $C_2$, because $I_r(P - r)$ is an even Böröczky example. Without loss of generality we can assume $p \in C_2$, so $C_1$ has $\frac{1}{2}(n - 1)$ points of $P$ and $C_2$ has $\frac{1}{2}(n + 1)$.

Next we show that for every $q \in C_1 \cap P$, $I_q(P - q)$ is not a Böröczky example. Suppose $q \in C_1 \cap P$ is chosen arbitrarily. Since $C_1 \cap C_2 = \emptyset$, $I_q(C_1 - q)$ and $I_q(C_2)$ are also disjoint and have $\frac{1}{2}(n - 3)$ and $\frac{1}{2}(n + 1)$ points of $I_q(P - q)$, respectively. Because $X_{n-1}$ consists of exactly the same number of points on a line and a circle, $I_q(P - q)$ cannot be projectively equivalent to $X_{n-1}$. So it is not a Böröczky example.

Thus for every $q \in C_1 \cap P$, we get at least $n - O(1)$ non-$q$ ordinary lines in $I_q(P - q)$. On the other hand, for every $q \in C_2 \cap P$, $I_q(P - q)$ has at least $\frac{1}{2}(n - 1) - O(1)$ non-$q$ ordinary lines, by Lemma 4.1. By double counting we have:

$$3oc(P) = \sum_{q \in C_1 \cap P} oc_q(P) + \sum_{q \in C_2 \cap P} oc_q(P) \geq \frac{1}{2}(n - 1)(n - O(1)) + \frac{1}{2}(n + 1) \left(\frac{1}{2}(n - 1) - O(1)\right) = \frac{3}{4}n^2 - O(n).$$

This completes the proof. □

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