Centripetal acceleration revisited with a simple physical derivation

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Abstract

This work shows how two simple derivations of the centripetal acceleration are related to original works of Newton and Huygens. A different, more physically motivated derivation is then given.
The simplest (no calculus, no vector, no trig function) derivation of the centripetal acceleration

\[ a = \frac{v^2}{r} \]  

(1)
is undoubtedly the two circle method,[1] which has been independently derived numerous times.[2–5] The above acceleration follows when the angular velocity \(2\pi/T\) in the circle traced out by the position vector

\[ v = \frac{2\pi r}{T} \]  

(2)
is substituted into the circle traced out by velocity vector

\[ a = \frac{2\pi v}{T}. \]  

(3)

However, despite its simplicity, (3) is usually not invoked in textbooks because a circle in “velocity space”[1] was deemed too abstract for most students.

The other misgiving about (2) and (3) is that they are just end products of an assumed circular motion. They do not convey the physical idea that in order to cause a circular orbit, one must provide a source of acceleration given by (1). Below we will first show Newton’s derivation of the required acceleration for generating a circular orbit.

Since constant angular velocity is the key, acceleration (1) can also be derived from any arc length of the circle in time \(\Delta t\) via

\[ v = \frac{\theta r}{\Delta t}, \quad a = \frac{\theta v}{\Delta t}. \]  

(4)

This is similar to methods used in most textbooks, by taking the limit \(\Delta t \to 0\). However, Newton derived (1) independent of the size of \(\Delta t\). According to Newton, in his Waste Book discussion[6], the acceleration (1) remains the same if one replaces the arc length by its corresponding chord. This is shown in Fig.1.

In Fig.1, one seeks to find the acceleration \(a\) such that the change in velocity \(\Delta v = a\Delta t\) will produce a polygonal orbit of uniform speed \(v\) on a circle of radius \(r\). Starting at \(A\), in time \(\Delta t\) and uniform speed \(v\), the particle moves to \(B\) along the chord \(AB = v\Delta t\) with center angle \(\theta\). If there were no force acting on it, it would move to \(C'\) in the next time interval unimpeded. However, if it were to land on \(C\), also on the circle, of equal chord length \(v\Delta t\) from \(B\), then there must be a change of velocity \(a\Delta t\) at \(B\), toward the center, resulting in a change of displacement from \(C'\) to \(C\) of length \((a\Delta t)\Delta t\). Since \(AB = v\Delta t\) is
FIG. 1: Newton’s polygonal orbit with triangle \( ABO \) similar to \( C'C'B \) because \( \theta + 2\phi = 180^\circ \).

the base of an isosceles triangle \( AOB \) with two side lengths \( r \), and \( C'C = (a\Delta t)\Delta t \) is the base of a similar isosceles triangle \( C'BC \) with two side lengths \( v\Delta t \), both must have the same base to side ratio

\[
\frac{a(\Delta t)^2}{v\Delta t} = \frac{v\Delta t}{r},
\]

thereby yielding (1) independent of \( \Delta t \). Thus (1) is true for any regular \( N \)-side polygon on the circle. The length of the polygon is \( Nv\Delta t \). In the limit of \( \Delta t \to 0 \), \( N\Delta t = T \), the polygon’s perimeter approaches \( 2\pi r \), and one has \( vT = 2\pi r \). Similarly \( N(a\Delta t) = 2\pi v \). The limit of \( \Delta t \to 0 \) has no effect on the acceleration (5); it only guarantees that the orbit is a true circle. Newton’s derivation makes it clear that (2) and (3) are mere consequences, the aftermaths of forcing a circular orbit by (5).

The other simple derivation, also independently found by many (including Tipler’s text [10]), was first given geometrically by Huygens [6, 11, 12], more than 350 years ago. Imagine a stone tied by a tiny string to the rim of a merry-go-around of radius \( r \) rotating at constant rim speed \( v \), as shown in Fig. 2. If the string were cut, the stone would fly off the tangent at constant speed \( v \) to \( B \), a distance of \( vt \) in a short time \( t \). For such a short time, the merry-go-around would have rotated from \( A \) to \( C \) and would view the stone as “fleeing”
FIG. 2: Huygens’ key diagram with triangle $ABC$ similar to $ABD$ because both have the same two angles $\gamma$ and $\alpha$.

away from the center to a distance $x$ from the rim given by

$$(r + x)^2 = (vt)^2 + r^2$$
$$(x + 2r)x = (vt)^2. \quad (6)$$

Neglecting $x$ relative to $2r$, this center-fleeing distance is

$$x = \frac{1}{2} \left( \frac{v^2}{r} \right) t^2. \quad (7)$$

Instead of algebra, Huygens cleverly observes that $ABC$ and $ABD$ are similar triangles and therefore immediately arrive at (neglecting $x$ in $BD = 2r + x$)

$$\frac{BC}{AB} = \frac{AB}{BD} \rightarrow x = \frac{(vt)^2}{2r}. \quad (8)$$

Huygens view this center-fleeing distance $x$ as due to a constant centrifugal acceleration $v^2/r$. This explanation from the rotating frame is clear and obvious. (It is not clear why Ref. 5 would state that Huygens’ method is “not elegant” yet praises the identical algebraic expression in Tipler’s text as “presented elegantly”.) On the other hand, if one stays in the
FIG. 3: The dropping distance $H$ due to earth’s curvature. Triangle $ABC$ is similar to $CBD$ because both are right triangles and $\alpha + \beta = 90^\circ$.

inertial frame, then $x$ is the distance prevented from happening by a centripetal acceleration so that the stone would stay put on the rim. This is a less obvious point-of-view and is the fundamental reason why students find centripetal acceleration more difficult to understand.

This work proposes the following more physical picture. From projectile motion, one learns that a baseball thrown horizontally with velocity $v$ would travel a horizontal distance $d$ in time $t = d/v$. During this time, the ball would free fall a height of

$$h = \frac{1}{2}gt^2 = \frac{1}{2}g \frac{d^2}{v^2}.$$  \hspace{1cm} (9)

If $h$ equals to the the initial height of the thrown ball, then the ball hits the ground at a distance $d$ from its original location. This is elementary to all students. However, the earth is not flat. As shown in Fig 3, over a horizontal distance $d$ earth’s “ground” drops below the horizon by a distance $H$. This $H$ can be solved from the obvious right triangle $BCO$. However, it is also easy to follow Huygens, by noting that triangle $ABC$ is similar to triangle $CBD$, and therefore

$$\frac{AB}{BC} = \frac{BC}{BD} \rightarrow H = \frac{d^2}{2R}.$$  \hspace{1cm} (10)
If a baseball were thrown with velocity $v$ such that the dropping of the ball $h$ exactly matches the dropping of the earth $H$, then the ball will never hit the ground. This velocity is

$$v = \sqrt{gR}. \quad (11)$$

At this velocity, the baseball becomes airborne and orbits the earth in a circular orbit with radius $R$, always free falling toward the center of earth with acceleration $g$. Generalizing $g$ to any center-seeking acceleration $a$ and earth’s radius $R$ to any radius $r$, then (11) reproduces the needed centripetal acceleration. In this derivation, the orbit is circular because the object is constantly falling toward the center, but just keep on missing the “ground”.

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