Iteration Variational Method for Solving Two-Dimensional Partial Integro-Differential Equations

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ABSTRACT

The two-dimensional integro-differential partial equations is one of the so difficult problems to be solved analytically and/or approximately, and therefore, a method that is efficient for solving such type of problems seems to be necessary. Therefore, in this paper, the iteration methods, which is so called the variational iteration method have been used to provide a solution to such type of problems approximately, in which the obtained results are very accurate in comparison with the exact solution for certain well selected examples which are constructed so that the exact solution exist. Main results of this work is to derive first the variational iteration formula and then analyzing analytically the error term and prove its convergence to zero as the number of iteration increases.

Keywords: Variational Iteration Method, Partial Integro-Differential Equations, Two-Dimensional Integro-Differential Partial Equation.

1. INTRODUCTION

In applied mathematics, an interesting attempts that concerning real life phenomena’s usually leads to functional equations, such as ordinary and differential partial equations, integro-differential and integral equations and others [1], [2]. Several formulations that are mathematical of such phenomena leads to integro-differential equations [3], [4], In some cases, the solution that is analytical could cause difficulty to evaluate; for this reason, approximate and numerical methods appear to be helpful to use which highlight the problem that is under consideration. Mathematicians focus their attention on the development of more efficient and advanced and methods for integro-differential and integral equations, such as semi numerical analytical techniques, Adomian’s decomposition method, method of homotopy perturbation. The Homotopy method perturbation and the method of Adomian’s decomposition were used for the solution of integral equation by Poushokouhi etal.[5], variational iteration method (VIM) have been used by Xu L. for the solution of Fredholm and Volterra Integral equations of the second type [6] and for solving Volterra integral equations by Abbasbandy [7], while for the two -
dimensional integro-differential and integral equations equation which is an extension of the previously proposed methods for solving one-dimensional cases. Also, there is many studies has been done for the solution of a class of two-dimensional problems for example using the VIM for solving mixed nonlinear Volterra-Fredholm integral equation [8], by using transform method that is deferential for the solution of nonlinear and linear two-dimension Volterra integral equations [9], solving two-dimensional Volterra integral equations by using iterated collocation and collocation method [10], providing a solution of a class of two-dimension nonlinear Volterra integral equations by using Legendre polynomials [11], providing a solution of mixed nonlinear Volterra-Fredholm integral equations with block-pulse functions that are two dimensional by using a method that is direct [12].

Whenever very little attempts have been paid to give a solution to the partial two-dimensional integral equations, for example, d’Halluin in 2004 [13] solved the integro-differential two-dimensional equations by using a semi-Lagrangian approach. The VIM that has been proposed by Ji-Huan recently. In 1998 he studied and used intensively by several engineers and scientists, which is favorably applied to several types of nonlinear and linear problems.

In this paper, the VIM will be used to provide a solution to partial two-dimensional integro-differential equations in which the analysis is based on deriving first the iterated formulas for evaluating the sequence of iterated approximate solutions, and then it will be used to prove the obtained sequence convergence to the precise solution.

The method may be considered as a modified approach to the method of General Lagrange multiplier into a method of iteration in correction with variational approach to derive the so called the correction functional, where the form of considered integro-differential two-dimensional equation is as follows:

$$\frac{\partial u(x,t)}{\partial t} = g(x,t) + \int_0^t k(u(s,y))dyds, x \in [0,b], t \in [0,T]$$

...(1)

with the condition that is initial:

$$u(x,0) = u_0(x)$$

...(2)

where $k$ is represents function of kernel, $g$ is the function that is given and $u$ stands for real unknown function to be evaluated.

Several studies were achieved to compare the method of VIM with available techniques, and it is reflected by all that this method gives precise solutions that are faster than other methods, in which the concept of convergence has been monstated to be an amount that is substantial for work of research and the studies of the VIM have been directed by many remarkable researchers, [14].

The VIM has been applied successfully to many kinds of problem, for instance, He first proposed the VIM to provide a solution for the nonlinear and linear integral and differential equations. In 1998, He used this method to solve some well known problems for example the classical Blasiu’s equation with more accurate results and then extensively used in 1999 by him to study and solve some non-linear well known problems. In 2000, VIM was used by him to solve systems of autonomous differential ordinary equations. In 2006, Soliman applied the VIM to solve equation of kdv-Burger and then to solve equation Lax’s seventh-order, Abulwafa and Momani used the VIM to give a solution to coagulation nonlinear problem that is with mass loss. In addition, in 2006, Odibat et al used the VIM to give a solution differential nonlinear equations of order that is fractional and the VIM has been used to give a solution to several types of problems, such as providing a solution to nonlinear PDE’s by Bildiki et al., for solving the equation of Fokker-Plank by Dehghan and Tateri, for solving differential equation of quadratic Riccati with constant coefficients by Abbasbandy. In 2007, Wang and He applied VIM to solve integro-differential equations, while Sweilam used VIM to solve boundary value problems of the nonlinear and linear fourth order equations that are integro-differential. In 2009, Wen-Hua Wang used the VIM to solve certain types of fractional integro-differential equations, [15], [16], [17]. Muhammet Kurrulay and
Adin Secer in 2011 used the VIM to solve nonlinear integro-differential equations of fractional order, [18] and A. Husaain et al in 2016 applied the VIM for solving one-dimensional partial integro-differential equations, [19].

2. The Main Aspects of the VIM for Solving Two-Dimensional Integro-Differential Partial Equations

As it is said previously, the VIM which was suggested has been illustrated to easily and effectively solve a large class of nonlinear and linear problems, where it may happen that one or two iteration may result in accurate high solutions. Generally, the procedure of solution of the VIM is very operative, convenient and straightforward for most problems given in advanced forms as a functional form, [20, 21].

The non-linear general equation below that is given in operator form could be regarded to show the basic idea of the VIM:

\[ Lu(x) + Nu(x) = g(x), \quad x \in [a, b] \]  

where \( L \) represents a linear operator, \( N \) stands for an operator nonlinear and \( g \) represents any function that is given and named the non-homogenous term.

Now, rewrite equation (3) as shown below

\[ Lu(x) + Nu(x) - g(x) = 0 \]  

and let \( u_n \) be the \( n \)-th equation approximate solution (4), and it is then shown as follows:

\[ Lu_n(x) + Nu_n(x) = g(x) \]  

and therefore the functional correction connected with equation (5), is provided by:

\[ u_{n+1}(x) = u_n(x) + \int_a^x \lambda(s) \{ L(u_n(s)) + N(\hat{u}_n(s)) - g(s) \} ds, \quad n = 0, 1, \ldots \]  

where \( \lambda \) is recognized as the general Lagrange multiplier, which can be optimally specified by the calculus of variation theory, and \( \hat{u}_n \) is regarded as a variation that is restricted that satisfy \( \delta \hat{u}_n = 0 \), [20].

Generally, it is plain now that the essential steps of the method of He’s variational iteration require first optimal determination of the multiplier value of Largrangian \( \lambda \). After recognizing the multiplier of Lagrang, the approximations that are successive \( u_{n+1} \), for all \( n = 0, 1, \ldots \) of the solution \( u \) will be obtained rapidly by the use of any function that is selective \( u_0 \), which is favored to be equal to the terms that are non homogenous for the integral equations. Thus, it could be demonstrated that the solution \( u_n \) show convergence to the exact solution \( u \) as \( n \rightarrow \infty \).

In the next theorem, the equation approximate solutions general form (1) by the use of the correction functional (6) is obtained which is based on the evaluation of the Lagrange multiplier that is general and that is connected with the integro-differential partial equation (1).

**Theorem (1):**

Consider the nonlinear partial two-dimension integro-differential equation (1) with initial condition (2). Then the sequence of iterative approximate solutions using VIM is provided by:

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^x \int_0^x \frac{\partial u_n}{\partial \xi}(\xi, \eta) - g(\xi, \eta) - \int_0^\xi k(\hat{u}_n(s), y)dyds \]  

for all \( n = 0, 1, \ldots \)

**Proof:**

The correction that is functional (6) connected with equation (1) is provided by:
\[ u_{n+1}(x,t) = u_n(x,t) + \lambda \left[ \frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^\xi \int_0^y k(\tilde{u}_n(s,y)) \, dy \, ds \right] d\xi \]  
\[ \lambda = \text{the general Lagrange multiplier, that must be evaluated using calculus that is variational, the subscript } n \text{ indicates the } n^{th} \text{ approximation and } \tilde{u}_n(t) \text{ is regarded as the variation that is restricted.} \]

Now, by having the first variation \( \delta \) with regard to \( u_n \) for the two sides of equation (8) and setting \( \delta u_n = 0 \), provides:

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \lambda \left[ \frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^\xi \int_0^y k(\tilde{u}_n(s,y)) \, dy \, ds \right] d\xi \]  

and noting that \( \delta \tilde{u}_n = 0 \), which will consequently reduce equation (9) to:

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \lambda \frac{\partial u_n}{\partial \xi}(x,\xi) d\xi \]  

Thus, by using the integration method by parts, equation (10) will have the form:

\[ \int_0^\xi \frac{\partial u_n}{\partial \xi}(x,\xi) d\xi = \lambda(x,\xi) u_n(x,\xi) - \int_0^{\xi} u_n(x,\xi) \lambda'(\xi) d\xi \]  

and substituting equation (11) back into equation (10) will give:

\[ \delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(x,\xi) u_n(x,\xi) - \delta \int_0^{\xi} u_n(x,\xi) \lambda'(\xi) d\xi \]  

Consequently, the following stationary conditions is gained:

\[ \lambda'(\xi) = 0 \]  
\[ \text{with initial condition:} \]

\[ 1 + \lambda(\xi)^{\text{add}} = 0 \]  

Now, providing solution to the ordinary differential equation (13) will provide the general Lagrange multiplier value connected with equation (1) to be:

\[ \lambda(\xi) = -1 \]  

Consequently, substituting \( \lambda(\xi) = -1 \) into the correction functional (8) will lead to the following approximate solution in the form that is iterated:

\[ u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\lambda} \left[ \frac{\partial u_n}{\partial \xi}(x,\xi) - g(x,\xi) - \int_0^\xi \int_0^y k(\tilde{u}_n(s,y)) \, dy \, ds \right] d\xi \]  

3. Convergence of the Sequence of Approximate Iterated Solutions

In this section, the sequence convergence of approximate iterated solution (7) using the VIM for solving partial integro-differential two-dimensional equation will be demonstrated. The central proof idea depends on the evaluation of the error term upper bound between the exact approximate solution of equation (1) which is demonstrated to be zero as \( n \to \infty \).
**Theorem (2):**

Let \( u, u_n \in C^n ([a,b] \times [0,T]) \) be the approximate and equation exact solutions (1) and (7), respectively. If \( E_n(x,t) = u_n(x,t) - u(x,t) \), for all \( n = 0, 1, \ldots \) and the kernel \( k \) satisfies Lipschitz condition with constant \( M \). Afterwards, the sequence of the approximate solutions \( \{u_n\}, n = 0, 1, \ldots \) shows convergence to the solution that is exact \( u \).

**Proof:**

From theorem (1), the approximate solution using the VIM is provided by:

\[
 u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[ \frac{\partial u_n(x,\xi)}{\partial \xi} - g(x,\xi) - \int_0^\xi k(u_n(s,y)) \, dy \right] d\xi 
\]  

...(16)

and since \( u \) is the exact solution of the equation (1), thus it satisfies VIM formula:

\[
 u(x,t) = u(x,t) - \int_0^t \left[ \frac{\partial u(x,\xi)}{\partial \xi} - g(x,\xi) - \int_0^\xi k(u(s,y)) \, dy \right] d\xi 
\]  

...(17)

Subtract \( u \) from \( u_{n+1} \) and recall that \( E_n(x,t) = u_n(x,t) - u(x,t) \), indicate:

\[
 u_{n+1}(x,t) - u(x,t) = u_n(x,t) - u(x,t) - \int_0^t \left[ \frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial u(x,\xi)}{\partial \xi} - g(x,\xi) - g(x,\xi) - \int_0^\xi k(u_n(s,y)) - k(u(s,y)) \, dy \right] d\xi 
\]  

...(18)

Thus:

\[
 E_{n+1}(x,t) = E_n(x,t) - \int_0^t \left[ \frac{\partial E_n(x,\xi)}{\partial \xi} - \int_0^\xi k(u_n(s,y)) - k(u(s,y)) \, dy \right] d\xi 
\]  

...(19)

\[
 = E_n(x,t) - E_n(x,t) - E_n(x,0) + \int_0^t \int_0^\xi \left[ k(u_n(s,y)) - k(u(s,y)) \right] ds \, dy \, d\xi 
\]  

...(20)

\[
 = \int_0^t \int_0^\xi \left[ k(u_n(s,y)) - k(u(s,y)) \right] ds \, dy \, d\xi , \quad \text{where } E_n(x,0) = 0 
\]  

...(21)

Taking the norm to the both equation sides (21), give:

\[
 \|E_{n+1}(x,t)\| = \left\| \int_0^t \int_0^\xi \left[ k(u_n(s,y)) - k(u(s,y)) \right] ds \, dy \, d\xi \right\| 
\]  

...(22)

\[
 \leq \int_0^t \int_0^\xi \left\| k(u_n(s,y)) - k(u(s,y)) \right\| ds \, dy \, d\xi 
\]  

\[
 \leq M \int_0^t \int_0^\xi \|u_n(s,y) - u(s,y)\| dy \, ds \, d\xi 
\]

Therefore:

\[
 \|E_{n+1}(x,t)\| \leq M \int_0^t \int_0^\xi \|E_n(s,y)\| dy \, ds \, d\xi , \quad \text{for all } n = 0, 1, \ldots 
\]

Now, if \( n = 0 \), then:
\[ \left\| E_1(x,t) \right\| \leq M \int_0^t \int_0^x \int_0^x \left\| E_0(s,y) \right\| \, dy \, ds \, d\xi \]
\[ = M \left\| E_0(s,y) \right\| \int_0^t \int_0^x \int_0^x \, dy \, ds \, d\xi \]
\[ = M \left\| E_0(s,y) \right\| \frac{x^2}{2!} \]
If \( n = 1 \), then:
\[ \left\| E_2(x,t) \right\| \leq M \int_0^t \int_0^x \int_0^x \left\| E_1(s,y) \right\| \, dy \, ds \, d\xi \]
\[ \leq M^2 \left\| E_0(s,y) \right\| \frac{x^4}{4} \]
If \( n = 2 \), and then:
\[ \left\| E_3(x,t) \right\| \leq M \int_0^t \int_0^x \int_0^x \left\| E_2(s,y) \right\| \, dy \, ds \, d\xi \]
\[ \leq M^3 \left\| E_0(s,y) \right\| \frac{x^6}{6} \]
If \( n = 3 \), then:
\[ \left\| E_4(x,t) \right\| \leq M \int_0^t \int_0^x \int_0^x \left\| E_3(s,y) \right\| \, dy \, ds \, d\xi \]
\[ \leq M^4 \left\| E_0(s,y) \right\| \frac{x^8}{12} \]
\[ \vdots \]
\[ \left\| E_n(x,t) \right\| \leq M^n \left\| E_0(x,t) \right\| \frac{x^{2n}}{n! (2n)!} \]
therefore having the supremum value of \( x \) and \( t \) over \([0,b]\) and \([0,T]\) respectively to obtain
\[ \left\| E_n(x,t) \right\| \leq M^n \left\| E_0(x,t) \right\| \frac{b^n T^{2n}}{n! (2n)!} \]
and as \( n \to \infty \) implies to \( E_n \to 0 \), i.e., \( u_n \to u \), as \( n \to \infty \). □
4. Illustrative Examples

In the present section, three examples that are illustrative are considered to examine the validity and illustrate the convergence of the variation iteration formula given by equation (8) for linear and nonlinear two-dimensional partial integro-differential equations.

**Example (1):**

Consider the linear partial integro-differential two-dimensional equation:

\[
\frac{\partial u(x,t)}{\partial t} = x + \frac{tx(t-x)(tx+3)}{6} + \int_0^x \int_0^t (s-y)u(s,y)\,dy\,ds, \quad (x,t) \in [0,1] \times [0,1]
\]  

...(23)

with initial condition:

\[ u(x,0) = 1, \quad 0 \leq x \leq 1 \]

For the purpose of comparison, the exact solution of equation (23) is provided by:

\[ u(x,t) = 1 + xt \]

Hence iteration formula of equation (23) that is related and variational is provided by:

\[
u_{n+1}(x,t) = u_n(x,t) - \int_0^x \int_0^t \frac{\partial u_n}{\partial \xi}(s,\xi) - x - \frac{\xi(x(\xi-x)(\xi x+3))}{6} - \int_0^x (s-y)u_n(s,y)\,dy\,ds \, d\xi
\]

and consider the initial approximation \( u_0(x) = u(x,0) = 1 \), then:

\[
u_1(x,t) = \frac{tx^2}{24} - \frac{tx^3}{18} + xt + 1
\]

\[
u_2(x,t) = \frac{tx^2}{24} - \frac{tx^3}{18} + xt - \frac{t^3x^2(100r^4x - 245r^3x^2 + 168r^2x^3 + 12600r - 16800x)}{302400} + 1
\]

Table (1) presents the results that are numerical for the approximate and exact solutions \( u, u_1, u_2, u_3 \) and \( u_4 \) for different values of \( x \) and \( t \) between 0 and 1. While table (2) shows the absolute error between \( u \) and \( u_1, u_2, u_3, u_4 \), respectively.

**Table (1):**

| \( x \) | \( T \) | \( u(x,t) \) | \( u_1(x,t) \) | \( u_2(x,t) \) | \( u_3(x,t) \) | \( u_4(x,t) \) |
|-------|-------|-----------|-----------|-----------|-----------|-----------|
| 0     | 0     | 1         | 1         | 1         | 1         | 1         |
| 0     | 0.25  | 1         | 1         | 1         | 1         | 1         |
| 0     | 0.5   | 1         | 1         | 1         | 1         | 1         |
| 0     | 0.75  | 1         | 1         | 1         | 1         | 1         |
| 0     | 1     | 1         | 1         | 1         | 1         | 1         |
| 0.25  | 0     | 1         | 1         | 1         | 1         | 1         |
| 0.25  | 0.25  | 1.0625    | 1.062947  | 1.0625    | 1.0625    | 1.0625    |
| 0.25  | 0.5   | 1.125     | 1.124932  | 1.125     | 1.125     | 1.125     |
| 0.25  | 0.75  | 1.1875    | 1.187225  | 1.1875    | 1.1875    | 1.1875    |
| 0.25  | 1     | 1.25      | 1.249295  | 1.25      | 1.25      | 1.25      |
| 0.5   | 0     | 1         | 1         | 1         | 1         | 1         |
| 0.5   | 0.25  | 1.125     | 1.125054  | 1.125     | 1.125     | 1.125     |
| 0.5   | 0.5   | 1.25      | 1.249783  | 1.25      | 1.25      | 1.25      |
| 0.5   | 0.75  | 1.375     | 1.373535  | 1.374999  | 1.375     | 1.375     |
| 0.5   | 1     | 1.5       | 1.49566   | 1.499993  | 1.5       | 1.5       |
| 0.75  | 0     | 1         | 1         | 1         | 1         | 1         |
| 0.75  | 0.25  | 1.1875    | 1.187958  | 1.1875    | 1.1875    | 1.1875    |
Example (2):

Consider the nonlinear partial two-dimensional integro-differential equation:

\[
\frac{\partial u(x,t)}{\partial t} = x - \frac{t^2 x^2 (4tx + 9)}{36} + \int_0^t \int_0^x sy + u^2(s,y) \, dy \, ds, (x,t) \in [0,1] \times [0,1]
\]  

(24)

with initial condition:

\[ u(x,0) = 0, \ 0 \leq x \leq 1 \]

For the purpose of comparison, the equation exact solution (24) is provided by:

\[ u(x,t) = xt \]
Thus the related the iteration variational formula of equation (24) is provided by:

\[ u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[ \frac{\partial u_n}{\partial \xi}(x,\xi) - x - \frac{\xi(x-x)(\xi x+3)}{6} \right] \int_0^\xi \left[ s y + u^2(s, y) \right] dy ds d\xi \]

and consider the approximation that is initial \( u_0(x) = u(x,0) = 0 \), then:

\[ u_1(x,t) = \frac{x(t^3 x^2 - 36)}{36} - \frac{t^3 x^3}{12} \]
\[ u_2(x,t) = \frac{t^4 x^3}{36} - \frac{t^7 x^5}{3780} + \frac{t^{10} x^7}{816480} - \frac{t x(t^3 x^2 - 36)}{36} \]

Table (3) shows results that numerical for the approximate and exact solutions \( u, u_1, u_2, u_3 \) and \( u_4 \) for different values of \( x \) and \( t \) between 0 and 1. While table (4) presents the error that is absolute between the exact solution \( u \) and solutions that are approximate \( u_1, u_2, u_3, u_4 \), respectively.

### Table (3)
**Numerical results of the approximate and exact solutions of example (2)**

| \( x \) | \( T \) | \( u(x,t) \) | \( u_1(x,t) \) | \( u_2(x,t) \) | \( u_3(x,t) \) | \( u_4(x,t) \) |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0.25 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0.5 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0.75 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.25 | 0.0625 | 0.062498 | 0.0625 | 0.0625 | 0.0625 |
| 0.25 | 0.5 | 0.125 | 0.124986 | 0.125 | 0.125 | 0.125 |
| 0.25 | 0.75 | 0.1875 | 0.187454 | 0.1875 | 0.1875 | 0.1875 |
| 0.25 | 1 | 0.25 | 0.249891 | 0.25 | 0.25 | 0.25 |
| 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.5 | 0.25 | 0.125 | 0.124973 | 0.125 | 0.125 | 0.125 |
| 0.5 | 0.5 | 0.25 | 0.249783 | 0.25 | 0.25 | 0.25 |
| 0.5 | 0.75 | 0.375 | 0.374268 | 0.375 | 0.375 | 0.375 |
| 0.5 | 1 | 0.5 | 0.498264 | 0.499998 | 0.5 | 0.5 |
| 0.75 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.75 | 0.25 | 0.1875 | 0.187363 | 0.1875 | 0.1875 | 0.1875 |
| 0.75 | 0.5 | 0.375 | 0.373901 | 0.374999 | 0.375 | 0.375 |
| 0.75 | 0.75 | 0.5625 | 0.558792 | 0.562492 | 0.5625 | 0.5625 |
| 0.75 | 1 | 0.75 | 0.741211 | 0.749965 | 0.75 | 0.75 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.25 | 0.25 | 0.249566 | 0.25 | 0.25 | 0.25 |
| 1 | 0.5 | 0.5 | 0.496528 | 0.499992 | 0.5 | 0.5 |
| 1 | 0.75 | 0.75 | 0.738281 | 0.749937 | 0.75 | 0.75 |
| 1 | 1 | 1 | 0.972222 | 0.999737 | 0.999999 | 1 |

### Table (4)
**The absolute error between the approximate and exact solutions of example (2)**

| \( x \) | \( T \) | \( |u(x,t) - u_1(x,t)|\) | \( |u(x,t) - u_2(x,t)|\) | \( |u(x,t) - u_3(x,t)|\) | \( |u(x,t) - u_4(x,t)|\) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0.25 | 0 | 0 | 0 | 0 |
| 0 | 0.5 | 0 | 0 | 0 | 0 |
| 0 | 0.75 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0.25 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0.25 | 0.000002 | 0 | 0 | 0 |
5. Conclusions

This paper has two main goals. The first goal is to employ the variational iteration method to investigate nonlinear and linear two-dimensional equations that are Volterra integro-differential and partial as well as studying the convergence of this method. The second goal is to show significant features of this method and its power. The VIM gives convergent that is rapid, successive, and approximate without any restrictive transformation or assumptions that could change physical behaviour of the problem. Generally, the procedure of VIM solution is very straightforward, convenient, and effective. Numerical results and a comparison with the exact solution are provided, which reveal its efficiency.

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