A note on vanishment of Fourier coefficients

Junyi Mo*

International Department of Chengdu Shude High School, Chengdu, Sichuan, 610065, China

*Corresponding author’s e-mail: nickmo0103@uchicago.edu

Abstract. A Fourier series is one of the methods to represent a function as a summation of orthogonal functions. In this paper, we give a detailed proof of a property regarding the behavior of a function at its continuity points. Finally, we show that an integrable function vanishes at its continuity points if all of its Fourier coefficients vanish.

1. Introduction
French mathematician Fourier discovered that any periodic function could be represented by an infinite series composed of a sine function and a cosine function (the sine function and the cosine function are chosen as the basis functions because they are orthogonal), which is later called Fourier Series is a special trigonometric series [1-5]. According to Euler's formula, trigonometric functions can be transformed into exponential form, which is also called Fourier series as a kind of exponential series. Fourier series play a vital role in signal processing, approximation theory, control theory, and partial differential equation [6-18]. In this short note, we give detailed proof of a property of a function at its continuity points.

2. Main works

2.1. Theorem
If the function $f(\theta)$ is integrable and continuous at $\theta_0$ with period of $2\pi$, for any positive integer $n$ and $\hat{f}(n) = 0$, $f(\theta_0) = 0$ whenever $f$ is continuous at the point $\theta_0$ [12].

In order to prove this theorem, we first show two lemmas to characterize $\cos^k \theta$ and $\sin^k(\theta)$ can be expressed as a linear combination of $\sin(n\theta), \cos(n\theta)$. With these two lemmas as a tool, we prove the theorem 1.1 by contradiction.

2.2. Lemma 1

- If $k$ is an odd number, then $\cos^k \theta = \frac{1}{2^k} \sum_{n=0}^{k} \binom{k}{n} \cos((2n-k)\theta)$.
- If $k$ is an odd number, then $\cos^k \theta = \frac{1}{2^k} \sum_{n=0}^{k} \binom{k}{n} \cos((2n-k)\theta) + \frac{1}{2^k \binom{k}{k/2}} \binom{k}{k/2}$.

2.3. Proof of lemma 1
Without loss of generality, we give a detailed proof for the case when $k$ is an odd number.
\[
\cos^k \theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^k = \frac{1}{2^k} \sum_{n=0}^{k} \binom{k}{n} (e^{i\theta})^n (e^{-i\theta})^{k-n} \\
= \frac{1}{2^{k+1}} \sum_{n=0}^{k} 2 \binom{k}{n} e^{i(2n-k)\theta} \\
= \frac{1}{2^{k+1}} \left( \sum_{n=0}^{k} \binom{k}{n} e^{i(2n-k)\theta} + \sum_{n=0}^{k} \binom{k}{k-n} e^{i(2n-k)\theta} \right) \\
= \frac{1}{2^{k+1}} \left( \sum_{n=0}^{k} \binom{k}{n} e^{i(2n-k)\theta} + \sum_{n=0}^{k} \binom{k}{n} e^{i(k-2n)\theta} \right) \\
= \frac{1}{2^{k}} \sum_{n=0}^{k} \binom{k}{n} \left( e^{i(2n-k)\theta} + e^{-i(2n-k)\theta} \right) = \frac{1}{2^{k}} \sum_{n=0}^{k} \binom{k}{n} \cos((2n-k)\theta).
\]

Using similar techniques above, we give a characterization of \(\sin^k(\theta)\) as below without proof.

2.4. Lemma 2
- If \(n\) is an even number, then
  \[
  \sin^k(\theta) = \left(\frac{-1}{2^{k-1}}\right)^{k/2} \left[ \sum_{n=0}^{\frac{k}{2}} (-1)^n \binom{k}{2n} \cos\left((k-2n)\theta\right) \right] + \frac{1}{2^n} \binom{k}{k/2}.
  \]
- If \(n\) is an odd number, then
  \[
  \sin^k(\theta) = \left(\frac{-1}{2^{k-1}}\right)^{k/2} \left[ \sum_{n=0}^{\frac{k}{2}} (-1)^n \binom{k}{2n} \cos\left((k-2n)\theta\right) \right].
  \]

2.5. Proof of the theorem 2.1
We split the proof of the theorem into two cases. We first prove the theorem holds if \(f\) is a real-valued function, and then show the theorem when \(f\) is a complex-valued function [12].

2.5.1. Case 1: \(f\) is a real number function.
Let the function \(g(\theta)\) be \(f(\theta + \theta_0)\). Then we can see that \(g(0) = f(\theta_0)\). Since \(f\) is a periodic function, \(g\) is also a periodic function. Notice that \(f\) is integrable on the circle. It follows that \(g\) is integrable on the circle.

By the definition of Fourier coefficient,
\[
\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta + \theta_0) e^{-in\theta} d\theta.
\] 

Let \(\theta' = \theta + \theta_0\). We have
\[
\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi+\theta_0}^{\pi+\theta_0} f(\theta') e^{-in(\theta'-\theta_0)} d\theta'
\]
\[
= \frac{1}{2\pi} \int_{-\pi+\theta_0}^{\pi+\theta_0} f(\theta') e^{-i\theta'} \cdot e^{in\theta_0} d\theta'.
\]
Thus, we have $\hat{g}(n) = \frac{1}{2\pi} e^{in\theta_0} \hat{f}(n)$.

Since $\hat{f}(n)$ is equal to 0, $\hat{g}(n)$ is equal to zero for each integer number $n$. Given any trigonometric polynomials $g_k$, we have that the integral $\int g_k(\theta) f(\theta) d\theta$ should be equal to 0.

Next, we show the proof by contradiction. From the augment above, it is safe to assume that $\theta_0$ is zero. Without loss of generality, we also assume that $f(0) > 0$. We aim to construct a family of trigonometric polynomials $\{p_k\}$ which blow up at the origin 0. As a result, we can get a contradiction that $\int p_k(\theta) f(\theta) d\theta \rightarrow \infty$, as $k \rightarrow \infty$.

Due to the condition that $f$ is continuous at 0, we can choose $\delta$ satisfying $0 < \delta \leq \frac{\pi}{2}$, so that $f(\theta) > \frac{f(0)}{2}$ whenever $|\theta| < \delta$. We let $p(\theta) = \epsilon + \cos \theta$ (4)

$p(\theta) = \epsilon + \cos \theta$ where the value of $\epsilon > 0$ is too small that $|p(\theta)| < 1 - \frac{\epsilon}{2}$, whenever $\pi \geq |\theta| \geq \delta$.

Then, we choose a positive number, $\eta$, smaller than $\delta$, and satisfying that $|p(\theta)| \geq 1 + \frac{\epsilon}{2}$, for $|\theta| < \eta$. And finally, we choose $p_k(\theta) = |p(\theta)|^k$.

And we assume B is an upper bounded for $|f(\theta)|$. That is $|f(\theta)| \leq B$ for all $\theta$. This is possible since $f$ is integrable, hence bounded. Thus, we have $\lim_{k \rightarrow \infty} p_k(\theta) = |p(\theta)|^k = (\epsilon + \cos \theta)^k = \infty$. Since $\hat{f}(n) = 0$ for all n shown by previous proof, we must have $\int_{\pi}^0 f(\theta)p_k(\theta) d\theta = 0$.

However, we have the estimate

$$\int_{|\theta| \leq \delta} f(\theta) p_k(\theta) d\theta \leq 2\pi B (1 - \frac{\epsilon}{2})^k$$ (5)

Also, our choice of $\delta$ guarantees that $p(\theta)$ and $f(\theta)$ are non-negative whenever $|\theta| < \delta$, thus $\int_{|\theta| \leq \delta} f(\theta)p_k(\theta)d\theta \geq 0$.

Finally,

$$\int_{|\theta| < \eta} f(\theta)p_k(\theta)d\theta \geq 2\eta \frac{f(0)}{2} (1 + \frac{\epsilon}{2})^k$$ (6)

$$\int_{|\theta| < \eta} f(\theta)p_k(\theta)d\theta \geq 2\eta \frac{f(0)}{2} (1 + \frac{\epsilon}{2})^k \therefore \int p_k(\theta)f(\theta)d\theta \rightarrow \infty$$, as $k \rightarrow \infty$. And we conclude that if the function $f(\theta)$ is integrable and continuous at $\theta_0$ with period of $2\pi$, then $f(\theta_0) = 0$.

2.5.2. Case 2: $f$ is a complex number function

We define $f$ is a complex number function with a range of $[\pi, -\pi]$ with $\theta = u + iv$, where $u$ and $v$ are both real-valued function. Notice that $u(\theta) = \frac{f(\theta) + \bar{f}(\theta)}{2}$ and $v(\theta) = \frac{f(\theta) - \bar{f}(\theta)}{2i}$. Due to the fact that $\hat{f}(n) = \bar{f}(-n)$, we have $\hat{u}(n)$ and $\hat{v}(n)$ are all zero. Thus we have that $f$ vanish at continuity points.

3. Conclusion

If all Fourier coefficients of an integrable function vanish, then it vanishes at its continuity points. This theorem has important applications in the fields of mathematics and physics, as well as in electronics. In the future, we will report more on the application of the Fourier series in the Partial differential equations.
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