Unitary Matrix Model of a Chiral $[SU(N)]^K$ Gauge Theory

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ABSTRACT

We build a matrix model of a chiral $[SU(N)]^K$ gauge theory (SQCD$_5$ deconstructed down to 4D) using random unitary matrices to model chiral bifundamental fields $(N, \bar{N})$ (without $(\bar{N}, N)$). We verify the duality by matching the loop equation of the matrix model to the anomaly equations of the gauge theory. Then we evaluate the matrix model’s free energy and use it to derive the effective superpotential for the gaugino condensates.

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1. Introduction

Three years ago, Robbert Dijkgraaf and Cumrun Vafa discovered a peculiar duality between the \( \mathcal{N} = 1 \) supersymmetric gauge theories in 4D and the bosonic matrix models without any spacetime at all. At first [1, 2], they showed that the effective superpotential for the gaugino condensates and the abelian gauge couplings follow from planar diagrams in the gauge theory (see also [3, 4]), and then they argued [5] that the very same perturbative series also gives the free energy of a bosonic random matrix model whose action is similar to the tree-level superpotential of the gauge theory. Shortly afterwards, Cachazo, Douglas, Seiberg, and Witten [6, 7, 8] pointed out that the planar diagrams describe the on-shell chiral ring of the gauge theory. They used a different technique to study the on-shell ring, namely the generalized Konishi anomaly equations, and those equations turned out to be exactly similar to the loop equations of the random matrix model. This confirmed the Dijkgraaf–Vafa gauge-matrix duality and made it more precise: the matrix model is dual to a subring of the gauge theory’s chiral ring comprising the gaugino condensates and the mesons; other operators of the gauge theory are invisible to the matrix model. In particular, the sphere-level free energy of the matrix model is dual to the effective prepotential of the gaugino condensates.

Gauge-matrix duality works for all kinds of theories, including quiver theories with multiple gauge groups and bifundamental matter [2]. Here is a brief summary of Dijkgraaf–Vafa rules for building the matrix model of a particular gauge theory:

1. A \( U(n) \) or \( SU(n) \) gauge symmetry of the field theory becomes a \( U(\hat{N}) \) global symmetry of the matrix model, and we take the \( \hat{N} \to \infty \) limit. If multiple gauge symmetries \( SU(n_i) \) are involved, we take all the \( \hat{N}_i \to \infty \) at the same rate: \( \hat{N}_i = t \times n_i \) for the same \( t \to \infty \). Similar rules apply to \( SO(n) \) or \( Sp(n) \) gauge group factors.

2. Chiral superfields becomes bosonic variables of the matrix model in similar multiplets of the symmetry group. Thus quark and antiquark superfields become complex vectors of length \( \hat{N} \), adjoint superfields become \( \hat{N} \times \hat{N} \) matrices, and the bi-fundamental multiplets of an \( SU(n) \times SU(m) \) symmetry become \( \hat{N} \times \hat{M} \) matrices.
3. A complex chiral superfield becomes a real bosonic variable of the matrix model (or rather \( t \) or \( t^2 \) real variables. Thus a complex adjoint superfield becomes a hermitian \( \hat{N} \times \hat{N} \) matrix. Likewise, a complex vector \( A \) of the matrix model and its conjugate \( B = A^\dagger \) together represent both the quark and the antiquark superfields of the gauge theory.

Rule 3 works well for non-chiral gauge theories with real matter multiples (or conjugate pairs like \( \Box + \bar{\Box} \)) but not for chiral theories. For example, consider an \( SU(n) \times SU(m) \) theory with a chiral bifundamental field \((n, m)\) without the conjugate \((\bar{n}, \bar{m})\) multiplets.\(^{(1)}\)

By rules 1 and 2 this field becomes an \( \hat{N} \times \hat{M} \) matrix transforming as a bifundamental of the \( U(\hat{N}) \times U(\hat{M}) \) symmetry, and since this representation is complex, the matrix must be complex too. But according to rule 3 such complex \( \hat{N} \times \hat{M} \) matrix corresponds to the non-chiral \((n, \bar{m}) + (\bar{n}, m)\) multiplet of fields rather than the chiral \((n, m)\) without the \((\bar{n}, \bar{m})\).

Lazaroiu et al. [9, 10] found a way out of this problem: instead of real bosonic variables integrated over \( \mathbb{R} \), one may use holomorphic variables, i.e. complex variables integrated over some contours in the complex plane rather than the whole plane \( \mathbb{C} \). More generally, one integrates \( \hat{N} \times \hat{M} \) matrix elements of a complex matrix over some variety \( \Gamma \subset \mathbb{C}^{\hat{N}\hat{M}} \) of real dimension \( \hat{N}\hat{M} \). The variety \( \Gamma \) must be consistent with the symmetries of the matrix model according to rules 1 and 2. For non-chiral models one may identify \( \Gamma \) with the real “axis” \( \mathbb{R}^{\hat{N}\hat{M}} \) of \( \mathbb{C}^{\hat{N}\hat{M}} \) and recover rule 3, but chiral models require non-linear integration varieties. Lazaroiu et al. found such non-linear \( \Gamma \) for the model of a single–\( SU(n) \) gauge theory with chiral \( \Box + (n - 4) \bar{\Box} \) matter spectrum [10], but in this article we interested in a different theory.

Specifically, we are interested in the chiral bifundamental fields, hence we are looking for varieties of \( \hat{N} \times \hat{M} \) complex matrices which are invariant under the \( U(\hat{N})_L \times U(\hat{M})_R \) symmetry action. To be precise, the variety \( \Gamma \) should satisfy

\[
\forall V_L \in U(\hat{N}), \forall V_R \in U(\hat{M}) : \quad V_L \Gamma V_R^\dagger \cong \Gamma
\]

where ‘\( \cong \)’ means equivalence as an integration variety: same topology with respect to the

\(^{(1)}\) The \((n, m)\) bifundamental must be accompanied by other chiral multiplets with opposite anomaly, but the present argument does not depend on those multiplets.
singularities of the integrand, and similar asymptotics when one or more matrix elements approach the complex infinity. For square matrices $\hat{N} \times \hat{N}$ there is a simple solution, namely the unitary group space $\Gamma = U(\hat{N})$ — which is actually invariant under the symmetry (1.1) $(V_L \Gamma V_R^\dagger = \Gamma)$ and has the right real dimension $= \hat{N}^2$. The problem is much more difficult for the rectangular matrices with $\hat{M} \neq \hat{N}$, and so we leave them for future research. In this article, we work with the square matrices integrated over $\Gamma = U(\hat{N})$.

On the gauge theory side, a unitary matrix $U \in U(\hat{N})$ is dual to an $(n, \bar{n})$ bifundamental field $\Omega$ with non-zero eigenvalues. That is, the vacuum states of the field theory which have matrix-model duals must have $\langle \det \Omega \rangle \neq 0$; there may also be vacua with $\langle \det \Omega \rangle = 0$, but the matrix model might not work for such vacua. Indeed, under the gauge-matrix duality, the allowed values of the fields correspond to matrix variables belonging to the integration variety $\Gamma$ or any of its allowed deformations $\Gamma' \cong \Gamma$. For example, a bosonic variable $z$ integrated over a unit circle with measure $\oint dz$ can also be integrated over any other loop surrounding the $z = 0$ origin. However, no such loop can go through the $z = 0$ point itself because of the measure singularity, and consequently the field $\varphi$ dual to $z$ can take any complex values except $\varphi = 0$. Likewise, the unitary matrix integral can be deformed to integral over a variety $\Gamma' \cong U(\hat{N})$ using an analytic extension

$$\int_{\Gamma'} d\omega[U] = \int_{\Gamma'} \prod_{(i,j)\text{ pairs}} (U^{-1} dU)_{i,j}$$

of the Haar measure. However, this extension becomes singular for $\det U = 0$, which limits the $\Gamma'$ integration varieties to the invertible matrices only. Hence, on the gauge theory side of the duality we should have $\det \Omega \neq 0$, otherwise the duality might break down.

The bifundamental fields with $\det \Omega \neq 0$ are common in dimensional deconstruction. Accordingly, in this article we build a unitary matrix model of an $[SU(N_c)]^K$ 4D supersymmetric gauge theory which deconstructs the 5D SQCD [11]; the chiral ring of this theory was studied in great detail in [12]. The matter fields of the theory are shown on the following quiver diagram:
In particular, the blue lines here denote chiral bifundamentals, which we shall model via unitary matrices $U_\ell$, $\ell = 1, 2, \ldots, K$. The red lines denote the quark and the antiquark fields; we shall model them in the usual way as complex $\hat{N} \times \hat{F}$ matrices $A_\ell$ and $B_\ell = A_\ell^\dagger$.

The article is organized as follows: in the next section §2 we build the matrix model of the gauge theory (1.3) and establish the gauge-matrix correspondence. In particular, we derive the the loop equations of the matrix model and see that they agree with the Konishi anomaly equations of the gauge theory. In §3 we evaluate the matrix integral and derive the free energy of the matrix model in terms of contour integrals on the spectral curve. We begin by reducing the problem to an integral over a single unitary matrix, and then we adapt the technology of Dijkgraaf and Vafa [5, 2, 1] and Cachazo et al. [6] to the unitary case. In §4 we derive the effective superpotential for the off-shell gaugino condensates of the dual gauge theory. Finally, in §5 we discuss open questions related to the present research.
2. The Unitary Matrix Model
and its Loop Equations

2.1 The \([SU(N_c)]^K\) Gauge Theory and its Matrix Model

Dimensional deconstruction of 5D SQCD leads to an \(\mathcal{N} = 1\) \([SU(N_c)]^K\) gauge theory in 4D as described in [13, 11]. The chiral ring of this \([SU(N_c)]^K\) theory was analyzed in much detail in [12], and in this article we build and study its matrix model. We begin with the basic structure of the 4D field theory as shown in the quiver diagram (1.3): the green circles denote simple factors of the net gauge group

\[
G_{4D} = \prod_{\ell=1}^{K} [SU(N_c)]_{\ell}
\]  

while the red and blue arrows denote the chiral superfields:

\[
\begin{align*}
\begin{array}{c}
\text{quarks } Q_{\ell,f} = (\Box_{\ell}) \\
\text{antiquarks } \tilde{Q}^f_{\ell} = (\overline{\Box}_{\ell}) \\
\text{link fields } \Omega_{\ell} = (\Box_{\ell+1}, \overline{\Box}_{\ell})
\end{array}
\end{align*}
\]

where \(f = 1, 2, \ldots, N_f\) and \(\ell = 1, 2, \ldots, K\) is understood modulo \(K\). Note that the link fields form chiral bifundamental multiplets of the gauge group.

The tree level superpotential has three types of terms serving different purposes,

\[
W_{\text{tree}} = W_{\text{OR}} + W_{\text{hop}} + W_{\text{def}}.
\]  

The O’Raifeartaigh terms

\[
W_{\text{OR}} = \beta \sum_{\ell} s_{\ell} \times (\det(\Omega_{\ell}) - v^{N_c})
\]  

\footnote{\text{For deconstruction purposes one takes the } K \rightarrow \infty \text{ limit, but from the 4D point of view } K \text{ is a fixed parameter of the field theory; in our analysis we shall assume } K \text{ to be largish but finite.}}
— where the $s_\ell$ are singlet fields not shown in the quiver diagram — turn each bifundamental field $\Omega_\ell$ into an $SL(N_c, \mathbb{C})$ linear sigma model. This is important for deconstruction purposes, and also allows us to model the $\Omega_\ell$ with unitary matrices without worrying about the zero eigenvalues. The hopping superpotential

$$W_{\text{hop}} = \gamma \sum_{\ell=1}^{K} \sum_{f=1}^{N_f} \left( \tilde{Q}_{\ell+1}^f \Omega_\ell Q_{\ell,f} - \mu_f \tilde{Q}_\ell^f Q_{\ell,f} \right)$$

(2.5)

describes quarks' masses and interactions which let them ‘hop’ between quiver nodes; in 5D terms, this allows quark propagation in the deconstructed $x^4$ direction. Finally, we have the deformation superpotential

$$W_{\text{def}} = \text{tr} \left( W(\Omega_K \Omega_{K-1} \cdots \Omega_2 \Omega_1) \right) \equiv \sum_{p=1}^{d} \frac{g_p}{p} \text{tr} \left( (\Omega_K \cdots \Omega_1)^p \right)$$

(2.6)

for some polynomial $W(X) = \sum_p \frac{g_p}{p} X^p$ with constant coefficients $g_1, \ldots, g_d$. The $W_{\text{def}}$ deforms the 4D theory away from deconstructed SQCD$_5$, breaks the Coulomb branch of the moduli space into a discrete set of vacua, and allows formation of the gaugino condensates. This deformation is analogous to the tree-level superpotential for the adjoint field in [6, 14, 7] and is essential for understanding the on-shell chiral ring of the theory. It is also a key ingredient of the matrix model.

In the matrix model, the $[SU(N_c)]^K$ gauge symmetry of the field theory becomes a $[U(\hat{N})]^K$ global symmetry and we take the $\hat{N} \to \infty$ limit. Consequently, the bifundamental fields $\Omega_\ell$ become $\hat{N} \times \hat{N}$ unitary matrices $U_\ell$, and for chirality’s sake we should integrate each matrix $U_\ell$ over the $U(\hat{N})$ group manifold or an equivalent variety. However, in light of the O’Raifeartaigh terms (2.4) we restrict the determinants of the $U_\ell$ matrices and integrate them over the $SU(\hat{N})$ group manifold. To accommodate the $v^{Nc}$ factor, we rescale the field-matrix correspondence according to $\Omega_\ell \leftrightarrow \hat{v} \times U_\ell$ where $\hat{v} = v + \text{quantum corrections}$. Such quantum corrections are computable in the field theory — cf. §4.3 of [12] for details — but in the matrix model they need to be put in by hand.

The quark sector of the field theory is non-chiral — for each quark $Q_\ell$ there is an anti-quark $\bar{Q}_\ell$ with opposite quantum numbers. In the matrix model, the quarks become complex
rectangular \( \hat{N} \times \hat{F} \) matrices \( A_\ell \) while the antiquark fields \( \tilde{Q}_\ell \) become conjugate \( \hat{F} \times \hat{N} \) matrices \( B_\ell = A_\ell^\dagger \). When we take the \( \hat{N} \to \infty \) limit, we have two options for the flavor number \( \hat{F} \) of the matrix model: we may keep it fixed (i.e., \( \hat{F} \equiv N_f \)), or we may let it grow while keeping the flavor/color ratio fixed, \( \hat{F}/\hat{N} \equiv N_f/N_c \) [15]. In the ’t Hooft limit of fixed \( \hat{F} \), the bifundamental sector dominates the matrix model in the large color limit, and the quark sector becomes quenched — its backreaction on the bifundamental sector becomes negligible. This limit oversimplifies the physics but makes for a simple \( 1/\hat{N} \) perturbation theory in terms of Feynman-like diagrams’ topology. On the other hand, in the un-quenched limit of \( \hat{F}, \hat{N} \to \infty \) the matrix model has rich flavor physics, but the \( 1/\hat{N} \) expansion becomes much more difficult. Consequently, we use the ’t Hooft limit in this article and leave the un-quenched flavor physics for future research.

Together, the \( U_\ell, A_\ell, \) and \( B_\ell \) matrices comprise the entire matrix model. Its partition function is defined as the following matrix integral:

\[
Z = C \prod_{\ell=1}^{K} \left\{ \int_{SU(\hat{N})} d\omega[U_\ell] \int B_\ell A_\ell \exp \left( -\frac{\hat{N}}{\hat{S}} \hat{W}(\text{all } U_\ell, B_\ell, A_\ell) \right) \right\}
\]

where the matrix potential is

\[
\hat{W}(U, B, A) = \hat{W}_{\text{def}}(U) + \hat{W}_{\text{hop}}(U, B, A)
\]

\[
= \sum_{p=1}^{d} \frac{g_p \hat{v}^{pK}}{p} \text{tr} \left( (U_K U_{K-1} \cdots U_2 U_1)^p \right) + \gamma \sum_{\ell=1}^{K} \left( \hat{v} \text{tr}(B_{\ell+1} U_\ell A_\ell) - \text{tr}(\hat{\mu} U_\ell B_\ell) \right)
\]

where \( \hat{\mu} \) is an \( \hat{F} \times \hat{F} \) matrix with eigenvalues \( \mu_1, \ldots, \mu_{\hat{F}} \). Note that the matrix potential does not contain O’Raifeartaigh terms analogous to the \( W_{\text{OR}} \) of the field theory, but integrating the \( U_\ell \) over the \( SU(\hat{N}) \) instead of \( U(\hat{N}) \) has the same effect.

The denominator \( \hat{S} \) in the exponent in eq. (2.7) is the overall coupling constant of the matrix model; as usual, \( \hat{S} \) is fixed while \( \hat{N} \to \infty \). Under gauge-matrix duality, \( \hat{S} \) is dual to the
net gaugino condensate \( S = \sum_i S_i \) of all subgroups of the \( SU(N_c)_{\text{diag}} = \text{diag}[\text{all } SU(N_c)] \). In the field theory, the \( S_i \) and hence the \( S \) emerge from the on-shell chiral ring, and then need to be integrated in to an effective off-shell superpotential, but in the matrix model the \( S \) is an input parameter and the effective superpotential emerges from a more direct calculation we shall perform in §3.

Finally, let us fix the overall normalization factor \( C \) in eq. (2.7). Each \( U_\ell \) is integrated over a compact manifold, while each \( A_\ell = B_\ell^T \) is integrated over the non-compact \( \mathbb{C}^{\hat{N} \hat{F}} \), but the integral is Gaussian. Hence, we let

\[
C = \left( \frac{1}{\text{Vol}[SU(\hat{N})]} \right)^K \times \frac{1}{\eta^{NFK}}
\]

where \( \eta \) is the Gaussian integral for a single quark mode of an average mass. The ‘average’ here does not have to be the arithmetic or the geometric mean, any representative value will do, and so we use \( \gamma \hat{v} \) because the modes vary in mass from \( \gamma(\hat{v} - \mu_f) \) to \( \gamma(\hat{v} + \mu_f) \). Hence,

\[
\eta = \int dx \, d\bar{x} \exp \left( -\frac{\hat{N}}{\hat{S}} \gamma \hat{v} |x|^2 \right) = \frac{2\pi \hat{S}}{N \gamma \hat{v}}.
\]

2.2 LOOP EQUATIONS OF THE MATRIX MODEL

Having defined our matrix model we now need to verify that it is indeed dual to the 4D gauge theory which deconstructs the SQCD\(_5\). In this section we shall verify that the loop equations of the matrix model are similar to the anomaly equations of the gauge theory. To be precise, the matrix model is dual to a rather small part of the gauge theory — namely the subring of its chiral ring involving either the gaugino condensates or the mesons\(^{3}\) — but the anomaly equations for that part of the gauge theory should be accurately reproduced by the loop equations of matrix model.

\(^{3}\) The matrix models also has analogues of the baryonic and antibaryonic operators of the gauge theory, but such “dual baryons” are not matrices and are rather difficult to handle. For this reason, we shall limit our analysis here to the duals of the gaugino condensates and mesons only.
The loop equations follow from infinitesimal holomorphic changes of integration variables. For a simplified example, consider a toy model of a single unitary $\hat{N} \times \hat{N}$ matrix $U$,

$$Z = \int_{U(\hat{N})} d\omega[U] \exp \left( -\frac{\hat{N}}{\hat{S}} \text{tr}(\hat{W}(U)) \right)$$  \hspace{1cm} (2.11)

where $d\omega[U]$ is the holomorphic form (1.2) of the Haar measure. Let us change

$$U \mapsto U' = U + \epsilon \times f(U)$$  \hspace{1cm} (2.12)

where $f$ is a holomorphic function of $U$. Generally, this breaks the unitarity of $U$, hence we should deform the integration variety from $\Gamma = U(\hat{N})$ to $\Gamma'$ which spans the $U'$ for $U \in U(\hat{N})$. However, $\Gamma' \cong \Gamma$ and hence this deformation does not affect the holomorphic integral (2.11).

On the other hand, the variable change (2.12) itself has a non-trivial Jacobian

$$J \equiv \frac{d\omega[U']}{d\omega[U]} = 1 + \sum_{i,j} \delta(U^{-1}dU)_{i,j} = 1 + \epsilon \left[ \sum_{jk} \frac{\partial f_{kj}(U)}{\partial U_{kj}} - \hat{N} \text{tr} \left( U^{-1} f(U) \right) \right].$$  \hspace{1cm} (2.13)

Also, the integrand of the matrix integral changes according to

$$\exp \left( -\frac{\hat{N}}{\hat{S}} \text{tr}(\hat{W}(U)) \right) \mapsto \text{same} \times \left[ 1 - \epsilon \frac{\hat{N}}{\hat{S}} \text{tr}(\hat{W}'(U)f(U)) \right].$$  \hspace{1cm} (2.14)

Altogether, we have changed the matrix integral by

$$\delta Z = \epsilon Z \times \left\{ \sum_{jk} \frac{\partial f_{kj}(U)}{\partial U_{kj}} - \hat{N} \text{tr} \left( U^{-1} f(U) \right) - \frac{\hat{N}}{\hat{S}} \text{tr}(\hat{W}'(U)f(U)) \right\}$$  \hspace{1cm} (2.15)

On the other hand, we have done nothing but changed the integration variable from $U$ to $U'$, hence the integral should not change at all. Therefore, for any holomorphic matrix $\mapsto$ matrix
function \( f(U) \) we must have
\[
\left\langle \sum_{jk} \frac{\partial f_{kj}(U)}{\partial U_{kj}} - \hat{N} \text{tr}(U^{-1} f(U)) - \frac{\hat{N}}{S} \text{tr}(\hat{W}'(U) f(U)) \right\rangle = 0. \tag{2.16}
\]

This toy example shows how to derive loop equations for unitary matrix models. Let us apply this technology to the unitary \( U_\ell \) matrices of our big matrix integral (2.7). Let us pick one matrix, say \( U_\ell \) and change
\[
U_\ell \mapsto U'_\ell = U_\ell + \epsilon \times f(U_\ell, \ldots) \tag{2.17}
\]
where \( f \) is a holomorphic function of the \( U_\ell \) and other matrices of the model (denoted by the \( \ldots \)). To preserve the symmetries of the model, \( f(U_\ell, \ldots) \) must be covariant, i.e. transform like \( U_\ell \) under the \( [U(\hat{N})]^K \). Also, because the \( U_\ell \) is integrated over the \( SU(\hat{N}) \) group manifold rather than \( U(\hat{N}) \), we must preserve the \( \det(U'_\ell) = \det(U_\ell) = 1 \) condition, hence \( f \) must satisfy
\[
\text{tr}(U_\ell^{-1} f(U_\ell, \ldots)) = 0. \tag{2.18}
\]
On the other hand, we do not need to preserve the unitarity of the \( U'_\ell \) because we may deform the integration variety from \( \Gamma = SU(\hat{N}) \) to a nearby \( \Gamma' \approx \Gamma \). Hence, proceeding exactly as in the toy example above, we find that for any covariant holomorphic \( f \) which satisfies eq. (2.18) we must have
\[
\left\langle \sum_{jk} \frac{\partial f_{kj}(U_\ell, \ldots)}{\partial (U_\ell)_{kj}} - \frac{\hat{N}}{S} \text{tr}\left( \frac{\partial \hat{W}(U, A, B)}{\partial U_\ell} f(U_\ell, \ldots) \right) \right\rangle = 0. \tag{2.19}
\]

Now let us focus on functions \( f \) which depend only on the unitary link matrices but not on the quark matrices. By covariance, products of the \( U_{\ell'} \) must be taken in the order of the quiver, hence \( f(U_\ell, \text{other } U_{\ell'}) \) must be a linear combination of \( (U_t U_{\ell-1} \cdots U_1 U_K \cdots U_{\ell+1})^p U_\ell \) for \( p = 0, 1, 2, \ldots \), which can be summarized in a power series
\[
\tilde{f}(X) = \sum_{p=0}^{\infty} X^{-1-p} \times \left( U_t U_{\ell-1} \cdots U_1 U_K \cdots U_{\ell+1} \right)^p \times U_\ell = \frac{1}{X - U_t U_{\ell-1} \cdots U_1 U_K \cdots U_{\ell+1}} \times U_\ell \tag{2.20}
\]
in an auxiliary complex variable \( X \). In field theory we use this series as it is, but in the
matrix model we must correct for the trace condition (2.18), thus

\[
f(U_\ell, \ldots) = \left[ \frac{1}{X - U_\ell U_{\ell-1} \cdots U_1 U_K \cdots U_{\ell+1}} - \frac{1}{S} R(X) \right] \times U_\ell \quad (2.21)
\]

where the second term inside the brackets assures \( \text{tr}[^1] = 0 \). Specifically,

\[
R(X) \overset{\text{def}}{=} \frac{\hat{S}}{N} \times \text{tr} \left( \frac{1}{X - U_\ell U_{\ell-1} \cdots U_1 U_K \cdots U_{\ell+1}} \right) \quad (2.22)
\]

which is same for all \( \ell \) and remains finite in the \( \hat{N} \to \infty \) limit. For \( f \) as in eq. (2.21)

\[
\sum_{jk} \frac{\partial f_{kj}(U_\ell, \ldots)}{\partial (U_\ell)_{kj}} = \frac{N^2}{S^2} \left[ X R^2(X) - \hat{S} R(X) + \frac{\hat{S}}{N^2} (X R'(X) + R(X)) \right] \quad (2.23)
\]

while

\[
\text{tr} \left( \frac{\partial \hat{W}_{\text{hop}}}{\partial U_\ell} f(U_\ell, \ldots) \right) = \frac{\hat{N}}{S} \left[ X W'(X) R(X) - \hat{S} W'(X) + \hat{S} P(X) - C R(X) \right] \quad (2.24)
\]

where

\[
W'(X) = \sum_{p=1}^{d} g_p \hat{v}^p K X^{p-1}, \quad (2.25)
\]

\[
P(X) = \frac{1}{\hat{N}} \text{tr} \left( (U_K \cdots U_1) \frac{W'(X) - W'(U_K \cdots U_1)}{X - U_K \cdots U_1} \right) \quad (2.26)
\]

(a polynomial of \( X \) of degree \( = d - 2 \))

and \( C = \frac{1}{\hat{N}} \text{tr} \left( (U_K \cdots U_1) W'(U_K \cdots U_1) \right) \).

(2.27)

Note that in the large \( \hat{N} \) limit \( C \) and \( P(X) \) remain finite, hence the right hand side of eq. (2.24) grows like \( \hat{N} \). By comparison,

\[
\text{tr} \left( \frac{\partial \hat{W}_{\text{hop}}}{\partial U_\ell} f(U_\ell, \ldots) \right) = O(\hat{F}), \quad (2.28)
\]
which is much smaller than $O(\hat{N})$ in the 't Hooft limit. Therefore, eq. (2.19) becomes

$$\langle XR^2(X) - (XW'(X) + \hat{S} - C) \times R(X) + \hat{S}W'(X) - \hat{S}P(X) \rangle = O(1/\hat{N}) \approx 0. \tag{2.29}$$

Moreover, for $\hat{N} \to \infty$ matrix averages factorize as

$$\langle R^2(X) \rangle \to \langle R(X) \rangle^2, \quad \langle C \times R(X) \rangle \to \langle C \rangle \times \langle R(X) \rangle, \tag{2.30}$$

and this gives us a loop equation for the $\langle R(X) \rangle$, namely

$$XR^2(X) - (XW'(X) + \hat{S} - \langle C \rangle) \times \langle R(X) \rangle + \hat{S}W'(X) + \hat{S} \langle P(X) \rangle = 0. \tag{2.31}$$

On the field theory side, we have a similar quadratic equation for the gaugino condensate resolvent

$$R(X) = \text{tr} \left( \frac{(W^\alpha W_\alpha)_\ell}{32\pi^2} \times \frac{1}{X - \Omega_\ell \Omega_{\ell-1} \cdots \Omega_1 \Omega_K \cdots \Omega_{\ell+1}} \right) \quad (\text{same for all } \ell). \tag{2.32}$$

In the on-shell chiral ring, this resolvent satisfies

$$XR^2(X) - \tilde{W}(X)R(X) + F(X) = 0 \tag{2.33}$$

where

$$\tilde{W}(X) = \sum_{p=1}^{d} g_p X^p + \beta v^{N_c} \langle s \rangle \quad (\text{same } \langle s_\ell \rangle \forall \ell) \tag{2.34}$$

and $F(X)$ is another polynomial (of degree $< d$) which depends on vacuum state of the field theory. Eqs. (2.31) and (2.33) are obviously dual to each other; to make them identical we simply need to identify

$$X^{\text{field theory}} \longleftrightarrow \hat{v}^K \times X^{\text{matrix model}}, \quad R(X)^{\text{field theory}} \longleftrightarrow \hat{v}^{-K} \times \langle R(X) \rangle^{\text{matrix model}}, \tag{2.35}$$

$$(\beta v^{N_c} \langle s \rangle)^{\text{field theory}} \longleftrightarrow (\hat{S} - \langle C \rangle)^{\text{matrix model}}, \quad F(X)^{\text{field theory}} \longleftrightarrow \hat{v}^{-K} \times \hat{S} (W'(X) + \langle P(X) \rangle)^{\text{matrix model}}. \tag{2.36}$$
This correspondence explains why $\hat{S}$ is dual to the net gaugino condensate

$$S_{\text{net}} = \frac{\langle \text{tr}(W^\alpha W_\alpha) \rangle}{32\pi^2} = \lim_{X \to \infty} (XR(X)).$$  

(2.37)

Solving the anomaly equation (2.33), we have

$$R(X) = \frac{\tilde{W}(X) - \sqrt{\tilde{W}^2(X) - 4XF(X)}}{2X} \xrightarrow{X \to \infty} \frac{F(X)}{\tilde{W}(X)}.$$  

(2.38)

According to eq. (2.36), the matrix dual of the right hand side is

$$\hat{v}^{-K} \hat{S} \times \frac{W'(X) + \langle (P(X)) \rangle}{XW'(X) + \text{const}} \xrightarrow{X \to \infty} \hat{S} \times \frac{\hat{v}^{-K}}{X_{\text{model}}} = \frac{\hat{S}}{X_{\text{theory}}}.$$  

(2.39)

(note that $P(X)$ has lower degree in $X$ than $W'(X)$), and therefore

$$(S_{\text{net}})^{\text{field}} \longleftrightarrow (\hat{S})^{\text{matrix}}_{\text{model}}.$$  

(2.40)

⋆ ⋆ ⋆

The mesonic resolvents of the gauge theory also have matrix duals. On the field theory side, chiral mesonic operators with quarks and antiquarks at different quiver nodes are packaged into a bunch of resolvents

$$\mathcal{M}_{\ell',\ell}(X) = \tilde{Q}_{\ell'}\Omega_{\ell'-1} \cdots \Omega_{\ell} \frac{1}{X - \Omega_{\ell-1} \cdots \Omega_1 \Omega_K \cdots \Omega_{\ell}} Q_{\ell'}$$  

(2.41)

subject to periodicity conditions

$$-\mathcal{M}_{\ell'=\ell+K} + X\mathcal{M}_{\ell=\ell} = M_{\ell} \equiv (\tilde{Q}_{\ell}Q_{\ell})$$  

(2.42)

where the right hand side is an ordinary mesonic operator which does not depend on $X$. Besides their quiver indices, the mesonic resolvents are also $N_f \times N_f$ matrices in the flavor
space. On the matrix-model side, they are dual to (the averages of) $\hat{F} \times \hat{F}$ matrices

$$M_{\ell',\ell}(X) = B_{\ell'} U_{\ell-1} \cdots U_1 X - U_{\ell-1} \cdots U_1 U_K \cdots U_{\ell-1} U_{\ell} A_{\ell}$$  \hspace{1cm} (2.43)

which satisfy a similar periodicity relation

$$-M_{\ell'=\ell+K}(X) + XM_{\ell'=\ell}(X) = M_{\ell} \equiv (B_{\ell} A_{\ell}).$$  \hspace{1cm} (2.44)

Because of the $\hat{v}$ factors in the $\Omega_{\ell} \leftrightarrow \hat{v}$ correspondence, in the mesonic sector we expect the gauge-matrix duality to work according to

$$X_{\text{field theory}} \leftrightarrow \hat{v}^K \times X_{\text{model}}, \quad M_{\ell'-\ell-K}(X) \leftrightarrow \hat{v}^{\ell'-\ell-K} \times (M_{\ell'-\ell}(X))_{\text{model}}.$$  \hspace{1cm} (2.45)

To derive loop equations for the mesonic resolvents of the matrix model, we need to vary the $A_{\ell}$ and the $B_{\ell}$ matrices independently of each other. Note that the conjugacy constraint $B_{\ell} = A_{\ell}^\dagger$ is just a special case of integrating the $2\hat{N}\hat{F}$ complex numbers comprising each $(A_{\ell}, B_{\ell})$ over a variety $\Gamma$ of real dimension $2\hat{N}\hat{F}$, we “just happened” to choose $\Gamma = \{B_{\ell} = A_{\ell}^\dagger\}$. But as long as the integrand is a holomorphic function of both $A_{\ell}$ and $B_{\ell}$ (views as independent variables), we may deform the integration variety without changing the integral,

$$\int\int_{B_{\ell} = A_{\ell}^\dagger} dA_{\ell} dB_{\ell} f(A_{\ell}, B_{\ell}, \ldots) = \int\int_{\Gamma'} dA_{\ell} dB_{\ell} f(A_{\ell}, B_{\ell}, \ldots) \quad \text{for} \quad \Gamma' \cong \{B_{\ell} = A_{\ell}^\dagger\}.$$  \hspace{1cm} (2.46)

Therefore, small variations of the $A_{\ell}$ and the $B_{\ell}$ matrices don’t need to be conjugate to each other — the discrepancy will deform the integration variety a bit, but small deformations do not affect the integral.

Minding this rule, let us vary any one $B_{\ell}$ matrix while the “conjugate” $A_{\ell}$ matrix remains unchanged. Specifically, let

$$B_{\ell} \mapsto B'_{\ell} = B_{\ell} + \epsilon \times B_{\ell} \times \frac{U_{\ell'-1} \cdots U_{\ell}}{X - U_{\ell'-1} \cdots U_{\ell}}$$  \hspace{1cm} (2.47)

where $\epsilon$ is an infinitesimal $\hat{F} \times \hat{F}$ matrix in the flavor space. The Jacobian for such change
The only difference is in the $\hat{v}$ factors — and that is in accordance with the correspondence rules (2.45) and (2.35).
Having two sets (2.51) and (2.53) of mesonic loop equations is not redundant. Combining both sets with the periodicity equations (2.44) allows us to solve for all the mesonic resolvents (or rather their averages) in terms of a single matrix $\langle M \rangle$. The solution works exactly as for the gauge theory, so instead of copying from [12] almost verbatim, let us simply state the result:

\[
\langle M_{\ell'=\ell}(X) \rangle = \frac{\langle M \rangle + \hat{v}^{-K} \mu^{K-1} \gamma^{-1} \langle R(X) \rangle \times 1}{X - \hat{v}^{-K} \hat{\mu}^K},
\]

(2.55)

\[
\langle M_{\ell'>\ell}(X) \rangle = \frac{\hat{\mu}^{\ell'-\ell} \times \hat{\mu} \times \langle M \rangle + \gamma^{-1} X \langle R(X) \rangle \times 1}{X - \hat{v}^{-K} \hat{\mu}^K},
\]

(2.56)

where $\langle M \rangle = \langle B_\ell A_\ell \rangle$ must be same for all $\ell$ and must commute with the quark mass matrix $\hat{\mu}$.\(^{(4)}\)

Finally, there one yet another bunch of loop equations for the mesonic resolvents stemming from $A$ and $B$ dependent variations of the $U$ matrices. Let us pick any one $U_\ell$ and vary it according to

\[
\delta U_\ell = A_{\ell+1} \times \epsilon \times B_\ell \times \frac{1}{X - U_{\ell-1} \cdots U_1 U_K \cdots U_\ell} - \frac{\text{tr}(U_\ell^{-1} \times \epsilon)}{N} \times U_\ell.
\]

(2.57)

where the second term assures $\delta \det(U_\ell) = 0$. The trace in the second term evaluates to

\[
\text{tr} \left( \epsilon \times \frac{M_{\ell+N,\ell+1}(X) - M_{\ell+N,\ell+1}(0)}{X} \right).
\]

(2.58)

The Jacobian of the variable change (2.57) is given by

\[
\delta J = \sum_{jk} \frac{\partial (\delta U_\ell)_{jk}}{\partial (U_\ell)_{jk}} = \frac{\hat{N}}{S} \text{tr} \left( \epsilon \times \begin{bmatrix} R(X) \times M_{\ell+K,\ell+1}(X) \\ - \hat{S} \times \frac{M_{\ell+K,\ell+1}(X) - M_{\ell+K,\ell+1}(0)}{X} \\ + O(1/\hat{N}^2) \end{bmatrix} \right),
\]

(2.59)

\(^{(4)}\) Strictly speaking, $\langle M \rangle$ must commute with the $\hat{\mu}^K$ matrix rather than with the $\hat{\mu}$ itself. Also, the matrix elements of $\langle B_\ell A_\ell \rangle$ which commute with $\hat{\mu}^K$ but not with $\hat{\mu}$ have $\ell$-dependent phases. To avoid this mess, we assume that all distinct mass eigenvalues also have distinct $K^{th}$ powers.
while the potential varies according to

$$\delta \hat{W}_{\text{hop}} = \gamma \hat{v} \text{tr} \left( \epsilon \times M_{\ell,\ell}(X) \times M_{\ell+1} \right) + O(1/\hat{N}), \quad (2.60)$$

$$\delta \hat{W}_{\text{def}} = \text{tr} \left( \epsilon \times \left[ \mathcal{W}'(X) \times M_{\ell+K,\ell+1}(X) \right. - C \times \left. \frac{M_{\ell+K,\ell+1}(X) - M_{\ell+K,\ell+1}(0)}{X} \right] - t_{\ell}(X) \right), \quad (2.61)$$

where $C$ is as in eq. (2.27) and $t_{\ell}(X)$ is an $\hat{F} \times \hat{F}$ matrix-valued polynomial of $X$ whose actual form does not affect the following argument. Altogether, these changes must cancel out of the matrix integral, hence demanding

$$\langle \delta J \rangle - \frac{\hat{N}}{S} \langle \delta \hat{W} \rangle = 0 \quad \forall \epsilon \quad (2.62)$$

we arrive at

$$\gamma \hat{v} \left( \langle M_{\ell,\ell}(X) \times M_{\ell+1} \rangle - \langle R(X) \times M_{\ell+K,\ell+1}(X) \rangle \right)$$

$$= \left[ \mathcal{W}'(X) + \frac{\hat{S} - C}{X} \right] \times \langle M_{\ell+K,\ell+1}(X) \rangle - \frac{\hat{S} - C}{X} \times \langle M_{\ell+K,\ell+1}(0) \rangle$$

$$= \langle t_{\ell}(X) \rangle + O(1/\hat{N}). \quad (2.63)$$

In the large $\hat{N}$ limit matrix averages factorize, and this gives us another family of loop equations, namely

$$\gamma \hat{v} \langle M_{\ell,\ell}(X) \rangle \times \langle M_{\ell+1} \rangle$$

$$= \left[ \mathcal{W}'(X) + \frac{\hat{S} - C}{X} - \langle R(X) \rangle \right] \times \langle M_{\ell+K,\ell+1}(X) \rangle$$

$$- \frac{\hat{S} - C}{X} \times \langle M_{\ell+K,\ell+1}(0) \rangle$$

$$= \text{a polynomial of } X. \quad (2.64)$$

Again, these loop equations are exactly similar to their field theory counterparts and we may solve them in a similar way. Making use of eqs. (2.55–56) and eq. (2.31), we transform
eq. (2.64) into a quadratic equation for the mesonic “VEV” \langle M \rangle, namely

\[
(\gamma \hat{\mu} \langle M \rangle)^2 + \frac{\hat{\mu}^K}{X \hat{v}^K} [X W'(X) + \hat{S} - C] \times (\gamma \hat{\mu} \langle M \rangle) + \frac{\hat{\mu}^K}{X \hat{v}^K} F(X) = 0.
\] (2.65)

Also, the \langle M \rangle matrix is block-diagonal in the eigenbasis of \hat{\mu} and does not depend on X. Hence, for each block we may substitute a different value of X into eq. (2.65) and apply the resulting equation to the block in question; our choice is \(X = \hat{\mu}_f^K / \hat{v}^K\) which kills the right hand side of eq. (2.65) regardless of the matrix-valued polynomial we didn’t spell out. Consequently, each block — and hence the whole matrix — satisfies

\[
(\gamma \hat{\mu} \langle M \rangle)^2 + \left( (\hat{\mu}/\hat{v})^K W'( (\hat{\mu}/\hat{v})^K ) + \hat{S} - C \right) \times (\gamma \hat{\mu} \langle M \rangle) + F((\hat{\mu}/\hat{v})^K) = 0. \] (2.66)

Note similarity between this equation and the loop equation (2.31); this allows us to write

\[
\langle M_\ell \rangle = -\frac{X \langle R(X) \rangle^\pm}{\gamma \hat{\mu}} \text{ evaluated for } X = (\hat{\mu}/\hat{v})^K \] (2.67)

where \(\langle R(X) \rangle^\pm\) on the right hand side indicates the two solutions of eq. (2.31).

This completes our analysis of the loop equations of our matrix model. Having seen that those equations are dual to the anomaly equations of the gauge theory of the quiver diagram (1.3), we can be positive that our model is indeed dual to that gauge theory.
3. Calculating the Matrix Integral

In this section we evaluate the matrix integral (2.7) in a sequence of simple steps. First we integrate over the $A_\ell$ and $B_\ell$ matrices dual to the quarks and the antiquarks. Second, we reduce the integral over $K \ell$ link matrices $U_\ell$ to an integral over a single unitary matrix $U = U_K U_{K-1} \cdots U_1$. After that we follow the Dijkgraaf–Vafa method adapted to a unitary rather than hermitian matrix: we reduce the integral over the whole matrix $U$ to an integral over its eigenvalues, and then we use the saddle-point approximation in the large $\hat{N}$ limit. In this limit, the eigenvalue spectrum becomes continuous with density $\rho(\lambda)$, the free energy has a $1/\hat{N}$ expansion where the sphere-level and the disk-level terms are given by spectral integrals, — and we relate the whole shmeer to the loop equation (2.31) and the period integrals of its Riemann surface.

Let us integrate over the quark matrices. The hopping potential (2.8) is bilinear with respect to the matrix elements of the $A_\ell$ and $B_\ell$:

\[
\hat{W}_{\text{hop}} = \sum_{\ell,\ell'} \sum_{i,i'} \sum_{f,f'} B_{\ell',i'} f' D_{\ell',i'f'} A_{\ell,i'f'}
\]

(3.1)

where

\[
D_{\ell',i'f'} = \hat{v} \delta_{\ell+1} (U_\ell)^{i'} f' - \delta_{\ell} \delta_i^{i'} \mu_{f'f}.
\]

(3.2)

This makes the quark-matrix integral a Gaussian integral over $K \hat{N} \hat{F}$ independent complex variables, which we evaluate as

\[
\prod_{\ell=1}^{K} \left\{ \int_{B_\ell = A_\ell} d^{\hat{N} \hat{F}} A_\ell d^{\hat{N} \hat{F}} B_\ell \right\} \exp \left( -\frac{\hat{N}}{\hat{S}} \hat{W}(U, A, B) \right) = \left( \frac{2 \pi \hat{S}}{\hat{N} \gamma} = \eta \hat{v} \right)^{K \hat{N} \hat{F}} \times \frac{1}{\text{Det } \mathcal{D}}
\]

(3.3)

where the determinant involves all indices: quiver, color, and flavor. In block form

\[
\mathcal{D} = \begin{pmatrix}
-\hat{\mu} & 0 & 0 & \cdots & \hat{v} U_K \\
\hat{v} U_1 & -\hat{\mu} & 0 & \cdots & 0 \\
0 & \hat{v} U_2 & -\hat{\mu} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{v} U_{K-1} & -\hat{\mu}
\end{pmatrix}
\]

(3.4)
and hence

\[
\text{Det}_{K \hat{N}_F \times K \hat{N}_F} (\mathcal{D}) = \text{Det}_{\hat{N}_F \times \hat{N}_F} (\hat{v}^K U K_{K-1} \cdots U_1 - \hat{\mu}^K) = \prod_{f=1}^{\hat{F}} \det_{\hat{N} \times \hat{N}} (\hat{v}^K U K_{K-1} \cdots U_1 - \hat{\mu}_f^K)
\]

(3.5)

where on the right hand side the \( \hat{\mu}_f \) are eigenvalues of the quark mass matrix \( \hat{\mu} \).

Note that the ‘quark’ integral (3.3) depends on the link matrices \( U_\ell \) only through their product \( U \equiv U_K U_{K-1} \cdots U_1 \). Likewise, the deformation superpotential \( \mathcal{W}_{\text{def}} \) depends on the \( U_\ell \) only through their product. Hence, at this stage, we may write the matrix integral (2.7) as

\[
Z = \frac{1}{(\text{Vol}[SU(\hat{N})])^K} \prod_{\ell=1}^{K} \left\{ \int_{SU(\hat{N})} d\omega[U_\ell] \right\} f(U \equiv U_K U_{K-1} \cdots U_1)
\]

(3.6)

where

\[
f(U) = \exp \left(-\frac{\hat{N}}{S} \hat{W}_{\text{def}}(U)\right) / \prod_{f=1}^{\hat{F}} \det_{\hat{N} \times \hat{N}} \left(U - (\mu_f / \hat{v})^K\right),
\]

(3.7)

and \( d\omega[U_\ell] \) is the Haar measure for \( U_\ell \in SU(\hat{N}) \). This measure is left-invariant — for any fixed \( V \in SU(\hat{N}) \), \( d\omega[VU] = d\omega[U] \) — and this makes it easy to change variables in the unitary matrix integrals. In particular, in an integral over \( K \) matrices such as (3.6) we can set \( U_1 = V U'_1 \) (for \( \ell = 1 \) only) and have

\[
d\omega[U_K] \times \cdots \times d\omega[U_2] \times d\omega[U_1 = V U'_1] = d\omega[U_K] \times \cdots \times d\omega[U_2] \times d\omega[U'_1]
\]

(3.8)

for any \( V \in SU(\hat{N}) \) which does not depend on the \( U'_1 \) matrix, even if \( V \) depends on the other unitary matrices \( U_2, \ldots, U_N \). Therefore

\[
\int_{SU(\hat{N})} d\omega[U_K] \times \cdots \times d\omega[U_1] f(U_K \cdots U_1)
\]

\[= \int_{SU(\hat{N})} d\omega[U_K] \times \cdots \int_{SU(\hat{N})} d\omega[U_2] \times \int_{SU(\hat{N})} d\omega[U'_1] f(U_K \cdots U_2 \times VU'_1)
\]

\[\langle\text{setting } V = (U_K \cdots U_2)^{-1} \Rightarrow U'_1 = V^{-1} U_1 = U_K \cdots U_2 U_1 \equiv U \rangle = \int_{SU(\hat{N})} d\omega[U_K] \times \cdots \int_{SU(\hat{N})} d\omega[U_2] \times \int_{SU(\hat{N})} d\omega[U'_1 = U] f(U)\]
\[
\left( \text{Vol}[\text{SU}(\hat{N})] \right)^{K-1} \times \int_{\text{SU}(\hat{N})} d\omega[U] f(U), \tag{3.9}
\]

and hence
\[
Z = \frac{1}{\text{Vol}[\text{SU}(\hat{N})]} \times \int_{\text{SU}(\hat{N})} d\omega[U] f(U). \tag{3.10}
\]

Furthermore, according to eq. (3.7), \( f(U) \) depends only on the eigenvalues of the unitary matrix \( U \) but not on its eigenvectors. Consequently, we decompose the \( \hat{N}^2 - 1 \) coordinates of the \( \text{SU}(\hat{N}) \) group manifold into \( \hat{N} - 1 \) independent eigenvalues \( e^{i\lambda_i} \) \((i = 1, \ldots, \hat{N}\) but \( \sum_i \lambda_i \equiv 0 \mod 2\pi \) and \( \hat{N}(\hat{N} - 1) \) angular variables \( \theta_{i\neq j} \) describing the eigenvectors. The Jacobian of this decomposition is given by
\[
\frac{d\omega}{d(\lambda_i, \theta_{i,j})} = \text{const} \times \prod_{i<j} 4 \sin^2 \frac{\lambda_i - \lambda_j}{2} \tag{3.11}
\]

— the unitary version of the Vandermonde determinant — which depends only on the eigenvalues \( \lambda_i \), hence integrating over the eigenvector variables \( \theta_{i,j} \) we arrive at
\[
Z = \hat{\nu}^{K\hat{N}\hat{F}} \frac{2\pi}{\hat{N}!} \int_{0}^{2\pi} d\lambda_1 \frac{2\pi}{2\pi} \int_{0}^{2\pi} d\lambda_{\hat{N}} f(\lambda_1, \ldots, \lambda_{\hat{N}}) \times \prod_{i<j} 4 \sin^2 \frac{\lambda_i - \lambda_j}{2} \times \sum_{L=-\infty}^{+\infty} e^{iL(\lambda_1 + \cdots + \lambda_{\hat{N}})} \tag{3.12}
\]
where the last factor is the \( \delta \)-function for the \( \sum_i \lambda_i \) modulo \( 2\pi \).

Thus far we made only exact calculation, but now we turn to approximations valid in the \( \hat{N} \to \infty \) limit. Re-writing the integrand of eq. (3.12) in exponential form
\[
\exp \left[ \frac{\hat{N}}{S} \sum_{j=1}^{\hat{N}} \mathcal{W}(e^{i\lambda_j}) - \sum_{j=1}^{\hat{N}} \sum_{f=1}^{\hat{F}} \log \left( e^{i\lambda_j} - (\mu_f / \hat{\nu})^K \right) \right. \\
\left. + \sum_{i<j} \log \left( 4 \sin^2 \frac{\lambda_i - \lambda_j}{2} \right) + iL \sum_{j=1}^{\hat{N}} \lambda_j \right] \tag{3.13}
\]
we see that all terms in exponent grow with \( \hat{N} \). Hence, in the large \( \hat{N} \) limit we may use the
saddle-point approximation:

\[ \log Z \approx - \frac{\hat{N}}{S} \sum_{j=1}^{\hat{N}} \mathcal{W}(e^{i\tilde{\lambda}_j}) - \sum_{j=1}^{\hat{N}} \sum_{f=1}^{\hat{F}} \log \left( e^{i\tilde{\lambda}_j} - (\mu_f / \hat{v})^K \right) \]

\[ + \sum_{i<j} \log \left( 4 \sin^2 \frac{\tilde{\lambda}_i - \tilde{\lambda}_j}{2} \right) + iL \sum_{j=1}^{\hat{N}} \lambda_j \]

where \((\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\hat{N})\) and \(L\) maximize the right hand side of this formula, or rather maximize its real part and extremize it imaginary part; generally, this requires moving away from the real axis into the complex plane. Or into some other complex space: since the \(\lambda_i\) are periodic variables on the circle \(\mathbb{R}/2\pi \mathbb{Z}\), the \(\tilde{\lambda}_i\) move into the complex cylinder \(\mathbb{C}/2\pi \mathbb{Z}\) rather than into the plane \(\mathbb{C}\). As to the \(L\), in the large \(\hat{N}\) limit the maximum happens for \(L = O(\hat{N})\) where the discreteness of \(L\) does not matter any more, hence \(L = \frac{\hat{N}}{S} \times \hat{C}\) where \(\hat{C}\) is finite and complex.

As usual for matrix models, for \(\hat{N} \to \infty\) the spectrum \(\Sigma\) of \((\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\hat{N})\) becomes continuous. In general, it comprises several continuous line segments \(\Sigma_1, \ldots, \Sigma_n\) on the complex cylinder. Denoting the spectral density over those lines by \(\frac{\hat{N}}{S} \rho(\lambda) d\lambda\), we write the free energy of the matrix model as

\[ \mathcal{F} \equiv - \frac{\hat{S}^2}{N^2} \log Z = \mathcal{F}_S + \frac{\hat{S}}{\hat{N}} \mathcal{F}_D + O(1/\hat{N}^2), \]

where

\[ \mathcal{F}_S = \int_{\Sigma} d\lambda \rho(\lambda) \left[ \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda \right] - \frac{1}{2} \int_{\Sigma_{\lambda}} d\lambda \rho(\lambda) \int_{\Sigma_{\lambda'}} d\lambda' \rho(\lambda') \log \left( 4 \sin^2 \frac{\lambda - \lambda'}{2} \right) \]

and

\[ \mathcal{F}_D = \sum_{f=1}^{\hat{F}} \int_{\Sigma} d\lambda \rho(\lambda) \log \left( e^{i\lambda} - (\mu^K_f / \hat{v})^K \right). \]

In “worldsheet” terms of the \(1/\hat{N}\) expansion [14], the leading contribution \(\mathcal{F}_S\) which comes from \(\mathcal{W}_{\text{def}}\) and the Vandermonde determinant corresponds to genus \(g = 0\) i.e. spherical

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topology, hence the notation. Likewise, the quarks’ sector contribution corresponds to the disk topology, hence the notation $\mathcal{F}_D$.

The spectral density $\rho(\lambda)$ minimizes the free energy $\mathcal{F} \approx \mathcal{F}_S$ under constraint

$$\#(\tilde{\lambda}_i) = \hat{N} \implies \int_{\text{whole } \Sigma} d\lambda \rho(\lambda) = \hat{S}, \quad (3.18)$$

and this gives us a variational equation

$$\frac{d}{d\lambda} \frac{\delta \mathcal{F}_S}{\delta \rho(\lambda)} = 0. \quad (3.19)$$

In light of eq. (3.16), this equation becomes

$$\frac{dW}{d\lambda} - i\hat{C} - \int_{\text{whole } \Sigma} d\lambda' \rho(\lambda') \cot \frac{\lambda - \lambda'}{2} = 0 \quad (3.20)$$

where the bar across the integral sign indicates principal-value integration over the pole at $\lambda' = \lambda$. However, it is more convenient to regulate the pole via bypassing it in the complex plane, so let us introduce the resolvent

$$\hat{R}(w) = \int_{\text{whole } \Sigma} d\lambda \rho(\lambda) \cot \frac{w - \lambda}{2} \quad (3.21)$$

which has branch cuts along the spectral segments $\Sigma_1, \ldots, \Sigma_n$ but is analytic elsewhere in the complex cylinder. In terms of this resolvent, eq. (3.20) applies to the average of the two sides of any branch cut, hence

$$\forall \lambda \in \Sigma : \hat{R}(\lambda \pm i\epsilon) = \frac{dW}{d\lambda} - i\hat{C} \mp 2\pi i\rho(\lambda). \quad (3.22)$$

Together, eqs. (3.21–22) lead to a quadratic equation for the resolvent $\hat{R}(w)$:

$$\hat{R}^2(w) - 2 \left[ \frac{dW(e^{i\epsilon})}{dw} - i\hat{C} \right] \times \hat{R}(w) + 2F(e^{i\epsilon}) + \hat{S}^2 = 0 \quad (3.23)$$
where $F$ is a polynomial function of $e^{iw}$.

Physically, eq. (3.23) is nothing but the loop equation (2.31) written in terms of the periodic coordinate $w$ of the complex cylinder instead of the flat coordinate $X = e^{iw}$. Indeed, in the saddle point approximation we have

$$\langle R(X) \rangle = \frac{\hat{S}}{N} \sum_i \frac{1}{X - \exp(i\lambda_i)} = \int_{\text{whole } \Sigma} d\lambda \frac{\rho(\lambda)}{X - e^{i\lambda}} \quad (3.25)$$

and hence

$$\hat{R}(w) = 2iX \langle R(X) \rangle - i\hat{S} \quad \text{for } X = e^{iw}. \quad (3.26)$$

Likewise

$$F(X = e^{iw}) = \hat{S} \times (\langle C \rangle - XW'(X) + 2X \langle P(X) \rangle), \quad (3.27)$$

and if we also identify $\hat{C} = \langle C \rangle$ then eq. (3.23) becomes the loop equation (2.31).

Solving the loop equations gives us

$$\langle R(X) \rangle = \frac{XW'(X) + \hat{S} - \hat{C} - Y}{2X} \Rightarrow \hat{R} = iXW'(X) - i\hat{C} - iY(X) \quad (3.28)$$

where

$$Y(X) = \pm \sqrt{\left(XW'(X) + \hat{S} - \hat{C}\right)^2 - 4\hat{S}X\left(W'(X) - \langle P(X) \rangle\right)} \quad (3.29)$$

has $n \leq d = \text{degree}(W)$ branch cuts which connect simple zeroes of the discriminant $Y^2(X)$.

Let us denote such zeroes $a_i^-$ and $a_i^+$ ($i = 1, \ldots, n$) and map them into the $w$ cylinder as

\__________{(5) Integrating $\int d\lambda \rho(\lambda) \int d\lambda'' \rho(\lambda'')$ the trigonometric identity

$$\cot \frac{\lambda + i\epsilon - \lambda'}{2} \times \cot \frac{\lambda' + i\epsilon - \lambda''}{2} - 1 = \cot \frac{\lambda + i\epsilon - \lambda'}{2} \times \left(\cot \frac{\lambda + i\epsilon - \lambda'}{2} + \cot \frac{\lambda' + i\epsilon - \lambda''}{2}\right),$$

we arrive after some work at eq. (3.33) where

$$F(w) = \int_{\Sigma} d\lambda \rho(\lambda) \cot \frac{w - \lambda}{2} \times \left(\frac{dW(e^{iw})}{dw} - \frac{dW(e^{i\lambda})}{d\lambda}\right). \quad (3.24)$$

\__________{(6) From now on we use index $i$ to label spectral segments $\Sigma_i$ and related parameters rather than individual eigenvalues.}
\(a_i^\pm = \exp(i\theta_i^\pm)\). Consequently, the \(\lambda\) spectrum \(\Sigma\) has precisely \(n\) segments \(\Sigma_i\) which begin at \(\theta_i^-\) and end at \(\theta_i^+\), the spectral density along each segment is given by

\[
\rho(\lambda) = \frac{1}{2\pi} Y(X = e^{i\lambda})
\]  

(3.30)

(it cf. eqs. (3.22) and (3.28)), and the shapes of the segments follow from \(\rho(\lambda)d\lambda\) being real and positive.

From the field theory point of view, the segments \(\Sigma_i\) correspond to confining (or rather pseudo-confining) subgroups \(SU(N_i)\) of the diagonal \(SU(N_c)\), and the integrals

\[
\hat{S}_i = \int_{\theta_i^-}^{\theta_i^+} d\lambda \rho(\lambda) = \frac{1}{4\pi i} \oint_{A_i} dw R(w) = \frac{1}{2\pi i} \oint_{A_i} dX \langle R(X) \rangle
\]  

(3.31)

are dual to the gaugino condensates

\[
S_i = \text{Tr} \left( \frac{W^\alpha W^\alpha}{32\pi^2} \right)_{SU(N_i)} = \frac{1}{2\pi i} \oint_{A_i} dX R(X).
\]  

(3.32)

In both cases the \(A_i\) are \(A\)-cycles of the Riemann surface of the \(R(X)\) field \(\leftrightarrow \langle R(X) \rangle\) matrix or their images on the \(w\) cylinder; the \(A_i\) surround individual segments of the spectrum as shown below:

\[
A_2 \quad A_1 \quad A_3 \quad A_4
\]

(3.33)
The advantage of the contour integral formalism is that we do not need to know the exact routes of all the segments but only the general locations of the $A$-cycles which surround them. This works for all integrals of the form $\int d\lambda \rho(\lambda) f(\lambda)$, for all kinds of $f(\lambda)$ functions. For example, eq. (3.17) for the disk-level free energy can be written as a sum of contour integrals

$$\mathcal{F}_D = \sum_{f=1}^{\hat{F}} \frac{1}{4\pi i} \sum_{i=1}^n \oint_{A_i} dw \hat{R}(w) \times \log \left( e^{iw} - (\mu_f/\hat{v})^K \right). \quad (3.34)$$

The sphere-level free energy can also be written as a sum of contour integrals, but this is more complicated because the logarithm in the second terms in eq. (3.16) has branch cuts of its own. To disentangle the branch cut structure, let us first rewrite eq. (3.16) as

$$\mathcal{F}_S = \frac{1}{2} \int_{\Sigma} d\lambda \rho(\lambda) \times \left[ \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda + \hat{E}(\lambda) \right] \quad (3.35)$$

where

$$\hat{E}(\lambda) = \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \int_{\Sigma} d\lambda' \rho(\lambda') \times \log \left( 4 \sin^2 \frac{\lambda - \lambda'}{2} \right). \quad (3.36)$$

Naively,

$$\frac{\delta \mathcal{F}_S}{\delta \rho(\lambda)} = \hat{E}(\lambda) \implies \frac{d\hat{E}}{d\lambda} = 0 \quad (3.37)$$

(cf. eq. (3.19)), but analytic continuation to the complex cylinder yields instead

$$\frac{d\hat{E}}{d\lambda} = \frac{d\mathcal{W}}{d\lambda} - i\hat{C} - \hat{R}(\lambda) = i\mathcal{Y}(X = e^{i\lambda}) \neq 0 \quad (3.38)$$

and hence the $\hat{E}(\lambda)$ itself is a non-trivial function with branch cuts along the $\Sigma_i$ segments. However, its average between the two sides of a cut is locally constant

$$\frac{\hat{E}(\lambda + i\epsilon) + \hat{E}(\lambda - i\epsilon)}{2} \equiv \text{constant } \hat{E}_i \quad \text{for } \lambda \in \Sigma_i \text{ only} \quad (3.39)$$

because the derivative (3.38) flips sign across the cut. Therefore, to make sure eq. (3.19) is consistent with the loop equation, we must disambiguate the logarithm in the second term.
of eq. (3.16) such that

\[
\frac{\delta \mathcal{F}_S}{\delta \rho(\lambda)} = \frac{\hat{E}(\lambda + i\epsilon) + \hat{E}(\lambda - i\epsilon)}{2}.
\]  

(3.40)

Hence in light of eq. (3.39),

\[
\mathcal{F}_S = \frac{1}{2} \sum_{i=1}^{n} \int_{\Sigma_i} d\lambda \rho(\lambda) \times \left[ \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda + \hat{E}_i \right]
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \left[ \hat{E}_i \times \hat{S}_i + \frac{1}{4\pi i} \oint_{A_i} dw \hat{R}(w) \times \left( \mathcal{W}(e^{iw}) - i\hat{C}w \right) \right].
\]

(3.41)

The \( \hat{E}_i \) can also be calculated as contour integrals. For general \( \lambda \),

\[
\hat{E}(\lambda) = \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{4\pi i} \sum_{i=1}^{n} \oint_{A_i} dw \hat{R}(w) \times \log \left( \frac{4\sin^2 \frac{\lambda - w}{2}}{2} \right)
\]

(3.42)

where the contours \( A_i \) should be drawn such as to exclude the branch cuts of the logarithm.

In particular, the \( \lambda \) point itself should be kept outside of all the \( A_i \) contours, and this prevents us from directly evaluating eq. (3.42) for \( \lambda \in \Sigma \). Instead, for the purpose of calculating an \( \hat{E}_i \) in eq. (3.41) we must first evaluate the integral eq. (3.42) for \( \lambda \notin \Sigma \), then take two limits of \( \lambda \) approaching the same point \( \lambda_i \in \Sigma_i \) from two opposite sides of the spectrum, and finally take the average of the two limits. Alternatively, we may take just one limit of \( \lambda \) going to an end point \( \theta^+_i \) or \( \theta^-_i \) of the spectral segment — at these points the difference between the two sides of the spectrum vanishes and the averaging becomes unnecessary. Thus,

\[
\hat{E}_i = \lim_{\lambda \rightarrow \theta^+_i} \left[ \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{4\pi i} \sum_{j=1}^{n} \oint_{A_j} dw \hat{R}(w) \times \log \left( \frac{4\sin^2 \frac{\lambda - w}{2}}{2} \right) \right].
\]

(3.43)

In the next section, these equations (as well as eq. (3.34)) will help us calculate the effective superpotential of the matrix model.

We conclude this section with a brief discussion of local versus global minima of the free energy. Eqs. (3.41) and (3.43) give us the \( \mathcal{F}_S \) minimized with respect to local variations of the
spectral density $\rho(\lambda)$, hence eq. (3.30). The global minimum requires further minimization with respect to the free parameters of eq. (3.29), namely $\hat{C}$ and $n-1$ independent coefficients of the $\langle P(X) \rangle$ polynomial.$^{(7)}$ Variationally, this implies that $\delta F_S/\delta \rho(\lambda)$ should be globally constant over the whole spectrum $\Sigma$ and not just individual segments, hence

$$\hat{E}_1 = \hat{E}_2 = \cdots = \hat{E}_n.$$  \hfill (3.44)

In addition, we should minimize with respect to the $\hat{C}$ parameter, hence

$$\frac{\delta F_S}{\delta \hat{C}} = \int_{\text{whole } \Sigma} d\lambda \rho(\lambda) \times (-i\lambda) = -\frac{1}{4\pi} \sum_{i=1}^n \oint_{A_i} dw \hat{R}(w) \times w = 0.$$  \hfill (3.45)

Note that eqs. (3.44) apply only to the global minimum of the free energy; in field theory terms this corresponds to taking the gaugino condensates $S_i$ on-shell. In the following section, we shall calculate the $W_{\text{eff}}$ for the off-shell $S_i$ or rather $\hat{S}_i$, and this means abandoning the global minimum and hence eqs. (3.44). Instead, eqs. (3.31) will determine the coefficients of $\langle P(X) \rangle$ in terms of the $\hat{S}_i$.

---

$^{(7)}$ $\langle P(X) \rangle$ has degree $d-2$ and hence $d-1$ coefficients, but if we want $n < d$ spectral segments, the polynomial $Y^2(X)$ (cf. eq. (3.29)) must have $d-n$ double zeroes, which imposes $d-n$ constraints on the coefficients of $\langle P(X) \rangle$. Consequently, only $n-1$ of those coefficients may vary independently of each other.
In this section we derive the effective superpotential of our matrix model. But first, a few general words about effective superpotentials for the off-shell gaugino condensates in field theory. In the single–$U(N)$ theory with adjoint matter, Cachazo et al. [6] found that the gauginos of the $U(1)$ center of the $U(N)$ generate auxiliary supersymmetries of the chiral ring of the gauge theory and hence

$$W_{\text{eff}}(S_1, \ldots, S_n) = \sum_{i=1}^{n} N_i \frac{\partial}{\partial S_i} \mathcal{F}(S_1, \ldots, S_n) + \sum_{i=1}^{n} (2\pi i \tau_0 + b_i) \times S_i$$

(4.1)

where $\mathcal{F}(S_1, \ldots, S_n)$ is a prepotential of the auxiliary SUSY, $\tau_0$ is the overall bare gauge coupling, and $b_i$ are integers distinguishing between specific vacua of the theory. Under gauge-matrix duality, $\tau_0$ is just an arbitrary parameter, but the prepotential $\mathcal{F}$ is dual to the sphere-level free energy $\mathcal{F}_S(\hat{S}_1, \ldots, \hat{S}_n)$ of the matrix model.

Adding the quarks — which couple to the $U(1)$ center — breaks the auxiliary supersymmetries, but the subring of the adjoint sector remains supersymmetric. Hence [14], integrating out the mesonic sector of the chiral ring yields

$$W_{\text{eff}}(S_1, \ldots, S_n) = \sum_{i=1}^{n} N_i \frac{\partial}{\partial S_i} \mathcal{F}(S_1, \ldots, S_n) + \sum_{i=1}^{n} (2\pi i \tau_0 + b_i) \times S_i + \mathcal{F}_Q(S_1, \ldots, S_n)$$

(4.2)

where $\mathcal{F}_Q$ is the quark sector’s contribution to the superpotential. Under gauge-matrix duality, the $\mathcal{F}_Q$ is dual to the disk-level part $\mathcal{F}_D$ of the matrix model’ free energy.

In our case, the $[SU(N)]^K$ theory does not have a $U(1)$ center(s), and we cannot promote the gauge symmetry to $[U(N)]^K$ because of gauge anomaly constraints, but we can add a single $U(1)$ factor common to all nodes of the quiver diagram (1.3). Since all the gaugino condensates of our theory belong to subgroups of the diagonal $SU(N)$, adding the common $U(1)$ effectively promotes the $SU(N)_{\text{diag}}$ to the $U(N)_{\text{diag}}$, with similar consequences for the effective superpotential. Thus, $W_{\text{eff}}(S_1, \ldots, S_n)$ is given by eq. (4.2) where the $\mathcal{F}(S_1, \ldots, S_n)$ is dual to the $\mathcal{F}_S$ of our matrix model and the $\mathcal{F}_Q(S_1, \ldots, S_n)$ is dual to its $\mathcal{F}_D$.

(8) By analogy, in the Veneziano–Yankielowicz superpotential $W = N S \log S + (2\pi i \tau_0 + b) S$, $b$ distinguishes between different vacua of the $SU(N)$. 
In the matrix model, taking the $\hat{S}_i$ off-shell and evaluating the free energy as a function $\mathcal{F}(\hat{S}_1, \ldots, \hat{S}_n)$ means imposing all $n$ eqs. (3.31) as constraints and then minimizing the $\mathcal{F} \approx \mathcal{F}_S$ under those constraints. Variationally, this implies that $\delta \mathcal{F}_S / \delta \rho(\lambda)$ should be locally constant along each continuous segment $\Sigma_i$ of the spectrum but may take different values for different segments. Therefore, eq. (3.30) and all the subsequent contour-integral formulæ of §3 remain unchanged, but we throw eqs. (3.44) out of the window and instead use eqs. (3.31) to determine the coefficients of $\langle P(X) \rangle$. Consequently, exactly as in [6]

$$\frac{\partial \mathcal{F}_S}{\partial \hat{S}_i} = \frac{\delta \mathcal{F}_S}{\delta \rho(\lambda)} \bigg|_{\lambda \in \Sigma_i} = \hat{E}_i,$$

(4.3)

and hence the effective superpotential of the matrix model is given by

$$W_{\text{eff}}(\hat{S}_1, \ldots, \hat{S}_n) = \sum_{i=1}^{n} \hat{N}_i \times \hat{E}_i(\hat{S}_1, \ldots, \hat{S}_n) + \sum_{i=1}^{n} (2\pi i \tau_0 + b_i) \times \hat{S}_i + \mathcal{F}_D(\hat{S}_1, \ldots, \hat{S}_n).$$

(4.4)

In the contour integral formalism, the $\hat{E}_i$ in this formula are given by eqs. (3.43). We can simplify the contour integrals by merging all the $A_i$ cycles together and then pushing the resulting loop outwards. The following picture illustrates how this works on the complex cylinder:

The new contour comprises two vertical lines on two sides of the logarithm’s branch cut, and
two loops around the cylinder. For the loops, we take \( \Omega \to \infty \) which gives us

\[
\hat{R}(w) \xrightarrow{\text{Im}(w) \to \pm \infty} \mp i \hat{S}
\]

and therefore

\[
\text{top loop} = 2\pi i \hat{S} (\Omega - \text{Im}(\lambda_i)),
\]
\[
\text{bottom loop} = 2\pi i \hat{S} (\Omega + \text{Im}(\lambda_i)).
\]

For the vertical lines, we have discontinuity

\[
\text{disc} \left[ \log \left( 4 \sin^2 \frac{\lambda - w}{2} \right) \right] = \begin{cases} 
+2\pi i & \text{for Im } w > \text{Im } \lambda, \\
-2\pi i & \text{Im } w < \text{Im } \lambda,
\end{cases}
\]

and hence

\[
\int_{\text{vert. lines}} dw \hat{R}(w) \times \log(\cdots) = 2\pi i \int_{\lambda}^{+i\Omega} dw \hat{R}(w) + 2\pi i \int_{\lambda}^{-i\Omega} dw \hat{R}(w).
\]

Putting the whole contour integral together, we arrive at

\[
\hat{E}(\lambda) = \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{2} \int_{\lambda}^{+i\Omega} dw \hat{R}(w) - \frac{1}{2} \int_{\lambda}^{-i\Omega} dw \hat{R}(w) - \hat{S} \times \Omega
\]

\[
= \mathcal{W}(e^{i\lambda}) - i\hat{C}\lambda - \frac{1}{2} \int_{\lambda}^{+i\infty} dw [\hat{R}(w) + i\hat{S}] - \frac{1}{2} \int_{\lambda}^{-i\infty} dw [\hat{R}(w) - i\hat{S}],
\]

and at this point, we may take \( \lambda = \theta_i^+ \) or \( \lambda = \theta_i^- \) to calculate the \( \hat{E}_i \).

We may also express the \( \hat{E}_i \) in term of the \( B \)–cycle periods of the Riemann surface of \( Y(X = e^{iw}) \). Indeed, plugging eq. (3.28) into formulæ (4.11), we obtain

\[
\hat{E}_i = \frac{i}{4} \int_{B_i^+}^{\text{reg}} dw Y(e^{iw}) - \frac{i}{4} \int_{B_i^-}^{\text{reg}} dw Y(e^{iw}) + \frac{1}{2} \mathcal{W}(e^{iw}) - \hat{S}\Omega
\]

\[
= -\frac{i}{4} \int_{B_i^+}^{\text{reg}} dw Y(e^{iw}) - \frac{i}{4} \int_{B_i^-}^{\text{reg}} dw Y(e^{iw}) + \frac{1}{2} \mathcal{W}(e^{iw}) - \hat{S}\Omega,
\]
where the $B_i^\pm$ cycle begins at $w = \pm i\infty$ on the physical sheet of the Riemann surface, crosses the $\Sigma_i$ branch cut to the other sheet, and then goes back to $w = \pm i\infty$ but on the unphysical sheet, and the integrals are regularized by starting and stopping at $w = \pm i\Omega$ instead of $w = \pm i\infty$. The cycles are illustrated on the following figure:

Note that the each sheet of our Riemann surface is a cylinder with two distinct infinities — and that’s why we have a double set of $B$–cycles. Fortunately, most of the extra $B$ cycles are redundant:

$\forall i, j : B_i^+ - B_i^- \equiv B_j^+ - B_j^-$ modulo $A$ cycles. \hspace{1cm} (4.14)

and hence

$\hat{E}_i = -\frac{i}{2} \int_{B_i^-}^{\text{reg}} d\omega Y(e^{i\omega}) + \text{terms common to all } \hat{E}_j + \text{a linear combination of } \hat{S}_j \text{ with integer coefficients.}$ \hspace{1cm} (4.15)

The last term here reflects the ambiguity of $B$–cycles modulo $A$ cycles: going from a branch cut $\Sigma_i$ to $\pm i\infty$ one may choose different passages between the other branch cuts $\Sigma_j$. In terms of the sphere-level free energy, this corresponds to different routing of the branch cuts of the log $\left(4 \sin^2 \frac{\lambda - \lambda'}{2}\right)$ around the spectral segments $\Sigma_j$ in both $\lambda$ and $\lambda'$ complex cylinders. Re-routing the log’s branch cuts changes the free energy by

$F_S(\hat{S}_1, \ldots, \hat{S}_n) \to F_S(\hat{S}_1, \ldots, \hat{S}_n) + \frac{1}{2} \sum_{i,j} c_{ij} \hat{S}_i \hat{S}_j$ \hspace{1cm} (4.16)
for some *integer* coefficients $c_{ij}$, and hence

\[ \hat{E}_i(\hat{S}_1, \ldots, \hat{S}_n) \rightarrow \hat{E}_i(\hat{S}_1, \ldots, \hat{S}_n) + \sum_j c_{ij} \hat{S}_j. \]  

(4.17)

Fortunately, from the superpotential (4.4) point of view, this change amounts to changing the integers $b_i$. In other words, we permute the branches belonging to different vacua, but the overall picture does not change.

Now consider the quark sector’s contribution $\mathcal{F}_D(\hat{S}_1, \ldots, \hat{S}_n)$ to the effective superpotential. Again, we are going to simplify eq. (3.34) by moving the integration contours, but it’s convenient to do it flavor-by-flavor. We need to distinguish between massive and massless flavors: they play different roles in field theory, both in 4D and in 5D\(^{(9)}\), and in the matrix model they have different contributions to the disk-level free energy. Let us begin with a massive flavor and let $e^{im_f} = (\hat{\mu}_f/\hat{v})^K$. Then for this flavor we have cuts and contours as shown on the left picture below:

Again, we merge the $A_i$ contours into a single loop and push it outward as shown on the right picture.

\[(9)\] In deconstruction [11], massive 4D flavors with $\mu_f \approx \hat{v}$ have light modes and give rise to light flavors in 5D; the massless 4D flavors do not have no 5D counterparts, but they are needed to adjust the Chern–Simons level of the 5D theory. From the purely 4D point of view [12], the massless flavors affect the Coulomb branch of the theory but do not give rise to Higgs branches; the on-shell mesonic and baryonic operators include only the massive flavors.
picture above. This time, we end up with two disconnected contours: a simple loop around the top of the cylinder, and a loop around the cylinder’s bottom attached to a vertical loop around the logarithm’s cut. The discontinuity across this cut is simply $+2\pi i$ and hence

$$\oint_{\text{vert. loop}} dw \hat{R}(w) \times \log(e^{iw} - e^{imf}) = 2\pi i \int_{-i\Omega}^{m_f} dw \hat{R}(w). \quad (4.19)$$

And for the loops around the cylinder we again take $\Omega \to \infty$ and use eq. (4.6); consequently

$$\text{top loop} = 2\pi i \hat{S} \times (im_f - \pi i), \quad \text{bottom loop} = 2\pi i \hat{S} \times (\Omega + \pi i). \quad (4.20)$$

Putting all the loops together, we find that one massive quark flavor contributes

$$\mathcal{F}_D^{\text{one flavor}} = \frac{1}{2} \int_{-i\Omega}^{m_f} dw \hat{R}(w) + \frac{\hat{S}}{2} \times (\Omega + im_f) = \frac{1}{2} \int_{-i\infty}^{m_f} dw [\hat{R}(w) + i\hat{S}] + i\hat{S} \times m_f. \quad (4.21)$$

For a massless flavor we have $\log(e^{iw} - (0/\hat{v})^K) = iw$, which does not have singularities on the complex cylinder but needs a branch cut anyway because $w$ is multi-valued. Consequently, we have contours and cuts as shown below:

Once again we merge the contours into a single loop and push it outward as shown on the
right picture above. This time we get a connected system of two loops around the cylinder and two vertical lines, and evaluating the integrals gives us

$$\text{vertical lines} : \quad \implies \quad 2\pi i \int_{-i\Omega}^{+i\Omega} dw \, \hat{R}(w), \quad (4.23)$$

$$\text{each horizontal loop} : \quad \implies \quad 2\pi i S \times (\Omega \pm \pi i), \quad (4.24)$$

hence altogether one massless flavor yields

$$F_D|_{\text{one } \mu=0} = \frac{1}{2} \int_{-i\infty}^{0} dw \left[ \hat{R}(w) + i \hat{S} \right] + \frac{1}{2} \int_{0}^{+i\infty} dw \left[ \hat{R}(w) - i \hat{S} \right]. \quad (4.25)$$

We conclude this section with a complete formula for the effective superpotential. Combining eqs. (4.4), (4.11), (4.21), and (4.25) together, we arrive at

$$W_{\text{eff}} \quad = \quad \sum_{i=1}^{n} \hat{N}_i \times \left( W(e^{i\theta_i^+}) - i \hat{C} \theta_i^+ + \int_{-i\Omega}^{+i\Omega} dw \, \hat{R}(w) \right)$$

$$\quad \quad \quad + \frac{1}{2} \sum_{f=1}^{m_f} \int_{-i\Omega}^{+i\Omega} dw \, \hat{R}(w) - \frac{\hat{N} - \hat{F}_2}{2} \times \int_{-i\Omega}^{+i\Omega} dw \, \hat{R}(w)$$

$$\quad \quad \quad + 2\pi i \tau \times \hat{S}_{\text{net}} + \sum_{i=1}^{n} b_i \hat{S}_i \quad (4.26)$$

where $\hat{F}_1$ is the number of massive flavors, $\hat{F}_2$ is the number of massless flavors, and $\tau$ is the renormalized gauge coupling according to

$$2\pi i \tau = 2\pi i \tau_0 + \frac{1}{2} \sum_{f=1}^{m_f} \log \frac{\mu_f^K}{v^K} - \Omega \times \left( \hat{N} - \frac{1}{2} \hat{F}_1 - \hat{F}_2 \right). \quad (4.27)$$

Note the coefficient of the cutoff $\Omega$ in this renormalization: The same combination $N - \frac{1}{2} F_1 - F_2$ appears in dimensional deconstruction as the coefficient of the 5D Chern–Simons term. We are not sure of the physical meaning of this coincidence, but it probably isn’t an accident.
Eq. (4.26) gives the effective superpotential in terms of the resolvent $\hat{R}(w)$, and to recast it as a function of the gaugino condensates $\hat{S}_1, \ldots, \hat{S}_n$ we need to solve eqs. (3.31) for the resolvent’s parameters. Since the resolvent with $n$ branch cuts has $n + 1$ independent parameters (including the $\hat{S} = \sum_i \hat{S}_i$), determining all of them requires one more equation besides $n$ eqs. (3.31). In the matrix model, such $n + 1$st equation is eq. (3.45), which follows from minimizing the free energy with respect to $\hat{C}$. The contour integrals in eq. (3.45) are similar to a massless flavor’s contribution to the $F_D$ and can be simplified in the same way: moving the contours according to fig. (4.22), we arrive at

$$\frac{\partial F_S}{\partial \hat{C}} = -\frac{1}{2} \int_{-\Omega}^{+\Omega} dw \, \hat{R}(w) - \Omega \hat{S} = 0. \hspace{1cm} (4.28)$$

The effective superpotential $W_{\text{eff}}(\hat{S}_1, \ldots, \hat{S}_n)$ as a function of the gaugino condensates follows from combined eqs. (4.28), (3.31), and (4.26).
5. Open Questions

We conclude this paper by discussing open questions raised by our unitary matrix model.

The first question is specific to the \([SU(N)]^K\) model: what, if anything, is the gauge-theory dual of eq. (4.28)? The chiral ring of the quiver (1.3) was studied in detail in [12], and none of the anomaly equations there looks remotely like eq. (3.45) or eq. (4.28). Instead, the \(n + 1\)st equation for the parameters of the \(R(X)\) resolvent comes from a completely different source, namely the loop equations for the link resolvent \(T(X) = \text{tr} \left( \frac{1}{X - \Omega_K \cdots \Omega_1} \right)\).

The analytic properties of the two resolvents \(R(X)\) and \(T(X)\) require \(Y(X)\) (cf. eq. (3.29)) to have the same branching points in the \(X\) plane as

\[
\sqrt{\prod_{j=1}^{N_c} (X - \varpi_j)^2 - 4 ((-\gamma)^{N_f} \Lambda^{2N_c-N_f})^K \times \prod_{f=1}^{N_f} (\mu^K_f - X)}
\]

(5.1)

where \(\varpi_1, \ldots, \varpi_{N_c}\) are the Coulomb moduli of the \([SU(N)]^K\) theory; only \(N_c - 1\) of these moduli are independent because of the \(SU(N_c)\) constraint

\[
\prod_{j=1}^{N_c} \varpi_j = \hat{v}^{KN_c}.
\]

(5.2)

Together, eqs. (5.1–2) impose one constraint on the parameters of the gaugino-condensate resolvent \(R(X)\); the remaining parameters are related to the individual condensates \(S_i\) according to eqs. (3.32), exactly as in the matrix model.

Unfortunately, the link resolvent \(T(X)\) and the Coulomb moduli \(\varpi_1, \ldots, \varpi_{N_c}\) of the gauge theory do not have any matrix duals. Consequently, eqs. (5.1–2) do not make sense on the matrix-model side of the duality, just like eq. (4.28) does not make any sense on the gauge-theory side. Ideally, these equations could be dual to each other, but we do not have any evidence for such duality. In fact, we do not even know whether eqs. (5.1–2) and (4.28) are even mutually consistent. This remains an open question we hope to answer as soon as possible.

A bigger open question concerns generalization of our unitary matrix model to other quiver theories. We believe that the random unitary matrices can be used to model all kinds
of chiral \((n, \bar{n})\) bifundamental fields with non-zero eigenvalues, but we would like to see how this works in different models. Also, it would be interesting to see what exactly goes wrong when an eigenvalue happens to vanish in some vacuum state of the gauge theory. We expect the gauge-matrix duality to fail for such vacuum, but we are not quite sure, and we certainly do not know the specifics (if any) of this failure.

Finally, we would like to build a matrix model of a chiral \((n, \bar{n})\) bifundamental with \(n \neq m\). As discussed in the introduction, this calls for complex \(\hat{N} \times \hat{M}\) matrices integrated over some variety \(\Gamma \subset \mathbb{C}^{\hat{N}\hat{M}}\) which has real dimension \(\hat{N}\hat{M}\) and satisfies the symmetry condition \((1.1)\). More generally, for a theory with several chiral bifundamentals, one may use correlated matrices: instead of independent matrices \(M_1, \ldots, M_K\) each integrated over a separate variety \(\Gamma_\ell (\ell = 1, \ldots, K)\), the array \((M_1, \ldots, M_K)\) of all the matrices is integrated over a combined variety \(\Gamma \subset \mathbb{C}^D\) where \(D = \sum_\ell \hat{N}_\ell \hat{M}_\ell\). Such combined variety should have real dimension \(D\) (same as the net dimension of all the \(\Gamma_\ell\)) and satisfy the generalized symmetry condition

\[
\forall g \in \text{(net symmetry group)}, \quad g : \Gamma \mapsto \Gamma'' \simeq \Gamma. \quad (5.3)
\]

We would like to construct such a variety for an interesting quiver theory, and then compare the loop equations of the matrix model to the anomaly equations for the field theory’s chiral ring.

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