FREE PENCILS ON DIVISORS

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1. Introduction

In algebraic geometry, it is rather typical that the embedding of a variety \( Y \) in another variety \( X \) forces strong constraints on the existence of free linear series on \( Y \). For example, a classical result in plane curve theory states that the gonality of a smooth plane curve of degree \( d \) is \( d - 1 \) ([ACGH]). It is then natural to look for general statements of this flavor.

One particular case, which is quite well understood, is the one where \( Y \) is a divisor in \( X \). This problem has been studied by several researchers. In particular, a wide range of situations is dealt with by the following result of Sommese ([So 76]):

**Theorem 1.1.** (Sommese) Let \( Y \subset X \) be an irreducible smooth ample divisor and let \( \phi: Y \rightarrow B \) be a morphism onto another projective manifold. If \( \dim(Y) \geq \dim(B) + 2 \) then \( \phi \) extends to a morphism \( \psi: X \rightarrow B \).

Serrano ([Se 87]) then studied the case where \( B \) is a smooth curve and \( \dim(X) = 2 \) or 3. Namely, he proved the two following theorems:

**Theorem 1.2.** (Serrano) Let \( C \) be an irreducible smooth curve contained in a smooth surface \( S \). Suppose that there exists a morphism \( \phi: C \rightarrow \mathbb{P}^1 \) of degree \( d \). If \( C^2 > (d+1)^2 \), then there exists a morphism \( \psi: S \rightarrow \mathbb{P}^1 \) extending \( \phi \).

and

**Theorem 1.3.** (Serrano) Let \( X \) be a smooth projective threefold, and let \( S \subset X \) be a smooth very ample surface. Let \( \phi: S \rightarrow \mathbb{P}^1 \) be a morphism with connected fibers. Let \( g(F) \) be the arithmetic genus of a fiber and set \( d = F \cdot S \). If \( S^3 > (d+1)^2 \) and \( \dim H^0(X, O_X(S)) \geq \mathbb{E}[+\mathbb{E}+\mathbb{E}](\mathcal{F}) \), then \( \phi \) extends to a morphism \( \psi: X \rightarrow \mathbb{P}^1 \).

Actually, Serrano proves more, in the sense that he shows how these statements imply analogous ones with \( \mathbb{P}^1 \) replaced by a general smooth curve \( B \), and he can also replace the above numerical conditions by weaker ones if \( S + K_X \) is a numerically even divisor. His argument
is based on Miyaoka’s vanishing theorem combined with a refinement of Bombieri’s method. Furthermore, Serrano applies the above results and methods to the study of the ampleness of the adjoint divisor.

On the other hand, in a celebrated theorem Reider \([\text{Re } 88]\) has shown how adjunction problems on surfaces can be exhaustively studied using vector bundle methods. His argument is based on an application of Bogomolov’s instability theorem. Furthermore, Reider himself has also given a proof along these lines of a statement close to Serrano’s theorem for surfaces (\([\text{Re } 89]\)). Also in light of Serrano’s result for threefolds, it is therefore reasonable to expect that methods of this type should be applicable to obtain some more general statement about the extension of linear series on a divisor. Our result in this direction is the following:

**Theorem 1.4.** (\(\text{char}(k) = 0\)) Let \(X\) be a smooth projective \(n\)-fold, and let \(Y \subset X\) be a reduced irreducible divisor. If \(n \geq 3\) assume that \(Y\) is ample, and if \(n = 2\) assume that \(Y^2 > 0\) (so that in particular it is at least nef). Let \(\phi : Y \to \mathbb{P}^1\) be a morphism, and let \(F\) denote the numerical class of a fiber.

(i) If
\[
F \cdot Y^{n-2} < \sqrt{Y^n} - 1,
\]
then there exists a morphism \(\psi : X \to \mathbb{P}^1\) extending \(\phi\). Furthermore, the restriction
\[
H^0(X, \psi^* \mathcal{O}_{\mathbb{P}^1}(\infty)) \to H'(Y, \phi^* \mathcal{O}_{\mathbb{P}^1}(\infty))
\]
is injective. In particular, \(\psi\) is linearly normal if \(\phi\) is.

(ii) If
\[
F \cdot Y^{n-2} = \sqrt{Y^n} - 1
\]
and \(Y^n \neq 4\), then either there exists an extension \(\psi : X \to \mathbb{P}^1\) of \(\phi\), or else we can find an effective divisor \(D\) on \(X\) such that \((D \cdot Y^{n-1})^2 = (D^2 \cdot Y^{n-2})Y^n\) and \(D \cdot Y^{n-1} = \sqrt{Y^n}\), and an inclusion
\[
\phi^* \mathcal{O}_{\mathbb{P}^1}(\infty) \subset \mathcal{O}_Y(D).
\]

When applied to \(n = 2\) and \(n = 3\), this gives the above statements of Serrano. However, the hypothesis are weaker, because we are not requiring \(Y\) to be smooth and we don’t need the assumption about the number of sections of \(\mathcal{O}_X(Y)\). Besides, we don’t require \(Y\) to be very ample (unlike Serrano’s statement for \(n = 3\)). We have furthermore a description of what happens in the boundary situation; for example, \((d + 1)^2 = C^2\) is the case of a minimal pencil on a smooth plane curve (of degree \(d + 1\)). With respect to Reider’s result on surfaces, the assumption that \(Y^2 \geq 19\) and that \(Y\) be smooth is not necessary.
The above furthermore shows that conclusion (a) in Proposition 2.15 of [Re 89] always occurs for $f < \sqrt{Y^2 - 1}$ (just take the Stein factorization of $X \rightarrow \mathbb{P}^1$), and therefore the other possibilities can only occur in the boundary case (ii). This in turn gives more information about this case. For example, comparison with Reider’s theorem shows that when $f = \sqrt{Y^2 - 1}$, under the additional hypothesis that the curve $Y$ be smooth and $Y^2 \geq 19$, if $\mathcal{O}_X(D)$ is not base point free then it has exactly one base point.

As to $n \geq 4$, this is clearly weaker than Sommese’s result except that we are not requiring $Y$ to be smooth.

The argument provides a direct geometric construction of the extension, as follows. If we let $A = \phi^* \mathcal{O}_{\mathbb{P}^1}(\infty)$, $V = \phi^* H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\infty))$, we can define a rank two vector bundle $\mathcal{F}$ by the exactness of the sequence:

$$0 \rightarrow \mathcal{F} \rightarrow V \otimes \mathcal{O}_X \rightarrow A \rightarrow !.$$

In light of Bogomolov’s instability theorem on an $n$-dimensional variety, the given numerical assumption implies that $\mathcal{F}$ is Bogomolov unstable with respect to $Y$, and so we have a saturated destabilizing line bundle $\mathcal{L} \subset \mathcal{F}$. Then $\mathcal{L} = \mathcal{O}_X(-D)$ for some effective divisor on $X$, and hence we are reduced to arguing that the numerology forces $D$ to move in a base point free pencil.

Using the relative version of the Harder-Narasimhan filtration ([Fl 84]) this concrete description can be adapted to families of morphisms, and one can also prove a more general statement about morphisms to arbitrary smooth curves.

Finally, using recent results of Moriwaki concerning a version of the Bogomolov-Gieseker inequality in prime characteristic, the above statements can be generalized to varieties defined over a field of characteristic $p$.

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2. Instability of rank two bundles

In this section we collect some statements about instability of rank two vector bundles on a smooth projective manifold. References in
this direction are, for example, [Bo 79], [Gi 79] and [Mi 85]. For the
statements in characteristic $p$, we shall be using results from [Mo 93].

Let us first assume $\text{char}(k) = 0$. We shall keep this convention until
otherwise stated. The basic result is given by the Bogomolov-Gieseker
inequality for semistable bundles:

**Theorem 2.1.** Let $S$ be a smooth projective surface, and let $E$ be a
rank two vector bundle on $X$ with Chern classes $c_1(E)$ and $c_2(E)$. If $H$
is any polarization and $E$ is $H$-semistable, then $c_1(E) - \Delta_\epsilon(E) \leq 1$.

**Definition 2.1.** Let $X$ be any projective $n$-dimensional manifold and
let $E$ be a rank two vector bundle on $X$. Let $c_i(E) \in A^i(X)$ be the
Chern classes of $E$, $i = 1$ and 2. Define the discriminant of $E$ as

$$ \Delta(E) = \int_{\infty}(E) - \Delta_\epsilon(E) \in A^\epsilon(X). $$

**Lemma 2.1.** Let $X$ be a smooth projective $n$-fold and fix a polar-
ization $H$ on $X$. Consider a rank two vector bundle $E$ on $X$ which
is $H$-unstable. Suppose that $L_\infty, L_\epsilon \subset E$ are line bundles and set $e = \text{deg}_H(E), l_1 = \text{deg}_H(L_\infty)$ and $l_2 = \text{deg}_H(L_\epsilon)$. Suppose that $2l_i > e$, $i = 1,2$ and that $L_\epsilon$ is saturated in $E$. Then $L_\infty \subset L_\epsilon$.

**Proof.** Set $l = \min\{l_1, l_2\}$. By assumption, we have $2l > e$. Let

$$ Q =: E/L_\epsilon. $$

Then $Q$ is a torsion free sheaf on $X$. If $L_\infty \not\subset L_\epsilon$, then the induced morphism $L_\infty \to Q$ is not identically zero, and therefore it is generically nonzero. This implies that the obvious morphism of vector bundles

$$ L_\infty \oplus L_\epsilon \to E $$

is generically surjective. Hence the line bundle $\wedge^2 E \otimes L_\infty^{-\infty} \otimes L_\epsilon^{-\infty}$ is effective, and therefore

$$ e \geq l_1 + l_2 \geq 2l, $$

a contradiction. ♦

**Remark 2.1.** The above argument still works if $2l_i \geq e$.

**Corollary 2.1.** Let $E$ be an $H$-unstable rank two vector bundle on $X$. If $L \subset E$ is a saturated destabilizing line bundle, then it is the maximal $H$-destabilizing line bundle of $E$. $L$ contains any $H$-destabilizing line bundle of $E$. 
Definition 2.2. Let $S$ be a smooth projective surface, and let $N^1(S)$ be the vector space of all numerical equivalence classes of divisors on $S$. The positive cone $K^+(S) \subset N^1(S)$ is described by the equations $D^2 > 0$ and $D \cdot H > 0$ for some (and hence for all) polarizations $H$ on $S$.

If we apply this to the situation of Bogomolov’s theorem, we have

Corollary 2.2. Let $S$ be a smooth projective surface and let $E$ be a rank two vector bundle on $S$ with $\Delta(E) > i$. Then there exists a sequence

$$0 \to A \to E \to B \otimes J_Z \to,$$

where $Z \subset S$ is local complete intersection codimension two subscheme and $A$ and $B$ are line bundles on $S$ such that $A - B \in K^+(S)$. Furthermore, $A$ is the maximal destabilizing line bundle of $E$ with respect to any polarization on $X$.

Proof. Fix any polarization $H$ on $S$. Since $E$ is $H$-unstable, there is an exact sequence

$$0 \to A \to E \to B \otimes J_Z \to,$$

where $A$ is the maximal destabilizing subsheaf of $E$, and in particular it is saturated. Hence we have $(A - B) \cdot H > 0$. On the other hand, since $c_1(E) = A + B$ and $c_2(E) = A \cdot B + [Z]$, we also have

$$0 < \Delta(E) = (A + B)^e - \triangle A \cdot B - \Delta[|Z|] \leq (A - B)^e.$$

This implies $A - B \in K^+(S)$, and therefore $A$ strictly destabilizes $E$ with respect to any polarization on $S$. On the other hand, being saturated, it then has to be the maximal destabilizing subsheaf of $E$ with respect to any polarization on $X$. ♦

We now want to generalize the above results to higher dimensional varieties. We start by recalling the following fundamental result of Mumford-Mehta-Ramanathan (cfr [Mi 85]):

Theorem 2.2. Let $X$ be a smooth projective $n$-fold, and let $H$ be a polarization on $X$. Suppose that $E$ is a vector bundle on $X$. If $m \gg 0$, and $Y \in |mH|$ is general, then the maximal destabilizing subsheaf of $E|_Y$ is the restriction to $Y$ of the maximal destabilizing subsheaf of $E$.

Remark 2.2. By the maximal destabilizing subsheaf of $E$ one means the first term $E_\infty$ of the Harder-Narashiman filtration of $E$. If $E$ is semistable, $E_\infty = E$.

We generalize definition 1.3 as follows:
Definition 2.3. Let $X$ be a smooth projective $n$-fold, and let $H$ be a polarization on $X$. Denote by $N^1(X)$ the vector space of all numerical equivalence classes of divisors on $X$. Then the $H$-positive cone $K^+(X, H) \subset N^1(X)$ is described by the equations $D^2 \cdot H^{n-2} > 0$ and $D \cdot H^{n-1} > 0$. Note that this implies $D \cdot H^{n-2} \cdot D > 0$ for any other polarization $L$ on $X$.

We then have:

Theorem 2.3. Let $X$ be a smooth projective $n$-fold and let $H$ be a fixed polarization on $X$. Consider a rank two vector bundle $E$ on $X$ of discriminant $\Delta(E)$. If $\Delta(E) \cdot H^{-1} > 1$, then there exists an exact sequence

$$0 \to A \to E \to B \otimes J \to W \to 1$$

where $W \subset X$ is a (possibly empty) codimension two local complete intersection subscheme, and $A$ and $B$ are line bundles on $X$ such that $A - B \in K^+(X, H)$.

Proof. For $n = 2$, this is the content of Corollary 2.2. For $n \geq 3$, let $V \in |mH|$ be general, with $m \gg 0$. We may assume that $V$ is a smooth irreducible surface, and that the maximal $H$-destabilizing subsheaf of $E|_S$ is the restriction of the maximal $H$-destabilizing subsheaf of $E$ (Theorem 2.2). By the hypothesis,

$$\Delta(E|_S) = \Delta(E) \cdot H^{-1} > 1.$$ 

Therefore, by induction $E|_V$ is Bogomolov-unstable with respect to $H|_V$, and so there exists an exact sequence

$$0 \to A \to E \to B \otimes J \to W \to 1,$$

satisfying the conclusions of theorem 1.1. Furthermore, by the above there is $A \subset E$ such that $A|_S = A$. Being normal of rank one, $A$ is a line bundle. 

Remark 2.3. Note the inequality $(A - B)^\varepsilon : H^\varepsilon \geq \Delta(E) : H^\varepsilon$.

Definition 2.4. Let $X$ be a smooth $n$-dimensional projective variety, and let $H$ be an line bundle on $X$. Consider a rank two vector bundle $E$ on $X$. We shall say that $E$ is Bogomolov-unstable with respect to $H$ if there exists a line bundle $L \subset E$ such that $2c_1(L) - L_\infty(E) \in K^+(X, H)$. Hence Theorem 2.3 can be rephrased by saying that if $\Delta(E) : H^\varepsilon > 1$, then $E$ is Bogomolov-unstable with respect to $H$. 


Let us now come to the case of positive characteristic. The basic result is given here by Moriwaki’s generalization of the Bogomolov-Gieseker inequality ([Mo 93]). Before stating his theorem, we need the following:

**Definition 2.5.** Let $X$ be a smooth projective $n$-fold and let $H$ be an ample line bundle on $X$. Let $E$ be a rank two vector bundle on $X$. We say that $E$ is weakly $\mu$-semistable w.r.t. $H$ if for any proper subsheaf $F \subset E$ there exists an ample divisor $D$ on $X$ such that $\mu(F, H, D) \leq \mu(E, H, D)$, where for a sheaf $G$ we set $\mu(G, H, D) = \inf \{ \frac{\langle K_X \cdot D, H^{n-2} \rangle}{D^2 \cdot H^{n-2}} \}$. 

**Remark 2.4.** In any characteristic, if $E$ is Bogomolov-unstable w.r.t. $H$ (Definition 2.4), then it is not $\mu$-semistable. On the other hand, if $E$ is not $\mu$-semistable w.r.t. $H$ and $\Delta(E) \cdot H^{n-2} > t$, then it is necessarily Bogomolov-unstable w.r.t. $H$.

**Definition 2.6.** Let $X$ be a smooth projective $n$-fold, and let $H$ be an ample line bundle on $X$, and let $Nef(X) \subset N^1(X)$ denote the nef cone of $X$. Set 

$$
\sigma(H) = \inf_{D \in Nef(X)} \left\{ \frac{(K_X \cdot D \cdot H^{n-2})^2}{D^2 \cdot H^{n-2}} \right\}. 
$$

We agree to take the above ratio equal to $\infty$ when the denominator vanishes.

**Theorem 2.4.** (Moriwaki) Let $X$ be a smooth projective $n$-fold over an algebraically closed field of characteristic $p > 0$. Assume that $X$ is not uniruled. Let $H$ be a polarization on $X$, and let $E$ be a rank two vector bundle on $X$. Suppose that for all $0 \leq i < r$ the Frobenius pull-back $E^{(i)}$ of $E$ is weakly $\mu$-semistable with respect to $H$. Then we have 

$$
\Delta(E) \cdot H^{n-2} \leq \frac{\sigma(H)}{(e - \infty)\varepsilon}. 
$$

Furthermore, Moriwaki proves the following powerful restriction lemma:

**Lemma 2.2.** (char($k$) $\geq 0$) Let $X$ be a smooth projective $n$-fold, and let $H$ be a very ample line bundle on $X$. Suppose that $E$ is a rank two vector bundle on $X$, which is weakly $\mu$-semistable w.r.t. $H$. Then for a general $Y \in |H|$ the restriction $E|_Y$ is weakly $\mu$-semistable w.r.t. $H|_Y$. 

**Definition 2.7.** Let $X$ be a smooth projective $n$-fold, and let $H$ be an ample line bundle on $X$. Define

$$\beta(H) = \inf_{D \in \text{Nef}(X)} \left\{ \frac{(D \cdot (H + K_X) \cdot H^{n-2})^2}{D^2 \cdot H^{n-2}} \right\}.$$

**Corollary 2.3.** Let $X$ be a smooth projective $n$-fold, with $n \geq 3$ on an algebraically closed field, and let $H$ be a very ample line bundle on $X$. Suppose that the general $Y \in |H|$ is not uniruled. Let $\mathcal{E}$ be a rank two vector bundle on $X$ such that

$$\Delta(\mathcal{E}) \cdot H^{n-\varepsilon} > \frac{\beta(H)}{(-\infty)^\varepsilon}.$$

Then $\mathcal{E}$ is Bogomolov-unstable with respect to $H$, i.e. there exists an exact sequence

$$0 \to A \to \mathcal{E} \to \mathcal{B} \otimes J_Z \to \mathcal{I},$$

where $A$ and $B$ are line bundles on $X$, $Z \subset X$ is a codimension two local complete intersection and $A - B \in K^+(X, H)$.

**Remark 2.5.** Although this is an immediate application of Moriwaki’s theorem 2.4, it is phrased in a way that makes it applicable to uniruled varieties. Furthermore, observe that if $k$ is an uncountable algebraically closed field and $X$ is a smooth non-uniruled projective variety over $k$ with a very ample line bundle $H$ on it, the general element of $|H|$ is not uniruled either. In fact, since $k$ is uncountable, a variety $X$ over $k$ is uniruled if and only if through a general point of $X$ there passes a rational curve ([MM 86]). But a general point in a general divisor of a very ample linear series is a general point of $X$.

**Proof.** By Lemma 2.2, it is sufficient to show that for general $Y \in |H|$ the restriction $\mathcal{E}|_Y$ is Bogomolov-unstable with respect to $H|_Y$. It is easy to deduce this fact from theorem 2.4 and the definition of $\beta(H)$. ♦

**Corollary 2.4.** Let $\mathcal{E}$ be a rank two vector bundle on $\mathbb{P}^r_k$, where $k$ is an algebraically closed field of characteristic $p$. If $\Delta(\mathcal{E}) > 1$, then $\mathcal{E}$ is unstable.

**Proof.** For $r = 2$, this is well-known. For $r \geq 3$, we apply corollary 2.3 taking the very ample line bundle in the statement to be $\mathcal{O}_{\mathbb{P}^r}(\Delta)$, so that $\beta(H) = 0$. We can also proceed inductively from the case $r = 2$ by applying Lemma 2.2. ♦
3. Extension Of Pencils

Let $Y \subset X$ be an inclusion of projective varieties, and let $|L|$ be a base point free pencil on $Y$. It is natural to look for conditions under which $|L|$ extends to $X$, in the spirit of the results of Sommese, Serrano and Reider ([Re 89], [So 76], [Se 87]). Our main result is the following:

**Theorem 3.1.** (char($k$) = 0) Let $X$ be a smooth projective $n$-fold, $n \geq 2$, and let $Y \subset X$ be a reduced irreducible divisor. If $n \geq 3$, assume that $Y$ is ample, and if $n = 2$ that $Y^2 > 0$ (so that in particular it is nef). Suppose given a morphism $\phi : Y \to \mathbb{P}^1$ and let $F$ denote the numerical class of a fiber of $\phi$.

(i) Suppose that $F \cdot Y^{n-2} < \sqrt{Y^n} - 1$. Then there exists a morphism $\psi : X \to \mathbb{P}^1$ extending $\phi$, and such that $H^0(X, \psi^*\mathcal{O}_{\mathbb{P}^1}(\infty)) \hookrightarrow H^0(Y, \phi^*\mathcal{O}_{\mathbb{P}^1}(\infty))$.

In particular, if $\phi$ is linearly complete, then so is $\psi$.

(ii) Suppose $F \cdot Y^{n-2} = \sqrt{Y^n} - 1$ and $Y^n \neq 4$. Then either $\phi^*\mathcal{O}_{\mathbb{P}^1}(\infty)$ extends to a base-point free pencil on $X$, or else there exists an effective divisor $D$ on $X$ such that

(a) the following equalities hold:

$$D^2 \cdot Y^{n-2} Y^n = (D \cdot Y^{n-1})^2$$

and

$$D \cdot Y^{n-1} = \sqrt{Y^n}.$$ (b) there is an inclusion

$$\phi^*\mathcal{O}_{\mathbb{P}^1}(\infty) \subset \mathcal{O}_Y(D).$$

**Remark 3.1.** If $Y$ is ample, the equalities in (a) of (ii) can be phrased as follows. If $S \subset X$ is a smooth complete intersection of $n-2$ divisor equivalent to multiples of $Y$, then

$$D - \frac{1}{\sqrt{Y^n}} Y \in Ker\{N(X) \to N(S)\}.$$ If $n = 2$, this is just saying that $D \equiv \frac{1}{\sqrt{Y^n}} Y$.

**Proof.** Set

$$A =: \phi^*\mathcal{O}_{\mathbb{P}^1}(\infty) \quad (1)$$

and let

$$V \subset H^0(Y, A) \quad (2)$$
be the pencil associated to $\phi$, i.e. $V = \phi^*H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\infty))$. Define a sheaf $\mathcal{F}$ on $X$ by the exactness of the sequence

$$0 \to \mathcal{F} \to V \otimes \mathcal{O}_X \to \mathcal{A} \to 0.$$  \hspace{1cm} (3)

Then $\mathcal{F}$ is a rank two vector bundle with Chern classes $c_1(\mathcal{F}) = -\mathcal{Y}$ and $c_2(\mathcal{F}) = [\mathcal{A}]$, where $\mathcal{Y}$ denotes the divisor class on $Y$ of an element of the pencil $|\mathcal{A}|$. In particular, $[\mathcal{A}]$ is represented by a fiber $F$ of $\phi$. Therefore the discriminant of $\mathcal{F}$ (definition 2.1) is

$$\Delta(\mathcal{F}) = \mathcal{Y}^e - \triangle[\mathcal{A}]$$ \hspace{1cm} (4)

and so

$$\Delta(\mathcal{F}) \cdot \mathcal{Y}^{1-e} = \mathcal{Y}^{1} - \Delta \mathcal{F} \cdot \mathcal{Y}^{1-e}. $$ \hspace{1cm} (5)

It is easy to check that

$$\sqrt{Y^n} - 1 \leq \frac{Y^n}{4} \hspace{1cm} (6)$$

and by assumption we then have in particular that $\Delta(\mathcal{F}) \cdot \mathcal{Y}^{1-e} > t$, and therefore $\mathcal{F}$ is Bogomolov-unstable with respect to $Y$ (definition 2.3). Hence there exists a saturated invertible subsheaf

$$\mathcal{L} \subset \mathcal{F}$$

which is the maximal destabilizing subsheaf of $\mathcal{F}$ with respect to $(Y, \cdots, Y, L)$, for any ample divisor $L$ on $X$ (theorem 2.3). Since $\mathcal{L} \subset \mathcal{F} \subset \mathcal{O}^e_X$, we can write

$$\mathcal{L} = \mathcal{O}_X(-D)$$

for some effective divisor $D$ on $X$. The instability condition then reads

$$(Y - 2D) \cdot Y^{n-1} \geq 0,$$ \hspace{1cm} (7)

with strict inequality holding if $Y$ is ample. Furthermore, using the fact that $\mathcal{L}$ is saturated one can see that

$$(Y - 2D)^2 \cdot Y^{n-2} \geq \Delta(\mathcal{F}) \cdot \mathcal{Y}^{1-e} \hspace{1cm} (8)$$

(see remark 2.3) and if we set $f =: F \cdot Y^{n-2}$ this can be rewritten as

$$f \geq D \cdot Y^{n-1} - D^2 \cdot Y^{n-2}. $$ \hspace{1cm} (9)

By assumption, we have $f < \sqrt{Y^n} - 1$ and together with (8) this gives

$$D^2 \cdot Y^{n-2} - 1 > D \cdot Y^{n-1} - \sqrt{Y^n}. $$

Applying the Hodge Index Theorem, we then get

$$\frac{(D \cdot Y^{n-1})^2}{Y^n} - 1 > D \cdot Y^{n-1} - \sqrt{Y^n}. $$ \hspace{1cm} (10)

Claim 3.1. $\mathcal{L}$ is saturated in $\mathcal{O}^e_X$. 

Proof. If not, there would exist an inclusion $O_X(Y - D) \subset O_X^\xi$ (here we use the fact that $Y$ is reduced and irreducible) and therefore we should have

$$(D - Y) \cdot Y^{n-1} \geq 0.$$ 

Together with (7), this would imply $Y^n \leq 0$, a contradiction.

Hence we have an exact sequence of the form

$$0 \to O_X(-D) \to O_X^\xi \to O_X(D) \otimes J_Z \to 1,$$

(11)

where $Z \subset X$ is a codimension two local complete intersection. Computing $c_2(O_X^\xi) = t$ from the above sequence we then get

$$D^2 = [Z]$$

(equivalently, one might just observe that $Z$ is the complete intersection of the two sections of $O_X(D)$ coming from the above sequence). Therefore under the assumptions of the theorem either $Z = \emptyset$, or else $D^2 \cdot Y^{n-2} > 0$.

Lemma 3.1. $Z = \emptyset$

Proof. Suppose, otherwise, that $D^2 \cdot Y^{n-2} > 0$. In this case the Hodge Index Theorem yields

$$(D \cdot Y^{n-1})^2 \geq (D^2 \cdot Y^{n-2}) Y^n \geq Y^n$$

and therefore

$$D \cdot Y^{n-1} \geq \sqrt{Y^n}.$$ 

Therefore the right hand side of (11) is nonnegative. We can rewrite (11) as

$$\frac{(D \cdot Y^{n-1})^2}{Y^n} - 1 > D \cdot Y^{n-1} - \sqrt{Y^n} = Y^n \left( \frac{D \cdot Y^{n-1}}{Y^n} - \frac{1}{\sqrt{Y^n}} \right).$$

Let us now make use of the destabilizing condition $Y^n \geq 2D \cdot Y^{n-1}$: we obtain

$$\frac{(D \cdot Y^{n-1})^2}{Y^n} - 1 > 2 \frac{(D \cdot Y^{n-1})^2}{Y^n} - 2 \frac{D \cdot Y^{n-1}}{\sqrt{Y^n}}$$

and this leads to $0 > \left( \frac{D \cdot Y^{n-1}}{\sqrt{Y^n}} - 1 \right)^2$, absurd. ♯
Since \( Z = \emptyset \), \( \mathcal{O}_X(-\mathcal{D}) \to \mathcal{O}_X^c \) never drops rank, and therefore neither does \( \mathcal{O}_X(-\mathcal{D}) \to \mathcal{F} \). Hence we have a commutative diagram

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & \downarrow & \\
0 & \mathcal{O}_X(-\mathcal{D}) & \to & \mathcal{F} \\
\downarrow & & \downarrow & \\
0 & \mathcal{O}_X(-\mathcal{D}) & \to & \mathcal{O}_X(\mathcal{D} - \mathcal{Y}) \\
\downarrow & & \downarrow & \\
A & \to & \mathcal{O}_Y(\mathcal{D}) \\
\downarrow & & \downarrow & \\
0 & 0 & & \\
\end{array}
\]

from which we see that

\[ A \cong \mathcal{O}_Y(\mathcal{D}). \]

Furthermore, since

\[ (D - Y) \cdot Y^{n-1} \leq 2D \cdot Y^{n-1} - Y^n < 0 \]

by the destabilizing condition (7), we have \( H^0(X, \mathcal{O}_X(\mathcal{D} - \mathcal{Y})) = 1 \) and therefore an injection

\[ H^0(X, \mathcal{O}_X(\mathcal{D})) \hookrightarrow \mathcal{H}^0(\mathcal{Y}, \mathcal{O}_Y(\mathcal{D} - \mathcal{Y})). \]

Since \( \mathcal{O}_X(\mathcal{D}) \) is a quotient of \( \mathcal{O}_X^c \), it is globally generated and \( V \) gives a base point free pencil of sections of \( \mathcal{O}_X(\mathcal{D}). \)

**Proof of (ii)** Suppose now that \( F \cdot Y^{n-2} = \sqrt{Y^n} - 1 \). It is easy to see that if \( Y^n \neq 4 \) then the inequality (8) is strict, and therefore \( \mathcal{F} \) is still Bogomolov unstable with respect to \( \mathcal{Y} \). Arguing exactly as in the proof of the previous lemma we get:

**Lemma 3.2.** Either \( Z = \emptyset \), or else the following equalities hold:

\[ (D \cdot Y^{n-1})^2 = (D^2 \cdot Y^{n-2})Y^n \]

and

\[ D \cdot Y^{n-1} = \sqrt{Y^n}. \]

To complete the argument, observe that a variant of the commutative diagram (12) gives the exact sequence

\[ 0 \to \mathcal{O}_X(\mathcal{D} - \mathcal{Y}) \otimes J_W \to \mathcal{O}_X(\mathcal{D}) \otimes J_Z \to \mathcal{O}_Y(\mathcal{D}) \otimes I_{Z \cap \mathcal{Y}} \to \mathcal{I}. \]
FREE PENCILS ON DIVISORS

where $\mathcal{I}$ denotes an ideal sheaf on $Y$. Therefore we also get an isomorphism $A \simeq \mathcal{O}_Y(D) \otimes \mathcal{I}_{\mathbb{Z} \cap Y}$. ♯

**Corollary 3.1.** (Serrano) Let $S$ be a smooth projective surface and let $C \subset S$ be an irreducible smooth curve with $C^2 > 0$. Then either

$$\text{gon}(C) \geq \sqrt{C^2} - 1$$

or else for every minimal pencil $A$ on $C$ there exists a base point free pencil $\mathcal{O}_S(D)$ on $S$ such that

$$A \simeq \mathcal{O}_C(D).$$

**Corollary 3.2.** Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d$. Then

$$\text{gon}(C) = d - 1.$$ 

Furthermore, any base point free pencil on $C$ is given by projecting through a point of $C$.

**Proof.** The bound in the theorem gives $\text{gon}(C) \geq d - 1$. On the other hand projecting from a point of $C$ shows that equality must hold. Let $A$ be any minimal pencil on $C$. We may assume that $d > 2$. We are then in the boundary situation $f = \sqrt{C^2} - 1$ (case (ii) of Theorem 3.1, $n = 2$). Hence we must have an inclusion $A \subset \mathcal{O}_C(H)$, which shows that $A$ has the form

$$A = \mathcal{O}_C(H - P)$$

for some $P \in C$. But (13) is saying exactly that $A$ is the pull back of the hyperplane bundle on $\mathbb{P}^1$ under the morphism given by projection from $P$. Hence all the minimal pencils are obtained in this way. ♯

**Example 3.1.** Let us apply the Theorem to the gonality of Castelnuovo extremal curves in $\mathbb{P}^3$. If $C$ has even degree $d = 2a$, then $C$ is the complete intersection of a quadric $S$ and an hypersurface of degree $a$. Suppose that $S$ is smooth. Then either $\text{gon}(C) \geq \sqrt{C \cdot S} - 1 = \sqrt{2a} - 1$, or else a minimal pencil is induced by a base point free pencil on $S$. $C$ on $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of type $(a,a)$, and restriction to it of the two rulings gives two pencils of degree $a = \frac{d}{2}$, which is the well-known answer. The argument is the same for even degree.

**Example 3.2.** For an example with $n = 3$, let $S \subset \mathbb{P}^3$ be a smooth surface of degree $s$ containing a line $L$, and let $\phi : S \rightarrow \mathbb{P}^1$ be induced by projection from $L$. Then a straightforward computation shows that $f > s\sqrt{s} - 1$. 

We now give an application to singular plane curves.

**Corollary 3.3.** Let \( C \subset \mathbb{P}^2 \) be a reduced irreducible curve of degree \( d \), and suppose that the only singularities of \( C \) are ordinary singular points \( P_1, \ldots, P_k \) of multiplicities \( m_1, \ldots, m_k \), respectively. Let \( m = \max\{m_i\} \) and denote by \( \tilde{C} \) the normalization of \( C \). Suppose that \( d^2 > \sum_i m_i^2 \). Then

\[
\text{gon}(\tilde{C}) \geq \min\left\{ \sqrt{d^2 - \sum_{i} m_i^2 - 1}, d - \sqrt{\sum_i m_i^2} \right\}.
\]

**Proof.** Let

\[
f : X \to \mathbb{P}^2
\]

be the blow up of \( \mathbb{P}^2 \) at \( P_1, \ldots, P_k \),

\[E_i = f^{-1}P_i\]

for \( i = 1, \ldots, k \) be the exceptional divisors, and let \( \tilde{C} \subset X \) be the proper transform of \( C \). Then \( \tilde{C} \) is an irreducible smooth curve and

\[
\tilde{C} \in |dH - \sum_{i=1}^{k} m_iE_i|.
\]

Therefore we have

\[
\tilde{C}^2 = d^2 - \sum_{i=1}^{k} m_i^2 > 0
\]

by assumption, and the hypothesis of the theorem are satisfied. Hence either

\[
\text{gon}(\tilde{C}) \geq \sqrt{\tilde{C}^2} - 1,
\]

or else there exists an effective divisor \( D \) on \( X \) moving in a base point free pencil and inducing a minimal pencil on \( \tilde{C} \). We may then assume that \( D \) has the form

\[D = xH - \sum_i a_iE_i\]

with \( x > 0 \) and all the \( a_i \geq 0 \). The condition \( D^2 = 0 \) then gives

\[
x = \sqrt{\sum_i a_i^2}.
\]

Hence

\[
D \cdot \tilde{C} = xd - \sum_i a_i m_i \geq xd - \sqrt{\sum_i a_i^2} \sqrt{\sum_i m_i^2} = xd - \sqrt{\sum_i m_i^2} \geq d - \sqrt{\sum_i m_i^2}.
\]

The statement follows. ♦
Example 3.3. Let us consider for example the case of a reduced irreducible plane curve $C \subset \mathbb{P}^2$ whose only singularities are nodes $P_1, \ldots, P_\delta$. Suppose that

$$4\delta < d^2.$$  

Then by the Corollary

$$\text{gon}(\tilde{C}) \geq \min\{\sqrt{d^2 - 4\delta} - 1, d - 2\sqrt{\delta}\}.$$  

For example, if we also assume that

$$\delta < d - 2$$

then

$$\sqrt{d^2 - 4\delta} - 1 > d - 3,$$

and for any effective divisor $D = xH - \sum a_iE_i$ with $D^2 = 0$ it is easy to see that $D \cdot \tilde{C} \geq d - 2$. Since projecting from a node gives a pencil of degree $d - 2$, we then have

$$\text{gon}(\tilde{C}) = d - 2.$$

We now show how theorem 2.1 applies to families of morphisms.

Proposition 3.1. Let $X$ be a smooth projective $n$-fold and $Y \subset X$ be a reduced irreducible divisor in $X$. Suppose that $Y$ is ample when $n > 2$ and that $Y^2 > 0$ when $n = 2$. Let $\Phi : Y \times B \to \mathbb{P}^1$ be a family of morphisms with $B$ smooth and set $\phi_b = \Phi|_{Y \times \{b\}}$. Denote by $F$ the numerical class of a fiber of $\phi_b$ (it is independent of $b \in B$), and suppose that

$$F \cdot Y^{n-2} < \sqrt{Y^n} - 1.$$  

Then there exists a nonempty open subset $T \subset B$ and a morphism

$$\Psi : X \times T \to \mathbb{P}^1$$

that restricts to $\Phi$ on $Y \times T$.

Proof. Let

$$A = \Phi^*O_{\mathbb{P}^1}(\infty)$$

and

$$V = :\Phi^*H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(\infty)).$$

Then we can define a rank two vector bundle on the smooth variety $X \times B$ in the usual guise, by the exactness of the sequence

$$0 \to \mathcal{F} \to V \otimes O_{X \times B} \to A \to t.$$  

(14)

For $b \in B$ let us set $X_b = X \times \{b\}$ and $A_b = A|_{X_b}$. Then we have $\mathcal{F}|_{X_1} \simeq \mathcal{F}|_1$, where $\mathcal{F}|_1 =: \mathcal{K}|\nabla\{V \otimes O_{X_1} \to A_1\}$. Then $\mathcal{F}$ can be seen as a family of vector bundles on $X$, with Chern classes $c_1(\mathcal{F}) = -V$ and $c_2(\mathcal{F}) = [A_1]$. As in the proof of the Theorem, these vector bundles are
Bogomolov unstable with respect to $Y$. Let $\mathcal{L}_i \subset \mathcal{F}_i$ be the maximal destabilizing line bundle of $\mathcal{F}_i$. By the construction in Theorem 3.1, the morphisms $\psi_b$ are associated to base point free pencils of sections of $\mathcal{L}_i^{-\infty}$ induced by $V$. Therefore, the proposition will follow once we show that the line bundles $\mathcal{L}_i$ can be glued to a line bundle $\mathcal{L} \subset \mathcal{F}|_{X \times \mathcal{T}}$ on some open subset $X \times T$. In fact, we have:

**Claim 3.2.** For some nonempty open subset $T \subset B$ there exists a line bundle $\mathcal{L} \subset \mathcal{F}|_{X \times \mathcal{T}}$ such that $\mathcal{L}$ restricts to $\mathcal{L}_i$ on $X_b$, for each $b \in T$.

**Proof** This follows from the relative version of the Harder-Narashiman filtration introduced in [FHS 80] and [FT 84]. ♦

This proves the statement of the proposition. ♦

In his paper ([Se 87]) Serrano expressed his results about extensions in terms of morphisms to arbitrary smooth curves. It seems in order to give here a corresponding generalization of theorem 2.1.

**Definition 3.1.** Let $B$ be a smooth curve. We shall denote by $s(B)$ the smallest degree of a nondegenerate plane birational model of $B$, i.e. the smallest $k$ for which $B$ has a birational $g_k^k$. Nondegenerate only means that we agree to take $s(\mathbb{P}^1) = 2$.

**Corollary 3.4.** Let $X$ and $Y$ satisfy the hypothesis of the theorem, and let $\phi : Y \to B$ be a morphism to a smooth curve. Denote by $F$ the numerical class of a fiber of $\phi$, and suppose that

$$(s(B) - 1)F \cdot Y^{n-2} < \sqrt{Y^n} - 1.$$ 

Then there exists a morphism $\psi : X \to B$ extending $\phi$.

**Proof.** We adapt the argument in [Se 87], Lemma 3.2. Let $f : B \to G \subset \mathbb{P}^2$ be a plane birational model of $B$, of degree $s = s(B)$, and let $B^* \subset B$ be the inverse image of the smooth locus of $G$. For $b \in B^*$, let $\pi_b : B \to \mathbb{P}^1$ be the projection from $f(b)$; $\pi_b$ is a morphism of degree $s - 1$. Consider the composition $\phi_b = \pi_b \circ \phi : Y \to \mathbb{P}^1$. A fiber of $\phi_b$ is numerically equivalent to a sum of $s - 1$ fibers of $\phi$, and the numerical hypothesis then imply, by theorem 3.1, that there exist extensions $\psi_b : X \to \mathbb{P}^1$. By Proposition 3.1, we can find a nonempty open subset $T \subset B$ and a morphism

$$\Psi : X \times T \to \mathbb{P}^1$$

extending the morphism

$$\Phi : Y \times T \to \mathbb{P}^1$$
given by \( \Phi(y, b) = \phi_b(y) \). From this one sees that, if

\[
X \xrightarrow{\gamma_b} \Delta_b \xrightarrow{\theta_b} \mathbb{P}^1
\]

is the Stein factorization of \( \phi_b \), then \( \Delta_b \cong \Delta \) for some fixed curve \( \Delta \) and all the morphisms \( \gamma_b \) can be identified. Consider the morphism

\[
h = (\gamma|_Y, \phi) : Y \to \Delta \times \mathbb{P}^1.
\]

It is easy to see that \( \pi_1 : h(Y) \to \Delta \) is an isomorphism. Hence we can define \( \psi = \pi_2 \circ \pi_1^{-1} \circ \gamma \).

Let us consider now the case of prime characteristic. We give the corresponding version of Theorem 3.1.

**Theorem 3.2.** Let \( k \) be an algebraically closed field of characteristic \( p \), and let \( X \) be a smooth projective \( n \)-fold over \( k \). Let \( Y \subset X \) be a reduced irreducible divisor, and suppose that there exists a morphism \( \phi : Y \to \mathbb{P}^1 \); let \( F \) denote the numerical class of a fiber of \( \phi \). Then \( \phi \) can be extended to a morphism \( \psi : X \to \mathbb{P}^1 \) in the following situations:

(i) \( \text{char}(k) \neq 2, 3, n = 2, Y^2 > 0, \deg(F) < \sqrt{Y^2 - 1}, X \text{ not of general type.} \)

(ii) \( n = 2, Y^2 > 0, X \text{ is not uniruled and} \)

\[
\deg(F) < \min \left\{ \sqrt{Y^2} - 1, \frac{1}{4} Y^2 - \frac{1}{(p - 1)^2} \sigma_S \right\}.
\]

(iii) \( n \geq 3, X \text{ is not uniruled and there exists a ample line bundle } \)

\( H \text{ on } X, \text{ such that } Y \equiv lH \text{ and} \)

\[
F \cdot Y^{n-2} < \min \left\{ \sqrt{Y^n} - 1, \frac{1}{4} Y^n - \frac{l^{n-2}}{(p - 1)^2} \sigma(H) \right\}.
\]

(iv) \( n \geq 3 \text{ and there exists a very ample line bundle } H \text{ on } X \text{ such that } Y \equiv lH, \text{ the general } Z \in |H| \text{ is not uniruled and} \)

\[
F \cdot Y^{n-2} < \min \left\{ \sqrt{Y^n} - 1, \frac{1}{4} Y^n - \frac{l^{n-2}}{(p - 1)^2} \beta(H) \right\}.
\]

**Remark 3.2.** The definitions of \( \sigma(H) \) and \( \beta(H) \) are given in section 2 (definitions 2.6 and 2.7). If \( n = 2, \sigma \text{ does not depend on } H, \text{ and we denote it by } \sigma_S. \)

**Proof.** As to (i), that \( \phi \) does not extend means that \( \Delta(F) > \iota \) (cfr eq. (\[3\])), but \( F \) is not Bogomolov-unstable. That this forces \( X \) to be of general type is the content of Theorem 7 of [SB 91]. For the other statements, the argument is exactly the same as in the characteristic zero case, the extra assumptions being needed to apply the results
about unstable rank two bundles from section 2 (e.g., Corollary 2.3).

Remark 3.3. For $n = 2$, we have to assume that $S$ is not uniruled to apply the characteristic $p$ version of Bogomolov’s theorem. However, in the case of $\mathbb{P}^2$ Bogomolov’s theorem still holds ([Sch 61]). We can therefore still argue as in Corollary 3.2 to deduce the classical statement about the gonality of plane curves.

Theorem 3.3 can be strengthened as follows (see Theorem 2.4).

Theorem 3.3. Let notation be as in Theorem 3.2, and suppose that $F \cdot Y^{n-2} < \sqrt{Y^n} - 1$. Assume that $X$ is not uniruled. Let $r$ be the smallest positive integer such that

$$F \cdot Y^{n-2} < \min \left\{ \sqrt{Y^n} - 1, \frac{Y^n}{4} - \frac{l^{n-2}}{(p^r - 1)^2} \sigma(H) \right\}.$$

Then if $X' \supset Y'$ denote the $(r-1)$-th Frobenius pull-backs of the varieties $X$ and $Y$, there exists $\psi : X' \to \mathbb{P}^1$ extending the induced morphism $\phi' : Y' \to \mathbb{P}^1$.

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