Quantifying the impact of network structure on speed and accuracy in collective decision-making

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Found in varied contexts from neurons to ants to fish, binary decision-making is one of the simplest forms of collective computation. In this process, information collected by individuals about an uncertain environment is accumulated to guide behavior at the aggregate scale. We study binary decision-making dynamics in networks responding to inputs with small signal-to-noise ratios, looking for quantitative measures of collectivity that control decision-making performance. We find that decision accuracy is controlled largely by three factors: the leading eigenvalue of the network adjacency matrix, the corresponding eigenvector’s participation ratio, and distance from the corresponding symmetry-breaking bifurcation. This allows us to predict how decision-making performance scales in large networks based on their spectral properties. Specifically, we explore the effects of localization caused by the hierarchical assortative structure of a “rich club” topology. This gives insight into the tradeoffs involved in the higher-order structure found in living networks performing collective computations.

\textbf{Keywords:} collective computation, neural networks, symmetry breaking transition, stochastic dynamical systems, rich club

\section*{Introduction}

Collective intelligence refers to the ability of groups of individual components to process environmental information and successfully perform adaptive functions at a larger collective scale. Building a coherent framework for understanding distributed functionality is challenging in that the internal structure of natural and engineered collectives varies strongly, from quasi-homogeneous systems like swarms of identical robots, to fish-schools consisting of similarly behaving individuals but with persistent behavioral differences (“personalities”) \cite{12} or different prior information \cite{34}, to strongly heterogeneous and hierarchical systems like primate societies \cite{56} or neurons in a brain \cite{78}. Facing this diversity, a key challenge for building a better abstract understanding of collective intelligence is to determine which details of such systems are most important to collective function and which are incidental and can be ignored. In this way,
we are searching for measures that usefully quantify “collectivity” across a broad continuum of complex systems.

In addition to diversity in heterogeneity and communication structure, myriad types of functions may be implemented in a collective system, ranging in complexity from simple majority consensus to high-level abstract information processing. Here we focus on a particularly simple function—making a correct binary decision about the sign of a noisy distributed input—and look for network statistics that delineate the full range of strategies that can be used to successfully perform this collective function.

Past experimental investigations of collective decision-making have mostly not addressed network structure, instead assuming all-to-all coupling and focusing on optimal rules for aggregating decisions made by individuals [9–13]. However, an increasing number of studies are beginning to investigate non-trivial network structures [14–17]. For example, Kearns et al [14] look at the effect of varying network structure in consensus formation in human groups. They find, e.g., that “preferential attachment” networks lead to faster consensus than Erdos–Renyi.

Theoretically, many examples of collective decision-making can be effectively described using networks of coupled dynamical components. Structural properties of such networks and how they affect self-organization and collective behavior have long been a focus of complex systems research (see e.g. [18–21]). Particularly well studied are effects of network structure on emergent dynamics in the context of synchronization and consensus formation [18,19].

Corresponding theoretical insights have driven the systematic analysis of structural properties of artificial and real-world networks, including node and degree heterogeneity [25] and structural hierarchies [26,27]. In particular, many real-world collective systems exhibit a “rich club” (core-periphery) structure, with examples coming from neuroscience [7,8,28–30], social science, and biochemistry [12,31]. The rich club refers to a subset of nodes that a) have a larger (in-)degree and b) are more likely to be connected to other rich club nodes than in an otherwise random wiring. It has been argued that such a core-periphery topology may play an important role for the function of complex information processing systems (see e.g. [7,30,32]).

The dynamical effects of network structure have been explored largely in the context of synchronization or consensus, the problem of collective agreement. Extending to the problem of decision-making also requires a notion of correctness: we want a system that not only produces collective agreement on any consensus state, but on the correct state, given a source of input information. Binary collective decision-making in this sense maps naturally onto “noisy integrator” models, such as leaky integration to bound (Ornstein–Ullenbeck [33]) and related models with stable attractors representing decision states [34,35]. A general constraint for any such decision-making system is the tradeoff between speed and accuracy [33,36,37]. Recently, it has been shown that the speed–accuracy tradeoff in simple collective decision models can be quantified in terms of distance from a bifurcation [35,38].

Motivated by the above findings, we focus in this work on the question of how a rich-club structure affects...
the speed and accuracy of collective decision-making. In particular, we look for network statistics that capture the most important properties controlling collective performance in decision dynamics.

Results

Collective decision-making model

A simple minimal model of distributed decision-making defines dynamics for the internal noisy states of individual components, each of which receives the same input signal $I$, recovers to a null state on a timescale $\tau$, and is affected by its neighbors through a saturating function of its neighbors’ states [35, 38]:

$$\frac{ds_i}{dt} = -\frac{s_i}{\tau} + \frac{\mu}{\tau} \sum_j A_{ij} \tanh(s_j) + \frac{I}{\tau} + \xi,\tag{1}$$

where $I$ is an input signal given uniformly to every node and $\xi$ is uncorrelated Gaussian noise with $\langle \xi(t)\xi(t+\Delta t) \rangle = \sigma^2 \tau^{-1} \delta(\Delta t)$. We explicitly write the differential equations in terms of an overall timescale $\tau$; in describing neural dynamics, for example, we expect $\tau$ to be on the order of tens of milliseconds.

We initialize the system in a state $\vec{s}_0$ that, in the case of zero noise, corresponds to a fixed point undergoing a pitchfork bifurcation as a function of the coupling strength $\mu$. This bifurcation separates the case of a single stable fixed point at $\vec{s}_0$ and the case of two distinct stable fixed points at $\vec{s}_0 \pm \epsilon \hat{e}_c$, which we treat as decision states (where $\hat{e}_c$ is the unit vector pointing in the direction in which the decision states emerge from $\vec{s}_0$ at the bifurcation). We focus here on the simplest such bifurcation, which occurs at $\vec{s}_0 = 0$.

To test how the existence of higher-order structure changes the decision-making performance, we vary the adjacency matrix $A$ to test symmetric networks with fixed size $N$ and total number of edges, changing only the degree distribution and higher-order structure (Figure 1). First, a "rich club" network is created through a random generation of a fixed number of edges from the $N(N-1)$ possible edges, where edges between $N_{\text{rich}}$ core nodes are biased to be more likely to appear by a factor $b_{\text{rich}}$. A corresponding network that has exactly the same degree distribution but no rich club is then created by randomly swapping existing edges. Finally, we test an Erdos–Renyi variant in which all possible edges are equally likely. We use the same three specific example networks shown in Figure 1 in simulations throughout the paper; results are qualitatively similar for other networks sampled from each ensemble.

Speed–accuracy tradeoff

We test each network in its ability to integrate information about a signal that is small compared to the noise and then retain that information after the signal is removed. We define accuracy in terms of whether the system ends the simulation near the correct decision state, the one that lies in the direction of the input $+I$ (instead of $-I$). Specifically, we test whether the sign of the final state along the unstable dimension, $(\vec{s}_{\text{final}} - \vec{s}_0) \cdot \hat{e}_c$, is the same as the sign of the input along that dimension, $I \hat{e}_c \cdot \hat{e}_c$.

As in previous studies using a similar model [35, 38] (and across a wide variety of systems in general [36, 37, 39]), we expect to find a speed–accuracy tradeoff: Slow dynamics should produce better accuracy.

\(^2\) Tuning a second control parameter can locate more general pitchfork bifurcations; see [38].
Figure 1: Three random networks with varying higher-order network structure. All networks share the same size ($N = 200$) and total number of edges (800). The “rich club” network is generated such that a core group of 10 nodes have increased probability of edges within the group ($b_{\text{rich}} = 257$); the “heterogeneous” network has identical degree distribution to the rich club network but with edges otherwise randomized; and the “Erdos–Renyi” network is selected such that all possible edges have equal probability. We characterize each network using its adjacency matrix (top row), network diagram (second row), degree distribution (third row), and dominant eigenvector and corresponding participation ratio $p$ (bottom row). Throughout the figure, nodes are ordered first by core and periphery groups and then sorted by degree. Core nodes are highlighted in red in the network diagrams.
Figure 2: Speed–accuracy tradeoff. Simulations show that maximal accuracy of a noisy collective decision occurs when the decision process happens over a longer timescale, lengthening the time over which the system remains sensitive to the input. Here and in other plots unless otherwise specified, \( \tau = 1, \sigma = 0.05, I = 0.001 \), the duration of the signal = 200\( \tau \), the total simulation time = 2000\( \tau \), \( \Delta \mu \) ranges from 0 to 0.04, and we average over 100 simulations for each network.

as the system is able to integrate the input over a longer time before fixating within a single decision state, whereas fast dynamics produce a decision based primarily on noise.

To quantify the speed of the decision, we measure a characteristic time \( t_D \) over which the system approaches the final decision fixed point. We first define the two decision states \( \vec{s}_\pm^\ast \) as the stable fixed points of the dynamics in Eq. (1) with zero input and zero noise.\(^3\) We then define the decision timescale \( t_D \) as the first time the state \( \vec{s} \) reaches halfway to the decision state \( \vec{s}^\ast \) along the dimension \( \hat{s}^\ast = (\vec{s}^\ast - \vec{s}_0)/|\vec{s}^\ast - \vec{s}_0| \).

As expected, our simulations show a speed–accuracy tradeoff as we vary the overall connection strength \( \mu \), shown in Figure 2. When tuned to a given decision timescale, the accuracy is largely unaffected by network structure. The highest accuracy is observed when the system supports long timescale dynamics.

Given that performance is largely controlled by the decision timescale \( t_D \), we would like to understand how the network structure, defined by the adjacency matrix \( A \), controls \( t_D \). We expect that the largest timescales will occur near the symmetry-breaking transition that creates the two decision states.

Locating the transition and decision states

First, we must locate the relevant pitchfork bifurcation, which controls the transition between dynamics in which node states are not correlated over long times (when \( \mu \) is small and interactions between nodes are weak) into dynamics in which a nodes can collectively store a long-term memory (when \( \mu \) is large enough that interactions support a self-reinforcing consensus state). With zero input and zero noise, it is straightforward to find this transition, by analyzing how a small perturbation \( \delta \vec{s} \) to the initial state \( \vec{s}_0 \)

\(^3\) In the case here with \( \vec{s}_0 = 0 \), the two decision states are related simply by an inversion symmetry: \( \vec{s}_+^\ast = -\vec{s}_-^\ast \). Sufficiently close to the bifurcation, we expect analogous results for the speed–accuracy tradeoff even in more complicated cases where this symmetry does not hold.\(^3\)
Figure 3: A local approximation for the location of decision state fixed points. Analytical approximations are compared to the simulated norm of the steady-state vector versus coupling strength. A local approximation using only the eigenvalue $\lambda_c$ and participation ratio $p$ [Eq. (5)] is shown as a dashed line. The numerical solution of the zero input, zero noise case [Eq. (1) with $I = \sigma^2 = 0$] is shown as a solid line. The average final state in simulations, including noise but no input, is shown as colored points, with errorbars (not visible for most points) corresponding to standard deviation of the mean.

The distance between each stable fixed point (decision state) and the unstable starting point $\vec{s}_0$ grows as a function of $\mu$: Near the transition [see Appendix Eq. (8)],

$$|| \vec{s}^* - \vec{s}_0 || \approx \sqrt{3\Delta \mu \lambda_c p} = \sqrt{3\bar{\mu}p},$$

where $\Delta \mu = \mu - \mu_c$, $\bar{\mu} = \Delta \mu / \mu_c$ is the reduced distance from the transition, and $p = 1/|\langle \hat{e}_c \rangle^4|$ characterizes the distributedness of the eigenvector $\hat{e}_c$ corresponding to the leading eigenvalue $\lambda_c$.

The value $p$ sets the scale of the distance between the collective decision states, and it corresponds roughly to the number of individual nodes contributing to the mode (see the bottom row of Figure 1). In general,
Figure 4: **Predicting the timescale of the decision.** Critical slowing down increases the timescale of motion toward the decision state (top row) when the coupling strength $\mu$ is tuned near the symmetry-breaking bifurcation. Simulation results are shown as colored points, and an analytical approximation of $t_D$ [Eq. (13)] is shown as dashed and solid black lines. The approximate maximum timescale $t_D^{\text{max}}$, expected to set an upper limit on accuracy, is proportional to $\sqrt{p}$. Accuracy also peaks near the transition (bottom row).

$p$ varies between 1 for a completely localized mode and $N$ for a completely delocalized mode (produced, for example, by homogeneous all-to-all coupling). The inverse of $p$ appears in studies of localization in random matrix theory, where it has been called the “inverse participation ratio” [41, 42]; we therefore call $p$ the participation ratio.

Figure 3 compares this zero-noise local approximation to the zero-noise numerical solution for the fixed point $\mathbf{s}^*$ and to the final state of the simulation including noise. For each network, the transition occurs at the expected $\mu_c$ and with the expected local dependence on $\Delta \mu$. Because the largest timescales also occur near the transition, this local analysis will allow us to approximate the maximal decision timescale in the next section.

**Predicting the timescale of the decision**

In the absence of noise, the timescale of the decision is expected to diverge at the transition, because the fixed point at the origin becomes marginally stable. With noise, this is smoothed out in a predictable way,

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1. Noise also affects the location of the transition. In our simulations here we use a small noise parameter $\sigma$ (with input signal $I$ even smaller to achieve a small signal-to-noise ratio); this means the effect of noise on the transition location is minimal on the scales we test. We calculate the lowest-order correction to $\mu_c$ in the Appendix and find that it is on the order of $10^{-3}$ for the plotted cases.

2. Note that, in the rich club network, increasing the coupling beyond the scale shown in Figure 3 can also create bistability in the peripheral nodes. These cases have four stable fixed points, two of which correspond to the core and periphery nodes coming to consensus on conflicting decisions, and two in which core and periphery disagree. In our current setup, these cases do not change our analysis because the core always decides first, biasing the remainder of the system.
leading to a simple equation for the timescale derived in the Appendix, Eq. (13). Roughly, the timescale is
determined by a combination of the distance between the two decision states (proportional to $\sqrt{\Delta \mu}$),
the characteristic timescale of exponential growth away from the unstable fixed point (proportional to $\Delta \mu^{-1}$),
and the characteristic speed of motion due solely to noise (proportional to $\sigma$).

In Figure 4, we demonstrate that the decision timescale is well-approximated by Eq. (13). As we saw before
in Figure 2, longer timescales correspond to better accuracy. Further, this analysis allows us to predict the
maximal decision timescale supported by a given network under a given level of noise $\sigma$; we find [see
Appendix Eq. (14)]

$$t_t^{\text{max}} \propto \sqrt{p}/\sigma;$$

that is, the timescale supported by the collective mode scales with the square root of the participation
ratio. Consequently, due to the fundamental speed–accuracy tradeoff, $p$ becomes a useful quantification
of higher order network structure that sets a limit on collective decision accuracy.

Discussion

We study here a collective decision process that relies on the phenomenon of critical slowing down, a mech-
anism for creating long-timescale dynamics. In the case of small signal-to-noise ratio, decision accuracy is
limited by the timescale of collective dynamics, and the system must be tuned near a symmetry-breaking
bifurcation to successfully integrate information into an accurate decision. Varying the distance from the
bifurcation $\Delta \mu$ traces out a speed–accuracy tradeoff (Figure 2).

In the spirit of quantifying collectivity, our aim is to characterize the aspects of network connectivity that
control this timescale and therefore place limits on collective decision accuracy. We find that the most
important factors characterize the normal mode of the network that is least stable. This is the mode that
first becomes unstable as interaction strengths are increased, thereby leading to bistability that encodes a
binary decision. First, the leading eigenvalue $\lambda_c$, effectively a measure of the connectivity of individuals
participating in the mode, sets the scale of the critical coupling $\mu_c$ required for reinforcing the decision
state. Second, the participation ratio $p$ of the corresponding eigenvector is a measure of the number of
individuals participating in the mode.

Our main result is to identify $\lambda_c$ and $p$ as important measures for quantifying collective behavior in het-
erogeneous networks. Given any detailed network structure, these two simple statistics encapsulate the
network’s ability to create long-timescale dynamics. This allows us to predict how collective timescales
behave across a variety of network structures. For instance, the maximal decision timescale that can be
produced by the critical slowing mechanism increases in a predictable way as more individual nodes are
allowed to participate in the unstable mode, scaling as $\sqrt{p}$. This fits with our rough intuition, as we expect
that the effects of noise will shrink in a group of $N$ individuals as $1/\sqrt{N}$.

Generally, our results resonate with recent studies that focus on low-dimensional collective modes control-
ling the most important aspects of distributed computation in biological networks [43,44]. The importance
of the principal eigenvalue and corresponding (inverse) participation ratio hints at connections between
our model of decision-making and related characterizations of disease spreading [45], correlations in fi-
nancial data [41], and Anderson localization in condensed matter physics [46].

Our motivation began with understanding the functional consequences of rich-club structure and criti-
cality in the brain. These results allow us to speculate about fundamental tradeoffs: What are potential
advantages and disadvantages to hierarchical rich-club structure? On the one hand, more distributed con-
Connectivity may be advantageous in that it leads to more distributed collective modes, longer timescales, and therefore better averages over the noisy knowledge of individuals. On the other hand, the localized modes created by a rich club structure could be advantageous for modularized function and localized control. In this way, the rich club could be a way to bring only a subset of the system supercritical, with consequently reduced noise-reduction benefits of collectivity.

We expect this framework and intuition to be useful in systems in which the interaction structure remains fixed over the timescale of a single decision process, but may vary over longer adaptive timescales. Besides neural dynamics, such a framework may be useful for describing genetic regulatory networks producing cell fate decisions during development [47], social networks producing consensus about dominance hierarchies [48], and networks of influence underlying decisions by political bodies [49][50]. In such systems, the computation of decisions happens on relatively fast timescales, while on longer adaptive timescales, there may be tuning of the network that could change the relevant parameters $\lambda_c$ and $p$.

To guide our intuition, our analysis focused on the simplest symmetry-breaking bifurcation, where the initial state of each individual is the same ($\vec{s}_0 = \vec{0}$). It will be useful in future work to focus on more complicated transitions (as explored in [38]), where we expect that differing states and therefore saturations across individuals will modify the calculation, perhaps leading to a generalized form of the participation ratio that weights individuals by their contributions.

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**Appendix**

**Derivation of distance between stable fixed points**

The normal form of a system undergoing a pitchfork bifurcation is

$$\frac{d\nu}{dt} = a\nu + b\nu^3/6.$$  \hspace{1cm} (7)

In a one-dimensional system with state $x$ and dynamics $dx/dt = F(x)$ that has a pitchfork bifurcation at $x = x_0$, the system is described by Eq. (7) near $x_0$, with $\nu = x - x_0$, $a = dF(x)/dx|_{x=x_0}$, and $b = d^3F(x)/dx^3|_{x=x_0}$. This is the Taylor series of $F(x)$ at $x = x_0$ up to third order, where the second-order term disappears due to the symmetry that is required for a pitchfork bifurcation: $F(x_0 + \delta) = -F(x_0 - \delta)$ near $\delta = 0$. The bifurcation happens when $a$ changes sign. We focus here on the case that creates two stable fixed points (decision states), which coincides with $b < 0$.

Solving Eq. (7) for $d\nu/dt = 0$, we find one fixed point at $\nu^* = \nu_0$ that changes from stable when $a < 0$ to unstable when $a > 0$, and two stable fixed points when $a > 0$ at

$$\nu^* = \pm \sqrt{6|a|/|b|}.$$  \hspace{1cm} (8)
For example, in a simple one-dimensional case where \( F(x) = -x + \mu \tanh x \), we have \( \nu = x \), \( x_0 = 0 \), \( a = \mu - 1 \), and \( b = -2\mu \). Inserting into Eq. (6), we find, for small \( \Delta \mu \equiv \mu - 1 \), \( \nu^* \approx \pm \sqrt{3}\Delta \mu \).

In the higher-dimensional context of Eq. (1), \( \nu \) becomes the linear combination of state \( \mathbf{s} \) along the dimension of the least-stable dimension \( \hat{e}_c; \nu = \mathbf{s} \cdot \hat{e}_c \). Then, to produce the Taylor series corresponding to Eq. (7), we take the relevant directional derivatives of the right-hand side of Eq. (1). Calling the zero-noise, zero-input part of the dynamics \( \bar{F} \) [that is, \( F_i(\mathbf{s}) = -s_i + \mu \sum_j A_{ij} \tanh(s_j) \)], we have

\[
\begin{align*}
a &= \hat{e}_c \cdot \frac{\partial \bar{F}(\mathbf{s})}{\partial \nu} |_{\mathbf{s}=0} = \hat{e}_c \cdot (\nabla F \cdot \hat{e}_c) |_{\mathbf{s}=0} = \mu \hat{e}_c^T A \hat{e}_c - 1 = \mu \lambda_c - 1; \\
b &= \hat{e}_c \cdot \frac{\partial^2 \bar{F}(\mathbf{s})}{\partial \nu^2} |_{\mathbf{s}=0} = \hat{e}_c \cdot (\nabla (\nabla F \cdot \hat{e}_c) \cdot \hat{e}_c) |_{\mathbf{s}=0} = -2\mu \lambda \sum_i (\hat{e}_c)_i^4. \tag{9}
\end{align*}
\]

Inserting this into Eq. (8) produces

\[
\nu^* = \pm \sqrt{3\mu - \mu_c} = \pm \sqrt{3\mu}\bar{\nu} + O(\Delta \mu^{3/2}), \tag{11}
\]

where \( \mu_c = 1/\lambda_c \), \( p = 1/\sum_i (\hat{e}_c)_i^4 \), \( \Delta \mu = \mu - \mu_c \), and \( \bar{\nu} = \Delta \mu/\mu_c \).

We note that the above result assumes that the adjacency matrix \( A \) is symmetric. In the asymmetric case,

\[
\nu^* \approx \sqrt{3\bar{\nu}\frac{\lambda_c}{(A^T \cdot \hat{e}_c) \cdot (\hat{e}_c)^T}}, \tag{12}
\]

where \( \hat{e}_c \) is the normalized eigenvector of \( A \) corresponding to \( \lambda_c \).

**Derivation of approximate decision timescale** \( t_D \)

We define the decision timescale \( t_D \) as the time for \( \nu = |\mathbf{s} \cdot \hat{\mathbf{s}}^*| \) to reach halfway to the fixed point \( \nu^* \). To approximate \( t_D \) in the presence of noise, we patch together two types of behavior. First, for sufficiently small times \( t \), we expect the average behavior along \( \hat{e}_c \) to be dominated by noise [the \( \xi \) term dominates in Eq. (1)]. Noise dominates here because the system is still close to the fixed point at the origin, where, at the bifurcation, the first two terms cancel up to second order in \( \nu \). Neglecting all terms other than the noise term, the average behavior is given by \( \langle \nu_1(t) \rangle = \sigma \sqrt{1/\tau} \). Then, after a crossing time \( t_{\text{cross}} \), the first two terms dominate and noise becomes unimportant. Now neglecting the noise term, given an initial condition of \( \nu_0 = \langle \nu_1(t_{\text{cross}}) \rangle \), and considering for simplicity only the lowest-order approximation of \( F \) near the unstable fixed point, the state simply grows exponentially: \( \langle \nu_2(t) \rangle = \nu_0 \exp(t - t_{\text{cross}})\bar{\nu}/\tau \).

We patch these two solutions together by defining \( t_{\text{cross}} \) as the time when their derivative matches:

\[
d(\nu_1(t))/dt |_{t=t_{\text{cross}}} = d(\nu_2(t))/dt |_{t=t_{\text{cross}}}. 
\]

Solving this produces \( t_{\text{cross}} = y\tau/\bar{\nu} \), where \( y \approx 0.3517 \) is the solution to \( 2 \exp y = y^{-1} \). Finally, we solve for \( t_D \), the time to reach \( \nu^*/2 \), as a function of the reduced distance from the transition \( \bar{\nu} \):

\[
t_D(\bar{\nu}) = \begin{cases} 3\bar{\nu} \frac{\sqrt{\pi y}}{2\gamma^2} & \bar{\nu} < \bar{\nu}_{\text{cross}}, \\ \frac{\tau}{2\bar{\nu}} \left(2y + \log \frac{3\bar{\nu}^2}{4\gamma y^2}\right) & \bar{\nu} \geq \bar{\nu}_{\text{cross}}, \end{cases} \tag{13}
\]

where \( \bar{\nu}_{\text{cross}} = 2\sigma\sqrt{y/3p} \). This approximation of the decision timescale is plotted in Figure 4 as dashed
and solid lines (dashed for $\bar{\mu} < \bar{\mu}_{\text{cross}}$ and solid for $\bar{\mu} > \bar{\mu}_{\text{cross}}$). The function $t_D(\bar{\mu})$ has a maximum at $t_{D,\text{max}} = \bar{\mu}_{\text{cross}} \exp(1 - y)$, producing the maximal decision timescale for a given transition as

$$t_{\text{D,\text{max}}} = \frac{\tau z \sqrt{3\rho}}{2\sigma},$$

where $z = e^{y-1}/\sqrt{y} \approx 0.882$.

### Impact of noise on the critical point

In order to assess the general impact of noise on the critical coupling strength $\mu_c$, we consider a mean-field approach (fully connected graph) in the absence of an external signal ($I = 0$). The general approach employed here is analogous to the one used in [51, 52], where a more detailed account can be found. The mean-field stochastic differential equation corresponding to Eq. 1 reads:

$$\frac{ds}{dt} = -\frac{s}{\tau} + \frac{\bar{\mu}}{\tau} \langle \tanh s \rangle + \xi$$

(15)

Here $\langle \tanh s \rangle$ represents the expectation value of the interaction term, and $\bar{\mu} = N\mu$ is the mean field coupling strength scaled by the number of nodes $N$. Assuming $s \ll 1$, we use the Taylor expansion $\tanh x \approx x - x^3/3$ to rewrite the interaction of the individual node with the mean field in terms of the first and third moment of $s$:

$$\frac{ds}{dt} = -\frac{s}{\tau} + \frac{\bar{\mu}}{\tau} \left( \langle s \rangle - \frac{1}{3} \langle s^3 \rangle \right) + \xi$$

(16)

From the above stochastic differential equation we can derive the following nonlinear Fokker-Planck equation for the probability density function (PDF) $p(s, t) = \langle \frac{1}{N} \sum_j \delta(s - s_j) \rangle$:

$$\tau \partial_t p(s, t) = -\partial_s \left\{ \left[ -s + \bar{\mu} \left( \langle s \rangle - \frac{1}{3} \langle s^3 \rangle \right) \right] p(s, t) \right\} + \frac{\sigma^2}{2} \partial_s^2 p(s, t)$$

(17)

Here, the crucial simplifying assumption in the derivation is that the $N$-particle distribution function factorizes, i.e. that the correlations between nodes can be neglected (mean-field ansatz).

Inserting Eq. [17] into

$$\partial_t \langle s^n \rangle = \partial_t \int_{-\infty}^{\infty} s^n p(s, t) ds = \int_{-\infty}^{\infty} s^n \partial_t p(s, t) ds$$

(18)

produces a hierarchy of coupled evolution equations for the different moments $\langle s^n \rangle$ of the PDF.

We rewrite the state variable as $s = u + \delta s$, where $u$ is the average state of the system and $\delta s$ is the fluctuation around the mean, and assume that $\langle \delta s^k \rangle = 0$ for all odd $k$ ($k = 1, 3, 5, \ldots$). This allows us to express the first three moments of $p(s, t)$ as:

$$\langle s \rangle = u$$

(19)

$$\langle s^2 \rangle = u^2 + \langle \delta s^2 \rangle$$

(20)

$$\langle s^3 \rangle = u^3 + 3u\langle \delta s^2 \rangle$$

(21)
In the following, we will use the notation \( T = \langle \delta s^2 \rangle \) for the variance of the fluctuations. It can be viewed as an effective “temperature” quantifying the intensity of the fluctuations around the mean.

Eventually combining equations \[17\] \[18\] and \[19\] we arrive at the following evolution equations for the mean \( u \) and the temperature \( T \):

\[
\begin{align*}
\partial_t u &= \frac{1}{\tau} \left[ \bar{\mu} - 1 - \frac{1}{3} (u^2 + 3T) \right] \\
\partial_t T &= \frac{1}{\tau} (\sigma^2 - 2T)
\end{align*}
\]

The corresponding stationary solutions can be easily obtained by solving the above equations for \( \partial_t u = \partial_t T = 0 \). The stationary temperature \( T^* \) is

\[
T^* = \frac{\sigma^2}{2}
\]

The cubic equation for the mean yields three stationary solutions \( u^* \), which correspond directly to the fixed points \( \nu^* \) discussed in the context of the general pitch-fork bifurcation above:

\[
\begin{align*}
u_1^* &= 0 \\
u_{2,3}^* &= \pm \sqrt{\frac{3}{\bar{\mu}} (\bar{\mu} - 1) - \frac{3\sigma^2}{2}}
\end{align*}
\]

Here, the \( u_1^* \) corresponds to the disordered solution below a critical coupling strength. If the coupling strength becomes too large, then this disordered solution becomes unstable. In the absence of an external signal (bias), we observe a spontaneous symmetry breaking, where \( u_{2,3}^* \) correspond to the two possible solutions, identical with the two different branches of the pitchfork. These two stationary solutions exist only if the argument in the square root is positive (as \( u^* \in \mathbb{R} \)).

For vanishing noise, \( \sigma^2 = 0 \), the critical mean-field coupling strength is \( \bar{\mu}_c = 1 \), which is consistent with our previous results if we consider a fully connected graph. For small noise \( \sigma \ll 1 \) the critical point is modified according to:

\[
\bar{\mu}_c = \frac{1}{1 - \frac{\sigma^2}{2}} \approx 1 + \frac{\sigma^2}{2}
\]

Thus, introducing noise leads effectively only to a shift of the critical coupling strength to larger values, without any qualitative change regarding general result obtained for the zero noise case. Furthermore, the shift is small for the regime we test in this study: with \( \sigma = 0.05 \) we expect corrections to \( \mu_c \) on the order of \( 10^{-3} \).

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