Electric-magnetic duality and topological order on the lattice

Oliver Buerschaper,1 Matthias Christandl,2,3 Liang Kong,4,5 and Miguel Aguado1

1Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, D-85748 Garching, Germany
2Fakultät für Physik, Ludwig-Maximilian-Universität, Theresienstraße 37, D-80333 Munich, Germany
3Institute für Theoretische Physik, Eidgenössische Technische Hochschule Zürich, Wolfgang-Pauli-Straße 27, CH-8093 Zürich, Switzerland
4California Institute of Technology, Center for the Physics of Information, Pasadena, CA 91125, USA
5Institute for Advanced Study (Science Hall), Tsinghua University, Beijing 100084, China

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Duality ranks among the deepest ideas in physics. Dualities are deep conceptual tools in many areas of physics and mathematics; in particular, electric-magnetic (EM) duality has a long tradition ranging from Dirac’s magnetic poles to S-duality in string and field theory. Here we investigate the duality structure in quantum lattice systems with topological order, a collective order also appearing in fractional quantum Hall systems. We define EM duality for all of Kitaev’s quantum double models based on discrete gauge theories with Abelian and non-Abelian groups, and identify its natural habitat as a new class of topological models based on Hopf algebras. We interpret these as extended string-net models, whereupon Levin and Wen’s string-nets, which describe all topological orders on the lattice with parity and time-reversal invariance, arise as magnetic and electric projections of the extended models. We conjecture that all string-net models can be extended in an analogous way, using a more general algebraic structure, such that EM duality continues to hold. Physical applications include topology probes in the form of pairs of dual tensor networks.

Duality ranks among the deepest ideas in physics. Dualities are powerful probes into the internal structure of mathematical and physical theories, including quantum many-body systems and field theories [1]. Frequently a duality relates weakly and strongly coupled regimes of physical systems, providing precious information beyond the reach of perturbative methods. Many physical instances of duality involve the exchange of electric and magnetic degrees of freedom. Such a symmetry for the Maxwell equations in vacuo led Dirac [2] to the introduction of pointlike sources of magnetic field to extend this electric-magnetic (EM) duality to matter, providing a unique argument for the quantisation of electric charge. Dirac’s insight has had profound influence in field theory and beyond (see, e. g., [3]).

Topological order [4] appears in a fascinating class of condensed matter phases, notably the fractional quantum Hall effect. Topological phases are sensitive only to global properties of the underlying space: their effective quantum field theories have only non-local degrees of freedom [5]. This non-local order has deep connections to high-energy physics and a rich mathematical structure. In 2D, it is associated with localised objects with fractional statistics, that is to say, anyons [6]. In topological lattice models, these anyons appear as excitations.

Kitaev’s proposal [7] to use topologically ordered quantum lattice systems to store and process quantum information linked topological order and quantum computation. His pioneering quantum double D(G) models are built in analogy to gauge theories with finite gauge groups. In this context, the topological degrees of freedom are Wilson loops; violations of gauge invariance and magnetic fluxes are described analogously, hinting at EM duality. Particularly simple are the models based on Abelian groups, including the toric code, the first laboratory for many ideas in topological order. Although there are deeper objects at play, D(G) models can be conveniently studied using group theory; there is an explicit local understanding of anyonic excitations and their interplay in terms of group representations. These models provide us with the first emergence of duality in topologically ordered systems, namely a self-duality of the toric code and Abelian D(G) models, involving the direct and dual lattices; an EM duality, in gauge theory language. Its importance has been stressed by Fendley [9].

Levin and Wen defined their landmark string-net (SN) models [8] from the physical intuition that topological phases are renormalisation group fixed points. SNs are built on ideas from category theory to describe all topological orders on the lattice with parity and time-reversal invariance. Technically, they use data from unitary tensor categories; in this spirit, dualities in SN models, in the sense of equivalences of unitary modular tensor categories, can be identified with invertible domain walls [10]. While SN models are broader in scope than Kitaev’s quantum double models, their excitations are, however, not understood as thoroughly as those of the D(G) models, especially locally.

So far the extension of the EM duality of the toric code to non-Abelian models has remained an open problem, and thus a key source of insight into the structure of topological order has been only available in the very restricted Abelian setting. The natural place to look for a well defined EM duality are Kitaev’s quantum double models, since the excitations there are clearly understood and can be assigned electric and magnetic quantum numbers, while this characterisation is absent in SN models.

Here we uncover EM duality for all of Kitaev’s D(G)’s,
The results lead us to propose the duality landscape of such a way from parent ESNs with a generalised algebraic language by identifying its natural setting as the quantum double of both Abelian and non-Abelian, and beyond. We do this by identifying its natural setting as the quantum double models $D(G)$, related by duality to quantum double models $D(G^*)$ based on algebras of functions on groups, whose intersection with $D(G)$ is $D(A)$; these are all instances of quantum double models $D(H)$ based on Hopf algebras, a larger class closed under duality, with cases of self-duality beyond groups [11]. We conjecture that EM duality can be defined for extended string-net models ESN, and that these can be identified as quantum double models $D(W)$ based on weak Hopf $G^*$-algebras. Arrows denote model identifications upon lattice dualisation.

**Kitaev’s Models Based on Groups**

Let us begin by reviewing the well-known self-duality in the toric code [7], the quantum double model based on the $\mathbb{Z}_2$ group. This is a model of qubits along the bonds of a $2D$ lattice $\Lambda$ (the dimension of the local Hilbert spaces being the size of the group), with a Hamiltonian of the form

$$\mathcal{H} = - \sum_v A_v - \sum_p B_p \ ,$$

where $v$ runs over vertices and $p$ runs over plaquettes of $\Lambda$. Mutually commuting vertex and plaquette operators are projectors involving Pauli matrices:

$$A_v = 1 + \otimes_{\epsilon_v} \sigma_i^\epsilon_v \ , \quad B_p = 1 + \otimes_{\epsilon_p} \sigma_j^\epsilon_p \ ,$$

with support on the bonds around the corresponding vertex or plaquette. Self-duality means that, upon a global unitary

$$U_{\Lambda} = \bigotimes_{i \in \Lambda} U_i \ , \quad U_i = \frac{(\sigma^x + \sigma^z)_i}{\sqrt{2}} \ ,$$

built from Hadamard maps on each edge $i$, the toric code on $\Lambda$ gets mapped to a toric code on the dual lattice $\Lambda^*$:

$$U_{\Lambda} A_v U_{\Lambda}^\dagger = \tilde{B}_v \ , \quad U_{\Lambda} B_p U_{\Lambda}^\dagger = \tilde{A}_p \ ,$$

where now $\tilde{B}_v$ corresponds to dual plaquette $v$, and $\tilde{A}_p$ to dual vertex $p$ in $\Lambda^*$. Ground states of the frustration-free Hamiltonian (1) minimise each term in the sum, that is, they satisfy all vertex and plaquette constraints $A_v |\psi\rangle = |\psi\rangle$, $B_p |\psi\rangle = |\psi\rangle$. Breakdown of any one such constraint means the state is in an excited level, and can be interpreted as the presence of a localised excitation, or particle, sitting at the vertex or plaquette that is broken; from the gauge theory interpretation, these are called electric and magnetic excitations. The electric-magnetic nature of duality (4) comes then from the interchange of vertices and plaquettes in going from the original to the dual lattice.

Hidden in the action of the global Hadamard map $U_{\Lambda}$ are, first, the mapping from $\Lambda$ to the dual lattice and the corresponding reinterpretation of vertex and plaquette operators, and second, a mapping of elements of the group $\mathbb{Z}_2$ to the functions from this group to the scalars. If we obtain again a toric code in the dual lattice it is because, for an Abelian group, the space of functions has the same structure as the group algebra, which is the space of linear combinations of group elements. One can exploit this algebraic fact to extend the self-duality of the toric code to all $D(G)$ models for Abelian groups in a straightforward way: this is the $D(A)$ region of figure 1.

For general $D(G)$ models, we start with a lattice with oriented bonds, whose Hilbert spaces have an orthonormal basis $\{|g\rangle\}$ labelled by elements of a finite group $G$. The Hamiltonian of the $D(G)$ model is still of the form (1), but now the mutually commuting vertex and plaquette operators are the projectors defined in figure 2. The obvious parallelism of plaquettes and vertices in the Abelian case is lost if $G$ is non-Abelian. Upon switching

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**Figure 1. A duality landscape.** We represent: Quantum doubles $D(A)$ based on Abelian groups, including the toric code, all of which are self-dual; general group QD models $D(G)$, related by duality to quantum double models $D(G^*)$ based on algebras of functions on groups, whose intersection with $D(G)$ is $D(A)$; these are all instances of quantum double models $D(H)$ based on Hopf algebras, a larger class closed under duality, with cases of self-duality beyond groups [11]. We conjecture that EM duality can be defined for extended string-net models ESN, and that these can be identified as quantum double models $D(W)$ based on weak Hopf $G^*$-algebras. Arrows denote model identifications upon lattice dualisation.
defines a set of operators on a lattice whose functions; this is shown as the arrow between the regions construction of the Abelian duality can be generalised as the group algebra itself. Hence, the extension of the on a non-Abelian group cease to have the same structure to the dual lattice, we cannot reinterpret the operators $A_v$ and $B_p$ as plaquette and vertex operators in a quantum double model based on a group, since the functions on a non-Abelian group cease to have the same structure as the group algebra itself. Hence, the extension of the EM duality to non-Abelian groups is not possible within the class of quantum doubles based on groups. Yet the construction of the Abelian duality can be generalised to the non-Abelian case, resulting in dual models which are no longer quantum double models based on groups, but rather quantum double models based on algebras of functions; this is shown as the arrow between the regions $D(G)$ and $D(G^*)$ in figure 1. To make sense of this, one has to widen the construction of quantum double lattice models beyond the group case.

As we will show, the natural habitat for the EM duality of $D(G)$ models is the class of quantum double models based on Hopf algebras [11]; this is the region $D(H)$ in figure 1. This is the smallest setting containing all $D(G)$ models and closed under tensor products and EM duality; the latter, moreover, takes a remarkably simple form in the language of Hopf algebras.

**THE $D(H)$ LATTICE MODELS**

When studying quantum many-body systems, more general notions of symmetry emerge than those furnished by group theory. In particular, linear transformations acting on tensor products of vector spaces lead naturally to Hopf algebras. For a detailed account of these in our context we refer to [11]. In the following we give an intuitive grasp of their structure, which is necessary to understand the $D(H)$ lattice models.

We consider Hopf algebras $H$ as spaces of transformations on a many-body Hilbert space. First of all, the linear nature of the target is naturally extended to its transformations, so Hopf algebras are vector spaces. We must be able to compose transformations and to include the identity transformation, so $H$ has a multiplication of vectors, and a unit. Most importantly, we must have a rule to *distribute* the action of an element of $H$ into a tensor product of target spaces; this is the so-called comultiplication. Additionally $H$ has a trivial representation $\varepsilon$, called counit, making precise the notion of spaces invariant under the action of $H$. Like for groups, the representation theory of Hopf algebras includes a notion of conjugate representation, implemented via an antipode mapping, which for groups is just the inversion $g \mapsto g^{-1}$.

In order to construct Hilbert spaces, we use Hopf $C^*$-algebras, which in addition have an inner product. We will only consider finite-dimensional Hopf $C^*$-algebras, which come equipped with a canonical, highly symmetric element, the *Haar integral* $h \in H$, invariant under multiplication in the sense $a \cdot h = \varepsilon(a) h$ for all elements $a$ of $H$. This canonical element is crucial for the construction of the lattice model and its ground states.

The root of the EM duality to be unveiled in the following is an algebraic fact: *The class of Hopf $C^*$-algebras is closed under dualisation*. That is, given a Hopf $C^*$-algebra $H$, its dual space $H^*$ of functions from $H$ to the scalars is again a Hopf $C^*$-algebra, whose structure is, moreover, determined by the structure of $H$, as illustrated by figure 3. This closure property is shared by the class of Abelian group algebras (which are all self-dual), but not for the whole class of group algebras. The landscape of figure 1 reflects these statements in the world of lattice models.

Figure 4 defines a set of operators on a lattice whose bonds carry elements of $H$, a finite-dimensional Hopf $C^*$-algebra. These operators depend on elements of the algebra and its dual, respectively. When these are taken as the Haar integrals $h$ of $H$ and $\phi$ of $H^*$, the resulting operators

$$A_v = A_{v,p}(h), \quad B_p = B_{p,v}(\phi)$$

(5)

define via (1) the Hamiltonian of a topological lattice model: the quantum double model based on $H$, or $D(H)$ model for short [11].

Kitaev’s models are a particular subclass of $D(H)$ models, recovered when $H$ is the group algebra of a finite group $G$. Both this algebra $CG$ and its dual $G^*$ have particularly simple Hopf $C^*$-algebra structures, summarised in figure 3. In particular, their Haar integrals read

$$h = \frac{1}{|G|} \sum_{g \in G} g \in CG, \quad \phi = \delta_e \in G^*.$$  

(6)
FIG. 3. Structure and dualities in Hopf algebras. The first row represents the structure maps of a Hopf algebra $H$. Solid lines represent elements of $H$, and dotted lines represent scalars in $C$. The second and third rows illustrate the algebraic duality within the class of Hopf algebras, taking as examples a group algebra $H = CG$ and its dual $H^*$, whose structure is defined implicitly in the figure. For each diagram, the upper expression is equal to the lower expression; the structure maps of $H$ and $H^*$ are represented by solid and dashed lines, respectively, according to the conventions of the first row. Hence, the multiplication in $H^*$ is determined by the comultiplication in $H$, the unit in $H^*$ is determined by the counit in $H^*$, etc.; this duality holds for general finite-dimensional Hopf algebras. We use the form of the structure maps in the basis \{g; g \in G\} of the group algebra, and in the basis \{\delta_\ell; \ell \in G\} of its dual, where $\delta_\ell: g \mapsto \delta_{\ell,g}$ are Kronecker delta functions. As for the extra structure needed to define our lattice models, the $C^*$-algebra structures of $H$ and $H^*$ are defined by declaring \{g\} to be an orthonormal basis, and \{$\delta_\ell$\} to be orthonormal up to a factor, $(\delta_\ell, \delta_m) = |G|^{-1} \delta_{\ell, m}$. The Haar integrals $h$ and $\phi$ in the group algebra and its dual are defined in equation (6).

Taking into account this equation, operators in figure 2, defining the Hamiltonian of the $D(G)$ model, can be checked to coincide with the general expression (5).

\begin{equation}
\begin{aligned}
A_{v, p}(u) &| a_2 \rangle \rangle = \sum_{\langle a_1 \rangle} \langle a_1 \rangle \langle \circlearrowright \circlearrowright | u^{(2)} a_2 \rangle, \\
B_{p, v}(f) &| a_2 \rangle \rangle = \sum_{\langle a_1 \rangle} f(a_1^{(2)} \ldots a_1^{(1)}) | a_1^{(1)} \rangle.
\end{aligned}
\end{equation}

FIG. 4. The core of $D(H)$ models. Topological models based on Hopf algebras are defined via these vertex and plaquette representations of $u \in H$ and $f \in H^*$ [28]. Oriented edges carry, in general, elements of a finite-dimensional Hopf $C^*$-algebra $H$; for Kitaev’s models, take $H$ to be the group algebra $CG$. Orientation can be reversed by using the antipode map $S$ referred to in the text. The sums denote the appropriate comultiplication or splitting of algebra elements into several tensor factors. Vertex operators $A_{v, p}$ now depend also on a plaquette $p$ marking the beginning of a loop around the vertex $v$, and correspondingly for the plaquettes. These operators probe the algebraic structure of the models: Given $v$ and $p$, $A_{v, p}(u)$ and $B_{p, v}(f)$ interact nontrivially, building representations of Drinfeld’s quantum double algebra $D(H)$ [14]. The irreducible representations of $D(H)$ classify the superselection sectors, and thus the anyonic structure of excitations, of the quantum double lattice models [7].

**ELECTRIC-MAGNETIC DUALITY**

We now define EM duality for general $D(H)$ models. Consider the following unitary map $U$ from $H$ to its dual:

\begin{equation}
\begin{aligned}
a &\mapsto U a, \\
f_a(b) &= \sqrt{|H|} \phi(a \cdot b),
\end{aligned}
\end{equation}

where $|H|$ is the dimension of $H$, and the function $f_a$ is constructed via the dual Haar integral $\phi \in H^*$ (for instance, for groups, $f_g = \sqrt{|G|} \delta_{g^{-1}}$). We associate this mapping with the transformation of $\Lambda$ into its dual lattice $\Lambda^*$ as shown in figure 5.

The vertex and plaquette representations in figure 4, associated with the $D(H)$ model on $\Lambda$, are mapped by $U_\Lambda = \bigotimes_{\ell \in \Lambda} U_\ell$ into precisely the representations associated with the $D(H^*)$ model on the dual lattice $\Lambda^*$ [29]:

\begin{equation}
\begin{aligned}
U_\Lambda B_{p, v}(f) U_\Lambda^\dagger &= \tilde{A}_{p, v}(f), \\
U_\Lambda A_{v, p}(u) U_\Lambda^\dagger &= \tilde{B}_{v, p}(u),
\end{aligned}
\end{equation}

where $p$ is a plaquette of $\Lambda$ and a vertex of $\Lambda^*$; this expression generalises equation (4) and identifies $U_\Lambda$ as the EM duality mapping.

Thus, $U_\Lambda$ identifies the $D(H)$ model on $\Lambda$ with the $D(H^*)$ model on $\Lambda^*$, as it transforms the Hilbert spaces and Hamiltonian of the former into those of the latter. We write this symbolically as $D(H)|_\Lambda = D(H^*)|_{\Lambda^*}$.

But EM duality is deeper than a mere identification of energy levels. The Hamiltonian involves just the opera-
tors $A_v(h)$ and $B_v(\phi)$, but, according to (8), $U_{\Lambda}$ induces a transformation of all operators of figure 4 that respects their algebraic structure, and relates the anyonic excitations of both models [13].

FROM QUANTUM DOUBLES TO STRING-NETS

We now make the connection with Levin and Wen’s string-net models [8]. In [12] the $D(G)$ model was identified, via a Fourier mapping, with an extended string-net model on the same lattice, with bond degrees of freedom given by irreducible representations (irreps) of $G$ and auxiliary matrix indices depending on each irrep (see also [15]). String-net vertex conditions amount to knitting the matrix indices together by means of $3j$ symbols, effectively mapping the local Hilbert spaces isometrically to those of a Levin-Wen SN model, whose degrees of freedom and underlying category theoretical structure are given solely by irreps of $G$ and their fusion properties. The ground levels of SN and their extended versions are identical.

Now the representation theory of finite Hopf $C^*$-algebras is essentially identical to that of finite groups (the only real difference being that the fusion of representations need not be commutative). Therefore the construction of [12] can be generalised immediately to any $D(H)$ model. That is, if $H$ has matrix irreps $D^{\alpha}$ of dimension $|\mu|$, we define a basis in $H$ as

$$[\mu ab] = \sqrt{|H||\mu|} \sum_{(h)} D^{\alpha}_{ab}(h^{(1)}) h^{(2)},$$

where $a, b = 1, \ldots, |\mu|$ are matrix indices for irrep $D^{\alpha}$. This is an orthonormal basis for each bond, and the analysis of [12] carries through intact to show that the $[D(H)]_{\Lambda}$ model can be written as an extended string-net model $[ESN_H^m]^*_{\Lambda}$. The latter reduces to a string-net model $[SN^m_{H}^*]_{\Lambda}$, whose degrees of freedom are the irreps of $H$, and whose ground level is identical to that of the extended model.

The same construction, as applied to the $[D(H^*)]_{\Lambda}$ model using the Fourier basis for $H^*$, yields an equivalent extended string-net $[ESN_{H^*}]_{\Lambda^*}$ on the dual lattice. This, in turn, reduces to another string-net model $[SN_{H^*}]_{\Lambda^*}$ on the dual lattice, whose bond degrees of freedom are irreps of $H^*$. We regard the two string-net models $[SN^m_{H}]_{\Lambda}$ and $[SN^m_{H^*}]_{\Lambda^*}$ as the magnetic and electric projections of the original quantum double model (see figure 6). By construction, they satisfy $[SN^m_{H}]_{\Lambda^*} = [SN^m_{H^*}]_{\Lambda^*}$.

For instance, in [12] the magnetic SN projection of the $D(G)$ model was analysed. Its electric SN projection, on the other hand, has the same local degrees of freedom as the $D(G)$ model, since group elements are the (one-dimensional) irreps of the dual of the group algebra [30].

Thus, EM duality can be given a meaning in the subclass of string-nets obtained by reduction of ESNs. This leads us to the following

Conjecture: All SN models can be extended into quantum double models based on a sufficiently general algebraic structure, the class of weak Hopf $C^*$-algebras [16][31]. EM duality can be defined for all extended SN models so that the pattern of figure 6 holds.

PROBING TOPOLOGY WITH TENSOR NETWORKS

Some physical consequences of EM duality are immediate. First of all, EM duality allows us to probe the topology of the surface underlying the lattice models by using only locally defined states.

As shown in [11], each $[D(H)]_{\Lambda}$ model has one canonical tensor network ground state $|\psi(H, \Lambda)\rangle$ constructed from identical tensors at each site, which are defined solely by the structure of $H$.

From the corresponding canonical state $|\psi(H^*, \Lambda^*)\rangle$ of the dual $[D(H^*)]_{\Lambda^*}$ model on the dual lattice we can obtain another ground state of the original $[D(H)]_{\Lambda}$ model by using the duality map (7):

$$|\tilde{\psi}(H, \Lambda)\rangle = U_{\Lambda}^\dagger |\psi(H^*, \Lambda^*)\rangle. \quad (10)$$

The relation between $|\psi(H, \Lambda)\rangle$ and $|\tilde{\psi}(H, \Lambda)\rangle$ depends on the topology of the surface underlying $\Lambda$. On the sphere, for instance, the ground level is nondegenerate, so both ground states are the same. On a topologically nontrivial surface such as the torus, these ground states can be shown to always be linearly independent.

For instance, in the toric code the tensor network construction of the canonical state $|\psi(H, \Lambda)\rangle$ coincides with the projected entangled-pair state (PEPS) ansatz in [18]. On the torus this yields the logical $|++\rangle$ state; the dual ground state $|\tilde{\psi}(H, \Lambda)\rangle$ is the logical $|00\rangle$ state. See [13]
DISCUSSION AND OUTLOOK

We have defined EM duality for non-Abelian D(G) models, showing how it arises naturally in the context of D(H) models. The connection to SN models comes from the Fourier construction of extended string-nets and their electric and magnetic projections. We conjecture that all SN models have extended versions, and that EM duality can be defined for all the extended models. EM duality offers nonlocal information about the systems, unveiling global characteristics of space, from tensor networks defined locally. Beyond this, it serves as an organizing principle for topological models; indeed, we envisage a net of dualities for topological phases (not unlike that connecting theories of strings and branes). This then reflects on the field theories underlying topological systems and on their mathematical structures, such as category theory.

We wish to stress the importance of the whole class of models based on Hopf algebras. This is indeed the natural class of models to study to understand electromagnetic duality for lattice models based on non-Abelian groups. We thus shift the focus away from lattice gauge theories based on groups, which are a primary concern in high energy theory, even when the Hopf algebra language is applied [24]. Ours is, moreover, the first proposal for a general EM duality governing topologically ordered systems.

The Hopf algebraic language makes the EM duality transparent and easy to grasp, instead of obscuring it; in addition, this language can be learnt here hands-on from the standpoint of quantum many-body systems, which can even be built in a laboratory. We also provide, with the Fourier construction, a way to understand some aspects of category theory in terms of representations, more familiar to many physicists.

Our next goal is defining the extended string-nets for all SN models [13], casting ESNs and their dualities in the language of categories. The nature of the anyonic models controlling the excitations of the D(H) models has a bearing on the experimental realisation of string-net models, which we contend would be most natural in their extended incarnations (protocols for anyonic manipulation in the spirit of [25] and [26] would then be possible). A key question to be explored is the meaning of EM upon perturbation of the models considered, that is, away from topological fixed points.

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On the other hand, the correspondence between string-net and quantum double models can be used to relate these PEPS to the tensor network descriptions of string-net ground states put forward in [20] and [21]. These come from the $|\psi(H, \Lambda)\rangle$ construction of the correspond-
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[27] The fact that anyon models in string-net models correspond to representations of algebraic structures was anticipated in Levin and Wen’s original paper [8].
[28] Technically, the $A_{v, p}(u)$ are representations of $H$, and the $B_{p, v}(f)$ are representations of a variant of the dual; namely, the dual $(H^{op})^*$ of the opposite algebra of $H$ (where the multiplication is flipped).
[29] Technically, representations $A_{v, p}(u)$ of $H$ and $B_{p, v}(f)$ of $(H^{op})^*$ act on bonds of $\Lambda$, that is, spaces of the form $H^{op_n}$. They get mapped by $U_{\Lambda}$ to representations $\tilde{A}_{v, p}(u)$ of $((H^{*})^{op})^*$ and $\tilde{B}_{p, v}(f)$ of $H^*$ on bonds of $\Lambda^*$, that is, spaces of the form $(H^*)^{op_n}$. The latter, for coinciding $v$ and $p$, define representations of the $D(H^*)$ algebra, building the $D(H^*)$ model on $\Lambda^*$.
[30] This connection of SN and $D(G)$ models had been recognised long ago by Héctor Bombín.
[31] The excitations in such models would be classified by representations of the double $D(W)$ of the weak Hopf $C^*$-algebra $W$, which is expected to be a weak quasi-Hopf algebra in the sense of [17].