Simple supersymmetric methods in neutron diffusion

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We present the supersymmetric Witten and double Darboux (strictly isospectral) constructions as applied to the diffusion of thermal neutrons from an infinitely long line source. While the Witten construction is just a mathematical scheme, the double Darboux method introduces a one-parameter family of diffusion solutions which are strictly isospectral to the stationary solution. They correspond to a Darboux-transformed diffusion length which is flux-dependent.

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Supersymmetric one-dimensional (1D) quantum mechanics has been considered by Witten in 1981 as a toy model for symmetry-breaking phenomena in quantum field theory [6]. With great speed its status has changed to a powerful research area as one can contemplate in the most recent review [7]. In a series of papers, we obtained interesting results for various physical problems by using Witten’s factorization procedure of the Schrödinger 1D operator and a more general supersymmetric double Darboux method [8] introduced by Mielnik [9]. The aim of this work is to apply these two simple, supersymmetric methods to the theory of diffusion of thermal neutrons.

We shall use the illustrative example of the neutron diffusion problem as presented in the textbook of Arfken [10] referring to an infinite line (Dirac delta) source of neutrons, thus providing the Green’s function for this case. The steady state continuity equation for the neutrons reads

$$D \nabla^2 \phi - \Sigma_a \phi + S = 0 ,$$

(1)

where the first term represents the diffusion, the second stands for the absorption losses and $S$ stands for the source strength. The diffusion constant $D$ is related to the neutron mean free path $\lambda_s$ as follows $D = \frac{\lambda_s}{3(1-\frac{2}{3}\lambda_s)}$, where $\lambda$ is the atomic number of the scattering nucleus and enters as a correction for the anisotropy of scattering in the laboratory system. The solution $\phi$ has the physical meaning of neutron flux being the product of neutron density times average velocity. Finally, $\Sigma_a$ is the macroscopic absorption cross section, i.e., the product of the microscopic (atomic) absorption cross section and the number of atoms per unit volume. Usually it is assumed that the absorption is small compared to the scattering. As we said, the neutron source is considered as an infinitely long line source that can be taken along the $z$-axis embedded in an infinite diffusing medium. Its strength is $S_0$ neutrons per unit length per unit time. Thus $S = S_0 \delta(r)$ where $\delta(r)$ is the cylindrical Dirac delta function. For $\rho \neq 0$, in view of the no $z$ and $\theta$ dependence, one gets the radial diffusion equation

$$\rho^2 \frac{d^2 \phi}{d\rho^2} + \rho \frac{d\phi}{d\rho} - \rho^2 k^2 \phi = 0 ,$$

(2)

where we have made use of the “diffusion length” $k^{-1} = \sqrt{D/\Sigma_a}$. Then one can write the general solution in terms of the modified Bessel functions $\phi = a_1 I_0(k\rho) + a_2 K_0(k\rho)$. The physical solution is only the $K_0$ term because the flux is supposed to decrease at large distances. The constant $a_2$ may be determined by requiring that $D$ times the integral of the negative gradient of the neutron flux around the edge of a small pillbox of unit height be equal to the production strength $S_0$ within the pillbox. The box is small ($\rho \to 0$) to eliminate absorption and gives

$$S_0 = \lim_{\rho \to 0} D a_2 \int [-\nabla K_0(k\rho) \cdot \rho_0] d\rho d\theta ,$$

(3)

which is a two-dimensional form of Gauss’s law. Using the series form of $K_0(k\rho)$ this turns into

$$S_0 = D a_2 \lim_{\rho \to 0} \frac{2\pi \rho}{\rho} ,$$

(4)

or

$$\phi = \frac{S_0}{2\pi D} K_0(k\rho)$$

(5)

for the final form of the common solution.

Let us pass now to the supersymmetric constructions. We have already presented these schemes in previous papers [3]. One needs a self-adjoint form of Eq. (2) that can be obtained by the change of function $\psi = \rho^{-1/2} \phi$ leading to

$$\psi'' - (k^2 - \frac{1}{4\rho^2}) \psi = 0 .$$

(6)

If Eq. (6) is interpreted as a Schrödinger equation at fixed zero energy with the potential $V_B(\rho) = k^2 - \frac{1}{4\rho^2}$, then one can think of supersymmetric quantum mechanical methods as follows.
(i). Witten’s construction has two steps. The first is the factorization of Eq. (6) by means of the operators \( A_1 = \frac{d}{d\rho} + W(\rho) \) and \( A_2 = \frac{d}{d\rho} - W(\rho) \), where \( W \) is the so-called superpotential function that can be determined from the initial (“bosonic”) Riccati equation \( V_B = W^2 - W' \), or even more directly as the negative of the logarithmic derivative of the stationary solution \( \psi (W = -\frac{d}{d\rho} \ln \psi) \). The second step of the Witten construction means to pass to a “fermionic” problem by merely changing the sign of the derivative term in the Riccati equation and determining the new potential \( V_F \), which is interpreted as a “fermionic” partner of the initial potential. Thus, \( V_F = W^2 + W' \). This partner potential enters a “fermionic” equation for which the factoring operators are applied in reversed order. On the other hand, the superpotential should be the negative of the logarithmic derivative of a “nodeless” solution, which is our case since modified Bessel functions of zero order occur. Moreover, the singularity at the origin of the superpotential function is not disturbing because we have an infinite line source there.

(ii). The double Darboux construction in the supersymmetric framework allows one to use the general solution \( W_{gen} \) of the “bosonic” Riccati equation and not the particular one as for the Witten construction. Thus, the factorization operators are now \( A_1 = \frac{d}{d\rho} + W_{gen} \) and \( A_2 = \frac{d}{d\rho} - W_{gen} \). In this way one can introduce a one-parameter family of “bosonic” potentials having the same “fermionic” partner, i.e.

\[
V_{iso} = V_B - 2\frac{d^2}{d\rho^2} \ln(\mathcal{I}(\rho) + \lambda),
\]

which is a general Darboux transform of \( V_B \). Eq. (7) can be written in the following form

\[
V_{iso}(\rho, \psi; \lambda) = V_B(\rho) - \frac{4\psi'\psi'}{\mathcal{I}(\rho) + \lambda} + \frac{2\psi^4}{(\mathcal{I}(\rho) + \lambda)^2},
\]

where \( \mathcal{I}(\rho) = \int_0^\rho \psi^2(r)dr \) and \( \lambda \) is the family parameter, which is a real positive quantity, being as a matter of fact the Riccati integration constant \( \mathcal{I} \). Besides, there is a modulational damping of the general solution, which reads

\[
\psi(\rho; \lambda) = \sqrt{\lambda(\lambda + 1)} \frac{\psi(\rho)}{\mathcal{I}(\rho) + \lambda},
\]

where the square root is a normalization constant. One can check easily that \( W_{gen} = -\frac{d}{d\rho} \ln \psi(\rho; \lambda) \). Some plots of the strictly isospectral potentials Eq. (8) and the isospectral solutions Eq. (9) are presented in Figures 1 and 2 for different \( \lambda \) values and \( a_1 = a_2 = 1 \); the case \( \lambda = \infty \) corresponds to the original “bosonic” solutions. The interesting point is that one can work with the general flux solution and still get physically meaningful, localized-type solutions (see Fig. 3). We mention that in quantum mechanics irregular wavefunctions have been found useful in state reconstruction [6]. To see the physical meaning of the isospectral construction for neutron diffusion we make the change of function \( \psi = \sqrt{\rho} \phi \) in the Schrödinger equation \( \psi'' - V_{iso} \psi = 0 \) to get

\[
\rho^2 \frac{d^2 \phi}{d\rho^2} + \rho \frac{d\phi}{d\rho} - \rho^2 k_{eff}^2 \phi = 0,
\]

where \( k_{eff}^{-2} = [k^2 - 2\frac{d^2}{d\rho^2} \ln(\mathcal{I}(\rho) + \lambda)]^{-1/2} \) is an effective, flux-dependent diffusion length, which is the Darboux transform of the constant diffusion length. Thus, one can see the connection between the common radial neutron diffusion equation and a similar isospectral diffusion.

In conclusion, we have shown that simple supersymmetric methods of 1D quantum mechanics may provide interesting results in neutron diffusion.

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Figure captions

Fig. 1: The original potential \( V_B = k^2 - \frac{1}{x^2} \) for \( k = 1 \) and four of the strictly isospectral potentials for \( x \) taking the values 1, 1000, 3000, 6000 respectively (the corresponding wells are from left to right, respectively).

Fig. 2: The original (nonphysical) solution \( \psi \) for the superposition constants \( a_1 = a_2 = 1 \) and the strictly isospectral solutions for the same values of \( \lambda \) as in Fig. 1.

Fig. 3: The flux solutions \( \phi \) corresponding to the \( \psi \) ones in Fig. 2. Unpublished note: In an inverted potential the isospectral “flux” solutions could be interpreted as a sort of “resonant” states.
