FLOQUET PROBLEM FOR ORDINARY DIFFERENTIAL OPERATORS WITH PERIODIC COEFFICIENTS IN HILBERT SPACES

VLADIMIR KOZLOV, JARI TASKINEN

1 Department of Mathematics, Linköping University, S-581 83 Linköping, Sweden
2 Department of Mathematics and Statistics, University of Helsinki, P.O.Box 68, 00014 Helsinki, Finland

E-mail: vladimir.kozlov@liu.se / jari.taskinen@helsinki.fi

ABSTRACT. A first order differential equation with a periodic operator coefficient acting in a pair of Hilbert spaces is considered. This setting models both elliptic equations with periodic coefficients in a cylinder and parabolic equations with time periodic coefficients. Our main result is a spectral splitting of the system into a finite dimensional system of ordinary differential equations with constant coefficients and an infinite dimensional part whose solutions have better properties in a certain sense. This gives a version of Floquet theorem for the infinite dimensional case and complements asymptotic results of S.Nazarov [9].

Keywords: Floquet theorem, differential equations with periodic coefficients, asymptotics of solutions to differential equations

1. INTRODUCTION

Consider a first order differential equation for an unknown function $x(t)$ with values in an infinite dimensional Hilbert space $X$,

$$\frac{dx(t)}{dt} = Ax(t) + f(t; x(t)), \quad t \in \mathbb{R},$$

where $A$ is an unbounded linear operator in $X$ which is constant in $t$, and $f : \mathbb{R} \times X \to X$ is given. If $P$ is a finite dimensional orthogonal projector in $X$ which commutes with $A$, then the system (1.1) with $f \equiv 0$ can be split into a finite dimensional system on the subspace $P(X)$ and an infinite dimensional system which may have better properties than the initial one. This reduction can be quite useful in the study of the large time behaviour of linear dynamical systems perturbed by a linear or non-linear perturbation $f \neq 0$. The main subject of this paper is to study similar splitting for the case when $A = A(t)$ is a periodic operator function with certain Fredholm properties. Our goal is to find a projector $P$ commuting with the operator of the periodic problem and reducing the problem to a finite dimensional problem with time independent operator and infinite dimensional problem having better properties than the original problem. This turns out to be possible although the projector $P$ itself is not of finite rank.

Our result can be considered as a generalization of the classical Floquet theorem to the infinite dimensional case. Indeed, to recall that, consider a system of ordinary
differential equations

\[ \frac{dx(t)}{dt} = A(t)x(t), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^d, \]

where \( A \) is a \( d \times d \)-matrix depending periodically on \( t \). Let \( G(t) \) be the fundamental matrix-solution of (1.2). The Floquet theorem says that there exists a constant matrix \( C \) and a periodic matrix \( P(t) \) such that \( G(t) = P(t)e^{Ct} \). This theorem allows to reduce the periodic system (1.2) to a system with constant coefficients.

The starting point of our study is the paper [9] (see also [11] and [10]), where an asymptotic theory for elliptic boundary value problems with periodic coefficients in a cylinder was developed. It extends similar results for elliptic boundary problems with constant coefficients in a cylinder (see [2], [11], [5] and references there) to the periodic coefficients case. Our contribution to the above periodic case is a construction of a projection operator and a spectral splitting of the problem. Also relevant to our work is the operator theoretic approach to periodic problems via Floquet-Bloch-transform techniques; we mention here the expositions [6], [7] and references there. Finally, we remark that [3] and [4] contain an analogous theory in the case \( A(t) \) is a perturbation of an operator \( A_0 \) independent of \( t \). We will use the same formalism of analytic Fredholm operator families, the theory of which is presented e.g. in the appendix of [3]. In particular in the treatment of the infinite dimensional part of the splitted system we use a technique developed [3], which allows us to avoid a choice of function spaces for estimating the remainder terms, since all of them can be treated from this ”pointwise estimate”, see Sect. 4 and 7 in [3].

The structure of the paper is the following. In Sect. 2 we formulate the problem, introduce the function spaces and present the main assumptions on the operator of the problem. In Sect. 3 we remind some basic definitions and properties of the eigenvalues, eigenvectors and generalized eigenvectors of the operator pencils associated with our periodic problem. The main result here is Lemma 3.3 which allows to introduce the projector operator. In the next section we derive some important properties of this projector. In Sect. 5 we collect known results on the solvability and asymptotics of solutions to periodic problems. These results are proved in [10] in the case of elliptic boundary value problems with periodic coefficients in periodic cylinders. Since we are dealing with an abstract setting we present proofs for the reader’s convenience. The main result of our paper is Theorem 6.1 in Sect. 6. We give a splitting of the the system into a finite dimensional system of ordinary differential equations with constant coefficients and an infinite dimensional part whose solutions have better properties in a certain sense.

2. Statement of the Problem

Let \( X \) and \( Y \) be Hilbert spaces such that \( X \) is compactly and densely imbedded in \( Y \). We denote the norms in \( X \) and \( Y \) by \( \| \cdot \|_X = \| \cdot \| : X \| \) and \( \| \cdot \|_Y = \| \cdot \| : Y \| \), respectively. We identify \( Y^* \) with \( Y \) by using the inner product \( (\cdot, \cdot) = (\cdot, \cdot)_Y \) and introduce for \( h \in Y \) the norm

\[ \| h \|_{X^*} = \text{sup}\{ |(g,h)| : g \in X, \| g \|_X = 1 \}. \]

The completion of \( Y \) with respect to this norm coincides with \( X^* \), and the sesquilinear form \( (g,h) \) can be extended for \( g \in X \) and \( h \in X^* \) such that the inequality \( |(g,h)| \leq \| g \|_X \| h \|_{X^*} \) holds. Clearly, \( Y \subset X^* \).
Given $a, b \in \mathbb{R}$, $a < b$, we denote by $\mathcal{X}(a,b)$ the space of functions $u : (a,b) \mapsto X$ such that the weak $t$-derivatives with values in $Y$ exist and are locally integrable (in the standard Bochner sense, see e.g. [1], Sect. 3.7.) and such that the norm

$$ (2.1) \quad \|u; \mathcal{X}(a,b)\| = \left( \int_a^b (\|u(t); X\|^2 + \|D_t u(t); Y\|^2) \, dt \right)^{1/2} $$

is finite. Here and elsewhere $D_t = \partial / \partial t$. Also, the space $\mathcal{Y}(a,b)$ consists of locally integrable functions $u : (a,b) \mapsto Y$ with finite norm

$$ (2.2) \quad \|f; \mathcal{Y}(a,b)\| = \left( \int_a^b \|f(t); Y\|^2 \, dt \right)^{1/2}. $$

The space $\mathcal{X}_{\text{loc}} := L^2_{\text{loc}}(\mathbb{R}; Y)$ consists of measurable functions defined on $\mathbb{R}$ with values in $Y$ with finite semi-norms (2.2) for all $a < b$, and the space $\mathcal{X}_{\text{loc}} := L^2_{\text{loc}}(\mathbb{R}; X)$ is defined analogously (cf. above); in particular for every $f \in \mathcal{X}_{\text{loc}}$, the semi-norms (2.1) are finite for all $a < b$.

Given $\beta \in \mathbb{R}$, the space $\mathcal{X}_\beta$ consists of functions $u \in L^2_{\text{loc}}(\mathbb{R}; X)$ such that $D_t u \in L^2_{\text{loc}}(\mathbb{R}; Y)$ and the norm

$$ (2.3) \quad \|u; \mathcal{X}_\beta\| = \left( \int_{\mathbb{R}} e^{2\beta t} (\|u(t); X\|^2 + \|D_t u(t); Y\|^2) \, dt \right)^{1/2} $$

is finite, and the space $\mathcal{Y}_\beta = L^2_{\beta}(\mathbb{R}; Y)$ consists of functions $f \in L^2_{\text{loc}}(\mathbb{R}; Y)$ with finite norm

$$ (2.4) \quad \|f; \mathcal{Y}_\beta\| = \left( \int_{\mathbb{R}} e^{2\beta t} \|f(t); Y\|^2 \, dt \right)^{1/2}. $$

In order to deal with periodic problems we follow [9], [11] and also introduce subspaces of $\mathcal{X}_{\text{loc}}$ and $\mathcal{Y}_{\text{loc}}$, which consist of periodic functions in $t$ of period 1 and which are denoted by $\mathcal{X}$ and $\mathcal{Y}$, respectively. The norms in these spaces are

$$ \|u; \mathcal{X}\| = \|u; \mathcal{X}(0,1)\|, \|f; \mathcal{Y}\| = \|f; \mathcal{Y}(0,1)\|. $$

Let $A(t)$ be a bounded operator from $X$ into $Y$ depending continuously on $t \in \mathbb{R}$ with respect to the operator norm. We assume that $A(t)$ is periodic with respect to $t$ with the period 1. For every $t$, we denote by $A(t)^* : Y \rightarrow X^*$ the adjoint operator with respect to the duality $(\cdot, \cdot)$, i.e.,

$$ (2.5) \quad (A(t) \varphi, \psi) = (\varphi, A(t)^* \psi) \quad \varphi \in X, \psi \in Y. $$

We also define the differential operators

$$ (2.6) \quad \mathcal{L} = \mathcal{L}(t,D_t) := D_t u(t) + A(t) u(t) \quad \text{and} \quad \mathcal{L}^*(t,D_t) := -D_t u(t) + A(t)^* u(t). $$

In the following we will consider the problem

$$ (2.7) \quad \mathcal{L}(t,D_t) u = f(t), $$

where $f \in L^2_{\text{loc}}(\mathbb{R}; Y)$ is a given function and $u \in \mathcal{X}_{\text{loc}}$ is a function to be found. Our aim is to introduce a reduction of this problem into a system consisting of a scalar valued, finite dimensional ODE-system and of another vector valued ODE, which
has better properties then the initial problem. The first main assumptions on \( L \) is the following local estimate (cf. \[4\], Sect. 2.2)

\[
\|u; X(0, 1)\| \leq C \left(\|L(t, D_t)u; Y(-1, 2)\| + \|u; Y(-1, 2)\|\right) \quad \text{for all } u \in X(-1, 2).
\]

(2.8)

To formulate the second assumption let us consider the following operator depending on a complex parameter \( \lambda \),

\[
A(\lambda) = L(t, D_t) + \lambda : \hat{X} \to \hat{Y}, \quad \lambda \in \mathbb{C}.
\]

(2.9)

The exact relation of \( A \) and \( L \) via the Floquet-Bloch-transform will be made clear in (5.4). Obviously, \( A \) is a holomorphic operator pencil with respect to the parameter \( \lambda \). The second main assumptions on \( L \) reads as (cf. \[9\]):

\[
\text{there exists } \lambda_0 \text{ for which } A(\lambda_0) : \hat{X} \to \hat{Y} \text{ is an isomorphism.}
\]

(2.10)

Remark. We have in mind some applications to parabolic and elliptic PDE-problems, which have been transformed into first order ODE-systems with respect to one of the variables in a canonical way. The assumptions (2.8), (2.10) are natural for such cases. The assumptions would in general fail for hyperbolic PDE-problems.

Lemma 2.1. If the assumptions (2.8) and (2.10) hold, then the families

\[
A(\lambda) : \mathbb{C} \to L(\hat{X}, \hat{Y}),
\]

\[
A^*(\lambda) := A(\bar{\lambda})^* : \mathbb{C} \to L(\hat{Y}, \hat{X}^*),
\]

where \( A^*(\lambda) = -D_t + A(t)^* + \bar{\lambda} \), are holomorphic Fredholm families. Moreover, there holds

\[
\left(\int_0^1 (\|u(t); X\|^2 + \|(D_t + \lambda)u(t); Y\|^2) dt\right)^{1/2} \leq Ce^{\|\Re\lambda\|} \left(\|A(\lambda)u; \hat{Y}\| + \|u; \hat{Y}\|\right)
\]

for \( u \in \hat{X} \).

(2.11)

Here, \( L(\hat{X}, \hat{Y}) \) denotes the Banach space of bounded linear operators from \( \hat{X} \) into \( \hat{Y} \).

Proof. Writing estimate (2.8) for the function \( e^{\lambda t}u, u \in \hat{X} \), we get

\[
\left(\int_0^1 e^{2t\Re\lambda} (\|u(t); X\|^2 + \|(D_t + \lambda)u(t); Y\|^2) dt\right)^{1/2} \leq C \left(\left(\int_{-1}^2 e^{2t\Re\lambda} \|L(t, D_t + \lambda)u(t); Y\|^2 dt\right)^{1/2} + \left(\int_{-1}^2 e^{2t\Re\lambda} \|u(t); Y\|^2 dt\right)^{1/2}\right),
\]

which implies estimate (2.11). Since the inclusion \( X \subset Y \) is compact, we can use the argument in \[8\], p. 20 or Theorem 2.1. to see that the embedding \( \hat{X} \subset \hat{Y} \) is also compact. Hence, estimate (2.11) implies that the kernel of \( A(\lambda) \) is finite dimensional and the image is closed for all \( \lambda \). This together with assumption (2.10) gives that the operator pencil is Fredholm with the index 0 for all \( \lambda \) (see \[3\], Section A.8).
The definition of the adjoint holomorphic family $A^*(\lambda)$ is as in [3], Section A.9, and its Fredholm property follows from that of the family $A(\lambda)$, as explained in the citation; see also the next lemma. □

Let us provide a description of the dual space $\hat{X}^*$. To this end we use the inner product to identify $\hat{Y}^* = \hat{Y}$, and we also denote by $L^2_{\text{per}}(\mathbb{R}; X^*)$ the subspace of $L^2_{\text{loc}}(\mathbb{R}; X^*)$ consisting of periodic functions $f: \mathbb{R} \to X^*$, endowed with the norm

$$\|f; L^2_{\text{per}}(\mathbb{R}; X^*)\| = \left( \int_0^1 \|f(t); X^*\|^2 dt \right)^{1/2}.$$ 

The proof of the following lemma is standard; it can be considered as a known fact.

**Lemma 2.2.** Under the dual pairing

$$\langle u, v \rangle_{\hat{Y}} := \int_0^1 (u(t), v(t)) dt,$$

the dual space $\hat{X}^*$ of $\hat{X}$ consists of periodic functions $w$ represented as

$$w = w_0 + D_tw_1,$$

where $w_0 \in L^2_{\text{per}}(\mathbb{R}; X^*)$ and $w_1 \in \hat{Y}$, and it is endowed with the norm

$$\|w; \hat{X}^*\| = \inf \left( \|w_0; L^2_{\text{per}}(\mathbb{R}; X^*)\| + \|D_tw_1; \hat{Y}\| \right),$$

where the infimum is taken over all representations (2.12). The adjoint $A^*(\lambda)$ of the operator $A(\lambda)$ satisfies

$$\langle A(\lambda)\varphi, \psi \rangle_{\hat{Y}} = \langle \varphi, A^*(\lambda)\psi \rangle_{\hat{Y}} \quad \varphi \in \hat{X}, \psi \in \hat{Y}.$$

We end this section with one more lemma. One can easily verify that

$$e^{2\pi i t}A(\lambda)u = A(\lambda - 2\pi i)(e^{2\pi it}u)$$

for $u \in \hat{X}$.

**Lemma 2.3.** Assume that the operator $A(\mu)$ is an isomorphism for $\mu = \beta + i\xi$ with a fixed $\beta \in \mathbb{R}$ and for all $\xi \in [0, 2\pi)$. Then, for all $u \in \hat{X}$ and $\lambda = \beta + i\xi$ with $\xi \in \mathbb{R}$,

$$\int_0^1 \left( \|u(t); X\|^2 + \|(D_t + \lambda)u(t); Y\|^2 \right) dt^{1/2} \leq C\|A(\lambda)u; \hat{Y}\|,$$

where $C$ may depend on $\beta$ but it is independent of $\xi$.

**Proof.** By (2.13) the optimal constant $c$ in the inequality

$$\|A(\lambda)u; \hat{Y}\| \geq c\|u; \hat{Y}\|$$

is the same for $\lambda$ and $\lambda - 2\pi ki$ for all $k = \pm 1, \pm 2, \ldots$. This together with (2.13) and the assumption of the lemma implies existence of a constant $c_0$ such that (2.15) is true for all $\lambda = \beta + i\xi$ with $\xi \in \mathbb{R}$. Using (2.15) we derive (2.14) from (2.11). □
3. Eigenvectors, generalized eigenvectors, Jordan chains

We recall some basic facts concerning the spectrum of the operator pencil $A(\lambda)$, see [3], Appendix, for more details. As in standard spectral theory of linear operators, the spectrum is the set of those $\lambda \in \mathbb{C}$ such that $A(\lambda) : \hat{X} \to \hat{Y}$ is not invertible; $\lambda$ is an eigenvalue, if the kernel of $A(\lambda)$ is not $\{0\}$.

Since $A(\lambda) : \hat{X} \to \hat{Y}$ is a holomorphic Fredholm family, its spectrum consists of isolated eigenvalues of finite algebraic multiplicity, see Proposition A.8.4 of [3]. From the relation (2.13) it follows that if $\lambda$ is an eigenvalue then the same is true for $\lambda + 2\pi i$ and their multiplicities coincide. In the following we denote for all $\beta \in \mathbb{R}$

$$\delta_{\beta} = \{ \lambda \in \mathbb{C} : \Re \lambda = \beta, \exists \lambda \in [0, 2\pi) \},$$

and we choose real numbers

$$\beta_1 < \beta_2$$

such that there are no eigenvalues of $A(\lambda)$ on the the intervals $\delta_{\beta_1}$ and $\delta_{\beta_2}$. We denote eigenvalues of $A(\lambda)$ in the set

$$\{ \lambda = \beta + i\xi ; \beta_1 < \beta < \beta_2, \xi \in [0, 2\pi) \}$$

by $\lambda_k$, $k = 1, \ldots, N$, and let $J_k$ and $m_{k,1}, \ldots, m_{k,J_k}$ be the geometric and partial multiplicities of $\lambda_k$. Assume that for every $k = 1, \ldots, N$,

$$\varphi_{j,m}^k, m = 0, \ldots, m_{k,j} - 1, j = 1, \ldots, J_k,$$

is a canonical system of Jordan chains of the linear pencil $A(\lambda)$ corresponding to $\lambda_k$ (see [3], Definition A.4.3, Propositions A.4.4, A.4.5.). The functions

$$\varphi_{j,0}^k, j = 1, \ldots, J_k,$$

form a linearly independent sequence of eigenvectors corresponding to the eigenvalue $\lambda_k$, while the functions (3.4) with $m \geq 1$ are associated vectors satisfying

$$A(\lambda_k)\varphi_{j,m}^k = 0, \quad A(\lambda_k)\varphi_{j,m}^k = -\varphi_{j,m-1}^k, \quad m = 1, \ldots, m_{k,j} - 1.$$

In the same way, the eigenfunctions and generalized eigenfunctions of the adjoint operator are the solutions of the equations

$$A^*(\lambda_k)\psi_{j,0}^k = 0, \quad A^*(\lambda_k)\psi_{j,m}^k = -\psi_{j,m-1}^k, \quad m = 1, \ldots, m_{k,j} - 1.$$

It will be important to specify the choice of the functions (3.5) and (3.7) such that certain orthogonality relations are satisfied. Notice that we consider a finite set of eigenvalues, which is fixed by the choice of the numbers $\beta_1, \beta_2$ above. The following assertion is known and its proof can be found in Remark A.10.3 in [3] (see formula (A.60) there).

**Lemma 3.1.** If the Jordan chains (3.4) are fixed, then there exist uniquely defined Jordan chains of the adjoint pencil $A^*(\lambda)$ corresponding to the eigenvalue $\lambda_k$

$$\psi_{j,m}^k, m = 0, \ldots, m_{k,j} - 1, j = 1, \ldots, J_k,$$

such that in addition to all equations (3.6) and (3.7) also the following hold true:

$$\langle \varphi_{j,m_{k,j-1}}, \psi_{j,m}^k \rangle_{\hat{Y}} = \delta_j^m \delta_0^m, \quad m = 0, \ldots, m_{k,j} - 1.$$
From now on we assume that the eigenfunctions and generalized eigenfunctions satisfy (3.9). It appears that the last relation implies some more orthogonality relations. For the convenience of the reader we present the proof of this fact in detail.

**Lemma 3.2.** Let the Jordan chains (3.3) and (3.8) be the same as in Lemma 3.1. Then the following biorthogonality relations hold:

\[
(\varphi_{j,m}^k, \psi_{j,m}^K, J_m, J_{M-1})_\delta = \delta^K_0 \delta^M_0
\]

for all \( k, K, j, J, m, M \).

Proof. Let first \( K = k \). Then (3.10) for \( m = m_{k,j} - 1 \) follows from (3.9).

Next we observe that for \( m = 1, \ldots, m_{k,j} - 1 \) and \( M = 1, \ldots, m_{k,j} - 1, J = 1, \ldots, J_k \), the relations (3.6) and (3.7) yield

\[
(\varphi_{j,m}^k, \psi_{j,m}^k, J_{M-1})_\delta = -(\varphi_{j,m}^k, A^*(\lambda_k)\psi_{j,m}^k, J_{M-1})_\delta
\]

(3.11)

Applying this relation with \( m = m_{k,j} - 1 \) and \( M = 1, \ldots, m_{k,j} - 1 \) and using that (3.10) is proved for \( m = m_{k,j} - 1 \), we arrived at (3.10) for \( m = m_{k,j} - 2 \) and \( M = 0, \ldots, m_{k,j} - 2 \). Since the relations

\[
(\varphi_{j,m}^k, \psi_{j,0}^k)_\delta = 0, \ m = 0, \ldots, m_{k,j} - 2,
\]

follow from the solvability of (3.6), we arrive at (3.10) for \( m = m_{k,j} - 2 \) and all \( M \). Repeating this argument we prove (3.10) for all \( m \) and \( M \).

We finally show that if \( k \neq K \), then the orthogonality in (3.10) automatically holds. For the two eigenfunctions we get the orthogonality \((\varphi_{j,0}^k, \psi_{j,0}^K)_\delta = 0 \) for all \( j = 1, \ldots, J_k, J = 1, \ldots, J_K \) by the simple classical argument, since the eigenvalues \( \lambda_k \) and \( \lambda_K \) are different. Then, we have, for all \( M = 0, \ldots, m_{k,j} - 2, j, J \),

\[
(\varphi_{j,0}^k, \psi_{j,M}^k)_\delta = -(\varphi_{j,0}^k, A^*(\lambda_k)\psi_{j,M+1}^K)_\delta = -(\varphi_{j,0}^k, A^*(\lambda_k)\psi_{j,M+1}^K)_\delta
\]

(3.12)

where the coefficient \( \lambda_k - \lambda_K \) is non-zero, so that the orthogonality \((\varphi_{j,0}^k, \psi_{j,M+1}^K)_\delta = 0 \) for all \( M = 0, \ldots, m_{k,j} - 1 \) and \( j, J \) follows by induction. In the same way one obtains \((\varphi_{j,m}^k, \psi_{j,0}^K)_\delta = 0 \) for all \( m = 0, \ldots, m_{k,j} - 1 \) and \( j, J \).

Then, one proves the following formulas in the same way as (3.12)

\[
(\varphi_{j,m}^k, \psi_{j,M}^k)_\delta = \begin{cases} 
(\varphi_{j,m+1}^k, \psi_{j,M-1}^K)_\delta + (\lambda_k - \lambda_K)(\varphi_{j,m+1}^k, \psi_{j,M}^K)_\delta \\
(\varphi_{j,m-1}^k, \psi_{j,M+1}^K)_\delta + (\lambda_k - \lambda_K)(\varphi_{j,m}^k, \psi_{j,M+1}^K)_\delta.
\end{cases}
\]

One can then proceed by induction to get the orthogonality for all indices. □

Let us still introduce some more notation with the help of the above introduced Jordan chains: we define

\[
\Phi_{j,m}^k(t) = e^{\lambda_k t} \sum_{n=0}^m \frac{t^n}{n!} \varphi_{j,m-n}^k = e^{\lambda_k t} \sum_{n=0}^m \frac{t^{m-n}}{(m-n)!} \varphi_{j,n}^k
\]

for all \( k = 1, \ldots, N, j = 1, \ldots, J_k, m = 0, \ldots, m_{k,j} - 1 \). It is known and one can verify it directly that these functions are solutions to the homogeneous equation
The binomial formula implies the following relation which will be needed later:

\[
e^{\lambda_k t} \sum_{n=0}^{m} \frac{(t - \tau)^n}{n!} \phi_{j,m-n}(t) = e^{\lambda_k t} \sum_{n=0}^{m} \sum_{\nu=0}^{n} \frac{(-\tau)^\nu}{(n - \nu)! \nu!} \phi_{j,m-n}(t)
\]

(3.14)

Moreover, by (3.6), (3.7),

\[
I(k,j,m) = e^{\lambda_k t} \sum_{n=0}^{m} \sum_{\nu=0}^{n} \frac{(-\tau)^\nu}{(m - \nu - n)! \nu!} \phi_{j,n}(t) = \sum_{\nu=0}^{m} \frac{(-\tau)^\nu}{\nu!} \Phi_{j,\nu}(t).
\]

The following, perhaps unexpected fact, is the key to the reduction of the problem

\[2.7\].

**Lemma 3.3.** We have for all \(k, K = 1, \ldots, N, j = 1, \ldots, J_k, J = 1, \ldots, J_K, m = 0, \ldots, m_{k,j} - 1, M = 0, \ldots, m_{k,j} - 1\),

\[
(\phi_{j,m}^k, \psi_{j,M}^K(t)) = \delta_{k,j} \delta_{m} \quad \text{for all } t \in \mathbb{R}.
\]

(3.15)

**Proof.** Introduce

\[
I(k,j,m; K,J,M)(t) = (\phi_{j,m}^k(t), \psi_{j,M}^K(t)).
\]

(3.16)

We have for all \(m, M = 0, \ldots, m_{k,j} - 1\)

\[
D_t I(k,j,m; K,J,M)(t) = (D_t \phi_{j,m}^k(t), \psi_{j,M}^K(t)) + (\phi_{j,m}^k(t), D_t \psi_{j,M}^K(t)).
\]

Moreover, by (3.6), (3.7),

\[
(D_t + A(t) + \lambda_k) \phi_{j,m}^k + \phi_{j,m-1}^k = 0, \quad (-D_t + A(t)^* + \lambda_K) \psi_{j,M}^K + \psi_{j,M-1}^K = 0,
\]

where we must agree that the functions with negative second lower index are zero. Thus,

\[
D_t I(k,j,m; K,J,M)(t) = -(A(t) + \lambda_k) \phi_{j,m}^k(t) + \phi_{j,m-1}^k(t), \psi_{j,M}^K(t)) + (\phi_{j,m}^k(t), (A(t)^* + \lambda_K) \psi_{j,M}^K(t) + \psi_{j,M-1}^K(t))
\]

After cancellation, we get

\[
D_t I(k,j,m; K,J,M)(t)
\]

\[
= - (\phi_{j,m-1}^k(t), \psi_{j,M}^K(t)) + (\phi_{j,m}^k(t), \psi_{j,M-1}^K(t))
\]

(3.17)

\[
= -I(k,j,m-1; K,J,M)(t) + I(k,j,m; K,J,M-1)(t).
\]

1°. We first prove the case \(K = k\) and \(J = j \in \{1, \ldots, J_k\}\); see the diagram below.

(i) Let first \(m = 0\) and \(M = 0\). Then, the right-hand side of (3.17) is zero by the convention made above, and therefore \(D_t I(k,j,0; k,j,0)(t) = 0\) and thus \(I(k,j,0; k,j,0)(t)\) does not depend on \(t\). By Lemma 3.2 we get for all \(t\)

(3.18)

\[
I(k,j,0; k,j,0)(t) = \int_0^1 I(k,j,0; k,j,0)(\tau)d\tau = (\phi_{j,0}^k, \psi_{j,0}^K)^2 = 0.
\]

(ii) We next consider the case \(m = 0, M = 1, \ldots, m_{k,j} - 2\). We use induction with respect to \(M\): assume that \(I(k,j,0; k,j,M)(t) = 0\) for some \(M < m_{k,j} - 2\) and all \(t \in [0, 1]\). By (3.17) we get

(3.19)

\[
D_t I(k,j,0; k,j,M+1)(t) = I(k,j,0; k,j,M)(t) = 0.
\]
hence, \( I(k, j, 0; k, j, M + 1)(t) \) is constant with respect to \( t \). Integrating this constant as in (3.18) and using Lemma 3.2 yield for all \( t \)

\[
I(k, j, 0; k, j, M)(t) = (\varphi^k_{j,0}, \psi^k_{j,M})_j = 0 \quad \forall M = 1, \ldots, m_{k,j} - 2.
\]

(Note that by Lemma 3.2 the inner product is not zero for \( M = m_{k,j} - 1 \).)

In the same way, using inductively

\[
D_t I(k, j, m + 1; k, j, 0)(t) = I(k, j, m; k, j, 0)(t) = 0
\]

for \( m = 0, \ldots, m_{k,j} - 2 \) instead of (3.19) we prove that

\[
I(k, j, m; k, j, 0) = 0 \quad \forall m = 1, \ldots, m_{k,j} - 2.
\]

\((iii)\) We next consider the case \( m + M \leq m_{k,j} - 2 \) by using a double induction: assume that for some index \( M \geq 0 \) with \( M \leq m_{k,j} - 3 \) the equality \( I(k, j, m; k, j, M) = 0 \) has been proven for all \( m = 0, \ldots, m_{k,j} - M - 3 \). Then, (3.17) implies, for \( m = 1, \ldots, m_{k,j} - M - 2 \),

\[
D_t I(k, j, m; k, j, M + 1) = -I(k, j, m - 1; k, j, M + 1) + I(k, j, m; k, j, M).
\]

We can thus proceed by induction with respect to \( m \) (using \( t \)-integration and Lemma 3.2 as above) to get

\[
I(k, j, m; k, j, M + 1) = 0 \quad \forall m = 0, \ldots, m_{k,j} - M - 2.
\]

Induction with respect to \( M \) yields (3.15) for all \( m + M \leq m_{k,j} - 2 \). In both induction procedures we use \((ii)\) for \( m = 0 \) and \( M = 0 \) to start with.

\((iv)\) Consider \( m + M = m_{k,j} - 1 \). Formula (3.17) and what we have proven until now again imply that for every \( m = 0, 1, \ldots, m_{k,j} - 1 \) the expression

\[
I(k, j, m; k, j, m_{k,j} - m - 1)
\]

is a constant, which by \( t \)-integration and Lemma 3.2 is equal to 1.

\((v)\) However, this and (3.17) again imply that for every \( m = 1, 2, \ldots, m_{k,j} - 1 \) we have

\[
I(k, j, m; k, j, m_{k,j} - m) = 0.
\]

\((vi)\) From here on we can continue in the same way as in \((iii)\) to get the result for \( M + m \geq m_{k,j} \).

Diagram on the progression of the previous proof:

| \( M = \) | \( m_{k,j} - 1 \) | \( m_{k,j} - 2 \) | \( m_{k,j} - 3 \) | \( m_{k,j} - 3 \) | \( 2 \) | \( 1 \) | \( 0 \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( (iv) \) \( (v) \) \( (vi) \) \( (vi) \) \( (vi) \) \( (vi) \) \( (vi) \) | \( (ii) \) \( (iv) \) \( (v) \) \( (vi) \) \( (vi) \) \( (vi) \) \( (vi) \) | \( (ii) \) \( (ii) \) \( (iii) \) \( (iv) \) \( (iv) \) \( (iv) \) \( (iv) \) | \( (ii) \) \( (ii) \) \( (iii) \) \( (ii) \) \( (ii) \) \( (ii) \) \( (ii) \) | \( (ii) \) \( (ii) \) \( (ii) \) \( (ii) \) \( (ii) \) \( (ii) \) \( (ii) \) | \( 0 \) | \( 1 \) | \( 2 \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( m_{k,j} - 4 \) | \( m_{k,j} - 3 \) | \( m_{k,j} - 2 \) | \( m_{k,j} - 1 \) | \( = m \) |
pairs \((m, M)\) with \(m = 0, M = 1, \ldots, m_{k,j} - 1\) and \(m = 1, \ldots, m_{k,j} - 1, M = 0\), by Lemma 3.2. Then, the procedure of \((iii)\) yields \(I(k, j, m; k, J, M) = 0\) for all remaining pairs \((m, M)\), since now we do not have the obstruction of the case \((iv)\) (the inner products \((\varphi^k_{j,m}, \psi^k_{j,M})\)) equal 0 instead of 1, by Lemma 3.2 for all indices in question).

\(3^\circ\). For the proof in the case \(K \neq k\) we need to introduce instead of \((3.16)\),

\[
I(k, j, m; K, J, M)(t) = e^{(\lambda_k - \lambda_K)t}(\varphi^k_{j,m}(t), \psi^k_{j,M}(t)),
\]

because \(\lambda_k \neq \lambda_K\). By a similar calculation as around \((3.17)\) we get for all \(j, J, m, M\)

\[
eq (\lambda_k - \lambda_K)(\varphi^k_{j,m}(t), \psi^k_{J,M}(t)) + (D_t\varphi^k_{j,m}(t), \psi^k_{J,M}(t)) + (\varphi^k_{j,m}(t), D_t\psi^k_{J,M}(t))
\]

\[(3.20)\]

\[
= -(\varphi^k_{j,m-1}(t), \psi^k_{J,M}(t)) + (\varphi^k_{j,m}(t), \psi^k_{J,M-1}(t)).
\]

The following argument shows that we can use \((3.20)\) instead of \((3.16)\) and repeat the proof of the case \(2^\circ\) (i.e. the steps \((i) - (iii)\) in \(1^\circ\)) for all \(j, J, m, M\) : assume that the right-hand side of \((3.20)\) equals 0 for all \(t\). Since \(e^{(-\lambda_k + \lambda_K)t} \neq 0\), this implies that \(D_tI(k, j, m; K, J, M)(t) = 0\) for all \(t\), hence, \(I(k, j, m; K, J, M)(t) = B\) for some constant \(B\), for all \(t\). Thus, by Lemma 3.2.

\[
0 = (\varphi^k_{j,m}, \psi^k_{J,M}) = \int_0^1 e^{(\lambda_K - \lambda_k)t}e^{(\lambda_k - \lambda_K)t}(\varphi^k_{j,m}(t), \psi^k_{J,M}(t))dt = B \int_0^1 e^{(\lambda_K - \lambda_k)t}dt.
\]

Here we have \(\int_0^1 e^{(\lambda_K - \lambda_k)t}dt \neq 0\), since \(\lambda_K - \lambda_k\) cannot equal a multiple of \(i2\pi\), see \((3.3)\). Hence, the constant \(B\) must be zero, and thus also \((\varphi^k_{j,m}(t), \psi^k_{J,M}(t)) = 0\) for all \(t\). \(\square\)

4. Pointwise projector.

The "pointwise projector" is now defined by

\[
\mathcal{P}u(t) = \sum_{k=1}^{N} \sum_{j=1}^{j_k} \sum_{m=0}^{m_{k,j} - 1} u^k_{j,m}(t)\varphi^k_{j,m_{k,j} - 1 - m}(t),
\]

where

\[
u^k_{j,m}(t) = (u(t), \varphi^k_{j,m}(t))
\]

and \(t \in \mathbb{R}\). That this is a projector in the spaces \(\mathcal{Y}_{\text{loc}}\) and \(\mathcal{X}_{\text{loc}}\) follows from \((3.15)\).

Since there are only finitely terms in the sums \((4.1)\) and

\[
|u^k_{j,m}(t)| \leq C\|u(t); Y\|\text{ and }|\partial_t u^k_{j,m}(t)| \leq C\|\partial_t u(t); Y\|,
\]

the operator \(\mathcal{P}\) is bounded as an operator in \(\mathcal{X}\) and \(\mathcal{Y}\).

Using \((3.6)\) we verify that for \(U = \mathcal{P}u\) we have

\[
(D_t + A(t))U(t)
\]

\[
= \sum_{k=1}^{N} \sum_{j=1}^{j_k} \sum_{m=0}^{m_{k,j} - 1} (D_t u^k_{j,m}(t))\varphi^k_{j,m_{k,j} - 1 - m}(t)
\]
also an isomorphism from $X$ to $L^p(4.3)$.

The left-hand side is evaluated in (4.3) so that to verify (4.4) we consider its right-hand side. We have

$$\mathcal{L} \mathcal{P} u = \mathcal{P} \mathcal{L} u \text{ for } u \in X_{\text{loc}}.$$  

The left-hand side is evaluated in (4.3) so that to verify (4.4) we consider its right-hand side. We have

$$\mathcal{P} \mathcal{L} u = \sum_{j,m} v^k_{j,m}(t) \psi^k_{j,m,k,j-1-m}(t) , \text{ where}$$

where

$$v^k_{j,m}(t) = ((D_t + A(t))u(t), \psi^k_{j,m}(t)) = D_t v^k_{j,m}(t) + (u(t), (-D_t + A(t)) \psi^k_{j,m}(t)).$$

Using (3.15) we obtain

$$v^k_{j,m}(t) = D_t v^k_{j,m}(t) - \lambda_k (u(t), \psi^k_{j,m}(t)) - (u(t), \psi^k_{j,m-1}(t)) = (D_t - \lambda_k) v^k_{j,m}(t) - u^k_{j,m-1}(t).$$

Therefore

$$\mathcal{P} \mathcal{L} u = \sum (D_t - \lambda_k) v^k_{j,m}(t) - u^k_{j,m-1}(t)) \psi^k_{j,m,k,j-1-m}(t),$$

which coincides with the right-hand side of (4.3) and relation (4.4) is proved.

5. SOME RESULTS ON SOLVABILITY AND ASYMPTOTICS FOR PROBLEM (2.7)

Here we present solvability and asymptotic results for problem (2.7), which are proved for general boundary value problems with periodic coefficients in a cylinder in [9], Sect. 4 and 5, or even a periodic quasi-cylinder [11], Sect. 4, Ch. 3. The proofs are similar to those in the references, but due to the technicalities it is necessary to present them in detail.

Given $\beta \in \mathbb{R}$ we introduce the following vector-valued Floquet-Bloch-, or, Gelfand-transform

$$\Theta_\beta u(\xi) := \Theta_\beta u(t; \xi) := U(t; \xi) := \sum_{n=-\infty}^{\infty} e^{-(\beta+i\xi)(n+t)} u(t+n),$$

where $u \in L^2_\beta(\mathbb{R}; Y)$ and $t \in [0, 1], \xi \in (0, 2\pi)$. We denote $\Theta := \Theta_0$. The inverse transform is defined by

$$\Theta^{-1}_\beta U(t; \xi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} e^{(\beta+i\xi)t} U(t - [t]; \xi) d\xi,$$

where $t \in \mathbb{R}$ and $[t]$ denotes the largest integer not bigger than $t$. We will need the following mapping properties of $\Theta_\beta$.

**Lemma 5.1.** For all $\beta \in \mathbb{R}$, the operator $\Theta_{-\beta}$ is a Hilbert-space isomorphism from $L^2_\beta(\mathbb{R}; Y)$ onto $L^2((0, 2\pi); \hat{Y})$, and (5.2) is its inverse operator. The same operator is also an isomorphism from $X_\beta$ onto $L^2((0, 2\pi); \hat{X})$, and (5.2) is its inverse operator.
This lemma is known (e.g. [7], Th. 4.2). Let us nevertheless comment the proof because of the special function spaces. If \( \beta = 0 \), the claim for a general Hilbert space \( Y \) can be proven as in the scalar case (where \( Y = \mathbb{C} \)), since the vector valued Fourier-series and coefficients are calculated by the usual scalar formulas. Indeed, if \( u \in L^2((0, 2\pi); Y) \), then the series
\[
u(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{ij\xi} u_j
\]
with coefficients \( u_j \in Y \) converges in \( L^2((0, 2\pi); Y) \), if and only if \( (\|u_j; Y\|)_{j=-\infty}^{\infty} \in l^2 \) (the standard Hilbert-space of square summable complex sequences). Then, one has the usual formulas
\[
u_j = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} e^{-ij\xi} u(\xi) \, d\xi,
\]
\[\|u; L^2((0, 2\pi); Y)\| = \|(\|u_j; Y\|)_{j=-\infty}^{\infty}\|_\ell^2.\]
The result in the case \( \beta = 0 \) follows from these. If \( \beta \neq 0 \), we note that
\[(5.3) \quad \Theta_{-\beta} u(t) = \Theta \mathcal{M}_\beta u \quad \text{and} \quad \Theta_{\beta}^{-1} u(t) = \mathcal{M}_\beta \Theta^{-1} u,
\]
where \( \mathcal{M}_\beta \) denotes the multiplication by the function \( e^{\beta t} \). Since \( \mathcal{M}_\beta \) is an isometry from \( L^2_\beta(\mathbb{R}; Y) \) onto \( L^2(\mathbb{R}; Y) \), the result for general \( \beta \) follows from the case \( \beta = 0 \).

To treat the mapping properties \( \Theta_\beta : X_\beta \to L^2((0, 2\pi); \hat{X}) \) needs some additional steps, but they are straightforward.

We next remark that for \( \lambda = i\xi \) the operators \( L(t, D_t) \) and \( A \) are related by
\[(5.4) \quad \mathcal{L}(t, D_t) u = \Theta^{-1}(A(i\xi)(\Theta u)(\xi));
\]
here and later in similar situations, e.g. \((5.7), (5.8)\), we consider \( \xi \) as the Floquet variable for the inverse transform \((5.2)\). We also remark that the expression
\[(5.5) \quad (\Theta_\beta u)(\xi) =: \Theta_\lambda u
\]
(which also depends on \( t \in [0, 1] \)) is an analytic function of the variable \( \lambda = i\xi + \beta \in \mathbb{C} \), see \((5.1)\).

The converse of the statement of the following theorem is likely to hold true, but we do not need it here.

**Theorem 5.2.** The mapping
\[(5.6) \quad \mathcal{L}(t, D_t) : X_{-\beta} \to Y_{-\beta}
\]
is isomorphic if the semi-interval \( \delta_\beta \) does not contain eigenvalues of the operator pencil \( \mathcal{A}(\lambda) \).

Proof. It is clear that \( L(t, D_t) \) is a bounded operator in the given spaces. By the assumption of the theorem, for all \( \xi \in [0, 2\pi] \) the resolvent \( \mathcal{R}(i\xi + \beta) \in \mathcal{L}(\hat{Y}, \hat{X}) \) exists and moreover the operator norm of \( \mathcal{R}(i\xi + \beta) \) is uniformly bounded in \( \mathcal{L}(\hat{Y}, \hat{X}) \) for all \( \xi \in [0, 2\pi] \). This implies that also the mapping
\[g(\xi) \mapsto \mathcal{R}(i\xi + \beta) g(\xi), \quad \xi \in (0, 2\pi),
\]
is bounded as a mapping from \( L^2((0, 2\pi); \hat{Y}) \) into \( L^2((0, 2\pi); \hat{X}) \), and consequently
\[(5.6) \quad g(\xi) \mapsto A(i\xi + \beta) \mathcal{R}(i\xi + \beta) g(\xi), \quad \xi \in (0, 2\pi),
\]
is the identity operator on $L^2((0,2\pi);\hat{Y})$.

We define the operator

\begin{equation}
G(t, \beta)u = \Theta^{-1}_\beta(\mathcal{R}(i\xi + \beta)\Theta\beta u(\xi)),
\end{equation}

cf. (5.4). The above remark on the resolvent implies that this operator is bounded $\mathcal{Y}_- = L^2_-(\mathbb{R};\mathcal{Y}) \to X_-$, see (5.3). Using the commutation relation

\[ \mathcal{L}(t, D_t)\mathcal{M}_\beta u = \mathcal{M}_\beta(\mathcal{L}(t, D_t) + \beta)u \]

and (5.3), (5.4) we get for $u \in L^2_-(\mathbb{R};\mathcal{Y})$

\[ \mathcal{L}(t, D_t)G(t, \beta)u = \mathcal{L}(t, D_t)\mathcal{M}_\beta\Theta^{-1}\mathcal{R}(i\xi + \beta)\Theta\beta u \]

\[ = \mathcal{M}_\beta(\mathcal{L}(t, D_t) + \beta)\Theta^{-1}\mathcal{R}(i\xi + \beta)\Theta\beta u \]

\[ = \mathcal{M}_\beta(\Theta^{-1}\mathcal{A}(i\xi)\Theta + \beta)\Theta^{-1}\mathcal{R}(i\xi + \beta)\Theta\mathcal{M}_\beta u. \]

Here we can write

\[ \Theta^{-1}\mathcal{A}(i\xi)\Theta + \beta = \Theta^{-1}\mathcal{A}(i\xi + \beta)\Theta \]

so that (5.8) readily reduces by (5.6) into $u$. The converse relation $G(t, \beta)\mathcal{L}(t, D_t)u = u$ for $u \in X_-$ can be proven in the same way. This completes the proof. \hfill \square

Let $g \in L^2_-(\mathbb{R};\mathcal{Y}) = \mathcal{Y}_-$. The operator $G(t, \beta)$, which gives the solution of the problem (2.7) with the right-hand side $g$, can be written as

\begin{equation}
G(t, \beta)g = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{it\xi + \beta t} \left( R(i\xi + \beta)(\Theta\beta g(\xi)) \right)(t - [t])d\xi.
\end{equation}

Apparently, the integrand depends analytically on the parameter $\beta + i\xi \in \mathbb{C}$ in the domain of the analyticity of the resolvent $\mathcal{R}$.

**Theorem 5.3.** Let $\beta_1 < \beta_2$ be real numbers such that the semi-intervals $\delta_{\beta_1}$ and $\delta_{\beta_2}$ do not contain eigenvalues of the operator pencil $\mathcal{A}(\lambda)$ and let $f \in \mathcal{Y}_- \cap \mathcal{Y}_{-2}$. Denote by $u_1$ and $u_2$ solutions to the problem (2.7) from the spaces $X_{-\beta_1}$ and $X_{-\beta_2}$ respectively (which exist according to Theorem 5.2). Then

\begin{equation}
u_2 - u_1 = \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_{kj}-1} c_{j,m}^k \Phi_{j,m}(t),
\end{equation}

where the functions $\Phi_{j,m}^k$, $k = 1, \ldots, N$, are all the functions (3.13) such that the eigenvalues $\lambda_k$ belong to the set

\[ \mathcal{Q}(\beta_1, \beta_2) := \{ \lambda = \beta + i\xi : \beta_1 < \beta < \beta_2, \ \xi \in [0,2\pi) \}, \]

and $c_{j,m}^k$ are constants.

Proof. By [3], Theorem A.10.2, for $\lambda \in \mathcal{Q}(\beta_1, \beta_2)$ the resolvent $\mathcal{R}(\lambda) : \hat{\mathcal{Y}} \to \hat{X}$ of $\mathcal{A}(\lambda)$ is

\begin{equation}
\mathcal{R}(\lambda)\varphi = \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_{kj}-1} \sum_{\mu=0}^{m} \frac{(\varphi, \psi_{j,m}^k)\hat{\mathcal{R}}_{j,m}^k}{(\lambda - \lambda_k)m_{kj}-m} + F(\lambda)\varphi, \ \varphi \in \hat{\mathcal{Y}},
\end{equation}

where $\lambda_k$ are eigenvalues of $\mathcal{A}(\lambda)$, and $\mathcal{R}_{j,m}^k$ are constants. The proof is similar to the proof of Theorem 2.2.
where the numbers $\lambda_k$ run through all eigenvalues in $Q(\beta_1, \beta_2)$ and $F : [\beta_1, \beta_2] \times [0, 2\pi] \to \mathcal{L}(\hat{Y}, \hat{X})$ is an analytic operator valued function. We consider the counterclockwise oriented closed contour $\Gamma \subset Q(\beta_1, \beta_2)$ which consists of the line segments

$$
\Gamma_1 = \{ \beta_1 + \varepsilon + i\xi : \xi \in [0, 2\pi] \}, \quad \Gamma_2 = \{ \beta_2 - \varepsilon + i\xi : \xi \in [0, 2\pi] \}, \\
\Gamma_\pm = \{ \beta + \pm \pi : \beta \in [\beta_1, \beta_2] \},
$$

where $\varepsilon > 0$ is so small that all eigenvalues $\lambda_k$ are still inside $\Gamma$. (We assume here that no eigenvalues with $\Im \lambda_k = 0$ exist; if they do, the contour has to be shifted by $-ih$ for a small $h > 0$. Using $2\pi$-periodicity, one can argue in the same way as we present here, but the details are left to the reader.)

We have

$$
(\Theta_{\lambda} f, \psi_{j,\mu}^k)_{\hat{Y}} = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\lambda(n+\tau)} f(\tau + n) \psi_{j,\mu}^k(\tau)d\tau
$$

(5.13)

$$
= \int_{-\infty}^{\infty} e^{-\lambda \tau} f(\tau) \psi_{j,\mu}^k(\tau)d\tau
$$

By the Cauchy integral formula and (5.5), (5.12),

$$
\frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda t} R(\lambda) \Theta_{\lambda} f(t - [t]) d\lambda
$$

$$
= \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_k,j-1} \sum_{\mu=0}^{m} \frac{d^{m_k,j-1}}{d\lambda^{m_k,j-1}} \left( e^{-\lambda t} (\Theta_{\lambda} f, \psi_{j,\mu}^k)_{\hat{Y}} \right)_{\lambda=\lambda_k} \times \frac{1}{(m_k - m - 1)!} \varphi_{j,m-\mu}^k(t - [t]).
$$

(5.14)

Here we use (5.13) to write

$$
\frac{d^{m_k,j-1}}{d\lambda^{m_k,j-1}} \left( e^{-\lambda t} (\Theta_{\lambda} f, \psi_{j,\mu}^k)_{\hat{Y}} \right)_{\lambda=\lambda_k} = e^{\lambda_k t} \int_{-\infty}^{\infty} e^{-\lambda_k \tau} (t - \tau)^{m_k,j-1} f(\tau) \psi_{j,\mu}^k(\tau)d\tau.
$$

Taking into account the periodicity of $\varphi_{j,m}^k$ and changing $m \mapsto m_k,j - 1 - m$ we get from (5.14)

$$
\frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda t} R(\lambda) \Theta_{\lambda} f(t - [t]) d\lambda
$$

$$
= \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_k,j-1} \sum_{\mu=0}^{m_k,j-m-1} \frac{1}{m!} \times \int_{-\infty}^{\infty} e^{\lambda_k (t-\tau)} (t - \tau)^{m} f(\tau) \psi_{j,\mu}^k(\tau)d\tau \varphi_{j,m_k,j-m-1-\mu}^k(t).
$$
\[
= \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{\mu=0}^{m_{k,j}-1} \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} e^{\lambda_k (t-\tau)} (t-\tau)^m f(\tau) \psi_{j,\mu}^k(\tau) d\tau \times \phi_{j,m,\mu^{-1}-m}(t)
\]

Now we employ the relation (3.14), which yields
\[
\int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} e^{\lambda_k (t-\tau)} (t-\tau)^m f(\tau) \psi_{j,\mu}^k(\tau) d\tau \phi_{j,m,\mu^{-1}-m}(t)
\]
\[
= \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} e^{-\lambda_k \tau} (-\tau)^m f(\tau) \psi_{j,\mu}^k(\tau) d\tau \Phi_{j,m}(t)
\]
\[
= \sum_{m=0}^{m_{k,j}-1} c_{j,\mu,m}^k \Phi_{j,m}(t)
\]

where for all \( k = 1, \ldots, N, j = 1, \ldots, J_k, \mu = 0, \ldots, m_{k,j}-1, m = 0, \ldots, \mu, \)
\[
c_{j,\mu,m}^k = \int_{-\infty}^{\infty} \frac{1}{m!} e^{-\lambda_k \tau} (-\tau)^m f(\tau) \psi_{j,\mu}^k(\tau) d\tau.
\]

We obtain
\[
(5.15) \quad \frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda \mathcal{R}(\lambda)} \Theta \lambda f(t-[t]) d\lambda = \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_{k,j}-1} c_{j,m}^k \Phi_{j,m}(t),
\]

where
\[
c_{j,m}^k = \sum_{\mu=0}^{m_{j,k}-1} c_{j,\mu,m}^k \quad (\text{with } c_{j,\mu,m}^k = 0, \text{ if } \mu + m > m_{j,k} - 1).
\]

On the other hand, we have
\[
(5.16) \quad \frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda \mathcal{R}(\lambda)} \Theta \lambda f(t-[t]) d\lambda = \sum_{j=2,3}^{1} \frac{1}{\sqrt{2\pi}} \int_{\Gamma_j} e^{\lambda \mathcal{R}(\lambda)} \Theta \lambda f(t-[t]) d\lambda
\]
since the integrals over \( \Gamma_{\pm} \) cancel out each other due to opposite integration directions and \( 2\pi \)-periodicity. Moreover, by (5.9), we get (by taking into account the proper direction)
\[
\frac{1}{\sqrt{2\pi}} \int_{\Gamma_1} e^{\lambda \mathcal{R}(\lambda)} \Theta \lambda f(t-[t]) d\lambda = -G(t, \beta_1) f = -u_1
\]
and the corresponding integral over the contour \( \Gamma_2 \) equals \( G(t, \beta_2) f = u_2 \). The result follows by combining these with (5.15) and (5.16). \( \square \)

A straightforward consequence of the last theorem is the following uniqueness result.
Corollary 5.4. Let \( \beta_1 \) and \( \beta_2 \) be the same as in Theorem 5.3. If \( u \in X_{loc} \) is a solution of (2.7) with \( f = 0 \) and

\[
\|u; X(t, t + 1)\| \leq C e^{\beta_1 t} \quad \text{for } t \geq 0 \quad \text{and} \quad \|u; X(t, t + 1)\| \leq C e^{\beta_2 t} \quad \text{for } t \leq 0
\]

for some positive constant \( C \), then

\[
\sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_{k,j}} c_{j,m}^k \Phi_{j,m}^k(t),
\]

where \( c_{j,m}^k \) are constants and \( \Phi_{j,m}^k \) are all functions (3.15) such that the eigenvalues \( \lambda_k \) belong to the set (5.11).

Proof. Let \( \beta_1' > \beta_1 \) and \( \beta_2' < \beta_2 \) be such that the intervals \( [\beta_1, \beta_1'] \) and \( [\beta_2', \beta_2] \) do not contain eigenvalues of the operator pencil \( A(\lambda) \). Let also \( \eta = \eta(t) \) be a smooth function of one variable such that \( \eta(t) = 1 \) for \( t > 1 \) and \( \eta(t) = 0 \) for \( t < 0 \). Consider the problem (2.7) with \( f = (D_t \eta) u \). This problem has two solutions, \( u_1 = \eta u \) and \( u_2 = (\eta - 1) u \). Since

\[
\int_0^\infty e^{-\beta_1' t} \|u_1(t)\|^2 dt \leq C \int_0^\infty e^{-\beta_2' t} \|u; X(t, t + 1)\|^2 dt \leq C \int_0^\infty e^{t(\beta_1 - \beta_1')} dt < \infty,
\]

\( u_1 \) belongs to \( X_{-\beta_1'} \). Similarly, \( u_2 \) belongs to \( X_{-\beta_2'} \). By Theorem 5.3, \( u = u_2 - u_1 \) is equal to the right-hand side of (5.10) and we arrive at (5.18). \( \square \)

6. Spectral splitting.

We assume that \( \beta_1 < \beta_2 \) are the same real numbers as in Theorem 5.3 so that in particular the semi-intervals \( \delta_{\beta_1}, \delta_{\beta_2} \) do not contain eigenvalues of the pencil \( A(\lambda) \) and its eigenvalues in the set \( Q(\beta_1, \beta_2) \) of (5.11) are \( \lambda_1, \ldots, \lambda_N \). We introduce the function

\[
\mu(t) = e^{-\beta_1 t} \quad \text{for } t \geq 0 \quad \text{and} \quad \mu(t) = e^{-\beta_2 t} \quad \text{for } t \leq 0.
\]

Using the projector \( P \) of (4.1) we represent a solution of (2.7) as

\[
u = U + V, \quad \text{where} \quad U = Pu, \quad V = Qu := (I - P)u.
\]

Due to the commutation relation (4.4) we have the following system of equations for \( U \) and \( V \):

\[
(6.1) \quad \mathcal{L}(t, D_t) U(t) = (D_t + A(t)) U(t) = Pf
\]

\[
(6.2) \quad \mathcal{L}(t, D_t) V(t) = (D_t + A(t)) V(t) = Qf.
\]

Using representations (4.1), (4.3) and writing

\[
P f(t) = \sum_{k=1}^{N} \sum_{j=1}^{J_k} \sum_{m=0}^{m_{k,j}} f_{j,m}^k(t) \psi_{j,m}^k(t),
\]

where

\[
f_{j,m}^k(t) = (f(t), \psi_{j,m}^k(t)),
\]

we can present (6.1) as a system of first order differential equations

\[
(6.3) \quad (D_t - \lambda_k) u_{j,m}^k(t) + u_{j,m-1}^k = f_{j,m}^k,
\]
Here, \( k = 1, \ldots, N, \ j = 1, \ldots, J_k, \ m = 0, \ldots, m_{k,j} - 1, \) and \( u_{j,m}^k \) is given by (4.2) and we assume that \( u_{j,m-1}^k = 0 \) if \( m = 0. \)

The equation (6.2) concerns the "remainder" term: here we have removed the spectrum \( \lambda_1, \ldots, \lambda_N \) from the operator using the projector, and hence it has better estimates. This property of the equation (6.2) is contained in the following assertion.

**Theorem 6.1.** Let \( f \in L^2_{\text{loc}}(\mathbb{R}; Y) \) and

\[
(6.4) \quad \int_{\mathbb{R}} \mu(t) \| f; \mathcal{Y}(t, t+1) \| dt < \infty.
\]

Then, the equation

\[
\mathcal{L}(t, D_t)u = f
\]

has a solution \( u = U + V \in \mathcal{X}_{\text{loc}} \) such that \( u \) is a solution of (6.1) and \( V \) is a solution of (6.2) satisfying the estimate

\[
(6.5) \quad \| V; \mathcal{X}(\tau, \tau + 1) \| \leq C \int_{\mathbb{R}} \mu(t - \tau) \| \mathcal{Q}f; \mathcal{Y}(t, t+1) \| dt
\]

for all \( \tau \in \mathbb{R} \).

Let \( f \) satisfy (6.4) and \( \mathcal{Q}f = 0. \) If the bounds (5.17) with some constant \( C \) hold for \( u \), then \( V = 0. \)

Proof. We need to prove here the existence of \( V \) satisfying (6.5) as well as the last uniqueness statement. We start by the uniqueness. Let \( \mathcal{Q}f = 0 \) and assume \( u \) satisfies (5.17). Let the coefficients of \( \mathcal{P}u \) in (4.1) satisfy (6.3). Then \( \mathcal{P}u \) is a solution to (2.7) and by analysing solutions of the ordinary differential equations (6.3) we conclude that this solution also satisfies (5.17) possibly with a slightly larger \( \beta_1 \) and smaller \( \beta_2 \). Then, according to the above uniqueness result from Corollary 5.4 we have \( u = \mathcal{P}u \) and hence \( V = \mathcal{Q}u = 0. \)

Let us turn to the existence. Let first \( f \) have a compact support and let \( g = \mathcal{Q}f. \) Applying Theorem 5.2 to the equation \( \mathcal{L}V = g, \) we get the estimates

\[
(6.6) \quad \int_{-\infty}^{\infty} e^{-\beta_k(t-\tau)} \| V(t); X \|^2 dt \leq C \int_{-\infty}^{\infty} e^{-\beta_k(t-\tau)} \| g(t); Y \|^2 dt,
\]

for \( k = 1, 2 \) and all \( \tau \in \mathbb{R}. \) Finally, assume \( f \) is as in (6.4) and write \( f = \sum_{j=-\infty}^{\infty} f_j, \) where

\[
f_j(t) = f \text{ on } (j, j+1) \text{ and } f_j = 0 \text{ for } t \text{ outside } (j, j+1), \ j = 0, \pm 1, \pm 2, \ldots.
\]

Using estimate (6.6) for the function \( V_j \) (corresponding to \( f_j \) and \( g_j = \mathcal{Q}f_j \)) we get

\[
\left( \int_{\tau}^{\tau+1} \| V_j(t); X \|^2 dt \right)^{1/2} \leq C \mu(j - \tau) \left( \int_{j}^{j+1} \| \mathcal{Q}f(t); Y \|^2 dt \right)^{1/2}.
\]

Summing up these relations, we obtain for all \( \tau \in \mathbb{R} \)

\[
\left( \int_{\tau}^{\tau+1} \| V(t); X \|^2 dt \right)^{1/2} \leq C \int_{\mathbb{R}} \mu(t - \tau) \| \mathcal{Q}f(t); Y \| dt,
\]

which is the same as (6.5). \( \square \)
Remark 6.2. We note that the function $\mu(t)$ is the Green function of the second order operator $-(D_t - \beta_1)(D_t - \beta_2)$ up to a positive constant factor. So the estimate (6.5) is similar to the representation of the solutions to the equation

$$-(D_t - \beta_1)(D_t - \beta_2)u(t) = f(t)$$

through the Green function and the right-hand side.

Estimates (6.4) and (6.5) imply estimate (5.17) possibly with some slightly larger (smaller) $\beta_1$ ($\beta_2$) and hence uniqueness and existence parts in Theorem 6.1 are in agreement.

Acknowledgement. V. K. was supported by the Swedish Research Council (VR), 2017-03837. J. T. was supported by a research grant from the Faculty of Science of the University of Helsinki.

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