Improved Lower Bound for the Union-Closed Sets Conjecture

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Abstract

We verify an explicit inequality conjectured in [5], thus proving that for any nonempty union-closed family $F \subseteq 2^{[n]}$, some $i \in [n]$ is contained in at least $3 - \sqrt{5}/2 \approx 0.38$ fraction of the sets in $F$. One case, an explicit one-variable inequality, is checked by computer calculation.

1 Introduction

Let $M_\phi$ be the set of probability measures $\mu \in P([0,1])$ with expectation $\phi$. Define

$$F(\mu) = \mathbb{E}_{(x,y) \sim \mu \times \mu} H(xy) - \mathbb{E}_{x \sim \mu} H(x)$$

(1)

where $H(x) = -x \log x - (1-x) \log(1-x)$ is the entropy function and log denotes natural logarithm. Note that $F$ is continuous in the weak topology and $M_\phi$ is compact, so $F$ has a minimizer over $M_\phi$. In this note, we will show the following results.

**Theorem 1.** For all $\phi \in [0,1]$, the minimum of $F(\mu)$ over $M_\phi$ is attained at some $\mu$ supported on at most two points. Furthermore, if a minimizer is supported on exactly two points, then one of the points is 0.

The case of $\mu$ supported on $\{0, x\}$ leads to the following definition:

$$S = \{ \phi \in [0,1] : \phi H(x^2) \geq x H(x) \ \forall x \in [\phi,1] \}, \ \ \phi^* = \min(S).$$

Note that the condition defining $S$ is monotone in $\phi$ and $S$ is clearly closed, so $\min(S)$ is well defined. As in the recent breakthrough [5] by Gilmer, a bound on the union-closed conjecture follows from the above. See [1, 7, 8, 2, 6] for past study of this conjecture or the survey [3].

**Theorem 2.** The union-closed conjecture holds with constant $1 - \phi^*$, i.e. for any non-empty union-closed family $F \subseteq 2^{[n]}$, some $i \in [n]$ is contained in at least $1 - \phi^*$ fraction of the sets in $F$.

Throughout this paper we set $\varphi = \frac{\sqrt{5} - 1}{2}$. In the Appendix, we give a numerical verification of the following claim. We require certain computer calculations (detailed in an attached Python file) to be accurate to within margin of error $10^{-3}$, which can be made completely rigorous using interval arithmetic.

**Claim 3.** If $x \in [\varphi,1]$, then $\varphi H(x^2) \geq x H(x)$, with equality if and only if $x \in \{\varphi,1\}$.

Assuming Claim 3, the following claim identifies the value of $\phi^*$. Then, Theorem 2 implies that the union-closed conjecture holds with constant $1 - \varphi = \frac{3 - \sqrt{5}}{2}$. This is a natural barrier for the method of [5] as explained therein.

**Claim 4.** We have that $\phi^* = \varphi$.

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2 Reduction to Two Point Masses

Lemma 5. F is concave on $\mathcal{M}_\phi$ for any $\phi \in [0,1]$, i.e.

$$pF(\mu_1) + (1 - p)F(\mu_2) \leq F(p\mu_1 + (1 - p)\mu_2) \quad \forall \mu_1, \mu_2 \in \mathcal{M}_\phi, \; p \in [0,1].$$  \hfill (2)

Proof. Let $\gamma(x) = \mu([0,x])$ be the cumulative distribution function of $\mu$. Thus $\gamma(1) = 1$ and

$$\phi = \int_0^1 x \mu(dx) = 1 - \int_0^1 \gamma(x) \, dx,$$

so

$$\int_0^1 \gamma(x) \, dx = 1 - \phi. \quad \hfill (3)$$

Using integration by parts,

$$\int_0^1 H(x) \mu(dx) = H(x)\gamma(x)\bigg|_0^1 - \int_0^1 H'(x)\gamma(x) \, dx = \int_0^1 \left(\log \frac{x}{1-x}\right) \gamma(x) \, dx.$$\

Similarly,

$$\int_0^1 H(xy) \mu(dy) = H(xy)\gamma(y)\bigg|_0^1 - \int_0^1 x H'(xy)\gamma(y) \, dy$$

$$\quad = H(x) + \int_0^1 \left(x \log \frac{xy}{1-xy}\right) \gamma(y) \, dy;$$

$$\int_0^1 \left(x \log \frac{xy}{1-xy}\right) \mu(dx) = \left(x \log \frac{xy}{1-xy}\right) \gamma(x)\bigg|_0^1 - \int_0^1 \frac{d}{dx} \left(x \log \frac{xy}{1-xy}\right) \gamma(x) \, dx,$$

$$\quad = \log \frac{y}{1-y} - \int_0^1 \left(1 - \frac{1}{1-xy} + \log \frac{xy}{1-xy}\right) \gamma(x) \, dx;$$

$$\int \int_{[0,1]^2} H(xy) \mu(dx) \mu(dy) = \int_0^1 H(x) \mu(dx) + \int_0^1 \gamma(y) \int_0^1 x \log \frac{xy}{1-xy} \mu(dx) \, dy,$$

$$\quad = 2 \int_0^1 \left(\log \frac{x}{1-x}\right) \gamma(x) \, dx - \int \int_{[0,1]^2} \left(1 - \frac{1}{1-xy} + \log \frac{xy}{1-xy}\right) \gamma(x) \gamma(y) \, dx \, dy.$$\

So, letting $F(\gamma) = F(\mu)$ by slight abuse of notation, we have

$$F(\gamma) = \int_0^1 \left(\log \frac{x}{1-x}\right) \gamma(x) \, dx - \int \int_{[0,1]^2} \left(\log x + \log y + \frac{1}{1-xy} + \log \frac{1}{1-xy}\right) \gamma(x) \gamma(y) \, dx \, dy.$$\

We will show this is concave in $\gamma$. The first integral is manifestly linear in $\gamma$, and the contributions of $\log x$ and $\log y$ are linear because, in light of (3),

$$\int \int_{[0,1]^2} \left(\log x\right) \gamma(x) \gamma(y) \, dx \, dy = (1 - \phi) \int_0^1 \left(\log x\right) \gamma(x) \, dx.$$

After removing these terms, we are reduced to showing convexity of

$$\int \int_{[0,1]^2} \left(\frac{1}{1-xy} + \log \frac{1}{1-xy}\right) \gamma(x) \gamma(y) \, dx \, dy.$$
Note that both \( \frac{1}{1-xy} \) and \( \log \frac{1}{1-xy} \) are of the form \( \sum_{k \geq 0} a_k x^k y^k \) for constants \( a_k \geq 0 \). Hence it suffices to prove convexity of
\[
\int \int_{[0,1]^2} x^k y^k \gamma(x) \gamma(y) \, dx \, dy = \left( \int_0^1 x^k \gamma(x) \, dx \right)^2
\]
for any \( k \geq 0 \). This is the square of a linear function of \( \gamma \), and hence is convex. (Note that all integrands are in \( L^1 \) and so there are no convergence issues.)

**Lemma 6.** \( \arg \min_{\mu \in \mathcal{M}_\phi} F(\mu) \) contains some \( \mu \) supported on at most two points.

**Proof.** This follows immediately from Lemma 5 and the Krein-Milman theorem since \( \mathcal{M}_\phi \) is compact in the weak topology and convex, and all extreme measures in \( \mathcal{M}_\phi \) are supported on 1 or 2 points.

We also include a more explicit and elementary version of this argument which proceeds as follows. First let \( \mu \in \mathcal{M}_\phi \) be any minimizer of \( F \) and note that \( \mu \) can be approximated arbitrarily well in the weak topology by \( \hat{\mu} \) with finite support. In particular for any \( \varepsilon > 0 \), there exists \( \hat{\mu} \in \mathcal{M}_\phi \) with \( F(\hat{\mu}) \geq F(\mu) - \varepsilon \) of the form
\[
\hat{\mu}(a_i) = b_i - b_{i-1}, \quad 1 \leq i \leq k
\]
for constants \( 0 \leq a_1 < \cdots < a_k \leq 1 \) and \( 0 = b_0 < b_1 < \cdots < b_k = 1 \). We claim that for any \( \varepsilon > 0 \), the minimal \( k \) such that such a \( \hat{\mu} \) exists is at most two. Indeed given such a \( \hat{\mu} \) with \( k \geq 3 \), we may consider \( \hat{\mu}_\eta \) defined by
\[
\hat{\mu}_\eta(a_1) = b_1 - b_0 + \eta(a_3 - a_2),
\]
\[
\hat{\mu}_\eta(a_2) = b_2 - b_1 - \eta(a_3 - a_1),
\]
\[
\hat{\mu}_\eta(a_3) = b_3 - b_2 + \eta(a_2 - a_1).
\]
Then \( \hat{\mu}_\eta \in \mathcal{M}_\phi \) if and only if \( -c_1 \leq \eta \leq c_2 \) for some \( c_1, c_2 > 0 \) and moreover the map \( \eta \mapsto F(\hat{\mu}_\eta) \) is concave by Lemma 5. It is easy to see that both \( \hat{\mu}_{-c_1}, \hat{\mu}_{c_2} \) have support size at most \( k - 1 \), and at least one of \( F(\hat{\mu}_{-c_1}), F(\hat{\mu}_{c_2}) \) is at most \( F(\hat{\mu}) \) by concavity. Iterating this argument, we find a \( \tilde{\mu} \in \mathcal{M}_\phi \) with support size at most 2 and with \( F(\tilde{\mu}) \geq F(\mu) - \varepsilon \). Taking a subsequential weak limit of the resulting \( \tilde{\mu} \) as \( \varepsilon \to 0 \) completes the proof.

3 Optimization over Two Point Masses

**Lemma 7.** If \( \mu \) is supported on exactly two points, neither of which is 0, then \( \mu \) is not a minimizer of \( F \) over \( \mathcal{M}_\phi \).

**Proof.** Suppose \( \mu = p \delta_x + (1-p) \delta_y \) is a minimizer for \( F \) over \( \mathcal{M}_\phi \) for \( 0 < y < x < 1 \) distinct and \( 0 < p < 1 \). Then any \( z \in [0,1] \) can be written as \( z = qx + (1-q)y \) for some \( q \in \mathbb{R} \) (which may be negative). We have
\[
\mu + t \delta_z - tq \delta_x - t(1-q) \delta_y \in \mathcal{M}_\phi
\]
for sufficiently small \( t \geq 0 \) and so
\[
\lim_{t \to 0^+} \frac{F(\mu + t \delta_z - tq \delta_x - t(1-q) \delta_y) - F(\mu)}{t} \geq 0.
\]
It is not difficult to see from the definition (1) of \( F \) that the left-hand limit equals
\[
f(z) - qf(x) - (1-q)f(y) \geq 0,
\]
for
\[
f(w) := 2[pH(xw) + (1-p)H(yw)] - H(w).
\]
Equation (4) implies that \( f \) lies above the line passing through \((x,f(x))\) and \((y,f(y))\). Since \( f \) is a smooth function and \( x, y \) are in the interior of \([0,1]\), we deduce that
However we compute using $H_w$.

By Theorem 1, it suffices to check $x = y$.
Moreover, (a) implies $\phi = 0$. 

Corollary 10. Suppose $\{c\} \subset [0,1]$ is a finite sequence of real numbers and $c$ is a random variable supported on $S$ such that $c \in [0,1]$. If $c'$ is independent of $c$, then

$$\mathbb{E}_{c,c'}[H(p_c + p_{c'} - p_c p_{c'})] \geq \mathbb{E}[H(p_c)].$$

4 Conclusion

**Proof of Theorem 1.** Follows from Lemmas 6 and 7.

**Lemma 8.** We have that $\phi^* \geq \varphi$. 

**Proof.** Note that $H(\varphi^2) = H(\phi)$. If $\phi < \varphi$, then $\phi H(\varphi^2) < \varphi H(\phi)$, and so $\phi \not\in S$.

**Corollary 9.** If $\phi \geq \phi^*$, then $F(\mu) \geq 0$ for all $\mu \in M_{\phi}$. 

**Proof.** By Theorem 1, it suffices to check $F(\mu) \geq 0$ for $\mu = \delta_{\phi}$ and $\mu = p\delta_x + (1-p)\delta_0$ with $p = \phi/x$ and $x \in [\phi,1]$. In the former case,

$$F(\mu) = H(\phi^2) - H(\phi) \geq 0$$

because $\phi \geq \phi^* \geq \varphi$ by Lemma 8. In the latter case,

$$F(\mu) = \frac{\phi^2}{x^2} H(x^2) - \frac{\phi}{x} H(x) = \frac{\phi}{x^2}(\phi H(x^2) - x H(x)) \geq 0.$$
Proof. Let $\mu$ be the distribution of $x = 1 - p_c$. Let $\phi = \mathbb{E}_{x \sim \mu}[x]$, so $\phi > \phi^*$. By Corollary 9,

$$
\mathbb{E}_{c,c'}[H(p_c + p_{c'} - p_c p_{c'})] - \mathbb{E}_c[H(p_c)] = \mathbb{E}_{(x,y) \sim \mu \times \mu} H(xy) - \mathbb{E}_x H(x) = F(\mu) \geq 0.
$$

Finally, we verify Claim 4 assuming Claim 3.

**Proof of Claim 4.** Claim 3 and monotonicity of the condition defining $S$ imply that $\phi^* \leq \phi$, while Lemma 8 gives $\phi^* \geq \phi$.

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**A Proof of Claim 3**

In this appendix, we prove Claim 3. Throughout this appendix, we use Claims to indicate results requiring the correctness of computer outputs within margin of error $10^{-3}$ or greater. The only computations which rely on a computer are the entries in Tables 1 and 2. Figure 1 plots the function

$$
G(x) = \phi H(x^2) - x H(x),
$$

from which Claim 3 can be checked visually. We show below that, assuming correctness of certain computer calculations to within margin of error $10^{-3}$,

$$
G(x) \geq 0, \quad \forall x \in [\phi, 1].
$$

The verification is done separately on the three intervals $I_1 = [\phi, 0.77], I_2 = [0.76, 0.98], I_3 = [0.98, 1]$.

**A.1 Verification on $I_1$**

We first compute the derivative of $G$:

$$
G'(x) = 2x \phi \log \frac{1 - x^2}{x^2} - H(x) - x \log \frac{1 - x}{x}
$$

$$
= 2x \phi \log \frac{1 - x^2}{x^2} + x \log x + (1 - x) \log(1 - x) + x \log x - x \log(1 - x)
$$

$$
= 2x \phi \log \frac{1 - x^2}{x^2} + 2x \log x + (1 - 2x) \log(1 - x)
$$
Figure 1: Plot of $G(x)$ for $x \in [0.6, 1]$. Claim 3 states the minimum value of 0 on $x \in [\varphi, 1]$ is achieved precisely at the endpoints $x \in \{\varphi, 1\}$.

Note that $G(\varphi) = G'(\varphi) = 0$, the latter since

$$G'(\varphi) = 2\varphi^2 \log(1/\varphi) + 2\varphi \log(\varphi) + (1 - 2\varphi) \log(\varphi^2)$$
$$= (-2\varphi^2 + 2\varphi + 2(1 - 2\varphi)) \log \varphi$$
$$= 2(1 - \varphi - \varphi^2) \log(\varphi) = 0.$$

**Claim 11.** Claim 3 holds on $I_1 = [\varphi, 0.77]$.

**Proof.** As $G(\varphi) = G'(\varphi) = 0$, it suffices to verify that $G$ is convex on $I_1$. It is not hard to check that its second derivative equals

$$G''(x) = \frac{L(x)}{1 - x^2},$$

where

$$L(x) := 2\varphi(1 - x^2) \log(x^{-2} - 1) - 4\varphi - 2x^2 \log x + 2(x^2 - 1) \log(1 - x) + x + 2 \log(x) + 1.$$ 

We now estimate the Lipschitz constant of each non-constant term of $L$ on $x \in I_1$. For the first term,

$$\left| \frac{d}{dx}(2\varphi(1 - x^2) \log(x^{-2} - 1)) \right| \leq 2\varphi \sup_{x \in I_1} \left( |2x^3| + 2|x \log(x^{-2} - 1)| \right)$$
$$\leq 2\varphi(1.1 + 1.6 \cdot \log(2))$$
$$\leq 2\varphi \cdot 2.3 \leq 3$$

since $\log(2) \leq 0.75$ and $\varphi \leq 5/8$. Next,

$$\left| \frac{d}{dx}(2x^2 \log(x)) \right| \leq \sup_{x \in I_1} |4x \log(x) + 2x|$$
$$\leq 1.6 \sup_{x \in I_1} |2 \log(x) + 1|$$
$$\leq 1.6$$

since $\log(x) \in [-1, 0]$ for all $x \in I_1$. Continuing, using $\log(5) \leq 2$,

$$\left| \frac{d}{dx}(2(x^2 - 1) \log(1 - x)) \right| \leq 2 \sup_{x \in I_1} |2x \log(1 - x) - \frac{x^2 - 1}{1 - x}|$$
$$\leq 2 \sup_{x \in I_1} |2x \log(1 - x) + x + 1|$$
$$\leq 2 \cdot \max(1.6 \log(5), 1.8)$$
$$\leq 2 \cdot 1.6 \cdot 2 = 6.4.$$
Finally \( \frac{d}{dx}(x) = 1 \) and \( \frac{d}{dx}(2\log x) = 2/x \leq 3.5 \). Moreover the derivative from the term \((5)\) is negative as both terms are positive and decreasing, while the derivative from \(2\log x\) is clearly positive. Combining, we find that \( L(x) \) restricted to \( I_1 \) has Lipschitz constant at most

\[ 1.6 + 6.4 + 1 + \max(3, 3.5) \leq 12.5. \]

Therefore to show \( G \) is convex and hence non-negative on \( I_1 = [\varphi, 0.77] \) it suffices to exhibit a \( \frac{1}{250} \)-dense subset of \( I_1 \) on which \( L(x) = (1 - x^2)G''(x) \geq \frac{12.5}{250} = 0.05 \). In Table 1 below we compute the values of \( L \) on each multiple of \( \frac{1}{250} \) from 0.6 to 0.77 inclusive. We in fact find that \( f(x) \geq 0.09 \) holds at all of these points, completing the numerical verification on \( I_1 \).

\[
\begin{array}{cccccccccc}
\hline
x & \mid & L(x) & x & \mid & L(x) & x & \mid & L(x) & x & \mid & L(x) \\
\hline
0.600 & 0.1020 & 0.630 & 0.1117 & 0.660 & 0.1173 & 0.690 & 0.1182 & 0.720 & 0.1137 & 0.750 & 0.1032 \\
0.605 & 0.1039 & 0.635 & 0.1130 & 0.665 & 0.1178 & 0.695 & 0.1178 & 0.725 & 0.1124 & 0.755 & 0.1069 \\
0.610 & 0.1057 & 0.640 & 0.1141 & 0.670 & 0.1182 & 0.700 & 0.1173 & 0.730 & 0.1109 & 0.760 & 0.0983 \\
0.615 & 0.1074 & 0.645 & 0.1151 & 0.675 & 0.1184 & 0.705 & 0.1167 & 0.735 & 0.1093 & 0.765 & 0.0955 \\
0.620 & 0.1089 & 0.650 & 0.1159 & 0.680 & 0.1185 & 0.710 & 0.1159 & 0.740 & 0.1075 & 0.770 & 0.0925 \\
0.625 & 0.1104 & 0.655 & 0.1167 & 0.685 & 0.1184 & 0.715 & 0.1149 & 0.745 & 0.1064 & & \\
\hline
\end{array}
\]

Table 1: Evaluations of \( L \) to precision \( 10^{-4} \). All values appear to be at least 0.09, and it suffices for all values to be at least 0.05.

### A.2 Verification on \( I_2 \)

Our verification for \( x \in I_2 \) is based on evaluating \( G \). We write \( G(x) = g_1(x) - g_2(x) \) for

\[
\begin{align*}
g_1(x) &= \varphi H(x^2), \\
g_2(x) &= xH(x).
\end{align*}
\]

Note that \( g_1 \) is clearly decreasing on \( I_2 \). The next lemma shows the same for \( g_2 \).

**Lemma 12.** \( g_2 \) is decreasing on \([5/7, 1] \supseteq I_2 \).

**Proof.** First we claim that it suffices to show \( g_2'(5/7) \leq 0 \). This is because

\[
g_2'(x) = H(x) + x \log \frac{1 - x}{x} = 2x \log \frac{1}{1 - x} - (2x - 1) \log \frac{1}{1 - x}
\]

so \( g_2'(x) \leq 0 \) if and only if

\[
\left( 1 - \frac{1}{2x} \right) \log \frac{1}{1 - x} \geq \log \frac{1}{x}
\]

and here both terms on the left-hand side are increasing while the right-hand side is decreasing.

It remains to show that \( g_2'(5/7) \leq 0 \) which in light of (6) is equivalent to showing

\[
\frac{3}{10} \log(7/2) \geq \log(7/5),
\]

i.e. \((7/5)^{10/3} \leq 7/2\). This holds because \((7/5)^3 \leq 2(7/5) = 14/5\) and \(7/5 \leq \left( \frac{5}{4} \right)^3 = \left( \frac{7/2}{14/5} \right)^3\). \(\square\)

**Claim 13.** Claim 3 holds for \( x \in I_2 \).

**Proof.** We computer-evaluate \( g_1, g_2 \) at a finite set of values \( x_1 < x_2 < \cdots < x_{94} \) with \( 5/7 < x_1 < 0.76 \) and \( x_{94} = 0.98 \) and verify that \( g_1(x_{i+1}) \geq g_2(x_i) \) for each \( i \). The values are shown in Table 2; note that in all cases \( g_1(x_{i+1}) - g_2(x_i) \geq \frac{2}{1000} \) holds, modulo rounding to four decimal places. The intervals \( [x_i, x_{i+1}] \) cover \( I_2 \), and for all \( x \in [x_i, x_{i+1}] \) we have

\[
g_2(x) \leq g_2(x_i) \leq g_1(x_{i+1}) \leq g_1(x).
\]

\(\square\)
Table 2: Evaluations of $g_1$ and $g_2$ to precision $10^{-4}$. We require that for consecutive inputs $x_i < x_{i+1}$ in the table, $g_1(x_{i+1}) - g_2(x_i) \geq 0$. The values shown in fact satisfy $g_1(x_{i+1}) - g_2(x_i) \geq \frac{2}{1000}$ modulo rounding.

### A.3 Verification on $I_3$

**Proposition 14.** Claim 3 holds for $x \in I_3$.

**Proof.** Taylor expansion of $\log(1-x)$ gives that for all $\varepsilon \in (0,1)$,

$$\epsilon \left( \log \frac{1}{\varepsilon} + 1 - \varepsilon \right) \leq H(\varepsilon) \leq \epsilon \left( \log \frac{1}{\varepsilon} + 1 \right).$$

Let $x = 1 - \varepsilon$ for $\varepsilon \in [0,0.02]$. Then

$$g_1(x) = \varphi H(2\varepsilon - \varepsilon^2) \geq \varphi \varepsilon (2 - \varepsilon) (\log \frac{1}{\varepsilon} - \log(2 - \varepsilon) + (1 - \varepsilon)^2),$$

$$g_2(x) = (1 - \varepsilon) H(\varepsilon) \leq \epsilon (1 - \epsilon) \left( \log \frac{1}{\varepsilon} + 1 \right).$$

Dividing by $\epsilon$, it suffices to prove

$$((2\varphi - 1) + (1 - \varphi)\varepsilon) \log \frac{1}{\varepsilon} \geq (1 - \epsilon) (1 - \varphi (1 - \epsilon)(2 - \epsilon)) + \varphi (2 - \epsilon) \log(2 - \epsilon).$$

Noting $\varphi(1-\varepsilon)(2-\varepsilon) \geq 1$ in the first line below, we next find

$$(1 - \epsilon) (1 - \varphi(1 - \epsilon)(2 - \epsilon)) + \varphi(2 - \epsilon) \log(2 - \epsilon) \leq 2 \varphi \log 2 = (\sqrt{5} - 1) \log 2,$$

$$((2\varphi - 1) + (1 - \varphi)\varepsilon) \log \frac{1}{\varepsilon} \geq (2\varphi - 1) \log \frac{1}{\varepsilon} \geq (\sqrt{5} - 2) \log 50.$$