Space-Time Codes Based on Rank-Metric Codes and Their Decoding

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Abstract—We propose a new class of space-time block codes based on finite-field rank-metric codes in combination with a rank-metric-preserving mapping to the set of Eisenstein integers. It is shown that these codes achieve maximum diversity order and improve upon certain existing constructions. Moreover, we present a new decoding algorithm for these codes which utilizes the algebraic structure of the underlying finite-field rank-metric codes and employs lattice-reduction-aided equalization. This decoder does not achieve the same performance as the classical maximum-likelihood decoding methods, but has polynomial complexity in the matrix dimension, making it usable for large field sizes and numbers of antennas.

Index Terms—Space-Time Codes, Gabidulin Codes, Eisenstein Integers, Decoding, Lattice Reduction

I. INTRODUCTION

Space-time (ST) codes were first introduced in [1] for multiple-input/multiple-output (MIMO) fading channels in point-to-point single-user (multi-antenna) scenarios. Several code constructions have been proposed so far, both ST convolutional and block codes. ST codes are usually maximum-likelihood decoded, yielding an exponential decoding complexity.

An important design criterion for ST codes is that the rank distance of two codewords must be as large as possible [1]. In [2, 3], finite-field rank-metric codes were used to construct ST block codes by mapping the finite-field elements to a modulation alphabet in the complex plane. It was shown that this mapping preserves the minimum rank distance of the finite-field code in case of binary phase shift keying and subsets of the Gaussian integers [4], as well as for other important constellations [5].

In this paper, we prove that there is a rank-metric-preserving mapping in the case of Eisenstein integers [6]. The use of this modulation alphabet promises to improve upon other modulation alphabets in C, since Eisenstein integers form the hexagonal lattice in C, the densest possible lattice in a 2-dimensional real vector space.

Furthermore, we present an alternative decoding method for these ST codes, using lattice-reduction-aided (LRA) equalization techniques in combination with a decoding algorithm of the underlying finite-field rank-metric code. This decoder is sub-optimal in terms of failure probability compared to the classical ML decoding methods, but has polynomial complexity and hence can be used for a larger set of parameters.

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The paper is organized as follows. In Section II, we describe the channel model and provide basics on Eisenstein integers and rank-metric codes. We propose a new ST code construction in Section III and present alternative decoding methods in Section IV. Section V concludes the paper.

II. PRELIMINARIES

A. Channel Model

We assume a flat-fading MIMO channel with additive Gaussian noise, i.e.

\[ Y = HX + N, \]

where \( N_{tx}, N_{rx}, N_{time} \in \mathbb{N} \) are the numbers of transmit antennas, receive antennas and time steps respectively, \( X \in \mathbb{C}^{N_{tx} \times N_{time}} \) is the sent codeword and \( Y \in \mathbb{C}^{N_{rx} \times N_{time}} \) is the received word (both over space (rows) and time (columns)). \( H \in \mathbb{C}^{N_{rx} \times N_{tx}} \) is the channel matrix, which is known at the receiver (perfect channel state information) and whose entries are drawn i.i.d. from the complex Gaussian distribution \( \mathcal{N}(0,1) \). Also, \( N \in \mathbb{C}^{N_{rx} \times N_{time}} \) is the noise matrix, which is unknown at the receiver and whose entries are sampled i.i.d. from the complex Gaussian distribution \( \mathcal{N}(0, \sigma_n^2) \) with some noise variance \( \sigma_n^2 \) [1]. The signal-to-noise (SNR) ratio is given by the transmit energy per information bit \( E_{0,TX} \) divided by the average noise power spectral density \( N_0 \).

B. Eisenstein Integers

Let \( \omega = e^{j \frac{2 \pi}{3}} \). Then the ring\n
\[ \mathbb{E} := \mathbb{Z}[\omega] = \{ a + \omega b : a, b \in \mathbb{Z} \} \subseteq \mathbb{C} \]

is called Eisenstein integers [6]. \( \mathbb{E} \) is a principal ideal domain (PID), a Euclidean domain and a lattice. The units of \( \mathbb{E} \) (seen as a ring) are the sixth roots of unity \( \mathbb{E}^\times = \{ \pm 1, \pm \omega, \pm \omega^2 \} \).

Let \( \Theta \in \mathbb{E} \setminus \{0\} \). Then \( \Theta \mathbb{E} \) is a sub-lattice of \( \mathbb{E} \) and for any \( z \in \mathbb{E} \) we can define a quantization function\n
\[ Q_{\Theta \mathbb{E}}(z) = \arg \min_{y \in \Theta \mathbb{E}} |z - y| \]

and a modulo function\n
\[ \text{mod}_{\Theta \mathbb{E}}(z) = z - Q_{\Theta \mathbb{E}}(z). \]

Both \( Q_{\Theta \mathbb{E}}(\cdot) \) and \( \text{mod}_{\Theta \mathbb{E}}(\cdot) \) can be extended to vector or matrix inputs by applying it component-wise. The Eisenstein constellation of \( \Theta \in \mathbb{E} \setminus \{0\} \) is the set\n
\[ \mathcal{E}_\Theta = \{ \text{mod}_{\Theta \mathbb{E}}(z) : z \in \mathbb{E} \}. \]
Note that \( \mathcal{E}_\Theta = \mathcal{R}_V(\Theta \mathbb{E}) \cap \mathbb{E} \), where \( \mathcal{R}_V(\Theta \mathbb{E}) \) is the Voronoi region of the lattice \( \Theta \mathbb{E} \). \( \mathcal{E}_\Theta \) contains \(|\Theta|^2\) elements. The resulting signal constellation has a hexagonal boundary region and is more densely packed than a signal constellation of the same cardinality over the Gaussian integers or quadrature amplitude modulation, cf. [8].

Besides its high packing density, Eisenstein integers have another major advantage compared to classical signal constellations: they possess algebraic structure. In order to use this fact, we need the following lemma.

**Lemma 1** ([6]). \( \Theta \) is a prime in \( \mathbb{E} \) if one of the following conditions is true.

(i) \( \Theta = u \cdot p \) for some \( u \in \mathbb{E}^* \) and \( p \) is a prime in \( \mathbb{N} \) with \( p \equiv 2 \mod 3 \) (Type I).

(ii) \(|\Theta|^2 = p \) is a prime in \( \mathbb{N} \) with \( p \equiv 1 \mod 3 \) or \( p = 3 \) (Type II).

We define multiplication and addition of \( a, b \in \mathcal{E}_\Theta \) as
\[
a \oplus b = \text{mod}_{\Theta \mathbb{E}}(a + b) \quad \text{and} \quad a \odot b = \text{mod}_{\Theta \mathbb{E}}(a \cdot b),
\]
where \( \oplus \) and \( \odot \) are the ordinary operations in \( \mathbb{C} \). Then the set \( \mathcal{E}_\Theta \) with these operations \((\mathcal{E}_\Theta, \oplus, \odot)\) is a ring and—even stronger—the following theorem holds.

**Theorem 1** ([6]). Let \( \Theta \) be a prime in \( \mathbb{E} \). Then \((\mathcal{E}_\Theta, \oplus, \odot)\) is a finite field. More precisely, the following isomorphisms hold.
\[
(\mathcal{E}_\Theta, \oplus, \odot) \cong \begin{cases} \mathbb{F}_p^\times, & \text{if } \Theta \text{ is of Type I,} \\ \mathbb{F}_p, & \text{if } \Theta \text{ is of Type II.}
\end{cases}
\]

A table of all Eisenstein constellations of size up to 127 can be found in [8, Table I], along with a list of Gaussian constellations. The table also states their resulting average power (mean squared absolute value of a constellation).

**C. Rank-Metric and Gabidulin Codes**

Rank-metric codes are sets of matrices where the distance of two elements is measured by the rank metric instead of the classical Hamming metric. The most famous class of rank-metric codes are Gabidulin codes, which were independently introduced in [9]–[11] and have found many applications such as random linear network coding [12] and cryptography [13].

In general, a rank-metric code \( C \) over a field \( \mathbb{K} \) is a subset of \( \mathbb{K}^{m \times n} \), along with the rank metric
\[
d_R : \mathbb{K}^{m \times n} \times \mathbb{K}^{m \times n} \to \{0, \ldots, \min\{m, n\}\},
\]
\[
(A, B) \mapsto \text{rk}(A - B).
\]

It has minimum rank distance
\[
d := \min_{C_1, C_2 \in C} d_R(C_1, C_2).
\]

Let \( q \) be a prime power and \( m \in \mathbb{N} \). Thus, \( \mathbb{F}_q^m \) can be seen as a vector space of dimension \( m \) over \( \mathbb{F}_q \) and for some \( n \in \mathbb{N} \), there is a mapping
\[
\text{ext} : \mathbb{F}_q^m \to \mathbb{F}_q^{m \times n}, \quad c \mapsto C,
\]
where each component of the vector \( c \) is extended into a fixed basis \( B \) of \( \mathbb{F}_q^m \) over \( \mathbb{F}_q \). The expansion of the \( i \)-th component of \( c \) is then the \( i \)-th column of \( C \). A linearized polynomial over \( \mathbb{F}_q^m \) of \( q \)-degree \( d_f \in \mathbb{N}_0 \) is a polynomial of the form
\[
f(X) = \sum_{i=0}^{d_f} f_i X^i, \quad f_i \in \mathbb{F}_q^m, \quad d_f \neq 0.
\]
The zero polynomial \( f(X) = 0 \) is also a linearized polynomial and has \( q \)-degree \( d_f = -\infty \). The set of linearized polynomials is denoted by \( L_{\mathbb{F}_q^m} \).

Let \( k, n \in \mathbb{N} \) be such that \( k < n \leq m \). We choose \( g_1, \ldots, g_n \in \mathbb{F}_q^m \) to be linearly independent over \( \mathbb{F}_q \). A Gabidulin code of length \( n \) and dimension \( k \) is given by
\[
C_G[n, k] = \{(f(g_1), \ldots, f(g_n)) : f \in L_{\mathbb{F}_q^m}, d_f < k\}.
\]
The codewords \( c = [f(g_1), \ldots, f(g_n)] \in \mathbb{F}_q^m \) can be interpreted as matrices \( C \in \mathbb{F}_q^{m \times n} \) using the ext mapping and thus, the rank metric is well-defined. The minimum rank distance of \( C_G[n, k] \) is \( d = n - k + 1 \) and therefore fulfills the rank-metric Singleton bound with equality [9]–[11].

It is shown in [14] that we can reconstruct \( C \in C_G \) from
\[
C + E + A_R B_R + A_C B_C,
\]
\[
E = \tau \in \mathbb{F}_q^m, \quad B_R \in \mathbb{F}_q^{m \times n}, \quad A_C \in \mathbb{F}_q^{m \times \delta}, \quad B_C \in \mathbb{F}_q^{\delta \times n},
\]
where \( \text{rk}(E) = \tau, A_R \in \mathbb{F}_q^{m \times \rho}, B_R \in \mathbb{F}_q^{\rho \times n}, A_C \in \mathbb{F}_q^{m \times \delta}, B_C \in \mathbb{F}_q^{\delta \times n}, \) whenever
\[
2\tau + \rho + \delta < d
\]
and \( A_R \) and \( B_C \) are known at the receiver (error and erasure decoder). The decoding complexity is \( O(m^3) \) operations in \( \mathbb{F}_q \) [14], or \( O^\sim(n^{1.69}m) \) using the algorithms in [15], where \( O^\sim \) is the asymptotic complexity neglecting \( \log(nm) \) factors.

A criss-cross error is a matrix that contains non-zero entries only in a limited number of rows and columns, cf. [11]. In general, if such a matrix can be covered with \( \tau_R \) rows and \( \tau_C \) columns such that outside the cover, there is no error, the matrix has rank \( \leq \tau_R + \tau_C \). Therefore, criss-cross and rank errors are closely related.

**III. A NEW CONSTRUCTION BASED ON EISENSTEIN INTEGERS**

In this section, we present a new construction method for ST codes based on finite-field rank-metric codes in combination with Eisenstein integers. The construction is similar to the one in [4], but uses a different embedding of the finite-field elements into the complex numbers. We give a proof that this mapping is rank-distance-preserving, which implies that the spacial diversity order of the ST code is lower bounded by the minimum rank distance of the finite-field code. Furthermore, we present simulation results that show a coding gain compared to the codes constructed in [4].

\footnote{E.g., \( B = (1, \alpha, \alpha^2, \ldots, \alpha^{m-1}) \), where \( \alpha \) is a primitive element of \( \mathbb{F}_q^m \).}
A. Code Construction

Let \( F_q \) be a finite field which is isomorphic to an Eisenstein constellation \( \mathcal{E}_3 \subseteq \mathbb{C} \) with modulo arithmetic \( \oplus \) and \( \otimes \), cf. Theorem 1. We choose an isomorphism \( \varphi : F_q \to \mathcal{E}_3 \) and extend the mapping to matrices by applying it entry-wise

\[
\Phi : F_q^{m \times n} \to \mathbb{C}^{m \times n},
\]

\[
[x_{ij}]_{i,j} \mapsto [\varphi(x_{ij})]_{i,j}.
\]

We can also define a generalized inverse

\[
\Phi^{-1} : \mathbb{C}^{m \times n} \to F_q^{m \times n},
\]

\[
[x_{ij}]_{i,j} \mapsto [\varphi^{-1}(\bmod_{\mathbb{Z}}(x_{ij}))]_{i,j}.
\]

The following theorem lays the foundation for a new class of ST codes based on Eisenstein integers.

**Theorem 2.** The mapping \( \Phi \) is minimum rank-distance-preserving, i.e., for any rank-metric code \( C \subseteq F_q^{m \times n} \) of minimum distance \( d \) the code \( C_E = \Phi(C) \subseteq \mathbb{C}^{m \times n} \) has minimum distance \( d \).

**Proof.** W.l.o.g. let \( n \leq m \); otherwise transpose all matrices. Let \( C^{(1)}, C^{(2)} \subseteq C, C^{(1)} \neq C^{(2)} \). Then, \( \rk(C^{(1)} - C^{(2)}) \geq d \). Take \( d \) linearly independent columns of \( C^{(1)} - C^{(2)} \), w.l.o.g. \( c_1^{(1)} - c_2^{(1)}, c_1^{(2)} - c_2^{(2)} \in F_q^n \). We can expand this set of vectors to a basis \( c_1^{(1)}, \ldots, c_d^{(1)} - c_d^{(2)}, c_{d+1}, \ldots, c_m \) of \( F_q^m \) and define the matrices

\[
\tilde{C}^{(1)} = [c_1^{(1)}, \ldots, c_d^{(1)}, c_{d+1}, \ldots, c_m] \in F_q^{m \times m},
\]

\[
\tilde{C}^{(2)} = [c_1^{(2)}, \ldots, c_d^{(2)}, 0, \ldots, 0] \in F_q^{m \times m}.
\]

Thus, \( \rk(\tilde{C}^{(1)} - \tilde{C}^{(2)}) = m \) and \( \det(\tilde{C}^{(1)} - \tilde{C}^{(2)}) \neq 0 \).

Since \( \varphi : F_q \to \mathbb{C} \) is an isomorphism, we know that \( \bmod_{\mathbb{Z}}'(\det(\Phi(C^{(1)}) - \Phi(C^{(2)}))) = \Phi(\det(\tilde{C}^{(1)} - \tilde{C}^{(2)})) \), implying that there is an \( A \in (\Theta \mathbb{Z})^{m \times m} \) such that \( \Phi(C^{(1)}) - \Phi(C^{(2)}) = A + \Phi(\tilde{C}^{(1)} - \tilde{C}^{(2)}) \). It follows from Lemmas 2 (**) and 3 (***) (in the appendix) that

\[
a := \bmod_{\mathbb{Z}}'(\det(\Phi(C^{(1)}) - \Phi(C^{(2)})))
\]

\[
= \bmod_{\mathbb{Z}}'(\det(A + \Phi(\tilde{C}^{(1)} - \tilde{C}^{(2)})))
\]

\[
(\Theta) \bmod_{\mathbb{Z}}'(\det(\Phi(C^{(1)}) - \Phi(C^{(2)})))
\]

\[
(****) \varphi(\det(\tilde{C}^{(1)} - \tilde{C}^{(2)})) \neq 0.
\]

Thus, \( \det(\Phi(C^{(1)}) - \Phi(C^{(2)})) = a + b \neq 0 \), for some \( b \in \Theta \mathbb{Z} \) \( (\Theta \neq a \text{ but } \Theta \mid b) \), and \( \rk(\Phi(C^{(1)}) - \Phi(C^{(2)})) = m \) (full rank). Hence, the first \( d \) columns of \( \Phi(C^{(1)}) - \Phi(C^{(1)}) \) are linearly independent and

\[
\rk(\Phi(C^{(1)}) - \Phi(C^{(2)})) \geq d,
\]

proving the claim. \( \square \)

It can be shown (one of the two design criteria in [1]) that the diversity order of an ST code is lower bounded by its rank distance. Since the mapping \( \Phi \) is rank-distance-preserving, we can design the diversity order of the ST code by choosing the finite-field rank-metric code accordingly.

**Example 1.** We can take a Gabidulin code \( C_G[n, k] \) over the field \( F_q^n \) with minimum distance \( d = n - k + 1 \) and obtain an ST code \( C_{ST} = \Phi(C_G) \) with spatial diversity \( d \). The resulting codewords \( \mathbf{X} \in C_{ST} \) are complex matrices of dimension \( m \times n \), we must choose \( m = N_{tx} \) and \( n = N_{time} \). If \( N_{tx} > N_{time} \), we can transpose the codewords and set \( m = N_{time} \) and \( n = N_{tx} \). In the special case of \( k = 1 \) (rank-metric repetition code equivalent), the resulting code \( C_{ST} \) has maximum diversity \( n \).

B. Numerical Results

Figure 1 shows simulation results (frame error rate (FER) over SNR) of a comparison of Gaussian integer ST codes from finite-field Gabidulin codes [4] and our construction presented in Section III-A. We use the channel model described in Section II-A with \( N_{tx} = N_{rx} = N_{time} = 4 \). The ST codes are defined using a complex Gaussian channel matrix \( H \), additive i.i.d. complex-Gaussian noise matrix \( N \).

It can be seen that in both scenarios, our Eisenstein construction provides a coding gain compared to the Gaussian ST codes from [4]. At FER = \( 5 \cdot 10^{-3} \), for \( q = 13 \), the gain is
approximately 0.3 dB and in the $q=37$ case, we are roughly 0.55 dB better. This gain is expected since Eisenstein integers are more densely packed in the complex plane than Gaussian integers, cf. [6].

IV. ALTERNATIVE DECODING

The above used ML-decoding method has complexity proportional to the number of ST codewords. For instance, the ST code constructed in Example 1 has $q^{N_{tx}}$ codewords and ML decoding is not possible in sufficiently short time already for relatively small field sizes $q$ or transmit antenna numbers $N_{tx}$.

It is interesting to note that although rank-metric codes have been used before to construct new ST codes, to the best of our knowledge, their decoding has not yet been employed. In this section, we propose a new decoding scheme which utilizes the decoding capabilities of Gabidulin codes in combination with a channel transformation based on LRA equalization.

For simplicity, we assume $N_{tx} = N_{rx}$, implying that $H$ is invertible with probability 1 (see, e.g., [17] on how LRA equalization works if $N_{tx} \neq N_{rx}$).

A. Channel Transformation using LRA Techniques

In LRA zero-forcing linear equalization [17], the inverse channel matrix $H^{-1}$ is decomposed into

$$H^{-1} = ZF$$

such that $Z \in \mathbb{E}$, $\det Z \in \mathbb{E}^\times$ (implying $Z^{-1} \in \mathbb{E}$), and the maximum of the row norms

$$\max_i \| f_i \|_2 \quad (\text{f}_i \text{ is the } i\text{th row of } F)$$

is minimal among all decompositions. The problem of finding such a decomposition is equivalent to solving the shortest basis problem (SBP) in a $\mathbb{E}$-lattice$^4$ (with the rows of $H^{-1}$ forming a basis of the lattice). The SBP is NP-hard. However, we can find an approximate solution using the LLL algorithm$^5$ in time $O(n^3)$. Since we know $F$, we can compute the alternative receive matrix

$$\hat{Y} = FY = Z^{-1}X + FN.$$  \hspace{1cm} (30)

Due to $Z^{-1}X \in \mathbb{E}^{m \times n}$, we can make a component-wise decision of the entries of $\hat{Y}$ to the closest point in $\mathbb{E}$ using the quantization function and obtain

$$\hat{Y} = Q_\mathbb{E}(\hat{Y}) = Z^{-1}X + Q_\mathbb{E}(FN) =: Z^{-1}X + E.$$ \hspace{1cm} (31)

Since $\hat{Y} \in \mathbb{E}^{m \times n}$, we can use the generalized random projection of $\Phi$ to get back to finite fields

$$\hat{Y}_F = \Phi^{-1}(\hat{Y}) = \Phi^{-1}(Z^{-1})\Phi^{-1}(X) + \Phi^{-1}(E)$$

$$=: Z^{-1}_{F}X_F + E_F. \hspace{1cm} (32)$$

Also, $\det(Z^{-1}_F) = \det(\Phi^{-1}(Z^{-1})) = \varphi^{-1}(\det(Z^{-1})) \neq 0$ (since $\det(Z)$ is a unit in $E$) and thus, $Z^{-1}_F$ is invertible and we can compute

$$\hat{Y}_F = Z_F^{-1}\hat{Y}_F = X_F + Z_FE_F.$$ \hspace{1cm} (34)

B. Decoding Using Rank-Metric Decoder

In order to see how rank-metric codes can be used to correct errors of the form $Z_FE_F$, we should have a closer look at the error matrix $E$. An entry of $E$ is non-zero if the corresponding entry in $FN$ is large enough (by absolute value) to be closer to some element of $\mathbb{E} \setminus \{0\}$ than to 0. It can be observed that the rows of $F$ have different norms $\|f_i\|_2$. Since the entries of $N$ are i.i.d. $\mathcal{N}(0, \sigma_n^2)$ distributed, an entry in the $i$th row of $FN$ is $\mathcal{N}(0, \|f_i\|_2^2\sigma_n^2)$ distributed and (i.i.d. to other entries in that row). Thus, those rows of $E$ with larger $\|f_i\|_2$ tend to contain more errors than others. Since the $\|f_i\|_2$'s might differ a lot, in general, non-zero entries of $E$ tend to occur row-wise.

Also, entries in columns are no longer independent and thus, if there is a relatively large entry in $N$, this value might influence the entries of the entire column in $FN$, or $E$.

$^1$In [16] a decoder of a generalized Gabidulin code is used. In their channel model, $H$ is always the identity matrix and $N$ naturally contains cross-criss error patterns. Hence, it differs significantly from the channel model for which ST codes were originally designed [1].

$^2$See [17] for an overview of different factorization criteria.

$^3$The same decomposition is possible for Gaussian integers. However, it performs better (in terms of max $\|f_i\|_2$) for Eisenstein integers, cf. [8].

$^4$For Eisenstein integers, the LLL algorithm has to be adapted, cf. [8], [18].

$^5$Finding the distribution of the row norms of $F$ is an open problem and is beyond the scope of this paper since it would involve a detailed analysis of the numerical properties of the LLL algorithm.
We can thus conclude that $E$ tends to contain criss-cross error patterns and therefore has low rank. We cannot use arbitrary criss-cross error correcting codes because the multiplication by $Z_F$ in the final error matrix destroys the criss-cross pattern. However, the rank is preserved, meaning that $Z_F E_F$ tends to have low rank and can be corrected using a rank-metric code.

**Example 2.** Let $N_{tx} = N_{rx} = N_{time} = 7$ and $6$ dB SNR. A realistic output of the channel matrix decomposition is $F \in \mathbb{C}^{7 \times 7}$ with squared row norms:

| $i$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\|f_i\|_2^2$ | $0.38$ | $0.21$ | $0.34$ | $0.19$ | $0.20$ | $0.28$ | $0.24$ |

For instance, the error matrix $E_F$ in the channel transformation procedure can have the form (here, * means that this entry is non-zero, all other entries are zero)

$$E_F = \begin{bmatrix}
    ** & * & * & ** \\
    ** & ** & ** & ** \\
    * & * & * & ** \\
\end{bmatrix} \Rightarrow \text{rk} E_F \leq 4,$$

(35)

where the rows which contain many errors ($i = 1, 3$ and $6$) are due to large values of $\|f_i\|_2^2$ and the corrupted column ($j = 6$) results from a large value in the $j$th column of the original noise matrix $N$, which spreads through the entire column due to the matrix multiplication $F N$.

**C. Improved Decoding Using GMD**

Since we know $F$, the knowledge of the squared row norms $\|f_i\|_2^2$ provides reliability information of the rows of $E$. Thus, we can use generalized minimum distance (GMD) decoding [19] in combination with an error- and erasure decoding algorithm for Gabidulin codes (cf. Section II-C) to obtain better results.

More exactly, we can start by trying to decode without erasures. Then, incrementally from $\ell = 1$ to $d - 1$, we estimate the likeliest $\ell$ rows of $E_F$ which are in error, using the soft information given by the $\|f_i\|_2^2$’s, say $E_\ell \subseteq \{1, \ldots, m\}$, $|E_\ell| = \ell$ (e.g., $E_2 = \{1, 6\}$ in Example 2). Then we can decompose the error into

$$Z_F E_F = Z_F E_F' + Z_F E_F'',$$

(36)

where $E_F''$ contains non-zero values only in the rows $E_\ell$ and $E_F'$ has zero rows in $E_\ell$. We can re-write $Z_F E_F' = [Z_F|_{E_\ell}] [E_F'|_{E_\ell}]$, where $[Z_F|_{E_\ell}] \in F_q^{m \times \ell}$ consists of the columns of $Z_F$ with indices in $E_\ell$ and the rows of $[E_F'|_{E_\ell}] \in F_q^{\ell \times m}$ are the non-zero rows of $E_F''$. The procedure is illustrated in the following example.

**Example 3.** Let $E_F$ be as in Example 2 and $k = 1$. Thus, our finite-field Gabidulin code has parameters $[7, 1]$, minimum rank distance $7$, and we cannot correct the rank error with a half-the minimum rank distance decoder since $\text{rk} E_F = 4 > 3 = \frac{d-1}{2}$. Using GMD, we can, e.g., declare $\ell = 2$ erasures as follows (recall that $E_2 = \{1, 6\}$):

$$Z_F \begin{bmatrix}
    ** & * & * & ** \\
    ** & ** & ** & ** \\
    * & * & * & ** \\
\end{bmatrix} = Z_F \begin{bmatrix}
    ** & * & * & ** \\
    ** & ** & ** & ** \\
    * & * & * & ** \\
\end{bmatrix} = Z_F E_F' + [Z_F|_{E_2}] [E_F'|_{E_2}] = 2 \cdot \text{rk}(Z_F E_F') + \text{rk}([Z_F|_{E_2}] E_2) = 4 + 2 < 7 = d.$$  

(37)

If we use a Gabidulin code of dimension $1$ as in [4] or Example 1, we need to know only one row in $Z_F X_F$ which does not contain an error for decoding successfully. Since there are only as many possibilities as there are rows, we can simply “try” all rows, meaning that iteratively for each row $i$ we declare an erasure in all other rows than the $i$th one, decode and obtain a candidate codeword. Among these candidates, we then find the one with minimum Frobenius norm difference to the received word as in (27). We call this method multi-trial (MT) GMD decoding here.

**D. Numerical Results**

Figure 3 shows simulation results. We use ST codes based on a $C_G[4,1]$ code of minimum distance $d = 4$ with an Eisenstein constellation of size $q = 13$, and the channel model described in Section II-A with $N_{tx} = N_{rx} = N_{time} = 4$. We compare ML decoding to the alternative decoding methods described in this section; BMD as in Section IV-B and (multi-trial) GMD as in Section IV-C.

For comparison, we perform factorization and equalization based on both zero-forcing (ZF) linear equalization (as described in Section IV-A) and the minimum mean-squared error (MMSE) criterion. The latter is not described here in detail for reasons of clarity, but can e.g. be found in [17].

It can be seen that all alternative decoding methods are sub-optimal compared to the ML case. The best of the alternatives, multi-trial GMD with MMSE factorization and equalization, is approximately 7 dB worse than ML decoding at FER $10^{-3}$. This effect can be expected due to the following reasons.

- The row norms of $F$ do not provide actual soft information. They merely describe a statistical tendency of the errors in $F Y$.
- GMD decoding of Gabidulin codes cannot fully utilize soft information. To our knowledge, there is no soft-information decoding algorithm for Gabidulin codes, yet.
- The LLL algorithm only finds an approximate solution to the shortest basis problem.

However, all alternative decoding methods share the advantage that their decoding has polynomial decoding complexity in the parameters $N_{tx}$, $N_{rx}$, and $N_{time}$ of the code. It can therefore be used for larger parameter sets.
The determinant is a finite sum of finitely many multiplications of matrix elements, so this relation extends to det as follows:
\[
\text{mod}_{\Theta}(\det(A + B)) = \text{mod}_{\Theta}(\det(\text{mod}_{\Theta}(B))) = \text{mod}_{\Theta}(\det(B)),
\]
which proves the claim (note that \(\text{mod}_{\Theta}(A) = 0\)).

**Lemma 3.** For any \(A \in \mathbb{F}_q^{m \times n}\),
\[
\varphi(\det(A)) = \text{det}_{\Theta, \odot}(\Phi(A)).
\]

**Proof.** Since \(\varphi : \mathbb{F}_q \rightarrow (\mathbb{E}, \odot, \odot)\) is an isomorphism, \(\varphi(\det(A)) = \text{det}_{\Theta, \odot}(\Phi(A))\), where \(\text{det}_{\Theta, \odot}\) is the determinant under modulo addition \(\oplus\) and multiplication \(\odot\). We obtain
\[
\text{mod}_{\Theta}(\det(\Phi(A))) = \text{det}_{\Theta, \odot}(\Phi(A)) = \varphi(\det(A)),
\]
where the first equality follows by (40) and (41).

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