Continuous-time Discontinuous Equations in Bounded Confidence Opinion Dynamics

Francesca Ceragioli * Paolo Frasca *

* Dipartimento di Matematica, Politecnico di Torino, Torino, Italy

Abstract: This paper deals with a continuous-time version of the well-known Hegselmann-Krause model of opinion dynamics with bounded confidence. The equations of such model turn out to have discontinuous righthand side, so that generalized solutions must be considered. We use Krasowskii solutions to perform our analysis. Presented results include existence and completeness of solutions, and asymptotical convergence to a “clusterization” of opinions.

1. INTRODUCTION

In the study of social dynamics, which has recently attracted much interest from physicists, mathematicians and control theorists, the effect of interactions between individuals is the core issue. Such interactions imply that it is useful to study social dynamics as a dynamical system on a graph. In many successful models of opinion evolution, interactions are assumed to take place only if the opinions of the interacting agents are close enough, say, closer than a certain threshold. Such models are usually called “bounded confidence” models. In this paper, we focus on the so-called “Hegselmann-Krause” model, and we study a continuous-time version of the latter, which has been introduced in Blondel et al. [2010]. Due to the bounded confidence assumption, the model features differential equations whose righthand side is discontinuous. This requires the introduction of some kind of generalized solutions in order to study the system’s behavior.

Contribution

The main novelty of this note is the use of Krasowskii solutions to solve a model of opinion dynamics with bounded confidence. The dynamics of the real-valued opinion \( x_i(t) \), for every “agent” \( i \) in \( I \), is given by the equation

\[
\dot{x}_i(t) = \sum_{j : |x_i - x_j| < 1} (x_j(t) - x_i(t)), \quad i \in I.
\]

Note that in the above model agent \( j \) influences agent \( i \) only if \( |x_i - x_j| < 1 \). This means that interactions can be represented by a graph whose topology depends explicitly on the state. From another point of view, we may see the righthand side of the system as a discontinuous vector field. This fact implies the need for defining solutions in suitable sense: in this paper, we use Krasowskii solutions, which are defined in the next section. On one hand, the advantage of using Krasowskii solutions is technical, as a complete existence and Lyapunov theory is available for such solutions, and this theory allows us to obtain interesting results without making the analysis cumbersome. Moreover the set of Krasowskii solutions is “large”, so that the results that we state for Krasowskii solutions also hold for other types of solutions as Filippov solutions and Carathéodory solutions, in case they exist. On the other hand, the convexification of the discontinuities, which is the main feature of Krasowskii’s definition, can also be seen as a refinement of the original model, which smooths out sharp decision thresholds.

Besides the novelty in the solution definition, a distinctive feature of this note is our focus on the fundamental properties of the model, in particular (a) average preservation, (b) order preservation, (c) contractivity, (d) existence of Lyapunov functions. In the literature about networked systems, these properties often play an important role in coordination problems: in our work, they are instrumental to derive the significant results about solutions, namely existence, completeness, and convergence to a clustered configuration. By clustered configuration we mean a state configuration such that, for every pair of agents, either their individual states coincide, or they differ by more than 1.

Related Works

In the last years, a vast literature has been produced about coordination and consensus problems. Many coordination problems in distributed systems involve a varying interaction topology; however, probably to keep the analysis simpler, few studies have assumed the topology to depend on the state; among the latter, see for instance Justh and Krishnaprasad [2004], Cucker and Smale [2007], Savla et al. [2009], Cristiani et al. [2011]. Nevertheless, the assumption of “bounded confidence”, that is of interactions limited to neighboring states, is very significant because it is considered as a key point to explain persistence of disagreement in societies. For this reason, bounded confidence models have been widely considered in opinion dynamics by sociologists and physicists, as accounted in the surveys by Castellano et al. [2009] and Lorenz [2007].

Most of the models in the literature evolve in discrete-time, probably because they were originally intended to be simulated on computers with an agent-based approach.

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The choice of a discrete time schedule for the Hegselmann-Krause model has several advantages, which are exploited for instance in Blondel et al. [2009a], but it also has the drawback of assuming the opinions updates to be instantaneous and synchronous. The latter synchronicity assumption is avoided in the probabilistic models of Como and Fagnani [2010], Acemoglu et al. [2010], where interactions are instantaneous but asynchronous and controlled by a certain stochastic process. Our work, instead, features continuous-time ODE models, in which no interaction schedule is required and the effect of interactions is integrated over time. The continuous-time approach has already been proposed in Blondel et al. [2010], and indeed our work is strictly related to the one presented in [Blondel et al., 2010, Section 2] and Blondel et al. [2009b]. In that pair of papers, the authors study the same bounded confidence model as here, but they assume a more restrictive definition of solutions: the present note contains an alternate proof and an extension of some of their results. In this paper, we study Krasowskii solutions, which include solutions starting at the “problematic points” defined in Blondel et al. [2009b]. The relationship of our results with the mentioned pair of papers is further discussed in Section 3.

The theory of Krasowskii solutions, and, more generally, of differential inclusions, has been a tool for control theorists since at least Filippov [1988]: the recent tutorial Cortés [2008] contains a survey of Krasowskii’s and other definitions of generalized solutions. Related recent applications include consensus stabilization with quantized communications in Ceragioli et al. [2010] and deployment algorithms in Cortés and Bullo [2009]. Finally, the “theory of indecision” developed in behavioral studies by Kaernbach [2001] and Bogacz et al. [2006] suggests that the Krasowskii convexification can be thought as a refinement of the bounded confidence model, intended to smooth out sharp decision thresholds, rather than as a mere mathematical tool.

2. PROBLEM AND RESULTS

Let us consider $N$ agents, indexed in a set $I$ of cardinality $N$, each of them having a time dependent real-valued “opinion” $x_i(t) \in \mathbb{R}$, which obeys the following dynamics

$$\dot{x}_i(t) = \sum_{j: |x_i - x_j| < 1} (x_j(t) - x_i(t)), \quad i \in I. \quad (1)$$

Before proceeding, let us introduce some useful notation. For every $i \in I$ and $x \in \mathbb{R}^N$, we let

$$\mathcal{N}_i(x) := \{k \in I : |x_i - x_k| < 1\},$$
$$\partial \mathcal{N}_i(x) := \{k \in I : |x_i - x_k| = 1\},$$

and correspondingly

$$\mathcal{E}(x) := \{(i,j), i,j \in I : |x_i - x_j| < 1\}$$
$$= \{(i,j), i,j \in I : j \in \mathcal{N}_i(x)\}$$

and

$$\partial \mathcal{E}(x) := \{(i,j), i,j \in I : |x_i - x_j| = 1\}$$
$$= \{(i,j), i,j \in I : j \in \partial \mathcal{N}_i(x)\}.$$

With the above definitions, we can define a state-dependent adjacency graph as

$$G(x) = (I, \mathcal{E}(x)).$$

Note that the graph $G(x)$ is symmetric, that is such that $i \in \mathcal{N}_j(x)$ if and only if $j \in \mathcal{N}_i(x)$. Analogously, we can define the graph $\tilde{G}(x) = (I, \tilde{\mathcal{E}}(x))$, with $\tilde{\mathcal{E}}(x) = \mathcal{E}(x) \cup \partial \mathcal{E}(x)$.

Using the above definitions, we can rewrite (1) as

$$\dot{x}_i(t) = f(x), \quad (2)$$

being

$$f_i(x) = \sum_{j \in \mathcal{N}_i(x)} (x_j(t) - x_i(t))$$

the components of the vector field $f(x)$ ($f_i(x) = 0$ in case $\mathcal{N}_i(x)$ is empty). Since the differential equation (2) has a discontinuous righthand side, we need to give a suitable definition of solution.

Definition 1. (Krasowskii solution). A Krasowskii solution to a differential equation $\dot{x} = g(x)$ is a map $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N$ such that

(1) $\phi$ is absolutely continuous

(2) for almost every $t$, $\phi$ satisfies

$$\dot{\phi}(t) \in K_g(\phi(t)),$$

where

$$K_g(x) = \bigcap_{\delta > 0} \text{co}(\{g(y) : y \text{ such that } \|x - y\| < \delta\}).$$

Remark 1. The most classical notion of solution to an equation with discontinuous righthand side is the notion of Carathéodory solution. More precisely a Carathéodory solution to a differential equation $\dot{x} = g(x)$ with the initial condition $x(t_0) = x_0$ on an interval $I \subset \mathbb{R}$ containing $t_0$, is a map $\phi : I \rightarrow \mathbb{R}^N$ such that

(1) $\phi$ is absolutely continuous on $I$,

(2) $\phi(t_0) = x_0$,

(3) $\phi = g(\phi)$ for almost every $t \in I$,

or, equivalently, a Carathéodory solution to $\dot{x} = g(x)$ is a solution to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} g(x(s))ds.$$

The main drawback of this notion is the fact that existence of solutions may be difficult to prove. This is also witnessed by the substantial work which is needed in Blondel et al. [2009b] to study the properties of a subset of Carathéodory solutions.

In the following, we consider Krasowskii solutions to (1), which we characterize as follows. For any $H \subset \partial \mathcal{E}(x)$, let us define

$$\mathcal{N}_i^H := \{k \in I : \{i,k\} \in H\},$$

and

$$f_i^H(x) = \sum_{j \in \mathcal{N}_i(x) \cup \mathcal{N}_i^H} (x_j(t) - x_i(t)).$$

It is easy to see that a Krasowskii solution to (1) satisfies at almost every time the inclusion

$$\dot{x} \in \text{co}(\{f^H(x) : H \subset \partial \mathcal{E}(x)\}),$$

or equivalently the inclusion

$$\dot{x} \in \{ \sum_{H \subset \partial \mathcal{E}(x)} \alpha_H f^H(x) : \alpha_H \geq 0, \forall H \subset \partial \mathcal{E}(x); \sum_{K \subset \partial \mathcal{E}(x)} \alpha_K = 1 \}.$$
Namely, for a given solution $\phi(\cdot)$,

$$
\dot{\phi}(t) = \sum_{H \subset \partial E(\phi(t))} \alpha_H(t) f_H(\phi(t))
$$

for almost every $t$, where the time-dependent coefficients $\alpha_H$ depend on the solution $\phi$ itself.

We now prove some basic properties of Krasowskii solutions to (1). First of all, note that since the right-hand side of (1) is locally essentially bounded, local existence of a Krasowskii solution is guaranteed (see for instance Hájek [1979]). Then, we show that any solution has the property that individuals can be sorted according to their opinion, and that their order is preserved along the solution.

**Proposition 1.** (Order preservation). Let $x(\cdot)$ be a Krasowskii solution to (1), on its domain of definition. For any $i,j \in I$, if $x_i(t_1) < x_j(t_1)$, then $x_i(t_2) < x_j(t_2)$ for any $t_2 > t_1$.

**Proof:** To prove the claim, we study the dynamics of the difference between $x_j$ and $x_i$. By continuity of the solutions, we can assume with no loss of generality that $x_j$ and $x_i$ are close, for instance that $x_j - x_i < 1$. For brevity, in the following we omit the explicit dependence of $N_i$ on $x$. For almost every time $t$, we have

$$
\frac{d}{dt}(x_j - x_i) = \sum_{H \subset \partial E(x)} \alpha_H \left[ \sum_{h \in N_j \cap N_j^H} (x_h - x_j) - \sum_{h \in N_i \cap N_i^H} (x_h - x_i) \right].
$$

Since if $h \in (N_j \cap N_j^H \setminus \partial E(x))$, then $x_h - x_i < 0$, whereas if $h \in (N_i \cup N_i^H \setminus \partial E(x))$, then $x_h - x_j > 0$, and since $|N_j \cap N_j^H| \leq |N_i|$, we get that

$$
\frac{d}{dt}(x_j - x_i) \geq -(|N_j \cap N_j^H| + |N_i| + |N_j^H|)(x_j - x_i).
$$

The obtained inequality ensures that $x_j - x_i$ can not reach zero in finite time, and yields our claim.

While the above proposition shows that strict inequalities between agents’ states are preserved by the dynamics, we have to remark that equalities are not. It is not in general true that if $x_i(t_1) = x_j(t_1)$, then $x_i(t_2) = x_j(t_2)$ for any $t_2 > t_1$. Indeed, we can observe that if $x_i(t_1) = x_j(t_1)$, then $x_i(t_1)$ and $x_j(t_1)$ have to satisfy to the same differential inclusion, but need not to be equal. However, it can be proven that it is always possible, given a solution $x(\cdot)$, to sort the states so that $x_i(t) \leq x_j(t) \leq \ldots \leq x_{i_n}(t)$, for every $t$. Note that this mapping $i(\cdot) : \{1, \ldots, N\} \rightarrow I$ depends on the solution and needs not to be unique. Nevertheless, it allows to define $x_{\min}(t) := x_{i_1}(t)$, and $x_{\max}(t) := x_{i_n}(t)$. This fact will be useful for the following proofs.

**Proposition 2.** (Contractivity). Let $x(\cdot)$ be a Krasowskii solution to (1) on its domain of definition. Then, for any $t > t_1$,

$$
\overline{\mathcal{O}}(\{x_i(t_2)\}_{i \in I}) \subset \overline{\mathcal{O}}(\{x_i(t_1)\}_{i \in I}).
$$

**Proof:** To prove the claim we show that the leftmost agent can only move to its right. More precisely, $x_i(t) - x_{\min}(t) \geq 0$ for every $t$ and every $i \in I$ and then, for almost every time $t$, $\frac{d}{dt}x_{\min}(t) \in [0, +\infty)$. Repeating an analogous argument for $x_{\max}$ implies the claim.

**Corollary 3.** (Completeness). Krasowskii solutions to (1) are complete.

**Proof:** Proposition 2 ensures that solutions are bounded. By standard arguments, this is enough to guarantee that local solutions can be extended for all $t > 0$.

**Proposition 4.** (Average preservation). Let $x(\cdot)$ be a Krasowskii solution to (1), and let $x_{\\text{ave}}(t) = N^{-1} \sum_{i=1}^N x_i(t)$. Then $x_{\\text{ave}}(t) = x_{\\text{ave}}(0)$, for $t > 0$.

**Proof:** The key remark to prove the claim is the symmetry of the adjacency graphs induced by the bounded confidence constraint. Indeed, for every $H \subset \partial E(x)$, and every $i,j \in I$, it holds that $x_j \in N_j \cup N_j^H$ if and only if $i \in N_j \cup N_j^H$. Then, for almost every time $t$,
\[
d\frac{d}{dt}x_{\text{ave}}(t) = N^{-1} \sum_{i \in I} \dot{x}_i(t)
\]

\[
= N^{-1} \sum_{i \in I} \sum_{H \subset \partial E(x(t))} \alpha_H(t) \sum_{j \in N_i(x(t)) \cup N_H^i} (x_j(t) - x_i(t))
\]

\[
= N^{-1} \sum_{H \subset \partial E(x(t))} \alpha_H(t) \sum_{i \in I} \sum_{j \in N_i(x(t)) \cup N_H^i} (x_j(t) - x_i(t))
\]

\[
= 0.
\]

This ensures \(x_{\text{ave}}(t) = x_{\text{ave}}(0)\) for every \(t > 0\).

We are now ready to prove the main result of the paper, which proves convergence to configurations in which agents are separated into clusters of agents which share the same opinion. First, we describe the set of equilibria of the system.

**Definition 2.** (Krasowskii equilibrium.) A point \(\tilde{x}\) is said to be a Krasowskii equilibrium of \(\dot{x} = g(x)\) if the function \(x(t) \equiv \tilde{x}\) is a Krasowskii solution to \(\dot{x} = g(x)\).

**Proposition 5.** (Set of Krasowskii equilibria.) The set of Krasowskii equilibria of \((1)\) is

\[
F = \left\{ x \in \mathbb{R}^N : \text{for every } (i, j) \in I \times I, \right. \\
\left. \quad \text{either } x_i = x_j \text{ or } |x_i - x_j| \geq \gamma \right\}.
\]

**Proof:** By definition, equilibria are points \(\tilde{x} \in \mathbb{R}^N\) such that \(0 \in \overline{co}\left\{ f^H(\tilde{x}) : H \subset \partial E(\tilde{x}) \right\}\). Clearly, every point in \(F\) is an equilibrium. To prove that there are no other equilibria, we proceed as follows. For a vector \(v \in \overline{co}\left\{ f^H(\tilde{x}) : H \subset \partial E(\tilde{x}) \right\}\) to be equal to zero, it is necessary that \(v_i = 0\). But since \(\tilde{x}_k - \tilde{x}_i \geq \gamma \) for every \(k \in I\), it is necessary that \(\tilde{x}_j - \tilde{x}_i \in \{0\} \cup [1, +\infty)\) for every \(j \in I\). Repeating this reasoning for \(i_2, i_3, \ldots\) we have that the set of equilibria actually coincides with the set \(F\).

Next, we prove our convergence result.

**Theorem 6.** (Convergence.) If \(x(t)\) is a Krasowskii solution to \((1)\), then \(x(t)\) converges, as \(t \rightarrow \infty\), to the set of Krasowskii equilibria \(F\).

**Proof:** We define the Lyapunov function

\[
V(x) = \frac{1}{2} \sum_{i \in I} x_i^2,
\]

and compute, using the symmetry of the graph \(G(x)\) and Proposition 4,

\[
\frac{d}{dt} V(x(t)) = \sum_{i \in I} x_i(t) \dot{x}_i(t)
\]

\[
= \sum_{i \in I} x_i(t) \left( \sum_{j : |x_i(t) - x_j(t)| < \gamma} (x_j(t) - x_i(t)) \\
+ \sum_{H \subset \partial E(x)} \alpha_H \sum_{j \in N_i^H} (x_j(t) - x_i(t)) \right)
\]

\[
= - \sum_{\{i,j\} \in E(x)} (x_j(t) - x_i(t))^2 \\
- \sum_{H \subset \partial E(x)} \alpha_H \sum_{\{i,j\} \in H} (x_j(t) - x_i(t))^2 \leq 0.
\]

Since the inequality is strict if \(x(t) \notin F\), and \(F\) is clearly closed and weakly invariant, we can apply a LaSalle invariance principle [Bacciotti and Ceragioli, 1999, Theorem 3] to conclude convergence to the set \(F\).

**Remark 2.** The definition of Krasowskii equilibrium we adopt in this paper is the usual one in the literature on differential inclusions. Note that in general there can be Krasowskii solutions which have Krasowskii equilibria as initial conditions, and leave the equilibria. Nevertheless, as a byproduct of the proof of Theorem 6, we have that Krasowskii solutions to \((1)\) starting in the set \(F\) cannot leave it. Moreover, we conjecture that Krasowskii solutions to \((1)\) actually converge to a point in \(F\).

### 3. KRASOWSKI AND CARATHÉODORY SOLUTIONS

The set of Krasowskii solutions is in general larger that the set of solutions intended in a Carathéodory sense. We now provide an example of a solution sliding on a discontinuity surface, proving that there are Krasowskii solutions to \((1)\) which are not Carathéodory solutions.

**Example 1.** (Sliding mode.) Let \(N = 3\) and consider a configuration \(x\) in which \(1 > x_2 - x_1 > 0\) and \(x_3 - x_2 = 1\). Then, \(x\) is on a discontinuity surface due to the disconnection between agents 2 and 3. Then, for almost every time

\[
\dot{x} \in \left\{ \alpha \left[ \begin{array}{c} x_2 - x_1 \\ 1 + x_1 - x_2 \\ -1 \end{array} \right] + (1 - \alpha) \left[ \begin{array}{c} x_2 - x_1 \\ x_1 - x_2 \\ 0 \end{array} \right] : \alpha \in [0, 1] \right\}.
\]

Since the normal vector to the discontinuity plane is \(v_\perp = [0, -1, 1]\), we have that

\[
v_\perp \cdot \dot{x} = -2\alpha + x_2 - x_1
\]

is equal to zero if \(\alpha = \frac{1}{2}(x_2 - x_1)\). Namely, the Krasowskii solution corresponding to such \(\alpha\) does not exit the discontinuity plane \(x_3 - x_2 = 1\) at time 0, but it slides on it. The sliding solution takes into account the fact that opinions \(x_3\) and \(x_2\) may remain for a while at the threshold distance before reaching the equilibrium configuration.

In Blondel et al. [2010] and Blondel et al. [2009b], the authors consider a carefully defined subset of Carathéodory solutions to \((1)\); they call proper solution any Carathéodory solution \(x(t)\) corresponding to an initial condition \(x_0\) (called proper initial condition) such that

a) \(x(t)\) is the unique Carathéodory solution to \((1)\) with initial condition \(x_0\) defined on \([0, +\infty)\);

b) the subset of \([0, +\infty)\) where \(x(t)\) is not differentiable is at most countable, and has no accumulation points;

c) if \(x_j(t) = x_j(t')\) for some \(t\), then \(x_j(t') = x_i(t')\) for all \(t' \geq t\).

Moreover they prove that almost all \(x \in \mathbb{R}^N\) are proper initial conditions, and that proper solutions are contractive in the sense of Proposition 2, preserve the average of states and converge to clusters. Our analysis shows that the most significant properties of proper solutions also hold in the larger set of Krasowskii solutions. Nevertheless, there are a few significant differences, which we detail in the following list.
• **Existence.** For any point \( x_0 \) in \( \mathbb{R}^N \), there is a Krasowskii solution \( x(t) \) such that \( x(0) = x_0 \). Instead, there are points, which belong to a certain set \( P \) of measure zero, such that Caratheodory solutions starting at these points may either not exist or not be unique, so that proper solutions may not exist. The set \( P \) includes points on the discontinuity surfaces, such that \( x_i = x_j \) for some \( i \neq j \). Note that Krasowskii solutions include sliding mode solutions as the one in Example 1: such solutions belong for a positive duration of time to a discontinuity surface.

• **Uniqueness.** Proper solutions are unique by definition, whereas Krasowskii solutions are in general not unique. Note, however, that the results obtained in this note, and in particular convergence to a clustered configuration, hold for every Krasowskii solution.

• **Regularity.** Krasowskii solutions are differentiable almost everywhere: proper solutions, by definition, are differentiable out of countable set with no accumulation point.

• **Order preservation.** Proper solutions preserve both inequalities and equalities between states, while we have remarked, after Proposition 1, that Krasowskii solutions only preserve inequalities.

• **Connectivity.** Along proper solutions, the number of connected components in \( \mathcal{G}(x(t)) \) is nondecreasing in time. Something similar can be shown for Krasowskii solutions: an argument similar to the proof of Proposition 2 implies that the number of connected components in \( \mathcal{G}(x(t)) \) is nondecreasing in time.

4. SUMMARY AND FUTURE WORK

In this paper, we have studied Krasowskii solutions to a bounded confidence model, extending the results in Blondel et al. [2010, 2009b] to a wider class of solutions. The study performed includes that of solutions starting at “problematic points” which may not admit proper solutions starting from them. Fundamental properties of solutions have been proved, and used as building blocks to obtain a convergence result. We believe that Krasowskii solutions can be a useful tool in opinion dynamics, whenever the model’s definition involves discontinuities. In the next future, we are keen on extending our analysis in two directions. First, we want to consider bounded confidence models in which the agents sit on the nodes of a given graph whose edges represent available communications: this constraint, combined with the constraint of bounded confidence, may lead to a richer dynamics and a finer cluserization. Second, we are interested in comparing the behavior of the linear law (1) with the behavior of more general, possibly nonlinear, update laws.

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