COMPETITIVE EROSION IS CONFORMALLY INVARIANT

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Abstract. We study a graph-theoretic model of interface dynamics called competitive erosion. Each vertex of the graph is occupied by a particle, which can be either red or blue. New red and blue particles are emitted alternately from their respective bases and perform random walk. On encountering a particle of the opposite color they remove it and occupy its position. We consider competitive erosion on discretizations of smooth planar simply connected domains. The main result of this article shows that at stationarity, with high probability the blue and the red regions are separated by an orthogonal circle on the disc and by a suitable hyperbolic geodesic on a general ‘smooth’ simply connected domain. This establishes conformal invariance of the model.

Figure 1. Competitive erosion on the unit disc, spontaneously forms an interface. $i$ and $-i$ form the red source and blue source respectively. In the initial state each vertex of a mesh of size $\frac{1}{50}$ is independently colored blue with probability $\frac{1}{3}$ and red otherwise. After 5000 time steps of competitive erosion, the red and blue regions have separated with the interface being an orthogonal circle.

1. Interface dynamics

In 2003, James Propp introduced competitive erosion: a graph-theoretic model of interface dynamics maintained in equilibrium by equal and opposing forces. The model has the following underlying data:

- A finite connected graph $G = (V, E)$ with vertex set $V$ and edge set $E$.
- Probability measures $\mu_1$ and $\mu_2$ on $V$.
- An integer $0 \leq k \leq \#V - 1$. 
Competitive erosion is a discrete-time Markov chain $(S(t))_{t\geq 0}$ on the space 
$$\{S \subset V : |S| = k\}.$$ 

One time step is defined by

$$S(t + 1) = (S(t) \cup \{X_t\}) - \{Y_t\}$$

(1.1)

where $X_t$ is the first site in $S(t) \cap$ visited by a simple random walk whose starting point has distribution $\mu_1$; and $Y_t$ is the first site in $S(t) \cup \{X_t\}$ visited by an independent simple random walk whose starting point has distribution $\mu_2$. As in Fig. 1 we will think of every vertex colored either ‘blue’ or ‘red’ and $S(t)$ will denote the set of blue vertices. If the distributions $\mu_1$ and $\mu_2$ have well-separated supports, one expects that these dynamics separate the graph into coherent red and blue territories.

The study of this competing particle system was initiated in [GLPP15] where the reader can find a detailed introduction of the process. The underlying graph considered in [GLPP15] was the cylinder (the product of a path and a cycle). However the original question asked by James Propp was in the setting where the underlying graph is a discrete approximation to a general smooth simply connected domain. The process was predicted to exhibit conformal invariance. Confirming this is the goal of the article.

1.1. Conformal Invariance and informal setup. We consider the case when the underlying graph is a discretization of a smooth planar simply connected domain $U$ i.e. $V = U_n := \frac{1}{n} \mathbb{Z}^2 \cap U$ and $k = \lfloor \alpha |U_n| \rfloor$ for some fixed $\alpha \in (0, 1/2]$. Consider the case when the underlying domain is the unit disc $\mathbb{D}$ thought of as a subset of the complex plane $\mathbb{C}$. Simulations (Fig. 1) show if the measures $\mu_1$ and $\mu_2$ are Dirac measures on $-i, i$, then after running the process for some time the blue and red regions are separated with the boundary being an orthogonal circle such that the blue region has $\alpha$ fraction of the total area. Let us call this region $D(\alpha)$. For a general domain $U$ with two points $(x_1, x_2)$ on the boundary, one can obtain the corresponding $U(\alpha)$ by conformally mapping $\mathbb{D}$ to $U$ with $-i, i$ mapped to $x_1, x_2$ respectively and using the area measure on $U$ to ensure that $U(\alpha)$ contains $\alpha$ fraction of area($U$), (see Fig 3). Precise definitions are provided in subsection 1.3.

It was predicted by James Propp in 2003 based on the conformally invariant nature of harmonic measure that competitive erosion should exhibit conformal invariance, that is, for any reasonably regular domain $\mathbb{D}$ and the erosion chain on $\mathbb{D}$ with blue and red sources being $x_1, x_2$ respectively the blue region should “converge” to $U(\alpha)$ as $n$ grows large. Our main result confirms this. However for technical reasons we need $\mu_1, \mu_2$ to be “smooth” and supported in the interior on $\mathbb{D}$ instead of being point masses. In this article we take them to be uniform measures on all lattice points inside small ‘blobs’ of radii $\delta/4$ lying inside $\mathbb{D}$, and also at distance $\delta$ from the points $x_1, x_2$ respectively. See Fig 2 i.

The main result then involves sending $\delta$ to 0 to “approximate” the Dirac measures at $x_1, x_2$. We now present the statement of our main result.

To improve readability at this point we postpone the precise description of the technical assumptions including the definitions of the blobs. These are discussed right after the statement of the theorem (subsection 1.3).

1.2. Main result. For any subset $S \subset U_n$, let

$$S^0 := S + \left[-\frac{1}{2n}, \frac{1}{2n}\right]^2$$

(1.2)
be the corresponding union of squares. Recall that $\alpha \in (0, \frac{1}{2}]$ is the fraction of blue vertices. Let a simply connected domain “$U \subset \mathbb{C}$ is smooth” mean that the boundary of $U$ is an analytic curve (equivalently the conformal map from $U$ to $D$ has a conformal extension across the boundary, see [Pom92, Prop 3.1]).

**Theorem 1.1.** (Main Result) Let $U \subset \mathbb{C}$ be a smooth simply connected domain. Consider the competitive erosion chain on $U_n = U \cap \frac{1}{n} \mathbb{Z}^2$ with blob radii $\delta/4$ and $\lfloor \alpha |U_n| \rfloor$ blue vertices. Then,

$$\lim_{\delta \to 0} \limsup_{m \to \infty} \frac{1}{2^m} \mathbb{E}_{\pi_{\delta,n}} \left[ \text{area}(S^\circ \triangle U_\alpha) \right] = 0,$$

where $S$ is the set of blue vertices sampled from $\pi_{\delta,n}$, the stationary measure of the chain. $\triangle$ denotes the symmetric difference of sets.

See Fig 1 and 2 ii. Note that for brevity we suppress the centers of the blobs in the notation $\pi_{\delta,n}$. The centers are points chosen at distance $\delta$, from $x_1, x_2$ respectively (see Fig 4), formally described in Section 1.3.

Informally the above theorem says the following: consider the erosion chain on the discretization of any smooth domain $U$ with points $x_1, x_2$ on the boundary. Suppose that $\alpha$ fraction of the vertices are blue. If the blobs from which the blue and red random walks start are small enough and close enough to $x_1$ and $x_2$ then as the mesh size goes to zero, after running the erosion chain for long enough, the blue region looks like the set $U_\alpha$. This is in the sense that the density of the blue region outside $U_\alpha$ is close to 0. As discussed above, the sets $U_\alpha$ for various domains $U$ can be obtained from each other via conformal maps. Thus Theorem 1.1 establishes that competitive erosion is conformally invariant.

**Note** that the mesh size goes to 0 at dyadic scales. This is for technical convenience. Reasons are elaborated in Section 1.5 iii.

### 1.3. Formal definitions and setup.

In this section we make precise all the definitions and the entire setup leading to the statement of Theorem 1.1. We will also present a more quantitative version of the result.

For any two points $x, y \in \mathbb{C}$, $d(x, y)$ will be used to denote the euclidean distance between them.
For any set $A \subset \mathbb{C}$ and any $x \in \mathbb{C}$ denote by $d(x, A)$, the distance between the point and the set. Throughout the article $\mathbb{D}$ will be used to denote the unit disc centered at the origin in the complex plane.

Throughout the rest of the article all our domains will be bounded, simply connected, smooth and hence we will drop the adjectives.

**Definition 1.** For any domain $B \subset \mathbb{C}$ denote by $\partial B$ the boundary of $B$.

For a domain $U$ and points $x_1, x_2 \in \partial U$ let

$$\phi : \mathbb{D} \to U$$

$$\psi : U \to \mathbb{D}$$

such that $\phi \circ \psi, \psi \circ \phi$ are the identity maps on the respective domains and $\psi(x_1) = -i, \psi(x_2) = i$. The existence of such maps is guaranteed by the Riemann Mapping Theorem. See for eg: [Ahl79, Chapter 6]. In fact there exists more than one pair of $(\phi, \psi)$ since a conformal map between domains has three degrees of freedom and here we have fixed the value at only two points. However we choose a particular pair $(\phi, \psi)$ assumed to be fixed throughout the rest of the article. For any $\beta \in \mathbb{R}$ define

$$U_\beta := \left\{ z \in U : \frac{32}{\pi} \log \frac{\psi(z) - i}{\psi(z) + i} \geq \beta \right\}.$$  

(1.5)

The constant $\frac{32}{\pi}$ is not important. It falls out of some natural integrals involving the Brownian motion heat kernel appearing later in the article. Thus for the disc, $\mathbb{D}_\beta$ is a region containing $-i$ and enclosed by a circular arc (hyperbolic geodesic) symmetric with respect to $i$ and $-i$. For a general domain $U$, we transfer the regions via conformal maps and the boundaries still remain geodesics as they are conformally invariant. See Fig. 3. However for our purposes we need an area parametrization of the regions $U_\beta$. Given $\alpha \in (0, 1)$, let $\beta = \beta(\alpha)$ be such that

$$\text{area}(U_\beta) = \alpha \text{ area}(U).$$  

(1.6)

Let

$$U_\alpha := U_{\beta(\alpha)}.$$  

(1.7)

We now describe the set up in the statement of Theorem 1.1 formally.

**Setup 1.** Given $U$ we take $U_n = U \cap (\frac{1}{n}\mathbb{Z})^2$, as our vertex set. As the edges of our graph we take the usual nearest-neighbor edges of $U_n$ though of as a subset of $\frac{1}{n}\mathbb{Z}^2$. However we delete every such edge which intersects $U^c$. Due to the smoothness of $U$, $U_n$ will be a connected for large enough $n$.  

![Figure 3. Hyperbolic geodesics are circular arcs on the disc. They are invariant under conformal maps. However since conformal maps are not area preserving $D_\alpha$ can get mapped to $U_{a'}$ for some $a' \neq \alpha$.](image)
Figure 4. $y_i$’s are points at distance $\delta$ from $x_i$’s. They are at distance at least $\frac{\delta}{2}$ from $\partial U$. The blobs are discs of radius $\frac{\delta}{4}$ centered at the $y_i$’s.

See Remark 1.1 below.

Fix $\alpha \in (0, \frac{1}{2}]$, and take $k = \lfloor \alpha |U_n| \rfloor$. Let $x_1, x_2 \in \partial U$. The following defines the blobs mentioned in the statement of Theorem 1.1. For small enough $\delta > 0$ let $y_1, y_2 \in U$ be such that
\[
\begin{align*}
    d(x_i, y_i) &= \delta \\
    d(y_i, \partial U) &> \frac{\delta}{2}.
\end{align*}
\]
For $i = 1, 2$, let $U_i = B(y_i, \frac{\delta}{4})$. As discrete approximations of $U_i$ we take
\[
U_{i,n} = B(z_{i,n}, \frac{\delta}{4}) \cap U_n
\]
where $z_{i,n} \in \frac{1}{n} \mathbb{Z}^2$ is the closest lattice point to $y_i$. Define $\mu_i$ to be the uniform measure on $U_{i,n}$.

Thus the blobs are the sets $U_1$ and $U_2$ and $\mu_i$’s are uniform over $U_{i,n}$. Note that in the above, $y_i$’s were just required to satisfy certain properties and other than that were completely arbitrary. Also we abuse notation a little in the definition of the blobs, $n$ should be thought of as large and hence $U_n$ (the underlying graph) should not be confused with the blobs $U_1$ and $U_2$.

**Remark 1.1.** The smoothness assumption on $U$ allows us to choose the $y_i$’s. This is formally proved in Corollary 8.1. See Fig. 4. The connectedness of $U_n$ for large enough $n$ follows due to the locally half plane like behavior, see (8.1).

1.3.1. *Quantitative version of Theorem 1.1.* As mentioned already to highlight the symmetrical roles played by $S(t)$ and its complement, we will think of the states as 2-colorings (blue and red) rather than sets. In this language $S(t)$ will denote the set of all vertices with a particular color (blue).

Formally define
\[
\begin{align*}
    \Omega := \Omega_n := \{\sigma \in \{1, 2\}^{U_n}, \# \{x \in U_n : \sigma(x) = 1\} = |\alpha |U_n|\} \\
    \Omega' := \Omega'_n := \{\sigma \in \{1, 2\}^{U_n}, \# \{x \in U_n : \sigma(x) = 1\} = |\alpha |U_n| + 1\}.
\end{align*}
\]
Also let
\[
\sigma_t(x) = \begin{cases} 
    1 & x \in S(t) \\
    2 & x \notin S(t).
\end{cases}
\]
Thus the competitive erosion chain $S(t)$, can be thought of as a Markov chain $\sigma_t$, on $\Omega_n$. Clearly a single time step of the chain consists of a step from $\Omega_n$ to $\Omega'_n$ followed by a step from $\Omega'_n$ back to
We now state a quantitative result which immediately implies Theorem 1.1. Given \( \epsilon > 0 \), define \( \Omega_\epsilon := \Omega_{n,\epsilon} \) to be the set of all configurations \( \sigma \in \Omega_n \) such that

\[
\# \{ x \in \bigcup_{(\alpha)} : \sigma(x) = 2 \} \leq \epsilon n^2
\]

\[
\# \{ x \in \bigcup_{(\alpha)}^c : \sigma(x) = 1 \} \leq \epsilon n^2,
\]

where \( \bigcup_{(\alpha)} \) was defined in (1.7). Thus informally \( \Omega_{n,\epsilon} \) is the set of all configurations where the amount of “dust” particles of the wrong color has density at most \( \epsilon \). See Fig. 5. The next result shows that the stationary measure of \( \Omega_{n,\epsilon} \) is large and hence implies Theorem 1.1.

**Theorem 1.2.** *(Quantitative version of Theorem 1.1).* Let \( \mathbb{U} \) be as in Setup 1 and \( \epsilon > 0 \). Then there exists \( \delta_0 = \delta_0(\epsilon) \) such that for all \( \delta \leq \delta_0 \) and \( n = 2^m > N(\delta) \),

\[
\pi_{\delta,n}(\Omega_{n,\epsilon}) \geq 1 - e^{-Dn^2}
\]

where \( D = D(\epsilon, \delta, \mathbb{U}) > 0 \), where \( \pi_{\delta,n} \), the stationary measure of the erosion chain on \( \mathbb{U}_n \) with \( \delta \) as in Setup 1.

1.4. **Comparison with IDLA.** Internal diffusion limited aggregation (IDLA) is a fundamental model of a random interface moving in a monotone (outward) fashion. IDLA involves only one species with an ever-growing territory

\[
I(t+1) = I(t) \cup \{ X_t \}
\]

where \( X_t \) is the first site in \( I(t)^c \) visited by a simple random walk whose starting point has distribution \( \mu_1 \). Competitive erosion can be viewed as a symmetrized version of IDLA: whereas \( I(t) \) and \( I(t)^c \) play asymmetric roles, \( S(t) \) and \( S(t)^c \) play symmetric roles in (1.1). IDLA on a finite graph is only defined up to the finite time \( t \) when \( I(t) \) is the entire vertex set. For this reason, the IDLA is usually studied on an infinite graph and the theorems about IDLA are limit theorems: asymptotic shape [LBG92], order of the fluctuations [AG13, AG14, JLS12], and distributional limit of the fluctuations [JLS14]. In contrast, competitive erosion on a finite graph is defined for all times,
so it is natural to ask about its stationary distribution. To appreciate the difference in character between IDLA and competitive erosion, note that the stationary distribution of the latter assigns tiny but positive probability to configurations that look very different to final figure in Fig. 1. Thus Competitive erosion will occasionally form these exceptional configurations.

1.5. Remarks about Theorems 1.1, 1.2 and Setup 1.

i. Since $U$ is smooth, using Schwarz reflection $\phi$ and hence $\psi$ can be extended conformally across the boundary onto some neighborhoods of $D$ and $U$. In particular this implies that $|\phi'|$ and $|\psi'|$ are bounded away from 0 and $\infty$ on $U$ and $D$ respectively. See [Pom92, Prop 3.1]. This bi-Lipschitz nature of the maps will be used in several distortion estimates throughout the rest of the article.

ii. Note that the blob sources $U_i$ lie entirely in the interior of $U$ and the measures $\mu_i$ should be thought of as “smooth” approximations to point masses at points $x_1, x_2$ as $\delta \to 0$. Also by choice $|U_{1,n}| = |U_{2,n}|$. This will be technically convenient.

iii. Lastly we discuss the choice of the dyadic mesh sizes in the statements of Theorems 1.1 and 1.2. The technical core of the proof of the theorems involve convergence of the random walk on $U_n$ under proper scaling of time to Reflected Brownian motion on $U$. The proof of this appears in [BC08] and subsequent local CLT estimates have been obtained in [CF13]. However both the above papers assume dyadic discretization in their proofs. Since the proof of our results rely heavily on the aforementioned convergence results we stick to the dyadic discretization in our statements as well. We end by mentioning that it has been pointed out (via personal communication) by the authors of [BC08] and [CF13] that the convergence holds true even when the mesh size at the $n^{th}$ step is $\frac{1}{n}$ instead of $\frac{1}{2^n}$ and the latter was chosen for technical convenience.

2. Sketch of the proofs and organization of the article

Theorem 1.1 is a simple corollary of Theorem 1.2. To prove the latter, i.e. that the stationary distribution concentrates on the small set $\Omega_{\epsilon} := \Omega_{n,\epsilon}$, we identify a Lyapunov function $w(\cdot)$ on the state space i.e. a function which attains its global maximum in $\Omega_{\epsilon}$ and increases in expectation when the process is outside $\Omega_{\epsilon}$, i.e.

$$\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma \notin \Omega_{\epsilon}) \geq a > 0.$$  \hspace{1cm} (2.1)

For more on Lyapunov functions see [FMM95]. To construct such a function $w(\cdot)$ we proceed by defining the following discrete Green function: Let for any $x \in U_n$

$$G_n(x) = \frac{2n^2}{|U_{1,n}|} \int_{t=0}^{\infty} \mathbb{P}_x(X(t) \in U_{1,n}) - \mathbb{P}_x(X(t) \in U_{2,n}) \, dt.$$ \hspace{1cm} (2.2)

where $\mathbb{P}_x(\cdot)$ is the heat kernel of the continuous time random walk on the graph $U_n$ started from $x$ with mean holding time $\frac{1}{2^n}$. This choice of holding time is made to ensure that the random walk converges to Reflected Brownian motion on $U$. Thus the above function is the difference in the amount of time random walk spends in $U_{1,n}$ and in $U_{2,n}$ respectively. Now for any $\sigma \in \Omega$ we define the weight function

$$w(\sigma) := \sum_{x \in B_i(\sigma)} G_n(x).$$

where for $\sigma \in \Omega \cup \Omega'$ and $i = 1, 2$

$$B_i(\sigma) := \{x \in U_n : \sigma(x) = i\}.$$ \hspace{1cm} (2.3)
Recalling $S(t)$ from the definition of competitive erosion we note that $S(t) = B_1(\sigma_t)$.

Theorem 5.1 states that: up to translation by a constant $G_n(x)$ is close to the function \(\frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right|\) if the blob sizes are small and $n$ is large.

This justifies why $w(\sigma)$ is maximized if all the blue vertices are in $U_{(a)}$ (see (1.7)). The technical core of the paper consists of the following:

- Proof of Theorem 5.1 which uses convergence of random walk on $U_n$ to Reflected Brownian motion in $U$, together with the fact that the Brownian motion heat kernel is a fundamental solution to the inhomogeneous Poisson problem on $U$ with Neumann boundary conditions. \(\frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right|\) is recovered as a limit of solutions of certain PDE’s with inhomogeneous Poisson initial condition on $U$ with Neumann boundary conditions.
- Proof of Theorem 4.1, which uses an electrical resistance argument involving Rayleigh’s monotonicity principle to establish the drift (2.1).

Once (2.1) is established we use Azuma’s inequality to argue that the process $w(\sigma_t)$ spends nearly all its time in a neighborhood of its maximum. These ingredients together with a general estimate relating hitting times to stationary distributions (Lemma 4.7) establish Theorem 1.2.

2.1. The level set heuristic. We now mention a heuristic that justifies the construction of the function $w(\cdot)$ outlined in the previous subsection and predicts the location of the competitive erosion interface. Consider the case of “well-separated” measures $\mu_1, \mu_2$. on general finite connected graph, which we assume for simplicity to be $r$-regular. Let $g$ be a function on the vertices satisfying

\[
\Delta g = \mu_1 - \mu_2
\]  

(2.4)

where $\Delta$ denotes the Laplacian $\Delta g(x) := \frac{1}{r} \sum_{y \sim x} (g(x) - g(y))$ and the sum is over vertices $y$ neighboring $x$. Since the graph is assumed connected, the kernel of $\Delta$ is one-dimensional consisting of the constant functions, so that equation (2.4) determines $g$ up to an additive constant. The boundary of a set of vertices $S \subset V$ is the set

\[
\partial S = \{ v \in V \setminus S : v \sim w \text{ for some } w \in S \},
\]  

(2.5)

where $v \sim w$ denotes that they are neighboring vertices. Consider a partition of the vertex set $V = S_1 \sqcup B \sqcup S_2$ where $\partial S_1 = B = \partial S_2$. (Think of $S_1$ and $S_2$ as the blue and red territories respectively, of the sort we might expect to see in equilibrium, and the sites in their common boundary $B$ have indeterminate color.)

Let $g_i$ for $i = 1, 2$ be the Green function for random walk started according to $\mu_i$ and stopped on exiting $S_i$. These functions satisfy

\[
\Delta g_i = \mu_i \quad \text{on } S_i, \\
g_i = 0 \quad \text{on } S_i^c.
\]

The probability that simple random walk started according to $\mu_i$ first exits $S_i$ at $x \in B$ is $-\Delta g_i(x)$. To maintain equilibrium in competitive erosion, we seek a partition such that $\Delta g_1 \approx \Delta g_2$ i.e. they are roughly equal, on $B$, that is

\[
\Delta(g_1 - g_2) \approx \mu_1 - \mu_2.
\]

(Exact equality holds except on $B$). Thus by (2.4), the function $g - (g_1 - g_2)$ is approximately constant. Since $g_i$ vanishes on $B$, the equilibrium interface $B$ should have the property that $g$ is approximately constant on $B$. 
The partition that comes closest to achieving this goal takes $S_1$ to be the level set
\[ S_1 = \{ x : g(x) < K \} \] (2.6)
for a cutoff $K$ chosen to make $\#S_1 = k$. An application of the maximum principle shows that for this choice of $S_1$, the maximum and minimum values of $g - (g_1 - g_2)$ differ by at most
\[ \max_{x \in S_1, y \notin S_1, x \sim y} |g(x) - g(y)|, \]
suggesting that the right notion of “well-separated” measures $\mu_1$ and $\mu_2$ is that the resulting function $g$ has small gradient.

In our case as in Setup 1, $\mu_i$ is the uniform measure on $U_{i,n}$. It is easy to check and shown in (5.4) that $\Delta G_n = \mu_1 - \mu_2$ where $G_n$ was defined in (2.2). This shows why the choice of the weight function is “natural”.

2.2. Organization of the paper. In Section 3 we show that the competitive erosion chain has a unique communicating class of recurrent states. The proof of the main result appear in Section 4. Theorem 1.1 is a straightforward corollary of Theorem 1.2. The key idea behind the proof of Theorem 1.2 is the construction of a suitable Lyapunov function as discussed in the previous subsection. This construction is done is Section 5. Theorem 4.1 states that the constructed function indeed can be used as a Lyapunov function by establishing the drift condition (2.1). The proof of this theorem poses the main technical challenge which we address in Section 6. Theorem 5.1 is one of the main convergence results that we use in this paper which states that the lyapunov function is asymptotically conformally invariant. We discuss a sketch of the proof in Section 7. The complete proof appears in [GP15].

Theorem 1.2 follows from Theorem 4.1 and hitting time estimates for submartingales and a general result in markov chains relation hitting times to stationary measure. Section 4 is devoted to this. Some basic geometric facts are proved in Section 8.

2.3. Assumptions, notations and conventions.

i. Recall $\alpha \in (0, \frac{1}{2}]$ in the statements of Theorems 1.1 and 1.2. For the rest of the article we assume $\alpha < 1/2$. This will be technically convenient in some of the later arguments. In section 2.4 we prove that proving the theorems for all $\alpha < \frac{1}{2}$ implies the theorems for $\alpha = \frac{1}{2}$.

ii. Throughout the article, by random walk on $U_n$, we will mean the continuous time random walk with $\exp(2n^2)$ waiting times unless specifically mentioned otherwise. This is done to ensure that the random walk density converges to that of Reflected Brownian motion. A fact which would be used heavily throughout the article.

We will denote the complex plane by $C$. For any two points $x, y \in C$, $d(x, y)$ will be used to denote the euclidean distance between them. Also for any set $A \subset C$ and any $x \in C$ denote by $d(x, A)$, the distance between the point and the set and similarly $d(A, B)$ denotes the distance between two sets $A, B \subset C$. $B(x, \epsilon)$ denotes the open euclidean ball of radius $\epsilon$ with center $x$. For any process and a subset $A$ of the corresponding state space $\tau(A)$ will denote the hitting time of that set (we drop the dependence on the process in the notation since it will be clear from context). Also $1(\cdot)$ will be used to denote the indicator function.

To avoid cumbersome notation, we will often use the same letter (generally $C$, $D$, $c$ or $d$) for a constant whose value may change from line to line. We use $\asymp$ to denote that two quantities that are equal up to universal constants. Also $O()$, $\Omega()$, $\Theta()$ are used to denote their usual meaning.
2.4. **α strictly less than 1/2. Claim:** Theorems 1.1 and 1.2 for all $\alpha < 1/2$ imply the corresponding theorems for the case $\alpha = 1/2$.

To show this we introduce a coupling of the competitive erosion chain for various values of $\alpha$.

2.4.1. **Diffusive sorting.** We define here a process called diffusive sorting, which couples the competitive erosion chains on a given graph for all values of $k$, using a single random walk to drive them all. It is convenient to imagine that the blue and red random walks start from vertices $v_1$ and $v_2$ respectively, which are external to the finite connected graph $G$ and have directed weighted edges into $G$: each edge $(v_1, v)$ has weight $\mu_i(v)$.

The state space of diffusive sorting consists of bijective labellings $\lambda$ of the vertex set of $G$ by the integers $\{1, \ldots, N\}$, where $N$ is the number of vertices. Consider an initial labelling $\lambda_0$. A random walker starts at $v_1$ carrying the label 0. Whenever it comes to a vertex whose label exceeds the label the walker is currently carrying, the walker swaps labels with that vertex, dropping its old (smaller) label and stealing the new (larger) label for itself. Eventually the walker will visit the vertex labelled $N$ and acquire the label $N$ for itself, at which point we may stop the walk, since the particle will carry the label $N$ forever after and no labels will change. The vertex labels are now $\{0, \ldots, N-1\}$. We now increase all those labels by 1, and call the result $\lambda_1/2$. (We have noticed that this process bears a strong resemblance to an algorithm in algebraic combinatorics called promotion [Sta09], but we have not explored this connection.) To get from $\lambda_1/2$ to $\lambda_1$, we apply the same process from the other side, releasing a random walker from $v_2$ bearing the label $N+1$, and letting it swap labels with any vertex it encounters whose label is smaller than its own, until it acquires the label 1; then we decrease the labels of all vertices by 1, obtaining $\lambda_1$. **Diffusive sorting** is the Markov chain $(\lambda_t)_{t \geq 0}$ on labellings, each of whose transitions from $\lambda_t$ to $\lambda_{t+1}$ is as described in the previous paragraph. To see how this chain relates to competitive erosion, for each integer $k = 0, \ldots, N-1$ let

$$S_k(t) = \{v : \lambda_t(v) \leq k\}.$$ 

Then it is not hard to check that $S_k$ has the law of the competitive erosion chain. (The key observation is that the vertices labelled 1 through $k$ in $\lambda_0$ must be a subset of the vertices labelled 1 through $k+1$ in $\lambda_1/2$, and similarly for going from $\lambda_1/2$ to $\lambda_1$.)

**Proof of the Claim.** We will use the coupling of the erosion chain for various values of $\alpha$ as denoted in Subsection 2.4.1. By subsection 4.4 Theorem 1.1 is a straightforward corollary of Theorem 1.2. Thus it suffices to just show that Theorem 1.2 for all $\alpha < 1/2$ implies the same theorem for $\alpha = 1/2$. Given $\epsilon > 0$ choose $\alpha < 1/2$ such that

$$\text{area} \left( \bigcup_{(1/2)} \setminus \bigcup_{(\alpha)} \right) \leq \epsilon/2.$$ 

Now think of the erosion processes corresponding to $\alpha$ and $1/2$ being coupled through the diffusive sorting coupling. Recall that in this coupling there is a labeling $\lambda$ assigning to every vertex a label in $\{1, \ldots, |U_n|\}$. Let

$$G_\alpha = \{v : \lambda(v) \leq |\alpha|U_n|\}$$

$$G_{1/2} = \{v : \lambda(v) \leq \frac{1}{2}|U_n|\}.$$ 

By Theorem 1.2 for small enough $\delta$, $\pi_{\delta,n} \left( |G_\alpha \cap U_n^{(a)}| \geq \epsilon n^2/2 \right) \leq e^{-Dn^2}$ for some $D = D(\epsilon, \alpha, \delta)$. However notice that $|G_\alpha \cap U_n^{(a)}| \leq \frac{\epsilon}{2}n^2 \implies |G_{1/2} \cap U_n^{(1/2)}| \leq \epsilon n^2$. To see this notice that
\[ \mathcal{G}_{1/2} \cap \mathbb{U}_{(1/2)}^c \subset \{ \mathcal{G}_{1/2} \setminus \mathcal{G}_\alpha \} \cup \{ \mathcal{G}_\alpha \cap \mathbb{U}_{(1/2)}^c \}. \] Hence we are done.

3. Connectivity properties

Implicit in the statement of Theorem 1.1 is the claim that competitive erosion has a unique stationary distribution. In this section we formally define the competitive erosion chain and prove this claim.

3.1. Formal definition of competitive erosion. Recall \( \mathbb{U}_n \) from Setup 1. We will use \( \mathbb{U}_n \) to denote both this graph and its set of vertices. For \( t = 0, 1, 2, \ldots \) let \((X_t^{(s)})_{s \geq 0}\) and \((Y_t^{(s+\frac{1}{2})})_{s \geq 0}\) be independent simple random walks in \( \mathbb{U}_n \) with

\[
P(X_0^{(t)} = z_1) = \frac{1}{|U_{1,n}|} = \frac{1}{|U_{2,n}|} = P(Y_0^{(t+\frac{1}{2})} = z_2)
\]

for all \( z_1, z_2 \in \mathbb{U}_{1,n}, \mathbb{U}_{2,n} \) respectively. That is, each walk \( X^{(t)} \) starts uniformly on \( U_{1,n} \) and each walk \( Y^{(t+\frac{1}{2})} \) starts uniformly on \( U_{2,n} \). Given the state \( S(t) \) of the competitive erosion chain at time \( t \), we build the next state \( S(t+1) \) in two steps as follows.

\[
S(t + \frac{1}{2}) = S(t) \cup \{X_t^{(t)}\}
\]

where \( \tau(t) = \inf\{s \geq 0 : X_s^{(t)} \notin S(t)\} \). Let

\[
S(t + 1) = S(t + \frac{1}{2}) - \{Y_{\tau(t+\frac{1}{2})}^{(t+\frac{1}{2})}\}
\]

where \( \tau(t + \frac{1}{2}) = \inf\{s \geq 0 : Y_s^{(t+\frac{1}{2})} \in S(t + \frac{1}{2})\} \). As already stated in (1.9) this formally defines the evolution of the process \( \sigma_t \) on \( \Omega_n \cup \Omega'_n \) (see (1.8)).

3.2. Blocking sets and transient states. We show that the erosion chain has an unique irreducible class and hence has a well defined stationary measure.

Definition 2. Call a subset \( A \subset \mathbb{U}_n \) blocking if \( \mathbb{U}_n \setminus A \) is disconnected and the subsets \( \mathbb{U}_{1,n} \setminus A \) and \( \mathbb{U}_{2,n} \setminus A \) lie in different components.

Definition 3. For two disjoint blocking subsets \( A, B \subset \mathbb{U}_n \) we say that \( A \) is over \( B \) if

1. \( A \) and \( \mathbb{U}_{1,n} \setminus B \) lie in different components of \( \mathbb{U}_n \setminus B \); and
2. \( B \) and \( \mathbb{U}_{2,n} \setminus A \) lie in different components of \( \mathbb{U}_n \setminus A \).

Lemma 3.1. The competitive erosion chain has exactly one irreducible class. Moreover any \( \sigma \in \Omega \) that has a blue blocking set over a red blocking set is transient.

Proof. We first claim the following: there exists \( \sigma_* \in \Omega_n \) such that for \( i = 1, 2 \), \( \sigma_*^{-1}(i) \) is a connected set containing \( U_{i,n} \). This is true in the continuous setting clearly by taking the sets to be \( \mathbb{U}_\alpha \) and \( \mathbb{U}_\alpha^c \) (see (1.7)). To find such subsets in \( \mathbb{U}_n \) we look at \( \mathbb{U}_n \cap \mathbb{U}_n^c \) and its complement. These sets might not be connected and might not have the exact number of points. However using the smoothness of \( U \), and considering the sets bounded by a lattice path approximating the boundary of \( \mathbb{U}_\alpha \) and making minor perturbations adding and removing only \( O(n) \) many vertices gives us such sets. The details are omitted.

Now to prove the first part notice that from any \( \sigma \) one can reach \( \sigma_* \). Since in the target configuration there is exactly one blue component \( B \) and red component \( R \), we look at the closest vertex with \( \sigma \)
value 2 from $U_{1,n}$ in $B$. Since there is a blue path from $U_{1,n}$ to that point the Markov chain allows us to change it to 1 and similarly at the other end in $R$. Thus we are done by repeating this.

To prove the second part we prove that starting from $\sigma_*$ one cannot reach any configuration like the one on the right which has one blue blocking set over another red blocking set. We formally prove this by contradiction. Let $\sigma$ be a configuration with a blue blocking set $B$ over a red blocking set $A$. Now the erosion chain evolves as $\sigma_0, \sigma_1/2, \sigma_1, \sigma_3/2, \ldots$ where for every non-negative integer $k$, $\sigma_k \in \Omega$ and $\sigma_{k+1/2} \in \Omega'$.

Assume now that there is a path $\sigma_0, \sigma_1/2, \sigma_1, \sigma_3/2, \ldots, \sigma_t = \sigma$. Let $\tau'$ and $\tau''$ be the last times along the path such that at least one vertex of $A$ is blue and similarly $B$ contains at least one red vertex respectively. If such a time does not exist let us call it $-\infty$. Since $\sigma_0$ does not have a blue blocking set over a red blocking set, $\max(\tau', \tau'') > -\infty$.

$\tau'$ must be a half integer (since at half integers there is one more blue particle) and similarly $\tau''$ must be an integer. Thus $\tau' \neq \tau''$. Next we see that we cannot have $\tau' < \tau''$. This is because then for all times greater than $\tau'$, $A$ is entirely red. Thus no blue walker crosses $A$ at any time greater than $\tau'$. So $B$ must already be blue at $\tau'$ and stays blue through till $t$ implying that $\tau'' < \tau'$. By similar argument we cannot have $\tau' > \tau''$. Hence we arrive at a contradiction. □

4. Proofs of the main results

The goal of this section is to prove Theorems 1.1 and 1.2 modulo some results whose proofs will be deferred to later sections. We first restate formally the crucial statistics mentioned in Section 2 to be used throughout the article. Recall from (2.2),

- The discrete Green function: for any $x \in U_n$,

\[
G_n(x) = \frac{2n^2}{|U_{1,n}|} \int_{t=0}^{\infty} \mathbb{P}_x(X(t) \in U_{1,n}) - \mathbb{P}_x(X(t) \in U_{2,n}) \, dt - c. \tag{4.1}
\]

where for $i = 1, 2$, $\mathbb{P}_x(X(t) \in U_{i,n})$ is the probability that random walk on $U_n$ started from $x$, is in $U_{i,n}$ at time $t$. $c = c(\delta)$ is a centering constant we introduce to ensure that if $\delta$ is small and $n$ is large then the Green function is close to the function

\[
\frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right|. \tag{4.2}
\]
This follows from Theorem 5.1 and Lemma 5.3. \( c(\delta) \) can be written in terms of the Reflected Brownian motion heat kernel on \( U \). The explicit form of \( c \) appears in (7.2).

- The weight function: Given \( \sigma \in \Omega \)

\[
w(\sigma) = \sum_{x \in B_1(\sigma)} G_n(x) \tag{4.3}
\]

where \( B_1(\sigma) \) is defined in (2.3).

### 4.1. Technical results

Recall \( \Omega_\epsilon \) from Table 1 and the erosion chain \( \sigma_t \). As mentioned in Section 2 the next result is one of the key technical ingredients of the paper. It shows that the weight function has a positive drift if the configuration is outside the set \( \Omega_\epsilon \).

**Theorem 4.1.** (Lyapunov Function) Given \( \epsilon > 0 \) there exists positive constants \( a = a(\epsilon), \delta_0 = \delta_0(\epsilon) \) such that for any \( \delta \leq \delta_0 \) and large enough \( n = 2^m > N(\delta) \) and any \( \sigma \in \Omega \setminus \Omega_\epsilon \),

\[
\mathbb{E}(w(\sigma_1) - w(\sigma_0) | \sigma_0 = \sigma) \geq a. \tag{4.4}
\]

The proof is quite involved and is deferred to Section 6. Next we state a few properties of the Greens function (see (4.1)) and the weight function \( w(\cdot) \) whose proofs are deferred to subsection 5.3. The following result proves an uniform upper bound of the weight function independent of \( \delta \).

**Lemma 4.1.** There exists a constant \( D \) such that for all \( \delta \) small enough and \( n = 2^m > N(\delta) \), for all \( \sigma \in \Omega \),

\[
|w(\sigma)| \leq Dn^2 \tag{4.5}
\]

The above fact is a consequence of the following fact: recall from (4.2) that as \( \delta \) goes to 0 the Green function approaches the function \( \frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| \). Now if \( n \) is chosen large enough the weight function divided by \( n^2 \) should approximate the integral of the above function over the blue region. The lemma now follows from the fact that the function \( \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| \) is integrable.

Let us denote by

\[
w_{\max} := \sup_{\sigma \in \Omega} w(\sigma). \tag{4.5}
\]

The next lemma shows that the weight function is uniformly close to its maximum value over the set \( \Omega_\epsilon \) (Table 1)

**Lemma 4.2.** For \( \epsilon > 0 \). There exists constants \( \delta_0(\epsilon), \zeta(\epsilon) \) such that for \( \delta \leq \delta_0 \) and large enough \( n = 2^m > N(\delta) \),

\[
\inf_{\sigma \in \Omega_\epsilon} w_\sigma \geq w_{\max} - \zeta n^2 \tag{4.6}
\]

where \( \zeta \) goes to 0 with \( \epsilon \).

As mentioned above the Green function approaches the function \( \frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| \), as \( n \to \infty \) followed by \( \delta \) going to 0. Thus in particular it blows up in the limit. The next lemma established the “logarithmic singularity” of the limit by determining the rate of blow up of the function as \( \delta \) goes to 0.

**Lemma 4.3.** For all small enough \( \delta \),

\[
\limsup_{n=2^m} \sup_{z \in \mathbb{U}_n} |G_n(z)| \asymp |\log(\delta)|. \tag{4.7}
\]
We need one more definition. Given a number $\epsilon_1$ let us define the following “good” set,

$$
\Gamma_{\epsilon_1} = \{ \sigma \in \Omega : w(\sigma) \geq w_{\text{max}} - 2\epsilon_1 n^2 \}.
$$

(4.8)

**Remark 4.1.** Given $\epsilon > 0$ there exists $\epsilon_2 < \epsilon < \epsilon_1$ such that for all $\delta < \delta_0(\epsilon)$ and $n = 2^m \geq N(\delta)$,

$$
\Gamma_{\epsilon_2} \subset \Omega_{\epsilon} \subset \Gamma_{\epsilon_1}.
$$

Moreover both $\epsilon_1, \epsilon_2$ go to 0 as $\epsilon$ goes to 0.

Thus the above remark strengthens Lemma 4.2. It says that not only does the weight function stay close to its maximum on $\Omega_{\epsilon}$, but $\Omega_{\epsilon}$ is roughly a level set of the weight function i.e. the actual level sets $\Gamma_{\epsilon}$’s can be approximated by the sets $\Omega_{\epsilon}$’s and vice versa.

For proofs of Lemmas 4.1, 4.2, 4.3 and Remark 4.1 see subsection 5.3.

**4.2. Hitting time estimates.** The next result proves an uniform bound on the hitting time of the set $\Omega_{\epsilon}$.

**Lemma 4.4.** Given $\epsilon > 0$ for all $\delta \leq \delta_0(\epsilon)$, there exists $C = C(\epsilon, \delta), d = d(\epsilon, \delta)$ such that for all $n = 2^m \geq N(\delta)$ and $\sigma \in \Omega$,

$$
\mathbb{P}_\sigma(\tau(\Omega_{\epsilon}) \geq Cn^2) \leq e^{-dn^2}
$$

(4.9)

where $\delta$ is the parameter appearing in Setup 1.

Now we show that once the process has hit $\Omega_{\epsilon}$ it tends to stay “close” to the set for a long time.

**Lemma 4.5.** Given $\epsilon_1$ there exists $\delta_0(\epsilon_1)$ such that for all $\delta \leq \delta_0$ there exists a positive constant $d = d(\epsilon_1, \delta)$ such that for all large enough $n = 2^m \geq N(\delta)$ and any $\sigma \in \Omega$

$$
\inf_{\sigma \in \Gamma_{\epsilon_1}} \mathbb{P}_\sigma(\tau(\Omega \setminus \Gamma_{\epsilon_1}) > e^{dn^2/2}) \geq 1 - e^{-dn^2/2}.
$$

The above results follow from a general lemma about hitting times of submartingale. The statement has a few parameters and could be difficult to read. However for subsequent applications we would plug in various values for the parameters.

Let $\omega(t)$ be a stochastic process taking values in an abstract set $\mathcal{D}$. Also let $g : \mathcal{D} \to \mathbb{R}$ be a real valued function. Let $\mathcal{F}_t$ be the filtration generated by the process up to time $t$ and $X_t := g(\omega(t))$.

**Lemma 4.6.** [GLPP15, Lemma 9] Let $A_1, A_2, a_1 > 0$. Suppose

$$
|g(\omega)| \leq A_1 \text{ for all } \omega \in \mathcal{D}
$$

(4.10)

$$
|X_t - X_{t-1}| \leq A_2 \text{ for all } t.
$$

(4.11)

Also suppose that $B \subset \mathcal{D}$ is such that for any time $t$

$$
\mathbb{E}(X_t - X_{t-1} | \mathcal{F}_{t-1}) \geq a_1 \mathbf{1}(\omega(t-1) \notin B)
$$

(4.12)

then
i. $P_{\omega}(\tau(B) \geq T) \leq \exp\left(-\frac{(a_2-a_1)T^2}{4A_2^2T}\right)$ for all $\omega \in \mathcal{D}$ such that $g(w) \geq A_1 - a_2$ for some $a_2 > 0$ and any $T$ such that $a_2 - a_1 T < 0$.

ii. Now consider the special case when $B$ is a level set i.e suppose for some $a_4 > 2A_2$, $B = \{\omega : g(w) \geq A_1 - a_4\}$ and $B' = \{\omega : g(w) \leq A_1 - 2a_4\}$. Then for all $\omega \in B$ and all $T > \frac{2A_2}{a_1}$

$$P_{\omega}(\tau(B') \geq T') \geq 1 - \left[\exp\left(-\frac{a_4^2}{32A_2^2T}\right) + \exp\left(-\frac{a_2^2T^2}{32A_2^2T}\right)\right]$$

where $T' = \exp\left(\frac{\min(a_4^2,a_2^2T^2)}{32A_2^2T}\right)$.

Using the above result the proofs of Lemmas 4.4 and 4.5 follows easily.

**Proof of Lemma 4.4.** The proof follows from Lemma 4.6 i. The stochastic process we consider is the erosion chain $\sigma_t$. $X_t = w(\sigma_t)$ is the weight function defined in (4.3). We make the following choice of parameters:

- $A_1 = Dn^2$ appearing in Lemma 4.1
- $A_2 = C \log(\delta)$ for a large enough universal constant $C$
- $a_1 = a(\epsilon)$ appearing in Theorem 4.1
- $a_2 = 2Dn^2$, $T = \frac{(3D+1)n^2}{a_1}$.

(4.10) is satisfied by Lemma 4.1. That (4.11) is satisfied by our choice of $A_2$ follows from Lemma 4.3.

Thus by Lemma 4.6 i.

$$P_{\sigma}\left(\tau(\Omega_\epsilon) \geq \frac{(3D+1)n^2}{a_1}\right) \leq e^{-Cn^2}$$

for some constant $C = C(D, \delta, a_1) > 0$. \hfill \Box

**Proof of Lemma 4.5.** The proof follows from Lemma 4.6 ii. Let

$$X(t) = w(\sigma_t)$$

where $w(\cdot)$ is the weight function defined in (4.3). We make the following choice of parameters:

- $B = \Gamma_{\epsilon_1}, B' = \Gamma_{2\epsilon_1}$, $a_4 = \epsilon_1 n^2$, $T = n^2$.
- $A_1 = w_{\max}$ (see (4.5)),
- $A_2 = C \log(\delta)$ for a large enough universal constant $C$,
- $a_1 = a(\epsilon)$ appearing in Theorem 4.1.

The drift condition (4.12) is satisfied by $a_1 = a(\epsilon)$. This follows from the lower containment in Remark 4.1 along with Theorem 4.1. Now by the above choice of parameters $T' = \frac{1}{2}e^{\frac{\min(\epsilon_1^2,a_1^2n^2)}{16}}$.

Thus by Lemma 4.6 ii. for all $\sigma \in \Gamma_{\epsilon_1}$,

$$P_{\sigma}(\tau(\Gamma_{2\epsilon_1}) \geq T') \geq 1 - \left[e^{-\frac{\epsilon_1^2n^2}{16}} + e^{-\frac{a(\epsilon)^2n^2}{16}}\right].$$

Hence the proof is complete. \hfill \Box
4.3. **Proof of Theorem 1.2.** We will use the following general lemma relating hitting times and stationary measure. The result roughly says the following: consider any Markov chain and subsets \( A \subset B \) of the state space. If the hitting time of the set \( A \) is uniformly “small” compared to the exit time of \( B \) after hitting \( A \), then \( B \) has large stationary measure.

**Lemma 4.7.** [GLPP15, Lemma 4] Let \( \omega(\cdot) \) be an irreducible Markov chain on a finite state space \( \mathcal{M} \). Suppose \( A \subset B \subset M \). Let \( t_1, t_2, p_1, p_2 \) be such that
\[
\max_{\omega \in M} \{ \mathbb{P}_\omega(\tau(A) \geq t_1) \} \leq p_1 \tag{4.13}
\]
\[
\min_{\omega \in A} \{ \mathbb{P}_\omega(\tau(B^c) \geq t_2) \} \geq 1 - p_2. \tag{4.14}
\]

Then
\[
\pi(B^c) \leq \frac{t_1}{t_2} + p_1 + p_2,
\]
where \( \pi \) is the stationary distribution of the Markov chain \( \omega(\cdot) \) on \( \mathcal{M} \).

We now proceed towards proving Theorem 1.2. Notice that by the lower containment in Remark 4.1 it suffices to prove that for given small enough \( \epsilon \) for all large enough \( n = 2^m \)
\[
\pi(\Gamma_{2\epsilon}) \geq 1 - e^{-cn^2} \tag{4.15}
\]
for some \( c = c(\epsilon, \delta, U) > 0 \). The proof now follows immediately from Lemma 4.7 with the following choices of the parameters:
\[
A = \Gamma_\epsilon, B = \Gamma_{2\epsilon}, t_1 = d_1 n^2, t_2 = e^{d_2 n}, p_1 = e^{-c_1 n^2}, p_2 = e^{-c_2 n^2},
\]
where \( c_1, c_2, d_1, d_2 \) are chosen such that the hypotheses (4.13) and (4.14) of Lemma 4.7 are satisfied. Lemmas 4.4 and 4.5 allow us to do that. \( \square \)

4.4. **Proof of Theorem 1.1.** Proof of Theorem 1.1 is now immediate. By Theorem 1.2 given \( \epsilon > 0 \) for small enough \( \delta \) and \( n = 2^m > N(\delta) \),
\[
\mathbb{E}_{\epsilon, n}[\text{area}(S^\cap U(\alpha))] \leq \epsilon(1 - e^{-Dn^2}) + \text{area(U)}e^{-Dn^2}.
\]
Since \( \epsilon \) is arbitrary the theorem follows. \( \square \)

5. **Technical Preliminaries**

In this section we will develop some preliminary tools needed for the proof of Theorem 4.1.

5.1. **Discrete Green Function.** Recall that we consider continuous time random walk on \( U_n \) (Section 2.3 ii). Call it \( X(t) \). For \( x, y \in U_n \) let
\[
\mathbb{P}_x(X(t) = y) \tag{5.1}
\]
denote the chance that the random walk on \( U_n \) starting from \( x \) is at \( y \) at time \( t \). For notational simplicity we suppress the \( n \) dependence in \( \mathbb{P} \) since the graph will be clear from context. Similarly for any set \( A \subset U_n \), let \( \mathbb{P}_x(X(t) \in A) \) denote the chance that the random walk is in \( A \) at time \( t \) starting from \( x \). Recall the definition of the function \( G_n \) on \( U_n \) from (4.1): for any \( x \in U_n \)
\[
G_n(x) = \frac{2n^2}{|U_{1,n}|} \int_0^\infty [\mathbb{P}_x(X(t) \in U_{1,n}) - \mathbb{P}_x(X(t) \in U_{2,n})]dt - c. \tag{5.2}
\]
where \( c = c(\delta) \) is some constant explicitly mentioned in (7.2). The fact that the integral in \( G_n \) is absolutely integrable follows immediately from the following lemma.
Lemma 5.1. [GP15, Lemma 3.1] Given $\mathbb{U}$ as in Setup 1 there exists a constant $D = D(\mathbb{U})$ and a time $T = T(\mathbb{U})$ such that, for all large enough $n$,
\[
\sup_{x \in \mathbb{U}_n} |P_x(X(t) \in \mathbb{U}_{1,n}) - P_x(X(t) \in \mathbb{U}_{2,n})| \leq 2e^{-Dt}
\]
for all $t \geq T$.

The above is a straightforward consequence of the mixing rate of random walk and the fact that since by choice $|\mathbb{U}_{1,n}| = |\mathbb{U}_{2,n}|$, $\pi_{RW}(\mathbb{U}_{1,n}) = \pi_{RW}(\mathbb{U}_{2,n})$, where $\pi_{RW}$ is the stationary measure for the random walk on $\mathbb{U}_n$.

For any function $f : \mathbb{U}_n \rightarrow \mathbb{R}$ define the laplacian $\Delta : \mathbb{U}_n \rightarrow \mathbb{R}$: for any $x \in \mathbb{U}_n$
\[
\Delta f(x) = f(x) - \frac{1}{d_x} \sum_{y \sim x} f(y)
\]
where $d_x$ is the degree of the vertex $x$ and $y \sim x$ denotes that $y$ is a neighbor of $x$.

Lemma 5.2. Consider the function $G_n(\cdot)$ on $\mathbb{U}_n$. Then,
\[
\Delta(G_n) = \frac{1}{|\mathbb{U}_{1,n}|}(1(\mathbb{U}_{1,n}) - 1(\mathbb{U}_{2,n})),
\]
where for any subset $A \subset \mathbb{U}_n$, $1(A)$ denotes the indicator of the set $A$.

Proof. Proof follows from definition of $G_n$ and looking at the first step of random walk which by definition is of expected duration $\frac{1}{2n^2}$. Thus we have
\[
G_n(x) = \frac{1}{|\mathbb{U}_{1,n}|}(1(\mathbb{U}_{1,n}) - 1(\mathbb{U}_{2,n}))(x) + \frac{1}{d_x} \sum_{y \sim x} G_n(y)
\]
and hence the lemma.

5.2. Conformal invariance of Green function. In this section we state a key technical result establishing conformal invariance of the Green function. We define the following function on the smooth domain $\mathbb{U}$,
\[
\tilde{f} := \frac{16}{\text{area}(\mathbb{U}_1)}(1(\mathbb{U}_2) - 1(\mathbb{U}_1)).
\]
Again 16 is a constant that appears for the same reason that 32 appears in (1.5) and is not important.

Recall the functions $\phi$ and $\psi$ from (1.4). Now let us call $\psi(\mathbb{U}_i) = A_i$. Then
\[
\tilde{f} \circ \phi = \frac{16}{\text{area}(\mathbb{U}_1)}(1(A_2) - 1(A_1)).
\]
For any $\mathbb{U}$ as in Setup 1, define the function $G_* : \mathbb{U} \rightarrow \mathbb{R}$ such that for all $z \in \mathbb{U}$ if $y \in \mathbb{D}$ is such that $\phi(y) = z$, then
\[
G_*(z) = \frac{1}{\pi} \int_{|\zeta| < 1} \tilde{f} \circ \phi(\zeta) |\phi'|^2(\zeta) \log(|z - y|) |1 - \zeta y| d\xi d\eta,
\]
where $\zeta = \xi + i\eta$. The above expression is well defined since by Section 1.5 i., the maps $\phi, \psi$ can be extended to neighborhoods containing $\mathbb{D}, \mathbb{U}$ respectively. Notice the dependence of $G_*$ on $\delta$ through $\tilde{f}$. However for brevity we choose to suppress the dependence on $\delta$ in the notation. The definition
of the function might be a little mysterious at this point. Nevertheless Lemma 5.3 shows that for small $\delta$, the function is essentially the same as the function
\[
\log \left| \frac{\psi(z) - i}{\psi(z) + i} \right|.
\]
Since the above function is conformally invariant by definition, this establishes that the function $G_*(\cdot)$ is “asymptotically” conformally invariant.

Even though the function $G_n$ in (5.2) is defined on the graph $\overline{U}_n$, in this section we use interpolation to think of it as a function on the closure of the whole domain, $\overline{U}$. We are now ready to state one of the main convergence results of this paper establishing asymptotic conformal invariance of $G_n(\cdot)$. We now define the interpolation scheme. We follow the scheme mentioned in [CF13, Pf of Theorem 2.12] (also appears in [Fan14, Pf of Theorem 2.2.8]).

For all $x \in \frac{1}{n} \mathbb{Z}^2 \setminus \overline{U}_n$, define $G_n(x) = 0$. Now having defined $G_n(x)$ for all $x$, $\in \frac{1}{n} \mathbb{Z}^2$ we define it on $C$ and hence by restricting on $\overline{U}$. This is done by interpolating $G_n(x)$ by a sequence of harmonic extensions along simplices,

i. First extend it along the edges using the value on the vertices in $\frac{1}{n} \mathbb{Z}^2$.

ii. Then extend it to the squares using the value on the edges.

Thus the function is extended to the entire complex plane $C$. By abuse of notation in the following result we call the extended function $G_n(\cdot)$ as well.

**Theorem 5.1.** [GP15, Theorem 3.1] For all small enough $\delta$,
\[
\lim_{m \to \infty} \sup_{z \in U} |G_n(z) - G_*(z)| = 0.
\]

The basic outline is sketched in Section 7. For the complete proof see [GP15]. We end this section by making a few observations about the function $G_*(\cdot)$ and hence by the aforementioned theorem, about the asymptotics of the function $G_n$.

Let us define the following one parameter family of curves for any $U$ which form the boundary of the sets $U_\beta$ defined in (1.5). For any $\beta \in \mathbb{R}$ define
\[
\gamma_\beta(U) = \{ z \in U : \frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| = \beta \}.
\]

(5.8)

We will often suppress the dependence on $U$ and denote the above just by $\gamma_\beta$ for brevity, if the underlying domain is clear from context. It is well known that for any $\beta \in \mathbb{R}$, $\gamma_\beta$ is a hyperbolic geodesic. In case of the disc, $\gamma_\beta$’s are circular arcs orthogonal to $\partial D$ symmetric with respect to $-i$ and $i$. For more details regarding the hyperbolic metric see [Ahl79].

The following is an uniform convergence result of $G_*(\cdot)$ away from the points $x_1, x_2$.

**Lemma 5.3.** Let $U$ be as in Setup 1. Given $a < 1$,
\[
\lim_{\delta \to 0} \sup_{z \in U} \left| G_*(z) - \frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| \right| = 0.
\]

**Proof.** Since the map $\phi$ is bi-Lipschitz by subsection 1.5 i. it suffices to show that
\[
\lim_{\delta \to 0} \sup_{z \in U} \left| G_* \circ \phi(z) - \frac{32}{\pi} \log \left| \frac{z - i}{z + i} \right| \right| = 0.
\]
Recall the sets $A_1, A_2 \subset \mathbb{D}$ from (5.6). Note that by hypothesis
\[ \mathbb{U}_1 \subset B(x_1, 2\delta) \cap \mathbb{U} \]
and hence
\[ A_1 \subset B(-i, C\delta) \cap \mathbb{D} \tag{5.9} \]
for some constant $C$ (where $C/2$ is the Lipschitz constant of $\psi$). Similar containment holds for $\mathbb{U}_2$ and $A_2$. Now notice that by the change of variable formula for $i = 1, 2$
\[ \frac{1}{\text{area}(\mathbb{U}_1)} \int_{A_i} \tilde{f} \circ \phi(\zeta)|\phi'|^2(\zeta)d\xi d\eta = 16. \tag{5.10} \]
These facts along with (5.7) clearly imply that
\[ G_\ast \circ \phi(z) = \frac{32}{\pi} \log \frac{|z - \bar{i}| + O(\delta)}{|z + \bar{i}| + O(\delta)}, \tag{5.11} \]
where the constant in the $O$ term depends only on $C$ in (5.9). Thus
\[ \left| G_\ast \circ \phi(z) - \frac{32}{\pi} \log \frac{|z - \bar{i}|}{|z + \bar{i}|} \right| = O \left( \log \left( 1 + \frac{O(\delta)}{|z - \bar{i}|} \right) + \log \left( 1 + \frac{O(\delta)}{|z + \bar{i}|} \right) \right) \]
Since $\lim_{\delta \to 0} \frac{\delta}{\delta} = 0$ by making $\delta$ go to 0 we see that the RHS uniformly converges to 0 on the set
\[ \{ z \in \overline{\mathbb{D}} : \min(|z - i|, |z + i|) \geq \delta^a \}. \]
Hence we are done. \qed

Since the function $\log \frac{|z - \bar{i}|}{|z + \bar{i}|}$, has logarithmic singularities at $i$ and $-i$ as $\delta \to 0$ by Lemma 5.3 the function $G_\ast(z)$ blows up near the singularities $x_1, x_2$. The following lemma determines the rate at which the blow up occurs.

**Lemma 5.4.** For $a < 1$ and all small enough $\delta$, for all $z \in \mathbb{U}$ such that $d(z, x_1) \leq \delta^a$

\[ G_\ast(z) \asymp |\log(\delta)| \]

and similarly if $d(z, x_2) \leq \delta^a$,

\[ G_\ast(z) \asymp -|\log(\delta)|. \tag{5.12} \]

The constants in the $\asymp$ notation depend on $a$ only.

**Proof.** We only show the first case since the argument for the other case is similar. Hence we assume $d(x_1, z) \leq \delta^a$. Also using the bi-Lipschitz property of $\phi$ it suffices to show that for all $z \in \overline{\mathbb{D}}$ such that $d(z, -\bar{i}) \leq \delta^a$,

\[ G_\ast \circ \phi(z) \asymp |\log(\delta)|. \]

Using (5.7) we do it in two steps. We first show the following:

(i).

\[ \frac{1}{\text{area}(\mathbb{U}_1)} \frac{1}{\pi} \int_{|\zeta| < 1} (1(A_2) - 1(A_1))(|\zeta|)|\phi'|^2(\zeta) \log |\zeta - z|d\zeta \asymp (|\log(\delta)|). \]

Clearly

\[ \frac{1}{\text{area}(\mathbb{U}_1)} \frac{1}{\pi} \int_{|\zeta| < 1} 1(A_2)(\zeta)|\phi'|^2(\zeta) \log |\zeta - z|d\zeta = O(1) \]

since $d(z, A_2) > d(x_1, x_2)/2$ for $\delta$ small enough. Since $\psi$ is bi-Lipschitz, $A_1$ is contained in a ball of radius $C\delta$ and contains a ball of radius $c\delta$ for some constants $C, c$ not depending on $\delta$ which implies that $\text{area}(A_1) = \Theta(\delta^2)$. Also $|\phi'|$ is bounded away from 0 and $\infty$. 

Thus for any \( z \in \overline{D} \) we have
\[
\frac{1}{\text{area}(U_1)} \frac{1}{\pi} \int_{|\zeta|<1} 1(A_1)(\zeta) |\phi|^2(\zeta) \log |\zeta - z|d\zeta = O\left(\int \frac{C \delta}{\delta^2} \int_{|\zeta-z|=r} \log |\zeta - z|d\theta dr\right)
\]
\[
= O\left(\frac{1}{\delta^2} \int_{0}^{C \delta} \int_{|\zeta-z|=r} \log(r)dr\right)
\]
\[
= O\left(\log(|\log(\delta)|)\right).
\]

The lower bound follows immediately since \( d(z, A_1) \leq \delta^a \).

(ii). We now prove a similar bound for the term
\[
\frac{1}{\text{area}(U_1)} \frac{1}{\pi} \int_{|\zeta|<1} 1(A_1) - 1(A_2)(\zeta) |\phi|^2(\zeta) \log |1 - \bar{\zeta} z|d\zeta.
\]

Again it is easy to check that
\[
\frac{1}{\text{area}(U_1)} \frac{1}{\pi} \int_{|\zeta|<1} 1(A_2)(\zeta) |\phi|^2(\zeta) \log |1 - \bar{\zeta} z|d\zeta = O(1).
\]

To find the exact order of the term
\[
\frac{1}{\text{area}(U_1)} \frac{1}{\pi} \int_{|\zeta|<1} 1(A_1)(\zeta) |\phi|^2(\zeta) \log |1 - \bar{\zeta} z|d\zeta
\]

recall from Setup 1 \( d(U_1, \partial U) = \Theta(\delta) \) and hence \( d(A_1, \partial \overline{D}) = \Theta(\delta) \). Thus for any \( \zeta \in A_1 \), we have \( C \delta \leq |1 - \bar{\zeta} z| \). Also since \( d(z, x_1) \leq \delta^a \) by hypothesis \( |1 - \bar{\zeta} z| < \delta^a \). Therefore,
\[
\frac{1}{\text{area}(U_1)} \frac{1}{\pi} \int_{|\zeta|<1} 1(A_1)(\zeta) |\phi|^2(\zeta) \log |1 - \bar{\zeta} z|d\zeta \times (|\log(\delta)|).
\]

Thus putting (i) and (ii) together we are done.

\( \square \)

5.3. **Proofs of some earlier statements.** *Proof of Lemma 4.3.* It follows immediately from Lemmas 5.3, 5.4 and Theorem 5.1.

*Proof of Lemma 4.1.* Clearly by (5.7) and Lemmas 5.3, 5.4, \( G_*(z) \) is a bounded continuous function on \( U \) (where the bound is \( O(|\log(\delta)|) \)). Thus approximating integral by Riemann sums we get
\[
\int_{U} |G_*(z)| d\xi d\eta = \lim_{n \to \infty} \sum_{z_n \in U_n} \frac{1}{n^2} |G_*(z_n)|
\]
where \( z = \xi + i\eta \). The proof will then immediately follow from Theorem 5.1 if we can show that
\[
\lim_{\delta \to 0} \sup_{U} \int |G_*(z)| d\xi d\eta < \infty.
\]

Now notice that
\[
\frac{\lim_{\delta \to 0} \int_{U} |G_*(z)| d\xi d\eta}{\int_{U} \pi} = \int_{U} \frac{32}{\pi} \left| \log \left( \frac{\psi(z) - i}{\psi(z) + i} \right) \right| d\xi d\eta
\]
\[
= \int_{\mathbb{D}} |\phi'(z)| \frac{22}{\pi} \left| \log \left( \frac{|z - i|}{|z + i|} \right) \right| d\xi d\eta
\]
\[
< \infty.
\]
The first equality is a consequence of Lemmas 5.3 and 5.4. The second equality is just the change of measure formula. The last inequality holds since $|\phi'(z)|$ is bounded and the function $\log|\frac{z-i}{z+i}|$ is an integrable function on the disc. Thus we are done. □

Proof of Lemma 4.2. Let us denote by $\sigma_*$ the configuration where every vertex in $\mathbb{U}(\alpha)$ is colored blue. By Lemma 5.3 for all $z \in \mathbb{U}$ such that $\min(d(z, x_1), d(z, x_2)) \geq \delta^a$ we have

$$G_*(z) = \frac{32}{\pi} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| + o(1). \tag{5.13}$$

Also by Lemma 4.3 on the remaining set of measure $O(\delta^a)$, $|G_*| = O(|\log(\delta)|)$. This along with Theorem 5.1 implies directly that given any number $c$ for small enough $\delta$ and $n = 2^m > N(\delta)$ we have

$$|w_{\max} - n \frac{32}{\pi} \int_{\mathbb{U}_n} \log \left| \frac{\psi(z) - i}{\psi(z) + i} \right| d\xi d\eta| \leq cn^2$$

$$|w(\sigma) - w_{\max}| \leq cn^2. \tag{5.14}$$

Also from Lemmas 5.3 and 5.4 it follows that the function $G_*(z)$ (as $\delta$ goes to 0) is uniformly integrable i.e. given any $a$ there exists $b = b(a) \ (b(a) \to 0 \text{ as } a \to 0)$ such that.

$$\limsup_{\delta \to 0} \sup_A \int_A G_*(z) d\xi d\eta \leq b$$

where the supremum is taken over all subsets $A$ of $\mathbb{U}$ of measure at most $a$. Using this and Theorem 5.1 we thus have the following : Given any constant $a$ there exists a constant $b = b(a)$ such that for all small enough $\delta$ and $n = 2^m > N(\delta)$:

$$\sup_{A \subseteq \mathbb{U}_n} \sum_{|A| \leq a n^2, z_n \in A} |G_n(z_n)| \leq bn^2 \tag{5.15}$$

where $b = b(a)$ tends to 0 as $a$ tends to 0. Thus for any $\sigma \in \Omega_\epsilon$ by (5.15)

$$|w(\sigma) - w(\sigma_*)| \leq b(2\epsilon)n^2.$$

Hence we are done by (5.14). □

Proof of Remark 4.1. The upper containment follows from (4.6). The lower containment follows from (5.13) and Theorem 5.1. □

We are now ready to prove the main technical result of this paper.

6. PROOF OF THEOREM 4.1

The proof is quite long and involved and constitutes the technical core of the paper. The argument has to be divided into cases depending on $\sigma$. Thus we split the result into two theorems. We will suppress the $\sigma$ dependence in the notations we define subsequently for brevity whenever there is no scope for confusion. We start with some definitions and notation.

Definition 4. Given $\sigma \in \Omega \cup \Omega'$ (see (1.8)) for $i = 1, 2$, let $R_i = R_i(\sigma)$ be the set of all points in $\mathbb{U}_n$ of color $i$ reachable by a monochromatic path of color $i$ from a point in $\mathbb{U}_{i,n}$ (we do not allow the endpoint to be the opposite color). Note that $R_1$ and $R_2$ are disjoint.
The blue connected region containing the blue blob is $R_1$ and similarly $R_2$. Note that the two red islands and the blue island are not included in either set. Also recall (even though not in figure) that if a part of the blue blob is colored red, it is not included in the set $R_1$.

Clearly by definition
\[ \mathbb{E}[w(\sigma_1) - w(\sigma_0) \mid \sigma = \sigma_0] = \mathbb{E}[G_n(X_{\tau(R_1(\sigma_0)))} - G_n(Y_{\tau(R_2(\sigma_{1/2}))})] \] (6.1)

where $X_{\tau(R_1(\sigma_0))}$ is the point at which the random walk exits $R_1(\sigma_0)$ and similarly the second term. The expectation in the first term is over $\mu_1$ (starting distribution of the blue random walk. See Setup 1.) Note the expectation in the second term is over $\mu_2$ as well as the random intermediate configuration $\sigma_{1/2}$.

For any $x \in \bar{U}, a > 0$ define
\[ C(x, a) := \partial B(x, a) \cap U, \text{ i.e the part of the boundary inside } U. \] (6.2)

Recall the weight function $w(\cdot)$ from (4.3). We now state the first theorem towards the proof of Theorem 4.1. The theorem roughly says that if either diameter of $R_1$ is “small” or $R_2$ connects a region “close” to $x_2$ to a region “close” to $x_1$ then the drift in Theorem 4.1 is like $|\log \delta|$. See Fig 8 and 9.

**Theorem 6.1.** Let $U$ be as in Setup 1. Then there exists constants
\[ 0 < a_2 < a_1 < 1 \]
and $\delta_0$ such that for $\delta \leq \delta_0$ and all $n = 2^m > N(\delta)$ if $\sigma \in \Omega$ is such that either
i. $R_1$ does not intersect $C(y_1, \delta_2)$ or,

ii. $R_1$ intersects $C(y_1, \delta_2)$ and $R_2$ intersects $C(y_1, \delta_1)$ or

switching the roles of $R_1$ and $R_2$ in the above two cases

iii. $R_2$ does not intersect $C(y_2, \delta_2)$ or,

iv. $R_2$ intersects $C(y_2, \delta_2)$ and $R_1$ intersects $C(y_2, \delta_1)$

then
\[ \mathbb{E}[w(\sigma_1) - w(\sigma_0) \mid \sigma_0 = \sigma] \asymp |\log(\delta)| \] (6.3)
where for \( i = 1, 2 \) \( \delta_i = \delta^{a_i} \).

The reason for the above is roughly the following: (we only discuss the first two cases.)

- In case \( i \), diameter of \( R_1 \) is small implies that the blue random walk stops before exiting a small ball. Thus the first term in (6.1) is roughly \(|\log(\delta)|\) by Lemma 5.4. Now by standard random walk estimates one can show that the red walk is likely to exit \( R_2 \) before coming close to \( x_1 \). Thus the second term in (6.1) is much smaller and hence the result follows.

- In case \( ii \), by hypothesis there is a red path which connects \( U_{2,n} \) to a small ball around \( y_1 \) of radius \( \delta^a \) for some \( a \). Then it becomes likely that the blue random walk starting from \( U_{1,n} \) will hit this path and hence exit \( R_1 \) before going too far from \( x_1 \) again makes the first term in (6.1) roughly \(|\log(\delta)|\). The rest of the arguments are then the same as in the previous case which shows that the second term in (6.1) is much smaller.

Before providing the formal proof we first state a lemma regarding random walk on \( U_n \). Let \( \mathbb{P}_x(\cdot) \) denote the random walk probability measure started from \( x \).

**Lemma 6.1.** [GP15, Lemma 5.2] For all \( U, x_1 \) as in Setup 1,

\[
\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sup_{\sigma \in \Omega \cup \Omega'} \left[ \sup_{z \in U_n \setminus B(x_1, \frac{1}{2})} \mathbb{P}_z \{ \tau(B(x_1, \epsilon)) \leq \tau(B_1) \} \right] = 0
\]

where \( B_1 = B_1(\sigma) \) was defined in (2.3).

The above lemma says that uniformly from any point \( z \) at distance \( 1/2 \) (any constant would work) from \( x_1 \) and all configurations \( \sigma \in \Omega \cup \Omega' \) the chance that the random walk does not hit \( B_1 \) before reaching at distance \( \epsilon \) from \( x_1 \) goes to 0 as \( \epsilon \) goes to 0. Note that since \( B_1 \) is the set of all blue particles by definition \( \pi_{RW}(B_1) = \alpha - o(1) \) where the \( o(1) \) goes to 0 as \( n \) goes to infinity. The proof of the above lemma for general sets of measure at least \( \alpha \) appears in [GP15]. We are now ready to prove Theorem 6.1.
Proof of Theorem 6.1.
By symmetry it suffices to prove the theorem for the first two cases.

Proof of case i. Let $a_3 < a_2 < a_1 < 1$ be constants to be specified later. Let $\delta_3 = \delta^{a_3}$ and $p(\delta_3)$ be the probability that the random walk started uniformly from $U_{2,n}$ hits $B(x_1, \delta_3)$ before hitting $B_1(\sigma)$ (the set of blue sites) which has size $\alpha \cdot \text{area}(U)^{n^2} + O(n)$. Now observe that in this case there exists universal $D_1, D_2 > 0$ (not depending on $\delta, n$) such that for all small enough $\delta$, and large enough $n$

$$E[w(\sigma_1) - w(\sigma_0) | \sigma_0] \geq D_1 |\log(\delta_2)| - [p(\delta_3)D_2 |\log(\delta)| + (1 - p(\delta_3))D_2 |\log(\delta_3)|].$$

(6.4)

To see (6.4) we first recall (6.1).

$$E[w(\sigma_1) - w(\sigma_0) | \sigma = \sigma_0] = E[G_n(X_{\tau(R_1(\sigma_0))}) - G_n(X_{\tau(R_2(\sigma_{1/2}))})].$$

By hypothesis $R_1$ does not intersect $B^c(x_1, \delta_2)$. See Fig 8. Hence the blue random walk stops before exiting $B(x_1, \delta_2)$. Thus from Theorem 5.1, Lemma 5.4 it follows that the first term is at least $D_1 |\log(\delta_2)|$. Now the second term is at most $O(|\log(\delta)|) by Theorem 5.1 and Lemma 4.3 if the red random walk enters the ball $B(x_1, \delta_3)$ before hitting $B_1(\sigma_{1/2})$ which happens with probability at most $p(\delta_3)$. Otherwise it is at most $D_2 |\log(\delta_3)|$ by Lemma 5.3 and Theorem 5.1.

Since $p(\delta_3)$ goes to 0 as $\delta$ goes to 0 we can suitably choose $a_3 < a_2$ to be done. Note that we do not need $a_1$ for this argument. It appears in the proof of the next case.

**Figure 9.** Illustrating the case when $R_2$ connects $B(y_2, \delta)$ to a point in the neighborhood of $x_1$.

**Proof of case ii.** By hypothesis there exists a path $\gamma$ of red vertices connecting $B(x_1, \delta_1)$ and $B^c(x_1, \delta_2)$. Thus it should be likely that the blue random walk started from $U_{1,n}$ will hit $\gamma$ before exiting $B(x_1, \delta_2)$. This is precisely the content of the next lemma which roughly says if a connected set $A$ of large enough diameter is close enough to $U_{1,n}$ then random walk starting from $U_{1,n}$ is more likely to hit the set $A$ before exiting a large enough ball.

**Lemma 6.2.** [GP15, Lemma 5.5] Let $0 < \epsilon_1 < \epsilon_2$. Assume $A \subset U_n$ is a connected set such that $d(U_{1,n}, A) \leq \epsilon_1$. 

Also assume \( A \cap (\mathbb{U}_n \setminus B(y_1, \epsilon_2)) \neq \emptyset \). Then
\[
\sup_{x \in \mathbb{U}_{1,n}} \mathbb{P}_x(\tau(\mathbb{U}_n \setminus B(y_1, \epsilon_2)) \leq \tau(A)) \leq C \log \left( \frac{\epsilon_2}{\epsilon_1} \right),
\]
for some \( C = C(\mathbb{U}) < 1 \) independent of \( n \).

Thus by the above lemma,
\[
\inf_{x \in \mathbb{U}_{1,n}} \mathbb{P}_x(\tau(\gamma) \leq \tau(B^c(x_1, \delta_2))) \geq 1 - C \log(\delta_2 / \delta_1)
\]
for some \( C = C(U) < 1 \). Thus
\[
\mathbb{E}(w(\sigma_1) - w(\sigma_0)|\sigma_0) \geq D_1|\log(\delta_2)| - C \log(\delta_2 / \delta_1) D_2|\log(\delta)| - [p(\delta_3)|\log(\delta)| + (1 - p(\delta_3))D_2|\log(\delta_3)|].
\]

This is because \( G_n(X_{\tau(R_i(\sigma_0))}) \) is at least \( |\log(\delta_2)| \) if the blue random walk stops before exiting \( B(x_1, \delta_2) \) which happens with chance at least \( 1 - C \log(\delta_2 / \delta_1) \) by (6.5). Otherwise it is at least \( -O(|\log(\delta)|) \) since that is the minimum value \( G_n \) takes by Theorem 5.1 and Lemma 4.3. Bound on \( G_n(X_{\tau(R_2(\sigma_{1/2})}) \) is exactly as in case \( i \).

Now one can suitably choose \( a_3 < a_2 < a_1 \) such that the RHS in both (6.4) and (6.6) are \( \Theta(|\log(\delta)|) \). Thus like case \( i \), the proof follows by suitably choosing \( a_i \)'s for \( i = 1, 2, 3 \). The proof of Theorem 6.1 is hence complete. \( \square \)

Now to finish the proof of Theorem 4.1 we have to consider the cases not considered in Theorem 6.1. Thus we first make the assumption,

**Assumption 1.** \( \sigma \) does not fall in any of the four cases in the statement of Theorem 6.1.

That is both \( R_1, R_2 \) extend beyond \( B(x_1, \delta_2), B(x_2, \delta_2) \) respectively and also \( R_2 \) does not reach \( B(x_1, \delta_1) \) and similarly \( R_1 \) does not reach \( B(x_1, \delta_2) \). Thus under Assumption 1,
\[
B(x_1, \delta_1) \cap R_2 = \emptyset
\]
\[
B(x_2, \delta_1) \cap R_1 = \emptyset.
\]

Notice that for \( i = 1, 2 \)
\[
\mathbb{U}_{i,n} \subset B(y_i, \delta_1)
\]
for small enough \( \delta \), since \( \delta_1 = \delta^a \) for some \( a < 1 \).

**Remark 6.1.** We now claim the following. Given any small enough \( \delta \) for all \( n \geq N(\delta) \) there exists a set \( A_{1,n} \subset \mathbb{U}_n \) such that \( A_{1,n} \) is connected and,
\[
B(y_1, \frac{\delta_1}{4}) \cap \mathbb{U}_n \subset A_{1,n} \subset B(y_1, \delta_1) \cap \mathbb{U}_n.
\]

Similarly one has \( A_{2,n} \) corresponding to \( y_2 \). This is a simple consequence of the fact that the locally near the boundary \( \mathbb{U} \) looks like a half plane, see (8.1). We omit the details.

Now notice that since
\[
\mathbb{U}_{i,n} \subset B(y_i, \frac{\delta_1}{4}) \cap \mathbb{U}_n \subset A_{i,n},
\]
it immediately follows from definition of \( R_i \) that
\[
R_i \cup A_{i,n}
\]
is a connected subset of $\mathbb{U}_n$. For technical reasons sometimes it will be convenient to work with the following larger set instead of $R_1$. Let $\mathbb{U}_n \setminus \{R_1 \cup A_{1,n}\} = \bigcup_{i=1}^k C_{1,i}$ where $C_{1,i}$ are the connected components of $\mathbb{U}_n \setminus \{R_1 \cup A_{1,n}\}$. Let $\{R_2 \cup A_{2,n}\} \subseteq C_{1,k}$. Define

$$R_1 = R_1 \bigcup_{i \neq k} A_{i,n} \bigcup C_{1,i}.$$  \hfill (6.8)

In words $\hat{R}_1$ is the set $R_1 \cup A_{i,n}$ along with all the red holes filled in. Similarly define $\hat{R}_2$. Note that for $i = 1, 2$, $\hat{R}_i$ is connected. See Fig. 10.

![Figure 10](image)

**Figure 10.** Illustrating the difference between the sets $R_i$ and $\hat{R}_i$ for $i = 1, 2$. Note that in the latter the holes are filled in.

Given $A \subset \mathbb{U}_n$ we define two types of boundary of $A$

$$\partial_{\text{out}} A := \{ y \in A^c : \exists x \in A \text{ such that } x \sim y \} \hfill (6.9)$$

$$\partial_{\text{in}} A := \{ x \in A : \exists y \in A^c \text{ such that } x \sim y \}.$$  

**Definition 5.** Let $\mathbb{U}_n^*$ denote the graph $\mathbb{U}_n$ along with all the diagonals of the squares that are entirely in $\mathbb{U}_n$. We will call connected subsets of $\mathbb{U}_n^*$ as $*$--connected subsets of $\mathbb{U}_n$.

**Remark 6.2.** Since both $\hat{R}_1$ and $\hat{R}_2^c$ are connected by Remark 6.1 it follows by [Tim13, Lemma 2] that both $\partial_{\text{out}} \hat{R}_1$ and $\partial_{\text{in}} \hat{R}_1$ are $*$--connected.

Recall that $\beta = \beta(\alpha) \in \mathbb{R}$ is such that

$$\text{area}(\mathbb{U}_\beta) = \alpha(\text{area}(\mathbb{U}))$$

as in (1.6). We need a few more notations. Given $a > 0$ we define the $a$--interior of $\mathbb{U}$,

$$\mathbb{U}^{(a)} := \{ x \in \mathbb{U} : d(x, \partial \mathbb{U}) > a \}. \hfill (6.10)$$

For any $\sigma \in \Omega \cup \Omega'$, define

$$\partial_{\text{in}}^{(a)} \hat{R}_1 = \partial_{\text{in}} \hat{R}_1 \cap \mathbb{U}^{(a)}.$$  

The following theorem roughly says that unless $\hat{R}_1$ looks like $\mathbb{U}_\beta$ there is positive drift. To argue this we consider a few cases: All the cases ignore the boundary of $\hat{R}_1$ which lies very close to $\partial \mathbb{U}$. 

The first four cases together say that unless $\partial_{\mathcal{m}} \hat{R}_1$ looks like $\gamma_\beta$ we get positive drift for the weight function. In the last case it says even if $\partial_{\mathcal{m}} \hat{R}_1$ is close to $\gamma_\beta$ if there are $O(n^2)$ vertices of wrong color one still gets positive drift. To measure how “far” the boundary is from the curve $\gamma_\beta$ we consider

$$\beta_{-5} > \beta_{-4} > \beta_{-3} > \beta_{-2} > \beta_{-1} = \beta > \beta_1 > \beta_2 > \beta_3 > \beta_4 > \beta_5$$

and look at the corresponding $\gamma_{\beta_i}$’s. See Fig. 11.

**Theorem 6.2.** Given $\epsilon > 0$, there exists $\epsilon_1, \epsilon_2, c > 0$ such that for all small enough $\delta$, and all $n = 2^m$ large enough, if $\sigma \in \Omega$ satisfying Assumption 1 satisfies any of the following cases,

i. $\partial_{\mathcal{m}}^{\{\epsilon_2\}} \hat{R}_1 = \emptyset$.

ii. $\partial_{\mathcal{m}}^{\{\epsilon_2\}} \hat{R}_1$ is a subset of $\mathbb{U}_{\beta_{-2}}$.

iii. $\partial_{\mathcal{m}}^{\{\epsilon_2\}} \hat{R}_1$ is a subset of $\mathbb{U}_{\beta_2}^c$.

iv. $\partial_{\mathcal{m}}^{\{\epsilon_2\}} \hat{R}_1$ intersects $\mathbb{U}_{\beta_i}$ and $\mathbb{U}_{\beta_{i+2}}^c$ for some $i \in \{-4, \ldots, 2\}$.

v. $\partial_{\mathcal{m}}^{\{\epsilon_2\}} \hat{R}_1 \in \mathbb{U}_{\beta_{-4}}^c \cap \mathbb{U}_{\beta_4}$ and one of the following holds

$$\# \{(x, y) \in \mathbb{U}_{\beta_{-5}} : \sigma(x, y) = 2\} \geq \epsilon n^2 / 2$$

$$\# \{(x, y) \in \mathbb{U}_{\beta_5}^c : \sigma(x, y) = 1\} \geq \epsilon n^2 / 2$$

then

$$\mathbb{E}(w(\sigma_1) - w(\sigma) \mid \sigma_0 = \sigma) > c$$

where

$$\beta_{-5} > \beta_{-4} > \beta_{-3} > \beta_{-2} > \beta_{-1} = \beta > \beta_1 > \beta_2 > \beta_3 > \beta_4 > \beta_5$$

(6.11)
such that for all $i = -5, -4, \ldots, 4,$

$$|\beta_i - \beta_{i+1}| = \epsilon_1.$$  

Like discussed before the statement of the theorem the first four cases consider situations where the boundary of $\tilde{R}_1$ is “far” from $\tilde{\gamma}_\beta$. The last case considers the situation that even in the case that the boundary is roughly $\tilde{\gamma}_\beta$, if $\tilde{R}_1$ contains $\Theta(n^2)$ red vertices or its complement contains $\Theta(n^2)$ blue vertices one still has positive drift. Before proving the above theorem we see how Theorem 4.1 immediately follows.

6.1. **Proof of Theorem 4.1.** The theorem is a straightforward corollary of Theorems 6.1, 6.2. This is because any $\sigma \in \Omega \setminus \Omega_\epsilon$ satisfies the hypotheses of at least one of the two results. To see this clearly the only case that needs discussion is when $\sigma$ does not satisfy either Theorem 6.1 or the first four cases of Theorem 6.2. We choose $\epsilon_1$ small enough compared to $\epsilon$ such that

$$\text{area}(U_{\beta_5} \setminus U_{\beta_{-5}}) \leq \frac{\epsilon}{4}.$$  

Now since by hypothesis $\sigma \in \Omega \setminus \Omega_\epsilon$ it must fall in case $v$ in the statement of Theorem 6.2. Thus we are done since the conclusion for Theorem 4.1 is the same as that of Theorems 6.1, 6.2. \hfill \square

We now proceed to proving Theorem 6.2. The proof will be based on an electrical resistance argument. We first review some basic facts about energy of flows on graphs.

6.2. **Energy and flows on finite graphs.** On a graph $G = (V,E)$ let $w \sim v$ signify that $w$ is a neighbor of $v$. Also let $E$ be the set of directed edges where each edge in $E$ corresponds to two directed edges in $\tilde{E}$, one in each direction (except for self-loops, which correspond to just one directed edge in $\tilde{E}$). A flow is an antisymmetric function $f : \tilde{E} \to \mathbb{R}$ (i.e. a function satisfying $f(w,v) = -f(v,w)$). Note that by definition the value of a flow on a self loop is 0. Define the energy of a flow $f$ by

$$\mathcal{E}(f) = \frac{1}{2} \sum_{(v,w) \in \tilde{E}} f(v,w)^2. \quad (6.12)$$

For any flow $f : \tilde{E} \to \mathbb{R}$ define the divergence $\text{div} f : V \to \mathbb{R}$ by

$$\text{div} f(v) = \sum_{w \sim v} f(w,v). \quad (6.13)$$

Note that for any flow

$$\sum_{x \in V} \text{div} f = \sum_{x,y \in V} f(x,y) + f(y,x) = 0 \quad (6.14)$$

since $f$ is antisymmetric. For disjoint subsets $A, B \subset V$ and a flow $f$ we say that the flow is from $A$ to $B$ if $\text{div} f(z) = 0$ for all vertices except vertices of $A$ and $B$ and the sum of the divergences across vertices of $A$ and $B$ are non negative and non positive respectively. For more about flows see [LPW09, Chapter 9]. For any function $F : V \to \mathbb{R}$ define the gradient $\nabla F : \tilde{E} \to \mathbb{R}$ by

$$\nabla F(v,w) = F(w) - F(v).$$

Recall the definition of laplacian from (5.3). Thus clearly for any $F : V \to \mathbb{R}$,

$$[\text{div} (\nabla F)](v) = d_v(\Delta F)(v). \quad (6.15)$$

The next result is a standard summation by parts formula.
Lemma 6.3. For any function $F : V \to \mathbb{R}$

$$
\mathcal{E}(\nabla F) = \sum_{v \in V} F(v) d_v \Delta F(v)
$$

where $d_v$ is the degree of the vertex $v$.

The proof follows by definition and expanding the terms.

We now discuss some energy interpretations of the Green function $G_n(\cdot)$ defined on $\mathbb{U}_n$ in (5.2). The next lemma states a standard result in our setting.

Lemma 6.4 (Thomson’s principle). $\mathcal{E}(\nabla G_n) \leq \mathcal{E}(\theta)$ for all flows $\theta$ on $\mathbb{U}_n$ such that $\text{div} \ \theta = \frac{4}{|\mathbb{U}_{1,n}|} [\mathbf{1}(\mathbb{U}_{1,n}) - \mathbf{1}(\mathbb{U}_{2,n})]$.

Proof. First observe that by (5.4) and (6.15), the flow $\nabla G_n$ has the same divergence condition as above. Note the 4 appears since every vertex in $\mathbb{U}_{1,n}$ and $\mathbb{U}_{2,n}$ has degree 4. The proof now follows by standard arguments. See [LPW09, Theorem 9.10]. We sketch the main steps. One begins by observing that the flow $\nabla G_n$ satisfies the cycle law i.e. sum of the flow along any cycle is 0. To see this notice that for any cycle $x_1, x_2, \ldots, x_k = x_1$ where $x_i$’s $\in \mathbb{U}_n$,

$$
\sum_{i=1}^{k-1} \nabla G_n(x_i, x_{i+1}) = \sum_{i=1}^{k-1} (G_n(x_{i+1}) - G_n(x_i)) = 0.
$$

The proof is then completed by first showing that the flow with the minimum energy must satisfy the cycle law, followed by showing that there is an unique flow satisfying the given divergence conditions and the cycle law which implies that $\nabla G_n$ is the unique minimizer.

□

Let

$$
\frac{1}{|\mathbb{U}_{1,n}|} \mathbf{1}(\mathbb{U}_{1,n}) = f_1 \quad (6.16)
$$

$$
\frac{1}{|\mathbb{U}_{1,n}|} \mathbf{1}(\mathbb{U}_{2,n}) = f_2 \quad (6.17)
$$

$$
f = f_1 - f_2. \quad (6.18)
$$

Thus rewriting (5.4) for brevity we get

$$
\Delta G_n = f. \quad (6.19)
$$

6.2.1. Stopped Green functions. For $i = 1, 2$ and $\sigma \in \Omega \cup \Omega'$ define

$$
G_{i,n}(x) := G_{i,n}(\sigma)(x) := 2n^2 \mathbb{E}_x \int_0^{\tau_i} f(X(t))dt, \quad (6.20)
$$

where $X(t)$ is the continuous time random walk on $\mathbb{U}_n$, as defined in (5.1). $B_i(\sigma)$ was defined in (2.3) and

$$
\tau_i := \tau_i(\sigma) := \tau(B_i(\sigma)). \quad (6.21)
$$
For $i = 1, 2$ observe that $\tau(B_i^c(\sigma)) = \tau(R_i^c(\sigma))$ where $R_i$ was defined in Definition 4. Also

$$\Delta G_{i,n}(\sigma)(x) = f(x) \text{ for } x \in R_i(\sigma) \quad (6.22)$$

$$G_{i,n}(\sigma)|_{U_n \setminus R_i(\sigma)} = 0. \quad (6.23)$$

Recalling notation for the energy of a flow from (6.12) we state the following key lemma, which translates Theorem 6.2 to a statement about difference in energies of flows.

**Lemma 6.5.** Under Assumption 1

$$E(w(\sigma_1) - w(\sigma_0) \mid \sigma_0 = \sigma) \geq \frac{1}{4} \left[ E(\nabla G_n) - E(\nabla G_{1,n}(\sigma_0)) - E(\nabla G_{2,n}(\sigma_0)) \right]. \quad (6.24)$$

Thus using the above result our approach to proving Theorem 6.2 involves lower bounding the RHS in the above equation.

Before proving Lemma 6.5 we make a few more observations about the functions $G_{i,n}$ to be used later. First notice that if $X_k$ was a discrete random walk then (6.20) would be the same as

$$\mathbb{E}_x \sum_{0}^{\tau_i-1} f(X_k), \quad (6.25)$$

where $\tau_i$ is now the exit time of the discrete random walk.

Thus for the rest of the section for notational convenience we will switch to the sum notation and hence think of the random walk as discrete time.

The next remark gives us a crude upper bound for the stopped Green functions $G_{i,n}$.

**Remark 6.3.** For $i = 1, 2$ and any $\sigma \in \Omega \cup \Omega'$,

$$\sup_{x \in U_{i,n}} G_{i,n}(x) = O\left(\frac{1}{\delta^2}\right).$$

A sharper upper bound of $O(\log(\frac{1}{\delta}))$ can be obtained with a little more work. However for our purposes this will suffice.

**Proof.** As a simple consequence of the fact that the $L_\infty$ mixing time of the discrete time random walk on $U_n$ is $O(n^2)$ one gets that for $i = 1, 2$

$$\sup_{x \in U_n} \mathbb{E}(\tau(B_i)) = O(n^2),$$

for a proof see [GP15, Lemma 5.6]). Since $B_1 = B_2^c$ from the above discussion we get that,

$$\sup_{x \in U_n} \mathbb{E}(\tau_i) = O(n^2).$$

Now the remark follows once we observe that by (6.25)

$$G_{i,n}(x) \leq \frac{1}{|U_{i,n}|} \mathbb{E}_x(\tau_i). \quad \Box$$

**Remark 6.4.** By (6.7) under Assumption 1, $R_1$ does not intersect $U_{2,n}$ which is the support of $f_2$ and similarly $R_2$ does not intersect $U_{1,n}$ which is the support of $f_1$. 

Thus (6.25) becomes
\[ G_{1,n}(\sigma)(x) = E_x \left[ \sum_{k=0}^{\tau_1(\sigma)-1} f_1(X_k) \right] \] (6.26)
\[ G_{2,n}(\sigma)(x) = E_x \left[ \sum_{k=0}^{\tau_2(\sigma)-1} f_2(X_k) \right], \]
and (6.22) and (6.23) become
\[ \Delta G_{i,n}(\sigma)(x) = f_i(x) \text{ for } x \in R_i(\sigma) \] (6.27)
\[ G_{i,n}(\sigma)|_{U_n \setminus R_i(\sigma)} = 0. \] (6.28)

An easy but useful observation is the following:

**Lemma 6.6.** Under Assumption 1 for any \( \sigma \in \Omega \),
\[ E[(G_{2,n}(\sigma_{1/2}))(x) | \sigma_0 = \sigma)] \leq G_{2,n}(\sigma)(x) \] (6.29)
for all \( x \in U_n \) where the expectation in the first term is over the random intermediate state \( \sigma_{1/2} \).

**Proof.** The proof immediately follows from (6.26) since \( \tau_2(\sigma_{1/2}) \leq \tau_2(\sigma_0) \) as \( \sigma_{1/2} \) has one more blue particle than \( \sigma_0 \). \( \square \)

6.2.2. **Proof of Lemma 6.5.**

**Proof.** Let us denote the random walk starting from \( U_{1,n} \) by \( X_k \) and the random walk starting from \( U_{2,n} \) by \( Y_k \). Recall (6.1),
\[ E(w(\sigma_1) - w(\sigma_0) | \sigma_0 = \sigma) = E(G_n(X_{\tau_1(\sigma_0)}) - G_n(Y_{\tau_2(\sigma_{1/2})})) \]
where the expectation on the right is over the random walks \( X_k, Y_k \). We start with the following telescopic sum,
\[ G_n(X_{\tau_1(\sigma_0)}) = G_n(X_0) + [G_n(X_1) - G_n(X_0)] + [G_n(X_2) - G_n(X_1)] + \ldots + [G_n(X_{\tau_1(\sigma_0)}) - G_n(X_{\tau_1(\sigma_0)-1})]. \]
Now clearly
\[ E(G_n(X_{i+1}) - G_n(X_i) | \mathcal{F}_i) = -\Delta G_n(X_i) 1(\tau_1(\sigma_0) > i) \]
where \( \mathcal{F}_i \) is the filtration generated by the random walk \( X_k \) up to time \( i \) and \( \sigma_0 \). Taking expectation on both sides we get
\[ E(G_n(X_{\tau_1(\sigma_0)})) = E[G_n(X_0) - \sum_{k \geq 0} \Delta G_n(X_{k\wedge \tau_1(\sigma_0)-1})]. \]
A similar equation holds for \( E(G_n(X_{\tau_2(\sigma_{1/2})})) \). Plugging in both of them we get
\[ E(G_n(X_{\tau_1(\sigma_0)}) - G_n(Y_{\tau_2(\sigma_{1/2})})) = E\left[ G_n(X_0) - \sum_{k \geq 0} \Delta G_n(X_{k\wedge \tau_1(\sigma_0)-1}) - G_n(Y_0) + \sum_{k \geq 0} \Delta G_n(Y_{k\wedge \tau_2(\sigma_{1/2})-1}) \right]. \] (6.30)

Now
\[ E(G_n(X_0) - G_n(Y_0)) = \sum_{x \in U_n} G_n(x)(f_1(x) - f_2(x)) = (G_n, \Delta G_n) = \frac{1}{4} \mathcal{E}(\nabla G_n). \] (6.31)
where all inequalities but the last are by definition. The last inequality is by Lemma 6.3 and the fact that all vertices in \( U_{1,n} \cup U_{2,n} \) have degree 4. Also,

\[
\mathbb{E}_x \sum_{k \geq 0} \Delta G_n(X_{k \wedge \tau_1(x_0) - 1}) = \mathbb{E}_x \sum_{k \geq 0} f_1(X_{k \wedge \tau_1(x_0) - 1}) = G_{1,n}(x_0)(x).
\]

where the first equality is by (6.27) and the second equality is by definition (6.26). Similarly

\[
\mathbb{E}_y \sum_{k \geq 0} \Delta G_n(Y_{k \wedge \tau_2(\sigma_{1/2}) - 1}) = -\mathbb{E}(G_{2,n}(\sigma_{1/2})(y)).
\]

Hence by the above discussion, notice in (6.30)

\[
\mathbb{E}[\sum_{k \geq 0} \Delta G_n(X_{k \wedge \tau_2(\sigma_{1/2}) - 1})] = \mathbb{E} \left[ \mathbb{E}[\sum_{k \geq 0} \Delta G_n(X_{k \wedge \tau_2(\sigma_{1/2}) - 1}) \mid X_0] \right]
\]

\[
= \mathbb{E}[G_{1,n}(\sigma_0)(X_0)].
\]

Similarly we have \( \mathbb{E}[\sum_{k \geq 0} \Delta G_n(Y_{k \wedge \tau_2(\sigma_0) - 1})] = -\mathbb{E}[G_{2,n}(\sigma_{1/2})(Y_0)]. \) Thus

\[
\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0) = \sum_{U_{1,n}} G_n(x) f_1(x) - \sum_{U_{2,n}} G_n(x) f_2(x) - \sum_{U_{1,n}} G_{1,n}(\sigma_0)(x) f_1(x) - \sum_{U_{2,n}} G_{2,n}(\sigma_{1/2})(x) f_2(x)
\]

\[
\geq \sum_{U_{1,n}} G_n(x) f_1(x) - \sum_{U_{2,n}} G_n(x) f_2(x) - \sum_{U_{1,n}} G_{1,n}(\sigma_0)(x) f_1(x) - \sum_{U_{2,n}} G_{2,n}(\sigma_0)(x) f_2(x)
\]

where for the inequality we replace the last term using Lemma 6.6. Now by Lemma 6.3 and (6.27)

\[
\sum_{U_{1,n}} G_{1,n}(x) f_1(x) = \frac{1}{4} \mathcal{E}(\nabla G_{1,n})
\]

\[
\sum_{U_{2,n}} G_{2,n}(x) f_2(x) = \frac{1}{4} \mathcal{E}(\nabla G_{2,n}).
\]

Thus we are done using these and (6.31) in the above inequality. \( \square \)

We need a few more definitions and results to use the inequality in Lemma 6.5 to prove Theorem 6.2. Recall the standard gluing operation on any multigraph \( G = (V,E) \) where certain subsets of vertices are identified ("glued") and they act as a single vertex. Any edge between two identified vertices now act as a self loop in the glued graph. For more on glued graphs see [LP05]. Also in the sequel we will often specify a graph by just referring to the set of vertices which will be a subset of the vertex set of \( \mathbb{U}_n \). The graph then would be the sub graph induced on the set of vertices from \( \mathbb{U}_n \).

For the following definitions recall (6.8) and (6.9).

**Definition 6.**

i. For \( i = 1, 2 \) let \( \tilde{R}^{{\text{out}}} \) denote the graph with vertex set \( \tilde{R}_i \cup \partial_{\text{out}} \tilde{R}_i \) with the following subset of vertices,

\[
\{ \tilde{R}_i \cup \partial_{\text{out}} \tilde{R}_i \} \setminus R_i
\]

glued.

ii. For \( i = 1, 2 \) let \( \hat{R}_i \) denote the graph with vertex set \( \hat{R}_i \) with the following subset of vertices,

\[
\{\hat{R}_i \setminus R_i\} \cup \partial_{in}\hat{R}_i
\]
glued.

The set of glued vertices now acting as a single vertex will be denoted by \( v \) (the underlying graph will be clear from context).

With the above definition \( \nabla(G_{i,n}) \) is a flow on \( \hat{R}_i^{out} \). (6.27) and (6.28) are the same as

\[
\Delta(G_{i,n})(x) = f_i \text{ for all } x \in R_i
\]

(6.32)

\[
G_{i,n}(\mathbf{v}) = 0.
\]

(6.33)

Analogous to \( G_{i,n} \) (see (6.26)) for \( i = 1, 2 \) we define \( G^*_{i,n} \) to be the function on \( \hat{R}_i^{in} \) such that

\[
G^*_{i,n}(x) = \mathbb{E}_x \sum_{k=0}^{\infty} f_i(X_k) 1((k < \tau(\hat{R}_i^c \cup \partial_{in}\hat{R}_i))).
\]

(6.34)

It is easy to verify that

\[
\Delta(G^*_{i,n})(x) = f_i \text{ for all } x \in R_i \setminus \partial_{in}\hat{R}_i
\]

(6.35)

\[
G^*_{i,n}(\mathbf{v}) = 0.
\]

(6.36)

Notice that by (6.14) the above information determines the laplacian (equivalently divergence) at \( \mathbf{v} \) in the glued graphs \( \hat{R}_i^{out}, \hat{R}_i^{in} \) respectively.

Let us form the following abbreviations

\[
\mathcal{E}(\nabla G_n) = \mathcal{E}.
\]

(6.37)

Also for \( i = 1, 2 \)

\[
\mathcal{E}(\nabla G_{i,n}) = \mathcal{E}_i
\]

(6.38)

\[
\mathcal{E}(\nabla G^*_{i,n}) = \mathcal{E}^*_i.
\]

(6.39)

For brevity we write (6.24) in terms of the above abbreviations

\[
\mathbb{E}(w(\sigma_1) - w(\sigma_0) \mid \sigma_0) \geq \frac{1}{4}[\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2].
\]

(6.40)

**Remark 6.5.** Notice that the arguments in the proof of Lemma 6.4 are not special to the potential function \( G_n \), i.e. [LPW09, Theorem 9.10] also gives us the following two facts:

i. \( \mathcal{E}_i = \min_\theta \mathcal{E}(\theta) \) where the minimum is taken over all flows \( \theta \) on \( \hat{R}_i^{out} \) such that

\[
\text{div } \theta(x) = 4f_i
\]

for all \( x \in R_i \).

ii. \( \mathcal{E}^*_i = \min_\theta \mathcal{E}(\theta) \) where the minimum is taken over all flows \( \theta \) on \( \hat{R}_i^{in} \) such that

\[
\text{div } \theta(x) = 4f_i(x)
\]

for all \( x \in R_i \setminus \partial_{in}\hat{R}_i \).

With the preceding preparation the proof of Theorem 6.2 follows from a series of lemmas which we prove next. We first prove a monotonicity result. It is standard but we include it for completeness.
Lemma 6.7. If $\sigma \in \Omega$ satisfies Assumption 1 then for $i = 1, 2$
\[ \mathcal{E}_i^* \leq \mathcal{E}_i, \]
where $\mathcal{E}_i, \mathcal{E}_i^*$ are defined in (6.38) and (6.39) respectively.

Proof. Wlog we take $i = 1$. By Lemma 6.3 and (6.32) we have $\mathcal{E}_1 = \sum_{\overset{\text{out}}{R_1}} G_{1,n}^4 f_1$, and similarly by (6.35) we have $\mathcal{E}_1^* = \sum_{\overset{\text{in}}{R_1}} G_{1,n}^* 4 f_1$. The lemma now follows since
\[ G_{1,n}^* \leq G_{1,n} \quad (6.41) \]
which immediately follows from definitions of $G_{1,n}, G_{1,n}^* ((6.26) and (6.34))$ and the simple fact that $\tau(R_1^c \cup \partial_{\overset{\text{in}}{R_1}}) \leq \tau(R_1^c)$.

The previous lemma shows that $\mathcal{E}_i \geq \mathcal{E}_i^*$. The next lemma says that nevertheless they are close to each other.

Lemma 6.8. There exists a $\beta > 0$ such that if $\sigma \in \Omega$ satisfies Assumption 1 then for $i = 1, 2$
\[ |\mathcal{E}_i - \mathcal{E}_i^*| = O\left( \frac{1}{n^{\beta}} \right). \]
where the constant in the $O$ depends on $\delta_2$ appearing in Theorem 6.1.

To prove the above we need a few preliminary lemmas first. Let,
\[ \theta_* = \nabla(G_{1,n}^*) \quad (6.42) \]

Thus (6.39) is the same as $\mathcal{E}_i^* := \mathcal{E}(\theta_*)$. Hence by Remark 6.5, $\theta_*$ has minimum energy among all the flows on $\overset{\text{in}}{R_1}$ with the same divergence condition. Even though $\overset{\text{in}}{R_1}$ is obtained from a subgraph of $\overset{\text{in}}{U_n}$ by identifying some vertices we will denote the edges of this graph by the corresponding edges of $\overset{\text{in}}{U_n}$. The next lemma is a conservation of flow result. It says that the total flow across $\partial_{\overset{\text{in}}{R_1}}$ is at most a constant.

Lemma 6.9. \[ \sum_{x \in \partial_{\overset{\text{in}}{R_1}}} \sum_{y \sim x} \theta_*(x,y) \leq 4. \]

Proof. We start with the observation that for all such $x, y$ as in the sum, $G_{1,n}^*(x) = 0$ and $G_{1,n}^*(y) \geq 0$ which follows by definition. Hence $\theta_*(x,y) \geq 0$ for all such $(x, y)$. Now by (6.14)
\[ \sum_{z \in \overset{\text{in}}{R_1}} \text{div}(\theta_*)(z) = 0. \]

By (6.35) for any $z \in \overset{\text{in}}{R_1}$ such that $z \in R_1 \setminus \partial_{\overset{\text{in}}{R_1}}$
\[ \text{div}(\theta_*)(z) = 4 f_1. \]

Note above that even though the graph is not regular the 4 appears since every vertex in the support of $f_1$ has degree 4. Also
\[ \sum_{z \in \overset{\text{in}}{R_1}} \text{div}(\theta_*)(z) = \sum_{x \in R_1 \setminus \partial_{\overset{\text{in}}{R_1}}} 4 f_1 + \sum_{x \in \partial_{\overset{\text{in}}{R_1}}} \sum_{y \sim x} \theta_*(y,x). \]

The LHS is 0 and the first sum on the RHS is at most 4. Thus we are done. \[ \square \]
We state two more lemmas. The first result is a Beurling type estimate which says that for any connected subset $A$ of $\mathbb{U}_n$ with large enough diameter which is at a certain distance away from $\mathbb{U}_{1,n}$, the probability that random walk started from a neighboring site of $A$, hits $\mathbb{U}_{1,n}$ before hitting $A$ decays as a power law in $n$. The result is standard on the whole lattice. The proof in our case with the necessary adaptations is provided in [GP15].

**Lemma 6.10.** [GP15, Lemma 5.1] Fix $c > 0$. Consider $A \subset \mathbb{U}_n$ be $*$-connected (Definition 5). Also suppose that $\min(diam(A), d(\mathbb{U}_1, A)) \geq c$ (here the distance is in terms of euclidean distance not graph distance). Then for large $n$, for all such $A$,

$$\sup_{x \sim A} P_x(\tau(\mathbb{U}_1) \leq \tau(A)) \leq C \frac{c}{n^\beta}$$

for some positive $\beta, C$ depending only on $c$ and $\mathbb{U}$. Here $x \sim A$ means that $x \notin A$ and there exists $y \in A$ such that $x$ is a neighbor of $y$.

The next result is a basic isoperimetry inequality saying that the boundaries of the set $\hat{R}_1$ is not too small.

**Lemma 6.11.** There exists a $c = c(\delta, \mathbb{U})$ such that

$$\min(diam(\partial_{in}\hat{R}_1), diam(\partial_{out}\hat{R}_1)) > c$$

where $\delta$ appears in Setup 1.

**Proof.** Clearly it suffices to just show that

$$\text{diam}(\partial_{in}\hat{R}_1) > c$$

since every vertex in $\partial_{in}\hat{R}_1$ has a neighbor in $\partial_{out}\hat{R}_1$. Recall that by definition $\mathbb{U}_{1,n} \subset \hat{R}_1$ and $\mathbb{U}_{2,n} \subset \hat{R}_1^*$. We first prove the lemma for the disc $\mathbb{D}$. It is easy to notice that in the disc there exists a $c = c(\delta)$ such that for any ball $B$ of radius $c$, $\mathbb{D}_1$ and $\mathbb{D}_2$ are connected by a lattice path in $\mathbb{D}_{n \setminus B}$. Now since $\partial_{in}\hat{R}_1$ separates $\mathbb{D}_{1,n}, \mathbb{D}_{2,n}$ this shows that $\text{diam}(\partial_{in}\hat{R}_1) > c$. The general proof now follows by using the bi-Lipschitz nature of the conformal map $\phi$ in (1.4).

We are now ready to prove Lemma 6.8. Roughly the following will be our strategy: wlog we take $i = 1$. Recall $\theta_*$ (6.42) is a flow on $\hat{R}_1^{in}$. We will construct a flow $\theta_1$ on $\hat{R}_1^{out}$ from $\theta_*$ by defining the value of the flow on the edges from $\partial_{in}\hat{R}_1$ to $\partial_{out}\hat{R}_1$ (the extra edges in $\hat{R}_1^{out}$ not in $\hat{R}_1^{in}$) and keeping the value of the flow on all other edges same as $\theta_*$. $\theta_1$ will have the same divergence as $\nabla G_{i,n}$. Thus

$$\mathcal{E}_i^* \leq \mathcal{E}_1 \leq \mathcal{E}(\theta_1), \quad (6.43)$$

where the first inequality is by Lemma 6.7 and the second inequality by Remark 6.5 i. Then we would show that

$$\mathcal{E}(\theta_1) - \mathcal{E}_i^* = O(\frac{1}{n^\beta})$$

for some $\beta$ and thus complete the proof. Formally we do the following.

**Proof of Lemma 6.8.** We construct a flow $\theta_1$ on $\hat{R}_1^{out}$: For each $y$ in $R_1 \cap \partial_{in}\hat{R}_1$ choose $x \in \partial_{out}\hat{R}_1$ such that $y \sim x$ (such an $x$ exists by definition of $\partial_{in}\hat{R}_1$). Let

$$\theta_1(y, x) = \sum_{\substack{z \in \hat{R}_1 \\ z \sim y}} \theta_* (z, y).$$

$$\theta_1(y', x') = 0 \text{ for all other edges with } y' \text{ in } \partial_{in}\hat{R}_1, x' \in \partial_{out}\hat{R}_1 \text{ everywhere else.}$$
We first claim that
\[ \mathcal{E}_1 \leq \mathcal{E}(\theta_1). \]  
(6.44)

By Remark 6.5 \(i\), this will follow if we can show for all \( y \in R_1 \)
\[ \text{div}(\theta_1)(y) = d_y \Delta G_{1,n}(y) = 4f_1(y), \]  
(6.45)
where \( d_y \) is the degree of \( y \) in \( \hat{R}^\text{out}_1 \). We know by definition that
\[ \text{div}(\theta^*)(y) = d_y \Delta G^*_{1,n}(y) = 4f_1(y) \text{ for all } y \in R_1 \setminus \partial_{in} \hat{R}_1. \]

Also by construction \( \theta_1 = \theta^* \) on all the edges except the boundary edges of \( \partial_{in} \hat{R}_1 \). Now by (6.8)
\[ d(\mathbb{U}_{1,n}, \partial_{in} \hat{R}_1) \geq \delta_1. \]  
(6.46)

Thus \( f_1 \mid_{\partial_{in} \hat{R}_1} = 0 \). Hence to verify (6.45) it suffices to show that
\[ \text{div}(\theta_1) \mid_{\partial_{in} \hat{R}_1 \cap R_1} = 0. \]

Let \( y \in \partial_{in} \hat{R}_1 \cap R_1 \). By construction we know that there exists exactly one \( x \in \partial_{out} \hat{R}_1 \) such that \( \theta_1(y, x) \neq 0 \).

Also
\[ \theta_1(y, x) = \sum_{z \in R_1 \atop z \sim y} \theta^*(z, y) = \sum_{z \in R_1 \atop z \sim y} \theta_1(z, y), \]
where the first equality is by definition and for the second inequality we use the fact that \( \theta_1 = \theta^* \) for all the edges in the sum by construction. Thus
\[ \text{div}(\theta_1)(y) = \sum_{w \in R_1 \setminus \partial_{out} \hat{R}_1 \atop w \sim y} \theta_1(w, y) = 0. \]

Thus (6.44) is verified. By (6.43) the proof of the lemma will be complete once we show that
\[ \mathcal{E}(\theta_1) - \mathcal{E}^*_1 \leq O\left(\frac{1}{n^{\beta}}\right). \]

Now we claim that
\[ \sup_{y \in \partial_{in} \hat{R}_1} \sum_{z \in R_1 \atop z \sim y} \theta^*(y, z) = O\left(\frac{1}{n^{\beta}}\right). \]  
(6.47)

First notice that \( \partial_{in} \hat{R}_1 \) satisfy the hypotheses of Lemma 6.10:
- By (6.46)
  \[ d(\partial_{in} \hat{R}_1, \mathbb{U}_1) \geq \delta_1. \]
- The connectedness hypothesis is satisfied by Remark 6.2.
- The diameter lower bound follows from Lemma 6.11.

Thus by Lemma 6.10, for any \( y \in \partial_{in} \hat{R}_1 \) and \( z \sim y \),
\[ \mathbb{P}_z(\tau(\mathbb{U}_{1,n}) < \tau(\partial_{in} \hat{R}_1)) = O\left(\frac{1}{n^{\beta}}\right), \]
where the constant in the $O$ term depends on $\delta$ and $\mathbb{U}$. Since $f_1$ is positive only on $\mathbb{U}_{1,n}$ and 0 everywhere else by (6.34) it follows that

$$G_{1,n}^*(z) \leq \mathbb{P}_z \left( \tau(\mathbb{U}_{1,n}) < \tau(\partial_{in} \hat{R}_1) \right) \sup_{x \in \mathbb{U}_{1,n}} G_{1,n}^*(x)$$

The last inequality follows since for all $x \in \mathbb{U}_{1,n} G_{1,n}^*(x) \leq G_{1,n}(x)$ by (6.41). Thus by Remark 6.3 we get,

$$\theta_*(y, z) = G_{1,n}^*(z) \leq O\left( \frac{1}{n^{\beta}} \right)$$

where the first equality follows from the fact that $\theta_*= \nabla G_{1,n}^*$ and $G_{1,n}^*(y) = 0$. Since $\theta_1 = \theta_*$ on all but the boundary edges,

$$\mathcal{E}(\theta_1) - \mathcal{E}(\theta_*) = O\left( \sum_{y \in \partial_{in} R_1} \sum_{z \sim y} |\theta_*(z, y)| \right)^2$$

where the last inequality is due to Lemma 6.9 and (6.47). Hence

$$\mathcal{E}_1 - \mathcal{E}_1^* \leq \mathcal{E}(\theta_1) - \mathcal{E}(\theta_*) = O\left( \frac{1}{n^{\beta}} \right)$$

where the first inequality is by (6.44). \qed

**Remark 6.6.** From Lemma 6.8 it follows that

$$\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2 \geq \mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2 - O\left( \frac{1}{n^{\beta}} \right).$$

(6.48)

By (6.40) and the above remark, to prove Theorem 6.2, it suffices to prove a lower bound for the quantity

$$\mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2.$$

We start with the following weaker version which shows it is always non negative.

**6.2.3. Weak version of Theorem 6.2.**

**Lemma 6.12.** If $\sigma \in \Omega$ satisfies Assumption 1 then

$$\mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2 \geq 0$$

**Proof.** Let us take the optimal flow $\theta$ for $\mathcal{E}$ on $\mathbb{U}_n$ i.e.

$$\theta = \nabla G_n.$$

(6.49)

Let for $i = 1, 2$

$$\tilde{\theta}_i = \theta |_{\hat{R}_i^{out}} \quad \text{and} \quad \tilde{\theta}_i^* = \theta |_{\hat{R}_i^{in}}$$
be the restrictions on \( R_i^{\text{out}} \) and \( R_i^{\text{in}} \) respectively. Now by (6.19) and Remark 6.4 it follows that
\[
\text{div}(\tilde{\theta}_i)(x) = 4f_i \quad \text{for all } x \in R_i
\]
(6.50)
\[
\text{div}(\tilde{\theta}_i^*)(x) = 4f_i \quad \text{for all } x \in R_i \setminus \partial R_i
\]
(6.51)
for \( i = 1, 2 \) where \( f_i \) was defined in (6.16) and (6.17). Thus the flows \( \tilde{\theta}_i \) and \( \tilde{\theta}_i^* \) on \( R_i^{\text{out}} \) and \( R_i^{\text{in}} \) have the same divergence as \( \nabla G_{i,n} \) and \( \nabla G_{i,n}^* \) respectively. Thus by Remark 6.5
\[
\mathcal{E}_i \leq \mathcal{E}(\tilde{\theta}_i), \mathcal{E}_i^* \leq \mathcal{E}(\tilde{\theta}_i^*).
\]
Recall the definitions of \( C_{1,k} \) from (6.8). Since \( R_1 = U \setminus C_{1,k} \) is connected and
\[
\{R_2 \cup B(y_2, \delta_1)\} \subseteq C_{1,k}
\]
by definition \( R_2 \subset C_{1,k} \). Thus the edge set of the graphs
\[
\tilde{R}_1^{\text{in}}, \tilde{R}_2^{\text{out}}
\]
are disjoint. Therefore
\[
\mathcal{E} = \mathcal{E}(\theta) \geq \mathcal{E}(\theta |_{\tilde{R}_1^{\text{in}}}) + \mathcal{E}(\theta |_{\tilde{R}_2^{\text{out}}}) = \mathcal{E}(\tilde{\theta}_1^*) + \mathcal{E}(\tilde{\theta}_2) \geq \mathcal{E}_1^* + \mathcal{E}_2.
\]
(6.52)
Hence we are done. \( \square \)

However as already mentioned before, proof of Theorem 6.2 demands a stronger quantitative version of Lemma 6.12. We now proceed to obtain that. Let \( U_n \) be the same graph as \( U_n \) but with the following set of vertices
\[
\partial R_1 \cup \{x \in R_1 : \sigma(x) = 2\} \cup \{x \in R_1^c : \sigma(x) = 1\}
\]
(6.53)
glued. The points are glues are all the red vertices in \( R_1 \) (which causes the blue random walk to stop) and similarly all the blue vertices in \( U_n \setminus R_1 \). Now in \( U_n \) the glued vertices act as a single vertex. Let us denote it by \( w \).

In the proof of Lemma 6.12 we took the flow \( \nabla G_n \) on \( U_n \) and restricted it to \( \tilde{R}_1^{\text{in}} \) and \( \tilde{R}_2^{\text{out}} \). To prove Theorem 6.2 instead of restricting \( \nabla G_n \) we will construct another flow on \( U_n \) and then restrict it. We first construct the flow on \( U_n \).

**Lemma 6.13.** There exists an unique flow \( ^* \theta \) on \( U_n \) with the following properties:

i.
\[
\text{div}(\theta)(w) = 4f \quad |_{(R_1 \cup R_2) \setminus w}
\]
\[
\text{div}(\theta)(w) \neq 0
\]
\[
\text{div}(\theta)(\cdot) = 0 \quad \text{otherwise.}
\]

ii.
\[
^* \theta = \min_g \mathcal{E}(g)
\]
where the infimum is taken over all flows \( g \) on \( U_n \) which has the same divergence as \( \theta \),

Note that using the above data and (6.14), \( \text{div}(\theta)(w) \) is determined exactly.
Proof. For brevity let us call the set of flows $g$ on $\text{wired } U_n$ satisfying the above divergence conditions as $\mathcal{F}$. That the set is non empty follows since clearly by (6.19) the flow $\nabla (G_n) \big|_{\text{wired } U_n}$ is in this set. Now let
\[ \mathcal{E}^* = \inf_{g \in \mathcal{F}} \mathcal{E}(g). \tag{6.54} \]
By standard arguments using compactness, see proof of [LPW09, Theorem 9.10], there exists a unique minimizer $\mathcal{E}^*$ of the energy minimization problem. \hfill \qed

**Remark 6.7.** Let $\mathcal{E}^*$ be as in Lemma 6.13. The restrictions,
\[ \dot{\theta}_1 = \mathcal{E}^* \bigg|_{\text{in } \hat{R}_1} \]
\[ \dot{\theta}_2 = \mathcal{E}^* \bigg|_{\text{out } \hat{R}_2} \]
under Assumption 1 satisfy the following divergence conditions
\[ \text{div} (\dot{\theta}_1) (x) = 4 f_1 (x) \text{ for all } x \in R_1 \setminus \partial_n \hat{R}_1 \]
\[ \text{div} (\dot{\theta}_2) (y) = 4 f_2 (y) \text{ for all } y \in R_2. \]

**Proof.** Note that no vertex in either $R_1 \setminus \partial_n \hat{R}_1$ or $R_2$ was in the set of glued vertices in (6.53). Now by Remark 6.4 for $i = 1, 2$, $\mathcal{E}^* |_{\text{in } \hat{R}_i} = f_i$. The remark thus follows directly since $\mathcal{E}^*$ by definition satisfies the divergence conditions. \hfill \qed

Thus by Remark 6.5 we have $\mathcal{E}_1^* \leq \mathcal{E}(\dot{\theta}_1), \mathcal{E}_2 \leq \mathcal{E}(\dot{\theta}_2).

**Remark 6.8.** Similar to (6.52) we have
\[ \mathcal{E}^* = \mathcal{E}(\dot{\theta}) \geq \mathcal{E}(\dot{\theta} \bigg|_{\text{in } \hat{R}_1}) + \mathcal{E}(\dot{\theta} \bigg|_{\text{out } \hat{R}_2}) = \mathcal{E}(\dot{\theta}_1) + \mathcal{E}(\dot{\theta}_2) \geq \mathcal{E}_1^* + \mathcal{E}_2. \]
Therefore we have
\[
\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2 \geq \mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2 - O \left( \frac{1}{n^\beta} \right) = \mathcal{E} - \mathcal{E}^* + \left[ \mathcal{E} - \mathcal{E}_1^* - \mathcal{E}_2 - O \left( \frac{1}{n^\beta} \right) \right]
\geq \mathcal{E} - \mathcal{E} - O \left( \frac{1}{n^\beta} \right)
\]
where the first inequality follows from (6.48). Hence to prove Theorem 6.2 by (6.40) it suffices to show that
\[ \mathcal{E} - \mathcal{E}^* \geq c \tag{6.55} \]
for some constant $c = c (\epsilon)$. Notice that $\mathcal{E} - \mathcal{E}^*$ is nothing but the drop in voltage when points at various voltages are glued together to get $\text{wired } U_n$ from $U_n$. This is the well known Rayleigh’s monotonicity principle. We make the necessary adaptations to prove it in our setting. For our purposes we also clearly need some quantitative version. For more on this see [LP05, Chap 2]. To estimate such voltage drops we state and prove two technical lemmas first. Let $\hat{U}_n$ be a graph obtained from $U_n$ by gluing certain pairs of points. Let $A$ be a set of $k$ pairs of points
\[ (z_1, z'_1), \ldots, (z_k, z'_k) \]
which are among the pairs glued to obtain $\tilde{U}_n$. Recall from (6.49),
\[ \theta = \nabla G_n. \]

Consider the restricted flow $\nabla G_n \mid_{\tilde{U}_n}$, (the flow that $\nabla G_n$ induces on $\tilde{U}_n$). Let $\tilde{\theta}$ denote the flow on $\tilde{U}_n$ such that
\[ \mathcal{E}(\tilde{\theta}) = \inf_g \{ \mathcal{E}(g) \}, \tag{6.56} \]
where the infimum is taken over all flows $g$ on $\tilde{U}_n$ which has the same divergence as $\theta \mid_{\tilde{U}_n}$. Existence of $\tilde{\theta}$ follows by standard compactness arguments as in the proof of [LPW09, Theorem 9.10], (this has already been mentioned in the proof of Lemma 6.13).

**Lemma 6.14.** Let $A \subset \mathbb{U}_n$ as defined above. Suppose that there exists simple paths in $\mathbb{U}_n$ joining $(z_i, z'_i)$ of length $d_i$ then
\[ \mathcal{E}(\tilde{\theta}) \leq \mathcal{E}(\theta) - \frac{\sum_{i=1}^{k} (G_n(z_i) - G_n(z'_i))^2}{D^2 \sum_{i} d_i} \]
where $D$ is the maximum number of paths that intersect some edge.

Thus the lemma measures the amount of voltage drop when points at different potentials are glued together.

**Proof.** Wlog assume $G_n(z_i) \geq G_n(z'_i)$. Take the path joining $z'_i$ to $z_i$ of length $d_i$. Think of it as oriented towards $z_1$. Since on $\tilde{U}_n$, $z_i$ and $z'_i$ are glued the path becomes a directed cycle. Let us denote it by $\tilde{\gamma}_i$. Also let us denote by $\tilde{\gamma}_i$ the same cycle in the reverse orientation. We first create a new flow $\theta_A$ on $\mathbb{U}_n$ by sending an additional amount of flow $\beta$ along $\tilde{\gamma}_i$ for all $i = 1, \ldots, k$. Thus
\[ \theta_A(\epsilon) = \theta(\epsilon) + \sum_{i=1}^{k} \beta (\epsilon \in \tilde{\gamma}_i) + \sum_{j=1}^{k} -\beta (\epsilon \in \tilde{\gamma}_j) \]
for all directed edges $\epsilon$, where every un-directed edge in $\mathbb{U}_n$ appears with both orientations. Now the additional flow along an edge is at most $D\beta$ since by hypothesis every edge is in at most $D$ many cycles. An easy computation by expanding the squares now shows that
\[ \mathcal{E}(\theta_A) \leq \mathcal{E}(\theta) + 2\beta \sum_{i=1}^{k} (G_n(z_i) - G_n(z'_i)) + D^2 \beta^2 \sum_{i} d_i. \]
Optimizing the RHS over $\beta$ we see that
\[ \inf_{\beta} \mathcal{E}(\theta_A) \leq \mathcal{E}(\theta) - \frac{\sum_{i=1}^{k} (G_n(z_i) - G_n(z'_i))^2}{D^2 \sum_{i} d_i}. \]
We first remark that $\theta_A$ and $\theta$ do not have the same divergence on $\mathbb{U}_n$. However since all the paths $\tilde{\gamma}_i$ become cycles in $\tilde{U}_n$ and sending additional mass along cycles does not change the divergence of the flow. Thus $\text{div}(\theta_A \mid_{\tilde{U}_n}) = \text{div}(\theta \mid_{\tilde{U}_n})$. Hence by (6.56), $\mathcal{E}(\tilde{\theta}) \leq \mathcal{E}(\theta_A \mid_{\tilde{U}_n}) \leq \mathcal{E}(\theta_A)$ and the proof is complete. \(\Box\)

To use Lemma 6.14 in the proof of Theorem 6.2, one has to estimate $d_i$’s and $D$ appearing in the statement of the lemma. That is the purpose of the next lemma which says that two large subsets of $\mathbb{U}_n$ which are away from the boundary can be connected using almost disjoint paths. The statement of the lemma is rather crude and has lots of room for strengthening. However for our purposes this will suffice. Recall the definition of $\mathbb{U}^{(a)}$ for any $a > 0$ from (6.10).
Lemma 6.15. Given a domain $\mathbb{U}$, for any $\epsilon, c > 0$ there are constants $d, D$ depending only on $\mathbb{U}, \epsilon, c$ such that for $n$ large enough the following is true: Suppose $A_1$ and $A_2$ are two subsets of $\mathbb{U}_n$ with
\[d(A_1, A_2) > c, A_1, A_2 \subset \mathbb{U}^{(\epsilon)}\]
and for both $i = 1, 2$ at least one of the statements hold
\[\#\{x : \exists (x, y) \in A_i\} \geq cn\]
\[\#\{y : \exists (x, y) \in A_i\} \geq cn.\]
Then there exists at least $dn$ pairs $(z_1, z'_1), \ldots, (z_{dn}, z'_{dn})$ with $z_j \in A_1$ and $z'_j \in A_2$ for $j = 1, \ldots, dn$ with paths in $\mathbb{U}_n$ between $z_j$ and $z'_j$ of length at most $Dn$ and such that no edge intersects more than 2 paths.

Proof. For $\epsilon, \epsilon' > 0$ recall the definitions of $\mathbb{U}^{(\epsilon)}$ and $\mathbb{U}^{(\epsilon')}$ from (6.10) and (6.58) respectively. $\epsilon$ is already fixed in the hypothesis of the lemma. We specify the value of $\epsilon'$ later. We let $n$ be much larger than $\epsilon'$. Clearly there are $O(\epsilon'^2)$ lattice squares of size $\epsilon'$ in $\mathbb{U}^{(\epsilon')} \cap \mathbb{U}^{(\epsilon)}$. Thus by hypothesis there exists two such squares $B_1$ and $B_2$ such that $A_1 \cap B_1$ and $A_2 \cap B_2$ satisfy conditions (6.57) for some different constant $a < c$. where $a$ depends on $\epsilon'$. Notice that all of these points can be projected on to one of the sides of the square using disjoint paths, Fig 12 i. Also for any square with $an$ points on one side, there exists disjoint paths of length at most $2\epsilon'n$ lying entirely in the square mapping it to $an$ points on any of the three other sides. See Fig 12 ii. If we now choose $\epsilon'$ small enough compared to $\epsilon$ by Lemma 6.16 $B_1$ and $B_2$ are connected by a path of of length at most $D = D(\mathbb{U})$ made of adjacent squares in $\mathbb{U}^{(\epsilon')}$ such that the path lies entirely in $\mathbb{U}^{(\epsilon')}$. The theorem now follows since one can connect the $an$ points of $B_1$ and $B_2$ using the path of squares along with the constructions of paths inside each squares as depicted in Fig 12 i., ii. See Fig 12 iii.

We state another basic lemma. The lemma essentially says that for any small $\epsilon$, if there are two points at distance $\epsilon$ away from the boundary then they are connected by a path which is also $c\epsilon$ distance away from the boundary for some universal constant $c = c(\mathbb{U})$. However for our purposes
we need it to be a lattice path in \( \mathbb{U} \cap \varepsilon \mathbb{Z}^2 \) for some small \( \varepsilon' \). Clearly this does not create any additional problem since any path has a lattice path approximation and if \( \varepsilon' \) is small enough then the lattice path is also at distance \( c\varepsilon \) away from the boundary.

For any \( \varepsilon' > 0 \) let

\[
\mathbb{U}(\varepsilon') = \mathbb{U} \cap \varepsilon \mathbb{Z}^2.
\]  

(6.58)

Note the notational difference from \( \mathbb{U}(\varepsilon) \).

**Lemma 6.16.** Let \( \mathbb{U} \) be as in Setup 1. Then there exists \( 0 < c = c(\mathbb{U}) < 1 \) such that given any small \( \varepsilon \), there exists \( \varepsilon_0 \) such that for all \( \varepsilon' \leq \varepsilon_0 \) and points \( x, y \in \mathbb{U}(\varepsilon') \cap \mathbb{U}(\varepsilon') \), \( x \) and \( y \) are connected by a path of length at most \( D = D(\mathbb{U}) \) in \( \mathbb{U}(\varepsilon') \) which lies in \( \mathbb{U}(\varepsilon) \).

**Proof.** This fact is clearly true for the disc. The general result follows directly by using the bi-Lipschitz nature of the the conformal maps (see (1.4)). The details are omitted.

As discussed above the proof of Theorem 6.2 will involve showing (6.55). We now consider the graph \( \mathbb{U}_n \). Recall the cases in the statement of Theorem 6.2. We will show that in most of the cases mentioned in the statement of the theorem, a subset of \( \Omega(n^2) \) vertices will be glued to get \( \mathbb{U}_n \) from \( \mathbb{U}_n \). We then use the following lemma which says that any such set can be divided into two sets which are at constant distance away from each other so that each of them have size \( \Omega(n^2) \) and the function \( G_n \) takes different values on the two sets. Formally we have the following lemma.

**Lemma 6.17.** For any \( d > 0 \) there exists \( c, c', a, \delta_0 > 0 \) such that for all \( \delta < \delta_0 \) and all \( n = 2^m > N(\delta) \) the following is true: for any \( A \subset \mathbb{U}_n \) with \( |A| \geq dn^2 \) there exists disjoint subsets \( A_1 \) and \( A_2 \) of \( A \) such that \( |A_1|, |A_2| \geq cn^2 \) and

\[
\inf_{z \in A_1} G_n(z) - \sup_{z \in A_2} G_n(z) > a, \quad d(A_1, A_2) > c'.
\]

**Proof.** Recall the notations \( \mathbb{U}_\beta, \mathbb{U}_\alpha, \gamma_\beta, A^\square \) from (1.5), (1.7), (5.8) and (1.2) respectively. Clearly area\( (\mathbb{U}_\beta) \) is a continuous function of \( \beta \) such that

\[
\lim_{\beta \to -\infty} \text{area}(\mathbb{U}_\beta) = 0, \quad \lim_{\beta \to \infty} \text{area}(\mathbb{U}_\beta) = \text{area}(\mathbb{U}).
\]

Thus for some \( c, c' > 0 \) and a positive integer \( k \), depending only on \( d \), there exists \( \beta_1 > \ldots > \beta_k \) with the following property:

\[
\text{area}(\mathbb{U}_{\beta_1}), \text{area}(\mathbb{U}_{\beta_j} \setminus \mathbb{U}_{\beta_{j-1}}), \text{area}(\mathbb{U}_{\beta_k}^\circ) \leq d/100, \quad d(\gamma_{\beta_{j-1}}, \gamma_{\beta_j}) \geq c, \quad \beta_{j-1} - \beta_j \geq c'
\]

for \( j = 2, \ldots, k \). By hypothesis \( |A| \geq dn^2 \), hence area\( (A^\square) \geq d \). Now look at \( 1 < j < k \) such that

\[
\text{area}(A^\square \cap \mathbb{U}_{\beta_j}) < \frac{d}{2}, \quad \text{area}(A^\square \cap \mathbb{U}_{\beta_{j+1}}) \geq \frac{d}{2}.
\]

Define

\[
A_1 = A \cap \mathbb{U}_{\beta_{j-1}} \cap \mathbb{U}_{\beta_k}^\circ, \quad A_2 = A \cap \mathbb{U}_{\beta_j}^\circ \cap \mathbb{U}_{\beta_k}.
\]

See Fig. 13. Since \( \beta_{j-1} - \beta_j \geq c' \) by Theorem 5.1 and Lemma 5.3 there exists \( \delta_0 \) such that given \( \delta < \delta_0 \)

\[
\lim_{n=2^m \to \infty} \inf_{z \in A_1} G_n(z) - \sup_{z \in A_2} G_n(z) > c'/2.
\]
Thus $A_1, A_2$ satisfy the required properties and hence we are done. □

We are now ready to prove Theorem 6.2.

6.3. Proof of Theorem 6.2. To prove Theorem 6.2 we will show: in all the cases mentioned in the statement of Theorem 6.2, there are sets $A_1, A_2$ which are subsets of the set in (6.53) (hence are glued to get wired $U_n$ from $U_n$) with the following properties:

- $A_1$ and $A_2$ satisfy the hypotheses of Lemma 6.15.
- For all $z \in A_1, z' \in A_2$
  $$G_n(z) - G_n(z') \geq c'$$
  for some $c' > 0$.

Before showing the above we first discuss why it suffices. Immediately using Lemma 6.15 we have pairs $(z_1, z'_1), \ldots, (z_k, z'_k)$ with $z_j \in A_1$ and $z'_j \in A_2$ for all $j = 1, \ldots, k$ which are glued and have paths connecting them of length $O(n)$ such that no edge appears in more than 2 paths. Also
  $$G_n(z_j) - G_n(z'_j) \geq c'$$
  for $j = 1 \ldots k$. Recall $\mathcal{E}$ from (6.54). Thus using Lemma 6.14 we get
  $$\mathcal{E} \leq \mathcal{E} - d$$
  for some constant $d = d(\epsilon)$. Hence we are done by (6.55).

Often instead of directly showing existence of $A_1$ and $A_2$ we will show a set $A$ of vertices with $|A| = Cn^2$ with $C = C(\epsilon)$, needs to be glued to get wired $U_n$ from $U_n$. By Lemma 6.17 this implies existence of $A_1$ and $A_2$ with the required properties. We now proceed to showing existence of such sets $A_1$ and $A_2$ for all the cases:

Recall the statement of the theorem. Observe that $\beta_{j-1} - \beta_j = \epsilon_1$ for some $\epsilon_1$. We now specify $\epsilon_1$. We would need
  $$\text{area}(U_{\beta_{j-1}}^c \cap U_{\beta_j}) \leq \epsilon/100$$
  for all $j = -4, \ldots, 5$. For one of the cases we also need the following
  $$\text{area}(U_{\beta_{j-1}}^c \cap U_{\beta_j}) \leq \frac{1 - 2\alpha}{100} \text{area}(U)$$
  (6.60)
for all \( j = -4, \ldots, 5 \). (100 is just a big enough number and is nothing special). Recall from Section 2.3 that \( \alpha < 1/2 \). As already remarked in the proof of Lemma 6.17, \( \text{area}(\mathbb{U}_\beta) \) is a continuous function of \( \beta \). Thus \( \epsilon_1 \) can be chosen such that both the above conditions are satisfied. Fix such an \( \epsilon_1 \). Let \( \epsilon' > 0 \) be such that

\[
\epsilon' < \text{area}(\mathbb{U}_{\beta_{i-1}} \cap \mathbb{U}_\beta)
\]

(6.61)

for all \( j = -4, \ldots, 5 \). We specify the choice of \( \epsilon_2 \) later.

**Case i.** \( \partial_{m_{\epsilon_2}} \hat{R}_1 = \emptyset \). By Lemma 6.16 for large enough \( n \), any two points in

\[
\bigcup_{\gamma} \mathbb{U}_\gamma \cap \mathbb{U}_n
\]

are connected by paths in \( \mathbb{U}_n \) which lie inside \( \mathbb{U}^{(\epsilon_2)} \) and hence do not intersect \( \partial_{m_{\epsilon_2}} \hat{R}_1 \). Since by definition \( \hat{R}_1 \) and \( \hat{R}_1^c \) are both connected subsets of \( \mathbb{U}_n \),

\[
\text{either } \bigcup_{\gamma} \mathbb{U}_\gamma \cap \mathbb{U}_n \subset \hat{R}_1 \text{ or } \bigcup_{\gamma} \mathbb{U}_\gamma \cap \mathbb{U}_n \subset \hat{R}_1^c.
\]

Now notice that

\[
\left| \bigcup_{\gamma} \mathbb{U}_\gamma \cap \mathbb{U}_n \right| = \left( \text{area}(\mathbb{U}) - O(\epsilon_2) \right)n^2.
\]

Thus in the first case at least \((1 - \alpha)\text{area}(\mathbb{U}) - O(\epsilon_2))n^2 \) (all the red vertices in \( \hat{R}_1 \)) are glued and in the latter case at least \( \alpha \text{area}(\mathbb{U}) - O(\epsilon_2))n^2 \) (all the blue vertices in \( \hat{R}_1 \)) are glued.

**Case iv.** In this case we will not show that the number of vertices glued is \( O(n^2) \). However by hypothesis roughly in this case \( \partial_{m_{\epsilon_2}} \hat{R}_1 \) (which is a connected set) intersects different level sets of the function \( G_n \). Thus the intersection with the different level sets could be taken to be \( A_1, A_2 \) in the usage of Lemma 6.15.

Formally we do the following: by hypothesis \( \partial_{m_{\epsilon_2}} \hat{R}_1 \) intersects both \( \mathbb{U}_{\beta_i} \) and \( \mathbb{U}_{\beta_{i+2}}^c \). Now there is some constant \( c = c(\beta, \epsilon_1, \mathbb{U}) \) such that for all \( i = -5, -4, \ldots, 5 \),

\[
\begin{align*}
\text{d}(\gamma(\beta_i - \epsilon_1/2), \gamma(\beta_i)) & \geq c \\
\text{d}(\gamma(\beta_i), \gamma(\beta_i + \epsilon_1/2)) & \geq c.
\end{align*}
\]

Since \( \partial_{m_{\epsilon_2}} \hat{R}_1 \) is \(*\)-connected (Remark 6.2) starting from \( \gamma(\beta_i + \epsilon_1) \cap \bigcup_{\gamma} \mathbb{U}_\gamma \) it either hits \( \bigcup_{\gamma} \mathbb{U}_{\beta_i} \) or \( \gamma(\beta_i + 1 - \epsilon_1/2) \) first. Similarly starting from \( \gamma(\beta_i) \cap \bigcup_{\gamma} \mathbb{U}_\gamma \) it hits \( \bigcup_{\gamma} \mathbb{U}_{\beta_i}^c \) or \( \gamma(\beta_i - \epsilon_1/2) \). Thus with \( d = \min(\frac{\epsilon_2}{2}, c) \), both

\[
\partial_{m_{\epsilon_2}} \hat{R}_1 \cap \bigcup_{\gamma(\beta_i - \epsilon_1/2)} \cup \bigcup_{\gamma} \mathbb{U}_\gamma 
\]

have a \(*\)-connected subset of diameter at least \( d \). Also for small enough \( \delta \), and \( n = 2^m > N(\delta) \), by Theorem 5.1 and Lemma 5.3 for all \( z \in \bigcup_{\beta_i - \epsilon_1/2} \mathbb{U}_{\beta_i} \)

\[
G_n(z) \geq \beta_i - 3\epsilon_1/4.
\]

Similarly for all \( z \in \bigcup_{\beta_i + 1 - \epsilon_1/2} \mathbb{U}_{\beta_i}^c \)

\[
G_n(z) \leq \beta_i + 1 - \epsilon_1/4.
\]

Thus the sets

\[
\begin{align*}
A_1 & := \partial_{m_{\epsilon_2}} \hat{R}_1 \cap \bigcup_{\beta_i - \epsilon_1/2} \cup \bigcup_{\gamma} \mathbb{U}_\gamma \\
A_2 & := \partial_{m_{\epsilon_2}} \hat{R}_1 \cap \bigcup_{\beta_i + 1 - \epsilon_1/2} \cup \bigcup_{\gamma} \mathbb{U}_\gamma
\end{align*}
\]

satisfy all the properties needed as stated in the beginning of the proof the and hence we are done in this case.
Cases **ii, iii** are exactly symmetric. Hence to avoid repetition we provide arguments only for case **ii**.

By hypothesis $\partial_{\text{n}}^*(\epsilon_2) \tilde{R}_1$ lies entirely in $\beta_{-2}$. Now by Lemma 6.16 for large enough $n$, any two points in $U_n \cap U_{\beta_1} \cap U_{\beta_2}$ are connected by a path in $U_n$ which lies entirely in $U_n \cap U_{\beta_1+\epsilon_1/2} \cap U_{\beta_2}$ (which does not intersect $\partial_n \tilde{R}_1$). Thus $U_n \cap U_{\beta_1} \cap U_{\beta_2}$ is either in $\tilde{R}_1$ or its complement. At this point we make the choice of $\epsilon_2$ such that

$$\text{area}(U \setminus U_{\beta_2}) \leq \epsilon'/100$$

(6.62)

where $\epsilon'$ was defined in (6.61). Recall by definition $\text{area}(U_{\beta_0}) = (1 - \alpha)\text{area}(U)$. Thus by the above choice of $\epsilon_2$, $\text{area}(U_{\beta_1} \cap U_{\beta_2}) \geq (1 - \alpha)\text{area}(U) + \epsilon' - \frac{\epsilon'}{100}n^2$. Hence number of vertices of both colors in $U_{n} \cap U_{\beta_1} \cap U_{\beta_2}$ is at least $\frac{99\epsilon'}{100}n^2$. Recall from (6.53) that both $\sigma^{-1}(2) \cap \tilde{R}_1$ and $\sigma^{-1}(1) \cap \tilde{R}_1^c$ are glued to obtain $\overset{\text{wired}}{\cup_{n}}$ from $U_{n}$. Thus at least $\frac{99\epsilon'}{100}n^2$ vertices are glued and hence again we are done.

**Case v.** By hypothesis $\partial_{\text{n}}^*(\epsilon_2) \tilde{R}_1 \subset U_{\beta_4} \cap U_{\beta_4}^c$. Now as argued in the previous case by Lemma 6.16 for large enough $n$, any two points in $D_1 := U_n \cap U_{\beta_5} \cap U_{\beta_2}$ are connected by a path in $U_n$ which lies in $U_n \cap U_{\beta_5+\epsilon_1/2} \cap U_{\beta_2}$ (does not intersect $\partial_n \tilde{R}_1$) and hence $D_1$ is either in $\tilde{R}_1$ or its complement. Similarly $D_2 := U_n \cap U_{\beta_5} \cap U_{\beta_2}$ also is either in $\tilde{R}_1$ or its complement.

Hence by the same arguments as we would be done once we show that in all the following four cases:

- $D_1 \cup D_2 \subset \tilde{R}_1$,
- $D_1 \cup D_2 \subset \tilde{R}_1^c$,
- $D_1 \subset \tilde{R}_1$ and $D_2 \subset \tilde{R}_1$,
- $D_1 \subset \tilde{R}_1$ and $D_2 \subset \tilde{R}_1^c$.

one glues $\Omega(n^2)$ vertices to obtain $\overset{\text{wired}}{\cup_{n}}$ from $U_{n}$. As in the previous case we now use the fact that $\sigma^{-1}(2) \cap \tilde{R}_1$ and $\sigma^{-1}(1) \cap \tilde{R}_1^c$ are the vertices glued. First consider the cases

$$D_1 \cup D_2 \subset \tilde{R}_1 \quad \text{or} \quad D_1 \cup D_2 \subset \tilde{R}_1^c.$$

Now by the choice of the $\beta_i$’s

$$\text{area}\left(\left\{U_{\beta_5} \cap U_{\beta_2}\right\}\right) \cup \left\{U_{\beta_5} \cap U_{\beta_2}\right\} \geq \text{area}(U) - O(\epsilon).$$

Thus $D_1 \cup D_2$ contains at least $(\text{area}(U) - O(\epsilon))n^2$ vertices. Since all the red vertices in $\tilde{R}_1$ and all the blue vertices in $\tilde{R}_1^c$ are glued, and since we assume that $\alpha < (1 - \alpha)$, in both the the cases at least $(\alpha \text{area}(U) - O(\epsilon))n^2$ vertices are glued. The remaining cases are

$$D_1 \subset \tilde{R}_1 \quad \text{and} \quad D_2 \subset \tilde{R}_1 \quad \text{or},$$

$$D_2 \subset \tilde{R}_1 \quad \text{and} \quad D_1 \subset \tilde{R}_1^c.$$

Recall that by hypothesis in this case $\cup_{n} \cap U_{\beta_5}$ has $\epsilon/2n^2$ vertices colored 2 or $\cup_{n} \cap U_{\beta_5}^c$ has $\epsilon/2n^2$ vertices colored 1. Therefore by the choice of $\beta_i$’s and $\epsilon_2$, either $D_1$ has $\left\lfloor \frac{\epsilon}{2} - \frac{\epsilon}{100}\right\rfloor n^2$ vertices colored 2 or $D_2$ has $\left\lfloor \frac{\epsilon}{2} - \frac{\epsilon}{100}\right\rfloor n^2$ vertices colored 1. Thus if $D_1 \subset \tilde{R}_1$ and $D_2 \subset \tilde{R}_1^c$ then at least $\epsilon n^2/4$
vertices are glued.

In the second case notice that by choice of $\epsilon_1$ (6.60) and $\epsilon_2$ (6.62),

$$\text{area}(\mathbb{U} \cap \mathbb{U}^{(\frac{c}{2})}) \geq \text{area}(\mathbb{U}) \left[ (1 - \alpha) - (1 - 2\alpha) \left( \frac{5}{100} + \frac{1}{100} \right) \right].$$

Thus $D_2$ has at least $[(1 - \alpha) - (1 - 2\alpha) \frac{6}{100}] \text{area}(\mathbb{U})n^2$ vertices and among them at least $\frac{1 - 2\alpha}{2} \text{area}(\mathbb{U})n^2$ vertices colored red since there are in total $\alpha \text{area}(\mathbb{U})n^2$ blue vertices. Recall from Section 2.3 that $\alpha < 1/2$. Hence in this case at least $\frac{1 - 2\alpha}{2} \text{area}(\mathbb{U})n^2$ vertices are glued and hence again as in the previous cases we are done.

The proof of Theorem 6.2 is thus complete. \hfill \square

7. Outline of the Proof of Theorem 5.1

The proof of Theorem 5.1 involves developing some tools using convergence of random walk on $\mathbb{U}_n$ to Reflected Brownian motion on $\mathbb{U}$. For a formal definition of Reflected Brownian motion on $\mathbb{U}$ see [CF13, Definition 2.7]. Also see [Bas95, Che93, BC08]. Throughout the rest of the article we will denote it by $B_t$.

The basic idea behind proving Theorem 5.1 is to show that the function $G^*(z)$ is the same as,

$$G(z) = \frac{2}{\text{area}(\mathbb{U}_1)} \int_0^\infty \left[ P_z (B_t \in \mathbb{U}_1) - P_z (B_t \in \mathbb{U}_2) \right] dt - c. \tag{7.1}$$

The constant $c$ is chosen such that the integral of $G$ along $\partial \mathbb{U}$ is 0. Formally, we parametrize the boundary $\partial \mathbb{U}$ by $\theta \in [0, 2\pi)$ via the conformal map $\phi$ (1.4). We then fix $c$ such that

$$\int_{|\zeta|=1} G \circ \phi(\zeta) \frac{d\zeta}{\zeta} = 0. \tag{7.2}$$

Compare the above expression with $G_n(\cdot)$ defined in (4.1) (recall that the constant $c$ is the same in both expressions). The proof of Theorem 5.1 has two parts. One part shows that $G_n$ converges to $G$, (see Lemma 5.1). This will follow by convergence of the random walk measure on $\mathbb{U}_n$ to $B_t$. For more on this see [BC08], [CF13] and the references therein. The remaining step then is to show that indeed $G_* = G$. This will be proved using the fact that the density for Reflected Brownian motion is a fundamental solution to the Neumann problem and hence the function $G$ is roughly satisfies,

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) G = \frac{4}{\text{area}(\mathbb{U}_1)} (1(\mathbb{U}_2) - 1(\mathbb{U}_1)) \tag{7.3}$$

with Neumann boundary conditions. One then checks that $G_*$ is a solution to the above problem. The proof is then complete by uniqueness of the solution of such a problem. For brevity we will adopt the following notation

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Recall (5.3). Thus we use $\Delta$ to denote the laplacian in both the continuous and discrete setting since there will be no scope of confusion.
7.1. Closed form of the limit.

**Theorem 7.1.** [GP15, Theorem 4.2] Let \( U \) be as in Setup 1. For all \( z \in U \),

\[
G(z) = G_*(z).
\]  

(7.4)

As discussed above to show this informally one identifies \( G \) as a solution to a second order partial differential equation also satisfied by \( G_* \). The result then follows by uniqueness of such a solution.

To give a general idea we quote a classical result in the theory of boundary value problems with Neumann boundary condition which is used in the proof.

**Lemma 7.1.** Let function \( w \in C^2(U) \cap C^1(\overline{U}) \) be a function on \( U \) satisfying the following properties

\[
\Delta(w) = \frac{f}{4} \quad \quad \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial U
\]

where \( \frac{\partial}{\partial \nu} \) denotes the normal derivative and \( f \in L^1(\mathbb{C}, U) \). Then

\[
w \circ \phi(z) = d + \frac{1}{\pi} \int_{|\zeta|<1} f \circ \phi(\zeta) |\phi'|^2(\zeta) \log(|\zeta - z||\zeta| - \bar{\zeta}z)|d\xi d\eta,
\]  

(7.5)

where

\[
\frac{1}{2\pi i} \int_{\partial D} \frac{w \circ \phi(\zeta)}{\zeta} d\zeta = d.
\]

**Proof.** The result for the case \( U = \mathbb{D} \) (the unit disc) is stated as [Beg05, Theorem 8]. The above lemma now follows by change of variable under composing with the map \( \phi \). \( \square \)

Theorem 7.1 now follows roughly by arguing that the function \( G(\cdot) \) satisfies Lemma 7.1 with

\[
\Delta G = \frac{4}{\text{area}(U_1)} (1(U_2) - 1(U_2))
\]

and noticing that \( G_* \) is nothing but \( f \) in Lemma 7.1 replaced by \( \frac{16}{\text{area}(U_1)} (1(U_2) - 1(U_2)) \). However formally one works with “smoother” versions of \( (1(U_2) - 1(U_2)) \). The detailed proof appears in [GP15].

**Remark 7.1.** Recall the constant \( c \) in (7.1) such that.

\[
\int_{|\zeta|=1} G \circ \phi(\zeta) \frac{d\zeta}{\zeta} = 0.
\]

Consider the the special case of the unit disc i.e. \( U = \mathbb{D} \) with point \( x_1 = -i, x_2 = i \) and \( y_1 = -(1 - \delta)i, y_2 = (1 - \delta)i \). Now owing to the invariance of Reflected Brownian motion on \( \mathbb{D} \) under the transformation \( \zeta \to -\zeta \), one sees that the integral in (7.1)

\[
\frac{2}{\text{area}(\mathbb{D}_1)} \int_0^\infty [P_z(B_t \in \mathbb{D}_1) - P_z(B_t \in \mathbb{D}_2)] dt
\]

is odd, (because with the choice of \( y_1, y_2 \mathbb{D}_1 = -\mathbb{D}_2 \)). Thus \( c \) is 0 and

\[
G(z) = \frac{2}{\text{area}(\mathbb{D}_1)} \int_0^\infty [P_z(B_t \in \mathbb{D}_1) - P_z(B_t \in \mathbb{D}_2)] dt.
\]
7.2. Convergence. The last missing piece is to show that the function $G_n$ converges to $G$. To this end we now state the following lemma which along with Theorem 7.1 proves Theorem 5.1.

Lemma 7.2. [GP15, Lemma 4.8]

$$\lim_{m \to \infty} \sup_{n=2^m} \sup_{x \in U} |G_n(x) - G(x)| = 0$$

where $G$ is defined in (7.1).

The proof of the lemma uses the local CLT estimates for random walk approximation of Reflected Brownian motion obtained in [CF13, Theorem 2.12]. The complete proof appears in [GP15].

Proof of Theorem 5.1. The proof follows immediately from Lemma 7.2 and Theorem 7.1. □

8. Basic geometric properties of $U$

In this section we prove some basic geometric properties of $U_n$ used throughout the article. We first state a standard property about the boundary of $U$ as in Setup 1. Since the boundary $\partial U$ is analytic there exists a $C > 0$ and an $\epsilon_0$ such that for all $x \in \partial U$ there exists an orthogonal system of coordinates centered at $x = (x_1, x_2)$ such that for all $\epsilon \leq \epsilon_0$

$$B(x, \epsilon) \cap U = \{(x'_1, x'_2) \in B(x, \epsilon) : x'_1 \in (x_1 - \epsilon, x_1 + \epsilon), x'_2 \geq f(x'_1)\}$$

and

$$|f(x'_1) - x_2| \leq C|x'_1 - x_1|^2.$$  

The above is a simple consequence of Taylor expansion up to second order of the curve locally near $x$. See Fig 14 i.

As a simple corollary of the above fact we see that $U$ satisfies the following property which shows that the $y_i$’s in Setup 1 can indeed be chosen.

Corollary 8.1. Let $U$ be as in Setup 1. Then there exists $\delta_0 = \delta_0(U)$ such that for all $x \in \overline{U}$ and $\delta < \delta_0$ there exists $y \in U$ such that

$$d(y, x) \leq \delta$$

and

$$B(y, \delta/2) \subset U.$$
Recall that \( d(y,x) \) is the euclidean distance between \( x \) and \( y \). \( B(y,\delta) \) denotes the euclidean ball of radius \( \delta \) with center at \( y \).

**Proof.** Choose \( \delta_0 \leq \epsilon_0/4 \) such that \( C\delta_0^2 \leq \frac{\delta_0}{100} \) where \( \epsilon_0 \) and \( C \) appear in (8.1). For any \( \delta < \delta_0 \) the lemma is immediate if \( d(x,\partial U) > \delta/2 \). since then we can choose \( y = x \).

Otherwise let \( z = (z_1,z_2) \in \partial U \) be the closest point on the boundary to \( x \). Now in the local coordinate system centered at \( z \) as in (8.1) choose \( y = (z_1,z_2 + \delta/2) \). Then

\[
\begin{align*}
    d(x,y) &\leq d(x,z) + d(z,y) \\
    &\leq \delta.
\end{align*}
\]

Also clearly \( B(y,\delta/2) \subset U \) and hence we are done. See Fig 14 ii. \( \Box \)

**Concluding remarks and future directions**

Observe that the main result of this article establishes existence of a macroscopic interface only up to an asymptotically 0 density set of “dust” particles, (see Fig 5). Thus Theorems 1.1 and 1.2 do not rule out the possibility that there could be a non empty but asymptotically zero density set of “dust” particles of the “wrong” color in either region. However Fig. 1 suggests that such dust particles are also unlikely. In [GLPP15] a stronger theorem ruling out the presence of such particles in either region was proven for the cylinder graph.

Thus a natural next step would be to have a similar stronger theorem in this setting and find the order of magnitude of fluctuations of the interface.

**Higher dimensions.** One can also study the process in higher dimensions and try to characterize the hyper surface separating the two colors at stationarity. It is natural to guess that it should be the level set of a suitable potential function which is harmonic in the interior of the corresponding domain and has positive and negative singularities at the two sources with Neumann boundary conditions.

**Acknowledgments**

We thank Krzysztof Burdzy, Zhen-Qing Chen, Wai-Tong Fan, Lionel Levine, and Steffen Rohde for valuable discussions at various stages of this work. We also thank Gerandy Brito, Christopher Hoffman, Matthew Junge and Brent Werness for several useful comments that helped improve the exposition. The work was initiated when S.G. was an intern with the Theory Group at Microsoft Research, Redmond. He thanks the group for its hospitality.

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