On Vanishing Properties of Polynomials on Symmetric Sets of the Boolean Cube, in Positive Characteristic

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Abstract

The finite-degree Zariski (Z-) closure is a classical algebraic object, that has found a key place in several applications of the polynomial method in combinatorics. In this work, we characterize the finite-degree Z-closures of a subclass of symmetric sets (subsets that are invariant under permutations of coordinates) of the Boolean cube, in positive characteristic.

Our results subsume multiple statements on finite-degree Z-closures that have found applications in extremal combinatorial problems, for instance, pertaining to set systems (Hegedűs, Stud. Sci. Math. Hung. 2010; Hegedűs, arXiv 2021), and Boolean circuits (Hrubes et al., ICALP 2019). Our characterization also establishes that for the subclasses of symmetric sets that we consider, the finite-degree Z-closures have low computational complexity.

A key ingredient in our characterization is a new variant of finite-degree Z-closures, defined using vanishing conditions on only symmetric polynomials satisfying a degree bound.

1 Introduction

The polynomial method is an ever-expanding set of algebraic techniques, which broadly entails capturing combinatorial objects by algebraic means, specifically using polynomials, and then employing algebraic tools to infer their combinatorial features. While several instances of the polynomial method have been part of the combinatorist’s toolkit for decades, development of this method has received more traction in recent times, owing to several breakthroughs like (i) Dvir’s solution [Dvi09] to the finite-field Kakeya problem, followed by an improvement by Dvir, Kopparty, Saraf, and Sudan [DKSS13], (ii) Guth and Katz [GK15] proving a conjecture by Erdős on the lower bound for the distinct distances problem, (iii) solutions to the capset problem by Croot, Lev, and Pach [CLP17], and Ellenberg and Gijswijt [EG17], to name a few. The surveys by Dvir [Dvi12] and Tao [Tao14], and the book by Guth [Gut16] provide detailed accounts of the polynomial method.

In this article, we are interested in one of the earliest avatars of the polynomial method, which has the following basic template:

(i) Associate a combinatorial object to a nonzero polynomial in a way that the degree of the polynomial is at most the size of the object.

(ii) Use the vanishing properties of the nonzero polynomial to assert a lower bound on its degree. This gives a lower bound on the size of the combinatorial object.

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For most applications, a study of this avatar, in fact, distills to a study of a classical algebraic object – the finite-degree Zariski closure. For any $S \subseteq \{0,1\}^n$ and $d \in \mathbb{N}$, the degree-$d$ Zariski ($\mathbb{Z}$-) closure of $S$, denoted by $Z\text{-}\text{cl}_{n,d}(S)$, is defined to be the common zero set, in $\{0,1\}^n$, of all polynomials with degree at most $d$, that vanish at each point in $S$. This is a closure operator\(^1\), and was defined by Nie and Wang [NW15] towards obtaining a better understanding of the applications of the polynomial method to combinatorial geometry. However, it has been studied implicitly even earlier. (See, for instance, Wei [Wei91], Heijnen and Pellikaan [HP98], Keevash and Sudakov [KS05], and Ben-Eliezer, Hod, and Lovett [BEHL12].)

1.1 Motivation

For any $a,b \in \mathbb{Z}$, $a \leq b$, by an abuse of notation, we will denote the integer interval of all integers between $a$ and $b$ by $[a,b]$. We also abbreviate $[n] := [1,n]$ for any $n \in \mathbb{Z}^+$. For any prime $p$, the finite field with $p$ elements is denoted by $\mathbb{F}_p$.

Let us begin by considering a few interesting instances of combinatorial problems solved by a closure operator, and (ii) $\text{cl}(\text{cl}(S))$ is any map $\text{cl} : P \to P$ satisfying: (i) $a \leq \text{cl}(a), \forall a \in P$, (ii) $\text{cl}(a) \leq \text{cl}(b), \forall a,b \in P, a \leq b$, and (iii) $\text{cl}(\text{cl}(a)) = \text{cl}(a), \forall a \in P$. This is a well-studied set operator. See, for instance, Birkhoff [Bir73, Chapter V, Section 1] for an introduction.

- Hegedüs [Heg10] proved a lower bound for a special case of a balancing problem for set systems using the following lemma.

**Lemma 1.1** ([Heg10]). Let $n = 4p$, where $p$ is a prime. If $f(\mathbb{X}) \in \mathbb{F}_p[\mathbb{X}]$ satisfies

(i) $f(x) = 0$ for all $x \in \{0,1\}^n$ with $|x| = 2p$, and

(ii) $f(y) \neq 0$ for some $y \in \{0,1\}^n$ with $|y| = 3p$,

then $\deg(f) \geq p$.

We know several proofs of Lemma 1.1 by now: Hegedüs [Heg10] gave a proof using Gröbner basis theory, Srinivasan (see [AKV20]) gave a simpler proof using Fermat’s Little Theorem and linear algebra, and Alon [Alo20] gave a proof using the Combinatorial Nullstellensatz [Alo99].

- The following lemma was proven by Hrubes, Ramamoorthy, Rao and Yehudayoff [HRRY19] to solve a different version of the balancing problem. They used this lemma to exploit a connection between balancing set systems and depth-2 threshold circuits, which are an important class of Boolean circuits studied in the theory of computation.

**Lemma 1.2** ([HRRY19]). Let $n = 2p$, where $p$ is a prime. If $f(\mathbb{X}) \in \mathbb{F}_p[\mathbb{X}]$ satisfies

(i) $f(x) = 0$ for all $x \in \{0,1\}^n$ with $|x| = p$, and (ii) $f(0^n) \neq 0$, then $\deg(f) \geq p$.

- Recently, Hegedüs [Heg21] proved the following lemma, and gave a lower bound for an $L$-balancing problem for set systems.

**Lemma 1.3** ([Heg21]). Let $p$ be a prime, $n, \ell \in \mathbb{Z}^+$ and $i \in [p^\ell - 1, n - p^\ell + 1]$. For any $f(\mathbb{X}) \in \mathbb{F}_p[\mathbb{X}]$ such that $\deg(f) \leq p^\ell - 1$, if $f(x) = 0$ for all $x \in \{0,1\}^n$ with $|x| = i$, then $f(x) = 0$ for all $x \in \{0,1\}^n$ with $|x| \in \{j \in [0,n] : j \equiv i \pmod{p^\ell}\}$.

\(^1\)A closure operator on a poset $(P, \leq)$ is any map $\text{cl} : P \to P$ satisfying: (i) $a \leq \text{cl}(a), \forall a \in P$, (ii) $\text{cl}(a) \leq \text{cl}(b), \forall a,b \in P, a \leq b$, and (iii) $\text{cl}(\text{cl}(a)) = \text{cl}(a), \forall a \in P$. This is a well-studied set operator. See, for instance, Birkhoff [Bir73, Chapter V, Section 1] for an introduction.
It should be noted that in each of the above results mentioned, we prescribe vanishing conditions on polynomials at all points having a fixed Hamming weight. This naturally introduces *symmetric sets* of the Boolean cube in our discussion. Let $\mathfrak{S}_n$ denote the symmetric group on $[n]$. A subset $S \subseteq \{0,1\}^n$ is said to be symmetric if

$$(x_1, \ldots, x_n) \in S, \sigma \in \mathfrak{S}_n \implies (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in S.$$  

It is easy to see that $S \subseteq \{0,1\}^n$ is symmetric if and only if

$$x \in S, y \in \{0,1\}^n, |y| = |x| \implies y \in S.$$ 

Thus, symmetric sets of the Boolean cube are determined by the Hamming weights of the points in them, and therefore, there is a one-to-one correspondence between subsets $E \subseteq [0, n]$ and symmetric sets $E := \{x \in \{0,1\}^n : |x| \in E\}$.

It is easy to check that the finite-degree $Z$-closure of a symmetric set is a symmetric set. So we will conveniently, wherever applicable, identify a symmetric set $S$ with its $Z$-cl $\text{Z-cl}_{n,d}(S)$.

Indeed, the results mentioned above are, in fact, statements about finite-degree $Z$-closures of special symmetric sets. In our notation, assuming we are working over the field $F_p$, Lemma 1.1 states that $\exists \not\in \text{Z-cl}_{4p,p-1}(2p)$, Lemma 1.2 states that $\not\in \text{Z-cl}_{2p,p-1}(p)$, and Lemma 1.3 states that $\text{Z-cl}_{n,p-1}(i) = \{j \in [0,n] : j \equiv i (\text{mod} \ p^\ell)\}$ for $i \in [p^\ell - 1, n - p^\ell + 1]$. In light of these results, we concern ourselves with the following question.

**Question 1.4.** Let $\mathbb{F}$ be a field with positive characteristic. Characterize (combinatorially) the finite-degree $Z$-closures $\text{Z-cl}_{n,d}(E)$, for all $E \subseteq [0, n], d \in [0, n]$.

### 1.2 Our results

Our first result subsumes Lemmas 1.1, 1.2 and 1.3. Fix any field $\mathbb{F}$ with positive characteristic $p$. By a layer in $\{0,1\}^n$, we mean a symmetric set $S$, $i \in [0, n]$. We determine the finite-degree $Z$-closures of single layers. This result could also be obtained from the proof techniques in Hegedűs [Heg10], but we give what we believe is a simpler proof, not involving any Gröbner basis or Hilbert function computations, and that is similar to the proof by Srinivasan (see [AKV20]) for Lemma 1.1.

For any $E \subseteq [0, n]$ and $\ell \in \mathbb{N}$, define

$$E \oplus p^\ell = \bigcup_{j \in E} \{t \in [0, n] : t \equiv j (\text{mod} \ p^\ell)\}.$$

For any $d \in \mathbb{N}$, define $\ell_p(d) = \lfloor \log_p(d + 1) \rfloor$. Thus, $\ell_p(d)$ is the unique integer $\ell \in \mathbb{N}$ such that $p^{\ell - 1} \leq d < p^\ell - 1$.

We have the following result, which answers Question 1.4 for single layers.

**Theorem 1.5** (Finite-degree $Z$-closure of a single layer). Let $i, d \in [0, n]$ and $\ell = \ell_p(d)$. Then

$$\text{Z-cl}_{n,d}(i) = \begin{cases} \{i\}, & i \not\in [d, n - d] \\ i \oplus p^\ell, & i \in [d, n - d] \end{cases}$$

We will then proceed to describe the finite-degree $Z$-closures of general symmetric sets. We do not manage to determine these for all symmetric sets, but for a large subclass. In this context, a variant of the finite-degree $Z$-closure shows itself very naturally.
Since our interest lies in symmetric sets, it begs the question whether vanishing conditions on just symmetric polynomial functions would suffice to understand the finite-degree Z-closures. Towards this, for any \( E \subseteq [0, n] \) and \( d \geq 0 \), we define the degree-\( d \) symmetric closure of \( E \), denoted by \( \text{sym-cl}_{n,d}(E) \), to be the common zero set, in \( \{0, 1\}^n \), of all symmetric polynomial functions with degree at most \( d \), that vanish at each point in \( E \). As in the case of finite-degree Z-closures, it is easy to see that the finite-degree symmetric closure of a symmetric set is symmetric, and so we will again identify the symmetric sets with subsets of \([0, n]\); in particular, we will identify (and denote) \( \text{sym-cl}_{n,d}(E) \subseteq \{0, 1\}^n \) by \( \text{sym-cl}_{n,d}(E) \subseteq [0, n] \).

Our second result is a characterization of Z-closures of symmetric sets in terms of their symmetric closures, under some conditions. Thus, we answer Question 1.4 for a special subclass of symmetric sets.

**Theorem 1.6 (Finite-degree Z-closures of symmetric sets).** Let \( d \in \mathbb{N} \), \( \ell = \ell_p(d) \). If \( n \geq 4p^\ell - 1 \), then for any \( E \subseteq [d, n - d] \), we have \( Z\text{-cl}_{n,d}(E) = \text{sym-cl}_{n,d}(E) \).

The finite-degree symmetric closures are also interesting due to them having *low computational complexity* relative to finite-degree Z-closures. It is known that in the worst-case, computing the finite-degree Z-closure of an arbitrary subset of \( \{0, 1\}^n \) will take time exponential in \( n \); in contrast, we will show by an easy linear algebraic argument that the finite-degree symmetric closure of any symmetric set in \( \{0, 1\}^n \) can be computed in time polynomial in \( n \). As a consequence, by Theorem 1.6, for \( d \in \mathbb{N} \), \( \ell = \ell_p(d) \), if \( n \geq 4p^\ell - 1 \), then for any \( E \subseteq [d, n - d] \), we can compute \( Z\text{-cl}_{n,d}(E) \) in time polynomial in \( n \).

### 1.3 Related work

We note here that prior to our work, there have been attempts to characterize other notions related to finite-degree Z-closures – namely, *Gröbner basis*, *standard monomials*, and *affine Hilbert function* of the vanishing ideal – for special cases of symmetric sets of the Boolean cube, over fields of both positive and zero characteristic. In fact, the Gröbner basis and the affine Hilbert function are stronger notions than the finite-degree Z-closures. For detailed introductions, refer for instance, Cox, Little, and O’Shea [CLO15] – Chapter 2 (for Gröbner basis), Chapter 5 (for standard monomials\(^2\)), and Chapter 9 (for affine Hilbert function).

Let \( \mathbb{F} \) be a field with either positive or zero characteristic. We will assume the basic definitions as given in [CLO15]. For any \( S \subseteq \{0, 1\}^n \), let \( \text{SM}_\leq(S) \) denote the set of *standard monomials* of the vanishing ideal of \( S \) with respect to a monomial order \( \leq \).\(^3\) Further, for any \( d \in [0, n] \), let \( \text{H}_d(S) \) denote the value of the degree-\( d \) affine Hilbert function for \( S \). Given a monomial order \( \leq \), let \( \text{LM}_\leq(P) \) denote the *leading monomial* (with respect to \( \leq \)) of the polynomial \( P(\bar{X}) \in \mathbb{F}[\bar{X}] \), where \( \bar{X} = (X_1, \ldots, X_n) \) are the indeterminates. For any \( \alpha \in \mathbb{N}^n \), we denote the monomial \( \bar{X}^{\alpha} = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \).

The following are basic facts that show the inter-relationships between the Gröbner bases, standard monomials, affine Hilbert functions, and the finite-degree Z-closures. These follow easily from the definitions, and results from the relevant chapters in [CLO15].

**Fact 1.7.** (a) Let \( \leq \) be any monomial order, and \( \mathcal{G}_\leq(S) \) be a Gröbner basis of the vanishing ideal of \( S \) with respect to \( \leq \). Then for any \( S \subseteq \{0, 1\}^n \),

\[
\text{SM}_\leq(S) = \{ \bar{X}^\alpha : \bar{X}^\alpha \text{ does not divide } \text{LM}_\leq(P), \text{ for any } P(\bar{X}) \in \mathcal{G}_\leq(S) \}.
\]

\(^2\)The terminology ‘standard monomials’, however, is not used in Cox, Little, and O’Shea [CLO15].

\(^3\)A linear order \( \leq \) on the set of all monomials in \( n \) indeterminates \( X_1, \ldots, X_n \), is a monomial order if (i) \( 1 \leq u \) for every monomial \( u \), and (ii) for monomials \( u, v \) with \( u \leq v \), we have \( uv \leq vw \) for every monomial \( w \).
(b) Let $\leq$ be any monomial order, and $G_{\leq}(S)$ be a Gröbner basis of the vanishing ideal of $S$ with respect to $\leq$. Then for any $S \subseteq \{0,1\}^n$, $x \in \{0,1\}^n$,

$$x \in \text{Z-cl}_{n,d}(S) \iff P(x) = 0, \text{ for all } P(\mathbf{x}) \in G_{\leq}(S), \deg(P) \leq d.$$ 

(c) For any $S \subseteq \{0,1\}^n$, $x \in \{0,1\}^n$, and $d \in [0,n]$,

$$x \in \text{Z-cl}_{n,d}(S) \iff H_d(S \cup \{x\}) = H_d(S).$$

Some immediate corollaries, for symmetric sets of the Boolean cube, are as follows.

**Corollary 1.8.** (a) Let $\leq$ be any monomial order, and $G_{\leq}(S)$ be a Gröbner basis of the vanishing ideal of $S$ with respect to $\leq$. Then for any $E \subseteq [0,n]$, $j \in [0,n]$,

$$j \in \text{Z-cl}_{n,d}(E) \iff P\big|_j = 0, \text{ for all } P(\mathbf{x}) \in G_{\leq}(E), \deg(P) \leq d.$$ 

(b) For any $E \subseteq [0,n]$, $j \in [0,n]$, and $d \in [0,n]$,

$$j \in \text{Z-cl}_{n,d}(E) \iff H_d(E \cup \{j\}) = H_d(E).$$

Some of the prior work on Gröbner bases, standard monomials, affine Hilbert functions, and finite-degree Z-closures for symmetric sets are as follows.

- Wilson [Wil90] determined the diagonal form (over all fields) for incidence matrices associated to certain symmetric sets. This was then used to determine the affine Hilbert function $H_d(\mathbf{x})$, for all $d, i \in [0,n]$, $i \in [d, n - d]$.

- Anstee, Rónyai, and Sali [ARS02] defined order shattering for set systems, and gave a characterization of standard monomials for any subset of the Boolean cube, with respect to all lexicographic orders, in terms of order shattered sets. Friedl and Rónyai [FR03] used this characterization and generalized the result of Wilson [Wil90] on incidence matrices.

- Hegedűs and Rónyai [HR03] characterized the reduced Gröbner basis for a single layer $i$, for all $i \in [0,n]$, with respect to all lexicographic orders (over all fields), and further generalized this characterization to linear Sperner families (over characteristic zero) in [HR18].

- Felszeghy, Ráth, and Rónyai [FRR06] studied a lex game to give a combinatorial criterion for a squarefree monomial to be a standard monomial of a symmetric set (over all fields).

- Felszeghy, Hegedűs, and Rónyai [FHR09] obtained characterizations of a Gröbner basis, standard monomials, as well as the affine Hilbert function, for the symmetric set $[d, d + \ell] \oplus p^k$, for $k \in \mathbb{Z}^+$ and $d, \ell \in [0,n]$, $d + \ell \leq n$ (over positive characteristic $p$).

- Over characteristic zero, Bernasconi and Egidi [BE99] determined the affine Hilbert functions of all symmetric sets. This can be used, via Corollary 1.8 (b), to determine the finite-degree Z-closures of all symmetric sets. Further, a more combinatorial characterization of the finite-degree Z-closures of all symmetric sets, independent of affine Hilbert function computations, was given by the second author [Ven21].
2 Preliminaries

Since we will work over fields of positive characteristic, and since we are only concerned with subsets of the Boolean cube, we can and will assume throughout that we have fixed the field $\mathbb{F}_p$, where $p$ is prime. So we have the Boolean cube $\{0,1\}^n \subseteq \mathbb{F}_p$. 

For any set of polynomials $\mathcal{P}$ in $\mathbb{F}_p[\mathbb{X}]$, where $\mathbb{X} = (X_1, \ldots, X_n)$ are the indeterminates, let $\mathcal{Z}(\mathcal{P}) = \{ x \in \{0,1\}^n : P(x) = 0, \text{ for all } P(\mathbb{X}) \in \mathcal{P} \}$. 

A fundamental result in our context is Alon’s Combinatorial Nullstellensatz, which we state here for the Boolean cube.

**Theorem 2.1** ([Alo99]). The set of monomials $\{ \mathbb{X}^\alpha : \alpha \in \{0,1\}^n \}$ is a basis of the vector space of all $\mathbb{F}_p$-valued functions on $\{0,1\}^n$.

Note that $\mathbb{X}^\alpha$, $\alpha \in \{0,1\}^n$ are precisely all the squarefree monomials in the indeterminates $\mathbb{X} = (X_1, \ldots, X_n)$. So Theorem 2.1 implies the following: for any polynomial $Q(\mathbb{X}) \in \mathbb{F}_p[\mathbb{X}]$, there exists a unique polynomial $\tilde{Q}(\mathbb{X}) \in \mathbb{F}_p[\mathbb{X}]$ which is a linear combination of squarefree monomials, such that $Q = \tilde{Q}$ as functions on $\{0,1\}^n$. Henceforth, for convenience, we will identify $Q(\mathbb{X})$ with $\tilde{Q}(\mathbb{X})$; in other words, for any polynomial $Q(\mathbb{X})$ that we define, that is not necessarily a linear combination of squarefree monomials, we will assume that $Q(\mathbb{X})$ has been immediately replaced by $\tilde{Q}(\mathbb{X})$, and denoted by $\tilde{Q}(\mathbb{X})$ itself, without mention. Also relevant is that, as a consequence, while considering the finite-degree $\mathbb{Z}$-closure $\mathbb{Z}\text{-cl}_{n,d}$, we can restrict $d \in [0,n]$. 

**Finite-degree $\mathbb{Z}$-closure.** For any $S \subseteq \{0,1\}^n$, let $\mathcal{I}_n,d(S)$ denote the vector space of all polynomials in $\mathbb{F}_p[\mathbb{X}]$ having degree at most $d$, that vanish at each point in $S$. Recall that for any $d \in [0,n]$ and $S \subseteq \{0,1\}^n$, the degree-$d$ Zariski ($\mathbb{Z}$-) closure is defined by $\mathbb{Z}\text{-cl}_{n,d}(S) = \mathcal{Z}(\mathcal{I}_n,d(S))$. It is easy to check that for any $E \subseteq [0,n]$, $\mathbb{Z}\text{-cl}_{n,d}(E)$ is a symmetric set. So we will stick to our identification of symmetric sets of $\{0,1\}^n$ with subsets of $[0,n]$, and use the notation $\mathbb{Z}\text{-cl}_{n,d}(E)$ instead. For any $E \subseteq \mathbb{Z}$ and $a,b \in \mathbb{Z}$, define $a + bE = \{ a + bx : x \in E \}$. We make the following preliminary observations.

**Proposition 2.2** (Properties of finite-degree $\mathbb{Z}$-closures of symmetric sets). Consider any $d \in [0,n]$ and $E \subseteq [0,n]$.

(a) If $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$, then $j \in \mathbb{Z}\text{-cl}_{m,d}(E)$, for all $m > n$.

(b) If $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$, then $n - j \in \mathbb{Z}\text{-cl}_{n,d}(n - E)$.

(c) If $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$, then $j + k \in \mathbb{Z}\text{-cl}_{n+k,d}(E + k)$, for all $k > 0$.

**Proof.** (a) Let $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$ and $m > n$. It is enough to show that $1^j 0^{n-j} \in \mathbb{Z}\text{-cl}_{m,d}(E)$. Consider any $f(X_1, \ldots, X_n) \in \mathcal{I}_m,d(E)$. Define $f^*(X_1, \ldots, X_n) = f(X_1, \ldots, X_n, 0^{n-m})$. Note that for any $x \in E$ in $\{0,1\}^n$, we have $x 0^{n-m} \in E$ in $\{0,1\}^m$. Also $\deg f^* \leq \deg f \leq d$ and hence $f^*(X_1, \ldots, X_n) \in \mathcal{I}_m,d(E)$. Then $f(1^j 0^{n-j}) = f^*(1^j 0^{n-j}) = 0$, since $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$. Thus $1^j 0^{n-j} \in \mathbb{Z}\text{-cl}_{m,d}(E)$.

(b) Let $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$. Consider any $f(X_1, \ldots, X_n) \in \mathcal{I}_n,d(n - E)$. It is enough to show that $1^n - j^i \in \mathbb{Z}\text{-cl}_{n,d}(n - E)$. Define $f^*(X_1, \ldots, X_n) = f(1 - X_1, \ldots, 1 - X_n)$. Then we have $f(x_1, \ldots, x_n) = 0$, where $(x_1, \ldots, x_n) \in n - E$ if and only if $f^*(1 - x_1, \ldots, 1 - x_n) = 0$, where $(1 - x_1, \ldots, 1 - x_n) \in E$. So $f^*(X_1, \ldots, X_n) \in \mathcal{I}_n,d(E)$. This gives $f^*(0^n - j^i) = 0$, since $j \in \mathbb{Z}\text{-cl}_{n,d}(E)$. Thus $f(1^n - j^i) = 0$, that is, $1^n - j^i \in \mathbb{Z}\text{-cl}_{n,d}(n - E)$.
(c) Let \( j \in \text{Z-cl}_{n,d}(E) \) and \( k > 0 \). It is enough to show that \( 1^j 0^{n-j} 1^k \in \text{Z-cl}_{n,k,d}(E + k) \). Consider any \( f \in \mathcal{I}_{n+k,d}(E + k) \). Define \( f^*(x) = f(x^k) \), for all \( x \in \{0,1\}^n \). Note that for any \( x \in E \) in \( \{0,1\}^n \), we have \( x1^k \in E + k \) in \( \{0,1\}^{n+k} \). Also \( \deg f^* \leq \deg f \leq d \) and hence \( f^* \in \mathcal{I}_{n,d}(E) \). Then \( f(1^j 0^{n-j} 1^k) = f^*(1^j 0^{n-j}) = 0 \), since \( j \in \text{Z-cl}_{n,d}(E) \). Thus \( 1^j 0^{n-j} 1^k \in \text{Z-cl}_{n,k,d}(E + k) \). \( \square \)

**p-ary representation of nonnegative integers.** For any \( d \in \mathbb{N} \), define \( \ell_p(d) = \lceil \log_p(d + 1) \rceil \). Thus, \( \ell_p(d) \) is the unique integer \( \ell \in \mathbb{N} \) such that \( p^{\ell-1} \leq d \leq p^\ell - 1 \). For any \( n \in \mathbb{N} \), we fix the following notation via the \( p \)-ary expansion of \( n \),

\[
n = \sum_{t \geq 0} n_t p^t, \quad \text{where} \quad n_t \in [0, p-1], \quad \text{for all} \quad t \geq 0.
\]

In other words, \( n_t \) denotes the \( t \)-th digit of \( n \) in its \( p \)-ary expansion, for all \( t \geq 0 \). The following observations are immediate.

**Observation 2.3.** Let \( m, n, \ell \in \mathbb{N} \).

(a) \( m = n \) if and only if \( m_t = n_t \), for all \( t \geq 0 \).

(b) If \( m < n \), then there exists \( t \in [0, \ell_p(n) - 1] \) such that \( m_t < n_t \).

(c) \( m \equiv n \pmod{p^\ell} \) if and only if \( m_t = n_t \), for all \( t \in [0, \ell - 1] \).

**Elementary symmetric polynomials.** Fix \( n \in \mathbb{Z}^+ \). For \( k \in [0, n] \), the elementary symmetric polynomial of degree \( k \) is a multilinear polynomial of degree \( k \) defined as

\[
\sigma_k(X) = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} X_i \in \mathbb{F}_p[X].
\]

It follows immediately from the definition that for any \( k \in [0, n] \) and \( x \in \{0,1\}^n \), we have \( \sigma_k(x) \equiv \binom{|x|}{k} \pmod{p} \).

The following result is crucial for us to work with the elementary symmetric polynomials.

**Theorem 2.4 (Lucas’s Theorem [Luc78]).** For any \( n, m \in \mathbb{N} \),

\[
\binom{n}{m} \equiv \prod_{t \geq 0} \binom{n_t}{m_t} \pmod{p}.
\]

So \( \binom{n}{m} \neq 0 \pmod{p} \) if and only if \( m_t \leq n_t \) for all \( t \geq 0 \).

As an immediate corollary, we get some properties of elementary symmetric polynomials.

**Corollary 2.5 (Properties of elementary symmetric polynomials).**

(a) For any \( t \geq 0 \) and \( x \in \{0,1\}^n \), we have \( \sigma_{p^t}(x) \equiv |x|_t \pmod{p} \).

(b) For any \( x, y \in \{0,1\}^n \), if \( |y| \equiv |x| \pmod{p^t} \) for some \( t \geq 0 \), then \( \sigma_d(x) \equiv \sigma_d(y) \pmod{p} \), for all \( d \in [0, p^t - 1] \).
Proof. (a) By definition, we get
\[
\sigma_{p^t}(x) \equiv \left\lfloor \frac{|x|}{p^t} \right\rfloor \pmod{p}
\equiv \left( \sum_{k \geq 0} |x| p^k \right) \pmod{p^t}
\equiv \left( \sum_{k \geq 0} (p^t)^k \right) \pmod{p^t}
\equiv \left( \frac{|x|}{1} \right) \pmod{p^t}
\equiv \left( \frac{|x|}{t} \right) \pmod{p^t}
\]
by Theorem 2.4

(b) Let \( x, y \in \{0, 1\}^n \) such that \( |y| \equiv |x| \pmod{p^t} \). So we have \( |y|_k = |x|_k \), for all \( k \in [0, t - 1] \).
Further, for any \( d \in [0, p^t - 1] \), we have \( d_k = 0 \) for all \( k \geq t \). So by definition, we get
\[
\sigma_d(y) \equiv \left( \frac{|y|}{d} \right) \pmod{p^t}
\equiv \prod_{k \geq 0} \left( \frac{|y|_k}{d_k} \right) \pmod{p^t}
\equiv \prod_{k = 0}^{t-1} \left( \frac{|y|_k}{d_k} \right) \pmod{p^t}
\equiv \prod_{k = 0}^{t-1} \left( \frac{|x|_k}{d_k} \right) \pmod{p^t}
\equiv \left( \frac{|x|}{d} \right) \pmod{p^t}
\equiv \sigma_d(x) \pmod{p^t}.
\]

\[\Box\]

**Integer-valued polynomials.** The integer-valued polynomials\(^4\) are precisely those polynomials \( P(Z) \in \mathbb{Q}[Z] \) such that \( P(Z) \subseteq \mathbb{Z} \). For any \( k \in \mathbb{N} \), consider the degree-\( k \) Newton polynomial defined as \( \binom{Z}{k} = \frac{1}{k!} \cdot Z(Z - 1) \cdots (Z - k + 1) \in \mathbb{Z}[Z] \). It is clear that as a function on \( \mathbb{Z} \), \( \binom{Z}{k} \) is \( \mathbb{Z} \)-valued, for all \( k \in \mathbb{N} \). The following lemma is folklore. The first mention of this result could be attributed to a letter by James Gregory to John Collins dated November 23, 1670 [tur59]. (See, for instance, Cahen and Chabert [CC97, Corollary I.1.2].)

**Lemma 2.6 (Folklore, [tur59],[CC97, Corollary I.1.2]).** Let \( d \in \mathbb{N} \) and \( I \subseteq \mathbb{N} \) be an interval with \( |I| = d + 1 \). For any function \( f : I \to \mathbb{N} \), there exists a unique polynomial \( Q_f(Z) \) that is a \( \mathbb{Z} \)-linear combination of \( \binom{0}{k}, \ldots, \binom{Z}{d} \) such that \( Q_f = f \).

Finally, yet another abbreviation. For any \( a_1, \ldots, a_k \in \{0, 1\} \) and \( n_1, \ldots, n_k \in \mathbb{N} \), we denote the binary vector \( a_1^{n_1} \cdots a_k^{n_k} \) := \( (a_1, \ldots, a_1, \ldots, a_k, \ldots, a_k) \in \{0, 1\}^{n_1+\cdots+n_k} \).

### 3 Finite-degree \( \mathbb{Z} \)-closure of a single layer

In this section, we will prove Theorem 1.5. Towards this, the following is an easy but important proposition.

---

\(^4\)See for instance, Cahen and Chabert [CC97] for a detailed account of integer-valued polynomials.
Proposition 3.1. Let $d \in [0, n]$ and $\ell = \ell_p(d)$.

(a) If $i \not\in [d, n - d]$, then $Z_{n,d}(i) = \{i\}$.

(b) If $i \in [d, n - d]$, then $Z_{n,d}(i) \subseteq i \oplus p^\ell$.

Proof. (a) Suppose $i < d$. For any $j > i$, the multilinear polynomial $X_1 \cdots X_{i+1}$ vanishes on $i$, does not vanish at $1^{i0^{n-j}} \in j$ and has degree at most $d$. So $j \not\in Z_{n,d}(i)$. Now consider any $j < i$. Then there exists $t \in [0, \ell - 1]$ such that $j_t < i_t$. Then $\sigma_{p^t}(X_1, \ldots, X_n) - i_t$ is a multilinear polynomial of degree $p_t^\ell \leq p^{\ell-1} \leq d$, which is zero on $i$ and nonzero on $j$, since $j_t \not\equiv i_t$. So $j \not\in Z_{n,d}(i)$. Thus $Z_{n,d}(i) = \{i\}$, if $i < d$. By Proposition 2.2 (b), we then also get $Z_{n,d}(i) = \{i\}$, if $i > n - d$. Hence $Z_{n,d}(i) \subseteq i \oplus p^\ell$.

(b) Consider any $j \not\in i \oplus p^\ell$. So $j \not\equiv (i \mod p^\ell)$. Then there exists $t \in [0, \ell - 1]$ such that $j_t \not\equiv i_t$. Then $\sigma_{p^t}(X_1, \ldots, X_n) - i_t$ is a multilinear polynomial which is zero on $i$ and nonzero on $j$, with degree at most $p_t^\ell \leq p^{\ell-1} \leq d$. So $j \not\in Z_{n,d}(i)$. Thus $Z_{n,d}(i) \subseteq i \oplus p^\ell$.

We will also need the following technical lemma.

Lemma 3.2. Let $n \in \mathbb{N}$, $i \in [0, n]$, and $\ell = \ell_p(i)$.

(a) There exists $h_i(X) \in \mathbb{F}_p[X]$ such that $\deg h_i \leq p^\ell - 1$ and

$$h_i(x) = 0 \iff |x| \equiv i \mod p^\ell.$$

(b) For any $j \in i \oplus p^\ell$, $j > i$, there exist $r_{i,j}(X) \in \mathbb{F}_p[X]$ such that $\deg r_{i,j} \leq j - i - p^\ell$ and

$$r_{i,j}(x) = \begin{cases} 0, & |x| \in [i + 1, j - 1], |x| \equiv i \mod p^\ell \\ \not\equiv 0, & |x| = j \end{cases}$$

Let us first assume Lemma 3.2 and prove Theorem 1.5. We will also need a result characterizing the duals of Reed-Muller codes over the Boolean cube.

The duals of Reed-Muller codes over $\mathbb{F}_q$ (where $q$ is a power of $p$) were determined by Delsarte, Goethals, and MacWilliams [DGM70], who remark that it could be readily obtained from Kasami, Lin, and Peterson [KLP68], and is also mentioned in some unpublished notes of Lin [Lin]. Beelen and Datta [BD18], using a different argument, described these duals more generally over finite grids in $\mathbb{F}_q$. The characterization of these duals over the Boolean cube could be obtained by the proofs in either of the above works; in fact, it is a special case of [BD18, Theorem 5.7].

For $d \in [0, n]$, the Reed-Muller code (over the Boolean cube $\{0, 1\}^n \subseteq \mathbb{F}_p$) with degree parameter $d$ is defined as

$$\text{RM}_p(n, d) = \left\{ P := [P(a)]_{a \in \{0,1\}^n} : P(X) \in \text{span}\{X^\alpha : \alpha \in \{0,1\}^n\}, \deg P \leq d \right\}.$$  

Theorem 3.3 (Dual of Reed-Muller code [DGM70, KLP68, Lin],[BD18, Theorem 5.7]).

For every $d \in [0, n]$,

$$\text{RM}_p(n, d)^\perp = \left\{ [(-1)^{|a|}Q(a)]_{a \in \{0,1\}^n} : Q(X) \in \text{span}\{X^\alpha : \alpha \in \{0,1\}^n\}, \deg Q \leq n - d - 1 \right\} = \left\{ \text{diag}((-1)^{|a|} : a \in \{0,1\}^n) \cdot [Q(a)]_{a \in \{0,1\}^n} : Q \in \text{RM}_p(n, n - d - 1) \right\}.$$  

For any $v \in \mathbb{F}_p^n$, define $\text{supp}(v) = \{i \in [n] : v_i \neq 0\}$. Similarly, for any function $f : A \to \mathbb{F}_p$ (where $A$ is some set), define $\text{supp}(f) = \{a \in A : f(a) \neq 0\}$.  

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Remark 3.4. Note that for any \( \lambda_i \in \mathbb{F}_p \setminus \{0\}, \ i \in [n], \) and \((v_1, \ldots, v_n) \in \mathbb{F}_p^n, \) we have \( \text{supp}(v_1, \ldots, v_n) = \text{supp}(\lambda_1 v_1, \ldots, \lambda_n v_n). \)

Consider the standard dot product for vectors in \( \mathbb{F}_p^n: \ v \cdot w = \sum_{i \in [n]} v_i w_i \) for \( v, w \in \mathbb{F}_p^n. \) The following is a standard fact from linear algebra, which follows from the properties of duals of linear subspaces with respect to the dot product.\(^5\)

**Fact 3.5.** Let \( W \subseteq \mathbb{F}_p^n \) be a linear subspace, and \( S \subseteq [n], \ j \in [n]. \) The following are equivalent.

- For any \( w \in W, \) if \( w_i = 0 \) for all \( i \in S, \) then \( w_j = 0. \)
- There exists \( v \in W^\perp \) such that \( j \in \text{supp}(v) \subseteq \{j\} \cup S. \)

We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** If \( i \not\in [d, n - d], \) then we are done by Proposition 3.1 (a).

Now suppose \( i \in [d, n - d]. \) By Proposition 3.1 (b), we know that \( Z\text{-cl}_{n,d}(i) \subseteq i \oplus p^\ell. \) We now prove that \( i \oplus p^\ell \subseteq Z\text{-cl}_{n,d}(i). \) By Proposition 2.2 (b), it is enough to prove that \( j \in Z\text{-cl}_{n,d}(i), \) for all \( j \in i \oplus p^\ell, \ j > i. \) Further, by Proposition 2.2 (c), it is enough to consider \( i = d, \ j = d + p^\ell k. \) Furthermore, by Proposition 2.2 (a), it is enough to consider \( i = d, \ n = j = d + p^\ell k. \) Therefore, we need to prove that \( n \in Z\text{-cl}_{n,d}(d). \)

Consider \( h_d(\mathcal{X}), r_{d,n}(\mathcal{X}) \in \mathbb{F}_p[\mathcal{X}] \) as given by Lemma 3.2. So we have

\[
\begin{align*}
  h_d(x) &= 0 \iff |x| \not\equiv d \pmod{p^\ell} \\
  \text{and} \quad r_{d,n}(x) &= \begin{cases} 
  0, & |x| \in [d + 1, n], |x| \equiv d \pmod{p^\ell} \\
  \neq 0, & x = 1^n 
  \end{cases}
\end{align*}
\]

and further, \( \deg h_d \leq p^\ell - 1 \) and \( \deg r_{d,n} \leq n - d - p^\ell. \) We now give two (essentially equivalent) arguments.

(a) We first note the following sequence of equivalences.

\[
1^n \in Z\text{-cl}_{n,d}(d)
\]

\[
\iff \exists f \in \text{RM}_p(n, d) : f(1^n) \neq 0, f|_d = 0
\]

\[
\iff \exists g \in \text{RM}_p(n, d)^+ : 1^n \in \text{supp}(g) \subseteq \{1^n\} \cup \{d\}
\]

\[
\iff \exists g \in \text{RM}_p(n, n - d - 1) : 1^n \in \text{supp}(g) \subseteq \{1^n\} \cup \{d\}
\]

by Fact 3.5

Now define \( g(\mathcal{X}) = h_d(\mathcal{X})r_{d,n}(\mathcal{X}). \) Then \( \deg g = \deg h_d + \deg r_{d,n} \leq (p^\ell - 1) + (n - d - p^\ell) = n - d - 1. \) So \( g \in \text{RM}_p(n, n - d - 1). \) Further, by (1) and (2), we have \( 1^n \in \text{supp}(g) \subseteq \{1^n\} \cup \{d\}. \)

Thus \( n \in Z\text{-cl}_{n,d}(d). \)

(b) Suppose \( n \not\in Z\text{-cl}_{n,d}(d). \) Let \( P(\mathcal{X}) \in \mathbb{F}_p[\mathcal{X}] \) be a polynomial such that \( \deg P \leq d, \ P|_d = 0 \) and \( P(1^n) \neq 0. \) Define \( Q(\mathcal{X}) = P(\mathcal{X})h_d(\mathcal{X})r_{d,n}(\mathcal{X}). \) Then \( Q(x) \neq 0 \) if and only if \( x = 1^n. \) So by the Combinatorial Nullstellensatz (Theorem 2.1),

\[
Q(\mathcal{X}) = \alpha \prod_{t \in [n]} X_t, \quad \text{for some } \alpha \neq 0,
\]

and so \( \deg Q = n. \) But

\[
\deg Q \leq \deg P + \deg h_d + \deg r_{d,n} \leq d + (p^\ell - 1) + (n - d - p^\ell) = n - 1,
\]

a contradiction. Thus \( n \in Z\text{-cl}_{n,d}(d). \)

\(\Box\)

Note that in characteristic \( p, \) the dot product is not an inner product. In fact, it is not even nondegenerate; there exist \( v \in \mathbb{F}_p^n, \ v \neq 0 \) such that \( v \cdot v = 0. \) But nevertheless, it is a bilinear form, with respect to which we can consider dual subspaces.
3.1 Proof of Lemma 3.2

Finally, we close this section by proving Lemma 3.2.

Proof of Lemma 3.2. (a) Let \( i \in [0, n] \) and \( \ell = \ell_p(i) \). Define

\[
h_i(X) = \prod_{t=0}^{\ell-1} (1 - (\sigma_p^t(x) - i_t)p^{-1}) \in \mathbb{F}_p[X].
\]

Then \( \deg h_i = (p - 1) \sum_{t=0}^{\ell-1} p^t = p^\ell - 1 \). Further, it is clear from the definition that \( h_i(x) = 0 \) if and only if \( |x| \equiv i \pmod{p^\ell} \).

(b) Let \( j \in i \oplus p^\ell \), \( j > i \), and so \( j = i + p^\ell k \), for some \( k \in \mathbb{Z}^+ \). Let \( I = \{i, i + p^\ell, \ldots, i + p^\ell k = j\} \). Also, let \( m = \ell_p(j) \). Then we have, for every \( r \in I \),

\[
r_t \begin{cases} = i_t, & t \in [0, \ell - 1] \\ \in [0, p - 1], & t \in [\ell, m - 1] \\ = 0, & t \geq m \end{cases}
\]

By Lemma 2.6, there exists a unique polynomial \( Q(Z) \) which is a \( \mathbb{Z} \)-linear combination of \((Z_0), \ldots, (Z_{k-1})\) such that \( Q(z) = 0 \) for all \( z \in [1, k-1] \), and \( Q(k) = 1 \). For any \( x \in \mathbb{L} \), let \( |x|^t := \sum_{t=\ell}^{m-1} |x|p^{t-\ell} \); then we have \( |x|^t = |x|_{\ell+t}, t \in [0, m - \ell - 1] \). Now for any \( u \in [0, k-1] \) and \( x \in \mathbb{L} \), we have

\[
\binom{|x|^t}{u} \equiv \prod_{t=0}^{\ell-1} \binom{|x|^t}{u_t} \pmod{p}
\]

by Theorem 2.4

\[
\equiv \prod_{t=0}^{m-\ell-1} \binom{|x|_{\ell+t}}{(p^\ell u)_{\ell+t}} \pmod{p}
\]

since \( u_t = (p^\ell u)_{\ell+t}, t \in [0, m - \ell - 1] \)

\[
\equiv \prod_{t=0}^{\ell-1} \binom{|x|_t}{(p^\ell u)_t} \prod_{t=\ell}^{m-1} \binom{|x|_t}{(p^\ell u)_t} \pmod{p}
\]

since \( (p^\ell u)_t = 0, t \in [0, \ell - 1] \)

\[
\equiv \binom{|x|^t}{p^\ell u} \pmod{p}
\]

by Theorem 2.4

\[
\equiv \sigma_{p^\ell u}(x) \pmod{p}.
\]

Suppose \( Q(Z) = \sum_{u=0}^{k-1} c_u (Z_u) \), where \( c_0, \ldots, c_{k-1} \in \mathbb{Z} \). Define \( r_{i,j}(X) = \sum_{u=0}^{k-1} c_u \sigma_{p^\ell u}(X) \in \mathbb{F}_p[X] \). Clearly, \( \deg r_{i,j} = p^\ell (k - 1) = j - i - p^\ell \).

Note that for any \( x \in \mathbb{L} \) and \( v \in [0, k] \), we have \( |x| = i + p^\ell v \) if and only if \( |x|^t = v \). So, for any \( x \in \mathbb{L} \), we have

\[
r_{i,j}(x) = \sum_{u=0}^{k-1} c_u \sigma_{p^\ell u}(x) = \sum_{u=0}^{k-1} c_u \binom{|x|^t}{u} = Q(|x|^t) = \begin{cases} 0, & |x| \in [i + 1, j - 1] \\ 1, & |x| = j \end{cases}
\]

This completes the proof. \( \square \)
4 Finite-degree Z-closures of arbitrary symmetric sets

In this section, we will proceed to show that the finite-degree Z-closures of symmetric sets are equal to the symmetric closures, under some conditions. This would imply that these Z-closures can be computed in polynomial time. However, a more explicit description of the finite-degree Z-closures of arbitrary symmetric sets à la Theorem 1.5 is still in want.

4.1 Some observations, a conjecture, and the main theorem

It is interesting to note that, in characteristic zero (that is, over the field \( \mathbb{R} \)), for any symmetric set \( E \subseteq [0, n] \), the event \( j \notin Z\text{-}cl_{n,d}(E) \) is witnessed by a polynomial \( P(X) \in \mathbb{R}[X] \) having a special form. This is immediate from the characterization of finite-degree Z-closures of symmetric sets, over \( \mathbb{R} \), given by the second author [Ven21]. Let XOR = \( \{X_i - X_j : i, j \in [n], i \neq j\} \).

**Observation 4.1 (Follows from [Ven21]).** Let \( d \in [0, n] \) and \( E \subseteq [0, n] \). Over the reals, if \( j \notin Z\text{-}cl_{n,d}(E) \), then there exists a polynomial \( P(X) = \ell_1(X) \cdots \ell_k(X)\sigma(X) \in \mathbb{R}[X] \), where \( \ell_1, \ldots, \ell_k \in \text{XOR} \) and \( \sigma \) is a symmetric polynomial, such that \( \deg P \leq d \), \( P|_E = 0 \) and \( P|_E \neq 0 \).

What we see now is that we can infer such statements, of the kind of Observation 4.1, for Z-closures over \( \mathbb{F}_p \) as well. More precisely, from Theorem 1.5 and the proof of Proposition 3.1, we have the following. For any \( S, T \subseteq [n] \), \( S \cap T = \emptyset \), define a generalized monomial to be the polynomial \( X^{(S; T)} := \prod_{s \in S} X_s \prod_{t \in T}(1 - X_t) \).

**Observation 4.2.** Let \( d \in [0, n] \), \( \ell = \ell_p(d) \).

(a) Let \( i \in [d, n - d] \). If \( j \notin Z\text{-}cl_{n,d}(i) \), then there exists a symmetric polynomial \( \sigma(X) \in \mathbb{F}_p[X] \) such that \( \deg \sigma \leq d \), \( \sigma|_E = 0 \) and \( \sigma|_E \neq 0 \). In other words, \( Z\text{-}cl_{n,d}(i) = \text{sym-cl}_d(n, i) \).

(b) Let \( i \notin [d, n - d] \). If \( j \notin Z\text{-}cl_{n,d}(i) \), then there exists a polynomial \( P(X) \in \mathbb{F}_p[X] \) that is either symmetric or a generalized monomial, such that \( \deg P \leq d \), \( P|_E = 0 \) and \( P|_E \neq 0 \).

Inspired by Observation 4.2, a reasonable conjecture about Z-closures of symmetric sets over \( \mathbb{F}_p \) is the following.

**Conjecture 4.3.** Let \( d \in [0, n] \) and \( E \subseteq [0, n] \). For \( j \in [0, n] \), if \( j \notin Z\text{-}cl_{n,d}(E) \), then there exists a polynomial \( P(X) = m_1(X) \cdots m_k(X)\sigma(X) \in \mathbb{F}_p[X] \), where \( m_1(X), \ldots, m_k(X) \) are generalized monomials and \( \sigma(X) \) is a symmetric polynomial, such that \( \deg P \leq d \), \( P|_E = 0 \) and \( P|_E \neq 0 \).

Unfortunately, Conjecture 4.3 is not true; the following is a counterexample.

**Counterexample 4.4.** We see that over \( \mathbb{F}_2 \), \( 0 \notin Z\text{-}cl_{5,2} \{1, 4\} \). A witness polynomial is the degree-2 polynomial

\[
(1 + X_1 + X_2 + X_3 + X_4)(1 + X_2 + X_3 + X_4 + X_5),
\]

which vanishes on \{1, 4\}, but does not vanish at \( 0^5 \). However, this witness polynomial is not of the form claimed in Conjecture 4.3. Let us now show that there is no witness polynomial of the claimed form. The following cover all the possibilities of a potential witness polynomial of the claimed form.

- Firstly, it is easy to see that there is no nonzero polynomial of degree at most 1 that vanishes on \{1, 4\}, and does not vanish at \( 0^5 \). So a potential witness polynomial must have degree 2.
• A generalized monomial of degree 2 has zero set of the form \(\{x \in \{0,1\}^n : x_i = a \text{ or } x_j = b\}\), for \(i, j \in [5], i \neq j\) and \(a, b \in \{0,1\}\), which clearly does not contain \(\{1,4\}\). A product of two generalized monomials of degree 1, which is not a generalized monomial of degree 2, is simply \(X_i(1 - X_i)\) for some \(i \in [5]\), which vanishes everywhere.

• A product of a generalized monomial of degree 1 and a symmetric polynomial of degree 1 has zero set of the form \(\{x \in \{0,1\}^5 : x_i = a \text{ or } |x| = b \pmod{2}\}\), for \(i \in [5], a, b \in \{0,1\}\), which again does not contain \(\{1,4\}\).

• A product of two distinct symmetric polynomial of degree 2 will vanish everywhere. The only other possibility of a symmetric polynomial of degree 2 has zero set of the form \(\{|x|_1 = a\}\), for \(a \in \{0,1\}\), which either contains \(\{0,1,4\}\) or does not contains \(\{1,4\}\).

This completes the analysis.

Nevertheless, the form of the above witness polynomial motivates the following weaker conjecture, which we leave open.

**Conjecture 4.5.** Let \(d \in [0,n]\) and \(E \subseteq [0,n]\). For \(j \in [0,n]\), if \(j \notin \text{Z-cl}_{n,d}(E)\), then there exists a polynomial \(P(X) = \ell_{\overline{1}}(X) \cdots \ell_{\overline{k}}(X)\sigma(X) \in F_p[X]\), where deg \(\ell_1 = \cdots = \deg \ell_k = 1\), and \(\sigma(X)\) is a symmetric polynomial, such that deg \(P \leq d\), \(P|_{\overline{i}} = 0\) and \(P|_{\overline{j}} \neq 0\).

Theorem 1.6 proves a special case of Conjecture 4.5, but with a stronger assertion about the witness polynomial. Specifically, Theorem 1.6 states that for \(d \in \mathbb{N}, \ell = \ell_p(d)\), if \(n > 4p^\ell - 1\), then the following is true for any \(E \subseteq [d,n-d]\): if \(j \notin \text{Z-cl}_{n,d}(E)\), then there exists a symmetric polynomial \(\sigma(X) \in F_p[X]\) such that deg \(\sigma \leq d\), \(\sigma|_{\overline{i}} = 0\) and \(\sigma|_{\overline{j}} \neq 0\).

**Remark 4.6.** The condition \(n > 4p^\ell - 1\) in the assumption of Theorem 1.6, where \(\ell = \ell_p(d)\), is simply a requirement for our proof. We believe the assertion of Theorem 1.6 is true even for all smaller values of \(n\), though we don’t have a proof yet.

An immediate corollary of Theorem 1.6 is the following.

**Corollary 4.7.** Let \(d \in \mathbb{N}, \ell = \ell_p(d)\). If \(n \geq 4p^\ell - 1\), then for any \(E \subseteq [d,n-d]\), \(\text{Z-cl}_{n,d}(E)\) can be computed in \(\text{poly}(n)\) time.

**Proof.** Let \(d \in [0,n]\), \(\ell = \ell_p(d)\). Consider any \(E \subseteq [0,n]\), \(j \in [0,n]\). By Theorem 1.6, it is enough to show that whether \(j \in \text{sym-cl}_{n,d}(E)\) can be decided in \(\text{poly}(n)\) time.

Clearly, every symmetric polynomial function, with degree at most \(d\), is a linear combination of \(\sigma_k, k \in [0,d]\). Appealing to Corollary 2.5 (a), the linear system of concern to us is

\[
\sum_{k=0}^{d} c_k \sigma_k(1^i0^{n-i}) = 0, \quad \text{for all } i \in E.
\]

This is a homogeneous system with \(d + 1 \leq n + 1\) variables \(c_k, k \in [0,d]\), and \(|E| \leq n\) constraints. The solution space of this system can thus be computed in \(\text{poly}(n)\) time (for instance, by Gaussian elimination). It is clear that \(j \in \text{sym-cl}_{n,d}(E)\) if and only if \(\sum_{k=0}^{d} c_k \sigma_k(1^i0^{n-j}) = 0\), for every solution \(c_k, k \in [0,d]\) of the system. Therefore, whether \(j \in \text{sym-cl}_{n,d}(E)\) can be decided in \(\text{poly}(n)\) time. 

\qed
4.2 The finite-degree symmetric closure in more detail

It is easy to see that again by Theorem 2.1, while considering the finite-degree symmetric closure sym-cl\(_{n,d}\), we can restrict \(d \in [0, n]\). For any \(d \in [0, n]\) and \(E \subseteq [0, n]\), let \(S_{n,d}(E)\) be the set of all symmetric polynomials with degree at most \(d\), that vanish at each point in \(E\). Recall that for any \(E \subseteq [0, n]\), we define sym-cl\(_{n,d}(E)\) = \(Z(S_{n,d}(E))\). Further, as mentioned earlier, sym-cl\(_{n,d}(E)\) is a symmetric set, and hence we identify (and denote) it by sym-cl\(_{n,d}(E)\) \(\subseteq [0, n]\).

Let us gather some interesting properties of the finite-degree symmetric closures. We begin with the following lemma.

**Lemma 4.8.** Let \(d \in [0, n]\) and \(f(X) = \sum_{u=0}^{d} c_u \sigma_u(X) \in \mathbb{F}_p[X]\). Define

\[
\begin{align*}
    f^+(X) &= \sum_{u=0}^{d} \left( \sum_{v=u}^{d} (-1)^{v-u} c_v \right) \sigma_u(X), \\
    f^-(X) &= \sum_{u=0}^{d} \left( \sum_{v=u}^{d} c_v \right) \sigma_u(X).
\end{align*}
\]

Then

(a) \(f(X) = (f^+)^-(X) = (f^-)^+(X)\).

(b) if \(j, j+1 \in [0, n]\), then \(f|_{j} = 0\) if and only if \(f^+|_{j+1} = 0\).

(c) if \(j, j-1 \in [0, n]\), then \(f|_{j} = 0\) if and only if \(f^-|_{j-1} = 0\).

**Proof.** (a) The assertion follows immediately from the following elementary fact, which is precisely Möbius inversion (see, for instance, Stanley [Sta11, Chapter 3, Section 3.7]) for \([0, d]\) with the obvious linear order.

**Fact.** Let \(d \in \mathbb{N}\), and for \(i, j \in [0, d]\), let \(a_{i,j} = 1\) if \(i \leq j\), and \(a_{i,j} = 0\) if \(i > j\). Then the integer matrices \(M_d = [a_{i,j}]_{i,j \in [0,d]}\) and \(N_d = [(-1)^{j-i}a_{i,j}]_{i,j \in [0,d]}\) both have determinant 1, and are inverses of each other.

(b) Consider any \(f(X) = \sum_{u=0}^{d} c_u \sigma_u(X) \in \mathbb{F}_p[X]\). By an abuse of notation, consider the integer representatives \(c_u \in [0, p-1]\), \(u \in [0, d]\), and let \(Q(Z) = \sum_{u=0}^{d} c_u \binom{Z}{u}\). Let

\[
Q^+(Z) := Q(Z-1)
\]

\[= \sum_{u=0}^{d} c_u \binom{Z-1}{u}
\]

\[= \sum_{u=0}^{d} c_u \left( \binom{Z}{u} - \binom{Z-1}{u-1} \right) \quad \text{by Pascal’s triangle, where } \binom{Z}{-1} := 0
\]

\[= \sum_{u=0}^{d} \left( \sum_{v=u}^{d} (-1)^{v-u} c_v \binom{Z}{u} \right).
\]
Let \( j \in [0, n] \) such that \( j + 1 \in [0, n] \). Then we have
\[
\begin{align*}
f(x) &= 0, \quad \text{for all } x \in j \\
\iff Q(j) &\equiv 0 \pmod{p} \\
\iff Q^+(j + 1) &= Q(j) \equiv 0 \pmod{p} \\
\iff f^+(x) &= 0, \quad \text{for all } x \in j + 1
\end{align*}
\]

(c) By Item (a), we have \( f(X) = (f^-)^+(X) \). So the assertion follows by Item (b). \( \square \)

The following property of finite-degree symmetric closures then follows quickly.

**Proposition 4.9.** For any \( d, k, j \in [0, n] \) and \( E \subseteq [0, n] \) such that \( E + k \subseteq [0, n] \), we have \( j \in \text{sym-cl}_{n,d}(E) \) if and only if \( j + k \in \text{sym-cl}_{n,d}(E + k) \).

**Proof.** It is enough to prove that for \( E \subseteq [0, n] \) such that \( E + 1 \subseteq [0, n] \), we have \( j \in \text{sym-cl}_{n,d}(E) \) if and only if \( j + 1 \in \text{sym-cl}_{n,d}(E + 1) \). Again, it is enough to show that if \( j \in \text{sym-cl}_{n,d}(E) \), then \( j + 1 \in \text{sym-cl}_{n,d}(E + 1) \). The argument for the converse is similar.

Let \( f(X) \in \mathcal{S}_{n,d}(E + 1) \). By Lemma 4.8, we have \( f^-(X) \in \mathcal{S}_{n,d}(E) \). So \( f^-|_{j+1} = 0 \). Again by Lemma 4.8, this implies \( f|_{j+1}^+ = (f^-)^+|_{j+1} = 0 \). This completes the proof. \( \square \)

Let us now gather some fairly straightforward lemmas, which are important for our results.

**Lemma 4.10.** Let \( d \in [0, n] \), \( \ell = \ell_p(d) \).

(a) For any \( E \subseteq [0, n] \), if \( j \in E \) such that \( j + p^\ell \in [0, n] \), then
\[
\text{sym-cl}_{n,d}(E) = \text{sym-cl}_{n,d}(n, E \cup \{j + p^\ell\} \setminus \{j\}).
\]

(b) For any \( E \subseteq [0, n] \), \( j \in [0, n] \), we have
\[
j \in \text{sym-cl}_{n,d}(E) \iff j \oplus p^\ell \subseteq \text{sym-cl}_{n,d}(E).
\]

(c) For any \( E \subseteq [d, n - d] \), if \( j \in E \) such that \( j + p^\ell \in [d, n - d] \), then
\[
\text{Z-cl}_{d}(E) = \text{Z-cl}_{d}(E \cup \{j + p^\ell\} \setminus \{j\}).
\]

(d) For any \( E \subseteq [d, n - d] \), \( j \in [d, n - d] \), we have
\[
j \in \text{Z-cl}_{n,d}(E) \iff j \oplus p^\ell \subseteq \text{Z-cl}_{n,d}(E).
\]

**Proof.** It is easy to note that every symmetric polynomial function, with degree at most \( d \), is a linear combination of \( \sigma_k \), \( k \in [0, d] \). So by Corollary 2.5 (b), for any symmetric polynomial function \( f \) with \( \deg f \leq d \), and any \( x, y \in \{0, 1\}^n \) with \( |y| \leq |x| \oplus p^\ell \), we have \( f(y) = f(x) \). This concludes the proof of Item (a) and Item (b).

By Theorem 1.5, for any polynomial \( f(X) \in \mathbb{F}_p[X] \) with \( \deg f \leq d \), and any \( x, y \in \{0, 1\}^n \) with \( |x| \in [d, n - d] \), \( |y| \leq |x| \oplus p^\ell \in [d, n - d] \), we have \( f(y) = 0 \) if and only if \( f(x) = 0 \). This concludes the proof of Item (c) and Item (d). \( \square \)

For any \( E \subseteq [d, n - d] \) and any interval \( I \subseteq \mathbb{N} \) with \( |I| = p^\ell \), define \( E_I \subseteq I \) as follows: for any \( j \in I \), define \( j \in E_I \) if \( j + kp^\ell \in E \) for some \( k \in \mathbb{Z} \). An immediate consequence of Lemma 4.10 is the following observation.

**Observation 4.11.** Let \( d \in [0, n] \) and \( \ell = \ell_p(d) \). For any \( E \subseteq [d, n - d] \) and any interval \( I \subseteq [d, n - d] \) with \( |I| = p^\ell \),
\[
\text{sym-cl}_{n,d}(E) = \text{sym-cl}_{n,d}(E_I) \quad \text{and} \quad \text{Z-cl}_{n,d}(E) = \text{Z-cl}_{n,d}(E_I).
\]
4.3 The main lemmas and proof of the main theorem

We will now characterize $Z\text{-cl}_{n,d}(E)$ for every $E \subseteq [d,n-d]$, when $n$ is large. Let us recall the main theorem.

**Theorem 1.6** (Finite-degree Z-closures of symmetric sets). Let $d \in \mathbb{N}$, $\ell = \ell_p(d)$. If $n \geq 4p^\ell - 1$, then for any $E \subseteq [d,n-d]$, we have $Z\text{-cl}_{n,d}(E) = \text{sym-cl}_{n,d}(E)$.

We will need the results obtained in Subsection 4.2, as well as a couple more. We will state these results, prove Theorem 1.6, and then finish the proofs of the results.

The first result characterizes the duals of a class of linear codes. This class of codes (called weighted Reed-Muller codes) was first introduced over $\mathbb{F}_q^n$ by Sørensen [Sør92], who also gave a description of their duals. The result we require is over a finite grid in $\mathbb{F}_p^n$, and is a special case of a result by Camps, López, Matthews and Sarmiento [CLMS20, Theorem 2.2]. Consider the indeterminates $\mathbb{T} = (T_0, \ldots, T_r)$. For any $P(\mathbb{T}) \in \mathbb{F}_p[\mathbb{T}]$, define

$$w\text{deg}_p(P) = \deg P(\sigma_{p^0}, \ldots, \sigma_{p^r}) = \max \left\{ \sum_{t=0}^{r} \alpha_t p^t : \text{coeff}(\mathbb{T}^\alpha, P) \neq 0 \right\}.$$  

**Theorem 4.13** ([CLMS20, Theorem 2.2]). Consider the finite grid $S = S_0 \times \cdots \times S_r = [0,p-1]^r \times [0,k] \subseteq \mathbb{F}_p^{r+1}$, for some $r \in \mathbb{N}$, $k \in [1,p-1]$. Let $N = \sum_{t \in [0,r-1]}(|S_t| - 1)p^t = (k + 1)p^r - 1$. Let

$$W(S,d) = \left\{ P := [P(t)]_{t \in S} : P(\mathbb{T}) \in \text{span}\{\mathbb{T}^\alpha : \alpha \in S\}, w\text{deg}_p(P) \leq d \right\}.$$  

Then there exists $\gamma_t \in \mathbb{F}_p \setminus \{0\}$ for every $t \in S$, such that

$$W(S,d)^\perp = \left\{ [\gamma_t Q(t)]_{t \in S} : Q(\mathbb{T}) \in \text{span}\{\mathbb{T}^\beta : \beta \in S\}, w\text{deg}_p(Q) \leq N - d - 1 \right\}$$

$$= \left\{ \text{diag}(\gamma_t : t \in S) \cdot [Q(t)]_{t \in S} : Q \in W(S,N - d - 1) \right\}.$$  

The second result characterizes, in a special case, when $1^n$ is in the Z-closure of a symmetric set in $\{0,1\}^n$.

**Lemma 4.14.** Let $n = (k + 1)p^r - 1$ for some $k \in [1,p-1]$, and $d \in [0,n]$. Then for any $E \subseteq [0,n]$, $n \in Z\text{-cl}_{n,d}(E)$ if and only if $n \in \text{sym-cl}_{n,d}(E)$.

**Remark 4.15.** It is easy to see that the assertion of Lemma 4.14 is not true for general $n \in \mathbb{Z}^+$. In Counterexample 4.4, we see that over $\mathbb{F}_2$, $0 \notin Z\text{-cl}_{5,2}([1,4])$. So by Proposition 2.2 (b), we get $5 \notin Z\text{-cl}_{5,2}([1,4])$. Note that $5 = 1 + 2^2$, and so 5 is not of the form required in the assumption of Lemma 4.14. Now trivially, we have $1 \in \text{sym-cl}_{5,2}([1,4])$. So by Lemma 4.10 (b), we get $5 \in 1 + 4 \subseteq \text{sym-cl}_{5,2}([1,4])$, since $\ell_2(2) = 2$.

We are now ready to prove our main theorem.

**Proof of Theorem 1.6.** Clearly $Z\text{-cl}_{n,d}(E) \subseteq \text{sym-cl}_{n,d}(E)$. Now let us prove that $\text{sym-cl}_{n,d}(E) \subseteq Z\text{-cl}_{n,d}(E)$. Since $n \geq 4p^\ell - 1$, we get $n \geq 2p^\ell + 2d - 1$; this means $[d,n-d] \geq 2p^\ell$.

Consider any $j \in \text{sym-cl}_{n,d}(E)$. Let $j' \in (j + p^\ell) \cap [d + p^\ell, d + 2p^\ell - 1]$. By Lemma 4.10 and Theorem 1.5, it is clear that $j' \in \text{sym-cl}_{n,d}(E)$, and further, that it is enough to show $j' \in Z\text{-cl}_{n,d}(E)$. Let $E' = E[j' - p^\ell, j' - 1]$. By Observation 4.11, we have $j' \in \text{sym-cl}_{d}(n, E')$ and we need to show $j' \in Z\text{-cl}_{n,d}(E')$.

We have two cases.
Case (i) \( j' - p^f \in E' \). Since \( j' \geq d + p^f \), we have \( j' - p^f \geq d \). Since \( j' \leq d + 2p^f - 1 \), we have \( j' - p^f \leq d + p^f - 1 \leq d + 2p^f - 1 \leq n - d \). Thus \( j' - p^f \in [d, n - d] \). So by Theorem 1.5, we have \( j' \in (j' - p^f) \oplus p^f \subseteq \text{Z-cl}_{n,d}(E') \).

Case (ii) \( j' - p^f \notin E' \). Then \( E' \subseteq [j' - (p^f - 1), j'] \), and so \( E' - (j' - (p^f - 1)) \subseteq [0, p^f - 2] \). Since \( j' \in \text{sym-cl}_d(n, E') \), we have \( j' \in \text{sym-cl}_d(E') \), and so by Proposition 4.9, \( p^f - 1 = (j' - (p^f - 1)) \in \text{sym-cl}_d(E' - (j' - (p^f - 1))) \). This gives \( p^f - 1 \in \text{sym-cl}_d(p^f - 1, E' - (j' - (p^f - 1))) \). By Lemma 4.14, we get \( p^f - 1 \in \text{Z-cl}_{p^f - 1, d}(E' - (j' - (p^f - 1))) \). Then by Proposition 2.2 (c), we get \( j' = p^f - 1 + (j' - (p^f - 1)) \in \text{Z-cl}_{j', d}(E') \). And finally, by Proposition 2.2 (a), we get \( j' \in \text{Z-cl}_{n,d}(E') \).

We conclude by proving Lemma 4.14. Towards this, let us prove yet another smaller result.

**Lemma 4.16.** Let \( n = (k + 1)p^r - 1 \) for some \( k \in [1, p - 1] \), and \( d \in [0, n] \). If \( n \in \text{sym-cl}_{n,d}(E) \), then for every \( P(T) \in F_p[T] \) satisfying \( P(i_0, \ldots, i_r) = 0 \) for all \( i \in E \), and \( \text{wdeg}_p(P) \leq d \), we have

\[
P\left(\left(\overbrace{(p-1, \ldots, p-1)}^{r \text{ times}}, k\right)\right) = 0.
\]

**Proof.** Consider any \( P(T) \in F_p[T] \) satisfying \( P(i_0, \ldots, i_r) = 0 \) for all \( i \in E \), and \( \text{wdeg}_p(P) \leq d \). Then \( P(\sigma_{p^u}(X), \ldots, \sigma_{p^u}(X)) \in F_p[X] \) is a symmetric polynomial with \( \text{deg}(P(\sigma_{p^u}, \ldots, \sigma_{p^u})) = \text{wdeg}_p(P) \leq d \). Further, for any \( x \in F \), we get \( P(\sigma_{p^u}(x), \ldots, \sigma_{p^u}(x)) = P([x]_0, \ldots, [x]_r) = 0 \). Since \( n \in \text{sym-cl}_{n,d}(E) \), this implies \( P(\sigma_{p^u}(1^n), \ldots, \sigma_{p^u}(1^n)) = 0 \). Further, since \( n = (k + 1)p^r - 1 = \sum_{u=0}^{r-1}(p-1)p^u + kp^r \), we have \( \sigma_{p^u}(1^n) = p - 1 \) for \( u \in [0, r - 1] \), and \( \sigma_{p^u}(1^n) = k \). This completes the proof.

We now have everything in place to prove Lemma 4.14.

**Proof of Lemma 4.14.** Clearly if \( n \in \text{Z-cl}_{n,d}(E) \), then \( n \in \text{sym-cl}_{n,d}(E) \).

Conversely, suppose \( n \in \text{sym-cl}_{n,d}(E) \). Note that since \( n = (k + 1)p^r - 1 \), we have \( \sigma_{p^u}(1^n) = n_u = (p - 1) \) for \( u \in [0, r - 1] \), and \( \sigma_{p^u}(1^n) = n_r = k \). By Lemma 4.16, we get

\[
P\left(\left(\overbrace{(p-1, \ldots, p-1)}^{r \text{ times}}, k\right)\right) = 0,
\]

for every \( P(T) \in F_p[T] \) satisfying \( P(i_0, \ldots, i_r) = 0 \) for all \( i \in E \), and \( \text{wdeg}_p(P) \leq d \). So by Theorem 4.13, Fact 3.5 and Remark 3.4, this implies that there exists \( Q(T) \in F_p[T] \) such that \( \text{wdeg}_p(Q) \leq n - d - 1 \) and

\[
\left(\overbrace{(p-1, \ldots, p-1)}^{r \text{ times}}, k\right) \in \text{supp}(Q) \subseteq \left\{ \left(\overbrace{(p-1, \ldots, p-1)}^{r \text{ times}}, k\right) \right\} \cup \{ (i_0, \ldots, i_r) : i \in E \}. \quad (4)
\]

Define \( R(X) = Q(\sigma_{p^u}(X), \ldots, \sigma_{p^u}(X)) \). Then \( \text{deg} R = \text{wdeg}_p(Q) \leq n - d - 1 \), and further by (4), we have

\[
1^n \in \text{supp}(R) \subseteq \{1^n\} \cup E.
\]

By Theorem 3.3, Fact 3.5 and Remark 3.4, this implies \( n \in \text{Z-cl}_{n,d}(E) \).
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