Perfect Italian domination on planar and regular graphs

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Abstract

A perfect Italian dominating function of a graph $G = (V,E)$ is a function $f : V \rightarrow \{0,1,2\}$ such that for every vertex $f(v) = 0$, it holds that $\sum_{u \in N(v)} f(u) = 2$, i.e., the weight of the labels assigned by $f$ to the neighbors of $v$ is exactly two. The weight of a perfect Italian function is the sum of the weights of the vertices. The perfect Italian domination number of $G$, denoted by $\gamma_{pI}(G)$, is the minimum weight of any perfect Italian dominating function of $G$. While introducing the parameter, Haynes and Henning (Discrete Appl. Math. (2019), 164–177) also proposed the problem of determining the best possible constants $c_G$ such that $\gamma_{pI}(G) \leq c_G \times n$ for all graphs of order $n$ when $G$ is in a particular class $\mathcal{G}$ of graphs. They proved that $c_G = 1$ when $\mathcal{G}$ is the class of bipartite graphs, and raised the question for planar graphs and regular graphs. We settle their question precisely for planar graphs by proving that $c_G = 1$ and for cubic graphs by proving that $c_G = 2/3$. For split graphs, we also show that $c_G = 1$. In addition, we characterize the graphs $G$ with $\gamma_{pI}(G)$ equal to 2 and 3 and determine the exact value of the parameter for several simple structured graphs. We conclude by proving that it is NP-complete to decide whether a given bipartite planar graph admits a perfect Italian dominating function of weight $k$.

1 Introduction

The motivation for the problem we study stems from the problem of deployment of military forces to guard several points of interest, modeled by an undirected graph. Such problems from different historical eras were described by ReVelle and Rosing [19] (see also Stewart [20]). For instance, the authors describe a defense-in-depth strategy by Emperor Constantine (Constantine the Great, 274–337) where units were deployed such that any city without a unit was to be neighbored by a city harbouring two units. The idea was that if the city without a unit was attacked, the neighboring city could dispatch a unit to protect it without becoming vulnerable itself. In this setting, the objective was to minimize the total number of units needed. Albeit overly simplified particularly for the modern era to be of practical use, these type of domination problems on graphs have resulted in interesting graph-theoretical problems that have attracted significant interest from the research community.

Let $G = (V,E)$ be a simple undirected graph. To reduce clutter, we can write an element $\{u,v\} \in E$ as $uv$. The open neighborhood of a vertex $v \in V$, denoted by $N(v)$, is the set of neighbors of $v$ excluding $v$ itself, i.e., $N(v) = \{u \mid uv \in E\}$. The degree of a vertex $v$ is the number of edges incident to it, i.e., $|N(v)|$. In particular, a vertex of degree one is a pendant

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and the vertex adjacent to a pendant vertex is a support. For the following discussion, let \( f : V \rightarrow \{0, 1, 2\} \) be a vertex-labeling of \( G \).

We say that \( f \) is a perfect Italian dominating function on \( G \), abbreviated a PID-function, when it holds that whenever \( f(v) = 0 \) for any \( v \in V \), it holds that \( \sum_{u \in N(v)} f(u) = 2 \), i.e., the accumulated weight assigned to the neighbors of \( v \) by \( f \) is exactly two. The weight of \( f \) is the sum of its labels, i.e., \( \sum_{v \in V} f(v) \). The perfect Italian domination number of \( G \), denoted by \( \gamma_{IP}(G) \), is the minimum weight of a PID-function on \( G \). This concept was introduced by Haynes and Henning [14] as a natural variant of similar, previously rather heavily-studied, parameters of so-called Roman domination introduced by Cockayne et al. [9]. We refer the interested reader to e.g., [12, Section 3.9] for a brief overview of some of these variants, but describe some relevant to our work in the following.

We say that \( f \) is a Roman dominating function, abbreviated an RDF-function, on \( G \) if every vertex \( v \in V \) for which \( f(v) = 0 \) is adjacent to at least one vertex \( u \) for which \( f(u) = 2 \). The Roman domination number of \( G \), denoted by \( \gamma_R(G) \), is the minimum weight of an RDF-function on \( G \). While introducing the concept, Cockayne et al. [9] also gave several bounds for \( \gamma_R(G) \) and determined its value for certain structured graph classes including paths, cycles and complete multipartite graphs. For example, the authors proved that \( \gamma(G) \leq \gamma_R(G) \leq 2\gamma(G) \) and that \( \gamma(G) = \gamma_R(G) \) implies \( G \) to be edgeless, where \( \gamma(G) \) is the domination number of \( G \). Further, they mentioned that is has been proved that deciding whether a graph \( G \) admits an RDF-function of weight at most \( k \) is NP-complete. For further combinatorial results on \( \gamma_R(G) \), see the survey [6, Section 5.7]. A possible application in network design is described by Chambers et al. [5], while Liedloff et al. [17] give algorithms for several structured graph classes.

Another variant of perfect Italian domination, introduced by Chellali et al. [8], is obtained by relaxing the constraint so that for every \( v \in V \), if \( f(v) = 0 \), then \( \sum_{u \in N(v)} f(u) \geq 2 \), i.e., the accumulated weight of \( f \) assigned to the neighbors of \( v \) is at least two. Such an \( f \) is known as a Roman \( \{2\} \)-dominating function of \( G \), also referred to as an Italian dominating function by Henning and Klostermeyer [15]. Here, the Roman \( \{2\} \)-domination number of \( G \), denoted by \( \gamma_{IR2}(G) \), is the minimum weight of a Roman \( \{2\} \)-dominating function on \( G \). In addition to various combinatorial results, Chellali et al. [8] also proved that deciding whether a graph \( G \) admits a Roman \( \{2\} \)-dominating function of weight at most \( k \) is NP-complete even when \( G \) is bipartite.

**Our results** We continue the study of perfect Italian domination initiated by Haynes and Henning [14] by giving the following results.

- In Section 2, we relate the perfect Italian domination number to other well-known Roman domination numbers. Further, we characterize the graphs \( G \) such that \( \gamma_{IP}(G) = 2 \) which includes connected threshold graphs, paths, cycles, and wheels. We proceed to give a characterization of graphs \( G \) such that \( \gamma_{IP}(G) = 3 \), and then conclude by determining the exact value of the parameter for complete multipartite graphs.

- In Section 3, we consider the question of Haynes and Henning [14] for finding best possible upper bounds on \( \gamma_{IP}(G) \) as a function of the order \( n \) when \( G \) is planar or regular. For planar graphs and split graphs, we prove that there is an infinite family of such connected graphs \( G \) such that \( \gamma_{IP}(G) = n \), meaning that no upper bound of the form \( c \cdot n \) exists, for any \( c < 1 \). For cubic graphs, we prove that \( \frac{2}{3} n \leq \gamma_{IP}(G) \leq \frac{2}{3} n \), and demonstrate that these bounds are tight.
In Section 4, we turn to complexity-theoretic questions. Specifically, we prove that deciding whether a given graph $G$ admits a PID-function of weight at most $k$ is NP-complete, even when $G$ is restricted to the class of bipartite planar graphs. We also strengthen the result of Chellali et al. [8] by showing that deciding whether $G$ admits a Roman $\{2\}$-dominating function of weight at most $k$ is NP-complete, even when $G$ is both bipartite and planar.

We conclude in Section 5 by giving some further open problems and conjectures arising from our work.

2 Basic bounds, properties and characterizations

In this section, we determine some basic properties of the perfect Italian domination number of a graph.

2.1 Graphs with perfect Italian domination number two

We begin with the following known bounds.

**Theorem 1** (Chellali et al. [8]). For every graph $G$, it holds that $\gamma(G) \leq R_2(G) \leq \gamma_R(G)$.

**Proposition 2.** For every graph $G$, it holds that $\gamma(G) \leq R_2(G) \leq \gamma_p(I)(G)$.

*Proof. Every PID-function of $G$ is a Roman $\{2\}$-dominating function of $G$, so the bound follows.*

Clearly, the optimal PID-function of a graph $G$ consists of optimal PID-functions of its components, as made precise in the following.

**Proposition 3.** If $G$ is a disconnected graph with components $G_1, G_2, \ldots, G_r$, then $\gamma_p(I)(G) = \sum_{i=1}^{r} \gamma_p(I)(G_i)$.

The lower bounds from Theorem 1 are tight for $n$-vertex paths $P_n$ and cycles $C_n$, and thus the following results are obtained via Chellali et al. [8 Corollary 10].

**Proposition 4.** For every integer $n \geq 1$, it holds that $\gamma_p(I)(P_n) = \lceil (n+1)/2 \rceil$ and $\gamma_p(I)(C_n) = \lceil n/2 \rceil$.

The following observation characterizes the graphs $G$ with $\gamma_p(I)(G) = 2$. Recall that the join of graphs $G$ and $H$ is the graph union of $G$ and $H$ with all the edges between $E(G)$ and $E(H)$ added.

**Proposition 5.** A non-trivial connected graph $G$ has $\gamma_p(I)(G) = 2$ precisely when $G$ can be written as the join of $G_1$ and $G_2$, where $G_1$ is either $K_1$, $2K_1$ or $K_2$.

*Proof. For $G$ to have $\gamma_p(I)(G) = 2$, there must exist a PID-function that labels (i) exactly one vertex 2 and the rest 0 or (ii) exactly two vertices 1 and the rest 0. If exactly one vertex $v$ has label 2, all vertices distinct from $v$ must be adjacent to it, i.e., $G_1$ must be $K_1$. Similarly, if there are two vertices $u$ and $v$ with label 1, $G_1$ must be either $2K_1$ or $K_2$ meaning that $u$ dominates at least $V(G) \setminus \{v\}$ and vice versa for $v$.\qed*
Several structured graph classes fall under the above characterization, as we will see next.

**Proposition 6.** A non-trivial connected threshold graph \( G \) has \( \gamma^p_I(G) = 2 \).

**Proof.** Every threshold graph \( G \) can be represented as a binary string \( s(G) \), read from left to right, where 0 denotes the addition of an isolated vertex and 1 denotes the addition of a dominating vertex (for a proof, see [18, Theorem 1.2.4]). Because \( G \) is connected, the last symbol of \( s(G) \) is a 1. As \( G \) has a dominating vertex, the proof follows by Proposition 5.

The following results are now immediate, where \( S_n \), \( K_n \), and \( W_n \) denote the star graph, complete graph, and wheel graph, respectively, on \( n \) vertices.

**Proposition 7.** For every integer \( n \geq 2 \), it holds that \( \gamma^p_I(S_n) = 2 \).

**Proposition 8.** For every integer \( n \geq 2 \), it holds that \( \gamma^p_I(K_n) = 2 \).

**Proposition 9.** For every integer \( n \geq 4 \), it holds that \( \gamma^p_I(W_n) = 2 \).

**Proposition 10.** For every integer \( n \geq 1 \), it holds that \( \gamma^p_I(K_{2,n}) = 2 \).

**Proof.** The graph \( K_{2,n} \) can be written as the join of \( 2K_1 \) and \( \overline{K}_n \) (i.e., the edgeless \( n \)-vertex graph), so the proof follows by Proposition 5.

### 2.2 Bounds via fair domination

In this subsection, we give a characterization of graphs \( G \) with \( \gamma^p_I(G) = 3 \). In order to do so, let us first introduce some concepts from domination.

Let \( G = (V,E) \) be a graph. For \( k \geq 1 \), a \( k \)-fair dominating set of \( G \) is a dominating set \( D \) such that \( |N(v) \cap D| = k \) for every \( v \in V \setminus D \). That is, every vertex not in \( D \) has precisely \( k \) neighbors in \( D \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( k \)-fair dominating set in \( G \). This concept was introduced by Caro et al. [2] (see also [13]). It is also captured by the concept of \([j,k]\)-domination as introduced by Chellali et al. [7]. Here, a subset \( S \subseteq V \) is a \([j,k]\)-set if for every vertex \( v \in V \setminus S \) it holds that \( j \leq |N(v) \cap S| \leq k \), that is, every vertex not in \( S \) has at least \( j \) but no more than \( k \) neighbors in \( S \). Clearly, a \( k \)-fair dominating set is equivalent to a \([k,k]\)-dominating set. Finally, such a set is also known as a perfect \( k \)-dominating set (see e.g., [3, 4]).

**Theorem 11.** For every graph \( G \), it holds that \( \gamma^p_I(G) \leq fd_2(G) \).

**Proof.** Let \( D \) be a 2-fair dominating set. Construct a vertex-labeling \( f \) such that \( f(v) = 1 \) for \( v \in D \) and \( f(u) = 0 \) for \( u \not\in D \). By definition, every \( u \) for which \( f(u) = 0 \) it holds that there are precisely two vertices \( v \) with \( f(v) = 1 \) in \( N(u) \), so \( f \) is a PID-function. The weight of \( f \) is \( |D| \) which can be as small as \( fd_2(G) \), completing the proof.

In order to exploit the previous theorem, we prove the following result regarding the structure of any PID-function \( f \) witnessing \( \gamma^p_I(G) = 3 \).

**Lemma 12.** Any PID-function \( f \) of a graph \( G \) witnessing \( \gamma^p_I(G) = 3 \) uses exactly three ones and no twos.
Proof. Suppose this was not the case, i.e., that instead \( f \) set \( f(u) = 2 \) and \( f(v) = 1 \) for some distinct \( u, v \in V(G) \). Now consider any \( v' \in N(v) \) such that \( v' \neq u \). Because \( f \) is a PID-function of weight three, it must hold that \( f(v') = 0 \). But because \( v' \) is adjacent to \( v \) and \( f(v) = 1 \), the labels on the neighbors of \( v' \) assigned by \( f \) cannot sum to exactly two, contradicting the fact that \( f \) is a PID-function. \( \square \)

We are now ready to prove the main result of the section.

**Theorem 13.** A graph \( G \) with \( \gamma^p_1(G) > 2 \) has \( \gamma^p_1(G) = 3 \) if and only if \( G \) has a 2-fair dominating set \( D \) of size 3.

Proof. Suppose that \( \gamma^p_1(G) = 3 \). By Lemma 12 any PID-function \( f \) of \( G \) has picked three vertices, say \( a, b, \) and \( c \) such that \( f(a) = f(b) = f(c) = 1 \) and labeled every other vertex 0. We claim that \( D = \{a, b, c\} \) is a 2-fair dominating set of size 3. Indeed, every vertex with label 0 must be adjacent to exactly two vertices of \( D \) since \( f \) is a PID-function, so the claim follows.

For the other direction, construct a PID-function \( f \) from a 2-fair dominating set \( D \) such that \( f(v) = 1 \) for \( v \in D \) and \( f(u) = 0 \) for \( u \notin D \). Clearly, as \( D \) is a 2-fair dominating set, every \( u \) is adjacent to exactly two vertices labeled 1. Further, because \( |D| = 3 \), we have that \( \gamma^p_1(G) \leq 3 \). As \( \gamma^p_1(G) > 2 \), we conclude that \( \gamma^p_1(G) = 3 \).

It is also possible to state the same result in a different way. To do this, we observe the following.

**Proposition 14.** Let \( G = (V, E) \) be a connected graph. A subset \( S \subseteq V \) of size \( s \) is an \( \ell \)-fair dominating set in \( G \) if and only if \( S \) is an \( (s - \ell) \)-fair dominating set in \( \overline{G} \).

A 1-fair dominating set is also known as a perfect dominating set (see Fellows and Hoover [11]).

**Corollary 15.** Let \( G = (V, E) \) be a connected graph. A subset \( S \subseteq V \) of size three is a 2-fair dominating set in \( G \) if and only if \( S \) is a perfect dominating set in \( \overline{G} \).

We can then restate our earlier theorem as follows.

**Theorem 16.** A graph \( G \) with \( \gamma^p_1(G) > 2 \) has \( \gamma^p_1(G) = 3 \) if and only if \( \overline{G} \) has a perfect dominating set of size 3.

Let us then proceed to determine the perfect Italian domination number of complete multipartite graphs.

**Lemma 17.** For every two integers \( n_1, n_2 \geq 3 \), it holds that \( \gamma^p_1(K_{n_1, n_2}) = 4 \).

Proof. Let us denote \( G = K_{n_1, n_2} \). As \( G \) does not have a pair of vertices that dominate every vertex (possibly excluding each other), it follows by Proposition 3 that \( \gamma^p_1(G) \geq 3 \). The complement \( \overline{G} \) of \( G \) is a disjoint union of two cliques \( K_{n_1} \) and \( K_{n_2} \). Thus, \( \overline{G} \) does not admit a perfect dominating set of size three, so \( \gamma^p_1(G) \geq 4 \). A matching upper bound is given by an \( f \) which assigns \( f(v) = 2 \) and \( f(u) = 2 \) for one \( v \in V_1 \) and one \( u \in V_2 \), while setting remaining labels to 0. This completes the proof. \( \square \)

**Lemma 18.** For every three integers \( n_1, n_2, n_3 \geq 3 \), it holds that \( \gamma^p_1(K_{n_1, n_2, n_3}) = 3 \).
Proof. Let us denote $G = K_{n_1,n_2,n_3}$. By Proposition $\ref{p1}$, $\gamma_1^p(G) \geq 3$. To give a matching upper bound, it suffices to notice that $G$ is a disjoint union of three cliques $K_{n_1}, K_{n_2}$, and $K_{n_3}$. A perfect dominating set of size three in $G$ is given by choosing exactly one vertex from each component. By Theorem $\ref{t6}$, we conclude that $\gamma_1^p(G) = 3$. □

Lemma $\ref{l19}$. For $k$ integers $n_1,n_2,\ldots,n_k \geq 3$, it holds that $\gamma_1^p(K_{n_1,n_2,\ldots,n_k}) = n$.

Proof. Let us denote $G = K_{n_1,n_2,\ldots,n_k}$. For the sake of contradiction, assume that a PID-function $f$ of $G$ with weight less than $n$ exists. In other words, there must exist a vertex $u$ in a set $V_i$ of the $k$-partition of $G$ for some $1 \leq i \leq k$ with $f(u) = 0$. Let us consider all the possibilities as to how the neighbors of $u$ must be labeled.

Case 1: There is a neighbor $v$ of $u$ in $V_j$ with $j \neq i$ such that $f(v) = 2$.

Because $f$ is a PID-function, it follows immediately that every vertex in $V \setminus (V_i \cup \{v\})$ must be labeled 0. Furthermore, every vertex with label 0 in $V_j$ must have neighbors of weight exactly two. Consequently, each vertex of $V_i \setminus \{u\}$ has label 2. But now a vertex of label 0 in $V_q$ for any $q \neq i \neq j$ has neighbors of weight four contradicting the fact that $f$ is a PID-function.

Case 2: There are neighbors $v$ and $v'$ of $u$ in $V_j$ with $j \neq i$ such that $f(v) = f(v') = 1$.

Because $n_j \geq 3$, there is a vertex $v''$ distinct from $v$ and $v'$ in $V_j$ whose neighbors in $V_i$ must have weight two. But similarly to Case 1, there is then a vertex in $V_q$ with label 0 whose neighbors have weight four, a contradiction.

Case 3: There are neighbors $v$ and $v'$ of $u$ in $V_j$ and $V_q$, respectively, with $j \neq q \neq i$ such that $f(v) = f(v') = 1$.

Similarly to Case 1, we again observe that every vertex in $V \setminus (V_i \cup \{v,v'\})$ must have label 0. Now, for instance, a vertex in $V_j$ with label 0 requires that a vertex in $V_i$ distinct from $u$ has label 1. But then a vertex of $V_q$ distinct from $v'$ has weight three, a contradiction. □

The previous lemmas together prove the following.

Theorem $\ref{t20}$. Let $G = K_{n_1,n_2,\ldots,n_k}$ be the complete $k$-partite graph, where $n_i \geq 3$ for each $1 \leq i \leq k$. Then

$$
\gamma_1^p(G) = \begin{cases} 
4, & \text{if } k = 2, \\
3, & \text{if } k = 3, \\
n, & \text{if } k \geq 4.
\end{cases}
$$

Remark $\ref{r21}$. The complete multipartite graph $G = K_{n_1,n_2,\ldots,n_k}$ for $k \geq 4$ shows that the difference between $\gamma_{\{R2\}}(G)$ and $\gamma_1^p(G)$ can be made arbitrarily large. Indeed, by Theorem $\ref{t20}$ we have that $\gamma_1^p(G) = n$, but $\gamma_{\{R2\}}(G) = 3$ as witnessed by labeling exactly one vertex 1 from three different sets of the $k$-partition of $G$ and labeling the remaining vertices 0.

Remark $\ref{r22}$. Let $G = K_{n_1,n_2,n_3}$ be the complete tripartite graph with $n_i \geq 3$ for $1 \leq i \leq 3$. By Lemma $\ref{l8}$, $\gamma_1^p(G) = 3$ while $\gamma_R(G) = 4$ (see $\ref{l8}$ Proposition 8). Thus, it is not true that $\gamma_R(G) \leq \gamma_1^p(G)$ in general (cf. Proposition $\ref{p3}$).
3 On upper bounds for restricted graph classes

Haynes and Henning [14] proposed the problem of determining the best possible constant $c_G$ such that $\gamma^p(G) \leq c_G \cdot n$ for all $n$-vertex graphs $G$ belonging to a particular class $\mathcal{G}$ of graphs. In particular, they showed that if $\mathcal{G}$ is the class of connected bipartite graphs, then $c_G = 1$, whereas if $\mathcal{G}$ is the class of trees (on at least 3 vertices), then $c_G = 4/5$. Further, the authors suggested to study the problem further when $\mathcal{G}$ would be e.g., the class of planar graphs or regular graphs.

In the following subsections, we settle precisely the question when $\mathcal{G}$ is the class of connected planar graphs by proving, perhaps surprisingly, that $c_G = 1$. In addition, we also completely settle the question when $\mathcal{G}$ is the class of connected cubic graphs by proving that $c_G = 2/3$. Further, when $\mathcal{G}$ is the class of $k$-regular graphs for $k \geq 6$, we show that $c_G = 1$. When $k \geq 9$, this family is also connected. We conclude by observing that $c_G = 1$ when $\mathcal{G}$ is the class of connected split graphs, implying that $c_G = 1$ also when $\mathcal{G}$ is any superclass of split graphs, like the class of chordal graphs.

3.1 Planar graphs

In this subsection, we describe an infinite family of connected planar graphs $G$ that have $\gamma^p(G) = n$, thus proving that $c_G = 1$ when $\mathcal{G}$ is the class of connected planar graphs.

Let $J_1$ be the connected 10-vertex planar graph that is formed by adding two dominating vertices to $2K_2$ and then finishing by connecting a pendant vertex to every vertex except for two vertices of degree three (see Figure 1). In particular, name the four support vertices of $J_1$ so that $u$ and $v$ are those with degree five, and $x$ and $y$ are those with degree four. The graph $J_2$ is obtained via widening $J_1$ by connecting both $u$ and $v$ with the pendants of $x$ and $y$, say $x'$ and $y'$, respectively, and by introducing a new pendant vertex to both $x'$ and $y'$. The widening of $J_1$ to obtain $J_2$ is illustrated in Figure 1. In total, a widening operation adds two vertices and six edges. In general, the graph $J_\ell$ for any $\ell \geq 3$ is obtained recursively by widening $J_{\ell-1}$, which in turn is obtained by widening $J_{\ell-2}$, and so on. Our goal is to show that $\gamma^p(J_\ell) = n$. To this end, we make the following claims concerning any PID-function with weight less than $n$.

**Lemma 23.** Let $f$ be a PID-function of $J_\ell$ with weight less than $n$. It must hold for the support vertices $u$ and $v$ that $f(u) + f(v) \leq 2$.

**Proof.** If this was not the case, i.e., if $f(u) + f(v) > 2$, none of the unlabeled non-pendant vertices could be labeled 0 because $u$ and $v$ are in the neighborhood of each such vertex. Thus, the weight of any PID-function would then be at least $n - 6 + 3 = n - 3$. Further, $f$ must label every remaining unlabeled vertex 0. This means that every support vertex must be labeled 2 (for otherwise $f$ is not a PID-function), but then the weight is $n$. \(\square\)

**Lemma 24.** Let $f$ be a PID-function of $J_\ell$ with weight less than $n$. The function $f$ must label $f(u) \neq 0$ and $f(v) \neq 0$.

**Proof.** For the sake of contradiction, suppose that $f$ has weight less than $n$ and $f(u) = 0$. Because $f$ is a PID-function, it holds that $\sum_{u' \in N(u)} f(u') = 2$. Clearly, the pendant of $u$ cannot be labeled 0, so first suppose that pendant of $u$ was labeled 2. It follows that every other vertex adjacent to $u$ must be labeled 0. But now it must be the case that $f(v) = 2$ and $f(y') = 2$, but $\{x, y\} \subseteq N(y)$, contradicting the fact that $f$ is a PID-function. So it must be
the case that the pendant of $u$ is labeled 1. It follows that precisely one unlabeled neighbor $a$ of $u$ is labeled 1 while the rest are labeled 0. Now, observe that there exists a non-neighbor of $a$ whose all neighbors have been labeled 0 except for $v$. Thus, it must be that $f(v) = 2$. But now a neighbor of $a$, labeled 0, is adjacent to $a$ (with label 1) and $v$ (with label 2), contradicting the fact that $f$ is a PID-function. We conclude that $f(u) \neq 0$. By a symmetric argument, $f(v) \neq 0$ under any valid PID-function $f$ whose weight is less than $n$.

**Lemma 25.** For any integer $\ell \geq 1$, it holds that $\gamma^p_I(J_\ell) = n$.

**Proof.** For the sake of contradiction, suppose that there is a PID-function $f$ for $J_\ell$ for any $\ell \geq 1$ with weight less than $n$. By combining Lemma 23 with Lemma 24, we know that any such $f$ must label $f(u) = f(v) = 1$. Consider any vertex $a$ that is a common neighbor of both $u$ and $v$. If $f(a) = 0$, all neighbors of $a$ must also be labeled 0. In particular, it now holds that $f(x) = f(y) = 0$ but then the pendant vertices $x'$ and $y'$ cannot receive any of the labels 0, 1, or 2 without violating the fact that $f$ is a valid PID-function, a contradiction. Otherwise, if there is no such $a$ with $f(a) = 0$, the weight of $f$ is at least $n - 4$ with only the pendants unlabeled. Clearly, the two pendants of $u$ and $v$ cannot be labeled 0, but can be labeled 1. For the pendants $x'$ and $y'$ of $x$ and $y$ there are two choices: either set (i) $f(x') = 0$ and $f(x) = 2$ or set (ii) $f(x') = f(x) = 1$, and similarly the same for $y$ and $y'$. In both cases $f$ has weight $n$, a contradiction. \qed

The previous lemma establishes the main result of this subsection.

**Theorem 26.** There is an infinite family of connected planar graphs $G$ such that $\gamma^p_I(G) = n$.

As a side remark, we can also see that for any $\ell \geq 1$, the treewidth of $J_\ell$ is three. Thus, unlike for e.g., chromatic number, it is not true that the perfect Italian domination number of a graph could be bounded as a function of treewidth.

### 3.2 Regular graphs

In this subsection, we shift our focus to regular graphs. As a main result here, we derive tight upper and lower bounds for the perfect Italian domination number of cubic graphs.
A strong matching, also known as an induced matching, is a set \( M \) of edges of a graph \( G \) such that no two edges in \( M \) are connected by an edge of \( G \). Viewed differently, an induced matching is an independent set in the square of the line graph \( G \). The strong matching number, denoted by \( \nu_s(G) \), is the size of a maximum induced matching of \( G \). For the next lemma, the key observation is that if \( M \) is a strong matching in a cubic graph \( G \), then \( V(G) \setminus V(M) \) is a 2-fair dominating set of \( G \).

**Lemma 27.** Every cubic graph \( G \) has \( \gamma_1^P(G) \leq n - 2 \nu_s(G) \).

**Proof.** Let \( M \) be any strong matching of \( G \). Construct a vertex-labeling \( f : V(G) \to \{0, 1, 2\} \) such that \( f(u) = f(v) = 0 \) for every \( \{u, v\} \in M \) and label all other vertices 1. Clearly, \( f \) is a PID-function since every vertex \( v \) with \( f(v) = 0 \) has two neighbors labeled 1 and one labeled 0. The weight of \( f \) is \( n - 2 |M| \), which is equal to \( n - 2 \nu_s(G) \) when \( |M| = \nu_s(G) \). □

The following bound for the strong matching number will be useful for us.

**Theorem 28** (Joos et al. [16]). A cubic graph with \( m \) edges has \( \nu_s(G) \geq m/9 \).

Before proceeding, we mention that Chellali et al. [8, Theorem 11] proved that \( \gamma_{\{R2\}}(G) \geq 2n/(\Delta + 2) \), where \( G \) is a connected \( n \)-vertex graph with maximum degree \( \Delta \). Combined with Proposition 2, we obtain the following.

**Theorem 29.** A connected graph on \( n \) vertices with maximum degree \( \Delta \) has \( \gamma_1^P(G) \geq 2n/(\Delta + 2) \).

We are now ready to establish the main result of this subsection.

**Theorem 30.** Every connected cubic graph with \( n \) vertices has \( 2/5 \leq \gamma_1^P(G) \leq 2/3 \). Moreover, these bounds are tight.

**Proof.** The lower bound follows from Theorem 29 by having \( \Delta = 3 \). The claimed upper bound follows by applying Lemma 27, for which we combine the fact that every cubic graph \( G \) with \( n \) vertices has \( 3/2n \) edges with Theorem 28. That is, we see that

\[
\gamma_1^P(G) \leq n - 2 \nu_s(G) \leq n - 2(m/9) = n - 2(n/6) = 2n/3.
\]

To see that the lower bound is tight, one can consider any connected cubic graph with 8 vertices. For instance, when \( G \) is the 8-vertex cubical graph, we have that \( \gamma_1^P(G) = 4 = \lceil 16/5 \rceil \). To see that the upper bound is tight, one can consider \( G \) defined as the Cartesian product of \( K_3 \) and \( K_2 \). Clearly, \( G \) does not satisfy the condition of Proposition 3. Further, \( G \) is isomorphic to the 6-cycle, which does not admit a perfect dominating set of size three, so by Theorem 16 it holds that \( \gamma_1^P(G) \geq 4 \). By our upper bound \( \gamma_1^P(G) \leq 4 \) as well, so both bounds are tight. □

Another example to see that \( \gamma_1^P(G) \leq 2/3 n \) is tight is the 6-vertex cubic graph obtained by taking a \( K_{3,2} \) and making a new vertex adjacent to each of the three vertices in the other set of the bipartition.

**Theorem 31.** For every \( k \geq 6 \) there is an infinite family of \( k \)-regular graphs \( G \) such that \( \gamma_1^P(G) = n \). For every \( k \geq 9 \), this family is connected.
Proof. For \( k \geq 9 \) the claim is clear as indicated by Theorem 20.

Now, for \( 6 \leq k \leq 8 \) we turn to a computer search with the help of House of Graphs [1], an online database for “interesting” graphs. Here, if we can find a \( k \)-regular graph \( G \) for which \( \gamma_1^p(G) = n \), an infinite (disconnected) family such graphs is obtained by taking multiple disjoint copies of \( G \). Indeed, by Proposition 3, an optimal PID-function for such a graph will also have weight \( n \). Below is a list of graphs represented in the well-known graph6 code:

\[
\text{KvyCJlmF}_{kN}
\]

The first of three is 6-regular, the second is 7-regular, and the third 8-regular, all with the property that they do not admit a PID-function of weight less than \( n \).

While for every \( k \geq 6 \), there are \( k \)-regular graphs \( G \) with \( \gamma_1^p(G) = n \), we conclude with the following observations.

**Lemma 32.** Let \( G \) be a \( k \)-regular graph for any \( k \geq 3 \) and let \( f \) be a PID-function of \( G \). Every \( v \) such that \( f(v) = 0 \) is adjacent to \( k - 2 \) or \( k - 1 \) vertices \( u \) such that \( f(u) = 0 \).

**Proof.** If this was not the case, the sum \( \sum_{u \in N(v)} f(u) \) would be not equal to two contradicting the fact that \( f \) is a PID-function.

**Theorem 33.** For every \( k \geq 3 \), there does not exist an \( n \)-vertex \( k \)-regular graph \( G \) with \( \gamma_1^p(G) = n - k - 2 \).

**Proof.** To reach a contradiction, let \( f \) be an optimal PID-function of \( G \) witnessing that \( \gamma_1^p(G) = n - k - 2 \). If such a graph \( G \) existed, then a vertex \( v \) with \( f(v) = 0 \) would be adjacent to exactly \( k - 3 \) vertices \( u \) with \( f(u) = 0 \), contradicting Lemma 32.

### 3.3 Split graphs

In this subsection, we consider split graphs defined as graphs whose vertex set can be partitioned into a clique and an independent set. Split graphs are highly restricted graphs forming a subclass of chordal graphs, which in turn are a subclass of perfect graphs.

For any \( \ell \geq 6 \), let \( S_\ell \) be the split graph obtained by starting from \( K_\ell \) and by choosing four distinct arbitrary vertices \( \{a, b, c, d\} \) of it and adding two new vertices \( x \) and \( y \) with the edges \( \{xa, xb, xc\} \cup \{yd\} \) (see Figure 2). That is, \( \{x, y\} \) forms an independent set, while \( V(S_\ell) \setminus \{x, y\} \) induces a clique of size \( \ell \).

**Lemma 34.** For any \( \ell \geq 6 \), it holds that \( \gamma_1^p(S_\ell) = n \).

**Proof.** For the sake of contradiction, suppose that \( \gamma_1^p(S_\ell) < n \) and that this is witnessed by a PID-function \( f \). Because \( f \) has weight less than \( n \), there must exist at least one vertex \( v \) such that \( f(v) = 0 \). Suppose that \( f(x) = 0 \). Then, without loss of generality, there are two possibilities: either (i) \( f(a) = 2 \) and \( f(b) = f(c) = 0 \) or (ii) \( f(a) = 0 \) and \( f(b) = f(c) = 1 \). In both cases, it follows that all the other vertices of the \( K_\ell \) must be labeled 0 by \( f \). In particular, it holds that \( f(d) = 0 \), but now there is no label \( f \) can assign to \( y \). Thus, \( f(x) \neq 0 \).
Without loss of generality, suppose that $f(a) = 0$. Now, if $f(x) = 2$, it must be that $f(b) = f(c) = 0$. Again, by the same argument as above, there is no label $f$ can assign to $y$. Thus, if $f(a) = 0$ then $f(x) = 1$ must hold. Now, $f$ must label exactly one vertex of the $\ell - 1$ vertices of the $K_\ell$ with 1 and the other with 0. But then there is always at least one vertex in $u \in V(K_\ell) \setminus \{b, c\}$, which is distinct from $d$ as $\ell \geq 6$, such that $\sum_{u' \in N(u)} f(u') = 1$, contradicting the fact that $f$ is a PID-function.

Because none of $a$, $b$, and $c$ can be labeled 0 by $f$, it follows that $f(a) + f(b) + f(c) \geq 3$, and thus $f(u) \neq 0$ for every $u \in K_\ell$. At this point, the only possibility is that $f(y) = 0$. It follows that $f(d) = 2$. As no other vertex can be labeled 0, we can label every remaining vertex 1. But now the weight of $f$ is $n$, a contradiction. We conclude that $\gamma^p_I(S_\ell) = n$, which is what we wanted to prove.

The previous lemma establishes the following result.

**Theorem 35.** There is an infinite family of connected split graphs $G$ such that $\gamma^p_I(G) = n$.

We can further contrast this result with the fact that threshold graphs, which are precisely the $P_4$-free split graphs, always admit a PID-function of weight at most 2 by Proposition 6.

## 4 Hardness of perfect Italian domination

In this section, we prove that perfect Italian domination is NP-complete, even when restricted to bipartite planar graphs. In all our hardness proofs, we omit explicitly showing membership to NP as it is an easy exercise.

To prove the claimed result, we give a polynomial-time reduction from Planar Exact Cover by 3-Sets in which we are given a finite set $X$ with $|X| = 3q$ and a family $C$ of 3-element subsets of $X$. The goal is to decide whether there is a subfamily $C'$ of $C$ such that every element of $X$ appears in exactly one element of $C'$. Every instance $(X, C)$ is associated with a bipartite incidence graph, in which the first set of the bipartition corresponds to elements in $X$ and the second to elements in $C$. The edge set is defined such that two vertices are connected precisely when an element of $X$ is contained in an element of $C$. In Planar Exact Cover by 3-Sets, we have the further constraint the incidence graph is both bipartite and planar. This problem was shown to be NP-complete by Dyer and Frieze [10].

**Theorem 36** (Dyer and Frieze [10]). Planar Exact Cover by 3-Sets is NP-complete.

Before describing our reduction, let us introduce the following gadget. For any positive integer $\ell \geq 1$, the fish gadget $F_\ell$ is constructed by starting from the disjoint union of $2\ell$
vertices partitioned into two equally-sized sets $T$ and $M$, and by adding two vertices $x$ and $y$ such that $y$ is adjacent to every vertex in $T \cup M$ and $x$ is adjacent to every vertex in $M$. Thus, $F$ has a total of $2\ell + 2$ vertices, with $\ell$ vertices of degree two and $\ell$ vertices of degree one. The fish gadget is illustrated in Figure 3.

**Proposition 37.** For any $\ell \geq 3$, any PID-function $f$ of $F$ has weight at least $\ell + 2$ if $f(x) = 1$. Similarly, if $f(x) = 2$, $f$ has weight at least $\ell + 4$.

We say that a vertex $v$ for which $f(v) = 0$ is *satisfied* if $\sum_{u \in N(v)} f(u) = 2$. Even more precisely, we say that such a $v$ is out-satisfied (with respect to some subgraph $H$ of $G$) if $\sum_{u \in N(v) \setminus V(H)} f(u) = 2$. Similarly, $v$ is in-satisfied if $\sum_{u \in N(v) \cap V(H)} f(u) = 2$. For the following statement, the subgraph $H$ is to be understood to be the gadget $F$ itself.

**Proposition 38.** For any $\ell \geq 3$, any PID-function $f$ of $F$ that sets $f(x) = 0$ has optimal weight 2 if $x$ is out-satisfied. Otherwise, if $x$ is in-satisfied, $f$ has optimal weight 4.

**Proof.** In the first case, set $f(y) = 2$ and label other vertices 0. In the second case, set $f(y) = 2$, label an arbitrary vertex in $M$ with 2, and label other vertices 0. □

Let us call **Perfect Italian Domination** the problem where we are given a graph $G$ and an integer $k$, and the goal is to decide whether $G$ admits a PID-function of weight at most $k$.

**Theorem 39.** Perfect Italian Domination is NP-complete for bipartite planar graphs.

**Proof.** Let $(X, C)$ be an instance of Planar Exact Cover by 3-Sets, such that $X = \{1, 2, \ldots, n\}$, $C = \{C_1, C_2, \ldots, C_t\}$, and $|X| = 3q$. We proceed by describing a polynomial-time reduction to Perfect Italian Domination as follows.

Let $H$ be the bipartite incidence graph of $(X, C)$, which we can also safely assume to be planar by Theorem 36. So more precisely, $V(H) = \{x_1, x_2, \ldots, x_n\} \cup \{c_1, c_2, \ldots, c_t\}$ with $x_i$ and $c_j$ adjacent precisely when $i$ is a member of $C_j \in C$. Let $k = 6q + 2t$. To obtain $G$ from $H$, identify $x_i$ for $i \in [n]$ with a fish gadget $F_k$ (at its vertex $x$) and attach to $c_j$ for $j \in [t]$ two pendants $c_j'$ and $c_j''$. We tacitly name $y_i$ the vertex $y$ of a fish gadget corresponding to the vertex $x_i$. Clearly, because the fish gadget is both bipartite and planar, $G$ is bipartite and planar as well. We claim that $(X, C)$ has an exact cover if and only if $(G, k)$ admits a PID-function of weight at most $k$.

Let $C'$ be an exact cover of $(X, C)$. We construct a vertex-labeling $f$ of $G$ such that $f(c_j) = 2$ for $C_j \in C'$; all other vertices $c_j$ not in $C'$ are labeled 0. Here, if $f(c_j) = 2$, we set...
At this point, the labels used by \( f \) have weight 2\( t \). For each \( i \in [n] \), we label \( f(x_i) = 0 \), \( f(y_i) = 2 \), and all the remaining vertices 0. As \(|X| = 3q\) and \( f(y_i) = 2 \), the weight of \( f \) is exactly \( 2 \cdot 3q + 2t = 6q + 2t = k \). Now, since \( C' \) is an exact cover, every \( x_i \) is out-satisfied by some \( c_j \) corresponding to a \( C_j \in C' \). For each \( j \in [t] \), if \( f(c_j) = 0 \), then \( c_j \) is satisfied by \( f(c'_j) + f(c''_j) = 2 \). It follows that \( f \) is a PID-function.

Conversely, suppose that \( f \) is a PID-function of weight \( k \). It holds for every \( i \in [n] \) that \( f(x_i) = 0 \) for otherwise \( f \) would have weight at least \( k + 2 > k \) by Proposition 37. Further, as \( S = \bigcup_{j \in [t]} \{ c_j, c'_j, c''_j \} \) requires labels of weight at least \( 2t \), it follows by Proposition 38 that each \( x_i \) must be out-satisfied for otherwise \( f \) would have weight at least \( 2t + 6q - 2 + 4 = k + 2 > k \). It follows that \( f \) has allocated labels of weight \( k - 6q = 2t \) to \( S \). Further, this is only possible if \( f(c_j) \neq 1 \) for \( j \in [t] \) for otherwise \( f \) would have weight at least \( 6q + 2t - 2 + 3 = k + 1 > k \). Therefore, since \( f \) is a PID-function, every \( x_j \) is out-satisfied by exactly one \( x_j \) for which \( f(x_j) = 2 \). Consequently, \( C' = \{ C_j \mid f(c_j) = 2 \} \) is an exact cover of \((X, C)\).

It is worth mentioning that the earlier result of Chellali et al. [8, Theorem 18] regarding the hardness of computing \( \gamma_{\{R2\}}(G) \) also works for bipartite planar graphs. Let us call \textsc{Roman \{2\}-Domination} the problem of deciding whether given a graph \( G \) and an integer \( k \), it is true that \( \gamma_{\{R2\}}(G) \leq k \).

**Theorem 40.** \textsc{Roman \{2\}-Domination} is \textsc{NP-complete} for bipartite planar graphs.

**Proof.** Chellali et al. [8, Theorem 18] prove \textsc{NP-completeness} of \textsc{Roman \{2\}-Domination} for bipartite graphs by a polynomial-time reduction from an arbitrary instance \((X, C)\) of \textsc{Exact Cover} by 3-Sets. In short, their reduction begins from the bipartite incidence graph \( H \) of \((X, C)\), but replaces every vertex corresponding to a \( C \in C \) with a \( C_0 \) with a chord followed by a 2-vertex path. Because this gadget is both bipartite and planar, we ensure that the instance \( G \) of \textsc{Roman \{2\}-Domination} is both bipartite and planar by assuming that \( H \) is planar. By Theorem 36 we can do this safely, so the result follows.

5 Open problems

In this section, we conclude by highlighting some open problems arising from our work.

We begin with the following complexity-theoretic statement.

**Conjecture 41.** For every \( k \geq 3 \), \textsc{Perfect Italian Domination} is \textsc{NP-complete} for the class of \( k \)-regular graphs.

In the light of our construction in the proof of Theorem 26, it might be interesting to consider other planar graphs \( G \) with \( \gamma_p(G) = n \). We verified by a computer search the smallest planar graph \( G \) with \( \gamma_p(G) = n \) to have \( n = 7 \) vertices, and there are no other such planar graphs on 7 vertices. Thus, one might ask the following.

**Problem 42.** Can we characterize the connected planar graphs \( G \) such that \( \gamma_p(G) = n \), or at least find some conditions for this to hold?

Also, after Theorem 26 it is natural to raise the question of Haynes and Henning [14] for the class of \textit{bipartite} planar graphs. At the same time, given our \textsc{NP-completeness} result Theorem 39, one should not expect a polynomial-time characterization for this class.
Problem 43. Determine the best possible constant $c_G$ such that $\gamma^p_1(G) \leq c_G \times n$ for all $n$-vertex graphs $G$ belonging to the class of connected bipartite planar graphs $\mathcal{G}$.

For this problem, we verified by an exhaustive computer search that $c_G = 1/2$ for every $n \leq 21$. However, there are larger bipartite planar graphs for which this is not true.

Similarly, the bounds we give in Theorem 30 are tight for cubic graphs. But what about quartic, that is, 4-regular graphs? In general, we find the further study of perfect Italian domination interesting for other regular graphs.

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