Pierre Hodara* and Ioannis Papageorgiou **
IME, Universidade de Sao Paulo

CONCENTRATION AND POINCARÉ TYPE INEQUALITIES FOR A DEGENERATE PURE JUMP MARKOV PROCESS.

PIERRE HODARA AND IOANNIS PAPAGEORGIU

ABSTRACT. We aim in proving Talagrand concentration and Poincaré type inequalities for a class of pure jump Markov processes inspired by the model introduced in [21] by Galves and Löcherbach to describe the behaviour of interacting brain neurons. As a result we obtain exponential rates of convergence to equilibrium. In particular we consider neurons with degenerate jumps, i.e. that lose their memory when they spike, while the probability of a spike depends on the actual position and thus the past of the whole neural system.

1. Introduction.

The aim of this paper is to study rates of convergence to equilibrium of the model introduced in [21] by Galves and Löcherbach, to describe the activity of a biological neural network. What is in particular interesting about the jump process in question is that it is characterized by degenerate jumps, in the sense that after a particle (neuron) spikes, it loses its memory by jumping to zero. Furthermore, the probability of a spike of a particular neuron at any time depends on its actual position and thus the past of the whole neural system.

For $P_t$ the associated semigroup and $\mu$ the invariant measure, we obtain concentration properties

$$\mu \left( \{ P_t f - \mu(f) \geq r^2 \} \right) \leq e^{-r}$$

which imply exponential fast convergence to equilibrium. In addition, we show further Talagrand type concentration inequalities,

$$\mu \left( \left\{ \sum_{i=1}^{N} x_i < r^2 \right\} \right) \geq 1 - e^{-r}.$$

2010 Mathematics Subject Classification. 60K35, 26D10, 60G99,

Key words and phrases. Talagrand inequality, Poincaré inequality, brain neuron networks.

Address: Neuromat, Instituto de Matematica e Estatistica, Universidade de Sao Paulo, rua do Matao 1010, Cidade Universitaria, Sao Paulo - SP- Brasil - CEP 05508-090.

Email: *hodarapierre@gmail.com ** ipapageo@ime.usp.br, papyannis@yahoo.com

This article was produced as part of the activities of FAPESP Research, Innovation and Dissemination Center for Neuromathematics (grant 2013/ 07699-0 , S.Paulo Research Foundation); This article is supported by FAPESP grant (2016/17655-8) * and (2017/15587-8)**.
To show the concentration properties, at first we prove some Poincaré type inequalities of the form

$$\frac{1}{c(t)} \mu(Var_{\mathcal{P}_t}(f)) \leq \mu(\Gamma(f, f)) + \mu(\Gamma(f, f)(\Delta(x)))$$

where the inequality constant $c(t)$ depends on the time and is a polynomial of order three.

Before we describe the model we present the neuroscience framework of the problem.

1.1. the neuroscience framework. The activity of one neuron is described by the evolution of its membrane potential. This evolution presents from time to time a brief and high-amplitude depolarisation called action potential or spike. The spiking probability or rate of a given neuron depends on the value of its membrane potential. These spikes are the only perturbations of the membrane potential that can be transmitted from one neuron to another through chemical synapses. When a neuron spikes, its membrane potential is reset to 0 and the post-synaptic neurons connected to it receive an additional amount of membrane potential.

From a probabilistic point of view, this activity can be described by a simple point process since the whole dynamic is characterised by the jump times. In the literature, Hawkes processes are often used in order to describe systems of interacting neurons, see [13], [19], [20], [21], [25] and [27] for example. The reset to 0 of the spiking neuron provides a variable length memory for the dynamic and therefore point-processes describing these systems are non-Markovian.

On the other hand, it is possible to describe the activity of the network with a process modelling not only the jump times but the whole evolution of the membrane potential of each neuron. This evolution needs then to be specified between the jumps. In [26] the process describing this evolution follows a deterministic drift between the jumps, more precisely the membrane potential of each neuron is attracted with exponential speed towards an equilibrium potential. This process is then Markovian and belongs to the family of Piecewise Deterministic Markov Processes introduced by Davis ([15] and [16]). Such processes are widely used in probability modeling of e.g. biological or chemical phenomena (see e.g. [14] or [31], see [4] for an overview). The point of view we adopt here is close to this framework, but we work without drift between the jumps. We therefore consider a pure jump Markov process and will make use of the abbreviation PJMP in the rest of the present work.

We consider a process $X_t = (X^1_t, ..., X^N_t)$, where $N$ is the number of neurons in the network and where for each neuron $i$, $1 \leq i \leq N$ and each time $t \in \mathbb{R}_+$, each variable $X^i_t$ represents the membrane potential of neuron $i$ at time $t$. Each
membrane potential $X^i_t$ takes value in $\mathbb{R}_+$. A neuron with membrane potential $x$ “spikes” with intensity $\phi(x)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a given intensity function. When a neuron $i$ fires, its membrane potential is reset to 0, interpreted as resting potential, while the membrane potential of any post-synaptic neuron $j$ is increased by $W_{i\to j} \geq 0$. Between two jumps of the system, the membrane potential of each neuron is constant.

Working with Hawkes processes allows to consider systems with infinitely many neurons, as in [21] or [27]. For our purpose, we need to work in a Markovian framework and therefore our process represents the membrane potentials of the neurons, considering a finite number $N$ of neurons. On the contrary of [26], a Lyapunov-type inequality allows us to get rid of the compact state-space assumption. Due to the deterministic and degenerate nature of the jumps, the process does not have a density continuous with respect to the Lebesgue measure. We refer the reader to [30] for a study of the density of the invariant measure. Here, we make use of the lack of drift between the jumps to work with discrete probabilities instead of density.

1.2. the model. Let $N > 1$ be fixed and $(N^i(ds,dz))_{i=1,...,N}$ be a family of i.i.d. Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ having intensity measure $dsdz$. We study the Markov process $X_t = (X^1_t, \ldots, X^N_t)$ taking values in $\mathbb{R}^N_+$ and solving, for $i = 1, \ldots, N$, for $t \geq 0$,

\begin{equation}
X^i_t = X^i_0 - \int_0^t \int_0^\infty X^i_{s-} 1_{\{z \leq \phi(X^i_{s-})\}}N^i(ds,dz)
+ \sum_{j \neq i} W_{j\to i} \int_0^t \int_0^\infty 1_{\{z \leq \phi(X^i_{s-})\}}N^j(ds,dz).
\end{equation}

In the above equation for each $j \neq i$, $W_{j\to i} \in \mathbb{R}_+$ is the synaptic weight describing the influence of neuron $j$ on neuron $i$. Finally, the function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is the intensity function.

The generator of the process $X$ is given for any test function $f : \mathbb{R}^N_+ \to \mathbb{R}$ and $x \in \mathbb{R}^N_+$ by

\begin{equation}
\mathcal{L}f(x) = \sum_{i=1}^{N} \phi(x^i) [f(\Delta_i(x)) - f(x)]
\end{equation}

where

\begin{equation}
(\Delta_i(x))_j = \begin{cases} 
x^j + W_{i\to j} & j \neq i \\
0 & j = i
\end{cases}
\end{equation}

Furthermore, we also assume the following conditions about the intensity function:

\begin{equation}
\phi(x) > cx \text{ for } x \in \mathbb{R}_+
\end{equation}
and
\begin{equation}
\phi(x) \geq \delta
\end{equation}
for some strictly positive constants $c$ and $\delta$.

1.3. Poincaré type inequalities. Our purpose is to show Poincaré type inequalities for our PJMP, whose dynamic is similar to the model introduced in [21], primarily with values in a non compact space, but also in a compact space.

Let us first describe the general framework and define the Poincaré inequalities on a discreet setting (see also [33], [18], [36], [3] and [12]). At first we should note a convention we will widely use. For a function $f$ and measure $\nu$ we will write $\nu(f)$ for the expectation of the function $f$ with respect to the measure $\nu$, that is
\[\nu(f) = \int f\,d\nu.\]

We consider a Markov process $(X_t)_{t \geq 0}$ which is described by the infinitesimal generator $L$ and the associated Markov semigroup $P_tf(x) = E_x^*(f(X_t))$. For a semigroup and its associated infinitesimal generator we will need the following well know relationships: $\frac{d}{ds}P_s = LP_s = P_sL$ (see for example [24]).

We define $\mu$ to be the invariant measure for the semigroup $(P_t)_{t \geq 0}$ if and only if $\mu P_t = \mu$.

Then, since $L^f := \lim_{t \to 0^+} \frac{P_t - f}{t}$, one also has $\mu(Lf) = 0$.

Furthermore, we define the ”carre du champ” operator by:
\[\Gamma(f, g) := \frac{1}{2}(L(fg) - fLg - gLf).\]

For the PJMP process defined above with the specific generator $L$ given by (1.2) a simple calculation shows that the carre du champ takes the following form.
\[\Gamma(f, f) = \frac{1}{2}(Lf^2 - 2fL(f)) = \frac{1}{2}\left(\sum_{i=1}^{N} \phi(x_i) \left[f(\Delta_i(x)) - f(x)\right]^2\right).\]

We say that $P_t$ satisfies a Poincaré inequality if there exists $C(t) > 0$ independent of $f$, which may however depend on $t$, such that
\[(SG) \quad \text{Var}_{P_t}(f) \leq C(t)P_t(\Gamma(f, f))\]
where the variance of a function $f$ with respect to a measure $\nu$ is defined with the usual way as: $\text{Var}_\nu = \nu(f - \nu(f))^2$.

In [36], [3] and [12], the Poincaré inequality (SG) has been shown for some point processes, for a constant that depends on time $t$, while the stronger log-Sobolev inequality, has been disproved. The general method used in these papers, that will
be followed also in the current work, is based on the so called semigroup method which shows the inequality for the semigroup $P_t$.

The main difficulty here is that, for the pure jump Markov process that we examine in the current paper, the translation property

$$E^{x+y}f(z) = E^x f(z+y)$$

used in [36] and [3] does not hold here. This appears to be important because the translation property is a key element in these papers, since it allows to bound the carre du champ by comparing the mean $E^x f(z)$ where the process starts from position $x$ with the mean $E^{\Delta_i(x)} f(z)$ where it starts from $\Delta_i(x)$, the jump-neighbour of $x$. However, we can still obtain Poincaré type inequalities, but with a constant $C(t)$ which is a polynomial of order three with respect to time $t$. This order is higher than the constant $C(t) = t$, the optimal obtained in [36] for a path space of Poisson point processes.

It should be noted, that the aforementioned translation property relates with the $\Gamma_2$ criterion (see [5] and [6]) for the Poincaré inequality (see discussion in subsection 2.2). Since this is not satisfied in our case we obtain a Poincaré type inequality instead.

We will investigate both the case of neurons with values in a compact set (bounded neurons) and with values in the whole of $\mathbb{R}_+$ (unbounded neurons). The Poincaré type inequalities we obtain for compact neurons, in addition to the energy referring to time $t$, involves also an energy down to the initial position. In the case of non compact neurons the inequality includes also a local term of the energy of the system after a jump within a compact set.

We will start with the compact case, since the general non compact case will follow by reduction to the compact one. Let’s first give an example of definition for the neuron process on a compact domain. For that we consider the generator of the process $X$ for any smooth test function $f : \mathbb{R}_+^N \to \mathbb{R}$ and $x \in \mathbb{R}_+^N$ to be given by

$$\mathcal{L}f(x) = \sum_{i=1}^N \phi(x^i) [f(\Delta_i(x)) - f(x)]$$

where

$$(\Delta_i(x))_j = \begin{cases} x^j + W_{i\to j} & j \neq i \text{ and } x^i + W_{i\to j} \leq m \\ x^j & j \neq i \text{ and } x^i + W_{i\to j} > m \\ 0 & j = i \end{cases}$$

for some $m > 0$. With this definition the process remains inside the compact set

$$D := \{x \in \mathbb{R}_+^N : x_i \leq m, 1 \leq i \leq N\}.$$ 

In the compact case we obtain a Poincaré type inequality as presented below.
Theorem 1.1. Assume the PJMP as described in (1.4)-(1.7). Then the following Poincaré type inequality holds.

\[ \text{Var}_{E_x}(f(x_t)) \leq \alpha(t)\Gamma(f,f)(x) + \beta(t)\sum_{i=1}^n \phi(x_i)\Gamma(f,f)(\Delta_i(x)) + \gamma(t)\int_0^t P_w\Gamma((f,f))(x)dw \]

with \(\alpha(t)\) and \(\beta(t)\) first order and \(\gamma(t)\) second order polynomial of the time \(t\) that do not depend on the function \(f\).

One notices that the right hand side of the statement in Theorem 1.1 can be divided into two distinct parts. The first, the local part, which consists of the first two terms of the right hand side, \(\Gamma(f,f)(x)\) and \(\Gamma(f,f)(\Delta_i(x))\), which refers to the energy of the system down to the initial position \(x\), and a second part that is the typical energy of the system up to time \(t\), \(\int_0^t P_w\Gamma((f,f))(x)dw\).

Since the coefficients \(\alpha(t)\) and \(\beta(t)\) of the local part, are polynomials of lower order than that of \(\gamma(t)\), the third term dominates over the first two for big time \(t\), as shown in the next corollary. This is not surprising since for \(t\) big enough the influence of the initial configuration decreases.

Corollary 1.2. Assume the PJMP as described in (1.4)-(1.7). For \(t\) sufficiently large, i.e. \(t > \zeta(f)\)

\[ \text{Var}_{E_x}(f(x_t)) \leq 3\gamma(t)\int_0^t P_w\Gamma((f,f))(x)dw \]

where \(\zeta(f)\) is a constant depending only on \(f\),

\[ \zeta(f) = \sup_{x \in D} \max \left\{ \frac{4t_0^2M^2}{C_1M^3} \sum_{i=1}^n \phi(x_i)\Gamma(f,f)(\Delta_i(x))}{\int_0^t P_w\Gamma((f,f))(x)dw}, \frac{N\delta\Gamma(f,f)(x)}{4C_1M^3 \int_0^t P_w\Gamma((f,f))(x)dw} \right\} \]

for some positive constants \(t_0\), \(C_1\) and \(M\).

One should notice that although the lower value \(\zeta(f)\) depends on the function \(f\), the coefficient of the inequality \(\gamma(t)\) does not depend on the function \(f\).

In the last results we considered bounded neurons for which we obtained Poincaré type inequalities for the semigroup \(P_t\), with an additional local energy term. In what follows we will see analogue results for neurons with non compact domain.

In this more general non compact case we will prove an alternative Poincaré type inequality, which is formulated by taking the expectation with respect to the invariant measure \(\mu\) for the semigroup \((P_t)_{t \geq 0}\) in the typical Poincaré inequality, that
is
\[
\int Var_{P_t}(f)(x)\mu(dx) \leq \delta(t) \int \Gamma(f,f)(x)\mu(dx) + \\
\beta(t) \int_D \sum_{i=1}^n \phi(x_i)\Gamma(f,f)(\Delta_i(x))\mu(dx),
\]
where on the right hand side we have also added a local term for some compact set $D$. The reasons why in the unbounded case we focus on this particular Poincaré type inequality rather than the classical one about $P_t$ presented above relate with the special features that characterise the behaviour of the PJMP process examined in the current paper. As it will be made more clear later during the proof of the local Poincaré inequality presented in section 2, there are two hindrances that pertain the jump behaviour of the neurons in the PJMP processes under investigation.

The first is the degenerate memorylessness behaviour of the neuron that spikes. Since whenever a neuron, i.e. one of the $N$ increments $x_i$ of $x = (x_1,\ldots,x_N)$ spikes it loses its memory by jumping at zero, together with the fact that the jump probabilities depend on the current position, as explained before, we cannot make use of the translation property $E_{x+y}f(z) = E_xf(z+y)$ used in [30] and [3]. This in the case of unbounded neurons has gravest implication than in the case of bounded neurons since we need in addition to control the spike probability of any neuron $i$ at position $x$, which relates to the unbounded intensity function $\phi$.

The need to control the unbounded intensity function $\phi$ presents the second difficulty. In order to handle the intensity functions we will use the Lyapunov methods presented in [11] and [7] which has the advantage of reducing the problem from the unbounded to the compact case where variables take value within the compact set $D$ defined in (1.8), that satisfies
\[
D \supset \left\{ \sum_{i=1}^N x_i \leq m \right\},
\]
where the set $\left\{ \sum_{i=1}^N x_i \leq m \right\}$ is the one involved in the Lyapunov method. Since the jump behaviour depends on the current position of a neuron this has the benefit of bounding the values of $\phi$ and thus controlling the spike behaviour of the neurons.

The Lyapunov method however, as we will see later in more detail in the proof of Proposition 3.2 requires the control of a Lyapunov function $V$, more specifically of $\dot{V}$. As it will be explained in more detail later, this is a problem that although can be solved relatively easy in the case of diffusions by choosing appropriate exponential
densities, in the case of jump processes it is more difficult and requires the use of invariant measures. The inequality which refers to the general case where neurons take values in the whole of $\mathbb{R}_+$ follows.

**Theorem 1.3.** Assume the PJMP as described in (1.2)-(1.5). Then the following Poincaré type inequality holds.

$$\int \text{Var}_{E^x}(f(x_t)) \mu(dx) \leq \delta(t) \int \Gamma(f, f)(x) \mu(dx) + \beta(t) \sum_{i=1}^{n} \phi(x_i) \Gamma(f, f)(\Delta_i(x)) \mu(dx)$$

for $\delta(t)$ a third order and $\beta(t)$ a first order polynomial of $t$ that do not depend on the function $f$, where the set $D$ is as in (1.8).

Furthermore, for $t$ sufficiently large, $t \geq \theta(f)$

$$\int \text{Var}_{E^x}(f(x_t)) d\mu \leq 2\delta(t) \int \Gamma(f, f)(x) d\mu$$

for $\mu$ invariant measure of $P_t$ and $\theta(f)$ a constant depending only on $f$,

$$\theta(f) = \left( \frac{4N\delta t^2_0 \int \Gamma(f, f)(\Delta_i(x)) d\mu}{C_1 M \int \Gamma((f, f)) (x) d\mu} \right)^{\frac{1}{2}}$$

for some positive constants $C_1, M, t_0$.

Again, the first assertion of the theorem states that the energy can be divided into two parts. The local one $\int_{D} \sum_{i=1}^{n} \phi(x_i) \Gamma(f, f)(\Delta_i(x)) d\mu$ that calculates the energy of the system in the compact set $D$, and the typical energy for the invariant measure $\mu$ at time $t$, $\int E^x \Gamma(f, f)(x_t) d\mu = \int \Gamma(f, f)(x) d\mu$.

As before, although the lower value $\theta(f)$ on the Poincaré type inequality of the corollary depends on the function $f$, the time constant $\delta(t)$ does not depend on the function $f$.

**1.4. Concentration and other Talagrand type inequalities.** Concentration inequalities play a vital role in the examination of a system’s convergence to equilibrium. Talagrand (see [34] and [35]) associated the log-Sobolev and Poincaré inequalities for exponential distributions through concentration properties with exponentially fast convergence to equilibrium (see also [10]), that is

$$\mu(P_t f - \mu(f) > r) \leq \lambda_0 e^{-ar^p}.$$  

In particular, when the log-Sobolev inequality holds, then (1.9) is true for $p = 2$, while in the case of the weaker Poincaré inequality, the exponent is $p = 1$. Furthermore, the modified log-Sobolev inequality that interpolates between the
two, investigated for example in [9], [22] and [32], gives convergence to equilibrium of speed $1 < p < 2$.

In the current paper, where Poincaré type inequalities are proven, we still obtain concentration inequalities, and exponential fast rate of convergence to equilibrium, of order $p = \frac{1}{3}$ (for $a = \frac{3}{2}$).

**Theorem 1.4.** Assume the PJMP as described either in (1.2)-(1.5) or in (1.4)-(1.7). For every function $f$, satisfying

$$D(f) := \max_{i \in \mathbb{N}} \sup_{x} |f(x) - f(\Delta_i(x))| < 1$$

there exists a constant $\lambda_0 > 0$ such that

$$\mu \left( f - \mu(f) > r^{\alpha + \frac{3}{2}} \right) \leq \lambda_0 e^{-r},$$

for any $a > 0$.

The problem of concentration properties for measures that satisfy a Poincaré inequality, or as in our case, the Poincaré type inequality, is closely related with exponential integrability of the measure, that is

$$\mu(e^{\lambda f}) < +\infty$$

for some appropriate class of functions $f$. This problem, is itself connected to bounding the carre du champ of the exponent of a function

$$\mu(\Gamma(e^{\lambda^2 f}, e^{\lambda f})) \leq \frac{\lambda_2^2}{4} \Psi(f) \mu(e^{\lambda f})$$

(1.10)

for some $\Psi(f)$ uniformly bounded. In the case of diffusion processes where the carre du champ is defined through a derivation, (1.10) is satisfied for $||\nabla f||_\infty < 1$ (see section 4 for more details). In the more general case of symmetric semigroups similar inequalities can be obtained, as shown in [2]. For a detailed discussion on the subject one can look at [29]. In our case where however the semigroup is not symmetric, the bounds can still be obtained for $D(f) < 1$. Furthermore, the last result can be derived for a different class of functions as described below. Consider

$$|||f|||_\infty = \sup \{ \mu(fg); \ g: \mu(g) \leq 1 \}.$$  

We can obtain exponential integrability and a bound (1.10) for functions $f$ such that $|||\phi(x^t)D(f)^2|||_\infty < 1$ and $|||\phi(x^t)e^{\lambda D(f)}D(f)^2|||_\infty < 1$. This, together with the Poincaré type inequality already obtained, can show concentration properties for a different class of functions than the ones assumed in Theorem 1.4, as presented in the next theorem.

**Theorem 1.5.** Assume the PJMP as described either in (1.2)-(1.5) or in (1.4)-(1.7). For every function $f$, satisfying

$$|||\phi(x^t)D(f)^2|||_\infty < 1 \ and \ |||\phi(x^t)e^{\lambda D(f)}D(f)^2|||_\infty < 1$$
there exists a constant $\lambda_0 > 0$ such that

$$\mu \left( \left\{ f - \mu(f) > r^{a + \frac{3}{2}} \right\} \right) \leq \lambda_0 e^{-r},$$

for any $a > 0$.

As a result of the last two theorems we obtain the following convergence to equilibrium property:

**Corollary 1.6.** Under the assumptions of any of the last two theorems, for any $a > 0$,

$$\mu \left( \left\{ P_t f - \mu(f) > r^{a + \frac{3}{2}} \right\} \right) \leq \lambda_0 e^{-r}$$

where $\mu$ is the invariant measure of the semigroup $P_t$.

Furthermore, for the case of unbounded neurons, we can obtain Talagrand inequalities in the spirit of the ones proven for the modified log-Sobolev in [9].

**Theorem 1.7.** Assume the PJMP as described in (1.2)-(1.5). Then the following Talagrand inequality holds.

$$\mu \left( \left\{ \sum_i x_i < r^{a + \frac{3}{2}} \right\} \right) \geq 1 - \lambda_0 e^{-r}$$

for any $a > 0$.

A few words about the structure of the paper. The proof of the non-compact result presented in Theorem 1.3 will trivially follow from Proposition 3.2 of section 3. The proof of Proposition 3.2 is based on two main conditions. The one is the existence of a Lyapunov inequality and the other a local Poincaré type inequality. The Lyapunov inequality will be shown in the same section, section 3, while the local Poincaré inequality will be the subject of section 2. In that same section the proof of the Poincaré inequalities of Theorem 1.1 and Corollary 1.2 for neurons with values in a compact set will also be proven. In the final section 4 the concentration inequalities are proven. At first in Proposition 4.1 we present the main tool that connects the Poincaré type inequality with the concentration properties. Then, in the next two subsection the required conditions are verified for the PJMP.

2. proof of the local Poincaré

In this section we focus on neurons that take values on compact sets as described by (1.4)-(1.7), and we prove the local Poincaré inequalities presented in Theorem 1.1 and corollary 1.2. Let us first state some technical results.
2.1. Technical results. We start by showing properties of the jump probabilities of the degenerate PJMP processes. Since the process is constant between jumps, the set of reachable positions \( y \) after a given time \( t \) for a trajectory starting from \( x \) is discrete. We therefore define

\[
\pi_t(x, y) := P_x(X_t = y) \quad \text{and} \quad D_x := \{ y \in D, \pi_t(x, y) > 0 \}.
\]

This set is finite for the following reasons. On one hand, for each neuron \( i \in I \), the set \( S_i = \{0\} \cup \{ \sum_{k=1}^{n} W_{j_k \rightarrow i}, n \in \mathbb{N}^*, j_k \in I \} \) is discrete and such that the intersection with any compact is finite. On the other hand, we have \( D_x \subset \left[ \prod_{i \in I} (S_i \cup (x_i + S_i)) \right] \cap D \).

The idea is that since the process is constant between jumps, elements of \( D_x \) are such that there exists a sequence of jumps leading from \( x \) to \( y \). Since we are only interested on the arrival position \( y \), among all jump sequences leading to \( y \), we can consider only sequences with minimal number of jumps and the number of such jump sequences leading to positions inside a compact is finite, due to the fact that each \( W_{j \rightarrow i} \) is non-negative. Since \( x \) is also in the compact \( D \), we can have an upper bound for the cardinal of \( D_x \) independent from \( x \).

For a given time \( s \in \mathbb{R}_+ \) and a given position \( x \in D \), we denote by \( p_s(x) \) the probability that starting at time 0 from position \( x \), the process has no jump in the interval \([0, s]\), and for a given neuron \( i \in I \) by \( p_s^i(x) \) the probability that the process has exactly one jump of neuron \( i \) and no jumps for other neurons. Introducing the notation \( \phi(x) = \sum_{j \in I} \phi(x_j) \) and given the dynamics of the model, we have that

\[
p_s(x) = e^{-s \bar{\phi}(x)}
\]

and

\[
(2.1) \quad p_s^i(x) = \int_0^s \phi(x_i) e^{-u \bar{\phi}(x)} e^{-(s-u) \bar{\phi}(\Delta^i(x))} du
= \begin{cases}
\frac{\phi(x_i)}{\phi(x) - \phi(\Delta^i(x))} \left( e^{-s \bar{\phi}(\Delta^i(x))} - e^{-s \bar{\phi}(x)} \right) & \text{if } \bar{\phi}(\Delta^i(x)) \neq \bar{\phi}(x) \\
\frac{s \phi(x_i) e^{-s \bar{\phi}(x)}}{\phi(x) - \phi(\Delta^i(x))} & \text{if } \bar{\phi}(\Delta^i(x)) = \bar{\phi}(x)
\end{cases}.
\]

Define

\[
(2.2) \quad t_0 = \begin{cases}
\frac{\ln(\bar{\phi}(x)) - \ln(\bar{\phi}(\Delta^i(x)))}{\phi(x) - \phi(\Delta^i(x))} & \text{if } \bar{\phi}(\Delta^i(x)) \neq \bar{\phi}(x) \\
\frac{1}{\phi(x)} & \text{if } \bar{\phi}(\Delta^i(x)) = \bar{\phi}(x)
\end{cases}.
\]

As a function of \( s \), \( p_s^i(x) \) is continuous, strictly increasing on \((0, t_0)\) and strictly decreasing on \((t_0, +\infty)\) and we have \( p_0^i(x) = 0 \).
Lemma 2.1. Assume the PJMP as described in (1.4)-(1.7). There exists positive constants $C_1$ and $C_2$ independent of $t, x$ and $y$ such that

- For all $t > t_0$, we have
  $$\sum_{y \in D_x} \frac{\pi_t^2(\Delta^i(x), y)}{\pi_t(x, y)} \leq C_1.$$
- For all $t \leq t_0$, we have
  $$\sum_{y \in D_x \setminus \Delta^i(x)} \frac{\pi_t^2(\Delta^i(x), y)}{\pi_t(x, y)} \leq C_1$$
  and
  $$\frac{\pi_t^2(\Delta^i(x), \Delta^i(x))}{\pi_t(x, \Delta^i(x))} \leq \frac{\pi_t(\Delta^i(x), \Delta^i(x))}{\pi_t(x, \Delta^i(x))} \leq C_2 t.$$

Proof. As said before, the set $D_x$ is finite so it is sufficient to obtain an upper bound for the ratio $\frac{\pi_t^2(\Delta^i(x), y)}{\pi_t(x, y)}$.

We have for all $s \in (0, t)$

\begin{equation}
\frac{\pi_t^2(\Delta^i(x), y)}{\pi_t(x, y)} \leq \frac{(\pi_{t-s}(\Delta^i(x), y)p_s(y) + \sup_{z \in D}(1 - p_s(z)))^2}{p_s(x)p_{t-s}(\Delta^i(x), y)}.
\end{equation}

Here we decomposed the numerator according to two events. Either $X_{t-s} = y$ and there no jump in the interval of time $[t-s, t]$ or there is at least one jump in the interval of time $[t-s, t]$, whatever the position $z \in D$ of the process at time $t-s$.

From the previous inequality, we then obtain

\begin{equation}
\frac{\pi_t^2(\Delta^i(x), y)}{\pi_t(x, y)} \leq \frac{(\pi_{t-s}(\Delta^i(x), y)p_s(y) + (1 - e^{-sN(\phi(m))})^2}{p_s(x)p_{t-s}(\Delta^i(x), y))}.
\end{equation}

where we recall that the constant $m$ appears in the definition of the compact set $D$ introduced in (1.8).

Let us first assume that $t > t_0$. Recall that $t_0$ is defined in (2.2).

If $\pi_{t-t_0}(\Delta^i(x), y) \geq p_{t_0}^i(x)$, we have

\begin{equation}
\frac{\pi_t^2(\Delta^i(x), y)}{\pi_t(x, y)} \leq \frac{1}{(p_{t_0}^i(x))^2}.
\end{equation}

Assume now that $\pi_{t-t_0}(\Delta^i(x), y) < p_{t_0}^i(x)$ and let us recall that as a function of $s$, $p_s^i(x)$ is continuous, strictly increasing on $(0, t_0)$ and $p_0^i(x) = 0$. 

On the other hand, as a function of $s$, $\pi_{t-s}(\Delta^i(x), y)$ is continuous and takes value $\pi_t(\Delta^i(x), y) > 0$ for $s = 0$.

We deduce from this that there exists $s_* \in (0, t_0)$ such that $p^i_{s_*}(x) = \pi_{t-s_*}(\Delta^i(x), y)$.

Now (2.4) with $s = s_*$ gives us

$$\frac{\pi^2_t(\Delta^i(x), y)}{\pi_t(x, y)} \leq (p_{s_*}(y))^2 + 2p_{s_*}(y) \frac{1 - e^{-s_*N\phi(m)}}{p^i_{s_*}(x)} + \left( \frac{1 - e^{-s_*N\phi(m)}}{p^i_{s_*}(x)} \right)^2.$$  

For all $s \in (0, t_0)$, $p_s(y) \leq 1$, and we then study $\frac{1 - e^{-sN\phi(m)}}{p^i_s(x)}$ as a function of $s \in (0, t_0)$.

Using the explicit value of $p^i_s(x)$ given in (2.1) and assumption (1.4), we obtain for all $s \in (0, t_0)$,

$$1 - e^{-sN\phi(m)} \leq \begin{cases} \frac{e^{t_0N\phi(m)}}{\delta} \frac{(\bar{\phi}(x) - \bar{\phi}(\Delta^i(x)))(1 - e^{-sN\phi(m)})}{1 - e^{-\phi(x) - \phi(\Delta^i(x))}} & \text{if } \bar{\phi}(\Delta^i(x)) \neq \bar{\phi}(x) \\ e^{t_0N\phi(m)} \frac{1 - e^{-s}}{s} & \text{if } \bar{\phi}(\Delta^i(x)) = \bar{\phi}(x) \end{cases}.$$  

Recall that $\delta > 0$ is defined in assumption (1.5) and satisfies $\phi(x) \geq \delta$ for all $x \in \mathbb{R}_+$.  

In both cases, when $s$ is far from zero, we can obtain an upper bound independent of $x$, and when $s$ goes to zero, the limit of the right hand term is $\frac{N\phi(m)e^{t_0N\phi(m)}}{s}$.

From this, we deduce that there exists a constant $M_D$ such that for all $s \in (0, t_0)$,

$$1 - e^{-sN\phi(m)} \leq M_D.$$  

Putting all together, we obtain the announced result for the case where $t > t_0$.

We now consider the case where $t \leq t_0$.

We start by considering the case where $y \neq \Delta^i(x)$ and go back to (2.4).

As a function of $s$, $\pi_{t-s}(\Delta^i(x), y)$ is continuous and takes values $\pi_t(\Delta^i(x), y) > 0$ and $\pi_0(\Delta^i(x), y) = 0$ respectively for $s = 0$ and $s = t$.

We deduce from this that there exists $s_* \in (0, t) \subset (0, t_0)$ such that $p^i_{s_*}(x) = \pi_{t-s_*}(\Delta^i(x), y)$ and we are back in the previous case so that the same result holds.

Let us assume now that $y = \Delta^i(x)$, we have

$$\frac{\pi^2_t(\Delta^i(x), y)}{\pi_t(x, y)} \leq \frac{\pi_t(\Delta^i(x), \Delta^i(x))}{\pi_t(x, \Delta^i(x))} \leq \frac{1}{p^i_t(x)}.$$
Recall the explicit expression of $p_i^t(x)$ given in (2.1) and use (1.5) to bound the intensity function

$$p_i^t(x) = t\phi(x) e^{-t\phi(x)} \geq t\delta e^{-t_0 \sup_{x \in D} \phi(x)} = Ct$$

for some constant $C$ independent of $t$ and $x$, which gives us the announced result.

\[\square\]

Taking under account the last result, we can obtain the first technical bound needed in the proof of the local Poincaré inequality, taking advantage of the bounds shown for times bigger than $t_0$.

**Lemma 2.2.** Assume the PJMP as described in (1.4)-(1.7). Then

$$\int_{t_0}^{t-s} \left( \mathbb{E}^{\Delta_i(x)} - \mathbb{E}^x \right) \mathbb{E}^x \sum_{j=1}^{N} \phi(x_u^j) (f(\Delta_j(x_u)) - f(x_u)) du \right)^2 \leq (t-s)(1+C_1)M \int_{t_0}^{t-s} \mathbb{E}^{\theta} \Gamma((f,f))(x_u).$$

**Proof.** Consider $\pi_t(x,y)$ to probability kernel of $\mathbb{E}^x$, i.e.

$$\mathbb{E}^x (f(x_u)) = \sum_y \pi_t(x,y) f(y).$$

Then we can write

$$\Pi_1 := \int_{t_0}^{t-s} \left( \mathbb{E}^{\Delta_i(x)} - \mathbb{E}^x \right) \mathbb{E}^x \sum_{j=1}^{N} \phi(x_u^j) (f(\Delta_j(x_u)) - f(x_u)) du \right)^2 =$$

$$= \int_{t_0}^{t-s} \sum_{y} \pi_u(x,y) \left( \frac{\pi_u(\Delta_i(x),y)}{\pi_u(x,y)} - 1 \right) \sum_{j=1}^{N} \phi_u(y^j) (f(\Delta_j(y)) - f(y)) du \right)^2.$$ 

To continue we will use Holder’s inequality to pass the second power inside the first integral, which will give

$$\Pi_1 \leq (t-s) \int_{t_0}^{t-s} \sum_{y} \pi_u(x,y) \left( \frac{\pi_u(\Delta_i(x),y)}{\pi_u(x,y)} - 1 \right) \sum_{j=1}^{N} \phi_u(y^j) (f(\Delta_j(y)) - f(y)) du \right)^2$$

and then we apply the Cauchy-Schwarz inequality for the measure $\mathbb{E}^x$ to get

$$\Pi_1 \leq (t-s) \int_{t_0}^{t-s} \mathbb{E}^x \left( \frac{\pi_u(\Delta_i(x),y)}{\pi_u(x,y)} - 1 \right) \sum_{j=1}^{N} \phi_u(y^j) (f(\Delta_j(y_u)) - f(y_u)) du \right)^2.$$
The first quantity involved in the above integral is bounded from Lemma 2.1 by a constant
\[ \mathbb{E}^x \left( \frac{\pi_u(\Delta_i(x), y)}{\pi_u(x, y)} - 1 \right)^2 \leq 1 + \mathbb{E}^x \left( \frac{\pi_u(\Delta_i(x), y)}{\pi_u(x, y)} \right)^2 = 1 + \sum_{y \in D_x} \frac{\pi^2_u(\Delta^i(x), y)}{\pi_u(x, y)} \leq 1 + C_1 \]
while for the sum involved in the second quantity, since \( \phi(x) \geq \delta > 0 \) we can use Holder’s inequality
\[ S = \left( \sum_{j=1}^N \phi(y_u^j) \right)^2 \left( \sum_{j=1}^N \frac{\phi(y_u^j)}{\sum_{j=1}^N \phi(y_u^j)} (f(\Delta_j(y_u^j)) - f(y_u^j)) \right)^2 \leq M \sum_{j=1}^N \phi(y_u^j)(f(\Delta_j(y_u^j)) - f(y_u^j))^2 = M \Gamma(f, f)(x_u) \]
where \( M = \sup_{x \in D} \sum_{i=1}^N \phi(x^i) \), so that
\[ \Pi_1 \leq (t - s)(1 + C_1)M \int_{t_0}^{t-s} \mathbb{E}^x \Gamma((f, f))(x_u) du. \]

We will now extend the last bound to an integral on a time domain starting at 0.

**Lemma 2.3.** For the PJMP as described in (1.4)-(1.5) and (1.6)-(1.7), we have
\[ \left( \int_0^{t-s} (\mathbb{E}^\Delta(x) (\mathcal{L}(f(x_u))) - \mathbb{E}^x (\mathcal{L}(f(x_u)))) du \right)^2 \leq 16t_0^2 M \Gamma(f, f)(\Delta_i(x)) + \]
\[ + c(t - s) \int_{t_0}^{t-s} \mathbb{E}^x \Gamma((f, f))(x_u) du \]
where \( c(t) = t_0^2 M(C_1 + 1) + 2t(1 + C_1)M. \)

**Proof.** In order to calculate a bound for
\[ \mathbb{E}^\Delta(x) (\mathcal{L}(f(x_u))) - \mathbb{E}^x (\mathcal{L}(f(x_u))) \]
we will need to control the ratio \( \frac{\pi^2_u(\Delta_i(x), y)}{\pi_u(x, y)} \). As shown in Lemma 2.1, this ratio depends on time when \( u \leq t_0 \), otherwise it is bounded by a constant. For this reason we will start by breaking the integration variable of the time \( t \) into two
domains, \((0, t_0)\) and \((t_0, t - s)\).

\[
I_1 := \left( \int_0^{t-s} (E \Delta_i(x)(\mathcal{L}f(x_u)) - E^x(\mathcal{L}f(x_u))) \, du \right)^2 \leq \nonumber
\]

\[
2 \left( \int_{t_0}^{t-s} (E \Delta_i(x) - E^x) \sum_{i=1}^{N} \phi(x_i^u)(f(\Delta_i(x_u)) - f(x_u)) \, du \right)^2 + \nonumber
\]

\[
(2.8) \quad + 2 \left( \int_0^{t_0} (E \Delta_i(x) - E^x) \sum_{i=1}^{N} \phi(x_i^u)(f(\Delta_i(x_u)) - f(x_u)) \, du \right)^2. \nonumber
\]

The first summand \(I_1\) is upper bounded by the previous lemma. To bound the second term \(I_2\) on the right hand side of (2.8) we write

\[
I_2 \leq \nonumber
\]

\[
2 \left( \int_0^{t_0} \left( \pi_u(\Delta_i(x), \Delta_i(x)) - \pi_u(x, \Delta_i(x)) \right) \sum_{i=1}^{N} \phi(\Delta_i(x))^i(f(\Delta_i(\Delta_i(x))) - f(\Delta_i(x))) \, du \right)^2 + \nonumber
\]

\[
(2.9) \quad + 2 \left( \int_0^{t_0} \left( \sum_{y \in D, y \neq \Delta_i(x)} \pi_u(\Delta_i(x), y) - \pi_u(x, y) \right) \sum_{i=1}^{N} \phi(y)^i(f(\Delta_i(y)) - f(y)) \, du \right)^2. \nonumber
\]

The distinction on the two cases, whether after time \(u\) the neurons configuration is \(\Delta_i(x)\) or not, relates to the two different bounds Lemma 2.1 provides for the fraction \(\frac{\pi_u^2(\Delta_i(x,y))}{\pi_u(x,y)}\) whether \(y\) is \(\Delta_i(x)\) or not. We will first calculate the second term on the right hand side. For this term we will work similar to Lemma 2.2. At first we will apply the Holder inequality on the time integral after we first divide with the normalisation constant \(t_0\). This will give

\[
I_2 \leq \nonumber
\]

\[
t_0 \int_0^{t_0} \left( \sum_{y \in D, y \neq \Delta_i(x)} \left( \frac{\pi_u(\Delta_i(x), y)}{\pi_u(x, y)} - 1 \right) \pi_u(x, y) \sum_{i=1}^{N} \phi(y)^i(f(\Delta_i(y)) - f(y)) \right)^2 \, du. \nonumber
\]
Now we will use the Cauchy-Schwarz inequality in the first sum. We will then obtain the following

\[ \mathbf{III}_2 \leq t_0 \int_0^{t_0} \left[ \sum_{y \in D, y \neq \Delta_i(x)} \pi_u(x, y) \left( \frac{\pi_u(\Delta_i(x), y)}{\pi_u(x, y)} - 1 \right)^2 \right] \, du. \]

(2.10)

The first term on the last product can be upper bounded from Lemma 2.1

\[ \sum_{y \in D, y \neq \Delta_i(x)} \pi_u(x, y) \left( \sum_{i=1}^{N} \phi(y^i) (f(\Delta_i(y)) - f(y)) \right)^2 \leq (C_1 + 1). \]

(2.11)

While for the second term involved in the product of (2.10) we can write

\[ \sum_{y \in D, y \neq \Delta_i(x)} \pi_u(x, y) \left( \sum_{i=1}^{N} \phi(y^i) (f(\Delta_i(y)) - f(y)) \right)^2 \leq M \sum_{y} \pi_u(x, y) \Gamma(f, f)(x_u) = M \mathbb{E}^x \Gamma(f, f)(x_u) \]

where for the first bound we made use once more of the Holder inequality, after we divided with the appropriate normalisation constant \( \sum_{i=1}^{N} \phi(x_u^i) \). If we put the last bound together with (2.11) into (2.10), we obtain

\[ \mathbf{III}_2 \leq t_0 M(C_1 + 1) \int_0^{t_0} \mathbb{E}^x \Gamma(f, f)(x_u) \, du. \]

(2.12)

We now calculate the first summand of (2.9). Notice that in this case we cannot use the analogue bound from Lemma 2.1, that is

\[ \frac{\pi^2(u(\Delta_i(x), \Delta_i(x)))}{\pi(x, \Delta_i(x))} \leq C_2, \]

as we did for \( \mathbf{III}_2 \), since that will lead to a final upper bound \( \mathbf{III}_1 \leq t_0 M(C_1 + 1) \int_0^{t_0} \frac{1}{2} \mathbb{E}^x \Gamma(f, f)(x_u) \, du \) which can diverge. Instead, we will bound the \( \mathbf{III}_1 \) by the carre du champ of the
function after the first jump. We can write
\[
\text{III}_1 \leq 4 \left( \int_0^t \left( \sum_{i=1}^N \phi(\Delta_i(x)^i)|f(\Delta_i(x))| - f(\Delta_i(x)) \right| du \right)^2
\]
\[
\leq 4t_0^2M \left( \sum_{i=1}^N \phi(\Delta_i(x)^i) \right)^2 \left( \sum_{i=1}^N \phi(\Delta_i(x)^i)|f(\Delta_i(x))| - f(\Delta_i(x)) \right)^2
\]
where above we divided with the normalisation constant \( \sum_{i=1}^N \phi(\Delta_i(x)^i) \), since \( \phi(x) \geq \delta \). We can now apply the Holder inequality on the sum, so that
\[
\text{III}_1 \leq 4t_0^2M \left( \sum_{i=1}^N \phi(\Delta_i(x)^i) \right)^2 \left( \sum_{i=1}^N \phi(\Delta_i(x)^i)|f(\Delta_i(x))| - f(\Delta_i(x)) \right)^2
\]
If we combine this together with (2.12) and (2.9) we get the following bound for the second term of (2.8)
\[
\text{II}_2 \leq 8t_0^2M \Gamma(f,f)(\Delta_i(x)) + 4(C_1 + 1)t_0M \int_0^t \mathbb{E}^x \Gamma(f,f)(x_u) du.
\]
The last one together with the bound shown in Lemma 2.2 for the first term \( \text{II}_1 \) of (2.8) gives
\[
\text{I}_1 \leq t_0^216M \Gamma(f,f)(\Delta_i(x)) + t_08M(C_1 + 1) \int_0^t \mathbb{E}^x \Gamma(f,f)(x_u) du +
\]
\[
+ 2(t - s)(1 + C_1)M \int_{t_0}^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du \leq
\]
\[
\leq t_0^216M \Gamma(f,f)(\Delta_i(x)) + 2M(C_1 + 1)(4t_0 + (t - s)) \int_0^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du.
\]

since the carre du champ is non negative, as shown below
\[
\Gamma(f,f) = \frac{1}{2}(\mathcal{L}(f^2) - 2f\mathcal{L}f) = \lim_{t \downarrow 0} \frac{1}{2t}P_t f^2 - (P_t f)^2 \geq 0
\]
by Cauchy-Swartz inequality. \( \Box \)

We have obtained all the technical results that we need in order to show the Poincaré inequality for bounded neurons.

2.2. proof of Theorem 1.1

Denote \( P_t f(x) = \mathbb{E}^x f(x_t) \). Then (2.13)
\[
P_t f^2(x) - (P_t f(x))^2 = \int_0^t \frac{d}{ds} P_s (P_{t-s} f)^2(x) ds = \int_0^t P_s \Gamma(P_{t-s} f, P_{t-s} f)(x) ds
\]
since $\frac{d}{ds}P_s = \mathcal{L}P_s = P_s\mathcal{L}$.

We can write

$$\Gamma(P_{t-s}f, P_{t-s}f)(x) = \sum_{i=1}^{N} \phi(x^i)(\mathbb{E}^x \Delta_i(x) f(x_{t-s}) - \mathbb{E}^x f(x_{t-s}))^2. \quad (2.14)$$

If we could use the translation property $\mathbb{E}^{x+y} f(z) = \mathbb{E}^x f(z + y)$ used for instance in proving Poincaré and modified log-Sobolev inequalities in [30] and [3], then we could bound relatively easy the carre du champ of the expectation of the functions by the carre du champ of the functions themselves, as demonstrated below

$$\sum_{i=1}^{N} \phi(x^i)(\mathbb{E}^x \Delta_i(x) f(x_{t-s}) - \mathbb{E}^x f(x_{t-s}))^2 = \sum_{i=1}^{N} \phi(x^i)(\mathbb{E}^x f(\Delta_i(x_{t-s})) - \mathbb{E}^x f(x_{t-s}))^2 \leq P_{t-s} \Gamma(f, f)(x)$$

The inequality $\Gamma(P_{t}f, P_{t}f) \leq P_{t}\Gamma(f, f)$ for $t > 0$ relates directly with the $\Gamma_2$ criterion (see [3] and [2]) which states that if $\Gamma_2(f) := \frac{1}{2}\mathcal{L}(\Gamma(f, f)) - 2\Gamma(f, \mathcal{L}f) \geq 0$ then the Poincaré inequality is true, since

$$\frac{d}{ds}(P_s \Gamma(P_{t-s}f, P_{t-s}f)) = \frac{1}{2} P_s(\mathcal{L} \Gamma(P_{t-s}f, P_{t-s}f)) - 2\Gamma(P_{t-s}f, \mathcal{L}P_{t-s}f) = P_s(\Gamma_2(P_{t-s}f)) \geq 0$$

implies $\Gamma(P_{t}f, P_{t}f) \leq P_{t}\Gamma(f, f)$ (see also [3]).

Unfortunately this is not the case with our PJMP where the degeneracy of jumps and the memoryless nature of them allows any neuron $x_i$ to jump to zero from any position, with a probability that depends on the current configuration of the neurons. Moreover, contrary on the case of Poisson processes, our intensity also depend on the position.

In order to obtain the carre du champ of the functions we will make use of the Dynkin’s formula which will allow us to bound the expectation of a function with the expectation of the infinitesimal generator of the function which is comparable to the desired carre du champ of the function.

So, from Dynkin’s formula

$$\mathbb{E}^x f(x_t) = f(x) + \int_{0}^{t} \mathbb{E}^x(\mathcal{L}f(x_u))du$$

we get

$$\left(\mathbb{E}^x \Delta_i(x) f(x_{t-s}) - \mathbb{E}^x f(x_{t-s})\right)^2 \leq 2 \left(f(\Delta_i(x)) - f(x)\right)^2 + 2 \left(\int_{0}^{t-s} \left(\mathbb{E}^x \Delta_i(x) \mathcal{L}f(x_u) - \mathbb{E}^x \mathcal{L}f(x_u)\right)du\right)^2.$$
In order to bound the second term above we will use the bound shown in Lemma 2.3

\[
(\mathbb{E}^{\Delta_i(x)} f(x_{t-s}) - \mathbb{E} f(x_{t-s}))^2 \leq 2(f(\Delta_i(x)) - f(x))^2 + 32\ell_0^2 M \Gamma(f,f)(\Delta_i(x)) + 2c(t-s) \int_0^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du
\]

This together with (2.14) gives

\[
\Gamma(P_{t-s} f, P_{t-s} f)(x) \leq 2\Gamma(f,f)(x) + 32\ell_0^2 M \sum_{i=1}^n \phi(x^i) \Gamma(f,f)(\Delta_i(x)) + 2Mc(t-s) \int_0^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du.
\]

Finally plugging this in (2.13) we obtain

\[
P_t f^2(x) - (P_t f(x))^2 \leq 2t\Gamma(f,f)(x) + 32\ell_0^2 M \sum_{i=1}^n \phi(x^i) \Gamma(f,f)(\Delta_i(x)) + 2M \int_0^t c(t-s) \int_0^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du ds.
\]

We can also write

\[
P_s \int_0^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du = P_s \int_s^t \mathbb{E}^x \Gamma((f,f))(x_{w-s}) dw.
\]

But the carre du champ is non negative, so

\[
P_s \int_0^{t-s} \mathbb{E}^x \Gamma((f,f))(x_u) du \leq P_s \int_0^t \mathbb{E}^x \Gamma((f,f))(x_{w-s}) dw = \int_0^t P_s P_{w-s} \Gamma((f,f))(x) dw = \int_0^t P_{w} \Gamma((f,f))(x) dw
\]

where above we used the property of the Markov semigroup $P_s P_{t-s} = P_t$ . We obtain

\[
P_t f^2(x) - (P_t f(x))^2 \leq 2t\Gamma(f,f)(x) + 32\ell_0^2 M \sum_{i=1}^n \phi(x^i) \Gamma(f,f)(\Delta_i(x)) + 2Mtc(t) \int_0^t P_{w} \Gamma((f,f))(x) dw
\]

\[
\leq 2t\Gamma(f,f)(x) + 32\ell_0^2 M \sum_{i=1}^n \phi(x^i) \Gamma(f,f)(\Delta_i(x)) + 2Mtc(t) \int_0^t P_{w} \Gamma((f,f))(x) dw
\]
where in the last inequality we used that $c(t - s) \leq c(t)$. And so, the theorem follows for constants

$$
\alpha(t) = 2t, \quad \beta(t) = 32tt_0^2M
$$

$$\gamma(t) = tc(t) = t(t_08 + 2t)M^2(C_1 + 1). \quad (2.15)
$$

### 2.3. proof of Corollary 1.2

To obtain the statement of Corollary 1.2 one should notice that $\gamma(t)$ as a function of $t$ is of higher order than both $\alpha(t)$ and $\beta(t)$ and that $\int_0^t P_w \Gamma((f, f))(x)dw$ is an increasing function of $t$ since $\Gamma(f, f)$ is positive. So simply by choosing $t \geq t_0$ sufficiently large so that

$$
2t^2(1 + C_1)M \int_0^{t_0} P_w \Gamma((f, f))(x)dw \Rightarrow
$$

and

$$
\alpha(t) \Gamma(f, f)(x) \leq 2t^2(1 + C_1)M \int_0^{t_0} P_w \Gamma((f, f))(x)dw \Rightarrow
$$

$$
\alpha(t) \Gamma(f, f)(x) \leq \gamma(t) \int_0^t P_w \Gamma((f, f))(x)dw
$$

one obtains

$$
\text{Var}_{E^*}(f(x_t)) \leq 3\gamma(t) \int_0^t P_w \Gamma((f, f))(x)dw
$$

which shows the second assertion for a constant $\gamma(t)$ as in (2.15) and $t \geq \zeta(f)$ for

$$
\zeta(f) = \sup_{x \in D} \max_{i=1}^n \left\{ \frac{16t_0^2 \sum_{i=1}^n \phi(x_i) \Gamma(f, f)(\Delta_i(x))}{(1 + C_1)M \int_0^{t_0} P_w \Gamma((f, f))(x)dw}, \frac{\Gamma(f, f)(x)}{(1 + C_1)M^2 \int_0^{t_0} P_w \Gamma((f, f))(x)dw} \right\}
$$

depending on $f$.

### 3. proof of the Poincaré inequality for unbounded neurons.

In this section we prove the main results of the paper for systems of neurons that take values on $\mathbb{R}_+$, presented in Theorem 1.3.

As mentioned in the introduction, the approach used will be to reduce the problem from the unbounded case to the compact case examined in the previous section 2. To do this we will follow closely the Lyapunov approach developed in [11] and [7] to prove superPoincaré inequalities.
Since we have already obtained local Poincaré inequalities, we will work towards deriving the Lyapunov inequality required. That will be the subject of the next lemma. Then, in Proposition 3.2 we show how the two conditions, the local Poincaré of Theorem 1.1 and the Lyapunov inequality, are sufficient for the Poincaré type inequality of Theorem 1.3.

We recall that under the framework of [26], the generator of our process is given, for any function \( f \), by

\[
\mathcal{L}f(x) = \sum_{i=1}^{N} \phi(x^i) (f(\Delta_i(x)) - f(x))
\]

where \( \Delta_i(x) \) is defined by \( (\Delta_i(x))^j := x_j + W_{i \to j} \) if \( j \neq i \) and \( (\Delta_i(x))^i := 0 \).

We assume that for all \( i, j, W_{i \to j} \geq 0 \), we can then consider that the state space is \( \mathbb{R}_+^N \).

We put \( W_i := \sum_{j \neq i} W_{i \to j} \).

**Lemma 3.1.** Assume that for all \( x \in \mathbb{R}_+ \), \( \phi(x) \geq cx \) and \( \delta \leq \phi(x) \) for some constants \( c \) and \( \delta > 0 \). Then if we consider the Lyapunov function:

\[
V(x) = 1 + \sum_{i=1}^{N} x_i,
\]

there exist positive constants \( \lambda, b \) and \( m \) so that the following Lyapunov inequality holds

\[
\mathcal{L}V \leq -\lambda V + bI_B
\]

for the set \( B = \left\{ \sum_{i=1}^{N} x^i \leq m \right\} \).

**Proof.** For the Lyapunov function \( V \) as stated before, we have

\[
\mathcal{L}V(x) = \sum_{i=1}^{N} \phi(x^i)(W_i - x^i)
\]

\[
\leq -\sum_{i : x^i > 1 + W_i} \phi(x^i) + \sum_{i : x^i \leq 1 + W_i} \phi(x^i)(1 + W_i) - \delta \sum_{i : x^i \leq 1 + W_i} x^i
\]

\[
\leq -(c \wedge \delta) \sum_{i=1}^{N} x^i + \sum_{i=1}^{N} \phi(1 + W_i) W_i.
\]
Putting $b := \sum_{i=1}^{N} \phi(i + W_i) W_i$, we have for any $\alpha \in [0, 1]$

$$\mathcal{L}V(x) \leq -\alpha(c \land \delta) \sum_{i=1}^{N} x^i - (1 - \alpha)(c \land \delta) \sum_{i=1}^{N} x^i + b$$

$$\leq -\alpha(c \land \delta)V(x) + b + \alpha(c \land \delta) - (1 - \alpha)(c \land \delta) \sum_{i=1}^{N} x^i$$

$$\leq -\alpha(c \land \delta)V(x) + (b + \alpha(c \land \delta)) 1_B(x),$$

with

$$B = \left\{ \sum_{i=1}^{N} x^i \leq \frac{b + \alpha(c \land \delta)}{(1 - \alpha)(c \land \delta)} \right\}$$

in which case $m = \frac{b + \alpha(c \land \delta)}{(1 - \alpha)(c \land \delta)}$. Since $\alpha$ can be chosen arbitrary close to 1, if we want to impose $\alpha(c \land \delta) > 1$, we need to assume that $c > 1$ and $\delta > 1$. 

Having obtained all the elements required, we can finish the proof of the theorem by showing how the local Poincaré obtained in the previous section together with the Lyapunov inequality shown in the last lemma lead to the desired Poincaré type inequality.

**Proposition 3.2.** Assume that for some $V \geq 1$ the Lyapunov inequality

$$\mathcal{L}V \leq -\lambda V + b I_B$$

holds and that for some $D \supset B$ we have the weighted local Poincaré

$$\mathbb{E}^x f^2(x_t) I_D \leq \alpha(t) \Gamma(f, f)(x) + \beta(t) \sum_{i=1}^{n} \phi(x^i) \Gamma(f, f)(\Delta_i(x)) I_D +$$

$$+ \gamma(t) \int_0^t \mathbb{P}^x \Gamma((f, f))(x) dw + (\mathbb{E}^x f(x_t) I_D)^2.$$

Then

$$\int Var_{E^t}(f(x_t)) d\mu \leq \delta(t) \int \mathbb{E}^t \Gamma(f, f)(x_t) d\mu + \beta(t) \int D \sum_{i=1}^{n} \phi(x^i) \Gamma(f, f)(\Delta_i(x)) d\mu.$$  

Furthermore, for $t$ sufficiently large, i.e. $t \geq \theta(f)$ for

$$\theta(f) = \left( \frac{3\delta^2 \int_D \sum_{i=1}^{n} \phi(x_i) \Gamma(f, f)(\Delta_i(x)) d\mu}{(1 + C_1) \int \Gamma((f, f))(x) d\mu} \right)^{\frac{1}{2}}$$

the following Poincaré type inequality holds

$$\int Var_{E^t}(f(x_t)) d\mu \leq 2\delta(t) \int \mathbb{E}^t \Gamma(f, f)(x_t) d\mu.$$
where \( \delta(t) = \alpha(t) + t\gamma(t) + \frac{d_1}{t} \), for some \( d_1 > 0 \).

**Proof.** At first, we can write
\[
\mathbb{E}^x f^2(X_t) = \mathbb{E}^x f^2(X_t) I_D + \frac{1}{\lambda} \mathbb{E}^x f^2(X_t) \lambda I_{D^c}
\]
where \( \lambda \) is the Lyapunov constant. If we take the expectation with respect to the invariant measure \( \mu \) on both sides we get
\[
\int \mathbb{E}^x f^2 d\mu = \int \mathbb{E}^x f^2 I_D d\mu + \frac{1}{\lambda} \int \mathbb{E}^x f^2 \lambda I_{D^c} d\mu. \tag{3.2}
\]
In order to bound the first term on the right hand side of \((3.2)\) we can use the local Poincaré inequality to obtain
\[
\int \mathbb{E}^x f^2 I_D d\mu \leq (t) \sum_{i=1}^n \phi(x_i) \Gamma(f, f)(\Delta_i(x)) d\mu + (\alpha(t) + t\gamma(t)) \int \Gamma((f, f))(x) d\mu + \int (\mathbb{E}^x f(x_t) I_D) d\mu. \tag{3.3}
\]
since by the definition of the invariant measure
\[
\int \mathbb{E}^x g(x_t) d\mu = \mu(P_t g(x)) = \mu(g(x)).
\]
One should notice that although the invariant measure has helped us simplify the last expression, essentially the bound which was based on the local Poincaré inequality did not require the use of it.

For the second term on the right hand side of \((3.2)\) we can use the Lyapunov inequality. That gives
\[
\int \mathbb{E}^x f^2 \lambda I_{D^c} d\mu \leq \int \mathbb{E}^x f^2 \frac{-L V}{V} I_{D^c} d\mu + b \int \mathbb{E}^x f^2 I_{B \cap D^c} d\mu.
\]
If we choose \( D \) large enough to contain the set \( B \), i.e. \( B \cap D^c = \emptyset \) the last one is reduced to
\[
\int \mathbb{E}^x f^2 \lambda I_{D^c} d\mu \leq \int f^2 \frac{-L V}{V} I_{D^c} d\mu.
\]
The need to bound the quantity \( \frac{-L V}{V} \) which appears from the use of the Lyapunov inequality is the actual reason why we need to make use of the invariant measure \( \mu \) and obtain the type of Poincaré inequality shown in our final result, rather than the Poincaré type inequality based exclusively on the \( P_t \) measure obtained in the previous section for the compact case. If we had not taken the expectation with respect to the invariant measure, we would had needed to bound
\[
\int f^2 \frac{-L V}{V} I_{D^c} dP_t
\]
instead. This, in the case of diffusions can be bounded by the carre du champ of the function $\Gamma(f, f)$ by making an appropriate selection of exponential decreasing density (see for instance [7], [8] and [11]). In the case of jump processes however, and in particular of PJMP as on the current paper where densities cannot been specified, a similar bound cannot be obtained. However, when it comes to the analogue expression involving the invariant measure there is a powerful result that we can use, which has been presented in [11] (see Lemma 2.12). According to this, when the expectation is taken with respect to the invariant measure, the desired bound holds as seen in the following lemma.

**Lemma 3.3.** ([11]: Lemma 2.12) For every $U \geq 1$ such that $-\frac{\mathcal{L}U}{U}$ is bounded from below, the following bound holds

$$
\mu f^2 - \frac{\mathcal{L}U}{U} \leq d_1 \mu (-\mathcal{L}f)
$$

where $\mu$ is the invariant measure of the process and $d_1$ is some positive constant.

Since $V \geq 1$ and for $x \in D$ we have from the Lyapunov inequality that $-\frac{\mathcal{L}V}{V} \geq \lambda$ we get the following bound

$$
\int \mathbb{E} f^2 \lambda \mathcal{I}_D d\mu \leq d_1 \int (f(-\mathcal{L})f)d\mu
$$

for some positive constant $d_1$. Since for the infinitesimal operator $\mu(\mathcal{L}f) = 0$ for every function $f$, we can write

$$
\int (f(-\mathcal{L})f)d\mu = \frac{1}{2} \int (\mathcal{L}(f^2) - 2f \mathcal{L}f)d\mu = \frac{1}{2} \int \Gamma(f, f)d\mu.
$$

So that,

$$
(3.4)
\int \mathbb{E} f^2 \lambda \mathcal{I}_D d\mu \leq \frac{d_1}{2} \int \Gamma(f, f)d\mu.
$$

Gathering together (3.2), (3.3) and (3.4) we finally obtain the desired inequality

$$
\int \mathbb{E} f^2 d\mu \leq \int (\mathbb{E} f(x_i) \mathcal{I}_D)^2 d\mu + \beta(t) \int_{D} \sum_{i=1}^{n} \phi(x^i) \Gamma(f, f)(\Delta_i(x))d\mu +
$$

$$
(3.5)
+ (\alpha(t) + t\gamma(t) + \frac{d_1}{2\lambda}) \int \Gamma((f, f))(x)d\mu
$$

which proves the first assertion of the proposition for constants

$$
\delta(t) = \alpha(t) + t\gamma(t) + \frac{d_1}{2\lambda}
$$

and $\beta(t)$ as in (2.15).

To obtain the second inequality of the proposition, we will work in the same manner as with the local inequality in the previous section. One notices that $\delta(t)$
as a function of $t$ is of higher order than $\beta(t)$. So simply by choosing $t$ sufficiently large so that $t \geq \theta(f)$, for

$$
\theta(f) = \left( \frac{4N\delta^2 t_0^2 \int \Gamma(f, f)(\Delta_i(x))d\mu}{C_1 M \int \Gamma((f, f))(x)d\mu} \right)^{\frac{1}{2}}
$$

we get

$$
\beta(t) \int \Gamma(f, f)(\Delta_i(x))d\mu \leq (\alpha(t) + t\gamma(t) + \frac{d}{2\lambda}) \int \Gamma((f, f))(x)d\mu.
$$

From this and (3.5) we obtain

$$
\int \text{Var}_{E^t}(f(x_i))d\mu \leq 2\delta(t) \int \Gamma((f, f))(x)d\mu
$$

which shows the second assertion of the proposition for constants $\alpha(t)$ and $\gamma(t)$ as in (2.15), $\delta(t) = \alpha(t) + t\gamma(t) + \frac{d}{2\lambda}$ and $t \geq \theta(f)$, for $\theta(f)$ depending on $f$.

\[\square\]

The last proposition together with the Lyapunov inequality from Lemma 3.1 and the local Poincaré inequality of Theorem 1.1 proves Theorem 1.3.

4. Proof of Concentration and Talagrand Inequality.

It is a well know result that the Poincaré inequality

$$
\text{Var}_\mu(f^2) \leq c\mu(\Gamma(f, f))
$$

implies exponential convergence to equilibrium

$$
\mu(\{f - \mu(f) \geq r\}) \leq e^{cr}
$$

(see for instance [10] and [28]). In the next proposition we show how the weaker Poincaré type inequalities obtained in theorems 1.1 and 1.3 can yield to similar exponentially fast convergence to equilibrium.

The concentration properties will be based on the following proposition, that follows closely the approach in [28] (see also [29], [10], [1] and [2]), which however assumed the typical Poincaré inequality (SG) for $C(t) = C$ that does not depend on time. We will also use elements from [9] since one of the main conditions (4.2), will refer to the bounded function $F_r = \min\{F, r\}$ instead of the function $F$, for $r > 0$.

Proposition 4.1. Assume that

\[4.1\]

$$
\mu \left( P_t f^2(x) - (P_t f(x))^2 \right) \leq \xi(t) \int \Gamma(f, f)(x)d\mu + \xi(t) \int_D \sum_{i=1}^{n} \phi(x_i) \Gamma(f, f)(\Delta_i(x))d\mu
$$
for some \( \xi(t) > 0 \), and that for some \( \lambda(t) > 0 \) and function \( F \),
\[
\mu(\Gamma(e^{\lambda(t)F_r/2}, e^{\lambda(t)F_r/2})) \leq C_1 \lambda(t)^2 \mu(e^{\lambda(t)F_r})
\]
and
\[
\int_D \sum_{i=1}^n \phi(x_i) \Gamma(F_r, F_r)(\Delta_i(x)) d\mu \leq C_2 \lambda(t)^2 \mu(e^{\lambda(t)F_r})
\]
where \( F_r = \min(F(x), r) \), for \( r := r(t) > 0 \). Then, for \( \lambda(t), \xi(t) \) such that \( \xi(t)(C_1 + C_2)\lambda(t)^2 < 1 \), there exists a constant \( \lambda_0 > 0 \) such that
\[
\mu(\{F > r(t)\}) \leq \lambda_0 e^{-\lambda(t)r(t)}.
\]

**Proof.** From the Poincaré type inequality, if we put \( e^{\lambda(t)F_r/2} \) in the place of \( f \) for \( \lambda(t) > 0 \) we have
\[
\mu(P_te^{\lambda(t)F_r}(x) - (P_te^{\lambda(t)F_r}(x))^2) \leq \xi(t) \mu(\Gamma(e^{\lambda(t)F_r}, e^{\lambda(t)F_r}))
\]
\[
+ \int_D \sum_{i=1}^n \phi(x_i) \Gamma(e^{\lambda(t)F_r}, e^{\lambda(t)F_r})(\Delta_i(x)) d\mu.
\]
To ease the notation we will write \( \lambda = \lambda(t) \) and \( \xi = \xi(t) \). Use the condition on the carre du champ to bound the right hand side, and set \( \Lambda(\lambda) = P_t(e^{\lambda F_r}) \), then
\[
\mu(\Lambda(\lambda)) - \mu(\Lambda(\frac{\lambda}{2})) \leq \xi C \lambda^2 \mu(\Lambda(\lambda))
\]
where \( C = C_1 + C_2 \). Since \( \psi := \xi C \lambda^2 < 1 \),
\[
\mu(\Lambda(\lambda)) \leq \frac{1}{1-\psi} \mu(\Lambda(\frac{\lambda}{2})^2).
\]
Iterate this inequality to obtain
\[
\mu(\Lambda(\lambda)) \leq \prod_{k=0}^{n-1} \left( \frac{1}{1-\psi^4^k} \right)^{2^k} \mu(\Lambda(\frac{\lambda}{2^n})^2).
\]
Since \( \Lambda(\lambda) = 1 + o(\lambda) \) one has that \( \Lambda(\frac{\lambda}{2^n})^2 \to 1 \) as \( n \to \infty \), and so
\[
\mu(\Lambda(\lambda)) \leq \prod_{k=0}^{\infty} \left( \frac{1}{1-\psi^4^k} \right)^{2^k}
\]
where above the product in the righthand side converges since \( \psi < 1 \), i.e. \( \lambda_0 < \infty \).

Since
\[
\{F_r < r\} = \{F < r\}
\]
we can apply Chebyshev’s inequality
\[
\mu(\{F > r\}) \leq e^{-\lambda r} \mu(e^{\lambda F_r})
\]
But because of the invariant measure property \( \mu P_t = \mu \), we have
\[
\mu(e^{\lambda F_r}) = \mu(P_t(e^{\lambda F_r})) = \mu(\Lambda(\lambda)) \leq \lambda_0
\]
because of (4.4), and so, we finally obtain
\[
\mu(\{F > r\}) \leq \lambda_0 e^{-\lambda r}.
\]

The concentration results presented in theorems 1.7, 1.4, and 1.5 follow directly from Proposition 4.1. The main condition (4.1) of Proposition 4.1 in the case of unbounded neurons is again guaranteed by Theorem 1.3 for \( \xi(t) = \delta(t) \), while for bounded neurons, it follows from the Poincaré type inequality of Theorem 1.1, after taking the expectation with respect to the invariant measure \( \mu \) in both sides of the inequality, for a time constant \( \xi(t) = a(t) + t\gamma(t) \). Then, choose \( r(t) \) such that \( r(t) \lambda(t) \to \infty \) as \( t \to \infty \) and \( \lambda(t)C\lambda(t)^2 < 1 \) so that \( \psi(t) < 1 \). For example, we can consider \( \lambda(t) = \frac{1}{2\sqrt{t}} t^{-\frac{3}{2}} \) and \( r(t) = t^{a+\frac{3}{2}} \), for any \( a > 0 \). Then we compute \( r = \frac{1}{2\sqrt{t}} t^a \to \infty \) as \( t \to \infty \), while \( \xi(t)C\lambda(t)^2 = \frac{1}{2} < 1 \), and the Talagrand inequality takes the form
\[
\mu(\{F > r^{a+\frac{3}{2}}\}) \leq \lambda_0 e^{-r}.
\]
for any \( a > 0 \). It remain to verify conditions (4.2) and (4.3).

In the following two subsections we show that the conditions (4.2) and (4.3) are satisfied for appropriate functions so that the desirable concentration inequalities will hold.

4.1. Talagrand inequality. In order to prove the Talagrand inequality presented in Theorem 1.7, it remains to show conditions (4.2) and (4.3). This will be the subject of the next two lemmata. We consider \( F(x) = \sum_{i=1}^{N} x^i \) for \( x = (x^1, ..., x^N) \in \mathbb{R}_+^N \).

We start with (4.2).

**Lemma 4.2.** Assume the PJMP as described in (1.2)-(1.5). Let \( F(x) = \sum_{i=1}^{N} x^i \) for \( x = (x^1, ..., x^N) \in \mathbb{R}_+^N \). Then
\[
\mu(\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})) \leq C_1 \lambda^2 \mu(e^{\lambda F_r})
\]
where \( F_r = \min(F(x), r) \) for \( r > 0 \).

**Proof.** From the definition of the carre du champ
\[
\mu(\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})) = \sum_i \mu\left(\phi(x^i)(e^{\lambda F_r(x)}/2 - e^{\lambda F_r(M_i(x))}/2)^2\right).
\]
To bound \( \mu(M_i) \) we will distinguish four cases:
a) Consider the set \( A := \{ x : F(x) \geq r \text{ and } F(\Delta_i(x)) \geq r \} \). Then, for \( x \in A \), \( F_r(\Delta_i(x)) = F_r(x) = r \) and so \( \mu(M_i^A) = 0 \).

b) Consider the set \( B := \{ x : F(x) \geq r \text{ and } F(\Delta_i(x)) \leq r \} \). Then, for \( x \in B \),

\[
F_r(\Delta_i(x)) = \sum_{j \neq i} \Delta_i(x)^j < r = F_r(x) \leq \sum_j x^j
\]

so that

\[
\mu(M_i^B) \leq \lambda^2 \mu \left( \phi(x^i)(x^i)^2 - \sum_{j \neq i} W_{i \rightarrow j} \right) = C^i \lambda^2 \mu \left( e^{F_r} \right) B
\]

where above we denoted \( C^i = \mu(\phi(x^i)(x^i)^2) + N_0^2 \mu(\phi(x^i)) \) and computed

\[
e^{AF_r} B = e^{\lambda r} B = \mu(e^{\lambda r} B) = \mu(e^{AF_r} B).
\]

c) Consider the set \( C := \{ F(\Delta_i(x)) \leq F(x) < r \} \). Then, for \( x \in C \),

\[
F_r(x) - F_r(\Delta_i(x)) = \sum_j x^j - \left( \sum_{j \neq i} x^j + \sum_{j \neq i} W_{i \rightarrow j} \right) = x^i - \sum_{j \neq i} W_{i \rightarrow j} \geq 0,
\]

so that

\[
\mu(M_i^C) \leq \lambda^2 \mu \left( \phi(x^i)(x^i)^2 - \sum_{j \neq i} W_{i \rightarrow j} \right) = C^i \lambda^2 \mu \left( e^{AF_r} C \right)
\]

Since \( F_r \leq r \) we know that \( \mu(e^{AF_r}) \leq e^{\lambda r} < \infty \) and so we can bound

\[
\mu(M_i^C) \leq \lambda^2 \left( \sup_{g : \mu(g) = 1} \{ \phi(x^i)(x^i)^2 g \} \right) \mu(e^{AF_r} C) \leq \lambda^2 \| \phi(x^i)(x^i)^2 \|_\infty \mu(e^{AF_r} C)
\]

where

\[
\| f \|_\infty = \sup_{g : \mu(g) = 1} \{ \mu(fg) \}.
\]
d) Consider the set \( D := \{ F(x) < r \text{ and } F(x) < F(\Delta_i(x)) \} \). Then, for \( x \in D \),
\[
\sum_j x^j = F_r(x) < F_r(\Delta_i(x)) \leq \sum_{j \neq i} W_{i \to j} + \sum_{j \neq i} x^j = F_r(x) + \left( \sum_{j \neq i} W_{i \to j} - x^i \right)
\]
which means that \( x^i \) is bounded by
\[
x^i \leq \sum_{j \neq i} W_{i \to j} \leq N_0
\]
and that
\[
0 \leq F_r(\Delta_i(x)) - F_r(x) \leq \sum_{j \neq i} W_{i \to j} - x^i.
\]
So, we can compute
\[
\mu(M, \mathcal{I}_D) \leq \lambda^2 \mu \left( \phi(x^i)e^{\lambda F_r(x)}(F_r(x) - F_r(\Delta_i(x)))^2 \right)
\]
\[
\leq \lambda^2 \mu \left( \phi(x^i)e^{\lambda F_r(\Delta_i(x))}(x^i - \sum_{j \neq i} W_{i \to j})^2 \right)
\]
\[
\leq \lambda^2 \mu \left( \phi(x^i)e^{\lambda F_r(x)}e^{\lambda(\sum_{j \neq i} W_{i \to j} - x^i)}(x^i - \sum_{j \neq i} W_{i \to j})^2 \right)
\]
\[
\leq N_0^2 \phi(N_0) \lambda^2 e^{\lambda N_0} \mu \left( e^{\lambda F_r} \mathcal{I}_D \right).
\]
If we gather all four cases together, we finally obtain
\[
\mu(\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})) \leq C\lambda^2 \mu \left( e^{\lambda F_r} \right)
\]
for a constant
\[
C_1 = \max \{ \mu(\phi(x^i)(x^i)^2) + N_0^2 \mu(\phi(x^i)), |||\phi(x^i)(x^i)^2||_\infty, N_0^2 \phi(N_0)e^{\lambda N_0} \}.
\]
\[\square\]

In the next lemma we show condition (1.3).

**Lemma 4.3.** Assume the PJMP as described in (1.2)-(1.5). Let \( F(x) = \sum_{i=1}^N x^i \) for \( x = (x^1, ..., x^N) \in \mathbb{R}_+^N \) and \( F_r = \min(F(x), r) \) for \( r > 0 \). Then,
\[
\mu \left( \sum_i \phi(x^i)\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})(\Delta_i(x))\mathcal{I}_D(x) \right) \leq C_1\lambda^2 \mu \left( e^{\lambda F_r} \right)
\]
where \( D \) a compact set such that \( D \supset \left\{ \sum_{i=1}^N x_i \leq m \right\} \).

**Proof.** Since \( D \) compact, we can assume that there exists some some \( \tilde{m} > 0 \) such \( \{ x : x_i \leq \tilde{m} \} \supset D \). Then
\[
\phi(x^i)\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})(\Delta_i(x))\mathcal{I}_D \leq \phi(\tilde{m})\Gamma(e^{\lambda F_r/2}, e^{\lambda F_r/2})(\Delta_i(x))\mathcal{I}_D(x)
\]
since $\phi$ is an increasing function. Similarly, since $W_{i \rightarrow j} \leq \sum_j W_{i \rightarrow j} \leq N_0$, for every $x \in D$ we have that $\Delta_i(x)^j \leq x^j + W_{i \rightarrow j} \leq \tilde{m} + N_0$, and so

$$\Gamma(e^{\lambda F_{i}^r/2}, e^{\lambda F_{i}^r/2})(\Delta_i(x))I_D(x) \leq \phi(\tilde{m} + N_0)\sum_j (e^{\lambda F_{i}^r(\Delta_i(x))/2} - e^{\lambda F_{i}^r(\Delta_i(x))/2})^2I_D(x).$$

In the case where $F(\Delta_i(x)^j) \geq r$ and $F(\Delta_i(x)) \geq r$ we trivially get $M_{ij} = 0$ since $F_r(\Delta_i(x)) = F_r(\Delta_i(x)^j) = r$. Otherwise,

$$M_{ij}(x) \leq (e^{\lambda F_{r}(\Delta_i(x))/2} - e^{\lambda F_{r}(\Delta_j(\Delta_i(x))))/2})^2 \leq \lambda^2 e^{\lambda F_{ij}^r(\hat{F}_{ij}^r)^2}$$

where above we have for simplicity denoted $\hat{F}_{ij}^r = \max\{F_r(\Delta_i(x)), F_r(\Delta_j(\Delta_i(x))))\}$.

As before, we compute

$$\Delta_j(\Delta_i(x))^k \leq \Delta_i(x)^k + W_{j \rightarrow k}(\Delta_i(x)) \leq x^k + W_{i \rightarrow k} + W_{j \rightarrow k}(\Delta_i(x)) \leq x^k + 2N_0$$

which together with $\Delta_i(x)^k \leq x^k + W_{i \rightarrow k} \leq x^k + \max\{N_0, \tilde{m}\}$ give the following bound

$$\hat{F}_{ij}^r \leq \sum_k x^k + N\tilde{m} + 2NN_0.$$

From the last one, we also obtain that $\hat{F}_{ij}^r \leq 4N\max\{N_0, \tilde{m}\}$ for $x \in D$, as well as that $\hat{F}_{ij}^r \leq F_r + 3N\max\{N_0, \tilde{m}\}$. Putting all together we finally obtain

$$\mu \left( \sum_i \phi(x^i)\Gamma(e^{\lambda F_{i}^r/2}, e^{\lambda F_{i}^r/2})(\Delta_i(x))I_D(x) \right) \leq C_2 \lambda^2 \mu(e^{\lambda F_{i}^r(x)})$$

where the constant $C_2 = e^{\lambda N_1}\phi(\tilde{m})N^2\sum_{i=1}^{\infty} 16N^2\max\{N_0, \tilde{m}\}^2\phi(\tilde{m} + N_0)$, for $N_1 = 3N\max\{N_0, \tilde{m}\}$. □

4.2. concentration inequalities. In this section we prove the main concentration properties of the paper, presented in theorems 1.4 and 1.5.

As one can see in the main tool to show concentration properties presented in Proposition 1.1 we need to bound $\mu(\Gamma(e^{\lambda M/2}, e^{\lambda M/2}))$. In the case of diffusions, where $\mu(\Gamma(f, f)) = \mu(||\nabla f||^2)$, for any smooth function $\psi$ one has

$$\mu(\Gamma(\psi(f), \psi(f))) \leq ||\nabla f||^2_{2, \infty}\mu(\psi^2(f)^2)$$

and so one can bound $\mu(\Gamma(e^{\lambda M/2}, e^{\lambda M/2})) \leq \frac{\lambda^2}{4}||\nabla f||^2_{2, \infty}\mu(e^{\lambda M})$, and so the condition follows for functions $f$ such that $||\nabla f||^2_{\infty} < 1$ (see [28] and [29]). In the case, as is in the current paper, of an energy expressed through differences, where the chain rule is not satisfied, this cannot hold. However, as demonstrated in [2] (see also [23] for applications), in the special situation where the semigroup is symmetric, one can have an analogue result, that is

$$\mu(\Gamma(e^{\lambda M/2}, e^{\lambda M/2})) \leq \frac{\lambda^2}{4}|||f|||^2_{2, \infty}\mu(e^{\lambda M(x)})$$
where now \( |||f|||_\infty \) can be considered as a generalised norm of the gradient (see also \(^{28}\)), given by the following expression
\[
|||f|||_\infty = \sup \left\{ \mathcal{E}(g, f) - \frac{1}{2} \mathcal{E}(g, f^2); g : ||g||_1 \leq 1 \right\}
\]
where \( \mathcal{E}(f, g) := \lim_{t \to 0} \frac{1}{2t} \int (f(x) - f(y))^2 p_t(x, dy) \mu(dx) \). Then, of course, for the concentration property to hold, one needs functions that satisfy the following condition
\[
|||f|||_\infty < 1.
\]
In our case however, where the semigroup is not symmetric, we can still obtain the desired property if we consider at first functions such that
\[
D(f) := \max \sup_{i \in \mathbb{N}} \{|f(x) - f(\Delta_i(x))|\} < 1
\]
as presented in Theorem 1.4. Furthermore, the result can be further obtained for a different class of functions, that satisfy
\[
|||\phi(x^i)D(f)^2|||_\infty < 1 \quad \text{and} \quad |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1
\]
where \( ||f|||_\infty = \sup_{g, \mu(g)=1} \{\mu(fg)\} \).

The proof is again based on Proposition 4.1. The desired result follows from Proposition 4.1 by putting \( f - \mu(f) \) in place of \( f \), for every function that satisfies conditions (4.2) and (4.3).

In the following lemma we show condition (4.2) under the hypothesis \( |||\phi(x^i)D(f)^2|||_\infty < 1 \) and \( |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1 \) of Theorem 1.5 for non-compact neurons as in (1.2)-(1.5).

**Lemma 4.4.** Assume the PJMP as described in (1.2)-(1.5). Assume functions \( f \) such that
\[
|||\phi(x^i)D(f)^2|||_\infty < 1 \quad \text{and} \quad |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty < 1
\]
Then
\[
\mu(\Gamma(e^{\lambda f_{x/2}}, e^{\lambda f_{x/2}})) \leq C_1 \lambda^2 \mu \left( e^{\lambda r} \right).
\]

**Proof.** From the definition of the carre du champ we compute
\[
\mu(\Gamma(e^{\lambda f_{x/2}}, e^{\lambda f_{x/2}})) = \sum_{i=1}^N \mu \left( \phi(x^i)(e^{\lambda f_{x}(x)} - e^{\lambda f_{\Delta_i(x)}(x)})^2 \right).
\]

a) Consider the set \( A := \{x : f(x) \geq r \text{ and } f(\Delta_i(x)) \geq r\} \). Then, for \( x \in A \), \( f(\Delta_i(x)) = f_r(x) = r \) and so \( \mu(M_i \lambda A) = 0 \).

b) Consider the set \( B := \{x : f(x) \geq r \text{ and } f(\Delta_i(x)) \leq r\} \). Then, for \( x \in B \),
\[
\mu(M_i \lambda B) \leq \lambda^2 \mu \left( \phi(x^i)e^{\lambda f_{x}(x)}D(f)^2 \right) \leq \lambda^2 e^{\lambda r} \mu \left( \phi(x^i)D(f)^2 \right)
\]
which leads to
\[ \mu(M_i I_B) \leq \mu(\phi(x^i)D(f)^2) \lambda^2 \mu(e^{\lambda f} I_B) \]

since
\[ e^{\lambda f} I_B = e^{\lambda r} I_B = \mu(e^{\lambda r} I_B) = \mu(e^{\lambda f} I_B). \]

c) Consider the set \( C := \{ f(\Delta_i(x)) \leq f(x) < r \} \). Then, for \( x \in C \),
\[
\begin{align*}
\mu(M_i I_C) & \leq \lambda^2 \mu(\phi(x^i)e^{\lambda f}(f_r(x) - f_r(\Delta_i(x)))^2) \\
& \leq \lambda^2 \mu(\phi(x^i)e^{\lambda r}D(f)^2).
\end{align*}
\]

Since \( f_r \leq r \) we know that \( \mu(e^{\lambda f}) \leq e^{\lambda r} < \infty \) and so we can bound
\[
\mu(M_i I_C) \leq \lambda^2 \left( \sup_{g, \mu(g) = 1} \{ \phi(x^i)D(f)^2 \} \right) \mu(e^{\lambda f} I_C) \leq \lambda^2 |||\phi(x^i)D(f)^2|||_\infty \mu(e^{\lambda f} I_C)
\]

where
\[
|||f|||_\infty = \sup \{ \mu(fg) : g \in \mu(g) \leq 1 \}.
\]

d) Consider the set \( D := \{ f(x) < r \text{ and } f(x) < f(\Delta_i(x)) \} \). Then, for \( x \in D \),
\[
\begin{align*}
\mu(M_i I_D) & \leq \lambda^2 \mu(\phi(x^i)e^{\lambda r}(\Delta_i(x)) (f_r(x) - f_r(\Delta_i(x)))^2) \\
& \leq \lambda^2 \mu(\phi(x^i)e^{\lambda r}e^{\lambda D(f)}D(f)^2) \\
& \leq \lambda^2 |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty \mu(e^{\lambda f} I_D).
\end{align*}
\]

where again we used that \( e^{\lambda f} I_D = e^{\lambda r} I_D = \mu(e^{\lambda r} I_D) = \mu(e^{\lambda f} I_D). \)

If we gather everything together we finally obtain
\[
\mu(\Gamma(e^{\lambda f/2}, e^{\lambda r/2})) \leq \lambda^2 \sum_{i=1}^{N} \left( 2|||\phi(x^i)D(f)^2|||_\infty + |||\phi(x^i)e^{\lambda D(f)}D(f)^2|||_\infty \right) \mu(e^{\lambda f})
\]
\[
\leq 3N \lambda^2 \mu(e^{\lambda f}).
\]

and the lemma follows for some constant \( C_2 = 3N \).

\[ \square \]

The analogue result under the hypothesis \( D(f) := \max_{i \in N} \sup_{x} \{|f(x) - f(\Delta_i(x))| \} < 1 \) of Theorem \ref{t3} takes the form

\begin{lemma}
Consider functions \( f \) such that
\[ D(f) := \max_{i \in N} \sup_{x} \{|f(x) - f(\Delta_i(x))| \} < 1. \]

Define \( f_r = \min(f(x), r) \). Then
\[
\mu(\Gamma(e^{\lambda f_r/2}, e^{\lambda f_r/2})) \leq C_1 \lambda^2 \mu(e^{\lambda f_r})
\]

for \( C_1 = 3|||\phi(x^i)|||_\infty \).
\end{lemma}
We omit the proof of this lemma, since this can be easily obtained from the proof of the previous lemma after the appropriate modifications.

It remain to verify condition (4.3). For the carre du champ for neurons in the compact $D$, we can work as in Lemma 4.3. Then

\[
\left( e^{\lambda f_r(\Delta_i(x))} - e^{\lambda f_r(\Delta_j(x)))} \right)^2 \leq \lambda^2 \left( e^{\lambda \max(f_r(\Delta_i(x))), f_r(\Delta_j(x)))} \right)^2 \leq \lambda^2 e^{\lambda^2 D(f)} e^{\lambda f_r(D(f))^2}
\]

since $f_r(\Delta_i(x)) \leq f_r(x) + D(f)$ and

\[
f_r(\Delta_j(\Delta_i(x))) \leq f_r(\Delta_i(x)) + D(f) \leq f_r(x) + 2D(f)
\]

and so similar bounds as the ones presented in lemmata 4.4 and 1.5 can be obtained for $\mu \left( \sum_i \phi(x^i) \Gamma(e^{\lambda F_r/2}, e^{\lambda^2 F_r/2})(\Delta_i(x)) I_D(x) \right)$.

Similarly, one can obtain the conditions (4.2) for the case of compact neurons as described in (1.4)-(1.7).

Acknowledgements

The authors thank Eva L" ocherbach for careful reading and valuable comments.

References

[1] S. Aida and T. Masuda and I. Shigekawa Logarithmic Sobolev inequalities and exponential integrability. J. Funct.Anal. 126, 83-101 (1994). 1, 75-86 (1994).
[2] S. Aida and D. Stroock Moment estimates derived from Poincaré and logarithmic Sobolev inequalities. Math. Res. Lett. 1, 75-86 (1994).
[3] C. Ane and M. Ledoux, Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. Probab. Theory Relat. Fields 116, 573-602 (2000).
[4] R. Azais, J.B. Bardet, A. Genadot, N. Krell and P.A. Zitt, Piecewise deterministic Markov process (pdmps). Recent results. Proceedings 44, 276-290 (2014).
[5] D. Bakry, L’hypercontractivité et son utilisation en théorie des semigroupes. Ecole d’Eté de Probabilités de St-Flour. Lecture Notes in Math., 1581, 1-114, Springer (1994).
[6] D. Bakry, On Sobolev and logarithmic Sobolev inequalities for Markov semigroups. New trends in Stochastic Analysis, 43-75, World Scientific (1997).
[7] D. Bakry, P. Cattiaux and A. Guillin, Lyapunov conditions for super Poincaré inequality. J Funct Anal 254 (3), 727-759 (2008).
[8] D. Bakry, I. Gentil, M. Ledoux, Analysis and geometry of Markov diffusion operators. Grundlehren der Mathematischen Wissenschaften 348, Springer, Berlin, (2014).
[9] F. Barth and C. Roberto Modified logarithmic Sobolev inequalities on $\mathbb{R}$. Potential Anal 29, 167-193 (2008).
[10] S. Bobkov and M. Ledoux, Poincaré’s inequalities and Talagrand’s concentration phenomenon for the exponential distribution. Probab Theory Relat Fields 107, 383-400 (1997).
[11] P. Cattiaux, A. Guillin, F-Y. Wang and L. Wu, Lyapunov conditions for super Poincaré inequality. J Funct Anal 256 (6), 1821-1841 (2009).
[12] D. Chafaï, Entropies, convexity, and functional inequalities. J. Math. Kyoto Univ. 44 (2), 325 - 363 (2004).
[13] J. Chevalier, Mean-field limit of generalized Hawkes processes. Stochastic Processes and their Applications 127 (12), 3870 - 3912 (2017).
[14] A. Crudu, A. Debussche, A. Muller and O. Radulescu Convergence of stochastic gene networks to hybrid piecewise deterministic processes. The Annals of Applied Probability 22, 1822-1859, (2012).
[15] M.H.A. Davis Piecewise-derministic Markov processes: a general class off nondiffusion stochastic models J. Roy. Statist. Soc. Ser. B, 46(3) 353 - 388 (1984).
[16] M.H.A. Davis Markov models and optimization Monographs on Statistics and Applied Probability, vol. 49 Chapman & Hall, London. (1993)
[17] J.D. Deuschel and D. Stroock, Large Deviations , Academic Press, San Diego (1989).
[18] P. Diaconis and L. Saloff-Coste Logarithmic Sobolev inequalities for finite Markov Chains. The Annals of Applied Probability 6, 695-750, (1996).
[19] A. Duarte, E. Löcherbach and G. Ost, Stability, convergence to equilibrium and simulation of non-linear Hawkes Processes with memory kernels given by the sum of Erlang kernels . (arxiv).
[20] A. Duarte and G.Ost, A model for neural activity in the absence of external stimuli Markov Processes and Related Fields 22, (2014).
[21] A. Galves and E. Löcherbach, Infinite Systems of Interacting Chains with Memory of Variable Length-A Stochastic Model for Biological Neural Nets. J Stat Phys 151, 896-921 (2013).
[22] I. Gentil, A. Guillin and L. Miclo, Modified logarithmic Sobolev inequalities and transport inequalities. Probab.Theor. Relat. Fields 13 3, 409-436 (2005).
[23] L. Gross and O. Rothaus, Herbst inequalities for supercontractive semigroups, J. Math. Kyoto Univ. 38, 295-318 (1998).
[24] A.Guionnet and B.Zegarlinski, Lectures on Logarithmic Sobolev Inequalities, IHP Course 98, 1-134 in Seminaire de Probabilite XXVI, Lecture Notes in Mathematics 1801, Springer (2003).
[25] N. Hansen, P. Reynaud-Bouret and V. Rivoirard Lasso and probabilistic inequalities for multivariate point processes. Bernoulli, 21(1) 83-143 (2015).
[26] P. Hodara, N. Krell and E. Löcherbach, Non-parametric estimation of the spiking rate in systems of interacting neurons. E. Stat Inference Stoch Process, 1-16 (2016).
[27] P. Hodara and E. Löcherbach, Hawkes processes with variable length memory and an infinite number of components. Adv. Appl. Probab 49, 84-107 (2017).
[28] M. Ledoux, The concentration of measure phenomenon. Mathematical Surveys and monographs 89, AMS (2001).
[29] M. Ledoux, Concentration of measure and logarithmic Sobolev inequalities. Seminaire de Probabilites XXXX. Lecture notes in Math. 1709, 120-216, Springer (1999).
[30] E. Löcherbach, Absolute continuity of the invariant measure in piecewise deterministic Markov Processes having degenerate jumps. Stochastic Processes and their Applications (2017).
[35] M. Talagrand, *A new isoperimetric inequality and concentration of measure phenomenon*. In: Lindenstrauss, J., Milman, V.D. (eds.) Geometric Aspects of Functional Analysis, Lecture notes in Math. 1469, 94-124, Springer-Verlag, Berlin (1991).

[36] F-Y. Wang and C. Yuan, *Poincaré inequality on the path space of Poisson point processes*. J Theor Probab 23 (3), 824-833 (2010).