Nonlinear Behavior in Ferromagnetism:
Simple Example and Possible Implications

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Abstract

Two cases of a phenomenological model for ferromagnetism are considered, discrete and continuous. And the relationship, in general, between discrete and continuous models explored. In a similar way to the logistic map behavior, the continuous case is exactly solvable while the discrete one contains the bifurcation route to chaos. Through the ferromagnetic interpretation I comment on the relevance of this to understanding evolution of systems in time, the role of the configuration space in chaotic behavior, and how this understanding may lead to new exotic magnetic phenomena.

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It has been known for some time in chaos theory that models behave in different ways when the dynamics is performed using different types of time evolution [1, 2]. A simple example of this is the logistic map, for which the continues time case is fully solvable for all finite parameter values while for discrete time the solution ranges from solvable to chaotic in a route known as the route to chaos. In this paper I analyze another simple example of this dual behavior, for a model based on two essential characteristics of ferromagnetism: symmetry breaking of the magnetization at the critical point and dissipating massless dynamics. The model considered is related to the Landau Ginzburg model of magnetism. By keeping the model simple the limit of the continuous case can be solved exactly and since the model contains the essential characteristics of ferromagnetism we have some tools that we can use to interpret the results.

The outline of the article is as follows. The model is introduced and then analyzed with the assumption of continuous time evolution. For this case one finds an initial-condition dependent, predictable relaxation toward the fixed points of the potential energy, for all range of parameters. Next, the basis for discrete time evolution and the relationship between discrete and continuous time evolution are discussed. Finally the discrete case is explored and the same relaxation behavior is found for some parameter regime but new types of behavior (leading to chaos) for other parameter regimes. I finish by interpreting this difference, that results from a different treatment of time, and comment on the importance of this to our understanding of the way processes evolve with time, and indicate systems where such behavior may be found.

The physical picture is as follows. Assume a bulk ferromagnet and let $\mathcal{F}(\Phi)$ be its total free energy, with $\Phi$ the total scalar magnetization in a given direction. If locally $\mathcal{F}(\Phi)$ is a proper physical function, it must have an appropriate expansion as a power series. The sign of $\Phi$ is an artifact of the choice of south magnetic pole, so $\mathcal{F}(\Phi) = \mathcal{F}(-\Phi)$ since reversing this choice should have no physical relevance. If we now take the fourth order approximation and neglect an additive term in the free energy we find,

$$\mathcal{F} = -\frac{a}{2}\Phi^2 + \frac{b}{4}\Phi^4$$

with $a$, $b$ the expansion coefficients. To ensure that infinite magnetization is excluded we must have $b > 0$. Since $b$ is a second order parameter it is assumed for the rest that it is a constant. Equation (1) has two characteristic forms as we vary $a$, for $a < 0$ there is a
single potential well around 0, we can identify this regime as paramagnetism. In the second regime, for \( a > 0 \), there is a double well potential. This broken symmetry regime is identified with ferromagnetism. Indeed for this model we can write \( a \propto (T_c - T) \). Where \( T, T_c \) are the temperature and the Curie temperature respectively.

Changing the magnetization is a strongly dissipative process that carries little to no inertia. To model this I use dissipative Newtonian dynamics for a massless particle and arrive at the following equation of motion for the magnetization \( \Phi \),

\[
\dot{\Phi} = M(a\Phi - b\Phi^3),
\]

where \( M \) is the mobility and \( \frac{1}{M} \) is related monotonically to a damping coefficient also known as the Khalatnikov damping constant. We can integrate this equation immediately to find the following solution for positive initial conditions,

\[
\Phi(t) = \frac{e^{2Mt}}{\frac{2}{a}e^{2Mt} + \left| 1 - \frac{a\Phi_0^2}{\Phi_0^2} \right|}.
\]

For negative initial conditions we find the solution to be \(-\Phi(t)\). For the trivial initial condition we find the trivial solution. The non-trivial long time behavior exhibits an asymptote located at the fixed point of the free-energy, \( \lim_{t \to \infty} \Phi(t) = Re(\pm \sqrt{\frac{a}{b}}) \). This solution is completely deterministic for all finite values of parameters and initial conditions, by this it is meant that for any desired accuracy of prediction past some time \( t_0 \), one can give a requirement on the accuracy of the initial conditions at \( t_0 \) which is good for all times \( t > t_0 \). As we shall see shortly this situation is rather different in the discrete case.

Before going into the discrete case we should explore the relationship between it and the continuous case. I claim there is a physical correspondence between the discrete and the continuous models. Usually the continuous solution would be called analytic and appear more pleasing to the trained physical eye. But there is no a-priori reason to assume that systems evolve continuously with time, passing through a continuous number of states. Indeed one can imagine how discrete processes are possible, where systems make jumps from state to state and between jumps remain in their state for some time interval \( \Delta t \).

To relate continues and discrete models one needs a mesh. For simplicity I assume a mesh with constant \( \Delta t \), i.e. we have a discrete linear relationship for the magnetization,
\[ \Phi_n = \Phi(t_0 + n\Delta t). \] (4)

In general an \( m \)th order differential model will be given by the condition \( f(\Phi, \dot{\Phi}, \ldots, \Phi^{(m)}(t)) = 0 \), with \( m \) initial conditions. Here, \( \Phi^{(n)}(t) = \frac{d^n \Phi}{dt^n} \). A general \( m \)th order difference model is given by, \( g_{\Delta t}(\Phi_n, \Phi_{n+1}, \ldots, \Phi_{n+m}) = 0 \), with \( m \) initial conditions.

Let \( g_{\Delta t} = 0 \) be the difference model that one gets when discretizing a differential model \( f = 0 \), by approximating the derivative \( \dot{\Phi} \approx \frac{\Phi_{n+1} - \Phi_n}{\Delta t} \) and similarly \( \Phi^{(m)} \) for higher derivatives. And let \( C_{\Delta t} \) the discretizing transformation. Symbolically this can be written as \( C_{\Delta t}(f = 0) = (g_{\Delta t} = 0) \). From analysis we know that \( C_{\Delta t} \rightarrow Id \) when \( \Delta t \rightarrow 0 \), where \( Id \) is the identity. From this general property one can show that for small enough \( \Delta t \), the \( C_{\Delta t} \) mapping preserves the physical essence of the model. Stated more precisely, for a given mesh of time there is a unique correspondence between the differential and the difference models. Because we know that going from the difference model by taking the limit \( \Delta t \rightarrow 0 \) one gets the corresponding differential model. If we show that this process is invertible, we can claim that the differential and difference equations of motion correspond to the same physical system with continuous or discrete evolution respectively. This invertibility for 1st order systems is now shown.

Assume for simplicity that a differential and a difference model are given by \( \dot{\Phi} = f(\Phi) \), \( \Phi_{n+1} = g(\Phi_n) \) respectively, where \( f, g \) are well behaved functions. For a given mesh size \( \Delta t > 0 \), let \( \Phi_n \) be given by equation \( (\Phi) \) and let \( C_{\Delta t}(\dot{\Phi} = f(\Phi)) = (\Phi_{n+1} = g(\Phi_n)) \) be the correspondence between differential and difference equations. Now if,

\[ C_{\Delta t}(\dot{\Phi} = f_1(\Phi)) = C_{\Delta t}(\dot{\Phi} = f_2(\Phi)) \]

it is easy to see that \( f_1 = f_2 \), just by allowing the initial conditions to vary continuously, i.e. \( C_{\Delta t} \) is 1-1. To show onto, we can take \( f(\Phi) = \frac{1}{\Delta t}[g(\Phi) - \Phi] \) where \( g \) is the difference model. Then,

\[ C_{\Delta t}(\dot{\Phi} = f(\Phi)) = (\Phi_{n+1} = g(\Phi_n)) \]

This shows that, for a given mesh of time, for every difference model there exists a unique differential model and vice versa. And thus, any changes that a given model will exhibit
when going from continuous to discrete time will result directly from this discretization and not from any other change to the underlying physical dynamics.

I now show the discrete case for this model of magnetism. If we discretize equation (2) we find the following difference equation for the magnetization,

\[ \Phi_{n+1} = g(\Phi_n) = \Phi_n + c_1[\Phi_n - c_2(\Phi_n)^3] \]  

(5)

where \( c_1 = Ma\Delta t \) and \( c_2 = \frac{b}{a}. \Delta t \) is a finite time element.

The fixed points for this map, given by the condition \( \Phi_{n+1} = \Phi_n \), are \( \Phi^* = 0 \) or \( \Phi^* = \pm \frac{1}{\sqrt{c_2}} \). To understand the nature of these fixed points their stability is analyzed,

\[ g'(\Phi^*) = 1 + c_1[1 - 3c_2(\Phi^*)^2].\]

Or,

\[ g'(0) = 1 + c_1, \]

\[ g'(\pm \frac{1}{\sqrt{c_2}}) = 1 - 2c_1. \]

Thus for \(-2 < c_1 < 0\) the \( \phi^* = 0 \) fixed point is stable which corresponds to the paramagnetic state. For \(0 < c_1 < 1\), the \( \Phi^* = \pm \frac{1}{\sqrt{c_2}} \) is a stable fixed point for positive and negative initial conditions respectively. We recognize this as the ferromagnetic state. For larger values of \( c_1 \) the fixes points become unstable and we find higher period fixed points. If we set \( c_1 \) even larger we find a dense set of unstable periodic orbits with total measure zero [1]. This last scenario is known as chaos and the route we have taken is the period doubling route to chaos.

Remembering that \( c_1 = Ma\Delta t, \ M > 0 \) and that for a forward evolving system \( \Delta t > 0 \), we can conclude again that \( a \) is the parameter that determines whether the system is a paramagnet or a ferromagnet. Note that for a backward evolving system (\( \Delta t < 0 \)) the roles of paramagnetism and ferromagnetism change and we get the transition from ferromagnetism to paramagnetism as we lower the temperature.

The standard conclusion derived from such a treatment is that as long as we keep \( \Delta t \) small enough such that \( c_1 \) is in the stable region \((-2, 1)\) we can analyze equation (2) with equation (5). But there is some thing more fundamental that we can conclude: the nature
of time evolution - whether discrete or continuous, leads to different observation in physical systems.

With this last conclusion in mind we are led quite directly to inferring from observations on the nature of ferromagnetism under discrete time evolution. For the system presented we consider two control parameters, the temperature and the mobility. Figure 1 is the bifurcation diagram as we vary the temperature, i.e. for a given temperature we iterate equation (5) many time steps (200) and we plot the final states it settles to (100 points). We start in the paramagnetic state with zero magnetization then at $T_c (a = 0)$ we have a transition to the ferromagnetic state. As we keep increasing $a$ we have more transitions until we reach the chaotic state. So, if time is discrete then we might expect that if we lower the temperature enough we will get a transition from the ferromagnetic state to a state for which the magnetization oscillates between two adjacent magnetization values. If we lower the temperature even further the magnetization difference between these two states will increase. However, it could turn out that time evolution is discrete and we would not observe such exotic magnetic states. The reason is that there is a physical bound on the value of $a$, set by the minimum possible temperature, $T = 0 K$. It could turn out, especially if we expect a small $\Delta t$, that in most cases the corresponding value of $c_1$ is well within the familiar ferromagnetic region.

The second control parameter is the mobility. Figure 2 is the bifurcation diagram keeping $a, b$ fixed, positive and varying the mobility. I remark that the part of the diagram for negative values of the mobility was left in mainly for esthetic reasons. As we increase the mobility we observe much the same bifurcation diagram as before. Indeed it is seen that there is a value of $M$ for which the familiar ferromagnetic state ends and we start a new phase for which the magnetization oscillates between nearby states. Moreover there seems to be no natural cutoff for the mobility and detection of these exotic magnetic states seems more plausible [3]. Another system where detection of exotic states may be possible is discussed in [2].

In the framework of magnetization these nonlinearities are intriguing and could also provide evidence about the nature of time evolution. Indeed as this and other work suggest, bifurcation routes and chaos rise naturally from discrete dynamics, and disappear once the continuous limit is taken. Interestingly enough, experimental and numerical work on a driven magnetic system do exhibit a period-doubling route to chaos [3, 4]. Another possible
system where the exotic nature of the magnetization may be found is in a super-critical ferromagnet where the critical temperature (and thus $a$) is very large compared to everyday ferromagnets. In such systems the distance between nearest neighbor magnetic moments is small such that the overlap between adjacent wave functions and hence the exchange integral are greatly increased. An example of such a system is a neutron star, where the distances between adjacent magnetic moments are of nuclear order. Such stars exhibit a large-scale ferromagnetic ordering at very high temperatures which gives a very high critical temperature. When such a star cools down $a$ becomes large and the possibility of exotic states arises in the realm of discrete time. Such exotic states will oscillate between different magnetization states with period of the order $\Delta t$. If our instruments make observations over a time much larger than $\Delta t$ we would observe only the average magnetization.

In conclusion, a simple model for the bulk magnetic state has been studied. It is found that under continuous time evolution assumption this model admits only a steady-state analytical solution that is fully deterministic for all model-parameters values. Under discrete time evolution assumption the model exhibits the same analytical solution for some parameter values but as part of the period doubling route to chaos. This different behavior resulted from the different assumption with regard to time evolution and thus such observation can aid in understanding the nature of time and of time evolution.

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There is some ambiguity in the choice of the discretization of the differentials. Since this ambiguity involves only translation of the order of $\Delta t$ in time I assume that we can safely neglect it.
FIG. 1: Temperature bifurcation diagram with constant mobility for the total magnetization in the
discretized Ginzburg-Landau model, $M\Delta t = 1$ and $b = 1$. Top: Positive initial condition Bottom:
Negative initial condition.
FIG. 2: Mobility bifurcation diagram with constant temperature for the total magnetization in the discretized Ginzburg-Landau model, $\alpha \Delta t = 1$ and $c_2 = 1$. Top: Positive initial condition. Bottom: Negative initial condition.