A New Singular Lift of Doi-Naganuma Type

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Introduction

The Shimura correspondence, assigning to certain cusp forms of half-integral weight \( k + \frac{1}{2} \) a modular form of weight \( 2k \), was initially defined in [Sh]. The papers [DN] and [Ng] discuss a relation, now referred to as the Doi–Naganuma lifting, between cusp forms of weight \( k \) and Hilbert modular forms of weight \( k \) over a real quadratic field. There is a map defined by Maaß, which takes cusp forms of weight \( k - \frac{1}{2} \) to Siegel modular forms of degree 2 and weight \( k \). The latter map was generalized in [G] to a correspondence sending modular forms of weight \( k + 1 - \frac{n}{2} \) to automorphic forms on \( O_{2,n} \). Later, [B] used the theta lift in order to unify all these maps into one concept: Given an even lattice \( L \) of signature \((2, b_-)\), an integral against the theta function of the lattice attached to a certain harmonic homogenous polynomial on \( L_\mathbb{R} \) maps vector-valued modular forms of weight \( 1 - b_- - m \) with the Weil representation \( \rho_L \) associated to the discriminant form of \( L \) to automorphic forms of weight \( m \) on the symmetric space \( G(L_\mathbb{R}) \) of \( O_{2,b_-}^+ \) with respect to the discriminant kernel of \( \text{Aut}^+(L) \). A certain regularization process appearing in [B] allows one to lift weakly holomorphic modular forms. The resulting automorphic forms have poles of order \( m \) along rational quadratic divisors.

The main goal of this paper is to obtain a new lift of this type. In this paper we present this new lift in the case in which the dimension \( b_- \) is even. Recall that given a primitive isotropic vector \( z \in L \), the Grassmannian \( G(L_\mathbb{R}) \) may be identified (as a complex manifold) with \( K_\mathbb{R} + iC \), where \( K = \mathbb{R}^+ / \mathbb{Z} \) is a Lorentzian lattice and \( C \) is a cone of positive vectors in \( K_\mathbb{R} \). The main result can now be stated.

**Theorem.** Let \( L \) be an even lattice of signature \((2, b_-)\) containing isotropic vectors, and assume that \( b_- \) is even. Given a weakly holomorphic modular form \( f(\tau) = \sum_{\gamma \in L^*/L} \sum_{n \gg -\infty} c_{\gamma,n} q^n e_{\gamma} \) of weight \( 1 - \frac{b_-}{2} - m \) and representation

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Theorem. Let \( f = \sum_{n>a>0, b>0} c_n q^n \) be a weakly holomorphic modular form of even weight \(-m\) with respect to \( SL_2(\mathbb{Z})\). Then the Fourier expansion

\[
2^{2m+1} \left[ \sum_{n>0} \sum_{n|b,a} c_{ab} \left( \frac{ab}{n} \right)^{m+1} q^n - \sum_{d=1}^\infty d^{m+1} c_{-d} \sum_{k,l=0} d \left( \frac{q^k - p^l}{p^k - p^l} \right)^m \right]
\]

describes a meromorphic Hilbert modular form \( \Psi \) on \( \mathcal{H} \times \mathcal{H} \), which has weight \((m+2, m+2)\) with respect to \( SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \), with known poles.

Here \( \varphi_{m+1} \) is a certain homogenous polynomial in two variables associated with the expansion \( \sum_{n=1}^\infty n^{m+1} w^n \) — see Lemma 5.1. Such a degenerate Hilbert modular form can be presented as the quotient of a symmetric holomorphic modular form of parallel weight \( m + 2 + 12n \) for some \( n \) divided by \( \Delta(\tau)^n \Delta(\sigma)^n \) and a product of expressions of the sort \( \Phi_d(j(\tau), j(\sigma))^{m+2} \), where \( \Phi_d \) is the classical modular polynomial. Meromorphic Hilbert modular forms for Hilbert modular groups arising from real quadratic number fields are obtained from appropriate signature \((2, 2)\) lattices which are not unimodular, but their explicit description is somewhat more involved.

The main tool used to construct this lift is the Borcherds-like singular theta lift constructed by the author in [Ze]. This lift takes a vector-valued modular form of weight \( 1 - \frac{b_-}{2} - m \), and produces an automorphic form of weight \( m \) which instead of being meromorphic, it is an eigenfunction of the weight \( m \) Laplacian operator on \( G(L_2) \) with eigenvalue \(-2mb_-\). As already observed in [Ze], for \( b_- = 1 \) (a case not considered here since the main result is asserted only for even \( b_- \)) this property suffices for the image of this lift under the Shimura–Maaß weight raising operator to be a meromorphic modular form of weight \( 2m + 2 \) on \( \mathcal{H} \) (i.e., an automorphic form of weight \( m + 1 \) on the symmetric space of \( O_{2,1}^+ \)).
with poles of order $m + 1$ at points of arithmetic significance. For even $b_-$ we show that applying an appropriate operator to the theta lift of $\mathbb{Z}e$ yields the desired meromorphic automorphic form $\Psi$.

The first task is therefore to find which kind of operators must be applied to the theta lift of $\mathbb{Z}e$ in order to obtain a meromorphic image. For this purpose we develop the theory of weight raising operators, denoted $R_{m}^{(b_-)}$, and the weight lowering operator $L_{m}^{(b_-)}$, for automorphic forms on Grassmannians in arbitrary dimension (either even or odd), a theory which is interesting in its own right. This is done by interpolating known operators for signatures (2, 1), (2, 2), and (2, 3). The former two symmetric spaces admit operators which are based on the Shimura–Maaß operators, while for the latter case, i.e., the Siegel upper half-space of degree 2, one uses certain operators which are defined in [Ma1] and [Ma2]. The operators $R_{m}^{(b_-)}$ and $L_{m}^{(b_-)}$ share many properties with the Shimura–Maaß operators, though in our case the theory is more complicated due to the fact that the Shimura–Maaß operators are order 1 differential operators, while we consider differential operators of order 2. The operator which we apply to the theta lift from $\mathbb{Z}e$ is the weight raising operator $R_{m}^{(b_-)}$ taken to the power $\frac{b_-}{2}$—this is why we need $b_-$ to be even. As $\mathbb{Z}e$ already provides a similar result for $b_- = 1$, we conjecture that appropriate operators should exist for other odd values of $b_-$ as well. In addition, as the result of $\mathbb{Z}e$ about $b_- = 1$ does not require the existence of the isotropic vector $z$, we conjecture that applying $R_{m}^{(2)}$ to the theta lift of $\mathbb{Z}e$ yields a meromorphic Hilbert modular form also when the signature (2, 2) lattice is non-isotropic (this is the only remaining case in which $L$ may not contain isotropic vectors, by Meyer’s Theorem).

The first half of the paper contains numerous statements whose proofs are delayed to later sections. We choose this way of presentation since most of the proofs consist of direct calculations, which may divert the reader’s attention from the main ideas. We mention at this point that the main feature of all the calculations is the eigenvalue under the appropriate Laplacian operator. Specifically, the paper is divided into 5 sections. In Section 1 we define the weight raising and weight lowering operators and state their properties. Section 2 presents the images of certain functions under the weight raising operators, and proves the main theorem. In Section 3 we present the interesting special case of a unimodular lattice of signature (2, 2) in detail, and pose some conjectures regarding whether our results might be extended to cases for which a proof is not yet available. Section 4 presents the proofs for the assertions of Section 1, while Section 5 contains the missing proofs of Section 2.

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1 Weight Changing Operators for Automorphic Forms on Orthogonal Groups

In this Section we present automorphic forms on complex manifolds arising as orthogonal Shimura varieties of signature \((2, b_-)\), introduce the weight raising and weight lowering operators on such forms, and give some of their properties. The proofs of most assertions are postponed to Section 4.

Let \(V\) be a real vector space with a non-degenerate bilinear form of signature \((b_+, b_-)\). The pairing of \(x\) and \(y\) in \(V\) is written \((x, y)\), and \(x^\perp\) stands for the norm \((x, x)\) of \(x\). For \(S \subseteq V\), \(S^\perp\) denotes the subspace of \(V\) which is perpendicular to \(S\). The Grassmannian \(G(V)\) of \(V\) is defined to be the set of all decompositions of \(V\) into the orthogonal direct sum of a positive definite space \(v_+\) and a negative definite space \(v_-\). In the case \(b_+ = 2\) (which is the only case we consider in this paper), it is shown in Section 13 of [B], Sections 3.2 and 3.3 of [Bru], or Subsection 1.2 of [Zc] (among others), that \(G(V)\) carries a complex structure and has several equivalent models, which we now briefly present. Let

\[
P = \{ Z_V = X_V + iY_V \in V_\mathbb{C} = V \otimes \mathbb{C} \mid Z_V\mathbb{C} = 0, \quad (Z_V, \overline{Z_V}) > 0 \}.
\]

\(Z_V \in V_\mathbb{C}\) lies in \(P\) if and only if \(X_V\) and \(Y_V\) are orthogonal and have the same positive norm. \(P\) has two connected components (which are interchanged by complex conjugation), and let \(P^+\) be one component. The map

\[
P^+ \to G(V), \quad Z_V \mapsto \mathbb{R}X_V \oplus \mathbb{R}Y_V
\]

is surjective, and multiplication from \(C^*\) acts freely and transitively on each fiber of this map. This realizes \(G(V)\) as the image of \(P^+\) in the projective space \(P(L_\mathbb{C})\), which is an analytically open subset of the (algebraic) quadric \(Z_V^2 = 0\) yielding a complex structure on \(G(V)\). This is the projective model of \(G(V)\).

Let \(z\) be a non-zero vector in \(V\) which is isotropic, i.e., \(z^2 = 0\). The vector space \(K_\mathbb{R} = z^+ / \mathbb{R}z\) is non-degenerate and Lorentzian of signature \((1, b_- - 1)\). Choosing some \(\zeta \in V\) with \((z, \zeta) = 1\) and restricting the projection \(z^+ \to K_\mathbb{R}\) to \(\{z, \zeta\}^\perp\) gives an isomorphism. We thus write \(V\) as \(K_\mathbb{R} \times \mathbb{R} \times \mathbb{R}\), in which

\[
(\alpha, a, b) = a\zeta + bz + (\alpha \in \{z, \zeta\}^\perp \cong K_\mathbb{R}), \quad (\alpha, a, b)^2 = \alpha^2 + 2ab + a^2\zeta^2.
\]

A (holomorphic) section \(s : G(V) \to P^+\) is defined by the pairing with \(z\) being 1. Subtracting \(\zeta\) from any \(s\)-image and taking the \(K_\mathbb{C}\)-image of the result yields a biholomorphism between \(G(V) \cong s(G(V))\) and the tube domain \(K_\mathbb{R} + iC\), where \(C\) is a cone of positive norm vectors in the Lorentzian space \(K_\mathbb{R}\). \(C\) is called the positive cone, and it is determined by the choice of \(z\) and the connected component \(P^+\). The inverse biholomorphism takes \(Z = X + iY \in K_\mathbb{C}\) to

\[
Z_{V,Z} = \left( Z, 1, \frac{-Z^2 - \zeta^2}{2} \right) = \left( X, 1, \frac{Y^2 - X^2 - \zeta^2}{2} \right) + i(Y, 0, -(X, Y)),
\]

with the real and imaginary parts denoted \(X_{V,Z}\) and \(Y_{V,Z}\) respectively. They are orthogonal and have norm \(Y^2 > 0\). This identifies \(G(V)\) with the tube domain.
model $K_\mathbb{R} + iC$. Taking the other connected component of $P$ corresponds to taking the other cone $-C$ to be the positive cone, and to complex conjugation.

The subgroup $O^+(V)$ consisting of elements of $O(V)$ preserving the orientation on the positive definite part acts on $P^+$ and $G(V)$, respecting the projection. Elements of $O(V) \setminus O^+(V)$ interchange the connected components of $P$. The action of $O^+(V)$ (and also of the connected component $SO^+(V)$) on $G(V)$ is transitive, with the stabilizer $K$ (or $SK \leq SO^+(V)$) of a point being isomorphic to $SO(2) \times O(n)$ (resp. $SO(2) \times SO(n)$). Therefore $G(V)$ is isomorphic to $O^+(V)/K$ and to $SO^+(V)/SK$. Given an isotropic $z$ as above, the action of $O^+(V)$ transfers to $K_\mathbb{R} + iC$, and for $M \in O^+(V)$ and $Z \in K_\mathbb{R} + iC$ we have

$$MZ_{V,Z} = J(M,Z)Z_{V,MZ}, \quad \text{with} \quad J(M,Z) = (MZ_{V,Z}, z) \in \mathbb{C}^*.$$  

$J$ is a factor of automorphy, namely the equality

$$J(MN, Z) = J(M, NZ)J(N, Z)$$

holds for all $Z \in K_\mathbb{R} + iC$ and $M$ and $N$ in $O^+(V)$. For such $M$ we define the slash operators of weight $m$, and more generally of weight $(m,n)$, by

$$\Phi[M]_{m,n}(Z) = J(M, Z)^{-m}J(M, Z)^{-n} \Phi(MZ), \quad [M]_m = [M]_{m,0}. $$

The fact that $(ZV, \overline{ZV}) = 2Y^2$ and the definition of $J(M, Z)$ yield the equalities

$$(3(MZ))^2 = \frac{Y^2}{|J(M,Z)|^2} \quad \text{and} \quad (F(Y^2)^t)[M]_{m,n} = F[M]_{m+t, n+t}(Y^2)^t \quad (1)$$

the latter holding for every $m, n, t$, and function $F$ on $K_\mathbb{R} + iC$ (see Lemma 3.20 of [Bru] for the first equality in Equation (1), and the second one follows immediately).

The invariant measure on $K_\mathbb{R} + iC$ is $\frac{dXdy}{(Y^2)^3}$ (see Section 4.1 of [Bru], but one can also prove this directly, using the generators of $O^+(V)$ considered in Section 4 below). Note that this measure depends on the choice of a basis for $K_\mathbb{R} + iC$, but changing the basis only multiplies this measure by a positive global scalar. Let $\Gamma$ be a discrete subgroup $\Gamma$ of $O^+(V)$ of cofinite volume. In most of the interesting cases $\Gamma$ will be either the $O^+$ or the $SO^+$ part of the orthogonal group of an even lattice $L$ in $V$, or the discriminant kernel of such a group. Given $m \in \mathbb{Z}$, an automorphic form of weight $m$ with respect to $\Gamma$ is defined to be a (complex valued) function $\Phi$ on $K_\mathbb{R} + iC$ for which the equation

$$\Phi(MZ) = J(M, Z)^m \Phi(Z), \quad \text{or equivalently} \quad \Phi[M]_m(Z) = \Phi(Z), $$

holds for all $M \in \Gamma$ and $Z \in K_\mathbb{R} + iC$. Using the standard argument, such a function is equivalent to a function on $P^+$ which is $-m$-homogenous (with respect to the action of $C^1$) and $\Gamma$-invariant, as considered, for example, in [Ei].

We now consider some differential operators on functions on $K_\mathbb{R} + iC$. Given a basis for $K_\mathbb{R}$, we write $\partial_{z_k}$ for $\frac{\partial}{\partial z_k}$ (for $1 \leq k \leq b_-$). Similarly, $\partial_{\bar{z}_k}$ stands for
the coordinates of the imaginary part from \( C \). The notation for the derivatives
\[
\partial_{z_k} = \frac{1}{2} (\partial_{x_k} - i \partial_{y_k}) \quad \text{and} \quad \partial_{\bar{z}_k} = \frac{1}{2} (\partial_{x_k} + i \partial_{y_k})
\]
will be further shortened to \( \partial_k \) and \( \partial_{\bar{\tau}} \) respectively.

The operator \( I = \sum_k x_k \partial_{z_k} \) multiplies a homogenous function on \( K_\mathbb{R} \) by its
homogeneity degree, and is thus independent of the choice of basis (indeed, it
has an intrinsic Lie-theoretic description). The operators
\[
D^* = \sum_k y_k \partial_k \quad \text{and} \quad \overline{D^*} = \sum_k y_k \partial_{\bar{\tau}}
\]
from \([Na]\) are intrinsic as well, and they are also invariant under translations
in the real part of \( K_\mathbb{R} + iC \). If the basis for \( K_\mathbb{R} \) is orthonormal, i.e., orthogonal with
the first vector having norm 1 and the rest having norm \(-1\), then the Laplacian
of \( K_\mathbb{R} \), denoted \( \Delta_{K_\mathbb{R}} \), is defined to be \( \partial_{z_1}^2 - \sum_{k=2}^{b_-} \partial_{z_k}^2 \). It is independent of the
choice of the orthonormal basis (though using a basis which is not orthonormal
it takes different forms), and it is invariant under the action of \( O(K_\mathbb{R}) \) as well as
under translations in \( K_\mathbb{R} \). With complex coordinates it has three counterparts,
\[
\Delta^h_{K_{C}} = \partial_1^2 - \sum_{k=2}^{b_-} \partial_k^2, \quad \Delta^\tau_{K_{C}} = \partial_1^2 - \sum_{k=2}^{b_-} \partial_k^2, \quad \text{and} \quad \Delta^R_{K_{C}} = \partial_1 \partial_{\tau} - \sum_{k=2}^{b_-} \partial_k \partial_{\bar{\tau}},
\]
which we call the holomorphic Laplacian of \( K_C \) (of Hodge weight \((2,0)\)), the
anti-holomorphic Laplacian of \( K_C \) (of Hodge weight \((0,2)\)), and the real Laplacian
of \( K_C \) (of Hodge weight \((1,1)\)), respectively. These operators have the
same invariance and independence properties as \( \Delta_{K_\mathbb{R}} \). Note that the appropri-
ate combinations appearing in \([Brm]\) and \([Na]\) can be identified as our operators
\( \frac{1}{2} \Delta^h_{K_{C}}, \frac{1}{2} \Delta^\tau_{K_{C}}, \) and \( \Delta^R_{K_{C}} \), respectively, expressed in a basis which is not orthonormal.
We shall indeed discuss and generalize the operators \( \Delta_1 \) and \( \Delta_2 \) of \([Na]\) in
Proposition 1.4 below.

The weight changing operators and their defining property are given in

**Theorem 1.1.** For any integer \( m \) define \( R^{(b_-)}_m \) to be the operator
\[
(Y^2)^{\frac{b_--m-1}{2}} \Delta^h_{K_C} (Y^2)^{m+1-b_-} = \Delta^h_{K_C} - \frac{i(2m + 2 - b_-)}{Y^2} D^* - \frac{m(2m+2-b_-)}{2Y^2}.
\]
In addition, define
\[
L^{(b_-)} = (Y^2)^{\frac{b_-}{2}} \Delta^R_{K_C} = (Y^2)^{\frac{b_-}{2}+1} \Delta^R_{K_C} (Y^2)^{1-b_-} = (Y^2)^2 \Delta^R_{K_C} + iY^2(2-b_-)\overline{D^*}.
\]
Then the equalities
\[
(R^{(b_-)}_m F)[M]_{m+2} = R^{(b_-)}_m (F[M]_m), \quad (L^{(b_-)} F)[M]_{m-2} = L^{(b_-)}(F[M]_m)
\]
hold for every \( C^2 \) function \( F \) on \( K_\mathbb{R} + iC \) and any \( M \in O^+(V) \).

The different descriptions of \( R^{(b_-)}_m \) and \( L^{(b_-)} \) coincide by Lemma 1.1 below.
Theorem 1.1 has the following standard
Corollary 1.2. If $\Phi$ is an automorphic form of weight $m$ on $G(V) \cong K_R + iC$ then $R_m^{(b_-)} \Phi$ and $L_m^{(b_-)} \Phi$ are automorphic forms on $K_R + iC$ which have weights $m + 2$ and $m - 2$ respectively.

In correspondence with Theorem 1.1 and Corollary 1.2 we call $R_m^{(b_-)}$ and $L_m^{(b_-)}$ the weight raising operator of weight $m$ and the weight lowering operator for automorphic forms on Grassmannians of signature $(2, b_-)$.

We shall make use of the operator

$$D^*D^* - \frac{D^*}{2t} = \frac{D^*}{2t} = \sum_{k,l} y_k y_l \partial_k \partial_l,$$

which we denote $|D^*|^2$. Lemma 1.2 of [Ze] shows that

$$\Delta_{m,n}^{(b_-)} = 8|D^*|^2 - 4Y^2 \Delta_{K_C} + 4imD^* + 4n(2m - b_-)$$

is the weight $(m, n)$ Laplacian on $K_R + iC$, and the weight $m$ Laplacian $\Delta_{m}^{(b_-)}$ is just $\Delta_{m,0}^{(b_-)}$ (this extends the corresponding assertion of [Na], since his operator $\Delta_1$ is our $\Delta_0^{(b_-)}$ divided by 8). The constants are normalized such that

$$\Delta_{m,n}^{(b_-)}(Y^2)^t = (Y^2)^t \Delta_{m+t,n+t}^{(b_-)} \quad (2)$$

holds for every $m$, $n$, and $t$ (see the remark after Lemma 1.1 below). The relations between $R_m^{(b_-)}$, $L_m^{(b_-)}$, and the corresponding Laplacians are given by

Proposition 1.3. The equalities

$$\Delta_{m+2}^{(b_-)} R_m^{(b_-)} - R_m^{(b_-)} \Delta_{m}^{(b_-)} = (2b_- - 4m - 4)R_m^{(b_-)}$$

and

$$\Delta_{m-2}^{(b_-)} L_m^{(b_-)} - L_m^{(b_-)} \Delta_{m}^{(b_-)} = (4m - 2b_- - 4)L_m^{(b_-)}$$

hold for every $m \in \mathbb{Z}$.

We recall that an automorphic form of weight $m$ on $K_R + iC$ is said to have eigenvalue $\lambda$ if it is annihilated by $\Delta_{m}^{(b_-)} + \lambda$ (i.e., eigenvalues are of $-\Delta_{m}^{(b_-)}$). Hence Proposition 1.3 has the following

Corollary 1.4. If $F$ is an automorphic form of weight $m$ on $K_R + iC$ which has eigenvalue $\lambda$ then the automorphic forms $R_m^{(b_-)} F$ and $L_m^{(b_-)} F$ have eigenvalues $\lambda + 4m - 2b_- + 4$ and $\lambda - 4m + 2b_- + 4$ respectively.

By evaluating compositions of the weight changing operators one shows

Proposition 1.5. The combination

$$\Xi_m^{(b_-)} = (Y^2)^2 \Delta_{K_C}^h \Delta_{K_C}^h - iY^2(2m + 2 - b_-)D^* \Delta_{K_C}^h + iY^2(2 - b_-)D^* \Delta_{K_C}^h +$$

...
\[
\frac{(2 - b_-)(2m + 2 - b_-)}{2} Y^2 R^b_{Kc} - m(2m + 2 - b_-) Y^2 R^b_{Kc} = \frac{2}{m} (2m + 2 - b_-) Y^2 R^b_{Kc}
\]

commutes with all the weight \( m \) slash operators as well as with the Laplacian \( \Delta_m^{(b_-)} \). The commutator of the global weight raising operator and the weight lowering operator is

\[
\left[ R_m^{(b_-)} , L_m^{(b_-)} \right] = \frac{m \Delta_m^{(b_-)}}{2} - \frac{mb_- (2m - 2 - b_-)}{4}.
\]

Proposition 1.5 provides another proof to Lemma 1.2 of \([Z\&]\) about \( \Delta_m^{(b_-)} \). It also implies that \( \Xi_m^{(b_-)} \) preserves the spaces of automorphic forms of weight \( m \) for all \( m \in \mathbb{Z} \) and for every discrete subgroup \( \Gamma \) of cofinite volume in \( O^+(V) \). It also commutes with \( \Delta_m^{(b_-)} \), hence preserves eigenvalues of such automorphic forms. By rank considerations, one can probably show that the ring of differential operators which commute with all the slash operators of weight \( m \) is generated by \( \Delta_m^{(b_-)} \) and \( \Xi_m^{(b_-)} \), hence is a polynomial ring in two variables (if \( b_- > 1 \)). As \( \Delta_0^{(b_-)} = 8 \Delta_1 \) and \( \Xi_0^{(b_-)} = 16 \Delta_2 \) in the notation of \([Na]\). Proposition 1.6 generalizes the main result of that reference to other weights. A similar argument yields results of the same sort for \( (m,n) \), where a possible normalization for \( \Xi_m^{(b_-)} \) is \( (Y^2)^{-n} \Xi_m^{(b_-)} (Y^2)^n \) for which an equality similar to Equation (12) holds. We shall not need these results in what follows.

We shall also need compositions of the weight raising operators. The natural \( l \)th power of \( R_m^{(b_-)} \) is the composition

\[
(R_m^{(b_-)})^l = R_m^{(b_-)} + 2l - 2 \circ \ldots \circ R_m^{(b_-)}.
\]

The general formulae for the resulting operator seems too complicated to write as a combination of \( \Delta^{Kc}_{Kc}, D^*, \) and \( \Delta^{Kc}_{2} \) with explicit coefficients, but for our applications it will suffice to know the properties given in the following

**Proposition 1.6.** (i) The operator \( (R_m^{(b_-)})^l \) takes automorphic forms of weight \( m \) on \( G(\mathbb{R}) \) to automorphic forms of weight \( m + 2l \). (ii) In case the former automorphic form is an eigenfunction with eigenvalue \( \lambda \), the latter is also an eigenfunction, and the corresponding eigenvalue is \( \lambda + l(4m + 4l - 2b_-) \). (iii) The operator \( (R_m^{(b_-)})^l \) can be written as

\[
(R_m^{(b_-)})^l = \sum_{c=0}^{l} \sum_{a=0}^{c} A_{a,c}^{(l)} \frac{(-D^*)^{c-a} (\Delta^{b_-}_{Kc})^{l-c}}{(-Y^2)^c},
\]

where \( A_{0,0}^{(l)} = 1 \) and given the coefficients \( A_{a,c}^{(l)} \) for given \( l \), the coefficient \( A_{a,c}^{(l+1)} \) of the next power \( l+1 \) is defined recursively as

\[
\sum_{s=0}^{a} \frac{(c - s)}{(a - s)} A_{s,c}^{(l)} + (2m + 4l - 2c + 4 - b_-) \left( A_{a,c-1}^{(l)} + \frac{m + 2l - c + 1}{2} A_{a-1,c-1}^{(l)} \right).
\]
For \( a = 0 \) the coefficients \( A_{0,c}^{(l)} \) are given by the explicit formula

\[
A_{0,c}^{(l)} = \frac{l! \cdot 2^c}{(l-c)!} \left( m + l - \frac{b_-}{c} \right).
\]

The binomial symbol appearing in part (iv) of Proposition 1.6 is the extended binomial coefficient: Indeed, for two non-negative integers \( x \) and \( n \) we have

\[
\binom{x}{n} = \frac{1}{n!} \prod_{j=0}^{n-1} (x-j),
\]

a formula which makes sense for \( x \in \mathbb{R} \) (and more generally). We discuss some simple properties of these extended binomial coefficients in Lemma 5.3 below.

Part (i) of Proposition 1.6 follows immediately from Corollary 1.2. For part (ii) Corollary 1.4 shows that the application of \( R_{m+2r} \) (for \( 0 \leq r \leq l-1 \)) to an eigenfunction adds \( 4m + 8r + 4 - 2b_- \) to the eigenvalue, so the assertion follows from evaluating

\[
\sum_{r=0}^{l-1} (4m + 8r + 4 - 2b_-) = l(4m + 4l - 2b_-).
\]

The proofs of parts (iii) and (iv) are given in Section 4.

We recall that \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \) defines the holomorphic map

\[
M : \tau \in \mathcal{H} = \{ \tau = x + iy \in \mathbb{C} | y > 0 \} \mapsto \frac{a\tau + b}{c\tau + d},
\]

with \( j(M, \tau) = c\tau + d \), the latter being the factor of automorphy of this action. Modular forms of weight \((k,l)\) (or just weight \( k \) if \( l = 0 \)) with respect to a discrete subgroup \( \Gamma \) of \( SL_2(\mathbb{R}) \) with cofinite volume (with respect to the invariant measure \( dx dy \)) are functions \( f : \mathcal{H} \to \mathbb{C} \) which are invariant under the corresponding weight \((k,l)\) slash operators for elements of \( \Gamma \). The weight \((k,l)\) Laplacian is

\[
\Delta_{k,l} = 4y^2 \partial_{\tau} \partial_{\tau} - 2i ky \partial_{\tau} + 2i ly \partial_{\tau} + l(k-1),
\]

normalized such that \( \Delta_k = \Delta_{k,0} \) annihilates holomorphic functions and the Laplacians commute with powers of \( y \) as in Equation (2). The Shimura–Maaß operators

\[
\delta_k = y^{-k} \partial_{\tau} y^k = \partial_{\tau} + \frac{k}{2i y} \quad \text{and} \quad y^2 \partial_{\tau}
\]

(note the different normalization from [Brz] and [Zw]) take modular forms of weight \( k \) to modular forms of weight \( k + 2 \) and \( k - 2 \) respectively, or more precisely, satisfy an appropriate commutation relation with the slash operators for all the elements of \( SL_2(\mathbb{R}) \). They also change Laplacian eigenvalues (again, with respect to \(-\Delta_k\) rather than \(\Delta_k\)): \( \delta_k \) adds \( k \) to the eigenvalue, while \( y^2 \partial_{\tau} \)
subtracts \( k - 2 \) from it. Moreover, the powers of the Shimura–Maaß operators are given by, e.g., Equation (56) in [Za], stating that

\[
\delta_k^l = \delta_{k+2l-2} \circ \ldots \circ \delta_k = \sum_{r=0}^{l} \frac{l!}{(l-r)!} \left( \frac{k + l - 1}{r} \right) \frac{\partial^{l-r}_r}{(2iy)^r}
\]

(for arbitrary \( k \), not necessarily integral and non-negative). Theorem 1.1 and Proposition 1.3 show that our weight changing operators \( R_{b-}^m \) and \( L_{b-}^m \) have similar properties. However, our operators are differential operators of order 2 while the Shimura–Maaß operators are of order 1. This is why the results of Propositions 1.5 and 1.6 are more complicated than the fact that \( \delta_{k-2}y^2\partial_\tau \) is just \( \Delta_{k} \), the commutator \( [\delta, y^2\partial_\tau] \) is simply \( \frac{k}{4} \), and Equation (56) of [Za].

Nonetheless, the operators \( R_{b-}^m \) and \( L_{b-}^m \) for small values of \( b_- \) are closely related to the Shimura–Maaß operators. Indeed, for \( b_- = 1 \) the group \( SO_{2,1}^+ \) is \( \text{PSL}_2(\mathbb{R}) \) and the tube domain \( K_{\mathbb{R}} + iC \) is just \( \mathcal{H} \). We have

\[
J(M, \tau) = j^2(M, \tau), \quad \text{hence} \quad [M]_m = [M]_{2m}^H \quad \text{and} \quad \Delta_m^{(1)} = \Delta_{2m}.
\]

The same assertions hold for the operators involving anti-holomorphic weights. Our operators \( R_m^{(1)} \) and \( L^{(1)} \) are squares of the Shimura–Maaß operators, namely

\[
R_m^{(1)} = \delta_{2m}^2 = \delta_{2m+2\delta_{2m}} \quad \text{and} \quad L^{(1)} = (y^2\partial_\tau)^2.
\]

Note that in this case

\[
\Xi_m^{(1)} = \frac{(\Delta_{2m})^2}{16} - \frac{m\Delta_{2m}}{8} \in \mathbb{C}[\Delta_m^{(1)} = \Delta_{2m}],
\]

in accordance with the rank of the group being 1 rather than 2 (in particular, in the notation of [Na] we have \( \Delta_2 = \frac{\Delta_4}{4} \) in this case).

For \( b_- > 1 \) many authors (including [Bru] and [Na]) take the basis for \( K_{\mathbb{R}} \) as two elements spanning a hyperbolic plane together with an orthogonal basis of elements of norm \(-2\). In elements of the positive cone \( C \), the first two coordinates are positive. In particular, for \( b_- = 2 \) we have \( K_{\mathbb{R}} + iC \cong \mathcal{H} \times \mathcal{H} \), with \( \tau = x + iy \) and \( \sigma = s + it \) being the two coordinates. The group \( SO_{2,2}^+ \) is an order 2 quotient of \( SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \), acting on \( G(V) \cong \mathcal{H} \times \mathcal{H} \) through

\[
(M, N) : (\tau, \sigma) \mapsto (M\tau, N\sigma) \quad \text{with} \quad J((M, N), (\tau, \sigma)) = j(M, \tau)j(N, \sigma).
\]

It follows that

\[
[M, N]_m = [M]_{m,\tau}^H [N]_{m,\sigma}^H \quad \text{and} \quad \Delta_m^{(2)} = 2\Delta_{m,\tau} + 2\Delta_{m,\sigma}
\]

(which extend to the operators with anti-holomorphic weights as well). Our operators are

\[
R_m^{(2)} = 2\delta_{m,\tau}\delta_{m,\sigma}, \quad L^{(2)} = 8y^2t^2\partial_\tau\partial_\sigma \quad \text{and} \quad \Xi_m^{(2)} = \Delta_{m,\tau}\Delta_{m,\sigma}.
\]
In both cases $b_-=1$ and $b_-=2$ the assertions of this Section follow from properties of the Shimura–Maaß operators (note that $Y^2$ is $2y^2$ for $b_-=1$). When $b_-=2$ the special orthogonal group of a negative definite subspace is also $SO(2)$, which makes the theory of automorphic forms more symmetric.

Working with $b_-=3$ in this model yields another coordinate $z = u + iv$. The positivity of $y, t,$ and $yt - v^2$ is equivalent to

$$\Pi = \left( \begin{array}{cc} \tau & z \\ \bar{z} & \sigma \end{array} \right) \text{ being in } \mathcal{H}_2 = \{ \Pi = X + iY \in M_2(\mathbb{C}) \mid \Pi = \Pi^t, \ Y > 0 \}.$$

Hence $K_R + iC$ is identified with the Siegel upper half-plane of degree 2. The group $SO^+_2,3$ is $PSp_4(\mathbb{R})$, with the symplectic action and the factor of automorphy (hence the slash operators) from the theory of Siegel modular forms. In this case

$$R_m^{(3)} = -\frac{M_m}{Y^2}, \quad L^{(3)} = -Y^2N_0, \quad \Delta_m^{(3)} = 2Tr(\Omega_{m,n})$$

in the notation of [Ma1] and [Ma2] for degree 2 (for weight $(m,n)$ the latter assertion extends to the modified Laplacian $\Delta_{m,n}^{(3)}$ of Section 4). The operator $\Delta_{K_R}$ is also a constant multiple of the operator $\mathcal{D}$ considered, for example, in [CE] and [Ch].

2 Meromorphic Images of Theta Lifts

The main technical result of [Ze] concerns certain theta lifts of weakly holomorphic modular forms of weight $1 - \frac{b_-}{2} - m$ (or, in fact, of their images under the order $m$ Shimura–Maaß operator $\left( \frac{1}{2\pi i} \right)^m \delta_{1 - \frac{b_-}{2} - m}$). For $b_- = 1$ the image of such a theta lift under $\left( \frac{1}{2\pi i} \right)^{b_-} 2m$ yields a meromorphic modular form of weight $2m + 2$ on $\mathcal{H}$ (i.e., a meromorphic automorphic form of weight $m + 1$ on $SO^+_2,1$). Our main goal is to prove that for even $b_-$, the image of these theta lifts under $(R_m^{(b_-)})^{b_-/2}$ also yields meromorphic automorphic forms on $SO^+_2,b_-$. We achieve this goal by evaluating the images of the various parts of the Fourier expansion of the theta lift of [Ze] under this operator. Again, we leave detailed proofs and derivations to Section 5.

We start with functions on $G(V) \cong K_R + iC$ depending only on the imaginary part of the variable. Let $\omega : C \to \mathbb{C}$ be a smooth function, and by a slight abuse of notation denote $\omega$ also the function taking $Z = X + iY \in K_R + iC$ to $\omega(Y)$. For such functions we have

**Lemma 2.1.** Assume that $Z = X + iY \mapsto \omega(Y)$ is an eigenfunction with respect to (minus) $\Delta_n^{(b_-)}$ with the eigenvalue $\lambda$, and that $\omega$ is homogenous of degree $d$. Then

$$R_n^{(b_-)}\omega = \frac{1}{2(d + n)(d + 2n + 2 - b_-) - 2d + \lambda} \cdot \frac{\omega}{-4Y^2}.$$
We remark that one can verify Corollary 1.4 directly for the functions considered in Lemma 2.1 as $D^* + D$ annihilates functions of $Y$. In any case, for multiple applications Lemma 2.1 has the following

**Corollary 2.2.** The composition $(R_{m,-})^l$ takes the function $\omega$ of Lemma 2.1 to

$$\prod_{r=0}^{l-1} [2(d + m + r)(d + 2m + 2r + 2 - b_-) - 2d(r + 1) + \lambda] \cdot \frac{\omega}{(-4Y^2)^r}.$$ 

**Proof.** The case $l = 0$ is trivial. If the assertion holds for $l$ then for $l+1$ we apply Lemma 2.1 for $R_{m+2}$, operating on a function of homogeneity degree $d - 2l$. As part (ii) of Proposition 1.6 shows that the latter function has eigenvalue $\lambda + l(4m + 4l - 2b_-)$, the assertion for $l+1$ follows. This proves the corollary. \(\square\)

We now consider functions of $Z \in K_\mathbb{R} + iC$ which involve the functions

$$f^{(p)}_{k,h,\rho}: C \to \mathbb{R}, \quad f^{(p)}_{k,h,\rho}(Y) = \frac{(\rho^2)^{p-k}(\rho, Y)^{k-h}}{\pi^{k+h}(Y^2)^k}$$

for $0 \neq \rho \in K_\mathbb{R}$ and integers $h$, $k$, and $p$ such that $0 \leq h \leq k \leq p$. In fact, the functions of which we have to analyze

$$g^{(p),+}_{k,h,\rho}(Z) = f^{(p)}_{k,h,\rho}(Y)e((\rho, Z)) \quad \text{and} \quad g^{(p),-}_{k,h,\rho}(Z) = f^{(p)}_{k,h,\rho}(Y)e((\rho, \overline{Z})),$$

where $e(w)$ is a shorthand for $e^{2\pi i w}$ for $w \in \mathbb{C}$. Theorem 2.7 of [Ze] constructs a theta lift whose Fourier expansion involves linear combinations of the functions $g^{(p),\pm}_{k,h,\rho}$, which are eigenfunctions for (minus) the Laplacian $\Delta^{(b_-)}_m$ with the eigenvalue $-2mb_-$. Observe that for $b_- > 1$ the functions $f^{(p)}_{k,h,\rho}$ are linearly independent for fixed $p$, while for $b_-=1$ all the functions with the same value of $k + h$ are multiples of $\frac{1}{\sqrt{\pi^{k+h}}}$. However, as the case $b_- = 1$ is already dealt with in [Ze], we assume $b_- > 1$ in what follows (in fact, from some point on we shall assume that $b_-$ is even). As we consider only functions $f^{(p)}_{k,h,\rho}$ with $0 \leq h \leq k \leq p$, we shall assume in what follows that any coefficient with indices $k$ and $h$ vanishes unless these inequalities between $k$, $h$, and $p$ hold. These statements extend to the functions $g^{(p),\pm}_{k,h,\rho}$.

Knowing that a linear combination of the functions $g^{(p),\pm}_{k,h,\rho}$ is an eigenfunction of a Laplacian $\Delta^{(b_-)}_m$ with a given eigenvalue determines in some cases (perhaps with some additional conditions) the combination up to a multiplicative scalar. We shall need this fact in two cases. The first one is

**Proposition 2.3.** For any $b_- \geq 2$ and $m \in \mathbb{N}$, if the function defined by

$$\sum_{k,h} B^-(m,-) B^-_{k,h} g^{(m),-}_{k,h,\rho}$$

has eigenvalue $-2mb_-$ under $\Delta^{(b_-)}_m$ then it is a constant multiple of $e(\rho, Y \overline{Z})/(Y^2)^l$, namely $B^-_{k,h} = 0$ unless $k = h = m$. 

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Note that the assertion of Proposition 2.3 extends to the case \( b_− = 1 \), since \( δ_{2m} = y^{−2m}∂_y y^{2m} \) annihilates \( 2 \) only for \( t = 2m \). We also remark that the combination \( \sum_{k,h} B^+_{k,h} g_{k,h,ρ}^{(m)+} \) is uniquely determined by the eigenvalue \(-2mb_−\) as well, though it is in general a more complicated combination. Now, the second linear combination which we wish to determine is

**Proposition 2.4.** Let \( b_− \) and \( m \) be as above, let \( p ∈ \mathbb{N} \), and assume that the function \( \sum_{k,h} B^+_{k,h} g_{k,h,ρ}^{(m)+} \) is harmonic (i.e., has eigenvalue \( 0 \)) with respect to the operator \( Δ_{m+b−} \). Assume further that \( B^+_{k,0} = 0 \) for all \( k > m \). Then only \( B^+_{0,0} \) may not vanish, namely this function is a constant times \( e((ρ, Z)) \).

We now consider the action of powers of our weight raising operators on linear combinations of the functions \( g_{k,h,ρ} \). Evaluating the full result is difficult, but it will suffice to determine the coefficients in front of \( f^{(q)}_{k,0,ρ} \) (for any \( k \) and the appropriate \( q \)) in this image. This is done in the following

**Proposition 2.5.** We can write

\[
\frac{1}{(-4π)^2} (F_m^{(b−)})^l \sum_{k,h} B^+_{k,h} g_{k,h,ρ}^{(p)+} \text{ as } \sum_{k,h} C^+_{k,h} g_{k,h,ρ}^{(p+l)+},
\]

where for all \( k \geq 0 \) we have

\[
C^+_{k,0} = \sum_{l=0}^{k} \frac{l!}{(l−r)!} (−1)^r \binom{m + l + r - b_−}{r} B^+_{k-r,0}.
\]

Given \( Z ∈ K_R + iC \cong G(V) \), let \( Z_{V,Z} \) be \( s(Z) ∈ P^+ \), and for \( µ ∈ V \) we define

\[
P_{r,s,t}(µ, Z) = \frac{(µ, Z_{V,Z})^r (µ, Z_{V,Z})^t}{(Y^2)^s}
\]

(as in Subsection 1.2 of \([Zζ]\)). These expressions are smooth if \( r, s, t \) are in \( \mathbb{N} \), but the singularities of the theta lift from Theorem 2.7 of \([Zζ]\) (which is cited as Theorem 2.11 in this paper) and its images under our weight raising operators involve these functions with negative \( r \) as well. We shall treat \( P_{r,s,t} \) as a function of \( Z ∈ K_R + iC \), with \( µ ∈ V \) playing the role of a parameter. We shorthand the pairing \( (µ, z) \) to \( µ_z \), and start our analysis with

**Lemma 2.6.** For every integers \( r, s, t, m \), and \( n \) and element \( µ ∈ V \), the function \( Δ_{m,n} P_{r,s,t}(µ, Z) \) equals

\[
-4rtµ^2 P_{r−1,s−1,t−1} + 2(t − s + n)(P_{r,s,t} − P_{r−1,s,t+1} + 2µ_z P_{r−1,s−1,t}) +
+ 2t(r−s+m)(P_{r,s,t} − P_{r+1,s−1,t−1} + 2µ_z P_{r,s−1,t−1}) + 2(s−n)(2(s−m)+b−) P_{r,s,t}.
\]
Equation (3) of \[Ze\] shows that the function \(P_{r,s,t}\) is automorphic of weight \((s-r,s-t)\). Indeed, with these parameters Lemma 2.6 is substantially simplified:

\[
\Delta_{s-r,s-t}^{(b,-)} P_{r,s,t} = -4rt\mu^2 P_{r-1,s-1,t-1} + 2t(2r + b_-) P_{r,s,t}.
\]

Moreover, the two remaining terms have the same value of \(s-r\) and \(s-t\).

When the operator in question is \(\Delta_{n}^{(b,-)}\), we should concentrate on linear combinations of the quotients \(P_{k-n,k,k}\) for \(k \in \mathbb{Z}\). We shall take only \(k < n\) (to consider singularities) but \(k \in \mathbb{N}\) (to have only \((\mu, Z_{V,Z})\) in the denominator but not \((\mu, Z_{V,Z})\)). It is obvious that if a function \(h\) describes the singularities of some function \(F\) which is annihilated by \(\Delta_{n}^{(b,-)} + \lambda\) then \((\Delta_{n}^{(b,-)} + \lambda)h\) is smooth. As in Propositions 2.3 and 2.4 this property determines the linear combination describing the singularity up to a scalar multiple, as is shown in Corollary 2.7.

**Corollary 2.7.** Assume that \(\sum a_k (\mu^2)^{n-k} P_{k-n,k,k}\) is an eigenfunction for (minus) \(\Delta_{n}^{(b,-)}\), with eigenvalue \(\lambda\), up to smooth functions. Then for every index \(0 \leq k \leq n-1\) we have

\[
a_k = \prod_{r=0}^{k-1} \left(2r(2n-2r-b_-) - \lambda\right) \cdot \frac{(n-k-1)!}{4^k(n-1)!} \cdot a_0.
\]

**Proof.** Comparing \(-\lambda \sum a_k (\mu^2)^{n-k} P_{k-n,k,k}\) with the image

\[
\sum_{k=0}^{n-1} a_k (\mu^2)^{n-k} [4k(n-k)\mu^2 P_{k-n-1,k-1,k-1} + 2k(b_- - 2n + 2k) P_{n-k,k,k}]
\]

of our combination under \(\Delta_n\) using (the simplified) Lemma 2.6 yields the equality

\[
(2k(2n-2k-b_-) - \lambda)a_k = 4(k+1)(n-k-1)a_{k+1}
\]

for all \(0 \leq k \leq n-2\). A simple inductive argument completes the proof.

The proof of Corollary 2.7 implies that for such a (non-trivial) combination to exist, the eigenvalue \(\lambda\) must come from a finite set of values \(2r(2n-2r-b_-)\) for some \(0 \leq r \leq n-1\) (see also the remark after the proofs of Propositions 2.3 and 2.4 in Section 5 below). A similar assertion holds for singularities as the one from Theorem 2.11 below, after a little extra work.

We now examine the images of \(P_{r,s,t}\) under the weight raising operators:

**Lemma 2.8.** The function \(R_{n}^{(b,-)} P_{r,s,t}\) equals

\[
\begin{align*}
& r \left( (r-1)\mu^2 P_{r-2,s,t} + \frac{2n - 2s + 2 - b_-}{2} P_{r-1,s+1,t+1} \right) + \\
& -(r+n-s) \left( 2r\mu^2 P_{r-1,s,t} + \frac{2n - 2s + 2 - b_-}{2} P_{r,s+1,t} \right).
\end{align*}
\]
Taking the correct weight of automorphy of \( P_{r,s,t} \) reduces Lemma 2.8 to
\[
R^{(b_{-})}_{s-r} P_{r,s,t} = r \left( (r-1)\mu^2 P_{r-2,s,t} + \frac{2 - b_{-} - 2r}{2} P_{r-1,s+1,t+1} \right)
\]
(with the same differences between the indices). Write \( s = k \) and \( r = k - m \), and divide by \( m - k \). The natural extension of the result to \( k = m \) appears in Lemma 2.9.

We have
\[
R^{(b_{-})}_{m} P_{0,m,t} \cdot \ln \left( \frac{|(\lambda, Z_V Z)|^2}{Y^2} \right) = \mu^2 P_{-2,m,t} - \frac{2 - b_{-} - 2}{2} P_{-1,m+1,t+1}.
\]

In addition, the simplified Lemma 2.8 has the following Corollary 2.10. We can write
\[
(R^{(b_{-})}_{m})^l \sum_{k=0}^{m-1} a_k (\mu^2)^{m-k} P_{k-m,k,k+t} \text{ as } \sum_{k=0}^{m+l-1} b_k (\mu^2)^{m+l-k} P_{k-m+2l,k,k+t},
\]
where the coefficient \( b_0 \) is \( \frac{(m+2l-1)!}{(m-1)!} a_0 \).

**Proof.** We argue by induction on \( l \), the case \( l = 0 \) being trivial. If the assertion holds for \( l \), then the simplified Lemma 2.8 shows that applying \( R_{m+2l} \) to the \( k \)th term in the result for \( l \) contributes to the terms with indices \( k \) and \( k + 1 \) in the asserted expression for \( l + 1 \). Hence only images the term with \( a_0 \) affect \( b_0 \), and the multiplier is easily seen to be \( \frac{(m+2l-1)!}{(m-1)!} \). This proves the corollary. \( \square \)

We now quote the main technical result of [Ze]. Let \( L \) be an even lattice of signature \((2,b_{-})\) with dual \( L^* \), and let \( \rho_L \) be the Weil representation of the metaplectic double cover \( Mp_2(Z) \) of \( SL_2(Z) \) on the space \( C[L^*/L] \) (see [B] or Section 2 of [Ze]) for the details). We denote the **discriminant kernel**, consisting of those elements of \( SO^+(L) \) operating trivially on \( L^*/L \), by \( \Gamma \). Given a modular form \( F \) of weight \( 1 - \frac{b_{-}}{2} + m \) (with \( m \in \mathbb{N} \)) and representation \( \rho_L \), the **regularized theta lift** of \( F \) is the integral
\[
\Phi_{L,m,0}(Z,F) = \int_{Mp_2(Z)/H} (F(\tau), \Theta_{L,m,0}(\tau,Z)) y^\frac{b_{-}}{2} dx dy
\]
regularized in the sense of [B]. Theorem 2.7 of [Ze] investigates the result for \( F \) being the image of a weakly holomorphic modular form \( f \) of weight \( 1 - \frac{b_{-}}{2} - m \) and representation \( \rho_L \) under the \( m \)th power of the Shimura–Maaß operator. The space of such weakly holomorphic modular forms \( f \), which means modular forms of weight \( 1 - \frac{b_{-}}{2} - m \) and representation \( \rho_L \) which are holomorphic on \( H \) but may have poles at the cusps, is denoted \( M_{1-\frac{b_{-}}{2}-m}(\rho_L) \).
Theorem 2.11. If

\[ f = \sum_{\gamma \in L^*/L} \sum_{n > -\infty} c_{\gamma,n} q^n e_\gamma \in M^1_{1 - \frac{b}{2} - m} (\rho_L) \quad \text{and} \quad F = \frac{1}{(2\pi i)^m} 5^m \sum_{1 - \frac{b}{2} - m} f \]

then \( \frac{i^m}{2} \Phi_{L,m,m,0} \) is an automorphic form of weight \( m \) with respect to \( \Gamma \) which is an eigenfunction of (minus) \( \Delta_{L,m}^{(b,-)} \) with eigenvalue \(-2mb_-\). Every negative norm \( \mu \in L^* \) contributes a singularity of

\[ \frac{1}{2} e_{\mu + L, n^2/4\pi} \sum_{k=0}^{m} (-1)^k \left( \frac{-b}{2} \right)^k (\mu^2)^{m-k} P_{k-m,k,k} \left\{ \frac{1}{\ln (|\mu Z_{\rho,\psi}|)^2} \right\} \quad k < m \]

along the divisor \( \mu^+ = \{ \nu \in G(L) | \mu \in \nu_\mu \} \), and these are the only singularities of \( \frac{i^m}{2} \Phi_{L,m,m,0} \). If \( L \) contains a (primitive) isotropic vector \( z \), \( K \) is the lattice \( z^+ / \mathbb{Z} z \), and we identify \( G(L) \) with \( K \mathbb{R} + iC \) using this vector \( z \), then for any \( Z \in K \mathbb{R} + iC \) which lies in a Weyl chamber \( W \) containing \( z \) in its closure, \( \frac{i^m}{2} \Phi_{L,m,m,0} \) admits the Fourier expansion of the sort

\[ \frac{i^m}{2} \Phi_{L,m,m,0}(Z,F) = \frac{\psi(\sqrt{y})}{\sqrt{y}} + \frac{\Lambda}{\sqrt{y}} + \sum_{\rho \in K^*} \sum_{k,h} B_{k,h,\rho} \left( \frac{g(m,\varepsilon)}{\varepsilon} \right) (Z). \]

Here \( \psi \) is a polynomial in \( \sqrt{y} \), \( \Lambda \) is some constant, \( \varepsilon = \text{sgn}(\rho, W) \), and

\[ B_{k,h,\rho} = a_{k,h,\varepsilon} \sum_{n > 0, n^2 \in L^*} \sum_{\mu \in \mu^+} \frac{c_{\mu,n}}{n^{m+1}} e(n(\gamma, \zeta)) \]

where \( a_{k,h,\varepsilon} \) are rational numbers with \( a_{0,0,1} = 2^m \).

Note that the coefficients \( A_{k,h,\rho} \) from Theorem 2.7 of \( [\text{Ze}] \) were replaced here by \( \rho^{m-k} B_{k,h,\rho} \). We also decomposed \( \sqrt{\frac{x^2}{\pi^2}} \) as \( \psi + \frac{\Lambda}{\sqrt{\pi x}} \), and the two expressions for the singularity are the same by the definition of \( \left( \frac{x}{a} \right) \) for real \( x \) and part (i) of Lemma \( [\text{Ze}] \).

We can now state and prove the main result of this paper.

Theorem 2.12. Let

\[ f = \sum_{\gamma \in L^*/L} \sum_{n > -\infty} c_{\gamma,n} q^n e_\gamma \in M^1_{1 - \frac{b}{2} - m} (\rho_L) \]

for some lattice \( L \) of signature \((2,b_-)\) with even \( b_- \), and let \( z \in L \) be a primitive isotropic vector. There exists an automorphic form \( \Psi \) on \( K \mathbb{R} + iC \), of weight \( m + b_- \) with respect to \( \Gamma \), whose Fourier expansion around \( z \) is

\[ \sum_{\rho \in K^* \setminus \{0, W\}} 2^m \left( \sum_{n > 0, n^2 \in L^*} \sum_{\mu \in \mu^+} \frac{c_{\mu,n}}{n^{m+1}} e(n(\gamma, \zeta)) \right) (\rho^2)^{m+b_-} e((\rho, Z)). \]
The singularities of $\Psi$ are poles of order $m+b_-$ along the divisors $\mu^+$ for negative norm elements $\mu \in L^*$, where each such element contributes to the pole along $\mu^+$ a singularity of the sort

$$(-i)^m(m+b_--1)! \sum_{\nu=0} \frac{|\mu|^2}{2^{m+b_--1}m+b_-} c_{\mu+L,\frac{2\pi}{Y}} (\mu, Z_{V,2})^{m+b_-}.$$ 

**Proof.** Apply $\frac{1}{(Y^2)^{l-2}} (R_m^{(b_-)})^{b_-}/2$ to the function $\frac{m}{2} \Phi_{L,m,m,0}$ from Theorem 2.11. The result is an an automorphic form $\Psi$ of weight $m+b_-$ with respect to $\Gamma$ by Part (i) of Proposition 1.6 and part (ii) of that Proposition shows that it is harmonic since $\lambda + l(4m + 4l - 2b_-) = 0$. We can analyze the $\frac{1}{(Y^2)^{l-2}} (R_m^{(b_-)})^{b_-}/2$-image of each part of the Fourier expansion of $\frac{m}{2} \Phi_{L,m,m,0}$ separately.

Each Fourier term with $g_{k,h,\rho}^{(m),-}$ satisfies the conditions of Proposition 2.3. Therefore these functions and the term $\frac{1}{(Y^2)^{l-2}}$ sum to an anti-holomorphic function divided by $(Y^2)^m + \Phi_m^{(b_-)}$ as $R_m^{(b_-)} = (Y^2)^{-m} R_{m}^{(b_-)} (Y^2)^m$ and $R_{m}^{(b_-)}$ annihilates anti-holomorphic functions, these parts do not contribute to $\Psi$. As $\phi$ depends only on $\psi$, we find that $\omega = \frac{\psi}{(Y^2)^{l-2}}$ is homogenous of degree $1 - m$. Corollary 2.2 shows that this function is also annihilated after applying $\frac{1}{(Y^2)^{l-2}} (R_m^{(b_-)})^{b_-}/2$, since the multiplier

$$2(d + m + l)(d + 2m + 2r + 2 - b_-) - 2d(r + 1) + \lambda$$

vanishes for $d = 1 - m$, $\lambda = -2mb_-$, and $r = \frac{b_--1}{2}$. Consider now the Fourier term involving $g_{k,h,\rho}^{(m),+}$. Proposition 2.3 shows that its image takes the form

$$\sum_{k,h} C_{k,h}^+(m+b_-+2)/2, $$

with $C_{k,0}^+$ explicitly evaluated in terms of the coefficients $B_{k,0,\rho}$. We claim that $C_{k,0}^+ = 0$ for all $k > m$. Indeed, for $0 < r < k - m$ the coefficient $B_{k-r,0,\rho}$ vanishes as $k - r > m$ (this covers the case $k > m + \frac{b_-}{2}$).

On the other hand, as $l = \frac{b_-}{2}$ the binomial coefficient from Proposition 2.3 is $(m+k+r)$, which vanishes for $k - m \leq r \leq \frac{b_-}{2}$ as $0 \leq m - k + r < r$. Thus, the combination in question satisfies the conditions of Proposition 2.3, which yields the holomorphicity and the desired Fourier expansion of $\Psi$ after substituting $f_{0,0,\rho}^{m+b_-}/2$ and the values of $B_{0,0,\rho}$ and $a_{0,0,1}$.

The singular part of $\Psi$ is the image of the singularity type of $\frac{m}{2} \Phi_{L,m,m,0}$ under $\frac{1}{(Y^2)^{l-2}} (R_m^{(b_-)})^{b_-}/2$. Successive applications of Lemma 2.8 together with Lemma 2.9 for the index $k = m$ in the first step, show that the resulting singular part is a linear combination of terms of the sort $(\mu^2)^{m-k+b_-}/2 P_{k-m-b_-,k,k}$, and this combination must be harmonic up to smooth functions. But Corollary 2.7 now implies that this sum contains only the term with $k = 0$, as the multiplier with $r = 0$ in that corollary is $-\lambda = 0$. The remaining coefficient is determined in Corollary 2.10 using the value of $a_0$ from Theorem 2.11 (recall that $\mu^2 < 0$ and the coefficient $\frac{1}{(Y^2)^{l-2}}$). This completes the proof of the theorem. \qed
3 A Special Case and Conjectural Generalizations

This Section discusses the special case of unimodular $L$ of signature $(2, 2)$ of Theorem 2.12 and poses conjectures for odd $b$- and lattices without isotropic vectors.

A special case of interest involves $b = 2$ and a unimodular lattice $L$. There exists only one such lattice (up to isomorphism), which can be realized as $M_2(\mathbb{Z})$ with the norm of a matrix being $-2$ times its determinant. In this case

$$K_R + iC = \mathcal{H} \times \mathcal{H}$$

and automorphic forms are Hilbert modular forms of parallel weight. The lattice $K$ is $\mathbb{Z}^2$, in which $(b, a)$ has norm $2ab$ and its pairing with $Z \in K_R + iC$ corresponding to $(\tau, \sigma) \in \mathcal{H} \times \mathcal{H}$ gives $ar + bs$. Hence $e((\rho, Z)) = q^m p^b$ for such $\rho \in K^* = K$, where $p = e(\sigma)$.

Our modular form $f$ should have weight $1 - b - 2 - m = -m$ and trivial representation, so that $m$ is even (otherwise $f = 0$). The Fourier expansion of the function $\Psi(\tau, \sigma)$ from Theorem 2.12 is

$$\frac{1}{2} \sum_{((b, a), W) > 0} \sum_{n|(b, a)} \frac{c_{ab}}{n} \left(\frac{ab}{n}\right)^{m+1} q^a p^b = \frac{1}{2} \sum_{((k, l), W) > 0} (4kl)^{m+1} c_{kl} \sum_{n=1}^{\infty} n^{m+1} q^{nk} p^{nl}.$$  

As small $z_{v, 1}$ means large $Y^2 = 2yt$, the Weyl chambers containing $z$ in their closure are characterized by large $y$ and $t$, and are separated by the real curves defined by the equations of the sort $dy = et$ for positive integers $d$ and $e$ with $c_{-de} \neq 0$. The condition $(\rho, W) > 0$ is equivalent to $ay + bt > 0$, a condition which is satisfied for all such Weyl chambers if $a \geq 0$ and $b \geq 0$ (but excluding $a = b = 0$). In addition, any $\rho \in K$ with norm 0 does not contribute to the Fourier expansion because of the positive power of $\rho^2$. The vectors in $K$ which remain to be considered have negative norms, and which one of $(a, -b)$ or $(-a, b)$ (with $a$ and $b$ positive) satisfies $(\rho, W) > 0$ depends on the Weyl chamber (though it is irrelevant for the final formula for $\Psi$—see below). It turns out more convenient to write the contribution from positive norm vectors $(b, a)$ using the expression with $\sum_{n|(b, a)}$, while for the negative norm vectors we use the infinite sum over $n$. In order to analyze the latter part we shall need the following

**Lemma 3.1.** Let $r \in \mathbb{N}$ and $w \in \mathbb{C}$ with $|w| < 1$ be given. Then the series

$$h_r(w) = \frac{1}{2} \delta_{r, 0} + \sum_{n=1}^{\infty} n^r w^n$$

equals

$$\frac{\varphi_r(w, 1)}{(1 - w)^{r+1}},$$

where $\varphi_{r+1}$ is a polynomial in two variables which is homogenuous of degree $r + 1$ and satisfies $\varphi_r(w, u) = \varphi_r(u, w)$.

A polynomial $\varphi_r$ satisfying the latter property will be called symmetric. $\delta_{r, 0}$ is the Kronecker delta symbol, and it is required for the symmetry of $\varphi_0$. 

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Proof. For \( r = 0 \) the series converges to \( \frac{1 + w}{1 - w} \), hence the assertion holds with \( \varphi_0(w, u) \) being \( \frac{w + u}{2} \). Assume now that the result holds for \( r \). The series with \( r + 1 \) is obtained from the series with \( r \) through the operator \( w \frac{d}{dw} \), which yields

\[
\varphi_{r+1}(w, 1) = w \left[ (1 - w) \partial_w \varphi_r(w, 1) + (r + 1) \varphi_r(w, 1) \right].
\]

Here \( \partial_w = \frac{\partial}{\partial w} \) is the partial derivative, and \( \partial_u = \frac{\partial}{\partial u} \) will be the derivative with respect to the other variable. Homogenizing to degree \( r + 2 \) and using the fact that \( w \partial_w + u \partial_u \) multiplies \( \varphi_r \) by its degree of homogeneity \( r + 1 \) yields the equality

\[
\varphi_{r+1}(w, u) = wu \left[ \partial_w \varphi_r(w, u) + \partial_u \varphi_r(w, u) \right],
\]

from which the assertion for \( r + 1 \) follows. This proves the lemma.

It follows from Lemma 3.1 that the expression \( \frac{\varphi'(w, 1)}{(1 - w)^{r+1}} \) is the substitution \( u = 1 \) in the function \( (w, u) \mapsto \frac{\varphi'(w, u)}{(1 - w)^{r+1}} \), which is homogenous of degree 0 on the subset of \( \mathbb{C}^2 \) consisting of elements with unequal coordinates. As interchanging \( u \) and \( w \) multiplies the expression by \((-1)^{r+1}\), we find that the function \( h_r \) from Lemma 3.1 satisfies \( h_r(\frac{1}{w}) = (-1)^{r+1} h(w) \) for all \( 1 \neq w \in \mathbb{C}^* \). In particular, if \( r \) is odd and \( \alpha \neq \beta \) are two non-zero complex numbers then \( h_r(\frac{\alpha}{\beta}) \) and \( h_r(\frac{\beta}{\alpha}) \) coincide, and the common value can be written as \( \frac{\varphi_{r}(\alpha, \beta)}{(\alpha - \beta)^{r+1}} \). Hence given \( k \) and \( l \) with negative product, the equality

\[
\sum_{n=1}^{\infty} n^{m+1} q^n k^p l^q = \left( \frac{\varphi_{m+1}(q^k, p^l)}{(q^k - p^l)^{m+2}} \right)
\]

holds either if \( k < 0 < l \) or if \( k > 0 > l \).

Given a matrix \( M \in L \) of determinant \( d > 0 \) and \( Z \in K \mathbb{R} + i \mathbb{C} \) corresponding to \( (\tau, \sigma) \in \mathcal{H} \times \mathcal{H} \) we find that

\[
(M, Z_{V,Z}) = j(M, \sigma)^{m+2}(\tau - M\sigma)^{m+2}.
\]

Substituting this into the singularity, Theorem 2.12 takes the form

**Theorem 3.2.** If \( f = \sum_{n >> -\infty} c_n q^n \in M_{-m}^1(SL_2(\mathbb{Z})) \) then the function \( \Psi(\tau, \sigma) \) on \( \mathcal{H} \times \mathcal{H} \) whose Fourier expansion is given for large \( y \) and \( t \) by

\[
2^{2m+1} \sum_{a > 0, b > 0} \sum_{n \in (b, a)} \frac{c_{ab}}{n} \left( \frac{ab}{n} \right)^{m+1} q^a p^b - \sum_{d=1}^{\infty} d^{m+1} c_{-d} \sum_{k,l,d} \varphi_{m+1}(q^k, p^l) (q^k - p^l)^{m+2}
\]

(the sum over \( d \) being essentially finite) is a meromorphic Hilbert modular form of weight \( (m + 2, m + 2) \) with respect to \( SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \). Its poles arise from positive determinant matrices \( M \in M_2(\mathbb{Z}) \), where such matrix gives a pole of the sort

\[
\frac{i^{m}(m + 1)!}{(2\pi)^{m+2}} \frac{(\det M)^{m+1} c_{-\det M}}{j(M, \sigma)^{m+2}(\tau - M\sigma)^{m+2}}.
\]

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In order to evaluate the Hilbert modular form obtained from Theorem 3.2, one first observes that if \( \Phi_d \) is the \( d \)th modular polynomial defining the modular curve \( X_0(d) \) then \( \Phi_d(j(\tau), j(\sigma)) \) vanishes precisely along the divisors \( M^\perp \) for primitive \( M \) of determinant \( d > 0 \). Hence multiplying \( \Psi \) by such expressions raised to the power \( m + 2 \) removes the singularities on \( \mathcal{H} \times \mathcal{H} \), and multiplying by an appropriate power of \( \Delta(\tau)\Delta(\sigma) \) (where \( \Delta \) is the normalized \( \Delta \) function of weight 12 for \( SL_2(\mathbb{Z}) \)) removes the poles which this operation created at the cusps of \( \Gamma \) (in fact, this is the only class of orthogonal Shimura varieties of dimension \( > 1 \) for which the Koecher principle fails). The resulting function is holomorphic, but of some higher weight \( k \), hence lies in \( M_k(SL_2(\mathbb{Z})) \otimes \mathbb{C} M_k(SL_2(\mathbb{Z})) \).

One can evaluate the Fourier expansion of this function and determine it using a convenient basis for \( M_k(SL_2(\mathbb{Z})) \). Unfortunately the expressions tend to be complicated rather quickly: The simplest possible case is \( m = -2 \) with \( f(\tau) \) being of the form \( \frac{1}{q} + O(1) \), hence \( f = \frac{E_k}{\Delta} \) where \( E_k \) is the weight \( k \) Eisenstein series for \( SL_2(\mathbb{Z}) \) normalized to attain 1 at the cusp. In this case

\[
\Psi(\tau, \sigma)(j(\tau) - j(\sigma))^4 \Delta(\tau)^3 \Delta(\sigma)^3
\]

is holomorphic of weight 40. We take the basis for \( M_{40}(SL_2(\mathbb{Z})) \) consisting of the functions \( F_k = E_4^{10 - 3k} \Delta^k \) with \( 0 \leq k \leq 3 \), and write \( F_{kl}(\tau, \sigma) = F_k(\tau)F_l(\sigma) \) for \( k \) and \( l \) between 0 and 3. Evaluating the Fourier coefficients one can show that the holomorphic weight 40 Hilbert modular form thus obtained is

\[
-F_{20} - 4F_{11} - 6F_{02} + 984F_{30} + 9384F_{21} + 9384F_{12} + 984F_{03} + 210F_{20} - 12607488F_{31} - 2654208F_{32} - 2654208F_{13}.
\]

Meromorphic Hilbert modular forms for Hilbert modular groups associated with arbitrary real quadratic fields can also be obtained from Theorem 2.12 using appropriate (non-unimodular) signature \((2,2)\) lattices. The Fourier expansion of these Hilbert modular forms will be a more complicated variant of the ones given in Theorem 3.2 and the singularities will be along Hirzebruch–Zagier divisors. On the other hand, these Hilbert modular varieties cannot have poles at cusps by to the Koecher principle, so that the determination of \( \Psi \) may be possible after multiplying by an appropriate Borcherds product. It is reasonable to expect that \( \Psi \) can be represented as the quotient of two holomorphic Borcherds products of large weights.

Returning to the general case, Theorem 2.12 shows that for any even \( b_\perp \), the image of \( \frac{\pi}{2} \Phi_{L,m,m,0} \) under \( \frac{1}{(-4\pi^2)^{b_-/2}} (R_m^{(b_-)})^{b_-/2} \) is automorphic of weight \( m + b_- \), admitting a Fourier expansion in which the coefficient of the term \( e((\rho, Z)) \) is some global constant times \( |\rho|^{2m+b_-} \) (where \( |\rho| \) stands for \( \sqrt{|\rho|^2} \)) times a combination of Fourier coefficients of the weakly holomorphic modular form \( f \) of weight \( 1 - b_- - m \). The singularity along \( v^\perp \) with \( v^2 = -2 \) is some global constant times

\[
\frac{(m + b_- - 1)!}{\pi^{m+b_-}} \sum_{\alpha \in L^*} \alpha^m c_{\alpha v, -v^2} \cdot \frac{1}{(v, Z v, Z)^{m+b_-}}
\]
Theorem 2.8 of [Ze] shows that for $b_- = 1$ applying $\frac{1}{\pi^{m+1}} \delta_{2m}$ to $\tilde{\psi}_m^{2m} \phi_{L,m,m,0}$ yields a meromorphic modular form of weight $2m + 2$, whose singularity at $\sigma = s + it \in \mathcal{H}$ is a global constant times

$$\frac{m!}{\pi^{m+1}} \sum_{\alpha \in L^*} \alpha^m c_{\alpha v,-\alpha^2} \left( \frac{-t}{(\tau - \sigma)(\bar{\tau} - \bar{\sigma})} \right)^{m+1}.$$

Now, $\frac{1}{\pi} \delta_{2m}$ is the “power $\frac{b_-}{2}$” of $\frac{1}{4\pi^2} R_m^{(1)} = \frac{\delta_{-b_-}^2}{4\pi^2}$, a modular form of weight $2m + 2$ is an automorphic form of weight $m + b_-$; $\{\sigma\}$ is $v_\perp$ for the norm $-2$ vector $v = J_\sigma = \frac{1}{2} \left( \begin{smallmatrix} s & -|s|^2 \\ 1 & -s \end{smallmatrix} \right)$, and $\left( \frac{-t}{(\tau - \sigma)(\bar{\tau} - \bar{\sigma})} \right)^{m+1}$ is $\frac{1}{(v, Z, Z)}^{m+b_-}$. Moreover, if the signature $(2,1)$ lattice $L$ contains an isotropic vector then the $r$th term in the Fourier expansion of $\Psi$ around it is $r^{2m+1}$ times the same combination of the Fourier coefficient of the modular form $f$ mentioned above (up to a global constant). As $K^*$ is cyclic and $r = |\rho|$ (up to a constant again), the analogy between the two cases is clear and leads us to formulate the following

**Conjecture 3.3.** For any dimension $b_-$ and any weight $m$ there exists an operator $S_m^{(b_-)}$ satisfying the following properties: (i) $S_m^{(b_-)}$ takes automorphic forms of weight $m$ on $K \mathbb{R} + i\mathbb{C}$ to automorphic forms of weight $m + b_-$ on that manifold. (ii) $S_m^{(b_-)}$ increases Laplacian eigenvalues by $2mb_-$. (iii) $S_m^{(b_-)}$ is a differential operator of order $b_-$ such that the combination $S_m^{(b_-)} S_m^{(b_-)}$ equals $\frac{1}{(-4\pi^2)^{-b_-}} \left( R_m^{(b_-)} \right)^{b_-}$. (iv) We have $S_m^{(b_-)} = (Y^2)^{-m} S_0^{(b_-)} (Y^2)^m$ and $S_0^{(b_-)}$ eliminates anti-holomorphic functions. (v) $S_m^{(b_-)}$ takes the unique combination $\sum_{k,h} B_{k,h}^{(m),+} g_{k,h,\rho}^{(m),+}$ with the eigenvalue $-2mb_-$ to $ia B_{0,0}^{+} (\rho^2)^{m+b_-} \epsilon((\rho, Z))$ for some $a$. (vi) $S_m^{(b_-)}$ takes the singularity type

$$\sum_{k=0}^{m-1} a_k (\mu^2)^{m-k} P_{k-m,k,k} + a_m P_{0,m,m} \cdot - \ln \frac{|(\mu, Z, Z)|^2}{Y^2}$$

having Laplacian eigenvalue $-2mb_-$ to

$$c \frac{(m+b_- - 1)!}{(-4\pi^2)^{b_-/2}(m-1)!} a_0 \frac{|\mu|^{2m+b_-} (\mu, Z, Z)^{m+b_-}}{|(\mu, Z, Z)|^{m+b_-}}$$

for some constant $c$. (vii) If $b_- > 1$ then $S_m^{(b_-)}$ annihilates functions of $Y$ alone which are homogenous of degree $1 - m$.

Indeed, we have proved that the operator $S_m^{(b_-)} = \frac{1}{(2\pi i)^b} \left( R_m^{(b_-)} \right)^{b_-/2}$ satisfies all the properties stated in Conjecture 3.3 if $b_-$ is even, while for $b_- = 1$ it follows from [Ze] that $\frac{1}{(2\pi i)^b}$ bears these properties. The proof of Theorem 2.12 shows that if such operators $S_m^{(b_-)}$ do exist for odd $b_-$ then the assertion of Theorem 2.12 will extend to the odd $b_-$ case.

The proof of the meromorphicity in Theorem 2.12 depends strongly on the Fourier expansion, hence on the existence of isotropic vectors in $L$. However, in
Theorem 2.8 of [Ze] the meromorphicity follows from simple properties of $\delta_{2m}$, hence is independent of the existence of such elements. By Meyer’s Theorem, isotropic vectors must exist for all $b_- \geq 3$, so that the only case in which meromorphicity is not guaranteed is where $b_- = 2$ and $L$ has no isotropic vectors. We thus pose also

**Conjecture 3.4.** The image of the weight $m$ automorphic form $\frac{m}{2} \Phi_{L,m,m,0}$ on $K_\mathbb{R} + iC \cong \mathcal{H} \times \mathcal{H}$ (with $b_- = 2$) under $\frac{1}{4\pi^2} R_m^{(2)}$ (namely $(\delta_m, \tau)_{\delta_m, \sigma} - \frac{2\pi}{\sqrt{2}}$) is meromorphic with the singularities given in Theorem 2.12 also when the signature (2,2) lattice $L$ contains no isotropic vectors.

The parts of Theorem 2.12 concerning the weight, the harmonicity, and the singularities of $\Psi$ do not depend on the the existence of isotropic vectors in $L$, hence hold also in the case considered in Conjecture 3.4. Thus only meromorphicity is not yet established. We mention that the eigenvalue $-4m$ property of $\frac{m}{2} \Phi_{L,m,m,0}$ implies that $\Psi$ must be annihilated by the operator $\partial_\tau \partial_\sigma$ (which is $\frac{L^{(2)}}{\sqrt{2}}$), which may be another hint suggesting that $\Psi$ is indeed meromorphic (though this assertion holds also when the modular form $f$ of weight $-m$ is a harmonic weak Maass form, a case in which we do not expect $\Psi$ to be meromorphic).

## 4 Proofs of the Properties of $R_{m(b^-)}$ and $L_{b^-}$

In this Section we include the proofs of the properties of the weight raising and weight lowering operators appearing in Section 1.

We first introduce (following [Na]) a convenient set of generators for $O^+(V)$. For $\xi \in K_\mathbb{R}$ we define the element $p_\xi \in SO^+(V)$ whose action is

$$[\mu \in K_\mathbb{R} = \{z, \zeta\}^+] \mapsto \mu - (\mu, \xi)z, \quad \zeta \mapsto \zeta + \xi - \frac{\zeta^2}{2}z, \quad z \mapsto z.$$

Furthermore, given an element $A \in O(K_\mathbb{R})$ and a scalar $a \in \mathbb{R}^*$ such that $a > 0$ if $A \in O^+(K_\mathbb{R})$ and $a < 0$ otherwise, we let $k_{a,A} \in O^+(V)$ be the element acting as

$$[\mu \in K_\mathbb{R} = \{z, \zeta\}^+] \mapsto A\mu, \quad \zeta - \frac{\zeta^2}{2} \mapsto \frac{1}{a}\left(\zeta - \frac{\zeta^2}{2}\right), \quad z \mapsto az.$$

For any $Z \in K_\mathbb{R} + iC$ we have

$$p_\xi Z = Z + \xi, \quad J(p_\xi, Z) = 1, \quad k_{a,A}Z = aAZ, \quad \text{and} \quad J(k_{a,A}, Z) = \frac{1}{a}.$$

Note that the relation between $A$ and the sign of $a$ is equivalent to preserving $C$ rather than mapping $Z$ into $K_\mathbb{R} - iC$—it appears that [Na] ignored this point. Choose now an element of $G(K_\mathbb{R})$ in which the positive definite space is
generated by the norm 1 vector \( u_1 \), and consider the involution \( w \in SO^+(K_{\mathbb{R}}) \) defined by

\[
\mu \in K_{\mathbb{R}} = \{ z, \zeta \} \rightarrow \mu - 2(\mu, u_1)u_1, \quad \zeta \rightarrow -z, \quad z \rightarrow -\left( \zeta - \frac{\zeta^2}{2} \right)
\]

\( (w \) inverts the positive definite space \( \mathbb{R}u_1 \)). Its action on \( K_{\mathbb{R}} + iC \) is through

\[
wZ = \frac{2}{Z^2} \left[ Z - 2(Z, u_1)u_1 \right] \quad \text{with} \quad J(w, Z) = \frac{Z^2}{2}.
\]

The elements \( k_{a, A} \) with \( (a, A) \) in the index 2 subgroup of \( R^* \times O(K_{\mathbb{R}}) \) thus defined and \( p_{\xi} \) for \( \xi \in K_{\mathbb{R}} \) generate the stabilizer \( St_{O^+(V)}(\mathbb{R}_Z) \) of the isotropic space \( \mathbb{R}_Z \) in \( O^+(V) \) as the semi-direct product of these groups. The fact that adding \( w \) to \( St_{O^+(V)}(\mathbb{R}_Z) \) generates \( O^+(V) \) is now easily verified by considering the action on isotropic 1-dimensional subspaces of \( V \).

Some useful relations are derived in the following Lemma 4.1.

**Lemma 4.1.** Let \( K_{\mathbb{R}} \) be a non-degenerate vector space of dimension \( b_- \), fix \( \alpha \in \mathbb{C} \), and let \( F \) be a \( C^2 \) function which is defined on a neighborhood of a point \( Z = X + iY \in K_{\mathbb{C}} \) with \( Y^2 > 0 \). Then the following relations hold:

\[
(Y^2)^{-\alpha} \Delta_{K_{\mathbb{C}}}^h ( (Y^2)^{\alpha} F)(Z) = \Delta_{K_{\mathbb{C}}}^{\alpha} F(Z) - \frac{2i\alpha}{Y^2} D^* F(Z) - \frac{\alpha(\alpha + 1 - b_-^2)}{Y^2} F(Z)
\]

and

\[
(Y^2)^{-\alpha} \Delta_{K_{\mathbb{C}}}^{\alpha} ( (Y^2)^{\alpha} F)(Z) = \Delta_{K_{\mathbb{C}}}^{-\alpha} F(Z) + \frac{2i\alpha}{Y^2} D^* F(Z) - \frac{\alpha(\alpha + 1 - b_-^2)}{Y^2} F(Z).
\]

We remark that Lemma 4.1 holds for \( K_{\mathbb{R}} \) of arbitrary arbitrary signature (not necessarily Lorentzian), but not negative definite (for \( Y^2 > 0 \) to be possible).

**Proof.** The proof is obtained by a straightforward calculation, using an orthonormal basis for \( K_{\mathbb{R}} \) and the action of \( \partial_k \) and \( \partial_{\bar{K}} \) on functions of \( Y \) alone. □

We remark that the third operator \( \Delta_{K_{\mathbb{C}}}^h \) bears a property similar to Lemma 4.1 which is used implicitly to prove Equation (2) in Section 2 of [Ze].

We can now present the

**Proof of Theorem 1.1.** Multiply both sides of the desired assertion for \( R_{m}^{(b_-)} \), as well as the function \( F \) there, by \( Y^m \). Lemma 4.1 the first definition of \( R_{m}^{(b_-)} \), and Equation 1 show that this yields the equivalent equality

\[
(R_{0}^{(b_-)} F)[M]_{2,-m} = R_{0}^{(b_-)} (F[M]_{0,-m}).
\]

Observe that conjugating the latter equation and multiplying by \( (Y^2)^2 \) yields the required equality for \( L^{(b_-)} \). Hence we are reduced to proving only this equality.
Moreover, $R_0^{(b_-)}$ involves only holomorphic differentiations, which means that it commutes with the power of $J(M, Z)$ coming from the anti-holomorphic weights. Hence we can take $m = 0$, which implies that proving the equation

$$(R_0^{(b_-)} F)[M]_2 = R_0^{(b_-)} (F[M]_0)$$

(which the assertion for $R_0^{(b_-)}$ in the formulation of the theorem) suffices for proving the theorem. Writing the arguments as $M^{-1}(Z)$ in both sides and using the cocycle condition brings the latter equation to the form

$$(R_0^{(b_-)} F)(Z) J(M^{-1}, Z)^2 = (R_0^{(b_-)} M^{-1} F)(Z).$$

(3)

By a standard argument it suffices to verify Equation (3) for $M^{-1}(Z)$ being one of the generators of $O^+(V)$ considered above. Equation (3) with $M^{-1} = p_k$ follows from the invariance of both $\Delta^{h_{K_C}}$ and $D^*$ under translations of $X = RZ$ and the fact that $J(p_k, Z) = 1$. The action of $M^{-1} = k_{a,A}$ divides $\Delta^{h_{K_C}}$ by $a^2$, leaves $D^*$ invariant, and divides $Y^2$ by $a^2$ (since $A \in O(K_\mathbb{R})$), which proves Equation (3) since $J(k_{a,A}, Z) = 1$. Finally, for $M^{-1} = w$ we have the equalities

$$(\Delta^{h_{K_C}})^w = \left(\frac{Z^2}{2}\right)^2 \Delta^{h_{K_C}} - (b_- - 2)\frac{Z^2}{2} D, \quad (D^*)^w = \frac{Z^2}{2} D^* - \frac{2i Y^2}{Z^2} D$$

with $D = \sum_k z_k \partial k$ from [Na] (the corresponding operator from [Na] is $\frac{1}{2} \Delta^{h_{K_C}}$ rather than $\Delta^{h_{K_C}}$, while $\delta = \frac{Z^2}{2}$, $\delta = \frac{Z^2}{2}$, and $d = \frac{Y^2}{2}$ there). Using Equation (3) we thus find that applying $M^{-1}$ to the sum of $\Delta^{h_{K_C}}$ and $\frac{4(b_- - 2)}{Y^2} D^*$ (which is $R_0^{(b_-)}$) multiplies it by $\left(\frac{Z^2}{2}\right)^2$ (as the coefficients in front of $D$ cancel), which establishes Equation (3) also using the value of $J(w, Z)$. This completes the proof of the theorem.

For calculational purposes it turns out convenient to introduce the operator

$$\tilde{\Delta}^{(b_-)}_{m,n} = \Delta^{(b_-)}_{m,n} - 2n(2m - b_-),$$

on which complex conjugation interchanges the indices $m$ and $n$. The operator

$$(D^*)^2 - \frac{D^*}{2i} = \sum_{k,l} y_k y_l \partial k \partial l$$

will also show up, so we denote it $\overline{(D^*)^2}$. We now turn to the

**Proof of Proposition 2.3** Conjugating the desired equality for $R^{(b_-)}_m$ by $(Y^2)^m$, applying Equation (3), and taking the differences between the operators $\Delta^{(b_-)}_{m,n}$ and $\tilde{\Delta}^{(b_-)}_{m,n}$ into consideration, we see that the asserted equality for $R^{(b_-)}_m$ is equivalent to

$$\tilde{\Delta}^{(b_-)}_{m,n} R^{(b_-)}_m - R^{(b_-)}_m \Delta^{(b_-)}_{m,n} = (2b_- + 4m - 4) R^{(b_-)}_m.$$
Moreover, multiplying the complex conjugate of the latter equation by \((Y^2)^2\) and comparing \(\Delta_{2,-m}^{(b,-)}\) with \(\Delta_{2,-m}^{(b,-)}\) yields the required property for \(L^{(b,-)}\) (with the index \(m\) replaced by \(-m\)). Hence, as in the proof of Theorem 1.1, we are reduced to proving this single equation. In addition, the dependence on \(m\) of the left hand side enters only through the difference \(-4imD^*\) between the operators \(\Delta_{l,-m}^{(b,-)}\) and \(\Delta_{l}^{(b,-)}\) with \(l \in \{0,2\}\). As a simple calculation yields

\[
[D^*, \Delta_{Kc}^h] = i\Delta_{Kc}^h \quad \text{and} \quad [D^*, \frac{D^*}{Y^2}] = \frac{iD^*}{Y^2},
\]

it suffices to prove the equality for \(m = 0\) (i.e., the original assertion for \(R_0^{(b,-)}\)):

\[
\Delta_{2,-}^{(b,-)} R_0^{(b,-)} - R_0^{(b,-)} \Delta_{2,-}^{(b,-)} = (2b_- - 4) R_0^{(b,-)}.
\]

The commutator of \(\Delta_0^{(b,-)}\) and \(R_0^{(b,-)}\) is evaluated using the equalities

\[
[|D^*|^2, \Delta_{Kc}^h] = i\overline{D^*}\Delta_{Kc}^h + iD^* \Delta_{Kc}^R + \frac{\Delta_{Kc}^R}{2},
\]

\[
[|D^*|^2, \frac{D^*}{Y^2}] = \frac{3i|D^*|^2 - i(\overline{D^*})^2 + D^*}{2Y^2},
\]

\[
|Y^2\Delta_{Kc}^R, \Delta_{Kc}^h| = 2iD^* \Delta_{Kc}^R + \frac{b_-}{2} \Delta_{Kc}^R,
\]

and

\[
[Y^2\Delta_{Kc}^R, \frac{D^*}{Y^2}] = \frac{2i|D^*|^2 - 2i(\overline{D^*})^2 + i\Delta_{Kc}^h + i\Delta_{Kc}^R + (2 - b_-)D^*}{2Y^2}
\]

(which all follow from straightforward calculations). Using the equalities

\[
\Delta_2^{(b,-)} = \Delta_0^{(b,-)} - 8i\overline{D^*} \quad \text{and} \quad \overline{D^*} \circ \left( \frac{D^*}{Y^2} \right) = \frac{2|D^*|^2 - iD^*}{2Y^2}
\]

and putting in the appropriate scalars now establishes the proposition.

Our next task is the

**Proof of Proposition 1.2** We begin by evaluating \(R_{m-2}^{(b,-)} L^{(b,-)}\) written as

\[
R_{m-2}^{(b,-)}(Y^2)^2 \Delta_{Kc}^R + R_{m-2}^{(b,-)}(2 - b_-)Y^2 \overline{D^*} = (Y^2)^2 R_{m}^{(b,-)} \Delta_{Kc}^R + i(2 - b_-)Y^2 R_{m-1}^{(b,-)} \overline{D^*}.
\]

Using the equalities

\[
[\Delta_{Kc}^R, \overline{D^*}] = -i\Delta_{Kc}^R \quad \text{and} \quad D^* \overline{D^*} = |D^*|^2 + \overline{D^*}^2
\]

we establish the equation

\[
R_{m-2}^{(b,-)} L^{(b,-)} = \Xi_{m}^{(b,-)} + \frac{(2 - b_-)(2m - b_-)}{8} \Delta_{m}^{(b,-)},
\]

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where \( \Xi_m^{(b_\pm)} \) is defined in the formulation of the proposition. We now decompose \( R_m^{(b_\pm)} \) in \( L^{(b_-)} R_m^{(b_-)} \) (which is \( (Y^2)^2 R_0^{(b_-)} R_m^{(b_-)} \)), yielding

\[
(Y^2)^2 R_0^{(b_-)} \Delta_{Kc}^{(b_-)} - i(2m + 2 - b_-) Y^2 R_{b_-1}^{(b_-)} D^* - \frac{m(2m + 2 - b_-) Y^2 R_{b_-1}^{(b_-)}}{2}.
\]

The formulae

\[
[\Delta_{Kc}^h, D^*] = i\Delta_{Kc}^g \quad \text{and} \quad \overline{D^*} D^* = |D^*|^2 - \frac{D^*}{2i}
\]

now show that

\[
L^{(b_-)} R_m^{(b_-)} = \Xi_m^{(b_-)} - \frac{b_-(2m + 2 - b_-)}{8} \Delta_{m}^{(b_-)} + \frac{mb_-(2m + 2 - b_-)}{4}.
\]

The required commutation relation follows. As Theorem 1.1 shows that the compositions \( R_m^{(b_-)} L^{(b_-)} \) and \( L^{(b_-)} R_m^{(b_-)} \) commute with all the slash operators of weight \( m \), and Proposition 1.3 implies that these operators commute with \( \Delta_{m} \), the assertion about \( \Xi_m^{(b_-)} \) is also established. This proves the proposition.

Finally, we come to the

**Proof of parts (iii) and (iv) of Proposition 1.6** We prove part (iii) by induction (the case \( l = 0 \) being trivial). If \( (R_m^{(b_-)})^l \) is presented by the asserted formula then \( (R_m^{(b_-)})^{l+1} \), which is \( R_m^{(b_-)}(R_m^{(b_-)})^l \), equals

\[
R_m^{(b_-)} R_m^{(b_-)} + \sum_{c=0}^{l} \sum_{s=0}^{c} A_s^{(l)} \left( \frac{iD^*)^{c-s} (\Delta_{Kc}^h)^{l-c} (-Y^2)^c}{(-Y^2)^c} \right) = \sum_{s,c} A_s^{(l)} \left( \frac{R_m^{(b_-)} (iD^*)^{c-s} (\Delta_{Kc}^h)^{l-c}}{(-Y^2)^c} \right).
\]

For each \( c \), the term involving \( \frac{D^*}{2} \) (resp. \( \frac{i}{2} \)) in \( R_m^{(b_-)} R_m^{(b_-)} \) takes the term with indices \( c \) and \( s \) (for \( l \)) to a multiple of the term with corresponding to \( c+1 \) and \( s \) (resp. \( c+1 \) and \( s+1 \)) for \( l+1 \). For \( \Delta_{Kc}^h \) we have

\[
[\Delta_{Kc}^h, iD^*] = \Delta_{Kc}^h \quad \text{hence} \quad \Delta_{Kc}^h (iD^*)^{c-s} = \sum_{a=s}^{c} \left( \frac{c-s}{a-s} \right) (iD^*)^{c-a} \Delta_{Kc}^h,
\]

and we multiply the latter sum by \( \left( \frac{\Delta_{Kc}^h (iD^*)^{c-s}}{(-Y^2)^c} \right) \). This shows that \( (R_m^{(b_-)})^{l+1} \) can be expressed by the asserted formula. Putting in the multipliers \( A_s^{(l)} \) from \( (R_m^{(b_-)})^l \) and the coefficients of \( \frac{D^*}{2} \) and \( \frac{i}{2} \) in \( R_m^{(b_-)} R_m^{(b_-)} \), summing over \( c \) and \( s \), and taking the coefficient in front of the term with indices \( c \) and \( a \) (and \( l+1 \)) in the result, we obtain the recursive relation asserted in part (iii). We now observe that for \( a = 0 \) the recursive formula reduces to

\[
A_{0,c}^{(l+1)} = A_{0,c}^{(l)} + (2m + 4l - 2c + 4 - b_-) A_{0,c-1}^{(l)}.
\]
Denote the asserted value of $A_{0,c}^{(l)}$ by $B_{0,c}^{(l)}$. As $A_{0,0}^{(0)} = 1 = B_{0,0}^{(0)}$, it suffices to show that the numbers $B_{0,c}^{(l)}$ satisfy the latter recursive formula. But the equality
\[
2(l-c+1)\left(m+l-c - \frac{b_-}{2} + 1\right) + c(2m+4l-2c+4 - b_-) = 2(l+1)\left(m + l - \frac{b_-}{2} + 1\right)
\]
holds for every $l$ and $c$ (and $m$ and $b_-$), and multiplication by $\frac{b_{2c-1}}{c(c+1)}$, and the binomial coefficient $\binom{m+l-b_-}{c-1}$ yields the required recursive relation for the numbers $B_{0,c}^{(l)}$. This completes the proof of the proposition. 

5 Proofs: Actions on Functions

This Section provides proofs for the claims of Section 2 which are used to prove Theorem 2.12.

We begin with the Proof of Lemma 2.14. On functions on $K_R + iC$ depending only on the imaginary part of the variable, the operators $D^+$ and $\overline{D}^+$ become $\frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial t}$ respectively, while $\Delta_{K_c}^h$, $\Delta_{K_c}^\overline{h}$, and $\Delta_{K_c}^\overline{h}$ reduce to $-\frac{1}{4}\Delta_{K_c}$, $\frac{1}{4}\Delta_{K_c}$, and $-\frac{1}{4}\Delta_{K_c}$ respectively. Hence the action of $\left|D^+\right|^2$ coincides with that of $\frac{t^2}{4}$, so that
\[
\Delta_n^{(b_-)}\omega(Y) = \left[2l^2 + 2(n-1) - Y^2\Delta_{K_c}\right]\omega(Y)
\]
and
\[
R_n^{(b_-)}\omega(Y) = \left[\frac{1}{4}\Delta_{K_c} - \frac{2n+2-b_-}{2Y^2}I - \frac{n(2n+2-b_-)}{2Y^2}\right]\omega(Y).
\]
If $\omega$ is homogeneous of degree $d$ then $2l^2 + 2(n-1)I$ multiplies $\omega$ by $2d(d+n-1)$. Hence the eigenfunction property shows that $\Delta_{K_c}^h$ multiplies $\omega$ by $\frac{2d(d+n-1)+\lambda}{Y^2}$. Divide this result by $-4$, and as the multiples of $\frac{1}{4}$ and $\frac{1}{4}$ from $R_n^{(b_-)}$ multiply $\omega$ by $\frac{2d(d+n-1)+2d+2-b_-}{-4Y^2}$, the lemma follows.

For analyzing eigenfunctions of Laplacians which are linear combinations of the functions $g_{k,h,\rho}^{(p),\pm}$ we shall use

Lemma 5.1. Fix $n \in \mathbb{Z}$, $p \in \mathbb{N}$, and an element $0 \neq \rho \in K_R$, and assume that the functions
\[
\varphi^+(Z) = \sum_{k,h} B_{k,h}^{+} g_{k,h,\rho}^{(p),+}(Z) \quad \text{and} \quad \varphi^-(Z) = \sum_{k,h} B_{k,h}^{-} g_{k,h,\rho}^{(p),-}(Z)
\]
on the Grassmannian of signature $(2, b_-)$ with $b_- \geq 2$ are eigenvalues of (minus) the weight $n$ Laplacian with some eigenvalue $\lambda$. Then the equations
\[
(k + 1 - h)(k + 2 - h)B_{k+1,h-1}^+ - 4(k + 1 - h)B_{k+1,h}^+ =
\]
\[
(1 + b) + 2h(h + 1 - n) + \lambda \beta_h^+ + 8(h + 1) \beta_{h+1}
\]
and
\[
(1 + b) + 2h(h + 1 - n) + \lambda \beta_h^- + 8(n - h - 1) \beta_{h+1}
\]
hold for every \( k \) and \( h \).

Recall that the sums are finite, since we assume that only coefficients \( \beta_{k,h}^\pm \)
with \( 0 \leq h \leq k \leq p \) may not vanish.

**Proof.** We begin by evaluating the result of applying \( \Delta_n \) to \( g_k^{(p),+} \) and to \( g_k^{(p),-} \).
We write \( \varepsilon \in \{ \pm 1 \} \) for the sign and let \( \delta \in \{ 0,1 \} \) be such that \(( -1)^\delta = \varepsilon \), and we carry out both evaluations together. Observe that

\[
\frac{f_{k,h,\rho}(\varepsilon)}{Y^2} = \pi^2 \rho^2 f_{k+1,h+1,\rho}(Y) \quad \text{and} \quad (\rho,Y)f_{k,h,\rho} = \frac{f_{k,h-1,\rho}(Y)}{\pi},
\]

and note that Leibniz’s rule implies

\[
|D^*|^2(g \cdot h) = (|D^*|^2 g)h + (D^* g)(\overline{D} h) + (\overline{D} g)(D^* h) + g(|D^*|^2 h).
\]

Moreover, \( D^* \) and \( \overline{D} \) multiply one of \( e((\rho, Z)) \) and \( e((\rho, \overline{Z})) \) by \( 2\pi i(\rho, Y) \) and annihilate the other one, and \( f_{k,h,\rho} \) is homogenous of degree \(-k - h\). It follows that \( 8|D^*|^2 \) and \( -4i\pi D^* \) take \( g_k^{(p),\varepsilon} \) to

\[
2(k + h)(k + h + 1)g_k^{(p),\varepsilon} + 8\varepsilon(k + h)g_{k-1,h,\rho}^{(p),\varepsilon}
\]
and

\[
-2n(k + h)g_k^{(p),\varepsilon} + 8\delta \delta \varepsilon g_{k-1,h,\rho}^{(p),\varepsilon}
\]
respectively, while a straightforward evaluation of \(-4Y^2 \Delta_{k,\rho} g_{k,h,\rho}^{(p),\varepsilon} \) yields

\[
4k(k - h)g_k^{(p),\varepsilon} - 4k(k + 1)g_{k+1,h,\rho}^{(p),\varepsilon} + 4\delta(k - h)g_{k-1,h,\rho}^{(p),\varepsilon} - 8\varepsilon k g_{k,h,\rho}^{(p),\varepsilon} +
\]
\[
-(k - h)(k - h - 1)g_{k-1,h+1,\rho}^{(p),\varepsilon} + 2k b_- g_{k,h,\rho}^{(p),\varepsilon}
\]
Multiplying the sum of these three expressions by \( \beta_k^{(p),\varepsilon} \), summing over \( k \) and \( h \), and comparing the result with \(-\lambda \sum_{k,h} \beta_k^{(p),\varepsilon} g_{k,h,\rho}^{(p),\varepsilon} \), we obtain the equality

\[
\sum_{k,h} \beta_k^{(p),\varepsilon} \left[ (2k(1 - n + h) + 2h(h + 1 - n) + \lambda)g_{k-1,h,\rho}^{(p),\varepsilon} +
\right.
\]
\[
+ 4\varepsilon(k - h)g_{k-1,h,\rho}^{(p),\varepsilon} - (k - h)(k - h - 1)g_{k-1,h+1,\rho}^{(p),\varepsilon} + 8(\delta n + \varepsilon h)g_{k,h-1,\rho}^{(p),\varepsilon} \right] = 0.
\]

Gathering the coefficients of each \( g_{k,h,\rho}^{(p),\varepsilon} \) and using the linear independence of the functions \( g_{k,h,\rho}^{(p),\varepsilon} \) implies, after substituting the values of \( \varepsilon \) and \( \delta \), the asserted equalities. This proves the lemma. □
Lemma 5.1 allows us to give the

**Proof of Proposition 2.3.** Assume first that \(B_{m,m}^- = 0\), and we claim that all the other coefficients \(B_{k,h}^-\) also vanish. We work by decreasing induction on \(k\), so that assume now that \(B_{l,h}^- = 0\) for all \(l > k\) (which holds for \(k = m\)), and consider the equation from Lemma 5.1 with our index \(k\) and index \(h\). It reads

\[
[2k(k - 1 - m) + 2h(h + 1 - m) - 2b_-(m - k)]B_{k,h}^- + 8(m - h - 1)B_{k,h+1}^- = 0
\]

after substituting \(n = m\) and the value of \(\lambda\). For any pair of \(h\) and \(k\) with \(0 \leq k \leq h \leq m\) excluding the case \(k = h = m\), the coefficient of \(B_{k,h}^-\) is strictly negative, as all three terms are non-positive and the two terms involving \(k\) cannot vanish together. Hence the vanishing of \(B_{k,h+1}^-\) implies the vanishing of \(B_{k,h}^-\) for every such \(k\) and \(h\). As \(B_{k,k+1}^- = 0\) we deduce that \(B_{k,h}^- = 0\) for all \(h\), establishing the induction step. This proves our claim. But we now observe that \(g_{m,m,\rho}^{(m),-}(Z)\), namely \(e((\rho, Z))\), has eigenvalue \(-2mb_-\) under (minus) \(\Delta_{m}^{(b,-)}\). Indeed, \(e((\rho, Z))\) is annihilated by \(\Delta_{0,-m}^{(b,-)}\), hence (minus) the operator \(\Delta_{0,-m}^{(b,-)}\) (which is \(\Delta_{0,-m}^{(b,-)} + 2mb_-\)) multiplies it by \(-2mb_-\), and we can transform to \(\Delta_{m,0}^{(b,-)}\) using Equation (2). Now, given a function as in the proposition, subtracting an appropriate multiple of \(g_{m,m,\rho}^{(m),-}\) yields again such a function but with vanishing \(B_{m,m}^-\). As the latter function vanishes by our claim, the original function must be a multiple of \(g_{m,m,\rho}^{(m),-}\). This proves the proposition.

We remark that a small variant of the latter proof establishes the uniqueness assertion regarding the combination \(\sum_{k,h} B_{k,h}^+ g_{k,h,\rho}^{(m),+}\) with eigenvalue \(-2mb_-\) as well. Another application of Lemma 5.1 gives us the

**Proof of Proposition 2.4.** We shall prove by decreasing induction on \(k\) that \(B_{k,h}^+ = 0\) for all \(k > 0\). If this assertion holds for any \(l > k\), then for any \(h\) the equality from Lemma 5.1 with \(n = m + b_-\) and \(\lambda = 0\) becomes

\[
[2k(k - 1 - m) + 2h(h + 1 - m - b_-)]B_{k,h}^+ + 8(h + 1)B_{k,h+1}^+ = 0.
\]

We shall use Equation (4) to show that \(B_{k,h}^+\) must vanish for all \(h\) if \(k > 0\), under our assumptions. We begin with the case \(k > m\). In this case we have \(B_{k,0}^+ = 0\) by assumption, and as the coefficient \(h + 1\) never vanishes for \(h \geq 0\), Equation (4) implies that if \(B_{k,h}^+\) vanishes then so does \(B_{k,h+1}^+\). The verifies our assertion for \(k > m\). For \(0 < k \leq m\) we argue in decreasing order of \(h\) (as in Proposition 2.3). The basis for this argument is the fact that \(B_{k,k+1}^+ = 0\) for all \(k\), and the fact that \(B_{k,k+1}^+ = 0\) implies \(B_{k,k}^- = 0\) follows from the fact that the coefficient of \(B_{k,h}^+\) in Equation (4) does not vanish for \(0 < k \leq m\) and \(0 \leq h \leq k\). Indeed, the product which is based on \(k\) is negative and the one which is based on \(h\) is non-positive for such \(k\) and \(h\). Hence all the coefficients \(B_{k,h}^+\) vanish for \(k > 0\), leaving only \(B_{0,0}^+\) to (possibly) be non-zero. This proves the proposition.

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As the holomorphic function $e((\rho, Z))$ is harmonic with respect to the Laplacian of every weight, the remaining coefficient $B_{k,n}^+$ may indeed not vanish. One can also show that for given $p$ and $n$ there are only finitely many $\lambda$ such that non-zero functions $\varphi^\pm$ as in Lemma 5.1 are eigenfunctions for (minus) $\Delta_n^{(k-\cdot)}$ with eigenvalue $\lambda$. This extends the well-known result that an almost holomorphic function on $\mathcal{H}$ of (exact) depth $d$ can be an eigenfunction for (minus) the modular Laplacian $\Delta_k$ only with the unique eigenvalue $\lambda = d(k-d-1)$.

For the proof of Proposition 2.3 we shall need the following

**Lemma 5.2.** The operators $\frac{1}{\ell} \Delta_{k,c}^\ell$, $\frac{1}{\ell} \pi_k^\pm$, and $\frac{1}{\ell} \pi_{k+1}^\pm$ take $g_{k,h,\rho}^{(p)+}$ to

$$-4g_{k,h,\rho}^{(p)+} - 4kg_{k+1,h,\rho}^{(p)+} + G_1, \quad 2g_{k,h,\rho}^{(p)+} + G_2, \quad \text{and} \quad 0 + G_3$$

respectively, where the $G_j$ are rational linear combinations of the functions $g_{1,j,\rho}^{(p)+}$ with $j > h$.

**Proof.** The proof of Lemma 5.1 shows that $\frac{1}{\ell} \pi_k^+$ takes $g_{k,h,\rho}^{(p)+}$ to $\pi_k^2 g_{k+1,h+1,\rho}^{(p)+}$, proving the assertion for $\frac{1}{\ell} \pi_k^+$. We also obtain

$$iD^* g_{k,h,\rho}^{(p)+} = -\frac{k + h}{2} g_{k,h,\rho}^{(p)+} - 2g_{k,h-1,\rho}^{(p)+}$$

(homogeneity), from which the assertion for this operator follows using the action of $\frac{1}{\ell} \pi_k^-$. One now evaluates $\frac{1}{\ell} \Delta_{k,c}^\ell g_{k,h,\rho}^{(p)+}$ as

$$k(k-h)g_{k+1,h+1,\rho}^{(p)+} - k(k+1)g_{k+1,h+1,\rho}^{(p)+} - \frac{(k-h)(k-h-1)}{4} g_{k,h+2,\rho}^{(p)+} +$$

$$+ \frac{k}{2} g_{k+1,h+1,\rho}^{(p)+} + 2(k-h)g_{k+1,h+1,\rho}^{(p)+} - 4k g_{k+1,h+1,\rho}^{(p)+} - 4g_{k,h,\rho}^{(p)+}$$

(straightforward), which implies the remaining assertion of the lemma. This completes the proof. \(\square\)

We shall also use properties the extended binomial expressions $\binom{x}{n}$:

**Lemma 5.3.** For real $x$ and $y$ and natural $n \in \mathbb{N}$ we have the equalities

$$(i) \quad \binom{x}{n} = (-1)^n \binom{n-1-x}{n} \quad \text{and} \quad (ii) \quad \binom{x+y}{n} = \sum_{s=0}^{n} \binom{x}{s} \binom{y}{n-s}.$$

**Proof.** Part (i) follows from inserting a minus sign in front of all the multipliers in the expression defining $\binom{x}{n}$. For part (ii) (which extends a well-known binomial identity) we work by induction on $n$, the case $n = 0$ being trivial. If the equality holds for $n$, then for $n+1$ we write $\binom{x+y}{n+1}$ as $\binom{x+y}{n+1}$, which using the induction hypothesis equals

$$\frac{x+y-n}{n+1} \sum_{s=0}^{n} \binom{x}{s} \binom{y}{n-s} = \sum_{s=0}^{n} \left[ \frac{x-s}{n+1} + \frac{y-n+s}{n+1} \right] \binom{x}{s} \binom{y}{n-s}.$$

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from operators h, index and the action of the modified operator C. As ≤ coefficient thus of the desired form, and it remains to prove that the coefficients of the desired result. Putting in the coefficients, we find that this operator takes a linear combination of the functions R

Change the summation index in the first term from s to s − 1 easily yields the desired result \( \sum_{n=0}^{n+1} \binom{n}{s} (n-s) \). This proves the lemma.

We are now ready for the

**Proof of Proposition 2.6** As \( R_{m}^{(b)} \) is a linear combination of the operators considered in Lemma 5.2 it follows that \( R_{m}^{(b)} \) takes the sum \( \sum_{k} B_{k,h}^{(p)} \) to a linear combination of the functions \( P_{k,h,b}^{(p+1),+} \). The image under \( \frac{1}{(4\pi^2)^l} (R_{m}^{(b)})^l \) is thus of the desired form, and it remains to prove that the coefficients \( C_{k,0}^{+} \) have the asserted value. Now, part (iii) of Proposition 1.6 shows that \( (R_{m}^{(b)})^l \) is a linear combination of products of \( \left( \frac{1}{(4\pi^2)^c} \right)^{a} \), \( (\Delta_{K_c}^h)^{l-c} \) for integers \( 0 \leq a \leq c \leq l \). By Lemma 5.2 none of these operators reduces the second index h, whereas if \( a > 0 \) then this product actually increases this index. Hence the coefficient \( C_{k,0}^{+} \) is constructed only from terms \( P_{j,h,b}^{(p)} \) with \( h = 0 \), and only from operators \( \left( \frac{1}{(4\pi^2)^c} \right)^{a} (\Delta_{K_c}^h)^{l-c} \) with \( a = 0 \). Moreover, when we evaluate the coefficient \( C_{k,0}^{+} \), in the remaining expression

\[
\frac{1}{(4\pi^2)^l} \sum_{c=0}^{l} A_{c,0}^{(l)} \left( \frac{iD^c}{-4\pi^2} \right) f_{j,0,h,b}^{(p),+} (Y) e((\rho, Z))
\]

we can use the expressions from Lemma 5.2 with the functions \( G_{j} \) omitted. We call the resulting operations the modified operators.

Now, simple induction on the power \( l - c \) shows that the modified operator \( \frac{1}{(4\pi^2)^l} (\Delta_{K_c}^h)^{l-c} \) takes \( f_{j,0,h,b}^{(p),+} \) to

\[
\sum_{c=0}^{l-c} \frac{(l-c)!}{(l-c-t)!} \binom{j+t-1}{t} g_{j+t,0,h,b}^{(p+t-c),+}
\]

and the action of the modified operator \( \left( \frac{1}{(4\pi^2)^c} \right)^{a} \) sends this expression to

\[
\frac{1}{(2)^c} \sum_{c=0}^{l-c} \frac{(l-c)!}{(l-c-t)!} \binom{j+t-1}{t} f_{j+t+c,0,h,b}^{(p+t),+}
\]

Putting in the coefficients, we find that this operator takes

\[
\sum_{j} B_{j,0}^{+} g_{j,0,h,b}^{(p),+} \quad \text{to} \quad \sum_{k} \frac{1}{(2)^c} \sum_{c=0}^{l-c} \frac{(l-c)!}{(l-c-t)!} \binom{k-c-1}{t} B_{k-c-t,0}^{+} g_{k,0,h,b}^{(p+t),+}.
\]

Multiplying by the coefficients \( A_{c,0}^{(l)} \) which are evaluated in part (iv) of Proposition 1.6 and summing over c establishes the formula

\[
C_{k,0}^{+} = \sum_{c=0}^{l} (-1)^c \sum_{t=0}^{l-c} \frac{l!}{(l-c-t)!} \binom{m+l-b}{c} \left( \frac{k-c-1}{t} \right) B_{k-c-t,0}^{+}.
\] 31
replaced by \( \mu, \zeta \), which a summation index change takes to

\[
C_{k,0}^+ = \sum_{r=0}^{l} \frac{l!}{(l-r)!} \left[ \sum_{c+t=r} (-1)^c \binom{m+l-b}{c} \binom{k-c-1}{t} \right] B_{k-r,0}^+.
\]

Write now \((-1)^c \binom{k-c-1}{t} \) as \((-1)^r \binom{m+l-r}{t} \) using part (i) of Lemma 5.3 and the condition \( c+t = r \). The inner sum now equals \((-1)^r \binom{m+l+r-1}{t} \) by part (ii) of Lemma 5.3. This completes the proof of the proposition.

When evaluating the action of our operators on the quotients \( P_{r,s,t} \), we shall need the projection of \( \mu \in V \) to \( \{z, \zeta\}^\perp \equiv K_R \), denoted \( \mu_K \), for short. In the coordinates given above, \( \mu \) is \( (\mu_K, \mu_z, (\mu, \zeta) - \zeta^2 \mu_z) \), from which the equality

\[
\mu^2 = \mu_K^2 + 2\mu_z (\mu, \zeta) - \zeta^2 \mu_z^2 \tag{5}
\]

easily follows. We now turn to the

Proof of Lemma 2.6. We first observe that \( D^* \) annihilates \( (\mu, \overline{Z_{V,Z}}) \) and \( \overline{D^*} \) eliminates \( (\mu, Z_{V,Z}) \), while the equalities

\[
D^*(\mu, Z_{V,Z}) = (\mu, Y_{V,Z}) - iY^2 \mu_z \quad \text{and} \quad \overline{D^*}(\mu, \overline{Z_{V,Z}}) = (\mu, Y_{V,Z}) + iY^2 \mu_z
\]

hold. As \( 2iY_{V,Z} \) is \( Z_{V,Z} - \overline{Z_{V,Z}} \) and multiplication by \( Y^2 \) reduces the middle index by 1, we find that

\[
-4im\overline{D}^*P_{r,s,t} = 2m(t - 2s)P_{r,s,t} - 2mtP_{r+1,s,t-1} + 4mt\mu_z P_{r,s-1,t-1}
\]

and

\[
+4in\overline{D}^*P_{r,s,t} = 2n(r - 2s)P_{r,s,t} - 2nrP_{r-1,s,t+1} + 4nr\mu_z P_{r-1,s-1,t}.
\]

Using Leibniz’ rule and these considerations we now evaluate \( 8|D^*|^2 P_{r,s,t} \) as

\[
4[s(2s+1) - rs - st + rt]P_{r,s,t} + 2t(2s-r)P_{r+1,s,t-1} + 2r(2s-t)P_{r-1,s,t+1} + 8st\mu_z P_{r-1,s-1,t-1} - 8rs\mu_z P_{r,s-1,t} + 8rt\mu_z^2 P_{r-1,s-2,t-1}.
\]

A straightforward calculation (using any orthonormal basis for \( K_R \) and the same considerations again) shows that the remaining term \(-4Y^2 \Delta^B_\mu P_{r,s,t} \) equals

\[
2s(r + t - 2s - 2 + b_{-}) P_{r,s,t} - 2stP_{r+1,s,t-1} + 2st\mu_z P_{r,s-1,t-1} - 2rsP_{r-1,s,t+1} + 4rs\mu_z P_{r-1,s-1,t-1} - 4rt \left[ \mu_K^2 - (\mu, Z) \mu_z - (\mu, \overline{Z}) \mu_z + (Z, \overline{Z}) \mu_z^2 \right] P_{r-1,s-1,t-1}.
\]

By Equation (5) and the formulae for \( Z_{V,Z} \) and \( \overline{Z_{V,Z}} \), the last term here can be replaced by

\[
-4rt\mu_z^2 P_{r-1,s-1,t-1} + 4rt\mu_z P_{r,s-1,t-1} + 4rt\mu_z P_{r-1,s-1,t} - 4rt\mu_z^2 P_{r-1,s-2,t-1}.
\]

Summing all the terms together with \( 2n(2m - b_{-}) P_{r,s,t} \) now yields the asserted expression. This completes the proof of the lemma. \( \square \)
The next step is the proof of Lemma 2.8. Write $R_n(b^-)_{P_{r,s,t}}$ as $\frac{R(b^-)_{\mu z}}{y_{r,s,t}}$. We have the equality

$$-2iD^n P_{r,0,t} = -rP_{r,0,t} + rP_{r-1,0,t+1} - 2r\mu z P_{r-1,-1,t}$$

from the proof of Lemma 2.6. In addition, arguments similar to the evaluation of $\Delta^h_{K_C}$ in the proof of that lemma (using Equation (5) again) show that

$$\Delta^h_{K_C} P_{r,0,t} = r(r-1)\mu^2 P_{r-2,0,t} - r(2r - 2 + b_-)\mu z P_{r-1,0,t}.$$ 

The assertion of the lemma now follows from simple algebra. \qed

A similar argument gives us the proof of Lemma 2.9. As in the proof of Lemma 2.8 we write

$$R_m(b^-) [P_{0,m,t} \cdot - \ln \frac{|(\lambda, Z_{V,Z})|^2}{Y^2}] = P_{0,m,t} R_0(b^-) \left( \ln Y^2 - \ln (\lambda, Z_{V,Z}) \right)$$

(we also used the fact that $R_0(b^-)$ eliminates anti-holomorphic functions). As

$$2iD^n \ln Y^2 = 2, \quad 2iD^n \ln (\lambda, Z_{V,Z}) = 1 - P_{-1,0,1} + 2\mu z P_{-1,-1,0},$$

$$\Delta^h_{K_C} \ln Y^2 = \frac{2 - b_-}{2Y^2}, \quad and \quad \Delta^h_{K_C} \ln (\lambda, Z_{V,Z}) = -\mu^2 P_{-2,0,0} + (2 - b_-)\mu z P_{-1,0,0}$$

(the latter equality uses Equation (5) as before), we obtain the desired result. \qed

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