IYAMA’S HIGHER AUSLANDER CORRESPONDENCE VIA THE HOMOLOGICAL THEORY OF IDEMPOTENT IDEALS.

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Abstract. A celebrated result in representation theory is that of higher Auslander correspondence. Let \( \Lambda \) an Artin algebra and \( X \) a \( d \)-cluster-tilting module. Iyama has shown that the endomorphism ring \( \Gamma \) of \( X \) is a \( d \)-Auslander algebra, and moreover this gives a correspondence between \( d \)-cluster-tilting modules and \( d \)-Auslander algebras. We present a self-contained and concise proof using the homological theory of idempotent ideals of Auslander–Platzeck–Todorov.

1. Introduction

The aim of this note is to present a concise proof of Iyama’s higher Auslander correspondence. For the benefit of the reader, it is as self-contained as possible. Let us stress that all the main arguments used are well known: Section 3 is based on the work of Iyama [4], Section 4 follows from the ideas of Auslander–Platzeck–Todorov [1], while Section 5 is also due to Iyama [3].

2. Preliminaries

Let \( K \) be a field and \( \Lambda \) a finite-dimensional \( K \)-algebra. We denote by \( \text{mod}\Lambda \) the category of (finitely-generated left) \( \Lambda \)-modules. For morphisms \( f : X \to Y \) and \( g : Y \to Z \), we denote the composition by \( fg \). Let \( \text{add}(M) \) be the full subcategory of \( \text{mod}\Lambda \) composed of all \( \Lambda \)-modules isomorphic to direct summands of finite direct sums of copies of \( M \). The functor \( D = \text{Hom}_K(\_, K) \) defines a duality.

Recall that the dominant dimension \( \text{dom.dim}(\Lambda) \) to be the number \( n \) such that for a minimal injective resolution of \( \Lambda \):

\[ 0 \to \Lambda \to I_0 \to \cdots \to I_{n-1} \to I_n \to \cdots \]

the modules \( I_0, \ldots, I_{n-1} \) are projective-injective and \( I_n \) is not projective. Equivalently, \( \text{dom.dim}(\Lambda) \) is the number \( n \) such that for a minimal projective resolution of \( D\Lambda \):

\[ \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to D\Lambda \to 0 \]

the modules \( P_0, \ldots, P_{n-1} \) are projective-injective and \( P_n \) is not.
Let \( \mathcal{C} \) be an additive subcategory of \( \text{mod}\Lambda \). A \( \mathcal{C} \)-module is a contravariant additive functor from \( \mathcal{C} \) to the category of abelian groups. A \( \mathcal{C} \)-module \( M \) is finitely presented if there exists a morphism \( f : X \to Y \) in \( \text{mod}\Lambda \) and an exact sequence

\[
\text{Hom}_\Lambda(C, X) \xrightarrow{\text{Hom}_\Lambda(C, f)} \text{Hom}_\Lambda(C, Y) \to M \to 0
\]

for all \( C \in \mathcal{C} \). Denote by \( \text{mod} \mathcal{C} \) the category of finitely-presented \( \mathcal{C} \)-modules. If \( \mathcal{C} = \text{add}(X) \) for some \( X \in \text{mod}\Lambda \), it is known that the categories \( \text{mod} \mathcal{C} \) and \( \text{mod} \text{End}_\Lambda(X) \) are equivalent.

A subcategory \( \mathcal{C} \) of \( \text{mod}\Lambda \) is precovering or contravariantly finite if for any \( M \in \text{mod}\Lambda \) there is an object \( C_M \in \mathcal{C} \) and a morphism \( f : C_M \to M \) such that \( \text{Hom}(C, -) \) is exact on the sequence

\[
C_M \to M \to 0
\]

for all \( C \in \mathcal{C} \). The module \( C_M \) is said to be a right \( \mathcal{C} \)-approximation. The dual notion of precovering is preenveloping or covariantly finite. A subcategory \( \mathcal{C} \) that is both precovering and preenveloping is called functorially finite. A right \( \mathcal{C} \)-resolution is a sequence

\[
\cdots \to C_1 \to C_0 \to M \to 0
\]

with \( C_i \in \mathcal{C} \) for each \( i \), and which becomes exact under \( \text{Hom}_\Lambda(C, -) \) for each \( C \in \mathcal{C} \). Define a left \( \mathcal{C} \)-resolution dually.

**Definition 3.1.** [4, Definition 2.2] A functorially-finite subcategory \( \mathcal{C} \subseteq \text{mod}\Lambda \) is a \( d \)-cluster-tilting subcategory if it satisfies the following conditions:

\[
\mathcal{C} = \{ X \in \text{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(C, X) = 0 \forall \ 0 < i < d, \ C \in \mathcal{C} \}.
\]

\[
\mathcal{C} = \{ X \in \text{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, C) = 0 \forall \ 0 < i < d, \ C \in \mathcal{C} \}.
\]

If \( \mathcal{C} = \text{add}(M) \), then we say \( M \) is a \( d \)-cluster-tilting module.

In particular, all projective and all injective \( \Lambda \)-modules are contained in \( \mathcal{C} \).

**Theorem 3.2.** [4, Theorem 3.6.1] Let \( \mathcal{C} \subseteq \text{mod}\Lambda \) be a \( d \)-cluster-tilting subcategory. Then

1. Any \( M \in \text{mod}\Lambda \) has a right \( \mathcal{C} \)-resolution

\[
0 \to C_{d-1} \to \cdots \to C_1 \to C_0 \to M \to 0.
\]

2. Any \( M \in \text{mod}\Lambda \) has a left \( \mathcal{C} \)-resolution

\[
0 \to M \to C_0 \to C_1 \to \cdots \to C_{d-1} \to 0.
\]
Proof. We give a proof for right resolutions. There is a right $C$-approximation $f : C_0 \to M$, since $C$ is precovering. The morphism $f$ is surjective, since every projective $\Lambda$-module is in $C$. Hence there is a short exact sequence

$$0 \longrightarrow K \longrightarrow C_0 \overset{f}{\longrightarrow} M \longrightarrow 0.$$ 

For any $C \in C$, there is a long exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(C, K) \longrightarrow \text{Hom}_\Lambda(C, C_0) \overset{\text{Hom}_\Lambda(C, f)}{\longrightarrow} \text{Hom}_\Lambda(C, M) \longrightarrow 0.$$ 

Since $\text{Hom}_\Lambda(C, f)$ is surjective, this implies $\text{Ext}_\Lambda^1(C, K) = 0$ and that there is an exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(C, K) \longrightarrow \text{Hom}_\Lambda(C, C_0) \overset{\text{Hom}_\Lambda(C, f)}{\longrightarrow} \text{Hom}_\Lambda(C, M) \longrightarrow 0.$$ 

For some $K' \in \text{mod}\Lambda$ there is a short exact sequence induced by the right $C$-approximation $C_N \to N$:

$$0 \longrightarrow K' \longrightarrow C_N \longrightarrow N \longrightarrow 0.$$ 

Now suppose there is a $\Lambda$-module $N$ such that for all $0 < i < d - 1$ we have $\text{Ext}_\Lambda^i(C, N) = 0$ and that for any $C \in C$ there is an exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(C, N) \longrightarrow \text{Hom}_\Lambda(C, C_{d-2}) \longrightarrow \cdots \longrightarrow \text{Hom}_\Lambda(C, C_0) \overset{\text{Hom}_\Lambda(C, f)}{\longrightarrow} \text{Hom}_\Lambda(C, M) \longrightarrow 0.$$ 

For any $C \in C$ and any $1 < i < d$, there is an exact sequence

$$\text{Ext}_\Lambda^{i-1}(C, N) \longrightarrow \text{Ext}_\Lambda^i(C, K') \longrightarrow \text{Ext}_\Lambda^i(C, C_N) = 0$$

which, together with the above argument for $d = 1$, implies $\text{Ext}_\Lambda^i(C, K') = 0$ for all $0 < i < d$. So $K' \in C$. Additionally there is an exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(C, K) \longrightarrow \text{Hom}_\Lambda(C, C_N) \longrightarrow \text{Hom}_\Lambda(C, C_{d-2}) \longrightarrow \cdots \longrightarrow \text{Hom}_\Lambda(C, C_0) \overset{\text{Hom}_\Lambda(C, f)}{\longrightarrow} \text{Hom}_\Lambda(C, M) \longrightarrow 0.$$ 

This finishes the proof. \qed
4. Homological theory of idempotent ideals

Let $\Gamma$ be a finite-dimensional algebra, and $P$ be a projective $\Gamma$-module. Let $\mathcal{I} = \tau_P(\Gamma)$, the trace of $P$ in $\Gamma$ which is the ideal generated by the homomorphic images of $P$ in $\Gamma$, be the idempotent ideal corresponding to $P$. When $P = \Gamma e$, then $\mathcal{I} = \langle e \rangle$, the two-sided ideal generated by the idempotent $e$. For a positive integer $d$, we define $P_{d-1}$ to be the full subcategory of mod$\Gamma$ consisting of the $\Gamma$-modules $X$ having a projective resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

with $P_i \in \text{add}(P)$ for all $0 \leq i \leq d - 1$. There is a characterisation of $P_{d-1}$.

**Proposition 4.1.** [1, Proposition 2.4] Let $P$ be a projective $\Lambda$-module. The following conditions are equivalent for a $\Gamma$-module $X$:

(i) $X$ is in $P_{d-1}$.
(ii) $\text{Ext}^i_{\Gamma}(X, Y) = 0$ for all $A/\mathcal{I}$-modules $Y$ and $0 < i < d$.
(iii) $\text{Ext}^i_{\Gamma}(X, Y) = 0$ for all injective $A/\mathcal{I}$-modules $Y$ and $0 < i < d$.

*Proof.* by induction on $d$. \hfill $\square$

We omit a more detailed proof of Proposition 4.1 since it is not used in the sequel. Note that in the special case $X = \Gamma$ (as used by Iyama [3, Lemma 3.5.1]) Proposition 4.1 reduces the proof of the following result, which also uses arguments from Sections 5 and 6 of [2].

**Theorem 4.2.** [1, Theorem 3.2] Let $P$ be a projective $\Gamma$-module and $\Lambda := \text{End}_{\Gamma}(P)$. For any $Y \in \text{mod}\Gamma$ and any $d \geq 1$, the functor $G := \text{Hom}_{\Gamma}(P, -)$ induces an isomorphism

$$\text{Ext}^{d-1}_{\Gamma}(X, Y) \rightarrow \text{Ext}^{d-1}_{\Lambda}(GX, GY)$$

provided $X \in P_d$.

*Proof.* We prove by induction, omitting the case $i = 2$ since the argument requires only minor modifications. Let $Y$ be a fixed, but arbitrary $\Gamma$-module. Recall that there is a canonical isomorphism $\text{Hom}_{\Gamma}(P, Y) \cong \text{Hom}_{\Lambda}(GP, GY)$. First suppose $X \in P_1$; so there is an exact sequence $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ such that $P_0, P_1 \in \text{add}P$. Then there is a commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_\Gamma(X, Y) & \rightarrow & \text{Hom}_\Gamma(P_0, Y) & \rightarrow & \text{Hom}_\Gamma(P_1, Y) \\
\downarrow & & \downarrow \cong & & \downarrow \cong & \\
0 & \rightarrow & \text{Hom}_\Lambda(GX, GY) & \rightarrow & \text{Hom}_\Lambda(GP_0, GY) & \rightarrow & \text{Hom}_\Lambda(GP_1, GY)
\end{array}$$
implying \( \text{Hom}_ \Gamma(X, Y) \cong \text{Hom}_ \Lambda(GX, GY) \) by the Five Lemma. Now suppose for some \( i \geq 2 \) that for all \( M \in P_i \) there are isomorphisms

\[
\text{Ext}^{i-1}_\Gamma(M, Y) \to \text{Ext}^{i-1}_\Lambda(GM, GY).
\]

Let \( X \in P_{i+1} \) and consider an exact sequence \( 0 \to K \to P_0 \to X \to 0 \) with \( P_0 \in \text{add}(P) \). Since \( K \in P_i \), we get \( \text{Ext}^{i-1}_\Gamma(K, Y) \cong \text{Ext}^{i-1}_\Lambda(GK, GY) \). From the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Ext}^{i-1}_\Gamma(K, Y) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ext}^{i-1}_\Lambda(GK, GY)
\end{array}
\]

we find also \( \text{Ext}^i_\Gamma(X, Y) \cong \text{Ext}^i_\Lambda(GX, GY) \).

\[\square\]

5. Higher Auslander Correspondence

Recall for a \( \Lambda \)-module \( X \) such that \( \Lambda, D \Lambda \in \text{add}(X) \), the endomorphism algebra \( \Gamma := \text{End}_\Lambda(X) \) has projective modules given by \( \text{add}(\text{Hom}_\Lambda(X, X)) \), and projective-injective modules given by \( \text{add}(\text{Hom}_\Lambda(X, D \Lambda)) \).

**Proposition 5.1.** Suppose \( \Lambda \) is an artin algebra and \( X \) is a \( d \)-cluster-tilting module in \( \text{mod} \Lambda \). Then \( \Gamma := \text{End}_\Lambda(X) \) satisfies

\[
\text{dom.dim}(\Gamma) \geq d + 1 \geq \text{gl.dim}(\Gamma),
\]

i.e. \( \Gamma \) is a \( d \)-Auslander algebra.

*Proof.* Let \( F := \text{Hom}_\Lambda(X, -) \) and \( C := \text{add}(X) \). We have that \( F \Gamma = FX \). Suppose \( X \) has an injective resolution:

\[
0 \to X \to I_0 \to I_1 \to \cdots
\]

Since \( X \) is \( d \)-cluster tilting, we have an exact sequence

\[
0 \to FX \to FI_0 \to \cdots \to FI_d
\]

whereby \( FI_j \) is projective-injective for each \( 0 \leq j \leq d \). Hence \( \text{dom.dim}(\Gamma) \geq d + 1 \).

For each \( M \in \text{mod} \Gamma \), we show \( \text{proj.dim}(M) \leq d + 1 \), and hence \( \text{gl.dim}(\Gamma) \leq d + 1 \).

Since \( \text{mod} \Gamma \) is equivalent to \( \text{mod} C \), there exist \( C_{-2}, C_{-1} \in C \) and an exact sequence:

\[
FC_{-1} \to FC_{-2} \to M \to 0.
\]

By Theorem 3.2, this extends to an exact sequence

\[
0 \to FC_d \to \cdots \to FC_0 \to FC_{-1} \to FC_{-2} \to M \to 0
\]

such that \( C_i \in C \) for all \( -2 \leq i \leq d \). Hence \( d + 1 \geq \text{gl.dim}(\Gamma) \) and we are done. \[\square\]

**Proposition 5.2.** Let \( \Gamma \) be a \( d \)-Auslander algebra, let \( P \) be a minimal projective-injective generator and let \( \Lambda := \text{End}_\Gamma(P) \). Then there is a \( d \)-cluster-tilting subcategory \( C \subseteq \text{mod} \Lambda \).
Proof. Let Λ := EndΓ(P) and G : HomΓ(P, −). We will show C := add(GΓ) ⊆ mod(Λ) is d-cluster tilting. By assumption DT ∈ Pd. So Theorem 4.2 implies an isomorphism
\[ \text{Ext}^i_\Lambda(X, Y) \to \text{Ext}^i_\Lambda(GX, GY) \]
for all X, Y ∈ add(Γ) and all 0 < i < d. Hence Ext^i_\Lambda(GX, GY) = 0 for all 0 < i < d. To show add(GΓ) is a d-cluster-tilting subcategory, we have to show maximality. So suppose on the other hand that there exists some M /∈ C such that Ext^i_\Lambda(C, M) = 0 for all 0 < i < d and all C ∈ C. Let F := Hom_\Lambda(GTΓ, −) and suppose M has an injective resolution:
\[ 0 \to M \to I_0 \to I_1 \to \cdots \]
Since Ext^i_\Lambda(C, M) = 0 for all 0 < i < d, we have an exact sequence
\[ 0 \to FM \to FI_0 \to \cdots \to FI_d \]
whereby FI_j is projective-injective for each 0 ≤ j ≤ d. Let N be the Λ-module N := coker(FI_{d-1} → FI_d). Since gl.dim(Γ) ≤ d + 1, the sequence
\[ 0 \to FM \to FI_0 \to \cdots \to FI_d \to N \to 0 \]
is a projective resolution of N, and hence FM is projective, in other words M ∈ C. Therefore C ⊆ modΛ is d-cluster tilting. □

Lemma 5.3. Let Λ, Γ, X as above. There are mutually inverse equivalences
\[ F : \text{add}(X) \to \text{add}(Γ) \]
\[ G : \text{add}(Γ) \to \text{add}(X) \]
Proof. That F is an equivalence has already been discussed. Moreover
\[ G \circ F = \text{Hom}_\Gamma(P, \text{Hom}_\Lambda(X, -)) \]
\[ = \text{Hom}_\Lambda(X \otimes_\Gamma P, -) \]
\[ = \text{Hom}_\Lambda(X \otimes_\Gamma \text{Hom}_\Gamma(X, \Lambda), -) \]
\[ = \text{Hom}_\Lambda(\Lambda, -) \]
\[ = 1 \]
\[ \Box \]

Let two d-cluster-tilting modules M, N ∈ modΛ be equivalent whenever the categories add(M) and add(N) are equivalent.

Theorem 5.4 (d-Auslander correspondence). [3, Theorem 0.2] For any d ≥ 1 there exists a bijection between the equivalence classes of d-cluster-tilting modules X ∈ modΛ and the set of Morita-equivalence classes of d-Auslander algebras, given by X ↦ End_Λ(X).

Proof. This follows from Proposition 5.1, Proposition 5.2 and Lemma 5.3 □
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