On the mean-field equations for ferromagnetic spin systems

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Abstract
We derive mean-field equations for a general class of ferromagnetic spin systems with an explicit error bound in finite volumes. The proof is based on a link between the mean-field equation and the free convolution formalism of random matrix theory, which we exploit in terms of a dynamical method. We present three sample applications of our results to Kać interactions, randomly diluted models, and models with an asymptotically vanishing external field.

1 Introduction

The subject of this note is a ferromagnetic spin system with Hamiltonian $H : \{-1,1\}^N \rightarrow \mathbb{R}$ defined by

$$H(\sigma) = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i, \quad J_{ij} \geq 0 \quad (1.1)$$

and Gibbs expectation of $f : \{-1,1\}^N \rightarrow \mathbb{R}$ given by

$$\langle f \rangle = \frac{1}{Z} \sum_\sigma f(\sigma) e^{-H(\sigma)} , \quad Z = \sum_\sigma e^{-H(\sigma)}.$$

There is no loss of generality in assuming that $J_{ij} = J_{ji}$, which we will do from now on. The results below will be meaningful in the setting where $\max J_{ij} \rightarrow 0$ as $N \rightarrow \infty$.

The prototypical example of such a system is the Curie-Weiss model, which corresponds to the choice $J_{ij} = \beta N^{-1}$ with $\beta \geq 0$ and $h_i = h \in \mathbb{R}$ (see [16] for a comprehensive discussion and bibliography). The phase diagram of the Curie-Weiss model can be obtained by studying the observable $M = N^{-1} \sum_i \sigma_i$, whose Gibbs expectation satisfies the mean-field equation

$$\langle M \rangle = \tanh(h + \beta \langle M \rangle) \quad (1.2)$$

in the thermodynamic limit $N \rightarrow \infty$. To understand (1.2) heuristically, we note that

$$\langle \sigma_i \rangle = \left\langle \tanh \left( h + \frac{\beta}{N} \sum_{j \neq i} \sigma_j \right) \right\rangle_i$$

where $\langle \cdot \rangle_i$ is the Gibbs expectation with the spin $\sigma_i$ removed. Since one expects $M$ to concentrate around its expectation both at high temperature (spins are approximately independent) and at low temperature when $h \neq 0$ (almost all spins take on the same value), it should be possible to pull the
Gibbs expectation inside the tanh. Although this intuition should extend also to the more general case (1.1), rigorous treatments of (1.2) rely strongly on the symmetries of the Curie-Weiss model. The classical approach (see [17, Ch. 2]) is to compute the entropy using that $H = \frac{N}{2} M^2 - h NM$ essentially depends on only one degree of freedom. Other proofs exploit this symmetry using the Hubbard-Stratonovich transformation [19, 32], Varadhan’s lemma [33], or the exchangeability of certain spin configurations [11, 12].

We propose a simple dynamical method for establishing explicit finite-volume versions of the mean-field equations that are valid for general interactions. Setting $m_i = \langle \sigma_i \rangle$, the analogue of (1.2) in the general setting is

$$\mathbf{m} = \tanh(\mathbf{h} + \mathbf{Jm})$$

(1.3)

where $\mathbf{m} = (m_i)$, $\mathbf{h} = (h_i)$, $\mathbf{J} = (J_{ij})$, and the tanh of a vector is defined in an entrywise sense. To state our main result, we also introduce the notation

$$\hat{h} = \min h_i, \quad ||\mathbf{J}||_{\infty,\infty} = \max_j \sum_i J_{ij}, \quad ||\mathbf{J}||_{1,\infty} = \max_{i,j} J_{ij}.$$ 

The typical mean-field setting corresponds to $||\mathbf{J}||_{\infty,\infty} = \mathcal{O}(1)$ and $||\mathbf{J}||_{1,\infty} = \mathcal{O}(N^{-1})$ as $N \to \infty$.

**Theorem 1.1.** Let $\mathbf{h} \geq 0$. Then

$$||\mathbf{m} - \tanh(\mathbf{h} + \mathbf{Jm})||_{\infty} \leq \frac{||\mathbf{J}||_{1,\infty}}{h} \left( 3 + ||\mathbf{J}||_{\infty,\infty} + \log \left( 1 + \frac{||\mathbf{J}||_{\infty,\infty}}{h} \right) \right).$$

Theorem 1.1 is only informative when $\hat{h} > 0$. However, when $||\mathbf{J}||_{1,\infty} \to 0$ as $N \to \infty$, the theorem is sufficient to study the physically relevant order of limits that first lets $N \to \infty$ and then lets $\mathbf{h} \to 0$. Moreover, one can arrange for both $\hat{h}$ and the right hand side in Theorem 1.1 to also vanish asymptotically if $||\mathbf{J}||_{1,\infty}$ does – a point we will return to in Section 3. To the best of our knowledge, quantitative bounds like Theorem 1.1 are relatively scarce in the standard literature on the Curie-Weiss model. Stein’s method for exchangeable pairs [11, 12] yields concentration bounds for $\mathcal{M}$ under the Gibbs measure with fluctuations of order $N^{-1/2}$. In particular, these bounds prove Theorem 1.1 for the Curie-Weiss model with an error rate of $N^{-1/2}$. When $\mathbf{h} > 0$, our result improves the error rate to $N^{-1}$ while remaining valid for models with general interactions $\mathbf{J}$. There have also been recent developments concerning the fluctuations of general nonlinear functions of Bernoulli random variables whose gradients are close to a low-dimensional manifold [13]. The application of these ideas to mean-field Gibbs measures was further explored in the works [4, 14, 20].

The general mean-field equations (1.3) are very common in the physics literature [27, Sec. 3.2]. Nevertheless, it was noted in [11, Sec. 3.4] that completely general interactions $\mathbf{J}$ seem to pose significant challenges for the existing mathematical strategies. It was also shown in [11, Thm. 3.5] that methods based on exchangeable pairs can yield analogues of the mean-field equations for certain conditional averages with high probability under the Gibbs measure, from which (1.3) follows at sufficiently high temperatures. Exchangeable pairs have also been used to analyze the rank-one case $\mathbf{J} = \mathbf{ww}^\top$ with a regular and non-negative $\mathbf{w} \in \mathbb{R}^N$ [15]. A different perspective can be found in the works [5, 6], which prove that the magnetizations of sufficiently high-dimensional or long-range systems approximately minimize the free energy of the associated mean-field theory. These are very strong results that, among many other things, imply the approximate validity of the mean-field equations but also rely on the powerful input of infrared bounds derived from reflection positivity.

There is a strong analogy between the mean-field equation and the subordination relations in random matrix theory and free probability. The central example of the latter is concerned with the
resolvent $G(t, z) = (A_t - z)^{-1}$ of an $N \times N$ matrix $A_t = A_0 + \sqrt{t} \Phi$, where $A_0$ is a diagonal matrix and $\Phi$ is drawn from the Gaussian orthogonal ensemble. The main assertion, that the limiting empirical eigenvalue distribution of $A_t$ is given by the free convolution of the limiting empirical eigenvalue distribution of $A_0$ and a semicircular element, can be captured by the fact that

$$s_0(z) = \frac{1}{N} \sum_{\lambda \in \sigma(A_0)} \frac{1}{\lambda - z}, \quad s_t(z) = \frac{1}{N} \sum_{\lambda \in \sigma(A_t)} \frac{1}{\lambda - z}$$

are analytic functions that map the half-plane $\text{Im } z > 0$ into itself and satisfy

$$s_t(z) = s_0(z + ts_t(z)) \quad (1.4)$$

in the $N \to \infty$ limit [28]. To illustrate the connection to the mean-field equation, we consider the simplest case of (1.3) when $h_i = h > 0$ and $\sum_j J_{ij} = \beta$ for all $i$. In this case, one can construct a solution of (1.3) by letting each entry solve the scalar equation

$$m = \tanh(h + \beta m). \quad (1.5)$$

It is not hard to show that the positive solution $m = m(h)$ of (1.5) extends to an analytic function of $h$ that maps the half-plane $\text{Re } h > 0$ into itself. It follows that $\tilde{m}(z) = im(-iz)$ is an analytic function of $\{\text{Im } z > 0\}$ into itself with

$$\tilde{m}(z) = \tan(z + \beta \tilde{m}(z)).$$

It is a consequence of the Herglotz trick [1] that

$$\tan(z) = \sum_{\lambda \in \Lambda} \frac{1}{\lambda - z}, \quad \Lambda = \pi(Z + 1/2),$$

so $\tilde{m}$ is the Stieltjes transform of the free convolution of $\sum_{\lambda \in \Lambda} \delta_{\lambda}$ and a semicircular element. Expanding on this theme, the assertion of Theorem 1.1 that the relation (1.5) is not only valid asymptotically but that the error is small even in finite volumes when $\tilde{h} \gg \|J\|_{1,\infty}$, is analogous to the local deformed semicircle law [24]. The proof of the local law depends crucially on the fact that

$$\left| \frac{\partial}{\partial z} G_{ii}(t, z) \right| \leq \sum_k |G_{ik}(t, z)|^2 = \frac{\text{Im } G_{ii}(t, z)}{\text{Im } z}. \quad (1.6)$$

Lemma 2.1, which rests on the Lee-Yang theorem [25], contains a similar inequality for $m_i$ that lies at the heart of Theorem 1.1.

It was observed in [28] that the subordination relation (1.4) is equivalent to the partial differential equation

$$\frac{\partial}{\partial h} s_t(z) = s_t(z) \frac{\partial}{\partial z} s_t(z). \quad (1.7)$$

In finite volumes, $s_t$ exactly satisfies a perturbed version of this transport equation, which enables a simple proof the local law by combining the analytic structure contained in (1.6) with an approximate characteristic curve [34, 35]. Our approach to Theorem 1.1 is to generalize this analysis by exploiting the identity

$$\frac{\partial}{\partial J_{ij}} (f) = m_j \frac{\partial}{\partial h_j} (f) + m_j \frac{\partial}{\partial h_i} (f) + \frac{\partial^2}{\partial h_i \partial h_j} (f). \quad (1.8)$$
The relationship between (1.8) and (1.7) is most apparent in the Curie-Weiss model, where (1.8) yields
\[ \frac{\partial}{\partial \beta} \langle M \rangle = \langle M \rangle \frac{\partial}{\partial h} \langle M \rangle + \frac{1}{2N} \frac{\partial^2}{\partial h^2} \langle M \rangle, \]
which is the evolution studied in [34] without the stochastic terms. We note that similar differential inequalities have been studied in a variety of models that go beyond the mean-field setting [2, 3].

This paper is structured as follows. In Section 2, we use (1.8) to derive a transport equation in the general setting and prove the analogue of (1.6). This allows us to extend the ideas of [34, 35] concerning approximate characteristic curves to the present setting and prove Theorem 1.1. Then, in Section 3, we present three sample applications of our method to Kac interactions, randomly diluted models, and models with an asymptotically vanishing external field.

### 2 Proof of Theorem 1.1

Rearranging indices shows that Theorem 1.1 follows if we can prove that
\[ |m_1 - \tanh(h_1 + \mathbf{J}_1^\top \mathbf{m})| \leq \frac{||\mathbf{J}_1||_{\infty}}{h} \left( 3 + ||\mathbf{J}_1||_1 + \log \left( 1 + \frac{||\mathbf{J}_1||_1}{h} \right) \right) \quad (2.1) \]
where \( \mathbf{J}_1 = (J_{1i}) \) is the first column of \( \mathbf{J} \). By symmetry, we can write \( H(t, \sigma) = H(1, \sigma) \), where
\[ H(t, \sigma) = t \sum_i J_{1i} \sigma_1 \sigma_i + \sum_i h_i \sigma_i + H_1(\sigma_2, \ldots, \sigma_N) \]
and \( H_1 \) does not depend on \( \sigma_1 \). Moreover, if \( \mathbf{m}(t, \mathbf{h}) \) denotes the vector of magnetizations under \( H(t, \cdot) \), the identity (1.8) yields
\[ \frac{\partial}{\partial t} \mathbf{m}(t, \mathbf{h}) = m_1(t, \mathbf{h}) \frac{\partial}{\partial \mathbf{J}_1} \mathbf{m}(t, \mathbf{h}) + \frac{\partial^2}{\partial h_1 \partial \mathbf{J}_1} \mathbf{m}(t, \mathbf{h}) + \langle \mathbf{J}_1^\top \mathbf{m} \rangle \frac{\partial}{\partial h_1} \mathbf{m}(t, \mathbf{h}), \quad (2.2) \]
where \( \frac{\partial}{\partial \mathbf{J}_1} = \mathbf{J}_1^\top \nabla_{\mathbf{h}} \) is the directional derivative with respect to \( \mathbf{h} \) in direction \( \mathbf{J}_1 \).

If we knew that the first two terms on the right hand side of (2.2) were negligible, then (2.2) would reduce to a transport equation that could be solved by varying \( h_1 \) along a suitable characteristic curve. We will prove the appropriate bounds for this purpose with the help of the following integral representation. Suppose that \( f \) is a holomorphic function defined on a half-plane \( \text{Re} \, z > -\kappa \) such that \( \text{Re} \, f \geq 0 \) and such that \( f \) remains bounded as \( z \to \infty \) along \( \mathbb{R} \). Then,
\[ f(z) = a + \int_{\mathbb{R}} \frac{1}{z + \kappa - i \lambda} \mu(d\lambda) \]
for some positive measure \( \mu \) and \( a \in \mathbb{C} [31, \text{Ch.} 5] \), which implies that
\[ |f'(z)| \leq \frac{\text{Re} \, f(z)}{\text{Re} \, z + \kappa}. \]
We note that the use of integral representations of holomorphic functions to bound correlations is a classical idea (see for instance [18, 29]).
Lemma 2.1. For every \( k \) and every \( s \geq 0 \),

\[
\left| \frac{\partial}{\partial s} m_k(t, h) \right| \leq \frac{\|s\|_\infty}{h} m_k(t, h)
\]

and

\[
\left| \frac{\partial^2}{\partial h_1 \partial s} m_k(t, h) \right| \leq \frac{\|s\|_\infty}{h} \frac{m_k(t, h)}{h_1}
\]

for all \( t \geq 0 \) and \( h > 0 \).

Proof. If we fix \( h_j \) with \( \Re h_j > 0 \) for \( j \neq k \), the meromorphic function \( f(h_k) = m_k(t, h) \) satisfies \( f'(h_k) = 1 - f^2(h_k) \) wherever it is analytic. Hence, combining the Picard-Lindelöf theorem with analytic continuation shows that there is some \( \Gamma \in \mathbb{C} \) such that

\[
f(h_k) = \tanh(h_k + \Gamma)
\]

for all \( h_k \in \mathbb{C} \). If \( \Re h > 0 \), the Lee-Yang theorem asserts that the partition function \( Z(t, h) = \sum_{\sigma} e^{-H(t, \sigma)} \) cannot vanish and therefore \( f \) cannot have a pole in this region. This is only possible when \( \Re \Gamma \geq 0 \) and therefore \( \Re m_k(t, h) = \Re f(h_k) \geq 0 \) whenever \( \Re h \geq 0 \).

To prove the first bound, we fix \( h \geq 0 \) and consider the function

\[
f(z) = m_k(t, h + zs).
\]

Then \( f > 0 \) on the half-plane \( \Re z > -\kappa \) with \( \kappa = \hat{h}/\|s\|_\infty \) so

\[
\left| \frac{\partial}{\partial s} m_k(t, h) \right| = |f'(0)| \leq \frac{\Re f(0)}{\kappa} = \frac{\Re m_k(t, h)}{\kappa}.
\]

Since \( m_k(t, h) \) is real when \( h \) is real, this is the first assertion of the lemma.

For the second bound, we use the auxiliary function

\[
g(z) = \partial_1 m_k(t, h + zs).
\]

Then \( g \) is also holomorphic on \( \Re z > -\kappa \) and considering the right hand side as a function of \( h_1 \) shows that

\[
|g(z)| \leq \frac{\Re m_k(t, h + zs)}{\Re h_1}.
\]

Letting \( C \) be a positively oriented circle about 0 of radius \( \kappa \), we have

\[
\frac{\partial^2}{\partial h_1 \partial s} m_k(t, h) = g'(0) = \oint_C \frac{g(\xi)}{\xi^2} d\xi
\]

so

\[
\left| \frac{\partial^2}{\partial h_1 \partial s} m_k(t, h) \right| \leq \frac{1}{2\pi \kappa} \int_0^{2\pi} \frac{\Re m_k(h + e^{i\theta}s)}{\Re h_1} d\theta = \frac{\Re m_k(t, h)}{\kappa \Re h_1}
\]

using the mean value property of the harmonic function \( z \to \Re m_k(t, h + zs) \). \( \square \)
With Lemma 2.1 in place, we now fix $h \geq 0$ and define an approximate characteristic curve $w = (w_i)$ for (2.2) by

$$\frac{\partial}{\partial t} w_i(t) = \begin{cases} -J_1^i m(t, w(t)) & i = 1 \\ 0 & \text{else} \end{cases}, \quad w(1) = h.$$  

Since $m$ is non-negative and uniformly Lipschitz continuous in $h \geq 0$, such a curve exists and satisfies $\dot{w}(t) \geq h$ for all $t \in [0, 1]$. The following lemma shows that any weighted average $s^Tm$ does not significantly change along the curve $w(t)$, provided that $\|s\|_{\infty}$ is small.

**Lemma 2.2.** Let $s \geq 0$. Then

$$\sup_{t \in [0, 1]} |s^Tm(h) - s^Tm(t, w(t))| \leq \frac{\|s\|_{\infty}}{h} \left(\|J_1\|_1 + \log \left(1 + \frac{\|J_1\|_1}{h}\right)\right).$$

**Proof.** Inserting the characteristic curve into (2.2) and multiplying by $s$, we obtain

$$s^Tm(h) - s^Tm(t, w(t)) = \int_t^1 m_1(r, w(r))J_1 s^Tm(r, w(r)) + \frac{\partial^2}{\partial h_1 \partial J_1} s^Tm(r, w(r)) dr. \quad (2.3)$$

Combining the identity

$$\frac{\partial}{\partial J_1} s^Tm = \frac{\partial}{\partial s} J_1^i m$$

with Lemma 2.1 shows that the first integral on the right hand side of (2.3) is bounded by

$$\int_t^1 \left| m_1(r, w(r))\frac{\partial}{\partial J_1} s^Tm(r, w(r)) \right| dr \leq \frac{\|s\|_{\infty} \|J_1\|_1}{h},$$

whereas the second integral is bounded by

$$\int_t^1 \left| \frac{\partial^2}{\partial h_1 \partial J_1} s^Tm(r, w(r)) \right| dr \leq \frac{\|s\|_{\infty}}{h} \int_t^1 \frac{J_1^i m(r, w(r))}{w_1(r)} dr. \quad (2.4)$$

The right hand side of (2.4) can be calculated explicitly since $\frac{\partial}{\partial t} w_1(t) = -J_1^i m(t, w(t))$, which yields a final bound of

$$\int_t^1 \left| \frac{\partial^2}{\partial h_1 \partial J_1} s^Tm(r, w(r)) \right| dr \leq \frac{\|s\|_{\infty}}{h} \log \left(\frac{w_1(t)}{h_1}\right) \leq \frac{\|s\|_{\infty}}{h} \log \left(1 + \frac{\|J_1\|_1}{h}\right).$$

The evolution of $m_1$ along the characteristic curve is given by

$$\frac{\partial}{\partial t} m_1(t, w(t)) = m_1(t, w(t))\frac{\partial}{\partial J_1} m_1(t, w(t)) + \frac{\partial^2}{\partial h_1 \partial J_1} m_1(t, w(t)).$$

By Lemma 2.1,

$$\left| m_1(t, w(t))\frac{\partial}{\partial J_1} m_1(t, w(t)) \right| \leq \frac{\|J_1\|_{\infty}}{h}.$$
and
\[ \left| \frac{\partial^2}{\partial h_1 \partial J_1} m_1(t, w(t)) \right| = 2 \left| m_1(t, w(t)) \frac{\partial}{\partial J_1} m_1(t, w(t)) \right| \leq \frac{2\|J_1\|_\infty}{h}, \]
so
\[ |m_1(h) - \tanh(w_1(0))| = |m_1(1, w(1)) - m_1(0, w(0))| \leq \frac{3\|J_1\|_\infty}{h}. \tag{2.5} \]
Since
\[ w_1(0) = h_1 + \int_0^1 J_1^\top m(t, w(t)) \, dt, \]
Lemma 2.2 with \( s = J_1 \) implies that
\[ |w_1(0) - h_1 - J_1^\top m(h)| \leq \frac{\|J_1\|_\infty}{h} \left( \|J_1\|_1 + \log \left( 1 + \frac{\|J_1\|_1}{h} \right) \right). \]
Inserting this into (2.5) and using the Lipschitz continuity of \( \tanh \) completes the proof of (2.1).

### 3 Three applications

Our first two applications consist of showing that two classes of models have the same thermodynamic behavior as the Curie-Weiss model. In the first, we consider a system in a box \( \Lambda \subset \mathbb{Z}^d \) with Kac interactions
\[ J_{ij} = \beta \lambda^d f(\lambda(i - j)), \quad h_i = h > 0, \quad i, j \in \Lambda \]
where \( f \) is a bounded Riemann integrable probability density on \( \mathbb{R}^d \) and \( \beta, \lambda > 0 \). Kac interactions are “physical” in the sense that they yield a convex free energy, but this free energy still converges to the convex envelope of the Curie-Weiss free energy as \( \lambda \to 0 \) [21, 23]. This fact can be used to provide a justification of Maxwell’s equal-area rule for the van der Waals isotherm (see also [17, Ch. 4] for further details). There is also an extensive literature on Kac interactions with fixed \( \lambda > 0 \) and related models – we refer the reader to [8–10, 22, 30] and references therein.

Writing \( m_\Lambda \) for the vector of magnetizations corresponding to the box \( \Lambda \), Theorem 1.1 asserts that
\[ \|m_\Lambda - \tanh(h + Jm_\Lambda)\|_\infty \leq C \frac{\lambda^d}{h \log h} \]
for some absolute constant \( C < \infty \). Translation invariance and standard convexity arguments show that there is some \( m \in \mathbb{R} \) such that for any fixed \( i \in \mathbb{Z}^d \) we have \( m_\Lambda, i \to m \) as \( \Lambda \to \mathbb{Z}^d \). By the dominated convergence theorem the limit \( m \) still satisfies
\[ \left| m - \tanh \left( h + \beta m \sum_i \lambda^d f(\lambda i) \right) \right| \leq C \frac{\lambda^d}{h \log h} \]
and therefore \( m = \tanh(h + \beta m) \) in the \( \lambda \to 0 \) limit.

The second model we consider is the randomly diluted model where
\[ J_{ij} = \frac{\beta}{Np} \epsilon_{ij}, \quad h_i = h > 0 \]
and \( \epsilon_{ij} \) are independent (up to symmetry) Bernoulli random variables with \( E \epsilon_{ij} = p \). In the case where \( p = p(N) \) is chosen such that \( Np \to \infty \) as \( N \to \infty \), it was shown in [7] that the limiting
magnetizations coincide with those of the Curie-Weiss model. Under this assumption, the variance of a weighted average is bounded by

$$E \left| J_1^T x - \frac{\beta}{N} \sum_i x_i \right|^2 \leq \frac{\beta^2 \|x\|_\infty^2}{Np} \to 0. \quad (3.1)$$

Applying the single-site bound (2.1), it follows that

$$m_1 - \tanh(h + J_1^T m) \to 0$$

in $L^2(\mathbb{P})$ as $N \to \infty$. We write $m = N^{-1} \sum_i m_i$ and let $m_i(1)$ denote the Gibbs mean of $\sigma_i$ with $\sigma_1$ removed. Using Lemma 2.2 and repeating the bound above yields

$$J_1^T m(h) - \sum_{i>1} J_i m_i(1)(h) \to 0$$

and

$$m - \frac{1}{N} \sum_{i>1} m_i(1)(h) \to 0$$

in $L^2(\mathbb{P})$. Since $m_i(1)$ is independent of $J_1$, combining this with the bound (3.1) shows that $J_1^T m - \beta m \to 0$ in $L^2(\mathbb{P})$. We conclude that both

$$m_1 - \tanh(h + \beta m) \to 0, \quad m - \tanh(h + \beta m) \to 0$$

in $L^2(\mathbb{P})$.

Finally, we consider a generic model with an asymptotically vanishing external field

$$\|J\|_{1,\infty} = o(1), \quad \hat{h} = \|J\|^{\frac{1}{2} - \delta}_{1,\infty}.$$

We assume a low-temperature condition of the form that there exists $\alpha > 1$, independent of $N$, such that for all $x \geq 0$ there is some index $i$ with

$$(Jx)_i \geq \alpha x_i. \quad (3.2)$$

Provided that $\|J\|_{\infty,\infty} = O(1)$, Theorem 1.1 implies that

$$\tanh(h + Jm) = m + \epsilon, \quad \|\epsilon\|_\infty = O\left( -\|J\|^{\frac{1}{2} + \delta}_{1,\infty} \log \|J\|_{1,\infty} \right).$$

There exists some $K > 0$ such that $\alpha \tanh(x) > x$ when $x \in (0, K)$. If it were true that $m \leq K$, combining this with the fact that $h, \epsilon \to 0$ and $h \gg \epsilon$ as $N \to \infty$ would imply that

$$Jm < \alpha m$$

for sufficiently large $N$, contradicting (3.2). We conclude that an external field strength of $\hat{h} = \|J\|^{\frac{1}{2} - \delta}_{1,\infty}$ is sufficient to select a positive Gibbs state in the sense that

$$\lim \inf_{N \to \infty} \max_i m_i > 0. \quad (3.3)$$

Degenerate examples like the case where $J_{1i} = 0$ for all $i$ demonstrate that, in general, one cannot hope for a stronger statement than (3.3). For the Curie-Weiss model, the previous argument shows that an external field strength of $h = N^{-\frac{1}{2} + \delta}$ is sufficient to select the positive Gibbs state below the critical temperature of $\beta = 1$. The book [26, Sec. III.1] mentions that actually an external field strength of $h = N^{-1 + \delta}$ already suffices, but we have not been able to locate a mathematical proof of this assertion in the literature.
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