Convergence rates for subcritical threshold-one contact processes on lattices

Xiaofeng Xue *
University of Chinese Academy of Sciences

Abstract:
In this paper we are concerned with threshold-one contact processes on lattices. We show that the probability that the origin is infected converges to 0 at an exponential rate $I$ in the subcritical case. Furthermore, we give a limit theorem for $I$ as the degree of the lattice grows to infinity. Our results also hold for classic contact processes on lattices.

Keywords: contact process, threshold, convergence rate, random walk.

1 Introduction

In this paper we are concerned with threshold-one contact processes on lattices $\mathbb{Z}^d$, $d = 1, 2, \ldots$ For any $x, y \in \mathbb{Z}^d$, we write $x \sim y$ when there is an edge connecting these two vertices. We say that $x$ and $y$ are neighbors when $x \sim y$.

The threshold-one contact process $\{\eta_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ is with state space $\{0, 1\}^{\mathbb{Z}^d}$. In other words, at each vertex of $\mathbb{Z}^d$ there is a spin taking value 0 or 1. For each $x \in \mathbb{Z}^d$ and $t > 0$, the spin at $x$ at moment $t$ is denoted by $\eta_t(x)$. Furthermore, we define $\eta_{t-}(x)$ as

$$\eta_{t-}(x) := \lim_{s \uparrow t, s < t} \eta_s(x).$$

Hence $\eta_{t-}(x)$ is the spin at $x$ at the moment just before $t$.

$\{\eta_t\}_{t \geq 0}$ evolves according to independent Poisson processes $\{N_x(t) : t \geq 0\}_{x \in \mathbb{Z}^d}$ and $\{Y_x(t) : t \geq 0\}_{x \in \mathbb{Z}^d}$. For each $x \in \mathbb{Z}^d$, $N_x$ is with rate 1 and $Y_x$ is with rate $\lambda$, where $\lambda > 0$ is a parameter called the infection rate. At $t = 0$, each spin takes a value from $\{0, 1\}$ according to some probability distribution. Then, for each $x \in \mathbb{Z}^d$, the spin at $x$ may flip only at event times of $N_x$ and $Y_x$. For

*E-mail: xuexiaofeng@ucas.ac.cn Address: School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.
any event time $s$ of $N_x$, $\eta_r(x) = 0$ no matter whatever $\eta_{r-}(x)$ is. For any event time $r$ of $Y_x$, if $\eta_{r-}(x) = 1$, then $\eta_r(x) = 1$. If $\eta_{r-}(x) = 0$, then $\eta_r(x) = 1$ when and only when at least one neighbor $y$ of $x$ satisfies $\eta_{r-}(y) = 1$.

Therefore, $\{\eta_t\}_{t \geq 0}$ is a spin system (see Chapter 3 of [6]) with flip rates function given by

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda & \text{if } \eta(x) = 0 \text{ and } \sum_{y:y \sim x} \eta(y) \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

for any $(x, \eta) \in \mathbb{Z}^d \times \{0, 1\}^{\mathbb{Z}^d}$.

Intuitively, the threshold-one contact process describes the spread of an infected disease. Vertices with spin 1 are infected individuals while vertices with spin 0 are healthy. An infected vertex waits for an exponential time with rate one to become healthy while a healthy vertex is infected by neighbors with rate $\lambda$ when at least one neighbor is infected.

Our main result in this paper about the threshold-one contact process $\{\eta_t\}_{t \geq 0}$ also holds for the classic contact process $\{\beta_t\}_{t \geq 0}$. The flip rates function of $\beta_t$ is given by

$$\tilde{c}(x, \beta) = \begin{cases} 1 & \text{if } \beta(x) = 1, \\ \lambda \sum_{y:y \sim x} \eta(y) & \text{if } \beta(x) = 0 \end{cases}$$

for any $(x, \beta) \in \mathbb{Z}^d \times \{0, 1\}^{\mathbb{Z}^d}$. The main difference between $\eta_t$ and $\beta_t$ is that for $\beta_t$, a healthy vertex is infected at rate proportional to the number of infected neighbors.

The threshold-one contact processes is introduced in [2] by Cox and Durrett as a tool to study threshold voter models (see Part two of [8] and [1, 5, 7, 10, 12]). [2] gives an important dual relationship between the threshold-one contact process and an additive Markov processes. According to this dual relationship, [2] shows that the critical value $\lambda_c(d)$ for the threshold-one contact process on $\mathbb{Z}^d$ satisfies $\lambda_c(d) \leq 2.18/d$. [11] develops this result by showing that $\lim_{d \to +\infty} 2d\lambda_c(d) = 1$. In recent years, there are some works concerned with threshold contact processes with threshold $K > 1$. [9] shows that the critical value $\lambda_c(d, K)$ for the threshold $K > 1$ contact process on $\mathbb{Z}^d$ satisfies $\lim_{d \to +\infty} \lambda_c(d, K) = 0$. [3] shows that the same conclusion holds for the case on regular trees $\mathbb{T}^N$ and gives the rate at which $\lambda_c(\mathbb{T}^N, K)$ converges to 0 as $N$ grows to infinity.
2 Main result

In this section, we will give the main result of this paper. First we introduce some notations. For \(d \geq 1\) and \(\lambda > 0\), we denote by \(P_{\lambda,d}\) the probability measure of the threshold-one contact process \(\{\eta_t\}_{t \geq 0}\) on \(\mathbb{Z}^d\) with infection rate \(\lambda\). We denote by \(E_{\lambda,d}\) the expectation operator with respect to \(P_{\lambda,d}\). We write \(\eta_t\) as \(\eta_t^\eta\) when \(P_{\lambda,d}(\eta_0 = \eta) = 1\) for some \(\eta \in \{0, 1\}^\mathbb{Z}^d\). We denote by \(\delta_1\) and \(\delta_0\) configurations in \(\{0, 1\}^\mathbb{Z}^d\) such that \(\delta_1(x) = 1, \delta_0(x) = 0\) for each \(x \in \mathbb{Z}^d\). We denote by \(O\) the origin of \(\mathbb{Z}^d\) and denote by \(e_1\) the unit vector \((1, 0, 0, \ldots, 0)\).

Since the threshold-one contact process is attractive (see Chapter 3 of [6]), for any \(t > s\) and \(\lambda_1 > \lambda_2\),

\[
P_{\lambda_1,d}(\eta_t^{\delta_1}(O) = 1) \geq P_{\lambda_2,d}(\eta_t^{\delta_1}(O) = 1).
\]

As a result, it is reasonable to define the following critical value.

\[
\lambda_c(d) := \sup \{\lambda : \lim_{t \to +\infty} E_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) = 0\}
\] (2.1)

for \(d \geq 1\).

When \(\lambda < \lambda_c(d)\), the process \(\eta_t\) converges weakly to \(\delta_0\) as \(t \to +\infty\), which is called the subcritical case.

In the subcritical case, we are concerned with the rate at which the probability that \(O\) is infected converges to 0 as the time \(t\) grows to infinity. To introduce our main result, we give a lemma at first.

**Lemma 2.1.** For any \(\lambda \geq 0\) and \(d \geq 1\), there exists \(I(\lambda, d) \in [-\infty, 0]\) such that

\[
\lim_{t \to +\infty} \frac{1}{t} \log P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) = I(\lambda, d).
\] (2.2)

After giving \(\lambda\) a proper scale, we obtain the following limit theorem of \(I(\lambda, d)\) as our main result.

**Theorem 2.2.** For any \(\lambda \geq 0\),

\[
\lim_{d \to +\infty} I\left(\frac{\lambda}{d}\right) = \begin{cases} 2\lambda - 1 & \text{if } \lambda \in [0, 1/2], \\ 0 & \text{if } \lambda > 1/2. \end{cases}
\] (2.3)
Theorem 2.2 shows that for subcritical threshold-one contact process with infection rate \( \lambda \), the probability that \( O \) is infected converges to 0 as \( t \to +\infty \) at an exponential rate approximate to \( 2\lambda d - 1 \) when the dimension \( d \) is sufficiently large.

According to the dual relationship introduced in [2], there is an intuitive explanation for Theorem 2.2. When the dimension \( d \) is large, the threshold-one contact process is similar with a branching process such that each particle generates \( 2d \) particles at rate \( \lambda \) or dies at rate one. The mean of the sum of the particles at \( t \) is \( \exp\{(2\lambda d - 1)t\} \).

In Theorem 2.2, the case where \( \lambda > 1/2 \) is trivial, since [11] shows that
\[
\lim_{d \to +\infty} 2d\lambda_c(d) = 1.
\]

Similar conclusion with Theorem 2.2 holds for the classic contact process \( \{\beta_t\}_{t \geq 0} \), the flip rate function of which is given in (1.2).

Theorem 2.3. For any \( \lambda \geq 0 \) and \( d \geq 1 \), there exists \( J(\lambda, d) \in [-\infty, 0] \) such that
\[
\lim_{t \to +\infty} \frac{1}{t} \log P_{\lambda,d}(\beta_t^{\delta_1}(O) = 1) = J(\lambda, d)
\]
and
\[
\lim_{d \to +\infty} J(\frac{\lambda}{d^2}, d) = \begin{cases} 
2\lambda - 1 & \text{if } \lambda \in [0, 1/2], \\
0 & \text{if } \lambda > 1/2.
\end{cases}
\]

In this paper, the proof of theorem about \( \beta_t \) is similar with that of the counterpart conclusion about \( \eta_t \). We will give all the details in the proof of theorem about \( \eta_t \) and give just a sketch for the proof of theorem about \( \beta_t \).

At the end of this section, we give the proof of Lemma 2.1. The proof of Theorem 2.2 is divide into Section 3 and Section 4.

Proof of Lemma 2.1. We utilize the dual relationship introduced in [2]. Let \( \{A_t\}_{t \geq 0} \) be a Markov process with state space
\[
2^{\mathbb{Z}^d} := \{ A : A \subseteq \mathbb{Z}^d \}
\]
and flip rate functions
\[
A_t \to \begin{cases} 
A_t \setminus x & \text{at rate 1}, \\
A_t \cup \{y : y \sim x\} & \text{at rate } \lambda
\end{cases}
\]
for any \( t \geq 0 \) and each \( x \in A_t \).

We write \( A_t \) as \( A_t^{\beta} \) when \( A_0 = A \). Then, according to [2], there is a dual relationship between \( \{\eta_t\}_{t \geq 0} \) and \( \{\beta_t\}_{t \geq 0} \) such that
\[
P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) = P_{\lambda,d}(A_t^{\beta} \neq \emptyset).
\]
As a result, according to strong Markov property,
\[
P_{\lambda,d}(\eta_{t+s}^\delta(O) = 1) = P_{\lambda,d}(A_{t+s}^O \neq \emptyset) = E_{\lambda,d}\left[P_{\lambda,d}(A_t^\delta \neq \emptyset; A_t^o \neq \emptyset)\right]. \tag{2.5}
\]
Since $A_t$ is symmetric for $\mathbb{Z}^d$ and is a monotone process under the partial order that $A \succeq B$ if and only if $A \supseteq B$,
\[
P_{\lambda,d}(A_{t+s}^\delta \neq \emptyset) \geq P_{\lambda,d}(A_{t}^O \neq \emptyset) \tag{2.6}
\]
on the event $\{A_t \neq \emptyset\}$.
By (2.5) and (2.6),
\[
P_{\lambda,d}(\eta_{t+s}^\delta(O) = 1) \geq P_{\lambda,d}(A_{t}^O \neq \emptyset) P_{\lambda,d}(\eta_{t+s}^\delta(O) = 1)
\]
and hence,
\[
\log P_{\lambda,d}(\eta_{t+s}^\delta(O) = 1) \geq \log P_{\lambda,d}(\eta_{t}^\delta(O) = 1) + \log P_{\lambda,d}(\eta_{s}^\delta(O) = 1) \tag{2.7}
\]
for any $t, s \geq 0$.

The existence of $I(\lambda, d)$ follows from (2.7) and Fekete’s Subadditive Lemma. By Fekete’s Subadditive Lemma,
\[
I(\lambda, d) = \sup_{t \geq 0} \frac{1}{t} \log P_{\lambda,d}(\eta_t^\delta(O) = 1).
\]

The proof of the existence of $J(\lambda, d)$ is nearly the same as that of $I(\lambda, d)$ by the self-duality of $\{\beta_t\}_{t \geq 0}$ introduced in Theorem 6.1.7 of [6].

\section*{3 Upper bound}

In this section we will give upper bounds for $I(\lambda/d, d)$ and $J(\lambda/d, d)$.

The proofs of Theorem 2.2 for cases where $\lambda = 0$ and $\lambda > 1/2$ are trivial. According to [11],
\[
\lim_{d \to +\infty} 2d\lambda_c(d) = 1.
\]
As a result, for \( \lambda > 1/2 \) and sufficiently large \( d \), \( \lambda_c(d) < \lambda/d \) and

\[
\lim_{t \to +\infty} P_{\lambda/d,d}(\eta_t^{\delta_1}(O) = 1) = K(\lambda, d) > 0.
\]

Therefore,

\[
I(\lambda/d, d) = \lim_{t \to +\infty} \frac{1}{t} \log P_{\lambda/d,d}(\eta_t^{\delta_1}(O) = 1) = 0
\]

for \( \lambda > 1/2 \) and sufficiently large \( d \).

The above analysis also holds for \( J(\lambda/d, d) \) since \( \ref{4} \) shows that the critical value \( \hat{\lambda}_c(d) \) for the classic contact process \( \{\beta_t\}_{t \geq 0} \) on \( \mathbb{Z}^d \) satisfies

\[
\lim_{d \to +\infty} 2d\hat{\lambda}_c(d) = 1.
\]

When \( \lambda = 0 \), \( O \) waits for an exponential time with rate one to become healthy and will never be infected again. Hence,

\[
P_{0,d}(\eta_t^{\delta_1}(O) = 1) = P_{0,d}(\beta_t^{\delta_1}(O) = 1) = e^{-t}
\]

and

\[
I(0, d) = J(0, d) = \lim_{t \to +\infty} \log e^{-t} = -1.
\]

Now we only need to deal with the case where \( \lambda \in (0, 1/2) \). The following lemma gives an upper bound for \( I(\lambda, d) \).

**Lemma 3.1.** For any \( \lambda > 0 \) and \( d \geq 1 \),

\[
\max \{ I(\lambda, d), J(\lambda, d) \} \leq 2\lambda d - 1.
\]

As a direct corollary,

\[
\max \{ \limsup_{d \to +\infty} I(\lambda/d, d), \limsup_{d \to +\infty} J(\lambda/d, d) \} \leq 2\lambda - 1 \]

for \( \lambda \in (0, 1/2) \).

**Proof of Lemma 3.1.** According to the flip rate functions of \( \{\eta_t\}_{t \geq 0} \) given in \( \ref{1.1} \) and Hille-Yosida Theorem,

\[
\frac{d}{dt} P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) = - P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1)
\]

\[
+ \lambda P_{\lambda,d}(\eta_t^{\delta_1}(O) = 0, \exists y \sim O, \eta_t^{\delta_1}(y) = 1)
\]

\[
\leq - P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) + \lambda P_{\lambda,d}(\exists y \sim O, \eta_t^{\delta_1}(y) = 1)
\]

\[
\leq - P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) + \lambda \sum_{y \sim O} P_{\lambda,d}(\eta_t^{\delta_1}(y) = 1)
\]

\[
= (2\lambda d - 1) P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1),
\]

for all \( t \geq 0 \) and

\[
\lim_{t \to +\infty} \frac{1}{t} \log P_{\lambda,d}(\eta_t^{\delta_1}(O) = 1) = 0.
\]
since $O$ has $2d$ neighbors and $\{\eta_t\}_{t \geq 0}$ is symmetric for $Z^d$.

Then, according to Grönwall’s inequality,

$$P_{\lambda,d}(\eta^\delta t_0(O) = 1) \leq e^{(2\lambda d - 1)t}P_{\lambda,d}(\eta^\delta 0(O) = 1) = e^{(2\lambda d - 1)t}$$

and hence

$$I(\lambda, d) = \lim_{t \to +\infty} \frac{1}{t} \log P_{\lambda,d}(\eta^\delta 1_t(O) = 1) \leq 2\lambda d - 1.$$

The analysis for $J(\lambda, d)$ is similar. According to the flip rate functions given in (1.2),

$$\frac{d}{dt}P_{\lambda,d}(\beta^\delta 1_t(O) = 1) = -P_{\lambda,d}(\beta^\delta 1_t(O) = 1) + \lambda \sum_{y : y \sim O} P_{\lambda,d}(\beta^\delta 1_t(y) = 1)$$

Then $J(\lambda, d) \leq 2\lambda d - 1$ follows from the same analysis as that of $I(\lambda, d)$.

4 Lower bound

In this section we will give lower bounds for $I(\lambda/d, d)$ and $J(\lambda/d, d)$ for $\lambda \in (0, 1/2)$. The main tool we utilize is a Markov process $\{\zeta_t\}_{t \geq 0}$ with state space $[0, +\infty)^{Z^d}$ introduced in (11). In other words, for $\{\zeta_t\}_{t \geq 0}$, there is a spin at each vertex of $Z^d$ taking a nonnegative value.

Let $\{N_x(t) : t \geq 0\}_{x \in Z^d}$ and $\{Y_x(t) : t \geq 0\}_{x \in Z^d}$ be the same Poisson processes as that in Section 1. $\{\zeta_t\}_{t \geq 0}$ evolves according to $\{N_x\}_{x \in Z^d}$ and $\{Y_x\}_{x \in Z^d}$. At $t = 0$, $\zeta_0(x) > 0$ for each $x \in Z^d$. For any event time $s$ of $N_x$, $\zeta_s(x) = 0$ no matter whatever $\zeta_{s-}(x)$ is. For any event time $r$ of $Y_x$, $\zeta_r(x) = \zeta_{r-}(x) + \sum_{y : y \sim x} \zeta_{r-}(y)$. Between any adjacent event times of Poisson processes, $\zeta_t(x)$ evolves according to deterministic ODE

$$\frac{d}{dt}\zeta_t(x) = (1 - 2\lambda d)\zeta_t(x).$$

In other words, if there is no event time of $N_x$ or $Y_x$ in $[t_1, t_2]$, then

$$\zeta_{t_2}(x) = \zeta_{t_1}(x) \exp\{(1 - 2\lambda d)(t_2 - t_1)\}. \tag{4.1}$$

It is useful for us to give the generator of $\{\zeta_t\}_{t \geq 0}$. For any $\zeta \in [0, +\infty)^{Z^d}$, $x \in Z^d$ and $m \in [0, +\infty)$, we define $U(\zeta, x) = \zeta(x) + \sum_{y : y \sim x} \zeta(y)$ and define
\( \zeta^{x,m} \in [0, +\infty)^{\mathbb{Z}^d} \) as
\[
\zeta^{x,m}(y) = \begin{cases} 
\zeta(y) & \text{if } y \neq x, \\
m & \text{if } y = x.
\end{cases}
\]

Then, the generator \( \Omega \) of \( \{\zeta_t\}_{t \geq 0} \) is given by
\[
\Omega f(\zeta) = \sum_{x \in \mathbb{Z}^d} \left[ f(\zeta^{x,0}) - f(\zeta) \right] + \lambda \sum_{x \in \mathbb{Z}^d} \left[ f(\zeta^{x,U(\zeta,x)}) - f(\zeta) \right]
+ (1 - 2\lambda d) \sum_{x \in \mathbb{Z}^d} f'_x(\zeta) \zeta(x),
\]
for \( f \in C([0, +\infty)^{\mathbb{Z}^d}) \), where \( f'_x(\zeta) \) is the partial derivative of \( f(\zeta) \) with respect to the coordinate \( \zeta(x) \).

The following lemmas are crucial for us to give lower bound for \( I(\lambda, d) \).

**Lemma 4.1.** There is a coupling of \( \eta^{\delta_1}_t \) and \( \zeta_t \) such that
\[
\eta^{\delta_1}_t(x) = \begin{cases} 
1 & \text{if } \zeta_t(x) > 0, \\
0 & \text{if } \zeta_t(x) = 0
\end{cases}
\]
for each \( x \in \mathbb{Z}^d \) and \( t \geq 0 \).

**Proof.** For any \( t \geq 0 \) and \( x \in \mathbb{Z}^d \), we define
\[
\overline{\eta}_t(x) = \begin{cases} 
1 & \text{if } \zeta_t(x) > 0, \\
0 & \text{if } \zeta_t(x) = 0.
\end{cases}
\]
Then, \( \overline{\eta}_0 = \delta_1 \). At any event time \( s \) of \( N_x \), \( \zeta_s(x) = 0 \) and hence \( \overline{\eta}_s(x) = 0 \).
At any event time \( r \) of \( Y_x \), \( \overline{\eta}_r(x) \) flips from 0 to 1 if and only if \( \zeta_r(x) = 0 + \sum_{y : y \sim x} \zeta_r(y) > 0 \). In other words, conditioned on \( \overline{\eta}_{r-}(x) = 0, \overline{\eta}_r(x) = 1 \) if and only if at least one neighbor \( y \) of \( x \) satisfies \( \zeta_{r-}(y) > 0 \) and meanwhile \( \overline{\eta}_{r-}(y) = 1 \). According to ODE (4.1), between any adjacent event times of Poisson processes \( N_x \) and \( Y_x \), \( \zeta_t(x) \) can not flip from positive value to zero or flip from zero to positive value, which makes \( \overline{\eta}_t(x) \) still.

Therefore, \( \{\overline{\eta}_t\}_{t \geq 0} \) evolves in the same way as that of \( \{\eta_t\}_{t \geq 0} \). Since \( \overline{\eta}_0 = \eta^{\delta_1}_0 = \delta_1, \{\overline{\eta}_t\}_{t \geq 0} \) and \( \{\eta_t^{\delta_1}\}_{t \geq 0} \) have the same distribution.

**Lemma 4.2.** When \( \zeta_0 = \zeta \) where \( \zeta(x) > 0 \) for each \( x \in \mathbb{Z}^d \), then there exists \( C(\lambda, d, \zeta) > 0 \) such that
\[
E_{\lambda,d} \zeta_t(0) \geq C(\lambda, d, \zeta) t^{-\frac{d}{4}}
\]
for any \( t \geq 0 \).
Proof. According to the generator of \( \{\zeta_t\}_{t \geq 0} \) given in (4.2) and Theorem 9.1.27 of [6],

\[
\begin{align*}
\frac{d}{dt} E_{\lambda,d}\zeta_t(x) &= -E_{\lambda,d}\zeta_t(x) + \lambda \sum_{y: y \sim x} E_{\lambda,d}\zeta_t(y) + (1 - 2\lambda d) E_{\lambda,d}\zeta_t(x) \\
&= \lambda \sum_{y: y \sim x} E_{\lambda,d}\zeta_t(y) - 2\lambda d E_{\lambda,d}\zeta_t(x)
\end{align*}
\]

(4.3)

for each \( x \in \mathbb{Z}^d \).

Let \( Q = (q(x,y))_{x,y \in \mathbb{Z}^d} \) be the Q-matrix of the continuous time simple random walk on \( \mathbb{Z}^d \) such that

\[
q(x,y) = \begin{cases} 
\lambda & \text{if } y \sim x, \\
-2\lambda d & \text{if } y = x, \\
0 & \text{else},
\end{cases}
\]

then, by (4.3),

\[
E_{\lambda,d}\zeta_t = P_t \zeta_0,
\]

where

\[
P_t = (p_t(x,y))_{x,y \in \mathbb{Z}^d} = e^{tQ} = \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}.
\]

In other words, \( P_t \) is the transition function of the simple random walk with Q-matrix \( Q \).

According to classic theory of continuous time simple random walk on \( \mathbb{Z}^d \), there exists \( C > 0 \) such that

\[
p_t(O, O) \geq [C(\lambda t)^{-\frac{1}{d}}]^d
\]

for any \( t \geq 0 \), where \( C \) does not depend on \( \lambda \) and \( d \).

Therefore,

\[
E_{\lambda,d}\zeta_t(O) = \sum_{x \in \mathbb{Z}^d} p_t(O, x) \zeta_0(x) \geq p_t(O, O) \zeta_0(O) \\
\geq \zeta_0(O) [C(\lambda t)^{-\frac{1}{d}}]^d.
\]

Let \( C(\lambda, d, \zeta) = \zeta_0(O)C^d\lambda^d \), then the proof is completed.

We define \( F : [0, +\infty) \to [0, +\infty)^{\mathbb{Z}^d} \) as

\[
F_t(x) = E_{\lambda,d} [\zeta_t(x)]
\]

(4.4)

for any \( t \geq 0 \) and \( x \in \mathbb{Z}^d \). Then, the following lemma holds for \( F \).
Lemma 4.3. For any \( t \geq 0 \),
\[
\frac{d}{dt}F_t = GF_t,
\] (4.5)
where \( G \) is a \( \mathbb{Z}^d \times \mathbb{Z}^d \) matrix such that
\[
G(x, y) = \begin{cases} 
-4\lambda d & \text{if } x = y \text{ and } x \neq O, \\
2\lambda & \text{if } y \sim x \text{ and } x \neq O, \\
1 - 2\lambda d & \text{if } x = y = O, \\
2\lambda d & \text{if } x = O \text{ and } y = e_1, \\
2\lambda d & \text{if } x = O, \ z \sim e_1 \text{ and } z \neq O, \\
0 & \text{otherwise},
\end{cases}
\] (4.6)
where \( e_1 = (1, 0, 0, \ldots, 0) \).

Proof. \( \Box \) follows directly from the generator of \( \{\zeta_t\}_{t \geq 0} \) given in (4.2) and Theorem 9.3.1 of [6].

Lemma 4.4. If \( \lambda \) satisfies
\[
GH = \mu H
\]
for some \( \mu > 0 \) and \( H : \mathbb{Z}^d \to \mathbb{R} \) such that \( H(x) > 0 \) for each \( x \in \mathbb{Z}^d \), then
\[
I(\lambda, d) \geq -\mu.
\]

Proof. Let \( \zeta_0(x) = H(x) \) for each \( x \in \mathbb{Z}^d \), then, according to Lemma 4.1, Lemma 4.2 and H"older’s inequality,

\[
P_{\lambda, d}(H^0(0) = 1) = P_{\lambda, d}(\zeta_t(0) > 0) \geq \frac{[E_{\lambda, d}\zeta_t^2(O)]^2}{E_{\lambda, d}\zeta_t^2(O)} \geq \frac{C^2(\lambda, d, \zeta) t^{-d}}{F_t(O)}. \tag{4.7}
\]

We denote by \( \| \cdot \|_{\infty} \) the \( l_\infty \) norm on \( \mathbb{R}^{\mathbb{Z}^d} \) such that
\[
\| \zeta \|_{\infty} = \sup_{x \in \mathbb{Z}^d} |\zeta(x)|
\]
for any \( \zeta \in \mathbb{R}^{\mathbb{Z}^d} \).

By direct calculation, for any \( \zeta_1, \zeta_2 \in \mathbb{R}^{\mathbb{Z}^d} \) such that \( \| \zeta_1 \|_{\infty}, \| \zeta_2 \|_{\infty} < +\infty \),
\[
\| G\zeta_1 - G\zeta_2 \|_{\infty} \leq (1 + 8\lambda d + 4\lambda d^2)\| \zeta_1 - \zeta_2 \|_{\infty}. \tag{4.8}
\]
By (4.8) and classic Theory for ODE on Banach Space, ODE (4.5) has the unique solution such that
\[ F_t = \Gamma_tF_0, \] (4.9)
where
\[ \Gamma_t = (\gamma_t(x, y))_{x, y \in \mathbb{Z}^d} = \exp\{tG\} = \sum_{n=0}^{+\infty} \frac{t^n G^n}{n!}. \] (4.10)
(4.8) ensures the sum in (4.10) is finite. By (4.9),
\[ F_t(O) = \sum_{x \in \mathbb{Z}^d} \gamma_t(O, x)H(x), \] (4.11)
since \( F_0 = H. \)

Since \( H \) is the eigenvector of \( G \) with respect to the eigenvalue \( \mu \), \( H \) is also the eigenvector of \( \Gamma_t = e^{tG} \) with respect to the eigenvalue \( \exp\{t\mu\} \).

As a result,
\[ F_t(O) = \sum_{x \in \mathbb{Z}^d} \gamma_t(O, x)H(x) = e^{t\mu}H(O). \] (4.12)

By (4.7) and (4.12),
\[ P_{\lambda, d}(\eta_t^{\hat{s}}(O) = 1) \geq C^2(\lambda, d, \zeta) \frac{t^{-d}}{e^{-t\mu}}, \]
and
\[ I(\lambda, d) = \lim_{t \to +\infty} \frac{1}{t} \log P_{\lambda, d}(\eta_t^{\hat{s}}(O) = 1) \geq -\mu. \]

To search \( \lambda \) and \( \mu \) satisfies the condition in Lemma 4.4, we introduce the simple random walk on \( \mathbb{Z}^d \cup \{\Delta\} \), where \( \Delta \not\in \mathbb{Z}^d \) is an absorbed state.

For \( d \geq 1 \) and \( p \in [0, 1] \), let \( \{S_n(d, p) : n = 0, 1, 2, \ldots\} \) be simple random walk on \( \mathbb{Z}^d \cup \{\Delta\} \) with transition probability
\[
\begin{align*}
P(S_{n+1}(d, p) = y | S_n(d, p) = x) &= \frac{p}{2d}, \\
P(S_{n+1}(d, p) = \Delta | S_n(d, p) = x) &= 1 - p, \\
P(S_{n+1}(d, p) = \Delta | S_n(d, p) = \Delta) &= 1
\end{align*}
\] (4.13)
for \( n \geq 0 \), each \( x \in \mathbb{Z}^d \) and each \( y \sim x \).

For \( d \geq 1, \ p \in [0, 1] \) and \( x \in \mathbb{Z}^d \), we define
\[ \tau(d, p) = \inf\{n \geq 0 : S_n(d, p) = O\} \]
and
\[ R(x, d, p) = P(\tau(d, p) < +\infty | S_0(d, p) = x). \]

We will give \( H(x) \) with the form \( R(x, d, p) \). For this purpose, we need the following lemma.
Lemma 4.5. For $d \geq 1$ and each $x \in \mathbb{Z}^d$, $R(x, d, p)$ is continuous in $p$.

Proof. The conclusion is trivial for $x = O$. For $x \neq O$ and $0 \leq p_1 < p_2 \leq 1$, we construct a coupling for $\{S_n(d, p_1)\}_{n \geq 0}$ and $\{S_n(d, p_2)\}_{n \geq 0}$ with $S_0(d, p_1) = S_0(d, p_2) = x$ such that $S_n(d, p_1) = \triangle$ or $S_n(d, p_1) = S_n(d, p_2) \neq \triangle$ for each $n \geq 1$.

The transition probability matrix $\hat{P}$ of the coupling process $\{S_n(d, p_1), S_n(d, p_2)\}$ is given by

$$
\hat{P}((x_1, y_1), (x_2, y_2)) = \begin{cases} 
\frac{p_1}{2d} & \text{if } x_1 = y_1 \neq \triangle, x_2 = y_2 \sim x_1, \\
1 - p_2 & \text{if } x_1 = y_1 \neq \triangle, x_2 = y_2 = \triangle, \\
\frac{p_2 - p_1}{2d} & \text{if } x_1 = y_1 \neq \triangle, x_2 = \triangle, y_2 \sim y_1, \\
\frac{p_2}{2d} & \text{if } x_1 = \triangle, y_1 \neq \triangle, x_2 = \triangle, y_2 \sim y_1, \\
1 - p_2 & \text{if } x_1 = \triangle, y_1 \neq \triangle, x_2 = y_2 = \triangle, \\
0 & \text{otherwise}. 
\end{cases}
$$ (4.14)

It is easy to check that $\hat{P}$ gives a coupling of $S_n(d, p_1)$ and $S_n(d, p_2)$ by direct calculation. The coupling ensures that $S_n(d, p_2) = S_n(d, p_1) \neq \triangle$ when $S_n(d, p_1) \neq \triangle$.

As a result, conditioned on $S_0(d, p_1) = S_0(d, p_2) = x$,

$$
R(x, d, p_2) - R(x, d, p_1) = P(\tau(d, p_1) = +\infty, \tau(d, p_2) < +\infty) \leq P(\exists n > 0, S_n(d, p_1) = \triangle, S_n(d, p_2) \neq \triangle). 
$$ (4.15)

Let

$$
\beta = \inf\{n \geq 1 : S_n(d, p_1) = \triangle, S_n(d, p_2) \neq \triangle\},
$$

then, by (4.14) and (4.15),

$$
R(x, d, p_2) - R(x, d, p_1) \leq P(\beta < +\infty) = \sum_{l=1}^{+\infty} P(\beta = l) = \sum_{l=1}^{+\infty} P(\beta > l - 1, S_l(d, p_1) = \triangle, S_l(d, p_2) \neq \triangle) = \sum_{l=1}^{+\infty} P(S_{l-1}(d, p_1) \neq \triangle)(p_2 - p_1) = \sum_{l=1}^{+\infty} p_1^{l-1}(p_2 - p_1) = \frac{p_2 - p_1}{1 - p_1}. 
$$ (4.16)

Lemma 4.5 follows from (4.16). \hfill \Box

Now we give a lower bound for $I(\lambda, d)$. 
Lemma 4.6. For each $d \geq 1$ and $\lambda < \frac{1}{2d}$, there exists unique $p = p(\lambda, d) \in (0, 1)$ such that
\[ 1 + 2\lambda d R(e_1, d, p) = \frac{4\lambda d}{p} [1 - d R(e, d, p)]. \] (4.17)
Furthermore,
\[ I(\lambda, d) \geq -4\lambda d \left[ \frac{1}{p(\lambda, d)} - 1 \right] \] (4.18)
for $\lambda < \frac{1}{2d}$.

Proof. For $p \in (0, 1]$, we define
\[ K(p) = \frac{4\lambda d}{p} \left[ 1 - d R(e_1, d, p) \right] - 1 - 2\lambda d R(e_1, d, p). \]
It is obviously that $K(p)$ is decreasing in $p$. By Lemma 4.5, $K(p)$ is continuous in $p$.

Since $\lambda < \frac{1}{2d}$ and $R(e_1, d, 1) \geq P(S_1(d, 1) = O|S_0(d, 1) = e_1) = 1/2d$,
\[ K(1) < 0. \] (4.19)
Since $R(e_1, d, 0) = 0$,
\[ \lim_{p \to 0} K(p) = +\infty. \] (4.20)
The existence and uniqueness of $p(\lambda, d)$ follows from (4.19), (4.20) and the fact that $K(p)$ is continuous and decreasing in $p$.

Let $\mu = 4\lambda d \left[ 1/p(\lambda, d) - 1 \right]$, $H(x) = R(x, d, p(\lambda, d))$ for each $x \in \mathbb{Z}^d$, then according to the fact that $p(\lambda, d)$ satisfies (4.17) and
\[ R(x, d, p) = \frac{p}{2d} \sum_{y: y \sim x} R(y, d, p) \]
for each $x \neq O$, it is easy to check that
\[ GH = \mu H. \]
As a result, (4.18) follows from Lemma 4.4.

To give a limit theorem of $p(\lambda, d)$, we need the following lemma.

Lemma 4.7. For $\{p_d\}_{d=1,2,...}$ such that $p_d \in (0, 1)$ for each $d \geq 1$, if
\[ \lim_{d \to +\infty} p_d = c, \]
then
\[ \lim_{d \to +\infty} 2d R(e_1, d, p_d) = c. \]
Proof.

\[
R(e_1, d, p_d) \geq P(\tau(d, p_d) = 1 | S_0(d, p_d) = e_1) \\
= P(S_1(d, p_d) = 0 | S_0(d, p_d) = e_1) = \frac{p_d}{2d}.
\] (4.21)

On the other hand,

\[
R(e_1, d, p_d) = \frac{p_d}{2d} + P(2 \leq \tau(d, p_d) < +\infty | S_0(d, p_d) = e_1) \\
\leq \frac{p_d}{2d} + P(2 \leq \tau(d, 1) < +\infty | S_0(d, 1) = e_1).
\] (4.22)

According to Lemma 5.3 of [11],

\[
\lim_{d \to +\infty} dP(2 \leq \tau(d, 1) < +\infty | S_0(d, 1) = e_1) = 0.
\] (4.23)

Lemma 4.7 follows from (4.21), (4.22) and (4.23). \(\square\)

Finally, we give the proof of \(\lim \inf_{d \to +\infty} I(\lambda/d, d) \geq 2\lambda - 1\) for \(\lambda \in (0, 1/2)\).

Proof. For any \(\lambda \in (0, 1/2)\), we define

\[
\overline{\tau}(\lambda) = \limsup_{d \to +\infty} p(\lambda/d, d)
\]

and

\[
\underline{c}(\lambda) = \liminf_{d \to +\infty} p(\lambda/d, d).
\]

Then by (4.17) and Lemma 4.7

\[
1 = \frac{4\lambda}{\overline{\tau}(\lambda)} \left[ 1 - \frac{\overline{\tau}(\lambda)}{2} \right]
\]

and

\[
1 = \frac{4\lambda}{\underline{c}(\lambda)} \left[ 1 - \frac{\underline{c}(\lambda)}{2} \right].
\]

Therefore,

\[
\overline{\tau}(\lambda) = \underline{c}(\lambda) = c(\lambda) = \frac{4\lambda}{1 + 2\lambda}
\]

and hence

\[
\lim_{d \to +\infty} p(\lambda/d, d) = c(\lambda).
\] (4.24)

By (4.18) and (4.24),

\[
\lim \inf_{d \to +\infty} I(\lambda/d, d) \geq -4\lambda \left[ \frac{1}{c(\lambda)} - 1 \right] = 2\lambda - 1
\]
for $\lambda \in (0, 1/2)$.

To finish the proof of Theorem 2.2 we only need to deal with the case where $\lambda = 1/2$.

**Proof of Theorem 2.2** For $\lambda \in (0, 1/2)$, we have shown that

$$\limsup_{d \to +\infty} I(\lambda/d, d) \leq 2\lambda - 1$$

in Section 3 and

$$\liminf_{d \to +\infty} I(\lambda/d, d) \geq 2\lambda - 1$$

in this section. Therefore,

$$\lim_{d \to +\infty} I(\lambda/d, d) = 2\lambda - 1$$

(4.25)

for $\lambda \in (0, 1/2)$.

In Section 3 we show that

$$\lim_{d \to +\infty} I(\lambda/d, d) = 0$$

(4.26)

for $\lambda > 1/2$. It is obviously that $I(\lambda, d)$ is increasing in $\lambda$. Therefore, by (4.25) and (4.26),

$$\lim_{d \to +\infty} I(1/2d, d) = 0.$$  

(4.27)

Theorem 2.2 follows from (4.25), (4.26) and (4.27).

Now the whole proof of Theorem 2.2 is completed. Furthermore, we show that

$$-4\lambda d\left[\frac{1}{p(\lambda, d)} - 1\right] \leq I(\lambda, d) \leq 2\lambda d - 1$$

for $\lambda < 1/2d$, where $p(\lambda, d)$ is the unique solution to

$$1 + 2\lambda dR(e_1, d, p) = \frac{4\lambda d}{p} [1 - dR(e, d, p)].$$

We give a sketch for the proof of Theorem 2.3.

**Proof of 2.3** We only need to show that $\liminf_{d \to +\infty} J(\lambda/d, d) \geq 2\lambda - 1$ for $\lambda \in (0, 1/2)$.
Let \( \{ \alpha_t \}_{t \geq 0} \) be Markov processes with state space \([0, +\infty)^d\) such that the generator \( \tilde{\Omega} \) of \( \{ \alpha_t \}_{t \geq 0} \) is given by

\[
\tilde{\Omega} f(\alpha) = \sum_{x \in \mathbb{Z}^d} \left[ f(\alpha_{x,0}) - f(\alpha) \right] + \lambda \sum_{x \in \mathbb{Z}^d} \sum_{y : y \sim x} \left[ f(\alpha_{x,\alpha(x)+\alpha(y)} - f(\alpha) \right] + \sum_{x \in \mathbb{Z}^d} (1 - 2\lambda d) f'_{x}(\alpha) \alpha(x),
\]

where

\[
\alpha_{x,m}(y) = \begin{cases} 
\alpha(y) & \text{if } y \neq x, \\
m & \text{if } y = x
\end{cases}
\]

for \( x \in \mathbb{Z}^d \) and \( m \geq 0 \).

When \( \alpha_0(x) > 0 \) for each \( x \in \mathbb{Z}^d \), then according to a similar analysis with that in the proof of Lemma 4.1, there exists unique \( \tilde{\beta}_t = \tilde{\beta}_t(\lambda, d) \) such that

\[
\frac{4\lambda d}{\tilde{p}} - 2\lambda d = 1 + 4\lambda d R(e_1, \tilde{p}).
\]

We define \( \tilde{F} : [0, +\infty) \to [0, +\infty)^d \) such that

\[
\tilde{F}_t(x) = E_{\lambda,d}[\alpha_t(O)\alpha_t(x)]
\]

for \( x \in \mathbb{Z}^d \) and \( t \geq 0 \). Then,

\[
\frac{d}{dt} \tilde{F}_t = \tilde{G} \tilde{F}_t,
\]

where \( \tilde{G} \) is a \( \mathbb{Z}^d \times \mathbb{Z}^d \) matrix such that

\[
\tilde{G}(x, y) = \begin{cases} 
2\lambda & \text{if } x \neq O, y \sim x, \\
-4\lambda d & \text{if } x \neq O, y = x, \\
1 - 2\lambda d & \text{if } x = y = O, \\
4\lambda d & \text{if } x = O, y = e_1, \\
0 & \text{otherwise}.
\end{cases}
\]

When \( \lambda < \frac{1}{2d} \), according to a similar analysis with that in the proof of Lemma 4.6, there exists unique \( \tilde{p} = \tilde{p}(\lambda, d) \) such that

\[
\frac{4\lambda d}{\tilde{p}} - 2\lambda d = 1 + 4\lambda d R(e_1, \tilde{p}).
\]
Let \( \tilde{\mu} = 4\lambda d [1/\tilde{p} - 1] \) and \( \tilde{H}(x) = R(x, d, \tilde{p}) \) for each \( x \in \mathbb{Z}^d \), then \( \tilde{G} \tilde{H} = \tilde{\mu} \tilde{H} \).

According to (4.28) and a similar analysis with that in the proof of Lemma 4.4,

\[ J(\lambda, d) \geq -\tilde{\mu} \]

for \( \lambda < \frac{1}{2} \).

Since \( R(e_1, d, \tilde{p}) \leq R(e_1, d, 1) \to 0 \) as \( d \to +\infty \),

\[ \lim_{d \to +\infty} \tilde{p}(\lambda/d, d) = \frac{4\lambda}{1 + 2\lambda} \]

for \( \lambda < 1/2 \). As a result,

\[ \liminf_{d \to +\infty} J(\lambda/d, d) \geq -4\lambda \left[ \frac{1 + 2\lambda}{4\lambda} - 1 \right] = 2\lambda - 1 \]

for \( \lambda < 1/2 \).

\[ \square \]

Acknowledgments. The author is grateful to the financial support from the National Natural Science Foundation of China with grant number 11171342.

References

[1] Andjel, E. D., Liggett, T. M. and Mountford, T. (1992). Clustering in one-dimensional threshold voter models. *Stochastic Processes and Their Applications* **42**, 73-90.

[2] Cox, J. T. and Durrett, R. (1991). Nonlinear voter models. In *Random Walks, Brownian Motion and Interacting Particle Systems*. A Festschrift in Honor of Frank Spitzer 189-201. Birkhäuser, Boston.

[3] Fontes, L. R., Schonmann, R. H. (2008). Threshold \( \theta \geq 2 \) contact processes on homogeneous trees. *Probability Theory and Related Fields* **141**, 513-541.

[4] Griffeath, D. (1983). The Binary Contact Path Process. *The Annals of Probability* **11** 692-705.

[5] Handjani, S. (1999). The complete convergence theorem for coexistent threshold voter models. *The Annals of Probability* **27** 226-245.

[6] Liggett, T. M. (1985). *Interacting Particle Systems*. Springer, New York.
[7] Liggett, T. M. (1994). Coexistence in threshold voter models. *The Annals of Probability*. **22**, 764-802.

[8] Liggett, T. M. (1999). *Stochastic interacting systems: contact, voter and exclusion processes*. Springer, New York.

[9] Mountford, T. and Schonmann, R. H. (2009) The survival of large dimensional threshold contact processes. *The Annals of Probability* **37**, 1483-1501.

[10] Xue, XF. (2012). Critical density points for threshold voter models on homogeneous trees. *Journal of Statistical Physics*. **146**, 423-433.

[11] Xue, XF. (2014). Asymptotic behavior of critical infection rates for threshold-one contact processes on lattices and regular trees. *Journal of Theoretical Probability*. Published online on February 2014.

[12] Xue, XF. (2015). Fluid limit of threshold voter models on tori. *Journal of Statistical Physics*. Published online on January 2015.