Wonders of chaos for wireless communication

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Driven by today’s huge demand for data, there is a desire to develop wireless communication systems that can handle several sources, each using different frequencies of the spectrum. Moreover, information must be transmitted fast, reliably, and lightly (low computational complexity). Chaos with its fascinating features has already provided promising solutions for speed, reliability and weight in a single transmitter and receiver configuration, operating within the same bandwidth. The main purpose of this letter is to show that chaos can also naturally provide a solution for a multi-source and multi-frequency wireless communication. I demonstrate that information combined from and to multiple users operating with different base frequencies and that is carried by chaotic waveforms is fully preserved after transmission in the wireless channel, and it can be trivially decoded with low probability of errors. This property that chaotic signals have is consequence of an equivalence principle that deterministic chaotic systems possess that permits that information combined from different sources can be effectively described by a single source. Finally, I provide simulations illustrating the performance of a proposed multi-user and multi-frequency communication system in prototyped wireless configurations.

Chaos has intrinsic properties that makes it attractive to sustain the modern design of communication systems. Take $x(t)$ to represent a controlled chaotic signal and that encodes information from a single transmitter. Let $r(t)$ to represent the transformed signal that is received. Chaos has offered communication systems whose information capacity could remain invariant by a small increase in the noise level, could be robust to filtering and multipath propagation, intrinsically present in the wireless communication. Decoding of $r(t)$ can be trivial, with the use of a simple threshold, with low-probability of errors. These properties of both signals $r(t)$ and $x(t)$ can be derived. Moreover, these systems have matched filters whose output maximises the signal to noise ratio (SNR) of $r(t)$, thus offering a practical and reliable way to decode transmitted information. Chaos allows for integrated communication protocols, it offers viable solutions for the wireless underwater communication, and simultaneously radar-communication. Chaotic communication has been experimentally shown to achieve higher bit rate in a commercial wired fiber-optic channel, and lower Bit-Error-Rate (BER) than conventional wireless non-chaotic baseband waveform methods. Moreover, chaos-based communication only requires equipment that is compatible with the today commonly-used ones.

Despite all these wonders that chaos offers for communication, for its broad adoption as a native signal to support wireless networked communication systems such as the Internet of Things or 5G, it remains to be shown that chaos can naturally allow for communication systems that operate in a multi-transmitter/receiver and multi-frequency scenario. In a scenario where the received signal $r(t)$ is composed by chaotic signals of two transmitters $x^{(1)}(t)$ and $x^{(2)}(t)$, each signal operating with different frequency bandwidths and each encoding different information contents, the scientific challenge would be to give $r(t) = \alpha^{(1)}x^{(1)}(t) + \alpha^{(2)}x^{(2)}(t) + w(t)$, with $\alpha^{(i)} \in \mathbb{R}$ and $w(t)$ representing Additive White Gaussian Noise (AWGS), is it possible to decompose both signals, $x^{(1)}(t)$ and $x^{(2)}(t)$, out of the received signal $r(t)$?

In this letter, I show that for the no noise scenario, the spectrum of positive Lyapunov exponents of $r(t)$ is the union of the set of the positive Lyapunov exponents (LEs) of both signals $x^{(1)}(t)$ and $x^{(2)}(t)$. And what is more, for the system proposed in the work of [11], the information content of $r(t)$ preserves all the information carried by both signals. Moreover, it is possible to determine appropriate power gains $\alpha^{(i)}$ such that the information content carried by the composed signal $r(t)$ can be trivially decoded by a simple threshold, with low-probability of errors. These properties that chaotic signals have can be demonstrated thanks to an equivalence principle based on fundamental properties of deterministic chaotic systems that permits that the combined signal from a configuration of several users can be effectively described by a signal departing from a single user.

I also show that the communication system proposed in the work of [11] can be roughly approximated by the unfolded-Baker’s map. This understanding provides insights into this system’s dynamical mechanisms and permits the conclusion that the matched filter of a single user can be used to extract its transmitted signal, from a received combined signal of several sources. Moreover, this low-dimension map description (exact mathematical description is provided in [24]) will permit academics and engineers to study complex and large configurations of wireless communication networks, in a computationally inexpensive manner. I then proceed to show that the filter of the approximated system is stable, and it is...
appropriate for medium with high levels of noise, because it has no positive LEs. Finally, I study the information capacity of the proposed communication system in prototype wireless network configurations.

The analysis will focus on two prototype wireless communication configurations: the uplink and the downlink. In the uplink communication, several users transmit their signals to a base station antenna (BS). In the downlink communication, a BS sends 1 signal containing information to be decoded by several users.

**Encoding:** A wonder of chaotic oscillations for communication is the system proposed in Ref. [11]. Originally, the system was set to operate with a frequency \( f_0 = 1 \), a natural period \( T_0 = 1 \), and an angular frequency \( \omega_0 = 2\pi \). With an appropriate rescaling of time to a new time-frame \( dt' = \gamma dt \), it can be rewritten as

\[
\ddot{x} - 2\beta \dot{x} + (\omega_0^2 + \beta^2)(x - s(t)) = 0, \tag{1}
\]

where \( s(t) \in (-1, 1) \) is a 2-symbols alphabet discrete state that switches value by the signum function \( s(t) = x(t)/|x(t)| \), whenever \( |x(t)| < 1 \) and \( \dot{x} = 0 \). If the information to be communicated is the binary stream \( b = \{b_0, b_1, b_2, \ldots\} \) (\( b_n \in \{0, 1\} \)) a signal can be created such that \( s(t) = (2b_n - 1) \), for \( nT \leq t < (n + 1)T \). In this new time-frame, the natural frequency is \( f(\gamma) = (1/\gamma)f_0(\gamma = 1) \) (angular frequency equals \( 2\pi f \)), or also \( f(\gamma) = (1/\gamma) \), the period \( T(\gamma) = 1/f(\gamma) = \gamma T_0(\gamma = 1) \), and \( \beta(\gamma) = \beta(\gamma = 1)/\gamma = \beta(1)f(\gamma) \), where \( 0 < \beta(1) \leq \ln(2) \). Any solution of the original non-rescaled dynamics \( \gamma(1) \) can be placed in the new time-frame by substituting \( \beta(1) \) by \( \beta(\gamma)/f(\gamma) \), which for simplification we denote as \( \beta/f \). Details of Eq. (1) can be seen in Sec. I of Sup. Material (SM).

The received signal in the noiseless wireless channel from user \( k \) can be modelled by

\[
r^{(k)}(t) = \sum_{l=0}^{L-1} \alpha_l(\gamma(1))x(t - \tau_l), \tag{2}
\]

where there are \( L \) propagation paths, each with an attenuation factor of \( \alpha_l \) and a time-delay \( \tau_l \) for the signal to arrive to the receiver along the path \( l \) (with \( 0 = \tau_0 < \tau_2 < \ldots < \tau_{L-1} \)), and \( \gamma(1) \) is an equalizing power gain. The noisy channel can thus be modelled by \( r(t) = w(t) \), where \( w(t) \) is an AWGN.

I propose a chaos-based communication system, named ”WiChaos”, that allows for multi-user communication, where each user operates with its own natural frequency. It is assumed other constraints of the wireless medium are present, such as multipath propagation and AWGN. WiChaos with 1 BS can be modeled by

\[
O(t)_u = \sum_{k=1}^{N} \sum_{l=0}^{L-1} \alpha_l^{(k)}(\gamma) \gamma(1) x(t - \tau_l) + w(t) \tag{3}
\]

\[
O^{(m)}(t)_d = \sum_{l=0}^{L-1} \alpha_l^{(m)}(\gamma) \gamma(1) x(t - \tau_l^{(m)}) + w^{(m)}
\]

\( O(t)_u \) represents the signal received at BS from all users in the uplink and \( O^{(m)}(t)_d \) that received by user \( m \) from the BS in the downlink. \( w(t) \) represents an AGWN at the base station, and \( w^{(m)}(t) \), for \( m = 1, \ldots, N \) represents AGWN at the user. \( \alpha_l^{(k)} \) is the attenuation factor between the BS and the user \( k \) along path \( l \), and \( \gamma(1) \) and \( \gamma(1) \) are power gains. \( L(1) \) are the number of propagation paths between user \( k \) and the BS.

For the curious reader, if 2 propagating paths would have been considered in the previous equations, one would have that \( r^{(1)}_{n+1} = 2r^{(1)}_{n} - \alpha_0^{(1)} s^{(1)}_{n} - \alpha_1^{(1)} s^{(1)}_{n-1} \).
At time \((n+1)T\), the signal received by BS from user \(k=2\) as a function of the signal received at \(nT\) is

\[
\begin{align*}
\gamma_{2n+2}^{(2)} &= 4\gamma_{2n}^{(2)} - \alpha_{0}^{(2)}[2\gamma_{2n}^{(2)} + s_{2n+1}^{(2)}],
\end{align*}
\]

where the \(\gamma_{2n}^{(2)}\) represents the value of \(\gamma^{(2)}(t=nT)\) (recall that at each period \(T\), user 2 chaotic system completes two full cycles). Notice that the LE of Eq. (9) per period (\(T\)) is equal to \(\ln(4)\), which is twice the LE of Eq. (8). This is because user 2 has a frequency twice larger than that of user 1 [20]. Since these two maps are full shift, their LE equals their Shannon Entropy, so their LE represents the encoding capacity (in units of nepit). Doing their LE equals their Shannon Entropy, so their LE represents the encoding capacity (in units of nepit). Doing

\[
\begin{align*}
\gamma^{(k)} &= 1/\alpha^{(k)},
\end{align*}
\]

for \(k=1\), Eq. (8) is simply the Bernoulli shift map, representing the discrete dynamics of user 1 (the signal received after equalizing for the attenuation), and Eq. (11) is the second iteration of the shift map representing the discrete dynamics of user 2, (after equalizing the attenuation). Figure 1(A)-(B) shows in red dots solutions for Eqs. (10) and (11), respectively. Corresponding return maps of the discrete set of points \(x_{n}^{(k)}\) constructed directly from the continuous solution of Eq. (11) with frequency given by \(f^{(k)} = kf\) by taking points at the time \(t = nT\), and doing the normalization as before \(x_{n}^{(k)} = 2x_{n}^{(k)} - 1\) (so, \(x_{n}^{(k)} \in [0,1]\)) is shown by the black crosses. The combined received signal at time \(nT\) and \(O_{n}\). The received return map, for \(w_{n} = 0\), is given by

\[
\begin{align*}
O_{n+1} &= 4O_{n} - 2\gamma^{(1)}u_{n}^{(1)} - 2\gamma^{(2)}b_{n}^{(2)} - \gamma^{(1)}b_{n}^{(1)},
\end{align*}
\]

where Eq. (14) is just Eq. (10). The system of Eqs. (13) and (14) has two distinct positive LEs, one along the direction \(v^{(1)} = (0 1)\) associated with the user 1 and equal to \(\chi^{(1)} = \ln(2)\) nepit per time \(T\) and another along the direction \(v^{(2)} = (1 0)\), which can be associated with the user 2 and equals \(\chi^{(2)} = \ln(4)\) nepit per time \(T\). To calculate the LEs of this system (see [25]) we consider the expansion of a unitary basis of orthogonal perturbation vectors \(v\) and calculate them by

\[
\chi = \lim_{n \to \infty} \frac{1}{n} \ln \|M \cdot v\|,
\]

where \(\|v\|\) is the norm of vector \(v\), \(M = J^{n}\), and

\[
J = \begin{pmatrix} 4 & -2\gamma^{(1)} \\ 0 & 2 \end{pmatrix}.
\]

Thus, combining chaotic signals with different frequencies as in Eq. (12) preserves the spectra of LEs of the signals from the users alone. This map is a full shift and will contain all combinations of symbolic sequences possible from both users. This will be shown further in our simulations. Consequently, the information received is equal to the sum of the information transmitted by both users, for the no noise scenario. Preservation of the spectrum of the LEs in the combined signal is a universal property of chaos. Demonstration is provided in Sec. IV of SM. This demonstration uses an equivalence principle.

Thanks to the deterministic properties of chaos, every wireless communication network with several users can be made equivalent to a single user in the presence of several imaginary propagating paths. Attenuation and power gain factors need to be recalculated to compensate for a signal that is in reality departing from user 2 but that is being effectively described as departing from user 1. Suppose the 2 users case, both with the same frequency \(f^{(k)} = f\), in the uplink scenario. The trajectory of user 2 at a given time \(t\), \(x^{(2)}(t)\), can be described in terms of the trajectory of the user 1 at a given time \(t - \tau\). So, Eq. (3) can be simply written as

\[
O(t) = \sum_{l=0}^{L} A_{l} x^{(2)}(t) + w(t).
\]

In practice, \(\tau\) can be very small, because of the sensitivity to the initial conditions and transitivity of chaos. For a small \(\tau\) and \(\epsilon\) it is true that \(|x^{(2)}(t) - x^{(1)}(t - \tau)| \leq \epsilon\), regardless of \(t\). This property of chaos is extremely valuable, since when extending the ideas of this work to arbitrarily large and complex communicating networks, one will want to be able to arrive to general expressions such as in Eqs. (13) and (14) to decode the information arriving at the BS. Details of how to use this principle to derive Eqs. (13) and (14) for two users with \(f^{(2)} = 2f^{(1)}\), and also when \(f^{(2)} = f^{(1)}\) are shown in Sec. V of SM.

In order to avoid interference, allowing one to discover the symbols \(b^{(1)}\) and \(b^{(2)}\) only by observing the 2-dimensional return map of \(O_{n+1} \times O_{n}\), we need to
appropriately choose the power gains $\tilde{\gamma}^{(k)}$. Looking at the mapping in Eq. \eqref{eq:13}, the term $2f^{(2)}O_n$ represents a piecewise linear map with $2f^{(2)}$ branches. The spatial domain for each piece has a length denoted by $\zeta(f^{(2)})$. The term $(2f^{(2)} - 2)\tilde{\gamma}^{(1)}u_n^{(1)}$ representing the dynamics for the smallest oscillatory frequency is described by a piecewise linear map with $(2f^{(2)} - 2)$ branches. To avoid interference, the return map for this term must occupy a length $\zeta(f^{(1)})$ that is fully embedded within the domain for the dynamics representing higher order frequencies. Assuming that for a given number of users $N$, all frequencies $f^{(i)}$ with $i = 1, \ldots, N$ are used, this idea can be expressed in terms of an equation where

$$\zeta(f^{(i)}) = 2f^{(i)}\zeta(i - 1), \quad i = \{1, \ldots, N\}. \quad (16)$$

Then, $\tilde{\gamma}^{(k)} = \zeta(k)$, but for a received map within the interval $[0, 1]$, normalization of the values $\tilde{\gamma}^{(k)}$ by

$$\tilde{\gamma}^{(k)} = \frac{\zeta(k)}{\sum_{i=1}^{N} \zeta(f^{(i)})}. \quad (17)$$

For $N = 2$ and $\zeta(1) = 0.2$, then $\tilde{\gamma}^{(1)} = 0.2$ and $\tilde{\gamma}^{(2)} = 0.8$. Using these values for $\tilde{\gamma}^{(1)}$ and $\tilde{\gamma}^{(2)}$ in Eq. \eqref{eq:12} and considering an AWGN $w_n$ with SNR of 40dB (with respect to the power of $O_n$) produces the return map shown by points in Fig. 2(A), with 8 branches.

The choice of the power gains for the downlink configuration is similarly done as in the uplink configuration, taking into consideration that each user has its own noise level. This is shown in Sec. VI of SM.

Decoding: Communication based on chaos offers several alternatives for decoding. The less complex approach to decode the digital message by inspecting the properties of the received signal, for example by a state space partition analysis, or by doing a simple “thresholding” analysis. Assuming the received signal is modelled as in Eq. \eqref{eq:11}, the optimal 2-dimensional partition to decode the digital information is described by the same map of Eqs. \eqref{eq:13} and \eqref{eq:14} with a translation. For the case of 2 users in the uplink scenario, this translates into a 7 lines partition

$$O_{n+1}(j) = 4O_n(j) - T_j, \quad (18)$$

$$T_j = \frac{1}{2} \left[ 3\tilde{\gamma}^{(1)} + (j - 1)\tilde{\gamma}^{(2)} \right], \quad j = \{1, \ldots, 7\}.$$ 

These partition lines for $\tilde{\gamma}^{(1)} = 0.2$ and $\tilde{\gamma}^{(2)} = 0.8$ are shown by the coloured straight lines in Fig. 2(A).

A more sophisticated approach to decode information is based on a matched filter \eqref{eq:11}. Details of the fundamentals presented in the following can be seen in Sec. VII of SM. If the equations describing the dynamics of the transmitted chaotic signal (in this case Eq. \eqref{eq:11}) possess no negative Lyapunov exponents - as it is shown Sec. III of SM - attractor estimation of a noisily corrupted signal can be done using its time-inverse dynamics that is stable and possess no positive LEs (shown in Sec. XIII of SM). The evolution to the future of the time-inverse dynamics is described by a system of ODE hybrid equations obtained by the time-rescaling $d/dt' = -d/dt$ applied to Eq. \eqref{eq:11} resulting in

$$\ddot{y} + 2\beta\dot{y} + (\alpha^2 + \beta^2)[y - \eta(t)] = 0, \quad (19)$$

where the variable $y$ represents the $x$ in time-inverse, and as shown in Sec. VII of SM, the auxiliar input variable of the matched filter $\eta(t)$ if it is defined as $\eta(t) = x(t) - x(t - T)$ (defined as $\eta(t) = x(t + T) - x(t)$ in Ref. \eqref{eq:11}) can be roughly approximated to be equal to the symbol $s(t)$. As recognized in Ref. \eqref{eq:18}, making $\eta(t) = s(t)$ in Eq. \eqref{eq:19}, makes this equation to describe what was called “reverse time chaos” in Ref. \eqref{eq:27}.

Taking the values of $y$ at discrete times at $nT$, writing $y(nT) = y_n$, and defining the new variable for users 1 and 2 as before $y^{(1)} = 2z^{(1)} - 1$ and $y^{(2)} = 2z^{(2)} - 1$ if Eqs. \eqref{eq:10} and \eqref{eq:11} are map solutions of Eq. \eqref{eq:11} (in the rescaled coordinate system, with appropriate $\gamma$ gains) for user $k$ with frequencies $f^{(k)} = k$, their inverse mapping the solution of Eq. \eqref{eq:19} is given by

$$z_{n+1}^{(k)} = 2^{-k}\{z_n^{(k)} - [2^{k}u_n^{(k)}]\}, \quad \{2^{k}u_n^{(k)}\} = b_{n}^{(k)}, \quad (20)$$

This map can be derived simply defining $z_{n+1}^{(k)} = u_n^{(k)}$ and $z_{n}^{(k)} = u_{n+1}^{(k)}$. We always have that $[2^{k}u_n^{(k)}] = b_n^{(k)}$. So, for any $z_n^{(k)} \in [0, 1]$ and which can be simply chosen to be equal to the received combined signal $O_n$ (normalized such that $\{0, 1\}$), it is also true that

$$[2^{k}z_{n+1}^{(k)}] = [2^{k}u_{n+1}^{(k)}] = b_{n+1}^{(k)}. \quad (21)$$

So, if we represent an estimation of the transmitted symbol of user $k$ by $\tilde{s}_{n}^{(k)}$, then decoding of the transmitted symbol of user $k$ can be done by calculating $z_{n+1}^{(k)}$ using the inverse dynamics of the user $k$

$$z_{n+1}^{(k)} = 2^{-k}\{z_n^{(k)} - \tilde{s}_{n}^{(k)}\}. \quad (22)$$

and applying this value to Eq. \eqref{eq:21}. This means that the system formed by the variables $u_n^{(k)}$, $z_n^{(k)}$ is a generalization (for $k \neq 1$) of the unfolded Baker’s map \eqref{eq:23}, being described by a time-forward variable $u_n^{(k)}$ (the Bernoulli shift for $k=1$), and its backward variable component $z_n^{(k)}$.

Figure 2(B) demonstrates that it is possible to extract the signal of a user ($k=2$) from the combined signal, $O_n$ (Eq. \eqref{eq:12}), by setting in Eq. \eqref{eq:22} that $z_n^{(2)} = O_n$, and $\tilde{s}_n^{(k)} = b_n^{(k)}$. Even thought $u_n^{(2)} \neq z_n^{(2)}$, decoding Eq. \eqref{eq:21} is satisfied.

I can now do an analysis of the performance of the WiChaos, for both the uplink and the downlink configurations, for 2 users modelled by Eqs. \eqref{eq:10} and \eqref{eq:11} with power gains $\tilde{\gamma}^{(1)} = 0.2$ and $\tilde{\gamma}^{(2)} = 0.8$. The information capacity for the user $k$ (in bits per iteration) is given by

$$C^{(k)} = 0.5 \log_2 \left( 1 + SNR^{(k)} \right), \quad (23)$$

where $SNR^{(2)} = P^{(2)}/P^{(2)}$ is the signal-to-noise ratio of user 2, and $SNR^{(1)} = P^{(1)}/P^{(1)}$ is the signal-to-noise ratio of user 1. Moreover, $P^{(k)}$ is the power of user $k$. 

the Mutual Information, realized at the BS (or at the receivers), quantified by to be compared to the actual rate of information being at the users (for the downlink configuration, assumed of the noise at the BS (for the uplink configuration) or received signal with \( \tilde{\gamma} \)). In (B) one sees a solution of the unfolded Baker’s map, where horizontal axis shows trajectory points from Eq. (11) and vertical axis trajectory points from Eq. (21), for the user \( k=2 \). In (C) is shown \( \sum C \) against \( \sum I \).

FIG. 2. [Colour online] (A) points shows the return map of the received signal with \( \tilde{\gamma}^{(1)} = 0.2 \) and \( \tilde{\gamma}^{(2)} = 0.8 \), and the lines the partitions from which received symbols are estimated. Inside the parenthesis, the first symbol is from user 2 and the second symbol is from user 2. In (B) one sees a solution of the unfolded Baker’s map, where horizontal axis shows trajectory points from Eq. (11) and vertical axis trajectory points from Eq. (21), for the user \( k=2 \). In (C) is shown \( \sum C \) against \( \sum I \).

(they power of the signal \( \tilde{\gamma}^{(k)}u^{(k)} \)), and \( P^w \) the power of the noise at the BS (for the uplink configuration) or at the users (for the downlink configuration, assumed to be the same). The capacity of user 1 is calculated assuming that decoding of user 2 could successfully resolve the contribution of \( \tilde{\gamma}^{(2)}u^{(2)} \) to the received signal \( O_n \), by treating \( \tilde{\gamma}^{(1)}u^{(1)} \) as noise. These capacities have to be compared to the actual rate of information being realized at the BS (or at the receivers), quantified by the Mutual Information, \( I(b_n^{(1)};\tilde{b}_n^{(1)}) \) between the symbols transmitted \( (b_n^{(1)}) \) and the decoded symbols \( \tilde{b}_n^{(1)} \) estimated by using partition in Eq. (18), defined as usual by \( I(b_n^{(k)};\tilde{b}_n^{(k)}) = H(b_n^{(k)}) - H(b_n^{(k)}|\tilde{b}_n^{(k)}) \) where \( H(b_n^{(k)}) \) denotes the Shannon’s Entropy of the user \( k \) which is equal to the positive LE of the user \( k \), for \( \beta(\gamma = 1) = \ln (2) \), and \( H(b_n^{(k)}|\tilde{b}_n^{(k)}) \) is the conditional Entropy.

Figure 2(C) shows in red squares the full theoretical capacity given by \( \sum C = C^{(1)} + C^{(2)} \) against the rate of information decoded given by \( \sum I = I(b_n^{(1)} ; \tilde{b}_n^{(1)}) + I(b_n^{(2)} ; \tilde{b}_n^{(2)}) \), in black circles. As it is to be expected, the information rate received is equal to the one transmitted (both equal to 3bits/period) for low noise levels, tough smaller than the theoretical limit. Notice that this analysis was carried out using the map version of the matched filter (11) in Eq. (19), and as such lacks the powerful use of the negativeness of the LE to filter noise. Moreover, decoding used the trivial 2D threshold by Eq. (18), and not higher-dimensions reconstructions.

Concluding, chaos offers a fertile field to not only do fundamental research but also to the development of innovative technology in wireless communication. Further improvement for the rate of information could be achieved by adding more transmitters (or receivers) at the expense of reliability. One could also consider similar ideas as in [4], which would involve more post-processing, at the expense of weight. Post-processing would involve the resetting of initial conditions in Eq. (20) all the time, and then using the inverse dynamics up to some specified number of backward iterations to estimate the past of \( u_n^{(k)} \).

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I. THE CONTINUOUS HYBRID-DYNAMICS CHAOTIC WAVE SIGNAL GENERATOR

A wonder of chaotic oscillations for communication is the system proposed in Ref. [1]. As originally proposed, the system is operating in a time frame whose infinitesimal is denoted by $dt$, has a natural frequency $f_0 = 1$, a natural period $T_0 = 1$, and an angular frequency $\omega_0 = 2\pi$. With an appropriate rescaling of time to a new time-frame $df = \gamma dt$, it can be rewritten as

$$\ddot{x} - 2\beta \dot{x} + (\omega^2 + \beta^2)(x - s(t)) = 0,$$  \hspace{1cm} (1)

where $s(t) \in (-1, 1)$ is 2-symbols alphabet discrete state that switches value by the signum function $s(t) = x(t)/|x(t)|$, whenever $|x(t)| < 1$ and $\dot{x} = 0$. In this new time-frame, the natural frequency will be $f(\gamma) = (1/\gamma)f_0(\gamma = 1)$ (angular frequency equals $2\pi f$), or also $f(\gamma) = (1/\gamma)$, the period $T(\gamma) = 1/f(\gamma) = \gamma T_0(\gamma = 1)$, and $\beta(\gamma) = \beta(\gamma = 1)/\gamma = \beta(\gamma = 1)f(\gamma)$, where $0 < \beta(\gamma = 1) \leq \ln(2)$. Any solution of the original non-rescaled dynamics ($\gamma = 1$) can be placed in the new time-frame by substituting $\beta(\gamma = 1)$ by $\beta(\gamma)/f(\gamma)$, which for simplification we denote as $\beta/f$.

Equation (1) has an analytical solution that links its continuous form to its symbolic encoding, provided by the discrete state $s_n$ obtained by sampling the time at $t = n/T$, where $n = \lceil ft \rceil$ is the floor function that extracts the integer part of $ft$. This solution can be written in terms of an infinite sum of basis functions $\frac{x_n}{f}$.

$$x(t) = s_n + \sum_{i=0}^{\infty} s_i + n e^{-\beta/f} \left( \cos \omega t - \frac{\beta}{\omega} \sin \omega t \right).$$  \hspace{1cm} (2)

In this equation, $s_n$ represents the binary symbol associated to the time interval $nT \leq t < (n+1)T$, where $s_n = s(t = nT)$. Sampling the time at this same rate a discrete mapping of $x(t)$ can be constructed

$$x_n = e^{n\beta/f} \left( x_0 - (1 - e^{-\beta/f}) \sum_{i=0}^{n-1} s_i e^{-\beta/f} \right).$$  \hspace{1cm} (3)

Moreover, this solution can be written in terms of an infinite sum of basis function whose coefficients are the symbolic encoding of the analogical trajectory ($s_n$). This representation allows for the creation of a matched filter which receives as the input the signal $x(t)$ corrupted by white Gaussian noise (AWGN) and produces as the output an estimation of $x(t)$.

It offers in a single system all the benefits of both the analogical and digital approaches to communicate. The continuous signal copes with the physical medium, and the digital representation provides a translation of the chaotic signal to the digital language that we and machines understand. Supposing the information to be communicated is a binary stream $b = \{b_0, b_1, b_2, \ldots\}$ a signal can be created (the source encoding phase) such that $s(t) = (2b_n - 1)$, for $nT \leq t < (n+1)T$. The so called source encoding phase is thus based on a digital encoding. Moreover, the discrete variable $s_n$ is the symbolic encoding of the chaotic trajectory in the space $x, \dot{x}$.

This kind of hybrid chaotic system to communicate is not unique. Corron and Blakely [6] and Corron, Cooper and Blakely [7] have recently proposed other similar chaotic systems to that of Eq. (1). It was hypothesized in Ref. [6] that the optimal waveform that allows for a stable matched filter is a chaotic waveform. In this work, we provide support for this conjecture, but by showing that stability for the recovery of the information can be cask in terms of the non-existence of negative Lyapunov exponents of Eq. (1). This is shown in section III of this Supplementary Material. Had this system negative LEs, its inverse dynamics - the matched filter - responsible to filter out noise of the received signal would possess a positive LE making it to become unstable to small perturbations in the input of the matched filter (the received signal).

II. THE RETURN MAPPING OF THE RECEIVED SIGNAL FOR A SINGLE USER AND AN ARBITRARY NUMBER OF PROPAGATION PATHS, IN THE NOISELESS CHANNEL

In Eq. (2), $s_n$ represents the binary symbol associated to the time interval $nT \leq t < (n+1)T$, where $s_n = s(t = nT)$. The received signal in the noiseless wireless channel with a single user can be modelled by

$$r(t) = \sum_{l=0}^{L-1} a_l(x(t - \tau_l)), \hspace{1cm} (4)$$

where there are $L$ propagation paths, each with an attenuation factor of $a_l$ and a time-delay $\tau_l$ for the signal to arrive to the receiver along the path $l$ (with $0 = \tau_0 < \tau_2 < \ldots < \tau_{L-1}$).

To obtain a map solution for Eq. (4), we need to understand which symbol $s_n$ is associated to the time $t - \tau_l$. 

Supplementary Material of paper “Wonders of chaos for wireless communication”

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Let us define the time-translated variable

$$t' = t - \tau_l,$$  \hspace{1cm} (5)

where $t$ represents the global time for all elements involved in the communication, the time that a certain signal $x(t)$ was generated by a user from the chaotic system in Eq. \(1\), the “transmitter”. $t'$ represents the delayed-time. The clock of the user, the “receiver”, is at time $t$ but it receives the signal $r(t')$. The receiver decodes for the symbol $s_{n'}$, where

$$n' = \lfloor ft' \rfloor.$$  \hspace{1cm} (6)

The operator $\lfloor \cdot \rfloor$ represents the floor function. The transmitter constructs a map at times $t = n/f$, so Eqs. \(6\) and \(7\) can be written as

$$n' = \lfloor n - ft_1 \rfloor = n + \lfloor -ft_1 \rfloor,$$  \hspace{1cm} (7)

$$t' = \frac{n}{f} - \tau_1,$$  \hspace{1cm} (8)

where the operator $\lfloor \cdot \rfloor$ represents the ceiling function.

In the time-frame of $t'$, the solution in \(12\) multiplied by an arbitrary attenuation factor can be written as

$$\alpha_t x(t') = \alpha_t s_{n'} + \alpha_t \left\{ -s_{n'} + (1 - e^{-\beta/f}) \sum_{i=0}^{\infty} s_{i+n'} e^{-i\beta/f} \right\}$$

$$\times e^{\beta/f(t' - n'T)} \left\{ \cos \omega t' - \frac{\beta}{\omega} \sin \omega t' \right\}. \hspace{1cm} (9)$$

Let us understand what happens to the oscillatory term $\left( \cos \omega t' - \frac{\beta}{\omega} \sin \omega t' \right)$ in Eq. \(9\). Using Eq. \(8\), we obtain that

$$\cos (\omega t') - \frac{\beta}{\omega} \sin (\omega t') = \cos \left( 2\pi \frac{n}{T} \right) + \frac{\beta}{\omega} \sin \left( \frac{2\pi n}{T} \right). \hspace{1cm} (10)$$

So,

$$\alpha_t x(t') = \alpha_t s_{n'} + \alpha_t \kappa_l \left\{ -s_{n'} + (1 - e^{-\beta/f}) \sum_{i=0}^{\infty} s_{i+n'} e^{-i\beta/f} \right\}, \hspace{1cm} (11)$$

where

$$\kappa_l = e^{\beta/f(t' - n'T)} \left\{ \cos \left( 2\pi \frac{n}{T} \right) + \frac{\beta}{\omega} \sin \left( 2\pi \frac{n}{T} \right) \right\}. \hspace{1cm} (12)$$

Let us calculate the previously shown quantities for a delayed-time $t''$ one period ahead in time of $t'$:

$$t'' = t' + T = t' + \frac{1}{f},$$  \hspace{1cm} (13)

$$n'' = \lfloor ft'' \rfloor = 1 + n'.$$  \hspace{1cm} (14)

It can be also be written that

$$t'' - n''T = t' - n'T.$$  \hspace{1cm} (15)

This equation can be derived by doing $t'' - n''T = t' + \frac{1}{f} - n'T - T$.

Using Eqs. \(10\), \(13\), and \(15\), it is possible to obtain

$$\cos (\omega t'') - \frac{\beta}{\omega} \sin (\omega t'') = \cos (\omega t') - \frac{\beta}{\omega} \sin (\omega t'), \hspace{1cm} (16)$$

$$= \cos \left( 2\pi \frac{n}{T} \right) + \frac{\beta}{\omega} \sin \left( 2\pi \frac{n}{T} \right),$$

$$e^{\beta/f(t'' - n''T)} = e^{\beta/f(t' - n'T)}. \hspace{1cm} (17)$$

So, the attenuated signal at time $t''$ is given by

$$\alpha_t x(t'') = \alpha_t s_{n'+1} - \alpha_t \kappa_l \left\{ (1 - e^{-\beta/f}) \sum_{i=0}^{\infty} s_{i+n'+1} e^{-i\beta/f} \right\}. \hspace{1cm} (18)$$

Returning to Eq. \(11\), notice that by a manipulation of the terms inside the summation, it can be written as

$$\alpha_t x(t') = \alpha_t s_{n'} + \alpha_t \kappa_l \left\{ -s_{n'} + (1 - e^{-\beta/f}) \sum_{i=0}^{\infty} s_{i+n'} e^{-i\beta/f} \right\}$$

$$= \alpha_t s_{n'} + \alpha_t \kappa_l \left\{ -s_{n'} e^{-\beta/f} + (1 - e^{-\beta/f}) \sum_{i=0}^{\infty} s_{i+n'+1} e^{-i\beta/f} \right\}. \hspace{1cm} (19)$$

Multiplying this equation by $e^{\beta/f}$ results in

$$e^{\beta/f} \alpha_t x(t') = e^{\beta/f} \alpha_t s_{n'} - \alpha_t \kappa_l s_{n'} + \alpha_t \kappa_l \left\{ (1 - e^{-\beta/f}) \sum_{i=0}^{\infty} s_{i+n'+1} e^{-i\beta/f} \right\}. \hspace{1cm} (20)$$

The received signal at time $t$ and $t + T$ are then given by

$$r(t) = \sum_{i=0}^{L-1} \alpha_t x(t'), \hspace{1cm} (21)$$

$$r(t + T) = \sum_{i=0}^{L-1} \alpha_t x(t''). \hspace{1cm} (22)$$
If we observe the received signal only at discrete times \( t = nT \), and defining the discrete variable \( r(nT) \equiv r_n \), we obtain that

\[
e^{\beta f} r_n = \sum_{l=0}^{L-1} \alpha_l \left( e^{\beta f} s_{n'} - K_l s_{n'} + K_l A \right), \tag{23}
\]

\[
r_{n+1} = \sum_{l=0}^{L-1} \alpha_l \left( s_{n'+1} - K_l s_{n'+1} + K_l A \right), \tag{24}
\]

where

\[
K_l = e^{-\beta f (\tau_l + [-\tau_l/T]T)} \left\{ \cos \left(2\pi \frac{\tau_l}{T} \right) + \frac{\beta}{\omega} \sin \left(2\pi \frac{\tau_l}{T} \right) \right\}. \tag{25}
\]

The variable \( K_l \) is derived by noticing that

\[
e^{-\beta f (\tau_l + [-\tau_l/T]T)} = e^{\beta f (t' - n'T)}. \tag{26}
\]

Comparing Eqs. (23) and (24), we finally arrive at a return map for the received signal with multipath propagation

\[
r_{n+1} = e^{\beta f} r_n - \sum_{l=0}^{L-1} \alpha_l \left( e^{\beta f} s_{n'} - K_l s_{n'} - s_{n'+1} + s_{n'+1} K_l \right). \tag{27}
\]

### III. CALCULATION OF THE LYAPUNOV EXPONENTS OF THE CONTINUOUS HYBRID SYSTEM

I now proceed to estimate the Lyapunov exponents (LEs) of the continuous hybrid chaotic system:

\[
\ddot{x} - 2\beta \dot{x} + (\omega^2 + \beta^2)x - s(t) = 0, \tag{28}
\]

where \( s(t) \in (-1, 1) \) is 2-symbols alphabet discrete state that switches value by \( s(t) = x(t)/|x(t)| \), whenever \( |x(t)| < 1 \) and \( \dot{x} = 0 \). In this new time-frame, the natural frequency will be \( f = (1/\gamma) f_0 \) (angular frequency equals \( 2\pi f \)), the period \( T = \gamma T_0 = 1/f \), and \( \beta \leq f \theta \) and \( 0 < \theta \leq \ln(2) \).

Firstly, I transform the hybrid system so that it is fully described by a set of first order Ordinary Differential Equations (ODEs). In order to do so, I first define the variable \( y(t) = x(t) \). Then, notice that the discrete variable \( \dot{y} \) assumes either 1 or 0 values, and the switch to either one of the values happen when \( y = 0 \). So, \( \dot{s} = 0 \), except when \( \dot{x} = 0 \). Thus, \( s \) is the function of the 2 variables \( (x, y) \). If we were able to find a continuous description of \( s \), its first-time derivative would be described by \( \dot{s} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y \). Making the approximation that \( s \) only depends on \( x \), such that it is described only by the signum function \( s(x) = \frac{x}{|x|} \), we would have that \( \dot{s} = 2y\delta(x) \), if \( y = 0 \) or \( \dot{s} = 0 \), otherwise, \( \dot{s} \neq 0 \), where \( \delta(x) \) is the delta’s Dirac function. Notice also that \( \frac{\partial}{\partial x} \) is unlikely to assume a value different than zero, since when \( s \) switches values from -1 to 1 (or from 1 to -1) \( x > 0 \) (\( x < 0 \)).

After these considerations, it is possible to write an effective system of ODEs for the Hybrid system as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= 2\beta y - (\omega^2 + \beta^2)(x - s), \tag{29}
\end{align*}
\]

\[
\dot{s} = 0.
\]

When calculating the Lyapunov exponents of a system of ODEs, the interest lies with the variational equations of the system that describe how perturbations propagate along the trajectory. Defining the vector \( Y(t) = (x(t), y(t), s(t)) \), a perturbed trajectory \( X(t) \) can be described by \( X(t) = Y(t) + \delta(t) \). Then, the ODE system describing how perturbations propagate is then given by the variational equation

\[
\dot{\delta}(t) = J \delta(t),
\]

where \( J \) represents the Jacobi matrix of partial derivatives of the vector flow in Eq. (29), and it is given by

\[
J = \begin{pmatrix} 0 & 1 & 0 \\ -\omega^2 - \beta^2 & 2\beta & (\omega^2 + \beta^2) \end{pmatrix}.
\]

This is consistent with the idea that there is no propagation of perturbations along the direction of the variable \( s \), once the system is a forced one, and defined as to have \( s \) as either -1 or 1. Moreover, any continuous system must possess a null LE.

Notice also that whereas \( \dot{y} \) is affected by, the propagation of perturbations does not depend on whether \( s = 1 \) or \( s = -1 \). The interest thus is to calculate how perturbations propagate along the plan \((x, y)\), which would mean that \( \delta \in \mathbb{R}^2 \) and

\[
J = \begin{pmatrix} 0 & 1 & 0 \\ -\omega^2 - \beta^2 & 2\beta \end{pmatrix}.
\]

I then search for a general solution for \( \delta(t) \) as

\[
\delta(t) = \Psi(t) v,
\]

where \( v \) is a constant initial perturbation vector, and \( \Psi(t) \) is the fundamental solution matrix of the system in Eq. (30) whose columns are independent solutions of this system of differential equations.

The 1-dimensional Lyapunov exponent of the system (24) in the direction of \( v \) along a trajectory with initial condition \( Y_0 \) is given by

\[
\chi(Y_0, v) = \lim_{t \to \infty} \frac{1}{t} \ln ||\Psi(t, Y_0) \cdot v||.
\]

The spectra of LEs can be obtained by setting \( v \) to be equal to an eigenvalue \( w_i \) of the matrix \( \Psi(t, Y_0)^T \Psi(t, Y_0) \). More specifically \( \chi \), Eq. (34) is algebraically equal to

\[
\chi(Y_0, v) = \lim_{t \to \infty} \frac{1}{2t} \ln [v^T \cdot \Psi^T \cdot (t, Y_0) \cdot \Psi(t, Y_0) \cdot v].
\]

(35)
And, the LE in the direction of \( \mathbf{w}_i \) (\( \mathbf{v} = \mathbf{w}_i \)) can then be calculated by

\[
\chi(Y_0, \mathbf{w}_i) = \lim_{t \to \infty} \frac{1}{2t} \ln \left[ \mathbf{w}_i^T \cdot \Psi^T \cdot (t, Y_0) \cdot \Psi(t, Y_0) \cdot \mathbf{w}_i \right],
\]

\[
= \lim_{t \to \infty} \frac{1}{2t} \ln \left[ \mathbf{w}_i^T \cdot \Lambda_i \cdot \mathbf{w}_i \right],
\]

\[
= \lim_{t \to \infty} \frac{1}{2t} \ln (\Lambda_i),
\]

(36)

since \( \frac{1}{2t} \ln ||\mathbf{w}_i||^2 \to 0 \), as \( t \to \infty \). Oseledec’s multiplicative ergodic theorem \(^{[9]}\) guarantees that the limit in Eq. (36) exists and moreover it does not depend on the initial condition, \( Y_0 \), for typical initial conditions. It is now clear to see that all that matters to calculate the LE is the matrix \( \Psi(t, Y_0) \). Typically, all that matters are the eigenvalues \( \Lambda_i \) of \( \Psi(t, Y_0)^T \Psi(t, Y_0) \), but I will show that for this system, it is only required the knowledge of \( \Psi(t, Y_0) \).

The fundamental solution matrix \( \Psi(t, Y_0) \) is given by

\[
\Psi = e^{\beta t} \mathbf{V}(t, V_1, V_2, k_1, k_2),
\]

(37)

where \( \mathbf{V}(t, V_1, V_2) \) is a time oscillatory matrix function \( (V(t, V_1, V_2, k_1, k_2) \in \mathbb{R}^2) \) that is bounded, and also a function of the components \((V_1, V_2)\) of the complex eigen-vectors of \( \mathbf{J} \), and \( k_1 \) and \( k_2 \) are constants set by the initial value.

Any unitary initial perturbation vector \( \mathbf{v} \) chosen on the plan \((x, y)\) will by Eq. (34) grow its magnitude by \( e^{\beta t} \). The matrix \( \mathbf{V}(t, V_1, V_2, k_1, k_2) \) will only be responsible for its rotation. So, Eq. (34) produces the same degenerated LE equal to

\[
\chi = \beta.
\]

(38)

IV. THE EQUIVALENCE PRINCIPLE FOR FLOWS

Let us assume that information is being encoded by using the Rössler attractor. User 1 encodes its information in the variable \( x_1(t) \) and user 2 in the variable \( x_2(t) \). User 2 has a base frequency \( Q \) times the one from user 1. User 1 chaotic signal \( x_1(t) \) is generated by

\[
\begin{align*}
\dot{x}_1 &= -y_1 - z_1(t), \\
\dot{y}_1 &= x_1 + ay_1, \\
\dot{z}_1 &= b + z_1(x_1 - c).
\end{align*}
\]

(39)

User 2 uses the signal \( x_2 \) generated by:

\[
\begin{align*}
\dot{x}_2 &= Q[-y_2 - z_2(t)], \\
\dot{y}_2 &= Q[x_2 + ay_2], \\
\dot{z}_2 &= Q[b + z_2(x_2 - c)],
\end{align*}
\]

(40)

where \( a, b \) and \( c \) represent the usual parameters of the Rössler attractors. Notice however that this demonstration would be valid to any nonlinear system, the Rössler was chosen simply to make the following calculation straightforward to follow. The system of equations in (40) are already in the transformed time-frame, so that user 2 has a basis frequency \( Q \) times larger than user 1.

The transmitted composed signal can be represented by

\[
O(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t),
\]

(41)

where \( \alpha_1 \) and \( \alpha_2 \) are attenuation, or power gain factors.

The interest is to derive a systems of ODEs that describes all variables involved in ODE system describing the received signal

\[
\dot{O}(t) = \alpha_1 \dot{x}_1(t) + \alpha_2 \dot{x}_2(t).
\]

(42)

Determinism in chaos allows us to write that at a time \( t \) there exists a \( \tau \) such that \( x_2(t) = x_1(t - \tau) \). More generally, if user 2 has a basic frequency \( Q \) times that of user 1, its trajectory at time \( t + nQ\delta t \) \( (n \in \mathbb{N}) \) can be written in terms of the user 1’s trajectory by

\[
x_2(t + nQ\delta t) = x_1(t + nQ\delta t - \tau).
\]

(43)

I now define a new set of transformation variables given by

\[
\begin{align*}
\mathbf{w}_n^1(t) &= x_1(t - Qn\delta t), \\
\mathbf{w}_n^2(t) &= x_1(t - Qn\delta t), \\
\mathbf{w}_n^3(t) &= z_1(t - Qn\delta t),
\end{align*}
\]

(44-46)

where

\[
n = \{0, \ldots, N\},
\]

\[
NQ\delta t = \tau.
\]

(47-48)

Defining the vectors \( \mathbf{X}_n(t) = \{x_1(t), y_1(t), z_1(t)\} \) and \( \mathbf{W}_n(t) = \{\mathbf{w}_n^1(t), \mathbf{w}_n^2(t), \mathbf{w}_n^3(t)\} \), I can write Eqs. (44-46) in a compact form

\[
\mathbf{W}_n(t) = \mathbf{X}_n(t - Qn\delta t).
\]

(49)

Notice also that by defining the vector \( \mathbf{X}_2(t) = \{x_2(t), y_2(t), z_2(t)\} \), I can write that at time \( t \)

\[
\mathbf{W}_n(t) = \mathbf{X}_2(t - \tau + (N - Qn)\delta t),
\]

\[
= \mathbf{X}_2 \left( t + \frac{(N - Qn)\delta t}{Q} \right).
\]

(50-51)

To facilitate the following calculations, I express some terms of Eqs. (50) and (51) along the variables \( x_1(t) \) and \( x_2(t) \) for \( n = \{0, 1, 2, \ldots, N - 1, N\} \):

\[
\begin{align*}
\mathbf{w}_N^1(t) &= x_1(t - \tau) = x_2(t), \\
\mathbf{w}_{N-1}^1(t) &= x_1(t - \tau + Q\delta t) = x_2(t + \delta t),
\end{align*}
\]

(53-54)

\[
\begin{align*}
\vdots
\mathbf{w}_2^1(t) &= x_1(t - 2Q\delta t), \\
\mathbf{w}_1^1(t) &= x_1(t - 2\delta t),
\end{align*}
\]

(55-56)

\[
\begin{align*}
\mathbf{w}_0^1(t) &= x_1(t) = x_2(t + \tau),
\end{align*}
\]

(57)
I express the transformation variables considering a small displacement $Q\delta t$ in time from the time $t$:

$$W_n^1(t + Q\delta t) = X_1(t - \tau + (N - Q_n)\delta t + Q\delta t) \quad (58)$$

$$= X_2\left(t + \frac{(N - Q_n)}{Q}\delta t + Q\delta t\right). \quad (59)$$

Then, time derivatives can now be defined by

$$\dot{W}_{n-1}^1(t) = \frac{(W_{n-1}^1 - W_{n}^1(t))}{Q\delta t}, \quad n \in [2, \ldots, N]. \quad (60)$$

For the variables of the user 2, we have that

$$\dot{x}_2(t) = \frac{x_2(t + \delta t) - x_2(t)}{\delta t} = \frac{w_{N-1}^2(t) - w_{N-1}^2(t)}{\delta t} \quad (61)$$

which takes us to

$$\dot{W}_n^2(t) = \frac{(W_{n-1}^2 - W_n^2(t))}{\delta t}, \quad n \in [1, \ldots, N - 1]. \quad (62)$$

The original variables of the Rössler system for user 1 can be written in terms of the new transformed variables by

$$W_0^1 = X_1(t), \quad (63)$$

$$\dot{W}_0^1 = \dot{X}_1(t), \quad (64)$$

and for the user 2

$$W_0^2 = X_2(t), \quad (65)$$

$$\dot{W}_0^2 = \dot{X}_2(t). \quad (66)$$

Before I proceed, it is helpful to do some considerations, regarding this transformation of variables. Notice that $\dot{x}_2 = \frac{wx_1(t) - wx_2(t)}{Q\delta t}$ and $\dot{x}_1 = \frac{wx_1(t) - wx_1(t)}{Q\delta t}$. So, $\dot{x}_2 = Q\dot{t}_1$. Moreover, $W_n^1(t) = X_1(t - \tau + (N - Q_n)\delta t)$ and $W_n^2(t) = X_2(t + (N - Q_n)/Q\delta t)$. So, $W_n^1(t) = W_n^2(t)$, but their derivatives are not equal.

In the new variables, the ODE system describing the received signal from user 1 is described by

$$\dot{w}_0^1x = -w_0^1y - w_0^1z, \quad (67)$$

$$\dot{w}_0^1y = w_0^1x + aw_0^1y, \quad (68)$$

$$\dot{w}_0^1z = b + w_0^1y(w_0^1 - c), \quad (69)$$

and user 2 is described by:

$$\dot{w}_N^2x = -w_N^2y - w_N^2z, \quad (70)$$

$$\dot{w}_N^2y = w_N^2x + aw_N^2y, \quad (71)$$

$$\dot{w}_N^2z = b + w_N^2y(w_N^2 - c). \quad (72)$$

The received signal is described by

$$O(t) = \alpha_1 w_0^1x + \alpha_2 w_N^2x, \quad (73)$$

and its first time derivative

$$\dot{O}(t) = \alpha_1 w_0^1x + \alpha_2 w_N^2x, \quad (74)$$

$$= \alpha_1(-w_0^1y - w_0^1z) + \alpha_2(-w_N^2y - w_N^2z). \quad (75)$$

A final equation is needed to describe the first time derivative of $\dot{w}_0^2x(t)$ in terms of the previously defined new variables. We have that

$$\dot{w}_0^2x = \frac{1}{\delta t}(w_0^2x(t + \delta t) - w_0^2x(t)). \quad (76)$$

Moreover,

$$w_0^2x(t + \delta t) = w_0^1x(t + Q\delta t), \quad (77)$$

$$w_0^1x(t + Q\delta t) = w_0^1x(t) + \dot{w}_0^1x \delta t. \quad (78)$$

Placing Eqs. (70) and (71) in Eq. (69), takes us to

$$w_0^2x = Q\dot{w}_0^1x = \frac{1}{\delta t}(w_0^1x - w_0^2x). \quad (79)$$

The variational equations of the systems formed by Eqs. (60), (62), (65), (66), (65), and (72) can be constructed by defining the perturbed variables $\tilde{w}_i^j = w_i^j + \delta w_i^j$, with the index representing $j \in \{1, 2, 3\}$, $k \in \{x, y, z\}$, $i \in \{0, \ldots, N\}$, whose first derivative is $\dot{\tilde{w}}_i^j = \dot{w}_i^j + \delta \dot{w}_i^j$. Also, $\dot{O} = O + \delta O$, for the received signal. The Jacobian matrix of the variational equations is thus given by
The upper-left diagonal block
\[
\begin{pmatrix}
-1 & -1 \\
1 & a \\
u^1_y & w^1_y - c
\end{pmatrix}
\]
is responsible to produce the same 3 Lyapunov exponents of the \(\dot{x} = f(x)\) combination of the signals. They are a consequence of the way the derivatives were defined, and they could have been made to have the same signs. These exponents represent a higher-dimensional dynamics that effectively does not participate in the low-dimensional ordinary dynamics of the measured received signal. They are a consequence of the transformation of a time-delayed system of differential equations into an ODE, without explicitly time dependence.

Concluding, the spectra of Lyapunov exponents of the dynamics generating the signals for user 1 and 2 are preserved in the received signal, and are not affected by the combination of the signals.

\[
\begin{array}{cccccccccccc}
\delta w^1_x & \delta w^1_y & \delta w^1_z & \delta w^2_x & \delta w^2_y & \delta w^2_z & \ldots & \delta w^N_x & \delta w^N_y & \delta w^N_z & \delta O \\
\delta u^1_x & -1 & -1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta u^1_y & 1 & a & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta u^1_z & w^1_y & w^1_y - c & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta u^2_x & \delta t^{-1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta u^2_y & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta u^2_z & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta u^N_x & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta w^N_x & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta w^N_y & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\delta w^N_z & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

V. THE EQUIVALENCE PRINCIPLE FOR MAPS

To understand how to use the equivalent principle to derive general returning map expressions to the signal arriving at the BS, I make the assumption that there is one user with a basic frequency \(f^{(1)} = f\), and then a second user with a frequency \(f^{(2)} = 2f\). Using the equivalence principle, the combined signal arriving at the BS can be written as

\[
O_n = \gamma^{(1)} u_n^{(1)} + \gamma^{(2)} u_n^{(2)} = \gamma^{(1)} u_n^{(1)} + \gamma^{(2)} u_n^{(2)}
\]

We now transform the delay system into an ordinary systems of equations, by defining a new set of \(\tau + 1\) variables

\[
W_n^{(T)} = u_n^{(1)} - \gamma^{(2)} W_n^{(\tau)}
\]

Noticing that \(W_n^{(T)} = u_n^{(1)} - \gamma^{(2)} W_n^{(\tau)} = u_n^{(2)}\), we can rewrite Eq. (73) as

\[
O_n = \gamma^{(1)} u_n^{(1)} + \gamma^{(2)} W_n^{(\tau)}
\]

and making 1 iteration in Eq. (75), using

\[
u_n^{(1)} = 2u_n^{(1)} - \gamma^{(1)} u_n^{(1)} - \gamma^{(1)} u_n^{(1)} - \gamma^{(1)} u_n^{(1)}
\]

\[
u_n^{(2)} = 4u_n^{(2)} - \gamma^{(2)} u_n^{(2)} - \gamma^{(2)} u_n^{(2)}
\]

and knowing that \(W_n^{(\tau)} = 2f^{(2)} W_n^{(\tau)} - 2f^{(2)} W_n^{(\tau)} = 2f^{(2)} W_n^{(\tau)} - 2f^{(2)} W_n^{(\tau)} = 2f^{(2)} W_n^{(\tau)} - 2f^{(2)} W_n^{(\tau)} = 2f^{(2)} W_n^{(\tau)} - 2f^{(2)} W_n^{(\tau)}
\]

We obtain that

\[
O_n + 2\gamma^{(1)} u_n^{(1)} - \gamma^{(1)} u_n^{(1)} + 2f^{(2)} \gamma^{(2)} W_n^{(\tau)} - \gamma^{(2)} W_n^{(\tau)}
\]

Comparing Eq. (79) with Eq. (76), we finally obtain a general expression for the return map

\[
O_n = 2f^{(2)} \gamma^{(1)} u_n^{(1)} - 2f^{(2)} \gamma^{(1)} u_n^{(1)} - \gamma^{(1)} u_n^{(1)} - \gamma^{(1)} u_n^{(1)}
\]

If the two users have the same frequency, then \(f^{(2)} = f^{(1)} = 1\), then \(O_n = 2O_n - \gamma^{(1)} b_n^{(1)} - \gamma^{(1)} b_n^{(1)}\).

VI. POWER GAINS IN THE DOWNLINK CONFIGURATION

Unlike the uplink, in which each user \(k\) decides on the power gain \(\gamma^{(k)}\) based on their knowledge of the attenuation factor \(\alpha^{(k)}\), in the downlink, the operator at the
BS must decide on a unique value for power gain $\gamma(k)$. The decision is done in such a way to compensate for the channel with the largest attenuation factor (or minimal value of $\alpha_i^{(k)}$).

$$\gamma(k) \equiv \gamma^* = \frac{1}{\min_i \{\alpha_i^{(k)}\}}.$$  \hfill (80)

Neglecting multipath and choosing for the uplink configuration that $\gamma(k) = [\alpha_0^{(k)}]^{-1}$ and for the downlink configuration as in Eq. (80), we arrive that

$$O(t)_u = \sum_{k=1}^{N} \tilde{\gamma}(k)x^{(k)}(t) + w(t) \hfill (81)$$

$$O^{(m)}(t)_d = \alpha_0^{(m)} \gamma^* \sum_{k=1}^{N} \tilde{\gamma}(k)x^{(k)}(t) + u^{(m)}(t) \hfill (82)$$

and so if $\alpha_0^{(m)} = \min_i \{\alpha_0^{(i)}\}$, then equation for the received uplink signal $O(t)_u$ is equal to the received signal in the downlink $O^{(m)}(t)_d$, except by the noise term. Otherwise, these 2 equations differ additionally by a constant factor $(\alpha_0^{(m)} \gamma^*)$ which will only actually result in that effectively the signal to power rate will change.

VII. THE MATCHED FILTER DECODING APPROACH

A more sophisticated approach to decode information is based on a matched filter [1]. In here I show that the system formed by Eq. (1) and its matched filter can be roughly approximately described by the unshadowed Baker’s map, a result that allows us to understand that the decoding of a message by a user from the combined signal solely depends on the inverse dynamics of this user. If the dynamical equations generating the chaotic signal to be transmitted (in this case Eq. (1)) possesses no negative Lyapunov exponents (demonstrated in Sec. VIII of this SM), then attractor estimation of a noisily corrupted signal can be done using its autonomous time-inverse dynamics that is stable and possess no positive LEs (shown in Sec. IX of SM). The evolution of the future of the time-inverse dynamics is described by a system of ODE hybrid equations obtained by the time-rescaling $d/dt' = -d/dt$ applied to Eq. (1) resulting in

$$\ddot{y} + 2\beta \dot{y} + (\omega^2 + \beta^2) |y - \eta(t)| = 0,$$  \hfill (83)

where the variable $y$ represents the $x$ in time-reverse.

Let us define the symbol $S(t)$ to be a discrete encoding of $y(t)$, by the same rules that $s(t)$ is calculated from $x(t)$. So, $S(t) \in \{-1, 1\}$ is a 2-symbols alphabet discrete state that switches value by the signum function $S(t) = y(t)/|y(t)|$, whenever $|y(t)| < 1$ and $x = 0$.

Let me do some introductory remarks, regarding Eqs. (83) and (1). There are numerous ways to set the initial conditions of Eq. (83), and depending on how that is done, one will obtain different results. If at time $t$ we set as an initial condition in Eq. (83) that $y(t) = x(t + T)$, then at the time $t + T$ it could be true that $y(t + T) = x(t)$, if other initial conditions are appropriately set. Notice that if $y(t + T) = x(t)$, then the symbol $S(t + T)$ encoding $y(t + T)$ should be a match for the symbol $s(t)$ encoding $x(t)$.

Let us now understand the dynamical mechanism responsible to allow partial reconstruction of the attractor produced by Eq. (1) by Eq. (83), which allows for the decoding of the message using the inverse dynamics. Since Eq. (83) has no positive Lyapunov Exponents, if there is ADWN in the variable $x(t)$, i.e., $x(t + T) + \xi(t + T)$, and we set in Eq. (83) that $y(t) = x(t + T) + \xi(t + T)$, then at the time $t + T$ we could have that $y(t + T) = x(t) + \epsilon$, more specifically that $(y(t + T) - x(t))e^{\lambda T} \propto \xi(t + T)$.

Any corrupted variable or set of variables that is transmitted can be plugged into the inverse dynamics in Eq. (83) in order to estimate a delayed version of the transmitted signal. In practice, we have not too many options. The transmitted variable is $x(t)$. This variable alone, however, might not contain sufficient information for the inverse dynamics to estimate the past of all the variables in Eq. (1). This is because the attractor of Eq. (1) lives effectively in two subspaces $s(t) = 1$ and $s(t) = -1$. To be able to determine what will be the past of the variable $x(t)$, one might need to additionally have a good estimation of either the values of $\dot{x}$, or the values of the discrete variable, basically the message to be transmitted. First-time derivative variables $\dot{x}$ corrupted by noise are typically non-optimal to signal processing. So, we remain with that to use the inverse dynamics in Eq. (83) a good estimation of either $x(t)$ or $s(t)$ (or both) are required. Then, driving the inverse dynamics with that estimated variable should provide a good estimation for the short-term past of the variable $x(t)$.

System (83) was originally derived in [1], but using approaches from signal analysis by searching for a matched filter for the signal generated by Eq. (1) that maximises signal to noise rate. In that work, a first order filter (also called the auxiliary variable) was defined by the first-time derivative of the variable $\eta(t)$ in the inverse dynamics of Eq. (83) by

$$\dot{\eta}(t) = \ddot{x}(t + T) - \dot{x}(t),$$  \hfill (84)

where $\dot{x}(t) = x(t) + \xi(t)$.

The use of this matched filter to communication channels possessing multipath propagation was done in a series of papers [3, 14, 11]. Estimation of the symbolic sequence can also be done directly from $x(t)$ [12] by noticing that if $s_n = 1$ ($s_n = -1$) then $\int_{t + T}^{t + 2T} \ddot{x}(t)dt > 0$ ($\int_{t + T}^{t + 2T} \ddot{x}(t)dt < 0$). These works have in common that their approach to decoding is based on a good estimation of the discrete variable $s(t)$.

The filter was derived using arguments of signal analysis, but its working dynamical mechanism can be understood if one notices that $\eta$ as defined represents a sum of
approximate delta functions whose integral are approximately unitary and therefore \( \eta(t) = s(0) + \int_0^t \dot{\eta} dt \equiv s(t + T) \), where I have adopted \( s(0) \) as the initial condition for \( \eta(0) \). So, basically, the integral of the auxiliary variable provides an oscillatory estimation for \( s(t + T) \). Notice that the subtraction of variables (\( \ddot{x}(t + T) - \ddot{x}(t) \)) per se already filters noise, since adding random variables effectively decreases their standard deviation. The integral of the auxiliary dynamics contributes further to the noise filtering, since the contribution of the noise to the integral can vanish.

The transmitted binary message is estimated by noticing that if \( \eta(t) \) is an estimation for \( s(t + T) \), and if \( y(t) \) is set to be equal to \( x(t + 2T) \), then, by integrating the inverse dynamics for 1 period, we obtain \( y(t + T) \), which should produce an estimation of \( x(t + T) \), and so, the binary encoding \( S(t + T) \) of \( y(t + T) \), should be equal to the binary encoding \( s(t + T) \) of \( x(t + T) \).

Based on the previous arguments, I redefine \( \dot{\eta} \) by a 1-period shift, and use

\[
\dot{\eta}(t) = x(t) - x(t - T), \quad (85)
\]

where the following analysis will assume that there is no noise, and \( x(t) \) in Eq. \( (85) \) is the trajectory produced by Eq. \( (11) \). Notice that using the auxiliary variable as defined in Eq. \( (33) \) in the inverse dynamics of Eq. \( (33) \), instead of using as in Eq. \( (34) \), only shifts the time of the variable \( y(t) \). But this time shift ensures that \( S(t + T) \) (the encoding of \( y(t + T) \)) is equal to \( s(t) \) (the encoding of \( x(t) \)), being that the \( \eta(t) \) is an estimation for \( s(t) \). Figure \( (1) \) shows that if \( \dot{\eta}(t) \) is defined as in Eq. \( (35) \), then \( \eta(t) \) (dashed blue line) is a good estimation for \( s(t) \) (red line), and not of \( s(t + T) \) if Eq. \( (34) \) had been used. In fact, \( \eta(t) \) is time-forwarded estimation of \( s(t) \). The variable \( x(t) \) of the forward dynamics is shown in black.

Figure \( (2) \) demonstrates, for the no-noise scenario, that the matched filter can successfully decode the transmitted symbol. This figure shows the signal of the forward dynamics \( x(t) \) and respective hybrid variable \( s(t) \) (Eq. \( (11) \), in black and red lines, respectively. Plus symbols in green colour indicate the values of the continuous hybrid variable \( S(t) \) for the matched filter of Eq. \( (33) \), with \( \eta(t) \) defined by \( (35) \).

Let me now argue that the system of Eqs. \( (11) \) and \( (33) \), coupled by making the rough approximation

\[
\eta(t) = s(t) \quad (86)
\]
can be approximately described by the unfolded-Baker’s map. Then, I make further conclusions about how decoding in the multiuser environment can be done. As recognized in Ref. [13], making \( q(t) = s(t) \) in Eq. (83), makes this equation to describe what was called “reverse time chaos” in Ref. [14].

Let us take the values of \( y \) in Eq. (83) at discrete times \( nT \), writing that \( y(nT) = y_n \), and define the new variable for users 1 and 2 as before \( y_1 = 2z_1 - 1 \) and \( y_2 = 2z_2 - 1 \).

Figure 3 shows in black circles and red squares that time-\( T \) mappings of solutions of Eq. (1) and of Eq. (83) for \( q(t) = s(t) \), \( f = 1 \), and for user \( k = 1 \) are the mappings presented in Eqs. (87) and (89), respectively.

If Equations
\[
\begin{align*}
\text{(1)}: & \quad u_{n+1}^{(1)} = 2u_n^{(1)} - 2u_n^{(1)} + b_n^{(1)}, \\
\text{(2)}: & \quad u_{n+1}^{(2)} = 4u_n^{(2)} - 4u_n^{(2)} + b_n^{(2)},
\end{align*}
\]

are map solutions of Eq. (1) (in the re-scaled coordinate system, with appropriate \( \gamma \) gains) for user \( k \) with frequencies \( f(k) = k \), their inverse mapping the solution of Eq. (83) is given by
\[
\begin{align*}
\text{(k)}: & \quad z_{n+1}^{(k)} = 2^{-k}\left(z_n^{(k)} - 2^k u_n^{(k)}\right), \quad \text{and} \quad 2^k u_n^{(k)} = b_n^{(k)}. \quad (89)
\end{align*}
\]

This map can be derived simply defining \( z_{n+1}^{(k)} = u_n^{(k)} \) and \( z_n = u_{n+1}^{(k)} \). We always have that \( 2^k u_n^{(k)} = b_n^{(k)} \). So, for any \( z_n^{(k)} \in [0, 1] \) and which can be simply chosen to be equal to the received combined signal \( O_n \) (normalized such that \( \in [0, 1] \)), it is also true that
\[
\begin{align*}
[2^k z_{n+1}^{(k)}] = [2^k u_n^{(k)}] = b_n^{(k)}. \quad (90)
\end{align*}
\]

So, if we represent an estimation of the transmitted symbol of user \( k \) by \( b_n^{(k)} \), then decoding of the transmitted symbol of user \( k \) can be done by calculating \( z_{n+1}^{(k)} \) using the inverse dynamics of the user \( k \)
\[
\begin{align*}
\text{(k)}: & \quad z_{n+1}^{(k)} = 2^{-k}\left(z_n^{(k)} - b_n^{(k)}\right), \quad (91)
\end{align*}
\]

and applying this value to Eq. (90).

Initial conditions in Eqs. (91) need to be set according to the decoding scheme and the uncertainty of the received signal. As long as \( b_n^{(k)} \in (0, 1) \) (so, can assume either the value 0 or 1), then regardless of \( z_n^{(k)} \in [0, 1] \), one will obtain that \( 2^k u_n^{(k)} = b_n^{(k)} \). So, if a symbol \( b_n^{(k)} \in (0, 1) \) is wrongly estimated, the use of Eq. (90) cannot correct that. However, \( b_n^{(k)} \in [0, 1/2] \) (or \( b_n^{(k)} \in [1/2, 1] \)) can be by the use of Eqs. (91) and with the decoding rule as in Eq. (90) correctly estimate the value of \( b_n^{(k)} \). The filter in Ref. [14] works based on this dynamical principle, in which the inverse dynamics is forced by the auxiliary variable, assumed to provide a good estimation of the transmitted symbol, with a shift in time depending on how \( \eta \) is defined.

The result in Eq. (89) for \( k = 1 \) can also be derived noticing that as shown in Ref. [14], Eq. (83) after the proper rescaling also done in here for the variables and symbolic sequence has a solution equal to \( z_n^{(1)} = \sum_{j=1}^{\infty} 2^{-(j+1)} b_{n-j} \). Similarly, a solution in terms of the symbols for the variable \( u_n \) in Eq. (87) can be written as \( u_n^{(k)} = \sum_{j=1}^{\infty} 2^{-(j+1)} b_n^{(k)} \). This means that the system formed by the variables \( u_n^{(k)}, z_n^{(k)} \) is a generalization (for \( k \neq 1 \)) of the unfolded Baker’s map [15], being described by a time-forward variable \( u_n^{(k)} \) (the Bernoulli shift for \( k = 1 \)), and its backward variable component \( z_n^{(k)} \).

VIII. THE NEGATIVE LYAPUNOV EXPONENTS OF THE INVERSE DYNAMICS

To time reverse the hybrid system in Eq. (1), we apply the time-rescaling \( \frac{dt}{dt'} = -d/dt \), which basically produces a system of ODEs that is the one in Eq. (1) for a time-reverse. This system of ODEs is the one in Eq. (83).

Such time-rescaling simply reverts the signs of the spectrum of LEs of the time-forward dynamics [8], thus it should possess a pair of degenerated exponents with a value of
\[
\chi = -\beta. \quad (92)
\]

IX. NON-ORTHOGONAL MULTIPLE ACCESS (NOMA)

The understanding of how to decompose periodic signals with equal or different frequencies combined together with appropriate power gains for each signal for communication purposes is not new [10, 18]. And it is in fact a promising approach to deliver the demand required for 5G systems. To cope with the expected demand in 5G wireless communication, non-orthogonal multiple access (NOMA) [16, 18] was proposed to allow all users to use the whole available frequency spectrum. One of the most popular NOMA scheme allocate different power gains to the signal of each user.

Suppose \( y_k(t) \) represents an orthogonal frequency-division multiplexing (OFDM) periodic wave signal for an user \( k \)
\[
y_k(t) = \Re \left\{ \sum_{k=0}^{N_s-1} s(k) e^{-j2\pi f_c k T_{OFDM}} t \right\}, \quad (93)
\]

where \( T_{OFDM} = N_s T_s \) represents the total length of the input signal, \( N_s \) the number of symbols to be transmitted by user \( k \), and \( s(k) = s(k T_s) \), with \( k = 0, \ldots, N_s - 1 \) are the symbols to be transmitted (in the frequency domain, so basically representing a discrete alphabet). Equation (91) is nothing but the inverse
Fourier transformation of the symbols. The user uses subcarrier frequency bands spaced by $1/20FDM$ and centered at $(f_c + k/(20FDM))$. Orthogonality of the signal between any two non-equal subcarriers means that $\sum_{k=0}^{N_c-1} e^{2\pi i (k/tOFDM)(f_c + k/(20FDM)T)} = 0$, for $k_1 \neq k_2$.

In the OFDM communication scheme, each user must use a different set of subcarriers. For example, user 1 communicates with $f_c = 1/20FDM$ and user 2 communicates with $f_c = N_c/20FDM$. For user 1, regardless of the subcarrier the bit rate is dominated by the lowest frequency. Moreover, users cannot use the same set of subcarriers, thus limiting the efficient use of the spectrum resources.

For the uplink scenario, this would translate into a signal being received in BS described by

$$O(t)_{up} = \sum_{k=1}^{N_c} \sqrt{\gamma^{(k)}} g_k y_k(t) + w(t), \quad (94)$$

where $\gamma^{(k)}$ is the average power of user $k$ signal, $g_k$ represents the attenuation for the shortest path link between the BS and the user $k$, and $w(t)$ is AWGN. The signal $y_k(t)$ is the one transmitted by user $k$. Other signal not propagating along the shortest path are treated as noise.

For the downlink communication, when a BS sends 1 signal to several users, the signal transmitted by the station is described by

$$O(t)_{down} = \sum_{k=1}^{N} \sqrt{\gamma^{(k)}} y_k(t), \quad (95)$$

and the signal received by user $k$ is described by

$$O^{(k)}(t)_{down} = g_k O(t)_{down} + w_k(t) \quad (96)$$

where $g_k$ is the attenuation factor between the BS and the user $k$ along the shortest path propagation.

Notice that considering only the direct path ($L = 1$), equations in the main manuscript modeling the uplink and downlink scenarios are equivalent to Eqs. (94), (95), and (96). There are however crucial differences between the WiChaos being here proposed and the NOMA. Each user in the WiChaos has a unique natural central frequency $f$, however $x(t)$ is a chaotic signal and as such it is naturally broadband. Signals from different users will always interfere. In the OFDM-based NOMA scheme, bits are "slowed" modulated. Each subcarrier modulates a symbol per channel use ($nT_s \leq t \leq (n+1)T_s$), whereas in the WiChaos, a user modulates $f$ digital symbols per channel use, or $f$ bits per channel use ($nN_s \leq t \leq (n+1)N_s$). In NOMA, encoding as well as decoding employs Fourier functions. In the WiChaos, decoding can be trivially done, or using simple filters. Finally, nothing impedes WiChaos to have users operating with arbitrary central frequencies and their multiples, using simultaneously the whole spectrum. This however is out of the scope of the present work.

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