EQUIVARIANT CARTAN HOMOTOPY FORMULAE FOR DG ALGEBRA

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Abstract. We study the equivariant Cartan Homotopy Formula for the DG-algebra obtained by a finite group action.

0. Introduction

The tool of paracyclic modules was used in [GJ1] and [BGJ] to understand the cyclic homology of the cross product algebras. It is an analogue of the Eilenberg-Zilber theorem for bi-paracyclic modules. Let $A$ be a unital DG algebra over a commutative ring $k$. Let $G$ be a finite discrete group which acts on $A$ by automorphisms. We then consider the cross product algebra $A times G$.

Theorem 0.1 ([GJ1]). If $G$ is finite and $|G|$ is invertible in $k$, then there is a natural isomorphism of cyclic homology and

$$HC_\bullet(A \rtimes G; W) = HC_\bullet(H_0(G, A^G); W),$$

where $H_0(G, A^G)$ is the cyclic module

$$H_0(G, A^G)(n) = H_0(G, k[G] \otimes A^{(n+1)}).$$

Where $W$ is a finite dimensional graded module over the polynomial ring $k[u]$, where $deg(u) = -2$; the above result when considered for different coefficients $W$ yield several theories, some are illustrated below:

1) $W = k[u]$ gives negative cyclic homology $HC^{-}(A)$;
2) $W = k[u, u^{-1}]$ gives periodic cyclic homology $HP_\bullet(A)$;
3) $W = k[u, u^{-1}]/uk[u]$ gives cyclic homology $HC_\bullet(A)$;
4) $W = k[u]/uk[u]$ gives the Hochschild homology $HH_\bullet(A)$.

The above decomposition has been studied and used by several to understand the homological properties of cross product algebra [CGGV] [P1] [P2] [P3] [Q1] [Q2] [Q3] [Q4] [ZH] The (co)homology modules of the cross product algebra decomposes into twisted (co)homology modules relative to the conjugacy classes of the group elements. Thus understanding the homological property of cross product algebra is simplified.

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The above result can also be extended to DG-algebra, we shall prove the existence of such a decomposition relative to its differential structure and then finally prove the Cartan homotopy formulae for the components. Let us first recall the definition of a DG-algebra. A unital DG algebra \((A, d)\) is a differentially graded algebra \(A\), with \(k\)-bilinear maps
\[
A_n \times A_m \to A_{n+m}, \text{ sending } (a, b) \mapsto ab \text{ such that }
\]
\[
d_{n+m}(ab) = d_n(a)b + (-1)^n ad_m(b)
\]
and such that \(\oplus A_n\) becomes an associative and unital \(k\)-algebra. Through out this article we demand that the finite group action on \((A, d)\) to preserve grading and to commute with the differential structure; i.e. for \(g \in G\), \(dg = gd\). The (co)homology theories on DG algebras have been studied extensively [T] [K] [GJ2]. The above algebra can also be considered as a special \(A_\infty\)-algebra \((A, m_i)\) with \(m_i = 0\) for \(i > 2\). Here \(m_2\) (2-cochain for \(A\)) defines product on \(A\) and \(m_1\) is the mixed differential. The Cartan homotopy formulae for algebras was first observed by Rinehart [R] in the case where \(D\) is a derivation on commutative algebra, and later in full generality by Getzler [G] for \(A_\infty\)-algebras. The formulae is stated below:

**THEOREM 0.2** (Cartan Homotopy Formula).
\[
[b, L_D] + L_{\delta_D} = 0, \quad [B, L_D] = 0 \quad \text{and} \quad [L_D, L_E] = L_{[D, E]}
\]
\[
[b + B, \iota_D + S_D] = L_D + \iota_D + S_{\delta_D}
\]
where \([,]\) is the graded Gerstenhaber bracket for \(C^\bullet(A, A)\). Along with the cup product it induces the Gerstenhaber algebra structure on the \(\widehat{H}^\bullet(A)\).

The above formulae was proved in generality for the \(A_\infty\)-algebras by Getzler. The case \(k = 1\) in his paper corresponds to the DG-algebra \((A, d)\). We refer to [G] for detailed expressions for \(A_\infty\)-algebras, here we shall prove the equivariant Cartan homotopy formulae for the twisted Hochschild complexes resulting by the paracyclic decomposition of \((A, d) \rtimes G\). We now introduce the statements of the article.

1. **Statement**

For \(G\) a finite discrete group acting on a DG-algebra \((A, d)\) over a ring \(k\) such that \(|G|\) is invertible in \(k\). For \(g \in G\) define a \(g\)-twisted \(A\)-left-module structure on \(A\) by the following formula:
\[
a \bullet (u_g m) := u_g g^{-1}(a) \cdot m,
\]
where \(a, m \in A\). We tag by \(u_g\) the twisted element of \(A\) with this left bimodule structure and the left twisted bimodule is denoted by \(A_g\) (refer [Q1], pp 332).

Define
\[
C_0(A)_g := A_g \quad \text{and} \quad C_n(A)_g := A_g \otimes A^\otimes n.
\]
Let \(b^g, B^g\) denote the chain differentials of the complex \(C^\bullet(A)_g\) and \(L_D^g\) be the twisted Lie derivative associated to \(D \in C^k(A, A)\). We shall observe that the paracyclic decomposition exists for DG-algebras and also describe the maps below explicitly.
**Theorem 1.1.** The Lie derivative $L^g_D$, chain maps $b^g$ and $B^g$ satisfy the following:

$[b^g, L^g_D] + L^g_D = 0 \quad [B^g, L^g_D] = 0 \quad [L^g_D, L^g_E] = L^g_{[D,E]}$ and $[b^g + B^g, \iota^g_D + S^g_D] = L^g_D + \iota^g_D + S^g_D$.

The $[,]$ above is the Gerstenhaber bracket for $C^\bullet(A, A)$ and $\delta$ is the Hochschild cochain map.

2. **Paracyclic decomposition for cross product $(A, d) \rtimes G$**

2.1. **Hochschild Chain Complex for $(A, d)$**. Define the differentials $d : C^\bullet(A) \to C^\bullet(A)$, $b : C^\bullet(A) \to C^\bullet(A)[-1]$ and $B : C^\bullet(A) \to C^\bullet(A)[1]$ as follows.

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} (-1)^{\sum_{k<i}(|a_k|+1)} (a_0 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_n)$$

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^{\sum_{i=0}^{k}(|a_i|+1)} (a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n)$$

$$+ (-1)^{|a_n| + (|a_n|+1)\sum_{i=0}^{n-1}(|a_i|+1)} (a_n a_0 \otimes \cdots \otimes a_{n-1})$$

and

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (-1)^{\sum_{i \leq k}(|a_i|+1)} \sum_{i \geq k}(|a_i|+1) (e \otimes a_{k+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_k)$$

The above formula satisfies $B(e, a_1, \ldots, a_k) = 0$. These sign conventions ensures that the elements $a_i$ for $i > 0$ occur with an implicit suspension reducing the degree to $|a_i| - 1$.

The homology of $(C^\bullet(A, A), b + d)$ is the Hochschild homology of $A$ with coefficients in $A$ endowed with the bimodule structure. While the module $H^\bullet(C^\bullet \otimes W, b + d + uB)$ is the cyclic homology of the mixed complex $(A, b + d, B)$ with coefficients in $W$.

2.2. **Paracyclic decomposition of $A \rtimes G$**. Consider the DG-algebra with a discrete group acting on it and preserving the grading. Abiding the notations of [GJ1], we have the following result

**Lemma 2.1.** Let $(A, d)$ be a unital algebra over the commutative ring $k$, and let $G$ be a discrete group acting by preserving the grading. There is a quasi-isomorphism, of mixed complexes

$$f_0 + uf_1 : Tot(N(A\sharp G)) \cong N((A \rtimes G)^2)$$

Hence we obtain the following isomorphism of cyclic homology groups

$$HC^\bullet(A \rtimes G; W) = HC^\bullet(Tot(N(A\sharp G)); W).$$
Proof. The proof of the above statement works mutatis mutandis with the proof in [GJ1, Theorem 3.1] by considering the bi-paracyclic module $N(A;G)$ derived from the complex $C_\bullet(A; m_1)$, where $m_1 = b + d + uB$. □

Hence $|G|$ invertible in $k$ [GJ1, Theorem 4.5] gives us the paracyclic decomposition of the cyclic homology of $A \rtimes G$.

3. PROOF OF THE THEOREM

Since, we know that a decomposition of the cyclic homology of $A \rtimes G$ exists and and we have the following isomorphism of complexes

$$HC_\bullet(A \rtimes G) = \bigoplus_{[g]} HC_\bullet(H_0(G^g, A^g)) = \bigoplus_{[g]} HC_\bullet(A^g)^G$$

with the maps; $b^g : C_\bullet(A^g) \rightarrow C_\bullet(A^g)[-1]$ and $B^g : C_\bullet(A^g) \rightarrow C_\bullet(A^g)[1]$ defined as:

$$b^g(u_g a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^{\sum_{i=0}^{n-1} |i| + 1} (u_g a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n)$$

and

$$B^g(u_g a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (-1)^{\sum_{i=0}^{n} |i| + 1} (u_g g^{-1}(a_0) a_0 \otimes \cdots \otimes a_{n-1})$$

One important point here to be noted is that since the action of $G$ on $(A, d)$ preserves grading, the crucial sign term $\epsilon_k = \sum_{i \leq k} (|a_i| + 1) \sum_{i \geq k} (|a_i| + 1)$ remains unchanged (in fact under this assumption the signs for $A_\infty$-algebras are also invariant under the unified algebraic structure for chains and cochains of $A_\infty$-algebras) and is a necessary condition to yield the Cartan homotopy formulae for twisted components.

For $D \in C^k(A, A)$, while the maps $t_D$ and $S_D^g$ can be computed using the signed cyclic permutation map $t$ [BGJ]. They are as follows:

$$t_D(u_g a_0 \otimes \cdots \otimes a_n) = (-1)^{|D| |a_0|} a_0 D(a_1, \ldots, a_n);$$

$$L_D^g(u_g a_0 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n-d} (-1)^{\nu_k(D, n)} u_g a_0 \otimes \cdots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \cdots \otimes a_n$$

$$+ \sum_{k=n+1-d}^{n} (-1)^{\nu_k(D, n)} u_g D(g^{-1}(a_{k+1}), \ldots, g^{-1}(a_n), a_0, \ldots) \otimes \cdots \otimes a_k.$$
where, \(|D| = \text{degree of the linear map } D\) + \(d\), \(D \in C^d(A, A)\) is being considered as a linear map \(D : A^d \to A\); and the sign coefficients are \(\nu_k(D, n) = (|D| + 1)(|a_0| + \sum_{i=1}^k (|a_i| + 1))\), \(\eta_k(D, n) = |D| + \sum_{i \leq k} (|a_i| + 1) \sum_{i \leq k} (|a_i| + 1)\) and

\[
\epsilon_{jk}(D, n) = (|D| + 1)(\sum_{i=k+1}^n (|a_i| + 1) + |a_0| + \sum_{i=1}^j (|a_i| + 1)).
\]

We briefly describe the Gerstenhaber algebra structure on the cohomology \(H^\bullet(A, A)\), the cup product is defined as below, for \(D \in C^d(A, A)\) and \(E \in C^e(A, A)\).

\[
(D \cup E)(a_1, \ldots, a_{d+e}) = (-1)^{|E|+\sum_{i\leq e}(|a_i|+1)} D(a_1, \ldots, a_e) E(a_{d+1}, \ldots, a_{d+e});
\]

and the product \(\circ\) is defined as:

\[
(D \circ E)(a_1, \ldots, a_{d+e}) = \sum_{j \geq 0} (-1)^{|E|+1} \sum_{i=1}^j (|a_i|+1) D(a_1, \ldots, a_j, E(a_{j+1}, \ldots, a_{j+e}), \ldots).
\]

The Lie bracket is hence \([D, E] = D \circ E - (-1)^{|D|+1}(|E|+1) E \circ D\). The cochain map \(\delta\) on \(C^\bullet(A, A)\) is

\[
(\delta D)(a_1, \ldots, a_{d+1}) = (-1)^{|a_1||D|+|D|+1} a_1 D(a_2, \ldots, a_{d+1}) + \sum_{j=1}^d (-1)^{|D|+1+\sum_{i=1}^j (|a_i|+1)} D(a_1, \ldots, a_j a_{j+1}, \ldots, a_{d+1}) + (-1)^{|D|} \sum_{i=1}^d (|a_i|+1) D(a_1, \ldots, a_d) a_{d+1}
\]

The operator \(\delta D\) can also be described as \(\delta D = [m_2, D]\).

**Proof of Theorem 1.1.**

\[
(3.1) \quad [b^\theta, L^\theta_D] + L^\theta_D b^\theta = b^\theta L^\theta_D - L^\theta_D b^\theta + L^\theta_{\delta D}.
\]

We evaluate the above expression on the element \((u_g a_0 \otimes \cdots \otimes a_n)\). We define the bar complex map on the equivariant Hochschild chain complex by \(b^\theta\), it is as follows

\[
b^\theta(u_g a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^{\sum_{i=0}^k (|a_i|+1)+1} (u_g a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n)
\]

Similarly we define the operator \(L^\theta_d\) as follows:

\[
L^\theta_d(u_g a_0 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n-d} (-1)^{\nu_k(D, n)} u_g a_0 \otimes \cdots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \cdots \otimes a_n
\]

The untwisted expression in (3.1) is \(b^\theta L^\theta_D - L^\theta_D b^\theta + L^\theta_{\delta D}\). We collect the coefficients and the terms do cancel subject to the matching of sign, expressions like below show up in order to justify the cancellation.

\[
\nu_w(D, n) + \sum_{i=0}^{w} (|a_i| + 1) + 1 + \nu_{w-1}(\delta D, n) + |a_w||D| + |D| + 1 \equiv 1 \pmod{2}.
\]

Using \(|\delta D| = |D| + 1\) we ascertain the above claim. In the equivariant case the elements are twisted by group action and (3.1) holds in the untwisted case hence the untwisted terms in
the equivariant case cancel each other. On the other hand the cancellation of the twisted terms in \((3.1)\) is non-trivial. To see this we firstly observe that

\[ |D(a_{i+1}, \ldots, a_{i+d})| = |D| \sum_{j=i+1}^{i+d} (|a_j| + 1). \]

Also, since the action of \(G\) is grade preserving we have \(|g^{-1}(a)| = |a|\). Hence the parity of signs remain the same as it were for the untwisted case. For example,

\[ \eta_k(\delta D, n) + |D| \sum_{i=1}^{d} (|b_i| + 1) + \eta_k(D, n) + |D(b_1, \ldots, b_d)| + 1 \equiv 1 \pmod{2}, \]

where \((b_1, \ldots, b_d) = (g^{-1}(a_{k+1}), \ldots, g^{-1}(a_n), a_0, \ldots, a_{d-n+k})\) and hence the terms of kind \(u_g D(g^{-1}(a_{k+1}), \ldots, g^{-1}(a_n), a_0, \ldots) a_{d-n+k+1} \otimes \cdots \otimes a_k\) cancel each other. Similarly, it is easy to check that all other types of twisted terms cancel each other and the appropriate parity of signs are ensured by the grade preserving group action.

\[ (3.2) \quad [L_D^g, L_E^g] = L_{[D,E]}^g \]

The proof of the above relation is straight forward, the bracket \([,]\) is the Gerstenhaber Lie algebra commutator as described above.

\[ (3.3) \quad [B^g, L_D^g] = B^g L_D^g - L_D^g B^g. \]

\[ B^g L_D^g(u_g a_0 \otimes \cdots \otimes a_n) = B^g \left\{ \sum_{l=1}^{n-d} (-1)^{\nu(D,n)} u_g a_0 \otimes \cdots \otimes D(a_{l+1}, \ldots, a_{l+d}) \otimes \cdots \otimes a_n \right\} \]

\[ = \sum_{l=1}^{n-d} (-1)^{\nu(D,n)} u_g D(g^{-1}(a_{l+1}), \ldots, g^{-1}(a_n), a_0, \ldots) \otimes \cdots \otimes a_l \]

\[ + \sum_{l=n+1-d}^{n} (-1)^{\nu(D,n)} u_g \sum_{k=0}^{n-d} (-1)^{\nu(D,n)} \sum_{l=1}^{n-d} \cdots \otimes a_k \]

such that any of \(a_i'\) could be \(D(\ldots)\).

\[ \sum_{l=1}^{n-d} (-1)^{\nu(D,n)} \sum_{k=0}^{n-d} (-1)^{\nu(D,n)} \sum_{l=1}^{n-d} \cdots \otimes a_k \]

such that for each \(l\) one of the \(a_i''\) in the sign expression is \(D(g^{-1}(a_{l+1}), \ldots, g^{-1}(a_n), a_0, \ldots)\) and the rest are remaining \(a_i's\) that do not appear in \(D(g^{-1}(a_{l+1}), \ldots, g^{-1}(a_n), a_0, \ldots)\).

While,

\[ L_D^g B^g(u_g a_0 \otimes \cdots \otimes a_n) = L_D^g \left\{ \sum_{k=0}^{n-d} \sum_{l=1}^{n-d} (-1)^{\nu(D,n)} \sum_{k=0}^{n-d} (-1)^{\nu(D,n)} \sum_{l=1}^{n-d} \cdots \otimes a_k \right\} \]

\[ = L_D^g \left\{ \sum_{k=0}^{n-d} \sum_{l=1}^{n-d} \cdots \otimes a_k \right\} + 0 \]
Observe that 
\[ \text{The sign for the term} \]
\[ \text{The above relation can be seen be comparing the parity of sign indices for the terms,} \]
\[ \text{each other. Hence we are left to show that} \]
\[ \text{Finally we want to show that:} \]
\[ (3.4) \]
\[ \text{The expression above can be written as below:} \]
\[ \text{The above relation can be seen be comparing the parity of sign indices for the terms, e.g.} \]
\[ \text{the sign for the term} \]
\[ \text{while for LHS such terms originate from} \]
\[ \text{Hence along with the term the sign also match up for cancellation to yield that} \]
\[ \text{Remark 3.1. We expect that the above twisted Cartan Homotopy formula to hold for} \]
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