Prepotential approach to quasinormal modes

Choon-Lin Ho
Department of Physics, Tamkang University, Tamsui 251, Taiwan, R.O.C.

In this paper we demonstrate how the recently reported exactly and quasi-exactly solvable models admitting quasinormal modes can be constructed and classified very simply and directly by the newly proposed prepotential approach. These new models were previously obtained within the Lie-algebraic approach. Unlike the Lie-algebraic approach, the prepotential approach does not require any knowledge of the underlying symmetry of the system. It treats both quasi-exact and exact solvabilities on the same footing, and gives the potential as well as the eigenfunctions and eigenvalues simultaneously. We also present three new models with quasinormal modes: a new exactly solvable Morse-like model, and two new quasi-exactly solvable models of the Scarf II and generalized Pöschl-Teller types.

I. INTRODUCTION

Quasinormal modes (QNM) has attracted great interest in recent years [1]. They arise as waves emitted by a perturbed neutron star or black hole that are outgoing to spatial infinity and the event horizon. Generally, the wave function of QNMs has discrete complex frequency, whose imaginary part leads to a damping behavior. QNM carry information of black holes and neutron stars, and thus are of importance to gravitational-wave astronomy. In fact, these oscillations, produced mainly during the formation phase of the compact stellar objects, can be strong enough to be detected by several large gravitational wave detectors under construction.

As black hole potentials are generally too complicated to allow analytic treatment, so in order to understand the origin of the discrete imaginary frequencies, one can try approximating the top region of the black hole potential by some inverted potentials which are solvable. This has been done by using the inverted harmonic oscillator [2], and the Pöschl-Teller potential [3].

Recently, in [4] we have extended the number of exactly solvable models that admit QNMs. We take QNMs to include both decaying and growing modes with complex energies. Furthermore, we have provided the first model with QNM that is quasi-exactly solvable (QES). A system is called QES if a part of its spectrum, but not the whole spectrum, can be determined analytically [5–12]. Our approach in [4] was to study solutions of QNM based on the sl(2)-Lie-algebraic approach to one-dimensional QES theory [5–9]. We demonstrated that, by suitably complexifying some parameters of the generators of the sl(2) algebra while keeping the Hamiltonian Hermitian, we could indeed obtain potentials admitting exact or quasi-exact QNMs. These models were later re-studied numerically by the asymptotic iteration method in [13].

In this paper we would like to show that exactly solvable and QES models with QNMs can be constructed much more simply without resorting to the machinery of Lie-algebra. This is achieved through the prepotential approach proposed recently [14–17]. This is a simple constructive approach, based on the so-called prepotential [7, 18, 19], which can give the potential as well as the eigenfunctions and eigenvalues simultaneously. The novel feature of the approach is that both exact and quasi-exact solvabilities can be solely classified by two integers, the degrees of two polynomials which determine the change of variables and the zero-th order prepotential. Hence this approach treats both quasi-exact and exact solvabilities on the same footing, and provides a simple way to determine the required change of variables, say \( x \), to a new one \( z = z(x) \). All the well-known exactly solvable models classified in supersymmetric quantum mechanics (SUSYQM) [20], the QES models discussed in [5–9], and some new QES ones (also for non-Hermitian Hamiltonians), can be generated by appropriately choosing the two polynomials. Our approach, unlike the Lie-algebraic approach, does not require any knowledge of the underlying symmetry of the system. Furthermore, our approach can generate the Coulomb, Eckart, Rosen-Morse type I and II models [16] which are not covered by the standard Lie-algebraic program [8, 9]. Compared with SUSYQM, our approach has the advantage that we do not have to assume the sufficient condition for integrability needed in SUSYQM, namely, shape invariance. In fact, shape invariance comes out automatically from this approach [17]. What is more, the transformation of the original variable \( z(x) \) is determined within the prepotential approach, whereas in SUSYQM this has to be taken as given from the known solutions of the respective models before one could solve the shape invariance condition.

We shall adopt the prepotential approach here to generate and classify all one-dimensional exactly solvable and QES models with QNMs based on sinusoidal coordinates, namely, those coordinates \( z(x) \) whose derivatives with respect to \( x \) squared are at most quadratic in \( z \). Our strategy is to suitably complexify some or all of the parameters in the prepotential while keeping the resulted potentials real. We find that all the systems reported in [4] can be very easily constructed. During the course of investigation, we also realize that there are three new QNM models which were missed in [4]. We thus take this opportunity to report on them.
The plan of the paper is as follows. In Sect. II we briefly review the essence of the prepotential approach. Sect. III to V then discuss the prepotential construction of QNM models of the Scarf II, the Morse, and the generalized Pöschl-Teller type, respectively. These three types of potentials have some interesting features, and serve as good examples to illustrate the procedure. Furthermore, there is one new QNM model in each of these three types that we would like to report. Sect. VI summarizes the paper. In Appendices A to C, we list all other cases for easy reference.

II. PREPOTENTIAL APPROACH

The main ideas of the prepotential approach \cite{14, 15} can be summarized as follows (we adopt the unit system in which \( \hbar \) and the mass \( m \) of the particle are such that \( \hbar = 2m = 1 \)). Consider a wave function \( \phi_N(x) \) (\( N \): non-negative integer) which is defined as

\[
\phi_N(x) = e^{-W_0(x)}p_N(z),
\]

with

\[
p_N(z) = \begin{cases} 
1, & N = 0; \\
\prod_{k=1}^{N}(z - z_k), & N > 0.
\end{cases}
\]

Here \( z = z(x) \) is some real function of the basic variable \( x \), \( W_0(x) \) is a regular function of \( z(x) \), and \( z_k \)'s are the roots of \( p_N(z) \). The variable \( x \) is defined on the full line, half-line, or finite interval, as dictated by the choice of \( z(x) \). The function \( p_N(z) \) is a polynomial in an \( (N + 1) \)-dimensional Hilbert space with the basis \( \{1, z, z^2, \ldots, z^N\} \). \( W_0(x) \) defines the ground state wave function.

The wave function \( \phi_N \) can be recast as

\[
\phi_N = \exp(-W_N(x, \{z_k\})),
\]

with \( W_N \) given by

\[
W_N(x, \{z_k\}) = W_0(x) - \sum_{k=1}^{N} \ln |z(x) - z_k|.
\]

Operating on \( \phi_N \) by the operator \(-d^2/dx^2\) results in a Schrödinger equation \( H_N \phi_N = 0 \), where

\[
H_N = -\frac{d^2}{dx^2} + V_N,
\]

\[
V_N = W''_N - W'_N.
\]

Here and below the prime represents derivative with respect to \( x \). Since the potential \( V_N \) is determined by \( W_N \), we thus call \( W_N \) the \( N \)th order prepotential. From Eq. \( 4 \), one finds that \( V_N \) has the form \( V_N = V_0 + \Delta V_N \):

\[
V_0 = W''_0 - W'_0,
\]

\[
\Delta V_N = -2 \left( W'_0 z' - \frac{z''}{2} \right) \sum_{k=1}^{N} \frac{1}{z - z_k} + \sum_{k,l} \frac{z'^2}{(z - z_k)(z - z_l)}.
\]

Thus the form of \( V_N \), and consequently its solvability, are determined by the choice of \( W_0(x) \) and \( z'^2 \) (or equivalently by \( z'' = (dz'^2/dz)/2 \)). Let \( W'_0 z' = P_m(z) \) and \( z'^2 = Q_n(z) \) be two polynomials of degree \( m \) and \( n \) in \( z \), respectively. The variables \( x \) and \( z \) are related by

\[
x(z) = \pm \int_{z}^{\infty} \frac{dz}{\sqrt{|Q_n(z)|}},
\]

and the prepotential \( W_0(x) \) is determined as

\[
W_0(x) = \left( \int_{z}^{\infty} \frac{P_m(z)}{Q_n(z)} dz \right)_{z = z(x)}.
\]

We assume \( 5 \) is invertible to give \( z = z(x) \). Eqs. \( 5 \) and \( 6 \) define the change of variables \( z(x) \) and the corresponding prepotential \( W_0(x) \). Thus, \( P_m(z) \) and \( Q_n(z) \) determine the quantum system. Of course, for bound state problems the choice of \( P_m \) and \( Q_n \) must ensure normalizability of \( \phi_0 = \exp(-W_0) \).

Now depending on the degrees of the polynomials \( P_m \) and \( Q_n \), we have the following situations \cite{14, 15}:
(i) if \( \max\{m, n - 1\} \leq 1 \), then in \( V_N(x) \) the parameter \( N \) and the roots \( z_k \)’s will only appear as an additive constant and not in any term involving powers of \( z \). Such system is then exactly solvable;

(ii) if \( \max\{m, n - 1\} = 2 \), then \( N \) may appear in the first power term in \( z \), but \( z_k \)’s only in an additive term. If \( N \) does appear before the \( z \)-term, then the system belongs to the so-called type 1 QES system defined in [6], i.e., for each \( N \geq 0 \), \( V_N \) admits \( N + 1 \) solvable states with the eigenvalues being given by the \( N + 1 \) sets of roots \( z_k \)’s. This is the main type of QES systems considered in the literature;

(iii) if \( \max\{m, n - 1\} \geq 3 \), then not only \( N \) but also \( z_k \)’s may appear in terms involving powers of \( z \). If \( z_k \)’s do appear before any \( z \)-dependent term, then for each \( N \geq 0 \), there are \( N + 1 \) different potentials \( V_N \), differing in several parameters in terms involving powers of \( z \), have the same eigenvalue (when the additive constant, or the zero point, is appropriately adjusted). When \( z_k \)’s appear only in the first power term in \( z \), such systems are called type 2 QES systems in [3]. We see that QES models of higher types are possible.

This gives a very simple algebraic classification of exact and quasi-exact solvabilities. Previously exact and QES systems were treated separately.

In the rest of this paper we shall consider only cases with \( m, n \leq 2 \). Coordinates with \( n \leq 2 \) are called the sinusoidal coordinates. Let \( P_2(z) = A_2z^2 + A_1z + A_0 \) and \( Q_2(z) = az^2 + \beta z + \gamma \). Hence the solvability of the system is determined solely by \( A_2 \): exactly solvable if \( A_2 = 0 \), or (type 1) QES otherwise. The potential \( V_N \) takes the form

\[
V_N = W_0'' - W_0' + \alpha N^2 - 2A_1N - 2A_2Nz - 2A_2 \sum_{k=1}^{N} z_k - 2 \sum_{k=1}^{N} \frac{1}{z - z_k} \left\{ P_2(z_k) - \frac{\alpha}{2}z_k - \frac{\beta}{4} \sum_{l \neq k} Q_2(z_l) \right\}.
\]

Demanding the residues at \( z_k \)’s vanish gives the Bethe ansatz equations satisfied by the roots \( z_k \)’s:

\[
P_2(z_k) - \frac{\alpha}{2}z_k - \frac{\beta}{4} \sum_{l \neq k} \frac{Q_2(z_l)}{z_k - z_l} = 0, \quad k = 1, 2, \ldots, N,
\]

or

\[
A_2z_k^2 + (A_1 - \frac{\alpha}{2})z_k + A_0 - \frac{\beta}{4} \sum_{l \neq k} \frac{\alpha z_l^2 + \beta z_l + \gamma}{z_k - z_l} = 0.
\]

Using

\[
W_0''(z) = \frac{P_2(z)}{\sqrt{Q_2(z)}}, \quad W_0'(z) = z' \frac{dW_0'}{dz} = \frac{Q_2 \frac{dP_2}{dz} - \frac{1}{2} P_2 \frac{dQ_2}{dz}}{Q_2},
\]

we arrive at the potential

\[
V_N(x) = P_2 - Q_2 \frac{dP_2}{dz} + \frac{1}{2} P_2 \frac{dQ_2}{dz} - \left( 2A_1N - \alpha N^2 + 2A_2Nz + 2A_2 \sum_{k=1}^{N} z_k \right)
\]

\[
= \left[ \frac{(A_2z^2 + A_1z + A_0)^2}{\alpha z^2 + \beta z + \gamma} - 2(N + 1)A_2z + \frac{1}{2} (A_2z^2 + A_1z + A_0) \frac{2\alpha z + \beta}{\alpha z^2 + \beta z + \gamma} \right]_{z = x(z)}
\]

\[
- \left( 2N + 1 \right) A_1 - \alpha N^2 + 2A_2 \sum_{k=1}^{N} z_k \right),
\]

and the wave function

\[
\psi_N \sim e^{-W_0} p_N(z) = e^{\int_{x(z)}^{z} \frac{dP_2}{dz}} \sqrt{Q_2} p_N(z).
\]

\[1\] We take this opportunity to correct a typographic error in [11, 13], where the condition for \( m \) and \( n \) for this case was erroneously written as \( \min\{m, n - 1\} \geq 3 \).
Eq. (14) gives the most general form of potential, based on sinusoidal coordinates, that cover both the exactly and quasi-exactly solvable systems.

It turns out that there are only three inequivalent canonical forms of the sinusoidal coordinates \[ 17 \], namely, (i) \( z^2 = \gamma \neq 0 \), (ii) \( z^2 = \beta z (\beta > 0) \), and (iii) \( z^2 = \alpha (z^2 + \delta) (\delta = 0, \pm 1 \text{ for } \alpha > 0, \text{ and } \delta = -1 \text{ if } \alpha < 0) \). Case (i) and (ii) correspond to one- and three-dimensional oscillator-type potentials, respectively. In case (iii), for \( \alpha > 0 \), the potentials are of the Scarf II (\( \delta = 1 \)), Morse (\( \delta = 0 \)) and generalized Pöschl-Teller (\( \delta = -1 \)) types, while for \( \alpha < 0 \) and \( \delta = -1 \), the potentials generated belong to the Scarf I type.

For clarity of presentation, in the main text we shall illustrate the prepotential construction of QNM models only for the Scarf II, Morse, and generalized Pöschl-Teller type potentials. Other cases are summarized in the Appendices. As mentioned in Sect. I, we choose to discuss these three cases because they have some interesting features that serve as good examples to illustrate the procedure, and because there are new QNM models in these types not realized in \[ 4 \].

### III. SCARF II: \( z^2 = \alpha (z^2 + 1) \)

Consider first the case \( z^2 = \alpha (z^2 + 1) (\alpha > 0) \), which is solved to give \( z(x) = \pm \sinh(\sqrt{\alpha} x) \). For definiteness we shall take \( z(x) = \sinh(\sqrt{\alpha} x) \). The case corresponding to the negative sign is simply the mirror image of the present case (i.e., by taking \( x \to -x \)). The same applies to the other sinusoidal coordinates discussed in the rest of the paper.

Putting \( \beta = 0 \) and \( \gamma = \alpha \) in (14), we get the potential \((-\infty < x < \infty)\)

\[
V_N = \frac{A_2^2}{\alpha} (z^2 + 1) + \left[ \frac{(A_0 - A_2)^2}{\alpha} - A_1 \left( \frac{A_1}{\alpha} + 1 \right) \right] \frac{1}{z^2 + 1} + (A_0 - A_2) \left( \frac{2A_1}{\alpha} + 1 \right) \frac{z}{z^2 + 1} + A_2 \left( \frac{2A_1}{\alpha} - 2N - 1 \right) z - \left[ 2A_1 N - \alpha N^2 - \frac{A_1^2}{\alpha} - \frac{2A_2}{\alpha} (A_0 - A_2) + 2A_2 \sum_{k=1}^{N} z_k \right].
\]

(16)

In terms of \( x \), it is

\[
V_N(x) = \frac{A_2^2}{\alpha} \cosh^2(\sqrt{\alpha} x) + \left[ \frac{(A_0 - A_2)^2}{\alpha} - A_1 \left( \frac{A_1}{\alpha} + 1 \right) \right] \text{sech}^2(\sqrt{\alpha} x)
\]

\[
+ (A_0 - A_2) \left( \frac{2A_1}{\alpha} + 1 \right) \tanh(\sqrt{\alpha} x) \text{sech}(\sqrt{\alpha} x) + A_2 \left( \frac{2A_1}{\alpha} - 2N - 1 \right) \sinh(\sqrt{\alpha} x)
\]

\[- \left[ 2A_1 N - \alpha N^2 - \frac{A_1^2}{\alpha} - \frac{2A_2}{\alpha} (A_0 - A_2) + 2A_2 \sum_{k=1}^{N} z_k \right].
\]

(17)

The prepotential \( W_0 \), obtained from (17), is

\[
W_0(x) = \frac{A_2}{\alpha} z + \frac{A_1}{2\alpha} \ln \left( z^2 + 1 \right) + \frac{A_0 - A_2}{\alpha} \tan^{-1} z
\]

\[= \frac{A_2}{\alpha} \sinh(\sqrt{\alpha} x) + \frac{A_1}{\alpha} \ln \cosh(\sqrt{\alpha} x) + \frac{A_0 - A_2}{\alpha} \tan^{-1} \sinh(\sqrt{\alpha} x). \]

(18)

Hence the ground state wave function \( \psi_0 \sim e^{-W_0} \) is

\[
\psi_0 \sim e^{-\frac{A_2}{\alpha} \sinh(\sqrt{\alpha} x)} \left( \cosh(\sqrt{\alpha} x) \right)^{-\frac{A_1}{\alpha}} e^{\frac{A_0 - A_2}{\alpha} \tan^{-1} \sinh(\sqrt{\alpha} x)}. \]

(19)

**A. \( A_2 = 0 \)**

Let us first discuss the exactly solvable case, with \( A_2 = 0 \). The potential becomes

\[
V_N = \left[ \frac{A_2^2}{\alpha} - A_1 \left( \frac{A_1}{\alpha} + 1 \right) \right] \text{sech}^2(\sqrt{\alpha} x) + A_0 \left( \frac{2A_1}{\alpha} + 1 \right) \tanh(\sqrt{\alpha} x) \text{sech}(\sqrt{\alpha} x)
\]

\[= 2A_1 N - \alpha N^2 - \frac{A_1^2}{\alpha}. \]

(20)
Suppose all $A_i$’s are real, the potential $V_N$ is just the Scarf II potential given in [20]. Recall that in our approach we have $H_N \phi_N = 0$, hence the eigenvalue of $H_N$ with potential $V_N$ is always zero. If we define the Scarf II potential $V(x)$ by the first two $N$-independent terms, then $V_N(x) = V(x) - E_N$, where $E_N = 2A_1N - \alpha N^2 - A_1^2/\alpha$ are the eigenvalues of $V(x)$. The corresponding eigenfunctions are given by (1), with $\phi_0$ given by (19), and $p_N(z)$ defined by the roots $z_k$'s of the corresponding Bethe ansatz equation (12), which gives the Jacobi polynomials [22].

If we take instead

$$A_0 = -\frac{i}{2} \frac{2A_1}{\alpha} + 1 = -i \frac{c}{\alpha},$$

then the potential is

$$V_N(x) = \frac{1}{4\alpha} (\alpha^2 + c^2 - d^2) \text{sech}^2(\sqrt{\alpha}x) - \frac{cd}{2\alpha} \tanh(\sqrt{\alpha}x)\text{sech}(\sqrt{\alpha}x)$$

$$-\left[ \frac{c^2}{4\alpha} - \left(N + \frac{1}{2}\right)^2 \alpha - ic \left(N + \frac{1}{2}\right) \right].$$

This is the exactly solvable case 1 hyperbolic QNM system discussed in [4]. The special case where $d = 0$ has been employed in [3] to study black hole's QNMs. If we take the potential to be defined by the first two terms, then the energies and ground state wave function are

$$E_N = \frac{c^2}{4\alpha} - \left(N + \frac{1}{2}\right)^2 \alpha - ic \left(N + \frac{1}{2}\right).$$

and

$$\psi_0(x) \sim (\cosh(\sqrt{\alpha}x))^{(ic+\alpha)/2\alpha} \exp(id \tan^{-1}(\sinh(\sqrt{\alpha}x)/2\alpha)).$$

Note that $E_N$ is independent of $d$, which is a general feature of the Scarf-type potentials. Also, the imaginary part is proportional to $N + 1/2$, which is characteristic of black hole QNMs.

### B. $A_2 \neq 0$

If $A_2 \neq 0$, then reality of the first term of $V_N$ in (17) requires that $A_2$ be real, or purely imaginary.

If $A_2$ is real, then it is clear from the first term of (19) that $\psi_0$, which governs the asymptotic behaviors of $\psi_N$, is not normalizable on the whole line. Hence there is no QES model with real energies in this case.

Suppose $A_2$ is purely imaginary, say $A_2 = ic \neq 0$ with real constant $c$. Then the fourth term of $V_N$ in (17) can be real provided that

$$\frac{2A_1}{\alpha} - 2N - 1 = id, \quad d : \text{real}.$$

If $d \neq 0$, then reality of the third term of $V_N$ demands that $A_0 - A_2 = \pm[2(N + 1) - id]/\alpha$. But then, as can be easily checked, with these values of $A_i$’s the second term of $V_N$ cannot be real, unless $d = 0$.

So we are left with the choice $A_2 = ic \neq 0$ and $2A_1/\alpha - 2N - 1 = 0$. This implies that $A_0 - A_2$ must be real for $V_N$ real. Let $A_0 - A_2 = a\alpha$ with real $a$. The potential of this system assumes the form

$$V_N(x) = -\frac{c^2}{\alpha} \cosh^2(\sqrt{\alpha}x) + \alpha \left[a^2 - \left(N + \frac{3}{2}\right) \text{sech}^2(\sqrt{\alpha}x) + 2a\alpha (N + 1) \tanh(\sqrt{\alpha}x)\text{sech}(\sqrt{\alpha}x) \right]$$

$$-\left( -\frac{\alpha}{4} - 2ica + 2ic \sum_{k=1}^{N} z_k \right).$$

The ground state wave function is

$$\psi_0 \sim e^{-i\frac{\alpha}{2} \sinh(\sqrt{\alpha}x)} \left(\cosh(\sqrt{\alpha}x)\right)^{-(N + \frac{3}{2})} e^{-a \tan^{-1}(\sinh(\sqrt{\alpha}x))}.$$

For $a \neq 0$, the potential [20] is a new QES model with QNMs which has been overlooked in our previous study based on the Lie-algebraic theory of [2].

The case $a = 0$ leads to a totally different model with real QES energies, which was first discussed in [21]. We briefly discuss it in the next subsection.
C. Singular potential with \( A_2 \neq 0 \)

For \( a = 0 \), i.e., \( A_2 = A_0 = ic \neq 0 \), the potential (26) becomes

\[
V_N(x) = -\frac{c^2}{\alpha} \cosh(\sqrt{\alpha}x) - \alpha \left( N + \frac{1}{2} \right) \left( N + \frac{3}{2} \right) \text{sech}^2(\sqrt{\alpha}x) - \left( -\frac{\alpha}{4} + 2ic \sum_{k=1}^{N} z_k \right). \tag{28}
\]

This is a singular potential unbounded from below. Yet it exhibits very peculiar features, such as the existence of QES bound states with real energies, and QES total transmission modes. We refer the reader to [21] for a detailed discussion of this potential.

At first sight it may seem strange that the system has QES real energies, owing to the term \( 2ic \sum_k z_k \). We prove below that this term is indeed real, as the sum \( \sum_k z_k \) is purely imaginary.

The Bethe ansatz equations (12) in this case are

\[
icz_k^2 + \alpha N z_k + ic - \alpha \sum_{l \neq k} \frac{z_l^2 + 1}{z_k - z_l} = 0, \quad k = 1, 2, \ldots, N. \tag{29}
\]

Multiplying (29) by \(-1\) and taking its complex conjugate, we have

\[
icz_k^2 + \alpha N (-\bar{z}_k) + ic - \alpha \sum_{l \neq k} \frac{(-\bar{z}_k)^2 + 1}{-\bar{z}_k - (-\bar{z}_l)} = 0, \quad k = 1, 2, \ldots, N. \tag{30}
\]

Here \( \bar{z} \) represent the complex conjugate of \( z \). Comparing (29) and (30), we conclude that if \( z_k \)'s are the solutions of (29), then so are their negative complex conjugates \(-\bar{z}_k \)'s. This implies that the sum \( \sum_k z_k \) is purely imaginary, and hence the QES energies are real.

IV. MORSE: \( z^2 = \alpha x^2 \) (\( \alpha > 0 \))

In this case the change of variables is given by \( z(x) = \exp(\pm \sqrt{\alpha}x) \). Again we shall only consider the positive case, i.e., we take \( z(x) = \exp(\sqrt{\alpha}x) \). The potential is \(( -\infty < x < \infty) \)

\[
V_N(x) = \frac{A_2^2}{\alpha} z^2 + A_2 \left( \frac{2A_1}{\alpha} - 2N - 1 \right) z + A_0 \left( \frac{2A_1}{\alpha} + 1 \right) \frac{1}{z} + A_0^2 \frac{1}{z^2} - \left( 2A_1 N - \alpha N^2 - \frac{A_1^2}{\alpha} - \frac{2A_2 A_0}{\alpha} + 2A_2 \sum_{k=1}^{N} z_k \right)
\]

\[
= \frac{A_2^2}{\alpha} e^{2\sqrt{\alpha}x} + A_2 \left( \frac{2A_1}{\alpha} - 2N - 1 \right) e^{\sqrt{\alpha}x} + A_0 \left( \frac{2A_1}{\alpha} + 1 \right) e^{-\sqrt{\alpha}x} + A_0^2 e^{-2\sqrt{\alpha}x}
\]

\[
- \left( 2A_1 N - \alpha N^2 - \frac{A_1^2}{\alpha} - \frac{2A_2 A_0}{\alpha} + 2A_2 \sum_{k=1}^{N} z_k \right). \tag{31}
\]

The prepotential \( W_0 \) is

\[
W_0 = \frac{1}{\alpha} \left( A_2 z + A_1 \ln z - \frac{A_0}{z} \right)
\]

\[
= \frac{A_2}{\alpha} e^{\sqrt{\alpha}x} + \frac{A_1}{\alpha} \sqrt{\alpha}x - \frac{A_0}{\alpha} e^{-\sqrt{\alpha}x}. \tag{32}
\]

Hence the ground state wave function is

\[
\phi_0 \sim e^{\frac{-A_0}{\alpha} e^{\sqrt{\alpha}x} - \frac{A_1}{\alpha} \sqrt{\alpha}x + \frac{A_1}{\alpha} e^{-\sqrt{\alpha}x}}. \tag{33}
\]

A. \( A_2 = 0 \)

For \( A_2 = 0 \), the potential reads

\[
V_N(x) = A_0 \left( \frac{2A_1}{\alpha} + 1 \right) e^{-\sqrt{\alpha}x} + \frac{A_0^2}{\alpha} e^{-2\sqrt{\alpha}x} - \left( 2A_1 N - \alpha N^2 - \frac{A_1^2}{\alpha} \right). \tag{34}
\]
To get a real potential defined by the first two terms of $V_N$, $A_0$ and $A_1$ must be both real, or both complex.

If $A_0$ is real, then the first term of $V_N$ requires that $A_1$ be real. The resulted model is the exactly solvable Morse potential.

If $A_0$ is complex, then the second term of $V_N$ demands that $A_0$ be purely imaginary, i.e., $A_0 = ic$ with $c$ real. The first term of $V_N$ then requires that $2A_1/\alpha + 1 = -id$ is also purely imaginary. The potential is

$$V_N(x) = cde^{-\sqrt{\alpha}x} - \frac{c^2}{\alpha}e^{-2\sqrt{\alpha}x} - \alpha \left[ \frac{d^2 - 1}{4} - N^2 - N - id \left(N + \frac{1}{2}\right) \right],$$

with ground state

$$\psi_0 \sim e^{i\frac{d}{2}x} e^{\sqrt{\alpha}x} + \frac{1}{2}(id+1)\sqrt{\alpha}x.$$

This gives a new exactly solvable QNM model. This system was overlooked in [4].

We note here that this system, or rather its mirror image, can be obtained by a different complexification of $A_0$, $A_1$ and $A_2$. We shall discuss this in the next subsection.

B. $A_2 \neq 0$

If $A_2 \neq 0$, then $A_2$ has to be real or purely imaginary.

If $A_2$ is real, then all $A_i$’s must be real. In this case, for $A_2 > 0$ and $A_0 < 0$, the potential $V_N$ in (31) defines a QES system.

For $A_2$ purely imaginary, $A_1/\alpha - (N + 1/2)$ must either be zero, or purely imaginary. If $A_1/\alpha - (N + 1/2) = 0$, we can have

$$A_2 = -i \frac{b}{2}, \quad A_1 = \left(N + \frac{1}{2}\right) \alpha, \quad A_0 = -d, \quad b, d : \text{real}.$$  

This leads to a potential

$$V_N(x) = -\frac{b^2}{4\alpha} e^{-\sqrt{\alpha}x} + (N + 1) de^{-\sqrt{\alpha}x} + \frac{d^2}{4\alpha} e^{-2\sqrt{\alpha}x} - \left( -\frac{\alpha}{4} - \frac{b}{2\alpha} - ib \sum_{k=1}^{N} z_k \right).$$

The system defined by the first three terms of $V_N$ is a QES system with complex energies given by the last term in bracket. This is indeed the very first QES QNM model presented in [4].

On the other hand, if $A_1/\alpha - (N + 1/2)$ is purely imaginary, then one can choose

$$A_2 = -ic, \quad A_1 = i \frac{d}{2} + \left(N + \frac{1}{2}\right), \quad A_0 = 0,$$

with $c$ and $d$ real constants. The resulted potential is

$$V_N(x) = cde^{\sqrt{\alpha}x} - \frac{c^2}{\alpha} e^{2\sqrt{\alpha}x} - \alpha \left[ \frac{d^2 - 1}{4} - i \frac{d}{2} - 2i \frac{c}{\alpha} \sum_{k=1}^{N} z_k \right].$$

In this case though $A_2 \neq 0$, the parameter $N$ and the roots $z_k$’s do not appear in any $x$-dependent term, and hence the system is exactly solvable.

This system is indeed the mirror image of the model described by (35), although the functional forms of the two potentials look very differently. To show this, we need to demonstrate that the potential, the eigenvalues and the eigenfunctions of one system are mapped into those of the other system under parity transformation $x \to -x$.

The wave function of the present system is

$$\psi_N \sim e^{i\frac{d}{2}x} e^{\sqrt{\alpha}x} - \frac{1}{2}(id+2N+1)\sqrt{\alpha}x \prod_{k=1}^{N} (z - z_k),$$

where the roots $z_k$’s satisfy the BAE (from (12))

$$-icz_k^2 + \alpha \left( i \frac{d}{2} + N \right) z_k - \alpha \sum_{l \neq k}^{N} \frac{z_k^2}{z_k - z_l} = 0, \quad k = 1, 2, \ldots, N.$$
We stress here that \( z = e^{\sqrt{\alpha} x} \) as before.

Under parity transformation \( x \rightarrow -x \), we have \( z \rightarrow z^{-1} \), \( z_k \rightarrow z_k^{-1} \). Eqs. \( (40) \), \( (41) \) and \( (42) \) are mapped, respectively, into

\[
V_N(x) = cd e^{-\sqrt{\alpha} x} - \frac{c^2}{\alpha} e^{-2\sqrt{\alpha} x} - \alpha \left[ \frac{d^2 - 1}{4} - i \frac{d}{2} - 2i \frac{c}{\alpha} \sum_k \frac{1}{z_k} \right],
\]

(43)

\[
\psi_N \sim e^{i \frac{\pi}{4} e^{-\sqrt{\alpha} x} + \frac{1}{2} (id + 2N + 1) \sqrt{\alpha} x} \prod_{k=1}^{N} \left( \frac{1}{z} - \frac{1}{z_k} \right),
\]

(44)

and

\[
-ic \frac{1}{z_k} + \alpha \left( i \frac{d}{2} + N \right) \frac{1}{z_k} - \alpha \sum_{l \neq k}^{N} \frac{z_l}{z_k(z_l - z_k)} = 0, \quad k = 1, 2, \ldots, N.
\]

(45)

Multiplying \( (45) \) by \( -z_k^2 \), and using the identity

\[
\sum_{l \neq k}^{N} \frac{z_l}{z_l - z_k} = N - 1 - \sum_{l \neq k}^{N} \frac{z_k}{z_k - z_l},
\]

(46)

we can transform \( (45) \) to

\[
- \alpha \left( i \frac{d}{2} + 1 \right) z_k + ic - \alpha \sum_{l \neq k}^{N} \frac{z_k^2}{z_k - z_l} = 0, \quad k = 1, 2, \ldots, N.
\]

(47)

This is just the BAE for the roots \( z_k \)'s of the eigenfunctions of the system defined by the potential \( (35) \). Hence the BAE of this system is mapped into the BAE of the system defined by \( (35) \) under parity.

Now the eigenfunctions \( (44) \) can be rewritten as

\[
\psi_N \sim e^{i \frac{\pi}{4} e^{-\sqrt{\alpha} x} + \frac{1}{2} (id + 2N + 1) \sqrt{\alpha} x} \left( e^{\sqrt{\alpha} x} \right)^N z^{-N} \prod_{k=1}^{N} \left( z - z_k \right),
\]

\[
\sim e^{i \frac{\pi}{4} e^{-\sqrt{\alpha} x} + \frac{1}{2} (id + 2N + 1) \sqrt{\alpha} x} \prod_{k=1}^{N} \left( z - z_k \right),
\]

(48)

where we have used the fact that \( (e^{\sqrt{\alpha} x})^N z^{-N} = 1 \). Eq. \( (48) \) is just the eigenfunction of the potential \( (35) \). Thus, together with the result of the last paragraph on BAE, one sees that under parity the eigenfunctions of the system given by \( (40) \) are mapped into those of given by \( (35) \).

Finally we show that the last term of the potential \( (40) \) is equal to the last term of \( (35) \). This proves the equality of the eigenvalues of both systems. Dividing the BAE \( (47) \) by \( z_k \), summing over all \( k \) and using the identity

\[
\sum_{k=1}^{N} \sum_{l \neq k}^{N} \frac{z_k}{z_k - z_l} = \frac{1}{2} N(N - 1),
\]

(49)

we get

\[
-2ic \alpha \sum_{k=1}^{N} \frac{1}{z_k} = -idN - N^2 - N.
\]

(50)

So the last term of the potential \( (40) \) becomes

\[
- \alpha \left[ \frac{d^2 - 1}{4} - i \frac{d}{2} - 2i \frac{c}{\alpha} \sum_k z_k \right] = -\alpha \left[ \frac{d^2 - 1}{4} - N^2 - N - id \left( N + \frac{1}{2} \right) \right].
\]

(51)

This is equal to the last term of \( (35) \). Hence under parity potential \( (40) \) is mapped into \( (35) \).
V. GENERALIZED PÖSCHL-TELLER: $z^2 = \alpha(z^2 - 1)$

In this case $z(x) = \cosh(\sqrt{\alpha}x)$. The potential is $(0 \leq x < \infty)$

$$V_N(x) = \frac{A_2^2}{\alpha} \sinh^2(\sqrt{\alpha}x) + \left[\frac{(A_0 + A_2)^2}{\alpha} + A_1 \left(\frac{A_1}{\alpha} + 1\right)\right] \cosh^2(\sqrt{\alpha}x)$$

$$+ (A_0 + A_2) \left(\frac{2A_1}{\alpha} + 1\right) \coth(\sqrt{\alpha}x) \cosech(\sqrt{\alpha}x) + A_2 \left(\frac{2A_1}{\alpha} - 2N - 1\right) \cosh(\sqrt{\alpha}x)$$

$$- \left[2A_1N - \alpha N^2 - \frac{A_1^2}{\alpha} - \frac{2A_2}{\alpha}(A_0 + A_2) + 2A_2 \sum_{k=1}^{N} z_k\right].$$

(52)

The ground state wave function $\psi_0 \sim \exp(-W_0)$ is

$$\psi_0 \sim e^{-\frac{A_2}{\alpha} \cosh(\sqrt{\alpha}x)} \left(\sinh(\sqrt{\alpha}x)\right)^{-\frac{A_1}{\alpha}} e^{\frac{A_2 + A_0}{\alpha} \coth^{-1} \cosh(\sqrt{\alpha}x)}$$

$$\sim e^{-\frac{A_2}{\alpha} \cosh(\sqrt{\alpha}x)} \left(\sinh(\sqrt{\alpha}x)\right)^{-\frac{A_1}{\alpha}} \left[\tanh(\sqrt{\alpha}x)\right]^{-\frac{A_2 + A_0}{\alpha}}.$$  (53)

Since $V_N \to \infty$ as $x \to 0$, we must have the boundary condition $\psi_N \to 0$ as $x \to 0$.

A. $A_2 = 0$

For $A_2 = 0$, the potential is

$$V_N = \left[\frac{A_0^2}{\alpha} + A_1 \left(\frac{A_1}{\alpha} + 1\right)\right] \cosech^2(\sqrt{\alpha}x) + A_0 \left(\frac{2A_1}{\alpha} + 1\right) \coth(\sqrt{\alpha}x) \cosech(\sqrt{\alpha}x)$$

$$- \left[2A_1N - \alpha N^2 - \frac{A_1^2}{\alpha}\right].$$

(54)

If all $A_i$’s are real, this gives the generalized Pöschl-Teller potential in [20]. If we take the Pöschl-Teller potential $V(x)$ to consist of the first two $N$-independent terms, then the eigen-energies are $E_N = 2A_1N - \alpha N^2 - A_1^2/\alpha$. These eigenvalues are the same as those of the Scarf II potential. Our approach makes it very easy to see why the eigenvalues are the same for the two apparently different systems.

We can also take

$$A_0 = -i\frac{d}{2}, \quad \frac{2A_1}{\alpha} + 1 = -i\frac{c}{\alpha},$$

(55)

which result in the potential

$$V_N(x) = -\frac{1}{4\alpha} \left(a^2 + c^2 + d^2\right) \cosech^2(\sqrt{\alpha}x) - \frac{cd}{2\alpha} \coth(\sqrt{\alpha}x) \cosech(\sqrt{\alpha}x)$$

$$- \left[\frac{e^2}{4\alpha} - \left(N + \frac{1}{2}\right)^2 \alpha - i\left(N + \frac{1}{2}\right)\right].$$

(56)

This is the exactly solvable case 2 hyperbolic QNM system discussed in [4].

B. $A_2 \neq 0$

As with the Scarf II case, if $A_2 \neq 0$, then reality of the first term of $V_N$ in [52] requires that $A_2$ be real, or purely imaginary.

If $A_2$ is real, then $A_0$ and $A_1$ have to be real. The boundary condition that $\psi_N \to 0$ as $x \to \infty$ is met if $A_2 > 0$. As $x \to 0$, we have

$$\psi_0 \sim x^{-\frac{A_2 + A_1 + A_0}{\alpha}}.$$  (57)
Thus $\psi_0 \to 0$ as $x \to 0$ is guaranteed if

$$A_2 + A_1 + A_0 < 0. \quad (58)$$

With $A_i$’s satisfying (58) and $A_2 > 0$, the potential $V_N$ gives a QES system with real energies.

If $A_2$ is imaginary, say $A_2 = ic \neq 0$ with real constant $c$, then the fourth term of $V_N$ in (59) can be real if $2A_1/\alpha - 2N - 1 = 0$ or $id \neq 0$. As in the Scarf II case, if $2A_1/\alpha - 2N - 1 = id \neq 0$, then reality of the third term of $V_N$ demands that $A_0 + A_2 = \pm [2(N + 1) - id\alpha]$. But then the second term of $V_N$ cannot be real, unless $d = 0$.

So we must have $2A_1/\alpha - 2N - 1 = 0$. In this case one has $A_2 + A_0 = -ia \neq 0$ with real $a$. The potential $V_N$ is

$$V_N(x) = -\frac{c^2}{\alpha} \sinh^2(\sqrt{\alpha}x) + \alpha \left[ a^2 + \left( N + \frac{1}{2} \right) \left( N + \frac{3}{2} \right) \right] \cosh^2(\sqrt{\alpha}x)$$

$$-2a\alpha (N + 1) \coth(\sqrt{\alpha}x) \cosech(\sqrt{\alpha}x)$$

$$- \left( -\frac{a}{4} + 2ica + 2ic \sum_{k=1}^{N} z_k \right), \quad (59)$$

and the ground state wave function

$$\psi_0 \sim e^{-i\frac{c}{2} \cosh(\sqrt{\alpha}x)} (\sinh(\sqrt{\alpha}x))^{-(N+\frac{1}{2})} \left[ \tanh(\frac{\sqrt{\alpha}}{2} x) \right]^a. \quad (60)$$

To satisfy the boundary condition $\psi_0 \to 0$ as $x \to 0$, we have from (58) the condition

$$a > N + \frac{1}{2}. \quad (61)$$

With these $A_i$’s, we have a new QES system with QNMs.

Unlike the Scarf II case, however, when $a = 0$, we do not have a QES model with real energy, since in this case $\psi_0$ is not normalizable at $x = 0$.

VI. SUMMARY

Exactly solvable models admitting QNM maybe useful in providing insights to QNMs emitted from more complicated systems such as black holes. In [4] we have found some new exactly solvable QNM models, and a new QES QNM model within the $sl(2)$ Lie-algebraic approach to QES theory.

In this paper we have demonstrated how exactly solvable and QES models admitting quasinormal modes can be constructed and classified very simply and directly by the prepotential approach. This approach, unlike the Lie-algebraic approach, does not require any knowledge of the underlying symmetry of the system. It treats both quasi-exact and exact solvabilities on the same footing, and gives the potential as well as the eigenfunctions and eigenvalues simultaneously. We also present a new exactly solvable Morse-like model with quasinormal modes, and two new quasi-exactly solvable models with quasinormal modes of the Scarf II and generalized Pöschl-Teller types.

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Appendix A: Shifted oscillator: $z'^2 = \gamma > 0$

The corresponding transformation $z(x)$ is $z(x) = \sqrt{\gamma}x + \text{constant}$. By an appropriate translation one can always set the constant to zero. Hence we shall take $z(x) = \sqrt{\gamma}x$ without loss of generality.

The potential is ($-\infty < x < \infty$)

$$V_N = A_2^2 \gamma x^4 + 2A_2A_1 \sqrt{\gamma} x^3 + \left( A_1^2 + 2A_2A_0 \right) x^2 + 2 \left[ \frac{A_1A_0}{\gamma} - A_2(N + 1) \right] \sqrt{\gamma} x$$

$$- \left[ 2A_1(N + \frac{1}{2}) - \frac{A_1^2}{\gamma} + 2A_2 \sum_{k=1}^{N} z_k \right]. \quad (A1)$$
The potential becomes start with degree six, i.e., the sextic oscillator \( [23] \). In fact, the potential (B1) with model with real energies. Together with the discussion in Appendix A, one concludes that one-dimensional QES models.

For \( A_2 = 0 \), the potential is

\[
V_N = \left( A_1 x + \frac{A_0}{\sqrt{N}} \right)^2 - 2A_1(N + \frac{1}{2}).
\]

(A3)

So \( A_1 \) and \( A_0 \) must both be real, this is just the potential of the shifted oscillator.

If we take

\[
A_1 = -\frac{c}{2}, \quad A_0 = 0,
\]

the potential becomes

\[
V_N = -\frac{1}{4} c^2 \beta x^2 + ic \left( N + \frac{1}{2} \right).
\]

(A5)

This is the case 5 exactly solvable QNM model discussed in [4].

**Appendix B: Radial oscillator** \( z'^2 = \beta z, \quad (\beta > 0) \)

For simplicity we take \( z(x) = \frac{\beta}{2} x^2 \). The potential is \( (0 \leq x < \infty) \)

\[
V_N = \frac{1}{64} A_2^2 \beta^2 x^6 + \frac{1}{8} A_2 A_1 \beta x^4 + \frac{1}{4} \left[ A_1^2 + 2A_2 A_0 - \frac{1}{2} A_2 \beta (4N + 3) \right] x^2
\]

\[
+ \frac{4A_0}{\beta} \left( \frac{A_0}{\beta} + \frac{1}{2} \right) \frac{1}{x^2} - \left[ A_1 \left( 4N + 1 - \frac{4A_0}{\beta} \right) + 2A_2 \sum_{k=1}^{N} z_k \right].
\]

(B1)

The ground state \( \psi_0 \sim \exp(-W_0) \) is

\[
\psi_0 \sim x^{-\frac{2A_0}{\beta}} e^{-\frac{1}{4} A_1 x^2 - \frac{1}{2} A_2 \beta x^4}.
\]

(B2)

It can be checked that if \( A_2 \neq 0 \), then all \( A_i \)’s have to be real. Hence there is no QES model with QNMs this case. For \( A_2 > 0 \) and \( A_0 < 0 \), the wave functions \( \psi_N \) are normalized on the positive half-line. The system is then a QES model with real energies. Together with the discussion in Appendix A, one concludes that one-dimensional QES models start with degree six, i.e., the sextic oscillator \([23]\). In fact, the potential (B1) with \( A_2 = 2a > 0, \quad A_1 = 2b, \quad A_0 = 0 \) and \( \beta = 4 \), namely,

\[
V_N = a^2 x^6 + 2ab x^4 + \left[ b^2 - (4N + 3)a \right] x^2
\]

\[
- \left[ (4N + 1)b + 4a \sum_{k=1}^{N} z_k \right],
\]

(B3)

is the very first QES model discussed in the literature [5].

If \( A_2 = 0 \), then (B1) becomes

\[
V_N = \frac{A_1}{4} x^2 + \frac{4A_0}{\beta} \left( \frac{A_0}{\beta} + \frac{1}{2} \right) \frac{1}{x^2} - A_1 \left( 2N + \frac{1}{2} - \frac{2A_0}{\beta} \right).
\]

For real \( A_1 \) and \( A_0 \), this is just the radial oscillator.

Suppose we take

\[
\beta = 4, \quad A_1 = -2ia, \quad \text{and} \quad A_0 = -2\gamma.
\]

(B5)

The potential takes the form

\[
V_N = -a^2 x^2 + \frac{\gamma(\gamma - 1)}{x^2} + ia (4N + 2\gamma + 1).
\]

(B6)

This is the Case 4 model with QNMs presented in [4].
Appendix C: Scarf I: $z^2 = \alpha(1 - z^2)$

In this case $z(x) = \sin(\sqrt{\alpha}x)$. The potential is

$$V_N(x) = \frac{A_2^2}{\alpha} \cos^2(\sqrt{\alpha}x) + \left(\frac{(A_0 + A_2)^2}{\alpha} + A_1 \left(\frac{A_1}{\alpha} - 1\right)\right) \sec^2(\sqrt{\alpha}x) + (A_0 + A_2) \left(\frac{2A_1}{\alpha} - 1\right) \tan(\sqrt{\alpha}x) \sec(\sqrt{\alpha}x)$$

$$- A_2 \left(\frac{2A_1}{\alpha} + 2N + 1\right) \sin(\sqrt{\alpha}x) - \left[2A_1 N + \alpha N^2 + \frac{A_1^2}{\alpha} + \frac{2A_2}{\alpha} (A_0 + A_2) + 2A_2 \sum_{k=1}^{N} z_k\right], \quad (C1)$$

with the ground state wave function

$$\psi_0 \sim e^{\frac{A_1}{\alpha}} \sin(\sqrt{\alpha}x) \left(\cos(\sqrt{\alpha}x) \right)^{\frac{A_1}{\alpha}} e^{-\frac{2A_1}{\alpha} \tanh^{-1} \sin(\sqrt{\alpha}x)}. \quad (C2)$$

The system is defined only on a finite interval, usually taken to be $\sqrt{\alpha}x \in [-\pi/2, \pi/2]$. Hence there are no QNMs.

If $A_2 \neq 0$, then all $A_i$’s have to be real. The third term of (C2) then implies that the wave functions $\psi_N$ are not normalizable. So there are no QES systems with real energies.

For $A_2 = 0$, the potential becomes

$$V_N = \left[\frac{A_2^2}{\alpha} + A_1 \left(\frac{A_1}{\alpha} - 1\right)\right] \sec^2(\sqrt{\alpha}x) + A_0 \left(\frac{2A_1}{\alpha} - 1\right) \tan(\sqrt{\alpha}x) \sec(\sqrt{\alpha}x)$$

$$- \left[2A_1 N + \alpha N^2 + \frac{A_1^2}{\alpha}\right]. \quad (C3)$$

For real $A_1$ and $A_0$, this is Scarf I potential given in [20].

[1] For a comprehensive review, see eg.: K.D. Kokkotas and B.G. Schmidt, Living Rev. Rel. 2, 2 (1999); S. Chandrasekhar, The mathematical theory of black holes (Clarendon, Oxford, 1983).

[2] S.P. Kim, J. Korean Phys. Soc. 49, 764 (2006); G. Barton, Ann. Phys. 166, 322 (1986).

[3] V. Ferrari and B. Mashhoon, Phys. Rev. Lett. 52, 1361 (1984).

[4] H.-T. Cho and C.-L. Ho, J. Phys. A40, 1325 (2007).

[5] A.V. Turbiner and A.G. Ushveridze, Phys. Lett. A 126, 181 (1987).

[6] A.V. Turbiner, Comm. Math. Phys. 118, 467 (1988).

[7] A.V. Turbiner, Sov. Phys. JETP 67, 230 (1988).

[8] A. González, N. Kamran and P.J. Olver, Comm. Math. Phys. 153, 117 (1993).

[9] M.A. Shifman, Int. J. Mod. Phys. A4, 2897 (1989).

[10] A.G. Ushveridze, Sov. Phys.-Lebedev Inst. Rep. 2, 50, 54 (1988); Quasi-exactly solvable models in quantum mechanics (IOP, Bristol, 1994).

[11] G. Post and A. Turbiner, Russian J. Math. Phys. 3, 113 (1995).

[12] N. Kamran, R. Milson and P.J. Olver, Invariant modules and the reduction of nonlinear partial differential equations to dynamical systems (1999), solv-int/9904014.

[13] O. Özer and P. Roy, Cent. Eur. J. Phys. 7, 747 (2009).

[14] C.-L. Ho, Ann. Phys. 323, 2241 (2008).

[15] C.-L. Ho, “Prepotential approach to exact and quasi-exact solvabilities of Hermitian and non-Hermitian Hamiltonians (Talk presented at: “Conference in Honor of CN Yang’s 85th Birthday”, 31 Oct - 3 Nov, 2007, Singapore). arXiv:0801.0944 [hep-th].

[16] C.-L. Ho, Ann. Phys. 324, 1095 (2009).

[17] C.-L. Ho, J. Math. Phys. 50, 042105 (2009).

[18] R. Sasaki and K. Takaesaki, J. Phys. A34, 9533 (2001).

[19] C.-L. Ho and P. Roy, J. Phys. A36, 4617 (2003); Ann. Phys. 312, 161 (2004); C.-L. Ho, Ann. Phys. 321, 2170 (2006).

[20] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995); G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, (Springer-Verlag, Berlin, 1996).

[21] H.-T. Cho and C.-L. Ho, J. Phys. A41, 172002 (2008); ibid., 255308 (2008).

[22] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publications Vol. 23 (Amer. Math. Soc., New York, 1939).

[23] If one allows non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians, then a QES polynomial potential can be quartic in its variable. See e.g.: C.M. Bender and S. Boettcher, J. Phys. A31, L273 (1998), and [13].