Abstract. Let \( k \) be a field, and let \( L \) be an étale \( k \)-algebra of finite rank. If \( a \in k^\times \), let \( X_a \) be the affine variety defined by \( N_{L/k}(x) = a \). Assuming that \( L \) has at least one factor that is a cyclic field extension of \( k \), we give a combinatorial description of the unramified Brauer group of \( X_a \).

0. Introduction

Let \( k \) be a field, let \( L \) be an étale \( k \)-algebra of finite rank, and let \( N_{L/k} : L \to k \) be the norm map. Let \( a \in k^\times \), and let \( X_a \) be the affine \( k \)-variety determined by \( N_{L/k}(t) = a \). Let \( X_a^c \) be a smooth compactification of \( X_a \). The aim of this paper is to describe the group \( \text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \) under the hypothesis that \( L \) has at least one cyclic factor. We first give a combinatorial description of a group associated to the étale algebra \( L \) (see §3), and then give an explicit isomorphism between this group and \( \text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \) (see §4, in particular Theorem 4.3).

Let us illustrate our results by a special case. Let \( p \) be a prime number and let \( n \geq 1 \) be an integer; assume that \( \text{char}(k) \neq p \) and let \( F \) be a Galois extension of \( k \) with Galois group \( \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \). Suppose that \( L \) is a product of \( r \) linearly disjoint cyclic subfields of \( F \) of degree \( p^n \). Then we have (see Theorem 4.9):

**Theorem.**
\[
\text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \simeq (\mathbb{Z}/p^n\mathbb{Z})^{r-2}.
\]

We also give explicit generators of this group, as follows. With the above notation, let \( K \) be one of the cyclic subfields of degree \( p^n \) of \( F \), and let \( \chi \) be an injective morphism from \( \text{Gal}(K/k) \) to \( \mathbb{Q}/\mathbb{Z} \). Let us write \( L = K \times K' \), with \( K' = \prod_{i \in I} K_i \), where \( K_i \) is a cyclic subfield of \( F \) of degree \( p^n \) of \( F \) for all \( i \in I \), and assume that \( K \) and the fields \( K_i \) are linearly disjoint in \( F \). For all \( i \in I \), set \( N_i = N_{K_i/k}(y_i) \), considered as elements of \( k(X_a)^\times \). Assume that the cardinal of \( I \) is \( r-1 \), so that \( L \) is a product of the \( r \) linearly independent cyclic subfields \( K \) and \( K_i \) of \( F \) of degree \( p^n \). Let \( I' \) be a subset of cardinal \( r-2 \) of \( I \).

Let \( (N_i, \chi) \) denote the class of the cyclic algebra over \( k(X_a) \) associated to \( \chi \) and the element \( N_i \in k(X_a)^\times \).

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*Date: October 13, 2021.*
Theorem. The group $\Br(X_\alpha^c)/\Im(\Br(k))$ is generated by the elements $(N_i, \chi)$ for $i \in I'$.

This is also proved in Theorem 4.9. Note that the above results are generalizations of [BP 20], Theorems 11.1 and 11.2.

The paper is organized as follows. Throughout the paper, $K$ is a finite cyclic extension of $k$, and $L = K \times K'$, where $K'$ is an étale $k$-algebra of finite rank. Sections 1 and 2 are preliminary: in particular, it is shown in §2 that we may assume $K$ be cyclic of prime power degree. Sections 3 and 4 contain the description of the unramified Brauer group. When $k$ is a global field, we obtain additional results concerning the “locally trivial” Brauer group (cf. §5). Finally, in §6 we apply Theorem 4.3 to give an alternative proof of [BLP 19] Theorem 7.1 for $k$ a global field with $\text{char}(k) \neq p$; we show that the Brauer-Manin map of [BLP 19] is the Brauer-Manin pairing, and hence deduce the Hasse principle from results of [Sa 81] and [DH 17].

1. Definitions and notation

Generalities

Let $k$ be a field, let $k_s$ be a separable closure of $k$ and let $G_k = \Gal(k_s/k)$ be the absolute Galois group of $k$. We fix once and for all this separable closure $k_s$, and all separable extensions of $k$ that will appear in the paper will be contained in $k_s$. We use standard notation in Galois cohomology; in particular, if $M$ is a discrete $G_k$-module and $i$ is an integer $\geq 0$, we set $H^i(G_k, M)$. A $G_k$-lattice will be a torsion free $\mathbb{Z}$-module of finite rank. Let $g \in G_k$, we denote by $\langle g \rangle$ the cyclic subgroup of $G_k$ generated by $g$. Let $M$ be a $G_k$-lattice. Set

$$\mathbb{H}^2_{\text{cyc}}(G, M) = \ker[H^2(G, M) \to \prod_{g \in G} H^2(\langle g \rangle, M)].$$

We recall a result of Colliot-Thélène and Sansuc (cf. [CTS 87] Prop. 9.5)

**Theorem 1.1.** Let $G$ be a finite group, let $T$ be a $k$-torus, and assume that the character group of $T$ is a $G$-lattice via a surjection $G_k \to G$. Let $T^c$ be a smooth compactification of $T$. We have $\Br(T^c)/\Br(k) \simeq \mathbb{H}^2_{\text{cyc}}(G, \hat{T})$.

**Proof.** See [BP 20], Theorem 1.3.

Norm equations

Let $L$ be an étale $k$-algebra of finite rank; in other words, a product of a finite number of separable extensions of $k$. Let $T_{L/k} = R_{L/k}(G_m)$ be the $k$-torus defined by

$$1 \to T_{L/k} \to R_{L/k}(G_m) \to \mathbb{G}_m \to 1.$$
Let $a \in k^\times$. Let $X_a$ be the affine $k$-variety associated to the norm equation

$$N_{L/k}(t) = a.$$ 

The variety $X_a$ is a torsor under $T_{L/k}$; let $X_a^c$ be a smooth compactification of $X_a$. We have a natural map $\text{Br}(k) \to \text{Br}(X_a^c)$; if $a = 1$ then $X_1 = T_{L/k}$, and the map $\text{Br}(k) \to \text{Br}(T_{L/k}^c)$ is injective, and moreover we have an injection

$$\text{Br}(X^c)/\text{Im}(\text{Br}(k)) \to \text{Br}(T^c)/\text{Br}(k)$$

(see for instance [BP 20], §5). Recall a result from [BP 20], Theorem 6.1:

**Theorem 1.2.** Assume that $L = K \times K'$, where $K/k$ is a cyclic extension and $K'$ an étale $k$-algebra. Then the map $\text{Br}(X^c)/\text{Im}(\text{Br}(k)) \to \text{Br}(T^c)/\text{Br}(k)$ is an isomorphism.

**Global fields**

If $k$ is a global field, we denote by $V_k$ be the set of all places of $k$; if $v \in V_k$, we denote by $k_v$ the completion of $k$ at $v$.

For any $k$-torus $T$, set

$$\Pi^i(k, T) = \text{Ker}(H^i(k, T) \to \prod_{v \in V_k} H^i(k_v, T)).$$

If $M$ is a $\mathcal{G}_k$-module, set

$$\Pi^i(k, M) = \text{Ker}(H^i(k, M) \to \prod_{v \in V_k} H^i(k_v, M)),$$

and let $\Pi^i_a(k, M)$ be the set of $x \in H^i(k, M)$ that map to $0$ in $H^i(k_v, M)$ for almost all $v \in V_k$.

**2. Norm equations and étale algebras**

In the sequel, we consider norm equations of étale algebras having at least one cyclic factor. The aim of this section is to introduce some notation and prove some results that will be used throughout the paper.

Let $K$ be a cyclic extension of $k$, and let $K'$ be an étale $k$-algebra of finite rank; set $L = K \times K'$. We first show that it is enough to consider the case when $K/k$ is cyclic of prime power degree.

**Reduction to the prime power degree case**

Let $\mathcal{P}$ be the set of prime numbers dividing $[K : k]$. For all $p \in \mathcal{P}$, let $K[p]$ be the largest subfield of $K$ such that $[K[p] : k]$ is a power of $p$, and set $L[p] = K[p] \times K'$. Recall from [BLP 19] the following result

**Proposition 2.1.** Assume that $k$ is a global field. We have

$$\Pi^2(k, \hat{T}_{L/k}) \simeq \bigoplus_{p \in \mathcal{P}} \Pi^2(k, \hat{T}_{L[p]/k}).$$
Proof. This follows from [BLP 19], Lemma 3.1 and Proposition 5.16.

Let $k'/k$ be a Galois extension of minimal degree splitting $T_L/k$, and let $G = \text{Gal}(k'/k)$.

Proposition 2.2. We have $\prod_{\text{cycl}}^2(G, \hat{T}_L/k) \cong \bigoplus_{p \in P} \prod_{\text{cycl}}^2(G, \hat{T}_{L[p]/k})$.

Proof. Let us write $K' = \prod_{i \in I} K_i$, where the $K_i$ are finite field extensions of $k$.

Let $H$ be the subgroup of $G$ such that $K = (k')^H$, and for all $i \in I$, let $H_i$ be the subgroup of $G$ such that $K_i = (k')^{H_i}$. Set $M = \hat{T}_L/k$. We have the exact sequence of $G$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[G/H] \oplus \bigoplus_{i \in I} \mathbb{Z}[G/H_i] \to M \to 0.$$ 

For all $p \in P$, let $H[p]$ be the subgroup of $G$ such that $K[p] = (k')^H[p]$. Set $M[p] = \hat{T}_{L[p]/k}$. We have the exact sequence of $G$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[G/H[p]] \oplus \bigoplus_{i \in I} \mathbb{Z}[G/H_i] \to M[p] \to 0.$$ 

Let $\ell'/\ell$ be an unramified extension of number fields with Galois group $G$ (cf. [F 62]). Set $L_0 = (\ell')^H$, $L_0[p] = (\ell')^H[p]$, and $L_i = (\ell')^{H_i}$. Let $E = L_0 \times \prod_{i \in I} L_i$ and $E[p] = L_0[p] \times \prod_{i \in I} L_i$. We have $\hat{T}_{E/\ell} \simeq M$ and $\hat{T}_{E[p]/\ell'} \simeq M[p]$. By Proposition 2.1, we have $\prod_{\text{cycl}}^2(\ell, M) \cong \bigoplus_{p \in P} \prod_{\text{cycl}}^2(\ell, M[p])$. Since $\ell'/\ell$ is unramified, we have $\prod_{\text{cycl}}^2(G, M) \cong \bigoplus_{p \in P} \prod_{\text{cycl}}^2(G, M[p]) = \bigoplus_{p \in P} \prod_{\text{cycl}}^2(G, M[p])$ (see [BP 20], Proposition 3.1), hence $\prod_{\text{cycl}}^2(G, M) \cong \bigoplus_{p \in P} \prod_{\text{cycl}}^2(G, M[p])$.

Proposition 2.3. Assume that $k$ is a global field. We have

$$\prod_{\text{cycl}}^2(k, \hat{T}_L/k) \cong \bigoplus_{p \in P} \prod_{\text{cycl}}^2(k, \hat{T}_{L[p]/k}).$$ 

Proof. This follows from Proposition 2.2 and [BP 20], Corollary 3.4.

The prime power degree case

Let $p$ be a prime number, and assume that $K/k$ is cyclic of degree a power of $p$. Let us write $K' = \prod_{i \in I} K_i$, where the $K_i$ are finite field extensions of $k$, and let $[K : k] = p^n$.

Notation 2.4. For all integers $1 \leq m \leq n$, let $K(m)$ be the unique subfield of $K$ of degree $p^m$ over $k$. The $K_i$-algebra $K(m) \otimes_k K_i$ is a product of cyclic extensions of $K_i$; let $p^{e_i(m)}$ be the degree of these extensions, and set $E(m) = \{e_i(m) \mid i \in I\}$. For all $i \in I$, let us chose one of the cyclic factors $E_i/K_i$ of $K \otimes_k K_i$. For all $m$ let $E_i(m)$ be the subfield of $E_i$ which corresponds to a cyclic factor of $K(m) \otimes_k K_i$. 

Let $\mathcal{K}$ be a Galois extension of $k$ containing $K$ and all the fields $K_i$, and let $G = \text{Gal}(\mathcal{K}/k)$. If $F$ is a subfield of $\mathcal{K}$, we denote by $G_F$ the subgroup of $G$ such that $F = \mathcal{K}^{G_F}$.

For all integers $0 \leq m \leq n$, let $\Gamma_i^m$ be the set of conjugacy classes of elements $g \in G$ such that $\langle g \rangle \cap (G_{E_i(n-m)} \setminus G_{E_i(n-m+1)}) \neq \emptyset$.

**Notation 2.5.** Assume moreover that $k$ is a global field. Let $V_i^m$ be the set of places $v$ of $k$ such that there exists a place $w$ of $K_i$ above $v$ having the property that $K \otimes_k (K_i)_w$ is a product of fields extensions of degree at least $p^m$ of $(K_i)_w$.

**Proposition 2.6.** Assume that $k$ is a global field. Let $V_{rm}$ be the set of places of $k$ which are ramified in $\mathcal{K}$. For all integers $0 \leq m \leq n$, sending a place $v \in V_i^m \setminus V_{rm}$ to the conjugacy class of its Frobenius element $f_v \in G$ gives rise to a surjection from $V_i^m \setminus V_{rm}$ onto $\Gamma_i^m$.

In order to prove the Proposition, we need the following lemma.

**Lemma 2.7.** Let $F$ be a field, and let $E$ be a cyclic extension of $F$ of prime degree. Let $M$ be an extension of $E$, and assume that $M$ is a Galois extension of $F$. Set $G_F = \text{Gal}(M/F)$ and $G_E = \text{Gal}(M/E)$. Let $v : M^\times \to \mathbb{Z}$ be a discrete valuation of $M$; assume that the restriction $v_F$ of $v$ to $F^\times$ is surjective, and that the residue field of $v$ is perfect.

Let $D_{M/F}$ be the decomposition group of $v$. Then $v_F$ is inert in $E$ if and only if $D_{M/F} \cap (G_F \setminus G_E) \neq \emptyset$.

**Proof.** Let $G_{E/F}$ be the Galois group of the extension $E/F$, and let $D_{E/F}$ be the decomposition group of $v_E$; note that $v_F$ is inert in $E$ if and only if $D_{E/F} = G_{E/F}$. Since $E/F$ is cyclic of prime degree, this amounts to saying that $D_{E/F}$ is not trivial.

We have the exact sequence

$$1 \to G_E \to G_{F} \to G_{E/F} \to 1.$$ 

The image of $D_{M/F}$ by the homomorphism $G_F \to G_{E/F}$ is equal to $D_{E/F}$ (see for instance [Se 79], Chap. I, Proposition 22). Hence $D_{E/F}$ is non trivial if and only if $D_{M/F} \cap (G_F \setminus G_E) \neq \emptyset$.

**Proposition 2.8.** Let $F$, $E$ and $M$ be as in Lemma 2.7. Assume moreover that $k$ is a subfield of $F$, and that $M$ is a Galois extension of $k$. Let $G = \text{Gal}(M/k)$. Let $v_k : k^\times \to \mathbb{Z}$ be a discrete valuation such that the extensions of $v_k$ to $M$ are unramified; let $\mathcal{D}$ be the set of corresponding decomposition groups. The following are equivalent

(a) There exists an extension of $v_k$ to $F$ that is inert in $E$.

(b) There exists $D \in \mathcal{D}$ such that $D \cap (G_F \setminus G_E) \neq \emptyset$.

**Proof.** Let us prove that (a) implies (b). Let $v_F$ be an extension of $v_k$ to $F$ that is inert in $E$, let $v$ be an extension of $v_F$ to $M$, and let $D$ be the decomposition group of $v$. By Lemma 2.7 we have (b). Conversely, assume that (b) holds. Let $D \in \mathcal{D}$ be as in (b), and let $v$ be the corresponding
valuation. Let \( v_F \) be the restriction of \( v \) to \( F \). By Lemma 2.7, we see that \( v_F \) is inert in \( E \), hence (a) holds.

**Proof of Proposition 2.6** If \( v \in V^m_i \setminus V_{rm} \), then by definition there exists a place of \( E_i(n - m) \) that is inert in the extension \( E_i(n)/E_i(n - m) \), and hence also in \( E_i(n - m + 1)/E_i(n - m) \); therefore by Proposition 2.8 the conjugacy class of its Frobenius element \( f_v \) belongs to \( \Gamma^m_i \).

Conversely, if the conjugacy class of \( g \in G \) belongs to \( \Gamma^m_i \), then by Chebotarev’s density theorem there exists an unramified place \( v \) such that its Frobenius element \( f_v \) is the conjugacy class of \( g \). Since the conjugacy class of \( g \) belongs to \( \Gamma^m_i \), there is a place \( w \) of \( E_i(n - m) \) above \( v \) such that \( w \) is inert in \( E_i(n - m + 1)/E_i(n - m) \). Therefore \( w \) is also inert in \( E_i(n)/E_i(n - m) \) as \( E_i(n)/E_i(n - m) \) is cyclic of \( p \)-power degree and is unramified at \( w \). This implies that \( v \in V^m_i \setminus V_{rm} \).

We get immediately the following corollary.

**Corollary 2.9.** For \( i, j \in I \), the map defined in Proposition 2.6 induces a surjection from \( V^m_i \cap V^m_j \setminus V_{rm} \) to \( \Gamma^m_i \cap \Gamma^m_j \).

**Remark 2.10.** Keep the notation in Proposition 2.6. For each conjugacy class in \( \Gamma^m_i \), by Chebotarev’s density theorem there are infinite many unramified places \( v \in V^m_i \) mapped to it.

### 3. Norm equations - unramified Brauer group

We keep the notation of the previous section. In particular, \( k \) is a field, \( K \) is a cyclic extension of \( k \), and \( L = K \times K' \) where \( K' \) is an étale \( k \)-algebra of finite rank. Let \( T_{L/k} = R^{(1)}_{L/k}(G_m) \) be the \( k \)-torus defined by

\[
1 \to T_{L/k} \to R_{L/k}(G_m) \xrightarrow{N_{L/k}} G_m \to 1.
\]

Let \( a \in k^\times \), and let \( X_a \) be the affine \( k \)-variety associated to the norm equation \( N_{L/k}(t) = a \). The variety \( X_a \) is a torsor under \( T_{L/k} \). Let \( T_{L/k}^{c} \) be a smooth compactification of \( T_{L/k} \), and let \( X_a^c \) be a smooth compactification of \( X_a \).

The aim of this section is to describe the group \( \text{Br}(X_a^c)/\text{Im}(\text{Br}(k)) \). Using the results of §2 we can assume that \( K/k \) is of degree \( p^n \), where \( p \) is a prime number.

We use the notation of §2 (see Notation 2.4). In addition, we need the following

**Notation 3.1.** For all integers \( n \geq 1 \), we denote by \( C(I, Z/p^nZ) \) the set of maps \( I \to Z/p^nZ \).

If \( 1 \leq m \leq n \), let \( \pi_{n,m} \) be the projection \( C(I, Z/p^nZ) \to C(I, Z/p^mZ) \).

For \( x \in Z/p^mZ \) and \( y \in Z/p^rZ \), we denote by \( \delta(x, y) \) the maximum integer \( d \leq \min\{m, r\} \) such that \( x = y \) (mod \( p^dZ \)).
We start with some special cases, in which the results are especially simple.

**$K/k$ cyclic of degree $p$**

Assume first that $[K : k] = p$, and that $K$ is not contained in any of the fields $K_i$. Then for all $i \in I$, $E_i$ is a cyclic field extension of degree $p$ of $K_i$. Let $\Gamma_i = \Gamma_i^1$ be the set of conjugacy classes of elements $g \in G$ such that $\langle g \rangle \cap (G_{K_i} - G_{E_i}) \neq \emptyset$ (cf. Notation 2.4).

Let $C(L)$ be the group

$$\{ c \in C(I, \mathbb{Z}/p\mathbb{Z}) \mid c(i) = c(j) \text{ if } \Gamma_i \cap \Gamma_j \neq \emptyset \},$$

and $D$ be the subgroup of constant maps $I \to \mathbb{Z}/p\mathbb{Z}$.

As a consequence of Theorem 3.8, we’ll show the following

**Proposition 3.2.** Assume that $K/k$ is cyclic of degree $p$, and that $K$ is not contained in any of the fields $K_i$. Then we have

$$\lim^{\oplus}_{\text{cycl}}(G, \hat{T}_{L/k}) \simeq C(L)/D.$$

By Theorem 1.1, this implies the following

**Corollary 3.3.** Assume that $K/k$ is cyclic of degree $p$, and that $K$ is not contained in any of the fields $K_i$. Then we have

$$\text{Br}(T^c)/\text{Br}(k) \simeq C(L)/D.$$
The general case

Recall that \( K/k \) is cyclic of degree \( p^n \), and that we use Notation 2.4. Recall that \( \mathcal{E} = \mathcal{E}(n) \).

**Notation 3.6.** For all \( e \in \mathcal{E} \), set \( I_e = \{ i \in I \mid e_i(n) = e \} \). Denote by \( \hat{e} \) the maximum element in \( \mathcal{E} \). Note that the index \( i \) belongs to \( I_e \) if and only if \( K \cap K_i \) is an extension of degree \( p^n \) of \( k \). As \( K \) is a cyclic extension, this means that given \( 0 \leq m \leq n \), the \( e_i(m) \) are the same for all \( i \in I_e \) and we denote it by \( e(m) \).

For all integers \( m \) with \( 1 \leq m \leq n \) set

\[
C^m = \{ c \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbb{Z}/p^{e-e(n-m)}\mathbb{Z}) \mid c(i) = c(j) \text{ if } \Gamma^m_i \cap \Gamma^m_j \neq \emptyset \}.
\]

We still denote by \( \pi_{n,m} \) the map from \( \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbb{Z}/p^{e-e(n-m)}\mathbb{Z}) \) to \( \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbb{Z}/p^{e-e(n-m)}\mathbb{Z}) \) induced by the natural projection.

Set

\[
C(L) = \{ c \in C^m \mid \pi_{n,m}(c) \in C^m \text{ for all } m \leq n \},
\]

and denote by \( D \) the image of constant maps \( I \to \mathbb{Z}/p^n\mathbb{Z} \) in \( C^m \) under the natural projection.

**Remark 3.7.** If \( E_i(n-m+1) \supseteq E_i(n-m) \), then \( K(n-m) \supseteq K \cap K_i \). In this case \( e_i(n) \geq m \) and \( e_i(n) - e_i(n-m) = m \).

The main results of this section are

**Theorem 3.8.** Assume that \( K/k \) is cyclic of degree \( p^n \). Then we have

\[
\text{III}^2_{\text{cycl}}(G, \hat{T}_{L/k}) \simeq C(L)/D.
\]

By Theorem 1.1, this implies the following

**Corollary 3.9.** Assume that \( K/k \) is cyclic of degree \( p^n \). then we have

\[
\text{Br}(T^c)/\text{Br}(k) \simeq C(L)/D.
\]

The proof of Theorem 3.8 will be given below, using some arithmetic results of [BLP 19]. We start by recalling and developing some results concerning global fields.

**Global fields**

Assume that \( k \) is a global field. Recall that \( K/k \) is cyclic of degree \( p^n \), and that we use notations 2.4 as well as 3.6. In addition, for global fields, we also use notation 2.5.

For all integers \( m \) with \( 1 \leq m \leq n \) set

\[
C^m_{\text{arith}} = \{ c \in \bigoplus_{e \in \mathcal{E}} C(I_e, \mathbb{Z}/p^{e-e(n-m)}\mathbb{Z}) \mid c(i) = c(j) \text{ if } V^m_i \cap V^m_j \neq \emptyset \}
\]
and
\[ C^m_\omega = \{ c \in \bigoplus_{e \in E} C(I_e, \mathbb{Z}/p^{e(n-m)}\mathbb{Z}) \mid c(i) = c(j) \text{ if } V^m_i \cap V^m_j \text{ is infinite} \} \]

Set
\[ C_{\text{arith}}(L) = \{ c \in C^m_\omega \mid \pi_{n,m}(c) \in C^m_\omega \text{ for all } m \leq n \} \]
and
\[ C_\omega(L) = \{ c \in C^m_\omega \mid \pi_{n,m}(c) \in C^m_\omega \text{ for all } m \leq n \} \].

**Theorem 3.10.** Assume that \( K/k \) is cyclic of degree \( p^n \). Then we have

1. \( \Pi^2(k, \hat{T}_L/k) \simeq C_{\text{arith}}(L)/D \).
2. \( \Pi^2_\omega(k, \hat{T}_L/k) \simeq C_\omega(L)/D \).

**Proof.** Recall some notation of [BLP 19].

For \( a = (a_i) \in \bigoplus_{e \in E} \bigoplus_{i \in I_e} (\mathbb{Z}/p^e\mathbb{Z}) \) and \( r \in \mathbb{Z}/p^r\mathbb{Z} \), set
\[ I_r(a) = \{ i \in I \mid a_i = r \mod p^{e_i(n)}\mathbb{Z} \} \].

Set
\[ G(K, K') = \{ a = (a_i) \in \bigoplus_{e \in E} \bigoplus_{i \in I_e} (\mathbb{Z}/p^e\mathbb{Z}) \mid \bigcap_{r \in \mathbb{Z}/p^r\mathbb{Z}} \bigcup_{i \notin I_r(a)} V^i_i = \emptyset \}; \]
\[ G_\omega(K, K') = \{ a = (a_i) \in \bigoplus_{e \in E} \bigoplus_{i \in I_e} (\mathbb{Z}/p^e\mathbb{Z}) \mid \bigcap_{r \in \mathbb{Z}/p^r\mathbb{Z}} \bigcup_{i \notin I_r(a)} V^i_i \text{ is finite} \}. \]

With the notation of [BLP 19], we have \( \Pi^2(k, \hat{T}_L/k) \simeq \Pi^2(K, K') = G(K, K')/D \), where \( D \) is the subgroup generated by \( (1,1,...,1) \) (see [BLP 19], Theorem 5.3 and Lemma 3.1). Similarly, it is shown in [Lee 21], Theorem 2.5 that \( \Pi^2_\omega(k, \hat{T}_L/k) \simeq G_\omega(K, K')/D \). Hence it suffices to show that \( G(K, K') \simeq C_{\text{arith}}(L) \) and that \( G_\omega(K, K') \simeq C_\omega(L) \).

We show that \( G(K, K') \simeq C_{\text{arith}}(L) \); the proof of \( G_\omega(K, K') \simeq C_\omega(L) \) is the same.

Let
\[ f : \bigoplus_{e \in E} \bigoplus_{i \in I_e} (\mathbb{Z}/p^e\mathbb{Z}) \to \bigoplus_{e \in E} C(I_e, \mathbb{Z}/p^e\mathbb{Z}) \]
be the map sending \( (a_i) \in \bigoplus_{i \in I_e} (\mathbb{Z}/p^e\mathbb{Z}) \) to \( c : I_e \to \mathbb{Z}/p^e\mathbb{Z} \) such that \( c(i) = a_i \).

We claim that the isomorphism \( f \) gives rise to an isomorphism
\[ G(K, K') \to C_{\text{arith}}(L) \].

For \( c \in \bigoplus_{e \in E} C(I_e, \mathbb{Z}/p^e\mathbb{Z}) \), we denote \( \pi_{n,m}(c) \) by \( c_m \).

Let \( a = (a_i) \in G(K, K') \) and \( c = f(a) \). We show that \( c_m \in C^m \) for \( 1 \leq m \leq n \).

Suppose that \( V^m_i \cap V^m_j \neq \emptyset \). By Remark 3.7, we have \( e_i - e_i(n-m) = e_j - e_j(n-m) = m \). Let \( v \in V^m_i \cap V^m_j \). As \( a \in G(K, K') \), there is \( r \in \mathbb{Z}/p^r\mathbb{Z} \) such that \( v \notin \bigcup_{l \notin I_r(a)} V^l_l \). If \( i \notin I_r(a) \), then \( \delta(r, a_i) + 1 > m \) since \( v \notin V^l_l \). Hence \( c(i) = r \mod p^m\mathbb{Z} \). If \( i \in I_r(a) \), then \( c(i) = r \mod \).
$p^r \cdot Z)$. In both cases we have $c_m(i) = r \mod p^m Z)$. The same argument
works for $j$. Therefore $c_m(i) = c_m(j)$ and $c_m \in C_m$.

Let $c \in C^m$ such that $c_m \in C^m$ for $1 \leq m \leq n$. Let $a = f^{-1}(c)$. If $c \in D$, then
clearly $a \in D$. Suppose that $c \notin D$. We claim that
\[ \bigcap_{r \in Z/p^mZ} \bigcup_{i \notin I_r(a)} V_r^{\delta (r, a_i) + 1} = \emptyset. \]

Suppose not. Let $v \in \bigcap_{r \in Z/p^mZ} \bigcup_{i \notin I_r(a)} V_r^{\delta (r, a_i) + 1}$. Choose $r_0 \in Z/p^mZ$. Since $c \notin D$, there is $r_1 \in Z/p^mZ$ and $i \in I_{r_1}(a) \setminus I_{r_0}(a)$ such that $v \in V_{r_1}^{\delta (r, a_i) + 1}$. For the same reason, there is $r_2 \in Z/p^mZ$ and $j \in I_{r_2}(a) \setminus I_{r_1}(a)$ such that $v \in V_{r_2}^{\delta (r, a_j) + 1}$.

By the choice of $i$ and $j$, we have $\delta (r_0, a_i) = \delta (r_0, r_1)$ and $\delta (r_1, a_j) = \delta (r_1, r_2)$. Suppose that $\delta (r_0, r_1) \geq \delta (r_1, r_2)$. Then $v \in V_{r_0}^{m} \cap V_{r_2}^{m}$, where $m = \delta (r_1, r_2) + 1$. Hence $c_m(i) = c_m(j)$ and $\delta (a_i, a_j) \geq m = \delta (r_1, r_2) + 1$, which contradicts that $\delta (r_1, r_2) \geq \delta (a_i, a_j)$. Therefore $\delta (r_0, r_1) < \delta (r_1, r_2)$.

We can continue the above process to get an infinite sequence of $r_i \in Z/p^mZ$ such that $\delta (r_i, r_{i+1}) < \delta (r_{i+1}, r_{i+2})$. It is a contradiction as $\delta (r_i, r_{i+1})$ ranges from 0 to $\hat{e}$. Hence $\bigcap_{r \in Z/p^mZ} \bigcup_{i \notin I_r(a)} V_r^{\delta (r, a_i) + 1} = \emptyset$ and $a \in G(K, K')$. As a consequence $f$ induces an isomorphism $G(K, K') \to C_{arith}(L)$.

**Corollary 3.11.** Let $k$ be a global field. Then $\III^2(k, \hat{T}_{L/k}) \simeq C(L)/D$.

**Proof.** By Corollary 2.9 and Remark 2.10, the two sets $C^m$ and $C^m_\omega$ are the same. Our claim then follows from Theorem 3.10.

**Proof of Theorem 3.8.** Recall that $K$ is a Galois extension of $k$ containing $K$ and all the fields $K_i$, and that $G = \text{Gal}(K/k)$; if $F$ is a subfield of $K$, we denote by $G_F$ the subgroup of $G$ such that $F = K^{G_F}$.

Note that $k$ is not necessarily a global field here. However there is always an unramified extension $\ell'/\ell$ with Galois group $\text{Gal}(\ell'/\ell) \simeq G \left( \begin{array}{c} \ell' \\ \ell \end{array} \right)$ (F.62). Hence we can regard $\hat{T}_{L/k} \in \text{Gal}(\ell'/\ell)$-module.

To be precise, set $F = (\ell')^{G_K}, L_i = (\ell')^{G_{K_i}}$ and $E = F \times \prod_{i \in I} L_i$. By construction, the $G$-lattices $\hat{T}_{E/\ell}$ and $\hat{T}_{L/k}$ are isomorphic.

Since the extension $\ell'/\ell$ is unramified, we have $\III^2(\ell, \hat{T}_{E/\ell}) \simeq \III^2(\ell, \hat{T}_{E/\ell}) \simeq \III^2_{cycl}(G, \hat{T}_{E/\ell})$. By Corollary 3.11 the group $\III^2_{cycl}(G, \hat{T}_{E/\ell})$ is isomorphic to $C(E)/D$. However $C(L)$ only depends on the group $G$. Hence $C(L) \simeq C(E)$. Therefore $\III^2_{cycl}(G, \hat{T}_{L/k}) \simeq \III^2_{cycl}(G, \hat{T}_{E/\ell}) \simeq C(L)/D$.

4. Unramified Brauer groups and generators

We keep the notation of the previous sections. Recall that $p$ is a prime number, $K/k$ a cyclic field extension of degree $p^n$, and $L = K \times K'$, where $K'$ is an étale $k$-algebra of finite rank. In the previous section, we introduced a group $C(L)$ and proved that $Br(X^c_o)/\text{Im}(Br(k)) \simeq C(L)/D$.

The aim of this section is to give more precise information about the isomorphism $C(L)/D \to Br(X^c_o)/\text{Im}(Br(k))$. 
Let $\text{Br}_{ur}(k(X_a))$ be the subgroup of $\text{Br}(k(X_a))$ consisting of all elements which are unramified at all discrete valuations of $k(X_a)$ with residue fields containing $k$ and with fields of fraction $k(X_a)$; recall that $\text{Br}_{ur}(k(X_a))$ is isomorphic to $\text{Br}(X_a^c)$ (see Cesnavius [C 19], Theorem 1.2).

As in the previous sections, let us write $K' = \prod_{i \in I} K_i$, where the $K_i$ are finite separable field extensions of $k$.

**Notation 4.1.** We denote by $G_k$ the absolute Galois group of $k$, $G_{k(X_a)}$ the absolute Galois group of $k(X_a)$. Let $R$ be a discrete valuation ring of $k(X_a)$ with residue field $\kappa_R$ containing $k$ and with field of fractions $k(X_a)$. We denote by $G_R$ the absolute Galois group of $\kappa_R$.

**Notation 4.2.** For all $i \in I$, let $\{\beta_{ij}\}$ be a basis of $K_i$ over $k$. Let $y_i = \sum_j \beta_{ij} x_{ij}$, where $x_{ij}$ are variables. Set $N_i = N_{K_i \otimes k(X_a)/k(X_a)}(y_i)$ considered as an element of $k(X_a)$. We define $N = N_{K \otimes k(X_a)/k(X_a)}(y)$ in a similar way. Fix an isomorphism $\chi : \text{Gal}(K/k) \to \mathbb{Z}/p^n\mathbb{Z}$. Then $\chi$ gives rise to a morphism $\tilde{\chi} : G_{k(X_a)} \to \mathbb{Z}/p^n\mathbb{Z}$ and a morphism $\chi_R : G_R \to \mathbb{Z}/p^n\mathbb{Z}$. Let $(N_i, \tilde{\chi})$ denote the class of the cyclic algebra over $k(X_a)$ associated to $\chi$ and the element $N_i \in k(X_a)$. (GS 06, Prop. 4.7.3)

The main result of this section is

**Theorem 4.3.** Assume $\text{char}(k) \neq p$. Then the map

$$u : C(L) \to \text{Br}(k(X_a))$$

given by

$$u(c) = \sum_{i \in I} c(i)(N_i, \tilde{\chi})$$

induces an isomorphism

$$C(L)/D \to \text{Br}(X_a^c)/\text{Im(Br(k))}.$$

**Remark 4.4.** Note that $(N_i, \tilde{\chi}) \in \text{Br}(k(X_a))$ has order at most $p^{e_i(n)}$, so $c(i)(N_i, \tilde{\chi})$ is well-defined for $c(i) \in \mathbb{Z}/p^{e_i(n)}\mathbb{Z}$ (ref. BLP 19, Lemma 6.1).

We start with following lemmas.

**Lemma 4.5.** The group $u(D)$ is contained in the image of $\text{Br}(k)$ in $\text{Br}(k(X_a))$.

**Proof.** Since $N \cdot \prod_{i \in I} N_i = c$, we have $\sum_{i \in I} (N_i, \tilde{\chi}) = (c/N, \tilde{\chi}) = (c, \tilde{\chi})$, which is the image of $(c, \chi)$ in $\text{Br}(k(X_a))$. Hence $u(D) \subseteq \text{Im(Br(k))}$.

The following lemma can be found in [Lee 21] §3. Here we use the notation $C(L)$ to simplify the proof.

**Lemma 4.6.**
(1) Let $c \in C(L) \setminus D$. Pick $i_0 \in I$. Let $\hat{c} \in D$ be the image of the constant map from $I$ to $c(i_0)$. Set $m$ to be the maximal integer such that $\pi_{n,m}(c) = \pi_{n,m}(\hat{c})$. Choose $r \in \mathbb{Z}/p^\ell\mathbb{Z}$ such that $\delta(r,c(i_0)) = m$. Consider the element $c' \in \oplus C(I_e, \mathbb{Z}/p^\ell\mathbb{Z})$ defined as follows:

\[
\begin{aligned}
\pi_{\bar{e},e}(n)(r), & \quad \text{if } e_i(n) > m \text{ and } m = \delta(c(i), c(i_0)); \\
\pi_{\bar{e},e}(n)(c(i_0)), & \quad \text{otherwise}.
\end{aligned}
\]

Then $c' \in C(L) \setminus D$.

(2) Suppose that $k$ is a global field and $c \in C_{arith}(L) \setminus D$. Then the $c'$ defined above is in $C_{arith}(L) \setminus D$.

**Proof.** As $c$ is not in $D$, by the choice of $m$ there is some $i \in I$ such that $e_i(n) > m$ and $\delta(c(i), c(i_0)) = m$. Hence $c'(i) \not\equiv c'(i_0) \pmod{p^m\mathbb{Z}}$ by our construction and $c' \not\in D$.

Now we show that $\pi_{n,l}(c') \in C^l(L)$ for $0 \leq l \leq n$. If $l \leq m$, then by the choice of $r$ we have $\pi_{n,l}(c(i_0)) = \pi_{n,l}(r)$. Clearly $\pi_{n,l}(c') \in C^l(L)$.

Suppose $l > m$. If $\Gamma_i \cap \Gamma_j \not= \emptyset$, then by Remark 3.7, $e_i(n)$ and $e_j(n)$ are at least $l$ and $c(i) = c(j) \pmod{p^l\mathbb{Z}}$. Hence $\delta(c(i_0), c(i)) = m$ if and only if $\delta(c(i_0), c(j)) = m$. By construction $c'(i) \equiv c'(j) \pmod{p^m\mathbb{Z}}$ and $\pi_{n,l}(c') \in C^l(L)$.

The proof of statement (2) is similar.

**Lemma 4.7.** Assume that $\text{char}(k) \not= p$. Let $R$ be a discrete valuation ring as in Notation [4.7]. Denote by $\partial_R$ the residue map from $Br(k(X_i))$ to $H^1(\kappa_R, \mathbb{Q}/\mathbb{Z})$. Suppose that the order of $\partial_R(N_i, \tilde{\chi})$ and the order of $\partial_R(N_j, \tilde{\chi})$ are both at least $p^{m_i}$. Then $\Gamma^m_i \cap \Gamma^m_j$ is nonempty.

**Proof.** Let $\nu_R$ be the discrete valuation associated to $R$. Denote the completion of $k(X_a)$ with respect to $\nu_R$ by $k(X_a)_R$. Choose an extension $\omega_R$ of $\nu_R$ to a separable closure of $k(X_a)_R$. By the construction of $\tilde{\chi}$, $\partial_R(N_i, \tilde{\chi}) = \nu_R(N_i)\chi_R$. (See [GS 06] 6.8.4 and 6.8.5.) Write $\nu_R(N_i)$ as $p^{m_i}q_i$ where $p \nmid q_i$. Let $p^{m_R}$ be the order of $\chi_R$. As the order of $\partial_R(N_i, \tilde{\chi}) \geq p^{m_i}$, we have $n_R - m_i \geq m_R$.

Since $\nu_R(N_i) = p^{m_i}q_i$, there is some factor $\bar{M}_i$ of $K_i \otimes k(X_a)_R$ such that $p^{m_i+1}$ does not divide $\nu_R(N_{M_i/k(X_a)_R}(y_{M_i}))$, where $y_{M_i}$ is the projection of $y_i$ in $M_i$. Let $\omega_{i,R}$ be the restriction of $\omega_R$ to $M_i$. Write the inert degree of $\omega_{i,R}$ over $\nu_R$ as $p^{f_i}q_i'$ where $p \nmid q_i'$. As $p^{m_i+1}$ does not divide $\nu_R(N_{M_i/k(X_a)_R}(y_{M_i}))$, we have $f_i \leq m_i$.

Choose a factor $M$ of $K \otimes k(X_a)_R$ and let $\overline{M}$ be its residue field. Let $M_i$ be the residue field of $w_{i,R}$. Both fields are considered as subfields of a separable closure $\kappa^{a}_{R}$ of $\kappa_R$.

As $f_i \leq m_i$ and $n_R - m_i \geq m$, the cyclic extension $\overline{M}\overline{M_i}/\overline{M_i}$ is of degree at least $p^{m_i}$.

Choose $g_{\ast} \in G_{\ast}$ such that $\chi_R(g_{\ast})$ generates the image of $\chi_R$ in $\mathbb{Q}/\mathbb{Z}$. Let $\mathcal{H}_i$ be the subgroup of $G_{\ast}$ which fixes $\overline{M_i}$. We claim that there are some $h_i \in G_{\ast}$ and some $\sigma \in \langle h_i^{-1} g_{\ast} h_i \rangle \cap \mathcal{H}_i$ such that $\chi_R(\sigma)$ is of order at least $p^{m_i}$.

Consider the group action of $\langle g_{\ast} \rangle$ on the set of left cosets of $\mathcal{H}_i$ in $G_{\ast}$. As $|G_{\ast}/\mathcal{H}_i| = p^{f_i}q_i'$ with $p \nmid q_i'$, there is some $h_i \in G_{\ast}$ such that $p^{f_i+1}$ does not divide the order of the orbit of $h_i\mathcal{H}_i$. Hence the stabilizer of $h_i\mathcal{H}_i$ is
\( \langle g_R^{p^{f_i r_i}} \rangle \) for some \( f'_i \leq f_i \) and some \( r_i \) coprime to \( p \). Let \( \sigma_R = h_i^{-1} g_R^{p^{f'_i r_i}} h_i \).

Then \( \chi_R(\sigma_R) = \chi_R(g_R^{p^{f_i r_i}}) \), which is of order \( p^{m - f_i} \). Since \( f'_i \leq f_i \leq m_i \) and \( n_R - m_i \geq m \), the order of \( \chi_R(\sigma_R) \) is at least \( p^m \).

Let \( g \) and \( \sigma \) be the image of \( g_R \) and \( \sigma_R \) in \( G \). Then \( \sigma \) fixes \( K \) and \( \sigma \) is an element of order at least \( p^m \) in \( \text{Gal}(K/k) \). Hence the conjugacy class of \( g \) belongs to \( \Gamma_{\ell}^m \). The same argument proves that the conjugacy class of \( g \) belongs to \( \Gamma_j^m \). Hence \( \Gamma_i^m \cap \Gamma_j^m \) is nonempty.

Next we prove that for all \( c \in C(L) \), the element \( \sum_{i \in I} c(i)(N_i, \chi) \) is unramified.

**Proposition 4.8.** Suppose that \( \text{char}(k) \neq p \). The image of \( u \) is an unramified subgroup of \( \text{Br}(k(X_a)) \).

**Proof.** By Lemma 4.5 we can assume that \( c(i) = 0 \) for some \( i \in I_\ell \).

Let \( R \) be a discrete valuation ring of \( k \) with residue field \( \kappa_R \) containing \( k \) and with field of fractions \( k(X_a) \). Let \( \nu_R \) be the discrete valuation associated to \( R \). Denote by \( \partial_R \) the residue map from \( \text{Br}(k(X_a)) \) to \( H^1(k, \mathbb{Q}/\mathbb{Z}) \). We claim that \( u(c) = \sum_{i \in I} c(i)(N_i, \chi) \) is unramified at \( R \).

Let \( J(c) = \{ i \in I \mid c(i) \neq 0 \text{ in } \mathbb{Z}/p^{f_i(n)} \mathbb{Z} \} \}. Let \( m \) be the maximum integer such that \( \pi_{n,m}(c) = 0 \) and set \( J_m(c) = \{ i \in J(c) \mid m = \delta(0, c(i)) \} \}. We prove by induction on \( |J(c)| \). If \( |J(c)| = 0 \), then \( c = 0 \) and our claim is trivial. Suppose that our claim is true for \( |J(c)| \leq h \).

Let \( |J(c)| = h + 1 \). Then \( c \notin D \) and \( J_m(c) \) is nonempty. Pick \( j \in J_m(c) \) and choose \( r \in \mathbb{Z}/p^{f_j(n)} \mathbb{Z} \) such that \( c(j) = r \) \( (\text{mod } p^{f_j(n)} \mathbb{Z}) \). Let \( c' \) be defined as in Lemma 4.6. We first prove the map \( u(c') \) is unramified at \( R \), i.e. \( \partial_R(u(c')) = 0 \).

Since \( c(i) = 0 \), by the definition of \( c' \) we have \( u(c') = \sum_{s \in J_m(c)} r(N_s, \chi) \). Hence \( \partial_R(u(c')) = \sum_{s \in J_m(c)} r \cdot \nu_R(N_s) \chi_R \). Suppose that \( \partial_R(u(c')) \) is not zero. Then there is some \( s \in J_m(c) \) such that \( r \cdot \nu_R(N_s) \chi_R \neq 0 \). As \( \delta(0, r) = m \), the order of \( \nu_R(N_s) \chi_R \) is at least \( p^m \).

By Lemma 4.5 there is some \( t \in I \setminus J_m(c) \) such that \( r \cdot \nu_R(N_t) \chi_R(g) \neq 0 \) and the order of \( \nu_R(N_t) \chi_R \) is at least \( p^m+1 \). By Lemma 4.7 the set \( \Gamma_{\ell}^{m+1} \cap \Gamma_j^{m+1} \) is nonempty. As \( \delta(0, r) = m \), we have \( c'(s) \neq c'(t) \) \( (\text{mod } p^{m+1} \mathbb{Z}) \). This contradicts that \( c' \in C(L) \). Therefore \( \partial_R(u(c')) = 0 \) and \( u(c') \) is unramified.

Consider the element \( c - c' \in C(L) \). By our construction of \( c' \), the cardinality of \( J(c - c') \) decreases by at least one. By induction hypothesis \( u(c - c') \) is unramified. Hence \( u(c) \) is unramified.

**Proof of Theorem 4.3** By Lemma 4.5 and Proposition 4.8 the map \( u : C(L)/D \to \text{Br}(X_a)/\text{Im(Br(k))} \) is well-defined.

A similar argument as in [BP 20] Thm. 12.2 proves the injectivity of \( u \). Consider the projection from \( X_a \) to the affine space \( A^d \), where \( d = \sum_{i \in I} [K_i; k] \) and the coordinates are given by \( x_{ij} \) defined in Notation 4.2.
Let $M$ be the function field of $\mathbb{A}^d$. Denote by $\chi_M$ the image of $\chi$ in $H^1(M, \mathbb{Q}/\mathbb{Z})$. Suppose that $u(c) = \alpha \in \text{Im}(\text{Br}(k))$. Then $u(c) - \alpha$ is in the kernel of $\text{Br}(M) \to \text{Br}(k(X_i))$, which is generated by $(a \prod_{i \in I} N^{-1}_i, \chi)$ (see [BlP 20]).

Lemma 12.3). Therefore $u(c) - \alpha = r(a \prod_{i \in I} N^{-1}_i, \chi)$.

Consider the discrete valuation $v_{N_i}$ on $M$ and let $\kappa_{N_i}$ be its residue field. Denote by $\chi_{N_i}$ the image of $\chi$ in $H^1(\kappa_{N_i}, \mathbb{Q}/\mathbb{Z})$. We claim that $\chi_{N_i}$ is of order $p^{\ell_i(n)}$. Let $M_i$ be the function field of the subvariety of $\mathbb{A}^{[K_i:K]}$ defined by $N_i$. Then $\kappa_{N_i} = M_i(x_{ij})$ where $x_{ij}$ are defined as in Notation 4.2 with $j \neq i$. Hence $\chi_{N_i}$ is purely transcendental over $M_i$. Let $\chi_{M_i}$ be the image of $\chi$ in $H^1(M_i, \mathbb{Q}/\mathbb{Z})$. It suffices to prove the order of $\chi_{M_i}$ is $p^{\ell_i(n)}$. Note that $K \otimes_k M_i \otimes_k K_i \cong K \otimes_k (\prod K_i(x)) \cong \prod E_i(n)(x)$, where $x$ is a variable (see Notation 2.4 for $E_i(n)$). On the other hand $K \otimes_k M_i \otimes_k K_i \cong (\prod M_i) \otimes_k K_i$, where $M_i$ is a factor of $K \otimes_k M_i$. If $[\tilde{M}_i : M_i] < p^{\ell_i(n)}$, then $\tilde{M}_i \otimes_k K_i = \tilde{M}_i \otimes_{M_i} M_i \otimes_k K_i$ is a product of extensions of $K_i(x)$ of degree less then $p^{\ell_i(n)}$, which is a contradiction. Hence $[\tilde{M}_i : M_i] = p^{\ell_i(n)}$ and $\chi_{M_i}$ is of order $p^{\ell_i(n)}$.

As $\chi_{N_i}$ is of order $p^{\ell_i(n)}$, after taking residue of $u(c) - \alpha$ at $v_{N_i}$, we see that $c(i) = -r \mod p^{\ell_i(n)} \mathbb{Z}$. Hence $c \in D$ and $u$ is injective.

Since $u$ is injective, $|C(L)/D| \leq |\text{Br}(X^c_i)/\text{Im}(\text{Br}(k))|$. By Theorem 3.3 and Theorem 1.1, the order of $C(L)/D$ is equal to the order of $\text{Br}(T^c_{L/k}/\text{Br}(k))$. By Theorem 1.2, the map $u$ is surjective and hence is an isomorphism.

We now prove the results announced in the introduction:

**Proposition 4.9.** Let $k$ be a field of char$(k) \neq p$. Let $F$ be a bicyclic extension of $k$ with Galois group $\mathbb{Z}/p^n \mathbb{Z} \times \mathbb{Z}/p^n \mathbb{Z}$. Let $K$ and $K_i$ be linearly disjoint cyclic subfields of $F$ with degree $p^n$ for $i = 1, \ldots, m$. Then

$$\text{Br}(X^c_i)/\text{Im}(\text{Br}(k)) \cong (\mathbb{Z}/p^n \mathbb{Z})^{m-1},$$

and is generated by $(N_i, \chi)$ for $i = 1, \ldots, m - 1$.

**Proof.** There is a number field $\ell$ and an unramified Galois extension $\ell'/\ell$ such that $\text{Gal}(\ell'/\ell) \simeq \text{Gal}(F/k)$ ([P 62]). Set $F = (\ell')^{G_K}$, $L_i = (\ell')^{G_{K_i}}$, and $E = F \times \prod L_i$. By construction, the $G$-lattices $\hat{T}_E/\ell$ and $\hat{T}_{L/k}$ are isomorphic.

By [Lee 21] Proposition 7.3, the group $\Pi_2^c(\ell, \hat{T}_{E/\ell}) \simeq (\mathbb{Z}/p^n \mathbb{Z})^{m-1}$. Since $\Pi_2^c(\ell, \hat{T}_{E/\ell}) \simeq \Pi_2^{cycl}(G, \hat{T}_{E/\ell}) \simeq \Pi_2^{cycl}(G, \hat{T}_{L/k})$, we have $C(L)/D \simeq (\mathbb{Z}/p^n \mathbb{Z})^{m-1}$ by Theorem 3.3. The assertion then follows from Theorem 4.3.

5. Global fields

We keep the notation of the previous section, and in addition we assume that $k$ is a global field. Denote by $B(X^c_i)$ the subgroup of $\text{Br}(X^c_i)/\text{Im}(\text{Br}(k))$ consisting of locally trivial elements; and by $B_\omega(X^c_i)$ the subgroup consisting of elements which are trivial at almost all places of $k$.

**Theorem 5.1.** Suppose that $k$ is a global field with char$(k) \neq p$. Then

1. $u$ induces an isomorphism between $C(L)/D$ and $B_\omega(X^c_i)$.
2. $u$ induces an isomorphism between $C_{\text{arith}}(L)/D$ and $B(X^c_i)$. 
Proposition 5.3. Let \( k \) be a global field of \( \text{char}(k) \neq p \). Let \( F \) be a bicyclic extension of \( k \) with Galois group \( \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \). Let \( K \) and \( K_1 \) be linearly disjoint cyclic subfields of \( F \) with degree \( p^n \) for \( i = 1, \ldots, m \). Assume moreover that \( F \otimes_k k_v \) is a product of cyclic extensions for all \( v \in V_k \). Then \( (N_v, \chi) \) generates \( \mathcal{B}(X^c_v) \).

Example 5.2. Let \( k = \mathbb{Q}(i) \). Let \( K = k(\sqrt[4]{17}) \), \( K_1 = k(\sqrt[4]{17} \times 13) \) and \( K_2 = k(\sqrt[4]{13}) \). By [Lee 21] Example 7.4, \( \mathbb{III}^2(k, \tilde{T}_{L/k}) \simeq \mathbb{Z}/4\mathbb{Z} \) and \( \mathbb{III}^2(k, \tilde{T}_{L/k}) \simeq \mathbb{Z}/2\mathbb{Z} \). By Theorem [4.3] and Theorem [5.1], the element \( (N_2, \chi) \) generates the group \( \mathcal{B}(X^c_\mathbb{Z}) \). Now for \( v \in V_k \). Then \( (N_v, \chi) \) generates \( \mathcal{B}(X^c_v) \).

More generally we have the following.

Proposition 5.5. Let \( k \) be a global field of \( \text{char}(k) \neq p \). Let \( F \) be a bicyclic extension of \( k \) with Galois group \( \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \). Let \( K \) and \( K_1 \) be linearly disjoint cyclic subfields of \( F \) with degree \( p^n \) for \( i = 1, \ldots, m \). Assume moreover that \( F \otimes_k k_v \) is a product of cyclic extensions for all \( v \in V_k \). Then \( (N_v, \chi) \) generates \( \mathcal{B}(X^c_v) \).
Proof. By [Lee 21] Prop. 7.3, we have \( \text{III}^2(k, \hat{T}_{L/k}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^{m-1} \). We apply Theorem 3.10 (1) and Theorem 5.1 (2) to conclude.

6. An application to Hasse principles

In this section we apply Theorem 4.3 to give an alternative proof of [BLP 19] Theorem 7.1 for \( k \) a global field with \( \text{char}(k) \neq p \). Moreover we can assume that \( K/k \) is a cyclic extension of \( p \)-power degree. (See §2 and [BLP 19] 6.3.)

We use the notation of the previous sections. In particular, \( X_a \) is the affine variety defined in the introduction, \( K/k \) is a cyclic extension of \( p \)-power degree, and \( K_i/k \) is a finite separable extension for all \( i \in I \). Recall that \( \chi \) an injective morphism from \( \text{Gal}(K/k) \) to \( \mathbb{Q}/\mathbb{Z} \).

Let \( \chi_v \) be the image of \( \chi \) in \( H^1(k_v, \mathbb{Q}/\mathbb{Z}) \). Let \( \text{inv} \) be the Hasse invariant map \( \text{inv} : \text{Br}(k_v) \rightarrow \mathbb{Q}/\mathbb{Z} \).

Denote by \( K_i^v \) the algebra \( K_i \otimes_k k_v \). Suppose that there is a local point \( (x_i^v) \in \prod_{v \in V_k} X_a(k_v) \), where \( x_i^v \in K_i^v \) for \( i \in I \) and \( x_0^v \in K \otimes_k k_v \). Define \( \alpha_a : C_{\text{arith}}(L)/D \rightarrow \mathbb{Q}/\mathbb{Z} \) as

\[
\alpha_a(c) = \sum_{v \in V_k} \sum_{i \in I} c(i) \text{inv}(N_{K_i^v/k_v}(x_i^v), \chi_v)
\]

**Theorem 6.1.** Suppose that there is a local point \( (x_i^v) \in \prod_{v \in V_k} X_a(k_v) \). Then the map \( \alpha_a \) is the Brauer-Manin pairing and \( X_a \) has a \( k \)-point if and only if \( \alpha_a = 0 \).

**Proof.** First we consider the case where \( k \) is a number field. By Theorem 5.1 and [Sa 81] Lemma 6.2, the map \( \alpha_a \) is the Brauer-Manin pairing of \( X_a^c \). Our claim then follows from Sansuc’s result [Sa 81] Cor. 8.7.

For \( k \) a global function field, we apply Theorem 5.1 and [DH 17] Theorem 2.5 to conclude.

**References**

[BLP 19] E. Bayer-Fluckiger, T.-Y. Lee, R. Parimala, *Hasse principles for multinorm equations*, Adv. Math. 356 (2019).

[BP 20] E. Bayer-Fluckiger, R. Parimala, *On unramified Brauer groups of torsors over tori*, Doc. Math. 25 (2020), 1263-1284.

[C 19] K. Cesnavicius, *Purity for the Brauer group*, Duke Math. J. 168 (2019), 1461-1486.

[CTHSk 03] J-L. Colliot-Thélène, D. Harari, A. Skorobogatov, *Valeurs d’un polynôme à une variable représentées par une norme*, Number theory and algebraic geometry, London Math. Soc. Lecture Note Ser. 303, Cambridge Univ. Press, Cambridge (2003), 69-89.

[CTHSk 05] J-L. Colliot-Thélène, D. Harari, A. Skorobogatov, *Compactification équivariante d’un tore* (d’après Brylinski et Künnemann), Expo. Math 23 (2005), 161-170.

[CTSk 19] J-L. Colliot-Thélène, A. Skorobogatov, *The Brauer-Grothendieck group*, Springer International Publishing (2021).

[CTS 77] J-L. Colliot-Thélène, J-J. Sansuc, *La R-équivalence sur les tores*, Ann. Sc ENS 10 (1977), 175-229.

[CTS 87] J-L. Colliot-Thélène, J-J. Sansuc, *Principal homogeneous spaces under flasque tori: applications*, J. Algebra 106 (1987), 148-205.
[DH 17] C. Demarche and D. Harari, *Local-global principles for homogeneous spaces of reductive groups over global fields*, preprint, https://arxiv.org/abs/2107.08906.

[F 62] A. Fröhlich, *On non-ramified extensions with prescribed Galois group*, Mathematika *9* (1962), 133-134.

[GS 06] P. Gille, T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge University Press (2006).

[Lee 21] T-Y. Lee, *The Tate-Shafarevich groups of multinorm-one tori*, to appear in J. Pure Appl. Algebra.

[Sa 81] J.-J. Sansuc, *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. *327* (1981) 12–80.

[Se 79] J-P. Serre, *Local fields*, Graduate Texts in Mathematics *67* Springer-Verlag, New York-Berlin (1979).

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