NONEXISTENCE OF GRADED UNITAL HOMOMORPHISMS BETWEEN LEAVITT ALGEBRAS AND THEIR CUNTZ SPLICES

GUIDO ARNONE AND GUILLERMO CORTIÑAS

Abstract. Let \( n \geq 2 \), let \( R_n \) be the graph consisting of one vertex and \( n \) loops and let \( R_n^- \) be its Cuntz splice. Let \( L_n = L(R_n) \) and \( L_n^- = L(R_n^-) \) be the Leavitt path algebras over a unital ring \( \ell \). Let \( C_m \) be the cyclic group on \( 2 \leq m \leq \infty \) elements. Equip \( L_n \) and \( L_n^- \) with their natural \( C_m \)-gradings. We show that under mild conditions on \( \ell \), which are satisfied for example when \( \ell \) is a field or a PID, there are no unital \( C_m \)-graded ring homomorphisms \( L_n \to L_n^- \) nor in the opposite direction.

1. Introduction

Let \( \ell \) be a unital ring and \( n \geq 1 \). The Leavitt algebra over \( \ell \) is the Leavitt path algebra [2, Definition 1.2.3] \( L_n = \ell \otimes L_\mathbb{Z}(R_n) \) of the graph \( R_n \) consisting of a single vertex and \( n \) loops. We write \( L_n^- \) for the Leavitt path algebra over \( \ell \) of the graph \( R_n^- \) whose adjacency matrix is

\[
A_{R_n^-} = \begin{pmatrix}
  n & 1 & 0 \\
  1 & 1 & 1 \\
  0 & 1 & 1 
\end{pmatrix}.
\]

The graph \( R_n^- \) is the Cuntz splice of \( R_n \) [1, Definition 2.11]. It is an open question [1, Hypothesis on page 24] whether the algebras \( L_2 \) and \( L_2^- \) over a field \( \ell \) are isomorphic or not. As with any Leavitt path algebras, \( L_n \) and \( L_n^- \) are graded over the infinite cyclic group \( C_\infty \), and therefore also over the cyclic group \( C_m \) of \( m \) elements for all \( m \geq 2 \), via the grading mod \( m \). The main result of this paper is the following theorem, which puts together Theorems 5.1 and 6.5.

**Theorem 1.2.** Let \( n \geq 2 \) and \( 2 \leq m \leq \infty \). Assume that \( \ell \) is regular supercoherent and that the canonical map \( \mathbb{Z} \to K_0(\ell) \) is an isomorphism. Then there are no unital \( C_m \)-graded ring homomorphisms \( L_n \to L_n^- \) nor in the opposite direction.

Theorem 1.2 generalizes a similar statement proved for \( n = m = 2 \) in [7, Proposition 5.6]. It implies that \( L_n \) and \( L_n^- \) are not graded isomorphic over any of the cyclic groups; this was well-known for the infinite cyclic group (see e.g. [5, Example 4.2], [6, end of Section 4.1]), and follows from [7, Proposition 5.6] in the case \( n = m = 2 \). The hypothesis on \( \ell \) in the theorem above are satisfied, for example, when \( \ell \) is a field, or a PID, or a noetherian regular local ring. They guarantee that for any directed graph \( E \) with finitely many vertices and edges and such that every vertex emits at least one edge, and any \( 2 \leq m \leq \infty \), the Grothendieck group \( K_0^{C_m \text{-gr}}(L(E)) \) of \( C_m \)-graded, finitely generated projective modules, equipped with the shift action, is isomorphic to the Bowen-Franks \( \mathbb{Z}[C_m] \)-module

\[
K_0^{C_m \text{-gr}}(L(E)) \cong \mathfrak{B}_m(E) := \text{coker}(I - \tau_m A_E^t).
\]

The second named author was supported by CONICET and partially supported by grants PICT 2017-1395 from Agencia Nacional de Promoción Científica y Técnica, UBACyT 0150BA from Universidad de Buenos Aires, and PGC2018-096446-B-C21 from the Spanish Ministerio de Ciencia e Innovación.
Here \( \tau_m \) is a generator of \( C_m \) and \( A_E \) is the adjacency matrix of \( E \). Under the isomorphism above, the class of the free module of rank one with its standard grading is mapped to \([1]_E := \sum_{v \in E^0} [v] \in \mathcal{B} \mathcal{F}_m(R) \). Thus if \( F \) is another such graph, the existence of a unital, \( C_m \)-graded ring homomorphism \( L(E) \rightarrow L(F) \) implies the existence of a homomorphism of \( \mathbb{Z}[C_m]\)-modules \( \mathcal{B} \mathcal{F}_m(E) \rightarrow \mathcal{B} \mathcal{F}_m(F) \) mapping \([1]_E \mapsto [1]_F \). Our proof of Theorem 1.2 consists in showing that, for \( \tau = \tau_m \), there are no such maps between

\[
\mathcal{B} \mathcal{F}_m(\mathcal{R}_n) = \mathbb{Z}[C_m] / (1 - n\tau) \quad \text{and} \quad \mathcal{B} \mathcal{F}_m(\mathcal{R}_n^{-}) = \mathbb{Z}[C_m] / (\tau^3 + (2n - 1)\tau^2 - (n + 2)\tau + 1).
\]

The rest of this article is organized as follows. In Section 2 the Bowen-Franks modules are introduced and some of their basic properties are proved, including a nontriviality criterion (Lemma 2.5). The latter is used in Section 3 to establish a lower bound on the number of elements of \( \mathcal{B} \mathcal{F}_m(\mathcal{R}_n^{-}) \) in Proposition 3.3. The modules \( \mathcal{B} \mathcal{F}_m(\mathcal{R}_n) \) and \( \mathcal{B} \mathcal{F}_m(\mathcal{R}_n^{-}) \) are also computed in this section, in Lemmas 3.1 and 3.4. The isomorphism (1.3) is proved in Section 4 as Lemma 4.2. In Section 5 we prove Theorem 5.1, which says that if \( n \geq 2 \) and \( 2 \leq m \leq \infty \) then there is no \( C_m \)-graded unital ring homomorphism \( L_n^- \rightarrow L_n \). The nonexistence of \( C_m \)-graded unital homomorphisms in the opposite direction is established in Section 6 as Theorem 6.5.

Acknowledgements. The computational results of Sections 5 and 6 were verified using SageMath [8]. This software was also used to obtain Equation (6.4) in Proposition 6.3. The first named author wishes to thank Lucas De Amorin and Fernando Martin for useful discussions. The second named author wishes to thank Pere Ara, whose manifestation of interest in the particular case \( n = m = 2 \) of our main theorem proved in [7] was a key encouragement to pursue the general case.

Notation 1.4. In this paper the natural numbers do not include 0. We write \( \mathbb{N} = \mathbb{Z}_{\geq 1} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( m \in \mathbb{N}_0 \), we write \( C_m \) for the cyclic group of order \( m \) and \( C_\infty \simeq \mathbb{Z} \). Having fixed \( m \in \mathbb{N}_0 \), the symbol \( \tau \) will refer to a generator of \( C_m \), written multiplicatively.

2. The Bowen-Franks modules of a graph

A (directed) graph \( E \) consists of a set \( E^0 \) of vertices and a set \( E^1 \) of edges together with source and range functions \( r, s : E^1 \rightarrow E^0 \). A vertex \( v \in E^0 \) is regular if it emits a finite, positive number of edges; we write \( \text{reg}(E) \subseteq E^0 \) for the set of regular vertices. The (reduced) adjacency matrix of a graph \( E \) is the matrix \( A_E \) with nonnegative integer coefficients, indexed by \( \text{reg}(E) \times E^0 \), whose \((v, w)\) entry is the number of edges with source \( v \) and range \( w \). The graph \( E \) is regular if \( E^0 = \text{reg}(E) \) and \( E^1 \) and both \( E^0 \) and \( E^1 \) are finite. If \( E \) is both finite and regular, then \( A_E \) is a finite square matrix with no zero rows.

Definition 2.1. Let \( m \in \mathbb{N}_0 \). The Bowen-Franks \( \mathbb{Z}[C_m] \)-module of a finite regular graph \( E \) with adjacency matrix \( A_E \) is \( \mathcal{B} \mathcal{F}_m(E) := \text{coker}(I - \tau \cdot A_E) \).

The following lemma shall be useful in what follows.

Lemma 2.2. Let \( 2 \leq m < \infty \) and let \( E \) be a finite regular graph. Equip \( \text{coker}(I - (A_E)^m) \) with the \( C_m \)-action \( \tau \cdot [x] = [(A_E)^{m-1}x] \). Then there is a \( \mathbb{Z}[C_m] \)-module isomorphism

\[ \mathcal{B} \mathcal{F}_m(E) \cong \text{coker}(I - (A_E)^m), \quad [v] \mapsto [v]. \]

Proof. Put \( M = \tau^{-1}(I - \tau A_E) \); clearly \( \mathcal{B} \mathcal{F}_m(E) \cong \text{coker}(M) \) as \( \mathbb{Z}[C_m] \)-modules. The lemma is immediate from the form of the matrix of multiplication by \( M \) with respect to the \( \mathbb{Z} \)-linear basis \( \{\nu \tau^i : v \in E^0, 0 \leq i < m - 1\} \) of \( \mathbb{Z}[C_m]^E \).

Corollary 2.3. The following are equivalent.
i) \( \mathcal{B} \mathcal{F}_m(E) \) is finite.
ii) \( \chi_{A_E^m}(1) \neq 0 \).
If these equivalent conditions hold, then \(|\mathfrak{B}_{m}(E)| = |\chi_{A_{E}^{m}}(1)|\).

Proof. Straightforward from Lemma 2.2 using the Smith normal form of the matrix \(I - (A_{E})^{m}\). □

Let \(E\) be a finite regular graph. In the following lemma and elsewhere, we write
\[
(2.4) \quad \chi_{E}(x) = \det(xI - A_{E}) \in \mathbb{Z}[x]
\]
for the characteristic polynomial associated to the adjacency matrix of \(E\).

Lemma 2.5. Let \(E\) be a finite regular graph. Assume that all complex roots of \(\chi_{E}\) are real. If \(\mathfrak{B}_{2}(E)\) is finite and nontrivial, then \(\infty > |\mathfrak{B}_{m}(E)| > |\mathfrak{B}_{2}(E)| > 1\) for all \(m > 2\).

Proof. Let \(\chi_{E} = (x - \alpha_{1}) \cdots (x - \alpha_{k})\) be the irreducible factorization of \(\chi_{E}\) in \(\mathbb{C}[x]\) (repetitions are allowed). For each \(m \geq 2\) we have
\[
|\chi_{A_{E}^{m}}(1)| = \prod_{i=1}^{k} |1 - \alpha_{i}|^{m} \geq \prod_{i=1}^{k} |1 - |\alpha_{i}|^{m}|.
\]
When \(m = 2\) we have \(|\alpha_{i}|^{2} = \alpha_{i}^{2}\), since all roots are real, and thus the inequality above is in fact an equality. In particular, by Corollary 2.3, the hypothesis that \(\mathfrak{B}_{2}(E)\) is finite and nontrivial implies that \(|\alpha_{i}|^{m} \neq 1\) for all \(m \geq 1\). Hence the right hand side of the inequality is strictly increasing, which shows that \(|\mathfrak{B}_{m}(E)| = |\chi_{A_{E}^{m}}(1)| > |\mathfrak{B}_{2}(E)| > 1\) for \(m > 2\) as desired. □

3. The Bowen-Franks groups of the rose and of its Cuntz splice

Let \(n \geq 1\); the rose of \(n\) petals is the graph \(R_{n}\) which consists of one vertex and \(n\) loops. The Cuntz splice of \(R_{n}\) ([1, Definition 2.11]) is the graph \(R_{n}^{-}\) with adjacency matrix (1.1).

Lemma 3.1. Let \(n, m \geq 1\). Then there is an isomorphism of \(\mathbb{Z}[C_{m}]\)-modules
\[
(3.2) \quad \mathfrak{B}_{m}(R_{n}) \simeq \mathbb{Z}/(n^{m} - 1)\mathbb{Z}.
\]
Here, the generator \(\tau\) of \(C_{m}\) acts on the right hand side of (3.2) as multiplication by \(n^{m-1}\).

Proof. Straightforward calculation. □

Next we turn our attention to \(\mathfrak{B}_{m}(R_{n}^{-})\). Before computing it, we may already use Lemma 2.5 to establish the following nontriviality result.

Proposition 3.3. Let \(n, m \geq 2\). Then \(\mathfrak{B}_{m}(R_{n}^{-})\) is finite and of order at least \(3n^{2} - 2n - 1\).

Proof. A direct computation shows that
\[
\chi_{R_{n}^{-}}(x) = x^{3} - (n + 2)x^{2} + (2n - 1)x + 1.
\]
One checks that the signs of the values of this polynomial at \(-1, 1, 2\) and \(n + 1\) alternate; hence all of its roots are real. Likewise, computing
\[
\chi_{A_{R_{n}^{-}}^{2}}(x) = x^{3} - (6 + n^{2})x^{2} + (4n^{2} - 2n + 5)x - 1
\]
we see that \(|\chi_{A_{R_{n}^{-}}^{2}}(1)| = |3n^{2} - 2n - 1| = 3n^{2} - 2n - 1 > 1\) if \(n \geq 2\). The result now follows from Lemma 2.5. □

Lemma 3.4. Let \(n \geq 1\) and \(2 \leq m \leq \infty\); put
\[
(3.5) \quad \xi_{n}(x) := x^{3} + (2n - 1)x^{2} - (n + 2)x + 1 \in \mathbb{Z}[x].
\]
There is an isomorphism of $\mathbb{Z}[C_m]$-modules
\begin{equation}
\mathcal{B}_m(L_{n-}) \simeq \frac{\mathbb{Z}[C_m]}{\langle \xi_n(\tau) \rangle}
\end{equation}
that sends $[(1, 1, 1)]$ to $[1 - n\tau]$.

**Proof.** Let $B = A_{R_{n-}}$; one checks that the matrices
\[
R = \begin{pmatrix}
1 & -n & 0 \\
\tau & 1 - n\tau & (n - 1)\tau - (n + 1) \\
\tau^2 & \tau - n\tau^2 & (n - 1)\tau^2 - (n + 1)\tau + 1
\end{pmatrix}
\]
and
\[
C = \begin{pmatrix}
1 & (1 - n)\tau + n & (n - 1)\tau^3 + (2n^2 - 4n + 1)\tau^2 + (1 - 3n^2)\tau + n(n + 1) \\
0 & 1 & (n + 1) + (1 - 2n)\tau - \tau^2 \\
0 & 0 & 1
\end{pmatrix}
\]
are invertible in $M_3(\mathbb{Z}[C_m])$ and satisfy
\[
R(I - \tau B)C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \tau^3 + (2n - 1)\tau^2 - (n + 2)\tau + 1
\end{pmatrix}.
\]
Hence we have a commutative diagram as follows, where all vertical arrows are isomorphisms.

The desired isomorphism is the composition $\pi_3R$. Indeed, from the fact that the quotient map $\mathbb{Z}[C_m]^3 \to \mathcal{B}_m(L_{n-})$ maps $(1, 1, 1)$ to $[(1, 1, 1)]$ and the commutativity of the diagram above, we conclude that
\[
\pi_3(R([(1, 1, 1)])) = \pi_3([(R(1, 1, 1)]) = \pi_3([(1 - n, -n, 1 - n\tau)]) = [1 - n\tau].
\]

4. **Bowen-Franks modules, graded $K_0$, and Leavitt path algebras**

A unital ring $R$ is **coherent** if the category of its finitely presented modules is abelian; it is **supercoherent** if the polynomial ring $R[t_1, \ldots, t_n]$ is coherent for every $n \geq 0$. We say that $R$ is **regular** if every (right) $R$-module has finite projective dimension, and **regular supercoherent** if it is both regular and supercoherent. This implies that $R[t_1, \ldots, t_n]$ is regular for all $n \geq 1$, by the argument of [3, beginning of Section 7].

**Standing assumption 4.1.** Throughout the rest of this paper, $\ell$ will be a unital, regular supercoherent ring such that the canonical map
\[
\mathbb{Z} \to K_0(\ell)
\]
is an isomorphism.
We write $L(E)$ for the Leavitt path algebra of $E$ over $\ell$ ([2]), which equals the tensor product $\ell \otimes LZ\ell(E)$ with the Leavitt path algebra over $Z$. The algebra $L(E)$ carries a canonical $C_\infty$-grading $L(E) = \bigoplus_{k \in Z} L(E)k$ that makes it a $C_\infty$-graded algebra. Hence $L(E)$ is also $C_m$-graded for every $m \geq 2$, where $L(E)i = \bigoplus_{k \in Z} L(E)_{mk+i}$ ($i \in Z/mZ \cong C_m$).

In general, if $G$ is a group and $R = \bigoplus_{g \in G} R_g$ a $G$-graded unital ring, we write $K_0^{G,-wR}(R)$ for the group completion of the monoid of finitely generated projective $G$-graded $R$-modules. Grading shifts equip $K_0^{G,-wR}(R)$ with a $Z[G]$-module structure.

In the case when $\ell$ is a field, the following lemma is immediate from a very particular case of [4, Proposition 5.7].

**Lemma 4.2.** Let $2 \leq m \leq \infty$ and let $E$ be a finite regular graph. Then there is an isomorphism of $Z[C_m]$-modules

$$K_0^{C_m,-wE}(L(E)) \cong \mathfrak{B}_m(E)$$

which maps $[L(E)] \mapsto \sum_{v \in E\ell} [v]$.

**Proof.** Let $E_m$ be the $C_m$-cover of $E$ as defined in [4, Section 5.2]. Observe that $C_m$ acts on $L(E_m)$ by algebra automorphisms, making $K_0(L(E_m))$ into a $C_m$-module. By [4, Proposition 2.5 and Corollary 5.3], there is a $Z[C_m]$-module isomorphism $K_0^{C_m,-wE}(L(E)) \cong K_0(L(E_m))$. By [3, Theorem 7.6], $K_0(L(E_m)) = \text{coker}(I - A_{E_m})$; one checks that this $Z[C_m]$-module is precisely $\mathfrak{B}_m(E)$. \hfill \Box

**Corollary 4.3.** Let $E$ and $F$ be finite regular graphs and let $m \geq 2$. Then any unital, $C_m$-graded ring homomorphism $L(E) \rightarrow L(F)$ induces a homomorphism of $Z[C_m]$-modules $\mathfrak{B}_m(E) \rightarrow \mathfrak{B}_m(F)$ that maps $\sum_{v \in E\ell} [v] \mapsto \sum_{w \in F\ell} [w]$. \hfill \Box

In the rest of this article, we shall concentrate on the Leavitt path algebras $L_n = L(R_n)$ and $L_{n^-} = L(R_{n^-})$.

5. NONEXISTENCE OF GRADED HOMOMORPHISMS $L_n^- \rightarrow L_n$

**Theorem 5.1.** Let $n \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Then there is no unital $C_m$-graded ring homomorphism $L_{n^-} \rightarrow L_n$.

**Proof.** Because any $C_\infty$-graded homomorphism is $C_2$-graded, we may assume that $m < \infty$. Suppose that there exists a $C_m$-graded unital morphism as in the theorem. Then by Lemmas 3.1 and 3.4 and Corollary 4.3, there must be a homomorphism of abelian groups

$$\phi : Z[C_m]/(\xi_n(\tau)) \rightarrow Z/(n^m - 1)$$

mapping $[1 - n \cdot \tau] \mapsto [1]$ and sending multiplication by $\tau$ into multiplication by $n^{m-1}$. But then

$$[1] = \phi([1 - n \cdot \tau]) = \phi(1) \cdot (1 - n\tau) = \phi(1)(1 - n) = 0,$$

a contradiction. \hfill \Box

6. NONEXISTENCE OF GRADED HOMOMORPHISMS $L_n \rightarrow L_{n^-}$

Let $n, m \geq 2$. Consider the ideal

$$(6.1) \quad I_{m,n} := \langle x^m - 1, \xi_n(x) \rangle \triangleleft \mathbb{Z}[x].$$

**Lemma 6.2.** Let $n, m \geq 2$. The existence of a unital $C_m$-graded ring homomorphism $L_n \rightarrow L_{n^-}$ would imply that $(1 - nx)^2 \in I_{m,n}$.

**Proof.** Lemma 3.4 says that $\mathfrak{B}_m(R_{n^-}) = \mathbb{Z}[x]/I_{m,n}$. The present lemma follows from this using Lemma 3.1 and Corollary 4.3. \hfill \Box
Proposition 6.3. Let \( m, n \geq 2 \). Assume that \((1-nx)^2 \in I_{m,n}\). Then
\[
I_{m,n} = ((x-1)^2, m(x-1), (3n-1)(x-1) + n - 1).
\]

Proof. The proof involves several steps, as follows.

Step 1: \((n-1)^2 \in I_{m,n}\). Let
\[
a_n = -n^5 + 2n^4 - 2n^3 + 3n^2, \\
b_n = -2n^6 + 5n^5 - 7n^4 + 10n^3 - 4n^2 + 2n, \\
c_n = n^6 - 2n^5 + 3n^4 - 4n^3 + n^2 - 2n + 1.
\]

Set \( p_n(x) = (n^7 - 2n^6 + 2n^5 - 3n^4)x - n^6 + 2n^5 - 3n^4 + 4n^3 \) and \( q_n(x) = a_n x^2 + b_n x + c_n \). A calculation shows that
\[
(6.4) \quad p_n(x)\xi_n(x) + q_n(x)(1 - n \cdot x)^2 = (n-1)^2.
\]

Step 2: \((n+1)x - 2 \in I_{m,n}\). Put
\[
g_n(x) = nx + 2n^2 - n + 2, \quad r_n(x) = (-n^4 + 2n^3 - 2n^2 + 3n)x + n^3 - 2n^2 + n - 2.
\]

One checks that
\[
n^3 \cdot \xi_n(x) = (1 - n \cdot x)^2 q_n(x) + r_n(x).
\]

This step is concluded once we observe that \( r_n \equiv (n+1)x - 2 \pmod{(n-1)^2} \).

Step 3: \((x-1)^2 \in I_{m,n}\). By the previous steps, \(1 - nx \equiv x - 1\) and so \(0 \equiv (1-nx)^2 \equiv (x-1)^2\).

Step 4: Conclusion. Observe that \( p \equiv (x-1) \cdot p'(1) + p(1) \pmod{(x-1)^2}\) for any \( p \in \mathbb{Z}[x] \).

Applying this to \(x^n - 1\) and \(\xi_n\), we get the proposition. \(\square\)

Theorem 6.5. Let \( m \in \mathbb{N}_{\geq 2} \cup \{\infty\} \) and let \( n \geq 2 \). Then there is no \( C_m\)-graded unital ring homomorphism \( L_n \to L_{n-1} \).

Proof. As in the proof of Theorem 5.1, we may assume that \( m < \infty \). Furthermore, we can reduce ourselves to the case in which \( m \) is prime, since for each \( d \mid m \) a \( C_m\)-graded ring homomorphism \( L_n \to L_{n-1} \) is also \( C_d\)-graded. Now, suppose that there exists a unital \( C_m\)-graded map \( L_n \to L_{n-1} \).

By Lemma 6.2, \((1-nx)^2 \in I_{m,n}\). Hence the identity of Proposition 6.3 tells us in particular that
\[
0 \equiv ((3n-1)(x-1) + n - 1)(x-1) \equiv (n-1)(x-1) \pmod{I_{m,n}}.
\]

First consider the case in which \( m \) does not divide \( n-1 \), which by primality of \( m \) means that \( m \) and \( n-1 \) are coprime. Hence there exist integers \( s, t \in \mathbb{Z} \) such that \((n-1)s + mt = 1 \) and so \((x-1) = s(n-1)(x-1) + tm(x-1) \equiv 0 \pmod{I_{m,n}}\). Therefore
\[
I_{m,n} = ((x-1)^2, m(x-1), (3n-1)(x-1) + n - 1, x - 1) \equiv (x - 1, n - 1),
\]

which in turn shows that \( \mathfrak{R}_m(L_{n-1}) \cong \mathbb{Z}/(n-1)\mathbb{Z} \) as abelian groups. However, this is a contradiction; the bound of Proposition 3.3 tells us that
\[
|\mathfrak{R}_m(L_{n-1})| \geq |\mathfrak{R}_2(L_{n-1})| = 3n^2 - 2n - 1 > n - 1.
\]

We are left to prove the case in which the prime \( m \) divides \( n-1 \). Write \( n = am + 1 \), so that
\[
I_{m,n} = I_{m,am+1} = ((x-1)^2, m(x-1), (3am + 2)(x-1) + am)
\]
\[
= ((x-1)^2, m(x-1), 2(x-1) + am).
\]
Setting $m = 2$, we obtain
\[ I_{2,2a+1} = \langle (x - 1)^2, 2(x - 1), 2(x - 1) + 2a \rangle \]
\[ = \langle (x - 1)^2, 2(x - 1), 2a \rangle. \]

In particular we have an epimorphism
\[ \frac{\mathbb{Z}[x]}{(x - 1)^2} \to \mathfrak{B}_2(\mathbb{R}_{(2a+1)^-}) \]
which tells us that $|\mathfrak{B}_2(\mathbb{R}_{(2a+1)^-})| \leq 4a^2$. This contradicts Proposition 3.3, since
\[ |\mathfrak{B}_2(\mathbb{R}_{(2a+1)^-})| = 3(2a + 1)^2 - 2(2a + 1) - 1 > (2a + 1)^2 > 4a^2. \]

We can therefore assume that $m$ is an odd prime. Write $m = 2b + 1$ for some $b \geq 1$. Then
\[ I_{m,am+1} \ni m(x - 1) - b(2(x - 1) + am) = (x - 1) - bam = x - (1 + am). \]
This shows that
\[ I_{m,am+1} = \langle (x - 1)^2, m(x - 1), 2(x - 1) + am, x - (1 + am) \rangle \]
\[ = \langle \beta^2 a^2 m^2, bam^2, 2abm + am, x - (1 + abm) \rangle. \]
Noting that $2abm + am = am(2b + 1) = am^2$, we get $I_{m,am+1} = \langle am^2, x - (1 + abm) \rangle$. Therefore, we obtain an isomorphism of abelian groups
\[ \mathfrak{B}_m(\mathbb{R}_{(am+1)^-}) \simeq \mathbb{Z}[x]/\langle x - (1 + abm), am^2 \rangle \simeq \mathbb{Z}/am^2\mathbb{Z}. \]
Once again we draw a contradiction from the bound of Proposition 3.3, since
\[ |\mathfrak{B}_m(\mathbb{R}_{(am+1)^-})| \geq |\mathfrak{B}_2(\mathbb{R}_{(am+1)^-})| = 3(am + 1)^2 - 2(am + 1) - 1 \geq (am + 1)^2 > am^2. \]
This completes the proof. \[ \square \]

Remark 6.6. The case $m = 2$ of Theorem 6.5 does not need the full force of Proposition 6.3; it can be derived after Step 3 of its proof. Indeed the existence of a graded unital ring a homomorphism $L_2 \to L_2^-$ would imply $1 = (2 - 1)^2 \equiv 0 \pmod{I_{m,2}}$, contradicting Proposition 3.3.

References

[1] Gene Abrams, Adel Lourly, Enrique Pardo, and Christopher Smith, Flow invariants in the classification of Leavitt path algebras, J. Algebra 333 (2011), 202–231. MR2785945

[2] Gene Abrams, Pere Ara, and Mercedes Siles Molina, Leavitt path algebras, Lecture Notes in Math., vol. 2008, Springer, 2017.\[ \dagger, 3 \]

[3] Pere Ara, Miquel Brustenga, and Guillermo Cortiñas, K-theory of Leavitt path algebras, Münster J. Math. 2 (2009), 5–33. MR2545605

[4] Pere Ara, Roozbeh Hazrat, Huanhuan Li, and Aidan Sims, Graded Steinberg algebras and their representations, Algebra Number Theory 12 (2018), 131–172. \[ \dagger, 5 \]

[5] P. Ara and E. Pardo, Towards a K-theoretic characterization of graded isomorphisms between Leavitt path algebras, J. K-Theory 14 (2014), no. 2, 203–245. MR3319704

[6] Luiz Gustavo Cordeiro, Daniel Gonçalvez, and Roozbeh Hazrat, The talented monoid of a directed graph with applications to graph algebras, Rev. Mat. Iberoam., to appear, available at arXiv:2003.09911.

[7] Guillermo Cortiñas, Classifying Leavitt path algebras up to involution preserving homotopy, available at arXiv:2101.08777.

[8] The Sage Developers, Sagemath, the Sage Mathematics Software System (Version 9.0), 2020. https://www.sagemath.org.

Dep. Matemática-IMAS, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina