Bayesian Sparsification Methods for Deep Complex-valued Networks

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Abstract

With continual miniaturization ever more applications of deep learning can be found in embedded systems, where it is common to encounter data with natural complex domain representation. To this end we extend Sparse Variational Dropout to complex-valued neural networks and verify the proposed Bayesian technique by conducting a large numerical study of the performance-compression trade-off of \( \mathbb{C} \)-valued networks on two tasks: image recognition on MNIST-like and CIFAR10 datasets and music transcription on MusicNet. We replicate the state-of-the-art result by Trabelsi et al. [2018] on MusicNet with a complex-valued network compressed by \( 50-100\times \) at a small performance penalty.

1 Introduction

Deep neural networks are an integral part of machine learning and data science toolset for practical data-driven problem solving. With continual miniaturization ever more applications can be found in embedded systems. Common embedded applications include on-device image recognition and signal processing. Despite recent advances in generalization and optimization theory specific to deep networks, deploying in actual embedded hardware remains a challenge due to storage, real-time throughput, and arithmetic complexity restrictions [He et al., 2018]. Therefore, compression methods for achieving high model sparsity and numerical efficiency without losing much in performance are especially relevant.

Inherently complex-valued nature of data in signal processing, specifically, acoustic or radio frequency signal analysis, has been the key driver behind the adoption of \( \mathbb{C} \)-networks. Hirose [2009] argues that \( \mathbb{C} \)-valued networks merit from the simultaneous phase rotation and amplitude adjustment, intrinsic to \( \mathbb{C} \) arithmetic, which reduce ineffective degrees of freedom in comparison with an double-dimensional \( \mathbb{R} \)-valued networks. Their study lists numerous applications related to acoustic and radio signal processing and demonstrates superiority of \( \mathbb{C} \)-networks in the task of shallow landmine detection using ground penetrating radar imaging, in spite of inherently higher arithmetic complexity of \( \mathbb{C} \)-networks, stemming from \( 3-4 \) floating point operations per single \( \mathbb{C} \) multiplication. More recently \( \mathbb{C} \)-networks have been applied to music transcription and speech recognition [Trabelsi et al., 2018]. Arjovsky et al. [2016] propose \( \mathbb{C} \)-valued unitary-RNNs to deal with exploding / vanishing gradients and apply the recurrent
models to long-term dependence prediction tasks. Wisdom et al. [2016] expand on their work by optimizing recurrent weights on the $\mathbb{C}^{n \times n}$ Stiefel manifold. And Wolter and Yao [2018] investigate different $\mathbb{C}$-valued gating mechanisms for unitary-RNNs and conduct experiments on motion prediction and music transcription. Yang et al. [2019] develop complex transformer, with $\mathbb{C}$-attention and complex encoder-decoder, and apply it to music transcription and wireless signal classification. Tarver et al. [2019] achieve significantly lower out-of-band performance with a $\mathbb{C}$-valued network in Digital Signal Predistortion task.

However, in spite of relative success of $\mathbb{C}$-valued networks, compression methods tailored to $\mathbb{C}$-domain remain a niche field of research. Computational efficiency improvement and model compression methods such as quantization [Uhlich et al., 2019], integer-based arithmetic [Lin et al., 2016] Chen et al. [2017], and knowledge distillation [Hinton et al., 2015] appear to be directly applicable to $\mathbb{C}$-valued networks. Certain model sparsification methods, e.g. sparsity inducing regularizers, and magnitude pruning [Zuo et al., 2019], which reacti-vates zeroed parameters based on gradients, can be adapted directly to $\mathbb{C}$-valued models. Other methods require additional considerations.

Bayesian Inference is a principled framework of reasoning about uncertainty, which updates prior beliefs in accordance with the likelihood of empirical observations into posterior belief. Under suitable modelling and distributional assumptions, Bayesian methods can be used towards model sparsification, e.g. Variational Dropout [Kingma et al., 2015] Molchanov et al. [2017]. Gale et al. [2019] compare Variational Dropout, magnitude pruning, and $\ell_0$ regularization [Louizos et al., 2017] on large-scale models. The latter is a probabilistic regularization method, which penalizes the expected number of non-zero entries in $[0,1]$-valued stochastic parameter masks with learnable hard-concrete distribution with atoms at 0 and 1 [Maddison et al., 2016] Jang et al. [2017]. Their results suggest that Variational Dropout may achieve good accuracy-sparsity balance and outperform pruning and $\ell_0$ in deep architectures, although pruning is preferred for simplicity, stability and speed. They also observe that dropout induces non-uniform sparsity throughout the model, which He et al. [2018] has shown to be essential for superior compression.

Inspired by variational and stochastic sparsification techniques we extend Bayesian dropout to $\mathbb{C}$-valued networks. We assess the performance-compression trade-off of the proposed Bayesian technique by conducting a large-scale numerical study of $\mathbb{C}$-valued networks on two tasks: image recognition on MNIST-like and CIFAR10 datasets and music transcription on MusicNet [Thickstun et al., 2017].

The paper is structured as follows. In sec. 2 we review Bayesian dropout techniques, and in sec. 3 we provide a brief summary of the inner working of complex-valued networks as functions of complex argument. The main contribution of this study is presented in sec. 4 where we consider different variational approximations and priors, outline the tricks and derive the penalty terms in the evidence lower bound. In sec. 5 we evaluate the sparsification rate, compare the resulting performance of various $\mathbb{C}$-networks proposed in prior work, and discuss the outcomes.
2 Variational Dropout

In general, the core of Bayesian Inference can be summarized as follows: given a prior distribution $\pi(m)$ on models (hypotheses) $\mathcal{M}$, utilize the empirical evidence $D$ to update the assumptions by considering the likelihood of the observations under each $m \in \mathcal{M}$. Models are represented by a parametric family indexed by parameters $\omega \in \Omega$ and each $m_{\omega}(\cdot)$ specifies the conditional distribution of the data $D = (z_i)_{i=1}^N$. The posterior distribution $p(\omega \mid D)$, derived using the Bayes rule $p(m \mid D) = \frac{p(D|m)\pi(m)}{p(D)}$ with $p(D) = \mathbb{E}_{\pi(m)} p(D \mid m)$, provides useful information about yet unobserved data and model parameters, e.g. classification uncertainty, predictive statistics, and parameter relevance.

Save for the relatively simple cases, the posterior distribution is intractable, and therefore exactness Bayesian Inference is traded for tractability and scalability of Variational Bayesian Inference. Instead of deriving the posterior $p(\omega \mid D)$, this approach postulates an optimization problem for finding an approximation $q$, which is close to $p(\omega \mid D)$ in terms of a proximity score $\rho$

$$q_\ast \in \arg\min_{q} \rho(q_\omega(\omega), p(\omega \mid D)),$$  \hspace{1cm} (1)  

over a tractable parametric family of distributions on the parameter space $Q = \{q_\theta(d\omega) : \theta \in \Theta\}$, $\theta$ - generic variational parameter. The variational approximation class $Q$ is chosen so that its members have tractable densities $q_\theta(\omega) d\omega$, can be sampled from and have either tractable log-derivatives $\omega \mapsto \nabla_\theta \log q_\theta(\omega)$ or can be represented as differentiable transformations of non-parametric noise [Williams, 1992, Kingma and Welling, 2014, Figurnov et al., 2019]. The most common choice for $\rho$ is Kullback-Leibler divergence of density $q$ from $p$,

$$KL(q\|p) = \mathbb{E}_{\omega \sim q} \log q(\omega) - \mathbb{E}_{\omega \sim q} \log p(\omega),$$ \hspace{1cm} (2) 

the main reason being that it satisfies the following identity for well-behaved densities:

$$\log p(D) - KL(q\|p(\omega \mid D)) = \mathbb{E}_{q(\omega)} \log p(D \mid \omega) - KL(q\|\pi) \hspace{1cm} (3)$$

This leads to an equivalent problem for (1) of maximizing the evidence lower bound (ELBO)

$$L(\theta; \phi, \lambda) = -KL(q_\theta\|\pi) + \mathbb{E}_{\omega \sim q_\theta} \log p_\phi(D \mid \omega),$$ \hspace{1cm} (4) 

where $\lambda$, $\theta$ and $\phi$ are optimization parameters of the prior $\pi$, approximation $q$ and the likelihood $p(D \mid \omega)$, respectively. Other objectives are possible, besides (4), provided $p(\omega \mid D)$ is evaluated only through $\log p(D \mid \omega)$ and $\log \pi(\omega)$, [Ranganath et al., 2018, Stochastic Gradient Variational Bayes (SGVB), proposed by [Kingma and Welling, 2014], replaces the Variational objective (4) by its unbiased differentiable Monte-Carlo estimator, which makes it possible to employ stochastic gradient optimization methods to solve (1). If $\omega \sim q_\theta$ is equivalent in distribution to $\omega = g(\varepsilon; \theta)$ for some differentiable $g(\varepsilon; \theta)$ and non-parametric random variable $\varepsilon \sim p_\varepsilon$, then the SGVB objective is

$$\tilde{L} = -KL(q_\theta\|\pi) + \frac{1}{L} \sum_{t=1}^{L} \log p_\phi(D \mid g(\varepsilon_t; \theta)),$$ \hspace{1cm} (5)
where $(\varepsilon_i)_{i=1}^N$ is an iid sample from $p_\varepsilon$, the KL-divergence term replaced by unbiased differentiable finite sample estimator if it is intractable. For the purposes of this study, we assume that the likelihood satisfies \( \log p_\phi(D | \omega) = \sum_{i} \log p_\phi(z_i | \omega) \), i.e. the observed data $D = (z_i)_{i=1}^N$ is conditionally independent, and consider (5) with $L = 1$ and random mini-batches from $D$:

\[
\mathcal{L} = -KL(q_\theta \| \pi_\lambda) + \frac{N}{M} \sum_{k=1}^M \log p_\phi(z_{ik} | g(\varepsilon_k; \theta)) ,
\]

for a random subsample $(z_{ik})_{k=1}^M$ and iid $\varepsilon_k \sim p_\varepsilon$.

With a special family of posterior approximation, Variational Inference can be used as a regularization method and as a model sparsification technique.

Dropout [Hinton et al., 2012] prevents overfitting by injecting multiplicative binary noise into layer’s weights, which breaks up co-adaptations that could occur during training. Wang and Manning [2013] argue that the overall effect of binary dropout on the intermediate outputs via the Central Limit Theorem can be approximated by a Gaussian with weight-input dependent mean and variance. Srivastava et al. [2014] propose using independent $\mathcal{N}(1, 1)$ noise, arguing that higher entropy of a Gaussian has better regularizing effect. In a recent study concerning multitask learning Cheung et al. [2019] have shown the possibility of storing task-specific parameters in non-destructive superposition within a single network. In particular, their argument implies that if the single task setting is viewed as multitask learning with replicated task, then by sampling uncorrelated binary masks Dropout acts as a superposition method, utilizing the learning capacity of the network better.

Dropout can be viewed as a Bayesian model averaging method, where models share parameters and are weighted equally. Thus, Kingma et al. [2015] consider Dropout, DropConnect [Wan et al., 2013], and Gaussian dropout [Wang and Manning, 2013] through the lens of Bayesian Inference methods and propose Variational Dropout. They argue that the multiplicative noise introduced by these methods induces a distribution equivalent to a fully factorized variational posterior of the form $q_\theta(\omega) = \prod_j q_\theta(\omega_j)$, where $q_\theta(\omega_j)$ is $\omega_j = \mu_j \xi_j$ with $\xi_j \sim p_\theta(\xi)$ iid from some $p_\theta(\xi)$.

Variational Dropout assumes fully factorized Gaussian approximate posterior $q_\theta(\omega) = \prod_j \mathcal{N}(\omega_j | \mu_j, \alpha_j \mu_j^2)$ and factorized log-uniform prior $\pi(\omega)$ with $\pi(\omega_j) \propto |\omega_j|^{-1}$. The divergence term in (6) unravels into the following sum:

\[
KL(q_\theta \| \pi) \propto \frac{1}{2} \sum_j \mathbb{E}_{\varepsilon \sim \mathcal{N}(0, 1)} \log \left[ \frac{1}{\sqrt{\alpha_j}} + \varepsilon \right]^2 .
\]

Kingma et al. [2015] approximate this expression by a non-linear polynomial regression over $\alpha \in (0, 1)$, Molchanov et al. [2017] refine the approximation by using a nonlinear regression based on sigmoid and soft-plus. In appendix A we verify the value and the derivative of their approximation against the Monte-Carlo estimate of (7) for $\alpha$ varying over a fine logarithmic grid and the gradient of (7), for which we obtain an exact expression.

The key role in the factorization is played by $\alpha_j$, which reflects the relevance of the parameter $\omega_j$, it is associated to, by being the ratio of its squared mean to its variance. Thus, Molchanov et al. [2017] focus on the model sparsification capabilities of Variational Dropout, optimizing $\alpha_j$ for each individual parameter,
or across groups of parameters for structured sparsity. Kharitonov et al. [2018] address theoretical issues with improper priors and propose a method based on Automatic Relevance Determination procedure, which falls under Empirical Bayes approach that fits a prior distribution while performing Variational Inference. In this method the log-uniform factor $\pi(\omega_j)$ is replaced by a proper Gaussian ARD prior $\mathcal{N}(\omega_j | 0, \tau_j^{-1})$ with learnable precision $\tau_j > 0$, [Neal, 1996].

Maximizing (6) over $\tau$, holding other parameters fixed, yields $\tau_j^* = (\mu_j^2 + \sigma_j^2)/\tau_j$, whence

$$KL(q_\theta \| \pi) = \frac{1}{2} \sum_j \log \left( 1 + \frac{1}{\alpha_j} \right). \tag{8}$$

Another contribution of Kingma et al. [2015] is the analysis of the effects of the local reparameterization trick on the variance of the gradient of (6). Proposed by Wang and Manning [2013] to speed up Gaussian dropout, this trick translates uncertainty from parameters of the model to intermediate outputs within the network: the stochasticity of $W \in \mathbb{R}^{n \times m}$ with $q(W) = \prod_{ij} \mathcal{N}(w_{ij} | \mu_{ij}, \alpha_{ij} \mu_{ij}^2)$ in a linear layer is pushed to its output $y = b + W^\top x$,

$$q(y) = \prod_i \mathcal{N}(y_i | b_i + \sum_j \mu_{ij} x_j, \sum_j \alpha_{ij} \mu_{ij}^2 x_j^2). \tag{9}$$

Kingma et al. [2015] show, that for the case of fully factorized Gaussian posterior approximations, using (9) decorrelates the estimators (6) within the mini-batch and makes the gradient estimator $\nabla_{\theta} \tilde{L}$ more statistically efficient. Specifically for the problem of selecting relevant parameters Molchanov et al. [2017] further improve the efficiency of the estimator of the $\nabla_{\mu} \tilde{L}$ by introducing additive noise parameterization. They revert the $(\mu, \alpha)$ parameterization back to the equivalent $(\mu, \sigma^2)$, since it stabilizes gradients of SGVB (6), by rendering the gradient with respect to $\mu$ independent from the local output noise, injected by (9). The relevance $\alpha$ is calculated using the ratio $\sigma_j^2/\mu_j^2$ when needed in the divergence term.

### 3 C-valued networks

Complex-valued networks are, in general, non-holomorphic, i.e. not C-differentiable, which is exacerbated by the loss being real-valued. Indeed a C-differentiable real-valued function is necessarily trivial by Cauchy-Riemann conditions. This issue is dealt with by employing Wirtinger, or CR calculus, which generalizes holomorphic calculus to non-holomorphic functions of C-argument [Adali et al., 2011, Boeddeker et al., 2019]. It satisfies the product and chain rules, respects complex conjugation and linearity, but most importantly the CR-differential of a function on $\mathbb{C}$ coincides with its classical differential as a function on $\mathbb{R}^2$. Essentially, it allows straightforward retrofitting of C-valued networks into existing $\mathbb{R}$ deep learning auto-differentiation frameworks. It was considered as a basis for C version of back-propagation by Benvenuto and Piazza [1992] (see appendix D).

More recently general development of C-valued networks has been continued by Trabelsi et al. [2018], who outline the essential building blocks for deep C-valued networks and describe suitable representation and operations,
including convolutional and dense layers, \( \mathbb{C} \)-valued activations, complex batch-normalization and weight initialization. Their approach coupled with CR calculus for backprop transforms a \( \mathbb{C} \)-valued network into an intricately connected \( \mathbb{R} \)-valued computational graph, constructed to respects \( \mathbb{C} \)-arithmetic. In particular, natural identification of \( \mathbb{C} \) and \( \mathbb{R}^2 \) implies that dense linear layers \( L: \mathbb{C}^m \to \mathbb{C}^n \) act upon their inputs as follows:

\[
(\mathbb{R}^n)^2 \to (\mathbb{R}^m)^2: (u,v) \mapsto (Pu - Qv, Pv + Qu),
\]

where \( P,Q: \mathbb{R}^n \to \mathbb{R}^m \) are unique operators such that \( L = P + jQ \). A \( \mathbb{C} \)-convolutional layer with kernel \( W \) can be implemented as two \( \mathbb{R} \)-convolutions with kernels \( \Re W \) and \( \Im W \) using (10). Activations include trigonometric functions and representation-dependent non-linearities [Hirose, 2009]: polar (phase-amplitude) \( z \mapsto \sigma(|z|)e^{j \text{arg} z} \) or planar (real-imaginary) \( z \mapsto \sigma(\Re z) + j\sigma(\Im z) \) for some \( \mathbb{R} \)-valued non-linearity \( \sigma \), possibly with learnable parameters [Trabelsi et al., 2018; Wolter and Yao, 2018].

4 \( \mathbb{C} \)-Variational Dropout

We propose to use fully factorized complex Gaussian posterior approximation and \( \mathbb{C} \) variant of the local reparameterization trick to extend Bayesian Dropout techniques to \( \mathbb{C} \)-valued networks. In this section we describe the approximation and derive the divergence penalties in (6). Similarly to \( \mathbb{R} \)-valued Variational Dropout, the proposed \( \mathbb{C} \)-valued extension can be straightforwardly enhanced to provide structured sparsity.

4.1 \( \mathbb{C} \)-Gaussian Distribution

A vector \( z \in \mathbb{C}^m \) has complex Gaussian distribution, \( q(z) = \mathcal{N}_m^C(\mu, \Gamma, C) \) with mean \( \mu \in \mathbb{C}^m \) and \( \mathbb{C}^m \times \mathbb{C}^m \) covariance and relation matrices \( \Gamma \) and \( C \), respectively, if

\[
\begin{pmatrix} \Re z \\ \Im z \end{pmatrix} \sim \mathcal{N}_{2m} \left( \begin{pmatrix} \Re \mu \\ \Im \mu \end{pmatrix}, \frac{1}{2} \begin{pmatrix} \Re(\Gamma + C) & \Im(C - \Gamma) \\ \Im(\Gamma + C) & \Re(\Gamma - C) \end{pmatrix} \right),
\]

provided \( \Gamma \succeq 0 \), \( \Gamma^\top = \Gamma \), \( C^\top = C \), and \( \Gamma \succeq C^\top \Gamma^{-1} C \). Here, \( M^\top \) is the matrix transpose, \( M^\dagger \) is elementwise complex conjugation, and \( M^\dagger = (M^\top)^\dagger \) denotes Hermitian conjugate. Matrices \( \Gamma \) and \( C \) are given by \( \mathbb{E}(z - \mu)(z - \mu)^\top \) and \( \mathbb{E}(z - \mu)(z - \mu)^\top \), respectively, and \( z \) is a proper \( \mathbb{C} \)-Gaussian vector if \( z \) and \( \bar{z} \) are uncorrelated, i.e. \( C = 0 \). The entropy of \( z \) terms of \( \Gamma \) and \( C \) is

\[
\text{H}(q) = -\mathbb{E}_{z \sim q} \log q(z) = \frac{1}{2} \log \det (\pi e \Gamma) \det (\pi e (\Gamma - C \Gamma^{-1} C)) \quad \text{for } C = 0.
\]

The \( \mathbb{C} \)-variant of the local reparameterization trick utilizes the fact that Gaussian vectors are closed under affine transformations:

\[
b + Az \sim \mathcal{N}_{n}^C (b + A \mu, A \Gamma A^\top, ACA^\top),
\]
for $A \in \mathbb{C}^{n \times m}$ and $b \in \mathbb{C}^n$. Therefore, if $W \in \mathbb{C}^{n \times m}$ is drawn from a fully factorized approximation

$$W \sim q(W) = \prod_{ij} \mathcal{N}(w_{ij} \mid \mu_{ij}, \Sigma_{ij}, C_{ij}),$$

with $\mu, C \in \mathbb{C}^{n \times m}$, $\Sigma \in \mathbb{R}^{n \times m}$, and $|C_{ij}|^2 \leq \Sigma_{ij}$, then for $x \in \mathbb{C}^m$ and $b \in \mathbb{C}^n$ the vector $y = b + Wx$ has independent components with

$$y_i \sim \mathcal{N}(b_i + \sum_{j=1}^m \mu_{ij}x_j, \sum_{j=1}^m \Sigma_{ij}|x_j|^2, \sum_{j=1}^m |C_{ij}|x_j^2).$$

The trick requires three operations, i.e. $b + \mu x$, $\Sigma |x|^2$ and $C|x|^2$ with the complex modulus and square applied elementwise, and can be applied to any layer, the output of which depends linearly on its parameters, such as convolutional, affine, and bilinear transformations ($(x, z) \mapsto x^T W^{(i)} z + b_i$). Similar to the real case, Variational Dropout for $\mathbb{C}$ convolutions draws independent realizations of $W$ for each spatial patch in the input [Molchanov et al. 2017]. This provides faster computations and better statistical efficiency of the SGVB gradient estimator.

4.2 The priors

4.2.1 VD prior

For a fully factorized approximation $q(\omega)$ and factorized prior belief $\pi(\omega) = \prod_{ij} \pi(\omega_{ij})$, the divergence term in ELBO (6) is

$$KL(q\|\pi) = -\sum_{ij} \mathbb{H}(q(\omega_{ij})) + \mathbb{E}_{q(\omega_{ij})} \log \pi(\omega_{ij}).$$

We consider two fully factorized priors: an improper prior for Variational Dropout and $\mathbb{C}$-Gaussian ARD prior. We omit subscripts $ij$ for brevity in the next sections.

4.2.1 VD prior

From (12) the KL-divergence for an improper prior $\pi(\omega) \propto |\omega|^{-\beta}$ with $\beta \geq 1$ is

$$KL(q\|\pi) \propto -\log \sigma^2 - \frac{\beta}{2} \left( \mathbb{E}_{\omega \sim q(\omega)} \log |\omega|^2 \right).$$

Property (13) implies that $\mathcal{N}(\mu, \sigma^2, 0) \sim \mu \cdot \mathcal{N}(1, \alpha, 0)$ for $\mu \neq 0$ and $\sigma^2 = \alpha |\mu|^2$, whence the expectation in brackets is given by

$$\log \alpha |\mu|^2 + \mathbb{E}_{\xi \sim \mathcal{N}(\xi(0,1),0)} \log \frac{1}{\sqrt{\alpha}} + |\xi|^2.$$
general integer \( m \geq 1 \) have been derived by Lapidoth and Moser [2003, p. 2466]. In particular, for \( m = 1 \) and \( \theta \in \mathbb{C} \) we have

\[
E_z \sim \mathcal{N}_\mathbb{C}(0,1,0) \log|\theta + z|^2 = \log|\theta|^2 - Ei(-|\theta|^2),
\]

where \( Ei(x) = \int_{-\infty}^{x} t^{-1} e^t dt \) for \( x < 0 \) is the Exponential Integral, which satisfies \( Ei(x) \leq \log(-x) \), \( Ei(x) \approx \log(-x) - \gamma \) as \( x \to 0 \) (\( \gamma \) is Euler’s constant) and \( Ei(x) \geq -e^x \) for \( x \leq -1 \). Although \( Ei \) is an intractable integral, requiring numerical approximations to compute, its derivative is exact: \( \frac{d}{dx} Ei(x) = \frac{e^x}{x} \) at \( x < 0 \).

From (18) and (19), the terms of the divergence that depend on the parameters are given by

\[
KL(q\|\pi) \propto \beta^{-2} \log|\mu|^2 + \log \frac{1}{\alpha} - \frac{\beta}{2} Ei(-\frac{1}{\alpha}).
\]

We set \( \beta = 2 \) to make the divergence term depend only on \( \alpha \) and add \( \gamma \) so that the right-hand side is nonnegative [Lapidoth and Moser, 2003, eq.(84)]. Since \( Ei(x) \) has simple analytic derivative and (6) depends additively on (17), it is possible to back-propagate through the divergence without forward evaluation, which speeds up gradient updates.

### 4.2.2 ARD prior

We consider the fully factorized proper \( \mathbb{C} \)-Gaussian ARD prior with \( \pi_\tau(\omega) \propto \mathcal{N}_\mathbb{C}(\omega|0,\tau^{-1},0) \). The per element divergence term in (16) is

\[
KL(q\|\pi_\tau) = -1 - \log(\tau \sigma^2) + \tau \left(\sigma^2 + |\mu|^2\right).
\]

In Empirical Bayes the prior adapts to the observed data, i.e. (6) is optimized w.r.t. \( \tau \) of each weight’s prior. Since divergence sits in (6) with negative sign, optimal \( \tau^* = (\sigma^2 + |\mu|^2)^{-1} \) is found by minimizing (20), which gives

\[
KL(q\|\pi_{\tau^*}) = \log \left(1 + \frac{|\mu|^2}{\sigma^2}\right) = \log \left(1 + \frac{1}{\alpha}\right).
\]

### 4.2.3 \( \mathbb{C} \)-Bayesian Dropout via \( R \)-scaling

Consider the following parameterization of \( W \): \( W_{ij} = \mu_{ij} \xi_{ij} \), \( \xi_{ij} \in \mathbb{R} \) with \( \xi_{ij} \sim \mathcal{N}(1, \alpha_{ij}) \), yet \( \mu \in \mathbb{C}^{n \times m} \). This case corresponds to inference regarding relevance multipliers \( \xi \) rather than the parameters themselves. Under this parameterization the weight distribution is fully factorized degenerate \( \mathbb{C} \)-Gaussian approximation (14) with \( \Sigma_{ij} = \alpha_{ij} |\mu_{ij}|^2 \) and \( C_{ij} = \alpha_{ij} \mu_{ij}^2 \). Unlike proper \( \mathbb{C} \)-Gaussian approximation, the complex relation parameter in (15) is nonzero in this case and equals \( \sum_j \alpha_{ij} (x_{ij} \mu_{ij})^2 \). The KL-divergence term coincides with (7), however the major drawback of this approximation is that the gradient of the loss with respect to \( \mu \) cannot be disentangled from the local output noise by additive reparameterization.

## 5 Experiments

Source code for the experiments conducted in this study and figures is available at [https://github.com/ivannz/complex_paper/tree/v2020.3](https://github.com/ivannz/complex_paper/tree/v2020.3). Source code
for a PyTorch extension which implements \( \mathbb{C} \)-valued Variational Dropout and ARD layers, and provides basic layers for \( \mathbb{C} \)-valued networks, including maskable sparsified layers, is available at \url{https://github.com/ivannz/cplxmodule}.

We study the performance-compression trade-off of the proposed Bayesian sparsification method for \( \mathbb{C} \)-valued networks presented above on image classification and music annotation tasks. Following Wolter and Yao [2018] and Trabelsi et al. [2018], the class logit scores are taken as real part of the complex-valued output of a network. Since image data is not naturally \( \mathbb{C} \)-valued, we preprocess it by either embedding \( \mathbb{R} \)-data into \( \mathbb{C} \) assuming \( \Im z = 0 \) (raw), or applying the two-dimensional Fourier Transform (fft), centering the lower frequencies. We do not train an auxiliary model to synthesize the imaginary component from the real data, proposed by Trabelsi et al. [2018].

For each dataset we conduct a set of experiments in which we try out all combinations of model kinds (\( \mathbb{R} \) or \( \mathbb{C} \)), sparsification methods (VD or ARD) and input features. Where reasonable, we also study the effects of halving (\( \frac{1}{2} \mathbb{C} \)) and doubling (\( 2 \mathbb{R} \)) of the size of intermediate layers in each network [Mönning and Manandhar, 2018]. Each experiment is replicated five times to take into account random initialization, random order of the mini-batch during SGD, stochastic outputs of the intermediate layers, and non-determinism of single-precision computations on GPUs.

Each network within every experiment passes through three successive stages of training: pre-training (“dense”), sparsification (“sparsify”), and fine-tuning (“fine-tune”) (sec. 5.1). The parameters and optimizer state are passed between stages, except for last one, which inherits just the parameters. Every experiment uses ADAM optimizer, resets the learning rate to \( 10^{-3} \) before each stage, and applies global \( \ell_2 \)-norm gradient clipping at 0.5.

The compression rate is calculated based on the number of IEEE754 floating point values needed to store the model and equals \( n_{par} - n_{zer} \), where \( n_{zer} \) is the number of explicit zeros, \( n_{par} \) is the total number of floats. In particular, each parameter is counted once in a \( \mathbb{R} \)-valued network, but twice in a \( \mathbb{C} \)-network. Each model has a compression limit, determined by biases, shift and scaling in \( \mathbb{R} \)- and \( \mathbb{C} \)-valued batch normalization layers.

\section{5.1 Stagewise training}

At the “dense” stage every network is fit “as-is”, retaining its original non-Bayesian architecture, and using only the likelihood term from (6) without variational dropout.

During the “sparsify” stage we make every layer Bayesian and apply \( \mathbb{C} \)-variational dropout sec. 4.2.1 (or sec. 4.2.2) or its \( \mathbb{R} \) counterpart. We inject a coefficient \( C \in (0, 1] \) at the KL divergence term in (6) to study the attainable sparsity levels:

\[
- \frac{C}{N} KL(q_0||\pi_\lambda) + \frac{1}{M} \sum_{k=1}^{M} \log p_\phi(z_{ik} | g(\varepsilon_k; \theta)).
\]  

By varying \( C \) we adjust the importance of the sparsifying prior in learning \( q_0(W) \), meaning that higher value yields more irrelevant parameters during in inference and makes layers sparser upon termination of this stage.
Between “sparsify” and “fine-tune” stages we compute masks of non-zero weights in each layer based on the relevance scores \( \alpha \) (sec. 2). Since the \( q_\theta \) factorizes into univariate distributions, a \( C \) or \( R \) parameter is considered non-zero if \( \log \alpha \leq \tau \) for \( \alpha = \frac{\sigma^2}{\mu^2} \), or zero otherwise.

The “fine-tune” stage reverts the network back to non-Bayesian layers, applies sparsity masks to its parameters and proceeds with training the kept parameters as during the “dense” stage. Unless specified otherwise the network is initialized to the mode of the learnt \( q_\theta(W) \).

The threshold \( \tau \) is picked so, that the remaining non-zero parameters are within \( \delta \) relative tolerance of their mode with high probability under the approximate posterior. Since \( q_\theta \) is fully factorized \( R \)- or a proper \( C \)-Gaussian, the random variable \( \frac{k|w-\mu|^2}{\alpha|\mu|^2} \) is \( \chi^2_k \) distributed with \( k = 1 \) (\( R \)) or \( 2 \) (\( C \)). For a generous tolerance \( \delta = 50\% \) all values of \( \log \alpha \) below \(-2.5\) yield at least 90% chance of a non-zero \( R/C \) parameter. We pick \( \tau = -\frac{1}{2} \) to retain parameters sufficiently concentrated around their mode, and encourage higher sparsity, at the same time being aware that \( q_\theta \) is merely an approximation. In comparison, \( \tau = 3 \) is commonly used as the threshold [Molchanov et al., 2017, Kingma et al., 2015].

5.2 MNIST-like datasets

We conduct a moderately sized experiment on MNIST-like datasets of \( 28 \times 28 \) greyscale images to study the performance-compression trade-off of the proposed C-variational dropout: MNIST [LeCun et al., 1998], KMNIST [Clanuwat et al., 2018], EMNIST [Cohen et al., 2017] and Fashion-MNIST [Xiao et al., 2017]. We deliberately use a fixed random subset of \( 10k \) images from the train split of each dataset to fit the networks and measure the performance with classification accuracy score on the usual test split.

We consider two simple architectures for \( R \) and \( C \) networks in this experiment, which have been chosen for the purpose of illustrating the compression and understanding the effects of experiment parameters. TwoLayerDenseModel is a wide dense ReLU network \( 784 \rightarrow 4096 \rightarrow n_{\text{out}} \), and SimpleConvModel is a ReLU net with two \( 2 \)d \( k_2s_2 \) convolutions with filters \( 20 \rightarrow 50 \), two \( k_2s_2 \) average pooling steps, and a dense classifier network \( 800 \rightarrow 500 \rightarrow n_{\text{out}} \). The number of intermediate feature is as specified in \( R \) and \( C \) models, is halved in complex models tagged \( \frac{1}{2}C \), and doubled in real models labelled \( 2R \).

Stages (sec. 5.1) last for 40, 75 and 40 epochs, respectively, in each experiment. The sparsification threshold \( \tau \) is fixed at \(-\frac{1}{2} \), the training batch size is set to 128 and the base learning rate \( 10^{-3} \) is reduced after the 10-th epoch to \( 10^{-4} \) at every stage. We let \( C \in \{ \frac{1}{2}2^{\frac{k}{2}} : k = 2, \cdots , 38 \} \) in \( C \), test VD and ARD methods, try raw or fft input features, and repeat each experiment 5 times to get a sample of compression-accuracy pairs.

Figures 1 and 2 depict the resulting compression-accuracy trade-off on MNIST for the simple models described above (other datasets are in appendix B). The points represent the trade-off of the compressed networks after fine-tuning, while their tails illustrate the impact of this stage on the accuracy. Transparent horizontal bands on each plot represent min-max performance spread of an uncompressed network on the test split.

The overarching pattern in all experiments on MNIST-like datasets is that
both ARD and VD yield compress similarly and ARD offers marginal advantage in terms of performance after fine-tuning. Fourier features perform marginally worse than raw image data for each model across all compression rates, and their trade-off for the dense network has a hat-like shape, contrasting the raw image features, for which the accuracy stays at one level for moderate rates, and then abruptly decays in high compression regime.

The impact of “fine-tune” is mixed, and depends on the dataset and features, although it frequently improves accuracy in high compression \( \times 50+ \) (high \( C \) in \([6]\)) for raw image input. On EMNIST Letters and MNIST datasets in the small
compression regime the dense network exhibits a trough in accuracy before fine-tuning stage.

These experiments demonstrate that \( \mathbb{C} \) Variational Dropout can offer adequate compression without much loss in performance for \( \mathbb{C} \)-valued networks. Although ARD performs on par with VD, unlike VD, it uses a proper prior and has a tractable analytic expression for the KL-divergence in both \( \mathbb{R} \) and \( \mathbb{C} \) cases. Wider \( \mathbb{R} \) and \( \mathbb{C} \) networks appear to have more favourable trade-off between compression and accuracy. However, it seems that due to more intrinsic degrees of freedom, \( \mathbb{R} \)-valued models perform and compress better than \( \mathbb{C} \), even
when the former uses the same number of features as the latter.

5.3 CIFAR10

We conduct an experiment on the CIFAR10 classification dataset [Krizhevsky 2009] comprising $32 \times 32$ color images of 10 classes to study the performance-compression trade-off of a moderately large deep network. The experiment is set up similarly to MNIST-like experiment, except that we train $R$ VGG16 network [Simonyan and Zisserman 2015] and its $C$ variant, in which we have replaced $R$-valued layers with their $C$-valued counterparts. $R$ and $C$ VGG16 have the same feature sizes in intermediate layers.

Unlike experiment in sec. 5.2, we consider the raw features only, use full training split, and allocate 20, 40, and 20 epochs to each stage. During training the mini-batch of 128 samples is randomly augmented by randomly flipping horizontally and cropping. Random cropping is done by zero-padding each side by four pixels and extracting a random $32 \times 32$ patch from the $40 \times 40$ intermediate image. We vary $C$ in (6') over \( \{2^2, 2^3 : k = 7, \ldots, 15\} \), compare VD and ARD methods, and measure accuracy on CIFAR10 test split.

Figure 3 shows that, although ARD method offers slightly less compression, it makes up for it by marginally better accuracy post fine-tune. VGG16 and $C$-VGG16 exhibit similar declining compression-accuracy trade-off, but due to higher capacity $C$-valued network yields higher accuracy [Mönning and Mandhar 2018]. Unlike MNIST, fine-tuning stage significantly improves the test performance, which, we speculate, might be due to dataset augmentation and deeper model.
5.4 MusicNet

We use MusicNet [Thickstun et al., 2017], an audio dataset of 330 annotated musical compositions, to investigate compressibility of the 1d VGG-like C-convent proposed by Trabelsi et al. [2018]. Similarly to the study, we downsample the audio from 44.1kHz to 11kHz, retain only 84 out of 128 labels, and hold-out the same validation and test compositions, on which we score the models with the pooled average precision. Each epoch lasts for 1000 random mini-batches of the remaining 321 pieces. The input features are C-valued Fourier transformed windows of 4096 samples from each waveform. The label vectors are taken from annotations at the middle of the window.

Experiments with the uncompressed model aimed at replicating the result of Trabelsi et al. [2018] have shown that early stopping almost always terminates much sooner, than 200 epochs used in their study: within the first 10 − 20 epochs with the validation performance peaking at 10 − 15 epochs and steadily declining afterwards. Thus we opt to use shorter stages: 12, 32 and 50 epochs (sec. 5.1), with early stopping activated only during the “fine-tune” stage. To keep the learning rate schedule consistent, we scale the learning rate of 10−3 after 5, 10, and 20-th epoch by 110, 120, and 1100, respectively.

We deviate from the set-up used by Trabelsi et al. [2018] by clipping ℓ2 norm of the gradients to 0.05 and shifting the low frequencies of the input to the centre to maintain spatial locality for convolutions.

We test complex ARD (sec. 4.2.2) and VD (sec. 4.2.1) methods and vary C ∈ {14, 12, 34, 1} · 10−k for k = 1, 2, 3, while keeping τ at −12. The performance is measured after “dense” stage, before the beginning and upon termination of fine-tuning. Additionally, we test the model of Trabelsi et al. [2018], in which we purposefully halve the receptive field of the first convolution from 6 to 3 (denoted by suffix k3). The motivation is to test if the handicap introduced by the forced compression of the most upstream layer can be alleviated by non-uniform compression, induced by variational dropout. We test only complex VD in this sub-experiment, since prior results have not demonstrated significant superiority of one method over another.

Figure 4 shows the resulting performance-compression frontier. ARD slightly outperforms VD in terms of performance, but both deliver similar compression rates. The trade-off demonstrates that with the proposed C-valued variational dropout methods (sec. 4) and post-sparsification fine-tuning it is possible to achieve average precision level comparable to the result of Trabelsi et al. [2018] with a model compressed 50×-200×. Furthermore, the k3 model, compressed 110× with C VD, outperforms its uncompressed baseline, but yields lower performance than the full model.

We provide the following interpretation of the apparent difference in performance impact of the “fine-tune” stage between less than 50× and higher than 100× compression regimes in figure 4, also observed in sec. 5.2. The value of C in (6’) is a good proxy for the ranking of the final compression rate since it directly affects the feedback from sparsifying prior. So, with low C models get less compressed during 50 epoch allotted for “sparsify” stage, which appears to be insufficient to move model away from the parameters after the dense stage. It is reasonable therefore to expect that for undercompressed models the fine-tuning stage acts essentially as a continuation of the pre-training stage. Since we have observed that longer training invariably deteriorates the validation performance,
Figure 4: Performance-compression curve for VD, ARD, and the $k_3$ version compressed with VD.

Figure 5: Early stopping epoch at fine-tuning stage.

the “fine-tune” stage leads to overfitting. Figure 5, which shows that the models, which have been sparsified with $C$ less than $\frac{1}{20}$ and have $50\times$ compression, need considerably less training epochs before early stopping terminates the process.

We take the models from experiments with $C \in \left\{ \frac{1}{20}, \frac{1}{200} \right\}$ and plot the compression and performance metrics before fine-tuning for various threshold levels on figure 6. From 6 and the relative positions of the compression curves it can be concluded that $C$ has much more substantial impact on the sparsity and performance, than the choice of $\tau$. Furthermore, for $\tau > 0$ the performance quickly saturates while compression creeps downwards.
Figure 6: Test split performance and compression rate for a $\mathbb{C}$-valued net on MusicNet.

6 Conclusion

In this study we have extended Variational Dropout methods of [Kingma et al. 2015], [Molchanov et al. 2017] and [Kharitonov et al. 2018] to $\mathbb{C}$-valued Bayesian networks. To validate the proposed technique we have conducted a large numerical study of complex-valued networks with simple architectures in image recognition tasks to assess the feasible performance-compression trade-off. At the cost of marginally lower performance, we have achieved $50 - 100 \times$ compression of the state-of-the-art deep $\mathbb{C}$-valued network of Trabelsi et al. [2018] on the MusicNet.

The results of this study have direct implications for real-time signal processing with embedded deep learning applications both in terms of lower storage requirements and higher throughput stemming from fewer floating-point multiplications, despite higher arithmetic complexity of $\mathbb{C}$-valued networks.

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A Gradient of the KL-divergence in $\mathbb{R}$ case

In this section we derive the exact gradient of (7). Even though, ultimately, the divergence involves a hypergeometric function, and its gradient requires evaluating the Gauss error function, the analysis provides independent verification of the approximation, proposed by Molchanov et al. [2017]. The logic of the derivation follows Lapidoth and Moser [2003].

For $(z_i)_{i=1}^m \sim \mathcal{N}(0, 1)$ iid and $(\mu_i)_{i=1}^m \in \mathbb{R}$, the random variable $W = \sum (\mu_i + z_i)^2$ is noncentral $\chi^2$ with shape $m$ and noncentrality parameter $\lambda = \sum \mu_i^2$, i.e. $W \sim \chi^2_m(\lambda)$. Its density is

$$f_W(x) = \frac{1}{2} e^{-\frac{x + \lambda}{2}} \left(\frac{x}{2}\right)^m \frac{1}{\Gamma(m + \frac{1}{2})} \sum_{n \geq 0} \frac{(\frac{x}{2})^n}{n!(n + m)}.$$  \hspace{1cm} (21)

By integrating the power series (21) with substitution $x \rightarrow 2u$, the expectation $E_W \sim \chi^2_m(\lambda) \log W$, needed in (7), equals

$$e^{-\frac{\lambda}{2}} \sum_{n \geq 0} \frac{(\frac{\lambda}{2})^n}{n!(n + m)} \int_0^\infty e^{-u} u^{n + \frac{m}{2} - 1} \log (2u) \, du = \log 2 + e^{-\frac{\lambda}{2}} \sum_{n \geq 0} \frac{(\frac{\lambda}{2})^n}{n!} \psi(n + \frac{m}{2}),$$ \hspace{1cm} (22)

where $\psi(x)$ is the digamma function, i.e.

$$\psi(x) = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} e^{-u} \log u \, du,$$  \hspace{1cm} (23)

which satisfies $\psi(z + 1) = \psi(z) + \frac{1}{z}$, $\psi(x) \leq \log x - \frac{1}{2x}$, and $\psi(\frac{1}{2}) = -\gamma - 2 \log 2$ ($\gamma$ is Euler’s constant). If we put

$$g_m(x) = e^{-x} \sum_{n \geq 0} \frac{x^n}{n!} \psi(n + \frac{m}{2}),$$  \hspace{1cm} (24)

then the desired expectation in (7) equals $\log 2 + g_1(\frac{1}{2\gamma})$. Formally differentiating the convergent power series within $g_m$ yields

$$\frac{d}{dx} g_m(x) = x^{-\frac{m}{2}} e^{-x} \sum_{n \geq 0} \frac{x^{n + \frac{m}{2}}}{n!(n + m)}.$$  \hspace{1cm} (25)

From $e^t = \sum_{n \geq 0} \frac{t^n}{n!}$ on $\mathbb{R}$ and $\frac{e^t}{\alpha} = \int_0^x t^{\alpha-1} \, dt$ for $\alpha \neq 0$ we get

$$\frac{d}{dx} g_m(x) = x^{-\frac{m}{2}} e^{-x} \sum_{n \geq 0} \frac{1}{n!} \int_0^x t^{n + \frac{m}{2} - 1} \, dt = x^{-\frac{m}{2}} e^{-x} \int_0^x t^{\frac{m}{2} - 1} e^t \, dt,$$  \hspace{1cm} (26)

Substitution $u^2 = t$ on $[0, \infty]$ with $2udu = dt$ yields

$$\frac{d}{dx} g_m(x) = 2x^{-\frac{m}{2}} e^{-x} \int_0^{\sqrt{x}} u^{m-1} e^{u^2} \, du.$$  \hspace{1cm} (27)
In particular, the derivative of \((\text{24})\) for \(m = 1\) is
\[
\frac{d}{dx} g_1(x) = 2 \frac{F(\sqrt{x})}{\sqrt{x}} ,
\] (28)
using Dawson’s integral \(F: x \mapsto e^{-x^2} \int_0^x e^{u^2} du\). Hence, the derivative of the expectation in \((\text{7})\) with respect to \(\alpha\) is
\[
\frac{d}{d\alpha} g_1 \left( \frac{1}{2\alpha} \right) = -\frac{1}{2\alpha^2} g_1' \left( \frac{1}{2\alpha} \right) = -\frac{1}{\alpha} \sqrt{\frac{\pi}{2\alpha}} F \left( \frac{1}{\sqrt{2\alpha}} \right) .
\]
Finally, since \(\alpha\) is nonnegative, it is typically parameterized via its logarithm. The gradient of \((\text{7})\) w.r.t \(\log \alpha\) is
\[
\frac{d}{d\log \alpha} \left( \frac{d}{d\alpha} \right) \approx -\frac{1}{\sqrt{2\alpha}} F \left( \frac{1}{\sqrt{2\alpha}} \right) .
\] (29)

We compute the Monte-Carlo estimator of \((\text{7})\) on a sample of \(10^7\) draws over an equally spaced grid of \(\log \alpha\) in \([-12, +12]\) of size 4096. The optimal coefficients of the regression, proposed by [Molchanov et al., 2017], are within 1% relative tolerance of the reported therein.
\[
(\text{7}) \approx \frac{1}{2} \log \left( 1 + e^{-\log \alpha} \right) + k_1 \sigma \left( -(k_2 + k_3 \log \alpha) \right),
\] (30)
The numerically estimated derivative of the penalty term with respect to \(\log \alpha\) using forward differences seems to be following \((\text{29})\) very closely (up to sampling error), see fig. 7. The derivative of \((\text{30})\) with respect to \(\log \alpha\) appears to be very close to \((\text{29})\). We compute a similar Monte-Carlo estimate for C variational dropout KL divergence term in \((\text{17})\) with \(\beta = 2\), fit the best approximation \((\text{30})\), and compare the derivatives.

### B MNIST-like experiments

The plots presented in this appendix support the conclusions made in the main text and provide an overview of the experiments conducted on MNIST-like datasets.
Each figure shows the compression-accuracy trade-off of a particular method and input features for *SimpleConvModel* and *TwoLayerDenseModel* models for all four of the studied datasets (described in the main text): EMNIST-Letters on the top-left, KMNIST – top-right, Fashion MNIST – bottom-left, and MNIST on the bottom-right. Figures [8] [9] [10] and [11] present R and C models with the same intermediate feature sizes.

We also report the trade-off comparison, when the argument by Mönning and Manandhar [2018] for higher intrinsic capacity of C-valued networks has been taken into account. We compare R networks against 1/2 C with half the number of parameters for raw input features on figures [14] and [15], and 2R with double the number of parameters against C for Fourier input features on figures [12] and [13].

C Complex-valued local reparameterization

By \( e_i \) we denote the \( i \)-th real unit vector of dimensionality conforming to the matrix-vector expression it is used in. Let \( M \) denote *row-major* flattening of \( M \) into a vector, i.e. in lexicographic order of its indices. \( \text{diag}(\cdot) \) embeds \( \mathbb{C}^n \) into \( n \times n \) diagonal matrices, and \( \otimes \) denotes the Kronecker product, for which we note the following identities:

\[
-\rightarrow PQR = (P \otimes R^\top) - \rightarrow Q,
\]

\[
(I \otimes x^\top) \text{diag} C (I \otimes x^\top)^\top = \sum_{i=1}^n (e_i e_i^\top) \sum_{j=1}^m C_{ij} x_j^2.
\]

Since \( 32 \) and \( 33 \) are diagonal, \( (y_i)_{i=1}^n \) are independent whence \( 13 \) implies

\[
y_i \sim \mathcal{N}^C(e_i^\top \mu x + b_i, \sum_{j=1}^m \Sigma_{ij} |x_j|^2, \sum_{j=1}^m C_{ij} x_j^2).
\]

Hence \( 15 \) follows.

D Backpropagation through C-networks

Wirtinger (CR) calculus relies on the natural identification of \( \mathbb{C} \) with \( \mathbb{R} \times \mathbb{R} \), and regards \( f: \mathbb{C} \to \mathbb{C} \) as an equivalent in algebraic sense function \( F: \mathbb{R}^2 \to \mathbb{C} \) defined \( f(z) = f(u + jv) = F(u, v) \). Within this framework the differential of \( f \) at \( z = u + jv \in \mathbb{C} \) is

\[
df(z) = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial v} dv = dF(u, v),
\]
with formally defined derivatives \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - j \frac{\partial}{\partial v} \right) \) and \( \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + j \frac{\partial}{\partial v} \right) \), and differentials \( dz = du + jdv \) and \( d\bar{z} = du - jdv \). This implies that the complex argument and its conjugate are treated as independent variables. Cauchy-Riemann conditions \(-j \frac{\partial F}{\partial v} = \frac{\partial F}{\partial u}\) can be expressed as \( \frac{\partial f}{\partial \bar{z}} = 0 \) in this notation.

Thus \( \text{CR} \) calculus subsumes the usual \( \mathbb{C} \)-calculus on holomorphic functions, when \( f(z) \), regarded as \( f(z, \bar{z}) \), is constant with respect to \( \bar{z} \). The usual rules of calculus, like chain and product rules, follow directly from the definition of the operators, e.g.

\[
\begin{align*}
\frac{\partial (f \circ g)}{\partial z} &= \frac{\partial f(g(z))}{\partial g(z)} \frac{\partial g(z)}{\partial z} + \frac{\partial f(g(z))}{\partial \bar{g}(z)} \frac{\partial \bar{g}(z)}{\partial \bar{z}}.
\end{align*}
\]

In machine learning tasks the target objective is typically empirical risk, which has to be real-valued to be minimized. Nevertheless, the expression of the \( \text{CR} \) gradient is compatible with what is expected, when \( f \) is treated like a \( \mathbb{R}^2 \) function. For a real-valued \( f : \mathbb{C} \to \mathbb{R} \) we have \( \text{Re} = f \), which implies \( \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \), whence

\[
\begin{align*}
\frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} = 2 \text{Re} \left( \frac{\partial f}{\partial x} dz \right) = 2 \text{Re} \left( \frac{\partial f}{\partial \bar{z}} dz \right).
\end{align*}
\]

Thus, the gradient of \( f \) at \( z \) is given by \( \nabla f(z) = \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} \).

Natural identification of \( \mathbb{C} \) with \( \mathbb{R} \times \mathbb{R} \), storing the real and imaginary parts in interleaved format, emulation of \( \mathbb{C} \)-algebra using \( \mathbb{R} \)-valued arithmetic, and Wirtinger calculus make it possible to reuse \( \mathbb{R} \) back-propagation and existing auto-differentiation frameworks [Trabelsi et al., 2018].

**E  \( \mathbb{C} \)-Linear operator representation**

Consider \( L : \mathbb{C}^m \to \mathbb{C}^n \) – linear in \( \mathbb{C} \). Then \( L(u + jv) = Lu + jLv \) for any \( u,v \in \mathbb{R}^m \), which implies that the effect of \( L \) as \( \mathbb{C} \)-linear operator is determined by its restriction onto \( \mathbb{R}^n \). Let \( F = L_{|_{\mathbb{R}^n}} : \mathbb{R}^m \to \mathbb{C}^n \) and \( F_r = \Re \circ F \) and \( F_i = \Im \circ F \) are \( \mathbb{R} \)-linear operators such that \( F = F_r + jF_i \) (pointwise). Indeed,

\[
F_r(a + \lambda b) = \Re L(a + \lambda b) = \Re La + \lambda \Re Lb = F_r a + \lambda F_r b.
\]

Therefore for any \( \mathbb{C} \)-linear operator \( L \) there are \( \mathbb{R} \)-linear operators \( U,V \) such that

\[
Lz = (U + jV)(\Re z + j\Im z) = U\Re z - V\Im z + j(V\Re z + U\Im z).
\]

Uniqueness of these operators follows, if this decomposition is applied to any \( z \) with \( \Im z = 0 \).
Figure 8: The trade-off of ARD method for $R$ and $C$ models using Fourier features.

Figure 9: The trade-off of VD method for $R$ and $C$ models using Fourier features.
Figure 10: The trade-off of ARD method for R and C models using raw features.

Figure 11: The trade-off of VD method for R and C models using raw features.
Figure 12: The trade-off of ARD method for $2\mathbb{R}$ and $C$ models using Fourier features.

Figure 13: The trade-off of VD method for $2\mathbb{R}$ and $C$ models using Fourier features.
Figure 14: The trade-off of ARD method for $R$ and $\frac{1}{2}C$ models using raw features.

Figure 15: The trade-off of VD method for $R$ and $\frac{1}{2}C$ models using raw features.