p-CAPACITY VS SURFACE-AREA

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ABSTRACT. This paper is devoted to exploring the relationship between the $[1, n) \ni p$-capacity and the surface-area in $\mathbb{R}^{n+2}$ which especially shows: if $\Omega \subseteq \mathbb{R}^n$ is a convex, compact, smooth set with its interior $\Omega^\circ \neq \emptyset$ and the mean curvature $H(\partial \Omega, \cdot) > 0$ of its boundary $\partial \Omega$ then

$$
\left(\frac{n(n-1)}{p(n-1)}\right)^{p-1} \leq \left(\frac{\text{cap}_p(\Omega)}{\text{area}((\partial \Omega)))} \right)^{p-1} \leq \left(\frac{\int_{\Omega} (H(\partial \Omega, \cdot))^p}{\sigma_{n-1}}\right)^{p-1} \forall \ p \in (1, n)
$$

whose limits $1 \leftarrow p$ & $p \to n$ imply

$$
1 = \frac{\text{cap}_p(\Omega)}{\text{area}(\partial \Omega)} \& \int_{\partial \Omega} (H(\partial \Omega, \cdot))^p \geq 1,
$$

thereby not only discovering that the new best known constant is roughly half as far from the one conjectured by Pólya-Szegő in [25 (2)] but also extending the Pólya-Szegő inequality in [25 (5)], with both the conjecture and the inequality being stated for the electrostatic capacity of a convex solid in $\mathbb{R}^3$.

1. Overview

Given a compact set $\Omega$ in the $2 \leq n$-dimensional Euclidean space $\mathbb{R}^n$ equipped with the standard volume and surface-area elements $dv$ and $d\sigma$. The variational $[1, n) \ni p$-capacity of $\Omega$ is defined by

$$
\text{cap}_p(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^p \ dv : f \in C_c^\infty(\mathbb{R}^n) \& f(x) \geq 1 \ \forall \ x \in \Omega \right\},
$$

where $C_c^\infty(\mathbb{R}^n)$ is the class of all infinitely differentiable functions with compact support in $\mathbb{R}^n$. Equivalently, the above infimum can be taken over either all $f \in C_c^\infty(\mathbb{R}^n)$ with $f = 1$ in a neighbourhood of $\Omega$, or all Lipschitz functions $u$ on $\mathbb{R}^n$ with $f = 1$ in a neighbourhood of $\Omega$ (cf. [11] pp. 27-28).

As a set function on compact subsets of $\mathbb{R}^n$, $\text{cap}_p(\cdot)$ enjoys the following basic properties (a) through (f) (cf. [11] pp. 28-32 and [20, Lemma 2.2.5]):

(a) Boundarization – if $\Omega$ is a compact subset of $\mathbb{R}^n$ with non-empty boundary $\partial \Omega$ then

$$
\text{cap}_p(\partial \Omega) = \text{cap}_p(\Omega).
$$

(b) Monotonicity – if $\Omega_1$ and $\Omega_2$ are compact subsets of $\mathbb{R}^n$ with $\Omega_1 \subseteq \Omega_2$ then

$$
\text{cap}_p(\Omega_1) \leq \text{cap}_p(\Omega_2).
$$

(c) Continuity – if $(\Omega_j)_{j=1}^\infty$ is a decreasing sequence of compact subsets of $\mathbb{R}^n$ then

$$
\text{cap}_p(\bigcap_{j=1}^\infty \Omega_j) = \lim_{j \to \infty} \text{cap}_p(\Omega_j).
$$

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(d) Ball capacity – if $B(x, r) = \{y \in \mathbb{R}^n : |y - x| \leq r\}$ and $\sigma_{n-1}$ is the surface area of the origin-centred unit ball $B(0, 1)$ then
\[
cap_{\rho}(B(x, r)) = r^{n-p}(\frac{p-1}{n-p}) \sigma_{n-1}.
\]
(e) Geometric endpoint – if $\Omega$ is a compact subset of $\mathbb{R}^n$ and $\text{area}(\cdot)$ stands for the surface-area of a set in $\mathbb{R}^n$ then
\[
\cap_2(\Omega) = \inf \{\text{area}(\partial \Lambda) : \Omega \subset \Lambda \cup \partial \Lambda \text{ with bound open } \Lambda \text{ and smooth } \partial \Lambda\}.
\]
(f) Physical interpretation – if $\Omega$ is a compact subset of $\mathbb{R}^{n+1}$, then $\cap_2(\Omega)$ is the maximal charge which can be placed on $\Omega$ when the electrical potential of the vector field created by this charge is controlled by 1, namely, $\cap_2(\Omega) = \sup \{\mu(\Omega) : \text{ measure } \mu \text{ with } \text{supp}(\mu) \subseteq \Omega \& \int_{\mathbb{R}^n} |x-y|^{2-n} \frac{d\mu(y)}{(n-2)\sigma_{n-1}} \leq 1 \forall x \in \mathbb{R}^n \setminus \Omega\}.$

Motivated by Pólya’s 1947 paper [25] as well as (a)&(e) above, this article stems from discovering the relationship between the $p$-capacity and the surface-area (via the mean curvature). The details for such a discovery are provided in §2 & §3 whose summary is shown in the sequel:

(h) Surface area to variational capacity (§2) – In Theorem 2.1 we use the convexity of level set of $(1,n) \ni p$-equilibrium potential and a minimizing technique to gain (2.3), a sharp convexity type inequality, linking the normalized variational capacity, the normalized surface area and the normalized volume and consequently deriving that $(\frac{p-1}{p(n-1)})^{p-1}$ times $(\frac{n-2}{n-1})$-th power of the normalized surface area is the asymptotically sharp lower bound of the normalized variational capacity, whence having half-solved [1] the Pólya-Szegö conjecture (for $\cap_2(\cdot)$ in $\mathbb{R}^3$) that of all convex bodies, with a given surface area, the circular disk has the minimum capacity.;

(i) Variational capacity to surface area (§3) – In Theorem 3.1 we employ a level set formulation of the inverse mean curvature flow (generated by a kind of $1$-equilibrium potential) to achieve (3.3), a log-convexity type inequality involving the normalized variational capacity, the normalized surface area and the normalized Willmore functional for the mean curvature and consequently revealing that the product of both $(\frac{p-1}{n-1})$-th power of the normalized Willmore functional for the mean curvature and $(\frac{n-2}{n-1})$-th power of the normalized surface area is the optimal upper bound of the normalized variational capacity, thereby extending the Pólya-Szegö principle (for $\cap_2(\cdot)$ in $\mathbb{R}^3$) that unless the convex solid is a ball the capacity is less than the mean-curvature-radius.

Naturally, a combination of (2.5) in Theorem 2.1 and (3.4) in Theorem 3.1 derives that if $\Omega \subset \mathbb{R}^n$ is a convex, compact, smooth set with its interior $\Omega^c \neq \emptyset$ and the mean curvature $H(\partial \Omega, \cdot) > 0$ of its boundary $\partial \Omega$ then
\[
\left(\frac{n(p-1)}{p(n-1)}\right)^{p-1} \leq \left(\frac{\cap_2(\Omega)}{\text{area}(\partial \Omega)}\right)^{\frac{p-1}{p(n-1)}} \leq \left(\frac{n-1}{\sigma_{n-1}} \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{n-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}}\right)^{\frac{1}{n-1}} \forall \ p \in (1, n)
\]
whose limiting cases $1 \leftarrow p & p \rightarrow n$ surprisingly yield the extremal case of (e) (cf. [19]) and the Willmore inequality (cf. [2, 29, 1]) as seen below:

1Namely, the new best known constant is roughly half as far from the conjectured one.
(k) \[ 1 = \frac{\text{cap}_1(\Omega)}{\text{area}(\partial \Omega)} \& \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{p-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \geq 1. \]

2. Surface-area to \( p \)-capacity

In [27, p.12] (cf. [25]) Pólya-Szegő conjectured that for any convex compact subset \( \Omega \) of \( \mathbb{R}^3 \) one has

\[ \text{cap}_2(\Omega) \geq \left( 4 \sqrt{\frac{2}{\pi}} \right) \sqrt{\text{area}(\partial \Omega)} \]

with equality if and only if \( \Omega \) is a two-dimensional disk in \( \mathbb{R}^3 \). Here it is perhaps worth pointing out that if \( \Omega \subset \mathbb{R}^2 \) then \( \text{area}(\partial \Omega) \) is replaced by two times of the two-dimensional Lebesgue measure of \( \Omega \).

The first remarkable result approaching the conjecture was obtained in Pólya-Szegő’s 1951 monograph: [27, p.165,(4)] (as a sequel to the work presented in their 1945 paper [26]) via suitable symmetrization and projection for any given convex compact set \( \Omega \subset \mathbb{R}^3 \):

\[ \text{cap}_2(\Omega) \geq \left( \frac{4}{\sqrt{\pi}} \right) \sqrt{\text{area}(\partial \Omega)}. \]

Since then, no improvement has been made on (2.2) and of course (2.1) has not yet been verified - see [16, 4, 5, 14] for an up-to-date report on this research. In the sequel, with the help of the isocapacity inequality for the volume \( \text{vol}(\cdot) \) of a level set of the equilibrium potential of an arbitrary convex compact set \( \Omega \subset \mathbb{R}^3 \) we show

\[ \text{cap}_2(\Omega) \geq \left( \frac{3 \sqrt{\pi}}{2} \right) \sqrt{\text{area}(\partial \Omega)}, \]

whence finding that (2.3) holds the nearly middle place between (2.1) and (2.2) in the sense of

\[
\begin{aligned}
4 \sqrt{\frac{2}{\pi}} &> \frac{3 \sqrt{\pi}}{2} > \frac{4}{\sqrt{\pi}}; \\
4 \sqrt{\frac{2}{\pi}} - \frac{3 \sqrt{\pi}}{2} &> 0.532857...; \\
\frac{3 \sqrt{\pi}}{2} - \frac{4}{\sqrt{\pi}} &> 0.401922....
\end{aligned}
\]

As a matter of fact, we discover the brand-new sharp convexity type inequality (2.4) (for the surface-area, the variational capacity and the volume) whose by-product (2.5) is much more general than (2.3).

**Theorem 2.1.** Let \( \Omega \) be a convex compact subset of \( \mathbb{R}^n \) with area(\( \partial \Omega \)) > 0. Then

\[ \frac{n(p-1)}{p(n-1)} \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{p}{n-1}} \leq 1 \quad \forall \ p \in (1, n) \]

holds with equality if and only if \( \Omega \) is a ball. Consequently

\[ \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{n-p}{n-1}} \leq \left( \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{\frac{1}{n-1}} \sigma_{n-1}} \right) \left( \frac{p(n-1)}{n(n-1)} \right)^{p-1} \quad \forall \ p \in (1, n), \]
which is asymptotically optimal in the sense that if \( p \to 1 \) or \( p \to n \) in (2.5) then
\[
\text{area}(\partial\Omega) = \text{cap}_1(\Omega) \quad \text{or} \quad 1 = 1.
\]

**Proof.** First of all, since \( \text{area}(\partial\Omega) > 0 \) and \( \Omega \) is convex, it follows from [19] that \( \text{cap}_1(\Omega) = \text{area}(\partial\Omega) > 0 \). In accordance with [22] Theorem 3.2, if \( 1 \leq p_1 < p_2 < n \) then there is a constant \( c(p_1, p_2, n) > 0 \) depending only on \( (p_1, p_2, n) \) such that
\[
\frac{(\text{cap}_p(\Omega))^\frac{1}{p}}{c(p_1, p_2, n)(\text{cap}_{p_2}(\Omega))^\frac{1}{p_2}}.
\]
Upon choosing \( p_1 = 1 < p_2 = p < n \), one gets \( \text{cap}_p(\Omega) > 0 \).

Next, we verify (2.4) through considering two situations.

**Situation 1:** suppose that the interior \( \Omega^0 \) of \( \Omega \) is not empty and the boundary \( \partial\Omega \) of \( \Omega \) is of \( C^1 \)-smoothness. In accordance with [3, 17], there is a unique \((1, n) \) \( p \)-equilibrium potential \( u \) of \( \Omega \) (not only smooth in \( \Omega^c = \mathbb{R}^n \setminus \Omega \) but also continuous in \( \mathbb{R}^n \setminus \Omega^0 \) ) such that:

- \( \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \) in \( \Omega^c \);
- \( u|_{\partial\Omega} = 1; \)
- \( \lim_{|x| \to \infty} u(x) = 0; \)
- \( 0 < u < 1 \) in \( \Omega^c \);
- \( |\nabla u| \neq 0 \) in \( \Omega^c \);
- \( \text{cap}_p(\Omega) = \int_{\mathbb{R}^n,\Omega} |\nabla u|^p \, dv = \int_{\{x \in \mathbb{R}^n; u(x)=t\} \setminus \{x \in \mathbb{R}^n; u(x)=0\}} |\nabla u|^{p-1} \, d\sigma \quad \forall \ t \in (0, 1); \)
- if \( u \) is set to be 1 on \( \Omega \) then \( \{x \in \mathbb{R}^n; u(x) \geq t\} \) is convex and \( \{x \in \mathbb{R}^n; u(x) = t\} \) is smooth for any \( t \in (0, 1) \).

Consequently, we can utilize the well-known monotonicity for the area function of convex domains, the Hölder inequality and the co-area formula to get
\[
\text{area}(\partial\Omega)
\leq \text{area}(\{x \in \mathbb{R}^n; u(x) = t\})
= \int_{\{x \in \mathbb{R}^n; u(x)=t\} \setminus \{x \in \mathbb{R}^n; u(x)=0\}} d\sigma
\leq \left( \int_{\{x \in \mathbb{R}^n; u(x)=t\} \setminus \{x \in \mathbb{R}^n; u(x)=0\}} |\nabla u|^{p-1} \, d\sigma \right)^\frac{1}{p-1}
= (\text{cap}_p(\Omega))^\frac{1}{p} \left( -\frac{d}{dt} \text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\}) \right)^\frac{p-1}{p},
\]
and accordingly,
\[
\text{(2.7)}
\left( \frac{\text{area}(\partial\Omega)}{(\text{cap}_p(\Omega))^\frac{1}{p}} \right)^\frac{p}{p-1} \leq -\frac{d}{dt} \text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\}),
\]
where
\[
\text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\})
\]
is the Lebesgue measure of the upper level set \( \{x \in \mathbb{R}^n; u(x) \geq t\} \). Recalling the Poincaré-Mazya isocapacitary inequality (cf. [27] for \( p = 2 \) and [20] for \( p \in (1, n) \))
\[
\text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\}) \leq \left( \text{cap}_p(\{x \in \mathbb{R}^n; u(x) \geq t\}) \right)^\frac{1}{p} \left( \frac{n-1}{n-p} \right)^{1-p} \text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\})^{\frac{n-p}{n-1}}
\]
\[
\text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\}) \leq \left( \text{cap}_p(\{x \in \mathbb{R}^n; u(x) \geq t\}) \right)^\frac{1}{p} \left( \frac{n-1}{n-p} \right)^{1-p} \text{vol}(\{x \in \mathbb{R}^n; u(x) \geq t\})^{\frac{n-p}{n-1}}.
and using (a) - the boundarization of $\text{cap}_p(\cdot)$ to achieve the following formula (cf. [27, 24] for $p = 2$)

$$
\text{cap}_p(x \in \mathbb{R}^n : u(x) \geq t) = \text{cap}_p(x \in \mathbb{R}^n : u(x) = t) = \int_{(x \in \mathbb{R}^n : u(x) = t)} (t^{-1} |\nabla u|)^{p-1} \, d\sigma = t^{1-p} \text{cap}_p(\Omega),
$$

we obtain via integrating both sides of (2.7) over the interval $(t, 1)$

$$(1 - t) \left( \frac{\text{area}(\partial \Omega)}{(\text{cap}_p(\Omega))^{\frac{1}{p}}} \right)^{\frac{p}{n}} \leq \text{vol}(x \in \mathbb{R}^n : u(x) \geq t) - \text{vol}(\Omega) \leq \left( \frac{\sigma_{n-1}}{n} \right) \left( \frac{\text{cap}_p((x \in \mathbb{R}^n : u(x) \geq t))}{(n-1)^{1-p} \sigma_{n-1}} \right)^{\frac{n}{p}} - \text{vol}(\Omega) \leq \left( \frac{\sigma_{n-1}}{n} \right) \left( \frac{1}{(n-1)^{1-p} \sigma_{n-1}} \right)^{\frac{n}{p}} \text{vol}(\Omega).$$

Note that the above estimate is valid for any $t \in [0, 1]$. But if

$$
t \in \left[ 1, \left( \frac{\text{vol}(\Omega)^{\frac{1}{n}}}{(\text{cap}_p(\Omega))^{\frac{1}{p}}} \right)^{\frac{n}{p}} \right],
$$

then

$$(1 - t) \left( \frac{\text{area}(\partial \Omega)}{(\text{cap}_p(\Omega))^{\frac{1}{p}}} \right)^{\frac{p}{n}} \leq 0 \leq \left( \frac{\sigma_{n-1}}{n} \right) \left( \frac{t^{1-p} \text{cap}_p(\Omega)}{(n-1)^{1-p} \sigma_{n-1}} \right)^{\frac{n}{p}} - \text{vol}(\Omega)$$

and hence one has:

$$(1 - t) \left( \frac{\text{area}(\partial \Omega)}{(\text{cap}_p(\Omega))^{\frac{1}{p}}} \right)^{\frac{p}{n}} \leq \left( \frac{\sigma_{n-1}}{n} \right) \left( \frac{t^{1-p} \text{cap}_p(\Omega)}{(n-1)^{1-p} \sigma_{n-1}} \right)^{\frac{n}{p}} - \text{vol}(\Omega) \quad \forall \quad t \in \left[ 0, \left( \frac{\text{vol}(\Omega)^{\frac{1}{n}}}{(\text{cap}_p(\Omega))^{\frac{1}{p}}} \right)^{\frac{n}{p}} \right].$$

Suppose $t_0$ is the critical point of the following function

$$
t \mapsto \phi(t) = (1 - t) \left( \frac{\text{area}(\partial \Omega)}{(\text{cap}_p(\Omega))^{\frac{1}{p}}} \right)^{\frac{p}{n}} - \left( \frac{\sigma_{n-1}}{n} \right) \left( \frac{t^{1-p} \text{cap}_p(\Omega)}{(n-1)^{1-p} \sigma_{n-1}} \right)^{\frac{n}{p}} + \text{vol}(\Omega).$$
Then solving \( \phi'(t_0) = 0 \) and using the classical isoperimetric inequality one gets

\[
t_0 = \left( \frac{\left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}}}{\left( \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{\frac{1}{p}}} \right)^{\frac{1}{p}}} \right)^{\frac{1}{n-p}},
\]

whence deriving

\[
(1 - t_0) \left( \frac{\text{area}(\partial \Omega)}{\left( \text{cap}_p(\Omega) \right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}} \leq \left( \frac{\sigma_{n-1}}{n} \right)^{\frac{n}{1}} \left( \frac{t_0}{n-1} \text{cap}_p(\Omega) \right)^{\frac{1}{n-p}} - \text{vol}(\Omega),
\]

which implies

\[
\frac{\text{vol}(\Omega)}{n^{-1} \sigma_{n-1}} \leq \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{n}{1}} - \left( \frac{1 - t_0}{n-1} \frac{\text{area}(\partial \Omega)}{\left( \text{cap}_p(\Omega) \right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}},
\]

namely,

\[
1 - t_0 \leq \left( \frac{n-p}{n(p-1)} \right) t_0 \left( 1 - \frac{\text{vol}(\Omega)}{\left( \text{area}(\partial \Omega) \right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}},
\]

and then (2.4) via a further computation with \( t_0 \).

**Situation 2:** suppose that \( \Omega \) is a general convex compact subset of \( \mathbb{R}^n \). For this setting there is a sequence of convex compact sets \( (\Omega_j)_{j=1}^{\infty} \) such that \( \Omega_j \neq \emptyset \), \( \partial \Omega_j \) is of \( C^1 \)-smoothness, and \( \Omega_j \) decreases to \( \Omega \). Since (2.4) and (2.5) are valid for \( \Omega_j \), an application of the continuity for area(\( \cdot \)), vol(\( \cdot \)), and cap\(_p(\cdot)\) acting on convex compact sets ensures that (2.4) is true for such \( \Omega \).

After that, we check the equality case of (2.4). If \( \Omega \) is a ball, then an application of both (d) and the identity

\[
\frac{n(p-1)}{p(p-1)} + \frac{n-p}{p(n-1)} = 1
\]

makes equality of (2.4) happen. Conversely, if equality of (2.4) occurs for all \( p \in (1, n) \), then

\[
\frac{n(p-1)}{p(n-1)} \left( \frac{\text{area}(\partial \Omega)}{\left( \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{\frac{1}{p}}} \right)^{\frac{1}{p}}} \right)^{\frac{n}{1}} + \frac{n-p}{p(n-1)} \left( \frac{\text{vol}(\Omega)}{n^{-1} \sigma_{n-1}} \right)^{\frac{1}{p}} = 1 \quad \forall \ p \in (1, n).
\]

Upon letting \( p \to 1 \) in this last equality and using the known fact that (cf. [22, 19])

\[
\lim_{p \to 1} \inf \text{cap}_p(\Omega) = \text{cap}_1(\Omega) = \text{area}(\partial \Omega)
\]

we obtain

\[
\left( \frac{\text{vol}(\Omega)}{n^{-1} \sigma_{n-1}} \right)^{\frac{1}{p}} = \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{1}{p}},
\]

namely, equality of the isoperimetric inequality holds for \( \Omega \), thereby finding that \( \Omega \) is a ball.

Finally, let us deal with (2.5) and its limiting cases. Note that the second term of the left-hand-side of (2.4) is non-negative. So, (2.5) follows immediately from (2.4). Moreover, the first identity of (2.6), as the limit case \( p \to 1 \) of (2.5), is well-known; see also [19, 8] and
Lemma 2.2.5]. To see the second identity of (2.6), let \( B(0, R_0) \) be an origin-symmetric ball containing \( \Omega \). Using (2.5) and (b)&(d) we find
\[
1 = \lim_{p \to n} \left( \frac{\text{cap}_p(\Omega)}{\sigma_{n-1}} \right)^{\frac{n-p}{p-1}} \leq \lim_{p \to n} \left( \frac{\text{cap}_p(\Omega)}{\sigma_{n-1}} \right)^{\frac{p(n-1)}{n(p-1)}} \leq \lim_{p \to n} R_0^{n-p} = 1,
\]
as desired. \( \square \)

Remark 2.2. Below are two comments on (2.5) of independent interest:

(i) In accordance with [15, Proposition 1.1], if \( \Omega \) is a convex compact subset of \( \mathbb{R}^{n+1} \) with \( \Omega^c \neq \emptyset \) and smooth \( \partial \Omega \), and \( u \) is the \( p \)-equilibrium potential of \( \Omega \), then an application of the fact that
\[
x \mapsto v(x) = \int_{\partial \Omega} |x-y|^{n-2} \frac{d\sigma(y)}{(n-2)\sigma_{n-1}}
\]
is harmonic in \( \mathbb{R}^n \setminus \partial \Omega \) (cf. [2]) gives
\[
v(x) = v_\infty((n-2)\sigma_{n-1})^{-1}|x|^{2-n} + O(|x|^{1-n}) \quad \text{as} \quad |x| \to \infty,
\]
where
\[
v_\infty = \int_{\partial \Omega} v|\nabla u| d\sigma.
\]
Note that (cf. [2])
\[
v(x) = ((n-2)\sigma_{n-1})^{-1} \text{area}(\partial \Omega)|x|^{2-n} + O(|x|^{1-n}) \quad \text{as} \quad |x| \to \infty.
\]
So, one has
\[
((n-2)\sigma_{n-1})^{-1} \text{area}(\partial \Omega) = v_\infty
\]
and
\[
(2.8) \quad v_\infty = \int_{\partial \Omega} v|\nabla u| d\sigma \leq (\max_{x \in \partial \Omega} v(x)) \int_{\partial \Omega} |\nabla u| d\sigma = (\max_{x \in \partial \Omega} v(x)) \text{cap}_2(\Omega).
\]

Using the well-known layer-cake formula under \( d\sigma \), one finds
\[
v(x)((n-2)\sigma_{n-1})^{-1} \text{area}(\partial \Omega)
\]
\[
= \int_{0}^{\infty} \sigma\left(\{y \in \partial \Omega : |x-y|^{2-n} \geq t\}\right) dt
\]
\[
= \left( \int_{0}^{r} + \int_{r}^{\infty} \right) \sigma\left(\{y \in \partial \Omega : |x-y|^{2-n} \geq t\}\right) dt
\]
\[
\leq \text{area}(\partial \Omega)r + (n-2)\sigma_{n-1}r^{\frac{2-n}{n}}.
\]
Minimizing the last quantity, one gets that
\[
r = \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{n}{n-1}}
\]
derives
\[
(2.9) \quad \int_{\partial \Omega} |x-y|^{2-n} \frac{d\sigma(y)}{\sigma_{n-1}} \leq (n-1) \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}}.
\]
This (2.9), along with (2.8), yields

\[(2.10) \quad \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right) \leq (n-1) \left( \frac{\text{cap}_2(\Omega)}{(n-2)\sigma_{n-1}} \right) \]

The inequality (2.10) is weaker than the case \( p = 2 \) of (2.5). However, (2.10) can be strengthened upon demonstrating the following conjecture

\[(2.11) \quad \int_{\Omega} |x-y|^{2-n} \frac{d\sigma(y)}{\sigma_{n-1}} \leq \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{1}{2}} \forall \ x \in \partial \Omega,
\]

with equality if and only if \( \Omega \) is a ball; see [18, p.249,(4)], [21] and [7] for some information related to (2.11).

(ii) The higher dimensional extension of the variational principle presented in [28, Theorem 1.1] derives that if \( \Omega \) is a convex compact subset of \( \mathbb{R}^n \) with \( \Omega^c \neq \emptyset \) and smooth \( \partial \Omega \) then

\[(2.12) \quad \frac{(n-2)\sigma_{n-1}}{\text{cap}_2(\Omega)} \leq \frac{\int_{\partial \Omega} \int_{\partial \Omega} |x-y|^{2-n} d\sigma(x)d\sigma(y)}{(\text{area}(\partial \Omega))^2}.
\]

A combination of (2.9) and (2.12) gives (2.10).

3. \( p \)-CAPACITY TO SURFACE-AREA

From [25, (5)] it follows that if \( n = 3 \) and \( \Omega \) is a convex compact subset of \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and its mean curvature \( H(\partial \Omega, \cdot) > 0 \) then one has the following Pólya-Szegö inequality for the electrostatic capacity and the mean radius:

\[(3.1) \quad \frac{\text{cap}_2(\Omega)}{4\pi} \leq \int_{\partial \Omega} H(\partial \Omega, \cdot) \frac{d\sigma(\cdot)}{4\pi}
\]

with equality if \( \Omega \) is a ball. This result has been extended by Freire-Schwartz to any outer-minimizing \( \partial \Omega \) in \( \Omega^c = \mathbb{R}^{n\geq3} \setminus \Omega \), i.e., \( \Omega \subseteq \Lambda \Rightarrow \text{area}(\partial \Omega) \leq \text{area}(\partial \Lambda) \) (cf. [6, Theorem 2]):

\[(3.2) \quad \frac{\text{cap}_2(\Omega)}{(n-2)\sigma_{n-1}} \leq \int_{\partial \Omega} H(\partial \Omega, \cdot) \frac{d\sigma(\cdot)}{\sigma_{n-1}}
\]

with equality if and only if \( \Omega \) is a ball. As a higher dimensional star-shaped generalization of (3.1), we have the following result whose (3.3) under \( p = 2 \) is a nice parallelism of (3.2) since the outer-minimizing and the star-shaped are not mutually inclusive; see also [10], and whose (3.4) discovers an optimal relation between the variational capacity and the surface area via the Willmore functional of the mean curvature (cf. [11] Corollary 2) for \( (p, n) = (2, 3) \).

**Theorem 3.1.** Let \( \Omega \) be a smooth, star-shaped, compact subset of \( \mathbb{R}^n \) with \( \Omega^c \neq \emptyset \) and \( H(\partial \Omega, \cdot) > 0 \). Then

\[(3.3) \quad \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{1-p} \sigma_{n-1}} \leq \left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^\frac{p-1}{p} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right) \frac{2}{n-1} \text{ as } 2 \leq p < n;
\]

\[\left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^\frac{q-1}{q} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right)^\frac{p-1}{p} \text{ as } 1 < p \leq 2 \leq q < n,
\]

where the first inequality becomes an equality if and only if \( \Omega \) is a ball. Consequently

\[(3.4) \quad \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{1-p} \sigma_{n-1}} \leq \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^\frac{p-1}{p-1} \left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^\frac{n-1}{n-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right) \forall \ p \in (1, n)
\]
holds with equality if and only if $\Omega$ is a ball. Moreover, the limit settings $p \to 1$ or $p \to n$ in (3.4) produce

\begin{equation}
\text{cap}_1(\Omega) \leq \text{area}(\partial\Omega) \quad \text{or} \quad 1 \leq \int_{\partial\Omega} (H(\partial\Omega, \cdot))^n \, d\sigma(\cdot) / \sigma^{n-1}.
\end{equation}

Proof. First of all, recall that a classic solution of inverse mean curvature flow in $\mathbb{R}^n$ is a smooth collection $F : M^{n-1} \times [0, T) \mapsto \mathbb{R}^n$ of closed hypersurfaces evolving by

\begin{equation}
\frac{\partial}{\partial t} F(x, t) = \frac{\tau(x, t)}{H(x, t)} \quad \forall \ (x, t) \in M^{n-1} \times [0, T),
\end{equation}

where

\[ H(x, t) = \text{div}(\tau(x, t)) > 0 \quad \text{and} \quad \tau(x, t) \]

are the mean curvature and the outward normal vector of the embedded hypersurface $M_t = F(M^{n-1}, t)$. According to Gerhardt [9] (or Urbas [30, 31]), one has that for any smooth, closed, star-sharped, initial hypersurface of positive mean curvature, equation (3.6) has a unique smooth solution for all times and the rescaled hypersurfaces $M_t$ converge exponentially to a unique sphere as $t \to \infty$.

According to Moser’s description (cf. [23]) of the inverse mean curvature flow (whose weak formulation was studied in Huisken-Illmanen’s papers [12, 13]), we see that a level set formulation was studied in Huisken-Illmanen’s papers \([12, 13]\), we see that a level set formulation of the above parabolic evolution problem for hypersurfaces in $\mathbb{R}^n$ with the initial hypersurface $M_0 = \Sigma = \partial\Omega$ produces a non-negative smooth function $u$ in $\Omega^c$ such that:

- $\text{div}(\frac{\nabla u}{|\nabla u|}) = |\nabla u| \in \Omega^c$;
- $u|_{\partial\Omega} = 0$;
- $u = t$ on $M_t = \Sigma_t$;
- $|\nabla u| \neq 0$ in $\Omega^c$;
- $H(\Sigma_t, \cdot) = (n - 1)^{-1}|\nabla u(\cdot)|$ on $\Sigma_t$;
- $\text{area}(\Sigma_t) = e^t\text{area}(\partial\Omega) \forall \ t \geq 0$.

This function $u$ may be treated as a kind of 1-equilibrium potential of $\Omega$ - more precisely - if $u_p = \exp \left(\frac{-u}{1-p}\right)$ obeys $\text{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0$ in $\Omega^c$ and $u_p|_{\partial\Omega} = 1$ then $(1 - p) \log u_p \to u$ locally uniformly in $\Omega^c$ as $p \to 1$; see [23] Theorem 1.1.

According to (a) and the determination of $\text{pcap}(\cdot)$ in terms of the $(1, n) \ni p$-equilibrium potential of $\Omega$, we have

\begin{equation}
\text{cap}_p(\Omega) = \text{cap}_p(\partial\Omega) \leq \inf_f \int_{\mathbb{R}^n \setminus \Omega^c} |\nabla f|^p \, dv
\end{equation}

where the infimum is taken over all functions $f = \psi \circ g$ that have the above-described level hypersurfaces $(\Sigma_t)_{t \geq 0}$ and enjoy the property that $\psi$ is a one-variable function with $\psi(0) = 0$ and $\psi(\infty) = 1$ and $g$ is a non-negative function on $\mathbb{R}^n \setminus \Omega^c$ with $g|_{\partial\Omega} = 0$ and $\lim_{|x| \to \infty} g(x) = \infty$. Note that the co-area formula yields

\[ \int_{\mathbb{R}^n \setminus \Omega^c} |\nabla f|^p \, dv = \int_0^\infty |\psi'(t)|^p \left(\int_{\Sigma_t} |\nabla g|^{p-1} \, d\sigma_t\right) \, dt. \]

In the above and below, $d\sigma_t$ is the surface-area-element on $\Sigma_t$. So, upon choosing

\[ \begin{cases} 
  g = u; \\
  U_p(t) = \int_{\Sigma_t} |\nabla u|^{p-1} \, d\sigma_t, \\
  \psi(t) = V_p(t) = \frac{\int_0^t \left(\frac{U_p(s)}{U_p(\infty)}\right)^{\frac{1}{p}} \, ds}{\int_0^\infty \left(\frac{U_p(s)}{U_p(\infty)}\right)^{\frac{1}{p}} \, ds}
\end{cases} \]
we utilize (3.7) to achieve
\[
\frac{\text{cap}_p(\Omega)}{\sigma_{n-1}} \leq \int_0^\infty U_p(t)\left| \frac{d}{dt}V_p(t) \right|^p dt,
\]
whence finding
\[
(3.8) \quad \frac{\text{cap}_p(\Omega)}{\sigma_{n-1}} \leq \left( \int_0^\infty (U_p(t))^{\frac{1}{1-p}} dt \right)^{1-p}.
\]

Next, let us work out the growth of \( U_p(\cdot) \).

Case 1: \( p \in [2, n) \). Under this assumption, utilizing [13, Lemma 1.2, (ii)&(v)], an integration-by-part, the inequality
\[
(H(\Sigma_t, \cdot))^2 - (n-1)|\Pi_t|^2 \leq 0
\]
with
\[
0 < H(\Sigma_t, \cdot) = (n-1)^{-1}|\nabla u|
\]
and \( \Pi_t \) being the mean curvature and the second fundamental form on \( \Sigma_t \) respectively, the differentiation under the integral, we obtain
\[
\frac{d}{dt}U_p(t) = \int_{\Sigma_t} (H(\Sigma_t, \cdot))^p d\sigma_t
\]
\[
= \int_{\Sigma_t} (p-1)(H(\Sigma_t, \cdot))^{p-2}\left( \frac{d}{dt}H(\Sigma_t, \cdot) \right) + (H(\Sigma_t, \cdot))^{p-1} d\sigma_t
\]
\[
= \int_{\Sigma_t} \left[ 1 - (p-1)(\frac{|\Pi_t|}{H(\Sigma_t, \cdot)})^2 - (p-2)|\nabla(H(\Sigma_t, \cdot))^{-1}|^2 \right] \cdot (H(\Sigma_t, \cdot))^{p-1} d\sigma_t
\]
\[
\leq \frac{n-p}{(n-1)\sigma_{n-1}} \int_{\Sigma_t} |\nabla u|^{p-1} d\sigma_t
\]
\[
= \left( \frac{n-p}{n-1} \right) U_p(t),
\]
whence discovering the following inequality through an integration
\[
(3.9) \quad U_p(t) \leq U_p(0) \exp\left( t\frac{n-p}{n-1} \right).
\]
Using (3.8)-(3.9) we get
\[
\frac{\text{cap}_p(\Omega)}{\sigma_{n-1}} \leq U_p(0) \left( \frac{n-1}{n-p} \right)^{1-p}
\]
whence reaching the inequality in (3.3) under \( 2 \leq p < n \).

Case 2: \( 1 < p \leq 2 \leq q < n \). Under this situation, we use the Hölder inequality to achieve
\[
\int_{\Sigma_t} |\nabla u|^{p-1} d\sigma_t \leq \left( \int_{\Sigma_t} |\nabla u|^q d\sigma_t \right)^{\frac{p-1}{q}} \left( \frac{\text{area}(\Sigma_t)}{\sigma_{n-1}} \right)^{\frac{q-p}{q}}
\]
Now, employing the estimate for \( q \in [2, n) \) and the definition of \( U_p \), we obtain
\[
U_p(t) \leq \left( \int_{\Omega} (H(\partial\Omega, \cdot))^q d\sigma(\cdot) \right)^{\frac{p-1}{q}} \left( \frac{\text{area}(\Sigma_t)}{\sigma_{n-1}} \right)^{\frac{q-p}{q}} \exp\left( t\frac{n-p}{n-1} \right).
\]
Bringing this last inequality into (3.8), along with
\[
\text{area}(\Sigma_t) = e^t \text{area}(\partial\Omega),
\]
we arrive at the second inequality of (3.3).

Case 3: equality of (3.3). If \( \Omega \) is a ball, then a direct computation gives equality of (3.3). Conversely, if the inequality \( \leq \) in (3.3) becomes an equality, then the above-established differential inequalities for \( U_p \) force

\[
(H(\Sigma_r, \cdot))^2 - (n-1)\|\Pi\|^2 = 0 \quad \text{on} \quad \Sigma_r,
\]

which in turn ensures that \( \Sigma_r \) consists of the union of disjoint spheres. Since \( \Sigma_r \) is generated by a smooth solution of the inverse mean curvature flow in \( \mathbb{R}^n \), \( \Sigma_r \) must be a single sphere. Consequently, \( \Omega \) is a ball.

After that, (3.4) and its equality case follow from (3.3) and its equality case as well as the following estimate (based on the Hölder inequality)

\[
\int_{\partial \Omega} (H(\partial \Omega, \cdot))^{q-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \leq \left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{q-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right)^{\frac{q}{q-1}} \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{q-1}{q}} \quad \forall \ q \in (1, n).
\]

Finally, let us check (3.5). On the one hand, letting \( p \to 1 \) in (3.4) yields the Mazya inequality (cf. [20] p.149, Lemma 2.2.5)):

\[
\text{cap}_1(\Omega) \leq \text{area}(\partial \Omega).
\]

On the other hand, choosing \( 0 < r < R \) with \( B(x_0, r) \subseteq \Omega \subseteq B(x_0, R) \), we utilize the properties (b)\&(d) of \( \text{cap}_p(\cdot) \) to derive

\[
r^{n-p} \leq \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{1-p} \sigma_{n-1}} \leq R^{n-p},
\]

whence achieving

\[
\lim_{p \to n} \frac{\text{cap}_p(\Omega)}{(\frac{p-1}{n-p})^{1-p} \sigma_{n-1}} = 1.
\]

This, together with letting \( p \to n \) in (3.4), derives the Willmore inequality (cf. [29] p. 87) or [2]

for immersed hypersurfaces in \( \mathbb{R}^n \):

\[
1 \leq \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{n-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}}.
\]

\[\square\]

Remark 3.2. Two comments are in order:

(i) Let \( \Omega \) be a smooth compact subset of \( \mathbb{R}^n \) with \( \Omega^c \neq \emptyset \) and \( H(\partial \Omega, \cdot) > 0 \). If \( \partial \Omega \) is outer-minimizing, then one has the (1, n) \( \ni \) p-Aleksandrov-Fenchel inequality:

\[
(3.10) \quad \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{p-n}{n-1}} \leq \left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{p-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \text{ as } 2 \leq p < n;
\]

\[
(3.11) \quad \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{q}{n-1}} \leq \int_{\partial \Omega} H(\partial \Omega, \cdot) \frac{d\sigma(\cdot)}{\sigma_{n-1}} \text{ as } 1 < p \leq 2 \leq q < n,
\]

where the first inequality becomes an equality if and only if \( \Omega \) is a ball.

In fact, using the known 2-Aleksandrov-Fenchel inequality (cf. [6] Theorem 2(b))

and the Hölder inequality, we gain

\[
\int_{\partial \Omega} H(\partial \Omega, \cdot) \frac{d\sigma(\cdot)}{\sigma_{n-1}} \leq \left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{p-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right)^{\frac{1}{p-1}} \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{q}{q-1}} \quad \forall \ p \in [2, n),
\]
whence implying (3.10). If the first inequality of (3.10) becomes equality, then equality of (3.11) is valid, and hence \( \Omega \) is a ball. Of course, the converse follows from a direct computation.

(ii) An application of (3.2), (3.10) and the Hölder inequality derives that if \( \Omega \subset \mathbb{R}^{n \geq 3} \) is a smooth compact set with \( \Omega^o \neq \emptyset \) and \( \partial \Omega \) being outer-minimizing as well as having \( H(\partial \Omega, \cdot) > 0 \) then one has the following log-convexity type inequality for the electrostatic capacity, the surface area and the Willmore functional:

\[
\frac{\text{cap}_2(\Omega)}{(n-2)\sigma_{n-1}} \leq \left( \frac{\text{area}(\partial \Omega)}{\sigma_{n-1}} \right)^{\frac{n-4}{n-2}} \left( \int_{\partial \Omega} (H(\partial \Omega, \cdot))^{n-1} \frac{d\sigma(\cdot)}{\sigma_{n-1}} \right)^{\frac{1}{n-1}}
\]

with equality if and only if \( \Omega \) is a ball. Interestingly and naturally, (3.12) and (3.4) under \( p = 2 \) complement each other thanks to the relative independence between the outer-minimizing and the star-shaped.

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