Possible Effects of Fierz Transformations on Vacua of Some Four-Fermion Interaction Models

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I. INTRODUCTION

The four-fermion interactions are very useful field theory models to describe dynamical spontaneous breaking of symmetries and their restoring at high temperature and high density, as well as the color superconducting phase transitions at low temperature and high density. For the involved four-fermion interaction models with dynamical symmetry breaking (from now on the fermion will be called quark), the ground states (vacua) could be in the antiquark-quark (q̅-q) condensate phase or in the diquark (q-q) condensate phase or in the coexistence phase of the above two condensates, depending on the interplay between the q̅-q and q-q condensates in the vacuum. The presupposition of such interplay is the coexistence of the scalar q̅-q and the scalar or pseudoscalar q-q channel couplings. On the other hand, for any given four-fermion couplings, the fermion fields entering them can always be rearranged by the Fierz transformations, thus, by the Fierz transformations, a q-q channel coupling will be led to some q-q channel coupling, and the opposite case will also occur. This will inevitably lead to the coexistence of the two kinds of couplings in the resulting effective Lagrangian. Thus a natural question would be drawn out: whether the Fierz transformations could change the feature of the vacuum of a given four-fermion interaction model? A systematical research on this topic has seemingly not appeared in the known literature.

The possibility that the diquark condensates could emerge from the vacuum has been researched or touched on by some phenomenological models, including the 2-flavor Quantum Chromodynamics (QCD) instanton-induced NJL model with any Nc, the random matrix model of 2 flavor and Nc color QCD and a 2 flavor color superconducting model. The main results show that such possibility has not been removed theoretically. To examine further this problem, we have made a more general analysis. Under the assumption that some q̅-q and q-q channel couplings coexist, by means of the effective potential method in the mean field approximation, we have researched the interplay between the q̅-q and q-q condensates in the vacuum respectively for 4D, 2D and 3D four-fermion interaction models with flavor Nf = 2 and color Nc = 3, and then extend the discussions to the case of any Nc, where some useful criterions by which the q̅-q and/or q-q condensates could emerge from the vacuum are obtained. However, in the above work, the coexistence of some q̅-q and q-q channel couplings is only an assumption, its possible origin was not be carefully considered. Certainly, the Fierz transformations could be one of the origins, and in fact, as a check of the derived criterions, the Fierz transformations were also briefly mentioned in the Conclusions of Ref. for some Nc = 3 models, e.g. 4D chiral-invariant model and the heavy gluon exchange models, however, they have not constituted a methodical research on possible effects of the Fierz transformations on the vacua.

In this paper, we will do a systematical research on such effects. In the case of Nf = 2 and keeping Nc to be arbitrary, when some four-fermion interaction couplings are given, we will examine how their Fierz transformations induce the couplings leading to q̅-q and q-q condensates and how this will affect the vacuum of the model. The given starting four-fermion couplings, which in 4D case are typical and in most cases, possibly relevant to QCD-like theory, besides the chiral-invariant model and the heavy gluon exchange model, will also include the diquark channel coupling which has never been considered before. Because the strengths of the given couplings are assumed to be known, by the Fierz transformations, we will be able to fix uniquely the strengths of the q̅-q and q-q channel couplings in the final effective Lagrangian, including their ratios. This makes it become possible, by means of the general criterions derived in Ref., to obtain some definite conclusions of that whether the vacua are actually in the q̅-q or...
q-q condensate phase or in the coexistence phase of the two condensates. The results will show that in 4D space-time, for given q-q channel couplings, the effects of the Fierz transformations are not enough to change the models’ feature that the vacuum are in the pure q-q condensate phases. The conclusion seem a little trivial, but it is demonstrated systematically for the first time. Furthermore, for a given q-q channel coupling, more interesting non-trivial effects will emerge from the Fierz transformations. In this case, the model’s vacuum could be in the expected q-q condensate phase only if the q-q channel coupling strength and the color number $N_c$ are small enough, otherwise, as the q-q channel coupling strength and $N_c$ increase, the vacuum would be first in a coexistence phase with q-q and q-q condensates and finally in a pure q-q condensate phase. Similar conclusions will also be derived from the 2D and 3D models. This shows some space-time dimensionality independence of the conclusions. It is emphasized that the basic ideas of the above research, including to relate the effects of the Fierz transformations to the vacuum of a class of given four-fermion interaction models with dynamical symmetry breaking, and working in the case of any $N_c$ and in the 4D, 2D and 3D space-time, are original and novel, and most of the obtained results appear in the literature for the first time.

In Sect.[II] we will analyze the effects of the Fierz transformations on scalar and pseudoscalar-isovector q-q channel couplings, the vectorial q-q channel couplings from heavy quon exchange and scalar q-q channel couplings in 4D space-time and in Sect.[III] and [IV] the discussions will be extended to the similarities of the above three couplings in 2D and 3D space-time. Finally, in Sect.[V] we come to our conclusions.

A brief introduction of the Fierz transformations and the explicit expressions of the Fierz transformation matrices and corresponding converse forms in spinor spaces of 4D, 2D and 3D space-time and in flavor or color $U(N)$ space will be given in Appendix. For a given q-q channel coupling $\mathcal{L}_{qq}$, its $qq \rightarrow \bar{q}q$ and $\bar{q}q \rightarrow qq$ channel Fierz rearrangements will be denoted respectively by $\mathcal{L}^{ex}_{qq}$ (exchange terms) and $\mathcal{L}^{int}_{qq}$. For a given qq channel coupling $\mathcal{L}_{qq}$, its $qq \rightarrow \bar{q}q$ channel Fierz rearrangements and corresponding exchange terms will be denoted respectively by $\mathcal{L}^{qq}$ and $\mathcal{L}^{qq-ex}$. For a given coupling, we will put down directly the total effective Lagrangian after the Fierz transformations and focus on its physical effects.

II. 4D FOUR-FERMION INTERACTIONS

1. Scalar and pseudoscalar-isovector q-q channel couplings.

The corresponding Lagrangian may be expressed by

$$\mathcal{L}_{4(S+P_{\tau})} = G[(\bar{q}q)^2 + (\bar{q}i\gamma_5\tau_0q)^2]$$

where $\tau_a(a = 1, \cdots, N_f - 1)$ are the generators of the flavor group $SU(N_f)$. In present paper, the summation of a Lorentz index is implied to combine into a Lorentz scalar and the summation of an index of the $SU(N)$ generator, unless specified otherwise, will always run over from 1 to $N^2 - 1$. When $N_f = 2$, the above $\mathcal{L}^{(S+P_{\tau})}_{4(S+P_{\tau})}$ is chiral $SU_f(2) \otimes SU_R(2)$-invariant. By using the transformations (A.8),(A.17),(A.9) and (A.18) in the Appendix, we can obtain respectively the Fierz-rearranged $\mathcal{L}^{ex}_{4(S+P_{\tau})}$ (exchange terms) and $\mathcal{L}^{qq}_{4(S+P_{\tau})}$, thus the total effective Lagrangian becomes

$$\mathcal{L}^{eff}_{4(S+P_{\tau})} = \mathcal{L}_{4(S+P_{\tau})} + \mathcal{L}^{ex}_{4(S+P_{\tau})} + \mathcal{L}^{qq}_{4(S+P_{\tau})}$$

$$= G_S(\bar{q}q)^2 + G_P(\bar{q}i\gamma_5\tau_0q)^2 + G_{\tau}(\bar{q}i\gamma_5\tau_0q)^2$$

$$+ H_S(\bar{q}i\gamma_5\tau_0\lambda_A\bar{q})\bar{q}) + \cdots$$

(2)

where we only display part of terms which could be physically interesting, $\tau_0$ and $\lambda_A$ are separately the anti-symmetric generators of the groups $SU_f(N_f)$ and $SU_c(N_c)$ and ellipsis stands for all the other possible couplings, where $\lambda_0(a = 1, \cdots, N_c - 1)$ are the generators of $SU_c(N_c)$. It is emphasized that $\mathcal{L}^{eff}_{4(S+P_{\tau})}$ must be used in Hartree approximation.

When $N_f = 2$, the coupling constants in Eq.(2) have the following explicit expressions:

$$G_S = G_P = (1 + 1/4N_c)G, \quad G_P = -G/4N_c,$$

$$H_S = -H_P = G/4.$$  

(3)

Eq.(3) shows that, for the two-flavor and $N_c$-color model, the induced scalar q-q channel interactions have a positive coupling constant $H_S$, however, compared with the scalar q-q channel interactions with the coupling constant $G_S$, we always have the ratio

$$G_S/H_S = (4N_c + 1)/N_c > 2/N_c.$$  

(4)

Thus, based on the general criterion of interplay between the q-q and q-q condensates, it is impossible to exist the scalar diquark condensates in the vacuum of this model. Such conclusion is also valid in the limit of $N_c = 2$. It is indicated that, when $N_c = 2$, the scalar q-q condensates $(\bar{q}q)$ and the scalar q-q condensates $(\bar{q}i\gamma_5\tau_0q)$ are both $SU_f(2) \otimes SU_c(2)$-singlets, however the former breaks $SU_f(2) \otimes SU_R(2)$ chiral symmetry but the latter conserves it. Hence in the case of $N_c = 2$ we also have spontaneous breaking of the chiral

\[^1\text{This chiral symmetry reproduces the one of QCD with } N_c \geq 3.\]

Hence Eq.(1) can be related to QCD with $N_c \geq 3$. However, it can not simulate $N_c = 2$ QCD with massless quarks, because the latter’s chiral symmetry is the higher $SU(4)$-invariant. It is easy to check that $\mathcal{L}_{4(S+P_{\tau})}$ in Eq.(1) does not have the $SU(4)$ symmetry. For instance, it is only a part of the whole instanton-induced four-fermion couplings which are $SU(4)$-invariant when $N_c = 2$. Based on the same grounds, the conclusions in $N_c = 2$ case in present section are merely applicable for the given models here and not for $N_c = 2$. In fact, the models considered here only the extensions of some $N_c = 3$ QCD-relevant four-fermion interaction models to any $N_c$ case and are not supposed to touch the very special $N_c = 2$ QCD theory.
symmetry. In addition, it is noted that, after the Fierz transformations, the largest attractive channel couplings are still the terms $(\bar{q}q)^2$ and $(\bar{q}i\gamma_5\sigma a)q^2$ with the same coupling strength $G_S = G_{\rho}$, this fact certainly keeps the basic feature of the original $\mathcal{L}_{4(S+P)\tau}$ in Eq.(1), including its chiral symmetry.

2. Four-fermion interactions from heavy gluon exchange.

The corresponding Lagrangian is assumed to be [13]

$$\mathcal{L}_{4(V\lambda)} = -g(\bar{q}\gamma^\mu \lambda_a q)(\bar{q}\gamma_\mu \lambda_a q)$$

with the constant $g$. It simulates the interactions induced by one gluon exchange in QCD. Similar to the steps taken in part 1, we can obtain the total effective Lagrangian for $N_f = 2$

$$\mathcal{L}_{4\text{eff}} = \mathcal{L}_{4(V\lambda)} + \mathcal{L}_{4(S\pi)} + \mathcal{L}_{4(S\pi)}^q = G_S(\bar{q}q)^2 + G_{\lambda}(\bar{q}\gamma^\mu \lambda_a q)^2 + H_S(\bar{q}i\gamma_5 \sigma \lambda_A q)(\bar{q}i\gamma_5 \sigma \lambda_A q) + \cdots$$

with

$$G_S = (N_c^2 - 1)g/N_c^2 = 2(N_c - 1)H_S/N_c,$$

$$G_{\lambda} = -(1 - 1/4N_c)g.$$  

Hence, the Fierz transformations have induced the scalar $\bar{q}q$ channel coupling and the scalar $q\bar{q}$ channel coupling, however, the corresponding coupling constants $G_S$ and $H_S$, in the case of $N_f = 2$ and any $N_c$, have the ratio

$$G_S/H_S = 2(N_c - 1)/N_c > 2/N_c, \text{ if } N_c > 2.$$  

This result was given in the Appendix A of Ref.[13]. Thus, based on the general criterion given in Ref.[18], if $N_c \geq 3$, the ground state (vacuum) of the model could only be in antiquark-quark condensate phase. In the limit of $N_c = 2$, $G_S/H_S = 1$. This implies that we will be at a critical point between breaking and restoring of the chiral symmetry. Once there are the other couplings included, such balance would be broken and the system could come to the phase of chiral symmetry breaking or chiral symmetry restoring, depending on the feature of the included couplings.

3. Scalar diquark channel interactions.

For describing two-flavor color superconductors, one introduces the pure scalar $q\bar{q}$ channel coupling with the Lagrangian [13, 18]

$$\mathcal{L}_{4(S\pi\pi)} = H_S(\bar{q}i\gamma_5 \tau_\lambda \lambda_A q^*) (\bar{q}i\gamma_5 \tau_\lambda \lambda_A q).$$  

Eq.(9) is used usually in the case with finite quark chemical potential, however, once it is put into a theory, then its Fierz transformations will be bound to induce some effects even in the case with zero quark chemical potential. In this paper we will only research such effects on the vacuum for given pure scalar $q\bar{q}$ couplings. In fact, based on the converse Fierz transformation matrices (A.10) and (A.19) we may put down the Fierz-rearranged $\mathcal{L}_{4(S\pi\pi)}^{\bar{q}q}$ from $q\bar{q}$ channel to $\bar{q}\bar{q}$ channel, and furthermore by using the transformation (A.8) obtain its exchange terms $\mathcal{L}_{4(S^\pi S^\pi)}^{\bar{q}q}$ which is in fact identical to $\mathcal{L}_{4(S\pi\pi)}^{\bar{q}q}$. Thus the effective Lagrangian for $N_f = 2$ becomes

$$\mathcal{L}_{4\text{eff}}^{S\pi\pi} = \mathcal{L}_{4(S\pi\pi)}^{\bar{q}q} + \mathcal{L}_{4(S\pi\pi)}^{\bar{q}q \rightarrow S^\pi S^\pi} = \mathcal{L}_{4(S\pi\pi)}^{\bar{q}q} + G_{\rho}(\bar{q}\gamma_5 \tau_\lambda q)^2 + G_{\lambda}(\bar{q}i\gamma_5 \sigma \lambda_A q)^2 + G_{\lambda}(\bar{q}\gamma^\mu \lambda_a q)^2 + \cdots,$$

$$G_S = G_{\rho} = (N_c - 1)H_S/4N_c,$$

$$G_{\lambda} = -H_S/8.$$  

Thus, as a result of the converse Fierz transformations, we are led from the pure scalar $q\bar{q}$ channel coupling (9) to the scalar and pseudoscalar-isovector coupling $(\bar{q}q)^2$ and $(\bar{q}i\gamma_5 \sigma q)^2$. When $N_f = 2$, they have the same coupling strengths $G_S = G_{\rho}$, and this means that the chiral $SU_{fL}(2) \otimes SU_{fR}(2)$ symmetry is maintained. Meanwhile, we are also led to the four-fermion interactions similar to the ones induced by heavy gluon exchange, but with weaker strength $G_{\lambda}^2 = -H_S/8$. It is interesting to make a comparison between the values of $G_S$ and $H_S$. For a given $G_S$ and $H_S$, Eq.(7) in Ref.[18] has given the possible least value points of the effective potential $V_4(\sigma, |\Delta|)$ (the ground states) of the model with the coupling terms corresponding to $G_S$ and $H_S$, where $\sigma$ and $\Delta$ represent the order parameters relevant to the scalar $\bar{q}q$ condensates and the scalar $q\bar{q}$ condensates respectively. In present case, the induced $G_S$ depends on $H_S$ and $N_c$, and Eq.(7) in Ref.[18] will be reduced to the following form: the ground state of the model will be at

$$(\sigma, |\Delta|) = \begin{cases} (0, \Delta_1) & \text{if } 1/2 < \tilde{H}_S < (3N_c + 1)/(N_c - 1)(N_c - 2), \quad N_c < 9 \\ (\sigma_2, \Delta_2) & \text{if } \tilde{H}_S > (3N_c + 1)/(N_c - 1)(N_c - 2), \quad N_c < 9 \\ (\sigma_1, 0) & \text{if } \tilde{H}_S > 4/(N_c - 1), \quad N_c > 9 \end{cases}$$

where we have used the denotations $\tilde{H}_S \equiv H_S\Lambda^2/\pi^2$ and $\Lambda$ is the 4D Euclidean momentum.
cutoff of the loop integrals. It is indicated that \( N_c < 9 \) and \( N_c > 9 \) correspond respectively to \( G_S/H_S < 2/N_c \) and \( G_S/H_S > 2/N_c \). Hence, when \( N_c < 9 \) i.e. \( G_S/H_S < 2/N_c \), for a given \( N_c \), the system can be in a pure \( q\bar{q} \) condensate phase only if the coupling strength \( \tilde{H}_S \) is less than the critical value \( (3N_c + 1)/(N_c - 1)(N_c - 2) \). Especially, in the limit of \( N_c = 2 \), the critical value of \( \tilde{H}_S \) goes to \( \infty \) and this implies that the system will only be in the chiral-invariant pure \( q\bar{q} \) condensate phase. Once \( 3 \leq N_c < 9 \) and \( \tilde{H}_S \) exceeds the above critical value, the \( q\bar{q} \) condensate phase will be changed into a coexistence phase with the \( q\bar{q} \) and \( \bar{q}q \) condensates, though the original purpose of our using \( L_{4(S\bar{S}q)} \) in Eq.(9) is only for expounding the pure \( q\bar{q} \) condensates. In particular, for the realistic case with \( N_c = 3 \) of QCD, the expected pure \( q\bar{q} \) condensate phase could appear only if \( \tilde{H}_S < 5 \). This is an interesting result. Once the strength \( \tilde{H}_S \) of the given \( q\bar{q} \) channel coupling is large enough, what could emerge from the vacuum will not be the expected diquark condensates, instead a coexistence of the \( q\bar{q} \) and \( \bar{q}q \) condensates. The critical value of \( \tilde{H}_S \) will decrease as the increase of \( N_c \), for example, it becomes 25/42 for \( N_c = 8 \).

On the other hand, when \( N_c > 9 \) i.e. \( G_S/H_S > 2/N_c \), for a sufficiently large \( \tilde{H}_S \), there could exist only the \( \bar{q}q \) condensates and no the \( q\bar{q} \) condensates. This statement will certainly keep to be valid until \( N_c \rightarrow \infty \), consistent with the general conclusion reached in Ref.\cite{18}. The present key point is that even if the starting point Eq.(9) is a pure scalar \( q\bar{q} \) channel coupling, as a result of the converse Fierz transformations, the above general conclusion is also true.

III. 2D FOUR-FERMION INTERACTIONS

1. Scalar and pseudoscalar-isovector \( \bar{q}q \) channel couplings.

Similar to the 4D case, we take the Lagrangian by

\[
L_{2(S+P\tau)} = G[(\bar{q}q)^2 + (\bar{q}i\gamma_5\tau_\sigma q)^2].
\] (13)

However, in 2D case, we need not to consider the continuous symmetries of a Lagrangian, since they can never be spontaneously broken based on Mermin-Wagner-Coleman theorem\cite{23}. Formally Eq.(13) is the same as Eq.(1), but now the \( \gamma_5 \) in it is a \( 2 \times 2 \) matrix. The steps to conduct the Fierz transformations are similar to the ones taken in 4D case in Sect.II. Based on the Fierz transformations (A.11),(A.17), (A.12) and (A.18), the resulting total effective Lagrangian \( L_{eff}^{S+P\tau} \) for \( N_f = 2 \) becomes

\[
L_{2(S+P\tau)}^{eff} = L_{2(S+P\tau)} + L_{2(S+P\tau)}^{qq} + L_{2(S+P\tau)}^{qq},
\] (14)

where \( \tau_S \) and \( \lambda_S \) are respectively symmetric generators of \( U_f(N_f) \) and \( U_c(N_c) \), including \( \tau_0 = \sqrt{2/N_f}1_f \) and \( \lambda_0 = \sqrt{2/N_c}1_c \). It indicated that when \( N_f = 2 \), the coupling terms \( (\bar{q}i\gamma_5\tau_\sigma q)^2 \) and \( (\bar{q}i\gamma_5\tau_\sigma q)^2 \) have disappeared. We see that in \( L_{eff}^{S+P\tau} \) the scalar and pseudoscalar-isovector channel couplings \( (\bar{q}q)^2 \) and \( (\bar{q}i\gamma_5\tau_\sigma q)^2 \) keep to be the maximal attractive ones. Denote the respective coupling strengths by \( G_S \) and \( G_P\tau \), then we will have

\[
G_S = G_P\tau = (1 + 1/2N_c)G.
\]

Consequently the model will maintain its original feature unchanged.

On the other hand, the Fierz transformations have also led to occurrence of the scalar and pseudoscalar \( q\bar{q} \) attractive channel couplings \( (\bar{q}i\gamma_5\tau_\sigma q)(\bar{q}i\gamma_5\tau_\sigma q) \) and \( (\bar{q}i\gamma_5\tau_\sigma q)(\bar{q}i\gamma_5\tau_\sigma q) \) with the coupling strength \( H_S = G/4 \). However, considering the ratio

\[
G_S/H_S = 2(2N_c + 1)/N_c > 2/N_c
\] (17)

we can affirm similarly based on the general criterion derived in Ref.\cite{18} that for the 2D four-fermion interaction model expressed by Eq.(13), only antiquark-quark condensates, rather than the diquark condensates, are possible in its vacuum.

2. Four-fermion interactions from heavy gluon exchange.

Take the Lagrangian to be

\[
L_{2(\lambda\lambda)} = -g(\bar{q}i\gamma_\mu\lambda_\sigma q)(\bar{q}i\gamma_\mu\lambda_\sigma q),
\] (18)
where $\gamma^\mu$ are $2 \times 2$ matrices. After the Fierz transformations, the total effective Lagrangian $\mathcal{L}^{eff}_{2(V A)}$ for $N_f = 2$ can be expressed as follows.

$$\mathcal{L}^{eff}_{2(V A)} = \mathcal{L}_{2(V A)} + \mathcal{L}^{ex}_{2(V A)} + \mathcal{L}^{qq}_{2(V A)}$$

with

$$\mathcal{L}^{ex}_{2(V A)} = G_S \sum_{a=0}^{3} [\bar{q} \gamma_\tau q]^2 - \frac{g}{2N_c} \sum_{a=0}^{3} [\bar{q} \gamma_\tau \lambda_q q]^2 + (\bar{q} \gamma_5 \tau_\lambda \lambda_q q]^2$$

$$\tau_0 = 1_f, \quad G_S = (N_c^2 - 1)g/N_c^2$$

and

$$\mathcal{L}^{qq}_{2(V A)} = -[(N_c - 1)g/2N_c] \sum_{a=S,A} [\bar{q} \gamma_5 \tau_\lambda \lambda_q A]^2 \left( q^5 i \bar{q} \gamma_5 \tau_\lambda \lambda_q A q + (i \gamma_5 \rightarrow 1_s) \right)$$

$$+ H_S \sum_{a=S,A} [\bar{q} \gamma_5 \tau_\lambda \lambda_q A]^2 \left( q^5 i \bar{q} \gamma_5 \tau_\lambda \lambda_q A q + (i \gamma_5 \rightarrow 1_s) \right), \quad H_S = (N_c + 1)g/2N_c.$$  

Since $H_S > 0$, so the corresponding coupling terms are attractive. However, the ratio of the strengths of the scalar $\bar{q}q$ channel coupling $(\bar{q}q)^2$ and the scalar $q-q$ channel coupling $(\bar{q} \gamma_5 \tau_\lambda \lambda_q A q)^2$ obtained to be

$$G_S/H_S = 2(N_c - 1)/N_c > 2/N_c, \text{ for } N_c > 2.$$  

Hence, if $N_c \geq 3$, there will be antiquark-quark condensates alone in the vacuum $[18]$. Eq.(22) is the same as Eq.(8) in 4D case.

3. Scalar diquark channel interactions.

The Lagrangian is given by $[18]$

$$\mathcal{L}_{2(S_{qq})} = H_S(\bar{q} \gamma_5 \tau_\lambda \lambda_q A^c q)(\bar{q} \gamma_5 \tau_\lambda \lambda_q A q).$$  

It is indicated that Eq.(23) is different from Eq.(9) with $\tau_S$ having replaced $\tau_A$ in Eq.(9), because in 2D case the matrix $C_{5\gamma}$ is symmetric. We may use the exchange matrices (A.13) and (A.19) in the Appendix to obtain the converse Fierz-rearranged $\mathcal{L}^{qq}_{2(S_{qq})}$ and furthermore use the transformation (A.11) in the Appendix to get its exchange terms

$$\mathcal{L}^{qq,ex}_{2(S_{qq})} = \mathcal{L}^{qq}_{2(S_{qq})},$$

thus the total effective Lagrangian for $N_f = 2$ becomes

$$\mathcal{L}^{eff}_{2(S_{qq})} = \mathcal{L}_{2(S_{qq})} + \mathcal{L}^{qq}_{2(S_{qq})} + \mathcal{L}^{qq,ex}_{2(S_{qq})}$$

$$= \mathcal{L}_{2(S_{qq})} + H_S \sum_{G^{8=1,5\gamma_5,\gamma^\mu} = 1} \left[\frac{3(N_c - 1)}{2N_c}(q^8 \Gamma^b q)^2 - \frac{3}{4} (q^8 \Gamma^b \lambda_q q)^2 + \frac{N_c - 1}{2N_c} (q^8 \Gamma^b \tau A q)^2 - \frac{1}{4} (q^8 \Gamma^b \tau \lambda_q A q)^2 \right].$$  

Eq.(24) contains the induced scalar channel term $(\bar{q}q)^2$ and pseudoscalar channel term $(\bar{q} \gamma_5 \tau_\lambda \lambda_q A q)^2$ which respectively have the coupling strengths $G_S = 3(N_c - 1)H_S/2N_c$ and $G_{P_{tau}} = (N_c - 1)H_S/2N_c$ and it may be seen that, among all the $\bar{q}q$ channel couplings of $\mathcal{L}^{eff}_{2(S_{qq})}$, the scalar channel term $(\bar{q}q)^2$ is maximal attractive. We note that the ratio of $G_S$ and the strength $H_S$ of the scalar $q-q$ channel coupling i.e. $\mathcal{L}_{2(S_{qq})}$ becomes

$$G_S/H_S = [3(N_c - 1)/4](2/N_c).$$  

Based on the general criterion given in Ref. [18], if there exist the $\bar{q}q$ condensates alone in the vacuum, then we must have the condition $G_S/H_S > 2/N_c$ to be satisfied, and from Eq.(25), this implies that $3(N_c - 1)/4 > 1$ and it leads to $N_c > 7/3$. Therefore, if $N_c \geq 3$, the vacuum of the system will in fact only in a $\bar{q}q$ condensate phase, even though originally given interaction (23) is a pure scalar $q-q$ channel coupling. On the other hand, if $N_c = 2 < 7/3$, we will have $G_S/H_S < 2/N_c$, however, owing to $G_S \neq 0$, theoretically one could just acquire a mixed phase with both $\bar{q}q$ and $q-q$ condensates, since in 2D case, it was proven that one could get a pure $\bar{q}q$ condensate phase in the vacuum only if $G_S = 0$ $[18].$
IV. 3D FOUR-FERMION INTERACTIONS

1. Scalar and isovector $\bar{q}q$ channel couplings.

Since there is not $\gamma_5$ matrix in 3D space-time, the similarities of Eqs.(1) and (13) in 4D and 2D case will be the Lagrangian expressed by

$$L_{3(S+S^r)} = G[(\bar{q}q)^2 + (\bar{q}^\mu q)^2]$$

which is SU$_c(N_c) \otimes$ SU$_f(N_f) \otimes U_f(1)$-invariant. For convenience, the coupling strengths of the two terms in $L_{3(S+S^r)}$ are assumed to be equal, but physically this is not essential. When $N_f = 2$, by (A.14) and (A.17), the Fierz-rearranged

$$L_{3(S+S^r)}^{ex} = -\frac{G}{N_c} [G(\bar{q}q)^2 + (\bar{q}^\mu q)^2] - \frac{G}{2} [(\bar{q}\lambda_\alpha q)^2 + (\bar{q}^\mu \lambda_\alpha q)^2]$$

and by (A.15) and (A.18), the Fierz-rearranged

$$L_{3(S+S^r)}^{qq} = \frac{G}{4} \sum_{a' = S', A'} \{[(\bar{q}^\tau A_\alpha a')q^\tau \tau A_\alpha a')(1 \rightarrow \gamma^\mu)] - [(\bar{q}\tau A_\alpha a')q^\tau \tau A_\alpha a')(1 \rightarrow \gamma^\mu)]\}.$$ (26)

Thus the total effective Lagrangian becomes

$$L_{3(S+S^r)}^{eff} = L_{3(S+S^r)} + L_{3(S+S^r)}^{ex} + L_{3(S+S^r)}^{qq}$$

$$= G_S(\bar{q}q)^2 + G_S(\bar{q}\tau A\lambda A^\tau q)^2 + \frac{H_P}{(\bar{q}^\tau A_\alpha A^\tau q)^2 + (\bar{q}^\mu \lambda_\alpha A^\tau q)^2} + \cdots, G_S = (1 - 1/N_c)G, G_{S^r} = G, H_P = G/4.$$ (27)

It should be indicated that, after the Fierz transformations, two maximal attractive channel couplings are still the term $(\bar{q}q)^2$ and $(\bar{q}^\tau q)^2$ contained in the original $L_{3(S+S^r)}$. However, the two terms with the same coupling constant $G_S$ in $L_{3(S+S^r)}$ now have different coupling strengths $G_S < G_{S^r}$. This implies that in the resulting $L_{3(S+S^r)}^{eff}$ the maximal attractive channel coupling will actually be the term $(\bar{q}^\tau q)^2$ rather than the term $(\bar{q}q)^2$. So it is more reasonable to assume that the condensates $(\bar{q}^\tau q)$ are formed more easily than the condensates $(\bar{q}q)$, and this will lead to spontaneous breaking of the flavor SU$_f(N_f)$ (for $N_f = 2$ i.e. isospin) symmetry. In this case we must replace the order parameter $\sigma = -2G_S(\bar{q}q)$ by $\sigma = -2G_{S^r}(\bar{q}^\tau q)$ (it is possible to fix the condensates in the $\tau_3$ direction through a rotation in isospin space). However, it may be proven that the derived expression for the effective potential of the model containing the new $\sigma$ will keep unchanged in form, hence the conclusions reached in Ref. [13] about interplay between the $\bar{q}q$ and $q-q$ condensates in the ground state (vacuum) is still true, the mere change is to replace the scalar channel coupling constant $G_S$ by the scalar-isovector channel constant $G_{S^r}$. Since

$$G_{S^r}/H_P = 4 > 2/N_c, \quad (28)$$

we can immediately conclude that although the Fierz transformations may bring about the $q-q$ channel coupling corresponding to $H_P$, it is still impossible to form the pseudoscalar diquark condensates $(\bar{q}^\tau A_\alpha A^\tau q)^2$ in the vacuum and the vacuum could only in the $(\bar{q}q)$ condensate phase.

2. Four-fermion interactions from heavy gluon exchange.

The Lagrangian is given by

$$L_{3(V\lambda\lambda)} = -g(\bar{q}^\mu \lambda_\alpha q)(\bar{q}^\mu \lambda_\alpha q),$$

where $\gamma^\mu$ is $2 \times 2$ matrices in 3D space-time. When $N_f = 2$, the Fierz-rearranged

$$L_{3(V\lambda\lambda)}^{ex} = \sum_{\lambda = 0}^3 \left[ G_S(\bar{q}\tau A_\lambda q)^2 - \frac{3g}{4N_c}(\bar{q}\tau A_\lambda q)^2 \right]$$

and

$$L_{3(V\lambda\lambda)}^{qq} = H_P(\bar{q}^\tau A_\lambda A^\tau q)^2(\bar{q}^\tau A_\lambda A^\tau q)$$

$$+ \frac{3(N_c^2 - 1)}{4N_c} g(\bar{q}^\mu \tau A_\lambda q)^2(\bar{q}^\mu \tau A_\lambda q)^2 + \cdots.$$ (29)

Thus the total effective Lagrangian becomes

$$L_{3(V\lambda\lambda)}^{eff} = L_{3(V\lambda\lambda)} + L_{3(V\lambda\lambda)}^{ex} + L_{3(V\lambda\lambda)}^{qq}$$

$$= G_S[(\bar{q}q)^2 + (\bar{q}^\mu q)^2] + H_P[(\bar{q}^\tau A_\lambda A^\tau q)^2(\bar{q}^\tau A_\lambda A^\tau q)^2 - \frac{1}{4N_c} (\bar{q}^\mu \lambda_\alpha q)^2 + \cdots.$$ (29)

Since the ratio

$$G_S/H_P = 2(N_c - 1)/2N_c, \quad \text{for} \quad N_c > 2, \quad (30)$$

we can affirm that there could not be the diquark condensates in the vacuum of the model [13].

3. Pseudoscalar diquark channel coupling.

The corresponding Lagrangian is given by

$$L_{3(P_{\mu\nu})} = H_P(\bar{q}^\tau A_\lambda A^\tau q)(\bar{q}^\tau A_\lambda A^\tau q).$$

By using the converse matrices (A.16) and (A.19) in the Appendix, we may obtain the Fierz-rearranged form $L_{3(P_{\mu\nu})}$ of Eq.(31) from $q-q$ channel to $\bar{q}q$ channel, and then by Eq. (A.14) in the Appendix get its exchange terms $L_{3(P_{\mu\nu})}^{ex} = L_{3(P_{\mu\nu})}$, thus when $N_f = 2$, their sum becomes

$$L_{3(P_{\mu\nu})}^{qq} + L_{3(P_{\mu\nu})}^{qq}$$

$$= \frac{N_c - 1}{2N_c} H_P[(\bar{q}^\tau q)^2 - (\bar{q}q)^2]$$

$$+ H_P[(\bar{q}^\mu \lambda_\alpha q)^2 - (\bar{q}^\mu \lambda_\alpha q)^2] - (1 \rightarrow \gamma^\mu)$$

$$= G_S(\bar{q}\tau A_\lambda q)^2 + G_S(\bar{q}q)^2 + \cdots.$$ (32)
where \( G_{S\tau} = (N_c - 1)H_P/2N_c = -G_S > 0 \), this implies that only the term \((\bar{q}\tau q)^2\) is a (maximally) attractive interaction which could induce the isovector condensates \(\langle \bar{q}\tau q \rangle\). As has been indicated in the sector of scalar and isovector \(\bar{q}q\) channel couplings, making the substitutions \(\langle \bar{q}q \rangle \rightarrow \langle \bar{q}\tau q \rangle\) and \(G_S \rightarrow G_{S\tau}\), we can conduct the same discussions and reach the same conclusions as the ones obtained in Ref. \[18, 24\] about interplay between the \(\bar{q}q\) and \(q\bar{q}\) condensates. A special feature is now that the induced coupling constant \(G_{S\tau}\) depends on \(H_P\) and \(N_c\). Let \(\sigma = -2G_{S\tau}\langle \bar{q}\tau q \rangle\) and \(\Delta\) represent the order parameters respectively corresponding to the \(\bar{q}q\) and \(q\bar{q}\) condensates, then the conclusion (34) in Ref. \[18, 24\] will be reduced to the following equation which shows the least value points of the effective potential \(V_\delta(\sigma, |\Delta|)\) being at

\[
(\sigma, |\Delta|) = \begin{cases} 
(0, |\Delta|), & \text{if } \tilde{H}_P > 1/8, \ N_c \leq 4 < 5 \\
(\sigma, 0), & \text{if } \tilde{H}_P > 2(N_c - 1), \ N_c > 5 
\end{cases}
\]

where \(\tilde{H}_P = H_P\lambda_3/\pi^2\), \(\lambda_3\) is the 3D Euclidean momentum cutoff of the loop integrals. It is indicated that \(N_c = 5\) corresponds to \(G_{S\tau}/H_P = 2/N_c\). We see from Eq.(33) that the four-fermion interactions used to describe pure pseudoscalar diquark condensates, after the converse Fierz transformations, will induce the \(\bar{q}q\) channel coupling term \(\langle \bar{q}\tau q \rangle^2\) and lead to interplay between the \(\bar{q}q\) and \(q\bar{q}\) condensates in the ground state. In this model, the \(q\bar{q}\) condensates could be formed only if \(N_c \leq 4 < 5\) and in that case the ground state could be in a pure \(q\bar{q}\) condensate phase. Once \(N_c > 5\), until \(N_c \rightarrow \infty\), we will be able to get merely the \(\bar{q}q\) condensates \(\langle \bar{q}\tau q \rangle\) instead of the diquark condensates.

### V. CONCLUSIONS

In this paper, we have theoretically analyzed possible effects of the Fierz transformations on the vacua of several given typical 4D, 2D and 3D two-flavor and \(N_c\)-color four-fermion interaction models. The results can be summarized as follows.

It is shown that, after the Fierz transformations, the 4D and 2D scalar and pseudoscalar-isovector couplings keep to be the maximal attractive ones with the strength \(G_S\), and some scalar diquark channel couplings with the strength \(H_S\) will be induced; in the case of 3D scalar and isovector coupling with equal strength, the isovector coupling will become the maximal attractive one with the strength \(G_{S\tau}\) and one also gets the induced pseudoscalar \(q\bar{q}\) channel coupling with the strength \(H_P\). However, it is found that the resulting ratios both \(G_S/H_S\) and \(G_{S\tau}/H_P\) are always greater than the critical value \(2/N_c\). This indicates that no diquark condensates could be generated in the vacua of these models, hence the above interaction models maintain to be the ones merely to describe possible \(\bar{q}q\) condensates. The above results are valid for any \(N_c\).

For the four-fermion interactions from heavy gluon exchange, no matter in 4D or 2D or 3D case, after the Fierz transformations, we will always get the ratio of the induced scalar \(\bar{q}q\) channel coupling strength \(G_S\) and the induced 4D and 2D scalar or 3D pseudoscalar \(q\bar{q}\) channel coupling strength \(H_S\) or \(H_P\) expressed by

\[
G_S/H_S = G_S/H_P = 2(N_c - 1)/N_c.
\]

On the same ground as the above, if \(N_c \geq 3\), this removes the possibility to emerge the diquark condensates from the vacua and only the \(\bar{q}q\) condensates could exist in the vacua.

When the starting points are the pure diquark channel scalar or pseudoscalar couplings with the strengths \(H_S\) or \(H_P\), the nontrivial effects of the Fierz transformations on the vacua will be displayed. Owing to the converse Fierz transformations, we will get the induced \(\bar{q}q\) channel couplings including the exchange terms with the strength \(G_S\) or \(G_{S\tau}\) and this is bound to lead to interplay between the \(\bar{q}q\) and \(q\bar{q}\) condensates. We have found that, independent of dimensionality of space-time, the expected \(q\bar{q}\) condensates could emerge from the vacua only if \(N_c < N_c^0\), a critical value determined by the conditions \(G_S/H_S < 2/N_c\) or \(G_{S\tau}/H_P < 2/N_c\) which is 9, 7/3 and 5 for 4D, 2D and 3D case respectively.

In 4D case, when \(N_c < 9\), only if the coupling strength \(H_S\) is less than some \(N_c\)-dependent critical value, we could just get a pure \(q\bar{q}\) condensate phase, otherwise, will obtain a coexistence phase with the \(q\bar{q}\) and \(\bar{q}q\) condensates. For the realistic case of \(N_c = 3\), the above critical value of \(H_S\) corresponds to \(H_S\lambda_3^2/\pi^2 < 5\). Hence, owing to the Fierz transformations, a sufficiently strong \(q\bar{q}\) channel coupling could lead to not the expected pure \(q\bar{q}\) condensate, instead only a coexistence of the \(q\bar{q}\) and \(\bar{q}q\) condensates in the vacuum. In 2D case, even if \(N_c < 7/3\), we could obtain only a coexistence phase with the two condensates. In 3D case, the condition \(N_c \leq 4 < 5\) will correspond to a pure \(q\bar{q}\) condensate phase.

Once \(N_c > N_c^0\), until \(N_c \rightarrow \infty\), in all the cases we will obtain only the \(\bar{q}q\) condensates and no the \(q\bar{q}\) condensates in the vacua, though the original purpose to introduce the pure \(q\bar{q}\) channel couplings is to deal with the diquark condensates.

The above results show that for the models which originally do not contain the diquark channel couplings, it seems that one needs not to worry about the occurrence of the diquark condensates in the vacua through the Fierz transformations. However, for a model of given diquark channel couplings, for example, in 4D space-time, one must note that the expected pure diquark condensates could appear only in the case of weak coupling and some small \(N_c\) and this is just the nontrivial effect of the Fierz transformations on the model’s vacuum to which now one must pay special attention. In addition, if one attempts to extend the above analysis based on the mean field approximation to higher order correction of the \(1/N_c\) expansion, then because the above effects induced by the Fierz transformations depend on the value of \(N_c\), more careful consideration must be conducted.

The analysis made in this paper can be extended to the case of finite temperature and finite quark chemical potential where the Fierz transformations will lead to the interplay between the thermal \(q\bar{q}\) and \(q\bar{q}\) condensates which could or could not affect the feature of the ground
state of a thermal four-fermion interaction model.

In this paper, we only research some special 2-flavor and $N_c$-color four-fermion models, however, the discussions may be of more general significance for any 2-flavor and $N_c$-color four-fermion model, since for any given such model, the Fierz transformations can always lead to scalar $\bar{q}q$ and scalar or pseudoscalar $gq$ channel coupling and induce the interplay between the corresponding condensates in the ground state. In addition, similar or possibly different effects could be assumed to emerge from more general four-fermion interaction models with dynamical symmetry breaking. It is just the above research and convenience of use, we will still give a brief introduction to the Fierz transformations with the ground states of a class of four-fermion interaction models with dynamical symmetry breaking thus provides us a new angle of view to inspect this class of models. In any case, theoretically, the possible ground state effects of the Fierz transformations should become an implicit factor to build and treat such class of models.

Appendix: Fierz Transformations

The Fierz transformations in Dirac spinor space of 4D space-time and in $U(N)$ space can be found in Appendix A of Ref. [13]. However, for this paper to be self-contained and convenience of use, we will still give a brief introduction of the Fierz transformations and list all the necessary explicit expressions for the transformations including the new results in spinor space of 2D and 3D space-time and the converse forms of all the non-self-converse Fierz transformations.

Consider a local four-fermion interactions of spinor fields $q \equiv q(x)$ with $N_f$ flavors and $N_c$ colors, the corresponding Lagrangian is

$$
\mathcal{L}_{int} = g(q\Gamma^a q)^2 = g\Gamma^{a}_{12}\Gamma^{a}_{34}\bar{q}_1 q_2 \bar{q}_3 q_4, \quad (A.1)
$$

where $g$ is the coupling constant, $\Gamma^a$ is outer product of the linearly independent matrices in spinor, flavor $U_f(N_f)$ and color $U_c(N_c)$ space, the numbers 1, 2, 3 and 4 represent the indices of the elements of $\Gamma^a$, for instance, the number 1 can represent $s_1 f_1 c_1$ for the product matrix, or $s_1, f_1$ and $c_1$ when $\Gamma^a$ is separately limited to the matrix acting on spinor, flavor and color space etc. and an index repeated always means its summing. In view of anticommutativity of the fermion fields $q$, Eq. (A.1) may be rewritten by

$$
\mathcal{L}_{int} = -g\Gamma^{a}_{12}\Gamma^{a}_{34}\bar{q}_1 q_4 \bar{q}_3 q_2 =: \mathcal{L}^{ex}_{int} \quad (A.2)
$$
or

$$
\mathcal{L}_{int} = g\Gamma^{a}_{12}\Gamma^{a}_{34}\bar{q}_1 q_4 \bar{q}_3 q_2 =: \mathcal{L}^{qq}_{int}. \quad (A.3)
$$

In the above expressions, we will restrict ourselves to Hartree-type approximation, for example, in Eq. (A.2), $\bar{q}_1$ is contracted with $q_4$ and $\bar{q}_3$ is contracted with $q_2$ thus $\mathcal{L}^{ex}_{int}$ will give exchange diagram of $\mathcal{L}_{int}$, and in Eq. (A.3), $\bar{q}_1$ is contracted with $q_3$ and $q_4$ is contracted with $q_2$ thus $\mathcal{L}^{qq}_{int}$ will give the coupling term of antiquark-antiquark ($\bar{q}q$) and quark-quark ($qq$). For this purpose, in Eq. (A.2) we must rewrite the matrices

$$
\Gamma^{a}_{12}\Gamma^{a}_{34} = \sum_b c^a_b \Gamma^b_{14}\Gamma^b_{32}, \quad (A.4)
$$

where $b$ runs over all the linearly independent matrices $\Gamma^b$. In this paper, Eq. (A.4) will be called the Fierz transformation of $\bar{q}q \rightarrow \bar{q}q$ channel. By means of Eq. (A.4), Eq. (A.2) becomes

$$
\mathcal{L}^{ex}_{int} = -g \sum_b c^a_b (\bar{q}\Gamma^b q)^2. \quad (A.5)
$$

On the other hand, in Eq. (A.3) we must rewrite the matrices

$$
\Gamma^{a}_{12}\Gamma^{a}_{34} = \sum_b d^a_b (\Gamma^b C)_{13}(\Gamma^b C)_{42}, \quad (A.6)
$$

where $C$ is the charge conjugate matrix in spinor space. Eq. (A.6) will be called the Fierz transformation of $\bar{q}q \rightarrow qq$ channel. By means of Eq. (A.6), Eq. (A.3) becomes

$$
\mathcal{L}^{qq}_{int} = g \sum_b d^a_b (\bar{q}\Gamma^b q^c)(\bar{q}^c\Gamma^b q), \quad (A.7)
$$

where $q^c = Cq^T$ and $\bar{q}^c = q^T C$ are charge conjugates of the fields $q$ and $\bar{q}$ respectively.

The problem to solve the Fierz transformations is reduced to calculate the expansion coefficients $c^a_b$ and $d^a_b$ in Eqs. (A.4) and (A.6). In view of the outer product feature of $\Gamma^a$ and the similarity of the groups $U_f(N_f)$ and $U_c(N_c)$, we can consider separately the cases of that $\Gamma^a$ are the matrices in spinor space and that $\Gamma^a = \{1, \tau_a\}$ with 1 as the unit matrix and $\tau_a (a = 1, \cdots, N-1)$ as the generators of the group $SU(N)$.

In the following we will give explicit expressions of the Fierz transformations of the matrices in spinor spaces in 4D, 2D and 3D space-time and of the $U(N)$ generators.

1. Matrices in spinor space.

(a) 4D space-time.

The independent $4 \times 4$ Dirac matrices are $1_s, i\gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu} (\mu = 0, 1, 2, 3)$. The Fierz transformations become
The independent 2 × 2 matrices in spinor space are

\[
(F^{-1}_{\bar{q}q \rightarrow qq}) = \begin{pmatrix}
1_s, \, \gamma^\mu (\mu = 0, 1), \, \gamma_5 = \gamma^0 \gamma^1
\end{pmatrix}
\]

and the charge conjugate matrix \( C = -\gamma^1 \). One adoption of \( \gamma^\mu \) is that \( \gamma^0 = \sigma^3, \, \gamma^1 = i\sigma^2 \) with Pauli matrices \( \sigma^i (i = 1, 2, 3) \). The Fierz transformations become

\[
(F^s_{\bar{q}q \rightarrow qq})^{-1} = F^s_{\bar{q}q \rightarrow qq}
\]

which corresponds to the converse of the transformation (A.9). Obviously, \( (F^s_{\bar{q}q \rightarrow qq})^{-1} \neq F^s_{\bar{q}q \rightarrow qq} \).
are
\[ 1_s, \gamma^\mu (\mu = 0, 1, 2) \]
and the charge conjugate matrix \( C = \gamma^2 \), but

\[
\begin{pmatrix}
(1_s)_{12}(1_s)_{34} \\
(\gamma^\mu)_{12}(\gamma^\mu)_{34}
\end{pmatrix}
= \begin{pmatrix}
\left( \frac{1}{2} \right) & \left( -\frac{1}{2} \right) \\
\left( \frac{3}{2} \right) & \left( -\frac{1}{2} \right)
\end{pmatrix}
\begin{pmatrix}
(1_s)_{14}(1_s)_{32} \\
(\gamma^\mu)_{14}(\gamma^\mu)_{32}
\end{pmatrix}
\quad (\bar{q}-q \rightarrow q\bar{q} \text{ channel})
\]
\[ \equiv F^{\text{eq}}_{\bar{q}q} 
\]
\[
\begin{pmatrix}
\left( -\frac{1}{2} \right) & \left( -\frac{1}{2} \right) \\
\left( -\frac{3}{2} \right) & \left( -\frac{1}{2} \right)
\end{pmatrix}
\begin{pmatrix}
(C)_{13}(C)_{42} \\
(\gamma^\mu C)_{13}(\gamma^\mu C)_{42}
\end{pmatrix}
\end{pmatrix}
\quad (\bar{q}-q \rightarrow q\bar{q} \text{ channel})
\]

Similarly, the matrix \( F^{\text{eq}}_{\bar{q}q\rightarrow qq} \) is self-converse, but \( F^{\text{eq}}_{\bar{q}q\rightarrow qq} \) is not. The converse of the latter is

\[
(F^{\text{eq}}_{\bar{q}q\rightarrow qq})^{-1} \equiv F^{\text{eq}}_{\bar{q}q\rightarrow qq} = \begin{pmatrix}
\left( -\frac{1}{2} \right) & \left( -\frac{1}{2} \right) \\
\left( -\frac{3}{2} \right) & \left( -\frac{1}{2} \right)
\end{pmatrix}
\]

which will generate the converse of the transformation (A.15).

2. Generators of \( U(N) \).

We denote the generators of the group \( SU(N) \) by

\[
\begin{pmatrix}
(1)_{12}(1)_{34} \\
(\tau_0)_{12}(\tau_0)_{34}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
2N^2-1 & 2N^2-1
\end{pmatrix}
\begin{pmatrix}
(1)_{14}(1)_{32} \\
(\tau_0)_{14}(\tau_0)_{32}
\end{pmatrix}
\quad (\bar{q}-q \rightarrow q\bar{q} \text{ channel})
\]
\[ \equiv F^{U(N)}_{\bar{q}q\rightarrow qq} 
\]
\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
N-1 & N-1
\end{pmatrix}
\begin{pmatrix}
(\tau S)_{13}(\tau S)_{42} \\
(\tau A)_{13}(\tau A)_{42}
\end{pmatrix}
\quad (\bar{q}-q \rightarrow q\bar{q} \text{ channel})
\]

In Eq.(18) \( \tau S \) (including \( \tau_0 \)) and \( \tau A \) are respectively symmetric and anti-symmetric generators of \( U(N) \). It is indicated that the matrix \( F^{U(N)}_{\bar{q}q\rightarrow qq} \) is self-converse, but \( F^{U(N)}_{\bar{q}q\rightarrow qq} \) is not. The converse of the latter is

\[
(F^{U(N)}_{\bar{q}q\rightarrow qq})^{-1} \equiv F^{U(N)}_{\bar{q}q\rightarrow qq} = \begin{pmatrix}
\frac{N+1}{2} & \frac{N+1}{2} \\
\frac{N}{2} & \frac{N}{2}
\end{pmatrix}
\]

which corresponds to the \( U(N) \) Fierz transformation of \( qq \rightarrow q\bar{q} \) channel.

It is emphasized that in the discussions of this paper, when \( U(N) \) is considered as the flavor group \( U_f(N_f) \), the generators will be denoted by \( (1_f, \tau_0) \) or \( (\tau S, \tau A) \) and when \( U(N) \) is considered as the color group \( U_c(N_c) \), the generators will be denoted by \( (1_c, \lambda_a) \) or \( (\lambda S, \lambda A) \).

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