The Jet Isomorphism Theorem of pseudo-Riemannian geometry

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Abstract

We give a detailed exposition of the Jet Isomorphism Theorem of pseudo-Riemannian geometry. In its weak form, this theorem states that the Taylor expansion up to order $k + 2$ of a pseudo-Riemannian metric in normal coordinates can be reconstructed in a universal way from suitable symmetrizations of the covariant derivatives of the curvature tensor up to order $k$. In its full generality it states the equivalence of Taylor expansions of metrics in normal coordinates up to order $k + 2$, abstract curvature $k$-jets and their symmetrizations as affine vector bundles. This theorem seems to be a cornerstone of local pseudo-Riemannian geometry which is hardly mentioned in the present literature anymore.

1 Overview

Let $(V, \langle \cdot, \cdot \rangle)$ be some pseudo-Euclidean space and $U$ be an open star-shaped neighborhood of the origin. Further, let $\text{Sym}^k V^*$ denote the $k$-th symmetric power of the dual space $V^*$ for any $k \geq 0$.

Definition 1. A pseudo-Riemannian metric $g: U \rightarrow \text{Sym}^2 V^*$ with $g(0) = \langle \cdot, \cdot \rangle$ is given in normal coordinates if the straight lines $t \mapsto t \xi$ are geodesics for any $\xi \in U$ and $t \in [0, 1]$.

Let a pseudo-Riemannian metric in normal coordinates on $U$ be given, $\nabla$ denote the Levi Civita connection on, $R$ the curvature tensor and $\nabla^k R$ its $k$-times iterated covariant derivative taken at the origin.

Definition 2. Let $R^k \in \text{Sym}^{k+2} V^* \otimes \text{Sym}^2 V^*$ be uniquely characterized (via polarization) through the polynomial function

$$R^k(\xi; x, y) := \nabla^k R(x, \xi, \xi, y)$$

for all $\xi, x, y \in V$. We will call $R^k$ the symmetrized $k$-th covariant derivative of the curvature tensor.

For a better understanding of $R^k$, let some geodesic $\gamma(t) = t \xi$ emanating from the origin be given and let $R_\gamma(t)$ be the usual Jacobi operator along $\gamma(t)$ defined by $R_\gamma(t)(x, y) := R(x, \dot{\gamma}(t), \dot{\gamma}(t), y)$ for all suitable small $t$ and all $x, y \in T_{\gamma(t)} M$. We denote by $R^k_\gamma(t)$ the $k$-th covariant derivative of $R_\gamma$ with respect to the parameter $t$ for any $k \geq 0$. Then

$$\forall x, y \in V : R^k_\gamma(0; x, y) = R^k(\xi; x, y).$$

In Section 2, we will give an explicit formula for tensorial expressions $Q_k$ in a set of free (non-commutative) generators $X_2, X_3, \cdots$ (see for example (30)) such that the substitution $X_{k+2} = R^k$ yields the $k$-th derivative of the metric. Therefore:
**Theorem 1** (Jet Isomorphism Theorem, weak version). The Taylor polynomial of order \( k + 2 \) of a pseudo-Riemannian metric in normal coordinates is a universal tensorial expression in the symmetrized curvature \( k \)-jet:

\[
g(\xi) = \sum_{\ell=0}^{k+2} \frac{1}{k!} Q_k(\mathcal{R}^0, \mathcal{R}^1, \cdots, \mathcal{R}^k)(\xi) + O(k+3) .
\]  

(3)

In particular, the pseudo-Riemannian structure can be asymptotically reconstructed from the Jacobi operators and their covariant derivatives along the geodesics through the origin of \( U \).

**Example 1.** For low values of \( k \)

\[
Q_0 = 1 , \quad Q_1 = 0 , \quad Q_2 = -\frac{2}{3} X_2 , \quad Q_3 = -X_3 , \quad Q_4 = -\frac{6}{5} X_4 + \frac{16}{15} X_2^2 , \quad Q_5 = -\frac{4}{3} X_5 + \frac{8}{3} (X_2 X_3 + X_3 X_2) .
\]  

(4)

Therefore, for any metric in normal coordinates the Taylor polynomial of degree five is (cf. [8, Cor. 229])

\[
\text{Id} - \frac{1}{3} \mathcal{R} - \frac{1}{6} \mathcal{R}^1 - \frac{1}{20} \mathcal{R}^2 + \frac{2}{45} (\mathcal{R}^0)^2 - \frac{1}{90} \mathcal{R}^3 + \frac{1}{45} (\mathcal{R}^0 \circ \mathcal{R}^1 + \mathcal{R}^1 \circ \mathcal{R}^0) .
\]  

(6)

In order to get a better feeling for the Taylor expansion of a metric in normal coordinates, set

\[
\mathcal{N}_k := \{ h \in \text{Sym}^k V^* \otimes \text{Sym}^2 V^* | \forall \xi \in V : h(\xi, \cdots, \xi) = 0 \} \quad (7)
\]

for any \( k \geq 1 \) (cf. [3, Def. 8.1].) Note that \( \mathcal{N}_1 = \{ 0 \} \). Further, \( \mathcal{R}^k \in \mathcal{N}_{k+2} \) by means of the algebraic properties of a curvature tensor.

According to Koszul’s formula for the Christoffel symbols, a pseudo-Riemannian metric satisfying \( g_0 = \langle \cdot, \cdot \rangle \) is given in normal coordinates if and only if \( g_\ell(\xi, u) = \langle \xi, u \rangle \) for all \( \xi \in U \) and \( u \in V \) (cf. [2, Theorem 2.3]). Therefore, the following is true:

**Lemma 1.** Let \((V, \langle \cdot, \cdot \rangle)\) be a pseudo-Euclidean vector space and \( h_\ell \in \text{Sym}^\ell V^* \otimes \text{Sym}^2 V^* \) for \( \ell \geq 1 \). The polynomial

\[
g := \langle \cdot, \cdot \rangle + \sum_{\ell=1}^{k} h_\ell
\]

(8)

defines a metric in normal coordinates on a small open star-shaped neighborhood of the origin if and only if \( h_\ell \in \mathcal{N}_\ell \) for \( \ell = 1, \ldots, k \), see (7).

Thus, the Taylor expansion up to order \( k + 2 \) of a metric in normal coordinates corresponds to an element of \( \bigoplus_{\ell=2}^{k+2} \mathcal{N}_\ell \). For a more sophisticated interpretation of this correspondence see [2, Theorem 2.4 b)]. For the following cf. also [9, p. 60]:

**Definition 3.** Let \( V \) be a vector space. A fiber bundle \( \tau : E \to M \) is called an affine vector bundle modelled on \( V \) if each fiber is an affine space with direction space \( V \) and such that the transition functions of local bundle trivializations are fibrewise affine isomorphisms. In particular, the fibers of \( \tau \) are non-empty.
For example, the projection map

\[ \bigoplus_{\ell=2}^{k+2} \mathcal{N}_\ell \to \bigoplus_{\ell=2}^{k+1} \mathcal{N}_\ell \] (omitting the \( k + 2 \)-th component). \hfill (9)

turns \( \bigoplus_{\ell=2}^{k+2} \mathcal{N}_\ell \) into an affine vector bundle over \( \bigoplus_{\ell=2}^{k+1} \mathcal{N}_\ell \) modeled on \( \mathcal{N}_{k+2} \).

### 1.1 Algebraic jets of curvature tensors

The curvature jet \((R, \nabla R, \cdots, \nabla^k R)\) has certain well known properties (cf. [3, Def. 8.2]):

**Definition 4.** Let \((V, \langle \cdot, \cdot \rangle)\) be a pseudo-Euclidean vector space and \(\otimes^k V^*\) denote the \(k\)-fold tensor product of the dual space \(V^*\).

(a) An algebraic curvature tensor \(R \in \otimes^4 V^*\) satisfies the algebraic identities

\[
R(x_1, x_2, y_1, y_2) = -R(x_2, x_1, y_1, y_2) = R(y_1, y_2, x_1, x_2) \quad \text{(i.e. } R \in \text{Sym}^2(\Lambda^2 V^*)) \hfill (10)
\]

\[
R(x_1, x_2, x_3, y_1) + R(x_2, x_3, x_1, y_1) + R(x_3, x_1, x_2, y_1) = 0 \quad \text{(first Bianchi identity)} \hfill (11)
\]

for all \(x_1, x_2, x_3, y_1, y_2 \in V\). We let \(C_0\) denote the linear space of algebraic curvature tensors.

(b) We call \(\nabla R \in \otimes^5 V^*\) an algebraic covariant derivative of a curvature tensor if the four-tensor \(\nabla_x R\) (obtained by inserting \(x\) into the first slot of \(R\) and leaving the other ones free) is an algebraic curvature tensor for each \(x \in V\) and

\[
\nabla_{x_1} R(x_2, x_3, x_1, y_2) + \nabla_{x_2} R(x_3, x_1, y_1, y_2) + \nabla_{x_3} R(x_1, x_2, y_1, y_2) = 0 \quad \text{(second Bianchi identity)} \hfill (12)
\]

for all \(x_1, x_2, x_3, y_1, y_2 \in V\). We let \(C_1\) denote the linear space of algebraic covariant derivatives of curvature tensors.

(c) An algebraic curvature 2-jet is a sequence \((R, \nabla R, \nabla^2 R)\) satisfying the following conditions:

- \(R \in C_0\), \(\nabla R \in C_1\) and \(\nabla^2 R \in \otimes^6 V^*\) satisfies \(\nabla^2_x R(\cdot, \cdot, \cdot, \cdot) \in C_1\) for all \(x \in V\).

- Let \(R(x_1, x_2)\) denote the skew-symmetric endomorphism of \(V\) defined by \(R(x_1, x_2)(y_1, y_2) := R(x_1, y_2, x_2, y_1)\). Further, let \(A \cdot V\) denote the usual derivation of some tensor \(V\) by an endomorphism \(A\). Then

\[
\nabla^2_{x_1, x_2} R - \nabla^2_{x_2, x_1} R = R(x_1, x_2) \cdot R \quad \text{(Ricci identity)} \hfill (13)
\]

for all \(x_1, x_2 \in V\).

(d) Algebraic curvature \(k\)-jets are defined in a similar way for \(k \geq 3\). They satisfy also higher Ricci identities, which are obtained by formally applying the Leibniz rule to (13). For \(k = 3\) for example,

\[
\nabla^3_{x_1, x_2, x_3} R - \nabla^3_{x_2, x_1, x_3} R = \nabla_{x_1} R(x_2, x_3) \cdot R + R(x_2, x_3) \cdot \nabla_{x_3} R \hfill (14)
\]

(see [11, Formula 12 on page 20] or [3, Def. 8.2]). We denote the space of algebraic curvature \(k\)-jets \((R, \nabla R, \cdots, \nabla^k R)\) by \(\text{Jet}^k C\) for any \(k \geq 0\).

(e) The vector space of linear algebraic curvature \(k\)-jets is defined by

\[
C_k := \{ \nabla^k R \in \otimes^{k+4} V^* | (0, \cdots, 0, \nabla^k R) \text{ is a curvature } k\text{-jet} \} \hfill (15)
\]
Although we used the suggestive “∇-notation”, the prefix algebraic emphasizes that such a curvature jet a priori needs not be induced by some metric.

However, it is a non-trivial consequence of Theorem 3 that in fact every algebraic curvature k-jet comes from a metric. For the following result cf. [2, Theorem 2.6 (a)]:

**Theorem 2** (Jet Isomorphism Theorem, linearized version). Let \((V, \langle \cdot, \cdot \rangle)\) be a pseudo-Euclidean vector space. The map \(C_k \rightarrow N_{k+2}\) which assigns to \(\nabla^k R \in C_k\) the symmetrization \(R^k\) is a linear isomorphism which is compatible with the actions of the orthogonal group \(\text{SO}(V, \langle \cdot, \cdot \rangle)\).

Since the arguments given in [2, 3] are a bit vague, we will give an explicit proof of the previous theorem in Sec. 3.

In order to state the Jet Isomorphism Theorem in its full generality, consider the following three maps

\[
R : \bigoplus_{\ell=2}^{k+2} N_\ell \rightarrow \text{Jet}^k C, \tag{16}
\]

\[
\mathcal{R} : \text{Jet}^k C \rightarrow \bigoplus_{\ell=2}^{k+2} N_\ell, \tag{17}
\]

\[
Q : \bigoplus_{\ell=2}^{k+2} N_\ell \rightarrow \bigoplus_{\ell=2}^{k+2} N_\ell, \tag{18}
\]

defined as follows:

- Given a collection \((h_2, \cdots, h_{k+2}) \in \bigoplus_{\ell=2}^{k+2} N_\ell\), the map \(R\) assigns to it the curvature k-jet \((R, \nabla R, \cdots, \nabla^k R)\) of the metric \(\langle \cdot, \cdot \rangle + \sum_{\ell=2}^{k+2} h_\ell\) (see Lemma 1.)
- Given an algebraic curvature k-jet \((R, \nabla R, \cdots, \nabla^k R)\), the map \(\mathcal{R}\) assigns to it the symmetrized curvature k-jet \((\mathcal{R}, \cdots, \mathcal{R}^k)\) (see (1).)
- Given a collection \((\mathcal{R}, \cdots, \mathcal{R}^k) \in \bigoplus_{\ell=2}^{k+2} N_\ell\), the map \(Q\) assigns to it the sum

\[
\sum_{\ell=2}^{k+2} \frac{1}{k!} Q_k(\mathcal{R}, \cdots, \mathcal{R}^k) \tag{3}
\]

**Theorem 3** (Jet Isomorphism Theorem, strong version). Let \((V, \langle \cdot, \cdot \rangle)\) be a pseudo-Euclidean vector space.

(a) With respect to the natural projection map

\[
\text{Jet}^k C \rightarrow \text{Jet}^{k-1} C, \tag{19}
\]

the jet space \(\text{Jet}^k C\) is an affine vector bundle over \(\text{Jet}^{k-1} C\) modeled on \(C_k\).

(b) The three maps given by (16), (17) and (18) are isomorphisms of affine vector bundles. They satisfy \(Q \circ R \circ R = \text{Id}\). Further, they commute with the natural actions of the orthogonal group \(\text{SO}(V, \langle \cdot, \cdot \rangle)\) on the total spaces.

The non-trivial assertion of Part (a) of the theorem is that every algebraic curvature k-jet can be extended to a \(k+1\)-jet.

**Corollary 1.** Every algebraic curvature k-jet comes in fact from a metric.
The author is aware that the ideas of this paper are general knowledge. In particular, the formula for the Taylor expansion of the metric in normal coordinates was already derived in [8] in quite a similar way as we do it here. For another derivation see [10]. A proof of the other parts of Theorem 3 is loosely sketched in [3, Ch. 8] with reference to [2, Sec. 2]. Nevertheless, this theorem seems to have sunk into obscurity over the past decades. To the authors best knowledge it can not be found in its full generality in the modern literature anymore. Possibly it has been better known by differential geometers of the first half of the previous century (freely cited from [2, p. 634].)

## 2 Taylor expansion of the metric in normal coordinates

We consider the subset of the natural numbers given by $\mathbb{N}_{\geq 2} := \{2, 3, \ldots\}$. The free tensor algebra $\mathbb{R}\langle \mathbb{N}_{\geq 2} \rangle$ is an associative algebra with elements $\{X_2, X_3, \cdots\}$ characterized by the following universal property:

Given another associative algebra $A$ with unit over $\mathbb{R}$ and a subset $A$ whose elements are labeled as $\{A_2, A_3, \cdots\}$, there is a unique homomorphism $\mathbb{R}\langle \mathbb{N}_{\geq 2} \rangle \to A, Q \mapsto Q(A_2, \cdots)$. We may think of the elements of $A$ as formal linear combinations of products in the free non-commutative variables $X_i$ and the homomorphism is obtained by substituting $X_i = A_i$.

Suppose additionally that $A$ is graded and $\deg(A_i) = i$. In order to understand the degree of $Q(A_2, A_3, \cdots)$, we turn $\mathbb{R}\langle X \rangle$ into a graded algebra over $\mathbb{R}$ as follows:

**Definition 5.** Different from the standard tensor grading, we define the degree of the simple tensorial expression $X_I = X_0^i \cdot X_1^i \cdots$ as $i_0 + \cdots + i_n$. As usual the unit element has degree zero. An arbitrary tensorial expression is called homogeneous if it can be written as a linear combination of simple tensorial expressions of the same degree. Thus the degree is well defined on homogeneous tensors.

Therefore, if the tensor $Q_d$ is homogeneous of degree $d$, the associative algebra $A$ is graded and $A_i$ is of degree $i$, then $Q_d(A_1, \cdots)$ is also homogeneous of the same degree $d$.

**Example 2.** The tensor $Q_4 := -\frac{1}{120} X_4^2 + \frac{2}{45} X_2^2$ is homogeneous of degree four.

Let $\text{End} V$ denote the space of homomorphisms of $V$. We now consider the associative algebra $A := \bigoplus_{k=0}^{\infty} \text{Sym}^k V \otimes \text{End} V$, whose elements will be seen as polynomial functions with values in the endomorphisms of $V$. The grading is given by the degree of the first factor, i.e. the polynomial grading. The multiplication is pointwise given by multiplication of endomorphisms.

Identifying the spaces of symmetric bilinear forms and symmetric endomorphisms on $V$ via the inner product $\langle \cdot, \cdot \rangle$, the Jacobi operator of any algebraic curvature tensor on $V$ can be seen as a homogeneous polynomial of degree two with values in $\text{End} V$, via

$$\langle \mathcal{R}(\xi; x), y \rangle = \mathcal{R}(\xi; x, y).$$

More generally, the symmetrized covariant derivative of a curvature tensor $\mathcal{R}^k$ belongs to $\text{Sym}^{k+2} V \otimes \text{End} V$, see also (7).

**Example 3.** Since $Q_4$ (defined above) is homogeneous of degree four, also the endomorphism valued polynomial $Q_4(\mathcal{R}^0, \mathcal{R}^2) = -\frac{1}{120} \mathcal{R}^2 + \frac{2}{45} (\mathcal{R}^0)^2$ is a homogeneous polynomial of degree four. The latter term enters into the Taylor polynomial of the metric up to order four.
2.1 Asymptotic expansion of the backwards parallel transport

This section follows ideas of [8, Sec. 2] and [11, Sec. 3]. Let a pseudo-Riemannian metric $g$ in normal coordinates on a star-shaped open neighborhood $U$ of the origin be given.

**Definition 6.** The backwards parallel transport $\Phi^{-1}$ assigns to a point $\xi \in U$ the parallel transport in the tangent bundle $TU$ from $\xi$ back to the origin along the straight line $t \mapsto t \xi$. Since $V \cong T_\xi \cong T_0 V$, we have $\Phi^{-1}: U \to \text{GL}(V)$.

We claim that there exist homogeneous tensors $\tilde{Q}_k$ of degree $k$ (see Def. 5) such that the asymptotic expansion of $\Phi^{-1}$ at the origin satisfies

$$\Phi^{-1} \sim \sum_{k=0}^{\infty} \frac{1}{k!} \tilde{Q}_k(R^0, R^1, \cdots).$$

(20)

In order to determine the $\tilde{Q}_k$'s, let some geodesic $\gamma(t) = t \xi$ emanating from the origin be given. Recall that a *Jacobi vector field* along $\gamma$ satisfies

$$Y^2(t) = -R_\gamma(t) Y(t).$$

(21)

If the metric is given in normal coordinates, then it is well known that the linear vector field $Y(t) := ty$ is a Jacobi vector field along any straight line $\gamma(t) := t \xi$. Further, as a special case of the product rule

$$Y^{\ell+1}(0) = (\ell + 1) \nabla^{\ell+1}_\xi y(0).$$

(22)

In order to find the asymptotic expansion of $\Phi^{-1}$ at the origin, we recursively define $\tilde{Q}_k \in \mathbb{R}(X)$ through

$$\tilde{Q}_{-1} := 0, \quad \tilde{Q}_0 := 1,$$

$$\tilde{Q}_k := -\frac{1}{k(k+1)} \sum_{\ell=2}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) (\ell - 1) \ell X^\ell \tilde{Q}_{k-\ell}$$

(23)

(24)

for all $k \geq 1$. Then we have the following (see [11, Theorem 3.2]):

**Proposition 1.** Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space and $U$ be an open star-shaped neighborhood of the origin on which the metric is given in normal coordinates. The asymptotic expansion of the backwards parallel transport is given by (20).

**Proof.** Interpreting a vector $y$ as a constant vector field, it is well known that $\Phi^{-1}$ has the asymptotic expansion

$$\Phi^{-1}y(\xi) \sim \sum_{k=0}^{\infty} \frac{1}{k!} \nabla^{(k)}_{\xi} y(0).$$

(25)

Hence, we have to show that

$$\tilde{Q}_k(R, R^1, \cdots)(\xi) = \nabla_{\xi}^{(k)} y(0)$$

(26)

for any constant vector field $y$: for $k = -1, 0$ this is immediately clear. For $k \geq 1$, we proceed by induction.
Suppose the claim holds for $\ell = -1, 0, \ldots, k - 2$. Let $\gamma$ be a geodesic with $\dot{\gamma}(0) = \xi$. Then

$$k(k + 1)\nabla^k(\xi, \ldots, \xi)g(0) = kY^{k+1}(0) = k(Y^2)^{k-1}(0) \equiv k(\mathcal{R}_\gamma Y)^{k-1}(0)$$

$$= -\sum_{\ell=0}^{k-2} k \binom{k-1}{\ell} \mathcal{R}^\ell Y^{k-\ell-1}(0) \binom{k-1}{\ell} \mathcal{R}^\ell (\xi)\nabla^{k-\ell-2}(\xi, \ldots, \xi)g(0)$$

$$= -\sum_{\ell=0}^{k-2} k \binom{k}{\ell+2} (\ell+1)(\ell+2)\mathcal{R}^\ell(\xi)\nabla^{k-\ell-2}(\xi, \ldots, \xi)g(0)$$

$$\equiv -\sum_{\ell=2}^{k-2} \binom{k}{\ell} (\ell+1)\ell [X_\ell \tilde{Q}_{k-\ell}] (\mathcal{R}, \mathcal{R}^1, \ldots)(\xi) \equiv k(k+1)\tilde{Q}_k(\mathcal{R}, \mathcal{R}^1, \ldots)(\xi).$$

The result follows. \(\square\)

In order to find the explicit solution to (23), (24), it seems suitable to rescale the variables like

$$\tilde{X}_j := -\frac{1}{(j-2)!}X_j.$$

Further, there is the canonical algebra anti-isomorphism $*: \mathbb{R}(X) \to \mathbb{R}(X)$ characterized by $X^*_i = X_i$ (such that a simple tensorial expression $X_{i_1, \ldots, i_r}$ gets mapped to $X_{i_r, \ldots, i_1}$.) Further, set $\deg(f) := i_1 + \cdots + i_r$ and

$$\Pi_I := i_1(i_1 + 1)(i_1 + i_2)(i_1 + i_2 + 1)\cdots(i_1 + \cdots + i_r)(i_1 + \cdots + i_r + 1)$$

and $\tilde{X}_I := \tilde{X}_{i_1} \cdots \tilde{X}_{i_r}$. The explicit solution to (23), (24) for $k \geq 1$ is given by\(^1\)

$$\tilde{Q}_k := \sum_{\deg(f)=k} \frac{k!}{\Pi_I} \tilde{X}_I.$$

Thus, for low values of $k$,

$$\tilde{Q}_1 = 0,$$

$$\tilde{Q}_2 = -\frac{1}{3}X_2,$$

$$\tilde{Q}_3 = -\frac{1}{2}X_3,$$

$$\tilde{Q}_4 = -\frac{3}{5}X_4 + \frac{1}{5}X_2^2,$$

$$\tilde{Q}_5 = -\frac{2}{3}X_5 + \frac{1}{3}X_2X_3 + \frac{2}{3}X_3X_2.$$

\(^1\)The formula was suggest by Gregor Weingart.
2.2 Proof of Theorem 1

We aim to calculate the summands \( h_\ell \) occurring on the r.h.s. of (8). Since these are supposed to be polynomials of degree \( \ell \) which take values in \( \text{Sym}^2 V^* \), we consider the canonical algebra anti-isomorphism \( * : \mathbb{R}(X) \to \mathbb{R}(X) \) from the last section. Now we define

\[
Q_k := \sum_{\ell=0}^{k} \binom{k}{\ell} \hat{Q}_\ell \hat{Q}_{k-\ell}
\]

for each \( k \geq 0 \). Then \( Q_k^* = Q_k \) by an elementary property of the binomial coefficients. Using the explicit formula (28) for \( \hat{Q}_k \),

\[
Q_k = \sum_{\ell=0}^{k} \sum_{\text{deg}(I) = \ell} \sum_{\text{deg}(J) = k - \ell} \frac{k!}{\prod I \prod J} \hat{X}_I \hat{X}_J^*.
\]

Corollary 2.

\[
Q_{k+2} = c_k X_{k+2} + \text{terms involving only } X_2, \ldots, X_{k+1}
\]

where the constant \( c_k \) is given by \(-2^{k+1} / k+3\).

The proof follows easily by induction.

Proposition 2. On the pseudo-Euclidean vector space \((V, \langle \cdot, \cdot \rangle)\), consider the tensorial expressions \( Q_k \) defined in (29).

(a) Given a metric \((g, U)\) in normal coordinates with \( g(0) = \langle \cdot, \cdot \rangle \), let \((R^0, R^1, \ldots)\) denote the symmetrized curvature jet at the origin. Using the canonical identification of symmetric two-tensors with symmetric endomorphisms via \( \langle \cdot, \cdot \rangle \), the asymptotic expansion of \( g \) at the origin is given by (3).

(b) Conversely, given \( R^\ell \in \mathcal{N}_{\ell+2} \) for \( \ell = 0, \ldots, k \) (see (7)), there exists \( \epsilon > 0 \) such that the polynomial (3) defines a metric in normal coordinates for \( \|\xi\| < \epsilon \). Moreover, the symmetrized \( k \)-jet of the curvature tensor at the origin is actually the prescribed one.

Proof. For (a),

\[
g_\xi(y, y) = \langle \Phi^{-1}(\xi)y, \Phi^{-1}(\xi)y \rangle
\]

since the metric is parallel. Hence

\[
g_\xi = \Phi^{-1}(\xi)^* \Phi^{-1}(\xi).
\]

Now the result follows from Proposition 1 and Definition (29).

For (b), let us show by induction over \( k \) that the symmetrized \( k \)-jet of the curvature tensor of the metric defined in (3) is the prescribed one:

for \( k = -1 \), the metric \( g \) is given by the identity map and there is nothing to prove.

For the inductive step, let \( k \geq -1 \) and \( R^\ell \in \mathcal{N}_{\ell+2} \) for \( \ell = 0, \ldots, k+1 \) be given. Define a polynomial metric \( g \) of polynomial degree \( k + 1 \) via (3). By the hypothesis of induction, and since the curvature \( k \)-jet of a
metric depends only on its Taylor expansion up to order \(k+2\), the symmetrized curvature \(k\)-jet of \(g\) is given by \((R, \ldots, R^k)\). Further, by means of (31), there exists a constant \(c_{k+1} \neq 0\) depending only on \(k\) such that

\[ Q_{k+3}(R^0, \ldots, R^{k+1}) = c_{k+1} R^{k+1} + \text{terms involving only } R^0, \ldots, R^k. \]

Using Theorem 1, we thus conclude that \(R^{k+1}\) is actually the symmetrized \(k+1\)-th covariant derivative of the curvature tensor. \(\square\)

3 Proof of Theorem 2

Lemma 2. The linear space (15) is non-trivial. In fact, with \(n := \dim(V)\),

\[ \dim(C_k) \geq \frac{n(k + 1)}{2} \binom{k + n + 1}{n - 2}. \] (32)

Proof. Obviously, the linear space \(C_1(15)\) has the alternate description

\[ C_k = \text{Sym}^k V^* \otimes C_0 \cap \text{Sym}^{k-1} V^* \otimes C_1. \] (33)

Let \(\text{SL}(V; \mathbb{C})\) denote the general linear group. Thus \(\text{Sym}^{k-1} V^* \otimes V^* \otimes C_0 \otimes \mathbb{C}\), \(\text{Sym}^{k-1} V^* \otimes C_1 \otimes \mathbb{C}\) and \(\text{Sym}^k V^* \otimes C_0 \otimes \mathbb{C}\) are \(\text{SL}(V; \mathbb{C})\)-modules in a natural way, respectively. Each of the three contains a unique irreducible summand of the same highest weight \((k+2, 2)\) (cf. [6]), called Cartan summand. Thus, by uniqueness, the Cartan summand of \(\text{Sym}^{k-1} V^* \otimes V^* \otimes C_0 \otimes C\) is already contained in the intersection of \(\text{Sym}^k V^* \otimes C_0 \otimes \mathbb{C}\) and \(\text{Sym}^{k-1} V^* \otimes C_1 \otimes \mathbb{C}\). So, in particular, this intersection is non-empty. Moreover, the dimension of this summand is given by r.h.s. of (32) according to Weyls dimension formula, which yields the lower dimension bound (32). \(\square\)

Further, the map

\[ R: C_k \to N_{k+2}, \nabla^k R \mapsto R^k \] (34)

used in Theorem 2 is well defined by means of the properties of an algebraic curvature tensor. In fact, we claim that the two spaces \(C_k\) and \(N_{k+2}\) are isomorphic via \(R\). In order to find the inverse map, recall the following:

Definition 7. The Kulkarni-Nomizu product is the linear map \(\otimes: \text{Sym}^{k+2} V^* \otimes \text{Sym}^2 V^* \to \text{Sym}^k V^* \otimes \text{Sym}^2 (\Lambda^2 V^*)\) given by

\[ \otimes(h_1 \otimes h_2)(x_1, \ldots, x_k; x_a, x_b, x_c, x_d) := \begin{cases} h_1(x_1, \ldots, x_k; x_a, x_c)h_2(x_b, x_d) \\ -h_1(x_1, \ldots, x_k; x_b, x_c)h_2(x_a, x_d) \\ -h_1(x_1, \ldots, x_k; x_a, x_d)h_2(x_b, x_c) \\ +h_1(x_1, \ldots, x_k; x_b, x_d)h_2(x_a, x_c). \end{cases} \] (35)

The following result is mentioned without proof in [2, proof of Theorem 2.6 a)]:

Proposition 3. A linear algebraic curvature \(k\)-jet \((0, \ldots, 0, \nabla^k R)\) can be reconstructed from its symmetrization (34) via

\[ \nabla^k_{x_1, \ldots, x_k} R(x_a, x_b, x_c, x_d) = -\frac{k + 1}{k + 3} \otimes R^k(x_1, \ldots, x_k; x_a, x_b, x_c, x_d). \] (36)
The proof of this proposition will be given by elementary calculations in the next section. For a more sophisticated approach towards (36) see [7, p.1162] and its discussion in [4, Theorem 6.1].

Proof of Theorem 2. In order to compute the dimension of $\mathcal{N}_k$, by the existence of a short exact sequence

$$0 \to \mathcal{N}_k \to \text{Sym}^k V^* \otimes \text{Sym}^2 V^* \to \text{Sym}^{k+1} V^* \otimes V^* \to 0$$

we have $\dim(\mathcal{N}_k) = \frac{n(n+1)}{2} \left( \binom{k+n-1}{p-1} - n \binom{k+n-1}{n-1} \right)$ with $n := \dim(V)$. Thus $\dim(\mathcal{N}_1) = 0$ and a straightforward calculation shows that $\dim(\mathcal{N}_{k+2})$ is equal to the r.h.s. of (32) for any $k \geq 0$. Further, Equation 36 implies that $\mathcal{R}$ is injective on $\mathcal{C}_k$. The result follows from dimensional reasons.

3.1 Proof of (36)

For this section see also [6, Sec.2 and 4] or [7, Sec. 4]. The Young tableau

$$\begin{array}{cccccc}
1 & 3 & 5 & \cdots & k & 4 \\
2 & 4 \\
\end{array}$$

describes the following “symmetrization-antisymmetrization”-rule for a tensorial expression $p(x_1, \cdots, x_{k+4})$: first, consider the row action

$$\tilde{p}(x_1, \cdots, x_{k+4}) := \sum_{\sigma, \tau} p(x_{\sigma(1)}, x_{\tau(2)}, x_{\sigma(3)}, x_{\tau(4)}, x_{\sigma(5)}, \cdots, x_{\sigma(k+4)})$$

where the sum runs over all permutations $\sigma$ and $\tau$ of $\{1, 3, \ldots, k+4\}$ and $\{2, 4\}$, respectively. By construction, $\tilde{p}(x_1, \cdots, x_{k+4})$ is symmetric in $x_1, x_3, \cdots, x_{k+4}$ and $x_2, x_4$, respectively. The full symmetrized-antisymmetrized tensor is obtained by applying the column action to $\tilde{p}$,

$$p(x_1, \cdots, x_{k+4}) := \begin{cases}
+\tilde{p}(x_1, x_2, x_3, x_4, x_5, \cdots, x_{k+4}) \\
-\tilde{p}(x_2, x_1, x_3, x_4, x_5, \cdots, x_{k+4}) \\
-\tilde{p}(x_1, x_2, x_4, x_3, x_5, \cdots, x_{k+4}) \\
+\tilde{p}(x_2, x_1, x_4, x_3, x_5, \cdots, x_{k+4})
\end{cases}.$$ 

This tensor is antisymmetric in $x_1, x_2$ and $x_3, x_4$. Let $\nabla^k R \in \mathcal{C}_k$ be given and

$$p(x_1, \cdots, x_{k+4}) := \nabla^k_{x_5, \cdots, x_{k+4}} R(x_1, x_2, x_3, x_4).$$

Recalling the definition of the Kulkarni-Nomizu product $\oplus$, (see (35)) it follows that

$$\nabla^k_{x_5, \cdots, x_{k+4}} R(x_1, x_2, x_3, x_4) = -2(k+2)! \oplus \mathcal{R}^k(x_5, \cdots, x_{k+4}; x_1, x_2, x_3, x_4).$$

Remark 1. It is known that there exists a natural number $c_k$ such that $\frac{1}{c_k}$ times the given Young symmetrizer is a projector. This number is equal to the product of the hook-numbers of the various boxes of the Young frame, see [6]. Hence, for a Young frame of shape $(k, 2)$,

$$c_k = (k + 3) \cdot (k + 2) \cdot k! \cdot 2 \cdot 1.$$ 

For example, we have $c_0 = 3 \cdot 2 \cdot 2 \cdot 1 = 12$, $c_1 = 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 24$ and $c_2 = 5 \cdot 4 \cdot 2 \cdot 2 \cdot 1 = 80$. Dividing $c_k$ by the factor $2(k+2)!$ this explains the occurrence of the fraction $\frac{k+3}{k+1}$ in (36).
We will now show that
\[
\nabla_{x_k, \ldots, x_{k+4}}^k R(x_1, x_2, x_3, x_4) = c_k \nabla_{x_5, \ldots, x_{k+4}}^k R(x_1, x_2, x_3, x_4). \tag{37}
\]

The cases \(k = 0\) and \(k = 1\) are handled separately. For \(k \geq 2\), calculations reduce to the previous two cases and Lemma 4.

For \(k = 0\), we claim that
\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} R(x_1, x_2, x_3, x_4) = 12 R(x_1, x_2, x_3, x_4)
\]
for any \(R \in \mathcal{C}_0\):

\textbf{Proof.} By definition of our Young symmetrizer
\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} R(x_1, x_2, x_3, x_4)
\]
\[
= R(x_1, x_2, x_3, x_4) + R(x_3, x_2, x_1, x_4) + R(x_1, x_4, x_3, x_2) + R(x_3, x_4, x_1, x_2)
\]
\[
- R(x_2, x_1, x_3, x_4) - R(x_3, x_1, x_2, x_4) - R(x_2, x_4, x_3, x_1)
\]
\[
- R(x_1, x_2, x_4, x_3) - R(x_4, x_2, x_1, x_3) - R(x_1, x_3, x_4, x_2)
\]
\[
+ R(x_2, x_1, x_4, x_3) + R(x_4, x_1, x_2, x_3) + R(x_2, x_3, x_4, x_1)
\]
\[
= 4 R(x_1, x_2, x_3, x_4) + 2 R(x_4, x_3, x_2, x_1) + 2 R(x_2, x_1, x_4, x_3) + 4 R(x_3, x_4, x_1, x_2),
\]
where we used the first Bianchi identity or its alternative version \(\sigma_{x_1, x_2, x_3} R(x_4, x_1, x_2, x_3) = 0\). Other symmetries of the curvature tensor give the claim.

For \(k = 1\), we need the following consideration:

\textbf{Lemma 3.} It holds that
\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} \nabla_{x_1} R(x_3, x_2, x_5, x_4) = \begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} \nabla_{x_3} R(x_5, x_2, x_1, x_4) = 6 \nabla_{x_5} R(x_1, x_2, x_3, x_4)
\]
\textit{(the variable \(x_5\) is kept fixed.)}

\textbf{Proof.} By means of pair symmetry,
\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} \nabla_{x_1} R(x_3, x_2, x_5, x_4) = \begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} \nabla_{x_1} R(x_5, x_2, x_3, x_4).
\]
Further, using the first Bianchi identity,
\[
\begin{bmatrix}
1 & 3 \\
2 & 4
\end{bmatrix} \nabla_{x_1} R(x_3, x_2, x_5, x_4)
\]
\[
= \nabla_{x_1} R(x_3, x_2, x_5, x_4) + \nabla_{x_1} R(x_1, x_2, x_5, x_4) + \nabla_{x_1} R(x_3, x_4, x_5, x_2) + \nabla_{x_1} R(x_1, x_4, x_5, x_2)
\]
\[
- \nabla_{x_2} R(x_3, x_1, x_5, x_4) - \nabla_{x_2} R(x_2, x_1, x_5, x_4) - \nabla_{x_2} R(x_3, x_4, x_5, x_1) - \nabla_{x_2} R(x_2, x_4, x_5, x_1)
\]
\[
- \nabla_{x_3} R(x_4, x_2, x_5, x_3) - \nabla_{x_3} R(x_1, x_2, x_5, x_3) - \nabla_{x_3} R(x_4, x_3, x_5, x_2) - \nabla_{x_3} R(x_1, x_3, x_5, x_2)
\]
\[
+ \nabla_{x_4} R(x_4, x_1, x_5, x_3) + \nabla_{x_4} R(x_2, x_1, x_5, x_3) + \nabla_{x_4} R(x_4, x_3, x_5, x_1) + \nabla_{x_4} R(x_2, x_3, x_5, x_1)
\]
\[
= + \nabla_{x_1} R(x_3, x_2, x_5, x_4) + 2 \nabla_{x_3} R(x_1, x_2, x_5, x_4) + 2 \nabla_{x_1} R(x_3, x_4, x_5, x_2) + 2 \nabla_{x_3} R(x_1, x_4, x_5, x_2)
\]
\[
+ \nabla_{x_2} R(x_4, x_3, x_5, x_1) + 2 \nabla_{x_4} R(x_2, x_1, x_5, x_3) + 2 \nabla_{x_2} R(x_4, x_3, x_5, x_1) + \nabla_{x_4} R(x_2, x_1, x_5, x_3).
\]
Making use of the second Bianchi identity this is equal to
\[
3 \nabla_{x_3} R(x_1, x_2, x_5, x_4) + 3 \nabla_{x_4} R(x_2, x_1, x_5, x_3) + 3 \nabla_{x_1} R(x_3, x_4, x_5, x_2) + 3 \nabla_{x_2} R(x_4, x_3, x_5, x_1) + 3 \nabla_{x_5} R(x_1, x_2, x_3, x_4),
\]

The result follows by means of further symmetries of an algebraic covariant derivative of a curvature tensor.

Now we are ready to handle the case \( k = 1 \):
\[
\nabla R \in C_1 : \sigma_{12345} \nabla_{x_5} R(x_1, x_2, x_3, x_4) = 24 \nabla_{x_5} R(x_1, x_2, x_3, x_4).
\]

**Proof.** In order to understand the row-actions, we recall that the symmetric group of \( \{1, 3, 5\} \) has six elements. We divide these into the fix group of \( \{5\} \) (which is given by the symmetric group of \( \{1, 3\} \)) and the pairs \( \{(1, 5), (1, 3, 5)\} \) and \( \{(1, 3), (5, 3, 1)\} \) (each consisting of a proper transposition and a proper three-cycle). Then
\[
\begin{aligned}
\nabla_{x_5} R(x_1, x_2, x_3, x_4) &= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \\ 3 \end{bmatrix} \nabla_{x_5} R(x_1, x_2, x_3, x_4) + \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 \end{bmatrix} \nabla_{x_1} R(x_3, x_2, x_5, x_4) \\
&+ \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 \end{bmatrix} \nabla_{x_3} R(x_5, x_2, x_1, x_4).
\end{aligned}
\]

Thus, using the case \( k = 0 \) and Lemma 3,
\[
\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \\ 3 \end{bmatrix} \nabla_{x_5} R(x_1, x_2, x_3, x_4) = (12 + 6 + 6) \nabla_{x_5} R(x_1, x_2, x_3, x_4)
\]

The case \( k = 2 \) is preceded by the following consideration: consider the permutations of \( \{1, 3, 5, 6\} \) given by \( (15)(36) \) and \( (16)(35) \) (decomposed as cycles).

**Lemma 4.** For all \( \nabla^2 R \in C_2 \):
\[
\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \\ 3 \end{bmatrix} \nabla^2_{x_1, x_3} R(x_5, x_2, x_6, x_4) = 4 \nabla^2_{x_3, x_6} R(x_1, x_2, x_3, x_4).
\]

**Proof.** L.h.s. of (41) is given by
\[
\begin{aligned}
2 \nabla^2_{x_1, x_3} R(x_5, x_2, x_6, x_4) + & 2 \nabla^2_{x_1, x_3} R(x_5, x_1, x_6, x_4) + \quad & 2 \nabla^2_{x_1, x_3} R(x_5, x_3, x_6, x_2) \\
\quad & 2 \nabla^2_{x_2, x_3} R(x_5, x_1, x_6, x_4) - & 2 \nabla^2_{x_2, x_3} R(x_5, x_4, x_6, x_1) \\
\quad & 2 \nabla^2_{x_1, x_3} R(x_5, x_2, x_6, x_3) - & 2 \nabla^2_{x_1, x_4} R(x_5, x_3, x_6, x_2) \\
\quad & 2 \nabla^2_{x_2, x_4} R(x_5, x_1, x_6, x_3) + & 2 \nabla^2_{x_2, x_4} R(x_5, x_3, x_6, x_1).
\end{aligned}
\]

Applying either the second Bianchi identity or its alternative version \( \sigma_{x_1, x_3, x_5} \nabla_{x_1} R(x_2, x_4, x_3, x_5) = 0 \) to any two summands which appear at the same horizontal positions in line one and three or two and four, this
Similarly, using once more the trivial Ricci identity, this gives the claimed result. We are now able to handle the case gives
\[ 2 \nabla_{x_1,x_5}^2 R(x_5,x_2,x_3,x_4) + 2 \nabla_{x_1,x_5}^2 R(x_3,x_4,x_6,x_2) + 2 \nabla_{x_2,x_5}^2 R(x_5,x_1,x_4,x_3) + 2 \nabla_{x_2,x_5}^2 R(x_4,x_3,x_6,x_1) \]
\[ = 2 \nabla_{x_1,x_5}^2 R(x_5,x_2,x_3,x_4) - 2 \nabla_{x_1,x_5}^2 R(x_2,x_6,x_3,x_4) - 2 \nabla_{x_2,x_5}^2 R(x_5,x_1,x_3,x_4) + 2 \nabla_{x_2,x_5}^2 R(x_1,x_6,x_3,x_4) . \]

Using the trivial Ricci identity \( \nabla_{x,y}^2 R = \nabla_{y,x}^2 R \) and the second Bianchi identity,
\[ 2 \nabla_{x_1,x_5}^2 R(x_5,x_2,x_3,x_4) - 2 \nabla_{x_2,x_5}^2 R(x_5,x_1,x_3,x_4) = 2 \nabla_{x_1,x_5}^2 R(x_5,x_2,x_3,x_4) \]
\[ = 2 \nabla_{x_1,x_5}^2 R(x_1,x_2,x_3,x_4) . \]

Similarly,
\[ -2 \nabla_{x_1,x_5}^2 R(x_2,x_6,x_3,x_4) + 2 \nabla_{x_2,x_5}^2 R(x_1,x_6,x_3,x_4) = 2 \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) \]

Using once more the trivial Ricci identity, this gives the claimed result.

We are now able to handle the case \( k = 2 \). Thus, we claim that
\[ \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 \end{bmatrix} \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) = 80 \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) . \]

Proof. We will reduce the calculation to the cases \( k = 0, 1 \) and Lemma 4:

for this, let \( S_{\{1,3,5,6\}} \) denote the symmetric group of \( \{1,3,5,6\} \). Further, let \( S_{\{5,6\}} \) denote the symmetric group of \( \{5,6\} \). Since \( \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) \) is already symmetric in \( x_5, x_6 \), complete symmetrization in \( x_1, x_3, x_5, x_6 \) factorizes over the right Cosets \( S_{\{5,6\}} \setminus S_{\{1,3,5,6\}} \). We may choose a set of representatives as follows:

the fixed-point groups of 5 and 6 are given by the symmetric groups \( S_{\{1,3,6\}} \) and \( S_{\{1,3,5\}} \), respectively. Their intersection is the symmetric group of \( \{1,3\} \) denoted by \( S_{\{1,3\}} \). This yields two times six minus two equal to ten different elements in \( S_{\{5,6\}} \setminus S_{\{1,3,5,6\}} \). Further, the remaining two right Cosets are represented by \( (15)(36) \) and \( (16)(35) \). In the free vector space over \( S_{\{5,6\}} \setminus S_{\{1,3,5,6\}} \)
\[ \sum_{[\pi] \in S_{\{5,6\}} \setminus S_{\{1,3,5,6\}}} [\pi] = \sum_{\pi \in S_{\{1,3,6\}}} [\pi] + \sum_{\pi \in S_{\{1,4,5\}}} [\pi] - \sum_{\pi \in S_{\{1,3\}}} [\pi] + [(15)(36)] + [(16)(35)] . \]

Hence, using the previous cases \( k = 0 \) and \( k = 1 \) together with Lemma 4,
\[
\begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 \end{bmatrix} \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) \\
= ( \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} ) \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1,x_3}^2 R(x_5,x_2,x_6,x_4) \\
= (24 + 24 - 12 + 4) \nabla_{x_5,x_6}^2 R(x_1,x_2,x_3,x_4) \\
= 40 .
\]

The claimed result (36) for \( k = 2 \) follows.
The general case \( k \geq 2 \) follows on the analogy of the case \( k = 2 \):

Proof. Let \( S_{1,3,5,\ldots,k+4} \) denote the symmetric group of \( \{1, 3, 5, \cdots, k + 4\} \). By invariance under a permutation of \( x_5, \ldots, x_{k+4} \), the action of \( S_{1,3,5,\ldots,k+4} \) factorizes over the space of right Cosets \( S_{1,3,5,\ldots,k+4} \setminus S_{5,\ldots,k+4} \). Similar as before, consider the the symmetric groups of \( \{1, 3, i\} \) denoted by \( S_{1,3,i} \) for \( i = 5, \ldots, k+4 \). Then \( S_{1,3,5,\ldots,k+4} \cap S_{1,3,i} \) is the symmetric group \( S_{1,3,i} \) for \( i \neq j \). Together with the \( k(k-1) \) permutations \( (1 i)(3 j) \) for \( 5 \leq i \neq j \leq k+4 \) we thus obtain \( 6 k - 2(k-1) + k(k-1) = k^2 + 3k + 2 = (k+2)(k+1) \) different elements of \( S_{1,3,5,\ldots,k+4} \) exhausting the space of right Cosets \( S_{5,\ldots,k+4} \setminus S_{1,3,5,\ldots,k+4} \). In the free vector space over this right Coset space

\[
\sum_{[\pi] \in S_{5,\ldots,k+4} \setminus S_{1,3,5,\ldots,k+4}} [\pi] = \sum_{i=5}^{k+4} \sum_{\pi \in S_{1,2,i}} [\pi] + \sum_{i \neq j=5}^{k+4} [(1 i)(2 j)] - (k - 1) \sum_{\pi \in S_{1,2}} [\pi].
\]

The same arguments as for \( k = 2 \) give

\[
\frac{1}{k!} \begin{bmatrix} 1 & 1 & 3 & \cdots & k & 4 \\ 2 & 4 & \end{bmatrix} \nabla^{k+2}_{x_5,\ldots,x_{k+4}} R(x_1, x_2, x_3, x_4) \\
= ( \sum_{j=5}^{k+4} \begin{bmatrix} 1 & 3 \\ 2 & j \end{bmatrix} - (k - 1) \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} ) \nabla^{k+2}_{x_5,\ldots,x_{k+4}} R(x_1, x_2, x_3, x_4) \\
- \sum_{5 \leq i < j}^{k+4} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla^{k+2}_{x_1,x_3,x_5,\ldots,x_i,\ldots,x_j,\ldots,x_{k+4}} R(x_i, x_2, x_j, x_4) \\
= \frac{24 k - 12(k - 1) + 4 k(k - 1)}{2} \nabla^{k+2}_{x_5,\ldots,x_{k+4}} R(x_1, x_2, x_3, x_4).
\]

This is the claimed result (36) for any \( k \geq 2 \).

\(\square\)

4 Proof of Theorem 3

Proof. We do the proof by induction over \( k \geq -1 \):

thus for \( k = -1 \), there is nothing to prove. For the inductive step, suppose the claim holds for \( k - 1 \). In particular, given an algebraic curvature \( k - 1 \)-jet, it is induced by some metric by means of induction hypothesis. Thus there exists some algebraic curvature \( k \)-jet extending the given \( k - 1 \)-jet. Part (a) of Theorem 3 now follows immediately from the definition of a curvature jet in combination with (33).

For Part (b), we claim that \( Q, R \) and \( R \) are affine isomorphisms:

Using Theorem 2 and that \( Q \circ R \circ R \) is the identity map according to Theorem 1, we thus see that the associated fiberwise linearized maps \( R_L, R_L \) and \( Q_L \) are the linear isomorphisms

\[
R_L : \mathcal{N}_{k+2} \to \mathcal{C}_k, \\
R_L(h_{k+2}) = -\frac{(k+2)!}{2} \otimes h_{k+2}, \\
R_L : \mathcal{C}_k \to \mathcal{N}_{k+2}, \\
R_L(\nabla^k) = \mathcal{R}^k, \\
Q_L : \mathcal{N}_{k+2} \to \mathcal{N}_{k+2}, \\
Q_L(\mathcal{R}^k) = -2 \frac{k+1}{(k+3)!} \mathcal{R}^k.
\]
respectively. Thus, the fiberwise affine maps $R$, $\mathcal{R}$ and $Q$ are isomorphisms, too (since an affine map is an isomorphism if and only if its linear part is an isomorphism.) This finishes the proof.

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