Nonsurjective epimorphisms in decomposable varieties of groups
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Abstract. A full characterization of when a subgroup $H$ of a group $G$ in a varietal product $N \mathcal{Q}$ is epimorphically embedded in $G$ (in the variety $N \mathcal{Q}$) is given. From this, a result of S. McKay is derived, which states that if $N \mathcal{Q}$ has instances of nonsurjective epimorphisms, then $N$ also has instances of nonsurjective epimorphisms. Two partial converses to McKay’s result are also given: when $G$ is a finite nonabelian simple group; and when $G$ is finite and $\mathcal{Q}$ is a product of varieties of nilpotent groups, each of which contains the infinite cyclic group.

Section 1. Introduction and notation

Given a category $\mathcal{C}$, a map $f: G \to K$ in $\mathcal{C}$ is an epimorphism if and only if it is right cancellable. When $\mathcal{C}$ is a full subcategory of the category of all algebras (in the sense of Universal Algebra) of a given type, it is not hard to verify that if $f$ is surjective, then it is an epimorphism. The converse, however, does not necessarily hold. For example, the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category of rings, but it is not surjective.

On the other hand, it is known that in the category of all groups, epimorphisms are surjective; for an elementary proof of this fact we direct the reader to [6]. Peter Neumann proved [15] that in a full subcategory of $\mathcal{G}roup$ in which all objects are solvable groups, and which is closed under taking quotients and subgroups, all epimorphisms are surjective. Susan McKay [7] later extended this result. On the other hand, an example of B.H. Neumann that appears in [15] shows that there are varieties of groups where there are nonsurjective epimorphisms. Specifically, the embedding $A_{4} \hookrightarrow A_{5}$ is an epimorphism in $\text{Var}(A_{5})$. For other examples of nonsurjective epimorphisms in varieties of groups, we direct the reader to [10] and [9].

In this work we study the question of when a varietal product $N \mathcal{Q}$, where $N$ and $\mathcal{Q}$ are varieties of groups, has instances of nonsurjective epimorphisms.

Isbell introduced the concept of dominions in [2] to study epimorphisms. Given a category $\mathcal{C}$ as above, and an algebra $A \in \mathcal{C}$, Isbell defines for a subalgebra $B$ of $A$ the dominion of $B$ in $A$ (in the category $\mathcal{C}$) as the intersection of all equalizer subalgebras of $A$ containing $B$. Explicitly,

$$\text{dom}_{\mathcal{C}}(B) = \{ a \in A \mid \forall C \in \mathcal{C}, \forall f, g: A \to C, \text{ if } f|_{B} = g|_{B} \text{ then } f(a) = g(a) \}. \tag{1}$$

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Note that given an arbitrary morphism of algebras \( \varphi: A' \to A \), \( \varphi \) is an epimorphism in \( C \) if and only if \( \text{dom}_A^C(\varphi(A')) = A \). Also, in a variety of algebras (and in any full category of algebras which is closed under subalgebras and quotients) we can factor \( \varphi \) into a surjection \( A' \to \varphi(A') \) and an embedding \( \varphi(A') \hookrightarrow A \). The description of surjective maps by means of congruence relations is well developed, so we may reduce the study of epimorphisms (at least in certain categories) to the study of dominions.

For the basic properties of dominions in varieties of groups we refer the reader to [11]. We recall the most important properties: \( \text{dom}^V_G(\cdot) \) is a closure operator on the lattice of subgroups of \( G \); the dominion construction respects finite direct products; and the dominion construction respects quotients. That is, if \( H \) is a subgroup of \( G \in \mathcal{V} \), where \( \mathcal{V} \) is a variety, and \( N \triangleleft G \), with \( N \) contained in \( H \), then

\[
\text{dom}^V_{G/N}(H/N) = \text{dom}^V_G(H) / N.
\]

Groups will be written multiplicatively unless otherwise stated. Given a group \( G \), the identity element of \( G \) will be written \( e_G \), although we will omit the subscript if it is understood from context. All maps will be assumed to be group morphisms unless otherwise specified. Given a group \( G \) and a subgroup \( H \), \( N_G(H) \) denotes the normalizer of \( H \) in \( G \); that is, the subgroup of all elements \( g \in G \) such that \( g^{-1}Hg = H \).

Given two groups, \( A \) and \( B \), we write \( A \wr B \) to denote the regular wreath product of \( A \) and \( B \); this is the semidirect product of \(|B|\)-copies of \( A \) (indexed by the elements of \( B \)) by \( B \), with \( B \) acting on the index set via the regular right action. The elements of \( A \wr B \) are written as \( b\phi \), where \( b \in B \) and \( \phi: B \to A \) is a set-theoretic function (that is, an element of \( A^B \)).

Recall that a variety of groups is a full subcategory of \( \text{Group} \) which is closed under taking quotients, subgroups, and arbitrary direct products. For the basic properties and facts about varieties of groups, we direct the reader to Hanna Neumann’s excellent book [13]. We will denote the variety of all groups by \( \mathcal{G} \), and the variety consisting only of the trivial group by \( \mathcal{E} \).

Given two varieties of groups \( \mathcal{N} \) and \( \mathcal{Q} \), their product variety \( \mathcal{N} \mathcal{Q} \) is the variety of all groups which are an extension of an \( \mathcal{N} \)-group by a \( \mathcal{Q} \)-group; that is, all groups \( G \) with a normal subgroup \( N \triangleleft G \) such that \( N \in \mathcal{N} \) and \( G/N \in \mathcal{Q} \). The product \( \mathcal{N} \mathcal{Q} \) is easily seen to be a variety, say by using Birkhoff’s HSP theorem [1]. Multiplication of varieties is associative.

The semigroup of varieties of groups has the structure of a cancellation semigroup with 0 and 1. The zero element is the variety of all groups, while the identity is the trivial variety, consisting only of the trivial group. We will say that a variety \( \mathcal{V} \) is nontrivial iff \( \mathcal{V} \neq \mathcal{G} \) and \( \mathcal{V} \neq \mathcal{E} \), and we will call it trivial otherwise. A variety \( \mathcal{V} \) factors nontrivially (or is decomposable) if it can be expressed as the product of two nontrivial varieties.
Furthermore, every variety other than $\mathcal{G}$ can be uniquely factored as a product of a finite number of indecomposable varieties (in a unique order), with $\mathcal{E}$ having the empty factorization, so that the semigroup with neutral element of varieties other than $\mathcal{G}$ is freely generated by the indecomposable varieties. See Theorems 21.72, 23.32 and 23.4 in [13].

Given a variety $\mathcal{V}$ and a group $G$ (not necessarily in $\mathcal{V}$), we will denote by $\mathcal{V}(G)$ the verbal subgroup of $G$ associated to $\mathcal{V}$. This is the fully invariant subgroup generated by all values of the words $v$ which are laws of $\mathcal{V}$. In particular, $G \in \mathcal{V}$ if and only if $\mathcal{V}(G) = \{e\}$. We also note the universal property associated to $\mathcal{V}(G)$: for any normal subgroup $N \triangleleft G$, $G/N \in \mathcal{V}$ if and only if $\mathcal{V}(G) \subseteq N$.

In Section 2 we will recall the result of Peter M. Neumann mentioned above. We will also recall a result from [8], which gives upper and lower bounds for the dominion of a subgroup in a decomposable variety. Next, in Section 3 we will characterize when a subgroup $H$ of $G$ is epimorphically embedded in $G$ in the variety $\mathcal{N}\mathcal{Q}$ in terms of the epimorphisms of $\mathcal{N}$, the laws of $\mathcal{Q}$, and the internal structure of $G$. Theorem 3.10 is the main result of this work. Finally, in Section 4 and Section 5 we will prove two partial converses to McKay’s theorem.

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**Section 2. Preliminary results**

In this section we generalize a theorem of P.M. Neumann about nonsurjective epimorphisms in certain classes of groups. The proof follows Neumann’s proof very closely.

**Theorem 2.1.** *(P.M. Neumann [15])* Let $\mathcal{X}$ be a full subcategory of Group, where $\text{Ob}(\mathcal{X})$ is closed under taking quotients and subgroups of objects of $\mathcal{X}$. Let $G \in \mathcal{X}$, and let $H$ be a proper subgroup of $G$. Suppose that there is a normal subgroup $N \triangleleft G$ such that $N$ is solvable and $NH = G$. Then

$$\text{dom}_{G}^{\mathcal{X}}(H) \subsetneq G.$$ 

**Proof:** Let $G \in \mathcal{X}$, and let $H$ be a proper subgroup of $G$. Let

$$\{e\} = N_{m} \triangleleft N_{m-1} \triangleleft \cdots \triangleleft N_{0} = N$$

be a normal series for $N$ such that $N_{i} \triangleleft G$ for each $i$, and $N_{i}/N_{i+1}$ is abelian for $i = 0, 1, \ldots, m - 1$ (we can obtain such a series by letting $N_{i}$ be the $i$-th derived subgroup of $N$).
Since \( N_{m}H = H \not\subseteq G \), and \( N_{0}H = NH = G \), there exists \( k \) such that \( N_{k+1}H \not\subseteq G \) and \( N_{k}H = G \). It will suffice to show that \( \text{dom}_{G}^{X}(N_{k+1}H) \not\subseteq G \).

Let \( H^{*} = N_{k+1}H \) and let \( M = N_{k} \cap H^{*} \). Then \( M \triangleleft H^{*} \), since \( N_{k} \triangleleft G \); and \( M \triangleleft N_{k} \), because \( N_{k+1} \subseteq M \), and so \( M \) corresponds to subgroup of \( N_{k}/N_{k+1} \), which is abelian. Therefore, \( M \triangleleft N_{k}H^{*} = G \).

By assumption, \( G/M \in \mathcal{X} \). Let \( \theta: G \to G/M \) be the quotient map. Then \( G/M \) is a semidirect product of \( \theta(N_{k}) \) by \( \theta(H^{*}) \), because \( \theta(N_{k}) \triangleleft \theta(G) = G/M \), and the kernel is contained in both \( H^{*} \) and \( N_{k} \), so

\[
\theta(H^{*}) \cap \theta(N_{k}) = \theta(H^{*} \cap N_{k}) = \theta(M) = \{e\}.
\]

Finally, note that \( \theta(H^{*})\theta(N_{k}) = \theta(H^{*}N_{k}) = \theta(G) = G/M \).

If we now compare the map \( \theta \) with the map obtained by composing \( \theta \) with the idempotent endomorphism of \( G/M \) with kernel \( \theta(N_{k}) \) and image \( \theta(H^{*}) \), we see that \( \text{dom}_{G}^{X}(H^{*}) = H^{*} \), and therefore we conclude that \( \text{dom}_{G}^{X}(H) \subseteq H^{*} \not\subseteq G \), as claimed.

**Corollary 2.2.** (P.M. Neumann [15]) Let \( \mathcal{X} \) be a full subcategory of \( \text{Group} \), where all objects in \( \mathcal{X} \) are solvable groups, and \( \text{Ob}(\mathcal{X}) \) is closed under taking quotients and subgroups of objects of \( \mathcal{X} \). Then all epimorphisms in \( \mathcal{X} \) are surjective.

**Corollary 2.3.** If \( \mathcal{V} \) is a variety of solvable groups, then all epimorphisms are surjective in \( \mathcal{V} \).

**Theorem 2.4.** (S. McKay [7]) Let \( \mathcal{V} = NQ \) be a variety, where \( N \) is a nontrivial variety. Let \( G \in \mathcal{V} \), and let \( N \triangleleft G \), with \( N \in N \) and \( G/N \in Q \). Then for all subgroups \( H \) of \( G \), \( \text{dom}_{G}^{\mathcal{V}}(H) \subseteq NH \). In particular,

\[
\text{dom}_{G}^{\mathcal{V}}(H) \subseteq Q(G)H.
\]

Finally, we recall the following result:

**Theorem 2.5.** (Theorem 3.12, [8]) Let \( \mathcal{V} = NQ \) be a nontrivial factorization of \( \mathcal{V} \), and let \( G \in \mathcal{V} \). Let \( H \) be a subgroup of \( G \). If \( D = \text{dom}_{G}^{\mathcal{V}}(Q(G) \cap H) \), then

\[
\langle H, D \rangle = HD \subseteq \text{dom}_{G}^{NQ}(H).
\]

Furthermore, if \( N_{G}(D)Q(G) = G \), then

\[
\text{dom}_{G}^{NQ}(H) = HD
\]

and \( \text{dom}_{G}^{NQ}(H) \cap Q(G) = D \).
Recall that two varieties $\mathcal{N}$ and $\mathcal{Q}$ are disjoint if and only if $\mathcal{N} \cap \mathcal{Q} = \mathcal{E}$.

**Corollary 2.6.** Let $\mathcal{V} = \mathcal{N} \mathcal{Q}$, where $\mathcal{N}$ and $\mathcal{Q}$ are disjoint nontrivial varieties of groups. Let $G \in \mathcal{N}$, and let $H$ be a subgroup of $G$. Then
\[
\text{dom}^\mathcal{N} \mathcal{Q}_G(H) = \text{dom}^\mathcal{N}_G(H).
\]

Recall that B.H. Neumann proved that the embedding $A_4 \hookrightarrow A_5$ is an epimorphism in the variety $\text{Var}(A_5)$ (see Example A in [15]); since there are uncountably many varieties of groups disjoint from $\text{Var}(A_5)$, it follows from Corollary 2.6 that there are uncountably many varieties of groups which contain instances of nonsurjective epimorphisms (see Theorem 4.27 in [8]).

**Section 3. Nonsurjective epimorphisms in decomposable varieties**

What we can say about nonsurjective epimorphisms in $\mathcal{N} \mathcal{Q}$ if we drop the requirement that $\mathcal{N} \cap \mathcal{Q} = \mathcal{E}$?

**Theorem 3.7.** Let $\mathcal{V} = \mathcal{N} \mathcal{Q}$ be a variety of groups, with $\mathcal{N}$ and $\mathcal{Q}$ nontrivial. Let $G \in \mathcal{V}$, and let $H$ be a subgroup of $G$. Then $\text{dom}^\mathcal{N} \mathcal{Q}_G(H) = G$ if and only if $H \mathcal{Q}(G) = G$ and $\text{dom}^\mathcal{N} \mathcal{Q}_{\mathcal{Q}(G)}(H \cap \mathcal{Q}(G)) = \mathcal{Q}(G)$.

**Proof:** Write $D = \text{dom}^\mathcal{N}_\mathcal{Q}(G)(H \cap \mathcal{Q}(G))$. We first prove the “if” part.

In this case we have
\[
G = H \mathcal{Q}(G) \quad \text{(by hypothesis)}
= HD \quad \text{(by hypothesis)}
\subseteq \text{dom}^\mathcal{N} \mathcal{Q}_G(H) \quad \text{(by Theorem 2.5)}
\subseteq G.
\]

Therefore, $G = \text{dom}^\mathcal{N} \mathcal{Q}_G(H)$, as claimed.

For the converse, first note that $\text{dom}^\mathcal{N} \mathcal{Q}_G(H) \subseteq H \mathcal{Q}(G)$, by Theorem 2.4. Therefore, $H \mathcal{Q}(G) = G$. Since $H$ normalizes itself, and everything normalizes $\mathcal{Q}(G)$, $H$ normalizes $H \cap \mathcal{Q}(G)$. Since the dominion construction respects automorphisms, it follows that $H$ must normalize $D$. We conclude that $H \subseteq N_G(D)$, and hence it follows that $N_G(D) \mathcal{Q}(G) = G$. Applying Theorem 2.5 we have
\[
\text{dom}^\mathcal{N} \mathcal{Q}_G(H) = \langle H, D \rangle = HD.
\]

We also note that $\text{dom}^\mathcal{N} \mathcal{Q}_G(H) \cap \mathcal{Q}(G) = D$. Since $\text{dom}^\mathcal{N} \mathcal{Q}_G(H) = G$ by hypothesis, $D = \mathcal{Q}(G)$, as claimed.

Since $\psi: H \to G$ is an epimorphism in the variety $\mathcal{V} = \mathcal{N} \mathcal{Q}$ if and only if
\[
\text{dom}^\mathcal{V}_G(\psi(H)) = G,
\]

5
Theorem 3.7 tells us for which groups $G \in \mathcal{N}$ there is a nonsurjective epimorphism with codomain $G$, in terms of the varieties $\mathcal{N}$ and $\mathcal{Q}$, and the structure of the group $G$. We also get the following corollary:

**Corollary 3.8.** (S. McKay [7]) Let $\mathcal{V} = \mathcal{N} \mathcal{Q}$ be a variety with $\mathcal{N}$ and $\mathcal{Q}$ nontrivial. If $\mathcal{V}$ has instances of nonsurjective epimorphisms then $\mathcal{N}$ has instances of nonsurjective epimorphisms.

**Proof:** By Theorem 3.7, if a subgroup $H$ of $G \in \mathcal{V}$ is epimorphically embedded in $G$ then $H \cap \mathcal{Q}(G)$ is epimorphically embedded (in $\mathcal{N}$) in $\mathcal{Q}(G)$. If $H$ contains $\mathcal{Q}(G)$, then $H = H \mathcal{Q}(G) = G$. Therefore, if $H$ is a proper subgroup of $G$, we must have $H \cap \mathcal{Q}(G) \not\subseteq \mathcal{Q}(G)$, so the embedding $H \cap \mathcal{Q}(G) \hookrightarrow \mathcal{Q}(G)$ is a nonsurjective epimorphism in the variety $\mathcal{N}$. \hfill $\square$

In a way, Theorem 3.7 tells us that the existence of nonsurjective epimorphisms is determined by the indecomposable varieties. For given a variety $\mathcal{V}$ and a group $G \in \mathcal{V}$, in order to find out if a subgroup $H$ of $G$ is epimorphically embedded in $G$ we only need to do the following: factor $\mathcal{V}$ into a product of indecomposable varieties $\mathcal{V}_1 \mathcal{V}_2 \cdots \mathcal{V}_n$, and test to see whether $H \mathcal{V}_2 \cdots \mathcal{V}_n(G) = G$ and $H \cap \mathcal{V}_2 \cdots \mathcal{V}_n(G)$ is epimorphically embedded in $\mathcal{V}_2 \cdots \mathcal{V}_n(G)$ in the variety $\mathcal{V}_1$.

On the other hand, carrying out these calculations may not be a trivial matter. Specifically, Kleıman has established that there does not exist an algorithm that determines whether an arbitrary finitely based variety decomposes into a product of two varieties (see Theorem 4.3 and introductory comments in [5]).

Before continuing, we give a variant of Theorem 3.7.

**Lemma 3.9.** Let $\mathcal{V}$ be a variety of groups, $G \in \mathcal{V}$ and $H$ a proper subgroup of $G$ such that $\text{dom}_{\mathcal{V}}^G(H) = G$. Let $N \lhd G$ and suppose that $\mathcal{A}$, $\mathcal{B}$ are two varieties such that $N \in \mathcal{A}$, $G/N \in \mathcal{B}$, and $\mathcal{A}\mathcal{B} \subseteq \mathcal{V}$. Then $HN = G$ and $\text{dom}_{\mathcal{A}}^G(H \cap N) = N$. In particular, $\mathcal{A}$ has instances of nonsurjective epimorphisms.

**Proof:** Since $G \in \mathcal{A}\mathcal{B} \subseteq \mathcal{V}$, it follows that $\text{dom}_{\mathcal{A}\mathcal{B}}^G(H) = G$. By Theorem 3.7, we must have $HB(G) = G$ and $\text{dom}_{\mathcal{B}(G)}^\mathcal{A}(H \cap B(G)) = B(G)$.

Since $G/N \in \mathcal{B}$, we also have $B(G) \subseteq N$. Hence

$$G = HB(G) \subseteq HN \quad \text{(since } B(G) \subseteq N \text{)} \subseteq G.$$ 

Therefore, $HN = G$, as claimed.

Since $B(G) \subseteq N$, and $\text{dom}_{\mathcal{B}(G)}^\mathcal{A}(H \cap B(G)) = B(G)$, it follows that

$$\langle H \cap N, B(G) \rangle = (H \cap N)B(G) \subseteq \text{dom}_{\mathcal{N}}^\mathcal{A}(H \cap N).$$
We claim that in fact $N = (H \cap N)B(G)$, which will prove the lemma. Indeed, let $n \in N$. Since $HB(G) = G$, there exist $h \in H$ and $b \in B(G)$ such that $hb = n$. Therefore, $h = nb^{-1}$. Since $B(G) \subseteq N$, it follows that $h \in N$. Therefore, $n \in (H \cap N)B(G)$, as claimed.

From Theorem 3.7 and Lemma 3.9, we conclude our main result:

**Theorem 3.10.** Let $\mathcal{V} = \mathcal{NQ}$ be a variety with $\mathcal{N}$ and $\mathcal{Q}$ nontrivial, and let $G$ be a group in $\mathcal{V}$. For a subgroup $H$ of $G$, the following are equivalent:

(i) $\text{dom}^\mathcal{V}_{G}(H) = G$.

(ii) $H$ is epimorphically embedded in $G$ in the variety $\mathcal{V}$.

(iii) $HQ(G) = G$ and $\text{dom}^\mathcal{N}_{G(Q)}(H \cap Q(G)) = Q(G)$.

(iv) For every normal subgroup $N$ of $G$ such that $N \in \mathcal{N}$ and $G/N \in \mathcal{Q}$, $HN = G$ and $\text{dom}^\mathcal{N}_{G(N)}(H \cap N) = N$.

**Proof:** By definition of dominion, (i) and (ii) are equivalent. The equivalence of (i) and (iii) follows from Theorem 3.7. Clearly, (iv) implies (iii). Finally, (i) implies (iv) by Lemma 3.9, setting $A = N$, and $B = Q$. 

**Remark 3.11.** We quickly note that in Theorem 3.10(iv), it is not enough to consider a single normal subgroup $N$. We defer an example for now.

We may ask whether the converse of Corollary 3.8 holds. That is, if a variety $\mathcal{N}$ has instances of nonsurjective epimorphisms, and $\mathcal{Q}$ is a nontrivial variety, does $\mathcal{NQ}$ have instances of nonsurjective epimorphisms? We will partially answer this question in the next sections.

**Section 4. Finite epimorphisms**

We will say that a variety $\mathcal{N}$ has instances of finite nonsurjective epimorphisms if there exists a finite group $G \in \mathcal{N}$ and a proper subgroup $H$ of $G$ with $\text{dom}^\mathcal{N}_{G}(H) = G$.

In this section we will prove two partial converses to Corollary 3.8. First, we will show that if $\mathcal{N}$ has instances of finite nonsurjective epimorphisms, then so does $\mathcal{NA}$, where $\mathcal{A}$ is the variety of abelian groups; then we will show that the same holds if $\mathcal{A}$ is replaced by any product of varieties of nilpotent groups, each of which contains the infinite cyclic group. We need some preliminary results regarding the commutator of a standard wreath product first.

Let $\mathbb{Z} = \langle x \rangle$ denote the infinite cyclic group, which we will write multiplicatively.

**Lemma 4.12.** Let $G$ be a group, and let $\mathbb{Z}$ be the infinite cyclic group. Then the commutator subgroup of $G \wr \mathbb{Z}$ is equal to $G^\mathbb{Z}$. In fact, letting $x$ denote a generator of $\mathbb{Z}$, every element of $G^\mathbb{Z}$ has the form $[\psi, x]$ for some $\psi \in G^\mathbb{Z}$.

**Proof:** The regular wreath $G \wr \mathbb{Z}$ is a semidirect product of $G^\mathbb{Z}$ with $\mathbb{Z}$. If we quotient out by $G^\mathbb{Z}$ we obtain a group isomorphic to $\mathbb{Z}$, hence abelian. Therefore, $[G \wr \mathbb{Z}, G \wr \mathbb{Z}] \subseteq G^\mathbb{Z}$.
Let $\phi : \mathbb{Z} \to G$ be an arbitrary element of $G^\mathbb{Z}$. We claim that there exists an element $\psi \in G^\mathbb{Z}$ such that $[\psi, x] = \phi$ in $G \wr \mathbb{Z}$.

We note that by definition of the wreath product,
\[ \psi^x(x^n) = \psi(x^n x^{-1}) = \psi(x^{n-1}). \]

In particular, we have
\[ [\psi, x](x^n) = \psi(x^n)^{-1} \psi(x^{n-1}). \tag{4.13} \]

We define $\psi$ recursively. Choose $\psi(e\mathbb{Z})$ arbitrarily. Assuming we have defined $\psi$ at $x^n$, $n \geq 0$, since we want $[\psi, x] = \phi$, from (4.13) we conclude that we must define
\[ \psi(x^{n+1}) = \psi(x^n) \phi(x^{n+1})^{-1}. \]

Similarly, if we have defined $\psi$ at $x^{-n}$, for some $n \geq 0$, again from $[\psi, x] = \phi$ and (4.13), we must define $\psi(x^{-n+1}) = \psi(x^{-n}) \phi(x^{-n})$.

This defines $\psi$ recursively, and by construction $[\psi, x] = \phi$, as desired. Therefore, $G^\mathbb{Z}$ is contained in $[G \wr \mathbb{Z}, G \wr \mathbb{Z}]$, giving equality.

**Remark 4.14.** This result also follows from work of P.M. Neumann on the wreath product, specifically Corollary 5.3 in [14].

**Theorem 4.15.** If $G$ is a group, then
\[ [(G \wr \mathbb{Z})^I, (G \wr \mathbb{Z})^I] = ([G \wr \mathbb{Z}, G \wr \mathbb{Z}])^I \tag{4.16} \]

for any set $I$.

**Proof:** Clearly, the left hand side is contained in the right hand side of (4.16). Conversely, let $(\psi_i)_{i \in I}$ be an element of $([G\wr \mathbb{Z}, G\wr \mathbb{Z}])^I$; thus $\psi_i \in [G\wr \mathbb{Z}, G\wr \mathbb{Z}] = G^\mathbb{Z}$ for each $i \in I$.

From Lemma 4.12 we know that every element of $[G \wr \mathbb{Z}, G \wr \mathbb{Z}]$ is equal to the commutator of two elements; therefore, for each $i \in I$ there exist elements $x_i, y_i$ in $G \wr \mathbb{Z}$ such that $\psi_i = [x_i, y_i]$. But this implies that
\[ [(x_i)_{i \in I}, (y_i)_{i \in I}] = ([x_i, y_i])_{i \in I} = (\psi_i)_{i \in I} \]

which shows that the right hand side of (4.16) is contained in the left hand side, proving equality. \qed

8
Lemma 4.17. Let \( \mathcal{N} \) be a variety of groups, and let \( G \in \mathcal{N} \) be a finite group. If \( H \) is a subgroup of \( G \) and \( I \) is an arbitrary set, then

\[
\text{dom}_{G^I}^N (H^I) = \left( \text{dom}_G^N (H) \right)^I.
\]

Proof: Clearly, we have \( \text{dom}_{G^I}^N (H^I) \subseteq \left( \text{dom}_G^N (H) \right)^I \). To prove the reverse inclusion, let \( \phi: I \to \text{dom}_G^N (H) \) be an element of \( \left( \text{dom}_G^N (H) \right)^I \).

For each \( g \in G \), let

\[
S_g = \{ i \in I \mid \phi(i) = g \}.
\]

Consider the group \( K = G^{[\text{dom}_G^N (H)]} \), and let \( M = H^{[\text{dom}_G^N (H)]} \). Since the number of direct factors is finite, and dominions respect finite direct products, we have

\[
\text{dom}_K^N (M) = (\text{dom}_G^N (H))^{[\text{dom}_G^N (H)]}.
\]

Let \( \eta: K \to G^I \) be the embedding that maps the \( g \)-coordinate of \( K \) diagonally to the \( S_g \) coordinate of \( G^I \). That is, \( \eta \) sends an element \( z \in K \) to the element \( \eta(z)_i = z_g \) if and only if \( i \in S_g \).

Clearly, \( \eta(M) \subseteq H^I \), and therefore

\[
\eta(\text{dom}_K^N (M)) \subseteq \text{dom}_{G^I}^N (H^I).
\]

We claim that \( \phi \in \eta(\text{dom}_K^N (M)) \). Indeed, let \( z \) be the element of \( K \) given by \( z_g = g \) for each \( g \in \text{dom}_K^N (H) \). By construction of \( \eta \), \( \eta(z) = \phi \), which proves the claim.

Therefore, \( \left( \text{dom}_G^N (H) \right)^I \subseteq \text{dom}_{G^I}^N (H^I) \), as claimed. \( \square \)

Theorem 4.18. Let \( \mathcal{N} \) be a variety of groups, and let \( G \in \mathcal{N} \) be a finite group. Let \( H \) be a subgroup of \( G \) such that \( \text{dom}_G^N (H) = G \). Then

\[
\text{dom}_{G \wr \mathbb{Z}}^N (H \wr \mathbb{Z}) = G \wr \mathbb{Z}
\]

where \( \mathcal{A} \) is the variety of abelian groups, and \( \mathbb{Z} \) is the infinite cyclic group.

Proof: Note that since \( H \) is a subgroup of \( G \), \( H \wr \mathbb{Z} \) is a subgroup of \( G \wr \mathbb{Z} \) in the obvious way, by considering only the functions \( \mathbb{Z} \to G \) which take values in \( H \).

By Theorem 3.10(iii), we need to verify that \( (H \wr \mathbb{Z}) \mathcal{A}(G \wr \mathbb{Z}) = G \wr \mathbb{Z} \), and that

\[
\text{dom}_{\mathcal{A}(G \wr \mathbb{Z})}^N ((H \wr \mathbb{Z}) \cap \mathcal{A}(G \wr \mathbb{Z})) = \mathcal{A}(G \wr \mathbb{Z}).
\]

We know that for any group \( K \), \( \mathcal{A}(K) = [K,K] \), the commutator of \( K \). By Lemma 4.12, \([G \wr \mathbb{Z}, G \wr \mathbb{Z}] = G^{\mathbb{Z}}\), so \( (H \wr \mathbb{Z}) \mathcal{A}(G \wr \mathbb{Z}) \) contains \( G^{\mathbb{Z}} \) and also contains
\(Z\) (since \(Z\) is a subgroup of \(H \wr Z\)). As these two subgroups generate \(G \wr Z\), we conclude that \((H \wr Z)A(G \wr Z) = G \wr Z\).

To verify the second condition, note that \(A(G \wr Z) = G^Z\), hence we have

\[(H \wr Z) \cap A(G \wr Z) = H^Z,\]

so we want to find \(\text{dom}^N_{G \wr Z}(H^Z)\). From Lemma 4.17 we conclude that

\[\text{dom}^N_{G \wr Z}(H^Z) = (\text{dom}^N_H)^Z = G^Z = A(G \wr Z).\]

By Theorem 3.10(iii), \(\text{dom}^N_{G \wr Z}(H \wr Z) = G \wr Z\), as claimed.

**Corollary 4.19.** Let \(\mathcal{N}\) be a variety of groups with instances of finite nonsurjective epimorphisms. If \(\mathcal{A}\) is the variety of abelian groups, then \(\mathcal{N} \mathcal{A}\) has instances of nonsurjective epimorphisms.

**Remark 4.20.** Note, however, that the result guarantees that \(\mathcal{N} \mathcal{A}\) has nonsurjective epimorphisms, but does not tell us whether it also has *finite* nonsurjective epimorphisms. Our proof certainly does not give a finite nonsurjective epimorphism, as \(H \wr Z\) has infinite index in \(G \wr Z\).

We can extend the result a bit more now:

**Lemma 4.21.** Let \(G\) be a group, \(n \geq 1\), and \(S_n\) be the variety of all solvable groups of solvability length at most \(n\). We define the groups \(K_{G,n}\) recursively, by letting

\[K_{G,1} = G \wr Z;\]

and

\[K_{G,n+1} = K_{G,n} \wr Z\]

Then \(S_n(K_{G,n}) \cong G^{Z^n}\).

**Proof:** We proceed by induction on \(n\). Note that \(S_1 = \mathcal{A}\), the variety of abelian groups, and \(\mathcal{A}(G \wr Z) = [G \wr Z, G \wr Z]\); the case \(n = 1\) now follows from Lemma 4.12. Assuming the result is true for \(n\), we have

\[S_{n+1}(K_{G,n+1}) = A(S_n(K_{G,n+1}))\]

\[= A(S_n(K_{G,n})))\]

\[= A\left((G \wr Z)^{Z^n}\right) \quad \text{(by the induction hypothesis)}\]

\[\cong \left(A(G \wr Z)^{Z^n}\right) \quad \text{(by Theorem 4.15)}\]

\[= \left(G^Z\right)^{Z^n}\]

\[\cong G^{Z^{n+1}}\]

as claimed.
Corollary 4.22. Let $\mathcal{N}$ be a variety of groups with instances of finite nonsurjective epimorphisms, and let $n > 0$. Let $S_n = \mathcal{A}^n$ be the variety of all solvable groups of solvability length at most $n$. Then $\mathcal{N}S_n$ has instances of nonsurjective epimorphisms.

Proof: As before, given any group $G$, we define recursively

$$K_{G,1} = G \wr \mathbb{Z};$$
$$K_{G,n+1} = K_{G,n} \wr \mathbb{Z}.$$ 

Let $G$ be a finite group in $\mathcal{N}$, and let $H$ a proper subgroup of $G$ which is epimorphically embedded into $G$. We want to show that

$$\text{dom}^{S_n}_{K_{G,n}}(K_{H,n}) = K_{G,n}.$$ 

This follows from Theorem 3.7, the fact that $S_n(K_{G,n}) \cong G^Z$ (by Lemma 4.21), and that

$$\text{dom}^\mathcal{N}_{G^Z}(H^Z) = (\text{dom}^\mathcal{N}_G(H))^Z$$ 

by Lemma 4.17. 

Next we prove a result similar to Corollary 4.19 with $\mathcal{A}$ replaced by an arbitrary variety of nilpotent groups that contains the infinite cyclic group $\mathbb{Z}$.

For $c > 0$, $\mathcal{N}_c$ denotes the variety of nilpotent groups of class at most $c$, defined by the single law $[x_1, x_2, x_3, \ldots, x_{c+1}]$. In particular, $\mathcal{N}_1 = \mathcal{A}$. We have the following lemma:

Lemma 4.23. Let $c > 0$, and let $G$ be a group. Then $\mathcal{N}_c(G \wr \mathbb{Z}) = G^Z$.

Proof: Since every element of $G^Z$ may be written in the form $[\psi, x]$, where $x$ is a generator for $\mathbb{Z}$ and $\psi \in G^Z$, by Lemma 4.12, it follows that $\mathcal{N}_2(G \wr \mathbb{Z}) = G^Z$. Proceeding by induction on $c$ we obtain the desired result.

Theorem 4.24. Let $\mathcal{N}$ be a variety of groups, $\mathcal{N}_c$ the variety of nilpotent groups of class at most $c$, $c > 0$, and let $G \in \mathcal{N}$ be a finite group. Let $H$ be a subgroup of $G$ such that $\text{dom}^\mathcal{N}_G(H) = G$. Then

$$\text{dom}^{\mathcal{N}_c}_{G \wr \mathbb{Z}}(H \wr \mathbb{Z}) = G \wr \mathbb{Z}.$$ 

Proof: The proof proceeds exactly as the proof of Theorem 4.18, after we note that $\mathcal{N}_c(G \wr \mathbb{Z}) = G^Z$ by Lemma 4.23.
Corollary 4.25. Let $\mathcal{N}$ be a variety with instances of finite nonsurjective epimorphisms, and let $c \geq 1$. Then $\mathcal{N} N c$ has instances of nonsurjective epimorphisms.

Corollary 4.26. Let $\mathcal{N}$ be a variety with instances of finite nonsurjective epimorphisms, and let $\mathcal{Q}$ be a variety of nilpotent groups such that $\mathbb{Z} \in \mathcal{Q}$. Then $\mathcal{N} \mathcal{Q}$ has instances of nonsurjective epimorphisms.

Proof: Let $G \in \mathcal{N}$ be a finite group and let $H$ be a subgroup of $G$ such that $\text{dom}_{G}^{\mathcal{N}}(H) = G$.

Since $\mathcal{Q}$ is a variety of nilpotent groups, there exists $c_{0} \geq 1$ such that $\mathcal{Q} \subseteq \mathcal{N} c_{0}$. By Theorem 4.24, it follows that $\text{dom}_{G \mathbb{Z}}^{\mathcal{N} c_{0}}(H \mathbb{Z}) = G \mathbb{Z}$. Since $G \mathbb{Z} \in \mathcal{N} \mathcal{Q}$, and $\mathcal{N} \mathcal{Q} \subseteq \mathcal{N} N_{c_{0}}$, we have

$$\text{dom}_{G \mathbb{Z}}^{\mathcal{N} \mathcal{Q}}(H \mathbb{Z}) \supseteq \text{dom}_{G \mathbb{Z}}^{\mathcal{N} c_{0}}(H \mathbb{Z}) = G \mathbb{Z}.$$ 

Therefore, $\mathcal{N} \mathcal{Q}$ also has instances of nonsurjective epimorphisms, as claimed.

In the statement of the next Corollary, note that $\mathcal{N} \mathfrak{N} \mathfrak{l}$ is not, in general, a variety.

Corollary 4.27. Let $\mathcal{N}$ be a variety with instances of finite nonsurjective epimorphisms, and let $\mathcal{N} \mathfrak{N} \mathfrak{l}$ be the category of all groups which are an extension of an $\mathcal{N}$-group by a nilpotent group. Then $\mathcal{N} \mathfrak{N} \mathfrak{l}$ has instances of nonsurjective epimorphisms.

Proof: Let $G \in \mathcal{N}$ be a finite group, and $H$ a proper subgroup of $G$ such that $\text{dom}_{G}^{\mathcal{N}}(H) = G$.

We claim that the embedding $H \mathbb{Z} \hookrightarrow G \mathbb{Z}$ is a nonsurjective epimorphism in the category $\mathcal{N} \mathfrak{N} \mathfrak{l}$.

Let $K \in \mathcal{N} \mathfrak{N} \mathfrak{l}$, and let $f, g: (G \mathbb{Z}) \rightarrow K$ be two maps which agree on $H \mathbb{Z}$.

Since $K$ is an extension of an $\mathcal{N}$-group by a nilpotent group, there exists a $c > 0$ such that $K \in \mathcal{N} c$. But $\text{dom}_{G \mathbb{Z}}^{\mathcal{N} c}(H \mathbb{Z}) = G \mathbb{Z}$, so $f|_{H \mathbb{Z}} = g|_{H \mathbb{Z}}$ implies $f = g$.

Therefore, the immersion $H \mathbb{Z} \hookrightarrow G \mathbb{Z}$ is an epimorphism in $\mathcal{N} \mathfrak{N} \mathfrak{l}$, which proves the corollary.

Theorem 4.28. Let $\mathcal{N}$ be a variety of groups, and assume that $\mathcal{N}$ has instances of finite nonsurjective epimorphisms. Let $\mathcal{Q}$ be a variety which is a finite product of varieties of nilpotent groups, each of which contains the infinite cyclic group $\mathbb{Z}$. Then $\mathcal{N} \mathcal{Q}$ also has instances of nonsurjective epimorphisms.

Proof: This is obtained in the same manner as Corollary 4.22 was obtained before, noting that a formula similar to the one in Lemma 4.21 holds for a variety $\mathcal{Q}$ as above.
Remark 4.29. Once again, the condition that $\mathcal{N}$ have finite nonsurjective epimorphisms is somewhat disappointing. It would be much better if we could drop this extra hypothesis, and I leave as an open question whether Theorem 4.28 holds without it.

Note that Theorem 4.28 implies the weaker Corollary 4.22.

Section 5. Another partial answer

In this section we will prove another partial converse to Corollary 3.8. Namely, we will show that if a variety $\mathcal{N}$ contains a nonsurjective epimorphism of the form $H \to S$, where $S$ is a finite nonabelian simple group, then for any nontrivial variety $\mathcal{Q}$, the variety $\mathcal{N}\mathcal{Q}$ also contains a (finite) nonsurjective epimorphism.

The idea behind the argument is simple. Suppose that we have a variety $\mathcal{N}$, a finite simple nonabelian group $S$ in $\mathcal{N}$, and a proper subgroup $H$ of $S$ which is epimorphically embedded into $S$ in the variety $\mathcal{N}$.

If $\mathcal{Q}$ is any variety, then either $S \in \mathcal{Q}$ or $S \notin \mathcal{Q}$. If $S \notin \mathcal{Q}$, then $\mathcal{Q}(S) = S$ (since $S$ is simple) and so by Theorem 3.7 the embedding $H \to S$ is also a nonsurjective epimorphism in $\mathcal{N}\mathcal{Q}$. On the other hand, if $S \in \mathcal{Q}$, we want to find some finite group $G$ in $\mathcal{Q}$ such that $S \wr G$ is not in $\mathcal{Q}$. We might expect such a $G$ to exist, say some finite group which is “barely” in $\mathcal{Q}$. Then we can hope that $H \wr G$ will be epimorphically embedded into $S \wr G$ in $\mathcal{N}\mathcal{Q}$.

Indeed, it turns out that this is the case. We need a few preparatory lemmas to establish the existence of a $G$ with the properties above.

Lemma 5.30. Let $\mathcal{V}$ be a variety of groups, and $p$ a prime. If $\mathcal{V}$ contains all finite $p$-groups, then $\mathcal{V}$ is the variety of all groups.

Proof: First, note that an absolutely free group can be embedded in a noncommuting formal power series ring, as the formal power series with constant term 1 over $\mathbb{Z}/p\mathbb{Z}$. Namely, the absolutely free group on $x_1, \ldots, x_n$ can be embedded into $\mathbb{Z}_p\langle\langle y_1, \ldots, y_n \rangle\rangle$ by sending $x_i$ into $1 + y_i$. One shows this is an embedding by noting that if $n > 0$ is an integer, and $n = ap^m$, where $\gcd(a, p) = 1$, then

$$(1 + y_i)^n = 1 + ay_i^p + y_i^{2p^m}h(y_i)$$

where $h(y_i)$ is a power series on $y_i$. Then, if $w = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ is a nontrivial reduced word on the $x_i$’s (so that $r_i \neq r_{i+1}$, and $a_j \neq 0$ for all $i$ and $j$), if we write $A_i = b_ip^{k_i}$, with $\gcd(b_i, p) = 1$, then the image of this word in $\mathbb{Z}_p\langle\langle y_1, \ldots, y_n \rangle\rangle$ under the map given above has a unique monomial of degree $p^{k_1} + \cdots + p^{k_s}$, namely

$$a_1 \cdots a_n y_1^{p^{k_1}} \cdots y_n^{p^{k_s}}$$

(see Theorem 5.6 in [12], for the details).

13
Next, note that every nontrivial identity on $n$ variables will fail in a truncated formal power series ring in finitely many noncommuting indeterminates,

$$\mathbb{Z}_p \langle \langle y_1, \ldots, y_n \rangle \rangle / (y_1, \ldots, y_n)^d$$

for some $d$, (namely, if the word is $w$ as above, setting $d > p^{k_1} + \cdots + p^{k_n}$ will guarantee that the image of $w$ is nontrivial). And the set of such truncated power series with constant term 1 is a finite $p$ group, since it has $p^{d-1}$ elements.

Therefore, if $\mathcal{V}$ contains all finite $p$ groups, then no nontrivial identity can be a law of $\mathcal{V}$, from which it follows that $\mathcal{V}$ is the variety of all groups.

**Corollary 5.31.** Let $\mathcal{V}$ be a proper subvariety of $\mathbf{Group}$. Then there exists a finite group $G \in \mathcal{V}$ such that $(\mathbb{Z}/p\mathbb{Z}) \wr G \notin \mathcal{V}$.

**Proof:** Suppose that for all finite groups $G \in \mathcal{V}$, $(\mathbb{Z}/p\mathbb{Z}) \wr G$ is also in $\mathcal{V}$.

Every finite $p$ group can be obtained from the trivial subgroup by successively extending $\mathbb{Z}/p\mathbb{Z}$. To see this, recall that a finite $p$ group always has nontrivial center, so the group must contain a normal subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Since any extension of $\mathbb{Z}/p\mathbb{Z}$ by $G$ can be realized as a subgroup of $(\mathbb{Z}/p\mathbb{Z}) \wr G$ by a theorem of Kaloujnine and Krasner [4], it follows that $\mathcal{V}$ must contain all finite $p$ groups. But in that case, Lemma 5.30 shows that $\mathcal{V}$ is the variety of all groups.

**Corollary 5.32.** Let $\mathcal{V}$ be a nontrivial variety of groups, and let $A \in \mathcal{V}$ be a nontrivial group. Then there exists a finite group $G \in \mathcal{V}$ such that $A \wr G \notin \mathcal{V}$.

**Proof:** Let $x$ be a nontrivial element of $A$, and let $p$ be a prime such that $\langle x \rangle$ has $\mathbb{Z}/p\mathbb{Z}$ as a homomorphic image. Then for every group $B$, $(\mathbb{Z}/p\mathbb{Z}) \wr B$ is a homomorphic image of a subgroup of $A \wr B$. By Corollary 5.31 there exists a finite group $G$ such that $(\mathbb{Z}/p\mathbb{Z}) \wr G \notin \mathcal{V}$. In particular $A \wr G$ cannot be in $\mathcal{V}$.

The final ingredient for the proof is the following lemma:

**Lemma 5.33.** Let $\mathcal{N}$ be a variety, and let $S$ be a finite nonabelian simple group in $\mathcal{N}$. Let $\mathcal{Q}$ be a nontrivial variety, and let $B$ be a finite group in $\mathcal{Q}$. Then either $Q(S \wr B) = S^B$ or $Q(S \wr B) = \{e\}$.

**Proof:** Since $(S \wr B)/S^B \cong B \in \mathcal{Q}$, we have $Q(S \wr B) \subseteq S^B$. Also $Q(S \wr B) \triangleleft S \wr B$, and so in particular $Q(S \wr B) \triangleleft S^B$. Since $S$ is a finite nonabelian simple group, and $B$ is a finite group, $Q(S \wr B)$ must equal the product of some of the copies of $S$ (see for example [16]).

Now we simply note that in $S \wr B$, $B$ acts transitively on the factors $S$ of $S^B$, hence given a subgroup of $S^B$ which is the product of a subset of these factors, and is invariant under the action of $B$ (as $Q(S \wr B)$ must be), it must be the product of all the factors, or of none.
Theorem 5.34. Let $\mathcal{N}$ be a nontrivial variety, $S$ a finite nonabelian simple group, and $H$ a proper subgroup of $S$ such that $S \in \mathcal{N}$ and $\text{dom}_S^N(H) = S$ (that is, the embedding $H \hookrightarrow S$ is a nonsurjective epimorphism in $\mathcal{N}$). Let $\mathcal{Q}$ be any nontrivial variety. Then $\mathcal{NQ}$ also has instances of finite nonsurjective epimorphisms. Namely, there exists a finite (possibly trivial) group $G \in \mathcal{Q}$ such that $H \wr G$ is epimorphically embedded in $S \wr G$, in the variety $\mathcal{NQ}$.

Proof: Let $H, S, \mathcal{N}$ and $\mathcal{Q}$ be as in the statement. By Corollary 5.32 there exists a finite group $G \in \mathcal{Q}$ such that $S \wr G \notin \mathcal{Q}$ ($G$ may be trivial, for example if $S \notin \mathcal{Q}$). By Lemma 5.33, we have $\mathcal{Q}(S \wr G) = S^G$, since $S \wr G \notin \mathcal{Q}$.

Consider the subgroup $H \wr G$ of $S \wr G$. $(H \wr G) \mathcal{Q}(S \wr G)$ equals the whole group, since it contains both $S^G$ and $G$, which together generate $S \wr G$. On the other hand, $(H \wr G) \cap \mathcal{Q}(S \wr G) = (H \wr G) \cap S^G = H^G$.

Since dominions respect finite direct products,

$$\text{dom}_S^N(H^G) = \left(\text{dom}_S^N(H)\right)^G = S^G = \mathcal{Q}(S \wr G).$$

By Theorem 3.10(iii), $\text{dom}_S^N(H \wr G) = S \wr G$, so the embedding $H \wr G \hookrightarrow S \wr G$ is a nonsurjective epimorphism in $\mathcal{NQ}$. Since all the groups involved are finite, it is an instance of a finite nonsurjective epimorphism, as claimed.

This provides a nice partial answer to the question at the end of Section 3, especially since all basic examples we have given so far (for example, in [10]) are precisely of the kind described in Theorem 5.34, namely the embedding of a proper subgroup into a simple nonabelian group. Since we also know that in a variety consisting only of solvable groups all epimorphisms are surjective, we might guess that finite nonabelian simple groups will play a large role whenever a nonsurjective epimorphism occurs.

We finish this section with the example promised in Remark 3.11.

Example 5.35. Let $\mathcal{N} = \mathcal{Q} = \text{Var}(A_5 \wr A_5)$, and let $G = (A_5 \wr A_5) \wr A_5 \in \mathcal{NQ}$. Let $\mathcal{N} = (A_5 \wr A_5)^{A_5}$ (that is, the base group of the last wreath product taken in the construction of $G$), and let $H$ be the subgroup of $G$ given by $H = (A_4 \wr A_5) \wr A_5$. Then $N \in \mathcal{N}$, $G/N \cong A_5 \in \mathcal{Q}$, $NH = G$, and $H \cap N = (A_4 \wr A_5)^{A_5}$, which is epimorphically embedded into $N$ in $\mathcal{N}$. However, $H$ is not epimorphically embedded in $G$ in the variety $\mathcal{NQ}$. To see this, we use the characterization in Theorem 3.7. Note that $\mathcal{Q}(G)$ is the subgroup $(A_5^{A_5})^{A_5}$, so $H \cap \mathcal{B}(G)$ is $(A_4^{A_5})^{A_5}$; we know $A_4$ is not epimorphically embedded into $A_5$ in $\mathcal{N}$, so $H \cap \mathcal{Q}(G)$ is not epimorphically embedded into $\mathcal{Q}(G)$. In particular, $H \hookrightarrow G$ is not an epimorphism in $\mathcal{NQ}$. This shows that in Theorem 3.10(iv), it is not enough to consider a single normal subgroup $N$. 

15


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