Positivity of mild solution to stochastic evolution equations with an application to forward rates

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Abstract

We prove a maximum principle for mild solutions to stochastic evolution equations with (locally) Lipschitz coefficients and Wiener noise on weighted $L^2$ spaces. As an application, we provide sufficient conditions for the positivity of forward rates in the Heath-Jarrow-Morton model, considering the associated Musiela SPDE on a homogeneous weighted Sobolev space.

1 Introduction

Consider stochastic evolution equations of the type

$$du(t) + Au(t) dt = f(u(t)) dt + B(u(t)) dW(t), \quad u(0) = u_0,$$

(1)

on a Hilbert function space $H$, where $A$ is a linear maximal monotone operator on $H$, $W$ is a cylindrical Wiener process, and the coefficients $f$ and $B$, which can be random and time-dependent, satisfy suitable measurability and Lipschitz continuity conditions. Precise assumptions on the data of the problem are given in §3 below. Our goal is to establish conditions on the coefficients $A$, $f$, and $B$ implying that the “flow” associated to (1) is positivity preserving, i.e., such that its solution is positive at all times provided the initial datum is positive.

Maximum principles for stochastic PDEs have a long history, and some references to the earlier literature can be found in [14]. Most recent contributions seem to consider equations that admits a variational formulation (in the sense of [16, 22]). As examples of such results, let us mention [13, 15], where large classes of second-order stochastic PDEs are considered, as well as [5] for the case of equations driven by certain classes of discontinuous noise. Further classes of second-order parabolic stochastic PDEs that can be treated by techniques reminiscent of the variational ones are discussed in [6, 7]. However, it is well known that not every (linear) maximal monotone operator admits a variational formulation (i.e., grosso modo, it cannot be represented as a bounded operator from a Banach space to its dual), hence, in particular, not every equation admitting a mild solution can be treated in the variational approach. The maximum principle for mild solutions to (1) obtained here is optimal, as far as the conditions imposed on $A$ is concerned, in the sense that we simply ask that the semigroup generated by $-A$ is positivity preserving. Large classes of operators are thus included, beyond second-order elliptic operators. On the other hand, we cannot treat, at least for the time being, time-dependent random operators, some of which are within the scope of the variational approach. Some results on the maximum principle for mild solutions to stochastic evolution equations can be found in the literature. One of the first results in this direction seems to be that in [13], where $A$ can also be time-dependent, the coefficients $f$ and $B$ are superposition operators, and the proof relies on some delicate discretization arguments (unfortunately it was shown in [24] that such arguments are

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not entirely correct). A different approach, again based on discretization, has been developed in [1]. Our proof seems conceptually quite simpler, as it relies only on approximation arguments and a version of Itô’s formula.

As an application of the maximum principle for mild solutions to stochastic evolution equations, we discuss the problem of positivity of forward rates in the Heath-Jarrow-Morton framework (see [12]), considering the associated Musiela SPDE on a weighted Sobolev space (see, e.g., [10, 18]). The same problem is treated in [11], also for models driven by Poisson random measures, by means of support theorems for Hilbert-space-valued SDEs, under a strong smoothness assumption on the diffusion coefficient $B$. We use instead a self-contained approximation argument allowing to infer the positivity of mild solutions to the Musiela SPDE in the above-mentioned weighted Sobolev space by the positivity of mild solutions to regularized equations on weighted $L^2$ spaces, which is in turn obtained by the maximum principle.

The rest of the text is organized as follows: in §2, after fixing some notation, we collect the main tools used in the proof of the maximum principle of §3. Basic facts about the Heath-Jarrow-Morton model of the term structure of interest rates, as well as about the well-posedness of Musiela’s SPDE in different function spaces, are contained in §4. Two different approximations of the volatility coefficient in Musiela’s SPDE are introduced and investigated in §§5–6. These are the crucial technical tools to deduce positivity of forward rates, discussed in §7.

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2 Preliminaries

2.1 Notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfying the usual conditions of right-continuity and completeness, with predictable $\sigma$-algebra $\mathcal{P}$. All random elements will be defined on this stochastic basis without further notice. We also stipulate that random variables equal outside an event of probability zero are considered equal, and that equality of stochastic processes is intended in the sense of indistinguishability. We shall denote by $W$ a cylindrical Wiener process on a separable Hilbert space $U$.

Let $H$ be a further Hilbert space. The spaces of linear bounded operators and of Hilbert-Schmidt operators from $U$ to $H$ will be denoted by $\mathcal{L}(U, H)$ and $\mathcal{L}^2(U, H)$, respectively. Given a bilinear form $\Phi$ and a linear bounded operator $L$ on $H$, denoting an orthonormal basis of $H$ by $(h_k)$, we set

$$\text{Tr}_L \Phi := \sum_{k=1}^{\infty} \Phi(Lh_k, Lh_k).$$

Given two Banach spaces $E$ and $F$, we shall write $E \hookrightarrow F$ to mean that $E$ is continuously embedded in $F$.

Let $m > 0$. The space of measurable functions $\phi: H \to \mathbb{R}$ such that

$$\sup_{x \in H} \frac{|\phi(x)|}{1 + \|x\|} < \infty$$

is denoted by $B_m(H)$. The subspace of $B_m(H)$ of continuous functions satisfying the same boundedness condition is denoted by $C_m(H)$. The first and second Fréchet derivatives of a function $\phi: E \to F$ are denoted by $\phi’$ and $\phi''$, respectively, while its Gâteaux derivative is denoted by $\delta \phi$. Let $k \in \mathbb{N}$ and $m > 0$. The space of $k$ times continuously (Fréchet or, equivalently,
Gâteaux) differentiable functions \( \phi : H \to \mathbb{R} \) such that

\[
\sup_{x \in H} \frac{\|\phi^{(j)}(x)\|}{1 + \|x\|^2} < \infty \quad \forall j \leq k,
\]

where \( \phi^{(k)} \) stands for the \( k \)-th order (Fréchet) derivative of \( \phi \), is denoted by \( C^k_\alpha(H) \). In the previous inequality, the norm of \( \phi^{(j)}(x) \) is the norm of the \( j \)-linear functions on \( H \) with values in \( \mathbb{R} \).

**2.2 Function spaces on \( \mathbb{R}_+ \)**

For any \( \alpha \in \mathbb{R} \), let \( \mu_\alpha \) be the measure on the Borel \( \sigma \)-algebra of \( \mathbb{R}_+ \) whose density with respect to Lebesgue measure is equal to \( x \mapsto e^{\alpha x} \). We shall denote \( L^2(\mathbb{R}_+, \mu_\alpha) \) by \( L^2_\alpha \). Note that \( \mu_{-\alpha}/\alpha \) is a probability measure if \( \alpha > 0 \). We shall use the elementary observation that, for any \( \alpha > 0 \), \( L^2_\alpha \) is continuously embedded in \( L^1(\mathbb{R}_+) \). In fact, for any \( f \in L^2_\alpha \), one has, by the Cauchy-Schwarz inequality,

\[
\int_0^\infty |f(x)| dx = \int_0^\infty |f(x)| e^{\frac{\alpha}{2} x} e^{-\frac{\alpha}{2} x} dx \leq \frac{1}{\sqrt{\alpha}} \left( \int_0^\infty |f(x)|^2 e^{\alpha x} dx \right)^{1/2}.
\]

For any \( \alpha > 0 \), let \( H_\alpha \) denote the vector space of functions \( \phi \in L^1_{\text{loc}}(\mathbb{R}_+) \) such that \( \phi' \in L^2_\alpha \), where \( \phi' \) stands for the derivative of \( \phi \) in the sense of distributions. Since \( L^2_\alpha \hookrightarrow L^1(\mathbb{R}_+) \), it follows that every \( \phi \in H_\alpha \) admits a finite limit at \( +\infty \). In fact, for any \( a \in \mathbb{R}_+ \) such that \( |\phi(a)| < \infty \), one has

\[
\phi(x) = \phi(a) + \int_a^x \phi'(y) dy,
\]

hence

\[
\phi(\infty) := \lim_{x \to +\infty} \phi(x) = \phi(a) + \int_a^\infty \phi'(y) dy < \infty.
\]

The vector space \( H_\alpha \) endowed with the scalar product

\[
\langle \phi, \psi \rangle_{H_\alpha} := \phi(\infty)\psi(\infty) + \int_0^\infty \phi'(x)\psi'(x) e^{\alpha x} dx,
\]

and corresponding norm

\[
\|\phi\|_{H_\alpha}^2 := \phi(\infty)^2 + \int_0^\infty |\phi'(x)|^2 e^{\alpha x} dx,
\]

is a separable Hilbert space. This is a slight modification of a construction suggested by Filipović in [10] (where an equivalent norm is adopted) already used, e.g., in [19, 23]. The spaces \( H_\alpha \) satisfy simple but crucial embedding properties.

**Lemma 2.1.** Let \( \alpha > 0 \). Then \( H_\alpha \) is continuously embedded in \( C_0(\mathbb{R}_+) \) and in \( L^p(\mathbb{R}_+) \oplus \mathbb{R} \) for every \( p \in [1, \infty[ \).

**Proof.** Let \( \phi \in H_\alpha \). The Cauchy-Schwarz inequality yields

\[
|\phi(x) - \phi(\infty)| \leq \int_x^\infty |\phi'(y)| dy = \int_x^\infty |\phi'(y)| e^{\frac{\alpha}{2} y} e^{-\frac{\alpha}{2} y} dy
\]

\[
\leq \left( \int_0^\infty |\phi'(y)|^2 e^{\alpha y} dy \right)^{1/2} \left( \int_x^\infty e^{-\alpha y} dy \right)^{1/2}
\]

\[
\leq \|\phi\|_{H_\alpha} \sqrt{\frac{1}{\sqrt{\alpha}} e^{-\frac{\alpha}{2} x}},
\]

where the last term, as a function of \( x \), belongs to \( L^p(\mathbb{R}_+) \) for all \( p \in [1, \infty[ \). The continuity of \( \phi \) can be established by a completely similar argument. \( \square \)
We shall also need basic properties of the translation semigroup $S = (S(t))_{t \geq 0}$, $S(t) : \phi \mapsto \phi(t)$.

**Lemma 2.2.** Let $\alpha > 0$. The translation semigroup $S$ is a strongly continuous semigroup of contractions on $H_\alpha$ with infinitesimal generator $-A$,

$$A : D(A) \subseteq H_\alpha \longrightarrow H_\alpha \quad \phi \mapsto -\phi',$$

where $D(A) = \{ \phi \in H_\alpha \cap C^1(\mathbb{R}_+) : \phi' \in H_\alpha \}$.

**Proof.** The strong continuity of $S$ in $H_\alpha$, equipped with an equivalent norm, has been proved in [10, pp. 78–79], hence it continues to hold in our case too. The identification of the negative generator $A$ is an immediate consequence thereof (cf. [10, Lemma 4.2.2]). The contractivity of $S$ is established as follows: for any $\phi \in H_\alpha$,

$$\|S(t)\phi\|_{H_\alpha}^2 \leq \phi(0)^2 + \int_0^\infty |\phi'(x + t)|^2 e^{-\alpha t} \, dx$$

$$= \phi(0)^2 + e^{-\alpha t} \int_t^\infty |\phi'(x)|^2 e^{\alpha x} \, dx$$

$$\leq \|\phi\|^2_{H_\alpha}.$$

The strong continuity of $S$ in $L^2_{\alpha}$ is most likely well known, but we have not been able to find a reference, hence we provide the (simple) proof for the reader’s convenience.

**Lemma 2.3.** Let $\alpha > 0$. The translation semigroup $S$ is strongly continuous in $L^2_{\alpha}$, and the semigroup $e^{-\frac{\alpha}{2} t} S$ is contractive in $L^2_{\alpha}$.

**Proof.** Let $f \in L^2_{\alpha}$. One has, for any $t > 0$,

$$\|S(t)f - f\|_{L^2_{\alpha}} = \|f(\cdot + t)e^{-\frac{\alpha}{2} t} - fe^{-\frac{\alpha}{2} t}\|_{L^2}$$

$$\leq \|f(\cdot + t)e^{-\frac{\alpha}{2} t} - f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)}\|_{L^2} + \|f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)} - fe^{-\frac{\alpha}{2} t}\|_{L^2},$$

where, since $f \in L^2_{\alpha}$ implies that $fe^{-\frac{\alpha}{2} t} \in L^2$, the last term converges to zero as $t \to 0$ by the strong continuity of $S$ in $L^2$ (see, e.g., [10, §I.4.16]). Moreover,

$$\|f(\cdot + t)e^{-\frac{\alpha}{2} t} - f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)}\|_{L^2}$$

$$= \|f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)} e^{\frac{\alpha}{2} t} - f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)}\|_{L^2}$$

$$= \left(e^{\frac{\alpha}{2} t} - 1\right) \|f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)}\|_{L^2}$$

$$\leq \left(e^{\frac{\alpha}{2} t} - 1\right) \|f\|_{L^2},$$

where the last term also converges to zero as $t \to 0$. Strong continuity of $S$ in $L^2_{\alpha}$ is thus proved. The contractivity of $(e^{-\frac{\alpha}{2} t} S(t))_{t \geq 0}$ is a consequence of the following entirely analogous computation:

$$\|S(t)f\|_{L^2_{\alpha}} = \|f(\cdot + t)e^{-\frac{\alpha}{2} t}\|_{L^2} = e^{\frac{\alpha}{2} t} \|f(\cdot + t)e^{-\frac{\alpha}{2} (\cdot + t)}\|_{L^2} \leq e^{\frac{\alpha}{2} t} \|f\|_{L^2_{\alpha}}.$$

The lemma implies that there exists a densely defined linear operator $A$ such that $-A$ is the infinitesimal generator of $S$, and that $A + \alpha/2$ is maximal monotone in $L^2_{\alpha}$. Moreover, one has $A\phi = -\phi'$ for every $\phi$ belonging to $C^\infty_c(\mathbb{R}_+)$, which is dense in $L^2_{\alpha}$. We shall use the same notation for the generators of $S$ in $H_\alpha$ and in $L^2_{\alpha}$, as they coincide on the intersection of their domains.
2.3 A hypoelliptic Ornstein-Uhlenbeck semigroup

Let $R = (R_\alpha)_{\alpha > 0}$ be the hypoelliptic Ornstein-Uhlenbeck semigroup on $C_b(H)$, the space of continuous bounded functions on $H$ with values in $\mathbb{R}$, defined as

$$R_\alpha g(x) := \int_H g(e^{\alpha C}x + y) \gamma_\alpha(dy),$$

where $C$ is a strictly negative self-adjoint operator with $C^{-1}$ of trace class, and $\gamma_\alpha$ is a centered Gaussian measure on $H$ with covariance operator

$$Q_\alpha := -\frac{1}{2} C^{-1}(I - e^{2\alpha C}).$$

We shall need the following properties of the semigroup $R$, for the proof of which we refer to the monographs [2, 3, 9].

**Lemma 2.4.** Let $m > 0$. The semigroup $R$ extends to a semigroup of continuous linear operators on $C_m(H)$. Moreover, the image of $B_m(H)$ through $R_\alpha$ is contained in $C^\infty(H)$.

The following pointwise continuity property of $\alpha \mapsto R_\alpha$ will play a fundamental role.

**Proposition 2.5.** Let $m > 0$ and $g \in C_m(H)$. Then $R_\alpha g$ converges pointwise to $g$ as $\alpha \to 0$.

**Corollary 2.6.** Let $g \in C^1_m(H)$ and $x \in H$. Then $R_\alpha g$ is Fréchet differentiable with

$$(R_\alpha g)'(x) : v \mapsto \int_H g'(e^{\alpha C}x + y)e^{\alpha C}v \gamma_\alpha(dy)$$

and $(R_\alpha g)'(x)v \to g'(x)v$ as $\alpha \to 0$ for every $v \in H$. Moreover, if $g'$ is Gâteaux differentiable with

$$\|\delta g'(x)\|_{L_2(H,\mathbb{R})} \lesssim 1 + \|x\|^m$$

and $x \mapsto \delta g'(x)(v, w)$ is continuous for all $v, w \in H$, then $R_\alpha g$ is twice Fréchet differentiable with

$$(R_\alpha g)''(x) : (v, w) \mapsto \int_H \delta g'(e^{\alpha C}x + y)(e^{\alpha C}v, e^{\alpha C}w) \gamma_\alpha(dy)$$

and $(R_\alpha g)''(x)(v, w) \to \delta g'(x)(v, w)$ as $\alpha \to 0$ for every $v, w \in H$.

**Proof.** Let $|t| \in [0, 1]$ and $x, v \in H$. One has

$$\frac{R_\alpha g(x + tv) - R_\alpha g(x)}{t} \to \int_H g(e^{\alpha C}(x + tv) + y) - g(e^{\alpha C}x + y) \gamma_\alpha(dy),$$

where, recalling that $g \in C^1(H)$,

$$\lim_{t \to 0} \frac{g(e^{\alpha C}(x + tv) + y) - g(e^{\alpha C}x + y)}{t} = g'(e^{\alpha C}x + y)e^{\alpha C}v.$$

Defining the continuously differentiable function $\tilde{g} : [0, 1] \to \mathbb{R}$ as $\tilde{g}(t) := g(e^{\alpha C}(x + tv) + y)$, the mean value theorem yields the existence of $t_0 \in [0, t]$ (dependent on $t$) such that

$$g(e^{\alpha C}(x + tv) + y) - g(e^{\alpha C}x + y) = g'(e^{\alpha C}(x + t_0v) + y)t e^{\alpha C}v,$$

from which it follows, recalling that $C$ is negative-definite,

$$|g(e^{\alpha C}(x + tv) + y) - g(e^{\alpha C}x + y)| \lesssim |t| \left(1 + \|x\|^m + \|v\|^m + \|y\|^m\right)\|v\|. $$

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Since $\gamma_\alpha$ has finite moments of all (positive) orders, the dominated convergence theorem implies that
\[
(R_\alpha g)'(x) : v \mapsto \int_H g'(e^{\alpha C} x + y) e^{\alpha C} v \gamma_\alpha(dy).
\]
Writing
\[
\int_H g'(e^{\alpha C} x + y) e^{\alpha C} v \gamma_\alpha(dy) = \int_H g'(e^{\alpha C} x + y) v \gamma_\alpha(dy) + \int_H g'(e^{\alpha C} x + y)(e^{\alpha C} v - v) \gamma_\alpha(dy),
\]
the first term on the right-hand side converges to $g'(x)v$ as $\alpha \to 0$ by proposition 2.5. It suffices then to show that the second term on the right-hand side converges to zero as $\alpha \to 0$. One has, using once again that $C$ is negative-definite,
\[
\left| \int_H g'(e^{\alpha C} x + y)(e^{\alpha C} v - v) \gamma_\alpha(dy) \right| \lesssim \|e^{\alpha C} v - v\| \int_H (\|x\|^m + \|y\|^m) \gamma_\alpha(dy),
\]
where the right-hand side converges to zero because the semigroup $\alpha \mapsto e^{\alpha C}$ is strongly continuous and the $m$-th moment of $\gamma_\alpha$, as a function of $\alpha$, is bounded.

Similarly, the Gâteaux differentiability of $g'$ implies
\[
\lim_{t \to 0} \frac{g'(e^{\alpha C}(x + tw) + y) e^{\alpha C} v - g'(e^{\alpha C} x + y) e^{\alpha C} v}{t} = \delta g'(e^{\alpha C} x + y)(e^{\alpha C} v, e^{\alpha C} w)
\]
for all $x, y, v, w \in H$, and, defining the differentiable function $\phi : [0, 1] \to \mathbb{R}$ as
\[
\phi(t) := g'(e^{\alpha C}(x + tw) + y)e^{\alpha C} v,
\]
the fundamental theorem of calculus yields
\[
g'(e^{\alpha C}(x + tw) + y)e^{\alpha C} v - g'(e^{\alpha C} x + y)e^{\alpha C} v = \phi(t) - \phi(0) = \int_0^t \delta g'(e^{\alpha C}(x + sw) + y)(e^{\alpha C} v, e^{\alpha C} w) ds,
\]
hence, as $C$ is negative definite,
\[
|g'(e^{\alpha C}(x + tw) + y)e^{\alpha C} v - g'(e^{\alpha C} x + y)e^{\alpha C} v| \lesssim |t| (1 + \|x\|^m + \|w\|^m + \|y\|^m) \|v\| \|w\|.
\]
The dominated convergence theorem then implies
\[
(R_\alpha g)''(x,v,w) = \lim_{t \to 0} \frac{(R_\alpha g)'(x + tw)v - (R_\alpha g)'(x)v}{t} = \int_H \delta g'(e^{\alpha C} x + y)(e^{\alpha C} v, e^{\alpha C} w) \gamma_\alpha(dy).
\]
Furthermore, writing $e^{\alpha C} v = v + e^{\alpha C} v - v$ and similarly for $e^{\alpha C} w$, it follows by bilinearity of $\delta g'$ that
\[
\delta g'(e^{\alpha C} x + y)(e^{\alpha C} v, e^{\alpha C} w) = \delta g'(e^{\alpha C} x + y)(v,w) + \delta g'(e^{\alpha C} x + y)(v,e^{\alpha C} w - v) + \delta g'(e^{\alpha C} x + y)(e^{\alpha C} v - v, w) + \delta g'(e^{\alpha C} x + y)(e^{\alpha C} v - v, e^{\alpha C} w - v).
\]
Since \( x \mapsto \delta g'(x)(v,w) \in C_m(H) \) for all \( v, w \in H \), proposition 2.5 implies

\[
\int_H \delta g'(e^{\alpha C}x + y)(v,w) \gamma_\alpha(dy) \to \delta g'(x)(v,w)
\]
as \( \alpha \to 0 \). It remains to show that the integrals over \( H \) with respect to \( \gamma_\alpha \) of the other three terms on the right-hand side of the previous identity converge to zero as \( \alpha \to 0 \). It follows by the polynomial boundedness of \( \delta g' \) and the contractivity of \( e^{\alpha C} \) that

\[
\left| \int_H \delta g'(e^{\alpha C}x + y)(e^{\alpha C}v - v, e^{\alpha C}w - w) \gamma_\alpha(dy) \right| \\
\lesssim \|e^{\alpha C}v - v\| \|e^{\alpha C}w - w\| \int_H (1 + \|x\|^m + \|y\|^m) \gamma_\alpha(dy),
\]
where the right-hand side converges to zero as \( \alpha \to 0 \) because, as before, \( e^{\alpha C} \) is strongly continuous and the \( m \)-th moment of \( \gamma_\alpha \) is bounded with respect to \( \alpha \). The remaining two terms can be treated in a completely similar way. \( \square \)

### 2.4 A differentiability result

The following result is most likely well known, but since we could not locate a proof in the literature, we include a proof for the convenience of the reader.

**Lemma 2.7.** Let \( E = L^2(X, \mathcal{A}, m) \), \( g \in C^2(\mathbb{R}) \) with \( g(0) = g'(0) = 0 \) and \( g'' \in C_b(\mathbb{R}) \), and \( G : E \to \mathbb{R} \) be defined as

\[
G : u \mapsto \int_X g(u) \, dm.
\]

Then \( G \) is continuously Fréchet differentiable on \( E \), with Fréchet derivative at \( u \in E \) given by

\[
G'(u) : v \mapsto \langle g'(u), v \rangle = \int_X g'(u)v \, dm.
\]

Moreover, \( G' \) is Gâteaux differentiable on \( E \), with Gâteaux derivative at \( u \in E \) given by

\[
\delta G'(u) : (v,w) \mapsto \langle g''(u), vw \rangle = \int_X g''(u)vw \, dm,
\]
and \( (u, v, w) \mapsto \delta G'(u)(v, w) \) is continuous.

**Proof.** Let us first show that \( G \), as well as the (for now just formal) expressions for \( G' \) and \( \delta G' \) are well defined: since \( g'(0) = 0 \) and \( g'' \) is bounded, the fundamental theorem of calculus yields \( |g'(x)| \lesssim |x| \) for all \( x \in \mathbb{R} \), hence also, as \( g(0) = 0 \), \( |g(x)| \lesssim |x|^2 \). This immediately implies that the integral defining \( G(u) \) is finite for every \( u \in E \). Similarly, in view of the Cauchy-Schwarz inequality, the linear bound on \( g' \) implies that \( G'(u) \) is a continuous linear map from \( E \) to \( \mathbb{R} \) for every \( u \in E \); and the boundedness of \( g'' \) implies that \( \delta G'(u) \) is a continuous bilinear map from \( E \times E \) to \( \mathbb{R} \) for every \( u \in E \).

In order to show the Fréchet differentiability of \( G \) at \( u \), note that, for any \( v \in E \), one has

\[
G(u + v) - G(u) - \langle g'(u), v \rangle = \left\langle v, \int_0^1 (g'(u + tv) - g'(u)) \, dt \right\rangle,
\]
hence

\[
\|G(u + v) - G(u) - \langle g'(u), v \rangle\| \leq \|v\| \left\| \int_0^1 (g'(u + tv) - g'(u)) \, dt \right\|.
\]
Taking into account that $g''$ is bounded, elementary calculus shows that the second term on the right-hand side is bounded by $\|g''\|_{C(R)} \|v\|$, so that

$$\frac{\|G(u + v) - G(u) - (g'(u), v)\|}{\|v\|} \to 0$$

as $v \to 0$ in $E$, i.e. the Fréchet derivative of $G$ at $u \in E$ is indeed the linear map $G'(u) : v \mapsto (g'(u), v)$. Let us prove its continuity: by the isomorphism $\mathcal{L}(E, \mathbb{R}) = E' \simeq E$, it is enough to show that, for any sequence $(u_n) \subset E$ converging to $u$ in $E$, $g'(u_n)$ converges to $g'(u)$ in $E$. But this follows immediately, in analogy to an argument already used, by the Lipschitz continuity of $g'$. We have thus established that $G \in C^1(E)$. To prove that $G'$ is Gâteaux differentiable, let us write, for any $u, v, w \in E$ and $t > 0$,

$$\frac{G'(u + tv)w - G'(u)w - (g''(u), v)w}{t} \to \int_X \left( \frac{g'(u + tv) - g'(u)}{t} - g''(u)v \right) w \, dm,$$

from which it follows, by the Cauchy-Schwarz inequality,

$$\left\| \frac{w \mapsto G'(u + tv)w - G'(u)w}{t} - (g''(u), v)w \right\|_{\mathcal{L}(E, \mathbb{R})} \leq \int_X \left| \frac{g'(u + tv) - g'(u)}{t} - g''(u)v \right|^2 \, dm.$$

Since $g'' \in C_b(\mathbb{R})$, hence $g'$ is Lipschitz continuous, the integrand on the right-hand side is bounded above by $|v|^2$, modulo a multiplicative constant, and $v \in L^2(m)$. Therefore the dominated convergence theorem immediately yields that $G'$ is Gâteaux differentiable on $E$ with Gâteaux derivative $\delta G'(u) : (v, w) \mapsto \langle g''(u), vw \rangle$. Let $u_n \to u$, $v_n \to v$, and $w_n \to w$ in $H$ as $n \to \infty$. Then $u_n \to u$ in $m$-measure, and $g''(u_n) \to g''(u)$ by the continuous mapping theorem. Setting $\bar{v}_n := g''(u_n)v_n$ and $\bar{v} = g''(u)v$, this implies that $\bar{v}_n \to \bar{v}$ in $L^2(m)$, hence in $m$-measure. In fact,

$$\|\bar{v}_n - \bar{v}\|_{L^2(m)} \leq \|g''(u_n)v_n - g''(u_n)v\|_{L^2(m)} + \|g''(u_n)v - g''(u)v\|_{L^2(m)},$$

where

$$\|g''(u_n)v_n - g''(u_n)v\|_{L^2(m)} \leq \|g''\|_{C(X)} \|v_n - v\|_{L^2(m)} \to 0,$$

and $g''(u_n)v - g''(u)v \to 0$ in $m$-measure and $|g''(u_n)v - g''(u)v| \leq \|g''\|_{C(X)} |v| \in L^2(m)$, hence

$$\|g''(u_n)v - g''(u)v\|_{L^2(m)} \to 0$$

by the dominated convergence theorem. Then we have

$$|\delta G'(u_n)(v_n, w_n) - \delta G'(u)(v, w)| = |\langle \bar{v}_n, w_n \rangle - \langle \bar{v}, w \rangle| \to 0,$$

thus completing the proof. □

Remark 2.8. The mapping $G$ is never twice Fréchet differentiable, unless $g(x) = ax^2$ for some constant $a \in \mathbb{R}$. In other words, $G$ is twice Fréchet differentiable if and only if it is proportional to the square of the norm. This is an immediate consequence of the fact that the superposition operator on $L^2(m)$ associated to a function $\phi \in C^\infty(\mathbb{R})$ is Fréchet differentiable if and only if $\phi$ is linear. In fact, identifying $\mathcal{L}(E, \mathbb{R})$ with $E$, the derivative $G'$ can be identified with the superposition operator on $E$ associated to the function $g' : \mathbb{R} \to \mathbb{R}$.

2.5 Approximation and convergence in locally Lipschitzian SPDEs

Consider the stochastic evolution equation on the Hilbert space $H$

$$du + Au dt + f(u) dt = B(u) dW, \quad u(0) = u_0,$$

(2)
where $A$ is a linear maximal monotone operator on $A$, $u_0$ is an $\mathcal{F}_0$-measurable $H$-valued random variable, and $f, B$ satisfy the same measurability conditions as in (b) of [23] below.

For any Banach space $E$ and any sequence of maps $(F_n)$, $F_n : \Omega \times \mathbb{R}_+ \to H \to E$, we shall say that $(F_n)$ is uniformly locally Lipschitz continuous if for every $R \in \mathbb{R}_+$ there exists a constant $N = N(R)$, independent of $n$, such that

$$\|F_n(\omega, t, x) - F_n(\omega, t, y)\|_E \leq N\|x - y\| \quad \forall x, y \in B_R(H) \quad (3)$$

and there exists $a \in H$ such that $(\omega, t, n) \mapsto F_n(\omega, t, a)$ is bounded. If the sequence $(F_n)$ reduces to a singleton $F$, we say that $F$ is locally Lipschitz continuous.

If $f : \Omega \times \mathbb{R}_+ \times H \to H$ and $B : \Omega \times \mathbb{R}_+ \times H \to \mathcal{L}^2(U, H)$ are locally Lipschitz continuous, there exists a unique local mild solution $u$ to (2) with lifetime $T$ (i.e., $T$ is a stopping time such that the norm of $u(t)$ tends to infinity as $t$ tends to $T$ from the left). Furthermore, let $(f_n)$ and $(B_n)$ be sequences of locally Lipschitz continuous maps with the same domains and codomains of $f$ and $B$, respectively, and $(u_{0n})$ a sequence of $\mathcal{F}_0$-measurable $H$-valued random variables. Let $u_n$, with lifetime $T_n$, be the unique local mild solution to the equation obtained replacing $f$, $B$, and $u_0$ in (2) with $f_n$, $B_n$, and $u_{0n}$, respectively. One has the following convergence result, proved (in a more general setting) in [17].

**Theorem 2.9.** Assume that

(i) $(f_n)$ and $(B_n)$ are uniformly locally Lipschitz continuous;

(ii) $f_n$ and $B_n$ converge pointwise to $f$ and $B$, respectively, as $n \to \infty$;

(iii) $u_{0n} \to u_0$ in probability as $n \to \infty$.

Then

$$u_n \mathbb{I}_{[0, T \land T_n]} \to u \mathbb{I}_{[0, T]}$$

in $L^p(\Omega \times \mathbb{R}_+; H)$ as $n \to \infty$.

3 Main result

The following hypotheses on the data of [11] will be in force throughout this section:

(a) $A$ is the generator of a positive $C_0$-semigroup $S$ on $H$.

(b) The mappings $f : \Omega \times \mathbb{R}_+ \times H \to H$ and $B : \Omega \times \mathbb{R}_+ \times H \to \mathcal{L}^2(U, H)$ are measurable and adapted, and there exist constants $C_f$ and $C_B$ such that

$$\|f(\omega, t, h_1) - f(\omega, t, h_2)\| \leq C_f \|h_1 - h_2\|,$$

$$\|f(\omega, t, h)\| \leq C_f (1 + \|h\|),$$

$$\|B(\omega, t, h_1) - B(\omega, t, h_2)\|_{\mathcal{L}^2(U, H)} \leq C_B \|h_1 - h_2\|,$$

$$\|B(\omega, t, h)\|_{\mathcal{L}^2(U, H)} \leq C_B (1 + \|h\|)$$

for all $\omega \in \Omega$, $t \in \mathbb{R}_+$, and $h, h_1, h_2 \in H$.

(c) One has, for every $\omega \in \Omega$, $t \in \mathbb{R}_+$, and $h \in H$,

$$-\langle h, f(\omega, t, h) \rangle + \|\mathbb{1}_{\{h \leq 0\}} B(\omega, t, h)\|_{\mathcal{L}^2(U, H)}^2 \lesssim \|h\|^2$$

By classical results on the well-posedness in the mild sense of stochastic evolution equations, it is known that, assuming $u_0 \in L^p(\Omega, \mathcal{F}_0, P)$ for some $p \geq 0$, for any $T > 0$ there exists a unique measurable, adapted, continuous (hence predictable) process

$$u \in L^p(\Omega; C([0, T]; H))$$
such that
\[ s \mapsto S(t-s)f(s,u(s)) \in L^1(0,t;H), \]
\[ s \mapsto S(t-s)B(s,u(s)) \in L^2(0,t;L^2(U,H)), \]
and
\[ u(t) = S(t)u_0 + \int_0^t S(t-s)f(s,u(s)) \, ds + \int_0^t S(t-s)B(s,u(s)) \, dW(s) \]
\( \mathbb{P}\)-a.s. for all \( t \in [0,T] \).

We can now formulate the main result of this section.

**Theorem 3.1.** Let \( u \) be a mild solution to (1). If \( u_0 \) is positive, then \( u(t) \) is positive for all \( t \in \mathbb{R}_+ \).

**Proof.** As a first step, we assume that \( A \) is a bounded operator, so that \( u \) is in fact a strong solution, in particular a semimartingale, that \( (Ah,h^-) \leq 0 \) for every \( h \in H \), and that \( \mathbb{E}\|u_0\|^2 < \infty \). Let \( g \) and \( G \) be as in lemma 2.7 and \( G_\alpha := R_\alpha G \), where \( (R_\alpha) \) is the hypoelliptic Ornstein-Uhlenbeck semigroup introduced in §2.3. Recalling that \( G_\alpha \in C^\infty(H) \) by lemma 2.4, Itô’s formula yields,

\[
G_\alpha(u) + \int_0^t \langle Au, G'_\alpha(u) \rangle \, ds = G_\alpha(u_0) + \int_0^t \langle f(u), G'_\alpha(u) \rangle \, ds + \int_0^t G'_{\alpha}(u)B(u) \, dW
\]
\[ + \frac{1}{2} \int_0^t \text{Tr } B(u) G''_{\alpha}(u) \, ds. \tag{4} \]

We are going to pass to the limit as \( \alpha \to 0 \) in each term of this identity. Proposition 2.3 implies that

\[ G_\alpha(u_0) \to G(u_0), \]
\[ G_\alpha(u(t)) \to G(u(t)) \quad \forall t > 0. \]

Moreover, \( G'_\alpha(u(s)) \to G'(u(s)) \) weakly for every \( s \geq 0 \) by corollary 2.4, hence

\[ \langle Au(s), G'_\alpha(u(s)) \rangle \to \langle Au(s), G'(u(s)) \rangle \]
\[ \langle f(u(s)), G'_\alpha(u(s)) \rangle \to \langle f(u(s)), G'(u(s)) \rangle \]

for every \( s \geq 0 \). The boundedness of \( g'' \) implies that \( G' \) grows at most linearly, therefore, recalling that \( C \) is negative-definite,

\[
\|G'_{\alpha}(u(s))\| \leq \int_H \|G'(e^{\alpha C}u(s) + y)\| \gamma_\alpha(dy) \lesssim \|u(s)\| + \int_H \|y\| \gamma_\alpha(dy).
\]

Since the last term on the right-hand side is finite for every \( \alpha > 0 \), \( A \) is bounded, and \( f \) also grows at most linearly, one has

\[
|\langle Au(s), G'_\alpha(u(s)) \rangle| + |\langle f(u(s)), G'_\alpha(u(s)) \rangle| \lesssim \alpha 1 + \|u(s)\|^2.
\]

As \( u \) is pathwise continuous, the dominated convergence theorem yields, for every \( t \geq 0 \),

\[
\int_0^t \langle Au(s), G'_\alpha(u(s)) \rangle \, ds \to \int_0^t \langle Au(s), G'(u(s)) \rangle \, ds
\]
\[ \int_0^t \langle f(u(s)), G'_\alpha(u(s)) \rangle \, ds \to \int_0^t \langle f(u(s)), G'(u(s)) \rangle \, ds.
\]
In order to have
\[ \int_0^t G'_\alpha(u(s))B(u(s)) \, dW(s) \longrightarrow \int_0^t G'(u(s))B(u(s)) \, dW(s) \]
in probability, it suffices to show that
\[ \int_0^t \|G'_\alpha(u)B(u) - G'(u)B(u)\|^2_{\mathcal{L}^2(U,R)} \, ds \longrightarrow 0 \]
in probability. Since \( G'_\alpha(h) \rightarrow G'(h) \) weakly for every \( h \in H \), i.e. the convergence takes place in \( \mathcal{L}(H,R) \), ideal properties of the space of Hilbert-Schmidt operators imply that
\[ \|G'_\alpha(u(s))B(u(s)) - G'(u(s))B(u(s))\|^2_{\mathcal{L}^2(U,R)} \longrightarrow 0 \]
for every \( s \in [0,t] \). We can thus conclude once again by the dominated convergence theorem, as
\[ \|G'_\alpha(u)B(u) - G'(u)B(u)\|_{\mathcal{L}^2(U,R)} \leq \|G'_\alpha(u) - G'(u)\| \|B(u)\|_{\mathcal{L}^2(U,H)} \lesssim_\alpha 1 + \|u\|^2. \]

Finally, denoting a complete orthonormal basis of \( U \) by \((e_k)\), corollary 2.6 also implies
\[ G''_\alpha(u(s))\{B(u(s))e_k, B(u(s))e_k\} \longrightarrow \delta G''(u(s))\{B(u(s))e_k, B(u(s))e_k\} \]
for every \( s \in [0,t] \) and \( k \in \mathbb{N} \). Moreover, since \( g'' \) is bounded and \( C \) is negative-definite, lemma 2.7 implies
\[ |G''_\alpha(u)(v, w)| = \left| \int_H \delta G''(e^{\alpha C}u + y)(e^{\alpha C}v, e^{\alpha C}w) \gamma_\alpha(dy) \right| \leq \int_H \|g''\|_{C(R)} \|v\| \|w\| \|\gamma_\alpha(dy)\| \lesssim \|v\| \|w\| \]
for every \( v, w \in H \). Therefore, since \( B(u(s)) \in \mathcal{L}^2(U,H) \), the dominated convergence theorem yields
\[ \int_0^t \text{Tr}_B(u) \, G''_\alpha(u) \, ds \longrightarrow \int_0^t \text{Tr}_B(u) \, \delta G''(u) \, ds \]
for every \( t \geq 0 \). We have thus shown that
\[ G(u) + \int_0^t \langle Au, G'(u) \rangle \, ds = G(u_0) + \int_0^t \langle f(u), G'(u) \rangle \, ds + \int_0^t G'(u)B(u) \, dW + \frac{1}{2} \int_0^t \text{Tr}_B(u) \, \delta G''(u) \, ds \]
\[ \tag{5} \]

Let \( (g_n) \subset C^2(\mathbb{R}) \) be a sequence of functions such that \( g_n(0) = g'_n(0) = 0 \) for all \( n \in \mathbb{N} \), \( g''_n \) is uniformly bounded, and
\[ g_n(x) \rightarrow \frac{1}{2}|x^-|^2, \quad g'_n(x) \rightarrow -x^-, \quad g''_n(x) \rightarrow \mathbb{1}_{\{x \leq 0\}} \]
for every \( x \in \mathbb{R} \). Defining the sequence of maps \((G_n)\) as
\[ G_n : H \longrightarrow \mathbb{R} \\
\quad v \longmapsto \int_X g_n(v(x)) \, m(dx), \]
The claim is thus proved under the additional assumptions that 

\[ \|u\|^2 - \int_0^T \langle A u, u \rangle \, ds = \frac{1}{2} \|u_0\|^2 - \int_0^T \langle f(u), u \rangle \, ds + \int_0^T u B(u) \, dW \]

\[ + \frac{1}{2} \int_0^T \text{Tr}B(u) \, \mathbb{1}_{\{u \leq 0\}} \, ds, \]

where

\[ \text{Tr}B(u) \, \mathbb{1}_{\{u \leq 0\}} = \|\mathbb{1}_{\{u \leq 0\}} B(u)\|_{L^2(U,H)}. \]

Since \( \langle A h, h^- \rangle \leq 0 \) for every \( h \in H \) by assumption, the second term on the left-hand side is positive. Moreover, thanks to assumption (c), the sum of the second and fourth term on the right-hand side can be estimated by

\[ C \int_0^T \|u\|^2 \, ds, \]

with \( C \) a positive constant. We are thus left with

\[ \|u\|^2 \leq \|u_0\|^2 + C \int_0^T \|u\|^2 \, ds + \int_0^T u B(u) \, dW. \]  

(6)

Let \( (T_n) \) be a localizing sequence for the continuous local martingale on the right-hand side \( u_n \). Introducing the stopped process \( u_n := u^{T_n} \), one has

\[ \|u_n\|^2 \leq \|u_0\|^2 + C \int_0^{\wedge T_n} \|u\|^2 \, ds + \int_0^{\wedge T_n} u B(u) \, dW \]

\[ \leq \|u_0\|^2 + C \int_0^{\wedge T_n} \|u_n\|^2 + \int_0^{\wedge T_n} u B(u) \, dW. \]

Recalling that \( \mathbb{E}\|u_0\|^2 < \infty \) by assumption, taking expectation on both sides and applying Tonelli’s theorem yields

\[ \mathbb{E}\|u_n\|^2 \leq \mathbb{E}\|u_0\|^2 + C \int_0^{\wedge T_n} \mathbb{E}\|u_n\|^2, \]

hence also, by Gronwall’s inequality,

\[ \mathbb{E}\|u_n(t)\|^2 \lesssim_t \mathbb{E}\|u_0\|^2 \quad \forall t \in \mathbb{R}_+. \]

Passing to the limit as \( n \to \infty \), Fatou’s lemma yields

\[ \mathbb{E}\|u^-(t)\|^2 \lesssim_t \mathbb{E}\|u^-\|^2 \quad \forall t \in \mathbb{R}_+. \]

The claim is thus proved under the additional assumptions that \( A \) is bounded, that \( \langle A h, h^- \rangle \leq 0 \) for every \( h \in H \), and that \( u_0 \) has finite second moment. We are now going to show that the result continues to hold also when these additional assumptions are not satisfied.

Let us assume that \( A \) is unbounded, and introduce its Yosida approximation

\[ A_\lambda := \frac{1}{\lambda}(I - (I + \lambda A)^{-1}), \quad \lambda > 0. \]

It is well known that \( A_\lambda \) is a bounded linear monotone operator. Let us show that the positivity preserving property of the semigroup \( S \) implies that \( \langle A_\lambda h, h^- \rangle \leq 0 \) for every \( h \in H \) and \( \lambda > 0 \): one has

\[ \lambda \langle A_\lambda h, h^- \rangle = \langle h, h^- \rangle - \langle (I + \lambda A)^{-1} h, h^- \rangle, \]
where $\langle h, h^- \rangle \leq -\|h^-\|^2$ and, recalling that the resolvent of $A$ is a positivity preserving contraction of $H$,

$$-\langle (I + \lambda A)^{-1} h, h^- \rangle = -\langle (I + \lambda A)^{-1} h^+, h^- \rangle + \langle (I + \lambda A)^{-1} h^-, h^- \rangle \leq \|h^-\|^2,$$

thus establishing the claim. Let us now consider the regularized equation

$$du_\lambda + A_\lambda u_\lambda \, dt = f(u_\lambda) \, dt + B(u_\lambda) \, dW, \quad u_\lambda(0) = u_0,$$

for which the first part of the proof implies that $u_\lambda \geq 0$. By virtue of the assumption that $u_0$ has finite second moment, one has, for any $T > 0$,

$$\lim_{\lambda \to 0} \mathbb{E} \sup_{t \leq T} \|u_\lambda(t) - u(t)\|^2 = 0,$$

where $u$ is the (unique) mild of (1) (see, e.g., [20]), from which it follows that the positivity of $u_0$ implies the positivity of $u(t)$ for all $t \geq 0$.

Finally, we can remove the assumption that $u_0 \in L^2(\mathbb{P})$. In fact, let $u_{0n} := 1_{\{\|u_0\| < n\}} u_0 \in L^2(\mathbb{P})$, and denote the unique mild solution to (1) with initial condition $u_{0n}$ by $u_n$. Then $u_n(t) \geq 0$ for all $t \geq 0$ by the previous reasoning, and theorem 2.9 implies that $u_n \to u$ in $L^0(\Omega \times [0, T]; H)$ for every $T > 0$. This in turn implies that $u(t) \geq 0$ for all $t \geq 0$, and the proof is completed.

\section{Heath-Jarrow-Morton models and Musiela’s equation}

As an application of the abstract positivity-preserving property of the previous section, we are going to provide sufficient conditions for the positivity of forward rates in the Heath-Jarrow-Morton (HJM) model (see [12]). We first briefly recall its origin and the reparametrization introduced by Musiela in [21].

Denoting the forward rate at time $t$ for date $T \geq t$ by $f(t, T)$, (a version of) the HJM model assumes that

$$f(t, T) = f(0, T) + \int_0^t \dot{\alpha}(s, T) \, ds + \sum_{k=1}^\infty \int_0^t \bar{\sigma}_k(s, T) \, dw^k(s), \quad (7)$$

where $(w^k)$ is a sequence of independent standard Wiener processes, $f(0, T)$ is an $\mathcal{F}_0$-measurable random variable, and $\dot{\alpha}(\cdot, T)$, $(\bar{\sigma}_k(\cdot, T))$ are predictable processes such that

$$\int_0^T |\dot{\alpha}(s, T)| + \|\bar{\sigma}(s, T)\|_{L_\infty}^2 \, ds < \infty$$

$\mathbb{P}$-almost surely. One of the major results of [12] is that the discounted bond price process $\tilde{B}(\cdot, T)$ implied by the forward rates $f(\cdot, T)$, i.e.

$$\tilde{B}(t, T) = \exp \left( -\int_0^t f(s, s) \, ds - \int_t^T f(t, s) \, ds \right),$$

is a local martingale (with respect to $\mathbb{P}$) if and only if

$$\dot{\alpha}(t, T) = \sum_{k=1}^\infty \bar{\sigma}_k(t, T) \int_t^T \dot{\bar{\sigma}}_k(t, s) \, ds. \quad (8)$$

Musiela observed in [21] that a simple change of variable allows to write (7), interpreted as a family of processes indexed by $T \in \mathbb{R}_+$, as the mild solution to a first-order stochastic PDE. In
particular, setting $x := T - t$ (which corresponds to considering the time to maturity rather than the time of maturity), (7) can be written as

$$f(t, t + x) = f(0, t + x) + \int_0^t \dot{\alpha}(s, t + x) \, ds + \sum_{k=1}^{\infty} \int_0^t \ddot{\sigma}_k(s, t + x) \, dw^k(s).$$

Introducing (for now in a purely formal way) the family of shift operators $(S(t))_{t \in \mathbb{R}_+}$ as

$$S(t): \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

and setting, for any function $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$, $S(t)\phi(x, y) := \phi(x, y + t)$, (7) can also be written as

$$f(t, t + x) = S(t)f(0, x) + \int_0^t S(t - s)\dot{\alpha}(s, s + x) \, ds + \sum_{k=1}^{\infty} \int_0^t S(t - s)\ddot{\sigma}_k(s, s + x) \, dw^k(s).$$

If there exists a Hilbert space of functions $H$ such that the family of shift operators $S(t)_{t \geq 0}$ is a strongly continuous semigroup and, on an event of probability one, $x \mapsto f(t, t + x) \in H$, $x \mapsto \alpha(\cdot, \cdot + x) \in L^1(0, t; H)$, and $x \mapsto \sigma(\cdot, \cdot + x) \in L^2(0, t; \mathbb{R}^2(H))$, setting

$$u(t) := f(t, t + \cdot), \quad \alpha(t) := \dot{\alpha}(t, t + \cdot), \quad \sigma(t) := \ddot{\sigma}(t, t + \cdot),$$

one has

$$u(t) = S(t)u(0) + \int_0^t S(t - s)\alpha(s) \, ds + \sum_{k=1}^{\infty} \int_0^t S(t - s)\sigma_k(s) \, dw^k(s), \quad (9)$$

e.i., in differential notation,

$$du + Au \, dt = \alpha(t) \, dt + \sum_k \sigma_k(t) \, dw^k(t),$$

where $-A$ is the infinitesimal generator of the translation semigroup $S$, hence an operator that acts on smooth functions as a first derivative.

It is by now well known that Hilbert spaces of functions indeed exist that permit to make the above reasoning rigorous. A particularly convenient choice of forward curves is the class of Hilbert spaces $H_\alpha$, $\alpha > 0$, defined in §2. In fact, elements of $H_\alpha$ are continuous functions, and constant functions belong to $H_\alpha$. These properties are essential, as empirical observations suggest that forward curves are smooth (at least continuous) and tend to flatten out for large times to maturity without necessarily decaying to zero. For technical reasons that will become apparent later, we shall also consider as state space the Hilbert space $L^2_{-\alpha}$ defined in §2.2.

In order to consider HJM models (in Musiela’s parametrization) for which the diffusion coefficient depends on the forward curve itself, we are naturally led to consider stochastic evolution equations, on the Hilbert spaces $H_\alpha$ and $L^2_{-\alpha}$, $\alpha > 0$, of the form

$$du + Au \, dt = \beta(t, u) \, dt + \sigma(t, u) \, dW, \quad u(0) = u_0, \quad (10)$$

to which we shall refer as Musiela’s equation. Here $W$ is a cylindrical Wiener process on a separable Hilbert space $U$, as in §3 and, writing $H$ for either $H_\alpha$ or $L^2_{-\alpha}$, $\sigma: \Omega \times \mathbb{R}_+ \times H \rightarrow \mathbb{L}^2(U, H)$ satisfies measurability, Lipschitz-continuity, and linear growth assumptions completely analogous to those imposed on $B$ in §3 and $\beta: \Omega \times \mathbb{R}_+ \times H \rightarrow H$ is such that, at least formally,

$$\beta(t, v) = \langle \sigma(t, v), I\sigma(t, v) \rangle_U,$$

where, for any separable Hilbert space $K$, $I$ is the operator defined by

$$I: L^1_{\text{loc}}(\mathbb{R}_+; K) \rightarrow L^1_{\text{loc}}(\mathbb{R}_+; K),$$

$$v \mapsto \int_0^\cdot v(y) \, dy.$$
Proof. Let for every \( \omega \) bounded, uniformly over \( \Omega \), one has
\[
\beta(t, v) = \sum_{k=1}^{\infty} \sigma_k(t, v) I \sigma_k(t, v) = \sum_{k=1}^{\infty} \sigma_k(t, v) \int_{0}^{\infty} [\sigma_k(t, v)](y) \, dy.
\]

4.1 Well-posedness on \( H_\alpha \)

It is not difficult to show (cf. [10, Corollary 5.1.2]) that the map \( h \mapsto h I h \) is a locally Lipschitz continuous map from \( H_\alpha \). More precisely, one has
\[
\| h I h - g I g \|_{H_\alpha} \lesssim (\| h \|_{H_\alpha} + \| g \|_{H_\alpha}) \| h - g \|_{H_\alpha} \quad \forall h, g \in H_\alpha.
\]

The following simple extension will be used several times.

**Lemma 4.1.** The map formally defined on \((H_\alpha)^N\) as
\[
h \mapsto \langle h, I h \rangle := \sum_{k=1}^{\infty} h_k I h_k
\]
is a locally Lipschitz continuous map from \( \ell^2(H_\alpha) \) to \( H_\alpha \). More precisely,
\[
\| \langle h, I h \rangle - \langle g, I g \rangle \|_{H_\alpha} \lesssim (\| h \|_{\ell^2(H_\alpha)} + \| g \|_{\ell^2(H_\alpha)}) \| h - g \|_{\ell^2(H_\alpha)}
\]
for every \( h, g \in \ell^2(H_\alpha) \).

**Proof.** Let \( h, g \in \ell^2(H_\alpha) \). The Minkowski and Cauchy-Schwarz inequalities imply
\[
\| \langle h, I h \rangle - \langle g, I g \rangle \|_{H_\alpha} = \left\| \sum_{k=1}^{\infty} (h_k I h_k - g_k I g_k) \right\|_{H_\alpha}
\leq \sum_{k=1}^{\infty} \| h_k I h_k - g_k I g_k \|_{H_\alpha}
\leq \sum_{k=1}^{\infty} (\| h_k \|_{H_\alpha} \| g_k \|_{H_\alpha} + \| g_k \|_{H_\alpha} \| h_k \|_{H_\alpha})
\leq (\| h \|_{\ell^2(H_\alpha)} + \| g \|_{\ell^2(H_\alpha)}) \| h - g \|_{\ell^2(H_\alpha)}.
\]

As an immediate corollary it follows that
\[
\| \beta(v_1) - \beta(v_2) \|_{H_\alpha} \lesssim (\| \sigma(v_1) \|_{\ell^2(U, H_\alpha)} + \| \sigma(v_2) \|_{\ell^2(U, H_\alpha)}) \| \sigma(v_1) - \sigma(v_2) \|_{\ell^2(U, H_\alpha)},
\]
for any \( v_1, v_2 \in H_\alpha \) (here and in the following we suppress the explicit indication of the dependence on \( \omega \) and \( t \)). In particular, \( \beta \) is Lipschitz continuous if \( \sigma \) is Lipschitz continuous and bounded, uniformly over \( \Omega \times \mathbb{R}_+ \). If \( \sigma \) is just locally Lipschitz continuous and locally bounded, uniformly over \( \Omega \times \mathbb{R}_+ \), the same holds for \( \beta \).

It turns out, however, that diffusion coefficients \( \sigma = (\sigma_k) \) given by superposition operators are not Lipschitz continuous and bounded, even for very regular functions, so that global well-posedness (in the mild sense) of [10] is not guaranteed. However, we are going to show that, under suitable conditions, they are locally Lipschitz continuous and bounded, so that [10] is locally well posed. Analogous results are proved in [10] §5.4, but we provide nonetheless a proof for several reasons: we use a different norm on \( H_\alpha \), our assumptions are slightly different, and we shall extensively employ these estimates later.

In the following, for any function \( \phi : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \), we shall denote by \( \partial_1 \phi \) and \( \partial_2 \phi \) the partial derivatives of \( \phi \) with respect to its third and fourth argument, respectively.

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Hypothesis 1. Let $\psi = (\psi_k) \in \ell^2(L^2_\alpha)$, with $\psi_k \geq 0$ for all $k \in \mathbb{N}$, and $\eta = (\eta_k) \subset \mathbb{R}^2$ be a sequence of positive increasing even functions such that $\tilde{\eta} := \|\eta(\cdot)\|_{\ell^2} : \mathbb{R} \to \mathbb{R}$ is bounded on bounded sets. The functions

$$\sigma_k : \Omega \times \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}, \quad k \in \mathbb{N},$$

are measurable with respect to the $\sigma$-algebra $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$, and $\sigma_k(\omega, t, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}_+)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$. Moreover, they satisfy the following conditions:

(a) $\lim_{x \to -\infty} \sigma_k(\omega, t, x, r) = 0$ for all $r \in \mathbb{R}$;
(b) $|\partial_1 \sigma_k(\omega, t, x, r)| \leq \psi_k(x) \eta_k(r)$ for all $(\omega, t, x, r) \in \Omega \times \mathbb{R}^2_+ \times \mathbb{R}$;
(c) $|\partial_2 \sigma_k(\omega, t, x, r)| \leq \eta_k(r)$ for all $(\omega, t, x, r) \in \Omega \times \mathbb{R}^2_+ \times \mathbb{R}$;
(d) $|\partial_1 \sigma_k(\omega, t, x, r_1) - \partial_1 \sigma_k(\omega, t, x, r_2)| \leq \psi_k(x)(\eta_k(r_1) + \eta_k(r_2))|r_1 - r_2|$ for all $(\omega, t, x) \in \Omega \times \mathbb{R}^2_+$ and $r_1, r_2 \in \mathbb{R}$;
(e) $|\partial_2 \sigma_k(\omega, t, x, r_1) - \partial_2 \sigma_k(\omega, t, x, r_2)| \leq (\eta_k(r_1) + \eta_k(r_2))|r_1 - r_2|$ for all $(\omega, t, x) \in \Omega \times \mathbb{R}^2_+$ and $r_1, r_2 \in \mathbb{R}$.

Proposition 4.2. Assume that Hypothesis 1 is satisfied. Then $\sigma$ is a well-defined map from $\Omega \times \mathbb{R}_+ \times H_\alpha$ to $L^2(U, H_\alpha)$, measurable with respect to the $\sigma$-algebra $\mathcal{P} \otimes \mathcal{B}(H_\alpha)$, and it satisfies the estimates

$$\|\sigma(\omega, t, v)\|_{L^2(U, H_\alpha)} \leq \tilde{\eta}(\delta(v)\|H_\alpha\|),$$

$$\|\sigma(\omega, t, v_1) - \sigma(\omega, t, v_2)\|_{L^2(U, H_\alpha)} \leq \alpha \left( \tilde{\eta}(\delta(v_1)\|H_\alpha\|) + \tilde{\eta}(\delta(v_2)\|H_\alpha\|) \right) \cdot \|v_1 - v_2\|_{H_\alpha}$$

for all $(\omega, t) \in \Omega \times \mathbb{R}_+$ and $v, v_1, v_2 \in H_\alpha$. In particular, $\sigma$ is locally bounded and locally Lipschitz continuous in its third argument, uniformly over $\Omega \times \mathbb{R}_+$.

Proof. Throughout the proof we shall omit the explicit indication of the first two arguments of $\sigma$ as well as of $\sigma_k$. Since $\sigma_k(\cdot, r)$ is zero at infinity for all $r \in \mathbb{R}$, one has, by the triangle inequality in $\ell^2(L^2_\alpha)$,

$$\|\sigma(v)\|_{L^2(U, H_\alpha)} \leq \left( \sum_{k=1}^\infty \|\partial_1 \sigma_k(\cdot, v) + \partial_2 \sigma_k(\cdot, v')\|^2_{L^2_\alpha} \right)^{1/2} \leq \left( \sum_{k=1}^\infty \|\partial_1 \sigma_k(\cdot, v)\|^2_{L^2_\alpha} \right)^{1/2} + \left( \sum_{k=1}^\infty \|\partial_2 \sigma_k(\cdot, v')\|^2_{L^2_\alpha} \right)^{1/2}$$

for every $v \in H_\alpha$. It follows by (b) that

$$\|\partial_1 \sigma_k(\cdot, v)\|_{L^2_\alpha} \leq \|\psi_k(\eta_k(v))\|_{L^2_\alpha} \leq \|\psi_k\|_{L^2_{\infty}} \|\eta_k(v)\|_{L^2_{\infty}}$$

where, denoting by $\delta = \delta(\alpha)$ the (operator) norm of the embedding $H_\alpha \hookrightarrow L^\infty$, the assumptions on $\eta_k$ imply

$$\|\eta_k(v)\|_{L^\infty} \leq \eta_k(\|v\|_{L^\infty}) \leq \eta_k(\delta(v)\|H_\alpha\|),$$

hence also

$$\|\partial_1 \sigma_k(\cdot, v)\|_{L^2_\alpha} \leq \|\psi_k\|_{L^2_{\infty}} \eta_k(\delta(v)\|H_\alpha\|).$$
Therefore, applying the Cauchy-Schwarz inequality and \( \| \cdot \|_{\ell^2} \leq \| \cdot \|_{\ell^2} \), one has
\[
\left( \sum_{k=1}^{\infty} \| \partial_t \sigma_k(\cdot, v) \|_{L^2_\alpha}^2 \right)^{1/2} \leq \left( \sum_{k=1}^{\infty} \| \psi_k \|_{L^2_\alpha}^2 \eta_k^2 (\delta v H_\alpha) \right)^{1/2} \\
\leq \left( \sum_{k=1}^{\infty} \| \psi_k \|_{L^2_\alpha}^4 \right)^{1/4} \left( \sum_{k=1}^{\infty} \eta_k^4 (\delta v H_\alpha) \right)^{1/4} \\
\leq \| \psi \|_{L^2(\mathbb{R}_+; \tilde{\eta} (\delta v H_\alpha))}.
\]

Similarly, (c) yields
\[
\| \partial_2 \sigma_k(\cdot, v) v' \|_{L^2_\alpha} \leq \| \eta_k(v) v' \|_{L^2_\alpha} \leq \| \eta_k(v) \|_{L^\infty} \| v \|_{H_\alpha} \leq \eta_k (\delta v H_\alpha) \| v \|_{H_\alpha},
\]
hence
\[
\left( \sum_{k=1}^{\infty} \| \partial_2 \sigma_k(\cdot, v) v' \|_{L^2_\alpha}^2 \right)^{1/2} \leq \tilde{\eta} (\delta v H_\alpha) \| v \|_{H_\alpha}.
\]

We have thus shown that
\[
\| \sigma(v) \|_{X^2(U, H_\alpha)} \leq \tilde{\eta} (\delta v H_\alpha) \| v \|_{H_\alpha} + \| \psi \|_{L^2(\mathbb{R}_+; \tilde{\eta} (\delta v H_\alpha))},
\]
from which it follows immediately that \( \sigma \) is well defined and locally bounded, recalling that \( \tilde{\eta} \) is bounded on bounded sets. Let us now establish the local Lipschitz continuity of \( \sigma \). In analogy to a previous computation, one has, for any \( v_1, v_2 \in H_\alpha \),
\[
\| \sigma(v_1) - \sigma(v_2) \|_{X^2(U, H_\alpha)} \leq \left( \sum_{k=1}^{\infty} \| \partial_1 \sigma_k(\cdot, v_1) - \partial_1 \sigma_k(\cdot, v_2) \|_{L^2_\alpha}^2 \right)^{1/2} \\
+ \left( \sum_{k=1}^{\infty} \| \partial_2 \sigma_k(\cdot, v_1)v'_1 - \partial_2 \sigma_k(\cdot, v_2)v'_2 \|_{L^2_\alpha}^2 \right)^{1/2},
\]
where, thanks to (d),
\[
\| \partial_1 \sigma_k(\cdot, v_1) - \partial_1 \sigma_k(\cdot, v_2) \|_{L^2_\alpha}^2 \leq \| \psi_k (\eta_k(v_1) + \eta_k(v_2)) v_1 - v_2 \|_{L^2_\alpha}^2 \\
\leq \| \psi_k \|_{L^2_\alpha}^2 \| (\eta_k(v_1) + \eta_k(v_2)) \|_{L^\infty} \| v_1 - v_2 \|_{L^\infty}^2 \\
\leq \delta \| \psi_k \|_{L^2_\alpha} \left( \eta_k (\delta v_1 H_\alpha) + \eta_k (\delta v_2 H_\alpha) \right) \| v_1 - v_2 \|_{H_\alpha} ^2,
\]
which implies, similarly to a previous computation,
\[
\left( \sum_{k=1}^{\infty} \| \partial_1 \sigma_k(\cdot, v_1) - \partial_1 \sigma_k(\cdot, v_2) \|_{L^2_\alpha}^2 \right)^{1/2} \leq \delta \| \psi \|_{L^2(\mathbb{R}_+; \tilde{\eta} (\delta v H_\alpha))} \left( \tilde{\eta} (\delta v_1 H_\alpha) + \tilde{\eta} (\delta v_2 H_\alpha) \right) \| v_1 - v_2 \|_{H_\alpha}.
\]

Moreover, one has
\[
\| \partial_2 \sigma_k(\cdot, v_1)v'_1 - \partial_2 \sigma_k(\cdot, v_2)v'_2 \|_{L^2_\alpha} \leq \| \partial_2 \sigma_k(\cdot, v_1)(v'_1 - v'_2) \|_{L^2_\alpha} + \| (\partial_2 \sigma_k(\cdot, v_1) - \partial_2 \sigma_k(\cdot, v_2))v'_2 \|_{L^2_\alpha},
\]
where, by (c),
\[
\| \partial_2 \sigma_k(\cdot, v_1)(v'_1 - v'_2) \|_{L^2_\alpha} \leq \| \eta_k(v_1) \|_{L^\infty} \| v_1 - v_2 \|_{H_\alpha} \leq \eta_k (\delta v H_\alpha) \| v_1 - v_2 \|_{H_\alpha}.
\]
and, by (e),
\[
\left\| (\partial_2 \sigma_k(\cdot, v_1) - \partial_2 \sigma_k(\cdot, v_2))v_2' \right\|_{L^2_\alpha} \\
\leq \left\| (\eta_k(v_1) + \eta_k(v_2))(v_1 - v_2)v_2' \right\|_{L^2_\alpha} \\
\leq \left\| \eta_k(v_1) + \eta_k(v_2) \right\|_{L^\infty} \left\| v_1 - v_2 \right\|_{L^\infty} \left\| v_2 \right\|_{H^\alpha} \\
\leq \delta \left( \eta_k(\delta \|v_1\|_{H^\alpha}) + \eta_k(\delta \|v_2\|_{H^\alpha}) \right) \|v_2\|_{H^\alpha} \left\| v_1 - v_2 \right\|_{H^\alpha},
\]
so that
\[
\left\| \partial_2 \sigma_k(\cdot, v_1)v_1' - \partial_2 \sigma_k(\cdot, v_2)v_2' \right\|_{L^2_\alpha} \\
\lesssim_\alpha \left( \eta_k(\delta \|v_1\|_{H^\alpha}) + \eta_k(\delta \|v_2\|_{H^\alpha}) \right) \|v_2\|_{H^\alpha} \left\| v_1 - v_2 \right\|_{H^\alpha},
\]
and
\[
\left( \sum_{k=1}^\infty \left\| \partial_2 \sigma_k(\cdot, v_1)v_1' - \partial_2 \sigma_k(\cdot, v_2)v_2' \right\|_{L^2_\alpha} \right)^{1/2} \\
\lesssim_\alpha \left( \tilde{\eta}(\delta \|v_1\|_{H^\alpha}) + \tilde{\eta}(\delta \|v_2\|_{H^\alpha}) \right) \|v_2\|_{H^\alpha} \left\| v_1 - v_2 \right\|_{H^\alpha}.
\]
We have thus proved that
\[
\left\| \sigma(v_1) - \sigma(v_2) \right\|_{L^2(U, H^\alpha)} \\
\lesssim_\alpha \left( \tilde{\eta}(\delta \|v_1\|_{H^\alpha}) + \tilde{\eta}(\delta \|v_2\|_{H^\alpha}) \right) \left( \|\psi\|_{L^2(U, H^\alpha)} + \|v_2\|_{H^\alpha} \right) \left\| v_1 - v_2 \right\|_{H^\alpha},
\]
which implies, thanks to the assumptions on \(\eta\), the asserted local Lipschitz continuity of \(\sigma\). \(\square\)

### 4.2 Well-posedness on \(L^2_{-\alpha}\)

Let us begin with the following estimate.

**Lemma 4.3.** Let \(\alpha > 0\), and \(f = (f_k)\), \(g = (g_k)\) be sequences of real-valued functions on \(\mathbb{R}_+\). Then
\[
\left\| \langle f, I^g \rangle_{\mathcal{L}} \right\|_{L^2_{-\alpha}} \lesssim_\alpha \left\| f \right\|_{L^2(U, H^\alpha)} \left\| g \right\|_{L^2_{-\alpha}}^2.
\]

**Proof.** One has
\[
\left\| \langle f, I^g \rangle_{\mathcal{L}} \right\|_{L^2_{-\alpha}}^2 = \int_0^\infty \left\| \sum_k f_k(x) \int_0^y g_k(y) \, dy \right\| e^{-\alpha x} \, dx \\
\leq \int_0^\infty \left\| \sum_k f_k(x) e^{-\frac{\alpha}{2} x} \int_0^y g_k(y) \, dy \right\|^2 \, dx \\
\leq \int_0^\infty \left\| \sum_k f_k(x) e^{\frac{\alpha}{2} x} \int_0^\infty g_k(y) \, dy \right\|^2 \, dx,
\]
where
\[
\int_0^\infty g_k(y) \, dy = \left\| g_k \right\|_{L^1_{-\alpha}} \lesssim_\alpha \left\| g_k \right\|_{L^2_{-\alpha}},
\]
thus also
\[
\sum_k f_k(x) e^{\frac{\alpha}{2} x} \int_0^\infty g_k(y) \, dy = \left\langle (f_k e^{\frac{\alpha}{2} x}), \left\| g_k \right\|_{L^2_{-\alpha}} \right\rangle_{L^2} \\
\leq \left\| (f_k e^{\frac{\alpha}{2} x}) \right\|_{L^2} \left\| g \right\|_{L^2_{-\alpha}},
\]

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Therefore
\[
\|\langle f, I g \rangle \|_{L^2_{\alpha}}^2 \lesssim_\alpha \|\|f\|_{L^2_{\alpha}}\|^2 \|g\|_{L^2(L^2_{\alpha})}^2 = \|f\|_{L^2_{\alpha}(\ell^2)}^2 \|g\|_{L^2(L^2_{\alpha})}^2. 
\]

We can now provide conditions on $\sigma$ implying that $\beta$ satisfies a suitable form of Lipschitz continuity.

**Lemma 4.4.** Assume that $\sigma(\omega, t, \cdot)$ is Lipschitz continuous and bounded from $L^2_{-\alpha}$ to $L^2(U, L^2_{\alpha})$ uniformly with respect to $(\omega, t) \in \Omega \times \mathbb{R}_+$, i.e. that there exists a constant $N$ such that
\[
\sum_{k=1}^{\infty} \|\sigma_k(v_1) - \sigma_k(v_2)\|_{L^2_{\alpha}}^2 \leq N \|v_1 - v_2\|_{L^2_{\alpha}}^2, \quad \sum_{k=1}^{\infty} \|\sigma_k(v)\|_{L^2_{\alpha}}^2 < N
\]
for all $v, v_1, v_2 \in L^2_{-\alpha}$, uniformly over $\Omega \times \mathbb{R}_+$. Then $\beta(\omega, t, \cdot)$ is a Lipschitz continuous endomorphism of $L^2_{\alpha}$, uniformly with respect to $(\omega, t) \in \Omega \times \mathbb{R}_+$.

**Proof.** For any $v_1, v_2 \in L^2_{\alpha}$ one has
\[
\|\beta(v_1) - \beta(v_2)\|_{L^2_{\alpha}} = \| \langle \sigma(v_1), I \sigma(v_1) \rangle_{\ell^2} - \langle \sigma(v_2), I \sigma(v_2) \rangle_{\ell^2} \|_{L^2_{\alpha}}
\]
\[
= \| \langle \sigma(v_1) - \sigma(v_2), I \sigma(v_1) \rangle_{\ell^2} - \langle \sigma(v_2), I \sigma(v_1) - I \sigma(v_2) \rangle_{\ell^2} \|_{L^2_{\alpha}}
\]
\[
= \| \langle \sigma(v_1) - \sigma(v_2), I \sigma(v_1) \rangle_{L^2_{\alpha}} \|_{L^2_{\alpha}} + \| \langle \sigma(v_2), I \sigma(v_1) - I \sigma(v_2) \rangle_{\ell^2} \|_{L^2_{\alpha}},
\]
where
\[
\| \langle \sigma(v_1) - \sigma(v_2), I \sigma(v_1) \rangle_{\ell^2} \|_{L^2_{\alpha}} \leq \|\|\sigma(v_1) - \sigma(v_2)\||_{\ell^2} \|I \sigma(v_1)\|_{L^2_{\alpha}}
\]
\[
\leq \|\sigma(v_1) - \sigma(v_2)\|_{L^2_{\alpha}(\ell^2, L^2_{\alpha})} \|I \sigma(v_1)\|_{L^2_{\alpha}(\ell^2, L^2_{\alpha})},
\]
and
\[
\|I \sigma(v_1)\|_{\ell^2} = \left(\sum_k \|I \sigma_k(v_1)\|^2\right)^{1/2} \leq \left(\sum_k \|\sigma_k(v_1)\|^2_{L^2}\right)^{1/2} = \|\sigma(v_1)\|_{\ell^2(L^2)},
\]
hence, recalling that $L^2_{\alpha} \hookrightarrow L^1$,
\[
\| \langle \sigma(v_1) - \sigma(v_2), I \sigma(v_1) \rangle_{\ell^2} \|_{L^2_{\alpha}} \leq N \|\sigma(v_1)\|_{\ell^2(L^2)} \|v_1 - v_2\|_{L^2_{\alpha}}.
\]
Furthermore, the previous lemma yields, by linearity of $I$,
\[
\| \langle \sigma(v_2), I \sigma(v_1) - I \sigma(v_2) \rangle_{\ell^2} \|_{L^2_{\alpha}} \lesssim_\alpha \|\sigma(v_2)\|_{\ell^2(L^2)} \|\sigma(v_1) - \sigma(v_2)\|_{\ell^2(L^2_{\alpha})} \leq N \|\sigma(v_2)\|_{\ell^2(L^2)} \|v_1 - v_2\|_{L^2_{\alpha}}.
\]
Recalling that $\sigma$ is bounded from $L^2_{-\alpha}$ to $L^2(\ell^2, L^2_{\alpha})$, we conclude that
\[
\|\beta(v_1) - \beta(v_2)\|_{L^2_{\alpha}} \lesssim \|v_1 - v_2\|_{L^2_{\alpha}},
\]
where the implicit constant depends on $\alpha$ and $N$.

We can now give a well-posedness result for Musielak’s equation on the state space $L^2_{-\alpha}$. 19
Proposition 4.5. Let $p > 0$. Assume that $\sigma$ satisfies the assumptions of lemma 4.4 and that $u_0 \in L^p(\Omega; \mathcal{F}_0; L_{-\alpha}^2)$. Then equation (13) admits a unique mild solution $u \in L^p(\Omega; C([0,T]; L_{-\alpha}^2))$ for every $T > 0$. Moreover, the solution $u$ depends continuously on the initial datum $u_0$. More precisely, the map

$$L^p(\Omega; \mathcal{F}_0; L_{-\alpha}^2) \rightarrow L^p(\Omega; C([0,T]; L_{-\alpha}^2))$$

$$u_0 \mapsto u$$

is Lipschitz continuous.

The proof follows by general well-posedness results in the mild sense for stochastic evolution equations in Hilbert spaces with Lipschitz continuous nonlinearities (see, e.g., [4]).

As in the previous subsection, we shall now focus on the case where, for every $k \in \mathbb{N}$, $\sigma_k$ is given by a (random) superposition operator acting on $u$: with a harmless abuse (or, better, overload) of notation, we assume that

$$\sigma_k(\omega, t, u) = \sigma_k(\omega, t, \cdot, u(t, \cdot)),$$

where the function $\sigma_k : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable with respect to the $\sigma$-algebra $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$.

Existence and uniqueness of a mild solution to Musiela’s equation is established next.

Proposition 4.6. Assume that the functions

$$\sigma_k : \Omega \times \mathbb{R}_+^2 \times \mathbb{R} \rightarrow \mathbb{R}, \quad k \in \mathbb{N},$$

satisfy the following conditions:

(a) there exists $\theta = (\theta_k) \in \ell^2(L^2_\alpha)$ such that $|\sigma_k(\omega, t, x, r)| \leq \theta_k(x)$ for all $(\omega, t, x, r) \in \Omega \times \mathbb{R}_+^2 \times \mathbb{R}$.

(b) there exists $c = (c_k) \in \ell^2$ such that $|\sigma_k(\omega, t, x, r_1) - \sigma_k(\omega, t, x, r_2)| \leq c_k |r_1 - r_2|$ for all $(\omega, t, x) \in \Omega \times \mathbb{R}_+^2$ and $r_1, r_2 \in \mathbb{R}$.

If $u_0 \in L^p(\Omega; \mathcal{F}_0; L_{-\alpha}^2)$, $p > 0$, then all assertions of proposition 4.5 hold.

Proof. It suffices to check that (a) and (b) imply that $\sigma$ satisfies the assumptions of lemma 4.4. In fact, by (a) it follows that

$$\|\sigma(v_1) - \sigma(v_2)\|_{\mathcal{L}^2(U; L_{-\alpha}^2)} = \left( \sum_k \|\sigma_k(v_1) - \sigma_k(v_2)\|_{L_{-\alpha}^2}^2 \right)^{1/2} = \left( \sum_k c_k^2 \right)^{1/2} \|v_1 - v_2\|_{L_{-\alpha}^2}$$

for all $v_1, v_2 \in L_{-\alpha}^2$, i.e. $\sigma(\omega, t, \cdot)$ is Lipschitz continuous from $L_{-\alpha}^2$ to $\mathcal{L}^2(U, L_{-\alpha}^2)$, uniformly with respect to its other arguments, with Lipschitz constant $\|c\|_{\ell^2}$. Moreover, (b) implies

$$\|\sigma(v)\|_{\mathcal{L}^2(U; L_{-\alpha}^2)}^2 = \sum_{k=1}^{\infty} \|\sigma_k(v)\|_{L_{-\alpha}^2}^2 \leq \sum_{k=1}^{\infty} \|\theta_k\|_{L_{-\alpha}^2}^2 = \|\theta\|_{\ell^2(L_{-\alpha}^2)}^2 < \infty$$

for all $v \in L_{-\alpha}^2$ and $(\omega, t) \in \Omega \times \mathbb{R}_+$. We have hence shown that the assumptions of lemma 4.4 are satisfied.\[\Box\]

Remark 4.7. Let $\alpha > 0$. It is proved in [10] §5.5] that Musiela’s equation is globally well posed on the state space $H_\alpha$, rather than just locally, if, for every $k \in \mathbb{N}$, $(\sigma(v), c_k)(x) = \sigma_k(x, \zeta_k(v))$, where $\zeta_k \in \mathcal{L}(H_\alpha, \mathbb{R})$, and
(a) \(|\partial_t \sigma_k(x, r)| \leq \psi_k(x)\);

(b) \(|\partial_t \sigma_k(x, r_1) - \partial_t \sigma_k(x, r_2)| \leq \psi_k(x)|r_1 - r_2|\).

Here \(\psi = (\psi_k)\) and the measurability assumptions on the functions \(\sigma_k\) are the same as in hypothesis 1. It is natural to ask whether the conditions (a) and (b) imply well-posedness on the state space \(L^2_{\alpha}\). We are going to show that a bit more integrability on \(\psi\) implies that this indeed the case. We point out, however, that (a) and (b) alone imply only the existence of a \(\psi\)-local mild solution on the state space \(L^2_{\alpha}\). We are going to show that (a) and (b) above, with \(\psi \in L^2_{\alpha+\varepsilon}(\ell^2)\), \(\varepsilon > 0\), imply that \(\sigma\) satisfies the assumptions of Lemma 4.4. In fact, the identity

\[
\sigma_k(\infty, r) - \sigma_k(x, r) = -\sigma_k(x, r) = \int_x^\infty \partial_t \sigma_k(y, r) \, dy
\]

implies

\[
\|\sigma(v)\|^2_{L^2(\mu, L^2_{\alpha})} = \sum_{k=1}^\infty \|\sigma_k(v)\|^2_{L^2_{\alpha}} \\
\leq \sum_{k=1}^\infty \int_0^\infty \left|\int_x^\infty \psi_k(y) \, dy\right|^2 e^{\alpha x} \, dx \\
\leq \sum_{k=1}^\infty \int_0^\infty \left|\int_x^\infty \psi_k(y)e^{\frac{\alpha x}{2}}e^{-\frac{\psi_1}{x}} \, dy\right|^2 \, dx,
\]

where, by the Cauchy-Schwarz inequality,

\[
\int_x^\infty \psi_k(y)e^{\frac{\alpha x}{2}}e^{-\frac{\psi_1}{x}} \, dy \leq \|\psi_k\|_{L^2_{\alpha+\varepsilon}} \left(\int_x^\infty e^{-\varepsilon y} \, dy\right)^{1/2} = \frac{1}{\sqrt{\varepsilon}}e^{\frac{\alpha x}{2}}\|\psi_k\|_{L^2_{\alpha+\varepsilon}},
\]

hence

\[
\|\sigma(v)\|^2_{L^2(\mu, L^2_{\alpha})} \leq \frac{1}{\varepsilon} \sum_{k=1}^\infty \|\psi_k\|^2_{L^2_{\alpha+\varepsilon}} \int_0^\infty e^{-\varepsilon x} \, dx = \frac{1}{\varepsilon^2}\|\psi\|^2_{L^2_{\alpha+\varepsilon}(\ell^2)}.
\]

Let us now consider Lipschitz continuity: from

\[
\sigma_k(\infty, r_1) - \sigma_k(\infty, r_2) - (\sigma_k(x, r_1) - \sigma_k(x, r_2)) = - (\sigma_k(x, r_1) - \sigma_k(x, r_2)) = \int_x^\infty (\partial_t \sigma_k(y, r_1) - \partial_t \sigma_k(y, r_2)) \, dy
\]

and condition (b) above it follows that

\[
|\sigma_k(x, r_1) - \sigma_k(x, r_2)| \leq |r_1 - r_2| \int_x^\infty |\psi_k(y)| \, dy \leq |r_1 - r_2| \int_0^\infty |\psi_k(y)|e^{\frac{\psi_1}{x}}e^{-\frac{\psi_1}{x}} \, dy \\
\leq \alpha \|\psi_k\|_{L^2_{\alpha}} |r_1 - r_2|.
\]

Since the \(\ell^2\)-norm of \(|\psi_k\|_{L^2_{\alpha}}\) is finite by assumption, \(\sigma_k\) satisfies hypothesis 2(a) with \(c_k := \|\psi_k\|_{L^2_{\alpha}}\), which in turn implies, as in the proof of proposition 4.6, the Lipschitz continuity of \(\sigma\) asserted in lemma 4.4.
5 Approximations I

Here we consider Musiela’s equation with \((\sigma, c_k)\) given by superposition operators associated to functions

\[
\sigma_k : \Omega \times \mathbb{R}_+ ^ 2 \times \mathbb{R} \to \mathbb{R}, \quad k \in \mathbb{N},
\]
we introduce approximations of such functions, and we prove several estimates and convergence results for the solutions to the approximated equations.

For each \(n \in \mathbb{N}\), let \(\chi_n \in C^\infty_c (\mathbb{R}_+)\) be a smooth cut-off function, i.e. \(\chi_n = 1\) on \([0, n]\), \(0 \leq \chi_n \leq 1\) and \(|\chi_n'| \leq 2\) in \([n, n+1]\), and \(\chi_n = 0\) on \([n + 1, \infty]\). For every \(k\) and \(n \in \mathbb{N}\), let us set

\[
\sigma_k^{(n)}(x, r) := \sigma_k(x, r) \chi_n(x) \quad \forall (x, r) \in \mathbb{R}_+ \times \mathbb{R}.
\]

Here and in the following, as already done before, we suppress the explicit indication of the dependence on \((\omega, t) \in \Omega \times \mathbb{R}_+\) where it is not essential.

The next lemma, while rather simple, is of crucial importance.

**Lemma 5.1.** Let \(\psi = (\psi_k) \in \ell^2(L_\alpha)\). For every \(n \in \mathbb{N}\), let \(\psi^{(n)} = (\psi_k^{(n)})\) and \(\tilde{\psi}^{(n)} = (\tilde{\psi}_k^{(n)})\) be defined as

\[
\psi_k^{(n)}(x) := \chi_n(x) \int_x^\infty \psi_k(y) \, dy, \quad \tilde{\psi}_k^{(n)}(x) := \chi_n(x) \int_x^\infty \psi_k(y) \, dy.
\]

Then \(\psi^{(n)}\) and \(\tilde{\psi}^{(n)}\) belong to \(\ell^2(L_\alpha^n)\) for every \(n \in \mathbb{N}\), and

\[
\lim_{n \to \infty} \|\tilde{\psi}^{(n)}\|_{\ell^2(L_\alpha^n)} = 0.
\]

**Proof.** Thanks to the estimate

\[
\int_x^\infty \psi_k(y) \, dy = \int_x^\infty \psi_k(y) e^{x y} e^{-\frac{1}{2} y^2} \, dy \lesssim \left( \int_x^\infty |\psi_k(y)|^2 e^{\alpha y} \, dy \right)^{1/2} e^{\frac{1}{2} x^2},
\]

one has

\[
\|\psi_k^{(n)}\|_{L_\alpha^n}^2 = \int_0^\infty \chi_n^2(x) \|\int_x^\infty \psi_k(y) \, dy\|^2 e^{\alpha x} \, dx \lesssim \alpha \int_0^{n+1} \|\psi_k\|_{L_\alpha^n}^2 \, dx = (n+1) \|\psi_k\|_{L_\alpha^n}^2,
\]

which implies

\[
\|\psi^{(n)}\|_{\ell^2(L_\alpha^n)} \lesssim \sqrt{n+1} \|\psi\|_{\ell^2(L_\alpha^n)} < \infty.
\]

Similarly, one has

\[
\|\tilde{\psi}_k^{(n)}\|_{L_{\alpha}^2}^2 = \int_0^\infty \chi_n^2(x) \left( \int_x^\infty \psi_k(y) \, dy \right)^2 e^{\alpha x} \, dx \lesssim \alpha \int_n^{n+1} \|1_{[n, \infty]} \psi_k\|_{L_\alpha^n}^2 \, dx \leq \|1_{[n, \infty]} \psi_k\|_{L_\alpha^n}^2,
\]

hence also

\[
\|\tilde{\psi}^{(n)}\|_{\ell^2(L_\alpha^n)} \lesssim \|1_{[n, \infty]} \psi\|_{\ell^2(L_\alpha^n)} \leq \|\psi\|_{\ell^2(L_\alpha^n)} < \infty.
\]

Moreover,

\[
\lim_{n \to \infty} \|1_{[n, \infty]} \psi\|_{\ell^2(L_\alpha^n)} = 0
\]

by the dominated convergence theorem. \(\square\)
Lemma 5.2. Assume that Hypothesis 1 is satisfied. Then, for every \( n \in \mathbb{N} \),
\[
\begin{align*}
&\text{(a)} \quad |\sigma_k^{(n)}(\omega, t, x, r)| \leq \psi_k^{(n)}(x) \eta_k(r) \quad \text{for all } (\omega, t, x, r) \in \Omega \times \mathbb{R}_+^2 \times \mathbb{R}; \\
&\text{(b)} \quad \left| \sigma_k^{(n)}(\omega, t, x, r_1) - \sigma_k^{(n)}(\omega, t, x, r_2) \right| \leq \psi_k^{(n)}(x) \left( \eta_k(r_1) + \eta_k(r_2) \right) |r_1 - r_2| \quad \text{for all } (\omega, t, x) \in \Omega \times \mathbb{R}_+^2 \text{ and } r_1, r_2 \in \mathbb{R}.
\end{align*}
\]
Proof. Since \( \sigma(\cdot, r) \) is zero at infinity for all \( r \in \mathbb{R} \), the fundamental theorem of calculus yields
\[
|\sigma_k(x, r)| \leq \int_x^\infty |\partial_1 \sigma_k(y, r)| \, dy \leq \eta_k(r) \int_x^\infty \psi_k(y) \, dy.
\]
Therefore, by definition of \( \psi_k^{(n)} \), it follows that
\[
|\sigma_k^{(n)}(x, r)| \leq \psi_k^{(n)}(x) \eta_k(r).
\]
The proof of (i) is thus complete, and the proof of (ii) is entirely similar, hence omitted. \( \square \)

Lemma 5.3. Assume that Hypothesis 1 is satisfied. Then, for every \( n \in \mathbb{N} \),
\[
\begin{align*}
&\text{(a)} \quad \lim_{x \to \infty} \sigma_k^{(n)}(\omega, t, x, r) = 0 \quad \text{for all } (\omega, t, x, r) \in \Omega \times \mathbb{R}_+^2 \times \mathbb{R}; \\
&\text{(b)} \quad |\partial_1 \sigma_k^{(n)}(\omega, t, x, r)| \leq (\psi_k(x) + \bar{\psi}_k^{(n)}(x)) \eta_k(r) \quad \text{for all } (\omega, t, x, r) \in \Omega \times \mathbb{R}_+^2 \times \mathbb{R}; \\
&\text{(c)} \quad |\partial_2 \sigma_k^{(n)}(\omega, t, x, r)| \leq \eta_k(r) \quad \text{for all } (\omega, t, x, r) \in \Omega \times \mathbb{R}_+^2 \times \mathbb{R}; \\
&\text{(d)} \quad |\partial_2 \sigma_k^{(n)}(\omega, t, x, r_1) - \partial_2 \sigma_k^{(n)}(\omega, t, x, r_2)| \leq \psi_k(x)(\eta_k(r_1) + \eta_k(r_2)) |r_1 - r_2| \quad \text{for all } (\omega, t, x) \in \Omega \times \mathbb{R}_+^2 \text{ and } r_1, r_2 \in \mathbb{R}; \\
&\text{(e)} \quad |\partial_2 \sigma_k^{(n)}(\omega, t, x, r_1) - \partial_2 \sigma_k^{(n)}(\omega, t, x, r_2)| \leq (\eta_k(r_1) + \eta_k(r_2)) |r_1 - r_2| \quad \text{for all } (\omega, t, x) \in \Omega \times \mathbb{R}_+^2 \text{ and } r_1, r_2 \in \mathbb{R}.
\end{align*}
\]
Proof. It is enough to prove (b) and (d), as (a), (c), and (e) are trivial. The chain rule yields
\[
|\partial_1 \sigma_k^{(n)}(x, r)| = \partial_1 \sigma_k(x, r) \chi_n(x) + \sigma_k(x, r) \chi_n'(x),
\]
hence, by definition of \( \bar{\psi}_k^{(n)} \),
\[
|\partial_1 \sigma_k^{(n)}(x, r)| \leq \chi_n(x) \psi_k(x) \eta_k(r) + \bar{\psi}_k^{(n)}(x) \eta_k(r).
\]
The proof of (b) is thus completed (recalling that \( \chi_n(x) \in [0, 1] \) for all \( x \in \mathbb{R} \)). By a similar computation,
\[
|\partial_2 \sigma_k^{(n)}(x, r_1) - \partial_2 \sigma_k^{(n)}(x, r_2)| \leq \left( \chi_n(x) \psi_k(x) + \bar{\psi}_k^{(n)}(x) \right) \left( \eta_k(r_1) + \eta_k(r_2) \right) |r_1 - r_2|,
\]
so that (d) is also proved. \( \square \)

Let \( \sigma^{(n)} \) be the map formally defined on \( \Omega \times \mathbb{R}_+ \times H_\alpha \) as
\[
\sigma^{(n)}(\omega, t, v) := \sum_{k=1}^{\infty} \sigma_k^{(n)}(\omega, t, \cdot, v(\cdot)) \epsilon_k.
\]
If hypothesis 1 is satisfied, Then \( \sigma^{(n)} \) is a well-defined map with values in \( \mathcal{L}^2(U, H_\alpha) \), and it is measurable with respect to the \( \sigma \)-algebra \( \mathcal{P} \otimes \mathcal{B}(H_\alpha) \). In fact, by lemma 5.3, \( \sigma_k^{(n)} \) satisfies hypothesis 1 with \( \psi_k \) replaced by \( \psi_k + \bar{\psi}_k^{(n)} \), where, by lemma 5.1, the \( \ell^2(I_\alpha^2) \) norm of \( \bar{\psi}_k^{(n)} \) is dominated by the norm of \( \psi_k \). The claim hence follows by (the proof of) proposition 4.2.

The same reasoning shows that
\[
\beta^{(n)} := \langle \sigma^{(n)}, I \rangle
\]
is a well-defined map from \( \Omega \times \mathbb{R}_+ \times H_\alpha \) to \( H_\alpha \), satisfying the same measurability properties of \( \beta \).
Lemma 5.4. Let Hypothesis 1 be satisfied. Then
\[
\lim_{n \to \infty} \|\sigma^{(n)}(\omega, t, v) - \sigma(\omega, t, v)\|_{L^2(U, H_\alpha)} = 0,
\]
\[
\lim_{n \to \infty} \|\beta^{(n)}(\omega, t, v) - \beta(\omega, t, v)\|_{H_\alpha} = 0
\]
for all \((\omega, t) \in \Omega \times \mathbb{R}_+\) and \(v \in H_\alpha\).

Proof. For any \(v \in H_\alpha\), one has (omitting the arguments \(\omega\) and \(t\) throughout for simplicity)
\[
\|\sigma^{(n)}(v) - \sigma(v)\|_{L^2(U, H_\alpha)} \leq \left( \sum_{k=1}^{\infty} \left\| \partial_1 \sigma_k^{(n)}(\cdot, v) - \partial_1 \sigma_k(\cdot, v) \right\|^2_{L^2_\alpha} \right)^{1/2}
\]
\[
+ \left( \sum_{k=1}^{\infty} \left\| \partial_2 \sigma_k^{(n)}(\cdot, v) - \partial_2 \sigma_k(\cdot, v) \right\|^2_{L^2_\alpha} \right)^{1/2}.
\]
It follows by (11) and by the triangle inequality in \(L^2(L^2_\alpha)\) that
\[
\left( \sum_{k=1}^{\infty} \left\| \partial_1 \sigma_k^{(n)}(\cdot, v) - \partial_1 \sigma_k(\cdot, v) \right\|^2_{L^2_\alpha} \right)^{1/2}
\]
\[
\leq \left( \sum_{k=1}^{\infty} \int_0^\infty \left\| \partial_1 \sigma_k(x, v(x))(1 - \chi_n(x)) \right\|^2 e^{\alpha x} \, dx \right)^{1/2}
\]
\[
+ \left( \sum_{k=1}^{\infty} \int_0^\infty \left\| \sigma_k(x, v(x))\chi_n(x) \right\|^2 e^{\alpha x} \, dx \right)^{1/2},
\]
where the first term on the right-hand side converges to zero by Hypothesis 1(b) and the dominated convergence theorem. Moreover, an argument entirely analogous to the proof of Lemma 5.2 yields
\[
|\sigma_k(x, r)\chi_n(x)| \leq \omega^{(n)}(\cdot) \eta_k(r) \quad \forall (x, r) \in \mathbb{R}_+ \times \mathbb{R},
\]
hence the second term on the right-hand side of the previous inequality is dominated by
\[
\left( \sum_{k=1}^{\infty} \left\| \psi_k^{(n)} \right\|^2_{L^2_\alpha} \|\eta_k(\cdot)\|_{L^\infty} \right)^{1/2} \lesssim \left\| \psi^{(n)} \right\|_{L^2(L^2_\alpha)} \|\delta(\cdot)\|_{H_\alpha},
\]
which converges to zero by Lemma 5.1. Furthermore, it follows by the identity
\[
(\partial_2 \sigma_k^{(n)}(\cdot, v) - \partial_2 \sigma_k(\cdot, v)) v' = (1 - \chi_n)\partial_2 \sigma_k(\cdot, v)v',
\]
Lemma 5.3(c), and the dominated convergence theorem, that
\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} \left\| (\partial_2 \sigma_k^{(n)}(\cdot, v) - \partial_2 \sigma_k(\cdot, v)) v' \right\|^2_{L^2_\alpha} = 0,
\]
thus proving the pointwise convergence of \(\sigma^{(n)}\) to \(\sigma\). The pointwise convergence of \(\beta^{(n)}\) to \(\beta\) is an immediate consequence of Lemma 4.1 which yields
\[
\|\beta^{(n)}(v) - \beta(v)\|_{L^2_\alpha} \lesssim \left( \|\sigma^{(n)}(v)\|_{L^2} + \|\sigma(v)\|_{L^2} \right) \|\sigma^{(n)}(v) - \sigma(v)\|_{L^2}.
\]
We are now going to show that the sequence of maps \((\sigma^{(n)})\) is uniformly locally Lipschitz continuous, hence, as a consequence, that the same holds for the sequence \((\beta^{(n)})\).
Lemma 5.5. For every $R \geq 0$ there exists a constant $N$, independent of $n$, such that
\[
\|\sigma^{(n)}(\omega, t, v_1) - \sigma^{(n)}(\omega, t, v_2)\|_{\mathcal{X}^2(U, H_\alpha)} \leq N\|v_1 - v_2\|_{H_\alpha},
\]
\[
\|\beta^{(n)}(\omega, t, v_1) - \beta^{(n)}(\omega, t, v_2)\|_{\mathcal{X}^2(U, H_\alpha)} \leq N\|v_1 - v_2\|_{H_\alpha},
\]
for all $(\omega, t) \in \Omega \times \mathbb{R}_+$ and $v_1, v_2 \in H_\alpha$ with $\|v_1\|_{H_\alpha}$, $\|v_2\|_{H_\alpha} \leq R$.

Proof. Lemma 5.3 implies that hypothesis 1 holds, for every $k \in \mathbb{N}$, with $\sigma_k$ replaced by $\sigma_k^{(n)}$ and $\psi_k$ replaced by $\psi_k + \tilde{\psi}_k^{(n)}$. Moreover, the $\mathcal{L}^2(L^2_\eta)$ norm of $\tilde{\psi}_k^{(n)}$ is bounded by the norm of $\psi_k$ thanks to lemma 5.1. Then proposition 4.2 implies
\[
\|\sigma^{(n)}(v_1) - \sigma^{(n)}(v_2)\|_{\mathcal{X}^2(U, H_\alpha)} \lesssim \tilde{\eta}(\delta v_1 H_\alpha) + \tilde{\eta}(\delta v_2 H_\alpha).
\]
which proves the uniform local Lipschitz continuity of $\sigma^{(n)}$. Again by lemmata 5.3 and 5.1 and (the proof of) proposition 4.2 it follows that
\[
\|\sigma^{(n)}(v)\|_{\mathcal{X}^2(U, H_\alpha)} \lesssim \tilde{\eta}(\delta v H_\alpha) (\|v\|_{H_\alpha} + \|\psi\|_{\mathcal{L}^2(L^2_\eta)})
\]
for all $v \in H_\alpha$. For any $v_1, v_2 \in H_\alpha$ with norm bounded by $R$, Lemma 1.1 then implies
\[
\|\beta^{(n)}(v_1) - \beta^{(n)}(v_2)\| \lesssim (\|\sigma^{(n)}(v_1)\|_{\mathcal{X}^2} + \|\sigma^{(n)}(v_2)\|_{\mathcal{X}^2})\|\sigma^{(n)}(v_1) - \sigma^{(n)}(v_2)\|_{\mathcal{X}^2}
\]
\[
\lesssim \tilde{\eta}(\delta R) (R + \|\psi\|_{\mathcal{L}^2(L^2_\eta)})\|v_1 - v_2\|_{H_\alpha},
\]
where the implicit constant depends on $\alpha$ and $R$, but not on $n$. $\square$

6   Approximations II

For every $n \in \mathbb{N}$, let $\phi_n \in C_0^\infty(\mathbb{R})$ be a primitive of the smooth cut-off function $\chi$, such that $\phi_n$ is odd and $\phi_n(0) = 0$. Then $\phi_n$ coincides with the identity function on $[-n, n]$ and is bounded from below and from above by $-(n + 1)$ and $n + 1$, respectively.

For every $k$ and $n \in \mathbb{N}$, let us define the functions $\sigma_k^n : \Omega \times \mathbb{R}^2_+ \times \mathbb{R} \to \mathbb{R}$ as
\[
\sigma_k^n(\omega, t, x, r) := \sigma_k(\omega, t, x, \phi_n(r)).
\]
We are going to show that, for each $n \in \mathbb{N}$, $(\sigma_k^n)$ satisfies hypothesis 1 with $\eta_k$ replaced by $3\eta_k \circ \phi_n$.

Lemma 6.1. Assume that hypothesis 1 is satisfied. For every $k$ and $n \in \mathbb{N}$ one has
(a) $\lim_{x \to \pm \infty} \sigma_k^n(\omega, t, x, r) = 0$ for all $(\omega, t, x, r) \in \Omega \times \mathbb{R}^2_+ \times \mathbb{R}$;
(b) $|\partial_1 \sigma_k^n(\omega, t, x, r)| \leq \psi_k(x)\eta_k(\phi_n(r))$ for all $(\omega, t, x, r) \in \Omega \times \mathbb{R}^2_+ \times \mathbb{R}$;
(c) $|\partial_2 \sigma_k^n(\omega, t, x, r)| \leq \eta_k(\phi_n(r))$ for all $(\omega, t, x, r) \in \Omega \times \mathbb{R}^2_+ \times \mathbb{R}$;
(d) $|\partial_1 \sigma_k^n(\omega, t, x, r_1) - \partial_1 \sigma_k^n(\omega, t, x, r_2)| \leq \psi_k(x)(\eta_k(\phi_n(r_1)) + \eta_k(\phi_n(r_2)))|r_1 - r_2|$ for all $(\omega, t, x) \in \Omega \times \mathbb{R}^2_+$ and $r_1, r_2 \in \mathbb{R}$;
(e) $|\partial_2 \sigma_k^n(\omega, t, x, r_1) - \partial_2 \sigma_k^n(\omega, t, x, r_2)| \leq (\eta_k(\phi_n(r_1)) + 3\eta_k(\phi_n(r_2)))|r_1 - r_2|$ for all $(\omega, t, x) \in \Omega \times \mathbb{R}^2_+$ and $r_1, r_2 \in \mathbb{R}$.

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Proof. Claim (a) is trivial. The remaining ones are based on the identities
\[ \partial_1 \sigma^n_k(x, r) = \partial_1 \sigma_k(x, \phi_n(r)), \quad \partial_2 \sigma^n_k(x, r) = \partial_2 \sigma_k(x, \phi_n(r)) \phi'_n(r), \]
where, as usual, we omit the arguments \( \omega \) and \( t \). In particular, \( (b) \) follows immediately, as well as \( (c) \), recalling that \( |\phi'_n| \leq 1 \). Similarly, one has
\[ |\partial_1 \sigma^n_k(x, r_1) - \partial_1 \sigma^n_k(x, r_2)| \leq \psi_k(x)(\eta_k(\phi_n(r_1)) + \eta_k(\phi_n(r_2)))|\phi_n(r_1) - \phi_n(r_2)|, \]
where \( |\phi_n(r_1) - \phi_n(r_2)| \leq |r_1 - r_2| \) because the Lipschitz constant of \( \phi_n \) is one, hence \( (d) \) is verified. It remains to show that \( (e) \) holds true: one has
\[ |2 \eta_k(\phi_n(r_2))| |\phi'_n(r_1) - \phi'_n(r_2)| \leq 2 \eta_k(\phi_n(r_2))|r_1 - r_2|, \]
and, thanks to the estimate \( |\phi'_n| \leq 2 \),
\[ |2 \eta_k(\phi_n(r_2))| |\phi'_n(r_1) - \phi'_n(r_2)| \leq 2 \eta_k(\phi_n(r_2))|r_1 - r_2|, \]
thus establishing \( (e) \) as well. \( \Box \)

The definition of \( \sigma^n \) as a map from \( \Omega \times \mathbb{R}_+ \times H_\alpha \to \mathcal{L}^2(U, H_\alpha) \), as well as the corresponding measurable properties, follows, mutatis mutandis, as for \( \sigma^{(n)} \) in the previous section.

**Lemma 6.2.** Let hypothesis 1 be satisfied. Then
\[
\lim_{n \to \infty} \|\sigma^n(\omega, t, v) - \sigma(\omega, t, v)\|_{\mathcal{L}^2(U, H_\alpha)} = 0,
\]
\[
\lim_{n \to \infty} \|\beta^n(\omega, t, v) - \beta(\omega, t, v)\|_{H_\alpha} = 0
\]
for all \( (\omega, t) \in \Omega \times \mathbb{R}_+ \) and \( v \in H_\alpha \).

**Proof.** By definition of the functions \( \sigma^n_k \) it follows that, for any \( v \in H_\alpha \),
\[
\|\sigma^n(v) - \sigma(v)\|_{\mathcal{L}^2(U, H_\alpha)} \leq \left( \sum_{k=1}^{\infty} \left\| \partial_1 \sigma_k(\cdot, \phi_n(v)) - \partial_1 \sigma_k(\cdot, v) \right\|_{L^2}^2 \right)^{1/2}
+ \left( \sum_{k=1}^{\infty} \left\| \partial_2 \sigma_k(\cdot, \phi_n(v)) \phi'_n(v) v' - \partial_2 \sigma_k(\cdot, v) v' \right\|_{L^2}^2 \right)^{1/2}.
\]
Recalling that, as \( n \to \infty \), \( \phi_n \) converges pointwise to the identity function and \( \phi'_n \) converges pointwise from below to the function identically equal to one, the claim follows by parts (b) and (c) of Lemma 6.1, the obvious estimate \( \eta_k \circ \phi_n \leq \eta_k \), and the dominated convergence theorem. The pointwise convergence of \( \beta^n \) follows exactly as in proof of lemma 6.4. \( \Box \)
Lemma 6.3. Let hypothesis 1 be satisfied. For every $R \geq 0$ there exists a constant $N$, independent of $n$, such that

$$
\|\sigma^n(\omega, t, v_1) - \sigma^n(\omega, t, v_2)\|_{\mathcal{L}_2(U, H_\alpha)} \leq N\|v_1 - v_2\|_{H_\alpha}
$$

$$
\|\beta^n(\omega, t, v_1) - \beta^n(\omega, t, v_2)\|_{H_\alpha} \leq N\|v_1 - v_2\|_{H_\alpha}
$$

for all $(\omega, t) \in \Omega \times \mathbb{R}_+$ and $v_1, v_2 \in H_\alpha$ with $\|v_1\|_{H_\alpha}$, $\|v_2\|_{H_\alpha} \leq R$.

Proof. Lemma 6.1 implies that hypothesis 1 holds with $\sigma_k^n$ and $\eta_k$ replaced by $3\eta_k \circ \phi_n$. Since $|\phi_n(r)| \leq |r|$ for all $r \in \mathbb{R}$ and $\eta_k$ is even and increasing, one has $\eta_k \circ \phi_n \leq \eta_k$, hence Proposition 4.2 yields

$$
\|\sigma^n(\omega, t, v_1) - \sigma^n(\omega, t, v_2)\|_{\mathcal{L}_2(U, H_\alpha)} \leq \eta(\delta\|v_1\|_{H_\alpha}) + \eta(\delta\|v_2\|_{H_\alpha}),
$$

$$
\cdot (\|v_1\|_{H_\alpha} + \|v_2\|_{H_\alpha}) \|v_1 - v_2\|_{H_\alpha}.
$$

Denoting the implicit constant in this inequality by $c(\alpha)$, setting

$$
N := 2c(\alpha) \bar{\eta}(\delta R)(R + \|v\|_{\mathcal{L}_2(U, H_\alpha)}),
$$

the claim regarding $\sigma^n$ follows. Moreover, again by the inequality $\eta_k \circ \phi_n \leq \eta_k$, lemma 6.1 and the (proof of) proposition 4.2 imply

$$
\|\sigma^n(\omega, t, v)\|_{\mathcal{L}_2(U, H_\alpha)} \leq \eta(\delta\|v\|_{H_\alpha}) (\|v\|_{H_\alpha} + \|v\|_{\mathcal{L}_2(U, H_\alpha)})
$$

for every $n \in \mathbb{N}$, $v \in H_\alpha$, and $(\omega, t) \in \Omega \times \mathbb{R}_+$. The claim about $\beta^n$ then follows by lemma 6.1 as in the proof of lemma 5.5.

7 Positivity of forward rates

Theorem 7.1. Assume that hypothesis 1 is fulfilled and that

$$
|\sigma_k(x, r)| I_{\{r \leq 0\}} \lesssim r^-, 
$$

or, more generally, that

$$
|\sigma_k(x, r)| \leq |r| |\eta_k(r)| \int_x^\infty \psi_k(y) dy
$$

For the latter to hold it is sufficient that $\sigma(x, 0) = 0$. Then forward rates are positive.

Proof. For every $k \in \mathbb{N}$ and $n, m \in \mathbb{N}$, let $\sigma_k^{n,m} := (\sigma_k^n)^m$, i.e.

$$
\sigma_k^{n,m}(x, r) = \sigma_k(x, \phi_n(r)) \chi_n(x).
$$

Lemmas 5.2 and 6.1 imply that, for each $n, m \in \mathbb{N}$, $\sigma_k^{n,m}$ satisfies hypothesis 1 with $\psi_k$ and $\eta_k$ replaced by $\psi_k + \psi_k^{(n)}$ and $3\eta_k \circ \phi_n$, respectively. Since, by lemma 5.1 the $\ell^2(L^2)$ norm of $\psi^{(n)}$ is dominated by the one of $\psi$, and $\eta_k \circ \phi_n \leq \eta_k$ for every $k, n \in \mathbb{N}$, proposition 4.2 implies that the corresponding map $\sigma^{n,m} : \Omega \times \mathbb{R}_+ \times H_\alpha \rightarrow \mathcal{L}_2(U, H_\alpha)$ is well defined for every $n, m \in \mathbb{N}$, and that it is locally bounded and locally Lipschitz continuous in its third argument, uniformly with respect to the other ones. In particular, setting

$$
\beta^{n,m} := (\sigma^{n,m}, I_{\sigma^{n,m}}),
$$

the equation

$$
dv + Av dt = \beta^{n,m}(v) dt + \sigma^{n,m}(v) dW, \quad u^{n,m}(0) = u_0,
$$

(12)
admits a unique $H_{\alpha}$-valued mild solution $u^{n,m}$ defined on a maximal stochastic interval $[0, T_{n,m}]$.

Let us show that, for every $n, m \in \mathbb{N}$, $(\sigma_k^{n,m})$ satisfies the assumptions of proposition 4.6. Lemmata 5.2 and 6.1 imply that

$$\|\sigma_k^{n,m}(x, r)\| \lesssim \psi_k^{(n)}(x) \eta_k(\phi_m(r))$$

as well as, recalling that $|\phi'_m| \leq 1$,

$$|\sigma_k^{n,m}(x, r_1) - \sigma_k^{n,m}(x, r_2)| \lesssim \psi_k^{(n)}(x) \left( \eta_k(\phi_m(r_1)) + \eta_k(\phi_m(r_2)) \right) |\phi_m(r_1) - \phi_m(r_2)| \leq \psi_k^{(n)}(x) \left( \eta_k(\phi_m(r_1)) + \eta_k(\phi_m(r_2)) \right) |r_1 - r_2|.$$ 

Since $|\phi_m| \leq m + 1$, it follows that

$$|\sigma_k^{n,m}(x, r)| \lesssim \psi_k^{(n)}(x) \eta_k(m + 1),$$

where $\|\psi_k^{(n)} \eta_k(m + 1)\|_{L_\alpha^2} = \eta_k(m + 1)\|\psi_k^{(n)}\|_{L_\alpha^2}$, hence, by the Cauchy-Schwarz inequality and the estimate $\|\| \leq \|\|_{L_\alpha^2}$,

$$\left\langle \psi_k^{(n)} \eta_k(m + 1) \right\rangle_{L_\alpha^2} = \left( \sum_{k=1}^{\infty} \eta_k^2(m + 1)\|\psi_k^{(n)}\|_{L_\alpha^2}^2 \right)^{1/2} \leq \left( \sum_{k=1}^{\infty} \eta_k^4(m + 1) \right)^{1/4} \left( \sum_{k=1}^{\infty} \|\psi_k^{(n)}\|_{L_\alpha^2}^4 \right)^{1/4} \leq \tilde{\gamma}(m + 1) \|\psi_k^{(n)}\|_{L_\alpha^2}.$$ 

Recalling that the $L^2(L_\alpha^2)$ norm of $\psi^{(n)}$ is finite by lemma 5.1, this shows that assumption (a) of proposition 4.6 is fulfilled. Similarly, one has

$$|\sigma_k^{n,m}(x, r_1) - \sigma_k^{n,m}(x, r_2)| \lesssim \psi_k^{(n)}(x) \eta_k(m + 1) |r_1 - r_2|,$$

where $\|\psi_k^{(n)} \eta_k(m + 1)\|_{L_\alpha^2} = \eta_k(m + 1)\|\psi_k^{(n)}\|_{L_\alpha^2}$ and

$$\|\psi_k^{(n)}\|_{L_\alpha^\infty} \leq \|\psi_k\|_{L_\alpha^1} \lesssim \|\psi_k\|_{L_\alpha^2}.$$ 

Since $\psi \in L^2(L_\alpha^2)$, assumption (b) of proposition 4.6 is also satisfied. Therefore $[\mathcal{X}]$ admits a unique $L_{\alpha,n}$-valued mild solution on any finite time interval. Since $H_{\alpha} \hookrightarrow L_{\alpha,n}$, it follows that the $H_{\alpha}$-valued solution $u^{n,m}$ coincides on $[0, T_{n,m}]$ with the global $L_{\alpha,n}$-valued solution, which we shall also denote by $u^{n,m}$.

One has

$$|\sigma_k^{n,m}(x, r)| = |\sigma_k(x, \phi_m(r)) \chi_n(x)| \leq |\sigma_k(x, \phi_m(r))|$$

and $r \leq 0$ if and only if $\phi_m(r) \leq 0$, hence

$$|\sigma_k^{n,m}(x, r)| 1_{\{r \leq 0\}} \leq |\sigma_k(x, \phi_m(r))| 1_{\{\phi_m(r) \leq 0\}} \lesssim \phi_m(r)^- \leq r^-.$$ 

Under the more general hypothesis on $(\sigma_k)$, one has

$$|\sigma_k^{n,m}(x, r)| 1_{\{r \leq 0\}} \leq \psi_k^{(n)}(x) \eta_k(\phi_m(r)) |r| 1_{\{r \leq 0\}} \leq \psi_k^{(n)}(x) \eta_k(m + 1) r^-.$$ 

Then, for any $v \in L_{\alpha,n}$,

$$\|\sigma_k^{n,m}(v) 1_{\{v \leq 0\}}\|_{L_{\alpha,n}} \leq \eta_k(m + 1) \|\psi_k^{(n)}\|_{L_\alpha^\infty} \|v^-\|_{L_{\alpha,n}}.$$ 

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where \( \| \psi^{(n)}_k \|_{L^\infty} \leq \| \psi^{(n)}_k \|_{L^1} \lesssim \| \psi^{(n)}_k \|_{L^2} \), hence

\[
\| \sigma^{n,m}(v) \mathbb{1}_{\{v \leq 0\}} \|_{\ell^2(L^2_x)} \lesssim \tilde{\eta}(m + 1) \| \psi^{(n)}_k \|_{\ell^2(L^2_x)} \| v^- \|_{L^2_x}.
\]

Let us now consider

\[
\langle v^-, \beta^{n,m}(v) \rangle_{L^2_x} = \sum_{k=1}^{\infty} \int_0^\infty \int_0^x e^{-\alpha s} v^-(x) \sigma^{n,m}_k(x, v(x)) \sigma^{n,m}_k(y, v(y)) \, dy \, dx,
\]

where

\[
\int_0^x \sigma^{n,m}_k(y, v(y)) \, dy \leq \| \sigma^{n,m}_k(v) \|_{L^1} \lesssim \| \sigma^{n,m}_k(v) \|_{L^2}.
\]

We recall that, as in the proof of proposition 4.6, one has

\[
\| \sigma^{n,m}_k(v) \|_{L^2} \leq \| \psi^{(n)}_k \|_{L^2}.
\]

Note that \( r^- = |r| \mathbb{1}_{\{r \leq 0\}} \) implies

\[
r^- \sigma^{n,m}_k(x, r) \leq \psi^{(n)}_k(x) \eta_k(m + 1) |r^-|^2,
\]

from which it follows, by the Hölder inequality,

\[
\langle v^-, \beta^{n,m}(v) \rangle_{L^2_x} \lesssim \sum_{k=1}^{\infty} \eta_k(m + 1) \| \sigma^{n,m}_k(v) \|_{L^2} \int_0^\infty \psi^{(n)}_k(x) |v^-(x)|^2 e^{-\alpha s} \, dx
\]

\[
\lesssim \| v^- \|_{L^2_x} \sum_{k=1}^{\infty} \eta_k(m + 1) \| \psi^{(n)}_k \|_{L^2} \| \sigma^{n,m}_k(v) \|_{L^2}
\]

\[
\leq \| v^- \|_{L^2_x} \sum_{k=1}^{\infty} \eta_k(m + 1) \| \psi^{(n)}_k \|_{L^2}^2 \lesssim \| v^- \|_{L^2_x} \| \psi^{(n)}_k \|_{\ell^2(L^2_x)} \| \sigma^{n,m}_k(v) \|_{\ell^2(L^2_x)}.
\]

Theorem 3.1 applied to equation (12) on the space \( L^2_x \) thus yields \( u^{n,m}(t) \geq 0 \) for every \( t \in \mathbb{R}_+ \).

As already observed, thanks to lemma 5.3, \( \sigma^{(n)} \) satisfies hypothesis 1 with \( \psi_k \) replaced by \( \psi \). Then lemmata 6.2 and 6.3 imply, in view of theorem 2.9, that

\[
u^{n,m}_0 \mathbb{1}_{[0,T_n \wedge T_\alpha, \ldots]} \rightarrow u^n \mathbb{1}_{[0,T_\alpha]}
\]

in \( L^0(\Omega \times \mathbb{R}_+; H_\alpha) \) as \( m \rightarrow \infty \), where \( u_n \) is the unique local mild solution on the maximal stochastic interval \([0, T_n]\) to the equation

\[
dv + \lambda(v) \, dt + \sigma(v) \, dW, \quad u^n(0) = u_0.
\]

By a completely similar argument,

\[
u_n \mathbb{1}_{[0,T_n \wedge T]} \rightarrow u \mathbb{1}_{[0,T]}
\]

in \( L^0(\Omega \times \mathbb{R}_+; H_\alpha) \) as \( n \rightarrow \infty \), where \( u \) is the unique local mild solution on the maximal stochastic interval \(([0, T]]\) to Musiela’s SPDE. Since convergence in \( H_\alpha \) implies convergence in \( C(\mathbb{R}_+) \), positivity of \( u^{n,m} \) implies positivity of \( u^n \) on \([0, T_n]\) for every \( n \in \mathbb{N} \), which in turns implies positivity of \( u \) on \([0, T]\).

\[ \square \]
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