ON $\gamma$-VECTORS AND THE DERIVATIVES OF THE TANGENT AND SECANT FUNCTIONS

SHI-MEI MA

Abstract. In this paper we consider the $\gamma$-vectors of the types $A$ and $B$ Coxeter complexes as well as the $\gamma$-vectors of the types $A$ and $B$ associahedrons. We show that these $\gamma$-vectors can be obtained by using derivative polynomials of the tangent and secant functions. A grammatical description for these $\gamma$-vectors is discussed. Moreover, we also present a grammatical description for the well known Legendre polynomials and Chebyshev polynomials of both kinds.

Keywords: $\gamma$-vectors; Tangent function; Secant function; Eulerian polynomials

2010 Mathematics Subject Classification: 05A05; 05A15

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. The hyperoctahedral group $B_n$ is the group of signed permutations of the set $\pm [n]$ such that $\pi(-i) = -\pi(i)$ for all $i$, where $\pm [n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ is alternating if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. Similarly, an element $\pi$ of $B_n$ is alternating if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. Denote by $E_n$ and $E_B^n$ the number of alternating elements in $\mathfrak{S}_n$ and $B_n$, respectively. It is well known (see [4, 24]) that

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \tan x + \sec x, \quad \sum_{n=0}^{\infty} E_B^n \frac{x^n}{n!} = \tan 2x + \sec 2x.$$  

Derivative polynomials are an important part of combinatorial trigonometry (see [1, 8, 10, 11, 12, 13, 14] for instance). Define

$$y = \tan(x), \quad z = \sec(x).$$

Denote by $D$ the differential operator $d/dx$. Clearly, we have $D(y) = 1 + y^2$ and $D(z) = yz$. In 1995, Hoffman [10] considered two sequences of derivative polynomials defined respectively by $D^n(y) = P_n(y)$ and $D^n(z) = zQ_n(y)$. From the chain rule it follows that the polynomials $P_n(u)$ satisfy $P_0(u) = u$ and $P_{n+1}(u) = (1 + u^2)P'_n(u)$, and similarly $Q_0(u) = 1$ and $Q_{n+1}(u) = (1 + u^2)Q'_n(u) + uQ_n(u)$.

As shown in [10], the exponential generating functions

$$P(u, t) = \sum_{n=0}^{\infty} P_n(u) \frac{t^n}{n!} \quad \text{and} \quad Q(u, t) = \sum_{n=0}^{\infty} Q_n(u) \frac{t^n}{n!}$$

are given by the explicit formulas

$$P(u, t) = \frac{u + \tan(t)}{1 - u \tan(t)} \quad \text{and} \quad Q(u, t) = \frac{\sec(t)}{1 - u \tan(t)}.$$  

(1)
Recall that a descent of a permutation $\pi \in S_n$ is a position $i$ such that $\pi(i) > \pi(i+1)$, where $1 \leq i \leq n-1$. Denote by $\text{des}(\pi)$ the number of descents of $\pi$. Then the equations

$$A_n(x) = \sum_{\pi \in S_n} x^{\text{des}(\pi)} = \sum_{k=0}^{n-1} \binom{n}{k} x^k,$$

define the Eulerian polynomial $A_n(x)$ and the Eulerian number $\binom{n}{k}$ (see [23, A008292]). For each $\pi \in B_n$, we define

$$\text{des}_B(\pi) = \# \{i \in \{0, 1, 2, \ldots, n-1\} | \pi(i) > \pi(i+1)\},$$

where $\pi(0) = 0$. Let

$$B_n(x) = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)} = \sum_{k=0}^{n} B(n, k) x^k.$$

The polynomial $B_n(x)$ is called an Eulerian polynomial of type $B$, while $B(n, k)$ is called an Eulerian number of type $B$ (see [23, A060187]).

Assume that

$$(Dy)^{n+1}(y) = (Dy)(Dy)^n(y) = D(y(Dy)^n(y)), \quad (Dy)^{n+1}(z) = (Dy)(Dy)^n(z) = D(y(Dy)^n(z)).$$

Note that $D(y) = z^2$ and $D(z) = yz$. Recently, we obtained the following result.

**Theorem 1** ([14]). For $n \geq 1$, we have

$$(Dy)^n(y) = 2^n \sum_{k=0}^{n-1} \binom{n}{k} y^{2n-2k-1} z^{2k+2}, \quad (Dy)^n(z) = \sum_{k=0}^{n} B(n, k) y^{2n-2k} z^{2k+1}.$$ 

Define

$$f = \sec(2x), \quad g = 2\tan(2x).$$

In this paper we will mainly consider the following differential system:

$$D(f) = fg, \quad D(g) = 4f^2. \quad (2)$$

Define $h = \tan(2x)$. Note that $f^2 = 1 + h^2$ and $g = 2h$. So the following result is immediate.

**Proposition 2.** For $n \geq 0$, we have $D^n(f) = 2^n f Q_n(h)$, $D^n(g) = 2^{n+1} P_n(h)$.

In the next section, we collect some notation and definitions that will be needed in the rest of the paper.

2. Notation, definitions and preliminaries

The $h$-polynomial of a $(d-1)$-dimension simplicial complex $\Delta$ is the generating function $h(\Delta; x) = \sum_{i=0}^{d} h_i(\Delta) x^i$ defined by the following identity:

$$\sum_{i=0}^{d} h_i(\Delta) x^i (1 + x)^{d-i} = \sum_{i=0}^{d} f_i(\Delta) x^i,$$
where \( f_i(\Delta) \) is the number of faces of \( \Delta \) of dimension \( i \). There is a large literature devoted to the \( h \)-polynomials of the form

\[
    h(\Delta; x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1 + x)^{d-2i},
\]

where the coefficients \( \gamma_i \) are nonnegative. Following Gal [9], we call \((\gamma_0, \gamma_1, \ldots)\) the \( \gamma \)-vector of \( \Delta \), and the corresponding generating function \( \gamma(\Delta; x) = \sum_{i\geq0} \gamma_i x^i \) is the \( \gamma \)-polynomial. In particular, the Eulerian polynomials \( A_n(x) \) and \( B_n(x) \) are respectively known as the \( h \)-polynomials of Coxeter complexes of types \( A \) and \( B \).

Let us now recall two classical results.

**Theorem 3** ([7] [21]). For \( n \geq 1 \), we have

\[
    A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a(n, k) x^k (1 + x)^{n-2k}.\]

**Theorem 4** ([5] [18] [19]). For \( n \geq 1 \), we have

\[
    B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n, k) x^k (1 + x)^{n-2k}.\]

It is well known that the numbers \( a(n, k) \) satisfy the recurrence

\[
    a(n, k) = (k + 1)a(n - 1, k) + (2n - 4k)a(n - 1, k - 1),
\]

with the initial conditions \( a(1, 0) = 1 \) and \( a(1, k) = 0 \) for \( k \geq 1 \) (see [23] A101280), and the numbers \( b(n, k) \) satisfy the recurrence

\[
    b(n, k) = (2k + 1)b(n - 1, k) + 4(n + 1 - 2k)b(n - 1, k - 1),
\]

with the initial conditions \( b(1, 0) = 1 \) and \( b(1, k) = 0 \) for \( k \geq 1 \) (see [5] Section 4).

The \( h \)-polynomials of the types \( A \) and \( B \) associahedrons are respectively given as follows (see [16] [17] [20] [22] for instance):

\[
    h(\Delta_{FZ}(A_{n-1}), x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} x^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1 + x)^{n-1-2k},
\]

\[
    h(\Delta_{FZ}(B_n), x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{n}{2k} x^k (1 + x)^{n-2k},
\]

where \( C_k = \frac{(2k)}{k+1} \binom{2k}{k} \) is the \( k \)th Catalan number and the coefficients of \( x^k \) of \( h(\Delta_{FZ}(A_{n-1}), x) \) is the Narayana number \( N(n, k + 1) \).

Define

\[
    F(n, k) = C_k \binom{n-1}{2k}, \quad H(n, k) = \binom{2k}{k} \binom{n}{2k}.\]

There are many combinatorial interpretations of the number \( F(n, k) \), such as \( F(n, k) \) is number of Motzkin paths of length \( n - 1 \) with \( k \) up steps (see [23] A055151). It is easy to verify that the numbers \( F(n, k) \) satisfy the recurrence relation

\[
    (n + 1)F(n, k) = (n + 2k + 1)F(n - 1, k) + 4(n - 2k)F(n - 1, k - 1),
\]
with initial conditions \( F(1, 0) = 1 \) and \( F(1, k) = 0 \) for \( k \geq 1 \), and the numbers \( H(n, k) \) satisfy the recurrence relation

\[
nH(n, k) = (n + 2k)H(n - 1, k) + 4(n - 2k + 1)H(n - 1, k - 1),
\]

with initial conditions \( H(1, 0) = 1 \) and \( H(1, k) = 0 \) for \( k \geq 1 \) (see \[23\] A089627).

3. \( \gamma \)-vectors

Define the generating functions

\[
a_n(x) = \sum_{k \geq 0} a(n, k)x^k, \quad b_n(x) = \sum_{k \geq 0} b(n, k)x^k.
\]

The first few \( a_n(x) \) and \( b_n(x) \) are respectively given as follows:

\[
a_1(x) = 1, \quad a_2(x) = 1, \quad a_3(x) = 1 + 2x, \quad a_4(x) = 1 + 8x;
\]

\[
b_1(x) = 1, \quad b_2(x) = 1 + 4x, \quad b_3(x) = 1 + 20x, \quad b_4(x) = 1 + 72x + 80x^2.
\]

Combining \( [1] \) and \[5\] Prop. 3.5, Prop. 4.10], we immediately get the following result.

**Theorem 5.** For \( n \geq 1 \), we have

\[
a_n(x) = \frac{1}{x}\left(\frac{\sqrt{4x - 1} - 1}{2}\right)^{n+1} P_n\left(\frac{1}{\sqrt{4x - 1}}\right), \quad b_n(x) = (4x - 1)^{\frac{n}{2}} Q_n\left(\frac{1}{\sqrt{4x - 1}}\right).
\]

Assume that

\[
(fD)^{n+1}(f) = (fD)(fD)^n(f) = fD((fD)^n(f)),
\]

\[
(fD)^{n+1}(g) = (fD)(fD)^n(g) = fD((fD)^n(g)).
\]

We can now present the main result of this paper.

**Theorem 6.** For \( n \geq 1 \), we have

\[
D^n(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n, k)f^{2k+1}g^{n-2k},
\]

\[
D^n(g) = 2^{n+1} \sum_{k=0}^{\lfloor n-1/2 \rfloor} a(n, k)f^{2k+2}g^{n-1-2k},
\]

\[
(fD)^n(f) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} H(n, k)f^{n+1+2k}g^{n-2k},
\]

\[
(fD)^n(g) = 2(n + 1)! \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} F(n, k)f^{n+2+2k}g^{n-1-2k}.
\]

**Proof.** We only prove the assertion for \( D^n(f) \) and the others can be proved in a similar way. It follows from \[2\] that \( D(f) = fg \) and \( D^2(f) = fg^2 + 4f^3 \). For \( n \geq 0 \), we define \( \tilde{b}(n, k) \) by

\[
D^n(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} \tilde{b}(n, k)f^{2k+1}g^{n-2k},
\]

(6)
Then \( \tilde{b}(1,0) = 1 \) and \( \tilde{b}(1,k) = 0 \) for \( k \geq 1 \). It follows from (6) that
\[
D(D^n(f)) = \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1)\tilde{b}(n,k)f^{2k+1}g^{n-2k+1} + 4 \sum_{k=0}^{\lfloor n/2 \rfloor} (n-2k)\tilde{b}(n,k)f^{2k+3}g^{n-2k-1}.
\]

We therefore conclude that \( \tilde{b}(n+1,k) = (2k+1)\tilde{b}(n,k) + 4(n+2-2k)\tilde{b}(n,k-1) \) and complete the proof by comparing it with (3). \( \square \)

Define
\[
N_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} (\binom{n}{k+1} (x+1)^k (x-1)^{n-1-k}),
\]
\[
L_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^k (x-1)^{n-k}.
\]

Taking \( f^2 = 1 + h^2 \) and \( g = 2h \) in Theorem 6 leads to the following result and we omit the proof of it, since it is a straightforward application of (4) and (5).

**Corollary 7.** For \( n \geq 1 \), we have
\[
(fD)^n(f) = n!f^{n+1}(-\imath)^nL_n(\imath h),
\]
\[
(fD)^n(g) = 2(n+1)!f^{n+2}(-\imath)^{n-1}N_n(\imath h),
\]
where \( \imath = \sqrt{-1} \).

It should be noted that the polynomial \( \frac{1}{2^n}L_n(x) \) is the famous Legendre polynomial [23, A100258]. Therefore, from Corollary 7 we see that the Legendre polynomial can be generated by \((fD)^n(f)\).

### 4. Context-free grammars

Many combinatorial objects permit grammatical interpretations (see [2, 3, 15] for instance). The grammatical method was systematically introduced by Chen [2] in the study of exponential structures in combinatorics. Let \( A \) be an alphabet whose letters are regarded as independent commutative indeterminates. A context-free grammar \( G \) over \( A \) is defined as a set of substitution rules that replace a letter in \( A \) by a formal function over \( A \). The formal derivative \( D \) is a linear operator defined with respect to a context-free grammar \( G \). For example, if \( G = \{ u \rightarrow uv, v \rightarrow 4u^2 \} \), then
\[
D(u) = uv, D(v) = v, D^2(u) = u(v + v^2), D^3(u) = u(v + 3v^2 + v^3).
\]

It follows from Theorem 6 that the \( \gamma \)-vectors of Coxeter complexes (of types \( A \) and \( B \)) and associahedrons (of types \( A \) and \( B \)) can be respectively generated by the grammars
\[
G_1 = \{ u \rightarrow uv, v \rightarrow 4u^2 \}
\]
and
\[
G_2 = \{ u \rightarrow u^2v, v \rightarrow 4u^3 \}. \tag{7}
\]

There are many sequences can be generated by the grammar \( G \). A special interesting result is the following.
Proposition 8. Let $G$ be the same as in [4]. Then

$$D^n(uv) = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} 4^k \binom{n+1}{2k} u^{n+1+2k} v^{n+1-2k}.$$ 

Let $T_n(x)$ and $U_n(x)$ be the Chebyshev polynomials of the first and second kind of order $n$, respectively. We can now conclude the following result, which is based on Proposition 8. The proof runs along the same lines as that of Theorem 6.

Theorem 9. If $G = \{ u \to u^2v, v \to u^3 \}$, then

$$D^n(uv) = n! \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n+1}{2k} u^{n+1+2k} v^{n+1-2k},$$

$$D^n(u^2) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} u^{n+2+2k} v^{n-2k}.$$ 

In particular,

$$D^n(uv) \big|_{u^2=x^2-1, v=x} = n!(x^2-1)^{\frac{n+1}{2}} T_{n+1}(x),$$

$$D^n(u^2) \big|_{u^2=x^2-1, v=x} = n!(x^2-1)^{\frac{n+2}{2}} U_{n}(x).$$

Taking $u = \sec^2(x)$ and $v = 2\tan(x)$, it is clear that $D(u) = uv$ and $D(v) = 2u$. One can easily verify another grammatical description of the $\gamma$-vectors of the type $A$ Coxeter complex.

Theorem 10. If $G = \{ u \to u^2v, v \to 2u \}$, then

$$D^n(u) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n+1,k) u^{k+1} v^{n-2k}. $$

Define

$$T(n, k) = \binom{n}{k} \cdot \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}. \quad (8)$$

It is well known that $T(n, k)$ is the number of paths of length $n$ with steps $U = (1, 1)$, $D = (1, -1)$ and $H = (1, 0)$, starting at $(0, 0)$, staying weakly above the $x$-axis (i.e. left factors of Motzkin paths) and having $k$ $H$ steps (see [23, A107230]). It follows from (8) that

$$(n+1)T(n, k) = (2n+1-k)T(n-1, k-1) + 2T(n-1, k) + 4(k+1)T(n-1, k+1). \quad (9)$$

We end our paper by giving the following result.

Theorem 11. If $G = \{ t \to tu^2, u \to u^2v, v \to 4u^3 \}$, then

$$D^n(t^2u^2) = (n+1)! t^2 \sum_{k=0}^{n} T(n, k) u^{2n+2-k} v^{k},$$

$$D^n(t^2u) = n! t^2 \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} u^{2n+1-k} v^{k}.$$
Proof. We only prove the assertion for $D^n(t^2u^2)$ and the corresponding assertion for $D^n(t^2u)$ can be proved in a similar way. It is easy to verify that $D(t^2u^2) = 2t^2(u^4 + u^3v)$ and $D^2(t^2u^2) = 3t^2(2u^6 + 2u^5v + u^4v^2)$. For $n \geq 0$, we define

$$D^n(t^2u^2) = (n+1)!t^2 \sum_{k=0}^{n} \tilde{T}(n, k) u^{2n+2-k} v^k.$$ 

Note that

$$\frac{D^{n+1}(t^2u^2)}{(n+1)!t^2} = \sum_k (2n + 2 - k) \tilde{T}(n, k) u^{2n+3-k} v^{k+1} + 2 \sum_k \tilde{T}(n, k) u^{2n+4-k} v^k + 4 \sum_k k \tilde{T}(n, k) u^{2n+5-k} v^{k-1}.$$ 

Thus, we get

$$(n+2) \tilde{T}(n+1, k) = (2n + 3 - k) \tilde{T}(n, k - 1) + 2 \tilde{T}(n, k) + 4(k+1) \tilde{T}(n, k + 1).$$

Comparing with [9], we see that the coefficients $\tilde{T}(n, k)$ satisfy the same recurrence relation and initial conditions as $T(n, k)$, so they agree. □

It should be noted that the numbers $\binom{n}{k} 2^{n-k}$ are elements of the $f$-vector for the $n$-dimensional cubes (see [23], A038207)

References

[1] K.N. Boyadzhiev, Derivative Polynomials for tanh, tan, sech and sec in Explicit Form, Fibonacci Quart. 45 (2007) 291–303.
[2] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.
[3] W.Y.C. Chen, R.X.J. Hao and H.R.L. Yang, Context-free Grammars and Multivariate Stable Polynomials over Stirling Permutations, arXiv:1208.1420v2
[4] C.-O. Chow, Counting involutory, unimodal, and alternating signed permutations, Discrete Math. 36 (2006) 2222–2228.
[5] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, Adv. in Appl. Math. 41 (2008) 133–157.
[6] K. Dilks, T.K. Petersen, J.R. Stembridge, Affine descents and the Steinberg torus, Adv. in Appl. Math. 42 (2009) 423–444.
[7] D. Foata and M. P. Schützenberger, Théorie géométrique des polynômes eulériens, Lecture Notes in Math. vol. 138, Springer, Berlin, 1970
[8] G.R. Franssens, Functions with derivatives given by polynomials in the function itself or a related function, Anal. Math. 33 (2007) 17–36.
[9] S.R. Gal, Real root conjecture fails for five and higher-dimensional spheres, Discrete Comput. Geom. 34 (2005) 269–284.
[10] M.E. Hoffman, Derivative polynomials for tangent and secant, Amer. Math. Monthly 102 (1995) 23–30.
[11] M.E. Hoffman, Derivative polynomials, Euler polynomials, and associated integer sequences, Electron. J. Combin. 6 (1999) #R21.
[12] S.-M. Ma, Derivative polynomials and enumeration of permutations by number of interior and left peaks, Discrete Math. 312 (2012) 405–412.
[13] S.-M. Ma, An explicit formula for the number of permutations with a given number of alternating runs, J. Combin. Theory Ser. A 119 (2012), 1660–1664.
[14] S.-M. Ma, A family of two-variable derivative polynomials for tangent and secant, Electron. J. Combin. 20(1) (2013), #P11.
[15] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, European J. Combin. 34 (2013) 1081–1091.
[16] T. Mansour, Y. Sun, Identities involving Narayana polynomials and Catalan numbers, Discrete Math. 309 (2009) 4079–4088.
[17] E. Marberg, Actions and identities on set partitions, Electron. J. Combin. 19 (2012), #P28.
[18] E. Nevo, T.K. Petersen, On $\gamma$-Vectors Satisfying the Kruskal-Katona Inequalities, Discrete Comput. Geom. 45 (2011) 503–521.
[19] T.K. Petersen, Enriched P-partitions and peak algebras, Adv. Math. 209 (2007) 561–610.
[20] A. Postnikov, V. Reiner, L. Williams, Faces of generalized permutohedra, Documenta Math. 13 (2008) 207–273.
[21] L.W. Shapiro, W.J. Woan, S. Getu, Runs, slides and moments, SIAM J. Algebraic Discrete Methods 4 (1983) 459–466.
[22] R. Simion, A type-B associahedron, Adv. in Appl. Math. 30 (2003) 2-25.
[23] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.
[24] R.P. Stanley, A Survey of Alternating Permutations, in Combinatorics and Graphs, R. A. Brualdi et. al. (eds.), Contemp. Math., Vol. 531, Amer. Math. Soc., Providence, RI, 2010, pp. 165–196.

School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066004, P. R. China

E-mail address: shimeima@yahoo.com.cn (S.-M. Ma)