A multipurpose Hopf deformation of the Algebra of Feynman-like Diagrams

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Abstract. We construct a three parameter deformation of the Hopf algebra L\textsuperscript{DIAG}. This new algebra is a true Hopf deformation which reduces to L\textsuperscript{DIAG} on one hand and to MQ\textsuperscript{Sym} on the other, relating L\textsuperscript{DIAG} to other Hopf algebras of interest in contemporary physics. Further, its product law reproduces that of the algebra of polyzeta functions.

1 Introduction

The complete journey between the first appearance of a product formula by Bender et al. \cite{1} and their related Feynman-like diagrams to the discovery of a Hopf algebra structure \cite{8} on the diagrams themselves, goes roughly as follows.

Firstly, Bender, Brody, and Meister \cite{1} introduced a special field theory which proved to be particularly rich in combinatorial links and by-products \cite{11} (not to mention the link with vector fields and one-parameter groups \cite{7,10}).

Secondly, the Feynman-like diagrams of this theory label monomials which combine naturally in a way compatible with monomial multiplication and co-addition (i.e. the standard Hopf algebra structure on the space of polynomials). This is the Hopf algebra DIAG \cite{8}. The (Hopf-)subalgebra of DIAG generated by the primitive graphs is the Hopf algebra BELL described in Solomon’s talk at this conference \cite{12}.

Thirdly, the natural noncommutative pull-back of this algebra, L\textsuperscript{DIAG}, has a basis (the labelled diagrams) which is in one-to-one correspondence with that of the Matrix Quasi-Symmetric Functions \cite{13} (the packed matrices of
with overlappings reminiscent of Hoffmann’s shuffle used in the theory of polyzeta functions \cite{2}. The superpositions and overlappings involved there are not present in (non-deformed) LDIAG and, moreover, the coproduct of LDIAG is co-commutative while that of MQSym is not.

The aim of this paper is to announce the existence of a Hopf algebra deformation which connects LDIAG to other Hopf algebras relevant to physics (Connes-Kreimer, Connes-Moscovici, Brouder-Frabetti, see \cite{6}) and other fields (noncommutative symmetric functions, Euler-Zagier sums).

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\section{Labelled Diagrams and Diagrams}

Product formula involves a summation over all diagrams of a certain type \cite{12} a labelled version of which is described below. Labelled diagrams can be identified with their weight functions which are mappings $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ such that the supporting subgraph

$$\Gamma_\omega = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid w(i, j) \neq 0\}$$

has specific projections i.e. $pr_1(\Gamma_\omega) = [1..p]$; $pr_2(\Gamma_\omega) = [1..q]$ for some $p, q \in \mathbb{N}$ (notice that when one of $p, q$ is zero so too is the other and the diagram is empty).

These graphs are represented by labelled diagrams as follows

\begin{center}
\begin{tikzpicture}[scale=0.8]
    
    \node (1) at (0,0) [shape=circle,draw] {1};
    \node (2) at (1,0) [shape=circle,draw] {2};
    \node (3) at (2,0) [shape=circle,draw] {3};
    \node (4) at (3,0) [shape=circle,draw] {4};

    \draw[thick] (1) to (2);
    \draw[thick] (2) to (3);
    \draw[thick] (3) to (4);

    \draw[thick] (1) to (3);
    \draw[thick] (1) to (4);

    \end{tikzpicture}
\end{center}

The labelled diagrams form the set \textbf{ldiag} and prescribe monomials through the formula $L^{\alpha(d)} \psi^{\beta(d)}$ where $\alpha(d)$ (resp. $\beta(d)$) is the “white spot type” (resp. the “black spot type”) i.e. the multi-index $(\alpha_i)_{i \in \mathbb{N}^+}$ (resp. $(\beta_i)_{i \in \mathbb{N}^+}$) such that $\alpha_i$ (resp. $\beta_i$) is the number of white spots (resp. black spots) of degree $i$ ($i$ lines connected to the spot).

There is a (graphically) natural multiplicative structure on \textbf{ldiag} such that the arrow

$$m_{(L, \psi)} : d \mapsto L^{\alpha(d)} \psi^{\beta(d)}$$

is a morphism.

It is clear that one can permute black spots, or white spots, of $d$ without changing the monomial $L^{\alpha(d)} \psi^{\beta(d)}$. The classes of (labelled) diagrams up to this equivalence (permutations of white - or black - spots among themselves) are naturally represented by unlabelled diagrams and will be denoted \textbf{diag} (including the empty one).
For both types of diagram the product consists of concatenating the diagrams i.e. placing $d_2$ on the right of $d_1$ \[8\] (the result, for $d_1, d_2$, will be denoted $[d_1|d_2]_D$ in diag and $[d_1|d_2]_L$ in ldiag). These products endow diag and ldiag with the structure of monoids, with the empty diagram as neutral element. The corresponding commutative diagram is as follows (where $X^2$ means the cartesian square of the set $X$).

\[
\begin{array}{c}
\text{Labelled diagrams}^2 \xrightarrow{\text{Unlabelling}^2} \text{Diagrams}^2 \xrightarrow{m(\mathcal{L},\mathcal{V}) \times m(\mathcal{L},\mathcal{V})} \text{Monomials}^2 \\
\text{product} \downarrow \quad \text{product} \downarrow \quad \text{product} \downarrow \\
\text{Labelled diagrams} \xrightarrow{\text{Unlabelling}} \text{Diagrams} \xrightarrow{m(\mathcal{L},\mathcal{V})} \text{Monomials}
\end{array}
\]

(3)

It is easy to see that the labelled diagram (resp. diagrams) form free monoids. We denote by DIAG and LDIAG the $K$-algebras of these monoids $\mathbb{R}$ ($K$ is a field).

One can shuffle the product in ldiag, counting crossings and superpositions. The definition of the deformed product is expressed by the diagrammatic formula

\[
\text{Labelled diagrams} \xrightarrow{\text{Unlabelling}} \text{Diagrams} \xrightarrow{m(\mathcal{L},\mathcal{V})} \text{Monomials}
\]

and the descriptive formula below.

\[
[d_1 | d_2]_{L(q_c,q_s)} = \sum_{c \in \text{at}(d_1|d_2)_L} q_c^{\text{weight}_c} q_s^{\text{weight}_1} q_s^{\text{weight}_2} c_s([d_1|d_2]_L)
\]

(4)

where
the exponent of $q^{nc \times \text{weight}_2}$ is the number of crossings of “what crosses” times its weight

- the exponent of $d^{\text{weight}_1 \times \text{weight}_2}$ is the product of the weights of “what is overlapped”

- terms $cs([d_1|d_2]_L)$ are the diagrams obtained from $[d_1|d_2]_L$ by the process of crossing and superposing the black spots of $d_2$ on those of $d_1$, the order and identity of the black spots of $d_1$ (resp. $d_2$) being preserved.

What is striking is that this law (denoted above $\uparrow$) is associative. Moreover, it can be shown [3, 4] that this process decomposes into two transformations: twisting and shifting. In fact, specialized to certain parameters, this law is reminiscent of others [2].

| Parameters | (0, 0) (shifted) | (1, 1) (shifted) | (1, 1) (unshifted) |
| LDIAG | MQSym | Hoffmann & Euler-Zagier |

3 Hopf Deformation

Using a total order on the monomials of $\text{ldiag}$, it can be shown that the algebra $\text{LDIAG}(q_c, q_s)$ is free. Thus one may construct a coproduct $\Delta; t \in K$ such that $\text{LDIAG}(q_c, q_s, t) = (\text{LDIAG}(q_c, q_s), \text{ldiag}, \Delta, \varepsilon, S)$ is a Hopf algebra. We have the following specializations

$$ (q_c, q_s, t) = (0, 0, 0) \quad (1, 1, 1) $$

4 Conclusion

The results which we announced here in this note can be illustrated by the following picture. All details will be given in forthcoming papers [3, 4].
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