ON ENTROPY TRANSMISSION FOR QUANTUM CHANNELS

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Abstract. In this paper a notion of entropy transmission of quantum channels is introduced as a natural extension of Ohya’s entropy. Here by quantum channel is meant unital completely positive mappings (ucp) of $B(H)$ into itself, where $H$ is an infinite dimensional Hilbert space. Using a representation theorem of ucp mapping we associate to every ucp map a uniquely determined state, and prove that entropy of ucp map is less then Ohya’s entropy of the associated state.

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1. Introduction

The concept of state in a physical system is a powerful weapon to study the dynamical behavior of that system. Since von Neumann introduced a quantum mechanical entropy of a state, many physicists have applied it in several dynamical systems and studied its general properties (see [8], [21]). For several reasons this entropy had been extended to $C^*$-dynamical systems by [2],[3],[8],[20]. In [10] the entropy of a state in quantum systems within $C^*$-algebraic framework was introduced and studied. Note that the mechanism of transmission is expressed by a so-called channel between input and output receivers. In the quantum information theory channels play an important role (see for example [6],[11, 12, 13],[14]). In [14],[15] the quantum mutual entropy and the quantum capacity were studied. In the algebraic framework a quantum channel can be expressed by so-called completely positive mapping from a $C^*$-algebra $M$ into another a $C^*$-algebra $N$. In the present paper we introduce a natural definition of entropy of unital completely positive (ucp) maps of $B(H)$. We show that this entropy coincides with Ohya’s entropy of a state (see [10]) when we take a state instead of ucp map. Note that in [1],[7] dynamical entropy of ucp maps were introduced and studied, which differs from ours. Since here introduced entropy in some sense, as well as, is an extension of von Neumann’s quantum mechanical entropy to ucp maps. On the other hand, roughly speaking, each ucp map $T$ can be represented as a convex combination of extremal ucp maps $T_k$, and hence, information is going through a channel $T$ depends on information going through channels $T_k$. The introduced entropy deals with uncertainties.
coming in this way and measures the amount of chaos within mixture of quantum channels.

2. Preliminaries

Let \( B(H) \) be the set of all linear bounded operators defined on a separable Hilbert space \( H \). An element \( x \in B(H) \) is called positive if there is an element \( y \in B(H) \) such that \( x = y^*y \). The set of all positive elements of \( B(H) \) we denote by \( B(H)_+ \). A linear functional \( \omega \) on \( B(H) \) is said to be a state if \( \omega(x) \geq 0 \) for all \( x \in B(H)_+ \) and \( \omega(1) = 1 \), here \( 1 \) stands the identity operator of \( H \). A state \( \omega \) is called faithful if \( \omega(x^*x) = 0 \) implies \( x = 0 \). A state \( \omega \) is called trace if the equality \( \omega(xy) = \omega(yx) \) is valid for all \( x,y \in B(H) \). A linear functional \( f \) is called normal if for every bounded increasing net \( \{x_\alpha\} \) of positive elements of \( B(H) \) the equality \( \sup f(x_\alpha) = f(\sup x_\alpha) \) is valid. By \( B(H)_* \) we denote the set all linear normal functionals on \( B(H) \).

By \( S \) the set of normal states on \( B(H) \) is denoted.

The set of linear continuous (in norm ) maps of \( B(H) \) into itself is denoted by \( BB(H) \). On \( BB(H) \) we define a weak topology by seminorms

\[
p_{\varphi,x}(T) = |\varphi(Tx)|, \quad x \in B(H), \varphi \in B(H)_*.
\]

(2.1)

A norm of \( T \in BB(H) \) is defined as usual by

\[
\|T\| = \sup_{x \in B(H), \|x\|=1} \|T(x)\|.
\]

Denote

\[
B_1 = \{T \in BB(H) : \|T\| = 1\}.
\]

In [9] the following theorem was proved.

**Theorem 2.1.** The set \( B_1 \) is weak compact.

Recall that a linear map \( T \in BB(H) \) is said to be completely positive if for any two collections \( a_1, \cdots, a_n \in B(H) \), \( b_1, \cdots, b_n \in B(H) \) the following relation holds

\[
\sum_{i,j=1}^n b_i^*T(a_i^*a_j)b_j \geq 0.
\]

(2.2)

A completely positive map \( T : B(H) \to B(H) \) with \( T1 = 1 \) is called unital completely positive (ucp) map. The set of all ucp maps defined on \( B(H) \) we denote by \( \Sigma(B(H)) \). For a ucp map \( T \) we have \( \|T\| = \|T(1)\| = 1 \), therefore \( \Sigma(B(H)) \subset B_1 \).

**Proposition 2.2.** The set \( \Sigma(B(H)) \) is weak convex compact.

**Proof.** Let a net \( \{T_\nu\} \subset \Sigma(B(H)) \) weakly converge to an operator \( T \). This means that for any state \( \varphi \in S \) we have

\[
\varphi(T(x)) = \lim_{\nu \to \infty} \varphi(T_\nu(x)) \quad \forall x \in B(H).
\]

(2.3)
Now show that $T$ is ucp map. From (2.3) one can see that $T \mathbf{1} = \mathbf{1}$. Since every $T_\nu$ is ucp map, so for them (2.2) holds, i.e.

$$\sum_{i,j=1}^{n} \varphi(b^*_i T_\nu(a^*_i a_j)b_j) \geq 0.$$  \hspace{1cm} (2.4)

Now passing to limit $\nu \to \infty$ from both sides of (2.4) we obtain that $T \in \Sigma(B(H))$. Therefore, $\Sigma(B(H))$ is a closed subset of $B_1$. Now Theorem 2.1 implies the assertion. \hfill \Box

Further $\text{extr}(\Sigma)$ denotes the set of all extremal points of a set $\Sigma := \Sigma(B(H))$. According to Proposition 2.2 and Krein - Milman Theorem the set $\text{extr}(\Sigma)$ is non empty. Note that in [16] certain properties of the set $\text{extr}(\Sigma)$ were studied.

3. An entropy of ucp maps

According to the compactness of $\Sigma$ we can apply the theory of decompositions of Choquet (see [4]).

According to that theory [4] for every operator $T \in \Sigma$ there is a probability measure $\mu$ on $\text{extr}(\Sigma)$ with a barycenter on $T$ such that

$$T = \int_{\text{extr}(\Sigma)} h d\mu(h) \hspace{1cm} (3.1)$$

If the measure $\mu$ is atomic, i.e. $\mu = \{\lambda_n\}$, $\lambda_n \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$ then (3.1) has a form

$$T = \sum_{n=1}^{\infty} \lambda_n T_n, \hspace{1cm} T_n \in \text{extr}(\Sigma). \hspace{1cm} (3.2)$$

An entropy of a ucp map $T$ is defined by

$$H(T) = \inf \left\{ -\sum_{n=1}^{\infty} \lambda_n \ln \lambda_n \right\}, \hspace{1cm} (3.3)$$

here inf is taken over for all possible discrete decompositions of $T$ because the measure $\mu$ is not always unique. If $\mu$ is not atomic then $H(T)$ is defined to be infinite, i.e. $H(T) = \infty$.

It is easy to see that $H(T) = 0$ if and only if $T \in \text{extr}(\Sigma)$.

Denote

$$\Sigma_f = \{T \in \Sigma : H(T) < \infty\}.$$

**Theorem 3.1.** The weak closure of $\Sigma_f$ is the set $\Sigma$.

The proof immediately follows from Proposition 2.2 and the Krein-Milman Theorem.

For a given state $\varphi \in S$ we define an ucp map $T_\varphi$ by

$$T_\varphi x = \varphi(x) \mathbf{1}, \hspace{1cm} x \in B(H).$$

**Lemma 3.2.** If $\varphi \in \text{extr}(S)$ then $T_\varphi \in \text{extr}(\Sigma(B(H)))$. 
Proof. Let us assume that the equality $T\varphi = \lambda T_1 + (1 - \lambda)T_2$ is valid, here $T_1, T_2 \in \Sigma(B(H))$ and $\lambda \in (0, 1)$. Whence for every state $\psi \in S$ we find

$$\varphi(x) = \lambda \psi(T_1 x) + (1 - \lambda)\psi(T_2 x), \quad x \in B(H).$$

According to the extremity of the state $\varphi$ we get $\psi(T_i x) = \varphi(x), \ i = 1, 2$. From the arbitrariness of $\psi$ we conclude that $T_i x = \varphi(x)I$. Thus $T\varphi \in \text{extr}(\Sigma(B(H)))$. The lemma is proved. \[\Box\]

We recall that an Ohya’s-entropy of a state $\varphi \in S$ introduced in [10] and denoted by $h(\varphi)$. Given a state $\varphi \in S$ there is a positive operator $\theta$ such that $\varphi(x) = \text{tr}(\theta x), \ x \in B(H)$. According to the spectral resolution theorem the operator $\theta$ can be expressed by

$$\theta = \sum_n \lambda_n \phi_n$$

where $\lambda_n$ are eigenvalues of $\theta$, and $\phi_n$ are projections to one dimensional subspaces generated by mutually orthogonal eigenvectors associated with $\lambda_n$, i.e.

$$\phi_n(\xi) = (\xi, \phi_n), \quad \xi \in H, \ n \in \mathbb{N}.$$ 

Then Ohya’s entropy of a state $\varphi$ is defined by

$$h(\varphi) = \inf \left\{ -\sum_{n=1}^{\infty} \lambda_n \ln \lambda_n \right\}, \quad (3.4)$$

here $\inf$ is taken over for all possible discrete decompositions of $\theta$.

According to Lemma 3.2 with (3.3), (3.4) we have

$$H(T\varphi) = h(\varphi).$$

This shows that the introduced entropy $H(T)$ is a generalization of Ohya’s one.

Recall that a ucp map $T \in Z$ is called normal if for every bounded increasing net of positive elements $\{x_\alpha\}$ of $B(H)$ the equality

$$T(\sup_\alpha x_\alpha) = \sup_\alpha T(x_\alpha).$$

is valid. This definition is equivalent to that an operator $T$ is continuous in $\sigma(B(H), B(H)_*)$ - weak topology.

Further by $\text{tr}$ we will denote a normalized trace on $B(H)$. Consider a set

$$\mathcal{T}_0 = \{ x \in B(H) \otimes B(H) : \text{tr} \otimes \text{tr}(|x|) < \infty \}$$

and denote $\|x\|_1 = \text{tr} \otimes \text{tr}(|x|), \ x \in \mathcal{T}_0$, which is a norm. By $\mathcal{T}(H \otimes H)$ we will denote the norm $\|\cdot\|_1$ closure of $\mathcal{T}_0$, and by $\text{tr}_H$ a conditional trace from $B(H) \otimes B(H)$ onto $B(H)$ defined on the elements of kind $x \otimes y$ by

$$\text{tr}_H(x \otimes y) = \text{tr}(y)x, \quad x, y \in B(H).$$

Now we formulate some auxiliary facts.
Lemma 3.3. If the following equality holds
\[ \text{tr}_H(p(1 \otimes x)) = 0, \] (3.5)
for any \( x \in B(H) \), then \( p = 0 \).

Proof. It follows from (3.5) that for every \( y \in B(H) \) the equality holds
\[ 0 = \text{tr}(y \otimes 1)\text{tr}_H(p(1 \otimes x)) = \text{tr} \otimes \text{tr}(p(y \otimes x)) \]
Hence from arbitrariness of \( x \) and \( y \) we find \( \text{tr} \otimes \text{tr}(pu) = 0 \) for any \( u \in B(H) \otimes B(H) \), which implies that \( p = 0 \). □

Let \( \{ \varphi_n \} \) be a complete orthonormal basis in the Hilbert space \( H \). One can see that a system \( \{ \varphi_m \otimes \varphi_n \} \) forms a complete orthonormal basis for \( H \otimes H \).

Denote
\[ H_0 = \left\{ \omega = \sum_{i,j=1}^n a_i b_j \varphi_i \otimes \varphi_j : \{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{C}, n \in \mathbb{N} \right\}. \] (3.6)

Lemma 3.4. The set \( H_0 \) is a dense subspace of \( H \otimes H \).

Proof. It is known [19] that the following set
\[ H_1 = \left\{ \sum_{i,j=1}^{n,m} a_i c_j \varphi_i \otimes \varphi_j : \{a_i\}_{i=1}^n, \{c_i\}_{i=1}^m \subset \mathbb{C}, n,m \in \mathbb{N} \right\} \]
is dense in \( H \otimes H \). Therefore, it is enough to show \( H_0 = H_1 \). Due to obvious inclusion \( H_0 \subset H_1 \), we have to show \( H_0 \supset H_1 \). Take any \( v \in H_1 \), i.e.
\[ v = \sum_{i,j=1}^{n,m} a_i c_j \varphi_i \otimes \varphi_j, \]
where \( \{a_i\}_{i=1}^n, \{c_j\}_{j=1}^m \subset \mathbb{C} \). Without loss of generality we may assume that \( n \geq m \). Let us put
\[ b_k = \begin{cases} c_k, & \text{if } k \leq m, \\ 0, & \text{if } m < k \leq n. \end{cases} \]
Then one can see that
\[ v = \sum_{i,j=1}^n a_i b_j \varphi_i \otimes \varphi_j, \]
hence \( v \in H_0 \). This proves the assertion. □

Theorem 3.5. Let \( T \in B_1 \). The following assertions are equivalent:

(i) \( T \) is a normal ucp map;

(ii) There exists a unique positive operator \( p \in \mathcal{T}(H \otimes H) \) with \( \text{tr}_H(p) = 1 \) such that \( Tx = \text{tr}_H(p(1 \otimes x)), \quad x \in B(H) \).
Proof. The implication (ii) ⇒ (i) is obvious. Therefore, consider (i) ⇒ (ii).

Define $e_{ij} : H \to H$ by

$$e_{ij}(\eta) = (\eta, \varphi_j)\varphi_i, \ \eta \in H, \ i,j \in \mathbb{N}. \quad (3.7)$$

It is clear that $e_{ij}(\varphi_k) = \delta_{kj}\varphi_i$ and $e_{ii}$ is a projection to one dimensional subspace of $H$ generated by the vector $\varphi_i$, where $\delta_{kj}$ stands for the symbol Kornecker. Moreover $\{e_{ij}\}$ forms a basis for $B(H)$.

Now define

$$p_{(ij)(kl)} = (T(e_{ij})\varphi_k, \varphi_l), \ i,j,k,l \in \mathbb{N}. \quad (3.8)$$

Recall some properties of $p_{(ij)(kl)}$ (see [17], Chap. 4):

(A) for every $n \in \mathbb{N}$ and any collection of numbers $\{a_k\}_{k=1}^n, \{b_k\}_{k=1}^n$ the following inequality holds

$$\sum_{i,j,k,l=1}^n a_i \overline{a_j} b_k \overline{b_l} p_{(ij)(kl)} \geq 0;$$

(B) for every $i,j,k,l \in \mathbb{N}$

$$p_{(ij)(kl)} = p_{(ji)(lk)};$$

(C) For every $k,l \in \mathbb{N}$

$$\sum_{i=1}^\infty p_{(ii)(kl)} = \delta_{k,l}.$$

Define an operator $p$ by

$$p = \sum_{(ij)(kl)} p_{(ij)(kl)} e_{kl} \otimes e_{ij}, \quad (3.8)$$

here convergence by the norm $\| \cdot \|_1$.

Now we calculate the conditional trace of $p$:

$$tr_H(p) = tr_H \left( \sum_{(ij)(kl)} p_{(ij)(kl)} e_{kl} \otimes e_{ij} \right)$$

$$= \sum_{(ij)(kl)} p_{(ij)(kl)} tr(e_{ij} e_{kl})$$

$$= \sum_{(ij)(kl)} p_{(ij)(kl)} \delta_{ij} e_{kl}$$

$$= \sum_k e_{kk} = 1,$$

here we have just used the property (C) of $p_{(ij)(kl)}$. 


Using (3.7) and (3.8) consider an action of \( p \) to the element \( \varphi_q \otimes \varphi_m \):

\[
p(\varphi_q \otimes \varphi_m) = \sum_{(ij)(kl)} p_{ijkl} e_{kl}(\varphi_q) \otimes e_{ij}(\varphi_m)
\]

\[
= \sum_{(ij)(kl)} p_{ijkl} \delta_{qj} \varphi_k \otimes \delta_{jm} \varphi_i
\]

\[
= \sum_{i,k} p_{ikm} e_k \otimes \varphi_i \quad (3.9)
\]

Now show that \( p \) is positive. Indeed, take any \( \omega \in H_0 \) (see (3.6)), i.e.

\[
\omega = \sum_{i,j} a_i b_j \varphi_i \otimes \varphi_j,
\]

where \( \{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{C}, n \in \mathbb{N} \).

Thanks to the positivity of operator \( T \), the property (A) and (3.9) we have

\[
(p(\omega), \omega) = \sum_{i,j=1}^n a_i b_j \varphi_i \otimes \varphi_j(p(\varphi_i \otimes \varphi_j), \varphi_k \otimes \varphi_l)
\]

\[
= \sum_{i,j,k,l=1}^n a_i a_k b_j b_l \sum_{r,s} p_{ijkl}(\varphi_s \otimes \varphi_r, \varphi_k \otimes \varphi_l)
\]

\[
= \sum_{i,j,k,l=1}^n a_i a_k b_j b_l p_{ijkl} \geq 0.
\]

From the density of \( H_0 \) in \( H \otimes H \) (see Lemma 3.4) we get the positivity of \( p \). Thus \( p \in \mathcal{T}(H \otimes H) \).

Consider

\[
p(1 \otimes e_{ij}) = \sum_{(kl)(rs)} p_{klrs} e_{rs} \otimes (e_{kl} \cdot e_{ij})
\]

\[
= \sum_{k,s} p_{ki} e_{rs} \otimes e_{kj}.
\]

Then we obtain

\[
tr_H(p(1 \otimes e_{ij})) = \sum_{k,s} p_{ki} tr(e_{kj}) e_{rs}
\]

\[
= \sum_{k,l} p_{(ij)kl} e_{rs} = T(e_{ij}).
\]

Using continuity of \( T \) for every \( x \in B(H) \) we get

\[Tx = tr_H(p(1 \otimes x)).\]

Let us show the uniqueness of \( p \). Assume that there is another operator \( q \) such that \( Tx = tr_H(q(1 \otimes x)) \). Then we have \( tr_H((p - q)(1 \otimes x)) = 0 \) for
every $x \in B(H)$. According to Lemma 3.3 we obtain $p = q$. This completes the proof. □

Remark 3.1. The proved Theorem generalizes a result of [18], where similar result has been obtained over finite-dimensional Hilbert spaces.

The operator $p$ in Theorem 3.5 is called a representative operator for $T$, and denoted by $\rho_T$. For each ucp map we can associate a state $\varphi_T$ on $B(H) \otimes B(H)$ defined by

$$
\varphi_T(x) = tr \otimes tr(\rho_T x), \quad x \in B(H) \otimes B(H).
$$

Theorem 3.6. Let $T$ be a ucp map on $B(H)$. Then the following inequality holds

$$
H(T) \leq h(\varphi_T) = -tr \otimes tr(\rho_T \ln \rho_T). \quad (3.10)
$$

Proof. Let us decompose the operator $\rho_T$ as follows

$$
\rho_T = \sum_k \lambda_k \rho_k, \quad (3.11)
$$

where $\lambda_k \geq 0$, $\sum_k \lambda_k = 1$ and $tr_H(\rho_k) = I$. Then (see (3.3))

$$
H(T) = \inf \left\{ -\sum_k \lambda_k \ln \lambda_k \right\}.
$$

According to the spectral resolution Theorem each operator $\rho_k$ can be decomposed as follows

$$
\rho_k = \sum_n \mu_{k,n} e_n^{(k)}. \quad (3.12)
$$

Keeping in mind (3.12) we can rewrite (3.11) by

$$
\rho_T = \sum_{k,n} \lambda_k \mu_{k,n} e_n^{(k)}.
$$

Thanks to $tr \otimes tr(\rho_k) = 1$ we get $\sum_n \mu_{k,n} = 1$. Hence, using the definition of Ohya’s entropy (see (3.4)) one gets

$$
\begin{align*}
\varphi_T(x) &= \inf \left\{ -\sum_{k,n} \lambda_k \mu_{k,n} \ln(\lambda_k \mu_{k,n}) \right\} \\
&= \inf \left\{ -\sum_{k,n} \lambda_k \mu_{k,n} \ln \lambda_k - \sum_{k,n} \lambda_k \mu_{k,n} \ln \mu_{k,n} \right\} \\
&= \inf \left\{ -\sum_k \lambda_k \ln \lambda_k - \sum_{k,n} \lambda_k \mu_{k,n} \ln \mu_{k,n} \right\} \\
&\geq H(T).
\end{align*}
$$

□

Remark 3.2. In Theorem 3.12 strict inequality can occur. Indeed, consider the following example.
Example. Let $H = \mathbb{C}^2$, then $B(H) = M_2(\mathbb{C})$, here $M_2(\mathbb{C})$ is the algebra of $2 \times 2$ matrices over complex field $\mathbb{C}$. By $e_{ij}$ we denote the matrix units of $M_2(\mathbb{C})$. A commutative algebra generated by the matrix units $e_{11}$ and $e_{22}$ is denoted by $CM_2$. We represent every element \( \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) of $M_2(\mathbb{C})$ as a vector of $\mathbb{C}^4$ by \((a_{11}, a_{22}, a_{12}, a_{21})\).

Now take $T \in \Sigma(CM_2)$ defined by
\[
T = \begin{pmatrix}
p & 1-p & 0 & 0 \\
q & 1-q & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
here $p, q \in (0, 1)$. One can see that that extreme elements of $\Sigma(CM_2)$ are the following ones
\[
T_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]
\[
T_3 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_4 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

A decomposition of $T$ into a convex combination of $T_i$, $i = 1, 4$ is given by
\[
T = (1 - q - d)T_1 + (p + q + d - 1)T_2 + (1 - p - d)T_3 + dT_4,
\]
here $d \in [\max\{0, 1 - p - q\}, \min\{1 - p, 1 - q\}]$.

Furthermore, we will assume that $p + q = 1$. Then from (3.13) one finds
\[
\begin{align*}
H(T) &= \inf_{d \in [0, \min p, q]} \left\{ -(p - d) \log(p - d) - (q - d) \log(q - d) - 2d \log d \right\} \\
&= -(p - q) \log(p - q) - 2q \log q.
\end{align*}
\]
Without loss of generality we may assume that $p \geq q$. Then investigating extremum of a function
\[
F(x) = -(p - x) \log(p - x) - (q - x) \log(q - x) - 2x \log x
\]
on $[0, q]$ we find that
\[
H(T) = -p \log p - q \log q.
\]
(3.14)

The representing operator $\rho_T$ can be written by
\[
\rho_T = \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & 1 - p & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & 1 - q
\end{pmatrix}.
\]
It is clear that
\[ d(\rho_T) = -(1 - p) \log(1 - p) - p \log p - q \log q - (1 - q) \log(1 - q). \]
From this and (3.14) one can see that \( H(T) < d(\rho_T). \)

4. Conclusions

In this paper a notion of entropy transmission of quantum channels is introduced. Here by quantum channel we mean unital completely positive (ucp) mappings of \( B(H) \) into itself, where \( H \) is an infinite dimensional separable Hilbert space. We have shown that this entropy coincides with Ohya’s entropy of a state (see [10]) if we take a state instead of ucp map. Therefore, the introduced entropy is a natural extension of Ohya’s one. Since Ohya’s entropy was a generalization of von Neumann quantum mechanical entropy, consequently the introduced entropy is also an extension of the quantum mechanical one. Besides, it measures the amount of chaos within mixture of quantum channels. Therefore, it differs from the dynamical entropy of ucp maps introduced in [1, 7]. Furthermore, using a representation theorem of ucp maps we associate to every ucp map a uniquely defined state, and prove that entropy of ucp map is less than Ohya’s entropy of the associated state. We hope that this entropy will have some relations with the Holevo capacity of quantum channel (see [5]).

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