A REINTERPRETATION OF EMERTON’S $p$-ADIC BANACH SPACES

RICHARD HILL

Abstract. It is shown that the $p$-adic Banach spaces introduced by Emerton are isomorphic to the cohomology groups of the sheaf of continuous $\mathbb{Q}_p$-valued functions on a certain space. Some applications of this result are discussed.

1. Introduction

Let $G$ be a reductive group over a number field $k$. We fix once and for all a maximal compact subgroup $K_\infty \subset G(k \otimes \mathbb{R})$, and we consider the “Shimura manifolds”:

$$Y(K_f) = G(k) \backslash G(\mathbb{A}) / K_\infty K_f,$$

where $K_f$ is a compact open subgroup of $G(k_f)$, and $K_\infty$ is the identity component in $K_\infty$. Fix once and for all a finite prime $p$ of $k$, and let $p$ be the rational prime below $p$.

In [2] Emerton introduced the following spaces:

$$\tilde{H}_{\ast}(K_P, \mathbb{Z}_p) = \lim_{\leftarrow} \lim_{\rightarrow} K_P H_{\ast}(Y(K_P K_P), \mathbb{Z}/p^s),$$

$$\tilde{H}_{\ast}^c(K_P, \mathbb{Q}_p) = \tilde{H}_{\ast}(K_P, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where $\ast$ is either the empty symbol, denoting usual cohomology, or “c”, denoting compactly supported cohomology. Here the $K_P$ ranges over the compact open subgroups of $G(k_P)$, and $K_P$ is a fixed compact open subgroup of $G(k_f)$. The spaces $H_{\ast}^c(K_P, \mathbb{Q}_p)$ are $p$-adic Banach spaces, and are central to Emerton’s construction of eigenvarieties in [2].

The aim of this paper is to give a more convenient interpretation of these spaces. To explain this interpretation, consider the topological space:

$$Y(K_P) = G(k) \backslash G(\mathbb{A}) / K_P K_\infty = \lim_{\leftarrow} K_P Y(K_P K_P).$$

Let $\mathcal{C}_{\mathbb{Z}_p}$ (resp. $\mathcal{C}_{\mathbb{Q}_p}$) be the sheaf of continuous $\mathbb{Z}_p$-valued (resp. $\mathbb{Q}_p$-valued) functions on $Y(K_P)$. The space $Y(K_P)$ need not be compact, but it is homotopic to a profinite simplicial complex, which we shall call $Y(K_P)^{B.S.}$. We shall also use the notation $\partial Y(K_P)^{B.S.} = Y(K_P)^{B.S.} \setminus Y(K_P)$.

Our main result is the following.

Theorem 1. There are canonical isomorphisms

$$\tilde{H}_{\ast}(K_P, \mathbb{Z}_p) = \tilde{H}_{\ast}(Y(K_P), \mathcal{C}_{\mathbb{Z}_p}), \quad \tilde{H}_{\ast}^c(K_P, \mathbb{Z}_p) = \tilde{H}_{\ast}^c(Y(K_P)^{B.S.}, \partial Y(K_P), \mathcal{C}_{\mathbb{Z}_p}),$$

and similarly for $\mathbb{Q}_p$. The right hand side of these equations is $\check{C}$ech cohomology, which in this case is equal to sheaf cohomology.

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One of our aims in proving this result is to compare the spaces $\tilde{H}^\bullet$ and $\tilde{H}_c^\bullet$. An immediate consequence of our result is a long exact sequence involving these two spaces
$$\tilde{H}_c^n(K^p, \mathbb{Z}_p) \to \tilde{H}_c^n(K^p, \mathbb{Z}_p) \to \tilde{H}_c^n(K^p, \mathbb{Z}_p) \to \tilde{H}_c^{n+1}(K^p, \mathbb{Z}_p) \to,$$
where we are using the notation $\tilde{H}_c^n = \tilde{H}_c^n(\partial Y(K^p), B.S., C.Z_p)$.

This is significant, since one can show that $\tilde{H}_\partial^n$ vanishes unless $n$ is quite small. For example, if $G$ has real rank 1 and $p$ is the only prime of $k$ above $p$, then only $\tilde{H}_c^0$ is non-zero. As a consequence, we know that for such groups, $\tilde{H}_c^n$ and $\tilde{H}_c^n$ are equal for $n \geq 2$. Generalizations of such results will be discussed in a forthcoming paper.

It is also envisioned that these results should give new insight into Eisenstein cohomology classes. To see why this might be the case, we recall that Emerton proved a spectral sequence
$$\operatorname{Ext}_G^p(W, \tilde{H}_c^q(K^p, k^p)_{\text{loc.an.}}) \Rightarrow H_c^{p+q}(Y(K^p), W),$$
where $W$ is a local system on $Y(K^p)$ given by a finite dimensional representation of $G$ over $k^p$. We remark that if $G$ is semi-simple then $\tilde{H}_c^n$ is zero for $n < \text{rank}_k(G)$; this follows easily by Poincaré duality. On the other hand $\tilde{H}_\partial^n$ vanishes for all but small values of $n$. Hence one might expect to be able to recover the boundary cohomology quite low down in the filtration given by the spectral sequence.

2. Some facts about Čech cohomology

Let $X$ be a topological space and $\mathcal{F}$ a presheaf on $X$. For an open cover $\mathcal{U} = \{U_i : i \in I\}$ of $X$, we define the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ by
$$\check{C}^n(\mathcal{U}, \mathcal{F}) = \{(f_{i_0, \ldots, i_n})_{i_0, \ldots, i_n} \in \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n}) : f_{i_0, \ldots, i_n} \in \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n})\}.$$ The cohomology groups of this complex are written $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$. The Čech cohomology groups are defined to be the direct limits of these cohomology groups:
$$\check{H}^n(X, \mathcal{F}) = \lim_{\mathcal{U}} \check{H}^n(\mathcal{U}, \mathcal{F}).$$
In fact $\check{H}^n(X, \mathcal{F})$ depends only on the sheafification of $\mathcal{F}$.

**Theorem 2** (Leray's Theorem). Let $\mathcal{F}$ be a sheaf on a topological space $X$ and $\mathcal{U}$ a countable open cover of $X$. If $\mathcal{F}$ is acyclic on every finite intersection of elements of $\mathcal{U}$, then
$$\check{H}^n(X, \mathcal{F}) = \check{H}^n(\mathcal{U}, \mathcal{F}).$$

**Theorem 3** (Thm. III.4.12 of [1]). If $\mathcal{F}$ is a sheaf on $X$ and $X$ is paracompact, then the Čech cohomology groups of $\mathcal{F}$ are equal to its sheaf cohomology groups, i.e. the derived functors of the global sections functor.

Given a presheaf $\mathcal{F}$ on a topological space $Y$, and a subspace $Z \subset Y$, we define presheaves $\mathcal{F}_Z$ and $\mathcal{F}^Z$ on $X$ by
$$\mathcal{F}_Z(U) = \begin{cases} \mathcal{F}(U) & \text{if } U \cap Z \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$
It turns out that \( \check{H}^\bullet(Z, \mathcal{F}) = \check{H}^\bullet(X, \mathcal{F}_Z) \), and one defines

\[
\check{H}^\bullet(Y, Z, \mathcal{F}) = \check{H}^\bullet(Y, \mathcal{F}^Z).
\]

There is a short exact sequence of presheafs:

\[
0 \rightarrow \mathcal{F}^A \rightarrow \mathcal{F} \rightarrow \mathcal{F}_A \rightarrow 0,
\]

This gives a long exact sequence:

\[
\check{H}^n(\mathcal{U}, \mathcal{F}^A) \rightarrow \check{H}^n(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^n(\mathcal{U}, \mathcal{F}_A) \rightarrow \check{H}^{n+1}(\mathcal{U}, \mathcal{F}^A).
\]

Passing to the direct limit, we obtain the long exact sequence of \( \check{\text{C}} \)ech cohomology groups:

\[
\check{H}^n(X, A, \mathcal{F}) \rightarrow \check{H}^n(X, \mathcal{F}) \rightarrow \check{H}^n(A, \mathcal{F}) \rightarrow \check{H}^{n+1}(X, A, \mathcal{F}).
\]

If \( A \) is an abelian group, then we shall also write \( A = \) for the sheaf of locally constant \( A \)-valued functions. Using Leray’s theorem, one easily proves the following:

**Theorem 4** (Comparison Theorem). Let \( Y \) be a finite simplicial complex, and \( Z \subset Y \) a subcomplex. For any abelian group \( A \), we have

\[
\check{H}^\bullet(Y, Z, A) = H^\bullet(Y, Z, A),
\]

where the right hand side is singular cohomology.

In fact the comparison theorem holds for much more general topological spaces (see for example [3]).

**Theorem 5** (Lem. 6.6.11, Cor 6.1.11 and Cor. 6.9.9 of [3]). Let \( Y \) be a finite simplicial complex and \( Z \) a subcomplex. For any abelian group \( A \), we have

\[
H^\bullet_c(Y \setminus Z, A) = H^\bullet(Y, Z, A).
\]

3. Proofs

Emerton used the following formalism to introduce the groups \( \check{H}^n_c \). Let \( G \) be a compact, \( \mathbb{Q}_p \)-analytic group, and fix a basis of open, normal subgroups:

\[
G = G_0 \supset G_1 \supset \ldots.
\]

Suppose we have a sequence of simplicial maps between finite simplicial complexes

\[
\cdots \rightarrow Y_2 \rightarrow Y_1 \rightarrow Y_0,
\]

and subcomplexes:

\[
\cdots \rightarrow Z_2 \rightarrow Z_1 \rightarrow Z_0,
\]

each equipped with a right action of \( G \), and satisfying the following conditions:

1. the maps in the sequence are \( G \)-equivariant;
2. \( G_r \) acts trivially on \( Y_r \);
3. if \( 0 \leq r' \leq r \) then the maps \( Y_r \rightarrow Y_{r'} \) and \( Z_r \rightarrow Z_{r'} \) are Galois covering maps with deck transformations provided by the natural action of \( G_{r'}/G_r \) on \( Y_r \).
Given this data, we let $Y$ be the projective limit of the spaces $Y_i$, and $Z$ be the projective limit of the spaces $Z_i$. We shall use the notation $Y^0 = Y \setminus Z$, $Y^0_i = Y_i \setminus Z_i$. Emerton defined the following spaces:

$$
\tilde{H}^n_c(Y^0, \mathbb{Z}_p) = \lim_{\leftarrow s} \lim_{\rightarrow r} \tilde{H}^n_c(Y^0_r, \mathbb{Z}/p^s) \quad \text{and} \quad \tilde{H}^n_c(Y^0, \mathbb{Q}_p) = \tilde{H}^n_c(Y^0, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
$$

In applications, $G$ will be a compact open subgroup of $\mathbb{G}(k_p)$, and the space $Y$ will be either $Y(K^p)^{B.S.}$ or $\partial Y^{B.S.}$. If $Y = Y(K^p)^{B.S.}$, then we may use the subspace $Z = \partial Y(K^p)^{B.S.}$.

**Theorem 6.** With the above notation,

$$
\tilde{H}^n_c(Y^0, \mathbb{Z}_p) = \tilde{H}^n(Y, Z, \mathbb{C}_{\mathbb{Z}_p}), \quad \tilde{H}^n_c(Y^0, \mathbb{Q}_p) = \tilde{H}^n(Y, Z, \mathbb{C}_{\mathbb{Q}_p}).
$$

**Proof.** We shall prove the case with coefficients in $\mathbb{Z}_p$. The $\mathbb{Q}_p$ case is a consequence. We shall write $\mathbb{C}$ instead of $\mathbb{C}_{\mathbb{Z}_p}$. To prove the theorem, we construct an acyclic cover of $Y$ and apply Leray’s Theorem.

### 3.1. A cover

We first choose a finite open cover $\mathcal{U}$ of $Y_0$ with the following properties:

1. If $U$ is an intersection of finitely many sets in $\mathcal{U}$ then either $U$ is empty or $U$ is contractible.
2. If $U$ is an intersection of finitely many sets in $\mathcal{U}$ and $U \cap Z_0$ is non-empty, then $U \cap Z_0$ is a deformation retract of $U$.

For each $U \in \mathcal{U}$, we let $U(r)$ be the preimage of $U$ in $Y_r$. The sets $U(r)$ form an open cover $\mathcal{U}(r)$ of $Y_r$, and have the following properties:

1. For every $U_1^{(r)}, \ldots, U_s^{(r)} \in \mathcal{U}(r)$ with non-empty intersection, the intersection $U_1^{(r)} \cap \cdots \cap U_s^{(r)}$ is isomorphic as a topological $G$-set to $(U_1 \cap \cdots \cap U_r) \times (G/G_r)$. In particular, the intersection is homotopic to a finite set.
2. If $U_1^{(r)}, \ldots, U_s^{(r)} \in \mathcal{U}(r)$ and $U_1^{(r)} \cap \cdots \cap U_s^{(r)} \cap Z_r$ is non-empty, then $U_1^{(r)} \cap \cdots \cap U_s^{(r)} \cap Z_r$ is a deformation retract of $U_1^{(r)} \cap \cdots \cap U_s^{(r)}$.

Furthermore, for each set $U \in \mathcal{U}$, we define $\tilde{U}$ to be the preimage of $U$ in $Y$. The sets $\tilde{U}$ form an open cover $\tilde{\mathcal{U}}$ of $Y$. We immediately verify the following:

1. If $\tilde{U}_1, \ldots, \tilde{U}_s \in \tilde{\mathcal{U}}$ have non-empty intersection, then their intersection is equivalent as a topological $G$-set to $(U_1 \cap \cdots \cap U_r) \times G$.
2. If $\tilde{U}_1, \ldots, \tilde{U}_s \in \tilde{\mathcal{U}}$ and $\tilde{U}_1 \cap \cdots \cap \tilde{U}_s \cap Z$ is non-empty, then $\tilde{U}_1 \cap \cdots \cap \tilde{U}_s \cap Z$ is a deformation retract of $\tilde{U}_1 \cap \cdots \cap \tilde{U}_s$.

### 3.2. $\mathcal{U}^{(r)}$ is $(\mathbb{Z}/p^s)^{Z_r}$-acyclic

Let $U$ be an intersection of finitely many sets in $\mathcal{U}$, and let $U^{(r)}$ be the preimage of $U$ in $Y_r$. We know that $U$ is contractible, and $U^{(r)} = U \times (G/G_r)$. The sheaf $(\mathbb{Z}/p^s)^{Z_r}$ on $Y_r$ consists of locally constant $\mathbb{Z}/p^s$-valued functions, which vanish on $Z_r$. It follows that $\tilde{H}^\bullet(U^{(r)}, (\mathbb{Z}/p^s)^{Z_r})$ is a direct sum of finitely many copies of $\tilde{H}^\bullet(U, (\mathbb{Z}/p^s)^{Z_0})$. We must therefore show that $\tilde{H}^n(U, (\mathbb{Z}/p^s)^{Z_0}) = 0$ for all $n > 0$.

If $U \cap Z_0$ is empty, then we have $\tilde{H}^n(U, (\mathbb{Z}/p^s)^{Z_0}) = \tilde{H}^n(U, \mathbb{Z}/p^s)$. By the comparison theorem, this is the same as singular cohomology, and therefore only depends on $U$ up to homotopy. Since $U$ is contractible, it follows that $\tilde{H}^n(U, \mathbb{Z}/p^s) = 0$ for $n > 0$. 

Suppose instead that \( U \cap Z_0 \) is non-empty. In this case, we know that \( U \cap Z_0 \) is a deformation retract of \( U \). It follows that the restriction map \( H^\bullet_{\text{sing}}(U, \mathbb{Z}/p^s) \to H^\bullet_{\text{sing}}(U \cap Z_0, \mathbb{Z}/p^s) \) is an isomorphism. By the comparison theorem, it follows that the map \( \hat{H}^\bullet(U, \mathbb{Z}/p^s) \to \hat{H}^\bullet(U \cap Z_0, \mathbb{Z}/p^s) \) is an isomorphism. The long exact sequence shows that \( \hat{H}^\bullet(U, U \cap Z_0, \mathbb{Z}/p^s) = 0 \).

In particular, using Leray’s Theorem, we have
\[
\hat{H}^\bullet(Y, Z, \mathbb{Z}/p^s) = \hat{H}^\bullet(\hat{\mathcal{U}}, (\mathbb{Z}/p^s)^\mathbb{Z}).
\]

### 3.3. \( \hat{\mathcal{U}} \) is \( \mathcal{C} \)-acyclic

Let \( U \) be an intersection of finitely many sets in \( \mathcal{U} \), and let \( \tilde{U} \) be the preimage of \( U \) in \( Y \). We know that \( U \) is contractible, and \( \tilde{U} = U \times G \). We must show that \( \hat{H}^n(\tilde{U}, \mathcal{C}) = 0 \) for \( n > 0 \).

Let \( \hat{\mathcal{U}} \) be an open cover of \( \tilde{U} \), and choose an element \( \sigma \in \hat{H}^n(\hat{\mathcal{U}}, \mathcal{C}) \) with \( n > 0 \). We shall find a refinement \( \hat{\mathcal{W}} \) of \( \hat{\mathcal{U}} \), such that the image of \( \sigma \) in \( \hat{H}^\bullet(\hat{\mathcal{W}}, \mathcal{C}) \) is zero. By passing to a refinement of \( \hat{\mathcal{U}} \) if necessary, we may assume that \( \hat{\mathcal{W}} \) is finite, and that each element of \( \hat{\mathcal{U}} \) is of the form \( V_i \times H_i \) for some open subset \( V_i \subseteq U \) and some open coset \( H_i \subseteq G \). By refining still further, we may assume that the cosets \( H_i \) are all cosets of the same open subgroup \( G_r \subseteq G \). This means that \( \hat{\mathcal{U}} \) is the pullback of an open cover \( \hat{\mathcal{Y}}^{(r)} \) of \( U^{(r)} \). For an open subset \( V^{(r)} \subseteq U^{(r)} \), we have
\[
\mathcal{C}(\hat{V}^{(r)}) = S(V^{(r)}),
\]
where \( S \) is the locally constant sheaf on \( U^{(r)} \) with values in \( \mathcal{C}(G_r, \mathbb{Z}_p) \) and \( \hat{V}^{(r)} \) is the preimage of \( V^{(r)} \) in \( \tilde{U} \). It follows that
\[
\hat{H}^\bullet(\hat{\mathcal{U}}, \mathcal{C}_{\mathbb{Z}_p}) = \hat{H}^\bullet(\hat{\mathcal{Y}}^{(r)}, S).
\]

Since \( U^{(r)} \) is homotopic to a finite set and \( S \) is a constant sheaf, it follows that \( \hat{H}^{>0}(U^{(r)}, S) \) is zero. This implies there is a refinement \( \hat{\mathcal{W}}^{(r)} \) of \( \hat{\mathcal{Y}}^{(r)} \), such that the image of \( \sigma \) in \( \hat{H}^\bullet(\hat{\mathcal{W}}^{(r)}, S) \) is zero. Pulling \( \hat{\mathcal{W}}^{(r)} \) back to \( \tilde{U} \), we have a refinement \( \hat{\mathcal{W}} \) of \( \hat{\mathcal{U}} \), such that the image of \( \sigma \) in \( \hat{H}^\bullet(\hat{\mathcal{W}}, \mathcal{C}_{\mathbb{Z}_p}) \) is zero.

### 3.4. \( \hat{\mathcal{U}} \) is \( \mathcal{C}^\mathbb{Z} \)-acyclic

Let \( U \) be an intersection of finitely many sets in \( \mathcal{U} \), and let \( \tilde{U} \) be the preimage of \( U \) in \( Y \). We know that \( U \) is contractible, and \( \tilde{U} = U \times G \). We must show that \( \hat{H}^n(\tilde{U}, \mathcal{U} \cap Z_0, \mathcal{C}) = 0 \) for \( n > 0 \). If \( U \) does not intersect \( Z_0 \), then this follows from the previous part of the proof. We therefore assume that \( U \) intersects \( Z_0 \). In this case, we know that \( U \cap Z_0 \) is a deformation retract of \( U \). In particular, \( U \cap Z_0 \) is contractible, and \( \tilde{U} \cap Z = (U \cap Z_0) \times G \). The previous part of the proof shows that \( \hat{H}^{>0}(\tilde{U}, \mathcal{C}) = 0 \) and \( \hat{H}^{>0}(\tilde{U} \cap Z, \mathcal{C}) = 0 \). Furthermore, one sees immediately that the restriction map \( \hat{H}^0(\tilde{U}, \mathcal{C}) \to \hat{H}^0(\tilde{U} \cap Z, \mathcal{C}) \) is an isomorphism. Hence by the long exact sequence, we have \( \hat{H}^\bullet(\tilde{U}, \mathcal{U} \cap Z, \mathcal{C}) = 0 \).

Thus by Leray’s Theorem, we have:
\[
\hat{H}^\bullet(Y, Z, \mathcal{C}) = \hat{H}^\bullet(\hat{\mathcal{U}}, \mathcal{C}^\mathbb{Z}).
\]
3.5. Fix for a moment a cohomological degree $n$, and let $U_1, \ldots, U_N$ be the non-empty intersections of $n + 1$-tuples of sets in $\mathcal{U}$, for which $U_i \cap Z_0 = \emptyset$. For each $U_i$, we let $U_i^{(r)}$ be the preimage of $U_i$ in $Y$. Let $\tilde{U}_i$ be the preimage of $U_i$ in $Y$.

Recall that $\tilde{H}^\bullet(\mathcal{U}^{(r)}, (\mathbb{Z}/p^s)^Z)$ is the cohomology of the chain complex

$$
\tilde{C}^n(\mathcal{U}^{(r)}, (\mathbb{Z}/p^s)^Z_r) = \prod_{i=1}^N (\mathbb{Z}/p^s)^Z_r(U_i^{(r)}),
$$

Each $U_i$ is contractible and disjoint from $Z_0$. Furthermore $U_i^{(r)} = U_i \times (G/G_r)$, so we have an isomorphism of $G$-modules: $(\mathbb{Z}/p^s)^Z_r(U_i^{(r)}) = (\mathbb{Z}/p^s)(G/G_r)$. This gives

$$
\tilde{C}^n(\mathcal{U}^{(r)}, (\mathbb{Z}/p^s)^Z) = (\mathbb{Z}/p^s)(G/G_r)^N.
$$

Similarly, we have

$$
\tilde{C}^n(\tilde{U}, (\mathbb{Z}/p^s)^Z) = (\mathbb{Z}/p^s)^N.
$$

Comparing the two formulae, it is clear that

$$
\tilde{C}^\bullet(\tilde{U}, (\mathbb{Z}/p^s)^Z) = \lim_r \tilde{C}^\bullet(\mathcal{U}^{(r)}, (\mathbb{Z}/p^s)^Z_r).
$$

Since the functor $\lim \rightarrow_r$ is exact, we have

$$
\tilde{H}^\bullet(\tilde{U}, (\mathbb{Z}/p^s)^Z) = \lim_r \tilde{H}^\bullet(\mathcal{U}^{(r)}, (\mathbb{Z}/p^s)^Z_r).
$$

3.6. Note also that $C^Z(\tilde{U}_i) = \mathcal{C}(G)$, and so we have

$$
\tilde{C}^n(\tilde{U}, \mathcal{C}^Z) = \mathcal{C}(G)^N.
$$

It follows that $\tilde{C}^n(\tilde{U}, \mathcal{C}^Z)$ is an admissible $\mathbb{Z}_p[G]$-module in the sense of [2]. Furthermore we have:

$$
\tilde{C}^\bullet(\tilde{U}, \mathcal{C}^Z) = \lim_s \tilde{C}^\bullet(\tilde{U}, (\mathbb{Z}/p^s)^Z), \quad \tilde{C}^\bullet(\tilde{U}, (\mathbb{Z}/p^s)^Z) = \tilde{C}^\bullet(\tilde{U}, \mathcal{C}^Z)/p^s.
$$

Hence by Proposition 1.2.12 of [2], we have:

$$
\tilde{H}^\bullet(\tilde{U}, \mathcal{C}^Z) = \lim_s \tilde{H}^\bullet(\tilde{U}, (\mathbb{Z}/p^s)^Z).
$$

By the previous part of the proof, we have:

$$
\tilde{H}^\bullet(\tilde{U}, \mathcal{C}^Z) = \lim_s \lim_r \tilde{H}^\bullet(\mathcal{U}^{(r)}, (\mathbb{Z}/p^s)^Z_r).
$$

Since our covers are acyclic, this translates to

$$
\tilde{H}^\bullet(Y, Z, \mathcal{C}) = \lim_s \lim_r \tilde{H}^\bullet(Y_r, Z_r, \mathbb{Z}/p^s).
$$

On the other hand, by Theorem [5], we have

$$
\tilde{H}^\bullet(Y_r, Z_r, \mathbb{Z}/p^s) = H^\bullet_c(Y_r \setminus Z_r, \mathbb{Z}/p^s).
$$

The result follows. □
Corollary 1. With the above notation,
\[ \tilde{H}^n(Y, \mathbb{Z}_p) = \check{H}^n(Y, \mathcal{C}_{\mathbb{Z}_p}), \quad \tilde{H}^n(Y, \mathbb{Q}_p) = \check{H}^n(Y, \mathcal{C}_{\mathbb{Q}_p}). \]

Proof. We apply the theorem in the case that \( Z \) is empty. Since \( Y^0 = Y \), which is compact, it follows that usual cohomology is the same as compactly supported cohomology on each \( Y_r \).

Corollary 2. In the notation of the introduction, there are long exact sequences:
\[ \tilde{H}^n_\ast(K^p, \mathbb{Z}_p) \to \tilde{H}^n_\ast(K^p, \mathbb{Q}_p) \to \check{H}^n_\ast(K^p, \mathbb{Z}_p) \to \check{H}^n_\ast(K^p, \mathbb{Q}_p), \]
\[ \tilde{H}^n_\ast(K^p, \mathbb{Q}_p) \to \tilde{H}^n_\ast(K^p, \mathbb{Q}_p) \to \check{H}^n_\ast(K^p, \mathbb{Q}_p). \]

Proof. For convenience, we shall write \( Y \) instead of \( Y(K^p) \). We have shown above that
\[ \tilde{H}^\ast_\ast(K^p, \mathbb{Q}_p) = \check{H}^\ast_\ast(Y, \mathcal{C}_{\mathbb{Q}_p}), \]
\[ \tilde{H}^\ast_\ast(K^p, \mathbb{Q}_p) = \check{H}^\ast_\ast(\partial Y^{B.S.}, \mathcal{C}_{\mathbb{Q}_p}), \]
\[ \check{H}^\ast_\ast(K^p, \mathbb{Q}_p) = \tilde{H}^\ast_\ast(Y^{B.S.}, \partial Y^{B.S.}, \mathcal{C}_{\mathbb{Q}_p}). \]

There is a long exact sequence in Čech cohomology:
\[ \tilde{H}^n_\ast(Y^{B.S.}, \mathcal{C}_{\mathbb{Q}_p}) \to \tilde{H}^n_\ast(\partial Y, \mathcal{C}_{\mathbb{Q}_p}) \to \tilde{H}^{n+1}_\ast(Y^{B.S.}, \partial Y^{B.S.}, \mathcal{C}_{\mathbb{Q}_p}) \to \tilde{H}^{n+1}_\ast(Y^{B.S.}, \mathcal{C}_{\mathbb{Q}_p}). \]

The same holds for \( \mathbb{Z}_p \).

References

[1] G. E. Bredon. Sheaf Theory, second edition. Graduate Texts in Mathematics vol. 170. Springer-Verlag 1997.

[2] M. Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. Invent. Math., 164(1):1–84, 2006.

[3] E. H. Spanier. Algebraic Topology. Springer-Verlag New York, inc., 1966.