LATTICES IN THE COHOMOLOGY OF $U(3)$ ARITHMETIC MANIFOLDS

DANIEL LE

ABSTRACT. Under hypotheses required for the Taylor-Wiles method, we prove for forms of $U(3)$ which are compact at infinity that the lattice structure on upper alcove algebraic vectors or on principal series types given by the $\lambda$-isotypic part of completed cohomology is a local invariant of the Galois representation attached to $\lambda$ when this Galois representation is residually irreducible locally at places dividing $p$. We combine Hecke theory and weight cycling with the Taylor-Wiles method to establish crucial mod $p$ multiplicity one results for upper alcove algebraic vectors and principal series types.

1. Introduction

We establish, to our knowledge, the first local-global compatibility result in the $p$-adic Langlands program for a group of semisimple rank greater than one. Recall classical local-global compatibility: if an automorphic representation $\Pi = \otimes_v \pi_v$ corresponds to a $p$-adic $n$-dimensional Galois representation $\rho$ via the cohomology of a unitary Shimura variety, then $\pi_v$ is obtained via the local Langlands correspondence from the Weil-Deligne representation attached to $\rho|_{D_v}$, which is independent of the Hodge filtration for $v | p$. Let $V_v$ be the algebraic representation corresponding to the Hodge-Tate weights of $\rho|_{D_v}$. The $p$-adic Langlands correspondence for $GL_2(Q_p)$ ([Col10, Pas13]) suggests the following question.

Question 1.0.1. Is there a natural $GL_n(F_v)$-invariant norm on $\pi_v \otimes V_v$ which corresponds to the Hodge filtration for $\rho|_{D_v}$?

Completed cohomology gives $\pi_v \otimes V_v$ an integral structure, and hence also a $GL_n(Q_p)$-invariant norm, which is of an a priori global nature. The following naive question extrapolates from $p$-adic local-global compatibility for $GL_2(Q_p)$ ([Eme11]).

Question 1.0.2. Does this integral structure depend only on $\rho|_{D_v}$?

The following is the $GL_3$-analogue of Breuil’s $GL_2$-conjecture (Conjecture 1.2 of [Bre]). Suppose that $n = 3$ and $p$ splits completely in $F$, the CM field over which the unitary group splits. Let $\tau_v \subset \pi_v$ be the $K$-type corresponding to $WD(\rho|_{D_v})$.

Conjecture 1.0.3. The lattice structure on $\tau_v \otimes V_v$ given by completed cohomology depends only on $\rho|_{D_v}$.

The following is the main theorem (see Theorems 6.1.4 and 6.2.5 for more details).

Theorem 1.0.4. Suppose that $\rho|_{D_v}$ is irreducible for all places $v | p$. Suppose further that $\rho|_{D_v}$ is crystalline with Hodge-Tate weights in the upper alcove or potentially crystalline of principal series type of Hodge-Tate weight $(0,1,2)$. Under mild hypotheses necessary for the Taylor-Wiles method and weight elimination, Conjecture 1.0.3 is true. In fact, the lattice can be described explicitly in terms of $\rho|_{D_v}$.

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Remark 1.0.5. If the highest weight of \( V_v \) is not in the lower alcove, then like a principal series type \( \tau_v \), \( V_v \) is residually reducible, and so has different possible lattices up to homothety. Furthermore, for \( n \geq 3 \), \( V_v \) and \( \tau_v \) have Jordan-Holder factors that do not lift to characteristic zero. If \( \sigma \) is a weight that does not lift to characteristic zero, modularity of weight \( \sigma \) has to our knowledge no known Galois theoretic interpretation.

1.1. Detailed summary of results. Let \( F \) be a CM field with totally real subfield \( F^+ \) and assume that \( F/F^+ \) is unramified at all finite places. Suppose that \( p \neq 2 \) splits completely in \( F \). Let \( E \) be a finite extension of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O}_E \), uniformizer \( \varpi_E \), and residue field \( k_E \). Let \( \rho : G_F \to \text{GL}_3(\mathcal{O}_E) \) be a continuous Galois representation that is crystalline at all places \( v \mid p \) with Hodge-Tate weights \( (c_v + p + 1, b_v + 1, a_v - p + 1) \) and principal series type \( \eta^{a_v} \oplus \eta^{b_v} \oplus \eta^{c_v} \) where \( \eta \) is the Teichmüller character. Furthermore suppose that \( \rho \) is ramified only at places that split in \( F/F^+ \).

Let \( \lambda \subset \mathbb{T} \) be the Hecke eigensystem attached to \( \rho \). Here \( \mathbb{T} \) is the spherical Hecke algebra over \( \mathcal{O}_E \) at places of \( F^+ \) where \( \rho \) is unramified. For each place \( v \) of \( F^+ \) that splits in \( F \), choose a place \( \mathfrak{v} | v \) of \( F \) and an isomorphism \( F_{\mathfrak{v}} \cong F_v^+ \). For each place \( w \) of \( F \), let \( \tau_w^0 \subset \tau_w \) be a lattice in the type corresponding to \( \rho|_{G_{F_w}} \) by results towards the inertial local Langlands correspondence (see Proposition 6.5.3 of [BC09]). Note that \( \tau_w \) is trivial for all but finitely many places \( w \), and in particular at all inert places.

Consider a form of \( U(3) \) over \( F^+ \) that splits over \( F \), is quasisplit at all finite places, and is compact at infinity. Let \( \tilde{H}^0 \) be the completed cohomology with \( \mathcal{O}_E \)-coefficients of the associated Shimura variety and assume that \( \tilde{H}^0[\lambda] \neq 0 \) (see Section 4.1.2 for more details). Let

\[
\pi = \text{Hom}_{K^p}(\otimes_{v \mid p}^\prime \tau_v^0, \tilde{H}^0[\lambda])
\]

where \( K^p \subset \text{GL}_3(\mathbb{A}_p^{\infty}) \) is a hyperspecial compact open subgroup away from \( p \infty \). Then

\[
\text{Hom}_{\prod_{v \mid p} \text{GL}_3(F_v^+)}(\otimes_{v \mid p} V_v \otimes \pi_v, \pi[p^{-1}]) \cong \text{Hom}_{\prod_{v \mid p} \text{GL}_3(\mathcal{O}_{F_v^+})}(\otimes_{v \mid p}^\prime V_v \otimes \tau_v, \pi[p^{-1}])
\]

is one-dimensional over \( E \) by Theorems 5.4 and 5.9 of [Lab1], where \( V_v \) is the algebraic representation of \( \text{GL}_3(F_v^+) \cong \text{GL}_3(\mathbb{Q}_p) \) of highest weight \( (c_v + p - 1, b_v, a_v - p + 1) \) in the crystalline case or trivial in the potentially crystalline case. One might hope that \( \pi \) depends only on \( \rho|_{G_{F_v}} \) at \( v \mid p \). The following theorem (see Theorems 6.1.4 and 6.2.5 for more precise statements) provides evidence of this.

Theorem 1.1.1. Suppose that \( \rho \) is a modular Galois representation which is crystalline at all places \( v \mid p \) of Hodge-Tate weights \( (c_v + p - 1, b_v, a_v - p + 1) \) or potentially crystalline of Hodge-Tate weights \( (0, 1, 2) \) and \( \eta^{a_v} \oplus \eta^{b_v} \oplus \eta^{c_v} \). Suppose that \( \bar{\rho} \) satisfies the Taylor-Wiles conditions, the product of the deformation spaces away from \( p \) with the types of \( \rho \) is regular, \( \bar{\rho}|_{G_{F_v}} \) is irreducible for places \( v \mid p \), and that \( a_v - b_v > 6, b_v - c_v > 6, \) and \( a_v - c_v < p - 5 \). Then the isomorphism class of the lattice \( (\otimes_v V_v \otimes \tau_v \otimes \mathbb{Q}_p, E) \cap \pi \subset (\otimes_v V_v \otimes \tau_v) \) depends only on \( \rho|_{G_{F_v}} \) for places \( v \mid p \) of \( F^+ \) in an explicit way.

To prove this theorem, we introduce a new technique of combining representation theory and Hecke operators with the Taylor-Wiles method. Modifications of the Taylor-Wiles patching method [TW95] are well suited for this result and play a
large role in its proof. Kisin’s local-global modification ([Kis09]) provides a natural setting to compare the local and global Langlands correspondences integrally. And the modification in [CEG+14] allows one to use Hecke theory and weight cycling as in [EGH13] to study lattices. We take the geometric perspective of [EG14] and prove the theorem in families where one can use that patched modules are Cohen-Macaulay. As a crucial ingredient, we prove (see Theorems 5.3.1 and 5.3.6) that Taylor-Wiles patched modules for (dual) Weyl modules of upper alcove weight and principal series types are cyclic, an analogue of modulo $p$ multiplicity one (as in [Dia97]).

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1.3. Notation. For a field $k$, $G_k$ denotes the absolute Galois group of $k$. We denote the cyclotomic character by $\epsilon$. Hodge-Tate weights are normalized so that $\epsilon$ has weight $-1$. We use $F^+ \subset F$ to denote the maximal totally real subfield in a CM field. The quadratic character of $F/F^{-}$ is used to denote places of $F^+$ while the letter $w$ is used to denote places of $F$. We use $E$ to denote a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, uniformizer $\varpi_E$, and residue field $k_E$.

Compact induction is denoted $\text{ind}^G_H$, while usual induction is denoted $\text{Ind}^G_H$. The symbols $\cdot^\vee$ and $\cdot^d$ are used to denote the Pontriagin dual and Schikhof dual, respectively (see Section 1.5 of [CEG+14]). The symbol $\cdot^*$ is used to denote the contragredient of a representation.

We label the elements of $S_3$ as $e = (\cdot)$, $s_1 = (12)$, $s_2 = (23)$, $r_1 = (132)$, $r_2 = (123)$, and $s_0 = (13)$. The ordered pair $(i,j)$ denotes a permutation of the ordered pair $(1,2)$. For $\nu \in X_*(T)$, let $W(\nu)$ be the $\mathbb{Z}_p$-points of the algebraic $\text{GL}_3$-representation $\text{Ind}_B^G s_0 \nu$. Let $\overline{W}(\nu)$ be the reduction of $W(\nu)$ modulo $p$, and let $F(\nu)$ be the (irreducible) socle of $\overline{W}(\nu)$. Every (absolutely) irreducible representation is isomorphic to $F(\nu)$ for some $\nu$, and the representations $F(\mu)$ and $F(\nu)$ are isomorphic if and only if $\mu \equiv \nu \pmod{p-1}$ (see Theorem 4.6 of [Her06]). The representations $F(\nu)$ are called (Serre) weights (of $\text{GL}_3(\mathbb{F}_p)$).

The triple $(a,b,c)$ with $a > b > c$ will be used to denote the highest weight of an algebraic representation of $\text{GL}_3(\mathbb{F}_p)$. We will assume throughout that $a-c < p-1$.

In Section 6 we assume that $a-b > 6$, $b-c > 6$, and $a-c < p-5$ (the strong genericity hypothesis of [EGH13]) to apply Theorem 7.5.5 of [EGH13].

By Theorem 2.8 and Section 4 of [And87], we have the following results. Note that [And87] describes $\text{SL}_3(\mathbb{F}_p)$-representations, but the proofs carry over verbatim to the $\text{GL}_3(\mathbb{F}_p)$-setting. If $\sigma$ and $\sigma'$ are weights, then

$$\dim_{\mathbb{F}_p} \text{Ext}^1_{\text{GL}_3(\mathbb{F}_p)}(\sigma', \sigma) = \dim_{\mathbb{F}_p} \text{Ext}^1_{\text{GL}_3(\mathbb{F}_p)}(\sigma, \sigma') \leq 1.$$
If \( F(x, y, z) \) is a weight, let \( \mathcal{E}(x, y, z) \) be the set of triples \((a, b, c)\) such that

\[
\text{Ext}^1_{GL_3(F_p)}(F(x, y, z), F(a, b, c)) \neq 0.
\]

Let \( x > y > z \) be integers such that \( x - z < p - 2 \). Note that \((x, y, z)\) is a lower alcove weight for \( GL_3 \) in the sense of the paragraph following Corollary 4.8 of [Her06]. Then

\[
\mathcal{E}(x, y, z) := \{(y + p - 1, x, z), (x, y, p + 1), (z + p - 2, y, x + p), (x + 1, z - 1, y - p + 1), (y + p - 1, x + 1, z - 1), (x, z - 1, y - p + 2), (y + p - 2, x + 1, z)\}.
\]

Moreover, \( \text{Ext}^1_{GL_3(F_p)}(\sigma', \sigma) = 0 \) if both \( \sigma \) and \( \sigma' \) are upper alcove weights. In Sections 3.2 and 5.1, \( W_1 \) and \( W_2 \) are chosen to be certain nonsplit extensions of weights, for example \( W(\nu) \) for \( \nu \) an upper alcove weight.

### 2. Lattices in locally algebraic representations

In this section, we classify lattices in upper alcove algebraic representations and principal series types and the maps between them.

#### 2.1. Lattices in algebraic vectors

We define two natural integral structures on upper alcove algebraic vectors. We follow the notation of Section 3 of [Her09]. Let \( G = GL_3 \) and \( B \) and \( T \) the algebraic subgroups over \( Z \) of upper triangular and diagonal matrices, respectively. Let \( K = G(Z_p) \) and \( K_1 \) be the kernel of the reduction map \( G(Z_p) \to G(F_p) \). For \( \nu \in X_s(T) \), let \( W(\nu) \) be the \( Z_p \)-points of the algebraic \( G \)-representation \( \text{Ind}_B^G \nu \) where \( \nu_0 \) is the longest Weyl element.

Fix \( a > b > c \) integers such that \( a - c < p \). Note that \((a - 1, b, c + 1)\) is a lower alcove weight for \( GL_3 \) in the sense of the paragraph following Corollary 4.8 of [Her06]. Let \( \nu = (c + p - 1, b, a - p + 1) \). Let \( W = W(\nu) \) and \( W^0 = W(-s_0\nu)^* \).

By Serre duality for \( G/B \) (see (8) of Chapter II.4 of [Jan03]), \( W^0 \cong R^3\text{Ind}_B^G(\nu - 2\rho)(Z_p) \), where \( * \) denotes the contragredient representation.

Let \( V = W[p^{-1}] = \text{Ind}_B^G s_0\nu(Q_p) \). By the Borel-Weil-Bott theorem, \( V \cong R^3\text{Ind}_B^G(\nu - 2\rho)(Q_p) \cong W^0[p^{-1}] \) (see II.5.3 of [Jan03]). However, the Borel-Weil-Bott theorem does not hold integrally, as one sees in the following proposition. Let \( \overline{W} \) and \( \overline{W}^0 \) be the reductions modulo \( p \) of \( W \) and \( W^0 \), respectively. Note that \( K_1 \) acts trivially on \( W \) and \( W^0 \).

**Proposition 2.1.1.** There are nonsplit exact sequences

\[
0 \to F(c + p - 1, b, a - p + 1) \to \overline{W} \to F(a - 1, b, c + 1) \to 0
\]

and

\[
0 \to F(a - 1, b, c + 1) \to \overline{W}^0 \to F(c + p - 1, b, a - p + 1) \to 0
\]

where \( F(a - 1, b, c + 1) \) and \( F(c + p - 1, b, a - p + 1) \) are irreducible \( GL_3(F_p) \)-modules defined as in Section 4 of [Her06].

**Proof.** The first exact sequence follows from Proposition 4.9 of [Her06]. The nonsplintness follows from Theorem 4.1 of [Her06]. The second nonsplit exact sequence follows from the same propositions and duality, where we use that the dual of \( F(x, y, z) \) is \( F(-z, -y, -x) \). \( \square \)
Let \( \sigma' = F(c + p - 1, b, a - p + 1) \) and \( \sigma = F(a - 1, b, c + 1) \). Fix an injection \( i : W^0 \hookrightarrow W \) that is nonzero modulo \( p \) and let \( i' : W \hookrightarrow W^0 \) be the unique map that is nonzero modulo \( p \) such that \( i \circ i' \) is a power of \( p \).

**Proposition 2.1.2.** The composition \( i \circ i' \) is \( p \).

**Proof.** See II.8.15 (3) and (4) and II.8.16 of [Jan03], where the context is modulo \( p \), but the proofs work integrally. \( \square \)

2.2. **Lattices in principal series types.** In this section, we classify lattices in generic principal series types which are residually multiplicity free.

2.2.1. **Integral structure on principal series types.** We begin by defining natural integral structures on principal series. Let \( I \subset K \) be the inverse image of \( B(F_p) \subset G(F_p) \) under the reduction map, and let \( I_1 \subset I \) be the inverse image of the maximal unipotent \( p \)-subgroup of \( B(F_p) \).

Let \( \eta : F_p^\times \to \mathbf{Z}_p^\times \) be the Teichmüller character. For integers \( a, b, \) and \( c \), we consider the character \( \chi = \eta^a \otimes \eta^b \otimes \eta^c \) of \( (F_p)^3 \), which we view as a character of \( I \) by inflation via the map \( I \to I/I_1 \cong (F_p)^3 \). Let \( \chi^s \) be the character with factors permuted by \( s \in S_3 \). Let \( e \) be the identity, \( s_0 = (13) \), \( r_1 = (132) \), \( r_2 = (123) \), \( s_1 = (12) \), and \( s_2 = (23) \). Then for example \( \chi^{r_1} = \eta^b \otimes \eta^c \otimes \eta^a \) and \( \chi^{r_2} = \eta^c \otimes \eta^a \otimes \eta^b \).

For each \( s \in S_3 \), consider the \( K \)-representation \( \tau^s = \text{ind}^K_I \chi^s \) over \( \mathbf{Z}_p \). The representations \( \tau^s \) are lattices in the principal series types \( \tau^s \otimes \mathbf{Z}_p \mathbf{Q}_p \). For \( s, s' \in S_3 \), the intertwiner \( \iota_{s/s'}^s : \text{ind}^K_I \chi^{s'} \hookrightarrow \text{ind}^K_I \chi^s \) from the classical representation theory of \( \text{GL}_3(F_p) \) is nonzero modulo \( p \) and induces isomorphisms \( \tau^s/s' \chi^{s'} \otimes \mathbf{Z}_p \mathbf{Q}_p \cong \tau^s \otimes \mathbf{Q}_p \).

Let \( v^s \in \text{ind}^K_I \chi^s \) be the function supported on \( I \) defined by extending \( \chi^s \) by 0. By Frobenius reciprocity, the intertwiner \( \iota_{s/s'}^s \) is determined by the image of \( v^{s'} \). Identify \( S_3 \) with the group of permutation matrices in \( \text{GL}_3 \). Then the intertwiner is given by

\[
(2.2.1) \quad \iota_{s/s'}^s : v^{s'} \mapsto \sum_{g \in I_1/(s'I_1s^{-1} \cap I_1)} gs'v^s,
\]

which is easily checked to be nonzero modulo \( p \). Let \( \ell \) denote the length function on \( S_3 \). Now suppose that \( a > b > c \) with \( a - c < p - 2 \).

**Proposition 2.2.1.** The composition \( \iota_{s'/s''}^s \circ \iota_{s''}^s : \text{ind}^K_I \chi^s \hookrightarrow \text{ind}^K_I \chi^{s'/s''} \) is \( p^{2\ell(s''') + \ell(s') + \ell(s''') - \ell(s''')} \iota_{s''}^s \).

**Proof.** A more general result can be obtained from a combination of Propositions 3.6 and 3.10 of [CL76]. We provide a short summary. Let the ordered pair \( (i, j) \) be a permutation of \((1, 2)\). By induction, it suffices to show that \( \iota_{s_is_j}^s \circ \iota_{s_j}^s = \iota_{s_i}^s \iota_{s_j}^s \) and \( \iota_{s_i}^s \circ \iota_{s_j}^s = \iota_{s_j}^s \circ \iota_{s_i}^s = p \).

Note that \( \cup_{g \in I_1/(s'I_1s^{-1} \cap I_1)} gs'I_1 = I_1s'I_1 \). To prove that \( \iota_{s_is_j}^s \circ \iota_{s_j}^s = \iota_{s_is_j}^s \), it suffices to show that the convolution of \( 1_{s_is_j}I_1 \) and \( 1_{t_is_i}I_1 \) is \( 1_{s_is_i}I_1 \). Since \( I_1s_is_jI_1 \subset I_1s_is_jI_1 \cdot I_1s_is_iI_1 \), we have that \( 1_{t_is_j}I_1 \leq 1_{t_is_j}I_1 * 1_{t_is_i}I_1 \). As \( \#I_1s_is_jI_1 = \#I_1s_is_iI_1 \) and \( 1_{t_is_i}I_1 * 1_{t_is_i}I_1 \) have the same integrals for any Haar measure. We conclude that they are equal. That \( \iota_{s_is_j}^s \circ \iota_{s_j}^s = p \) follows from Lemme 2.2 of [Bre]. \( \square \)
2.2.2. Submodule structure of principal series types. Let $\tau = \tau^e \otimes Z_p Q_p$. Now suppose that $a > b > c$ with $a - c < p - 2$, so that in particular $\tau$ is absolutely irreducible. Florian Herzig provided the argument for the following proposition, which describes the submodule structure of $\tau^e$.

**Proposition 2.2.2.** The socle and cosocle filtrations of $\tau^e$ agree and have associated graded pieces

\[
F(a, b, c) =
\]

\[
F(a, c, b - p + 1) \oplus F(b - p - 1, a, c)
\]

\[
F(c + p - 1, a, b) \oplus F(b - 1, c, a - p + 2) \oplus F(a - 1, b, c + 1)
\]

\[
\oplus F(c + p - 2, a, b + 1) \oplus F(b, c, a - p + 1)
\]

\[
F(c + p - 1, b, a - p + 1),
\]

where any number of bottom rows are the graded pieces of a submodule. Furthermore, all nontrivial extensions that can occur do occur (see Section 1.3). The same statement is true for $\tau^s$ except that the order of the rows is reversed. In particular, $\tau^s$ is residually multiplicity free for all $s \in S_3$.

**Proof.** A proof is sketched in Section 8 of [CL76] in the case of $SL_3(F_p)$. This proof works without modification for $GL_3(F_p)$ as described below.

As $a, b$, and $c$ are distinct mod $p - 1$, the Jordan-Holder factors of $\tau^e$ are distinct by Theorems 4.1 and 5.1 of [Her06]. Using Frobenius reciprocity and Lemma 2.3 of [Her11a], the socle and cosocle of $\tau^e$ are $F(c + p - 1, b, a - p + 1)$ and $F(a, b, c)$, respectively. Hence, $F(c + p - 1, b, a - p + 1)$ and $F(a, b, c)$ are pieces of the associated graded for both the socle and cosocle filtrations. Satz 4.4 of [Jan84] gives a filtration with associated graded pieces as described above arising from the intertwiners $\iota_e$ (see also the last lemma of Section 2 and Section 5.2 of [Jan84] and Section 8 of [CL76]). We conclude that the socle and cosocle filtrations have length 3 or 4. It remains to show that the socle and cosocle filtrations have length 4, to describe the associated graded pieces, and show that all nontrivial extensions that can occur between the inner layers do occur.

Let $P_1 = \left\{ \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} \right\}$, $P_2 = \left\{ \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 0 \end{pmatrix} \right\} \subset GL_3(F_p)$. Consider $\text{Ind}_{P_1}^{GL_3(F_p)} F(b + p - 1, a) \otimes F(c) \subset \tau^e$, which has Jordan Holder factors $F(b + p - 1, a, c), F(b - 1, c, a - p + 2), F(a - 1, b, c + 1), F(c + p - 2, a, b + 1), F(b, c, a - p + 1),$ and $F(c + p - 1, b, a - p + 1)$ by Lemma 6.1.1 of [EGH13]. Again using Frobenius reciprocity and Lemma 2.3 of [Her11a], $\text{Ind}_{P_2}^{GL_3(F_p)} (F(b + p - 1, a) \otimes F(c))$ has socle and cosocle $F(c + p - 1, a, b - p + 1)$ and $F(b + p - 1, a, c)$, respectively. This shows that the socle and cosocle filtrations of $\tau$ must have length 4 and further that $F(b + p - 1, a, c)$ is in the second layer and $F(b - 1, c, a - p + 2), F(a - 1, b, c + 1), F(c + p - 2, a, b + 1)$, and $F(b, c, a - p + 1)$ are in the third with all possible nontrivial extensions occurring.

Similarly examining $\text{Ind}_{P_2}^{GL_3(F_p)} F(a) \otimes F(c, b - p + 1)$ shows that $F(a, c, b - p + 1)$ is in the second layer and $F(c + p - 1, a, b)$ is in the third. Furthermore, all nontrivial
extensions of \( F(a,c,b - p + 1) \) occur. The statement for \( \tau^s \) is obtained similarly or by duality. That \( \tau^s \) is residually multiplicity free follows from the fact that the Jordan-Holder factors listed are distinct by Theorem 4.1 of [Her06]. □

For \( s \in S_3 \), let \( s(a,b,c) \) be a \( p \)-restricted (see Definition 4.2 of [Her06]) ordered triple congruent mod \( p - 1 \) to the \( s \)-permutation of \( (a,b,c) \).

**Proposition 2.2.3.** The lattices \( \tau^s \) are the unique lattices up to homothety with cosocle \( F(s(a,b,c)) \). The map \( \iota^s : \tau^s \to \tau^s \) is the unique up to scalar map which is nonzero modulo \( p \).

**Proof.** Since \( \tau^s \) is residually multiplicity free by Proposition 2.2.2, uniqueness follows from Lemma 4.1.1 of [EGS13]. The second statement follows similarly. □

Having described the lattices in \( \tau \) with cosocle \( F(s(a,b,c)) \), let \( \tau^1, \tau^2, \) and \( \tau^3 \subset \tau \) be the unique lattices up to homothety with cosocle \( F(b-1,c,a-p+2), F(a-1,b,c+1), \) and \( F(c+p-2,a,b+1) \), respectively. It remains to describe the submodule structure of \( \tau^1, \tau^2, \) and \( \tau^3 \).

**Proposition 2.2.4.**

1. The socle and cosocle filtrations of \( \tau^1 \) agree and have associated graded pieces

\[
F(b-1,c,a-p+2)
\]

\[
F(b+p-1,a,c) \oplus F(a,c,b-p+1) \oplus F(c+p-1,b,a-p+1)
\]

\[
F(a,b,c) \oplus F(b,c,a-p+1) \oplus F(c+p-1,a,b)
\]

\[
\oplus F(c+p-2,a,b+1) \oplus F(a-1,b,c+1).
\]

2. The socle and cosocle filtrations of \( \tau^2 \) agree and have associated graded pieces

\[
F(a-1,b,c+1)
\]

\[
F(b+p-1,a,c) \oplus F(a,c,b-p+1) \oplus F(c+p-1,b,a-p+1)
\]

\[
F(a,b,c) \oplus F(b,c,a-p+1) \oplus F(c+p-1,a,b)
\]

\[
\oplus F(c+p-2,a,b+1) \oplus F(b-1,c,a-p+2).
\]

3. The socle and cosocle filtrations of \( \tau^3 \) agree and have associated graded pieces

\[
F(c+p-2,a,b+1)
\]

\[
F(b+p-1,a,c) \oplus F(a,c,b-p+1) \oplus F(c+p-1,b,a-p+1)
\]
\[ F(a, b, c) \oplus F(b, c, a - p + 1) \oplus F(c + p - 1, a, b) \]
\[ \oplus F(b - 1, c, a - p + 2) \oplus F(a - 1, b, c + 1). \]

Again, any number of bottom rows are the graded pieces of a submodule. Moreover, all nontrivial extensions that can occur do occur (see Section 1.3).

**Proof.** We will prove the proposition for \( \tau^1 \), the other cases being similar. The weight \( F(b - 1, c, a - p + 2) \) is in the first layer for both filtrations by construction. Recall that in the proof of Lemma 4.1.1 of \([EGS13]\), \( \tau^1 \subset \tau^{s_0} \) can be taken to be the inverse image of the minimal submodule of \( \tau^{s_0} \) containing \( F(b - 1, c, a - p + 2) \) as a Jordan-Holder factor. Proposition 2.2.2 shows that \( F(a, c, b - p + 1) \) and \( F(b + p - 1, a, c) \) are in the second layer of the cosocle filtration of \( \tau^1 \). Similarly, looking at the inclusion \( \tau^1 \subset \tau^{s_1} \), we see that \( F(c + p - 1, b, a - p + 1) \) is also in the second layer of the cosocle filtration of \( \tau^1 \). In fact, \( F(a, c, b - p + 1) \oplus F(b + p - 1, a, c) \oplus F(c + p - 1, b, a - p + 1) \) is the second layer of the cosocle filtration since these weights are the only Jordan-Holder factors of \( \tau^1 \) that nontrivially extend \( F(b - 1, c, a - p + 2) \) (see Section 1.3).

The other five weights are all in the third layer of the cosocle filtration since there are no nontrivial extensions between them (see Section 1.3). We now show that all possible nontrivial extensions between the bottom two rows occur. Since the map \( \tau^1 \to \tau^{s_1} \) has image with Jordan-Holder factors \( F(b - 1, c, a - p + 2) \), \( F(a, c, b - p + 1) \), \( F(c + p - 1, b, a - p + 1) \), and \( F(c + p - 1, a, b) \) by Proposition 2.2.2, these are the Jordan-Holder factors of the cokernel of \( \tau^{s_1} \to \tau^1 \), the nonzero reduction of an inclusion \( \tau^{s_1} \to \tau^1 \). Hence all possible nontrivial extensions of \( F(b + p - 1, a, c) \) occur in \( \tau^1 \) since they occur in \( \text{im}(\tau^{s_1} \to \tau^1) \) by Proposition 2.2.2. All other nontrivial extensions are established analogously. From this we see that the cosocle and socle filtrations agree.

We now use Proposition 2.2.4 to complete the classification of maps between lattices. For \( i = 1, 2, 3 \), \( s \in S_3 \), let \( \iota^s_i : \tau^i \to \tau^s \) be fixed inclusions of lattices that are nonzero modulo \( p \). For \( i \) and \( j \) in the set \( \{1, 2, 3\} \) with \( i < j \), let \( \iota^s_i : \tau^i \to \tau^j \) be fixed inclusions of lattices that are nonzero modulo \( p \). These inclusions are unique up to unit scalar by Lemma 4.1.1 of \([EGS13]\). Let \( \iota^s_i : \tau^s \to \tau^i \) and \( \iota^s_i : \tau^j \to \tau^i \) be the unique inclusions of lattices that are nonzero modulo \( p \) such that \( \iota^s_i \circ \iota^s_i \) and \( \iota^s_i \circ \iota^s_i \) are powers of \( p \).

**Proposition 2.2.5.** The compositions of inclusions of lattices are given as follows:

1. \( \iota^s_j \circ \iota^s_i = p^2 \);
2. if \( s = e, r_1, \) or \( r_2 \), then \( \iota^s_i \circ \iota^s_i = p^3 \);
3. and if \( s = s_1, s_2, \) or \( s_0 \), then \( \iota^s_i \circ \iota^s_i = p \).

**Proof.** We begin with (1). We use the symbol \( \sim \) to denote equality up to a unit. By Propositions 2.2.2 and 2.2.4, we have \( \iota^{s_0}_i \sim \iota^{s_0}_i \circ \iota^{s_1}_j \), \( \iota^{s_1}_i \sim \iota^{s_0}_j \circ \iota^{s_1}_j \), \( \iota^{s_0}_j \sim \iota^{s_1}_i \circ \iota^{s_1}_j \), and \( \iota^{s_0}_j \sim \iota^{s_0}_j \circ \iota^{s_0}_i \). Hence \( \iota^{s_0}_i \circ \iota^{s_0}_i \sim \iota^{s_0}_i \circ \iota^{s_0}_i \circ \iota^{s_1}_j \circ \iota^{s_0}_i \sim \iota^{s_1}_i \circ \iota^{s_1}_j \sim p \iota^{s_0}_i \).

For (2), by symmetry we can assume without loss of generality that \( s = e \). From Propositions 2.2.2 and 2.2.4, we see that \( \iota^{e}_i \sim \iota^{e}_i \circ \iota^{s_0}_i \) and \( \iota^{e}_i \sim \iota^{e}_i \circ \iota^{s_0}_i \). Hence \( \iota^{e}_i \circ \iota^{e}_i \sim \iota^{e}_i \circ \iota^{s_0}_i \circ \iota^{e}_i \sim \iota^{e}_i \circ \iota^{s_0}_i \sim p^3 \) by Proposition 2.2.1.
For (3), by symmetry we can assume without loss of generality that \( s = s_0 \).

By Proposition \( \text{2.2.4} \), the image of the composition \( i_{s_1} \circ i_{s_0} \) is zero modulo \( p \), and hence a positive power of \( p \) times \( i_{s_0} \) up to a unit. Then by substitution, \( p^2 = i_{s_1} \circ i_{s_0} = i_{s_1} \circ i_{s_1} \circ i_{s_0} \) is a positive power of \( p \) times \( i_{s_0} \circ i_{s_0} \). We conclude that \( i_{s_0} \circ i_{s_0} = p \).

\[ \square \]

2.3. Lattices via Morita theory. In this subsection, we use Morita theory to explicitly describe the moduli of lattices in \( K \)-representations. The idea to use Morita theory to describe the moduli of lattices was suggested by David Helm (see Section 1.6 of [EGS13]) and carried out in [CEGS14].

2.3.1. The abelian category generated by lattices. Let \( \tau \) be a continuous irreducible (finite dimensional) \( K \)-representation over \( \mathbb{Q}_p \), which is residually multiplicity free. Let \( \mathcal{L} \) be the category of all finitely generated \( \mathbb{Z}_p \)-modules with a \( K \)-action which are isomorphic to subquotients of \( \mathbb{Z}_p \)-lattices in \( \tau^{\otimes n} \) for some \( n \in \mathbb{N} \). The irreducible objects of \( \mathcal{L} \) are \( \sigma \) where \( \sigma \) is a Jordan-Holder factor of \( \tau \). Let \( \tau_\sigma \subset \tau \) be a lattice (unique up to homothety by Lemma 4.1.1 of [EGS13]) with cosocle \( \sigma \).

Proposition 2.3.1. The lattices \( \tau_\sigma \) are projective.

Proof. We show that \( \tau_\sigma \) satisfies the lifting property. Let \( M \rightarrow N \) be a surjection in \( \mathcal{L} \). We wish to lift a map \( \tau_\sigma \rightarrow N \) to a map \( \tau_\sigma \rightarrow M \). By replacing \( N \) by the image of \( \tau_\sigma \) in \( N \) and \( M \) by the inverse image in \( M \), we assume without loss of generality that \( \tau_\sigma \) surjects onto \( N \). By replacing \( M \) by a lattice that surjects onto it, we can assume without loss of generality that \( M \) is \( p \)-torsion free. We can and do replace \( M \) by a minimal submodule of \( M \) that surjects onto \( N \).

Since \( N \) has cosocle isomorphic to \( \sigma \), the cosocle of \( M \) contains \( \sigma \). By minimality, the cosocle of \( M \) must be isomorphic to \( \sigma \). In particular, \( M \) is generated by one element, so that \( M \otimes \mathbb{Z}_p \mathbb{Q}_p \) is isomorphic to \( \tau \). By Lemma 4.1.1 of [EGS13], \( \tau_\sigma \) and \( M \) are isomorphic. Since \( \tau_\sigma \) is generated by one element, the surjection \( \tau_\sigma \rightarrow N \) is unique up to multiplication by an element of \( \mathbb{Z}_p \). We conclude that there is an isomorphism \( \tau_\sigma \cong M \) that lifts \( \tau_\sigma \rightarrow N \).

\[ \square \]

Let \( \mathcal{P} = \oplus_\sigma \tau_\sigma \) where \( \sigma \) runs over the distinct Jordan-Holder factors of \( \tau \). Let \( \mathcal{E} = \text{End}_K(\mathcal{P}) \).

Proposition 2.3.2. The functor \( M \mapsto \text{Hom}_K(\mathcal{P}, M) \) gives an equivalence of categories \( \mathcal{L} \rightarrow \text{f.g.} \mathcal{E}^{\text{op}}\text{-modules} \).

Proof. By proposition 2.3.1 \( \mathcal{P} \) is projective generator of the category \( \mathcal{L} \). The equivalence follows from Morita theory with quasi-inverse given by \( (\cdot) \otimes \mathcal{E}^{\text{op}} \mathcal{P} \).

\[ \square \]

2.3.2. Moduli of lattices in algebraic vectors. Let \( \mathcal{E}^\text{alg} := \text{End}_K(W^0 \oplus W) \). Using Proposition 2.1.2 it is easy to calculate that there is an isomorphism

\[ \mathcal{E}^\text{alg} \cong \left( \begin{array}{c} \mathbb{Z}_p \\ \mathbb{Z}_p \\ \mathbb{Z}_p \end{array} \right), \]

where \( \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \) and \( \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \) restrict to the identity on \( W^0 \) and \( W \), respectively, and \( \left( \begin{array}{c} p \\ 0 \\ 0 \end{array} \right) \) and \( \left( \begin{array}{c} 0 \\ p \\ 0 \end{array} \right) \) correspond to \( i^\vee \) and \( i \), respectively.
We now describe the moduli space of lattices in $V$ (with rigidifications). Let $\mathcal{M}$ be the functor which takes a $\mathbb{Z}_p$-algebra $R$ to the set

$$(\mathcal{P}_R, \psi_1, \psi_2)|_{\mathcal{P}_R}$$

is a free $R$-module with an $\mathcal{E}_{\text{alg}}$-action,

$$\psi_1 : R \xrightarrow{\sim} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \mathcal{P}_R, \psi_2 : R \xrightarrow{\sim} \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \mathcal{P}_R$$

We consider $\mathcal{E}_{\text{alg}}$-modules rather than $\mathcal{E}_{\text{alg,op}}$-modules because of the variance of the patching construction in Section 4.

**Proposition 2.3.3.** The functor $\mathcal{M}$ is represented by the ring $\mathbb{Z}_p[x,y]/(xy - p)$ where $x \mapsto \psi_1^{-1} \circ \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \circ \psi_2$ and $y \mapsto \psi_2^{-1} \circ \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \circ \psi_1$.

**Proof.** Let $A = \mathbb{Z}_p[x,y]/(xy - p)$. Then we define $\mathcal{P}_A$ to be $A^2$ where $\mathcal{E}_{\text{alg}} \to M_2(A)$ is given by

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \mapsto \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$$

$$\left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \mapsto \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$$

$$\left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \mapsto \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$$

The triple $(P_A, \text{id}_A, \text{id}_A)$ is universal since $\psi_1 \oplus \psi_2 : (R^2, \text{id}_R, \text{id}_R) \to (\mathcal{P}_R, \psi_1, \psi_2)$ is an isomorphism. Here the $\mathcal{E}_{\text{alg}}$-action of $R^2$ is given by the map $A \to R$ described in the statement of the proposition. \hfill $\square$

## 3. Hecke algebras

In this section, we describe integral Hecke operators for some locally algebraic types, which play a key role in determining lattice structures in cohomology. We also relate these Hecke operators to classical $T_p$ and $U_p$ Hecke operators.

### 3.1. Hecke algebras for split reductive groups

We describe integral Hecke algebras for representations of the $\mathbb{Z}_p$-points of split reductive groups. Let $G$ be a split reductive group over $\mathbb{Z}_p$, $B$ a Borel subgroup, and $T$ a maximal torus. Let $X_*(T)$ denote the group of cocharacters of $T$. Let $K = G(\mathbb{Z}_p)$ be a maximal compact open hyperspecial subgroup of $G = G(\mathbb{Q}_p)$ and

$$K_1 = \ker(G(\mathbb{Z}_p) \to G(\mathbb{F}_p)).$$

Let $W$, $W_1$, and $W_2$ be $K$-representations over $\mathbb{Z}_p$. Define the Hecke bimodule

$$\mathcal{H}(W_1, W_2) := \text{Hom}_G(\text{ind}_K^G W_1, \text{ind}_K^G W_2)$$

$$\cong \{ \text{compactly supported } f : G \to \text{Hom}_{\mathbb{Z}_p}(W_1, W_2) \}$$

$$\forall k_1, k_2 \in K, g \in G, f(k_1 g k_2) = k_1 \circ f(g) \circ k_2 \}.$$ 

If $W$ is a $K$-representation, let $\mathcal{H}(W) = \mathcal{H}(W, W)$ be the Hecke algebra of $W$. If $\pi$ is a $G$-representation over $\mathbb{Z}_p$, then $\text{Hom}_K(W, \pi)$ is a right $\mathcal{H}_G(W)$-module.

Recall the Cartan decomposition

$$G = \sqcup_{\mu \in X_*(T)_-} K t_\mu K$$

where $t_\mu = \mu(p)$. This gives the decomposition

$$\text{ind}_K^G W = \bigoplus_{\mu \in X_*(T)_-} \text{Ind}_K^{K t_\mu K} W$$

(3.1.1)
as $K$-representations where

$$\text{Ind}^{KtK}_K W = \{ f : KtK \to W | f(kx) = kf(x) \ \forall k \in K, x \in KtK \}$$

and the $K$-action is right regular. We can simplify this as follows:

$$\text{Ind}^{KtK}_K W \cong \text{Ind}^{KtK}_K W \cong \text{Ind}^{KtK}_K W^{(t)}$$

where $W^{(t)}$ is identified with $W$ as vector spaces, but the superscript denotes the twisted action on $W^{(t)}$ defined by $kw^{(t)} = (kt^{-1}w)^{(t)}$ for $w \in W^{(t)}$ and $k \in t^{-1}Kt$.

Given a coweight $\mu \in X_*(T)$, let $\mathcal{H}(W_1, W_2)_{\mu} = \text{Hom}_K(W_1, \text{Ind}_K^{KtK} W_2) \subset \mathcal{H}(W_1, W_2)$ denote the subspace of elements supported on the double coset $Kt_{\mu}K$. Let $\tilde{N}_\mu$ and $\tilde{P}_\mu$ be $t^{-1}_1Kt_{\mu} \cap K$ and $t^{-1}_2Kt_{\mu} \cap K$, respectively. The following proposition is contained in the proof of Theorem 1.2 of [Her11b].

**Proposition 3.1.1.** Suppose that $K_1$ acts trivially on $W_1$ and $W_2$. We have a natural isomorphism $\mathcal{H}(W_1, W_2)_{\mu} \cong \text{Hom}_{\tilde{P}_\mu}((W_1)_{\tilde{N}_\mu}, (W_2^{\tilde{N}_\mu})^{(t_{\mu})})$.

**Proof.** We have the injection

$$\text{Hom}_{\tilde{P}_\mu}((W_1)_{\tilde{N}_\mu}, (W_2^{\tilde{N}_\mu})^{(t_{\mu})}) \hookrightarrow \text{Hom}_{\tilde{P}_\mu}(W_1, W_2^{(t_{\mu})}) \cong \text{Hom}_K(W_1, \text{Ind}_{\tilde{P}_\mu} W_2^{(t_{\mu})})$$

where the first isomorphism follows from Frobenius reciprocity and the second isomorphism follows from (3.1.2). It suffices to show surjectivity of the first map.

Let $f \in \text{Hom}_{\tilde{P}_\mu}(W_1, W_2^{(t_{\mu})})$. Since $W_1$ is $K_1 \cap t^{-1}_1Kt_{\mu}$-invariant, the image of $f$ is contained in

$$(W_2^{(t_{\mu})})^{K_1 \cap t^{-1}_1Kt_{\mu}} = (W_2^{t_{\mu}Kt_{\mu}^{-1}\cap K})^{(t_{\mu})} = (W_2^{\tilde{N}_\mu})^{(t_{\mu})}.$$ 

Since $W_2$ is $K_1 \cap t^{-1}_1Kt_{\mu}$-invariant, $W_2^{(t_{\mu})}$ is $\tilde{N}_\mu$-invariant. Therefore the map $f$ factors through $(W_1)_{\tilde{N}_\mu}$. \hfill \square

We can rephrase Proposition 3.1.1 as follows. Given a $\tilde{P}_\mu$-morphism $(W_1)_{\tilde{N}_\mu} \to (W_2^{\tilde{N}_\mu})^{(t_{\mu})}$, we obtain a map $\text{Ind}_{\tilde{P}_\mu}^{KtK}(W_1)_{\tilde{N}_\mu} \to \text{Ind}_{\tilde{P}_\mu}^{KtK}(W_2^{\tilde{N}_\mu})^{(t_{\mu})}$. By Frobenius reciprocity, we have a natural map $W_1 \to \text{Ind}_{\tilde{P}_\mu}^{KtK}(W_1)_{\tilde{N}_\mu}$. The composition

$$W_1 \to \text{Ind}_{\tilde{P}_\mu}^{KtK}(W_1)_{\tilde{N}_\mu} \to \text{Ind}_{\tilde{P}_\mu}^{KtK}(W_2^{\tilde{N}_\mu})^{(t_{\mu})} \subset \text{Ind}_{\tilde{P}_\mu}^{KtK} W_2^{(t_{\mu})} \cong \text{Ind}_{\tilde{P}_\mu}^{KtK} W_2$$

is the corresponding element of $\mathcal{H}(W_1, W_2)_{\mu}$.

We can further simplify this when $\mu$ is minuscule, which we now suppose. See Section 3 and particularly Proposition 3.8 of [Her11b] for a more general context. Let $N_{\mu}$ and $P_{\mu}$ be the images of $\tilde{N}_\mu$ and $\tilde{P}_\mu$ in $G(F_p)$, respectively. Then $N_{\mu}$ and $P_{\mu}$ are the usual unipotent and parabolic subgroups in $G(F_p)$, respectively, corresponding to $\mu$. Let $M_{\mu} = P_{\mu}/N_{\mu}$. Note that $t_{\mu}P_{\mu}t_{\mu}^{-1} = \tilde{P}_\mu$ and $t_{\mu} \tilde{N}_\mu K t_{\mu}^{-1} = \tilde{N}_\mu K_1$, and so conjugation by $t_{\mu}$ gives an isomorphism $M_{\mu} \cong M_{\mu}$. 

**Proposition 3.1.2.** Suppose that $\mu$ is minuscule and that $K_1$ acts trivially on $W_1$ and $W_2$. We have a natural isomorphism $\mathcal{H}(W_1, W_2)_{\mu} \cong \text{Hom}_{M_{\mu}}((W_1)_{N_{\mu}}, (W_2^{N_{\mu}}))$. 


Proposition 2.2.2. Let \( i \) in Section 2.1. Let \( M_\mu \) acts on \( W_{2 \sigma} \) through the isomorphism \( M_\mu \mapsto M_{-\mu} \) given by conjugation by \( t_\mu \).

Proof. Since \( \mu \) is minuscule, \( K_1 \subset \tilde{P}_\mu \). Using Proposition 3.1.1 we have

\[
\mathcal{H}(W_1, W_2)_\mu \cong \text{Hom}_{\tilde{P}_\mu}((W_1)_{N_\mu}, (W_2^{\tilde{N}_\mu})^{(t_\mu)})
\]
\[
\cong \text{Hom}_{\tilde{P}_\mu}((W_1)_{N_\mu K_1}, (W_2^{\tilde{N}_\mu K_1})^{(t_\mu)})
\]
\[
\cong \text{Hom}_{M_\mu}((W_1)_{N_\mu}, W_2^{N_\mu})
\]

where \( M_\mu \) acts on \( W_2^{N_\mu} \) through the isomorphism \( M_\mu \mapsto M_{-\mu} \) given by conjugation by \( t_\mu \). \( \square \)

We specialize the above discussion to the case when \( W_1 = W_2 = \sigma \) is an irreducible \( G(F_p) \)-representation over \( F_p \) (and hence a \( K \)-representation by inflation) and \( \mu \) is a minuscule coweight. Consider \( \sigma^{N_\mu} \subset \text{ind}_{\tilde{P}_\mu}^{G} \sigma^{N_\mu} \), a \( \tilde{P}_\mu \)-representation. By Lemma 2.3 of [Her11a], \( \sigma_{N_\mu} \) is irreducible and isomorphic to \( \sigma^{N_\mu} \) as \( M_\mu \)-representation. In other words, \( t_\mu \cdot \sigma^{N_\mu} \subset \text{ind}_{\tilde{P}_\mu}^{G} \sigma^{N_\mu} \) is isomorphic to \( \sigma_{N_\mu} \) as a \( \tilde{P}_\mu \)-representation. This defines a nonzero map \( T_{\sigma, \mu} : \text{ind}_{\tilde{P}_\mu}^{G} \sigma_{N_\mu} \cong \text{ind}_{\tilde{P}_\mu}^{G} \text{Ind}_{\tilde{P}_\mu}^{K} \sigma_{N_\mu} \to \text{ind}_{\tilde{P}_\mu}^{G} \text{Ind}_{\tilde{P}_\mu}^{K} \sigma^{N_\mu} \cong \text{ind}_{\tilde{P}_\mu}^{G} \sigma^{N_\mu} \). As \( \mathcal{H}(\sigma)_\mu \) is one-dimensional by Proposition 3.1.2, the composition of \( T_{\sigma, \mu} \) with the natural inclusion \( \text{ind}_{\tilde{P}_\mu}^{G} \sigma^{N_\mu} \cong \text{ind}_{\tilde{P}_\mu}^{K} (\sigma^{N_\mu})^{(t_\mu)} \subset \text{ind}_{K}^{K_{t_\mu}K} \sigma \) is a generator of \( \mathcal{H}(\sigma)_\mu \). The following is a rephrasing of weight cycling as in Proposition 2.3.1 of [EGH13].

**Proposition 3.1.3.** The map \( T_{\sigma, \mu}^\prime \) is an isomorphism.

Proof. The composition \( T_{\sigma, \mu}^\prime \circ T_{\sigma, \mu}^\prime \) is multiplication by \( t_\mu t_\mu^{-1} = 1 \). \( \square \)

We specialize \( \mu \) from the notation if it is clear from context. The following proposition follows from the discussion after Proposition 3.1.1.

**Proposition 3.1.4.** If \( T' \in \mathcal{H}(W_1, W_2)_\mu \) corresponds via Proposition 3.1.2 to a map that factors through \( \sigma_{N_\mu} \cong \sigma^{N_\mu} \), then \( T' \) factors through \( T_{\sigma, \mu} \) as \( \text{ind}_{\tilde{P}_\mu}^{G} W_1 \to \text{ind}_{\tilde{P}_\mu}^{G} \text{Ind}_{\tilde{P}_\mu}^{K} \sigma_{N_\mu} \to \text{ind}_{\tilde{P}_\mu}^{G} \text{Ind}_{\tilde{P}_\mu}^{K} \sigma^{N_\mu} \to \text{ind}_{K}^{K_{t_\mu}K} W_2 \).

### 3.2. Hecke algebras for upper alcove algebraic vectors of \( GL_3 \).

We specialize the ideas of Section 3.1 to the case of certain extensions of weights including the case of upper alcove algebraic vectors. Let \( \mu \) be a minuscule coweight. Let \( W_1 \) be a (nontrivial) extension of weights \( 0 \to \sigma' \to W_1 \to \sigma \to 0 \) such that \( W_1 \) injects into \( \text{Ind}_{P_\mu}^{G} \sigma'_{N_\mu} = \text{Ind}_{P_\mu}^{G} (\sigma')_{N_\mu} \). Let \( W_2 \) be the (unique up to isomorphism) nontrivial extension \( 0 \to \sigma \to W_2 \to \sigma' \to 0 \) by Section 1.3. Then we have a surjection \( \text{Ind}_{P_\mu}^{G} (\sigma')^{N_\mu} \to W_2 \) by Lemma 6.1.1 of [EGH13], which describes the Jordan-Hölder factors of \( \text{Ind}_{P_\mu}^{G} (\sigma')^{N_\mu} \). Proposition 3.1.4 gives an element \( T' \in \mathcal{H}(W_1, W_2)_\mu \). Fix a (unique up to scalar) nonzero map \( i : W_2 \to W_1 \), which induces maps \( \mathcal{H}(W_1, W_2) \to \mathcal{H}(W_1) \) and \( \mathcal{H}(W_1, W_2) \to \mathcal{H}(W_2) \). We denote the images of \( T' \) by \( T_{W_1} \) and \( T_{W_2} \), respectively.

Now let \( W_1 \) and \( W_2 \) be \( W \) and \( W' \), respectively, where \( \mathcal{W} \) and \( \mathcal{W}' \) are as in Section 2.1. Let \( i : W' \to W \) be the inclusion from Section 2.1. We have \( \mathcal{W} \mapsto \text{Ind}_{P_\mu}^{G} (\sigma')_{N_\mu} \) for antidominant minuscule \( \mu \) by Lemma 6.1.1 of [EGH13] and Proposition 2.2.2.
**Proposition 3.2.1.** For $\mu \in X_*(T)_-$, $\mathcal{H}(W, W_0)_{\mu}$, $\mathcal{H}(W^0)_{\mu}$, and $\mathcal{H}(W)_{\mu}$ are free of rank one over $\mathbb{Z}_p$.

**Proof.** Since $V = W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is naturally a $G$-representation, $\text{ind}_K^G V \cong V \otimes \text{ind}_G^G 1$. Since $V$ is an irreducible $G$-representation, $\mathcal{H}(V)$ is isomorphic to $\mathcal{H}(1)$ and is therefore one-dimensional over $\mathbb{Q}_p$ by the Satake isomorphism. This completes the proof since $\mathcal{H}(W, W_0)_{\mu}$, $\mathcal{H}(W^0)_{\mu}$, and $\mathcal{H}(W)_{\mu}$ are $\mathbb{Z}_p$-lattices. □

**Proposition 3.2.2.** We have $\overline{W}_{N_\mu} \cong \sigma_{N_\mu} \oplus \sigma'_{N_\mu}$, $\overline{W}^{N-\mu} \cong (\sigma')^{N-\mu}$, $\overline{W}^0_{N_\mu} \cong \sigma'_{N_\mu}$, and $(\overline{W}^0)^{N-\mu} \cong \sigma^{N-\mu} \oplus (\sigma')^{N-\mu}$. The spaces $\mathcal{H}(\overline{W})_{\mu}$ and $\mathcal{H}(\overline{W}^0)_{\mu}$ are both one-dimensional over $\mathbb{F}_p$.

**Proof.** From Proposition 2.1.1 we have exact sequences

$$0 \to (\sigma')^{N-\mu} \to \overline{W}^{N-\mu} \to \sigma^{N-\mu}$$
$$0 \to \sigma^{N-\mu} \to (\overline{W}^0)^{N-\mu} \to (\sigma')^{N-\mu}$$

$$\sigma'_{N_\mu} \to \overline{W}^0_{N_\mu} \to \sigma'_{N_\mu} \to 0$$

By Lemma 2.3 of [Her11a] and the fact that $a - c < p$, $(\sigma')^{N-\mu} \cong \sigma'_{N_\mu}$ and $\sigma^{N-\mu} \cong \sigma_{N_\mu}$ have different actions of the GL$_1$ factor of $M_\mu$. Thus the corresponding short exact sequences split. Recall from Proposition 2.1.1 that $\sigma'$ is isomorphic to the socle of $\overline{W}$ and the cosocle of $\overline{W}^0$. Since $\sigma'$ is not a Jordan-Holder factor of $\text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} \sigma^{N-\mu}$ (resp. $\text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} \sigma_{N_\mu}$) by Lemma 6.1.1 of [EGH13], any map $\text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} \sigma^{N-\mu} \to \overline{W}$ (resp. $\overline{W}^0 \to \text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} \sigma_{N_\mu}$) must be zero. By Frobenius reciprocity, we conclude that $\overline{W}^{N-\mu} \cong (\sigma')^{N-\mu}$ and $\overline{W}^0_{N_\mu} \cong \sigma'_{N_\mu}$.

On the other hand, we have an inclusion $\overline{W} \subset \text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} \sigma'_{N_\mu}$ and a surjection $\text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} (\sigma')^{N-\mu} \to \overline{W}^0$ as described at the beginning of Section 3.2. By Frobenius reciprocity, we conclude that $\overline{W}_{N_\mu} \cong \sigma_{N_\mu} \oplus \sigma'_{N_\mu}$ and $(\overline{W}^0)^{N-\mu} \cong \sigma^{N-\mu} \oplus (\sigma')^{N-\mu}$. For the second part, use Proposition 3.1.2. □

**Proposition 3.2.3.** The inclusion $i$ induces natural maps $\mathcal{H}(W, W^0)_{\mu} \to \mathcal{H}(W^0)_{\mu}$ and $\mathcal{H}(W, W^0)_{\mu} \to \mathcal{H}(W)_{\mu}$, which are isomorphisms between rank one $\mathbb{Z}_p$-modules. Furthermore, the natural map $\mathcal{H}(\overline{W}, \overline{W}^0)_{\mu} \to \mathcal{H}(\sigma')_{\mu}$ induced by the maps $\sigma' \hookrightarrow \overline{W}$ and $\overline{W}^0 \to \sigma'$ is an isomorphism.

**Proof.** By Proposition 3.2.1 and Nakayama’s lemma, it suffices to show that maps induced by $i$ are nonzero modulo $p$. By Proposition 3.2.1 $\mathcal{H}(W, W^0)_{\mu}$ has rank one. By Proposition 3.2.2 the reduction modulo $p$ of the generator of $\mathcal{H}(W, W^0)_{\mu}$ must be $T'$. By Proposition 3.1.2 and the fact that the compositions $\overline{W}^0 \to \overline{W} \hookrightarrow \text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} \sigma'_{N_\mu}$ and $\text{Ind}_{P_{\mu}}^{G(\mathbb{F}_p)} (\sigma')^{N-\mu} \to \overline{W}^0 \to \overline{W}$ are nonzero, the compositions $i \circ T'$ and $T' \circ i$ are nonzero. Similarly, the map $\mathcal{H}(\overline{W}, \overline{W}^0)_{\mu} \to \mathcal{H}(\sigma')_{\mu}$ is nonzero. The last statement of the proposition follows from the fact that $\mathcal{H}(\sigma')_{\mu}$ is one-dimensional by Lemma 2.3 of [Her11a]. □
Let \( T' \) be a generator of \( \mathcal{H}(W, W^0)_\mu \). By abuse of notation, let \( T \) denote the generator which is the image of \( T' \) in \( \mathcal{H}(W)_\mu, \mathcal{H}(W^0)_\mu, \mathcal{H}(W^0)_\mu, \mathcal{H}(\sigma')_\mu \) via the maps in Proposition 3.2.3. We record this in the following proposition.

**Proposition 3.2.4.** The morphism \( T \in \mathcal{H}(W^0)_\mu \) factors as \( \text{ind}_K^G W^0 \rightarrow \text{ind}_K^G W \xrightarrow{T'} \text{ind}_K^G W^0 \).

### 3.3. Hecke algebras for principal series types

In this section, we describe Hecke operators for principal series types, and relate them to Hecke operators for extensions in Proposition 3.3.3. Keep the notation of Section 2.2.1. Let Hecke operators for principal series types, and relate them to Hecke operators for principal series types. induction isomorphisms \( \nu_{r,s} : \text{ind}_I^G \chi^{r,s} \rightarrow \text{ind}_I^G \chi^s \).

**Proof:** The proof is similar to that of Proposition 3.1.3. First, \( r_i t_i v^s \) normalizes \( I_1 \), and so \( r_i t_i v^s \) spans an \( I_1 \)-invariant subspace. Second, we have that \( v^s = r_i t_i v^s = r_i t_i t_i^{-1} t_i v^s = r_i t_i(t_i^{-1} t_i) v^s = \chi^{r,s}(t) r_i t_i v^s \) for any \( t \in T(\mathbb{Z}_p) \).

For the second statement, note that the composition

\[
\nu_{r,s} \circ \nu_{s}^s : \text{ind}_I^G \chi^s \rightarrow \text{ind}_I^G \chi^{r,s} \rightarrow \text{ind}_I^G \chi^s
\]

maps \( v^s \) to \( r_i t_i r_i t_i v^s = \sum_{g \in I_1/(r_i t_i^{-1} t_i)} g t_j v^s \) which is invertible and hence an isomorphism.

Let \( U_j^s \) be the operator on \( \text{ind}_I^G \chi^s \) given by \( v^s \mapsto \sum_{g \in I_1/(t_j t_i^{-1} \cap I_1)} g t_j v^s \).

**Proposition 3.3.2.** The composition

\[
\nu_{r,s}^s \circ \nu_{r,s}^s : \text{ind}_I^G \chi^{r,s} \rightarrow \text{ind}_I^G \chi^{r,s} \rightarrow \text{ind}_I^G \chi^{r,s}
\]

is the operator \( U_j^{r,s} \).

**Proof.** From the respective formulas in (2.2.1) and Proposition 3.3.1, we see that the composition is defined by

\[
v^{r,s} \mapsto \sum_{g \in I_1/(r_i t_i^{-1} t_i)} g r_i t_j v^{r,s} = \sum_{g \in I_1/(r_i t_i^{-1} t_i)} g t_j v^{r,s} = \sum_{g \in I_1/(t_j t_i^{-1} \cap I_1)} g t_j v^{r,s} = U_j^{r,s} v^{r,s}.
\]

Let \( \mu \) be the minuscule coweight such that \( t_i = t_\mu \). Suppose that \( \sigma' \) is a submodule of \( \pi^{r,s} \) so that \( \text{Ind}_{\mathbb{Z}_p}^K \sigma'_{N_\mu} \subset \pi^{r,s} \). Let \( W_1 \) and \( W_2 \) be as in the beginning of Section 3.2

**Proposition 3.3.3.** There is a commutative diagram of \( K \)-morphisms
where the composition of the top row is \( U_i^s \) and the composition of the bottom row is \( T_{W_2} \).

**Proof.** Note that the support of \( U_i^s \) is contained in \( Kt_{\mu}K \). The top row is the composition of \( \nu_s \) and \( \iota_s \), which is \( U_i^s \) by Proposition 3.3.2. The second row is the reduction of the top row modulo \( p \). The third row is the restriction of the second row to parabolic inductions which are naturally submodules in principal series. By the discussion preceding Proposition 3.1.3, the second map of the third row is a restriction of \( T_{\sigma'} \) to \( \text{Ind}_{\tilde{P}}^K \sigma'_{N_s} \). The bottom row then gives \( T_{W_2} \) by definition as discussed in the beginning of Section 3.2. □

## 4. Patching

Following [EGS13], we use Kisin’s local-global modification of the Taylor-Wiles method to study lattices in completed cohomology in families on local deformation rings. In this section, we introduce the Taylor-Wiles patching method as in [CHT08] and [CEG+14], in order to deduce Theorem 4.1.4.

### 4.1. Galois representations and automorphic forms.

In this section, we describe some Galois representations and automorphic forms relevant to our implementation of the Taylor-Wiles patching method. Let \( F \) be a CM field with \( F^+ \) its totally real subfield such that \( F/F^+ \) is unramified at all finite places and \( p \) splits completely in \( F \).

#### 4.1.1. Galois representations.

Following Section 2 of [CHT08], let \( \mathcal{G}/\mathbb{Z} \) be the group scheme defined to be the semidirect product of \( \text{GL}_3 \times \text{GL}_1 \) by the group \( \{ 1, j \} \), which acts on \( \text{GL}_3 \times \text{GL}_1 \) by \( j(g, s) = (s \cdot (g^{-1})^t, s) \). Let \( E \) be a finite extension of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O}_E \), uniformizer \( \varpi_E \), and residue field \( k_E \). We consider a continuous Galois representation \( r : G_{F^+} \to \mathcal{G}(\mathcal{O}_E) \) such that \( r^{-1}((\text{GL}_3(E) \times \text{GL}_1(E)) = G_F \). Then the composition \( \rho : G_F \to \text{GL}_3(E) \times \text{GL}_1(E) \to \text{GL}_3(E) \) of \( r|_{G_F} \) and the first projection is an essentially conjugate self-dual Galois representation. Assume that \( \rho^c \cong \rho^\vee \tau^{-2} \) where \( \cdot^c \) denotes complex conjugation.

Suppose that \( \rho \) ramifies only at places that are split in \( F/F^+ \) and that \( \rho|_{G_w} \) is potentially crystalline at places \( w|p \) of \( F \). As usual, we let \( \bar{\rho} : G_F \to \text{GL}_3(k_E) \) denote the reduction of \( \rho \).

**Axiom 4.1.1.** We say that \( \bar{\rho} \) satisfies the Taylor-Wiles conditions (TW) if

- \( \bar{\rho} \) has image containing \( \text{GL}_3(F) \) for some \( F \subset k_E \) with \( \# F > 9 \), and
- \( \bar{\rho} \) does not contain \( F(\zeta_p) \).
Note that the first condition, which is stronger than the usual condition of adequacy (see Definition 2.3 of [THO12]), allows us to choose a place $v_1$ of $F^+$ which is prime to $p$ satisfying the following properties (see Section 2.3 of [CEG+14]):

- $v_1$ splits in $F$ as $v_1 = w_1 w_1^c$.
- $v_1$ does not split completely in $F(F_p)$.
- $\rho(Frob_{w_1})$ has distinct $F$-rational eigenvalues, no two of which have ratio $\pm 1$.

See Remark 1.2.1 for an explanation of the choice of $v_1$.

Let $\Sigma$ be a finite set of places of $F^+$ away from $p$ such that $\rho$ is unramified away from $\Sigma$ and $p$. Let $S$ be the set of places $v_1$, those in $\Sigma$, and those dividing $p$ in $F^+$. For each place $v$ in $S$, fix a place $\tilde{v}$ in $F$ lying over $v$ and let $\tilde{S}$ be the set of these places $\tilde{v}$. For places $w$ in $F$, let $R^\univ_w$ be the universal $O_E$-lifting ring of $\rho|_{G_{F_w}}$. For $\tilde{v}$ with $v \in \Sigma$, let $R^\loc_{\tilde{v}, v}$ be the reduced $p$-torsion free quotient of $R^\univ_{\tilde{v}}$ corresponding to lifts of the inertial type of $\rho|_{G_{F_v}}$, $\tau_v$. Recall that $\epsilon$ denotes the cyclotomic character and $\delta_{F/F^+}$ denotes the quadratic character of $F/F^+$. Let

$$S = \left( F/F^+, S, \tilde{S}, O_E, \rho, \epsilon^{-2}\delta_{F/F^+}, \{ R^\loc_w \} \cup \{ R^\univ_{\tilde{v}, v} \} \cup \{ R^\loc_{\tilde{v}, v} \} \}_{v \in \Sigma}$$

be the deformation problem in the terminology of [CHT08]. There is a universal deformation ring $R^\univ_S$ and a universal $S$-framed deformation ring $R^\univ_S$ in the sense of Definition 1.2.1 of [CHT08].

Let

$$R^\loc = \bigotimes_{v \in \Sigma} R^\loc_{\tilde{v}, v} \otimes \left( \bigotimes_{v \in \Sigma} R^\loc_{\tilde{v}, v} \right) \otimes R^\loc_{v_1}$$

where all completed tensor products are taken over $O_E$. Choose an integer $q$ as in Section 2.5 of [CEG+14]. We introduce the ring

$$R_{\infty} = R^\loc[\left[ x_1, \ldots, x_q \right]],$$

over which we construct a patched module of automorphic forms in Section 4.2.

4.1.2. Automorphic forms on definite unitary groups. As in Section 2.3 of [CEG+14], we let $U/O_{F^+}$ be a model for a definite unitary group that is quasisplit at all finite places of $F^+$ and splits over $F$. For each place $v$ of $F^+$ that splits as $v = w w^c$ in $F$, let $O_v$ and $O_{w^c}$ denote $O_{F_v^+}$ and $O_{F_{w^c}}$, respectively. Fix an isomorphism $\iota_v : U(O_v) \cong \text{GL}_3(O_{w^c})$. For $\rho$ as in Section 4.1.1, one can define a corresponding ideal $\lambda \subset T$, where $T = T^\univ_{S, \rho}$ is defined in Section 2.3 of [CEG+14]. The ideal $\lambda$ is the Hecke eigensystem corresponding to $\rho$.

Let the Hodge-Tate weights of $\rho|_{G_{F_{w^c}}}$ be $x_v > y_v > z_v$. Let $V_v$ be the algebraic $U(O_{F_v})$-representation given by $\iota_v$ of highest weight $(x_v, y_v, 0, 0)$. Then $V := \bigotimes_{v \in \Sigma} V_v$ defines a local system on $U(F^+) \setminus U(A_{F^+})/U$ for $U = \prod_v U_v \subset U(A_{F^+})$ a compact open subgroup.

**Axiom 4.1.2. We say that $\rho$ is modular (M) if**

$$H^0(U(F^+) \setminus U(A_{F^+})/U, V \otimes Q_v |\lambda| \neq 0$$

for some compact open subgroup $U = \prod_v U_v \subset U(A_{F^+})$ for which $U_v$ is ramified (not hyperspecial) at only finitely many $v$, all of which split over $F$.

**Remark 4.1.3.** If $\rho$ satisfies (M), then in particular $\rho^c \cong \rho^c \epsilon^{-2}$ where $\cdot^c$ denotes complex conjugation.
We recall the definition of $\pi$ from Section 1.1. For $v \in \Sigma$, let $\tau_v$ be the $U(\mathcal{O}_v) \rightarrow \text{GL}_3(\mathcal{O}_w)$-representation over $E$ which is the type corresponding to $\rho|_{G_{F_v}} \cong G_{F_w}$ by results towards inertial local Langlands (see Proposition 6.5.3 of [BC09]). For $v \nmid p$, let $U'_v$ be the maximal principal congruence subgroup such that $U'_v \mapsto \tau_v$. For $v|p$, let $U_{m,v}$ be the kernel of the map $U(\mathcal{O}_v) \rightarrow U(\mathcal{O}_v/p^m)$. Let $\hat{H}^0$ and $\mathcal{H}^0$ be the completed cohomology groups
\[
\lim_{\tau \rightarrow m} H^0(U(F^+) \backslash U(A_F^+) / \prod_{v \mid p} U_{m,v} \times \prod_{v \nmid p} U'_v, \mathcal{O}_E / \mathcal{O}_E^r)
\]
and
\[
\lim_{m \rightarrow \tau} H^0(U(F^+) \backslash U(A_F^+) / \prod_{v \mid p} U_{m,v} \times \prod_{v \nmid p} U'_v, \mathcal{O}_E / \mathcal{O}_E^r),
\]
respectively. For each $v \mid p$, let $\tau'_v$ be a $GL_3(\mathcal{O}_w)$-invariant $\mathcal{O}_E$-lattice in $\tau_v$, and $\tau_v$ be its reduction. Then we let
\[
\pi = \text{Hom}_{U^p}(\otimes'_{v \mid p} \tau'_v, \hat{H}^0[\lambda])
\]
and
\[
\pi = \text{Hom}_{U^p}(\otimes'_{v \mid p} \tau_v, \mathcal{H}^0[\lambda])
\]
where $U^p$ is the maximal compact open subgroup away from $p$. Let $G_v$ denote $U(F_v)$ and $K_v \subset G_v$ denote a maximal compact subgroup $U(\mathcal{O}_v)$. Then $\pi$ and $\pi$ are naturally $\prod_{v \mid p} G_v$-representations, which are nonzero precisely when $\rho|_{G_{F_v}}$ satisfies (M).

The following theorem summarizes some of the main results of [CEG+14]. The proof is postponed to Section 4.2.

**Theorem 4.1.4.** There exists an $R_\infty[\prod_{v \mid p} G_v]$-module $M_\infty$ satisfying the following properties. For a $\prod_{v \mid p} K_v$-representation $\otimes_{v \mid p} W_v$ over $\mathbb{Z}_p$, let $M_\infty(\otimes_{v \mid p} W_v)$ denote $\text{Hom}_{\prod_{v \mid p} K_v}(\otimes_{v \mid p} W_v, M_\infty)^\vee$, where $\cdot^\vee$ denotes the Pontrjagin dual.

1. $M_\infty$ is projective as an $\mathcal{O}_E[\prod_{v \mid p} K_v]$-module.
2. For a locally algebraic $\prod_{v \mid p} K_v$-representation $\otimes_{v \mid p} W_v$, $M_\infty(\otimes_{v \mid p} W_v)$ is maximal Cohen-Macaulay over its support in $R_\infty$ or is 0.
3. Let $\varphi \subset m \subset R_\infty$ be ideals corresponding to $\rho$ and $\rho$, respectively. Then $\pi = M_{\infty}[\varphi]$ and $\pi = M_{\infty}[m]$.

**Remark 4.1.5.** By Lemma 4.13 of [CEG+14], if $\otimes_{v} W_v$ has no $\mathcal{O}_E$-torsion, then $M_\infty(\otimes_{v} W_v) \cong \text{Hom}_K(\otimes_{v} W_v, M_\infty)^d$ where $\cdot^d$ denotes the Schikhof dual (see Section 1.5 of [CEG+14]). Note that in [CEG+14], $M_\infty(\otimes_{v} W_v)$ is only defined for $\mathcal{O}_E$-torsion-free objects and the definition with Schikhof duality is used instead.

**Remark 4.1.6.** As explained in the section following Remark 2.7 of [CEG+14], one chooses a surjection $R_\infty \rightarrow R_{\infty}^\square$. Then the ideals $\varphi \subset m \subset R_\infty$ corresponding to $\varphi$ and $\varphi$, respectively, are just the inverse images of the corresponding ideals of $R_{\infty}^\square$.

4.2. **The construction of patched modules.** In this section, we describe the construction of the patched module $M_\infty$ satisfying the properties in Theorem 4.1.4. The main difference from [CEG+14] is that we allow ramification away from $p$, and we patch at all places dividing $p$ simultaneously. The necessary modifications are straightforward and the proofs are identical.
4.2.1. Compact open subgroups and Taylor-Wiles primes. Assume (M) with compact open subgroup $U$. Let $\Sigma$ be the set of places $v \nmid p$ in $F^+$ where $U$ is ramified, or in other words where $U_v$ is not a hyperspecial compact open subgroup. For $m \in \mathbb{N}$, let $U_m = \prod_v U_{m,v} \subset U(A_F^\infty)$ be the compact open subgroup where

- $U_{m,v} = U(O_v)$ for all places $v$ which split in $F$ other than $v_1$ and those dividing $p$;
- $U_{m,v_1}$ is the preimage of the upper triangular matrices under the map
  $$U(O_v) \to U(k_v) \xrightarrow{\sim} \text{GL}_3(k_{w_1});$$
- $U_{m,v}$ is the kernel of the map $U(O_v) \to U(O_v/p^m)$ for $v | p$;
- $U_{m,v}$ is a hyperspecial maximal compact open subgroup of $U(F_v)$ if $v$ is inert in $F$.

Remark 4.2.1. Note that $v_1$ was chosen in Section 4.1.1 so that for all $t \in U(A_F^\infty)$, $t^{-1}U(F^+)t \cap U$ does not contain an element of order $p$, necessary for Theorem 4.1.1.

We now choose Taylor-Wiles primes. Recall that in Section 4.1.1 we chose an integer $q$. As in Section 2.5 of [CEG+14], for each $N \geq 1$ we use (TW) to choose $q$ places $Q_N$ and $\tilde{Q}_N$ of $F^+$ and $F$, respectively, which are in particular disjoint from $p$ and $\Sigma$. For $v \in Q_N$, we let $U_1(Q_N)_v \subset U(O_v)$ be the corresponding parahoric compact open subgroup defined in Section 5.5 of [EG14]. Let $U_1(Q_N)_{m,v} = U_{m,v}$ for $v \notin Q_N$ and $U_1(Q_N)_{m,v} = U_1(Q_N)_v$ for $v \in Q_N$. We define the compact open subgroup $U_1(Q_N)_m = \prod_v U_1(Q_N)_{m,v} \subset U(A_F^\infty)$.

4.2.2. Algebraic modular forms. Recall from Section 4.2.1 the definition of the set of places $\Sigma$ and the lattice in types $\tau_v^0$ for $v \in \Sigma$. Let $S_\tau(U_1(Q_N)_m)$ be the functions
  $$f : U(F^+) \setminus U(A_F^\infty) \to \otimes_{v \in \Sigma} \tau_v^0$$
such that $f(gu) = u^{-1}f(g)$ for all $g \in U(A_F^\infty)$ and $u \in U_1(Q_N)_m$ where $U_1(Q_N)_m$ acts on $\otimes_{v \in \Sigma} \tau_v^0$ by its projection to $\prod_v U_{m,v}$.

For each $v \in Q_N$, let $\mathcal{R}^{\text{rig}}_v$ be the quotient of $\mathcal{R}^{\text{rig}}_v$ defined in Section 5.5 of [EG14]. We let $S_{Q_N}$ be the deformation problem
  $$S_{Q_N} = (F/F^+, S \cup Q_N, \overline{s} \cup \tilde{Q}_N, O_E, \overline{\rho}, \epsilon^{2\delta_{F/F^+}},$$
  $$\{\mathcal{R}^{\text{rig}}_{v_1}\} \cup \{\mathcal{R}^{\text{rig}}_v\}_{v \mid p} \cup \{\mathcal{R}^{\text{rig},\tau}_v\}_{v \in \Sigma} \cup \{\mathcal{R}^{\text{rig}}_{v_1}\}_{v \in Q_N}.$$ Again, we have the universal deformation ring $R_{S_{Q_N}}^{\text{univ}}$ and the universal $S$-framed deformation ring $R_{S_{Q_N}}^{\text{framed}}$. Let $M_{1,Q_N} = \text{pr}_! \left( S_\tau(U_1(Q_N)_{2N}, O_E/\mathcal{O}_{Q_N}) \right)^\vee$ where $\cdot^\vee$ denotes the Pontrjagin dual, $\text{pr}_!$ is defined in Section 2.5 of [CEG+14], and $m_{Q_N}$ is the maximal ideal of the spherical Hecke algebra at places away from $S \cup Q_N$ corresponding to $\overline{\rho}$ (see Section 2.3 of [CEG+14]). Let $M_{1,Q_N} = M_{1,Q_N} \otimes_{R_{S_{Q_N}}^{\text{univ}}} R_{S_{Q_N}}^{\text{framed}}$. Let
  $$S_{Q_N} = O_E[[\zeta_1, \ldots, \zeta_9, s, y_1, \ldots, y_9]].$$ As in Section 2.6 of [CEG+14], we patch the spaces $M_{1,Q_N}$ together to get an $R_\infty \times S_\infty[[\prod_{v \mid p} K_v]]$-module $M_\infty$. 


Proof of Theorem 4.1.4. For part (1) of Theorem 4.1.4, the proof of Proposition 2.8 of [CEG+14] shows that $M_\infty$ is in fact projective as an $S_\infty[[\prod_{v \nmid p} K_v]]$-module and admits a $\prod_{v \mid p} G_v$-action that extends the action of $\prod_{v \mid p} K_v$.

Assume $M_\infty(\otimes_v W_v)$ is nonzero. Part (2) follows from the inequalities
\[
\text{depth}_{R_\infty} M_\infty(\otimes_v W_v) \geq \text{depth}_{S_\infty} M_\infty(\otimes_v W_v) = \dim S_\infty 
\geq \dim \text{Supp}_{R_\infty} M_\infty(\otimes_v W_v),
\]
where the first inequality comes from the fact that the $S_\infty$-action factors through $R_\infty$, the equality comes from the fact that $M_\infty$ is a projective $S_\infty$-module and the last inequality follows from the local algebraicity of $\otimes_v W_v$, Theorem 3.3.4 of [Kis08], and Lemma 4.16 of [CEG+14]. Part (3) follows directly from the construction of $M_\infty$ (see Section 2.6 of [CEG+14]).

\[
\square
\]

5. A KEY ISOMORPHISM AND CYCLICITY OF PATCHED MODULES

We deduce mod $p$ multiplicity one results (see Theorems 5.3.1 and 5.3.2) for upper alcove algebraic vectors and principal series types.

5.1. A key isomorphism. In this section, we prove Lemma 5.1.4 which provides the key input in proving cyclicity of patched modules. Suppose that $\bar{\sigma}$ satisfies (TW) and $\rho$ satisfies (M) from Sections 4.1.1 and 4.1.2. Fix a place $v \nmid p$ of $F^+$. From now on, for a $K_v \cong \mathrm{GL}_3(\mathbb{Z}_p)$-representation $V$, we write $M_\infty(V)$ to denote $M_\infty(V \otimes_{\nu \neq v} V_{v'})$ where $V_{v'}$ is a fixed $K_{v'}$-representation for each $v' | p$ with $v' \neq v$. The representations $V_{v'}$ are specialized to certain representations in Section 5.3.

We say that $\bar{\sigma}$ is modular of weight $\sigma$ at $v$ if $\text{Hom}_{K_v}(\sigma, \pi)$ is nonzero. Denote by $W_v(\bar{\sigma})$ the set of modular Serre weights for $\bar{\sigma}$ at $v$. The following is an immediate corollary of Theorem 4.1.4(3).

Proposition 5.1.1. A weight $\sigma \in W(\bar{\sigma})$ if and only if $M_\infty(\sigma) \neq 0$ for some choice of $V_{v'}$.

As in Section 3.2, we consider extensions $0 \to \sigma' \to W_1 \to \sigma \to 0$ and $0 \to \sigma \to W_2 \to \sigma' \to 0$ such that $W_1 \to \text{Ind}_{F_\mu}^K \sigma'_{\bar{\nu}}$, for some minuscule coweight $\bar{\nu}$. We introduce the following axiom.

Axiom 5.1.2. We say that $\bar{\sigma}$ satisfies weight elimination (WE) for $(W_1, \mu)$ at $v$ if $\text{Ind}_{F_\mu}^K \sigma'_{\bar{\nu}} / W_1$ has no Jordan-Hölder factors in $W_v(\bar{\sigma})$.

Remark 5.1.3. By Proposition 5.1.1 and Theorem 4.1.4(1), $\bar{\sigma}$ satisfies (WE) for $(W_1, \mu)$ at $v$ if and only if $M_\infty(\text{Ind}_{F_\mu}^K \sigma'_{\bar{\nu}} / W_1) = 0$ for all choices of $V_{v'}$, if and only if
\[
M_\infty(\ker(\text{Ind}_{F_\mu}^K \sigma'_{\bar{\nu}} \to W_2)) = 0
\]
for all choices of $V_{v'}$.

The map $T' : \text{ind}_K^G W_1 \to \text{ind}_K^G W_2$ from Section 3.1 induces a map
\[
\text{Hom}_C(\text{ind}_K^G W_2, M_\infty) \to \text{Hom}_C(\text{ind}_K^G W_1, M_\infty)
\]
and hence by Frobenius reciprocity and duality, a map $T' : M_\infty(W_1) \to M_\infty(W_2)$.

Lemma 5.1.4. Suppose that $\bar{\sigma}$ satisfies (WE) for $(W_1, \mu)$ at $v$. Then the $R_\infty$-module homomorphism $T' : M_\infty(W_1) \to M_\infty(W_2)$ is an isomorphism.
Proof. By Proposition 3.1.4 and Theorem 4.1.4, the map $T : M_{\infty}(W_1) \to M_{\infty}(W_2)$ can be written as the composition

$$M_{\infty}(W_1) \to M_{\infty}(\text{Ind}_{F_{\mu}}^{K}\sigma_{N_{\mu}}) \to M_{\infty}(\text{Ind}_{F_{\mu}}^{K}\sigma^{N_{\mu}}) \to M_{\infty}(W_2).$$

By Proposition 3.1.3, the second morphism is an isomorphism. By the exactness of $M_{\infty}(\cdot)$ from Theorem 4.1.4[1] and Remark 5.1.3, the first and third morphisms are isomorphisms. □

5.2. Multiplicity one for extensions. From Lemma 5.1.4, we deduce cyclicity of certain patched modules, extending the method of Diamond and Fujiwara.

Axiom 5.2.1. We say that $p$ has regular type (RT) if $\hat{s}_{v}\in E_{p}^{\sigma_{w}}$ is a regular local ring.

Suppose that for each place $v|p$ of $F$, $\sigma_v$ is a Serre weight $F(x_v, y_v, z_v)$ with $x_v > y_v > z_v$ and $x_v - z_v < p - 3$. The following theorem is a well-known consequence of the method of Diamond and Fujiwara.

Theorem 5.2.2. The $R_{\infty}$-module $M_{\infty}(\otimes_v \sigma_v)$ is cyclic.

Remark 5.2.3. Note that we include the possibility that $M_{\infty}(\otimes_v \sigma_v) = 0$. With mild hypotheses, the module $M_{\infty}(\otimes_v \sigma_v)$ is nonzero if and only if $p|G_{F_{\mu}}$ is Fontaine-Laffaille of weight $(x_v + 2, y_v + 1, z_v)$ for all $v|p$ by Lemma 1.4.3 and Corollary 4.5.2 of [BGT14]. Alternatively, one could check Axiom A1 of [EGH13].

Proof. Since the Hodge-Tate weights $(x_v + 2, y_v + 1, z_v)$ are in the Fontaine-Laffaille range, the corresponding crystalline deformation rings at places dividing $p$ are formally smooth by Lemmas 2.4.27 and 2.4.28 of [CHT08]. By Theorem 2.1 of [Dia97], the assumption (RT), and local-global compatibility at $p$ from Lemma 4.16 of [CEG+14], $M_{\infty}(\otimes_v \sigma_v)$ is a free module over its support. We conclude that $M_{\infty}(\otimes_v \sigma_v)$ has rank one because $M_{\infty}(\otimes_v \sigma_v)/p$ has rank one over $O_E$ by Theorem 4.1.4[3] and Theorems 5.4 and 5.9 of [Lab11]. □

Let $W_1$ and $W_2$ be as in the beginning of Section 3.2. Suppose that $\tilde{p}$ satisfies (WE) for $(W_1, \mu)$ for some minuscule coweight $\mu$. We fix a place $v|p$ of $F^+$ and continue to use the notation of Section 5.1. Let $R$ be a ring over $R_{\infty}$ which acts on $M_{\infty}(W_1)$, $M_{\infty}(W_2)$, $M_{\infty}(\sigma)$, and $M_{\infty}(\sigma')$ and whose image in $\text{End}_{R_{\infty}}(M_{\infty}(W_1))$, $\text{End}_{R_{\infty}}(M_{\infty}(W_2))$, and $\text{End}_{R_{\infty}}(M_{\infty}(\sigma'))$ contains $T W_1$, $T W_2$, and $T \sigma'$. We conclude that $M_{\infty}(\otimes_v \sigma_v)$ is cyclic and nonzero by assumption. We conclude that $M_{\infty}(W_2)$ is cyclic by Nakayama’s lemma. The natural map $M_{\infty}(W_2) \to M_{\infty}(\sigma')$ shows that $M_{\infty}(\sigma')$ is cyclic. By Lemma 5.1.4, $M_{\infty}(W_1)$ is also cyclic. □
5.3. Multiplicity one for locally algebraic types. In this section, we prove some mod $p$ multiplicity one results for upper alcove algebraic vectors and principal series types.

5.3.1. The case of algebraic vectors. We again fix a place $v | p$ of $F^+$ and continue to use the notation of Section 5.3.1. Let $W'_v = W$ or $W^0$. Let $R_{\text{alg}} \subset \prod \text{End}_{R_{\text{alg}}} (M_{\infty}(\otimes_v W'_v))$ be the subring generated by the images of $R_{\infty}$ and $T_v$ where the product runs over all choices of $W'_v$. Note that the image of $R_{\text{alg}}$ in $\text{End}_{R_{\text{alg}}} (M_{\infty}(\otimes_v \overline{W}'_v))$ contains $T_{\overline{W}'_v}$ and for every place $v' | p$.

**Theorem 5.3.1.** Let $W = W(c+p-1,b,a-p+1)$, $W^0 = W(-a+p-1,-b,-c-p+1)$, and $\sigma = F(a-1,b,c+1)$. For each place $v$, let $\mu_v$ be some minuscule coweight. Suppose that $\overline{W}_v$ satisfies (RT) and (WE) for $(\overline{W}, \mu_v)$ at $v$. Suppose further that $M_{\infty}(\otimes_v \overline{W}'_v) \neq 0$. Let $\overline{W}_v = \overline{W}, \overline{W}'$, or $\sigma$ and $W'_v = W$ or $W^0$. Then $M_{\infty}(\otimes_v \overline{W}'_v)$ is a cyclic $R_{\text{alg}}$-module, and $M_{\infty}(\otimes_v W'_v)$ is free of rank one over $R_{\text{alg}}$.

**Proof.** To prove that $M_{\infty}(\overline{W}'_v)$ is cyclic, we induct on the number of places dividing $p$ for which $\overline{W}'_v \neq \sigma$. The base case is provided by Theorem 5.2.2. The induction step is given by Lemma 5.2.4. By Nakayama’s lemma, $M_{\infty}(\otimes_v \overline{W}'_v)$ is cyclic as well. Freeness follows from the fact that $R_{\text{alg}}$ was defined to act faithfully on $M_{\infty}(\otimes_v W'_v)$. \hfill \Box

5.3.2. The case of principal series types. For places $v | p$ of $F^+$, let $\tau_v = \text{ind}_K^v \chi_v$, where $\chi_v = \eta_v^a \otimes \eta_v^b \otimes \eta_v^c$, be a principal series type representation of $K_v$.

**Axiom 5.3.2.** We say that $\overline{\sigma}$ satisfies weight elimination (WE) for $\tau_v$ if
\[
W_v(\overline{\sigma}) \cap \text{JH}(\tau_v) \subset \{ F(a_v - 1, b_v, c_v + 1), F(b_v, c_v, a_v - p + 1), F(b_v + p - 1, a_v, c_v), F(c_v + p - 1, b_v, a_v - p + 1) \}.
\]

Suppose that $\overline{\sigma}$ satisfies (WE) for $\tau_v$.

**Proposition 5.3.3.** The map $\iota^{s_1,s_2}_{\overline{\tau}_{s_1,s_2}} : M_{\infty}(\overline{\tau}_{s_1,s_2}) \rightarrow M_{\infty}(\tau^{s_1,s_2})$ is an isomorphism.

**Proof.** The map is injective by Theorem 4.1.4([1]). By Lemma 2.2.2, Proposition 2.2.2 and (WE) for $\tau_v$, the cokernel of $\iota^{s_1,s_2}_{\overline{\tau}_{s_1,s_2}} : \overline{\tau}_{s_1,s_2} \approx \tau^{s_1,s_2}$ contains no Serre weights in $W_v(\overline{\sigma})$. By Proposition 5.1.1, the cokernel of $\iota^{s_1,s_2}_{\overline{\tau}_{s_1,s_2}} : M_{\infty}(\tau^{s_1,s_2}) \rightarrow M_{\infty}(\overline{\tau}_{s_1,s_2})$ is 0. By Nakayama’s lemma, the cokernel of $\iota^{s_1,s_2}_{\overline{\tau}_{s_1,s_2}} : M_{\infty}(\tau^{s_1,s_2}) \rightarrow M_{\infty}(\tau^{s_1,s_2})$ is 0. \hfill \Box

**Proposition 5.3.4.** The induced map $\iota^2_{\tau} : M_{\infty}(\tau^2) \rightarrow M_{\infty}(\tau^2)$ is an isomorphism.

**Proof.** The map is injective again by Theorem 4.1.4([1]). By the proof of Lemma 4.1.1 of [EGSI3], the image of $\overline{\tau}_2$ is the minimal submodule containing $F(a-1, b, c+1)$ as a Jordan-Holder factor. By Proposition 2.2.2 and (WE) for $\tau_v$, we see that the cokernel of $\iota^2_{\tau} : \tau^2 \rightarrow \tau^2$ contains no Jordan-Holder factors in $W_v(\overline{\sigma})$. Again by Nakayama’s lemma, the cokernel of $M_{\infty}(\tau^2) \rightarrow M_{\infty}(\tau^2)$ is 0. \hfill \Box

For $i = 1, 2, 3$, let $\epsilon^i : \tau^{s_1} \otimes \tau^{s_0} \rightarrow \tau^i$ denote the sum of $\epsilon^i_{s_1}$ and $\epsilon^i_{s_0}$.

**Proposition 5.3.5.** For $i = 1, 3$, the map $\epsilon^i : M_{\infty}(\tau^{s_1} \otimes \tau^{s_0}) \rightarrow M_{\infty}(\tau^i)$ is surjective.
Proof. By Propositions 2.2.2, Proposition 2.2.4, and (WE) for \( \tau_v \), the cokernel of \( \tau^s : \tau^{s_1} \oplus \tau^{s_0} \to \tau^1 \) has two Jordan-Holder factors, both of which are not in \( W_s(\mathfrak{p}) \). By Proposition 5.1.1, the cokernel of \( \tau^s : M_\infty(\tau^{s_1} \oplus \tau^{s_0}) \to M_\infty(\tau^1) \) is 0. By Nakayama’s lemma, \( \nu \) is surjective. \( \square \)

Let \( R^{ps} \subset \prod_{s, s' \in S_3} \text{End}_{R_\infty}(M_\infty(\otimes_{v' | p} \tau_{s', v'})) \) denote the subring generated by the image of \( R_\infty \) and the Hecke operators \( U_{i,v'}^s \) for \( i = 1, 2 \) and \( s, v' \in S_3 \). Here \( U_{i,v'}^s \) acts on \( \text{ind}_{K_{v'}} G_{v'} \tau_{s, v'} \) via the inclusion \( \iota_{s, v'}^i \), so that \( M_\infty(\cdot) \) gives a functor from \( K \)-subquotients of \( \otimes_{v' | p} \tau_{s', v'} \) to \( R^{ps} \)-modules.

Theorem 5.3.6. Assume that \( \mathfrak{p} \) satisfies (RT), (WE) for \( \tau_{v'} \) for all \( v' | p \), and \( M_\infty(\otimes_{v' | p} F(a_{v'} - 1, b_{v'}, c_{v'} + 1)) \neq 0 \). Then the \( R^{ps} \)-module \( M_\infty(\otimes_{v' | p} \tau_{s', v'}) \) for \( s, v' \in S_3 \) is free of rank 1.

Proof. By Proposition 3.3.1, \( \nu_{s_2, s_1}^{s_2} \) induces an isomorphism \( M_\infty(\tau^{s_1, s_2}) \cong M_\infty(\tau^{s_2}) \), and \( \nu_{s_0, s_2}^{s_1} \) induces an isomorphism \( M_\infty(\tau^{s_0, s_2}) \cong M_\infty(\tau^{s_1, s_2}) \). By Proposition 5.3.3, we have isomorphisms \( \iota_{s_2, s_1}^{s_2} : M_\infty(\tau^{s_1, s_2}) \cong M_\infty(\tau^{s_2}) \). We conclude that \( M_\infty(\otimes_{v' | p} \tau_{s', v'}) \) are isomorphic for all combinations \( (s, v') \). Suffices to show that \( M_\infty(\otimes_{v' | p} \tau_{s', v'}) \) is free of rank one over \( R^{ps} \) for some choice of \( s, v' \in S_3 \).

By Proposition 3.3.2, we have an isomorphism \( \psi_{s_2, s_1}^{s_2} : M_\infty(\otimes_{v' | p} \tau_{s_2} \otimes_{v' | p} \tau_{s_1}(\tau^{s_1})) \to M_\infty(\otimes_{v' | p} \tau_{s_2} \otimes_{v' | p} \tau_{s_1}(\tau^{s_1})) \). By (WE) for \( \tau_{v'} \), the natural inclusion

\[
M_\infty(\otimes_{v' | p} \tau_{s', v'}) \to M_\infty(\otimes_{v' | p} \tau_{s_2} \otimes_{v' | p} \tau_{s_1}(\tau^{s_1}))
\]

is an isomorphism. If we let \( \mu = (0, 0, 1) \), then again by (WE) for \( \tau_{v'} \), \( \mathfrak{p} \) satisfies (WE) for \( (\tau_{v'}, \mu) \) at all \( v' | p \). By Proposition 3.3.3, \( T_{\tau_{v'}} \) is in the image of \( R^{ps} \) in \( \text{End}_{R_\infty}(\otimes_{v' | p} \tau_{v'}) \). We conclude by Theorem 5.3.1 that \( M_\infty(\otimes_{v' | p} \tau_{s', v'}) \cong M_\infty(\otimes_{v' | p} \tau_{s_2} \otimes_{v' | p} \tau_{s_1}(\tau^{s_1})) \). Freeness follows from the fact that \( R^{ps} \) was defined to act faithfully on \( M_\infty(\otimes_{v' | p} \tau_{s', v'}) \). \( \square \)

5.3.3. An application of cyclicity. In this subsection, we deduce a lemma which allows one to use multiplicity one to relate the reduction of lattices to modularity of Serre weights. We again fix a place \( v' | p \) of \( F^+ \) and continue to use the notation of Section 5.1.

Lemma 5.3.7. Let \( \tau \) be a locally algebraic type, and let \( \tau_1, \tau_2 \subset \tau \) be a lattice with irreducible cosocle \( \sigma \) where \( M_\infty(\sigma) \neq 0 \). Suppose that \( M_\infty(\tau_1) \) is cyclic and \( \tau_2 \subset \tau_1 \) is a sublattice. Then \( M_\infty(\tau_2) \subset mM_\infty(\tau_1) \).

Proof. The inclusion \( \tau_2 \hookrightarrow \tau_1 \) and surjection \( \tau_1 \twoheadrightarrow \sigma \), whose composition is 0, induce the diagram

\[
\begin{array}{ccc}
M_\infty(\tau_2) & \longrightarrow & M_\infty(\tau_1) \\
\downarrow & & \downarrow \\
M_\infty(\tau_1)/m & \longrightarrow & M_\infty(\sigma)/m
\end{array}
\]

where the composition of the top row is 0. The map \( M_\infty(\tau_1) \to M_\infty(\sigma) \) is surjective by exactness of \( M_\infty(\cdot) \), and so the bottom row is surjective. By assumption,
$M_\infty(\tau_1)/m$ is one dimensional and $M_\infty(\sigma) \neq 0$, and so the bottom row is an isomorphism. We conclude that the composition $M_\infty(\tau_2) \to M_\infty(\tau_1) \to M_\infty(\tau_1)/m$ is 0. \hfill \square

6. Lattices in patched modules

In this section, we deduce our main theorem on lattices in cohomology, Theorem 1.1.1 (see Theorems 6.1.4 and 6.2.5). We again fix a place $v|p$ of $F^+$ and continue to use the notation of Section 5.1.

6.1. Lattices in patched modules for upper alcove algebraic vectors. Recall the definitions of $W$ and $W^0$ from Section 2.1.4.2 and $R^{alg}$ from Section 5.3.

Lemma 6.1.1. Suppose that $\bar{p}$ satisfies (WE) for $(W, \mu)$ at $v$ for some minuscule coweight $\mu$. Then the $R_\infty$-module homomorphism $T' : M_\infty(W) \to M_\infty(W^0)$ is an isomorphism.

Proof. The map $T' : M_\infty(\overline{W}) \to M_\infty(\overline{W}^0)$ is an isomorphism by Lemma 5.1.4. By Nakayama’s lemma, $T' : M_\infty(W) \to M_\infty(W^0)$ is surjective.

We now prove that $T' : M_\infty(W) \to M_\infty(W^0)$ is injective. Let $I$ be the kernel of the map $M_\infty(W) \to M_\infty(W^0)$. Suppose that $M_\infty(W)$ and $M_\infty(W^0)$ have support of dimension $d$ as $R_\infty$-modules (see the remark after Definition 2.2.4 of [CHT08] and Theorem 3.3.4 of [Kis08] for a formula for $d$). Let $Z_{d-1}$ be the functor that assigns to an $R^{alg}$-module the associated $d-1$-dimensional cycle (see Definition 2.2.5 of [EG14]). We have

$$Z_{d-1}(M_\infty(W/p^n)) = Z_{d-1}(M_\infty(W^0/p^n))$$

by exactness of $M_\infty(\cdot)$ from Theorem 4.1.4, the fact that $\overline{W}$ and $\overline{W}^0$ have the same Jordan-Holder factors, and additivity of $Z_{d-1}$ in exact sequences.

Let $I_n$ be the image of $I/p^n$ in $M_\infty(W/p^n)$. Then we have the exact sequence

$$0 \to I_n \to M_\infty(W/p^n) \to M_\infty(W^0/p^n) \to 0.$$

Again by additivity of $Z_{d-1}$, we see that $Z_{d-1}(I_n) = 0$ and so $I_n$ has support of dimension less than $d-1$. Since $M_\infty(W/p^n)$ is maximal Cohen-Macaulay over $R_\infty$, there are no embedded associated primes by Theorem 17.3(i) of [Mat89], and so $I_n = 0$ and the map $I/p^n \to M_\infty(W/p^n)$ must be 0 for all $n$. We conclude that the map $I \to M_\infty(W)$ is 0 and so $I = 0$. \hfill \square

Let $V_{v'} = W_{v'}$ or $W^0_{v'}$ for $v' \neq v$ and $v'|p$ a place of $F^+$. With this fixed choice of $V_{v'}$, we continue to use the notation of Section 5.1. Suppose that $M_\infty(\otimes_{v'|p} F(a_{v'} - 1, b_{v'}, c_{v'} + 1)) \neq 0$. By Theorem 5.3.1, we can and do fix an isomorphism $s : R^{alg} \cong M_\infty(W^0)$, which also gives an isomorphism $(T')^{-1} \circ s : R \cong M_\infty(W)$ by Lemma 6.1.1. This gives an element

$$(M_\infty(W^0) \oplus M_\infty(W), s, (T')^{-1} \circ s) \in M(R^{alg}).$$

Let $\Lambda : \mathbb{Z}_p[x, y]/(xy - p) \to R^{alg}$ be the corresponding morphism from Proposition 2.3.3.

Theorem 6.1.2. We have $\Lambda(y) = s^{-1} \circ T \circ s$. In other words, in the composition $\mathbb{Z}_p[x, y]/(xy - p) \to R^{alg} \to \text{End}_{R^{alg}}(M_\infty(W^0))$, the image of $y$ is $T$. 

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Proof. By Proposition 2.3.3 we have that
\[ \Lambda(y) = ((T')^{-1} \circ s)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ s = s^{-1} \circ T' \circ i \circ s = s^{-1} \circ T \circ s. \]

Remark 6.1.3. If \( F(a_v - 1, b_v, c_v + 1) \notin W_v(\overline{\rho}) \), then \( M_\infty(W_0) \rightarrow M_\infty(W) \) is an isomorphism and the question of lattices is trivial. We now assume that \( (a_v - 1, b_v, c_v + 1) \in W_v(\overline{\rho}) \) and that \( \overline{\rho}|_{G_{F_v^+}} \) is irreducible. Then by Theorems 3.3.13(i) and 7.5.6 of [EFGH13], \( \overline{\rho}|_{I_v} \cong \psi \oplus \psi^p \oplus \psi^{p^2} \) where \( \psi \cong \omega_3^{a_v-1+p(b_v+p^2(c_v+1))} \) or \( \psi \cong \omega_3^{a_v-1+p(b_v+p^2(c_v+1))} \) (see Theorem 5.1 and Conjecture 6.6 of [Her06] for a formula for \( W^2(\overline{\rho}) \)). We assume that \( \psi \cong \omega_3^{a_v-1+p(b_v+p^2(c_v+1))} \), the other case being symmetric (one has to use the other Hecke operator).

Theorem 6.1.4. Let \( F \) be a CM field in which \( p \) splits completely. Let \( F^+ \) be its totally real subfield and assume that \( F/F^+ \) is unramified at all finite places. Let \( \rho : G_F \rightarrow GL_3(C_E) \) be a Galois representation satisfying (M). Assume further that
- \( \rho \) satisfies (RT);
- for all places \( v | p \) of \( F^+ \), \( \rho|_{F_v^+} \) is a lattice in a crystalline representation with Hodge-Tate weights \( (c_v + p + 1, b_v + 1, a_v - p + 1) \) with \( a_v - b_v > 6 \), \( b_v - c_v > 6 \), and \( a_v - c_v < p - 5 \);
- for all places \( v | p \) of \( F \), \( \overline{\rho}|_{F_v^+} \) is irreducible with \( \overline{\rho}|_{I_v} \cong \psi \oplus \psi^p \oplus \psi^{p^2} \) where \( \psi \cong \omega_3^{a_v-1+p(b_v+p^2(c_v+1))} \);
- and the reduction \( \overline{\rho} : G_F \rightarrow GL_3(k_E) \) satisfies (TW).

Let \( \lambda_v \) be the trace of \( \varphi \) acting on \( D_{\text{cris}}(\rho|_{F_v^+}) \). Let \( V \) be as in Section 4.1.2. Then the lattice \( V \cap \pi \subset V \) is isomorphic to \( \otimes_{v | p} (i(W_0^0) + p^{-a_v+p-1}\lambda_vW_v) \subset \otimes_{v | p} (W_v \otimes_{Z_p} E) \).

Proof. Let \( \theta : R_{alg}^S \rightarrow E \) be the map corresponding to \( \rho \), defined as follows. Recall that in Remark 4.1.6 we fixed a surjection \( R_\infty \rightarrow R_{S_{\Sigma}}^S \). The composition of this map and the map \( R_{S_{\Sigma}}^S \rightarrow E \) defining \( \rho \) gives a map \( \theta : R_\infty \rightarrow E \). Let \( \mu = \mu_1 = (0, 0, 1) \). Then further, \( \theta(T_v) \) is defined to be \( p^{-a_v} + p^{-1}\lambda_v \). These maps are compatible by Corollary 4.4.3 of [EFGH13], Theorem 1.2 of [BLGGT12], Theorem 4.1.3([4]), and the unramified local Langlands correspondence. The map \( \theta : R_\infty \rightarrow E \) factors through \( R_{alg}^E \) if and only if \( M_\infty(\otimes_v W_v)/\ker(\theta) \) is nonzero. The space \( M_\infty(\otimes_v W_v)/\ker(\theta) \) is nonzero by Theorem 4.1.3([4]), axiom (M), and local-global compatibility at \( p \) as in Theorem 1.2 of [BLGGT12].

Note that by the inequalities that \( a_v, b_v, \) and \( c_v \) satisfy, \( (a_v - 1, b_v, c_v + 1) \) is a strongly generic weight in the sense of Definition 6.2.2 of [EFGH13]. By Proposition 6.1.3(ii) and Theorem 7.5.5 of [EFGH13], \( \overline{\rho} \) satisfies (WE) for \( (W, \mu_1) \) (see Theorem 5.1 and Conjecture 6.6 of [Her06] for a formula for \( W^2(\overline{\rho}) \)). By Theorem 7.5.6 of [EFGH13], \( M_\infty(\otimes_v F(a_v - 1, b_v, c_v + 1)) \neq 0 \). By Theorem 6.1.2 the composition \( \theta \circ \Lambda \) takes \( y_v \) to \( p^{-a_v} + p^{-1}\lambda_v \). By Theorem 4.1.3([4]) and Remark 4.1.5 Hom_K(\( \otimes_v W_v^0 \otimes \otimes_v W_v, \pi \)) \cong (\( M_\infty(\otimes_v W_v^0) \otimes M_\infty(\otimes_v W_v) \)) \otimes E \ as \( \mathcal{E} \)-modules. We conclude that the \( \mathcal{E}^{op} \)-module \( \text{Hom}_K(\otimes_v W_v^0 \otimes \otimes_v W_v, \pi) \) is isomorphic to the \( \mathcal{E}^{op} \)-module corresponding to the lattice \( \otimes_v F(i(W_v^0) + p^{-a_v} + p^{-1}\lambda_v W_v) \subset \otimes_v (W_v \otimes_{Z_p} E) \) for each place \( v | p \).
Corollary 6.1.5. Keep the assumptions of Theorem 6.1.4. Then \(a_v - p + 1 < \text{val}(\lambda_v) < a_v - p + 2\).

Proof. Note that the first inequality follows from Corollary 4.4.3 and Proposition 4.5.2 of \textbf{EGHS13}. Lemma 5.3.7 shows that the reduction of the algebraic vectors modulo \(\varpi_E\) must be semisimple since \(\tau_v\) and \(\tau_v^f\) are both modular Serre weights of \(\varpi\) by Theorem 7.5.6 of \textbf{EGHS13}. Hence \(0 < \text{val}(p^{-a_v+p-1}\lambda) < 1\). \hfill \square

6.2. Lattices in patched modules for principal series types. In this section, we prove Theorem 1.1.1 in the case of principal series types. We keep the notation of Section 2.2.1 such as \(\tau_v^f\) for \(s \in S_3\) and \(\tau_v^i\) for \(i = 1, 2, 3\). We continue to assume that \(\varpi\) satisfies (RT), (WE) for \(\varpi'\) for all \(\varpi' \mid p\), and \(M_{\infty}(\otimes_{\varpi'} F(1_v, b_v', c_v + 1)) \neq 0\).

Fix a place \(v \mid p\) of \(F^+\). Fix lattices \(\tau_v^f \subset \tau_v\) for places \(v \mid p\) of \(F^+\) with \(v' \neq v\). From now on we denote \(M_{\infty}(V_\varpi \otimes \otimes_{\varpi'} \otimes_{\tau_v^i} \otimes_{\varpi'} \psi)\) by \(M_{\infty}(V)\) where \(V\) is a \(K_v \cong GL_3(\mathcal{O}_v)\)-representation. Using Theorem 5.3.3 we can and do choose an isomorphism \(\psi_{i, \varpi} : R^{\varpi} \rightarrow M_\infty(\tau_{i, \varpi})\). The compositions \(\psi_{i, \varpi} : (\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} : R^{\varpi} \rightarrow M_\infty(\tau_{i, \varpi}) \rightarrow M_\infty(\tau_{i, \varpi})\) are also isomorphisms.

For each \(i\), Proposition 5.3.3 gives an isomorphism \(\psi_{i, \varpi} : R^{\varpi} \rightarrow M_\infty(\tau_{i, \varpi}) \rightarrow M_\infty(\tau_{i, \varpi})\). This also gives an isomorphism \(\psi_{i, \varpi} : U_{i, \varpi} \rightarrow M_\infty(\tau_{i, \varpi})\).

Lemma 6.2.1. (1) The map \((\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = 1\).
(2) The map \((\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} = \psi_{i, \varpi}\).
(3) The map \((\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} = \psi_{i, \varpi}\).

Proof. For (1), we note that \((\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = (\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = (\psi_{i, \varpi})^{-1} \psi_{i, \varpi} = U_{i, \varpi}\).

Using Proposition 3.3.2 For (2), Proposition 2.2.1 gives \((\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = (\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = (\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi}\), which is \(U_{i, \varpi}\) by (1). For (3), \((\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = (\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} \circ \psi_{i, \varpi} = (\psi_{i, \varpi})^{-1} \circ \psi_{i, \varpi} = U_{i, \varpi}\).

Proposition 6.2.2. Let \(\alpha_1, \alpha_2,\) and \(\alpha_3\) be the eigenvalues of \(\varphi\) acting on the \(\eta_{\alpha_1}, \eta_{\alpha_2},\) and \(\eta_{\alpha_3}\)-eigenspaces, respectively, of the crystalline Dieudonné module \(D\) over \(\mathcal{O}_{\mathcal{C}_{\varpi}}\). Then \(U_1\) and \(U_2\) act on \(\text{Hom}_{\mathcal{O}_{\mathcal{C}_{\varpi}}} (\tau_{\varpi}, \pi)\) by \(\alpha_1, \alpha_2,\) and \(\alpha_3\) respectively.

Proof. This follows from Theorem 1.2 of \textbf{BLGHS12} and the Langlands correspondence for representations with vectors fixed by the pro-\(p\) Iwahori subgroup. \hfill \square

Proposition 6.2.3. Given an \(\mathcal{O}_{\mathcal{E}}\)-lattice \(L \subset \tau \otimes_{\mathcal{Z}_{\mathcal{E}}} E,\) \(L\) is the sum of the saturations of each lattice with irreducible cosocle as described in Lemma 4.1.1 of \textbf{EGHS13}.

Proof. The image of this sum in \(L\) modulo \(\varpi_L\) is \(L\) since the Jordan-Hölder factor \(\sigma\) is in the image of the lattice with cosocle \(\sigma\). By Nakayama’s lemma, the image of the sum is \(L\). \hfill \square
Remark 6.2.4. Assume that \( \rho \) is modular of type \( \tau_v \) and \( \mathfrak{p}(\mathcal{O}_E) \) is irreducible.

Then after possibly relabelling \((a_v, b_v, c_v)\) by \(r_i(a_v, b_v, c_v)\) in the notation following Proposition 2.2.2 we can assume that \( F(a_v - 1, b_v, c_v + 1) \in W_v(p) \) by Theorem 7.5.6 of [LGH13]. As in Remark 6.1.3 we assume without loss of generality that \( \mathfrak{p}|_{\mathcal{I}_v} \cong \psi \oplus \psi' \oplus \psi^{p^2} \) where \( \psi \cong \omega_3^{-a_v - 1 + pb_v + p^2(c_v + 1)} \) since the other case is symmetric (one has to use the opposite Hecke operators).

Theorem 6.2.5. Let \( F \) be a CM field in which \( p \) splits completely. Let \( F^+ \) be its totally real subfield and assume that \( F/F^+ \) is unramified at all finite places. Let \( \rho : G_F \to \text{GL}_3(\mathcal{O}_E) \) be a Galois representation satisfying (M). Assume further that

- \( \rho \) satisfies (RT);
- for all places \( v' | p \) of \( F^+ \), \( \rho|_{F^{v'}} \) is a lattice in a potentially crystalline representation with Hodge-Tate weights \((0, 1, 2)\) with tame type \( \eta^{v'} \oplus \eta^{h_{v'}} \oplus \eta^{v'} \) where \( a_{v'} - b_{v'} > 6 \), \( b_{v'} - c_{v'} > 6 \), and \( a_{v'} - c_{v'} < p - 5 \);
- for all places \( w | p \) of \( F \), \( \mathfrak{p}|_{F_w} \) is irreducible with \( \mathfrak{p}|_{\mathcal{I}_w} \cong \psi \oplus \psi'' \oplus \psi^{p^2} \) where \( \psi \cong \omega_3^{-a_v - 1 + pb_v + p^2(c_v + 1)} \);
- and its reduction \( \overline{\rho} : G_F \to \text{GL}_3(k_E) \) satisfies (TW).

Then the lattice \( (\otimes_{v'} \tau_{v'} \otimes \mathbb{Z}_p E) \cap \pi \subset \otimes_{v'} \tau_{v'} \otimes \mathbb{Z}_p E \) is isomorphic to a tensor product of lattices with factor at \( v \) given by

\[
\tau^{s_v} + (\alpha_1 \alpha_3/p) - 1 l_v^2(\tau^1) + \alpha_1^{-1} l_v^2(\tau^2) + (\alpha_2 \alpha_3/p) - 1 l_v^2(\tau^1) + \alpha_2^{-1} l_v^2(\tau^2) + (\alpha')^{-1} l_3(\tau^2),
\]

where \( \alpha' \) is the one of \( \alpha_1 \alpha_3/p \) and \( \alpha_1 \) which has smaller valuation.

Proof. Let \( \theta : R^p \to \mathcal{O}_E \) be the map corresponding to \( \rho \), defined as follows. Recall that by Remark 4.1.6 we chose a surjection \( R_{\infty} \to \mathcal{R}^{(p)}_{\infty} \). The composition of this map and the map \( R_{\mathcal{R}^{(p)}_{\infty}} \to \mathcal{O}_E \) defining \( \rho \) gives a map \( \theta : R_{\infty} \to \mathcal{O}_E \). The images of the Hecke operators are given by Proposition 6.2.3. The map \( \theta : R_{\infty} \to \mathcal{O}_E \) factors through \( R^{(p)}_\infty \) if and only if \( M_{\infty}(\otimes_{v'} \tau_{v'}) / \ker(\theta) \) is nonzero. The space \( M_{\infty}(\otimes_{v'} \tau_{v'}) / \ker(\theta) \) is nonzero by Theorem 4.1.4[3], axiom (M), and local-global compatibility at \( p \) as in Theorem 1.2 of [BLGTT12].

By Proposition 6.2.3 the factor at \( v \) of the lattice \( (\otimes_{v'} | p \tau_{v'} \otimes \mathbb{Z}_p E) \cap \pi \) takes the form

\[
\tau^{s_v} + (\alpha_1 \tau^1) - 1 l_v^2(\tau^1) + (\alpha_2 \tau^2) - 1 l_v^2(\tau^2) + (\alpha_3 \tau^3) - 1 l_v^2(\tau^3) + (\alpha')^{-1} l_3(\tau^2) + \alpha_1^{-1} l_3(\tau^3) + \alpha_2^{-1} l_3(\tau^3) + \alpha_3^{-1} l_3(\tau^3) + \alpha'' l_3(\tau^3),
\]

By Proposition 4.1.4[3], the isomorphisms \( \psi^p : R_{\mathcal{R}^{(p)}_{\infty}} \to M_{\infty}(\tau^p) \) in Section 6.2 induce identifications \( \text{Hom}_{\Pi_{L'}} K_{\nu'}(\tau^p \otimes \otimes_{v'} \not\tau_{v'}^p, \pi) \to \mathcal{O}_E \otimes \otimes \mathcal{O}_E \) via \( \otimes \theta \mathcal{O}_E \) and Schikhof duality.

By Lemma 6.2.1 the map

\[
\mathcal{O}_E \cong \text{Hom}_{\Pi_{L'}} K_{\nu'}(\tau^p \otimes \otimes_{v'} \not\tau_{v'}^p, \pi) \to \text{Hom}_{\Pi_{L'}} K_{\nu'}(\tau^{p'} \otimes \otimes_{v'} \not\tau_{v'}^{p'}, \pi) \cong \mathcal{O}_E
\]

induced by \( \tau^{p} \) is multiplication by \( \alpha_1 \alpha_3/p \); and so \( \alpha_3 = \alpha_1 \alpha_3/p \). The values for \( \alpha^2, \alpha'^2, \alpha'^3, \) and \( \alpha'' \) are obtained similarly.

We deduce that \( \alpha'' \) can be taken to be 1 by Proposition 5.3.4. By Proposition 5.3.5 for \( i = 1, 3 \), either the map

\[
(6.2.1) \quad \text{Hom}_{\Pi_{L'}} K_{\nu'}(\tau^i \otimes \otimes_{v'} \not\tau_{v'}^p, \pi) \to \text{Hom}_{\Pi_{L'}} K_{\nu'}(\tau^i \otimes \otimes_{v'} \not\tau_{v'}^{p'}, \pi)
\]
or the map

$$\text{Hom}_\Pi K_{i'}(\tau^t \otimes \phi \tau^t, \pi) \to \text{Hom}_\Pi K_{i'}(\tau^s \otimes \phi \tau^s, \pi)$$

is an isomorphism. As the maps $i^s_0$ and $i^s_1$ factor through $i$, we see that $\alpha'$ and $\alpha''$ can be taken to be one of $\alpha_1$ and $\alpha_1 \alpha_3 / p$. Finally, $\text{val}(\alpha') = \text{val}(\alpha'') \leq \text{val}(\alpha_1), \text{val}(\alpha_1 \alpha_3 / p)$ since the maps in (6.2.1) and (6.2.2) are given by elements of $\mathcal{O}_E$. We conclude that $\alpha'$ and $\alpha''$ can be taken to be the one of $\alpha_1$ and $\alpha_1 \alpha_3 / p$ which has smaller valuation.

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E-mail address: le@math.uchicago.edu

Department of Mathematics, The University of Chicago