FUNCTION CLASSES AND RELATIONAL CONSTRAINTS
STABLE UNDER COMPOSITIONS WITH CLONES

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Abstract. The general Galois theory for functions and relational constraints
over arbitrary sets described in the authors’ previous paper is refined by im-
posing algebraic conditions on relations.

1. Introduction

In this paper we extend the results obtained in [3] by considering more general
closure conditions on classes of functions of several variables, and by restricting
relational constraints to consist of invariant relations. In fact, the Theorems 2.1
and 3.2 in [3] correspond to Theorems 1 and 3 below, respectively, in the particular
case \( C_1 = C_2 = \mathcal{P} \), where \( \mathcal{P} \) denotes the smallest clone containing only projections.

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2. Basic notions and preliminary results

Throughout the paper, let \( A, B, E \) and \( G \) be arbitrary nonempty sets. Given a
nonnegative integer \( m \), the elements of \( A^m \) are viewed as unary functions on the
von Neumann ordinal \( m = \{0, \ldots, m - 1\} \) to \( A \).

A function of several variables on \( A \) to \( B \) (or simply, function on \( A \) to \( B \)) is
a map \( f : A^n \to B \), for some positive integer \( n \) called the arity of \( f \). A class
of functions on \( A \) to \( B \) is a subset \( F \subseteq \bigcup_{n \geq 1} B^{A^n} \). For a fixed arity \( n \), the \( n \) different
projection maps \( a_t \mid t \in n \to a_i \), \( i \in n \), are also called variables. For
\( A = B = \{0, 1\} \), a function on \( A \) to \( B \) is called a Boolean function.

If \( f \) is an \( n \)-ary function on \( B \) to \( E \) and \( g_1, \ldots, g_n \) are all \( m \)-ary functions on \( A \) to
\( B \) then the composition \( f(g_1, \ldots, g_n) \) is an \( m \)-ary function on \( A \) to \( E \), and its value
on \( (a_1, \ldots, a_m) \in A^m \) is \( f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)) \). If \( I \subseteq \bigcup_{n \geq 1} E^{B^n} \)
and \( J \subseteq \bigcup_{n \geq 1} B^{A^n} \) we define the composition of \( I \) with \( J \), denoted \( IJ \), by

\[ IJ = \{ f(g_1, \ldots, g_n) \mid n, m \geq 1, f \text{\ n-ary in } I, g_1, \ldots, g_n \text{\ m-ary in } J \}. \]

If \( I \) is a singleton, \( I = \{ f \} \), then we write \( fJ \) for \( \{ f \}J \). We say that a class
\( I \) of functions of several variables is stable under right (left) composition with \( J \)
if, whenever the composition is well defined, \( IJ \subseteq I \) (\( JI \subseteq I \), respectively).

A clone on \( A \) is a set \( \mathcal{C} \subseteq \bigcup_{n \geq 1} A^{A^n} \) that contains all projections and satisfies
\( \mathcal{C}C \subseteq \mathcal{C} \) (or equivalently, \( \mathcal{C}C = \mathcal{C} \)). Note that if \( \mathcal{J} \) is a clone on \( A \) (on \( B \)) and
\( I \subseteq \bigcup_{n \geq 1} B^{A^n} \), then \( IJ \subseteq I \) if and only if \( IJ = I \) (\( JI \subseteq I \) if and only if
\( JI = I \), respectively). Note that stability under right composition with the clone

\( \mathcal{C} \) is always well-defined. The clone \( \mathcal{C} \) of all functions on \( A \) to \( A \) is the smallest clone containing all functions on \( A \) to \( A \), and \( \mathcal{P} \) denotes the smallest clone containing only projections.
\( \mathcal{P} \subseteq \bigcup_{n \geq 1} A^A \) of all projections on \( A \) subsumes the operations of identification of variables, permutation of variables and addition of inessential variables.

**Associativity Lemma.** Let \( A, B, E \) and \( G \) be arbitrary nonempty sets, and consider function classes \( \mathcal{I} \subseteq \bigcup_{n \geq 1} G^E \), \( \mathcal{J} \subseteq \bigcup_{n \geq 1} E^B \), and \( \mathcal{K} \subseteq \bigcup_{n \geq 1} B^A \). The following hold:

(i) \( (\mathcal{I}, \mathcal{J}) \mathcal{K} \subseteq \mathcal{I}(\mathcal{J} \mathcal{K}) \);

(ii) If \( \mathcal{J} \) is stable under right composition with the clone of projections on \( B \), then \( \mathcal{I}(\mathcal{J}) \mathcal{K} = \mathcal{I}(\mathcal{J} \mathcal{K}) \).

**Proof.** The inclusion (i) is a direct consequence of the definition of function class composition. Property (ii) asserts that the converse inclusion also holds if \( \mathcal{J} \) is stable under right composition with projections. This hypothesis means in particular that all functions obtained from members of \( \mathcal{J} \) by permutation of variables and addition of inessential variables are also in \( \mathcal{J} \). A typical function in \( \mathcal{I}(\mathcal{J} \mathcal{K}) \) is of the form

\[
\text{f}(g_1(h_{11}, \ldots, h_{1m_1}), \ldots, g_n(h_{n1}, \ldots, h_{nm_n}))
\]

where \( f \) is in \( \mathcal{I} \), the \( g_i \)'s are in \( \mathcal{J} \), and the \( h_{ij} \)'s are in \( \mathcal{K} \). By taking appropriate functions \( g_1', \ldots, g_n' \) obtained from \( g_1, \ldots, g_n \) by permutation of variables and addition of inessential variables, the function above can be expressed as

\[
\text{f}(g_1'(h_{11}, \ldots, h_{1m_1}), \ldots, g_n'(h_{n1}, \ldots, h_{nm_n}))
\]

which is easily seen to be in \( (\mathcal{I}, \mathcal{J}) \mathcal{K} \). \( \square \)

Note that statement (ii) of the Associativity Lemma applies, in particular, if \( \mathcal{J} \) is any clone on \( E = B \).

Let \( \mathcal{F} \) be a set of functions on \( A \) to \( B \). If \( \mathcal{P} \) is the clone of all projections on \( A \), then \( \mathcal{FP} = \mathcal{F} \) expresses closure under taking minors in \( \mathcal{F} \), or closure under simple variable substitutions in the terminology of \( \mathcal{F} \). If \( A = B = \{0, 1\} \) and \( \mathcal{L}_{01} \) is the clone (Post class) of constant preserving linear Boolean functions, then \( \mathcal{FL}_{01} = \mathcal{F} \) is equivalent to closure under substitution of triple sums \( x+y+z \) for variables, while \( \mathcal{L}_{01} \mathcal{F} = \mathcal{F} \) is equivalent to closure under taking triple sums of Boolean functions \( f + g + h \) (see \( \mathcal{F} \)).

An \( m \)-ary relation on \( A \) is a subset \( R \) of \( A^m \). Thus the relation \( R \) is a class (set) of unary maps on \( m \) to \( A \). A function \( f \) of several variables on \( A \) to \( A \) is said to preserve \( R \) if \( fR \subseteq R \).

For a class \( \mathcal{F} \subseteq \bigcup_{n \geq 1} A^A \) of functions on \( A \), an \( m \)-ary relation \( R \) on \( A \) is called an \( \mathcal{F} \)-invariant if \( \mathcal{FR} \subseteq R \). In other words, \( R \) is an \( \mathcal{F} \)-invariant if every member of \( \mathcal{F} \) preserves \( R \). If two classes of functions \( \mathcal{F} \) and \( \mathcal{G} \) generate the same clone, then the \( \mathcal{F} \)-invariants are the same as the \( \mathcal{G} \)-invariants. (See Pöschel \( \mathcal{F} \) and \( \mathcal{G} \).)

Observe that we always have \( R \subseteq \mathcal{FR} \) if \( \mathcal{F} \) contains the projections, but we can have \( R \subseteq \mathcal{FR} \) even if \( \mathcal{F} \) contains no projections. (Take the Boolean triple sum \( x_1 + x_2 + x_3 \) as the only member of \( \mathcal{F} \).)

For a clone \( \mathcal{C} \), the intersection of \( m \)-ary \( \mathcal{C} \)-invariants is always a \( \mathcal{C} \)-invariant and it is easy to see that, for an \( m \)-ary relation \( R \), the smallest \( \mathcal{C} \)-invariant containing \( R \) in \( A^m \) is \( \mathcal{CR} \), and it is said to be generated by \( R \). (See \( \mathcal{F} \) and \( \mathcal{G} \), where Pöschel denotes \( \mathcal{CR} \) by \( \Gamma_{\mathcal{C}}(R) \)).
3. Classes of Functions Definable by Constraints Consisting of Invariant Relations

Consider arbitrary nonempty sets $A$ and $B$. An $m$-ary $A$-to-$B$ constraint (or simply, $m$-ary constraint, when the underlying sets are understood from the context) is a couple $(R, S)$ where $R \subseteq A^m$ and $S \subseteq B^m$. The relations $R$ and $S$ are called the antecedent and consequent, respectively, of the relational constraint (Pippenger [6]). Let $C_1$ and $C_2$ be clones on $A$ and $B$, respectively. If $R$ is a $C_1$-invariant and $S$ is a $C_2$-invariant, we say that $(R, S)$ is a $(C_1, C_2)$-constraint. A function $f : A^n \rightarrow B$, $n \geq 1$, is said to satisfy an $m$-ary $A$-to-$B$ constraint $(R, S)$ if $fR \subseteq S$.

The following result generalizes Lemma 1 in [1]:

Lemma 1. Consider arbitrary nonempty sets $A$ and $B$. Let $f$ be a function on $A$ to $B$ and let $C$ be a clone on $A$. If every function in $fC$ satisfies an $A$-to-$B$ constraint $(R, S)$, then $f$ satisfies $(CR, S)$.

Proof. The assumption means that $(fC)R \subseteq S$. By the Associativity Lemma, $(fC)R = f(CR)$, and thus $f(CR) \subseteq S$. \qed

A class $K \subseteq \bigcup_{n \geq 1} B^{A^n}$ of functions on $A$ to $B$ is said to be locally closed if for every function $f$ on $A$ to $B$ the following holds: if every finite restriction of $f$ (i.e., a restriction to a finite subset) coincides with a finite restriction of some member of $K$, then $f$ belongs to $K$.

A class $K \subseteq \bigcup_{n \geq 1} B^{A^n}$ of functions on $A$ to $B$ is said to be definable by a set $T$ of $A$-to-$B$ constraints, if $K$ is the class of all those functions which satisfy every constraint in $T$.

Theorem 1. Consider arbitrary nonempty sets $A$ and $B$ and let $C_1$ and $C_2$ be clones on $A$ and $B$, respectively. For any function class $K \subseteq \bigcup_{n \geq 1} B^{A^n}$ the following conditions are equivalent:

(i) $K$ is locally closed and it is stable both under right composition with $C_1$ and under left composition with $C_2$;

(ii) $K$ is definable by some set of $(C_1, C_2)$-constraints.

Proof. To show that (ii) $\Rightarrow$ (i), assume that $K$ is definable by some set $T$ of $(C_1, C_2)$-constraints. For every $(R, S)$ in $T$, we have $KR \subseteq S$. Since $R$ is a $C_1$-invariant, $KR = K(C_1R)$. By the Associativity Lemma, $K(C_1R) = (KC_1)R$, and therefore $(KC_1)R = KR \subseteq S$. Since this is true for every $(R, S)$ in $T$ we must have $KC_1 \subseteq K$.

For every $(R, S)$ in $T$, we have $KR \subseteq S$, and therefore $C_2(KR) \subseteq C_2S$. By the Associativity Lemma, $(C_2K)R \subseteq C_2(KR) \subseteq C_2S$, and $C_2S = S$ because $S$ is a $C_2$-invariant. Thus $(C_2K)R \subseteq S$ for every $(R, S)$ in $T$, and we must have $C_2K \subseteq K$.

To see that $K$ is locally closed, consider $f \notin K$, say of arity $n \geq 1$, and let $(R, S)$ be an $m$-ary $(C_1, C_2)$-constraint that is satisfied by every function $g$ in $K$ but not satisfied by $f$. Hence for some $a^1, \ldots, a^n$ in $R$, $f(a^1, \ldots, a^n) \notin S$ but $g(a^1, \ldots, a^n) \in S$, for every $n$-ary function $g$ in $K$. Thus the restriction of $f$ to the finite set $\{(a^1(i), \ldots, a^n(i)) : i \in m\}$ does not coincide with that of any member of $K$.

To prove (i) $\Rightarrow$ (ii), we show that for every function $g$ not in $K$, there is a $(C_1, C_2)$-constraint $(R, S)$ which is satisfied by every member of $K$ but not satisfied by $g$. The class $K$ will then be definable by the set $T$ of those $(C_1, C_2)$-constraints that are satisfied by all members of $K$. 

Note that $K$ is a fortiori stable under right composition with the clone containing all projections, that is, $K$ is closed under simple variable substitutions. We may assume that $K$ is non empty. Suppose that $g$ is an $n$-ary function on $A$ to $B$ not in $K$. Since $K$ is locally closed, there is a finite restriction $g_F$ of $g$ to a finite subset $F \subseteq A^n$ such that $g_F$ disagrees with every function in $K$ restricted to $F$. Suppose that $F$ has size $m$, and let $a^1, \ldots, a^n$ be $m$-tuples in $A^m$, such that $F = \{(a^1(i), \ldots, a^n(i)) : i \in m\}$. Define $R_0$ to be the set containing $a^1, \ldots, a^n$, and let $S = \{f(a^1), \ldots, a^n) : f \in K, f \ n$-ary}. Clearly, $(R_0, S)$ is not satisfied by $g$, and it is not difficult to see that every member of $K$ satisfies $(R_0, S)$. As $K$ is stable under left composition with $C_2$, it follows that $S$ is a $C_2$-invariant. Let $R$ be the $C_1$-invariant generated by $R_0$, i.e. $R = C_1R_0$. By Lemma 1, the constraint $(R, S)$ constitutes indeed the desired separating $(C_1, C_2)$-constraint.

This generalizes the characterizations of closed classes of functions given by Pippenger in [3] as well as in [1] and [3] by considering arbitrary underlying sets, possible infinite, and more general closure conditions. We obtain as special cases of Theorem 1 the characterizations given in Theorem 2.1 of [3] and, in the finite case, in Theorem 3.2 of [6], by considering $C_1 = C_2 = \mathcal{P}$, and $C_1 = \mathcal{U}$ and $C_2 = \mathcal{P}$, respectively, where $\mathcal{U}$ is a clone containing only functions having at most one essential variable, and $\mathcal{P}$ is the clone of all projections. Taking $A = B = \{0, 1\}$ and $C_1 = C_2 = \mathcal{L}_{0,1}$, we get the characterization of classes of Boolean functions definable by sets of affine constraints given in [1].

4. Sets of Invariant Constraints Characterized by Functions of Several Variables

In order to discuss sets of constraints determined by functions of several variables, we need to recall the following concepts and constructions introduced in [6] and [3].

Given maps $f : A \to B$ and $g : C \to D$, their composition $g \circ f$ is defined only if $B = C$. Removing this restriction, the concatenation of $f$ and $g$, denoted simply $gf$, is defined as the map with domain $f^{-1}[B \cap C]$ and codomain $D$ given by $(gf)(a) = g(f(a))$ for all $a \in f^{-1}[B \cap C]$. Clearly, if $B = C$ then $gf = g \circ f$, thus concatenation subsumes and extends functional composition.

Let $(g_i)_{i \in I}$ be a family of maps, $g_i : A_i \to B_i$ such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$. The (piecewise) sum of the family $(g_i)_{i \in I}$, denoted $\Sigma_{i \in I} g_i$, is the map from $\cup_{i \in I} A_i$ to $\cup_{i \in I} B_i$ whose restriction to each $A_i$ agrees with $g_i$. If $I$ is finite, we may use the infix $+$ notation.

For $B \subseteq A$, $\iota_{AB}$ denotes the canonical injection (inclusion map) from $B$ to $A$. Note that the restriction $f|B$ of any map $f : A \to C$ to the subset $B$ is given by $f|B$ is the concatenation $f\iota_{AB}$.

Let $=_A$ be the equality relation on a set $A$. The binary $A$-to-$B$ equality constraint is simply $(=_A, =_B)$. A constraint $(R, S)$ is called the empty constraint if both antecedent and consequent are empty. For every $m \geq 1$, the constraints $(A^m, B^m)$ are said to be trivial. Note that every function on $A$ to $B$ satisfies each of these constraints.

A constraint $(R, S)$ is said to be a relaxation of a constraint $(R_0, S_0)$ if $R \subseteq R_0$ and $S \supseteq S_0$. Given a non-empty family of constraints $(R, S_j)_{j \in J}$ of the same arity (and antecedent), the constraint $(R, \cap_{j \in J} S_j)$ is said to be obtained from $(R, S_j)_{j \in J}$ by intersecting consequents.
Let $m$ and $n$ be positive integers (viewed as ordinals, i.e., $m = \{0, \ldots, m - 1\}$). Let $h : n \rightarrow m \cup V$ where $V$ is an arbitrary set of symbols disjoint from the ordinals called existentially quantified indeterminate indices, or simply indeterminates, and $\sigma : V \rightarrow A$ any map called a Skolem map. Then each $m$-tuple $a \in A^m$, being a map $a : m \rightarrow A$, gives rise to an $n$-tuple $(a + \sigma)h \in A^n$.

Let $H = (h_j)_{j \in J}$ be a non-empty family of maps $h_j : n_j \rightarrow m \cup V$, where each $n_j$ is a positive integer (recall $n_j = \{0, \ldots, n_j - 1\}$). Then $H$ is called a minor formation scheme with target $m$, indeterminate set $V$ and source family $(n_j)_{j \in J}$. Let $(R_j)_{j \in J}$ be a family of relations (of various arities) on the same set $A$, each $R_j$ of arity $n_j$, and let $R$ be an $m$-ary relation on $A$. We say that $R$ is a restrictive conjunctive minor of the family $(R_j)_{j \in J}$ via $H$, or simply a restrictive conjunctive minor of the family $(R_j)_{j \in J}$, if for every $m$-tuple $a$ in $A^m$, the condition $R(a)$ implies that there is a Skolem map $\sigma : V \rightarrow A$ such that, for all $j$ in $J$, we have $R_j[(a + \sigma)h_j]$. On the other hand, if for every $m$-tuple $a$ in $A^m$, the condition $R(a)$ holds whenever there is a Skolem map $\sigma : V \rightarrow A$ such that, for all $j$ in $J$, we have $R_j[(a + \sigma)h_j]$, then we say that $R$ is an extensive conjunctive minor of the family $(R_j)_{j \in J}$ via $H$, or simply an extensive conjunctive minor of the family $(R_j)_{j \in J}$. If $R$ is both a restrictive conjunctive minor and an extensive conjunctive minor of the family $(R_j)_{j \in J}$ via $H$, then $R$ is said to be a tight conjunctive minor of the family $(R_j)_{j \in J}$ via $H$, or tight conjunctive minor of the family. Note that given a scheme $H$ and a family $(R_j)_{j \in J}$, there is a unique tight conjunctive minor of the family $(R_j)_{j \in J}$ via $H$.

If $(R_j, S_j)_{j \in J}$ is a family of $A$-to-$B$ constraints (of various arities) and $(R, S)$ is an $A$-to-$B$ constraint such that for a scheme $H$

(i) $R$ is a restrictive conjunctive minor of $(R_j)_{j \in J}$ via $H$,
(ii) $S$ is an extensive conjunctive minor of $(S_j)_{j \in J}$ via $H$,

then $(R, S)$ is said to be a conjunctive minor of the family $(R_j, S_j)_{j \in J}$ via $H$, or simply a conjunctive minor of the family of constraints.

If both $R$ and $S$ are tight conjunctive minors of the respective families via $H$, the constraint $(R, S)$ is said to be a tight conjunctive minor of the family $(R_j, S_j)_{j \in J}$ via $H$, or simply a tight conjunctive minor of the family of constraints. Note that given a scheme $H$ and a family $(R_j, S_j)_{j \in J}$, there is a unique tight conjunctive minor of the family via the scheme $H$.

We say that a class $T$ of relational constraints is closed under formation of conjunctive minors if whenever every member of the nonempty family $(R_j, S_j)_{j \in J}$ of constraints is in $T$, all conjunctive minors of the family $(R_j, S_j)_{j \in J}$ are also in $T$.

The following lemma was first obtained in [3] and it shows that closure under formation of conjunctive minors is a necessary condition to describe those sets of constraints determined by functions of several variables.

**Lemma 2.** Let $(R, S)$ be a conjunctive minor of a non-empty family $(R_j, S_j)_{j \in J}$ of $A$-to-$B$ constraints. If $f : A^n \rightarrow B$ satisfies every $(R_j, S_j)$ then $f$ satisfies $(R, S)$.

A set $T$ of relational constraints is said to be locally closed if for every $A$-to-$B$ constraint $(R, S)$ the following holds: if every relaxation of $(R, S)$ with finite antecedent coincides with some member of $T$, then $(R, S)$ belongs to $T$. The following result was shown in [3] (see Theorem 3.2) and it provides necessary and
sufficient conditions for a set of constraints to be determined by functions of several variables.

**Theorem 2.** Consider arbitrary non-empty sets $A$ and $B$. Let $T$ be a set of $A$-to-$B$ relational constraints. Then the following are equivalent:

(i) $T$ is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors;

(ii) There is a set of functions on $A$ to $B$ which satisfy exactly those constraints in $T$.

Let $C_1$ and $C_2$ be clones on arbitrary nonempty sets $A$ and $B$, respectively. Among all $A$-to-$B$ constraints, observe that the empty constraint and the equality constraint are $(C_1, C_2)$-constraints.

The following Lemma is essentially a restatement, in a variant form, of the closure condition given by Szabó in [10, 11] on the set of relations preserved by a clone of functions. We indicate a proof via Lemma 2 above.

**Lemma 3.** (Szabó) Let $C$ be a clone on an arbitrary nonempty set $A$. If $R$ is a tight conjunctive minor of a nonempty family $(R_j)_{j \in J}$ of $C$-invariants, then $R$ is a $C$-invariant.

**Proof.** Let $R$ be a tight conjunctive minor of a nonempty family $(R_j)_{j \in J}$ of $C$-invariants. We have to prove that every function in $C$ preserves $R$ or, equivalently, that every function in $C$ satisfies the $A$-to-$A$ constraint $(R, R)$. Since $(R_j)_{j \in J}$ is a nonempty family of $C$-invariants, every function in $C$ preserves every member of the family $(R_j)_{j \in J}$, that is, every function in $C$ satisfies every member of the family $(R_j, R_j)_{j \in J}$ of $A$-to-$A$ constraints. From Lemma 2 above, it follows that every member of $C$ satisfies $(R, R)$, that is, $R$ is a $C$-invariant. 

Thus every tight conjunctive minor $(R, S)$ of a nonempty family $(R_j, S_j)_{j \in J}$ of $(C_1, C_2)$-constraints is a $(C_1, C_2)$-constraint. However, not all relaxations of $(C_1, C_2)$-constraints are $(C_1, C_2)$-constraints and so not all conjunctive minors of a nonempty family $(R_j, S_j)_{j \in J}$ of $(C_1, C_2)$-constraints are $(C_1, C_2)$-constraints. A relaxation $(R, S)$ of an $A$-to-$B$ constraint $(R_0, S_0)$ is called a $(C_1, C_2)$-relaxation of $(R_0, S_0)$ if $(R, S)$ is a $(C_1, C_2)$-constraint. Similarly, a conjunctive minor $(R, S)$ of a nonempty family $(R_j, S_j)_{j \in J}$ of $A$-to-$B$ constraints is called a $(C_1, C_2)$-conjunctive minor of the family $(R_j, S_j)_{j \in J}$, if $(R, S)$ is a $(C_1, C_2)$-constraint.

A set $T$ of $(C_1, C_2)$-constraints is said to be closed under formation of $(C_1, C_2)$-conjunctive minors if whenever every member of the nonempty family $(R_j, S_j)_{j \in J}$ of constraints is in $T$, all $(C_1, C_2)$-conjunctive minors of the family $(R_j, S_j)_{j \in J}$ are also in $T$. The following result extends Lemma 1 in [2].

**Lemma 4.** Let $C_1$ and $C_2$ be clones on arbitrary nonempty sets $A$ and $B$, respectively. Let $T_0$ be a set of $(C_1, C_2)$-constraints, closed under $(C_1, C_2)$-relaxations. Define $T$ to be the set of all relaxations of the various constraints in $T_0$. Then $T_0$ is the set of $(C_1, C_2)$-constraints which are in $T$, and the following are equivalent:

(a) $T_0$ is closed under formation of $(C_1, C_2)$-conjunctive minors;

(b) $T$ is closed under taking conjunctive minors.

**Proof.** Clearly, the first claim holds, and it is easy to see that (b) ⇒ (a). To prove implication (a) ⇒ (b), assume (a). Let $(R, S)$ be a conjunctive minor of a
nonempty family \((R_j, S_j)_{j \in J}\) of A-to-B constraints in \(T\) via a scheme \(H = (h_j)_{j \in J}\), \(h_j : n_j \to m \cup V\). We have to prove that \((R, S) \in T\).

Since for every \(j \in J\) \((R_j, S_j) \in T\), there is a nonempty family \((R^0_j, S^0_j)_{j \in J}\) of \((C_1, C_2)\)-constraints in \(T_0\) such that, for each \(j \in J\), \((R_j, S_j)\) is a relaxation of \((R^0_j, S^0_j)\). So let \((R_0, S_0)\) be the tight conjunctive minor of the family \((R^0_j, S^0_j)_{j \in J}\) via the scheme \(H\). From Lemma 2, it follows that \(R_0\) is a \(C_1\)-invariant and \(S_0\) a \(C_2\)-invariant, and since \(T_0\) is closed under formation of \((C_1, C_2)\)-conjunctive minors, we have \((R_0, S_0) \in T_0\).

Let us prove that \((R, S)\) is a relaxation of \((R_0, S_0)\) and, thus, that \((R, S) \in T\). Since \(R\) is a restrictive conjunctive minor of the family \((R_j)_{j \in J}\) via the scheme \(H = (h_j)_{j \in J}\), we have that for every \(m\)-tuple \(a\) in \(R\) there is a Skolem map \(\sigma : V \to A\) such that, for all \(j \in J\), the \(n_j\)-tuple \((a + \sigma)h_j\) is in \(R_j\). Since \(R_j \subseteq R^0_j\) for every \(j \in J\), it follows that \((a + \sigma)h_j\) is in \(R^0_j\) for every \(j \in J\). Thus \(a\) is in \(R_0\) and we conclude \(R \subseteq R_0\).

By analogous reasoning one can easily verify that \(b\) is in \(S\) whenever \(b\) is in \(S_0\), i.e. that \(S \supseteq S_0\). Thus \((R, S)\) is a relaxation of \((R_0, S_0)\) and so \((R, S) \in T\), and the proof of \((a)\) is complete. 

Let \(T_0\) be a set of \((C_1, C_2)\)-constraints. We say that \(T_0\) is \((C_1, C_2)\)-locally closed if the set \(T\) of all relaxations of the various constraints in \(T_0\) is locally closed.

We can now extend Theorem 2 above to sets of \((C_1, C_2)\)-constraints.

**Theorem 3.** Let \(C_1\) and \(C_2\) be clones on arbitrary nonempty sets \(A\) and \(B\), respectively, and let \(T_0\) be a set of \((C_1, C_2)\)-constraints. Then the following are equivalent:

(i) \(T_0\) is \((C_1, C_2)\)-locally closed, contains the binary equality constraint, the empty constraint, and it is closed under formation of \((C_1, C_2)\)-conjunctive minors;

(ii) There is a set of functions on \(A\) to \(B\) which satisfy exactly those \((C_1, C_2)\)-constraints that are in \(T_0\).

**Proof.** To prove implication \((ii) \Rightarrow (i)\), assume \((ii)\). Let \(\mathcal{K}\) be the set of all functions satisfying every constraint in \(T_0\). Note that \(T_0\) is closed under \((C_1, C_2)\)-relaxations. By Theorem 1, we have \(C_2\mathcal{K} = \mathcal{K}\), and \(C_1\mathcal{K} = \mathcal{K}\). We may assume that \(\mathcal{K} \neq \emptyset\). Let \(\mathcal{T}\) be the set of all those constraints (not necessarily \((C_1, C_2)\)-constraints) satisfied by every function in \(\mathcal{K}\). Observe that \(T_0\) is the set of all \((C_1, C_2)\)-constraints which are in \(\mathcal{T}\). We show that \(\mathcal{T}\) is the set of all relaxations in \(T_0\).

Let \((R, S)\) be a constraint in \(\mathcal{T}\). From the definition of \(\mathcal{T}\), it follows that \(KR \subseteq S\). Note that \(\mathcal{K}\) is stable under right composition with the clone of projections on \(A\), because \(C_1\mathcal{K} = \mathcal{K}\). Thus by the Associativity Lemma it follows that \(C_2(KR) = (C_2K)R\). Since \(C_2\mathcal{K} = \mathcal{K}\), we have that \(C_2(KR) = KR\), i.e. \(KR\) is a \(C_2\)-invariant. Also, again because \(C_1\mathcal{K} = \mathcal{K}\), by Lemma 1 we conclude that every function in \(\mathcal{K}\) satisfies \((C_1R, KR)\). Clearly, \((C_1R, KR)\) is a \((C_1, C_2)\)-constraint, therefore it belongs to \(T_0\). Thus every constraint \((R, S)\) in \(\mathcal{T}\) is a relaxation of a member of \(T_0\), namely, a relaxation of \((C_1R, KR)\).

By Theorem 2 above, we have that \(\mathcal{T}\) is locally closed and contains the binary equality constraint, the empty constraint, and it is closed under formation of conjunctive minors. Since the binary equality constraint and the empty constraint are \((C_1, C_2)\)-constraints, it follows from Lemma 4 that \((i)\) holds.
To prove implication (i) ⇒ (ii), it is enough to show that for every \((C_1, C_2)\)-constraint \((R, S)\) not in \(T_0\), there is a function \(g\) which satisfies every constraint in \(T_0\), but does not satisfy \((R, S)\).

Let \(T\) be the set of relaxations of the various \((C_1, C_2)\)-constraints in \(T_0\). Observe that \((R, S) \not\in T\), otherwise \((R, S)\) would be a \((C_1, C_2)\)-relaxation of some \((C_1, C_2)\)-constraint in \(T_0\), contradicting the fact implied by (i) that \(T_0\) is closed under taking \((C_1, C_2)\)-relaxations. Clearly, \(T\) is locally closed, contains the binary equality constraint, and the empty constraint. From Lemma 4 it follows that \(T\) is closed under taking conjunctive minors. By Theorem 2, there is a function \(g\) which does not satisfy \((R, S)\) but satisfies every constraint in \(T\) and so, in particular, \(g\) satisfies every constraint in \(T_0\). Thus we have (i) ⇒ (ii).

\(\square\)

Theorem 3 generalizes the characterizations of closed classes of constraints given in Pippenger \cite{pippenger2002} and also in \cite{couceiro2003} as well as \cite{couceiro2005} by considering both arbitrary, possibly infinite, underlying sets, and more general closure conditions on classes of relational constraints.

Theorems \cite{couceiro2003} and \cite{couceiro2005} may also be viewed as analogues, with constraints instead of relations, of the characterization given by Pöschel, as part of Theorem 3.2 in \cite{poschel1999}, of the closed sets in a class of Galois connections between operations and relations of a prescribed type on a set \(A\).

REFERENCES

[1] M. Couceiro, S. Foldes. Definability of Boolean Function Classes by Linear Equations over GF(2), Discrete Applied Mathematics, 142 (2004) 29–34.
[2] M. Couceiro, S. Foldes. On Affine Constraints Satisfied By Boolean Functions, Rutcor Research Report 3-2003, Rutgers University, \url{http://rutcor.rutgers.edu/~rrr/}
[3] M. Couceiro, S. Foldes. On Closed Sets of Relational Constraints and Classes of Functions Closed under Variable Substitutions, Algebra Universalis, 54 (2005) 149–165.
[4] O. Ekin, S. Foldes, P.L. Hammer, L. Hellerstein, Equational Characterizations of Boolean Functions Classes, Discrete Mathematics, 211 (2000) 27–51.
[5] D. Geiger, Closed Systems of Functions and Predicates, Pacific Journal of Mathematics, 27 (1968) 95–100.
[6] N. Pippenger, Galois Theory for Minors of Finite Functions, Discrete Mathematics, 254 (2002) 405–419.
[7] R. Pöschel, Concrete Representation of Algebraic Structures and a General Galois Theory, Contributions to General Algebra, Proceedings Klagenfurt Conference, May 25-28 (1978) 249–272. Verlag J. Heyn, Klagenfurt, Austria 1979.
[8] R. Pöschel, A General Galois Theory for Operations and Relations and Concrete Characterization of Related Algebraic Structures, Report R-01/80, Zentralinstitut für Math. und Mech., Berlin 1980.
[9] R. Pöschel, Galois Connections for Operations and Relations, In Galois connections and applications, K. Denecke, M. Erné, S.L. Wismath (eds.), Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht 2004.
[10] L. Szabó, Concrete Representation of Related Structures of Universal Algebras, Acta Sci. Math. (Szeged), 40 (1978) 175–184.