Manifestly covariant canonical quantization I: the free scalar field

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Abstract

Classical physics is reformulated as a constrained Hamiltonian system in the history phase space. Dynamics, i.e. the Euler-Lagrange equations, play the role of first-class constraints. This allows us to apply standard methods from the theory of constrained Hamiltonian systems, e.g. Dirac brackets and cohomological methods. In analogy with BRST quantization, we quantize in the history phase space first and impose dynamics afterwards. To obtain a truly covariant formulation, all fields must be expanded in a Taylor series around the observer’s trajectory, which acquires the status of a quantized physical field. The formalism is applied to the harmonic oscillator and to the free scalar field. Standard results are recovered, but only in the approximation that the observer’s trajectory is treated as a classical curve.

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1 Introduction

The path-integral formulation of quantum mechanics has one major advantage over the Hamiltonian formalism, namely that it preserves manifest covariance. However, the canonical formalism may justly be considered more fundamental. It would therefore be desirable to have a Hamiltonian formulation of quantum mechanics which maintains manifest covariance. This paper initiates a program towards such a formulation.

The first observation is the well-known fact that phase space is really a covariant concept; it is the space of solutions to the classical equations of motion, i.e. the space of histories. A phase space point \((q, p)\) corresponds to the history \((q(t), p(t))\), such that \((q(0), p(0)) = (q, p)\). Since \(p(t)\) is completely determined from \(q(t)\) and Hamilton’s equations, one usually drops reference to \(p(t)\) and define the covariant phase space \(\Sigma\) to be the space of histories \(q(t)\) that solve the Euler-Lagrange (EL) equations.

We are not interested so much in \(\Sigma\) itself, but rather in the space of functions over \(\Sigma\), \(C(\Sigma)\), which will become our Hilbert space after quantization. This space can be identified with the space of functions over history space \(Q\), which is spanned by arbitrary trajectories \(q(t)\), modulo the ideal \(N\) generated by the EL equations: \(C(\Sigma) = C(Q)/N\). It is a standard result in the antifield formalism that \(C(\Sigma)\) can described as a resolution of a certain differential complex \([1]\). In the absence of gauge symmetries, this complex is known as the Koszul-Tate (KT) complex.

Alas, the antifield formalism is not suited for canonical quantization, although this is of course no problem if we only want to do path integrals. After adding antifields, the history space \(Q\) is replaced by an extended history space \(Q^*\). We can define an antibracket in \(Q^*\), but in order to do canonical quantization we need an honest Poisson bracket. To this end, we introduce canonical momenta conjugate to the history and its antifield, and obtain an even larger space \(P^*\), which may be thought of as the phase space corresponding to the extended history space \(Q^*\). It turns out to be possible to define a KT complex also in \(C(P^*)\). The KT differential can now be written as a bracket, \(\delta F = [Q, F]\). Under a technical assumption, all momenta are killed in cohomology, and we obtain a different description of \(C(\Sigma)\).

Since \(C(P^*)\) is naturally equipped with a Poisson bracket, we can now do canonical quantization on \(P^*\) by replacing Poisson brackets with commutators. However, we also need to represent the graded Heisenberg algebra on a Hilbert space, in a way which makes the Hamiltonian bounded from below. At this step we must single out a privileged time direction and give up manifest covariance. The Hamiltonian is simply defined as the generator
of rigid time translations. Since it commutes with the KT operator $Q$ even after normal ordering, the Hamiltonian is well defined in cohomology. This process defines a non-covariant quantization in the covariant phase space.

Covariant quantization requires a covariant definition of the Hamiltonian. To this end, we introduce the observer’s trajectory in spacetime, $q^\mu(t)$. All fields and antifields are expanded in a Taylor series around $q^\mu(t)$, before canonical momenta are introduced and the KT complex is constructed. The parameter $t$ has no physical significance, so the generator of rigid $t$ translations cannot serve as a Hamiltonian; indeed, the fields in $\Sigma$ are $t$-independent. Instead the Hamiltonian is the operator which translates the fields relative to the observer. This definition turns out to agree with the usual Hamiltonian in the limit that $q^\mu(t)$ can be regarded as a classical variable.

Hence the key idea is to regard the physical phase space as the constraint surface in the history phase space $\mathcal{P}$, with the EL equations playing the role of a first-class constraint. One way to treat a constrained system is to introduce a gauge-fixing condition, which relates momenta to velocities, replace Poisson by Dirac brackets, and eliminate the constraints (dynamics) prior to quantization. This strategy is investigated in Section 2.

In the following two sections, we describe how to construct a cohomological model for the covariant space space, and how to do non-covariant and covariant canonical quantization, respectively, before imposing dynamics. In the next three sections, the formalism is applied to the harmonic oscillator and the free scalar field, in non-covariant and covariant form, respectively. The result deviates from conventional canonical quantization in two respects, one technical and one substantial.

1. The momenta are only eliminated in cohomology provided that the Hessian, i.e. the second functional derivative matrix of the action, is non-singular. This technical assumption is not true for the harmonic oscillator. I argue that one can avoid this problem, which is present in the standard antifield treatment as well, by adding a small perturbation to the action, making the Hessian non-singular. In the limit that the perturbation vanishes, the non-covariant quantization yields the correct Hilbert space. Without this trick, the Hilbert space describes quanta of the right energy, but there would be too many types of quanta.

2. The energy of a plane wave is the projection of its momentum $k_\mu$ along the observer’s velocity, i.e. $\hat{q}^\mu(t)k_\mu$. In general, this is a quantum operator which creates an observer quantum from the vacuum. This is a $c$-number (and the correct $c$-number) only if the observer can be regarded as a classical object, i.e. if the reference state is a mixed state with many
observer quanta. But this is only an approximation. In a fundamental theory, everything must be quantized, including the observer’s trajectory. Classical observation is merely a special limit of quantum interaction. It is therefore encouraging that treating canonical quantization in a manifestly covariant way forces us to introduce a quantized observer trajectory.

In section 8, the algebra of reparametrizations of the observer’s trajectory is discussed. This plays no role if we explicitly break reparametrization freedom by requiring the observer to move along a straight line in Minkowski space. In applications to general-covariant theories the observer will move along a geodesic instead, keeping reparametrization invariance intact. The reparametrization algebra will acquire quantum corrections, making it into a Virasoro algebra. A well-defined central extension can only be constructed in three dimensions and with twice as many fermionic as bosonic fields.

The last section contains a discussion of the results and an outlook for future work.

2 Phase space as a constraint surface in history phase space

Consider a classical dynamical system with action $S$ and degrees of freedom $\phi^\alpha$. As is customary in the antifield literature, we use an abbreviated notation where the index $\alpha$ stands for both discrete indices and spacetime coordinates. Dynamics is governed by the Euler-Lagrange (EL) equations,

$$\mathcal{E}_\alpha = \partial_\alpha S \equiv \frac{\delta S}{\delta \phi^\alpha} = 0. \quad (2.1)$$

An important role is also played by the Hessian, i.e. the symmetric second functional-derivative matrix

$$K_{\alpha\beta} = K_{\beta\alpha} = \partial_\beta \mathcal{E}_\alpha \equiv \frac{\delta \mathcal{E}_\alpha}{\delta \phi^\beta} = \frac{\delta^2 S}{\delta \phi^\alpha \delta \phi^\beta}. \quad (2.2)$$

The Hessian is assumed non-singular, so it has an inverse $M^{\alpha\beta}$ satisfying

$$K_{\beta\gamma} M^{\gamma\alpha} = M^{\alpha\gamma} K_{\gamma\beta} = \delta^\alpha_\beta. \quad (2.3)$$

In the Hamiltonian formulation we split the index $\alpha = (i, t)$ and consider the phase space $\Sigma$ with coordinates $(\phi^i, \pi_j)$. $\Sigma$ is equipped with a Poisson bracket satisfying the commutation relations

$$[\pi_j, \phi^i] = \delta^i_j, \quad [\phi^i, \phi^j] = [\pi_i, \pi_j] = 0. \quad (2.4)$$
The time evolution of the system is described by a curve \((\phi^i(t), \pi_j(t)) \in \Sigma\), governed by Hamilton’s equations,

\[
\frac{d\phi^i(t)}{dt} = [\phi^i(t), H], \quad \frac{d\pi_j(t)}{dt} = [\pi_j(t), H].
\] (2.5)

The Hamiltonian formalism breaks manifest covariance, due to the special role played by the time coordinate. However, the phase space is a covariant concept; it is the space of solutions to the classical equations of motions \((\phi^i(t), \pi_j(t)) = (\phi^\alpha, \pi_\alpha)\). The standard way to coordinatize such a curve is to use the initial values \((\phi^i, \pi_j) = (\phi^i(0), \pi_j(0))\), but physics does not depend on this particular way to put coordinates on \(\Sigma\). Since \(\pi_\alpha\) is completely determined from \(\phi^\alpha\) and Hamilton’s equations, we may drop reference to \(\pi_\alpha\) and define the covariant phase space \(\Sigma\) to be the space of histories \(\phi^\alpha\) modulo the EL equations.

Let \(Q\) be the space of histories \(\phi^\alpha\). For each history \(\phi^\alpha\) we introduce the canonical momentum \(\pi_\alpha = \delta/\delta \phi^\alpha\), and define \(P\) to be the space of histories \((\phi^\alpha, \pi_\beta)\). \(P\) has a natural symplectic structure, given by the canonical commutation relations

\[
[\pi_\beta, \phi^\alpha] = \delta^\alpha_\beta, \quad [\phi^\alpha, \phi^\beta] = [\pi_\alpha, \pi_\beta] = 0.
\] (2.6)

Spelled out in detail, it reads

\[
[\pi_j(t), \phi^i(t')] = \delta^i_j \delta(t - t'), \quad [\phi^i(t), \phi^j(t')] = [\pi_i(t), \pi_j(t')] = 0.
\] (2.7)

Observe that this Heisenberg algebra holds in the history phase space, not in the physical phase space to be constructed. In particular, the factor \(\delta(t - t')\) means that the Poisson brackets are not only defined at equal times.

This suggests that we may regard the EL equation \(E_\alpha = 0\) as a constraint in \(P\), and use standard methods for constrained Hamiltonian systems to recover the physical phase space \(\Sigma\). However, some care must be exercised. The EL equations do not determine a history uniquely, but only up to initial conditions. We therefore redefine \(\phi^\alpha \rightarrow (\phi^\alpha, \dot{\phi}^\alpha)\), where \(\dot{\phi}^i\) are the initial conditions (typically, \(\phi(0)\) and \(\dot{\phi}(0)\)), and \(\phi^\alpha\) are the remaining variables \((\phi(t)\) for \(t > 0))\).

We thus impose dynamics as a constraint \(E_\alpha \approx 0\) in \(P\). As is customary, \(\approx\) denotes weak equality, i.e. equality when all constraints have been taken into account. This constraint is first class; \([E_\alpha, E_\beta] = 0 \approx 0\) since \(E_\alpha\) only depends on the \(\phi\)’s but not on the \(\pi\)’s. To make the constraint second
class, we must introduce more constraints which make the Poisson bracket matrix non-singular. We take \( \pi_\alpha \approx C_\alpha \), where \( C_\alpha(\phi) \) is some function which is independent of \( \pi_\alpha \). The precise choice is not important; it will affect the definition of canonical momenta but dynamics for the fields is always given by the same EL equation. There is no field equations for the initial conditions, but their canonical momenta are constrained by \( \pi_i \approx C_i \).

A set of constraints \( \chi_A \approx 0 \) are second class if the Poisson bracket matrix \( \Delta_{AB} = [\chi_A, \chi_B] \) is invertible; denote the inverse by \( \Delta^{AB} \). The Dirac bracket

\[
[F, G]_\ast = [F, G] - [F, \chi_A]\Delta^{AB}[\chi_B, G]
\]

defines a new Lie bracket which is compatible with the constraints: \( [F, \chi_A]_\ast = 0 \) for every \( F \in C(\mathcal{P}) \). Since this is a strong equality, i.e. it holds throughout \( C(\mathcal{P}) \) and not only modulo constraints, we can now solve the constraints \( \chi_A = 0 \). The Dirac bracket becomes the Poisson bracket on the reduced phase space.

The second-class constraints in the history phase space are

\[
\begin{align*}
\mathcal{E}_\alpha & \approx 0, \\
\pi_\alpha - C_\alpha & \approx 0, \\
\pi_i - C_i & \approx 0.
\end{align*}
\]

The Poisson-bracket matrix reads

\[
\Delta_{AB} = \left[ \begin{array}{ccc}
\mathcal{E}_\alpha & \pi_\beta - C_\beta & \pi_j - C_j \\
\pi_\alpha - C_\alpha & 0 & 0 \\
\pi_i - C_i & 0 & 0
\end{array} \right]
\]

(2.10)

Assuming that \( \partial_i \mathcal{E}_\beta = \partial_i C_\beta = 0 \), it has the inverse

\[
\Delta^{AB} = \left( \begin{array}{ccc}
\Gamma^{\alpha\beta} & M^{\alpha\beta} & 0 \\
-M^{\alpha\beta} & 0 & 0 \\
0 & 0 & \Omega^{ij}
\end{array} \right),
\]

(2.11)

where \( M^{\alpha\beta} \) is the inverse of the Hessian, \( \Omega^{ij} \) is the inverse of \( \Omega_{ij} = \partial_j C_i - \partial_i C_j \), and \( \Gamma^{\alpha\beta} = M^{\alpha\gamma}(\partial_\beta C_\gamma - \partial_\gamma C_\beta)M^{\delta\beta} = M^{\alpha\gamma}\Omega_{\gamma\delta}M^{\delta\beta} \).

One easily verifies that the Dirac brackets commute with the constraints,

\[
[F, \mathcal{E}_\alpha]_\ast = [F, \pi_\alpha - C_\alpha]_\ast = [F, \pi_i - C_i]_\ast = 0,
\]

(2.12)
for every $F \in C(\mathcal{P})$. We can thus solve the constraints for $\phi^i$, $\pi_\alpha$ and $\pi_i$. The constraint surface $\Sigma$ is identified with the physical phase space. It has coordinates $\phi^i$ and the Poisson bracket is given by

$$[\phi^i, \phi^j] = \Omega^{ij}.$$  

(2.13)

Alternatively, we can solve the constraint $\pi_i - C_i \approx 0$ for $\phi^i$, and describe $\Sigma$ in terms of coordinates $\pi_i$ with Poisson bracket

$$[\pi_i, \pi_j] = \partial_i C_k \Omega^{k\ell} \partial_j C_{\ell}.$$  

(2.14)

The Hamiltonian in $\mathcal{P}$ is simply the generator of rigid time translations along the trajectories. When we pass to Dirac brackets, it acquires knowledge about the dynamics of the system.

This section illustrates how the physical phase space $\Sigma$ can be obtained from the history phase space $\mathcal{P}$:

1. Introduce the Euler-Lagrange equations as first class constraints.

2. Introduce gauge conditions which relate momenta to velocities, making the constraints second class.

3. Pass to Dirac bracket and eliminate the constraints.

We will not pursue this strategy further, because relating momenta to velocities necessarily breaks covariance. In the next section we introduce the analogue of the BRST formalism, where $\Sigma$ is constructed by cohomological methods, and no gauge-fixing condition is necessary at any stage.

3 Koszul-Tate cohomology

In the previous section we described dynamics as a constraint in the history phase space, and recovered the physical phase space by solving this constraint. A more elegant way to treat constrained systems is the BRST approach. Here one never solves any constraint. Instead, a nilpotent BRST operator is introduced in an extended phase space including ghosts, and the physical phase space is identified with the zeroth cohomology group of the BRST complex. We saw in the previous section that the EL equations may be viewed as a first-class constraint in the history phase space. Although we assume that there are no gauge symmetries, the situation is nevertheless very similar here, with the EL equation playing the role of a gauge symmetry. There should thus be an analogue of the BRST formalism in the history
phase space. In particular, we want to construct a differential complex in $C(\mathcal{P})$, which is a resolution of the space of functions over the true phase space, $C(\Sigma) = C(\mathcal{Q})/\mathcal{N}$.

Let us recall how this is done in the antifield formalism \[1\]. Introduce an antifield $\phi^{*\alpha}$ for each EL equation $E_\alpha = 0$, and replace the space of $\phi$-histories $\mathcal{Q}$ by the extended history space $\mathcal{Q}^*$, spanned by both $\phi$ and $\phi^*$. In $\mathcal{Q}^*$ we define the Koszul-Tate (KT) differential $\delta$ by

$$
\delta\phi^\alpha = 0, \quad \delta\phi^{*\alpha} = E_\alpha.
\tag{3.1}
$$

One checks that $\delta$ is nilpotent, $\delta^2 = 0$. Let the antifield number $\text{afn} \phi^\alpha = 0$, $\text{afn} \phi^{*\alpha} = 1$. The KT differential clearly has antifield number $\text{afn} \delta = -1$.

The space $C(\mathcal{Q}^*)$ decomposes into subspaces $C^k(\mathcal{Q}^*)$ of fixed antifield number

$$
C(\mathcal{Q}^*) = \sum_{k=0}^{\infty} C^k(\mathcal{Q}^*)
\tag{3.2}
$$

The KT complex is

$$
0 \leftarrow^\delta C^0 \leftarrow^\delta C^1 \leftarrow^\delta C^2 \leftarrow^\delta \ldots
\tag{3.3}
$$

The cohomology spaces are defined as usual by $H^\bullet_{cl}(\delta) = \ker \delta/\text{im} \delta$, i.e. $H^k_{cl}(\delta) = (\ker \delta)_k/(\text{im} \delta)_k$, where the subscript $cl$ indicates that we deal with a classical phase space. It is easy to see that

$$
\begin{align*}
(\ker \delta)_0 &= C(\mathcal{Q}), \\
(\text{im} \delta)_0 &= C(\mathcal{Q})E_\alpha \equiv \mathcal{N}.
\end{align*}
\tag{3.4}
$$

Thus $H^0_{cl}(\delta) = C(\mathcal{Q})/\mathcal{N} = C(\Sigma)$. Since we assume that there are no non-trivial relations among the $E_\alpha$, the higher cohomology groups vanish. This is a standard result \[1\]. The complex \[3.3\] thus gives us a resolution of the covariant phase space $C(\Sigma)$, which by definition means that $H^0_{cl}(\delta) = C(\Sigma)$, $H^k_{cl}(\delta) = 0$, for all $k > 0$.

The extended history space $\mathcal{Q}^*$ is not a phase space, because it has no Poisson bracket. It does admit an antibracket, defined on the coordinates by

$$
(\phi^\alpha, \phi^{*\beta}) = \delta^{\alpha}_\beta.
\tag{3.5}
$$

However, the antibracket is not an proper Poisson bracket because it is symmetric, $(F, G) = +(G, F)$, so is can not be used for canonical quantization.
Moreover, as we will see in the next section, the antibracket can not even be defined for jets.

The key observation is now that the same space $C(\Sigma)$ admits a different resolution. Introduce canonical momenta $\pi_\alpha = \delta / \delta \phi^\alpha$ and $\pi_\alpha = \delta / \delta \phi^\alpha$ for both the fields and antifields. The momenta satisfy by definition the graded canonical commutation relations

$$[\pi^\beta, \phi^\alpha] = \delta^\alpha_\beta, \quad [\phi^\alpha, \phi^\beta] = [\pi^\alpha, \pi^\beta] = 0,$$

$$[\pi^\alpha_*, \phi_*^\alpha] = \delta^\alpha_\alpha, \quad [\phi_*^\alpha, \phi_*^\beta] = [\pi^\alpha_*, \pi^\beta_*] = 0,$$

where $[\cdot, \cdot]_+$ is the symmetric bracket. Let $P$ be the phase space of histories with basis $(\phi^\alpha, \pi^\beta)$, and let $P^*$ be the extended phase space with basis $(\phi^\alpha, \pi^\beta, \phi^\ast_\alpha, \pi^\ast_\beta)$. The following table summarizes the different spaces that we have defined.

| Space  | Basis               | Name                        |
|--------|---------------------|-----------------------------|
| $\Sigma$ | $\phi^\alpha : \mathcal{E}_\alpha = 0$ | Physical phase space        |
| $Q$     | $\phi^\alpha$       | History space               |
| $Q^*$   | $\phi^\alpha, \phi^\ast_\alpha$ | Extended history space      |
| $\mathcal{P}$ | $\phi^\alpha, \pi^\beta$ | History phase space         |
| $\mathcal{P}^*$ | $\phi^\alpha, \pi^\beta, \phi^\ast_\alpha, \pi^\ast_\beta$ | Extended history phase space |

The definition of the KT differential extends to $\mathcal{P}^*$ by requiring that $\delta F = [Q, F]$ for every $F \in C(\mathcal{P}^*)$, where the KT operator is

$$Q = \mathcal{E}_\alpha \pi^\alpha_*.$$  

(3.7)

It acts on the various fields as

$$\delta \phi^\alpha = 0,$$

$$\delta \phi^\ast_\alpha = \mathcal{E}_\alpha,$$

$$\delta \pi^\alpha_\alpha = -\delta \mathcal{E}_\alpha \pi^\beta_\ast = -K_{\alpha \beta} \pi^\beta_*,$$

$$\delta \pi^\ast_\alpha = 0,$$

where $K_{\alpha \beta}$ is the Hessian [2.2]. We check that $\delta$ is still nilpotent: $\delta^2 = [Q, Q]_+ = 0$.

Like $C(Q)$, the function space $C(\mathcal{P}^*)$ decomposes into subspaces of fixed antifield number, $C(\mathcal{P}^*) = \sum_{k=-\infty}^{\infty} C^k(\mathcal{P}^*)$. We can therefore define a KT complex in $C(\mathcal{P}^*)$

$$\ldots \xleftarrow{Q^2} C^{-2} \xleftarrow{Q} C^{-1} \xleftarrow{Q} C^0 \xleftarrow{Q} C^1 \xleftarrow{Q} C^2 \xleftarrow{Q} \ldots$$

(3.9)
Because the Hessian (2.2) is non-singular by assumption with inverse $M^{\alpha \beta}$, we can invert the relation $\delta \pi_\alpha = -K_{\alpha \beta} \pi_\beta^*$ and get 
\[ \pi_\alpha^* = -M^{\alpha \beta} \delta \pi_\beta = \delta(-M^{\alpha \beta} \pi_\beta^*), \quad (3.10) \]

since $M^{\alpha \beta}$ depends on $\phi$ alone.

Let us now compute the cohomology. Any function which contains $\pi_\alpha$ is not closed, so $\text{ker} \delta = C(\phi, \phi^*, \pi_\alpha)$. Moreover, $\text{im} \delta$ is generated by the two ideals $\mathcal{E}_\alpha$ and $\pi_\alpha^\alpha$. The momenta $\pi_\alpha$ and $\pi_\alpha^\alpha$ thus vanish in cohomology, and the part with zero antifield number is thus still $H_{cl}^0(\delta) = C(\mathcal{Q})/\mathcal{N} = C(\Sigma)$. The higher cohomology groups $H_{cl}^k(\delta) = 0$ by the same argument as above. Hence the complex (3.9) yields a different resolution of the same phase space $\Sigma$.

It is important that the spaces $C^k$ in (3.9) are phase spaces, equipped with the Poisson bracket (3.6). Unlike the resolution (3.3), the new resolution (3.9) therefore allows us to do canonical quantization: replace Poisson brackets by commutators and represent the graded Heisenberg algebra (3.6) on a Hilbert space. However, the Heisenberg algebra can be represented on different Hilbert spaces. To pick the correct one, we must impose the physical condition that there is an energy which is bounded on below.

To define the Hamiltonian, we must single a privileged variable $t$ among the $\alpha$’s, and declare it to be time. Thus replace $\alpha = (i, t)$, so e.g. $\phi^\alpha = \phi^i(t)$, $\mathcal{E}_\alpha = \mathcal{E}_i(t)$, etc. This step means of course that we sacrifice covariance. The Hamiltonian reads 
\[ H = -i \int dt \dot{\phi}^i(t) \pi_i(t) + \dot{\phi}^*_i(t) \pi^*_i(t). \quad (3.11) \]

It satisfies 
\[ [H, \phi^i(t)] = -i \dot{\phi}^i(t), \quad [H, \pi_i(t)] = -i \dot{\pi}_i(t), \quad (3.12) \]
\[ [H, \phi^*_i(t)] = -i \dot{\phi}^*_i(t), \quad [H, \pi^*_i(t)] = -i \dot{\pi}^*_i(t). \]

Expand all fields in a Fourier series with respect to time, e.g., 
\[ \phi^i(t) = \int_{m=-\infty}^{\infty} dm \, \phi^i(m) e^{imt}. \quad (3.13) \]
The Fourier modes $\pi_i(m)$, $\phi^*_i(m)$ and $\pi^*_i(m)$ are defined analogously. The Hamiltonian acts on the Fourier modes as 
\[ [H, \phi^i(m)] = m \phi^i(m), \quad [H, \pi_i(m)] = m \pi_i(m), \quad \quad (3.14) \]
\[ [H, \phi^*_i(m)] = m \phi^*_i(m), \quad [H, \pi^*_i(m)] = m \pi^*_i(m). \]
Now quantize. In the spirit of BRST quantization, our strategy is to quantize first and impose dynamics afterwards. In the extended history phase space $\mathcal{P}^*$, we define a Fock vacuum $|0\rangle$ which is annihilated by all negative frequency modes, i.e.

$$\phi^i(-m)|0\rangle = \pi_i(-m)|0\rangle = \phi_i^*(-m)|0\rangle = \pi_i^*(-m)|0\rangle = 0,$$

for all $-m < 0$. We must also decide which of the zero modes that annihilate the vacuum, but the decision is not important unless zero-momentum modes will survive in cohomology, and even then it will not affect the eigenvalues of the Hamiltonian.

The Hamiltonian does not act in a well-defined manner, because it assigns an infinite energy to the Fock vacuum. To correct for that, we replace the Hamiltonian by

$$H = -i \int dt \, :\dot{\phi}^i(t)\pi_i(t): + :\dot{\phi}_i^*(t)\pi_i^*(t):,$$

where normal ordering $:\cdot:\,$ moves negative frequency modes to the right and positive frequency modes to the left. The vacuum has zero energy as measured by the normal-ordered Hamiltonian, $H|0\rangle = 0$. The Hilbert space can be identified with

$$\mathcal{H}(\mathcal{P}^*) = C(\phi^i(m > 0), \pi_i(m > 0), \phi_i^*(m > 0), \pi_i^*(m > 0)).$$

The energy of a state in $\mathcal{H}(\mathcal{P}^*)$ follows from

$$H \phi^{i_1}(m_1)...\pi_i^{i_n}(m_n)|0\rangle = (m_1 + ... + m_n)\phi^{i_1}(m_1)...\pi_i^{i_n}(m_n)|0\rangle.$$  

It is important that the KT operator

$$Q = \mathcal{E}_\alpha \pi_\alpha^* = \int dt \, \mathcal{E}_i(t)\pi_i^*(t) = \int_{-\infty}^{\infty} dm \, \mathcal{E}_i(m)\pi_i^*(-m)$$

is already normal ordered, because $\mathcal{E}_\alpha$ and $\pi_\alpha^*$ commute. This means that $Q^2 = 0$ also quantum mechanically; there are no anomalies. Moreover, $Q$ still commutes with the Hamiltonian, $[Q, H] = 0$, and this property is not destroyed by normal ordering. Hence the Hilbert space $\mathcal{H}(\mathcal{P}^*)$ has also a well-defined decomposition into subspaces of definite antifield number,

$$\mathcal{H}(\mathcal{P}^*) = ... + \mathcal{H}^{-2} + \mathcal{H}^{-1} + \mathcal{H}^0 + \mathcal{H}^1 + \mathcal{H}^2 + ...$$

There is a KT complex in $\mathcal{H}(\mathcal{P}^*)$

$$... \xleftarrow{Q} \mathcal{H}^{-2} \xleftarrow{Q} \mathcal{H}^{-1} \xleftarrow{Q} \mathcal{H}^0 \xleftarrow{Q} \mathcal{H}^1 \xleftarrow{Q} \mathcal{H}^2 \xleftarrow{Q} ...$$
The physical Hilbert space is identified with $\mathcal{H}(\Sigma) = \mathcal{H}_{qm}^0(Q) = (\ker Q)_0/(\text{im } Q)_0$. The action of the Hamiltonian on the physical Hilbert space is still given by (3.18), restricted to $\mathcal{H}(\Sigma) \subset \mathcal{H}(\mathcal{P}^*)$, and that coincides with the conventional action of the Hamiltonian.

Hence we have quantized the theory given by the EL equation (2.1) by first quantizing the space of phase space histories $\mathcal{P}^*$, and then imposing dynamics through KT cohomology.

4 Covariant phase space, covariant quantization

A covariant definition of the phase space was given in the previous section, but the Hamiltonian and thus the quantum Hilbert space broke covariance, due to the selection of the privileged time coordinate. In this section we correct this defect.

The compact notation in Section 3 is not very useful here, because the notion of covariance does not make sense unless some indices are identified with spacetime coordinates. So we assume that we have some fields $\phi^\alpha(x)$, where $x = (x^\mu) \in \mathbb{R}^N$ is the spacetime coordinate. The EL equations read

$$ E^\alpha(x) = \frac{\delta S}{\delta \phi^\alpha(x)} = 0. \quad (4.1) $$

We also need the Hessian

$$ K_{\alpha\beta}(x, x') = K_{\beta\alpha}(x', x) = \frac{\delta E^\alpha(x)}{\delta \phi^\beta(x')} = \frac{\delta^2 S}{\delta \phi^\alpha(x) \delta \phi^\beta(x')} = \frac{\delta^2 S}{\delta \phi^\alpha(x) \delta \phi^\beta(x')}. \quad (4.2) $$

which we assume is non-singular.

Now let all fields depend on an additional parameter $t$. It will eventually be identified with time, but so far it is completely unrelated to the $x^\mu$. Upon the substitution $\phi^\alpha(x) \rightarrow \phi^\alpha(x, t)$, the EL equations are replaced by

$$ E^\alpha(x, t) = 0. \quad (4.3) $$

The Hessian (4.2) becomes

$$ K_{\alpha\beta}(x, t, x', t') = K_{\beta\alpha}(x', t', x, t) = \frac{\delta E^\alpha(x, t)}{\delta \phi^\beta(x', t')} = \frac{\delta E^\alpha(x, t)}{\delta \phi^\beta(x', t')}. \quad (4.4) $$

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which has the inverse \( M^{\alpha\beta}(x, t, x', t') \) satisfying

\[
\int d^N x'' \int dt'' K_{\beta\gamma}(x, t, x'', t'') M^{\gamma\alpha}(x'', t'', x', t') = \int d^N x'' \int dt'' M^{\alpha\gamma}(x, t, x'', t'') K_{\gamma\beta}(x'', t'', x', t') = \delta_\beta^\alpha \delta(x - x') \delta(t - t').
\] (4.5)

To remove the condition (4.3) in cohomology we introduce antifields \( \phi^*_\alpha(x, t) \). But the fields in the physical phase space do not depend on the parameter \( t \), which gives rise to the extra condition

\[
\partial_t \phi^\alpha(x, t) \equiv \frac{\partial \phi^\alpha(x, t)}{\partial t} = 0. \tag{4.6}
\]

We can implement this condition by introducing new antifields \( \overline{\phi}_\alpha(x, t) \). However, the identities \( \partial_t \mathcal{E}_\alpha(x, t) \equiv 0 \) give rise to unwanted cohomology. To kill this condition, we must introduce yet another antifield \( \overline{\phi}_\alpha(x, t) \). The KT differential \( \delta \) is defined by

\[
\begin{align*}
\delta \phi^\alpha(x, t) &= 0, \\
\delta \phi^*_\alpha(x, t) &= \mathcal{E}_\alpha(x, t), \\
\delta \overline{\phi}_\alpha(x, t) &= \partial_t \phi^\alpha(x, t), \\
\delta \overline{\phi}^*_\alpha(x, t) &= \partial_t \phi^*_\alpha(x, t) - \int d^N x' \int dt' K_{\alpha\beta}(x, t, x', t') \phi^*_\beta(x', t').
\end{align*} \tag{4.7}
\]

The zeroth cohomology group \( H^0_{\delta}(\delta) \) equals \( C(\phi) \), modulo the ideals generated by \( \mathcal{E}_\alpha(x, t) \) and \( \partial_t \phi^\alpha(x, t) \). Moreover, the wouldbe cohomology related to the identity

\[
\delta \left( \partial_t \phi^*_\alpha(x, t) - \int d^N x' \int dt' \frac{\partial \mathcal{E}_\alpha(x, t)}{\partial \overline{\phi}^*_\beta(x', t')} \phi^*_\beta(x', t') \right) \equiv 0 \tag{4.8}
\]

is killed because the RHS equals \( \delta \overline{\phi}_\alpha(x, t) \).

Introduce canonical momenta for all fields and antifields: \( \pi_\alpha(x, t) = \delta/\delta \phi^\alpha(x, t) \), \( \pi^*_\alpha(x, t) = \delta/\delta \phi^*_\alpha(x, t) \), \( \overline{\pi}_\alpha(x, t) = \delta/\delta \overline{\phi}_\alpha(x, t) \), and \( \overline{\pi}^*_\alpha(x, t) = \delta/\delta \overline{\phi}^*_\alpha(x, t) \), with non-zero commutation relations

\[
\begin{align*}
\left[ \pi_\beta(x, t), \phi^\alpha(x', t') \right] &= \delta_\beta^\alpha \delta(x - x') \delta(t - t'), \\
\left[ \pi^*_\beta(x, t), \phi^*_\alpha(x', t') \right]_+ &= \delta_\beta^\alpha \delta(x - x') \delta(t - t'), \\
\left[ \overline{\pi}_\beta(x, t), \overline{\phi}_\alpha(x', t') \right]_+ &= \delta_\beta^\alpha \delta(x - x') \delta(t - t'), \\
\left[ \overline{\pi}^*_\beta(x, t), \overline{\phi}^*_\alpha(x', t') \right] &= \delta_\beta^\alpha \delta(x - x') \delta(t - t'). \tag{4.9}
\end{align*}
\]
The KT operator takes the explicit form

\[
Q = \int d^N x \int dt \left( \mathcal{E}_\alpha(x,t) \pi_\alpha^*(x,t) + \partial_t \phi_\alpha(x,t) \pi_\alpha(x,t) \right) + \left( \partial_t \phi_\alpha^*(x,t) - \int d^N x' \int dt' \mathcal{K}_{\alpha\beta}(x,t,x',t') \phi_\beta^*(t') \pi_\alpha^*(x,t) \right).
\]  

From this we can read off the action of \( \delta \) on the momenta. As in the previous section, the zeroth cohomology group consists of functions \( \phi_\alpha(x,t) \) which satisfy \( \mathcal{E}_\alpha(x,t) = 0 \) and \( \partial_t \phi_\alpha(x,t) = 0 \). Hence \( H_0^0(\delta) = \mathcal{C}(\Sigma) \), as desired.

At this point, we must define a Hamiltonian. The candidate

\[
H_0 = -i \int dt \left( \partial_t \phi_\alpha(x,t) \pi_\alpha(x,t) + \partial_t \phi_\alpha^*(x,t) \pi_\alpha^*(x,t) \right)
\]

might seem natural, but it is not acceptable. The action of the Hamiltonian is KT exact, e.g.

\[
[H_0, \phi_\alpha(x,t)] = \partial_t \phi_\alpha(x,t) = \delta \phi_\alpha(x,t),
\]

and thus \( H_0 \approx 0 \). This \( H_0 \) is not a genuine Hamiltonian, but rather a Hamiltonian constraint \( H_0 \approx 0 \), familiar from canonical quantization of general relativity.

However, we can construct a well-defined and physical Hamiltonian with some extra work. The crucial idea is to introduce the observer’s trajectory \( q^\mu(t) \in \mathbb{R}^N \), and then expand all fields in a Taylor series around this trajectory. The Taylor coefficients and the observer’s trajectory together constitute a jet, more precisely the infinite jet, corresponding to the field. We can now define a genuine Hamiltonian which moves the fields relative to the observer. As explained in \[2\], the same step is also crucial in the representation theory of algebras of diffeomorphisms and gauge transformations.

Hence we make the Taylor expansion

\[
\phi_\alpha(x,t) = \sum_m \frac{1}{m!} \phi_\alpha^m(t)(x - q(t))^m,
\]

where \( m = (m_1, m_2, ..., m_N) \), all \( m_\mu \geq 0 \), is a multi-index of length \(|m| = \sum_{\mu=1}^N m_\mu \), \( m! = m_1!m_2!...m_N! \), and

\[
(x - q(t))^m = (x^1 - q^1(t))^{m_1}(x^2 - q^2(t))^{m_2}...(x^N - q^N(t))^{m_N}.
\]

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Denote by $\mu$ a unit vector in the $\mu$:th direction, so that $\mathbf{m} + \mu = (m_1, ..., m_\mu + 1, ..., m_N)$, and let

$$\phi^\alpha_{\mathbf{m}}(t) = \partial_\mathbf{m} \phi^\alpha(q(t), t) = \partial_{m_1} \partial_{m_2} \cdots \partial_{m_N} \phi^\alpha(q(t), t)$$

be the $|\mathbf{m}|$:th order derivative of $\phi^\alpha(x, t)$ evaluated on the observer’s trajectory $q^\mu(t)$.

The Taylor coefficients $\phi^\alpha_{\mathbf{m}}(t)$ are referred to as jets; more precisely, infinite jets. Similarly, we define a $p$-jet by truncation to $|\mathbf{m}| \leq p$. We will not need finite $p$-jets in this paper, but they play an important role as a regularization of symmetry generators. A note on nomenclature may be appropriate at this point. A $p$-jet is usually defined as an equivalence class of functions; two functions are equivalent if all mixed partial derivatives at a given point $q$, up to order $p$, agree. Since each equivalence class has a unique representative which is a polynomial of order at most $p$, namely the truncated Taylor series around $q$, we will identify the jet with the Taylor series. Moreover, the function $\phi^\alpha_{\mathbf{m}}(t)$ defines a trajectory in jet space, but for brevity we will simply refer to $\phi^\alpha_{\mathbf{m}}(t)$ itself as a jet.

Expand also the Euler-Lagrange equations and the antifields in a similar Taylor series,

$$\mathcal{E}_\alpha(x, t) = \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} \mathcal{E}_{\alpha, \mathbf{m}}(t)(x - q(t))^\mathbf{m},$$

$$\phi^\ast_\alpha(x, t) = \sum_{\mathbf{m}} \frac{1}{\mathbf{m}!} \phi^\ast_{\alpha, \mathbf{m}}(t)(x - q(t))^\mathbf{m},$$

etc. These relations define the jets $\mathcal{E}_{\alpha, \mathbf{m}}(t)$, $\phi^\ast_{\alpha, \mathbf{m}}(t)$, $\overline{\phi}^\ast_{\alpha}(t)$ and $\overline{\mathcal{E}}_{\alpha, \mathbf{m}}(t)$. Jets of antifields will sometimes be called antijets.

The equation of motion and the time-independence condition translate into

$$\mathcal{E}_{\alpha, \mathbf{m}}(t) = 0,$$

$$D_t \phi^\alpha_{\mathbf{m}}(t) \equiv \frac{d}{dt} \phi^\alpha_{\mathbf{m}}(t) - \sum_\mu q^\mu(t) \phi^\alpha_{\mathbf{m} + \mu} = 0. \tag{4.17}$$
The KT differential $\delta$ which implements these conditions is

\[
\begin{align*}
\delta \phi^\alpha_m(t) &= 0, \\
\delta \phi^*_{\alpha,m}(t) &= \mathcal{E}_{\alpha,m}(t), \\
\delta \phi_{\alpha,m}^*(t) &= D_t \phi^\alpha_m(t), \\
\delta \phi^*_{\alpha,m}(t) &= D_t \phi^*_{\alpha,m}(t) - \sum_n \int dt' K^n_{m,\alpha,\beta}(t, t') \phi^\beta_n(t').
\end{align*}
\] (4.18)

The cohomology group $H^0_{\text{cl}}(\delta)$ consists of linear combinations of jets $\phi^\alpha_m(t)$ satisfying $\mathcal{E}_{\alpha,m}(t) = 0$ and $D_t \phi^\alpha_m(t) = 0$.

The Taylor expansion requires that we introduce the observer’s trajectory as a physical field, but what equation of motion does it obey? The obvious answer is the geodesic equation, which we compactly write as $\mathcal{G}_\mu(t) = 0$. The geodesic operator $\mathcal{G}_\mu(t)$ is a function of the metric $g_{\mu\nu}(q(t), t)$ and its derivatives on the curve $q^\mu(t)$. To eliminate this ideal in cohomology we introduce the trajectory antifield $q^*_\mu(t)$, and extend the KT differential to it:

\[
\begin{align*}
\delta q^\mu(t) &= 0, \\
\delta q^*_\mu(t) &= \mathcal{G}_\mu(t).
\end{align*}
\] (4.19)

For models defined over Minkowski spacetime, the geodesic equation simply becomes $\dddot{q}^\mu(t) = 0$, and the KT differential reads

\[
\delta q^*_\mu(t) = \eta_{\mu\nu} q''^\nu(t).
\] (4.20)

$H^0_{\text{cl}}(\delta)$ only contains trajectories which are straight lines,

\[
q^\mu(t) = u^\mu t + a^\mu,
\] (4.21)

where $u^\mu$ and $a^\mu$ are constant vectors. We may also require that $u^\mu$ has unit length, $u_\mu u^\mu = 1$. This condition fixes the scale of the parameter $t$ in terms of the Minkowski metric, so we may regard it as proper time rather than as an arbitrary parameter.

Now introduce the canonical momenta $\pi^m_\alpha(t)$, $\pi^*_{\alpha,m}(t)$, $\pi^m_\alpha(t)$, $\pi^*_{\alpha,m}(t)$ for the jets and antijets (jet and antijet momenta), and momenta $p_\mu(t)$ and $p^*_\mu(t)$ for the observer’s trajectory and its antifield. The defining relations
are
\[ [\pi^m_\alpha(t), \phi^\beta_n(t')] = \delta^\beta_\alpha \delta^m_n \delta(t - t'), \]
\[ [\pi^{\alpha,m}_\beta(t), \phi^\gamma_n(t')] = \delta^\alpha_\beta \delta^{\alpha,m}_n \delta(t - t'), \]
\[ [\pi^m_\alpha(t), \phi^\beta_n(t')] = \delta^\alpha_\beta \delta^m_n \delta(t - t'), \]
\[ [\pi^{\alpha,m}_\beta(t), \phi^\gamma_n(t')] = \delta^\alpha_\beta \delta^{\alpha,m}_n \delta(t - t'). \]  (4.22)

The advantage of this formalism is that we can now define a genuine Hamiltonian \( H \), which translates the fields relative to the observer or vice versa. Since the formulas are shortest when \( H \) acts on the trajectory but not on the jets, we make that choice, and define
\[
H = i \int dt \ (\dot{q}^\mu(t)p_\mu(t) + \dot{q}^*_\mu(t)p^*_\mu(t)).
\]  (4.23)

Note the sign; moving the fields forward in \( t \) is equivalent to moving the observer backwards. This Hamiltonian acts on the jets as
\[
[H, \phi^\alpha_m(t)] = -i \frac{\partial}{\partial x^0} \phi^\alpha_m(x, t),
\]  (4.25)

This a crucial result, because it allows us to define a genuine energy operator in a covariant way. In Minkowski space, the trajectory is a straight line \( (4.21) \), and \( \dot{q}^\mu(t) = u^\mu \). If we take \( u^\mu \) to be the constant four-vector \( u^\mu = (1, 0, 0, 0) \), then \( (4.25) \) reduces to
\[
H, \phi^\alpha_m(t) = -i \frac{\partial}{\partial x^0} \phi^\alpha_m(x, t).
\]  (4.26)

Equation \( (4.25) \) is thus a genuine covariant generalization of the energy operator.

Is there an analogue of the antibracket \( [\ ] \) in jet space? The answer is no. The obvious candidate would be
\[
(\phi^\alpha_m(t), \phi^\beta_n(t')) = \delta^\alpha_\beta \delta^m_n \delta(t - t').
\]  (4.27)
However, the RHS contains an object $\delta_{m,n}$ with two lower multi-indices, and such an object transforms non-trivially. Hence the antibracket in jet space is not well defined.

Now we quantize the theory. Since all operators depend on the parameter $t$, we can define the Fourier components, e.g.

$$\phi^\alpha_m(t) = \int_{-\infty}^{\infty} dm \phi^\alpha_m(m) e^{i m t}, \quad q^\mu(t) = \int_{-\infty}^{\infty} dm q^\mu(m) e^{i m t}. \quad (4.28)$$

The Fock vacuum $|0\rangle$ is defined to be annihilated by all negative frequency modes, i.e.

$$\phi^\alpha_m(-m)|0\rangle = \phi^\alpha_{a,m}(-m)|0\rangle = \bar{\phi}^\alpha_m(-m)|0\rangle = \bar{\phi}^\alpha_{a,m}(-m)|0\rangle = 0,$$

$$\pi^m(-m)|0\rangle = \pi^m_{a,m}(-m)|0\rangle = \bar{\pi}^m(-m)|0\rangle = \bar{\pi}^m_{a,m}(-m)|0\rangle = 0, \quad (4.29)$$

$$q^\mu(-m)|0\rangle = q^\mu_{a,m}(-m)|0\rangle = p_\mu(-m)|0\rangle = p^\mu_{a,m}(-m)|0\rangle = 0,$$

for all $-m < 0$.

The normal-ordered form of the Hamiltonian (4.23) reads, in Fourier space,

$$H = -\int_{-\infty}^{\infty} dm \ m (:q^\mu(m)p_\mu(m): + :q^\mu_{a,m}(m)p_\mu(-m):), \quad (4.30)$$

where double dots indicate normal ordering with respect to frequency. This ensures that $H|0\rangle = 0$. The classical phase space $H^0\omega(\delta)$ is thus the the space of fields $\phi^\alpha(x)$ which solve $E^\alpha(x) = 0$, and trajectories $q^\mu(t) = u^\mu t + a^\mu$, where $u^2 = 1$. After quantization, the fields and trajectories become operators which act on the physical Hilbert space $\mathcal{H} = H^0_{qm}(Q)$, which is the space of functions of the positive-energy modes of the classical phase space variables.

This construction differs technically from conventional canonical quantization, but there is also a physical difference. Consider the state $|\phi^\alpha(x)\rangle = \phi^\alpha(x)|0\rangle$ which excites one $\phi$ quantum from the vacuum. The Hamiltonian yields

$$H|\phi^\alpha(x)\rangle = -i q^\mu(t) \partial_\mu \phi^\alpha(x)|0\rangle = -i |\phi^\alpha(x)\rangle.$$  \quad (4.31)

If $u^\mu$ were a classical variable, the state $|\phi^\alpha(x)\rangle$ would be a superposition of energy eigenstates:

$$H|\phi^\alpha(x)\rangle = -iu^\mu \partial_\mu |\phi^\alpha(x)\rangle.$$  \quad (4.32)
In particular, let \( u^\mu = (1, 0, 0, 0) \) be a unit vector in the \( x^0 \) direction and \( \phi^\mu(x) = \exp(ik \cdot x) \) be a plane wave. We then define the state \( |0; u, a\rangle \) by
\[
q^\mu(t)|0; u, a\rangle = (u^\mu t + a^\mu)|0; u, a\rangle.
\]
(4.33)
Now write \( |k; u, a\rangle = \exp(ik \cdot x)|0; u, a\rangle \) for the single-quantum energy eigenstate.
\[
H|k; u, a\rangle = k^\mu u^\mu|k; u, a\rangle,
\]
(4.34)
so the eigenvalue of the Hamiltonian is \( k^\mu u^\mu = k_0 \), as expected. Moreover, the lowest-energy condition \( \text{(4.29)} \) ensures that only quanta with positive energy will be excited; if \( k^\mu u^\mu < 0 \) then \( |k; u, a\rangle = 0 \).

However, the present analysis shows that it is in principle wrong to consider \( u^\mu \) and \( a^\mu \) as classical variables. The definition \( \text{(4.33)} \) means that the reference state \( |0; u, a\rangle \) is a very complicated, mixed, macroscopic state where the observer moves along a well-defined, classical trajectory. This is of course an excellent approximation in practice, but it is in principle wrong.

5 Harmonic oscillator
The action and Euler-Lagrange equations read
\[
S = \frac{1}{2} \int dt \left( \ddot{q}^2(t) - \omega^2 q^2(t) \right)
\]
\[
E(t) \equiv -\frac{\delta S}{\delta q(t)} = \ddot{q}(t) + \omega^2 q(t) = 0.
\]
(5.1)
Introduce antifields \( q^*(t) \) and canonical momenta \( p(t) = \delta/\delta q(t) \) and \( p^*(t) = \delta/\delta q^*(t) \). The space spanned by \( q(t), q^*(t), p(t) \) and \( p^*(t) \) is the extended phase space \( \mathcal{P}^* \). The non-zero brackets are
\[
[p(t), q(t')] = [p^*(t), q^*(t')]_+ = \delta(t - t').
\]
(5.2)
The KT differential and the Hamiltonian in \( \mathcal{P}^* \) read
\[
Q = \int dt \left( \ddot{q}(t) + \omega^2 q(t) \right)p^*(t),
\]
\[
H = -i \int dt \left( \dot{q}(t)p(t) + \dot{q}^*(t)p^*(t) \right).
\]
(5.3)
$Q$ acts as $\delta F = [Q, F]$, where
\begin{align*}
\delta q(t) &= 0, \\
\delta q^*(t) &= \ddot{q}(t) + \omega^2 q(t), \\
\delta p(t) &= -(\ddot{p}^*(t) + \omega^2 p^*(t)), \\
\delta p^*(t) &= 0.
\end{align*}
\tag{5.4}

After Fourier transformation, $\mathcal{P}^*$ is spanned by modes $q_m, q_m^*, p_m$ and $p_m^*$, and the EL equations read
\begin{equation}
\mathcal{E}_m = -(m^2 - \omega^2)q_m = 0.
\tag{5.5}
\end{equation}

The non-zero Poisson brackets in $\mathcal{P}^*$ are
\begin{equation}
[p_m, q_n] = [p_m^*, q_n^*] = \delta_{m+n}.
\tag{5.6}
\end{equation}

The KT differential and the Hamiltonian are
\begin{align*}
Q &= \sum_{m=-\infty}^{\infty} (m^2 - \omega^2)q_mp_{-m}^*, \\
H &= \sum_{m=-\infty}^{\infty} m(q_mp_{-m} + q_m^*p_{-m}^*).
\tag{5.7}
\end{align*}

$Q$ acts as $\delta F = [Q, F]$, where
\begin{align*}
\delta q_m &= 0, \\
\delta q_m^* &= (m^2 - \omega^2)q_m, \\
\delta p_m &= -(m^2 - \omega^2)p_m^*, \\
\delta p_m^* &= 0.
\end{align*}
\tag{5.8}

The cohomology is computed as follows. Since the equations (5.8) decouple, we can consider each value of $m$ separately. First assume that $m^2 \neq \omega^2$, i.e. $m \neq \pm \omega$. $q_m$ and $p_m^*$ are closed for all $m$, but $q_m^*$ and $p_m$ are not closed since $\delta q_m^* \neq 0$, etc. We can invert the second and third equations to read
\begin{align*}
q_m &= \frac{1}{m^2 - \omega^2}\delta q_m^*, \\
p_m^* &= -\frac{1}{m^2 - \omega^2}\delta p_m.
\tag{5.9}
\end{align*}
Hence \( q_m \) and \( p^*_m \) lie in the image of \( \delta \), and the cohomology vanishes completely: only \( q_m \) and \( p^*_m \) lie in the kernel, but they also lie in the image.

Now turn to the case \( m^2 = \omega^2 \), say \( m = \omega \). Clearly, \( \delta q_m = \delta p_m = \delta q^*_m = \delta p^*_m = 0 \), so all four variables lie in the kernel but not in the image. This clearly that the cohomology spaces are too big; the classical cohomology spaces can be identified with \( H^*_{cl}(\delta) = C(q_{\pm \omega}, p_{\pm \omega}, q^*_{\pm \omega}, p^*_{\pm \omega}) \). The zeroth cohomology space consists of such functions with total antifield number zero, i.e. \( H^0_{cl}(\delta) = C(q_{\pm \omega}, p_{\pm \omega}, (q^*_{\pm \omega}p^*_{\pm \omega})) \).

Now quantize by introducing a Fock vacuum \( \mid 0 \rangle \) satisfying

\[
q_{-m} \mid 0 \rangle = p_{-m} \mid 0 \rangle = q^*_{-m} \mid 0 \rangle = p^*_{-m} \mid 0 \rangle = 0,
\]

for all \( -m < 0 \). These conditions eliminate all modes with negative frequency \( -\omega \). The quantum cohomology thus consists of Hilbert spaces built from the modes with positive frequency \( \omega \), \( H^*_{qm}(Q) = C(q_{\omega}, p_{\omega}, q^*_{\omega}, p^*_{\omega}) \). In particular, the zeroth cohomology space consists of such functions with total antifield number zero, i.e. \( H^0_{qm}(Q) = C(q_{\omega}, p_{\omega}, (q^*_{\omega}p^*_{\omega})) \).

This Hilbert space is also too big. It is spanned by states of the form

\[
\mid n_q, n_p, n^*_q, n^*_p \rangle = (q_{\omega})^{n_q}(p_{\omega})^{n_p}(q^*_{\omega})^{n^*_q}(p^*_{\omega})^{n^*_p} \mid 0 \rangle.
\]

The energy of this state is \( (n_q + n_p + n^*_q + n^*_p)\omega \), because

\[
H \mid n_q, n_p, n^*_q, n^*_p \rangle = (n_q + n_p + n^*_q + n^*_p)\omega \mid n_q, n_p, n^*_q, n^*_p \rangle.
\]

The zeroth cohomology again consists of states with antifield number zero, i.e. \( n^*_q = n^*_p \).

One way to avoid the unwanted cohomology is to add a small perturbation, so the Hessian \( K_{mn}(q) = \delta^2 S/\delta q_m \delta q_n \) is non-singular and has an inverse \( M_{mn}(q) \). The \( p \)-part of (5.8) is replaced by

\[
\delta p_m = -\sum_n K_{mn}(q)p^*_n,
\]

\[
\delta p^*_m = 0,
\]

i.e.

\[
p^*_m = -\delta(\sum_n M_{mn}(q)p_n),
\]

\[
\delta p^*_m = 0,
\]

In the perturbed theory, \( p^*_m \) is both closed and exact for all \( m \), so it vanishes in cohomology. Moreover, \( p_m \) is never closed for any \( m \). After removing
the small perturbation, the classical cohomology space can thus be identified with \( H^*_{cl}(Q) = C(q_\omega, q_{-\omega}, q^*_\omega, q^*_{-\omega}) \), and the zeroth cohomology space is \( H^0_{cl}(Q) = C(q_\omega, q_{-\omega}) \). After quantization, the negative frequency modes \( q_{-\omega} \) and \( q^*_{-\omega} \) annihilate the vacuum, so the total quantum cohomology is \( H^*_{qm}(Q) = C(q_\omega, q^*_\omega) \). In particular, \( H^0_{qm}(Q) = C(q_\omega) = \mathcal{H} \) is the Hilbert space of the harmonic oscillator. It has a basis consisting of \( n \)-quanta states, \( |n\rangle = (q_\omega)^n |0\rangle \) with the right energy, \( H |n\rangle = n\omega |n\rangle \).

One may note that the standard antifield treatment of the harmonic oscillator, without momenta, suffers from an analogous problem. Since \( \delta q_{\pm\omega} = \delta q^*_{\pm\omega} = 0 \), the \( k \):th cohomology group, rather than being zero, is spanned by functions of the form \( \sum_j (k_j) f_j(q_\omega, q_{-\omega})(q^*_\omega)^j(q^*_{-\omega})^{k-j} \). The zeroth cohomology space is thus the right physical phase space, \( H^0_{cl}(\delta) = C(q_\omega, q_{-\omega}) = C(\Sigma) \), but this is not a resolution because the higher cohomology groups do not vanish.

### 6 Free scalar field: non-covariant quantization

The action, Euler-Lagrange equations, and Hessian read

\[
S = \frac{1}{2} \int d^N x \left( \partial_\mu \phi(x) \partial^\mu \phi(x) - \omega^2 \phi^2(x) \right),
\]

\[
\mathcal{E}(x) \equiv -\frac{\delta S}{\delta \phi(x)} = \Delta \phi(x) + \omega^2 \phi(x) = 0,
\]

\[
K(x, x') = -\frac{\delta^2 S}{\delta \phi(x) \delta \phi(x')} = \Delta \delta(x - x') + \omega^2 \delta(x - x')
\]

where \( \Delta = \partial_\mu \partial^\mu \).

Introduce antifields \( \phi^*(x) \) and canonical momenta \( \pi(x) = \delta / \delta \phi(x) \) and \( \pi_*(x) = \delta / \delta \phi^*(x) \). The non-zero brackets are

\[
[\pi(x), \phi(x')] = [\pi_*(x), \phi^*(x')] = \delta(x - x').
\]

The KT differential reads

\[
Q = \int d^N x \left( \Delta \phi(x) + \omega^2 \phi(x) \right) \pi_*(x).
\]

\( Q \) acts as \( \delta F = [Q, F] \), where

\[
\begin{align*}
\delta \phi(x) &= 0, \\
\delta \phi^*(x) &= \Delta \phi(x) + \omega^2 \phi(x), \\
\delta \pi(x) &= -(\Delta \pi_*(x) + \omega^2 \pi_*(x)), \\
\delta \pi_*(x) &= 0.
\end{align*}
\]
Now we do a Fourier transformation. The extended phase space $P^*$ is spanned by modes $\phi(k)$, $\phi^*(k)$, $\pi(k)$ and $\pi_*(k)$, and the EL equation becomes
\[ E(k) = -(k^2 - \omega^2)\phi(k) = 0. \] (6.5)
The non-zero brackets are
\[ [\pi(k), \phi(k')] = [\pi_*(k), \phi^*(k')] = \delta(k - k'). \] (6.6)
The KT differential is
\[ Q = \int d^N k \ (k^2 - \omega^2)\phi(k)\pi_*(-k). \] (6.7)
$Q$ acts as $\delta F = [Q, F]$, where
\[ \delta \phi(k) = 0, \]
\[ \delta \phi^*(k) = (k^2 - \omega^2)\phi(k), \]
\[ \delta \pi(k) = -(k^2 - \omega^2)\pi_*(k), \]
\[ \delta \pi_*(k) = 0. \] (6.8)

To quantize the theory we must specify a Hamiltonian. Let it be
\[ H = -i \int d^N x \ (\partial_0 \phi(x)\pi(x) + \partial_0 \phi^*(x)\pi_*(x)) \]
\[ = \int d^N k \ k_0(\phi(k)p(-k) + \phi^*(k)\pi_*(-k)). \] (6.9)
Note that at this stage we break Poincaré invariance, since the Hamiltonian treats the $x^0$ coordinate differently from the other $x^\mu$. Quantize by introducing a Fock vacuum $|0\rangle$ satisfying
\[ \phi(k)|0\rangle = \pi(k)|0\rangle = \phi^*(k)|0\rangle = \pi_*(k)|0\rangle = 0, \] (6.10)
for all $k$ such that $k_0 < 0$.

The rest proceeds as for the harmonic oscillator. After adding a small perturbation to make the Hessian invertible, $\pi(k)$ and $\pi_*(k)$ vanish in cohomology, as do the off-shell components of $\phi(k)$ and $\phi^*(k)$. The classical cohomology $H^*_\text{cl}(Q) = C(\phi(k; k^2 = \omega^2), \phi^*(k; k^2 = \omega^2))$ consists of functions of the on-shell components of $\phi$ and $\phi^*$, and $H^{\text{on}}_0(Q) = C(\phi(k; k^2 = \omega^2))$ is the classical phase space. The quantization step eliminates the components $\phi(k)$ with $k_0 < 0$, which leaves us with the physical Hilbert space $\mathcal{H} = H^{\text{on}}_0(Q) = C(\phi(k; k^2 = \omega^2$ and $k_0 > 0)$). A basis for $\mathcal{H}$ consists of multi-quanta states
\[ |k, k', \ldots, k^{(n)}\rangle = \phi(k)\phi(k')\ldots\phi(k^{(n)})|0\rangle \] (6.11)
with energy $H = k + k' + \ldots + k^{(n)}$. 23
7 Free scalar field: covariant quantization

Following the prescription in Section 4, we make the replacement \( \phi(x) \rightarrow \phi(x, t) \), where \( t \in \mathbb{R} \) is a parameter. The EL equation (6.1) becomes

\[
\mathcal{E}(x, t) \equiv \Delta \phi(x, t) + \omega^2 \phi(x, t) = 0. \tag{7.1}
\]

To remove this condition in cohomology we introduce antifields \( \phi^*(x, t) \). But there is an extra condition

\[
\partial_t \phi(x, t) \equiv \partial \phi(x, t) = 0. \tag{7.2}
\]

We can implement this condition by introducing new antifields \( \phi^*(x, t) \). However, the identities \( \partial_t \mathcal{E}(x, t) \equiv 0 \) give rise to unwanted cohomology. To kill this condition, we must introduce a second-order antifield \( \phi^{**}(x, t) \). The full KT differential \( \delta \) is now defined by

\[
\delta \phi(x, t) = 0, \quad \delta \phi^*(x, t) = \Delta \phi(x, t) + \omega^2 \phi(x, t), \quad \delta \phi^*(x, t) = \partial_t \phi(x, t), \quad \delta \phi^{**}(x, t) = \partial_t \phi^*(x, t) - (\Delta \phi(x, t) + \omega^2 \phi(x, t)). \tag{7.3}
\]

Introduce canonical momenta for all fields and antifields: \( \pi(x, t) = \delta/\delta \phi(x, t) \), \( \pi^*(x, t) = \delta/\delta \phi^*(x, t) \), \( \pi^*(x, t) = \delta/\delta \phi^{**}(x, t) \), and \( \pi^*(x, t) = \delta/\delta \phi^{**}(x, t) \). The KT differential can now be expressed as a bracket, \( \delta F = [Q, F] \), where the KT operator is

\[
Q = \int d^N x \int dt \left( (\Delta \phi(x, t) + \omega^2 \phi(x, t)) \pi^*(x, t) + \partial_t \phi(x, t) \pi(x, t) + (\partial_t \phi^*(x, t) - (\Delta \phi(x, t) + \omega^2 \phi(x, t))) \pi^*(x, t) \right). \tag{7.4}
\]

Make a Fourier transform in \( t \), e.g.

\[
\phi(x, t) = \int_{-\infty}^{\infty} dm \, \phi(x, m) e^{imt}. \tag{7.5}
\]

which gives us Fourier-transformed fields \( \phi(x, m), \phi^*(x, m), \bar{\phi}(x, m), \bar{\phi}^*(x, m) \), on which the KT differential acts as

\[
\delta \phi(x, m) = 0, \quad \delta \phi^*(x, m) = \Delta \phi(x, m) - \omega^2 \phi(x, m), \quad \delta \bar{\phi}(x, m) = m \phi(x, m), \quad \delta \bar{\phi}^*(x, m) = m \phi^*(x, m) - (\Delta \phi(x, m) + \omega^2 \phi(x, m)). \tag{7.6}
\]

24
The candidate Hamiltonian \( H_0 \) acts on the fields as

\[
[H_0, \phi(x, m)] = m\phi(x, m), \quad [H_0, \phi^*(x, m)] = m\phi^*(x, m),
\]

\[
[H_0, \overline{\phi}(x, m)] = m\overline{\phi}(x, m), \quad [H_0, \overline{\phi}^*(x, m)] = m\overline{\phi}^*(x, m).
\] (7.7)

Alas, this Hamiltonian is identically zero on the physical phase space. The third equation in (7.6) can be rewritten as \( \phi(x, m) = m^{-1}\delta\overline{\phi}(x, m) \), which means that \( \phi(x, m) \) is KT exact and thus vanishes in cohomology unless \( m = 0 \), and in that case \( H_0 = 0 \). \( H_0 \) is thus a Hamiltonian constraint rather than a proper Hamiltonian.

To construct the physical Hamiltonian, we introduce the observer’s trajectory \( q^\mu(t) \in \mathbb{R}^N \), and then expand all fields in a Taylor series around this trajectory, i.e. we pass to jet data. Hence e.g.,

\[
\phi(x, t) = \sum_m \frac{1}{m!} \phi_m(t)(x - q(t))^m.
\] (7.8)

The equation of motion and the time-independence condition translate into

\[
\sum_\mu \phi_{m+2\mu}(t) + \omega^2 \phi_m(t) = 0,
\]

\[
D_t \phi_m(t) \equiv \frac{d}{dt}\phi_m(t) - \sum_\mu \dot{q}^\mu(t)\phi_{m+\mu} = 0.
\] (7.9)

We introduce anti-jets \( \phi^*_m(t) \), \( \overline{\phi}_m(t) \) and \( \overline{\phi}^*_m(t) \) and the KT differential \( \delta \) to implement these conditions:

\[
\delta \phi_m(t) = 0,
\]

\[
\delta \phi^*_m(t) = \sum_\mu \phi_{m+2\mu}(t) + \omega^2 \phi_m(t),
\]

\[
\delta \overline{\phi}_m(t) = D_t \phi_m(t),
\]

\[
\delta \overline{\phi}^*_m(t) = D_t \phi^*_m(t) - \left( \sum_\mu \overline{\phi}_{m+2\mu}(t) + \omega^2 \overline{\phi}_m(t) \right).
\] (7.10)

The classical cohomology group \( H^0_{cl}(\delta) \) consists of linear combinations of jets satisfying

\[
\phi_m(t) = e^{ik\cdot q(t)}(ik)^m
\] (7.11)

where \( k^2 = \omega^2 \), \( k \cdot q = k_\mu q^\mu \) and the power \( k^m \) is defined in analogy with (4.14). It is hardly surprising that the Taylor series (7.8) can be summed,
The Fock vacuum gives
\[ \phi(x,t) = e^{i k \cdot q(t)} \sum_{m} \frac{1}{m!} (i k)^m (x - q(t))^m \]
\[ = e^{i k \cdot q(t)} e^{i k \cdot (x - q(t))} \]
\[ = e^{i k \cdot x}. \] (7.12)

The physical Hamiltonian \( H \) is defined as in Equation (4.30). The classical phase space \( H_{cl}^0(\delta) \) is thus the space of plane waves \( e^{i k \cdot x} \), cf (7.12), and trajectories \( q^\mu(t) = u^\mu t + a^\mu \). The energy is given by
\[ [H, e^{i k \cdot x}] = k_\mu \dot{q}^\mu(t)e^{i k \cdot x} = k_\mu u^\mu e^{i k \cdot x}, \]
\[ [H, q^\mu(t)] = i \dot{q}^\mu(t). \] (7.13)

This is a covariant description of phase space, because the energy \( k_\mu u^\mu \) is Poincaré invariant.

We now quantize the theory before imposing the dynamics. To this end, we introduce the canonical momenta \( \pi^m(t), \pi^m_*(t), \pi^m_{\pi}(t), \pi^m_\pi(t) \) for the jets and antijets, and \( p_\mu(t) \) and \( p_\mu^\pi(t) \) for the observer’s trajectory and its antifield. The defining relations are
\[ [\pi^m(t), \phi_n(t')] = \delta^m_n \delta(t - t'), \]
\[ [\pi^m_*(t), \phi_n^*(t')] = \delta^m_n \delta(t - t'), \]
\[ [\pi^m_{\pi}(t), \phi_n^\pi(t')] = \delta^m_n \delta(t - t'), \]
\[ [\pi^m_\pi(t), \phi^\pi_n(t')] = \delta^m_n \delta(t - t'). \] (7.14)

Since the jets also depend on the parameter \( t \), we can define their Fourier components, e.g.
\[ \phi^m_\mu(t) = \int_{-\infty}^{\infty} dm \phi^m_\mu(m)e^{i m t}, \quad q^\mu(t) = \int_{-\infty}^{\infty} dm \ q^\mu(m)e^{i m t}. \] (7.15)

The Fock vacuum \( |0\rangle \) is defined to be annihilated by the negative frequency modes of the jets and antijets, i.e.
\[ \phi^m_\mu(-m)|0\rangle = \phi^*_m(-m)|0\rangle = \pi^m_\mu(-m)|0\rangle = \pi^m_{\pi}(m)|0\rangle = \pi^m_\pi(-m)|0\rangle = \pi^m_\pi(-m)|0\rangle = q^\mu_\mu(-m)|0\rangle = p^\mu_\mu(-m)|0\rangle = p^\mu_\pi(-m)|0\rangle = p^\mu_\pi(-m)|0\rangle = 0. \] (7.16)

for all \(-m < 0\). The quantum Hamiltonian is still defined by \( (1.30) \), where double dots indicate normal ordering with respect to frequency, ensuring that \( H|0\rangle = 0 \).
The rest proceeds as in the end of Section 4. We can consider the one-quantum state with momentum $k$ over the true Fock vacuum, $|k⟩ = \exp(ik·x)|0⟩$. This state is not an energy eigenstate, because the Hamiltonian excites a quantum of the observer’s trajectory: $H|k⟩ = k_\mu u^\mu|k⟩$. We may treat the observer’s trajectory as a classical variable and introduce the macroscopic reference state $|0; u, a⟩$, on which $q^\mu(t)|0; u, a⟩ = (u^\mu t + a^\mu)|0; u, a⟩$. We can then consider a state $|k; u, a⟩ = \exp(ik·x)|k; u, a⟩$ with one quantum over the reference state. The Hamiltonian gives $H|k; u, a⟩ = k_\mu u^\mu|k; u, a⟩$. In particular, if $u^\mu = (1, 0, 0, 0)$, then the eigenvalue of the Hamiltonian is $k_\mu u^\mu = k_0$, as expected. Moreover, the lowest-energy condition (7.14) ensures that only quanta with positive energy will be excited; if $k_\mu u^\mu < 0$ then $|k; u, a⟩ = 0$.

8 Reparametrization algebra

In the covariant approach we introduced an auxiliary parameter $t$, which is related to physical time through the geodesic equation (4.19). In Minkowski space, the condition $\ddot{q}^\mu(t) = 0$ explicitly breaks reparametrization invariance, and the reparametrization group does hence not act in a well-defined manner on the cohomology. However, if we would apply the formalism to a general-covariant theory such as general relativity, the geodesic equation would not break reparametrization invariance. The reason is that the metric, as one of the physical fields $\phi^\alpha(x)$, is replaced by a parametrized field $\phi^\alpha(x, t)$. It is therefore of interest to study the group of reparametrizations.

The infinitesimal generators are $L_f$, where $f = f(t)d/dt$ is a vector field on the line. The reparametrization algebra acts on the parametrized fields as

$$[L_f, \phi^\alpha(x, t)] = -f(t)\partial_t \phi^\alpha(x, t),$$

$$[L_f, \phi^\star_\alpha(x, t)] = -f(t)\partial_t \phi^\star_\alpha(x, t),$$

$$[L_f, q^\mu(t)] = -f(t)\dot{q}^\mu(t),$$

etc. This translates into an action on the jet data:

$$[L_f, \phi^\alpha_m(t)] = -f(t)\frac{d}{dt}\phi^\alpha_m(t),$$

$$[L_f, \phi^\star_\alpha_m(t)] = -f(t)\frac{d}{dt}\phi^\star_\alpha_m(t),$$

$$[L_f, \pi^\alpha_m(t)] = -f(t)\frac{d}{dt}\pi^\alpha_m(t),$$

$$[L_f, \pi^\star_\alpha_m(t)] = -f(t)\frac{d}{dt}\pi^\star_\alpha_m(t),$$

$$[L_f, q^\mu(t)] = -f(t)\dot{q}^\mu(t),$$

$$[L_f, p_\mu(t)] = -f(t)\dot{p}_\mu(t),$$

(8.2)
etc. A crucial observation is that $L_f$ is not a well defined operator on the Hilbert space $\mathcal{H}(\mathcal{P}^*)$, because infinities arise when it acts on the Fock vacuum. To remedy this, we normal order. However, $L_f$ is still not a well-defined operator, because normal ordering formally gives rise to an infinite central extension. We must therefore regularize the generators further. Fortunately, a natural regularization is available in jet space: simply truncate the Taylor expansion at some fixed, finite order $p$. In other words, we pass from the space of infinite jets to the space of $p$-jets (or rather trajectories in $p$-jet space). This regularization respects all relevant symmetries, such as a Poincaré, diffeomorphism or gauge symmetry, and it is in fact the unique regularization with this property.

A normal-ordered and regularized generator of the reparametrization algebra acting only on the proper fields is thus

$$L_f = -\int dt \ f(t)(\dot{q}^\mu(t)p_\mu(t) + \sum_{|m|\leq p} \frac{d}{dt}\phi_\mu^m(t)\pi_\nu^m(t))$$  \hspace{1cm} (8.3)

Such operators generate a Virasoro algebra,

$$[L_f, L_g] = L_{[f,g]} + \frac{c}{24\pi i} \int dt (\dot{f}(t)\dot{g}(t) - \dot{f}(t)g(t)),$$  \hspace{1cm} (8.4)

with central charge

$$c = 2 + 2n\binom{N + p}{N},$$  \hspace{1cm} (8.5)

where $n$ is the number of components $\phi^\alpha(x, t)$ and $N$ is the dimension of spacetime. The first term is the contribution from the trajectory $q^\mu(t)$ and its momentum $p_\mu(t)$, and the second term comes from the $p$-jets $\phi_\mu^m(t)$; one verifies that a multi-index $m$ can take on $\binom{N + p}{N}$ different values with $|m| \leq p$ in $N$ dimensions.

A regularization must be removed at the end of the process, i.e. we must take the limit $p \to \infty$ in (8.3). A necessary condition for taking this limit is that the central charge converges to a finite value in this limit. Taken at face value, the prospects for succeeding appear bleak. When $p$ is large, $c = \binom{N + p}{N} \approx p^N/N!$, so the central charge diverges. However, there is a way out of this problem. A bosonic $p$-jet contributes $2\binom{N + p}{N}$ to $c$, but a fermionic one makes the same contribution with opposite sign. In a theory with both bosonic and fermionic jets of different order, the leading divergences can cancel, leaving us with a finite $p \to \infty$ limit.
Consider \( n_i \) \((p-i)\)-jets, where the sign of \( n_i \) decides the Grassmann parity; \( n_i > 0 \) for bosons and \( n_i < 0 \) for fermions. The reparametrization algebra acts as a direct sum on the space of collections of jets, with different values of the jet order \( p \). Take the sum of \( r + 1 \) terms like those in (8.5), with \( p \) replaced by \( p, p-1, \ldots, p-r \), respectively, and with \( n \) replaced by \( n_i \) in the \( p-i \) term. The total central charge is

\[
c_{TOT} = n_0 \binom{N + p}{N} + n_1 \binom{N + p}{N - 1} + \ldots + n_r \binom{N + p}{N - r}.
\] (8.6)

Using the recurrence formula,

\[
\binom{n}{i} = \binom{n}{i-1} + \binom{n-1}{i-1},
\] (8.7)

it is straightforward to show that

\[
c_{TOT} = \binom{N - r}{N} n_0,
\] (8.8)

provided that

\[
n_i = n_0 (-)^i \binom{r}{i}.
\] (8.9)

Such a sum of contributions arises naturally from the KT complex, because the antifields are only defined up to an order smaller than \( p \); if \( \phi^{\alpha}_{\text{in}}(t) \) is defined for \( |\mathbf{m}| \leq p \), \( \phi^{\ast}_{\text{in}, \mathbf{m}}(t) \) is only defined for \( |\mathbf{m}| \leq p - 2 \), because the EL equations are second order. The EL jets \( \mathcal{E}_{\alpha, \mathbf{m}}(t) \) with \( |\mathbf{m}| \geq p - 1 \) involves the field jets \( \phi^{\alpha}_{\text{in}}(t) \) with \( |\mathbf{m}| \geq p + 1 \), which were truncated by the regularization. In the full reparametrization algebra, the generator (8.3) is replaced by

\[
L_f = - \int dt \, f(t) \left( : \dot{q}^{\mu}(t) p_{\mu}(t) : + : \ddot{q}^{\mu}(t) p^{\ast}_{\mu}(t) : ight)
+ \sum_{|\mathbf{m}| \leq p} : \frac{d}{dt} \phi^{\alpha}_{\text{m}}(t) \pi^{\mathbf{m}}_{\alpha}(t) : + \sum_{|\mathbf{m}| \leq p - 2} : \frac{d}{dt} \phi^{\ast}_{\alpha, \mathbf{m}}(t) \pi^{\alpha, \mathbf{m}}_{\ast}(t) :
+ \sum_{|\mathbf{m}| \leq p - 1} : \frac{d}{dt} \phi^{\alpha}_{\text{m}}(t) \pi^{\mathbf{m}}_{\alpha}(t) : + \sum_{|\mathbf{m}| \leq p - 3} : \frac{d}{dt} \phi^{\ast}_{\alpha, \mathbf{m}}(t) \pi^{\alpha, \mathbf{m}}_{\ast}(t) : 
\] (8.10)

assuming that the EL equations are second order.

Let us now consider the solutions to (8.9) for the numbers \( n_i \), which can be interpreted as the number of fields and anti-fields. First assume that the
field $\phi^\alpha_m(t)$ is fermionic with $n_F$ components, which gives $n_0 = -n_F$. We may assume, by the spin-statistics theorem, that the EL equations are first order, so the bosonic antifields $\phi^{*\alpha}_{o,m}(t)$ contribute $n_F$ to $n_1$. The barred antifields $\bar{\phi}^\alpha_m(t)$ are also defined up to order $p - 1$, and so give $n_1 = n_F$, and the barred second-order antifields $\bar{\phi}^{*\alpha}_{o,m}(t)$ give $n_2 = -n_F$. For bosons the situation is analogous, with two exceptions: all signs are reversed, and the EL equations are assumed to be second order. Hence $\phi^{*\alpha}_o(t)$ yields $n_2 = -n_B$, and the barred antifields are one order higher.

The situation is summarized in the following tables, where the upper half is valid if the original field is fermionic and the lower half if it is bosonic:

| $i$ | Jet  | Order | $n_i$  |
|-----|------|-------|--------|
| 0   | $\phi^\alpha_m(t)$ | $p$     | $-n_F$ |
| 1   | $\phi^\alpha_{o,m}(t)$ | $p - 1$ | $n_F$  |
| 1   | $\phi^{*\alpha}_{o,m}(t)$ | $p - 1$ | $n_F$  |
| 2   | $\bar{\phi}^\alpha_m(t)$ | $p - 2$ | $-n_F$ |
| 0   | $\phi^\alpha_m(t)$ | $p$     | $n_B$  |
| 1   | $\phi^\alpha_{o,m}(t)$ | $p - 1$ | $-n_B$ |
| 1   | $\phi^{*\alpha}_{o,m}(t)$ | $p - 2$ | $-n_B$ |
| 2   | $\bar{\phi}^\alpha_m(t)$ | $p - 3$ | $n_B$  |

If we add all contributions of the same order, we see that relation in (8.9) can only be satisfied provided that

$$
\begin{align*}
    p : & \quad -n_F + n_B = n_0 \\
    p - 1 : & \quad 2n_F - n_B = r n_0, \\
    p - 2 : & \quad -n_B - n_F = \left(\frac{r}{2}\right) n_0, \\
    p - 3 : & \quad n_B = -\left(\frac{r}{3}\right) n_0, \\
    p - 4 : & \quad 0 = \left(\frac{r}{4}\right) n_0, ...
\end{align*}
$$

(8.12)

The last equation holds only if $r \leq 3$ (or trivially if $n_0 = 0$). On the other hand, if we demand that there is at least one bosonic field, the $p-3$ equation yields $r \geq 3$. Thus we are unambiguously guided to consider $r = 3$ (and thus
The specialization of (8.12) to three dimensions reads

\begin{align*}
    p & : -n_F + n_B = n_0 \\
    p - 1 & : 2n_F - n_B = -3n_0, \\
    p - 2 & : -n_B - n_F = 3n_0, \\
    p - 3 & : n_B = -n_0,
\end{align*}

(8.13)

Clearly, the unique solution to these equations is

\begin{align*}
    n_F &= -2n_0, \\
    n_B &= -n_0.
\end{align*}

(8.14)

The negative sign implies that the sign of the central charge is negative. Thus, in three dimensions $c_{TOT} = -n_0$ independent of the truncation order $p$, provided that the bosonic fields have $n_0$ components and the fermionic fields have twice as many components. Moreover, this is the only case where the total central charge has a finite, non-zero limit when the regulator is removed, i.e. the limit $p \to \infty$ can be taken.

The counting changes in the presence of gauge symmetries. It was observed in [3] that if all gauge symmetries are irreducible, then the $p \to \infty$ limit exists only in four dimensions. This can be viewed as a non-trivial prediction for the spacetime dimension.

9 Conclusion

The key observation in this paper is that we may obtain the physical phase space by reduction from the phase space of histories $\mathcal{P}$. The dynamics, i.e. the Euler-Lagrange equations, play the role of first-class constraints. This allows us to apply standard methods from the theory of constrained Hamiltonian systems, e.g. Dirac brackets and cohomological methods. To obtain a truly covariant formulation, we expand all fields in a Taylor series around the observer’s trajectory, which acquires the status of a physical field, whose dynamics is governed by the geodesic equation.

These methods were then applied to the harmonic oscillator and to the free scalar field. The expected results were recovered if the vacuum is a macroscopic reference state where the observer’s velocity is a classical variable. However, like all other physical objects, the observer’s trajectory is fundamentally quantum, and must be in principle be treated as a quantum object. Observation is a special case of quantum interaction.

The reparametrization algebra can only act in a well-defined manner provided that spacetime has three dimensions and there are twice as many
fermions as bosons. In the presence of gauge symmetries, the counting changes so that spacetime must have four dimensions, apparently in good agreement with experiments.

There are several directions in which the present work can be extended. The present construction is exact but not explicit; extracting numbers will not be simpler than with standard methods. To extract numbers, we need to formulate perturbation theory and understand renormalization from this new viewpoint. Another generalization is to consider theories with gauge symmetries, such as Yang-Mills theory and general relativity. Essential calculations were done already in [3], although with some errors, particularly in nomenclature and physical interpretation.

A most striking new feature is the appearance of new anomalies, which can not be seen in field theory because they are functionals of the observer’s trajectory. E.g., consider the algebra of maps from $\mathbb{R}^N$ to a finite-dimensional Lie algebra $\mathfrak{g}$ with structure constants $f^{ab}_{c}$ and Killing metric $\delta^{ab}$. Let $X = X_{a}(x)J^{a}$ be a generator of this current algebra, and $[X, Y]_{c} = f^{ab}_{c}X_{a}Y_{b}$. It will acquire an extension of the form

$$[\mathcal{J}_{X}, \mathcal{J}_{Y}] = \mathcal{J}_{[X, Y]} - \frac{k}{2\pi i} \delta^{ab} \int dt \frac{\partial}{\partial \rho} X_{a}(q(t))Y_{b}(q(t)),$$

which clearly generalizes affine algebras to several dimensions. The constant $k$ is the value of the second Casimir operator. It vanishes in the special case of the adjoint representation of $u(1)$, so quantization of the free Maxwell field will go through as usual, but interacting gauge fields will necessarily acquire such new quantum corrections. This clearly indicates that making the observer’s trajectory physical is not only an option, but absolutely necessary; a missed anomaly is a serious oversight. This will be the subject of an upcoming publication.

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