Electrodynamics in accelerated frames revisited

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Abstract

Maxwell’s equations are formulated in arbitrary moving frames by means of tetrad fields, which are interpreted as reference frames adapted to observers in space-time. We assume the existence of a general distribution of charges and currents in an inertial frame. Tetrad fields are used to project the electromagnetic fields and sources on accelerated frames. The purpose is to study several configurations of fields and observers that in the literature are understood as paradoxes. For instance, are the two situations, (i) an accelerated charge in an inertial frame, and (ii) a charge at rest in an inertial frame described from the perspective of an accelerated frame, physically equivalent? Is the electromagnetic radiation the same in both frames? Normally in the analysis of these paradoxes the electromagnetic fields are transformed to (uniformly) accelerated frames by means of a coordinate transformation of the Faraday tensor. In the present approach coor-
dinate and frame transformations are disentangled, and the electromagnetic field in the accelerated frame is obtained through a frame (local Lorentz) transformation. Consequently the fields in the inertial and accelerated frames are described in the same coordinate system. This feature allows the investigation of paradoxes such as the one mentioned above.

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1 Introduction

The electromagnetic theory defined by Maxwell’s equations is a remarkable theory developed more than a century ago. From the classical point of view, the limits of the theory seem to be related to phenomena that involve electromagnetic radiation. The electromagnetic radiation emitted by a classical electron in circular orbit is at the roots of the quantum theory. And the radiation of a linearly accelerated charged particle is a beautiful result of the theory that still nowadays is object of discussion. As viewed from a single inertial frame, the electromagnetic radiation of an accelerated charged particle is a well established result of the theory, except for the fact that so far it has not been verified experimentally. However, our intuition of this phenomenon becomes less clear when we consider such radiation field from the point of view of an accelerated frame. Does an accelerated observer measure electromagnetic radiation due to an equally accelerated charged particle? The purpose of this paper is to try to answer this question, as well as to address the two situations described in the Abstract, namely, (i) an accelerated charge in an inertial frame, and (ii) a charge at rest with respect to an accelerated frame. Is the electromagnetic radiation the same in both frames?

The electromagnetic field is described by the Faraday tensor $F^{\mu\nu}$. In the present analysis we will consider that \{\(F^{\mu\nu}\}\ are just tensor components in the flat Minkowski space-time described by arbitrary coordinates $x^\mu$. The projection of $F^{\mu\nu}$ on inertial or noninertial frames yield the electric and magnetic fields $E_x$, $E_y$, $E_z$, $B_x$, $B_y$ and $B_z$. The projection is carried out with the help of tetrad fields $e_\mu^\alpha$. For instance, $E_x = -cF^{(0)(1)}$, where $c$ is the speed of light and $F^{(0)(1)} = e^{(0)}_\mu e^{(1)}_\nu F^{\mu\nu}$. 

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Tetrad fields are considered as reference frames adapted to observers that follow trajectories described by functions $x^\mu(s)$ in space-time. These fields project vectors and tensors in space-time on the local frame of observers. The local projection of the vector $A^\mu(x)$ in space-time, for instance, is defined by $A^a(x) = e^a_\mu(x)A^\mu(x)$, and the projection of the Faraday tensor is $F^{ab}(x) = e^a_\mu(x)e^b_\nu(x)F^{\mu\nu}(x)$. Note that the right hand side and left hand side of these expressions are evaluated at the same space-time event $x^\mu$. Therefore the projection is carried out in the same coordinate system. The measurable quantities are those that are projected on the frame. Thus the laboratory quantities are $F^{ab}$.

In this paper we will write down equations for $F^{ab}$ that are completely equivalent to the well known Maxwell’s equations. These equations hold in any frame, inertial or noninertial frames. This formalism ensures that the procedure for projecting electromagnetic fields on noninertial frames is mathematically and physically consistent. Consequently we may investigate the paradoxes mentioned above. The comparison of the electromagnetic fields in inertial and noninertial frames is possible because these fields are defined in the same coordinate system. We will conclude that the radiation of an accelerated charged particle in an inertial frame is different from the radiation of the charged particle at rest, as viewed from an equally accelerated frame. Consequently, the accelerated motion in space-time is not relative, and the radiation of an accelerated charged particle is an absolute feature of the theory.

Electromagnetic radiation in accelerated systems has been addressed by Anderson and Ryon [1]. They analyzed the three possible cases: I. observer inertial, medium accelerated; II. observer accelerated, medium inertial; III. observer and medium co-accelerated. The subject has also been investigated by other authors [2, 3, 4, 5]. A common feature to all these approaches is that the accelerated frame is determined by means of a coordinate transformation of the Faraday tensor. Therefore in these investigations coordinate transformations and Lorentz transformations stand on equal footing. This is not the point of view that we adopt in this paper. Coordinate and Lorentz transformations are mathematically different transformations, and we bring this difference to the physical realization of the theory.

Notation: space-time indices $\mu, \nu, ...$ and Lorentz (SO(3,1)) indices $a, b, ...$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i, \ a = (0), (i)$. The space-time is flat, and therefore the metric tensor
is $g_{\mu\nu} = (-1, +1, +1, +1)$ in cartesian coordinates. The flat, tangent space Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = (-1, +1, +1, +1)$. The frame components are given by the inverse tetrads $e_{a}^{\mu}$, although we may as well refer to $\{e_{a}^{\mu}\}$ as the frame. The determinant of the tetrad field is represented by $e = \det(e_{a}^{\mu})$.

2 Reference frames in space-time

Tetrad fields constitute a set of four orthonormal vectors in space-time, $\{e^{(0)}_{\mu}, e^{(1)}_{\mu}, e^{(2)}_{\mu}, e^{(3)}_{\mu}\}$, that establish the local reference frame of an observer that moves along a trajectory $C$, represented by functions $x^\mu(s)$ [6, 7, 8] ($s$ is the proper time of the observer). The tetrad field yields the space-time metric tensor $g_{\mu\nu}$ by means of the relation $e^{a}_{\mu}e^{b}_{\nu}\eta_{ab} = g_{\mu\nu}$, and $e^{(0)}_{\mu}$ and $e^{(i)}_{\mu}$ are timelike and spacelike vectors, respectively. We identify the timelike component of the frame with the observer’s velocity $u_{\mu} = dx^\mu/ds$ along the trajectory: $e^{(0)}_{\mu} = u_{\mu}$.

The acceleration $a_{\mu}$ of the observer is given by the absolute derivative of $u_{\mu}$ along $C$,

$$a_{\mu} = \frac{D u_{\mu}}{ds} = \frac{D e^{(0)}_{\mu}}{ds} = u^{a}_{\nu}\nabla_{\alpha}e^{(0)}_{\mu},$$

where the covariant derivative is constructed out of the Christoffel symbols. Thus the derivative of $e^{(0)}_{\mu}$ yields the acceleration along the worldline of an observer adapted to the frame. Therefore a set of tetrad fields for which $e^{(0)}_{\mu}$ describes a congruence of timelike curves is adapted to a class of observers characterized by the velocity field $u_{\mu} = e^{(0)}_{\mu}$ and by the acceleration $a_{\mu}$. If $e^{a}_{\mu} = \delta^{a}_{\mu}$ everywhere in space-time, then $e^{a}_{\mu}$ is adapted to inertial observers, and $a_{\mu} = 0$.

The acceleration of the whole frame is determined by the absolute derivative of $e_{a}^{\mu}$ along $x^{\mu}(s)$. Thus, assuming that the observer carries an orthonormal tetrad frame $e_{a}^{\mu}$, the acceleration of the latter along the path is given by [9, 10]

$$\frac{D e_{a}^{\mu}}{ds} = \phi^{b}_{a}e^{b}_{\mu},$$

where $\phi_{ab}$ is the antisymmetric acceleration tensor. According to Refs. [9, 10], in analogy with the Faraday tensor we may identify $\phi_{ab} \rightarrow (a, \Omega)$, where $a$
is the translational acceleration \((\phi_{(0)(i)} = a_{(i)})\) and \(\Omega\) is the angular velocity of the local spatial frame with respect to a nonrotating (Fermi-Walker transported \([6, 8]\)) frame. It follows from Eq. (2) that

\[
\phi^a_b = e^b_\mu \frac{De_a^\mu}{ds} = e^b_\mu u_\lambda \nabla_\lambda e_a^\mu. \tag{3}
\]

The accelerations \(a^\mu\) and \(\phi_{(0)(i)}\) are related via \(e^{(i)}_\mu a^\mu = e^{(i)}_\mu u^\alpha \nabla_\alpha e_{(0)}^\mu = \phi_{(0)}^{(i)}\).

For a given frame determined by the set of tetrad fields \(e^a_\mu\), the object of anholonomity \(T^\lambda_{\mu\nu}\) is given by \(T^\lambda_{\mu\nu} = e_\alpha^\lambda T^a_{\mu\nu}\), where

\[
T^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu. \tag{4}
\]

Note that \(T^\lambda_{\mu\nu}\) is also the torsion tensor of the Weitzenböck space-time. It is possible to show that in terms of \(T^a_{\mu\nu}\), the acceleration tensor may be written as \([7, 8]\)

\[
\phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}], \tag{5}
\]

where \(T_{abc} = e^a_\mu e^b_\nu e^c_\lambda T_{\mu\nu\lambda}\).

The expression for \(\phi_{ab}\) is not covariant under local Lorentz (SO(3,1) or frame) transformations, but is invariant under coordinate transformations. The noncovariance under local Lorentz transformations allows us to take the values of \(\phi_{ab}\) to characterize the frame. The acceleration tensor \(\phi_{ab}\) represent the inertial accelerations on the frame along \(x^\mu(s)\) \([7, 8]\). As an example, let us consider the tetrad fields adapted to observers at rest in Minkowski space-time. It is given by \(e^a_\mu(ct, x, y, z) = \delta^a_\mu\). We then consider a time-dependent boost in the \(x\) direction, say, after which the tetrad field reads

\[
e^a_\mu(ct, x, y, z) = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{6}
\]

where \(\gamma = (1 - \beta^2)^{-1/2}\), \(\beta = v/c\) and \(v = v(t)\). The frame above is adapted to observers whose four-velocity is \(u^\mu = \dot{e}_{(0)}^\mu(ct, x, y, z) = (\gamma, \beta \gamma, 0, 0)\). After simple calculations we obtain \([7]\)
\[
\begin{align*}
\phi_{(0)(1)} &= \frac{d}{dx^0} \beta \gamma = \frac{d}{dt} \left[ \frac{v/c^2}{\sqrt{1 - v^2/c^2}} \right], \\
\phi_{(0)(2)} &= 0, \\
\phi_{(0)(3)} &= 0,
\end{align*}
\]
and \( \phi_{(i)(j)} = 0 \). The usual hyperbolic motion (uniform acceleration) is characterized by \( \phi_{(0)(1)} = a = \text{constant} \).

For a static object whose four-velocity is given by \( V^\mu = (c, 0, 0, 0) \) we may compute its frame components \( V^a = e^a_\mu V^\mu \) with the help of eq. (6). We find \( V^a = (\gamma c, -\beta \gamma c, 0, 0) \). Thus in the classical limit \( v/c << 1 \) the velocity of the object with respect to the accelerated frame is \( V^{(1)} = -v(t) \), as expected.

3 Maxwell’s equations in moving frames

Electrodynamics is formulated in terms of vector and tensor quantities, the vector potential \( A^\mu \) and the Faraday tensor \( F^{\mu\nu} \) which are related by \( F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The sources are denoted by the four-vector current \( J^\mu \). Space-time indices are raised and lowered by means of the flat space-time metric tensor \( g^{\mu\nu} = (-1, +1, +1, +1) \). On a particular frame the electromagnetic quantities are projected according to \( A^a(x) = e^a_\mu(x) A^\mu(x) \) and \( F^{ab}(x) = e^a_\mu(x) e^b_\nu(x) F^{\mu\nu}(x) \).

An inertial frame is characterized by the vanishing of the acceleration tensor \( \phi_{ab} \). For instance, \( e^a_\mu(t, x, y, z) = \delta^a_\mu \) describes an inertial frame because it satisfies \( \phi_{ab} = 0 \). More generally, all tetrad fields that are function of space-time independent parameters (boost and rotation parameters) determine inertial frames. Suppose that \( A^a \) are components of the vector potential in an inertial frame, i.e., \( A^a = (e^a_\mu)_{in} A^\mu = \delta^a_\mu A^\mu \). The components of \( A^a \) in a noninertial frame are obtained by means of a local Lorentz transformation,

\[
\tilde{A}^a(x) = \Lambda^a_b(x) A^b(x),
\]
where \( \Lambda^a_b(x) \) are space-time dependent matrices that satisfy

\[
\Lambda^a_c(x) \Lambda^b_d(x) \eta_{ab} = \eta_{cd}.
\]
Likewise, we have $\tilde{A}_a(x) = \Lambda^b_a(x)A_b(x)$. An alternative and completely equivalent way of obtaining the field components $\tilde{A}_a(x)$ consists in performing a frame transformation by means of a suitable noninertial frame $e^a_\mu$, namely, in projecting $A^\mu$ on the noninertial frame,

$$\tilde{A}^a(x) = e^a_\mu(x)A^\mu(x).$$  \hfill (10)

The covariant derivative of $A_a$ may be defined as

$$D_aA_b = e^\mu_\mu D_\mu A_b = e^\mu_a(\partial_\mu A_b - e^c_\mu c^b_\mu A_c),$$  \hfill (11)

where

$$e^{c\mu}(\Omega_{abc} - \Omega_{bac} - \Omega_{cab}),$$  \hfill (12)

is the metric-compatible Levi-Civita connection. Note that we are considering the flat space-time, and yet this connection may be nonvanishing. In particular, for noninertial frames it is nonvanishing. The Weitzenböck torsion tensor $T^a_{\mu\nu}$ is also nonvanishing. However, the curvature tensor constructed out of $e^{\mu\nu}_{\mu\nu}$ vanishes: $R^a_{\beta\gamma\delta}(e^{\mu\nu}_{\mu\nu}) = 0$. Under a local Lorentz transformation we have

$$\tilde{e}^{\mu}_{\mu\nu} = \frac{1}{2}e^{c\mu}(\Omega_{abc} - \Omega_{bac} - \Omega_{cab}) = e^{c\alpha}(e^\beta_\mu \partial_\mu e^\gamma_\alpha - e^\gamma_\mu \partial_\mu e^\beta_\gamma),$$  \hfill (13)

It follows from eqs. (8), (12) and (13) that under a local Lorentz transformation we have

$$\tilde{D}_a\tilde{A}_b = \Lambda^c_a(x)\Lambda^d_b(x)D_cA_d.$$  \hfill (14)

The Faraday tensor in a noninertial frame is defined as

$$F_{ab} = D_aA_b - D_bA_a.$$  \hfill (15)

In view of eq. (14) we find that the tensors $F_{ab}$ and $\tilde{F}_{ab}$ in two arbitrary frames are related by

$$\tilde{F}_{ab} = \Lambda^c_a(x)\Lambda^d_b(x)F_{cd}.$$  \hfill (16)
The Faraday tensor defined by eq. (15) is related to the standard expression defined in inertial frames. By substituting (11) in (15) we find

$$F_{ab} = e^a_\mu (\partial_\mu A_b - 0^{\omega_\mu m_b} A_m) - e^b_\mu (\partial_\mu A_a - 0^{\omega_\mu m_a} A_m)$$  \hspace{1cm} (17)

$$= e^a_\mu (\partial_\mu A_b) - e^b_\mu (\partial_\mu A_a) + (0^{\omega_{abm}} - 0^{\omega_{bam}}) A^m.$$  \hspace{1cm} (18)

We make use of the identity

$$0^{\omega_{abm}} - 0^{\omega_{bam}} = T_{mab},$$  \hspace{1cm} (19)

where $T_{mab}$ is given by eq. (4), and write

$$F_{ab} = e^a_\mu e^b_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + T_{mab} A^m$$  \hspace{1cm} (20)

$$+ e^a_\mu (\partial_\mu e^b_\nu) A_\nu - e^b_\mu (\partial_\mu e^a_\nu) A_\nu.$$  \hspace{1cm} (21)

In view of the orthogonality of the tetrad fields we have

$$\partial_\mu e^b_\nu = -e^c_\lambda (\partial_\mu e^\nu_\lambda) e^\nu_c.$$  \hspace{1cm} (22)

With the help of (20) we find that the last two terms of eq. (19) may be rewritten as

$$e^a_\mu (\partial_\mu e^b_\nu) A_\nu - e^b_\mu (\partial_\mu e^a_\nu) A_\nu = -T_{mab} A^m.$$  \hspace{1cm} (23)

Therefore the last three terms of (19) cancel out and finally we have

$$F_{ab} = e^a_\mu e^b_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu).$$  \hspace{1cm} (24)

Thus $F_{ab}$ is just the projection of the Faraday tensor $F_{\mu\nu}$ in the noninertial frame determined by $e^a_\mu$. The scheme characterized by eqs. (10-16) and (24) is in agreement with the procedure developed by Mashhoon [11] in the investigation of electrodynamics of accelerated systems, except that we deal with local fields, contrary to Mashhoon, who considers a nonlocal representation of electromagnetic fields. A physical theory that is constructed out of local fields predicts phenomena whose measurements are pointwise. As argued by Mashhoon, the Bohr-Rosenfeld principle implies that only averages of field components over a finite space-time region are physically meaningful, and therefore a nonlocal formulation of electrodynamics is necessary for
an improvement of the theory. The nonlocal formulation of electrodynamics is still being developed, and does not seem to be mandatory in the present analysis. Note, however, that an ideal accelerated observer (to be discussed in section 4) is described by a one-dimensional timelike trajectory in space-time. Therefore the present formalism may admit nonlocality in time (but not in space). The possible nonlocality in time will pose no problem to the analyses in section 4, since we will be interested in total quantities such as the total radiated power and the total radiated energy.

The covariant derivative of \( F_{ab} \) is defined by

\[
D_a F_{bc} = e_a^\mu D_\mu F_{bc}
\]

\[
= e_a^\mu (\partial_\mu F_{bc} - 0_\mu^m b F_{mc} - 0_\mu^m c F_{bm}).
\]

Making extensive use of relations (18) and (20) we find that the source free Maxwell’s equations in an arbitrary noninertial frame are given by

\[
D_a F_{bc} + D_b F_{ca} + D_c F_{ab} = e_a^\mu e_b^\nu e_c^\lambda (\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu}) = 0.
\]

Maxwell’s equations with sources are obtained from an action integral whose Lagrangian density is given by

\[
L = -\frac{1}{4} e F^{ab} F_{ab} - \mu_0 e A_b J^b,
\]

where \( e = \det(e^a_\mu) \), \( J^b = e^b_\mu J^\mu \) and \( \mu_0 \) is the magnetic permeability constant. Although in flat space-time we have \( e = 1 \), we keep \( e \) in the expressions below because it allows a straightforward inclusion of the gravitational field. Note that in view of eq. (22) we have

\[
F^{ab} F_{ab} = F^{\mu\nu} F_{\mu\nu},
\]

and therefore \( L \) is frame independent. The field equations derived from \( L \) are

\[
\partial_\mu (e F^{\mu b}) + e F^{\mu c} (0^b_\mu c) = \mu_0 e J^b,
\]

or

\[ e_b^\nu [\partial_\mu (e F^{\mu b}) + e F^{\mu c} (0^b_\mu c)] = \mu_0 e J^\nu, \]
where $F^{\mu c} = e_b^{\mu} F^{bc}$. In view of eq. (26) it is clear that the equations above are equivalent to the standard form of Maxwell’s equations in flat space-time.

Equations (24) and (27) are equations for the electromagnetic field components $F_{ab}$ in flat space-time, in arbitrary noninertial frames. They correspond to projections of the standard Maxwell’s equations on an arbitrary frame determined by $e_a^{\mu}$.

The definition of a Lagrangian density such as eq. (25) is not unique. One could instead define the Faraday tensor as

\[ F_{ab} = \partial_a A_b - \partial_b A_a \]

\[ = e_a^{\mu} \partial_{\mu} (e_b^{\nu} A_{\nu}) - e_b^{\mu} \partial_{\mu} (e_a^{\nu} A_{\nu}). \] (29)

Out of the expression above one would consider the Lagrangian density $L'$ defined by

\[ L' = -\frac{1}{4} e F^{ab} F_{ab} - \mu_0 e A_b J^b, \] (30)

The field equations derived from $L'$ read

\[ \partial_{\mu} (e F^{\mu b}) = \mu_0 e J^b. \] (31)

With the help of expression (20) we may rewrite the field equations above with only space-time indices. It reads

\[ \partial_{\mu} F^{\mu \nu} + \frac{1}{2} F^{\mu \lambda} T_{\mu \lambda}^{\nu} = \mu_0 J^\nu. \] (32)

This is precisely the equation presented in ref. [12] (eq. (B.4.33)) in the analysis of Maxwell’s equations in an arbitrary noninertial frame. In view of the discussion above it is clear that eq. (32) is not just the projection of the standard form of Maxwell’s equations on an arbitrary noninertial frame. Moreover, eq. (22) does not hold in this framework. The field equation (27) is derived from a Lagrangian density constructed out of $F_{ab} F_{ab}$ given by (26), and therefore it is clear that if we make $J^\mu = 0$ everywhere in space-time we necessarily arrive at $F_{ab} = 0$. On the other hand, considering eq. (32), it is not immediately clear that $J^\mu = 0$ implies $F_{ab} = 0$ in arbitrary noninertial frames for which $T_{\nu \mu \lambda}^{\nu} \neq 0$. Equation (32) could lead to nontrivial vacuum solutions, which would be a very interesting but unexpected and improbable result of the theory.
As a straightforward consequence of eq. (28), we consider the formulation of Gauss law in the frame determined by eq. (6), where \( v = v(t) \). We assume the existence of the current \( J^\mu = (c\rho(r,t), 0, 0, 0) \), where \( c \) is the speed of light. In an inertial frame we have \( (J^\mu)_m = \delta_\mu^a J^\mu = (c\rho, 0, 0, 0) \). We will denote \( F^{ab} \) the components of the Faraday tensor in the accelerated frame, and (to simplify the notation) \( F^{\mu\nu} \) the components in the inertial frame where the source \( \rho(r,t) \) is defined.

Gauss law is obtained by taking the \( \nu = 0 \) component of eq. (28). In view of the notation above we have

\[
F^{ab} = \begin{pmatrix}
0 & -\tilde{E}_x/c & -\tilde{E}_y/c & -\tilde{E}_z/c \\
\tilde{E}_x/c & 0 & -\tilde{B}_z & \tilde{B}_y \\
\tilde{E}_y/c & \tilde{B}_z & 0 & -\tilde{B}_z \\
\tilde{E}_z/c & -\tilde{B}_y & -\tilde{B}_x & 0
\end{pmatrix}
\] (33)

The components of \( F^{\mu\nu} \) will be denoted without the tilde. The only nonzero component of the Levi-Civita connection \( _0\omega_{0ab} \) is given by

\[
_0\omega_{0(0)(1)} = -\frac{1}{c^2} \gamma^2 \frac{d\beta}{dt}.
\] (34)

Substitution of (33) and (34) into the \( \nu = 0 \) component of (28) yields, after a number of simplifications,

\[
\partial_x \tilde{E}_x + \gamma(\partial_y \tilde{E}_y + \partial_z \tilde{E}_z) + \beta c\gamma(\partial_y \tilde{B}_z - \partial_z \tilde{B}_y) = \frac{\rho}{\varepsilon_0}.
\] (35)

The electric field in the inertial frame is related, by means of local Lorentz transformations, to the fields in the accelerated frame according to

\[
\begin{align*}
E_x &= \tilde{E}_x \\
E_y &= \gamma \tilde{E}_y + \beta c\gamma \tilde{B}_z \\
E_z &= \gamma \tilde{E}_z - \beta c\gamma \tilde{B}_y.
\end{align*}
\] (36)

After substitution of these expressions in eq. (35) we obtain the usual form of Gauss law \( \nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \), as expected. Recall that \( J^\mu = (c\rho(r,t), 0, 0, 0) \), and consequently \( J^a = (\gamma c\rho, -\beta c\gamma\rho, 0, 0) \), by means of eq. (6).

We may instead consider the charge density to be “at rest” in the accelerated frame. In this case \( J^a = (c\rho, 0, 0, 0) \), which is obtained from \( J^\mu = (\gamma c\rho, \beta c\gamma\rho, 0, 0) \), and therefore Gauss law in the accelerated frame in
which the charge density is “at rest” (i.e., the charge density is accelerated with respect to the inertial frame at rest) reads

\[
\partial_x \tilde{E}_x + \gamma (\partial_y \tilde{E}_y + \partial_z \tilde{E}_z) + \beta \gamma (\partial_y \tilde{B}_z - \partial_z \tilde{B}_y) = \frac{\gamma \rho}{\varepsilon_0},
\]

(37)

which is similar to eq. (35), except that the charge density \( \rho \) is increased by a factor \( \gamma \). Written in terms of the inertial frame components, eq. (37) reads

\[ \nabla \cdot \mathbf{E} = (\gamma \rho)/\varepsilon_0. \]

4 Electromagnetic radiation in accelerated frames

An ideal observer in space-time is defined by a timelike trajectory \( x^\mu(s) \), where \( s \) is the proper time, and \( u^\mu = dx^\mu/ds \) is the observer’s velocity. Thus the (one-dimensional) four-velocity \( e_0^\mu = u^\mu \) describes the observer, and \( e_a^\mu \) describes the whole frame. We assume that such ideal observer is equipped with gyroscopes that determine the orientation of the frame and with instruments that perform pointwise measurements. The representation of the observer by a single world line allows to simplify the analysis, and is not a fundamental limitation. We will be ultimately interested in total values of field quantities such as the total radiated power, and thus the present setting is suitable for addressing the qualitative differences that arise in the calculations carried out in inertial and noninertial frames.

The electric and magnetic field components \( (\mathbf{E}, \mathbf{B}) \) and \( (\tilde{\mathbf{E}}, \tilde{\mathbf{B}}) \) in the inertial and accelerated frames, respectively, are related through the expression

\[ F_{ab} = e_a^\mu e_b^\nu F_{\mu\nu}, \]

where \( e_a^\mu \) is given by eq. (6). The relations read

\[
\begin{align*}
\tilde{E}_x &= E_x, \\
\tilde{E}_y &= \gamma E_y - \beta c \gamma B_z, \\
\tilde{E}_z &= \gamma E_z + \beta c \gamma B_y, \\
\tilde{B}_x &= B_x, \\
\tilde{B}_y &= \gamma B_y + \frac{1}{c} \beta \gamma E_z, \\
\tilde{B}_z &= \gamma B_z - \frac{1}{c} \beta \gamma E_y.
\end{align*}
\]

(38)

These relations will be used in the consideration of two known configurations of electromagnetic fields.
4.1 An accelerated point charge in an inertial frame

The first configuration is the field of an accelerated charged particle. Let \( x(t) \) represent the trajectory of a particle of charge \( q \) restricted to move along the \( x \) direction in an inertial frame. We define

\[
b(t) = \frac{v(t)}{c} = \frac{1}{c} \frac{dx(t)}{dt} \hat{x},
\]

such that \( \dot{b} \neq 0 \). The point of observation in space is denoted by \( r \). We also define the vector

\[
R(t) = r - x(t)\hat{x},
\]

and \( \hat{R} = \frac{R}{R} \). The electric and magnetic fields at the space-time event \((r, t)\) are given by (see, for instance, Ref. 13)

\[
E(r, t) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{\hat{R} - b}{\gamma^2 R^2(1 - b \cdot \hat{R})^3} + \frac{\hat{R} \times [(\hat{R} - b) \times b]}{cR(1 - b \cdot \hat{R})^3} \right],
\]

\[
B(r, t) = \frac{1}{c} [\hat{R} \times E]_{\nu'},
\]

where \( t' \) is the retarded time, obtained as the solution of the equation

\[
t' = t - \frac{1}{c} |r - x(t')\hat{x}|.
\]

The frame will be co-moving with the accelerated charged particle, i.e., the frame and the charged particle will be equally accelerated, if we require the vector \( b(t) \) in eqs. (40) and (41) and \( \beta(t) \) in eq. (38) to satisfy \( |b(t)| = \beta(t) \), so that the charged particle will be at rest in the accelerated frame. It is clear that the magnetic field \( \hat{B} \), calculated out of (38) and (41), does not vanish in the accelerated frame (it can be easily calculated), and both \( (\hat{E}, \hat{B}) \) generate a nontrivial Poynting vector \( \hat{S} \). The total power radiated by the point charge is nonvanishing in the co-moving frame.

Note that the Poynting vector is related to the \( T^{\mu\nu} \) components of the energy-momentum tensor \( T^{\mu\nu} \) of the electromagnetic field 14. For arbitrary components of \( T^{\mu\nu} \), a frame transformation defined by eq. (6) (or a local Lorentz transformation) in general leads to nonvanishing \( T^{(0)(i)} \) components. For instance, for the \( T^{(0)(1)} \) component we have
\[ T^{(0)(1)} = \frac{1 + \beta^2}{1 - \beta^2} T^{01} - \beta \gamma^2 (T^{00} + T^{11}). \]

The right hand side of the expression above is clearly nonvanishing in the limit \( \beta \ll 1. \)

In the analysis above we have assumed that the interval between the point charge and a particular observer (both accelerated) is timelike, and that they are not separated by a horizon. If the interval is spacelike, the observer will not detect radiation.

### 4.2 A point charge at rest observed from the point of view of an accelerated frame

The second configuration of electromagnetic field consists in the field of a point charge at rest, at the origin (say) of an inertial frame. It generates only the Coulomb field \( \mathbf{E} = (E_x, E_y, E_z) \). The electric field \( \mathbf{E} \) varies with the radial distance as \( 1/r^2 \). Let \( (\tilde{\mathbf{E}}, \tilde{\mathbf{B}}) \) represent the fields obtained in the accelerated frame by means of eq. (6) and of \( F^{ab} = \varepsilon^a_{\mu} \varepsilon^b_{\nu} F^{\mu\nu} \). The Poynting vector in the accelerated frame is

\[ \tilde{\mathbf{S}} = \frac{1}{\mu_0} \tilde{\mathbf{E}} \times \tilde{\mathbf{B}}, \]

whose components are given by

\[ \tilde{S}_x = -\frac{1}{\mu_0 c} \beta \gamma^2 (E_y^2 + E_z^2) \]

\[ \tilde{S}_y = -\frac{1}{\mu_0 c} \beta \gamma E_x E_y \]

\[ \tilde{S}_z = \frac{1}{\mu_0 c} \beta \gamma E_x E_z, \]

in view of (38). It is clear from the expressions above that the Poynting vector \( \tilde{\mathbf{S}} \) varies with the radial distance as \( 1/r^4 \), and therefore the total power due to \( \tilde{\mathbf{S}} \), measured in the accelerated frame, vanishes. Thus this situation is not physically equivalent to that in which the point charge is accelerated with respect to an inertial frame. The two situations are not relative to each other.

The difference between the two physical situations discussed above becomes more clear in the nonrelativistic limit where \( v(t) \) is finite but \( \beta << 1. \)
For the two physical situations the integral of the Poynting vector over a two-dimensional spherical surface of constant radius $r_0$ around the observer can be easily calculated and compared to each other. Note that the tetrad field for an inertial observer $e^a_{\mu}(t, x, y, z) = \delta^a_{\mu}$ and for the accelerated observer given by eq. (6) are written in the same coordinate system, and therefore a (spacelike) spherical surface of constant radius may be taken to be the same for both observers. In the evaluation of total quantities we require $r_0 \rightarrow \infty$.

5 Conclusion

In this paper we have investigated the formulation of electrodynamics in accelerated frames in flat space-time. The Faraday tensor and Maxwell’s equations are considered as vector and tensor quantities in the space-time described by arbitrary coordinates \{\(x^\mu\}\}, and are projected on the frame of an accelerated observer by means of tetrad fields. We are then able to obtain a consistent formulation of Maxwell’s equations in any noninertial frame in flat space-time. The advantage of our approach is that the Faraday tensor (as well as Maxwell’s equations) in the inertial and noninertial frames are written in the same coordinate system, a feature that allows the comparison of the fields in the two frames.

The introduction of the gravitational field is straightforward. It amounts to replacing the flat space-time tetrad field by the one that yields the gravitational field according to $e^a_{\mu}e_{a\nu} = g_{\mu\nu}$, and that is adapted to an observer, as described in section 2. The tetrad field describes both a noninertial frame and the gravitational field.

The conclusion of the two situations discussed in the section 4 is that the accelerated motion in space-time is intrinsically absolute, not relative. The accelerated motion of a point charge in an inertial frame is not physically equivalent to a point charge at rest with respect to an accelerated frame. Moreover, the radiation field of an accelerated point charge is measurable even in a co-moving frame. The relative motion in space-time seems to be verified only in the realm of inertial frames in Special Relativity. This conclusion holds as long as the interpretation of the tetrad field as a geometrical quantity that projects vectors and tensors on frames is valid.

Equations (24) and (27) may be worked out to yield the equations for electromagnetic waves in arbitrary accelerated frames. This issue will be investigated elsewhere.
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