Abstract

Our aim in this note is to show that, for any $\epsilon > 0$, there exists a union-closed family $F$ with (unique) smallest set $S$ such that no element of $S$ belongs to more than a fraction $\epsilon$ of the sets in $F$. More precisely, we give an example of a union-closed family with smallest set of size $k$ such that no element of this set belongs to more than a fraction $(1 + o(1))\log_{2}k$ of the sets in $F$.

We also give explicit examples of union-closed families containing 'small' sets for which we have been unable to verify the Union-Closed Conjecture.

Mathematics Subject Classifications: 05D05

1 Introduction

If $X$ is a set, a family $F$ of subsets of $X$ is said to be union-closed if the union of any two sets in $F$ is also in $F$. The Union-Closed Conjecture (a conjecture of Frankl [5]) states that if $X$ is a finite set and $F$ is a union-closed family of subsets of $X$ (with $F \neq \emptyset$), then there exists an element $x \in X$ such that $x$ is contained in at least half of the sets in $F$. Despite the efforts of many researchers over the last forty-five years, and a recent Polymath project [7] aimed at resolving it, this conjecture remains wide open. It has only been proved under very strong constraints on the ground-set $X$ or the family $F$; for example, Balla, Bollobás and Eccles [3] proved it in the case where $|F| \geq \frac{3}{4}2^{|X|}$; more recently, Karpas [6] proved it in the case where $|F| \geq (\frac{1}{2} - c)2^{|X|}$ for a small absolute...
constant $c > 0$; and it is also known to hold whenever $|X| \leq 12$ or $|F| \leq 50$, from work of Vučkođ and Živković [11] and of Roberts and Simpson [9]. Note that the Union-Closed Conjecture is not even known to hold in the weaker form where we replace the fraction 1/2 by any other fixed $\epsilon > 0$.

For general background and a wealth of further information on the Union-Closed Conjecture see the survey of Bruhn and Schaudt [4].

As usual, if $X$ is a set we write $\mathcal{P}(X)$ for its power-set. If $X$ is a finite set and $F \subset \mathcal{P}(X)$ with $F \neq \emptyset$, we define the frequency of $x$ (with respect to $F$) to be $\gamma_x = |\{A \in F : x \in A\}|/|F|$, i.e., $\gamma_x$ is the proportion of members of $X$ that contain $x$. If a union-closed family contains a ‘small’ set, what can we say about the frequencies in that set?

If a union-closed family $F$ contains a singleton, then that element clearly has frequency at least 1/2, while if it contains a set $S$ of size 2 then, as noted by Sarvate and Renaud [10], some element of $S$ has frequency at least 1/2. However, they also gave an example of a union-closed family $F$ whose smallest set $S$ has size 3 and yet where each element of $S$ has frequency below 1/2. Generalising a construction of Poonen [8], Bruhn and Schaudt [4] gave, for each $k \geq 3$, an example of a union-closed family with (unique) smallest set of size $k$ and with every element of that set having frequency below 1/2.

However, in these and all other known examples, there is always some element of a minimal-size set having frequency at least 1/3. So it is natural to ask if there is really a constant lower bound for these frequencies.

Our aim in this note is to show that this is not the case.

**Theorem 1.** For any positive integer $k$, there exists a union-closed family in which the (unique) smallest set has size $k$, but where each element of this set has frequency

$$\left(1 + o(1)\right) \frac{\log k}{2k}.$$

(All logarithms in this paper are to base 2. Also, as usual, the $o(1)$ denotes a function of $k$ that tends to zero as $k$ tends to infinity.)

Theorem 1 is proved by an explicit construction. It is asymptotically sharp, in view of results of Wójcik [12] and Balla [2]: Wójcik showed that if $S$ is a set of size $k \geq 1$ in a finite union-closed family, then the average frequency of the elements in $S$ is at least $c_k$, where $k \cdot c_k$ is defined to be the minimum average set-size over all union-closed families on the ground-set $[k]$, and Balla showed that $c_k = (1 + o(1)) \frac{\log k}{2k}$, confirming a conjecture of Wójcik from [12].

Remarkably, there are union-closed families containing small sets, even sets of size 3, for which we have been unable to verify the Union-Closed Conjecture. We give some examples at the end of the paper.

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1Note added in proof: shortly before the acceptance of this manuscript, Gilmer [arXiv:2211.09055] obtained a breakthrough on the Union-Closed Conjecture, showing that it holds in the weaker form with the fraction 1/2 replaced by 1/100.
2 Proof of main result

For our construction, we need the following ‘design-theoretic’ lemma.

Lemma 2. For any positive integers \( k > t \) there exist infinitely many positive integers \( d \) such that \( t \) divides \( dk \) and the following holds. If \( X \) is a set of size \( dk/t \), then there exists a family \( A = \{A_1, \ldots, A_k\} \) of \( k \) \( d \)-element subsets of \( X \), such that each element of \( X \) is contained in exactly \( t \) sets in \( A \), and for \( 2 \leq r \leq t \), any \( r \) distinct sets in \( A \) have intersection of size

\[
\frac{d(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)},
\]

i.e.

\[
|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}| = \frac{d(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)}
\]

for any \( 1 \leq i_1 < i_2 < \cdots < i_r \leq k \).

Proof. Let \( q \) be a positive integer, and set \( d = \binom{k-1}{t-1}q^t \); we will take \( |X| = \binom{k}{t}q^t \). Partition \([k]q\) into \( k \) sets, \( B_1, B_2, \ldots, B_k \) say, each of size \( q \); we call these sets ‘blocks’. We let \( X \) be the set of all \( t \)-element subsets of \([k]q\) that contain at most one element from each block. For each \( i \in [k] \) we let \( A_i \) be the family of all sets in \( X \) that contain an element from the block \( B_i \). Clearly, \( |A_i| = \binom{k-1}{t-1}q^t = d \) for each \( i \in [k] \), and each element of \( X \) appears in exactly \( t \) of the \( A_i \). Also, for example \( A_i \cap A_j \) consists of all sets in \( X \) that contain both an element from the block \( B_i \) and an element from the block \( B_j \), so

\[
|A_i \cap A_j| = \binom{k-2}{t-2}q^t = \binom{k-1}{t-1}q^t \frac{t-1}{k-1} = d \frac{t-1}{k-1}.
\]

It is easy to check that the other intersections also have the claimed sizes. \( \Box \)

We remark that, in what follows, it is vital that the integer \( d \) in Lemma 2 can be taken to be arbitrarily large as a function of \( k \) and \( t \).

Proof of Theorem 1. We define \( n = dk/t + k \), we take \( d \in \mathbb{N} \) as in the above lemma, and we let \( X = [dk/t] \); the claim yields a family \( A = \{A_1, \ldots, A_k\} \) of \( k \) \( d \)-element subsets of \( X = [dk/t] \) such that each element of \([dk/t]\) is contained in exactly \( t \) of the sets in \( A \), and for any \( 2 \leq r \leq t \), any \( r \) distinct sets in \( A \) have intersection of size

\[
\frac{d(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)}.
\]

Write \( m = dk/t \). We take \( F \subset \mathcal{P}([n]) \) to be the smallest union-closed family containing the \( k \)-element set \( \{m+1, m+2, \ldots, m+k\} \) and all sets of the form \( \{m+i\} \cup (X \setminus \{x\}) \) where \( i \in [k] \) and \( x \in A_i \).

For brevity, we write \( S_0 = \{m+1, m+2, \ldots, m+k\} \). We will show that each element of \( S_0 \) has frequency

\[
(1 + o(1)) \frac{\log k}{2k},
\]

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provided \( t \) and \( d \) are chosen to be appropriate functions of \( k \); moreover, with these choices, \( S_0 \) will be the smallest set in \( \mathcal{F} \).

Clearly, \( \mathcal{F} \) contains \( S_0 \), all sets of the form \( S_0 \cup (X \setminus \{x\}) \) for \( x \in X \), all sets of the form \( R \cup X \) where \( R \) is a nonempty subset of \( S_0 \), and finally all sets of the form \( R \cup (X \setminus \{x\}) \), where \( R = \{m + i_1, \ldots, m + i_r\} \) is a nonempty \( r \)-element subset of \( S_0 \) and \( x \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r} \), for \( 1 \leq r \leq t \). It is easy to see that the family \( \mathcal{F} \) contains no other sets.

It follows that

\[
|\mathcal{F}| = 1 + dk/t + (2^k - 1) + \sum_{r=1}^{t} \binom{k}{r} d \frac{(t-1)(t-2) \cdots (t-r+1)}{(k-1)(k-2) \cdots (k-r+1)}
\]

\[
= dk/t + 2^k + \frac{d}{t} \sum_{r=1}^{t} \binom{t}{r}
\]

\[
= dk/t + 2^k + \frac{d}{t} (2^t - 1)
\]

\[
= 2^k + \frac{dk2^t}{t}.
\]

On the other hand, the number of sets in \( \mathcal{F} \) that contain the element \( m + 1 \) is equal to

\[
1 + dk/t + 2^{k-1} + \sum_{r=1}^{t} \binom{k-1}{r-1} d \frac{(t-1)(t-2) \cdots (t-r+1)}{(k-1)(k-2) \cdots (k-r+1)}
\]

\[
= 1 + dk/t + 2^{k-1} + d \sum_{r=1}^{t} \binom{t-1}{r-1}
\]

\[
= 1 + dk/t + 2^{k-1} + 2^{t-1} d.
\]

It follows that the frequency of \( m + 1 \) (or, by symmetry, of any other element of \( S_0 \)) equals

\[
\frac{1 + kd/t + 2^{k-1} + 2^{t-1} d}{2^k + dk2^t/t} = \frac{(1 + 2^{k-1})/d + k/t + 2^{t-1}}{2^k/d + k2^t/t}.
\]

To (asymptotically) minimise this expression, we take \( t = \lfloor \log k \rfloor \) and \( d \to \infty \) (for fixed \( k \)); this yields a union-closed family in which the (unique) smallest set (namely \( S_0 \)) has size \( k \), and every element of that set has frequency

\[
(1 + o(1)) \frac{\log k}{2k},
\]

proving the theorem.

\[\square\]

### 3 An open problem

We now turn to some explicit examples of union-closed families containing small sets for which we have been unable to establish the Union-Closed Conjecture. For simplicity, we
concentrate on the most striking case, when the family contains a set of size 3, and indeed is generated by sets of size 3.

Our families live on ground-set $\mathbb{Z}_n^2$, the $n \times n$ torus.

**Question 3.** Let $n \in \mathbb{N}$ and let $R \subset \mathbb{Z}_n$ with $|R| = 3$. Does the Union-Closed Conjecture hold for the union-closed family $\mathcal{F}$ of subsets of $\mathbb{Z}_n^2$ generated by all the translates of $R \times \{0\}$ and of $\{0\} \times R$?

(Here, as usual, we say a union-closed family $\mathcal{F}$ is generated by a family $\mathcal{G}$ if it consists of all unions of sets in $\mathcal{G}$.)

Perhaps the most interesting case is when $n$ is prime. In that case we may assume that $R = \{0, 1, r\}$ for some $r$, and so one feels that the verification of the Union-Closed Conjecture should be a triviality, but it seems not to be. Note that all the families in Question 3 are transitive families, in the sense that all points ‘look the same’, so that the Union-Closed Conjecture is equivalent to the assertion that the average size of the sets in the family is at least $n^2/2$.

We mention that the corresponding result in $\mathbb{Z}_n$ (in other words, the special case of the Union-Closed Conjecture for the union-closed family on ground-set $\mathbb{Z}_n$ generated by all translates of $R$) is known to hold: this is proved in [1].

We have verified the special case of Question 3 where $R = \{0, 1, 2\}$. A sketch of the proof is as follows. Assume that $n \geq 6$, and let $\mathcal{F} \subset \mathcal{P}(\mathbb{Z}_n^2)$ be the union-closed family generated by all translates of $\{0, 1, 2\} \times \{0\}$ and of $\{0\} \times \{0, 1, 2\}$ (we call these translates 3-tiles, for brevity). Let $C = \{0, 1, 2, 3\}^2$, a $4 \times 4$ square. Consider the bipartite graph $H = (\mathcal{X}, \mathcal{Y})$ with vertex-classes $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{X}$ consists of all subsets of $C$ with size less than 8 that are intersections with $C$ of sets in $\mathcal{F}$, and $\mathcal{Y}$ consists of all subsets of $C$ with size greater than 8 that are intersections with $C$ of sets in $\mathcal{F}$, and we join $S \in \mathcal{X}$ to $S' \in \mathcal{Y}$ if $|S'| + |S| \geq 16$ and $S' = S \cup U$ for some union $U$ of 3-tiles that are contained within $C$. It can be verified (by computer) that $H$ has a matching $m : \mathcal{X} \to \mathcal{Y}$ of size $|\mathcal{X}| = 16520$. Such a matching $m$ gives rise to an injection

$$f : \{S \in \mathcal{F} : |S \cap C| < |C|/2\} \to \{S \in \mathcal{F} : |S \cap C| > |C|/2\}$$

given by

$$f(S) = (S \setminus C) \cup m(S \cap C)$$

with the property that $|S \cap C| + |f(S) \cap C| \geq |C|$ for all $S \in \mathcal{F}$ with $|S \cap C| < |C|/2$. It follows that a uniformly random subset of $\mathcal{F}$ has intersection with $|C|$ of expected size at least $|C|/2$, which in turn implies that there is an element of $C$ with frequency at least 1/2 (and in fact, since $\mathcal{F}$ is transitive, every element has frequency at least 1/2).

We remark that this proof does not work if one tries to replace $C = \{0, 1, 2, 3\}^2$ by $\{0, 1, 2\}^2$, as the resulting bipartite graph $H' = (\mathcal{X}', \mathcal{Y}')$ does not contain a matching of size $|\mathcal{X}'|$.

We remark also that it would be nice to find a non-computer proof of the above result.

**Acknowledgements**

We are very grateful to Igor Balla for bringing the papers [2] and [12] to our attention.
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