Cutoff for Almost All Random Walks on Abelian Groups

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Abstract

Consider the random Cayley graph of a finite group $G$ with respect to $k$ generators chosen uniformly at random, with $1 \ll \log k \ll \log |G|$; denote it $G_k$. A conjecture of Aldous and Diaconis [1] asserts, for $k \gg \log |G|$, that the random walk on this graph exhibits cutoff. Further, the cutoff time should be a function only of $k$ and $|G|$, to sub-leading order.

This was verified for all Abelian groups in the ’90s. We extend the conjecture to $1 \ll k \lesssim \log |G|$. We establish cutoff for all Abelian groups under the condition $k - d(G) \gg 1$, where $d(G)$ is the minimal size of a generating subset of $G$, which is almost optimal. The cutoff time is described (abstractly) in terms of the entropy of random walk on $Z^k$. This abstract definition allows us to deduce that the cutoff time can be written as a function only of $k$ and $|G|$ when $d(G) \ll \log |G|$ and $k - d(G) \asymp k \gg 1$; this is not the case when $d(G) \asymp \log |G| \asymp k$.

For certain regimes of $k$, we find the limit profile of the convergence to equilibrium.

Wilson [46] conjectured that $Z^d_2$ gives rise to the slowest mixing time for $G_k$ amongst all groups of size at most $2^d$. We give a partial answer, verifying the conjecture for nilpotent groups. This is obtained via a comparison result of independent interest between the mixing times of nilpotent $G$ and a corresponding Abelian group $G'$, namely the direct sum of the Abelian quotients in the lower central series of $G$. We use this to refine a celebrated result of Alon and Roichman [3]: we show for nilpotent $G$ that $G_k$ is an expander provided $k - d(G) \gtrsim \log |G|$. As another consequence, we establish cutoff for nilpotent groups with relatively small commutators, including high-dimensional special groups, such as Heisenberg groups.

The aforementioned results all hold with high probability over the random Cayley graph $G_k$.

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1 Introduction and Statement of Results

1.1 Motivation, Brief Overview of Results and Notation

1.1.1 Motivating Conjectures of Aldous and Diaconis and Wilson

We analyse properties of the random walk (abbreviated \(RW\)) on a Cayley graph of a finite group. The generators of this graph are chosen independently and uniformly at random. Precise definitions are given in §1.4.1; for now, let \(G\) be a finite group, let \(k\) be an integer (allowed to depend on \(G\)) and denote by \(G_k\) the Cayley graph of \(G\) with respect to \(k\) independently and uniformly random generators. We consider values of \(k\) with \(1 \ll k \ll \log |G|\) for which \(G_k\) is connected with high probability (abbreviated \(whp\)), ie with probability tending to 1 as \(|G|\) grows.

Since pioneering work of Erdős, it has been understood that the typical behaviour of \(random\) objects in some class can shed valuable light on the class as a whole. Thus, when considering some class of combinatorial objects, it is natural to ask questions such as the following.

* What does a typical object in this class ‘look like’?
* If an object is chosen uniformly at random, which properties hold with high probability?

Aldous and Diaconis [1] applied this philosophy to the study of random walks on groups.

Aldous and Diaconis [1, 2] coin the phrase \(cutoff\) phenomenon: this occurs when the total variation distance (TV) between the law of the RW and its invariant distribution drops abruptly from close to 1 to close to 0 in a time-interval of smaller order than the mixing time. The material in this article is motivated by a conjecture of theirs regarding ‘universality of cutoff’ for the RW on the random Cayley graph \(G_k\): it is given in [1, Page 40], which is an extended version of [2].

Conjecture (Aldous and Diaconis, 1985). For any group \(G\), if \(k \gg \log |G|\) and \(k \ll \log |G|\), then the random walk on \(G_k\) exhibits cutoff \(whp\). Further, the cutoff time, to leading order, is independent of the algebraic structure of the group: it can be written as a function only of \(k\) and \(|G|\).

This conjecture spawned a large body of work, including [19, 20, 30, 31, 32, 43, 46]; see §1.3. It has been established in the Abelian set-up by Dou and Hildebrand [20, 30]; see §1.3.1 and §7.2 where we give a short proof. Save [32] which considers the cyclic group \(\mathbb{Z}_p\) for prime \(p\) and [46] which considers \(\mathbb{Z}_d^2\) (which enforces \(k \geq d = \log_2 |G|\)), focus has been on \(k \gg \log |G|\).

We establish cutoff for all Abelian groups when \(1 \ll k \ll \log |G|\) under almost optimal conditions in terms of group-generation. We also give simple conditions under which the cutoff time is independent of the algebraic structure of the group.

The second part of this article is motivated by a conjecture of Wilson. Wilson [46] establishes cutoff for the RW on \(G_k\) when \(G = \mathbb{Z}_d^2\) and then conjectures that \(\mathbb{Z}_d^2\) is the slowest amongst all groups of size at most \(2^d\), asymptotically as \(d \to \infty\); see [46, Theorem 1 and Conjecture 7].

Conjecture (Wilson, 1997). For all diverging \(d\) and \(n\) with \(n \leq 2^d\) and all groups \(G\) of size \(n\), if \(k - \log_2 n \gg 1\) and \(k \ll \log n\), then \(t_{\text{mix}}(\varepsilon, G_k)/t_{\text{mix}}(\varepsilon', H_k) \leq 1 + o(1)\) whp for all \(\varepsilon, \varepsilon' \in (0, 1)\) where \(H := \mathbb{Z}_d^2\)—ie, the mixing time for \(G_k\) is at most that of \(H_k\) whp up to smaller order terms.

We establish a comparison between the mixing times for nilpotent and Abelian groups. Wilson’s conjecture in the nilpotent set-up is an immediate consequence of this general comparison result. We apply our nilpotent–Abelian comparison theorem to establish cutoff for various examples of non-Abelian groups, including \(p\)-groups with ‘small’ commutators and Heisenberg groups.

1.1.2 Brief Overview of Results

Our focus is on mixing properties of the RW on the random Cayley graph \(G_k\). We consider the limit as \(n := |G| \to \infty\) under the assumption that \(1 \ll k \ll \log |G|\). The condition \(1 \ll k \ll \log |G|\) is necessary for cutoff on \(G_k^n\) for all nilpotent \(G\); see Remark A.6.

We establish cutoff when \(G\) is any Abelian group, requiring only \(k - d(G) \gg 1\), where \(d(G)\) is the minimal size of a generating subset of \(G\). We show that the leading order
1.1.3 Notation and Terminology

Cayley graphs. Motivation for this model and an overview of historical work is given in Cayley graphs. Up to a slightly adjusted definition of simple, with high probability, groups. Likewise, the quantities ≍ are a cutoff window, time-interval, known as the cutoff time. We establish cutoff w.p. for the RW on G_k where G is a nilpotent group with a relatively small commutator. Examples of such groups include high-dimensional extra special or Heisenberg groups.

Lastly, we show that the random Cayley graph of a nilpotent group G is an expander whp whenever k ≳ log |G| and k − d(G) ≳ k. (If G is Abelian, then G = G.)

Introduce by Aldous and Diaconis [1], there has been a great deal of research into these random Cayley graphs. Motivation for this model and an overview of historical work is given in §1.3.

1.1.3 Notation and Terminology

Cayley graphs are either directed or undirected; we emphasise this by writing G_k+ and G_k-, respectively. When we write G_k or G_k^+, this means “either G_k^+ or G_k^−”, corresponding to the undirected, respectively directed, graphs with generators chosen independently and uniformly at random.

Conditional on being simple, G_k+ is uniformly distributed over the set of all simple degree-k Cayley graphs. Up to a slightly adjusted definition of simple for undirected Cayley graphs, our results hold with G_k replaced by a uniformly chosen simple Cayley graph of degree k; see §1.4.2.

Our results are for sequences (G_N)_N∈N of finite groups with |G_N| → ∞ as N → ∞. For ease of presentation, we write statements like “let G be a group” instead of “let (G_N)_{N∈N} be a sequence of groups”. Likewise, the quantities d(G) and, of course, k appearing in the statements all correspond to sequences, which need not be fixed (or bounded) unless we explicitly say otherwise. In the same vein, an event holds with high probability (abbreviated whp) if its probability tends to 1.

We use standard asymptotic notation: “≪” or “o(·)” means “of smaller order”; “≪” or “O(·)” means “of order at most”; “≈” means “of the same order”; “≈” means “asymptotically equivalent”.

1.2 Statements of Main Results

We analyse mixing in the total variation (abbreviated TV) distance. The uniform distribution on G, denoted π_G, is invariant for the RW. Let S = (S(t))_{t≥0} denote the RW on G_k; its law is denoted P_{G_k}(S(t) ∈ ·). For t ≥ 0, denote the TV distance between the law of S(t) and π_G by

\[ d_{G_k}(t) := \| P_{G_k}(S(t) ∈ ·) − π_G \|_{TV} := \max_{A ∈ G} | P_{G_k}(S(t) ∈ A) − |A|/|G| |. \]

Throughout, unless explicitly specified otherwise, we use continuous time: t ≥ 0 means t ∈ [0, ∞).

1.2.1 Cutoff for All Abelian Groups

We use standard notation and definitions for mixing and cutoff; see, eg, [34, §4 and §18].

**Definition.** A sequence (X^N)_{N∈N} of Markov chains is said to exhibit cutoff when, in a short time-interval, known as the cutoff window, the TV distance of the distribution of the chain from equilibrium drops from close to 1 to close to 0, or more precisely if there exists (t_N)_{N∈N} with

\[ \lim_{N → ∞} \sup_{N → ∞} d_N(t_N(1 − ε)) = 1 \quad \text{ and } \quad \lim_{N → ∞} \sup_{N → ∞} d_N(t_N(1 + ε)) = 0 \quad \text{ for all } \ ε ∈ (0, 1), \]

where d_N(·) is the TV distance of X^N(·) from its equilibrium distribution for each N ∈ N.

We say that a RW on a sequence of random graphs (H_N)_{N∈N} exhibits cutoff around time (t_N)_{N∈N} whp if, for all fixed ε, in the limit N → ∞, the TV distance at time (1 + ε)t_N converges in distribution to 0 and at time (1 − ε)t_N to 1, where the randomness is over H_N.
To extend the Aldous–Diaconis conjecture to $1 \ll k \lesssim \log |G|$, one needs additional assumptions. For an Abelian group $G$, write $d(G)$ for the minimal size of a generating set of $G$. If $k < d(G)$, then the group cannot be generated by any choice of generators. Pomerance [42] shows that the expected number of independent, uniform generators required to generate the group is at most $d(G) + 3$. (That is, if $Z_1, Z_2, \ldots \sim \text{iid } \text{Unif}(G)$ and $\kappa \in \mathbb{N}$ is minimal with $(Z_1, \ldots, Z_\kappa) = G$, then $d(G) \leq \mathbb{E}(\kappa) \leq d(G) + 3$.) Thus $k - d(G) \gg 1$ is always sufficient for $G$ to be generated by $\{Z_k^+, \ldots, Z_k^-\}$ whp (by Markov’s inequality); we assume this throughout. In many cases, $k - d(G) \gg 1$ is necessary to generate the group whp, and so this assumption cannot be removed. For a characterisation of these cases and related discussion, see [25, Lemma 8.1]. The condition $k - d(G) \asymp k$ is particularly relevant for the Aldous–Diaconis conjecture; see Remark A.4.

We use an entropic method, which involves defining entropic times; see §1.3.5 for a high-level description of the method and §2.1 for the specific application. The main idea is to use an auxiliary process $W_t$ to generate the walk $S_t$; one then studies the entropy of the process $W$. Write $Z = [Z_1, \ldots, Z_k]$ for the (multiset of) generators of the Cayley graph; then $G_k$ corresponds to choosing $Z_1, \ldots, Z_k \sim \text{iid } \text{Unif}(G)$. Here, $W_i(t)$ is, for each $i$, the number of times generator $Z_i$ has been applied minus the number of times $Z_i^{-1}$ has been applied; $W$ is a rate-1 RW on $\mathbb{Z}^k$. Then $S(t) = W(t) \cdot Z$ when the group is Abelian. (This auxiliary process $W$ is key even when studying nilpotent groups.) For undirected graphs, $W$ is the usual simple RW (abbreviated SRW): a coordinate is selected uniformly at random and incremented/decremented by 1 each with probability $\frac{1}{2}$. For directed graphs, inverses are never applied, so a step of $W$ is as follows: a coordinate is selected uniformly at random and incremented by 1; we term this the directed RW (abbreviated DRW).

**Definition A.** For $\gamma \in \mathbb{N} \cup \{\infty\}$, let $t^\pm_\gamma(k, G)$ be the time at which the entropy of rate-1 RW (ie SRW or DRW, as appropriate) on $\mathbb{Z}^k$ is $\log |G|/\gamma |G|$, where $\gamma G := \{\gamma g \mid g \in G\}$; we use the convention, $Z_\infty := Z$ and $\infty G := |G| G = \{\text{id}\}$. Set $t^+_\gamma(k, G) := \max_{\gamma \in \mathbb{N}} t^\pm_\gamma(k, G)$.

We establish cutoff for all Abelian groups, under almost optimal conditions on $k$ in terms of $G$. This gives an affirmative answer for Abelian groups in a strong sense to the primary part of the conjecture (occurrence of cutoff) of Aldous and Diaconis [13]; we discuss the secondary part (time depending only on $k$ and $|G|$) in Remark A.4.

Cutoff has already been established for Abelian groups when $k \gg \log |G|$ with $k \ll \log |G|$, as mentioned above; see §1.3.1. We thus restrict our statements to $1 \ll k \lesssim \log |G|$. For $1 \ll k \lesssim \log |G|$, only two groups had been considered previously: $\mathbb{Z}_2^k$ in [46] and $\mathbb{Z}_p^k$ with $p$ prime in [32]. Recall that $1 \ll \log k \ll \log |G|$ is necessary for cutoff for nilpotent $G$, eg Abelian $G$; see Remark A.6. More refined statements are given in Theorems 2.4, 3.6 and 4.1.

**Theorem A.** Let $G$ be an Abelian group and $k$ an integer with $1 \ll k \lesssim \log |G|$. Suppose that $k - d(G) \gg 1$. Then the RW on $G^k$ exhibits cutoff at time $t^+_\gamma(k, G)$ whp.

Moreover, if $k - d(G) \asymp k$ and $d(G) \ll \log |G|$, then $t_\gamma(k, G) \sim t_{\infty}(k, \log |G|) \sim |G|^{2/k}/(2\pi e)$. If $k > d(G)$, then $t_\gamma(k, G) \leq k |G|^{2/k} \log k$. If $k - d(G) \asymp \log |G|$, then $t_\gamma(k, G) \sim k |G|^{2/k}$.

We now give some remarks on this theorem. Further remarks are deferred to §1.2.4.

**Remark A.1.** For certain regimes of $k$ we find the limit profile of the convergence to equilibrium: we define entropic times $t_\alpha$ and show that $dG_t(t_\alpha) \to^\Psi \Psi(\alpha)$, where $\Psi$ is the standard Gaussian tail; see Definition 2.1, Proposition 2.2 and Theorem 2.4. This holds for any Abelian group if, for example, $k - d(G) \asymp k$ and $1 \ll k \ll \log |G|/\log \log \log |G|$ or $k - d(G) \gg 1$ and $1 \ll k \ll \sqrt{\log |G|/\log \log \log |G|}$. The result holds for any $1 \ll k \ll \log |G|$ under some constraints on the group. In [25, Theorem A] we show the same for $k \asymp \log |G|$, again with some constraints on $G$. \(\triangle\)

**Remark A.2.** From the abstract entropic definition, amongst Abelian groups $\mathbb{Z}_2^k$ is the slowest:

$$\max \{t_\gamma(k, G) \mid G \text{ Abelian group with } |G| \leq 2^d \} = t_\gamma(k, \mathbb{Z}_2^d).$$

This thus verifies Wilson’s conjecture in the Abelian set-up. \(\triangle\)
Remark A.3. The entropic time $t_{\infty}$ arises naturally; see §2.5 for an outline. In essence, we desire
\[ W_t := \{ w \in \mathbb{Z}^k \mid \mathbb{P}(W(t) = w) \ll 1/|G| \} = \{ w \in \mathbb{Z}^k \mid -\log \mathbb{P}(W(t) = w) - \log |G| \gg 1 \} \]
to satisfy $\mathbb{P}(W(t) \in W_t) = 1 - o(1)$. We thus want the entropy of $W(t)$ to be at least $\log |G|$.

The arisal of the entropic times $t_\gamma$ is more delicate. We outline this in §3.5. \hfill \triangle

One can also consider cutoff in $L_2$ distance, instead of TV (ie $L_1$). For time $t \geq 0$, define
\[ d_{G_k}^{(2)}(t) := \left\| \mathbb{P}_{G_k} (S(t) \in \cdot) - \pi_G \right\|_{2,\pi_G} := (|G|^{-1} \sum_{g \in G} (|G| \mathbb{P}_{G_k}(S(t) = g) - 1)^2)^{1/2}. \]
One can then define mixing and cutoff with respect to $L_2$ analogously to TV ($L_1$) distance.

It turns out that $L_2$ mixing time is at least a constant larger than the TV. Similar considerations to those in Remark A.3 suggest that for the $L_2$ mixing the key condition is $\mathbb{P}(W(2t) = 0) \ll 1/|G|$. This leads us to the following conjecture for the $L_2$ mixing time, which we state informally.

Conjecture A. For $\gamma \in \mathbb{Z} \cup \{ \infty \}$, let $\hat{t}_\gamma^k := \hat{t}_\gamma^k(k, G)$ be the time $t$ at which the return probability for RW on $\mathbb{Z}^k$ at time $2t$ is $|G|^{-1}$. Set $\hat{t}_\gamma^k(k, G) := \max_{\gamma \in \mathbb{Z}} \hat{t}_\gamma^k(k, G)$. Then, under similar conditions to those of Theorem A, whp, the RW on $G_k$ exhibits cutoff in the $L_2$ metric at time $\hat{t}_\gamma^k(k, G)$.

We also consider cutoff in separation distance. For time $t \geq 0$, define
\[ s_{G_k}(t) := \max_{g \in G} \left\{ 1 - \mathbb{P}_{G_k}(S(t) = g) \right\}. \]
One can then define mixing and cutoff with respect to separation distance analogously to TV.

It is standard that, under reversibility, the TV and separation mixing times differ by up to a factor 2; see, eg, [34, Lemmas 6.16 and 6.17]. However, Hermon, Lacoin and Peres [22, Theorem 1.1] showed that TV and separation cutoff are not equivalent, and that neither one implies the other.

We analyse the regime $k - d(G) \gg \log |G|$; in this regime, we show that separation cutoff occurs whp, and moreover that the cutoff time is the same, up to subleading order, as for TV.

A more refined statement is given in Theorem 5.1.

Theorem B. Let $G$ be an Abelian group and $k$ an integer. Suppose that $1 \ll \log k \ll \log |G|$ and $k - d(G) \gg \max\{(\frac{k}{2}) \log |G|)^2, (\log |G|)^{1/2}\}$. Then the RW on $G_k$ exhibits cutoff in separation distance at time $t_*(k, G)$ whp.

Remark B. The conditions hold if $k \gtrsim (\log |G|)^{3/4}, \log k \ll \log |G|$ and $k - d(G) \gg (\log |G|)^{1/2}$. Analogously to Remark A.2, the slowest amongst Abelian groups for separation mixing is $\mathbb{Z}_2^d$.

1.2.2 Comparison of Mixing Times Between Different Groups

The previous results established cutoff. The next results are of a slightly different flavour. They consider nilpotent groups: these are groups $G$ whose lower central series is the sequence $(G_\ell)_{\ell \geq 0}$ defined by $G_0 := G$ and $G_\ell := [G_{\ell-1}, G]$ for $\ell \geq 1$, stabilises at the trivial group. The results compared the mixing times between different groups; these mixing times are random.

Definition. For $\varepsilon \in (0,1)$ and a Cayley graph $H$, write $t_{\text{mix}}(\varepsilon, H) := \inf\{ t \geq 0 \mid d_H(t) \leq \varepsilon \}$.

For two sequences $H := (H_N)_{N \in \mathbb{N}}$ and $H' := (H'_N)_{N \in \mathbb{N}}$ of random Cayley graphs, say that $t_{\text{mix}}(H)/t_{\text{mix}}(H') \leq 1 + o(1)$ whp if there exist non-random sequences $(\gamma_N)_{N \in \mathbb{N}}$ and $(\delta_N)_{N \in \mathbb{N}}$ with $\lim_{N \to \infty} \delta_N = 0$ such that, for all $\varepsilon, \varepsilon' \in (0,1)$, we have
\[ \lim_{N \to \infty} \mathbb{P}(t_{\text{mix}}(\varepsilon, H_N) \leq (1 + \delta_N)\gamma_N) = 1 = \lim_{N \to \infty} \mathbb{P}((1 - \delta_N)\gamma_N \leq t_{\text{mix}}(\varepsilon', H'_N)). \]

We establish Wilson’s conjecture in the nilpotent set-up, as the following theorem describes.

Theorem C. For all diverging $d$ and $n$ with $n \leq 2^d$ and all nilpotent groups $G$ of size $n$, if $k - \log_2 n \gg 1$ and $\log k \ll \log n$, then $t_{\text{mix}}(G_k)/t_{\text{mix}}(H_k) \leq 1 + o(1)$ whp where $H := \mathbb{Z}_2^d$. 5
As noted in Remark A.2, for Abelian groups this follows from our cutoff result and the abstract entropic definition of the cutoff time $t_*(k, G)$ for Abelian $G$. The extension to nilpotent groups is then established by Theorem D below, which is of independent interest. It is quite significantly stronger than Wilson’s conjecture in the nilpotent set-up. We can use it to establish cutoff for a class of nilpotent groups with ‘small commutator’; see Corollary D.

**Theorem D.** Let $G$ be a nilpotent group. Set $G := \oplus \mathbb{T} (G_{\ell} / G_{d})$ where $(G_{\ell})_{\ell \geq 0}$ is the lower central series of $G$ and $L := \min \{ \ell \geq 0 \mid G_{\ell} = \{ \mathrm{id} \} \}$. Suppose that $1 \ll k \ll |G|$ and $k - d(G) \gg 1$. Then $t_{\text{mix}}(G_{k}) / t_{\text{mix}}(G_{k}) \leq 1 + o(1)$ whp.

Besides being the key ingredient in the proof of Wilson’s conjecture in the nilpotent case, we demonstrate that this result is tight enough to establish cutoff for some nilpotent groups. For a group $G$, denote by $G^\text{com} := [G, G]$ its commutator and by $G^{ab} := G / G^\text{com}$ its Abelianisation.

**Corollary D.1.** Let $G$ be a finite, non-Abelian, nilpotent group and $k$ such that $1 \ll k \ll |G|$. Then

- If $k \lesssim \log |G|$, then suppose that $k \gg d(G, G) \log |G, G|$ and $k - d(G^{ab}) \gg d(G, G)$.
- If $k \gg \log |G|$, then suppose only that $\log |G, G| \ll \log |G^{ab}|$.

Then the RW on $G_{k}$ exhibits cutoff at $t_{*}(k, G^{ab})$ whp.

For step-2 nilpotent groups, $[G, G]$ is Abelian and hence $[G, G] = [G, G]$. The above corollary is thus particularly applicable for these groups. A particular example of such groups is special groups with small commutator. For a prime $p$, a $p$-group is special if it is step-2 and has centre $Z(G)$, Frattini subgroup $\Phi(G)$ and commutator subgroup $[G, G]$ all equal and elementary Abelian (ie isomorphic to $\mathbb{Z}_p^s$ for some $s$). In this case, also $G^{ab} = \mathbb{Z}_p^s$, where $r := \ell - s$ and $\ell := \log_{p} |G|$.

Using this particular form of the Abelianisation and commutator, we can relax the conditions on $k$. Note that $t_{p}(k, \mathbb{Z}_{p}^r)$ is the time at which the entropy of RW on $\mathbb{Z}_{p}^k$ reaches $\log(p^r) = \log |\mathbb{Z}_{p}^r|$.

**Corollary D.2.** Let $p$ be prime, $G$ be a non-Abelian, special $p$-group and $k$ such that $1 \ll \log k \ll |G|$. Let $r := \log_{p} |G^{ab}|$, $s := \log_{p} |G^{com}|$ and $\ell := r + s = \log_{p} |G|$. Suppose that $k \geq \ell$.

- If $k \lesssim \log |G|$, then suppose that $k \gg s^2 \log p$ and $k - r \gg s$.
- If $k \gg \log |G|$, then suppose only that $s \ll r$.

Then the RW on $G_{k}$ exhibits cutoff at $t_{*}(k, G^{ab}) = t_{p}(k, \mathbb{Z}_{p}^r)$ whp conditional that $G_{k}$ is connected. If $(k - r)p \gg 1$, then $G_{k}$ is connected whp. If $k - r \asymp k$ and $p \gg 1$, then $t_{p}(k, \mathbb{Z}_{p}^r) \asymp t_{\infty}(k, \mathbb{Z}_{p}^r)$.

Special groups are ubiquitous amongst $p$-groups of a given size in a precise, quantitative sense. Hence Corollary D.2 is applicable to many groups. See Remark 6.11 for a precise statement as well as some asymptotic expressions. Sims [44] gives, for given $(p, \ell, s)$, a simple, explicit description of all special groups of size $p^s$ whose commutator is of size $p^\ell$.

*Extra special* groups satisfy $G^{\text{com}} \cong \mathbb{Z}_p$ (so $d(G^{\text{com}}) = 1$) and $|G| = p^{2d - 3}$ for some integer $d \geq 3$. For given $d$ and $p \neq 2$, up to isomorphism there are only two extra special groups. One of these is the Heisenberg group, which can be defined for $p$ not prime also. For (not necessarily prime) $m, d \in \mathbb{N}$, the Heisenberg group $H_{m, d}$ is the set triples $(x, y, z) \in \mathbb{Z}_{m}^{d - 2} \times \mathbb{Z}_{m}^{d - 2} \times \mathbb{Z}_{m}$ with

$$(x, y, z) \circ (x', y', z') := (x + x', y + y', z + z + x \cdot y'),$$

where $x \cdot y'$ is the usual dot product for vectors in $\mathbb{Z}_{m}^{d - 2}$. We have $H_{m, d}^{ab} \cong \mathbb{Z}_{m}^{2d - 4}$ and $H_{m, d}^{\text{com}} \cong \mathbb{Z}_{m}$.

For $p$ prime, $H_{p, d}$ with $d \gg 1$ falls into the class analysed in Corollary D.2 with $r = 2d - 4$ and $s = 1$. The following corollary thus focusses on $H_{m, d}$ with $m$ not (necessarily) prime. Note that $t_{\infty}(k, \mathbb{Z}_{m}^r)$ is the time at which the entropy of RW on $\mathbb{Z}_{m}^d = \mathbb{Z}_{m}^{d - 2}$ reaches $\log(m^r) = \log |\mathbb{Z}_{m}^r|$.

**Corollary D.3.** Let $m, d \in \mathbb{N}$ with $d \gg 1$. Suppose that $k - 2d \gg 1$, $k \gg \log m$ and $k \ll d \log m \ll \log |H_{m, d}|$. Then whp, the RW on $(H_{m, d})_{k}$ exhibits cutoff at $t_{*}(k, H_{m, d}^{ab}) \cong \mathbb{Z}_{m}^{2d - 4}$. If additionally $k - 2d \asymp k$ and $m \gg 1$, then $t_{*}(k, \mathbb{Z}_{m}^{2d - 4}) = t_{m}(k, \mathbb{Z}_{m}^{2d - 4}) \asymp t_{\infty}(k, \mathbb{Z}_{m}^{2d - 4}) \approx km^{2d - 4} / (2me)$.

If $m$ is fixed (and thus $d \gg 1$), then the condition $k \gg \log m$ is absorbed into $k \gg 1$. Thus this corollary handles arbitrary Heisenberg groups $H_{m, d}$ with $m$ fixed and $k - 2d \gg 1$. 

6
We now give some remarks on Theorem D and Corollary D.

**Remark D.1.** Wilson’s conjecture requires \( k - \log_2 |G| \gg 1 \) and compares \( t_{\text{mix}}(G_k) \) with \( t_*(k, Z_k^2) \). We have \( d(G) \leq \max \{ \ell \in \mathbb{N} \mid p^\ell \text{ divides } |G| \text{ for some prime } p \} \leq \log_2 |G|; \) often \( d(G) \) is much smaller than \( \log_2 |G| \). (In fact, in some precise sense of choosing an Abelian group \( H \) uniformly, typically \( d(H) \ll \log_2 |H| \).) Further, \( t_{\text{mix}}(G_k) \) may be significantly smaller than \( t_*(k, Z_k^2) \).

The bounds on \( t_*(k, G) \), for Abelian \( G \), described in Theorem A complement the upper bound \( t_{\text{mix}}(G_k) \leq t_{\text{mix}}(G_k^*) \) to give explicit bounds on \( t_{\text{mix}}(G_k) \) which hold whp. \( \triangle \)

**Remark D.2.** In the course of proving this theorem, we prove an exact relation between the \( L_2 \) mixing time for the RWs on \( G_k \) and \( G_k^* \), namely \( E(d(G_k^*)^2(t)) \leq E(d(G_k^*)^2(t)) \). We actually prove a more refined version of this which allows us to compare the modified \( L_2 \) distances, i.e. the \( L_2 \) distances conditional that \( W(t) \) is in some ‘typical set’, i.e. \( W(t) \in W_t \); recall Remark A.3. We use precisely this modified \( L_2 \) calculation to upper bound the TV mixing time in the proof of Theorem A. The comparison of TV distances, namely Theorem D, follows since \( P(W(t) \in W_t) = 1 - o(1) \). \( \triangle \)

As explained below, it is natural to conjecture that Theorem D does not require \( G \) to be nilpotent. The definition of the Abelian group \( G \) corresponding to \( G \) required \( G \) to be nilpotent. We extend this definition to allow general group \( G \). (The definitions are equivalent if \( G \) is nilpotent.)

The following conjecture extends Theorem D; it contains, as a special case, Wilson’s conjecture.

**Conjecture D.** Let \( G \) be a group. Let \( (G_t)_{t \geq 0} \) be its lower central series and \( L := \min \{ \ell \geq 0 \mid G_\ell = \{ \text{id} \} \} \). Let the prime decomposition of \( |G_L| \) be \( |G_L| = \prod_j p_j \). Set \( G := \langle \oplus_j \langle G_{t-1}/G_t \rangle \rangle \otimes \langle \oplus_j Z_{p_j} \rangle \). Suppose that \( 1 \ll \log k \ll \log |G| \) and \( k - d(G) \gg 1 \). Then \( t_{\text{mix}}(G_k)/t_{\text{mix}}(G_k^*) \leq 1 + o(1) \) whp.

We are showing in Theorem D, for nilpotent groups, that being non-Abelian can only speed up the mixing. Finite nilpotent groups are intuitively thought of as ‘almost Abelian’; this is (partially) because two elements having co-prime orders must commute. Thus removing the nilpotent property should only mean the group is ‘farther from Abelian’ and speed up the mixing.

### 1.2.3 Expander Graphs of Nilpotent Groups

Our last result considers the expansion properties of the random Cayley graph.

**Definition E.** The isoperimetric constant of a finite \( d \)-regular graph \( G = (V, E) \) is defined as

\[
\Phi_\ast := \min_{1 \leq |S| \leq \frac{|V|}{2}} \Phi(S) \quad \text{where} \quad \Phi(S) := \frac{1}{|S|} \left| \{ (a, b) \in E \mid a \in S, b \in S^c \} \right|.
\]

**Theorem E.** Let \( G \) be a nilpotent group. Set \( G := \langle \oplus_j \langle G_{t-1}/G_t \rangle \rangle \) where \( (G_t)_{t \geq 0} \) is the lower central series of \( G \) and \( L := \min \{ \ell \geq 0 \mid G_\ell = \{ \text{id} \} \} \). For all \( c > 0 \), there exists a \( c' > 0 \) so that if \( k - d(G) \geq c \log |G| \), then \( \Phi_\ast(G_k) \geq c' \) whp.

**Remark E.** This theorem is already known when \( k - \log_2 |G| \gg k \), without the nilpotent restriction; it is a celebrated result of Alon and Roichman [3]. It thus suffices to consider only \( k \approx \log |G| \). \( \triangle \)

### 1.2.4 Further Remarks on Theorem A

Here we make some remarks on Theorem A in addition to the three in §1.2.1.

**Remark A.4.** When \( d(G) \ll \log |G| \) and \( k - d(G) \gg k \), one can check that \( t_*(k, G) \) is the same as the time at which the entropy of rate-1 RW on \( Z_k^d \) is \( \log |G| \). When \( 1 \ll k \ll \log |G| \), this is \( k|G|^{2/k}/(2\pi e) \), up to smaller order terms; see Proposition 3.2b. This means that the natural extension of the Aldous-Diaconis conjecture to the regime \( 1 \ll k \ll \log |G| \) is verified in full for Abelian groups when \( d(G) \ll \log |G| \) and \( k - d(G) \gg k \).

However, when \( k \approx \log |G| \gg d(G) \), while cutoff is still exhibited whp, the cutoff time does not depend only on \( k \) and \( |G| \). Eg, if \( k \approx 2 \log(4^r) \), then \( Z_k^2 \) and \( Z_3^3 \) give rise to mixing times which
differ by a constant factor. There are even regimes with $1 \ll k \ll \log |G|$ where the claim does not hold, provided $1 \ll k - d(G) \ll k$; see [25, Proposition 3.2 and Theorem 3.4] where $\mathbb{Z}_p$ is studied (with both of $p$ and $d$ allowed to diverge).

**Remark A.5.** For $k \gg \log |G|$ with $\log k \ll \log |G|$, cutoff has been established for all Abelian groups, at an explicit time, and this time is an upper bound on the mixing time for arbitrary (not just Abelian) groups; see §1.3.1. It is not difficult to show the explicit time given is asymptotically equivalent to $t_\infty(k, |G|)$, which in turn is equivalent to $t_\star(k, |G|)$; see, eg, [27, Proposition B.19].

**Remark A.6.** This article establishes cutoff in a variety of set-ups, but always in the regime $1 \ll \log k \ll \log |G|$. This leaves the regimes $k \asymp 1$ and $\log k \asymp \log |G|$, for which there is no cutoff for any choice of generators: when $k \asymp 1$, this holds whenever the group is nilpotent; when $\log k \asymp \log |G|$, this holds for all groups. The former result is due to Diaconis and Saloff-Coste [16]; we give a short exposition of this in [25, §4]. The latter result is proved in §7.2 below: the mixing time is order 1. Dou [19, Theorems 3.3.1 and 3.4.7] establishes a more general result for $\log k \asymp \log |G|$. 

**Remark A.7.** Our approach lifts the walk $S$ from the Abelian Cayley graph $G(Z)$ to a walk $W$ on the free Abelian group with $k = |Z|$ generators. Note that the walk $W$ is independent of $Z$, ie of which $k$ generators are used. We then study the lifted walk $W$, in particular its entropic profile, before projecting back from $W$ to $S$. This gives us a candidate mixing time; see §1.3.5 and §2.1. 

**Remark A.8.** The theorem is established via two distinct approaches: the former applies for $k$ not growing too rapidly; the second can be seen as a refinement of the first, optimised for larger $k$, where the first breaks down. We combine the two approaches to analyse an interim regime of $k$.

We separate the exposition of the approaches: they are given in §2, §3 and §4, respectively. In the first two a concept of entropic times is defined; see §2.2 and §3.2. A precise statement for each approach is given; see §2.4, §3.4 and §4.1, respectively. In summary, Theorem A is a direct consequence of Propositions 2.2 and 3.2 and Theorems 2.4, 3.6 and 4.1.

### 1.3 Historic Overview

In this subsection, we give a fairly comprehensive account of previous work on mixing and cutoff for random walk on random Cayley graphs; we compare our results with existing ones. The occurrence of cutoff in particular has received a great deal of attention over the years. We also mention, where relevant, other results which we have proved in companion papers; see also §1.4.3.

#### 1.3.1 Universal Cutoff: The Aldous–Diaconis Conjecture

Aldous and Diaconis [1, Page 40] stated their conjecture for $k \gg \log |G|$. A more refined version is given by Dou [19, Conjectures 3.1.2 and 3.4.5]; see also [30, 43]. An informal, more general, variant was reiterated by Diaconis in [13, Chapter 4G, Question 8]; he gave some related open questions recently in [14, §5]. Towards the conjecture, an upper bound, valid for arbitrary groups, was established by Dou and Hildebrand [20, Theorem 1] and later Roichman [43, Theorems 1 and 2], who simplified their argument. A matching lower bound, valid only for Abelian groups, was given by Hildebrand [30, Theorem 3]; see also Hildebrand [31, Theorem 5]. Dou and Hildebrand [20, Theorem 4] modify the proof of [30, Theorem 3] to extend from Abelian groups to some families of groups with irreducible representations of bounded degree. Combined, this established the Aldous–Diaconis conjecture for Abelian groups and such groups with low degree irreducible representations.

Moreover, the cutoff time was determined explicitly: it is at

$$T(k, |G|) := \log |G| / \log (k / \log |G|) = \frac{\log k}{\rho-1} \log |G|$$

where $\rho$ is defined by $k = (\log |G|)^\rho$.

(To have $k \gg \log |G|$, one needs $\rho - 1 \gg 1 / \log \log |G|$.) See also Dou [19] and Hildebrand [31].

There is a trivial diameter-based lower bound of $\log_k |G|$. If $\rho \gg 1$, ie $k$ is super-polylogarithmic in $|G|$, then $T(k, |G|) \approx \log_k |G|$. Thus cutoff is established for all groups for such $k$.

In [24, Theorem B], using the group $U_{m,d}$ of $d \times d$ unit upper triangular matrices with entries in $\mathbb{Z}_m$, we disprove the part of the conjecture concerning the independence of the cutoff time
from the algebraic structure of the group: if $d \geq 3$ is fixed and $k = (\log |U_{m,d}|)^{1+1/d}$, then there is cutoff at $\frac{4}{3} T(k, |U_{m,d}|)$. In fact, $T(k, |U_{m,d}|)$ does not even capture the correct order: letting $d \to \infty$ sufficiently slowly, we still have $k = (\log |U_{m,d}|)^{1+1/d} \gg \log |U_{m,d}|$ and the cutoff time is still shown to be $\frac{4}{3} T(k, |U_{m,d}|)$, which is $o(T(k, |U_{m,d}|))$.

There has been some investigation into the regime $1 < k \leq \log |G|$, but with much less success. Hildebrand [30, Theorem 4] showed that the mixing time must be super-polynomial, unlike for $k \gg \log |G|$. Wilson [46, Theorem 1] established cutoff for $\mathbb{Z}_2^d$; this naturally requires $k \geq d = \log_2 |G|$. Regarding $1 < k \ll \log |G|$, a breakthrough came recently when Hough [32, Theorem 1.7] established cutoff for $\mathbb{Z}_p$ with $1 < k \leq \log p/\log \log p$ and $p$ a (diverging) prime. The techniques were specialised to their respective cases; we consider arbitrary Abelian groups.

### 1.3.2 Comparison of Mixing Times

In the direction of comparison of mixing times, there has been much less work. The only work of note (of which we are aware) is by Pak [39]. There he studies universal mixing bounds (ie ones valid for all groups), but his bounds are not tight; they are always at least a constant factor away from those conjectured by Wilson [46] (and by us above).

A related universal bound in which $\mathbb{Z}_2^d$ is the worst case is given by Pak [40]. Let $\varphi_k(G) := \mathbb{P}(G_k$ is connected), ie the probability that the group $G$ is generated by $k$ uniformly chosen generators. Then Pak [40, Lecture 1, Theorem 6] proves that if $|G| \leq 2^d$ then $\varphi_k(G) \geq \varphi(\mathbb{Z}_2^d)$ for all $k$.

### 1.3.3 Random Walks on Upper Triangular Matrix Groups

The study of random walks on Heisenberg groups and other groups of upper triangular matrices has a rich history. We give a detailed historical account in [24, §1.3.2].

As noted above, in [24] we study $d \times d$ unit upper triangular matrices with entries in $\mathbb{Z}_m$. By viewing the Heisenberg group as $d \times d$ matrices (see §1.1.3), these $d \times d$ unit upper triangular matrices can be seen as a supergroup of the $d$-dimensional Heisenberg group.

### 1.3.4 Expander Graphs for Nilpotent Groups

A celebrated result of Alon and Roichman [3, Corollary 1] asserts that, for any finite group $G$, the random Cayley graph with at least $C_\varepsilon \log |G|$ random generators is whp an $\varepsilon$-expander, provided $C_\varepsilon$ is a sufficiently large (in terms of $\varepsilon$). (A graph is an $\varepsilon$-expander if its isoperimetric constant is bounded below by $\varepsilon$; up to a reparametrisation, this is equivalent to the spectral gap of the graph being bounded below by $\varepsilon$.) There has been a considerable line of work building upon this general result of Alon and Roichman. (Pak [38] proves a similar result.) Their proof was simplified and extended, independently, by Koh and Schulman [35] and Landau and Russell [33]; both were able to replace $\log_2 |G|$ by $\log_2 D(G)$, where $D(G)$ is the sum of the dimensions of the irreducible representations of the group $G$; for Abelian groups $D(G) = |G|$. A ‘derandomised’ argument for Alon–Roichman is given by Chen, Moore and Russell [10]. Both [10, 33] use some Chernoff-type bounds on operator valued random variables.

Christofides and Markström [11] improve these further by using matrix martingales and proving a Hoeffding-type bound on operator valued random variables. They also improved the quantification for $C_\varepsilon$, showing that one may take $C_\varepsilon := 1+c_\varepsilon$ with $c_\varepsilon \to 0$ as $\varepsilon \to 0$; this means that, whp, the graph is an $\varepsilon$-expander whenever $k \geq (1+c_\varepsilon) \log_2 D(G)$ and $c_\varepsilon \to 0$ as $\varepsilon \to 0$. They also generalise Alon–Roichman to random cotet graphs. The proofs use tail bounds on the (random) eigenvalues.

It is well-known that $D(G) \geq \sqrt{|G|}$. Thus all these results require at least $k \geq \frac{1}{2} \log_2 |G|$. Our result, on the other hand, applies $k \geq c \log |G|$ for any constant $c > 0$, provided the underlying group is suitable—eg, this is the case if $G$ is Abelian and $d(G) \ll \log |G|$; another example is given by $d \times d$ unit upper triangular matrix groups with entries in $\mathbb{Z}_m$ if $m \gg 1$.

Hough [32, Theorem 1.1] showed, for all diverging (sequences of) primes $p$, that the order of the relaxation time of the RW on the cyclic group $\mathbb{Z}_p$ is $p^{2/k}$ when $1 < k \leq \log p/\log \log p$.

In [26, Theorem D], we restrict to Abelian groups under the assumption $k - 2d(G) \asymp k$ and determine (via an altogether different method) the order of the relaxation time whenever $1 < k \ll \log |G|$; it is $|G|^{2/k}$ whp. Thus $k = \log |G|$ and $k - 2d(G) \asymp k$ gives relaxation time order 1, which
is equivalent to being an expander by the Cheeger inequalities. Further, we show for ‘most $G$’ (in a precise sense) that $k - d(G) \asymp k$ suffices. This extends, in the Abelian set-up, the above results.

1.3.5 Cutoff for ‘Generic’ Markov Chains and the Entropic Method

We now put our results into a broader context. A common theme in the study of mixing times is that ‘generic’ instances often exhibit the cutoff phenomenon. In this set-up, a family of transition matrices chosen from a certain family of distributions is shown to, whp, give rise to a sequence of Markov chains which exhibits cutoff. A few notable examples include random birth and death chains [17, 45], the simple or non-backtracking random walk on various models of sparse random graphs, including random regular graphs [37], random graphs with given degrees [5, 6, 7, 8], the giant component of the Erdős–Rényi random graph [7] (where the authors consider mixing from a ‘typical’ starting point) and a large family of sparse Markov chains [8], as well as random walks on a certain generalisation of Ramanujan graphs [9] and random lifts [9, 12].

A recurring idea in the aforementioned works is that the cutoff time can be described in terms of entropy. One can look at some auxiliary random process which up to the cutoff time can be coupled with, or otherwise related to, the original Markov chain—often in the above examples this is the RW on the corresponding Benjamini–Schramm local limit. The cutoff time is then shown to be (up to smaller order terms) the time at which the entropy of the auxiliary process equals the entropy of the invariant distribution of the original Markov chain. It is a relatively new technique, and has been used recently in [7, 8, 9, 12]. For ‘most’ regimes of $k$, this is the case for us too; further, for the non-Abelian groups considered in [24] we use a similar idea. As our auxiliary random process, we use a SRW, respectively DRW, in the undirected, respectively directed, case.

With the exception of the very recent [28], to the best of our knowledge, all previous instances where the entropic method was used the graphs were tree-like. This is not the case for us: in the Abelian set-up, $G_k$ has cycles of length 4 (potentially up to the direction of edges). Admittedly, this has less of an impact on the walk since each vertex is of diverging degree.

1.4 Additional Remarks

1.4.1 Precise Definition of Cayley Graphs

Consider a finite group $G$. Let $Z$ be a multiset of $G$. We consider geometric properties, namely through distance metrics and the spectral gap, of the Cayley graph of $(G, Z)$; we call $Z$ the generators. The undirected, respectively directed, Cayley graph of $G$ generated by $Z$, denoted $G^{-}(Z)$, respectively $G^{+}(Z)$, is the multigraph whose vertex set is $G$ and whose edge multiset is $\{(g, g \cdot z) \mid g \in G, z \in Z\}$, respectively $\{(g, g \cdot z) \mid g \in G, z \in Z\}$.

If the walk is at $g \in G$, then a step in $G^{+}(Z)$, respectively $G^{-}(Z)$, involves choosing a generator $z \in Z$ uniformly at random and moving to $gz$, respectively one of $gz$ or $gz^{-1}$ each with probability $\frac{1}{2}$.

We focus attention on the random Cayley graph defined by choosing $Z_1, \ldots, Z_k \sim \text{Unif}(G)$; when this is the case, denote $G^+_k := G^+(Z)$ and $G^-_k := G^-(Z)$. While we do not assume that the Cayley graph is connected (ie, $\tilde{Z}$ may not generate $G$), in the Abelian set-up the random Cayley graph $G_k$ is connected whp whenever $k - d(G) \gg 1$; see [25, Lemma 8.1]. In the nilpotent set-up, this is the case whenever $k - d(G/[G, G]) \gg 1$; see [24, Remark E.1].

The graph depends on the choice of $Z$. Sometimes it is convenient to emphasise this; we use a subscript, writing $P_{G(z)}(\cdot)$ if the graph is generated by the group $G$ and multiset $z$. Analogously, $P_{G_k}(\cdot)$ stands for the random law $P_{G(Z)}(\cdot)$ where $Z = [Z_1, \ldots, Z_k]$ with $Z_1, \ldots, Z_k \sim \text{Unif}(G)$.

1.4.2 Typical and Simple Cayley Graphs

The directed Cayley graph $G^+(z)$ is simple if and only if no generator is picked twice, ie $z_i \neq z_j$ for all $i \neq j$. The undirected Cayley graph $G^-(z)$ is simple if in addition no generator is the inverse of any other, ie $z_i \neq z_j^{-1}$ for all $i, j \in [k]$. In particular, this means that no generator is of order 2, as any $s \in G$ of order 2 satisfies $s = s^{-1}$—this gives a multiedge between $g$ and $gs$ for each $g \in G$.

The RW on $G^{-}(Z)$ is equivalent to an adjusted RW on $G^{+}(Z)$ where, when a generator $s \in z$ is chosen, instead of applying a generator $s$, either $s$ or $s^{-1}$ is applied, each with probability $\frac{1}{2}$.
Abusing terminology, we relax the definition of simple Cayley graphs to allow order 2 generators, i.e., remove the condition $z_i \neq z_i^{-1}$ for all $i$.

Given a group $G$ and an integer $k$, we are drawing the generators $Z_1, ..., Z_k$ independently and uniformly at random. It is not difficult to see that the probability of drawing a given multiset depends only on the number of repetitions in that multiset. Thus, conditional on being simple, $G_k$ is uniformly distributed on all simple degree-$k$ Cayley graphs. Since $k \ll \sqrt{|G|}$, the probability of simplicity tends to 1 as $|G| \to \infty$. So when we say that our results hold “whp (over $Z$)”, we could equivalently say that the result holds “for almost all degree-$k$ simple Cayley graphs of $G$”.

Our asymptotic evaluation does not depend on the particular choice of $Z$, so the statistics in question depend very weakly on the particular choice of generators for almost all choices. In many cases, the statistics depend only on $G$ via $|G|$ and $d(G)$. This is a strong sense of ‘universality’.

### 1.4.3 Overview of Random Cayley Graphs Project

This paper is one part of an extensive project on random Cayley graphs. There are three main articles [23, 24, 26] (including the current one [23]), a technical report [25] and a supplementary document [27]. *Each main article is readable independently.*

The main objective of the project is to establish cutoff for the random walk and determining whether this can be written in a way that, up to subleading order terms, depends only on $k$ and $|G|$; we also study universal mixing bounds, valid for all, or large classes of, groups. Separately, we study the distance of a uniformly chosen element from the identity, i.e., typical distance, and the diameter; the main objective is to show that these distances concentrate and to determine whether the value at which these distances concentrate depends only on $k$ and $|G|$.

- [23] Cutoff phenomenon (and Aldous–Diaconis conjecture) for general Abelian groups; also, for nilpotent groups, expander graphs and comparison of mixing times with Abelian groups.
- [26] Typical distance, diameter and spectral gap for general Abelian groups.
- [24] Cutoff phenomenon and typical distance for upper triangular matrix groups.
- [25] Additional results on cutoff and typical distance for general Abelian groups.
- [27] Deferred technical results mainly regarding random walk on $\mathbb{Z}$ and the volume of lattice balls.

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## 2 TV Cutoff: Approach #1

In this section, we prove the first part of the upper bound on mixing for arbitrary Abelian groups. The main result of the section is Theorem 2.4. The outline of the section is as follows.

- $\S 2.1$ introduces the entropic method.
- $\S 2.2$ defines entropic times and states a CLT.
- $\S 2.3$ sketches arguments to evaluate these entropic times.
- $\S 2.4$ states precisely the main theorem of the section.
- $\S 2.5$ outlines the argument.
- $\S 2.6$ is devoted to the lower bound.
- $\S 2.7$ is devoted to the upper bound.

### 2.1 Entropic Times: Methodology

We use an ‘entropic method’, as mentioned in $\S 1.3$; cf [7, 8, 9, 12]. The method is fairly general; we now explain the specific application in a little more depth.
We define an auxiliary random process \((W(t))_{t \geq 0}\), recording how many times each generator has been used: for \(t \geq 0\), for each generator \(i = 1, \ldots, k\), write \(W_i(t)\) for the number of times that it has been picked by time \(t\). By independence, \(W(\cdot)\) forms a rate-1 DRW on \(\mathbb{Z}_k^+\). For the undirected case, recall that we either apply a generator or its inverse; when we apply the inverse of generator \(i\), increment \(W_i \to W_i - 1\) (rather than \(W_i \to W_i + 1\)). In this case, \(W(\cdot)\) is a SRW on \(\mathbb{Z}^k\).

Since the underlying group is Abelian, the order in which the generators are applied is irrelevant and generator-inverse pairs cancel; hence we can write
\[
S(t) = \sum_{i=1}^k W_i(t)Z_i = W(t) \cdot Z.
\]

Recall that the invariant distribution is uniform, regardless of the group. For an Abelian group \(G\), we propose as the mixing time the time at which the auxiliary process \(Q\) has been picked by time \(t\); hence
\[
\text{whp; \ 
\sum_{\tau = 0}^t |(\tau + 1) \cdot W| \leq \log |G|; \ 
\text{then, \ for any constant} \ 
\|W(t) - \mathbb{E}[W(t)]\|_1 \geq 1/n \ 
\text{whp.}
\]

2.2 Entropic Times: Definition and Concentration

We now define precisely the notion of entropic times. Write \(\mu_t\), respectively \(\nu_s\), for the law of \(W(t)\), respectively \(W_1(sk)\); so \(\mu_t = \nu_{t/k} \circ \mu_{t/k}^{-1}\). Define
\[
Q_i(t) := -\log \nu_{t/k}(W_i(t)) \quad \text{and set} \quad Q(t) := -\log \mu_t(W(t)) = \sum_{i=1}^k Q_i(t).
\]
So \(\mathbb{E}(Q(t))\) and \(\mathbb{E}(Q_1(t))\) are the entropies of \(W(t)\) and \(W_1(t)\), respectively. Observe that \(t \mapsto \mathbb{E}(Q(t))\) is a smooth, increasing bijection.

**Definition 2.1 (Entropic and Times).** For all \(k, n \in \mathbb{N}\) and all \(\alpha \in \mathbb{R}\), define \(t_\alpha := t_\alpha(k, n)\) so that
\[
\mathbb{E}(Q_1(t_\alpha)) = (\log n + \alpha \sqrt{vk})/k \quad \text{and} \quad s_\alpha := t_\alpha/k, \quad \text{where} \ v := \text{Var}(Q_1(t_\alpha)),
\]
assuming that \(\log n + \alpha \sqrt{vk} \geq 0\). We call \(t_\alpha\) the entropic time and the \(\{t_\alpha\}_{\alpha \in \mathbb{R}}\) cutoff times.

Direct calculation with the Poisson distribution and SRW on \(\mathbb{Z}\) gives the following relations. These calculations are sketched below in §2.3; rigorous arguments are given in [27, §A].

**Proposition 2.2 (Entropic and Cutoff Times; [27, Proposition A.2]).** Assume that \(1 \ll k \ll \log n\). For all \(\alpha \in \mathbb{R}\), we have \(t_\alpha \approx t_0\) and furthermore
\[
t_0 \approx k \cdot n^{2/k}/(2\pi e) \quad \text{and} \quad (t_\alpha - t_0)/t_0 \approx \alpha \sqrt{2/k}.
\]

Since \(Q = \sum_{i=1}^k Q_i\) is a sum of \(k\) iid random variables, \(Q(t_0)\) concentrates around \(\log N\). One can show that if the time is multiplied by a factor \(1+\xi\) for any constant \(\xi > 0\) then the entropy increases by a significant amount; similarly, if \(\xi < 0\) then the entropy decreases by a significant amount. Further, the change is by an additive term of larger order than the standard deviation \(\sqrt{\text{Var}(Q(t_0))}\). Thus \(Q((1+\xi)t_0)\) concentrates around this new value. In particular, the following hold:
\[
\mu_{(1+\xi)t_0}(W((1+\xi)t_0)) = \exp(-Q((1+\xi)t_0)) \ll 1/n \quad \text{whp};
\]
\[
\mu_{(1-\xi)t_0}(W((1-\xi)t_0)) = \exp(-Q((1-\xi)t_0)) \gg 1/n \quad \text{whp}.
\]

The following proposition quantifies this change in entropy and this concentration; see [27, §A].
Proposition 2.3 (CLT; [27, Proposition A.3]). Assume that \( 1 \ll k \ll \log n \). For all \( \alpha \in \mathbb{R} \), we have

\[
\mathbb{P}(Q(t_0) \leq \log n \pm \omega) \to \Psi(\alpha) \quad \text{for} \quad \omega := \text{Var}(Q(t_0))^{1/4} = (vk)^{1/4}.
\]

(There is no specific reason for choosing this \( \omega \). We just need some \( \omega \) with \( 1 \ll \omega \ll (vk)^{1/2} \).)

2.3 Entropic Times: Sketch Evaluation

In this subsection, we sketch details towards a proof of Proposition 2.2. The full, rigorous details can be found in [27, Proposition A.2], where all of the approximations below are justified.

Recall that \( t_0 \) is the time \( t \) at which the entropy of \( W_1(t) \), which is a rate-1/\( k \) process, is \( \log n/k \). We need to find the variance \( \text{Var}(Q_1(s_0k)) \), as this is used in the definition of \( t_0 \), given in Definition 2.1. In the sketch below, we replace \( \text{Var}(Q_1(t_0)) \) by an approximation.

For \( s \geq 0 \), denote \( X_s := W_1(sk) \) for \( s \geq 0 \) and the entropy of \( X_s \) as \( H(s) \). The target entropy \( \log n/k \gg 1 \), and so \( s_0 \gg 1 \). For \( s \gg 1 \), we find that \( X_s \) has approximately the normal \( N(\mathbb{E}(X_s), s) \) distribution. Translating the random variable has no affect on its entropy, and so we approximate the entropy of \( X_s \), which we denoted \( H(s) \), by the entropy of a \( N(0,s) \) random variable, which we denoted \( \overline{\mathcal{H}}(s) \). Direct calculation with the normal distribution shows that

\[
\overline{\mathcal{H}}(s) = \frac{1}{2} \log(2\pi es) \quad \text{and hence} \quad \overline{\mathcal{H}}(s) = 1/(2s).
\]

Define \( \overline{\tau}_s \) as the entropic times for the approximation:

\[
\overline{\mathcal{H}}(\overline{\tau}_s) = (\log n + \alpha \sqrt{vk})/k \quad \text{where} \quad \overline{\tau} := \text{Var}(\overline{Q}_1(\overline{\tau}_s k)),
\]

where \( \overline{Q}_1(sk) \) is the analogue of \( Q_1(sk) \), except with \( W_1(sk) \) replaced by \( N(0,s) \). Hence \( \overline{\tau}_0 = n^{2/k} / (2\pi e) \gg 1 \). By direct calculation, specific to the normal distribution, for \( s \gg 1 \) one finds

\[
\text{Var}(\overline{Q}_1(sk)) = \frac{1}{2}.
\]

As mentioned above, for this sketch, to ease the calculation of \( t_0 \) in Definition 2.1, we replace \( \text{Var}(Q_1(t_0)) \) by its approximation \( \frac{1}{2} \), and assume the above normal distribution approximation.

In order to find the window, assuming for the moment that \( \alpha > 0 \), we write

\[
s_\alpha - s_0 = \int_0^{s_\alpha} \frac{d\tau}{\text{Var}(\overline{\mathcal{H}}(\tau))} \, d\tau.
\]

Again, we replace \( s_\alpha \) with \( \overline{\tau}_s \). By definition, \( \overline{\tau}_s \) satisfies

\[
\overline{\mathcal{H}}(\overline{\tau}_s) = \log n/k + \alpha / \sqrt{2k}, \quad \text{and hence} \quad \frac{d\overline{\tau}}{d\tau} = 1 / \sqrt{2k}.
\]

Using the expressions for \( d\overline{\tau}_s / d\tau \) and \( \overline{\mathcal{H}}(s) \) above, we find that

\[
\overline{\tau}_s - \overline{\tau}_0 = (2k)^{-1/2} \int_0^{\alpha} 2\overline{\tau}_0 \, d\tau \approx (2k)^{-1/2} \int_0^{\alpha} 2\overline{\tau}_0 \, d\alpha = \alpha \overline{\tau}_0 \sqrt{2/k},
\]

since \( \overline{\tau}_0 \) only varies by subleading order terms over \( \alpha \in [0, \alpha] \). The argument is analogous for \( \alpha < 0 \).

We have now shown the desired result for \( \overline{\tau}_s \), i.e when approximating \( W_1(sk) \) by \( N(\mathbb{E}(X_s), s) \). It will turn out that this approximation is sufficiently good for the results to pass over to the original case, i.e to apply to \( s_0 \) and \( t_0 = s_0 k \). This is made rigorous with a local CLT.

2.4 Precise Statement and Remarks

In this subsection, we state precisely the main theorem of the section. There are some simple conditions on \( k \), in terms of \( d(G) \) and \( |G| \), needed for the upper bound.

Hypothesis A. The sequence \( (k_N, G_N)_{N \in \mathbb{N}} \) satisfies Hypothesis A if the following hold:

\[
\lim_{N \to \infty} |G_N| = \infty, \quad \lim_{N \to \infty} (k_N - d(G_N)) = \infty \quad \text{and} \quad \lim_{N \to \infty} (k_N - d(G_N)) = \infty \quad \text{and} \quad \frac{k_N - d_N(G_N) - 1}{k_N} \geq \frac{5}{5} \frac{k_N}{|G_N|} + \frac{2}{2} \frac{d_N(G_N) \log |G_N|}{|G_N|} = \frac{1}{|G_N|} \quad \text{for all} \quad N \in \mathbb{N}.
\]
In Remark 2.5 below, we give some sufficient conditions of Hypothesis A to hold. Throughout the proofs, we drop the subscript-N from the notation, e.g., writing k or n, considering sequences implicitly. Recall that we abbreviate the TV distance from uniformity at time t as

\[ d_{G_k,N}(t) = \left\| P_{G_k}(S(t) \in \cdot) - \pi_{G_k} \right\|_{TV}, \]

where \( Z_1, \ldots, Z_{kN} \sim \text{Unif}(G_N) \).

We now state the main theorem of this section. Recall that \( \Psi \) is the standard Gaussian tail.

Theorem 2.4. Let \((k_N)_{N \in \mathbb{N}}\) be a sequence of positive integers and \((G_N)_{N \in \mathbb{N}}\) a sequence of finite, Abelian groups; for each \( N \in \mathbb{N} \), define \( Z_{(N)} := [Z_1, \ldots, Z_{kN}] \) by drawing \( Z_1, \ldots, Z_{kN} \sim \text{Unif}(G_N) \).

Suppose that the sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis A. For all \( \alpha \in \mathbb{R} \) and \( N \in \mathbb{N} \), write \( t_{\alpha,N} := t_\alpha(\alpha, |G_N|) \). Let \( \alpha \in \mathbb{R} \). Then

\[ t_{\alpha,N}/t_{0,N} \to 1 \quad \text{and} \quad d_{G_{k,N}}(t_{\alpha,N}) \to^P \Psi(\alpha) \quad \text{(in probability)} \quad \text{as } N \to \infty. \]

That is, whp there is cutoff at time \( t_0 \) with profile given by \( t_\alpha \) for all \( \alpha \in (0, 1) \), the difference in the mixing times \( t_{\text{mix}}(\varepsilon) - t_{\text{mix}}(1/2) \) is given, up to subleading order terms, by \( t_\Psi^{-1}(\varepsilon) - t_0 \). Moreover, the implicit lower bound on the TV distance holds deterministically, i.e., for all choices of generators.

Remark. Using Proposition 2.2, we can write the cutoff statement in the form

\[ (t_{\text{mix}}(\varepsilon) - t_0)/w \to^P \Psi^{-1}(\varepsilon) \quad \text{for all } \varepsilon \in (0, 1), \]

where \( t_0 \approx |G|^2 k/(2\pi \varepsilon) \) is the mixing time and \( w \approx \sqrt{k}|G|^2 k/(\sqrt{2}\pi \varepsilon) \) the window. \( \triangle \)

Remark 2.5. Write \( n := |G| \). Note that the final condition of Hypothesis A implies that \( k \leq \frac{1}{2} \log n \); so we are in the regime \( 1 \ll k \ll \log n \). Any of the following conditions imply Hypothesis A:

1. \( 1 \ll k \ll \sqrt{\log n / \log \log n} \) and \( k - d \gg 1 \);
2. \( 1 \ll k \ll \sqrt{\log n} \) and \( k - d \gg \log k \);
3. \( 1 \ll k \ll \log n / \log \log n \) and \( k - d \geq \delta k \) for some suitable \( \delta = o(1) \);
4. \( d \ll \log n / \log \log n \) and \( k - d \approx k \ll \log n \). \( \triangle \)

Remark. The CLT, Proposition 2.3, will give the dominating term in the TV distance:

- on the event \( \{Q(t_\alpha) \leq \log n - \omega\} \), we lower bound the TV distance by \( 1 - o(1) \);
- on the event \( \{Q(t_\alpha) \geq \log n + \omega\} \), we upper bound the expected TV distance by \( o(1) \).

Combined with the CLT, we deduce that the \( d_{G_k}(t_\alpha) \to \Psi(\alpha) \) in probability. \( \triangle \)

Remark. Observe that Hypothesis A does not cover the regime \( k \gg \log |G| \). Under fairly mild conditions on the group we can apply a variation on the argument given below to obtain a limit profile result for any \( 1 \ll k \ll \log |G| \). We do not carry out the analysis here; see [25, §2]. \( \triangle \)

2.5 Outline of Proof

We now give a high-level description of our approach, introducing notations and concepts along the way. No results or calculations from this section will be used in the remainder of the document. Recall the definitions from the previous sections.

In all cases we show that cutoff occurs around the entropic time. As \( Q(t) \) is a sum of iid random variables, we expected it to be concentrated around its mean. Loosely speaking, we show that the shape of the cutoff, i.e., the profile of the convergence to equilibrium, is determined by the fluctuations of \( Q(t) \) around its mean, which in turn, by the CLT (Proposition 2.3), are determined by \( \text{Var}(Q(t)) \), for \( t \) ‘close’ to \( t_0 \); note that \( \text{Var}(Q(t)) = k \text{Var}(Q_1(t)) \) since the \( Q_i \) are iid.

Throughout this section (§2.5), as is standard, we write 0 for the identity element of the Abelian group \( G \). We now outline the proof in more detail. We often drop \( t \)-dependence from the notation.

We start by discussing the lower bound. If \( Q \) is sufficiently small, then \( W \), and hence also \( S \), is restricted to a small set. Indeed, \( Q \ll \log n - \omega \) if and only if \( \mu(W) \geq n^{-1}e^{-\omega} \), and thus if this is
the case then $W \in \{ w \mid \mu(w) \geq n^{-1}e^{\omega} \}$. Since we generate $S$ via $W$, it is thus also the case that

$$S \in E := \{ g \in G \mid \Pr(S = g) \geq n^{-1}e^{\omega} \}.$$ 

But clearly $|E| \leq ne^{-\omega}$. Choosing the time $t$ slightly smaller than the entropic time $t_0$ and $\omega \gg 1$ suitably, the event $(Q(t) \leq \log n - \omega)$ will hold whp. Thus, whp, $S(t)$ is restricted to a set of size $o(n)$. It hence cannot be mixed. This heuristic applies for any choice of generators.

Precisely, we show for any $\omega$ with $1 \leq \omega \leq \log n$, all $t$ and all $Z = [Z_1, \ldots, Z_k]$, that

$$d_{L_2}(t) \geq \Pr(Q(t) \leq \log n - \omega) - e^{-\omega}.$$ 

Observe that the probability on the right-hand side is independent of $Z$. Thus we are naturally interested in the fluctuations of $Q(t)$ for $t$ close to $t_0$. Using the CLT application above, ie Proposition 2.3 with $\omega \equiv \Var(Q(t_0))^{1/4}$, we deduce the lower bound in Theorem 2.4.

We now turn to discussing the upper bound. The lower bound was valid for any choice of generators $Z$. Here we exploit the independence and uniform randomness of the elements of $Z$.

Let $W'(t)$ be an independent copy of $W(t)$, and let $V(t) := W(t) - W'(t)$. Observe that, in both the undirected and directed case, the law of $V(t)$ is that of the rate-2 SRW in $\Z^k$, evaluated at time $t$. The standard $L_2$ calculation (using Cauchy-Schwarz) says that

$$2 \| \zeta - \pi_G \|_{TV} \leq \| \zeta - \pi_G \|_2 = \sqrt{n \sum_{x \in G} (\zeta(x) - \frac{1}{n})^2},$$

recalling that $\pi_G(x) = 1/n$ for all $x \in G$. A standard, elementary calculation shows that

$$\|[\Pr_{G_k}(S(t) \in \cdot) - \pi_G]_2 = \sqrt{n \Pr(V(t) \cdot Z = 0 \mid Z) - 1}.$$ 

Unfortunately, writing $X = (X(s))_{s \geq 0}$ for a rate-1 SRW on $\Z$, a simple calculation shows that

$$\Pr(V(t_0) \cdot Z = 0 \mid Z) \geq \Pr(V(t_0) = (0, \ldots, 0) \in \Z^k) = \Pr(X(2t_0/k) = 0)^k \gg 1/n.$$ 

(This calculation differs amongst the regimes of $k$.) Moreover, the $L_2$-mixing time can then be shown to be larger than the TV-mixing time by at least a constant factor; hence this is insufficiently precise for showing cutoff in TV. (We drop the $t$-dependence from the notation from now on.)

This motivates the following ‘modified $L_2$ calculation’. First let $W \subseteq \Z^k$, and write

$$\text{typ} := \{ W, W' \in W \}, \quad \Pr(\cdot) := \Pr(\cdot \mid \text{typ}) \quad \text{and} \quad \mathbb{E}(\cdot) := \mathbb{E}(\cdot \mid \text{typ});$$

note that here we are (implicitly) averaging over $Z$. The set $W \subseteq \Z^k$ will be chosen later so that

$$\mathbb{P}(V = 0) \ll 1/n \quad \text{and} \quad \Pr(W \notin W) = o(1);$$

we call this typicality. We now perform the same $L_2$ calculation, but for $\mathbb{P}$ rather than $\Pr$:

$$d_{L_2}(t) = \|[\Pr_{G_k}(S \in \cdot) - \pi_G]_{TV} \leq \|\Pr_{G_k}(S \in \cdot \mid W \in W) - \pi_G\|_{TV} + \Pr(W \notin W);$$

$$4\mathbb{E}(\|[\Pr_{G_k}(S \in \cdot \mid W \in W) - \pi_G]_{TV}^2 \leq \mathbb{E}([G] \Pr(V \cdot Z = 0 \mid Z) - 1) = |G|\mathbb{P}(V \cdot Z = 0) - 1;$$

see Lemma 2.6. By taking expectation over $Z$ and doing a modified $L_2$ calculation, we transformed the quenched estimation of the mixing time into an annealed calculation concerning the probability that a random word involving random generators is equal to the identity. This is a key step.

To have $w \in W$, we impose local and global typicality requirements. The global part says that

$$-\log \mu(w) \geq \log n + \omega \quad \text{for all} \quad w \in W,$$

where $\omega := (vk)^{1/4}$ as above; the local part will come later. For a precise statement of the typicality requirements, see Definition 2.7. These have the property that $\Pr(W \notin W) = \Psi(\alpha) + o(1) \asymp 1$ when $t = t_\alpha$; see Proposition 2.8. This has the advantage that now

$$\mathbb{P}(V = (0, \ldots, 0)) \asymp \Pr(W = W' \mid W' \in W) \leq n^{-1}e^{-\omega},$$

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since $-\log x \geq \log n + \omega$ if and only if $x \leq n^{-1} e^{-\omega}$.

Of course, there are other scenarios in which we may have $V \cdot Z \equiv 0$. To deal with these, since linear combinations of independent uniform random variables in an Abelian group are uniform on their support, we have $v \cdot Z \sim \text{Unif}(g, G)$ where $g = \gcd(v_1, \ldots, v_k, n)$; see Lemma 2.11. Then

$$|G| \mathbb{P}(V \cdot Z = 0, V \neq 0) = |G| \mathbb{P}(1 \neq 0)/|gV G|).$$

(Recall that $V$ and $Z$ are independent.) We use the local typicality conditions to ensure that $\max_{\gamma} |W_i| \leq r_*$, for some explicit $r_*$ which diverges a little faster than $n^{1/k}$. This allows us to consider only values $g \in [2r_*]$ for the gcd. It is here where the two approaches ($\S$2 and $\S$3) diverge.

In the first approach ($\S$2) we use a rather direct approach. First, it is elementary that

$$\mathbb{P}(gV = \gamma) \lesssim (2/n^{1/k} + 1/\gamma)^k;$$

see Lemma 2.14. Provided at least one of $d(G)$ or $k$ is not too close to $\log n$, we are able to use this inequality to control the expectation, showing $\mathbb{E}(gV), 1(V \neq 0)) = 1 + o(1)$; see Corollary 2.15.

Combining these two analyses, we deduce that

$$n \mathbb{P}(V \cdot Z = 0) \leq n \mathbb{P}(V \cdot Z = 0, V \neq 0) + n \mathbb{P}(V = 0) = 1 + o(1).$$

The modified $L^2$ calculation then says that the TV distance is roughly $\Psi(\alpha)$ plus a term $o(1)$, ie tending to 0 in probability. This establishes a matching limiting upper bound of $\Psi(\alpha)$ in probability.

The second approach ($\S$3) analyses the term $\mathbb{P}(gV = \gamma)$ and uses it to kill $|G|/\gamma G|$ directly in

$$|G| \mathbb{E}(1(V \neq 0)/|gV G|) = \sum_{\gamma \in \mathbb{N}} \mathbb{P}(gV = \gamma)|G/\gamma G|.$$

We outline in more detail the adaptation in $\S$3.5, including where Approach #1 breaks down.

This concludes the outline; we now move onto the formal proofs.

### 2.6 Lower Bound on Mixing

In this subsection, we prove the lower bound on mixing, which holds for every choice of $Z$.

**Proof of Lower Bound in Theorem 2.4.** For this proof, assume that $Z$ is given, and suppress it. We convert the CLT, Proposition 2.3, from a concentration statement about $Q$ into one about $W$: for all $\alpha \in \mathbb{R}$, by the CLT, we have

$$\mathbb{P}(E_\alpha) \approx \Psi(\alpha) \quad \text{where} \quad E_\alpha := \{\mu(W(t_\alpha)) \geq n^{-1} e^{\omega}\} = \{Q(t_\alpha) \leq \log n - \omega\};$$

recall that $\omega \gg 1$. Fix $\alpha \in \mathbb{R}$. Consider the set

$$E_\alpha := \{x \in G \mid \exists w \in \mathbb{Z}^d \text{ st } \mu_\alpha(w) \geq n^{-1} e^{\omega} \text{ and } x = w \cdot Z\}.$$

Since we use $W$ to generate $S$, we have $\mathbb{P}(S(t_\alpha) \in E_\alpha | E_\alpha) = 1$. Every element $x \in E_\alpha$ can be realised as $x = w_x \cdot Z$ for some $w_x \in \mathbb{Z}^k$ with $\mu_\alpha(w_x) \geq n^{-1} e^{\omega}$. Hence, for all $x \in E_\alpha$, we have

$$\mathbb{P}(S(t_\alpha) = x) \geq \mathbb{P}(W(t_\alpha) = w_x) = \mu_\alpha(w_x) \geq n^{-1} e^{\omega}.$$

Taking the sum over all $x \in E_\alpha$, we deduce that

$$1 \geq \sum_{x \in E_\alpha} \mathbb{P}(S(t_\alpha) = x) \geq |E_\alpha| \cdot n^{-1} e^{\omega}, \quad \text{and hence } |E_\alpha|/n \leq e^{-\omega} = o(1).$$

Finally we deduce the lower bound from the definition of TV distance:

$$\|\mathbb{P}(S(t_\alpha) \in \cdot | Z) - \pi_G\|_{TV} \geq \mathbb{P}(S(t_\alpha) \in E_\alpha) - \pi_G(E_\alpha) \geq \mathbb{P}(E_\alpha) - \frac{1}{n} |E_\alpha| \geq \Psi(\alpha) - o(1).$$

**Remark.** Given a general (not necessarily Abelian) group $G$, one can project to its Abelianisation $G^{ab} = G/[G, G]$, which is an Abelian group. The lower bound $t_0(k, |G^{ab}|)$ then holds for the projected walk on $G^{ab}$. But projection cannot increase the TV distance. Thus $t_0(k, |G^{ab}|)$ gives a lower bound for the mixing of the original walk on $G$. 


2.7 Upper Bound on Mixing

It is often easier to consider $L_2$ distances than $L_1$; roughly, squares are easier to deal with than absolute values; moreover, $L_2$ admits several exact formulas, eg involving return probability, the spectral decomposition and representation theory. TV has a significant advantage, though: it is uniformly bounded (by 1). As such, we can condition on high probability events and upper bound by the TV distance 1 when this event fails.

We use a ‘modified $L_2$ calculation’: first conditioning that $W$ is ‘typical’; then using a standard $L_2$ calculation on the conditioned law. Let $W'$ be an independent copy of $W$; then $S' := W' \cdot Z$ has the same law as $S$ and is conditionally independent of $S$ given $Z$.

Lemma 2.6. For all $t \geq 0$ and all $W \subseteq \mathbb{Z}^k$, the following inequalities hold:

$$d_{G_t}(t) = \| \mathbb{P}_{G_t} (S(t) \in \cdot) - \pi_G \|_{TV} \leq \| \mathbb{P}_{G_t} (S(t) \in \cdot | W(t) \in W) - \pi_G \|_{TV} + \mathbb{P}(W(t) \notin W);$$

$$4 \mathbb{E}(\| \mathbb{P}_{G_t} (S(t) \in \cdot | W(t) \in W) - \pi_G \|_{TV}^2) \leq n \mathbb{P}(S(t) = S'(t) | W(t), W'(t) \in W) - 1.$$

Proof. The first claim follows immediately from the triangle inequality. For the second, using Cauchy–Schwarz, we upper bound the TV distance of the conditioned law by its $L_2$ distance:

$$4 \mathbb{E}(\| \mathbb{P}_{G_t} (S(t) \in \cdot | W(t) \in W) - \pi_G \|_{TV}^2) \leq n \sum_x (\mathbb{P}_{G_t}(S = x | W(t) \in W) - \frac{1}{n})^2$$

$$= n \sum_x \mathbb{P}_{G_t}(S = x | W(t) \in W)^2 - 1 = n \sum_x \mathbb{P}_{G_t}(S = x | W(t) \in W) - 1 - 1,$$

as $S = W \cdot Z$ and $S' = W' \cdot Z$. The claim follows by taking expectations.

We now make the specific choice of the ‘typical’ set $W$; we make a different choice for each $\alpha \in \mathbb{R}$. The collection $\{W_\alpha\}_{\alpha \in \mathbb{R}}$ of sets will satisfy $\mathbb{P}(W(t) \notin W_\alpha) \approx \Psi(\alpha)$, using the CLT (Proposition 2.3); see Proposition 2.8. (Recall that $\Psi$ is the standard Gaussian tail.) We show that the modified $L_2$ distance (given typicality) is $o(1)$; see Proposition 2.9. Applying Lemma 2.6, we find that $d_{G_t}(t) \leq \Psi(\alpha) + o(1)$ whp over $Z$. This matches the lower bound from §2.6.

By considering all $\alpha \in \mathbb{R}$, we are able to find the shape of the cutoff. If we only desire the order of the window, then we need only consider the limit $\alpha \to \infty$; in this case, $\mathbb{P}(W(t) \notin W_\alpha) \approx \Psi(\alpha) \approx 0$, which explains the use of the word ‘typical’ in describing $W_\alpha$.

The typicality conditions will be a combination of ‘local’ (coordinate-wise) and ‘global’ ones.

Definition 2.7. For all $\alpha \in \mathbb{R}$, define the local and global typicality conditions, respectively:

$$W_{\alpha, loc} := \{ w \in \mathbb{Z}^k \mid |w_i - \mathbb{E}(W_i(t_\alpha))| \leq r_*, \forall i = 1, ..., k \} \quad \text{where} \quad r_* := \frac{1}{2} n^{1/k} (\log k)^2;$$

$$W_{\alpha, glo} := \{ w \in \mathbb{Z}^k \mid \mathbb{P}(W(t_\alpha) = w) \leq n^{-1} e^{-\omega} \}.$$

Define $W_\alpha := W_{\alpha, loc} \cap W_{\alpha, glo}$, and say that $w \in \mathbb{Z}^k$ is $(\alpha)$-typical if $w \in W_\alpha$.

The following proposition determines the probability that $W(t_\alpha)$ lies in $W_\alpha$, ie of typicality.

Proposition 2.8. For each $\alpha \in \mathbb{R}$, we have

$$\mathbb{P}(W(t_\alpha) \notin W_\alpha) \to \Psi(\alpha).$$

Proof. By our CLT, Proposition 2.3, the probability that the global conditions hold converges to $1 - \Psi(\alpha)$. Proposition 2.2 along with [27, Definitions C.1 and C.2 and Proposition C.3] together say that the probability that a single coordinate fails the local condition is at most $k^{-3/2}$. By the union bound, the probability that local typicality fails to hold is then at most $k^{-1/2} = o(1)$.

Herein, we fix $\alpha \in \mathbb{R}$ and frequently suppress the time $t_\alpha$ from the notation, eg writing $W$ for $W(t_\alpha)$ or $W$ for $W_\alpha$. Let $V := W - W'$, so $\{ W \cdot Z = W' \cdot Z \} = \{ V \cdot Z = 0 \}$. Write

$$D := D(t_\alpha) := n \mathbb{P}(V(t_\alpha) \cdot Z = 0 \mid \text{typ}_\alpha) - 1 \quad \text{where} \quad \text{typ} := \text{typ}_\alpha := \{ W(t_\alpha), W'(t_\alpha) \in W_\alpha \}.$$
It remains to show that \( D(t_\alpha) = o(1) \) for all \( \alpha \in \mathbb{R} \). Recall Hypothesis A, the crux of which is that

\[
\frac{k - d - 1}{k} - 2 \frac{d \log \log k}{\log n} \geq 5 \frac{k}{\log n} \quad \text{and} \quad k - d \gg 1.
\]

For \( r_1, \ldots, r_\ell \in \mathbb{Z} \setminus \{0\} \), we use the convention \( \gcd(r_1, \ldots, r_\ell, 0) := \gcd(|r_1|, \ldots, |r_\ell|) \).

**Proposition 2.9.** Suppose that \((d, n, k)\) jointly satisfy Hypothesis A. (Recall that, implicitly, \((d, n, k)\) is a sequence of triples of integers.) Write \( g := \gcd(V_1, \ldots, V_k, n) \). Then, for all \( \alpha \in \mathbb{R} \), we have

\[
0 \leq D(t_\alpha) = \sum_{\gamma \in \mathbb{N}} \mathbb{P}(g = \gamma \mid \text{typ}) \cdot |G|/|\gamma G| - 1 = o(1).
\]

Given this proposition, we can prove the upper bound in the main theorem, Theorem 2.4.

**Proof of Upper Bound in Theorem 2.4 Given Proposition 2.9.** Fix \( \alpha \in \mathbb{R} \) and consider the TV distance at time \( t_\alpha \). Apply the modified \( L_2 \) calculation, i.e., Lemma 2.6 and Definition 2.7, at time \( t_\alpha \); by Proposition 2.9, the modified \( L_2 \) distance (given typicality) is \( o(1) \) in expectation; by Markov’s inequality, it is thus \( o(1) \) whp. Proposition 2.8 says that typicality holds with probability \( \Psi(t_\alpha) \) asymptotically. Combined, this all says that \( d_{\gamma G}(t_\alpha) \leq \Psi(\alpha) + o(1) \) whp.

It remains to prove Proposition 2.9, i.e., to bound the modified \( L_2 \) distance. The remainder of the section is dedicated to this goal. To do this, we are interested in the law of \( V \cdot Z \).

Obviously, when \( V = 0 \), we have \( V \cdot Z = 0 \). The following auxiliary lemma controls this probability; its proof is deferred to the end of the subsection.

**Lemma 2.10.** We have

\[
n \mathbb{P}(V = 0 \mid \text{typ}) \leq e^{-\omega}/\mathbb{P}(\text{typ}) \lesssim e^{-\omega} = o(1).
\]

Linear combinations of independent uniform random variables in an Abelian group are themselves uniform on their support. Hence the distribution of \( V \cdot Z \) is uniform on \( \gcd(v_1, \ldots, v_k, n)G \). This is proved in [27, Lemma F.1]; we state the version which we desire here.

**Lemma 2.11.** For all \( v \in \mathbb{Z}^k \), we have

\[
v \cdot Z \sim \text{Unif}(\gamma G) \quad \text{where} \quad \gamma := \gcd(v_1, \ldots, v_k, n).
\]

We thus need to control \( |\gamma G| \), since Lemma 2.11 implies that

\[
\mathbb{P}(V \cdot Z = 0 \mid \text{typ}) = \sum_{\gamma \in \mathbb{N}} \mathbb{P}(g = \gamma \mid \text{typ})/|\gamma G| \quad \text{where} \quad g := \gcd(V_1, \ldots, V_k, n).
\]

**Lemma 2.12.** For all Abelian groups \( G \) and all \( \gamma \in \mathbb{N} \), we have

\[
|G|/|\gamma G| \leq \gamma^{d(G)}.
\]

**Proof.** Decompose \( G \) as \( \bigoplus_1^d \mathbb{Z}_{m_j} \) with \( d = d(G) \) and some \( m_1, \ldots, m_d \in \mathbb{N} \). Then \( \gamma G \) can be decomposed as \( \bigoplus_1^d \gcd(\gamma, m_j)\mathbb{Z}_{m_j} \). Hence \( |\gamma G| = \prod_1^d (m_j/\gcd(\gamma, m_j)) \geq \prod_1^d (m_j/\gamma) = |G|/\gamma^d \).

These lemmas combine to produce a simple, but key, corollary.

**Corollary 2.13.** We have

\[
n \mathbb{P}(V \cdot Z = 0, V \neq 0 \mid \text{typ}) \leq \mathbb{E}(g^{d(G)} 1(V \neq 0 \mid \text{typ}).
\]

**Proof.** The conditioning does not affect \( Z \). The corollary follows from Lemmas 2.11 and 2.12.

In order to control this \( \gcd \), we need to determine the probability that an individual coordinate is a multiple of a given number. We evaluate the RW around the entropic time \( t_\alpha \). The proof of the following auxiliary lemma is deferred to the end of the subsection. This and Corollary 2.13 are the key elements to the proof of Proposition 2.9.
Lemma 2.14. For all \( \gamma \in \mathbb{N} \), we have
\[
\mathbb{P}(V_1 \in \gamma \mathbb{Z} \mid V_1 \neq 0) \leq 1/\gamma \quad \text{and} \quad \mathbb{P}(g = \gamma \mid \text{typ}) \leq (1/\gamma + 2/n^{1/k})^k.
\]

From this, using the conditions of Hypothesis A, we can deduce that \( \mathbb{E}(g^{d(G)} 1(V \neq 0) \mid \text{typ}) = 1 + o(1) \). We refer to this as a “corollary”, since its proof is purely technical, not relying on any properties of the RW or the generators, just algebraic manipulation. Its proof is briefly deferred.

Corollary 2.15. Given Hypothesis A, we have \( \mathbb{E}(g^{d(G)} 1(V \neq 0) \mid \text{typ}) = 1 + o(1) \).

Proposition 2.9 now follows immediately from Lemma 2.10 and Corollaries 2.13 and 2.15.

Proof of Proposition 2.9. By Lemma 2.10 and Corollaries 2.13 and 2.15, we have
\[
n \mathbb{P}(V \cdot Z = 0 \mid \text{typ}) \leq n \mathbb{P}(V = 0 \mid \text{typ}) + n \mathbb{P}(V \cdot Z = 0, V \neq 0 \mid \text{typ}) \\
\leq n \mathbb{P}(V = 0 \mid \text{typ}) + \mathbb{E}(g^{d(G)} 1(V \neq 0) \mid \text{typ}) = 1 + o(1).
\]

We now give the deferred proof of Corollary 2.15.

Proof of Corollary 2.15. Let \( d = d(G) \). By local typicality, \( g \leq 2r_\ast = n^{1/k} (\log k)^2 \) if \( V \neq 0 \). Thus
\[
\mathbb{E}(g^d 1(V \neq 0) \mid \text{typ}) = \sum_{\gamma \in \mathbb{N}} \gamma^d \mathbb{P}(g = \gamma \mid \text{typ}) \leq 1 + \sum_{\gamma=2}^{n^{1/k} (\log k)^2} \gamma^d \mathbb{P}(g = \gamma \mid \text{typ}).
\]

For \( \gamma \geq 2 \), we use Lemma 2.14. Let \( \delta \in (0, 1) \). For \( 2 \leq \gamma \leq \delta n^{1/k} \), we use the bound
\[
\mathbb{P}(g = \gamma \mid \text{typ}) \leq (1/\gamma + 2/(\gamma/\delta))^k = (1 + 2\delta)^k/\gamma^{2k}.
\]

For \( \gamma \geq \delta n^{1/k} \), we use the slightly crude bound \( (a + b)^k \leq 2^k (a^k + b^k) \) for \( a, b \geq 0 \) to deduce that
\[
\mathbb{P}(g = \gamma \mid \text{typ}) \leq 2^k (1/\gamma^k + 2^{k}/n) = 2^k/\gamma^k + 4^k/n.
\]

Dividing the appropriate sum over \( \gamma \) into two parts according to whether or not \( \gamma \leq \delta n^{1/k} \) and using the above inequalities, elementary algebraic manipulations can be used to deduce that
\[
\mathbb{E}(g^d 1(V \neq 0) \mid \text{typ}) - 1 \leq e^{2dk}2^{d+1-k} + 2^k \delta^{d+1-k/n}(d+1-k)/k + 4^k n^{(d+1)/k} (\log k)^2 (d+1)/n.
\]

This is \( o(1) \), by the conditions of Hypothesis A, as we now outline. Write \( \eta := (k-d-1)/k \in (0, 1) \). We wish to choose \( \delta \) as large as possible so that the first term is \( o(1); \) set \( \delta := 1/\eta \). With this definition, it is not difficult to see that the assumption \( \eta \geq 4k/\log n \), which follows immediately from Hypothesis A, is sufficient to make the middle term small. Finally, the inequality in Hypothesis A is designed precisely so that the final term is \( o(1) \), noting that \( \eta k \geq k - d - 1 \geq 1/\eta (k - d) \).

Remark. We have always been assuming that \( k - d(G) \gg 1 \). Our analysis does apply if \( M := k - d(G) \geq 2 \) is fixed (ie not diverging) too. Then, however, it is not necessarily the case that the group is generated whp—eg if \( G = \mathbb{Z}_2^d \) then it is not. Our analysis shows that with probability bounded away from 0 the mixing time is of order \( t_0 \); by choosing \( M \) sufficiently large, this probability can be made arbitrarily close to 1—but to be \( 1 - o(1) \) one requires \( M \gg 1 \).

It remains to prove the auxiliary lemmas, namely Lemmas 2.10 and 2.14.

Proof of Lemma 2.10. By direct calculation, since \( W \) and \( W' \) are independent copies,
\[
\mathbb{P}(V = 0, \text{typ}) = \mathbb{P}(W = W', W \in \mathcal{W}) = \sum_{w \in \mathcal{W}} \mathbb{P}(W = w)^2.
\]

Recall global typicality: \( \mathbb{P}(W = w) \leq n^{-1} e^{-\omega} \) for all \( w \in \mathcal{W} \). Thus
\[
n \mathbb{P}(V = 0 \mid \text{typ}) \leq n \sum_{w \in \mathcal{W}} \mathbb{P}(W = w)^2 / \mathbb{P}(\text{typ}) \leq e^{-\omega} / \mathbb{P}(\text{typ}).
\]
**Proof of Lemma 2.14.** Let $X = (X_s)_{s \geq 0}$ be a rate-1 SRW on $\mathbb{Z}$. To calculate the expectation, we use that $V = W - W'$ has the distribution of a SRW run at twice the speed; in particular, $V_i(t) \sim X_{2t/k}$, and that coordinates of $V$ are independent. (This holds for both the undirected and directed cases.) Clearly the distribution of $X$ is symmetric about 0.

It is easy to see that any non-increasing distribution on $\mathbb{N}$ can be written as a mixture of $\text{Unif}([1, \ldots, Y])$ distributions, for different $Y \in \mathbb{N}$. Observe that the map $m \mapsto \mathbb{P}(|X_s| = m) : \mathbb{N} \to [0, 1]$ is non-increasing for any $s \geq 0$. Hence $|V_1|$ conditional on $V_1 \neq 0$ has such a distribution. Thus

$$|V_1| \sim \text{Unif}(1, \ldots, Y) \quad \text{conditional on } V_1 \neq 0,$$

where $Y$ has some distribution. Hence we have

$$\mathbb{P}(V_1 \in \gamma \mathbb{Z} \mid V_1 \neq 0) = \mathbb{E}(\lfloor Y/\gamma \rfloor / Y) \leq 1/\gamma.$$

If the gcd $\gcd = \gamma$, then $V_i \in \gamma \mathbb{Z}$ for all $i \in [k]$. Hence, by independence of coordinates, we obtain

$$\mathbb{P}(\gamma = \gamma \mid \text{typ}) \leq \mathbb{P}(\gamma = \gamma)/\mathbb{P}(\text{typ}) \leq \mathbb{P}(V_1 \in \gamma \mathbb{Z})^k \leq (\mathbb{P}(V_1 = 0) + \mathbb{P}(V_1 \in \gamma \mathbb{Z} \mid V_1 \neq 0))^k,$$

noting that $\mathbb{P}(\text{typ}) > 1$. Using Proposition 2.2 to argue that $\mathbb{P}(V_1 = 0) \leq 2/n^{1/k}$, we deduce that

$$\mathbb{P}(\gamma = \gamma \mid \text{typ}) \leq (2/n^{1/k} + 1/\gamma)^k. \quad \square$$

## 3 TV Cutoff: Approach #2

Recall that Theorem A is established via distinct approaches. In the previous section we used one approach to deal with the case that $k$ is ‘not too large’. In this section we use a new approach to deal with the case that $k$ is ‘not too small’. The main result of the section is Theorem 3.6. The outline of the section is roughly the same as that of the previous one.

- §3.1 discusses the new, refined entropic methodology.
- §3.2 defines the new entropic times.
- §3.3 states bounds on the growth rate of the entropy and concentration.
- §3.4 states precisely the main theorem of the section.
- §3.5 outlines the differences between this argument and the previous approach.
- §3.6 is devoted to the lower bound.
- §3.7 is devoted to the upper bound.

### 3.1 Entropic Times: New Methodology and Definition

The underlying principles of the method used in this section (§3) are the same as those of the previous (§2). We adjust the method slightly to deal with the cases which were not covered in §2.

We first discuss where the previous approach broke down and how we might fix it. The primary issue was when $d$ was very large. E.g consider $\mathbb{Z}_2^d$. Since all elements are of order 2, instead of looking at $W$, a RW on $\mathbb{Z}$, we could equally have looked at $W$ taken modulo 2. The entropy of $W$ mod 2 is significantly smaller than that of $W_1$ at the original entropic time $t_0$.

We saw that $V \cdot Z \sim \text{Unif}(\gamma G)$ when $\gcd(V_1, \ldots, V_k, n) = \gamma$. (This assumes that the group $G$ is Abelian.) This motivates defining $t_\gamma$ to be the time at which the entropy of $W_1$ mod $\gamma$ is $\log |G/\gamma G|$. The proposed upper bound is then given by $t_* := \max_{\gamma \in \mathbb{N}} t_\gamma$.

While this method will be able to handle arbitrary Abelian groups, we only get an abstract definition of the cutoff time, which is not easily calculable for many groups. For some, though, it is: eg $t_*(k, \mathbb{Z}_2)$ is the time at which the entropy of RW on $\mathbb{Z}_2^k$ reaches $\log |G|$.

As in the previous sections, not only are we interested in the entropy at this proposed mixing time $t_*$, but we also desire quantitative information about the rate of change of entropy at this time and the variance of the ‘random entropy’, denoted $Q$. 

20
3.2 Entropic Times: Definition and Concentration

In this section, we redefine entropic times. There is some overlap notation from before, but all entropic definitions from §2.2 should be forgotten; all terms will be defined below.

We now define precisely the (updated) notion of entropic times. Let \( W = (W_i(t) \mid i \in [k], t \geq 0) \) be a RW on \( \mathbb{Z} \), counting the uses of generators, as in the previous sections. (This can be either a SRW on \( \mathbb{Z} \) or DRW on \( \mathbb{Z}^+ \).) As before, \( S(t) = W(t) \mod \gamma \). For \( \gamma \in \mathbb{N} \), define \( W_\gamma \) via \( W_\gamma_i(t) := W_i(t) \mod \gamma \); write \( W_\infty := W \). Then \( W_\gamma \) is a RW on \( \mathbb{Z}_\gamma^k \); so \( W_{\gamma,i} := (W_{\gamma,i}(t))_{t \geq 0} \) forms an iid sequence (over \( i \in [k] \)) of rate-1 RWs on \( \mathbb{Z}_\gamma \).

Write \( \mu_{\gamma,t} \), respectively \( \nu_{\gamma,s} \), for the law of \( W_\gamma(t) \), respectively \( W_{\gamma,1}(sk) \); so \( \mu_{\gamma,t} = \nu_{\gamma,sk}^{\otimes k} \). Define

\[
Q_{\gamma}(t) := -\log \mu_{\gamma,t}(W_\gamma(t)) \quad \text{and} \quad Q_{\gamma,i}(t) := -\log \nu_{\gamma,sk}(W_{\gamma,i}(t));
\]

so \( Q_{\gamma,i} \) forms an iid sequence over \( i \in [k] \) and

\[
Q_{\gamma}(t) = \sum_{i=1}^k Q_{\gamma,i}(t), \quad h_{\gamma}(t) := \mathbb{E}(Q_{\gamma}(t)) \quad \text{and} \quad \nu_{\gamma}(s) := \mathbb{E}(Q_{\gamma,1}(sk)).
\]

So \( h_{\gamma}(t) \) and \( \nu_{\gamma}(s) \) are the entropies of \( W(t) \) and \( W_1(sk) \), respectively. Note that \( h_{\gamma}(t) = kH_{\gamma}(t/k) \) and that \( h_{\gamma} : [0, \infty) \to [0, \log(\gamma^k)) \) is a strictly increasing bijection.

Some of these expressions, such as \( h_{\gamma} \), depend on \( k \); we usually suppress this from the notation.

**Definition 3.1.** For \( N < k^k \), define the entropic time

\[
t_0(\gamma, N) := h_{\gamma}^{-1}(\log N) \quad \text{and} \quad s_0(\gamma, N) := t_0(\gamma, N)/k = H_{\gamma}^{-1}(\log N/k).
\]

We are interested primarily in \( N := |G/\gamma G| \). For an Abelian group \( G \), define

\[
t_*(k, G) := \max_{\gamma \mid |G|} t_0(\gamma, |G/\gamma G|).
\]

Our next result determines the asymptotics of \( t_* \). The first part is for \( k-d(G) \ll k \): it shows that here the mixing time is the same order as that from Approach #1, ie \( k|G|^2/k \). Combining the two approaches, this means that all Abelian groups have mixing time order \( kn^2/k \) when \( 1 \ll k \ll \log |G| \) and \( k - d(G) \ll k \). The second part allows \( k - d(G) \) to diverge arbitrarily slowly: in this case the mixing time can be as large as order \( k|G|^2/k \log k \). The final part evaluates \( t_* \) up to smaller order terms when \( d(G) \ll \log |G| \) and \( k - d(G) \gg k \). The proofs are given in [27, §B.3.2].

**Proposition 3.2a ([27, Proposition B.17]).** Suppose that \( 1 \ll k \ll \log |G| \). The following hold:

- if \( k - d(G) \ll k \), then \( t_*(k, G) = k|G|^2/k \);
- if \( k - d(G) > 1 \), then \( k|G|^2/k \ll t_*(k, G) \ll k|G|^2/k \log k \).

**Proposition 3.2b ([27, Proposition B.18]).** Suppose that \( d(G) \ll \log |G| \) and \( k - d(G) \gg k \). Then \( t_*(k, G) \approx t_0(\infty, G) \). (Note that \( t_0(\infty, |G|) = t_0(k, |G|) \) in the notation of Definition 2.1.)

**Heuristics Behind Proofs.** For the RW on \( \mathbb{Z}_\gamma \), until time \( \gamma^2 \) the walk looks roughly the same as if it were on \( \mathbb{Z} \). In particular, the entropy growth rates are comparable. From this, we are able to see that \( s_* \) is the same order as the entropic time \( s_0 \) from §2 when \( k - d \gg k \).

For \( k - d > 1 \), by Lemma 2.12, we have \( s_0(\gamma, |G/\gamma G|) \leq s_0(\gamma, \mathbb{Z}_\gamma^d) = R_{\gamma}^{-1}(\zeta) \). So the worst case is studying relative entropy for the RW on \( \mathbb{Z}_\gamma^d \). In [25, §3] we analyse in detail RWs on random Cayley graphs of \( \mathbb{Z}_p^d \). In particular, we analyse this entropic time for \( 1 \ll k - d \ll k \).

The same heuristics hold for the regime \( 1 \ll k \ll \log |G| \), except that now one checks that any extremal \( \gamma \) (in the sense of attaining the maximum in the definition of \( t_* \)) satisfies \( \gamma \gg 1 \) and \( s_0(\gamma, |G/\gamma G|) \ll \gamma^2 \). In this case, the RW on \( \mathbb{Z}_\gamma \) is almost indistinguishable from that on \( \mathbb{Z} \). Hence the entropic times are asymptotically equivalent.

For \( k \ll \log |G| \), the mixing time is order \( k \ll \log |G| \). As such one expects each generator to be picked an order 1 number of times. Separate into large \( \gamma \) and small \( \gamma \): upper bound \( |G/\gamma G| \leq |G| \) in the former and \( |G/\gamma G| \leq \gamma^d(G) \) in the latter. Any extremal \( \gamma \) must be small, ie satisfy \( \gamma \approx 1 \). □
In §3.6, we show that $t_0(\gamma, |G/\gamma G|)$ is a lower bound on mixing for all $\gamma$, for all $Z$. Throughout this section, we work under the assumption that $k \leq \log |G|$. (Recall from §1.3 that cutoff had already been established for all Abelian groups when $k \gg \log |G|$.) As a result of this, taking $\gamma := |G|$, we see that the mixing time is at least order $k$. There hence exists a $\varsigma > 0$ so that the mixing time is at least $2\gamma k$. (This is true for all choices of $Z$, not just whp over $Z$.)

For technical reasons, for $\gamma$ with $t_0(\gamma, |G/\gamma G|) \leq \varsigma$, it is convenient to replace $t_0(\gamma, |G/\gamma G|)$ with an adjusted entropic time $t_\gamma$ defined below. Crucially, $\max_{\gamma \in \mathbb{N}} t_\gamma = \max_{\gamma \in \mathbb{N}} t_0(\gamma, |G/\gamma G|)$.

**Definition 3.3.** For $s \geq 0$ and $\gamma \geq 2$, define the (adjusted) entropic time and relative entropy via

$$s_\gamma := s_0(\gamma, |G/\gamma G|) \vee \varsigma, \quad t_\gamma := s_\gamma k \quad \text{and} \quad R_\gamma(s) := \log \gamma - H_\gamma(s).$$

The maximal entropy of a distribution on $Z_\gamma$, is $\log \gamma$, obtained uniquely by the uniform distribution $\text{Unif}(Z_\gamma)$. Hence $R_\gamma(s) \to 0$ as $s \to \infty$ since the RW converges to $\text{Unif}(Z_\gamma)$.

### 3.3 Entropic Times: Entropy Growth Rate and Concentration

In the previous approach, we had a CLT for the random variable $Q$. Here we do not give such precise results; this means that while we show cutoff, we do not find the profile. (Even if we knew such refined information, it would be difficult to calculate $\max_{\gamma \in \mathbb{N}} t_\gamma$, as this is highly dependent on the group.) Instead, we determine the rate of change of the entropy around the entropic time and establish concentration estimates on the ‘random entropy’, i.e. the $Q_\gamma$ random variable, at a time shortly after the entropic time.

The first lemma controls the rate of change of the entropy near the entropic time; see [27, §B].

**Lemma 3.4 ([27, Lemma B.20]).** There exists a continuous function $\bar{\gamma} : (0, 1) \to (0, 1)$ so that, for all $\gamma \geq 2$, all $\xi \in (-1, 1) \setminus \{0\}$ and all $s \geq \varsigma$, we have

$$|H_\gamma(s(1 + \xi)) - H_\gamma(s)| \geq 2\bar{\gamma}(\varsigma)(R_\gamma(s) \land 1).$$

Given that we know how much the entropy, i.e. the expectation of $Q_\gamma$, changes, we now want a concentration result, giving upper and lower tail estimates. The upper tail is used for the lower bound on mixing: it says that $Q_\gamma$ is at most some value whp. Similarly, the lower tail is used for the upper bound on mixing. These are given in Proposition 3.5, which is proved in [27, §B].

Recall that $t_\gamma = \max_{\gamma \in \mathbb{N}} t_\gamma$ and $d = d(G)$. For $\gamma \in \mathbb{N}$, write $\varsigma_\gamma := \frac{1}{k}(k - d(G))\log \gamma$.

**Proposition 3.5 ([27, Proposition B.21]).** Assume that $k > d$. There exists a continuous function $c : (0, 1) \to (0, 1)$ so that, for all $\gamma \geq 2$ and all $\epsilon \in (0, 1)$, the following hold:

$$\mathbb{P}(Q_\gamma(t_\gamma(1 + \epsilon)) \leq \log |G/\gamma G| + c_\epsilon(\varsigma_\gamma \land 1)k \leq \exp(-c_\epsilon(\varsigma_\gamma \land 1)k);$$

$$\mathbb{P}(Q_\gamma(t(1 - \epsilon)) \geq \log |G/\gamma G| - c_\epsilon(\varsigma_\gamma \land 1)k) = o(1) \quad \text{for all} \quad t \leq t_\gamma.$$

**Heuristics Behind Proof.** The proof of this proposition is given in [27, §B]. We give a brief outline here. Recall that $Q_\gamma(t) = \sum_{i=1}^t Q_{\gamma,i}(t)$ is a sum of iid terms, each of which has mean $H_\gamma(t/k)/k$. Applying the entropy growth rate lemma, i.e Lemma 3.4, we see, for any $\xi \in (-1, 1) \setminus \{0\}$, that the change in entropy between times $s$ and $(1 + \xi)s$ is order $R_\gamma(s) \land 1$ (with implicit constant depending on $|\xi|$). Taking $s := s_0(\gamma, |G/\gamma G|)$, recalling that $|G/\gamma G| \leq \gamma^d(G)$ by Lemma 2.12, gives

$$R_\gamma(s) = \log \gamma - H_\gamma(s) = \log \gamma - (\log |G/\gamma G|)/k \geq \frac{1}{k}(k - d(G))\log \gamma = \varsigma_\gamma.$$

(We are interested in the times $s_\gamma$, not $s_0(\gamma, |G/\gamma G|)$; this is a minor technical complication.)

Regarding the concentration, the non-quantitative part is then an application of Chebyshev’s inequality, once one has shown that the variance $\text{Var}(Q_{\gamma,1}(sk))$ is uniformly bounded over $s \geq \varsigma$. The quantitative part requires first arguing that $E(Q_{1,\gamma}) - Q_{1,\gamma}$ is deterministically bounded (from above); we then apply a (one-sided) variant of Bernstein’s inequality for a sum of iid, deterministically bounded random variables.
3.4 Precise Statement and Remarks

In this subsection, we state precisely the main theorem of the section. There are some simple conditions on \( k \), in terms of \( d(G) \) and \( |G| \), needed for the upper bound.

**Hypothesis B.** The sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis B if the following hold:

\[
\limsup_{N \to \infty} \frac{k_N}{\log |G_N|} < \infty, \quad \liminf_{N \to \infty} \left( k_N - d(G_N) \right) = \infty \quad \text{and} \quad \liminf_{N \to \infty} \frac{k_N}{\log |\mathcal{H}_N|} = \infty,
\]

where \( \mathcal{H}_N := \{ \gamma G_N \mid \gamma \in [2, n_{*,N}] \} \) and \( n_{*,N} := \lceil |G_N|^{1/k_N} (\log k_N)^2 \rceil \).

In Remark 3.7 below, we give a sufficient condition for Hypothesis B to hold. Throughout the proofs, we drop the subscript-\( N \) from the notation, e.g., writing \( k \) or \( n \), considering sequences implicitly. Recall that we abbreviate the TV distance from uniformity at time \( t \) as

\[
d_{G,k,N}(t) = \| P_{G_N}(Z_1, \ldots, Z_{k_N}) \left( S(t) \in \cdot \right) - \pi_{G,N} \|_{\text{TV}} \quad \text{where} \quad Z_1, \ldots, Z_{k_N} \sim \text{iid Unif}(G_N).
\]

We now state the main theorem of this section. Recall that \( t_* = \max_{\gamma \in \mathbb{N}} t_0(\gamma, |G/\gamma G|) \).

**Theorem 3.6.** Let \((k_N)_{N \in \mathbb{N}}\) be a sequence of positive integers and \((G_N)_{N \in \mathbb{N}}\) a sequence of finite, Abelian groups; for each \( N \in \mathbb{N} \), define \( Z_{(N)} := [Z_1, \ldots, Z_{k_N}] \) by drawing \( Z_1, \ldots, Z_{k_N} \sim \text{iid Unif}(G_N) \).

Suppose that the sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis B. Let \( c \in (-1, 1) \setminus \{0\} \). Then

\[
d_{G,k,N}^\pm \left( (1 + c)t_*^\pm (k_N, G_N) \right) \to^P 1(c < 0) \quad \text{(in probability)} \quad \text{as} \quad N \to \infty.
\]

That is, whp, there is cutoff at \( \max_k t_*^\pm (\gamma, |G/\gamma G|) \). Moreover, the implicit lower bound holds deterministically, i.e., for all choices of generators.

**Remark 3.7.** If \( k \gg \sqrt{\log n} \), then \( k \gg \log |\mathcal{H}| \), since \( |\mathcal{H}| \leq n_* \leq n^{1/(\log k)^2} \). \( \triangle \)

3.5 Outline of Proof

The general outline of this approach is the same as that of the previous; see §2.5 for an outline of the previous approach. The previous approach failed once either \( d \) or \( k \) became too large or \( k - d \) became too small. We outline here the ideas used to cover these cases.

For the lower bound, we project the walk from \( G \) to \( G/\gamma G \). This can only decrease the TV distance. The idea, then, is that where before we looked at a RW on \( \mathbb{Z}^k \), we now look at a RW on \( \mathbb{Z}^k/\gamma G \) and wait until it has entropy \( \log |G/\gamma G| \); see Definition 3.1. We then take a worst-case over \( \gamma \in \mathbb{N} \).

For the upper bound, fundamentally, we still wish to bound the same expression:

\[
D(t) = \sum_{\gamma \in \mathbb{N}} P(g = \gamma \mid \text{typ}) \cdot |G/\gamma G| - 1;
\]

see Propositions 2.9 and 3.13. In §2.7, we upper bounded \( |G/\gamma G| \leq \gamma^{d(G)} \). In certain situations, this is too crude. Instead, observe that if \( g = \gamma \) then \( V \equiv 0 \mod \gamma \). But \( W_\gamma := W \mod \gamma \) and \( W_\gamma' := W' \mod \gamma \) are simply RWs on \( \mathbb{Z}^k/\gamma \). Just as we used the entropic time (and typicality) to get

\[
P(W = W' \mid \text{typ}) \ll 1/|G|
\]

in Lemma 2.10, here we adjust the entropic time (and typicality) so that

\[
P(g = \gamma \mid \text{typ}) \leq P(W_\gamma = W_\gamma' \mid \text{typ}) \ll |G/\gamma G|/|G|;
\]

see Definitions 3.1 and 3.8 and the proof of Proposition 3.13.
3.6 Lower Bound on Mixing

In this subsection, we state and prove the lower bound, matching the upper bound of Theorem 3.6; it holds not only for all groups \( G \) but also for all choices of \( Z \), not just whp over \( Z \).

The idea is to quotient out by \( \gamma G \), and show that the walk on this quotient is not mixed at time \((1 - \varepsilon)t_0(\gamma, |G/\gamma G|)\), and hence the original walk is not mixed on \( G \) either. We use the same idea as in §2.6 to show that, for each \( \gamma \), the walk is not mixed on \( G/\gamma G \) at time \((1 - \varepsilon)t_0(\gamma, |G/\gamma G|)\).

In §3.6 we used a CLT to control the entropic variables. Here we use the entropy growth rate and variance bounds, detailed in Proposition 3.5.

Proof of Lower Bound in Theorem 3.6. For this proof, assume that \( Z \) is given, and suppress it.

We first convert the statement from one about \( Q_\gamma \) to one about \( W_\gamma \). Let \( \varepsilon \in (0, 1) \) and set \( t := (1 - \varepsilon)t_0(\gamma, |G/\gamma G|) \). Write \( \zeta_\gamma := R_\gamma(s_0(\gamma, |G/\gamma G|)) \). From Proposition 3.5, we obtain
\[
\mathbb{P}(\mathcal{E}) = 1 - o(1) \quad \text{where} \quad \mathcal{E} := \{ \mu_{\gamma,t}(W_\gamma(t)) \geq \delta_{\gamma,t}^{-1}/|G/\gamma G| \} \quad \text{and} \quad \delta_{\gamma,t} := \exp(-c_\varepsilon(\zeta_\gamma \wedge 1)k).
\]

From Lemma 2.12, we have \( |G/\gamma G| \leq \gamma^{d(G)} \). Thus
\[
\zeta_\gamma = R_\gamma(s_0(\gamma, \log |G/\gamma G|)) = \log \gamma - \log |G/\gamma G|/k \geq \frac{1}{k}(k - d(G)) \log \gamma;
\]
also, \( k - d(G) \gg 1 \). Thus \( \delta_{\gamma,t} = o(1) \) uniformly in \( \gamma \). Consider the set
\[
A := \{ x \in G/\gamma G \mid \exists w \in \mathbb{Z}_k^+ \text{ s.t. } \mu_{\gamma,t}(w) \geq \delta_{\gamma,t}^{-1}/|G/\gamma G| \text{ and } x = (w \cdot Z)\gamma G \}.
\]

Define \( S_\gamma \) to be the projection of \( S \) to \( G/\gamma G \). Since we use \( W \) to generate \( S_\gamma \), we have \( \mathbb{P}(S_\gamma(t) \in A \mid G/\gamma G) = 1 \). Every element \( x \in A \) can be realised as \( x = w_x \cdot Z \) for some \( w_x \in \mathbb{Z}_k^+ \) with \( \mu_{\gamma,t}(w_x) \geq \delta_{\gamma,t}/|G/\gamma G| \). Hence, for all \( x \in A \), we have
\[
\mathbb{P}(S_\gamma(t) = x) \geq \mathbb{P}(W_\gamma(t) = w_x) = \mu_{\gamma,t}(w_x) \geq \delta_{\gamma,t}^{-1}/|G/\gamma G|,
\]
recalling that \( S_\gamma \) lives in the quotient \( G/\gamma G \). Summing over \( x \in A \), we deduce that
\[
1 \geq \sum_{x \in A} \mathbb{P}(S_\gamma(t) = x) \geq |A| \cdot \delta_{\gamma,t}^{-1}/|G/\gamma G|, \quad \text{and hence} \quad |A|/|G/\gamma G| \leq \delta_{\gamma,t} = o(1).
\]

Projecting onto \( G/\gamma G \) (which can only decrease the TV distance), we see that
\[
\left\| \mathbb{P}_{G_\gamma}(S(t) \in \cdot) - \pi_{G/\gamma G} \right\|_{TV} \geq \mathbb{P}(S_\gamma(t) \in A) - \pi_{G/\gamma G}(A) \geq \mathbb{P}(\mathcal{E}) - |A|/|G/\gamma G| = 1 - o(1).
\]

Finally, recall that \( \max_{\gamma \in \mathbb{N}} t_\gamma = \max_{\gamma \in \mathbb{N}} t_0(\gamma, |G/\gamma G|) \). This completes the proof. \( \square \)

3.7 Upper Bound on Mixing

To upper bound the mixing time, we use a ‘modified \( L_2 \) calculation’, as in the previous approach. This involves first conditioning that \( W \) has some ‘typical’ properties, laid out in the following definition, and then performing a standard \( TV-L_2 \) upper bound on the conditioned law.

Abbreviate \( t_{\gamma,\varepsilon} := t_\gamma(1 + \varepsilon) \). Recall that \( d = d(G) \) and \( \zeta_\gamma = \frac{1}{k}(k - d) \log \gamma \); set \( \zeta_\gamma := \zeta_\gamma \wedge 1 \).

Definition 3.8. Let \( \varepsilon > 0 \); recall the constant \( c_\varepsilon > 0 \) from Proposition 3.5. The following depend on \( \varepsilon \); we suppress this in the notation. Define global typical sets for \( \gamma \in \mathbb{N} \) by
\[
W_{\gamma, glo} := \{ w \in \mathbb{Z}_k^+ \mid \mathbb{P}(W_\gamma(t_{\gamma,\varepsilon}) = w) \leq \delta_{\gamma,\varepsilon}/|G/\gamma G| \} \quad \text{where} \quad \delta_{\gamma} := \delta_{\gamma,\varepsilon} := e^{-c_\varepsilon \zeta_\gamma k}.
\]

Also define \( \delta_\infty := \delta_{\infty,\varepsilon} := e^{-c_\varepsilon k} \). Define the local typicality set by
\[
W_{loc} := \{ w \in \mathbb{Z}_k \mid |w_i - \mathbb{E}(W_i(t_{\gamma,\varepsilon}))| \leq r_i, \forall i \in [k] \} \quad \text{where} \quad r_i := \frac{1}{2}|G|^{1/k}(\log k)^2.
\]

When \( W' \) is an independent copy of \( W \), define typicality by
\[
typ := \{ W(t_{\gamma,\varepsilon}), W'(t_{\gamma,\varepsilon}) \in W_{loc} \} \cap \{ \gamma \in \mathbb{R} \} \{ W_\gamma(t_{\gamma,\varepsilon}), W'_\gamma(t_{\gamma,\varepsilon}) \in W_{\gamma, glo} \}
\]

where \( \Gamma \) is a subset of \([2, |G|]\) to be defined below in Definition 3.11. 24
We are going to do a union bound over $\gamma \in \Gamma$, so desire control on $\sum_{\gamma \in \Gamma} \delta_{\gamma}$.

**Lemma 3.9.** For all $\Gamma \subseteq \mathbb{N} \setminus \{1\}$, we have $\sum_{\gamma \in \Gamma} \delta_{\gamma} \leq \delta_{\infty,\varepsilon} |\Gamma| + o(1)$.

**Proof.** Since $\min \Gamma \geq 2$ and $k - d > 0$, we have

$$\sum_{\gamma \in \Gamma} \delta_{\gamma} \leq \sum_{\gamma \in \Gamma} (e^{-c_\gamma k} + e^{-c_\gamma \zeta k}) = e^{-c_\gamma k} |\Gamma| + \sum_{\gamma \in \Gamma} \gamma^{-c_\gamma (k-d)} = \delta_{\infty} |\Gamma| + o(1).$$

**Proposition 3.10.** For all $\varepsilon > 0$ and any subset $\Gamma \subseteq \mathbb{N} \setminus \{1\}$, we have

$$\mathbb{P}(\text{typ}) \geq 1 - 2\delta_{\infty,\varepsilon} |\Gamma| - o(1).$$

**Proof.** Suppress the time-dependence from the notation, eg writing $W$ for $W(t_{*,\varepsilon})$.

Consider global typicality. First, observe that

$$Q_\gamma = -\log \mu_\gamma(W_\gamma) \geq \log |G/\gamma G| + c_\gamma \zeta k$$

if and only if $\mu_\gamma(W_\gamma) \leq e^{-c_\gamma \zeta k} / |G/\gamma G|$. Hence, recalling that $\delta_{\gamma,\varepsilon} = \exp(-c_\gamma \zeta k)$, by Proposition 3.5, we have

$$\mathbb{P}(\mu_\gamma(W_\gamma) \leq \delta_{\gamma,\varepsilon} / |G/\gamma G|) \leq \delta_{\gamma}, \quad \text{and hence } \mathbb{P}(\bigcap_{\gamma \in \Gamma} \{ W_\gamma \in W_{\gamma,\zeta \alpha} \}) \geq 1 - \sum_{\gamma \in \Gamma} \delta_{\gamma},$$

by the union bound. Recall that $\zeta_\gamma = \frac{1}{k}(k - d) \log \gamma$. Applying Lemma 3.9, we deduce that

$$\mathbb{P}(\bigcap_{\gamma \in \Gamma} \{ W_\gamma \in W_{\gamma,\zeta \alpha} \}) \geq 1 - \delta_{\infty,\varepsilon} |\Gamma| - o(1) \quad \text{where } \delta_{\infty,\varepsilon} = e^{-c_\gamma k}.$$

Now consider local typicality. Proposition 3.2a says that $t/k \leq |G|^{2/k} \log k$. Then [27, Definitions C.1 and C.2 and Proposition C.3] together give

$$\mathbb{P}(\bigcap \{ |W_i - \mathbb{E}(W_i)| \leq r_\ast \}) = 1 - o(1); \quad \text{hence } \mathbb{P}(W \in W_{\text{loc}}) = 1 - o(1).$$

The claim follows by combining local and global typicality and applying the union bound.

We now choose the set $\Gamma$, to make sense of typicality. Recall that $\alpha \vdots \beta$ means that $\alpha$ divides $\beta$.

**Definition 3.11.** Abbreviate $n_* := (n - 1) \land |2r_*|$. Define $\Delta := \{ \gamma \in [2, n_*] \mid \gamma \vdots n \}$. Write $\mathcal{H}$ for the set of all proper subgroups $H$ of $G$ which can be represented as $H = \gamma G$ for some $\gamma \in \Delta$:

$$\mathcal{H} := \{ H \mid H = \gamma G \neq G \text{ for some } \gamma \vdots n \text{ with } 2 \leq \gamma \leq n_* \}.$$  

Given $H \in \mathcal{H}$, write $\Gamma_H := \{ \gamma \in \Delta \mid H = \gamma G \}$ and denote by $\gamma_H$ the minimal $\gamma \vdots n$ so that $H = \gamma G$, ie $\gamma_H := \inf \Gamma_H$. Finally, define $\Gamma := \{ \gamma_H \mid H \in \mathcal{H} \cup \{n\} \mid \text{for all } H \in \mathcal{H} \cup \{n\} \}$; so $\Gamma \subseteq \Delta \cup \{n\} \subseteq [2, n_*] \cup \{n\}$.

The following lemma, whose proof is deferred to the end of the subsection, will also be needed.

**Lemma 3.12.** For all $H \in \mathcal{H}$ and all $\gamma \in \Gamma_H$, we have $\gamma_H \vdots \gamma$.

As shown below, we can combine our results to control the $L_2$ distance conditioned on typicality. In analogy with §2.7 and Proposition 2.9, write

$$D := D(t) := n \mathbb{P}(V(t) \cdot Z = 0 \mid \text{typ}) - 1.$$  

**Proposition 3.13.** Write $g := \gcd(V_1, \ldots, V_k, n)$. Then, for all $\varepsilon \in (0, 1)$, we have

$$0 \leq D(t(1 + \varepsilon)) = \sum_{\gamma \in \mathbb{N}} \mathbb{P}(g = \gamma \mid \text{typ}) \cdot |G/\gamma G| - 1 \leq (\delta_{\infty,\varepsilon} |\mathcal{H}| + o(1)) / \mathbb{P}(\text{typ}).$$

(The conditions of Hypothesis B imply immediately that this last term is $o(1)$.)

From Propositions 3.10 and 3.13, it is straightforward to deduce the upper bound on mixing.
Proof of Upper Bound in Theorem 3.6. We use a modified $L_2$ calculation at time $(1+\varepsilon)\max, t_\gamma$.

- Condition that $W$ satisfies typicality; see Definition 3.8 and Proposition 3.10.
- Perform the standard TV–$L_2$ upper bound on the law of $S$ conditioned that $W$ is typical.
- Upper bound the expected $L_2$ distance by $(\delta_{\infty,\varepsilon}[H]+o(1))/P(\text{typ})$; see Proposition 3.13.
- This gives an upper bound on the expected TV distance of $(\delta_{\infty,\varepsilon}[H]+o(1))/P(\text{typ}) + P(\text{typ}^\ast)$.
- From the definition of $\Gamma$, it is clear that $|\Gamma| \leq |H| + 1$. Since $\delta_{\infty,\varepsilon} = e^{-c_\varepsilon k} = o(1)$, with $c_\varepsilon$ an arbitrary constant, the assumed condition $k \gg \log |H|$ gives a final bound of $o(1)$ on the expected TV distance, recalling that $P(\text{typ}) = 1-o(1)$ by Proposition 3.10.
- By Markov’s inequality, this means that the TV distance is $o(1)$ whp.

We now prove Proposition 3.13. To ease exposition, while all terms are evaluated at time $t_{\ast,\varepsilon} = (1+\varepsilon)\max, t_\gamma$, we suppress this from the notation.

Proof of Proposition 3.13. Write $V := W - W'$ and $g := \gcd(V_{\gamma,1}, \ldots, V_{\gamma,k}, n)$. If $g = \gamma$, which must have $\gamma \not\mid n$ as the gcd is with $n$, then $V \cdot Z \sim \text{Unif}(\gamma G)$ by Lemma 2.11. Then
\[
D = n P(V \cdot Z = 0 \mid \text{typ}) - 1 = |G| \sum_{\gamma | n} P(g = \gamma \mid \text{typ})/|\gamma G| - 1.
\]

We consider various cases. Combining together all $\gamma$ such that $\gamma G = G$, we upper bound
\[
|G| P(g \in \{\gamma \mid \gamma G = G\})/|\gamma G| \leq 1.
\]

If $V_{\infty} = 0$ in $Z^k$, then $g = \gamma = n$, which gives $\gamma G = \{\text{id}\}$; using the definition of typicality,
\[
|G| P(V \cdot Z = 0 \mid \gamma G) / |\gamma G| = |G| E(P(W_{\infty} = W'_{\infty} \mid W_{\infty}', \text{typ}) \mid \text{typ}) \leq \delta_{\infty}/P(\text{typ});
\]
cf Lemma 2.10. If $V_{\infty} \neq 0$, then, given (local) typicality, $g \leq n_* = (n-1) \wedge |2_r|$. So it remains to study $\gamma \in \Delta$. As a consequence of Lemma 3.12, for any $H \in \mathcal{H}$, we have
\[
\{V_\gamma = 0 \text{ for some } \gamma \in \Gamma_H\} \subseteq \{V_{\gamma m} = 0\}.
\]
(Recall that $V_\gamma \in Z_k^H$ for each $\gamma$.) This is key: it allows us to collapse the consideration of all $\gamma \in \Gamma_H$ down to the single element $\gamma_H$. Indeed, using the above we have
\[
\sum_{\gamma \in \Gamma_N} P(g = \gamma \mid \text{typ}) / |\gamma G| = P(\bigcup_{\gamma \in \Gamma_N} \{g = \gamma \mid \text{typ}\}) / |H| \leq P(V_\gamma = 0 \text{ for some } \gamma \in \Gamma_H) / |H| \leq P(V_{\gamma N} = 0 \mid \text{typ}) / |H| \leq (\delta_{\gamma N} / |G|) / P(\text{typ}),
\]
with the final inequality using typicality, as above. We decompose $\sum_{\gamma \in \Delta}$ into $\sum_{H \in \mathcal{H}} \sum_{\gamma \in \Gamma_H}$:
\[
|G| \sum_{\gamma \in \Delta} P(g = \gamma \mid \text{typ}) / |\gamma G| = |G| \sum_{H \in \mathcal{H}} \sum_{\gamma \in \Gamma_H} P(g = \gamma \mid \text{typ}) / |\gamma G| \leq \sum_{H \in \mathcal{H}} \delta_{\gamma N} / P(\text{typ})
\]

(Note that every $\gamma$ gives rise to a unique $H$ such that $\gamma G = H$ and, by definition, $\mathcal{H}$ is the set of all $H$ which can be obtained as $\gamma G$ for some $\gamma$; hence this decomposition neither overcounts nor undercounts $\gamma \in \Delta$.) Combining all these and using Lemma 3.9, we deduce the proposition:
\[
0 \leq n P(V \cdot Z = 0 \mid \text{typ}) - 1 = |G| \sum_{\gamma | n} P(g = \gamma \mid \text{typ}) / |\gamma G| - 1 \leq (\delta_{\infty} |\mathcal{H}| + o(1)) / P(\text{typ}).
\]

It remains to give the deferred proof of the divisibility lemma, namely Lemma 3.12.

Proof of Lemma 3.12. Consider any decomposition of $G$ as $\bigoplus_{i} Z_{m_i}$; this does not require $r = d(G)$. Fix some $\beta \in \Gamma_H$. Since $\alpha G = \beta G$ if and only if $\gcd(\alpha, m_i) = \gcd(\beta, m_i)$ for all $i$, we may decompose $H$ as $\bigoplus_{i} h_i Z_{m_i}$ where $h_i := \gcd(\beta, m_i)$ for all $i$. Set $\gamma_* := \text{lcm}(h_1, \ldots, h_r)$. We show that $\gamma_* H = H$ and that $\gamma_* \mid \alpha$ for all $\alpha \in \Gamma_H$; this proves the lemma.

Fix $j \in [r]$. Now, $h_j \mid \gamma_*$, and $h_j \mid m_i$ by assumption. Hence $h_j \mid \gcd(\gamma_*, m_j)$. Conversely, if $x \mid z$ and $y \mid z$ then $\text{lcm}(x, y) \mid z$, and so $\gamma_* = \text{lcm}(h_1, \ldots, h_r) \mid \beta$ since $h_j \mid \beta$. Hence $\gcd(\gamma_*, m_j) \mid \gcd(\beta, m_j) = h_j$. Thus $h_j = \gcd(\gamma_*, m_j)$. Hence $\gamma_* G = H$. Now consider any $\alpha$ with $\alpha G = H$; so $h_j = \gcd(\alpha, m_j)$ for all $j$. Hence $h_j \mid \alpha$ for all $j$, and so $\text{lcm}(h_1, \ldots, h_r) \mid \alpha$, i.e. $\gamma_* \mid \alpha$. \(\square\)
4 TV Cutoff: Combining Approaches #1 and #2

In this section we combine the analysis from the previous two approaches to study the regime
\[ \frac{\sqrt{\log |G|}}{\log \log \log |G|} \lesssim k \lesssim \sqrt{\log |G|} \quad \text{with} \quad 1 \ll k - d(G) \ll k. \]

We use the more refined notion of the entropic times; see §3.2.

4.1 Precise Statements and Remarks

In this subsection, we state precisely the main theorem of the section. There are some simple conditions on \( k \), in terms of \( d(G) \) and \( |G| \), needed for the upper bound.

**Hypothesis C.** The sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis C if the following hold:

\[
\liminf_{N \to \infty} k_N / \sqrt{\log |G_N| / \log \log \log |G_N|} > 0, \quad \limsup_{N \to \infty} k_N / \sqrt{\log |G_N|} < \infty, \]
\[
\liminf_{N \to \infty} (k_N - d(G_N)) = \infty \quad \text{and} \quad \limsup_{N \to \infty} (k_N - d(G_N))/k_N = 0.
\]

Throughout the proofs, we drop the subscript-\( N \) from the notation, eg writing \( k \) or \( n \), considering sequences implicitly. Recall that we abbreviate the TV distance from uniformity at time \( t \) as
\[
d_{G_k,N}(t) = \| P_{G_N}(z_1, \ldots, z_{k_N}) \{ S(t) \in \cdot \} - \pi_{G_N} \|_{TV}
\]
where \( Z_1, \ldots, Z_{k_N} \sim \text{iid Unif}(G_N) \).

We now state the main theorem of this section. Recall that \( t_* = \max_{\gamma \in \mathbb{N}} t_0(\gamma, |G/\gamma G|) \).

**Theorem 4.1.** Let \((k_N, G_N)_{N \in \mathbb{N}}\) be a sequence of positive integers and \((G_N)_{N \in \mathbb{N}}\) a sequence of finite, Abelian groups; for each \( N \in \mathbb{N} \), define \( Z_{(N)} := [Z_1, \ldots, Z_{k_N}] \) by drawing \( Z_1, \ldots, Z_{k_N} \sim \text{iid Unif}(G_N) \).

Suppose that the sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis C. Let \( c \in (-1, 1) \setminus \{0\} \). Then
\[
d_{G_k,N}^\pm (1 + c)k_{\pm}^\pm (k_N, G_N) \overset{P}{\rightarrow} 1(c < 0) \quad \text{(in probability)} \quad \text{as} \quad N \to \infty.
\]
That is, whp, there is cutoff at max, \( t_*^\pm (\gamma, |G/\gamma G|) \). Moreover, the implicit lower bound holds deterministically, ie for all choices of generators.

**Remark 4.2.** In short, the conditions of Hypothesis C say that
\[ \frac{\sqrt{\log |G|}}{\log \log \log |G|} \lesssim k \lesssim \sqrt{\log |G|} \quad \text{and} \quad 1 \ll k - d(G) \ll k. \]

The regime of smaller \( k \) is covered by Approach #1 and of larger \( k \) by Approach #2. \( \triangle \)

**Remark.** Recall that the lower bound from §3 is valid whenever \( 1 \ll k \ll \log |G| \) and \( k - d(G) \gg 1 \).

It thus suffices to consider only the upper bound here. \( \triangle \)

4.2 Outline of Proof

Fundamentally, we still wish to bound the same expression that we did in previously:
\[
\sum_{\gamma \mid |G|} \mathbb{P}(g = \gamma \mid \text{typ}) \cdot |G/\gamma G| - 1;
\]
see Propositions 2.9 and 3.13. In §2.7 we used \(|G/\gamma G| \leq \gamma^{d(G)}\). In §3.7 we used typicality to get
\[
\mathbb{P}(g = \gamma \mid \text{typ}) \leq \mathbb{P}(W_\gamma = W'_\gamma \mid \text{typ}) \ll 1/|G/\gamma G|.
\]
The upper bound \(|G/\gamma G| \leq \gamma^{d(G)}\) is fairly crude. Roughly the idea here is to show, for this interim regime of \( k \) around \( \sqrt{\log |G|} \), that for all but \( e^{o(k)} \) of the \( \gamma \) we can improve it; for the remaining \( \gamma \), we use the second approach. (Before we considered \( |\mathcal{H}| \) different \( \gamma \), and so required \( |\mathcal{H}| = e^{o(k)} \).)
4.3 Upper Bound on Mixing

Let $G$ be an Abelian group; set $n := |G|$. One can find a decomposition $\oplus_1^d \mathbb{Z}_{m_i}$ of $G$ such that $d = d(G)$, the minimal size of a generating set, and $m_i m_j$ for all $i \leq j$. (This can be proved by induction. Alternatively, write $G$ as a direct sum of $p$-groups then merge the $p$-groups appropriately.) For the remainder of this section we fix such a decomposition.

We use the more refined concept of typicality from Approach #2. Let $\varepsilon > 0$ and let $t := (1 + \varepsilon)t_*(k, G)$. We frequently suppress the $t$ and $\varepsilon$ dependence in the notation. Let $c := c_\varepsilon > 0$ be the constant from Proposition 3.5. Recall some notation:

$$
\zeta_\gamma := \frac{1}{k}(k - d) \log \gamma, \quad \hat{\zeta}_\gamma := \zeta_\gamma \land 1 \quad \text{and} \quad \delta_\gamma := e^{-c\hat{\zeta}_\gamma k}.
$$

Note that $k - d \gg 1$ and $k \leq \log n$, so $\hat{\zeta}_n = 1$; set $\tilde{\zeta}_\infty := 1$. Recall that $W$ is a RW on $\mathbb{Z}$ and we define $W_\gamma$ by $W$ mod $\gamma$; set $W_\infty := W$. We now define typicality for this section precisely.

Definition 4.3 (cf Definition 3.8). Define typical sets for $\gamma \in \mathbb{N} \cup \{\infty\}$ by the following:

$$
\begin{align*}
W_{\gamma, \text{glob}} & := \{w \in \mathbb{Z}_\gamma^k \mid \mathbb{P}(W_\gamma(t) = w) \leq \delta_\gamma/|G/\gamma G|\} \quad \text{where} \quad \delta_\gamma := e^{-c\hat{\zeta}_\gamma k}; \\
W_{\text{loc}} & := \{w \in \mathbb{Z}_\gamma^k \mid |w_i - E(W_\gamma(t))| \leq r_s \quad \forall i \in [k]\} \quad \text{where} \quad r_s := \frac{1}{2} n^{1/k}(\log k)^2.
\end{align*}
$$

Choose $L$ to be the maximal integer in $[1, d]$ with $m_L \leq M$ where

$$
M := \exp\left(\sqrt{\log n/ \log \log n}\right) \quad \text{set} \quad \Gamma := \{rm \mid r \in [k^{1/2}], m \mid m_L, rm \mid n\} \setminus \{1\}.
$$

When $W'$ is an independent copy of $W$, define typicality by

$$
\text{typ} := \{W(t), W'(t) \in W_{\text{loc}} \cap (\cap_{\gamma \in \Gamma} \{W_\gamma(t), W'_\gamma(t) \in W_{\gamma, \text{glob}}\})\}.
$$

Lemma 4.4. We have $\log |\Gamma| \ll k$. In particular, $\delta_\infty |\Gamma| = o(1)$.

Proof. We have $|\Gamma| \leq k^{1/2} \div m_L$, where $\div m$ is the number of divisors of $m \in \mathbb{N}$. By [21, §18.1], we have $\log \div m \ll \log m/ \log \log m$ uniformly in $m \in \mathbb{N}$. By the definition of $m_L$ and the assumption that $k \geq \sqrt{\log n/ \log \log n}$, we obtain

$$
\log \div m_L \ll \log M/ \log \log M \ll \sqrt{\log n/ \log \log n} \ll k.
$$

Thus $\log |\Gamma| \ll k$. Recall that $\log(1/\delta_\infty) \asymp k$. \hfill \square

In §3, typicality was initially defined for a general subset of $\mathbb{N} \setminus \{1\}$; see Definition 3.8. The following result is an immediate consequence of Lemmas 3.9 and 4.4 and Proposition 3.10.

Lemma 4.5 (cf Lemma 3.9 and Proposition 3.10). We have $\sum_{\gamma \in \Gamma} \delta_\gamma = o(1)$ and $\mathbb{P}(\text{typ}) = 1 - o(1)$.

Thus, by applying the modified $L_2$ calculation, it suffices to prove the following result.

Proposition 4.6. Let $\varepsilon > 0$ be fixed and set $t := (1 + \varepsilon)t_*(k, G)$. Then

$$
|G| \mathbb{P}(S = S' \mid \text{typ}) - 1 = \sum_{\gamma \in \Gamma} |G/\gamma G| \mathbb{P}(g = \gamma \mid \text{typ}) - 1 = o(1).
$$

In order to prove this, we first show that $L \asymp d \asymp k$.

Lemma 4.7. We have $0 \leq d - L \leq \sqrt{\log n/ \log \log n} \ll k$. In particular, $L \asymp d \asymp k$.

Proof. Since $n = m_1 \cdots m_d$, $m_1 \leq \cdots \leq m_d$ and $m_L \leq M$, if $L < d$ then $M^{d-L} \leq m_{L+1}^{d-L} \leq n$. Rearranging gives the inequality. Finally, recall that $k \geq \sqrt{\log n/ \log \log n}$ and $k \asymp d$. \hfill \square

We prove Proposition 4.6 by separating the sum over $\gamma$ into two parts according to $\Gamma$.  

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Proof of Proposition 4.6. Observe that \(|G/\gamma G|\mathbb{P}(g = \gamma \mid \text{typ}) \leq 1\) when \(\gamma = 1\). Also, \(g \not\equiv n\). Thus
\[
\sum_{\gamma \in \Gamma} |G/\gamma G|\mathbb{P}(g = \gamma \mid \text{typ}) - 1 \leq \sum_{\gamma \in \Gamma} |G/\gamma G|\mathbb{P}(g = \gamma \mid \text{typ}) + \sum_{\gamma \in \Gamma} |G/\gamma G|\mathbb{P}(g = \gamma \mid \text{typ})
\]
where \(\Gamma := \{\gamma \in [2, n] \mid \gamma n \not\equiv n\} \setminus \Gamma\). We analyse these sums with Approach #1 and #2, respectively. Namely we show below that both sums are \(o(1)\), when \(t := (1+c)t_*(k, G)\) with \(c > 0\) a constant.

Analysis via Approach #1. Suppose that \(\gamma \in \Gamma\), so \(\gamma \not\equiv \Gamma \cup \{1\}\). We improve the inequality \(|G/\gamma G| \leq \gamma^d\) via the following argument. For each \(j \in [L]\), we may write
\[
\gamma = r_j \cdot \gcd(\gamma, m_j) \quad \text{and} \quad m_j = r_j' \cdot \gcd(\gamma, m_j) \quad \text{where} \quad \gcd(r_j, r_j') = 1.
\]
By definition of \(\Gamma\), if \(\gamma = r \cdot m\) for some \(m \mid m_j\), then \(r > k^{1/2}\). Hence \(\gcd(\gamma, m_j) = \gamma / r_j \leq \gamma / k^{1/2}\) for \(j \in [L]\). Applying this to the first \(L\) terms of the product gives \(|G/\gamma G| \geq \prod_{j = 1}^{L} \gcd(\gamma, m_j) \leq k^{d/L^2} / k^{d/L}\).

Exactly the same analysis as in the proof of Corollary 2.15 then leads us to
\[
\sum_{\gamma \in \Gamma} |G/\gamma G|\mathbb{P}(g = \gamma \mid \text{typ}) \leq e^{24k}2^{d+1-k} + 2^k d^{d+1-k} k^{(d+1-k)/k} + 4^k (\log k)^{2(d+1)/k} / k^{d/L^2},
\]
where \(\delta\) is any value in \((0, 1)\). We set \(\delta := 1/(k - d - 1)/k\); the calculation showing that the first two terms are \(o(1)\) if \(k - d \gg 1\) and \(1 \ll k \ll \log n\) for this choice is the same as the one from Corollary 2.15. For the third term, \(4^k (\log k)^{2(d+1)/k} / k^{d/L} \ll 1\) as \(L \approx k \approx d\); thus the final term is also \(o(1)\). We thus deduce that the sum over \(\gamma \in \Gamma\) is \(o(1)\).

Analysis via Approach #2. The typicality conditions set out in Definition 4.3 imply that
\[
\mathbb{P}(g = \gamma \mid \text{typ}) \leq \mathbb{P}(W_\gamma = W'_\gamma \mid \text{typ}) \leq \delta_\gamma / |G/\gamma G|;
\]
cf Lemma 2.10. Combining this with Lemma 4.5, we deduce that the sum over \(\gamma \in \Gamma\) is \(o(1)\):
\[
\sum_{\gamma \in \Gamma} |G/\gamma G|\mathbb{P}(g = \gamma \mid \text{typ}) \leq \sum_{\gamma \in \Gamma} \delta_\gamma = o(1).
\]

5 Separation Cutoff

In this section we prove Theorem B, namely establish cutoff in the separation metric for an appropriate regime of \(k\). Recall that, for \(t \geq 0\), the separation distance is defined by
\[
s(t) := \max_{x,y} \{1 - P_t(x, y) / \pi(y)\},
\]
where \(P_t(x, y)\) is the time-\(t\) transition probability from \(x\) to \(y\) and \(\pi\) the invariant distribution. We write \(s_{d_{G_k, N}}\) when considering sequences \((k_N, G_N)_{N \in \mathbb{N}}\), analogously to \(d_{G_k, N}^+\).

5.1 Precise Statement and Remarks

We now state the main theorem; as for the previous theorems, there are conditions on \((k, G)\).

**Hypothesis D.** The sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis D if the following hold:
\[
\liminf_{N \to \infty} \frac{k_N - d(G_N)}{\max \{ (\log |G_N|/k_N)^2, (\log |G_N|)^{1/2}\}} = \infty \quad \text{and} \quad \limsup_{N \to \infty} \frac{\log k_N}{\log |G_N|} = 0.
\]

**Theorem 5.1.** Let \((k_N)_{N \in \mathbb{N}}\) be a sequence of positive integers and \((G_N)_{N \in \mathbb{N}}\) a sequence of finite, Abelian groups; for each \(N \in \mathbb{N}\), define \(Z_{1,N} := \{Z_1, ..., Z_{k_N}\}\) by drawing \(Z_1, ..., Z_{k_N} \sim\text{Ind} \text{Unif}(G_N)\).

Suppose that the sequence \((k_N, G_N)_{N \in \mathbb{N}}\) satisfies Hypothesis D. Let \(c \in (-1, 1) \setminus \{0\}\). Then
\[
s_{d_{G_k, N}}^+((1 + c)t_*(k_N, G_N)) \to^p 1(c < 0) \quad (\text{in probability}) \quad \text{as} \quad N \to \infty.
\]
That is, there is cutoff in the separation metric at \(t_*(k, G)\) whp. Moreover, the implicit lower bound on the separation distance holds deterministically, ie for all choices of generators.

**Remark 5.2.** It is easy to check that Hypothesis D is satisfied when
\[
k \geq (\log |G|)^{3/4}, \quad k - d(G) \geq (\log |G|)^{1/2} \quad \text{and} \quad \log k \ll \log |G| \quad (\text{simultaneously}).
\]

The proof uses the previously established TV mixing time bound as a building block.
5.2 Lower Bound

Since TV is a lower bound on separation (see, e.g., [34, Lemma 6.16]), the lower bound follows from the TV result. References for the TV result are as follows. See Theorem 3.6, specifically §3.6 for the lower bound on mixing, for the regime $k < \log |G|$. For $k \gg \log |G|$, TV cutoff had already been established at time $t_*(k, G)$; see §1.3.1 and [27, Proposition B.19].

5.3 Upper Bound

We analyse the upper bound in Theorem 5.1 via a sequence of lemmas. From these, the upper bound in Theorem 5.1 follows immediately. Throughout, Hypothesis D should be assumed.

Preliminaries. For $y, z \in G$ and $t \geq 0$, write $P_t(y, z) := P_y(S(t) = z)$ for the time-$t$ transition probability from $y$ to $z$. Write $n := |G|$. We want to show, for fixed $\xi > 0$, that

$$\min_{x \in G} P_t^x(0, x) \geq \frac{1}{n}(1-o(1)) \quad \text{for some} \quad t \leq (1+2\xi)t^+_{\ast}(k, G).$$

Abbreviate $d := d(G)$. Let $\chi = o(1)$ to be specified later. Throughout the proof, we impose conditions on $\chi$: at the end of the proof, we show that these are equivalent to Hypothesis D. Set $k' := k - \chi(k-d)$; then $k' \approx k$ and $k' - d = (1 - \chi)(k-d) \approx k-d \gg 1$. Let $A = [Z_1, ..., Z_{k'}]$ be the first $k'$ generators and $B := [Z_{k' + 1}, ..., Z_k]$ be the remaining $k - k' = \chi(k-d)$. Since $G$ is Abelian, $P_t = P_t^A P_t^B$ where in $P_t^A$, respectively $P_t^B$, we pick each generator of $A$, respectively $B$, at rate $1/k$ independently. (In words, first apply the generators from $A$ and then those from $B$.) □

Let $\xi > 0$ be a constant; let $t' := (1+\xi)t_*(k', G)$. Since there is cutoff at $t_*(k', G)$, we can then choose $\delta = o(1)$ so that $t'$ is larger than the $\delta^2$-TV mixing time for the rate-1 RW on $G(A)$ for a typical choice of $A$. In the regime $k \gg \log n$, simply having $\delta = o(1)$ will be sufficient. In the regime $k \lesssim \log n$, we quantify this $\delta$; since $k \gg \sqrt{\log n}$, by Hypothesis D, Approach #2 (§3) applies. We also compare $t_*(k', G)$ and $t_*(k, G)$, the whp-cutoff times for $G(A)$ and $G(Z)$, respectively.

The following two auxiliary lemmas have their proofs deferred to §5.4.

Lemma 5.3. Assume Hypothesis D. When $k \lesssim \log n$ there exists a constant $c > 0$ so that we may choose $\delta := e^{-2c(k-d)}$, ie the $e^{-4c(k-d)}$ mixing time of the RW on $G(A)$ is at most $t'$ whp.

Lemma 5.4. We have $t_*(k', G) \approx t_*(k, G)$ if and only if $\chi(k-d)k^{-2}\log n \ll 1$.

Assume that $\chi(k-d)k^{-2}\log n \ll 1$ so that $t_*(k', G) \approx t_*(k, G)$. To relate this to the rate-1 RW on $G(Z)$, rescale time by $k/|A| = 1/(1 - \chi(k-d)/k)$: set $t := t'/((1 - \chi(k-d)/k)$. Thus $t \approx (1 + \xi)t_*(k, G)$ as $\chi \ll k/(k-d)$; in particular, $t \leq (1 + 2\xi)t_*(k, G)$. By monotonicity of the separation distance with respect to time, it thus suffices to show that

$$\min_{x \in G} P_t(0, x) \geq \frac{1}{n}(1-o(1)).$$

Lemma 5.5. Assume that $\chi(k-d)k^{-2}\log n \ll 1$. Suppose that we can choose $\delta, \chi, \eta \ll 1$ so that, for all (deterministic) sets $D \subseteq G$ with $|G \setminus D| \leq \delta|G|$ and all $x \in G$ uniformly, we have

$$\Pr(Q_B(x, D) \leq 1 - \eta) = o(1/|G|) \quad \text{where} \quad Q_B(y, z) := |B_\pm|^{-1} \sum_{b \in B_\pm} 1(y + b^{-1} = z)$$

for $y, z \in G$ where $B_+ := B$ and $B_- := B \cup B^{-1}$ (as multisets). Then

$$\min_{x \in G} P_t(0, x) \geq \frac{1}{n}(1-o(1)) \quad \text{whp}.$$

Proof. We condition on a typical realisation of $A$, namely write $A := \{a \mid t_{\max}(\delta^2; G(a)) \leq t'\}$ and condition on $A = a$ for a fixed $a \in A$. We have $\Pr(A \in A) = 1-o(1)$. Given $A = a \in A$, the set

$$D := \{z \in G \mid P_{t,a}(0, z) \geq \frac{1}{n}(1-\delta)\}$$

satisfies $|D| \geq n(1-\delta)$.

For the undirected case (ie the RW on $G_k^u$), by reversibility, conditional on $A$, we have

$$P_t^x(0, x) \geq P_{t,B}^x(x, D) \cdot \frac{1}{n}(1-\delta).$$
While the RW on $G_k^+$ is not reversible, Cayley graphs have the special property that a step ‘backwards’ with a generator $z$ corresponds to a step ‘forwards’ with $z^{-1}$. Thus

$$P^+_t(0, x) \geq Q^+_{t,B}(x, D) \cdot \frac{1}{t}(1 - \delta)$$

where $Q^+_{t,B}$ is the heat kernel for the RW on $G^+(B^{-1})$ where $B^{-1} := [z^{-1} \mid z \in B]$, rather than on $G(B)$. For the RW on $G_k^+$, replacing the generators with their inverses has no effect on the graph (or RW); set $Q_{t,B} := P^+_{t,B}$. We want to show that $Q_{t,B}(x, D) = 1 - o(1)$ uniformly in $x \in G$ whp.

This is a RW on $G^+(B^{-1})$ run for time $t$. By considering just the final step of this RW, we now argue that the hypothesis of the lemma is sufficient. Indeed, first note that

$$\min_x Q_{t,B}(x, D) \geq (1 - e^{-t|B|/k}) \cdot \min_x Q_B(x, D),$$

where $e^{-t|B|/k}$ is the probability that none of the generators in $B$ are applied by time $t$. We impose below the condition that $\chi(k - d)^2 \gg \log n$; since $k - d \leq k \leq \log n$, this in particular implies that $|B| = \chi(k - d) \gg 1$. Since $t \gg k$, we deduce that $t \gg k/|B|$, i.e. $e^{-t|B|/k} = o(1)$. Thus the above failure probability allows us to perform a union bound to say, conditional on $A = a \in \mathcal{A}$, that

$$P(\min_x Q_{t,B}(x, D) \leq 1 - 2\eta \mid A = a) = o(1),$$

where the randomness is over the generators $B$, provided $\eta$ decays sufficiently slowly. For $A \in \mathcal{A}$ we have the desired lower bound on $\min_x P_t(0, x)$. Finally we average over $A$ and use $P(A \in \mathcal{A}) = 1 - o(1)$ to show that $\min_x P_t(0, x) \geq \frac{1}{t}(1 - o(1))$ whp.

We next find conditions under which the supposition of the lemma is satisfiable.

**Lemma 5.6.** Assume that $\chi(k - d)k^{-2} \log n \ll 1$ and $\chi(k - d)^2 \gg \log n$. We can choose $\delta, \chi, \eta < 1$ so that, for all (deterministic) sets $B \subseteq G$ with $|G \setminus D| \leq \delta|G|$ and all $x \in G$ uniformly, we have

$$P(Q_B(x, D) \leq 1 - \eta) = o(1/|G|)$$

where $Q_B(y, z) := |B_k|^{-1} \sum_{b \in B_k} 1(y + b^{-1} = z)$ for $y, z \in G$ where $B_+ := B$ and $B_- := B \cup B^{-1}$ (as multisets).

**Proof.** Fix an arbitrary $x \in G$. We desire at least a proportion $1 - \eta$ of the generators in $B$ to connect $x$ to $D$. The generators are chosen independently, and each connect with probability $|D||G| \geq 1 - \delta$. Since there are $\chi(k - d)$ generators, it thus suffices to choose $\eta \ll 1$ so that

$$P(\text{Bin}(\chi(k - d), 1 - \delta) \leq \chi(k - d)(1 - \eta)) = o(1/|G|).$$

Let $L := \chi(k - d)$. Direct calculation, using standard inequalities, gives

$$P(\text{Bin}(L, 1 - \delta) \leq L(1 - \eta)) = P(\text{Bin}(L, \delta) \geq \eta L) \leq \left(\frac{E^L}{\eta L}\right)^{\eta L} \leq (\delta e/\eta)^{\eta L} = (\delta e/\eta)^{\eta/k}.$$  

We require this to be $o(1/|G|)$. Here we separate the regimes $k \ll \log n$ and $k \gg \log n$.

Consider first $k \gg \log n$; necessarily, $k - d \approx k$. In this case, we do not quantify $\delta$; we simply know that $\delta = o(1)$. By choosing $\eta$ and $\chi$ to vanish sufficiently slowly (compared with $\delta$) gives $(\delta e/\eta)^{\eta/k} = o(1)$. Raising this to the power $k - d \approx k \gg \log n$, we obtain super-polynomial decay.

Consider now $k \ll \log n$. We use the quantification of $\delta = e^{-2c(k - d)}$ from Lemma 5.3. Choosing $\eta$ to satisfy $\eta \geq e^{-c(k - d)^{k - d + 1}}$, we deduce that $(\delta e/\eta)^{\eta/k} \approx \exp(-c\eta\chi(k - d)^2)$. Choosing $\eta$ vanishing sufficiently slowly so that $\eta(k - d)^2 \gg \log n$ gives

$$P(Q_B(x, D) \leq 1 - \eta) = P(\text{Bin}(\chi(k - d), \delta) \geq \eta \chi(k - d)) \leq (\delta e/\eta)^{\eta(k - d)} = o(1/|G|).$$

This bound is independent of $x$, and hence holds for all $x \in G$ uniformly, as required.

It remains to show that the above conditions are satisfied under Hypothesis D.

**Lemma 5.7.** Suppose that Hypothesis D holds. Then we can choose some $\chi = o(1)$ satisfying

$$\chi(k - d)k^{-2} \log n \ll 1 \quad \text{and} \quad \chi(k - d)^2 \gg \log n.$$

In fact, Hypothesis D are equivalent to being able to pick such a $\chi$.

**Proof.** We defer the proof of this auxiliary lemma to §5.4.

These lemmas combine immediately to establish the upper bound in Theorem 5.1.
5.4 Auxiliary Lemmas

It remains to give the deferred proofs of the auxiliary lemmas.

Proof of Lemma 5.3. By Hypothesis D, we have \( k \gtrsim (\log n)^{2/3} \) and \( k - d \gtrsim (\log n)^{1/2} \); hence
\[
-\log(\delta_\infty|\Gamma|) \approx k - d \gtrsim -\log(\delta_\infty|\mathcal{H}|) \quad \text{and} \quad \log|\Gamma| \leq \log|\mathcal{H}| \lesssim \frac{1}{k} \log n \log \log n \ll k - d.
\]
Also, \( -\log \delta_\infty \approx k - d \). By Proposition 3.13, this means that the TV distance conditioned on typicality is at most \( e^{-3c(k-d)} \) for some constant \( c > 0 \), as desired. It remains to check that typicality holds with sufficiently high probability, i.e. with probability at least \( 1 - e^{-3c(k-d)} \) for some constant \( c > 0 \). Then following Proof of Upper Bound in Theorem 3.6 then gives the quantification.

Quantifying the \( o(1) \)-error in Lemma 3.9 using the above relations, we see that global typicality fails with probability at most \( e^{-3c(k-d)} \) for some constant \( c > 0 \). Lastly, we used [27, Definitions C.1 and C.2 and Proposition C.3] to say that local typicality failed with probability \( o(1) \); this can be quantified using [27, Propositions C.5 and C.6]. Replace \( r_* \) in the definition of typicality by \( r_* \log n \).

The claim follows by Proposition 3.2a for \( 1 \ll k \lesssim \log n \). On the other hand, if \( k \gg \log n \), then \( t_*(k,G) \approx T(k,n) := \log n / (\log(k) / \log n) \);

see §1.3.1. Hence \( T(k,n) \approx T(\alpha k,n) \) for all \( \alpha \in (0,\infty) \). Thus \( T(k,n) \approx T(k',n) \) as \( k \approx k' \).

Proof of Lemma 5.7. Rearranging the conditions, they are equivalent to
\[
\sqrt{\log n} / \chi \ll k - d \ll k^2 / (\chi \log n) \quad \text{for some} \quad 1 / (k - d) \leq \chi \ll 1.
\]
placing \( \chi \) with \( \chi \omega \) for some \( \omega \gg 1 \) diverging sufficiently slowly, replacing the relation \( \chi \geq 1 / (k - d) \) with \( \chi \gg 1 / (k - d) \) gives an equivalent set of conditions. Let \( \varepsilon \in (0,\infty) \) and set
\[
\chi := \frac{\varepsilon k^2}{(k - d) \log n}; \quad \text{then} \quad \sqrt{\frac{\log n}{\chi}} = \frac{\sqrt{k - d} \log n}{\sqrt{\varepsilon k}}.
\]
The conditions on \( \chi \) then, in terms of \( \varepsilon \), become
\[
\frac{(\log n)^2}{(k - d)k^2} \ll \varepsilon \ll 1 \quad \text{and} \quad \frac{\log n}{k^2} \ll \varepsilon \ll \frac{(k - d) \log n}{k^2}.
\]
We can find such an \( \varepsilon \in (0,\infty) \), implicitly a sequence, if and only if
\[
\max\left\{ \frac{(\log n)^2}{(k - d)k^2}, \frac{\log n}{k^2} \right\} \ll \min\left\{ 1, \frac{(k - d) \log n}{k^2} \right\}.
\]
Some case analysis shows that this condition is equivalent to the first condition of Hypothesis D.

6 Nilpotent Groups: Mixing Comparison and Expansion

In this section we compare the mixing time of a general nilpotent group \( G \) with a ‘corresponding’ Abelian group \( \overline{G} \): we show that \( t_{\text{mix}}(G_k) / t_{\text{mix}}(\overline{G}_k) \leq 1 + o(1) \) whp. We apply this to upper bound the \( 1/n^2 \)-mixing times, for some constant \( c > 0 \), for \( G_k \): we show that it is order \( \log n \), from which we deduce that the graph is an expander, whp.
6.1 Precise Statements

We compare the mixing time for $G$ with that for $\overline{G}$. Specifically, we prove Theorem D, which we recall here for the reader’s convenience.

**Theorem 6.1.** Let $G$ be a nilpotent group. Set $\overline{G} := \oplus_1^\ell \left( (G_{\ell-1}/G_\ell) \right)$ where $(G_\ell)_{\ell \geq 0}$ is the lower central series of $G$ and $L := \min\{\ell \geq 0 \mid G_\ell = \{\text{id}\}\}$. Suppose that $1 \ll \log k \ll \log |G|$ and $k - d(\overline{G}) \gg 1$. Let $\varepsilon > 0$ and let $t \geq (1 + \varepsilon)t_*(k, \overline{G})$. Then $d_{\overline{G}}(t) = o(1)$ whp.

**Remark.** An upper bound valid for all groups has already been established in the regime $k \gg \log |G|$ at $T(k, |G|) \approx t_*(k, \overline{G})$; recall Remark A.5. Thus we need only consider $1 \ll k \lesssim \log |G|$. $\triangle$

We use this mixing time bound to show that $G_k$ for nilpotent $G$ is an expander whp when $k - d(\overline{G}) \gtrsim \log |G|$. The isoperimetric constant was defined in Definition E for $d$-regular graphs:

$$\Phi_\ast := \min_{1 \leq |S| \leq \frac{1}{2}|V|} \Phi(S) \quad \text{where} \quad \Phi(S) := \frac{1}{|V|} \left| \{(a, b) \in E \mid a \in S, b \in S'\} \right|.$$

Specifically, we prove Theorem E, which we recall here for the reader’s convenience.

**Theorem 6.2.** Let $G$ be a nilpotent group. Set $\overline{G} := \oplus_1^\ell \left( (G_{\ell-1}/G_\ell) \right)$ where $(G_\ell)_{\ell \geq 0}$ is the lower central series of $G$ and $L := \min\{\ell \geq 0 \mid G_\ell = \{\text{id}\}\}$. Suppose that $k - d(\overline{G}) \gtrsim \log |G|$. Then $\Phi_\ast(G_k) \approx 1$ whp.

The isoperimetric constant is defined more generally for Markov chains; see [34, §7.2]. The isoperimetric constant for a graph is that of the nearest-neighbour RW on the graph. For a given stochastic matrix, it is easy to see that the original chain, its time-reversal and its additive symmetrisation all have the same isoperimetric profile. The additive symmetrisation of the transition matrix of the RW on a directed Cayley graph is precisely the transition matrix of the RW on the corresponding undirected Cayley graph. Thus the isoperimetric constant for a directed Cayley graph is the same as that for the undirected version. It thus suffices to work with undirected graphs.

6.2 Outline of Proof

Let $L$ be the minimal integer such that $G_L$ is the trivial group. Consider the series of quotients $(Q_\ell := G_{\ell-1}/G_\ell)_{\ell=1}^L$. For each $\ell \in [L]$, choose a set $R_\ell \subseteq G_{\ell-1}$ of representatives for $Q_\ell = G_{\ell-1}/G_\ell$.

In order to sample $Z_\ell \sim \text{Unif}(G)$ it suffices to sample $Z_{i,\ell} \sim \text{Unif}(R_\ell)$ for each $\ell$ independently and then take the product: $Z_i := Z_{i,1} \cdots Z_{i,L}$; see Lemma 6.3. Then $Z_{i,\ell}G_\ell \sim \text{Unif}(Q_\ell)$ independently for each $i$ and $\ell$; see Corollary 6.4.

Suppose that $M$ steps are taken; let $\sigma : [M] \to [k]$ indicate which generator is used in each step. Let $S := \prod_{m=1}^M Z_{\sigma(m)}$. For each $\ell \in [L]$, let $S_\ell := \prod_{m=1}^M Z_{\sigma(m),\ell}$; this is the projection of $S$ to $Q_\ell$. Then each $S_\ell G_\ell \equiv \text{RW on } Q_\ell$, which is an Abelian group, but all using the choice $\sigma$.

Since these are RWs on Abelian groups, the ordering in $\sigma$ will not matter. For each $i \in [k]$, let $W_i$ be the number of times in $\sigma$ that generator $Z_i$ has been applied minus the number of times that $Z_i^{-1}$ has been applied. Let $\sigma'$ be an independent copy of $\sigma$ and define $S'$ and $W'$ via $\sigma'$ and $Z$; for each $\ell \in [L]$, define $S'_\ell := \prod_{m=1}^M Z_{\sigma'(m),\ell}$. Then $S$ and $S'$ are iid conditional on $Z$.

To compare the RW on the nilpotent group with one on an Abelian group, we show that

$$n \mathbb{P}(S = S' \mid (W, W')) \leq n \prod_{\ell=1}^L \mathbb{P}(S_\ell G_\ell = S'_\ell G_\ell \mid (W, W')) = \left| \overline{G}/g\overline{G} \right|,$$

where $g := \gcd(W_1 - W'_1, \ldots, W_k - W'_k, n)$; see Proposition 6.6 and Corollary 6.9. Via analysing $\left| \overline{G}/g\overline{G} \right|$, we showed in §2–§4 that the RW on $G_k$ is mixed whp shortly after $t_*(k, \overline{G})$; see specifically Lemma 2.11. From this and the inequality above, we are able to deduce that the RW on $G_k$ is mixed whp shortly after the same time.

6.3 Reduction to Abelian-Type Calculations

Let $L$ be the minimal integer such that $G_L$ is the trivial group. Consider the series of quotients $(Q_\ell := G_{\ell-1}/G_\ell)_{\ell=1}^L$. For each $\ell \in [L]$, choose a set $R_\ell \subseteq G_{\ell-1}$ of representatives for $Q_\ell = G_{\ell-1}/G_\ell$.  

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is a set \( R_\ell \) with \( |R_\ell| = |Q_\ell| \) and \( \{ rG_\ell \}_{r \in R_\ell} = G_{\ell-1}/G_\ell = Q_\ell. \)

We want to sample the uniform generators by using uniform random variables on each of the quotients. In this way, projecting to one of the quotients, we get a RW on this quotient. The following two proofs are deferred to [27, Lemma F.5 and Corollary F.6], respectively.

**Lemma 6.3.** For each \( \ell \in [L] \), let \( Y_\ell \sim \operatorname{Unif}(R_\ell) \) independently. Then \( Y := Y_1 \cdots Y_L \sim \operatorname{Unif}(G) \).

**Corollary 6.4.** For each \((i, \ell) \in [k] \times [L]\), sample \( Z_{i, \ell} \sim \operatorname{Unif}(R_\ell) \) independently and set \( Z_i := Z_{i,1} \cdots Z_{i,L} \). Then \( Z_1, \ldots, Z_L \sim \operatorname{Ind} \operatorname{Unif}(G) \). Further, \( Z_i G_\ell \sim \operatorname{Unif}(Q_\ell) \) independently for each \((i, \ell) \).

For the remainder of the section, assume that \( Z \) is drawn in this way. The next main result (Proposition 6.6) is the key element of the proof of Theorem 6.1. Informally, it reduces the problem to a collection of Abelian calculations, the like of which were handled when we established cutoff when the underlying group was Abelian. We first need a preliminary ‘worst-case’ lemma.

As is standard, we write 0 for the identity of an Abelian group.

**Lemma 6.5.** Let \( H \) be an Abelian group. Let \( Z_1, \ldots, Z_k \sim \operatorname{Ind} \operatorname{Unif}(H) \). Let \( v \in Z_k \). Then
\[
\max_{h \in H} \Pr(v \cdot z = h) = \Pr(v \cdot z = 0).
\]

**Proof.** Let \( h \in H \). Write \( A(h) := \{ z \in H^k \mid v \cdot z = h \} \). If \( w \in A(h) \), then \( B := \{ z - w \mid z \in A(h) \} \subseteq A(0) \); also, clearly, \( |B| = |A(h)| \), so \( |A(h)| \leq |A(0)| \). Hence
\[
\Pr(v \cdot z = h) = |A(h)|/|H|^k \leq |A(0)|/|H|^k = \Pr(v \cdot z = 0).
\]

We now prove the decomposition theorem. It crucially uses the nilpotency of the group.

**Proposition 6.6.** Let \( M, M' \in \text{N} \). Let \( \sigma : [M] \to [k] \) and \( \sigma' : [M'] \to [k] \). Let \( \eta \in \{ \pm 1 \}^M \) and \( \eta' \in \{ \pm 1 \}^{M'} \). For \( \ell \in [L] \), set
\[
S_\ell := \prod_{m=1}^M Z_{\sigma(m), \ell}^{\eta_m}, \quad S'_\ell := \prod_{m=1}^{M'} Z_{\sigma'(m), \ell}^{\eta'_m}, \quad S := \prod_{m=1}^M Z_{\sigma(m)}^{\eta_m} \quad \text{and} \quad S' := \prod_{m=1}^{M'} Z_{\sigma'(m)}^{\eta'_m}.
\]

For \( i \in [k] \), write \( v_i := \sum_{m \in [M]: \sigma'(m) = i} \eta'_m - \sum_{m \in [M]: \sigma(m) = i} \eta_m \). Then
\[
\Pr(S = S') \leq \prod_{\ell=1}^L \Pr(S_\ell G_\ell = S'_\ell G_\ell) = \prod_{\ell=1}^L \Pr(\sum_{i=1}^k v_i Z_{i, \ell} G_\ell = \text{id}(Q_\ell)).
\]

**Proof.** The claimed equality follows immediately from the fact that \( Q_\ell \) is Abelian.

We now set up a little notation. Write \( A_{i, \ell} := Z_{i,1} \cdots Z_{i,\ell-1} \) and \( B_{i, \ell} := Z_{i,\ell+1} \cdots Z_{i,L} \); then \( Z_1 = A_1 \cdots Z_{i,1} B_{i,1} \) (here, \( A_{i,1} := \text{id} \) and \( B_{i,1} := \text{id} \)). Note that \( B_{j, \ell} \in G_\ell \) for all \( j \in [k] \) and \( \ell \in [L] \).

Let \( \mathcal{E}_\ell := \{ S'S^{-1} \in G_\ell \} \). Then
\[
\Pr(S = S') = \prod_{\ell=1}^L \Pr(\mathcal{E}_\ell | \mathcal{E}_{\ell-1}).
\]

For all \( g \in G \) and \( h \in G_{\ell-1} \), we have \([g, h] \in G_\ell \) and \( hg = gh[h^{-1}, g^{-1}] = gh[g, h]^{-1} \). We can hence write \( S'S^{-1} \) in the following way:
\[
S'S^{-1} = M_\ell N_\ell \cdot \left( \prod_{m=1}^M B_{\sigma(m), \ell}^{\eta_m} C_{\sigma'(m), \ell}^{\eta'_m} \right) \cdot \left( \prod_{m=1}^{M'} B_{\sigma(m+1-M), \ell}^{-\eta_m} C_{\sigma(m+1-M), \ell}^{\eta'_m} \right)
\]
for some \( C_{j, \ell}, C'_j \in G_\ell \) and \( M_\ell \) and \( N_\ell \) defined as follows:
\[
M_\ell := \left( \prod_{m=1}^M A_{\sigma(m), \ell}^{\eta_m} \right) \cdot \left( \prod_{m=1}^{M'} A_{\sigma(m+1-M), \ell}^{-\eta_m} \right)
\]
\[
N_\ell := \left( \prod_{m=1}^M Z_{\sigma(m), \ell}^{\eta_m} \right) \cdot \left( \prod_{m=1}^{M'} Z_{\sigma(m+1-M), \ell}^{-\eta_m} \right) \in G_{\ell-1}.
\]

We thus see that \( \mathcal{E}_{\ell-1} = \{ S'S^{-1} \in G_{\ell-1} \} \) holds if and only if \( \{ M_\ell \in G_{\ell-1} \} \) holds. Crucially, this implies that the indicator \( 1(\mathcal{E}_{\ell-1}) \) of this event is independent of \( N_\ell \).

We claim the following:

given that \( S'S^{-1} \in G_{\ell-1} \), we have \( S'S^{-1} \in G_\ell \) if and only if \( M_\ell N_\ell \in G_\ell \).

To prove this, first make the following observations, recalling that \( G_{\ell-1}/G_\ell \) is Abelian:
• for all \( \alpha \in G_{\ell-1} \), we have \( \alpha G_{\ell} = G_{\ell} \) and \( (\alpha \beta)G_{\ell} = (\alpha G_{\ell})(\beta G_{\ell}) \) for all \( \beta \in G \);
• \( B_{j,\ell}, C_{j,\ell} G_{\ell} \in G_{\ell} \) for all \( j \in [k] \) and \( N_\ell \in G_{\ell-1} \);
• \( S'S^{-1} \in G_{\ell-1} \) if and only if \( M_\ell N_\ell \in G_{\ell-1} \), and so \( M_\ell N_\ell \in G_{\ell-1} \).

Assume that \( S'S^{-1} \in G_{\ell-1} \). Applying these observations in the above formula above gives

\[
S'S^{-1}G_{\ell} = (M_\ell N_\ell G_{\ell}) \cdot (\prod_{m=1}^{M'} (B_{\sigma(m),\ell} G_{\ell}) (C_{\sigma'(m),\ell} G_{\ell})) = (M_\ell N_\ell G_{\ell}).
\]

Thus \( S'S^{-1} \in G_{\ell-1} \) if and only if \( M_\ell N_\ell \in G_{\ell-1} \), as claimed.

Now, \( M_\ell \) is independent of \( N_\ell \) and so \( N_\ell \) is independent also of \( 1(\mathcal{E}_{\ell-1}) \). Thus

\[
\mathbb{P}(\mathcal{E}_\ell | \mathcal{E}_{\ell-1}) = \mathbb{P}(M_\ell N_\ell \in G_{\ell} | \mathcal{E}_{\ell-1}) \leq \max_{x \in G_{\ell-1}} \mathbb{P}(x N_\ell \in G_{\ell}).
\]

Now, \( G_{\ell-1}/G_{\ell} \) is Abelian and \( N_\ell \) is a product of generators \( Z_{j,\ell} \) for different \( j \in [k] \).

Hence we are in the set-up of Lemma 6.5. Applying said lemma, we deduce that

\[
\mathbb{P}(\mathcal{E}_\ell | \mathcal{E}_{\ell-1}) \leq \mathbb{P}(N_\ell \in G_{\ell}) = \mathbb{P}(S_{\ell} G_{\ell} = S_\ell' G_{\ell}),
\]

using the definition of \( N_\ell \). This proves the desired inequality. \( \square \)

6.4 Evaluation of Abelian-Type Calculations

When establishing cutoff for RWs on Abelian groups, we had to bound a very similar expression to those in the product of Proposition 6.6. In particular, since the \( Q_\ell \) are Abelian groups, it does not matter in which order the generators are applied. So instead of considering the exact sequence \( \sigma : [M] \to [k] \), it suffices to consider \( W \) where \( W_i := \sum_{m=1}^{M} (\sigma(m) = i) \) for each \( i \in [k] \).

Key in analysing these Abelian-type terms are gcds: for all \( w, w' \in \mathbb{Z}^k \), define

\[
g_{(w, w')} := \gcd(w_1 - w'_1, w_2 - w'_2, \ldots, w_k - w'_k, |G|).
\]

We use this to evaluate the right-hand side of Proposition 6.6, culminating in Corollary 6.9.

Lemma 6.7. Let \( \ell \in [L] \). For all \( w, w' \in \mathbb{Z}^k \), we have

\[
\sum_{i=1}^{k} v_i Z_{\ell, i} G_{\ell} \sim \text{Unif}(g_{(w, w')} Q_{\ell}).
\]

Proof. Corollary 6.4 says that each \( Z_{\ell, i} G_{\ell} \) is an independent \( \text{Unif}(Q_{\ell}) \). [27, Lemma F.1] in the supplementary material says that linear combinations of independent random variables in an Abelian group are also uniform, but the subgroup given by the gcd of the coefficients.

This leads us to a bound on \( \mathbb{P}_{(w, w')} (S = S') \) in terms of a product of \( |Q_{\ell}|/|\gamma Q_{\ell}| \) over \( \ell \in [L] \), for some \( \gamma \) which is a suitable gcd. The following lemma controls this product.

Lemma 6.8. For all \( \gamma \in \mathbb{N} \), we have \( \prod_{\ell=1}^{L} |\gamma Q_{\ell}| = |\gamma G| \).

Proof. For any Abelian groups \( A \) and \( B \) and any \( \gamma \in \mathbb{N} \), we have \( \gamma (A \oplus B) = (\gamma A) \oplus (\gamma B) \) and \( |A \oplus B| = |A||B| \). Since \( G \) was defined to be a direct sum of the \( Q_{\ell} \), the claim now follows. \( \square \)

Let \( (S', W') \) be an independent copy of \( (S, W) \). Combining Proposition 6.6 and Lemmas 6.7 and 6.8 gives the following corollary. For \( w, w' \in \mathbb{Z}^k \), write \( \mathbb{P}_{(w, w')}(\cdot) := \mathbb{P}(\cdot | (W, W') = (w, w')) \).

Corollary 6.9. For all \( w, w' \in \mathbb{Z}^k \), we have

\[
n \mathbb{P}_{(w, w')} (S = S') \leq \prod_{\ell=1}^{L} |\gamma Q_{\ell}|/|g_{(w, w')} G_{\ell}| = |\gamma G|/|g_{(w, w')} G_{\ell}| = |\gamma G|/|g_{(w, w')} G_{\ell}|.
\]

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Proof. Note that $|Q_v|$ divides $|G|$, and so $\gcd(v_1, \ldots, v_k, |Q_v|) \leq \gcd(v_1, \ldots, v_k, |G|)$ for all $v \in \mathbb{Z}^k$. Also, for any Abelian subgroup $H$ of $G$, if $\alpha \not| H$ and $\alpha \not| \beta$, then $\alpha H \not= \beta H$. Combined with Proposition 6.6 and Lemma 6.7, this proves the inequality. The first equality follows immediately from Lemma 6.8. The second equality follows from Lagrange’s theorem.

Observe that the right-hand side of this corollary depends only on the Abelian group $\bar{G}$. By applying the results used for Abelian groups, we can prove Theorem 6.1; we explain this now. Here, as there, we use a modified $L_2$ calculation; see Lemma 2.6.

Lemma 6.10 (Lemma 2.6). For all $t \geq 0$ and all $W \subseteq \mathbb{Z}^k$, the following inequalities hold:

\[ d_{G_k}(t) = \|P_{G_k}(S(t) \in \cdot) - \pi_G\|_{TV} \leq \|P_{G_k}(S(t) \in \cdot | W(t) \in W) - \pi_G\|_{TV} + \mathbb{P}(W(t) \not\in W); \]

\[ 4E(\|P_{G_k}(S(t) \in \cdot | W(t) \in W) - \pi_G\|_{TV}^2) \leq n\mathbb{P}(S(t) = S'(t) | W(t), W'(t) \in W) - 1. \]

Proof of Theorem 6.1. Let $W \subseteq \mathbb{Z}^k$ be arbitrary for the moment. Set

\[ D := n\mathbb{P}(S = S' | \text{typ}) - 1 \quad \text{where} \quad \text{typ} := \{W, W' \in W\}. \]

Abbreviate $g := g_{(W, W')}$. Applying now Corollary 6.9, we obtain

\[ D \leq \sum_{\gamma \in \mathbb{N}} \mathbb{P}(g = \gamma | \text{typ}) \cdot |\bar{G}/\gamma\bar{G}| - 1. \]

This latter expression is purely a statistics of the Abelian group $\bar{G}$, we established the upper bound on mixing by looking at precisely this quantity. Bounding it was one of the main challenges. There were three different arguments for bounding it, corresponding to different regimes of $k$. We briefly outline these arguments now. The choice of $W$ varies from argument to argument.

- In §2.7 we upper bounded $|\bar{G}/\gamma\bar{G}| \leq \gamma d(\bar{G})$; we then used unimodularity to show that $\mathbb{P}(\gamma \not| W_i | W_i \not= 0) \leq 1/\gamma$, and convert this into $\mathbb{P}(g = \gamma | \text{typ}) \leq (1/\gamma + \mathbb{P}(W_i = 0 | \text{typ}))^k$.
- In §3.7 we analysed $(W, W')$ taken modulo $\gamma$, for each $\gamma$; we then used entropic considerations to bound $\mathbb{P}(g = \gamma | \text{typ}) \ll |\bar{G}/\gamma\bar{G}|$ in a quantitative sense.
- In §4.3 we combined these two approaches. Instead of reconstructing these arguments, we reference the appropriate places in the previous sections. For each approach, there are conditions on $(k, \bar{G})$; see Hypotheses A to C. At least one of these is satisfied if $1 \ll k \lesssim \log |G|$ and $k - d(\bar{G}) \gg 1$; see Remarks 2.5, 3.7 and 4.2.

We need to choose the set $W$; see Definitions 2.7 and 3.8 for the respective definitions. (In those definitions, replace $G$ with $\bar{G}$.) See Propositions 2.9, 3.13 and 4.6 specifically for the results bounding this sum. The conclusion of these results is that

\[ D \leq \sum_{\gamma \in \mathbb{N}} \mathbb{P}(g = \gamma | \text{typ}) \cdot |\bar{G}/\gamma\bar{G}| - 1 = o(1). \]

Combined with the modified $L_2$ calculation of Lemma 6.10 this completes the proof.

6.5 Cutoff for Nilpotent Groups with Small Commutators

In this subsection we prove Corollaries D.1 and D.3. Namely, we establish cutoff for the RW on a nilpotent group with small commutator—this includes high-dimensional Heisenberg groups.

Proof of Corollary D.1. The lower bound is a relatively straightforward projection argument. We outline it here; more details are given in [24, §3.3]. Projection can only decrease the TV distance. Thus if the RW projected to the Abelianisation is not mixed then neither is the original RW. The Abelianisation $G^{\text{ab}}$ is Abelian, so a lower bound of $t_\ast(k, G^{\text{ab}})$ holds by Theorem A.

We now argue that it suffices to establish an upper bound of $t_\ast(k, G)$ and $\bar{G}$, instead of $G^{\text{ab}}$. The conditions of Corollary D.1 are precisely those of [27, Proposition B.30] with $A := G^{\text{ab}} \oplus [G, G]$. Said proposition implies that $t_\ast(k, G) \approx t_\ast(k, G^{\text{ab}})$. Hence an upper bound of $t_\ast(k, G)$ indeed suffices. We now establish this upper bound, analysing $k \gg \log |G|$ and $k \lesssim \log |G|$ separately.

Consider first $k \gg \log |G|$. Here it is known that $T(k, |G|) = \log |G|/\log(k/\log |G|) \approx t_\ast(k, G)$ gives an upper bound, regardless of the underlying group; see §1.3.1, [20] and [27, Proposition B.19].

Consider now $k \lesssim \log |G|$. The upper bound of $t_\ast(k, G)$ is immediate from Theorem D.
Recall that $t_m(k, Z_m^n)$ is the time at which the entropy of RW on $Z_m^n$ reaches $\log(m^r) = \log |Z_m^n|$. 

**Proof of Corollary D.2.** The lower bound argument is exactly the same as for Corollary D.1.

For the upper bound, we slightly refine the argument used to prove Theorem D. First, we claim that $G^{ab} \cong Z_p^\ell$. Indeed, the Frattini subgroup $\Phi(G)$ satisfies $\Phi(G) = [G, G]G^p$ when $G$ is a $p$-group where $G^p := \langle g^p \mid g \in G \rangle$. By definition of being special, $\Phi(G) = [G, G]$, thus $G^p \leq [G, G]$. In particular, the Abelianisation is of exponent $p$, as required. Thus $\mathcal{T} \cong Z_p^\ell$ as $G^{com} \cong Z_p^\ell$.

To prove that general upper bound of Theorem D, we cited three approaches used in establishing the upper bound for general Abelian groups in Theorem A, in §2.7, §3.7 and §4.3. Here, though, $\mathcal{H} \cong Z_p^\ell$, not simply a general Abelian group. Thus we do not need the full generality of these cited approaches. We instead apply the approach of [25, Theorem B]; here we study $Z_p^\ell$ in detail, using exactly the same modified $L_2$ method, but now specialised to this group. The conditions for this approach are only $k \geq \ell$—rather than $k - r \gg s$ (ie $k - \ell \gg s$, as $s \ll r$) previously.

We turn to the entropic time. We have $\gamma Z_p^\ell = Z_p^\ell$ unless $p \not| \gamma$. Thus the worst-case $\gamma$ in $t_s(k, Z_p^\ell) = \max_{\gamma | m} t_s(k, Z_p^\ell)$ is $\gamma = p$. Thus $t_s(k, Z_p^\ell) = t_p(k, Z_p^\ell)$.

Regarding group generation, it is standard that to generate a nilpotent group it suffices to be the upper bound for general Abelian groups in Theorem 2.1] by counting step-2 groups whose Frattini group is equal to the centre and of index $p$. This allows us to evaluate the upper bound for general Abelian groups in Theorem 2.1] by counting step-2 groups whose Frattini group is equal to the centre and of index $p$. This allows us to evaluate

The lower bound argument is exactly the same as for Corollary 2.5, so $t_m(k, Z_m^n) = t_m(k, Z_m^n)$ under these conditions.

Finally, after Corollary D.2 we mentioned that special groups are ubiquitous amongst $p$-groups of a given size. We elaborate on this claim in the following remark.

**Remark 6.11.** In his classical work [29], Higman gave upper and lower bounds on the number groups of size $p^r$ for a prime $p$. The upper bound was later refined by Sims [44]. Together they show that this number is $p^{2(r+1)/3} + O(\sqrt{r})$. The lower bound $p^{2(r+1)/3} + O(\sqrt{r})$ is obtained from Higman [29, Theorem 2.1] by counting step-2 groups whose Frattini group is equal to the centre and is elementary Abelian of size $p^2$ and of index $p$, where $r = \ell - s$. It is classical that such a group is special if and only if it has exponent $p$, ie every element other than the identity has order $p$.

Higman [29] showed that the number of such groups of size $p^r$ is between $p^{(1/2)sr(r+1)-r^2-s^2}$ and $p^{(1/2)sr(r+1)-s(r-1)}$ if $s \leq \frac{1}{2}r(r+1)$ and 0 otherwise. A small variant of his argument shows that the number of special groups of size $p^{r+s}$ whose commutator subgroup is of size $p^s$ is between $p^{(1/2)sr(r+1)-r^2-s^2}$ and $p^{(1/2)sr(r+1)-s(r-1)}$ for $s \leq \frac{1}{2}r(r+1)$.

6.6 Expander Graphs of Nilpotent Groups with $k \geq \log |G|$ 

We analyse the spectral gap via considering the $1/n$-mixing time for some $c > 0$.

**Proposition 6.12.** Let $G$ be a nilpotent group. Suppose that $k - d(G) \gg k \gg \log |G|$. Let $\epsilon > 0$ and set $t := (1+\epsilon)t^-_m(k, G)$. Then there exists a constant $c > 0$ so that $d_{G_k}(t) \leq |G|^{-c}$ whp.

**Proof.** Consider first Abelian $G$; here, $G = Z_m^n$. Since Hypothesis D is satisfied, $d_{G_k}(t) \leq e^{-c'k(1-d(G))}$ whp for some constant $c' > 0$ by Lemma 5.3. The claim now follows as $k - d(G) \gg \log |G|$ here.
Consider now nilpotent $G$; here, $G \neq \overline{G}$. We apply our nilpotent-to-Abelian method. There we upper bounded the modified $L_2$ distance for the RW on $G$ (at time $t$) by the modified $L_2$ distance for the RW on $\overline{G}$ (at time $t$); see specifically Proposition 6.6, Lemma 6.7 and Corollary 6.9. For Abelian groups we used the modified $L_2$ calculation (in §2–§4). Thus the nilpotent case is an immediate application of the nilpotent-to-Abelian method and Abelian case.

We apply Proposition 6.12 along with standard results relating the mixing time, spectral gap and the isoperimetric constant. From these, the expansion estimate Theorem 6.2 follows easily.

**Proof of Theorem 6.2.** As noted in Remark E, it suffices to consider $k \approx \log n$.

First, use the well-known discrete analogue of Cheeger’s inequality, discovered independently by multiple authors: for a discrete-time, finite, reversible Markov chain, writing $\gamma$ for its spectral gap,

$$\frac{1}{2}\gamma \leq \Phi_* \leq \sqrt{2\gamma};$$

see, eg, [34, Theorem 13.10]. The spectral gap for the discrete-time chain is up to constants the same as that of the associated (rate-1) continuous time chain. Thus to show that $\Phi_* \approx 1$ it suffices to show that the spectral gap $\gamma \approx 1$ for either the discrete- or continuous-time chain.

Next, use a standard relation between the mixing time and spectral gap: for a continuous-time reversible Markov chain on a state space of size $n$ with uniform invariant distribution, writing $t_{\text{mix}}(\cdot)$ for its mixing time $t_{\text{mix}}(\cdot)$ and $\gamma$ for its spectral gap,

$$t_{\text{mix}}(1/n^c) \approx \gamma^{-1}\log n \quad \text{for any constant } c > 0;$$

see, eg, [34, Theorem 20.6 and Lemma 20.11]. Thus to show that $\gamma \approx 1$ it suffices to show that $t_{\text{mix}}(1/n^c) \lesssim \log n$ for some constant $c > 0$. This then implies that $\Phi_* \approx 1$.

The mixing claim follows immediately from Proposition 6.12 and the fact that $k \approx \log |G|$. 

7 Concluding Remarks and Open Questions

§7.1 We discuss some statistics in the regime where $k$ is a fixed constant.

§7.2 We give a very short proof, which is a small variant on Roichman’s argument [43, Theorem 2], establishing an upper bound on mixing, for arbitrary groups and any $k \gg \log |G|$.

§7.3 We briefly discuss cutoff in other metrics, namely $L_2$ and relative entropy.

§7.4 To conclude, we discuss some questions which remain open and gives some conjectures. Throughout this section, we only sketch details.

7.1 Lack of Cutoff When $k$ Is Constant

Throughout the paper we have always been assuming that $k \to \infty$ as $|G| \to \infty$. It is natural to ask what happens when $k$ does not diverge. This case has actually already been covered by Diaconis and Saloff-Coste [16], using their concept of *moderate growth*. There is no cutoff.

Diaconis and Saloff-Coste establish this not only for Abelian groups, but for nilpotent groups. Recall that a group $G$ is called nilpotent of step at most $L$ if its lower central series terminates in the trivial group after at most $L$ steps: $G_0 = G$ and $G_{\ell} = [G_{\ell-1}, G]$ for $\ell \in \mathbb{N}$ with $G_L = \{\text{id}\}$.

For a Cayley graph $G(Z)$, use the following notation. Write $\Delta := \text{diam } G(Z)$ for its diameter. For the lazy simple random walk on $G(Z)$, write $t_{\text{rel}} := t_{\text{rel}}(G(Z))$ for the relaxation time (ie the inverse of the spectral gap) and $t_{\text{mix}} := t_{\text{mix}}(\varepsilon; G(Z))$ for the (TV) $\varepsilon$-mixing time, for $\varepsilon \in (0, 1)$. When considering sequences $(G_N, (Z(N))_{N \in \mathbb{N}}$, add an $N$-sub/superscript.

We phrase the result of Diaconis and Saloff-Coste [16] in our language.

**Theorem 7.1** (cf [16, Corollary 5.3]). Let $(G_N, N \in \mathbb{N})$ be a sequence of finite, nilpotent groups. For each $N \in \mathbb{N}$, let $Z(N)$ be a symmetric generating set for $G_N$ and write $L_N$ for the step of $G_N$. Suppose that $\sup_N |Z(N)|/N < \infty$ and $\sup_N L_N \to \infty$. Then $t_{\text{mix}}^N / k_N \lesssim \Delta^2_N \lesssim t_{\text{rel}}^N \lesssim t_{\text{mix}}^N$ as $N \to \infty$; in particular, $(t_{\text{mix}}^N)_{N \in \mathbb{N}}$ does not exhibit the cutoff phenomenon.
We give a very brief exposition of the results of Diaconis and Saloff-Coste [16], including the definition of moderate growth, leading to this conclusion in [25, §4].

7.2 A Variant on Roichman’s Argument

In this subsection we give a very short argument upper bounding the mixing time for arbitrary groups and \( k \gg \log |G| \); it is a small modification of Roichman’s argument [43, Theorem 2], but it applies in both the undirected and directed cases. (Roichman [43, Theorem 1] deals with the directed case, but requires additional matrix algebra machinery.)

The proof proceeds as follows. Assume that \( k \gg \log |G| \) and \( \log k \ll \log |G| \); let \( \varepsilon > 0 \) and let \( t := (1 + \varepsilon) \log |G| / \log(k / \log |G|) \). Note that \( 1 \ll k \ll t \). Choose some \( \omega \gg 1 \), diverging arbitrarily slowly; set \( t_\pm := \lfloor t(1 \pm \omega / \sqrt{7}) \rfloor \) and \( L := \omega \lfloor t^2 / k \rfloor \). Whp the number of generators picked at most once is at least \( t - L \); whp of these the number picked exactly once lies in \([t_-, t_+]\). Take typ to be the event that these two conditions hold for two independent copies, \( W \) and \( W' \). We use a modified \( L_2 \) calculation (see, eg, Lemma 2.6) meaning that we need to control

\[
|G| \mathbb{P}(S = S' \mid W = W', \text{typ}) < 1.
\]

Let \( \mathcal{E} \) be the event that some generator is used once in \( W \) and not at all in \( W' \) or vice versa, ie

\[
\mathcal{E} := \bigcup_{i \in |G|} \bigl( |W_i| = 1, |W'_i| = 0 \bigr) \cup \bigl( |W'_i| = 1, |W_i| = 0 \bigr).
\]

Then \( S' \cdot S^{-1} \sim \text{Unif}(G) \) on \( \mathcal{E} \). Indeed, if \( Z \sim \text{Unif}(G) \) and \( X, Y \in G \) are independent of \( Z \), then \( XZY \sim \text{Unif}(G) \); here \( Z \) corresponds to one of the generators used once in \( W \) and not in \( W' \) or vice versa, with the obvious choice of \( X \) and \( Y \) so that \( XZY = S' \cdot S^{-1} \). Off \( \mathcal{E} \), every generator picked once in \( W \) must be picked at least once in \( W' \) and vice versa. There are at most \( L \) generators which are picked more than once in \( W' \). Thus

\[
\mathbb{P}(\mathcal{E} \mid \text{typ}) \leq \min_{a \in [t_-, t_+], b \leq L} 1/(a-b)^{1/2 - L}.
\]

An application of Stirling’s approximation shows that this probability is \( o(1/|G|) \) when \( \omega \) diverges sufficiently slowly. Combined with the modified \( L_2 \) calculation, this proves the upper bound.

Finally, consider the case \( k = |G|^{\alpha} \) for some fixed \( \alpha \in (0, 1) \). The discrete-time chain cannot be mixed at time \( [1/\alpha] - 1 \) by considering the size of its support, but noting that \( \binom{n}{k} \gg |G| \) for \( t := [1/\alpha] + 1 \), by the above argument we see that the walk is mixed whp after \( t \) steps.

Dou proves a more general statement than this which allows the generators to be picked from a distribution other than the uniform distribution; see [19, Theorems 3.3.1 and 3.4.7].

7.3 Mixing in Different Metrics

One can also consider cutoff in the \( L_2 \) distance. For time \( t \geq 0 \), define

\[
d^{(2)}_{\pi_G}(t) := \| \mathbb{P}_{\pi_G}(S(t) \in \cdot) - \pi_G \|_{2, \pi_G} := (|G|^{-1} \sum_{g \in G} (|G| \mathbb{P}_{\pi_G}(S(t) = g) - 1)^2)^{1/2}.
\]

One can then define mixing and cutoff with respect to \( L_2 \) analogously to \( TV \) \( (L_1) \) distance.

**Definition.** Let \( \gamma \in \mathbb{Z} \cup \{\infty\} \). Let \( \tilde{t}_\gamma(k, G) \) be the time \( t \) at which the return probability for SRW on \( \mathbb{Z}^d_k \) at time \( 2t \) is \( |G|/\gamma|G|^{-1} \). Equivalently, \( \tilde{t}_\gamma(k, G) := ks \) where \( s \) is the unique solution to \( \mathbb{P}(X_{2s} = 0) = |G|/\gamma|G|^{-1/k} \) where \( (X_s)_{s \geq 0} \) is a rate-1 SRW on \( \mathbb{Z}^d_k \). Set \( \tilde{t}_\gamma(k, G) := \max_{\gamma \in \mathbb{N}} \tilde{t}_\gamma(k, G) \).

For reasons explained below, we strongly believe that the following is true—and can be proved in the framework which we have developed in this article.

Let \( G \) be an Abelian group and suppose that \( 1 < k \leq \log |G| \). Suppose that \( k - d(G) \gg 1 \). Then, whp, the RW on \( G^d_k \) exhibits cutoff in the \( L_2 \) metric at time \( \tilde{t}_\gamma(k, G) \).
Lastly, the combination of the two approaches works when $|H|$ has lower central series terminating at the trivial group and this sequence is of bounded length. In fact, we conjecture this whenever $G|H|$. Approach #2, we replace $r_*$ as we have low effect on the proof, in essence because $(\log n)^d = n^{o(1)}$. In Approach #2, we replace $|H|$ by $|H|\log |G|$, but still $k \gg \log |G|$ implies that $k \gg \log(|H|\log |G|)$. Lastly, the combination of the two approaches works when $\sqrt{\log |G|/\log \log |G|} \ll k \ll \sqrt{\log |G|}$.

Using somewhat similar adaptations, we believe that cutoff in the relative entropy (abbreviated $RE$) distance can be established. In this case, we quantify the probability with which global typicality holds: the maximal relative entropy of a measure on $G$ with respect to $\pi_G$ is $\log |G|$; thus, naively at least, to condition on global typicality we desire it to hold with probability $1 - o(1/\log |G|)$—for $L_2$ we had $1 - o(1/|G|)$. Also, one should modify local typicality as previously. This gives conditions on $k$ and $d(G)$. Under such conditions, the $RE$ and TV cutoff times should then be the same.

We believe that with more effort these conditions can be improved via obtaining some estimates on the relative entropy given that global typicality fails.

### 7.4 Open Questions and Conjectures

We close the paper with some questions which are left open.

1: Does the Product Condition Imply Cutoff?

The problem of singling out abstract conditions under which the cutoff phenomenon occurs has drawn considerable attention. For a reversible Markov chain $X$, write $t_{\text{mix}}(X)$ for its mixing time and $\gamma_{\text{gap}}(X)$ for its spectral gap. In 2004, Peres [41] proposed a simple spectral criterion for a sequence $(X^N)_{N \in \mathbb{N}}$ of reversible Markov chains, known as the product condition:

$$\text{cutoff} \quad \text{equivalent to} \quad t_{\text{mix}}(X^N)\gamma_{\text{gap}}(X^N) \to \infty \text{ as } N \to \infty.$$  

It is well-known that the product condition is a necessary condition for cutoff; see, eg, [34, Proposition 18.4]. It is relatively easy to artificially create counter-examples, but these are not ‘natural’; see, eg, [34, §18] where constructions due to Aldous and due to Pak are described. The product condition is widely believed to be sufficient for “most” chains.

We conjecture that the product condition implies cutoff for random Cayley graph of Abelian groups. In fact, we conjecture this whenever $G$ is nilpotent of bounded step (denoted step$G$), ie has lower central series terminating at the trivial group and this sequence is of bounded length.

**Conjecture 1.** Let $(G_N)_{N \in \mathbb{N}}$ be a sequence of finite, nilpotent group and $(Z_N)_{N \in \mathbb{N}}$ a sequence of subsets with $Z_N \subseteq G_N$ for all $N \in \mathbb{N}$. For each $N \in \mathbb{N}$, write $t_{\text{mix}}^N$, respectively $\gamma_{\text{gap}}^N$, for the mixing time, respectively spectral gap, of the SRW on $G_N(Z_N)$.

Suppose that $\limsup_{N \to \infty} \text{step} G_N < \infty$ and that the product condition holds, ie $t_{\text{mix}}^N\gamma_{\text{gap}}^N \to \infty$ as $N \to \infty$. Then the sequence of SRWs exhibits cutoff.

An equivalence between the product condition and cutoff has been established for birth-and-death chains by Ding, Lubetzky and Peres [18] and, more generally, for RWs on trees by Basu, Herman and Peres [4]. It is believed to imply cutoff for the SRW on transitive expanders of bounded degree, but this is known only in the case of Ramanujan graphs, due to Lubetzky and Peres [36].

2: An Explicit Choice of Generators

We have shown that if one chooses the generators $Z$ uniformly, then one obtains cutoff whp, at a time which does not depend on $Z$. In particular, this means that there is cutoff for almost all choices of generators at a time independent of the choice of generators. This ‘almost universal’ mixing time is given by $t_*(k, G)$ from Definition 3.1. A question raised to us by Diaconis [15] is to find explicit sets of generators for which cutoff occurs; see also [13, Chapter 4G, Question 2].
Open Problem 2. Let $G$ be an Abelian group and $1 \ll k \lesssim \log |G|$. Find an explicit choice of generators $Z$ so that the RW on $G(Z)$ exhibits cutoff. Further, find generators so that the cutoff time is $t_*(k, G)$.

Hough [32, Theorem 1.11] shows for the cyclic group $\mathbb{Z}_p$ with $p$ prime that the choice $Z := \{0, \pm 1, \pm 2, \ldots, \pm 2^{\lceil \log_2 p \rceil - 1}\}$, which he describes as “an approximate embedding of the classical hypercube walk into the cycle”, gives rise to a random walk on $\mathbb{Z}_p$ which has cutoff. The cutoff time is not the entropic time, however. Although the entropic time is the mixing time for ‘most’ choice of generators, finding an explicit choice of generators which gives rise to cutoff at the entropic time is still open—even for the cyclic group of prime order.

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