Research Article

Numerical Treating of Mixed Integral Equation Two-Dimensional in Surface Cracks in Finite Layers of Materials

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The goal of this paper is study the mixed integral equation with singular kernel in two-dimensional adding to the time in the Volterra integral term numerically. We established the problem from the plane strain problem for the bounded layer medium composed of different materials that contains a crack on one of the interface. Also, the existence of a unique solution of the equation proved. Therefore, a numerical method is used to translate our problem to a system of two-dimensional Fredholm integral equations (STDFIEs). Then, Toeplitz matrix (TMM) and the Nystrom product methods (NPM) are used to solve the STDFIEs with Cauchy kernel. Numerical examples are presented, and their results are compared with the analytical solution to demonstrate the validity and applicability of the methods. The codes were written in Maple.

1. Introduction

Many problems of engineering, mathematical physical, and contact problems in the theory of elasticity lead to singular integral equations. Integral equations provide an important tool for solving the ordinary and partial differential equations. Therefore, many different methods are used to obtain the solution of the linear and nonlinear integral equations. Brunner and Kauthen [1] introduced collocation and iterated collocation methods for solving the two-dimensional Volterra integral equation (T-DVIE). In [2], authors proposed a class of explicit Runge-Kutta-type methods of order 3 for solving nonlinear T-DVIE. In [3], authors studied the approximate solution of T-DVIEs by the two-dimensional differential transform method. Abdou, in [4, 5], used different methods to obtain the solution of F-VIE of the first and second kinds in which the Fredholm integral term is considered in position while the Volterra integral term is considered in time. EL-Borai et al., in [6], studied the numerical solution for the T-DFIE with weak singular kernel, but they have studied the problem on a rectangular path of the parties only. AL-Bugami in [7] studied and discussed the solution of the T-DFIE with applications in contact problems. AL-Bugami in [8] studied and discussed the solution of the two-dimensional singular Fredholm integral equation (T-DFIE) with time. The solution of a large of mixed boundary value problem of a great variety of contact and crack problems in solid mechanics, physical, and engineering can be related to a system of the singular IEs have a simple Cauchy-type singularity (Ref. [9]). In [10], the authors studied the linear two-dimensional Volterra integral equation with continuous kernel numerically. In [11], the authors discussed continuous Fredholm-Volterra integral equation and solved numerically. In [12], the author studied the surface cracks of finite layers of fractional materials.

In this work, we consider a mixed integral equation with singular kernel in two-dimensional (MIE)

\[
\eta w(x, t) + \frac{\lambda}{\pi} \int_{-1}^{1} \int_{-1}^{1} p_{1}(x - u)p_{2}(y - v)|w(u, v; t)|\, du\, dv \\
+ \lambda \int_{-1}^{1} k(x, y)w(y, t)\, dy + \int_{0}^{\infty} \xi(t, \tau)w(x, \tau)\, d\tau = f(x, t).
\]

Formula (1) is called the MIE with singular kernel in
two-dimensional of the second kind with Cauchy kernel in
\((L_2(-1, 1) \times L_2(-1, 1)) \times C(0; Y) \); \(Y < 1\), where the FI term
is considered in position with singular kernel, and the VI term
is considered in time with a positive and continuous
kernel \(f(t, \tau)\). \(f(x, t)\) is known function, while \(w(x, t)\) is
unknown function to be determined. The numerical coe-
cfficient \(\lambda\) is called the parameter of the IE.

2. The Basic Formulas of the Problem

Consider the plane strain problem for the bounded layer
medium (Figure 1), composed of three di-

rematerial, the half length of the

is considered in time with a positive and continuous
kernel \(f(t, \tau)\). \(f(x, t)\) is known function, while \(w(x, t)\) is
unknown function to be determined. The numerical coe-
cfficient \(\lambda\) is called the parameter of the IE.

Figure 1: The layers of materials.

\[ F(t) = \text{known function of } t. \] Hence, using (7) and (8) in
(5) and (6), we get

\[ (\lambda_i + 2\mu_i) \frac{\partial^2 U_i}{\partial x^2} + \mu_i \frac{\partial^2 U_i}{\partial y^2} + (\lambda_i + \mu_i) \frac{\partial^2 V_i}{\partial x \partial y} = 0, \]

\[ (\lambda_i + 2\mu_i) \frac{\partial^2 V_i}{\partial y^2} + \mu_i \frac{\partial^2 V_i}{\partial x^2} + (\lambda_i + \mu_i) \frac{\partial^2 U_i}{\partial x \partial y} = 0, \]

\[ \frac{d^2 F(t)}{dt^2} = \frac{\mu_i}{\rho} F(t). \]

Formula (11) has a solution

\[ F(t) = B e^{-\sqrt{2} \lambda t}, F(\infty) \rightarrow 0. \]

For solving the two formulas (9) and (10), we use the
Fourier integral transform:

\[ U_i(x, y) = \frac{2}{\pi} \int_0^\infty \psi_i(\alpha, y) \sin \alpha x \alpha d\alpha, (i = 1, 2, 3), \]

\[ V_i(x, y) = \frac{2}{\pi} \int_0^\infty \psi_i(\alpha, y) \cos \alpha x \alpha d\alpha. \]

Then, we have

\[ -\alpha^2 (\lambda_i + 2\mu_i) \psi_i + \mu_i \frac{d^2 \psi_i}{dy^2} + \alpha (\lambda_i + \mu_i) \frac{d\psi_i}{dy} = 0, \]

\[ -\alpha^2 (\lambda_i + 2\mu_i) \psi_i + \mu_i \frac{d^2 \psi_i}{dy^2} + \alpha (\lambda_i + \mu_i) \frac{d\psi_i}{dy} = 0. \]

After solving the system of Eqs. (13) and (14), and then
using the two formulas (13) and (14), we get

\[ U_i(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ A_{ii} + A_{i\gamma} e^{-\gamma} + (A_{ii} + A_{i\gamma}) e^{\gamma} \right\} \sin \alpha x \alpha d\alpha, \]

\[ V_i(x, y) = \frac{2}{\pi} \int_0^\infty \left\{ A_{ii} + \left( \frac{K_i}{\mu} + \gamma \right) A_{ii} e^{-\gamma} + \left( -A_{ii} \right) e^{\gamma} \right\} \cos \alpha x \alpha d\alpha, \]

where \( K_i \) has physical meaning and \( K_i = 3 - 4\nu_1 \) for
plane strain and \( K_i = (3 - 3\nu_1)/(1 + \nu_1) \) for
generalized plane stress, \( \nu_i \) is Poisson’s coefficient for each material, and \( A_{ij} \)
\( \frac{d^2 V_i}{\partial y^2} + \mu_i \frac{\partial^2 V_i}{\partial x^2} + (\lambda_i + \mu_i) \frac{\partial^2 U_i}{\partial x \partial y} = 0, \]

\[ d^2 F(t) \]

\[ \frac{d^2 F(t)}{dt^2} = \frac{\mu_i}{\rho} F(t). \]

\[ \frac{d^2 F(t)}{dt^2} = \frac{\mu_i}{\rho} F(t). \]

\[ F(t) = B e^{-\sqrt{2} \lambda t}, F(\infty) \rightarrow 0. \]
The continuity requires that on the interfaces, the stress and displacement vectors in the adjacent layers be equal, i.e.,

\[ u_{i+1} - u_i = 0, \quad v_{i+1} - v_i = 0, \quad \sigma_{ij}^{i+1} - \sigma_{ij}^i = 0, \quad \sigma_{ij}^{i+2} - \sigma_{ij}^{i+1} = 0, \quad i = 1, 2, 3. \]  

Now, to obtain the integral equation, we first assume that at \( y = 0 \), the bond between the two adjacent layers is perfect except for the dislocations at \( y = 0 \) and \( x = y \) defied by

\[ \frac{\partial}{\partial x}(u_x^+ - u_x^-) = f_1(x, t), \quad \frac{\partial}{\partial x}(v_y^+ - v_y^-) = f_2(x, t), \]  

where the superscripts + and − refer to the limiting values of the displacement as \( y \) approaches zero from + and − sides, respectively. In addition to (21), on the interface \( y = 0 \), we have the following conditions

\[ \sigma_{xx}^2 - \sigma_{yy}^2 = 0, \quad \sigma_{xy}^2 - \sigma_{yy}^3 = 0, \quad (0 \leq x < \infty, y = 0). \]  

The components of the stress vector at \( y = 0 \) and \( x > 0 \)

| \( T \) | \( N \) | \( x \) | Exact sol. | Approx. \( T \) | Error. \( T \) | Approx. \( N \) | Error. \( N \) |
|---|---|---|---|---|---|---|---|
| -1.00 | 10 | 0.000400000 | 0.00045299 | 0.0000029901 | 0.00445921 | 0.00045921 |
| -0.60 | -0.20 | 0.00014400 | 0.00014029 | 0.370969 \times 10^{-5} | 0.00208274 | 0.00188474 |
| -0.20 | 0.20 | 0.00016000 | 0.00020222 | 0.422877 \times 10^{-5} | 0.00264946 | 0.00263346 |
| 0.60 | 0.60 | 0.00014400 | 0.00014029 | 0.370969 \times 10^{-5} | 0.00797219 | 0.00782819 |
| 1.00 | 1.00 | 0.00400000 | 0.0045299 | 0.0000529901 | 0.00238274 | 0.00188474 |
| -1.00 | -0.60 | 0.000400000 | 0.00047952 | 0.0000795200 | 0.00443268 | 0.0043268 |
| -0.60 | -0.20 | 0.00014400 | 0.00014029 | 0.370969 \times 10^{-5} | 0.00208274 | 0.00188474 |
| -0.20 | 0.20 | 0.00016000 | 0.00023484 | 0.748577 \times 10^{-5} | 0.00264620 | 0.00263020 |
| 0.60 | 0.60 | 0.00014400 | 0.00014029 | 0.370969 \times 10^{-5} | 0.00797219 | 0.00782819 |
| 1.00 | 1.00 | 0.00400000 | 0.00479520 | 0.0000795200 | 0.00241519 | 0.00155840 |
Fourier transforms of $A_\alpha(\mu)$ may be expressed as

$$
1+K_\mu^2\varepsilon_{\gamma\gamma}(x,0,t) = \lim_{\rho \to 0} \int_0^{\infty} \varepsilon^e \left\{ a_{11}A_1(\alpha,t) + a_{12}A_2(\alpha,t) \right\} \cos \alpha d\alpha + \frac{2}{\pi} \int_0^{\infty} \left\{ H_{11}(\alpha)A_1(\alpha,t) + H_{12}(\alpha)A_2(\alpha,t) \right\} \cos \alpha d\alpha + \int_0^\infty F(\tau)f_1(x,0,\tau) d\tau,
$$

and

$$
-\frac{1+K_\mu^2}{\mu^2}\sigma_{\gamma\gamma}^o(x,0,t) = \lim_{\rho \to 0} \int_0^{\infty} \varepsilon^e \left\{ a_{11}A_1(\alpha,t) + a_{22}A_2(\alpha,t) \right\} \sin \alpha d\alpha + \frac{2}{\pi} \int_0^{\infty} \left\{ H_{21}(\alpha)A_1(\alpha,t) + H_{22}(\alpha)A_2(\alpha,t) \right\} \sin \alpha d\alpha + \int_0^\infty F(\tau)f_2(x,0,\tau) d\tau,
$$

where $H(\alpha)$ is the Heaviside functions, and $A_1$ is the Fourier transforms of $f_1$ defined as follows:

$$
A_1(\alpha,t) = \int_0^\infty f_1(z,t) \cos \alpha dz,
A_2(\alpha,t) = \int_0^\infty f_2(z,t) \sin \alpha dz.
$$

Table 2: The values of exact, approximate, and absolute error values by TMM and PNM for (Plutonium $\nu = 0.21$).

| $T$ | $N$ | $x$ | Exact sol. | Approx. $T$ | Error $T$ | $\lambda_1$ | $\lambda_2$ | Error $\lambda_1$ | Error $\lambda_2$ |
|-----|-----|-----|------------|-------------|-----------|----------|----------|--------------|--------------|
| 10  | 0.02| -1.00| 0.00400000| 0.00443078  | 0.00043078| 0.0046912| 0.00046912| 0.003316366| 0.0002819 | 0.00188419 |
|     |     | -0.60| 0.00014400| 0.00140836  | 0.316366 \times 10^{-5} | 0.0020819 | 0.000481703 |
|     |     | -0.20| 0.00016000| 0.00020081  | 0.408164 \times 10^{-5} | 0.00264960 | 0.000263360 |
|     |     | 0.20  | 0.00016000| 0.00020081  | 0.408164 \times 10^{-5} | 0.00483303 | 0.000481703 |
|     |     | 0.60  | 0.00014400| 0.00140836  | 0.316366 \times 10^{-5} | 0.0079164  | 0.000782764 |
|     |     | 1.00  | 0.00400000| 0.00443078  | 0.00043078 | 0.00247804 | 0.000421964 | 0.00446912 | 0.000481402 |
|     |     | -1.00| 0.00400000| 0.00463333  | 0.00063339 | 0.00448877 | 0.000448877 |
|     |     | -0.60| 0.00014400| 0.00140836  | 0.843735 \times 10^{-6} | 0.0020249 | 0.00188019 |
|     |     | -0.20| 0.00001600| 0.00020081  | 0.7095783 \times 10^{-5} | 0.00264659 | 0.00263059 |
|     |     | 0.20  | 0.00016000| 0.00020081  | 0.7095783 \times 10^{-5} | 0.00483002 | 0.00481402 |
|     |     | 0.60  | 0.00014400| 0.00140836  | 0.843735 \times 10^{-6} | 0.00796764 | 0.00782364 |
|     |     | 1.00  | 0.00400000| 0.00463333  | 0.00063339 | 0.00245778 | 0.00154221 |
| 20  | 0.20| -1.00| 0.04000000| 0.07863996  | 0.038639964| 0.07372776 | 0.03372776 |
|     |     | -0.60| 0.01440000| 0.02389663  | 0.009466379 | 0.02172759 | 0.00732759 |
|     |     | -0.20| 0.00160000| 0.00258811  | 0.00988111 | 0.00008157 | 0.00151842 |
|     |     | 0.20  | 0.00160000| 0.00258811  | 0.00988111 | 0.00226500 | 0.00066500 |
|     |     | 0.60  | 0.01440000| 0.02389663  | 0.009496637 | 0.01578414 | 0.00138414 |
|     |     | 1.00  | 0.04000000| 0.07863996  | 0.038639964 | 0.07517884 | 0.03571884 |
| 20  | 0.20| -1.00| 0.04000000| 0.08232299  | 0.04232299 | 0.07741079 | 0.03741079 |
|     |     | -0.60| 0.01440000| 0.02462094  | 0.010220934 | 0.02245190 | 0.00805190 |
|     |     | -0.20| 0.00160000| 0.00312894  | 0.001528933 | 0.00172417 | 0.00012417 |
|     |     | 0.20  | 0.00160000| 0.00312894  | 0.001528933 | 0.00172417 | 0.00012417 |
|     |     | 0.60  | 0.01440000| 0.02462094  | 0.010220934 | 0.01650845 | 0.00210845 |
|     |     | 1.00  | 0.04000000| 0.08232299  | 0.04232299 | 0.07940187 | 0.03940187 |

The constants $a_{ij}$ depend on the elastic properties of the materials adjacent to the crack only and are given by

$$
a_{11} = -a_{22} = \left( 1 + \lambda_2 \lambda_3 \right) / \lambda_4, \quad a_{12} = -a_{21} = -\left( 1 + 2 \lambda_4 - \lambda_2 \lambda_3 \right) / \lambda_4,
\lambda_2 = (K_2 \mu_3 - K_3 \mu_2) / (\mu_2 + K_3 \mu_3), \quad \lambda_4 = (\mu_3 + \mu_2 K_3) / (\mu_2 + \mu_3),
$$

Figure 2: The value absolute error by TMM at $N = 10$ and $T = 0.02$.
where $\mu_i$ is the shear modulus, and $\lambda$'s is Lame's constants.

Note that once the dislocations $f_i(x)$ on the interface are specified, formulas (23) and (24) give the stresses for all values of $x$. The crack problem under consideration $f_i(x)$ is zero for $|x| > 1$ and is unknown for $|x| < 1$. On the other hand, the stress vector on the interface $y = 0$ is unknown for $|x| > 1$ that is given by the following known functions

for $|x| < 1$, i.e.,

$$f_1(x, t) = f_1(-x, t), f_2(x, t) = -f_2(-x, t).$$

Then,

$$\int_0^\infty H(\alpha) \cos \alpha a dx \int_0^t f_1(z, t) \cos \alpha zdz$$

$$= \frac{1}{2} \int_{-1}^1 \int_{-1}^1 f_1^*(z, t) f_1^*(z, t) dx dz \int_0^\infty H(\alpha) \cos \alpha(z - x) da.$$

Hence, we obtain

$$-\frac{1 + K_4}{\mu_s} p_i(x, t) = \lim_{\eta \to 0} \left[ \frac{a_0}{\pi} \int_{-1}^1 \int_{-1}^1 f_1^*(z, t) f_i^*(z, t) dz dx \int_0^\infty \left( \eta \cos \alpha(z - x) da \right. \right.$$

$$+ \frac{a_0}{\pi} \int_{-1}^1 \int_{-1}^1 f_2^*(z, t) f_i^*(z, t) dz dx \int_0^\infty (\eta \sin \alpha(z - x) da$$

$$+ \frac{1}{\pi} \sum_{p=1}^{\infty} \int_{-1}^1 k_p^*(x, z) f_i^*(z, t) f_i^*(z, t) dz dx$$

$$+ \int_0^1 F(\eta) f_i(x, \eta) d\eta \right].$$

Evaluating the infinite integrals in (30), passing to the
Cauchy theorems, we have

\[
\begin{align*}
-\frac{1 + K_3}{a_{12}} p_1(x, t) &= \eta w_1(x, t) + \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{w_1(u, v; t)}{(x - u)(y - v)} du dv \\
- \frac{1}{a_{12}} \int_{-1}^{1} \int_{-1}^{1} K_1(x, y) w(x, t) dy + \frac{1}{a_{12}} \int_{-1}^{1} F(t) w(x, t) dt, \\
- \frac{1 + K_3}{a_{12}} p_1(x, t) &= \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{a_{11} w_1(u, v; t)}{(x - u)(y - v)} du dv \\
- \eta w_2(x, t) - \frac{1}{a_{12}} \int_{-1}^{1} \int_{-1}^{1} K_2(x, y) w(x, t) dy + \frac{1}{a_{12}} \int_{-1}^{1} F(t) w(x, t) dt, \\
-\frac{1 + K_3}{a_{12}} p_1(x, t) &= \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{a_{11} w_1(u, v; t)}{(x - u)(y - v)} du dv \\
- \left( \frac{\mu_2 + K_3 \mu_2}{a_{12}} \right) \eta w_1(x, t) - \frac{1}{a_{12}} \int_{-1}^{1} \int_{-1}^{1} K_1(x, y) w(x, t) dy + \frac{1}{a_{12}} \int_{-1}^{1} F(t) w(x, t) dt, \\
-\frac{1 + K_3}{a_{12}} p_1(x, t) &= \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{a_{11} w_1(u, v; t)}{(x - u)(y - v)} du dv \\
- \left( \frac{\mu_2 + K_3 \mu_2}{a_{12}} \right) \eta w_2(x, t) - \frac{1}{a_{12}} \int_{-1}^{1} \int_{-1}^{1} K_2(x, y) w(x, t) dy + \frac{1}{a_{12}} \int_{-1}^{1} F(t) w(x, t) dt.
\end{align*}
\]

(31)

The two formulas of (31) represent a system of MIE with Cauchy kernel. For one layer, we can have the following MIE, on noting the difference notations.

\[
\begin{align*}
\eta w(x, t) - \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{w(u, v; t)}{(x - u)(y - v)} du dv - \frac{1}{\pi a_{12}} \int_{-1}^{1} k(x, y) w(y, t) dy \\
+ \frac{1}{a_{12}} \int_{0}^{t} F(t, \tau) w(x, \tau) d\tau = f(x, t).
\end{align*}
\]

(33)

In general, we can write Eq. (33) in the form:

\[
\begin{align*}
\eta w(x, t) + \lambda \int_{-1}^{1} p_1(x - u)|p_2(x)| y - \nu w(u, v; t) du dv \\
+ \lambda \int_{-1}^{1} k(x, y) w(y, t) dy + \lambda \int_{0}^{t} \zeta(t, \tau) w(x, \tau) d\tau = f(x, t),
\end{align*}
\]

where \( p_1(x - u)|p_2(x)| y - \nu = \frac{1}{(x - u)(y - \nu)} \)

(34)

3. The Existence and Uniqueness of the Solution

We write this formula in the integral operator form

\[
\begin{align*}
W w(x, t) &= \frac{1}{\eta} f(x, t) - W w(x, t), (\eta \neq 0), \\
W w &= H w + D w + \zeta w,
\end{align*}
\]

(35)
The value absolute error by TMM at Figure 11: The value absolute error by PNM at N = 10 and T = 0.02.

The value absolute error by TMM at N = 20 and T = 0.02.

The value absolute error by PNM at N = 20 and T = 0.2.

\[
Hw = \frac{\lambda}{n \pi} \int_{-1}^{1} \int_{-1}^{1} p_{1} |x - u| p_{2} |y - v| w(u, v; t) dudv, Dw \\
= \frac{\lambda}{n} \int_{-1}^{1} k(x, y) w(y, t) dy, \xi w = \frac{\lambda}{n} \int_{0}^{t} \zeta(t, \tau) w(x, \tau) d\tau.
\]

We assume the following conditions:

1. The singular kernel of FI term satisfies in \( L_{2}[−1, 1] \times L_{2}[−1, 1] \) the discontinuity condition

\[
\left[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \| p_{1} |x - u| p_{2} |y - v| \| dx dudv \right]^{2} = M, (M \text{ is a constant}).
\]

2. The kernel of VI term \( \zeta(t, \tau) \) is continuous in the Banach space \( C[0, T], 0 \leq \tau \leq t \leq T < \infty \) and satisfies

\[|\zeta(t, \tau) \leq N, \ \forall t, \tau \in [0, T]|.\]

3. The continuous kernel \( |k(x, y)| \leq L \)

\[\| f(x, t) \| = \max_{0 \leq t \leq T} \left[ \int_{-1}^{1} f^{2}(x, \tau) dx \right]^{1/2} \leq R, \quad (R \text{ is a constant}).\]

4. \( w(x, t) \) in the space, \( L_{2}[−1, 1] \times L_{2}[−1, 1] \times C[0, T] \), behaves as the known function \( f(x, t) \)

**Theorem 1.** Eq. (34) has an exact unique solution in \( L_{2}[−1, 1] \times L_{2}[−1, 1] \times C[0, T] \), under the condition

\[|\eta| > |\lambda| \left( \frac{M}{\pi} + \sqrt{2T(L + N)} \right), \quad T = \max_{0 \leq t \leq T} t.\]

**Lemma 1.** The integral operator \( \bar{W} \) maps \( L_{2}[−1, 1] \times L_{2}[−1, 1] \times C[0, T] \) into itself.
From (35) and (40), the normality of the integral operator takes the forms

Applying Cauchy-Schwarz inequality, we have

\[
||Hw|| \leq \left( \frac{\lambda}{\eta} \right) \left( \int_{-1}^{1} \int_{-1}^{1} |p_1^u x - u| |p_2^v y - v| |dudv| \right)^{1/2} \left( \int_{-1}^{1} \int_{-1}^{1} w^2(u, v; t) |dudv| \right)^{1/2}.
\]

Using the definition of the norm in the space \( L_2[-1, 1] \times L_2[-1, 1] \times C[0, T] \), we get

\[
||Hw|| \leq \left( \frac{\lambda}{\eta} \right) \left( \int_{-1}^{1} \int_{-1}^{1} |p_1^u x - u| |p_2^v y - v| |dudv| \right)^{1/2} \max_{0 \leq t \leq T} \int_{-1}^{1} \int_{-1}^{1} w^2(u, v; t) |dudv| \right)^{1/2} \frac{\tau}{\eta} L \cdot \int_{-1}^{1} \int_{-1}^{1} w^2(u, v; t) |dudv| \right)^{1/2} \frac{T}{\eta} \max_{0 \leq t \leq T} \int_{-1}^{1} \int_{-1}^{1} w^2(u, v; t) |dudv| \right)^{1/2}.
\]

Then, using condition (1), we obtain

\[
||Hw|| \leq \left( \frac{\lambda}{\eta} \right) M ||w||.
\]

Also, the term \( Dw \) takes the form

\[
||Dw|| \leq \left( \frac{\lambda}{\eta} \right) \left( \int_{-1}^{1} \int_{-1}^{1} k(x, y) |w(y, t)| |dy| \right).
\]

Using condition (3), we get

\[
||Dw|| \leq \left( \frac{\lambda}{\eta} \right) L \max_{0 \leq t \leq T} \left( \int_{-1}^{1} \int_{-1}^{1} w^2(y, \tau) |dy| \right)^{1/2} \frac{T}{\eta} \max_{0 \leq t \leq T} \int_{-1}^{1} \int_{-1}^{1} w^2(y, \tau) |dy| \right)^{1/2}.
\]

Hence,

\[
||Dw|| \leq \left( \frac{\lambda}{\eta} \right) \sqrt{2L} ||w||.
\]

In the same manner, we can write

\[
||\zeta w|| \leq \left( \frac{\lambda}{\eta} \right) \left( \int_{0}^{T} |\zeta(t, \tau)| w(x, \tau) |d\tau| \right).
\]

Using condition (2), we obtain

\[
||\zeta w|| \leq \left( \frac{\lambda}{\eta} \right) N \max_{0 \leq t \leq T} \left( \int_{0}^{T} \int_{-1}^{1} w^2(y, \tau) |dy| \right)^{1/2} \frac{T}{\eta} \max_{0 \leq t \leq T} \int_{0}^{T} \int_{-1}^{1} w^2(y, \tau) |dy| \right)^{1/2}.
\]

Thus, one has

\[
||\zeta w|| \leq \left( \frac{\lambda}{\eta} \right) \sqrt{2NT} ||w||, \quad T = \max_{0 \leq t \leq T}.
\]

Hence, with the aid of conditions (5), (44), (47), and
The inequality (51) involves the boundedness of the operators \( W \) and \( \tilde{W} \).

**Lemma 2.** The integral operator (35) under the condition (40) is continuous and contraction operator.

**Proof.** For the functions \( w_1(x, t), w_2(x, t) \) in the space \( L^2[-1, 1] \times L^2[-1, 1] \times C[0, T], \) formula (35) yields

\[
\| (\tilde{W} w_1 - \tilde{W} w_2) (x, t) \| = \| W (w_1 - w_2) (x, t) \|. \tag{52}
\]

Hence, we have

\[
\| (\tilde{W} w_1 - \tilde{W} w_2) (x, t) \| \leq a \| w_1 - w_2 \|, \quad a = |\eta^{-1} \lambda| \left( \frac{M}{\pi} + \sqrt{2T} (L + N) \right). \tag{54}
\]

Hence, \( \tilde{W} \) is a continuous operator in the space \( L^2[-1, 1] \times L^2[-1, 1] \times C[0, T], \) and under the condition \( (\alpha < 1), \) \( \tilde{W} \) is a contraction operator.

### 4. THE STDFIEs

Consider Eq. (34). In this section, we divide the interval \([0, T] \), \( 0 \leq t \leq T < \infty, 0 = t_0 < t_1 < \cdots < t_n = T \), where \( t = t_i, i = 0, 1, 2, \ldots, n \), to get

\[
\eta w(x, t_i) + \lambda \int_{-1}^{1} \frac{w(u, v, t_i)}{(x-u)(y-v)} dudv + \lambda \int_{-1}^{1} k(x, y) w(y, t_i) dy \\
+ \lambda \int_{0}^{1} \zeta(t_i, r) w(x, r) dr = f(x, t_i). \tag{55}
\]

Using the quadrature formula, the Volterra term becomes

\[
\int_{0}^{1} \zeta(t_i, r) w(x, r) dr = \sum_{j=0}^{i} \kappa_j \zeta(t_i, t_j) w(x, t_j) + R(x, t_i). \tag{56}
\]

Using formula (53) with the conditions (1), (2), and (3), then applying Cauchy-Schwarz inequality, we obtain

\[
\| W w_1 - W w_2 \| \leq \| w_1 - w_2 \|, \quad a = |\eta^{-1} \lambda| \left( \frac{M}{\pi} + \sqrt{2T} (L + N) \right). \tag{54}
\]

Then, the general form of Eq. (58) can be represented as

\[
\mu_i w_i (x) + \lambda \int_{-1}^{1} p_1 (|x-u|) p_2 (|y-v|) w_i (u, v) dudv \\
+ \lambda \int_{-1}^{1} k(x, y) w_i (y) dy = \psi_i (x), \tag{59}
\]

where

\[
\mu_i = (\eta + \lambda \kappa \zeta_i), \quad \psi_i = f_i (x) - \lambda \sum_{j=0}^{i} \kappa_j \zeta_j w_j (x), \quad i = 0, 1, \ldots, n. \tag{60}
\]

Formula (59) represents a linear system of TDFIEs of the second kind, which contains \((n + 1)\) equation of \((n + 1)\) unknown functions of \(w_i (x)\) corresponding to the time interval \([0, T]\).

### 5. Some Numerical Methods

#### 5.1. The TMM

We present the TMM to obtain numerical solution of TDFIE of the second kind with Cauchy form, which it expresses in the form

\[
\mu_i w_i (x) + \lambda \int_{-1}^{1} p_1 (|x-u|) p_2 (|y-v|) w_i (u, v) dudv \\
+ \lambda \int_{-1}^{1} k(x, y) w_i (y) dy = \psi_i (x), \tag{61}
\]

which it may be adapted as

\[
\mu w (x) = \psi (x) - \lambda \int_{-1}^{1} F(x-u, y-v) w(u, v; t) dudv, \tag{62}
\]

where

\[
\mu = (\eta + \lambda \kappa \zeta), \quad \psi = f (x) - \lambda \sum_{j=0}^{i} \kappa_j \zeta_j w_j (x), \quad i = 0, 1, \ldots, n.
\]
where

\[ F(x - u, y - v) = \frac{(1 + \pi k(x, y)(y - x))}{(x - u)(y - v)} = \left( \frac{1}{(x - u)(y - v)} \right) (1 + \pi k(x, y)(y - x)). \]  

Then, write the integral term in Eq. (62) as the form

\[ \int_{-1}^{1} \int_{-1}^{1} F(x - u, y - v) w(u, v; t) \, du \, dv = \sum_{n = -N}^{N} \sum_{m = -M}^{M} F(x - u, y - v) w(u, v; t) \, du \, dv \left( h = \frac{1}{N} \right). \]

Formula (64) reduces as

\[ \int_{-1}^{1} \int_{-1}^{1} F(x - u, y - v) w(u, v; t) \, du \, dv = \sum_{n = -N}^{N} \sum_{m = -M}^{M} F(x - u, y - v) w(u, v; t) \, du \, dv \left( h = \frac{1}{N} \right). \]

Then,

\[ \int_{nh}^{nh+h} \int_{mh}^{mh+h} F(x - u, y - v) w(u, v; t) \, du \, dv = A_{n,m}(x, y) w(nh, mh) + B_{n,m}(x, y) w(nh + h, mh + h) + R. \]

Then, we put \( w(u, v) = 1.1, uv \) in Eq. (66), and then we obtain

\[ A_{n,m}(x, y) = \frac{1}{h} \left[ \left( \frac{(nh + h)(mh + h)}{(nh + mh + h)} \right) I - \frac{f}{(nh + mh + h)} \right], \]

\[ B_{n,m}(x, y) = \frac{1}{h} \left[ \frac{f}{(nh + mh + h)} - \frac{(nh)(mh)I}{(nh + mh + h)} \right]. \]

where

\[ I(x, y) = \int_{nh}^{nh+h} \int_{mh}^{mh+h} k(t - \tau, |x - y|) \, du \, dv, \]

\[ J(x, y) = \int_{nh}^{nh+h} \int_{mh}^{mh+h} uv \cdot k(|x - u|, |y - v|) \, du \, dv. \]

Eq. (65) becomes

\[ \int_{-1}^{1} \int_{-1}^{1} F(x - u, y - v) w(u, v; t) \, du \, dv = \sum_{n = -N}^{N} \sum_{m = -M}^{M} \left[ A_{n,m}(x, y) w(nh, mh; t) + B_{n,m}(x, y) w(nh + h, mh + h; t) \right] \]

\[ = \sum_{n = -N}^{N} \sum_{m = -M}^{M} A_{n,m}(x, y) w(nh, mh; t) + \sum_{n = -N}^{N} \sum_{m = -M}^{M} B_{n,m}(x, y) w(nh, mh; t) \]

\[ = \sum_{n = -N}^{N} \sum_{m = -M}^{M} D_{n,m}(x, y) w(nh, mh; t). \]

Then, the IE (62) becomes

\[ \mu w(x, y) + \frac{\lambda}{n} \sum_{n = -N}^{N} \sum_{m = -M}^{M} D_{n,m}(x, y) w(nh, mh; t) = f(x, y). \]

If we put \( x = kh, y = lh \), then we get

\[ \mu w_{kl} + \frac{\lambda}{n} \sum_{n = -N}^{N} \sum_{m = -M}^{M} D_{kn,m}(x, y) w_{nm} = f_{kl} \quad -N \leq k \leq N, -M \leq l \leq M, \]

where

\[ D_{kn,m} = \begin{cases} \frac{A_{n,m}(kh, lh)}{n} & n = m = -N, \\ \frac{A_{n,m}(kh, lh)}{n} + \frac{B_{n,m}(kh, lh)}{n} & -N < n = m < N, \\ \frac{B_{n,m}(kh, lh)}{n} & n = m = N. \end{cases} \]

The matrix \( D_{kn,m} \) may be written as \( D_{kn,m} = G_{kn,m} - E_{kn,m} \), where

\[ G_{kn,m} = A_{n,m}(kh, lh), \quad -N \leq k, l, n, \leq N \]

is the TM of order \( 2N + 1 \), and the matrix

\[ E_{kn,m} = \begin{cases} B_{-N-1}(kh, lh) & n = m = -N, \\ 0 & -N < n = m < N, \\ A_{N}(kh, lh) & n = m = N. \end{cases} \]

However, the solution of the system can be obtained in
the form
\[ w_{ij} = \left[ \mu I - \lambda (G_{kn} - E_{kn}) \right]^{-1} f_{kl}, \]
where \( I \) is the identity matrix and \( [\mu \lambda (G_{kn} - E_{kn})] \neq 0 \).

5.2. The PNM. Consider
\[ p(x) = \psi(x) \frac{1}{\pi} \int_{-1}^{1} p(x - u; y ; v) \tilde{k}(x - u; y ; v) \psi(u, v ; t) \, du \, dv, \]
where
\[ p(x - u; y ; v) \tilde{k}(x - u; y ; v) = \left( \frac{1}{\pi} \frac{\pi}{\pi} \right) (1 + nk(x,y)(y-x)), \]
where \( n \) and \( \tilde{k} \) are badly behaved and well-behaved functions of their arguments, respectively. We approximate the integral term in (77) when \( \{ \psi(x, y) \} \) by
\[ \int_{-1}^{1} \int_{-1}^{1} p(x - u; y ; v) \tilde{k}(x - u; y ; v) \psi(u, v ; t) \, du \, dv = \sum_{j=0}^{N} \sum_{j=0}^{N} \kappa_{ij} \tilde{k}(x - u; y ; v) \psi(u, v ; t), \]
where \( \kappa_{ij} \) is the weights. Also, we approximate the integral term in (77) in the form:
\[ \int_{-1}^{1} \int_{-1}^{1} p(x - u; y ; v) \tilde{k}(x - u; y ; v) \psi(u, v ; t) \, du \, dv = \sum_{j=0}^{N} \sum_{j=0}^{N} \mu_{ij} \int_{-1}^{1} p(x - u; y ; v) \tilde{k}(x - u; y ; v) \psi(u, v ; t) \, du \, dv, \]
where \( x_i = u_i = y_i = 1 + ih \), \( i = 0, 1, \ldots, N \) with \( h = b - a/N \) and \( N \) even. Now, if we approximate the nonsingular part of the integrand over each interval \( [u_{j-1}, u_{j+1}], ] \), \( [v_{j-1}, v_{j+1}], \) by the second degree Lagrange interpolation polynomial that interpolates, we find
\[ \int_{-1}^{1} \int_{-1}^{1} p(x - u; y ; v) \tilde{k}(x - u; y ; v) \psi(u, v ; t) \, du \, dv = \sum_{j=0}^{N} \sum_{j=0}^{N} \mu_{ij} \int_{-1}^{1} p(x - u; y ; v) \tilde{k}(x - u; y ; v) \psi(u, v ; t) \, du \, dv, \]
where \( u_j = jh, u_{j+1} = (j+1)h, u_j - u_{j+1} = v_1 - v_{j+1} = -h. \)

If we define
\[ a_{ij}(u, v) = \frac{1}{4h^2} \sum_{j=1}^{n} \int_{u_{j-1}}^{u_{j+1}} p(u, v; t) (u - u_{j-1}) (v - v_{j+1}) \, du \, dv, \]
\[ b_{ij}(u, v) = \frac{1}{4h^2} \sum_{j=1}^{n} \int_{u_{j-1}}^{u_{j+1}} p(u, v; t) (u - u_{j-1}) (v - v_{j+1}) \, du \, dv, \]
\[ y_{ij}(u, v) = \frac{1}{4h^2} \sum_{j=1}^{n} \int_{u_{j-1}}^{u_{j+1}} p(u, v; t) (u - u_{j-1}) (v - v_{j+1}) \, du \, dv. \]

In general, assume \( K_0 \) thus (82) become
\[ a_{ij}(u, v) = \frac{h^2}{4} \int_{0}^{\pi} \int_{0}^{\pi} \xi (\xi - 1) \rho(u, v; t) \, d\xi \, d\eta, \]
\[ b_{ij}(u, v) = \frac{h^2}{4} \int_{0}^{\pi} \int_{0}^{\pi} (\xi - 1) (\xi - 2) \rho(u, v; t) \, d\xi \, d\eta, \]
\[ y_{ij}(u, v) = \frac{h^2}{4} \int_{0}^{\pi} \int_{0}^{\pi} (\xi - 2) \rho(u, v; t) \, d\xi \, d\eta. \]

If we define \( \psi_k = \int_{0}^{\pi} \int_{0}^{\pi} \xi^2 \sigma_x^2 \rho(u, v; t) \, d\xi \, d\eta, \) the absolute error of \( \xi = 1, 0, 1, 2 + 2h, \) \( v_i - v_{i-2} = (i - 2 + 2h), \) \( v_i = \int_{0}^{\pi} \int_{0}^{\pi} \xi (\xi - \xi) \rho(u, v; t) \, d\xi \, d\eta, \)
\[ \psi_k - \int_{0}^{\pi} \int_{0}^{\pi} \xi^2 \sigma_x^2 \rho(u, v; t) \, d\xi \, d\eta, \]
\[ \psi_k - \int_{0}^{\pi} \int_{0}^{\pi} \xi^2 \sigma_x^2 \rho(u, v; t) \, d\xi \, d\eta. \]

6. Numerical Applications and Discussions

In this section, we state some applications and numerical results to discuss the approximate solution (i.e., the treat of this method) to the problems that occurs in the materials as a result, the constant in finite of materials, which the deformation increases as well as the time increases in the interval \([0, T]\). The TMM and PNM are used to get numerical solution for values of \( \mu = 1, \) and for different materials, plutonium \( v = 0.21 \) and fiber \( v = 0.22, \) where the Poisson ratio is \( 0 \leq v < 1, \lambda = \lambda = 2G, \) \( \lambda = 2G(1 - 2v), \) (G shear modulus). We divided the position interval by \( N = 10, 20, 20 \) units. Since \( 0 \leq t < T < \infty, \) we choose the time \( T = 0.02, 0.2, \) Tables 1 and 2 are as follows: exact sol. \( \rightarrow \) the exact solution, Approx. T. \( \rightarrow \) approximate solution of TMM, error T. \( \rightarrow \) the absolute error of TMM, Approx. N. \( \rightarrow \) approximate solution of PNM, and error. N. \( \rightarrow \) the absolute error of PNM. Figures 2–17 show the value absolute error by TMM and PNM at \( N = 10, 20 \) and the time \( T = 0.02, 0.2, \) for the materials plutonium \( v = 0.21 \) and fiber \( v = 0.22. \)
Consider

\[ \mu w(x, t) = f(x, t) - \frac{\lambda}{\mu} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{(x-u)(y-v)} w(u, v; t) du dv \]
\[ - \lambda \int_{-1}^{1} |x-y| w(y, t) dy - \int_{0}^{t} t w(x, \tau) d\tau, (\mu = 1). \]

The exact solution \( \phi(x, y) = x^2 y^2 \).

7. The Conclusion

We have presented a successful technique for the numerical solution of MIE with singular kernel in two-dimensional by using TMM and PNM which is established from the plane strain problem for the bounded layer medium composed of different materials. From Tables 1 and 2 and Figures 2–17, we note that the errors due to the TMM are less than the errors due to PNM. In addition, we note that \( N \) increases for the two different materials (fiber \( v = 0.22 \)) (plutonium \( v = 0.21 \)), the values of \( k(\{g(x) - g(y)\}) = \cot (g(y) - g(x)/2) \) and \( x \in [-\pi, \pi], \phi(x, \pi, t) = 0. \) are fixed, and the error values increase. The approximate solution is nearly coincident with the exact solution for \( t > 0 \) at each value of \( x \in [-1, 1] \).

Data Availability

All the data are available within the article and also as the references.

Conflicts of Interest

The author declares that he/she has no conflicts of interest.

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