Multivariate Alexander quandles, IV. The medial quandle of a link

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Abstract
Joyce observed that the Alexander invariant and the medial quandle of a classical knot are equivalent to each other, as invariants. In the present paper, we discuss the rather complicated extension of Joyce’s observation to several different medial quandles and reduced (one-variable) Alexander modules associated with classical links. The theme is that for links, medial quandles provide stronger invariants than reduced Alexander modules.

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1 Introduction
If \( L = K_1 \cup \cdots \cup K_\mu \) is a classical, oriented link of \( \mu \) components in \( S^3 \), then its (multivariate) Alexander module \( M_A(L) \) is a module over the ring \( \Lambda_\mu = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}] \) of Laurent polynomials in \( \mu \) variables, with integer coefficients. The Alexander module is often described by a presentation with generators and relations corresponding to arcs and crossings of a diagram of \( L \); see Sec. 2 for details.

A useful part of the Alexander module theory is the Crowell map, a module epimorphism \( \phi_L : M_A(L) \to I_\mu \) introduced in [1]. Here \( I_\mu \) is the augmentation ideal of \( \Lambda_\mu \), i.e., the kernel of the augmentation map \( \epsilon : \Lambda_\mu \to \mathbb{Z} \) given by \( \epsilon(t_i^{\pm 1}) = 1 \ \forall i \in \{1, \ldots, \mu\} \). If \( D \) is a diagram of \( L \), then an element of \( M_A(L) \) corresponding to an arc of \( K_i \) in \( D \) is mapped to \( t_i - 1 \) by \( \phi_L \). We say that two links \( L, L' \) are Crowell equivalent if there is a module isomorphism \( f : M_A(L) \to M_A(L') \) with \( \phi_L = \phi_{L'} \circ f \). Following Rolfsen [9], we refer to \( \ker \phi_L \) as the Alexander invariant of \( L \). The Alexander invariant corresponds to the first homology group of the universal abelian cover of \( S^3 - L \), while the Alexander module corresponds to the first relative homology group of the covering space with respect to its fiber.

In the first paper of this series, \( \phi_L \) was used to define an operation \( \triangleright \) on \( M_A(L) \). The operation \( \triangleright \) defines quandle structures on various subsets of the
Alexander module. One of these subsets yields \( Q_A(L) \), the fundamental multivariate Alexander quandle of \( L \). In the third paper of the series we completed a proof of the following.

**Theorem 1.** ([17] [18]) As an invariant of classical oriented links, the fundamental multivariate Alexander quandle \( Q_A(L) \) (up to quandle isomorphism) is strictly stronger than \( \phi_L \) (up to Crowell equivalence and permutation of component indices).

Here are three comments about Theorem 1.

(a) The theorem fails if index permutations are not allowed. For instance, let \( L \) be a link whose Alexander polynomial is not symmetric with respect to a permutation of \( \{1, \ldots, \mu\} \), and let \( L' \) be obtained from \( L \) by applying that permutation to the component indices. Then \( Q_A(L) \cong Q_A(L') \), as \( Q_A(L) \) does not explicitly reflect component indices, but \( M_A(L) \not\cong M_A(L') \).

(b) The theorem implies that \( Q_A(L) \) (up to quandle isomorphism) is also strictly stronger than both the Alexander invariant and the Alexander module (up to module isomorphism and permutation of \( \{1, \ldots, \mu\} \)).

(c) The fundamental quandle \( Q(L) \) is a strictly stronger link invariant than \( Q_A(L) \).

The purpose of the present paper is to discuss the reduced (one-variable) version of the theory involved in Theorem 1. Let \( \Lambda = \mathbb{Z}[t^{\pm 1}] \) be the ring of Laurent polynomials in the variable \( t \), with integer coefficients. If \( \tau : \Lambda_{\mu} \to \Lambda \) is the homomorphism of rings with unity given by \( \tau(t_i) = t \forall i \in \{1, \ldots, \mu\} \), then \( \tau \) defines a \( \Lambda_{\mu} \)-module structure on \( \Lambda \), with scalar multiplication given by \( x \cdot y = \tau(x)y \forall x \in \Lambda_{\mu} \forall y \in \Lambda \). The reduced Alexander module of \( L \) is the tensor product \( M_A^\text{red}(L) = M_A(L) \otimes_{\Lambda_{\mu}} \Lambda \), considered as a \( \Lambda \)-module via multiplication in the second factor. The tensor product of \( \phi_L \) with the identity map of \( \Lambda \) is a \( \Lambda \)-linear map \( \phi_\tau : M_A^\text{red}(L) \to I_\mu \otimes_{\Lambda_{\mu}} \Lambda \).

**Definition 2.** Two links \( L, L' \) are \( \phi_\tau \)-equivalent if there is a \( \Lambda_{\mu} \)-module isomorphism \( f : M_A^\text{red}(L) \to M_A^\text{red}(L') \) that is compatible with the \( \phi_\tau \) maps of \( L \) and \( L' \), i.e., \( \phi_\tau = \phi_\tau' \circ f \).

**Definition 3.** If \( L \) is a classical link, let

\[
Q_A^\text{red}(L) = \{ x \otimes 1 \mid x \in Q_A(L) \} \subset M_A^\text{red}(L).
\]

It is not hard to verify that the quandle operation \( \triangleright \) of \( Q_A(L) \) defines a quandle structure on \( Q_A^\text{red}(L) \) in a natural way: \( (x \otimes 1) \triangleright (y \otimes 1) = (x \triangleright y) \otimes 1 \). Also, \( Q_A^\text{red}(L) \) is a subquandle of the standard Alexander quandle on the \( \Lambda \)-module \( M_A^\text{red}(L) \). In fact, \( Q_A^\text{red}(L) \) is an invariant subquandle of \( M_A^\text{red}(L) \), in this sense: if \( L \) and \( L' \) are ambient isotopic oriented links, then there is an isomorphism \( M_A^\text{red}(L) \cong M_A^\text{red}(L') \) that maps \( Q_A^\text{red}(L) \) isomorphically onto \( Q_A^\text{red}(L') \).

In Section 3 we verify the following.
Theorem 4. As an invariant of classical links, $Q^\text{red}_A(L)$ (up to quandle isomorphism) is equivalent to $\phi_\tau$ (up to $\phi_\tau$-equivalence and permutation of component indices).

Modified versions of the earlier comments (a), (b), (c) hold for Theorem 4. The easiest one to state is (a): like Theorem 1, Theorem 4 fails if index permutations are disallowed. See Sec. 4 for details.

In order to state the reduced version of comment (b), it is convenient to let 

\[ \eta : I_\mu \otimes_{\Lambda_\mu} \Lambda \to \Lambda \]

be the map with $\eta((t_i - 1) \otimes 1) = 1 \forall i \in \{1, \ldots, \mu\}$. The composition $\phi^\text{red}_L = \eta \circ \phi_\tau : M^\text{red}_A(L) \to \Lambda$ appears in the reduced version of Crowell’s link module sequence, i.e. the homology sequence of the total linking number cover of $S^3 - L$. We call $\phi^\text{red}_L$ the reduced Alexander invariant of $L$. For more information regarding the properties of reduced link module sequences, we refer to Hillman [3, Sec. 5.4].

We now have three $\Lambda$-modules associated with $L$: the reduced Alexander module $M^\text{red}_A(L)$, the reduced Alexander invariant $\ker \phi^\text{red}_L$, and $\ker \phi_\tau$, for which we do not have a special name. The situation may seem complicated, but it turns out that one module determines the other two.

Proposition 5. The $\Lambda$-module $\ker \phi^\text{red}_L$ determines both $M^\text{red}_A(L)$ and $\ker \phi_\tau$, up to isomorphism: $M^\text{red}_A(L) \cong \ker \phi^\text{red}_L \oplus \Lambda$ and $\ker \phi_\tau = (t - 1) \cdot \ker \phi^\text{red}_L$.

Proof. The epimorphism $\phi^\text{red}_L : M^\text{red}_A(L) \to \Lambda$ must split, so $M^\text{red}_A(L) \cong \ker \phi^\text{red}_L \oplus \Lambda$. The equality $\ker \phi_\tau = (t - 1) \cdot \ker \phi^\text{red}_L$ is not so obvious; see Proposition 9.

Here is the reduced version of comment (b).

Theorem 6. In general, the quandle $Q^\text{red}_A(L)$ (up to quandle isomorphism) is a strictly stronger link invariant than the module $\ker \phi^\text{red}_L$. For knots, though, $Q^\text{red}_A(L)$ and $\ker \phi^\text{red}_L$ are equivalent invariants.

In addition to being the image of $Q_A(L)$ in $M^\text{red}_A(L)$, $Q^\text{red}_A(L)$ is also the image in $M^\text{red}_A(L)$ of a link invariant introduced by Joyce [6], namely, the fundamental medial quandle $MQ(L)$. (Joyce denoted this quandle $\text{AbQ}(L)$ rather than $MQ(L)$, and he called it the “abelian link quandle” of $L$.) Joyce proved that when $\mu = 1$, $MQ(L)$ and $\ker \phi_L$ are equivalent invariants [6, Sec. 17]. This property is reflected in the reduced version of comment (c):

Theorem 7. For knots, $MQ(L)$ and $Q^\text{red}_A(L)$ are isomorphic quandles. In general, though, $MQ(L)$ is a strictly stronger link invariant than $Q^\text{red}_A(L)$.

We should mention that early versions of the present paper, posted on the arxiv, included the incorrect assertion that $Q^\text{red}_A(L)$ and $MQ(L)$ are always isomorphic. We are grateful to Kyle Miller for helping us understand the mistake.

Here is an outline of our discussion. In Sec. 2 we present some basic properties of Alexander modules and Crowell maps. In Sec. 3 we discuss several quandles associated with Alexander modules, and prove Theorem 4. Theorem 6 is proven in Sec. 4.

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In the rest of the paper, we do more than just prove Theorem 7; we try to provide as much insight as we can into the quandles $Q_{\text{red}}(L)$ and $MQ(L)$, and their connections with the $\Lambda$-module $M_{\text{red}}(L)$. Sec. 5 is a brief account of some of the basic theory of general quandles and medial quandles; most of the material is drawn from the work of Jedlička, Pilitowska, Stanovský and Zamojska-Dzienio [4, 5]. In Sec. 6, we discuss Joyce’s description of $MQ(L)$ as a quandle “augmented” by a group [6]; we denote this group $MG(L)$. We also verify two assertions of Theorem 7: $MQ(L)$ determines $Q_{\text{red}}(L)$, and if $\mu = 1$, then $MQ(L) \cong Q_{\text{red}}^A(L)$. In Sec. 7, we complete the proof of Theorem 7 by providing examples distinguished by $MQ(L)$ but not by $Q_{\text{red}}(L)$. In the last two sections of the paper, we show that $Q_{\text{red}}(L)$ is isomorphic to a quandle contained in the group $MG(L)$.

2 Alexander Modules and Crowell Maps

We follow the usual conventions for diagrams of classical links. A diagram $D$ consists of piecewise smooth closed curves in the plane, whose only (self-) intersections are crossings, i.e., transverse double points. The set of crossings in $D$ is denoted $C(D)$. At each crossing, two short segments are removed, to indicate which of the intersecting curves is the underpasser. Removing these short segments cuts the curves into separate parts, the arcs of $D$. The set of arcs is denoted $A(D)$. If $D$ is a diagram of $L = K_1 \cup \cdots \cup K_{\mu}$, then there is a function $\kappa_D : A(D) \to \{1, \ldots, \mu\}$, with $\kappa_D(a) = i$ if $a$ is part of the image of $K_i$ in $D$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{crossing.png}
\caption{A crossing.}
\end{figure}

Let $D$ be a link diagram, and let $\Lambda_{\mu}^{A(D)}$ and $\Lambda_{\mu}^{C(D)}$ be the free $\Lambda_{\mu}$-modules on the sets $A(D)$ and $C(D)$. There is a $\Lambda_{\mu}$-linear map $\rho_D : \Lambda_{\mu}^{C(D)} \to \Lambda_{\mu}^{A(D)}$ given by

$$\rho_D(c) = (1 - t_{\kappa_D(a_2)})a_1 + t_{\kappa_D(a_1)}a_2 - a_3$$

whenever $c \in C(D)$ is a crossing of $D$ as indicated in Fig. [1] and there is an exact sequence

$$\Lambda_{\mu}^{C(D)} \xrightarrow{\rho_D} \Lambda_{\mu}^{A(D)} \xrightarrow{\gamma_D} M_A(L) \rightarrow 0.$$ 

The Crowell map $\phi_L : M_A(L) \to I_{\mu}$ has $\phi_L \gamma_D(a) = t_{\kappa_D(a)} - 1 \forall a \in A(D)$.

The reduced Alexander module of $L$ is $M_{\text{red}}(L) = M_A(L) \otimes_{\Lambda_{\mu}} \Lambda$. It is equivalent to say that $M_{\text{red}}(L)$ is the quotient $M_A(L)/(J \cdot M_A(L))$, where $J$ is the ideal of $\Lambda_{\mu}$ generated by the elements $t_i - t_j$. (An isomorphism between
\(M_A(L)/(J \cdot M_A(L))\) and \(M_A(L) \otimes_{\Lambda} \Lambda\) is given by \(x + J \cdot M_A(L) \leftrightarrow x \otimes 1\). Alternatively, \(M_A^{\text{red}}(L)\) can be obtained by setting all \(t_i\) equal to \(t\) in any description of \(M_A(L)\) (e.g., the description in the preceding paragraph).

Every element of \(M_A^{\text{red}}(L)\) is of the form \(x \otimes 1\) for some \(x \in M_A(L)\), for the following reasons: (a) if \(x \in M_A(L)\) and \(\lambda \in \Lambda_\mu\) then \(x \otimes \tau(\lambda) = (\lambda \cdot x) \otimes 1\), and (b) if \(x_1, x_2 \in M_A(L)\) then \(x_1 \otimes 1 + x_2 \otimes 1 = (x_1 + x_2) \otimes 1\). It follows that formulas involving elements of \(M_A^{\text{red}}(L)\) can be specified using elements of the form \(x \otimes 1\), with \(x \in M_A(L)\). For instance, the map \(\phi_\tau : M_A^{\text{red}}(L) \to I_\mu \otimes_{\Lambda_\mu} \Lambda\) is defined by \(\phi_\tau(x \otimes 1) = \phi_L(x) \otimes 1\) \(\forall x \in M_A(L)\), and if \(D\) is a diagram of \(L\), then \(\phi_\tau(\gamma_D(a) \otimes 1) = (t_{\kappa_D(a)} - 1) \otimes 1\) \(\forall a \in A(D)\).

When thinking about the map \(\phi_\tau\), it is helpful to have in mind an explicit description of the \(\Lambda\)-module \(I_\mu \otimes_{\Lambda_\mu} \Lambda\). Let \(\epsilon : \Lambda \to \mathbb{Z}\) be the augmentation map given by \(\epsilon(t) = 1\), and let \(\mathbb{Z}_\epsilon\) be the \(\Lambda\)-module obtained from \(\mathbb{Z}\) using the scalar multiplication given by \(\lambda \cdot n = \epsilon(\lambda)n\) \(\forall \lambda \in \Lambda\) \(\forall n \in \mathbb{Z}\). Notice that \(\mathbb{Z}_\epsilon \cong \Lambda / (t - 1)\), where \((t - 1) = \ker \epsilon\) is the augmentation ideal of \(\Lambda\).

**Lemma 8.** As a \(\Lambda\)-module, \(I_\mu \otimes_{\Lambda_\mu} \Lambda\) is isomorphic to

\[
\Lambda \oplus \mathbb{Z}_\epsilon \oplus \cdots \oplus \mathbb{Z}_\epsilon
\]

with the \(\Lambda\) summand generated by \((t_1 - 1) \otimes 1\) and the \(\mathbb{Z}_\epsilon\) summands generated by \((t_2 - t_1) \otimes 1, \ldots, (t_\mu - t_1) \otimes 1\).

**Proof.** It is well known that as a \(\Lambda_\mu\)-module, \(I_\mu\) is generated by the elements \(t_1 - 1, \ldots, t_\mu - 1, \) subject to the defining relations \((t_i - 1) \cdot (t_j - 1) = (t_j - 1) \cdot (t_i - 1)\) \(\forall i, j\). (See [3] p. 71, for instance.) It follows that as a \(\Lambda\)-module, \(I_\mu \otimes_{\Lambda_\mu} \Lambda\) is generated by the elements \((t_1 - 1) \otimes 1, \ldots, (t_\mu - 1) \otimes 1\), subject to the defining relations

\[
(t - 1) \cdot ((t_j - 1) \otimes 1) = (t_j - 1) \otimes (t - 1) = (t_j - 1) \otimes \tau(t_i - 1) = ((t_j - 1)(t_i - 1)) \otimes 1
\]

\[
= ((t_i - 1)(t_j - 1)) \otimes 1 = (t_i - 1) \otimes \tau(t_j - 1) = (t_j - 1) \otimes (t - 1) = (t - 1) \cdot ((t_i - 1) \otimes 1)
\]

for all values of \(i\) and \(j\). Equivalently, \(I_\mu \otimes_{\Lambda_\mu} \Lambda\) is generated by \((t_1 - 1) \otimes 1\) and the \(\mu - 1\) elements

\[
(t_2 - 1) \otimes 1 - (t_1 - 1) \otimes 1, \ldots, (t_\mu - 1) \otimes 1 - (t_1 - 1) \otimes 1,
\]

subject to the defining relations \((t - 1) \cdot ((t_i - 1) \otimes 1 - (t_i - 1) \otimes 1) = 0\) \(\forall i \in \{2, \ldots, \mu\}\). As \((t_i - 1) \otimes 1 - (t_i - 1) \otimes 1 = (t_i - t_1) \otimes 1\) \(\forall i \in \{2, \ldots, \mu\}\), the result follows.

The map \(\eta : I_\mu \otimes_{\Lambda_\mu} \Lambda \to \Lambda\) was defined in the introduction by the formula \(\eta((t_i - 1) \otimes 1) = 1\) \(\forall i \in \{1, \ldots, \mu\}\). Taking the isomorphism of Lemma 8 into account, \(\eta : \Lambda \oplus (\mathbb{Z}_\epsilon)^{\mu - 1} \to \Lambda\) is simply the projection onto the first coordinate of the direct sum. Therefore \(\phi^{\text{red}}_L = \eta \circ \phi_\tau : M_A^{\text{red}}(L) \to \Lambda\) is the first coordinate of \(\phi_\tau : M_A^{\text{red}}(L) \to \Lambda \oplus (\mathbb{Z}_\epsilon)^{\mu - 1}\). As \(\phi^{\text{red}}_L(\gamma_D(a) \otimes 1) = 1\) \(\forall a \in A(D)\), it is easy to see that \(\ker \phi^{\text{red}}_L\) is the submodule of \(M_A^{\text{red}}(L)\) generated by the elements \((\gamma_D(a) - \gamma_D(a')) \otimes 1\) with \(a, a' \in A(D)\). It is not much harder to describe \(\ker \phi_\tau\).
Proposition 9. The kernel of \( \phi_\tau \) is \((t-1) \cdot \ker \phi_L^{\text{red}}\).

Proof. Identify \( I_\mu \otimes_{\Lambda_\mu} \Lambda \) with \( \Lambda \oplus (\mathbb{Z}_e)^{\mu-1} \) using the isomorphism of Lemma 8. If \( x \in \ker \phi_L^{\text{red}} \), then \( \phi_\tau(x) = (0, n_1, \ldots, n_{\mu-1}) \in \Lambda \oplus (\mathbb{Z}_e)^{\mu-1} \) for some integers \( n_1, \ldots, n_{\mu-1} \). As \( t \cdot n_i = n_i \in \mathbb{Z}_e \) for each \( i \), it follows that \( \phi_\tau((t-1)x) = (t-1)\phi_\tau(x) = 0 \). Thus \((t-1) \cdot \ker \phi_L^{\text{red}} \subseteq \ker \phi_\tau \).

To verify the opposite inclusion, suppose \( a_1, \ldots, a_n \in A(D), \lambda_1, \ldots, \lambda_n \in \Lambda \), and

\[
x = \sum_{j=1}^n \lambda_j \cdot (\gamma_D(a_j) \otimes 1) \in \ker \phi_\tau.
\]

We claim that \( x \in (t-1) \cdot \ker \phi_L^{\text{red}} \).

Every coordinate of \( \phi_\tau(x) \) in \( \Lambda \oplus (\mathbb{Z}_e)^{\mu-1} \) is 0, so

\[
\begin{align*}
(i) \quad & \sum_{j=1}^n \lambda_j = 0 \quad \text{and} \\
(ii) \quad & \text{for each } i \in \{2, \ldots, \mu\}, \sum_{\kappa_D(a_j) = i} \epsilon(\lambda_j) = 0.
\end{align*}
\]

Notice that property (i) implies that property (ii) holds also when \( i = 1 \).

Suppose first that \( \epsilon(\lambda_1), \ldots, \epsilon(\lambda_n) \) are all 0. The kernel of \( \epsilon \) is the principal ideal \((t-1)\) of \( \Lambda \), so for each \( j \), there is a \( \lambda_j' \in \Lambda \) with \( \lambda_j = \lambda_j' \cdot (t-1) \). Property (i) implies that \( \sum_{j=1}^n \lambda_j' = 0 \), so if \( a^* \) is any fixed element of \( A(D) \),

\[
x = (t-1) \sum_{j=1}^n \lambda_j' \cdot (\gamma_D(a_j) \otimes 1) - (t-1) \left( \sum_{j=1}^n \lambda_j' \right) \cdot (\gamma_D(a^*) \otimes 1)
\]

\[
= \sum_{j=1}^n \lambda_j' \cdot (t-1) \cdot ((\gamma_D(a_j) - \gamma_D(a^*)) \otimes 1).
\]

This satisfies the claim.

Now, suppose at least one of \( \epsilon(\lambda_1), \ldots, \epsilon(\lambda_n) \) is not 0. For convenience, introduce a new summand \( 0 \cdot (\gamma_D(a) \otimes 1) \) for each \( a \in A(D) \), and collect all the appearances of each \( a_j \) into one summand, so that every \( a \in A(D) \) appears precisely once in \([1]\). If all values of \( \epsilon(\lambda_j) \) are now 0, the earlier argument applies. Otherwise, re-index the elements of \( A(D) \) so that \( \epsilon(\lambda_1) \neq 0 \). Let \( \kappa = \kappa_D(a_1) \). Re-index the elements of \( A(D) \) so that for some \( k \in \{1, \ldots, n\} \), \( a_1, \ldots, a_k \) are the arcs of \( D \) with \( \kappa_D(a_k) = \kappa \), and the arcs \( a_1, \ldots, a_k, a_1 \) are encountered in this order as we walk along the image of \( K_\kappa \) in \( D \). Notice that according to property (ii) above, \( \epsilon(\lambda_1) \neq 0 \) implies that \( k > 1 \), so \( a_1, \ldots, a_k \) are all distinct.

For \( 1 \leq i < k \), let \( a'_i \) be the overpassing arc at the crossing \( c_i \) of \( D \) that separates \( a_i \) from \( a_{i+1} \). Then depending on the orientation of \( a'_i \), one of these two formulas is equal to 0 in \( M_\Lambda^{\text{red}}(L) \).

\[
\gamma_D \rho_D(c_i) \otimes 1 = (1-t)(a'_i \otimes 1) + t(a_i \otimes 1) - a_{i+1} \otimes 1
\]
\[-\gamma_D \rho_D(c_i) \otimes 1 = -(1 - t)(a'_i \otimes 1) - t(a_{i+1} \otimes 1) + a_i \otimes 1\]

Let 0_\text{i} denote one of the two displayed formulas that does equal 0 in \(M^\text{red}_\Lambda(L)\). Notice that if we add 0_\text{i} to the sum (1), the only effect on the values of \(\epsilon(\lambda_1), \ldots, \epsilon(\lambda_n)\) is to add 1 to the value of \(\epsilon(\lambda_i)\), and add \(-1\) to the value of \(\epsilon(\lambda_{i+1})\). Of course if we add \(-0_i\), instead, we produce the opposite effects.

It follows that by repeatedly adding \(\pm 0_i\) to the sum in (1), we can obtain a sum still equal to \(x\), in which \(\epsilon(\lambda_1)\) is 0. Doing the same thing for \(i = 2, \ldots, k-1\), we obtain a sum still equal to \(x\), in which \(\epsilon(\lambda_1), \ldots, \epsilon(\lambda_{k-1})\) are all 0. Property (ii) then implies that \(\epsilon(\lambda_k)\) is 0 too, so every arc \(a_j \in A(D)\) with \(\kappa_D(a_j) = \kappa\) has \(\epsilon(\lambda_j) = 0\).

Repeating this argument for each component of \(L\) that has some arc \(a_j\) with \(\epsilon(\lambda_j) \neq 0\), we ultimately obtain a sum (1) equal to \(x\) in which \(\epsilon(\lambda_1), \ldots, \epsilon(\lambda_n)\) are all 0. Then the earlier argument tells us that the claim holds for \(x\).  

We end this section with a well-known property of the Alexander invariants of knots.

**Corollary 10.** Suppose \(\mu = 1\). Then scalar multiplication by \(t - 1\) defines an automorphism of \(\ker \phi^\text{red}_L\) as a \(\Lambda\)-module.

**Proof.** Of course, scalar multiplication by any element of \(\Lambda\) defines an endomorphism of any \(\Lambda\)-module. As \(\mu = 1\), the map \(\eta : I_\mu \otimes_{\Lambda_\mu} \Lambda \to \Lambda\) is an isomorphism, so \(\ker \phi_* = \ker(\eta \phi_*) = \ker \phi^\text{red}_L\); hence Proposition 9 tells us that \((t - 1) \cdot \ker \phi^\text{red}_L = \ker \phi^\text{red}_L\). Therefore, scalar multiplication by \(t - 1\) defines a surjective endomorphism of \(\ker \phi^\text{red}_L\). As a submodule of the finitely generated module \(M^\text{red}_\Lambda(L)\) over the Noetherian ring \(\Lambda\), \(\ker \phi^\text{red}_L\) is Noetherian; therefore a surjective endomorphism of \(\ker \phi^\text{red}_L\) must be an automorphism. \(\Box\)

3 **Theorem 4**

Recall the definition of a quandle:

**Definition 11.** A **quandle** is a set \(Q\) equipped with a binary operation \(\triangleright\), which satisfies the following properties.

1. \(x \triangleright x = x \ \forall x \in Q\).
2. For each \(y \in Q\), the formula \(\beta_y(x) = x \triangleright y\) defines a permutation \(\beta_y\) of \(Q\).
3. \((x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) \ \forall x, y, z \in Q\).

There is a traditional way to associate a quandle to any \(\Lambda\)-module, which was mentioned by both Joyce and Matveev when they introduced quandles as link invariants \([6, 8]\).

**Proposition 12.** ([6, 8]) If \(M\) is a \(\Lambda\)-module, then the operations \(x \triangleright y = tx + (1 - t)y\) and \(x \triangleright^{-1} y = t^{-1}x + (1 - t^{-1})y\) define a quandle structure on \(M\).
The quandles described in Proposition 12 are called Alexander quandles in the literature. In order to distinguish them from other quandles associated with Alexander modules, we refer to them as standard Alexander quandles. Notice that every standard Alexander quandle is a whole Λ-module. Also, if $M$ is a standard Alexander quandle and $w, x, y, z \in M$, then

$$(w \triangleright x) \triangleright (y \triangleright z) = t^2w + t(1-t)x + (1-t)ty + (1-t)^2z$$

That is, all standard Alexander quandles satisfy the medial property $(w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z)$. It follows that all subquandles of standard Alexander quandles are medial, too.

In [11] we introduced the operation $x \triangleright y = (\phi_L(y) + 1)x - \phi_L(x)y$ on $M_A(L)$, and showed that $\triangleright$ defines a quandle structure on the subset

$$U(L) = \{ x \in M_A(L) \mid \phi_L(x) + 1 \text{ is a unit of } \Lambda_\mu \}.$$ 

The subquandle of $U(L)$ generated by $\gamma_D(A(D))$ is the fundamental multivariate Alexander quandle, $Q_A(L)$. The following result was proven in [11].

**Theorem 13.** ([11]) If $L$ and $L'$ are equivalent links with diagrams $D$ and $D'$, then there is an isomorphism $f : M_A(L) \rightarrow M_A(L')$ which maps the quandle $U(L)$ isomorphically onto $U(L')$, maps the quandle $Q_A(L)$ isomorphically onto $Q_A(L')$, and is compatible with the Crowell maps of $L$ and $L'$, i.e., $\phi_L = \phi_{L'} \circ f$.

Multivariate Alexander quandles differ from standard Alexander quandles in several regards. For one thing, $Q_A(L)$ corresponds to a proper subset of the $\Lambda_\mu$-module $M_A(L)$, not a whole $\Lambda$-module. For another, $Q_A(L)$ is not a medial quandle, in general.

In the introduction, we defined $Q_A^{\text{red}}(L)$ to be $\{ x \otimes 1 \mid x \in Q_A(L) \} \subset M_A^{\text{red}}(L)$, and stated that it is a quandle under the operation $(x \otimes 1) \triangleright (y \otimes 1) = (x \triangleright y) \otimes 1$. Here is an equivalent description.

**Proposition 14.** Let $D$ be a diagram of a link $L$. Then $Q_A^{\text{red}}(L)$ is the subquandle of the standard Alexander quandle on $M_A^{\text{red}}(L)$ generated by the elements $\gamma_D(a) \otimes 1, a \in A(D)$.

**Proof.** If $x, y \in Q_A(L)$ then $\phi_L(x), \phi_L(y) \in \{ t_1 - 1, \ldots, t_\mu - 1 \}$, so

$$(x \otimes 1) \triangleright (y \otimes 1) = (x \triangleright y) \otimes 1 = ((\phi_L(y) + 1)x - \phi_L(x)y) \otimes 1$$

$$= ((\phi_L(y) + 1)x) \otimes 1 - (\phi_L(x)y) \otimes 1 = x \otimes \tau(\phi_L(y) + 1) - y \otimes \tau(\phi_L(y))$$

$$= x \otimes t - y \otimes (t - 1) = t \cdot (x \otimes 1) + (1 - t) \cdot (y \otimes 1).$$

This equals $(x \otimes 1) \triangleright (y \otimes 1)$ in the standard Alexander quandle on $M_A^{\text{red}}(L)$.

The result follows, as $Q_A(L)$ is generated by $\gamma_D(A(D))$. 

\[\square\]
Having both descriptions of $Q^\text{red}_A(L)$ is convenient because it makes it unnecessary to provide new proofs for many properties of $Q^\text{red}_A(L)$. Instead, we can simply refer to established properties of $Q_A(L)$ and $M^\text{red}_A(L)$. For instance, Propositions[12] and [14] tell us that $Q^\text{red}_A(L)$ is indeed a quandle.

Here are some other properties of $Q^\text{red}_A(L)$.

**Theorem 15.** Suppose $L$ and $L'$ are oriented links of the same link type, with diagrams $D$ and $D'$, and associated maps

$$\phi_\tau : M^\text{red}_A(L) \to I_\mu \otimes_{\Lambda_{\mu}} \Lambda \quad \text{and} \quad \phi'_\tau : M^\text{red}_A(L') \to I_\mu \otimes_{\Lambda_{\mu}} \Lambda.$$  

Then there is an isomorphism $M^\text{red}_A(L) \cong M^\text{red}_A(L')$, which maps $Q^\text{red}_A(L)$ isomorphically onto $Q^\text{red}_A(L')$, and has $\phi_\tau = \phi'_\tau \circ f$.

**Proof.** The result follows from Theorem[13] using the right exactness of tensor products. A direct proof can be obtained by setting all $t_i$ equal to $t$ in the discussion of[11] Sec. 3.

Recall that an *orbit in a quandle* $Q$ is an equivalence class under the equivalence relation generated by $x \sim x \triangleright^{-1} y \sim x \trianglerighthand y \forall x,y \in Q$.

**Theorem 16.** Let $L = K_1 \cup \cdots \cup K_\mu$ be a link with a diagram $D$, and let $Q$ be $Q^\text{red}_A(L)$, or $Q_A(L)$, or the fundamental quandle $Q(L)$. Then there are surjective functions $\kappa_D : Q \to \{1, \ldots, \mu\}$ and $\hat{\sigma}_\tau : \Lambda^Q \to M^\text{red}_A(L)$ with the following properties.

1. The orbits of $Q$ are the sets $\kappa_D^{-1}([1]), \ldots, \kappa_D^{-1}([\mu])$. Moreover, after an appropriate permutation of the indices of $K_1, \ldots, K_\mu$, it will be true that $\kappa_D(q) = \kappa_D(a)$ whenever $q \in Q$ corresponds to an arc $a \in A(D)$.

2. The restriction $\sigma_\tau = \hat{\sigma}_\tau|Q$ is a quandle homomorphism onto $Q^\text{red}_A(L)$, and if $a \in A(D)$ then the image under $\sigma_\tau$ of the element of $Q$ corresponding to $a$ is $\gamma_D(a) \otimes 1$. In particular, if $Q = Q^\text{red}_A(L)$ then $\sigma_\tau$ is the identity map of $Q$.

3. The map $\hat{\sigma}_\tau$ is $\Lambda$-linear, and ker $\hat{\sigma}_\tau$ is the submodule of $\Lambda^Q$ generated by

\[\{tx + (1-t)y - (x \trianglerighthand y), t^{-1}x + t^{-1}(t-1)y - (x \trianglerighthand y) \mid x, y \in Q\}.\]

**Proof.** This result follows from the discussion of[11] Sec. 4], along with the right exactness of the functor $- \otimes_{\Lambda_{\mu}} \Lambda$. Alternatively, set all $t_i$ equal to $t$ in that discussion.

Part 2 of Theorem[16] implies that once the function $\kappa_D : Q \to \{1, \ldots, \mu\}$ is adjusted as in part 1, the map $\phi_\tau : M^\text{red}_A(L) \to I_\mu$ will be determined by the fact that $\phi_\tau(\sigma_\tau(q)) = (t_i - 1) \otimes 1 \forall q \in \kappa_D^{-1}([i])$. We deduce the “forward” direction of Theorem[4] if $L$ and $L'$ are links and $f : Q^\text{red}_A(L) \to Q^\text{red}_A(L')$ is an isomorphism, then after adjusting component indices in $L$ and $L'$ so that $f$ maps the $\kappa_D^{-1}(i)$ orbit of $Q^\text{red}_A(L)$ to the $\kappa_D^{-1}(i)$ orbit of $Q^\text{red}_A(L')$ for each
$i \in \{1, \ldots, \mu\}$, $f$ will extend to an isomorphism $M_A^\text{red}(L) \cong M_A^\text{red}(L')$ that is compatible with the $\phi_t$ maps of $L$ and $L'$.

The next two results give us the “backward” direction of Theorem 4.

**Lemma 17.** Suppose $0 \in W \subseteq \Lambda$. Then $W = \Lambda$ if and only if $W$ is closed under the following operations.

1. $(w_1, w_2) \mapsto tw_1 + (1 - t)w_2$
2. $(w_1, w_2) \mapsto 1 + tw_1 + (t - 1)w_2$
3. $(w_1, w_2) \mapsto t^{-1}w_1 + (1 - t^{-1})w_2$
4. $(w_1, w_2) \mapsto -t^{-1} + t^{-1}w_1 + (t^{-1} - 1)w_2$

**Proof.** If $W = \Lambda$ then $W$ is closed under all binary operations defined on $\Lambda$. For the converse, suppose $W$ is closed under the four listed operations. Using $w_2 = 0$ in operations 1 and 3, we see that $W$ is closed under multiplication by $t^{\pm 1}$. Combining operation 2 with multiplication by $t^{-1}$, we see that $W$ contains the following elements: $(0, 0) \mapsto 1, 1 \mapsto t^{-1}, (t^{-1}, 0) \mapsto 2, 2 \mapsto 2t^{-1}, (2t^{-1}, 0) \mapsto 3$, and so on. Combining operation 4 with multiplication by $t$, we see that $W$ contains the following elements: $(0, 0) \mapsto -t^{-1}, -t^{-1} \mapsto -1, (-1, 0) \mapsto -2t^{-1}, -2t^{-1} \mapsto -2, (-2, 0) \mapsto -3t^{-1}, -3t^{-1} \mapsto -3$, and so on. We conclude that $\mathbb{Z} \subseteq W$.

Closure under multiplication by $t^{\pm 1}$ implies that $W$ contains every monomial $mt^n$ with $m, n \in \mathbb{Z}$. Now, suppose $\lambda \in \Lambda$ is not a monomial. Say $\lambda = n_1t^a + n_2t^{a+1} + \cdots + n_kt^{a+k}$ for some $k \geq 1 \in \mathbb{Z}$ and some $n_1, \ldots, n_k \in \mathbb{Z}$, with $n_1 \neq 0 \neq n_k$. Using induction on $k$, we may presume that $W$ contains $m_1t^b + m_2t^{b+1} + \cdots + m_jt^{b+j}$ whenever $j < k$. Then $W$ contains both $w_1 = n_1t^{a-1} + n_2t^a + \cdots + n_k-2t^{a+k-3} + (n_{k-1} + n_k)t^{a+k-2}$ and $w_2 = -n_kt^{a+k-1}$. As $W$ is closed under operation 1, $W$ contains $tw_1 + (1 - t)w_2 = \lambda$.

**Proposition 18.** $Q_A^\text{red}(L) = (\phi_t)^{-1}((t_1 - 1) \otimes 1, \ldots, (t_{\mu} - 1) \otimes 1))$.

**Proof.** Let $S = (\phi_t)^{-1}((t_1 - 1) \otimes 1, \ldots, (t_{\mu} - 1) \otimes 1))$, and let $D$ be a diagram of $L$. Suppose $s \in S$ and $a \in A(D)$ have $\phi_t(s) = (t_i - 1) \otimes 1$ and $\kappa_D(a) = j$. Then

$\phi_t(s \circ (\gamma_D(a) \otimes 1)) = \phi_t(ts + (1 - t) \cdot (\gamma_D(a) \otimes 1))$

$= t \cdot ((t_1 - 1) \otimes 1) + (1 - t) \cdot ((t_j - 1) \otimes 1) = (t_i - 1) \otimes t(t_j - 1) \otimes (1 - t)$

$= (t_j(t_i - 1)) \otimes 1 + ((1 - t_i)(t_j - 1)) \otimes 0 = (t_i - 1) \otimes 1,$

so $s \circ (\gamma_D(a) \otimes 1) \in S$. Clearly $\gamma_D(a) \otimes 1 \in S \forall a \in A(D)$, so $S$ contains the subquandle of the standard Alexander quandle on $M_A^\text{red}(L)$ generated by $\{\gamma_D(a) \otimes 1 \mid a \in A(D)\}$. That is, $Q_A^\text{red}(L) \subseteq S$.

Verifying the opposite inclusion is more difficult. Recall that $Q_A^\text{red}(L)$ is a subquandle of the standard Alexander quandle on $M_A^\text{red}(L)$, so $Q_A^\text{red}(L)$ is closed under the operations $\circ, \circ^{-1}$ mentioned in Proposition 12.
Suppose \( a_1 \) and \( a_2 \) are any two arcs of \( D \). Let

\[
W(a_1, a_2) = \{ w \in \Lambda \mid \gamma_D(a_1) \otimes 1 + w(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1 \} \subseteq Q^\text{red}_Q(L),
\]

and let \( W = W(a_1, a_2) \cap W(a_2, a_1) \).

Note that if \( w_1, w_2 \in W \) then \( Q^\text{red}_Q(L) \) contains both

\[
(\gamma_D(a_1) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
\triangleright (\gamma_D(a_1) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) =
\]

\[
t \cdot (\gamma_D(a_1) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
+ (1-t) \cdot (\gamma_D(a_1) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) =
\]

\[
\gamma_D(a_1) \otimes 1 + (tw_1 + (1-t)w_2)(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

and the element obtained from this by interchanging \( a_1 \) and \( a_2 \). It follows that \( tw_1 + (1-t)w_2 \in W \). That is, \( W \) is closed under operation 1 of Lemma 17.

For operation 2 of Lemma 17, note that if \( w_1, w_2 \in W \) then \( Q^\text{red}_Q(L) \) contains both

\[
(\gamma_D(a_1) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
\triangleright (\gamma_D(a_2) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_2) - \gamma_D(a_1)) \otimes 1) =
\]

\[
t \cdot (\gamma_D(a_2) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
+ (1-t) \cdot (\gamma_D(a_2) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_2) - \gamma_D(a_1)) \otimes 1) =
\]

\[
\gamma_D(a_1) \otimes 1 + (1-tw_2 + t(t-1)w_2) \cdot (t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

and the element obtained from this by interchanging \( a_1 \) and \( a_2 \). It follows that \( W \) is closed under the operation \( (w_1, w_2) \mapsto 1 + tw_1 + (t-1)w_2 \).

To show that \( W \) is closed under operations 3 and 4 of Lemma 17 use \( \triangleright^{-1} \) instead of \( \triangleright \):

\[
(\gamma_D(a_1) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
\triangleright^{-1} (\gamma_D(a_1) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) =
\]

\[
t^{-1} \cdot (\gamma_D(a_1) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
+ (1-t^{-1}) \cdot (\gamma_D(a_1) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) =
\]

\[
\gamma_D(a_1) \otimes 1 + (t^{-1}w_2 + (1-t^{-1})w_2)(t-1) \cdot (\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)
\]

\[
\triangleright^{-1} (\gamma_D(a_2) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_2) - \gamma_D(a_1)) \otimes 1) =
\]

\[
t^{-1} \cdot (\gamma_D(a_2) \otimes 1 + w_1(t-1) \cdot (\gamma_D(a_2) - \gamma_D(a_1)) \otimes 1)
\]

\[
+ (1-t^{-1}) \cdot (\gamma_D(a_2) \otimes 1 + w_2(t-1) \cdot (\gamma_D(a_2) - \gamma_D(a_1)) \otimes 1) =
\]
Lemma \[\text{(17)}\] now tells us that $W = \Lambda$. It follows that for every choice of $a_1, a_2 \in A(D)$ and $\lambda \in \Lambda$,

$$q(a_1, a_2, \lambda) = \gamma_D(a_1) \otimes 1 + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) \in Q^\text{red}_A(L).$$

Now, suppose that $x \in Q^\text{red}_A(L)$, $a_1, a_2 \in A(D)$ and $\lambda \in \Lambda$. Then $Q^\text{red}_A(L)$ contains $x$, $q(a_1, a_2, \lambda)$ and $\gamma_D(a_2) \otimes 1$, so since $Q^\text{red}_A(L)$ is closed under $\triangleright$ and $\triangleright^{-1}$, $Q^\text{red}_A(L)$ also contains

$$r(x, a_1, a_2, \lambda) = (x \triangleright^{-1} q(a_1, a_2, \lambda) \triangleright (\gamma_D(a_2) \otimes 1)$$

$$= t \cdot (x \triangleright^{-1} q(a_1, a_2, \lambda)) + (1 - t) \cdot (\gamma_D(a_2) \otimes 1)$$

$$= t \cdot ((t^{-1} x + (1 - t^{-1})q(a_1, a_2, \lambda)) - (t - 1) \cdot (\gamma_D(a_2) \otimes 1)$$

$$= x + (t - 1) \cdot (q(a_1, a_2, \lambda) - (\gamma_D(a_2) \otimes 1))$$

$$= x + (t - 1) \cdot (1 + \lambda(t - 1)) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1).$$

Our next claim is that $x + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) \in Q^\text{red}_A(L)$ for all choices of $x \in Q^\text{red}_A(L)$, $a_1, a_2 \in A(D)$ and $\lambda \in \Lambda$. If $\epsilon(\lambda) = 1$, then $\lambda = 1 + \lambda' \cdot (t - 1)$ for some $\lambda' \in \Lambda$, and

$$x + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) = r(x, a_1, a_2, \lambda') \in Q^\text{red}_A(L),$$

so the claim is satisfied. If $\epsilon(\lambda) > 1$ and

$$x' = x + (\lambda - 1)(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) \in Q^\text{red}_A(L),$$

then

$$r(x', a_1, a_2, 0) = x + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) \in Q^\text{red}_A(L),$$

and the claim is satisfied. Using induction on $\epsilon(\lambda)$, we conclude that whenever $\epsilon(\lambda) \geq 1$, $x + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) \in Q^\text{red}_A(L)$. As $a_1$ and $a_2$ are arbitrary, it follows that the claim is also satisfied when $\epsilon(\lambda) \leq -1$, because

$$x + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1) = x + (-\lambda)(t - 1) \cdot ((\gamma_D(a_2) - \gamma_D(a_1)) \otimes 1).$$

If $\epsilon(\lambda) = 0$ then $\epsilon(\lambda - 1) = -1$, so it follows that $Q^\text{red}_A(L)$ contains both $x' = x + (\lambda - 1)(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)$ and $x + \lambda(t - 1) \cdot ((\gamma_D(a_1) - \gamma_D(a_2)) \otimes 1)$

$$= r(x', a_1, a_2, 0).$$

Again, the claim is satisfied.

According to Proposition \[\text{(9)}\] this claim tells us that whenever $x \in Q^\text{red}_A(L)$ and $y \in \ker \phi_\tau$, $x + y \in Q^\text{red}_A(L)$. As $\phi_\tau(a) = (t_i - 1) \otimes 1$ for every $a \in A(D)$ with $\kappa_D(a) = i$, it follows that $Q^\text{red}_A(L)$ contains every element of the set $S$. \[\Box\]

Proposition \[\text{(18)}\] implies that if $L$ and $L'$ are $\phi_\tau$-equivalent links, then there is an isomorphism $\tilde{M}^\text{red}_A(L) \cong \tilde{M}^\text{red}_A(L')$ of $\Lambda$-modules, under which $Q^\text{red}_A(L)$ and $Q^\text{red}_A(L')$ correspond. This gives us the "backward" direction of Theorem \[\text{(4)}\].
4 Theorem 6

Suppose $L$ and $L'$ are links with $Q^\text{red}_A(L) \cong Q^\text{red}_A(L')$. Then Theorem 4 implies that after permuting the indices of components, $L$ and $L'$ will be $\phi_\text{t}$-equivalent. An isomorphism $f : M^\text{red}_A(L) \to M^\text{red}_A(L')$ with $\phi_\text{t} = \phi'_\text{t} \circ f$ will have $\phi^\text{red}_L = \eta \circ \phi_\text{t} = \eta \circ \phi'_\text{t} \circ f = \phi^\text{red}_L \circ f$, so $\ker \phi^\text{red}_L \cong \ker \phi^\text{red}_L$. We conclude that as a link invariant, $Q^\text{red}_A$ is at least as strong as the module $\ker \phi^\text{red}_L$. Examples verifying that $Q^\text{red}_A$ is strictly stronger than $\ker \phi^\text{red}_L$ are presented in the subsections below.

Before discussing these examples, we prove that for knots, $Q^\text{red}_A(L)$ and $\ker \phi^\text{red}_L$ are equivalent invariants. We have already seen that $Q^\text{red}_A(L)$ determines $\ker \phi^\text{red}_L$. For the converse, recall that as noted in the introduction, every link $L$ has $M^\text{red}_A(L) \cong \ker \phi^\text{red}_L \times \Lambda$. If we identify these two modules according to this isomorphism, then the map $\phi^\text{red}_L : \ker \phi^\text{red}_L \times \Lambda \to \Lambda$ is given by a simple formula: $\phi^\text{red}_L(x, y) = y$.

It follows that if $L$ and $L'$ are links and there is an isomorphism $f : \ker \phi^\text{red}_L \to \ker \phi^\text{red}_{L'}$, then the resulting isomorphism $f \circ \id : \ker \phi^\text{red}_L \times \Lambda \to \ker \phi^\text{red}_{L'} \times \Lambda$ has $\phi^\text{red}_{L'} = \phi^\text{red}_L \circ (f \circ \id)$. If $\mu = 1$, then the map $\eta : I_\mu \times \Lambda \to \Lambda$ is an isomorphism, and

$$\phi_\text{t} = \eta^{-1} \circ \phi^\text{red}_L \circ \eta^{-1} \circ \phi^\text{red}_{L'} \circ (f \circ \id) = \phi'_\text{t} \circ (f \circ \id),$$

so $L$ and $L'$ are $\phi_\text{t}$-equivalent. Theorem 4 then tells us that $Q^\text{red}_A(L) \cong Q^\text{red}_A(L')$.

To complete a proof of Theorem 6, it suffices to exhibit a pair of links $L, L'$ with $\ker \phi^\text{red}_L \cong \ker \phi^\text{red}_{L'}$ and $Q^\text{red}_A(L) \not\cong Q^\text{red}_A(L')$. In the rest of this section we present two such pairs.

4.1 Two 4-component links

Let $L$ be the link illustrated in Fig. 2. Then $M^\text{red}_A(L)$ is generated by the four elements $\gamma_D(a) \otimes 1, \gamma_D(b) \otimes 1, \gamma_D(c) \otimes 1$ and $\gamma_D(d) \otimes 1$, subject to the crossing relations $(1-t) \cdot (\gamma_D(a) \otimes 1) = (1-t) \cdot (\gamma_D(b) \otimes 1)$ and $(1-t) \cdot (\gamma_D(c) \otimes 1) = (1-t) \cdot (\gamma_D(d) \otimes 1)$. It follows that

$$M^\text{red}_A(L) \cong \Lambda \oplus (\Lambda/(1-t)) \oplus \Lambda \oplus (\Lambda/(1-t)),$$

with the four summands generated by $\gamma_D(a) \otimes 1, (\gamma_D(b) - \gamma_D(a)) \otimes 1, (\gamma_D(c) - \gamma_D(a)) \otimes 1$ and $(\gamma_D(d) - \gamma_D(c)) \otimes 1$, respectively. The map $\phi^\text{red}_L : M^\text{red}_A \to \Lambda$ maps all of $\gamma_D(a) \otimes 1, \gamma_D(b) \otimes 1, \gamma_D(c) \otimes 1, \gamma_D(d) \otimes 1$ to 1, so

$$\ker \phi^\text{red}_L \cong (\Lambda/(1-t)) \oplus \Lambda \oplus (\Lambda/(1-t)).$$

Now, let $L'$ be the link with the diagram $D'$ pictured in Fig. 3. Then $M^\text{red}_A(L')$ is generated by $\gamma_D'(v) \otimes 1, \gamma_D'(w) \otimes 1, \gamma_D'(x) \otimes 1, \gamma_D'(y) \otimes 1$ and $\gamma_D'(z) \otimes 1$. The crossing relations from the two crossings on the left tell us that

$$\gamma_D'(v) \otimes 1 = (1-t) \cdot (\gamma_D'(w) \otimes 1) + t \cdot (\gamma_D'(v) \otimes 1)$$

and

$$\gamma_D'(w) \otimes 1 = (1-t) \cdot (\gamma_D'(v) \otimes 1) + t \cdot (\gamma_D'(x) \otimes 1),$$
so \( t \cdot (\gamma D'(w) \otimes 1) = t \cdot (\gamma D'(x) \otimes 1) \), and hence \( \gamma D'(w) \otimes 1 = \gamma D'(x) \otimes 1 \). Taking this into account, the two relations from the crossings on the right are the same: 
\[(1-t) \cdot (\gamma D'(x) \otimes 1) = (1-t) \cdot (\gamma D'(y) \otimes 1) \]. It follows that 
\[M_{\text{red}}^A(L) \cong \Lambda \oplus (\Lambda/(1-t)) \oplus (\Lambda/(1-t)) \oplus \Lambda, \] (3)
with the four summands generated by \( \gamma D'(x) \otimes 1 \), \( (\gamma D'(v) - \gamma D'(x)) \otimes 1 \), \( (\gamma D'(y) - \gamma D'(x)) \otimes 1 \) and \( (\gamma D'(z) - \gamma D'(x)) \otimes 1 \), respectively. Therefore 
\[\text{ker} \phi_{\text{red}}^L \cong (\Lambda/(1-t)) \oplus (\Lambda/(1-t)) \oplus \Lambda.\]

It is apparent that \( \text{ker} \phi_{\text{red}}^L \cong \text{ker} \phi_{\text{red}}^L' \). Nevertheless, we have the following.

**Proposition 19.** No matter how their components are indexed, \( L \) and \( L' \) are not \( \phi_\tau \)-equivalent.

**Proof.** Recall that Lemma \[\] tells us 
\[I_4 \otimes \Lambda_4 \Lambda \cong \Lambda \oplus \mathbb{Z}_e \oplus \mathbb{Z}_e \oplus \mathbb{Z}_e, \]
with the direct summands generated by \( (t_1 - 1) \otimes 1 \), \( (t_2 - t_1) \otimes 1 \), \( (t_3 - t_1) \otimes 1 \) and \( (t_4 - t_1) \otimes 1 \), respectively. We use this isomorphism to identify \( I_4 \otimes \Lambda_4 \Lambda \) with \( \Lambda \oplus \mathbb{Z}_e \oplus \mathbb{Z}_e \oplus \mathbb{Z}_e \).

Given an indexing of the components of \( L \), the resulting map \( \phi_\tau : M_{A_{\text{red}}}^L \rightarrow I_4 \otimes \Lambda_4 \Lambda \) will send \( \gamma D(a) \otimes 1 \), \( \gamma D(b) \otimes 1 \), \( \gamma D(c) \otimes 1 \) and \( \gamma D(d) \otimes 1 \) to \( (1,0,0,0) \), \( (1,1,0,0) \), \( (1,0,1,0) \) and \( (1,0,0,1) \), in some order. No matter what order is
used, the image of \((\gamma_D(b) - \gamma_D(a) + \gamma_D(d) - \gamma_D(c)) \otimes 1\) will be of the form 
\((0, \pm 1, \pm 1, \pm 1)\). Notice that according to (2), 
\((\gamma_D(b) - \gamma_D(a) + \gamma_D(d) - \gamma_D(c)) \otimes 1\) is an element of \(M_{\text{red}}^A(L)\) that is annihilated by \(t - 1\).

Given an indexing of the components of \(L'\), \(\phi' : M_{\text{red}}^A(L') \rightarrow I_4 \otimes \Lambda, \Lambda\) will map \(\gamma_D(v) \otimes 1, \gamma_D(x) \otimes 1, \gamma_D(y) \otimes 1\) and \(\gamma_D(z) \otimes 1\) to \((1, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0)\) and \((1, 0, 0, 1)\), in some order. According to (3), every element of \(M_{\text{red}}^A(L')\) annihilated by \(t - 1\) is of the form

\[\lambda_1(\gamma_D(v) - \gamma_D(x)) \otimes 1 + \lambda_2(\gamma_D(y) - \gamma_D(x)) \otimes 1\]

for some \(\lambda_1, \lambda_2 \in \Lambda\). It is easy to see that the image under \(\phi'\) of such an element cannot be of the form \((0, \pm 1, \pm 1, \pm 1)\).

Therefore, no isomorphism \(f : M_{\text{red}}^A(L) \rightarrow M_{\text{red}}^A(L')\) has \(\phi' \circ f = \phi_{\tau}\).

The proof of Proposition 19 is derived from an argument in the second paper of this series [12, Sec. 15.4], where we observed that the links \(L\) and \(L'\) are distinguished by their involutory medial quandles. Results of [12] imply that any two links with isomorphic reduced Alexander invariants and nonisomorphic involutory medial quandles must have \(\mu \geq 4\).

In the next subsection we see that in contrast, it is possible for medial quandles to distinguish 3-component links with isomorphic reduced Alexander invariants. We do not know of any analogous examples of 2-component links.

### 4.2 Two 3-component links

In this subsection, \(L\) denotes the link pictured in Fig. 4. We are grateful to Livingston, Moore and other researchers who developed and maintained the LinkInfo website [7], where we found this interesting example.

![Diagram of link L](image)

Figure 4: The link \(L = L10n93\{0,1\}\) in the LinkInfo table [7].

To describe \(M_{\text{red}}^A(L)\), we use the crossings not marked * in Fig. 4 to eliminate generators other than \(\overline{a} = \gamma_D(a) \otimes 1, \overline{e} = \gamma_D(e) \otimes 1\) and \(\overline{j} = \gamma_D(j) \otimes 1\), as follows.

\[
\gamma_D(b) \otimes 1 = (1 - t)\overline{e} + t\overline{a}
\]

\[
\gamma_D(c) \otimes 1 = (1 - t)\overline{j} + t(\gamma_D(b) \otimes 1) = t^2\overline{a} + (t - t^2)\overline{e} + (1 - t)\overline{j}
\]
\[ \gamma_D(f) \otimes 1 = (1 - t)\alpha + t\tau \]
\[ \gamma_D(d) \otimes 1 = (1 - t)(\gamma_D(f) \otimes 1) + t(\gamma_D(c) \otimes 1) \]
\[ = (1 - 2t + t^2 + t^3)\alpha + (t - t^3)\tau + (t - t^2)\tau_j \]
\[ \gamma_D(g) \otimes 1 = (1 - t)\tau_j + t\tau \]
\[ \gamma_D(h) \otimes 1 = (1 - t)\tau + t\tau_j \]
\[ \gamma_D(i) \otimes 1 = (1 - t^{-1})(\gamma_D(f) \otimes 1) + t^{-1}(\gamma_D(h) \otimes 1) \]
\[ = (1 - t)^2 + (1 - t)\tau + \tau_j \]

The three remaining crossings are marked * in Fig. 4. These three crossings provide three relations:
\[ \gamma_D(g) \otimes 1 = (1 - t)(\gamma_D(c) \otimes 1) + t(\gamma_D(f) \otimes 1) \]
\[ \tau_j = (1 - t)(\gamma_D(c) \otimes 1) + t(\gamma_D(i) \otimes 1) \]
\[ \alpha = (1 - t)(\gamma_D(h) \otimes 1) + t(\gamma_D(d) \otimes 1) \]

The first two of these remaining crossing relations are both equivalent to
\[ 0 = (1 - t)\cdot((t^2 + t)\alpha - t^2\tau - \tau_j), \quad (4) \]
and the third is equivalent to
\[ 0 = (t^2 - 1)\cdot((t^2 + t)\alpha - t^2\tau - \tau_j), \]
which is a consequence of (4). Therefore
\[ M_{A_0}^\text{red}(L) \cong \Lambda \oplus \Lambda \oplus (\Lambda/(1 - t)), \quad (5) \]
with the three summands generated by $\alpha, \tau - \alpha$ and $\tau_j = (t^2 + t)\alpha - t^2\tau - \tau_j$ (respectively). The map $\phi_{L_0}^\text{red} : M_{A_0}^\text{red} \to \Lambda$ maps $\alpha, \tau$ and $\tau_j$ to 1, so
\[ \ker \phi_{L_0}^\text{red} \cong \Lambda \oplus (\Lambda/(1 - t)). \]

Now, let $L'$ be the link with the diagram $D'$ pictured in Fig. 5. Then it is easy to see that
\[ M_{A_0}^\text{red}(L') \cong \Lambda \oplus (\Lambda/(1 - t)) \oplus \Lambda, \quad (6) \]
with the three summands generated by $\gamma_{D'}(x) \otimes 1$, $(\gamma_{D'}(y) - \gamma_{D'}(x)) \otimes 1$ and $(\gamma_{D'}(z) - \gamma_{D'}(x)) \otimes 1$ (respectively). As $\phi_{L_0}^\text{red} : M_{A_0}^\text{red}(L') \to \Lambda$ maps $\gamma_{D'}(x) \otimes 1$, $\gamma_{D'}(y) \otimes 1$ and $\gamma_{D'}(z) \otimes 1$ to 1, it follows that
\[ \ker \phi_{L_0}^\text{red} \cong \Lambda \oplus (\Lambda/(1 - t)). \]

The relationship between the links denoted $L$ and $L'$ in this subsection is similar to the relationship between the links denoted $L$ and $L'$ in Sec. 4.1.

**Proposition 20.** No matter how their components are indexed, $L$ and $L'$ are not $\phi_\tau$-equivalent.
Given an indexing of the components of \(L\), the resulting map \(\phi : M^\text{red}_A(L) \to I_3 \otimes \Lambda_3 \Lambda\) will send \(\pi, \tau\) and \(\gamma\) to \((1, 0, 0), (1, 1, 0),\) and \((1, 0, 1)\), in some order.

According to [9], every element \(m \in M^\text{red}_A(L)\) with \((1 - t)m = 0\) is of the form \(m = \lambda j = \lambda((t^2 + t)\pi - t^2\pi - \tau j)\) for some \(\lambda \in \Lambda\). If \(\phi_\tau(\pi) = (1, 0, 0)\), then \(\phi_\tau(\lambda j) = \epsilon(\lambda)\phi_\tau((t^2 + t)\pi - t^2\pi - \tau j) = (0, -\epsilon(\lambda), -\epsilon(\lambda))\). If \(\phi_\tau(\pi) = (0, 1, 0)\), then \(\phi_\tau(\lambda j) = \epsilon(\lambda)\phi_\tau((t^2 + t)\pi - t^2\pi - \tau j) = (0, 2\epsilon(\lambda), -\epsilon(\lambda))\). If \(\phi_\tau(\pi) = (0, 0, 1)\), then \(\phi_\tau(\lambda j) = \epsilon(\lambda)\phi_\tau((t^2 + t)\pi - t^2\pi - \tau j) = (0, -\epsilon(\lambda), 2\epsilon(\lambda))\). No matter how the components of \(L\) are indexed, if \(m \in M^\text{red}_A(L)\) has \((1 - t)m = 0\), then \(\phi_\tau(m) \neq (0, 1, 0)\).

Suppose the components of \(L' = K'_1 \cup K'_2 \cup K'_3\) are indexed so that \(x, y\) and \(z\) are the images in \(D'\) of \(K'_1, K'_2\) and \(K'_3\), in order. Then \(\phi'_\tau : M^\text{red}_A(L') \to I_3 \otimes \Lambda_3 \Lambda\) has \(\phi'_\tau(\gamma_{D'}(x) \otimes 1) = (1, 0, 0)\), \(\phi'_\tau(\gamma_{D'}(y) \otimes 1) = (1, 1, 0)\) and \(\phi'_\tau(\gamma_{D'}(z) \otimes 1) = (1, 0, 1)\). It follows that \(\phi'_\tau((\gamma_{D'}(y) - \gamma_{D'}(x)) \otimes 1) = (0, 1, 0)\).

According to [10], \(m = (\gamma_{D'}(y) - \gamma_{D'}(x)) \otimes 1\) is an element of \(M^\text{red}_A(L')\) with \((1 - t)m = 0\). Every isomorphism \(f : M^\text{red}_A(L) \to M^\text{red}_A(L')\) maps elements of \(M^\text{red}_A(L)\) annihilated by \(1 - t\) to elements of \(M^\text{red}_A(L')\) annihilated by \(1 - t\), so every isomorphism \(f : M^\text{red}_A(L) \to M^\text{red}_A(L')\) has \(\phi_\tau \neq \phi'_\tau \circ f\).

All of the links discussed in this section illustrate a fact mentioned in the introduction: Theorem 4 does not hold without allowing for re-indexing of link components. The reasoning is simple. For instance, suppose the components of the link denoted \(L'\) in this subsection are \(K'_1, K'_2, K'_3\) in \(x, y, z\) order, as in the proof of Proposition 20. The isomorphism [10] implies that if \(m \in M^\text{red}_A(L')\) has \((1 - t)m = 0\), then \(m = \lambda \cdot ((\gamma_{D'}(y) - \gamma_{D'}(x)) \otimes 1)\) for some \(\lambda \in \Lambda\), so \(\phi'_\tau : M^\text{red}_A(L') \to I_3 \otimes \Lambda_3 \Lambda\) has \(\phi'_\tau(m) = (0, \epsilon(\lambda), 0)\). Now, let \(L'' = K''_1 \cup K''_2 \cup K''_3\) be the same link, but with \(K''_1, K''_2, K''_3\) in \(x, z, y\) order. Then the same analysis leads to the conclusion that if \(m \in M^\text{red}_A(L'')\) has \((1 - t)m = 0\), the map \(\phi'_\tau : M^\text{red}_A(L'') \to I_3 \otimes \Lambda_3 \Lambda\) has \(\phi''(m) = (0, 0, n)\) for some integer \(n\). It follows that no isomorphism between the modules \(M^\text{red}_A(L')\) and \(M^\text{red}_A(L'')\) is compatible with \(\phi_\tau\) and \(\phi'_\tau\), so \(L'\) and \(L''\) are not \(\phi_\tau\)-equivalent. However \(Q^\text{red}_A(L') \cong Q^\text{red}_A(L'')\), because the only difference between \(L'\) and \(L''\) is the indexing of their components, and the quandles do not detect the indices.
5 Medial Quandles

In this section, we summarize some ideas from the general theory of quandles. We refer to the work of Jedlička, Pil
towska, Stanovský and Zamojska-Dzienio [14,15] for a more thorough discussion. Proofs are included for the reader’s convenience.

If \( Q \) is a quandle and \( y \in Q \), then the map \( \beta_y \) is a translation of \( Q \). Part 2 of Definition [11] implies that \( \beta_y \) has an inverse function; the notation \( \beta_y^{-1}(x) = x \triangleright y \) is often used. Notice that part 3 of Definition [11] can be written as \( \beta_z(x \triangleright y) = \beta_z(x) \triangleright \beta_z(y) \), so every translation of \( Q \) is a quandle automorphism. (Indeed, some authors call translations “inner automorphisms.”) Of course it follows that the inverse function of a translation is an automorphism too, so \( \beta_z^{-1}(x \triangleright y) = \beta_z^{-1}(x) \triangleright^{-1} \beta_z(y) \forall x,y,z \in Q \). This implies that \( \triangleright^{-1} \) also defines a quandle structure on the set \( Q \).

The fact that \( \beta_z \) is an automorphism of \( Q \) also implies that \( \beta_z(x \triangleright^{-1} y) = \beta_z(x) \triangleright^{-1} \beta_z(y) \forall x,y,z \in Q \). That is, \( \beta_z \beta_y^{-1}(x) = \beta_{\beta_z(y)} \beta_z(x) \forall x,y,z \in Q \).

It follows that there is a special way to express conjugation of translations in \( \text{Aut}(Q) \): \( \beta_z \beta_y \beta_z^{-1} = \beta_{\beta_z(y)} \).

If \( y,z \in Q \) then the composition \( \beta_y \beta_z^{-1} \) is an elementary displacement of \( Q \). The subgroup of the automorphism group \( \text{Aut}(Q) \) generated by the elementary displacements is denoted \( \text{Dis}(Q) \), and its elements are called displacements. (Some references use the term “transvections” instead.)

**Proposition 21.** Here are six properties of displacements and translations.

1. If \( d \in \text{Dis}(Q) \) then \( d = \beta_{y_1} \beta_{y_2}^{-1} \cdots \beta_{y_{2n-1}} \beta_{y_{2n}}^{-1} \) for some \( y_1, \ldots, y_{2n} \in Q \).
2. The elementary displacements also include products of the form \( \beta_y^{-1} \beta_z \).
3. If \( y_1, \ldots, y_n \in Q \), \( m_1, \ldots, m_n \in \mathbb{Z} \) and \( \sum m_i = 0 \), then \( \prod \beta_{y_i}^{m_i} \in \text{Dis}(Q) \).
4. If \( f \in \text{Aut}(Q) \) and \( y \in Q \), then \( \beta_{f(y)} = f \beta_y f^{-1} \).
5. \( \text{Dis}(Q) \) is a normal subgroup of \( \text{Aut}(Q) \).
6. If \( f : Q_1 \to Q_2 \) is a surjective quandle map, then \( f \) induces a surjective homomorphism \( \text{Dis}(f) : \text{Dis}(Q_1) \to \text{Dis}(Q_2) \), defined by

\[
\text{Dis}(f)(\prod \beta_{y_i}^{m_i}) = \prod \beta_{f(y_i)}^{m_i}
\]

whenever \( y_1, \ldots, y_n \in Q_1 \), \( m_1, \ldots, m_n \in \mathbb{Z} \) and \( \sum m_i = 0 \).

**Proof.** For the first property, note that the inverse of an elementary displacement is an elementary displacement. It follows that every element of \( \text{Dis}(Q) \) is a product of elementary displacements.

For the second property, replace \( y \) with \( y' = \beta_z^{-1}(y) \) in the equality \( \beta_z \beta_y^{-1} = \beta_{\beta_z(y)} \beta_z \) mentioned above. The result is \( \beta_z \beta_y^{-1} = \beta_y^{-1} \beta_z \).

For the third property, introduce repetitions in the list \( y_1, \ldots, y_n \), so that \( m_1, \ldots, m_n \in \{-1,1\} \). If \( m_1 \neq m_2 \) then \( \prod \beta_{y_i}^{m_i} = (\beta_{y_1}^{m_1} \beta_{y_2}^{m_2}) (\prod_{i>2} \beta_{y_i}^{m_i}) \), and...
induction on \( n \) applies. If \( m_1 = m_2 \) then find the least index \( j \) with \( m_1 \neq m_j \), and apply the second property to replace \( \beta_{y_j}^{m_j-1} \) and \( \beta_{y_j}^{m_j} \) with \( \beta_{y_j}^{m_j} \) and \( \beta_{y_j}^{m_j-1} \), respectively, so that \( \beta_{y_j}^{m_j} \beta_{y_j}^{m_j-1} = \beta_{y_j}^{m_j-1} \beta_{y_j}^{m_j} \). Now induction on \( j \) applies.

For the fourth property, notice that for each \( x \in Q \), \( f \beta_y(x) = f(x \triangleright y) = f(x) \triangleright f(y) = \beta_f(y) f(x) \). Hence \( f \beta_y = \beta_f(y) f \).

For the fifth property, notice that if \( \beta_y \beta_z^{-1} \) is an elementary displacement, then for every \( f \in \text{Aut}(Q) \),

\[
f \beta_y \beta_z^{-1} f^{-1} = f \beta_y (f \beta_z)^{-1} = \beta_f(y) f (\beta_f(z)) f^{-1} = \beta_f(y) f f^{-1} \beta_f(z) = \beta_f(y) \beta_f(z)
\]

is also an elementary displacement.

For the sixth property, note that as \( f \) is a quandle map, \( f(x \triangleright y) = f(x) \triangleright f(y) \) \( \forall x, y \in Q_1 \). Also, if \( x, y \in Q_1 \) then \( f(x \triangleright y) = f((x \triangleright y) \triangleright y) = f(x) \), so \( f(x \triangleright y) = f(x) \triangleright f(y) \). We deduce that \( f \circ \beta_y^\pm = \beta_f(y) \circ f \) for every \( y \in Q_1 \). This implies that \( f \circ \beta_y^n = \beta_f^n \circ f \forall y \in Q_1 \forall n \in \mathbb{Z} \).

Suppose \( y_1, \ldots, y_n, z_1, \ldots, z_p \in Q_1, m_1, \ldots, m_n, \ell_1, \ldots, \ell_p \in \mathbb{Z} \), and

\[
\prod_{i=1}^n \beta_{y_i}^{m_i} = \prod_{j=1}^p \beta_{z_j}^{\ell_j}
\]

Then

\[
\left( \prod_{i=1}^n \beta_{f(y_i)}^{m_i} \right) \circ f = \left( \prod_{i=1}^{n-1} \beta_{f(y_i)}^{m_i} \right) \circ f \circ \beta_{y_n}^{m_n} = \cdots = f \circ \left( \prod_{i=1}^n \beta_{y_i}^{m_i} \right)
\]

\[
= f \circ \left( \prod_{j=1}^p \beta_{z_j}^{\ell_j} \right) = \beta_{f(z_1)}^{\ell_1} \circ f \circ \left( \prod_{j=2}^p \beta_{z_j}^{\ell_j} \right) = \cdots = \left( \prod_{j=1}^p \beta_{f(z_j)}^{\ell_j} \right) \circ f.
\]

As \( f \) is surjective, we deduce that

\[
\prod_{i=1}^n \beta_{y_i}^{m_i} = \prod_{j=1}^p \beta_{z_j}^{\ell_j} \implies \prod_{i=1}^n \beta_{f(y_i)}^{m_i} = \prod_{j=1}^p \beta_{f(z_j)}^{\ell_j},
\]

That is, we have a well-defined function mapping products of powers of translations of \( Q_1 \) to products of powers of translations of \( Q_2 \), according to the formula

\[
\prod_{i=1}^n \beta_{y_i}^{m_i} \mapsto \prod_{i=1}^n \beta_{f(y_i)}^{m_i}.
\]

The restriction of this function to products with \( \sum m_i = 0 \) is the map \( \text{Dis}(f) \) mentioned in the sixth property of the proposition. As the function maps products to products, it is obvious that it is a homomorphism. The image of \( \text{Dis}(f) \) includes every elementary displacement of \( Q_2 \), so \( \text{Dis}(f) \) is surjective.

If \( x \) is an element of a quandle \( Q \), then the orbit of \( x \) in \( Q \) is the smallest subset that contains \( x \) and is preserved by \( \beta_y \) and \( \beta_y^{-1} \), for every \( y \in Q \).
Proposition 22. If \( x \in Q \), then the orbit of \( x \) in \( Q \) is \( \{d(x) \mid d \in \text{Dis}(Q)\} \).

Proof. A displacement is a composition of translations and their inverses, so the orbit of \( x \) includes \( d(x) \) for every displacement \( d \).

Now, suppose \( y \) is an element of the orbit of \( x \). Then there are \( y_1, \ldots, y_n \in Q \) and \( \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\} \) such that \( y = \beta_{y_n}^\epsilon_n \cdots \beta_{y_1}^\epsilon_1(x) \). For \( 1 \leq i \leq n \), let \( x_i = \beta_{y_i}^\epsilon_i \cdots \beta_{y_1}^\epsilon_1(x) \). Then \( \beta_{x_i}(x_i) = \beta_{x_i}^{-1}(x_i) = x_i \) for every \( i \), so the function
\[
d = (\beta_{x_n}^{-\epsilon_n} \beta_{y_n}^\epsilon_n)(\beta_{x_{n-1}}^{-\epsilon_{n-1}} \beta_{y_{n-1}}^\epsilon_{n-1}) \cdots (\beta_{x_1}^{-\epsilon_1} \beta_{y_1}^\epsilon_1)
\]
has \( y = d(x) \). Each product \( \beta_{x_i}^{-\epsilon_i} \beta_{y_i}^\epsilon_i \) is an elementary displacement, so \( d \in \text{Dis}(Q) \).

Definition 23. A quandle is semiregular if the only displacement with a fixed point is the identity map.

According to Proposition 22, if \( Q \) is semiregular then for each \( x \in Q \), the map \( d \mapsto d(x) \) is a bijection from \( \text{Dis}(Q) \) to the orbit of \( x \) in \( Q \).

Definition 24. A quandle is medial if it has the property that \( (w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z) \) \( \forall w, x, y, z \in Q \).

Proposition 25. These three properties of a quandle \( Q \) are equivalent.

1. \( Q \) is medial.
2. \( \beta_q \beta_r^{-1} \beta_s = \beta_s \beta_r^{-1} \beta_q \) \( \forall q, r, s \in Q \).
3. \( \text{Dis}(Q) \) is an abelian group.

Proof. Suppose first that \( Q \) is medial. Recall that as mentioned in the third paragraph of this section, if \( q, r, s \in Q \) then the formula \( \beta_{q,r} = \beta_{r,q} = \beta_r \beta_q \beta_r^{-1} \) holds. Also, if \( q, r, s \in Q \) then for every \( x \in Q \),
\[
\beta_{q,r}(\beta_s(x)) = (x \triangleright s) \triangleright (q \triangleright r) = (x \triangleright q) \triangleright (s \triangleright r) = \beta_{s,q,r}(\beta_q(x)),
\]
so \( \beta_{q,r} \beta_s = \beta_{s,q,r} \beta_q \). Using the first formula twice, we obtain the second property:
\[
\beta_{q,r}^{-1} \beta_s = \beta_r^{-1} \cdot \beta_r \beta_q \beta_r^{-1} \beta_s = \beta_r^{-1} \cdot \beta_{q,r} \beta_s
\]
\[
= \beta_r^{-1} \cdot \beta_{s,q,r} \beta_q = \beta_r^{-1} \cdot \beta_r \beta_s \beta_r^{-1} \beta_q = \beta_s \beta_r^{-1} \beta_q.
\]

Now, suppose the second property holds. Let \( \beta_q \beta_r^{-1} \) and \( \beta_r \beta_t^{-1} \) be elementary displacements of \( Q \). Then using the second property twice,
\[
(\beta_q \beta_r^{-1})(\beta_s \beta_t^{-1}) = (\beta_q \beta_r^{-1} \beta_s) \beta_t^{-1} = (\beta_s \beta_r^{-1} \beta_q) \beta_t^{-1} = \beta_s(\beta_r^{-1} \beta_q \beta_t^{-1})
\]
\[
= \beta_s(\beta_q \beta_r^{-1} \beta_t)^{-1} = \beta_s(\beta_r \beta_q^{-1} \beta_t)^{-1} = \beta_s(\beta_r^{-1} \beta_q \beta_t^{-1}) = (\beta_s \beta_r^{-1})(\beta_q \beta_t^{-1}).
\]
That is, the elementary displacements \( \beta_q \beta_r^{-1} \) and \( \beta_r \beta_t^{-1} \) commute with each other. As the group \( \text{Dis}(Q) \) is generated by elementary displacements, it follows that \( \text{Dis}(Q) \) is abelian.
For the last part of the proof, suppose \( \text{Dis}(Q) \) is abelian. Let \( a, b, c, d \in Q \), and let \( x = \beta_d^{-1}\beta_b(a) \). Then using part 3 of Definition 11 twice,

\[
(a \triangleright b) \triangleright (c \triangleright d) = (x \triangleright d) \triangleright (c \triangleright d) = (x \triangleright c) \triangleright d = \beta_d\beta_c\beta_d^{-1}\beta_b(a)
\]

\[
= \beta_d(\beta_c\beta_d^{-1})(\beta_b\beta_d^{-1}\beta_d(a)) = \beta_d(\beta_b\beta_d^{-1})(\beta_c\beta_d^{-1})\beta_d(a) = \beta_d\beta_b\beta_d^{-1}\beta_c(a)
\]

\[
= (\beta_d^{-1}(a \triangleright c) \triangleright b) \triangleright d = (\beta_d^{-1}(a \triangleright c) \triangleright d) \triangleright (b \triangleright d) = (a \triangleright c) \triangleright (b \triangleright d),
\]

so \( Q \) is medial.

**Corollary 26.** Let \( Q \) be a medial quandle, and let \( q^* \) be a fixed element of \( Q \). Then \( \text{Dis}(Q) \) is a \( \Lambda \)-module, with scalar multiplication given by

\[
t \cdot d = \beta_{q^*} d \beta_{q^{-1}}^{-1} \forall d \in \text{Dis}(Q).
\]

Changing the choice of \( q^* \) does not change the \( \Lambda \)-module structure of \( \text{Dis}(Q) \).

**Proof.** If \( A \) is an abelian group with an automorphism \( \alpha : A \rightarrow A \), we obtain a \( \Lambda \)-module structure on \( A \) by setting \( t \cdot a = \alpha(a) \forall a \in A \). The first assertion follows, because conjugation by \( \beta_{q^*} \) is an automorphism of \( \text{Aut}(Q) \), and it defines an automorphism of the normal abelian subgroup \( \text{Dis}(Q) \subset \text{Aut}(Q) \).

Suppose \( q^* \) and \( q^{**} \) are fixed elements of \( Q \). As \( \text{Dis}(Q) \) is abelian, we have

\[
\beta_{q^*} d \beta_{q^{-1}}^{-1} = \beta_{q^*} (\beta_{q^{**}}^{-1} \beta_{q^*}) (\beta_{q^{**}}^{-1} \beta_{q^*}) d \beta_{q^{-1}}^{-1} = \beta_{q^{**}} (\beta_{q^{**}}^{-1} \beta_{q^*}) d (\beta_{q^{**}}^{-1} \beta_{q^*}) \beta_{q^{-1}}^{-1}
\]

\[
= \beta_{q^{**}} d \beta_{q^{-1}}^{-1}
\]

for every \( d \in \text{Dis}(Q) \).

In the proof of the sixth part of Proposition 21, we showed that if \( f : Q_1 \rightarrow Q_2 \) is a surjective quandle map, then \( f \) induces a well-defined function mapping products of translations of \( Q_1 \) to products of translations of \( Q_2 \), given by

\[
\prod_{i=1}^{n} \beta_{y_i}^{m_i} \rightarrow \prod_{i=1}^{n} \beta_{f(y_i)}^{m_i}.
\]

If \( Q_1 \) and \( Q_2 \) are medial quandles, then the induced function \( \text{Dis}(f) : \text{Dis}(Q_1) \rightarrow \text{Dis}(Q_2) \) is not only a homomorphism of abelian groups. Corollary 26 defines scalar multiplication in \( \text{Dis}(Q_1) \) and \( \text{Dis}(Q_2) \) using multiplication of translations, so \( \text{Dis}(f) \) is actually a homomorphism of \( \Lambda \)-modules.

Recall that as discussed in Sec. 3 if \( M \) is a \( \Lambda \)-module then the standard Alexander quandle on \( M \) is given by the operation \( x \triangleright y = tx + (1-t)y \).

**Proposition 27.** If \( M \) is a \( \Lambda \)-module, then the standard Alexander quandle on \( M \) is a semiregular medial quandle. Also, \( \text{Dis}(M) \cong (1-t)M \) as \( \Lambda \)-modules.
Proof. It is easy to see that $M$ is a quandle, with $\triangleright^{-1}$ given by $x \triangleright^{-1} y = t^{-1}x + (1-t^{-1})y$. The medial property is verified at the beginning of Sec. 3.

Notice that if $y, z \in M$ then for every $x \in M$,

$$
\beta_y \beta_z^{-1}(x) = \beta_y(t^{-1}x + (1-t^{-1})z) = t(t^{-1}x + (1-t^{-1})z) + (1-t)y = x + (1-t)(y - z).
$$

It follows that there is a well-defined function $g : (1-t)M \to \text{Dis}(M)$, with $g((1-t)m)$ being the displacement given by $g((1-t)m)(x) = x + (1-t)m$. (That is, $g((1-t)m) = \beta_m \beta_0^{-1}$.) It is obvious that $g$ is injective, as $(1-t)m = g((1-t)m)(0)$.

If $m_1, m_2 \in M$ then $g((1-t)m_1 + (1-t)m_2)$ is the function with

$$
g((1-t)m_1 + (1-t)m_2)(x) = x + (1-t)m_1 + (1-t)m_2
$$

so $g((1-t)m_1 + (1-t)m_2) = g((1-t)m_1) \circ g((1-t)m_2)$. That is, $g$ is a homomorphism of abelian groups.

Moreover, the image of $g$ contains every elementary displacement, because $\beta_y \beta_z^{-1} = g((1-t)(y - z))$. It follows that $g$ is surjective.

Choose a fixed element $q^* \in M$, and define a $\Lambda$-module structure on $\text{Dis}(M)$ using $t \cdot d = \beta_q \cdot d \beta_q^{-1}$, as in Corollary 26. Then for all $m, x \in M$,

$$
g(t \cdot (1-t)m)(x) = t(1-t)m + x = x + (t-1)q^* + t(1-t)m + (1-t)q^*
$$

$$
= (t^{-1} \cdot (x + (t-1)q^* + t(1-t)m)) \triangleright^{-1} q^* = \beta_q(t^{-1} \cdot (x + (t-1)q^*) + (1-t)m)
$$

$$
= (\beta_q \circ g((1-t)m))(t^{-1} \cdot (x + (t-1)q^*)) = (\beta_q \circ g((1-t)m))(x \triangleright^{-1} q^*)
$$

$$
= (\beta_q \circ g((1-t)m) \circ \beta_q^{-1})(x) = (t \cdot g((1-t)m))(x).
$$

Hence $g(t \cdot (1-t)m) = t \cdot g((1-t)m)$, so $g$ is an isomorphism of $\Lambda$-modules.

To verify semiregularity of $M$, suppose $d \in \text{Dis}(M)$. Then $d = g((1-t)m)$ for some $m \in M$. If there is an $x \in M$ with $x = d(x)$, then $x = g((1-t)m)(x) = x + (1-t)m$, so $(1-t)m = 0$. Therefore $d = g(0)$ is the identity map of $M$.

Notice that if $M$ is a standard Alexander quandle then the surjectivity of the map $g$ used in the proof of Proposition 27 implies that every displacement of $M$ is of the form $g((1-t)m) = \beta_m \beta_0^{-1}$ for some $m \in M$. In the terminology of Jedlička, Pilitowska, Stanovský and Zamojska-Dzienio, standard Alexander modules have “tiny” displacement groups.

Standard Alexander quandles are the building blocks of medial quandles.

**Proposition 28.** Let $x$ be an element of a medial quandle $Q$, and let $Q_x$ be the orbit of $x$ in $Q$. Then $\text{Fix}(x) = \{d \in \text{Dis}(Q) \mid d(x) = x\}$ is a $\Lambda$-submodule of $\text{Dis}(Q)$, and $Q_x$ is isomorphic, as a quandle, to the standard Alexander quandle on the quotient module $\text{Dis}(Q)/\text{Fix}(x)$. A quandle isomorphism $e : \text{Dis}(Q)/\text{Fix}(x) \to Q_x$ is given by $e(d + \text{Fix}(x)) = d(x).$
Proof. It is easy to see that Fix(x) is a subgroup of Dis(Q). According to Corollary 26 if \( d \in \text{Fix}(x) \) then \((t \cdot d)(x) = \beta_x d \beta_x^{-1}(x) = \beta_x d(x) = \beta_x(x) = x \) and \((t^{-1} \cdot d)(x) = \beta_x^{-1} d \beta_x(x) = \beta_x^{-1} d(x) = \beta_x^{-1}(x) = x \). Therefore Fix(x) is closed under scalar multiplication, so Fix(x) is a \( \Lambda \)-submodule of Dis(Q).

According to Proposition 22 there is a surjection mapping Dis(Q) onto \( Q_x \), given by \( d \mapsto d(x) \). This surjection induces a bijection \( e : \text{Dis}(Q)/\text{Fix}(x) \) onto \( Q_x \). To verify that \( e \) is an isomorphism of quandles, notice that if \( d_1, d_2 \in \text{Dis}(Q) \) then according to Corollary 26

\[
e((d_1 + \text{Fix}(x)) \triangleright (d_2 + \text{Fix}(x))) = e(t \cdot d_1 + (1-t) \cdot d_2 + \text{Fix}(x))
\]

\[
= (t \cdot d_1 + (1-t) \cdot d_2)(x) = (d_2 + t \cdot (-d_2) + t \cdot d_1)(x)
\]

\[
= d_2(t \cdot d_2^{-1})(t \cdot d_1)(x) = d_2(\beta_x d_2^{-1} \beta_x^{-1})(\beta_x d_1 \beta_x^{-1})(x)
\]

\[
= d_2 \beta_x d_2^{-1} d_1 \beta_x^{-1}(x) = d_2 \beta_x d_2^{-1} d_1(x).
\]

According to property 4 of Proposition 21 it follows that

\[
e((d_1 + \text{Fix}(x)) \triangleright (d_2 + \text{Fix}(x))) = \beta_{d_2(x)} d_1(x) = d_1(x) \triangleright d_2(x)
\]

\[
= e(d_1 + \text{Fix}(x)) \triangleright e(d_2 + \text{Fix}(x)).
\]

A generalization of Corollary 10 follows.

Corollary 29. If Q is a medial quandle with only one orbit, then Q is isomorphic to the standard Alexander quandle on the \( \Lambda \)-module \( \text{Dis}(Q) \). For any \( x_0 \in Q \), a quandle isomorphism \( e : \text{Dis}(Q) \rightarrow Q \) is given by \( e(d) = d(x_0) \).

Moreover, if Q is finitely generated, then scalar multiplication by \( 1-t \) defines an automorphism of \( \text{Dis}(Q) \).

Proof. Suppose \( x_0 \in Q \), and \( d_0 \in \text{Fix}(x_0) \). If x is any element of Q then as Q has only one orbit, there is a \( d \in \text{Dis}(Q) \) with \( x = d(x_0) \). The group \( \text{Dis}(Q) \) is abelian, so it follows that \( d_0(x_0) = d_0(d(x_0)) = d(d_0(x_0)) = d(x_0) = x \). That is, the only element of \( \text{Fix}(x_0) \) is the identity map of Q. Proposition 28 tells us that the map e is a quandle isomorphism.

It follows that the \( \Lambda \)-module \( \text{Dis}(Q) \) has only one orbit in its standard Alexander quandle. According to the formulas \( x \triangleright y = tx + (1-t)y \) and \( x \triangleright^{-1} y = t^{-1}x + (1-t^{-1})y \), the orbit of 0 is contained in the submodule \( (1-t) \cdot \text{Dis}(Q) \), so this submodule must be all of \( \text{Dis}(Q) \). That is, scalar multiplication by \( 1-t \) defines a surjective \( \Lambda \)-linear endomorphism \( \text{Dis}(Q) \rightarrow \text{Dis}(Q) \).

Using the equality \( \beta_x \beta_y \beta_x^{-1} = \beta_{\beta_x(y)} \) mentioned in the third paragraph of this section, it is easy to see that if \( F \) is a finite generating set for the quandle \( Q \), then the \( \Lambda \)-module \( \text{Dis}(Q) \) is generated by the elementary displacements \( \beta_{f_1} \beta_{f_2}^{-1} \) with \( f_1, f_2 \in F \). As \( \Lambda \) is a Noetherian ring, it follows that a surjective endomorphism of \( \text{Dis}(Q) \) must be an isomorphism.

Another consequence of Proposition 27 is the following.
Corollary 30. If \( Q \) is a subquandle of a standard Alexander quandle \( M \), then \( Q \) is medial and semiregular, and \( \text{Dis}(Q) \) is isomorphic to the \( \Lambda \)-submodule of \( (1-t)M \) generated by \( \{(1-t)(q-q') \mid q, q' \in Q\} \).

Proof. Subquandles inherit both the medial and semiregularity properties.

Each translation \( \beta_q \) of \( Q \) extends to the corresponding translation \( \beta_q \) of \( M \). This obvious correspondence provides a homomorphism \( \text{ext} : \text{Dis}(Q) \to \text{Dis}(M) \), defined by \( \text{ext}(\prod \beta_q^{m_i}) = \prod \beta_q^{m_i} \). Semiregularity implies that \( \text{ext} \) is well-defined, and it is obviously injective. If \( g : (1-t)M \to \text{Dis}(M) \) is the isomorphism that appears in the proof of Proposition [27], then the composition \( g^{-1} \circ \text{ext} \) maps a displacement \( \prod \beta_q^{m_i} \in \text{Dis}(Q) \) to \( (1-t)(\sum m_i q_i) \). Proposition [21] tells us that \( \sum m_i = 0 \); it follows that \( (1-t)(\sum m_i q_i) \) is an element of the \( \Lambda \)-submodule of \( (1-t)M \) generated by \( \{(1-t)(q-q') \mid q, q' \in Q\} \).

\[ \square \]

6 The Fundamental Medial Quandle of a Link

Here is the definition of the fundamental medial quandle of a link, \( \text{AbQ}(L) \) in Joyce’s notation [6].

Definition 31. If \( D \) is a diagram of a link \( L \), then \( \text{MQ}(L) \) is the medial quandle generated by the set \( \{q_a \mid a \in A(D)\} \), subject to the requirement that at each crossing \( c \in C(D) \) as pictured in Fig. 4, \( q_{a_2} > q_{a_1} = q_{a_1} \).

Part of Theorem 7 follows readily from Definition 31 and results of Sec. 3.

Proposition 32. Suppose \( L \) and \( L' \) are links with \( \text{MQ}(L) \cong \text{MQ}(L') \). Then \( \text{Q}_A^\text{red}(L) \cong \text{Q}_A^\text{red}(L') \).

Proof. The definition of the fundamental quandle \( Q(L) \) is the same as Definition 31 but with the word “medial” removed. Therefore, there is a surjective quandle map \( Q(L) \to \text{MQ}(L) \), under which the image of the \( q_a \) element of \( Q(L) \) is the \( q_a \) element of \( \text{MQ}(L) \), for each \( a \in A(D) \).

Also, if we replace the phrase “the medial quandle” in Definition 31 with “a medial quandle,” and we replace each occurrence of \( q_a \) or \( q_{a_1} \) with \( \gamma_D(a) \otimes 1 \) or \( \gamma_D(a_1) \otimes 1 \), then the resulting sentence is true of \( \text{Q}_A^\text{red}(L) \). Therefore, there is a surjective quandle map \( \text{MQ}(L) \to \text{Q}_A^\text{red}(L) \), under which \( q_a \mapsto \gamma_D(a) \otimes 1 \) \( \forall a \in A(D) \).

Theorem 16 holds for both \( Q = Q(L) \) and \( Q = \text{Q}_A^\text{red}(L) \), so it follows that Theorem 16 also holds for \( Q = \text{MQ}(L) \). As discussed after Theorem 16, we deduce that if \( L \) and \( L' \) are links and \( f : \text{MQ}(L) \to \text{MQ}(L') \) is a quandle isomorphism, then after adjusting component indices in \( L \) and \( L' \) so that \( f \) matches quandle orbits corresponding to link components with the same indices, \( f \) will define an isomorphism \( \text{M}_A^\text{red}(L) \cong \text{M}_A^\text{red}(L') \) that is compatible with the \( \phi_e \) maps of \( L \) and \( L' \). That is, \( L \) and \( L' \) will be \( \phi_e \)-equivalent. Then Theorem 4 tells us that \( \text{Q}_A^\text{red}(L) \cong \text{Q}_A^\text{red}(L') \).

In addition to what is stated in Proposition 32, Theorem 7 asserts that the converse of Proposition 32 holds for knots, and fails in general. Before
verifying these assertions, we present some results that will help us describe the translations and displacements of MQ(L).

**Proposition 33.** MQ(L) has $\mu$ orbits, one for each component of L.

**Proof.** The proposition follows from the corresponding properties of $Q(L)$ and $Q_{\text{red}}(L)$, because of the surjective quandle maps $Q(L) \rightarrow MQ(L)$ and $MQ(L) \rightarrow Q_{\text{red}}^A(L)$ mentioned in the proof of Proposition 32.

A very useful notion discussed by Joyce [6, Sec. 9] involves describing a quandle through an “augmentation,” i.e. a group action. For MQ(L), an appropriate group is defined as follows.

**Definition 34.** Let $D$ be a diagram of a link L. Then the medial group $MG(L)$ is generated by $\{g_a | a \in A(D)\}$, with two types of relations.

1. If $a_1, a_2, a_3$ are any elements of $A(D)$ and $x_1, x_2, x_3 \in MQ(L)$ are conjugates of $g_{a_1}, g_{a_2}, g_{a_3}$ (respectively), then $x_1 x_2^{-1} x_3 = x_3 x_2^{-1} x_1$.

2. If $a_1, a_2, a_3$ are arcs appearing at a crossing of $D$ as pictured in Fig. 7, then $g_{a_1} g_{a_2} g_{a_1}^{-1} = g_{a_3}$.

**Proposition 35.** There is a homomorphism $\beta : MQ(L) \rightarrow Aut(MQ(L))$ with $\beta(g_a) = \beta_{g_a}$ $\forall a \in A(D)$.

**Proof.** We must show that the $\beta$ maps of MQ(L) satisfy the two kinds of relations given in Definition 34. The relations of type 1 follow from the medial property of MQ(L); see Proposition 25. For the relations of type 2, suppose $a_1, a_2, a_3$ are arcs appearing at a crossing of $D$ as pictured in Fig. 7. Then $q_{a_3} = q_{a_2} \triangleright q_{a_1} = \beta_{q_{a_2}}(q_{a_1})$ and as noted in the third paragraph of Sec. 5, it follows that $\beta_{q_{a_3}} = \beta_{q_{a_1}} \beta_{q_{a_2}} \beta_{q_{a_1}}^{-1}$.

Two subsets of $MG(L)$ will be particularly important for us.

**Definition 36.** Let $QM(L)$ be the subset of $MG(L)$ that includes all conjugates of elements $g_a$, where $a \in A(D)$.

**Definition 37.** Let $MG^0(L)$ be the subset of $MG(L)$ consisting of all products $g_{a_1}^{n_1} \cdots g_{a_k}^{n_k}$ such that $a_1, \ldots, a_k \in A(D)$, $n_1, \ldots, n_k \in \mathbb{Z}$ and $\sum n_i = 0$.

The next three results show that $QM(L)$ is a semiregular medial quandle under conjugation. Later, we will see that $QM(L)$ is isomorphic to $Q_{\text{red}}^A(L)$.

**Proposition 38.** $QM(L)$ is a medial quandle under conjugation: if $x, y \in QM(L)$, then $x \triangleright y = yxy^{-1}$.

**Proof.** The first two defining properties of a quandle – $x \triangleright x = x \forall x$ and the fact that for each $y$, $\beta_y(x) = x \triangleright y$ defines a permutation – follow from elementary properties of conjugation in groups. For the third defining property, we have

$$(x \triangleright z) \triangleright (y \triangleright z) = zy z^{-1} z x z^{-1} (zy z^{-1})^{-1} = zy \cdot x z^{-1} z y^{-1} z^{-1}$$
The medial property holds in QMG\(L\) because
\[
(w \triangleright x) \triangleright (y \triangleright z) = zyx^{-1}w(x^{-1}yz^{-1})^{-1} = z(y^{-1}x)w(x^{-1}zy^{-1})^{-1} = zyx^{-1}w(yz^{-1}x)^{-1}z^{-1} = zyx^{-1}w(yz^{-1}x)^{-1}z^{-1} = zyx^{-1}w(yz^{-1}x)^{-1}z^{-1} = (w \triangleright y) \triangleright (x \triangleright z).
\]
\[\square\]

**Lemma 39.** Suppose \(n \geq 3\) is an odd integer, and \(y_1, \ldots, y_n \in QMG(L)\). Then \(y_1y_2^{-1}y_3 \cdots y_{n-1}^{-1}y_n = y_ny_{n-1}^{-1}y_{n-2} \cdots y_2^{-1}y_1\) in MG\(L\).  

*Proof.* If \(n = 3\), the assertion of the lemma is the same as part 1 of Definition 34. Note that the \(n = 3\) case implies \(y_1^{-1}y_2y_3^{-1} = (y_3y_2^{-1}y_1)^{-1} = (y_1y_2^{-1}y_3)^{-1} = y_3^{-1}y_2y_1\). If \(n > 3\), we have
\[
y_1y_2^{-1}y_3 \cdots y_{n-1}^{-1}y_n = y_3^{-1}y_2^{-1}y_1y_4^{-1}y_5 \cdots y_{n-1}^{-1}y_n = \cdots = y_3^{-1}y_2^{-1}y_1y_4^{-1}y_5 \cdots y_n^{-1}y_{n-1}y_1 = \cdots = y_3^{-1}y_2^{-1}y_1 \cdots y_n^{-1}y_{n-1}y_1.
\]
The assertion of the lemma follows, by applying an inductive hypothesis to \(y_3^{-1}y_2^{-1}y_1 \cdots y_n^{-1}y_n\).  \[\square\]

**Proposition 40.** QMG\(L\) is semiregular.

*Proof.* Let \(d \in \text{Dis}(QMG(L))\). Proposition 21 implies \(d = \beta_{y_1}^{-1} \beta_{y_2}^{-1} \cdots \beta_{y_{n-1}}^{-1} \beta_{y_2^{-1} y_1^{-1}}\) for some \(y_1, \ldots, y_{2n} \in QMG(L)\). As the quandle operation of QMG\(L\) is given by \(\beta_z(x) = zx^{-1}\), it follows that \(d\) is defined by conjugation by \(y = y_1y_2^{-1} \cdots y_{2n-1}y_2^{-1}\).

If \(d(x_0) = x_0\), then according to Lemma 39,
\[
1 = x_0^{-1}d(x_0) = x_0^{-1}y_1^{-1}y_2y_3^{-1} \cdots y_{n-1}^{-1}y_n^{-1}x_0 \cdot y_n^{-1}y_{n-1}^{-1}y_{n-2}^{-1} \cdots y_2^{-1}y_1^{-1} = x_0^{-1} \cdot (x_0y_2^{-1}y_{2n-1} \cdots y_2^{-1}y_1) \cdot y_n^{-1}y_{n-1}^{-1}y_{n-2}^{-1} \cdots y_2^{-1}y_1^{-1} = y_2^{-1}y_n^{-1}y_{n-1}^{-1}y_1 \cdot y_n^{-1}y_{n-1}^{-1}y_{n-2}^{-1} \cdots y_2^{-1}y_1^{-1}.
\]
Using Lemma 39 again, we deduce that for every \(x \in QMG(L)\),
\[
d(x) = y_1y_2^{-1} \cdots y_{2n-1}y_2^{-1} x \cdot y_n^{-1}y_{n-1}^{-1}y_{n-2}^{-1} \cdots y_2^{-1}y_1^{-1} = x \cdot y_2^{-1}y_{2n-1} \cdots y_2^{-1}y_1 \cdot y_n^{-1}y_{n-1}^{-1}y_{n-2}^{-1} \cdots y_2^{-1}y_1^{-1} = x \cdot 1 = x.
\]
That is: if \(d\) has a fixed point, then \(d\) is the identity map.  \[\square\]

Now we turn our attention to MG\(^0\)(\(L\)).

**Proposition 41.** MG\(^0\)(\(L\)) is the subgroup of MG\(L\) generated by products \(xy^{-1}\) such that \(x, y \in QMG(L)\).
Then certainly
\[ H \subseteq \text{MG}(L). \] We claim that \( \text{MG}(L) \subseteq H. \)

Let \( x \in \text{MG}(L) \), and let \( \ell(x) \) be the smallest integer \( \ell \) such that there exist \( x_1, \ldots, x_\ell \in \text{QM}(L) \) and \( \epsilon_1, \ldots, \epsilon_\ell \in \{-1, 1\} \) with \( \prod x_i^{\epsilon_i} = x \) and \( \sum \epsilon_i = 0 \). If \( \ell(x) = 0 \), then \( x = 1 \in H \). The argument proceeds using induction on \( \ell(x) \geq 2 \).

Case 1. If \( \epsilon_1 = 1 \neq \epsilon_2 \), then \( y = x_1x_2^{-1} \in H \), and the inductive hypothesis implies that \( z = \prod_{i=3}^{\ell(x)} x_i^{\epsilon_i} \in H \). It follows that \( x = yz \in H \).

Case 2. If \( \epsilon_1 = -1 \neq \epsilon_2 \), then \( y = x_1^{-1}x_2 = (x_1^{-1}x_2x_1)x_1^{-1} \in H \). Again, the inductive hypothesis implies that \( z = \prod_{i=3}^{\ell(x)} x_i^{\epsilon_i} \in H \), and hence \( x = yz \in H \).

Case 3. Suppose \( \epsilon_1 = \epsilon_2, \) and \( j > 2 \) is the smallest integer with \( \epsilon_j \neq \epsilon_1 \). Let \( g = \prod_{i=1}^{j-1} x_i^{\epsilon_i} \). Then

\[
x = \prod_{i=1}^{\ell(x)} x_i^{\epsilon_i} = gx_j^{\epsilon_j} \cdot g^{-1} \cdot \prod_{i=j+1}^{\ell(x)} x_i^{\epsilon_i} = (gx_j g^{-1})^{\epsilon_j} \cdot x_1^{\epsilon_1} \cdot \prod_{i=2}^{j-1} x_i^{\epsilon_i} \cdot \prod_{i=j+1}^{\ell(x)} x_i^{\epsilon_i},
\]

and since \( \epsilon_1 \neq \epsilon_j \), the latter product falls under case 1 or case 2. \( \square \)

**Theorem 42.** The following properties hold for \( \text{MG}(L) \).

1. \( \text{MG}(L) \) is a normal subgroup of \( \text{MG}(L) \).

2. The map \( \beta : \text{MG}(L) \to \text{Aut}(\text{MQ}(L)) \) has \( \beta(\text{MG}(L)) = \text{Dis}(\text{MQ}(L)) \).

3. \( \text{MG}(L) \) is commutative.

4. Let \( z^* \) be a fixed element of \( \text{QM}(L) \). Then \( \text{MG}(L) \) is a \( \Lambda \)-module, with addition given by multiplication in \( \text{MG}(L) \) and scalar multiplication given by

\[
t \cdot x = z^* x (z^*)^{-1} \forall x \in \text{MG}(L).
\]

5. The module structure on \( \text{MG}(L) \) is independent of the choice of \( z^* \). That is: if \( z^*, z^{**} \in \text{QM}(L) \) then \( z^* x (z^*)^{-1} = z^{**} x (z^{**})^{-1} \forall x \in \text{MG}(L) \).

6. Let \( a^* \in A(D) \) be a fixed element. Then \( \text{MG}(L) \) is generated, as a \( \Lambda \)-module, by the elements \( h_a = g_a g_0^{-1} \) with \( a \in A(D) \).

7. For any crossing of \( D \) as pictured in Fig. 7, \( h_a = (1-t)h_{a_1} + th_{a_2} \).

**Proof.** The first property follows immediately from Definition 37 and the second follows from the third property mentioned in Proposition 21.

Proposition 41 tells us that \( \text{MG}(L) \) is generated by elements of the form \( xy^{-1} \) with \( x, y \in \text{QM}(L) \). To verify commutativity of \( \text{MG}(L) \), then, it suffices to show that such elements commute:

\[
(vw^{-1})(xy^{-1}) = v(w^{-1}xy^{-1}) = v(yx^{-1}w)^{-1} = v(wx^{-1}y)^{-1} = (vy^{-1}x)w^{-1} = (xy^{-1}v)w^{-1} = (xy^{-1})(vw^{-1})
\]
The fourth property follows from commutativity of $\text{MG}^0(L)$ and the fact that
conjugation by $x^*$ defines an automorphism of the normal subgroup $\text{MG}^0(L) \subset \text{MG}(L)$. For the fifth property, note that if $z^*, z^{**} \in \text{QMG}(L)$ then $z^* (z^{**})^{-1} \in \text{MG}^0(L)$, so if $x \in \text{MG}^0(L)$ then
\[
z^* x (z^*)^{-1} = z^* (z^{**})^{-1} \cdot z^{**} x (z^{**})^{-1} = z^{**} x (z^{**})^{-1}.
\]

To verify the sixth property, let $S$ be the $A$-submodule of $\text{MG}^0(L)$ generated
by $\{h_a \mid a \in A(D)\}$. We claim that if $a_1, \ldots, a_{2n} \in A(D), \epsilon_1, \ldots, \epsilon_{2n} \in \{-1, 1\}$
and $\sum \epsilon_i = 0$, then $x = g_{a_1}^{\epsilon_1} \ldots g_{a_{2n}}^{\epsilon_{2n}} \in S$.

Before presenting the argument, we should mention that the operation $\text{MG}^0(L)$
can be written using either additive or multiplicative notation. Additive notation is more natural when we think of $\text{MG}^0(L)$ as a $A$-module, and
multiplicative notation is more natural when we think of $\text{MG}^0(L)$ as a subgroup
of $\text{MG}(L)$. For example, we can write
\[
g_{a_1} g_{a_2}^{-1} g_{a_3}^{-1} g_{a_4} = g_{a_1} g_{a_2}^{-1} + g_{a_3}^{-1} g_{a_4} = g_{a_1} g_{a_2}^{-1} - g_{a_3} g_{a_4}^{-1}.
\]

However, the expression $g_{a_1} g_{a_2}^{-1} g_{a_3}^{-1} + g_{a_4}$ is meaningless, because $+$ is not used
for the group operation in $\text{MG}(L)$ outside of $\text{MG}^0(L)$. As usual, the scalar
multiplication operation in a module is written using $\cdot$ or juxtaposition.

Returning to the claim, let $m$ be the number of occurrences of arcs $a_i \neq a^*$
in the list $a_1, \ldots, a_{2n}$. If $m = 0$, then $x = g_{a_1}^{\epsilon_1} \ldots g_{a_{2n}}^{\epsilon_{2n}} = (g_{a^*})^0$ is the identity
element of $\text{MG}^0(L)$. The identity element is included in every submodule.

The argument proceeds using induction on $m > 0$. If $a_1 = a^*$, then
\[
t^{-\epsilon_1} \cdot x = t^{-\epsilon_1} \cdot g_{a_1}^{\epsilon_1} \cdot g_{a_2}^{\epsilon_2} \ldots g_{a_{2n}}^{\epsilon_{2n}} = g_{a^*}^{-\epsilon_1} g_{a_1}^{\epsilon_1} \cdot g_{a_2}^{\epsilon_2} \ldots g_{a_{2n}}^{\epsilon_{2n}} = g_{a_2}^{\epsilon_2} \ldots g_{a_{2n}}^{\epsilon_{2n}} g_{a_1}^{\epsilon_1}.
\]

Repeating this as many times as necessary, we will ultimately obtain a scalar $t^k \cdot x$ which is equal to a product of elements of the form $g_{a_i}^{\epsilon_i}$, in which
the first term involves an arc other than $a^*$. As $t^k \cdot x \in S$ if and only if $x \in S$,
we may as well assume that $x = g_{a_1}^{\epsilon_1} \ldots g_{a_{2n}}^{\epsilon_{2n}}$ and $a_1 \neq a^*$.

If $a_1 \neq a^*$ and $\epsilon_1 = 1$, then
\[
-h_{a_1} + x = h_{a_1}^{-1} x = g_{a^*} g_{a_1}^{-1} g_{a_1}^{\epsilon_1} \ldots g_{a_{2n}}^{\epsilon_{2n}} = g_{a^*} g_{a_2}^{\epsilon_2} \ldots g_{a_{2n}}^{\epsilon_{2n}}.
\]

As the latter product involves only $m - 1$ occurrences of arcs not equal to $a^*$,
the inductive hypothesis guarantees that $-h_{a_1} + x \in S$. Of course $h_{a_1} \in S$, so
it follows that $x \in S$ too.

Similarly, if $a_1 \neq a^*$ and $\epsilon_1 = -1$, then
\[
t^{-1} \cdot h_{a_1} + x = (t^{-1} \cdot h_{a_1}) \cdot x = g_{a^*}^{-1} g_{a_1} g_{a_1}^{-1} g_{a^*} \cdot g_{a_2}^{\epsilon_2} \ldots g_{a_{2n}}^{\epsilon_{2n}} = g_{a^*}^{-1} g_{a_2}^{\epsilon_2} \ldots g_{a_{2n}}^{\epsilon_{2n}},
\]

and the latter product involves only $m - 1$ occurrences of arcs not equal to $a^*$.

Once again, the inductive hypothesis applies, and it tells us that $t^{-1} \cdot h_{a_1} + x \in S$.
As $h_{a_1} \in S$, it follows that $x \in S$. 28
Turning to property 7, let \( c \in C(D) \) be a crossing as pictured in Fig. 1. Then \( g_a = g_{a_1}g_{a_2}g_{a_3}^{-1} \) in \( MG(L) \), so

\[
\begin{align*}
 h_{a_3} &= g_{a_3}g_{a_3}^{-1} = g_{a_3}g_{a_3}g_{a_3}^{-1}g_{a_3}^{-1} = (g_{a_3}g_{a_3}^{-1})(g_{a_3}g_{a_3}^{-1})g_{a_3}^{-1} \nonumber \\
 &= g_{a_3}g_{a_3}^{-1} + g_{a_3}g_{a_3}^{-1}g_{a_3}^{-1} = h_{a_3} + g_{a_3}^{-1}(g_{a_3}g_{a_3}^{-1})g_{a_3}^{-1} + (g_{a_3}g_{a_3}^{-1})g_{a_3}^{-1} \nonumber \\
 &= h_{a_3} + t \cdot h_{a_2} - g_{a_3}^{-1}(g_{a_3}g_{a_3}^{-1})g_{a_3}^{-1} = h_{a_3} + t \cdot h_{a_2} - t \cdot h_{a_3}. \nonumber
\end{align*}
\]

Recall that if \( D \) is a diagram of a link \( L \), then \( \phi^A_{\text{red}} : M\text{red}^A(L) \to A \) is the \( A \)-linear map with \( \phi^A_{\text{red}}(\gamma_D(a) \otimes 1) = 1 \forall a \in A(D) \). Of course, the kernel of \( \phi^A_{\text{red}} \) is the submodule of \( M\text{red}^A(L) \) generated by the elements \((\gamma_D(a) - \gamma_D(a^*)) \otimes 1 \) with \( a, a' \in A(D) \).

**Corollary 43.** In the situation of Theorem 42 there is a \( A \)-linear epimorphism \( e_D : \ker \phi^A_{\text{red}} \to MG^0(L) \) given by \( e_D((\gamma_D(a) - \gamma_D(a^*)) \otimes 1) = h_a \forall a \in A(D) \).

**Proof.** As discussed in Sec. 2, \( M\text{red}^A(L) \) is the \( A \)-module generated by the elements \( \gamma_D(a) \otimes 1 \) with \( a \in A(D) \), subject to the defining relations

\[
0 = \gamma_Dp_D(c) \otimes 1 = (1 - t)(\gamma_D(a_1) \otimes 1) + t(\gamma_D(a_2) \otimes 1) - (\gamma_D(a_3) \otimes 1)
\]

whenever \( c \in C(D) \) is a crossing as pictured in Fig. 1. It follows from property 7 of Theorem 42 that there is a \( A \)-linear map \( M\text{red}^A(L) \to MG^0(L) \) given by \( \gamma_D(a) \otimes 1 \mapsto h_a \forall a \in A(D) \). Restricting this map to \( \ker \phi^A_{\text{red}} \) yields \( e_D \). The stated formula \( e_D((\gamma_D(a) - \gamma_D(a^*)) \otimes 1) = h_a \) reflects the fact that \( h_{a^*} = 0 \) in \( MG^0(L) \).

The fact that \( e_D \) is surjective follows from property 6 of Theorem 42.

Notice that the fixed element \( a^* \in A(D) \) is helpful in stating a definition for \( e_D \), and in referencing Theorem 42 for the fact that \( e_D \) is surjective. But once we know \( e_D \) is well defined, we can calculate a formula for \( e_D \) that does not require \( a^* \): if \( a, a' \) are any elements of \( A(D) \), then

\[
\begin{align*}
 e_D((\gamma_D(a) - \gamma_D(a')) \otimes 1) &= e_D((\gamma_D(a) - \gamma_D(a^*)) \otimes 1) - e_D((\gamma_D(a') - \gamma_D(a^*)) \otimes 1) \nonumber \\
 &= h_a(g_{a'})^{-1}g_{a^*}(g_{a^*})^{-1} = g_a^{-1}. \quad (7)
\end{align*}
\]

We are now ready to prove the special assertion of Theorem 7 for knots.

**Corollary 44.** If \( \mu = 1 \), then \( MQ(L) \cong Q\text{red}^A(L) \).

**Proof.** As noted in the proof of Proposition 29 it follows from Definition 31 that for any link \( L \) there is a surjective quandle map \( MQ(L) \to Q\text{red}^A(L) \), with \( q_a \mapsto \gamma_D(a) \otimes 1 \forall a \in A(D) \). As observed after Corollary 26, this surjective quandle map induces a surjective \( A \)-linear map \( \text{Dis}(MQ(L)) \to \text{Dis}(Q\text{red}^A(L)) \).

Theorem 42 and Corollary 43 provide a surjective homomorphism \( \beta \phi_{\text{red}} : \ker \phi_{\text{red}} \to \text{Dis}(MQ(L)) \). The map \( e_D \) is \( A \)-linear, and comparing item 4 of
Theorem 42 to Corollary 26 we see that \( \beta : MG^0(L) \to \text{Dis}(MQ(L)) \) is \( \Lambda \)-linear too. Therefore \( \beta \in L : \text{ker} \phi_L^{red} \to \text{Dis}(MQ(L)) \) is surjective and \( \Lambda \)-linear.

Corollary 30 tells us that \( \text{Dis}(Q_A^{red}(L)) \cong (1-t) \cdot \text{ker} \phi_L^{red} \), and if \( \mu = 1 \), Corollary 10 tells us that \( (1-t) \cdot \text{ker} \phi_L^{red} = \text{ker} \phi_L^{red} \). Therefore, if \( \mu = 1 \) we have surjective \( \Lambda \)-linear maps \( \text{Dis}(MQ(L)) \to \text{Dis}(Q_A^{red}(L)) \) and \( \text{Dis}(Q_A^{red}(L)) \to \text{Dis}(MQ(L)) \). As the displacement groups are finitely generated \( \Lambda \)-modules and \( \Lambda \) is Noetherian, it follows that both surjections are isomorphisms. As each of \( \text{Dis}(MQ(L)) \) and \( \text{Dis}(Q_A^{red}(L)) \) is a medial quandle with only one orbit, Corollary 29 tells us that both \( \Lambda \)-module isomorphisms are also quandle isomorphisms.

\[ \square \]

7 Examples for Theorem 7

In this section we complete the proof of Theorem 7 by providing a pair of links distinguished by their MQ quandles but not by their \( Q_A^{red} \) quandles.

We begin with some examples of small medial quandles. Let \( \text{id} : Z \to Z \) be the identity map, and let \( s^+, s^- : Z \to Z \) be the two unit shift maps, given by \( s^+(n) = n \pm 1 \). Let \( Q \) be a set consisting of three disjoint copies of \( Z \), denoted \( Q_1, Q_2 \) and \( Q_3 \). Choose three triples \( t_1, t_2, t_3 \) in such a way that each triple \( t_i = (t_{i1}, t_{i2}, t_{i3}) \) has \( t_{ij} \in \{\text{id}, s^+, s^-\} \) \( \forall i, j \in \{1, 2, 3\} \), \( t_{ii} = \text{id} \) \( \forall i \in \{1, 2, 3\} \), and for each \( j \in \{1, 2, 3\} \), at least one \( t_{ij} \) is \( s^+ \) or \( s^- \). Use the triples \( t_i \) to define bijections \( \beta_i : Q \to Q_i \), with \( \beta_1 | Q_j = t_{ij} : Q_j \to Q_j \). Notice that \( \beta_i \beta_j = \beta_j \beta_i \) \( \forall i, j \in \{1, 2, 3\} \).

![Proof](image)

Lemma 45. Under these circumstances, \( Q \) is a medial quandle under the operation \( \triangleright \) defined by: if \( x \in Q_j \) and \( y \in Q_i \), then \( x \triangleright y = \beta_i(x) \). The quandle \( Q \) has three orbits, \( Q_1, Q_2 \) and \( Q_3 \).

Proof. If \( x \in Q_i \), then \( x \triangleright x = \beta_i(x) = t_{ii}(x) = \text{id}(x) = x \). Also, each \( \beta_i \) is a permutation of \( Q \). If \( x \in Q_k, y \in Q_j \) and \( z \in Q_i \), then

\[
(x \triangleright y) \triangleright z = \beta_i \beta_j(x) \beta_i(x) = \beta_j(x) \triangleright (y \triangleright z) = (x \triangleright z) \triangleright (y \triangleright z),
\]

so \( Q \) is a quandle.

As \( \beta_i \beta_j = \beta_j \beta_i \forall i, j \in \{1, 2, 3\} \), the permutations \( \beta_1, \beta_2, \beta_3 \) generate a commutative group \( G \) of permutations of \( Q \). Every \( \beta \) map of the quandle \( Q \) is one of the three permutations \( \beta_1, \beta_2, \beta_3 \), so \( \text{Dis}(Q) \) is a subgroup of \( G \). It follows that \( \text{Dis}(Q) \) is abelian, so \( Q \) is a medial quandle. The fact that the orbits of \( Q \) are \( Q_1, Q_2 \) and \( Q_3 \) follows from the hypothesis that for each \( j \), at least one \( t_{ij} \) is \( s^+ \) or \( s^- \). \[ \square \]

Let \( L \) be the link pictured in Fig. 4. Then \( MG(L) \) is generated by \( g_a, g_b, g_c \) and \( g_d \), with two types of defining relations: \( xy^{-1}z = zy^{-1}x \) for all conjugates of \( g_a, g_b, g_c \) and \( g_d \), and four crossing relations: \( gb = g_agbg_a^{-1}, ga = gbagb^{-1}, g_c = g_dg_c^{-1}, gb = gdg_c^{-1} \). Using the last relation to eliminate \( gb \), we are left with generators \( g_a, g_c \) and \( g_d \), and relations \( ga = ga^{-1}, g_c = g_dg_c^{-1}, g_d = g_d^{-1}g_ag_d^{-1}g_d^{-1} \).
The third relation implies that \( g_c \) and \( g_d \) commute with each other. The first relation then implies \( g_\beta g_c g^{-1}_\beta = g_c \); i.e. \( g_a \) and \( g_c \) commute with each other. It follows that \( g_a \) and \( g_d \) commute with each other:

\[
g_a g_d = g_a g_d \cdot g^{-1}_c g_c = g_a \cdot g_d g^{-1}_c g_c = g_a \cdot g^{-1}_c g_d = g_a g^{-1}_c g_d \cdot g_c
\]

\[
= g_d g^{-1}_c g_a \cdot g_c = g_d g^{-1}_c g_a g_c = g_d g^{-1}_c g_c g_a = g_d g_a.
\]

Therefore, \( MG(L) \) is the free abelian group on the generators \( g_a, g_c, g_d \). According to Proposition 33, it follows that the subgroup of \( Aut(MQ(L)) \) generated by \( \{ \beta_q \mid q \in MQ(L) \} \) is commutative.

**Lemma 46.** Let \( Q \) be a quandle with \( \beta_q \beta_r = \beta_r \beta_q \forall q,r \in Q \). Then (a) \( \beta_q = \beta_{q'} \) whenever \( q \) and \( q' \) are elements of the same orbit in \( Q \), and (b) if \( f : Q \to Q \) is a composition of \( \beta \) maps of \( Q \) that fixes an element \( x \) of \( Q \), then the restriction of \( f \) to the orbit of \( x \) is the identity map.

**Proof.** The displacement group of \( Q \) is generated by products \( \beta_q \beta^{-1}_z \), with \( y, z \in Q \). Therefore, the hypothesis \( \beta_q \beta_r = \beta_r \beta_q \forall q,r \in Q \) implies that \( Dis(Q) \) is an abelian group; according to Proposition 21, it follows that \( Q \) is a medial quandle.

In the third paragraph of Sec. 5 it was observed that the identity \( \beta_{q \circ r} = \beta_r \beta_q \beta^{-1}_r \) holds for all elements of any quandle. In \( Q \) the \( \beta \) maps commute with each other, so we have \( \beta_{q \circ r} = \beta_q \forall q,r \in Q \). It follows that (a) holds. For (b), note that if \( f(x) = x \) then \( f(\beta_q^n(x)) = \beta_q^n(f(x)) = \beta_q^n(x) \forall q \in Q \forall n \in \mathbb{Z} \).

Lemma 46 tells us that \( \beta_{qa}, \beta_q, \) and \( \beta_{qa} \) are the only \( \beta \) maps of the quandle \( MQ(L) \); in particular, \( \beta_{qa} = \beta_q \). As \( \beta_{qa}(qa) = qa = \beta_q(qa) \), the restrictions of \( \beta_{qa} \) and \( \beta_q \) to the orbit of \( MQ(L) \) containing \( qa \) are both the identity map, so the orbit of \( qa \) is \( \{ \beta^n_{qa}(qa) \mid n \in \mathbb{Z} \} \). Similarly, the equalities \( \beta_q(q_d) = q_d = \beta_d(q_d) \) imply that both \( \beta_q \) and \( \beta_{qa} \) restrict to the identity map of the orbit of \( q_d \), so this orbit is \( \{ \beta^n_q(q_d) \mid n \in \mathbb{Z} \} \). The equalities \( \beta_q(q_c) = q_c = \beta_q^{-1}(q_c) = \beta_d^{-1}_d \beta_q(q_c) \) imply that both \( \beta_q \) and \( \beta_d^{-1} \beta_q \) restrict to the identity map of the orbit of \( q_c \), and this orbit is \( \{ \beta^n_{qa}(q_c) \mid n \in \mathbb{Z} \} \), with \( \beta^n_{qa}(q_c) = \beta^n_{qa}(q_c) \forall n \in \mathbb{Z} \).

Let \( Q \) be a quandle of the type described in Lemma 15 corresponding to the triples \( t_1 = (id, s^+, s^+) \), \( t_2 = (id, id, id) \) and \( t_3 = (s^+, s^+, id) \). Then the observations of the preceding paragraph imply that there is a surjective quandle
map $Q \to MQ(L)$, under which the 0 elements of $Q_1$, $Q_2$ and $Q_3$ are mapped to $q_a$, $q_c$ and $q_d$ (respectively). (The element of $Q_2$ corresponding to 1 is mapped to $q_b$.) As the relations required by the crossings of Fig. 6 hold in $Q$, and $MQ(L)$ is the largest medial quandle generated by $g_a$, $g_c$ and $q_d$ in which these crossing relations hold, the surjective quandle map $Q \to MQ(L)$ must be an isomorphism.

![Figure 7: This link is denoted $L'$ in Sec. 7](image)

Now, let $L'$ be the link pictured in Fig. 7. Then $MG(L')$ is generated by $g_a, g_b, g_c$ and $g_d$, with $xy^{-1}z = zy^{-1}x$ for all conjugates of $g_a, g_b, g_c$ and $g_d$, and these four crossing relations: $g_b = g_a g_c g_a^{-1}, g_a = g_d g_b g_d^{-1}, g_d = g_c g_d g_c^{-1}$, and $g_c = g_d g_b g_d^{-1}$. Using the last relation to eliminate $g_c$, we are left with generators $g_a, g_b$ and $g_d$, and relations $g_b = g_a g_d g_b g_d^{-1} g_d^{-1}, g_a = g_b g_d g_b^{-1}$ and $g_d = g_d g_b g_d^{-1} g_d g_b^{-1} g_d^{-1}$.

The second relation implies that $g_a$ and $g_b$ commute with each other. The first implies $g_a^{-1} g_b g_a = g_d g_b g_d^{-1}$; as $g_a$ and $g_b$ commute with each other, this implies $g_b = g_a g_b g_d^{-1}$, so $g_b$ and $g_d$ commute with each other. Then

$$g_a g_d = g_a g_d \cdot g_b^{-1} g_b = g_a \cdot g_d g_b^{-1} \cdot g_b = g_a \cdot g_b^{-1} g_d \cdot g_b = g_a g_b^{-1} g_d \cdot g_b$$

$$= g_d g_b^{-1} g_a \cdot g_b = g_d g_b^{-1} \cdot g_a g_b = g_d g_b^{-1} \cdot g_b g_a = g_d g_a,$$

so $g_a$ and $g_d$ commute with each other. That is, $MG(L')$ is commutative. It follows that $MG(L')$ is the free abelian group on the generators $g_a, g_b, g_d$. According to Proposition 45, all of the $\beta$ maps of $MG(L')$ commute with each other, and according to Lemma 45, the only $\beta$ maps of the quandle $MQ(L')$ are $\beta_{q_a}, \beta_{q_b}, \beta_{q_c}$ and $\beta_{q_d}$. The equalities $\beta_{q_a}(q_a) = q_a = \beta_{q_b}(g_a)$ and $\beta_{q_c}(g_d) = g_d = \beta_{q_d}(q_d)$ imply that the restrictions of $\beta_{q_a}$ and $\beta_{q_c}$ to the $q_a$ orbit of $MQ(L')$ are both the identity map, and the restrictions of $\beta_{q_b}$ and $\beta_{q_d}$ to the $q_d$ orbit are both the identity map. The equalities $\beta_{q_b}(q_c) = q_c = \beta_{q_d}(q_b) = \beta_{q_d} \beta_{q_c}(q_c)$ imply that both $\beta_{q_b}$ and $\beta_{q_d} \beta_{q_c}$ restrict to the identity map of the $q_c$ orbit.

Let $Q'$ be the quandle from Lemma 45 corresponding to the triples $t'_1 = (id, s^+, s^+)$, $t'_2 = (id, id, id)$ and $t'_3 = (s^-, s^-, id)$. Then the discussion above implies that there is a surjective quandle map $Q' \to MQ(L')$, under which the 0 elements of $Q'_1, Q'_2, Q'_3$ are mapped to $q_a, q_c$ and $q_d$ respectively. The crossing relations from Fig. 7 hold in $Q'$, so this quandle map is an isomorphism.

Notice that the quandle $MQ(L)$ has these properties: there are only three $\beta$ maps, one from each orbit; one $\beta$ map is the identity; and the other two $\beta$ maps
have the same restriction to the orbit whose \( \beta \) map is the identity. In contrast, \( \text{MQ}(L') \) has these properties: there are only three \( \beta \) maps, one from each orbit; one \( \beta \) map is the identity; and the restrictions of the other two \( \beta \) maps to the orbit whose \( \beta \) map is the identity are inverses of each other. It is clear that \( \text{MQ}(L) \) and \( \text{MQ}(L') \) are not isomorphic.

To complete the proof of Theorem 7, we need to show that \( Q_{A}^{\text{red}}(L) \) and \( Q_{A}^{\text{red}}(L') \) are isomorphic. According to Theorem 4, it is enough to show that \( L \) and \( L' \) are \( \phi_{\tau} \)-equivalent. In fact, \( L \) and \( L' \) are related through the stronger multivariate version of \( \phi_{\tau} \)-equivalence, which we call Crowell equivalence.

**Proposition 47.** There is a \( \Lambda_{\mu} \)-linear isomorphism \( f : M_{A}(L) \to M_{A}(L') \), with \( \phi_{L} = \phi_{L'} \circ f : M_{A}(L) \to I_{\mu} \).

**Proof.** We refer to Sec. 3 for definitions.

Abusing notation, we use \( D \) for the link diagrams in both Fig. 6 and Fig. 7, and we use \( a, b, c, d \) for the arcs in both diagrams. The components of \( L \) and \( L' \) are indexed in order, from left to right in Figs. 6 and 7.

The two crossings on the left in Figs. 6 and 7 provide the relations \( \gamma_{D}(b) = (1 - t_{2})\gamma_{D}(a) + t_{1}\gamma_{D}(c) \) and \( \gamma_{D}(a) = (1 - t_{1})\gamma_{D}(b) + t_{2}\gamma_{D}(a) \), in both \( M_{A}(L) \) and \( M_{A}(L') \). The second relation is equivalent to \( (1 - t_{2})\gamma_{D}(a) = (1 - t_{1})\gamma_{D}(b) \), and with this equality, the first relation is equivalent to \( \gamma_{D}(b) = \gamma_{D}(c) \).

Keeping in mind that \( \gamma_{D}(b) = \gamma_{D}(c) \) in both \( M_{A}(L) \) and \( M_{A}(L') \), the two crossings on the right in Figs. 6 and 7 provide the same two relations in \( M_{A}(L) \) and \( M_{A}(L') \): \( \gamma_{D}(d) = (1 - t_{3})\gamma_{D}(b) + t_{2}\gamma_{D}(d) \) and \( \gamma_{D}(b) = (1 - t_{2})\gamma_{D}(d) + t_{3}\gamma_{D}(b) \). Both of these relations are equivalent to \( (1 - t_{3})\gamma_{D}(b) = (1 - t_{2})\gamma_{D}(d) \).

In summary, \( M_{A}(L) \) and \( M_{A}(L') \) are both generated by \( \gamma_{D}(a), \gamma_{D}(b), \gamma_{D}(c), \) and \( \gamma_{D}(d), \) subject to the relations \( (1 - t_{2})\gamma_{D}(a) = (1 - t_{1})\gamma_{D}(b) \) and \( (1 - t_{3})\gamma_{D}(b) = (1 - t_{2})\gamma_{D}(d) \). The obvious isomorphism \( f : M_{A}(L) \to M_{A}(L') \) satisfies the statement.

To describe the \( \Lambda \)-modules \( M_{A}^{\text{red}}(L) \) and \( M_{A}^{\text{red}}(L') \), we modify the descriptions of \( M_{A}(L) \) and \( M_{A}(L') \) by replacing each indeterminate \( t_{i} \) with \( t \), and replacing each element \( \gamma_{D}(x) \) with \( \gamma_{D}(x) \otimes 1 \). We conclude that

\[
M_{A}^{\text{red}}(L) \cong M_{A}^{\text{red}}(L') \cong \Lambda \oplus (\Lambda/(1 - t)) \oplus (\Lambda/(1 - t))
\]

with the direct summands generated by \( \gamma_{D}(b) \otimes 1 \), \( (\gamma_{D}(a) - \gamma_{D}(b)) \otimes 1 \) and \( (\gamma_{D}(d) - \gamma_{D}(b)) \otimes 1 \), respectively. It is apparent that \( L \) and \( L' \) are \( \phi_{\tau} \)-equivalent.

### 8 Longitudes

We would like to have more information about the map \( e_{D} \). To obtain this information it will be useful to consider a special type of link diagram, defined using the familiar notion of writhe. See Fig. 8.

**Definition 48.** Let \( D \) be a link diagram. Then \( D \) has alternating writhe if every arc \( a \in A(D) \) occurs as the underpassing arc of two crossings, one of writhe \(-1\) and the other of writhe \(1\).
w = −1 \quad w = 1

Figure 8: The writhe of a crossing is denoted \( w \).

**Proposition 49.** Every classical link has a diagram with alternating writhes.

*Proof.* Start with any diagram \( D \) of \( L \). If \( L \) has a component \( K_i \) which is not the underpassing component of any crossing of \( D \), insert a trivial crossing into the one arc of \( D \) that represents \( K_i \). (Trivial crossings are pictured in Fig. 9.) We now have a diagram \( D' \) in which every arc appears as the underpassing arc of at least one crossing.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{trivial_crossing.png}
\end{array}
\]

Figure 9: A trivial crossing of either writhe may be inserted into an arc.

For every arc \( a \in A(D') \) that appears as the underpassing arc only at crossing(s) of one writhe value \( w \), insert a trivial crossing of writhe \(-w\) into \( a \). The effect is to split \( a \) into two distinct arcs, each of which appears as the underpassing arc at two crossings of opposite writhe.

Now, suppose \( L \) is a link, and \( D \) is a diagram of \( L \) with alternating writhes. For each component \( K_i \) of \( L \), we choose an arbitrary arc \( b_{i0} \in A(D) \) with \( \kappa_D(b_{i0}) = i \), and we start walking along \( b_{i0} \), in the direction given by the orientation of \( K_i \). When we reach the end of \( b_{i0} \), we index that crossing as \( c_{i0} \), the overpassing arc at that crossing as \( a_{i0} \), and the next arc of \( K_i \) as \( b_{i1} \). The alternating writhes condition guarantees that as we walk along \( K_i \), we pass under an even number of crossings. By the time we get back to \( b_{i0} \), we will have indexed these crossings as \( c_{i0}, \ldots, c_{i(2n_i-1)} \), indexed the arcs of \( K_i \) as \( b_{i0}, \ldots, b_{i(2n_i-1)} \), and indexed the overpassing arcs of the crossings over \( K_i \) as \( a_{i0}, \ldots, a_{i(2n_i-1)} \). We consider the second indices of \( a_{ij}, b_{ij} \) and \( c_{ij} \) modulo \( 2n_i \).
Let \( j \in \{1, \ldots, 2n_i\} \), and let \( w_{ij} \) be the writhe of the crossing \( c_{ij} \). We have the following equalities in \( M^\text{red}_i(L) \) and \( MQ(L) \) (respectively).
\[
\gamma_D(b_{ij}) \otimes 1 = (1 - t^{-w_{ij}})(\gamma_D(a_{ij}) \otimes 1) + t^{-w_{ij}}(\gamma_D(b_{i(j+1)}) \otimes 1) \tag{8}
\]
\[
q_{aij} = q_{b_{i(j+1)}} b^{-w_{ij}} q_{bij} = \beta^{-w_{ij}} q_{bj_{i+1}} \tag{9}
\]

**Definition 50.** Under these circumstances, for each \( i \in \{1, \ldots, \mu\} \) the \( i \)th longitude of \( L \) is
\[
\chi_i = \sum_{j=1}^{2n_i} (-w_{ij})(\gamma_D(a_{ij}) \otimes 1) \in \ker \phi^\text{red}_L.
\]

**Proposition 51.** For each \( i \in \{1, \ldots, \mu\} \), \( \chi_i \) has the following properties.

1. \((1 - t) \cdot \chi_i = 0.\)

2. The image of \( \chi_i \) under the map \( e_D : \ker \phi^\text{red}_L \to MG^0(L) \) is an element of the center of \( MG(L) \).

3. The image of \( \chi_i \) under the composition \( \beta e_D : \ker \phi^\text{red}_L \to \text{Aut}(MQ(L)) \) is a displacement of \( MQ(L) \) whose restriction to the \( K_i \) orbit of \( MQ(L) \) is the identity map.

4. If \( \mu = 1 \), then \( \chi_1 = 0.\)

**Proof.** For convenience, we assume that the arcs and crossings of \( D \) have been indexed so that \( w_{11} = -1 \). \( D \) has alternating writhes, so \( w_{ij} \) is always \((-1)^j\).

According to (8), if \( i \in \{1, \ldots, \mu\} \) and \( j \in \{1, \ldots, 2n_i\} \) then we have
\[
(\gamma_D(b_{ij}) \otimes 1) = (1 - t^{-w_{ij}})(\gamma_D(a_{ij}) \otimes 1) + t^{-w_{ij}}(\gamma_D(b_{i(j+1)}) \otimes 1).
\]

If \( j \) is odd, it follows that
\[
(1 - t) \cdot (-w_{ij})(\gamma_D(a_{ij}) \otimes 1)) = (\gamma_D(b_{ij}) \otimes 1) - t(\gamma_D(b_{i(j+1)}) \otimes 1).
\]

If \( j \) is even, it follows that
\[
(1 - t^{-1})(\gamma_D(a_{ij}) \otimes 1) = (\gamma_D(b_{ij}) \otimes 1) - t^{-1}(\gamma_D(b_{i(j+1)}) \otimes 1)
\]
and hence
\[
(1 - t) \cdot (-w_{ij})(\gamma_D(a_{ij}) \otimes 1)) = t(\gamma_D(b_{ij}) \otimes 1) - (\gamma_D(b_{i(j+1)}) \otimes 1).
\]

Therefore,
\[
(1 - t) \cdot \chi_i = \sum_{j=1}^{2n_i} (1 - t) \cdot (-w_{ij})(\gamma_D(a_{ij}) \otimes 1))
\]
\[
= \sum_{\text{odd } j} ((\gamma_D(b_{ij}) \otimes 1) - t(\gamma_D(b_{i(j+1)}) \otimes 1))
\]
\[
+ \sum_{\text{even } j} (t(\gamma_D(b_{ij}) \otimes 1) - (\gamma_D(b_{i(j+1)}) \otimes 1)).
\]

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The total is 0 because when \( j \) is odd, \( \gamma_D(b_{ij}) \otimes 1 \) occurs with coefficient 1 in the penultimate sum and coefficient \(-1\) in the last sum, and when \( j \) is even, \( \gamma_D(b_{ij}) \otimes 1 \) occurs with coefficient \(-t\) in the penultimate sum and coefficient \( t\) in the last sum.

For the second property, recall that \( e_D \) is \( \Lambda \)-linear by Corollary 43. Therefore

\[
e_D(x_i) = (1-t+t) \cdot e_D(x_i) = e_D((1-t) \cdot \chi_i) + t \cdot e_D(x_i) = 0 + t \cdot e_D(x_i) = t \cdot e_D(x_i).
\]

According to Theorem 42 it follows that for every \( z^* \in QMG(L) \),

\[
e_D(x_i) = t \cdot e_D(x_i) = z^* \cdot e_D(x_i) \cdot (z^*)^{-1}.
\]

That is, \( e_D(x_i) \) commutes with every \( z^* \in QMG(L) \). The elements of \( QMG(L) \) generate the group \( MG(L) \), so \( e_D(x_i) \) commutes with every element of \( MG(L) \).

For the third property, notice that formula (7) implies

\[
e_D(x_i) = \sum_{k=1}^{n_i} e_D((\gamma_D(a_i(2k-1)) \otimes 1) - (\gamma_D(a_i(2k)) \otimes 1))
\]

\[
= \prod_{k=1}^{n_i} (g_{ai(2k-1)}^{-w_{1i}} g_{ai(2k)}^{-w_{2i}} \cdots g_{ai(2n_{i-1})}^{-w_{ni}}).
\]

According to Proposition 35 \( \beta : MG(L) \to \text{Aut}(MQ(L)) \) is the homomorphism with \( \beta(g_a) = \beta_{g_a} \forall a \in A(D) \). As \( b_{i(2n_i+1)} = b_{i1} \), the relations (9) imply that

\[
\begin{align*}
\beta e_D(x_i)(q_{b_{i1}}) &= \beta(g_{a_{i1}})^{-w_{1i}} \beta(g_{a_{i2}})^{-w_{2i}} \cdots \beta(g_{a_{i(2n_i-1)}})^{-w_{ni}} (q_{b_{i(2n_i+1)}}) \\
&= \beta(g_{a_{i1}})^{-w_{1i}} \beta(g_{a_{i2}})^{-w_{2i}} \cdots \beta(g_{a_{i(2n_i-1)}})^{-w_{ni}} (q_{b_{i(2n_i)}}) \\
&= \cdots = \beta(g_{a_{i1}})^{-w_{1i}} (q_{b_{i2}}) = q_{b_{i1}}.
\end{align*}
\]

The group \( \text{Dis}(MQ(L)) \) is commutative, so every \( d \in \text{Dis}(MQ(L)) \) has \( \beta e_D(x_i)(d(q_{b_{i1}})) = d(\beta e_D(x_i)(q_{b_{i1}})) = d(q_{b_{i1}}) \). According to Proposition 22 every element of the \( K_i \) orbit of \( MQ(L) \) is \( d(q_{b_{i1}}) \) for some \( d \in \text{Dis}(MQ(L)) \). It follows that \( \beta e_D(x_i) \) fixes every element of the \( K_i \) orbit of \( MQ(L) \).

The fourth property follows immediately from the first property and Corollary 10. \( \square \)

**Proposition 52.** Let \( L \) be a diagram of \( L \), with alternating writhes. Then \( \text{ann}(1-t) = \{ x \in M_A^{\text{red}}(L) \mid (1-t)x = 0 \} \) is the submodule of \( M_A^{\text{red}}(L) \) generated by \( \chi_1, \ldots, \chi_{\mu} \).

**Proof.** The first property of Proposition 51 implies that \( \chi_1, \ldots, \chi_{\mu} \in \text{ann}(1-t) \).

Now, suppose that \( x \in M_A^{\text{red}}(L) \) has \( (1-t)x = 0 \). We must show that \( x \) is equal to a \( \Lambda \)-linear combination of \( \chi_1, \ldots, \chi_{\mu} \). Recall the exact sequence

\[
\Lambda_\mu^{C(D)} \xrightarrow{\rho_D} \Lambda_\mu^{A(D)} \xrightarrow{\gamma_D} M_A(L) \to 0,
\]
discussed in Sec. 2. If \( \text{id} \) denotes the identity map of \( \Lambda \), then the right exactness of tensor products yields an exact sequence

\[
\Lambda^C(D) \otimes_{\Lambda} \Lambda \xrightarrow{\rho_D \otimes \text{id}} \Lambda^A(D) \otimes_{\Lambda} \Lambda \xrightarrow{\gamma_D \otimes \text{id}} M^\text{red}_A(L) \to 0.
\]

Therefore, there is a function \( f_x : A(D) \to \Lambda \) such that \( x = (\gamma_D \otimes \text{id})(x') \), where

\[
x' = \sum_{a \in A(D)} f_x(a)(a \otimes 1). \tag{10}
\]

As \( (1 - t)x' \in \ker(\gamma_D \otimes \text{id}) \), there is also a function \( g_x : C(D) \to \Lambda \) such that

\[
(1 - t)x' = \sum_{c \in C(D)} g_x(c) \cdot (\rho_D(c) \otimes 1). \tag{11}
\]

The functions \( f_x \) and \( g_x \) are not unique. If \( c \in C(D) \) then for any element \( \lambda_c \in \Lambda \), we may add \( \lambda_c \cdot (\rho_D(c) \otimes 1) \) to the sum on the right-hand side of \( \text{(10)} \) without changing the fact that \( x = (\gamma_D \otimes \text{id})(x') \), and \( \text{(11)} \) will remain valid so long as \( (1 - t)\lambda_c \) is added to \( g_x(c) \). In particular, for each \( c \in C(D) \) there is a \( \lambda_c \in \Lambda \) such that \( (1 - t)\lambda_c = \epsilon g_x(c) - g_x(c) \). The result of adding \( \epsilon g_x(c) - g_x(c) \) to \( g_x(c) \) is to replace \( g_x(c) \) with \( \epsilon g_x(c) - g_x(c) + g_x(c) = \epsilon g_x(c) \), which is an integer. It follows that we may assume \( g_x(c) \in \mathbb{Z} \forall c \in C(D) \), without loss of generality.

We now claim that for every arc \( a \in A(D) \), the values of \( g_x(c) \) for the two crossings at which \( a \) is an underpassing arc are negatives of each other. To see why the claim is true, notice first that \( \text{(10)} \) and \( \text{(11)} \) yield

\[
\sum_{a \in A(D)} (1 - t)f_x(a)(a \otimes 1) = \sum_{c \in C(D)} g_x(c) \cdot (\rho_D(c) \otimes 1). \tag{12}
\]

This equality holds in the \( \Lambda \)-module \( \Lambda^A(D) \otimes_{\Lambda} \Lambda \), which is freely generated by the elements \( a \otimes 1 \) with \( a \in A(D) \). Therefore, for each \( a \in A(D) \) the coefficients of \( a \otimes 1 \) on the two sides of \( \text{(12)} \) are precisely equal.

Suppose \( c \in C(D) \) has \( g_x(c) \neq 0 \), and \( a \in A(D) \) is the arc that corresponds to \( a_2 \), when \( c \) is pictured as in Fig. 1. Then the contribution of the term \( g_x(c) \cdot (\rho_D(c) \otimes 1) \) to the coefficient of \( a \otimes 1 \) on the right-hand side of \( \text{(12)} \) is \( g_x(c) \cdot t \). Let \( c' \) be the other crossing of \( D \) at which \( a \) is one of the underpassing arcs. It is easy to see that the alternating writhes property guarantees that \( a \) plays the same role at \( c' \), i.e., \( a_2 \) rather than \( a_3 \), as pictured in Fig. 1. Therefore the contribution of the term \( g_x(c') \cdot (\rho_D(c') \otimes 1) \) to the coefficient of \( a \otimes 1 \) on the right-hand side of \( \text{(12)} \) is \( g_x(c') \cdot t \). Aside from \( c \) and \( c' \), this arc \( a \) is incident only at crossings \( c'' \) where it is the overpassing arc, and for such a crossing \( c'' \), the contribution of the term \( g_x(c'') \cdot (\rho_D(c'') \otimes 1) \) to the coefficient of \( a \otimes 1 \) on the right-hand side of \( \text{(12)} \) is divisible by \( 1 - t \). The coefficient of \( a \otimes 1 \) on the left-hand side of \( \text{(12)} \) is divisible by \( 1 - t \), so it follows that \( g_x(c) = -g_x(c') \), as claimed.
The argument for an arc \( a \) that plays the role of \( a_3 \) in Fig. 1 is almost the same. The only difference is that the contributions from \( c \) and \( c' \) are \( g_x(c) \cdot (-1) \) and \( g_x(c') \cdot (-1) \), rather than \( g_x(c) \cdot t \) and \( g_x(c') \cdot t \). This completes the proof of the claim.

As \( D \) has alternating writhes, the claim can also be stated as follows. For each arc \( a \in A(D) \), there is an integer \( m_a \) such that the two crossings at which \( a \) is an underpassing arc both satisfy the equality \( g_x(c) = w(c)m_a \). At each crossing there is only one value of \( g_x(c) \), so the two underpassing arcs must have the same value of \( m_a \). Walking from crossing to crossing along the arcs of \( D \), we deduce that the value of \( m_a \) is constant on each component \( K_i \) of \( L \). We denote this constant value \( m_i \).

While proving the claim we showed that on the right-hand side of (12), all of the contributions from underpassing arcs cancel each other. This leaves only the contributions from overpassing arcs. That is, if we recall the indexing convention for \( a_{ij}, b_{ij}, c_{ij} \) mentioned after Proposition 49, then

\[
\sum_{a \in A(D)} (1-t) f_x(a)(a \otimes 1) = \sum_{c \in C(D)} g_x(c) \cdot (\rho_D(c) \otimes 1) = \mu \sum_{i=1}^{\mu} \frac{2n_i}{2} \sum_{j=1}^{n_i} w(c_{ij}) \cdot (1-t)(a_{ij} \otimes 1).
\]

Once again, this equality holds in the free \( \Lambda \)-module \( \Lambda_{\mu}^{A(D)} \otimes_{\Lambda} \Lambda \), so for every \( a \in A(D) \), the coefficients of \( a \otimes 1 \) in the first and last displayed sums must be precisely equal. It follows that the factors of \( 1-t \) may be canceled, so

\[
x' = \sum_{a \in A(D)} f_x(a)(a \otimes 1) = \mu \sum_{i=1}^{\mu} \frac{2n_i}{2} \sum_{j=1}^{n_i} w(c_{ij}) m_i \cdot (a_{ij} \otimes 1)
\]

and hence \( x = (\gamma_D \otimes \text{id})(x') = -\sum m_i X_i \). \( \square \)

**9 \( Q^{\text{red}}_A(L) \) and QMG\((L)\)**

Our final result is that the quandles \( Q^{\text{red}}_A(L) \) and QMG\((L)\) are always isomorphic. The first part of the proof involves the Alexander module of the group MG\((L)\). We give a brief summary of the theory regarding this module, and refer to Crowell [2] for a thorough account.

The integral group ring \( \mathbb{Z}(\text{MG}(L)) \) consists of formal linear combinations \( \sum n_i g_i \), where the \( n_i \) are integers and the \( g_i \) are elements of \( \text{MG}(L) \). The integral group ring \( \mathbb{Z}(\text{MG}(L)/\text{MG}(L)^\prime) \) of the abelianization of \( \text{MG}(L) \) is defined analogously, and it is made into a \( \mathbb{Z}(\text{MG}(L)) \)-module using the multiplication...
of \( Z(MG(L)) \). The augmentation ideal \( I \subset Z(MG(L)) \) is the ideal generated by the elements \( g-1 \), where \( g \in MG(L) \). The Alexander module of \( MG(L) \) is the tensor product

\[
Z(MG(L)/MG(L)') \otimes_{Z(MG(L))} I = M,
\]

considered as a \( Z(MG(L)/MG(L)') \)-module via multiplication in the first factor of the tensor product. That is, if \( \alpha : MG(L) \to MG(L)/MG(L)' \) is the canonical map onto the quotient, \( g, h \in MG(L) \) and \( i \in I \), then

\[
\alpha(g) \cdot (\alpha(h) \otimes i) = \alpha(gh) \otimes i = \alpha(g) \otimes (hi) = 1 \otimes (ghi).
\]

Definition \( \ref{def:alexander-module} \) implies that \( MG(L)/MG(L)' \) is a free abelian group of rank \( \mu \), with one generator for each component \( K_i \) of \( L \). The generator corresponding to \( K_i \) is \( \alpha(g_a) \), for all \( a \in A(D) \) with \( \kappa_D(a) = i \). There is then a natural isomorphism between \( Z(MG(L)/MG(L)') \) and \( \Lambda_\mu \), under which the generator of \( Z(MG(L)/MG(L)') \) corresponding to \( K_i \) is mapped to \( t_i \).

As explained by Crowell \( \ref{crown} \), the Alexander module \( M \) is a finitely presented \( \Lambda_\mu \)-module. The generators in the presentation are the elements \( 1 \otimes (g_a - 1) \), where \( a \in A(D) \). The relations in the presentation of the module \( M \) are obtained by taking the free derivatives of the relators in the presentation of the group \( MG(L) \) given in Definition \( \ref{def:presentation} \) and then applying the abelianization map \( \alpha \).

We recall the definition of the free derivatives. Suppose \( F \) is the free group on the set \( X \), and

\[
w = \prod_{i=1}^{n} x_i^{\epsilon_i} \in F,
\]

where \( x_1, \ldots, x_n \in X \) and \( \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\} \). For \( 1 \leq i \leq n \), define the \( i \)th initial segment of \( w \) as follows:

\[
w_i = \begin{cases} 
\prod_{j=1}^{i-1} x_j^{\epsilon_j}, & \text{if } \epsilon_i = 1 \\
\prod_{j=1}^{i-1} x_j^{\epsilon_j}, & \text{if } \epsilon_i = -1.
\end{cases}
\]

Then for each \( x \in X \), the free derivative of \( w \) with respect to \( x \) is

\[
\frac{\partial w}{\partial x} = \sum_{x_i = x} \epsilon_i w_i \in \mathbb{Z}F.
\]

Recall that there are two types of relators in Definition \( \ref{def:presentation} \): if \( b, c, d \) are conjugates of \( g_a \) elements then there is a relator \( bc^{-1}db^{-1}cd^{-1} \), and if \( a_1, a_2, a_3 \) are arcs that appear at a crossing of \( D \) as pictured in Fig. \( \ref{fig:crossing} \) then there is a relator \( g_{a_1}g_{a_2}g_{a_3}^{-1}g_{a_4}^{-1} \).

**Theorem 53.** If \( M \) is the Alexander module of the group \( MG(L) \), then there is an isomorphism of \( \Lambda \)-modules

\[
\tilde{\gamma}_D : M \otimes_{\Lambda_\mu} \Lambda \to M_{\text{red}}^\Lambda(L),
\]

with \( \tilde{\gamma}_D((1 \otimes (g_a - 1)) \otimes 1) = \gamma_D(a) \otimes 1 \forall a \in A(D) \).
Proof. Right exactness of tensor products implies that a presentation of the \( \Lambda \)-module \( M \otimes_{\Lambda} \Lambda \) can be obtained from the presentation of the \( \Lambda_{\mu} \)-module \( M \) described above, by applying \( \tau \) to all coefficients.

We claim that for a relator of the form \( r = bc^{-1}db^{-1}cd^{-1} \), all of the resulting \( \tau \) values are 0. There are several different places in the relator where a generator might appear; we consider two of them, and leave it to the reader to consider the rest. Suppose first that \( c \) is a conjugate of \( g \), say \( c = eg_{a}e^{-1} \). This appearance of \( g \) in the middle of \( c \) contributes two terms to the free derivative \( \partial r/\partial g \), namely, \(-beg_{a}^{-1} \) and \( bc^{-1}db^{-1}e \). Under the composition of the abelianization map \( \alpha : \text{MG}(L) \to \text{MG}(L)/\text{MG}(L)^{\prime} \) and the map \( \tau : Z(\text{MG}(L)/\text{MG}(L)^{\prime}) \cong \Lambda_{\mu} \to \Lambda \), each of \( b, c, d, g \) is mapped to \( t \). Hence the image of \(-beg_{a}^{-1} + bc^{-1}db^{-1}e \) is the same as the image of \(-e + e \); of course, this image is 0. For another example, suppose there is an appearance of \( g \) in \( c \) that is not in the middle of \( c \); say \( c = e_{1}g_{a}^{-1}e_{2}g_{a}e_{2}^{-1}g_{a}e_{1}^{-1} \). These two appearances of \( g \) in \( c \) also provide two corresponding appearances of \( g \) in \( c^{-1} \), and the total contribution to \( \partial r/\partial g \) of these four appearances of \( g \) is

\[-be_{1}g_{a}^{-1} + be_{1}g_{a}^{-1}e_{2}g_{a}e_{2}^{-1} - be^{-1}db^{-1}e_{1}g_{a}^{-1} + be^{-1}db^{-1}e_{1}g_{a}^{-1}e_{2}g_{a}e_{2}^{-1} \]

As \( b, c, d, g \) and \( g' \) are all mapped to \( t \), the image of this total in \( \Lambda \) is the same as the image of

\[-e_{1} + e_{2}g_{a}^{-1} - e_{1}g_{a}^{-1} + e_{1} \]

which is 0.

The claim implies that the relators of the form \( r = bc^{-1}db^{-1}cd^{-1} \) do not contribute in a significant way to the presentation of the \( \Lambda \)-module \( M \otimes_{\Lambda} \Lambda \). For a relator \( g_{a_{1}}g_{a_{2}}g_{a_{1}}^{-1}g_{a_{3}}^{-1} \) corresponding to a crossing, instead, the nonzero free derivatives are \( 1 - g_{a_{1}}g_{a_{2}}g_{a_{1}}^{-1} \) with respect to \( g_{a_{1}}, g_{a_{1}} \) with respect to \( g_{a_{2}}, g_{a_{2}} \), and \(-g_{a_{1}}g_{a_{2}}g_{a_{1}}^{-1}g_{a_{3}} \) with respect to \( g_{a_{3}} \). The images in \( \Lambda \) are \( 1 - t \) with respect to \( g_{a_{1}}, t \) with respect to \( g_{a_{2}}, -1 \) with respect to \( g_{a_{3}} \). These coefficients provide the relation

\[0 = (1 - t)((1 \otimes (g_{a_{1}} - 1)) \otimes 1) + t(((1 \otimes (g_{a_{2}} - 1)) \otimes 1) - ((1 \otimes (g_{a_{3}} - 1)) \otimes 1)\]

in \( M \otimes_{\Lambda} \Lambda \), and this relation matches precisely with the crossing relation for \( M^{\text{rel}}_{\Lambda}(L) \) that appears in the definition given at the beginning of Sec. 2.

Corollary 54. If \( L \) is a link with a diagram \( D \), then there is a surjective quandle map \( f_{D} : \text{QMG}(L) \to Q_{A}^{\text{rel}}(L) \), under which \( g_{a} \mapsto \gamma_{D}(a) \otimes 1 \forall a \in A(D) \).

Proof. The isomorphism \( \hat{\gamma}_{D} : M \otimes_{\Lambda} \Lambda \to M^{\text{rel}}_{A}(L) \) can be composed with the function \( \text{MG}(L) \to M \otimes_{\Lambda} \Lambda \) given by \( g \mapsto (1 \otimes (g - 1)) \otimes 1 \forall g \in \text{MG}(L) \). The function \( f_{D} \) is obtained by restricting this composition to \( \text{QMG}(L) \). The composition certainly has \( g_{a} \mapsto \gamma_{D}(a) \otimes 1 \forall a \in A(D) \). The \( \gamma_{D}(a) \otimes 1 \) elements generate the quandle \( Q_{A}^{\text{rel}}(L) \), so in order to show that \( f_{D} \) is a surjective quandle map, it is enough to show that \( f_{D} \) is a quandle map.

If \( x, y \in \text{QMG}(L) \), then

\[f_{D}(y \triangleright x) = f_{D}(yx^{-1}) = \hat{\gamma}_{D}((1 \otimes (yx^{-1} - 1)) \otimes 1)\]
the proof of Corollary 30 maps a displacement under \( \tau a \) before. Let \( \text{Aut}(QMG(L)) \) be fixed.

Every element of \( QMG(L) \) is a conjugate of some \( g_a \) element, so its image under \( \tau a \) is \( t \). Therefore \( \tau a(xy^{-1}) = t^2 t^{-1} = t = \tau a(x) \), so

\[
\gamma_D(1 \otimes (xy^{-1} - xy)) = \gamma_D((1 \otimes (xy - x)) \otimes 1) + \gamma_D((1 \otimes (x - 1)) \otimes 1)
\]

\[
= \gamma_D((\alpha(xyx^{-1}) \otimes (1 - x)) \otimes 1) + \gamma_D((\alpha(x) \otimes (y - 1)) \otimes 1) + \gamma_D((1 \otimes (x - 1)) \otimes 1)
\]

\[
= \gamma_D((1 \otimes (x - 1)) \otimes \tau a(xy^{-1})) + \gamma_D((1 \otimes (y - 1)) \otimes \tau a(x)) + \gamma_D((1 \otimes (x - 1)) \otimes 1).
\]

Corollary 55. Let \( D \) be a diagram of \( L \) with alternating writhes. Then the map \( f_D \) induces an isomorphism \( \text{Dis}(f_D) : \text{Dis}(QMG(L)) \to \text{Dis}(Q^\text{red}_A(L)) \) of abelian groups, given by the formula

\[
\text{Dis}(f_D)(\prod \beta_{m_i}^{m_i}) = \prod \beta_{f_D(m_i)}^{m_i}.
\]

Proof. Corollary 30 and property 6 of Proposition 21 tell us that \( f_D \) induces an epimorphism \( \text{Dis}(f_D) \), with the given formula. In order to show that \( \text{Dis}(f_D) \) is injective, we assemble an inverse function from pieces that have been discussed before. Let \( a^* \in A(D) \) be fixed.

According to Corollary 30, \( \text{Dis}(Q^\text{red}_A(L)) \) is isomorphic to the \( \Lambda \)-submodule of \( (1-t)M^A_M(L) \) generated by \( \{ (1-t)(q-q') \mid q, q' \in Q^\text{red}_A(L) \} \). This submodule is \( (1-t) \ker \phi^\text{red}_L \). The isomorphism \( g : \text{Dis}(Q^\text{red}_A(L)) \to (1-t) \ker \phi^\text{red}_L \) given in the proof of Corollary 30 maps a displacement \( \prod \beta_{m_i}^{m_i} \) to the module element \( (1-t)\sum m_i q_i \). Therefore, if \( a \in A(D) \) then this isomorphism \( g \) has

\[
g(\beta_{\gamma_D(a) \otimes 1}^{m_i} \beta_{(a^*) \otimes 1}^{-1}) = (1-t)((\gamma_D(a) \otimes 1) - (\gamma_D(a^*) \otimes 1))
\]

\[
= (1-t)((\gamma_D(a) - \gamma_D(a^*)) \otimes 1).
\]

Let \( e_D : \ker \phi^\text{red}_L \to MG^0(L) \) be the \( \Lambda \)-linear map of Corollary 43, and let \( \text{ann}(1-t) = \{ m \in \ker \phi^\text{red}_L \mid (1-t)m = 0 \} \). Proposition 32 tells us that \( \text{ann}(1-t) \) is the submodule of \( \ker \phi^\text{red}_L \) generated by the longitudes \( \chi_1, \ldots, \chi_m \).

An argument just like the proof of Proposition 35 provides a homomorphism \( \beta : MG(L) \to \text{Aut}(QMG(L)) \), with \( \beta(g_a) = \beta_{g_a} \forall a \in A(D) \). This homomorphism maps each product \( \prod g_{a_i}^{m_i} \) to \( \prod \beta_{g_{a_i}}^{m_i} \), so it maps \( MG^0(L) \) onto \( \text{Dis}(QMG(L)) \). According to Corollary 26 and Theorem 12, the scalar multiplication operations in the \( \Lambda \)-modules \( MG^0(L) \) and \( \text{Dis}(QMG(L)) \) can be defined using conjugation by \( g_a \) and \( \beta_{g_a} \), respectively. Clearly then the homomorphism \( \beta \) is \( \Lambda \)-linear.
An argument just like the proof of the third property of Proposition 31 shows that for each \( i \in \{1, \ldots, \mu\} \), the restriction of \( \beta e_D(\chi_i) \) to the \( K_i \) orbit of \( QMG(L) \) is the identity map. Proposition 40 tells us that \( QMG(L) \) is semiregular, so it follows that each \( \beta e_D(\chi_i) \) is the identity map of \( QMG(L) \). That is, \( \chi_1, \ldots, \chi_\mu \in \ker(\beta e_D) \). Then Proposition 52 implies that \( \ker\text{ann}(1-t) \subseteq \ker(\beta e_D) \), so \( \beta e_D : \ker\phi_L^{\text{red}} \to \text{Dis}(QMG(L)) \) induces a \( \Lambda \)-linear map \( \ker\phi_L^{\text{red}}/\text{ann}(1-t) \to \text{Dis}(QMG(L)) \), given by \( x + \text{ann}(1-t) \mapsto \beta e_D(x) \).

There is a \( \Lambda \)-linear epimorphism \( \ker\phi_L^{\text{red}} \to (1-t) \cdot \ker\phi_L^{\text{red}} \) defined using scalar multiplication by \( 1-t \), and the kernel of this epimorphism is \( \ker\text{ann}(1-t) \). Hence \( (1-t) \cdot \ker\phi_L^{\text{red}} \) is isomorphic to \( \ker\phi_L^{\text{red}}/\text{ann}(1-t) \), and an isomorphism is given by \( (1-t) \cdot x \mapsto x + \text{ann}(1-t) \). Composing this isomorphism with the map induced by \( \beta e_D \) that was mentioned at the end of the preceding paragraph, we obtain a map

\[
\widehat{\beta e_D} : (1-t) \cdot \ker\phi_L^{\text{red}} \to \text{Dis}(QMG(L)),
\]
given by \( \widehat{\beta e_D}((1-t)x) = \beta e_D(x) \quad \forall x \in \ker\phi_L^{\text{red}} \).

Recall the isomorphism \( g \) mentioned in the second paragraph of the proof. The composition

\[
\widehat{\beta e_D} \circ g : \text{Dis}(Q_A^{\text{red}}(L)) \to \text{Dis}(QMG(L))
\]
is a \( \Lambda \)-linear map, and for every \( a \in A(D) \) it has

\[
(\widehat{\beta e_D} \circ g)(\text{Dis}(f_D)(\beta_{g_a}\beta_{g_a}^{-1})) = (\widehat{\beta e_D} \circ g)(\beta f_D(g_a)\beta f_D(g_a)^{-1})
\]
\[
= \widehat{\beta e_D}(g(\beta_{f_D(a)}\otimes_1\beta_{f_D(a)}^{-1})) = \widehat{\beta e_D}((1-t)((\gamma_D(a) - \gamma_D(a^*) \otimes 1))
\]
\[
= \beta e_D((\gamma_D(a) - \gamma_D(a^*) \otimes 1)) = \beta(h_a) = \beta(g_a^{-1}g_a^{-1}) = \beta_{g_a}\beta_{g_a}^{-1}.
\]

The elementary displacements \( \beta_{g_a}\beta_{g_a}^{-1} \) generate \( \text{Dis}(QMG(L)) \), so the composition \( \widehat{\beta e_D} \circ g \circ \text{Dis}(f_D) \) is the identity map of \( \text{Dis}(QMG(L)) \). As \( \text{Dis}(f_D) \) is surjective, it follows that \( \text{Dis}(f_D) \) is an isomorphism. \( \square \)

**Theorem 56.** If \( D \) is a diagram of \( L \) with alternating writhes, then the quandle map \( f_D : QMG(L) \to Q_A^{\text{red}}(L) \) is an isomorphism.

**Proof.** Suppose \( x \neq y \in QMG(L) \), and \( f_D(x) = f_D(y) \).

According to Definition 30 there are arcs \( a, b \in A(D) \) such that \( x \) is a conjugate of \( g_a \) and \( y \) is a conjugate of \( g_b \). Then the orbit of \( Q_A^{\text{red}}(L) \) containing \( f_D(x) = f_D(y) \) contains both \( \gamma_D(a) \otimes 1 \) and \( \gamma_D(b) \otimes 1 \), so \( \kappa_D(a) = \kappa_D(b) \). Considering the crossing relations that appear in the description of \( MG(L) \) in Definition 23, we deduce that \( g_a \) and \( g_b \) are conjugates of each other. Therefore \( x \) and \( y \) belong to the same orbit of \( QMG(L) \).

According to Proposition 22 \( y = d(x) \) for some \( d \in \text{Dis}(QMG(L)) \). As \( x \neq y \), \( d \) is not the identity map of \( QMG(L) \). According to Corollary 55 it follows that \( \text{Dis}(f_D)(d) \) is not the identity map of \( Q_A^{\text{red}}(L) \). As \( Q_A^{\text{red}}(L) \) is semiregular, this
implies that \((\text{Dis}(f_D)(d))(f_D(x)) \neq f_D(x)\). However, if \(d = \prod \beta_{y_i}^{m_i}\) then using the fact that \(f_D\) is a quandle map, we calculate

\[
(\text{Dis}(f_D)(d))(f_D(x)) = \left( \prod \beta_{f_D(y_i)}^{m_{f_D(y_i)}} \right) (f_D(x)) = f_D \left( \left( \prod \beta_{y_i}^{m_i} \right)(x) \right) = f_D(d(x)) = f_D(y) = f_D(x),
\]
a contradiction. We conclude that \(f_D\) is injective. \(\Box\)

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