On $q$-de Rham cohomology via $\Lambda$-rings

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Abstract
We show that Aomoto’s $q$-deformation of de Rham cohomology arises as a natural cohomology theory for $\Lambda$-rings. Moreover, Scholze’s $(q - 1)$-adic completion of $q$-de Rham cohomology depends only on the Adams operations at each residue characteristic. This gives a fully functorial cohomology theory, including a lift of the Cartier isomorphism, for smooth formal schemes in mixed characteristic equipped with a suitable lift of Frobenius. If we attach $p$-power roots of $q$, the resulting theory is independent even of these lifts of Frobenius, refining a comparison by Bhatt, Morrow and Scholze.

Introduction

The $q$-de Rham cohomology of a polynomial ring is a $\mathbb{Z}[q]$-linear complex given by replacing the usual derivative with the Jackson $q$-derivative $\nabla_q(x^n) = [n]_q x^{n-1} dx$, where $[n]_q$ is Gauss’ $q$-analogue $\frac{q^n - 1}{q - 1}$ of the integer $n$. In [13], Scholze discussed the $(q - 1)$-adic completion of this theory for smooth rings, explaining relations to $p$-adic Hodge theory and singular cohomology, and conjecturing that it is independent of co-ordinates, so functorial for smooth algebras over a fixed base [13, Conjectures 1.1, 3.1 and 7.1].

We show that $q$-de Rham cohomology with $q$-connections naturally arises as a functorial invariant of $\Lambda$-rings (Theorems 1.17, 1.23 and Proposition 1.25), and that its $(q - 1)$-adic completion depends only on a $\Lambda p$-ring structure (Theorem 2.8), for $P$ the set of residue characteristics; a $\Lambda p$-ring has a lift of Frobenius for each $p \in P$. This recovers the known equivalence between de Rham cohomology and complete $q$-de Rham cohomology over the rationals, while giving no really new functoriality statements for smooth schemes over $\mathbb{Z}$. However, in mixed characteristic, it means that
complete $q$-de Rham cohomology depends only on a lift $\Psi^p$ of absolute Frobenius locally generated by co-ordinates with $\Psi^p(x_i) = x_i^p$. Given such data, we construct (Proposition 2.10) a quasi-isomorphism between Hodge cohomology and $q$-de Rham cohomology modulo $[p]_q$, extending the local lift of the Cartier isomorphism in [13, Proposition 3.4].

Taking the Frobenius stabilisation of the complete $q$-de Rham complex of $A$ yields a complex resembling the de Rham–Witt complex. We show (Theorem 3.11) that up to $(q^{1/p^\infty} - 1)$-torsion, the $p$-adic completion of this complex depends only on the $p$-adic completion of $A[\zeta_{p^\infty}]$ (where $\zeta_n$ denotes a primitive $n$th root of unity), with no requirement for a lift of Frobenius or a choice of co-ordinates. The main idea is to show that the stabilised $q$-de Rham complex is in a sense given by applying Fontaine's period ring construction $A_{\inf}$ to the best possible perfectoid approximation to $A[\zeta_{p^\infty}]$. As a consequence, this shows (Corollary 3.13) that after attaching all $p$-power roots of $q$, $q$-de Rham cohomology in mixed characteristic is independent of choices, which was already known after base change to a period ring, via the comparisons of [4] between $q$-de Rham cohomology and their theory $A\Omega$.

The cohomology theories we construct thus depend either on Adams operations at the residue characteristics (for de Rham) or on $p$-power roots of $q$ (for variants of de Rham–Witt), establishing correspondingly weakened versions of the conjectures of [13]; in Remark 3.15, we suggest a possible candidate for a theory without those restrictions. The essence of our construction of $q$-de Rham cohomology of $A$ over $R$ is to set $q$ to be an element of rank 1 for the $\Lambda$-ring structure, and to look at flat $\Lambda$-rings $B$ over $R[q]$ equipped with morphisms $A \to B/(q - 1)$ of $\Lambda$-rings over $R$. If these seem unfamiliar, reassurance should be provided by the observation that $(q - 1)B$ carries $q$-analogues of divided power operations (Remark 1.4). For the variants of de Rham–Witt cohomology in Sect. 3, the key to giving a characterisation independent of lifts of Frobenius is the factorisation of the tilting equivalence for perfectoid algebras via a category of $\Lambda_p$-rings, leading to constructions similar to [4].

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1 Comparisons for \(\Lambda\)-rings

We will follow standard notational conventions for \(\Lambda\)-rings. These are commutative rings equipped with operations \(\lambda^i\) resembling alternating powers, in particular satisfying \(\lambda^k(a + b) = \sum_{i=0}^k \lambda^i(a)\lambda^{k-i}(b)\), with \(\lambda^0(a) = 1\) and \(\lambda^1(a) = a\). For background, see [5] and references therein. The \(\Lambda\)-rings we encounter are all torsion-free, in which case [16] shows the \(\Lambda\)-ring structure is equivalent to giving ring endomorphisms \(\Psi^n\) for \(n \in \mathbb{Z}_{>0}\) with \(\Psi^{mn} = \Psi^m \circ \Psi^n\) and \(\Psi^p(x) \equiv x^p \mod p\) for all primes \(p\). If we write \(\lambda_t(f) := \sum_{i \geq 0} \lambda^i(f)t^i\) and \(\Psi_t(f) := \sum_{n \geq 1} \Psi^n(f)t^n\), then the families of operations are related by the formula \(\Psi_t = -t\log \lambda_t\).

We refer to elements \(x\) with \(\lambda^i(x) = 0\) for all \(i > 1\) (or equivalently \(\Psi^n(x) = x^n\) for all \(n\)) as elements of rank 1.

1.1 The \(\Lambda\)-ring \(\mathbb{Z}[q]\)

Definition 1.1 Define \(\mathbb{Z}[q]\) to be the \(\Lambda\)-ring with operations determined by setting \(q\) to be of rank 1.

We now consider the \(q\)-analogues \([n]_q := \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q]\) of the integers, with \([n]_q! = [n]_q[n - 1]_q \ldots [1]_q\), and \(\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}\).

Remark 1.2 To see the importance of regarding \(\mathbb{Z}[q]\) as a \(\Lambda\)-ring observe that the binomial expressions

\[\lambda^k(n) = \binom{n}{k}, \quad \lambda^k(-n) = (-1)^k \binom{n+k-1}{k}\]

have as \(q\)-analogues the Gaussian binomial theorems

\[\lambda^k([n]_q) = q^{k(k-1)/2}\binom{n}{k}_q, \quad \lambda^k([-n]_q) = (-1)^k \binom{n+k-1}{k}_q,\]

as well as Adams operations

\[\Psi^i([n]_q) = [n]_{q^i}.\]

For any torsion-free \(\Lambda\)-ring, localisation at a set of elements closed under the Adams operations always yields another \(\Lambda\)-ring, since \(\Psi^p(a^{-1}) - a^{-p} = (\Psi^p(a)a^p)^{-1}(a^p - \Psi^p(a))\) is divisible by \(p\).

Lemma 1.3 For the \(\Lambda\)-ring structure on \(\mathbb{Z}[x, y]\) with \(x, y\) of rank 1, the elements

\[\lambda^n \left( \frac{y-x}{q-1} \right) \in \mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}, x, y]\]
are given by

$$\lambda^k \left( \frac{y-x}{q-1} \right) = \frac{(y-x)(y-qx)\ldots(y-q^{k-1}x)}{(q-1)^k[k]_q!},$$

$$= \sum_{j=0}^{k} \frac{q^j(j-1)/2(-x)^j y^{k-j}}{[j]_q! [k-j]_q!}.$$

**Proof** The second expression comes from multiplying out the Gaussian binomial expansions. The easiest way to prove the first is to observe that $\lambda^k(\frac{y-x}{q-1})$ must be a homogeneous polynomial of degree $k$ in $x, y$, with coefficients in the integral domain $\mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}]$, and to note that

$$\lambda^k \left( \frac{q^n x - x}{q-1} \right) = \lambda^k ([n]_q x) = q^{k(k-1)/2} \binom{n}{k}_q \lambda^k.$$

Thus $\lambda^k(\frac{y-x}{q-1})$ agrees with the homogeneous polynomial above for infinitely many values of $\frac{y}{x}$, so must be equal to it. \qed

**Remark 1.4** Note that as $q \to 1$, Lemma 1.3 gives $(q-1)^k \lambda^k(\frac{y-x}{q-1}) \to \frac{(x-y)^k}{k!}$. Indeed, for any rank 1 element $x$ in a $\Lambda$-ring we have

$$\lambda_{(q-1)t} \left( \frac{x}{q-1} \right) = \sum_{k \geq 0} \frac{(xt)^k}{[k]_q!},$$

which is just the $q$-exponential $e_q(xt)$. Multiplicativity and universality then imply that $\lambda_{(q-1)t}(\frac{a}{q-1})$ is a $q$-deformation of $\exp(at)$ for all $a$. Thus $(q-1)^k \lambda^k(\frac{a}{q-1})$ is a $q$-analogue of the $k$th divided power $(a^k/k!)$. An explicit expression comes recursively from the formula

$$[k]_q (q-1) \lambda^k \left( \frac{a}{q-1} \right) = \sum_{i \geq 0} \lambda^i(a) \lambda^{k-i} \left( \frac{a}{q-1} \right),$$

obtained by subtracting $\lambda_i(\frac{a}{q-1})$ from each side of the expression $\lambda_{qt}(\frac{a}{q-1}) = \lambda_t(a) \lambda_t(\frac{a}{q-1})$, which arises because $q$ is of rank 1 and $\frac{aa}{q-1} = a + \frac{a}{q-1}$.

**Lemma 1.5** For elements $x, y$ of rank 1, the $\Lambda$-subring of $\mathbb{Z}[q, \{(q^n - 1)^{-1}\}_{n \geq 1}, x, y]$ generated by $q, x, y, \frac{y-x}{q-1}$ has basis $\lambda^k(\frac{y-x}{q-1})$ as a $\mathbb{Z}[q, x]$-module.

**Proof** The $\Lambda$-subring clearly contains the $\mathbb{Z}[q, x]$-module $M$ generated by the elements $\lambda^k(\frac{y-x}{q-1})$, which are also clearly $\mathbb{Z}[q, x]$-linearly independent. Since $\mathbb{Z}[q, x]$ is a $\Lambda$-ring, it suffices to show that $M$ is closed under multiplication.

By Lemma 1.3, we know that

$$\lambda^i \left( \frac{y-x}{q-1} \right) \lambda^j \left( \frac{y-q^i x}{q-1} \right) = \binom{i+j}{i}_q \lambda^{i+j} \left( \frac{y-x}{q-1} \right).$$
We can rewrite \( \frac{y-x}{q-1} = \frac{y-x}{q-1} - [i] \cdot y \), so \( \lambda^i \left( \frac{y-x}{q-1} \right) - \lambda^j \left( \frac{y-x}{q-1} \right) \) lies in the \( \mathbb{Z}[q, x] \)-module spanned by \( \lambda^m \left( \frac{y-x}{q-1} \right) \) for \( m < j \). By induction on \( j \), it thus follows that

\[
\lambda^i \left( \frac{y-x}{q-1} \right) \lambda^j \left( \frac{y-x}{q-1} \right) - \lambda^j \left( \frac{y-x}{q-1} \right) \lambda^i \left( \frac{y-x}{q-1} \right) \in M,
\]

so the binomial expression above implies \( \lambda^i \left( \frac{y-x}{q-1} \right) \lambda^j \left( \frac{y-x}{q-1} \right) \in M. \)

\[\square\]

### 1.2 q-cohomology of \( \Lambda \)-rings

**Definition 1.6** Given a \( \Lambda \)-ring \( R \), say that \( A \) is a \( \Lambda \)-ring over \( R \) if it is a \( \Lambda \)-ring equipped with a morphism \( R \to A \) of \( \Lambda \)-rings. We say that \( A \) is a flat \( \Lambda \)-ring over \( R \) if \( A \) is flat as a module over the commutative ring underlying \( R \).

**Definition 1.7** Given a morphism \( R \to A \) of \( \Lambda \)-rings, we define the category \( \text{Strat}^q_{A/R} \) to consist of flat \( \Lambda \)-rings \( B \) over \( R[q] \) equipped with a compatible morphism \( f : A \to B/(q-1) \), such that \( f \) admits a lift to \( B \); a choice of lift is not taken to be part of the data, so need not be preserved by morphisms.

More concisely, \( \text{Strat}^q_{A/R} \) is the Grothendieck construction of the set-valued functor

\[
(Spec \, A)^q_{\text{strat}} : B \mapsto \text{Im} \left( \text{Hom}_{\Lambda, R}(A, B) \to \text{Hom}_{\Lambda, R}(A, B/(q-1)) \right)
\]

on the category \( f \Lambda(R[q]) \) of flat \( \Lambda \)-rings over \( R[q] \).

**Definition 1.8** Given a flat morphism \( R \to A \) of \( \Lambda \)-rings, define \( \text{qDR}(A/R) \) to be the cochain complex of \( R[q] \)-modules given by taking the homotopy limit (in the sense of [6]) of the functor

\[
\text{Strat}^q_{A/R} \to \text{Ch}(R[q])
\]

\( B \mapsto B \).

The cochain complex \( \text{qDR}(A/R) \) naturally carries \( (R[q], \Psi^n) \)-semilinear operations \( \Psi^n \) coming from the morphisms \( \Psi^n : B \otimes_{R[q], \Psi^n} R[q] \to B \) of \( R[q] \)-modules, for \( B \in \text{Strat}^q_{A/R} \).

Equivalently, can we follow the approach of [8,14] towards the stratified site and de Rham stack by regarding \( \text{qDR}(A/R) \) as the quasi-coherent cohomology complex of \( (Spec \, A)^q_{\text{strat}} \), as follows.

**Definition 1.9** Given a category \( \mathcal{C} \), write \([\mathcal{C}, \text{Set}]\) and \([\mathcal{C}, \text{Ab}]\) for the categories of functors on \( \mathcal{C} \) taking values in sets and abelian groups, respectively. For any functor \( X : \mathcal{C} \to \text{Set} \), we then denote by \( \text{RHom}_{[\mathcal{C}, \text{Set}]}(X, -) \) the functor from \([\mathcal{C}, \text{Ab}]\) to cochain complexes given by taking the right-derived functor of the functor

\[
\text{Hom}_{[\mathcal{C}, \text{Set}]}(X, -) : [\mathcal{C}, \text{Ab}] \to \text{Ab}
\]

of natural transformations with source \( X \).
For the forgetful functor $\mathcal{O} : f \Lambda(R[q]) \to \text{Mod}(R[q])$ to the category of $R[q]$-modules, we then have

$$q\text{DR}(A/R) = \text{RHom}_{f \Lambda(R[q]).\text{Set}}((\text{Spec } A)^q, \mathcal{O}),$$

with Adams operations $\Psi^n : \mathcal{O} \otimes_{R[q], \psi^n} R[q] \to \mathcal{O}$ giving the $(R[q], \Psi^n)$-semilinear operations $\Psi^n$ on $q\text{DR}(A/R)$.

**Remark 1.10** The cochain complex $q\text{DR}(A/R)$ naturally carries much more structure than these Adams operations. Whenever we can factor the functor $\mathcal{O}$ through a model category $\mathcal{C}$ equipped with a forgetful functor to Ch($R[q]$) preserving weak equivalences and homotopy limits, we can regard $q\text{DR}(A/R)$ as an object of the homotopy category of $\mathcal{C}$ by taking the defining homotopy limit in $\mathcal{C}$.

The universal such example for $\mathcal{C}$ is given by the model category of cosimplicial $\Lambda$-rings over $R[q]$, with weak equivalences being quasi-isomorphisms (i.e. cohomology isomorphisms) and fibrations being surjections; the underlying cochain complex has differential $\sum (-1)^j \partial j$. That this determines a model structure follows from Kan’s transfer theorem [9, Theorem 11.3.2] applied to the cosimplicial Dold–Kan normalisation functor taking values in unbounded chain complexes with the projective model structure; the conditions of that theorem are satisfied because the left adjoint functor sends acyclic cofibrant complexes to cosimplicial $\Lambda$-rings which automatically have a contracting homotopy in the form of an extra codegeneracy map.

In particular, $q\text{DR}(A/R)$ naturally underlies a quasi-isomorphism class of cosimplicial $\Lambda$-rings over $R[q]$; forgetting the $\lambda$-operations gives a cosimplicial commutative $R[q]$-algebra, and stabilisation then gives an $E_\infty$-algebra over $R[q]$, all with underlying cochain complex $q\text{DR}(A/R)$.

**Definition 1.11** Given a polynomial ring $R[x]$, recall from [13] that the $q$-de Rham (or Aomoto–Jackson) cohomology $q-\Omega^\bullet_{R[x]/R}$ is given by the complex

$$R[x][q] \xrightarrow{\nabla_q} R[x][q]dx, \quad \text{where} \quad \nabla_q(f) = \frac{f(qx) - f(x)}{x(q - 1)} dx,$$

so $\nabla_q(x^n) = [n]_q x^{n-1} dx$.

Given a polynomial ring $R[x_1, \ldots, x_d]$, the $q$-de Rham complex $q-\Omega^\bullet_{R[x_1,\ldots,x_d]/R}$ is then set to be

$$q-\Omega^\bullet_{R[x_1]/R} \otimes R[q] q-\Omega^\bullet_{R[x_2]/R} \otimes R[q] \ldots \otimes R[q] q-\Omega^\bullet_{R[x_d]/R},$$

so takes the form

$$R[x_1, \ldots, x_d][q] \xrightarrow{\nabla_q} \Omega^1_{R[x_1,\ldots,x_d]/R}[q] \xrightarrow{\nabla_q} \ldots \xrightarrow{\nabla_q} \Omega^d_{R[x_1,\ldots,x_d]/R}[q].$$

**Definition 1.12** Given a flat morphism $R \to A$ of $\Lambda$-rings with $X = \text{Spec } A$, define the functor $X^q_{\text{strat}}$ from flat $\Lambda$-rings over $R[q]$ to simplicial sets by taking the Čech nerve of $\text{Hom}_{\Lambda,R}(A, B) \to \text{Hom}_{\Lambda,R}(A, B/(q - 1))$, so

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\[(\mathcal{X}^q_{\text{strat}})_n(B) := \underbrace{\text{Hom}_{\Lambda,R}(A, B) \times \text{Hom}_{\Lambda,R}(A, B/q(1)) \cdots \times \text{Hom}_{\Lambda,R}(A, B/q(1))}_{n+1} \text{Hom}_{\Lambda,R}(A, B) \]

\[= \text{Hom}_{\Lambda,R}(A, B \times B/(q-1) \cdots \times B/(q-1) B),\]

with simplicial operations

\[\partial_j(f_0, f_1, \ldots, f_n) := (f_0, f_1, \ldots, f_{j-1}, f_{j+1}, f_{j+2}, \ldots, f_n),\]

\[\sigma_j(f_0, f_1, \ldots, f_n) := (f_0, f_1, \ldots, f_j, f_j, f_{j+1}, \ldots, f_n).\]

**Definition 1.13** Given a cosimplicial abelian group \(V^\bullet\), we write \(NV\) for the Dold–Kan normalisation of \(V\) ([15, Lemma 8.3.7] applied the opposite category). This is a cochain complex with \(N^r V = V^r \cap \ker \sigma^r\) and differential \(d = \sum_{j=0}^{r+1} (-1)^j \partial_j : N^r V \to N^{r+1} V\).

**Lemma 1.14** If, for \(X = \text{Spec } A\), the functors \((\mathcal{X}^q_{\text{strat}})_n\) are represented by flat \(\Lambda\)-rings \(\Gamma((\mathcal{X}^q_{\text{strat}})_n, \mathcal{O})\) over \(R[q]\), then a model for \(q\text{DR}(A/R)\) is given by the Dold–Kan normalisation of the cosimplicial module \(n \mapsto \Gamma((\mathcal{X}^q_{\text{strat}})_n, \mathcal{O}).\)

**Proof** The set-valued functor \(X^q_{\text{strat}} = (\text{Spec } A)^q_{\text{strat}}\) of Definition 1.7 is resolved by the simplicial functor \(\mathcal{X}^q_{\text{strat}}\) of Definition 1.12. In the notation of Definition 1.9, this implies that the functor \(\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}(X^q_{\text{strat}}, -)\) on \([f\Lambda(R[q]), \text{Ab}]\) is resolved by the cochain complex

\[N\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}(\mathcal{X}^q_{\text{strat}}, -).\]

Although \(X^q_{\text{strat}}\) is not representable on the category of flat \(\Lambda\)-rings over \(R[q]\), our hypotheses ensure that each functor \((\mathcal{X}^q_{\text{strat}})_n\) is so. Thus the functors \(\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}(X^q_{\text{strat}}, -)\) and their direct summands \(N^n\text{Hom}_{[f\Lambda(R[q]), \text{Set}]}(\mathcal{X}^q_{\text{strat}}, -)\) are exact, and are their own right-derived functors. This implies that the cochain complex of functors above models \(\text{RHom}_{[f\Lambda(R[q]), \text{Set}]}(X^q_{\text{strat}}, -)\), and the result follows by evaluation at \(\mathcal{O}\). \(\square\)

**Proposition 1.15** If \(R\) is a \(\Lambda\)-ring and \(x\) of rank 1, then \(q\text{DR}(R[x]/R)\) can be calculated by Dold–Kan normalisation of the cosimplicial \(R[q]\)-module \(U^\bullet\) given by setting \(U^n\) to be the \(\Lambda\)-subring

\[U^n \subset R[q, \{(q^m - 1)^{-1}\}_{m \geq 1}, x_0, \ldots, x_n]\]

generated by \(q\) and the elements \(x_i\) and \(\frac{x_i - x_j}{q-1}\), with cosimplicial operations

\[\partial^i x_i := \begin{cases} x_i & j > i \\ x_{i+1} & j \leq i \end{cases}, \quad \sigma^i x_i := \begin{cases} x_i & j \geq i \\ x_{i-1} & j < i \end{cases}.\]
Proof We verify the conditions of Lemma 1.14 by showing that each $U^n$ is a flat $\Lambda$-ring over $R[q]$ representing $(\hat{X}^q_{\text{strat}})_n$. Taking $X = \text{Spec } R[x]$, observe that any element of $(\hat{X}^q_{\text{strat}})_n(B)$ gives rise to a morphism $f : R[q, x_0, \ldots, x_n] \to B$ of $\Lambda$-rings over $R[q]$, with the image of $x_i - x_j$ divisible by $(q - 1)$. Flatness of $B$ then gives a unique element $f(x_i - x_j)/(q - 1) \in B$, so we have a map $f$ to $B$ from the free $\Lambda$-ring $L$ over $R[q, x_0, \ldots, x_n]$ generated by elements $z_{ij}$ with $(q - 1)z_{ij} = x_i - x_j$.

Since $B$ is flat, it embeds in $B[\{(q^m - 1)^{-1}\}_{m \geq 1}]$ (the only hypothesis we really need) implying that the image of $f$ factors through the image $U^n$ of $L$ in $R[q, \{(q^m - 1)^{-1}\}_{m \geq 1}, x_0, \ldots, x_n]$. To see that $(\hat{X}^q_{\text{strat}})_n$ is represented by $U^n$, we only now need to check that $U^n$ is itself flat over $R[q]$, which follows because the argument of Lemma 1.5 gives a basis

$$x_0^{r_0} \lambda_1^{r_1} \left( \frac{x_1 - x_0}{q-1} \right) \cdots \lambda_n^{r_n} \left( \frac{x_n - x_{n-1}}{q-1} \right)$$

for $U^n$ over $R[q]$. We therefore have $\text{qDR}(R[x]/R) \simeq NU^\bullet$. \hfill \Box

In fact, the proofs of Lemma 1.14 and Proposition 1.15 show that the natural cosimplicial $\Lambda$-ring structure on $U^\bullet$ gives a model for the cosimplicial $\Lambda$-ring structure on $\text{qDR}(R[x]/R)$ coming from Remark 1.10.

Definition 1.16 Following [13, Proposition 5.4], we denote by $L\eta_{(q-1)}$ the décalage functor with respect to the derived $(q - 1)$-adic filtration. This is given on complexes $C^\bullet$ of $(q - 1)$-torsion-free $R[q]$-modules by

$$(\eta_{(q-1)}C)^n := \{c \in (q - 1)^n C^n : dc \in (q - 1)^{n+1} C^{n+1}\},$$

and is extended to the derived category of $R[q]$-modules by taking torsion-free resolutions.

Theorem 1.17 If $R$ is a $\Lambda$-ring and if the polynomial ring $R[x_1, \ldots, x_n]$ is given the $\Lambda$-ring structure for which the elements $x_i$ are of rank 1, then there are $R[q]$-linear zigzags of quasi-isomorphisms

$$\text{qDR}(R[x_1, \ldots, x_n]/R) \simeq (\Omega^\bullet_{R[x_1, \ldots, x_n]}/R[q], (q - 1)\nabla_q)$$

$$L\eta_{(q-1)} \text{qDR}(R[x_1, \ldots, x_n]/R) \simeq q^{-}\Omega^\bullet_{R[x_1, \ldots, x_n]/R}.$$

Proof It suffices to prove the first statement, the second following immediately by décalage. We have \( (\text{Spec } A \otimes_R A')_{\text{strat}}^q(B) = (\text{Spec } A')_{\text{strat}}^q(B) \times (\text{Spec } A')_{\text{strat}}^q(B) \), and similarly for the simplicial functor \( (\text{Spec } A \otimes_R A')_{\text{strat}}^q \). Since coproduct of flat $\Lambda$-rings over $R[q]$ is given by $\otimes_{R[q]}$, it follows from Lemma 1.14 and Proposition 1.15 that $\text{qDR}(R[x_1, \ldots, x_n]/R)$ can be calculated as the Dold–Kan normalisation of $(U^\bullet)^{\otimes_{R[q]} n}$ (given by the $n$-fold tensor product $(U^m)^{\otimes_{R[q]} n}$ in cosimplicial level $m$), for the cosimplicial module $U^\bullet$ of Proposition 1.15.

The proof now proceeds in a similar fashion to the comparison between crystalline and de Rham cohomology in [3]. We consider the cochain complexes $\tilde{\Omega}^\bullet(U^m)$ given by

\begin{itemize}
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\end{itemize}
$$U^m \xrightarrow{(q-1)\nabla_q} \bigoplus_i U^m dx_i \xrightarrow{(q-1)\nabla_q} \bigoplus_{i<j} U^m dx_i \wedge dx_j \xrightarrow{(q-1)\nabla_q} \ldots.$$ 

In order to see that this differential takes values in the codomains given, observe that

$$(q-1)\nabla_q y^k \left( \frac{y-x}{q-1} \right) = y^{-1} \left( \lambda^k \left( \frac{qy-x}{q-1} \right) - \lambda^k \left( \frac{y-x}{q-1} \right) \right) dy$$

$$= y^{-1} \left( \lambda^k \left( y + \frac{y-x}{q-1} \right) - \lambda^k \left( \frac{y-x}{q-1} \right) \right) dy$$

$$= \lambda^{k-1} \left( \frac{y-x}{q-1} \right) dy,$$

and similarly

$$(q-1)\nabla_q x^i \lambda^k \left( \frac{y-x}{q-1} \right) = \sum_{i \geq 1} (-1)^i x^{i-1} \lambda^{k-i} \left( \frac{y-x}{q-1} \right) dx.$$

The first calculation also shows that the inclusion $\tilde{\Omega}^\bullet(U^{m-1}) \hookrightarrow \tilde{\Omega}^\bullet(U^m)$ is a quasi-isomorphism, since for $\omega \in \tilde{\Omega}^\bullet(U^{m-1})$, we have

$$(q-1)\nabla_q x^m \omega \lambda^k \left( \frac{x_m-x_{m-1}}{q-1} \right) = \omega \lambda^{k-1} \left( \frac{x_m-x_{m-1}}{q-1} \right) dx_m$$

for $k \geq 1$, allowing us to define a contracting homotopy

$$h \left( \omega \lambda^{k-1} \left( \frac{x_m-x_{m-1}}{q-1} \right) dx_m \right) := \omega \lambda^k \left( \frac{x_m-x_{m-1}}{q-1} \right),$$

$$h \left( \omega \lambda^{k-1} \left( \frac{x_m-x_{m-1}}{q-1} \right) \right) := 0.$$ 

Since contracting homotopies interact well with tensor products, it also follows that the inclusion $\tilde{\Omega}^\bullet(U^{m-1}) \otimes_{R[q]} \n \hookrightarrow \tilde{\Omega}^\bullet(U^m) \otimes_{R[q]} \n$ is a quasi-isomorphism. By induction on $m$ we deduce that the inclusions $\tilde{\Omega}^\bullet(U^0) \otimes_{R[q]} \n \hookrightarrow \tilde{\Omega}^\bullet(U^m) \otimes_{R[q]} \n$, and hence their retractions given by diagonals $U^m \to U^0$, are quasi-isomorphisms. These combine to give a quasi-isomorphism

$$\text{Tot} N(\tilde{\Omega}^\bullet(U^\bullet)) \to \tilde{\Omega}^\bullet(U^0) \otimes_{R[q]} \n = \tilde{\Omega}^\bullet(R[x]) \otimes_{R[q]} \n$$

on total complexes of normalisations.

Now, the cosimplicial module $\tilde{\Omega}^\bullet(U^\bullet)$ is given by the cosimplicial (i.e. levelwise) tensor product of $U^\bullet$ with the cosimplicial $\mathbb{Z}$-module

$$j \mapsto \bigoplus_{0 \leq i_1 < i_2 < \ldots < i_r \leq j} \mathbb{Z} dx_{i_1} \wedge \cdots \wedge dx_{i_r}.$$
with operations induced by those in Proposition 1.15. For $r > 0$, this cosimplicial $\mathbb{Z}$-module is contractible, via the extra codegeneracy map given by

$$\sigma^{-1}(dx_{i_1} \wedge \cdots \wedge dx_{i_r}) = \begin{cases} dx_{i_1 - 1} \wedge \cdots \wedge dx_{i_r - 1} & i_1 > 0, \\ 0 & i_1 = 0. \end{cases}$$

The Eilenberg–Zilber theorem ([15, §8.5] applied to the opposite category) ensures that the normalisation of a cosimplicial tensor product is quasi-isomorphic to the tensor product of the normalisations. Tensoring with a complex which has an extra codegeneracy map always produces an acyclic complex, so $\tilde{\Omega}^r(U^\bullet)$ and its tensor powers are all acyclic for $r > 0$.

The brutal truncation maps

$$\text{Tot} N(\tilde{\Omega}^\bullet(U^\bullet)^{\otimes R[q]^n}) \to N(U^\bullet)^{\otimes R[q]^n} \simeq \text{qDR}(R[x_1, \ldots, x_n]/R)$$

are therefore quasi-isomorphisms of flat cochain complexes over $R[q]$, so

$$\text{qDR}(R[x_1, \ldots, x_n]/R) \simeq \tilde{\Omega}^\bullet(R[x])^{\otimes R[q]^n},$$

and we just observe that $\tilde{\Omega}^\bullet(R[x]) = (\Omega^*_R[x]/R, (q - 1)\nabla_q)$. \hfill \Box

**Remark 1.18** Note that Theorem 1.17 and Remark 1.10 together imply that $q$-$\text{qDR}(x_1, \ldots, x_n)/R$ naturally underlies the décalage of a cosimplicial $\Lambda$-ring over $R[q]$. Even the underlying cosimplicial commutative ring structure carries more information than an $E_{\infty}$-structure when $\mathbb{Q} \not\subseteq R$.

### 1.3 Completed $q$-cohomology

**Definition 1.19** Given a morphism $R \to A$ of $\Lambda$-rings, we define the category $\text{Strat}^q_A/R \subset \text{Strat}^q \Lambda_{/R}$ to consist of those objects which are $(q - 1)$-adically complete.

Equivalently, $\text{Strat}^q_A/R$ is the Grothendieck construction of the functor

$$(\text{Spec } A)^q_{\text{strat}} : B \mapsto \text{Im } (\text{Hom}_{\Lambda_R}(A, B) \to \text{Hom}_{\Lambda,R}(A, B/(q - 1))).$$

on the category of flat $(q - 1)$-adically complete $\Lambda$-rings over $R[q]$.

**Definition 1.20** Given a flat morphism $R \to A$ of $\Lambda$-rings, define $\text{qDR}(A/R)$ to be the cochain complex of $R[q - 1]$-modules given by taking the homotopy limit of the functor

$$\text{Strat}^q_{A/R} \to \text{Ch}(R[q - 1])$$

$$B \mapsto B.$$

The following is immediate:
Lemma 1.21 Given a flat morphism $R \to A$ of $\Lambda$-rings, the complex $\widehat{q\text{DR}}(A/R)$ is the derived $(q - 1)$-adic completion of $q\text{DR}(A/R)$.

Definition 1.22 As in [13, §3], given a formally étale map $\square : R[x_1, \ldots, x_d] \to A$, define $q\widehat{\Omega}_{A/R, \square}$ to be the complex

$$A(q - 1) \xrightarrow{\nabla_1} \Omega_{A/R}[q - 1] \xrightarrow{\nabla_2} \cdots \xrightarrow{\nabla_d} \Omega_{A/R}[q - 1],$$

where $\nabla_q$ is defined as follows. First note that the $R[q - 1]$-linear ring endomorphisms $\gamma_i$ of $R[x_1, \ldots, x_d][q - 1]$ given by $\gamma_i(x_j) = q^b_{ij} x_j$ extend uniquely to endomorphisms of $A[q - 1]$ which are the identity modulo $(q - 1)$, then set

$$\nabla_q(f) := \sum_i \frac{\gamma_i(f) - f}{(q - 1)x_i} dx_i.$$

Note that $q\widehat{\Omega}_{R[x_1, \ldots, x_d]/R}$ is just the $(q - 1)$-adic completion of $q\widehat{\Omega}_{R[x_1, \ldots, x_d]/R}$.

Theorem 1.23 If $R$ is a flat $\Lambda$-ring over $\mathbb{Z}$ and $\square : R[x_1, \ldots, x_d] \to A$ is a formally étale map of $\Lambda$-rings, the elements $x_i$ having rank 1, then there are zigzags of $R[q]$-linear quasi-isomorphisms

$$\widehat{q\text{DR}}(A/R) \simeq (\Omega^*_{A/R}[q - 1], (q - 1)\nabla_q), \quad L\eta_{(q - 1)}\widehat{q\text{DR}}(A/R) \simeq q\widehat{\Omega}_{A/R, \square}.$$

The induced quasi-isomorphisms

$$\widehat{q\text{DR}}(A/R) \otimes R[q - 1] R \simeq (\Omega^*_{A/R}, 0), \quad (L\eta_{(q - 1)}\widehat{q\text{DR}}(A/R)) \otimes R[q - 1] R \simeq \Omega^*_{A/R}$$

are independent of the choice of framing.

Proof Since the framing $\square$ is formally étale, for any $(q - 1)$-adically complete commutative $R[q]$-algebra $B$, any commutative square

$$\begin{array}{ccc}
R[x_1, \ldots, x_d] & \to & B \\
\square & & \downarrow \\
A & \to & B/(q - 1).
\end{array}$$

of $R$-algebra homomorphisms admits a unique dashed arrow as shown.

For any $(q - 1)$-adically complete flat $\Lambda$-ring $B$ over $R$, we then have the same property for $\Lambda$-ring homomorphisms over $R$ instead of $R$-algebra homomorphisms: the diagram above gives a unique dashed $R$-algebra homomorphism, and uniqueness of lifts ensures that it commutes with Adams operations, so is a $\Lambda$-ring homomorphism ($R$ being flat over $\mathbb{Z}$). Similarly (taking $B = A[q - 1]$) uniqueness of lifts ensures that the operations $\gamma_i$ are $\Lambda$-ring endomorphisms of $A[q - 1]$. 

We can now proceed as in the proof of Theorem 1.17. The complex $q\hat{\Omega}^\bullet(\hat{U}(A))$ can be realised as the cochain complex underlying a cosimplicial $\Lambda$-ring $\hat{U}(A)$, representing the functor $\hat{X}^q_{\text{strat}}$ of Definition 1.12 for $X = \text{Spec } A$, restricted to $(q - 1)$-adically complete $\Lambda$-rings $B$. By the consequences of formal étaleness, we have

$$\text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(A, B/(q-1))} \text{Hom}_{\Lambda, R}(A, B) \cong \text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(A, B/(q-1))} \text{Hom}_{\Lambda, R}(R[x_1, \ldots, x_d], B),$$

giving $(\hat{X}^q_{\text{strat}})_n \cong \text{Hom}_{\Lambda, R}(A, B) \times_{\text{Hom}_{\Lambda, R}(R[x_1, \ldots, x_d], B)} (\hat{Y}^q_{\text{strat}})_n$ for each $n$, where $Y = \text{Spec } R[x_1, \ldots, x_d]$ and the fibre product is given via the projection of $(\hat{Y}^q_{\text{strat}})_n$ onto the first factor.

In particular, this means that $\hat{U}(A)^n$ is the $(q - 1)$-adic completion of

$$A \otimes_{R[x_1, \ldots, x_d]} (U(R[x_1])^n \otimes_{R[q]} \cdots \otimes_{R[q]} U(R[x_d])^n),$$

where each $U(R[x_i])$ is a copy of the cosimplicial ring $U$ from Proposition 1.15. This isomorphism respects the cosimplicial operations; note that $\partial^0$ is not linear for the left multiplication by $A$, but is still determined via formal étaleness of the framing.

We now define a cosimplicial cochain complex $\hat{\Omega}^\bullet(\hat{U}(A))$ by setting $\hat{\Omega}^\bullet(\hat{U}(A))^n$ to be the $(q - 1)$-adic completion of

$$(A \otimes_{R[x_1, \ldots, x_d]} (\hat{\Omega}^\bullet(U(R[x_1])^n) \otimes_{R[q]} \cdots \otimes_{R[q]} \hat{\Omega}^\bullet(U(R[x_d])^n)), (q - 1)\nabla_q) \cong (\hat{U}(A)^n \otimes_{A^{\otimes(n+1)}} (\hat{\Omega}^\bullet_{\Lambda/R})^{\otimes(n+1)}, (q - 1)\nabla_q)).$$

where each $\hat{\Omega}^\bullet(U(R[x_i]))$ is a copy of the complex $\hat{\Omega}^\bullet(U^n)$ from the proof of Theorem 1.17. Compatibility of this construction with the cosimplicial operations follows because the $\gamma_i$ are $\Lambda$-ring homomorphisms.

The calculations contributing to the proof of Theorem 1.17 are still valid after base change, with contracting homotopies giving quasi-isomorphisms

$$(\Omega^\bullet_{\Lambda/R}[q]), (q - 1)\nabla_q) \leftrightarrow \text{Tot } N \hat{\Omega}^\bullet(\hat{U}(A))^\bullet \rightarrow N \hat{U}(A)^\bullet.$$

Reduction of this modulo $(q - 1)^2$, or of its décalage modulo $(q - 1)$ (cf. [4, Proposition 6.12]), replaces $\nabla_q$ with $d$ throughout, removing any dependence on co-ordinates. □

As in [13, Definition 7.3], there is a notion of $q$-connection $\nabla_q = (\nabla_{1,q}, \ldots, \nabla_{d,q})$ on a finite projective $A[q - 1]$-module $M$, in the form of commuting $R[q - 1]$-linear operators $\nabla_{i,q}$ on $M$, with each $\nabla_{i,q}$ satisfying $\nabla_{i,q}(av) = \nabla_{q,x_i}(a)v + \gamma_i(a)\nabla_{i,q}(v)$ for $a \in A$, $v \in M$.

**Definition 1.24** Given a flat morphism $R \rightarrow A$ of $\Lambda$-rings with $X := \text{Spec } A$, denote the forgetful functor $(B, f) \mapsto B$ from $	ext{Strat}^\bullet_{A/R}$ to rings by $\hat{\mathcal{O}}_{\hat{X}^q_{\text{strat}}}$.

There is then a notion of $\hat{\mathcal{O}}_{\hat{X}^q_{\text{strat}}}$-modules in the category of functors from $	ext{Strat}^\bullet_{A/R}$ to abelian groups; we will simply refer to these as $\hat{\mathcal{O}}_{\hat{X}^q_{\text{strat}}}$-modules. Given a property
P of modules, we will say that an $\mathcal{O}_{\hat{X}^q, \text{strat}}$-module $\mathcal{F}$ has the property $P$ if for each $(B, f) \in \text{Strat}^q_{A/R}$, the $B$-module $\mathcal{F}(B, f)$ has property $P$.

We say that an $\mathcal{O}_{\hat{X}^q, \text{strat}}$-module $\mathcal{F}$ is Cartesian if for each morphism $(B, f) \to (B', f')$ in $\text{Strat}^q_{A/R}$, the map $\mathcal{F}(B, f) \otimes_B B' \to \mathcal{F}(B', f')$ is an isomorphism.

Given an $\mathcal{O}_{\hat{X}^q, \text{strat}}$-module $\mathcal{F}$, we define $\Gamma(\hat{X}^q_{\text{strat}}, \mathcal{F}) := \lim_{\text{Strat}^q_{A/R}} \mathcal{F}$.

In [13, Conjecture 7.5], Scholze predicted that the category of $q$-connections on finite projective $A[q - 1]$-module is independent of co-ordinates on $A$. The following proposition gives the weaker statement that the category depends only on the $\Lambda$-ring structure on $A$.

**Proposition 1.25** Under the conditions of Theorem 1.23, with $X := \text{Spec } A$, the category of finite projective $A[q - 1]$-modules $(M, \nabla)$ with $q$-connection is equivalent to the category of those finite projective $\mathcal{O}_{\hat{X}^q, \text{strat}}$-modules $\mathcal{N}$ for which the map

$$\Gamma(\hat{X}^q_{\text{strat}}, \mathcal{N}/(q - 1)) \otimes_A (\mathcal{O}_{\hat{X}^q, \text{strat}}/(q - 1)) \to \mathcal{N}/(q - 1)$$

is an isomorphism.

**Proof** The restriction on $\mathcal{N}/(q - 1)$ ensures that it is Cartesian; this also implies that $\mathcal{N}$ is Cartesian, because finite projective modules are flat and $(q - 1)$-adically complete.

Now, the cosimplicial $\Lambda$-ring $\hat{U}(A)$ realising $\hat{\text{DR}}(A/R)$ in the proof of Theorem 1.23 admits a natural map $A \to \hat{U}(A)/(q - 1)$ from the constant cosimplicial diagram. Thus $\hat{U}(A)$ defines a cosimplicial diagram in $\text{Strat}^q_{A/R}$. Since the functor $\hat{X}^q_{\text{strat}}$ of Definition 1.12 resolves $X^q_{\text{strat}}$, it follows that the functor $\hat{U}(A) : \Delta \to \text{Strat}^q_{A/R}$ from the simplex category is initial in the sense of [11, §IX.3].

In particular, this means that the category of Cartesian $\mathcal{O}_{\hat{X}^q, \text{strat}}$-modules $\mathcal{N}$ is equivalent to the category of Cartesian cosimplicial $\hat{U}(A)$-modules $\mathcal{N}$, where the Cartesian condition amounts to saying that the maps $\mathcal{N}^m \otimes_{\hat{U}(A)^m, \partial^i} \hat{U}(A)^{m + 1} \to \mathcal{N}^{m + 1}$ are all isomorphisms. Setting $M = N^0$, Cartesian $\hat{U}(A)$-modules are equivalent to $\hat{U}(A)^0 = A[q - 1]$-modules $M$ with isomorphisms $\Delta : (\partial^1)^*M \cong (\partial^0)^*M$ satisfying the cocycle condition $\partial^1 \Delta = (\partial^0 \Delta) \circ (\partial^2 \Delta) : (\partial^2 \partial^0)^*M \to (\partial^0 \partial^0)^*M$.

The map $\Delta$ is determined by its restriction to $M$, so using the basis for $U^1$ from Lemma 1.5, and taking $v \in M$, we have

$$\Delta(v) = \sum_{k \in \mathbb{N}_0^d} \partial^0(\Delta_k(v)) \lambda^{k_1} \left( \frac{\partial^1 x_1 - \partial^0 x_1}{q - 1} \right) \cdots \lambda^{k_d} \left( \frac{\partial^1 x_d - \partial^0 x_d}{q - 1} \right)$$

for $A[q - 1]$-linear endomorphisms $\Delta_k$ of $M$. Since $\lambda_1(a + b) = \lambda_1(a) \lambda_1(b)$, the cocycle condition becomes $\Delta_{j + k} = \Delta_j \circ \Delta_k$, meaning $\Delta$ is determined by the operators $\Delta_k$ at the basis vectors, which must moreover commute.

Linearity of $\Delta$ with respect to $\hat{U}(A)^1$ then reduces to the condition that $\Delta(au) = \partial^1(a) \Delta(u)$ for $a \in A$, $u \in M$. Writing $A$ for $\partial^0 A$ and $h^k_i : = \lambda^k \left( \frac{\partial^1 x_i - \partial^0 x_i}{q - 1} \right)$, the ideal
J := (h_i^{[2]}; h_i; h_j^\neq j) satisfies U^1 = A \oplus \bigoplus_i A h_i \oplus J. The proof of Theorem 1.23 gives \partial^1(a) = a + (q - 1) \sum_i \nabla_{q, x_i}(a)h_i \mod J, and in U^1 / J we have \([h_i]^2 \equiv x_i[h_i].\) Comparing coefficients of \(h_i\) in the equation \(\Delta(a) = \partial^1(a)\Delta(v) \mod J\) then gives

\[
\Delta_{e_i}(av) = (q - 1)\nabla_{q, x_i}(a)v + a\Delta_{e_i}(v) + (q - 1)x_i\nabla_{q, x_i}(a)\Delta_{e_i}(v)
\]

\[
= (q - 1)\nabla_{q, x_i}(a)v + \gamma_i(a)\Delta_{e_i}(v).
\]

Finally, note that the condition that \(N/(q - 1)\) be the pullback of an \(A\)-module (necessarily \(\Gamma(\hat{X}^{q}_{\text{strat}}, N/(q - 1))\)) is equivalent to saying that \(\partial^0_N \equiv \partial^1_N \mod (q - 1),\) or that \((q - 1)\) divides \(\Delta_e\), and setting \(\nabla_{i,q} := (q - 1)^{-1}\Delta_{e_i}\) gives a \(q\)-connection \((\nabla_{i,q})_{1 \leq i \leq d}\) on \(M = N^0\) uniquely determining \(\Delta\).

The inverse construction is given by \(\Delta_k = (q - 1)\sum k_i \nabla_{1,q}^{k_i} \circ \cdots \circ \nabla_{d,q}^{k_d}.\)

2 Comparisons for \(\Lambda_P\)-rings

Since very few étale maps \(R[\{x_1, \ldots, x_d\}] \rightarrow A\) give rise to \(\Lambda\)-ring structures on \(A,\) Theorem 1.23 is fairly limited in its scope for applications. We now show how the construction of \(\hat{q}\text{DR}\) and the comparison quasi-isomorphism survive when we weaken the \(\Lambda\)-ring structure by discarding Adams operations at invertible primes.

2.1 \(q\)-cohomology for \(\Lambda_P\)-rings

Our earlier constructions for \(\Lambda\)-rings all carry over to \(\Lambda_P\)-rings, as follows.

**Definition 2.1** Given a set \(P\) of primes, we define a \(\Lambda_P\)-ring \(A\) to be a \(\Lambda_{\mathbb{Z}, P}\)-ring in the sense of [5]. This means that it is a coalgebra in commutative rings for the comonad given by the functor \(W^{(P)}\) of \(P\)-typical Witt vectors. When a commutative ring \(A\) is flat over \(\mathbb{Z},\) giving a \(\Lambda_P\)-ring structure on \(A\) is equivalent to giving commuting Adams operations \(\Psi^P\) for all \(p \in P,\) with \(\Psi^P(a) \equiv a^P \mod p\) for all \(a.\)

Thus when \(P\) is the set of all primes, a \(\Lambda_P\)-ring is just a \(\Lambda\)-ring; a \(\Lambda_{q}\)-ring is just a commutative ring; for a single prime \(p,\) we write \(\Lambda_p := \Lambda_{\{p\}},\) and note that a \(\Lambda_P\)-ring is a \(\delta\)-ring in the sense of [10].

**Definition 2.2** Given a \(\Lambda_P\)-ring \(R,\) say that \(A\) is a \(\Lambda_P\)-ring over \(R\) if it is a \(\Lambda_P\)-ring equipped with a morphism \(R \rightarrow A\) of \(\Lambda_P\)-rings. We say that \(A\) is a flat \(\Lambda_P\)-ring over \(R\) if \(A\) is flat as a module over the commutative ring underlying \(R.\)

**Definition 2.3** Given a morphism \(R \rightarrow A\) of \(\Lambda_P\)-rings, we define the category \(\text{Strat}_{A/R}^{q, P}\) to consist of flat \(\Lambda_P\)-rings \(B\) over \(R[q]\) equipped with a compatible morphism \(A \rightarrow B/(q - 1),\) such that the map \(A \rightarrow B/(q - 1)\) admits a lift to \(B.\) We define the category \(\text{Strat}_{A/R}^{q, P} \subset \text{Strat}_{A/R}^{q}\) to consist of those objects which are \((q - 1)\)-adically complete.
More concisely, Strat\textsuperscript{q,P}_{A/R} (resp. \hat{\text{Strat}}\textsuperscript{q,P}_{A/R}) is the Grothendieck construction of the functor (Spec A)\textsuperscript{q,P}_{\text{strat}} (resp. (Spec A)\textsuperscript{q,P}_{\hat{\text{strat}}}) given by

\[
B \mapsto \text{Im} (\text{Hom}_{\Lambda_{p,R}}(A, B) \rightarrow \text{Hom}_{\Lambda_{p,R}}(A, B/(q-1)))
\]

on the category of flat \Lambda_{p}\text{-rings} (resp. \((q-1)\text{-adically complete flat }\Lambda_{p}\text{-rings}) over \(R[\![q]\!]\).

**Definition 2.4** Given a flat morphism \(R \rightarrow A\) of \(\Lambda_{p}\text{-rings}, define \(q\text{DR}_{p}(A/R)\) to be the cochain complex of \(R[\![q]\!]\)\-modules given by taking the homotopy limit of the functor

\[
\text{Strat}_{A/R}^{q,P} \rightarrow \text{Ch}(R[\![q]\!])
\]

\[
B \mapsto B.
\]

Define \(\hat{q}\text{DR}_{p}(A/R)\) to be the cochain complex of \(R[\![q-1]\!]\)\-modules given by the corresponding homotopy limit over \(\hat{\text{Strat}}_{A/R}^{q,P}\).

For \(p \in P\), the cochain complex \(q\text{DR}_{p}(A/R)\) naturally carries \((R[\![q]\!], \Psi^{p})\)\-semilinear operations \(\Psi^{p}\) coming from the morphisms \(\Psi^{p} : B \otimes_{R[\![q]\!]} R[\![q]\!] \rightarrow B\) of \(R[\![q]\!]\)\-modules, for \(B \in \text{Strat}_{A/R}^{q,P}\).

Thus when \(P\) is the set of all primes, we have \(q\text{DR}_{p}(A/R) = q\text{DR}(A/R)\). At the other extreme, for \(A\) smooth, \(q\text{DR}_{\emptyset}(A/R)\) is the Rees construction of the Hodge filtration on the infinitesimal cohomology complex \([8]\) of \(A\) over \(R\), with formal variable \((q-1)\). In more detail, there is a decreasing filtration \(F\) of \(\mathcal{O}_{\text{inf}}\) given by powers of the augmentation ideal of \(\mathcal{O}_{\text{inf}} \rightarrow \mathcal{O}_{\text{Zar}}\) (with \(F_{\nu} \mathcal{O}_{\text{inf}} = \mathcal{O}_{\text{inf}}\) for \(\nu \leq 0\)), and then

\[
\hat{q}\text{DR}_{\emptyset}(A/R) \simeq \prod_{\nu \in \mathbb{Z}} (q-1)^{-\nu} \mathbb{R}\Gamma(\text{Spec }A, F^{\nu} \mathcal{O}_{\text{inf}}).
\]

**Lemma 2.5** For a set \(P\) of primes, the forgetful functor from \(\Lambda\text{-rings}\) to \(\Lambda_{p}\text{-rings}\) has a right adjoint \(W(\#P)\). There is a canonical ghost component morphism

\[
W(\#P)(B) \rightarrow \prod_{n \in \mathbb{N}; (n,p)=1 \forall p \in P} B,
\]

which is an isomorphism when \(P\) contains all the residue characteristics of \(B\).

**Proof** Existence of a right adjoint follows from the comonadic definitions of \(\Lambda\text{-rings}\) and \(\Lambda_{p}\text{-rings}. The ghost component morphism is given by taking the Adams operations \(\Psi^{p}\) coming from the \(\Lambda\text{-ring structure on }W(\#P)(B), followed by projection to \(B\). When \(P\) contains all the residue characteristics of \(B\), a \(\Lambda\text{-ring structure is the same as a }\Lambda_{p}\text{-ring structure with compatible commuting Adams operations for all primes not in }P, leading to the description above. \(\square\)
Note that the big Witt vector functor $W$ on commutative rings thus factorises as $W = W(\mathcal{O}_P) \circ W(P)$, for $W(P)$ the $P$-typical Witt vectors.

**Proposition 2.6** Given a morphism $R \to A$ of $\Lambda$-rings, and a set $P$ of primes, there are natural maps

$$qDR_P(A/R) \to qDR(A/R), \quad \widehat{qDR}_P(A/R) \to \widehat{qDR}(A/R),$$

and the latter map is a quasi-isomorphism when $P$ contains all the residue characteristics of $A$.

**Proof** We have functors

$$(\text{Spec } A)_{\text{strat}}^q \circ W(\mathcal{O}_P) : B \mapsto \text{Im} (\text{Hom}_{\Lambda,R}(A, W(\mathcal{O}_P)B) \to \text{Hom}_{\Lambda,R}(A, (W(\mathcal{O}_P)B)/(q - 1)))$$

$$(\text{Spec } A)_{\text{strat}}^{q,P} : B \mapsto \text{Im} (\text{Hom}_{\Lambda,P,R}(A, B) \to \text{Hom}_{\Lambda,P,R}(A, B/(q - 1)))$$

on the category of flat $\Lambda_P$-rings over $R[q]$. There is an obvious map

$$(W(\mathcal{O}_P)B)/(q - 1) \to W(\mathcal{O}_P)(B/(q - 1)),$$

and hence a natural transformation $(\text{Spec } A)_{\text{strat}}^q \circ W(\mathcal{O}_P) \to (\text{Spec } A)_{\text{strat}}^{q,P}$, which induces the morphism $qDR_P(A/R) \to qDR(A/R)$ on cohomology.

When $P$ contains all the residue characteristics of $A$, the map $(W(\mathcal{O}_P)B)/(q - 1) \to W(\mathcal{O}_P)(B/(q - 1))$ is just

$$\prod_{(n,p) = 1 \forall p \in P} \frac{B}{(q^n - 1)} \to \prod_{(n,p) = 1 \forall p \in P} \frac{B}{(q - 1)},$$

since the morphism $R[q] \to W(\mathcal{O}_P)B$ is given by Adams operations, with $\Psi^n(q - 1) = q^n - 1$.

We have $(q^n - 1) = (q - 1)[n]_q$, and $[n]_q$ is a unit in $\mathbb{Z}[\frac{1}{n}][q - 1]$, hence a unit in $B$ when $n$ is coprime to the residue characteristics. Thus the map $(W(\mathcal{O}_P)B)/(q - 1) \to W(\mathcal{O}_P)(B/(q - 1))$ gives an isomorphism whenever $B$ is $(q - 1)$-adically complete and admits a map from $A$, so the transformation $(\text{Spec } A)_{\text{strat}}^q \circ W(\mathcal{O}_P) \to (\text{Spec } A)_{\text{strat}}^{q,P}$ is a natural isomorphism on the category of flat $(q - 1)$-adically complete $\Lambda_P$-rings over $R[q]$, and hence $\widehat{qDR}_P(A/R) \cong \widehat{qDR}(A/R)$. \hfill \square

**Remark 2.7** Remark 1.10 shows that $qDR(A/R)$ can naturally be promoted to a cosimplicial $\Lambda$-ring, and the same reasoning promotes $qDR_P(A/R)$ to a cosimplicial $\Lambda_P$-ring. The proof of Proposition 2.6 then ensures that the map $qDR_P(A/R) \to qDR(A/R)$ is naturally a morphism of cosimplicial $\Lambda_P$-rings.

Over $\mathbb{Z}[[\frac{1}{p} : p \in P]]$, every $\Lambda_P$-ring can be canonically made into a $\Lambda$-ring, by setting all the additional Adams operations to be the identity. However, this observation is of limited use in establishing functoriality of $q$-de Rham cohomology, because the resulting $\Lambda$-ring structure will not satisfy the conditions of Theorem 1.23. We now give a more general result which does allow for meaningful comparisons.
Theorem 2.8 If R is a flat \( \Lambda_p \)-ring over \( \mathbb{Z} \) and \( \square : R[x_1, \ldots, x_d] \to A \) is a formally étale map of \( \Lambda_p \)-rings, the elements \( x_i \) having rank 1, then there are zigzags of \( R[1 - q] \)-linear quasi-isomorphisms

\[
\widehat{\text{qDR}}_p(A/R) \simeq (\Omega^*_A/R[1 - q], (q - 1) \nabla_q), \quad L\eta_{(q-1)}\widehat{\text{qDR}}_p(A/R) \simeq q - \Omega^*_{A/R, \square}.
\]

whenever \( P \) contains all the residue characteristics of \( A \).

Proof The key observation to make is that formally étale maps have a unique lifting property with respect to nilpotent extensions of flat \( \Lambda_p \)-rings, because the Adams operations must also lift uniquely. In particular, this means that the operations \( \gamma_i \) featuring in the definition of \( q \)-de Rham cohomology are necessarily endomorphisms of \( A \) as a \( \Lambda_p \)-ring.

Similarly to Theorem 1.23, \( \widehat{\text{qDR}}_p(A/R) \) is calculated using a cosimplicial \( \Lambda_p \)-ring given in level \( n \) by the \( (q - 1) \)-adic completion \( \hat{U}_{p, A}^n \) of the \( \Lambda_p \)-ring over \( R[q] \) generated by \( A^{\otimes R(n+1)}[q] \) and \( (q - 1)^{-1} \ker(A^{\otimes R(n+1)} \to A)[q] \). The observation above shows that \( \hat{U}_{p, A}^n \simeq \hat{U}_{p, R[x_1, \ldots, x_d]}^n \otimes_{R[x_1, \ldots, x_d]} A \), changing base along \( \square \) applied to the first factor.

As in Proposition 2.6, \( \hat{U}_{p, R[x_1, \ldots, x_d]}^n \) is just the \( (q - 1) \)-adic completion of the complex \( U^* \) from Proposition 1.15. Further application of the key observation above then allows us to adapt the constructions of Theorem 1.17, giving the desired quasi-isomorphisms.

\( \square \)

2.2 Cartier isomorphisms in mixed characteristic

In [13, Conjecture 7.1], Scholze predicted that \( q - \Omega^*_{A/R, \square} \) is a functorial invariant of the \( R \)-algebra \( A \), independent of the choice of framing, so extends to all smooth schemes. Theorem 2.8 shows that \( q - \Omega^*_{A/R, \square} \) is functorial invariant of the \( \Lambda_p \)-ring \( A \) over \( R \).

The only setting in which Theorem 2.8 leads to results close to Scholze’s conjecture is when \( R = W^{(p)}(k) \), the \( p \)-typical Witt vectors of a perfect field of characteristic \( p \), and \( A = \lim_n A_n \) is a formal deformation of a smooth \( k \)-algebra \( A_0 \). Then any formally étale morphism \( W^{(p)}(k)[x_1, \ldots, x_d] \to A \) of topological rings gives rise to a unique compatible lift \( \Psi^p \) of absolute Frobenius on \( A \) with \( \Psi^p(x_i) = x_i^p \), so gives \( A \) the structure of a topological \( \Lambda_p \)-ring. The framing still affects the choice of \( \Lambda_p \)-ring structure, but at least such a structure is guaranteed to exist, giving rise to a complex \( \text{qDR}_p(A/R)^{\wedge p} := R \lim_n \text{qDR}_p(A/R) \otimes_R R_n \) depending only on the choice of \( \Psi^p \), where \( R_n = W^{(p)}(k) \).

Our constructions now allow us to globalise the quasi-isomorphism

\[
(q - \Omega^*_{A/R, \square})^{\wedge p} / [p]q \simeq (\Omega^*_A/R)^{\wedge p} [q - 1] / [p]q
\]

of [13, Proposition 3.4], where \( \Omega^*_A/R \) denotes the complex \( A \to \Omega^1_{A/R} \to \Omega^2_{A/R} \to \ldots \).
Lemma 2.9  Under the quasi-isomorphism $\hat{\text{qDR}}_p(A/R) \simeq (\Omega^*_{A/R}[q-1], (q-1)\nabla_q)$ from Theorem 2.8, the semilinear Adams operation $\Psi^p$ on $\hat{\text{qDR}}_p(A/R)$ described in Definition 1.8 corresponds to the operation on $\Omega^*_{A/R}[q-1]$ given by setting  

$$\Psi^p(adx_{i_1} \wedge \cdots \wedge dx_{i_m}) := \Psi^p(a)x_p^{p-1} \cdots x_{i_m}^{p-1} dx_{i_1} \wedge \cdots \wedge dx_{i_m}.$$  

for $a \in A[q-1]$.

Proof  Just observe that this expression defines a chain map on $(\Omega^*_{A/R}[q-1], (q-1)\nabla_q)$ (for instance $\Psi^p((q-1)\nabla_q x_i) = (q^p - 1)\Psi^p(dx) = (q-1)\nabla_q x_i^p$), and that the quasi-isomorphisms in the proof of Theorem 1.23 commute with these operations.

As in [13, §4], we refer to formal schemes over $W^{(p)}(k)$ as smooth if they are flat deformations of smooth schemes over $k$. We refer to morphisms of such schemes as étale if they are flat deformations of étale morphisms over $k$.

Proposition 2.10  Take a smooth formal scheme $\mathcal{X}$ over $R = W^{(p)}(k)$ equipped with a lift $\Psi^p$ of Frobenius which étale locally admits co-ordinates $\{x_i\}_i$ as above with $\Psi^p(x_i) = x_i^p$. Then there is a global quasi-isomorphism  

$$C^{-1}_q : (\Omega^*_{\mathcal{X}/R})^{\wedge^p}[q-1]/[p]_q \to (L\eta_{(q-1)\hat{\text{qDR}}_p(\mathcal{O}_\mathcal{X}/R)})^{\wedge^p}/[p]_q$$  

in the derived category of étale sheaves on $\mathcal{X}$.

Proof  The unique lifting property of formally étale morphisms ensures that each affine étale scheme $\mathcal{U}$ étale over $\mathcal{X}$ has a unique lift $\Psi^p|_{\mathcal{U}}$ of Frobenius compatible with the given operation $\Psi^p$ on $\mathcal{X}$. Functoriality of the construction $\hat{\text{qDR}}_p$ for rings with Frobenius lifts thus gives us an étale presheaf $\hat{\text{qDR}}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p}$ of complexes on $\mathcal{X}$. As in Definition 1.8, the Adams operation $\Psi^p$ on $\mathcal{O}_\mathcal{X}$ then extends to $(R[q-1], \Psi^p)$-semilinear maps  

$$\Psi^p : \text{qDR}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p} \to \text{qDR}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p}$$  

$$\text{qDR}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p}/(q-1) \to \text{qDR}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p}/(q^p - 1),$$  

and thus, denoting good truncation by $\tau$,

$$(q-1)^j\Psi^p : \tau^{\leq i}(\text{qDR}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p}/(q-1)) \to (L\eta_{(q-1)\hat{\text{qDR}}_p(\mathcal{O}_\mathcal{X}/R)^{\wedge^p}}/[p]_q;$$  

the left-hand side is quasi-isomorphic to $\bigoplus_{j \leq i} (\Omega^j_{\mathcal{O}_\mathcal{X}/R})^{\wedge^p}/[-j]$ by Theorem 1.23.

Extending the construction $R[q]$-linearly and restricting to top summands therefore gives us the global map $C^{-1}_q$. For a local choice of framing, Lemma 2.9 gives equivalences  

$$(q-1)^j\Psi^p \simeq \sum_{j \leq i} (q-1)^{i-j}(\tilde{C}^{-1})^j$$
for Scholze’s locally defined lifts $(\tilde{\mathcal{C}}^{-1})^j : (\Omega^j_{A^p/R})^{\wedge^n}/[-j] \to (\tilde{\mathcal{O}}^\bullet_{A^p/R,\square})^{\wedge^n}/[p]_q$

of the Cartier quasi-isomorphism. The local calculation of [13, Proposition 3.4] then ensures that $C^{-1}_q$ is a quasi-isomorphism.

\[ \square \]

### 3 Functionality via analogues of de Rham–Witt cohomology

In order to obtain a cohomology theory for smooth commutative rings rather than for $\Lambda^p$-rings, we now consider $q$-analogues of de Rham–Witt cohomology. Our starting point is to observe that if we allow roots of $q$, we can extend the Jackson differential to fractional powers of $x$ by the formula

\[ \nabla_q(x^{m/n}) = \frac{q^{m/n} - 1}{q - 1} x^{m/n} d \log x, \]

where $d \log x = x^{-1} dx$, so terms such as $[n]_{q^{1/n}} x^{m/n}$ have integral derivative, where $[n]_{q^{1/n}} = \frac{q^{1/n} - 1}{q^{1/n} - 1}$.

#### 3.1 Motivation

**Definition 3.1** Given a $\Lambda^p$-ring $B$, define $\Psi^{1/p^\infty} B$ to be the smallest $\Lambda^p$-ring which is equipped with a morphism from $B$ and for which the Adams operations are automorphisms.

In the case $P = \{p\}$, the $\Lambda^p$-ring $\Psi^{1/p^\infty} B$ is thus the colimit of the diagram

\[ B \xrightarrow{\psi^p} B \xrightarrow{\psi^p} B \xrightarrow{\psi^p} \ldots \]

By Remark 2.7, $\hat{\text{DR}}_p(A/R)$ naturally underlies a cosimplicial $\Lambda^p$-ring, so applying $\Psi^{1/p^\infty}$ levelwise gives another cosimplicial $\Lambda^p$-ring. For the Adams operation $\Psi^p$ of Definition 2.3, the underlying cochain complex is just $\Psi^{1/p^\infty} \hat{\text{DR}}_p(A/R) := \lim_{\to} \psi^p \hat{\text{DR}}_p(A/R)$. As an immediate consequence of Lemma 2.9, we have:

**Lemma 3.2** If $R$ is a flat $\Lambda^p$-ring over $\mathbb{Z}_{(p)}$ with $\Psi^p$ an isomorphism, then $\Psi^{1/p^\infty} \hat{\text{DR}}_p(R[x]/R)$ is quasi-isomorphic to the complex

\[ (R[x^{1/p^\infty}, q^{1/p^\infty}], (q-1)^{-1} \nabla_q x^{1/p^\infty} R[x^{1/p^\infty}, q^{1/p^\infty}] d \log x)^{\wedge^{(q-1)}}, \]

so the décalage $L_{q^{(q-1)}} \Psi^{1/p^\infty} \hat{\text{DR}}_p(R[x]/R)$ and the complex

\[ \{a \in R[x^{1/p^\infty}, q^{1/p^\infty}] : \nabla_q a \in R[x^{1/p^\infty}, q^{1/p^\infty}] d \log x \} \]

\[ \xrightarrow{\nabla_q} (x^{1/p^\infty}) R[x^{1/p^\infty}, q^{1/p^\infty}] d \log x. \]
are quasi-isomorphic after \((q - 1)\)-adic completion.

Thus in level 0 (resp. level 1), \(L \eta_{(q - 1)} \Psi^{1/p^\infty} \widehat{qDR}(R[x]/R)\) is spanned by elements of the form \([p^n], q^{1/p^n} x^{m/p^n}\) (resp. \(x^{m/p^n} d \log x\)), so setting \(q^{1/p^\infty} = 1\) gives a complex whose \(p\)-adic completion is the \(p\)-typical de Rham–Witt complex.

**Lemma 3.3** Let \(R\) and \(A\) be flat \(p\)-adically complete \(\Lambda_p\)-algebras over \(\mathbb{Z}_p\), with \(\Psi^p\) an isomorphism on \(R\). For elements \(x_i\) of rank 1, take a map \(\square \colon R[x_1, \ldots, x_d]^{\vee_p} \to A\) of \(\Lambda_p\)-rings which is a flat \(p\)-adic deformation of an étale map. Then the map

\[
\left( R[q^{1/p^\infty}] \otimes_{R[q]} L \eta_{(q - 1)} \widehat{qDR}\, p(A/R) \right)^{\wedge_p} \to L \eta_{(q - 1)} \left( \Psi^{1/p^\infty} \widehat{qDR}\, p(A/R) \right)^{\wedge_p}
\]

is a quasi-isomorphism.

**Proof** The map \(\Psi^p : A \otimes_{R[x_1, \ldots, x_d]} R[x_1^{1/p}, \ldots, x_d^{1/p}] \to A\) becomes an isomorphism on \(p\)-adic completion, because \(\square\) is flat and we have an isomorphism modulo \(p\). Thus

\[
\Psi^{1/p^\infty} A \cong A[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge_p} := \left( A \otimes_{R[x_1, \ldots, x_d]} R[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}] \right)^{\wedge_p}
\]

Combined with the calculation of Lemma 2.9, this gives us a quasi-isomorphism between \((\Psi^{1/p^\infty} \widehat{qDR}\, p(A/R))^{\wedge_p}\) and the \((p, q - 1)\)-adic completion of

\[
\left( \bigoplus_I \bigoplus_\alpha A[q - 1] x_1^{\alpha_1} \ldots x_d^{\alpha_d} dx^I [-|I|], (q - 1)\nabla_q \right),
\]

where \(I\) ranges over finite subsets of \(\{1, \ldots, d\}\) and \(\alpha\) ranges over elements of \(p^{-\infty} \mathbb{Z}^d\) with \(0 \leq \alpha_i < 1\) if \(i \not\in I\) and \(-1 < \alpha_i \leq 0\) if \(i \in I\).

We then observe that the contributions to the décalage \(\eta_{(q - 1)}\) from terms with \(\alpha \neq 0\) must be acyclic, via a contracting homotopy defined by the restriction to \(\eta_{(q - 1)}\) of the \(q\)-integration map

\[
f x_1^{\alpha_1} \ldots x_d^{\alpha_d} dx^I \mapsto f x_1^{\alpha_1} \ldots x_d^{\alpha_d} \sum_{i \in I} \pm x_i [\alpha_i]^{-1}_q dx^{(I \setminus i)},
\]

where \(\left[ \frac{m}{p^n} \right]^{-1} = \left[ \frac{1}{q^{1/p^n}} \right]^{-1} \left[ \frac{m}{p^n} \right]_{q^{1/p^n}} \) for \(m\) coprime to \(p\), noting that \([m]_{q^{1/p^n}}\) is a unit in \(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p, q - 1)}}\). \(\square\)

**Remark 3.4** The endomorphism given on \(\Psi^{1/p^\infty} \widehat{qDR}\, p(A/R)\) by

\[
a \mapsto \Psi^{1/n} ([n]_q a) = [n]_{q^{1/n}} \Psi^{1/n} a
\]

descends to an endomorphism of \(H^0(\Psi^{1/p^\infty} \widehat{qDR}\, p(A/R)/(q - 1))\), which we may denote by \(V_n\) because it mimics Verschiebung in the sense that \(\Psi^n V_n = n \cdot \text{id}\) (since
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For $A$ smooth over $\mathbb{Z}$, we then have

$$H^0 \left( \Psi^{1/P_\infty} qdr, (A/\mathbb{Z})/(q - 1) \right)/(V_p : p \in P) \cong A[q^{1/P_\infty}]/([p]_{q^{1/p}} : p \in P) \cong A[\zeta_{p_\infty}],$$

for $\zeta$ a primitive $n$th root of unity.

By adjunction, this gives an injective map

$$H^0 \left( \Psi^{1/P_\infty} qdr, (A/\mathbb{Z})/(q - 1) \right) \hookrightarrow W^p A[\zeta_{p_\infty}]$$

of $\Lambda$-rings, which becomes an isomorphism on completing $\Psi^{1/P_\infty} qdr, (A/\mathbb{Z})$ with respect to the system $\{(n[q^{1/n}])_{n \in P_\infty} : n \in P \}$ of integers whose prime factors are all in $P$. This implies that the cokernel is annihilated by all elements of $(q^{1/P_\infty} - 1)$, so leads us to consider almost mathematics as in [7].

### 3.2 Almost isomorphisms

From now on, we consider only the case $P = \{p\}$. Combined with Lemma 3.3, Remark 3.4 allows us to regard $L_{\eta(q - 1)} \Psi^{1/P_\infty} qdr, (A/\mathbb{Z})^\wedge_p$ as being almost a $q^{1/P_\infty}$-analogue of $p$-typical de Rham–Witt cohomology.

The ideal $(q^{1/P_\infty} - 1)^\wedge(p,q^{-1}) = \ker((\mathbb{Z})^{1/P_\infty})^{\wedge(p,q^{-1})} \to \mathbb{Z}_p)$ is equal to the $p$-adic completion of its square, since we may write it as the kernel $W^p(m)$ of $W^p(\mathbb{F}_p[q^{1/P_\infty}]^{\wedge(q^{-1})} \to W^p(\mathbb{F}_p)$, for the idempotent maximal ideal $m = ((q - 1)^{1/P_\infty})^{\wedge(q^{-1})}$ in $\mathbb{F}_p[q^{1/P_\infty}]^{\wedge(q^{-1})}$. If we set $h^{1/P_\infty}$ to be the Teichmüller element

$$[q^{1/P_\infty} - 1] = \lim_{r \to \infty} (q^{1/P_{r\infty}} - 1) \in \mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})},$$

then $W^p(m) = (h^{1/P_\infty})^{\wedge(h^{-1})}$. Although $W^p(m)/p^{n}$ is not maximal in $\mathbb{Z}[h^{1/P_\infty}]^{\wedge(h^{-1})}$, it is idempotent and flat, so gives a basic setup in the sense of [7, 2.1.1]. We thus regard the pair $(\mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})}, W^p(m))$ as an inverse system of basic setups for almost ring theory.

We then follow the terminology and notation of [7], studying $p$-adically complete $(\mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})})^a$-modules (almost $\mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})}$-modules) given by localising at almost isomorphisms, the maps whose kernel and cokernel are $W^p(m)$-torsion.

**Definition 3.5** The obvious functor $(-)^a$ from modules to almost modules has a right adjoint $(-)_*$ given by $N_* := \text{Hom}_{\mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})}}(W^p(m), N)$, the module of almost elements.

Since the counit $(M_*)_a \to M$ of the adjunction is an (almost) isomorphism, we may also regard almost modules as a full subcategory of the category of modules, consisting of those $M$ for which the natural map $M \to (M^a)_*$ is an isomorphism. We can define $p$-adically complete $(\mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})})^a$-algebras similarly, forming a full subcategory of $\mathbb{Z}[q^{1/P_\infty}]^{\wedge(q^{-1})}$-algebras.
3.3 Perfectoid algebras

We now relate Scholze’s perfectoid algebras to a class of $\Lambda_p$-rings, by factorising the tilting equivalence. For simplicity, we work over $\mathbb{Z}[\zeta_p^{\infty}]^\wedge_p$, although Lemma 3.8 has natural analogues over the ring $K^\rho \subset K$ of power-bounded elements of any perfectoid field $K$ in the sense of [12].

**Definition 3.6** Define Fontaine’s period ring functor $\mathcal{A}_{\text{inf}}$ from commutative rings to $\Lambda_p$-rings by $\mathcal{A}_{\text{inf}}(C) := \lim_{\leftarrow} \Psi_p W(p)(C)$.

**Definition 3.7** Define a perfectoid $\Lambda_p$-ring to be a flat $p$-adically complete $\Lambda_p$-algebra over $\mathbb{Z}_p$, on which the Adams operation $\Psi_p$ is an isomorphism.

By analogy with [2, Notation 1.4], we say that a perfectoid $\Lambda_p$-ring over $\mathbb{Z}[q^{1/p^{\infty}}]^{\wedge(p,q-1)}$ is integral if the morphism $B \to B_*$ of Definition 3.5 is an isomorphism.

**Lemma 3.8** We have equivalences of categories

$$
\begin{array}{ccc}
\text{perfectoid almost } \mathbb{Z}[\zeta_p^{\infty}]^\wedge_p\text{-algebras} & \xrightarrow{\mathcal{A}_{\text{inf}}(-)_*} & \text{integral perfectoid } \Lambda_p\text{-rings over } \mathbb{Z}[q^{1/p^{\infty}}]^{\wedge(p,q-1)} \\
-/[p]^{1/p} & \uparrow & -/p \\
\text{perfectoid almost } \mathbb{F}_p[q^{1/p^{\infty}}]^{\wedge(q-1)}\text{-algebras} & \xleftarrow{W(p)(-)_*} & \end{array}
$$

**Proof** A perfectoid $\Lambda_p$-ring $B$ is a deformation of the perfect $\mathbb{F}_p$-algebra $B/p$. As in [12, Proposition 5.13], a perfect $\mathbb{F}_p$-algebra $C$ has a unique deformation $W(p)(C)$ over $\mathbb{Z}_p$, to which Frobenius must lift uniquely; this shows that $W(p)$ gives an equivalence between perfect $\mathbb{F}_p$-algebras and perfectoid $\Lambda_p$-rings. To obtain the bottom equivalence of the diagram, we will show that the functor $W(p)$ commutes with the respective functors $C \mapsto C_*$ of almost elements, then appeal to the tilting equivalence.

Because the idempotent ideals of the basic setups in each of our three categories are generated by the rank 1 elements $h^{p^{-n}}$ constructed before Definition 3.5, we can write $C_* = \bigcap_n h^{-p^{-n}} C$ in each setting. For a Teichmüller element $[c] \in W(p)(C)$, the standard isomorphism $W(p)(C) \cong C^{\mathbb{N}_0}$ of sets gives an isomorphism $[c]W(p)(C) \cong \prod_{m \geq 0} c^{p^m} C$. Thus the natural map $W(p)(C)_* \to W(p)(C_*)$ of $\Lambda_p$-rings is an isomorphism, since

$$W(p)(C)_* \cong \bigcap_{n \geq 0} \prod_{m \geq 0} h^{-p^{m-n}} C \cong \prod_{m \geq 0} C_* \cong W(p)(C_*),$$

and taking inverse limits with respect to $\Psi_p$ gives $\mathcal{A}_{\text{inf}}(C)_* \cong \mathcal{A}_{\text{inf}}(C_*)$ as well.

Next, we observe that since $B := \mathcal{A}_{\text{inf}}(C)$ is a perfectoid $\Lambda_p$-ring for any flat $p$-adically complete $\mathbb{Z}_p$-algebra $C$, we must have $B \cong W(p)(B/p)$. Comparing rank
1 elements then gives a monoid isomorphism \((B/p) \cong \lim_{x \mapsto x^p} C\), from which it follows that

\[
\mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathcal{A}_{\text{inf}}(C) \cong \lim_{\Phi} (C/p) = C^b
\]

whenever \(C\) is perfectoid. Since tilting gives an equivalence of almost algebras by [12, Theorem 5.2], this completes the proof. \(\square\)

### 3.4 Functoriality of q-de Rham cohomology

Since \((\Psi^{1/p^\infty}_* q\text{DR}_p(A/\mathbb{Z}_p)^{\wedge_p})\) is represented by a cosimplicial perfectoid \(\Lambda_p\)-ring over \(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{p,q-1}}\) for any \(\Lambda_p\)-ring \(A\) over \(\mathbb{Z}_p\), it corresponds under Lemma 3.8 to a cosimplicial perfectoid \((\mathbb{Z}[\zeta_p^\infty]^{\wedge_p})^\Lambda\)-algebra, representing the following functor:

**Lemma 3.9** For a perfectoid \((\mathbb{Z}[\zeta_p^\infty]^{\wedge_p})^\Lambda\)-algebra \(C\), and a \(\Lambda_p\)-ring \(A\) over \(\mathbb{Z}_p\) with \(X = \text{Spec} A\), there is a canonical isomorphism

\[
X^{q,p}_{\text{strat}}(\mathcal{A}\text{inf}(C)_*) \cong \text{Im} \left( \lim_{\Psi^p} X(C_*) \to X(C_*) \right),
\]

for the ring \(C_*\) of almost elements.

**Proof** By definition, \(X^{q,p}_{\text{strat}}(\mathcal{A}\text{inf}(C)_*)\) is the image of

\[
\text{Hom}_{\Lambda_p}(A, \mathcal{A}\text{inf}(C)_*) \to \text{Hom}_{\Lambda_p}(A, (\mathcal{A}\text{inf}(C)_*)/(q - 1)).
\]

Since right adjoints commute with limits and \(\mathcal{A}\text{inf} = \lim_{\Psi^p} W(p)\), we may rewrite the first term as \(\lim_{\Psi^p} \text{Hom}_{\Lambda_p}(A, W(p)(C_*)) = \lim_{\Psi^p} X(C_*)\).

Setting \(B := \lim_{\Psi^p} W(p)(C)_*\), observe that because \([p^n]_{q^{1/p^n}}(q^{1/p^n} - 1) = (q - 1)\), we have \(\bigcap_n [p^n]_{q^{1/p^n}} B = (q - 1)B\), any element on the left defining an almost element of \((q - 1)B\), hence a genuine element since \(B = B_*\) is flat. Then note that since the projection map \(\theta : B \to C_*\) has kernel \(([p]_{q^{1/p^n}}\), the map \(\theta \circ \Psi p^{n-1}\) has kernel \(([p]_{q^{1/p^n}}\), and so \(B \to W(p)(C)_*\) has kernel \(\bigcap_n [p^n]_{q^{1/p^n}} B\). Thus

\[
\text{Hom}_{\Lambda_p} \left( A, \left( \lim_{\Psi^p} W(p)(C)_* \right)/(q - 1) \right) \hookrightarrow \text{Hom}_{\Lambda_p}(A, W(p)(C)_*) = X(C_*).
\]

\(\square\)

In fact, the tilting equivalence gives \(\lim_{\Psi^p} X(C_*) \cong X(C^b_*)\), so the only dependence of \(X^{q,p}_{\text{strat}}(\mathcal{A}\text{inf}(C)_*)\), and hence \((\Psi^{1/p^\infty}_* q\text{DR}_p(A/\mathbb{Z}_p)^{\wedge_p})^\Lambda\), on the Frobenius lift \(\Psi^p\) is in determining the image of \(X(C^b_*) \to X(C_*)\) as \(C\) varies.

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Although the map $X(C_+^b) \to X(C_+)$ is not surjective, it is almost so in a precise sense, which we now use to establish independence of $\Psi^p$, showing that, up to faithfully flat descent, $q\text{DR}_p(A/\mathbb{Z}_p)^{\wedge p}/[p]_{q^{1/p}}$ is the best possible perfectoid approximation to $A[\xi_{q^{\infty}}]^{\wedge p}$.

**Definition 3.10** Given a functor $X$ from $(\mathbb{Z}[\xi_{q^{\infty}}])^a$-algebras to sets and a functor $\mathcal{A}$ from perfectoid $(\mathbb{Z}[\xi_{q^{\infty}}])^a$-algebras to abelian groups, we write

$$R\Gamma_{\text{Pfd}}(X, \mathcal{A}) := R\text{Hom}_{\text{Pfd}((\mathbb{Z}[\xi_{q^{\infty}}])^a), \text{Set}}(X, \mathcal{A}),$$

where $\text{Pfd}(S^a)$ denotes the category of perfectoid almost $S$-algebras, and $R\text{Hom}_{C, \text{Set}}(\_,-)$ is as in Definition 1.9.

When $X$ is representable by a $(\mathbb{Z}[\xi_{q^{\infty}}])^a$-algebra $C$, we simply denote $R\Gamma_{\text{Pfd}}(X, \mathcal{A})$ by $R\Gamma_{\text{Pfd}}(C, \mathcal{A})$ — when $C$ is perfectoid, this will just be $\mathcal{A}(C)$.

Thus $R\Gamma_{\text{Pfd}}(C, \mathcal{A})$ is the homotopy limit of the functor $\mathcal{A}$ (regarded as taking values in cochain complexes) on the category of perfectoid $(\mathbb{Z}[\xi_{q^{\infty}}])^a$-algebras equipped with a map from $C$. This is closely related to the pushforward from the pro-étale site of the generic fibre, whose décalage for $\mathcal{A} = \mathcal{A}_{\inf}$ is the complex $A\Omega$ of [4, Definition 9.1].

**Theorem 3.11** If $R$ is a $p$-adically complete $\Lambda_p$-ring over $\mathbb{Z}_p$, and $A$ a formal $R$-deformation of a smooth ring over $(R/p)$, then the complex

$$R\Gamma_{\text{Pfd}}((A[\xi_{q^{\infty}}] \otimes_R \Psi^{1/p^{\infty}}) \wedge R, \mathcal{A}_{\inf})$$

of $(\Psi^{1/p^{\infty}} R[q])^{\wedge (p,q-1)}$-modules is almost quasi-isomorphic to $(\Psi^{1/p^{\infty}} q\text{DR}_p (A/R))^{\wedge p}$ for any $\Lambda_p$-ring structure on $A$ coming from a framing over $R$ as in Theorem 2.8.

**Proof** Since passage to almost modules is an exact functor, it follows from the definition of $q\text{DR}_p$ that the cochain complex $(\Psi^{1/p^{\infty}} q\text{DR}_p (A/R))^{\wedge p}$ is given by $R\text{Hom}_{[f \hat{\Lambda}_p(R[\mathbb{Q}-1]), \text{Set}]}(X^{q,p}_{\text{strat}}, ((\Psi^{1/p^{\infty}} \mathcal{O}))^{\wedge p})$ in the notation of Definition 1.9, where $f \hat{\Lambda}_p(R[\mathbb{Q}-1])$ denotes the category of flat $(p,q-1)$-adically complete $\Lambda_p$-algebras over $R[q-1]$.

Now note that $C \mapsto ((\Psi^{1/p^{\infty}} C)^{\wedge p})_*$ is left adjoint to the inclusion functor $i : \text{Pfd}_{\Lambda_p}(R[q-1]) \to f \hat{\Lambda}_p(R[q-1])$ from the category of integral perfectoid $\Lambda_p$-rings over $\Psi^{1/p^{\infty}} R[q-1]^{\wedge p}$. Thus $i^* : \text{Ch}([f \hat{\Lambda}_p(R[q-1]), \text{Ab}]) \to \text{Ch}([\text{Pfd}_{\Lambda_p}(R[q-1]), \text{Ab}])$ has exact right adjoint $\mathcal{F} \mapsto (\mathcal{F} \circ (\Psi^{1/p^{\infty}})^{\wedge p})_*$. We therefore have

$$R\text{Hom}_{[f \hat{\Lambda}_p(R[q-1]), \text{Set}]}(X^{q,p}_{\text{strat}}, ((\Psi^{1/p^{\infty}} \mathcal{O}))^{\wedge p}) \simeq R\text{Hom}_{[\text{Pfd}_{\Lambda_p}(R[q-1]), \text{Set}]}(i^* X^{q,p}_{\text{strat}}, \mathcal{O}^a).$$

It thus follows that the cochain complex $(\Psi^{1/p^{\infty}} q\text{DR}_p (A/R))^{\wedge p}$ is the homotopy limit of the functor $(B, x, y) \mapsto B^a$ on the category of triples $(B, x, y)$ for integral perfectoid $\Lambda_p$-rings $B$ over $\mathbb{Z}[q^{1/p^{\infty}}]^{\wedge (p,q-1)}$ and
(x, y) ∈ X^q_p, B × Y^q_p, B Y(B),

where X = Spec A and Y = Spec R.

By Lemma 3.8, such Λ_p-rings B are uniquely of the form \( \mathcal{A}_\inf(C_*) \) for C ∈ Pfd((\( \mathbb{Z}_p[\xi_\infty]^p \))^a), so this homotopy limit becomes

\[
\left( \left( \psi^{1/p_\infty}_p \mathfrak{qDR}_p(A/R) \right)^{a} \right) \cong \mathcal{R}\Gamma_{\text{Pfd}} \left( (X^q_p \times Y^q_p) \circ (\mathcal{A}_\inf)_*, (\mathcal{A}_\inf)^a \right).
\]

Writing \( X^\infty(C) := \text{Im} (\lim_{\psi_p} X(C_*) \to X(C_*)) \), Lemma 3.9 then combines with the description above to give

\[
\left( \mathfrak{qDR}_p(A/R) \right)^{a} \cong \mathcal{R}\Gamma_{\text{Pfd}} \left( X^\infty \times y^\infty \lim_{\psi_p} Y, (\mathcal{A}_\inf)^a \right),
\]

\[
\cong \mathcal{R}\Gamma_{\text{Pfd}} \left( X^\infty \times y^\infty \lim_{\psi_p} Y, (\mathcal{A}_\inf)^a \right).
\]

We now introduce a Grothendieck topology on the category \([\text{Pfd}_{(\mathbb{Z}[\xi_\infty]^p)}^a, \text{Set}]\) by taking covering morphisms to be those maps \( C \to C' \) of perfectoid algebras which are almost faithfully flat modulo p. Since \( C^0 = \lim_{\Phi} (C/p) \), the functor \( \mathcal{A}_\inf \) satisfies descent with respect to these coverings, so the map

\[
\mathcal{R}\Gamma_{\text{Pfd}} \left( X^\infty \times y \lim_{\psi_p} Y, (\mathcal{A}_\inf)^a \right) \to \mathcal{R}\Gamma_{\text{Pfd}} \left( X^\infty \times y \lim_{\psi_p} Y, (\mathcal{A}_\inf)^a \right)
\]

is a quasi-isomorphism, where \((-)^\sharp\) denotes sheafification.

In other words, the calculation of \( (\mathfrak{qDR}_p(A/R)^{a})^a \) is not affected if we tweak the definition of \( X^\infty \) by taking the image sheaf instead of the image presheaf. We then have

\[
(X^\infty)^\sharp(C) = \bigcup_{C \to C'} \text{Im} \left( X(C_*) \times X(C'_*) \lim_{\psi_p} X(C'_*) \to X(C_*) \right),
\]

where \( C \to C' \) runs over all covering morphisms.

Now, \( \lim_{\psi_p} X \) is represented by the perfectoid algebra \( (\psi^{1/p_\infty}_p A)^{a} \), which is isomorphic to \( A[x_1^{1/p_\infty}, \ldots, x_d^{1/p_\infty}]^p \) as in the proof of Lemma 3.3. This allows us to appeal to André’s results [1, §2.5] as generalised in [2, Theorem 2.3]. For any morphism \( f : A \to C \), there exists a covering morphism \( C \to C_i \) such that \( f(x_i) \) has arbitrary \( p \)-power roots in \( C_i \). Setting \( C' := C_1 \otimes C \ldots \otimes C_d \), this means that the composite \( A \to C \to C' \) extends to a map \( (\psi^{1/p_\infty}_p A)^{a} \to C' \), so \( f \in (X^\infty)^\sharp(C) \). We have thus shown that \( (X^\infty)^\sharp = X \), giving the required equivalence.

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\[
\left( \left( \Psi^{1/p^{\infty}} q\text{DR}_{p}(A/R) \right)^{\wedge_{p}} \right)^{a} \cong R\Gamma_{\text{Pfd}} \left( X \times_{Y} \lim_{\Psi_{p}} X_{p}, (\mathcal{A}_{\text{inf}})^{a} \right).
\]

Finally, compatibility of these equivalences with the \( (\Psi^{1/p^{\infty}} R[q])^{\wedge_{(p,q-1)}} \)-module structures is given by functoriality, multiplicativity and the identification \( (\Psi^{1/p^{\infty}} R[q])^{\wedge_{(p,q-1)}} \cong (\Psi^{1/p^{\infty}} q\text{DR}_{p}(R/R))^{\wedge_{p}} \).

**Remark 3.12** Corresponding to the cohomology theory \( ((\Psi^{1/p^{\infty}} q\text{DR}_{p}(A/R))^{\wedge_{p}})^{a} \), it is natural to consider \( q \)-connections on finite projective modules \( M \) over

\[
\eta_{(q-1)}^{0} \left( \left( \Psi^{1/p^{\infty}} \left( \Omega_{A\times R}^{*} \right) \right)^{\wedge_{p,a}} \right)
\]

\[
= \left\{ a \in \left( \Psi^{1/p^{\infty}} (A[q - 1]) \right)^{\wedge_{(p,q-1),a}} : \nabla_{q} a \in \left( \Psi^{1/p^{\infty}} \left( \Omega_{A}^{1}[q - 1] \right) \right)^{\wedge_{(p,q-1),a}} \right\}
\]

\[
= \left( \left( \sum_{n} [p^{n}]_{q^{1/p^{n}}} \Psi^{1/p^{n}} A[q^{1/p^{n}}] \right)^{\wedge_{(p,q-1)}} \right)^{a}.
\]

It follows from the proof of Proposition 1.25 that these are equivalent, for \( X = \text{Spec} A \), to finite projective almost \( (\Psi^{1/p^{\infty}} \mathcal{O}_{X_{q,\text{strat}}})^{\wedge_{p}} \)-modules \( \mathcal{N} \) for which \( \mathcal{N} \cap (q - 1) \) is the pullback of the almost \( H^{0}(\Psi^{1/p^{\infty}} q\text{DR}_{p}(A/R))^{\wedge_{p}} / (q - 1) \)-module \( \Gamma(\mathcal{X}_{q,\text{strat}}, \mathcal{N} / (q - 1)) =: M_{0} \).

Up to almost isomorphism, these correspond via the proof of Theorem 3.11 to those finite projective \( \mathcal{A}_{\text{inf}} \)-modules \( N \) on the site of integral perfectoid algebras \( C \) over \( A[\xi_{p^{\infty}}]^{\wedge_{p}} \otimes R \Psi^{1/p^{\infty}} R \) for which there exists a \( W^{(p)}(A[\xi_{p^{\infty}}]^{\wedge_{p}}) \)-module \( M_{0} \) with \( W^{(p)}(C) \)-linear isomorphisms

\[
N(C) \otimes_{\mathcal{A}_{\text{inf}}(C)} W^{(p)}(C) \cong M_{0} \otimes_{W^{(p)}(A[\xi_{p^{\infty}}]^{\wedge_{p}})} W^{(p)}(C),
\]

functorial in \( C \).

This establishes a weakened form of [13, Conjecture 7.5] on co-ordinate independence of the category of \( q \)-connections, giving the statement for almost \( (\sum_{n} [p^{n}]_{q^{1/p^{n}}} \Psi^{1/p^{n}} A[q^{1/p^{n}} - 1])^{\wedge_{p}} \)-modules rather than \( A[q - 1] \)-modules.

The following gives a slight partial refinement of [4, Theorem 1.17]:

**Corollary 3.13** If \( R \) is a \( p \)-adically complete \( \Lambda_{p} \)-ring over \( \mathbb{Z}_{p} \), and \( A \) a formal \( R \)-deformation of a smooth ring over \( (R/p) \), then the \( q \)-de Rham cohomology complex \( (q - \Omega_{A/R}^{*} \otimes_{R[q]} (\Psi^{1/p^{\infty}} R)[q^{1/p^{\infty}}])^{\wedge_{p}} \) is, up to almost quasi-isomorphism, independent of a choice of co-ordinates \( \square \). As such, it is naturally an invariant of the commutative \( p \)-adically complete \( (\Psi^{1/p^{\infty}} R)[\xi_{p^{\infty}}]^{\wedge_{p}} \)-algebra \( (A[\xi_{p^{\infty}}] \otimes R \Psi^{1/p^{\infty}} R)^{\wedge_{p}} \).

**Proof** Since

\[
\Psi^{1/p^{\infty}} q\text{DR}_{p}(A/R) = \Psi^{1/p^{\infty}} q\text{DR}_{p}(A \otimes_{R} \Psi^{1/p^{\infty}} R)/\Psi^{1/p^{\infty}} R,
\]

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Theorem 2.8 combines with Lemma 3.3 to give

\[(q\Omega^*_A/R, \square \otimes_{R[q]} (\Psi^1/p^\infty R)[q^1/p^\infty])^\wedge_p \simeq L_{n(1)} \left( (\Psi^1/p^\infty qDR_p(A/R))^\wedge_p \right),\]

and by Theorem 3.11, we know that this depends only on \((A[\xi^\infty] \otimes_R \Psi^1/p^\infty R)^\wedge_p\) up to almost quasi-isomorphism.

**Remark 3.14** The almost quasi-isomorphism in Corollary 3.13 should be a genuine quasi-isomorphism when we impose some conditions on the base ring \(R\). By [4, Lemma 8.11], it would suffice to verify that \(H^*(\Psi^1/p^\infty (\Omega^*_A/R[q^{-1}]/(q-1)\nabla_q))^\wedge_p\) and its quotient by \((q-1)\) have no \((q^1/p^\infty - 1)\)-torsion, which should follow for \(R\) smooth by an argument similar to [4, Proposition 8.9].

**Remark 3.15** (Eliminating roots of \(q\)) The key feature of the comparison results in this section is that, up to faithfully flat descent, the functor \(X^{q,p}_{\text{strat}}\) does not depend on Adams operations when restricted to the category of integral perfectoid \(\Lambda_p\)-rings \(B\) over \(\mathbb{Z}[q]\), since the proof of Theorem 3.11 gives \((X^{q,p}_{\text{strat}})^\wedge(B) \cong X(B/[p]_{q^{-1}})\). We can extend the latter functor to more general \(\Lambda_p\)-rings over \(\mathbb{Z}[q]\) by setting

\[X^{q,p}(B) := X(B/(\Psi^p)^{-1}([p]_q B)),\]

which does not depend on any Adams operations on \(X\).

When \(\mathcal{O}_X\) has a \(\Lambda_p\)-ring structure, there is then a natural map \(\alpha : X^{q,p}_{\text{strat}} \to X^{q,p}\) because \(\Psi^p((q-1)B) \subset [p]_q B\). This induces a transformation

\[\alpha^* : \mathbf{RHom}_{(\text{f\,\mathcal{A}t}, \mathbb{R}[\mathcal{A}]/[q^{-1}]), \text{Set}}(X^{q,p}, \mathcal{O}) \to \widehat{qDR}_p(A/R)^\wedge_p\]

for \(X = \text{Spec} A\). But for integral perfectoid \(\Lambda_p\)-rings \(B\), we know that \(X^{q,p}(B) = (X^{q,p}_{\text{strat}})^\wedge(B)\), so by adjunction, as in the proof of Theorem 3.11, \(\alpha^*\) becomes an almost quasi-isomorphism on applying a form of completed stabilisation \(\Psi^1/p^\infty (-)^\wedge_p\). Thus \(H^*(X^{q,p}, \mathcal{O})\) might be a candidate for the co-ordinate independent \(q\)-de Rham cohomology theory proposed in [13]. It naturally carries an Adams operation \(\Psi^p\), which would correspond to the operation \(\phi_p\) of [13, Conjecture 6.1].

Any \(a \in A\) defines an element of \(H^0(X^{q,p}, \mathcal{O}/(\Psi^p)^{-1}([p]_q \mathcal{O}))\) so \(\Psi^p(a) \in H^0(X^{q,p}, \mathcal{O}/[p]_q)\) and applying the connecting homomorphism associated to \([p]_q : \mathcal{O} \to \mathcal{O}\) gives an element \(\beta_{[p]_q} \Psi^p(a) \in H^1(X^{q,p}, \mathcal{O})\) whose image under \(H^1(\alpha^*)\) is

\[[p]_q^{-1} \Psi^p((q-1)\nabla_q a) = (q-1) \Psi^p(\nabla_q a).\]

Moreover, to \(a \in A\) we may associate elements \(a_n \in H^0(X^{q,p}, \mathcal{O}/[p^n]_q)\) for \(n \geq 1\), determined by the property that \(a_n \equiv \Psi^p a^{p^n-1} \mod [p]_q^{n+1}\) for \(1 \leq i \leq n\), and these give rise to elements \(\beta_{[p^n]_q} a_n \in H^1(X^{q,p}, \mathcal{O})\). Explicitly, if we define operations \(\varepsilon_i\) on \(\mathcal{O}\) by \(\varepsilon_0 = \text{id}\) and \(\varepsilon_{i+1}(a) := (a^{p^{i+1}} - \Psi^p(a^{p^{i}}))/p^{i+1}\), then for a local lift \(\tilde{a} \in \mathcal{O}\) of \(a \in \mathcal{O}/(\Psi^p)^{-1}([p]_q)\), we have

\[a_n = \sum_{i=0}^{n-1} [p^i]_{q^{p^n-i}} \Psi^{p^n-i}(\varepsilon_i \tilde{a} + [p^n]_q \mathcal{O}),\]
\[ H^1(\alpha^*)(\beta_i p^n \lambda_q d_n) = (q - 1) \sum_{i=0}^{n-1} \Psi p^{n-i} (\nabla q \varepsilon_i \tilde{a}). \]

In particular, for \( A = R[x] \) these include all the elements \((q - 1)[m] q p^n x^{p^n - 1} dx\), since \( \varepsilon_i(x^m) = 0 \) for all \( i > 0 \), \( x^m \) having rank 1. This suggests that in general the image of \( H^1(\alpha^*) \) might be \((q - 1)H^1 qDR_p(A/R)^{\ast, p}\), tying in well with \((q - 1)\)-adic décalage. Explicit descriptions for much of the functoriality from Corollary 3.13 can also be inferred from this analysis, since it implies that the transformations
\[
\sum_{i=0}^{n-1} \Psi p^{n-i} \circ \nabla_q \circ \varepsilon_i : A \to H^1(q \Omega_{\ast}^\bullet_A / R, \square)
\]
are all natural in \( A \).

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