Derivation of the simplest transport equations for dissipation and vorticity from a modified equation for small-scale velocity

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Abstract. Transport equations for dissipation and vorticity are derived from a transport equation for small-scale velocity, using expression for turbulent viscosity through dissipation and gradients of large-scale velocity, assuming validity of local turbulent energy balance. These equations are of simple form. The equations can be used for parametrization in the various turbulence models. The approach can be generalized for magnetohydrodynamics, the flow with variable fluid density, the non-newton flow, etc.

1. Introduction
Transport equations for vorticity and dissipation rate are of an important role in modern theory of hydrodynamic turbulence. For example, the vorticity transport equation was used for the derivation of Lundgren’s vortex [1], as a fine small-scale structure of a turbulent flow. Semi-empirical transport equations for dissipation rate are important part of several turbulence models [2-3]. There are debatable methods of derivation these models from the Navier-Stokes equations [4-6]. Here we confine ourselves to a derivation of the simplest forms of these transport equations from the simple small-scale transport equation [7].

2. Approximate dissipation rate transport equation for anisotropic turbulence.
Following to [8], dissipation rate of turbulent energy is defined as:

$$\varepsilon = \frac{1}{2} \nu < \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 >$$

(1)

or

$$\varepsilon = \nu < \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} > .$$

(2)
where \( \nu \) - molecular viscosity, \( u_i \) - i-component of small-scale velocity, \( i=1,2,3; \)
\( x_i \) - spatial rectangular coordinate, \( i=1,2,3; \) \(<\ldots>\) - averaging over volume \( V \).

\( V \approx L^3 \), where \( L \) - integral scale of turbulence. Sometimes, the transport equation for isotropic dissipation is used:

\[
D = \nu < \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} > ,
\]

(3)

It is assumed that \( \varepsilon \approx D \) at large Reynolds numbers. The equation for transport of small-scale velocity is [7]:

\[
\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + U_k \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_k} (u_i u_k - < u_i u_k >) = - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} .
\]

(4)

Differentiating this equation with respect to \( x_m \) and taking into account that derivative of large-scale velocity with respect to \( x_m \) can be considered as zero, we obtain an equation:

\[
\frac{\partial}{\partial x_m} \left( \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j} \frac{\partial U_j}{\partial x_k} + U_k \frac{\partial^2 u_i}{\partial x_m \partial x_k} + \frac{\partial^2}{\partial x_m \partial x_k} (u_i u_k - < u_i u_k >) \right) = - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^3 u_i}{\partial x_m \partial x_j \partial x_k} .
\]

(5)

Introducing the instantaneous dissipation rate of turbulent energy as:

\[
\varepsilon_{ins} = \frac{1}{2} \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 ,
\]

(6)

we have: \( \varepsilon = <\varepsilon_{ins}> \).

First, let us to differentiate (6) with respect to time and to a spatial coordinate. Next, we are exchanging the order of differentiation with respect to time and space. At the third, we are taking into account the equality of the mixed derivatives:

\[
\frac{\partial^2 u_i}{\partial x_j \partial t} = \frac{\partial^2 u_j}{\partial x_i \partial t} .
\]

Then we have:

\[
\frac{\partial \varepsilon_{ins}}{\partial t} = 2\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} .
\]

(7)

Inserting relation (4) in relation (7), we obtain a transport equation of the instantaneous dissipation rate:

\[
\frac{\partial \varepsilon_{ins}}{\partial t} = 2\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left( \frac{\partial u_k}{\partial x_j} \frac{\partial U_i}{\partial x_k} - U_i \frac{\partial^2 u_j}{\partial x_j \partial x_k} - \frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_j} - \frac{\partial^3 (u_i u_k)}{\partial x_j \partial x_k \partial x_k} + \nu \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_k} \right) .
\]

(8)

We assume that we use averaging \(<\ldots>\) over cubic cell with side, which is equal to the integral scale of turbulence: \( L = e^{1/2} S^{-3/2} \) and period \( T = 1/S \). If turbulent flow is sweeping through cell with constant large-scale velocity \( U \), then in moving frame of reference with this velocity, this large-scale velocity can be put zero. It is obvious, that the instantaneous dissipation rate (6) and the
mean dissipation rate (12) do not change. Averaging equation (8) and returning to original immovable system of reference, we obtain an equation:

\[
\frac{\partial \varepsilon}{\partial t} + U_i \frac{\partial \varepsilon}{\partial x_i} = T_1 + T_2 + T_3 + T_4,
\]

where \( T_1 = -2\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial^2 \rho}{\partial x_j \partial x_j} \) - mixed production,

\[
T_2 = -2\nu \frac{\partial^2 (u_i u_k)}{\partial x_j \partial x_j} \frac{\partial^2 p}{\partial x_i \partial x_i} \frac{\partial^2 F}{\partial x_k \partial x_k} \frac{\partial^2 p}{\partial x_k \partial x_k} \]

- tendency to isotropy by pressure,

\[
T_3 = -2\nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial^2 (u_i u_k)}{\partial x_j \partial x_k} \frac{\partial^2 (u_i u_k)}{\partial x_k \partial x_k} \]

- turbulent production,

\[
T_4 = 2\nu^2 \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial^2 (u_i u_k)}{\partial x_j \partial x_k} \frac{\partial^2 (u_i u_k)}{\partial x_k \partial x_k} \]

- dissipation.

This is the simplest form of the dissipation rate equation, derived previously. It should note that the last term \( T_4 \) can be written down in more symmetrical form:

\[
T_4 = 2\nu^2 \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial^2 (u_i u_k)}{\partial x_j \partial x_k} \frac{\partial^2 (u_i u_k)}{\partial x_k \partial x_k} \]

By integrating by parts, the last expression can be rewrite as:

\[
T_4 = 2\nu^2 \left[ \frac{\partial}{\partial x_k} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]^2
\]

that shows that \( T_4 \) dissipates of the dissipation rate.

3. Derivation of simplified transport equation for the small-scale vorticity.

Considering the equation (4), we assume that

\[
\frac{\partial <u u_k>}{\partial x_k} = 0, \quad i = 1, 2, 3
\]

since we are interested in only the small-scale velocity. Using identity:

\[
(u \cdot \nabla)u = \frac{1}{2} \nabla u^2 - u \times w
\]

where vorticity is defined as \( w = \nabla \times u \), the equation (4) can be written down in the Gromeka–Lamb form:

\[
\frac{\partial u}{\partial t} + u \cdot \nabla U + U \cdot \nabla u + \frac{1}{2} \nabla u^2 - u \times w = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u
\]

(13)
We assume in equation (13) that \( U = \text{const} \), \( \frac{\partial U_j}{\partial x_j} = \text{const} \). Let us vector multiply operator \( \nabla \) on the left and the right hand parts of the equation (23), taking into account the identity
\[
\nabla \times (\nabla \Phi) = 0,
\]
that is valid for arbitrary scalar function \( \Phi \). As a result, we obtain a transport equation for vorticity:
\[
\frac{\partial \omega}{\partial t} + U \cdot \nabla \omega + w \cdot \nabla U - \nabla \times (u \times w) = \nu \nabla^2 w
\]
(14)

Then we use identity
\[
\nabla \times (u \times w) = u(\nabla \cdot w) - w(\nabla \cdot u) + (w \cdot \nabla)u - (u \cdot \nabla)w,
\]
(15)
which is simplified for incompressible fluid (\( \nabla \cdot u = 0 \), \( \nabla \cdot w = 0 \)):
\[
\nabla \times (u \times w) = (w \cdot \nabla)u - (u \cdot \nabla)w.
\]
(16)

Inserting the relation (16) in the equation (14), we obtain:
\[
\frac{\partial \omega}{\partial t} + U \cdot \nabla \omega + w \cdot \nabla U + (u \cdot \nabla)w = (w \cdot \nabla)u + \nu \nabla^2 w
\]
(17)

The equation (17) is a simplified transport equation for vorticity under the influence of large-scale velocity gradients. Assuming \( U = 0 \) and \( \frac{\partial U_j}{\partial x_j} = 0 \), the equation (17) transforms into the Helmholtz transport equation for eddy [9].

4. Results and discussion
We obtained the simplest forms of transport equations for dissipation rate and vorticity under the influence of large-scale velocity gradients. They are distinct from the usual equations, assuming that gradients of large-scale velocity are constants instead of linear dependence on spatial coordinates. It will help to obtain more simple turbulent models than used today. Especially, we can obtain the simpler solution from the vorticity equation than Lundgren’s vortex [1].

5. Conclusion
Here, we derived the simplest forms of dissipation rate and transport of vorticity equations. These equations can be used in various applications of turbulence. We assume that spatial scale of the mean velocity variability is much greater than the spatial scale of the small-scale velocity variability. Thus, the approach cannot be used in vicinity of the walls, where the known boundary functions should be used.

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