Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process.

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Abstract

Let \((X_t)\) be a one-dimensional Ornstein-Uhlenbeck process with initial density function \(f: \mathbb{R} \to \mathbb{R}\), which is a regularly varying function with exponent \(-1 - \eta\), \(\eta \in (0, 1)\). We prove the existence of a probability measure \(\nu\) with a Lebesgue density, depending on \(\eta\), such that for every \(A \in \mathcal{B}(\mathbb{R})\):

\[
\lim_{t \to \infty} P_f(X_t \in A \mid T_0^X > t) = \nu(A).
\]

1 Introduction

Let \(\Omega = C([0, \infty), \mathbb{R})\) be the space of real continuous functions, and \(\mathcal{F}\) the standard Borel \(\sigma\)-field on \(\Omega\). For a probability measure \(\mu\) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) we denote by \(P_\mu\) the probability measure on \((\Omega, \mathcal{F})\) such that \(B_t(w) = w(t)\) is a Brownian Motion with initial distribution \(\mu\). If \(\mu = \delta_x\) is the Dirac mass at \(x \in \mathbb{R}\) we denote \(P_x\) instead of \(P_{\delta_x}\). Similarly, if \(\mu\) has a density \(f\) we use the notation \(P_f\) instead of \(P_\mu\).

Consider a one-dimensional diffusion process \((X_t)\) which in differential form may be written as

\[
dX_t = dB_t - \alpha(X_t)dt, \quad X_0 = B_0,
\]

where the drift \(\alpha: \mathbb{R} \to \mathbb{R}\) is a given function. We denote by \(\mathcal{L}\) the infinitesimal operator of the process \((X_t)\). That is

\[
\mathcal{L}f := \frac{1}{2} \partial_{xx} f - \alpha \partial_x f,
\]

(2)

We also denote by \(\mathcal{L}^*\) the formal adjoint operator of \(\mathcal{L}\) with respect to the Lebesgue measure. In other words

\[
\mathcal{L}^*f := \frac{1}{2} \partial_{xx} f + \partial_x (\alpha f).
\]

(3)

The hitting time of zero \(T_0^X\) is defined as the first time that the process \((X_t)\) reaches zero. That is \(T_0^X := \inf \{t \geq 0 : X_t = 0\}\). A similar notation will be used for the hitting time of zero for \((B_t)\): \(T_0^B\).

A probability measure \(\nu\) is said to be a quasi-stationary distribution (qsd) if

\[
\forall t, \forall A \in \mathcal{B}(\mathbb{R}) \quad P_\nu(X_t \in A \mid T_0^X > t) = \nu(A).
\]

(4)

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If the drift is regular, this condition is equivalent to the existence of $\lambda \in \mathbb{R}_+$ such that
\[
\forall t, \forall A \in \mathcal{B}(\mathbb{R}) \quad P_\nu(X_t \in A, T^X_0 > t) = e^{-\lambda t} \nu(A),
\]
which implies that $\nu = \nu_\lambda$, the probability measure concentrated on $\mathbb{R}_+$ with smooth density proportional to the unique solution of the differential problem
\[
\begin{align*}
\mathcal{L}_* \varphi_\lambda &= -\lambda \varphi_\lambda \\
\varphi_\lambda(0) &= 0, \quad \varphi'_\lambda(0) = 1.
\end{align*}
\]
That is $\forall A \in \mathcal{B}(\mathbb{R}_+)$ we have $\nu_\lambda(A) := (\int_0^\infty \varphi_\lambda(x) dx)^{-1} \int_A \varphi_\lambda(x) dx$.

We remark that in particular from (5) the absorption time is exponentially distributed when the initial distribution is $\nu_\lambda$, that is $P_{\nu_\lambda}(T^X_0 > t) = e^{-\lambda t}$. Usually the set of values of $\lambda$ for which (6) has a positive and integrable solution is an interval $(0, \lambda]$ and moreover $\lambda$ coincides with the ground state of $\mathcal{L}_*$. We shall prove this result in the context of a Ornstein-Uhlenbeck process, but this holds in many others situations (see [5], [6] and [9] for example).

This paper deals with the domain of attraction of the qsd. We say that $\mu$ is in the domain of attraction of the qsd $\nu$ if
\[
\lim_{t \to \infty} P_{\mu}(X_t \in \cdot | T^X_0 > t) = \nu(\cdot),
\]
where the limit is taken in the weak topology. We notice that from (4) $\nu$ belongs to its own domain of attraction. We work with absolutely continuous initial distributions $\mu$, whose density $f$ is a regularly varying function (see definitions on section 3). Our main result is the following theorem, where we assume that the drift is linear $\alpha = ax$ with $a > 0$, that is $(X_t)$ is an Ornstein-Uhlenbeck process.

**Theorem**

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a density function with exponent $-(1 + \eta)$ where $\eta \in (0, 1)$. Then for all $A \in \mathcal{B}(\mathbb{R}_+)$
\[
\lim_{t \to \infty} P_f(X_t \in A | T^X_0 > t) = \nu_\lambda^*(A)
\]
where $\lambda^* := a\eta \in (0, a)$.

We remark that, in the case that the density function $f$ is smooth, the condition $\lim_{u \to \infty} \frac{uf'(u)}{f(u)} = -(1 + \eta)$ is enough to ensure that $f$ has the desired exponent.

In the literature there are two works directly related with the problem mentioned above. The first one was published by P. Mandl [8] who consider general drift assumptions. He proved that the bottom of the spectrum of $\mathcal{L}_*$ is given by $\lambda^* = \sup \{\lambda \in \mathbb{R} : \varphi_\lambda$ does not change sign $\}$. Also he proved that under certain hypothesis on the behaviour of the Fourier transform of the initial density function $f$ around the point $\lambda^*$ the limit in (7) exists and $\nu$ is the probability measure $\nu_\lambda^*$. The measure $\nu_\lambda^*$ is called the minimal qsd. Because of this, we say that Mandl’s result only deals with the domain of attraction of the minimal qsd. For the Ornstein-Uhlenbeck process it is not hard to see that $\lambda^* = a$. 

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The other related result is due to S. Martinez et al. [9]. They studied the domain of attraction of \( qsd \) for a Brownian Motion with constant drift \( \alpha(x) \equiv a \) with \( a > 0 \). In this setting \( \Lambda = a^2/2 \). They prove that the limit in (7) exists if \( \lim_{u \to \infty} -\frac{\ln f(u)}{u} = \beta > 0 \). Moreover, when \( \beta \in (0,a] \) \( \nu \) is the probability measure \( \nu_{\lambda^*} \) where \( \lambda^* = a\beta - \beta^2/2 \). When \( \beta \geq a \) then \( \nu = \nu_{\lambda^*} \).

The present paper is organized as follows. In section 2 we present general and well known facts about the Ornstein-Uhlenbeck process which we include for the sake of completeness. In section 3 we present the proof of the main result of this paper.

## 2 Some facts about the Ornstein-Uhlenbeck process

From now on, \((X_t)\) denotes an Ornstein-Uhlenbeck process, that is to say, the process that solves the stochastic differential equation (1) for a linear drift \( \alpha(x) := ax \) with \( a > 0 \) constant.

As usual \( P_x(B_t \in dy)/dy := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \) denotes the transition density of a Brownian Motion starting from \( x \). We define the functions \( h \) and \( g \) by the formulas

\[
h(t) := 1 - e^{-2at}/2a, \quad g(t) := \frac{e^{2at} - 1}{2a}.
\]

The transition density of the process \((X_t)\) starting from \( x \) can be computed as

\[
p(t,x,y) := P_x(X_t \in dy)/dy = P_{e^{-at}x}(B_{h(t)} \in dy)/dy
\]

On the other hand, an Ornstein-Uhlenbeck process satisfies a reflection principle. In words, conditioning on \( T_0^X \), using the strong Markov property and the fact that \((X_t)\) and \((-X_t)\) have the same law under \( P_0 \), one obtains for \( x,t > 0 \) and \( A \in \mathcal{B}(\mathbb{R}_+) \)

\[
P_x(X_t \in A, T_0^X > t) = P_x(X_t \in A) - P_x(-X_t \in A).
\]

Consequently, using formulas (9) and (10), and the reflection principle for a Brownian Motion, we obtain for every \( x,t > 0 \),

\[
P_x(T_0^X > t) = P_{e^{-at}x}(T_0^B > h(t)).
\]

We also have a formula for \( q \) the transition density of the submarkovian process \((X_t 1_{T_0^X > t})\) given by

\[
q(t,x,y) = p(t,x,y) - p(t,x,-y) = \sqrt{\frac{2}{\pi h(t)}} e^{-\frac{(e^{-at}x)^2 + y^2}{2h(t)}} \sinh \left( \frac{e^{-at}xy}{h(t)} \right),
\]

where it is assumed that \( xy > 0 \).

We remark that formulas (8) and (11) will help us to rewrite probabilities about the process \((X_t)\) in terms of the probabilities about the Brownian motion \((B_t)\). Also, before ending the section, we notice that the reflection principle for the process \((X_t)\) essentially holds for any diffusion process \((Y_t)\) that starting from zero has the same distribution as \((-Y_t)\). Certainly, this is the case when a process \((Y_t)\) solves an stochastic differential equation, of the type (1), for which the drift \( \alpha \) is an odd function and uniqueness in distribution holds.
3 Proof of the main result

Our main theorem relies on the concept of regularly varying functions (for a complete reference, see [1], [3]). A non-negative measurable function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be regularly varying with exponent \( \beta \) (briefly, \( f \) has exponent \( \beta \)), if for all \( c > 0 \)

\[
\lim_{u \to \infty} \frac{f(cu)}{f(u)} = c^\beta.
\]

In order to make a discussion in the same terms of our main result, we deal with a function \( f \) with exponent of the form \(-(1 + \eta)\) with \( \eta \in \mathbb{R} \). Since the function \( f(u)u^{1+\eta} \) varies slowly, we have the following asymptotic for \( \ln f \) (see [1] Proposition 1.3.6)

\[
\lim_{u \to \infty} \frac{\ln f(u)}{\ln u} = -(1 + \eta).
\]

We start the proof of the main Theorem by proving that the set of measures \( \{ P_f(X_t \in \cdot \mid T_0^X > t) \}_{t \geq 1} \) is tight. The next Lemma is technically important for that purpose.

**Lemma 3.1**

If \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a density function with exponent \(-(1 + \eta)\) for some \( \eta \in (0,1) \) then for all \( \gamma \in (0, \eta) \)

\[
\lim_{u \to \infty} u^{1-\gamma} \frac{\int_0^u x^{\gamma}f(x)dx}{\int_0^u xf(x)dx} = \frac{1-\eta}{\eta-\gamma} < \infty.
\]

**Proof:** From the hypothesis assumed on \( f \) it follows immediately that \( x^{\gamma}f(x) \) is integrable near \( \infty \). Therefore from [3], Theorem 1 on section VIII.9, we have the following limits exist

\[
\lim_{u \to \infty} \frac{u^{\gamma+1}f(u)}{\int_u^\infty x^{\gamma}f(x)dx} = \eta - \gamma > 0,
\]

and

\[
\lim_{u \to \infty} \frac{u^2f(u)}{\int_0^u xf(x)dx} = 1-\eta > 0,
\]

from which the result follows. \( \square \)

**Lemma 3.2**

If \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a density function with exponent \(-(1 + \eta)\) for some \( \eta \in (0,1) \), then the set of probability measures: \( \{ P_f(X_t \in \cdot \mid T_0^X > t) \}_{t \geq 1} \) is tight.
**Proof:** It is enough to check that \( \limsup_{t \to \infty} E_f \left( X_t^\gamma \mid T_0^X > t \right) < \infty \) for some \( 1 \geq \gamma > 0 \), where

\[
E_f \left( X_t^\gamma \mid T_0^X > t \right) = \frac{I(0, \infty, 0, \infty)}{J(0, \infty, 0, \infty)},
\]

and

\[
I(A, B, C, D) := \int_A^B \int_C^D f(x) y^\gamma q(t, x, y) \, dy \, dx, \quad J(A, B, C, D) := \int_A^B \int_C^D f(x) q(t, x, y) \, dy \, dx.
\]

It can be checked using (12) that there exists a constant \( c_1 = c_1(a) > 0 \) such that for every \( t \geq 1 \) and \( x \in [0, e^a t] \)

\[
1 \leq \frac{\int_0^t y^\gamma q(t, x, y) \, dy}{\int_0^t y q(t, x, y) \, dy} \leq c_1.
\]

From (15) and the last inequality we have that

\[
E_f \left( X_t^\gamma \mid T_0^X > t \right) \leq \frac{I(0, e^a t, 0, \infty)}{J(0, e^a t, 0, 1)} \left\{ 1 + \frac{I(e^a t, 0, \infty)}{I(0, e^a t, 0, \infty)} \right\} \leq c_1 \left\{ 1 + \frac{I(e^a t, 0, \infty)}{I(0, e^a t, 0, 1)} \right\}.
\]

Therefore to prove the Lemma is enough to find \( 1 \geq \gamma > 0 \) such that

\[
\limsup_{t \to \infty} \frac{\int_0^t \int_0^t f(x) y^\gamma q(t, x, y) \, dy \, dx}{\int_0^t \int_0^t f(x) y q(t, x, y) \, dy \, dx} < \infty.
\]

But since \( 0 < h(1) \leq h(t) \leq \frac{1}{2a} \) for every \( t \geq 1 \) and \( \sinh(z) \geq z \) for all \( z \geq 0 \), we see from (12) that

\[
\int_0^{e^a t} \int_0^1 f(x) y^\gamma q(t, x, y) \, dy \, dx \geq \frac{4a^{3/2}}{\sqrt{\pi} e^a} \int_0^{e^a t} x f(x) e^{-\frac{(e^a - x)^2}{2a^2}} \, dx \int_0^1 y^{\gamma + 1} e^{-\frac{y^2}{2a^2}} \, dy.
\]

On the other hand \( q(t, x, y) \leq p(t, x, y) \). Thus, using (9) and the last inequality, it follows the existence of a constant \( c_2 = c_2(a) > 0 \) such that

\[
\frac{\int_0^\infty \int_0^\infty f(x) y^\gamma q(t, x, y) \, dy \, dx}{\int_0^\infty \int_0^1 f(x) y^\gamma q(t, x, y) \, dy \, dx} \leq c_2 e^a \frac{\int_0^\infty \int_0^\infty f(x) \left( \int_0^\infty y^\gamma e^{-\frac{(e^a - x-y)^2}{2a^2}} \, dy \right) \, dx}{\int_0^\infty f(x) \, dx}.
\]

Now, if \( 0 < \gamma \leq 1 \) there exists a constant \( c_3 > 0 \) such that for \( x \geq e^a t \)

\[
\int_0^\infty y^\gamma e^{-\frac{(e^a - x-y)^2}{2a^2}} \, dy \leq c_3 (e^{-a t} x)^\gamma,
\]

hence from (17) we can conclude that
\[
\limsup_{t \to \infty} \int_0^\infty f(x) e^{-\gamma q(t, x, y)} dy dx \leq c_2 c_3 \limsup_{t \to \infty} e^{at} \int_0^\infty f(x) x^\gamma dx \int_0^1 f(x) dx
\]

To finish the proof we notice that Lemma 3.1 ensures that the right hand side in the last inequality is finite for any \(0 < \gamma < \eta\). This proves assertion (16), and therefore the desired result follows. \(\square\)

**Lemma 3.3**

Let \(0 < b < c\) and \(x > 0\). Then

\[
1 \leq \frac{P_x(T^B_0 > b)}{P_x(T^B_0 > c)} \leq \left(\frac{c}{b}\right)^{3/2}.
\]

**Proof:** The first inequality is direct. On the other hand (see [7], page 197) after a linear substitution we get

\[
\frac{P_x(T^B_0 > b)}{P_x(T^B_0 > c)} = \frac{\int_b^\infty \frac{1}{u} e^{-\frac{x^2}{2u}} du}{\int_c^\infty \frac{1}{u} e^{-\frac{x^2}{2u}} du} \leq \frac{\int_b^\infty \frac{1}{u} e^{-\frac{x^2}{2u}} du}{\int_b^\infty \frac{1}{u} e^{-\frac{x^2}{2u}} du} \leq \left(\frac{c}{b}\right)^{3/2}.
\]

The last inequality follows since the function \(u \to \left(1 + \frac{c-b}{u}\right)\) is decreasing. \(\square\)

**Lemma 3.4**

If \(f : \mathbb{R}_+ \to \mathbb{R}_+\) is a density function with exponent \(-(1 + \eta)\) where \(\eta \in (0, 1)\), then for every \(t > 0\) and \(c > 0\)

\[
\lim_{u \to \infty} \frac{\int_0^{ln(u)/c} f(x) P_{c/u}(T^B_0 > t) dx}{\int_0^{ln(u)/c} f(x) P_{c/u}(T^B_0 > t) dx} = 0.
\]

**Proof:** Since the function \(x \to P_x(T^B_0 > t)\) is increasing on \(\mathbb{R}_+\) we obtain

\[
\int_0^{ln(u)/c} f(x) P_{c/u}(T^B_0 > t) dx \leq P_{ln(u)/c}(T^B_0 > t) \int_0^{ln(u)/c} f(x) dx \leq P_{ln(u)/c}(T^B_0 > t).
\]

But the function \(x \to P_x(T^B_0 > t)\) is differentiable at \(x = 0\), hence, there exists a constant \(c_1 > 0\), which depends on \(t\), such that for \(u > 0\) sufficiently large

\[
\int_0^{ln(u)/c} f(x) P_{c/u}(T^B_0 > t) dx \leq c_1 \frac{ln(u)}{u}.
\]
On the other hand, since \( f \) is regularly varying, from (13) if \( \kappa \in (\eta, 1) \) we have for large \( x \) the inequality \( f(x) \geq \frac{1}{x^{1+\kappa}} \). In particular, for \( u > 0 \) large enough we have that

\[
\int_{\ln(u)/c}^{\infty} f(x) P_{\mu}^{\kappa}(T_0^B > t)dx \geq \left( \frac{c}{u} \right)^{\kappa} \int_{\ln(u)/u}^{\infty} \frac{1}{x^{1+\kappa}} P_x(T_0^B > t)dx.
\]

Now, since the function \( u \to \frac{\ln(u)}{u} \) is asymptotically decreasing to 0, there exists a constant \( c_2 > 0 \), which also depends on \( t \), such that for big \( u > 0 \)

\[
\int_{\ln(u)/c}^{\infty} f(x) P_{\mu}^{\kappa}(T_0^B > t)dx \geq \frac{c_2}{u^{\kappa}}. \tag{20}
\]

From (19) and (20), and the fact that \( \kappa < 1 \), it follows the result. \( \square \)

**Lemma 3.5**

*If \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a density function with exponent \(-(1 + \eta)\) where \( \eta \in (0, 1) \), then for every \( s \geq 0 \)

\[
\lim_{t \to \infty} \frac{P_f(T_0^X > t + s)}{P_f(T_0^X > t)} = e^{-(an)s}. \tag{21}
\]

**Proof**: Let \( s > 0 \) and \( h \) be the function defined on (8). Then, using (11), it follows for every \( t > 0 \) that

\[
\frac{P_f(T_0^X > t + s)}{P_f(T_0^X > t)} = \frac{\int_0^\infty f(x) P_{\mu}(-a(t+s)x)(T_0^B > h(t+s))dx}{\int_0^\infty f(x) P_{\mu}(-ax)(T_0^B > h(t))dx}.
\]

Notice that \( h(t) < h(t+s) < \frac{1}{2a} \) and \( \lim_{t \to \infty} h(t) = \frac{1}{2a} \). Therefore, using the bound obtained in Lemma 3.3 and setting \( u = e^{at} \), it can be easily seeing that (21) is equivalent to

\[
\lim_{u \to \infty} \frac{\int_0^\infty f(x) P_{\mu}(-a\frac{x}{u})(T_0^B > \frac{1}{2a})dx}{\int_0^\infty f(x) P_{\mu}(T_0^B > \frac{1}{2a})dx} = e^{-(an)s}. \tag{22}
\]

Writing

\[
\frac{\int_0^\infty f(x) P_{\mu}(-a\frac{x}{u})(T_0^B > \frac{1}{2a})dx}{\int_0^\infty f(x) P_{\mu}(T_0^B > \frac{1}{2a})dx} = \frac{\left( \int_0^{\infty} \int \frac{1}{e^{a\ln(u)}} f(x) P_{\mu}(-a\frac{x}{u})(T_0^B > \frac{1}{2a})dx \right)}{\left( \int_0^{\ln(u)} \frac{1}{e^{a\ln(u)}} f(x) P_{\mu}(T_0^B > \frac{1}{2a})dx \right)},
\]
from lemma 3.4 and using the substitution $y = e^{-as}x$ in the numerator, to check (22) it is sufficient to prove that

$$\lim_{u \to \infty} \frac{\int_{\ln(u)}^{\infty} f(e^{as}x)P_u(T_0^B > \frac{1}{2a})dx}{\int_{\ln(u)}^{\infty} f(x)P_u(T_0^B > \frac{1}{2a})dx} = e^{-a(1+\eta)s}.$$ 

But, $f$ is regularly varying therefore $\lim_{x \to \infty} \frac{f(e^{as}x)}{f(x)} = e^{-a(1+\eta)s}$ for all $x > 0$. From this fact, it follows the result. \hfill \Box

Lemma 3.6

(a) $\{ \lambda \mid \varphi_\lambda \text{ does not change sign } \} = (-\infty, a]$;

(b) For every $\lambda \in (0, a]$, $\int_0^\infty \varphi_\lambda(x)dx < \infty$.

Proof: Let $\varphi_\lambda$ be the solution of (6). Then $\psi_\lambda(u) = e^{au^2}\varphi_\lambda(u)$ is the unique solution of the equation

$$\begin{cases} L\psi_\lambda = -\lambda \psi_\lambda \\ \psi_\lambda(0) = 0 , \quad \psi'_\lambda(0) = 1. \end{cases}$$

Hence, to prove 3.6(a), we just need to concentrate our attention on $\psi_\lambda$. Note that $\psi_\lambda$ is an analytic function. Setting $\psi_\lambda(u) = \sum b_k u^k$ one sees that $b_0 = 0$, $b_1 = 1$ and one obtains a recursion for $b_{k+2}$ in terms of $b_k$ which shows that $\forall k \geq 0$ $b_{2k} = 0$ and for $k \geq 1$

$$b_{2k+1} = \frac{a^k}{(2k+1)k!} \prod_{i=0}^{k-1} \left(1 - \frac{\lambda a^{-1}}{2i+1}\right). \quad (23)$$

In particular $\psi_\lambda$ cannot change sign if $\lambda \leq a$. Notice also that $\psi_\lambda(u) = u$.

On the other hand, if $\lambda \in (a, 3a)$ then from (23) we get that $b_{2k+1} < 0$ for every $k \geq 1$. Thus, $\lim_{k \to \infty} \psi_\lambda(u) = -\infty$. But $\psi_\lambda(0) = 0$ and $\psi'_\lambda(0) > 0$, hence, from the last limit, we see that there exists $x_0 > 0$ such that $\psi_\lambda > 0$ on $(0, x_0)$ and $\psi_\lambda(x_0) = 0$.

Let $\lambda > a$. We prove then that $\psi_\lambda$ has to change its sign. Letting $k \in (a, min\{3a, \lambda\})$ we just proved the existence of some $x_0 > 0$ such that $\psi_\kappa > 0$ on $(0, x_0)$ and $\psi_\kappa(x_0) = 0$. But, simultaneously

$$\begin{cases} 
(e^{-au^2}\psi'_\lambda(u))'' + 2\lambda e^{-au^2}\psi_\lambda(u) = 0 \\
(e^{-au^2}\psi'_\kappa(u))'' + 2\kappa e^{-au^2}\psi_\kappa(u) = 0.
\end{cases}$$

Since $\lambda > \kappa$, by the Sturm-Liouville’s theorem (see [12], page 104), there exists $y_0 \in (0, x_0)$ such that $\psi_\lambda(y_0) = 0$. Hence, to prove that $\psi_\lambda$ changes sign, it is sufficient to check that
If this were not the case it can be easily checked out that \( \psi^{(k)}(y_0) = 0 \) for all \( k \geq 0 \). Thus, \( \psi_\lambda \equiv 0 \) on \([0, \infty)\). This can not occur because \( \psi_\lambda'(0) \neq 0 \). Consequently, for \( \lambda > a \), \( \psi_\lambda'(y_0) \neq 0 \) which implies that \( \psi_\lambda \) changes its sign. This proves 3.6(a).

Now, we prove part 3.6(b). For \( \lambda \in (0, a] \), we have

\[
\int_0^\infty \varphi_\lambda(u) du = \frac{1}{2a} \sum_{k \geq 0} k! b_{2k+1} = \frac{1}{2a} \left\{ 1 + \sum_{k \geq 1} \frac{1}{2k+1} \prod_{i=0}^{k-1} \left( 1 - \frac{\lambda a^{-1}}{2i+1} \right) \right\}
\]

\[
\leq \frac{1}{2a} \left\{ 1 + \sum_{k \geq 1} \frac{1}{2k+1} \exp \left( -\lambda a^{-1} \sum_{i=0}^{k-1} \frac{1}{2i+1} \right) \right\}
\]

\[
\leq \frac{1}{2a} \left\{ 1 + \sum_{k \geq 1} \frac{1}{2k+1} \exp \left( -\frac{\lambda}{2a} \sum_{i=1}^{k} \frac{1}{i} \right) \right\}
\]  _(24)_

where we have used the fact \( 1 - z \leq e^{-z} \), for \( z \in [0, 1] \). Let \( d := \sup_{k>0} \left| \ln(k) - \sum_{i=1}^{k} \frac{1}{i} \right| \). Since \( d < \infty \), _(25)_ yields

\[
\int_0^\infty \varphi_\lambda(u) du \leq \frac{1}{2a} \left\{ 1 + \frac{e^{\frac{\lambda d}{2a}}}{2} \sum_{k \geq 1} \frac{1}{k^{1 + \frac{d}{2a}}} \right\}
\]

The previous inequality shows that \( \int_0^\infty \varphi_\lambda(u) du < \infty \), which proves 3.6(b).

Proof of the Theorem

Let \( t'_n \to \infty \). From lemma 3.2, we know that there exits a subsequence \( t_n \to \infty \) and a probability measure \( \mu \), defined on \( B(\mathbb{R}_+) \), such that

\[
\lim_{n \to \infty} P_f(X_{t_n} \in \cdot \mid T_0^X > t_n) = \mu.
\]  _(26)_

Now, the function \( P_x(T_0^X > s) \) is continuous and bounded on \( x \in \mathbb{R}_+ \). The strong Markov property allows us to show that for every \( n \)

\[
\frac{P_f(T_0^X > t_n + s)}{P_f(T_0^X > t_n)} = E_f \left( P_{X_{t_n}}(T_0^X > s) \mid T_0^X > t_n \right).
\]

Therefore, taking limit as \( n \to \infty \) and using lemma 3.5, we deduce that for every \( s > 0 \)

\[
\int_0^\infty P_x(T_0^X > s) \mu(dx) = e^{-(a \eta)s} = e^{-\lambda^* s},
\]  _(27)_

where \( \lambda^* := a \eta \in (0, a) \). On the other hand, by lemma 3.6(b), we see that \( \varphi_{\lambda^*} \geq 0 \) is integrable on \([0, \infty)\). This allows us to define for each \( s > 0 \) and \( y > 0 \) the function

\[
\Lambda(s, y) := \int_0^\infty q(s, x, y) \varphi_{\lambda^*}(x) dx.
\]
A simple computation shows
\[ \frac{\partial}{\partial s} q(s, x, y) = \mathcal{L}_x q(s, x, y), \]
which is nothing but the classical Kolmogorov’s equation. Now, from the given definitions we have
\[ \frac{\partial}{\partial s} \Lambda(s, y) = \int_0^\infty \mathcal{L}_x q(s, x, y) \varphi_\lambda (x) dx = \int_0^\infty q(s, x, y) (\mathcal{L}_x^* \varphi_\lambda) (x) dx \]
\[ = -\lambda^* \int_0^\infty q(s, x, y) \varphi_\lambda(x) dx = -\lambda^* \Lambda(s, y). \]
Therefore \( \Lambda(s, y) = \Lambda(0, y) e^{-\lambda^* s} = \varphi_\lambda(y) e^{-\lambda^* s}. \) Hence, integrating this equality over \( y \) leads to
\[ \int_0^\infty \int_0^\infty q(s, x, y) dy \varphi_\lambda(x) dx = \int_0^\infty P_x(T_{x_0}^B > s) \varphi_\lambda(x) (dx) = ce^{-\lambda^* s}, \quad (28) \]
where \( c = \int_0^\infty \varphi_\lambda(x) dx \in (0, \infty). \)

Now, define on \( \mathcal{B}(\mathbb{R}_+) \) the finite signed-measure \( \rho(A) := \mu(A) - \frac{1}{c} \int_A \varphi_\lambda(x) dx. \) It follows that \( \rho([0, \infty)) = 0. \) From (27) and (28), and then (11), we see that for every \( s > 0 \)
\[ \int_0^\infty P_{e^{-as}} (T_{x_0}^B > h(s)) \rho(dx) = 0. \quad (29) \]
Using the well known distribution for the running maxima of a Brownian motion one has
\[ P_x(T_{x_0}^B > t) = \frac{2}{\sqrt{2\pi t}} \int_0^t e^{-y^2/2} dy. \] This identity, a simple substitution and integrating by parts allow us to conclude from (29) that for every \( s > 0 \)
\[ \int_0^\infty \rho[0, x] - \frac{1}{\sqrt{2\pi g(s)}} e^{-x^2/2g(s)} dx = 0, \]
where \( g \) is defined as in (8). Finally, because of the set of density functions \( \left\{ \frac{1}{\sqrt{2\pi \theta}} e^{-x^2/2}\theta \right\}_{\theta > 0} \) is a complete family and the range of \( g \) is \([0, \infty)\), we deduce that \( \rho[0, x] = 0 \), for every \( x > 0. \) Therefore, for all \( A \in \mathcal{B}(\mathbb{R}_+) \), \( \mu(A) = \frac{1}{c} \int_A \varphi_\lambda(x) dx = \nu_{\lambda^*}(A). \) Notice that the limiting measure does not depend on the initially chosen sequence, from which the result follows. \( \square \)

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