Recent years have seen a profusion of hat puzzles, which seek strategies that some prisoners can use to gain their freedom. These riddles are attractive because there are often strategies that have much higher success rates than one would think possible. Here is the classic case:

There are \( n \) prisoners, Alice, Bob, Charlie, ..., under the care of a warden who lines them up in order. He has \( n \) hats—each is either red or blue—and randomly places one hat on each prisoner’s head. Each prisoner can see the hats of only the prisoners in front of him or her: Alice sees all but her own, Bob sees \( n - 2 \) hats, and so on; the last prisoner sees no hats at all. Alice then guesses her color; all prisoners can hear the guesses. Then Bob guesses his hat color, and so on for all prisoners. If they all are correct, they will all be freed. Otherwise, none are freed.

The prisoners know the rules and can devise a strategy in advance; no communication other than the guesses is allowed once the hats are placed. Find their best strategy.

Simply guessing randomly gives the prisoners a \( 2^{-n} \) chance of getting them all correct. Alice is in a bad position, because for her the probability of success is only 50%: she is guessing from two equally likely choices. But the chance of correctness for everyone else can be increased to absolute certainty if they use a clever parity strategy. Alice will declare “red” if the number of reds she sees is even; otherwise, she says “blue.” On hearing this, Bob can easily determine, from the colors he sees, whether his hat is red or blue, and the same for all the other prisoners.

There are many variations on this folklore puzzle, several of which are discussed in [6] (see also [3] for a version in which each prisoner sees all other prisoners’ hats). For example, there might be three hat colors for the \( n \) prisoners. Or there might even be infinitely many prisoners, a topic discussed in detail in the book by Hardin and Taylor [5]. Here we study the variation in which the hats all have distinct colors. The case of \( n \) prisoners and \( n \) differently colored hats is uninteresting, since Alice can immediately deduce her color. But Tanya Khovanova [6] has presented the case in which there are \( n + 1 \) hat colors and the warden just discards the unused hat, a nice variation due to K. Knop and A. Shapovalov.

For the extra-hat case, Alice must make a choice from the two hats she does not see, and so on average, she will succeed half the time. But again the other prisoners can be certain of their color if they all agree to a parity strategy. The twist is that it is the parity of a permutation that they must analyze. The prisoners, numbered 1 (Alice) through \( n \), imagine a ghost prisoner, number \( n + 1 \), who wears the missing hat. The colors are identified with 1, 2, ..., \( n + 1 \), and the prisoners assume that the permutation \( \pi \) of 1, 2, ..., \( n + 1 \) induced by the hats is an even permutation.
That assumption gives each prisoner only one possibility for his or her declaration. So they succeed when \( \pi \) is in fact even; half the permutations are even, so they succeed half the time. This is a perfect situation, because the strategy wins 50% of the time, and that is best possible.

**Two Extra Hats**

The preceding puzzle leads naturally to the case of two or more extra hats, and there are some surprises as well as interesting connections to graph theory, Steiner systems, Latin squares, and ordered designs. We use \( n \) for the number of prisoners and \( k \) for the number of extra hats. Because Alice must choose from \( k + 1 \) hats (she sees \( n - 1 \) of the \( n + k \) hats), the chance of any strategy’s success is never greater than \( 1/(k + 1) \). We consider here only deterministic strategies (as opposed to probabilistic ones).

A natural conjecture is that a perfect strategy exists in all cases.

We begin with \( k = 2 \) and introduce two ghosts (numbered \( n + 1 \), \( n + 2 \)) who wear the unused hats. If there are only two prisoners, it is easy to find a perfect strategy. The prisoners will assume that the hat assignment is \((1, 2), (2, 3), (3, 4), \) or \((4, 1)\). Equivalently, Alice subtracts 1 from the color she sees, while Bob adds 1 to what he hears, working modulo 4. Because there are 12 possible hat states (in general, \((n + k)!/k!\) states), the success rate is \( 4/12 \), or 1/3. For more than two prisoners, the problem becomes complicated and there are several types of strategies. We start with a natural arithmetic strategy, leaving the details as an exercise.

**Modular arithmetic strategy (Larry Carter).** The prisoners assume that the sum of the numbers of the hats on their heads is 1 (mod \( n + 2 \)) if \( n \equiv 2 \) (mod 4) and 0 (mod \( n + 2 \)) otherwise. We omit the details, but the probability that the modular assumption is correct is \( 1/(2\lceil n/2 \rceil + 1) \), and when it is correct, they win. For two prisoners, this strategy has success probability 1/3; for three prisoners, it is 1/5.

Because the hat sums are not equidistributed modulo \( n + 2 \), using a single residue such as 0 is not optimal. While the use of the two residues 0 and 1 maximizes the success probability for this strategy, the modular arithmetic strategy is far from optimal when \( n > 2 \).

The next strategy uses parity for both numbers and permutations; it shows that the prisoners, regardless of how many there are, always have at least a 1/4 chance of winning. When dealing with the parity of the full permutation of the colors, we must make some assumption about the order of the ghost colors. Because the ghosts can exchange hats at will, any assumption is allowed. For the following strategy we assume that an even ghost color always precedes an odd. This turns any hat assignment into a permutation \( \pi \) of \( \{1, 2, \ldots, n, n + 1, n + 2\} \).

**Double parity strategy.** The prisoners assume that the hat assignment satisfies the following conditions:

1. The unused colors have opposite parity, and they are in the order (even, odd) in positions \( n + 1 \) and \( n + 2 \).
2. The permutation \( \pi \) is even.

If the two unused colors have the same parity and Alice or another prisoner can deduce that, then this strategy is undefined, and the prisoners lose.

**Claim.** This strategy wins when the assumption holds.

**Proof.** Consider Alice, who because of what she sees, knows the three missing colors. By condition 1 above, her choices are either two evens and an odd, or two odds and an even; assume the first. The ghost color in position \( n + 1 \) is even, and the other ghost color is odd. So Alice knows that she is wearing an even hat and only one of her two choices will lead to \( \pi \) being even. The other case is similar. Argue the same way for Bob, because he knows that Alice’s declaration is correct, and so on inductively for all the prisoners.

**Claim.** The probability of success is at least 1/4.

**Proof.** Suppose \( n \) is even. The probability that the two ghost colors have different parity is \( m/(2m - 1) \), where \( n + 2 = 2m \); this is because once the first ghost has a hat, the second can any of \( m \) from the \( 2m - 1 \) remaining colors. Because the probability of the permutation being even is 1/2, the probability of success is \( 1/2 \cdot \frac{m}{2m} \), which is \( 1/4 + \frac{1}{4(n+1)} \).

The odd case is similar, yielding \( 1/4 + \frac{1}{4(n+2)} \). The limiting probability is 1/4.

This asymptotic success rate of 25% is the best such result we know of, but far from the 1/3 that perfect strategies attain. For three prisoners the hat assignments in the assumed set of the double parity strategy are

| 123 | 134 | 145 | 152 | 215 | 231 | 253 | 312 | 325 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 341 | 354 | 413 | 435 | 451 | 514 | 521 | 532 | 543 |

with size 18 and success probability 18/60, or 3/10. We shall see in a moment that a perfect strategy exists when \( n = 3 \). But for large \( n \), the double parity strategy is the best known.

In many cases there are perfect strategies that outperform the preceding ones. A perfect strategy for \( n \) prisoners and \( k \) extra hats has an important connection to the arrangement graph \( A_{n+k,n} \): this graph has as vertices all ordered \( n \)-tuples consisting of distinct integers chosen from 1 through \( n + k \), with two vertices being adjacent if the corresponding triples differ in exactly one position. Thus the vertex set consists of all possible hat assignments. We use \( \alpha_{m,n} \) to denote the independence number (size of the largest independent set) of \( A_{m,n} \).

**Theorem 1.** For \( n \) prisoners and \( k \) extra hats, a perfect strategy exists if and only if

\[
\alpha_{n+k,n} = \frac{(n+k)!}{(k+1)!}
\]
(i.e., there is an independent set in $A_{n+k,n}$ having size that is $1/(k + 1)$ of the vertex count).

**Proof.** If $X$ is an independent set, the prisoners can assume that the hat assignment lies in $X$; if it does in fact do so, then the color of each prisoner’s hat is uniquely determined, and the prisoners will win; if $X$’s size is $1/(k + 1)$ of the vertex count, then the resulting strategy is perfect. Conversely, any strategy leads to the set of all hat assignments for which the strategy wins; this set is an independent set in $A_{n+k,n}$ because an edge in this set would mean that one prisoner’s color was not uniquely determined. If the strategy is perfect, then the size of the independent set is as claimed.

An independent set as in Theorem 1 is called a perfect independent set. Figure 1 shows $A_{6,2}$ (the 30 vertices are ordered pairs from 1 through 6; edges are all vertical and horizontal connections, not just the nearby ones shown, e.g., there is an edge from (5, 1) to (5, 4)). A perfect independent set of size six is shown; if the hat assignment is one of these six, then if Alice sees color $i$, she knows that her color is $i − 1$ (mod 6), and the same for Bob with $i + 1$. The larger the independent set, the higher the probability of success, and so the best possible strategy requires computing $x_{n+k,n}$; we use $V_{n+k,n}$ for the vertex count of $A_{n+k,n}$, which is $(n + k)!/k!$. The graph $A_{n+1,n}$ (the case $k = 1$) is bipartite, with parts defined by the parity of the permutation $\pi$. This gives $x_{n+1,n} = V_{n+1,n}/2$ and so yields a perfect strategy, identical to the one-hat-too-many solution given earlier. When $k = 2$, this graph is a Cayley graph of the alternating group graph $AG_{n+2}$ [11]; this graph has vertices for every group element and an edge connecting permutations that differ by either (1, $i$, 2) or (1, 2, $i$), where these are in cycle notation, $i \geq 3$, and the first is used if $i$ is odd and the second if $i$ is even.

Further, a perfect independent set in $A_{n+k,n}$ is exactly an ordered design OD$_1(n − 1, n, n + k)$; see [1]. An OD$_1(t, n, v)$ is an $n \times \binom{v}{t}$ array with entries from 1 to $v$ such that the following conditions are satisfied:

- each column has $n$ distinct entries;
- each collection of $t$ rows contains each possible $t$-tuple exactly once among its columns.

For example, the $3 \times 20$ array of Table 1 is an OD$_1(2, 3, 5)$. There are $\binom{5}{2} = 20$ ordered pairs from \{1, 2, 3, 4, 5\}, and for each way of selecting two rows, all 20 pairs appear as columns in the table.

The existence of an OD$_1(n − 1, n, n + k)$ is equivalent to the existence of a perfect independent set in $A_{n+k,n}$. Each column of the array corresponds to a vertex in $A_{n+k,n}$; the number of columns is $\binom{n+k}{n-1}$!, which is the same as $\frac{(n+k)!}{k!}$, the size of a perfect independent set, and two columns cannot differ in one position only, since that would violate the last point in the definition. Thus the existence of such designs leads to perfect hat strategies (and the converse is also true). The cases $n \leq 4$ of the next result were known to researchers in ordered designs, but the results for $n \geq 5$ are new.

**Theorem 2.** When $k = 2$, perfect strategies exist for $n \leq 6$ and for no other values.

**Proof.** $n = 2$: $x_{4,2} = 4$ by (1, 2), (2, 3), (3, 4), (4, 1), a perfect independent set because $12/3 = 4$.

$n = 3$: If the prisoners assume that the hat-color vector is in

$S = \{(a, b, (3a + 3b) \mod 5) : 1 \leq a, b \leq 5, b \neq a\}$

then they win whenever the assumption is correct, because any two of $a$, $b$, $3(a+b)$ determine the third, and because $3a + 3b \neq a$ or $b$ (mod 5). Note that $S$ is invariant under permutation of the first two elements. This set $S$ is identical to the columns of the earlier example of an OD$_1(2, 3, 5)$. The 20 triples—hat assignments—show that $x_{5,5} = 20$ and $S$ is a perfect independent set, because 20 is one-third of $V_{5,5} = 60$.

---

**Table 1.** This $3 \times 20$ array is an OD$_1(2, 3, 5)$ with $2\binom{5}{2} = 20$ ordered pairs from \{1, 2, 3, 4, 5\}, and for each way of selecting two rows, all 20 pairs appear as columns in the table.

|      | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 |
|------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1    | 2 | 3 | 4 | 5 | 1 | 3 | 4 | 5 | 1 | 2 | 4 | 5 | 1 | 2 | 3 | 5 | 1 | 2 | 3 | 4 |
| 2    | 3 | 4 | 5 | 1 | 3 | 4 | 5 | 1 | 2 | 4 | 5 | 1 | 2 | 3 | 5 | 1 | 2 | 3 | 4 | 2 |
Consider the following matrix, derived from Teirlinck’s work [9] on ordered designs:

\[
M = \begin{pmatrix}
5 & 3 & 1 & 4 & 2 & 0 \\
3 & 5 & 4 & 2 & 0 & 1 \\
1 & 4 & 5 & 0 & 3 & 2 \\
4 & 2 & 0 & 5 & 1 & 3 \\
2 & 0 & 3 & 1 & 5 & 4 \\
0 & 1 & 2 & 3 & 4 & 5 \\
\end{pmatrix}.
\]

Note that \(M\) is a symmetric Latin square with constant diagonal. The method of construction of \(M\) is given in the \(n = 4\) case in the next section. Let \(S\) be the set of \(4\)-vectors \((a, b, c, d)\) from \(\{1, 2, 3, 4, 5, 6\}\) such that \(M_{a,b} = M_{c,d}\). We claim that \(S\) is a perfect independent set. Two vectors in \(S\) cannot differ in exactly one coordinate, because each row (and column) of \(M\) has distinct entries. The common value in the defining equation can be any of 0 through 4; assume it is 0.

We claim that there are 24 vectors in \(S\) for this value. Suppose \((a, b, c, d) \in S\). Then, because \(M\) is symmetric, \(S\) contains \((a, b, d, c, h, a, d, c)\), \((c, d, a, b)\), \((c, d, b, a)\), and \((d, c, b, a)\). That is, \(S\) is invariant under the eight-element group \(G\) generated by \((1 2)\) and \((1 4)(2 3)\). And there are three possibilities for the orbit generators: 1625, 1634, and 2534; these correspond to the three zeros in the upper right quadrant. So there are 24 in all, as claimed. This count works for any entry in place of 0, three zeros in the upper right quadrant. So there are 24 in all, as claimed. This count works for any entry in place of 0, three zeros in the upper right quadrant. So there are 24 in all, as claimed. This count works for any entry in place of 0, three zeros in the upper right quadrant.

Table 3. The 56 elements for the case \(n = 5\).

| 12345 12346 12347 12348 12349 12354 12357 12367 12387 12397 | 12456 12457 12458 12459 12467 12468 12476 12478 12487 12486 | 12547 12548 12549 12567 12578 12587 12647 12648 12657 12687 | 12746 12748 12749 12758 12786 12787 12857 12858 12867 13457 | 13458 13467 13468 13476 13486 13487 13547 13548 13567 13578 | 13568 13576 13587 13647 13648 13657 13678 13687 13746 13748 | 13749 13758 13786 13847 13848 13857 13867 13947 13948 13958 | 13967 13976 13987 14345 14346 14347 14348 14356 14357 14367 14376 14387 14397 | 14456 14457 14458 14459 14467 14468 14476 14478 14487 14486 | 14546 14547 14548 14549 14567 14568 14576 14578 14587 14597 | 14646 14647 14648 14649 14657 14658 14676 14678 14687 14697 | 14746 14747 14748 14749 14756 14758 14786 14787 14798 14797 | 14846 14847 14848 14849 14857 14858 14876 14878 14897 14898 | 14946 14947 14948 14949 14957 14958 14976 14978 14987 14998 |

\(n = 6\): The same ideas used for \(n = 5\) work. The set we found is invariant under the group of order 120 generated by \((1 2)(4 5)\) and \((2 6 3 5)\). The group orbit of the 56 elements in Table 3 gives the perfect independent set of size 6720.

When \(n = 7\), the double parity strategy leads to 50400 vectors, but this can be improved to 50880 (hence a 28% chance of success) by an ILP approach that assumes that the set is invariant under the 120 permutations of indices generated by \((12)(37)\) and \((2654)(37)\). A perfect strategy requires 60480 vectors; it came as a surprise when a backtracking search showed that such a 60480-sized set does not exist (see Table 4), thus disproving the conjecture that perfect strategies always exist when \(k = 2\). This result means that a perfect strategy does not exist for \(n \geq 7\), because such a strategy for \(n\) easily leads to one for \(n - 1\) (delete \(n + 2\) from all vectors ending in \(n + 2\)).

**More Hats**

The problem can be studied when there are three or more unused hats. The double parity strategy for \(k = 2\) extends to show that for \(k \geq 2\) there is a strategy for \(n\) prisoners having success probability greater than \(1/(ek^2)\), independent of \(n\). For this extension, we again imagine that ghosts wear the unused hats, and that the ghosts are in the order specified in condition 1 below. Let \(t = [k^2/2]\). Then the prisoners make the following three assumptions:

1. The unused colors are distinct modulo \(t\), and the unused colors are assumed to be in the order of their mod-\(t\) residues.
2. The permutation \(\pi\) is even.
3. The mod-\(t\) sum of the unused colors is \(\sigma\) (where \(\sigma\) is chosen to maximize the success rate).

Note that assumption 3 follows from assumption 1 when \(k = 2\) and \(t = 2\) (\(\sigma\) being 1). The proof that this assumption’s truth leads to a win is the same as for the \(k = 2\) case discussed earlier: Assumptions 1 and 3 narrow Alice’s possibilities to one or two colors; if two, then assumptions 1 and 2 yield the correct color. Calculating the probability that the assumption holds for the hat assignment requires a little work. The key is to first study the probability of assumption 1, a problem identical to the classic birthday puzzle (with \(t\) days and \(k\) people). We omit the details, but the probability that assumption 1 holds is at least \(t!/((t-k)!t^k)\); standard factorial approximations and bounds show that this is at least \(1/e\) for our choice of \(t\). Now an averaging argument implies that there is some \(\sigma\) such that the probability of assumptions 1 and 2 is at least \(1/er\), yielding the lower bound \(2/(ek^2)\). Assumption 2 then reduces this to \(1/(ek^2)\). The details of this analysis show why, in the choice of \(t\), the exponent 2 and
Table 4. The independence number $\chi_{n,k,n}$. Black entries are $(n + k)!/ (k + 1)!$ and indicate perfect strategies; the black non-bold entries follow from known results about ordered designs; bold entries are new. Red intervals indicate new results that give bounds on $\chi_{n,k,n}$. In the last row, $a_p = all perfect$, $a_p = all perfect$ except $k = 3$, $a_p = infinitely many perfect.

| $k$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ |
|-----|--------|--------|--------|--------|--------|--------|--------|
| 1   | 1      | 3      | 12     | 60     | 360    | 2520   | 20160  |
| 2   | 1      | 4      | 20     | 120    | 840    | 56720  | [50,880, 60,479] |
| 3   | 1      | 5      | 30     | [204, 206] | 1648   | <14,832 | <149,320 |
| 4   | 1      | 6      | 42     | 336    | 2024   |        |        |
| 5   | 1      | 7      | 56     | 504    |        |        |        |
| 6   | 1      | 8      | 72     | 720    | 7920   | 95,040 |        |

**Table 5.** The set $S$ of 126 hat assignments for the case $n = 5$.

| $X$ | $Y$ | $Z$ | $W$ | $V$ | $U$ | $S$ |
|-----|-----|-----|-----|-----|-----|-----|
| 12349 | 12356 | 12367 | 12378 | 12384 | 12395 | 12458 |
| 12597 | 12679 | 12685 | 12698 | 12786 | 12794 | 12893 |
| 13582 | 13594 | 13674 | 13689 | 13692 | 13785 | 13796 |
| 14697 | 14789 | 14793 | 14895 | 15672 | 15687 | 15693 |
| 23451 | 23468 | 23476 | 23485 | 23497 | 23569 | 23574 |
| 23986 | 24567 | 24579 | 24586 | 24593 | 24671 | 24689 |
| 25781 | 25796 | 25894 | 26784 | 26793 | 26897 | 27895 |
| 34781 | 34795 | 34892 | 35671 | 35684 | 35697 | 35786 |
| 45681 | 45698 | 45782 | 45791 | 45897 | 46785 | 46792 |

The entries are given in $[1, k + 1]$.

When $n = 4$, perfect strategies exist for every $k$, except $k = 3$, as proved by Teirlinck [9, pp. 370–372] (he used the language of orthogonal arrays and quasigroups). The negative result when $k = 3$ is that $\chi_{4,3} \leq 209$; this was proved by C. Colbourn. Using ILP, we found an independent set of size 204, and then more computer searching eliminated 207; therefore, $204 \leq \chi_{4,3} \leq 207$. When $k$ is even, Teirlinck’s methods yield a perfect strategy as follows, extending the method presented earlier for $k = 2$. Define the symmetric Latin square $M$ as follows, where the fourth case is reduced modulo $k + 3$, with residue from $\{1, \ldots, k + 3\}$:

\[
M_{ij} = \begin{cases} 
    k + 3 & \text{if } i = j, \\
    j & \text{if } i = k + 3, \\
    i & \text{if } j = k + 3, \\
    \frac{1}{2} (k + 4) (i + j) & \text{otherwise}. 
\end{cases}
\]

The last case uses the fact that $(k + 4)/2$ is the inverse of 2 (mod $k + 3$). Then the same proof as for $k = 2$ works; the size of $S$ is $8(k + 3) (k + 4)/2$, which simplifies to the perfect count

\[
(k + 4)(k + 3)(k + 2).
\]

The case of $k$ odd is quite a bit more complicated.

When $n = 5$, the negative result for $n = 4$, $k = 3$ gives the same for $n = 5$, $k = 3$ by the method mentioned in the $k = 2$, $n = 7$ case. For $k = 4$, there is a perfect independent set. We used ILP to find such a set with an interesting symmetry property. Consider the set $S$ of 126 hat assignments that appears in Table 5.

Let $X$ be the result of permuting the first four entries in each entry of $S$ all $4!$ possible ways. Then $|X| = 24 \cdot 126 = 3024$, and $X$ is a perfect independent set in $A_{0,5}$ (and hence a new ordered design $OD_1(4, 5, 9)$). The existence of a perfect strategy in the case of $n = k = 5$ is an interesting open question. The result we discuss next shows that for $n = 5$, perfect strategies exist when $k$ is one of $6, 10, 12, 16, 18, 22, 24, 26, 28, 30, 31, 36$, and infinitely many others.
The following result of Teirlinck [7, p. 36] gives, for any value of \( n \), infinitely many values of \( k \) admitting perfect strategies. His theorem, translated from the language of ordered designs, is that a perfect strategy exists for \( k \) and \( n \) whenever the prime factorization \( \prod p_i^{\alpha_i} \) of \( k + 1 \) satisfies \( \prod \alpha_i (p_i - 1) \geq n \). In particular, this holds whenever \( k \geq n \) and \( k + 1 \) is prime.

Another technique for getting perfect strategies involves Steiner systems \( S(n - 1, n, m) \); see [4]. Such a system is a set of \( n \)-subsets of \( \{1, 2, \ldots, m\} \) such that every \( (n - 1) \)-set appears exactly once in one of the \( n \)-sets. If a Steiner system \( S(n - 1, n, n + k) \) exists, then one can permute all its elements in all possible ways to get a perfect strategy.

For example, \( S(4, 5, m) \) exists when \( m = 11 \), giving a perfect strategy for \( n = 5 \) and \( k = 6 \). However, Teirlinck [8] proved that whenever a Steiner system \( S(n - 1, n, n + k) \) exists, then his prime factorization theorem just given applies to the parameters. Therefore, a Steiner system cannot give a new perfect strategy.

It is worth noting that the strategies from Steiner systems are stronger than the others in the sense that a prisoner need not see which of the other prisoners has which hats; he or she need see only the set. More precisely, if the rules were changed so that the prisoners saw only the hat colors and not see which of the other prisoners has which hats; he or she would be improved so that the prisoners saw only the hat colors and could not identify other prisoners by sight or by their voices, then strategies based on Steiner systems would still work.

Further Questions

It is remarkable that a simple hat puzzle has connections to several different areas of mathematics. Several intriguing open questions remain. The main question arises from the natural, but false, conjecture that the best strategy wins with probability \( 1/(ek^2) \). We have found a strategy, based on an error-correcting code in [10] that succeeds with probability \( 1/O(k \log k) \). But can this be improved?

**QUESTION 1.** Is there a strategy for each \( k \) and \( n \) such that the overall success rate in all cases is \( 1/O(k) \)?

**QUESTION 2.** Can anything more be said about the cases \( n, k \) for which a perfect strategy exists? In particular, is there a perfect strategy when \( n = k = 5 \)?

**QUESTION 3.** Can \( 1/4 \) be improved on as an asymptotic success probability when there are two extra hats?

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