A FINITENESS THEOREM FOR ALGEBRAIC CYCLES

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Abstract. Let \( X \) be a smooth projective variety. Starting with a finite set of cycles on powers \( X^m \) of \( X \), we consider the \( \mathbb{Q} \)-vector subspaces of the \( \mathbb{Q} \)-linear Chow groups of the \( X^m \) obtained by iterating the algebraic operations and pullback and push forward along those morphisms \( X^l \to X^m \) for which each component \( X^l \to X \) is a projection. It is shown that these \( \mathbb{Q} \)-vector subspaces are all finite-dimensional, provided that the \( \mathbb{Q} \)-linear Chow motive of \( X \) is a direct summand of that of an abelian variety.

1. Introduction

Let \( X \) be a smooth projective variety over a field \( F \). Starting with a finite set of cycles on powers \( X^m \) of \( X \), consider the \( \mathbb{Q} \)-vector subspaces \( C_m \) of the \( \mathbb{Q} \)-linear Chow groups \( CH(X^m)_{\mathbb{Q}} \) formed by iterating the algebraic operations and pullback \( p^* \) and push forward \( p_* \) along those morphisms \( p : X^l \to X^m \) for which each component \( X^l \to X \) is a projection. It is plausible that the \( C_m \) are always finite-dimensional, because when \( F \) is finitely generated over the prime field it is plausible that the \( CH(X^m)_{\mathbb{Q}} \) themselves are finite-dimensional. In this paper we prove the finite-dimensionality of the \( C_m \) when the \( \mathbb{Q} \)-linear Chow motive of \( X \) is a direct summand of that of an abelian variety.

More precisely, suppose given an adequate equivalence relation \( \sim \) on \( \mathbb{Q} \)-linear cycles on smooth projective varieties over \( F \). We say that \( X \) is a Kimura variety for \( \sim \) if, in the category of \( \mathbb{Q} \)-linear Chow motives modulo \( \sim \), the motive of \( X \) is the direct sum of one for which some exterior power is 0 and one for which some symmetric power is 0. A Kimura variety for \( \sim \) is also a Kimura variety for any coarser equivalence relation. It is known (e.g. [4], Corollary 4.4) that if the \( \mathbb{Q} \)-linear Chow motive of \( X \) is a direct summand of that of an abelian variety, then \( X \) is a Kimura variety for any \( \sim \). The main result is now Theorem 1.1 below. In addition to a finiteness result, it contains also a nilpotence result. By a filtration \( C^* \) on a graded \( \mathbb{Q} \)-algebra \( C \) we mean a descending sequence \( C = C^0 \supset C^1 \supset \ldots \) of graded ideals of \( C \) such that \( C^r \cdot C^s \subseteq C^{r+s} \) for every \( r \) and \( s \). The morphisms \( p : X^l \to X^m \) in Theorem 1.1(a) are exactly those defined by maps \([1,m] \to [1,l]\).

Theorem 1.1. Let \( X \) be a smooth projective variety over \( F \) which is a Kimura variety for the equivalence relation \( \sim \). For \( n = 0,1,2,\ldots \), let \( Z_n \) be a finite subset of \( CH(X^n)_{\mathbb{Q}}/\sim \), with \( Z_n \) empty for \( n \) large. Then there exists for each \( n \) a graded \( \mathbb{Q} \)-subalgebra \( C_n \) of \( CH(X^n)_{\mathbb{Q}}/\sim \), and a filtration \( (C_n)^* \) on \( C_n \), with the following properties.

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(a) If $p : X^l \to X^m$ is a morphism for which each component $X^l \to X$ is a projection, then $p^*$ sends $C_m$ to $C_l$ and $p_*$ sends $C_l$ to $C_m$, and $p^*$ and $p_*$ respect the filtrations on $C_l$ and $C_m$.

(b) For every $n$, the $\mathbb{Q}$-algebra $C_n$ is finite-dimensional and contains $Z_n$.

(c) For every $n$, the cycles in $C_n$ which are numerically equivalent to 0 lie in $(C_n)^1$, and we have $(C_n)^r = 0$ for $r$ large.

The finiteness result of Theorem 1.1 is non-trivial only if $Z_n$ is non-empty for some $n > 1$. Indeed it will follow from Proposition 2.1 below that for any smooth projective variety $X$ over $F$ and finite subset $Z_1$ of $CH(X)_\mathbb{Q}$, there is a finite-dimensional graded $\mathbb{Q}$-subalgebra $C_n$ of $CH(X)_\mathbb{Q}$ for each $n$ such that $C_1$ contains $Z_1$ and $p^*$ sends $C_m$ to $C_l$ and $p_*$ sends $C_l$ to $C_m$ for every $p$ as in (a) of Theorem 1.1.

If $X$ is a Kimura variety for $\sim$, then the ideal of correspondences numerically equivalent to 0 in the algebra $CH(X \times_F X)_\mathbb{Q}/\sim$ of self-correspondences of $X$ has been shown by Kimura ([4], Proposition 7.5) to consist of nilpotent elements, and by Andrés and Kahn ([1], Proposition 9.1.14) to be in fact nilpotent. The nilpotence has been shown by Kimura ([4], Proposition 7.5) to consist of nilpotent elements, and by Andrés and Kahn ([1], Proposition 9.1.14) to be in fact nilpotent. The nilpotence result of Theorem 1.1 implies that of Kimura, but neither implies nor is implied by that of Andrés and Kahn.

If $\sim$ is numerical equivalence, then the $CH(X^m)_\mathbb{Q}/\sim = \overline{CH}(X^m)_\mathbb{Q}$ all finite dimensional. The following result shows that they are generated in a suitable sense by $(\overline{CH}(X^m)_\mathbb{Q}$ for $m$ not exceeding some fixed $n$.

**Theorem 1.2.** Let $X$ be a smooth projective variety over $F$ which is a Kimura variety for numerical equivalence. Then there exists an integer $n \geq 0$ with the following property: for every $m$, the $\mathbb{Q}$-vector space $\overline{CH}(X^m)_\mathbb{Q}$ is generated by elements of the form

$$p_*(p_1)^*(z_1).(p_2)^*(z_2).\cdots.(p_r)^*(z_r),$$

where $z_i$ lies in $\overline{CH}(X^{m_i})_\mathbb{Q}$ with $m_i \leq n$, and $p : X^l \to X^m$ and the $p_i : X^l \to X^{m_i}$ are morphisms for which each component $X^l \to X$ is a projection.

Theorem 1.1 will be proved in Section 7 and Theorem 1.2 in Section 8. Both theorems are deduced from the following fact. Given a Kimura variety $X$ for $\sim$, there is a reductive group $G$ over $\mathbb{Q}$, a finite-dimensional $G$-module $E$, a central $\mathbb{Q}$-point $\rho$ of $G$ with $\rho^2 = 1$, and a commutative algebra $R$ in the tensor category $REP(G, \rho)$ of $G$-modules with symmetry modified by $\rho$, with the following property. If we write $M_{\sim}(F)$ for the category of ungraded motives over $F$ modulo $\sim$ and $E_R$ for the free $R$-module $R \otimes_k E$ on $E$, then there exist isomorphisms

$$\xi_{r,s} : \text{Hom}_{G,R}(E_R)^{\otimes r}, (E_R)^{\otimes s}) \sim \text{Hom}_{M_{\sim}(F)}(h(X)^{\otimes r}, h(X)^{\otimes s}),$$

which are compatible with composition and tensor products of morphisms and with symmetries interchanging the factors $E_R$ and $h(X)$. These isomorphisms arise because there is a fully faithful tensor functor from the category of finitely generated free $R$-modules to $M_{\sim}(F)$, which sends $E_R$ to $h(X)$ (see [7], Lemma 3.3 for a similar result). However to keep the exposition as brief as possible, the $\xi_{r,s}$ will simply be constructed directly here, in Sections 5 and 6. Now we have an equality

$$CH(X^m)_\mathbb{Q}/\sim = \text{Hom}_{M_{\sim}(F)}(1, h(X)^{\otimes r})$$

of $\mathbb{Q}$-algebras, and pullback along a morphism $f : X^l \to X^m$ is given by composition with $h(f)$. There is a canonical autoduality $h(X)^{\otimes 2} \to 1$ on $h(X)$, and push
forward along \( f \) is given by composing with the transpose of \( h(f) \) defined by this autoduality. Using the isomorphisms \( \xi_{r,s} \), Theorems 1.1 and 1.2 then reduce to easily solved problems about the \( G \)-algebra \( R \).

2. Generated spaces of cycles

The following result gives an explicit description of the spaces of cycles generated by a given set of cycles on the powers of an arbitrary smooth projective variety \( X \). By the top Chern class of \( X \) we mean the element \( (\Delta_X)^*(\Delta_X)_*(1) \) of \( CH(X) Q \), where \( \Delta_X : X \to X^2 \) is the diagonal. We define the tensor product \( z \otimes z' \) of \( z \) in \( CH(X) Q \) and \( z' \) in \( CH(X') Q \) as \( pr_1^*(z), pr_2^*(z') \) in \( CH(X \times_F X') Q \). The push forward of \( z \otimes z' \) along a morphism \( f \times f' \) is then \( f_*(z) \otimes f'_*(z') \).

**Proposition 2.1.** Let \( X \) be a smooth projective variety over \( F \). For \( m = 0, 1, 2, \ldots \) let \( Z_m \) be a subset of \( CH(X^m) Q \), such that \( Z_1 \) contains the top Chern class of \( X \). Denote by \( C_m \) the \( Q \)-vector subspace of \( CH(X^m) Q \) generated by elements of the form

\[
p_\ast((p_1)^*(z_1) \cdot (p_2)^*(z_2) \cdot \cdots \cdot (p_n)^*(z_n)),
\]

where \( z_i \) lies in \( Z_{m_i} \), and \( p : X^j \to X^m \) and the \( p_i : X^j \to X^m \) are morphisms for which each component \( X^j \to X \) is a projection. Then \( C_m \) is a \( Q \)-subalgebra of \( CH(X^m) Q \) for each \( m \). If \( q : X^l \to X^m \) is a morphism for which each component \( X^l \to X \) is a projection, then \( q^\ast \) sends \( C_m \) into \( C_l \) and \( q_* \) sends \( C_l \) into \( C_m \).

**Proof.** Write \( P_{l,m} \) for the set of morphisms \( X^l \to X^m \) for which each component \( X^l \to X \) is a projection. Then the composite of an element of \( P_{j,l} \) with an element of \( P_{l,m} \) lies in \( P_{j,m} \). Thus \( q_*(C_l) \subset C_m \) for \( q \) in \( P_{l,m} \). Similarly the product of two elements of \( P_{j,m} \) lies in \( P_{2j,2m} \). Thus the tensor product of two elements of \( C_m \) lies in \( C_{2m} \). Since the product of two elements of \( CH(X^m) Q \) is the pullback of their tensor product along \( \Delta X^m \in P_{m,2m} \), it remains only to show that

\[
q^\ast(C_m) \subset C_l
\]

for \( q \) in \( P_{l,m} \). This is clear when \( l = m \) and \( q \) is a symmetry \( \sigma \) permuting the factors \( X \) of \( X^m \), because \( \sigma^\ast = (\sigma^{-1})_\ast \). An arbitrary \( q \) factors for some \( l' \) as \( q' \circ q' \) with \( q' \) in \( P_{l',l} \) a projection and \( q'' \) in \( P_{l',m} \) a closed immersion. It thus is enough to show that (2.2) holds when \( q \) is a projection or \( q \) is a closed immersion.

Suppose that \( q \) is a projection. If we write

\[
w_{s,n} : X^{s+n} \to X^n
\]

for the projection onto the last \( n \) factors, then \( q \) is a composite of a symmetry with \( w_{l-m,m} \). Thus (2.2) holds because \( w_{l-m,m}^\ast = 1 \otimes \) sends \( C_m \) into \( C_l \).

Suppose that \( q \) is a closed immersion. Then \( q \) is a composite of the

\[
e_s = X^{s-2} \times \Delta_X : X^{s-1} \to X^s
\]

for \( s \geq 2 \) and symmetries. To prove (2.2), we may thus suppose that \( m \geq 2 \) and \( q = e_m \).

Denote by \( W_m \) the \( Q \)-subalgebra of \( CH(X^m) Q \) generated by the \( v^\ast(Z_{m'}) \) for any \( m' \) and \( v \) in \( P_{m,m'} \). Then \( u^\ast \) sends \( W_m \) into \( W_l \) for any \( u \) in \( P_{l,m} \), and by the projection formula \( C_m \) is a \( W_m \)-submodule of \( CH(X^m) Q \). Since \( C_m \) is the sum of the \( p_\ast(W_j) \) with \( p \) in \( P_{j,m} \), it is to be shown that

\[
(e_m)^\ast p_\ast(W_j) \subset C_{m-1}
\]
for every $p$ in $\mathcal{P}_{j,m}$. We have $p = w_{j,m} \circ \Gamma_p$ with $\Gamma_p$ the graph of $p$, and

$$(e_m)^* \circ (w_{j,m})_* = (w_{j,m-1})_* \circ (e_{j+m})^*.$$  

Thus (2.3) will hold provided that $(e_{j+m})^*(\Gamma_p)_*(W_j) \subset C_{j+m-1}$. Replacing $m$ by $j + m$ and $p$ by $\Gamma_p$, we may thus suppose that $p$ has a left inverse in $\mathcal{P}_{m,j}$. In that case any $y$ in $W_j$ is of the form $p^\tau(x)$ with $x$ in $W_m$, and then

$$(e_m)^* p_*(y) = (e_m)^* (p_*(1).x) = (e_m)^* p_*(1). (e_m)^*(x).$$

Thus (2.3) will hold provided that $(e_m)^* p_*(1)$ lies in $C_{m-1}$.

To see that $(e_m)^* p_*(1)$ lies in $C_{m-1}$, note that $e_m$ has a left inverse $f$ in $\mathcal{P}_{m,m-1}$. Then

$$(e_m)^* p_*(1) = f_*(e_m)_*(e_m)^* p_*(1) = f_*(p_*(1).(e_m)_*(1)) = f_*(p_*(1).e_m)(1).$$

Since $(e_m)_*(1) = (w_{m-2,2})^*(\Delta_X)_*(1)$, we reduce finally to showing that $h^*(\Delta_X)_*(1)$ lies in $C_j$ for every $h$ in $\mathcal{P}_{j,2}$. Such an $h$ factors as a projection followed by either a symmetry of $X^2$ or $\Delta_X$, so we may suppose that $j = 1$ and $h = \Delta_X$. Then $h^*(\Delta_X)_*(1)$ is the top Chern class of $X$, which by hypothesis lies in $Z_1 \subset C_1$. $\square$

3. Group representations

Let $k$ be a field. By a $k$-linear category we mean a category equipped with a structure of $k$-vector space on each hom-set such that the composition is $k$-bilinear. A $k$-linear category is said to be pseudo-abelian if it has a zero object and direct sums, and if every idempotent endomorphism has an image. A $k$-tensor category is a pseudo-abelian $k$-linear category $\mathcal{C}$, together with a structure of symmetric monoidal category on $\mathcal{C}$ ([9], VII 7) such that the tensor product $\otimes$ is $k$-bilinear on hom-spaces. Thus $\mathcal{C}$ is equipped with a unit object $1$, and natural isomorphisms

$$(L \otimes M) \otimes N \sim L \otimes (M \otimes N),$$

the associativities,

$$M \otimes N \sim N \otimes M,$$

the symmetries, and $1 \otimes M \sim M$ and $M \otimes 1 \sim M$, which satisfy appropriate compatibilities. We assume in what follows that $1 \otimes M \sim M$ and $M \otimes 1 \sim M$ are identities: this can be always arranged by replacing if necessary $\otimes$ by an isomorphic functor. Brackets in multiple tensor products will often be omitted when it is of no importance which bracketing is chosen. The tensor product of $r$ copies of $M$ will then be written as $M^{\otimes r}$, and similarly for morphisms. There is a canonical action $\tau \mapsto M^{\otimes \tau}$ of the symmetric group $\mathfrak{S}_r$ on $M^{\otimes r}$, defined using the symmetries. It extends to a homomorphism of $k$-algebras from $k[\mathfrak{S}_r]$ to $\text{End}(M^{\otimes r})$. When $k$ is of characteristic 0, the symmetrising idempotent in $k[\mathfrak{S}_r]$ maps to an idempotent endomorphism $e$ of $M^{\otimes r}$, and we define the $r$th symmetric power $S^r M$ of $M$ as the image of $e$. Similarly we define the $r$th exterior power $\wedge^r M$ of $M$ using the antisymmetrising idempotent in $k[\mathfrak{S}_r]$

Let $G$ be a linear algebraic group over $k$. We write $\text{REP}(G)$ for the category of $G$-modules. The tensor product $\otimes_k$ over $k$ defines on $\text{REP}(G)$ a structure of $k$-tensor category. Recall ([9], 3.3) that every $G$-module is the filtered colimit of its finite-dimensional $G$-submodules. If $E$ is a finite-dimensional $G$-module, then regarding $\text{REP}(G)$ as a category of comodules ([9], 3.2) shows that $\text{Hom}_G(E, -)$ preserves filtered colimits. When $k$ is algebraically closed, a $k$-vector subspace of a
$G$-module is a $G$-submodule provided it is stable under every $k$-point of $G$. This is easily seen by reducing to the finite-dimensional case.

We suppose from now on that $k$ has characteristic 0. Let $\rho$ be a central $k$-point of $G$ with $\rho^2 = 1$. Then $\rho$ induces a $\mathbb{Z}/2$-grading on $\text{REP}(G)$, with the $G$-modules pure of degree $i$ those on which $\rho$ acts as $(-1)^i$. We define as follows a $k$-tensor category $\text{REP}(G, \rho)$. The underlying $k$-linear category, tensor product and associativities of $\text{REP}(G, \rho)$ are the same as those of $\text{REP}(G)$, but the symmetry $M \otimes N \xrightarrow{\tau} N \otimes M$ is given by multiplying that in $\text{REP}(G)$ by $(-1)^{ij}$ when $M$ is of degree $i$ and $N$ of degree $j$, and then extending by linearity. When $\rho = 1$, the $k$-tensor categories $\text{REP}(G)$ and $\text{REP}(G, \rho)$ coincide.

An algebra in a $k$-tensor category is defined as an object $R$ together with a multiplication $R \otimes R \to R$ and unit $1 \to R$ satisfying the usual associativity and identity conditions. Since the symmetry is not used in this definition, an algebra in $\text{REP}(G, \rho)$ is the same as an algebra in $\text{REP}(G)$, or equivalently a $G$-algebra. An algebra $R$ in $\text{REP}(G, \rho)$ will be said to be finitely generated if its underlying $k$-algebra is. It is equivalent to require that $R$ be generated as a $k$-algebra by a finite-dimensional $G$-submodule.

A (left) module over an algebra $R$ is an object $N$ equipped with an action $R \otimes N \to N$ satisfying the usual associativity and identity conditions. If $R$ is an algebra in $\text{REP}(G, \rho)$ or $\text{REP}(G)$, we also speak of a $(G, R)$-module. A $(G, R)$-module is said to be finitely generated if it is finitely generated as a module over the underlying $k$-algebra of $R$. It is equivalent to require that it be generated as a module over the $k$-algebra $R$ by a finite-dimensional $G$-submodule.

An algebra $R$ in a $k$-tensor category is said to be commutative if composition with the symmetry interchanging the factors $R$ in $R \otimes R$ leaves the multiplication unchanged. If $R$ is an algebra in $\text{REP}(G, \rho)$, this notion of commutativity does not in general coincide with that of the underlying $k$-algebra, but it does when $\rho$ acts as $1$ on $R$.

Coproducts exist in the category of commutative algebras in a $k$-tensor category: the coproduct of $R$ and $R'$ is $R \otimes R'$ with multiplication the tensor product of the multiplications of $R$ and $R'$ composed with the appropriate symmetry. To any map $[1, m] \to [1, l]$ and commutative algebra $R$ there is then associated a morphism $R^{\otimes m} \to R^{\otimes l}$, defined using symmetries $R^{\otimes r}$ and the unit and multiplication of $R$ and their tensor products and composites, such that each component $R \to R^{\otimes l}$ is the embedding into one of the factors.

Let $R$ be a commutative algebra in $\text{REP}(G, \rho)$. Then the symmetry in $\text{REP}(G, \rho)$ defines on any $R$-module a canonical structure of $(R, R)$-bimodule. The category of $(G, R)$-modules has a structure of $k$-tensor category, with the tensor product $N \otimes_R N'$ of $N$ and $N'$ defined in the usual way as the coequaliser of the two morphisms

$$N \otimes_k R \otimes_k N' \to N \otimes_k N'$$

given by the actions of $R$ on $N$ and $N'$, and the tensor product $f \otimes_R f'$ of $f : M \to N$ and $f' : M' \to N'$ as the unique morphism rendering the square

$$\begin{array}{ccc}
M \otimes_R M' & \xrightarrow{f \otimes_R f'} & N \otimes_R N' \\
\uparrow & & \uparrow \\
M \otimes_k M' & \xrightarrow{f \otimes_k f'} & N \otimes_k N'
\end{array}$$
Let $P$ be an object in $\text{REP}(G, \rho)$. We write $P_R$ for the object $R \otimes_k P$ in the $k$-tensor category of $(G, R)$-modules. A morphism of commutative algebras $R' \to R$ in $\text{REP}(G, \rho)$ induces by tensoring with $P$ a morphism of $R'$-modules $P_{R'} \to P_R$. For each $l$ and $m$, extension of scalars along $R' \to R$ then gives a $k$-linear map

\[(3.1) \quad \text{Hom}_{G,R'}((P_R)^{\otimes m}, (P_{R'})^{\otimes l}) \to \text{Hom}_{G,R}((P_R)^{\otimes m}, (P_R)^{\otimes l})\]

Explicitly, (3.1) sends $f'$ to the unique morphism of $(G, R)$-modules $f$ that renders the square

\[
\begin{array}{ccc}
(P_R)^{\otimes m} & \to & (P_R)^{\otimes l} \\
\uparrow & & \uparrow \\
(P_{R'})^{\otimes m} & \to & (P_{R'})^{\otimes l}
\end{array}
\]

commutative, where the vertical arrows are those defined by $P_{R'} \to P_R$. If $P$ is finite-dimensional, then for given commutative algebra $R$ and $f$, there is a finitely generated $G$-subalgebra of $R'$ of $R$ such that $f$ is in the image of (3.1). This can be seen by writing $R$ as the filtered colimit $\text{colim}_\lambda R_\lambda$ of its finitely generated $G$-subalgebras, and noting that since $P^{\otimes m}$ is finite-dimensional, the composite of $P^{\otimes m} \to (P_R)^{\otimes m}$ with $f$ factors through some $(P_{R'})^{\otimes l}$.

Suppose that $G$ is reductive, or equivalently that $\text{Hom}_G(P, -)$ is exact for every $G$-module $P$. Then $\text{Hom}_G(P, -)$ preserves colimits for $P$ finite-dimensional. In particular $(-)^G = \text{Hom}_G(k, -)$ preserves colimits. If $R$ is a commutative algebra in $\text{REP}(G, \rho)$ with $R^G = k$, then $R$ has a unique maximal $G$-ideal. Indeed $J^G = 0$ for $J \neq R$ a $G$-ideal of $R$, while $(J_1)^G = 0$ and $(J_2)^G = 0$ implies $(J_1 + J_2)^G = 0$.

**Lemma 3.1.** Let $G$ be a reductive group over a field $k$ of characteristic 0 and $\rho$ be a central $k$-point of $G$ with $\rho^2 = 1$. Let $R$ be a finitely generated commutative algebra in $\text{REP}(G, \rho)$ with $R^G = k$, and $N$ be a finitely generated $R$-module.

(i) The $k$-vector space $N^G$ is finite-dimensional.

(ii) For every $G$-ideal $J \neq R$ of $R$, we have $(J^rN)^G = 0$ for $r$ large.

**Proof.** Every object $P$ of $\text{REP}(G, \rho)$ decomposes as $P_0 \oplus P_1$ where $\rho$ acts as $(-1)^i$ on $P_i$. In particular $R = R_0 \oplus R_1$ with $R_0$ a $G$-subalgebra of $R$. Suppose that $R$ is generated as an algebra by the finite-dimensional $G$-submodule $M$. Then $R_0$ is generated as an algebra by $M_0 + M_1^2$, and hence is finitely generated. Since $R$ is a commutative algebra in $\text{REP}(G, \rho)$, it is generated as an $R_0$-module by $M_1$. Hence any finitely generated $R$-module is finitely generated as an $R_0$-module.

To prove (i) we reduce after replacing $R$ by $R_0$ to the case where $R = R_0$. Then $R$ is a commutative $G$-algebra in the usual sense. In this case it is well known that $N^G$ is finite-dimensional over $k = R^G$ (e.g. [S], II Theorem 3.25).

To prove (ii) note that $J_0 \neq R_0$ is an ideal of $R_0$. Since $R$ is a finitely generated $R_0$-module, so also is $R_1$. If $x_1, x_2, \ldots, x_s$ generate $R_1$ over $R_0$, then since each $x_i$ has square 0 we have $R_i^r = 0$ and hence $J_i^r = 0$ for $r > s$. Thus for $r > s$ we have

$$J^rN = (J_0 + J_1)^rN = J_0^rN + J_0^{r-1}J_1N + \cdots + J_0^{r-s}J_1^sN \subset J_0^{r-s}N.$$ 

Replacing $R$ by $R_0$ and $J$ by $J_0$, we thus reduce again to the case where $R = R_0$ is a commutative $G$-algebra in the usual sense. We may suppose further that $k$ is algebraically closed.
Lemma 3.2. (i) if \( e_k \) in the sum of the two elements of \( Z \) in (3.6) for \( \rho \), the group \( \Gamma ^{\rho} \) generated by \( \rho \) and \( \rho ^2 \), it follows that \( k[\Gamma _r] \) and \( \rho \) is algebraically closed is a \( G \)-ideal of \( R \). Thus \( J + p' \neq R \), because \( J \) and \( p' \) are contained in the unique maximal \( G \)-ideal of \( R \). Since each \( gp \) lies in the finite set of associated primes of \( N \), it follows that \( J + gp \neq R \) for some \( g \) in \( G(k) \). Applying \( g^{-1} \) then shows that \( J + p \neq R \). □

Let \( l_0 \) and \( l_1 \) be integers \( \geq 0 \). Write
\[
(3.2) \quad G = GL_{l_0} \times_k GL_{l_1},
\]
\( E_i \) for the standard representation of \( GL_{l_i} \), regarded as a \( G \)-module, and
\[
(3.3) \quad E = E_0 \oplus E_1.
\]
We may identify the endomorphism of \( E \) that sends \( E_i \) to itself and acts on it as \((-1)^i\) with a central \( k \)-point \( \rho \) of \( G \) with \( \rho ^2 = 1 \).

Consider the semidirect product
\[
(3.4) \quad \Gamma _r = (\mathbb{Z}/2)^r \rtimes \mathfrak{S}_r,
\]
where the symmetric group \( \mathfrak{S}_r \) acts on \( (\mathbb{Z}/2)^r \) through its action on \([1, r]\). For each \( r \), the group \( \Gamma _r \) acts on \( E^\otimes r \), with the action of \( (\mathbb{Z}/2)^r \) the tensor product of the actions \( i \mapsto \rho ^i \) of \( \mathbb{Z}/2 \) on \( E \), and the action of \( \mathfrak{S}_r \) that defined by the symmetries in \( \text{REP}(G, \rho) \). Thus we obtain a homomorphism
\[
(3.5) \quad k[\Gamma _r] \to \text{End}_G(E^\otimes r)
\]
of \( k \)-algebras.

For \( r \leq r' \) we may regard \( \Gamma _r \) as a subgroup of \( \Gamma _{r'} \), and hence \( k[\Gamma _r] \) as a \( k \)-subalgebra of \( k[\Gamma _{r'}] \), by identifying \( (\mathbb{Z}/2)^r \) with the subgroup of \( (\mathbb{Z}/2)^{r'} \) with the last \( r' - r \) factors the identity and \( \mathfrak{S}_r \) with the subgroup of \( \mathfrak{S}_{r'} \) which leaves the last \( r' - r \) elements of \([1, r']\) fixed. Write \( e_0 \) for the idempotent of \( k[\mathbb{Z}/2] \) given by half the sum of the two elements of \( \mathbb{Z}/2 \), and \( e_1 \) for \( 1 - e_0 \). Given \( \pi = (\pi _1, \pi _2, \ldots, \pi _r) \)
in \( (\mathbb{Z}/2)^r \), we then have an idempotent
\[
e_\pi = e_{\pi _1} \otimes e_{\pi _2} \otimes \cdots \otimes e_{\pi _r}
\]
in \( k[\mathbb{Z}/2]^\otimes r = k[(\mathbb{Z}/2)^r] \subset k[\Gamma _r] \). When every component of \( \pi \) is \( i \in \mathbb{Z}/2 \), we write \( e_{i, r} \) for \( e_\pi \). We also write \( a_{0, r} \) for the antisymmetrising idempotent and \( a_{1, r} \) for the symmetrising idempotent in \( k[\mathfrak{S}_r] \), and for \( i \in \mathbb{Z}/2 \) we write
\[
(3.6) \quad x_{i, r} = e_{i, r} + a_{1, r} + e_{i, r} + a_{0, r} + e_{i, r} = a_{i, r} + e_{i, r} + e_{i, r} + a_{0, r} + e_{i, r} + e_{i, r} = a_{i, r} + e_{i, r} \in k[\Gamma _{i+1}] \subset k[\Gamma _r]
\]
if \( r > l_i \) and \( x_{i, r} = 0 \) otherwise.

Lemma 3.2. (i) If \( r \neq r' \) then \( \text{Hom}_G(E^\otimes r, E^\otimes r') = 0 \).

(ii) The homomorphism (3.5) is surjective, with kernel the ideal of \( k[\Gamma _r] \) generated by \( x_{0, r} \) and \( x_{1, r} \).}

Proof. (i) The action of \( G \) on \( E \) restricts along the appropriate \( G_m \) to \( G \) to the homothetic action of \( G_m \) on \( E \).

(ii) Write \( I \) for the ideal of \( k[\Gamma _r] \) generated by \( x_{0, r} \) and \( x_{1, r} \). The image of \( e_\pi \) under (3.5) is the projection onto the direct summand
\[
E_\pi = E_{\pi _1} \otimes_k E_{\pi _2} \otimes_k \cdots \otimes_k E_{\pi _r}
\]
of $E^{\otimes r}$. The $e_{\pi}$ give a decomposition of the identity of $k[\Gamma_r]$ into orthogonal idempotents, and (3.5) is the direct sum over $\pi$ and $\pi'$ of the homomorphisms

$$e_{\pi'}k[\Gamma_r]e_{\pi} \to \text{Hom}_G(E_{\pi}, E_{\pi'})$$

it induces on direct summands of $k[\Gamma_r]$ and $\text{End}_G(E^{\otimes r})$. It is thus enough to show that (3.7) is surjective, with kernel $e_{\pi'}Ie_{\pi}$.

Restricting to the centre of $G$ shows that the target of (3.7) is 0 unless $\pi'$ and $\pi$ have the same number of components 0 or 1, or equivalently $\pi' = \pi \tau^{-1}$ for some $\tau \in \mathcal{S}_r$. The same holds for the source of (3.7), because

$$\tau e_{\pi}e_{\pi^{-1}} = e_{\tau \pi \tau^{-1}}$$

for every $\tau$ and $\pi$. Since further the image of $\tau \in \mathcal{S}_r$ under (3.5) induces an isomorphism from $E_\pi$ to $E_{\tau \pi \tau^{-1}}$, to show that (3.5) has the required properties we may suppose that $\pi' = \pi$ and that $r = r_0 + r_1$ where the first $r_0$ components of $\pi$ are 0 and the last $r_1$ are 1. Then the source of (3.7) has a basis $e_{\pi}e_{\pi} = e_{\tau} \tau$ with $\tau$ in the subgroup $\mathcal{S}_{r_0} \times \mathcal{S}_{r_1}$ of $\mathcal{S}_r$ that permutes the first $r_0$ and last $r_1$ elements of $[1, r]$ among themselves. Thus we may identify

$$k[\mathcal{S}_{r_0}] \otimes_k k[\mathcal{S}_{r_1}] = k[\mathcal{S}_{r_0} \times \mathcal{S}_{r_1}]$$

with the (non-unitary) $k$-subalgebra $e_{\pi}k[\Gamma_r]e_{\pi}$ of $k[\Gamma_r]$. Similarly we may identify

$$\text{End}_G(E_0^{\otimes r_0}) \otimes_k \text{End}_G(E_1^{\otimes r_1}) = \text{End}_G(E_0^{\otimes r_0} \otimes_k E_1^{\otimes r_1})$$

with the (non-unitary) $k$-subalgebra $\text{End}_G(E_{\pi})$ of $\text{End}_G(E)$. Now given $\tau$ and $\tau'$ in $\mathcal{S}_r$, the element $e_{\tau'}x_{i, \tau^{-1}}e_{\tau}$ is 0 unless both $\tau$ and $\tau'$ send $[1, l_i + 1]$ into $[1, r_0]$ if $i = 0$ or into $[r_0 + 1, r]$ if $i = 1$. With the above identifications, $e_{\pi}Ie_{\pi}$ is thus the ideal of $k[\mathcal{S}_{r_0}] \otimes_k k[\mathcal{S}_{r_1}]$ generated by $y_0 \otimes 1$ and $1 \otimes y_1$, where $y_i$ is $a_{i, l_i + 1}$ in $k[\mathcal{S}_{l_i + 1}]$ if $r_i > l_i$ and $y_i = 0$ otherwise. Further (3.7) is the tensor product of the homomorphisms

(3.8)

$$k[\mathcal{S}_{r_0}] \to \text{End}_G(E_i^{\otimes r_1})$$

of $k$-algebras sending $\tau \in \mathcal{S}_{r_1}$ to $E_i^{\otimes r_1}$ in $\text{REP}(G, \rho)$. It will thus suffice to prove that (3.8) is surjective with kernel generated by $y_i$. If $i = 0$, (3.8) may be identified with the homomorphism defined by the action of $\mathcal{S}_{r_1}$ by symmetries on the $r_1$th tensor power in $\text{REP}(GL_{l_1})$ of the standard representation of $GL_{l_1}$, while if $i = 1$, the composite of the automorphism $\tau \mapsto \text{sgn}(\tau)\tau$ of $k[\mathcal{S}_{r_1}]$ with (3.8) may be so identified. The required result is thus classical (e.g. [3], Theorem 6.3).

4. Duals

Let $C$ be a $k$-tensor category. By a duality pairing in $C$ we mean a quadruple $(L, L', \eta, \varepsilon)$ consisting of objects $L$ and $L'$ of $C$ and morphisms $\eta : 1 \to L' \otimes L$, the unit, and $\varepsilon : L \otimes L' \to 1$, the counit, satisfying triangular identities analogous for those of an adjunction ([5], p. 85). Explicitly, it is required that, modulo associativities, the composite of $L \otimes \eta$ with $\varepsilon \otimes L$ should be $1_L$, and of $\eta \otimes L'$ with $L' \otimes \varepsilon$ should be $1_{L'}$. When such an $(L, L', \eta, \varepsilon)$ exists for a given $L$, it is said to be a duality pairing for $L$, and $L$ is said to be dualisable, and $L'$ to be dual to $L$. We then have a dual pairing $(L', L, \eta', \varepsilon')$ for $L'$, with $\eta'$ and $\varepsilon'$ obtained from $\eta$ and $\varepsilon$ by composing with the appropriate symmetries.

In verifying the properties of duals recalled below, it is useful to reduce to the case where $C$ is strict, i.e. where all associativities of $C$ are identities. This can be
fixed, by composing on the left and right with appropriate morphisms 
\( L \) and \( \tau \).

Suppose given duality pairings \((L, L^\vee, \eta, \varepsilon)\) for \( L \) and \((L', L'^\vee, \eta', \varepsilon')\) for \( L' \). Then we have a tensor product duality pairing for \( L \otimes L' \), with dual \( L^\vee \otimes L'^\vee \), and unit and counit obtained from \( \eta \otimes \eta' \) and \( \varepsilon \otimes \varepsilon' \) by composing with the appropriate symmetries. Further any morphism \( f : L \to L' \) has a transpose \( f'^\vee : L'^\vee \to L^\vee \), characterised by the condition

\[ \varepsilon \circ (L \otimes f'^\vee) = \varepsilon' \circ (f \otimes L^\vee), \]

or by a similar condition using \( \eta \) and \( \eta' \). Explicitly, \( f'^\vee \) is given modulo associativities by the composite of \( \eta \otimes L^\vee \) with \( L'^\vee \otimes f \otimes L^\vee \) and \( L^\vee \otimes \varepsilon \). We have \((1_L)^\vee = 1_{L^\vee}\) and \((f' \otimes f)^\vee = f'^\vee \otimes f^\vee\), and, with the transpose of \( f^\vee \) taken using the dual pairing, we have \( f'^{\vee} = f \). In particular taking \( L = L' \) shows that a duality pairing for \( L \) is unique up to unique isomorphism.

Let \( L \) be a dualisable object of \( \mathcal{C} \). Then we have a \( k \)-linear map

\[ \text{tr}_L : \text{Hom}_\mathcal{C}(N \otimes L, N' \otimes L) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(N, N'), \]

natural in \( N \) and \( N' \), which sends \( f \) to its contraction \( \text{tr}_L(f) \) with respect to \( L \), defined as follows. Modulo associativities, \( \text{tr}_L(f) \) is the composite of \( N \otimes \overline{\eta} \) with \( f \otimes L^\vee \) and \( N' \otimes \varepsilon \), with \( L^\vee \) and \( \varepsilon \) as above and \( \overline{\eta} \) the composite of \( \eta \) with the symmetry interchanging \( L^\vee \) and \( L \). It does not depend on the choice of duality pairing for \( L \). When \( N = N' = 1 \), the contraction \( \text{tr}_L(f) \) is the trace \( \text{tr}(f) \) of the endomorphism \( f \) of \( L \). The rank of \( L \) is defined as \( \text{tr}(1_L) \). Modulo associativities, \( \text{tr}_{L \otimes L'} \) is given by successive contraction with respect to \( L' \) and \( L \), and \( \text{tr}_L \) commutes with \( M \otimes - \). By the appropriate triangular identity for \( L \) we have

\[ (g'' \otimes g') \circ \text{tr}_L = \text{tr}_L((g'' \otimes g') \circ \sigma) \]

for \( g' : M' \to L \) and \( g'' : L \to M'' \), with \( \sigma \) the symmetry interchanging \( M' \) and \( L \).

Let \( L \) be a dualisable object of \( \mathcal{C} \), and let \( \tau \) be a permutation of \([1, r + 1]\) and \( f_1, f_2, \ldots, f_{r+1} \) be endomorphisms of \( L \). Write \( \tau' \) for the permutation of \([1, r]\) obtained by omitting \( r + 1 \) from the cycle of \( \tau \) containing it, and define endomorphisms \( c \) of \( 1 \) and \( f'_1, f'_2, \ldots, f'_{r} \) of \( L \) as follows. If \( \tau \) leaves \( r + 1 \) fixed, then \( c = \text{tr}(f_{r+1}) \) and \( f'_i = f_i \) for \( i \leq r \). If \( \tau \) sends \( r + 1 \) to \( i_0 \), then \( c = 1 \), and \( f'_i = f_i \) when \( i \neq i_0 \) and \( f_{i_0} \circ f_{r+1} \) when \( i = i_0 \). We then have

\[ \text{tr}_L((f_1 \otimes f_2 \otimes \cdots \otimes f_{r+1}) \circ L^{\otimes \tau}) = c((f'_1 \otimes f'_2 \otimes \cdots \otimes f'_{r}) \circ L^{\otimes \tau'}). \]

To see this, reduce to the case where \( \tau \) leaves all but the last two elements of \([1, r+1]\) fixed, by composing on the left and right with appropriate morphisms \( L^{\otimes \gamma_0} \otimes L \) with \( \gamma_0 \) a permutation of \([1, r] \).

Let \( L, L', M \) and \( M' \) be objects in \( \mathcal{C} \), and \((L, L^\vee, \eta, \varepsilon)\) and \((L', L'^\vee, \eta', \varepsilon')\) be duality pairings for \( L \) and \( L' \). Then we have a canonical isomorphism

\[ \text{Hom}_\mathcal{C}(M, M' \otimes L) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(M \otimes L^\vee, M') \]

which modulo associativities sends \( f : M \to M' \otimes L \) to the composite of \( f \otimes L^\vee \) with \( M' \otimes \varepsilon \). Its inverse is defined using \( \eta \). We also have a canonical isomorphism

\[ \text{Hom}_\mathcal{C}(M' \otimes M, M') \xrightarrow{\sim} \text{Hom}_\mathcal{C}(M, L'^\vee \otimes M') \]

defined using \( \eta' \). Replacing \( M \) by \( L' \otimes M \) in the first of these isomorphisms and by \( M \otimes L^\vee \) in the second, and using the symmetries interchanging \( M \) and \( L' \) and \( L'^\vee \)
and \( M' \), then gives a canonical isomorphism

\[
\delta_{M,L;M',L'} : \text{Hom}_\mathcal{C}(M \otimes L', M' \otimes L) \cong \text{Hom}_\mathcal{C}(M \otimes L^\vee, M' \otimes L'^\vee).
\]

Modulo associativities, \( \delta_{M,L;M',L'} \) sends \( f \) to the composite of \( M \otimes \tilde{\eta}' \otimes L^\vee \), the tensor product of \( f \) with the symmetry interchanging \( L^\vee \) and \( L' \), and \( M' \otimes \varepsilon \otimes L'^\vee \), where \( \tilde{\eta}' \) is \( \eta' \) composed with the symmetry interchanging \( L'^\vee \) and \( L' \).

With the transpose taken using the chosen duality pairings for \( L \) and \( L' \), we have

\[
(4.3) \quad \delta_{M,L;M',L'}(h \otimes g) = h \otimes g'^\vee.
\]

With the duality pairing \((1,1,1_1,1_1)\) for \( 1 \), we have

\[
(4.4) \quad \delta_{M,L;1,1}(f) = \varepsilon \circ (f \otimes L^\vee).
\]

With the tensor product duality pairings for \( L_1 \otimes L_2 \) and \( L'_1 \otimes L'_2 \), we have

\[
(4.5) \quad \sigma''' \circ (\delta_{M_1,L;M'_1,L'_1}(f_1) \otimes \delta_{M_2,L_2;M'_2,L'_2}(f_2)) \circ \sigma'' =
\delta_{M_1 \otimes M_2,L_1 \otimes L_2;M'_1 \otimes M'_2,L'_1 \otimes L'_2}(\sigma' \circ (f_1 \otimes f_2) \circ \sigma)
\]

where each of \( \sigma, \sigma', \sigma'' \) and \( \sigma''' \) is a symmetry interchanging the middle two factors in a tensor product \(( - \otimes - ) \otimes ( - \otimes - )\).

If \( M' \) is dualisable, we have

\[
(4.6) \quad \delta_{M,L;M';L'}(f') \circ \delta_{M,L;M',L'}(f) = \delta_{M,L;M';L'}(\text{tr}_{M \otimes L'}(\sigma_2 \circ (f' \otimes f) \circ \sigma_1))
\]

for \( \sigma_1 \) the symmetry interchanging \( M \) and \( M' \) and \( \sigma_2 \) the symmetry interchanging \( L' \) and \( L \). This can be seen by showing that modulo associativities both sides of (4.6) coincide with a morphism obtained from

\[
f' \otimes f \otimes L^\vee \otimes L'' \otimes M' \otimes L'^\vee
\]

as follows: compose on the left and right with appropriate symmetries, then on the left with the tensor product of \( M'' \otimes L'^\vee \) and the counits for \( L, L' \) and \( M' \) and on the right with the tensor product of \( M \otimes L^\vee \) with the units for \( L'', L' \) and \( M' \). To show this in the case of the left hand side of (4.6), write it as a contraction with respect to \( M' \otimes L'^\vee \) using (4.1) and contract first with respect to \( L'^\vee \), using the triangular identity.

With the duality pairing \((1,1,1_1,1_1)\) for \( 1 \) and the tensor product duality pairing for \( L \otimes N \), we have

\[
(4.7) \quad \delta_{M,N;1,1}(g \circ f) = (\delta_{M,L;1,1}(f) \otimes \delta_{L,N;1,1}(g)) \circ \sigma \circ \delta_{M,N;M \otimes L,N \otimes N}(\alpha),
\]

with \( \alpha : M \otimes (L \otimes N) \cong (M \otimes L) \otimes N \) the associativity and \( \sigma \) the symmetry interchanging \( L \) and \( L^\vee \) in the tensor product of \( M \otimes L \) and \( L^\vee \otimes N^\vee \). Indeed modulo associativities \( \sigma \circ \delta_{M,N;M \otimes L,N \otimes N}(\alpha) \) is \( 1_M \otimes \eta \otimes 1_{N^\vee} \) by the triangular identity for \( N \), and (4.7) then follows by the triangular identity for \( L \).

Let \((L,L^\vee,\eta,\varepsilon)\) be a duality pairing for the object \( L \) of \( \mathcal{C} \). Its \( r \)th tensor power \((L^\otimes r,(L^\vee)^\otimes r,\eta_r,\varepsilon_r)\) is a duality pairing for \( L^\otimes r \). We write

\[
L^{r,s} = L^\otimes r \otimes (L^\vee)^\otimes s.
\]
Then $L^{r,0} = L^\otimes r$. We define a $k$-bilinear product $\widetilde{\otimes}$ on morphisms between the $L^{r,s}$ by requiring that the square
\[
\begin{array}{c}
L^{r_1,s_1} \otimes L^{r_2,s_2} \\ f_1 \otimes f_2 \\
\downarrow \quad \downarrow
\end{array}
\xrightarrow{\sim} \quad
\begin{array}{c}
L^{r_1+r_2,s_1+s_2} \\
f_1 \otimes f_2
\end{array}
\]
(4.8)
\[
\text{commute, with the top isomorphism the symmetry interchanging the two factors (}L^\vee)^{\otimes s_1}\text{ and }L^{\otimes r_2}\text{ and the bottom that interchanging } (L^\vee)^{\otimes s_1}\text{ and }L^{\otimes r_2}. \text{ Then } \widetilde{\otimes}\text{ preserves composites, is associative, and we have}
\]
\[
f_2 \widetilde{\otimes} f_1 = \sigma' \circ (f_1 \widetilde{\otimes} f_2) \circ \sigma^{-1},
\]
(4.9)
where $\sigma$ interchanges the first $r_1$ with the last $r_2$ factors $L$ and the first $s_1$ with the last $s_2$ factors $L^\vee$ of $L^{r_1+r_2,s_1+s_2}$, and similarly for $\sigma'$. We define an isomorphism
\[
\delta_{L,r,s;r',s':} : \text{Hom}_C(L^{\otimes (r+s')}, L^{\otimes (r'+s)}) \xrightarrow{\sim} \text{Hom}_C(L^{r,s}, L^{r',s'})
\]
(4.10)
by taking $L^{\otimes r}, L^{\otimes s}, L^{\otimes r'}, L^{\otimes s'}$ for $M, L, M', L'$ in $\delta_{M,L;M',L'}$.

It follows from (4.3) that
\[
\delta_{L,r,s;r',s'}(h \otimes g) = h \otimes g^\vee.
\]
(4.11)
and from (4.4) that
\[
\delta_{L,r,s;0,0}(f) = \varepsilon_s \circ (f \otimes (L^\vee)^{\otimes s}).
\]
(4.12)
By (4.5), we have
\[
\delta_{L;r_1,s_1};r_2,s_2'}(f_2) = \delta_{L;r_2,s_2};r_2',s_2'}(f_2) = \delta_{L;r_1+r_2,s_1+s_2};r_2',s_2'}(\sigma' \circ (f_1 \otimes f_2) \circ \sigma)
\]
(4.13)
for appropriate symmetries $\sigma$ and $\sigma'$. By (4.7), we have
\[
\delta_{L;r_2,s_2};r_2',s_2'}(f') \circ \delta_{L;r_1,s_1;0;0}(g) = \delta_{L;r_1,s_1};r_2,s_2'}(f') \circ \delta_{L;r_2,s_2};0;0}(g)
\]
(4.14)
for appropriate symmetries $\sigma_1$ and $\sigma_2$. We have
\[
\delta_{L;r,t;0,0}(g) = \delta_{L;r,t;0,0}(f) \otimes \delta_{L;r,t;0,0}(g)
\]
(4.15)
by (4.7).

Let $G$ be a linear algebraic group over $k$ and $\rho$ be a central $k$-point of $G$ with $\rho^2 = 1$. Let $E$ be a finite-dimensional $G$-module and $R$ be a commutative algebra in REP$(G, \rho)$. Then $E$ in REP$(G, \rho)$ and $E_R$ in the $k$-tensor category of $(G, R)$-modules are dualisable. Suppose chosen duality pairings for $E$ and $E_R$. Then we have a $G$-module $E^{r,s}$ and a $(G, R)$-module $(E_R)^{r,s}$ for every $r$ and $s$. We have canonical embeddings $E \to E_R$ and $E^\vee \to (E_R)^\vee$, which are compatible with the units and counits of the chosen duality pairings for $E$ and $E_R$. They define a canonical embedding $E^{r,s} \to (E_R)^{r,s}$, which induces an isomorphism of $(G, R)$-modules $(E^{r,s})_R \xrightarrow{\sim} (E_R)^{r,s}$. Given $u : E^{r,s} \to E^{r',s'}$, we write $u_{R;r,s;r',s'}$ for the unique morphism of $(G, R)$-modules for which the square
\[
\begin{array}{ccc}
(E_R)^{r,s} & \xrightarrow{u_{R;r,s;r',s'}} & (E_R)^{r',s'} \\
\uparrow & & \uparrow \\
E^{r,s} & \xrightarrow{u} & E^{r',s'}
\end{array}
\]
(4.16)
commutes, with the vertical arrows the canonical embeddings. Then \((-)_{R,r,s;r',s'}\) preserves identities and composites, counts \(E^{r,r} \to E^{0,0}\) and \((E_R)^{r,r} \to (E_R)^{0,0}\) and (with the identification \(E^{r,0} = E^{\otimes r}\)) commutes with the isomorphisms \(\delta_E\) and \(\delta_{E_R}\). For each \(r\) and \(s\) we have an isomorphism

\[
\psi_{r,s} : \text{Hom}_{G,R}( (E_R)^{r,s}, R) \overset{\sim}{\to} \text{Hom}_G( E^{r,s}, R),
\]

given by composing with the canonical embedding \(E^{r,s} \to (E_R)^{r,s}\). Then

\[
\psi_{r,s}(u' \circ u_{R,r,s;r',s'}) = \psi_{r',s'}(u') \circ u
\]

for every \(w' : (E_R)^{r',s'} \to R\) and \(u : E^{r,s} \to (E_R)^{r,s}\).

Suppose that \(G\) is reductive and that \(R^G = k\), so that \(R\) has a unique maximal \(G\)-ideal \(J\). Let \(N\) be a dualisable \((G,R)\)-module and \(f : R \to N\) be a morphism of \((G,R)\)-modules which does not factor through \(JN\). Then \(f\) has a left inverse. Indeed \(f^\vee\) does not factor through \(J\), because \(f\) is the composite of the unit for \(N^\vee\) with \(N \otimes_R f^\vee\). Hence \(f^\vee\) is surjective, and there is an \(x\) in its source fixed by \(G\) with \(f^\vee(x) = 1\). Thus \(f^\vee\) has a unique right inverse \(g\) with \(g(1) = x\), and \(g^\vee\) is left inverse to \(f = f^\vee\).

5. Kimura objects

Let \(k\) be a field of characteristic 0 and \(C\) be a \(k\)-tensor category with \(\text{End}_C(1) = k\).

An object \(L\) of \(C\) will be called positive (resp. negative) if it is dualisable and \(\bigwedge^{r+1} L\) (resp. \(S^{r+1} L\)) is 0 for some \(r\).

An object of \(C\) will be called a Kimura object if it is the direct sum of a positive and a negative object of \(C\).

Let \(L\) be a Kimura object of \(C\). Then \(L = L_0 \oplus L_1\) with \(L_0\) positive and \(L_1\) negative. Denote by \(l_0\) (resp. \(l_1\)) the least \(r\) such that \(\bigwedge^{r+1} L_0\) (resp. \(S^{r+1} L_1\)) is 0, and let \(G\) and \(E\) be as in \((\text{3.2})\) and \((\text{3.3})\), and \(\rho\) be the central \(k\)-point of \(G\) which acts as \((-1)^i\) on \(E_i\). The goal of this section is to construct a commutative algebra \(R\) in \(\text{REP}(G,\rho)\) and an isomorphism

\[
(\text{5.1})\quad \xi_{r,s} : \text{Hom}_{G,R}( (E_R)^{\otimes r}, (E_R)^{\otimes s}) \overset{\sim}{\to} \text{Hom}_C( L^{\otimes r}, L^{\otimes s})
\]

every \(r\) and \(s\), such that the \(\xi\) preserve composites and symmetries and are compatible with \(\otimes_R\) and \(\otimes\).

Given an object \(M\) of \(C\), write \(a_{M,0,r}\) (resp. \(a_{M,1,r}\)) for the image of the antisymmetrising (resp. symmetrising) idempotent of \(k[\mathfrak{G}_r]\) under the \(k\)-homomorphism to \(\text{End}(M^{\otimes r})\) that sends \(\sigma\) in \(\mathfrak{G}_r\) to \(M^{\otimes r}\). If \(M\) is dualisable of rank \(d\), then applying \((\text{4.2})\) with the \(f_j\) the identities shows that

\[
(r + 1) \text{tr}_M(a_{M,i,r+1}) = (d - (-1)^i) a_{M,i,r}
\]

for \(i = 0, 1\). If \(M\) is positive (resp. negative), it follows that \(d\) (resp. \(-d\)) is the least \(r\) for which \(\bigwedge^{r+1} M\) (resp. \(S^{r+1} M\)) is 0. Thus \(L_i\) has rank \((-1)^i l_i\).

Write \(b\) for the automorphism of \(L\) that sends \(L_i\) to \(L_i\) and acts on it as \((-1)^i\).

Then for every \(r\), the group \(\Gamma_r\) of \((\text{3.4})\) acts on \(L^{\otimes r}\) with the action of \((\mathbb{Z}/2)^r\) the \(r\)th tensor power of the action \(i \mapsto b^i\) of \(\mathbb{Z}/2\) on \(L\), and the action of \(\mathfrak{G}_r\) that given by \(\tau \mapsto L^{\otimes r}\). Thus we obtain a homomorphism

\[
\alpha_r : k[\Gamma_r] \to \text{End}_C(L^{\otimes r})
\]

of \(k\)-algebras. If \(l_i < r\), then \(a_{\alpha_r}\) sends the element \(x_{i,r}\) of \((\text{3.6})\) to the projection onto the direct summand \(\bigwedge^{l_0+1} L_0 \otimes_k L^{\otimes (r-l_0-1)}\) when \(i = 0\) and \(S^{l_1+1} L_1 \otimes_k L^{\otimes (r-l_0-1)}\) when \(i = 1\). Thus both \(x_{0,r}\) and \(x_{1,r}\) lie in the kernel of \(\alpha_r\). If we write \(\beta_r\) for \((\text{5.5})\),
it follows by Lemma \[\text{Lemma 3.2[iii]}\] that the kernel of $\alpha_r$ contains that of $\beta_r$. Hence by Lemma \[\text{Lemma 3.2}\] there is for each $r$ and $r'$ a unique $k$-linear map

$$\varphi_{r,r'} : \Hom_G(E^\otimes r, E^\otimes r') \to \Hom_G(L^\otimes r, L^\otimes r')$$

such that

$$\alpha_r = \varphi_{r,r'} \circ \beta_r$$

for every $r$. By construction, the $\varphi_{r,r'}$ preserve symmetries, identities and composites, and they are compatible with $\otimes_k$ and $\otimes$. Applying \[\text{Lemma 3.2}\] $t$ times shows that for $v : E^\otimes (r+t) \to E^\otimes (r'+t)$ we have

$$\varphi_{r,r'}(\tr_{E^\otimes t}(v)) = \tr_{L^\otimes t}(\varphi_{r+t;r'+t}(v)),$$

because $\tr(\rho) = \tr(b') = l_0 - (-1)^t l_1$.

For every $r, s$ and $r', s'$, we define a $k$-linear map $\varphi_{r,s;r',s'}$ by requiring that the square

$$\begin{array}{ccc}
\Hom_G(E^\otimes (r+s), E^\otimes (r's')) & \xrightarrow{\varphi_{r,s;r',s'}} & \Hom_G(L^\otimes (r+s), L^\otimes (r's')) \\
\delta_{E^\otimes (r+s), s'} & \uparrow \delta_{L^\otimes (r+s), s'} & \\
\Hom_G(E^\otimes (r+s), E^\otimes (r'+s)) & \xrightarrow{\varphi_{r+s;r'+s}} & \Hom_G(L^\otimes (r+s), L^\otimes (r'+s))
\end{array}$$

commute, with the $\delta$ the isomorphisms of \[\text{Lemma 3.2}\]. Then by \[\text{Lemma 3.2}\] the $\varphi_{r,s;r',s'}$ preserve identities, and by \[\text{Lemma 3.2}\] and \[\text{Lemma 3.2}\] they preserve composites. By \[\text{Lemma 3.2}\], they are compatible with the bilinear products, defined as in \[\text{Lemma 3.2}\], $\otimes_k$ on $G$-homomorphisms between the $E^\otimes r$ and $\otimes$ on morphisms between the $L^\otimes r$. By \[\text{Lemma 3.2}\], they send symmetries permuting the factors $E$ or $E'$ of $E^\otimes r$ to the corresponding symmetries of $L^\otimes r$.

We now define as follows a commutative algebra $R$ in $\text{REP}(G, \rho)$. Consider the small category $\mathcal{L}$ whose objects are triples $(r, s, f)$ with $r$ and $s$ integers $\geq 0$ and $f : L^\otimes r \to 1$, where a morphism from $(r, s, f)$ to $(r', s', f')$ in $\mathcal{L}$ is a morphism $u : E^\otimes r \to E^\otimes r'$ such that

$$f = f' \circ \varphi_{r,s;r',s'}(u).$$

Then we define $R$ as the colimit

$$R = \colim_{(r,s,f) \in \mathcal{L}} E^\otimes r$$

in $\text{REP}(G, \rho)$. Write the colimit injection at $(r, s, f)$ as

$$i_{(r,s,f)} : E^\otimes r \to R.$$

We define the unit $1 \to R$ of $R$ as $i_{(0,0,0)}$. We define the multiplication $R \otimes_k R \to R$ by requiring that for every $((r_1, s_1, f_1), (r_2, s_2, f_2))$ in $\mathcal{L} \times \mathcal{L}$ the square

$$\begin{array}{ccc}
E^\otimes (r_1 + r_2 + s_1 + s_2) & \xrightarrow{i_{(r_1 + s_1, f_1)} \otimes_k i_{(r_2, s_2, f_2)}} & E^\otimes (r_1 + r_2, s_1 + s_2) \\
R \otimes_k R & \xrightarrow{i_{(r_1, s_1, f_1)} \otimes_k i_{(r_2, s_2, f_2)}} & R
\end{array}$$

(5.3)

should commute, where the top isomorphism is that of \[\text{Lemma 3.2}\] with $E$ for $L$. Such an $R \otimes_k R \to R$ exists and is unique because the left vertical arrows of the squares \[\text{Lemma 3.2}\] form a colimiting cone by the fact that $\otimes_k$ preserves colimits, while their top right legs form a cone by the compatibility of the $\varphi_{r,s;r',s'}$ with $\otimes_k$ and $\otimes$. The associativity of the multiplication can be checked by writing $R \otimes_k R \otimes_k R$ as a
form a colimiting cone of $k$-vector spaces. Thus for every $r$ and $s$ there is a unique homomorphism
\[ \theta_{r,s} : \Hom_G(E^{r,s}, R) \to \Hom_G(L^{r,s}, 1) \]
whose composite with $\Hom_G(E^{r,s}, i_{(v',s'),(v)})$ sends $u : E^{r,s} \to E^{r',s'}$ to
\[ f' \circ \varphi_{r,s;r',s'}(u). \]
Further $\theta_{r,s}$ is an isomorphism, with inverse sending $f : L^{r,s} \to 1$ to $i_{(r,s,f)}$. Thus every $E^{r,s} \to R$ can be written uniquely in the form $i_{(r,s,f)}$. It follows that
\[ \theta_{r,s}(v' \circ u) = \theta_{r',s'}(v') \circ \varphi_{r,s;r',s'}(u) \]
for $v' : E^{r',s'} \to R$, that $\theta_{0,0}$ sends the identity $k \to R$ of $R$ to $1_1$, and that
\[ \theta_{r_1+r_2,s_1+s_2}(v) = \theta_{r_1,s_1}(v_1) \otimes \theta_{r_2,s_2}(v_2) \]
for $v_1 : E^{r_1,s_1} \to R$ and $v_2 : E^{r_2,s_2} \to R$, where $v$ is defined by a diagram of the form (5.3) with left arrow $v_1 \otimes_{k} v_2$ and right arrow $v$.

Composing the isomorphisms $\psi_{r,s}$ of (4.10) and $\theta_{r,s}$ gives an isomorphism
\[ \tilde{\theta}_{r,s} = \theta_{r,s} \circ \psi_{r,s} : \Hom_G((E_R)^{r,s}, R) \cong \Hom_G(L^{r,s}, 1). \]

Then with $u_{R;r,s;r',s'}$ as in (4.11), we have by (4.18) and (5.4)
\[ \tilde{\theta}_{r,s}(w' \circ u_{R;r,s;r',s'}) = \tilde{\theta}_{r',s'}(w') \circ \varphi_{r,s;r',s'}(u) \]
for every $w' : (E_R)^{r',s'} \to R$ and $u : (E_R)^{r,s} \to (E_R)^{r',s'}$. Also $\tilde{\theta}_{0,0}(1_R) = 1_1$, and
\[ \tilde{\theta}_{r_1+r_2,s_1+s_2}(w_1 \otimes_{R} w_2) = \tilde{\theta}_{r_1,s_1}(w_1) \otimes \tilde{\theta}_{r_2,s_2}(w_2) \]
for every $w_1 : (E_R)^{r_1,s_1} \to R$ and $w_2 : (E_R)^{r_2,s_2} \to R$, by (5.5).

We now define the isomorphism (5.1) by requiring that the square
\[ \begin{array}{ccc}
\Hom_G((E_R)^{r,s}, (E_R)^{r,s}) & \xrightarrow{\xi_{r,s}} & \Hom_G(L^{r,s}, L^{r,s}) \\
\downarrow{\delta_{E_r,r,s,0,0}} & & \downarrow{\delta_{L^{r,s},0,0}} \\
\Hom_G((E_R)^{r,s}, R) & \xrightarrow{\tilde{\theta}_{r,s}} & \Hom_G(L^{r,s}, 1)
\end{array} \]
commute. The $\xi$ preserve composites by (4.15), (5.6), (5.7), and the fact that $(-)_{R;r,s;r',s'}$ preserves identities and is compatible with $\delta_{E}$ and $\delta_{E_R}$. They are compatible with $\otimes_{R}$ and $\otimes$ by (4.13), where the relevant $\sigma$ and $\sigma'$ reduce to associativities, and (5.7). They are compatible with the symmetries by (5.6) with $w' = 1_R$ and $u$ the composite of $\sigma \otimes_k (E^{r'})^{r,s}$ for $\sigma$ a symmetry of $E^{r,s}$ with the counit $E^{r,r} \to k$, using (4.12) and the compatibility of $(-)_{R;r,s;r',s'}$ with symmetries, composites, and counits.
6. Kimura varieties

We denote by $\mathcal{M}_\sim(F)$ the category of ungraded $\mathbb{Q}$-linear motives over $F$ for the equivalence relation $\sim$. It is a $\mathbb{Q}$-tensor category. There is a contravariant functor $h$ from the category $\mathcal{V}_F$ of smooth projective varieties over $F$ to $\mathcal{M}_\sim(F)$, which sends products in $\mathcal{V}_F$ to tensor products in $\mathcal{M}_\sim(F)$. We then have

$$\text{Hom}_{\mathcal{M}_\sim(F)}(h(X'), h(X)) = CH(X' \times_F X)_{\mathbb{Q}/\sim},$$

and the composite $z \circ z'$ of $z' : h(X'') \to h(X')$ with $z : h(X') \to h(X)$ is given by

$$z \circ z' = (\text{pr}_{13})_*(\text{pr}_{12})^*(z').$$

where the projections are from $X'' \times_F X' \times_F X$. Further $h(q)$ for $q : X \to X'$ is the push forward of $1$ in $CH(X)_{\mathbb{Q}/\sim}$ along $X \to X' \times_F X$ with components $q$ and $1_X$.

The images under $h$ of the structural morphism and diagonal of $X$ define on $h(X)$ a canonical structure of commutative algebra in $\mathcal{M}_\sim(F)$. With this structure (6.1) reduces when $X' = \text{Spec}(F)$ to an equality of algebras

$$\text{Hom}_{\mathcal{M}_\sim(F)}(1, h(X)) = CH(X)_{\mathbb{Q}/\sim}.$$ 

Also $h(X)$ is canonically autodual: we have canonical duality pairing

$$(h(X), h(X), \eta_X, \epsilon_X),$$

with both $\eta_X$ and $\epsilon_X$ the class in $CH(X \times_F X)_{\mathbb{Q}/\sim}$ of the diagonal of $X$. The canonical duality pairing for $h(X \times_F X')$ is the tensor product of those for $h(X)$ and $h(X')$. The canonical duality pairings define a transpose $(\sim)$ for morphisms $h(X') \to h(X)$, given by pullback of cycles along the symmetry interchanging $X$ and $X'$. For $q : X \to X'$ and $z \in CH(X)_{\mathbb{Q}/\sim}$ and $z' \in CH(X')_{\mathbb{Q}/\sim}$, we have

$$(6.2) \quad q^*(z') = h(q) \circ z'$$

and

$$(6.3) \quad q_*(z) = h(q)^\sim \circ z.$$ 

A Kimura variety for $\sim$ is a smooth projective variety $X$ over $F$ such that $h(X)$ is a Kimura object in $\mathcal{M}_\sim(F)$. If the motive of $X$ in the category of graded motives for $\sim$ is a Kimura object, then $X$ is a Kimura variety for $\sim$. The converse also holds, as can be seen by factoring out the tensor ideals of tensor nilpotent morphisms, but this will not be needed.

Let $X$ be a Kimura variety for $\sim$. We may apply the construction of Section 5 with $k = \mathbb{Q}$, $C = \mathcal{M}_\sim(F)$ and $L = h(X)$. For appropriate $l_0$ and $l_1$, we then have with $G$, $E$ and $\rho$ as in Section 5 a commutative algebra $R$ in $\text{REP}(G, \rho)$ and isomorphisms

$$\xi_{r,s} : \text{Hom}_{G,R}(\eta_r \otimes_s (E_R)^{\otimes r}, (E_R)^{\otimes s}, h(X)^{\otimes r}, h(X)^{\otimes s})$$

which are compatible with composites, tensor products, and symmetries.

The homomorphisms of $R$-modules $\iota$ and $\mu$ with respective images under $\xi_{0,1}$ and $\xi_{2,1}$ the unit and multiplication of $h(X)$ define a structure of commutative $R$-algebra on $E_R$. Also the homomorphisms $\eta_1$ and $\epsilon_1$ with respective images $\eta_X$ and $\epsilon_X$ under $\xi_{0,2}$ and $\xi_{2,0}$ are the unit and counit a duality pairing $(E_R, E_R, \eta_1, \epsilon_1)$ for $E_R$. We denote by

$$((E_R)^{\otimes r}, (E_R)^{\otimes r}, \eta_r, \epsilon_r)$$
its rth tensor power. Then $\xi_{0,2r}(\eta_r) = \eta_{X^r}$ and $\xi_{2r,0}(\varepsilon_r) = \varepsilon_{X^r}$. For any $(G, R)$-homomorphism $f$ from $(E_R)^{\otimes m}$ to $(E_R)^{\otimes \ell}$ we have
\[
\xi_{l,m}(f^\vee) = \xi_{m,l}(f)^\vee,
\]
where the transpose of $f$ is taken using duality pairings just defined. Further
\[
\xi_{0,n} : \text{Hom}_{G,R}(R_c, (E_R)^{\otimes n}) \xrightarrow{\sim} \text{Hom}_{M_n(F)}(1, h(X)^{\otimes n})
\]
is an isomorphism of $Q$-algebras. We note that
\[
R^G = \text{Hom}_{G,R}(R, R) = CH(\text{Spec}(F))_Q/\sim = Q,
\]
by the isomorphism $\xi_{0,0}$.

7. Proof of Theorem 1.1

To prove Theorem 1.1 we may suppose that $Z_1$ contains the classes of the equidimensional components of $X$, and that $Z_n$ contains the homogeneous components of each of its elements for the grading of $CH(X^n)_Q/\sim$. Denote by $A$ the set of those families $C = ((C_n)^0)_{n \in \mathbb{N}}$ with $(C_n)^0$ a $Q$-subalgebra $C_n$ of $CH(X^n)_Q/\sim$ and $((C_n)^i)_{n \in \mathbb{N}}$ a filtration of the algebra $C_n$, such that [a] [b] and [c] of Theorem 1.1 hold. It is to be shown that there is a $C$ in $A$ which is graded, i.e. such that $(C_n)^i$ is a graded $Q$-vector subspace of $CH(X^n)_Q/\sim$ for each $n$ and $i$. For $\lambda \in Q^*$, define an endomorphism $z \mapsto \lambda \ast z$ of the algebra $CH(X^n)_Q/\sim$ by taking $\lambda \ast z = \lambda^j z$ when $z$ is homogeneous of degree $j$. Then the graded subspaces of $CH(X^n)_Q/\sim$ are those that are stable under each $\lambda \ast -$ . For each $C$ in $A$ we have a $\lambda \ast C$ in $A$ with $((\lambda \ast C)^i)_i$ the image under $\lambda \ast -$ of $(C_n)^i$. Indeed $(\lambda \ast C)^i$ contains $C_n$ by the homogeneity assumption on $Z_n$, and $p_r$ sends $((\lambda \ast C)^i)_i$ to $(\lambda \ast C)^i$ for $p$ as in [a] because $C_1$ contains the classes of the equidimensional components of each factor $X$ of $X^i$ by the assumption on $Z_1$. The $C$ in $A$ that are graded are those fixed by each $\lambda \ast -$ . Now if $A$ is non-empty, it has at least one for the ordering of the $C$ by inclusion of the $(C_n)^i$'s. Such a least element will be fixed by the $\lambda \ast -$ , and hence graded. It will thus suffice to show that $A$ is non-empty.

Let $G$, $E$, $\rho$, $R$, $\xi_{r,s}$, $\eta_r$, $\varepsilon_r$, $t$ and $\mu$ be as in Section 8. With the identification
\[
(7.1) \quad \text{Hom}_{M_n(F)}(1, h(X)^{\otimes n}) = CH(X^n)_Q/\sim,
\]
there exists a finitely generated $G$-subalgebra $R'$ of $R$ such that if we write $\beta_{n,n}$ for the homomorphism (3.1) with $P = E$, then $(\xi_{0,n})^{-1}(Z_n)$ is contained in the image of $\beta_{0,n}$ for every $n$, and $\eta_1 = \beta_{0,2}(\eta'_1)$, $\varepsilon_1 = \beta_{2,0}(\varepsilon'_1)$, $t = \beta_{0,1}(t')$ and $\mu = \beta_{2,1}(\mu')$ for some $\varepsilon'_1$, $\eta'_1$, $t'$ and $\mu'$. We then have duality pairing $(E_R', E_R', \eta'_1, \varepsilon'_1)$ for $E_R'$, and if its $r$th tensor power is
\[
((E_R')^{\otimes r}, (E_R')^{\otimes r}, \eta'_r, \varepsilon'_r),
\]
we have $\eta_r = \beta_{0,2r}(\eta'_r)$ and $\varepsilon_r = \beta_{2r,0}(\varepsilon'_r)$. Further $t'$ and $\mu'$ define a structure of commutative $(G, R')$-algebra on $E_R'$. The $\beta_{m,1}$, and hence their composites
\[
\xi_{m,1} : \text{Hom}_{G,R'}((E_R')^{\otimes m}, (E_R')^{\otimes n}) \to \text{Hom}_{M_n(F)}(h(X)^{\otimes n}, h(X)^{\otimes n})
\]
with the $\xi_{m,n}$, preserve identities, composition, tensor products, and transposes defined using the $\eta'_r$ and $\varepsilon'_r$. Further $\xi'_{0,0}$ is a homomorphism of $Q$-algebras.

We have $R'^G = R'^G = Q$. Thus by Lemma 3.1 (ii) the $Q$-algebra
\[
\text{Hom}_{G,R'}(R', (E_R')^{\otimes n}) \xrightarrow{\sim} \text{Hom}_{Q}(k, (E_R')^{\otimes n}) = ((E_R')^{\otimes n})^G
\]
is finite-dimensional for every \( n \). Denote by \( J' \) the unique maximal \( G \)-ideal of \( R' \). Then we have for each \( n \) a filtration of the \((G,R')\)-algebra \((E_{R'})^\otimes n\) by the \( G \)-ideals
\[
J'^r(E_{R'})^\otimes n, 
\]
and hence a filtration of the \( \mathbb{Q} \)-algebra \( \text{Hom}_{G,R'}(R', (E_{R'})^\otimes n) \) by the ideals
\[
(7.2) \quad \text{Hom}_{G,R'}(R', J'^r(E_{R'})^\otimes n). 
\]
Since \( (7.2) \) is isomorphic to \((J'^r(E_{R'})^\otimes n)^G\), it is 0 for \( r \) large, by Lemma \( \text{[6.1 (ii)]} \).

We now define an element \( C \) of \( \mathcal{A} \) as follows. With the identification \( (7.1) \), take for \( C_n \) the image of \( \xi_{0,n} \), and for \((C_n)^r\) the image under \( \xi_{0,n} \) of \( (7.2) \). Then \( (6) \) holds.

Let \( z = \xi_{0,n}(x) \) be an element of \( C_n \) which does not lie in \((C_n)^1\). Then \( x \) does not factor through \( J'(E_{R'})^\otimes n \). As was seen at the end of Section 4, this implies that \( x \) has a left inverse. Hence \( z \) has a left inverse \( y : h(X)^\otimes n \to 1 \). Identifying \( y \) with an element of \( CH(X^n)_{Q/\sim} \), the composite \( y \circ z = 1_1 \) is the push forward of \( y.z \) along the structural morphism of \( X^n \). Thus \( z \) is not numerically equivalent to 0. The first statement of \( [c] \) follows. The second statement of \( [c] \) follows from the fact that \( (7.2) \) is 0 for \( r \) large.

Let \( p : X^l \to X^m \) be as in \( [a] \). If \( p \) is defined by \( \nu : [1,m] \to [1,l] \), then
\[
h(p) : h(X)^\otimes m \to h(X)^\otimes l
\]
is the morphism of commutative algebras in \( \mathcal{M}_{\sim}(F) \) defined by \( \nu \). Thus
\[
h(p) = \xi_{m,l}(f)
\]
for \( f : (E_{R'})^\otimes m \to (E_{R'})^\otimes l \) the morphism of commutative \((G,R')\)-algebras defined by \( \nu \). That \( p^* \) sends \( C_m \) to \( C_l \) and respects the filtrations now follows from \( (6.2) \) and the compatibility of the \( \xi_{m,l} \) with composites. That \( p_* \) sends \( C_l \) to \( C_m \) and respects the filtrations follows from \( (6.3) \) and the compatibility of the \( \xi_{m,l} \) with composites and transposes. Thus \( [a] \) holds.

8. Proof of Theorem 1.2

Let \( G, E, \rho, R, \xi_{r,s}, \eta_r \) and \( \varepsilon_r \) be as in Section 4 and suppose that the equivalence relation \( \sim \) is numerical equivalence. We show first that \( R \) is \( G \)-simple, i.e. has no \( G \)-ideals other than 0 and \( R \). Any non-zero \( z : h(X)^\otimes m \to 1 \) has a right inverse \( y \), because \( z \circ y \) is the push forward of \( z.y \) along the structural isomorphism of \( X^m \). The isomorphisms \( \xi \) then show that any non-zero \((E_R)^\otimes m \to R \) has a right inverse, and is thus surjective. Let \( J \neq 0 \) be a \( G \)-ideal of \( R \). Since \( G \) is reductive and \( E \) is a faithful representation of \( G \), the category of finite-dimensional representations of \( G \) is the pseudo-abelian hull of its full subcategory with objects the \( E^{r,s} \) (\( [9] \), 3.5). Thus for some \( r,s \) there is a non-zero homomorphism of \( G \)-modules from \( E^{r,s} \) to \( R \) which factors through \( J \). It defines by the isomorphism \( \text{[4.14]} \) a non-zero homomorphism of \((G,R)\)-modules \( f : (E_R)^{r,s} \to R \) which also factors through \( J \). Since \((E_R)^{r,s} \) is isomorphic by autoduality of \( E_R \) to \((E_R)^{r,s} \), it follows that \( f \) is surjective, so that \( J = R \). Thus \( R \) has no \( G \) ideals other than 0 and \( R \).

If \( R_1 \) is the \( G \)-submodule of \( R \) on which \( \rho \) acts as \(-1 \), then the ideal of \( R \) generated by \( R_1 \) is a \( G \)-ideal \( \neq R \), because the elements of \( R_1 \) have square 0. Thus \( R_1 = 0 \), so that \( R \) is commutative as an algebra in \( \text{REP}(G) \). By a theorem of Magid ([6], Theorem 4.5), the \( G \)-simplicity of \( R \) and the fact that \( R^G = Q \) then imply that \( \text{Spec}(R) \) is isomorphic to \( G_k/H \) for some extension \( k \) of \( Q \) and closed.
subgroup $H$ of $G_k$. Thus $R$ is a finitely generated $Q$-algebra. Hence there exists an $n$ such that a set of generators of $R$ is contained in the sum of the images of the $G$-homomorphisms $E^{r,s} \to R$ for $r + s \leq n$. We may suppose that $n \geq 2$. We show that $n$ satisfies the requirements of Theorem 1.2.

Denote by $U_m$ the $Q$-vector subspace of $\overline{CH(X^m)}_Q = CH(X^m)_Q/\sim$ generated by the elements $(8.1)$, and by

$$U_{m,l} \subset \text{Hom}_{G,R}((E_R)^{\otimes m}, (E_R)^{\otimes l})$$

the inverse image of

$$U_{m+1} \subset \text{Hom}_{M_n(F)}(h(X)^{\otimes m}, h(X)^{\otimes l}) = \overline{CH(X^{m+l})}_Q$$

under $\xi_{m,l}$. The symmetries of $(E_R)^{\otimes m}$ lie in $U_{m,m}$, because by Proposition 2.1 the symmetries of $h(X)^{\otimes m}$ lie in $U_{2m}$. Similarly the composite of an element of $U_{m,m'}$ with an element of $U_{m',m''}$ lies in $U_{m,m''}$, the tensor product of an element of $U_{m,l}$ with an element of $U_{m',l'}$ lies in $U_{m+m',l+l'}$, and $\eta_m$ lies in $U_{0,2m}$ and $\varepsilon_m$ lies in $U_{2m,0}$. Also $U_{m,l}$ coincides with $\text{Hom}_{G,R}((E_R)^{\otimes m}, (E_R)^{\otimes l})$ for $m + l \leq n$.

Since $E_R$ is canonically autodual, we may identify $(E_R)^{\otimes (r+s)}$. The morphism $u_{R,r,s;r',s'}$ of (4.10) may then be identified with a morphism of $R$-modules

$$u_{R,r,s;r',s'} : (E_R)^{\otimes (r+s)} \to (E_R)^{\otimes (r'+s')}$$

and the isomorphism $\psi_{r,s}$ of (4.17) with an isomorphism $\psi_{r,s} : \text{Hom}_{G,R}((E_R)^{\otimes (r+s)}, R) \cong \text{Hom}_{G}(E^{r,s}, R)$. Then (4.18) still holds. Also we have a commutative square

$$\begin{array}{ccc}
E^{r_1,s_1} \otimes_Q E^{r_2,s_2} & \xrightarrow{\sim} & E^{r_1+r_2,s_1+s_2} \\
\psi_{r_1,s_1}(f_1) \otimes_Q \psi_{r_2,s_2}(f_2) \downarrow & & \downarrow \psi_{r_1+r_2,s_1+s_2}(f) \\
R \otimes_Q R & \longrightarrow & R
\end{array}$$

(8.1)

where the top isomorphism is that of (4.8) with $E$ for $L$, the bottom arrow is the multiplication of $R$, and $f$ is the composite of the appropriate symmetry of $(E_R)^{\otimes (r_1+r_2+s_1+s_2)}$ with $f_1 \otimes R f_2$.

By Lemma 3.2 a non zero $w : E^{r',0} \to E^{r,0}$ exists only if $r = r'$, when any such $w$ is a composite of symmetries and tensor products of endomorphisms of $E$. Thus $w_{R,r',0} \circ r,0$ lies in $U_{r',r}$ for such a $w$, because $n \geq 2$. Since $(-)_{R,r,s;r',s'}$ commutes with the isomorphisms of (4.10), it follows that $w_{R,r',s';r,s}$ lies in $U_{r'+s',r+s}$ for any $w : E^{r',s'} \to E^{r,s}$.

To prove Theorem 1.2 write

$$W_{r,s} = \psi_{r,s}(U_{r+s,0})$$

Consider the smallest $G$-submodule $R'$ of $R$ such that $a : E^{r,s} \to R$ factors through $R'$ for each $r, s$, and $a$ in $W_{r,s}$. By (3.1), $R'$ is a subalgebra of $R$. Since every $E^{r,s} \to R$ lies in $W_{r,s}$ when $r + s \leq n$, the algebra $R'$ contains a set of generators of $R$. Hence $R' = R$. Given $a : E^{r,s} \to R$, there are thus $a_i$ in $W_{r_i,s_i}$ for $i = 1, 2, \ldots, t$ such that the image of $a$ lies in the sum of the images of the $a_i$. By semisimplicity of REP$(G, \rho)$, it follows that

$$a = a_1 \circ w_1 + a_2 \circ w_2 + \cdots + a_t \circ w_t$$
for some \(w_i\). Hence by [1.18], \(a\) lies in \(W_{r,s}\). Thus \(W_{r,s} = \text{Hom}_G(E^{r,s}, R)\) for every \(r\) and \(s\). It follows that \(U_{m,0} = \text{Hom}_{G,R}((E^R)^{\otimes m}, R)\), and hence \(U_m = \overline{CH}(X^m)_Q\), for every \(m\). This proves Theorem 1.2.

9. Concluding remarks

Theorem 1.1 is easily generalised to the case where instead of cycles on the powers of a single Kimura variety \(X\) for \(\sim\), we consider also cycles on products of a finite number of such varieties: it suffices to take for \(X\) their disjoint union and to include in \(Z_1\) their fundamental classes. Similarly in the condition on \(X^l \to X^m\) in [a] we may consider a finite number of morphisms \(X^l \to X\) additional to the projections: it suffices to include in the \(Z_i\) the classes of their graphs. Suppose for example that \(X\) is an abelian variety, and let \(\Gamma\) be a finitely generated subgroup of \(X(k)\). Then we may consider in [a] pullback and push forward along any morphism \(X^l \to X^m\) which sends the identity of \(X(k)^l\) to an element of \(\Gamma^m\).

More generally, we can construct a small category \(\mathcal{V}\), an equivalence \(T\) from \(\mathcal{V}\) to the category Kimura varieties over \(F\) for \(\sim\), a filtered family \((\mathcal{V}_\lambda)_{\lambda \in A}\) of (not necessarily full) subcategories \(\mathcal{V}_\lambda\) of \(\mathcal{V}\) with union \(\mathcal{V}\), and for each \(\lambda \in A\) and \(V\) in \(\mathcal{V}_\lambda\) a finite-dimensional graded \(Q\)-subalgebra \(C_\lambda(V)\) of \(CH(T(V))_Q/\sim\) and a filtration \(C_\lambda(V)^\bullet\) on \(C_\lambda(V)\), with the following properties.

(a) Finite products exist in the \(\mathcal{V}_\lambda\), and the embeddings \(\mathcal{V}_\lambda \to \mathcal{V}\) preserve them.

(b) We have \(C_\lambda(V)^r \subset C_{\lambda'}(V)^r\) for \(\lambda \leq \lambda'\) and \(V\) in \(\mathcal{V}_\lambda\), and \(CH(T(V))_Q/\sim\) for \(V\) in \(\mathcal{V}_\lambda\) is the union of the \(C_{\lambda'}(V)\) for \(\lambda' \geq \lambda\).

(c) \(T(f)^*\) sends \(C_\lambda(V')\) into \(C_\lambda(V)\) and \(T(f)_*\) sends \(C_\lambda(V)\) into \(C_{\lambda'}(V')\) for \(f: V \to V'\) in \(\mathcal{V}_\lambda\), and \(T(f)^*\) and \(T(f)_*\) preserve the filtrations.

(d) For \(V\) in \(\mathcal{V}_\lambda\), the projection from \(C_\lambda(V)\) to \(\overline{CH}(T(V))_Q\) is surjective with kernel \(C_\lambda(V)^l\), and \(C_\lambda(V)^r\) is 0 for \(r\) large.

By applying the usual construction for motives (say ungraded) to \(\mathcal{V}\) and the \(CH(T(V))_Q/\sim\) we obtain a \(Q\)-tensor category \(\mathcal{M}\) and a cohomology functor from \(\mathcal{V}\) to \(\mathcal{M}\), and \(T\) defines a fully faithful functor from \(\mathcal{M}\) to \(\mathcal{M}/(F)\). Similarly we obtain from \(\mathcal{V}_\lambda\) and the \(C_\lambda(V)\) a (not necessarily full) \(Q\)-tensor subcategory \(\mathcal{M}_\lambda\) of \(\mathcal{M}\). Then each \(\mathcal{M}_\lambda\) has finite-dimensional hom-spaces, and \(\mathcal{M}\) is the filtered union of the \(\mathcal{M}_\lambda\). A question involving a finite number of Kimura varieties, a finite number of morphisms between their products, and a finite number of morphisms between the motives of such products, thus reduces to a question in some \(\mathcal{V}_\lambda\) and \(\mathcal{M}_\lambda\). By [d] the projection from \(\mathcal{M}_\lambda\) to the quotient of \(\mathcal{M}_\lambda/F\) by its unique maximal tensor ideal is full, with kernel the unique maximal tensor ideal \(J_\lambda\) of \(\mathcal{M}_\lambda\). Further \(\mathcal{M}_\lambda\) is the limit of the \(\mathcal{M}_\lambda/(J_\lambda)^\vee\). Thus we can argue by lifting successively from the semisimple abelian category \(\mathcal{M}_\lambda/J_\lambda\) to the \(\mathcal{M}_\lambda/(J_\lambda)^\vee\).

Theorems 1.1 and 1.2 extend easily to the case where the base field \(F\) is replaced by a non-empty connected smooth quasi-projective scheme \(S\) over \(F\). For the category of ungraded motives over \(S\) we then have \(\text{End}(1) = CH(S)_Q/\sim\), which is a local \(Q\)-algebra with residue field \(Q\) and nilpotent maximal ideal. All the arguments carry over to this case, provided that Lemma 3.1 is proved in the more general form where the hypothesis “\(R^G = k\)” is replaced by “\(R^G\) is a local \(k\)-algebra with residue field \(k\)”.


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