Completeness of “Good” Bethe Ansatz Solutions of a Quantum Group Invariant Heisenberg Model

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Abstract

The $sl_q(2)$-quantum group invariant spin $1/2$ XXZ-Heisenberg model with open boundary conditions is investigated by means of the Bethe ansatz. As is well known, quantum groups for $q$ equal to a root of unity possess a finite number of “good” representations with non-zero $q$-dimension and “bad” ones with vanishing $q$-dimension. Correspondingly, the state space of an invariant Heisenberg chain decomposes into “good” and “bad” states. A “good” state may be described by a path of only “good” representations. It is shown that the “good” states are given by all “good” Bethe ansatz solutions with roots restricted to the first periodicity strip, i.e. only positive parity strings (in the language of Takahashi) are allowed. Applying Bethe’s string counting technique completeness of the “good” Bethe states is proven, i.e. the same number of states is found as the number of all restricted path’s on the $sl_q(2)$-Bratteli diagram. It is the first time that a “completeness” proof for an anisotropic quantum invariant reduced Heisenberg model is performed.

1 Introduction

The Bethe ansatz method has been applied to a large number of integrable models as one-dimensional Heisenberg spin chains and statistical lattice models in two dimensions. The

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underlying Yang Baxter Algebra is responsible for the integrability of these models. Yang Baxter Algebras are related to new mathematical structures often referred to as quantum groups which were introduced by Drinfeld (1986) and Jimbo (1985). On the other hand, these quantum groups have attracted attention as a powerful tool for studying properties of solvable systems.

The isotropic XXX-Heisenberg model solved by Bethe (1931) corresponds to a rational solution of the Yang Baxter equation (Baxter 1982) and is $SU(2)$ symmetric. A deformation of the XXX model leads to an anisotropic XXZ-Heisenberg model related to trigonometric Yang Baxter solutions. A version of the model with open boundary conditions is quantum group invariant. The Bethe ansatz method was used to solve the quantum invariant spin chain for open boundary conditions by several authors (see e.g. Alcaraz et al 1987, Cherednik 1984, Sklyanin 1988, Mezincescu and Nepomechie 1991, Martin and Rittenberg 1992, Destri and de Vega 1992, Foerster and Karowski 1993). A quantum group invariant version with periodic boundary conditions has been constructed and analyzed in (Karowski and Zapletal 1993, 1994). We restrict here our interest to a chain with open boundary conditions.

For generic values of $q$ the representations of $sl_q(2)$ are known to be equivalent to the ordinary $SU(2)$ representations (Luztig 1989, Rosso 1988). However, this correspondence holds only if the deformation parameter $q$ is not a root of unity ($q^r \neq \pm 1, r = \text{integer}$).

In contrast to this generic case the quantum group representation theory for $q^r = \pm 1$ is more complicated (Luztig 1989, Reshetikin and Smirnov 1989, Pasquier and Saleur 1990, Reshetikin and Turaev 1991), because there exits two types of representations. The representations with non-zero $q$-dimension (see section 2 equation (16)) are called “good” (Reshetikin and Smirnov 1989) or of type-II (Pasquier and Saleur 1990). Moreover, all of them have positive q-dimension, if and only if $q = e^{i\pi/r}$. They are irreducible and possess the same structure as the usual $SU(2)$ ones. There are only a finite number of “good” representations, namely, those with spin $j = 0, 1/2, 1, \ldots, r/2 - 1$ if $q^r = \pm 1$. Those with vanishing q-dimension called “bad” or type-I representations. Some of them are irreducible and have spin $j = (nr - 1)/2$. The others can be described as a mixing of two representations of the generic case. They are reducible but indecomposable. This phenomenon has a consequence for the eigenstates of an quantum group invariant Hamiltonian. If $q$ approaches a root of unity, some eigenstates which correspond to “bad” representations become dependent and will mix. Due to the coincidence of two originally independent eigenstates the whole eigenspace is not complete, i.e. the Hamiltonian may not be completely diagonalizable. This phenomenon was investigated by Bo-Yu Hou et al (1991).

The existence of “good” and “bad” representations implies the decomposition of the state space of an Heisenberg chain with $N$ sites into “good” and “bad” states. We consider the state iteratively fused by the $N$ spin 1/2-representations. The “good” ones are characterized by the condition that all intermediate representations have also to be “good” ones. This means that the “good” states are described by a restricted ($j \leq r/2 - 1$) path on an $sl_q(2)$-Bratteli diagram. A projection of the full Hilbert space to the subspace of “good” states is analog to the restriction of the solid-on-solid model (SOS) to the so-called
RSOS model (Andrews et al 1984), where the local height variables of the SOS model are restricted to the finite set \( \{1, 2, \ldots, r-1\} \). This provides a connection to the minimal models of conformal field theories related to critical phenomena of 2D systems with second order phase transitions. Indeed, the analysis of finite size corrections of quantum group invariant XXZ spin chains (Hamer et al 1987) and RSOS-models (Karowski 1988) lead to conformal charges smaller than one.

In section 2 we present the model in terms of Pauli matrices and Temperly-Lieb operators. In addition we write some \( sl_q(2) \)-formulas which will be used later. In section 3 we formulate the model in terms of the path basis. This formulation is closely related to the RSOS model. It leads to an explicit “quantum group reduction”.

In section 3 we investigate the model by means of the Bethe ansatz method. The eigenstates and eigenvalues of the Hamiltonian are described by sets of parameters, the Bethe ansatz roots. Such set of roots satisfy a system of algebraic equations, the Bethe ansatz equations (BAE). As a generalization of Bethe’s (1931) work Takahashi (1971, 1972) introduced for the XXZ-Heisenberg model the general string picture. This means that a solution of the BAE consists of a series of strings in the form

\[
\lambda = \Lambda + im, \quad \Lambda = \text{real}, \quad m = -M, -M + 1, \ldots, M, \quad M = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots
\]

with a positive parity or strings with a negative parity

\[
\lambda = \Lambda + ir/2+im, \quad \Lambda = \text{real}, \quad m = -M, -M+1, \ldots, M, \quad M = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots
\]

In this context the question arises how “good” and “bad” eigenstates are determined by the solutions of the Bethe ansatz equations. We conjecture that “good” states are exactly given by strings of positive parity restricted to the first periodicity strip \( \exists \lambda < r/2 \) and whose total spin fulfill \( j \leq r/2-1 \). We have no rigorous proof for this conjecture, but we are able to show the following coincidence. The total number of all possible string configurations of restricted positive parity coincides with the number of “good” states in the path picture of the model, which is counted by the number of all “good” path’s, i.e. all restricted path’s on the spin 1/2 \( sl_q(2) \)-Bratteli diagram. This completeness will be shown in section 4. Such “proofs of completeness” of Bethe ansatz solutions are well known for models with group symmetry. Already Bethe applied this procedure to the XXX-Heisenberg model. For other models see Essler et al (1992a) for the \( SU(2) \times SU(2) \) symmetric Hubbard model and Foerster and Karowski (1992, 1993) for the \( spl(2,1) \)-t-J-model. After having completed this paper we received a preprint (Kirillov and Liskova 1994) in which a “completeness proof” is treated for XXZ spin chains with periodic boundary conditions which have no quantum group symmetry. In contrast to these models, we here present for the first time a “completeness proof” for an anisotropic quantum group symmetric (spin 1/2) model where the number of “good” states is smaller than \( 2^N \).
2 Tensor basis description of $H$

The $sl_q(2)$ invariant XXZ Hamiltonian with open boundary condition can be expressed in terms of Pauli matrices

$$H = \sum_{i=1}^{N-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2} \left( \sigma_i^z \sigma_{i+1}^z - 1 \right) \right) + \frac{q - q^{-1}}{2} \left( \sigma_1^z - \sigma_N^z \right).$$

(1)

This expression may be rewritten in terms of Temperly Lieb operators

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(2)

which satisfy the relations

$$T_k T_{k+1} T_k = T_k,$$
$$T_k^2 = (q + q^{-1}) T_k,$$
$$T_k T_l = T_l T_k, \quad |k - l| \geq 2.$$

(3)

The Hamiltonian $H$ reads

$$H = -2 \sum_{k=1}^{N-1} T_k.$$  

(4)

This formula suggests the notation of open boundary conditions.

The model is quantum group symmetric, since the Hamiltonian $H$ commutes with the generators $S^\pm, S^z$ of the $q$-deformed algebra $U_q(sl(2))$

$$[H, S^\pm] = 0, \quad [H, S^z] = 0.$$  

(5)

We list here some formulas which will be used later. The generators possess the following properties

$$[S^+, S^-] = [2S^z]_q, \quad q^{S^z} S^+ S^- S^z = q^{\pm 1} S^\pm.$$  

(6)

The $q$-number $[x]_q$ is defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$  

(7)

In the limit $q \to 1$ one finds $[x]_q = x$ and (5) tends to the usual relations of $SU(2)$. For $q$ equal to a root of unity one has in addition

$$(S^\pm)^r = 0 \quad \text{for} \quad q^r = \pm 1.$$  

(8)

The underlying Hopf algebra structure (Drinfeld 1986, Jimbo 1985) is described by coproduct, antipode and counit

$$\Delta(q^{S^z}) = q^{S^z} \otimes q^{S^z}, \quad \Delta(q^{S^\pm}) = q^{S^z} \otimes S^\pm + S^\pm \otimes q^{-S^z},$$
$$\gamma(q^{S^z}) = q^{S^z}, \quad \gamma(q^{S^\pm}) = -q^{\pm 1} S^\pm,$$
$$\varepsilon(q^{S^z}) = 1, \quad \varepsilon(q^{S^\pm}) = 0.$$  

(9)
The configuration space \( (C^2)^\otimes N \) of a quantum spin 1/2 chain with \( N \) lattice points has the natural tensor basis
\[
|\alpha\rangle = |\alpha_N, \ldots, \alpha_1\rangle = |\alpha_N\rangle \otimes \cdots \otimes |\alpha_1\rangle
\]
with \(|-\frac{1}{2}\rangle = (\downarrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) (spin down) and \(|\frac{1}{2}\rangle = (\uparrow) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) (spin up). The representation of the quantum group generators on the configuration space is given by
\[
q^{S^z} = q^{\sigma^z/2} \otimes \cdots \otimes q^{\sigma^z/2},
\]
(10)
\[
S^\pm = \sum_i S^\pm_i,
\]
(11)
\[
S^\pm_i = q^{\sigma^z/2} \otimes \cdots \otimes q^{\sigma^z/2} \otimes \sigma^\pm_i/2 \otimes q^{-\sigma^z/2} \otimes \cdots \otimes q^{-\sigma^z/2}.
\]
(12)

The q-deformed Casimir operator is
\[
S^2 = S^- S^+ + [S^z + 1/2]^2 - [1/2]^2 q_r.
\]
(13)

For generic values of \( q \) all representations \( \rho_j \) are irreducible and highest weight ones with spin \( j = 0, 1/2, 1, \ldots \) (see e.g. Lusztig 1989, Rosso 1988). If \( q \) tends to a root of unity \( (q^r = \pm 1) \) the Casimir eigenvalue
\[
S^2 = [j + 1/2]^2 q - [1/2]^2 q
\]
(14)
can take identical values for different spins \( j \) and \( j' \) in case
\[
j' = j + nr \quad \text{or} \quad j' = r - 1 - j + nr, \quad n \in \mathbb{Z}.
\]
(15)

The representations \( \rho_j \) and \( \rho_{j'} \), which are different for generic \( q \), mix for \( q^r = \pm 1 \) and build reducible but indecomposable representations. Together with the irreducible representations with spin \( j = (nr - 1)/2 \) they are characterized by vanishing q-dimension defined by
\[
\dim_q \rho = \text{tr}_{V^\rho}(q^{-2S^+})
\]
(16)
where \( V^\rho \) is the representation space. Therefore they are called “bad” or type I representation. On the other hand, there exits a finite number of representations which are in one-to-one correspondence to \( SU(2) \) representations. They are irreducible, have spin \( j = 0, 1/2, \ldots, r/2 - 1 \) and non-vanishing q-dimension
\[
d_j = \dim_q \rho_j = [2j + 1]_q.
\]
(17)

They are called “good” or type-II representations. Their q-dimension is always positive if \( q = e^{i\pi/r} \).

Reshetikhin and Turaev (1991) have shown that for the “good” and the “bad” representation spaces the following structure holds
\[
V_{\text{good}} \otimes V_{\text{good}} = \bigoplus V_{\text{good}} \oplus \left( \bigoplus V_{\text{bad}} \right),
\]
(18)
\[
V_{\text{good}} \otimes V_{\text{bad}} = \bigoplus V_{\text{bad}}.
\]
(19)
This means that “unitarity” in the “good” subspace (see also Reshetikin and Smirnov 1989) is fulfilled for “good” covariant operators

\[
\langle \text{good}' | A_{j_{\text{good}}} B_{k_{\text{good}}} | \text{good} \rangle = \sum_{\text{good}''} \langle \text{good}' | A_{j_{\text{good}}} | \text{good}'' \rangle \langle \text{good}'' | B_{k_{\text{good}}} | \text{good} \rangle.
\]

In the next section the “good” representations in the state space \((\mathbb{C}^2)^{\otimes N}\) of the XXZ-Heisenberg model are characterized in terms of the “path picture”. For \(q\) equal to a root of unity the state space may be reduced to the subspace of all “good” states. Changing the metric in this subspace the Hamiltonian \([1]\) will become selfadjoint.

### 3 Path basis formulation

For generic values of the deformation parameter \(q\) the representations of \(sl_q(2)\) are irreducible and classified as those of the undeformed group \(SU(2)\). The space \(V^j\) of the spin \(j\) representation is spanned by a set of basis vectors \(|j, m\rangle\), \(m = j, j - 1, \ldots, -j\) with

\[
S^\pm |j, m\rangle = \sqrt{|j \mp m|_q |j \pm m + 1|_q} |j, m \pm 1\rangle \quad S^z |j, m\rangle = m |j, m\rangle.
\]

The tensor product space \(V^{j_1} \otimes V^{j_2}\) decomposes into a direct sum of irreducible spaces

\[
V^{j_1} \otimes V^{j_2} = \bigoplus_{j = |j_1 - j_2|} V^j
\]

given by Clebsch-Gordan coefficients (see e.g. Kirillov and Reshetikin 1989)

\[
|j, m\rangle_{j_1, j_2} = \sum_{m_1, m_2} |j_1 m_1\rangle \otimes |j_2 m_2\rangle |j_1 \alpha_{j_1} \alpha_{j_2}\rangle.
\]

By successive fusion of the \(N\) spin \(1/2\) states associated to the lattice sites we construct the state

\[
|\cdot, m\rangle = \sum_\alpha |\alpha\rangle \langle \alpha | \cdot, m\rangle
\]

which is labeled by the “path” of spins \(\cdot = (j_N, j_{N-1}, \ldots, j_2, j_1 = 1/2)\) and the magnetic quantum number \(m = \sum_i \alpha_i\). The matrix element is a product of Clebsch-Gordan coefficients

\[
\langle \alpha | \cdot, m\rangle = \sum_{m_2, \ldots, m_{N-1}} |j_N 1/2 j_{N-1} 1/2 \ldots j_3 1/2 j_2 1/2 j_1 1/2 \rangle_{\alpha_N \alpha_{N-1} \alpha_3 \alpha_2 \alpha_1}.
\]

The spins \(j_k\) are restricted by the fusion rule

\[
j_{k+1} = j_k \pm 1/2
\]
and $j = j_N$ describes the total spin of the state. The space $V_{1/2}^N$ is decomposed into a direct product of a space described by paths $W_j$ and a space $V_j$ where the generators $S^\pm, S^z$ act

$$V^\otimes N = \sum_j W_j \otimes V_j.$$  \hfill (26)

Quantum group invariant operators as the Hamiltonian only act in the path space $W_j$. By the Wigner-Eckart theorem the magnetic quantum number $m$ is not be changed and in addition the path space (or reduced) matrix elements do not depend on $m$. Therefore we will omit it in the following.

For example the Temperly Lieb operators (2) in path space act as

$$T_k \mid \ldots, j_k, \ldots \rangle = \delta_{j_{k-1}j_{k+1}} \sum_{j'_k=j_{k+1}+1/2} \mid \ldots, j'_k, \ldots \rangle \sqrt{\frac{d_{j_k}d_{j'_k}}{d_{j_{k+1}}}}.$$  \hfill (27)

Thus, we easily obtain the matrix in path space of the Hamiltonian $H$, which decomposes into different blocks for different total spin $j$.

Now we turn to the case where the deformation parameter $q$ is a root of unity ($q^r = \pm 1$). In the path picture it is very simple to characterize the “good” states. From section 2 it is obvious that the “good” states are given by all restricted paths

$$\mid j_{\text{good}} \rangle = \mid j_n, \ldots, j_1 \rangle, \quad 2j_k + 1 < r, \quad k = 1, \ldots, N.$$  \hfill (28)

For generic values of $q$ the number of all states with total spin $j$ is equal to the number of all possible unrestricted paths (25) on the $sl(2)$-Bratteli diagram

$$\Gamma_j = \begin{pmatrix} N \\ N/2 - j \end{pmatrix} - \begin{pmatrix} N \\ N/2 + 1 + j \end{pmatrix}.$$  \hfill (29)

For $q^r = \pm 1$ the number of all “good” states with total spin $j < r/2 - 1$ is equal to the number of all possible restricted paths (28) on the cut $sl_q(2)$-Bratteli diagram

$$\Omega_j = \sum_{k=-\infty}^{\infty} \Gamma_{j+rk}.$$  \hfill (30)

In the next section, it turns out that this number coincides with the number of, what we will introduce, the “good” Bethe ansatz states. Note that only a finite number of terms contribute, because the binomial coefficient $\binom{m}{n}$ is defined to be zero for integer $n < 0$ or $n > m$.

The Hamiltonian does not lead to a transition from a “good” state to a “bad” one. This follows from relation (27). The only possible “good” path’s $\mid j_{\text{good}} \rangle$ which lead by an action of $H = -2 \sum T_k$ to a “bad” state $\mid j'_{\text{bad}} \rangle$ by

$$T_k \mid j_{\text{good}} \rangle = c \mid j_{\text{good}} \rangle + c' \mid j'_{\text{bad}} \rangle.$$  \hfill (31)

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must have one or more values \( j_{k+1} = j_{k-1} = j_k + 1/2 \) with \( 2j_{k+1} + 1 = r - 1 \). Then the “bad” path would have the value of \( j'_k = j_k + 1 = (r - 1)/2 \). But the coefficient \( c' \) vanishes because of eq. (27) and \( d_{j'_k} = [r]_q = 0 \). That means, the matrix \( H \) in path space decomposes into a “good” submatrix and a “bad” part. Note that these arguments cannot be directly translated to the tensor picture, because the transition matrix elements (24) become singular for “bad” states.

We now introduce a new scalar product in the “good” subspace in the path formulation, such that the Hamiltonian becomes selfadjoint

\[
\langle j', m' | j, m \rangle = \delta_{m'm} \delta_{j'j}.
\] (32)

Note that this scalar product coincides with the natural one in the tensor picture only for real values of \( q \). The selfadjointness of the Hamiltonian follows from eq. (27)

\[
H^\dagger = H \quad \text{for} \quad q = e^{i\pi/r}, \quad r = 2, 3, \ldots
\] (33)

since the q-dimensions \( d_j = [2j+1]_q \) are positive for \( 2j + 1 < r \).

### 4 Bethe ansatz method

The eigenstates and eigenvalues of the XXZ-Hamiltonian (1) with open boundary conditions are described by sets of distinct spectral parameters \( \{\lambda_1, \ldots, \lambda_l\} \), the roots of the Bethe ansatz equations (BAE)

\[
\left( \frac{\sinh \gamma(\lambda_j + i/2)}{\sinh \gamma(\lambda_j - i/2)} \right)^{2N} = \prod_{k=1,k\neq j}^l \frac{\sinh \gamma(\lambda_j - \lambda_k + i) \sinh \gamma(\lambda_j + \lambda_k + i)}{\sinh \gamma(\lambda_j - \lambda_k - i) \sinh \gamma(\lambda_j + \lambda_k - i)}.
\] (34)

where \( q = \exp(i\gamma) \). Because of the open boundary conditions it is sufficient to consider only roots with positive real parts \( 0 < \Re \lambda_k < \infty \). The energy is

\[
E = -4 \sum_{k=1}^l \frac{\sin^2 \gamma}{\cosh 2\gamma \lambda_k - \cos \gamma}.
\] (35)

The total spin \( j \) of an eigenstate is related to the number of roots \( l \) by

\[
j = N/2 - l, \quad l = 0, \ldots, N/2
\] (36)

where the lattice length \( N \) is assumed to be even. Because of the quantum group invariance the Bethe ansatz solutions are highest weight states (Destri and de Vega 1992). The aim of our investigation is to find a concrete criterion for the Bethe ansatz solutions to be “good” states in the sense of section 2 and 3.

A detailed numerical analysis of the BAE motivated also by the observation of Destri and de Vega (1992) that when \( q \) approaches a root of unity the appearance of a “bad” state correspond to a root tending to infinity (see also Appendices A and B) leads us to the following

**Conjecture 1:** For \( q = \exp(i\pi/r) \) a Bethe ansatz state is the highest weight vector of a “good” representation (in the sense of section \( 2 \) and \( 3 \) see eq. (28), if and only if
(i) the total spin \( j \) is restricted by \( 2j + 1 < r \), i.e. the number of roots \( l \) must be larger than \((N + 1 - r)/2\).

(ii) the roots are restricted to the first periodicity strip \(| \Im \lambda_k | < r/2\).

The first condition (i) is obvious from the definition of “good” states. We have no rigorous proof for the second condition (ii), but we can show completeness in the sense that the number of “good” Bethe ansatz solutions defined by Conjecture 1 leads to the correct number of all “good” states on the lattice counted by all paths on the Bratteli diagram (see eq. (30)). To “prove” Conjecture 1 we proceed as follows. We classify the Bethe ansatz roots by means of Takahashi’s string picture. A configuration of strings is given by sets of integers. In Conjecture 2 we give upper bounds for these integers. This is also for the group symmetric case a nontrivial (but simpler) problem. The upper bounds are smaller than a naive estimate would suspect. The reason for this phenomenon is that the string picture is only a very rough approximation. The exact set of roots are given by deformed (sometimes even degenerated) strings. This leads to the fact that less roots are possible than the exact string picture would allow. For the group symmetric case already Bethe (1931) was able solve this problem, because there is a natural way to fix the bounds in order to get the correct number of states. This is much more complicated for the quantum group case. Only guided by a lot of numerical calculations, we were able to solve this problem.

It is convenient to classify solutions by the so-called string hypothesis (Takahashi 1971, 1972). Any solution of the BAE consists approximatively of a series of strings in the form

\[
\lambda_M = \Lambda_k + im, \quad m = -M, -M + 1, \ldots, M, \quad M = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots
\]

with positive parity or strings with negative parity \( \lambda = \Lambda + \frac{ir}{2} + im \). According to Conjecture 1 (ii) we must take into account only the positive type \([37]\).

A Bethe vector is characterized by the sets of real string centers

\[
\{\{\Lambda_0\}, \{\Lambda_1\}, \ldots\} \quad \text{where} \quad \{\Lambda_M\} = \{\Lambda_{M,1}, \ldots, \Lambda_{M,\nu_M}\}.
\]

The length of a string is \(2M + 1\) and \(\nu_M\) is the number of \((2M + 1)\)-strings with different centers \(\Lambda_{M,k}\). The total number of roots writes as

\[
l = \sum_{M=0}^{2N} (2M + 1)\nu_M.
\]

As usual we rewrite the Bethe ansatz equations in terms of the strings centers

\[
V_{2N}^{M+1/2}(\Lambda_{M,i}) = \prod_{m=1}^{2M} V_m^2(2\Lambda_{M,i})
\]

\[
\times \prod_{M'} \prod_{k=1}^{\nu_{M'}} V_{M,M'}(\Lambda_{M,i} - \Lambda_{M',k}) V_{M,M'}(\Lambda_{M,i} + \Lambda_{M',k}),
\]
where
\[ V_{M,M'}(\lambda) = \prod_{m=|M-M'|}^{M+M'} V_m(\lambda)V_{m+1}(\lambda) \quad \text{and} \quad V_m(\lambda) = \frac{\sinh \gamma(\lambda + im)}{\sinh \gamma(\lambda - im)}. \] (41)

Taking the logarithm of eq. (40) sets of integers \( \{Q_M\} \) occur which determine the solutions
\[ 2N\Psi_{M+1/2}(\Lambda_{M,i}) = 2\pi Q_{M,i} + \sum_{m=1}^{2M} 2\Psi_m(2\Lambda_{M,i}) \]
\[ + \sum_{M'} \sum_{k=1}^{\nu_{M'}} \left( \Psi_{M,M'}(\Lambda_{M,i} - \Lambda_{M',k}) + \Psi_{M,M'}(\Lambda_{M,i} + \Lambda_{M',k}) \right) \] (42)
where
\[ \Psi_{M,M'}(\lambda) = \sum_{m=|M-M'|}^{M+M'} (\Psi_m(\lambda) + \Psi_{m+1}(\lambda)) \quad \text{and} \]
\[ \Psi_m(\lambda) = 2 \arctan \left( \cot(\gamma m) \tanh(\gamma \lambda) \right), \quad m > 0, \quad \Psi_0(\lambda) = 0. \] (43)

A solution of the BAE is parameterized by a configuration of distinct integers
\[ 0 < Q_{M,1} < Q_{M,2} < \ldots < Q_{M,\nu_M} \leq Q_{M}^{\max}, \quad M = 0, 1/2, 1, \ldots \] (45)
which can be used to count the total number of possible solutions. The upper bounds \( Q_{M}^{\max} \) are defined by the following condition. All roots corresponding to the integers fulfilling (45) are “good” ones in the sense of Conjecture 1. However, if one \( Q_{M,\nu_M} = Q_{M}^{\max} + 1 \) then one root would be pushed to \( \infty \) or it would return as a “bad” one with \(|\Im \lambda| \geq r/2\). This is meant in the sense of deforming the model with respect to \( q \). The counting of the “good” states may be performed by the following

Conjecture 2: For “good” Bethe ansatz solutions:

(iii) the upper bounds in relation (43) are given by
\[ Q_{M}^{\max} = 2j + \nu_M + 2 \sum_{M' > M}^{M+M'} 2(M' - M)\nu_{M'} - G_M, \quad G_M = \max(2j + 2M + 3 - r, 0) \] (46)
(iv) and the string length is restricted by
\[ 2M + 1 \leq r - 2. \] (47)

Note that this condition (iv) is stronger than expected from Conjecture 1 (ii) and (46), which would allow also strings of length \( r - 1 \). However, (iv) follows from (iii), since for the maximal string the sum in eq. (46) is empty and by relation (45)
\[ \nu_{\max} \leq Q_{M}^{\max} \leq 2j + \nu_{\max} - G_{\max} \leq \nu_{\max} - 2M_{\max} - 3 + r. \]
Note also that a naive estimate would lead to a larger upper bound than that of (iii), namely to the largest integer smaller than $Q_M^\infty$, which is obtained by putting $\Lambda_{M,i} \to \infty$ in eq. (12) (using $\Psi_m(\infty) = \pi - 2\gamma m$)

$$Q_M^\infty = 2j + \nu_M + 2 \sum_{M' > M} 2(M' - M)\nu_{M'} + (2M + 1) \left(1 - \frac{1}{r}(N - 2l + 2M + 2)\right). \quad (48)$$

The upper bound of (iii) is obviously smaller than that obtained from this equation, even for the isotropic case $q = 1$ or $r = \infty$ where $Q_M^{\text{max}}(q = 1) = Q_M^\infty - 2M - 1$, already used by Bethe (1931) with $J(M, M') = 2 \min(M, M') + 1 - \delta_{M,M'}/2$ and

$$Q_M^{\text{max}}(q = 1) = 2j + \nu_M + 2 \sum_{M' > M} 2(M' - M)\nu_{M'} \quad (49)$$

$$= N - 2\sum_{M'} \nu_{M'} J(M, M'). \quad (50)$$

We are not able to prove (iii) rigorously, however, we performed many numerical calculations (see Appendices A and B) and justify it by the following counting of “good” states leading to the correct result.

We must solve the following combinatorial problem to compute the number of Bethe ansatz states. The integer $Q_M^{\text{max}}$ denotes the number of vacancies for a string of given length $2M + 1$. The number of possible configurations (45) is

$$\left(\frac{Q_M^{\text{max}}}{\nu_M}\right)^{\nu_M}$$

and therefore the number of combinations of a given set $\{\nu_M\}$ reads

$$Z(N, \{\nu_M\}, r) = \prod_M \left(\frac{Q_M^{\text{max}}}{\nu_M}\right). \quad (51)$$

Now we obtain the total number of Bethe ansatz states with fixed $l$ by taking the sum over all configurations $\{\nu_M\}$

$$Z(N, l, r) = \sum_{\{\nu_M\}} Z(N, \{\nu_M\}, r) \quad \text{with fixed} \quad l = \sum_M (2M + 1)\nu_M. \quad (52)$$

Extending the calculations made by Bethe (1931) one can show for $M > 0$

$$Q_M^{\text{max}}(N, \{\nu_M\}, r) = Q_{M-1/2}^{\text{max}}(N - 2\mu, \{\nu_M'\}, r - 1), \quad \mu = \sum_M \nu_M, \quad \nu_M' = \nu_{M+1/2}. \quad (53)$$

This implies with eqs. (51) and (46)

$$Z(N, \{\nu_M\}, r) = \left(\frac{Q_0^{\text{max}}}{\nu_0}\right) Z(N - 2\mu, \{\nu_M'\}, r - 1) \quad \text{with} \quad Q_0^{\text{max}} = N - 2\mu + \nu_0 - G_0(r). \quad (54)$$

We introduce the partial number of configurations depending on the number of strings $\mu$

$$Z(N, l, \mu, r) = \sum_{\sum (2M + 1)\nu_M = l, \sum \nu_M = \mu} Z(N, \{\nu_M\}, r). \quad (55)$$
From eq. (54) we obtain the following recurrence relation

\[
Z(N, l, \mu, r) = \sum_{\nu_0=0}^{\mu-1} \left( \frac{Q_{\nu_0}^{\max}}{\nu_0} \right) Z(N - 2\mu, l - \mu, \mu - \nu_0, r - 1). \tag{56}
\]

This relation differs from that of Bethe for the XXX-model by the additional dependence on \(r\). Note that the equation (56) holds for \(\mu < l\).

The initial values of this recurrence relations are given by

\[
l = \mu \quad \text{where only real roots exist (} \nu_0 = l \text{ and } \nu_M = 0 \text{ for } M > 0),
\]

therefore from eqs. (51) and (55)

\[
Z(N, l, \mu = l, r) = \left( \frac{Q_0^{\max}}{l} \right) = \left( \frac{2j + l - G_0}{l} \right), \tag{57}
\]

\[
Z(N, l, \mu, r = 2) = 0. \tag{58}
\]

The second initial condition for \(r = 2\) follows from eq. (46).

We now introduce the functions \(f_{k,d}\) depending on the integers \(k\) and \(d = 0, 1\)

\[
f_{k,d}(N, l, \mu, r) = \left( \frac{N - l - k(r - 2) - G_0(r + 1) + d}{N - l + 1 - k(r - 1) - \mu - G_0(r + 1)} \right) \left( \frac{l + k(r - 2) - d}{l + k(r - 1) - \mu} \right). \tag{59}
\]

**Lemma:** The function

\[
Z(N, l, \mu, r) = \sum_{k=-\infty}^{\infty} (f_{k,1} - f_{k,0}) \tag{60}
\]

solves the recurrence relation (54) and fulfills the initial condition (57). Thus it is equal to the partial number of states defined in eq. (53).

The proof of this lemma is performed in Appendix C.

In the isotropic limit \(r \to \infty\) all the functions \(f_{k,1} - f_{k,0}\) vanish except that for \(k = 0\) which coincides with the solution already found by Bethe (1931). One can calculate the total number of Bethe ansatz states for a given number of roots \(l\) (52) by

\[
Z(N, l, r) = \sum_{\mu=1}^{l} Z(N, l, \mu, r). \tag{61}
\]

For the interesting case \(2j + 1 < r\) (with \(j = N/2 - l\)) it has the form

\[
Z(N, l, r) = \sum_{k=-\infty}^{\infty} \Gamma_{j+kr}, \quad \Gamma_{j+kr} = \sum_{\mu=1}^{l} (f_{k,1} - f_{k,0}) = \left( \frac{N}{l - kr} \right) - \left( \frac{N}{l - 1 - kr} \right) \tag{62}
\]

which indeed coincides with the number of “good” path’s (30). Thus, we have shown that the conjecture about Bethe ansatz states of the “good” type yields the correct number of states with respect to the configuration space of the quantum invariant spin chain restricted to the subspace of “good” representations.
5 Conclusions

The configuration space of the quantum group invariant XXZ Heisenberg model with open boundary conditions decomposes into a “good” and a “bad” part, if and only if the deformation parameter $q$ is a root of unity. Furthermore, if $q$ takes the values $q = \exp(i\pi/r)$ ($r = 3, 4, \ldots$) a new metric may be introduced in the “good” subspace such that the Hamiltonian becomes self-adjoint. This is reminiscent of minimal models of conformal field theory. The central charge of the Virasoro algebra is known to be restricted by unitarity (Friedan et al 1984) to the values

$$c = 1 - 6/r(r - 1), \quad r = 3, 4, 5, \ldots$$

Indeed, by finite size computations (Hamer et al 1987) of the spin chain one obtains the identification to $c$.

In this paper the completeness of the Bethe ansatz of the quantum group symmetric spin $1/2$ Heisenberg model was proved for a configuration space reduced to the “good” representations at $q = \exp(i\pi/r)$ with $r = 3, 4, 5, \ldots$. The “good” Bethe ansatz solutions (in terms of strings of positive parity) are parameterized by integers which are bounded by the upper values $Q^\text{max}_M$ (46). The conjecture for $Q^\text{max}_M$ in the quantum group case introduced in this paper yields the correct number of states. Furthermore, the “good” Bethe ansatz solutions were checked numerically for small lattice length ($N \leq 12$) by solving the BAE exactly. In addition the eigenvalues obtained by these solutions were compared with those obtained by diagonalization of the Hamiltonian in the path basis (see eq. (27)).

Appendix A: Non-string solutions of the BAE

The Bethe ansatz equations in the logarithmic form provide a determination of solutions by a set of integers $\{Q_M\}$ if one assumes that complex roots consist of strings $\lambda = \Lambda \pm im$ with known imaginary part $m$. However, this assumption is known to be only an approximation. In general we have complex pairs $\lambda = x \pm iy$ where the imaginary part is not a integer or a half-integer (in case of finite lattice length $N$). To consider exact solutions we analyze the BAE without this conjecture. Unfortunately, for $l > 2$ ($l$ . . . number of roots) such an investigation can be made only numerically. But analyzing a large amount of numerical solutions we were lead to some rules which should be viewed as general properties of the Bethe ansatz equations.

We have found (see appendix B) that by a successive increasing of $\gamma$ to integer values $r = \pi/\gamma$ ($q = \exp(i\gamma)$) the largest admissible value of $Q$ must be reduced in order to obtain finite rapidities of positive parity (which are assumed to be related to “good” states). Especially, if a 2-string ($l = 2$) tends to infinity at $Q$ and $r$ then a 2-string at $Q' = Q - 1$ degenerate at $r' = r - 1$ into two 1-strings where one 1-string tends to infinity. This leads to the corrections $G_{1/2}$ and $G_0$ of the upper boundary $Q^\text{max}_M$. In the general case $l > 2$, we can also assume that the largest value $Q^\text{max}_M$ for a set of integers is caused by
non-string effects. By numerical computations we have observed the following important property.

If a k-string (string of length k=2M+1) tends to infinity at $Q_M$ and $r$ then a k-string at $Q'_M = Q_M - 1$ and $r' = r - 1$ degenerates into a divergent (k-1)-string and a 1-string. Moreover, a k-string at $Q''_M = Q_M - 2$ and $r'' = r - 2$ degenerates into a divergent (k-2)-string and a 2-string. In general, a k-string at $Q_{M}' = Q_M - n$ and $r_{M}' = r - n$ degenerates into a divergent (k-n)-string or the n-string becomes infinite.

Due to this conjecture we obtain the condition

$$G_M(r) = G_M(r - 1) - 1.$$  \hspace{1cm} (64)

Considering the value $Q_M^\infty$ in equation (48)

$$Q_M^\infty = N - 2 \sum_{M'} \nu_{M',J(M,M')} + (2M + 1)(1 - \frac{\gamma}{\pi}(N - 2l + 2M + 2))$$

one can assume that the initial value $G_M(r) = 1$ is given by

$$\frac{\gamma}{\pi}(N - 2l + 2M + 2) = 1$$

which leads to

$$G_M(r) = 2j + (2M + 1) - r + 2$$ \hspace{1cm} (66)

as was suggested in (46).

We point out that this picture of admissible strings holds only in the case $2j + 1 < r$. For $2j + 1 \geq r$ there are more configurations which leads to finite solutions of positive parity. Nevertheless, such states always belong to the “bad” type and have to be excluded.

**Appendix B: Numerical computation of the BAE**

Considering the two-magnon case $l = 2$ one obtains two kinds of possible solutions. Due to (34) there are either two distinct real roots \{$\Lambda_1, \Lambda_2$\} or a complex pair \{$x + iy, x - iy$\}. The two real roots satisfy the following Bethe ansatz equations

$$2\pi Q_1 = 2N\Psi_{1/2}(\Lambda_1) - \Psi_1(\Lambda_1 - \Lambda_2) - \Psi_1(\Lambda_1 + \Lambda_2)$$ \hspace{1cm} (67)

$$2\pi Q_2 = 2N\Psi_{1/2}(\Lambda_2) - \Psi_1(\Lambda_2 - \Lambda_1) - \Psi_1(\Lambda_2 + \Lambda_1)$$ \hspace{1cm} (68)

corresponding to two 1-strings. Because of the zero imaginary part the solution is exactly given by (12) with $\nu_0 = 2$ and $\nu_k = 0 \ (k > 0)$. If one root tends to infinity the value $Q$ reads

$$Q^\infty = 2j + 2 - \frac{\gamma}{\pi}(2j + 2).$$ \hspace{1cm} (69)
Thus, for \((2j + 2)\pi/\gamma < 1\) the largest admissible integer is given by
\[
Q_0^{\text{max}} = 2j + 2
\]  
(70)
and for \(\pi/\gamma = r = (2j + 2)\) one root tends to infinity. The condition
\[
Q_0^{\text{max}} = 2j + 2 - G_0, \quad G_0 = 1 = 2j + 3 - r
\]  
(71)
ensures finite solutions. For example, the case \(N = 8\) \((l = 2, j = 2)\) and \(r > 2j + 2\) provides configurations of integers \(\{Q_1, Q_2\}\) which are in one-to-one correspondence to the solutions of the BAE
\[
\{\Lambda_1, \Lambda_2\} \leftrightarrow \{Q_1, Q_2\}
\]
with \(1 \leq Q_1 < Q_2 \leq Q_0^{\text{max}} = 2j + 2\). At \(r = 2j + 2\) the number of configurations is reduced by \(Q_0^{\text{max}} = 6 - 1\). For \(r < 2j + 2\) we have no “good” state.

The case of complex roots \((l = 2)\) is more complicated. Using the string conjecture we would have the following equation for the complex pair \((42)\)
\[
\pi Q_{1/2} = N\Psi_1(\Lambda_{1/2}) - \Psi_1(2\Lambda_{1/2})
\]  
(72)
with the configuration \(\nu_{1/2} = 1, \nu_0 = 0, \nu_1 = 0 \ldots\). As a function of \(\Lambda\) the r.h.s. is monotone and all integers \(Q\) up to the asymptotic value
\[
Q_{1/2}^\infty = 2j + \nu_{1/2} + 2 - 2(2j + 3)/r
\]
would lead to a finite value \(\Lambda_{1/2}\). This contradicts the condition that the largest admissible integer is given by
\[
Q_{1/2}^{\text{max}} = 2j + \nu_{1/2} - G_{1/2} = 2j + 1 - G_{1/2}
\]  
(73)
Even in the XXX case \((G_{1/2} = 0)\) there are integers \(Q\) with \(Q_{1/2}^{\text{max}} < Q < Q_{1/2}^\infty\) which are allowed in (72) but forbidden as Bethe ansatz states. We have already pointed out that for such large values \(Q\) the imaginary part of the root tends away from the assumed (half) integer value and therefore the string conjecture does not hold shown by the exact solutions of the BAE. A complex root denoted by \(\lambda = x + iy\) satisfies the equations \((34)\)
\[
\left(\frac{\sinh \gamma(x + i(\pm y + 1/2))}{\sinh \gamma(x + i(\pm y - 1/2))}\right)^{2N} = \frac{\sin \gamma(\pm 2y + 1) \sinh \gamma(2x + i)}{\sin \gamma(\pm 2y - 1) \sinh \gamma(2x - i)}.
\]  
(74)
Considering the phase of these relations
\[
\pi Q_{1/2} = -N\phi_{1/2\pm y}(x) - N\phi_{1/2\mp y}(x) + \phi_1(2x) + \pi(N - 1)
\]  
(75)
\[
\phi_y(x) = 2 \arctan(\tan \gamma y \coth \gamma x)
\]  
(76)
we obtain in the limit \(y \to 1/2\) (with \(\phi_y(x) = \pi - \Psi_y(x)\)) the equation for a two-string \((72)\). Thus, these integers \(Q\) are related to \((72)\). The magnitude reads
\[
\left(\frac{\sinh^2 \gamma x + \sinh^2 \gamma(\pm y + 1/2)}{\sinh^2 \gamma x + \sinh^2 \gamma(\pm y - 1/2)}\right)^{2N} = \frac{\sin^2 \gamma(\pm 2y + 1)}{\sin^2 \gamma(\pm 2y - 1)}.
\]  
(77)
which is used to eliminate the real part $x$ in (75). With

$$ x = b(y) $$

$$ \sinh^2 \gamma b(y) = \frac{W_2(y)^{1/2N} - W_1(y)}{W_1(y)(1 - W_2(y)^{1/2N})} \sinh^2 \gamma (y + 1/2) $$

$$ W_m(y) = \frac{\sinh^2 \gamma m(y + 1/2)}{\sinh^2 \gamma m(y - 1/2)} $$

the counting function $z(y)$ depending on $y$ reads

$$ \pi z(y) = -N \left( \phi_{1/2+y}(b(y)) + \phi_{1/2-y}(b(y)) \right) + \phi_1(2b(y)) + \pi(N - 1). $$

It turns out that a solution for $|y| < 1/2$ and $|y| > 1/2$ is given by even and odd integers, respectively

$$ Q_{1/2} = z(y). $$

The general behavior of $z$ drawn in figure 1 shows that for small values of $Q$ the imaginary part $y$ of a complex pair is approximated very well by $y = 1/2$ with the correspondence

$$ \{x + iy, x - iy\} \leftrightarrow \{Q_{1/2}\}. $$

But for large values $Q$ the imaginary part of the root tends away from the line $y = 1/2$. Now we discuss the example $N = 8$ but it can be easily generalized. The counting

![Figure 1: Counting function for a complex pair $\lambda = x \pm iy$ with $N = 8$, $l = 2$ and $r = 14$ as a function of $y$. Full (open) circles correspond to odd (even) values of $Q$.](image)
function $z(y)$ takes its maximum at $y = 0$ (for $y < 1/2$) and at $y = \pi/4\gamma = r/4$. One can show that for $\gamma \to 0$ (XXX case) the maximal integer is determined by $Q_{1/2}^{\text{max}} = 2j + 1$ which corresponds to the upper restriction stated above. Although the maximum $z_{\text{max}}$ (at $y = r/4$) is larger than $6$

$$6 < z_{\text{max}} < 7, \quad y > 1/2$$

the largest admissible integer is related to

$$Q_{1/2}^{\text{max}} = 2j + 1 = 5.$$ 

Note that the odd integers $Q_{1/2} = 1, 3, 5$ determine the solutions with $y > 1/2$. The even integers $Q_{1/2} = 2, 4$ correspond to roots with $y < 1/2$ because of

$$4 < z_{\text{max}} < 5, \quad y < 1/2.$$ 

The maximum of $z(y)$ for both $y < 1/2$ and $y > 1/2$ depends on the anisotropy $\gamma$. It decreases with increasing $\gamma$. The picture described above remains unchanged up to values $r > 2j + 3$. In the case $r = 8$ for example we have $z_{y>1/2}^{\text{max}} = 5.25$ and $z_{y<1/2}^{\text{max}} = 4.07195\ldots$.

Thus, the integers $Q_{1/2} = 1, 2, 3, 4, 5$ are assumed to correspond to “good” state if $r \geq 8$.

Increasing $\gamma$ we have at

$$\pi/\gamma = r = 2j + 3 = 7$$

(according to (81)) a divergent complex pair with $Q_{1/2} = 2j + 1 = 5$ ($x \to \infty, y = \pm r/4$) because of $z_{y>1/2}^{\text{max}} = 5$. Therefore, the value $Q_{1/2}^{\text{max}}$ for finite roots must be reduced by one

$$Q_{1/2}^{\text{max}} = 2j + 1 - 1 = 4$$

which leads to $G_{1/2} = 1$. The odd value $Q_{1/2} = 2j + 1 = 5$ is now forbidden. Furthermore, by increasing $\gamma$ or increasing $N$ we reach a critical value where the maximum of $z(y)$ for $y < 1/2$ is smaller than the even integer $Q_{1/2} = 2j = 4$

$$z_{y<1/2}^{\text{max}} = 3.93162\ldots < 2j, \quad r = 7.$$ 

The imaginary part $y$ tends to zero. This fact was observed by Vladimirov (1984) and Efßler et al (1991b) in the XXX case. The complex pair was found to be replaced by two additional real roots. The dependence on $\gamma$ was investigated by Jüttner et al (1993). Here we have a similar situation. The additional real solution is considered as a degenerate complex pair corresponding to $Q_{1/2} = 2j = 4$. The roots are finite and therefore remain “good”. Such a degenerate solution is described by (68) having identical integers

$$Q_1 = Q_2 = Q_0^{\text{max}} = 2j + 2 = 6$$

at $Q_0^{\text{max}}$ although their rapidities are distinct $\Lambda_1 \neq \Lambda_2$. This correspondence is denoted by

$$\{\Lambda_1, \Lambda_2\} \leftrightarrow \{Q_0^{\text{max}}, Q_0^{\text{max}}\} \leftrightarrow \{Q_{1/2}\}.$$ 

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Note that there is a difference between $Q_{0}^{\text{max}}$ of real roots and the value $Q_{1/2}$ for a 2-string. At $r = 7$ we consider the values $Q_{1/2} = 1, 2, 3, 4$ as “good” integers.

Increasing $\gamma$ to $r = 6$ we will see that a solution related to $Q_{1/2} = 4$ becomes “bad”. It can be shown that in the limit $\pi/\gamma \rightarrow 2j + 2$ a complex pair at $Q_{1/2} = 2j = 4$ always degenerate. But at $r = 2j + 2 = 6$ a real root related to $Q = 2j + 2 = Q_{0}^{\text{max}} = 6$ is known to tend to infinity. This happens also for a degenerate complex pair. Because one part of this degenerate string is divergent we must exclude the corresponding integer of the 2-string

$$Q_{1/2}^{\text{max}} = 2j - 1$$

which leads to $G_{1/2} = 2$ and “good” values $Q_{1/2} = 1, 2, 3$.

A further increasing of $\gamma$ to $r \leq 5$ leads to the case where any state belongs to the “bad” sector. Namely, the total spin ($j = 2$) is related to $2j + 1 \geq r$. One can check that at $r = 5$ the admissible numbers for finite solutions of positive parity read $Q_{1/2} = 1, 2, 3$. Thus we have no further reduction of $Q_{1/2}^{\text{max}}$. But this is of no interest for our consideration of “good” states.

The case $l > 2$ is treated as follows. Fixing a given set of integers $\{Q_M\}$ we first calculate the centers of strings $\{\Lambda_M\}_{\gamma_0}$ for $0 < \gamma_0 << 1$ by fixed point iterations of the BAE (42)

$$\Lambda_{M,i} = \frac{1}{\gamma} \tanh^{-1} \left( \tan \gamma m \tan \left( 2\pi Q_{M,i} + \sum_{m=1}^{2M} 2\Psi_m(2\Lambda_{M,i}) + \sum_{M'=0}^{\nu_{M'}} \sum_{k=1}^{\ldots} \right) / 4N \right) \quad (83)$$

It turns out that the iteration converges (nearly) independently on the initial values for $\Lambda_{M,i}$. The set of centers $\{\Lambda_M\}_{\gamma_0}$ is now a function of $\{Q_M\}$. The next step consists in an exact computation of the roots $\{\lambda\}$ by the original BAE (34) which is numerically solved by the Newton method. As initial values we use the string centers $\{\Lambda_M\}_{\gamma_0}$ where the corresponding imaginary part of each member reads

$$\lambda = \Lambda_M + im + i\delta$$

with $0 < |\delta| << 1$. With a careful choice of $\delta$ the Newton method converges which yields the accurate solutions of the BAE for $\gamma_0$. Thus, we have a correspondence between the set $\{Q_M\}$ and $\{\lambda\}$

$$\{\lambda\}_{\gamma_0} = f(\{Q_M\}; \gamma_0).$$

Now the anisotropy $\gamma$ is increased in small steps

$$\gamma_{k+1} = \gamma_k + \Delta.$$

For each $\gamma_k$ the Newton method is applied. In contrast to the first step we now take initial values which are accurate solutions in the step $k - 1$. If $\Delta$ is small enough one can assume that the roots $\{\lambda\}_{\gamma_k}$ keep the relation to $\{\lambda\}_{\gamma_{k-1}}$ with respect to the correspondence to the integers $\{Q_M\}$. This provides

$$\{\lambda\}_{\gamma_k} = f(\{Q_M\}; \gamma_k)$$
Table 1: Exact solutions for $N = 8$, $l = 3$.

| $\pi/\gamma$ | $Q = 1$ |  | $Q = 2$ |  | $Q = 3$ |
|---------------|---------|---------|---------|---------|---------|
|               | $\lambda_1$ | $\lambda_{2,3}$ | $\lambda_1$ | $\lambda_{2,3}$ | $\lambda_1$ | $\lambda_{2,3}$ |
| 7             | 0.672828   | 0.672829 ± 1.000377 | 1.305862 | 1.338141 ± 1.035500 | 2.379542 | 2.183152 ± 1.348160 |
| 6             | 0.716146   | 0.716398 ± 1.00649  | 1.407535 | 1.489400 ± 1.060994 | $\infty$ | $\infty$ ± $\pi/3\gamma$ |
| 5             | 0.810005   | 0.811995 ± 1.001089 | 1.681845 | $\infty$ ± $\pi/4\gamma$ |           |           |
| 4             | $\infty$   | 1.376812 ± i0.569724 |           |           |           |           |

which can be used to investigate the behavior of the solutions as a function of the integers $\{Q_M\}$.

Now we discuss some numerical examples. Considering a 3-string for $N = 8$ and $l = 3$ ($\nu_1 = 1$) the integers $Q = 1, 2, 3$ are allowed ($\pi/\gamma > 6$) and the exact solutions are listed in table 1. The numbers $Q = 1, 2, 3$ are associated to the following picture where the real and imaginary part of the roots are plotted in the x-y-plane. At $\gamma = \pi/7$ we obtain

```

Q=1

1
0

x

1
0

x

Q=2

1
0

x

Q=3

1
0

x
```

The roots are arranged as 3-strings very well. Increasing $\gamma$ to $\pi/6$ the roots related to $Q = 3$ tend to infinity - the real root as well as the real part of the complex pair.

```

Q=1

1
0

x

Q=2

1
0

x

Q=3

1
0

x
```

Thus, the admissible numbers are reduced to $Q = 1, 2$. Now, in the limit $\gamma \to \pi/5$ the string at $Q = 2$ degenerate. The real root remains finite - but the difference to the real part of the complex pair becomes larger. This 3-string degenerate into a 2-string and a 1-string in such a manner that at $\pi/5$ the complex pair tends to infinity.

```

Q=1

1
0

x

Q=2

1
0

x

Q=3

1
0

x

"bad"
```


Now we have only one allowed integer $Q = 1$.

The behavior of a 4-string is similar. If $N = 10$ and $l = 4$, $\nu_{3/2} = 1$ we have 3 admissible values $Q = 1, 2, 3$ ($\gamma < \pi/7$).

At $\gamma = \pi/7$ the configuration related to $Q = 3$ becomes infinite and the allowed integers are reduced to $Q = 1, 2$. If the anisotropy is increased to a critical value between $\pi/7 > \gamma > \pi/6$, the imaginary part of one complex pair becomes smaller and vanishes at the critical value. This complex pair is replaced by a pair of two real roots.

Thus, the 4-string degenerates. The whole configuration can be viewed as a set of one finite 1-string and one 3-string which tends to infinity at $\gamma \to \pi/6$. 

---

![Diagram](image)
In this case only \( Q = 1 \) is permitted. However, in the limit \( \gamma \to \pi/5 \) this set degenerates into two (2-string like) complex pairs. One 2-string is divergent.

Therefore, no integer is allowed for a 4-string \((\pi/\gamma = r \leq 5)\).

These examples and similar ones lead us to the conjectures about admissible configurations of finite strings of positive parity stated in section 4.

Appendix C: Proof of the Lemma in Section 4

We define the function

\[
\tilde{Z}(N, l, \mu, r) = \sum_{k=-\infty}^{\infty} (f_{k,1} - f_{k,0})
\]  

and show that it solves the recurrence relation (56) and fulfills the initial condition (57). Only a finite number of terms contribute because the binomial coefficients \( \binom{m}{n} \) are zero for \( m < 0 \) or \( n < 0 \).

The right hand side of the recurrence relation reads for \( f_{k,d} \)

\[
F_{k,d}(N, l, \mu, r) = \sum_{\nu_0=0}^{\mu-1} \frac{(N - 2\mu + \nu_0 - G_0(r))}{\nu_0} f_{k,d}(N - 2\mu, l - \mu, \mu - \nu_0, r - 1)
\]  

\[
= \sum_{\nu_0=0}^{\mu-1} \frac{(N - 2\mu + \nu_0 - G_0(r))}{\nu_0} \left( \frac{N - l - k(r - 3) - G_0(r) + d - \mu}{N - l + 1 - k(r - 2) - 2\mu - G_0(r) + \nu_0} \right)
\]

\[
\times \frac{l + k(r - 3) - d - \mu}{l + k(r - 2) - 2\mu + \nu_0}
\]  

Using the sum rule of Binomial coefficients

\[
\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}
\]  

one has (with \( g = G_0(r) \) and \( \nu = \nu_0 \))

\[
\binom{N - 2\mu - g + \nu}{\nu} = \sum_{\omega=0}^{\nu} \binom{N - 2\mu + \nu - l - g + 1 - k(r - 2)}{\nu - \omega} \binom{l - 1 + k(r - 2)}{\omega}
\]  

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which is inserted in \( F_{k,d} \). With the identity

\[
\left( N - 2\mu + \nu - l - g + 1 - k(r - 2) \right) \left( N - l - g + d - k(r - 3) - \mu \right) \\
\nu - \omega \\
\left( N - l - g + d - k(r - 3) - \mu \right) \left( d + k + \mu - 1 - \omega \right)
\]

we have

\[
F_{k,d} = \sum_{\omega=0}^r \left( \frac{l + 1 + k(r - 2)}{\omega} \right) \left( \frac{N - l + d - g - k(r - 3) - \mu}{d + k + \mu - 1 - \omega} \right) \\
\sum_{\nu=\omega}^r \left( \frac{d + k + \mu - 1 - \omega}{\nu - \omega} \right) \left( \frac{l + d + k(r - 3) - \mu}{l + k(r - 2) - 2\mu + \nu} \right).
\]

With the help of the sum rule (87) this equation reads

\[
F_k = \sum_{\omega=0}^r \left( \frac{l + 1 + k(r - 2)}{\omega} \right) \left( \frac{N - l + d - g - k(r - 3) - \mu}{d + k + \mu - 1 - \omega} \right) \\
\left( \frac{1 + k(r - 2) - \omega}{l + 1 + k(r - 2) - \omega} \right) (l + 1 + k(r - 1) + d - \mu)
\]

\[
= \sum_{\omega=0}^r \left( \frac{l + 1 + k(r - 2)}{\omega} \right) \left( \frac{\mu - d - k}{\mu - d} \right) \left( \frac{N - l + d - g - k(r - 3) - \mu}{d + k + \mu - 1 - \omega} \right)
\]

with the final result

\[
F_{k,d} = \left( \frac{N - l - k(r - 2) - g}{N - l + 1 - k(r - 1) - g - \mu - d} \right) \left( \frac{l + 1 + k(r - 2)}{l + k(r - 1) - \mu - 1 + d} \right).
\]

In the “good” sector \( 2j + 1 < r - 1 \), where by (46) \( G_0(r) = G_0(r + 1) = g = 0 \) it is easy to show the recurrence relation (56) for term in (60) separately

\[
F_{k,1} - F_{k,0} = f_{k,1} - f_{k,0}.
\]

Thus the function \( \tilde{Z}(N,l,\mu,r) \) (84) fulfills the recurrence relation. But in order to use the initial conditions (57) and (58) we also have to consider the sector \( 2j + 1 \geq r - 1 \) where \( G_0(r) > 0 \) and \( G_0(r) = G_0(r + 1) + 1 \) which yields additional terms

\[
F_{k,1} - F_{k,0} = f_{k,1} - f_{k,0} + K_{k,1} - K_{k,0}
\]

with

\[
K_{k,d} = \left( \frac{N - l - k(r - 2) - G_0(r + 1) - d}{N - l + 1 - k(r - 1) - G_0(r + 1) - \mu} \right) \left( \frac{l + k(r - 2) + d - 1}{l + k(r - 1) - \mu} \right).
\]

The function \( \tilde{Z} \) (84) fulfills the recurrence relation also for \( 2j + 1 \geq r - 1 \) due to the following identity

\[
\sum_k (K_{k,1} - K_{k,0}) = 0.
\]
We have no analytic proof of this identity but we have verified it numerically for all possible cases for \( N < 60 \). It is easy to show that \( \tilde{Z} \) assume the initial conditions. Therefore, the function \( \tilde{Z}(N, l, \mu, r) = Z(N, l, \mu, r) \) is equal to the partial number of states defined in eq. (55).

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