SHARP GLOBAL WELL-POSEDNESS FOR A HIGHER ORDER SCHRÖDINGER EQUATION

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Abstract. Using the theory of almost conserved energies and the “I-method” developed by Colliander, Keel, Staffilani, Takaoka and Tao, we prove that the initial value problem for a higher order Schrödinger equation is globally well-posed in Sobolev spaces of order $s > 1/4$. This result is sharp.

1. Introduction

In this paper we will describe a sharp result of global well-posedness for solutions of the initial value problem (IVP)

$$\begin{cases}
\partial_t u + ia \partial_x^2 u + b \partial_x^3 u + ic |u|^2 u + d |u|^2 \partial_x u + e u^2 \partial_x \bar{u} = 0, & x, t \in \mathbb{R}, \\
u(x, 0) = \varphi(x),
\end{cases}$$

(1.1)

where $u$ is a complex valued function and $a, b, c, d$ and $e$ are real parameters with $be \neq 0$.

This model was proposed by Hasegawa and Kodama in [17, 21] to describe the nonlinear propagation of pulses in optical fibers. In literature, this model is called as a higher order nonlinear Schrödinger equation or also Airy-Schrödinger equation.

We consider the following gauge transformation

$$v(x, t) = \exp \left( i \frac{a}{3b} x + i \left( a \lambda^2 - 2b \lambda^3 \right) t \right) u(x + \left( 2a \lambda - 3b \lambda^2 \right) t, t),$$

(1.2)

then, $u$ solves (1.1) if and only if $v$ satisfies the IVP

$$\begin{cases}
\partial_t v + ia \partial_x^2 v + b \partial_x^3 v + i(c - \lambda(d - e)) |v|^2 v + d |v|^2 \partial_x v + e v^2 \partial_x \bar{v} = 0, \\
v(x, 0) = \exp(i \lambda x) u(x, 0).
\end{cases}$$

(1.3)

Thus, if we take $\lambda = a/3b$ in (1.2) and $c = (d - e)a/3b$, then the function

$$v(x, t) = \exp \left( i \frac{a}{3b} x + i \frac{a^3}{27b^2} t \right) u(x + \frac{a^2}{3b} t, t),$$

(1.4)

satisfies the complex modified Korteweg-de Vries type equation

$$\begin{cases}
\partial_t v + b \partial_x^3 v + d |v|^2 \partial_x v + e v^2 \partial_x \bar{v} = 0, \\
v(x, 0) = \exp(i a x/3b) u(x, 0).
\end{cases}$$

(1.5)

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Was shown in [22] that the flow associated to the IVP (1.1) leaves the following quantity
\[ I_1(u) = \int_{\mathbb{R}} |u|^2(x,t) \, dx, \] (1.6)
conserved in time. Also, when \( be \neq 0 \) we have the following conserved quantity
\[ I_2(u) = c_1 \int_{\mathbb{R}} |\partial_x u|^2(x,t) \, dx + c_2 \int_{\mathbb{R}} |u|^4(x,t) \, dx + c_3 \text{Im} \int_{\mathbb{R}} u(x,t) \partial_x u(x,t) \, dx, \] (1.7)
where \( c_1 = 3be, \ c_2 = -e(e + d)/2 \) and \( c_3 = (3be - a(e + d)) \). We may suppose \( c_3 = 0 \). In fact, when \( c_3 \neq 0 \) we can take in the gauge transformation (1.2)
\[ \lambda = -\frac{c_3}{6be}. \]
Then, \( u \) solves (1.1) if and only if \( v \) satisfies (1.3) and in this new IVP we have the constant \( c_3 = 0 \).

We say that the IVP (1.1) is locally well-posed in \( X \) (Banach space) if the solution uniquely exists in certain time interval \([-T,T]\) (unique existence), the solution describes a continuous curve in \( X \) in the interval \([-T,T]\) whenever initial data belongs to \( X \) (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e. continuity of application \( u_0 \mapsto u(t) \) from \( X \) to \( C([-T,T];X) \). We say that the IVP (1.1) is globally well-posed in \( X \) if the same properties hold for all time \( T > 0 \). If some hypotheses in the definition of local well-posed fail, we say that the IVP is ill-posed.

Particular cases of (1.1) are the following:

• Cubic nonlinear Schrödinger equation (NLS), \( (a = \mp1, \ b = 0, \ c = -1, \ d = e = 0) \).
\[ iu_t \mp u_{xx} + |u|^2 u = 0, \quad x,t \in \mathbb{R}. \] (1.8)
The best known local result for the IVP associated to (1.8) is in \( H^s(\mathbb{R}), \ s \geq 0 \), obtained by Tsutsumi [31]. Since the \( L^2 \) norm is preserved in (1.8), one has that (1.8) is globally well-posed in \( H^s(\mathbb{R}), \ s \geq 0 \).

• Nonlinear Schrödinger equation with derivative \( (a = -1, \ b = 0, \ c = 0, \ d = 2e) \).
\[ iu_t + u_{xx} + i\lambda(|u|^2u)_x = 0, \quad x,t \in \mathbb{R}. \] (1.9)
The best known local result for the IVP associated to (1.9) is in \( H^s(\mathbb{R}), \ s \geq 1/2 \), obtained by Takaoka [30]. Colliander et al. [10] they proved that (1.9) is globally well-posed in \( H^s(\mathbb{R}), \ s > 1/2 \).

• Complex modified Korteweg-de Vries (mKdV) equation \( (a = 0, \ b = 1, \ c = 0, \ d = 1, \ e = 0) \).
\[ u_t + u_{xxx} + |u|^2u_x = 0, \quad x,t \in \mathbb{R}. \] (1.10)
If \( u \) is real, (1.10) is the usual mKdV equation. Kenig et al. [19] proved that the IVP associated to it is locally well-posed in \( H^s(\mathbb{R}) \), \( s \geq 1/4 \) and Colliander et al. [11], proved that (1.10) is globally well-posed in \( H^s(\mathbb{R}) \), \( s > 1/4 \).

- When \( a \neq 0 \) is real and \( b = 0 \), we obtain a particular case of the well-known mixed nonlinear Schrödinger equation

\[
    u_t = iau_{xx} + \lambda(|u|^2)_{xx} + g(u), \quad x, t \in \mathbb{R},
\]

where \( g \) satisfies some appropriate conditions and \( \lambda \in \mathbb{R} \) is a constant. Ozawa and Tsutsumi in [24] proved that for any \( \rho > 0 \), there is a positive constant \( T(\rho) \) depending only on \( \rho \) and \( g \), such that the IVP (1.11) is locally well-posed in \( H^{1/2}(\mathbb{R}) \), whenever the initial data satisfies

\[
    \|v_0\|_{H^{1/2}} \leq \rho.
\]

There are other dispersive models similar to (1.1), see for instance [1, 8, 25, 26, 28] and the references therein.

Regarding the IVP (1.11), Laurey in [22] showed that the IVP is locally well-posed in \( H^s(\mathbb{R}) \) with \( s > 3/4 \), and using the quantities (1.6) and (1.7) she proved the global well-posedness in \( H^s(\mathbb{R}) \) with \( s \geq 1 \). In [27] Staffilani established the local well-posedness in \( H^s(\mathbb{R}) \) with \( s \geq 1/4 \), for the IVP associated to (1.11), improving Laurey’s result.

In the IVP (1.1), when \( a, b \) are real functions of \( t \), in [4, 6] was prove the local well-posedness in \( H^s(\mathbb{R}) \), \( s \geq 1/4 \). Also, in [17] was study the unique continuation property for the solution of (1.1).

**Remark 1.1.** 1) Using (1.4) and the results obtained in [11] we have that the PVI (1.1) is globally well-posed in \( H^s(\mathbb{R}) \) with \( s > 1/4 \), for initial data of the form:

\[
    \exp \left\{ -i \frac{\partial}{\partial x} \right\} v_0(x), \quad \exp \left\{ -i \frac{\partial}{\partial t} \right\} \{ v_0(x) + i v_0(x) \},
\]

where \( v_0 \in H^s \), \( s > 1/4 \), \( v_0 \in \mathbb{R} \). Therefore it suggests us to improve the result and obtain the global existence for the general case in \( H^s(\mathbb{R}) \), \( s > 1/4 \).

2) If \( e = 0 \), \( bd > 0 \) and \( c = (a/3b)d \) in (1.1), then the equation

\[
    \partial_t u + i a \partial_x^2 u + b \partial_x^3 u + i \frac{\alpha}{3b} d|u|^2 u + d|u|^2 \partial_x u = 0,
\]

(1.12)

have the following solution with two parameters

\[
    u_{\eta,N}(x,t) = f_{\eta}(x + \psi(\eta,N)t) \exp \left\{ N x + \phi(\eta,N)t \right\},
\]

(1.13)

where \( f_{\eta}(x) = \eta f(\eta x) \), \( f(x) = (A \cosh x)^{-1} \), \( A = \sqrt{d/(6b)} \), \( \psi(\eta,N) = 2aN + 3bN^2 - \eta^2 b \) and \( \phi(\eta,N) = aN^2 + bN^3 - 3\eta^2 bN - a\eta^2 \).
Using the transformation (1.4) we can obtain other family of solutions for

\[ \frac{\partial w + \partial_x^3 w + |w|^2 \partial_x w}{w(x,0) = w_0(x) = f_1(x) \exp i\{N x\} = \left(\frac{1}{\sqrt{6}} \cosh x\right)^{-1} \exp i\{N x\}, \] (1.14)

given by (1.13). If \( w \) is a solution of (1.14), then

\[ v(x,t) = \frac{1}{\alpha} w(b^{-1/3} x, t), \quad \alpha = \sqrt{\frac{d}{b^{1/3}}} \]

is a solution of

\[ \begin{cases} \partial_t v + b \partial^3 x v + d |v|^2 \partial_x v = 0, & x, t \in \mathbb{R}, \\
 v(x,0) = v_0(x), \end{cases} \] (1.15)

with initial data \( v_0(x) = (1/\alpha)w(b^{-1/3} x, 0) \) and if \( v \) is a solution of (1.15) then, using the transformation (1.4)

\[ u(x,t) = v(x - \frac{a^2}{3b} t, t) \exp \{2a^3/27b^2 - \frac{a}{3b} x\} \]

is a solution of (1.12) with initial data \( u_0(x) = v(x,0) \exp \{-i(a/3b)x\} \), therefore other solution of (1.12) with two parameters is

\[ u_{\eta,N}(x,t) = g_\eta(b^{-1/3} x + \psi(\eta,N)t) \exp \{ix(b^{-1/3} N - \frac{a}{3b}) + it \phi(\eta,N)\}, \] (1.16)

where \( g(x) = (\tilde{\alpha} \cosh x)^{-1}, \quad \tilde{\alpha} = \alpha/\sqrt{6}, \quad \phi(\eta,N) = 2a^3/(27b^2) - 3N\eta^2 + N^3 - N a^2 b^{-1/3}/(3b), \quad \psi(\eta,N) = -a^2 b^{-1/3}/(3b) - \eta^2 + 3N^2 \) and

\[ u_{\eta,N}(x,0) = u_{0\eta,N}(x) = g_\eta(b^{-1/3} x) \exp \{ix(b^{-1/3} N - \frac{a}{3b})\}. \]

When \( a = 0 \) and \( b = d = 1 \) in (1.12), this solution coincide with the solution obtained in [20].

3) If \( e \neq 0 \) and \( b(d + e) > 0 \), then (1.13) have solutions with one parameter:

\[ u_\eta(x,t) = g_\eta(x + \psi(\eta,w)t) \exp i\{wx + \phi(\eta,w)t\}, \]

where \( w = (c-2aA^2)/(2e), g_\eta(x) = \eta g(\eta x), g(x) = (A \cosh x)^{-1}, A = \sqrt{(e+d)/(6b)}, \psi \) and \( \phi \) as in (1.15).

We have also that if \( u \) is a solution of (1.4) then, \( v = \alpha u \) is a solution of (1.4), where \( \alpha \in \mathbb{C}, |\alpha| = 1 \), and if \( d \neq e \) in (1.4) then \( u(x,t) = \exp i\{Cx + Dt + C_0\} \) is a solution of (1.4), where \( D = aC^2 + bC^3 \) and \( C = c/(e - d) \).

Recently there appeared several papers devoted to the global solution of the dispersive type equation, where the framework is based on almost conserved laws and the I-method, see [9] [10] [11] [12] [13]. In this paper we adopt this way in order to obtain our results.

Our aim in this paper is to extend the local solution to a global one. Now, we state our main theorem of global existence:
Theorem 1.2. The IVP (1.1), with \( c = (d - e)a/3b \), is global well-posedness in \( H^s \), \( s > 1/4 \).

Notation. The notation to be used is mostly standard. We will use the space-time Lebesgue \( L^p_x L^q_t \) endowed with the norm

\[
\|f\|_{L^p_x L^q_t} = \left( \int_{\mathbb{R}} \left( \int_0^T |f(x,t)|^q dt \right)^{p/q} dx \right)^{1/p}.
\]

We will use the notation \( \|f\|_{L^p_t L^q_t} \) when the integration in the time variable is on the whole real line. In order to define the \( X_{s,\beta} \) spaces we consider the following IVP

\[
\begin{aligned}
u_t + i a \nu_{xx} + b \nu_{xxx} &= 0, & x, t \in \mathbb{R}, & b \neq 0, \\
u(0) &= u_0.
\end{aligned}
\]

whose solution is given by \( u(x, t) = U(t)u_0(x) \), where the unitary group \( U(t) \) is defined as

\[
\hat{U}(t)u_0(\xi) = e^{it(b\xi^3 + a\xi^2)}\hat{u}_0(\xi).
\]

For \( s, \beta \in \mathbb{R} \), \( X_{s,\beta} \) denotes the completion of the Schwartz space \( S(\mathbb{R}^2) \) with respect to the norm

\[
\|u\|_{s,\beta} \equiv \|u\|_{X_{s,\beta}} \equiv \|U(-t)u\|_{H_{s,\beta}} \equiv \langle \tau \rangle^\beta \langle \xi \rangle^s \left| \hat{U}(-t)u(\xi, \tau) \right|_{L^p_t L^q_t}_2 \\
= \left| \langle \tau - (b\xi^3 + a\xi^2) \rangle^\beta \langle \xi \rangle^s \hat{u}(\xi, \tau) \right|_{L^p_t L^q_t}_2,
\]

where

\[
\hat{u}(\xi, \tau) \equiv \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)}u(x, t)dx dt.
\]

For any time interval \([0, \rho]\), we define the space \( X^p_{s, b} \) by the norm

\[
\|u\|_{X^p_{s, b}} = \inf\{\|U\|_{X_{s, b}} : U|_{[0, \rho]\times \mathbb{R} = u}\}.
\]

The notation \( A \lesssim B \) means there exist a constant \( C \) such that \( A \leq CB \), and \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \). The notations \( \xi_{ij} \) means \( \xi_i + \xi_j \), \( \xi_{ijk} \) means \( \xi_i + \xi_j + \xi_k \), etc. Also we use the notation \( m(\xi_i) := m_i \), \( m(\xi_{ij}) := m_{ij} \), etc.

The notations for multilinear expressions is the same as in [9, 10]. We define a spatial n-multiplier to be any function \( M_n(\xi_1, \ldots, \xi_n) \) on the hyperplane

\[
\Gamma_n := \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n ; \xi_1 + \ldots + \xi_n = 0\},
\]

which we endow with the dirac measure \( \delta(\xi_1 + \ldots + \xi_n) \). We define the n-linear functional as

\[
\Lambda(M_n; f_1, \ldots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \ldots, \xi_n) \prod_{1}^{n} \hat{f}_j(\xi_j),
\]

where \( f_1, \ldots, f_n \) are complex functions on \( \mathbb{R} \). We shall denote

\[
\Lambda(M_n; f) := \Lambda(M_n; f, \overline{f}, \overline{f}, \ldots, \overline{f}).
\]
For $1 \leq j \leq n$, $k \geq 1$ we define the elongation $X_j^k(M_n)$ of $M_n$ to be the multiplier of order $n + k$ given by

$$X_j^k(M_n)(\xi_1, \ldots, \xi_{n+k}) := M_n(\xi_1, \ldots, \xi_{j-1}, \xi_j, \ldots, \xi_{j+k}, \xi_{j+k+1}, \ldots, \xi_{n+k}).$$

### 2. Almost Conservations Laws

From [14] we have

$$\begin{align*}
\partial_t w + i a \partial^2_x w + b \partial^3_x w + d (\partial_x w) \bar{w}w + e w (\partial_x \bar{w}) w &= 0, \\
\partial_t \bar{w} - i a \partial^2_x \bar{w} + b \partial^3_x \bar{w} - ic \bar{w}w + d (\partial_x \bar{w}) w + e \bar{w} (\partial_x w) \bar{w} &= 0.
\end{align*}$$

Taking Fourier transformation in the above equalities we obtain the following result

**Proposition 2.1.** Let $n \geq 2$, be an even integer, and let $M_n$ be a multiplier of order $n$, then

$$\begin{align*}
\partial_t \Lambda_n(M_n; w) &= i \Lambda_n(M_n Y_n^{a,b}; w) - i \Lambda_{n+2} \left( \sum_{j=1}^{n/2} X_{2j-1}^2(M_n) \xi_{2j-1} + \sum_{j=1}^{n/2} X_{2j}^2(M_n) \xi_{2j+2}; w \right), \\
&= -id \Lambda_{2n+2} \left( \sum_{j=1}^{n/2} X_{2j-1}^2(M_n) \xi_{2j-1} + \sum_{j=1}^{n/2} X_{2j}^2(M_n) \xi_{2j+2}; w \right), \quad (2.1)
\end{align*}$$

where $Y_n^{a,b} = \sum_{j=1}^n ((-1)^{j-1} a \xi_j^2 + b \xi_j^3)$ and $Y_j^{c,e} = (-1)^{j-1} c + e \xi_j + 1$.

We define the first modified energy as

$$E_1 = k_1 \Lambda_2(M_2; w), \quad M_2(\xi_1, \xi_2) = \xi_1 \xi_2 m(\xi_1)m(\xi_2), \quad (2.2)$$

where $k_1 = 3be$, and the second modified energy as

$$E_2 = E_1 + \Lambda_4(\delta_4), \quad (2.3)$$

where the 4-multiplier $\delta_4$ will be chosen after. By [24] we get

$$\begin{align*}
\partial_t E_2 = & \partial_t E_1 + \partial_t \Lambda_4(\delta_4) = k_1 \Lambda_2(M_2 Y_2^{a,b}) - ik_1 \Lambda_4 \left( \sum_{j=1}^{2} Y_{j,4}^c X_j^2(M_2) \right) \\
& - idk_1 \Lambda_4(X_1^2(M_2) \xi_1 + X_2^2(M_2) \xi_4) + i \Lambda_4(\delta_4 Y_4^{a,b}) + i \Lambda_6 \left( \sum_{j=1}^{4} Y_{j,6}^c X_j^2(\delta_4) \right) \\
& - id \Lambda_6 \left( \sum_{j=1}^{2} X_{2j-1}^2(\delta_4) \xi_{2j-1} + \sum_{j=1}^{2} X_{2j}^2(\delta_4) \xi_{2j+2} \right), \quad (2.4)
\end{align*}$$

it is clear that $\Lambda_2(M_2 Y_2^{a,b}) = 0$. Now if $\tilde{M}_n$ is an $n$-multiplier $\Lambda_n(\tilde{M}_n)$ is invariant under permutations of the even $\xi_j$ indices or of the odd $\xi_j$ indices, therefore for
achieve a cancellation of the 4-linear expression, we choose $\delta_4$ such that
\[
\Upsilon_4^{a,b}\delta_4 = \frac{ck_1}{2}(\xi_1^2m_1^2 - \xi_2^2m_2^2 + \xi_3^2m_3^2 - \xi_4^2m_4^2) - \frac{ck_1}{2}(\xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2)
\]
\[- \frac{dc_1}{2}(\xi_1\xi_4^2m_4^2 + \xi_4\xi_1^2m_1^2 + \xi_3\xi_2^2m_2^2 + \xi_2\xi_3^2m_3^2),
\] (2.5)
consequently from (2.6) we get
\[
\partial_t E_2 = \Lambda_6(\delta_6),
\] (2.6)
with
\[
\delta_6 = \frac{-ie}{36} \sum_{\{(k,m,o)\in\{1,3,5\}\}} \sum_{\{(l,n,p)\in\{2,4,6\}\}} [\xi_6\delta_4(\xi_{klm},\xi_{o},\xi_{p}) + \xi_m\delta_4(\xi_{k},\xi_{lmn},\xi_{o},\xi_{p}) + \xi_6\delta_4(\xi_{k},\xi_{l},\xi_{mno},\xi_{p})
\]
\[+ \xi_{o}\delta_4(\xi_{k},\xi_{l},\xi_{m},\xi_{nop})] - \frac{id}{36} \sum_{\{(k,m,o)\in\{1,3,5\}\}} \sum_{\{(l,n,p)\in\{2,4,6\}\}} [\xi_6\delta_4(\xi_{klm},\xi_{o},\xi_{p}) + \xi_m\delta_4(\xi_{k},\xi_{l},\xi_{mno},\xi_{p})
\]
\[+ \xi_{o}\delta_4(\xi_{k},\xi_{lmn},\xi_{o},\xi_{p}) + \xi_{p}\delta_4(\xi_{k},\xi_{l},\xi_{m},\xi_{nop})].
\]

**Proposition 2.2.** If $m(\xi) = 1$ for all $\xi$, then
\[
\partial_t E_2 = 0.
\]

**Proof.** From definition of $E_2$, we have
\[
E_2 = 3be\Lambda_2(\xi_1,\xi_2; w) + \Lambda_4(\delta_4; w),
\] (2.7)
where
\[
\Upsilon_4^{a,b}\delta_4 = \frac{ck_1}{2}(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2) - \frac{ck_1}{2}(\xi_2\xi_4^2 + \xi_4\xi_2^2 + \xi_1\xi_3^2 + \xi_3\xi_1^2)
\]
\[- \frac{dc_1}{2}(\xi_1\xi_4^2 + \xi_4\xi_1^2 + \xi_3\xi_2^2 + \xi_2\xi_3^2).
\]
If $\xi_1 + \cdots + \xi_4 = 0$, then $\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 = 2\xi_{12}\xi_{14}$ and $\xi_1^3 + \cdots + \xi_4^3 = 3\xi_{12}\xi_{13}\xi_{14}$, therefore
\[
\Upsilon_4^{a,b} = 2a\xi_{12}\xi_{14} + 3b\xi_{12}\xi_{13}\xi_{14} = \xi_{12}\xi_{14}(2a + 3b\xi_{13}).
\] (2.8)
On the other hand $\xi_2\xi_4^2 + \xi_4\xi_2^2 + \xi_1\xi_3^2 + \xi_3\xi_1^2 = -\xi_{12}\xi_{13}\xi_{14}$ (see Lemma 3.5 in [9] and Remark 3.6 in [10]), similarly $\xi_1\xi_4^2 + \xi_4\xi_1^2 + \xi_3\xi_2^2 + \xi_2\xi_3^2 = -\xi_{12}\xi_{13}\xi_{14}$, hence
\[
\delta_4 = \frac{3be}{2} \frac{2e\xi_{12}\xi_{14} + (d + e)\xi_{12}\xi_{13}\xi_{14}}{\xi_{12}\xi_{14}(2a + 3b\xi_{13})}
\]
\[= \frac{e(d + e)}{2}.
\]
And from (2.7) we get
\[
E_2 = 3be\Lambda_2(\xi_1, \xi_2; w) + \frac{e(d + e)}{2} \Lambda_4(1; w)
\]
\[
= -3be \int_\mathbb{R} |w_x|^2 + \frac{e(d + e)}{2} \int_\mathbb{R} |w|^4
\]
\[
= -I_2(w).
\]
This concludes the proof of the proposition. \(\square\)

In the following sections we will consider \(a = c = 0\) in the IVP (1.1) (see (1.4) and (1.5)).

3. Preliminary results

For the estimates on the multipliers we use the following elementary results.

**Lemma 3.1.** 1) *(Double mean value theorem DMVT)*
Let \(f \in C^2(\mathbb{R})\), and \(\max\{|\eta|, |\lambda|\} \ll \xi\), then
\[
|f(\xi + \eta + \lambda) - f(\xi + \eta) - f(\xi + \lambda) + f(\xi)| \lesssim |f''(\theta)||\eta||\lambda|,
\]
where \(|\theta| \sim |\xi|\).

2) *(Triple mean value theorem TMVT)*
Let \(f \in C^3(\mathbb{R})\), and \(\max\{|\eta|, |\lambda|, |\gamma|\} \ll \xi\), then
\[
|f(\xi + \eta + \lambda + \gamma) - f(\xi + \lambda + \eta) - f(\xi + \eta + \gamma) - f(\xi + \lambda + \gamma) + f(\xi + \eta)|
\]
\[
+ f(\xi + \lambda) + f(\xi + \gamma) - f(\xi)| \lesssim |f'''(\theta)||\eta||\lambda||\gamma|,
\]
where \(|\theta| \sim |\xi|\).

And for the proof of Proposition 5.2 shall be fundamental the improved Strichartz estimate.

**Lemma 3.2.** Let \(s > 1/4\), \(v_1, v_2 \in S(\mathbb{R} \times \mathbb{R})\) such that \(\text{supp} \widehat{v}_1 \subset \{ |\xi| \sim N \}\) and \(\text{supp} \widehat{v}_2 \subset \{ |\xi| \ll N \}\), then
\[
\|v_1 v_2\|_{L^4_t L^2_x} \lesssim \frac{1}{(1 - 4s)^{1/4}} \frac{1}{N} \|v_2\|_{X^s_{s,1/2+}} \|v_1\|_{X^c_{0,1/2+}}.
\]

**Proof.** As in [9] is sufficient to prove
\[
\|v_1 v_2\|_{L^4_t L^2_x} \lesssim \frac{1}{(1 - 4s)^{1/4}} \frac{1}{N} \|\phi\|_{H^s} \|\psi\|_{L^2},
\]
where \( v_1 = U(t) \psi \) and \( v_2 = U(t) \phi \). By duality, definition of \( v_1, v_2 \), Fubini theorem and Plancherel identity in the spatial variable we have

\[
\|v_1v_2\|_{L^4_x L^4_y} \lesssim \frac{1}{N^2} \sup_{\|F\|_{L^{4/3}_x L^{4/3}_y} \leq 1} \int_{R^2} |\hat{\phi}(y) \hat{\psi}(z) \hat{F}(z+y, \xi)|^2 dy dz
\]

where we used the change of variable \( z + y = s \), \( \xi = y^3 + \xi^3 \), which has Jacobian of size \( N^2 \). Now if we applying Hölder inequality and a change of variables back for \( z \) and \( y \), we obtain

\[
\|v_1v_2\|_{L^4_x L^4_y} \lesssim \frac{1}{N} \sup_{\|F\|_{L^{4/3}_x L^{4/3}_y} \leq 1} \|\hat{\phi}(y(s, r)) \hat{\psi}(z(s, r))\|_{L^{4/3}_x L^{4/3}_y} \|\hat{F}\|_{L^4_x L^4_y},
\]

where the Fourier transform of \( F \) is taking only in the space variable. Using Hölder inequality we obtain for \( s > 1/4 \)

\[
\int_{R} |\hat{\phi}|^{4/3} \leq \left( \int_{R} \langle \xi \rangle^{2s} |\hat{\phi}|^2 \right)^{2/3} \left( \int_{R} \frac{1}{\langle \xi \rangle^2} \right)^{1/3},
\]

therefore

\[
\|\hat{\phi}\|_{L^{4/3}} \lesssim \frac{1}{(1 - 4s)^{1/4}} \|\phi\|_{H^s},
\]

and by Hausdorff-Young inequality and Minkowsky integral inequality, we get

\[
\|\hat{F}\|_{L^4_x L^4_y} \leq \|\hat{F}\|_{L^4_x L^4_y} \leq \|F\|_{L^{4/3}_x L^{4/3}_y} \leq \|F\|_{L^{4/3}_x L^{4/3}_y} \leq 1.
\]

This completes the proof. \( \square \)

We define the Fourier multiplier operator \( I \) with symbol

\[
m(\xi) = \begin{cases} 1, & |\xi| < N, \\ \frac{N^{1-s}}{\xi^{1-s}}, & |\xi| > 2N. \end{cases}
\]

We have \( I : H^s \mapsto H^1 \). For the local result we define the Fourier multiplier operator \( L \), with symbol

\[
l(\xi) = m(\xi) \langle \xi \rangle^{1-s} = \begin{cases} \langle \xi \rangle^{1-s}, & |\xi| < N, \\ \langle \xi \rangle^{1-s} \frac{N^{1-s}}{|\xi|^{1-s}}, & |\xi| > 2N. \end{cases}
\]

Is obvious that

\[
\|Iu\|_{H^1} = \|Lu\|_{H^1}, \quad \|Iu\|_{X_{1,b}} = \|Lu\|_{X_{1,b}}, \quad (3.2)
\]

and for \( s \in [0, 1) \) is \( 1 \leq l(\xi) \lesssim N^{1-s} \), therefore

\[
\|u\|_{s', \nu} \lesssim \|Iu\|_{s'-s+1, \nu} \lesssim N^{1-s} \|u\|_{s', \nu}, \quad s \in [0, 1),
\]

\[
|y| \lesssim 1, \quad \|\phi\|_{H^N} \lesssim 1
\]
observe that if $V|_{[0,\rho] \times \mathbb{R}} = Iu$, $V \in X_{s'-s+1,b'}$, then $U$ defined by $\hat{U} = (1/m)\hat{V}$, satisfies

$$\|U\|_{s',b'} \lesssim \|V\|_{s'-s+1,b'},$$

moreover in $[0,\rho]$ is $U|_{[0,\rho] \times \mathbb{R}} = u$, therefore

$$\|u\|_{X^s_{s',b'}} \lesssim \|Iu\|_{X^s_{s'-s+1,b'}}. \quad (3.3)$$

Also we have

$$l(\xi_1 + \xi_2) \lesssim l(\xi_1) + l(\xi_2). \quad (3.4)$$

In fact, for see this, without lost of generality we can assume $|\xi_1| \geq |\xi_2|$, we consider two cases:

i) If $|\xi_1| \leq N$, then we have $|\xi_1 \pm \xi_2| \leq 2N$, this implies

$$l(\xi_1 + \xi_2) \sim (\xi_1 + \xi_2)^{1-s} \leq (\xi_1)^{1-s} + (\xi_2)^{1-s} = l(\xi_1) + l(\xi_2).$$

ii) If $|\xi_1| \geq N$, then we have $l(\xi_1) \sim N^{1-s}$, thus for all $\xi$, $l(\xi) \lesssim l(\xi_1)$, in particular

$$l(\xi_1 + \xi_2) \lesssim l(\xi_1) \leq l(\xi_1) + l(\xi_2).$$

In the proof of Theorem 1.2 we will use the following local result.

**Theorem 3.3.** Let $s \geq 1/4$, then the IVP (1.1) is locally well-posed for data $\varphi$, with $I\varphi \in H^1$ where the time of existence satisfies

$$\delta \sim \|I\varphi\|_{H^1}^{-\theta}, \quad (3.5)$$

with $\theta > 0$. Moreover the solution of the IVP (1.1), is such that

$$\|Iu\|_{X^s_{s,1/2^+}} \lesssim \|Iu_0\|_{H^1}. \quad (3.6)$$

**Proof.** The Theorem 3.3 is practically done in [29] (see also [32]), in fact, is sufficient to prove

$$\|L(u \varphi_{x})\|_{X^s_{s,-1/2^+}} \lesssim \|Lu\|_{X^s_{s,1/2^+}} \|Lv\|_{X^s_{s,1/2^+}} \|Lw\|_{X^s_{s,1/2^+}}, \quad (3.7)$$

$$\|L(u \varphi_{x})\|_{X^s_{s,-1/2^+}} \lesssim \|Lu\|_{X^s_{s,1/2^+}} \|Iv\|_{X^s_{s,1/2^+}} \|Lw\|_{X^s_{s,1/2^+}}, \quad (3.8)$$

in order to prove the first inequality we make the following decomposition

$$l(\xi)u\varphi_{x}(\xi,\tau) = l(\xi) \int_{|\xi_1| > 2N} \zeta + l(\xi) \int_{|\xi_1| \leq 2N} \zeta + l(\xi) \int_{|\xi_1| \leq 2N} \zeta$$

$$+ l(\xi) \int_{|\xi_1| \leq 2N} \zeta,$$
where $\zeta := \hat{u}(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2)\hat{v}(\xi_1, \tau_1)\hat{w}_x(\xi_2, \tau_2)$, thus
\[
|I(\xi)\hat{w}_x(\xi, \tau)| = |\hat{u}_1(\xi_1, \tau_1)| + |\hat{u}_2(\xi_2, \tau_2)| + |\hat{u}_3(\xi_3, \tau_3)| + |\hat{u}_4(\xi_4, \tau_4)|.
\]
with
\[
\begin{align*}
\hat{u}_1(\xi, \tau) &= \chi_{|\xi| > 2N}\hat{v}(\xi, \tau)|l(\xi), \\
\hat{u}_2(\xi, \tau) &= \chi_{|\xi| \leq 2N}\hat{v}(\xi, \tau), \\
\hat{v}_3(\xi, \tau) &= \chi_{|\xi| > 2N}\hat{w}_x(\xi, \tau)|l(\xi), \\
\hat{v}_4(\xi, \tau) &= \chi_{|\xi| \leq 2N}\hat{w}_x(\xi, \tau),
\end{align*}
\]
and applying Proposition 2.7 in [29] (see also Theorem 2.1 in [32]) we obtain (3.7).

For the other inequalities we make an analogous decomposition. \qed

We also have a result of local well-posed with the interval of existence as in (3.5), without the use of the theory of the spaces $X_{s,b}$.

In fact, let
\[
|||u|||_{\Delta T, s} = \|\partial_x u\|_{L^\infty_t L^2_x} + \|D_x^s \partial_x u\|_{L^\infty_t L^2_x} + \|D_x^{s-1/4} \partial_x u\|_{L^{4/7}_{1/3} L^8_{1/3}} + \|u\|_{L^1_t L^{16}_{16}}
\]
\[
+ \|D_x^s u\|_{L^2_t L^4_{1/2}} + \|u\|_{L^1_t L^\infty_{1/2}} + \|u\|_{L^4_t L^8_{1/2}} + \|D_x^s u\|_{L^2_t L^4_{1/2}}.
\]

Theorem 3.4. Let $s \geq 1/4$, and $a, b \in \mathbb{R}, b \neq 0, c, d, e \in \mathbb{C}$, then the IVP (1.1) is locally well-posed for data $\varphi$, with $I\varphi \in H^1$. Moreover the solution is such that
\[
|||u|||_{s, \delta} \lesssim ||I\varphi||_{H^1},
\]
where $\delta$ satisfies (3.9) with $\theta = 4$.

Proof. The theorem follows from the proof in [27] if we prove
\[
|||U(t)^u_0|||_{\delta, s} \lesssim ||Iu_0||_{H^1} = ||Lu_0||_{H^s},
\]
we consider the first term in the definition of $||\cdot||_{T}$, we will prove
\[
||\partial_x U(t)^u_0||_{L^\infty_t L^2_x} \lesssim ||Lu_0||_{L^2}.
\]
(3.9)
The inequality (3.9) is equivalent with
\[
||L^{-1}\partial_x U(t)^u_0||_{L^\infty_t L^2_x} \lesssim ||u_0||_{L^2},
\]
where the Fourier multiplier operator $L^{-1}$ have symbol $1/l(\xi) \leq 1$, it is easy to see that
\[
||L^{-1}\partial_x U(t)^u_0||_{L^\infty_t L^2_x} \leq ||\partial_x U(t)L^{-1}u_0||_{L^\infty_t L^2_x} \leq ||L^{-1}u_0||_{L^2} \leq ||u_0||_{L^2}.
\]
(3.10)
We proceed similarly with the others terms. \qed
Lemma 3.5. For any $s_1 \geq 1/4$, $s_2 \geq 0$ and $b > 1/2$ we have

\[
\|u\|_{L^4_t L^\infty_x} \lesssim \|u\|_{X^{s_1,b}}; \tag{3.11}
\]
\[
\|u\|_{L^4_t L^2_x} \lesssim \|u\|_{X^{s_2,b}}; \tag{3.12}
\]
\[
\|u\|_{L^4_t L^8_x} \lesssim \|u\|_{X^{s_0,b}}. \tag{3.13}
\]

Proof. The inequalities (3.11) and (3.12) follows from

\[
\|U(t)u_0\|_{L^4_t L^\infty_x} \lesssim \|u_0\|_{H^{1/4}}, \quad \|U(t)u_0\|_{L^4_t L^2_x} \lesssim \|u_0\|_{L^2}
\]

and from a standard argument, see for example [2, 10, 14].

The inequality (3.13) follows by interpolation between \(\|v\|_{L^2_t L^8_x} \lesssim \|v\|_{X^{0,1/2+}}\) and the trivial estimate \(\|v\|_{L^2_t L^2_x} \leq \|v\|_{0,0}\). \hfill \Box

Remark 3.6. Actually the inequality (3.13) is valid for all \(s > -1/4\) (see [5]).

4. Estimates for \(\delta_4\) and \(\delta_6\)

From here onwards we will consider the notation \(|\xi_i| = N_i, m(N_i) = m_i, |\xi_{ij}| = N_{ij}, m(N_{ij}) = m_{ij}\), etc. Given a number \(N_1, N_2, N_3, N_4, \mathcal{C} = \{N_1, \ldots, N_4\}\), we will note \(N_s = \max \mathcal{C}, N_a = \max \mathcal{C} \setminus \{N_s\}, N_t = \max \mathcal{C} \setminus \{N_s, N_a\}, N_b = \min \mathcal{C}\), in this way

\[N_s \geq N_a \geq N_t \geq N_b.\]

Proposition 4.1. Let \(m\) defined as in (3.1), then

\[
\|\delta_4\| \lesssim m^2(N_s) \tag{4.1}
\]

and

\[
\|\delta_6\| \lesssim N_s m^2(N_s). \tag{4.2}
\]

In order to prove (4.1) we will use the following proposition in similar form like when \(m = 1\).

Proposition 4.2. Let \(m\) defined as in (3.1), then

\[
|\xi_2 \xi_3^2 m_2^4 + \xi_4 \xi_2^2 m_2^2 + \xi_1 \xi_3^2 m_1^2 + \xi_3 \xi_1^2 m_1^2| \lesssim m^2(N_s)|\xi_2 \xi_3 \xi_1|, \tag{4.3}
\]

and

\[
|\xi_4 \xi_3^2 m_2^4 + \xi_4 \xi_1^2 m_2^2 + \xi_2 \xi_3^2 m_1^2 + \xi_3 \xi_2^2 m_1^2| \lesssim m^2(N_s)|\xi_2 \xi_3 \xi_1| \tag{4.4}
\]

Proof. Without lost of generality we can assume \(|\xi_1| = N_s\), and by symmetry \(|\xi_{12}| \leq |\xi_4|\). In [10] (Lemma 4.1) they proved that

\[
|\xi_2 \xi_3^2 m_2^4 + \xi_4 \xi_2^2 m_2^2 + \xi_1 \xi_3^2 m_1^2 + \xi_3 \xi_1^2 m_1^2| \lesssim m^2(N_s)|\xi_2 \xi_3 \xi_1| N_s,
\]
therefore we can suppose $|\xi_{13}| \ll N_s$, this implies $|\xi_1| \sim |\xi_3|$. Let $f(\xi) = \xi m(\xi)$, observing that $\xi_{12}\xi_{14} = \xi_2\xi_4 - \xi_1\xi_3$, we have
\[
\xi_2\xi_4 m_3^2 + \xi_4\xi_2 m_3^2 + \xi_1\xi_3 m_3^2 + \xi_3\xi_1 m_3^2 = \xi_2\xi_4(f(\xi_2) + f(\xi_4)) + \xi_1\xi_3(f(\xi_1) + f(\xi_3)) - \xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4)) - \xi_1\xi_3(f(\xi_1) + f(\xi_3)).
\] (4.5)

In the second term of (4.5) we can use the medium value theorem (MVT) for to obtain
\[
|\xi_{12}\xi_{14}(f(\xi_1) + f(\xi_3))| = |\xi_{12}\xi_{14}(f(\xi_1) - f(-\xi_3))| \lesssim |\xi_{12}\xi_{14}\xi_{13}| m^2(N_s),
\]
where we used that $|\xi_{13}| \ll N_s$, and $|f'(\xi_1)| \sim |m^2(\xi_1)|$. Therefore we will only estimate the first term in (4.5).

We consider two cases:
1) $|\xi_{14}| \gg |\xi_1|$, in this case we consider two sub-cases
a) If $|\xi_{12}| \ll |\xi_1|$, then using the DMVT (Lemma 3.1) with $\xi = -\xi_1$, $\lambda = \xi_{12}$ and $\eta = \xi_{13}$
\[
|\xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4))| \lesssim |\xi_{14}| N_s |\xi_{12}\xi_{13}| \frac{m^2(N_s)}{N_s},
\]
where we also used that $|\xi_2| \leq |\xi_1| \sim |\xi_3| \lesssim |\xi_{14}|$ and $|f''(\xi_1)| \lesssim m^2(\xi_1)/|\xi_1|$. 

b) If $|\xi_{12}| \gg |\xi_1|$, here we proceed similarly as in (10) (Lemma 4.1). Using the fact that $N_s \lesssim |\xi_{12}| \leq |\xi_{14}|$, $(m^2(\xi)f''(\xi) = m^2(\xi)\xi_1$, $m^2(\xi)\xi_3$ is nondecreasing and the MVT we have
\[
|\xi_2\xi_4 m_3^2 + \xi_4\xi_2 m_3^2 + \xi_1\xi_3 m_3^2 + \xi_3\xi_1 m_3^2| = |\xi_2 m_2^2 - m_1^2 - 13\xi_1^2| \lesssim m_3^2 \xi_3^2 (N_s) \xi_1^2.
\]
Hence we obtain (4.3) in this sub-case.

2) $|\xi_{14}| \ll |\xi_3|$, using the TMVT considering in Lemma 3.1 $\xi = -\xi_1$, $\lambda = \xi_{12}$, $\eta = \xi_{13}$ and $\gamma = \xi_{14}$ we have
\[
|\xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4))| \lesssim N_s^2 |\xi_{12}\xi_{13}| \frac{m^2(N_s)}{N_s},
\]
where we also used that $|f'''(\xi_1)| \lesssim m^2(\xi_1)/|\xi_1|^2$.

Now, in order to obtain (4.1), using (4.3) we get
\[
|\xi_1\xi_4^2 m_3^2 + \xi_4\xi_1^2 m_3^2 + \xi_2\xi_2^2 m_3^2 + \xi_3\xi_3^2 m_3^2| \lesssim m^2(N_s)|\xi_{21}\xi_{23}\xi_{24}| = m^2(N_s)|\xi_{12}\xi_{14}\xi_{13}|.
\]
This completes the proof.

By (3.6), (2.5) and Proposition 4.2 we have (4.1). The estimate (4.2) is obvious.
5. Estimates 4-lineal and 6-lineal

The following lemma will be used frequently in the estimates 4-lineal and 6-lineal.

**Lemma 5.1.** Let \( n \geq 2 \) an even integer, \( w_1, \ldots, w_n \in S(\mathbb{R}) \), then

\[
\int_{\xi_1 + \ldots + \xi_n = 0} \hat{w}_1 \hat{w}_2 \ldots \hat{w}_{n-1} \hat{w}_n = \int_{\mathbb{R}} w_1 w_2 \ldots w_{n-1} w_n. \quad (5.1)
\]

In the proof of our global result, we will need the following properties.

**Proposition 5.2.** Let \( w \in S(\mathbb{R} \times \mathbb{R}) \), then we have

\[
\left| \int_0^\rho \Lambda_6(\delta_6; w(t)) dt \right| \lesssim N^{-3} \| w \|_{H^\frac{1}{2},1}^6
\]

and

\[
|\Lambda_4(\delta_4; w(t))| \lesssim \| w \|_{H^1}. \quad (5.3)
\]

**Proof.** As in \([9, 10, 11]\), we first perform a Littlewood-Paley decomposition of the six factors \( w \), so that the \( \xi_i \) are essentially the constants \( N_i, i = 1, \ldots, 6 \). To recover the sum at the end we borrow a \( N^{-\epsilon} \) from the large denominator \( N \) and often this will not be mentioned. Also without loss of generality we can assume that the Fourier transforms in the left-side of (5.2) and (5.3) are real and nonnegative.

Let \( I = \{s, a, t, b\} \) the set of indices such that \( N_s \geq N_a \geq N_t \geq N_b \). We will proved first (5.2), we divide the proof into two cases.

1) \( N_b \gtrsim N \), by definition of \( m \) we have \( N_i m_i \gtrsim N \) and \( N_b m_b \gtrsim N \), therefore

\[
N_s m_s^2 \lesssim N^{-3} N_s m_s N_a m_a N_t m_t N_b m_b,
\]

and consequently by \([7, 11]\), Hölder inequality, \([8, 9]\) and Lemma \([3, 5]\) we have

\[
\left| \int_0^\rho \Lambda_6(\delta_6; w(t)) dt \right| \lesssim N^{-3} \int_0^\rho \prod_{j \in I} D_x^i I w_j \prod_{j \notin I} w_j dx dt \lesssim N^{-3} \prod_{j \in I} \| D_x^i I w_j \|_{L^2} \| w_j \|_{L^6} \lesssim N^{-3} \prod_{j \in I} \| I w_j \|_{X^{\frac{1}{2},1/2+}} \| I w_j \|_{X^{\frac{1}{2},1/2+}} \lesssim N^{-3} \| w \|_{H^\frac{1}{2},1}^6.
\]

2) \( N_b \ll N \), by \([26]\) and Proposition \([22]\) if \( N_s \ll N \), then \( \Lambda_6(\delta_6) = 0 \), therefore we can assume \( N_s \gtrsim N \), and for \( \xi_1 + \ldots + \xi_6 = 0 \) this implies \( N_s \sim N_a \gtrsim N \), hence

\[
N_s m_s^2 \lesssim N^{-1} N_s m_s N_a m_a,
\]
by \(5.1\), Hölder inequality, \(3.3\) and Lemmas \(3.3\) and \(3.2\) one obtains
\[
\left| \int_0^\rho \Lambda_6(\delta_6; w(t)) dt \right| \lesssim N^{-1} \int_\mathbb{R} D_x I w_a D_x I w_a \prod_{j \notin \{s,a\}} w_j dx \ dt
\]
\[
\lesssim N^{-1} \| (D_x I w_a) w_b \|_{L^4_x L^4_t} \| (D_x I w_a) w_p \|_{L^4_x L^4_t} \| w_t \|_{L^4_x L^2_t} \| w_q \|_{L^4_x L^\infty_t}
\]
\[
\lesssim N^{-3} \| I w \|_{X^{1/4,1/2}}^6.
\]
where \(1/4 < s_0 < 1\).

For to prove \(5.3\), by \(4.1\) and \(5.1\) we have
\[
|A_4(\delta_4; w(t))| \lesssim \int_{\xi_1 + \cdots + \xi_n} \delta_4(\xi_1, \ldots, \xi_n) \hat{w}_1 \hat{w}_2 \hat{w}_3 \hat{w}_4
\]
\[
\lesssim \int_\mathbb{R} |w(t)|^4 dx \lesssim \| w(t) \|_{H^{1/4}}^4
\]
\[
\lesssim \| I w \|_{H^{1/4}}^4.
\]
Which finished the proof. \(\Box\)

6. PROOF OF THEOREM 1.2

We will use the following results.

**Lemma 6.1.** If \(u\) is a solution of IVP \((1.1)\), then
\[
\| I u(t) \|_{L^2} \leq \| I \varphi \|_{H^{1-s}}.
\]
for \(0 \leq s < 1\).

**Proof.** The lemma follows from definition of \(I\), the conservation law in \(L^2\) and definition of \(l(\xi)\). \(\Box\)

**Lemma 6.2.** If \(u\) is a solution of IVP \((1.1)\), then
\[
|E_2(t) - E_1(t)| \leq c \| I \varphi \|_{H^1}^2 + c E_1(t)^4.
\]
If \(k\) is a positive integer and \(u(t)\) is defined in the time interval \([0,k]\), then
\[
E_2(k) = E_1(0) + A_4(\delta_4)(0) + \sum_{j=1}^k \int_{j-1}^j A_6(\delta_6)(t) dt.
\]
**Proof.** The inequality \(6.1\) is obvious from \(2.3\), \(3.3\) and Lemma \(6.1\).

By \(2.6\) we have
\[
E_2(k) = E_2(0) + \sum_{j=1}^k \int_{j-1}^j A_6(\delta_6)(t) dt.
\]
and by \(2.3\) we obtain \(6.2\). \(\Box\)
6.1. Rescaling. We know that if \( u(x,t) \) is a solution of (1.5) with initial data \( u(x,0) = \varphi \), then

\[
u_\lambda(x,t) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right),
\]
is also a solution of (1.5) with initial data

\[
u_\lambda(x,0) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, 0\right) = \frac{1}{\lambda} \varphi\left(\frac{x}{\lambda}\right) := \varphi_\lambda.
\]

Let \( c_0 \in (0,1) \) a constant to be chosen later, we have

\[
\|I\varphi_\lambda\|_{H^1} \lesssim \|\partial_x I\varphi_\lambda\|_{L^2} + \|I\varphi_\lambda\|_{L^2} \lesssim \frac{N^{1-s}}{\lambda^{2-s}} \|D_x^s \varphi\|_{L^2} + \frac{1}{\lambda^{2-s}} \|\varphi\|_{L^2} < c_0,
\]

taking

\[
\lambda \sim N^{2(1-s)/(1+2s)} \left(\frac{\|D_x^s \varphi\|_{L^2}}{c_0}\right)^{2/(1+2s)} \quad \text{and} \quad N > \left(\frac{\|\varphi\|_{L^2}}{c_0}\right)^{2s/(1-s)}. \tag{6.3}
\]

6.2. Iteration. Without lost of generality we can assume \( k_1 = 1 \) in (2.2). We consider our solution rescaled with initial data

\[
\|I\varphi\|_{H^1} = \epsilon_0 < c_0 < 1,
\]
then by Theorem 3.3 we have a solution of (1.1) defined in the time interval \([0, 1]\). For to extend the solution of local theorem in the time interval \([0, \lambda^3 T]\) we need to prove that \(\|Iu(n)\|_{H^1} \lesssim \epsilon_0\), for all \( n \in \{1, 2, \ldots, m_{\lambda,T}\} = W \), where \( m_{\lambda,T} \sim \lambda^3 T \).

Indeed we will prove that

\[
\|Iu(n)\|_{H^1}^2 \leq 3 \epsilon_0^2, \quad n \in W, \tag{6.4}
\]

but as \(\|Iu(t)\|_{H^1}^2 = \|Iu(t)\|_{L^2}^2 + \|\partial_x Iu(t)\|_{L^2}^2\), by Lemma 6.1 is sufficient to prove

\[
\|\partial_x Iu(n)\|_{L^2}^2 \leq 2 \epsilon_0^2, \quad n \in W. \tag{6.5}
\]

We will prove (6.5) by induction.

1) When \( k = 1 \), we suppose by contradiction that \(\|\partial_x Iu(1)\|_{L^2}^2 > 2 \epsilon_0^2\), then there exist \( t_0 \in (0,1) \) such that \(\|\partial_x Iu(t_0)\|_{L^2}^2 = 2 \epsilon_0^2\), from (6.1) we have

\[
|E_2(t_0) - 2 \epsilon_0^2| \leq 5c \epsilon_0^4,
\]

and using (2.8), (2.9), (3.0) and (5.2) we obtain

\[
E_2(t_0) = E_1(0) + \Lambda_4(\delta_4)(0) + \int_0^{t_0} \Lambda_6(\delta_6)(0),
\]

and from here

\[
|E_2(t_0) - \epsilon_0^2| \leq 5c \epsilon_0^4 + \frac{1}{N^3} 8c \epsilon_0^6.
\]
hence if \( \epsilon_0^2 < \frac{1}{2c} \), we have
\[
\epsilon_0^2 \leq |2\epsilon_0^2 - E_2(t_0)| + |E_2(t_0) - \epsilon_0^2| \leq 5c\epsilon_0^4 + 8c\epsilon_0^6 + 5c\epsilon_0^4 < \epsilon_0^2,
\]
but this is a contradiction.

2) Now, we suppose (6.5) for \( n = 1, 2 \ldots, k \), with \( k \geq 2 \) a positive integer, then we also will prove (6.5) for \( n = k + 1 \). In fact, in similar way as in case 1), we suppose by contradiction that \( \|\partial_x Iu(k+1)\|_{L^2}^2 > 2\epsilon_0^2 \), then there exist \( t_0 \in (0, k+1) \) such that \( \|\partial_x Iu(t_0)\|_{L^2}^2 = 2\epsilon_0^2 \). Similarly as in the case 1), from (6.1) we have
\[
|E_2(t_0) - 2\epsilon_0^2| \leq 5c\epsilon_0^4,
\]
by (2.6) and (6.2) we get
\[
|E_2(t_0) - E_1(0)| \leq |A_4(\delta_4)(0)| + \left| \sum_{j=1}^{t_0} \int_{t_{j-1}}^{t_j} A_6(\delta_6)(t) \, dt \right| + \left| \int_{t_0}^{t_0} A_6(\delta_6)(t) \, dt \right|,
\]
therefore by (3.6) and (5.2) we easily deduce that
\[
|E_2(t_0) - \epsilon_0^2| \leq 5c\epsilon_0^4 + (1 + |t_0|) \frac{8c}{N^3} \epsilon_0^6
\]
\[
\leq 5c\epsilon_0^4 + \lambda^3T \frac{8c}{N^3} \epsilon_0^6.
\]
As in the case \( k = 1 \), by (6.6) and (6.7) we obtain a contradiction if \( \lambda^3T \sim N^3 \), consequently we can to iterate this process \( m_{\lambda,T} \sim \lambda^3T \) times if \( T \sim \lambda^{-3}N^3 \) and by (6.3) if
\[
T \sim N(12s - 3)/(1 + 2s).
\]
Hence \( u \) is globally well-posed in \( H^s \) for all \( s > 1/4 \).

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