Faà di Bruno Hopf Algebra of the Output Feedback Group for Multivariable Fliess Operators

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Abstract
Given two nonlinear input-output systems written in terms of Chen-Fliess functional expansions, it is known that the feedback interconnected system is always well defined and in the same class. An explicit formula for the generating series of a single-input, single-output closed-loop system was provided by the first two authors in earlier work via Hopf algebra methods. This paper is a sequel. It has four main innovations. First, the full multivariable extension of the theory is presented. Next, a major simplification of the basic set up is introduced using a new type of grading that has recently appeared in the literature. This grading also facilitates a fully recursive algorithm to compute the antipode of the Hopf algebra of the output feedback group, and thus, the corresponding feedback product can be computed much more efficiently. The final innovation is an improved convergence analysis of the antipode operation, namely, the radius of convergence of the antipode is computed.

Key words: formal power series, functional series, Hopf algebras, output feedback, nonlinear systems

1. Introduction
Given two nonlinear input-output systems written in terms of Chen-Fliess functional expansions \[7\], it was shown in \[16, 20\] that the feedback interconnected system is always well defined and in the same class. An explicit formula for the generating series of a single-input, single-output (SISO) closed-loop system was later provided in \[11\] using Hopf algebra methods. In particular, the so-called feedback product of the two generating series for the component systems can be computed in terms of the antipode of a Faà di Bruno type Hopf algebra. This antipode was described in terms of a sequence of polynomials of increasing degree. While explicit, this somewhat brute force formula is not ideal for software implementation \[13\]. Nevertheless, this antipode can be used to provide a tractable formula for nonlinear system inversion from a purely input-output point of view, i.e., no state space model is required \[14, 15\].

This paper is a sequel to \[11\]. It has four main innovations. First, the full multivariable extension of the theory in \[11\] is presented, which makes it more relevant to practical control problems. The second innovation is more technical, but it greatly simplifies the basic set up. Specifically, it was shown recently in \[9\] that the Hopf algebra for the SISO output feedback group is connected under a grading that is distinct from the one described in \[11\]. This important observation implies that the bialgebra presented in the original paper is automatically a Hopf algebra, and therefore, much of the technical analysis concerning the existence of the antipode can now be omitted. So here the method in \[9\] is extended to the multivariable case and applied throughout. The third innovation is related to the existence of this new grading. Namely, the partially recursive formula for the antipode of any connected graded Hopf algebra in \[6\] is exploited here to produce a fully recursive antipode algorithm for the Hopf algebra of the output feedback group. This in turn allows one to compute the feedback product much more efficiently. The approach involves carefully combining results from \[6, 9\] and \[26\].

The SISO version of this algorithm was presented in \[13\] and compared against other existing methods. In a Mathematica implementation, this new algorithm provided an order of magnitude reduction in execution times. For the multivariable case, such gains are likely to be even larger, but this analysis is beyond the scope of this paper. The final innovation is an improved convergence analysis of the antipode operation, specifically, the radius of convergence of the antipode is computed using techniques presented in \[28\]. In \[11\] it was only shown that this radius of convergence is positive.

The paper is organized as follows. In the next section, some mathematical preliminaries and background are summarized, mainly for notational purposes. More complete treatments can be found in \[11, 12, 16, 28\]. In Sec-
tion the Hopf algebra of the multivariable output feedback group is presented, including the recursive algorithm for the antipode and the radius of convergence for this operation. In the subsequent section, these results are used to define the multivariable feedback product, and the corresponding convergence analysis is presented. The theory is demonstrated on a simple physical example. The conclusions are given in the final section.

2. Preliminaries

A finite nonempty set of noncommuting symbols \( X = \{x_0, x_1, \ldots, x_m\} \) is called an alphabet. Each element of \( X \) is called a letter, and any finite sequence of letters from \( X \) is called a word over \( X \). The length of \( \eta \), \( |\eta| \), is the number of letters in \( \eta \). Let \( |\eta|_2 \) denote the number of times the letter \( x_i \in X \) appears in the word \( \eta \). The set of all words including the empty word, \( \emptyset \), is designated by \( X^* \). It forms a monoid under catenation. Any mapping \( c : X^* \to \mathbb{R}^d \) is called a formal power series. The value of \( c \) at \( \eta \in X^* \) is written as \( (c, \eta) \) and called the coefficient of \( \eta \) in \( c \). Typically, \( c \) is represented as the formal sum \( c = \sum_{\eta \in X^*} (c, \eta) \). The collection of all formal power series over \( X \) is denoted by \( \mathbb{R}^d(X^*) \). It forms an associative \( \mathbb{R} \)-algebra under the catenation product and an associative and commutative \( \mathbb{R} \)-algebra under the shuffle product, which is uniquely specified by the shuffle product of two words: \( (x_i, \eta) \shuffle (x_j, \xi) = (x_i x_j, (x_i \eta) \shuffle (x_j \xi)), \)

where \( x_i, x_j \in X \), \( \eta, \xi \in X^* \) and with \( \emptyset \shuffle \emptyset = \emptyset \). Its restriction to polynomials over \( X \) is \( \text{sh} : \mathbb{R}(X) \otimes \mathbb{R}(X) \to \mathbb{R}(X) : p \otimes q \mapsto p \shuffle q \).

The corresponding adjoint map \( \text{sh}^* \) is the unique \( \mathbb{R} \)-linear map of the form \( \mathbb{R}(X) \to \mathbb{R}(X) \otimes \mathbb{R}(X) \) which satisfies the identity

\[ \text{sh}(p \otimes q, r) = (p \otimes q, \text{sh}^*(r)) \]

for all \( p, q, r \in \mathbb{R}(X) \). The following theorem states an important duality.

**Theorem 1.** [28] The adjoint map \( \text{sh}^* \) is an \( \mathbb{R} \)-algebra morphism for the catenation product \( \otimes \) : \( p \otimes q \mapsto pq \). That is,

\[ \text{sh}^*(pq) = \text{sh}^*(p) \text{sh}^*(q) \]

for all \( p, q \in \mathbb{R}(X) \) with \( \text{sh}^*(1) = 1 \otimes 1 \). In particular, for \( x_i \in X \) and \( \eta \in X^* \)

\[ \text{sh}^*(x_i \eta) = (x_i \otimes 1 + 1 \otimes x_i) \text{sh}^*(\eta) \].

One can formally associate with any series \( c \in \mathbb{R}^d(X^*) \) a causal \( m \)-input, \( \ell \)-output operator, \( F_c \), in the following manner. Let \( p \geq 1 \) and \( t_0 < t_1 \) be given. For a Lebesgue measurable function \( u : [t_0, t_1] \to \mathbb{R}^m \), define \( \|u\|_p = \max\{\|u_i\| : 1 \leq i \leq m\} \), where \( \|u\|_p \) is the usual \( L_p \)-norm for a measurable real-valued function, \( u \), defined on \([t_0, t_1]\). Let \( L^m_p[t_0, t_1] \) denote the set of all measurable functions defined on \([t_0, t_1]\) having a finite \( \| \cdot \|_p \) norm and \( B^m_p(R)[t_0, t_1] := \{u \in L^m_p[t_0, t_1] : \|u\|_p \leq R\} \). Assume \( C[t_0, t_1] \) is the set of continuous functions in \( L^m_1[t_0, t_1] \). Define inductively for each \( \eta \in X^* \) the map \( E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1] \) by setting \( E_\emptyset[u] = 1 \) and letting

\[ E_{x_i \eta}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_\eta[u](\tau, t_0) \, d\tau, \]

where \( x_i \in X \), \( \eta \in X^* \), and \( u_0 = 1 \). The input-output operator corresponding to \( c \) is the Fliess operator

\[ F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0) \]

and then \( F_c \) constitutes a well defined mapping from \( B^m_p(R)[t_0, t_0+T] \) into \( B^m_q(S)[t_0, t_0+T] \) for sufficiently small \( R, T > 0 \), where the numbers \( p, q \in [1, \infty) \) are conjugate exponents, i.e., \( 1/p + 1/q = 1 \).}

(Here, \( |z| := \max|z_i| \) when \( z \in \mathbb{R}^\ell \).) The set of all such locally convergent series is denoted by \( \mathbb{R}^d_{LC}(X) \). In particular, when \( p = 1 \), the series \( \{F_c\} \) converges absolutely and uniformly if \( \max\{R, T\} < 1/M_c(m+1) \). It is important in applications to identify the smallest possible geometric growth constant, \( M_c \), in order to avoid over restricting the domain of \( F_c \). So let \( \pi : \mathbb{R}^d_{LC}(X) \to \mathbb{R}^+ \cup \{0\} \) take each series \( c \) to the infimum of all \( M_c \) satisfying \( \{F_c\} \). Therefore, \( \mathbb{R}^d_{LC}(X) \) can be partitioned into equivalence classes, and the number \( 1/M_c(m+1) \) will be referred to as the radius of convergence for the class \( \pi^{-1}(M_c) \). This is in contrast to the usual situation where a radius of convergence is assigned to individual series. When \( c \) satisfies the more stringent growth condition

\[ |(c, \eta)| \leq K_c M_c^{\|\eta\|}, \quad \eta \in X^*, \]

the series \( \{F_c\} \) defines an operator from the extended space \( L^m_{p,c}(t_0) \) into \( C[t_0, t_1] \), where

\[ \mathbb{R}^m_{p,c}(t_0) := \{u : [t_0, t_1] \to \mathbb{R}^m : u|_{[t_0, t_1]} \in L^m_p[t_0, t_1] \}, \]

\[ \forall t_1 \in (t_0, \infty) \}

and \( u|_{[t_0, t_1]} \) denotes the restriction of \( u \) to \([t_0, t_1] \). Given Fliess operators \( F_c \) and \( F_d \), where \( c, d \in \mathbb{R}^\ell(X^*) \), the parallel and product connections satisfy \( F_c + F_d = F_{c+d} \) and \( F_c F_d = F_{c \odot d} \), respectively. When Fliess operators \( F_c \) and \( F_d \) with \( c \in \mathbb{R}^\ell(X^*) \) and \( d \in \mathbb{R}^m(X^*) \) are interconnected in a cascade fashion, the composite system
3. Hopf Algebra for Multivariable Output Feedback Group

Consider the set of operators \( \mathcal{F}_\delta := \{ I + F_c : c \in \mathbb{R}^m(\langle X \rangle) \} \), where \( I \) denotes the identity operator. It is convenient to introduce the symbol \( \delta \) as the (fictitious) generating series for the identity map. That is, \( F_\delta := I \) such that \( I + F_c := F_{\delta+c} = F_{c_\delta} \) with \( c_\delta := \delta + c \). The set of all such generating series for \( \mathcal{F}_\delta \) will be denoted by \( \mathbb{R}^m(\langle X \rangle) \). The first theorem describes the multivariable output feedback group which is at the heart of all the analysis in this paper. The group product is described in terms of the modified composition product of \( c \in \mathbb{R}^l(\langle X \rangle) \) and \( d \in \mathbb{R}^m(\langle X \rangle) \), namely,

\[
c \circ d = \sum_{\eta \in X^*} (c, \eta) \phi_d(\eta)(1),
\]

where \( \phi_d \) is the continuous (in the ultrametric sense) algebra homomorphism from \( \mathbb{R}(\langle X \rangle) \) to \( \text{End}(\mathbb{R}(\langle X \rangle)) \) uniquely specified by \( \phi_d(x, \eta) = \phi_d(x_1) \circ \phi_d(\eta) \) with \( \phi_d(x_1)(e) = x_1 e + x_0(d_i \omega e) \).

\[
i = 0, 1, \ldots, m \text{ for any } e \in \mathbb{R}(\langle X \rangle), \text{ and where } d_0 := 0, \text{ and } \phi_d(\emptyset) \text{ is the identity map on } \mathbb{R}(\langle X \rangle). \text{ It can be easily shown that for any } x_i \in X \]

\[
(x_i c) \circ d = x_i (c \circ d) + x_0(d_i \omega (c \circ d)). 
\]

The following (non-associativity) identity was proved in \(^2\)

\[
(c \circ d) \circ e = c \circ (d \circ e + e) 
\]

for all \( c \in \mathbb{R}^l(\langle X \rangle) \) and \( d, e \in \mathbb{R}^m(\langle X \rangle) \). The lemma below will be also useful. Its proof is deferred to Section 3, when all the appropriate tools are available.

**Lemma 1.** Let \( d \in \mathbb{R}^m(\langle X \rangle) \) be fixed. Then \( c \circ d = K \in \mathbb{R}^l \) if and only if \( e = K \).

The central idea is that \( \mathcal{F}_\delta \circ I \) forms a group of operators under the composition

\[
F_{c \circ d} = (I + F_c) \circ (I + F_d) = F_{c_\circ d},
\]

where \( c_\circ d := \delta + c \circ d \). Given the uniqueness of generating series of Fliess operators, this assertion is equivalent to the following theorem.

**Theorem 2.** The triple \( (\mathbb{R}^m(\langle X \rangle), \circ, \delta) \) is a group.

**Proof.** By design, \( \delta \) is the identity element of the group. The associativity of the product can be established in a manner similar to the SISO case addressed in \(^1\). (See \(^1\) for an alternative approach.) The existence of an inverse will be handled differently here (more directly) via Lemma 1. Specifically, for a fixed \( c_\delta \in \mathbb{R}^m(\langle X \rangle) \), the composition inverse, \( c_\delta^{-1} = \delta + c^{-1} \), must satisfy \( c \circ c_\delta^{-1} = \delta \) and \( c_\delta^{-1} \circ c_\delta = \delta \), which reduce, respectively, to

\[
c^{-1} = (\delta - c) \circ c^{-1} 
\]

\[
e = (-c^{-1}) \circ \delta. 
\]

It was shown in \(^1\) that \( e \mapsto (\delta - c) \circ e \) is always a contraction in the ultrametric sense on \( \mathbb{R}^m(\langle X \rangle) \) as a complete ultrametric space and thus has a unique fixed point. So it follows directly that \( c_\delta^{-1} \) is a right inverse of \( c_\delta \), i.e., satisfies \(^3\). To see that this same series is also a left inverse, observe that \(^6\) is equivalent to:

\[
c^{-1} \circ 0 + c \circ c^{-1} = 0
\]

\footnote{For notational convenience, \( c = K \emptyset \) is written as \( c = K \).}

\footnote{The same symbol will be used for composition on \( \mathbb{R}^m(\langle X \rangle) \) and \( \mathbb{R}^m(\langle X \rangle) \). As elements in these two sets have a distinct notation, i.e., \( e \) versus \( \delta \), respectively, it will always be clear which product is at play.}

Figure 1: Feedback connection
where $\eta$ is defined as $\eta = \text{id} \circ \delta - \delta \circ \text{id}$, which is the multivariable version of Proposition [9] and the specific nature of the factors $\delta^i$.

The second coproduct is $\hat{\Delta}(H) \subset V \otimes H$, which (following the notation of Sweedler [21]) is induced by the identity

$$\hat{\Delta} a^i_0 (c, d) = a^i_0 (c \hat{\otimes} d) = (c_i \hat{\otimes} d, \eta)$$

where $c, d \in \mathbb{R}^m \langle \{X\} \rangle$, $a^{(1)}_0 \in V$ and $a^{(2)}_0 \in H$. The summation is taken over all terms that appear in $(c_i \hat{\otimes} d, \eta)$, and the specific nature of the factors $a^{(1)}_0$ and $a^{(2)}_0$ is not important here. A key observation is that this coproduct can be computed recursively as described in the following lemma, which is the multivariable version of Proposition 3 in [4].

**Lemma 2.** The following identities hold:

1. $\hat{\Delta} a^0_0 = a^0_0 \otimes 1$
The Einstein summation notation is used in item (3) and throughout to indicate summations from either 0 or 1 to m, e.g., \( \sum_{i=1}^{m} a_i b_i = a_i b_i \). It will be clear from the context which lower bound is applicable.

The third coproduct is defined as \( \Delta a^i_0 = \Delta a^i_0 + 1 \otimes a^i_0 \). The coassociativity of \( \Delta \) follows from the associativity of the product \( c \circ d = c \circ d + d \). Specifically, for any \( c, d, e \in \mathbb{R}^m((X)) \),

\[
(\text{id} \otimes \Delta) \circ \Delta a^i_0(c, d, e) = (c_1 \otimes (d \circ e), \eta) = ((c \circ d) \otimes e, \eta) = (\Delta \otimes \text{id}) \circ \Delta a^i_0(c, d, e).
\]

Therefore, \( (\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta \) as required. This coproduct is used in the following central result.

**Theorem 3.** \((H, \mu, \Delta)\) is a connected graded commutative unital Hopf algebra.

**Proof.** From the development above, it is clear that \((H, \mu, \Delta)\) is a bialgebra. Here it is shown that this bialgebra is graded and connected. Therefore, \(H\) automatically has an antipode, and thus, is a Hopf algebra \([\text{[2]}]\). The antipode, \(S\), and the group (composition) inverse are necessarily related by

\[
a_{\eta} c_{\eta}^{-1} = (SA^i_{\eta}) c, \quad \forall \eta \in X^*, \quad i = 1, 2, \ldots, m.
\]

Specifically, it needs to be shown for any \( \eta \in X^* \) with \( \deg(a^i_{\eta}) = n \) that

\[
\Delta a^i_{\eta} \otimes (V \otimes H)_n := \bigoplus_{j+k+m=0} V_j \otimes H_k.
\]

This fact is evident from the first few terms computed via Lemma \([\text{[2]}]\).

\[
n = 1 : \Delta a^i_0 = a^i_0 \otimes 1
\]
\[
n = 2 : \Delta a^i_{x_0} = a^i_{x_0} \otimes 1 + a^i_{x_1} \otimes a^i_0
\]
\[
n = 3 : \Delta a^i_{x_0 x_1} = a^i_{x_0 x_1} \otimes 1 + a^i_{x_1 x_0} \otimes a^i_0 + a^i_{x_1 x_1} \otimes a^i_0
\]
\[
n = 4 : \Delta a^i_{x_0 x_1 x_2} = a^i_{x_0 x_1 x_2} \otimes 1 + a^i_{x_1 x_0 x_2} \otimes a^i_0 + a^i_{x_1 x_1 x_0} \otimes a^i_0
\]
\[
n = 5 : \Delta a^i_{x_0} = a^i_{x_0} \otimes 1 + a^i_{x_1 x_0} \otimes a^i_0 + a^i_{x_1 x_0} \otimes a^i_0 + a^i_{x_1 x_0} \otimes a^i_0
\]
\[
\]
\[
= \bigoplus_{j+k+m=0} V_j \otimes H_k.
\]

where \(i, j, k, l = 1, 2, \ldots, m\). In which case, via the identities \(\Delta(a^i_{\eta}) = \Delta a^i_{\eta} \otimes a^i_{\eta}\) and \(\Delta a^i_{\eta} = \Delta a^i_{\eta} + 1 \otimes a^i_{\eta}\) it follows that \(\Delta H_n \subseteq (H \otimes H)_n\), and this would complete the proof. To prove \([\text{[3]}]\), the following facts are essential:

1. \(\deg(\theta a^i_{\eta}) = \deg(a^i_{\eta}) + 1, \quad l = 1, 2, \ldots, m\)
2. \(\deg(\theta a^i_{\eta}) = \deg(a^i_{\eta}) + 2\)
3. \(\Delta a^i_{\eta} \in (V \otimes V)_{n+1}, \quad n = \deg(a^i_{\eta}).\)
0. Let \( n = \deg(a_{i}^{\prime}) \). There are two ways to increase the length of \( \eta \). First consider \( a_{i}^{\prime} \eta \) for some \( i \neq 0 \). From item 1 above \( \deg(a_{i}^{\prime} \eta) = n + 1 \), and from Lemma 2 \( \Delta \alpha_{x_{i} \eta} = (\theta_{i} \otimes \id) \circ \Delta a_{i}^{\prime} \). Therefore, using the induction hypothesis, \( \Delta a_{i}^{\prime} \eta \in \bigoplus_{j+k=n} V_{j+1} \otimes H_{k} \subset (V \otimes H)_{n+1} \), which proves the assertion. Consider next \( a_{2}^{\prime} \eta_{0} \). From item 2 above \( \deg(a_{2}^{\prime} \eta_{0}) = n + 2 \). Lemma 2 is employed as in the first case. First note that item 3 gives \( \Delta_{i}^{l} a_{j}^{\prime} \in (V \otimes V)_{n+1} \), and so using the induction hypothesis it follows that \( (\Delta \otimes \id) \circ \Delta a_{i}^{\prime} \in (V \otimes H \otimes V)_{n+2} \). In which case, \( \theta_{i} \otimes \id \circ \Delta_{i}^{l} a_{0}^{\prime} \in (V \otimes H)_{n+2} \). By a similar argument, \( \theta_{0} \otimes \id \circ \Delta a_{i}^{\prime} \in (V \otimes H)_{n+2} \). Thus, \( \Delta a_{2}^{\prime} \eta_{0} \in (V \otimes H)_{n+2} \), which again proves the assertion and completes the proof.

The deferred proof from Section 2 is presented next.

**Proof of Lemma 2.** The only non trivial claim is that \( c \in \mathbb{K} \) implies \( c = \mathbb{K} \). If \( c \in \mathbb{K} \) then clearly \( K_{i} = a_{i}^{\prime} \in \mathbb{K} \). Furthermore, for any \( x_{i} \in X \), \( c \in \mathbb{K} \). Therefore, \( a_{i}^{\prime} \in \mathbb{K} \). The antipode of any graded connected Hopf algebra can be computed as described in the following theorem. It can be viewed as being partially recursive in that the coproduct needs to be computed first before the antipode recursion can be applied.

**Theorem 4.** The antipode, \( S \), of any graded connected Hopf algebra \( (H, \mu, \Delta) \) can be computed for any \( a \in H_{k} \), \( k \geq 1 \) by

\[
Sa = -a - \sum_{a_{i}^{\prime} \eta(1)} a_{i}^{\prime} \eta(2) - a_{i}^{\prime}(1) Sa_{i}^{\prime}(2),
\]

where the reduced coproduct is defined as \( \Delta a = \Delta a - a \otimes 1 - 1 \otimes a = a_{i}^{\prime}(1) a_{i}^{\prime}(2) \). In addition, \( S(aa^{\prime}) = Sa Sa \).

The next theorem provides a fully recursive algorithm to compute the antipode for the output feedback group.

**Theorem 5.** The antipode, \( S \), of any \( a_{i} \in V_{k} \) in the output feedback group can be computed by the following algorithm:

i. Recursively compute \( \Delta_{i}^{l} a_{i}^{\prime} \) via 1.

ii. Recursively compute \( \Delta_{i} \) via Lemma 3.

iii. Recursively compute \( S \) via Theorem 4 with \( \Delta a_{i}^{\prime} \) equal to \( \Delta a_{i}^{\prime} = \Delta a_{i}^{\prime} - a_{i}^{\prime} \otimes 1 \).
where
\[
L_{g_i} h_i := L_{g_{i1}} \ldots L_{g_{iK}} h_i, \quad \eta = x_{jK} \ldots x_{j1},
\]
the Lie derivative of \( h_i \) with respect to \( g_j \) is defined as
\[
L_{g_j} h_i : W \to \mathbb{R} : z \mapsto \frac{\partial h_i}{\partial z} (g_j(z)),
\]
and \( L_{g_i} h_i = h_i \). It is not difficult to see that the composition inverse of the return difference operator \( I + H \), that is, \( (I + H)^{-1} = I + F_{c,-1} \) : \( u' \mapsto y' \), is described by the feedback system in Figure 2. A straightforward calculation gives a realization for \( F_{c,-1} \), namely, \( \{g_0 - \sum_{j=1}^{m} g_j h_j, g_1, \ldots, g_m, -h_i, z_0 \} \). Using this realization and (10), it can be readily verified that \( \textbf{S} \) holds. For example,
\[
\begin{align*}
(c^{-1}, x_0) &= L_{g_0} - \sum_{j=1}^{m} g_j h_j (-h_i)(z_0) \\
&= -L_{g_0} h_i(z_0) + \sum_{j=1}^{m} (L_{g_j} h_i(z_0)) h_j(z_0) \\
&= -(c_i, x_0) + \sum_{j=1}^{m} (c_i, x_j)(c_j, \emptyset) \\
&= (-a_{x_0}^i + a_{x_j}^i a_{x_j}^0) c \\
&= (\text{Sa}_x^i c).
\end{align*}
\]
In the special case of a linear time-invariant system with strictly proper \( m \times m \) transfer function \( H(s) \) and state space realization \((A, B, C)\), the corresponding components of the linear generating series are \( c_i = \sum_{k \geq 0} \sum_{j=0}^{m} c_{ij} x_{ij}^k \), where \( (c_i, x_{ij}^k) = C_i A^k B_j \), \( k \geq 0 \), and \( C_i, B_j \) denote the \( i \)-th row of \( C \) and the \( j \)-th column of \( B \), respectively. The composition inverse of return difference matrix \( I + H(s) \) is computed directly as
\[
(I + C(sI - A)^{-1}B)^{-1} = I - C(sI - (A - BC))^{-1}B.
\]
Therefore, it follows that
\[
(c^{-1}, x_0 x_j) = -C_i (A - BC)^k B_j, \quad k \geq 0, \quad i, j = 1, 2, \ldots, m.
\]
Expanding this product gives the expected antipode formulas. For example,
\[
(c^{-1}, x_0 x_j) = -C_i (A - BC) B_j
\]
where the fact that \( (c, x_i x_j) = (c, 0) = 0 \) has been used in the second to the last line. It is worth repeating that the antipode formulas derived at the beginning of this section required no state space setting. Hence, they still apply even when \( c \) does not have finite Lie rank.

The next theorem establishes that local convergence is preserved by the composition inverse operation. This fact was proved for the SISO case in [11] using only a grading of \( H \). But here a different approach is taken, one that produces the exact radius of convergence for the operation.

**Theorem 6.** For any \( c \in \mathbb{R}^n_{GC}(\langle X \rangle) \) with growth constants \( K_c, M_c \geq 0 \) it follows that
\[
||c^{-1}, \eta||_2 \leq K(A(K_c)M_c)^{|\eta|} ||\eta||, \quad \eta \in X^*, \quad (11)
\]
for some \( K > 0 \) and
\[
A(K_c) = \frac{1}{1 - mK_c \ln \left(1 + \frac{1}{mK_c}\right)}.
\]
Therefore, \( c^{-1} \in \mathbb{R}^n_{GC}(\langle X \rangle) \). Furthermore, no geometric growth constant smaller than \( A(K_c)M_c \) can satisfy (11), so the radius of convergence for this operator is \( 1/A(K_c)M_c(m + 1) \).

**Proof.** It was shown in [28, Corollary 2] that the generating series for the unity feedback system \( c \odot \delta \) has exactly the properties described above, and therefore, so does \( (-c) \odot \delta \). The present theorem is thus proved by showing that \( c^{-1} = (-c) \odot \delta \). Recall that in the proof of Theorem 2 it was shown in general that \( c^{-1} = (-c) \odot \delta \). But it is also known that \( (-c) \odot \delta \) satisfies the fixed point equation \( -c) \odot \delta = (-c) \odot ((-c) \odot \delta) \). Therefore, since \( c \mapsto (-c) \odot \delta \) is a contraction on a complete ultrametric space, the identity in question must hold.

It is worth noting that the growth constants determined in Theorem 6 must hold for every series \( c \) with growth constants \( K_c, M_c \). Thus, it tends to be conservative for specific series in this class (see [28] for further discussion on this topic). A similar approach yields the global counterpart of this theorem.

**Theorem 7.** For any \( c \in \mathbb{R}^n_{GC}(\langle X \rangle) \) with growth constants \( K_c, M_c \geq 0 \) it follows that
\[
||c^{-1}, \eta||_2 \leq K(B(K_c)M_c)^{|\eta|} ||\eta||, \quad \eta \in X^*, \quad (12)
\]
for some $K > 0$ and
\[ B(K_c) = \frac{1}{\ln \left(1 + \frac{c}{mK_c}\right)}. \]

Therefore, $c^{-1} \in \mathbb{R}_{LC}^m(\langle X \rangle)$. Furthermore, no geometric growth constant smaller than $B(K_c)M_c$ can satisfy \[\text{(12)},\] so the radius of convergence for this operator is $1/B(K_c)M_c(m + 1)$.

It is known that feedback does not in general preserve global convergence (see \[\text{[13]}\] for a specific example). Thus, there is no reason to expect that the composition inverse will do so either.

4. Feedback Product

The goal of this section is to derive an explicit formula for the multivariable feedback product $c@d$ using the Faà di Bruno Hopf algebra described in the previous section. Given two Fliess operators $F_c$ and $F_d$, which are linear time-invariant systems with $\ell_c \times m_c$ transfer function $G_c$ and $\ell_d \times m_d$ transfer function $G_d$, respectively, the closed-loop transfer function is clearly
\[ G_c(I - G_dG_c)^{-1} = G_c \sum_{k=0}^{\infty} (G_dG_c)^k, \tag{13} \]
where necessarily $\ell_c = m_d$ and $\ell_d = m_c$. There is no a priori requirement that the systems be square, that is, $m_c = \ell_c$ or $m_d = \ell_d$. But to handle the most general case here, the series composition products introduced in Sections \[\text{2} and \text{3} have to be generalized to accommodate two alphabets, $X_c = \{x_0, x_1, \ldots, x_m\}$ and $X_d = \{x_0, x_1, \ldots, x_m\}$. This offers no serious technical issues as described in \[\text{[17, Example 3.5]},\] just a bit more bookkeeping. The inverse is computed easily in this special case because all the underlying series are rational. The next theorem gives the nonlinear generalization of \[\text{(13)}\].

**Theorem 8.** For any $c \in \mathbb{R}_{m}^m(\langle X_c \rangle)$ and $d \in \mathbb{R}_{m}^m(\langle X_d \rangle)$, it follows that $c@d = c \circ (d \circ c)^{-1}$. \[\tag{14} \]

**Proof.** The proof is not significantly different from the SISO case presented in \[\text{[11]}\]. Since it is short, it is presented here for completeness. Clearly the function $v$ in Figure \[\text{4} must satisfy the identity
\[ v = u + F_{doc}[v]. \]
Therefore, \[ (I + F_{-doc})[v] = u, \]
where in the notation of Section \[\text{3} the operator on the left-hand side is an element of $\mathcal{F}_d$ with $m = m_c = \ell_d$.

Applying the composition inverse $(I + F_{(-doc)^{-1}})$ on the left gives
\[ v = (I + F_{(-doc)^{-1}})[u], \]
and thus, \[ F_{c@d}[u] = F_c[v] = F_c \circ (d \circ c)^{-1}[u] \]
as desired. The second identity in the theorem is a formal way of expressing the first identity since $c \circ (d \circ c)^{-1} = c \circ (\delta \circ (-d \circ c)^{-1})$ and by definition $(\delta \circ (-d \circ c)^{-1} = \delta + (-d \circ c)^{-1}$.

As noted earlier, \[\text{(14)}\] also makes sense when either $c = \delta$ or $d = \delta$, namely, $\delta \circ d = (\delta - d)^{-1} = \delta + (-d)^{-1}$ and $c \circ d = c \circ (\delta - c)^{-1} = (-c)^{-1}$. In addition, it was shown in \[\text{[14, Theorem 4.3]}\] that $c@d$ satisfies the fixed point equation $c@d = c \circ (d \circ d \circ (c@d))$. So if $c$ is a linear series then
\[ c@d = c + c \circ d \circ (c@d) \]
\[ (\delta - c \circ d) \circ (c@d) = c \]
\[ c@d = (\delta - c \circ d)^{-1} \circ c. \]
But in general, even in the SISO case, $c@d \neq (\delta - c \circ d)^{-1} \circ c$.

Next it is shown that feedback preserves local convergence. But the following preliminary result is needed first.

**Theorem 9.** The triple $(\mathbb{R}_{LC}^m(\langle X_s \rangle), \circ, \delta)$ is a subgroup of $(\mathbb{R}_{m}^m(\langle X_s \rangle), \circ, \delta)$.

**Proof.** The set of series $\mathbb{R}_{LC}^m(\langle X_s \rangle)$ is closed under composition since the set $\mathbb{R}_{LC}^m(\langle X \rangle)$ is closed under modified composition \[\text{[16, 22]}\]. In light of Theorem \[\text{3} the \mathbb{R}_{LC}^m(\langle X \rangle)$ is also closed under inversion. Hence, the theorem is proved.

**Theorem 10.** If $c \in \mathbb{R}_{LC}^m(\langle X_c \rangle)$ and $d \in \mathbb{R}_{LC}^m(\langle X_d \rangle)$ then $c@d \in \mathbb{R}_{LC}^m(\langle X_c \rangle)$.

**Proof.** Since the composition product, the modified composition product, and the composition inverse all preserve local convergence, the claim follows directly from Theorem \[\text{3}.\]

**Example 3.** Consider the differential axle shown in Figure \[\text{6} This device has zero mass and moves in the plane with independent angular velocities $u_t$ and $u_l$ corresponding to the right and left wheels, respectively. The dynamics of this system are
\[ \dot{z}_1 = \frac{r}{2}(u_l + u_r) \cos(z_3) \]
\[ \dot{z}_2 = \frac{r}{2}(u_l + u_r) \sin(z_3) \]
\[ \dot{z}_3 = \frac{l}{L}(u_r - u_l). \]
In particular, if $u_l = u_r > 0$ then the axle moves forward in the direction the wheels are pointing, and if $u_l = -u_r > 0$ the axle rotates clockwise because the wheels are turning in
opposite directions. For simplicity, define $u_1 = \frac{1}{2}(u_l + u_r)$
and $u_2 = (u_r - u_l)$, and let $L = r = 1$. Choosing outputs
$y_i = z_i$, $i = 1, 2$, the corresponding two-input, two-output
state space realization is

\[
\begin{align*}
\dot{z}_1 &= \cos(z_3) u_1 + 0 u_2 \\
\dot{z}_2 &= \sin(z_3) u_1 + 1 u_2 \\
y_1 &= z_1 \\
y_2 &= z_2.
\end{align*}
\]

Its generating series, $c$, can be computed directly from
using the vector fields and output function given in
with the help of the Mathematica software package
NCAlgebra, this calculation gives

\[
c = \left( x_1 - x_1 x_2^2 + x_1 x_3^4 - x_1 x_2^6 + x_1 x_2^8 - x_1 x_2^{10} + \cdots \\
x_1 x_2 - x_1 x_2^2 + x_1 x_3^6 - x_1 x_2^3 + x_1 x_2^5 - x_1 x_2^{11} + \cdots \right)
\]

when $z_3(0) = z_3(0) = 0$. This series is clearly
in $\mathbb{R}_\infty((X))$ with $X = \{x_0, x_1, x_2\}$ and growth constants $K_c = 1$ and $M_c = 1$.

Consider now the problem of steering the differential
axle around a circle. For this purpose, a two channel
proportional-integral controller is used in the feedback
path so that one obtains a closed-loop system as shown in
Figure 3. The dynamics of the controller are

\[
\begin{align*}
\dot{z}_4 &= 0 u_1 + 0 u_2 \\
\ddot{z}_5 &= k_1 z_4 \\
\ddot{y}_1 &= k_2 z_5.
\end{align*}
\]

For gains $k_1 = 2$ and $k_2 = 10$, the corresponding generating
series is

\[
d = \left( \begin{array}{c} 4 + 2 x_1 \\ 20 + 10 x_2 \end{array} \right)
\]

when $z_3(0) = z_3(0) = 2$. Here $d \in \mathbb{R}_\infty((X))$ with growth constants $K_d = 20$ and $M_d = 0.5$.

From Theorem 5 the feedback product of $c$ and $d$ is
given by the series

\[
\begin{align*}
(c \circ d)_1 &= 4 x_0 + x_1 - 1,592 x_3^4 + 2 x_1^2 x_2 - 80 x_0 x_0 x_2 - 80 x_0 x_2 x_2 - 4 x_0 x_2 x_2 - 400 x_0 x_0 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2 - 20 x_1 x_2 x_2
\end{align*}
\]

and

\[
\begin{align*}
(c \circ d)_2 &= 80 x_0^2 + 4 x_0 x_2 + 20 x_1 x_2 - x_1 x_2 - 31,520 x_0^2 + 80 x_2^3 - 80 x_0^2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2 - 400 x_0 x_0 x_2 x_2
\end{align*}
\]

These series were computed up to degree nine using a
current off-the-shelf personal computer. The outputs of
the closed-loop system for a zero reference input are then

\[
y_1(t) = F_{(c \circ d)_1}(0)(t)
\]
and

\[ y_2(t) = F_{(c \circ d)}[0](t) = 40t^2 - \frac{3940t^4}{3} + \frac{80t^5}{3} + \frac{49340t^6}{3} - \frac{254560t^7}{63} - \frac{5496593t^8}{63} + \frac{25525609t^9}{189} + \cdots \]

Various estimates of the natural response of the closed-loop differential axle system are shown in Figure 4. (The tick marks along the circle indicate time.) Specifically, the numerically computed nonlinear closed-loop response of (15) is compared against Fliess operator responses whose generating series are computed from the feedback product truncated to degrees 7, 8 and 9. Also shown in Figure 4 is the response of the small angle approximation system

\[
\begin{align*}
\dot{z}_1 &= 1 - 25t^3 + 400t^6 - 13528 + 79879t^7 + \cdots \\
\dot{z}_2 &= \frac{25734t^5}{3} - \frac{9}{5}t^9 + \frac{1653800t^{10}}{63} + \frac{59450001t^{11}}{5670} + \cdots \\
\dot{z}_3 &= 0 \\
y_1 &= \frac{0}{1}z_1 + \frac{0}{1}z_2 \\
y_2 &= \frac{z_1}{z_2}
\end{align*}
\]

steered by the same proportional-integral controller. It is evident that this system underestimates the correct position of the differential axle almost immediately. On the other hand, the Fliess operator approximations clearly improve as additional terms are added to the approximation.

One way to get some insight into the convergence characteristics of \(F_{(c \circ d)}[0]\) is to empirically estimate the geometric growth constants for the natural response portion of the series \(c \circ d\), i.e., the series \(\sum_{k \geq 0}(c \circ d)_N \cdot \bar{x}^k \cdot \bar{x}^0\), by plotting \(\ln(k(c \circ d)_N / |y|)\) versus \(|y|\) for the local case and \(\ln((c \circ d)_N)\) versus \(|y|\) for the global case as shown in Figure 5. To improve the quality of the estimates, coefficients above order nine were computed using (14). In each case, the corresponding growth constant can be estimated by linearly fitting the data (see (12) for more discussion concerning this methodology). The parameter \(R^2\) is the square of Pearson’s correlation coefficient, so the closer this statistic is to unity, the better the linear fit. In this case, the data appears to match better the global growth rate with \(M_{(c \circ d)} = \exp(3.1157) = 22.549\). But as will be discussed shortly, the series can fall somewhere in between being locally convergent and globally convergent as defined by (2) and (3), respectively. There is also the option of constructing a piecewise analytic approximation of the response using a sequence of closed-loop generating series computed by brute force or via analytic extension (24). This approach has the additional advantage that lower order approximations of each piece are likely to suffice. For example, it appears here that two degree nine approximations joined at the midpoint of the path could easily traverse the entire circle.

An alternative method to exploring the nature of the convergence of \(F_{(c \circ d)}[0]\) is to use Theorems 6 or 7 in conjunction with what is known at present about the convergence of interconnected Fliess operators as reported in (28). First observe that \((-d \circ c, \eta) = \pm(\frac{2}{10})\) when \(\eta \neq 0\), and therefore, \(-d \circ c\) is globally convergent with \(K_{doc} = 20\) and \(M_{doc} = 1\). Applying Theorem 7 provides an upper bound on the local geometric growth constant of \(e := (-d \circ c)^{-1}\), specifically,

\[
M_e = \frac{M_{doc}}{\ln \left(1 + \frac{1}{2K_{doc}}\right)} = 40.50.
\]

Repeating the empirical method used above for \(e\) gives the data shown in Figure 5. It indicates that \(e\) is more globally convergent in nature than locally convergent, but if it were assumed to be locally convergent, the corresponding geometric growth constant would be \(\exp(1.0253) = 2.7879 < 40.50\). On the other hand, if it were taken to be globally convergent then (since \(F_{doc}[0] = F_{coc}[0]\) it follows that \((c \circ d)_N = c \circ e\) is the composition of two globally convergent series. As discussed in (28, p. 2800), the resulting series can lie strictly in between locally and globally convergent. But independent of this fact, it is still known in this instance that the series \(F_{coc}[0]\) will converge over any finite interval (28, Theorem 9).

5. Conclusions

The main thrust of this paper was to provide the full multivariable extension of a theory to explicitly compute the generating series of a feedback interconnection of two systems represented as Fliess operators. This was largely
facilitated by utilizing a new type of grading for the underlying Hopf algebra. This grading also provided a fully recursive algorithm to compute the antipode of the algebra and thus, the corresponding feedback product can be computed much more efficiently. Finally, an improved convergence analysis of the antipode operation was presented, one that gives the radius of convergence for this operation.

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