Abstract

In this paper we consider an interacting Bose gas at zero temperature, in a finite box and in the mean field limiting regime. The \( N \) gas particles interact through a pair potential of positive type and with an ultraviolet cut-off. Its (nonzero) Fourier components are sufficiently large with respect to the corresponding kinetic energies of the modes. Using the multi-scale technique in the occupation numbers of particle states introduced in [Pi1], we provide a convergent expansion of the ground state of the \textit{particle number preserving} Bogoliubov Hamiltonian in terms of the bare operators. In the limit \( N \to \infty \) the expansion is up to any desired precision.

Summary of contents

- In Section 1, the model of a Bose gas is defined along with the notation used throughout the paper. In particular, the \textit{particle number preserving} Bogoliubov Hamiltonian (from now on Bogoliubov Hamiltonian) is defined.

- In Section 2, we review the main ideas and results of the multi-scale analysis in the occupation numbers of particle states implemented for the three-modes Bogoliubov Hamiltonian in [Pi1]. In fact, the full Bogoliubov Hamiltonian can be thought of as a collection of three-modes systems, and we can iteratively apply the multi-scale analysis to them. The corresponding Feshbach-Schur flows are described informally in Section 2.1.

- In Section 3, in the mean field limiting regime (i.e., at fixed box volume \(|\Lambda|\), \( N \) sufficiently large, and for a coupling constant inversely proportional to the particle density) a convergent expansion of the ground state vector of the Bogoliubov Hamiltonian is provided as a byproduct of subsequent Feshbach-Schur flows, each of them associated with a couple of (interacting) modes.

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1 Interacting Bose gas in a box

We study a gas of (spinless) nonrelativistic Bose particles that, at zero temperature, are constrained to a $d$-dimensional box of side $L$ with $d \geq 1$. The particles interact through a pair potential with a coupling constant proportional to the inverse of the particle density $\rho$. The rigorous description of this system has many intriguing mathematical aspects not completely clarified yet. In spite of the progress in recent years, some important problems are still open to date, in particular in connection to the thermodynamic limit and the exact structure of the ground state vector. In this paper we focus on the problem of the construction of the ground state vector for the gas system in the mean field limiting regime. For the convenience of the reader, we recall some of the results closer to our present work.

The low energy spectrum of the Hamiltonian in the mean field limit was predicted by Bogoliubov [Bo1], [Bo2]. As for the ground state energy of a Bose gas, rigorous estimates have been provided for certain systems in [LS1], [LS2], [ESY],[YY], and [LSSY]. Concerning the excitation spectrum, in Bogoliubov theory it consists of elementary excitations whose energy is linear in the momentum for small momenta. After some important results restricted to one-dimensional models (see [G], [LL], [L]), this conjecture was proven by Seiringer in [Se1] (see also [GS]) for the low-energy spectrum of an interacting Bose gas in a finite box and in the mean field limit, where the pair potential is of positive type. In [LNSS] it has been extended to a more general class of potentials and the limiting behavior of the low energy eigenstates has been studied. Later, the result of [Se1] has been proven to be valid in a sort of diagonal limit where the particle density and the box volume diverge according to a prescribed asymptotics; see [DN]. Recently, Bogoliubov’s prediction on the energy spectrum in the mean field limit has been shown to be valid also for the high energy eigenvalues (see [NS]).

These results are based on energy estimates starting from the spectrum of the corresponding Bogoliubov Hamiltonian.

A different approach to studying a gas of Bose particles is based on renormalization group. In this respect, we mention the paper by Benfatto, [Be], where he provided an order by order control of the Schwinger functions of this system in three dimensions and with an ultraviolet cut-off. His analysis holds at zero temperature in the infinite volume limit and at finite particle density. Thus, it contains a fully consistent treatment of the infrared divergences at a perturbative level. This program has been later developed in [CDPS1], [CDPS2], and, more recently, in [C] and [CG] by making use of Ward identities to deal also with two-dimensional systems where some partial control of the renormalization flow has been provided; see [C] for a detailed review of previous related results.

Within the renormalization group approach, we also mention some results towards a rigorous construction of the functional integral for this system contained in [BFKT1], [BFKT2], and [BFKT].

Important results concern Bose-Einstein condensation for a system of trapped Bose particles interacting each other via a potential of the type $N^{3\beta-1} v(N^{\beta} x)$, with $0 \leq \beta \leq 1$, that for $\beta = 1$ corresponds to the so called Gross-Pitaevskii limiting regime: see [LSY], [LS], [LSSY], [NRS], and [LNR].

Both in the grand canonical and in the canonical ensemble approach (see [Se1]), starting from the Hamiltonian of the system one can define an approximated one, the Bogoliubov Hamiltonian. For a finite box and a large class of pair potentials, upon a unitary transformation\(^1\)

\(^1\)In the canonical ensemble approach, the used unitary transformation yields the diagonalization of the (particle preserving) Bogoliubov Hamiltonian only in the mean field limit (see [Se1]).
the Bogoliubov Hamiltonian describes a system of non-interacting bosons with a new energy dispersion law, which is in fact the correct description of the energy spectrum of the Bose particles system in the mean field limit.

In the companion paper [Pi1], we construct the ground state of so called three-modes Bogoliubov Hamiltonians by means of a multi-scale analysis in the particle states occupation numbers. Here, we explain how the new multi-scale scheme can be applied to a Bogoliubov Hamiltonian involving a finite number of interacting modes. More precisely, we implement the multi-scale scheme in the mean field limiting regime and under the strong interaction potential assumption considered in [Pi1] where the interaction potential is strong with respect to the kinetic energy of the interacting modes; see Definition 1.1. The results of the present paper are preliminary ingredients for the construction and expansion of the ground state of the complete Hamiltonian of the system in [Pi3].

We recall the Hamiltonian describing a gas of (spinless) nonrelativistic Bose particles that are constrained to a $d$-dimensional box, $d \geq 1$, and interact through a pair potential. Though the number of particles is fixed we use the formalism of second quantization. The Hamiltonian corresponding to the pair potential $\phi(x - y)$ and to the coupling constant $\lambda > 0$ is

$$
\mathcal{H} := \int \frac{1}{2m}(\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2} \int \int a^*(x)a^*(y)\phi(x - y)a(x)a(y)dxdy, \quad (1.1)
$$

where: $\hbar$ has been set equal to 1; reference to the integration domain $\Lambda := \{x \in \mathbb{R}^d | |x_i| \leq L/2, \text{ for } i = 1, \ldots, d\}$ is omitted and periodic boundary conditions are assumed; $dx$ is the Lebesgue measure in $d$ dimensions. Here, the operators $a^*(x), a(x)$ are the usual operator-valued distributions on the bosonic Fock space

$$
\mathcal{F} := \Gamma \left( L^2(\Lambda, \mathbb{C}; dx) \right)
$$

that satisfy the canonical commutation relations (CCR)

$$
[a^#(x), a^#(y)] = 0, \quad [a(x), a^*(y)] = \delta(x - y)\mathbb{1},
$$

with $a^# := a$ or $a^*$. In terms of the field modes, they read

$$
a(x) = \sum_{j \in \mathbb{Z}^d} a_j e^{ik_j \cdot x}, \quad a^*(x) = \sum_{j \in \mathbb{Z}^d} a_j^* e^{-ik_j \cdot x},
$$

where $k_j := \frac{2\pi j}{L}, j = (j_1, \ldots, j_d)$, $j_1, \ldots, j_d \in \mathbb{Z}$, and $|\Lambda| = L^d$, with CCR

$$
[a_j^#, a_j^#] = 0, \quad [a_j, a_j^*] = \delta_{j,j'}, \quad a_j^# = a_j \text{ or } a_j^*.
$$

The (nondegenerate) vacuum vector of $\mathcal{F}$ is denoted by $\Omega (||\Omega|| = 1)$.

Given any function $\varphi \in L^2(\Lambda, \mathbb{C}; dx)$, we express it in terms of its Fourier coefficients $\varphi_j$, i.e.,

$$
\varphi(z) = \frac{1}{|\Lambda|} \sum_{j \in \mathbb{Z}^d} \varphi_j e^{ik_j \cdot z}. \quad (1.2)
$$

**Definition 1.1.** The potential $\phi(x - y)$ is a bounded, real-valued function that is periodic, i.e., $\phi(z) = \phi(z + jL)$ for $j \in \mathbb{Z}^d$, and satisfies the following conditions:

1. $\phi(z)$ is an even function, in consequence $\phi_j = \phi_{-j}$. 

3
2. \( \phi(z) \) is of positive type, i.e., the Fourier components \( \phi_j \) are nonnegative.

3. The pair interaction has a fixed but arbitrarily large ultraviolet cutoff (i.e., the nonzero Fourier components \( \phi_j \) form a finite set \( \{\phi_0, \phi_{\lambda_1}, \ldots, \phi_{\lambda_M}\} \)) with the requirement below to be satisfied:

\[
\text{(Strong Interaction Potential Assumption)} \quad \text{The ratio} \quad \xi_j \quad \text{between the kinetic energy of the modes} \quad \pm j \neq 0 \quad \text{and the corresponding Fourier component,} \quad \phi_j(\neq 0), \quad \text{of the potential, i.e.,} \quad \frac{k^2}{\phi_j} =: \xi_j, \quad \text{is required to be small enough to ensure the estimates used in} \ [Pt1].
\]

**Remark 1.2.** Other regimes for the ratios \( \xi_j \) can be explored with the same method by suitable modifications of some estimates (see \([CP]\)).

We restrict \( \mathcal{H} \) to the Fock subspace \( \mathcal{F}^N \) of vectors with \( N \) particles

\[
\mathcal{H} \uparrow_{\mathcal{F}^N} = \left( \int \frac{1}{2m} (\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2} \int \int \bar{a}^*(x) a^*(y) \phi(x-y) \phi(y) a(x) dxdy \right) \uparrow_{\mathcal{F}^N}. \tag{1.3}
\]

From now on, we shall study the Hamiltonian

\[
H := \int \frac{1}{2m} (\nabla a^*)(\nabla a)(x)dx + \frac{\lambda}{2} \int \int \bar{a}^*(x) a^*(y) \phi(x-y) \phi(y) a(x) dxdy + \xi N \mathbb{1} \tag{1.4}
\]

where \( \xi N := \frac{\lambda \phi_0}{2N^2} N - \frac{\lambda \phi_0}{2N^2} N^2 \) with \( 0 = (0, \ldots, 0) \). Hereafter, the operator \( H \) is meant to be restricted to the subspace \( \mathcal{F}^N \). Note that \( \mathcal{H} \uparrow_{\mathcal{F}^N} = (H - \xi N \mathbb{1}) \uparrow_{\mathcal{F}^N} \).

### 1.1 The Hamiltonian \( H \) and the Hamiltonian \( H^{Bog} \)

Using the definitions

\[
a_j^+(x) := \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \frac{a_j}{|\Lambda|^\frac{d}{2}} e^{i\lambda_j x} \quad \text{and} \quad a_0^+(x) := \frac{a_0}{|\Lambda|^\frac{d}{2}}, \tag{1.1}
\]

the Hamiltonian \( H \) reads

\[
H = \sum_{j \in \mathbb{Z}^d} \frac{k^2}{2m} a_j^+ a_j \tag{1.2}
\]

\[
+ \frac{\lambda}{2} \int \int a_j^+(x) a_j^+(y) \phi(x-y) a_j(x) a_j(y) dxdy \tag{1.3}
\]

\[
+ \lambda \int \int a_j^+(x) a_j^+(y) \phi(x-y) a_j^+(x) a_0(y) + \text{h.c.} dxdy \tag{1.4}
\]

\[
+ \frac{\lambda}{2} \int \int a_0^+(x) a_0^+(y) \phi(x-y) a_j^+(x) a_j^+(y) + \text{h.c.} dxdy \tag{1.5}
\]

\[
+ \lambda \int \int a_0^+(x) a_j^+(y) \phi(x-y) a_0(x) a_j(y) dxdy \tag{1.6}
\]

\[
+ \lambda \int \int a_0^+(x) a_j^+(y) \phi(x-y) a_0(y) a_j(x) dxdy \tag{1.7}
\]

\[
+ \frac{\lambda}{2} \int \int a_0^+(x) a_0^+(y) \phi(x-y) a_0(x) a_0(y) dxdy \tag{1.8}
\]

\[
+ \xi N \mathbb{1}. \tag{1.9}
\]
Because of the implicit restriction to $\mathcal{F}^N$ and the subtraction of the constant $-\epsilon_N$ in (1.9), it turns out that

$$H = \sum_{j \in \mathbb{Z}^d} \frac{k_j^2}{2m} a_j^* a_j$$

(1.10)

$$+ \frac{\lambda}{2} \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) dxdy$$

(1.11)

$$+ \lambda \int \int \{ a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) + h.c. \} dxdy$$

(1.12)

$$+ \frac{\lambda}{2} \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) dxdy$$

(1.13)

$$+ \lambda \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(y) a_j(x) dxdy$$

(1.14)

where $\phi_{(\neq 0)}(x - y) := \phi(x - y) - \phi_{(0)}(x - y)$ with $\phi_{(0)}(x - y) := \frac{\phi_0}{|x|}$.

Next, we define the particle number preserving Bogoliubov Hamiltonian

$$H_{\text{Bog}} := \sum_{j \in \mathbb{Z}^d} \frac{k_j^2}{2m} a_j^* a_j$$

(1.15)

$$+ \frac{\lambda}{2} \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) dxdy$$

(1.16)

$$+ \frac{\lambda}{2} \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) dxdy$$

(1.17)

$$+ \lambda \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(y) a_j(x) dxdy$$

(1.18)

that we can express in terms of the field modes,

$$H_{\text{Bog}} = \sum_{j \in \mathbb{Z}^d \backslash \{0\}} \left( \frac{k_j^2}{2m} + \frac{\lambda}{|x|} a^*_j a_j \right) a_j^* a_j + \frac{\lambda}{2} \sum_{j \in \mathbb{Z}^d \backslash \{0\}} \frac{\phi_0}{|x|} \left[ a^*_j a_j a_j^* a_j + a^*_j a^*_j a_j^* a_j \right].$$

(1.19)

Hence, the Hamiltonian $H$ corresponds to

$$H = H_{\text{Bog}} + V$$

(1.20)

with

$$V := \lambda \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) dxdy$$

(1.21)

$$+ \lambda \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(x) a_j(y) dxdy$$

(1.22)

$$+ \frac{\lambda}{2} \int \int a^*_j(x) a^*_j(y) \phi_{(\neq 0)}(x - y) a_j(y) a_j(x) dxdy.$$  

(1.23)

Following the convention of [Pi1], we set

$$\lambda = \frac{1}{\rho} , \quad m = \frac{1}{2} , \quad N = \rho|\Lambda| \quad \text{and even},$$

(1.24)

where $\rho > 0$ is the particle density.
In this paper we are interested in the mean field limiting regime. This means that we keep the box fixed and let the particle density \( \rho \) be (arbitrarily) large. Consequently, in the mean field limiting regime the number of particles \( N \) is independent of \( L \).

The main result of the paper is the recursive formula in (3.41)-(3.42) by which in (3.44) we construct the groundstate of the Hamiltonian \( H^{Bog} \) assuming the conditions on the potential specified in Definition 1.1. In Corollary 4.6 we re-expand the operators entering the formula in (3.41)-(3.42), and obtain, in turn, an expansion of the groundstate vector in terms of the interaction terms \( W^*_{j} + W^j \) and of the resolvents \( \frac{1}{\hat{H}^j_{Bog} - E_{Bog}^j} \) (see (2.4) and (2.3), respectively) applied to \( \eta \) (the state with all the particles in the zero mode).

The paper relies on some results obtained in [Pi1] that are listed in Section 2 along with an overview of the multi-scale technique applied to three-modes Bogoliubov Hamiltonians. Before providing a rigorous proof of the algorithm for the construction of the groundstate in Theorem 4.3 (Section 4), in Section 3 we introduce the definitions entering the scheme and outline the procedure leading to formula (3.41)-(3.42).

**Notation**

1. The symbol \( \mathbb{1} \) stands for the identity operator. If helpful we specify the Hilbert space where it acts, e.g., \( \mathbb{1}_{\mathcal{F}^N} \). For \( c \)-number operators, e.g., \( z \mathbb{1} \), we may omit the symbol \( \mathbb{1} \).
2. The symbol \( \langle \,,\, \rangle \) stands for the scalar product in \( \mathcal{F}^N \).
3. The word mode is used for the wavelength \( \frac{2\pi}{L_j} \) (or simply for \( j \)) when we refer to the field mode associated with it.
4. The symbol \( O(\alpha) \) stands for a quantity bounded in absolute value by a constant times \( \alpha \) (\( \alpha > 0 \)). The symbol \( o(\alpha) \) stands for a quantity such that \( o(\alpha)/\alpha \to 0 \) as \( \alpha \to 0 \). Throughout the paper the implicit multiplicative constants are always independent of \( N \).
5. In some cases we use explicit constants, e.g., \( C_l \), if the same quantity is used in later proofs. Unless otherwise specified, or unless it is obvious from the context, the explicit constants may depend on the size of the box and on the details of the potential, in particular on the number, \( M \), of couples of nonzero frequency components in the Fourier expansion of the pair potential.
6. The symbol \( |\psi\rangle\langle\psi| \), with \( \|\psi\|=1 \), stands for the one-dimensional projection onto the state \( \psi \).
7. Theorems and lemmas from the companion paper [Pi1] are underlined, quoted in italic, and with the numbering that they have in the corresponding paper; e.g., *Theorem 3.1 of [Pi1]*.

**2 Multi-scale analysis in the particle states occupation numbers for a three-modes Bogoliubov Hamiltonian: Review of results**

In [Pi1] we consider the *three-modes Bogoliubov Hamiltonian*

\[
\hat{H}^{Bog}_j := \sum_{j \in \mathbb{Z}^d \setminus \{0; \pm j\}} k_j^2 a_j^* a_j + \hat{H}^{Bog}_j
\]  

(2.1)
where
\[ H_{\text{Bog}}^j := \hat{H}_j^0 + W_j + W_j^* \] (2.2)

involves the three modes \( j = 0, \pm j \), only \((j_0 \neq 0)\); see the definitions in (2.3)-(2.4). Therefore, \( H_{\text{Bog}}^j \) is the sum of:

- The operator
  \[ \hat{H}_j^0 := (k_j^2 + \phi_j \frac{a_j a_j^*}{N}) a_j a_j^* + (k_j^2 + \phi_j \frac{a_j a_j^*}{N}) a_{-j} a_{-j}^* \] (2.3)

commuting with each number operator \( a_j^* a_j \);

- The interaction terms
  \[ \phi_j \frac{a_j a_j^* a_{-j} a_{-j}^*}{N} =: W_j, \quad \phi_j \frac{a_j a_j^* a_{-j} a_{-j}^*}{N} =: W_j^* \] (2.4)

that change the number of particles in the three modes \( j = 0, \pm j \);

- The kinetic energy \( \sum_{j \in \mathbb{Z}^d \setminus \{0\}} k_j^2 a_j^* a_j \) of the noninteracting modes.

The following identity follows from the definitions above:
\[ H_{\text{Bog}}^j = \frac{1}{2} \sum_{j \in \mathbb{Z}^d \setminus \{0\}} \hat{H}_{\text{Bog}}^j. \] (2.5)

**Remark 2.1.** Notice that \( H_{\text{Bog}}^j \) contains the kinetic energy corresponding to all the modes whereas \( \hat{H}_{\text{Bog}}^j \) contains the kinetic energy associated with the interacting modes only.

### 2.1 Feshbach-Schur projections and Feshbach-Schur Hamiltonians associated with \( H_{\text{Bog}}^j \)

In this section, we briefly summarize the main results of the strategy introduced in \([\text{Pi1}]\) where we construct the ground state of \( H_{\text{Bog}}^j \) by implementing a multi-scale analysis in the occupation numbers of the modes \( \pm j \). The multi-scale analysis relies on a novel application of Feshbach-Schur map that we outline in the next lines. We recall the use of the Feshbach-Schur map for the spectral analysis of quantum field theory systems starting with the seminal work by V. Bach, J. Fröhlich, and I.M. Sigal, \([\text{BFS}]\), followed by refinements of the technique and variants (see \([\text{BCFS}]\) and \([\text{GH}]\)). In those papers, the use of the Feshbach-Schur map is in the spirit of the functional renormalization group, and the projections \((P, \overline{\mathcal{P}})\) (see the paragraph below equation (2.6)) are directly related to energy subspaces of the free Hamiltonian.

For the Hamiltonian \( H_{\text{Bog}}^j \) applied to \( \mathcal{F}^N \) we define:

- \( Q_j^{(0,1)} := \) the projection (in \( \mathcal{F}^N \)) onto the subspace generated by vectors with \( N - 0 = N \) or \( N - 1 \) particles in the modes \( j \) and \(-j\), i.e., the operator \( a_j^* a_j + a_{-j}^* a_{-j} \) has eigenvalues \( N \) and \( N - 1 \) when restricted to \( Q_j^{(0,1)} \mathcal{F}^N \);

- \( Q_j^{(>1)} := \) the projection onto the orthogonal complement of \( Q_j^{(0,1)} \mathcal{F}^N \) in \( \mathcal{F}^N \).
Therefore, we have
\[ Q_{j,+}^{(0,1)} + Q_{j,-}^{(i+1)} = 1_{F^N}. \]

Analogously, starting from \( i = 2 \) up to \( i = N-2 \) with \( i \) even, we define \( Q_{j,+}^{(i,i+1)} \) the projection onto the subspace of \( Q_{j,-}^{(i-1)}F^N \) spanned by the vectors with \( N-i \) or \( N+i-1 \) particles in the modes \( j \) and \(-j \). Furthermore, \( Q_{j,+}^{(i+1)} \) is the projection onto the orthogonal complement of \( Q_{j,+}^{(i,i+1)} \) in \( Q_{j,-}^{(i-1)}F^N \), i.e.,
\[ Q_{j,+}^{(i+1)} + Q_{j,-}^{(i,i+1)} = Q_{j,-}^{(i-1)} \quad \text{(2.6)}. \]

We recall that given the (separable) Hilbert space \( \mathcal{H} \), the projections \( \mathcal{P}, \overline{\mathcal{P}} \) \((\mathcal{P} = \mathcal{P}^2 \overline{\mathcal{P}} = \mathcal{P}^2)\) where \( \mathcal{P} + \overline{\mathcal{P}} = 1_\mathcal{H} \), and a closed operator \( K - z, z \) in a subset of \( \mathbb{C} \), acting on \( \mathcal{H} \), the Feshbach-Schur map associated with the couple \( \mathcal{P}, \overline{\mathcal{P}} \) maps \( K - z \) to the operator \( \mathcal{F}(K - z) \) acting on \( \mathcal{P}\mathcal{H} \) where (formally)
\[ \mathcal{F}(K - z) := \mathcal{P}(K - z) \mathcal{P} - \mathcal{P}K\overline{\mathcal{P}} \frac{1}{\mathcal{P}(K - z)\mathcal{P}} \overline{\mathcal{P}}K\mathcal{P}. \quad \text{(2.7)} \]

We iterate the Feshbach-Schur map starting from \( i = 0 \) up to \( i = N-2 \) with \( i \) even, using the projections \( \mathcal{P}^{(i)} \) and \( \overline{\mathcal{P}}^{(i)} \) for the \( i \)-th step\(^2\) of the iteration where
\[ \mathcal{P}^{(i)} := Q_{j,-}^{(i+1)} \quad \overline{\mathcal{P}}^{(i)} := Q_{j,+}^{(i,i+1)}. \quad \text{(2.8)} \]

We denote by \( \mathcal{F}^{(i)} \) the Feshbach-Schur map at the \( i \)-th step (even) of the iteration. We start applying \( \mathcal{F}^{(0)} \) to \( H_{j,-}^{\text{Bog}} - z \) where \( z(\in \mathbb{R}) \) ranges in the interval \((-\infty, z_{\text{max}})\) with \( z_{\text{max}} \) larger but very close to
\[ E_{j}^{\text{Bog}} := -\left(k_j^2 + \phi_j - \sqrt{(k_j^2)^2 + 2\phi_jk_j^2}\right). \quad \text{(2.9)} \]

Hence, we define
\[ \mathcal{H}_{j,-}^{\text{Bog}(0)}(z) := \mathcal{F}^{(0)}(H_{j,-}^{\text{Bog}} - z) \quad \text{(2.10)} \]
\[ = Q_{j,-}^{(1)}(H_{j,-}^{\text{Bog}} - z)Q_{j,-}^{(1)} - Q_{j,-}^{(1)}H_{j,-}^{\text{Bog}}Q_{j,-}^{(0,1)} - \frac{1}{Q_{j,-}^{(0,1)}(H_{j,-}^{\text{Bog}} - z)Q_{j,-}^{(0,1)}}Q_{j,-}^{(0,1)}H_{j,-}^{\text{Bog}}Q_{j,-}^{(1)} \quad \text{(2.11)} \]
\[ = Q_{j,-}^{(1)}(H_{j,-}^{\text{Bog}} - z)Q_{j,-}^{(1)} - Q_{j,-}^{(1)}W_jQ_{j,-}^{(0,1)} - \frac{1}{Q_{j,-}^{(0,1)}(H_{j,-}^{\text{Bog}} - z)Q_{j,-}^{(0,1)}}Q_{j,-}^{(0,1)}W_jQ_{j,-}^{(1)}. \quad \text{(2.12)} \]

Next, by recursion we define
\[ \mathcal{H}_{j,-}^{\text{Bog}(i)}(z) := \mathcal{F}^{(i)}(\mathcal{H}_{j,-}^{\text{Bog}(i-2)}(z)), \quad i = 2, \ldots, N-2 \quad \text{even}, \quad \text{(2.13)} \]
that acts on \( Q_{j,-}^{(i+1)}F^N \). In the derivation of the Feshbach-Schur Hamiltonians \( \mathcal{H}_{j,-}^{\text{Bog}(i)}(z) \) we employ the following convenient notation
\[ W_{j,-;i,i'} := Q_{j,-}^{(i,i+1)}W_j Q_{j,-}^{(i,i+1)}, \quad W_{j,-;i,i'}^{\ast} := Q_{j,-}^{(i,i+1)}W_j Q_{j,-}^{(i,i+1)}, \]
and
\[ R_{j,-;i,i'}^{(i)}(z) := Q_{j,-}^{(i,i+1)} \frac{1}{Q_{j,-}^{(i,i+1)}(H_{j,-}^{\text{Bog}} - z)Q_{j,-}^{(i,i+1)}}Q_{j,-}^{(i,i+1)}. \]

\(^2\)We use this notation though the number of steps is in fact \( i/2 + 1 \) being \( i \) an even number.
Due to the selection rules of the operators $W_j$ and $W_j^*$, the Feshbach-Schur Hamiltonian at the $i$-th step is

\[ \mathcal{H}^{(i)}_{\text{Bog}}(z) = Q^{(i+1)}_j (H^{\text{Bog}}_{j} - z) Q^{(i+1)}_j \]

\[ - \sum_{l=0}^{\infty} Q^{(i+1)}_j W_j Q^{(i+1)}_{j;l,i}(z) \left[ W_j^*;i-2,l \right] \left[ R^{\text{Bog}}_{j;i-2,l} \right] \left[ Q^{(i+1)}_j \right] \]

\[ \times \sum_{l_{i-2}=0}^{\infty} \left[ W_j^*;i-2,l \ldots W_j^*;i-4,l \right] \left[ R^{\text{Bog}}_{j;i-4,l} \right] \left[ \ldots \right] \left[ W_j^*;i-2,l \right] \left[ R^{\text{Bog}}_{j;i-2,l} \right] \left[ Q^{(i+1)}_j \right] \]

where the expressions corresponding to \ldots in (2.17) are made precise in Theorem 3.1 of [Pil] reported below.

**Remark 2.2.** The ratio $\epsilon_j$ between the kinetic energy of the modes $\pm j$, $\neq 0$ and the corresponding Fourier component, $\phi_{j}$, of the potential, i.e., $\frac{\epsilon_j^2}{\phi_j}$, is required to be small to ensure the estimates used in [Pil]. In particular, $\epsilon_j$ is assumed small enough so that $E_{j}^{\text{Bog}} + \phi_j \sqrt{\epsilon_j^2 + 2\epsilon_j} < 0$ and, likewise, the spectral variable $z \in \mathbb{R}$ is restricted to negative values. Notice that $\epsilon_j$ small corresponds either to a low energy mode $\pm \frac{2\epsilon_j}{\epsilon_j}$ or to a large potential $\phi_j$.

Here, we recall the main result from [Pil] regarding the control of the flow of Feshbach-Schur Hamiltonians up to the step $i = N - 2$, along with some related tools.

**Theorem 3.1 of [Pil]**

For

\[ z \leq E_{j}^{\text{Bog}} + (\delta - 1) \phi_j \sqrt{\epsilon_j^2 + 2\epsilon_j} \] \hspace{1cm} (2.18)

with $\delta \leq 1 + \frac{1}{\epsilon_j^2} \epsilon_j^{p}$ for some $\nu > \frac{11}{8}$, and $\epsilon_j$ sufficiently small, the operators $\mathcal{H}^{(i)}_{\text{Bog}}(z)$, $0 \leq i \leq N - 2$ and even, are well defined. For $i = 0$, it is given in (2.12). For $i = 2, 4, 6, \ldots, N - 2$ they correspond to

\[ \mathcal{H}^{(i)}_{\text{Bog}}(z) = Q^{(i+1)}_j (H^{\text{Bog}}_{j} - z) Q^{(i+1)}_j \]

\[ - Q^{(i+1)}_j W_j R^{\text{Bog}}_{j;i,l}(z) \sum_{l=0}^{\infty} \left[ R^{\text{Bog}}_{j;i,l}(z) R^{\text{Bog}}_{j;i,l}(z) \right]^{l-1} W_j^* Q^{(i+1)}_j \]

where:

- $\Gamma^{\text{Bog}}_{j;i,2;2}(z) := W_{j;i,2;0} R^{\text{Bog}}_{j;i,0;0}(z) W_{j;i,0;2}$ \hspace{1cm} (2.20)

- for $N - 2 \geq i \geq 4$,

\[ \Gamma^{\text{Bog}}_{j;i,l}(z) := W_{j;i-2,l} R^{\text{Bog}}_{j;i-2,l}(z) \sum_{l_{i-2}=0}^{\infty} \left[ R^{\text{Bog}}_{j;i-2,l}(z) R^{\text{Bog}}_{j;i-2,l}(z) \right]^{l_{i-2}-1} W_{j;i-2,l} \]

\[ = W_{j;i-2} (R^{\text{Bog}}_{j;i-2,l}(z))^{l_{i-2}-1} \sum_{l_{i-2}=0}^{\infty} \left[ (R^{\text{Bog}}_{j;i-2,l}(z))^{l_{i-2}-1} \left[ R^{\text{Bog}}_{j;i-2,l}(z) \right]^{l_{i-2}-1} \right] \times \] \hspace{1cm} (2.21)
Hence, Theorem 3.1 of [Pi1] states that the flow of Feshbach-Schur Hamiltonians (up to the index value \(N-2\)) can be defined for spectral values \(z\) up to \(E_{j,\phi_j}^{\text{Bog}} + (\delta - 1)\phi_j \sqrt{\epsilon_j^2 + 2\epsilon_j}\) with \(\delta > 1\) but very close to 1. The Hamiltonian \(\mathcal{H}_{j,\phi_j}^{\text{Bog}}(N-2)(z)\) acts on the Hilbert space \(\mathcal{H}(\mathbb{R})^{(N-1)}\).

The key estimates to prove Theorem 3.1 of [Pi1] are derived in Lemma 3.4 and Lemma 3.6 of [Pi1]. In fact, the Feshbach-Schur map is implementable as long as

\[
\| (R_{j,\phi_j}^{\text{Bog}}(z))^{1/2} \sum_{l=0}^{\infty} \left[ (R_{j,\phi_j}^{\text{Bog}}(z))^{1/2} (R_{j,\phi_j}^{\text{Bog}}(z))^{1/2} \right]^{l} \| < \infty.
\]

The norm in (2.23) can be related to the inverse of the \((N-i+2)-th\) term of the sequence studied in:

**Lemma 3.6 of [Pi1]**

Assume \(\epsilon > 0\) sufficiently small. Consider for \(j \in \mathbb{N}_0\) the sequence defined by

\[
X_{2j+2} := 1 - \frac{1}{4(1 + a_\epsilon - \frac{2b_\epsilon}{N-2j-1} - \frac{1-\epsilon}{(N-2j-1)^2})X_{2j}},
\]

with initial condition \(X_0 = 1\), up to \(X_{2j=N-2}\) where \(N \geq 2\) is even. Here,

\[
a_\epsilon := 2\epsilon + O(\epsilon^\nu), \quad \nu > \frac{11}{8},
\]

\[
b_\epsilon := (1 + \epsilon)\delta \chi_{[0,2]}(\delta) \sqrt{\epsilon^2 + 2\epsilon} \bigg|_{\delta = 1+\sqrt{\epsilon}}
\]

and

\[
c_\epsilon := -(1 - \epsilon^2 \chi_{[0,2]}(\epsilon))(\epsilon^2 + 2\epsilon) \bigg|_{\delta = 1+\sqrt{\epsilon}}
\]

with \(\chi_{[0,2]}\) the characteristic function of the interval \([0,2]\). Then, the following estimate holds true for \(2 \leq N-2j \leq N\),

\[
X_{2j} \geq \frac{1}{2} \left[ 1 + \sqrt{\eta a_\epsilon} - \frac{b_\epsilon / \sqrt{\eta a_\epsilon}}{N-2j-\epsilon^\Theta} \right] (> 0)
\]

with \(\eta = 1 - \sqrt{\epsilon}\), where \(\Theta := \min\{2(\nu - \frac{11}{8}); \frac{1}{4}\}\).

Indeed, the inequality

\[
\| \sum_{l=0}^{\infty} \left[ (R_{j,\phi_j}^{\text{Bog}}(z))^{1/2} (R_{j,\phi_j}^{\text{Bog}}(z))^{1/2} \right]^{l} \| \leq \frac{1}{X_{i-2}}
\]

is used to control the Feshbach-Schur flow. It relies on the crucial estimate derived in the next lemma.

**Lemma 3.4 of [Pi1]**

Let

\[
z \leq E_{j,\phi_j}^{\text{Bog}} + (\delta - 1)\phi_j \sqrt{\epsilon_j^2 + 2\epsilon_j} \quad (< 0)
\]
with $\delta < 2$, $\frac{1}{N} \leq \epsilon_j^{(i)}$ for some $\nu > 1$, and $\epsilon_j$ sufficiently small. Then
\[
\| R_{\tilde{J}_j}^{Bog}(z) \|^{\frac{1}{2}} W_{\tilde{J}_j \cdot \tilde{J}_j - 2} \left[ R_{\tilde{J}_j}^{Bog}(z) \right]^{\frac{1}{2}} \left\| R_{\tilde{J}_j}^{Bog}(z) \right\|^2 \| W_{\tilde{J}_j \cdot \tilde{J}_j - 2} \left[ R_{\tilde{J}_j}^{Bog}(z) \right]^{\frac{1}{2}} \| \leq \frac{1}{4(1 + a_{\epsilon_j} - \frac{2b_{\epsilon_j}^{(i)}}{N - i + 1} - \frac{1 - \epsilon_j^{(i)}}{(N - i + 1)^2})}
\]
holds for all $2 \leq i \leq N - 2$ (i even). Here,
\[
a_{\epsilon_j} := 2\epsilon_j + O(\epsilon_j^2),
\]
\[
b_{\epsilon_j}^{(i)} := (1 + \epsilon_j)\delta \chi(0,2) \sqrt{\epsilon_j^2 + 2\epsilon_j},
\]
and
\[
c_{\epsilon_j}^{(i)} := -(1 - \delta^2 \chi(0,2)\epsilon_j^2 + 2\epsilon_j),
\]
with $\chi(0,2)$ the characteristic function of the interval $(0, 2)$.

A posteriori, one can observe that the choice of the Feshbach-Schur projections yields a rather refined control of the Neumann expansion in (2.22) thanks to Lemma 3.4 of [Pi1]. The price to be paid is the iteration from $i = 0$ up to $i = N - 2$ and the nontrivial control of the sequence $\{X_{2j}\}$ studied in Lemma 3.6 of [Pi1].

### 2.2 Fixed point, construction of the ground state of $H_{\tilde{J}_j}^{Bog}$, and algorithm for the re-expansion

We implement the Feshbach-Schur flow as described in the previous section with the purpose to get a simple final effective Hamiltonian, namely a multiple of the one-dimensional projection $|\eta\rangle\langle\eta|$ where $\eta$ is the state with all the $N$ particles in the zero mode, i.e.,
\[
\eta = \frac{1}{\sqrt{N!}} a_0^* \cdots a_0^* \Omega.
\]

This way we can reconstruct the ground state of $H_{\tilde{J}_j}^{Bog}$ starting from $\eta$, by means of Feshbach-Schur theory. Consequently, for the last step of the Feshbach-Schur flow we employ the couple of projections: $\mathcal{P}_\eta := |\eta\rangle\langle\eta|$ and $\overline{\mathcal{P}}_\eta$ such that
\[
\mathcal{P}_\eta + \overline{\mathcal{P}}_\eta = 1 \cdot O_{N-1}^{(N-1)} \cdot \Omega.
\]

We remind that for $i = N - 2$ the projection $O_{j_k}^{(N-i+1,N-1)}$ coincides with the projection onto the subspace where less than $N - i + 1 = N - N + 1 = 1$ particles in the modes $j_*$ and $-j_*$ are present, i.e., where no particle in the modes $j_*$ or $-j_*$ is present.

Starting from the formal expression
\[
\mathcal{K}_{\tilde{J}_j}^{Bog(N)}(z)
\]
\[
:= \mathcal{K}^{(N)}(\mathcal{K}_{\tilde{J}_j}^{Bog(N-2)}(z))
\]
\[
= \mathcal{P}_\eta (H_{\tilde{J}_j}^{Bog} - z) \mathcal{P}_\eta
\]
\[ -\mathcal{P}_\eta W_j \mathcal{R}^{Bog}_{j;N-2,N-2}(z) \sum_{l_{N-2}=0}^{\infty} \left[ \mathcal{R}^{Bog}_{j;N-2,N-2}(z) \mathcal{R}^{Bog}_{j;N-2,N-2}(z)^\dagger \mathcal{P}_\eta \right] \]
\[ -\mathcal{P}_\eta W_j \mathcal{P}_\eta \frac{1}{\mathcal{P}_\eta \mathcal{R}^{Bog}_{j;N-2}(z) \mathcal{P}_\eta} \mathcal{P}_\eta W_j \mathcal{P}_\eta, \]

in [Pi1] we study the invertibility of
\[ \mathcal{P}_\eta \mathcal{R}^{Bog}_{j;N-2}(z) \mathcal{P}_\eta \]
and show that the R-H-S of (2.40) is well defined for
\[ z < \min \left\{ z_0 + \frac{\Delta_0}{2} ; E^{Bog}_j + \sqrt{\phi_j} \sqrt{e_j^2 + 2e_j} \right\}, \]
where \( \Delta_0 := \min \left\{ (k_j)^2 | j \in \mathbb{Z}^d \setminus \{0\} \right\} \) and \( z_0 \) is the unique solution of \( f_j(z) = 0 \) with
\[ f_j(z) := -z - \langle \eta, W_j \mathcal{R}^{Bog}_{j;N-2,N-2}(z) \sum_{l_{N-2}=0}^{\infty} \left[ \mathcal{R}^{Bog}_{j;N-2,N-2}(z) \mathcal{R}^{Bog}_{j;N-2,N-2}(z)^\dagger \mathcal{P}_\eta \right] \]
In order to determine \( z_0, \) in Section 4.1.1. of [Pi1], we derive the identity (for \( z \leq E^{Bog}_j + \sqrt{\phi_j} \sqrt{e_j^2 + 2e_j} < 0 \))
\[ \langle \eta, W_j \mathcal{R}^{Bog}_{j;N-2,N-2}(z) \sum_{l_{N-2}=0}^{\infty} \left[ \mathcal{R}^{Bog}_{j;N-2,N-2}(z) \mathcal{R}^{Bog}_{j;N-2,N-2}(z)^\dagger \mathcal{P}_\eta \right] \]
\[ = (1 - \frac{1}{N}) \frac{\phi_j}{2e_j - \frac{4}{N} + 2 - \frac{z}{\phi_j}} \mathcal{G}_{j;N-2,N-2}(z) \]
where \( \mathcal{G}_{j;N-2,N-2}(z) \) is defined (for \( i \) even and \( N - 2 \geq i \geq 2 \)) by
\[ \mathcal{G}_{j;N-2,N-2}(z) := \sum_{l_{N-2}}^{\infty} \left[ W_{j;N-2,N-2}(z) W_{j;N-2,N-2}(z)^\dagger \mathcal{G}_{j;N-2,N-2}(z) \right], \quad \mathcal{G}_{j;0,0}(z) = 1, \]
and
\[ W_{j;N-2,N-2}(z) W_{j;N-2,N-2}(z)^\dagger := \frac{(n_{j_0} - 1)n_{j_0}}{N^2} \phi_j \frac{(n_{j_0} + 1)(n_{j_0} - 1)}{\left( \frac{n_{j_0}}{N} \phi_j + (k_{j_0}^2) \right)(n_{j_0} + n_{j_0} - 2) - z} \]
\[ \times \frac{1}{\left( \frac{2n_{j_0} - 1}{N} \phi_j + (k_{j_0}^2) \right)(n_{j_0} + n_{j_0} - 2) - z} \]
with
\[ n_{j_0} + n_{j_0} = N - i \quad n_{j_0} = n_{j_0} \quad n_{j_0} = i. \]

Remark 2.3. In [Pi1], by induction \( \mathcal{G}_{j;N-2,N-2}(z) \) is shown to be nondecreasing in \( z. \)

Finally, the identities
\[ \mathcal{P}_\eta (H^{Bog}_{j;N-2,N-2} - z) \mathcal{P}_\eta = -z \mathcal{P}_\eta, \quad \mathcal{P}_\eta W_j \mathcal{P}_\eta = 0, \]

12
\[ \mathcal{H}_{Bog}^{(N)}(z) = f_j(z)\langle \eta | \eta \rangle. \] (2.49)

In Corollary 4.6 of [Pi1] we show that in the limit \( N \to \infty \) the ground state energy \( \varepsilon_0 \) tends to \( \mathcal{H}_{Bog}^{(N)} \) with a spectral gap that (in the mean field limit) can be estimated\(^3\) larger than \( \frac{\Delta_0}{N} \) (where \( \Delta_0 := \min \{ (k_j)^2 : j \in \mathbb{Z}^d \setminus \{0\} \} \)). More importantly, the ground state vector of the Hamiltonian \( \mathcal{H}_{Bog}^{(N)} \) is derived exploiting Feshbach-Schur theory:

\[
\psi_{Bog} = \eta
\]

\[
- \frac{1}{Q_{j}^{(N-2,N-1)}} \mathcal{H}_{Bog}^{(N-4)}(z_0)Q_{j}^{(N-2,N-1)}W^*_{j\eta} (\text{by (2.50)})
\]

\[
- \sum_{j=2}^{N/2} \sum_{l=j}^{2} \frac{1}{Q_{j}^{(N-2,r,N-2,r+1)}} \mathcal{H}_{Bog}^{(N-2,r-2)}(z_0)Q_{j}^{(N-2,r,N-2,r+1)}W^*_{j,N-2,r,N-2,r+1} (\text{by (2.53)})
\]

Remark 2.4. The sum in (2.54) is controlled in norm by a multiple of

\[
\sum_{j=2}^{\infty} c_j := \sum_{j=2}^{\infty} \left\{ \sum_{l=j}^{2} \frac{1}{Q_{j}^{(N-2,r,N-2,r+1)}} \mathcal{H}_{Bog}^{(N-2,r-2)}(z_0)Q_{j}^{(N-2,r,N-2,r+1)}W^*_{j,N-2,r,N-2,r+1} \right\} (2.55)
\]

which is convergent for \( \varepsilon_0 > 0 \) because \( \frac{c_j}{c_j} < 1 \) for \( j \) sufficiently large. The series diverges in the limit \( \varepsilon_j \to 0 \). Hence, for any \( \varepsilon_0 > 0 \) sufficiently small there is a convergent expansion of \( \psi_{Bog} \) controlled by the parameter \( \varepsilon_0 := \frac{1}{1 + \sqrt{\varepsilon_0 \cdot \text{other terms}}} \).

In Section 4 of [Pi1] we also show how to expand (for a gas in a fixed finite box) the vector \( \psi_{Bog} \) in terms of finite sums of finite products of the operators \( W_j, W^*_j \), and \( \frac{1}{\mathcal{H}_{Bog}^{(N)} - E_{Bog}^{(N)}} \) (see (2.3)) applied to the vector \( \eta \), up to any desired precision in the limit \( N \to \infty \) at fixed box size.

Remark 2.5. We observe that \( \psi_{Bog} \) is also eigenvector of \( \mathcal{H}_{Bog}^{(N)} \) (see the definition in (2.2)) with the same eigenvalue \( z_0 \).

\(^3\)In [Se1] the results concerning the spectrum provide a much more accurate estimate of the spectral gap.
3 Ground state of $H^{\text{Bog}}$: Outline of the proof

From now on we consider the system in the mean field limiting regime: fixed box $\Lambda$ and a number of particles $N$ independent of $|\Lambda| = L^d$. Since we have assumed that an ultraviolet cut-off is imposed on the interaction potential, there are $M$ couples of (nonzero) interacting modes with $M < \infty$. The strategy to construct the ground state of $H^{\text{Bog}}$ consists in three operations:

1. We define intermediate Bogoliubov Hamiltonians obtained by adding (to the interaction Hamiltonian) a couple of modes, $(\{j_m, -j_m\})$ with $1 \leq m \leq M$, at a time;
2. At each step, i.e., for each intermediate Hamiltonian, we use the Feshbach-Schur map flow described in Section 2 and associated with the new couple of modes, $(\{j_m, -j_m\})$, that has been considered;
3. We use the projection onto the ground state of the intermediate Bogoliubov Hamiltonian at the $(m-1) - th$ step as the final projection $\mathcal{P}$ of the Feshbach-Schur map flow at the $m - th$ step.

In Section 3.1 we outline the procedure, and in Section 4 we provide the results that are needed to make the construction rigorous.

3.1 The Feshbach-Schur flows associated with $H^{\text{Bog}}$: The intermediate Hamiltonians $H^{\text{Bog}}_{j_1, \ldots, j_m}$

We start from $H^{\text{Bog}}_{j_1}$ and construct

$$\mathcal{K}^{\text{Bog}(N)}_{j_1}(z) = \mathcal{P}_\eta(H^{\text{Bog}}_{j_1} - z)\mathcal{P}_\eta - \mathcal{P}_\eta W_{j_1} \sum_{l_{j_2} = 0}^{\infty} R^{\text{Bog}}_{j_1,N-2,N-2}(z) \left[ \Gamma^{\text{Bog}}_{j_1,N-2,N-2}(z) R^{\text{Bog}}_{j_1,N-2,N-2}(z) \right]^{N-2} W^*_j \mathcal{P}_\eta. \tag{3.1}$$

Next, we determine the ground state energy, $z^{\text{Bog}}_{j_1}$, of $H^{\text{Bog}}_{j_1}$ by imposing

$$z^{\text{Bog}}_{j_1} = \langle \eta, W_{j_1} \sum_{l_{j_2} = 0}^{\infty} R^{\text{Bog}}_{j_1,N-2,N-2}(z^{\text{Bog}}_{j_1}) \left[ \Gamma^{\text{Bog}}_{j_1,N-2,N-2}(z^{\text{Bog}}_{j_1}) R^{\text{Bog}}_{j_1,N-2,N-2}(z^{\text{Bog}}_{j_1}) \right]^{N-2} W^*_j \eta \rangle. \tag{3.2}$$

Hence, the ground state of $H^{\text{Bog}}_{j_1}$ is given by (2.52)-(2.54) replacing $j_1$ with $j_1$. This first step just requires the results of [Pil1] that have been summarized in Section 2.1.

In the next step, we consider the intermediate Hamiltonian

$$H^{\text{Bog}}_{j_1, j_2} := \sum_{j \in \mathbb{Z}^d \setminus \{j_1 : \pm j_2\}} k_j^2 a_j^* a_j + \hat{H}^{\text{Bog}}_{j_1, j_2} := \sum_{j \in \mathbb{Z}^d \setminus \{j_1 : \pm j_2\}} k_j^2 a_j^* a_j + \sum_{l=1}^{2} \hat{H}^{\text{Bog}}_{j_2} \tag{3.3}$$

and construct the Feshbach-Schur Hamiltonians,

$$\mathcal{K}^{\text{Bog}(0)}_{j_1, j_2}(z^{\text{Bog}}_{j_1} + z) \tag{3.4}$$

$$:= Q^{(1)}_{j_2}(H^{\text{Bog}}_{j_1, j_2} - z^{\text{Bog}}_{j_1} - z)Q^{(1)}_{j_2} - Q^{(1)}_{j_2} W_{j_2} R^{\text{Bog}}_{j_1, j_2} : 0, 0 (z^{\text{Bog}}_{j_1} + z) W^*_j Q^{(1)}_{j_2}. \tag{3.5}$$
and, for $2 \leq i \leq N - 2$ and even,

$$
\mathcal{H}_{\text{Bog}}^{(i)}(z) := \sum_{j=0}^{\infty} \left[ \Gamma_{\text{Bog}}^{j} (\zeta_{j_{1}}^{\text{Bog}} + z) R_{j_{1},j_{2}:i,j}^{\text{Bog}} (\zeta_{j_{1}}^{\text{Bog}} + z) \right]^{j_{1},j_{2}} W_{j_{1}}^{\text{Bog}} Q_{j_{2}}^{(i+1)}
$$

(3.6)

where we use the definitions:

$$
R_{j_{1},j_{2}:i,j}^{\text{Bog}} (\zeta_{j_{1}}^{\text{Bog}} + z) := \sum_{l_{i}=0}^{\infty} \left[ \Gamma_{\text{Bog}}^{l_{i}} (\zeta_{j_{1}}^{\text{Bog}} + z) R_{j_{1},j_{2}:i,l_{i}}^{\text{Bog}} (\zeta_{j_{1}}^{\text{Bog}} + z) \right]^{l_{i},j_{2}} W_{j_{1}}^{\text{Bog}} Q_{j_{2}}^{(i+1)}
$$

(3.7)

and, for $N - 2 \geq i \geq 4$ and even,

$$
\Gamma_{\text{Bog}}^{j} (\zeta_{j_{1}}^{\text{Bog}} + z) := \sum_{l_{i}=0}^{\infty} \left[ \Gamma_{\text{Bog}}^{l_{i}} (\zeta_{j_{1}}^{\text{Bog}} + z) R_{j_{1},j_{2}:i-2,l_{i}}^{\text{Bog}} (\zeta_{j_{1}}^{\text{Bog}} + z) \right]^{l_{i},j_{2}} W_{j_{1}}^{\text{Bog}} Q_{j_{2}}^{(i+1)}
$$

(3.8)

In the last implementation of the Feshbach-Schur map we use the projections

$$
\mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} := \frac{\langle \psi_{j_{1}}^{\text{Bog}} \rangle_{\psi_{j_{1}}^{\text{Bog}}}^{\text{Bog}}}{\langle \psi_{j_{1}}^{\text{Bog}} \rangle_{\psi_{j_{1}}^{\text{Bog}}}^{\text{Bog}}}, \quad \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} := 1 - \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}}
$$

(3.9)

where $\mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}}^{(N-1)}\mathcal{P}^{N}$ is the subspace of states in $\mathcal{P}^{N}$ with no particles in the modes $\pm j_{2}$, and we define

$$
\Gamma_{\text{Bog}}^{j} (\zeta_{j_{1}}^{\text{Bog}} + z) := \sum_{l_{i}=0}^{\infty} \left[ \Gamma_{\text{Bog}}^{l_{i}} (\zeta_{j_{1}}^{\text{Bog}} + z) R_{j_{1},j_{2}:i-2,l_{i}}^{\text{Bog}} (\zeta_{j_{1}}^{\text{Bog}} + z) \right]^{l_{i},j_{2}} W_{j_{1}}^{\text{Bog}} Q_{j_{2}}^{(i+1)}
$$

(3.10)

For the derivation of $\mathcal{H}_{\text{Bog}}^{(N)}(z)_{\text{Bog}}$, we point out that (see Remark 2.5)

$$
(\hat{H}_{j_{1}}^{\text{Bog}} - \zeta_{j_{1}}^{\text{Bog}}) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = 0, \quad \sum_{j_{1} \in \mathbb{Z} \setminus \{0, \pm j_{1}\}} a_{j_{1}}^{*} a_{j_{1}} \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = 0,
$$

and

$$
\mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} (\hat{H}_{j_{2}}^{\text{Bog}} - \zeta_{j_{2}}^{\text{Bog}}) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} (\hat{H}_{j_{2}}^{\text{Bog}} - \zeta_{j_{2}}^{\text{Bog}}) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} (W_{j_{2}} + W_{j_{2}}^{*} - \zeta) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = -z \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}},
$$

$$
\mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} (\hat{H}_{j_{2}}^{\text{Bog}} - \zeta_{j_{2}}^{\text{Bog}}) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} (\hat{H}_{j_{2}}^{\text{Bog}} - \zeta_{j_{2}}^{\text{Bog}}) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} (W_{j_{2}} + W_{j_{2}}^{*} - \zeta) \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}} = -z \mathcal{P}_{\psi_{j_{1}}^{\text{Bog}}}.
$$

(3.11)
Formally, we get

\[ \mathcal{H}^{\text{Bog}}(N)_{J_1,J_2}(z_{J_1} + z) \leq \mathcal{H}^{\text{Bog}}(H_{J_1,J_2} - z_{J_1} - z) \mathcal{P}_{\psi_{J_1}} \]  

(3.15)

where

\[ \mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} = \mathcal{P}_{\psi_{J_1}} \]  

(3.16)

These identities follow from the definitions of \( \mathcal{P}_{\psi_{J_1}} \), \( \mathcal{P}_{\psi_{J_1}} \), and \( H_{J_1,J_2}^{\text{Bog}} \), combined with the fact that \( Q_{J_1,J_2}^{(N-1)} \) is the projection onto the subspace of states with no particles in the modes \( \pm J_2 \).

Formally, we get

\[ \mathcal{H}^{\text{Bog}}(N)_{J_1,J_2}(z_{J_1} + z) \leq \mathcal{H}^{\text{Bog}}(H_{J_1,J_2} - z_{J_1} - z) \mathcal{P}_{\psi_{J_1}} \]  

(3.17)

\[ \leq -z \mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}(N-1)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}} \]  

(3.18)

\[ = -z \mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}(N-1)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}} \]  

(3.19)

\[ - \mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}(N-1)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}} \]  

(3.20)

We determine the ground state energy, \( \varepsilon^{(2)}_{J_1,J_2} := \varepsilon^{(2)}_{J_1} + \varepsilon^{(2)} \), of \( H_{J_1,J_2}^{\text{Bog}} \) by imposing

\[ \varepsilon^{(2)}_{J_1,J_2} = -\left( \frac{\mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}(N-1)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}}}{||\psi_{J_1}||} \right) \frac{\mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}}{||\psi_{J_1}||} \]  

(3.21)

Hence, the ground state vector of \( H_{J_1,J_2}^{\text{Bog}} \) is (up to normalization)

\[ \psi_{J_1,J_2}^{\text{Bog}} := \{ \sum_{j=2}^{N/2} \prod_{r=1}^{2} \left( \frac{1}{Q_{J_2}^{(N-2r,N-2r+1)} \mathcal{H}^{\text{Bog}}(N-2r)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}} - Q_{J_2}^{(N-2r,N-2r+1)} W_{J_2}^{(N-2r,N-2r+1)} \right) \} \times \]  

(3.22)

\[ \left[ \mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}(N-1)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}} \right] \]  

(3.23)

\[ \times \left[ \mathcal{P}_{\psi_{J_1}} \mathcal{P}_{\psi_{J_1}} \mathcal{H}^{\text{Bog}}(N-1)_{J_1,J_2} \mathcal{P}_{\psi_{J_1}} \right] \]  

(3.24)

where \( H_{J_1,J_2}^{\text{Bog}} := H_{J_1,J_2} - \varepsilon_{J_1,J_2}^{(2)} \).
At the $m-th$ step, we define
\[
H_{j_1, \ldots, j_m}^{\text{Bog}} := \sum_{j \in \mathbb{Z}^d} k_j^2 \hat{a}_j^* \hat{a}_j + \sum_{l=1}^m \hat{H}_{l}^{\text{Bog}},
\]
where the reader should note that the kinetic energy of the interacting nonzero modes, $\pm j_1, \ldots, \pm j_m$, is contained in $\sum_{l=1}^m \hat{H}_{l}^{\text{Bog}}$. Then, we construct (for $0 \leq i \leq N - 2$ and even)
\[
\mathcal{K}_{j_1, \ldots, j_m}^{\text{Bog}}(i)(z_{j_1, \ldots, j_m}) := Q_{j_m}^{(i+1)}(H_{j_1, \ldots, j_m}^{\text{Bog}} - z_{j_1, \ldots, j_m}) Q_{j_m}^{(i+1)}
\]
\[
- Q_{j_m}^{(i+1)} W_{j_m}^{R_{j_1, \ldots, j_m}}(z_{j_1, \ldots, j_m} + z) \sum_{l=0}^{\infty} \left[ \Gamma_{j_1, \ldots, j_m ; l}^{(i)}(z_{j_1, \ldots, j_m} + z) R_{j_1, \ldots, j_m ; l}^{(i)}(z_{j_1, \ldots, j_m} + z) \right] W_{j_m}^{*} Q_{j_m}^{(i+1)}
\]
and, eventually,
\[
\mathcal{K}_{j_1, \ldots, j_m}^{\text{Bog}}(N)(z_{j_1, \ldots, j_m-1}) := -Q_{j_m}^{(N)}(H_{j_1, \ldots, j_m}^{\text{Bog}} - z_{j_1, \ldots, j_m-1}) Q_{j_m}^{(N)} \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times \times 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\[
\Gamma_{j_1,\ldots,j_m}^{\text{Bog}, N,N}(z) := W_{j_m} R_{j_1,\ldots,j_{m-1}}^{\text{Bog},N} (z_{j_1,\ldots,j_{m-1}} + z) \times \sum_{l_k, z=0} \left[ \Gamma_{j_1,\ldots,j_m}^{\text{Bog},N-2,N-2}(z_{j_1,\ldots,j_{m-1}} + z) R_{j_1,\ldots,j_m}^{\text{Bog},N-2} (z_{j_1,\ldots,j_{m-1}} + z) \right]^{N-2} W_{j_m}. \tag{3.38}
\]

We compute the ground state energy, \( z_{j_1,\ldots,j_m}^{\text{Bog}} := z_{j_1,\ldots,j_{m-1}}^{\text{Bog}} + \zeta(m) \), of \( H_{j_1,\ldots,j_m}^{\text{Bog}} \) by solving the equation in \( z \):

\[
j_{j_1,\ldots,j_m}^{\text{Bog}} (z + z_{j_1,\ldots,j_{m-1}}^{\text{Bog}}) = 0. \tag{3.40}
\]

Hence, the ground state vector of \( H_{j_1,\ldots,j_m}^{\text{Bog}} \) is (up to normalization)

\[
\psi_{j_1,\ldots,j_m}^{\text{Bog}} := \left\{ \sum_{j=2}^{N/2} \prod_{r=j}^{2} \left[ -\frac{1}{Q_{j_m}^{(N-2r,N-2r+1)}} H_{j_1,\ldots,j_{m-1}}^{\text{Bog}, (N-2r-2)} (z_{j_1,\ldots,j_{m-1}}) Q_{j_m}^{(N-2r,N-2r+1)} W_{j_m} \right] + 1 \right\} \times \nabla_{j_1,\ldots,j_{m-1}}^{\text{Bog}} \nabla_{j_1,\ldots,j_{m-1}}^{\text{Bog}, (N-2)} \psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}}, \tag{3.41}
\]

where \( H_{j_1,\ldots,j_m}^{\text{Bog}, (N-2)} (z_{j_1,\ldots,j_m}) := H_{j_1,\ldots,j_m}^{\text{Bog}} - z_{j_1,\ldots,j_m}^{\text{Bog}} \). Thus, we have derived the formula

\[
\psi_{j_1,\ldots,j_M}^{\text{Bog}} = T_M \ldots T_1 \eta. \tag{3.44}
\]

Remark 3.1. We point out that by construction

\[
\| \psi_{j_1,\ldots,j_M}^{\text{Bog}} \| \geq \| \psi_{j_1,\ldots,j_{M-1}}^{\text{Bog}} \| \geq \ldots \geq \| \eta \| = 1. \tag{3.45}
\]

In Corollary 4.6, we show how to approximate the vector \( \psi_{j_1,\ldots,j_M}^{\text{Bog}} \) constructed in (3.44) up to any arbitrarily small error \( \zeta \) (for \( N \) sufficiently large) with a vector \( (\psi_{j_1,\ldots,j_M}^{\text{Bog}})_{\zeta} \) corresponding to a \( \zeta \)-dependent finite sum of finite products of the interaction terms \( W_{j_1}^{\text{Bog}} + W_{j_i} \), and of the resolvents \( \frac{1}{B_{j_1}^{\text{Bog}} - E_{j_1}^{\text{Bog}}} \) (see (2.3)), \( 1 \leq l \leq M \), applied to \( \eta \).

4 Rigorous construction of the Feshbach-Schur Hamiltonians \( \mathcal{H}_{j_1,\ldots,j_m}^{\text{Bog},(i)}(z) \)

The derivation in Section 3.1 can be made rigorous if we show that at the \( m \)-th step (for \( \epsilon_{j_m} \) sufficiently small and \( N \) sufficiently large):

\( \text{(i)} \) The Feshbach-Schur Hamiltonian \( \mathcal{H}_{j_1,\ldots,j_m}^{\text{Bog},(i)}(z + z_{j_1,\ldots,j_{m-1}}^{\text{Bog}}), 0 \leq i \leq N - 2, \) is well-defined provided

\[
z \leq E_{j_m}^{\text{Bog}} + (\delta - 1) \phi_{j_m} \sqrt{\epsilon_{j_m}^2 + 2 \epsilon_{j_m}} (< 0) \tag{4.1}
\]

with \( \delta = 1 + \sqrt{\epsilon_{j_m}} \).
\( \Phi \) The expression \( \mathcal{H}^{Bog}_{j_1,...,j_m}(z + z^{Bog}_{j_1,...,j_m-1}) \) is well defined and the final Feshbach-Schur Hamiltonian is

\[
\mathcal{H}^{Bog(N)}_{j_1,...,j_m}(z + z^{Bog}_{j_1,...,j_m-1}) = \mathcal{H}^{Bog}_{j_1,...,j_m}(z + z^{Bog}_{j_1,...,j_m-1}) \rho^{Bog}_{j_1,...,j_m-1}(z^{Bog})
\]

provided

\[
z \leq z_m + \gamma \Delta_{m-1} - O\left( \frac{1}{(\ln N)^2} \right) < E_{j_m}^{Bog} + \sqrt{\epsilon_{j_m}^2 + 2 \epsilon_{j_m}} < 0,\quad \gamma = \frac{1}{2},
\]

where \( z_m \) is the ground state energy of \( H_{j_1,...,j_m}^{Bog} \) and \( \Delta_{m-1} > 0 \) is a lower bound to the spectral gap above the ground state energy of \( H_{j_1,...,j_m-1}^{Bog} \). Both \( \Delta_{m-1} \) and \( O\left( \frac{1}{(\ln N)^2} \right) \) are specified later in Theorem 4.3.

Furthermore, there is a (unique) fixed point, \( z^{(m)} \), in the given range of \( z \) (see (4.3)) that solves \( \mathcal{H}^{Bog}_{j_1,...,j_m}(z^{(m)} + z^{Bog}_{j_1,...,j_m-1}) = 0 \).

Concerning requirement \((\text{II})\) we recall that

\[
H_{j_1,...,j_m}^{Bog} - z^{Bog}_{j_1,...,j_m-1} - z = \sum_{j \in \mathbb{Z}^d(\pm j_1,...,\pm j_m)} k_j^2 a_j^* a_j + \sum_{l=1}^{m-1} H_{j_l}^{Bog} - z^{Bog}_{j_1,...,j_m-1} + H_{j_m}^{Bog} - z.
\]

If we assume

\[
\text{infspec}\left\{ \sum_{l=1}^{m-1} H_{j_l}^{Bog} - z^{Bog}_{j_1,...,j_m-1} \geq -O\left( \frac{1}{(\ln N)^2} \right) \right\}
\]

we can reproduce a result analogous to Lemma 3.4 of [Pil] (reported in Section 2.1) for the Hamiltonian \( H_{j_1,...,j_m}^{Bog} \); see Corollary 5.1. Indeed, thanks to this input, the operator norm estimate (5.3) in Corollary 5.1 can be derived as if the modes \( \pm j_1,\ldots,\pm j_{m-1} \) were absent.

Consequently, the counterpart of Theorem 3.1 of [Pil] (see Section 2.1) can be proven for \( \mathcal{H}^{Bog(i)}_{j_1,...,j_m}(z + z^{Bog}_{j_1,...,j_m-1}) \):

**Theorem 4.1.** Assume condition a) of Corollary 5.1. Then, for

\[
z \leq E_{j_m}^{Bog} + (\delta - 1) \phi_{j_m} \sqrt{\epsilon_{j_m}^2 + 2 \epsilon_{j_m}} (< 0)
\]

with \( \delta = 1 + \sqrt{\epsilon_{j_m}}, \epsilon_{j_m} \) sufficiently small and \( N \) sufficiently large, the operators \( \mathcal{H}^{Bog(i)}_{j_1,...,j_m}(z + z^{Bog}_{j_1,...,j_m-1}), 0 \leq i \leq N - 2 \) and even, are well defined. For \( i = 2, 4, 6, \ldots, N - 2 \) they correspond to

\[
\mathcal{H}^{Bog(i)}_{j_1,...,j_m}(z + z^{Bog}_{j_1,...,j_m-1}) = Q_{j_m}^{(i)}(H_{j_1,...,j_m}^{Bog} - z - z^{Bog}_{j_1,...,j_m-1})Q_{j_m}^{(i+1)},
\]

\[
= Q_{j_m}^{(i+1)}H_{j_1,...,j_m}^{Bog}Q_{j_m}^{(i+1)} - Q_{j_m}^{(i+1)}W_{j_m}^{Bog}R_{j_1,...,j_m;i_i;i_j}^{Bog}R_{j_1,...,j_m;i_i;i_j}^{Bog}(z + z^{Bog}_{j_1,...,j_m-1})^i \sum_{l=0}^{\infty} \left[ R_{j_1,...,j_m;i_i;i_j}(z + z^{Bog}_{j_1,...,j_m-1})^l W_{j_m}^{Bog}Q_{j_m}^{(i+1)}
\]

where \( R_{j_1,...,j_m;i_i;i_j}^{Bog}(z + z^{Bog}_{j_1,...,j_m-1}) \) and \( R_{j_1,...,j_m;i_i;i_j}^{Bog}(z + z^{Bog}_{j_1,...,j_m-1}) \) are defined in (3.33)-(3.37).

\( ^4 \mathcal{H}^{Bog(i)}_{j_1,...,j_m}(z) \) is self-adjoint on the domain of the operator \( Q_{j_m}^{(i+1)}(H_{j_1,...,j_m}^{Bog} - z)Q_{j_m}^{(i+1)} \).
The following estimates hold true for $2 \leq i \leq N - 2$ and even:

$$
\| \Gamma_{j_1,\ldots,j_n}^{\text{Bog}} (z + z_{j_1,\ldots,j_n}^{\text{Bog}}) \| \leq \frac{1}{X_i} \psi_{j_i}
$$

(4.9)

where

$$
\Gamma_{j_1,\ldots,j_n}^{\text{Bog}} (z + z_{j_1,\ldots,j_n}^{\text{Bog}}) := \sum_{l=0}^{\infty} [ (R_{j_1,\ldots,j_n}^{\text{Bog}} (z) ) \Gamma_{j_1,\ldots,j_n}^{\text{Bog}} (z + z_{j_1,\ldots,j_n}^{\text{Bog}}) ]_{j_1,\ldots,j_n}^{\text{Bog}} (R_{j_1,\ldots,j_n}^{\text{Bog}} (z + z_{j_1,\ldots,j_n}^{\text{Bog}}) )^{l}]
$$

(4.10)

and $X_i$ is defined in Lemma 3.6 of [Pi1] (that is reported in Section 2.1) and fulfills the bound

$$
X_{2j} \geq \frac{1}{2} \left[ 1 + \sqrt{\eta a_{e}} - \frac{b_{\epsilon} / \sqrt{\eta a_{e}}}{N - 2j - \Theta} \right] > 0
$$

(4.11)

with $\eta = 1 - \sqrt{\epsilon}$, $\Theta := \min \{ 2(\nu - \frac{1}{8}) ; \frac{1}{4} \}$, and $a_{e}$, $b_{\epsilon}$ are those defined in Corollary 5.1.

Proof

The proof is identical to Theorem 3.1 of [Pi1] using the estimates provided in Corollary 5.1. $\square$

Requirement U2) has been proven to hold for the first step (i.e., for the Hamiltonian $H_{j_1}^{\text{Bog}}$) in Corollary 4.6 of [Pi1] (see Section 2.1) with $\Delta_0 := \min \{ \langle k_j \rangle ^2 ; j \in \mathbb{Z}^d \setminus \{0\} \}$. The proof for the successive steps (i.e., for the Hamiltonians $H_{j_1,\ldots,j_n}^{\text{Bog}}$) is not straightforward, rather it requires an inductive procedure implemented in Theorem 4.3. To better understand the strategy of Theorem 4.3 we must explain the new difficulties in some detail.

In the first step (i.e., for $H_{j_1}^{\text{Bog}}$) the function

$$
f_{j_1}^{\text{Bog}} (z) := -z - \langle \eta, W_{j_1}^{\text{Bog}} : N_2, N_2 \rangle (z) \sum_{l=0}^{\infty} [ ( \Gamma_{j_1}^{\text{Bog}} (N_2) ) \Gamma_{j_1}^{\text{Bog}} (z + z_{j_1}^{\text{Bog}}) ]_{j_1}^{\text{Bog}} (N_2) W_{j_1}^{\text{Bog}} (z)]
$$

(4.12)

is well defined as a result of the construction of $\mathcal{G}_{j_1}^{\text{Bog}(N_2)} (z)$. Next, according to the scheme used in [Pi1], starting from the existence of the (unique) solution $z^{(1)} \equiv z_1 < E_{j_1}^{\text{Bog}} + \sqrt{\epsilon_{j_1}}, \sqrt{\epsilon_{j_1}^2 + 2 \epsilon_{j_1}}$ of $f_{j_1} (z) = 0$ we show the invertibility on $\mathcal{P}_{\eta} F^{N_2}$ of the operator

$$
\mathcal{P}_{\eta} \mathcal{G}_{j_1}^{\text{Bog}(N_2)} (z) \mathcal{P}_{\eta}
$$

(4.13)

for $z \leq \min \{ z_1 + \frac{\Delta_0}{2} ; E_{j_1}^{\text{Bog}} + \sqrt{\epsilon_{j_1}} \sqrt{\epsilon_{j_1}^2 + 2 \epsilon_{j_1}} \}$. On the contrary, starting from $m = 2$ the definition of $f_{j_1,\ldots,j_n}^{\text{Bog}} (z + z_{j_1,\ldots,j_n}^{\text{Bog}})$ (see (3.31)) requires the existence of

$$
\mathcal{P}_{\psi_{j_1,\ldots,j_n}^{\text{Bog}}} \mathcal{G}_{j_1,\ldots,j_n}^{\text{Bog}(N_2)} (z + z_{j_1,\ldots,j_n}^{\text{Bog}}) \mathcal{P}_{\psi_{j_1,\ldots,j_n}^{\text{Bog}}}.
$$

(4.14)

The latter is proven in Lemma 4.5 for $z$ in the interval specified in (4.3). Then, we can define the function $f_{j_1,\ldots,j_n}^{\text{Bog}} (z + z_{j_1,\ldots,j_n}^{\text{Bog}})$.
In order to determine the solution, \( z^{(m)} \), to the equation in \( z \)

\[
f_{j_{1},...,j_{m}}^{Bog} (z + z_{j_{1},...,j_{m-1}}^{Bog}) = 0 , \tag{4.15}
\]

we exploit that the fixed point problem at the \( m - th \) step boils down to the fixed point problem for a three-modes system if in the formula that defines \( f_{j_{1},...,j_{m}}^{Bog} (z + z_{j_{1},...,j_{m-1}}^{Bog}) \) the vector \( \psi_{j_{1},...,j_{m-1}}^{Bog} \)
is replaced with \( R^{Bog} \) and

\[
R_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z_{j_{1},...,j_{m-1}}^{Bog} + z) \sum_{l=2}^{\infty} \left[ \Gamma_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z_{j_{1},...,j_{m-1}}^{Bog} + z) R_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z_{j_{1},...,j_{m-1}}^{Bog} + z) \right]^{l_{N-2}}
\]

is replaced with

\[
R_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z) \sum_{l=2}^{\infty} \left[ \Gamma_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z) R_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z) \right]^{l_{N-2}} \tag{4.16}
\]

where \( \tilde{\cdot} \) means that

\[
\sum_{l=1}^{m-1} R_{j_{l}}^{Bog}
\]
is omitted in the resolvents \( R_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z) \) and in all the other resolvents entering the definition of the operators \( \Gamma_{j_{1},...,j_{m};i_{d}}^{Bog} (z) \). In fact, with the help of Lemma 4.4 we show that

\[
f_{j_{1},...,j_{m}}^{Bog} (z + z_{j_{1},...,j_{m-1}}^{Bog}) = f_{j_{m}}^{Bog} (z) + o_{N \rightarrow \infty} (1).
\]

The result concerning \( \Pi(2) \) is finally proven in Theorem 4.3. To this end, various preliminary ingredients are needed. Definition 4.1 and Proposition 4.2 deal with the re-expansion of \( \Gamma_{j_{1},...,j_{m};N-2,N-2}^{Bog} (z_{j_{1},...,j_{m-1}}^{Bog} + z) \) and are generalizations of structures already encountered in [P1]. This re-expansion is crucial to implement the mechanisms behind the proofs of Lemma 4.4 and Lemma 4.5, that are explained in the Outline of the proof provided in this section for both.

To streamline formulae, in Definition 4.1, Proposition 4.2, and Remarks 4.2 and 4.3 below, we write \( W_{j_{1},j_{2}}^{*}, W_{j_{2},j_{1}}^{*}, R_{j_{1},j_{2}}^{Bog} (w) \), and \( \Gamma_{j_{1},j_{2}}^{Bog} (w) \) instead of \( W_{j_{1},j_{2}}^{*}, W_{j_{2},j_{1}}^{*}, R_{j_{1},j_{2}}^{Bog} (z_{j_{1},...,j_{m-1}}^{Bog} + z) \), and \( \Gamma_{j_{1},...,j_{m};j_{1},...,j_{m-1}}^{Bog} (z_{j_{1},...,j_{m-1}}^{Bog} + z) \), respectively.

**Definition 4.1.** Let \( h \in \mathbb{N} \), \( h \geq 2 \), and

\[
w \leq z_{j_{1},...,j_{m-1}}^{Bog} + \epsilon_{j_{m}}^{Bog} + (\delta - 1) \phi_{j_{m}} \sqrt{\epsilon_{j_{m}}^{2} + 2 \epsilon_{j_{m}}} . \tag{4.18}
\]

with \( \delta \leq 1 + \sqrt{\epsilon_{j_{m}}} \). Assume condition a) of Corollary 5.1, and let \( \epsilon_{j_{m}} \) be sufficiently small and \( N \) sufficiently large. We define:

1. For \( N - 2 \geq j \geq 4 \) and even

\[
[\Gamma_{j_{1},j_{2}}^{Bog} (w)]_{j_{1},...,j_{m},j_{2},h_{m}} := [\Gamma_{j_{1},j_{2}}^{Bog} (w)]_{j_{1},...,j_{m},j_{2},h_{m}}^{(0)} + [\Gamma_{j_{1},j_{2}}^{Bog} (w)]_{j_{1},...,j_{m},j_{2},h_{m}}^{(0)} \tag{4.19}
\]

where

\[
[\Gamma_{j_{1},j_{2}}^{Bog} (w)]_{j_{1},...,j_{m},j_{2},h_{m}}^{(0)} := W_{j_{1},j_{2}} R_{j_{1},j_{2}}^{Bog} (w) W_{j_{1},j_{2}}^{*} \quad \text{for } j \geq 2 \tag{4.20}
\]
3. For $N - 2 \geq j \geq 4$ and even

$$\Gamma_{j,j+1}^{Bog}(w)_{(j-2, h_2)} := W_{j,j-2} \left( R_{j-2,j-2}^{Bog}(w) \right)^{\frac{1}{2}} \times$$

$$\times \sum_{l,j=1}^{h-1} \left[ (R_{j-2,j-2}^{Bog}(w))^2 W_{j-2,j-4} R_{j-4,j-2}^{Bog}(w) W_{j-4,j-2} R_{j-2,j-2}^{Bog}(w) \right]^{l-2} (R_{j-2,j-2}^{Bog}(w))^2 W_{j-2,j}^* \right]^{l-2} (R_{j-2,j-2}^{Bog}(w))^2 W_{j-2,j}^* ,$$

for $N - 2 \geq j \geq 4$ and even

$$\Gamma_{j,j}^{Bog}(w)_{(j-2, h_2)} := W_{j,j-2} \left( R_{j-2,j-2}^{Bog}(w) \right)^{\frac{1}{2}} \times$$

$$\times \sum_{l,j=1}^{h-1} \left[ (R_{j-2,j-2}^{Bog}(w))^2 \Gamma_{j,j}^{Bog}(w)_{(j-2, h_2)} R_{j-2,j-2}^{Bog}(w) \right]^{l-2} \times$$

$$\times (R_{j-2,j-2}^{Bog}(w))^2 W_{j-2,j}^*. \quad (4.24)$$

2. For $N - 2 \geq j \geq 6, 2 \leq l \leq j - 4$ (both even numbers)

$$\Gamma_{j,j+1}^{Bog}(w)_{(l,h; j+2, h; \ldots; j-4, h; j-2, h_2)} := W_{j,j-2} \left( R_{j-2,j-2}^{Bog}(w) \right)^{\frac{1}{2}} \times$$

$$\times \sum_{l,j=1}^{h-1} \left[ (R_{j-2,j-2}^{Bog}(w))^2 R_{j-2,j-2}^{Bog}(w) \right]^{l-2} \times$$

$$\times (R_{j-2,j-2}^{Bog}(w))^2 W_{j-2,j}^*. \quad (4.25)$$

Here, the symbol $\sum_{l,j=1}^{h-1}$ stands for a sum of terms resulting from operations A1 and A2 below:

A1) At fixed $1 \leq l,j-2 \leq h - 1$ summing all the products

$$\left[ (R_{j-2,j-2}^{Bog}(w))^2 X(R_{j-2,j-2}^{Bog}(w)) \right]^{l-2} \quad (4.27)$$

that are obtained by replacing $X$ for each factor with the operators (defined by iteration) of the type $\Gamma_{j,j}^{Bog}(w)_{(s,h; s+2,h; \ldots; j-4,h; j-2, h_2)}$ with $l \leq s \leq j-4$ and even, with the constraint that if $l \leq j-6$ then $X$ is replaced with $\Gamma_{j,j}^{Bog}(w)_{(l,h; j+2, h; \ldots; j-4, h; j-2, h_2)}$ in one factor at least, whereas if $l = j-4$ then $X$ is replaced with $\Gamma_{j,j}^{Bog}(w)_{(j-4, h; j-2, h_2)}$ in one factor at least;

A2) Summing from $l = 1$ up to $l = j-2 \leq h - 1$.

3. For $N - 2 \geq j \geq 6, 2 \leq l \leq j - 4$ and even

$$\Gamma_{j,j+1}^{Bog}(w)_{(l,h; j+2, h; \ldots; j-4, h; j-2, h_2)} := W_{j,j-2} \left( R_{j-2,j-2}^{Bog}(w) \right)^{\frac{1}{2}} \times$$

$$\sum_{l,j=1}^{h-1} \left[ (R_{j-2,j-2}^{Bog}(w))^2 \right]^{l-2} \times$$

$$\times (R_{j-2,j-2}^{Bog}(w))^2 W_{j-2,j}^*. \quad (4.29)$$

Here, the symbol $\sum_{l,j=1}^{h-1}$ stands for a sum of terms resulting from operations B1 and B2 below:
B1) At fixed \(1 \leq l_{j-2} \leq h - 1\), summing all the products

\[
\left[ (R_{j-2,j-2}^{Bog}(w))^\frac{1}{2} X(R_{j-2,j-2}^{Bog}(w))^\frac{1}{2} \right]_{l_{j-2}}^2
\]

(4.30)

that are obtained by replacing \(X\) for each factor with the operators (iteratively defined) of the type \([\Gamma_{j-2,j-2}^{Bog}(z)](s_{i,j};i+2,h_{i};i-2,h_{i})\) and \([\Gamma_{j-2,j-2}^{Bog}(z)](s_{i,j};i+2,h_{i};i-2,h_{i})\) with \(l \leq s \leq j - 4\) and \(2 \leq s' \leq j - 4\) where \(s\) and \(s'\) are even, and with the constraint that \(X\) is replaced with \([\Gamma_{j-2,j-2}^{Bog}(z)](l_{h_{i}},i+2,h_{i};i-2,h_{i})\) in one factor at least.

B2) Summing from \(l_{j-2} = 1\) up to \(h - 1\).

The definitions of above can be adapted in an obvious manner to the case \(h = \infty\), in particular the terms \([\Gamma_{j,j}^{Bog}(w)](l_{h_{i}},i+2,h_{i};i-2,h_{i})\) are absent.

**Proposition 4.2.** Assume condition a) of Corollary 5.1, and let \(\epsilon_{m} \equiv \epsilon\) be sufficiently small and \(N\) sufficiently large. For any fixed \(2 \leq h \in \mathbb{N}\) and for \(N - 2 \geq i \geq 4\) and even, the splitting

\[
\Gamma_{i,j}^{Bog}(w) = \sum_{l=2, l \text{ even}}^{i-2} \left[ \Gamma_{i,j}^{Bog}(w) \right]_{l_{h_{i}},i+2,h_{i};i-2,h_{i}} + \sum_{l=2, l \text{ even}}^{i-2} \left[ \Gamma_{i,j}^{Bog}(w) \right]_{l_{h_{i}},i+2,h_{i};i-2,h_{i}}
\]

(4.31)

holds true for \(w \leq \epsilon_{j,h_{i}-1}^{Bog} + E_{m}^{Bog} \) and \((\delta - 1)\phi_{m} \sqrt{e_{m}^{2} + 2\epsilon_{m}}\) and \(\delta \leq 1 + \sqrt{\epsilon_{m}}\). Moreover, for \(2 \leq i \leq 2 - 2\) and even, the estimates

\[
\left\| (R_{i,j}^{Bog}(w))^{\frac{1}{2}} \left[ \Gamma_{i,j}^{Bog}(w) \right]_{l_{h_{i}},i+2,h_{i};i-2,h_{i}} (R_{i,j}^{Bog}(w))^{\frac{1}{2}} \right\| \leq \prod_{f=l+2, f-\text{even}}^{i} \frac{K_{f,\epsilon}}{(1 - Z_{f-2,\epsilon})^{2}}
\]

(4.32)

and

\[
\left\| (R_{i,j}^{Bog}(w))^{\frac{1}{2}} \left[ \Gamma_{i,j}^{Bog}(w) \right]_{l_{h_{i}},i+2,h_{i};i-2,h_{i}} (R_{i,j}^{Bog}(w))^{\frac{1}{2}} \right\| \leq (Z_{i,\epsilon})^{h} \prod_{f=l+2, f-\text{even}}^{i} \frac{K_{f,\epsilon}}{(1 - Z_{f-2,\epsilon})^{2}}
\]

(4.33)

hold true, where

\[
K_{i,\epsilon} := \frac{1}{4(1 + a_{\epsilon} - \frac{2b_{\epsilon}}{N-i+1} - \frac{1-c_{\epsilon}}{(N-i+1)^{2}})} \quad , \quad Z_{i-2,\epsilon} := \frac{1}{4(1 + a_{\epsilon} - \frac{2b_{\epsilon}}{N-i+3} - \frac{1-c_{\epsilon}}{(N-i+3)^{2}})} \left[ 1 + \sqrt{\eta_{\epsilon} - \frac{h_{2}/\sqrt{\epsilon}}{N-i+4}} \right].
\]

(4.34)

where \(a_{\epsilon}, b_{\epsilon}, c_{\epsilon}\) are those defined in Corollary 5.1, and \(\Theta\) is defined in Lemma 3.6 of [Pi1] (reported in Section 2.1).

**Proof**

The proof is a straightforward generalization of Proposition 4.10 of [Pi1], which is possible thanks to estimate (5.3) in Corollary 5.1 that implies (4.9) through Lemma 3.6 of [Pi1] (reported in Section 2.1). □

**Remark 4.2.** From the definitions in (4.34) and the \(\epsilon\)-dependence of \(a_{\epsilon}, b_{\epsilon},\) and \(c_{\epsilon}\) (see (5.5)-(5.6)-(5.7)), it is evident that there exist constants \(C, c > 0\) such that (assuming \(\epsilon\) sufficiently small)

\[
\frac{K_{f,\epsilon}}{(1 - Z_{f-2,\epsilon})^{2}} \leq \frac{1}{1 + c \sqrt{\epsilon}}
\]

(4.35)
for $N - f > \frac{C}{\sqrt{\varepsilon}}$. With a similar computation, one can check that for $N - 2 \geq i > N - \frac{C}{\sqrt{\varepsilon}}$ and some $c' > 0$

$$
K_{f,e} \leq (1 + c' \frac{\sqrt{\varepsilon}}{N - f} + O(\frac{1}{(N - f)^2}))
$$

(4.36)

In consequence, for $N - 2 \geq i > N - \frac{C}{\sqrt{\varepsilon}}$ (and assuming for simplicity that $N - \frac{C}{\sqrt{\varepsilon}}$ is an even number) the inequality

$$
\prod_{f=N - \frac{C}{\sqrt{\varepsilon}}, f \text{ even}}^{i} \frac{K_{f,e}}{(1 - Z_{f-2,e})^2}
$$

(4.37)

$$
\leq \prod_{f=N - \frac{C}{\sqrt{\varepsilon}}, f \text{ even}}^{i} (1 + c' \frac{\sqrt{\varepsilon}}{N - f} + O(\frac{1}{(N - f)^2}))
$$

(4.38)

$$
= \prod_{f=N - \frac{C}{\sqrt{\varepsilon}}, f \text{ even}}^{i} \exp[\ln (1 + c' \frac{\sqrt{\varepsilon}}{N - f} + O(\frac{1}{(N - f)^2}))] \leq O(1)
$$

(4.39)

holds true. Therefore, we can conclude that:

1) If $N - \frac{C}{\sqrt{\varepsilon}} \geq i \geq l + 2$

$$
\prod_{f=l+2, f \text{ even}}^{i} \frac{K_{f,e}}{(1 - Z_{f-2,e})^2} \leq O((\frac{1}{1 + c \sqrt{\varepsilon}})^{i-l-2}) ;
$$

(4.40)

2) If $N - 2 \geq i > N - \frac{C}{\sqrt{\varepsilon}} \geq l + 2$, then

$$
\prod_{f=l+2, f \text{ even}}^{i} \frac{K_{f,e}}{(1 - Z_{f-2,e})^2}
$$

(4.41)

$$
= [\prod_{f=l+2, f \text{ even}}^{N - \frac{C}{\sqrt{\varepsilon}}} \frac{K_{f,e}}{(1 - Z_{f-2,e})^2}] [\prod_{f=N - \frac{C}{\sqrt{\varepsilon}}, f \text{ even}}^{i} \frac{K_{f,e}}{(1 - Z_{f-2,e})^2}]
$$

(4.42)

$$
\leq O((\frac{1}{1 + c \sqrt{\varepsilon}})^{N - \frac{C}{\sqrt{\varepsilon}} - l - 2}).
$$

(4.43)

**Remark 4.3.** In this remark we explain how to provide an estimate of

$$
\|(R_{i,j}^{\text{Bog}}(w))^{\frac{1}{2}} \sum_{l=2, l \text{ even}}^{i-2} \prod_{l=2, l \text{ even}}^{i-2} \|F_{i,j}^{\text{Bog}}(w)\|_{(l; l+2, h; \ldots; l-2, h)} (R_{i,j}^{\text{Bog}}(w))^{\frac{1}{2}}\|, \ i \leq N - 2,
$$

(4.44)

without using (4.32) and the computations in Remark 4.2. Indeed, this would make the estimate worse.

The operator in (4.44) can be expressed as a sum of products of operators of the type in (2.31). We call “blocks” the operators of the type in (2.31) and define

$$
\mathcal{E}(\|(R_{i,j}^{\text{Bog}}(z))^{\frac{1}{2}} \sum_{l=2, l \text{ even}}^{i-2} \prod_{l=2, l \text{ even}}^{i-2} \|F_{i,j}^{\text{Bog}}(w)\|_{(r; r+2, h; \ldots; l-2, h)} (R_{i,j}^{\text{Bog}}(z))^{\frac{1}{2}}\|)
$$

(4.45)
the upper bound (to the operator norm in the argument of $E$) obtained estimating the norm of the sum (of the operators) with the sum of the norms of the summands, and the norm of each operator product with the product of the norms of the blocks. The estimate of the norm of each block is provided by Corollary 5.1 in the Appendix.

Next, we point out that

- by using the decomposition in (4.31) of Proposition 4.2 for $h' \equiv \infty$, we get
\[
(R_{i,i}^{\text{Bog}}(w))^{\frac{1}{2}} \Gamma_{i,i}^{\text{Bog}}(w)(R_{i,i}^{\text{Bog}}(w))^{\frac{1}{2}}
\]
(4.46)
and
\[
(R_{i,i}^{\text{Bog}}(w))^{\frac{1}{2}} \sum_{l=2,1 \text{even}}^{i-2} [\Gamma_{i,i}^{\text{Bog}}(w)]_{(l,h',d+2,h',\ldots;i-2,h')} (R_{i,i}^{\text{Bog}}(w))^{\frac{1}{2}},
\]
(4.47)
where the last step follows for $\epsilon$ sufficiently small from the identity
\[
(R_{i,i}^{\text{Bog}}(w))^{\frac{1}{2}} \Gamma_{i,i}^{\text{Bog}}(w)(R_{i,i}^{\text{Bog}}(w))^{\frac{1}{2}}
\]
(4.52)
and from estimates (5.4), (4.9), and (4.11).

In the next lemma we develop some tools for Theorem 4.3. More precisely, the results in Lemma 4.4 are used later to prove that
\[
f_{j_1,\ldots,j_m}(z + z_{j_1,\ldots,j_m}^{\text{Bog}}) = f_{j_m}^{\text{Bog}}(z) + o_{N\to\infty}(1).
\]

**Lemma 4.4.** Let $2 \leq m \leq M$ and assume that the Hamiltonian $H_{j_1,\ldots,j_m}^{\text{Bog}}$ has ground state vector $\psi_{j_1,\ldots,j_m}^{\text{Bog}}$ (see (3.41)-(3.42)) with ground state energy $z_{j_1,\ldots,j_m}^{\text{Bog}}$. Furthermore, assume:
\[ \text{STEP III) From Step II), an expression consisting of a finite sum of (finite) products with inner factors of the type} \]
\[
(R_{j_1,\ldots,j_m; l_i, m-2}^{\text{Bog}}(w))^\frac{1}{2} W_{j_1,\ldots,j_m; i, l-2, i-2, l-2}(w)^\frac{1}{2} \quad \text{and} \quad (R_{j_1,\ldots,j_m; i, l-2, i-2, l-2}^{\text{Bog}}(w))^\frac{1}{2} W_{j_1,\ldots,j_m; i-2, l}(R_{j_1,\ldots,j_m; i, l})^\frac{1}{2}
\]
and the two outer factors

\[ W_{j_0}(R_{j_1,\ldots,j_m;N-2,N-2}(w)) \frac{1}{2}, \quad (R_{j_1,\ldots,j_m;N-2,N-2}(w))^\frac{1}{2} W_{j_0}, \] (4.61)

is left. Then, we make use of the main mechanism of the proof that we highlight below:

After the truncations in Steps I) and II), in the resulting expression the resolvents \( R_{j_1,\ldots,j_m;N-2,N-2}(w) \) are sufficiently “close” to the vector \( \psi_{j_1,\ldots,j_m}^{Bog} \), hence they can be replaced with \( \tilde{R}_{j_1,\ldots,j_m;i}(z) \) (see the definition after (4.17)) up to a small remainder, by exploiting the identity

\[ \sum_{j=1}^{m-1} \hat{H}_{j}^{Bog} \psi_{j_1,\ldots,j_m}^{Bog} = z_{j_1,\ldots,j_m}^{Bog} \psi_{j_1,\ldots,j_m}^{Bog}, \]

**STEP IV** In this step the resolvents \( \tilde{R}_{j_1,\ldots,j_m;i}(z) \) are replaced with \( z \)-dependent \( c \) – numbers and the vector \( \psi_{j_1,\ldots,j_m}^{Bog} \) is replaced with \( \eta \). One more remainder term is produced.

Next lemma deals with the invertibility of

\[ \mathcal{P}_{\partial_{j_1,\ldots,j_m-1}}^{\psi_{j_1,\ldots,j_m}^{Bog}(N-2)}(z + z_{j_1,\ldots,j_m}^{Bog}) \mathcal{P}_{\partial_{j_1,\ldots,j_m-1}}^{\psi_{j_1,\ldots,j_m}^{Bog}} \]

on \( \mathcal{P}_{\partial_{j_1,\ldots,j_m-1}} \mathcal{F}^N \).

**Lemma 4.5.** Let \( 2 \leq m \leq M \) and assume that the Hamiltonian \( H_{j_1,\ldots,j_m}^{Bog} \) has nondegenerate ground state energy \( z_{j_1,\ldots,j_m}^{Bog} \) and ground state vector \( \psi_{j_1,\ldots,j_m}^{Bog} \) with the property in (4.56). Furthermore, assume that:

1. There exists \( \Delta_0 \geq \Delta_{m-1} > 0 \) such that

\[ \inf \text{spec} \left[ \left( \hat{H}_{j_1,\ldots,j_m}^{Bog} + \sum_{j \in \mathbb{Z} \setminus \{j_1,\ldots,j_m\}} (k_j)^2 a_j^* a_j \right) \cal{P}_{\partial_{j_1,\ldots,j_m-1}}^{(N-2)}(\cal{F}^N) \cap \cal{C} \psi_{j_1,\ldots,j_m}^{Bog} \right] - z_{j_1,\ldots,j_m}^{Bog} \geq \Delta_{m-1} \]

(4.62)

where \( \cal{O}_{\partial_{j_1,\ldots,j_m-1}}^{(N-2)} \cal{F}^N \) is the subspace of states in \( \cal{F}^N \) with no particles in the modes \( \pm j_m \), and \( \cal{C} \psi_{j_1,\ldots,j_m}^{Bog} \) is the subspace generated by the vector \( \psi_{j_1,\ldots,j_m}^{Bog} \). (Notice that

\[ \hat{H}_{j_1,\ldots,j_m}^{Bog} + \sum_{j \in \mathbb{Z} \setminus \{j_1,\ldots,j_m\}} (k_j)^2 a_j^* a_j = H_{j_1,\ldots,j_m}^{Bog} - (k_{j_m})^2 (a_{j_m}^* a_{j_m} + a_{j_m}^* a_{j_m} - a_{j_m}), \]

i.e., the kinetic energy associated with the modes \( \pm j_m \) is absent.)

2.

\[ \inf \text{spec} \left[ \sum_{j=1}^{m-1} \hat{H}_{j}^{Bog} \right] - z_{j_1,\ldots,j_m}^{Bog} \geq \frac{m-1}{(\ln N)^2}. \] (4.63)

Let \( \epsilon_{j_m} \) be sufficiently small and \( N \) sufficiently large such that:

a) for

\[ z \leq E_{j_m}^{Bog} + \sqrt{\epsilon_{j_m} \phi_{j_m} \epsilon_{j_m}^2 + 2 \epsilon_{j_m}} (< 0) \]

the Feshbach-Schur flow associated with the couple of modes \( \pm j_m \) is well defined (see Theorem 4.1) up to the index value \( i = N - 2; \)
b) \[ \frac{\ln N}{N} \ll 1 \quad \text{and} \quad \frac{U_{j_m}}{\sqrt{N}} < \frac{\Delta_{m-1}}{2} \quad \text{where} \quad U_{j_m} := k_{j_m}^2 + \phi_{j_m}; \] (4.64)

c) \[ z_m + \gamma \Delta_{m-1} < E_{j_m}^{Bog} + \frac{1}{2} \sqrt{\epsilon_{j_m} \phi_{j_m}} \sqrt{\epsilon_{j_m}^2 + 2 \epsilon_{j_m}}, \quad \gamma = \frac{1}{2}, \] (4.65)

where \( z_m \) is the ground state energy of \( H_{j_m}^{Bog} \).

Then, there exists a constant \( C^\perp > 0 \) such that for
\[ z \leq z_m - \frac{C^\perp}{(\ln N)^{\frac{1}{2}}} + \gamma \Delta_{m-1} \] (4.66)

the following estimate holds true:
\[ \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \mathcal{H}_{j_1 \ldots j_m}^{Bog} (N-2)(z + z_{j_1 \ldots j_{m-1}}) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \geq (1 - \gamma) \Delta_{m-1} \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} . \] (4.67)

Outline of the proof. Like for Lemma 4.4, the detailed proof is deferred to the Appendix; see Lemma 5.3. Here, we outline the procedure. We start observing that due to the definition in (3.32)

\[ \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \mathcal{H}_{j_1 \ldots j_m}^{Bog} (N-2)(z + z_{j_1 \ldots j_{m-1}}) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \]
\[ = \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \mathcal{H}_{j_1 \ldots j_m}^{Bog} (z - z_{j_1 \ldots j_{m-1}}) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} - \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \mathcal{H}_{j_1 \ldots j_m}^{Bog} (z - z_{j_1 \ldots j_{m-1}}) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \]
\[ = \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} (H_{j_1 \ldots j_m}^{Bog} + \sum_{j \neq \{j_1 \ldots j_{m-1}\}} (k_j)^2 a_j a_j - z_{j_1 \ldots j_{m-1}} - z) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} - \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} (H_{j_1 \ldots j_m}^{Bog} + \sum_{j \neq \{j_1 \ldots j_{m-1}\}} (k_j)^2 a_j a_j - z_{j_1 \ldots j_{m-1}} - z) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} . \]

Hence, the result in (4.67) is proven if
\[ \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \mathcal{H}_{j_1 \ldots j_m}^{Bog} (N-2)(z + z_{j_1 \ldots j_{m-1}}) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \]
\[ = \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} (H_{j_1 \ldots j_m}^{Bog} + \sum_{j \neq \{j_1 \ldots j_{m-1}\}} (k_j)^2 a_j a_j - z_{j_1 \ldots j_{m-1}} - z) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} - \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} (H_{j_1 \ldots j_m}^{Bog} + \sum_{j \neq \{j_1 \ldots j_{m-1}\}} (k_j)^2 a_j a_j - z_{j_1 \ldots j_{m-1}} - z) \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} \]
\[ \geq (1 - \gamma) \Delta_{m-1} \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}} . \] (4.73)

Assuming \( \epsilon_{j_m} \) sufficiently small and \( N \) sufficiently large, if \( z < -4\phi_{j_m} \), the inequality in (4.75) follows easily from the assumption in (4.62), the estimates in (4.9) and (4.11), and the norm bound
\[ \| W_{j_m} (R_{j_1 \ldots j_m}^{Bog} + (z_{j_1 \ldots j_{m-1}} + z))^2 \| < \frac{\sqrt{\phi_{j_m}}}{2} . \] (4.76)

Assume that for
\[ -4\phi_{j_m} \leq z < z_m + \gamma \Delta_{m-1} \left( E_{j_m}^{Bog} + \frac{1}{2} \sqrt{\epsilon_{j_m} \phi_{j_m}} \sqrt{\epsilon_{j_m}^2 + 2 \epsilon_{j_m}} < 0 \right) \] (4.77)

\[ \text{This condition holds for } N \text{ sufficiently large because in Corollary 4.6 of [Pil] we show that in the limit } N \to \infty \text{ the ground state energy, } z_m \text{, of } H_{j_m}^{Bog} \text{ tends to } E_{j_m}^{Bog} \text{ (see Section 2.2), and because } \gamma \Delta_{m-1} \leq \frac{\Delta}{2}. \]
we can show

\[
\left\| \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}}^{Bog} \right\| = \left\| \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}}^{Bog} W_{j_1 \ldots j_{m-1}}^{Bog} R_{j_1 \ldots j_{m-1}}^{Bog} (z) \right\| \geq \frac{m}{N} \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} \sqrt{\frac{z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}}{\phi_{j_m}}}} \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}) + O\left( \frac{1}{(\ln N)^{\frac{1}{2}}} \right)
\]

where \( \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z) \) is introduced in (2.45)-(2.48) (with \( j_m \equiv j_m \)) and \( U_{j_m} \) is defined in (4.64). Then, we can readily set a positive constant \( C^\perp \) such that also for \( z \) in the intersection of the two intervals in (4.77) and (4.66),

\[
(4.71) + (4.72) \geq \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}}^{Bog} (\hat{H}_{j_1 \ldots j_{m-1}}^{Bog} + \sum_{j \neq \lfloor j_1, \ldots, j_m \rfloor} (k_j)^2 a^*_ja_j - z_{j_1 \ldots j_{m-1}} - z) \geq \frac{m}{N} \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} \sqrt{\frac{z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}}{\phi_{j_m}}}} \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}) + C^\perp \frac{1}{(\ln N)^{\frac{1}{2}}} \mathcal{P}_{\phi_{j_1 \ldots j_{m-1}}}^{Bog}
\]

where in the steps from (4.83) to (4.84) and from (4.84) to (4.85) we have used the following ingredients:

1. \( \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z) \) is nondecreasing (see Remark 2.3) in the given range of \( z \), therefore

\[
-(1 - \frac{1}{N}) \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} \sqrt{\frac{z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}}{\phi_{j_m}}}} \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}})
\]

(4.86)

(4.87)

because \( z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}} < z - \Delta_{m-1} \frac{1}{2} < z_m \) (see (4.64) and (4.77));

2. By definition

\[
f_{j_m}(z) := -z - (1 - \frac{1}{N}) \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} \sqrt{\frac{z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}}{\phi_{j_m}}}} \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z) = 0,
\]

(4.88)

and \( z_m \) solves the fixed point equation \( f_{j_m}(z) = 0 \), therefore

\[
-(1 - \frac{1}{N}) \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} \sqrt{\frac{z - \Delta_{m-1} + \frac{U_{j_m}}{\sqrt{N}}}{\phi_{j_m}}}} \mathcal{G}_{j_1 \ldots j_{m-1}}^{N-2,N-2}(z_m) = z_m;
\]

(4.89)
3. Due to the assumption in (4.62),
\[
\mathcal{P}_{\psi_{j_1,\ldots,j_m}}(\hat{H}_{j_1,\ldots,j_m}^{Bog} + \sum_{j \notin \{j_1,\ldots,j_m\}} (k_j)^2 a_j^* a_j - z_{j_1,\ldots,j_m}) \mathcal{P}_{\psi_{j_1,\ldots,j_m}} \geq \Delta_{m-1} \mathcal{P}_{\psi_{j_1,\ldots,j_m}};
\]
due to (4.66), \(-z + z_m < \frac{C_l}{(\ln N)^2} \geq \frac{\Delta_{m-1}}{2}.
\]
In order to prove the key inequality in (4.81), in Lemma 5.3 we implement an \(h\)-dependent truncation\(^6\) of the sum over \(l_{N-2}\) and of \(\Gamma_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog}),\)
\[
= \| \mathcal{P}_{\psi_{j_1,\ldots,j_m}} W_{j_1,\ldots,j_m} R_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog}) \|
\times \sum_{l_{N-2} = 0}^{h-1} \left[ \Gamma_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog}) \right] r_h R_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog}) \left[ l_{N-2} W_{j_1,\ldots,j_m}^{*} \mathcal{P}_{\psi_{j_1,\ldots,j_m}} \right]
+ O((\frac{4}{5})^h) + O((\frac{1}{1 + \epsilon \sqrt{\eta_v}})^h)
\]
(4.93)
where \([\Gamma_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog})] r_h\) is defined in Lemma 5.2; see (5.26). This truncation relies on the results of Proposition 4.2. Next, we exploit the following mechanism:

For \(h\) sufficiently “small” with respect to \(N\) all the resolvents in expression (4.92) are sufficiently “close” to \(\mathcal{P}_{\psi_{j_1,\ldots,j_m}}\) so that up to a small error the operator
\[
\sum_{j \in \mathbb{Z}^d \setminus \{j_1,\ldots,j_m\}} (k_j)^2 a_j^* a_j + \hat{H}_{j_1,\ldots,j_m}^{Bog} - z_{j_1,\ldots,j_m}^{Bog}
\]
(4.94)
contained in each resolvent is bounded below by
\[
\Delta_{m-1} \mathcal{P}_{\psi_{j_1,\ldots,j_m}}
\]
in a sense specified in the detailed proof (Lemma 5.3). The implementation requires some lengthy technical steps; see a), b), c), d), e), and f) in Lemma 5.3.

The construction of the Feshbach-Schur flow and of the ground state for each Hamiltonian \(H_{j_1,\ldots,j_m}^{Bog}, 1 \leq m \leq M,\) is the content of Theorem 4.3 below that concerns five properties proven by induction. For the convenience of the reader, we list the five properties and outline the structure of the proof.

Property 1. ensures the construction of the Feshbach-Schur flow \(\mathcal{M}_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog})\) \((0 \leq i \leq N)\) up to \(i = N.\)

Property 2. provides the existence of the unique solution of the fixed point equation associated with the Feshbach-Schur Hamiltonian \(\mathcal{M}_{j_1,\ldots,j_m}^{Bog}(z + z_{j_1,\ldots,j_m}^{Bog})\) defined with Property 1. Hence, the (non-degenerate) ground state energy of \(H_{j_1,\ldots,j_m}^{Bog}\) and the corresponding ground state vector are determined.

\(^6\)Following the argument of Corollary 5.9 of [PiI] the second term in (4.93) is \(O(\frac{1}{\sqrt{\ln \epsilon}} (\frac{1}{1 + \epsilon \sqrt{\eta_v}})^h).\) However, in the present paper the multiplicative constants may depend on the size of the box and the details of the potential.
Property 3. is concerned with the spectral gap of the Hamiltonian
\[
(H_{\mathbf{j}1,\ldots,\mathbf{j}_m}^{Bog} + \sum_{\mathbf{j} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} (k_j)^2 a_j^* a_j) \mathbb{P}_{\mathbf{j} \in (N-1)\mathbb{N}^N}.
\]
The gap condition at step \( m \) provided by Property 3. is needed to apply Lemma 4.5 and derive Property 1. and 2. at step \( m+1 \).

Property 4. provides the information on
\[
\text{infspec } \left[ \sum_{j=1}^{m} H_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog} \right] - z_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog}
\]
that is assumed in Corollary 5.1 and Lemma 4.5. Then, Property 4. is needed to derive Property 1. at step \( m+1 \). Thanks to this input, the operator norm estimate (5.3) in Corollary 5.1 can be derived as if the modes \( \pm \mathbf{j}_1, \ldots, \pm \mathbf{j}_{m-1} \) were absent.

Property 5. provides the bound on the expectation value of the number operator \( N_a := \sum_{\mathbf{j} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} a_j^* a_j \) in the ground state of \( H_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog} \). This information is needed to control the fixed point equation associated with \( H_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog} \).

**Theorem 4.3.** Let \( \max_{1 \leq m \leq M} \|q_m\| \) be sufficiently small and \( N \) sufficiently large. Then the following properties hold true for all \( 1 \leq m \leq M \):

1) The Feshbach-Schur Hamiltonian \( \mathcal{H}_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog}(N) (z + z_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog}) \) in (3.28)-(3.31) is well defined for
\[
z \leq z_m + \gamma \Delta_m - \frac{C^+}{(\ln N)^{\frac{3}{2}}} < E_{\mathbf{j}_m}^{Bog} + \frac{1}{2} \sqrt{\epsilon_{\mathbf{j}_m}} \phi_{\mathbf{j}_m} \sqrt{\epsilon_{\mathbf{j}_m}^2 + 2 \epsilon_{\mathbf{j}_m}} < 0 \quad , \quad \gamma = \frac{1}{2} \tag{4.96}
\]
where:
- \( z_{\mathbf{j}_1,\ldots,\mathbf{j}_{m-1}}^{Bog} \) is the ground state energy of \( H_{\mathbf{j}_1,\ldots,\mathbf{j}_{m-1}}^{Bog} \) and is defined recursively in point 2) below;
- \( z_m \) is the ground state energy of \( H_{\mathbf{j}_m}^{Bog} \);
- \( \Delta_m \) is defined recursively by \( \Delta_0 := \min \{ (k_j)^2 | \mathbf{j} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \} \) and
\[
\Delta_m := \gamma \Delta_{m-1} - \frac{C^+}{(\ln N)^{\frac{3}{2}}} - \frac{2}{\gamma} \frac{C_{III}}{(\ln N)^{\frac{3}{2}}} > 0 \tag{4.97}
\]
with \( C_{III} := C_I + \frac{C_{II}}{1 + \gamma \Delta_0} \) where the constants \( C_I, C_{II} \) are introduced in Lemma 4.4, and \( C^+ \) is introduced in Lemma 4.5.

2) For \( z \) as in (4.96), there exists a unique value \( z^{(m)} \) such that
\[
\mathcal{H}_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog}(z + z_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog})|_{z=\gamma(m)} = 0.
\]
The inequality
\[
|z^{(m)} - z_m| \leq \frac{2}{\gamma} \frac{C_{III}}{(\ln N)^{\frac{3}{2}}} \tag{4.98}
\]
holds true.

The Hamiltonian \( H_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog} \) has (nondegenerate) ground state energy
\[
z_{\mathbf{j}_1,\ldots,\mathbf{j}_m}^{Bog} := z_{\mathbf{j}_1,\ldots,\mathbf{j}_{m-1}}^{Bog} + z^{(m)}
\]
where \( z_{\mathbf{j}_1,\ldots,\mathbf{j}_{m-1}}^{Bog} \mid_{m=1} = 0 \). The corresponding eigenvector is given in (3.41)-(3.42).
3) The spectral gap of the two operators

\[ H_{j_1 \ldots j_m}^{\text{Bog}}, \quad (\hat{H}_{j_1 \ldots j_m}^{\text{Bog}} + \sum_{j \in \mathbb{Z} \setminus \{\pm j_1, \ldots, \pm j_{m+1}\}} (k_j)^2 a_j^* a_j) \upharpoonright Q_{j_1 \ldots j_m}^{(N-1)} \]  

above the (common) ground state energy \( z_{j_1 \ldots j_m}^{\text{Bog}} \) is larger or equal to \( \Delta_m \).

4) The lower bound

\[
\inf \text{spec} \left\{ \sum_{l=1}^{m} \hat{H}_{j_l}^{\text{Bog}} \right\} - z_{j_1 \ldots j_m}^{\text{Bog}} \geq - \frac{m}{(\ln N)^2} 
\]  

holds true.

5) For \( \tilde{C}_m = \frac{\sum_{j=1}^{m} \phi_j}{\Delta_0} \), the upper bound

\[
\frac{\langle \psi_{j_1 \ldots j_m}^{\text{Bog}} \rangle}{\| \psi_{j_1 \ldots j_m}^{\text{Bog}} \|}, \quad \sum_{j \in \mathbb{Z} \setminus \{0\}} a_j^* a_j \frac{\langle \psi_{j_1 \ldots j_m}^{\text{Bog}} \rangle}{\| \psi_{j_1 \ldots j_m}^{\text{Bog}} \|} \leq \tilde{C}_m
\]  

holds true.

Proof

For \( m = 1 \), by construction \( z_1 \equiv z_{j_1}^{\text{Bog}} \) and Property 1) and 2) have been proven in Corollary 4.6 of [Pl1] that is reported in Section 2.2. As far as Property 3) is concerned, we observe that the vector \( \psi_{j_1}^{\text{Bog}} \) belongs to \( Q_{j_1}^{(N-1)} \) and is eigenvalue of \( \hat{H}_{j_1}^{\text{Bog}} + \sum_{j \in \mathbb{Z} \setminus \{\pm j_1, \pm j_2\}} (k_j)^2 a_j^* a_j \upharpoonright Q_{j_1}^{(N-1)} \).

Thus, for any \( \psi \in Q_{j_1}^{(N-1)} \), \( \| \psi \| = 1 \), that is orthogonal to \( \psi_{j_1}^{\text{Bog}} \) we can derive

\[
\langle \psi, (\hat{H}_{j_1}^{\text{Bog}} + \sum_{j \in \mathbb{Z} \setminus \{\pm j_1\}} (k_j)^2 a_j^* a_j) \psi \rangle = \langle \psi, (\hat{H}_{j_1}^{\text{Bog}} + \sum_{j \in \mathbb{Z} \setminus \{\pm j_1\}} (k_j)^2 a_j^* a_j) \psi \rangle 
\]  

\[
= \langle \psi, H_{j_1}^{\text{Bog}} \psi \rangle \geq z_{j_1}^{\text{Bog}} + \Delta_1
\]  

where the last step follows from the result in Corollary 4.6 of [Pl1].

Concerning Property 4), we recall that \( \psi_{j_1}^{\text{Bog}} \) is also eigenstate of \( \hat{H}_{j_1}^{\text{Bog}} \) with the same eigenvalue \( z_{j_1}^{\text{Bog}} \). Furthermore, we can restrict \( \hat{H}_{j_1}^{\text{Bog}} \) to any subspace \( \mathbb{F}^{N-j} \) of \( \mathbb{F}^N \) with a fixed number of particles, \( j \), in the modes different from \( \pm j_1 \) and \( 0 \). We assume that \( j \leq N-2 \) (if \( j = N-2 \) the property is trivially proven) and even, but the final results hold also for \( j \) odd.

By adapting Corollary 5.1 and Theorem 4.1 to the Hamiltonian \( \hat{H}_{j_1}^{\text{Bog}} \), the Feshbach-Schur flow can be implemented in the same way it has been done for \( \hat{H}_{j_1}^{\text{Bog}} \) starting from the projections

\[
\phi_{j_1}^{(j)} := Q_{j_1}^{(j+1)} := \mathbb{I}_{\mathbb{F}^{N-j} \setminus \{j_1\}}, \quad \phi_{j_1}^{(j)} := Q_{j_1}^{(j+1)} := \mathbb{I}_{\mathbb{F}^{N-j} \setminus \{j_1\} \setminus \{j\}} \]  

(4.106)

\[7\] In this case we re-define the first couple of projections \( \phi_{j_1}^{(j)} := Q_{j_1}^{(j+1)} := \mathbb{I}_{\mathbb{F}^{N-j} \setminus \{j_1\}} \) and \( \phi_{j_1}^{(j)} := Q_{j_1}^{(j+1)} := \mathbb{I}_{\mathbb{F}^{N-j} \setminus \{j_1\} \setminus \{j\}} \), where \( Q_{j_1}^{(j)} \) is the projection onto the subspace of \( \mathbb{F}^N \) with \( N-j \) particles in the modes \( \pm j_1 \). Then, we proceed for \( i \geq j+1 \) (and even) up to \( N-2 \) according to the definitions in (4.107).
and proceeding for \( i > j \) (and even) up to \( i = N - 2 \) with the definitions

\[
\hat{\mathcal{H}}^{(i)} := \hat{Q}^{(i+1)}_{ji} := \hat{Q}^{(i+1)}_{ji} \hat{Q}^{(i-1)}_{ji} \quad \text{and} \quad \hat{\mathcal{H}}^{(i)} := \hat{Q}^{(i+1)}_{ji} := \hat{Q}^{(i-1)}_{ji} - \hat{Q}^{(i+1)}_{ji}. \tag{4.107}
\]

We call \( \hat{\mathcal{H}}^{Bog}(i) \) the Feshbach-Schur Hamiltonians so defined. Note that \( \hat{Q}^{(N-1)}_{ji} \) is the projection onto the subspace of states with \( N - j \) particles in the zero mode state and \( j \) particles in the modes different from \( \pm j_1 \) and \( 0 \). Setting \( w \equiv z \) with \( z \) in the range defined in (4.96), we obtain

\[
\hat{\mathcal{H}}^{Bog} (N-2) (z) \tag{4.108}
\]

\[
= -z \hat{Q}^{(N-1)}_{ji} \tag{4.109}
\]

\[
- \hat{Q}^{(N-1)}_{ji} W_j \hat{R}^{Bog}_{ji; N-2, N-2} (z) \sum_{l_{N-2}=0}^{\infty} [\hat{\Gamma}^{Bog}_{ji; N-2, N-2} (z) \hat{R}^{Bog}_{ji; N-2, N-2} (z)]^{l_{N-2}} W_j \hat{Q}^{(N-1)}_{ji} \tag{4.110}
\]

where \( \hat{R}^{Bog}_{ji; i;i} (z) \) and \( \hat{\Gamma}^{Bog}_{ji; i;i} (z) \) have the same definition of \( \hat{R}^{Bog}_{ji; i;i} (z) \) and \( \hat{\Gamma}^{Bog}_{ji; i;i} (z) \) but in terms of \( \hat{H}^{Bog}_{ji} \) and of the new projections. Next, we notice that since \( \hat{Q}^{(N-1)}_{ji} \) projects onto a subspace of states with no particles in the modes \( \pm j_1 \) the operators

\[
\left[ \hat{R}^{Bog}_{ji; i;i} (z) \right]^{1/2} \hat{R}^{Bog}_{ji; j-i,j-i} (z) \left[ \hat{R}^{Bog}_{ji; i;i} (z) \right]^{1/2} \tag{4.111}
\]

appearing in (4.110) and entering the definition of \( \hat{\Gamma}^{Bog}_{ji; N-2, N-2} (z) \) are in fact restricted to states with an even number of particles in the modes \( \pm j_1 \). Then, we use a procedure analogous to Lemma 5.3 by implementing Steps a), b), d), and e); see Remark 5.5. We conclude that for \( N \) sufficiently large

\[
\left\| \hat{Q}^{(N-1)}_{ji} W_j \hat{R}^{Bog}_{ji; N-2, N-2} (z) \sum_{l_{N-2}=0}^{\infty} [\hat{\Gamma}^{Bog}_{ji; N-2, N-2} (z) \hat{R}^{Bog}_{ji; N-2, N-2} (z)]^{l_{N-2}} W_j \hat{Q}^{(N-1)}_{ji} \right\| \tag{4.112}
\]

\[
\leq \left( 1 - \frac{1}{N} \right) \frac{\phi_{j_1}}{2 \epsilon_{j_1} + 2 - \frac{4 N}{\phi_{j_1}^2} \left. \frac{U_{j_1}}{\sqrt{N}} \right. + \frac{1}{2 \ln N} \epsilon_{j_1}^2 + 2 \epsilon_{j_1} \right). \tag{4.112}
\]

Note that, for \( N \) sufficiently large and \( z \) in the range (4.96), we have

\[
z + \frac{U_{j_1}}{\sqrt{N}} + \frac{1}{2 \ln N} \left. \frac{1}{\epsilon_{j_1}^2} \right. + 2 \epsilon_{j_1} \leq E^{Bog}_{j_1} + \epsilon_{j_1}^2 \frac{1}{2 \ln N} \tag{4.112}
\]

therefore \( \tilde{\mathcal{G}}_{ji; N-2, N-2} (z + \frac{U_{j_1}}{\sqrt{N}} + \frac{1}{2 \ln N} \epsilon_{j_1}^2) \) and \( \hat{\mathcal{G}}_{ji; N-2, N-2} (z + \frac{U_{j_1}}{\sqrt{N}} + \frac{1}{2 \ln N} \epsilon_{j_1}^2) \) are well defined; see (2.42)-(2.48). Since \( \tilde{\mathcal{G}}_{ji; N-2, N-2} (z) \) is nondecreasing (see Remark 2.3) the inequality in (4.112) implies

\[
\left\| \hat{Q}^{(N-1)}_{ji} W_j \hat{R}^{Bog}_{ji; N-2, N-2} (z) \sum_{l_{N-2}=0}^{\infty} [\hat{\Gamma}^{Bog}_{ji; N-2, N-2} (z) \hat{R}^{Bog}_{ji; N-2, N-2} (z)]^{l_{N-2}} W_j \hat{Q}^{(N-1)}_{ji} \right\| \tag{4.113}
\]

\[
\leq \left( 1 - \frac{1}{N} \right) \frac{\phi_{j_1}}{2 \epsilon_{j_1} + 2 - \frac{4 N}{\phi_{j_1}^2} \left. \frac{U_{j_1}}{\sqrt{N}} \right. + \frac{1}{2 \ln N} \epsilon_{j_1}^2 + 2 \epsilon_{j_1} \right). \tag{4.113}
\]
and, consequently,

$$-z - \| \hat{Q}_{\mathcal{J}}^{(N-j-1)} W_{\mathcal{J}} \hat{R}_{\mathcal{J}}^{Bog} (z) \sum_{l_{\mathcal{J}}=0}^{\infty} [ \hat{R}_{\mathcal{J}}^{Bog} (z) \hat{R}_{\mathcal{J}}^{Bog} (z) ]^{l_{\mathcal{J}}=2} W_{\mathcal{J}} \hat{Q}_{\mathcal{J}}^{(N-j-1)} \| \geq \frac{U_{\mathcal{J}}}{\sqrt{N}} + \frac{1}{2(\ln N)^{\#}} - (z + \frac{U_{\mathcal{J}}}{\sqrt{N}} + \frac{1}{2(\ln N)^{\#}})$$

$$(4.114)$$

$$- (1 - \frac{1}{N}) \frac{\phi_{\mathcal{J}}}{\sqrt{N}} \frac{1}{2(\ln N)^{\#}} \hat{G}_{\mathcal{J}}^{Bog} (z) \geq 2\varepsilon_{\mathcal{J}} + 2 - \frac{4N}{\phi_{\mathcal{J}}} \frac{1}{2(\ln N)^{\#}} \phi_{\mathcal{J}}$$

$$(4.115)$$

Since, due to Remark 2.3, the derivative with respect to $z$ of

$$z + (1 - \frac{1}{N}) \frac{\phi_{\mathcal{J}}}{\sqrt{N}} \frac{1}{2(\ln N)^{\#}} \hat{G}_{\mathcal{J}}^{Bog} (z) = -f_{\mathcal{J}}^{Bog} (z)$$

$$(4.116)$$

is not smaller than 1, for $N$ large enough and

$$z < z_{1}^{Bog} - \frac{1}{(\ln N)^{\#}} < z_{1}^{Bog} - \frac{U_{\mathcal{J}}}{\sqrt{N}} - \frac{1}{2(\ln N)^{\#}} \Rightarrow z + \frac{U_{\mathcal{J}}}{\sqrt{N}} + \frac{1}{2(\ln N)^{\#}} < z_{1}^{Bog}$$

we deduce that

$$- (z + \frac{U_{\mathcal{J}}}{\sqrt{N}} + \frac{1}{2(\ln N)^{\#}})$$

$$(4.117)$$

$$- (1 - \frac{1}{N}) \frac{\phi_{\mathcal{J}}}{\sqrt{N}} \frac{1}{2(\ln N)^{\#}} \hat{G}_{\mathcal{J}}^{Bog} (z) \geq 2\varepsilon_{\mathcal{J}} + 2 - \frac{4N}{\phi_{\mathcal{J}}} \frac{1}{2(\ln N)^{\#}} \phi_{\mathcal{J}}$$

$$(4.118)$$

$$\geq f_{\mathcal{J}}^{Bog} (z_{1}^{Bog})$$

$$(4.119)$$

$$= 0.$$  

$$(4.120)$$

Hence, $(4.114)+(4.115) > 0$ and, consequently, the operator $\mathcal{K}_{\mathcal{J}}^{Bog (N-2)} (z)$ acting on $\hat{Q}_{\mathcal{J}}^{(N-1)}$ is bounded invertible for $z < z_{1}^{Bog} - \frac{1}{(\ln N)^{\#}}$. Since this holds for any subspace $\{ F_{\mathcal{J}} \} \mathcal{J}$ of $\mathcal{F}$, and due to the isospectrality of the Feshbach-Schur map, also $\hat{H}_{\mathcal{J}}^{Bog} - z$ is bounded invertible for $z < z_{1}^{Bog} - \frac{1}{(\ln N)^{\#}}$.

Property 5) follows from the identity below

$$\hat{H}_{\mathcal{J}}^{Bog} = \sum_{j=\mathcal{J}}^{\infty} k_{j}^{2} a_{j}^{\dagger} a_{j} + \left\{ \frac{\phi_{\mathcal{J}}}{N} (a_{0}^{\dagger} a_{\mathcal{J}} + a_{0} a_{-\mathcal{J}}) (a_{0}^{\dagger} a_{\mathcal{J}} + a_{0} a_{-\mathcal{J}}) \right\} \left( \phi_{\mathcal{J}} a_{\mathcal{J}}^{\dagger} a_{-\mathcal{J}} + a_{0} a_{0}^{\dagger} \right).$$

$$(4.121)$$

Indeed, since

$$a_{0}^{\dagger} a_{\mathcal{J}} + a_{0} a_{-\mathcal{J}} \geq 0$$

$$(4.122)$$

and

$$a_{0}^{\dagger} a_{\mathcal{J}} + a_{0} a_{-\mathcal{J}} \leq N,$$

$$(4.123)$$
from
\begin{equation}
\phi_{j_1}^{\text{Bog}} \left( \sum_{j=j_1} a_j^* a_j \right) \psi_{j_1}^{\text{Bog}} ||\psi_{j_1}^{\text{Bog}}|| \right)
\end{equation}

\begin{equation}
\frac{\psi_{j_1}^{\text{Bog}}}{||\psi_{j_1}^{\text{Bog}}||}, \quad \left( \frac{\phi_{j_1}(a_0^* a_{j_1} + a_0 a_{j_1}^*) (a_0^* a_{j_1} + a_0 a_{j_1}^*) - \frac{\phi_{j_1}}{N} [a_{j_1}^* a_{j_1} - a_{j_1}^* a_{j_1}] ||\psi_{j_1}^{\text{Bog}}|| \right)
\end{equation}

we readily derive
\begin{equation}
\frac{\psi_{j_1}^{\text{Bog}}}{||\psi_{j_1}^{\text{Bog}}||}, \quad \left( \sum_{j=j_1} a_j^* a_j \right) \leq \frac{z_{j_1}^{\text{Bog}} + \phi_{j_1} ||\psi_{j_1}^{\text{Bog}}||}{k_{j_1}^2} \leq \frac{\phi_{j_1}}{k_{j_1}^2} \leq \frac{\phi_{j_1}}{k_{j_1}^2}
\end{equation}

where we have used that $z_{j_1}^{\text{Bog}} < 0$. Furthermore, by construction of $\psi_{j_1}^{\text{Bog}}$

\begin{equation}
\frac{\psi_{j_1}^{\text{Bog}}}{||\psi_{j_1}^{\text{Bog}}||}, \quad \left( \sum_{j=j_1} a_j^* a_j ||\psi_{j_1}^{\text{Bog}}|| \right) = 0.
\end{equation}

Now, we assume that Properties 1), 2), 3), 4), and 5) hold for $1 \leq m - 1 \leq M - 1$ and show that they are true also for $m$.

**Property 1)** Given Property 4) at step $m - 1$, we can apply Corollary 5.1. Then, Theorem 4.1 ensures that for $z \leq E_{j_m}^{\text{Bog}} + \sqrt{\epsilon_{j_m}} \sqrt{\epsilon_{j_m}} + 2\epsilon_{j_m}$ the Feshbach-Schur Hamiltonian $\mathcal{H}_{j_1,..,j_m}^{\text{Bog}(N-2)}(z + z_{j_1,..,j_m})$ is well defined. Furthermore, thanks to Properties 3) and 4) at step $m - 1$, we can apply Lemma 4.5 and conclude that for $N$ sufficiently large and

\begin{equation}
\begin{aligned}
z \leq z_{m} - \frac{C_1}{(\ln N)^2} + \gamma \Delta_{m-1} < E_{j_m}^{\text{Bog}} + \frac{1}{2} \sqrt{\epsilon_{j_m}} \sqrt{\epsilon_{j_m}} + 2\epsilon_{j_m} \leq 0 \quad , \quad \gamma = \frac{1}{2},
\end{aligned}
\end{equation}

the operator
\begin{equation}
\mathcal{H}_{j_1,..,j_{m-1}}^{\text{Bog}(N-2)}(z + z_{j_1,..,j_{m-1}})
\end{equation}

is bounded invertible on $\mathcal{F}_{j_1,..,j_{m-1},N}^{\text{Bog}(N-2)}$. Note that the second inequality in (4.129) holds because from **Lemma 5.5** of [Pi1] we can estimate $|z_m - E_{j_m}^{\text{Bog}}| \leq O(\frac{1}{N})$ for any $0 < \beta < 1$. Thus, for $z$ in the interval given in (4.129) we can define (w = $z + z_{j_1,..,j_{m-1}}$)

\begin{align}
\mathcal{H}_{j_1,..,j_{m}}^{\text{Bog}(N)}(w) &= -z \mathcal{H}_{j_1,..,j_{m-1}}^{\text{Bog}(N)}(w) - \mathcal{H}_{j_1,..,j_{m}}^{\text{Bog}(N, N, N)(w)} - \mathcal{H}_{j_1,..,j_{m}}^{\text{Bog}(N, N, N)(w)} \\
&= \mathcal{H}_{j_1,..,j_{m-1}}^{\text{Bog}(N-2)}(w) \mathcal{H}_{j_1,..,j_{m-1}}^{\text{Bog}(N-2)}(w) \\
&= \mathcal{F}_{j_1,..,j_{m-1}}^{\text{Bog}(N-2)}(w) \mathcal{F}_{j_1,..,j_{m-1}}^{\text{Bog}(N-2)}(w)
\end{align}

\begin{equation}
= \int_{j_1,..,j_{m-1}}^{\text{Bog}}(w) \mathcal{F}_{j_1,..,j_{m-1}}^{\text{Bog}(N-2)}(w)
\end{equation}
Property 2) The Hamiltonian $\mathcal{H}^{Bog}(N)(z + z^{Bog}_{1,\ldots,J_m})$ has eigenvalue zero if there is a solution, $z \equiv z^{(m)}$, to $f^{Bog}_{1,\ldots,J_m}(z + z^{Bog}_{1,\ldots,J_m}) = 0$ in the range given in (4.129). This equation can also be written

$$z = -\left(\frac{\psi^{Bog}_{1,\ldots,J_{m-1}}}{\|\psi^{Bog}_{1,\ldots,J_{m-1}}\|} \right) \Gamma^{Bog}_{1,\ldots,J_{m},N,N}(z + z^{Bog}_{1,\ldots,J_{m-1}}) \frac{\psi^{Bog}_{1,\ldots,J_{m-1}}}{\|\psi^{Bog}_{1,\ldots,J_{m-1}}\|}$$

(4.134)

$$-\left(\frac{\psi^{Bog}_{1,\ldots,J_{m-1}}}{\|\psi^{Bog}_{1,\ldots,J_{m-1}}\|} \right) \Gamma^{Bog}_{1,\ldots,J_{m},N,N}(z + z^{Bog}_{1,\ldots,J_{m-1}}) \frac{\psi^{Bog}_{1,\ldots,J_{m-1}}}{\|\psi^{Bog}_{1,\ldots,J_{m-1}}\|} \times$$

$$\times \left(\frac{1}{\|\psi^{Bog}_{1,\ldots,J_{m-1}}\|} \right) \frac{\Gamma^{Bog}_{1,\ldots,J_{m},N,N}(z + z^{Bog}_{1,\ldots,J_{m-1}})}{\|\psi^{Bog}_{1,\ldots,J_{m-1}}\|} \right).$$

(4.135)

Properties 1), 2), 4) and 5) at step $m - 1$ enable us to derive estimates (4.57), (4.58) in Lemma 4.4. We recall that Properties 3) and 4) at step $m - 1$ yield the inequality in (4.67). Consequently, combining (4.57)-(4.58) with (4.67), we can rewrite the fixed point equation in (4.134)-(4.135) in the following form

$$z = -\langle \eta, \Gamma^{Bog}_{j_m;N,N}(z) \eta \rangle + \mathcal{Y}(z)$$

(4.136)

with

$$|\mathcal{Y}(z)| \leq \frac{C_I}{(\ln N)^\gamma} + \frac{C^2_{II}}{(\ln N)^\gamma(1 - \gamma)\Delta_{m-1}}$$

(4.137)

$$\leq \frac{C_I}{(\ln N)^\gamma} + \left(\frac{\gamma}{\gamma - 1}\right)^m \left(\frac{\gamma}{\gamma - 1}\right)^2 \Delta_{m-1} \leq \frac{C^2_{II}}{(\ln N)^\gamma(1 - \gamma)\Delta_0}$$

(4.138)

where we have used that for $1 \leq m \leq M$ and $N$ large enough

$$\Delta_{m-1} \geq \Delta_0 \left(\frac{\gamma}{\gamma - 1}\right)^m;$$

see the definition of $\Delta_m$ in (4.97). Consequently, we have reduced the fixed point equation

$$f^{Bog}_{1,\ldots,J_m}(z + z^{Bog}_{1,\ldots,J_{m-1}}) = 0$$

to the fixed point equation, $f^{Bog}_{1,\ldots,J_m}(z) = 0$, of a three-modes system, up to the small error $\mathcal{Y}(z)$. Therefore, the same argument of Theorem 4.1 of [Pi1] implies that for $N$ large enough there exists a $z^{(m)}$ in the range (4.129) that solves the equation in (4.136). Now, we show the inequality

$$|z^{(m)} - z_m| \leq \left(\frac{\gamma}{2}\right)^m \frac{C_{III}}{(\ln N)^\gamma}, \quad C_{III} := C_I + \frac{C^2_{II}}{(1 - \gamma)\Delta_0}.$$  (4.139)

Since, by construction,

$$z_m = -\langle \eta, \Gamma^{Bog}_{j_m;N,N}(z_m) \eta \rangle,$$

(4.140)

after subtracting the equation in (4.136) we get

$$z_m + \langle \eta, \Gamma^{Bog}_{j_m;N,N}(z_m) \eta \rangle - z^{(m)} = -\langle \eta, \Gamma^{Bog}_{j_m;N,N}(z^{(m)}) \eta \rangle = -\mathcal{Y}(z^{(m)}).$$

(4.141)
Hence, the inequality in (4.139) follows from (4.138) and the mean value theorem applied to the L-H-S of (4.141), because the derivative with respect to $z$ of

$$z + \langle \eta, \Gamma_{j_{m}, N-2, N-2}(z) \eta \rangle = z + (1 - \frac{1}{N}) \phi_{j_{m}} \mathcal{J}_{j_{m}, N-2, N-2}(z)$$

is not smaller than 1; see Remark 2.3.

Using the isospectrality of the Feshbach-Schur map, we deduce that $H_{j_{m}, j_{m}}^{Bog}$ has the eigenvalue

$$z^{(m)} + z_{j_{m}, j_{m}}^{Bog} = z_{j_{m}, j_{m}}^{Bog}.$$ 

The corresponding eigenvector is given in (3.41)-(3.42).

The uniqueness of $z^{(m)}$ in the considered range of $z$ holds because for any other value, $(z^{(m)})'$, solving the fixed point problem we can state the inequality

$$|z^{(m)} - z_{m}| \leq O\left(\frac{1}{(\ln N)^{\frac{1}{2}}}\right)$$

by the same argument used to prove the inequality in (4.139). Now, assuming that there are two distinct eigenvalues $z_{j_{m}, j_{m}}^{Bog} + z_{j_{m}, j_{m}}^{Bog}$ and $(z^{(m)})'$, the corresponding eigenvectors, $\psi_{j_{m}, j_{m}}^{Bog}$ and $(\psi_{j_{m}, j_{m}})'$, are given by the formula in (3.41)-(3.42) replacing $z_{j_{m}, j_{m}}^{Bog}$ with $z_{j_{m}, j_{m}}^{Bog} + z_{j_{m}, j_{m}}^{Bog}$ and $(z^{(m)})'$, respectively. Next, starting from the expression of $\mathcal{J}_{j_{m}, j_{m}}^{Bog}(N-2r-2)(w)$ (see (3.26)) and using the relation in (3.35)-(3.37), we exploit the expansion

$$\frac{1}{Q_{j_{m}}^{(N-2r,N-2r+1)}(N-2r-2)} R_{j_{m}, j_{m}}^{Bog}(N-2r-2)(w) Q_{j_{m}}^{(N-2r,N-2r+1)} = \sum_{I_{N-2r} = 0}^{\infty} R_{j_{m}, j_{m}}^{Bog}(N-2r-2)(w) \Gamma_{j_{m}, j_{m}}^{Bog}(N-2r,N-2r-2)(w) R_{j_{m}, j_{m}}^{Bog}(N-2r,N-2r-2)(w)$$

(4.142)

(4.143)

where $w = z_{j_{m}, j_{m}}^{Bog} + z_{j_{m}, j_{m}}^{Bog}$ or $w = z_{j_{m}, j_{m}}^{Bog} + (z^{(m)})'$ depending on the considered eigenvalue. Due to the control of the series in (2.55) and by means of a procedure analogous to STEP I and II in Lemma 4.4, we can re-expand the operators $\Gamma_{j_{m}, j_{m}}^{Bog}(N-2r,N-2r-2)(w)$ and estimate (with a suitable choice of $h$)

$$||\psi_{j_{m}, j_{m}}^{Bog} - (\psi_{j_{m}, j_{m}})'|| = O\left(\frac{1}{(\ln N)^{\frac{1}{2}}}\right)$$

(4.144)

because $|z^{(m)} - (z^{(m)})'| \leq |(z^{(m)})' - z_{m}| + |(z^{(m)}) - z_{m}| \leq O\left(\frac{1}{(\ln N)^{\frac{1}{2}}}\right)$ (see section 0.1 in supporting-file-Bose2.pdf'). Hence, since

$$||\psi_{j_{m}, j_{m}}^{Bog}||, ||(\psi_{j_{m}, j_{m}})'|| \geq 1$$

by construction (see Remark 3.1), for $N$ large enough the two eigenvectors are not orthogonal and the corresponding eigenvalues must coincide.

Property 3) In Property 2) we have derived that in the interval

$$z \leq z_{m} - \frac{C \gamma}{(\ln N)^{\frac{1}{2}}} + \gamma \Delta m - \frac{1}{2} \frac{\phi_{j_{m}}}{\phi_{j_{m}}^{2}} \phi_{j_{m}} \left[ \sqrt{\varepsilon_{j_{m}}^{2} + 2 \varepsilon_{j_{m}} (\varepsilon_{j_{m}} > 0)} \right], \quad \gamma = \frac{1}{2},$$

(4.145)
the Hamiltonian \( \mathcal{H}_B^{\text{Bog}(N)}(z + z_{j_1 \ldots j_{m-1}}^{\text{Bog}}) \) is bounded invertible except for \( z \equiv z^{(m)} \). This together with (4.139) imply that for \( N \) large enough

\[
\text{infspec} \left[ H_B^{\text{Bog}}_{j_1, \ldots, j_m} \right] - z_{j_1, \ldots, j_m}^{\text{Bog}} \geq \Delta_m := \gamma \Delta_{m-1} - \frac{C}{(\ln N)^2} - \frac{2}{\gamma} \Delta_{III}.
\]

By means of the same argument used for \( m = 1 \), we deduce that

\[
\text{infspec} \left[ \left( \hat{H}_B^{\text{Bog}}_{j_1, \ldots, j_m} + \sum_{j \not\in \{j_1, \ldots, j_{m-1}\}} (k_j)^2 a_j^* a_j \right) \right] - z_{j_1, \ldots, j_m}^{\text{Bog}} \geq \Delta_m.
\] (4.148)

**Property 4)** The argument is analogous to the case \( m = 1 \), making use of the restriction to subspaces with fixed number, \( j \), (that we assume less than \( N - 2 \) and even) of particles in the modes different from \( \pm j_1, \ldots, \pm j_{m-1} \). Invoking Property 4) at step \( m - 1 \) we can adapt Corollary 5.1 and Theorem 4.1 to the Hamiltonian \( \sum_{l=1}^m \hat{H}_B^{\text{Bog}}_{j_l} \). Hence, we define the Feshbach-Schur Hamiltonian \( (w \equiv z + z_{j_1, \ldots, j_{m-1}}^{\text{Bog}}) \)

\[
\mathcal{H}_B^{\text{Bog}(N-2)}(w) = \hat{Q}_{j_m}^{(N-1)} \left( \sum_{l=1}^m \hat{H}_{j_l}^{\text{Bog}} - z_{j_1, \ldots, j_{m-1}}^{\text{Bog}} - z \right) \hat{Q}_{j_m}^{(N-1)} - \hat{Q}_{j_m}^{(N-1)} W_{j_m} \hat{R}_{j_1, \ldots, j_m; N-2, N-2}^{\text{Bog}}(w) \times \sum_{l=2}^\infty \left[ \hat{R}_{j_1, \ldots, j_m; N-2, N-2}^{\text{Bog}}(w) \hat{R}_{j_1, \ldots, j_m; N-2, N-2}^{\text{Bog}}(w) \right]^{l_{N-2}} W_{j_m} \hat{Q}_{j_m}^{(N-1)}.
\] (4.150)

Due to Property 4) at step \( m - 1 \), by implementing **Steps a)**, **b)**, **d)**, and **e)** of Lemma 5.3 (see Remark 5.5), we can estimate for \( w \equiv z + z_{j_1, \ldots, j_{m-1}}^{\text{Bog}} \) with \( z \) in the interval defined in (4.96)

\[
\left\| \hat{Q}_{j_m}^{(N-1)} W_{j_m} \hat{R}_{j_1, \ldots, j_m; N-2, N-2}^{\text{Bog}}(w) \sum_{l=2}^\infty \left[ \hat{R}_{j_1, \ldots, j_m; N-2, N-2}^{\text{Bog}}(w) \hat{R}_{j_1, \ldots, j_m; N-2, N-2}^{\text{Bog}}(w) \right]^{l_{N-2}} W_{j_m} \hat{Q}_{j_m}^{(N-1)} \right\|
\leq \left( 1 - \frac{1}{N} \right) \phi_{j_m} \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{\phi_{j_m}}} \frac{\phi_{j_m}}{\phi_{j_m} + \frac{U_{j_m}}{\sqrt{N}}} + \frac{m - 1 + \frac{1}{2}}{(\ln N)^2} + O\left( \frac{1}{(\ln N)^2} \right).
\] (4.152)

where we have also made use of the property

\[
\bar{G}_{j_m; N-2, N-2}(z + \frac{U_{j_m}}{\sqrt{N}} + \frac{m - 1}{(\ln N)^2}) \leq \bar{G}_{j_m; N-2, N-2}(z + \frac{U_{j_m}}{\sqrt{N}} + \frac{m - 1 + \frac{1}{2}}{(\ln N)^2});
\]

see Remark 2.3. Using Property 4) at step \( m - 1 \) one more time in order to estimate the infimum of (4.150), we deduce that the spectrum of the Feshbach-Schur Hamiltonian \( \mathcal{H}_B^{\text{Bog}(N-2)}(w) \) is
bounded from below by

\[-z - \frac{m-1}{(\ln N)\tau}\]

\[\leq \left\| \sum_{j_{-2}=0}^{\infty} [R_{j_{1},...,j_{m}}^{Bog}; N-2, N-2] (w) \right\|_{\infty} - 2 \ln N - \frac{1}{2} \frac{1}{\sqrt{N}} + O(1) (\ln N)\tau]\]

\[\geq \frac{1}{2(\ln N)\tau} + \frac{U_{j_{m}}}{\sqrt{N}} + O(1) (\ln N)\tau\]

\[= \left( \frac{z + m-1 + \frac{1}{2}}{(\ln N)\tau} \right)\]

\[-\left( 1 - \frac{1}{N} \right) \frac{\phi_{j_{m}}}{2\epsilon_{j_{m}} + 2 - \frac{4}{N} \frac{1}{\phi_{j_{m}}} + \frac{m-1}{(\ln N)\tau} (\ln (\phi_{j_{m}}))} \frac{\hat{G}_{j_{m}; N-2, N-2}(z + \frac{U_{j_{m}}}{\sqrt{N}} + \frac{m-1}{(\ln N)\tau})}{1} \right].

Next, we observe that, for \(N\) sufficiently large and

\[z + m-1 + \frac{1}{2} \frac{U_{j_{m}}}{\sqrt{N}} \leq z_{m} \Rightarrow z \leq z_{m} - \frac{m-1}{(\ln N)\tau} - \frac{U_{j_{m}}}{\sqrt{N}},\]

the inequalities (4.155) > 0 and (4.156) + (4.157) ≥ 0 hold true. Hence, the isospectrality of the Feshbach-Schur map implies that

\[\sum_{l=1}^{m} \hat{R}_{j_{l}}^{Bog} - w\]

is bounded invertible if (recall \(z^{(m)} + z_{j_{1},...,j_{m-1}}^{Bog} \equiv z_{j_{1},...,j_{m}}^{Bog}\))

\[w = z + z_{j_{1},...,j_{m-1}}^{Bog} \leq z_{m} - \frac{m-1}{(\ln N)\tau} - \frac{U_{j_{m}}}{\sqrt{N}} - z^{(m)} + z^{Bog}_{j_{1},...,j_{m-1}} = (z_{m} - z^{(m)}) + z^{Bog}_{j_{1},...,j_{m-1}} - \frac{m-1}{(\ln N)\tau} - \frac{U_{j_{m}}}{\sqrt{N}}.\]

The difference \(|z_{m} - z^{(m)}|\) is estimated in (4.98), from which we conclude that

\[|z_{m} - z^{(m)}| + \frac{U_{j_{m}}}{\sqrt{N}}\]

is bounded by \(\frac{1}{2(\ln N)\tau}\) for \(N\) large. Thus, \(\sum_{l=1}^{m} \hat{R}_{j_{l}}^{Bog} - w\) is bounded invertible for \(w \leq \frac{z_{j_{1},...,j_{m}}^{Bog} - \frac{m}{(\ln N)\tau}}{2(\ln N)\tau}\).

Property 5. The argument is analogous to the case \(m = 1\). □

The very last result of this section concerns the expansion of the ground state vector \(\psi_{j_{1},...,j_{m}}^{Bog}\) in terms of the bare quantities.

**Corollary 4.6.** Assume that the hypotheses of Theorem 4.3 are satisfied. Then, at fixed \(M\), for any arbitrarily small \(\zeta > 0\), there exist \(N_{\zeta} < \infty\) and a vector \((\psi_{j_{1},...,j_{m}}^{Bog})_{\zeta}\), corresponding to a \((\zeta\text{-dependent})\) finite sum of (finite) products of the interaction terms \(W_{j_{l}} + W_{j_{l}}\) and of the resolvents \(\frac{1}{\hat{R}_{j_{l}}^{Bog} - \frac{m}{(\ln N)\tau}}\) applied to \(\eta\), such that

\[||\psi_{j_{1},...,j_{m}}^{Bog} - (\psi_{j_{1},...,j_{m}}^{Bog})_{\zeta}|| \leq \zeta, \quad 1 \leq m \leq M,\]

for \(N > N_{\zeta}\).
Proof

Making use of the formulae in (3.41)-(3.42) and (3.44), we implement the following operations on each $T_m\psi^B_{j_1,\ldots,j_m}$ starting from $m \leq M$ down to $m = 1$:

i) We replace

$$\left[ \mathcal{P}^B_{j_1,\ldots,j_{m-1}} \psi^B_{j_1,\ldots,j_m} \right] = \mathcal{P}^B_{j_1,\ldots,j_{m-1}} \mathcal{P}^B_{j_1,\ldots,j_m} \mathcal{P}^B_{j_1,\ldots,j_{m-1}} \psi^B_{j_1,\ldots,j_m}$$

with $\psi^B_{j_1,\ldots,j_{m-1}}$ up to a remainder with norm less than $O\left(\frac{1}{\Delta_{mn}(\ln N)^2}\right)$ thanks to (4.58) in Lemma 4.4 and (4.67) in Lemma 4.5.

ii) We truncate the sum in (3.42) at some $\tilde{j}$ using the convergence of the series in (2.55) with $j$, replaced with $j_m$, and where $a_{\tilde{q}_j}, b_{\tilde{q}_j}, c_{\tilde{q}_j}$, with $\tilde{q}_j \equiv q_{j_m}$ are those defined in Corollary 5.1 and $\Theta$ is defined in Theorem 4.1. The remaining expression is (see the procedure in section 0.1 of supporting-file-Bose2.pdf):

$$\psi^B_{j_1,\ldots,j_{m-1}} + \sum_{j=2}^j \left( \prod_{r=j}^2 \left[ - \frac{1}{Q_{j_m}^{(N-2r,N-2r+1)}} \mathcal{P}^B_{j_1,\ldots,j_m} \mathcal{P}^B_{j_1,\ldots,j_m} \right] \right)$$

The remainder can be estimated in norm less than $O\left(\frac{1}{1+\sqrt{\tilde{q}_{j_m}}}\right)$ for some $c > 0$ (see (2.55)).

iii) For each factor appearing in (4.160) and (4.161) we exploit the expansion (see the definition in (3.26) and the relation in (3.35)-(3.37))

$$\frac{1}{Q_{j_m}^{(N-2r,N-2r+1)}} \mathcal{P}^B_{j_1,\ldots,j_m} \mathcal{P}^B_{j_1,\ldots,j_m}$$

with $w \equiv z_{j_1,\ldots,j_m} = \tilde{z}^B_{j_1,\ldots,j_{m-1}} + \tilde{z}^{(m)}$. For each summand in (4.161) it is clear that we obtain an expression on which we can implement a procedure analogous to Steps I, II, III, and IV of Lemma 5.2 because we have just a finite number of factors of the type in (4.162) and the complete product of operators is applied to $\psi^B_{j_1,\ldots,j_{m-1}}$. This way, for some (even) $h$ to be determined later, we replace: a) each factor in (4.162) with a truncation of the sum in (4.163) at $l_{N-2r} = h$; b) each $\mathcal{P}^B_{j_1,\ldots,j_{m-1}} : N-2r,N-2r, w$ with $\tilde{z}^B_{j_1,\ldots,j_{m-1}} : N-2r,N-2r, (w)$

$$\left[ \tilde{z}^B_{j_1,\ldots,j_{m-1}} : N-2r,N-2r, (w) \right]$$

that is defined in Step III of Lemma 5.2; c) each $\mathcal{P}^B_{j_1,\ldots,j_{m-1}} : N-2r,N-2r, (w)$ with $\tilde{z}^{(m)}_{j_1,\ldots,j_{m-1}} : N-2r,N-2r, (w)$.

The norm of the remainder produced in this step is less than $O\left(\frac{\tilde{z}^{(2)}_{h(2h)\frac{\tilde{z}}{2}}}{\sqrt{\tilde{z}}}\right)$ (see an analogous procedure in section 0.1 of supporting-file-Bose2.pdf).
iv) In the leading term (that consists of a finite number of products) resulting from the previous operations, we replace $ς^{(m)}$ with $E_{\text{Bog}}^{B_{j}}$ up to a remainder term that can be estimated thanks to (4.139) combined with Lemma 5.5 of [PiI]. Indeed, this lemma provides the estimate $|ς_{m} - E_{\text{Bog}}^{B_{j}}| \leq O(\frac{1}{N^{\beta}})$ for any $0 < \beta < 1.

Hence, it is evident that repeating the procedure for each $m$ down to $m = 1$, the leading term that is obtained is of the form described in the statement of the lemma, and the norm of the sum of all the remainder terms is less than any $\zeta > 0$ by setting a suitable $\zeta$-dependence for $j$, $h$, and $N$. □

# 5 Appendix

**Corollary 5.1.** For $M \geq m \geq 1$ assume:

(a)

$$\inf_{\text{spec}} \left\{ \sum_{j=1}^{m-1} H_{j}^{B_{j}} - z_{j_{1},...,j_{m-1}} \right\} \geq - \frac{(m - 1)}{(\ln N)^{\frac{1}{2}}}$$

(5.1)

where $z_{j_{1},...,j_{m-1}}$ is the ground state energy of $H_{j_{1},...,j_{m-1}}$ for $m \geq 2$ and $z_{j_{1},...,j_{m-1}}|_{m=1} \equiv 0$;

(b)

$$w := z + z_{j_{1},...,j_{m-1}} \leq z_{j_{1},...,j_{m-1}} + E_{j_{m}}^{B_{j}} + (\delta - 1)φ_{j_{m}} \sqrt{2} j_{m}^{2} + 2e_{j_{m}}$$

(5.2)

with $\delta \leq 1 + \sqrt{e_{j_{m}}}$ and $e_{j_{m}}$ sufficiently small.

Then, for $N$ sufficiently large

$$||R_{j_{1},...,j_{m-1}}; i, i-2 \left[ R_{j_{1},...,j_{m-1}}; i-2, i-2 \right](w) \|^{\frac{1}{2}} \left( N^{i-2} \right) \leq \frac{1}{4(1 + a_{j_{m}} - \frac{2b_{j_{m}}}{N^{i-2}} - \frac{1}{2} \frac{e_{j_{m}}}{(N-i+2)^{2}})} \leq \frac{1}{4(1 + a_{j_{m}} - \frac{2b_{j_{m}}}{N^{i-2}} - \frac{1}{2} \frac{e_{j_{m}}}{(N-i+2)^{2}})}$$

(5.3)

holds for $2 \leq i \leq N - 2$ and even. Here,

$$a_{j_{m}} := 2e_{j_{m}} + O(e_{j_{m}}^{v}) \quad v \geq \frac{11}{8}$$

(5.5)

$$b_{j_{m}}^{(\delta)} := (1 + e_{j_{m}})\frac{\delta}{\sqrt{2}} \frac{\chi(0,2)(\delta)}{j_{m}} + 2e_{j_{m}} \quad b_{j_{m}} := (1 + e_{j_{m}})\frac{\delta}{\sqrt{2}} \frac{\chi(0,2)(\delta)}{j_{m}}$$

$$c_{j_{m}}^{(\delta)} := -(1 - \delta^2)\frac{\chi(0,2)(\delta)}{e_{j_{m}}^2 + 2e_{j_{m}}} \quad c_{j_{m}} := -(1 - \delta^2)\frac{\chi(0,2)(\delta)}{e_{j_{m}}^2 + 2e_{j_{m}}}$$

(5.6)

(5.7)

and

with $\chi(0,2)$ the characteristic function of the interval $\{0,2\}$.

**Proof**

We recall that

$$R_{j_{1},...,j_{m-1}}; i, i-2 \left[ R_{j_{1},...,j_{m-1}}; i-2, i-2 \right](w) := \frac{1}{\mathcal{J}_{j_{m}}^{(i-1)}} \mathcal{J}_{j_{m}}^{(i-1)} + \frac{1}{\mathcal{J}_{j_{m}}^{(i-1)}} \left( R_{j_{1},...,j_{m-1}}^{B_{j}} + \sum_{k \in \mathbb{Z} \setminus \{\pm j_{1},...,j_{m-1}\}} (k^2) a_{k}^{B_{j}} - z_{j_{1},...,j_{m-1}}^{B_{j}} + H_{j_{m}}^{B_{j}} - z \right) \mathcal{J}_{j_{m}}^{(i-1)} \mathcal{J}_{j_{m+1}}^{(i-1)}$$

(5.8)
Consider $\epsilon_{j_m}$ sufficiently small and $N$ sufficiently large so that $-z - \frac{(m-1)}{(\ln N)^{2_k}} > 0$. Consequently, the operator
\[
S_{j_m;i,i}(z) := Q_{ij}^{(i+1)} J_m(Q_{ij}^{(i+1)} H_{j_m}^{Bog} - z - \frac{(m-1)}{(\ln N)^{2_k}}) J_m^{(i+1)} Q_{ij}^{(i+1)}
\]
is well defined. Next, we observe that
\[
\left\| \frac{1}{S_{j_m;i,i}^{Bog}(z)} \right\| \leq 1
\]
due to the inequality in (5.1). Using the procedure of Lemma 3.4 of [Pi], for $z \leq E_{j_m}^{Bog} + (\delta - 1)\phi_{j_m} \sqrt{\epsilon_{j_m}^2 + 2\epsilon_{j_m}}$, and $N$ large and $\epsilon_{j_m}$ small (recall that we have assumed $-z - \frac{(m-1)}{(\ln N)^{2_k}} > 0$) we can bound
\[
\left\| S_{j_m;i,i-2} \right\| = \left\| S_{j_m;i,i-2} \frac{1}{S_{j_m;i,i}} \right\| = \left\| S_{j_m;i,i} \right\| \left\| S_{j_m;i,i-2} \right\|
\]
\[
\leq \sup_{2 \leq n \leq N} \frac{n_{i-1} - 1}{N - N^i} \frac{\phi_{j_m}(N - i + 1)}{\phi_{j_m}(N - i)}\]
For some $0 < C_1, C_H < \infty$, where $\Gamma_{\varphi}^{B, \text{II}}_{j_1, \ldots, j_m; N, N}(\varphi)$ is defined starting from $\Gamma_{\varphi}^{B, \text{II}}_{j_1, \ldots, j_m; N, N}(\varphi + \varepsilon_{j_1, \ldots, j_m})$ (see (3.38)) by replacing (4.16) with (4.17).

Proof
At first, we show the result contained in (5.15). For (5.16) the argument is partially the same and is provided afterwards.

We implement STEPS I-IV outlined in Outline of the proof; see Lemma 4.4.

**STEP I**
For $\mu_n$ sufficiently small and $N$ sufficiently large, we can estimate (recall $w \equiv z + \varepsilon_{j_1, \ldots, j_m}$)

\[
\left\langle \mathcal{P}_{\mu_n}^{B, \varphi} \gamma_{j_1, \ldots, j_m; N, N}(z + \varepsilon_{j_1, \ldots, j_m}) \mathcal{P}_{\mu_n}^{B, \varphi} \right\rangle_{j_1, \ldots, j_m; N, N} \leq C_H \left( \ln N \right)^{\frac{1}{2}}
\]  

(5.16)

for some $0 < C_1, C_H < \infty$, where $\Gamma_{\varphi}^{B, \text{II}}_{j_1, \ldots, j_m; N, N}(\varphi)$ is defined starting from $\Gamma_{\varphi}^{B, \text{II}}_{j_1, \ldots, j_m; N, N}(\varphi + \varepsilon_{j_1, \ldots, j_m})$ (see (3.38)) by replacing (4.16) with (4.17).

\[
\left\langle \mathcal{P}_{\mu_n}^{B, \varphi} \gamma_{j_1, \ldots, j_m; N, N}(z + \varepsilon_{j_1, \ldots, j_m}) \mathcal{P}_{\mu_n}^{B, \varphi} \right\rangle_{j_1, \ldots, j_m; N, N} \leq C_H \left( \ln N \right)^{\frac{1}{2}}
\]  

(5.16)

**STEP II**
From the expansion of $\Gamma_{j_1, \ldots, j_m; N, N}(\varphi)$ discussed in Proposition 4.2 where $h$ is an even natural number, $N - 4 \geq h \geq 2$, that is fixed a posteriori, we write

\[
\left\langle \mathcal{P}_{\mu_n}^{B, \varphi} \gamma_{j_1, \ldots, j_m; N, N}(z + \varepsilon_{j_1, \ldots, j_m}) \mathcal{P}_{\mu_n}^{B, \varphi} \right\rangle_{j_1, \ldots, j_m; N, N} \leq C_H \left( \ln N \right)^{\frac{1}{2}}
\]  

(5.16)

for some $0 < C_1, C_H < \infty$, where $\Gamma_{\varphi}^{B, \text{II}}_{j_1, \ldots, j_m; N, N}(\varphi)$ is defined starting from $\Gamma_{\varphi}^{B, \text{II}}_{j_1, \ldots, j_m; N, N}(\varphi + \varepsilon_{j_1, \ldots, j_m})$ (see (3.38)) by replacing (4.16) with (4.17).
for some $c > 0$ thanks to the estimates in (4.32), (4.33), and the observations in Remarks 4.2 and 4.3. For the details we refer the reader to Corollary 5.9 of [Pi1] where a similar argument is implemented.

**STEP III**

Concerning the quantity in (5.24), in Lemma 5.6 we show that

$$
\sum_{l=N-2-h, l \text{ even}}^{N-4} \left[ \Gamma_{J_1,\ldots,J_m}^{Bog} : N-2,N-2(w) \right]_{l;J_1,\ldots,J_m} \equiv \left[ \Gamma_{J_1,\ldots,J_m}^{Bog} : N-2,N-2(w) \right]_{r_h} \quad (5.26)
$$

where the R-H-S is defined recursively by

$$
\left[ \Gamma_{J_1,\ldots,J_m; i,i}(w) \right]_{r_h} := W_{j_0} R_{J_1,\ldots,J_m; i-2,i-2}(w) \sum_{l=-2}^{h-1} \left[ \Gamma_{J_1,\ldots,J_m; i-2,i-2}(w) \right]_{r_h} R_{J_1,\ldots,J_m; i-2,i-2}(w) \bigg|_{l=2} \bigg|_{w} \quad (5.27)
$$

for $N-2 \geq i \geq N-h$, and

$$
\left[ \Gamma_{J_1,\ldots,J_m; N-2-h,N-2-h}(w) \right]_{r_h} := W_{j_0} R_{J_1,\ldots,J_m; N-4,N-4}(w) \bigg|_{w} \quad (5.28)
$$

In the sequel, we make use of the quantity $\left[ \Gamma_{J_1,\ldots,J_m; i,i}(w) \right]_{r_h}$ defined like $\left[ \Gamma_{J_1,\ldots,J_m; i,i}(w) \right]_{r_h}$ in (5.27)-(5.28) but with each operators $R_{J_1,\ldots,J_m; i,i}(w)$, $w = z + z^{Bog}_{j_1,\ldots,j_m} \equiv \sum_{h=1}^{N} \left( \Gamma_{J_1,\ldots,J_m; i,i}(w) \right)_{r_h}$ replaced with the corresponding $R_{J_1,\ldots,J_m; i,i}(z)$.

Next, we observe that:

**III** The expansion in (5.27) of the operator

$$
\left[ \Gamma_{J_1,\ldots,J_m; i,i}(w) \right]_{r_h} \quad (5.29)
$$

produces $O(h^2)$ operators $\left[ \Gamma_{J_1,\ldots,J_m; i-2,i-2}(w) \right]_{r_h}$. Then, using iteratively (5.27) from $i = N-2$ down to $i = N-h$ we get

$$
\left[ \Gamma_{J_1,\ldots,J_m; N-2,N-2}(w) \right]_{r_h} =: \sum_{r=1}^{\tilde{r}} \left[ \Gamma_{J_1,\ldots,J_m; N-2,N-2}(w) \right]_{r_h} \quad (5.30)
$$

for some $h$–dependent $\tilde{r} < \infty$ and each summand $\left[ \Gamma_{J_1,\ldots,J_m; N-2,N-2}(w) \right]_{r_h}$ corresponds to $W_{j_0}$ multiplying on the right a finite product of “RW-blocks”, i.e., operators of the type

$$
R_{J_1,\ldots,J_m; i,i}(w) W_{j_0} \quad , \quad R_{J_1,\ldots,J_m; i,i}(w) W_{j_0} \quad (5.31)
$$

where $i$ is even and ranges from $N-4-h$ to $N-4$. The number of the RW-blocks for each $\left[ \Gamma_{J_1,\ldots,J_m; N-2,N-2}(w) \right]_{r_h}$ is bounded by $O((2h)^{\tilde{r}})$ (see section 0.2 in [supporting-file-Bose2.pdf]).

---

8Following the argument of Corollary 5.9 of [Pi1] the second term in (5.25) is $O\left( \frac{1}{\sqrt{n_m}} \left( \frac{1}{1+\frac{1}{\sqrt{n_m}}} \right)^h \right)$. However, in the present paper the multiplicative constants may depend on the size of the box and the details of the potential.

9For the expression $R_{J_1,\ldots,J_m; i,i}(w) W_{j_0}$, the index $i$ ranges from $N-2-h$ to $N-4$.

10It is enough to show by induction taking into account the maximum number of factors $\left[ \Gamma_{J_1,\ldots,J_m; i-2,i-2}(w) \right]_{r_h}$ and $\left[ \Gamma_{J_1,\ldots,J_m; i-2,i-2}(w) \right]_{r_h}$ in $W_{j_0} R_{J_1,\ldots,J_m; i-2,i-2}(w) \left[ \Gamma_{J_1,\ldots,J_m; i-2,i-2}(w) \right]_{r_h} R_{J_1,\ldots,J_m; i-2,i-2}(w) W_{j_0}$ and that $R_{J_1,\ldots,J_m; N-4,N-4}(w) \left[ \Gamma_{J_1,\ldots,J_m; N-4,N-4}(w) \right]_{r_h}$ contains two RW-blocks.
III\(_2\) We can write

\[
\begin{align*}
&\langle \psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}} \rangle, W_{jn} \times \\
&\times \sum_{l=0}^{\bar{j}-1} \sum_{r_1=1}^{p} \cdots \sum_{r_{\bar{j}-1}=1}^{p} \|\psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}}\|, W_{jn} \times \\
&\times \left[ \prod_{i=1}^{\bar{j}} \left[ R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(r_i) R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(r_i) \right] W_{jn}^{*} \psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}} \right] \|\psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}}\|
\end{align*}
\]

where each \([1_{j_1,\ldots,j_n;N-2,N-2}(w)]^{(r_i)}\) is a summand in (5.30) and \(\prod_{i=1}^{\bar{j}} \ldots \equiv 1\). The operator

\[
W_{jn} \left[ R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(w) \prod_{i=1}^{\bar{j}} \left[ 1_{j_1,\ldots,j_n;N-2,N-2}(w)]^{(r_i)} R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(w) \right] W_{jn}^{*} \right]
\]

corresponds to \(W_{jn}\) multiplying on the right a finite product of RW-blocks

\[
R_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w) W_{jn}^{*}, \quad R_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w) W_{jn},
\]

where \(i\) is even and ranges\(^{11}\) from \(N-4-h\) to \(N-2\). The number of RW-blocks in (5.35) is \(O(\bar{j}(2h)^{z_{\frac{1}{2}}} )\) at most because \(0 \leq \bar{j} \leq \bar{j} - 1\).

III\(_3\) We implement the procedure described below on the operator in (5.35) starting from the RW-block on the very right, i.e., \(R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(w) W_{jn}\). We make use of the resolvent equation

\[
R_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w) = \tilde{R}_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w) - R_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w) \mathcal{H}_{j_1} + \ldots + \mathcal{H}_{j_{m-1}} - z_{j_1,\ldots,j_{m-1}} \tilde{R}_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w)
\]

and pull \(\mathcal{H}_{j_1} + \ldots + \mathcal{H}_{j_{m-1}} - z_{j_1,\ldots,j_{m-1}}\) to the right until it hits the vector \(\psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}}\) in the expression

\[
W_{jn} \left[ R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(w) \prod_{i=1}^{\bar{j}} \left[ 1_{j_1,\ldots,j_n;N-2,N-2}(w)]^{(r_i)} R_{j_1,\ldots,j_n;N-2,N-2}^{\text{Bog}}(w) \right] W_{jn}^{*} \psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}} \right] \|\psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}}\|
\]

Recall that, by construction,

\[
(\mathcal{H}_{j_1} + \ldots + \mathcal{H}_{j_{m-1}} - z_{j_1,\ldots,j_{m-1}}) \psi_{j_1,\ldots,j_{m-1}}^{\text{Bog}} = 0.
\]

\(^{11}\)For the expression \(R_{j_1,\ldots,j_n;i,i}^{\text{Bog}}(w) W_{jn}\), the index \(i\) ranges from \(N - 2 - h\) to \(N - 2\).
Assuming that the resolvent $R_{j_1,\ldots,j_m;\ i}(w)$ to be replaced with $\tilde{R}_{j_1,\ldots,j_m;\ i}(z)$ is contained in a RW-block of the type $R_{j_1,\ldots,j_m;\ i}(w)W_{j_n}^*$, we consider the identity

$$\left\{ \sum_{l=1}^{m-1} \hat{H}_{j_l}^B - z_{j_1,\ldots,j_{l-1}} \right\} \tilde{R}_{j_1,\ldots,j_m;\ i}(z)W_{j_n}^*$$

$$(5.41)$$

$$= \tilde{R}_{j_1,\ldots,j_m;\ i}(z)W_{j_n}^* \left\{ \sum_{l=1}^{m-1} \hat{H}_{j_l}^B - z_{j_1,\ldots,j_{l-1}} \right\}$$

$$(5.42)$$

$$-\tilde{R}_{j_1,\ldots,j_m;\ i}(z)\Delta_{R_{j_1,\ldots,j_m;\ i}} \tilde{R}_{j_1,\ldots,j_m;\ i}(z)W_{j_n}^*$$

$$(5.43)$$

$$+\tilde{R}_{j_1,\ldots,j_m;\ i}(z)\Delta W_{j_n}$$

$$(5.44)$$

where

$$\Delta W_{j_n} := \left\{ \sum_{l=1}^{m-1} \hat{H}_{j_l}^B, W_{j_n}^* \right\}$$

$$(5.45)$$

and

$$-\tilde{R}_{j_1,\ldots,j_m;\ i}(z)\Delta_{R_{j_1,\ldots,j_m;\ i}} \tilde{R}_{j_1,\ldots,j_m;\ i}(z) := \left\{ \sum_{l=1}^{m-1} \hat{H}_{j_l}^B, \tilde{R}_{j_1,\ldots,j_m;\ i}(z) \right\}.$$ 

$$(5.46)$$

The terms proportional to (5.43) and (5.44) are the remainders produced by pulling the operator $\sum_{l=1}^{m-1} \hat{H}_{j_l}^B - z_{j_1,\ldots,j_{l-1}}^B$ through the RW-block

$$\tilde{R}_{j_1,\ldots,j_m;\ i}(z)W_{j_n}^*.$$ 

An analogous procedure applies to a RW-block of the type $R_{j_1,\ldots,j_m;\ i}(w)W_{j_n}$.

III$_4$ We proceed with the computation of the commutators in (5.45) and (5.46):

$$\left\{ \sum_{l=1}^{m-1} \hat{H}_{j_l}^B, W_{j_n}^* \right\}$$

$$(5.47)$$

$$= \sum_{l=1}^{m-1} \left[ \phi_j \frac{a_0^* a_0}{N} (a_j^* a_j + a_{-j}^* a_{-j}) + \phi_j \frac{a_0 a_0}{N} a_j^* a_{-j} + \phi_j \frac{a_0^* a_0}{N} a_j a_{-j} + \phi_j \frac{a_0 a_0}{N} a_j a_{-j} \right]$$

$$= \sum_{l=1}^{m-1} \left[ \phi_j \frac{a_0^* a_0}{N} (a_j^* a_j + a_{-j}^* a_{-j}) + \phi_j \frac{a_0 a_0}{N} a_j^* a_{-j} a_{-j} + \phi_j \frac{a_0 a_0}{N} a_j a_{-j} a_{-j} \right]$$

$$= -2 \sum_{l=1}^{m-1} \phi_j \frac{a_0 a_0}{N^2} (a_j^* a_j + a_{-j}^* a_{-j}) + \phi_j \frac{a_0 a_0}{N} a_j^* a_{-j} a_{-j} - \phi_j \frac{a_0 a_0}{N} a_j a_{-j} a_{-j}$$

$$= -2 \phi_j a_j^* a_{-j} a_{-j} \frac{a_0 a_0}{N} \sum_{l=1}^{m-1} \phi_j (a_j^* a_j + a_{-j}^* a_{-j})$$

$$- \phi_j a_j^* a_{-j} a_{-j} \frac{a_0 a_0}{N} \sum_{l=1}^{m-1} \phi_j a_j a_{-j} a_{-j} + \frac{a_0 a_0}{N} \sum_{l=1}^{m-1} \phi_j (a_j^* a_j + a_{-j}^* a_{-j})$$

$$- \phi_j a_j^* a_{-j} a_{-j} \frac{a_0 a_0}{N} \sum_{l=1}^{m-1} \phi_j \frac{4}{N} a_j a_{-j} a_{-j} + \frac{a_0 a_0}{N} \sum_{l=1}^{m-1} \phi_j \frac{2}{N^2} a_j a_{-j} a_{-j}.$$
and (using \([\hat{Q}^{(i,i+1)}_{jm} \sum_{i=1}^{m-1} \hat{H}^{Bog}_{l}] = 0\))

\[
\sum_{l=1}^{m-1} \hat{H}^{Bog}_{l}, \ R^{Bog}_{j_{1},...,j_{m}:l_{i}}(z) \]

\[
= -\hat{R}^{Bog}_{j_{1},...,j_{m}:l_{i}}(z) \sum_{l=1}^{m-1} \hat{H}^{Bog}_{l}, \ \Phi_{jm} \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \hat{R}^{Bog}_{j_{1},...,j_{m}:l_{i}}(z)
\]

where

\[
\sum_{l=1}^{m-1} \hat{H}^{Bog}_{l}, \ \Phi_{jm} \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \]

\[
= \sum_{l=1}^{m-1} \left[ \Phi_{jm} \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \right] + \sum_{l=1}^{m-1} \left[ \Phi_{jm} \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \right] \]

\[
= \sum_{l=1}^{m-1} \Phi_{jm} \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) + 2 \sum_{l=1}^{m-1} \Phi_{jm} \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm})
\]

\[
= -2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) + 2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm})
\]

\[
= -2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) + 2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm})
\]

\[
= -2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) + 2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm})
\]

\[
= -2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) + 2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm})
\]

\[
= -2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) + 2 \frac{a^{\ast}_{jm}a_{jm} - a^{\ast}_{jm}a_{jm}}{N} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm}) \sum_{l=1}^{m-1} \Phi_{jm} (a^{\ast}_{jm}a_{jm} + a^{\ast}_{jm}a_{jm})
\]

III\textsubscript{5} Using the previous mechanism, step by step in expression (5.39) we replace each \(R^{Bog}_{j_{1},...,j_{m}:l_{i}}(w)\) with \(\hat{R}^{Bog}_{j_{1},...,j_{m}:l_{i}}(z)\) by pulling the operator \(\hat{H}^{Bog}_{j_{1},...,j_{m}:l_{i}}(z)\) through each block until it hits the vector \(\psi^{Bog}_{j_{1},...,j_{m}}\). This yields a (leading) product of blocks denoted by

\[
W_{jm} \left\{ \hat{R}^{Bog}_{j_{1},...,j_{m};N-2,N-2}(z) \right\} \prod_{l=1}^{j} \left[ \hat{H}^{Bog}_{j_{1},...,j_{m};N-2,N-2}(z) \right] \psi^{Bog}_{j_{1},...,j_{m}} \left\| \psi^{Bog}_{j_{1},...,j_{m}} \right\|
\]

where each \(R^{Bog}_{j_{1},...,j_{m}:l_{i}}(w)\) has been replaced with the corresponding \(\hat{R}^{Bog}_{j_{1},...,j_{m}:l_{i}}(z)\) and \([\hat{H}^{Bog}_{j_{1},...,j_{m};l_{i}}(z)]^{(r)}_{l_{i}}\) stands for \([\hat{H}^{Bog}_{j_{1},...,j_{m};l_{i}}(w)]^{(r)}_{l_{i}}\) after these replacements. In this process, due to the estimate of the number of RW-blocks in (5.35), each RW-block in (5.39) produces \(O(j(2h)^{\frac{1}{2}})\) remainder terms at most. Thus, the total number of remainder terms associated with (5.39) is bounded by \(O(j(2h)^{\frac{1}{2}})\).

III\textsubscript{6} In general (unless \(i'\) in (5.51) is equal to \(N - 2\)), each remainder term can be written as

\[
W_{jm} \left\{ R^{Bog}_{j_{1},...,j_{m};N-2,N-2}(w) \right\} \prod_{l=1}^{j} \left[ \hat{H}^{Bog}_{j_{1},...,j_{m};N-2,N-2}(z) \right] \left( -R^{Bog}_{j_{1},...,j_{m};l_{i}}(w) x (5.51) \right)
\]
for some \( i', N - 4 - h \leq i' \leq N - 2 \), so that the original product of RW-blocks has been split into the product of three subproducts:

1. the symbol “Left” stands for a subproduct where the original RW-blocks are unchanged. Note that if \( i' = N - 2 \) the “Left” part is absent;
2. the “separating resolvent” \((-\mathbf{R}_{Bog}^{\text{\neg}})^{i_1, \ldots, j_m; \iota; \iota'}(w)\) that separates the “Left” part from the “Right” part;
3. the symbol “Right” stands for a subproduct where, with respect to the original expression, all the RW-blocks minus one – the “exceptional RW-block” – are unchanged except for the fact that the operators \(\mathbf{R}_{Bog}^{\text{\neg}}\) have been replaced with \(\mathbf{R}_{Bog}^{\text{\neg}}\). Assume that the exceptional RW-block is of the type \(\mathbf{R}_{Bog}^{\text{\neg}}(w)\). (The other case is analogous.) Then, it is replaced either with

\[
\mathbf{R}_{Bog}^{\text{\neg}}(z) \Delta \mathbf{W}^\star_{J_m}
\]

or with

\[
- \mathbf{R}_{Bog}^{\text{\neg}}(z) \Delta \mathbf{R}_{J_m} \mathbf{R}_{Bog}^{\text{\neg}}(z) \mathbf{W}^\star_{J_m}.
\]

Furthermore, the exceptional RW-block splits the subproduct labeled by “Right” into two subproducts of RW-blocks labeled by “Right 1” and “Right 2”, respectively, unless:

1) \( i' \equiv i'' \), in this case the factor “Right 1” is just the identity; 2) \( i'' \equiv N - 2 \), in this case the factor “Right 2” is just the identity. Therefore, assuming for example that the replacement of the exceptional RW-block is with (5.53) (see also (5.45) and (5.47)) we get

\[
W_{J_m}\left[\mathbf{R}_{Bog}^{\text{\neg}}(w) \prod_{l=1}^{l} \left[\left[\mathbf{R}_{Bog}^{\text{\neg}}\right]_{J_1, \ldots, J_m; N-2, N-2}(w)\right]^{(r)} \mathbf{R}_{Bog}^{\text{\neg}}\right]_{J_1, \ldots, J_m; N-2, N-2}(w)\right]\times
\]

\[
\mathbf{R}_{Bog}^{\text{\neg}}(z) \Delta \mathbf{W}^\star_{J_m}
\]

\[
- \mathbf{R}_{Bog}^{\text{\neg}}(z) \Delta \mathbf{R}_{J_m} \mathbf{R}_{Bog}^{\text{\neg}}(z) \mathbf{W}^\star_{J_m}.
\]
\begin{align}
&+ W_{j_0} \left\{ R_{j_1 \ldots j_m ; N-2, N-2} (w) \prod_{l=1}^{j} \left[ \left[ R_{j_1 \ldots j_m ; N-2, N-2} (w) \right]^{(r_l)} R_{j_1 \ldots j_m ; N-2, N-2} (w) \right] \right\} \left( - \right) R_{j_1 \ldots j_m ; l' \ldots l'} (w) \times \\
&\left( \begin{array}{c}
\times \left( R_{j_1 \ldots j_m ; N-2, N-2} (z) \prod_{l=1}^{j} \left[ \left[ R_{j_1 \ldots j_m ; N-2, N-2} (z) \right]^{(r_l)} R_{j_1 \ldots j_m ; N-2, N-2} (z) \right] \right) \times \\
\times \left( R_{j_1 \ldots j_m ; l' \ldots l'} (z) \phi_{j_0} \frac{a_{j_0}^* a_{j_0} - 2}{N} \left( - 4 a_{j_0}^* a_{j_0} - 2 \right) \sum_{l=1}^{m-1} \phi_{j_l} a_{j_l} a_{j_l} - 1 \right) \times \\
\times \left( R_{j_1 \ldots j_m ; N-2, N-2} (z) \prod_{l=1}^{j} \left[ \left[ R_{j_1 \ldots j_m ; N-2, N-2} (z) \right]^{(r_l)} R_{j_1 \ldots j_m ; N-2, N-2} (z) \right] \right) W_{j_0}^* \times \\
\times \frac{\sum_{l=1}^{m-1} a_{j_l}^* a_{j_l} + 1}{N} \frac{\psi_{j_0}}{||\psi_{j_0}||} \right) \right) \right) \times \\
\left( \begin{array}{c}
\text{Left} \\
\text{Right 1} \\
\text{Right 2}
\end{array} \right) \right) \right)
\end{align}

(5.61)

(5.62)

(5.63)

(5.64)

(5.65)

(5.66)

where we have used that the operators \( \{a_{j_l}, a_{j_l}^* \mid l = 1, \ldots, m - 1\} \) commute with “Right 2”. An analogous structure is obtained if we consider (5.54) replacing the exceptional RW-block, or in the cases where the exceptional RW-block is of type \( R_{j_1 \ldots j_m ; l' \ldots l'} (z) W_{j_0} \) (see point 3. above).

III) Here, we estimate the set of remainder terms of the type (5.51)-(5.52) that have been produced in the previous step. Due to the assumption in (5.14), the norm of the vectors of the type in (5.60) is bounded by \( O(\frac{1}{N}) \). The rest of the expression can be controlled by considering the operator norm estimates in Corollary 5.1, elementary bounds like \( \sum_{l \in \mathbb{Z}^+} a_{j_l}^* a_{j_l} \leq N \), and invoking the argument in Remark 4.3. To this end, we recall that due to Remark 4.3 for \( \sigma_m \) small enough we can estimate

\begin{align}
&\| \sum_{l=0}^{j-1} \sum_{r_l=1}^{j} \sum_{r_1=1}^{j} \prod_{l=1}^{j} \left[ \left[ R_{j_1 \ldots j_m ; N-2, N-2} (w) \right]^{(r_l)} R_{j_1 \ldots j_m ; N-2, N-2} (w) \right] \int \right\| \right) \right)
\end{align}

(5.67)

(5.68)

(5.69)

(5.70)

(5.71)

where the symbol \( E \) is defined in (4.45) and the step from (5.70) to (5.71) follows from (4.49)-(4.54).
For the moment, assume that $i' < N - 2$ and $i'' < N - 2$. Consider the expression

$$\sum_{l=0}^{j-1} \sum_{r_1=1}^{p} \cdots \sum_{r_l=1}^{p} \prod_{l=1}^{l} [(R_{j_1, \ldots, j_m; N-2,N-2}(w))^{\frac{1}{2}} [\Gamma_{j_1, \ldots, j_m; N-2,N-2}(w)]_{r_{\theta}} [R_{j_1, \ldots, j_m; N-2,N-2}(w)]^{\frac{1}{2}}]_{#}$$ (5.72)

where the symbol $[...]_{#}$ means that some of the resolvents $R_{j_1, \ldots, j_m; i_{l_\theta}}(w)$ might have been replaced with $\tilde{R}_{j_1, \ldots, j_m; i_{l_\theta}}(z)$, and one separating resolvent (with $i' < N - 2$) and one exceptional RW-block (with $i'' < N - 2$) are present, with the exceptional RW-block replaced according to the rules explained in STEP III$\delta$: In the example of above, the exceptional RW-block is replaced with the operators in (5.58) + (5.64). An analogous structure can be shown in all other cases different from the given example. We observe that:

- In Corollary 5.1 we can replace one or both the two resolvents $R_{j_1, \ldots, j_m; i_{l_\theta}}(w)$, $R_{j_1, \ldots, j_m; i_{-2,l-2}}(w)$ with $\tilde{R}_{j_1, \ldots, j_m; i_{l_\theta}}(z)$ and $\tilde{R}_{j_1, \ldots, j_m; i_{-2,l-2}}(z)$, respectively, and yet we can provide the same estimate from above for the operator norms in (4.4);

- Concerning the operator replacing the exceptional RW-block in (5.72) we note that, in the given example,

$$\|(\tilde{R}_{j_1, \ldots, j_m; i_{l'\theta},i_{l''\theta}}(z))^{-\frac{1}{2}}(\tilde{R}_{j_1, \ldots, j_m; i_{l'+2,l''+2}}(z))^\frac{1}{2}\| \leq C E(\|\tilde{R}_{j_1, \ldots, j_m; i_{l'\theta},i_{l''\theta}}(z)\|^2 W_{j_m}(\tilde{R}_{j_1, \ldots, j_m; i_{l'+2,l''+2}}(z))^\frac{1}{2}\|)$$

and the same holds for the operator in (5.64). This type of estimate holds for the operators replacing any possible exceptional block;

- The norm of the separating resolvent is less than a universal constant.

Hence, we can estimate the norm of (5.72) as we estimate (5.67) with the help of $E(\ldots)$ defined in Remark 4.3, i.e., for some $C' > 0$

$$\left\|\sum_{l=0}^{j-1} \sum_{r_1=1}^{p} \cdots \sum_{r_l=1}^{p} \prod_{l=1}^{l} [(R_{j_1, \ldots, j_m; N-2,N-2}(w))^{\frac{1}{2}} [\Gamma_{j_1, \ldots, j_m; N-2,N-2}(w)]_{r_{\theta}} [R_{j_1, \ldots, j_m; N-2,N-2}(w)]^{\frac{1}{2}}]_{#}\right\|$$ (5.73)

$$\leq C' E\left\|\sum_{l=0}^{j-1} \sum_{r_1=1}^{p} \cdots \sum_{r_l=1}^{p} \prod_{l=1}^{l} [(R_{j_1, \ldots, j_m; N-2,N-2}(w))^{\frac{1}{2}} [\Gamma_{j_1, \ldots, j_m; N-2,N-2}(w)]_{r_{\theta}} [R_{j_1, \ldots, j_m; N-2,N-2}(w)]^{\frac{1}{2}}]_{#}\right\|$$ (5.74)

$$= C' E\left\|\sum_{l=0}^{j-1} \sum_{r_1=1}^{p} \cdots \sum_{r_l=1}^{p} E\left(\prod_{l=1}^{l} [(R_{j_1, \ldots, j_m; N-2,N-2}(w))^{\frac{1}{2}} [\Gamma_{j_1, \ldots, j_m; N-2,N-2}(w)]_{r_{\theta}} [R_{j_1, \ldots, j_m; N-2,N-2}(w)]^{\frac{1}{2}}]\right)\right\|.$$ (5.75)

Then, we can draw the conclusion: The norm of each remainder term (where the exceptional RW-block has index $i'' < N - 2$ and the separating resolvent corresponds to an index $i' < N - 2$) that is produced in expression (5.34) from the summand

$$\frac{\psi_{j_1, \ldots, j_m; i_{l_1}}}{\|\psi_{j_1, \ldots, j_m; i_{l_1}}\|} \times W_{j_m} \times$$ (5.76)

$$\times (R_{j_1, \ldots, j_m; N-2,N-2}(w) \prod_{l=1}^{l} [(\Gamma_{j_1, \ldots, j_m; N-2,N-2}(w)]_{r_{\theta}} [R_{j_1, \ldots, j_m; N-2,N-2}(w)]^{\frac{1}{2}}]_{#})^\frac{1}{2}$$ (5.77)
is surely bounded by

\[
O\left(\frac{1}{\sqrt{N}}\right) \times \mathcal{E}\left(\prod_{t=1}^{T} \left\| \left(\tilde{R}_{j_{1},\ldots,j_{m}}; N-2,N-2(w)\right)^{r_{t}} \left[\Gamma_{j_{1},\ldots,j_{m}}; N-2,N-2(w)\right]_{\tilde{r}_{t}} \right\|_{\|\cdot\|}ight) \times 5.78
\]

\[
\leq O\left(\frac{1}{\sqrt{N}}\right) \times \mathcal{E}\left(\prod_{t=1}^{T} \left\| \left(\tilde{R}_{j_{1},\ldots,j_{m}}; N-2,N-2(w)\right)^{r_{t}} \left[\Gamma_{j_{1},\ldots,j_{m}}; N-2,N-2(w)\right]_{\tilde{r}_{t}} \right\| \right) \times 5.79
\]

where the factor \(O\left(\frac{1}{\sqrt{N}}\right)\) comes from the norm of the vectors of the type in (5.60) and (5.66). By a similar argument we derive the same estimate (5.79) for each remainder term coming from the summand in (5.76) but corresponding to the cases \(i' = N - 2\) and/or \(i'' = N - 2\). Next, we make use of the estimates in (5.67)-(5.71) and (5.73)-(5.75) to control the sums \(\sum_{j=0}^{j-1} \sum_{r_{1}=1}^{\rho} \ldots \sum_{r_{p}=1}^{\rho}\). Since the total number of remainder terms that are produced in (5.34) out of each summand (5.76) is bounded by \(O(\tilde{j}(2\bar{h})^{|r_{2}+r_{2}} \cdot j(2\bar{h})^{|r_{2}+r_{2}}))\), we conclude that the total contribution of the error terms produced in \textbf{STEP III} is bounded by \(O\left(\frac{1}{\sqrt{N}}\right)\).

\textbf{STEP IV)

} Collecting the results in \textbf{STEP I}, \textbf{STEP II}, and \textbf{STEP III}, and setting \(\tilde{j} \equiv h\), we conclude that

\[
\langle \phi_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N,N_{\theta}(w) \rangle_{\|\phi_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N,N_{\theta}(w)\|} \times 5.80
\]

\[
= \langle \phi_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N,N_{\theta}(w) \rangle_{\|\phi_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N,N_{\theta}(w)\|} \times 5.81
\]

\[
\times \left[ \tilde{W}_{j_{m}} \sum_{l_{N-2}=0}^{h-1} \tilde{R}_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z) \left[\Gamma_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z)\right]_{\tilde{r}_{t}} \tilde{R}_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z) \right]^{l_{N-2}} \times 5.82
\]

\[
+ O\left(\frac{h^{2}(2\bar{h})^{h+2}}{\sqrt{N}}\right) + O\left(\left(\frac{4}{5}\right)^{h}\right) + O\left(\left(\frac{1}{1+c\sqrt{\epsilon_{j_{m}}}}\right)^{h}\right).
\]

For convenience, we define

\[
[\Gamma_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N,N_{\theta}(z)]_{\tilde{r}_{t}} \times 5.83
\]

\[
:= W_{j_{m}} \sum_{l_{N-2}=0}^{h-1} \tilde{R}_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z) \left[\Gamma_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z)\right]_{\tilde{r}_{t}} \tilde{R}_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z) \right]^{l_{N-2}} W_{j_{m}}
\]

where \([\Gamma_{j_{1},\ldots,j_{m}}^{Bog}, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z)]_{\tilde{r}_{t}}\) and \(\tilde{R}_{j_{1},\ldots,j_{m}}^{Bog}; N-2,N-2(z)\) have been defined in \textbf{STEP III} after (5.28).

Now, we observe that the leading term from \textbf{STEP III}, i.e., the term in (5.81), can be replaced with

\[
\langle \eta, \Gamma_{j_{1},\ldots,j_{m}}^{Bog}; N,N_{\theta}(z) \rangle
\]

up to a small error that vanishes as \(N \to \infty\). To this purpose, we denote by \(\chi_{N-h}\) the projection onto the subspace of states with at least \(N - h\) particles in the zero mode. Then, using the
In the sequel, we estimate the assumption in (5.14) we get

$$
\|(1 - \chi N^{-h}) \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|} \| \leq \|(N_e)_{\frac{h}{\sqrt{h}}} (1 - \chi N^{-h}) \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|} \| \leq \mathcal{O}\left(\frac{1}{\sqrt{h}}\right)
$$

(5.84)

where $N_e := \sum_{j=0}^{\infty} a_j^* a_j$. Consequently,

$$
\left\langle \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|}, \left[\Gamma_{j_1, \ldots, j_{N-1}}^{\text{Bog}}(\zeta)\right]_{\tau_h} \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|}\rightangle = \langle \chi N^{-h} \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|}, \left[\Gamma_{j_1, \ldots, j_{N-1}}^{\text{Bog}}(\zeta)\right]_{\tau_h} \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|}\rangle + \mathcal{O}\left(\frac{1}{\sqrt{h}}\right).
$$

(5.85)

(5.86)

In the sequel, we estimate the difference between

$$
\langle \chi N^{-h} \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|}, \left[\Gamma_{j_1, \ldots, j_{N-1}}^{\text{Bog}}(\zeta)\right]_{\tau_h} \chi N^{-h} \frac{\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}}{\|\psi_{\text{Bog}}^{\text{Bog}}_{j_1, \ldots, j_{N-1}}\|}\rangle
$$

(5.87)

and

$$
\langle \eta, \left[\Gamma_{j_1, \ldots, j_{N-1}}^{\text{Bog}}(\zeta)\right]_{\tau_h} \eta \rangle.
$$

(5.88)

In passing, it is helpful to recall the identity (see (2.44))

$$
\langle \eta, \left[\Gamma_{j_1, \ldots, j_{N-1}}^{\text{Bog}}(\zeta)\right]_{\tau_h} \eta \rangle = (1 - \frac{1}{N}) \frac{\delta_{j_k}}{2\epsilon_{j_k} + \frac{2}{N} + 2 - \frac{1}{\epsilon_{j_k}}} \mathcal{G}_{j_k : N-2, N-2}(\zeta)
$$

(5.89)

and to express (5.87) in a similar way. For this reason, in Lemma 5.7 we consider the modified expression $[\tilde{\mathcal{G}}_{j_k : i_i(z)}]_{\tau_h : \Delta n_{j_k}}$ defined for $i$ even, $N - h - 2 \leq i \leq N - 2$, $0 \leq \Delta n_{j_k} \leq h$, $i - \Delta n_{j_k} - 2 \geq 0$, by the relation

$$
[\tilde{\mathcal{G}}_{j_k : i_i(z)}]_{\tau_h : \Delta n_{j_k}} := \sum_{i=0}^{h-1} ([W_{j_k : i-j-2}(\zeta)W_{j_k : i-j-2}^*(\zeta)]_{\Delta n_{j_k}}) [\tilde{\mathcal{G}}_{j_k : i-2, i-2}(\zeta)]_{\tau_h : \Delta n_{j_k}}
$$

(5.90)

where

$$
[\tilde{\mathcal{G}}_{j_k : N-h-4, N-h-4}(\zeta)]_{\tau_h : \Delta n_{j_k}} = 1
$$

(5.91)

and

$$
[W_{j_k : i-j-2}(\zeta)W_{j_k : i-j-2}^*(\zeta)]_{\Delta n_{j_k}} := \frac{(i - \Delta n_{j_k} - 1)(i - \Delta n_{j_k})}{N^2} \frac{(N - i + 2)^2}{4\left((k_{j_k})^2(N - i) - z\right)}.
$$

(5.92)

(5.93)

(5.94)

For $N$ large enough, $(0 \leq) \Delta n_{j_k} \leq h$, and

$$
\frac{z}{(h + 4)\phi_{j_k}} (\leq 0),
$$

(5.95)
in Lemma 5.7 we estimate
\[ \left| \frac{\partial \tilde{\mathcal{G}}_{j_1, j_1}(\xi)}{\partial \Delta n_{j_0}} \right|_{\tau_h; \Delta n_{j_0}} \leq K h \cdot \frac{g^{j-N+h}}{\sqrt{N}} \] (5.96)

where \( g \) is not larger than 4 and \( K \) is a universal constant.

Going back to expression (5.87), we observe that the operator
\[ \left[ \tilde{\Gamma}^{\text{Bog}}_{j_1, \ldots, j_0; N,N(z)} \right]_{\tau_h} \] (5.97)
preserves the number of particles for any mode. Therefore,
\[ \langle \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} [\tilde{\Gamma}^{\text{Bog}}_{j_1, \ldots, j_0; N,N(z)}]_{\tau_h} \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \rangle \]
\[ = \sum_{n_{j_0} = N-h}^{N} \sum_{n_{j_0} = N-h}^{N} \langle \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} [\tilde{\Gamma}^{\text{Bog}}_{j_1, \ldots, j_0; N,N(z)}]_{\tau_h} \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \rangle \]
\[ = \sum_{n_{j_0} = N-h}^{N} \sum_{n_{j_0} = N-h}^{N} \frac{\phi_{j_0}}{N^2} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} \right|^2. \] (5.100)

Invoking Lemma 5.7, we can write
\[ \sum_{n_{j_0} = N-h}^{N} \sum_{n_{j_0} = N-h}^{N} \frac{\phi_{j_0}}{N^2} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} \right|^2 \]
\[ = (1 - \frac{1}{N}) \frac{\phi_{j_0}}{2 \phi_{j_0}} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} \right|^2 + S(z) \] (5.103)

with \( |S(z)| \leq O\left( \frac{h^2 g^h}{\sqrt{N}} \right) \). In turn, we conclude that
\[ \langle \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} [\tilde{\Gamma}^{\text{Bog}}_{j_1, \ldots, j_0; N,N(z)}]_{\tau_h} \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \rangle \]
\[ = (1 - \frac{1}{N}) \frac{\phi_{j_0}}{2 \phi_{j_0}} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} + O\left( \frac{1}{\sqrt{N}} \right) + O\left( \frac{h^2 g^h}{\sqrt{N}} \right) \] (5.105)

\[ = (1 - \frac{1}{N}) \frac{\phi_{j_0}}{2 \phi_{j_0}} \left( \chi^{N-h}_{j_1, \ldots, j_{n_0}-1} \psi^{\text{Bog}}_{j_1, \ldots, j_{n_0}-1} \right)_{\tau_h; \Delta n_{j_0}} + O\left( \frac{1}{\sqrt{N}} \right) + O\left( \frac{h^2 g^h}{\sqrt{N}} \right) \] (5.106)

\[ + O\left( \frac{1}{\sqrt{N}} \right) + O\left( \frac{h^2 g^h}{\sqrt{N}} \right) + O\left( \frac{4}{5} \right) + O\left( \frac{1}{1 + c \sqrt{\phi_{j_0}}} \right) \] (5.107)

\[ = \langle \eta, \tilde{\Gamma}^{\text{Bog}}_{j_1, \ldots, j_0; N,N(z)} \eta \rangle + O\left( \frac{1}{\sqrt{h}} \right) + O\left( \frac{h^2 g^h}{\sqrt{N}} \right) + O\left( \frac{4}{5} \right) + O\left( \frac{1}{1 + c \sqrt{\phi_{j_0}}} \right) \] (5.108)
where in the step from (5.105) to (5.106) we essentially implement the inverse of STEP I and II on the quantity

\[
(1 - \frac{1}{N}) \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} - \frac{c}{e_{j_m}}} [\tilde{G}_{j_m; N-2,N-2}(z)]_{\tau_h; 0} \equiv \langle \eta, [\Gamma_{Bog}^{j_m, N-2,N-2}(z)]_{\tau_h} \rangle.
\]

As far as (5.16) is concerned, with the same arguments we arrive at

\[
\| \mathcal{P}_{j_m}^{Bog} [\Gamma_{Bog}^{j_m, N,N}(z)]_{\tau_h} \psi_{Bog}^{j_m, N,N}(z) \| \leq \| \mathcal{P}_{j_m}^{Bog} [\Gamma_{Bog}^{j_m, N,N}(z)]_{\tau_h} \| + O(\frac{h^2(2h)^{h+2}}{\sqrt{N}}) + O((\frac{4}{5})^h) + O((\frac{1}{1 + c \sqrt{e_{j_m}}})^h)
\]

\[
\leq \| \mathcal{P}_{j_m}^{Bog} [\Gamma_{Bog}^{j_m, N,N}(z)]_{\tau_h} \| + O(\frac{1}{\sqrt{h}}) + O(\frac{h^2g^h}{\sqrt{N}}) + O((\frac{4}{5})^h) + O((\frac{1}{1 + c \sqrt{e_{j_m}}})^h)
\]

\[
= (1 - \frac{1}{N}) \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} - \frac{c}{e_{j_m}}} [\tilde{G}_{j_m; N-2,N-2}(z)]_{\tau_h; 0} \| \mathcal{P}_{j_m}^{Bog} [\Gamma_{Bog}^{j_m, N,N}(z)]_{\tau_h} \| + O(\frac{1}{\sqrt{h}}) + O(\frac{h^2g^h}{\sqrt{N}}) + O((\frac{4}{5})^h) + O((\frac{1}{1 + c \sqrt{e_{j_m}}})^h)
\]

\[
= (1 - \frac{1}{N}) \frac{\phi_{j_m}}{2e_{j_m} + 2 - \frac{4}{N} - \frac{c}{e_{j_m}}} [\tilde{G}_{j_m; N-2,N-2}(z)]_{\tau_h; 0} \| \mathcal{P}_{j_m}^{Bog} [\Gamma_{Bog}^{j_m, N,N}(z)]_{\tau_h} \| + O(\frac{1}{\sqrt{h}}) + O(\frac{h^2g^h}{\sqrt{N}}) + O((\frac{4}{5})^h) + O((\frac{1}{1 + c \sqrt{e_{j_m}}})^h)
\]

\[
= O(\frac{1}{\sqrt{h}}) + O(\frac{h^2g^h}{\sqrt{N}}) + O((\frac{4}{5})^h) + O((\frac{1}{1 + c \sqrt{e_{j_m}}})^h)
\]

Then, we set \( h \equiv \{ \lfloor \ln N \rfloor \}^{\frac{1}{2}} \) where \( \{ \lfloor \ln N \rfloor \}^{\frac{1}{2}} \) is assumed to be even and determine two positive constants \( C_I \) and \( C_{II} \) such that the inequalities in (5.15) and (5.16) hold true.

\[ \square \]

**Lemma 5.3.** Let \( 2 \leq m \leq M \) and assume that the Hamiltonian \( H_{Bog}^{j_m, N,N} \) has nondegenerate ground state energy \( \zeta_{Bog}^{j_m, N,N} \) and ground state vector \( \psi_{Bog}^{j_m, N,N} \) with the property in (5.14). Furthermore, assume that:

1. There exists \( \Delta_0 \geq \Delta_{m-1} > 0 \) such that

\[
\inf \text{spec} \left[ (\tilde{H}_{Bog}^{j_m, N,N} + \sum_{j \in \mathbb{Z}^d \setminus \{z_1, \ldots, z_{j_m}\}} (k_j)^2 a_j^* a_j) \Gamma_{Bog}^{j_m, N,N}(z) \right]_{\tau_h} \geq \delta_{Bog}^{j_m, N,N} \geq \Delta_{m-1}
\]
where \( Q_{(N-1)}^{(N)} \) is the subspace of states in \( \mathcal{F}^N \) with no particles in the modes \( \pm j_m \), and \( \Omega_{\{j_1, \ldots, j_{N-1}\}} \) is the subspace generated by the vector \( \psi_{j_1, \ldots, j_{N-1}}^{\text{Bog}} \).

(Notice that)

\[
\hat{H}_{j_1, \ldots, j_{N-1}}^{\text{Bog}} + \sum_{j \in \mathbb{Z}^d \setminus \{j_1, \ldots, j_{N-1}\}} (k_j)^2 a_j^\dagger a_j = H_{j_1, \ldots, j_{N-1}}^{\text{Bog}} - (k_j)^2 (a_{j_m}^\dagger a_{j_m} + a_{-j_m}^\dagger a_{-j_m}),
\]

i.e., the kinetic energy associated with the modes \( \pm j_m \) is absent.

2. \[
\inf \text{spec} \left[ \sum_{l=1}^{m-1} \hat{H}_{j_l}^{\text{Bog}} \right] - z_{j_1, \ldots, j_{N-1}}^{\text{Bog}} \geq -\frac{m-1}{(\ln N)^{\frac{1}{2}}},
\]

Let \( \epsilon_m \) be sufficiently small and \( N \) sufficiently large such that:

a) for
\[
z \leq E_{j_m}^{\text{Bog}} + \sqrt{\epsilon_m} \phi_{j_m} \sqrt{\epsilon_m^2 + 2 \epsilon_m} < 0
\]

the Feshbach-Schur flow associated with the couple of modes \( \pm j_m \) is well defined (see Theorem 4.1);

b) \[
\frac{\ln N}{N} \ll 1 \quad \text{and} \quad \frac{U_{j_m}}{\sqrt{N}} < \frac{\Delta_{m-1}}{2}
\]

where \( U_{j_m} := \phi_{j_m}^2 + \epsilon_{j_m} \).

c) \[
\gamma \Delta_{m-1} < E_{j_m}^{\text{Bog}} + \frac{1}{2} \sqrt{\epsilon_m} \phi_{j_m} \sqrt{\epsilon_m^2 + 2 \epsilon_m} < 0
\]

where \( \epsilon_m \) is the ground state energy of \( H_{j_m}^{\text{Bog}} \).

Then, there exists a constant \( C^\perp > 0 \) such that for
\[
z \leq \epsilon_m - \frac{C^\perp}{(\ln N)^{\frac{1}{2}}} + \gamma \Delta_{m-1}
\]

the following estimate holds true:
\[
\mathcal{P}_{\psi_{j_1, \ldots, j_{N-1}}^{\text{Bog}}} \mathcal{A}_{j_1, \ldots, j_{N-1}}^{\text{Bog}} (z + \epsilon_{j_1, \ldots, j_{N-1}}^{\text{Bog}}) \geq (1 - \gamma) \Delta_{m-1} \mathcal{P}_{\psi_{j_1, \ldots, j_{N-1}}^{\text{Bog}}}.
\]

**Proof**

As we have explained in the **Outline of the proof** (see Lemma 4.5), the result is proven if the key inequality in (4.81) holds true in the interval (4.77). In order to prove (4.81), we implement the same truncations of **Step I** and **Step II** in Lemma 5.2, and make use of (5.26). Hereafter, we set \( w = z + \epsilon_{j_1, \ldots, j_{N-1}}^{\text{Bog}} \). For \( h \geq 2 \) and even, we get

(4.78)

\[
= \| \mathcal{P}_{\psi_{j_1, \ldots, j_{N-1}}^{\text{Bog}}} W_{j_m} R_{j_1, \ldots, j_{N-2}}^{\text{Bog}} (w) \times
\]

\[
\times \sum_{l_m=0}^{j_m-1} \left[ \left[ \Gamma_{j_1, \ldots, j_{N-2}}^{\text{Bog}} (w) \right]_{l_m} R_{j_1, \ldots, j_{N-2}}^{\text{Bog}} (w) \right]^{j_m-2} W_{j_m} \mathcal{P}_{\psi_{j_1, \ldots, j_{N-1}}^{\text{Bog}}} \|
\]

+ \mathcal{O}\left( \frac{4}{5} \right) + \mathcal{O}\left( \frac{1}{1 + c \sqrt{\epsilon_m}} \right).
\]

\[\tag{5.122}\]

\[\tag{5.120}\]

\[\tag{5.119}\]

\[\tag{5.118}\]

\[\tag{5.117}\]

\[\tag{5.116}\]

\[\tag{5.115}\]

\[\tag{5.81}\]
We set $\tilde{j} \equiv h$. From (5.121)-(5.122) we derive the inequality in (4.81) through steps a), b), c), d), e), and f) described below.

**STEP a)**

In expression (5.121), using the definition of $[\Gamma_{j_1,\ldots,j_m}^{Bog}; N-2, N-2(w)]_{\tau_h}$ given in (5.27)-(5.28) and the decomposition

$$[\Gamma_{j_1,\ldots,j_m}^{Bog}; N-2, N-2(w)]_{\tau_h} = \sum_{r=1}^{\tilde{j}} [\Gamma_{j_1,\ldots,j_m}^{Bog}; N-2, N-2(w)]_{\tau_h}^{(r)}$$

in (5.30), to each (left) operator of the type

$$(R_{j_1,\ldots,j_m; i,i}^{Bog})(w)\frac{i}{2} W_{j_m;i,i-2} (R_{j_1,\ldots,j_m; i-2,i-2}^{Bog})(w)\frac{i}{2}, \quad N - 4 \geq \tilde{i} \geq N - 2 - h \quad \text{and even,}$$

that pops up from the re-expansion of

$$(R_{j_1,\ldots,j_m; N-2, N-2}^{Bog})(w)\frac{i}{2} [\Gamma_{j_1,\ldots,j_m; N-2, N-2}^{Bog}; \tau_h](R_{j_1,\ldots,j_m; N-2, N-2}^{Bog})(w)\frac{i}{2}$$

we can assign a (right) companion contained in the same $[\Gamma_{j_1,\ldots,j_m; N-2, N-2}^{Bog}; \tau_h]^{(r)}$, precisely the closest operator

$$(R_{j_1,\ldots,j_m; i-2,i-2}^{Bog})(w)\frac{i}{2} W_{j_m;i,i-2} (R_{j_1,\ldots,j_m; i,i}^{Bog})(w)\frac{i}{2}$$

(5.124)

that is placed on the right of (5.123) in $[\Gamma_{j_1,\ldots,j_m; N-2, N-2}^{Bog}; \tau_h]^{(r)}$. The companion operator always exists because of the structure of $[\Gamma_{j_1,\ldots,j_m; N-2, N-2}^{Bog}; \tau_h]$ (see (5.27)-(5.28)). Then, starting from the two companions

$$(R_{j_1,\ldots,j_m; N-2-h, N-2-h}^{Bog})(w)\frac{i}{2} [\Gamma_{j_1,\ldots,j_m; N-2, N-2-h}^{Bog}; \tau_h](R_{j_1,\ldots,j_m; N-2-h, N-2-h}^{Bog})(w)\frac{i}{2} (5.125)$$

in (5.28), and using the relation in (5.27), it is not difficult to show by induction that, in expression (5.121), all operators of the type (5.123) and (5.124) can be replaced with operator-valued functions of the “variables” $a_{j_m}^* a_{j_m}$ and $a_{j_m}^* a_{-j_m}$, and the (number) operators $a_{j_m}^* a_{j_m}, a_{j_m}^* a_{-j_m}$ can be replaced with c-numbers; see section 0.3 in supporting-file-Bose2.pdf. In essence, this is due to: 1) the projections contained in the definition of $R_{j_1,\ldots,j_m; i,i}^{Bog} (w)$ (see (3.33)); 2) the form of the interaction terms $W_{j_m}, W_{j_m;}^*3$ the operator $\mathcal{P}_{j_1,\ldots,j_{m-1}}^{Bog}$ (see (5.121)) projecting onto states with no particles in the modes $\pm j_m$. More precisely, since

$$a_{j_m}^* a_{-j_m} a_{-j_m}^* a_{j_m} = (a_{j_m}^* a_{j_m} + 1)(a_{j_m}^* a_{j_m} + 1)$$

(5.126)

it turns out that (for $N - 2 \leq i \leq N - h - 2$

$$((R_{j_1,\ldots,j_m; i,i}^{Bog})(w)\frac{i}{2} \phi_{j_m}^* a_{j_m}^* a_{j_m}^* a_{-j_m} a_{-j_m} N (R_{j_1,\ldots,j_m; i-2,i-2}^{Bog})(w)\frac{i}{2}$$

(5.127)

is replaced with

$$[\Gamma_{j_1,\ldots,j_m; i,i}^{Bog}; \tau_h](w)\frac{i}{2} W_{j_m;i,i-2} (R_{j_1,\ldots,j_m; i-2,i-2}^{Bog})(w)\frac{i}{2}]_1$$

$$:= \phi_{j_m} \frac{1}{\sum_{j \neq j_1,\ldots,j_m} (k_j)^2 a_{j_m}^* a_{j_m} + R_{j_1,\ldots,j_m; i-2}^{Bog} - \phi_{j_m}^* a_{j_m}^* a_{j_m} + (\frac{a_{j_m}^* a_{j_m}}{N} \phi_{j_m} + (k_j^2))(n_{j_m} + n_{-j_m}) - \epsilon)} \times$$

$$\times \frac{a_{j_m}^* a_{j_m}^*}{N} \left( n_{j_m} + 1 \right)^{\tilde{j}} \left( n_{-j_m} + 1 \right)^{\tilde{j}}$$

$$\frac{1}{\sum_{j \neq j_1,\ldots,j_m} (k_j)^2 a_{j_m}^* a_{j_m} + R_{j_1,\ldots,j_m; i-2}^{Bog} - \phi_{j_m}^* a_{j_m}^* a_{j_m} + (\frac{a_{j_m}^* a_{j_m}}{N} \phi_{j_m} + (k_j^2))(n_{j_m} + n_{-j_m} + 2)}$$

(5.128)

(5.129)

(5.130)
where
\[ n_{j_0} + n_{-j_0} = N - i , \quad n_{j_m} = n_{-j_m} , \] (5.131)

We stress that the resolvents in (5.129)-(5.130) act on Fock subspaces \( \mathcal{F}^M \) with \( M < N \) and even. The operator in (5.128) maps \( \mathcal{F}^M \) to \( \mathcal{F}^{M+2} \) with \( M < N \). Therefore, it is convenient to think of the Fock subspaces \( \mathcal{F}^M \), \( M < N \), embedded in \( \mathcal{F}^N := \oplus_{n=0}^N \mathcal{F}^n \). Nevertheless, the inequality in (5.115) implies\(^\text{13}\) that the operator
\[ \tilde{H}_{j_1, \ldots, j_{m-1}}^{\text{Bog}} - \zeta_{j_1, \ldots, j_{m-1}}^{\text{Bog}} \]
restricted to \( \mathcal{F}^M \) with \( M < N \) is also bounded below by the R-H-S of (5.115). Hence, the resolvents in (5.129) and (5.130) are well defined for \( \epsilon_{j_m} \) sufficiently small, \( N \) sufficiently large, and \( z \) in the interval (4.77).

Analogously, the operator
\[ (R_{j_1, \ldots, j_m ; i-2, i-2}^{\text{Bog}}(w))^\frac{1}{2} \phi_{j_m} \frac{a_0 a_0^* a_{j_m}^*}{N} (R_{j_1, \ldots, j_m ; i, i}(w))^\frac{1}{2} \] is replaced with the hermitian conjugate of (5.129)-(5.130) that we denote by
\[ [(R_{j_1, \ldots, j_m ; i-2, i-2}^{\text{Bog}}(w))^\frac{1}{2} W_{j_1, \ldots, j_m ; i, i}(R_{j_1, \ldots, j_m ; i-2, i-2}^{\text{Bog}}(w))^\frac{1}{2}]_1 . \] (5.133)

Similar replacements hold for the outer companion operators:
\[ \phi_{j_m} \frac{a_0^* a_0 a_{-j_m}^*}{N} (R_{j_1, \ldots, j_m ; N-2, N-2}^{\text{Bog}}(w))^\frac{1}{2} = \phi_{j_m} \frac{a_0^* a_0}{N} (R_{j_1, \ldots, j_m ; N-2, N-2}^{\text{Bog}}(w))^\frac{1}{2} \] (5.134)
\[ \phi_{j_m} (R_{j_1, \ldots, j_m ; N-2, N-2}^{\text{Bog}}(w))^\frac{1}{2} \frac{a_0 a_0^* a_{-j_m}^*}{N} = \phi_{j_m} (R_{j_1, \ldots, j_m ; N-2, N-2}^{\text{Bog}}(w))^\frac{1}{2} \frac{a_0 a_0}{N} \] (5.135)

where it is assumed that in \( R_{j_1, \ldots, j_m ; N-2, N-2}^{\text{Bog}}(w) \) the operator \( a_{j_m}^* a_{-j_m} + a_{-j_m}^* a_{j_m} \) is replaced with 2.

**Remark 5.4.** For the details of the replacement of the operators \( a_{j_m}^* a_{-j_m} \) with c-numbers described in lines (5.127)-(5.135) we refer the reader to Proposition 5.7 of [P11] where a similar procedure is implemented by induction.

By means of the implemented operations we have shown that (more details in section 0.3 of *supporting-file-Bose2.pdf*)
\[ \mathcal{D}_{\phi_{j_1, \ldots, j_{m-1}}^{\text{Bog}}} W_{j_m} R_{j_1, \ldots, j_m ; N-2, N-2}^{\text{Bog}}(w) \sum_{l_{N-2}=0}^{h-1} \left( [I_{j_1, \ldots, j_m ; N-2, N-2}(w)]_{l_N=0} R_{j_1, \ldots, j_m ; N-2, N-2}(w) \right)^{l_{N-2}} W_{j_m} \mathcal{D}_{\phi_{j_1, \ldots, j_{m-1}}^{\text{Bog}}} \] (5.136)

coincides with
\[ \left[ \mathcal{D}_{\phi_{j_1, \ldots, j_{m-1}}^{\text{Bog}}} W_{j_m} R_{j_1, \ldots, j_m ; N-2, N-2}(w) \sum_{l_{N-2}=0}^{h-1} \left( [I_{j_1, \ldots, j_m ; N-2, N-2}(w)]_{l_N=0} R_{j_1, \ldots, j_m ; N-2, N-2}(w) \right)^{l_{N-2}} W_{j_m} \mathcal{D}_{\phi_{j_1, \ldots, j_{m-1}}^{\text{Bog}}} \right]_1 \] (5.137)

\[^{13} \text{It is enough to consider } \tilde{H}_{j_1, \ldots, j_{m-1}}^{\text{Bog}} - \zeta_{j_1, \ldots, j_{m-1}}^{\text{Bog}} \text{ applied to the subspace of } \mathcal{F}^N \text{ generated by the vectors with at least } N - M \text{ particles in the modes different from } 0, \pm j_1, \ldots, \pm j_{m-1}. \]
where the symbol $\left[ \right]$ means that each couple of companions has been transformed according to the rules described in lines (5.127)–(5.135). Concerning notation, in the following steps, we make use of the definitions

$$
[R_{j_1,\ldots,j_m; i, \ldots, i}^{Bog}(w)]_1 := \frac{1}{\sum_{j \notin \{z_{j_1}, \ldots, z_{j_m}\}} (k_j^2) a_j a_j^* + \hat{\mathcal{H}}_{j_1,\ldots,j_{m-1}}^{Bog} - \bar{z}_{j_{m-1}}^{Bog} + \left( \frac{\alpha_{a_0} u_0}{N} \phi_{j_m} + (k_j^2) \right)(N - i) - z}
$$

and

$$
[(R_{j_1,\ldots,j_m; i, \ldots, i}^{Bog}(w))^\dagger]_1 := \left[ \frac{1}{\sum_{j \notin \{z_{j_1}, \ldots, z_{j_m}\}} (k_j^2) a_j a_j^* + \hat{\mathcal{H}}_{j_1,\ldots,j_{m-1}}^{Bog} - \bar{z}_{j_{m-1}}^{Bog} + \left( \frac{\alpha_{a_0} u_0}{N} \phi_{j_m} + (k_j^2) \right)(N - i) - z} \right]^\dagger.
$$

**STEP b)**

Next, we consider each couple of left and right companion operators contained in (5.137). Up to a small remainder to be estimated, we replace the operator $a_0^* a_0^\ast$ appearing in the numerator of (5.128) with $a_0^* a_0^\ast - 1$, and the operator $a_0^\ast_0 a_0^\ast_0$ appearing in the numerator of (5.133) with $a_0^* a_0^\ast$. Roughly speaking, we take the operator $\frac{a_0^* a_0^\ast}{N}$ (belonging to the right companion) next to the operator $\frac{a_0^* a_0^\ast}{N}$ (of the left companion), and we write

$$
\frac{a_0^* a_0^\ast a_0^* a_0^\ast}{N^2} = \frac{(a_0^* a_0^\ast - 1) a_0^* a_0^\ast}{N}.
$$

The operator $\frac{a_0^* a_0^\ast}{N}$ is then taken back to the original position (of $a_0^* a_0^\ast$) in the right companion. Before providing a rigorous description of this mechanism, we describe in detail the final leading expression that we obtain.

As a result of the previous operations we get as leading term an expression identical to (5.137) but where each operator $[(R_{j_1,\ldots,j_m; i, \ldots, i}^{Bog}(w))^\dagger W_{j_0}; i, i-2 (R_{j_1,\ldots,j_m; i, i-2, i-2}^{Bog}(w))^\dagger]_1$ is replaced with

$$
[(R_{j_1,\ldots,j_m; i, \ldots, i}^{Bog}(w))^\dagger W_{j_0}; i, i-2 (R_{j_1,\ldots,j_m; i, i-2, i-2}^{Bog}(w))^\dagger]_1 \times \frac{1}{(N - i + 2)^\dagger (N - i + 2)^\dagger}
$$

and its right companion $[(R_{j_0,\ldots,j_m; i, \ldots, i}^{Bog}(w))^\dagger W_{j_0}; i, i-2, i, i-2 (R_{j_0,\ldots,j_m; i, i-2, i-2}^{Bog}(w))^\dagger]_1$ with the operator

$$
[R_{j_0,\ldots,j_m; i, \ldots, i}^{Bog}(w))^\dagger W_{j_0}; i, i-2 (R_{j_0,\ldots,j_m; i, i-2, i-2}^{Bog}(w))^\dagger]_1 \times \frac{1}{(N - i + 2)^\dagger (N - i + 2)^\dagger}
$$

14In (5.141)-(5.142) we use that $n_{j_0} + n_{j_m} = N - i$ and $n_{j_0} = n_{j_m}$. 

58
We point out that the resolvents in (5.141)-(5.142) and in (5.144)-(5.145) are now operators from $\mathcal{F}^N$ to $\mathcal{F}^N$. Analogous replacements are implemented on the outer companions in (5.134)-(5.135), and we denote them by $\left[ W_{\jmath_{m}}(R_{\jmath_{1},...,\jmath_{m};N-2,N-2})^{\frac{1}{2}} \right]_2$ and $\left[ (R_{\jmath_{1},...,\jmath_{m};N-2,N-2})^{\frac{1}{2}} W_{\jmath_{m}} \right]_2$, respectively.

We denote this (leading) collection of terms as

$$
\mathcal{P}_{\psi_{\jmath_{1},...,\jmath_{m-1}}} W_{\jmath_{m}} R_{\jmath_{1},...,\jmath_{m};N-2,N-2}(w) \sum_{l_{N-2}=0}^{h-1} \left[ (R_{\jmath_{1},...,\jmath_{m};N-2,N-2}(w))_{\tau_{h}} R_{\jmath_{1},...,\jmath_{m};N-2,N-2}(w) \right]^{l_{N-2}} W_{\jmath_{m}} \mathcal{P}_{\psi_{\jmath_{1},...,\jmath_{m-1}}} \bigg]_2.
$$

(5.146)

Now, we describe the procedure that yields (5.146) and the way we organize the remainder terms corresponding to (5.137) – (5.146). Starting from the definition in (5.27), for $N - 2 \geq i \geq N - h$ we can write

$$
\left[ R_{\jmath_{1},...,\jmath_{i},i}(w) \right]_{\tau_{h}} R_{\jmath_{1},...,\jmath_{i},i}(w) \bigg]_{1} \tag{5.147}
$$

$$
\left[ R_{\jmath_{1},...,\jmath_{i},i}(w) \right]_{\tau_{h}} W_{\jmath_{m}} R_{\jmath_{1},...,\jmath_{i},i}(w) \bigg]_{1} \tag{5.148}
$$

$$
\times \sum_{l_{i-2}=0}^{h-1} \left[ (R_{\jmath_{1},...,\jmath_{i-2},i-2}(w))_{\tau_{h}} (R_{\jmath_{1},...,\jmath_{i-2},i-2}(w))^{l_{i-2}} \right] \tag{5.149}
$$

$$
\times (R_{\jmath_{1},...,\jmath_{i-2},i-2}(w))^{l_{i-2}} W_{\jmath_{m}} (R_{\jmath_{1},...,\jmath_{i},i}(w)) \bigg]_{1}
$$

where

$$
\left[ \Gamma_{\jmath_{1},...,\jmath_{i},i-2}(N-2-h,N-2-h) \right]_{\tau_{h}} := W_{\jmath_{m}} R_{\jmath_{1},...,\jmath_{i},i}(N-4-h,N-4-h) W_{\jmath_{m}}. \tag{5.150}
$$

**Warning.** As no confusion can arise, to make our formulae shorter in the remaining part of STEP b) we omit the label $\jmath_{1},...,\jmath_{m}$ and the argument $w$ in $R_{\jmath_{1},...,\jmath_{m};i}(w)$, and the label $\psi_{\jmath_{1},...,\jmath_{m-1}}$ in $\mathcal{P}_{\psi_{\jmath_{1},...,\jmath_{m-1}}}$. Using the formulae of above, we can write (in the new notation)

$$
\mathcal{P} W R_{N-2,N-2} \sum_{l_{N-2}=0}^{h-1} \left[ \left[ R_{N-2,N-2} \psi_{\jmath_{1},...,\jmath_{m-1}} \right]_{\tau_{h}} R_{N-2,N-2}^{\frac{1}{2}} \bigg] W^{*} \mathcal{P} \bigg]_{1}
$$

$$
= \mathcal{P} W \left[ \left[ R_{N-2,N-2}^{\frac{1}{2}} \bigg]_{1} \sum_{l_{N-2}=0}^{h-1} \left[ \left[ R_{N-2,N-2} \psi_{\jmath_{1},...,\jmath_{m-1}} \right]_{\tau_{h}} R_{N-2,N-2}^{\frac{1}{2}} \right] W^{*} \mathcal{P} \bigg]_{2}
$$

$$
= \mathcal{P} W \left[ R_{N-2,N-2}^{\frac{1}{2}} \bigg] \sum_{l_{N-2}=0}^{h-1} \left[ \left[ R_{N-2,N-2} \psi_{\jmath_{1},...,\jmath_{m-1}} \right]_{\tau_{h}} R_{N-2,N-2}^{\frac{1}{2}} \right] W^{*} \mathcal{P} \bigg]_{2} \tag{5.151}
$$

$$
+ \mathcal{P} W \left[ R_{N-2,N-2}^{\frac{1}{2}} \bigg]_{1} \sum_{l_{N-2}=0}^{h-1} \left[ \left[ R_{N-2,N-2} \psi_{\jmath_{1},...,\jmath_{m-1}} \right]_{\tau_{h}} R_{N-2,N-2}^{\frac{1}{2}} \right] W^{*} \mathcal{P} \bigg]_{2} \tag{5.152}
$$

where the symbol $\mathcal{P}$ means that the embraced expression (including the outer $W$ and $W^*$) is replaced with

$$
\frac{a_{\psi_{\jmath_{1},...,\jmath_{m-1}}} N}{N} \left[ \left[ R_{N-2,N-2} \psi_{\jmath_{1},...,\jmath_{m-1}} \right]_{\tau_{h}} R_{N-2,N-2}^{\frac{1}{2}} \right]_{1}, \frac{a_{\psi_{\jmath_{1},...,\jmath_{m-1}}} N - i + 2}{2} \bigg]_{l_{N-2}}
$$

59
\[ + \frac{(a_0^* a_0 - 1)}{N} \times \]
\[ \times \left[ \frac{a_0^* a_0}{N} \left( \frac{N - i + 2}{2} \right) \right] \left[ \left( R_{N-2,N-2}^{Bog} \right)^{\frac{1}{2}} \sum_{l_{i-2} = 0}^{h-1} \left[ \left( R_{N-2,N-2}^{Bog} \right)^{\frac{1}{2}} \left( R_{N-2,N-2}^{Bog} \right)^{\frac{1}{2}} \right] \right] \right]. \]

Next, in a similar way, for \( N - h - 2 \leq i \leq N - 2 \), we define\(^{15}\) the operation \( \mathcal{D}_i \) where \( i \) stands for level \( i \) (recall that \( i \) is an even number):

\[
\mathcal{D}_i \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \]
\[ := \left[ (R_{i,j}^{Bog})^{\frac{1}{2}} W(R_{i-2,i-2}^{Bog})^{\frac{1}{2}} \right] \sum_{l_{i-2} = 0}^{h-1} \left[ \left( R_{i-2,i-2}^{Bog} \right)^{\frac{1}{2}} \left( R_{i-2,i-2}^{Bog} \right)^{\frac{1}{2}} \right] \left[ (R_{i-2,i-2}^{Bog})^{\frac{1}{2}} W(R_{i,j}^{Bog})^{\frac{1}{2}} \right], \tag{5.153} \]

where the symbol \( \mathcal{T} \) means that the embraced expression is replaced with

\[
\frac{N - i + 2}{2} \frac{a_0^* a_0}{N} \times \]
\[ \times \left[ \frac{a_0^* a_0}{N} \left( \frac{N - i + 2}{2} \right) \right] \left[ \left( R_{N-2,N-2}^{Bog} \right)^{\frac{1}{2}} \sum_{l_{i-2} = 0}^{h-1} \left[ \left( R_{N-2,N-2}^{Bog} \right)^{\frac{1}{2}} \left( R_{N-2,N-2}^{Bog} \right)^{\frac{1}{2}} \right] \right] \right]. \tag{5.154} \]

On a product of \( n \) operators \( \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \), by definition \( \mathcal{D}_i \) acts as a “derivative” according to the Leibniz rule:

\[
\mathcal{D}_i \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \]
\[ := \sum_{j=0}^{n-1} \mathcal{L}_j \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \mathcal{D}_{j+1} \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \right] \right] \right] \right] \right] \right], \tag{5.156} \]

where

\[
\mathcal{L}_j \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \]
\[ := \left[ (R_{i,j}^{Bog})^{\frac{1}{2}} W(R_{i-2,i-2}^{Bog})^{\frac{1}{2}} \right] \sum_{l_{i-2} = 0}^{h-1} \left[ \left( R_{i-2,i-2}^{Bog} \right)^{\frac{1}{2}} \left( R_{i-2,i-2}^{Bog} \right)^{\frac{1}{2}} \right] \left[ (R_{i-2,i-2}^{Bog})^{\frac{1}{2}} W(R_{i,j}^{Bog})^{\frac{1}{2}} \right], \tag{5.157} \]

On a product of \( n \) operators \( \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \) not necessarily contiguous (i.e., the operators \( \left[ \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \left( R_{i,j}^{Bog} \right)^{\frac{1}{2}} \right] \) can be factors of a larger product of operators, and as factors of this product they may be not contiguous) we define the action of \( \mathcal{D}_i \) as in (5.156)

\(^{15}\)If \( i = N - h - 2 \) the operator \( \sum_{l_{i-2} = 0}^{h-1} \left[ \left( R_{i-2,i-2}^{Bog} \right)^{\frac{1}{2}} \left( R_{i-2,i-2}^{Bog} \right)^{\frac{1}{2}} \right] \) is absent.
with $D_{i}(A) = 0$ and $L_{i}(A) = A$ if $A \neq [[R_{ij}^{Bog}, \frac{1}{2}[[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}}$, e.g.,

$$D_{i}(\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1}) = D_{i}(\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1}) + L_{i}(\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1})$$

(5.158)

$$D_{i}(\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1}) = D_{i}(\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1}) + L_{i}(\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1})$$

(5.159)

By definition, $D_{i}$ acts linearly with respect to the sum. Using the definitions of above (see (5.153), (5.154), (5.155), (5.156), and (5.157)), it is straightforward to check that for any product of $n$ operators $\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1}$ the following identity holds (for the details see section 0.4 in supporting-file-Bose2.pdf)

$$\left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\} = \left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\}$$

(5.161)

$$\left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\} = \left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\} + \left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\}$$

(5.162)

The same property is true for finite products where the factors $\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]_{1}$ are not contiguous, e.g.,

$$\left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\} = \left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\}$$

(5.163)

(5.164)

$$\left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\} = \left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\}$$

(5.165)

$$\left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\}$$

(5.166)

Therefore, we can re-write

$$\left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\} = \left\{\left[(R_{ij}^{Bog})^{\frac{1}{2}}[\Gamma_{ij}^{Bog}, \tau_{h}(R_{ij}^{Bog})^{\frac{1}{2}}]_{1}^{n}\right]\right\}$$

(5.167)

(5.168)

(5.169)

We observe that the expression in (5.168) coincides with the original one in (5.137) with the only difference that all the companion operators $[\ldots]_{1}$ on level $N-2$ (i.e., the highest index amongst the two resolvents of a companion operator) are replaced with the corresponding $[\ldots]_{2}$. Therefore, we re-write (5.168) as follows

$$\overline{\mathcal{F}}\left\{W(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}\right\}^{2}_{2} \sum^{l_{N-2}=0}_{l_{N-2}=0} D_{i-2}\left\{\left[(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}[\Gamma_{N-2,N-2}^{Bog}, \tau_{h}(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}]_{1}^{n-2}\right]\right\}^{l_{N-2}}\left[(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}W^{\ast}\right]^{2}_{2} \overline{\mathcal{F}}$$

(5.170)

(5.171)

where, in general, we denote by

$$\left[(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}[\Gamma_{N-2,N-2}^{Bog}, \tau_{h}(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}]_{1}^{n-2}\right]_{2, l_{N-2}}$$

(5.172)

the expression $\left[(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}[\Gamma_{N-2,N-2}^{Bog}, \tau_{h}(R_{N-2,N-2}^{Bog})^{\frac{1}{2}}]_{1}^{n-2}\right]_{2, l_{N-2}}$ where the companion operators $[\ldots]_{1}$ on all levels from $N-2$ down to $i$ are replaced with the corresponding operators $[\ldots]_{2}$. We keep
(5.152) and (5.169) aside and implement the decomposition (5.161)-(5.163) on level \(N - 4\) for the expression in (5.168). We iterate this scheme down to level \(i = N - 2 - h\). By induction, it is immediate to verify that

\[
\mathcal{P} \left[ W(R_{N-2,N-2}^\uparrow) \right] = \mathcal{P} \left[ W(R_{N-2,N-2}^\uparrow) \right] + \sum_{j=N-4, \text{even}}^{N-2-h} \mathcal{P} \left[ W(R_{N-2,N-2}^\uparrow) \right] \left[ D_i \left( \left[ \left( R_{N-2,N-2}^\uparrow \right)^2 \right] \right) \right] \times \left( R_{N-2,N-2}^\uparrow \right) \mathcal{P} \left[ W(R_{N-2,N-2}^\uparrow) \right]
\]

(5.174)

\[
+ (5.152).
\]

The structure of the remainder terms displayed in (5.174)-(5.176) is important to estimate them in STEP e).

**STEP e)**

We proceed by implementing Steps III\(_1\) and III\(_2\) of Lemma 5.2 on (5.146). Hence, we express (5.146) as a sum of expressions of the type

\[
\left[ \mathcal{P}_{\phi_{1\ldots j_m+1}} \right] \left( \phi_{1\ldots j_m+1}^2 \right) \left( R_{j_1\ldots j_m+1,N-2,N-2}^\uparrow \right) \left[ \left[ \left[ R_{j_1\ldots j_m+1,N,N}^\uparrow \right] \left( R_{j_1\ldots j_m+1,N-2,N-2}^\uparrow \right) \right] \right] \left( a_0 \alpha_0 \right) \left( \frac{N}{\mathcal{P}_{\phi_{1\ldots j_m+1}}} \right)
\]

(5.177)

where the symbol \( \left[ \left[ \left[ \right] \right] \right] \) means that each couple of companion operators has been replaced according to the rules of above. Next, starting from the very right of (5.177), in front (i.e., on the right) of each resolvent we split the identity (in \(Q_{j_m}^{(N-1)}F_N\)) into

\[
1_{Q_{j_m}^{(N-1)}F_N} = \mathcal{P}_{\phi_{1\ldots j_m+1}} + \mathcal{P}_{\phi_{1\ldots j_m+1}} \mathcal{P}_{\phi_{1\ldots j_m+1}}
\]

(5.178)

Due to the operator \( \mathcal{P}_{\phi_{1\ldots j_m+1}} \), for the very first resolvent from the right we have

\[
\frac{1}{T_{j \neq \{j_1, \ldots, j_m\}}} + \mathcal{P}_{\phi_{1\ldots j_m+1}} \mathcal{P}_{\phi_{1\ldots j_m+1}} \mathcal{P}_{\phi_{1\ldots j_m+1}} \mathcal{P}_{\phi_{1\ldots j_m+1}} = \mathcal{P}_{\phi_{1\ldots j_m+1}} + \mathcal{P}_{\phi_{1\ldots j_m+1}} \mathcal{P}_{\phi_{1\ldots j_m+1}} \mathcal{P}_{\phi_{1\ldots j_m+1}}
\]

(5.179)

where \( T_{j \neq \{j_1, \ldots, j_m\}} := \sum_{j \neq \{j_1, \ldots, j_m\}} k_j^2 a_j^* a_j \). Note that the property in (5.14) implies

\[
\| \mathcal{P}_{\phi_{1\ldots j_m+1}} \sum_{j \neq 0} a_j^* a_j \| \leq O(\sqrt{N}).
\]

(5.180)
Consequently, the norm of the operator that in (5.179) is proportional to \( \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \frac{a_0^*a_0}{N} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \) is small, given that

\[
\left\| \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \frac{a_0^*a_0}{N} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \right\| = \left\| \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \frac{N - \sum_{j=0}^\infty a_j^*a_j}{N} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \right\| \leq O\left( \frac{1}{\sqrt{N}} \right). \tag{5.181}
\]

Using the property in (5.14) once more we can estimate\(^{16}\)

\[
\left\| \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \left[ T_{j\neq \{\pm 1, \ldots, \pm j\}} + \mathcal{H}_{\text{Bog}}^{\text{Bog}} \right] - z_{\text{Bog}} \right\| \leq O\left( \frac{N - i}{\sqrt{N}} \right). \tag{5.182}
\]

With regard to the operator that in (5.179) is proportional to \( \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \frac{a_0^*a_0}{N} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \), combining (5.182) with the (formal) resolvent identity

\[
\frac{1}{A} = \frac{1}{B} + \frac{1}{A} \left( B - A \right) \frac{1}{B} \tag{5.183}
\]

where

\[
A \equiv T_{j\neq \{\pm 1, \ldots, \pm j\}} + \mathcal{H}_{\text{Bog}}^{\text{Bog}} - z_{\text{Bog}} \left( \frac{a_0^*a_0}{N} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \right), \quad B \equiv \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \left[ T_{j\neq \{\pm 1, \ldots, \pm j\}} + \mathcal{H}_{\text{Bog}}^{\text{Bog}} \right] - z_{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}},
\]

up to an operator whose norm is bounded by \( O\left( \frac{N - i}{\sqrt{N}} \right) \), we can replace

\[
\frac{1}{T_{j\neq \{\pm 1, \ldots, \pm j\}} + \mathcal{H}_{\text{Bog}}^{\text{Bog}} - z_{\text{Bog}} \left( \frac{a_0^*a_0}{N} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \right)} \tag{5.184}
\]

with

\[
\mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \left[ T_{j\neq \{\pm 1, \ldots, \pm j\}} + \mathcal{H}_{\text{Bog}}^{\text{Bog}} - z_{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \right] - z_{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \tag{5.185}
\]

Proceeding from the right to the left, an analogous argument can be repeated for all the resolvents in (5.177). Hence, up to an operator whose norm is bounded by \( O\left( \frac{N - i}{\sqrt{N}} \right) \), each operator of the type

\[
[ (R_{\text{Bog}}^{\text{Bog}};_{i\neq i-2}(w)) \right]_2 \tag{5.186}
\]

is replaced with

\[
\left[ \left( R_{\text{Bog}}^{\text{Bog}};_{i\neq i-2}(w) \right) \right]_2 W_{j\neq \{\pm 1, \ldots, \pm j\}} - 2 \left( R_{\text{Bog}}^{\text{Bog}};_{i\neq i-2}(w) \right) \right]_2 \times \tag{5.187}
\]

\[
\times \left[ \frac{1}{\mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}}} \left( \frac{a_0^*a_0}{N} \right) (N - i + 2) \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \times \times \left( \frac{a_0^*a_0 - 1}{2N} \right) (N - i + 2) \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \right]^\frac{1}{2} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}}.
\]

\(^{16}\)Recall that \( \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} \left[ T_{j\neq \{\pm 1, \ldots, \pm j\}} + \mathcal{H}_{\text{Bog}}^{\text{Bog}} \right] - z_{\text{Bog}} \mathcal{P}_{\psi_{\text{Bog}}}^{\text{Bog}} = 0.\)
The companion operator of (5.186) is replaced with
\[
\left[ (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2} W_{J_1,...,J_m; i-2, i-2 (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2}} \right]_{\psi}
\]
\[= \frac{1}{\mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi}} \left( T_{J_1,...,J_m} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \phi_{J_m} \right) \times \frac{1}{\mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi}} \frac{a_0^2 a}{2N} (N-i+2) \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \times \left( \frac{1}{\mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi}} \Delta_{m-1} + \frac{a_0^2 a}{N} \phi_{J_m} + k_J^2 (N-i) - z \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \right)^\frac{1}{2} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi},
\]
(5.190)

Analogous replacements hold for the outer companions (i.e., the operators in (5.134) and (5.135)).

**STEP d)**

Here, we estimate from above the norm of the leading expression resulting from STEPS a), b), c). We observe that, under the assumptions on \(\epsilon_{J_m}, N, \) and \(z,\)
\[
\| (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2} W_{J_1,...,J_m; i-2, i-2 (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2}} \| \leq 1
\]
(5.191)

thanks to (5.114). (See a similar argument in Corollary 5.1.) Hence, we can estimate
\[
\| (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2} W_{J_1,...,J_m; i-2, i-2 (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2}} \| \leq \frac{1}{\mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi}} \left( \Delta_{m-1} + \frac{a_0^2 a}{N} \phi_{J_m} + k_J^2 (N-i) - z \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \right)^\frac{1}{2} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi},
\]
(5.193)

We set \(h = \lfloor (\ln N)^{\frac{1}{2}} \rfloor \) with \(\ln N \ll N.\) The (even) index \(i\) ranges from \(N - 2 - \lfloor (\ln N)^{\frac{1}{2}} \rfloor \) to \(N - 2\) where, for simplicity of the notation, we have assumed that \(\lfloor (\ln N)^{\frac{1}{2}} \rfloor \) is even.

With the help of the spectral theorem for commuting self-adjoint operators (i.e., \(\mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi}\) and \(a_0^2 a_0 \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi} \mathcal{P}_{\phi_{J_1,...,J_m-1}}^{\psi}\) and the bound \(a_0^2 a_0 \leq N,\) we deduce (see section 0.5 of the file supporting-file-Bose2.pdf) that
\[
\| (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2} W_{J_1,...,J_m; i-2, i-2 (R_{J_1,...,J_m}; i-2, i-2 (w))^\frac{1}{2}} \| \leq \frac{1}{2} \left[ \Delta_{m-1} + (\phi_{J_m} + k_J^2 (N-i) - z \right] \frac{1}{2} \left[ \Delta_{m-1} + (\phi_{J_m} + k_J^2 (N-i) - z \right] \frac{1}{2}
\]
(5.194)
for $N$ large enough and $z$ in the interval (4.77). The same bound holds for

$$\| (R_{1;\ldots;i,m}^{\text{Bog}})^{i} W_{1;\ldots;i,m}^{*} (R_{1;\ldots;i,m}^{\text{Bog}})_{i} \|_p. $$

Finally, we collect the leading terms obtained for each expression (5.177) after the implementation of STEP e), and recall that, by construction, the sum of expressions (5.177) coincides with (5.146). Therefore, due to (5.195) and to an analogous operator norm estimate for the outer companions, the norm of the sum of the collected leading terms associated with (5.146) is bounded by

$$\frac{\phi_{1m}^2}{[\Delta_{m-1} + 2(\phi_{1m} + k_{1m}^2) - z]} \tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z).$$

(5.196)

where $\tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z)$ is defined by recursion:

$$\tilde{\mathcal{K}}_{m;i,N-4-h,N-4-h}^{(h)}(z) \equiv 1$$

(5.197)

and for $N - 2 \geq i \geq N - 2 - h$ (and even)

$$\tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z) := \sum_{l=0}^{h-1} \left[ \frac{(N - i + 2)^2}{4[\Delta_{m-1} + (\phi_{1m} + k_{1m}^2)(N - i + 2) - z] \Delta_{m-1} + (\phi_{1m} + k_{1m}^2)(N - i) - z} \right] \mathcal{K}_{m;i,N-2,N-2}^{(h)}(z).$$

(5.198)

Eventually, we want to show that, for $z$ in the interval in (4.77) and assuming the condition in (5.116), for the chosen $h$ the quantity in (5.196) is bounded by

$$\left(1 - \frac{1}{N}\right) \frac{\phi_{1m}}{2\epsilon_{1m} + 2 - \frac{4}{N} - \frac{\epsilon_{1m}}{\phi_{1m}}} \tilde{G}_{m;i,N-2,N-2}(z - \Delta_{m-1} + \frac{U_{m}}{\sqrt{N}}).$$

(5.199)

where $U_{m} = k_{1m}^2 + \phi_{1m}$. (Recall that we have set $h = [(\ln N)^{\frac{1}{2}}]$ with $\ln N \ll N$. The (even) index $i$ ranges from $N - 2 - [(\ln N)^{\frac{1}{2}}]$ to $N - 2$ where, for simplicity of the notation, we have assumed that $[(\ln N)^{\frac{1}{2}}]$ is even.) Then, for $z$ in the interval in (4.77), and assuming the condition in (5.116), a direct computation (see section 0.6 in supporting-file-Bose2.pdf) shows that the following inequality holds true (for $N - 2 \geq i \geq N - 2 - h$)

$$\frac{(N - i + 2)^2}{4[\Delta_{m-1} + (\phi_{1m} + k_{1m}^2)(N - i + 2) - z] \Delta_{m-1} + (\phi_{1m} + k_{1m}^2)(N - i) - z} \leq W_{m;i,N-2,N-2}(z - \Delta_{m-1} + \frac{U_{m}}{\sqrt{N}}) \leq \tilde{G}_{m;i,N-2,N-2}(z - \Delta_{m-1} + \frac{U_{m}}{\sqrt{N}}).$$

(5.200)

where $W_{m;i,N-2,N-2}(z) \leq \tilde{G}_{m;i,N-2,N-2}(z)$ enters the definition of $\tilde{G}_{m;i,N-2,N-2}(z)$ (see (2.45)-(2.48)). Now, we define $\tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z)$ similarly to $\tilde{G}_{m;i,N-2,N-2}(z)$ but replacing $\sum_{l=0}^{h-1}$ with $\sum_{l=0}^{\infty}$ in the recursive definition (see (2.45)). Then, for $N$ sufficiently large we readily obtain

$$\tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z) \leq \tilde{\mathcal{G}}_{m;i,N-2,N-2}^{(h)}(z - \Delta_{m-1} + \frac{U_{m}}{\sqrt{N}}) \leq \tilde{\mathcal{G}}_{m;i,N-2,N-2}^{(h)}(z - \Delta_{m-1} + \frac{U_{m}}{\sqrt{N}}) \leq \tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z).$$

(5.201)

and

$$\frac{\phi_{1m}^2}{[\Delta_{m-1} + 2(\phi_{1m} + k_{1m}^2) - z]} \tilde{\mathcal{K}}_{m;i,N-2,N-2}^{(h)}(z) \leq (5.199).$$

(5.202)
STEP e)

Now, we estimate (all) the remainder terms produced implementing **STEP b** on the expression in (5.137). We claim that each of the $h/2 + 2$ remainder terms in (5.174)-(5.176) is bounded in norm by

$$O(h \frac{2^h}{N}).$$

Concerning notation, we remind that in **STEP b** we omit the label $j_1, \ldots, j_m$ and the argument $w$ in $R^{Bog}_{j_1, \ldots, j_m; i}(w)$ and the label $\psi^{Bog}_{j_1, \ldots, j_m}$ in $\mathcal{P}^{Bog}_{j_1, \ldots, j_m}$. In the present step where we estimate the norm of the (remainder) terms produced in **STEP b** the notation is consistent with that choice, except for points 4. and 5. where it is useful to re-introduce the complete notation.

To estimate the norm of the $j-th$ summand in (5.174), i.e.,

$$\mathcal{P}[W(R^{Bog}_{N-2,N-2})^{1/2} \{ \sum_{l_{N-2}=0}^{h-1} \mathcal{D}_j \left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2} \}^{l_{N-2}} \}^{1/2},$$

at first we express

$$\sum_{l_{N-2}=0}^{h-1} \left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$$

as a sum of “monomials” similarly to the procedure in **STEP III** of Lemma 5.2. Next,

- for each of these monomials we evaluate how many new monomials are created due to the action of $\mathcal{D}_j$;
- we estimate the norm of each resulting term;
- we apply the argument of Remark 4.3 and conclude that the norm of the whole expression in (5.203) is bounded by $O(N^2)$ times the maximum number of new terms created by the action of $\mathcal{D}_j$ on a single monomial.

Below we explain the procedure.

1. Like in **STEP III** of Lemma 5.2, we use formula (5.27) iteratively in order to re-express

$$\sum_{l_{N-2}=0}^{h-1} \left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$$

In detail, similarly to (5.30), we get (for some $j-$dependent $\tilde{r}$)

$$\left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2} =: \sum_{r=1}^{j} \left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$$

where each $\left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$ is a “monomial” that corresponds to $\left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$ multiplying (on the right) factors of the type

$$\left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$$

$$\left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$$

$$\left[ \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \left[ \Gamma^{Bog}_{N-2,N-2} \right] \left( R^{Bog}_{N-2,N-2} \right)^{1/2} \right]_{l_{N-2}; j+2}$$

(5.207)
with \( j + 2 \leq s \leq N - 2 \) and even. Therefore, (5.205) can be written

\[
\sum_{i=0}^{h-1} \sum_{\tau_1=1}^j \cdots \sum_{\tau_{j+2}=1}^j \prod_{i=1}^l \left\{ \left[ (R_{N-2,N-2}^{Bog})^{\frac{1}{2}} \right] \right\}_2 j_{j+2} \tag{5.208}
\]

where each \( \left[ (R_{N-2,N-2}^{Bog})^{\frac{1}{2}} \right] \) is a summand in (5.206) and \( \prod_{i=1}^{l=0} \cdots \equiv 1 \).

We define

\[ #_{RFR} := \text{the maximum number of factors} \left\{ \left[ (R_{N-2,N-2}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \text{ contained in each}
\]

\[ \text{product} \prod_{i=1}^l \left\{ \left[ (R_{N-2,N-2}^{Bog})^{\frac{1}{2}} \right] \right\}_2 j_{j+2} \]

2. Consider the action of \( \mathcal{D}_l \) on a product like the one in (5.156) but possibly with extra-factors \( A_r, \ldots, B_r, r = 1, \ldots, l_{i-1} \), namely

\[
\left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \prod_{i=1}^l \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 A_1 \ldots B_1 \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \prod_{i=1}^l \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 A_{l_{i-1}} \ldots B_{l_{i-1}} \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \prod_{i=1}^l \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \tag{5.209}
\]

Out of the product in (5.209), the operation \( \mathcal{D}_l \) yields a sum of products (see (5.156)) where:

a) the number of summands equals the number, \( l_i \), of factors \( \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \) in the product (5.209);

b) each summand is a product like the one in (5.209) where some of the factors \( \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \) may have been replaced with \( \mathcal{L}_i \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \), and one single factor \( \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \) has been replaced with \( \mathcal{D}_i \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \).

Therefore, the maximum number of new summands that are produced (due to (5.156)) from a monomial of (5.208) is \#_{RFR} (defined in point 1.).

3. As specified in point 2., each summand produced in point 2. is a product containing only one

\[
\mathcal{D}_i \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \prod_{i=1}^l \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \tag{5.210}
\]

The symbol \( \sum \) stands for two commutators (see (5.154)-(5.155)). After computing the commutators, the R-H-S of (5.210) can be written as a sum of product of operators.

We want to estimate the number of the new summands due to the commutators as specified below after (5.214). To this purpose, using formula (5.27), we re-write \( \left[ (R_{j,i}^{Bog})^{\frac{1}{2}} \right] \) as

\[
\left[ (R_{j,i}^{Bog})^{\frac{1}{2}} \right] \prod_{i=1}^l \left\{ \left[ (R_{i,i}^{Bog})^{\frac{1}{2}} \right] \right\}_1 \tag{5.211}
\]

(for some \( h \)-dependent \( \bar{r} \)) where each summand \( \left[ (R_{j,i}^{Bog})^{\frac{1}{2}} \right] \) corresponds to \( \phi_{N-2} \phi_{N-2} \) multiplying on the right a finite product of modified "RW-blocks", i.e., operators of the
\[ \begin{align*}
[R_{ij}^{Bog} W]_1 & := [R_{ij}^{Bog}]_1 \phi_j \frac{a_j^0 a_j^0}{N} (N - i + 2) \frac{N}{2}, \quad [R_{ij}^{Bog} W^*]_1 := [R_{ij}^{Bog}]_1 \phi_j a_j^0 \frac{N - i}{N} \frac{2}{2}, \\
\text{where } i \text{ is even and ranges from } j - 4 - h \text{ to } j - 4. \text{ The number of modified RW-blocks for each } [R_{ij}^{Bog} W]_1 \text{ is bounded by } O((2h)^{b_2}). \text{ Then, we re-express}
\end{align*} \]

\[ \begin{align*}
& \sum_{l_j = 0}^{h-1} \left\{ \left[ (R_{ij}^{Bog} W)_{j-2,j-2}^2 \right]_{1} \left[ (R_{ij}^{Bog} W^*_{j-2,j-2})_{1} \right]_1 \right\} \\
& \text{as a sum of "monomials"}
\end{align*} \]

\[ \begin{align*}
& \sum_{l_0 = 0}^{p} \sum_{l_1 = 1}^{q} \cdots \sum_{l_{t-1} = 1}^{s} \left\{ \left[ (R_{ij}^{Bog} W)_{j-2,j-2}^2 \right]_{1} \left[ (R_{ij}^{Bog} W^*_{j-2,j-2})_{1} \right]_1 \right\}
\end{align*} \]

where each \( [R_{ij}^{Bog} W]_1 \) is a summand in (5.211) and \( \prod_{l_0 = 0}^{p} \cdots \prod_{l_{t-1} = 1}^{s} \equiv 1 \). Out of each monomial in expression (5.214) the commutators on the R-H-S of (5.210) create new monomials the number of which can be estimated less than \( O(\#_{RW}) \) where

\[ \#_{RW} := \text{ the maximum number of modified "RW" blocks contained in each product } \prod_{l_1 = 1}^{t} \left\{ \left[ (R_{ij}^{Bog} W)_{j-2,j-2}^2 \right]_{1} \left[ (R_{ij}^{Bog} W^*_{j-2,j-2})_{1} \right]_1 \right\}. \]

4. Starting from the assumption in (5.115), we derive the inequality

\[ \begin{align*}
\| & \left[ \sum_{j \geq 1} \phi_j \frac{a_j^0}{N} a_j^0 (N - i + 2) \right] \times
\end{align*} \]

\[ \begin{align*}
\times & \left\{ \left[ (R_{ij}^{Bog} W)_{j-2,j-2}^2 \right]_{1} \left[ (R_{ij}^{Bog} W^*_{j-2,j-2})_{1} \right]_1 \right\} \right\}
\end{align*} \]

where the operator norm is referred to the restriction of the operator to \( \mathcal{F}^{N} \). (Likewise, this restriction is assumed below whenever we consider operators of the type \([ \ldots ]_1\), otherwise the operator acts from \( \mathcal{F}^{N} \) to \( \mathcal{F}^{N} \).) Then, we observe that for \( w = \frac{R_{ij}^{Bog}}{j_1, \ldots, j_m} + z \)

\[ \begin{align*}
& \| (R_{ij}^{Bog} W)_{j-2,i-2}^2 (R_{ij}^{Bog} W^*_{j-2,i-2})_{1} \| \leq \| \phi_j \| \times
\end{align*} \]

\[ \begin{align*}
& \left\{ \left[ (R_{ij}^{Bog} W)_{j-2,i-2}^2 \right]_{1} \left[ (R_{ij}^{Bog} W^*_{j-2,i-2})_{1} \right]_1 \right\} \right\}
\end{align*} \]

\[ \begin{align*}
& \leq E(\| (R_{ij}^{Bog} W)_{j-2,i-2}^2 (R_{ij}^{Bog} W^*_{j-2,i-2})_{1} \|)
\end{align*} \]

\[ ^{17} \text{For the expression } [R_{ij}^{Bog} W]_1, \text{ the index } i \text{ ranges from } j - 2 - h \text{ to } j - 4. \]

68
where for the step from (5.219) to (5.221) we bound (5.219)-(5.220) by $\sqrt{5.13}$ and proceed as in Corollary 5.1, recall that, by definition, $\mathcal{E}([a_{R_1}^{Bog} a_{R_2}^{Bog} a_{R_3}^{Bog} a_{R_4}^{Bog} a_{R_5}^{Bog} a_{R_6}^{Bog} a_{R_7}^{Bog} a_{R_8}^{Bog}])$ coincides with the R-H-S of (5.4). In the same way we bound $\|((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|_{2}$. Analogously, we can state

\[
\|(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|_{2} \times \|((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|_{2} \leq \mathcal{E}((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} .
\]

Regarding the commutators corresponding to the symbol $\tilde{a}_{i_{1},l_{1},i_{2},l_{2}}^{\frac{1}{2}}$, we estimate

\[
\|((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} \leq \mathcal{E}((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} .
\]

Here, we have used

\[
\|((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} \leq \mathcal{E}((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} .
\]

that follows from a standard computation and the bound

\[
\hat{H}_{1,j_{1},...j_{m}}^{Bog} \geq \Delta_{0} \sum_{j_{1},...j_{m-1}} a_{j_{1}}^{\dagger} a_{j_{1}} - \sum_{l=1}^{m-1} \sum_{i_{1},...i_{m-1}} a_{j_{1}}^{\dagger} a_{j_{l}} - \sum_{l=1}^{m-1} \phi_{j_{l}}
\]

(see (4.121), (4.122), and (4.123) and extend the same inequalities to the case of $m-1$ couples of modes). With the same argument, we get

\[
\|((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} \leq \mathcal{E}((R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{m}};i_{2},l_{2}(R_{1,j_{1},...j_{m}}^{Bog};i,l,d(m))\hat{w}_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})\|^{2} .
\]

We get analogous estimates for the commutators with an operator of the type $[R_{i,l,d}^{Bog} W_{j_{m}};i_{2},l_{2}(R_{i,l,d}^{Bog} W_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})]$ in the first case (see (5.224)), and of type $[R_{i,l,d}^{Bog} W_{j_{m}};i_{2},l_{2}(R_{i,l,d}^{Bog} W_{j_{1}};i_{1},l_{1}(w)^{\frac{1}{2}})]$ in the second case (see (5.231)).
5. We follow the fate through the operations described in points 2. and 3. of each monomial
\[
\prod_{j=1}^{l} \left[ \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \right]_{\tau_{h}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{2; j+2}
\]
that appears in (5.208) (see point 1.), and estimate the ratio between the total number of resulting monomials and the number (of monomials) that is obtained if:

a) we just expand one single factor
\[
\left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}}
\]
(contained in \( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{2; j+2} \)) according to the identity
\[
\left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{1} = \left[ \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{1} \sum_{l_{j}=0}^{h-1} \left( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{1} \right)^{l_{j}-2} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} W' \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} ;
\]

b) we replace \( \sum_{l_{j}=0}^{h-1} \left( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{1} \right)^{l_{j}-2} \) contained in (5.233) with (5.214).

This ratio is bounded by \( O(\# RW) \times O(\# RT) \) because:

- In point 2., out of \( \prod_{j=1}^{l} \left[ \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{2; j+2} \) we produce at most \( O(\# RT) \) new monomials of the same type except that some factors \( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \) have been replaced with \( L_{j} \left( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right)_{1} \) and one single factor \( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \) with \( D_{j} \left( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right)_{1} \) of the type

- In point 3., for each monomial (see (5.214)) of the type
\[
\prod_{l=1}^{l} \left[ \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{1} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{1}
\]

the commutators create at most \( O(\# RW) \) more terms of the same type except for the factor coming from a commutator; see (5.224)) and (5.231)) in point 4..

We intend to make use of the same argument exploited at STEP III\( _{7} \) of Lemma 5.2. When implementing this argument, due to the estimates in (5.221)-(5.223) combined with the constraint on \( z \) assumed in (5.118), we can ignore that in
\[
\left[ \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right]_{2; j+2} \right]_{1}^{l_{j}-2}
\]
the original companion operators have been replaced with \([...],_{2}\) from level \( N-2 \) down to level \( j+2 \) and with \([...],_{1}\) from level \( j-4 \) down to \( N-2-h \). Likewise, as far as the operator norm is concerned, we can ignore the replacement of factors \( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \Gamma_{j,j}^{\text{Bog}} \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \) with \( L_{j} \left( \left( R_{j,j}^{\text{Bog}} \right)^{\frac{1}{2}} \right)_{1} \) which is due to the action of \( D_{j} \). Furthermore, due to the result in point 4. the estimate of the norm of each resulting monomial carries an extra factor bounded by \( O \left( \frac{h_{j}}{N} \right) \). Using the same rationale of STEP III\( _{7} \) of Lemma 5.2, after the previous analysis concerning the structure of the monomials we can conclude
that the norm of the sum of monomials resulting from the operations in point 2. and 3. applied to
\[
\prod_{l=1}^{\tilde{t}} \left\{ \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \left[ I_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right]_r^{(r)} \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \right\}_{l_2 ; j_2 + 2}
\]
can be estimated less than
\[
O\left( \frac{N - j}{N} \right) \times O(\#_{RW}) \times O(\#_{RT}) \times \epsilon \prod_{l=1}^{\tilde{t}} \left\{ \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \left[ I_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right]_r^{(r)} \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \right\}_{l_2 ; j_2 + 2}. \]

By invoking Remark 4.3, and making use of (4.49)-(4.54), we get
\[
\sum_{l=0}^{h-1} \sum_{r_1=1}^{\tilde{t}} \ldots \sum_{r_{l_2-2}=0}^{\tilde{t}} \epsilon \prod_{l_1=1}^{\tilde{t}} \left\{ \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \left[ I_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right]_r^{(r)} \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \right\}_{l_2 ; j_2 + 2} \leq O(1)
\]
Finally, we conclude that the norm of the \( j - \)th summand in (5.174) is bounded by
\[
O\left( \frac{N - j}{N} \right) \times O(\#_{RW}) \times O(\#_{RT}) \times \epsilon \prod_{l=1}^{\tilde{t}} \left\{ \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \left[ I_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right]_r^{(r)} \left( R_{Bog}^{j_1, \ldots, j_{m-1}; N-2, N-2} (w) \right)^{\frac{1}{2}} \right\}_{l_2 ; j_2 + 2} \leq O(1)
\]
where \( N - j \leq O(h) \). Likewise, we can bound (5.175) and (5.176).

6. The numbers \( \#_{RW} \) and \( \#_{RT} \) are less than \( O(h (2h)^{\frac{1}{2}}) \) (see STEP III, of Lemma 5.2).

**STEP f**
Due to the inequalities in (5.195), (5.200), and the estimates in **STEP e**, we can argue as in **STEP e** and bound the norm of the sum of the remainder terms produced in **STEP e** by
\[
O\left( \frac{h}{\sqrt{N}} \right) \times O(h (2h)^{\frac{1}{2}}).
\]

As a final result of **STEPS a), b), c), d), e), f**, we have derived
\[
\left\| \psi_{\phi_{Bog}}^{j_1, \ldots, j_{m-1}; N, N} (z) \right\|_{\psi_{\phi_{Bog}}^{j_1, \ldots, j_{m-1}; N, N}} \leq (5.199) + O\left( \frac{4}{5} h \right) + O\left( \frac{1}{1 + c \sqrt{\eta_{NN}}} \right) + O\left( \frac{h^2 (2h)^{\frac{1}{2}}}{\sqrt{N}} \right) \]
\[
\leq (5.199) + O\left( \frac{1}{\sqrt{(\ln N)}} \right)
\]
where in the last step we have used that \( h = O((\ln N)^{\frac{1}{2}}) \).
Remark 5.5. For the estimate of the norm of (4.151) (see Property 4 in Theorem 4.3), we can adapt the steps of Lemma 5.3 because the operator \( \hat{Q}\) projects onto a subspace of states with fixed number, say \( j \), of particles in the modes different from \( \pm j_1, \ldots, \pm j_m \) and 0, and without particles in the modes \( \pm j_m \). Hence, we can implement Steps a) and b) but the \( \tilde{\psi} \) in formula (5.121) in general does not not coincide with \( h \) because, depending on the number of particles, \( j \), in the modes different from \( \pm j_1, \ldots, \pm j_m \) and 0, the number \( h \) to be chosen may be strictly smaller than \( \lfloor (\ln N)^2 \rfloor \). Analogously, the sums over \( l_{N-2} \) are up to \( \tilde{j} - 1 \). \( \tilde{j} \) is set equal to \( \lfloor (\ln N)^2 \rfloor \). We skip Step c) because we do not split \( \hat{Q}_{j_m}^{(N-1)} \) into the sum of two projections. For the estimate of the leading terms resulting from Steps a) and b) we proceed like in Step d) but replacing \( \Delta_{m-1} \) with \( \frac{m-1}{(\ln N)^2} \) due to the inequality in (4.100) that is assumed to hold at the inductive step \( m = 1 \). Furthermore, the projection \( \hat{P}_{j_{m-1}}^{(N-1)} \) is replaced with \( \hat{Q}_{j_m}^{(N-1)} \) and the sum over \( l_i \) in formula (5.198) is up to \( \tilde{j} - 1 \). Step e) is essentially the same. Step f) is absent because Step c) is absent.

Lemma 5.6. On the basis of Definition 4.1, for \( N - 4 \geq h \geq 2 \) and even, the identity below holds true

\[
\sum_{l=N-2-h, l_{even}}^{N-4} [\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2, N-2(w)]_{l, h, \ldots, h} = [\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2, N-2(w)]_{r_h} \tag{5.241}
\]

where the R-H-S is defined by

\[
[\Gamma^{Bog}_{j_1, \ldots, j_m}; i, j; v (w)]_{r_h} := W_{j_h} \sum_{i=0}^{l_{h-2}} R^{Bog}_{j_1, \ldots, j_m, i-2, j-2}([\Gamma^{Bog}_{j_1, \ldots, j_m}; i-2, j-2 (w)]_{r_h} R^{Bog}_{j_1, \ldots, j_m}; i-2, j-2 (w)]_{r_h} \tag{5.242}
\]

for \( N - 2 \geq i \geq N - h \) with

\[
[\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2-h, N-2-h (w)]_{r_h} := W_{j_h} R^{Bog}_{j_1, \ldots, j_m}; N-4-h, N-4-h (w) W_{j_m}. \tag{5.243}
\]

Proof

If \( h = 2 \) the identity in (5.241) follows directly from the definitions in (4.19), (4.20), (5.242), and (5.243). Therefore, from now on we assume \( h \geq 4 \).

For \( r = 0 \) we define (see (4.20))

\[
[\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2-h, N-2-h (w)]_{r_h} := [\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2-h, N-2-h (w)]^{(0)}_{N-4-h, h} \tag{5.244}
\]

and, for \( 2 \leq r \leq h \) with \( r \) even,

\[
[\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2-h, N-2-h (w)]_{r_h} := \hat{\pi}_r [\Gamma^{Bog}_{j_1, \ldots, j_m}; N-4-h, h (w)]_{(N-2-h, N-2-h)} + \ldots + \hat{\pi}_r [\Gamma^{Bog}_{j_1, \ldots, j_m}; N-4-h, h (w)]_{(N-2-h, N-2-h)} \tag{5.245}
\]

We shall show by induction that for \( N - 2 \geq i \geq N - h - 2 \)

\[
[\Gamma^{Bog}_{j_1, \ldots, j_m}; i, j; v (w)]_{r_h} = \hat{\pi}_r [\Gamma^{Bog}_{j_1, \ldots, j_m}; N-2-h, N-2-h (w)]_{r_h} \tag{5.250}
\]
This readily implies
\[
\left[ \Gamma_{J_1, \ldots, J_m : N-2, N-2}^{\text{Bog}}(w) \right]_{r_h} \equiv \left\{ \left[ \Gamma_{J_1, \ldots, J_m : N-h-2+r_h, N-h-2}^{\text{Bog}}(w) \right]_{r_h} \right\}_{j=r_h} ^{r_h} \left( \Gamma_{J_1, \ldots, J_m : N-2, N-2}^{\text{Bog}}(w) \right)_{l=2}^{r_h} W_{J_0}^* \tag{5.251}
\]
\[
= \sum_{l=N-2-h, l \text{ even}}^{N-4} \left[ \Gamma_{J_1, \ldots, J_m : N-2, N-2}^{\text{Bog}}(w) \right]_{l, h: l, h+2, \ldots: N-4, h-..} \cdot \tag{5.252}
\]

For \( i = N - h - 2 \) the statement in (5.250) is true because the R-H-S of (5.243) is equal to the R-H-S of (5.244). Next, we assume that it holds for \( i - 2 \geq N - h - 2 \) and show that it is also true for \( i \). To this end, for \( N - 2 \geq i \geq N - h \) and \( r_i := i - N + h + 2 \) (\( \Rightarrow h \geq r_i \geq 2 \)), we make use of the definition in (5.242) and assume (5.250) for \( i - 2 \). We get:
\[
\left[ \Gamma_{J_1, \ldots, J_m : i}^{\text{Bog}}(w) \right]_{r_h} \equiv \left\{ \left[ \Gamma_{J_1, \ldots, J_m : i-2, j-2}^{\text{Bog}}(w) \right]_{r_h} \right\}_{r_i} ^{r_i} R_{J_1, \ldots, J_m : i-2, j-2}^{\text{Bog}}(w) W_{J_0}^* \tag{5.254}
\]
\[
= \sum_{l_i=2}^{h-1} W_{J_0} R_{J_1, \ldots, J_m : i-2, j-2}^{\text{Bog}}(w) W_{J_0}^* \tag{5.255}
\]
\[
+ \sum_{l_i=2}^{h-1} W_{J_0} R_{J_1, \ldots, J_m : i-2, j-2}^{\text{Bog}}(w) \left( \left[ \Gamma_{J_1, \ldots, J_m : i-2, j-2}^{\text{Bog}}(w) \right]_{r_h} \right)_{r_i} ^{r_i} R_{J_1, \ldots, J_m : i-2, j-2}^{\text{Bog}}(w) W_{J_0}^* \tag{5.256}
\]
\[
\left[ \left[ \Gamma_{J_1, \ldots, J_m : i}^{\text{Bog}}(w) \right]_{r_h} \right]_{r_i} ^{r_i} \tag{5.257}
\]
where \( \left[ \left[ \Gamma_{J_1, \ldots, J_m : i}^{\text{Bog}}(w) \right]_{r_h} \right]_{r_i} \) is given in (5.245) with \( r \equiv r_i \). The latter identity is evident if we take into account that:
- by definition (see (4.19), (4.21), and (4.23))
\[
\left[ \Gamma_{J_1, \ldots, J_m : N-h-2+r_h, N-h-2}^{\text{Bog}}(w) \right]_{(N-h-4+r_h, h)} \tag{5.258}
\]
\[
= \sum_{l_i=2}^{h} \sum_{l_i=2}^{h} W_{J_0} \times \left( \left[ \Gamma_{J_1, \ldots, J_m : N-h-2+r_h, N-h-2}^{\text{Bog}}(w) \right]_{(N-h-4+r_h, h)} \right)_{(l_i, h)} ^{(l_i, h)} \times \left( \left[ \Gamma_{J_1, \ldots, J_m : N-h-2+r_h, N-h-2}^{\text{Bog}}(w) \right]_{(N-h-4+r_h, h)} \right)_{(l_i, h)} ^{(l_i, h)} \tag{5.259}
\]
- each other term in (5.257) of the type
\[
\left[ \Gamma_{J_1, \ldots, J_m : N-h-2+r_h, N-h-2}^{\text{Bog}}(w) \right]_{(N-h-4+q_h, h, \ldots, N-h-4+q_h, h)} \tag{5.260}
\]
with \( 0 \leq q \leq r_i - 2 \) (and even) is obtained from the sum in (5.256) according to the definition in (4.25).

\[\square\]

**Lemma 5.7.** Assume that \( \varepsilon_j \) is small enough and \( N \) large enough such that for \( z \leq E_{J_0}^{\text{Bog}} + \sqrt{\varepsilon_j \phi_j} \sqrt{\varepsilon_j^2 + 2\varepsilon_j} \) the functions \( \tilde{G}_{J_0, i_0}(z) \) (see (2.45)) are well defined and fulfill the
bound \( \tilde{G}_{j,:i}(z) \leq \frac{1}{X_i} \) where \( X_i \) is defined in (2.24). Then, for \( N \) large enough, \( 0 \leq \Delta n_{j_0} \leq h \), and

\[
z \leq E_{j_0}^{Bog} + \sqrt{\epsilon_j} \phi_j \sqrt{e_j^2 + 2\epsilon_j} - \frac{(h + 4)\phi_j}{N}
\]

(5.263)

the estimate below holds true for some \( g \geq 0 \) not larger than 4

\[
\left| \frac{\partial [\tilde{G}_{j,:i}(z)]_{\tau_0 : \Delta n_{j_0}}}{\partial \Delta n_{j_0}} \right| \leq K h \cdot g^{i + h - 2} \frac{N^2}{N^2}
\]

(5.264)

where \( [\tilde{G}_{j,:i}(z)]_{\tau_0 : \Delta n_{j_0}} \) is defined in (5.90)-(5.94) for \( i \) even, \( N - h - 2 \leq i \leq N - 2 \), and \( i - \Delta n_{j_0} - 2 \geq 0 \). \( K \) is a universal constant.

**Proof**

First, we observe that we can apply **Theorem 3.1 of [Pi1]** (see Section 2.1) and the following inequality holds (see (2.29))

\[
\tilde{G}_{j,:i}(z) \leq \| \sum_{l=0}^{\infty} \left( (R_{j,:i}^{Bog}(z)) \frac{1}{l} (R_{j,:i}^{Bog}(z)(R_{j,:i}^{Bog}(z))) \right)^l \| \leq \frac{1}{X_i}
\]

(5.265)

where \( X_i \) is defined in **Lemma 3.6 of [Pi1]**. Starting from the definition

\[
[\tilde{G}_{j,:i}(z)]_{\tau_0 : \Delta n_{j_0}} := \sum_{l=0}^{b-1} \left( [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \frac{[\tilde{G}_{j,:i-2}(z)]_{\tau_0 : \Delta n_{j_0}}}{X_i} \right)
\]

(5.266)

with \( [\tilde{G}_{j,:N-h-4,N-h-4}(z)]_{\tau_0 : \Delta n_{j_0}} = 1 \), the derivative with respect to \( \Delta n_{j_0} \) yields the recursive relation

\[
\frac{\partial [\tilde{G}_{j,:i}(z)]_{\tau_0 : \Delta n_{j_0}}}{\partial \Delta n_{j_0}} = \left( \sum_{l=0}^{b-1} l [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \frac{[\tilde{G}_{j,:i-2}(z)]_{\tau_0 : \Delta n_{j_0}}}{X_i} \right) \times
\]

(5.267)

\[
\times \left[ \frac{\partial [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}}}{\partial \Delta n_{j_0}} \frac{[\tilde{G}_{j,:i-2}(z)]_{\tau_0 : \Delta n_{j_0}}}{X_i} \right] + [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \frac{\partial [\tilde{G}_{j,:i-2}(z)]_{\tau_0 : \Delta n_{j_0}}}{\partial \Delta n_{j_0}} \right).
\]

(5.268)

We recall \( 0 \leq \Delta n_{j_0} \leq h \) and observe (see (5.92)-(5.94)) that for \( N - h - 2 \leq i \leq N - 2 \) (see the explicit computation in section 0.7 of supporting-file-Bose2.pdf)

\[
[W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \leq [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \frac{(h + 4)\phi_j}{N} \sqrt{e_j^2 + 2\epsilon_j} - \frac{(h + 4)\phi_j}{N}.
\]

(5.270)

The inequality in (5.270) together with the assumption on \( z \) (see (5.263)) imply that

\[
[W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \leq [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \frac{(h + 4)\phi_j}{N} \sqrt{e_j^2 + 2\epsilon_j} - \frac{(h + 4)\phi_j}{N}.
\]

(5.271)

\[
\leq \sum_{l=0}^{b-1} [W_{j,:i-2}(z)W_{j,:i-2}(z)]_{\Delta n_{j_0}} \frac{(h + 4)\phi_j}{N} \sqrt{e_j^2 + 2\epsilon_j} - \frac{(h + 4)\phi_j}{N}.
\]

(5.272)
Since $[\tilde{G}_j: N \rightarrow N, \Delta \eta_0] r_0 = 1$, by induction the inequality in (5.271) implies
\[
[\tilde{G}_j : i, i(z) |_{r_0} : \Delta \eta_0] 
\leq [\tilde{G}_j : i, i(z + \frac{(h + 4)\phi_j}{N}) |_{r_0}] \quad (5.273)
\]
\[
\leq \tilde{G}_j : i, i(z + \frac{(h + 4)\phi_j}{N}) \quad (5.274)
\]
\[
\leq \tilde{G}_j : i, i(z + \frac{(h + 4)\phi_j}{N}) \quad (5.275)
\]
Next, we make use of (5.270) and (5.271)-(5.275) to estimate
\[
\sum_{i=0}^{n-1} l_i[\tilde{W}_j : i, i, i-2(z) \tilde{W}^*_{i-2, i}(z)]_{r_0} \Delta \eta_0 \quad (5.276)
\]
\[
\leq \left[ 1 - \tilde{W}^*_{i-2, i}(z + \frac{(h + 4)\phi_j}{N}) \tilde{G}_j : i, i-2(z + \frac{(h + 4)\phi_j}{N}) \right] \quad (5.277)
\]
\[
\leq \left[ 1 - \tilde{W}^*_{i, i-2}(z + \frac{(h + 4)\phi_j}{N}) \tilde{G}_j : i, i-2(z + \frac{(h + 4)\phi_j}{N}) \right] \quad (5.278)
\]
\[
= \left( \tilde{G}_j : i, i(z + \frac{(h + 4)\phi_j}{N}) \right)^2 \quad (5.279)
\]
Hence, we go back to (5.267) and, thanks to (5.279) we estimate
\[
\left| \frac{\partial[\tilde{G}_j : i, i(z) |_{\Delta \eta_0}]}{\partial \Delta \eta_0} \right| \quad (5.280)
\]
\[
\leq \left( \tilde{G}_j : i, i(z + \frac{(h + 4)\phi_j}{N}) \right)^2 \times \quad (5.281)
\]
\[
\times \left| \left[ \frac{\partial[\tilde{W}_j : i, i, i-2(z) \tilde{W}^*_{i-2, i}(z)]_{\Delta \eta_0}}{\partial \Delta \eta_0} \tilde{G}_j : i, i-2(z) \right]_{r_0} : \Delta \eta_0 \right| \quad (5.282)
\]
\[
+ \left[ \tilde{W}^*_{i, i-2}(z + \frac{(h + 4)\phi_j}{N}) \tilde{G}_j : i, i-2(z + \frac{(h + 4)\phi_j}{N}) \right] \quad (5.283)
\]
Furthermore, using (5.270) and the constraint on $z$ contained in (5.263), we can estimate
\[
[\tilde{W}_j : i, i-2(z) \tilde{W}^*_{i-2, i}(z)]_{\Delta \eta_0} \leq W_j : i, i-2(z + \frac{(h + 4)\phi_j}{N}) \tilde{W}^*_{i-2, i}(z + \frac{(h + 4)\phi_j}{N}) \leq \frac{1}{2} + O(\sqrt{\eta_j}) \quad (5.284)
\]
and, with the help of (5.265) and Lemma 3.6 of [Pil] (see Section 2.1),
\[
\tilde{G}_j : i, i(z) \leq \tilde{G}_j : i, i(z + \frac{(h + 4)\phi_j}{N}) \leq \frac{1}{X_j} \leq \frac{8}{3} + O(\sqrt{\eta_j}) \quad (5.285)
\]
Next, from (5.284) and the computation (see the definition in (5.92)-(5.94))
\[
\frac{\partial[\tilde{W}_j : i, i-2(z) \tilde{W}^*_{i-2, i}(z)]_{\Delta \eta_0}}{\partial \Delta \eta_0} \quad (5.286)
\]
\[
= -2i + 2\Delta \eta_0 + 1 \quad (5.287)
\]
\[
\times \left( \frac{N}{N-i+2} \phi_j \right)^2 \quad (5.288)
\]
\[
\frac{\phi_{N}(N+i)}{N}(N-i+2)\frac{\phi_{N}(N-i)}{N}(N-i+2)
\]
\[
\frac{4\left((i-\Delta n_{j})^{2} \phi_{N} + (k_{j})^{2} \phi_{N} + (k_{j})^{2} \phi_{N} + (N-i+2-\Delta n_{j})(N-i+2-\Delta n_{j})\right)}{N^{2}}
\]
\[
\frac{\phi_{N}(N+i+2)}{N}(N-i+2)^{2}
\]
we derive
\[
\left| \frac{\partial \left[ W_{j}^{n}(i,i-2,\tau \mid z) \right]}{\partial \Delta n_{j}} \right| \leq O\left( \frac{\hbar}{N} \right).
\]

Therefore, for \( z \) in the interval \( (5.263) \), by using \( (5.279) \), \( (5.285) \), and \( (5.293) \), we derive the bound
\[
\left| \frac{\partial \left[ \tilde{G}_{j}^{n}(i,i-2) \right]}{\partial \Delta n_{j}} \right| \leq \left( \frac{8}{3} + O\left( \sqrt{\epsilon_{j}} \right) \right)^{2}
\]
\[
\times \left[ O\left( \frac{\hbar}{N} \right)^{2} + O\left( \sqrt{\epsilon_{j}} \right) + \left[ \frac{1}{2} + O\left( \sqrt{\epsilon_{j}} \right) \right] \left\| \frac{\partial \left[ \tilde{G}_{j}^{n}(i-2,i-2,\tau \mid \Delta n_{j}) \right]}{\partial \Delta n_{j}} \right\| \right].
\]

Finally, for \( N \) sufficiently large and \( \sqrt{\epsilon_{j}} \) sufficiently small, by induction we can conclude that
\[
\left| \frac{\partial \left[ \tilde{G}_{j}^{n}(i,i-2) \right]}{\partial \Delta n_{j}} \right| \leq K \frac{\hbar \cdot g^{N+h}}{\sqrt{N}}
\]
for some universal constants \( K \) and \( g \in (3, 4) \).

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Notation: The formulae that are not of the type (0...) are those in the manuscript (file BOSE2-2017.pdf). Theorems and corollaries are those in the manuscript (file BOSE2-2017.pdf) unless otherwise specified.

0.1 Estimate of $\|\psi_{j_1,...,j_{m-1}}^{\text{Bog}} - (\psi_{j_1,...,j_{m-1}}^{\text{Bog}})'\|$ in (4.144)

We recall that $z_{j_1,...,j_{m-1}}^{\text{Bog}} + (z_{j_1,...,j_{m}}^{\text{Bog}})' = (z_{j_1,...,j_{m-1}}^{\text{Bog}})'$ and $z_{j_1,...,j_{m-1}}^{\text{Bog}} + z_{j_1,...,j_{m-1}} = z_{j_1,...,j_{m-1}}^{\text{Bog}}$. We proceed as in step i) and ii) of Corollary 4.6, hence for some $j < \frac{N}{2}$ to be fixed later we write

$$\psi_{j_1,...,j_{m-1}}^{\text{Bog}})' = \psi_{j_1,...,j_{m-1}}^{\text{Bog}}$$  

$$+ \left[ -\frac{1}{Q_{j_1,...,j_{m-1}}^{\text{Bog}}}(z_{j_1,...,j_{m-1}}^{\text{Bog}})' Q_{j_{m-1}}^{\text{Bog}}(z_{j_1,...,j_{m}}^{\text{Bog}})' Q_{j_{m}}^{\text{Bog}}(z_{j_1,...,j_{m-1}}^{\text{Bog}}) W_{j_{m}}^s \right] \psi_{j_1,...,j_{m-1}}^{\text{Bog}}$$  

$$+ \sum_{j=2}^{j} \left( \prod_{r=j}^{2} \left[ -\frac{1}{Q_{j_1,...,j_{m-1}}^{\text{Bog}}}(z_{j_1,...,j_{m-1}}^{\text{Bog}})' Q_{j_{m-1}}^{\text{Bog}}(z_{j_1,...,j_{m}}^{\text{Bog}})' Q_{j_{m}}^{\text{Bog}}(z_{j_1,...,j_{m-1}}^{\text{Bog}}) W_{j_{m}}^s \right] \psi_{j_1,...,j_{m-1}}^{\text{Bog}} \right)$$  

$$+ \mathcal{O}\left( \frac{1}{1 + c \sqrt{N}} \right) + \mathcal{O}\left( \frac{1}{\Delta_{m-1} (\ln N)^2} \right)$$  

for some $c > 0$. This is possible because, like in Corollary 4.6 of [Pi1], we can control the norm of the sum in (0.3) in terms of a geometric series. Indeed, by construction, for $w = (z_{j_1,...,j_{m}}^{\text{Bog}})'$, we have

$$\left\| \prod_{l=j}^{2} \left[ -\frac{1}{Q_{j_1,...,j_{m-1}}^{\text{Bog}}}(z_{j_1,...,j_{m-1}}^{\text{Bog}})' Q_{j_{m-1}}^{\text{Bog}}(z_{j_1,...,j_{m}}^{\text{Bog}})' Q_{j_{m}}^{\text{Bog}}(z_{j_1,...,j_{m-1}}^{\text{Bog}}) W_{j_{m}}^s \right] \psi_{j_1,...,j_{m-1}}^{\text{Bog}} \right\|$$  

$$\leq \left\| R_{j_1,...,j_{m} : N-2j ; N-2}^{\text{Bog}}(w) \right\|^\frac{1}{2} \times$$  

$$\times \left\{ \prod_{l=j}^{2} \left[ -\frac{1}{Q_{j_1,...,j_{m-1}}^{\text{Bog}}}(z_{j_1,...,j_{m-1}}^{\text{Bog}})' Q_{j_{m-1}}^{\text{Bog}}(z_{j_1,...,j_{m}}^{\text{Bog}})' Q_{j_{m}}^{\text{Bog}}(z_{j_1,...,j_{m-1}}^{\text{Bog}}) W_{j_{m}}^s \right] \psi_{j_1,...,j_{m-1}}^{\text{Bog}} \right\}$$  

where

$$\hat{r}_{j_1,...,j_{m} : i}^{\text{Bog}}(w) := \sum_{l=0}^{\infty} \left[ (R_{j_1,...,j_{m} : i}^{\text{Bog}}(w))^{\frac{1}{2}} \hat{r}_{j_1,...,j_{m} : i}^{\text{Bog}}(w)(R_{j_1,...,j_{m} : i}^{\text{Bog}}(w))^{\frac{1}{2}} \right]^l,$$  

and the estimates below hold true:

$$\left\| R_{j_1,...,j_{m} : N-2j ; N-2l}^{\text{Bog}}(w) \right\|^\frac{1}{2} \times$$  

$$\times \left\{ \prod_{l=j}^{2} \left[ -\frac{1}{Q_{j_1,...,j_{m-1}}^{\text{Bog}}}(z_{j_1,...,j_{m-1}}^{\text{Bog}})' Q_{j_{m-1}}^{\text{Bog}}(z_{j_1,...,j_{m}}^{\text{Bog}})' Q_{j_{m}}^{\text{Bog}}(z_{j_1,...,j_{m-1}}^{\text{Bog}}) W_{j_{m}}^s \right] \psi_{j_1,...,j_{m-1}}^{\text{Bog}} \right\}.$$
\[ \leq \frac{1}{2\left[ 1 + a_{\epsilon_{jn}} - \frac{2b_{\epsilon_{jn}}}{2l-1} - \frac{1-c_{\epsilon_{jn}}}{(2l-1)^2} \right]^2} \]  \hspace{1cm} (0.9)

\[ \|I_{j_1,\ldots,j_{n}}^{\text{Bog}} : N-2l, N-2r(w)\| \leq \frac{2}{1 + \sqrt{\eta} a_{\epsilon_{jn}} - \frac{b_{\epsilon_{jn}}}{\sqrt{\eta} a_{\epsilon_{jn}}} - \frac{1-c_{\epsilon_{jn}}}{(2l-1)^2}}, \]  \hspace{1cm} (0.10)

and,

\[ \|I_{j_1,\ldots,j_{n}}^{\text{Bog}} : N-2l, N-2r(w)\|^2 \|I_{j_1,\ldots,j_{n}}^{\text{Bog}} : N-2l, N-2r(w)\| \leq O(1) \]  \hspace{1cm} (0.11)

where \( \eta = 1 - \sqrt{\epsilon_{jn}}, a_{\epsilon_{jn}}, b_{\epsilon_{jn}}, c_{\epsilon_{jn}} \) are those defined in Corollary 5.1, and \( \Theta \) is defined in Theorem 4.1. Therefore, we can bound \( \| (0.3) \| \) with a multiple of the series

\[ \sum_{j_2} c_j : \sum_{j_2} \left\{ \sum_{l=j_2}^{\infty} \frac{1}{\left[ 1 + \sqrt{\eta} a_{\epsilon_{jn}} - \frac{b_{\epsilon_{jn}}}{\sqrt{\eta} a_{\epsilon_{jn}}} - \frac{1-c_{\epsilon_{jn}}}{(2l-1)^2} \right]^2} \right\} \]  \hspace{1cm} (0.12)

which is convergent because for \( j \) sufficiently large

\[ \frac{c_j}{c_{j-1}} = \frac{1}{1 + \sqrt{\eta} a_{\epsilon_{jn}} - \frac{b_{\epsilon_{jn}}}{\sqrt{\eta} a_{\epsilon_{jn}}} - \frac{1-c_{\epsilon_{jn}}}{(2l-1)^2}} < \frac{1}{1 + c \sqrt{\epsilon_{jn}}} < 1 \]  \hspace{1cm} (0.13)

for some \( c > 0 \).

Now, starting from the very left in (0.3) we replace each operator of the type

\[ \sum_{l_{n-2r}=0}^{h-1} R_{j_1,\ldots,j_{n-2r}}^{\text{Bog}} ((z_{j_1,\ldots,j_{n}}^{\text{Bog}}))^2 (\Gamma_{j_1,\ldots,j_{n}}^{\text{Bog}})^2 \]  \hspace{1cm} (0.14)

as

\[ R_{j_1,\ldots,j_{n-2r}}^{\text{Bog}} ((z_{j_1,\ldots,j_{n}}^{\text{Bog}}))^2 \]  \hspace{1cm} (0.15)

with

\[ \sum_{l_{n-2r}=0}^{h-1} R_{j_1,\ldots,j_{n-2r}}^{\text{Bog}} ((z_{j_1,\ldots,j_{n}}^{\text{Bog}}))^2 \]  \hspace{1cm} (0.16)

where \( h \) (even) is determined later. We estimate the difference between (0.15) and (0.16). To this end, we define (for \( 2 \leq r \leq j \))

\[ T_{N-2r, N-2r} := (R_{j_1,\ldots,j_{n-2r}}^{\text{Bog}} ((z_{j_1,\ldots,j_{n}}^{\text{Bog}}))^2)^{\frac{1}{2}} (\Gamma_{j_1,\ldots,j_{n}}^{\text{Bog}})^{\frac{1}{2}} \]  \hspace{1cm} (0.17)

and

\[ S_{N-2r, N-2r} := (R_{j_1,\ldots,j_{n-2r}}^{\text{Bog}} ((z_{j_1,\ldots,j_{n}}^{\text{Bog}}))^2)^{\frac{1}{2}} \sum_{l_{n-2r}=0}^{h-1} (\Gamma_{j_1,\ldots,j_{n}}^{\text{Bog}})^{\frac{1}{2}} \times \]  \hspace{1cm} (0.17)

\[ \times (R_{j_1,\ldots,j_{n-2r}}^{\text{Bog}} ((z_{j_1,\ldots,j_{n}}^{\text{Bog}}))^2)^{\frac{1}{2}} \]
For \(2j \geq 2l \geq 4\), the estimates in (0.8) and (0.10) imply \(|(0.8)|^2 \leq \frac{4}{3} + o(1)\) and \(|(0.10)| \leq \frac{8}{3} + o(1)\). Then, using the relation in (4.52)-(4.54) (in Remark 4.3), and invoking the argument of Remark 4.3, we can estimate \(|T_{N-2r,N-2r} - S_{N-2r,N-2r}| \leq \frac{4}{3}\). Furthermore, the estimates in (4.32), (4.33) (in Remark 4.2), combined with the argument in Remarks 4.3, imply

\[
\|T_{N-2r,N-2r} - S_{N-2r,N-2r}\| \leq O\left(\frac{1}{1 + c \sqrt{\epsilon}}\right)^{\frac{j}{h}}
\]

for some \(c > 0\). Then, we can write

\[
\left\| \sum_{l_N-2r=0}^{N-2r-2} \left( (R_{l_N-2r}^{Bog} (z_{j_1,...,j_m}^{Bog}))^{1/2} I_{l_N-2r}^{Bog} (z_{j_1,...,j_m}^{Bog}) \right) \right\|_{N-2r} \leq \left\| \sum_{l_N-2r=0}^{N-2r-2} (R_{l_N-2r}^{Bog} (z_{j_1,...,j_m}^{Bog}))^{1/2} \left[ \sum_{l=3-2r}^{N-2r-2} \left( I_{l_N-2r-1}^{Bog} (z_{j_1,...,j_m}^{Bog}) \right) \right] \right\|_{N-2r} \]

\[
\leq \left\| \sum_{l_N-2r=0}^{N-2r-2} (R_{l_N-2r}^{Bog} (z_{j_1,...,j_m}^{Bog}))^{1/2} \left[ \sum_{l=3-2r}^{N-2r-2} \left( I_{l_N-2r-1}^{Bog} (z_{j_1,...,j_m}^{Bog}) \right) \right] \right\|_{N-2r} \]

\[
\leq O\left(\frac{1}{1 + c \sqrt{\epsilon}}\right)^{\frac{j}{h}} + O\left(\frac{4}{5}\right)^{h}
\]

We implement the same operations on \(\psi_{l_N-2r}^{Bog} (z_{j_1,...,j_m}^{Bog})\) defined like \(\psi_{l_N-2r}^{Bog} (z_{j_1,...,j_m}^{Bog})\) but with \(z_{j_1,...,j_m}^{Bog}\) instead of \((z_{j_1,...,j_m}^{Bog})\). The approximated expressions obtained for the two vectors coincide up to the replacement of \(z_{j_1,...,j_m}^{Bog}\) with \((z_{j_1,...,j_m}^{Bog})\). In order to estimate the total error term, it is enough to estimate the number of resolvents, bounded by \(O((2h)^{\frac{3s}{2}})\), contained in each

\[
\sum_{l=3-2r}^{N-2r-2} \left( I_{l_N-2r-1}^{Bog} (z_{j_1,...,j_m}^{Bog}) \right) \]

(see section 0.2 below), and recall that \((z_{j_1,...,j_m}^{Bog}) - z_{j_1,...,j_m}^{Bog} = O\left(\frac{1}{(\ln N)^{\frac{3s}{2}}}\right)\). Finally, thanks to the control of the sum in (0.3) in terms of a convergent geometric series, we can conclude that, for \(\epsilon\) small but strictly positive, the norm of the sum of all remainder terms can be estimated

\[
O\left(h(2h)^{\frac{3s}{2}} \frac{1}{(\ln N)^{\frac{3s}{2}}} \right) + O\left(\frac{1}{1 + c \sqrt{\epsilon}}\right)^{h}
\]

By choosing \(N\) large enough and \(\tilde{h} = h = [(\ln(\ln N))]^{\frac{1}{2}}\) the error term in (0.20) is bounded by \(O\left(\frac{1}{[\ln(\ln N)]^{\frac{1}{2}}}\right)\).
0.2 Estimate of the number of RW-blocks in $[\Gamma_{\text{Bog}}^{\text{Bog}};N-2,N-2(w)]_{\tau_h}$

Using the definition (with $h \geq 2$ and even) in (5.27), we can write

$$[\Gamma_{\text{Bog}}^{\text{Bog}};N-2,N-2(w)]_{\tau_h} = W_{\text{Bog}} R_{\text{Bog}}^{\text{Bog}}(N-2,N-2(w)) \sum_{l_0=0}^{h-1} \{[\Gamma_{\text{Bog}}^{\text{Bog}};N-2,N-2(w)]_{\tau_h} R_{\text{Bog}}^{\text{Bog}}(N-2,N-2(w))\}_{l_0} W_{\text{Bog}}^*$$

with

$$[\Gamma_{\text{Bog}}^{\text{Bog}};N-4,N-4(w)]_{\tau_h} = W_{\text{Bog}} R_{\text{Bog}}^{\text{Bog}}(N-4,N-4(w)) W_{\text{Bog}}^*.$$  

(0.23)

Each summand in (0.22) is a monomial containing at most

$$1 + 2(h-1) < 2h$$

RW-blocks. Now, assume that for $j$ even and $h \geq j$, $[\Gamma_{\text{Bog}}^{\text{Bog}};N-4,jN-4(j+2)]_{\tau_h}$ is a sum of monomials of the form $W_{\text{Bog}}$ times a product of RW-blocks where the number of RW-blocks is $(2h)^{\frac{j}{2}}$ at most. Since, by the definition in (5.27),

$$[\Gamma_{\text{Bog}}^{\text{Bog}};N-4,jN-4(j+2)]_{\tau_h} = W_{\text{Bog}} R_{\text{Bog}}^{\text{Bog}}(N-4,jN-4(j+2)) \sum_{l_0=0}^{h-1} \{[\Gamma_{\text{Bog}}^{\text{Bog}};N-4,jN-4(j+2)]_{\tau_h} R_{\text{Bog}}^{\text{Bog}}(N-4,jN-4(j+2))\}_{l_0} W_{\text{Bog}}^*$$

we derive that each summand in (0.25) is a monomial containing at most

$$(h-1)(2h)^{\frac{j}{2}} + 1 < (2h)^{\frac{j+2}{2}}$$

RW-blocks.

0.3 Proof by induction to show the identity (5.136)=(5.137) in Step a) of Lemma 5.3

We recall the definition

$$[\Gamma_{\text{Bog}}^{\text{Bog}};i-2,i-2(w)]_{\tau_h} = W_{\text{Bog}} R_{\text{Bog}}^{\text{Bog}}(i-2,i-2(w)) \sum_{l_0=0}^{h-1} \{[\Gamma_{\text{Bog}}^{\text{Bog}};i-2,i-2(w)]_{\tau_h} R_{\text{Bog}}^{\text{Bog}}(i-2,i-2(w))\}_{l_0} W_{\text{Bog}}^*$$

for $N-2 \geq i \geq N-h$, and

$$[\Gamma_{\text{Bog}}^{\text{Bog}};N-2,hN-2-h(w)]_{\tau_h} := W_{\text{Bog}} R_{\text{Bog}}^{\text{Bog}}(N-2,hN-2-h(w)) W_{\text{Bog}}^*.$$  

(0.27)

By the normal ordering of the operators $a_{\text{Bog}}$ and $a_{\text{Bog}}^*$, it is evident that

$$\hat{Q}_{\text{Bog}}^{(N-2-h)}(R_{\text{Bog}}^{\text{Bog}}(N-2,hN-2-h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)}(R_{\text{Bog}}^{\text{Bog}}(N-2,hN-2-h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)}$$

$$\times (R_{\text{Bog}}^{\text{Bog}}(N-4,hN-4(h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)})$$

$$\times [(R_{\text{Bog}}^{\text{Bog}}(N-4,hN-4(h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)}$$

$$\times [(R_{\text{Bog}}^{\text{Bog}}(N-4,hN-4(h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)}$$

$$\times [(R_{\text{Bog}}^{\text{Bog}}(N-4,hN-4(h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)}$$

$$\times [(R_{\text{Bog}}^{\text{Bog}}(N-4,hN-4(h(w)) + \frac{1}{2} \hat{Q}_{\text{Bog}}^{(N-2-h)}$$

(0.30)
where \( \bar{Q}^{(j)}_{j_m} \) projects onto the subspace of vectors with exactly \( \frac{N-j}{2} \) particles in the modes \( j_m \) and \( -j_m \), respectively. Next, for \( N \geq 4 \geq i - 2 \), we assume

\[
\bar{Q}^{(i-2)}_{j_m}(R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2}[\Gamma_{j_1,...,j_m;i-2}^{\text{Bog}}(w)]_\tau (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i-2)}_{j_m}
\]

and use the relation in (0.26). We get

\[
\bar{Q}^{(j)}_{j_m}(R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2}[\Gamma_{j_1,...,j_m;i,j}^{\text{Bog}}(w)]_\tau (R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i)}_{j_m}
\]

\[
= \bar{Q}^{(i)}_{j_m}(R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2} W_{\tau} (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i)}_{j_m}
\]

\[
\times \sum_{l=0}^{h-1} [(R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2}[\Gamma_{j_1,...,j_m;i-2}^{\text{Bog}}(w)]_\tau (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i-2)}_{j_m}]^{l-2}\times
\]

\[
\times (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} W_{\tau} (R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i)}_{j_m}
\]

\[
= \bar{Q}^{(i)}_{j_m}(R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2} W_{\tau} (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i)}_{j_m}
\]

\[
\times \sum_{l=0}^{h-1} [(R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2}[\Gamma_{j_1,...,j_m;i-2}^{\text{Bog}}(w)]_\tau (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i-2)}_{j_m}]^{l-2}\times
\]

\[
\times (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} W_{\tau} (R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i)}_{j_m}
\]

The normal ordering of the variables \( a_{j_m} \) and \( a^{+}_{j_m} \) contained in (0.38) and (0.40) can be implemented as if the operator in (0.39) were absent, because (due to the inductive hypothesis) in (0.39) the operators \( a_{j_m} \) and \( a^{+}_{j_m} \) can be replaced with c-numbers. Thus, eventually we get

\[
\bar{Q}^{(i)}_{j_m}(R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2}[\Gamma_{j_1,...,j_m;i,j}^{\text{Bog}}(w)]_\tau (R_{j_1,...,j_m;i,j}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i)}_{j_m}
\]

Using the previous result for \( i = N - 2 \), with the same arguments we get that

\[
\mathcal{P}_{j_1,...,j_m} W_{\tau} (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \times
\]

\[
\sum_{l=0}^{h-1} [(R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2}[\Gamma_{j_1,...,j_m;i-2}^{\text{Bog}}(w)]_\tau (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \bar{Q}^{(i-2)}_{j_m}]^{l-2}\times
\]

\[
\times (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} W_{\tau} \mathcal{P}_{j_1,...,j_m} \]

\[
= \mathcal{P}_{j_1,...,j_m} W_{\tau} (R_{j_1,...,j_m;i-2}^{\text{Bog}}(w))^\frac{1}{2} \times
\]
We use the shorthand notation
\[
Q := \left[ (R_{i,i}^{(Bog)})^2 \Gamma_{i,i}^{(Bog)} \right]_{1} \left[ (R_{i,i}^{(Bog)})^2 \right]_{1} \quad (0.55)
\]
so that the definition in (5.156) reads
\[
\mathcal{D}(Q) := \sum_{j=0}^{n-1} \left[ \mathcal{L}(Q) \right]^{j} \mathcal{D}(Q) Q^{n-j-1}. \quad (0.56)
\]

For \( n = 1 \) the identity in (5.161)-(5.163) corresponds to
\[
Q = \mathcal{L}(Q) + \mathcal{D}(Q) \quad (0.57)
\]
and follows from the definitions in (5.151)-(5.153). Now, assuming that the property holds for \( n \), i.e.,
\[
Q^n = \left[ \mathcal{L}(Q) \right]^{n} + \mathcal{D}(Q^n), \quad (0.58)
\]
we show that it is also true for \( n + 1 \). Using the inductive hypothesis and the definition in (0.56), we can write
\[
Q^{n+1} = Q^n Q = \left[ \left[ \mathcal{L}(Q) \right]^{n} + \mathcal{D}(Q^n) \right] Q = \left[ \mathcal{L}(Q) \right]^{n} Q + \mathcal{D}(Q^n) Q \quad (0.60)
\]
We recall that

\[ \| (R^{Bog}_{j_1,\ldots,j_m} : i, \{w\})^2 W_{j_m : i-2} (R^{Bog}_{j_1,\ldots,j_m} : i-2, \{w\})^2 \|_{\mathcal{G}} \]

\leq \left( \frac{1}{\mathcal{P}_{Bog}^{\phi_{j_1,\ldots,j_m-1}} \Delta_{m-1} + \left( a^0_{\phi j_m} N \phi_j + k^2_j \right)(N-i) - \frac{z}{2} \right) \mathcal{P}_{Bog}^{\phi_{j_1,\ldots,j_m-1}} \times

\mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} \left( \Delta_{m-1} + \left( a^0_{\phi j_m} N \phi_j + k^2_j \right)(N-i) - \frac{z}{2} \right) \mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} \times

\left( \frac{1}{\mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} \Delta_{m-1} + \left( a^0_{\phi j_m} N \phi_j + k^2_j \right)(N-i) - \frac{z}{2} \right) \mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} \right)^\frac{1}{2} \mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} \|

where \( N - 2 \geq i \geq N - 2 - h \), with \( h = \lfloor (\ln N)^\frac{1}{2} \rfloor \) and even, \( w = z_{Bog}^{j_1,\ldots,j_m-1} + z \), and

\[ -4 \phi_{j_m} \leq z < z_m + \gamma \Delta_{m-1} \left( < E_{j_m}^{\text{Bog}} + \frac{1}{2} \sqrt{\bar{\epsilon}_{j_m} \phi_j} \sqrt{\overline{\epsilon}_{j_m}^2 + 2 \bar{\epsilon}_{j_m}} < 0 \right). \]

We call \( r \) the spectral variable associated with \( 0 \leq \mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} a^0_{\phi j_m} N \mathcal{P}_{\phi_{j_1,\ldots,j_m-1}}^{\phi_{j_1,\ldots,j_m-1}} \leq 1 \) and we study

\[ \sup_{1 \geq r \geq 0} \left| \left( \frac{1}{\Delta_{m-1} + (r \phi_{j_m} + k^2_j)(N-i) - \frac{z}{2}} \right)^\frac{1}{2} \times \right. \]

\[ \left. \phi_{j_m} \frac{(r - \frac{1}{N})}{2} (N-i+2) \left( \frac{1}{\Delta_{m-1} + (r \phi_{j_m} + k^2_j)(N-i+2) - \frac{z}{2}} \right)^\frac{1}{2} \right|. \]

If \( N \) is sufficiently large,

\[ \sup_{1 \geq r \geq 0} \left| \left( \frac{1}{\Delta_{m-1} + (r \phi_{j_m} + k^2_j)(N-i) - \frac{z}{2}} \right)^\frac{1}{2} \times \right. \]

\[ \left. \phi_{j_m} \frac{(r - \frac{1}{N})}{2} (N-i+2) \left( \frac{1}{\Delta_{m-1} + (r \phi_{j_m} + k^2_j)(N-i+2) - \frac{z}{2}} \right)^\frac{1}{2} \right|. \]
is trivially bounded by the R-H-S of (5.195). For \(\frac{1}{N} < r \leq 1\) we observe that

\[
\sup_{1 \geq r \geq \frac{1}{N}} \left( \frac{1}{\Delta_{m-1} + (r \phi_{m_n} + k_{m_n}^2)(N - i) - z} \right)^{\frac{1}{2}} x \times \\
\times \phi_{m_n} \left( \frac{r - \frac{1}{N}}{2} \right)^2 (N - i + 2) \left( \frac{1}{\Delta_{m-1} + (r \phi_{m_n} + k_{m_n}^2)(N - i) - z} \right)^{\frac{1}{2}}
\]

\[
= \sup_{1 \geq r \geq \frac{1}{N}} \left( \frac{\Delta_{m-1}}{(r - \frac{1}{N})^2} + \frac{r \phi_{m_n} + k_{m_n}^2}{(r - \frac{1}{N})^2} \right)^{\frac{1}{2}} x \times \\
\times \phi_{m_n} \left( \frac{r - \frac{1}{N}}{2} \right)^2 (N - i + 2) \left( \frac{1}{\Delta_{m-1} + (r \phi_{m_n} + k_{m_n}^2)(N - i) - z} \right)^{\frac{1}{2}}
\]

\[
\leq \frac{(N - i + 2)^\frac{1}{2} (N - i + 2)^\frac{1}{2}}{2 \left[ \Delta_{m-1} + (\phi_{m_n} + k_{m_n}^2)(N - i + 2) - z \right]^{\frac{1}{2}}} \left[ \Delta_{m-1} + (\phi_{m_n} + k_{m_n}^2)(N - i) - z \right]^{\frac{1}{2}} \phi_{m_n}
\]

### 0.6 Inequality in (5.200)

The inequality to be proven is

\[
\frac{(N - i + 2)^2}{4 \left[ \Delta_{m-1} + (\phi_{m_n} + k_{m_n}^2)(N - i + 2) - z \right]} \phi_{m_n}^2 \leq \mathcal{W}_{m_n; i, i-2} (z - \Delta_{m-1} + \frac{U_{m_n}}{\sqrt{N}}) \mathcal{W}_{m_n; i, i-2, i} (z - \Delta_{m-1} + \frac{U_{m_n}}{\sqrt{N}})
\]

for

\[-4 \phi_{m_n} \leq z < z_m + \gamma \Delta_{m-1} \left( < E_{m_n}^{\text{Bog}} + \frac{1}{2} \sqrt{\varepsilon_{m_n}} \phi_{m_n} \sqrt{\varepsilon_{m_n}^2 + 2 \varepsilon_{m_n}} < 0 \right)
\]

and for \(N - 2 \geq i \geq N - 2 - O(\ln N)^{\frac{1}{2}}\) with \(N\) sufficiently large.

We recall that

\[
\mathcal{W}_{m_n; i, i-2} (z - \Delta_{m-1} + \frac{U_{m_n}}{\sqrt{N}}) \mathcal{W}_{m_n; i, i-2, i} (z - \Delta_{m-1} + \frac{U_{m_n}}{\sqrt{N}}) = \frac{(i - 1) i}{N^2} \frac{(N - i + 2)^2}{4 \left[ \frac{(i - 2)}{N} \phi_{m_n} + k_{m_n}^2 \right] (N - i + 2) - z + \Delta_{m-1} - \frac{U_{m_n}}{\sqrt{N}}} \times \\
\times \phi_{m_n}^2 \left[ \phi_{m_n} + k_{m_n}^2 \right] (N - i) - z + \Delta_{m-1} - \frac{U_{m_n}}{\sqrt{N}}
\]

\[
= \frac{(i - 1) i}{N^2} \frac{(N - i + 2)^2}{4 \left[ \frac{(i - 2)}{i - 1} \phi_{m_n} + \frac{N^2}{i - 1} k_{m_n}^2 \right] (N - i + 2) + \frac{N^2}{i - 1} [z + \Delta_{m-1} - \frac{U_{m_n}}{\sqrt{N}}]} \times \\
\times \phi_{m_n}^2 \left[ \phi_{m_n} + \frac{N^2}{i - 1} k_{m_n}^2 \right] (N - i) + \frac{N^2}{i - 1} [z + \Delta_{m-1} - \frac{U_{m_n}}{\sqrt{N}}].
\]
Hence, it is enough to observe that for $N - 2 > i > N - 2 - O((\ln N)^{1/2})$ and for
\[
-4\phi_{j_n} \leq z < z_m + \gamma \Delta_{m-1} \left( < E_{j_m}^{\text{Bog}} + \frac{1}{2} \sqrt{\epsilon_{j_m}} \phi_{j_m} \sqrt{\epsilon_{j_m}^2 + 2\epsilon_{j_m}} < 0 \right) \tag{0.86}
\]
we have
\[
\frac{(i - 2)}{i - 1} \phi_{j_n} + \frac{N}{i - 1} k_{j_n}^2(N - i + 2) + \frac{N}{i - 1} [-z + \Delta_{m-1} - \frac{U_{j_n}}{\sqrt{N}}]
\]
\[
= \Delta_{m-1} + (\phi_{j_n} + k_{j_n}^2)(N - i + 2) - z - \frac{U_{j_n}}{\sqrt{N}} + O\left(\frac{\ln N}{N}\right) \tag{0.87}
\]
and
\[
\frac{N}{i} \phi_{j_n} + \frac{N}{i} k_{j_n}^2(N - i) + \frac{N}{i} [-z + \Delta_{m-1} - \frac{U_{j_n}}{\sqrt{N}}]
\]
\[
= \Delta_{m-1} + (\phi_{j_n} + k_{j_n}^2)(N - i) - z - \frac{U_{j_n}}{\sqrt{N}} + O\left(\frac{\ln N}{N}\right) \tag{0.88}
\]
with $U_{j_n} > 0$.

### 0.7 Inequality in (5.270)

We recall that by assumption
\[
i \geq N - h - 2, \quad i - \Delta n_{j_0} - 2 \geq 0, \quad \Delta n_{j_0} \geq 0.
\]

Then, we estimate
\[
[W_{j: i-2}^{-1}(z)W_{j: i-2}^{-1}(z)]_{\Delta n_{j_0}} = (i - \Delta n_{j_0} - 1)(i - \Delta n_{j_0}) \phi_{j_n}^2 \tag{0.91}
\]
\[
\times \frac{(N - i + 2)^2}{4\left[\frac{i - \Delta n_{j_0}}{N} \phi_{j_n} + (k_{j_n})^2(N - i) - \frac{z}{(i - \Delta n_{j_0})}\right]\left[\frac{i - 2 - \Delta n_{j_0}}{N} \phi_{j_n} + (k_{j_n})^2(N - i + 2) - \frac{z}{(i - \Delta n_{j_0})}\right]} \tag{0.92}
\]
\[
= \frac{1}{N^2} \phi_{j_n}^2 \times \frac{(N - i + 2)^2}{4\left[\frac{i - \Delta n_{j_0}}{N} \phi_{j_n} + (k_{j_n})^2(N - i) - \frac{z}{(i - \Delta n_{j_0})}\right]\left[\frac{i - 2 - \Delta n_{j_0}}{N} \phi_{j_n} + (k_{j_n})^2(N - i + 2) - \frac{z}{(i - \Delta n_{j_0})}\right]} \tag{0.93}
\]
\[
= \frac{4N^2}{(\frac{i}{N} \phi_{j_n} + \frac{1}{(i - \Delta n_{j_0})}(k_{j_n})^2(N - i) - \frac{z}{(i - \Delta n_{j_0})})\left[\frac{i - 2 - \Delta n_{j_0}}{N} \phi_{j_n} + (k_{j_n})^2(N - i + 2) - \frac{z}{(i - \Delta n_{j_0})}\right]} \tag{0.94}
\]
\[
= \frac{4N^2}{(\frac{i}{N} \phi_{j_n} + \frac{1}{(i - \Delta n_{j_0})}(k_{j_n})^2(N - i) - \frac{z}{(i - \Delta n_{j_0})})\left[\frac{i - 2 - \Delta n_{j_0}}{N} \phi_{j_n} + (k_{j_n})^2(N - i + 2) - \frac{z}{(i - \Delta n_{j_0})}\right]} \tag{0.95}
\]
\[
\leq \frac{\phi_{j_n}^{(N + 2)}}{N} \leq \frac{\phi_{j_n}^{(h+4)}}{N}. \tag{0.96}
\]

Now we observe that for $i \geq N - h - 2$ we have $\frac{\phi_{j_n}^{(N + 2)}}{N} \leq \frac{\phi_{j_n}^{(h+4)}}{N}$. Therefore, if
\[
z \leq E_{j_n}^{\text{Bog}} + \sqrt{\epsilon_{j_n} \phi_{j_n}} \sqrt{\epsilon_{j_n}^2 + 2\epsilon_{j_n}} - \frac{(h + 4)\phi_{j_n}}{N} \tag{0.97}
\]
and $\epsilon_j$ is small, we have $\frac{\phi_j(N-i+2)}{N} + z \leq \frac{\phi_j(h+4)}{N} + z < 0$ and we can bound

\[(0.96)\]

\[
\leq \frac{1}{N^2} \phi_j^2 \times \frac{(N-i+2)^2}{4\left[\left(\frac{\phi_j}{N} + \frac{1}{i}(k_j)^2\right)(N-i) - \frac{1}{i}\left(\frac{\phi_j}{N} + \frac{(k_j)^2}{(i-1)}\right)(N-i-2) - \frac{1}{i-1}\left(\frac{\phi_j}{N} + \frac{(h+4)}{i-1}\right)\right]} \]

\[(0.98)\]

\[
\leq \frac{1}{N^2} \phi_j^2 \times \frac{(N-i+2)^2}{4\left[\left(\frac{\phi_j}{N} + \frac{1}{i}(k_j)^2\right)(N-i) - \frac{1}{i}\left(\frac{\phi_j}{N} + \frac{(k_j)^2}{(i-1)}\right)(N-i-2) - \frac{1}{i-1}\left(\frac{\phi_j}{N} + \frac{(h+4)}{i-1}\right)\right]} \]

\[(0.99)\]

\[
= W_{j,i,i-2}(z + \frac{\phi_j(h+4)}{N})W_{j,i-2,i}(z + \frac{\phi_j(h+4)}{N}) \]

\[(0.100)\]