Abstract

To capture a multidimensional consistency feature of integrable systems in terms of the geometry, we give a condition called \textit{geodesic compatibility} that implies the existence of integrals in involution of the geodesic flow. The geodesic compatibility condition is constructed from a concrete example namely the integrable Calogero’s Goldfish system through the Poisson structure and the variational principle. The geometrical view of the geodesic compatibility gives a compatible parallel transport between two different Hamiltonian vector fields.

1 Introduction

In recent years, the multidimensional consistency feature of integrable systems has been interested by many people in the field. This intriguing feature first arose in the level of discrete integrable systems namely the consistency around the cube (CAC) \cite{1,2,3,4} such that there exists a set of compatible equations defined in each subspace corresponding to the number of independent variables. This means that it allows us to embed the difference equations in a multidimensional discrete space consistently. In the context of Hamiltonian systems, the Liouville integrability is a natural criterion to test the system in question \cite{5}. The important feature is called the \textit{Hamiltonian commuting flows} which actually can be considered as the multidimensional consistency in the level of the Poisson structure. The multidimensional consistency can also be captured in the Lagrangian description known as the Lagrangain multi-forms \cite{6}. The main feature for integrability in this context is called the \textit{closure relation} which implies the existence of infinite paths on the space of independent variables corresponding to a single path on the space of dependent variables with a critical action.

The Calogero-Moser (CM) type systems, Ruijsanaars-Schneider (RS) type systems and Calogero’s Goldfish (GF) type systems are well known integrable one-dimensional many-body systems \cite{7,8,9} in the context of Liouville integrability. Furthermore, their integrability can also be exhibited through the Lagrangian 1-form structure \cite{10,11,12,13,14,15,16,17}. Intriguingly, for the GF systems, Hamiltonians are all written with exponential of conjugate momenta and their equations of motion are perfectly in a form of geodesic representation. The geodesic interpretation of GF models was first investigated in \cite{18} and it was found that the Riemann curvature tensor for the case of rational GF models vanishes which implies that the evolution of this system is indeed a free geodesic motion in Cartesian-like coordinates under the coordinate transformation.

As we mentioned earlier that the GF models are the integrable systems exhibiting the multidimensional consistency through the Hamiltonian commuting flows and the closure relation and, since GF models are also integrable geodesic flows, in the present paper, we would like to capture the multidimensional consistency from the geometrical point of view namely through the metric tensors. In section 2, we provide a brief review on the geodesic flows and a criteria for their integrability. The Goldfish systems are also
presented together with its geodesic interpretation. In section 3, the condition on metric tensors called the geodesic compatibility is derived from the commuting Hamiltonian flows and the variational principle on the space of time variables. Both rational and hyperbolic GF models are explicitly used to verify the geodesic compatibility. In section 4, the interpretation of the geodesic compatibility condition in the geometrical point of view is presented. In section 5, the summary is given as well as remarks.

2 Preliminaries

2.1 Integrable geodesic flows

Suppose there is an $N$-dimensional manifold $M$ equipped with the metric tensor $g(q)$, where $q = (q^1, q^2, ..., q^N)$ is a set of local coordinates, and a pair $(M, g)$ forms a well known (smooth) Riemannian manifold. Let a smooth curve $\gamma(q(t), \frac{d}{dt}q(t))$ be a geodesic defined on the tangent bundle $TM$. Mathematically, a geodesic flow is a family of the diffeomorphisms $\phi_t$ of the tangent bundle such that each point on the geodesic can be expressed as

$$\phi_t \left( q(0), \frac{d}{dt}q(0) \right) := (q(t), \frac{d}{dt}q(t)) .$$

(2.1)

Let us now define $SM$ as a unit tangent bundle which is a subset of $TM$ such that $\frac{dq}{dt}$ has a unity norm. We find that (2.1), $SM$ is preserved under this map along the curve $\gamma$ which means that, for any $(q, \frac{dq}{dt}) \in SM$, $\phi_t(q, \frac{dq}{dt}) \in SM$.

In Hamiltonian context, the geodesic flow is the trajectory that describes the evolution for a system of equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} , \quad -\frac{dp_i}{dt} = \frac{\partial H}{\partial q^i} , \quad i = 1, 2, ..., N ,$$

(2.2)

on the cotangent bundle $T^*M$. The Hamiltonian $H(q, p)$ is given in the form

$$H(q, p) = \frac{1}{2} \sum_{i,j} g^{ij}(q)p_ip_j = \frac{1}{2} g^{ij}p_ip_j ,$$

(2.3)

where $(q, p) \equiv (q^1, q^2, ..., q^N, p_1, p_2, ..., p_N)$ is the canonical coordinates on a $2N$-dimensional phase space and $g^{ij}$ are the elements in the metric tensor such that $p_j = g_{ij} \frac{dq^i}{dt}$. With a given Hamiltonian (2.3), equation (2.2) reads

$$\frac{dq^i}{dt} = g^{ij}p_j , \quad -\frac{dp_i}{dt} = \frac{1}{2} \frac{\partial g^{jk}}{\partial q^i}p jp_k ,$$

(2.4)

resulting in the geodesic equations

$$\frac{d^2q^i}{dt^2} + \Gamma^i_{jk} \frac{dq^j}{dt} \frac{dq^k}{dt} = 0 ,$$

(2.5)

where $\Gamma^i_{jk}$ are the affine connection given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{jm}}{\partial q^k} + \frac{\partial g_{km}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^m} \right) .$$

(2.6)

In general, the geodesic on a closed Riemannian manifold can be globally complicated, but still regular, and of course not chaotic. Global regular behaviour is a main characteristic property of integrable geodesic flows which are defined as follows.

Definition: The geodesic flow is said to be completely Liouville integrable if there exists a set of $N$ functions defined on the phase space $\{F_1(q, p), F_2(q, p), ..., F_N(q, p)\}$ which satisfies the following requirements:
They are integrals of the geodesic flow, i.e., constant along each geodesic line.

They are commuting with respect to Poisson bracket on $T^*\mathcal{M}$, i.e., $\{F_i, F_j\} = 0$, where $i \neq j = 1, 2, ..., N$.

They are functionally independent on $T^*\mathcal{M}$. In other words, the gradients of every integrals are linearly independent.

We find that it is not difficult to obtain the Lagrangian associated with the Hamiltonian (2.3)

\[ L(q, dq/dt) = \frac{1}{2} g_{ij} dq^i dq^j , \] (2.7)

and the Euler-Lagrange equations

\[ \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial (dq^i/dt)} = 0 \] (2.8)

give us again (2.5).

We end this section with some well known examples of topological objects admitting integrable geodesic flows. The 2-Sphere $S^2 := \{(q^1)^2 + (q^2)^2 + (q^3)^2 = 1\}$ whose geodesics are equators (the curves on the great circles) and the torus with a flat metric $ds^2 = (dq^1)^2 + (dq^2)^2$, whose angle coordinates $\theta_i(t)$ defining the surface is quasi-periodic, i.e., $\theta_i(t) = c_i t$ of period $2\pi$, where $i = 1, 2$, are classical examples of two-dimensional surfaces with integrable geodesic flows. However, surfaces of revolution admitting non-trivial linear constants of motion called Clairaut integrals and surfaces with Liouville metrics $ds^2 = (f(x) + g(x))(dq^1)^2 + (dq^2)^2$ admitting non-trivial quadratic integrals are also examples. \[20\]

### 2.2 Calogero’s Goldfish models as geodesic Hamiltonian flow

The Calogero’s Goldfish models are the Hamiltonian system \[9, 21\] and the Hamiltonian is given by

\[ \mathcal{H}(q, p) = \sum_{i=1}^{N} e^{a p_i} \prod_{j=1, j \neq i}^{N} f(q^i - q^j) , \] (2.9)

where $a$ is a parameter and

\[ f(q) = \begin{cases} \frac{1}{q} : & \text{rational case} \\ \frac{1}{\sinh(q)} : & \text{hyperbolic case} \\ \wp(q) : & \text{elliptic case} \end{cases} \]

respectively. Using Hamilton’s equations, the equations of motion are given by

\[ \frac{d^2 q^i}{dt^2} = \sum_{j \neq i}^{N} \frac{dq^i}{dt} \frac{dq^j}{dt} W(q^i - q^j) \text{ for } i = 1, 2, ..., N , \] (2.10)

where

\[ W(q) = \begin{cases} \frac{2}{q} : & \text{rational case} \\ 2 \coth(q) : & \text{hyperbolic case} \\ \wp'(\gamma)/\wp(\gamma) : & \text{elliptic case} \end{cases} \]

and $\gamma$ is an arbitrary parameter. It accidentally turns out that (2.10) are in the form of geodesic equations with the affine connection given by \[18\]

\[ \Gamma^i_{jk} = \delta^i_j w_{ik} + \delta^i_k w_{ij} , \text{ and } w_{ij} = -\frac{1}{2}(1 - \delta_{ij})W(q^i - q^j) . \] (2.11)
In the rational case, it has been shown that all components of the curvature tensor (Riemann tensor)
\[
R_{ijkl} = \frac{\partial \Gamma_{ij}^k}{\partial q^l} - \frac{\partial \Gamma_{ij}^k}{\partial q^l} + \Gamma_{km}^i \Gamma_{lj}^m - \Gamma_{km}^i \Gamma_{lj}^m
\]  
(2.12)
vanish identically. This means that goldfish evolution is indeed free geodesic and there exists the Cartesian-like coordinates
\[
x_n[q] = \frac{1}{n!} \sum_{(i_1, i_2, \ldots, i_N)} q^{i_1}q^{i_2} \ldots q^{i_n},
\]  
(2.13)
where \( n = 1, 2, \ldots, N \), and ‘\( \prime \)’ indicates that the all indices are different, such that the goldfish equations becomes \( d^2x_n/dt^2 = 0 \), where \( n = 1, 2, \ldots, N \).

The geodesic interpretation for the Ruijsenaars-Schneider systems and Toda systems had been investigated further. In the case of rational Ruijsenaars-Schneider, there is the same structure with the rational Goldfish system. In the hyperbolic Ruijsenaars-Schneider system and relativistic Toda systems, it turns out to be that they are linked to non-metric connections  \[22,23\].

### 3 Compatible geodesic flows

It is known that the Goldfish models  \[2.9\] are completely integrable  \[21\] and certainly the systems possess the Hamiltonian hierarchies. The first three Hamiltonians of the Goldfish system are
\[
\mathcal{H}_1 = \mathcal{H}_1^i = g_1^{ij} \pi_i \pi_j, \quad \mathcal{H}_2 = g_2^{ij} \pi_i \pi_j, \quad \mathcal{H}_3 = g_3^{ij} \pi_i \pi_j, \quad i, j = 1, 2, \ldots, N,
\]  
(3.14)
where \( \pi_i \equiv e^{p_i/2} \) and \( p_i \) is the conjugate momentum in canonical coordinates \((p,q)\). The first three metric tensors for the rational case are given by
\[
g_1^{ij} = \delta_{ij} \frac{1}{\prod_{a \neq i}^N (q^i - q^a)} , \quad g_2^{ij} = \delta_{ij} \frac{1}{\prod_{a \neq i}^N (q^i - q^a)} , \quad g_3^{ij} = \delta_{ij} \frac{1}{\prod_{a \neq i}^N (q^i - q^a)} ,
\]  
(3.15)
and the first three metric tensors for the hyperbolic case are given by
\[
g_1^{ij} = \delta_{ij} \frac{1}{\prod_{a \neq i}^N \sinh (q^i - q^a)} , \quad g_2^{ij} = \delta_{ij} \frac{\sum_{b \neq i}^N e^{-2q^b}}{\prod_{a \neq i}^N \sinh (q^i - q^a)} , \quad g_3^{ij} = \delta_{ij} \frac{\sum_{b \neq i}^N e^{-2q^b}}{\prod_{a \neq i}^N \sinh (q^i - q^a)} .
\]  
(3.16)

It is well known that the Hamiltonian is a time generator and here we definitely have different time variables for each Hamiltonian. The geodesic equations for the Hamiltonian \( \mathcal{H}_k \) are given by
\[
\frac{d^2q_i}{dt^2_k} = \sum_{j=1,j \neq i}^N \frac{dq^i}{dt^k} \frac{dq^j}{dt^k} W(q^i - q^j) \quad \text{for} \quad i, k = 1, 2, \ldots, N,
\]  
(3.17)
where
\[
W(q) = \begin{cases} 
\frac{q}{2} & : \text{rational case} \\
\frac{2}{q} \coth(q) & : \text{hyperbolic case}
\end{cases}
\]

However, we would like to point that the Hamiltonians in  \[3.14\] are in the pseudo-geodesic form. What we mean by “pseudo” is that actually this Hamiltonian hierarchy is not explicitly in the form given in  \[2.3\] since the momenta \( \pi_i \) are not canonical variables and of course the Poisson bracket \( \{ \pi_i, q^j \} \neq \delta_i^j \). Another point is that with the structure  \[3.17\] the Riemann tensor for each geodesic flows is again zero for
the rational case and there must exist transformations such that all geodesic equations \(3.17\) become free geodesic flows: \(d^2x_n/dt_k^2 = 0\). This implies that there is an \(N\)-dimensional flat manifold governed by these compatible free geodesic flows.

In this section, we are interested to construct the relation that implies integrability through the structure of the \(g\) metric tensors. We first set out to derive the condition directly from the involution of the Hamiltonians and then we look for the condition from different perspective namely from the variational principle.

**The Poisson structure:** Given two arbitrary Hamiltonians in the hierarchy

\[
\mathcal{H}_l = g^{ij}_l \pi_i \pi_j \quad \text{and} \quad \mathcal{H}_s = g^{ij}_s \pi_i \pi_j ,
\]

(3.18)

the Poisson bracket between them gives

\[
\{\mathcal{H}_l, \mathcal{H}_s\} = \frac{\partial \mathcal{H}_l}{\partial q^m} \frac{\partial \mathcal{H}_s}{\partial p_m} - \frac{\partial \mathcal{H}_s}{\partial q^m} \frac{\partial \mathcal{H}_l}{\partial p_m} = \left( \frac{\partial g^{ij}_l}{\partial q^m} \right) \left( g^{nk}_s \pi_k + g^{nk}_l \pi_n \right) \pi_i \pi_j \pi_n \pi_k .
\]

(3.19)

This equation can be called as the *geodesic compatibility* and can be used as integrability criterion for the geodesic Hamiltonian systems.

**The variational principle:** The action functional of a system with \(N\) independent variables is given by

\[
S = \int_C \sum_{k=1}^N \left( p_i q^{ij}_k - g^{ij}_k \pi_i \pi_j \right) dt_k ,
\]

(3.20)

where \(C\) is a curve on the space of independent variables and \(q_{t_k} \equiv \partial q/\partial t_k\). Now we introduce a time-parameterised variable \(s\), \(s_0 \leq s \leq s_1\), such that

\[
S = \int_{s_0}^{s_1} \left[ \sum_{k=1}^N (p_i q^{ij}_k - g^{ij}_k \pi_i \pi_j) \frac{dt_k}{ds} \right] ds .
\]
The variation of action according to the local deformation on \( t_l-t_s \) plane \((l \neq s)\) is

\[
\delta S = \int \left\{ \left[ \frac{p_i q_i}{dt_s} + \frac{\partial p_i}{\partial t_l} q_i^l t_l + \frac{\partial q_i}{\partial t_l} \delta t_l + p_i \frac{\partial g_{ij}}{\partial t_l} \pi_i \pi_j \delta t_l - \frac{\partial g_{ij}}{\partial t_s} \pi_i \pi_j \delta t_s \right] dt_l - \frac{1}{2} g_{ij} \frac{\partial p_i}{\partial t_l} \pi_i \pi_j \delta t_l - \frac{1}{2} g_{ij} \frac{\partial p_j}{\partial t_l} \pi_i \pi_j \delta t_l \right\} ds .
\]

Integrating by parts the first and last terms inside each square bracket, cancellation among the terms will give

\[
\delta S = \int \left\{ \left[ \frac{1}{2} \left( \frac{\partial p_i}{\partial t_l} q_i^l + \frac{\partial p_i}{\partial t_s} \pi_i \right) - \frac{\partial q_i}{\partial t_s} \pi_i \right] dt_l + \left[ \frac{1}{2} \left( \frac{\partial p_i}{\partial t_l} q_i^l - \frac{\partial q_i}{\partial t_l} \pi_i \right) \pi_i \right] dt_l \right\} ds .
\]

Using the equations of motion,

\[
\frac{dq^k}{dt_l} = g_{lk}^{ik} \pi_i \pi_k , \quad \frac{dp_k}{dt_l} = - \frac{\partial q_{ij}}{\partial g_{lk}^{ik}} \pi_i \pi_j , \quad \frac{dq^k}{dt_s} = g_{lk}^{ik} \pi_i \pi_k , \quad \frac{dp_k}{dt_s} = - \frac{\partial q_{ij}}{\partial g_{lk}^{ik}} \pi_i \pi_j ,
\]

we obtain

\[
\sum_{i,j,n,k=1}^{N} \left( \frac{\partial g_{ij}}{\partial q^n} g_{nk}^{ik} - \frac{\partial g_{ij}}{\partial q^n} g_{nk}^{ik} \right) \pi_i \pi_j \pi_n \pi_k = 0 ,
\]

which is actually identical to \((3.19)\). One may find that it is straightforward to show that this condition holds true for every pair of metric tensors in the case of \(N\) degrees of freedom.

**Proposition:** For integrable pseudo-geodesic Hamiltonian systems, the following identity

\[
\sum_{i,j,n,k=1}^{N} \left( \frac{\partial g_{ij}}{\partial q^n} g_{nk}^{ik} - \frac{\partial g_{ij}}{\partial q^n} g_{nk}^{ik} \right) \pi_i \pi_j \pi_n \pi_k = 0
\]

holds true on solutions of the Hamilton's equations.

Above statement can be verified with explicit computation. Next, we will give direct computation on the compatibility between \(g_1\) and \(g_2\) for the rational and hyperbolic Calogero's goldfish systems.

The rational case: For simplicity, we consider first the case of three particles. We found that, in rational case, the whole inside the bracket of \((3.22)\) vanish naturally independent of others under the summation. Therefore, the general case of \(N\) particles can be proved as follows. Calculating the term inside the bracket, we obtain

\[
\frac{\partial g_{ij}^{ik}}{\partial q^b} g_{jk}^{nk} = \sum_{i \neq j} \left[ \prod_{i \neq j} (q^i - q^n) \right] \left( \delta_{ik} - \delta_{jk} \right) \sum_{b \neq n} q^b \frac{\prod_{a \neq i} (q^i - q^n)^2 \prod_{a \neq n} (q^n - q^a)}{\prod_{a \neq n} q^a - q^n} ,
\]

\[(3.23)\]
and
\[
\frac{\partial g_{ij}^{nk}}{\partial q^k} g_1^{nk} = \frac{\prod_{a \neq i}^N (q^i - q^a)(\sum_{b \neq i}^N \delta_{bk}) - \sum_{b \neq i}^N q^b \left\{ \sum_{l \neq i}^N \left[ \prod_{i \neq a \neq l}^N (q^i - q^a) \right] (\delta_{lk} - \delta_{lk}) \right\}}{\prod_{a \neq i}^N (q^i - q^a)^2 \prod_{a \neq n}^N (q^n - q^a)}.
\]  

(3.24)

Here, we have suppressed the initial condition of every summation and product since they are all starting from one. We observe that (3.23) and (3.24) are different by just the term that contains \( \delta_{bk} \). So, we divide our proof into two parts.

Part 1 for the case \( i = k \): Consider (3.24), the condition for the summation on \( b \) becomes \( b \neq k \) which means that the kronecker delta functions \( \delta_{bk} \) are all zero. Therefore, changing every \( i \) appearing in (3.24) into \( k \) and since \( n \) is always equal to \( k \) for Goldfish models, we find that (3.23) and (3.24) are exactly the same.

Part 2 for the case \( i \neq k \): \( \delta_{ik} \) always vanish and the kronecker delta functions \( \delta_{bk} \) will be one only the term that \( l = k \). Also, the kronecker delta functions \( \delta_{bk} \) can only be one since there is only one term where \( b = k \). Then, (3.24) is reduced to

\[
\frac{\partial g_{ij}^{nk}}{\partial q^k} g_1^{nk} = \frac{(q^i - q^k) \prod_{i \neq a \neq k}^N (q^i - q^a) + (q^k + \sum_{i \neq b \neq k}^N q^b) \prod_{i \neq a \neq k}^N (q^i - q^a)}{\prod_{a \neq i}^N (q^i - q^a)^2 \prod_{a \neq n}^N (q^n - q^a)}
\]

\[
= \frac{(q^i + \sum_{i \neq b \neq k}^N q^b) \prod_{i \neq a \neq k}^N (q^i - q^a)}{\prod_{a \neq i}^N (q^i - q^a)^2 \prod_{a \neq n}^N (q^n - q^a)}
\]

(3.25)

Also, (3.23) becomes

\[
\frac{\partial g_{ij}^{nk}}{\partial q^k} g_2^{nk} = \frac{\prod_{b \neq k}^N (q^b) \prod_{i \neq a \neq k}^N (q^i - q^a)}{\prod_{a \neq i}^N (q^i - q^a)^2 \prod_{a \neq n}^N (q^n - q^a)}
\]

(3.26)

The equation (3.25) and (3.26) are identical. Then this completes the verification of the geodesic compatibility for rational case.

The hyperbolic case: We start the computation as the same fashion with the rational case. Substituting the metric tensors into (3.22), we get

\[
\frac{\partial g_1^{ij}}{\partial q^k} g_2^{nk} = -\sum_{l \neq i}^N \left[ \prod_{i \neq a \neq l}^N \sinh(q^i - q^a) \cosh(q^i - q^l)(\delta_{ik} - \delta_{lk}) \right] \sum_{b \neq n}^N \prod_{a \neq i}^N \sinh(q^i - q^a)^2 \prod_{a \neq n}^N \sinh(q^n - q^a)
\]

(3.27)
Then, (3.30) becomes

\[
\frac{\partial g_{ij}^{nk}}{\partial q^k g_1} = \frac{\prod_{a \neq i}^N \sinh(q^i - q^a)(-2 \sum_{b \neq i}^N e^{-2q_i^b} \delta_{bk}) - \sum_{b \neq i}^N e^{-2q_i^b} \left\{ \sum_{l \neq 1}^N \left( \prod_{j \neq a \neq \ell}^N \sinh(q^i - q^a) \right) \cosh(q^i - q^l)(\delta_{ik} - \delta_{ll}) \right\}}{\prod_{a \neq i}^N \sinh(q^i - q^a)^2 \prod_{a \neq n}^N \sinh(q^n - q^a)}
\]

(3.28)

Again, for the case of \(i = k\), it can be easily seen that both (3.27) and (3.28) are identical as we do have in the rational type.

For the case of \(i \neq k\), we start with an observation that for a pair \((11 - 22)\) we have

\[
\left( \frac{\partial g_{11}^{22}}{\partial q^2} g_2^{22} - \frac{\partial g_{11}^{22}}{\partial q^2} g_1^{11} \right) \pi_1 \pi_2 \pi_2 + \left( \frac{\partial g_{22}^{11}}{\partial q^1} g_2^{11} - \frac{\partial g_{22}^{11}}{\partial q^1} g_2^{11} \right) \pi_1 \pi_2 \pi_1 = 0
\]

(3.29)

and this also holds true for other pairs, i.e., \((11 - 33), (22 - 44)\), etc. Then, simplifying (3.27) and (3.28) as in the rational case and subtracting them, we obtain

\[
\frac{\partial g_{ij}^{nk}}{\partial q^k g_2} - \frac{\partial g_{ij}^{nk}}{\partial q^k g_1} = \frac{2e^{-2q_k} \prod_{a \neq i}^N \sinh(q^i - q^a) + \prod_{b \neq k}^N \sinh(q^i - q^b) \cosh(q^i - q^k) \left( \sum_{b \neq k}^N e^{-2q_k^b} - \sum_{b \neq i}^N e^{-2q_i^b} \right)}{\prod_{a \neq i}^N \sinh(q^i - q^a)^2 \prod_{a \neq k}^N \sinh(q^k - q^a)}
\]

(3.30)

The term in the numerator inside the last bracket can be simplified as

\[
\sum_{b \neq k}^N e^{-2q_k^b} = \sum_{b \neq i}^N e^{-2q_i^b} - \sum_{b \neq i}^N e^{-2q_k^b} - \sum_{b \neq k}^N e^{-2q_i^b} = e^{-2q_i^i} - e^{-2q_k^k}.
\]

Then, (3.30) becomes

\[
\frac{\partial g_{ij}^{nk}}{\partial q^k g_2} - \frac{\partial g_{ij}^{nk}}{\partial q^k g_1} = \frac{2e^{-2q_k} \sinh(q^i - q^k) + (e^{-2q_i^i} - e^{-2q_k^k}) \cosh(q^i - q^k)}{\prod_{a \neq i}^N \sinh(q^i - q^a)^2 \prod_{a \neq k}^N \sinh(q^k - q^a)}
\]

\[
= \frac{2e^{-2q_k} \sinh(q^i - q^k) - 2 \sinh(q^i - q^k) \cosh(q^i - q^k)}{\prod_{a \neq i}^N \sinh(q^i - q^a)^2 \prod_{a \neq k}^N \sinh(q^k - q^a)}
\]

\[
= \frac{2e^{-2q_k} - 2 \cosh(q^i - q^k)}{\prod_{a \neq i}^N \sinh(q^i - q^a)^2 \prod_{a \neq k}^N \sinh(q^k - q^a)}
\]

(3.31)
The summations appear in (3.19) will generate another term similar to (3.31) but the indices are interchanged such that (3.31) becomes

$$\frac{\partial g_{nk}^{ij}}{\partial q^i} g_{kj}^{ij} - \frac{\partial g_{nk}^{ij}}{\partial q^j} g_{kj}^{i} = \frac{\left(2e^{-2q^i} - 2 \cosh(q^k - q^i)\right)}{\prod_{a \neq k} \sinh(q^k - q^a) \prod_{a \neq i} \sinh(q^i - q^a)}.$$  \hspace{1cm} (3.32)

Since the cosine hyperbolic is an even function and the denominators are the same, adding (3.31) and (3.32) up as suggested by the observation, we get

$$(2e^{-2q^k} - 2 \cosh(q^j - q^k)) + (2e^{-2q^j} - 2 \cosh(q^k - q^j)) = 2(e^{-2q^k} + e^{-2q^j}) - 4 \cosh(q^k - q^j) = 0.$$  

This completes the verification for the hyperbolic case.

### 4 Geometrical interpretation

To see what we would interpret the geodesic compatibility in terms of geometry, the present form of the relation (3.22) is not so trivial. To unravel the geometrical insight, we may start with the relation between the derivative with respect to coordinates $q^i$ and the affine (Levi-Civita) connection

$$\partial_k g_{ij} = -\Gamma_{kh}^i g_{hj}^{ij} - \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}.$$  \hspace{1cm} (4.33)

Then we find that the term inside the bracket of the geodesic compatibility condition (3.22) becomes

$$\partial_k g_{ij}^{nk} - \partial_k g_{ij}^{i} g_{kl}^{nk} = (\Gamma_{kh}^i g_{hj}^{ij} - \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij} - (\Gamma_{kh}^i g_{hj}^{ij} - \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij}.$$  \hspace{1cm} (4.34)

Then, the geodesic compatibility becomes

$$\sum_{i,j,n,k=1}^{N} \left( \frac{\partial g_{ij}}{\partial q^k} g_{nk}^{ij} - \frac{\partial g_{ij}}{\partial q^j} g_{nk}^{ij} \right) \pi_i \pi_j \pi_n \pi_k = \sum_{i,j,n,k=1}^{N} \left[ (\Gamma_{kh}^i g_{hj}^{ij} + \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij} - (\Gamma_{kh}^i g_{hj}^{ij} + \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij} \right] \pi_i \pi_j \pi_n \pi_k = 0.$$  \hspace{1cm} (4.35)

Recalling the covariant derivative of the metric tensor

$$\nabla_k g_{ij} = \partial_k g_{ij} + \Gamma_{kh}^i g_{hj}^{ij} + \Gamma_{j}^{ij}_{kh} g_{ih}^{jk},$$  \hspace{1cm} (4.36)

and substituting (4.36) into (4.34), we get

$$\partial_k g_{ij}^{nk} - \partial_k g_{ij}^{i} g_{kl}^{nk} = (\nabla_k g_{ij}^{nk} - \Gamma_{kh}^i g_{hj}^{ij} - \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij} - (\nabla_k g_{ij}^{ij} - \Gamma_{kh}^i g_{hj}^{ij} - \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij}$$

$$= \left[ (\nabla_k g_{ij}^{ij}) g_{nk}^{ij} - (\nabla_k g_{ij}^{ij}) g_{nk}^{ij} \right] - \left[ (\Gamma_{kh}^i g_{hj}^{ij} + \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij} - (\Gamma_{kh}^i g_{hj}^{ij} + \Gamma_{j}^{ij}_{kh} g_{ih}^{jk}) g_{nk}^{ij} \right].$$  \hspace{1cm} (4.37)

The second bracket in (4.37) effectively vanishes according to (4.35) resulting in

$$\sum_{i,j,n,k=1}^{N} \left( \frac{\partial g_{ij}}{\partial q^k} g_{nk}^{ij} - \frac{\partial g_{ij}}{\partial q^j} g_{nk}^{ij} \right) \pi_i \pi_j \pi_n \pi_k = \sum_{i,j,n,k=1}^{N} \left( (\nabla_k g_{ij}^{ij}) g_{nk}^{ij} - (\nabla_k g_{ij}^{ij}) g_{nk}^{ij} \right) \pi_i \pi_j \pi_n \pi_k = 0.$$  \hspace{1cm} (4.38)

We know that (4.38) is a direct consequence of the Hamiltonian commuting flows, see section 3; $\{\mathcal{H}_l, \mathcal{H}_s\} = 0$. Suppose that $X_{\mathcal{H}_l}$ and $X_{\mathcal{H}_s}$ are vector fields associated with the Hamiltonians $\mathcal{H}_l$ and $\mathcal{H}_s$, respectively. We have now a condition that the Lie bracket of these two vector fields vanishes

$$[X_{\mathcal{H}_l}, X_{\mathcal{H}_s}] = 0.$$  \hspace{1cm} (4.39)

It is well known that (4.39) gives a compatibility between two Hamiltonian vector fields. This means that compatibility between flows forms a perfect parallelogram. The covariant derivative (4.36) gives the parallel transport of the vector on the manifold. Then (4.38) represents the compatible parallel transport of two different Hamiltonian vector fields, see figure [1].
5 Conclusion

We successfully construct the geodesic compatibility condition to capture the multidimensional consistency on the level of geometry. This compatibility between metric tensors, which is equivalent to the Hamiltonian commuting flows and the closure relation, can be possibly treated as integrability feature for the system. The rational and hyperbolic GF systems, hierarchy geodesic flows (3.14), are used as concrete examples to explicitly verify the geodesic compatibility condition. The condition can be geometrically interpreted as a compatible parallel transports between two different directions corresponding to two Hamiltonian vector fields. We put here a remark on the RS type systems. The geodesic interpretation holds only for the first flow in the RS hierarchy since the second equation of motion in the hierarchy is not in the geodesic form [11]. Then the RS type systems is not applicable for the geodesic compatibility test. Another point is that we do not have the geodesic interpretation in the Lagrangian description for both GF and RS systems, see the Lagrangian hierarchy in [11][12].

References

[1] F. W. Nijhoff and A. J. Walker, 2001, The discrete and continuous Painlevé VI hierarchy and the Garnier systems, Glasgow Mathematical Journal, 43A, pp. 109-123.
[2] F. W. Nijhoff, R. Ramani, B. Grammaticos and Y. Ohta, 2001, On discrete Painleve Equations associated with the lattice KdV systems and the Painlevé VI equation, Studies in Applied Mathematics, 106(3), pp. 261-314.
[3] F. W. Nijhoff, 2002, Lax pair for the Adler (lattice Krichever-Novikov) system, Physics Letters A, 297(1-2), pp. 49-58.
[4] V. E. Adler, A. I. Bobenko and Yu. B. Suris, 2003, Classification of integrable equations on quad-graphs. The consistency approach, Communications in Mathematical Physics, 233(3), pp. 513-543.
[5] V. I. Arnold, 1978, Mathematical methods of classical mechanics, Springer.
[6] S. B. Lobb and F. W. Nijhoff, 2009, Lagrangian multiforms and multidimensional consistency, Journal of Physics A: Mathematical and Theoretical, 42(45).
[7] F Calogero, 1969, Solution of a three-body problem in one dimension, Journal of Mathematical Physics 10, 2191; 1971, Solution of the One-Dimensional N-Body Problem with Quadratic and/or Inversely Quadratic Pair Potentials, Journal of Mathematical Physics 12, 418.
[8] S N M Ruijsenaars and H Schneider, 1986, A new class of integrable systems and its relation to solitons, *Annal Physics*, **170**, pp. 370-405.

[9] F Calogero, 2001, The neatest many-body problem amenable to exact treatments (a goldfish?), *Physica D: Nonlinear Phenomena*, **152**, pp. 78-84.

[10] S. Yoo-Kong, S. B. Lobb and F. W. Nijhoff, 2011, Discrete-time Calogero-Moser system and Lagrangian 1-form structure, *Journal of Physics A: Mathematical and Theoretical*, **44(36)**.

[11] S. Yoo-Kong and F. W. Nijhoff, 2013, Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure, arXiv:1112.4576v2 nlin.SI.

[12] U. Jairuk, S. Yoo-Kong and M. Tanasittikosol, 2015, On the Lagrangian structure of Calogero’s Goldfish model, *Theoretical and Mathematical Physics*, **183(2)**, pp. 665-683.

[13] U. Jairuk, S. Yoo-Kong and M. Tanasittikosol, 2017, On the Lagrangian structure of the Hyperbolic Calogero-Moser System, *Reports on Mathematical Physics*, **79(3)**, pp. 299-330.

[14] R Boll, M Petrera and Yu B Suris, 2013, Multi-time Lagrangian 1-forms for families of Bäcklund transformations Toda-type systems, *Journal of Physics A: Mathematical and Theoretical*, **46(27)**.

[15] R Boll, M Petrera and Yu B Suris, 2015, Multi-time Lagrangian 1-forms for families of Bäcklund transformations Relativistic Toda-type systems, *Journal of Physics A: Mathematical and Theoretical*, **48(8)**.

[16] Yu B Suris, 2013, Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms, *Journal Geometric Mechanics*, **5(3)**, pp. 365-379.

[17] C Puttarprom, W Piensuk and S Yoo-Kong, 2019, Integrable Hamiltonian hierarchies and Lagrangian 1-forms, arXiv:1904.00582.

[18] J Arnlind, M Bordemann, J Hoppe and C Lee, 2008, Goldfish geodesics and Hamiltonian reduction of matrix dynamics, *Letters in Mathematical Physics*, **84**, pp. 89-98.

[19] G P Paternain, 1999, Geodesic Flows, *Progress in Mathematics*, **180**, Birkhäuser Basel.

[20] A V Bolsinov and B Jovanović, 2004, Integrable geodesic flows on Riemannian manifolds: Construction and Obstructions, *Contemporary geometry and related topics*, pp. 57?103, World Sci. Publ., River Edge, NJ.

[21] Yu B Suris, 2005, Time Discretization of F. Calogeros Goldfish System, *Journal of Nonlinear Mathematical Physics*, **12**, pp. 633-647.

[22] A J Galajinsky, 2018, Ruijsenaars-Schneider three-body models with $N = 2$ supersymmetry, High Energ. Phys, 079.

[23] A J Galajinsky, 2018, $N = 2$ supersymmetric extensions of relativistic Toda lattice, High Energ. Phys, 061.