Weak conformality of stable stationary maps
for a functional related to conformality

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ABSTRACT. Let $(M, g)$, $(N, h)$ be compact Riemannian manifolds without boundary, and let $f$ be a smooth map from $M$ into $N$. We consider a covariant symmetric tensor $T_f = f^*h - \frac{1}{m} \|df\|^2 g$, where $f^*h$ denotes the pull-back metric of $h$ by $f$. The tensor $T_f$ vanishes if and only if the map $f$ is weakly conformal. The norm $\|T_f\|$ is a quantity which is a measure of conformality of $f$ at each point. We are concerned with maps which are critical points of the functional $\Phi(f) = \int_M \|T_f\|^2 dv_g$. We call such maps $C$-stationary maps. Any conformal map or more generally any weakly conformal map is a $C$-stationary map. It is of interest to find when a $C$-stationary map is a (weakly) conformal map.

In this paper we prove the following result. If $f$ is a stable $C$-stationary maps from the standard sphere $S^m$ ($m \geq 5$) or into the standard sphere $S^n$ ($n \geq 5$), then $f$ is a weakly conformal map.

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1 Introduction

Let $(M, g)$, $(N, h)$ be Riemannian manifolds. A smooth map $f$ from $M$ into $N$ is a conformal map if and only if there exists a smooth positive function $\varphi$ on $M$ such that $f^*h = \varphi g$, where $f^*h$ denotes the pullback metric of $h$ by $f$, i.e.,

$$(f^*h)(X, Y) = h(df(X), df(Y)).$$

In this situation we utilize a covariant tensor

$$T_f : = f^*h - \frac{1}{m} \|df\|^2 g.$$

where

$$\|df\|^2 = \sum_i h(df(e_i), df(e_i)).$$

Then $f$ is a conformal map if and only if $T_f = 0$, unless $df \neq 0$. We consider a functional

$$\Phi(f) = \int_M \|T_f\|^2 dv_g,$$
where $dv_g$ denotes the volume form of $(M, g)$, and

$$||T_f||^2 = \sum_{i,j} T_f(e_i, e_j)^2.$$ 

($e_i$ is a local orthonormal frame on $(M, g)$. ) Minimizers of $\Phi$ are close to conformal maps, even if there does not exist any conformal map from $M$ into $N$. In [5], the second author introduced the above functional $\Phi$ and proved the first variation formula, the second variation formula, a quasi-monotonicity formula and a Bochner type formula. We call a map $f$ C-stationary if it is a critical point of the functional $\Phi$, i.e., if the first variation of $\Phi$ at $f$ vanishes. Any conformal map or more generally any weakly conformal map is a C-stationary map. It is of interest to find when a C-stationary map is a (weakly) conformal map. In this paper we prove the following two theorems for stable C-stationary maps.

**Theorem 1.** Let $f$ be a stable C-stationary map from the standard sphere $S^m$ into a Riemannian manifold $N$. If $m \geq 5$, then $f$ is a weakly conformal map.

**Theorem 2.** Let $f$ be a stable C-stationary map from a Riemannian manifold $M$ into the standard sphere $S^n$. If $n \geq 5$, then $f$ is a weakly conformal map.

The contents of this paper are as follows:

1. Introduction
2. Weakly conformal maps and the functional $\Phi$
3. Preliminaries
4. Stable C-stationary maps from spheres
5. Stable C-stationary maps into spheres

### 2 Weakly conformal maps and the functional $\Phi$

Let $f$ be a smooth map from a Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$. In this section we give a tensor $T_f$ of conformality for any smooth map $f$. We recall here the following two notions.

**Definition 1.** (i) A map $f$ is conformal if there exists a smooth positive function $\varphi$ on $M$ such that

$$f^* h = \varphi g.$$ 

(ii) A map $f$ is weakly conformal if there exists a smooth non-negative function $\varphi$ on $M$ satisfying (1).

The condition (1) is equivalent to

$$f^* h = \frac{1}{m}||df||^2 g,$$ 

where $df$ denotes the volume form of $(M, g)$, and
since taking the trace of the both sides of (1) (w.r.t. the metric $g$), we have $\|df\|^2 = m\varphi$, i.e., $\varphi = \frac{1}{m}\|df\|^2$. Then $f$ is weakly conformal if and only if it satisfies (2). Note that $f$ is weakly conformal if and only if for any point $x \in M$, $f$ is conformal at $x$ or $df_x = 0$.

Taking the above situation into consideration, we utilize the covariant tensor

$$T_f \overset{\text{def}}{=} f^*h - \frac{1}{m}\|df\|^2g,$$

i.e.,

$$T_f(X, Y) \overset{\text{def}}{=} (f^*h)(X, Y) - \frac{1}{m}\|df\|^2g(X, Y) = h(df(X), df(Y)) - \frac{1}{m}\|df\|^2g(X, Y),$$

where $f^*h$ denotes the pull-back of the metric $h$.

**Remark 1.** In the case of $m = 2$, the quantity $T_f$ is equal to the stress energy tensor (up to the sign)

$$S_f = f^*h - \frac{1}{2}\|df\|^2g$$

in the harmonic map theory. (See Eells-Lemaire [3], p.392.)

**Lemma 1.**

(a) $T_f$ is symmetric, i.e., $T_f(X, Y) = T_f(Y, X)$.

(b) $f$ is weakly conformal if and only if $T_f = 0$.

(c) $T_f$ is trace-free (with respect to the metric $g$), i.e.,

$$\langle g, T_f \rangle = \text{Trace}_g T_f = \sum_i T_f(e_i, e_i) = 0.$$ 

(d) The pairing of the pull-back metric $f^*h$ and the tensor $T_f$ is equal to the norm $\|T_f\|$, i.e.,

$$\langle f^*h, T_f \rangle = \sum_{i,j} (f^*h)(e_i, e_j)T_f(e_i, e_j) = \|T_f\|^2.$$ 

(e) $\|T_f\|^2 = \|f^*h\|^2 - \frac{1}{m}\|df\|^4$.

In the above equalities, The product $\langle \ , \ \rangle$ denotes the pairing of the covariant 2-tensors, i.e.,

$$\langle A, B \rangle = \sum_{i,j=1}^n A(e_i, e_j)B(e_i, e_j)$$

for any covariant 2-tensors $A, B$, where $e_i$ ($i = 1, \cdots, m$) is an orthonormal frame.
Proof. (a) follows directly from the definition of $T_f$.
(b) follows easily from the argument mentioned above, i.e.,

$f$ is a weakly conformal
\iff
There exists a smooth non-negative function $\varphi$ s.t. $f^*h = \varphi g$
\iff
$f^*h = \frac{1}{m} \|df\|^2 g$
\iff
$T_f = 0$.

Here if $f^*h = \varphi g$, then taking the trace of the both sides, we have $\|df\|^2 = m \varphi$, i.e., $\varphi = \frac{1}{m} \|df\|^2$.

(c) Note $(g, T_f) = \text{Trace}_g T_f$. Moreover we have

\[
\text{Trace}_g T_f = \sum_i T_f(e_i, e_i) \]
\[
= \sum_i \left\{ h(df(e_i), df(e_i)) - \frac{1}{m} \|df\|^2 g(e_i, e_i) \right\} \]
\[
= \sum_i h(df(e_i), df(e_i)) - \frac{1}{m} \|df\|^2 \sum_i g(e_i, e_i) \]
\[
= \sum_i h(df(e_i), df(e_i)) - \|df\|^2 \]
\[
= \|df\|^2 - \|df\|^2 \]
\[
= 0. \]

(d)
\[
(f^*h, T_f) = (T_f + \frac{1}{m} \|df\|^2 g, T_f) \]
\[
= (T_f, T_f) - \frac{1}{m} \|df\|^2 (g, T_f) \]
\[
= \|T_f\|^2. \]

(e)
\[
\|T_f\|^2 = (f^*h, T_f) \]
\[
= (f^*h, f^*h - \frac{1}{m} \|df\|^2 g) \]
\[
= \|f^*h\|^2 - \frac{1}{m} \|df\|^2 (f^*h, g) \]
\[
= \|f^*h\|^2 - \frac{1}{m} \|df\|^4. \]

Thus we obtain Lemma 1. □
In this paper, we are concerned with the functional of the norm of $T_f$

$$\Phi(f) = \int_M \|T_f\|^2 dv_g.$$ 

This quantity $\Phi(f)$ gives a measure of the conformality of maps $f$. Note that if $f$ is a conformal map, then $\Phi(f)$ vanishes.

3 Preliminaries

In this section we give a technical lemma (Lemma 2), the first variation formula and the second variation formula with some notations and definitions. The two formulas are obtained in the second author’s paper [6]. For reader’s convenience we give their proofs of these two formula in the appendix at the end of the paper.

Take any smooth deformation $F$ of $f$, i.e., any smooth map $F : (-\varepsilon, \varepsilon) \times M \rightarrow N$ s.t. $F(0, x) = f(x)$, where $\varepsilon$ is a positive constant. Let $f_t(x) = F(t, x)$. Then we have $f_0(x) = f(x)$. We often say a deformation $f_t(x)$ instead of a deformation $F(t, x)$. Let $X = dF(\frac{\partial}{\partial t})|_{t=0}$ denotes the variation vector field of the deformation $F$.

We define an $f^{-1}TN$-valued 1-form $\sigma_f$ on $M$ by

$$(3) \quad \sigma_f(X) = \sum_j T_f(X, e_j)df(e_j)$$

where $\{e_j\}$ is an orthonormal frame. The 1-form $\sigma_f$ plays an important role in our arguments.

We first give the following lemma, which we often use in our arguments.

Lemma 2.

$$(4) \quad \sum_j h(Z, dF(e_j)) T_F(W, e_j) = h(Z, \sigma_F(W)).$$

In particular

$$(5) \quad \sum_j h(Z, df(e_j)) T_f(W, e_j) = h(Z, \sigma_f(W)).$$

$$(6) \quad \|T_f\|^2 = \sum_i h(df(e_i), \sigma_f(e_i)).$$

Proof of Lemma 2. The equality (4) easily follows from the definition of $\sigma_F$. Indeed, since $h(A, B)T_F(C, D) = h(A, T_F(C, D)B)$, we have

$$\sum_j h(Z, dF(e_j)) T_F(W, e_j) = h(Z, \sum_j T_F(W, e_j)dF(e_j))$$

$$= h(Z, \sigma_F(W)).$$
The equality (5) follows from (4) at \( t = 0 \). Furthermore let \( Z = df(e_i) \) and let \( W = e_i \) in (5), and sum with respect to \( i \). Then we have
\[
\sum_{i, j} h(df(e_i), df(e_j)) T_f(e_i, e_j) = \sum_i h(df(e_i), \sigma_f(e_i)).
\]

Since \((f^*h)(e_i, e_j) = h(df(e_i), df(e_j))\), the above equality and Lemma 1 (d) imply (6).

The first variation of the functional \( \Phi \) is given by the following formula which is given and proved in [5].

**Proposition 1** (first variation formula).
\[
\frac{d\Phi(f_t)}{dt} \bigg|_{t=0} = -4 \int_M \langle X, \text{div}_g \sigma_f \rangle \, dv_g,
\]
where \( \text{div}_g \sigma_f \) denotes the divergence of \( \sigma_f \), i.e., \( \text{div}_g \sigma_f = \sum_{i=1}^m (\nabla_{e_i} \sigma_f)(e_i) \).

We give here the notion of C-stationary maps.

**Definition 2.** We call a smooth map \( f \) **C-stationary** if the first variation of \( \Phi(f) \) (at \( \bar{f} \)) identically vanishes, i.e.,
\[
\frac{d\Phi(f_t)}{dt} \bigg|_{t=0} = 0
\]
for any smooth deformation \( f_t \) of \( f \). By Proposition 1, a smooth map \( f \) is C-stationary if and only if it satisfies the equation
\[
\text{div}_g \sigma_f = 0,
\]
which is called the Euler-Lagrange equation for the functional \( \Phi(f) \), where \( \sigma_f \) is the covariant tensor defined by (3).

We give the second variation formula for the functional \( \Phi(f) \). Take any smooth deformation \( F \) of \( f \) with two parameters, i.e., any smooth map
\[
F : (-\delta, \delta) \times (-\varepsilon, \varepsilon) \times M \rightarrow N \quad \text{s.t.} \quad F(0, 0, x) = f(x).
\]

Let \( f_{s, t}(x) = F(s, t, x) \), and we often say a deformation \( f_{s, t}(x) \) instead of a deformation \( F(s, t, x) \). Let
\[
X = \left. \frac{dF(\frac{\partial}{\partial s})}{dt} \right|_{s, t=0}, \quad Y = \left. \frac{dF(\frac{\partial}{\partial t})}{dt} \right|_{s, t=0}
\]
denote the variation vector fields of the deformation \( f_{s, t} \). Then we have the following second variation formula.
Proposition 2 (second variation formula).

\[
\frac{1}{4} \left. \frac{\partial^2 \Phi(f, s, t)}{\partial s \partial t} \right|_{s, t=0} = \int_M h(\text{Hess}_f \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right), \text{div}_g \sigma_f) \, dv_g
\]

+ \int_M \sum_{i,j} h(\nabla e_i X, \nabla e_j Y) T_f(e_i, e_j) \, dv_g

+ \int_M \sum_{i,j} h(\nabla e_i X, df(e_j)) h(\nabla e_j, df(e_j)) \, dv_g

+ \int_M \sum_{i,j} h(\nabla e_i X, df(e_j)) h(\nabla e_j, df(e_j)) \, dv_g

- \frac{2}{m} \int_M \sum_i h(\nabla e_i X, df(e_i)) \sum_j h(\nabla e_j, df(e_j)) \, dv_g

+ \int_M \sum_{i,j} h(NR(X, df(e_i)) Y, df(e_j)) T_f(e_i, e_j) \, dv_g,

where \( \text{Hess}_f \) denotes the Hessian of \( f \), i.e., \( \text{Hess}_f(Z, W) = (\nabla_Z df)(W) = (\nabla_W df)(Z) \).

Remark 2. Note that the first term in the right hand side vanishes if \( f \) is a C-stationary map.

Remark 3. The last term of the right hand side in Proposition 2 is equal to

\[ \int_M \sum_i h(NR(X, df(e_i)) Y, \sigma_f(e_i)) \, dv_g, \]

since

\[
\sum_{i,j} h(NR(X, df(e_i)) Y, df(e_j)) T_f(e_i, e_j)
= \sum_i h(NR(X, df(e_i)) Y, \sum_j T_f(e_i, e_j) df(e_j))
= \sum_i h(NR(X, df(e_i)) Y, \sigma_f(e_i)).
\]

Definition 3. We call a C-stationary map stable if the second variation of \( \Phi(f) \) (at \( f \)) is nonnegative, i.e.,

\[
\frac{d^2 \Phi(f_1)}{dt^2} \bigg|_{t=0} \geq 0.
\]

We give an example of the stable C-stationary map which is not weakly conformal.

Example. Let us define a map

\[
f : M = S^1 \times S^1 \times \cdots \times S^1 \to N = S^1_k \times S^1 \times \cdots \times S^1
\]
Denoting unit parallel vector fields $e_1,e_2 \ldots e_{\ell}$ implies that $\text{div } \sigma = 0$ at the origin of $\mathbb{R}^2$. Obviously $f$ is not weakly conformal if $k \neq 1$. Let us take orthonormal unit parallel vector fields $e_1,e_2 \ldots e_{\ell}$ on $M$ where $e_i$ is tangent to the $i$-th factor, and $E_1,E_2 \ldots E_{\ell}$ on $N$ where $E_j$ is tangent to the $j$-th factor. Hence we have

$$df(e_1) = kE_1, \quad df(e_i) = E_i \quad (i \neq 1).$$

Every vector field $X$ on $N$ is written as $X = \sum \varphi_i E_i$ where $\varphi_i$ is a function on $N$. It is easy to show

$$T_f(v,w) = C\langle v, w \rangle$$

where

$$C = \begin{cases} \frac{(k^2-1)(\ell-1)}{\ell} & (v \text{ and } w \text{ are tangent to the first factor}) \\ -\frac{k^2-1}{\ell} & \text{(otherwise)} \end{cases}$$

Hence we obtain for a tangent vector $v$ on $M$,

$$\sigma_f(v) = \frac{k(k^2-1)(\ell-1)}{\ell} \langle v, e_1 \rangle E_1 - \frac{k^2-1}{\ell} \sum_{j=2}^{\ell} \langle v, e_j \rangle E_j,$$

which implies that $\text{div } \sigma_f = 0$, i.e., the map $f$ is $C$-stationary.

We have only to prove that $f$ is stable, i.e., the second variation at $f$ is non-negative. Denoting $\varphi_i \circ f$ by $\psi_i$, we calculate terms of the right hand side of the second variation formula as follows:

$$\int_M \sum_{i,j} \langle \nabla e_i X, \nabla e_j X \rangle T_f(e_i, e_j) = \int_M \frac{(k^2-1)(\ell-1)}{\ell} \langle \nabla e_1 X, \nabla e_j X \rangle - \frac{k^2-1}{\ell} \sum_{i=2}^{\ell} \langle \nabla e_i X, \nabla e_j X \rangle$$

$$= \int_M \frac{(k^2-1)(\ell-1)}{\ell} \sum_j (\nabla e_j \psi_j)^2 - \frac{k^2-1}{\ell} \sum_{i=2}^{\ell} \sum_j (\nabla e_i \psi_j)^2,$$

$$\int_M \sum_{i,j} \langle \nabla e_i X, df(e_j) \rangle \langle \nabla e_i X, df(e_j) \rangle = \int_M \sum_i k^2 (\nabla e_i \psi_1)^2 + \sum_{i,j \geq 2} (\nabla e_i \psi_j)^2,$$

$$\int_M \sum_{i,j} \langle \nabla e_i X, df(e_j) \rangle \langle df(e_i), \nabla e_j X \rangle = \int_M k^2 (\nabla e_i \psi_1)^2 + 2k \sum_{i \geq 2} (\nabla e_i \psi_1)(\nabla e_i \psi_1) + \sum_{i,j \geq 2} (\nabla e_i \psi_j)(\nabla e_j \psi_1),$$

$$-\frac{2}{\ell} \int_M \left( \sum_i \langle \nabla e_i X, df(e_i) \rangle \right)^2 = -\frac{2}{\ell} \int_M k^2 (\nabla e_i \psi_1)^2 + 2k \sum_{i \geq 2} (\nabla e_i \psi_1)(\nabla e_i \psi_1) + \sum_{i,j \geq 2} (\nabla e_i \psi_j)(\nabla e_j \psi_1).$$
Note that the following identities hold:
\[
(\nabla_{e_i} \psi_j)(\nabla_{e_j} \psi_i) = \nabla_{e_i} (\psi_j \nabla_{e_j} \psi_i) - \psi_j (\nabla_{e_i} \nabla_{e_j} \psi_i) \\
(\nabla_{e_i} \psi_i)(\nabla_{e_j} \psi_j) = \nabla_{e_j} (\psi_j \nabla_{e_i} \psi_i) - \psi_j (\nabla_{e_i} \nabla_{e_j} \psi_i).
\]
Hence exchanging the orders of iterated integrals, we obtain for each \(i\) and \(j\),
\[
\int_{S^1 \times \cdots \times S^1} (\nabla_{e_i} \psi_j)(\nabla_{e_j} \psi_i) = -\int_{S^1 \times \cdots \times S^1} \psi_j (\nabla_{e_i} \nabla_{e_j} \psi_i),
\]
\[
\int_{S^1 \times \cdots \times S^1} (\nabla_{e_i} \psi_i)(\nabla_{e_j} \psi_j) = -\int_{S^1 \times \cdots \times S^1} \psi_j (\nabla_{e_i} \nabla_{e_j} \psi_i).
\]
Thus these integrals coincide because \([e_i, e_j] = 0\). Especially we get
\[
\int_M (\nabla_{e_i} \psi_1)(\nabla_{e_i} \psi_1) = \int_M (\nabla_{e_i} \psi_1)(\nabla_{e_j} \psi_1).
\]
Let us denote the function \(\nabla_{e_i} \psi_j\) by \(a_{ij}\) for simplicity. From the fact above, the right hand side of the second variation formula is equal to
\[
\begin{align*}
\int_M & \left( \frac{(k^2 - 1)(\ell - 1)}{\ell} \sum_j (\nabla_{e_i} \psi_j)^2 - \frac{k^2 - 1}{\ell} \sum_{i \geq 2} \sum_j (\nabla_{e_i} \psi_j)^2 \\
& + k^2 \sum_i (\nabla_{e_i} \psi_1)^2 + \sum_i \sum_{j \geq 2} (\nabla_{e_i} \psi_j)^2 + k^2(\nabla_{e_i} \psi_1)^2 \\
& + 2k \sum_{i \geq 2} (\nabla_{e_i} \psi_i)(\nabla_{e_i} \psi_1) + \sum_{i \geq 2} \sum_{j \geq 2} (\nabla_{e_i} \psi_j)(\nabla_{e_i} \psi_1) \\
& - \frac{2}{\ell} k^2(\nabla_{e_i} \psi_1) - \frac{4k}{\ell} \sum_{i \geq 2} (\nabla_{e_i} \psi_1)(\nabla_{e_i} \psi_1) - \frac{2}{\ell} \sum_{i \geq 2} \sum_{j \geq 2} (\nabla_{e_i} \psi_i)(\nabla_{e_j} \psi_j) \right) \\
& = \int_M \left( \frac{\ell - 1}{\ell} (3k^2 - 1)a_{i1}^2 + \sum_{j \geq 2} \frac{1 - k^2}{\ell} a_{ij}^2 + \sum_{i \geq 2} \sum_{j \geq 2} 2k \left(1 - \frac{2}{\ell}\right) a_{i1}a_{i1} \\
& + \sum_{j \geq 2} \left( k^2(\ell - 1) + 1 \right) a_{i1}^2 + \sum_{i \geq 2} \frac{1}{\ell} (k^2(\ell - 1) + 1) a_{i1}^2 \\
& + \left\{ \sum_{i \geq 2} \sum_{j \geq 2} a_{ij}^2 + \sum_{i \geq 2} \sum_{j \geq 2} a_{ij}a_{ji} - \frac{2}{\ell} \sum_{i \geq 2} \sum_{j \geq 2} a_{i1}a_{j1} \right\} \right) \\
\end{align*}
\]
Note that
\[
\sum_{i \geq 2} 2k \left(1 - \frac{2}{\ell}\right) a_{i1}a_{i1} \geq -k \left(1 - \frac{2}{\ell}\right) \sum_{i \geq 2} (a_{i1}^2 + a_{i1}^2) \\
= -k \left(1 - \frac{2}{\ell}\right) \sum_{i \geq 2} a_{i1}^2 - k \left(1 - \frac{2}{\ell}\right) \sum_{i \geq 2} a_{i1}^2.
\]
If \(k\) is close to 1, then we have
\[
\frac{1}{\ell} (k^2(\ell - 1) + 1) > k \left(1 - \frac{2}{\ell}\right).
\]
Thus the sum of the third, fourth and fifth terms of the equation (*) is non-negative. Since 
\[ \|A\|^2 + \text{Tr}(A^2) - \frac{2}{\ell-1} \text{Tr}(A)^2 \geq 0 \]
for every \((\ell-1) \times (\ell-1)\) matrix \(A\), we have 
\[ \sum_{i \geq 2, j \geq 2} a^2_{ij} + \sum_{i \geq 2, j \geq 2} a_{ij} a_{ji} - \frac{2}{\ell} \sum_{i \geq 2, j \geq 2} a_{ij} a_{jj} \geq 0. \]
Hence if \(k \leq 1\) and \(k\) is close to 1, then the second variation at \(f\) is non-negative.

As is seen, there exist stable C-stationary maps which are not weakly conformal. We shall show in the next section that this is not the case when the domain or the range is a standard sphere.

## 4 Stable C-stationary maps from spheres

In this section, we prove Theorem 1.

**Proof of Theorem 1.** Since the standard sphere \(S^m\) is a submanifold of the Euclidean space \(\mathbb{R}^{m+1}\), we may consider that the tangent space of the standard sphere \(S^m\) at \(x \in S^m\) is a subspace of the linear space \(\mathbb{R}^{m+1} \simeq T_x \mathbb{R}^{m+1}\). Let \(p = p_x\) denote the canonical projection from \(\mathbb{R}^{m+1}\) onto the tangent space \(T_x S^m\). Let \(E\) be a unit parallel vector field on \(\mathbb{R}^{m+1}\), and let \(Z\) be the vector field which is the image by the projection \(p\) of \(E\), i.e.,
\[ Z_x = p_x(E) \]
for \(x \in S^m\). Then we can verify
\[ \nabla_{e_i} Z = -\varphi e_i, \]
where
\[ \varphi = \langle E, \nu \rangle \]
(the notation \(\langle , \rangle\) denotes the inner product on \(\mathbb{R}^{m+1}\) and \(\nu\) is the unit outer normal vector field on \(S^m\) in \(\mathbb{R}^{m+1}\). Then we have
\[
\nabla_{e_i} (df(Z)) = (\nabla_{e_i} df)(Z) + df(\nabla_{e_i} Z)
\]
\[
= (\nabla_{e_i} df)(Z) - \varphi df(e_i)
\]
(10)

Take orthonormal parallel vector fields \(E_1, \ldots, E_{m+1}\) on \(\mathbb{R}^{m+1}\), and set \(Z_k = p(E_k)\) \((k = 1, \ldots, m+1)\). Then by (10), we see
\[
\nabla_{e_i} (df(Z_k)) = (\nabla_{e_i} df)(Z_k) - \varphi_k df(e_i)
\]
(11)
where
\[ \varphi_k = \langle E_k, \nu \rangle. \]

The stability of the C-stationary map \(f\) implies the inequality
\[ L(df(Z_k), df(Z_k)) \geq 0, \quad \text{hence} \quad \sum_{k=1}^{m+1} L(df(Z_k), df(Z_k)) \geq 0, \]
Lemma 1 (a), (e), we have

We denote terms on the right hand side by $A$, $B$, $C$, $D$ and $E$ respectively. By (11) and Lemma 1 (a), we get

\[
L(df(Z_k), df(Z_k)) = \int_M \sum_{i,j} h(\nabla e_i(df(Z_k)), \nabla e_j(df(Z_k))) T_f(e_i, e_j) \, dv_g
+ \int_M \sum_{i,j} h(\nabla e_i(df(Z_k)), df(e_j)) h(\nabla e_i(df(Z_k)), df(e_j)) \, dv_g
+ \int_M \sum_{i,j} h(\nabla e_i(df(Z_k)), df(e_j)) h(df(e_i), \nabla e_j(df(Z_k))) \, dv_g
- \frac{2}{m} \int_M \sum_i h(\nabla e_i(df(Z_k)), df(e_i)) \sum_j h(\nabla e_j(df(Z_k)), df(e_j)) \, dv_g
+ \int_M \sum_{i,j} h(NR(df(Z_k), df(e_i)) df(Z_k), df(e_j)) T_f(e_i, e_j) \, dv_g.
\]

We denote terms on the right hand side by $A$, $B$, $C$, $D$ and $E$ respectively. By (11) and Lemma 1 (a), (e), we have

\[
A = \int_M \sum_{i,j} h(\nabla e_i(df)(Z_k), (\nabla e_j df)(Z_k)) T_f(e_i, e_j) \, dv_g
- 2 \int_M \phi_k \sum_{i,j} h((\nabla e_i df)(Z_k), df(e_j)) T_f(e_i, e_j) \, dv_g
+ \int_M \phi_k^2 \|T_f\|^2 \, dv_g.
\]

By (11) and Lemma 1 (a), we get

\[
B = \int_M \sum_{i,j} h(\nabla e_i(df)(Z_k), df(e_j)) h((\nabla e_i df)(Z_k), df(e_j)) \, dv_g
- 2 \int_M \phi_k \sum_{i,j} h((\nabla e_i df)(Z_k), df(e_j)) h(df(e_i), df(e_j)) \, dv_g
+ \int_M \phi_k^2 \|f^* h\|^2 \, dv_g,
\]

\[
C = \int_M \sum_{i,j} h((\nabla e_i df)(Z_k), df(e_j)) h(df(e_i), (\nabla e_j df)(Z_k)) \, dv_g
- 2 \int_M \phi_k \sum_{i,j} h((\nabla e_i df)(Z_k), df(e_j)) h(df(e_i), df(e_j)) \, dv_g
+ \int_M \phi_k^2 \|f^* h\|^2 \, dv_g.
\]
and

\begin{align}
\text{(16)} \quad D &= -\frac{2}{m} \int_M \sum_i h((\nabla e_i, df)(Z_k), df(e_i)) \sum_j h((\nabla e_j, df)(Z_k), df(e_j)) \, dv_g \\
&\quad + \frac{4}{m} \int_M \varphi_k \sum_i h((\nabla e_i, df)(Z_k), df(e_i)) \|df\|^2 \, dv_g \\
&\quad - \frac{2}{m} \int_M \varphi_k^2 \|df\|^4 \, dv_g
\end{align}

Then by (12), (13), (14), (15), (16) and Lemma 1 (c), we have

\begin{align}
\text{(17)} \quad L(df(Z_k), df(Z_k)) &= \int_M \sum_{i,j} h((\nabla e_i, df(Z_k)), (\nabla e_j, df(Z_k))) \, T_f(e_i, e_j) \, dv_g \\
&\quad + \int_M \sum_{i,j} h((\nabla e_i, df(Z_k)), df(e_j)) h((\nabla e_i, df(Z_k)), df(e_j)) \, dv_g \\
&\quad + \int_M \sum_{i,j} h((\nabla e_i, df(Z_k)), df(e_j)) h(df(e_i), (\nabla e_j, df(Z_k))) \, dv_g \\
&\quad - \frac{2}{m} \int_M \sum_i h((\nabla e_i, df(Z_k)), df(e_i)) \sum_j h((\nabla e_j, df(Z_k)), df(e_j)) \, dv_g \\
&\quad + \int_M \sum_{i,j} h(\mathcal{N}R(df(Z_k), df(Z_k), df(Z_k), df(Z_k)), df(e_i), df(e_j)) \, T_f(e_i, e_j) \, dv_g \\
&\quad - 6 \int_M \varphi_k \sum_{i,j} h((\nabla e_i, df(Z_k)), df(e_j)) T_f(e_i, e_j) \, dv_g \\
&\quad + 3 \int_M \varphi_k^2 \|T_f\|^2 \, dv_g.
\end{align}

Using the following formula, we replace the curvature term of $N$ by the Ricci curvature term of $M$.

**Lemma 3.** Let $f$ be a smooth map from $M$ into $N$. For any vector fields $X$ and $Y$ on
where \( \{ e_i \} \) denotes an orthonormal frame on \( M \).

**Remark 4.** Using integration by parts, we can verify that the integral of the first term of the right hand side in Lemma 3 over \( M \) vanishes for C-stationary maps \( f \).

**Proof of Lemma 3.** By Lemma 1, We see

\[
(18) \quad \frac{1}{4} \nabla_X \nabla_Y \| T_f \|^2 = \frac{1}{4} \nabla_X \nabla \left( \sum_{i, j} T_f(e_i, e_j) \right) = \sum_{i, j} h((\nabla_X \nabla_Y df)(e_i), df(e_j)) T_f(e_i, e_j) \\
+ \sum_{i, j} h((\nabla_X df)(e_i), (\nabla_Y df)(e_j)) T_f(e_i, e_j) \\
+ \sum_{i, j} h((\nabla_X df)(e_i), df(e_j)) h((\nabla_Y df)(e_i), df(e_j)) \\
+ \sum_{i, j} h((\nabla_Y df)(e_i), df(e_j)) h(df(e_i), (\nabla_X df)(e_j)) \\
- \frac{2}{m} \sum_i h((\nabla_X df)(e_i), df(e_i)) \sum_j h((\nabla_Y df)(e_j), df(e_j)) \\
- \sum_{i, j} h(df(MR(X, e_i)(Y)), df(e_j)) T_f(e_i, e_j) \\
+ \sum_{i, j} h(NR(df(X), df(e_i)) df(Y), df(e_j)) T_f(e_i, e_j).
\]

By the Ricci formula, we have

\[
(19) \quad (\nabla_X \nabla_Y df)(e_i) = (\nabla_X \nabla_{e_i} df)(Y) \\
= (\nabla_{e_i} \nabla_X df)(Y) - df(MR(X, e_i)(Y)) + NR(df(X), df(e_i)) df(Y)
\]

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Furthermore by Lemma 2 we have

\[(21) \quad \sum_{i,j} h((\nabla_{e_i} \nabla_X df)(Y), df(e_j)) T_f(e_i, e_j) = \sum_i h((\nabla_{e_i} \nabla_X df)(Y), \sigma_f(e_i)).\]

By (19), (20) and (21), we have the equality in Lemma 3. □

By Lemma 3 with \(X = Y = Z_k\), we have

\[
\sum_{i,j} h((\nabla_{Z_k} df)(e_i), (\nabla_{Z_k} df)(e_j)) T_f(e_i, e_j) \\
+ \sum_{i,j} h((\nabla_{Z_k} df)(e_i), df(e_j)) h((\nabla_{Z_k} df)(e_i), df(e_j)) \\
+ \sum_{i,j} h((\nabla_{Z_k} df)(e_i), df(e_j)) h(df(e_i), (\nabla_{Z_k} df)(e_j)) \\
- \frac{2}{m} \sum_i h((\nabla_{Z_k} df)(e_i), df(e_i)) \sum_j h((\nabla_{Z_k} df)(e_j), df(e_j)) \\
+ \sum_{i,j} h(df^{(3)} R(df(Z_k), df(e_i)) df(Z_k), df(e_j)) T_f(e_i, e_j). \]

\[
= \frac{1}{4} \nabla_{Z_k} \nabla_{Z_k} \|T_f\|^2 \\
- \sum_i h((\nabla_{e_i} \nabla_{Z_k} df)(Z_k), \sigma_f(e_i)) \\
+ \sum_{i,j} h(df^{(m)} R(Z_k, e_i)(Z_k), df(e_j)) T_f(e_i, e_j) \]

Then by the equalities (17)

\[(22) \quad 0 \leq \sum_{k=1}^{m+1} L(df(Z_k), df(Z_k)) \]

\[
= \frac{1}{4} \int_M \sum_{k=1}^{m+1} \nabla_{Z_k} \nabla_{Z_k} \|T_f\|^2 \, dv_g \\
- \int_M \sum_{k=1}^{m+1} \sum_i h((\nabla_{e_i} \nabla_{Z_k} df)(Z_k), \sigma_f(e_i)) \, dv_g \\
+ \int_M \sum_{k=1}^{m+1} \sum_{i,j} h(df^{(m)} R(Z_k, e_i)(Z_k), df(e_j)) T_f(e_i, e_j) \, dv_g \\
- 6 \int_M \sum_{k=1}^{m+1} \phi_k \sum_{i,j} h((\nabla_{e_i} df)(Z_k), df(e_j)) T_f(e_i, e_j) \, dv_g \\
+ 3 \int_M \sum_{k=1}^{m+1} \phi_k^2 \|T_f\|^2 \, dv_g. \]

We write terms on the right hand side as I, II, III, IV and V respectively. To calculate these terms, we use the following lemma.
Lemma 4.

(a) \( \sum_{k=1}^{m+1} \nabla Z_k \nabla Z_k = \triangle \), where \( \triangle \) denotes the Laplacian on \( M = S^m \).

(b) \( \sum_{k=1}^{m+1} g(e_i, Z_k)Z_k = e_i \).

(c) \( \sum_{k=1}^{m+1} \sum_i h(df(Z_k), df(e_i)) T_f(Z_k, e_i) = ||T_f||^2 \).

Proof of Lemma 4. We first prove that \( \sum_{k=1}^{m+1} \nabla Z_k \nabla Z_k \) does not depend on the choice of \( E_k \). Take any two sets of orthonormal parallel vector fields \( \{E_k\}, \{E'_k\} \) on \( \mathbb{R}^{m+1} \). Then there exists an orthogonal matrix \( (a_{kp}) \) such that \( E_k = \sum_{p=1}^{m+1} a_{kp} E_p \). Hence

\[ Z_k = \sum_{p=1}^{m+1} a_{kp} Z_k \text{, where } Z_k = p(E_k). \]

Thus we have

\[ \sum_{k=1}^{m+1} \nabla Z_k \nabla Z_k = \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} a_{kp} a_{kq} \nabla Z_p \nabla Z_q = \sum_{\ell=1}^{m+1} \nabla Z_\ell \nabla Z_\ell , \]

which implies that this operator does not depend on the choice of \( E_k \). At a point \( x \) on \( S^m \), take orthonormal parallel vector fields \( E_1, \cdots, E_{m+1} \) on \( \mathbb{R}^{m+1} \) such that \( Z_k(x) \) \( (k=1, \cdots, m) \) is an orthonormal basis of \( T_x S^m \) and \( Z_{m+1}(x) = 0 \). Then the equality in (a) holds clearly.

To prove (b) and (c), we only have to prove \( \sum_{k=1}^{m+1} g(X, Z_k)Z_k \) and \( \sum_{k=1}^{m+1} \sum_i h(df(Z_k), df(e_i)) T_f(Z_k, e_i) \) do not depend on the choice of \( E_k \), which we can verify easily. □

By Lemma 4 (a), we have

\[ I = \frac{1}{4} \int_M \sum_{k=1}^{m+1} \nabla Z_k \nabla Z_k ||T_f|| \, dv_g = \frac{1}{4} \int_M \triangle ||T_f|| \, dv_g = 0. \]

Using the integration by parts, we have

\[ \text{II} = \int_M \sum_i h((\nabla e_i \nabla Z_k, df)(Z_k), \sigma_f(e_i)) \, dv_g \]
\[ = - \int_M h((\nabla Z_i, df)(Z_k), \text{div}_g \sigma_f) \, dv_g \]
\[ = 0, \]
since \( f \) is a C-stationary map, i.e., \( \text{div}_g \sigma_f = 0 \).

Considering the fact that \( S^*R(U, V)W = g(V, W)U - g(U, W)V \), we get by Lemma 4 (b)

\[
\text{III} = \int_M \sum_{i,j}^{m+1} h \left( df \left( S^*R(Z_k, e_i)Z_k \right), df(e_j) \right) T_f(e_i, e_j)
\]

\[
= \int_M \sum_{i,j}^{m+1} h \left( df(g(e_i, Z_k)Z_k - g(Z_k, Z_k)e_i), df(e_j) \right) T_f(e_i, e_j)
\]

\[
= - (m - 1) \int_M \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j)
\]

\[
= - (m - 1) \int_M \|T_f\|^2.
\]

We used here

\[
\sum_{k=1}^{m+1} g(e_i, Z_k)Z_k = p \left( \sum_{k=1}^{m+1} g(e_i, E_k)E_k \right) = p(e_i) = e_i.
\]

For the calculation of the term IV, let us define

\[
\gamma_k(X) = h(df(Z_k), \sigma_f(X)).
\]

Then by (11) and Lemma 2 we have

\[
\sum_{i,j} h((\nabla_{e_i} df)(Z_k), df(e_j)) T_f(e_i, e_j)
\]

\[
= \sum_i h((\nabla_{e_i} df)(Z_k), \sigma_f(e_i))
\]

\[
= \sum_i \left\{ h(\nabla_{e_i} df(Z_k), \sigma_f(e_i)) + \varphi_k \sum_i h(df(e_i), \sigma_f(e_i)) \right\}
\]

\[
= \sum_i (\nabla_{e_i} \gamma_k)(e_i) - h(df(Z_k), \sum_i (\nabla_{e_i} \sigma_f)(e_i)) + \varphi_k \sum_i h(df(e_i), \sigma_f(e_i))
\]

\[
= \text{div} \gamma_k + \varphi_k \|T_f\|^2,
\]

since \( \sum_i (\nabla_{e_i} \sigma_f)(e_i) = \text{div} \sigma_f = 0 \). Hence we have

\[
(23) \quad \sum_{k=1}^{m+1} \varphi_k \sum_{i,j}^{m+1} h((\nabla_{e_i} df)(Z_k), df(e_j)) T_f(e_i, e_j)
\]

\[
= \sum_{k=1}^{m+1} \varphi_k \text{div} \gamma_k + \sum_{k=1}^{m+1} \varphi_k^2 \|T_f\|^2
\]

\[
= \text{div} \left( \sum_{k=1}^{m+1} \varphi_k \gamma_k \right) - \sum_{k=1}^{m+1} \sum_i e_i(\varphi_k) \gamma_k(e_i) + \sum_{k=1}^{m+1} \varphi_k^2 \|T_f\|^2.
\]
We see
\[ \sum_i e_i(\varphi_k) e_i = \sum_i (\langle E_k, \nu \rangle) e_i = \sum_i \langle E_k, e_i \rangle = p(E_k) = Z_k, \]
since \( E \) is parallel and \( \mathbb{R}^{n+1} \nabla_{e_i} \nu = e_i \) where \( \mathbb{R}^{n+1} \nabla \) denotes the standard connection on \( \mathbb{R}^{n+1} \). Hence by Lemma 4 (c), we have
\[
\begin{align*}
\sum_{k=1}^{m+1} \sum_i e_i(\varphi_k) \gamma_k(e_i) &= \sum_{k=1}^{m+1} \gamma_k(\sum_i e_i(\varphi_k) e_i) \\
&= \sum_{k=1}^{m+1} \gamma_k(Z_k) = \sum_{k=1}^{m+1} h(df(Z_k), \sigma_f(Z_k)) \\
&= \sum_{k=1}^{m+1} \sum_i h(df(Z_k), df(e_i)) T_f(Z_k, e_j) = \|T_f\|^2.
\end{align*}
\]

We see
\[
\begin{align*}
\sum_{k=1}^{m+1} \varphi_k^2 &= \sum_{k=1}^{m+1} g(E_k, \nu)^2 = g(\nu, \sum_{k=1}^{m+1} g(\nu, E_k) E_k) = g(\nu, \nu) = 1.
\end{align*}
\]

Then by (23), (24) and (25), we obtain IV = 0. By (25), we have
\[
V = 3 \int_M \|T_f\|^2 dv_g.
\]

Finally substituting I through V into (22), we have
\[
0 \leq \sum_{k=1}^{m+1} L(df(Z_k), df(Z_k)) \\
= -(m-1) \int_M \|T_f\|^2 dv_g + 3 \int_M \|T_f\|^2 dv_g \\
= (4 - m) \int_M \|T_f\|^2 dv_g,
\]
i.e.,
\[
(m - 4) \int_M \|T_f\|^2 dv_g \leq 0.
\]

Hence, if \( m \geq 5 \), then we get \( \|T_f\| = 0 \), i.e. \( f \) is a weakly conformal map. \( \square \)

5 Stable maps into spheres

In this section we prove Theorem 2.

Proof of Theorem 2. We use notations similar to those in the proof of Theorem 1. Since the standard sphere \( S^n \) is a submanifold of the Euclidean space \( \mathbb{R}^{n+1} \), we may
consider that the tangent spaces of standard sphere $S^n$ at $y \in S^n$ is a subspace of the linear space $\mathbb{R}^{n+1} \simeq T_y\mathbb{R}^{n+1}$. Let $p = p_y$ denotes the canonical projection from $\mathbb{R}^{n+1}$ onto the tangent space $T_yS^n$.

Let $E$ be normal parallel vector field on $\mathbb{R}^{n+1}$, and let $Z$ be the vector field which is the image by the projection $p$ of $E$. Then we define a smooth section $W$ of the pull-back bundle $f^{-1}T\mathbb{S}^n$ by

$$W_x := Z_{f(x)} = p_{f(x)}(E_{f(x)})$$

for $x \in M$. Then we see

$$(26) \quad \nabla^{e_i}W = f^{-1}T\mathbb{S}^n \nabla^{e_i}W = S^n \nabla^{d f(e_i)}Z = -\varphi d f(e_i),$$

where

$$\varphi = h(E, \nu)$$

and $\nu = \nu_x$ is the unit outer normal vector at $f(x)$. (Note that $\nu_x = f(x)$ in the case of $N = S^n$.)

For simplicity, we use the notation $Z$ instead of $W$, and then by (26) we have

$$(27) \quad \nabla^{e_i}Z = -h(E, \nu) d f(e_i).$$
Then by (27), we have

\[
L(Z, Z) = \int_M \sum_{i,j} h(\nabla e_i Z, \nabla e_j Z) T_f(e_i, e_j) \, dv_g \\
+ \int_M \sum_{i,j} h(\nabla e_i Z, df(e_j))^2 \, dv_g \\
+ \int_M \sum_{i,j} h(\nabla e_i Z, df(e_j)) h(df(e_i), \nabla e_j Z) \, dv_g \\
- \frac{2}{m} \int_M \left\{ \sum_i h(\nabla e_i Z, df(e_i)) \right\}^2 \, dv_g \\
+ \int_M \sum_{i,j} h(S_n R(Z, df(e_i)) Z, df(e_j)) T_f(e_i, e_j) \, dv_g
\]

\[
= 3 \int_M h(E, \nu)^2 \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j) \, dv_g \\
- \int_M h(Z, Z) \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j) \, dv_g \\
+ \int_M \sum_{i,j} h(Z, df(e_i)) h(Z, df(e_j)) T_f(e_i, e_j) \, dv_g
\]

\[
= 3 \int_M h(E, \nu)^2 ||T_f||^2 \, dv_g \\
- \int_M h(Z, Z)||T_f||^2 \, dv_g \\
+ \int_M \sum_{i,j} h(Z, df(e_i)) h(Z, df(e_j)) T_f(e_i, e_j) \, dv_g.
\]

Take orthonormal parallel vector fields \(E_1, \cdots, E_{n+1}\) on \(\mathbb{R}^{n+1}\). The stability of the C-stationary map \(f\) implies the inequality

\[
L(Z_k, Z_k) \geq 0, \text{ hence } \sum_{k=1}^{n+1} L(Z_k, Z_k) \geq 0.
\]

Then by (28), we have

\[
0 \leq \sum_{k=1}^{n+1} L(Z_k, Z_k)
\]

\[
= 3 \int_M \sum_{k=1}^{n+1} h(E_k, \nu)^2 ||T_f||^2 \, dv_g \\
- \int_M \sum_{k=1}^{n+1} h(Z_k, Z_k)||T_f||^2 \, dv_g \\
+ \int_M \sum_{i,j} \sum_{k=1}^{n+1} h(Z_k, df(e_i)) h(Z_k, df(e_j)) T_f(e_i, e_j) \, dv_g.
\]
To calculate $L(Z_k, Z_k)$, we first give the following lemma.

**Lemma 5.** \[ \sum_{k=1}^{n+1} h(Z_k, Z_k) = n \]

**Proof of Lemma 5.** We first prove that \[ \sum_{k=1}^{n+1} h(Z_k, Z_k) \] does not depend on the choice of $E_k$. Take any two sets of orthonormal parallel vector fields \( \{E_k\}, \{\overline{E}_k\} \) on \( \mathbb{R}^{n+1} \). Then there exists an orthogonal matrix \((a_{kp})\) such that \[ Z_k = \sum_{p=1}^{n+1} a_{kp} \overline{Z}_p. \] Thus we have

\[ \sum_{k=1}^{n+1} h(Z_k, Z_k) = \sum_{p=1}^{n+1} \sum_{q=1}^{n+1} a_{kp} a_{kq} h(\overline{Z}_p, \overline{Z}_q) = \sum_{\ell=1}^{n+1} h(\overline{Z}_\ell, \overline{Z}_\ell) \]

which implies that this quantity does not depend on the choice of $E_k$. At a point $x$ on $S^m$, take orthonormal parallel frame \( \{E_1, \cdots, E_{n+1}\} \) on \( \mathbb{R}^{n+1} \) such that \( \{Z_1(x), \cdots, Z_m(x)\} \) is an orthonormal base of $T_xS^m$ and $Z_{n+1}(x) = 0$, and then we have Lemma 5. □

Since $Z_k$ is the projection of $E_k$ onto the tangent space of the sphere $S^n$, we have

\[ \sum_{k=1}^{n+1} h(E_k, \nu)^2 = h(\nu, \nu) = 1 \]

and

\[ \sum_{k=1}^{n+1} \sum_{i,j} h(Z_k, df(e_i)) h(Z_k, df(e_j)) T_f(e_i, e_j) \]

\[ = \sum_{i,j} h(df(e_i), \sum_{k=1}^{n+1} h(df(e_j), Z_k) T_f(e_i, e_j) \]

\[ = \sum_{i,j} h(df(e_i), df(e_j)) T_f(e_i, e_j) = \|T_f\|^2. \]

Therefore we get by Lemma 5 and (29)

\[ 0 \leq \sum_{k=1}^{n+1} L(Z_k, Z_k) \]

\[ = 3 \int_M \|T_f\|^2 dv_g - n \int_M \|T_f\|^2 dv_g + \int_M \|T_f\|^2 dv_g \]

\[ = (4 - n) \int_M \|T_f\|^2 dv_g. \]

Thus we obtain

\[ (n - 4) \int_M \|T_f\|^2 \leq 0. \]

Hence, if $n \geq 5$, then we get $\|T_f\| = 0$, i.e. $f$ is a weakly conformal map. □
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Appendix

In this appendix, we give, for reader’s convenience, proofs of the first variation formula (Proposition 1) and the second variation formula (Proposition 2).

Proof of Proposition 1. We calculate \( \frac{\partial}{\partial t} \| f^*_t h \|^2 \) at any fixed point \( x_0 \in M \). The connection \( \nabla \) is trivially extended to a connection on \( (-\varepsilon, \varepsilon) \times M \). The frame \( e_i \) is also trivially extended to a frame on \( (-\varepsilon, \varepsilon) \times M \). Using a normal coordinate at \( x_0 \), we can assume \( \nabla_{e_i} e_j = 0 \) at \( x_0 \) for any \( i, j \). Since \( (dF)_{(t, x)}((e_i)_{(t, x)}) = (dF)_x((e_i)_x) \), we denote them by \( dF(e_i) \) simply. Note

\[
\nabla \frac{\partial}{\partial t} (dF(e_i)) = \nabla_{e_i} (dF(\frac{\partial}{\partial t})) ,
\]

since \([\frac{\partial}{\partial t}, e_i] = 0\). Then using Lemma 1 (a) and (d), we have

\[
\frac{\partial}{\partial t} \| T_t \|^2 = \frac{\partial}{\partial t} \sum_{i, j} T_F(e_i, e_j)^2
\]

\[
= 2 \sum_{i, j} \frac{\partial T_F(e_i, e_j)}{\partial t} T_F(e_i, e_j)
\]

\[
= 2 \sum_{i, j} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) - \frac{1}{m} \frac{\partial \|dF\|^2}{\partial t} g(e_i, e_j) \right\} T_F(e_i, e_j)
\]

\[
= 2 \sum_{i, j} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j) - \frac{2}{m} \frac{\partial \|dF\|^2}{\partial t} \sum_{i, j} g(e_i, e_j) T_F(e_i, e_j)
\]

\[
= 2 \sum_{i, j} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j)
\]

\[
= 4 \sum_{i, j} h(\nabla \frac{\partial}{\partial t} (dF(e_i)), dF(e_j)) T_F(e_i, e_j)
\]

\[
= 4 \sum_{i, j} h(\nabla_{e_i} (dF(\frac{\partial}{\partial t})), dF(e_j)) T_F(e_i, e_j)
\]

\[
= 4 \sum_{i} h(\nabla_{e_i} (dF(\frac{\partial}{\partial t})), \sigma_F(e_i)).
\]

The last equality follows from Lemma 2 for \( Z = \nabla_{e_i} (dF(\frac{\partial}{\partial t})) \) and for \( W = e_i \). Integrate it over \( M \) and let \( t = 0 \). Then using integration by parts, we obtain the first variation formula. \( \square \)

Proof of Proposition 2. We calculate \( \frac{\partial^2}{\partial s \partial t} \| f^*_s h \|^2 \) at any fixed point \( x_0 \in M \). The connection \( \nabla \) is trivially extended to a connection on \( (-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M \). We use the same notation \( \nabla \) for this connections. The frame \( e_i \) is also trivially extended to a frame on \( (-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M \). Integrating this formula over \( M \) and using integration by parts, we obtain the second variation formula.
Then we see
\[
\nabla \frac{\partial}{\partial s} e_i = \nabla e_i \frac{\partial}{\partial s} = 0,
\]
\[
\nabla \frac{\partial}{\partial t} e_i = \nabla e_i \frac{\partial}{\partial t} = 0,
\]
\[
\nabla \frac{\partial}{\partial s} \frac{\partial}{\partial t} = \nabla \frac{\partial}{\partial t} \frac{\partial}{\partial s} = 0
\]
on \((-\varepsilon, \varepsilon) \times (-\delta, \delta) \times M\). Take and fix any point \(x_0 \in M\). We calculate \(\frac{\partial}{\partial t} \| f^* h \|^2 \) at \(x = x_0\). Since we can assume \(\nabla e_i e_j = 0\) at \(x_0\) for any \(i, j\), we see at \(x_0\),
\[
\nabla \frac{\partial}{\partial s} (dF(e_i)) = \nabla e_i (dF(\frac{\partial}{\partial s})) ,
\]
\[
\nabla \frac{\partial}{\partial t} (dF(e_i)) = \nabla e_i (dF(\frac{\partial}{\partial t})) .
\]

Then we have
\[
(31) \quad \frac{1}{4} \frac{\partial^2}{\partial s \partial t} \| T_{f^*} h \|^2
\]
\[
= \frac{1}{4} \frac{\partial^2}{\partial s \partial t} \sum_{i,j} T_F(e_i, e_j)^2
\]
\[
= \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} (T_F(e_i, e_j)) T_F(e_i, e_j) \right\} + \frac{1}{2} \sum_{i,j} \frac{\partial T_F(e_i, e_j)}{\partial s} \frac{\partial T_F(e_i, e_j)}{\partial t}
\]
\[
\quad \overset{\text{def}}{=} I_1 + I_2
\]

We have
\[
(32) \quad I_1 = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial s \partial t} \left\{ h(dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j)
\]
\[
= \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j) - \frac{1}{m} \| dF \|^2 g(e_i, e_j) T_F(e_i, e_j)
\]
\[
= \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j) \quad (\text{by Lemma 1 (d)})
\]
\[
= \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial^2}{\partial s \partial t} h(dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j)
\]
\[
= \sum_{i,j} \left\{ h(\nabla \frac{\partial}{\partial s} dF(e_i), dF(e_j)) \right\} T_F(e_i, e_j)
\]
\[
+ \sum_{i,j} \left\{ h(\nabla \frac{\partial}{\partial t} dF(e_i), \nabla \frac{\partial}{\partial s} dF(e_j)) \right\} T_F(e_i, e_j)
\]

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We get

\begin{equation}
(33) \quad \nabla \frac{\partial}{\partial s} \nabla \frac{\partial}{\partial t} (dF(e_i)) \quad = \quad (\nabla \frac{\partial}{\partial s} \nabla \frac{\partial}{\partial t} dF(e_i)) = \quad (\nabla \frac{\partial}{\partial s} \nabla e_i, dF) \quad (\frac{\partial}{\partial t}) \\
\quad = \quad (\nabla e_i, \nabla \frac{\partial}{\partial s} dF) \quad (\frac{\partial}{\partial t}) \quad - \quad N_R (dF(e_i), dF(\frac{\partial}{\partial s})) dF(\frac{\partial}{\partial t})
\end{equation}

\( \therefore \quad \nabla \frac{\partial}{\partial s} \frac{\partial}{\partial t} = \nabla e_i \frac{\partial}{\partial t} = 0 \)

\( \quad = \quad \nabla e_i, \text{Hess}_F (\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) - N_R (dF(e_i), dF(\frac{\partial}{\partial s})) dF(\frac{\partial}{\partial t}) \).

Then by (32) and (33), we have

\begin{equation}
(34) \quad I_1 = \sum_{i,j} h(\nabla e_i, \text{Hess}_F (\frac{\partial}{\partial s}, \frac{\partial}{\partial t}), \sigma_F(e_j)) \\
- \sum_{i,j} h (N_R (dF(e_i), X) Y, dF(e_j)) T_F(e_i, e_j) \\
+ \sum_{i,j} h(\nabla e_i (dF(\frac{\partial}{\partial s})), \nabla e_j (dF(\frac{\partial}{\partial t}))) T_F(e_i, e_j)
\end{equation}

In the last equality, we used Lemma 2 for \( Z = \nabla e_i, \text{Hess}_F (\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) \) and \( W = e_i \). On the other hand since

\[ \sum_{i,j} g(e_i, e_j) \frac{\partial T_F(e_i, e_j)}{\partial t} = \frac{\partial}{\partial t} \left( \sum_{i,j} g(e_i, e_j) T_F(e_i, e_j) \right) = 0 \]

by Lemma 1 (d) and

\[ \frac{\partial}{\partial t} \frac{\|dF\|^2}{\partial t} = \frac{\partial}{\partial t} \sum_j h(dF(e_j), dF(e_j)) = \sum_j \frac{\partial}{\partial t} h(dF(e_j), dF(e_j)), \]
we have

\[ I_2 = \frac{1}{2} \sum_{i,j} \frac{\partial}{\partial s} \left\{ h(dF(e_i), dF(e_j)) - \frac{1}{m} \|dF\|^2 g(e_i, e_j) \right\} \frac{\partial T_F(e_i, e_j)}{\partial t} \]

\[ = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} \left\{ h(dF(e_i), dF(e_j)) - \frac{1}{m} \|dF\|^2 g(e_i, e_j) \right\} \right\} \]

\[ - \frac{1}{2m} \sum_{i,j} \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \sum_{i,j} g(e_i, e_j) \frac{\partial}{\partial t} \frac{\partial T_F(e_i, e_j)}{\partial t} \]

\[ = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) - \frac{1}{m} \frac{\partial}{\partial t} \|dF\|^2 g(e_i, e_j) \right\} \]

\[ = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \]

\[ - \frac{1}{2m} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \sum_{i,j} g(e_i, e_j) \frac{\partial}{\partial t} \frac{\partial T_F(e_i, e_j)}{\partial t} \]

\[ = \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \]

\[ - \frac{1}{2m} \sum_{i,j} \left\{ \frac{\partial}{\partial s} h(dF(e_i), dF(e_j)) \right\} \sum_{i,j} \left\{ \frac{\partial}{\partial t} h(dF(e_i), dF(e_j)) \right\} \]

\[ = : I_3 + I_4 \]

We have

\[ I_3 = \frac{1}{2} \sum_{i,j} \left\{ h(\nabla \frac{\partial}{\partial s} (dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla \frac{\partial}{\partial s} (dF(e_j))) \right\} \]

\[ \times \left\{ h(\nabla \frac{\partial}{\partial t} (dF(e_i)), dF(e_j)) + h(dF(e_i), \nabla \frac{\partial}{\partial t} (dF(e_j))) \right\} \]

\[ = \sum_{i,j} h(\nabla \frac{\partial}{\partial s} (dF(e_i)), dF(e_j)) h(\nabla \frac{\partial}{\partial t} (dF(e_i)), dF(e_j)) \]

\[ + \sum_{i,j} h(\nabla \frac{\partial}{\partial s} (dF(e_i)), dF(e_j)) h(dF(e_i), \nabla \frac{\partial}{\partial t} (dF(e_j))) \]

\[ = \sum_{i,j} h(\nabla e_i (dF(\frac{\partial}{\partial s})), dF(e_j)) h(\nabla e_i (dF(\frac{\partial}{\partial t})), dF(e_j)) \]

\[ + 4 \sum_{i,j} h(\nabla e_i (dF(\frac{\partial}{\partial s})), dF(e_j)) h(dF(e_i), \nabla e_i (dF(\frac{\partial}{\partial s}))). \]
We see

\[ I_4 = -\frac{2}{m} \sum_i h(\nabla \frac{\partial}{\partial s} (dF(e_i)), dF(e_i)) \sum_j h(\nabla \frac{\partial}{\partial t} (dF(e_j)), dF(e_j)) \]

\[ = -\frac{2}{m} \sum_i h(e_i, (dF(\frac{\partial}{\partial s}))), dF(e_i)) \sum_j h(e_j, (dF(\frac{\partial}{\partial t}))). \]

Then by (34), (35), (36) and (37), we have

\[ \frac{1}{4} \frac{\partial^2 \Phi(f_{s,t})}{\partial s \partial t} \bigg| _{s,t=0} = \int_M h(\text{Hess}_F(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}), \text{div}_g \sigma_f) dv_g + \int_M \sum_{i,j} h(\nabla e_i, (dF(\frac{\partial}{\partial s}))) \nabla e_j, (dF(\frac{\partial}{\partial t}))) T_f(e_i, e_j) dv_g \]

\[ + \int_M \sum_{i,j} h(\nabla e_i, (dF(\frac{\partial}{\partial s})), df(e_j)) h(\nabla e_j, (dF(\frac{\partial}{\partial t})), df(e_j)) dv_g \]

\[ + \int_M \sum_{i,j} h(\nabla e_i, (dF(\frac{\partial}{\partial s})), df(e_j)) h(\nabla e_j, (dF(\frac{\partial}{\partial t})), df(e_j)) dv_g \]

\[ -\frac{2}{m} \int_M \sum_i h(\nabla e_i, (dF(\frac{\partial}{\partial s})), df(e_i)) \sum_j h(\nabla e_j, (dF(\frac{\partial}{\partial t})), df(e_j)) dv_g \]

\[ + \int_M \sum_{i,j} h(NR (dF(\frac{\partial}{\partial s})), df(e_i)) dF(\frac{\partial}{\partial t}), df(e_j)) T_f(e_i, e_j) dv_g. \]

Integrate it over $M$ and let $t = 0$. Then using the integration by parts, we obtain the second variation formula. □