Abstract. We obtain a necessary and sufficient condition for a composition of two simple rotations in $E^4$ to be a simple rotation. Geometric parameters of the composition of the rotations are calculated and the analogous to the Rodrigies formula in $E^3$ is achieved in $E^4$.

1. Introduction

In this note we are studying a composition of simple rotations in the Euclidean space $E^4$, the four dimensional real vector space with the standard scalar product.

There are several significant differences between rotations in $E^4$ and in the three dimensional space. The group of rotations in $E^3$ essentially comprises one type, that is, a rotation about an axis, while in $E^4$ there are three types of rotations: (i) A simple rotation, it leaves a plane (two dimensional subspace of $E^4$) point-wise fixed and induces a two dimensional rotation in the orthogonal plane; (ii) Clifford translation (isoclinic rotation), here each vector in $E^4$ turns through the same angle; (iii) A double rotation, here $E^4$ is decompose into two orthogonal planes and points in the first plane rotate through an angle $\alpha$, while points in the second plane rotate through an angle, $\beta \neq \alpha$.

The quaternionic representation of rotations in $E^4$ was discovered by Cayley in 1855 [1]. Many years later, in 1945, Coxeter gave a very elegant treatment of rotations in $E^4$ by means of reflections and quaternions [3]. Using his approach we explore the composition of simple rotations in $E^4$ and achieve analytical and geometrical characterizations of the subgroup of simple rotations.

The known Rodrigies formula provides the information of the axis and angle of the composition of two rotations in $E^3$ (see e.g. [4] p.58, 71)). We obtain formulas that allow to calculate the orthogonal planes and angles of the composed rotation in terms of the corresponding
characteristics of each one of the rotations in \( \mathbb{E}^4 \). However, while in \( \mathbb{E}^3 \) the Rodrigues formula requires one Gibbs vector, the situation in \( \mathbb{E}^4 \) is more complicated and two Gibbs vectors are needed in order to obtain the geometrical characterization of composed rotation.

When the paper was written the authors were not aware that in 1890 Cole explored the subgroup of simple rotations in \( \mathbb{E}^4 \) \cite{2}. The present paper has the same main result as Cole, namely, the composition of two simple rotations is simple if and only if the fixed–point planes contain a common line. However, the methods and the analytical characterizations are different. Cole essentially worked out Cayley’s parameters of the subgroup and this method demands laborious computations. He also obtained formulas for the orthogonal planes and the angle of the composed rotation, but they are complicated and lengthy, and inapplicable in a particular example.

The outline of the paper is as follows. In the next section we shall give a brief account of Coxeter’s approach. Section 3 deals with the compositions and ends with an example.

2. Representation of reflections and rotations by quaternions

We first fix the notations. A quaternion \( x \) is expressed as
\[
x = x_0 + x_1i + x_2j + x_3k \equiv Sx + Vx,
\]
where \( Sx = x_0 \) is called the real part, \( Vx = x_1i + x_2j + x_3k \) is the vector part, and the coefficients \( x_0, x_1, x_2 \) and \( x_4 \) are real numbers. The quaternions multiplication is defined by the table
\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]
Since a quaternion is completely defined by its four real components, it determines a point (vector) in the four dimensional real space \( \mathbb{R}^4 \). Similarly, \( Vx = x_1i + x_2j + x_3k \) can be identified with a point (vector) in three dimensional space \( \mathbb{R}^3 \). The multiplication formula
\[
xy = SxSy - Vx \cdot Vy + SxVy + SyVx + Vx \times Vy,
\]
is a consequence of the distributive property the quaternionic algebra. Here \( Vx \cdot Vy \) and \( Vx \times Vy \) are the scalar cross products in \( \mathbb{E}^3 \) respectively.

Let \( x \) be a quaternion, then \( \bar{x} = Sx - Vx \) is the conjugate quaternion, and \( Nx = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 \) is the square of the norm. Any unit quaternion \( a \) can be written in a polar form \( a = \cos \alpha + p \sin \alpha \), where \( p = p_1i + p_2j + p_3k \). The vector \( p \) is called a pure quaternion and we consider it both as a quaternion and as a vector \( (p_1, p_2, p_3) \) in \( \mathbb{E}^3 \), depending on the circumstances.
We recall now the quaternionic representation of reflections and rotations in $\mathbb{E}^4$.

**Reflections:** Notice that any two quaternions $y$ and $x$ are orthogonal if and only if $xy + yx = 0$. Hence if $Ny = 1$, then the transformation

$$x \rightarrow -yxy$$

is a reflection with respect to a hyperplane having $y$ as its normal (see [3, §5]).

**Simple rotations:** A rotation in the plane through the angle $\alpha$ and all the points in the orthogonal plane remain fixed. Its representation is given by

$$x \rightarrow axb, \quad Na = Nb = 1, \quad Sa = Sb = \cos \frac{\alpha}{2}.$$  

It is known that any rotation in $\mathbb{E}^4$ can be decomposed to two reflections (see e.g. [6]). Furthermore, it is shown in [3] there are two unit quaternions $y$ and $z$ such that $a = z\bar{y}, b = \bar{y}z$, and the simple rotation (2) is the product of the reflections $x \rightarrow -y\bar{xy}$ and $x \rightarrow -\bar{zx}z$, that is,

$$x \rightarrow axb = (z\bar{y})x(\bar{y}z).$$

Obviously, $x \rightarrow (z\bar{y})x(\bar{y}z) = x$, whenever $x$ is orthogonal both to $y$ and $z$. Hence, the point–wise invariant plane of the simple rotation (3) is the intersection of the hyperplanes

$$\sum_{\mu=0}^{3} x_{\mu}y_{\mu} = \sum_{\mu=0}^{3} x_{\mu}z_{\mu} = 0.$$

**Clifford translations:** Each vector in $\mathbb{E}^4$ turns through the same angle. Let $a$ be a unit quaternion, then $x \rightarrow ax$ is a left Clifford translation and $x \rightarrow xa$ is a right Clifford translation. The angle $\alpha$ of both types satisfies $\cos \alpha = Sa$.

**Double rotations:** Two orthogonal planes rotate simultaneously with two different angles. Let $p$ and $q$ be two pure quaternions such that $p^2 = q^2 = -1$, then a double rotation has the representation

$$x \rightarrow (\cos \alpha + p \sin \alpha)x(\cos \beta + q \sin \beta) =: f(x).$$

The rotations’ angles are $\alpha \pm \beta$. 
The planes of the double and simple rotations: Coxeter proved that the orthogonal planes of the rotation (4) are

\[ \Pi_1 = \text{Span} \{ p - q, 1 + pq \} =: \{ s_1(p - q) + s_2(1 + pq), s_1, s_2 \in \mathbb{R} \} \]

and

\[ \Pi_2 = \text{Span} \{ p + q, 1 - pq \} . \]

Points in the plane \( \Pi_1 \) rotate through the angle \( \alpha + \beta \) and in the plane \( \Pi_2 \) through the angle \( \alpha - \beta \), see [3, §9]. It follows that if \( \beta = \alpha \), then (4) is a simple rotation with the points-wise invariant plane \( \Pi_2 \), while \( \Pi_1 \) is the fixed-plane when \( \beta = -\alpha \).

If one of the the quaternions \( p \mp q, 1 \pm q \) is zero, then Coxeter’s characterization of the invariant planes is not valid and we have to treat these cases separately. First of all note that the equalities \( p = \pm q \) and \( pq = \mp 1 \) are equivalent and, in fact, there are only two special cases: \( q = \pm p \). It is readily follows from formula (4) that in these cases

\[ f(1) \in \text{Span} \{ 1, p \} \quad \text{and} \quad f(p) \in \text{Span} \{ 1, p \} . \]

Therefore \( \Lambda_1 = \text{Span} \{ 1, p \} \) is one of the rotating planes. Since the second one \( \Lambda_2 \), is the orthogonal complement of \( \Lambda_1 \), we conclude that \( \Lambda_2 = \{ x : p \cdot x = 0, Sx = 0 \} \).

The pointwise invariant planes of simple rotations when \( q = \pm p \) can be found by the following considerations. If \( q = p \), then by \( f(1) \neq 1 \), and therefore \( \Lambda_2 \) is the fixed plane. While if \( q = -p \), then \( f(1) = 1 \), and hence \( \Lambda_1 \) is the fixed plane.

3. Composition of simple rotations in \( \mathbb{E}^4 \)

Let \( f : x \rightarrow axb \) and \( g : x \rightarrow cxd \) be two simple rotations and consider the composition \( g \circ f : x \rightarrow (ac)x(bd) \). We ask under which conditions \( g \circ f \) will be again a simple rotation.

It follows from (2) that the rotation \( g \circ f \) is simple if and only if \( S(ca) = S(bd) \). According to the multiplication formula (1), this condition is equivalent to

\[ ScSa - Vc \cdot Va = SbSd - Vb \cdot Vc. \]

Since both \( f \) and \( g \) are simple rotations, \( S(a) = S(b) \) and \( S(c) = S(d) \), hence the above condition results in

\[ Vc \cdot Va - Vb \cdot Vd = 0. \]

We now recall that each rotation is decomposed into two reflections by formula (3). Thus there exists four unit quaternions \( y, z, u, w \) such that

\[ a = z\bar{y}, b = \bar{y}z, c = w\bar{u}, d = \bar{u}w. \]
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Hence
$$V_c \cdot V a = V(w\bar{u}) \cdot V(z\bar{y})$$
$$= [-w_0Vu + u_0Vw + Vu \times Vw] \cdot [-z_0Vy + y_0Vz + Vy \times Vz]$$
and
$$V b \cdot V d = V(\bar{yz}) \cdot V(\bar{uw})$$
$$= [y_0Vz - z_0Vy - Vy \times Vz] \cdot [u_0Vw - w_0Vu - Vu \times Vw].$$

Thus (5) becomes
$$V c \cdot V a - V b \cdot V d = -2 \{y_0V z \cdot (V u \times V w) - z_0V y \cdot (V u \times V w)$$
$$+ u_0V w \cdot (V y \times V z) - w_0V u \cdot (V y \times V z)\}$$
$$= -2 \text{det}[y, z, u, w] = 0,$$
where $[y, z, u, w]$ denote the matrix with column vectors $y, z, u$ and $w$. These vectors are the normals of the hyperplanes of the reflections that generate the rotations. The point-wise invariant planes of the rotations $f$ and $g$ are
$$\Pi_f := \{x : \sum_{\mu=0}^{3} y_\mu x_\mu = \sum_{\mu=0}^{3} z_\mu x_\mu = 0\},$$
$$\Pi_g := \{x : \sum_{\mu=0}^{3} u_\mu x_\mu = \sum_{\mu=0}^{3} w_\mu x_\mu = 0\}$$
respectively. Obviously the intersection $\Pi_f \cap \Pi_g$ contains a non–zero vector if and only if $\text{det}[y, z, u, w] = 0$. Thus we have obtained the following theorem.

**Theorem.** Let $f : x \rightarrow axb$ and $g : x \rightarrow cxd$ be two simple rotations, and $\Pi_f$ and $\Pi_g$ be the point-wise invariant planes of the rotations $f$ and $g$ respectively. Then the composition $h = g \circ f$ is a simple rotation if and only if $\Pi_f \cap \Pi_g \neq \{0\}$.

We turn now to the calculations of the angle and the planes of the composed rotation, under the assumption that it is a simple rotation. In order to do it we shall represent the unit quaternions $a, b, \ldots$ in a polar form, that is,
$$f : x \rightarrow (\cos \alpha_1 + p_1 \sin \alpha_1)x(\cos \alpha_1 + q_1 \sin \alpha_1)$$
$$g : x \rightarrow (\cos \alpha_2 + p_2 \sin \alpha_2)x(\cos \alpha_2 + q_2 \sin \alpha_2),$$
and the composition
$$h = g \circ f : x \rightarrow (\cos \alpha + p \sin \alpha)x(\cos \alpha + q \sin \alpha),$$
where \((\cos \alpha + p \sin \alpha) = (\cos \alpha_1 + p_1 \sin \alpha_1) (\cos \alpha_2 + p_2 \sin \alpha_2)\), \((\cos \alpha + q \sin \alpha) = (\cos \alpha_1 + q_1 \sin \alpha_1) (\cos \alpha_2 + q_2 \sin \alpha_2)\) and \(p_1, q_1, \ldots\) are unit pure quaternions. In accordance to the multiplication formula \([1]\), we get that
\[
\cos \alpha = \cos \alpha_1 \cos \alpha_2 - (p_1 \cdot p_2) \sin \alpha_1 \sin \alpha_2
\]
and
\[
(7) \quad p \sin \alpha = p_2 \sin \alpha_2 \cos \alpha_1 + p_1 \sin \alpha_1 \cos \alpha_2 + (p_2 \times p_1) \sin \alpha_2 \sin \alpha_1.
\]
Note that \(\cos \alpha = 0\) only when \(p_1 = \pm p_2\), \(q_1 = \pm q_2\) and \(\cos(\alpha_1 \pm \alpha_2) = 0\), then \(h(x) = (\pm p_1) x (\pm q_1)\). Otherwise we may divide \(7\) by \(\cos \alpha\) and get that
\[
p \tan \alpha = \frac{p_2 \tan \alpha_2 + p_1 \tan \alpha_1 + (p_2 \times p_1) \tan \alpha_2 \tan \alpha_1}{1 - (p_1 \cdot p_2) \tan \alpha_1 \tan \alpha_2},
\]
and in a similar manner
\[
q \tan \alpha = \frac{q_1 \tan \alpha_1 + q_2 \tan \alpha_2 + (q_1 \times q_2) \tan \alpha_1 \tan \alpha_2}{1 - (q_1 \cdot q_2) \tan \alpha_1 \tan \alpha_2}.
\]
Setting \(p = \tilde{p} \cot \alpha, \quad q = \tilde{q} \cot \alpha\) etc. and using Coxeter’s calculations of the axises of the rotations’ planes \([3, \S 9]\), we conclude:

**Conclusion.** If the composed rotation \(h\) is simple, then the angle is equal to \(2\alpha\), where
\[
(8) \quad \cos \alpha = \cos \alpha_1 \cos \alpha_2 - (p_1 \cdot p_2) \sin \alpha_1 \sin \alpha_2,
\]
and the point-wise invariant plane of \(h\) is spanned by the quaternions \(\tilde{p} - \tilde{q}, 1 + \tilde{p} \tilde{q}\), where
\[
(9) \quad \tilde{p} = \frac{\tilde{p}_2 + \tilde{p}_1 + \tilde{p}_2 \times \tilde{p}_1}{1 - \tilde{p}_2 \cdot \tilde{p}_1}, \quad \tilde{q} = \frac{\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_1 \times \tilde{q}_2}{1 - \tilde{q}_1 \cdot \tilde{q}_2}.
\]

Formulas \([9]\) were obtained by Fedorov, but in a matrices form \([4]\). They resemble the known Rodrigues formula for the composition of rotations in \(\mathbb{E}^3\) (see e.g. \([5\) p. 58, 71\]): Suppose \(x \rightarrow (\cos \alpha_1 + c_1 \sin \alpha_1) x (\cos \alpha_1 + c_1 \sin \alpha_1)\) and \(x \rightarrow (\cos \alpha_2 + c_2 \sin \alpha_2) x (\cos \alpha_2 + c_2 \sin \alpha_2)\) are two rotations in \(\mathbb{E}^3\), then the axis of the composed rotation is given by the Gibbs vector
\[
\tilde{c} = \frac{\tilde{c}_1 + \tilde{c}_2 + \tilde{c}_1 \times \tilde{c}_2}{1 - \tilde{c}_1 \cdot \tilde{c}_2}.
\]
Here the quaternion \(\tilde{c}\) has the same meaning as \(\tilde{p}\) and \(\tilde{q}\) above.
Remark. The computations of the angles and planes for double require slight modification of the above formulas. Suppose

\[ f : x \rightarrow (\cos \alpha_1 + p_1 \sin \alpha_1) x \left( \cos \beta_1 + q_1 \sin \beta_1 \right) \]
\[ g : x \rightarrow (\cos \alpha_2 + p_2 \sin \alpha_2) x \left( \cos \beta_2 + q_2 \sin \beta_2 \right) \]

are two double rotations with angles \( \alpha_1 \pm \beta_1 \) and \( \alpha_2 \pm \beta_2 \) respectively. We can calculate the angles \( \gamma_1 \pm \gamma_2 \) of their composition \( g \circ f : x \rightarrow (ca)x(bd) \) in the same way as it done in the case of simple rotations:

\[
\cos \gamma_1 = S(ca) = \cos \alpha_1 \cos \alpha_2 - (p_1 \cdot p_2) \sin \alpha_1 \sin \alpha_2, \\
\cos \gamma_2 = S(bd) = \cos \beta_1 \cos \beta_2 - (q_1 \cdot q_2) \sin \beta_1 \sin \beta_2.
\]

The planes of the composition rotation are spanned by the vectors \( \bar{p} \pm \bar{q}, 1 \mp \bar{p} \bar{q} \), where the vectors \( \bar{p}, \bar{q} \) are calculated by formulas (9).

We illustrate the Conclusion in the following example.

Example. Let

\[ f : x \rightarrow \frac{1 + i}{\sqrt{2}} x \frac{1 + j}{\sqrt{2}} \quad \text{and} \quad g : x \rightarrow \frac{1 + j}{\sqrt{2}} x \frac{1 + k}{\sqrt{2}} \]

be two simple rotations. Then \( \Pi_f = \text{Span} \{i - j, 1 + k\} \) is the fixed-points plane of the rotation \( f \) and \( \Pi_g = \text{Span} \{j - k, 1 + i\} \) of \( g \). We see that \( \Pi_f \cap \Pi_g \neq \{0\} \), hence the composition should be a simple rotation. And indeed,

\[ h = g \circ f : x \rightarrow \frac{1 + i + j - k}{2} x \frac{1 + i + j + k}{2} =: pxq, \]

is simple, since \( Sp = Sq \). We shall now verify formulas (8) and (9). In order to it we express these rotation in a polar form:

\[ f : x \rightarrow \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) x \left( \cos \frac{\pi}{4} + j \sin \frac{\pi}{4} \right), \]
\[ g : x \rightarrow \left( \cos \frac{\pi}{4} + j \sin \frac{\pi}{4} \right) x \left( \cos \frac{\pi}{4} + k \sin \frac{\pi}{4} \right) \]

and

(10) \[ h : x \rightarrow \left( \cos \frac{\pi}{3} + \frac{i + j - k}{\sqrt{3}} \sin \frac{\pi}{3} \right) x \left( \cos \frac{\pi}{3} + \frac{i + j + k}{\sqrt{3}} \sin \frac{\pi}{3} \right). \]

Let \( p_f, q_f, p_g, q_g, p_h, q_h \) the pure quaternions that represent the corresponding rotations, and \( \alpha_f, \alpha_g, \alpha_h \) be the angles. Then

\[
\cos \frac{\alpha_h}{2} = \cos \frac{\alpha_f}{2} \cos \frac{\alpha_g}{2} - (i \cdot j) \sin \frac{\alpha_f}{2} \sin \frac{\alpha_g}{2} = \frac{1}{2}.
\]
Thus $\alpha_h = \frac{2\pi}{3}$, and that coincides with the polar representation of the rotation $h$. Set now $\tilde{p}_f = p_f \tan \frac{\pi}{4} = i$, $\tilde{q}_f = q_f \tan \frac{\pi}{4} = j$, $\tilde{p}_g = p_g \tan \frac{\pi}{4} = j$ and $\tilde{q}_g = q_g \tan \frac{\pi}{4} = k$, then according to formula (9)

$$\tilde{p}_h = \frac{j + i + j \times i}{1 - j \cdot i} = i + j - k, \quad \tilde{q}_h = \frac{i + k + j \times k}{1 - j \cdot i} = i + j + k.$$ 

Taking the corresponding unit quaternions, we have that $p_h = \frac{i + j - k}{\sqrt{3}}$ and $q_h = \frac{i + j + k}{\sqrt{3}}$, which agrees with (10).

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