Quantum Disturbance without State Change: 
Defense of State-Dependent Error-Disturbance Relations

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The uncertainty principle states that a measurement inevitably disturbs the system, while it is often supposed that a quantum system is not disturbed without state change. Korzekwa, Jennings, and Rudolph [Phys. Rev. A 89, 052108 (2014)] pointed out a conflict between those two views, and concluded that state-dependent formulations of error-disturbance relations are untenable. Here, we reconcile the conflict by showing that a quantum system is disturbed without state change, in favor of the recently obtained universally valid state-dependent error-disturbance relations.

I. INTRODUCTION

Heisenberg’s error-disturbance relation (EDR)

\[ \varepsilon(A)\eta(B) \geq \frac{1}{2}|[A, B]| \]  
(1)

for the mean error \( \varepsilon(A) \) of a measurement of an observable \( A \) in any state and the mean disturbance \( \eta(B) \) caused on an observable \( B \), originally introduced by the \( \gamma \)-ray microscope thought experiment [1], has been commonly believed as a dynamical aspect of Heisenberg's uncertainty principle, which is formally represented by the rigorously proven relation

\[ \sigma(A)\sigma(B) \geq \frac{1}{2}|[A, B]| \]  
(2)

for the indeterminacies, defined as the standard deviations \( \sigma(A) \) and \( \sigma(B) \), of arbitrary observables \( A, B \) in any state [1–3]. There have been longstanding research efforts to prove Heisenberg’s EDR [4–8], while the universal validity has not been reached. Instead, a recent study [9, 10] revealed a universal valid form of EDR

\[ \varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|[A, B]|, \]  
(3)

where \( \sigma(A) \) and \( \sigma(B) \) are the standard deviations just before the measurement, and made Heisenberg’s EDR testable [10, 11] to observe its violations [12–14], confirming the validity of the new relation as well. Subsequently, stronger EDRs were derived [15–18], and confirmed experimentally [19–24].

In order to define the error \( \varepsilon(A) \) and disturbance \( \eta(B) \) in Eq. (3), we suppose that the measurement \( M \) of \( A \) is described by an interaction from time \( t = 0 \) to \( t = \tau \) between the system \( S \) in a state \( |\psi\rangle \) and the probe \( P \) with the meter observable \( M \) prepared in a fixed state \( |\xi\rangle \). In the Heisenberg picture, we shall write \( X(0) = X \otimes I, X(\tau) = U^\dagger X(0)U \), \( Y(0) = I \otimes Y, Y(\tau) = U^\dagger Y(0)U \) for observables \( X \) in \( S \) and \( Y \) in \( P \), where \( U \) is the unitary evolution operator for \( S + P \) from \( t = 0 \) to \( t = \tau \). The error \( \varepsilon(A) = \varepsilon_O(A, M, |\psi\rangle) \) and disturbance \( \eta(B) = \eta_O(B, M, |\psi\rangle) \) in Eq. (3) are state-dependently defined by

\[ \varepsilon_O(A, M, |\psi\rangle) = \langle \psi, \xi| [M(\tau) - A(0)]^2 |\psi, \xi\rangle^{1/2}, \]  
(4)

\[ \eta_O(B, M, |\psi\rangle) = \langle \psi, \xi| [B(\tau) - B(0)]^2 |\psi, \xi\rangle^{1/2}, \]  
(5)

where \( |\psi, \xi\rangle = |\psi\rangle \otimes |\xi\rangle \); see Ref. [10] for details. We call \( \varepsilon_O \) and \( \eta_O \) as the operator-based error measure and the operator-based disturbance measure. We shall write \( \varepsilon_O(A) = \varepsilon_O(A, M, |\psi\rangle) \) and \( \eta_O(B) = \eta_O(B, M, |\psi\rangle) \) when no confusion may occur.

Korzekwa, Jennings, and Rudolph (KJR) [25] criticized the use of the operator-based disturbance measure, based on their definition of “operationally” non-disturbing measurements proposed as follows.

Definition: Operational disturbance. A measurement is “operationally” non-disturbing to an observable \( B \) in the system state \( |\psi\rangle \) if \( B(0) \) and \( B(\tau) \) have identical probability distributions in \( |\psi, \xi\rangle \).

Based on this, they posed the following requirement.

Operational requirement (OR) for disturbance measures. Any disturbance measure should assign the value 0 to “operationally” non-disturbing measurements.

KJR [25] called the OR “the commonly accepted and operationally motivated requirement that all physically meaningful notions of disturbance should satisfy.” They claimed that the operator-based disturbance measure does not satisfy the OR and has even an ‘unphysical’ property, since it takes a positive value for a measurement that does not change the state at all. Further, they concluded that state-dependent formulations of EDRs are not tenable.

In this paper, we examine the validity of the OR. For this purpose, we consider a more fundamental principle in quantum mechanics, the correspondence principle, stating that if the classical description is available, quantized concepts should be consistent with the classical description. We argue that the OR violates the correspondence principle. We generally show that even if the measurement does not change the state, the disturbance is operationally detectable as long as the operator-based disturbance measure takes a positive value. Thus, the claim made by KJR that the operator-based disturbance measure has an ‘unphysical’ property is groundless. The OR requires that disturbance measures only count

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the change of the probability distribution in time, but according to the correspondence principle, valid disturbance measures should also count the change of the observable that manifest in the time-like correlation, as the operator-based disturbance measure does. We show that the operator-based disturbance measure satisfies the correspondence principle, and conclude that state-dependent formulations of EDRs based on the operator-based disturbance measure reliably represent the originally motivated dynamical aspect of Heisenberg’s uncertainty principle.

II. CORRESPONDENCE PRINCIPLE

The correspondence principle generally states that quantum theory should be consistent with classical theories in those cases where the classical descriptions are also available. In fact, it is a common practice to apply classical descriptions to commuting observables through their joint probability distributions.

It is well-known that any commuting observables $X, Y$ have their joint probability distribution in any state. Here, for a given state $|\Psi\rangle$, the joint probability distribution (JPD) of any two observables $X, Y$ is defined as a probability distribution $\mu(u, v)$ satisfying

$$
\langle \Psi | f(X, Y) | \Psi \rangle = \sum_{u,v} f(u, v) \mu(u, v)
$$

for every (non-commutative) polynomial $f(X, Y)$ of $X$ and $Y$. In general, two observables $X, Y$ have their JPD in a state $|\Psi\rangle$ if and only if they commute in $|\Psi\rangle$ in the sense that $[P^X(u), P^Y(v)]|\Psi\rangle = 0$, where $P^X(u)$ and $P^Y(v)$ denote the spectral projections of $X$ and $Y$. In this case, the JPD $\mu$ is uniquely determined by

$$
\mu(u, v) = \langle \Psi | P^X(u) P^Y(v) | \Psi \rangle,
$$

for their eigenvalues $u, v$ (Ref. [18], Theorem 1).

Suppose that $B(\tau)$ and $B(0)$ in Eq. (5) have their JPD $\mu$ in the initial state $|\psi, \xi\rangle$. Then their JPD $\mu$ determines the (classical) root-mean-square deviation (RMSD) $\delta_G(\mu)$ between $B(\tau)$ and $B(0)$ by

$$
\delta_G(\mu) = \left( \sum_{u,v} (u-v)^2 \mu(u, v) \right)^{1/2}.
$$

We say that a disturbance measure $\eta$ satisfies the correspondence principle (CP) if $\eta(B) = \delta_G(\mu)$ provided that $B(\tau)$ and $B(0)$ have their JPD $\mu$ in the initial state $|\psi, \xi\rangle$. An important property of the operator-based disturbance measure $\eta_O$ is that it satisfies the CP, as easily follows from Eq. (6).

III. DISTURBING OBSERVABLES WITHOUT DISTURBING STATES

KJR [26] identified as ‘unphysical’ the property of the operator-based disturbance measure $\eta_O$ that it may not assign the value 0 when the state has not changed at all. In such a case, the probability distribution of every observable has not changed, so that this is a stronger violation of the OR. However, we shall show here that this is not a peculiarity of the operator-based disturbance measure, but a straightforward consequence of the CP.

Consider a qubit measurement. The projective measurement of $A = \sigma_z$ in the state $|0\rangle := |\sigma_z = +1\rangle$ does not change the initial state $|\psi\rangle = |0\rangle$. In this case, it was shown that the operator-based disturbance measure indicates that $B = \sigma_x$ is disturbed by the amount $\eta_O(\sigma_x) = \sqrt{2}$ [26], and this value was actually obtained by a neutron optical experiment [19]. However, according to the OR, every disturbance measure $\eta$ should assign the value 0. In contrast, we shall show that every disturbance measure $\eta$ satisfying the CP assigns the same value $\sqrt{2}$.

It is well-known that the projective measurement of $\sigma_z$ is carried out by the controlled-NOT operation

$$
U = |0\rangle \langle 0 | \otimes I + |1\rangle \langle 1 | \otimes \sigma_x
$$

for the measured qubit $S$ and the probe qubit $P$ prepared in the fixed state $|\xi\rangle = |0\rangle$ from $t = 0$ to $t = \tau$ and by the subsequent meter measurement for $M = \sigma_z$ in $P$ (Figure 1). The Schrödinger time evolution satisfies

$$
U(|0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle,
$$

$$
U(|1\rangle \otimes |0\rangle) = |1\rangle \otimes |1\rangle.
$$

For $B = \sigma_x$ the Heisenberg time evolution is given by

$$
\sigma_x(0) = \sigma_x \otimes I,
$$

$$
\sigma_x(\tau) = \sigma_x \otimes \sigma_x.
$$

Here, Eq. (13) follows from

$$
U^\dagger (\sigma_x \otimes I) U = |1\rangle \langle 1 | \sigma_x |0\rangle \otimes \sigma_x + |0\rangle \langle 0 | |0\rangle \langle 0 | \sigma_x |1\rangle \langle 1 | \otimes \sigma_x
$$

$$
= \sigma_x \otimes \sigma_x.
$$

FIG. 1. “Operationally” non-disturbing measurements violate the correspondence principle. A projective measurement of $\sigma_z$ in $|\psi\rangle = |0\rangle$ is ‘operationally’ non-disturbing to the observable $\sigma_z$. Thus, the OR requires any disturbance measure $\eta$ to assign the value as $\eta(\sigma_z) = 0$. However, the CP requires any disturbance measure $\eta$ to assign the value as $\eta(\sigma_z) = \sqrt{2}$.
It follows that $\sigma_x(\tau)$ and $\sigma_x(0)$ commute and they have the JPD $\mu(u, v)$ in the state $|\psi, \xi\rangle = |0, 0\rangle$ as

$$\mu(u, v) = (0, 0)P^{\sigma_x(\tau)}(u)P^{\sigma_x(0)}(v)|0, 0\rangle = (0, 0)P^{\sigma_x(\tau)}(u)P^{\sigma_x(0)}(v)|0, 0\rangle.$$  \hfill (14)

Then we obtain

$$\mu(u, v) = \frac{1}{4}$$  \hfill (15)

(see Appendix A for derivation). Thus, if the disturbance measure $\eta$ satisfies the CP, we have

$$\eta(\sigma_x)^2 = \delta_G(\mu)^2 = \sum_{u, v = \pm 1} (u - v)^2 \mu(u, v) = 2.$$  \hfill (16)

Therefore we conclude $\eta(\sigma_x) = \sqrt{2}$. Thus, the non-zero value $\eta(\sigma_x) = \sqrt{2}$ is not a peculiar property of the operator-based disturbance measure.

This conclusion might sound counter-intuitive, as the pure state has the “maximal information” about the system. However, the unchanged pure state does not imply unchanged the observable, because the “maximal information” about the system does not include the “maximal information” about the observable, analogously with the fact that the “maximal information” about the whole system does not include the “maximal information” about subsystems.

In fact, according to the available classical description, the conditional probability

$$\Pr\{\sigma_x(\tau) = u|\sigma_x(0) = v\} = \mu(u|v) = \frac{1}{2}$$  \hfill (17)

shows that the value of $\sigma_x(0)$ has been completely randomized, although their marginals have not changed at all as

$$\Pr\{\sigma_x(\tau) = u\} = \Pr\{\sigma_x(0) = u\} = \frac{1}{2}.$$  \hfill (18)

Thus, the OR neglects the disturbance caused by the randomization by measurement without changing the probability distribution.

### IV. STATE-DEPENDENT DEFINITION FOR NON-DISTURBING MEASUREMENTS

We have shown that the OR with the notion of “operationally” non-disturbing measurements contradicts the CP. To reconcile the conflict, we shall characterize non-disturbing measurements from the two fundamental requirements: the CP and the operational accessibility.

Consider the following condition.

(S) $B(\tau)$ and $B(0)$ have their JPD $\mu$ in $|\psi, \xi\rangle$ satisfying that $\mu(u, v) = 0$ if $u \neq v$.

From the point of view of the CP, if condition (S) holds, we should conclude that the measurement $M$ is non-disturbing to $B$ in $|\psi\rangle$. Thus, condition (S) is considered as a sufficient condition for a proper definition of non-disturbing measurements.

On the other hand, it is convenient to consider the weak joint distribution (WJD) $\nu(u, v)$ of $B(\tau)$ and $B(0)$ in $|\psi, \xi\rangle$ defined by

$$\nu(u, v) = \langle \psi, \xi | P^{\tau}(u)P^{B(0)}(v) | \psi, \xi \rangle.$$  \hfill (19)

The WJD always exists, though possibly takes negative or complex values, and is operationally accessible by weak measurement and post-selection [27–29]; see also Ref. [30] for a short survey. Then it is natural to consider the following condition.

(W) The WJD of $B(\tau)$ and $B(0)$ in $|\psi, \xi\rangle$ satisfies that $\nu(u, v) = 0$ if $u \neq v$.

If the measurement $M$ is non-disturbing to the observable $B$ in $|\psi\rangle$, any operational tests for witnessing the disturbance should fail. Since measuring WJD is one of such operational tests for which the disturbance is detected if $\nu(u, v) \neq 0$ for some $u \neq v$ [31–32], condition (W) is considered as a necessary condition for a proper definition of non-disturbing measurements.

Obviously, (W) is logically weaker than or equivalent to (S). However, the following theorem shows that both conditions are actually equivalent.

**Theorem 1.** Let $M$ be a measurement of a system $S$ in a state $|\psi\rangle$ carried out by a measuring interaction with a probe $P$ prepared in a fixed state $|\xi\rangle$ from $t = 0$ to $t = \tau$. Then for any observable $B$ in $S$, the following conditions are equivalent.

(i) Condition (W) holds.

(ii) The relation

$$P^{B(\tau)}(u)|\psi, \xi\rangle = P^{B(0)}(u)|\psi, \xi\rangle$$

holds for any $u$.

(iii) Condition (S) holds.

According to the theory of quantum perfect correlations [33–34], both conditions (S) and (W) equivalently require that $B(\tau)$ and $B(0)$ are perfectly correlated in the state $|\psi, \xi\rangle$ [30]. A direct proof of the above theorem for the present context is given in Appendix B. The above theorem justifies the following definition of non-disturbing measurements. We say that the measurement $M$ is *properly non-disturbing* to an observable $B$ in $|\psi\rangle$ if one of the conditions (S) or (W) is satisfied. Since the WJD is operationally accessible, this definition is also operationally accessible.

### V. RELIABILITY OF THE OPERATOR-BASED DISTURBANCE MEASURE

To consider the reliability of the operator-based disturbance measure, we examine the following requirements: (i) the CP, (ii) soundness, (iii) operational accessibility, and (iv) completeness.

We have already shown that the operator-based disturbance measure $\eta_0$ satisfies the CP, i.e., $\eta_0(B) = \delta_G(\mu)$ if $B(\tau)$ and $B(0)$ have the JPD $\mu$. We introduce the *soundness* requirement: Any disturbance measure $\eta$ should assign the value 0 to
any properly non-disturbing measurements. It is interesting to see that the CP implies soundness. To show this suppose that the measurement is properly non-disturbing to \( B \) in \(|\psi\rangle\). Then \( B(\tau) \) and \( B(0) \) have the JPD \( \mu \) satisfying that \( \mu(u,v) = 0 \) if \( u \neq v \). It follows that \( \varepsilon_C(\mu) = 0 \) and by the CP we have \( \eta(B) = \varepsilon_C(\mu) = 0 \). Accordingly, the operator-based disturbance measure \( \eta_O \) satisfies the soundness requirement. We conclude, therefore, that even if the measurement does not change the state, the disturbance is operationally detectable as long as the operator-based disturbance measure takes a positive value.

It has been known that the operator-based disturbance measure \( \eta_O \) is operationally accessible in the two ways: (i) the tomographic three state method, proposed by Ozawa \([10]\) and experimentally realized by Erhart et al. \([12]\) and others \([14,19,22]\) and (ii) the weak measurement method, proposed by Lund and Wiseman \([11]\) and experimentally realized by Rozenb. \([13]\) and others \([20,22]\).

As the converse of soundness, a disturbance measure \( \eta \) is said to be complete if \( \eta \) assigns the value 0 only to properly non-disturbing measurements. There is an example in which \( \eta_O \) does not satisfy completeness (Ref. \([34]\), p. 750). However, it is known that \( \eta_O \) satisfies completeness if (i) (commutative case) \( B(\tau) \) and \( B(0) \) commute in \(|\psi,\xi\rangle\) or if (ii) (dichotomic case) \( B^2 = I \) (Ref. \([18]\), Theorem 3).

We have seen that the operator-based disturbance measure satisfies all requirements (i)–(iii), and partially satisfies requirement (iv) above.

Analogously from an argument for the operator-based error measure \( \varepsilon_O \) in Ref. \([18]\), it follows that \( \eta_O \) can be modified to satisfy completeness by defining the operator-based locally uniform disturbance measure \( \eta_O \) as

\[
\eta_O(B, M, |\psi\rangle) = \sup_{}\eta_O(B, M, e^{-itB}|\psi\rangle).
\]

Then the error measure \( \eta_O \) satisfies requirements (i) – (iv) and also (v) (Dominating property) \( \eta_O(B, M, |\psi\rangle) \leq \eta_O(B, M, |\psi\rangle) \) for any \(|\psi\rangle\), and (vi) (Conservation property for dichotomic measurements) \( \eta_O(B) = \eta_O(B) \) if \( B^2 = I \).

VI. DEFENSE OF STATE-DEPENDENT FORMULATIONS OF ERROR-DISTURBANCE RELATIONS

In order to examine the reliability of the operator-based disturbance measure, KJR \([25]\) introduced the following definition. A state \(|\psi\rangle\) is called a zero-noise zero-disturbance (ZNZD) state with respect to observables \( A \) and \( B \) if the projective measurement of \( A \) in the state \(|\psi\rangle\), which always satisfies \( \varepsilon(A) = 0 \), is “operationally” non-disturbing to \( B \). Then they proved that for every pair of non-commuting observables \( A \) and \( B \), there exists a ZNZD state \(|\psi\rangle\) such that \(|\langle\psi|[A,B]|\psi\rangle| \neq 0 \). Thus, if the disturbance measure \( \eta \) satisfies the OR, any expression of the form

\[
\sum_{m,n=0}^{\infty} f_{mn}(A,B)\varepsilon(A,\rho)^m\eta(B,\eta)^n \geq \langle[A,B]\rangle,
\]

where \( f_{00}(A,B) = 0 \), must be violated. From this, KJR \([25]\) concluded that any state-dependent EDR, based on the expectation value of the commutator as a lower bound, is not tenable, and that state-independent formulations are inevitable.

We have two objections to their claims. First of all, the universally valid relation \( \ref{eq:universal} \) leads to the relation

\[
\eta_O(B) \geq \frac{\langle[A,B]\rangle}{2\sigma(A)} > 0
\]

for any projective measurement of \( A \) in any ZNZD state such that \(|\langle\psi|[A,B]|\psi\rangle| \neq 0 \). Thus, the measurement is properly disturbing to \( B \) by the soundness of \( \eta_O \), and consequently the disturbance is operationally detectable by the operational accessibility of the definition of properly non-disturbing measurements, so that the assumption by KJR \([25]\) that \( \eta(B) = 0 \) in any ZNZD state is unfounded.

Secondly, they concluded that state-independent formulations are inevitable for alternative formulations. However, currently proposed state-independent formulations of EDRs \([35,37]\) do not appear to capture the essence of Heisenberg’s original idea. Recall that Heisenberg derived his EDR by the \( \gamma \)-ray microscope thought experiment, in which the EDR is derived from the relation between the resolution power and the Compton recoil, reciprocally relating to the wave length of the incident light. Since the wave length is independent of the state of the object, the above formulation might be considered as state-independent. However, the analysis is valid only state-dependently, since the resolution power of the microscope can be defined by the wave length only in the limited situation in which the object is properly placed in the scope of the microscope. Thus, we can adequately define the error of the \( \gamma \)-ray microscope only state-dependently. In the state-independent formulations, currently one defines the state-independent error as the worst case of the state-dependent error, which must diverge to infinity as the object wave function spreads out or moves far apart. Such state-independent definitions would facilitate to reproduce the form of Heisenberg’s original idea, but do not keep the physics underlying it.

Thus, state-dependent formulations are inevitable to represent Heisenberg’s original idea underlying the uncertainty principle.

VII. CONCLUSION

In this paper, we gave a definition of non-disturbing measurement from the point of view of the correspondence principle and operational accessibility. Subsequently, we established the reliability of the operator-based disturbance measure. We have already discussed the reliability of the operator-based error measure in our previous work \([18]\). Both accounts ensure that universally valid EDRs \([3,15,17]\) reliably represent a dynamical aspect of Heisenberg’s uncertainty principle.
Besides the well-established relation for the indeterminacy in quantum states representing a kinetic aspect of the principle. Thus, the objections to state-dependent formulations of EDRs are unfounded, although those views appear to still prevail in the literature [38–39]. We conclude that the theory [9–11, 15–18] and experiments [12,14,19,21] for state-dependent formulations of EDRs are reliable and state-dependent formulations are inevitable to represent Heisenberg’s original idea underlying the uncertainty principle.

The new quantitative methods developed in this paper for universally valid EDRs with the well-defined operator-based disturbance measure incorporating with the methods of weak values and weak measurements will provide new quantitative methods to understand the change, transfer, or disturbance of observables in time, which does not manifest in the change of the probability distribution, but which does manifest in the time-like correlation. This quantity will be useful and even inevitable for exploring foundational problems in quantum physics including the long-lasting controversy over the roles of uncertainty principle in which-way measurements for interferometers (Refs. [31,32,40,42] and the references therein). In addition to the foundational problems, it will be expected that universally valid EDRs call for new research interests in exploring various frontiers in physics including fault-tolerant quantum computing [43–45], quantum metrology [46–48], and multi-messenger astronomy [49], in which technological limits would be overcome by the fundamental principle independent of particular models. We hope that the methods of operator-based disturbance measures as well as operator-based error measures will be accepted for broad areas of quantum physics.

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Appendix A: Derivation of Eq. (15)

Here, we shall give a derivation of Eq. (15). The JPD $\mu$ of $\sigma_\tau = \sigma_x \otimes \sigma_x$ and $\sigma_0 = \sigma_x \otimes I$ in the state $|0,0\rangle$ is given by

$$\mu(u,v) = \langle 0,0|P^{\sigma_x \otimes \sigma_x}(u)P^{\sigma_x \otimes I}(v)|0,0\rangle.$$

We have

$$P^{\sigma_x \otimes \sigma_x}(+1)P^{\sigma_x \otimes I}(\pm1) = [P^{\sigma_x}(+1) \otimes P^{\sigma_x}(+1) + P^{\sigma_x}(-1) \otimes P^{\sigma_x}(-1)](P^{\sigma_x}(\pm1) \otimes I)$$

$$= P^{\sigma_x}(\pm1) \otimes P^{\sigma_x}(\pm1),$$

$$P^{\sigma_x \otimes \sigma_x}(-1)P^{\sigma_x \otimes I}(\pm1) = [P^{\sigma_x}(+1) \otimes P^{\sigma_x}(-1) + P^{\sigma_x}(-1) \otimes P^{\sigma_x}(+1)](P^{\sigma_x}(\pm1) \otimes I)$$

$$= P^{\sigma_x}(\pm1) \otimes P^{\sigma_x}(\mp1).$$

Consequently,

$$\mu(+1,\pm1) = \langle 0,0|P^{\sigma_x}(\pm1) \otimes P^{\sigma_x}(\pm1)|0,0\rangle = \langle 0|P^{\sigma_x}(\pm1)|0\rangle \langle 0|P^{\sigma_x}(\pm1)|0\rangle = \frac{1}{4},$$

$$\mu(-1,\pm1) = \langle 0,0|P^{\sigma_x}(\pm1) \otimes P^{\sigma_x}(\mp1)|0,0\rangle = \langle 0|P^{\sigma_x}(\pm1)|0\rangle \langle 0|P^{\sigma_x}(\mp1)|0\rangle = \frac{1}{4}.$$

Therefore, we obtain Eq. (15), i.e.,

$$\mu(u,v) = \frac{1}{4}.$$

Appendix B: Proof of Theorem 1

The assertion was generally proved in Refs. [33,34] after a lengthy argument. We give a direct proof for the present context.

(i) $\Rightarrow$ (ii): Suppose (i) holds. Then the WJD $\nu(u,v)$ of $B(\tau)$ and $B(0)$ in $|\psi,\xi\rangle$ satisfies $\nu(u,v) = 0$ if $u \neq v$. It follows that $\nu(u,u) = \sum_v \nu(u,v)$. Thus,

$$\langle \psi,\xi|P^{B(\tau)}(u)P^{B(0)}(u)|\psi,\xi\rangle = \langle \psi,\xi|P^{B(0)}(u)|\psi,\xi\rangle,$$

$$\langle \psi,\xi|P^{B(0)}(u)P^{B(\tau)}(u)|\psi,\xi\rangle = \langle \psi,\xi|P^{B(\tau)}(u)|\psi,\xi\rangle.$$

Consequently,

$$\|P^{B(\tau)}(u)|\psi,\xi\rangle - P^{B(0)}(u)|\psi,\xi\rangle\|^2 = 0,$$
and
\[ P^{B(\tau)}(u) |\psi, \xi \rangle = P^{B(0)}(u) |\psi, \xi \rangle. \]

Thus, condition (ii) holds and the implication (i)⇒(ii) follows.

(ii)⇒(iii): Suppose (ii) holds. Then
\[ P^{B(0)}(u) P^{B(\tau)}(v) |\psi, \xi \rangle = \delta_{u,v} P^{B(0)}(u) |\psi, \xi \rangle, \]
\[ P^{B(\tau)}(v) P^{B(0)}(u) |\psi, \xi \rangle = \delta_{u,v} P^{B(0)}(u) |\psi, \xi \rangle. \]

Consequently,
\[ P^{B(0)}(u) P^{B(\tau)}(v) |\psi, \xi \rangle = P^{B(\tau)}(v) P^{B(0)}(u) |\psi, \xi \rangle. \]

It follows that \( B(0) \) and \( B(\tau) \) commute in \( |\psi, \xi \rangle \) and condition (S) holds. Thus the implication (ii)⇒(iii) follows.

Since the implication (iii)⇒(i) holds obviously, all conditions (i) – (iii) are equivalent.