Inequalities between ground-state energies of Heisenberg models

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Abstract
The Lieb-Schupp inequality is the inequality between ground state energies of certain antiferromagnetic Heisenberg spin systems. In our paper, the numerical value of energy difference given by Lieb-Schupp inequality has been tested for spin systems in various geometries: chains, ladders and quasi-two-dimensional lattices. It turned out that this energy difference was strongly dependent on the class of the system. The relation between this difference and a fall-off of a correlation function has been empirically found and formulated as a conjecture.

1 Introduction
The list of general results on the area of quantum spin systems is not too large. Among them, there is a result due to Schupp [1], establishing the inequality between a ground-state energies of an antiferromagnetic Heisenberg chains. (The simplest example is the difference $2E_{m+n} + E_{2m} - E_{2n}$, where $E_k$ is the ground state energy of the Heisenberg chain with $k$ sites). Later on, this inequality has been extended to more general class of Heisenberg models [2].

These inequalities are rigorous ones. They are based on the matrix inequality proved by Kennedy, Lieb and Shastry (KLS) [3]. Authors applied this inequality to proving the existence of the Long Range Order in a class of anisotropic Reflection-Positive $d \geq 2$ quantum Heisenberg models. The KLS inequality has also been applied to establish certain properties of ground states of Hubbard models [4][5][6] as well as to prove the absence of orderings in Heisenberg models on pyrochlore lattices [7][8].
Although the Schupp inequalities between ground-state energies of Heisenberg models are rigorous, they do not give any information about the actual value of this difference. It would be very interesting to test how close to each other are both sides of the inequality. This was one of the goals of our paper: To test a numerical value of the difference between the both sides of the inequality.

We have tested this difference numerically. We used exact diagonalization procedure and in some cases the DMRG method. The ARPACK++ and ALPS packages have been used [9][10]. We considered systems in various geometries: chains, ladders, and rectangles, up to 27 - 28 sites\(^1\) (exact diagonalization) and 200 - 256 sites (DMRG). As a rule, the differences between both sides of inequality were very small. More quantitative considerations allowed us to pose certain conjecture between strength of spin correlations in the system and the difference between both sides of inequality.

The outline of the paper is as follows.

In the Sec. 2, the ground-state energy inequalities have been formulated.

In the Sec. 3, the differences between both sides of inequality for various systems have been numerically tested. The chains, ladders, and rectangles in various geometries: square, pyrochlore and squares with crossing bonds, have been analysed. The results and their implications (both rigorous and conjectural) have been presented.

The Sec. 4 contains summary, conclusions as well as perspectives for future research.

## 2 Formulation of inequalities

### 2.1 Structure of the system

We consider finite lattice spin systems with all spins being finite, so all spaces are finite-dimensional and operator are matrices. We assume that all spins are identical (this assumption can be relaxed).

We consider system which can be divided into two subsystems: \( L \) (‘left’) and \( R \) (‘right’) parts. Corresponding division of the total Hilbert space \( \mathcal{H} \) is:

\[
\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R,
\]

where \( \mathcal{H}_L \) and \( \mathcal{H}_R \) are Hilbert spaces for the \( L \) and \( R \) subsystems, respectively. Let \( H_L \) be the Hamiltonian of the \( L \) subsystem acting on \( \mathcal{H}_L \), and analogously \( H_R \) the Hamiltonian of the \( R \) subsystem acting on \( \mathcal{H}_R \). They can be lifted to operators acting on the whole \( \mathcal{H} \):

\[
H_L \to H_L \otimes \text{Id}_R, \quad H_R \to \text{Id}_L \otimes H_R.
\]

Then, the Hamiltonian is of the following general form:

\[
H = H_L \otimes \text{Id}_R + \text{Id}_L \otimes H_R + H_I,
\]

where \( H_I \) is the ‘interaction’ Hamiltonian, acting on \( \mathcal{H} \).

\(^1\)Depending on the lattice type.
2.2 Assumptions on Hamiltonian

Hamiltonians $H_L$ and $H_R$ are arbitrary spin-interaction Hamiltonians. The crucial assumption is the one concerning the interaction Hamiltonian $H_I$.

We assume that it is the sum of antiferromagnetic Heisenberg interactions. More precisely:

Let $A \subset L$ be some $m$-site subset of $L$: $m = |A|$. Let us write: $A = \{i_1, i_2, \ldots, i_m\}$. Let us define $S_A$, being operator acting on $H_L$: $S_A = \sum_{k=1}^m J_k s_{i_k}$, where coefficients $J_k$ are real, and $s$ is total spin operator: $s = (s^x, s^y, s^z)$. Let $A' \subset R$ be some subset of $R$, let $m = |A'|$, let $A' = \{i'_1, i'_2, \ldots, i'_m\}$, and define $S_{A'}$, an operator acting on $H_R$: $S_{A'} = \sum_{k=1}^m J_k s_{i'_k}$. (So, we can say that $S_{A'}$ is a 'mirror image' of $S_A$; but we do not assume that the $L$ system is a mirror image of $R$ system). Then we assume that the interaction Hamiltonian is of the form

$$H_I = \sum_A S_A \cdot S_{A'}, \quad (4)$$

Figure 1: Illustration of the structure of the interaction Hamiltonian. Interactions within $L$ and $R$ subsystems are arbitrary, but the interactions between them are antiferromagnetic, i.e. they have form $[4]$. The simplest example of $A$ is one site; it leads to ordinary Heisenberg AF interaction. For $A$ being two-site set, an example is the pyrochlore lattice. We will consider these two sorts of interactions.

2.3 Formulation of the main inequality

Consider now the following Hilbert spaces and systems ('LL' and 'RR' ones):

$$\mathcal{H}_{LL} = \mathcal{H}_L \otimes \mathcal{H}_L, \quad \mathcal{H}_{RR} = \mathcal{H}_R \otimes \mathcal{H}_R; \quad (5)$$

and Hamiltonians:

$$H_{LL} = H_L \otimes \text{Id}_L + \text{Id}_L \otimes H_L + \sum_A S_A \cdot S_A, \quad (6)$$
\begin{equation}
H_{RR} = H_R \otimes \text{Id}_R + \text{Id}_R \otimes H_R + \sum_{A'} S_{A'} \cdot S_{A'}
\end{equation}  
(7)

Let \( E_{LL} \) be the ground-state energy of the Hamiltonian (6), \( E_{RR} \) – the ground-state energy of (7) and \( E_{LR} \) – the ground-state energy of (3) with (4).

Under assumptions above, it has been proved in [1] that the following inequality between ground-state energies \( E_{LL} \), \( E_{RR} \), \( E_{LR} \) holds:
\begin{equation}
E_{LL} + E_{RR} \leq 2E_{LR}.
\end{equation}  
(8)

Figure 2: Illustration of interrelations between initial \( L - R \) system and resulting \( L - L \) and \( R - R \) systems. In all three cases structures of the interaction Hamiltonians are identical.

3 Magnitudes of differences between two sides of inequality

In this Section we present results for one-half spin Heisenberg models in chosen geometries. We test numerically how close to each other are the two sides of inequality (8).

We have examined the following systems: chains, ladders (square, pyrochlore and with crossing bonds) as well as rectangles.

The main numerical method we have applied are: ED (Exact Diagonalization) as implemented in the ARPACK++ package based on Lanczos algorithm with Arnoldi modification [9]. We present results of exact diagonalization for systems up to 28 sites. The precision of the method is close to numeric limitation, which is \( 10^{-14} \).

\footnote{For some small system (up to 16 sites) ED was also performed with the ALPS library for a test purpose.}
Moreover, for chosen class of systems (chains and ladders) we have also applied the DMRG method as developed in the ALPS package. We considered systems up to 256 sites. However, the price paid was some lack of precision, but still it was about $10^{-10}$.

We have checked that in all tested systems the inequality was fulfilled in the range of precision of the method (it must be so because the inequality is rigorous). However it turns out that the magnitude of difference between two sides of inequality strongly depend of the kind of the system under consideration. Below we present results we have obtained in more details.

## 3.1 Chains

In this subsection we present results of the analysis done for open chains. The coupling constant is equal to 1.

The results of exact diagonalization for chains of length up to 28 sites are presented in Table I in Appendix A. They are fully consistent with existing data for smaller chains (see for instance [11]). We have also performed DMRG calculations for chains up to 200 sites. We have checked correctness of DMRG calculations by comparison with ED results wherever it was possible and they were identical within numerical precision.

To compare both sides of the Schupp inequality (8), let us denote by $E_k$ the ground-state energy for the $k$-site chain. Then the inequality (8) takes the form: (see Fig. 3 for illustration)

$$2E_{m+n} \geq E_{2n} + E_{2m}. \quad (9)$$

To better illustrate the results, let us rewrite the inequality (9) into the following form

$$\Delta_{mn} = 2E_{m+n} - E_{2n} - E_{2m} \geq 0, \quad (10)$$

and then slightly redefine $\Delta_{mn}$ into

$$\Delta_{mn} \equiv \Delta_L^{(d)} = 2E_L - E_{L-2d} - E_{L+2d} \geq 0, \quad (11)$$

where $L = m + n$ is the length of the initial chain, and $d$ is a distance from the middle of the chain where we are making a slice. For example: in Fig. 3 we have $L = 6$ and $d = 1$.

In Fig. 4 we are making a plot of LHS of the inequality (11). It is clearly seen that the larger systems are, the difference $\Delta_L^{(d)}$ is smaller. Analogously, the more similar left and right subsystems are, the difference $\Delta_L^{(d)}$ is smaller too. The the difference $\Delta_L^{(d)}$ ranges from $10^{-2}$ (for few-site systems) to $10^{-6}$ (for hundred site systems divided about the middle).

One can ask how the differences $\Delta_L^{(d)}$ depend of the length $L$. One can guess that this dependence can be power law or exponential. It turns out the first possibility is realized with good accuracy (see Fig. 4). Assuming power-law dependence:

$$\Delta_L^{(d)} \sim L^{-\theta} \quad (12)$$

we have found by least-square fitting that for $d = 1, 2, 3, 4$ the value of $\theta = 2.83(4)$
Figure 3: Scheme of the open chain lattice and demonstration of the division. Chain of the 6 sites is divided into chains with 4 and 2 sites which further are 'glued' together in combinations: 4+4 and 2+2, so two chains of length of 8 and 4 sites are formed.

Let us make also the following comment about content of the Fig. 4. Presented points refer to the situation where lengths of initial chains are even. (Also all other systems that will be presented later are referred to lattices of even length.) We have also examined chains of the odd length. In this case, the data obtained by the exact diagonalization methods show that values of $\Delta^{(d)}_L$ are also greater than zero and have magnitude around $10^{-1}$. This behaviour is the same as claimed by the folk knowledge on the area of antiferromagnetic spin systems: For even number of spins, they exhibit tendency to be paired, so in general the average energy per spin is lower for systems with even number of spins than for systems with odd number. We confirm this rule for chains.

Remark. One can ask whether the inequality (9) can be extended to the inequality

$$2E_{(m+n)/2} \geq E_n + E_m, \quad (13)$$

where $m$ and $n$ are odd numbers. The answer is no. For instance, we have (see Table II): $2E_6 = -4.97$, $E_5 + E_7 = -4.76$, so the inequality \((13)\) cannot be fulfilled in general.

### 3.2 Square ladders

The results of this subsection are based on exact diagonalization for ladders of length up to 14 (i.e. 28-sites systems) and DMRG results for ladders of length up to 126 (i.e. 252-sites systems). All coupling constants are equal to 1. The ED results are collected in Table II in Appendix A.
Figure 4: Open chains: plot of $\Delta_{L}^{(d)}$ for as a function of $L$ for different values of $d$. The scales on both axes are logarithmic. The illustrated data have been obtained by exact diagonalization and DMRG.

Figure 5: Scheme of the open ladder lattice and demonstration of the division. Ladder of the $L = 6$ is divided into ladders with $L = 4$ and $L = 2$ and by doubling the length we are creating two ladders of $L = 8$ and $L = 4$.  

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We examine here the value of the difference $\Delta_{L}^{(d)}$ defined by (11), similarly as we did in the case of the open chain systems. The only difference is that now by ladder of length $L$ we refer to square lattice of size $2 \times L$ (for details, see Fig. 5). The results are presented in Fig. 6.

It is seen that the values of $\Delta_{L}^{(d)}$ are very small. Typically, they are $10^3$ – $10^4$ times smaller than corresponding values for chains of the same size.

For not-too-large systems (30-site systems divided near the middle) the difference $\Delta_{L}^{(d)}$ become smaller than the numerical error of the DMRG algorithm. Within the method precision, the difference always was greater than zero.

Figure 6: Square ladders: plot of $\Delta_{L}^{(d)}$ as a function of $L$ for different values of $d$. The illustrated data have been obtained by exact diagonalization and DMRG. For not-too-large systems (30-site systems divided near the middle) the difference $\Delta_{L}^{(d)}$ become smaller than the numerical error of the DMRG algorithm.
3.3 Pyrochlore and ladders with crossing bonds

We test here another systems to which inequality (8) applies. They are: ladders with crossing bonds and 'pyrochlore' ladders. Similarly to the previous cases of chains and open ladders, data were obtained using exact diagonalization and DMRG method. The ED results are collected in Table II in Appendix A.

3.3.1 Ladders with crossing bonds

For the ladder with crossing bonds, the horizontal and vertical couplings are 1, whereas diagonal couplings are equal to $\frac{1}{2}$. For details of the system division, see Fig. 7.

Numerical results are presented in Fig. 8. Values of $\Delta^{(d)}_L$ – as a rule – are larger than for ordinary ladders, but still are $10^4 - 10^5$ times smaller than for chains.

Figure 7: Scheme of the open ladder with crossing bonds and demonstration of the division. Ladder of the $L = 6$ is divided into ladders with $L = 4$ and $L = 2$ and by doubling the length we are creating two ladders of $L = 8$ and $L = 4$. 

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Figure 8: Ladders with crossing bonds: plot of $\Delta^{(d)}_L$ as a function of $L$ for different values of $d$. The illustrated data have been obtained by exact diagonalization and DMRG. For not-too-large systems (40-site systems divided near the middle) the difference $\Delta^{(d)}_L$ become smaller than the numerical error of the DMRG algorithm.

### 3.3.2 Pyrochlore ladders

For pyrochlores, all coupling constants are equal to 1, and this was motivated by the fact that for rectangular lattices of larger width, the inequality (8) applies when diagonal $J_d$ and horizontal/vertical $J$ couplings satisfy:

$$\frac{J_d^2}{J^2} \leq \frac{1}{2} \quad [1, 2].$$

Fig. 9 shows a family of system divisions that was chosen. They exhaust all interesting possibilities; for details, see Appendix B.

Ground-state energies are collected in Table III whereas values of $\Delta_L$ (defined analogously as for square ladders) are illustrated in Fig. 10.

One can ask how the difference $\Delta$ depends of the system size in the case of ladders. Figures 6, 8 and 10 suggest that in all cases this dependence is exponential one. We have fitted the data to exponential function:

$$\Delta \sim \exp(-\alpha L) \quad (14)$$

The fit was moderately good for simple ladders (here $\alpha = 0.68(5)$) and very good for ‘crossed’ ($\alpha = 0.541(7)$) and pyrochlore ($\alpha = 0.630(5)$) ladders.
Figure 9: Possible slices for pyrochlore lattices. They are discussed in the Appendix B.

Figure 10: Pyrochlore ladders: plot of $\Delta^d_L$ as a function of $L$ for different values of $d$. The illustrated data have been obtained by exact diagonalization and DMRG. For not-too-large systems (40-site systems divided near the middle) the difference $\Delta^d_L$ become smaller than the numerical error of the DMRG algorithm.
We have observed that for all three sorts of ladders, values of $\Delta^{(d)}_{L}$ are much smaller than for chains. Which is the origin of this difference? One can guess that the more the $L$ and $R$ subsystems are independent (uncorrelated), the smaller difference $\Delta^{(d)}_{L}$ is. So it is natural to compare the value of $\Delta^{(d)}_{L}$ with behaviour of correlations in system of consideration. We calculated some (selected) correlation functions $\langle S_i \cdot S_j \rangle$ in ladders (see Fig. 11).

We have checked that in all cases the decay of correlations was exponential one. Such a behaviour is consistent with (extension of) Haldane rule \[12\], \[13\], \[14\]. On the other hand, it is well known that in the Heisenberg chain the decay of correlations is algebraic. So correlations for chains are stronger than for ladders, and it turns out that $\Delta^{(d)}_{L}$ is larger for chains than for ladders.

We made also another observation. Namely, for some pyrochlore systems, the difference $\Delta^{(d)}_{L}$ is equal to zero (within numerical precision), what means that the inequality saturates! It is a consequence of the fact that for pyrochlore systems, the addition of the pair of sites at the end of the ladder changes the energy by a constant value (equal to $-0.75$), independent of the ladder size (see Table II). In this case, the correlation function of the boundary spin with all other spins is exactly zero (again within numerical precision). It would be very interesting to understand these facts theoretically.

Figure 11: Correlation function $\langle S_1 \cdot S_j \rangle$ as a function of inter-site distance $j$. The correlations tend to zero in algebraic manner for the chain, whereas for ladders, they fall-off exponentially.
3.4 Rectangles of width 3, 4 and 5

The results of exact diagonalization for small rectangular subsets of square, pyrochlore and square-with-crossing-bonds lattices are collected in the Table II. The differences between two sides of inequality are presented in Fig. 12.

It can be seen that differences are larger for square lattice systems than for ladders and chains. It can be again related with behaviour of correlation functions, which are expected to tend for large distances to non-zero constant [17], [18]. So the correlations are larger here, and the values of $\Delta^{(d)}_L$ are larger, too.

Figure 12: Differences of energies for strips of width 3, 4 and 5

3.5 Some conjectures

Let us summarize our observations:

The smallest values of $\Delta_L^{(d)}$ is observed for i) all ladders and for pyrochlore as well as X lattice. The larger value of $\Delta_L^{(d)}$ takes place for ii) chains, and the largest one – for iii) quasi-2d square lattice.

These results seem to be related with the behaviour of spin correlations in the system considered. For the i) case, the spin correlations $\langle S_i \cdot S_j \rangle$ tends to zero exponentially with the distance $|i - j|$. For ii), the fall-off of correlations has power law, whereas for iii), correlations are expected to tend to a constant.

It seems for us that this inter-relation can be true in the more general situations and we formulate the following conjecture.

Conjecture rough version. The faster fall-off of the spin correlations in the system, the smaller difference between two sites of the inequality
Conjecture more precise version for chains and ladders. If the fall-off of the correlation function is exponential, then the difference $\Delta^{(d)}_L$ as a function of $L$ behaves in an analogous manner. If the correlation function decays with the distance according to the power law, then the difference $\Delta^{(d)}_L$ as a function of $L$ also decreases according to the power law.

4 Summary

In the paper, the numerical value of Lieb-Schupp inequality has been tested for spin systems in various geometries: chains, ladders and quasi-two-dimensional lattices. The tools used were Exact Diagonalization (for all classes of systems up to 28 spins) and DMRG (for ladders and chains).

It has been checked that Lieb-Schupp inequality has been fulfilled in all cases (it must be so as this inequality is rigorous). But the value of difference between two sides of inequality $\Delta^{(d)}_L$ was strongly dependent of the class of system. The largest value was observed for quasi-2d systems, smaller – for chains, and the smallest – for ladders. It has been conjectured that the value of $\Delta^{(d)}_L$ is related with the fall-off of correlations: The smaller value of $\Delta^{(d)}_L$, the faster fall-off of two-point correlation functions.

In some cases (pyrochlore ladders) the value of $\Delta$ was equal to zero – i.e. the inequality saturates. In these cases, also certain correlation functions has been exactly equal to zero.

It would be very interesting to understand these facts theoretically. Moreover, in the cases where $\Delta^{(d)}_L = 0$ and correlation functions are equal to zero, perhaps it would be possible to construct the ground state of the system in the spirit of Matrix Product States [14], [15], [16].
Appendix A

| $N$ | $E_N$          |
|-----|---------------|
| 2   | -0.750       |
| 3   | -1.000       |
| 4   | -1.616       |
| 5   | -1.928       |
| 6   | -2.494       |
| 7   | -2.836       |
| 8   | -3.374       |
| 9   | -3.736       |
| 10  | -4.258       |
| 11  | -4.632       |
| 12  | -5.142       |
| 13  | -5.525       |
| 14  | -6.027       |
| 15  | -6.417       |
| 16  | -6.912       |
| 17  | -7.307       |
| 18  | -7.797       |
| 19  | -8.197       |
| 20  | -8.682       |
| 21  | -9.086       |
| 22  | -9.568       |
| 23  | -9.974       |
| 24  | -10.454      |
| 25  | -10.863      |
| 26  | -11.339      |
| 27  | -11.751      |
| 28  | -12.225      |

Table I: Ground state energies in the open chain lattice. Coupling constant $J = 1$ and $N$ represents lengths of the chains.
Table II: Ground-state energies in quasi-two-dimensional systems: square, pyrochlore (two kinds) and X (square-with-crossing-bonds) lattices

| Nx | Ny | Square              | Pyrochlore, case A | Pyrochlore, case B | X lattice  |
|----|----|---------------------|--------------------|--------------------|------------|
| 2  | 2  | -2.000000000000    | -1.500000000000    | -2.000000000000    | -1.750000000000 |
| 2  | 3  | -3.1293852415718   | -2.750000000000    | -2.750000000000    | -2.677862533180 |
| 2  | 4  | -4.2930664566570   | -3.500000000000    | -4.0277505942154   | -3.641829874567 |
| 2  | 5  | -5.4467120643352   | -4.777505942154    | -4.777505942154    | -4.5873084880937 |
| 2  | 6  | -6.6034724753869   | -5.5277505942154   | -5.607411404916    | -5.5431961748705 |
| 2  | 7  | -7.7593260611500   | -6.8107411404916   | -6.8107411404916   | -6.4933181096298 |
| 2  | 8  | -8.9154711235558   | -7.5607411404916   | -8.0949932311853   | -7.4467788894730 |
| 2  | 9  | -10.0715341613484  | -8.8449932311853   | -8.8449932311853   | -8.3983424061390 |
| 2  | 10 | -11.2276251173666  | -9.5949932311853   | -10.129577877187   | -9.3510128680377 |
| 2  | 11 | -12.3837088687482  | -10.879577877187   | -10.879577877186   | -10.3030475515486 |
| 2  | 12 | -13.5397954074066  | -11.629577877187   | -12.1642537019725  | -11.2554538123736 |
| 2  | 13 | -14.6958813681522  | -12.9142537019725  | -12.9142537019725  | -12.207645356328 |
| 2  | 14 | -15.8519676317127  | -13.6642537019726  | -14.1985949790545  | -13.1599626306679 |
| 3  | 3  | -4.7493272585528   | -4.0087848535303   | -4.0087848535303   | -3.959339973975 |
| 3  | 4  | -6.6916801935149   | -5.6617068232824   | -5.6617068232824   | -5.5345034217058 |
| 3  | 5  | -8.3876285183968   | -6.9910226671666   | -6.9910226671666   | -6.8685484091210 |
| 3  | 6  | -10.2835182238578  | -8.5954218204916   | -8.5954218204916   | -8.4037660947387 |
| 3  | 7  | -12.007230867337   | -9.9647656528603   | -9.9647656528603   | -9.7629674917248 |
| 3  | 8  | -13.8813610992452  | -11.5380986367773  | -11.5380986367773  | -11.276504616748 |
| 4  | 4  | -9.1892070651929   | -7.3280745721674   | -8.102252720723   | -7.505556590810 |
| 4  | 5  | -11.6515708351550  | -9.7410214922860   | -9.7410214922860   | -9.407787759888 |
| 4  | 6  | -14.129146864446   | -11.381745234209   | -12.185727796133   | -11.350974919405 |
| 5  | 5  | -14.6961464371187  | -12.0391686609399  | -12.0391686609399  | -11.7667640193786 |
Appendix B

The analysis of the Lieb-Schupp inequality in the case of the pyrochlore ladder system may seem to be a complicated task. First of all there exist two types of system – one that begin with the crossing bonds (denoted as $A$) and one that begin with simple bonds (denoted as $B$). Secondly the line of division initial system may go through a crossing bonds (like in the ladder with the crossing bonds) or through only vertical bonds (like in a simple square ladder case).

Fortunately analysing data more closely and taking into account behaviour of a correlation functions allows us to choose only one family of possible divisions, which is equivalent to all others.

The goal of this appendix is to present an argumentation based of the empirical observations that the choice of this particular family of divisions is justified.

Observation 1  On the beginning let take a look on the systems of the even length. We will denote this systems with $A_s$ and $B_s$, because they have a reflection symmetry – if the system begining with a crossing bonds (type $A$) it also end with a crossing bonds or if the system begin with a simple vertical bond (type $B$) it also end the same way.

Furthermore let denote by $S_1$ a slice that goes through a simple vertical bonds and by $S_2$ a slice that goes through a crossing bonds. Making plot of a value of $\Delta^{(d)}_L$ for this families [Fig.13] we notice that slices $S_1$ and $S_2$ can be treated during analyse as one family.

For an example in Fig. 9 we present system of the type $A_s$ with slices of the type $S_1$ in the top and bottom situations and of the type $S_2$ in the middle situation.

![Figure 13: Illustration of the observation 1. Different types of slice lies on the same line which allows them to be analysed as a one family of divisions.](image)

Observation 2  System of an odd length $A$ is identical as a system $B$. Let denote this system as $A_a$ to make is distinguished from $A_s$ and $B_s$. From the data in Table II we can observe that adding two sites on
the end of a system $A_\alpha$ (for example coming from $2 \times 3$ to $2 \times 4$ system or from $2 \times 5$ to $2 \times 6$ system) the ground state energy change by a constant value $-0.75$. It is also worth to notice that adding two spins on the edge of the $B_\alpha$ type system also change ground state energy only by the same constant.

This observation is consistent with the fact that correlation of the edge spins in the type $A_\alpha$ systems is zero with every other spin in the system\footnote{But not between the two spins on the same edge.}

It is very easily to proof that if ground state energy of the system that ends with simple bond differs only by a constant from the situation, that this edge is extended with two spins to form a crossing bond, the value of $\Delta_L^{(d)}$ do not change.

The illustration on this may be seen in Fig. 13 where the $B_\alpha$ system can be extended to a system $A_\alpha$ by adding two sites on the both system edges.
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