The $A_5$ and the pion field

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In this talk, an SU($N_f$) x SU($N_f$) Yang-Mills model with a compact extra-dimension is used to describe the spin-1 mesons and pions of massless QCD in the large-$N_c$. The right 4D symmetry and symmetry-breaking pattern is produced by imposing appropriate boundary conditions. The Goldstone boson (GB) fields are constructed using a Wilson line. We derive the low-energy limit (chiral lagrangian), discuss $\rho$-meson dominance, sum rules between resonance couplings and the relation with the QCD high-energy behavior. Finally, we provide an analytic expression for the two-point function of vector and axial currents.

1. Introduction

Based on [1], we present a model for a sector of large-$N_c$ QCD in the chiral limit: that of pions and the infinite tower of spin-1 mesons. In view of a leading $1/N_c$ approximation, we limit ourselves to tree-level. The model never refers to quarks, but rather to mesons. These will be described by Kaluza-Klein (KK) excitations of five-dimensional (5D) fields.

Consider SU($N_f$) x SU($N_f$) Yang-Mills fields in a compact extra dimension. From the four-dimensional (4D) point of view, these are seen as two infinite towers of resonances. Imposing appropriate boundary conditions (BCs) on one boundary will induce spontaneous breaking of the 4D SU($N_f$) x SU($N_f$) symmetry acting on the other boundary (where the standard 4D electroweak interactions are located).

We define the model in Section 2. We concentrate in Section 3 on the predictions for low-energy physics, i.e., pion interactions. At higher energies, resonances play the leading role: we consider the constraints on their interactions in Section 4. Sum rules for their couplings are obtained, relying on 5D gauge structure: they imply a soft high-energy behavior. This allows a partial matching to QCD. Up to this point, the geometry (5D metric) influences none of the qualitative features of the model. We discuss $\rho$-meson dominance in Section 5. In Section 6, we focus on the AdS$_5$ metric and give analytic results for the vector and axial two-point functions.

2. The model

Using conformally flat coordinates $ds^2 = w(z)^2 (\eta_{\mu\nu}dz^\mu dz^\nu - dz^2)$, we consider a compact extra dimension extending from $z = l_0$ (UV brane) to $z = l_1$ (IR brane). The standard SU($N_f$) x SU($N_f$) Yang-Mills action in 5D reads $M = (\mu, 5)$

$$S_{YM} = \frac{1}{4g_5^2} \int d^4x \int_{l_0}^{l_1} dz w(z) \eta^{MN} \eta^{RS} \times \langle L_{MR} L_{NS} + R_{MR} R_{NS} \rangle.$$  (1)

Having a larger (i.e., 5D instead of 4D) SU($N_f$) x SU($N_f$) symmetry will have important consequences on the high-energy behavior of the model, see Section 4. As for now, we discuss the fate of

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the elements of the gauge group acting on the two boundaries, i.e. the BCs.

- **UV brane**: The fields at \( z = l_0 \) are taken to be classical sources: this defines a generating functional. The 4D SU \((N_f) \times SU(N_f)\) local symmetry at \( z = l_0 \) (a subgroup of the 5D gauge invariance) then guarantees the Ward identities of a 4D SU \((N_f) \times SU(N_f)\) global symmetry.

- **IR brane**: Of the SU \((N_f) \times SU(N_f)\) transformations acting at \( z = l_1 \), we allow only the vector subgroup as a symmetry of the whole theory. This is done by imposing the BCs

\[
\begin{align*}
R_\mu(x, z = l_1) - L_\mu(x, z = l_1) &= 0, \quad (2) \\
R_{5\mu}(x, z = l_1) + L_{5\mu}(x, z = l_1) &= 0. \quad (3)
\end{align*}
\]

3. Goldstone bosons

With the above BCs, the spectrum contains a massless 4D spin-0 mode, related to the (axial combination of) the fifth component of the gauge fields. The corresponding covariant object is obtained by constructing the following Wilson line

\[
U(x) = \mathcal{P} \left\{ e^{i \int_{l_1}^{l_0} dz R_5(x, z)} \right\} \times \mathcal{P} \left\{ e^{i \int_{l_1}^{l_0} dz L_5(x, z)} \right\}. \quad (4)
\]

\( U \) depends on the first four coordinates only. Moreover, of the whole (5D) group, \( U \) is only affected by those transformations acting on the UV brane, in a manner that is precisely appropriate for GBs of the 4D chiral symmetry defined in \( z = l_0 \): the breaking is spontaneous.

The interactions of the massless modes (the pions) at low energy can then be obtained. Paying attention to preserve the 4D chiral symmetries, one performs the identification with the chiral lagrangian operator by operator. The pion decay constant can be read off as \( 1/f_{\pi}^2 = g_\rho^2 \int_{l_0}^{l_1} \frac{dz}{w(z)} \).

The \( O(p^4) \) low-energy constants of the \( \chi PT \) lagrangian can be computed. In addition, the following relations hold independently of the metric (the first two are familiar from [11]):

\[
\begin{align*}
L_2 &= 2L_1 = -1/3L_3, \quad (5) \\
L_{10} &= -L_9. \quad (6)
\end{align*}
\]

4. Resonances

Having extracted the massless scalar, we perform redefinitions to eliminate the fifth components. Also, due to the BCs (2,3), the action will be diagonal in terms of vector and axial combinations of the original gauge fields. One then follows the standard KK procedure: a 5D field \( V_\mu(x, z) \) is decomposed as a sum of 4D modes \( V_\mu^{(n)}(x) \), each possessing a profile \( \varphi_\mu^n(z) \) in the fifth dimension, i.e.

\[
V_\mu(x, z) = \sum_{n=1}^{\infty} V_\mu^{(n)}(x) \varphi_\mu^n(z).
\]

The wave-functions \( \varphi(z) \) are entirely determined by the metric, as solutions of \(-\partial_z^2 \varphi_n(z) \varphi_n^Z = M^2_n w(z) \varphi_n^Z \), with BCs deduced from those of Section 2. The BCs are found to be, for the vector case \( \varphi_\mu^n(z) \big|_{z=l_0} = \partial_z \varphi_\mu^n(z) \big|_{z=l_1} = 0 \), and for the axial one \( \varphi_\mu^n(z) \big|_{z=l_0} = \varphi_\mu^n(z) \big|_{z=l_1} = 0 \).

This yields an alternating tower of vector and axial resonances. The masses of the heavy resonances behave as \( M_{V_n, A_n} \sim n \) as a consequence of the 5D Lorentz invariance broken by the finite size of the extra-dimension. This conflicts with the expectation \( M_{V_n, A_n} \sim \sqrt{n} \) from the quasiclassical hadronic string [11].

Couplings of resonances are expressed as bulk integrals over the fifth coordinate. One derives sum rules using the completeness relation for the KK wave-functions:

\[
\sum_{n=1}^{\infty} f_{V_n} g_{V_n} = 2L_3, \quad (7)
\]

\[
\sum_{n=1}^{\infty} f_{V_n} g_{V_n} M_{V_n}^2 = f_\pi^2, \quad (8)
\]

\[
\sum_{n=1}^{\infty} g_{V_n}^2 = 8L_1, \quad (9)
\]

\[
\sum_{n=1}^{\infty} g_{V_n}^2 M_{V_n}^2 = \frac{f_\pi^2}{3}, \quad (10)
\]

\[
\sum_{n=1}^{\infty} (f_{V_n}^2 - f_{A_n}^2) = -4L_{10}. \quad (11)
\]

The first three relations are reminiscent of those considered in [11], but generalized to infinite sums. They ensure that the vector form factor (VFF) and GB forward elastic scattering amplitudes.

\footnote{For definitions and derivations see [11], and also [9].}
amplitude respectively satisfy an unsubtracted and once-subtracted dispersion relation, as expected in QCD. The sum rule (11) shows that the natural value for the KSFR II ratio is 3 here, rather than 2.

5. $\rho$-meson dominance

Before extracting numerical values, we consider the question of $\rho$-meson dominance. As can be expected in any 5D model, only the light resonances will have non-negligible overlap integrals with the pion field, since the GBs are non-local objects in the fifth dimension [7]. As a consequence, the sums (7-10) are dominated by the first resonance, the $\rho$. In Table 1 we focus on the VFF, which can be put in the form $F(q^2) = \sum_{n=1}^{\infty} \frac{f_{Vn} g_{Vn} M_{Vn}^2}{f_\pi^2 M_{Vn}^2-q^2}$; two metrics are considered, flat space and AdS.

| $w(z) = 1$ | $w(z) = \frac{q_1}{z}$ |
|-----------------|-----------------|
| $f_{Vn} g_{Vn} M_{Vn}^2 / f_\pi^2$ | $\sim \left(\frac{-1)^n}{\sqrt{n}}$ |

Table 1: Behavior of the quantity $f_{Vn} g_{Vn} M_{Vn}^2 / f_\pi^2$; compare with the results quoted in [8].

The situation is then the following: independently of the metric, the 5D gauge structure ensures two properties: soft HE behavior through the sum rules (7-9) and approximate saturation of these sums by the first term ($\rho$-meson dominance). Approximating the sum rules by keeping only the first term, we would recover exactly the same relations as in [7]. As in this reference, we are therefore assured that, once the two input parameters are matched to $M_\rho \approx 776$ MeV and $f_\pi \approx 87$ MeV, the predictions for the low-energy constants and decays of the $\rho$ will provide good estimates, see Table 2. This result does not depend strongly on the chosen metric. On the other hand, the masses of the resonances above the $a_1$ grow too fast with $n$, as mentioned before.

\begin{tabular}{|c|c|c|c|}
\hline
& $w(z) = 1$ & $w(z) = \frac{q_0}{z}$ & Experiment \\
\hline
$10^4 L_1$ & 0.5 & 0.5 & 0.4 $\pm$ 0.3 \\
$10^4 L_2$ & 1.0 & 1.1 & 1.35 $\pm$ 0.3 \\
$10^4 L_3$ & $-3.1$ & $-3.1$ & $-3.5$ $\pm$ 1.1 \\
$10^4 L_9$ & 5.2 & 6.8 & 6.9 $\pm$ 0.7 \\
$10^4 L_{10}$ & $-5.2$ & $-6.8$ & $-5.5$ $\pm$ 0.7 \\
$\Gamma_{\rho \rightarrow \pi\pi}$ & 4.4 & 8.1 & 7.02 $\pm$ 0.11 \\
$M_{A_1}$ & 1.6 $\times$ 10^4 & 1.2 $\times$ 10^4 & 1230 $\pm$ 40 \\
$M_{V_1}$ & 2.3 $\times$ 10^4 & 1.8 $\times$ 10^4 & 1465 $\pm$ 25 \\
$M_{V_2}$ & 3.9 $\times$ 10^3 & 2.8 $\times$ 10^3 & 1688.1 $\pm$ 2.1 \\
\hline
\end{tabular}

Table 2: Numerical outputs. Experimental values for the $L_i$'s correspond to a renormalization scale $\mu = M_\rho$ [8]. Dimensionful quantities are in MeV, except for $\Gamma_{\rho \rightarrow \pi\pi}$ in KeV.

6. Two-point functions

We particularize to an exactly conformal metric $w(z) = l_0 / z$: conformal invariance is then broken only by the presence of the IR brane. The vector and axial two-point functions contain an infinite number of poles, and can be expressed for timelike momenta as

$$\Pi_{V,A}(q^2) = -\frac{l_0}{g_5^2} \left( \log \frac{q^2}{\mu^2} + \lambda \right) + 2\pi \sum_{n=1}^{\infty} \frac{V_{n1}(q_{l})}{J_{n1}(q_{l})}$$

where $\lambda = \log(\mu^2 l_0^2/4) + 2\gamma_E$.

The first term in this expression reproduces the partonic logarithm, as already emphasized in [7,8]. The last piece in the correlator (12) comes from the breaking of conformality produced by (different) BCs on the IR brane. Correspondingly, in the euclidean $Q^2 = -q^2 > 0$, it is suppressed by an exponential factor $e^{-\lambda q_{l}}$ at high energies. Therefore, (12) does not reproduce any of the condensates that appear in the QCD operator product expansion (OPE) [8].

The absence of these power corrections (in the difference of vector and axial two-point functions) translates as an infinite number of generalized Weinberg sum rules [10]: this result will hold as

5A similar result was noted in [7,8].
long as the symmetry-breaking effects are localized at a finite distance from the UV brane.

Expanding now at low energies $q l_1 \ll 1$, the logarithmic contribution is canceled by part of the last term in (12) to end up with an analytic function in powers of $q l_1$

$$\frac{g_5^2}{l_0} \Pi_V (q^2) \simeq c + \frac{1}{2} l_0^2 q^2 + \frac{5}{64} l_0^4 q^4 + O (q^6) \quad (13)$$

The constant term is the expected one from the chiral expansion: in RS1, we can verify that $c = -10 l_0 = 4 L_{10} - 8 H_1$. We have seen that the result (12) lacked power corrections to reproduce QCD results. Also, our matching at low-energy — to $f_\pi$ and $M_\rho$ — corresponds to $N_c \approx 4.3$ (identifying the factor $l_0 / g_5^2$ in (12)). To correct these flaws, one can modify the metric $^7$. A metric that behaves near the UV brane as $\frac{1}{z} w(z) - 1 \sim (z - l_0)^{2d}$ will generate a $1/Q^{2d}$ correction to (12). This introduces new parameters, allowing to impose $N_c = 3$ while preserving numerical agreements for low-energy quantities. The case $d = 2$ will represent a gluon condensate. Modifications with $d = 3$, if felt differently by the vector and axial fields, will describe the $(q^2 / Q^6$ term.

7. Conclusions

An $SU(N_f) \times SU(N_f)$ Yang-Mills model with a compact extra-dimension implements some essential properties of the pions and vector and axial resonances of large-$N_c$ QCD. The model in its simplest form contains two parameters: the 5D gauge coupling and another parameter related to the geometry.

The simplicity of our model comes from the way the spontaneous breaking of the 4D global $SU(N_f) \times SU(N_f)$ symmetry is implemented: by BCs. As a consequence, the 5D Yang-Mills fields yield a multiplet of pions descending from the fifth component of the axial vector field $A_5$, without the need for any other degree of freedom.

Our results can be summarized in four points:

1. We have derived, for a generic metric, the low-energy limit of such a model, while at the same time preserving chiral symmetry. This ensures that the lagrangian obtained is that of Chiral Perturbation Theory.

2. We have extracted resonance interactions and demonstrated the ensuing soft HE behavior. This result is independent of the 5D metric. It allows to match the behavior of QCD for some amplitudes.

3. $\rho$-meson dominance holds due to the 5D structure. It ensures that low-energy quantities give good estimates of the QCD ones, and are not very sensitive to the geometry. However, it should be noted that the way $\rho$-meson dominance is realized depends on the precise form of the metric.

4. We have derived an analytic expression for the vector and axial two-point functions at all momenta. This is done for AdS$_5$ space, which enables a further matching with the partonic logarithms of QCD $^7$. We also propose modifications of the metric as a means of improving the matching with the QCD OPE.

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