Uniqueness and regularity of unbounded weak solutions to a class of cross diffusion systems

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Abstract. We establish the uniqueness and regularity of weak (and very weak) solutions to a class of cross diffusion systems which is inspired by models in mathematical biology/ecology, in particular the Shigesada–Kawasaki–Teramoto model in population biology. No boundedness assumption on these solutions is supposed here as known techniques for scalar equations such as maximum/comparison principles are generally unavailable for systems. Furthermore, for planar domains we show that unbounded weak solutions satisfying mild integrability conditions are in fact smooth.

Mathematics Subject Classification. 35J70, 35B65, 42B37.

Keywords. Cross diffusion systems, Hölder regularity, Global existence.

1. Introduction

In this paper, we study the following parabolic system of $m$ equations ($m \geq 2$) for the unknown vector $u = [u_i]_{i=1}^m$ for the unknown vector $u = [u_i]_{i=1}^m$

$$u_t = \Delta(P(u)) + f(u), \quad (x, t) \in Q := \Omega \times (0, T_0). \quad (1.1)$$

Here, $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a $C^2$ map and $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a $C^1$ map. $\Omega$ is a bounded domain with smooth $C^2$ boundary in $\mathbb{R}^N$, $N \geq 2$, and $T_0 > 0$.

The system is equipped with boundary and initial conditions

$$\begin{cases} u = 0 \text{ on } \partial\Omega \times (0, T_0), \\
u(x, 0) = u_0(x), \quad x \in \Omega. \end{cases} \quad (1.2)$$

The consideration of (1.1) is motivated by the extensively studied model in population biology introduced by Shigesada et al. in [18]

$$\begin{cases} u_t = \Delta(d_1 u + a_{11} u^2 + a_{12} uv) + k_1 u + \beta_{11} u^2 + \beta_{12} uv, \\
v_t = \Delta(d_2 v + a_{21} uv + a_{22} v^2) + k_2 v + \beta_{21} uv + \beta_{22} v^2. \end{cases} \quad (1.3)$$
Here, $d_i, \alpha_{ij}, \beta_{ij}$ and $k_i$ are constants with $d_i > 0$. Dirichlet or Neumann boundary conditions were usually assumed for (1.3). This model was used to describe the population dynamics of two species densities $u, v$ which move and react under the influence of population pressures.

Under suitable assumptions on the constant parameters $\alpha_{ij}$’s, $\beta_{ij}$’s and that $\Omega$ is a planar domain ($N = 2$), Yagi proved in [19] the global existence of (strong) positive solutions, with positive initial data. In this paper, among other general settings, we will investigate weak solutions to multi-species versions of (1.3) for more than two species and consider much more general structural conditions. Naturally, we will replace the quadratics in the Laplacians and $f_i$ of (1.3) by polynomials of order $k + 1$ for some $k > 0$. Obviously, the system (1.3) is a special case of (1.1) with $m = 2$ and $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ being a quadratic map.

Let us describe the multi-species version of (1.3). For $m \geq 2$ let $u : \Omega \times (0,T_0) \rightarrow \mathbb{R}^m$ and $\alpha_i, \beta_i \in \mathbb{R}^m, i = 1, \ldots, m$. The multi-species version of (1.3) is then (1.1) with

$$P_i(u) = d_i u_i + u_i \langle \alpha_i, u \rangle, \quad f_i(u) = k_i u_i + u_i \langle \beta_i, u \rangle.$$  \hspace{1cm} (1.4)

The generalized multi-species version of (1.3) is naturally obtained by replacing $\alpha_i, \beta_i$ by maps on $\mathbb{R}^m$ with certain polynomial growth. That is, $|\alpha_i(u)|, |\beta_i(u)| \sim |u|^\kappa$ for $u \in \mathbb{R}^m$ and some $\kappa \geq 0$. In this paper, with this setting, we will refer to (1.1) as the SKT system if $\kappa = 1$ and the generalized SKT one if $\kappa > 0$ in general. We should mention a recent work [15] where a similar setting was considered.

We can write (1.1) in its divergence form

$$u_t = \text{div}(A(u) Du) + f(u),$$

where $A(u) = P_u(u)$, the Jacobian of $P(u)$. For appropriate $\alpha_{ij}$’s (see [19] for (1.3)) this system is parabolic in the sense that $P_u$ is normally elliptic (see [1]). That is, there are a constant $\lambda_0 > 0$ and a linear function $\lambda(u) = \lambda_0 + |u|$ such that $\lambda(u) \geq \lambda_0$ and

$$\langle P_u(u) \zeta, \zeta \rangle \geq \lambda(u) |\zeta|^2, \text{ for all } u \in \mathbb{R}^m \text{ with } u_i \geq 0, \zeta \in \mathbb{R}^{Nm}.$$  

This will be the structural main assumption on (1.1) (see the condition A) and the remark follows it) and we note that, in this paper, the above condition can be assumed for $u \in K$ in general where $K$ is a cone in $\mathbb{R}^m$, an invariant set of the weak solution $u$. In [19], $K = \mathbb{R}^m_+$. For simplicity we take $K = \mathbb{R}^m$ here. Furthermore, ellipticity function $\lambda(u)$ can be of polynomial growth in $u$.

Our first goal is to establish the uniqueness of unbounded weak solutions of (1.1). To our best knowledge, this is the first time this problem is treated for cross diffusion systems like (1.1) with no boundedness is assumed. The problem was addressed in [4] for scalar equations, where maximum principles were available, and it was nontrivial already. Here, we are working with cross diffusion systems and no invariant/maximum principles are available so that the uniqueness question must be treated in a completely different way. Also, without boundedness, the matrix $A(u) = P_u(u)$ is not regular elliptic as it is not bounded from above. Under very weak integrability conditions
on $u$, we obtain the uniqueness of weak (and very weak) solution of (1.1) in Theorem 2.3. Its immediate consequences then apply to the generalized weak solutions from $V_2(Q)$ (following the common definition of [9] in literature) of the SKT system (1.3) and its generalized versions. For examples, \textit{nonnegative and unbounded} weak solution of SKT considered in [19] is unique if it satisfies our integrabilities and assumptions.

Next, we consider the regularity properties of weak solutions (again, unbounded). This is a long standing and hard problem in the theory of pde’s, especially for strongly coupled parabolic systems like (1.1) or even (1.3). Here, by combining our uniqueness results with the theory in [12,13] (see also [10]) which dealt with strong solutions, we consider (1.1) defined on planar domains ($N = 2$) and show that \textit{unbounded} generalized solutions from $V_2(Q)$ of the SKT systems and its generalized versions are in fact classical. This result is new and our indirect approach (a combination of the studies of uniqueness of weak solutions and existence of strong ones) may come as a surprise in comparison with direct methods which worked only with \textit{bounded} weak solutions but not in our case (see Remark 5.3). The paper is organized as follows. In Sect. 2, we state the general structural conditions on (1.1), the main uniqueness and regularity results and some of their immediate applications to the SKT systems. More examples will be discussed later in Sects. 4 and 5 when we complete the proof of the main results because some improvements can be obtained by slight modifications of the proof thanks to additional and special structures of the models and need more detailed discussion. Technical tools will be presented in Sect. 3. Section 4 is devoted to the proof of the uniqueness result. We prove and discuss the regularity results in Sect. 5.

2. The main results

We state our main results in this section. We first discuss the uniqueness of weak solutions. This has been done for scalar parabolic equations for \textit{bounded} weak solutions. But this is not the case for systems because the boundedness of solutions to systems generally is an open problem and the arguments for scalar equations are not applicable here. For the system (1.1) we will establish this result for \textit{unbounded} weak solutions which is defined in a very general sense and satisfy very mild integrability conditions.

The system (1.1) with boundary and initial condition (1.2) is a special case of the parabolic system in divergent form with $A(u) = P_u(u)$, the Jacobian of $P(u)$,

$$u_t = \text{div}(A(u)Du) + f(u).$$

(2.1)

Following the standard definition, we say that
Definition 2.1. $u$ is a weak solution on $\Omega \times (0, T_0)$ the system (2.1) with boundary and initial condition (1.2) if $u \in L^\infty((0, T_0), L^1(\Omega))$ and $A(u) Du \in L^1(\Omega \times (0, T_0))$ and for a.e. $T \in (0, T_0)$ and any $\phi \in C^1(\Omega \times (0, T))$ we have
\[
\int_\Omega \langle u(T), \phi(T) \rangle - \langle u_0, \phi(0) \rangle \, dx = \int_{\Omega \times (0, T)} [(u, \phi_t) - \langle A(u) Du, D\phi \rangle + \langle f(u), \phi \rangle] \, dz.
\]

(2.2)

In this paper, as $A(u) Du = D(P(u))$, from the Eq. (2.2) with a simple integration by parts in $x$, assuming that $P(u) = 0$ on $\partial\Omega \times (0, T_0)$ and using the homogeneous Dirichlet boundary condition, we also have the following weaker definition for weak solution $u$ of (1.1) with no integrability assumption on (weak) derivatives of $u$ and more restrictive admissible test functions.

Definition 2.2. $u$ is a very weak solution on $\Omega \times (0, T_0)$ the system (1.1) with boundary and initial condition (1.2) if $u \in L^\infty((0, T_0), L^1(\Omega))$ and $P(u) \in L^1(\Omega \times (0, T_0))$ and for a.e. $T \in (0, T_0)$ and any $\phi \in C^2(\Omega \times (0, T))$ we have
\[
\int_\Omega \langle u(T), \phi(T) \rangle - \langle u_0, \phi(0) \rangle \, dx = \int_{\Omega \times (0, T)} [(u, \phi_t) + \langle P(u), \Delta \phi \rangle + \langle f(u), \phi \rangle] \, dz.
\]

(2.3)

We note that a admissible test function $\phi$ in this definition, as a minimum requirement, needs only that $\phi_t, \Delta \phi \in L^\infty(\Omega \times (0, T_0))$. Also, in both definitions, we can consider initial data $u_0 \in L^1(\Omega)$.

In order for (2.1) to be regular parabolic with $A(u) = P(u)$, we naturally impose our main assumption on the structure of the system is

A) $P(0) = 0$. $P_u$ is regular elliptic. That is there are function $\lambda$ and constant $\lambda_0 > 0$ such that $\lambda(u) \geq \lambda_0$ and
\[
\langle P_u(u) \zeta, \zeta \rangle \geq \lambda(u)|\zeta|^2, \text{ for all } u \in \mathbb{R}^m, \zeta \in \mathbb{R}^{N \times m}.
\]

Here, with a slightly abuse of notations, the dot product of $P_u(u) \zeta$ and $\zeta$ for $P = [P(u)_{i,j}]_1^m$, $u = [u_i]_1^m$ and $\zeta = [\zeta_i]_1^m$ (with $u_i \in \mathbb{R}$ and $\zeta_i \in \mathbb{R}^N$) should be understood as $\sum_{i,j} P_{u_{ij}} \langle \zeta_i, \zeta_j \rangle$. It means the Jacobian of $P$ in $u$ is the normal elliptic as described in [1].

Furthermore, we assume further that

F) there is a convex function $\hat{F}$ such that $|\partial_u f(u)|^2\lambda(u)^{-1} \leq \hat{F}(u)$ on $\mathbb{R}^m$.

We also introduce the following notations for our theorem statements.

\[
p_r := \frac{r'p}{p - r'}, \text{ where } r' = r/(1 - r), \text{ the conjugate of } r,
\]

(2.4)

\[
\sigma_N = \begin{cases} 
\text{any number in } (1, \infty) & \text{if } N = 2, \\
\text{any number in } (1, 6 + \frac{10}{N - 2}) & \text{if } N = 3, \\
\frac{2(N + 2)}{N - 2} & \text{if } N \geq 4.
\end{cases}
\]

(2.5)

The main result of this paper is the following uniqueness theorem.

Theorem 2.3. Assume A) and F). For some $p > 2$ we also assume the following integrability continuity conditions (with $Q = \Omega \times (0, T_0)$ and the notations in (2.4) and (2.5)):
(i) The map \( u \mapsto \partial_u P(u) \) is continuous from \( L^p(Q) \) to \( L^{p^2}(Q) \),
(ii) The map \( u \mapsto \partial_u f(u) \) is continuous from \( L^p(Q) \) to \( L^{p+\infty}(Q) \).

If \( u \) is a very weak solution, in the sense of Definition 2.2, and satisfies \( u \in L^p(Q) \) and for some \( q_0 \geq N/2 \)

\[
\sup_{t \in (0,T_0)} \| \hat{F}(u(t)) \|_{L^{q_0}(\Omega)} < \infty, \tag{2.6}
\]
then \( u \) is unique.

We will see that the conditions of this theorem can be verified in many models in application. To discuss this matter further, we note that there are many ways to define the concept of weak solution and they all start with the Eq. (2.2) in Definition 2.1 (or (2.3) in Definition 2.2) and there is a trade off among these definitions in order that the integrals of (2.2) are all finite. The main difference lies in the choices of admissible test function \( \phi \) in (2.2). If the space of admissible test functions is more restrictive then the space of weak solutions will be in wider and it is harder to obtain the uniqueness result and vice versa. We would like to remark this fact here for future references.

**Remark 2.4.** Of course, Definition 2.2 is weaker than Definition 2.1 so that Theorem 2.3 also applies to weak solutions in the sense of Definition 2.1.

Still, our Definition 2.1 is an enough general one as we needs only that the first order derivatives of \( \phi \) are defined and \( \phi \in C^1(Q) \). Consequently, a weak solution \( u \) is this sense needs only satisfy \( u \in L^\infty((0,T_0),L^1(\Omega)) \) and \( D(P(u)), f(u) \) are in \( L^1(Q) \) in order that the integrals in (2.2) are all finite. Of course, our class of weak solutions is sufficiently wide and the checking of their integrability conditions of Theorem 2.3 seems to be already not an easy condition under such limited information.

On the other hand, if we allow more general test function \( \phi \) then the space of weak solutions will be smaller and the uniqueness result can be applied easily and almost immediate in some cases.

Following [9, Chapter III], which has been used widely in literature, we say that \( u \) is a generalized solution from \( V_2(Q) \), the Banach space with norm

\[
\| u \|_{V_2(Q)} = \sup_{t \in (0,T_0)} \| u \|_{L^2(\Omega \times \{t\})} + \| Du \|_{L^2(Q)},
\]

if \( u \) satisfies (2.2) for any test function \( \phi \in W_2^{1,1}(Q) \), the Hilbert space with scalar product

\[
\langle u,v \rangle_{W_2^{1,1}(Q)} = \iint_Q [\langle u,v \rangle + \langle u_t,v_t \rangle + \langle Du,Dv \rangle] \, dz.
\]

Adopting this concept of weak solutions, we have

**Corollary 2.5.** Generalized solutions from \( V_2(Q) \) to the SKT system on domains in \( \mathbb{R}^N \) with \( N \leq 4 \) are unique.

Much more on the generalized version of the SKT systems will be discussed in Sect. 4.
Next, we will consider the regularity of unbounded weak (and very weak) solutions of (1.1). This is a very hard problem in the theory of pdes, especially for strongly coupled parabolic systems with (1.1) as a special case. There is a vast literature on this problem, see [6], and all assume that the considered weak solution is bounded and satisfies a crucial condition that its BMO norm is small in small balls. Here, for any ball $B_R$ of radius $R$ in $\mathbb{R}^N$ and $\Omega_R = B_R \cap \Omega$ the BMO norm of a (vector valued) function $u$ is defined by

$$\|u\|_{BMO(\Omega_R)} := \sup_{B_r \subset \Omega_R} \left( \int_{B_r} |u - u_r| \, dx + \int_{\Omega_R} |u| \, dx \right),$$

where $u_r$ is, as usual, the average of $u$ over $B_r$. Beside the boundedness assumption, which is already a very hard problem for weak solutions to cross diffusion systems, the smallness of the norm (2.7) when $R$ is small is even a harder problem and, to the best of our knowledge, none were done in literature to address either question in this general setting.

In this paper, we follow an indirect and novel approach to this regularity problem and the idea is very simple: If we can show that there exists a strong solution $u$ to (1.1) then this solution is of course also a weak one and satisfies the integrability conditions of Theorem 2.3. By the uniqueness result for such weak solutions, any weak solution satisfying sufficient integrability of Theorem 2.3 is exactly this strong solution $u$ and therefore it is in fact classical (or smooth). Thus, we immediately have the following statement.

**Corollary 2.6.** Assume that (1.1) possesses a strong solution. If $u$ is a (very) weak solution of (1.1) and satisfies the integrability conditions of Theorem 2.3 then $u$ is a classical one.

Of course, the existence of a strong solution to (1.1) is also one of the hardest problems in the theory of cross diffusion systems so that the hypothesis of Corollary 2.6 is a very bold one. This existence problem was considered in the pioneering work by Amann [1] (see also [19]), using semigroup theory. However, his hypothesis on a priori estimates for the gradients of (strong) solutions was also equivalent to the investigation of Hölder continuity of the solutions, again a difficult task for systems and only few works were done and relied on very ad hoc techniques which applied only to special cases of (1.1). Recently, in [13], we introduced an alternative approach using fixed point theories to establish the existence of strong solutions under a set of a priori integrability conditions and, again crucially, the smallness of the norm (2.7). One of the advantages of the theory is that one can remove the boundedness assumption in [1] and replace it by certain mild integrability ones.

Yet, establishing the smallness of the norm (2.7) eludes many efforts for general dimension $N$. We present in Sect. 5 the conditions $S)$ and $S')$ which can be verified for (1.1) in applications to affirm both integrability and smallness of the norm (2.7) when $N = 2$. We restrain ourselves from giving the details of these conditions as they need some technical preparations and discussion. Here, we just state the following consequence for the SKT systems and its generalizations.
Corollary 2.7. Suppose further that the reaction term \( f(u) \) satisfies
\[ \langle f(w), w \rangle \leq \varepsilon_0 \lambda(w)|w|^2 + C|w|^2 \] (2.8)
for some positive constants \( C, \varepsilon_0 \). If \( \varepsilon_0 \) is sufficiently small, in terms of the diameter of \( \Omega \), then the generalized solution from \( V_2(Q) \) of the SKT system and its generalized versions (for \( k < 2 \)) on planar domains are classical.

Even in this special case \( (N = 2) \), the result is new and remarkable because no boundedness is assumed.

3. Some technical lemmas

In this section, we collect some technical lemmas some of which may be elementary to experts and others are subtle will play crucial roles in the proof of our main results and examples.

First of all, in the proof we will frequently make use of the following interpolation Sobolev inequality

**Lemma 3.1.** For any \( \varepsilon > 0, \beta \in (0, 1], \ p \geq 1 \) and \( W \in W^{1,p}(\Omega) \) we can find a constant \( C(\varepsilon, \beta) \) such that
\[ \|W\|_{L^q(\Omega)} \leq \varepsilon \|DW\|_{L^p(\Omega)} + C(\varepsilon, \beta)\|W^\beta\|_{L^1(\Omega)} \] for any \( q \in [1, p_*) \). (3.1)

Here and throughout this paper, \( p_* \) as usual, denotes the Sobolev conjugate of \( p \).

\[ p_* = \begin{cases} Np/(N-p) \quad \text{if } p < N, \\ \text{any number in}(1, \infty) \text{ otherwise}. \end{cases} \]

**Proof.** By contradiction, assume that (3.1) is not true then we can find \( \varepsilon_0 > 0 \) and a sequence \( \{W_n\} \) such that
\[ \|W_n\|_{L^q(\Omega)} > \varepsilon_0 \|DW_n\|_{L^p(\Omega)} + n\|W^\beta_n\|_{L^1(\Omega)} \] for any \( n \). (3.2)

By scaling we can suppose that \( \|W_n\|_{L^q(\Omega)} = 1 \). The above implies that \( \|DW_n\|_{L^p(\Omega)} < 1/\varepsilon_0 \) for all \( n \). We see that \( \{W_n\} \) is bounded in \( W^{1,p}(\Omega) \) so that, by compactness (as \( q < p_* \)), we can assume that it converges to some \( W \) in \( L^q(\Omega) \). Of course, \( \|W\|_{L^q(\Omega)} = 1 \). Meanwhile, (3.2) implies \( \|W^\beta_n\|_{L^1(\Omega)} \to 0 \) so that \( \|W^\beta\|_{L^1(\Omega)} = 0 \), which can be easily seen by Hölder’s inequality and the Hölder continuity of the function \( |x|^{\beta} \). Thus \( W = 0 \) a.e. on \( \Omega \) and contradicts the fact that \( \|W\|_{L^q(\Omega)} = 1 \). The proof is complete. \( \Box \)

Next, we need some estimates of solutions to the following linear parabolic system
\[ \begin{cases} \Psi_t = A\Delta \Psi + \mathcal{G}\Psi \quad \text{on } Q = \Omega \times (0,T), \\ \Psi = 0 \quad \text{on } \partial \Omega \times (0,T), \\ \Psi(x,0) = \psi(x). \end{cases} \] (3.3)

Here, \( A(x,t), \mathcal{G}(x,t) \) are matrices of sizes \( m \times m \) and \( m \times 1 \) respectively and satisfy the following condition
AG) $\mathcal{A}, \mathcal{G}$ are smooth in $Q$ and $\mathcal{A}$ is regular elliptic. That is, there are function $\lambda_*$ on $Q$ and a constant $\lambda_0 > 0$ such that $\lambda_*(x,t) \geq \lambda_0$ and
\[ \lambda_*(x,t)|\zeta|^2 \leq \langle \mathcal{A}(x,t)\zeta, \zeta \rangle \text{ for all } \zeta \in \mathbb{R}^m \text{ and all } (x,t) \in Q. \] (3.4)

Furthermore, $\|\mathcal{A}(\cdot, t)\|_{L^\infty(\Omega)}$ is continuous at $t = 0$ and the $L^\infty(\Omega)$ norms of $DA(\cdot, t)$ and $\mathcal{G}(\cdot, t)$ are bounded near $t = 0$.

Next, we present a lemma which concerns the behavior of $\Psi$ near $t = 0$ and is in the spirit of the Hille-Yoshida theorem (e.g., see [3, Theorem 7.8]) on the continuity of the (weak) derivatives of weak solutions to (3.3) when $\mathcal{A}$ is a constant. Otherwise, this theorem does not apply directly to our case because $\mathcal{A}$ depends on both $A$ and $G$ and test the system with $H = 0$ that $s > 0$ where
\[ A(x,0) = \mathcal{A}(x,0) \text{ and } B(x,t) = \mathcal{A}(x,t) - A(x). \]
Also, $h(0) = \psi$ and $H(0) = 0$. We rewrite the equation for $H$ as
\[ H_t = \text{div}(ADH + BDh) - DADH - DBDH + \mathcal{G}H \]
and test the system with $H$ and use the fact that $H(0) = 0$ to obtain for any $s > 0$ that
\[ \int_{\Omega \times \{s\}} |H|^2 \, dx + \int_0^s \int_{\Omega} \langle ADH, DH \rangle \, dx \
\leq -\int_0^s \int_{\Omega} (\langle BDh, DH \rangle + \langle DADH + DBDH, H \rangle + \langle G, H \rangle) \, dx. \] (3.6)
Applying Young’s inequalities to the integrands involving $DH$ on the right hand side and using the ellipticity assumption (3.4), we easily get
\[ \int_0^s \int_{\Omega} |DH|^2 \, dxdt \leq C \int_0^s \int_{\Omega} |B|^2 |Dh|^2 \, dxdt \]
\[ + C \int_0^s \int_{\Omega} (|DA|^2 + |G|)|H|^2 \, dxdt + C \int_0^s \int_{\Omega} |DB|Dh|H| \, dxdt. \]
We now divide the about inequality by $s$ to have
\[ \frac{1}{s} \int_0^s \int_{\Omega} |DH|^2 \, dxdt \leq C \frac{1}{s} \int_0^s \int_{\Omega} |B|^2 |Dh|^2 \, dxdt \]
\[ + C \frac{1}{s} \int_0^s \int_{\Omega} (|DA|^2 + |G|)|H|^2 \, dxdt + C \frac{1}{s} \int_0^s \int_{\Omega} |DB|Dh|H| \, dxdt. \] (3.7)

**Lemma 3.2.** Assume AG) and $\psi \in W^{1,2}(\Omega)$. If $\Psi$ is a strong solution to (3.3) then
\[ \liminf_{t \to 0} \|D\Psi\|_{L^2(\Omega \times \{t\})} \leq \|D\psi\|_{L^2(\Omega)}. \] (3.5)

**Proof.** We split $\Psi = h + H$ with $h, H$ solving
\[ h_t = a(x)\Delta h + \mathcal{G}h, \quad H_t = A\Delta H + B\Delta h + \mathcal{G}H, \]
where $a(x) = \mathcal{A}(x,0)$ and $B(x,t) = \mathcal{A}(x,t) - a(x)$. Also, $h(0) = \psi$ and $H(0) = 0$. We rewrite the equation for $H$ as
\[ H_t = \text{div}(ADH + BDh) - DADH - DBDH + \mathcal{G}H \]
and test the system with $H$ and use the fact that $H(0) = 0$ to obtain for any $s > 0$ that
\[ \int_{\Omega \times \{s\}} |H|^2 \, dx + \int_0^s \int_{\Omega} \langle ADH, DH \rangle \, dx \
= -\int_0^s \int_{\Omega} (\langle BDh, DH \rangle + \langle DADH + DBDH, H \rangle + \langle G, H \rangle) \, dx. \] (3.6)
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We now divide the about inequality by $s$ to have
\[ \frac{1}{s} \int_0^s \int_{\Omega} |DH|^2 \, dxdt \leq C \frac{1}{s} \int_0^s \int_{\Omega} |B|^2 |Dh|^2 \, dxdt \]
\[ + C \frac{1}{s} \int_0^s \int_{\Omega} (|DA|^2 + |G|)|H|^2 \, dxdt + C \frac{1}{s} \int_0^s \int_{\Omega} |DB|Dh|H| \, dxdt. \] (3.7)
We will let $s \to 0$ and need to investigate the limits of the terms on the right hand side.

From the definition of $B$, $B(x,t) = A(x,t) - A(x,0)$. As $\|A(\cdot,t)\|_{L^\infty(\Omega)}$ is continuous at $t = 0$, we see that $\lim_{t \to 0} \|B(\cdot,t)\|_{L^\infty(\Omega)} = 0$. Also, by the Hille-Yoshida theorem (e.g., see [3, Theorem 7.8], note that $a(x) = A(x,0)$ is elliptic, smooth and independent of $t$), $Dh$ belongs to $C([0,T_0],L^2(\Omega))$. In particular, $\|Dh(t)\|_{L^2(\Omega)}$ is continuous at $t = 0$. Hence, the limit of the first term on the right hand side of (3.7) when $s \to 0$ is zero.

Meanwhile, as $H$ is a strong solution and $H(0) = 0$, we have
\[
\frac{1}{s} \int_0^s \int_\Omega |H|^q \, dx \, dt \to 0.
\]
By the assumption AG), the $L^\infty(\Omega)$ norms of $DA(\cdot,t)$ (and then $DB(\cdot,t)$) and $G(\cdot,t)$ are bounded near $t = 0$. We see that the last two terms tend to 0. Hence, letting $s \to 0$ in (3.7), we derive $\lim_{t \to 0} \|DH\|_{L^2(\Omega \times \{t\})} = 0$. Meanwhile, as we explained above, $\lim_{t \to 0} \|Dh(t)\|_{L^2(\Omega)} = \|D\psi\|_{L^2(\Omega)}$. Because $\Psi = h + H$, we obtain (3.5) of the lemma.

**Remark 3.3.** Lemma 3.2 still holds if the $L^p(\Omega)$ norms of $DA(\cdot,t)$ and $G(\cdot,t)$ are bounded near $t = 0$ for some finite $p > 1$. Indeed, as $H$ is a strong solution and $H(0) = 0$, we have
\[
\frac{1}{s} \int_0^s \int_\Omega |H|^q \, dx \, dt \to 0
\]
for any $q$ as $s \to 0$. It is easy to see that a simple use of Hölder’s inequality shows that the last two integrals of (3.7) still tend to 0.

**Remark 3.4.** The assertion of Lemma 3.2 also applies to the system $\Psi_t = \text{div}(AD\Psi) + G\Psi$ which is in divergence form. The systems for $h, H$ now are
\[
\begin{aligned}
\dot{h}_t &= \text{div}(a(x)Dh) + Gh, \\
\dot{H}_t &= \text{div}(ADH) + \text{div}(BDh) + GH.
\end{aligned}
\]
We then repeat the same argument and obtain (3.7) without the terms involving $DA, DB$ so that the assumption on the boundedness of $DA$ (and so $DB$) near $t = 0$ can be dropped.

We then have the following estimates for derivatives of $\Psi$ which will play an important role in our proof of uniqueness results.

**Lemma 3.5.** Assume AG) and let $\psi \in C^1(\Omega, \mathbb{R}^m)$. Then there is a classical solution $\Psi$ to (3.3). In addition, assume that there are constants $C_0, q_0$ such that $q_0 \geq N/2$ ($q_0 > 1$ if $N = 2$) and
\[
\sup_{(0,T)} \|g_s\|_{L^\infty(\Omega \times \{s\})} \leq C_0, \quad \text{where } g_s := |G|^2 \lambda_s^{-1}. \tag{3.8}
\]
Then there is a constant $C(T, C_0)$ such that
\[
\int_{\Omega \times \{s\}} |D\Psi|^2 \, dx \leq C(T, C_0) \|D\psi\|_{L^2(\Omega)} \text{ for all } s \in [0, T], \tag{3.9}
\]
\[
\iint_Q |\Delta\Psi|^2 \, dz \leq \lambda_0^{-1} C(T, C_0) \|D\psi\|_{L^2(\Omega)}. \tag{3.10}
\]
One should note that the constant $C(T, T_0)$ in (3.9) and (3.10) is independent of the norms of $A, G$ and their derivatives.

**Proof.** The existence of $\Psi$ is clear because $A, G$ are smooth and $A$ is regular elliptic. We just need to establish (3.9) and (3.10). Multiplying (3.3) with $\Delta \Psi$ and integrating by parts ($\Psi_t = 0$ on the boundary as homogeneous Dirichlet condition is considered for $\Psi$), we get for any $0 < s < T$ and $Q(s) = \Omega \times (0, s)$

$$
\int_{Q(s)} \frac{d}{dt} |D\Psi|^2 \, dz + \int_{Q(s)} \langle A \Delta \Psi, \Delta \Psi \rangle \, dz = - \int_{Q(s)} \langle G \Psi, \Delta \Psi \rangle \, dz.
$$

By the ellipticity of $A$ we get for any $s' \in (0, s)$

$$
\int_{\Omega \times \{s\}} |D\Psi|^2 \, dx + \int_{Q(s)} \lambda_* |\Delta \Psi|^2 \, dz \leq \int_{\Omega \times \{s'\}} |D\Psi|^2 \, dx + \int_{Q(s)} \langle G \Psi, \Delta \Psi \rangle \, dz.
$$

Next, by Young’s inequality

$$
\langle G \Psi, \Delta \Psi \rangle \leq \varepsilon \lambda_* |\Delta \Psi|^2 + C(\varepsilon)|g_*|^2, g_* := |G|^2 \lambda_*^{-1}.
$$

Therefore, for small $\varepsilon > 0$ we deduce from (3.11) the following inequality

$$
\int_{\Omega \times \{s\}} |D\Psi|^2 \, dx + \int_{Q(s)} \lambda_* |\Delta \Psi|^2 \, dz \leq \int_{\Omega \times \{s'\}} |D\Psi|^2 \, dx + C \int_{Q(s)} g_* |\Psi|^2 \, dz.
$$

Because $N/2 \leq q_0, 2q'_0 \leq 2s_*$, the Sobolev conjugate of 2, so that we can use Hölder and Sobolev’s inequalities (Ψ = 0 on the boundary) to estimate the integral of $g_* |\Psi|^2$ over $\Omega \times \{\tau\}$ by

$$
\left( \int_{\Omega \times \{\tau\}} g_*^{q_0} \, dx \right)^{\frac{1}{q_0}} \left( \int_{\Omega \times \{\tau\}} |\Psi|^{2q'_0} \, dx \right)^{\frac{1}{2q'_0}} \leq C \int_{\Omega \times \{\tau\}} |D\Psi|^2 \, dx, \quad \tau \in (0, T).
$$

Here, we used the assumption (3.8) on $g_*$. Hence,

$$
\int_{Q(s)} g_* |\Psi|^2 \, dz \leq C \int_{Q(s)} |D\Psi|^2 \, dz.
$$

Using this in (3.12) we deduce

$$
\int_{\Omega \times \{s\}} |D\Psi|^2 \, dx + \int_{Q(s)} \lambda_* |\Delta \Psi|^2 \, dz \leq \int_{\Omega \times \{s'\}} |D\Psi|^2 \, dx + C \int_{Q(s)} |D\Psi|^2 \, dz.
$$

Replacing $s'$ in (3.14) by $s_k$ where $s_k$ belongs to a sequence in $(0, s)$ such that the limit of $\|D\Psi(\cdot, s_k)\|_{L^2(\Omega)}$ is $\liminf_{t \to 0} \|D\Psi(\cdot, t)\|_{L^2(\Omega)}$ and using (3.5), we then derive

$$
\int_{\Omega \times \{s\}} |D\Psi|^2 \, dx + \int_{Q(s)} \lambda_* |\Delta \Psi|^2 \, dz \leq \|D\psi\|_{L^2(\Omega)} + C \int_0^s \int_{\Omega \times \{t\}} |D\Psi|^2 \, dx \, dt.
$$

(3.15)
Dropping the second term on the left side of (3.15), we obtain an integral Gronwall inequality for \( \| D\Psi \|_{\Omega \times \{s\}}^2 \) which yields \( \| D\Psi \|_{\Omega \times \{s\}} \leq C(T) \| D\psi \|_{L^2(\Omega)} \) for some constant \( C(T) \) and any \( s \in (0, T) \). This is (3.9). We also obtain the estimate (3.10) for \( \Delta \Psi \) from (3.15) and (3.9), using the fact that \( \lambda_* \geq \lambda_0 \) a positive constant. This completes the proof of the lemma. □

In addition, we have the following integrability result for \( \Psi \).

**Lemma 3.6.** Assume as in Lemma 3.5. Then \( \Psi \in L^{\sigma_N}(Q) \) with \( \sigma_N \) given by

\[
\sigma_N = \begin{cases} 
\text{any number in } (1, \infty) & \text{if } N = 2, \\
\text{any number in } (1, \frac{10}{3}) & \text{if } N = 3, \\
\frac{2(N+2)}{N-2} & \text{if } N \geq 4.
\end{cases}
\]  

(3.16)

Note that the number \( \sigma_N \) here is exactly the one defined in (2.5) and used in Theorem 2.3. The proof of this lemma based on the bounds (3.9) and (3.10) and the following parabolic Sobolev imbedding inequality.

**Lemma 3.7.** Let \( r^* = \frac{p}{N} \) if \( N > p \) and \( r^* \) be any number in \( (0, 1) \) if \( N \leq p \). For any sufficiently nonegative smooth functions \( g, G \) and any time interval \( I \) there is a constant \( C \) such that

\[
\int_{\Omega \times I} g^{r^*} G^p \, dz \leq C \sup_I \left( \int_{\Omega \times \{t\}} g \, dx \right)^{r^*} \int_{\Omega \times I} (|DG|^p + G^p) \, dz
\]  

(3.17)

If \( G = 0 \) on the parabolic boundary \( \partial \Omega \times I \) then the integral of \( G^p \) over \( \Omega \times I \) on the right hand side can be dropped.

Furthermore, if \( r < r^* \) then for any \( \varepsilon > 0 \) we can find a constant \( C(\varepsilon) \) such that

\[
\int_{\Omega \times I} g^{r} G^p \, dz \leq C \sup_I \left( \int_{\Omega \times \{t\}} g \, dx \right)^{r} \int_{\Omega \times I} (\varepsilon |DG|^p + C(\varepsilon) G^p) \, dz
\]  

(3.18)

**Proof.** For any \( r \in (0, 1) \) and \( t \in I \) we have via Hölder’s inequality

\[
\int_{\Omega} g^{r} G^p \, dx \leq \left( \int_{\Omega} g \, dx \right)^{r} \left( \int_{\Omega} G^{\frac{p}{1-r}} \, dx \right)^{1-r}.
\]  

(3.19)

If \( r = r^* \) then \( p/(1 - r) = N_* = pN/(N - p) \), the Sobolev conjugate of \( p \) if \( N > p \) (the case \( N \leq p \) is obvious), so that the Sobolev inequality gives

\[
\left( \int_{\Omega} G^{\frac{p}{1-r}} \, dx \right)^{1-r} \leq \int_{\Omega} (|DG|^p + G^p) \, dx.
\]

Using the above in (3.19) and integrating over \( I \), we easily obtain (3.17). On the other hand, if \( r < r^* \), then \( p/(1 - r) < N_* \). A simple contradiction argument (similar to that of the proof of Lemma 3.1) and the compactness of the imbedding of \( W^{1,p}(\Omega) \) into \( L^{p/(1-r)}(\Omega) \) imply that for any \( \varepsilon > 0 \) there is \( C(\varepsilon) \) such that

\[
\left( \int_{\Omega} G^{\frac{p}{1-r}} \, dx \right)^{1-r} \leq \varepsilon \int_{\Omega} |DG|^p \, dx + C(\varepsilon) \int_{\Omega} G^p \, dx.
\]
We then obtain (3.18).

\( \square \)

**Proof of Lemma 3.6.** We now have from (3.9) and (3.10) that
\[
\sup_{(0,T)} \| D\Psi \|_{L^2(\Omega)} < \infty. \tag{3.20}
\]

The case \( N = 2 \) is then obvious. Indeed, the bound for \( D\Psi \) in (3.20) and Sobolev’s embedding inequality (\( \Psi = 0 \) on \( \partial\Omega \)) imply that \( \sup_{(0,T)} \| \Psi^{q_1} \|_{L^1(\Omega)} \) is finite for any \( q_1 \in (1, \infty) \). We need only consider \( N \geq 3 \) below.

The system (3.3) is a linear system with smooth coefficients so that by a similar study of the fundamental solutions for systems as in [9, Chapter IV] we can apply [9, Theorem 9.1]. Indeed, by AG) \( \mathcal{A}, \mathcal{G} \) is continuous on \( Q \) so that \( \mathcal{A}, \mathcal{G} \in L^1_{\text{loc}}(s,Q) \) for any \( s \in (1, \infty) \) and all the conditions of [9, Theorem 9.1] are verified (in particular the boundary of \( \Omega \) is of class \( O^2 \) as required in [9, Theorem 9.1]).

Thus, with \( q = 2 \), we have from [9, (9.3) of Theorem 9.1] (with \( f = 0 \)) that
\[
\| \Psi \|_{2, Q}^{(2)} \leq C \| \psi \|_{2, Q}^{(1)},
\]
where \( \psi \) is the initial data of \( \Psi \) and \( \| W \|_{2, Q}^{(l)} = \sum_{2l+j=l} \| D_1^l D_2^j W \|_{L^2(Q)} \). The above estimate implies \( \| D^2 \Psi \|_{L^2(Q)} < \infty \).

We first apply Lemma 3.7 with \( g = |D\Psi|^2, G = |D\Psi| \) and \( p = 2 \). From the previous estimate for \( D^2 \Psi \), we then see that \( \| DG \|_{L^2(Q)} = \| D^2 \Psi \|_{L^2(Q)} < \infty \). Thus with \( p = 2 \) and \( r_1 = 2/N \) as \( N > 2 \), we get
\[
\int_Q |D\Psi|^{2r_1+2} \, dz = \int_Q g^{r_1} G^2 \, dz < \infty. \tag{3.21}
\]

The bound for \( D\Psi \) in (3.20) and Sobolev’s embedding inequality (\( \Psi = 0 \) on \( \partial\Omega \)) imply that \( \sup_{(0,T)} \| \Psi^{q_1} \|_{L^1(\Omega)} \) is finite for \( q_1 = \frac{2N}{N-2} \). We then apply Lemma 3.7 again with \( g = |\Psi|^{q_1}, G = |\Psi|, \) and \( p = 2r_1 + 2 \). Note that \( p > N \) if \( N = 3 \) only and \( r^* \) can be any number in \((0,1)\) in this case. Otherwise, \( r^* = p/N \). We use (3.21) to obtain an estimate for the integral of \( g^{r^*} G^p \) over \( Q \). A straightforward calculation with these parameters implies that \( \Psi \in L^{\sigma_N}(Q) \) with \( \sigma_N = q_1 r^* + p \) given by
\[
\sigma_N = \begin{cases} 
\text{any number in } (1, 6 + \frac{10}{3}) & \text{if } N = 3, \\
\frac{2(N+2)}{N-2} & \text{if } N \geq 4.
\end{cases}
\]

This is the exponent defined in (3.16) for \( N \geq 3 \) and the proof is complete.

**4. Proof of uniqueness of unbounded (very) weak solutions**

In this section we present the proof of Theorem 2.3 on the uniqueness of very weak solutions. Let us recall its assumptions. First of all, for some \( p > 2 \) we assume the following continuity conditions (the definitions of \( p_2, p_{\sigma_N}, \sigma_N \) are given in (2.4) and (2.5)).

(i) The map \( u \to \partial_u P(u) \) is continuous from \( L^p(Q) \) to \( L^{p_2}(Q) \),
(ii) The map \( u \to \partial_u f(u) \) is continuous from \( L^p(Q) \) to \( L^{p_{\sigma_N}}(Q) \).
The above continuity conditions forces $P, f$ to have polynomial growths as in [2, Chapter 3].

We then consider very weak solutions $u$ in $L^p(Q)$ which satisfy for some $q_0 \geq N/2$
\[ \sup_{t \in (0, T_0)} \| \hat{F}(u(t)) \|_{L^q_0(\Omega)} < \infty, \]  
\tag{4.1}

where $\hat{F}$ is a convex function in $F)$ and satisfies
\[ \left| \frac{\partial_u f(u)}{\lambda(u)} \right|^2 \leq \hat{F}(u), \quad \text{for all } u \in \mathbb{R}^m. \]

According to Definition 2.2, we recall that a very weak solution $u$ satisfies (2.3) for all admissible initial data $u_0$.

\[ \text{Lemma 4.2 following this proof that there is a strong solutions } \Psi \]

\[ \text{where } \hat{0} \geq N/\rho \int_1 u_i, n \phi(T) - \langle u_0, \phi(0) \rangle \ dx = \iint_{\Omega \times (0, T)} [(u, \phi_t) + (P(u), \Delta \phi) + (f(u), \phi)]
\]
\[ \hspace{1cm} \ dx. \]  
\tag{4.2}

\[ \text{Proof of Theorem 2.3:} \] For any $u_1, u_2$ we can write

\[ P(u_1) - P(u_2) = a(u_1, u_2)(u_1 - u_2), \quad a(u_1, u_2) := \int_0^1 \partial_u P(su_1 + (1 - s)u_2) \ ds, \]

\[ f(u_1) - f(u_2) = g(u_1, u_2)(u_1 - u_2), \quad g(u_1, u_2) := \int_0^1 \partial_u f(su_1 + (1 - s)u_2) \ ds. \]

Using these notations, if $u_1, u_2$ are two very weak solutions with the same initial data $u_0$ then we subtract the two equations (4.2) for $u_1, u_2$ to see that $w = u_1 - u_2$ satisfies

\[ \int_{\Omega} \langle w(T), \phi(T) \rangle \ dx = \iint_{\Omega \times (0, T)} \langle w, \phi \rangle + a(u_1, u_2)^T \Delta \phi + g(u_1, u_2)^T \phi \rangle \ dx. \]  
\tag{4.3}

We consider the sequences $\{u_{1,n}\}, \{u_{2,n}\}$ of mollifications of $u_1, u_2$. That is, we consider $C^\infty$ functions $\eta(t)$ and $\rho(x)$ whose supports are $(-1, 1)$ and $B_1(0)$ and $\|\eta\|_{L^1(\mathbb{R})} = \|\rho\|_{L^1(\mathbb{R}^N)} = 1$. Denote $\eta_n(t) = n\eta(t/n)$ and $\rho_n(x) = n^N \rho(x/n)$. For $i = 1, 2$ define

\[ u_{i,n}(t, y) = (\eta_n \phi_n) * u_i(t, y) = \int_{\mathbb{R}} \int_{\mathbb{R}^N} \eta_n(s - t) \phi_n(x - y) u_i(t, x) \ dx \ ds. \]

For each $n$ and any given $\psi \in C^1(\Omega)$ and $T \in (0, T_0)$ we will show in Lemma 4.2 following this proof that there is a strong solutions $\Psi_n$ to the systems

\[ \begin{cases}
\Psi_t + a(u_{1,n}, u_{2,n})^T \Delta \Psi + g(u_{1,n}, u_{2,n})^T \Psi = 0 \text{ in } Q := \Omega \times (0, T), \\
\Psi = 0 \text{ on } \partial \Omega \times (0, T), \\
\Psi(x, T) = \psi(x) \text{ on } \Omega.
\end{cases} \]  
\tag{4.4}

Furthermore, there is a constant $C(\|D\psi\|_{L^2(\Omega)})$ such that for all $n$

\[ \sup_{(0, T)} \|D\Psi_n\|_{L^2(\Omega)}, \ |\Delta \Psi_n|_{L^2(Q)}, \ |\Psi_n|_{L^{\infty}(Q)} \leq C(\|D\psi\|_{L^2(\Omega)}). \]  
\tag{4.5}
One should note that the above bounds are independent of $n$ ($\sigma_N$ is defined in (2.5)).

From the equation of $\Psi_n$ and (4.3) with $\phi = \Psi_n$ (this is eligible because $\Psi_n$ is a strong solution) we derive

$$
\int_{\Omega} \langle w(T), \psi \rangle \, dx = -\iint_{Q} \langle w, [a(u_{1,n}, u_{2,n})^T - a(u_1, u_2)^T] \Delta \Psi_n \rangle \, dz
- \iint_{Q} \langle w, [g(u_{1,n}, u_{2,n})^T - g(u_1, u_2)^T] \Psi_n \rangle \, dz.
$$

This is

$$
\int_{\Omega} \langle w(T), \psi \rangle \, dx = -\iint_{Q} \langle [a(u_{1,n}, u_{2,n}) - a(u_1, u_2)]w, \Delta \Psi_n \rangle \, dz
- \iint_{Q} \langle [g(u_{1,n}, u_{2,n}) - g(u_1, u_2)]w, \Psi_n \rangle \, dz. \tag{4.6}
$$

Letting $n \to \infty$, we will see that the integrals on the right hand side tend to zero. Indeed, we consider the first integral. The bound (4.5) implies $\|\Delta \Psi_n\|_{L^2(Q)}$ is bounded uniformly in $n$ so that we need only to show that $[a(u_{1,n}, u_{2,n}) - a(u_1, u_2)]w$ converges strongly to zero in $L^2(Q)$. By Hölder’s inequality with $q = p_2 = 2p/(p - 2)$

$$
\| [a(u_{1,n}, u_{2,n}) - a(u_1, u_2)]w \|_{L^2(Q)} \leq \| a(u_{1,n}, u_{2,n}) - a(u_1, u_2) \|_{L^0(Q)} \| w \|_{L^p(Q)}.
$$

As we are assuming in i) that the map $u \to \partial_u P(u)$ is continuous from $L^p(Q)$ to $L^0(Q)$ and because $u_{i,n} \to u_i$ in $L^p(Q)$, it is clear from the definition of $a$ that $a(u_{1,n}, u_{2,n})$ converges to $a(u_1, u_2)$ in $L^q(Q)$. Thus, $[a(u_{1,n}, u_{2,n}) - a(u_1, u_2)]w$ converges strongly to zero in $L^2(Q)$. Thus, the first integral on the right hand side of (4.6) tends to zero as $n \to \infty$.

Similar argument applies to the second integral to obtain the same conclusion. Using (4.5) we see that $\|\Psi_n\|_{L^p(Q)}$ are uniformly bounded. We then need only to show that $[g(u_{1,n}, u_{2,n}) - g(u_1, u_2)]w$ converges strongly to zero in $L^{\sigma_N}(Q)$. This is case by the assumption ii) on $\partial_u f$ and Hölder’s inequality with $q = p_{\sigma_N} = (\sigma_N)^p/(p - (\sigma_N)^p)$

$$
\| [g(u_{1,n}, u_{2,n}) - g(u_1, u_2)]w \|_{L^{\sigma_N}(Q)} \leq \| g(u_{1,n}, u_{2,n}) - g(u_1, u_2) \|_{L^0(Q)} \| w \|_{L^p(Q)}.
$$

We just prove that the right hand side of (4.6) tends to zero. We then have

$$
\int_{\Omega} \langle w(T), \psi \rangle \, dx = 0 \text{ for any } \psi \in C^1(\Omega).
$$

We conclude that $w(T) = 0$ for all $T \in (0, T_0)$. Hence $u_1 \equiv u_2$ on $Q$.

**Remark 4.1.** The continuity conditions i) and ii) are not needed if we discuss bounded weak solutions. Indeed, as the sequences $\{u_{1,n}\}$, $\{u_{2,n}\}$ converge to $u_1, u_2$ in $L^\infty(Q)$ we see that $a(u_{1,n}, u_{2,n}) \to a(u_1, u_2)$ and $g(u_{1,n}, u_{2,n}) \to g(u_1, u_2)$ strongly in $L^\infty(Q)$.

We now provide the following lemma which establish the claim (4.5) in the proof.
Lemma 4.2. For all $n$ there exist strong solutions $\Psi_n$ to the system (4.4) and a constant $C(\|D\psi\|_{L^2(\Omega)})$ independent of $n$ such that

$$\sup_{(0,T)} \|D\Psi_n\|_{L^2(\Omega)}, \|\Delta \Psi_n\|_{L^2(Q)}, \|\Psi_n\|_{L^{\sigma N}(Q)} \leq C(\|D\psi\|_{L^2(\Omega)}). \quad (4.7)$$

Proof. By a change of variables $t \rightarrow T - t$, the system (4.4) is equivalent to the following linear parabolic system for $\hat{\Psi}(x, t) = \Psi(x, T - t)$

$$\begin{aligned}
\hat{\Psi}_t &= a(u_{1,n}, u_{2,n})^T \Delta \hat{\Psi} + g(u_{1,n}, u_{2,n})^T \hat{\Psi} \text{ in } Q := \Omega \times (0, T), \\
\hat{\Psi}(x, t) &= 0 \text{ on } \partial \Omega \times (0, T), \\
\hat{\Psi}(x, 0) &= \psi(x) \text{ on } \Omega.
\end{aligned} \quad (4.8)$$

We will apply Lemma 3.5 here with $\mathcal{A} = a^T(u_{1,n}, u_{2,n})$ and $\mathcal{G} = g^T(u_{1,n}, u_{2,n})$. We need to verify the condition AG) for such $\mathcal{A}, \mathcal{G}$. These functions are smooth and bounded on $Q$, because $u_{1,n}, u_{2,n}$ are, so that AG) is clearly satisfied with

$$\lambda_s = \int_0^1 \lambda(su_{1,n} + (1 - s)u_{2,n}) \, ds \geq \lambda_0 > 0.$$ 

Thus, the existence of $\Psi_n$ is clear and we need only to establish (4.7) by checking the condition (3.8) of Lemma 3.5. To this end, we have to consider the function $g_* = |\mathcal{G}|^2/\lambda_s$ and show that $\|g_*\|_{L^{q_0}(\Omega)}$ is bounded for some $q_0 \geq N/2$. Here,

$$\mathcal{G} = g(u_{1,n}, u_{2,n})^T = \int_0^1 \partial_u f(su_{1,n} + (1 - s)u_{2,n})^T \, ds$$

Denote $w(s) = su_{1,n} + (1 - s)u_{2,n}$. Writing $|\partial_u f(w(s))| = |\partial_u f(w(s))\lambda^\frac{2}{3}(w(s))$ and using Hölder’s inequality, we have

$$\int_0^1 |\partial_u f(w(s))| \, ds \leq \left( \int_0^1 \frac{|\partial_u f(w(s))|^2}{\lambda(w(s))} \, ds \right)^\frac{1}{2} \left( \int_0^1 \lambda(w(s)) \, ds \right)^\frac{1}{2}.$$ 

This implies for each $t \in (0, T)$ that

$$\frac{|\mathcal{G}|^2}{\lambda_s} \leq \int_0^1 \frac{|\partial_u f(w(s))|^2}{\lambda(w(s))} \, ds \leq \int_0^1 \hat{F}(s) \, ds,$$

where $\hat{F}$ is the convex function specified in $\mathcal{F})$ of the theorem and satisfying $|\partial_u f(w)|^2 \leq \hat{F}(w)$. Therefore,

$$\left\| \frac{|\mathcal{G}|^2}{\lambda_s} \right\|_{L^{q_0}(\Omega)} \leq \int_0^1 \left\| \hat{F}(su_{1,n} + (1 - s)u_{2,n}) \right\|_{L^{q_0}(\Omega)} \, ds.$$ 

As $\hat{F}$ is convex, by Jensen’s inequality $\hat{F}(su_{1,n} + (1 - s)u_{2,n}) \leq s\hat{F}(u_{1,n}) + (1 - s)\hat{F}(u_{2,n})$. Similarly, because $\|\eta_n \rho_n\|_{L^1(\mathbb{R}^N)} = 1$, for $i = 1, 2$ $\hat{F}(u_{i,n}) \leq (\eta_n \rho_n) \ast \hat{F}(u_i)$ so that

$$\left\| \hat{F}(u_{i,n}(t)) \right\|_{L^{q_0}(\Omega)} \leq \int_{\mathbb{R}} \eta_n(s-t) \|\rho_n \ast_x \hat{F}(u_i(t))\|_{L^{q_0}(\Omega)} \, ds.$$
Here, \(*_x\) denotes the convolution in \(\mathbb{R}^N\). Obviously, \(\|\rho_n *_x \hat{F}(u_i(t))\|_{L^q_0(\Omega)} \leq \|\hat{F}(u_i(t))\|_{L^q_0(\Omega)}\). As \(\|\eta_n\|_{L^1(\mathbb{R})} = 1\), we find a constant \(C_0\), by the assumption (4.1), such that
\[
\|\hat{F}(u_{i,n}(t))\|_{L^q_0(\Omega)} \leq \|\hat{F}(u_i(t))\|_{L^q_0(\Omega)} \leq C_0 \text{ for all } t \in (0, T) \text{ and integer } n.
\]
(4.9)

Hence, \(\|G\|^{2\lambda_n^{-1}}\|_{L^q_0(\Omega)} \leq C_0\) on \((0, T)\) for some \(q_0 \geq N/2\) and we can apply Lemma 4.2 to obtain a constant \(C(\|D\psi\|_{L^2(\Omega)})\), which is also dependent of \(C_0\) and \(T\) but independent of \(n\) such that
\[
\sup_{(0, T)} \|D\hat{\Psi}_n\|_{L^2(\Omega)}, \|\Delta\hat{\Psi}_n\|_{L^2(\Omega)} \leq C(\|D\psi\|_{L^2(\Omega)}).
\]

Because \(\Psi_n(x, t) = \hat{\Psi}_n(x, T - t)\), the above estimate implies the bound for the first two terms in (4.7). The bound for the last term in (4.7) comes from Lemma 3.6.

In many models in application, the components of the maps \(P, f\) of (1.1) are polynomials of \(u \in \mathbb{R}^m\). Our uniqueness theorem then applies to these models to show that unbounded weak solutions are unique if they satisfy sufficient integrability as in the following

**Corollary 4.3.** For some \(k, l > 0\) with \(2l - k \geq 1\) assume that \(\partial_u P(u)\) and \(\partial_u f(u)\) have polynomial growths
\[
|\partial_u P(u)| \leq C(|u|^k + 1), \ |\partial_u f(u)| \leq C(|u|^l + 1).
\]
Then the uniqueness conclusion of Theorem 2.3 applies to weak solutions in the space \(L^p(Q) \cap L^\infty((0, T_0), L^r(\Omega))\) if \(p \geq \max\{2(1 + k), (\sigma_N)'(1 + l)\}\) and \(r \geq (2l - k)N/2\).

**Proof.** We need to verify first the the assumptions i) and ii) of Theorem 2.3. It is clear from its proof that we need only to establish the convergences \(a(u_{1,n}, u_{2,n}) \to a(u_1, u_2)\) and \(g(u_{1,n}, u_{2,n}) \to g(u_1, u_2)\) in \(L^q(Q)\) for appropriate \(q\)'s along some subsequences of \(\{u_{1,n}\}, \{u_{2,n}\}\) which converge to \(u_1, u_2\) in \(L^p(Q)\). In particular, by the Riesz-Fisher theorem we can find subsequences of \(\{u_{i,n}\}\) and functions \(\hat{u}_i \in L^p(Q)\) such that, after relabeling, \(u_{i,n} \to u_i\) and \(|u_{i,n}| \leq \hat{u}_i\) a.e. in \(Q\). The growth condition of \(\partial_u P(u)\) then implies \(|a(u_{1,n}, u_{2,n})| \leq |\hat{u}_1|^k + |\hat{u}_2|^k + 1\), a function in \(L^{p/k}(Q)\). Furthermore, \(a(u_{1,n}, u_{2,n}) \to a(u_1, u_2)\) a.e. in \(Q\) because \(a\) is continuous. By the Dominated convergence theorem, we see that \(a(u_{1,n}, u_{2,n}) \to a(u_1, u_2)\) in \(L^{p/k}(Q)\). This also yields the convergence in \(L^q(Q)\) for \(q = p_2 = 2p/(p - 2)\) because \(q \leq p/k\) (as \(p \geq 2(1 + k)\)). Similarly, from the growth condition of \(\partial_u f\), we replace \(2, k\) respectively by \((\sigma_N)'(1 + l)\) in the argument to see that \(g(u_{1,n}, u_{2,n}) \to g(u_1, u_2)\) in \(L^{p/l}(Q)\) and thus in \(L^q(Q)\) for \(q = p\sigma_N\).

Finally, from the growth assumptions on \(\partial_u P\) and \(\partial_u f\), we see that \(|\partial_u f(u)|/X(u) \leq C|u|^{2l - k}\) for \(|u|\) large so that we can take the function \(\hat{F}\) defined in F) to be \(|C|u|^{2l - k}\). Because \(2l - k \geq 1\), \(\hat{F}\) is convex. We then have \(|\hat{F}(u)|^{q_0} \leq C|u|^{(2l-k)q_0}\). Because \(u \in L^\infty((0, T_0), L^r(\Omega))\) for some \(r \geq (2l - k)N/2\) from the assumption of the corollary, we have \(\sup_{(0, T_0)} \|\hat{F}(u)\|_{L^{q_0}(\Omega)}\) is
finite for some \( q_0 \geq N/2 \). We see that all assumptions of of Theorem 2.3 are verified here. This completes the proof. □

Remark 4.4. If \( P_u(u) \) is bounded, i.e., \( k = 0 \) then we can allow \( p = 2 \) in Theorem 2.3 (the condition i) is then dropped) and the above corollary. Indeed, from the proof of the theorem, we need the sequence \( \{ |a(u_{1,n}, u_{2,n}) - a(u_1, u_2)|w \} \), \( w = u_1 - u_2 \), converges strongly in \( L^2(Q) \). But this is obvious because this sequence converges pointwise in \( Q \) and is bounded by \( C|w| \), a function in \( L^2(Q) \). The Dominated convergence theorem applies again to give the corollary.

As we mentioned in Sect. 2, our Definition 2.1 is the most general one as one needs at least that the derivatives of the test function \( \phi \) are defined and bounded so that we need only \( \phi \in C^1(Q) \) so that we need only that \( u \in L^\infty((0,T_0), L^1(\Omega)) \) and \( D(P(u)) \in L^2(Q) \). This assumption is too weak in order to verify the integrability conditions of Theorem 2.3.

On the other hand, if we allow more general test function \( \phi \) then the space of weak solutions will be smaller and the uniqueness result can be applied easily and almost immediate in some cases. As an example, we will discuss an application of Corollary 4.3 to the SKT system and its generalizations. We now prove Corollary 2.5 concerning the generalized solutions from \( V^2(Q) \).

Proof of Corollary 2.5:. Let \( u \) be a generalized solution from \( V^2(Q) \). Now, the space of admissible test functions is \( W^{1,1}_2(Q) \) and it is clear that in order for the integrals in (2.2) to be finite for all \( \phi \in W^{1,1}_2(Q) \) we should assume further that \( u \in L^\infty((0,T), L^2(\Omega)) \) and \( D(P(u)) \in L^2(Q) \). The first assumption is satisfied because \( u \in V^2(Q) \). We now let \( g = |u| \), \( G = |P(u)| \) and \( p = 2 \) in (3.17). As \( |P(u)| = |u|^{k+1} \), we see that \( u \in L^{2r^{*}+2k+2}(Q) \) for some \( r^{*} > 0 \). The condition in Corollary 4.3 that \( u \in L^p(Q) \) with \( p \geq 2 + 2k \) is then obvious and this also implies \( p \geq (\sigma_N)'(1 + l) \) as \( l = k \) and \( (\sigma_N)' < 2 \). From Corollary 4.3, we need only that \( \sup_{(0,T_0)} \|u\|_{L^r(\Omega)} \) is finite for some \( r \geq kN/2 \). This condition is clearly satisfied for generalized solutions \( r = 2 \) to the usual SKT system, where \( P(u) \) has quadratic growth in \( u \) (so that \( k = 1 \)), in domains with dimension \( N \leq 4 \). Thus, \( u \) is unique.

The above proof also implies

Corollary 4.5. Generalized solutions to generalized SKT systems are unique as long as \( 1 \leq k \leq 4/N \).

5. Regularity

In this section, we consider the regularity problem of weak solutions of (1.1) when \( \Omega \) is a planar domain, i.e., \( N = 2 \). The main idea of the proof is simple. We will show that there exists a strong solution \( u \) to (1.1). This solution is of course a weak one. By the uniqueness result for weak solutions in the previous section, any weak solution satisfying sufficient integrability is then exactly this strong solution \( u \) and is in fact classical.
We first establish the existence of strong solutions of (1.1). To this end, we embed (1.1) in the following family of systems with \( \sigma \in [0, 1] \)

\[
\begin{aligned}
  w_t - \Delta (P(w)) &= \sigma^2 f(w), \ (x, t) \in \Omega \times (0, T_0), \\
  w &= 0 \text{ on } \partial \Omega \times (0, T_0), \\
  w(x, 0) &= \sigma u_0(x), \ x \in \Omega.
\end{aligned}
\]  

(5.1)

We then assume the following main a priori integrability condition on (5.1).

S) There are \( q_0 > 1 \) and a constant \( C_1 \), which may depend on \( T_0 \) but independent of \( \sigma \in [0, 1] \), such that any strong solutions \( w \) of this (5.1) satisfy

\[
\sup_{(0,T_0)} \| \lambda(w) \|_{L^{q_0}(\Omega \times \{t\})}, \sup_{(0,T_0)} \| w \|_{L^{q_0}(\Omega \times \{t\})} \leq C_1.
\]  

(5.2)

Furthermore, \( \lambda(u), f(u) \) have polynomial growths in \( |u| \) and for some constant \( C_1 \) all all \( u \in \mathbb{R}^m \)

\[
|\lambda_u(u)||u| \leq C \lambda(u) \text{ and } |f(u)| \leq C(1 + |u|)(1 + \lambda(u)).
\]  

(5.3)

We should note that the integrability condition (5.2) is a very mild one, especially it needs only be satisfied for strong solutions, and can be verified in many models.

We first establish the existence of a strong solution to (1.1). To this end, we will use the theory in [13] which needs smoother initial data \( u_0 \).

**Proposition 5.1.** Assume S) and \( u_0 \in W^{1,p}_0(\Omega) \) for some \( p > 2 \). Then there is a strong (classical) solution to (1.1).

**Proof.** The system (1.1) can be written as, with \( A(u) = P_u(u) \)

\[
\begin{aligned}
  u_t - \text{div}(A(u)Du) &= f(u), \ (x, t) \in \Omega \times (0, T_0), \\
  u &= 0, \ (x, t) \in \partial \Omega \times (0, T_0), \\
  u(x, 0) &= u_0(x), \ x \in \Omega.
\end{aligned}
\]  

(5.4)

To establish the existence of a strong solution to this system, we apply [13, Theorem 3.4.1] here by verifying its assumptions. First of all, we need to show that the number \( \Lambda = \sup_{u \in \mathbb{R}^m} \Lambda(u) \), with \( \Lambda(u) = |\lambda_u(u)|/\lambda(u) \), is finite. Since \( \lambda(u) \geq \lambda_0 > 0 \), if \( |u| \) is bounded then so is \( \Lambda(u) \). For large \( |u| \) we use the assumption in (5.2) that \( |\lambda_u(u)| \leq C \lambda(u)/|u| \) to see that \( \Lambda(u) \leq C/|u| \) is also bounded. Hence, the number \( \Lambda \) is finite.

Next, also following [13, Theorem 3.4.1], we embed (5.4) in the following family

\[
\begin{aligned}
  u_t - \text{div}(A(\sigma u)Du) &= \sigma f(\sigma u), \ (x, t) \in \Omega \times (0, T_0), \\
  u &= 0, \ (x, t) \in \partial \Omega \times (0, T_0), \\
  u(x, 0) &= \sigma u_0(x), \ x \in \Omega.
\end{aligned}
\]  

(5.5)

The most non trivial condition of the theory in [13] needs to be checked is that the strong solutions of this family has small BMO norm in small ball uniformly. Namely,
(Sbmo) For any given $\mu > 0$ there is $R > 0$ depending only on $\mu$ and the parameters of the system (1.1) (but not on $\sigma$) such that any strong solution $u$ to the family (5.5) satisfies: for any ball $B_R$ in $\mathbb{R}^N$ with $\Omega_R = B_R \cap \Omega \neq \emptyset$

$$\sup_{t \in (0, T_0)} \|u(\cdot, t)\|_{BMO(\Omega_R)} \leq \mu.$$ 

We is going to verify this property. Multiplying $\sigma > 0$ to the equation in (5.5), we see easily that $w = \sigma u$ is a strong solution to (5.1). 

For $\sigma = 1$ and a strong solution $w$ of (5.1) we can multiply the system of $w$ by $P(w)_t$ and follow the proof of [13, Lemma 5.3.2] and use the fact that $\frac{d}{dt} |D(P(u))|^2 = 2 \langle DP(u), (D(P(u))_t) \rangle = 2 \langle DP(u), D(P(u)_t) \rangle$,

to prove that: There is an absolute constant $C^*$ such that or any $t \in (0, T_0)$

$$\int_{\Omega \times \{t\}} \lambda(w)|w_t|^2 \, dx + \frac{d}{dt} \int_{\Omega \times \{t\}} |A(w)Dw|^2 \, dx \leq C^* \int_{\Omega \times \{t\}} \lambda(w)|f(w)|^2 \, dx. \tag{5.6}$$ 

This yields a Gronwall inequality for $\|A(w)Dw\|_{L^2(\Omega)}$. Indeed, as in the proof of [13, Proposition 5.3.1] (see also Theorem 5.4 in this paper) for general $N$, under the growth condition (5.3) in $S$ and its assumption that there is a constant $C_1$ such that

$$\sup_{(0, T_0)} \|\lambda(w)\|_{L^{q_0}(\Omega)}, \sup_{(0, T_0)} \|w\|_{L^{q_1}(\Omega)} \leq C_1 \tag{5.7}$$

for some $q_0 > N/2$ and $q_1 > 2N/(N + 2)$, which is fulfilled here by (5.2) as $N = 2$, we can prove that there is a constant $C$ such that

$$\int_{\Omega \times \{t\}} \lambda(w)|f(w)|^2 \, dx \leq C \int_{\Omega \times \{t\}} |A(w)Dw|^2 \, dx + C(C_1). \tag{5.8}$$

This and (5.6) imply

$$\frac{d}{dt} \int_{\Omega \times \{t\}} |A(w)Dw|^2 \, dx \leq C^* \left[ C \int_{\Omega \times \{t\}} |A(w)Dw|^2 \, dx + C(C_1) \right]. \tag{5.9}$$

On the other hand, we can apply Lemma 3.2 (and Remark 3.4) to the system (5.4) (in divergence form) here, with

$$A = A(w), \ G = \int_0^1 \partial_u f(sw) \, ds$$

to see that $\liminf_{t \to 0} \|Dw\|_{L^2(\Omega)} \leq \|Du_0\|_{L^2(\Omega)}$. The condition AG is satisfied here because $w$ is a strong solution and continuous at $t = 0$. Also, by Remark 3.4, we do not need the boundedness of $DA = DA(w)$ here for systems in divergence form like (5.5). It then follows

$$\liminf_{t \to 0} \|A(w)Dw\|_{L^2(\Omega)} \leq \|A(u_0)\|_{L^\infty(\Omega)} \|Du_0\|_{L^2(\Omega)}.$$
Therefore, from the Gronwall inequality (5.9) for $\|A(w)Dw\|_{L(\Omega)}^2$ and the fact that $\lambda(w)|Dw|$ is comparable to $|A(w)Dw|$, we have

$$\sup_{t \in [0,T_0]} \int_{\Omega} \lambda^2(w)|Dw|^2 \, dx \leq \|A(u_0)\|_{L^\infty(\Omega)}^2 \|Du_0\|_{L^2(\Omega)}^2 + C^* C(C_1),$$

where $C_1$ is the constant in (5.2).

For $\sigma \neq 1$ we replace $f(w), u_0$ respectively by $\sigma^2 f(w), \sigma u_0$ in the above argument. The constant $C^*$ in (5.6) and the above estimates is now $\sigma^2 C^*$ accordingly. As $w = \sigma u$, the above argument shows that

$$\sup_{t \in [0,T_0]} \int_{\Omega} \lambda(\sigma u)^2 |D(\sigma u)|^2 \, dx \leq \sigma^2 C(\|D u_0\|_{L^2(\Omega)}, C_1).$$

Using the fact that $\lambda(\sigma u) \geq \lambda_0$, we obtain

$$\lambda_0^2 \sup_{t \in [0,T_0]} \int_{\Omega} |D \sigma u|^2 \, dx \leq \sigma^2 C(\|D u_0\|_{L^2(\Omega)}, C_1).$$

Thus, the strong solutions $u$ of (5.5) satisfy

$$\sup_{t \in [0,T_0]} \int_{\Omega} |Du|^2 \, dx \leq \lambda_0^{-2} C(\|Du_0\|_{L^2(\Omega)}, C_1). \quad (5.10)$$

As $N = 2$, a simple use of Poincaré’s inequality, the continuity of integral and the above estimate show that strong solutions $u$ to (5.5) satisfy the (Sbmo) condition in [13] uniformly in $\sigma \in (0,1]$ (see also [13, Corollary 3.4.4] for more details on the implication of the property Sbmo) from (5.10).

By Sobolev’s inequality and because $u = 0$ on the boundary, (5.10) implies that $\|u\|_{L^q(\Omega)}$ is uniformly bounded for any $q \geq 1$. From the polynomial growths of $\lambda$ and $f$, we now see that $\lambda(u), |f(u)|\lambda^{-1}(u)$ are in bounded by powers of $|u|$ so that their integrability conditions in [13, Theorem 3.4.1] are verified. The last condition needs to be checked is

$$\int_0^{T_0} \int_{\Omega} |Du|^2 \, dx \, dt \leq C(T_0) \quad (5.11)$$

for some constant $C(T_0)$. But this is an immediate consequence of (5.10).

We thus verified all conditions of [13, Theorem 3.4.1] and therefore obtain the existence of a strong solution of (1.1).

We now turn to the regularity problem of unbounded weak solutions. The following theorem shows that unbounded weak solutions are in fact smooth if they satisfy sufficient higher integrability. We would like to emphasize that the common and crucial assumption on the smallness of their BMO norms in small balls is not needed here.

**Theorem 5.2.** Assume the conditions of Theorem 2.3 and S). If $u$ is a very weak solution of (1.1), with initial data $u_0 \in W^{1,p_0}(\Omega)$ for some $p_0 > 2$, in the sense of Definition 2.2, and satisfies the integrability conditions of Theorem 2.3 then $u$ is a classical one.
Proof. By S), we already show in Proposition 5.1 that there exists a strong solution $u$ to (1.1). This solution is of course a bounded weak one and satisfies the integrability of Theorem 2.3. Now, any (very) weak solution satisfying sufficient integrability Theorem 2.3 must be this strong solution $u$ by the uniqueness. Thus, these weak solutions are in fact classical and the theorem is proved. □

**Remark 5.3.** Even for systems like (1.1) on planar domains, our regularity result asserted in Theorem 5.2 is remarkable because the direct approach in literature (e.g., see [6]) cannot apply here as the basic Caccioppoli and Poincaré inequalities are not available for weak solutions in the sense of Definition 2.2 (one cannot test the systems (4.2) with $u$, which is not admissible, to obtain such inequalities). In addition, the key estimate (5.10) is not available for weak solutions. In this proposition, we worked with strong solutions so that the integrals in starting inequality (5.6) (and those follow) were all finite in order to derive (5.10).

Let us consider an alternate version of S) and connect the condition sets of Theorem 2.3 and Proposition 5.1. Following Theorem 2.3 and with a slight abuse of notations, we define (without the convexity assumption in F))

$$\hat{F}(u) := |\partial_u f(u)|^2 \lambda^{-1}(u).$$

(5.12)

S’) There are $q_0 \geq 1$, $\beta \in (0,1)$ and constants $C_0, C_1$, which may depends on $T_0$ but independent of $\sigma \in [0,1]$, such that any strong solutions $w$ of (5.1) satisfies

$$\sup_{(0,T_0)} \|\hat{F}(w)\|_{L^{q_0}(\Omega \times \{t\})} \leq C_0, \sup_{(0,T_0)} \|\lambda(w)|w|\|_{L^1(\Omega \times \{t\}))} \leq C_1.$$

(5.13)

Furthermore, $\lambda(u), f(u)$ have polynomial growths in $|u|$ and for some positive constants $C,M$ if $|u| \geq M$ then

$$|\lambda(u)||u| \leq C \lambda(u),$$

(5.14)

$$|f(u)| \leq C |u| |\partial_u f(u)|.$$

(5.15)

**Theorem 5.4.** The conclusion of Theorem 5.2 still holds if S) is replaced by S’.

Proof. We just need to show that the condition S’) will lead to the same Gronwall inequality (5.9) for $\|A(w)Dw\|_{L^2(\Omega \times \{t\})}^2$ so that the proof of Proposition 5.1 can continue to provide the existence of a unique strong solution. Together with the uniqueness of Theorem 5.2 we obtain our assertion here.

We can argue as in the proof of Proposition 5.1 until we get (5.8). We need to estimate the last integral of $\lambda(w)|f(w)|^2$ in this inequality to obtain (5.9).

To this end, for each for $t \in (0,T)$ we denote $\Omega_{M,t} := \{(x,t) : |w(x,t)| \geq M\}$. On $\Omega \times \{t\} \setminus \Omega_{M,t}, |w| < M$ so that the integrand $\lambda(w)|f(w)|^2$ is bounded by some constant depending on $M$. On $\Omega_{M,t}$, we use the condition (5.15) and the notation (5.12) to see that $\lambda(w)|f(w)|^2 \leq C(\lambda(w)|w|)^2|\partial_u f(w)|^2 \lambda^{-1}(w) \leq$
C(\lambda(w)|w|^2 \hat{F}(w))\). Thus, we will estimate the integral of \((\lambda(w)|w|^2 \hat{F}(w))\) below. If \(q_0 \geq N/2\) then we can find \(q \in [1, 2\_]\) such that \(N/2 \leq (q/2)'</sup> \(q_0\) and apply the Hölder inequality and the (5.13) to have

\[
\int_{\Omega_{M,t}} (\lambda(w)|w|^2 \hat{F}(w) \, dx \leq \left( \int_{\Omega_{M,t}} (\lambda(w)|w|)^q \, dx \right)^{\frac{2}{q}} \| \hat{F}(w) \|_{L((\frac{q}{2})', \Omega)} ^{\frac{2}{q}} \leq C_0 \left( \int_{\Omega_{M,t}} (\lambda(w)|w|)^q \, dx \right)^{\frac{2}{q}} .
\]

Because \(q \leq 2\_), we can apply Lemma 3.1 with \(p = 2\) and \(W = \lambda(w)|w|\) to have

\[
\int_{\Omega_{M,t}} (\lambda(w)|w|)^q \, dx \leq \int_{\Omega_{M,t}} |D(\lambda(w)|w|)|^2 \, dx + C(\beta) \left( \int_{\Omega_{M,t}} (\lambda(w)|w|)^\beta \, dx \right)^{\frac{2}{\beta}}
\]

where \(\beta \in (0, 1)\). Let \(\beta\) be the exponent in (5.13). The last integral is then bounded by \(C_1\). On the other hand, by (5.14) we have \(|\lambda_u(w)||w| \leq C\lambda(w)|w|\) so that

\[
|D(\lambda(w)|w|)| \leq \lambda(w)|Dw| + |\lambda_u(w)||w||Dw| \leq C\lambda(w)|Dw|.
\]

Putting these estimates together and using \(|\lambda(w)Dw| \sim |A(w)Dw|\), we see that

\[
\int_{\Omega \times \{t\}} \lambda(w)|f(w)|^2 \, dx \leq C \int_{\Omega \times \{t\}} |A(w)Dw|^2 \, dx + C(C_0, C_1) + C(M).
\]

Hence, the Gronwall inequality (5.9) for \(|A(w)Dw|^2_{L(\Omega)}\) continues to hold and the proof of the theorem can go on as before. \(\square\)

**Remark 5.5.** The assumption (5.14) implicitly implies that \(\lambda(u)\) must have a polynomial growth in \(|u|\). Meanwhile, (5.15) does not require such growth for \(f(u)\). However, as we see in the proof of Proposition 5.1, we need that \(\|f(u)\lambda^{-1}(u)\|_{L^q(\Omega)}\) is bounded from some large \(q\) so that a polynomial growth for \(f(u)\) seems to be necessary.

**Remark 5.6.** Our argument shows that the Gronwall inequality (5.9) holds under \(S')\) with \(q_0 \geq N/2\) but eventually we still have to assume \(N = 2\) in order for the property Sbmo) can be verified in accordance with the theory in [13] to establish the existence of strong solutions.

We conclude the paper by presenting some examples. In particular, we consider (1.1) with polynomial growth for its data as in Corollary 4.3 and for simplicity we assume that \(k = l\) that is \(P, f\) have the same growth \(k + 1\). Immediately, we have the following result.

**Corollary 5.7.** Assume the growth conditions as in Corollary 4.3 with \(k = l\). Assume also \(S\) (or \(S')\)). If \(u\) is a weak solution of (1.1) in \(L^p(Q) \cap L^\infty((0, T_0), L^r(\Omega))\) for some \(p \geq 2(1 + k)\) and \(r \geq k\) then \(u\) is classical.
Proof. The proof is almost obvious. We apply Theorem 5.2 (or Theorem 5.2) with Corollary 4.3 in place of Theorem 2.3. The integrability condition of $u$ in Corollary 4.3 is already assumed here, noting that $\sigma_N > 2$ so that $(\sigma_N)' < 2$ and the condition $p \geq 2(1 + k)$ alone is sufficient.

The stated Corollary 2.7 in Sect. 2 on the regularity of generalized weak solutions to the SKT systems now follows easily.

Proof of Corollary 2.7: We recall the assumption (2.8)

$$\langle f(w), w \rangle \leq \varepsilon_0 \lambda(w)|w|^2 + C|w|^2$$ (5.17)

for some positive constants $C, \varepsilon_0$. We will show that if $\varepsilon_0$ is sufficiently small Corollary 5.7 applies to give that the generalized solutions from $V_2(Q)$ of the SKT system and its generalized versions (with $k < 2$) on planar domains are classical.

The description of the SKT system and its generalized versions clearly implies the growth condition (5.3) of $S). We will show that the integrability condition (5.2) also holds under (2.8) if $\varepsilon_0$ is sufficiently small.

Let $w$ be a strong solution of (5.1). Testing the system with $w$, we obtain for any $T \in (0, T_0)$ and $Q_T = \Omega \times (0, T)$

$$\sup_{t \in (0, T)} \int_{\Omega} |w|^2 \, dx + \int_{Q_T} \lambda(w)|Dw|^2 \, dz \leq C\sigma^2 \int_{Q_T} \langle f(w), w \rangle \, dz + \int_{\Omega} |u_0|^2 \, dx.$$ 

From the assumption (2.8) and the polynomial growth $\lambda(w) \sim (\lambda_0 + |w|)^k$ we deduce

$$\sup_{t \in (0, T)} \int_{\Omega} |w|^2 \, dx + \int_{Q_T} |w|^k|Dw|^2 \, dz \leq C\sigma^2 \int_{Q_T} \varepsilon_0|w|^{k+2} + |w|^2 \, dz + C.$$ 

Because $w = 0$ on $\partial\Omega$ and $N = 2$, using the Poincaré inequality, we can find a constant $C$ depending on the diameter of $\Omega$ such that

$$\int_{\Omega} |w|^{k+2} \, dx \leq C \int_{\Omega} |w|^k|Dw|^2 \, dx.$$ 

Hence, if $\varepsilon_0$ is sufficiently small then we can deduce from the above two inequalities that

$$\sup_{t \in (0, T)} \int_{\Omega} |w|^2 \, dx \leq C \int_{0}^{T} \int_{\Omega} |w|^2 \, dxdt + C.$$ 

This is a Gronwall inequality for $\|w\|_{L^2(\Omega)}$ and yields a bound for $\|w\|_{L^2(\Omega)}$.

As we are assuming $\lambda(u) \sim (\lambda_0 + |u|)^k$ and $k < 2$, the integrability condition (5.2) is verified for some $q_0 > 1$. The assertion then follows as in Corollary 5.7 and the proof is complete.

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References

[1] Amann, H.: Dynamic theory of quasilinear parabolic systems III. Global existence. Math. Z 202, 219–250 (1989)

[2] Appell, J., Zabrejko, P.: The superposition operator in Lebesgue spaces. In: Nonlinear Superposition Operators (Cambridge Tracts in Mathematics), pp. 89–118. Cambridge University Press, Cambridge (1990)

[3] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2010)

[4] Brezis, O.H., Crandall, M.: Uniqueness of solutions of the initial value problem for $u_t - \Delta \phi(u) = 0$. J. Math. Pures Appl. 58, 153–163 (1979)

[5] Friedman, A.: Partial Differential Equations. Holt, Rinehart and Winston, New York (1969)

[6] Giaquinta, M., Struwe, M.: On the partial regularity of weak solutions of nonlinear parabolic systems. Math. Z 179, 437–451 (1982)

[7] John, O., Stara, J.: On the regularity of weak solutions to parabolic systems in two spatial dimensions. Commun. Partial Differ. Equ. 27, 1159–1170 (1998)

[8] Kőnig, K.H.W.: Invariant regions for quasilinear reaction–diffusion systems and applications to a two population model. NoDEA 3, 421–444 (1996)

[9] Ladyzhenskaya, O.A., Solonnikov, V.A., Uraltseva, N.N.: Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs. AMS (1968)

[10] Le, D.: Regularity of BMO weak solutions to nonlinear parabolic systems via homotopy. Trans. Am. Math. Soc. 365(5), 2723–2753 (2013)

[11] Le, D.: Global Existence for Large Cross Diffusion Systems on Planar Domains. submitted

[12] Le, D.: Weighted Gagliardo–Nirenberg inequalities involving BMO norms and solvability of strongly coupled parabolic systems. Adv. Nonlinear Stud. 16(1), 125–146 (2016)

[13] Le, D.: Strongly Coupled Parabolic and Elliptic Systems: Existence and Regularity of Strong/Weak Solutions. De Gruyter, Berlin (2018)

[14] Le, D., Nguyen, V.: Global and blow up solutions to cross diffusion systems on 3D domains. Proc. AMS 144(11), 4845–4859 (2016)
[15] Lepoutre, T., Moussa, A.: Entropic structure and duality for multiple species cross-diffusion systems. Nonlinear Anal. 159, 298–315 (2017)

[16] Necas, J., Sverak, V.: On regularity of solutions of nonlinear parabolic systems. Ann. Scuola Norm Sup. Pisa Cl. Sci. (4) 18(1), 1–11 (1991)

[17] Redlinger, R.: Existence of the global attractor for a strongly coupled parabolic system arising in population dynamics. J. Differ. Equ. 118, 219–252 (1995)

[18] Shigesada, N., Kawasaki, K., Teramoto, E.: Spatial segregation of interacting species. J. Theor. Biol. 79, 83–99 (1979)

[19] Yagi, A.: Global solution to some quasilinear parabolic systems in population dynamics. Nonlinear Anal. 21, 603–630 (1993)

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Received: 18 June 2020.
Accepted: 19 February 2021.