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To cite this version:
Daniel Barlet. Meromorphic quotients for some holomorphic G-actions. 2015. hal-01151383

HAL Id: hal-01151383
https://hal.science/hal-01151383
Preprint submitted on 12 May 2015

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Meromorphic quotients for some holomorphic G-actions.

Daniel Barlet.*

12/05/15

Abstract. Using mainly tools from [B.13] and [B.15] we give a necessary and sufficient condition in order that a holomorphic action of a connected complex Lie group $G$ on a reduced complex space $X$ admits a strongly quasi-proper meromorphic quotient. We apply this characterization to obtain a result which assert that, when $G = K.B$ with $B$ a closed complex subgroup of $G$ and $K$ a real compact subgroup of $G$, the existence of a strongly quasi-proper meromorphic quotient for the $B$–action implies, assuming moreover that there exists a $G$–invariant Zariski open dense subset in $X$ which is good for the $B$–action, the existence of a strongly quasi-proper meromorphic quotient for the $G$–action on $X$.

AMS classification. 32 M 05, 32 H 04, 32 H 99, 57 S 20.

Key words. Strongly quasi-proper map, strongly quasi-proper meromorphic quotient, Holomorphic G-action, finite type cycles.

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1 Introduction

In this article we explain how the tools developed in [M.00], [B.08], [B.13] and [B.15]
can be applied to produce in suitable cases a meromorphic quotient of a holomorphic
action of a connected complex Lie group $G$ on a reduced complex space $X$. This
uses the notion of strongly quasi-proper map introduced in loc. cit. and our
first goal is to give three hypotheses, called [H.1], [H.2], [H.3] on the group action
which are equivalent to the existence of a strongly quasi-proper meromorphic
quotient, notion defined in the section 1.2.

The proof of this equivalence is the content of proposition 2.5.1 and theorem 2.6.1.
Then we discuss these hypotheses and give a simple sufficient condition [H.1str]
which implies [H.1]. The existence theorem for a strongly quasi-proper meromorphic
quotient under our three assumptions is applied to prove the following results :

**Theorem 1.0.1** Assume that we have a holomorphic action of a connected com-
plex Lie group $G$ on a reduced complex space $X$. Assume that $G = K.B$ where $K$
is a compact (real) subgroup of $G$ and $B$ a connected complex closed subgroup of $G$.
Assume that the action of $B$ on $X$ satisfies the condition [H.1str] on a $G$—invariant
Zariski open dense subset $\Omega$ in $X$, and the conditions [H.2] and [H.3]. Then the
$G$—action satisfies [H.1str], [H.2] and [H.3] ; so it has a strongly quasi-proper mero-
orphic quotient.

ACKNOWLEDGEMENTS. An important part of this article comes from discussions
with Peter Heinzner during a stay in Bochum. I want to thank him for his help and
for his hospitality.

2 Strongly quasi-proper meromorphic quotients.

2.1 Preliminaries.

For the definition of the topology on the space $C^f_n(X)$ of finite type $n$—cycles in $X$
and its relationship with the (topological) space $C^{\text{loc}}_n(X)$ we refer to [B-M], [B.13]
and [B.15].

For the convenience of the reader we recall shortly here the definition of a geometri-
cally f-flat map (GF map) and of a strongly quasi-proper map (SQP map) between
irreducible complex spaces and we give a short summary on some properties of the
SQP maps. For more details on these notions see [B.13] and [B.15].
Definition 2.1.1 A holomorphic map $f : M \to N$ between two irreducible irreducible complex spaces is called a geometrically f-flat map (a GF-map for short) if the following conditions are fullfilled:

i) The map is quasi-proper equidimensionnal and surjective.
   Let $n : \dim M - \dim N$.

ii) There exists a holomorphic map $\varphi : N \to C_f^1(M)$\(^1\) such that for $y$ generic in $N$ the cycle $\varphi(y)$ is reduced and equal to the set-theoretic fiber $f^{-1}(y)$ of $f$ at $y$.

A holomorphic map $f : M \to N$ between two irreducible irreducible complex spaces will be strongly quasi-proper (SQP map for short) if there exists a modification\(^2\) $\tau : \tilde{N} \to N$ such that the strict transform\(^3\) $\tilde{f} : \tilde{M} \to \tilde{N}$ of $f$ by $\tau$ is a GF map. A meromorphic map $M \dashrightarrow N$ will be called strongly quasi-proper when the projection on $N$ of its graph is a SQP map.

Note that a GF map has, by definition, a holomorphic fiber map and that a SQP holomorphic (or meromorphic) map has a meromorphic fiber map via the composition of the holomorphic fiber map of $\tilde{f}$ with the (holomorphic) direct image map $\tau_* : C^f_\alpha(\tilde{M}) \to C^f_\alpha(M)$. Of course, a SQP holomorphic map is quasi-proper, but the converse is not true. The notion of strongly quasi-proper map is stable by modification of the target space, property which is not true in general for a quasi-proper map having “big fibers” (see [B.15]).

Let $\pi : M \to N$ be a SQP map between irreducible complex spaces and define $n := \dim M - \dim N$. By definition of a SQP map, we can find a Zariski open dense subset $N_0$ in $N$ and a holomorphic map $\varphi_0 : N_0 \to C^f_\alpha(M)$ such that

i) For each $y$ in $N_0$ we have the equality of subsets $|\varphi_0(y)| = \pi^{-1}(y)$.

ii) For $y$ generic in $N_0$ the cycle $\varphi_0(y)$ is reduced.

Let $\Gamma \subset N_0 \times C^f_\alpha(M)$ be the graph of $\varphi_0$. Then also by definition of a SQP map, the closure $\bar{\Gamma}$ of $\Gamma$ in $N \times C^f_\alpha(M)$ is proper over $N$. Then, using the semi-proper direct image theorem 2.3.2 of [B.15], this implies that $\tilde{N} := \bar{\Gamma}$ is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on $N \times C^f_\alpha(M)$. Moreover the natural projection $\tau : \tilde{N} \to N$ is a (proper) modification.

Let $\tilde{M} := M \times_{N,\text{str}} \tilde{N}$ the strict transform of $M$ by $\tau$, that is to say the irreducible component of $M \times_N \tilde{N}$ containing the graph of $\pi_0$, the restriction of $\pi$ to the open set $\pi^{-1}(N_0)$\(^4\). Then let $\tilde{\pi} : \tilde{M} \to \tilde{N}$ the strict transform of $\pi$ by the modification $\tau$, which is induced on $\tilde{M}$ by the natural projection of $M \times_N \tilde{N}$ onto $\tilde{N}$. The

\(^1\)that is to say a f-analytic family of finite type $n$-cycles in $M$ parametrized by $N$.
\(^2\)a modification is, by definition, always proper.
\(^3\)By definition $\tilde{M}$ is the irreducible component of $M \times_N \tilde{N}$ which surjects on $\tilde{N}$ and $\tilde{f}$ is induced by the projection.
\(^4\)this graph is a Zariski open set in $M \times_N \tilde{N}$ which is irreducible as $N$ is irreducible.
The set-theoretical fiber at $\tilde{y} := (y,C) \in \tilde{N}$ of $\tilde{\pi}$ is the subset $|C| \times \tilde{y}$ in $\tilde{M}$. The map $\psi : \tilde{N} \to C'_h(M)$ given by $(y,C) \mapsto C \times \{\tilde{y}\}$ is holomorphic and satisfies $|\psi(\tilde{y})| = \tilde{\pi}^{-1}(\tilde{y})$ for all $\tilde{y}$ in $\tilde{N}$. Moreover $\psi(\tilde{y})$ is a reduced cycle for generic $\tilde{y}$ in $\tilde{N}$. So the map $\tilde{\pi}$ is geometrically f-flat. It is the canonical GF-flattning of $\pi$.

Then we have an isomorphism induced by $\psi$

$$\psi : \tilde{N} \to C'_h(\tilde{\pi})$$

where $C'_h(\tilde{\pi}) := \{C \in C'_h(\tilde{M}) / \exists \tilde{y} \in \tilde{N} \; s.t. \; |C| \subset \tilde{\pi}^{-1}(\tilde{y})\}$ is a closed analytic subset of $C'_h(M)$ (see [B.15] proposition 2.1.7.); the inverse map is induced by the holomorphic map $\tilde{\pi} : C'_h(\tilde{\pi}) \to \tilde{N}$ which associates to $\gamma \in C'_h(\tilde{\pi})$ the point in $\tilde{N}$ whose $\tilde{\pi}$–fiber contains $\gamma$ (see the proposition 2.1.7 of [B.15]).

The direct image of $n$–cycles by $\tau$ gives a holomorphic map $\tau_\ast : C'_h(\tilde{M}) \to C'_h(M)$ which sends $\tilde{N} \simeq C'_h(\tilde{\pi})$ to $C'_h(\pi)$. Let us show that it is an isomorphism of $\tilde{N}$ onto its image in $C'_h(\pi)$:

We have an obvious holomorphic map $\tilde{N} \to C'_h(\pi)$ given by $(y,C) \mapsto C$. We have also a holomorphic map $C'_h(\pi) \to N \times C'_h(M)$ given by $C \mapsto (\tilde{\pi}, C)$ where $\tilde{\pi} : C'_h(\pi) \to N$ is the map associating to $C \in C'_h(\pi)$ the point $y \in N$ such that $|C| \subset \tilde{\pi}^{-1}(y)$. This proves our claim.

Remark that $C'_h(\pi)$ is not, in general, a complex space (locally of finite dimension).

### 2.2 Action of $G$ on $C'_h(X)$.

Let $G$ be a Lie group. We shall say that $G$ acts continuously holomorphically on the reduced complex space $X$ when the action $f : G \times X \to X$ is a continuous map such that for each $g \in G$ fixed, the map $x \mapsto f(g, x)$ is a (biholomorphic) automorphism of $X$. Then there is a natural action of $G$ induced on the set $C'_h(X)$ of finite type $n$–cycles given by $(g,C) \mapsto g_\ast(C)$ where we denote $g_\ast(C)$ the direct image of the cycle $C$ by the automorphism of $X$ associated to $g \in G$. When $G$ is a complex Lie group and the map $f$ is holomorphic we shall say that the action is completely holomorphic.

**Proposition 2.2.1** The action of $G$ on $C'_h(X)$ is continuously holomorphic. This precisely means that the map $G \times C'_h(X) \to C'_h(X)$ given by $(g,C) \mapsto g_\ast(C)$ is continuous and that, for any $f$–analytic family of $n$–cycles $(C_s)_{s \in S}$ in $X$ parametrized by a reduced complex space $S$, the family $g_\ast((C_s)_{(g,s) \in G \times S}$ is $f$–analytic for each fixed $g \in G$.

If $G$ is complex Lie group and the action is completely holomorphic, the action of $G$ on $C'_h(X)$ is completely holomorphic, so for any $f$–analytic family of $n$–cycles $(C_s)_{s \in S}$ in $X$ parametrized by a reduced complex space $S$, the family $g_\ast((C_s)_{(g,s) \in G \times S}$ is $f$–analytic.

**Proof.** First we prove the continuity of the action of $G$ on $C'_{h \text{ loc}}(X)$. To apply the theorem IV 2.5.6 de [B-M] it is enough to see that the map $F : G \times X \to G \times X$ given by $(g,x) \mapsto (g,g.x)$ is proper. But if $L \subset G$ and $K \subset X$ are compacts sets,
we have $F^{-1}(L \times K) \subset L \times (L^{-1}K)$ which is a compact set in $G \times X$.

The only point left to prove the continuity statement for the topology of $C^f_n(X)$, assuming that the continuity for the topology of $C^{loc}_n(X)$ is obtained as follows:

Let $W$ be a relatively compact open set in $X$ and $\mathcal{W}$ be the open set in $C^f_n(X)$ of cycles $C$ such any irreducible component of $C$ meets $W$. Then we want to show that the set of $(g, s) \in G \times S$ such that $g_s(C)$ lies in $W$ is an open set in $G \times S$.

As the topology of $C^f_n(X)$ has a countable basis it is enough to show that if a sequence $(g_\nu, s_\nu)$ converges to $(g, s)$ with $g_s(C_s) \in W$ then for $\nu \gg 1$ we have also $(g_\nu)_s(C_{s_\nu}) \in W$. If this not the case, we can choose for infinitely many $\nu$ an irreducible component $\Gamma_\nu$ of $(g_\nu)_s(C_{s_\nu})$ which does not meet $W$. Up to pass to a sub-sequence, we may assume that the sequence $\Gamma_\nu$ converges in $C^{loc}_n(X)$ to a cycle $\Gamma$ which does not meet $W$ and is contained in $g_s(C_s)$. This is a simple consequence of the continuity of the $G-$action on $C^{loc}_n(X)$ and the characterization of compact subsets in $C^{loc}_n(X)$ (see [B-M] ch.IV). As any irreducible component of $g_s(C_s)$ meets $W$ this implies that $\Gamma$ is the empty $n-$cycle. This means that for any compact $K$ in $X$ there exists an integer $\nu(K)$ such that for $\nu \geq \nu(K)$ we have $\Gamma_\nu \cap K = \emptyset$. Choose now a compact neighbourhood $L$ of $g(K)$. For $\nu$ large enough we shall have $K \subset g_\nu^{-1}(L)$. This comes from the fact that the automorphisms $g_\nu^{-1}$ converge to $g^{-1}$ in the compact-open topology. Then this implies that for $\nu \geq \nu(L)$ the irreducible component $g_\nu^{-1}(\Gamma_\nu)$ of $C_{s_\nu}$ does not meet $K$. Then, when $s_\nu \to s$ the cycles $C_{s_\nu}$ does not converge to $C_s$ for the topology of $C^f_n(X)$ because we have some “escape at infinity” in a well chosen sub-sequence. Contradiction.

Lemma 2.2.2 Assume now that in the situation above we have a Lie group $H$ acting continuously holomorphically on $M$. Assume that $\pi$ satisfies $\pi(h.x) = \pi(x)$ for each $x \in M$. Then the strict transform $\tilde{M}$ of $M$ by the canonical modification $\tau : \tilde{N} \to N$ giving the canonical GF flattning of $\pi$ has a natural continuous holomorphic action of $H$ such that the projection $\tilde{M} \to M$ is $H-$equivariant. Moreover the GF map $\tilde{\pi} : \tilde{M} \to \tilde{N}$ satisfies $\tilde{\pi}(h.\tilde{x}) = \tilde{\pi}(\tilde{x})$ for each $(h, \tilde{x}) \in H \times \tilde{M}$.

Proof. We have a natural action of $H$ on $C^f_n(M)$ which is continuous and holomorphic for each fixed $h \in H$. As the action of $H$ is trivial on the set of fibers of $\pi$, the action of $H$ is trivial on $\tilde{N}$. Define the action of $H$ on $M \times_N \tilde{N}$ by the formula

$$h.(x, (\pi(x), C)) = (h.x, \pi(x), h_s(C)).$$

It is easy to see that this is a continuous holomorphic action, and that it leaves $\tilde{M}$ globally invariant. Now we have for each $h \in H$ and each $(x, \pi(x), C) \in \tilde{M} :$

$$\tilde{\pi}\left(h.(x, (\pi(x), C))\right) = \tilde{\pi}(h.x, (\pi(x), h_s(C))) = (\pi(x), h_s(C)) = (\pi(x), C) = \tilde{\pi}(x, (\pi(x), C))$$

because a limit of $H-$invariant cycles is an $H-$invariant cycle. \hfill \blacksquare
Proposition 2.2.3 Let $M$ be a reduced complex space and $(X_s)_{s \in S}$ a $f$-continuous family of $d-$dimensional finite type cycles parametrized by a compact subset $S$ in $\mathcal{C}_d^f(M)$. Let $(C_t)_{t \in T}$ be a $f$-continuous family of finite type non empty $n-$dimensional cycles in $M$ parametrized by a subset $T$ in $\mathcal{C}_n^f(M)$ which is compact in $\mathcal{C}_n^{\text{loc}}(M)$. We assume the following condition:

- There exists an open dense set $T'$ in $T$ such that each $C_t, t \in T'$ is equal to the union some $X_s$. 

Then $T$ is a compact subset in $\mathcal{C}_d^f(M)$.

Proof. First remark that, as $S$ is compact, there exists a compact set $L \subset M$ such that any irreducible component of any $X_s$ meets $L$.

Let $(t_m)_{m \in \mathbb{N}}$ be a sequence of points in $T'$ converging to a point $t \in T$ and denote by $C_m$ the cycle $C_{t_m}$ for short and $C_t = C_\infty$. Now choose for each $m$ an irreducible component $\Gamma_m$ of some $X_{s_m}$ contained in $C_m$. Up to pass to a subsequence, we may assume that $\Gamma_m$ converges in $\mathcal{C}_d^f(M)$ to a cycle $\Gamma$ which is non empty (it contains a point in $L$) and included in $|C_\infty|$. So $C_\infty$ is not the empty cycle.

Let $x$ be a generic point of an irreducible component $D$ of $C_\infty$. Then, up to pass to a subsequence, we may choose a sequence $(x_m)$ of points respectively in $C_m$ which converges to $x$. Choose for each $m$ an irreducible component $\Gamma_m$ of some $X_{s_m} \subset |C_m|$ which contains $x_m$. This is possible because of condition $(@@)$. Now, again up to pass to a subsequence, we may assume that the sequence $(\Gamma_m)_{m \in \mathbb{N}}$ converges in $\mathcal{C}_d^f(M)$ to a cycle $\Gamma$ containing the point $x$ and contained in $|C_\infty|$. Note that $|\Gamma|$ is contained in some $|X_{s_\infty}|$ as we may assume, by compactness of $S$, that the sequence $(s_m)$ converges to $s_\infty \in S$. Then we have $|X_{s_\infty}| \subset |C_\infty|$. As $D$ is the only irreducible component of $C_\infty$ containing $x$, it contains at least an irreducible component of $|X_{s_\infty}|$ containing $x$, and so $D$ meets $L$. So we have proved that $C_\infty$ is not the the empty $n-$cycle and that any irreducible component of $C_\infty$ meets the compact set $L$. This is enough to conclude thanks to the proposition 3.2.2 in [B.15].

2.3 Definition of SQP meromorphic quotient.

We shall consider a complex connected Lie group $G$ and a completely holomorphic action of $G$ on an irreducible complex space $X$. It is given, by definition, by a holomorphic map $f : G \times X \to X$ such that $f(g.g', x) = f(g, f(g'.x))$ for all $g, g' \in G$ and $x \in X$, assuming that for each $g \in G$ the holomorphic map $x \mapsto f(g, x)$ is an automorphism of $X$, and that $f(1, x) = x$ for all $x \in X$.

A strongly quasi-proper meromorphic quotient (we shall say a SQP-meromorphic quotient for short) for such an action $f : G \times X \to X$ will be the following data:
1. a $G$–modification$^6 \tau : \tilde{X} \rightarrow X$ with center $\Sigma$.

2. a holomorphic $G$–invariant GF map $q : \tilde{X} \rightarrow Q$ where $Q$ is an irreducible complex space.

3. an analytic $G$–invariant subset $Y \subset X$ containing $\Sigma$, with no interior point in $X$. We shall denote $\tilde{Y} := \tau^{-1}(Y)$, $\Omega := X \setminus Y$, $\tilde{\Omega} := \tau^{-1}(\Omega)$ and $Q' := q(\tilde{\Omega})$. Note that, as $q$ is an open surjective map, $Q'$ is open and dense in $Q$.

Now we ask that these data satisfy the following properties :

i) The restriction to $\Omega$ of the map $q \circ \tau^{-1}$ is a GF map onto the dense open set $Q'$ in $Q$ and there is an open dense set $Q''$ in $Q'$ such that each fiber of $q \circ \tau^{-1}$ at a point in $Q''$ is equal to a $G$–orbit in $\Omega$.

ii) There exists an open dense subset $\Omega_0 \subset \Omega$ such that for each $\tilde{x}$ in $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$ the closure $\overline{G.\tilde{x}}$ of $G.\tilde{x}$ in $\tilde{X}$ is exactly the set $q^{-1}(q(\tilde{x}))$.

Proposition 2.3.1 Let $G$ be a complex connected Lie group which acts completely holomorphically on an irreducible complex space $X$. Assume that we have a SQP meromorphic quotient for this action, given by a modification $\tau : \tilde{X} \rightarrow X$ and a $G$–invariant GF map $q : \tilde{X} \rightarrow Q$. Then let $\psi : Q \rightarrow C^n(X)$ be the holomorphic map obtained by the composition of the fiber map of the GF map $q$ and the direct image map for $n$–cycles by the modification $\tau$. Define $Q_u := \psi(Q)$. Then we have the following properties:

1. $Q_u$ is a closed analytic subset in $C^n(X)$ which is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on $C^n(X)$.

2. Let $\tilde{X}_u$ be the graph of the meromorphic map $q_u : X \rightarrow Q_u$ given by the holomorphic map $\psi \circ q : \tilde{X} \rightarrow Q_u$ and let $\tau_u : \tilde{X}_u \rightarrow X$ and $q_u : \tilde{X}_u \rightarrow Q_u$ be the projections on $X$ and $Q_u$ respectively of this graph. Then $(\tau_u, q_u)$ is also a SQP meromorphic quotient for the given $G$–action.

3. For any SQP meromorphic quotient $(\tau, q)$ there exists a unique holomorphic surjective map $\eta : Q \rightarrow Q_u$ such that the meromorphic maps $q : X \rightarrow Q$ and $q_u : X \rightarrow Q_u$ satisfies $\eta \circ q = q_u$.

4. For any $G$-invariant holomorphic map $h : X \rightarrow Y$ there exists a holomorphic map $H : Q_u \rightarrow Y$ such that $h \circ \tau_u = H \circ q_u$.

Definition 2.3.2 In the situation of the previous theorem the SQP meromorphic quotient for the given $G$–action defined by $(\tau_u, q_u)$ will be called the minimal SQP meromorphic quotient of this $G$–action.

$^6$This means that we have a completely holomorphic $G$–action on $\tilde{X}$ and that the modification $\tau$ is $G$–equivariant.
So the proposition above says that the existence of a SQP meromorphic quotient for the given $G$–action implies the existence and uniqueness of a minimal meromorphic quotient for this $G$–action.

**Proof.** To prove the point 1) we shall prove that the map $\psi \circ q : \tilde{X} \to \mathcal{C}_n'(X)$ is semi-proper.

Let $C \neq \emptyset$ be in $\mathcal{C}_n'(X)$ and fix a relatively compact open set $W$ in $X$ meeting all irreducible components of $C$. The subset $\mathcal{W}$ of $\mathcal{C}_n'(X)$ of cycles $C'$ such that any irreducible component of $C'$ meets $W$ is an open set containing $C$. Now $q(\tau^{-1}(W))$ is a compact set in $Q$, as $\tau$ is proper. Take any $y \in Q$ such that $C' := \psi(y)$ is in $\mathcal{W}$. The point $y$ is the limit in $Q$ of points $y_\nu \in q(\hat{\Omega}_0)$ such that the fiber of $Q$ at $y$ is limit in $\mathcal{C}_n'(\tilde{X})$ of the fibers $q^{-1}(y_\nu) = \overline{G.x_\nu}$ where, for $\nu \gg 1$, we can choose $\tilde{x}_\nu$ in $\hat{\Omega}_0 \cap \tau^{-1}(W)$. Up to pass to a sub-sequence, we may assume that $\tilde{x}_\nu$ converges to a point $\tilde{x}$ in $\tau^{-1}(W)$. Then the continuity of $q$ implies that $q(\tilde{x}) = y$ and $C'$ is the limit of $G.\tilde{x}_\nu$. So $|C'|$ is in the image by $\psi$ of the compact set $q(\tau^{-1}(W))$ and this gives the semi-properness of $\psi \circ q$.

Now the direct image theorem 2.3.2 in [B.15] shows that $Q_u$ is an irreducible complex space (locally of finite dimension) and the point 1) is proved.

To prove the second point we have to show that the map $q_u : \tilde{X}_u \to Q_u$ is a GF map. By definition $\tilde{X}_u$ is the closure in $X \times Q_u$ of the graph of the map $q|_{\Omega_0}$ where $\Omega_0$ is an open dense set in $X$ such that for any point $x \in \Omega_0$ we have $\psi(q(x)) = \overline{G.x}$ (as $\Omega_0$ is disjoint from the center of the modification $\tau$ we identify here $\Omega_0$ and $\hat{\Omega}_0 := \tau^{-1}(\Omega_0)$).

Then by irreducibility of $Q_u$ and $X$ the closed analytic subset $\tilde{X}_u \subset X \times Q_u$ is equal to the graph of the tautological family of cycles in $X$ parametrized by $Q_u \subset \mathcal{C}_n'(X)$. This proves the point 2).

Now consider a SQP meromorphic quotient of the given action given by the maps $\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$. Let $\psi : \tilde{X} \to \mathcal{C}_n'(X)$ the composition of the holomorphic map $q_u$ with the direct image of $n$–cycles by the modification $\tau$. Then, by the construction in the proof of the point 1), we know that $\psi(Q) = Q_u$ and then the map $\psi$ induces a surjective holomorphic map $\eta : Q \to Q_u$. We may assume that $\tilde{X}$ is in fact the graph of the meromorphic map $q : X \dasharrow Q$. Then, as $\tilde{X}_u$ is the graph of the meromorphic map $q_u : X \dasharrow Q_u$ the holomorphic map $\text{id}_X \times \eta : X \times Q \to X \times Q_u$ sends $\tilde{X}$ to $\tilde{X}_u$ because this is true over a dense open set in $X$ where the maps $q$ and $q_u$ are holomorphic and satisfy $q^{-1}(q(x)) = q_u^{-1}(q_u(x)) = \overline{G.x}$. This complete the proof of 3).

Consider now a $G$–invariant holomorphic map $h : X \to Y$. Recall (see [B.15] proposition 2.1.7) that $\mathcal{C}_n'(h)$, the subset of cycles $C \in \mathcal{C}_n'(X)$ which are contained in a fiber of $h$ is a closed analytic subset in $\mathcal{C}_n'(X)$. On an open dense subset of $Q_u$ we know that a point corresponds to a cycle with support $\overline{G.x}$ for some $x \in X$. Then this means that an open dense subset in $Q_u$ is contained in $\mathcal{C}_n'(h)$. As this subset is closed we obtain $Q_u \subset \mathcal{C}_n'(h)$. Now there is a holomorphic map $\hat{h} : \mathcal{C}_n'(h) \to Y$ associating to a cycle $C$ the point in $Y$ such $|C| \subset h^{-1}(y)$ (see [B.15] proposition 2.1.7). This induces a holomorphic map $H : Q_u \to Y$, and it is clear that the relation $h \circ \tau_u = H \circ q_u$ is true on a dense open set, so everywhere. ■
2.4 Good points, good open set.

Let $X$ be a reduced complex space and $G$ be a connected complex Lie group. Let $f : G \times X \to X$ be a holomorphic action of $G$ on $X$.

**Definition 2.4.1** We shall say that a point $x \in X$ is a **good point** for the action $f$ if the following condition is satisfied:

- For each compact set $K$ in $X$ there exists an open neighbourhood $V$ of $x$ and a compact set $L$ in $G$ such that if $y \in V$ and $g \in G$ are such that $g.y \in K$, there exists $\gamma \in L$ with $\gamma.y = g.y$.

We shall say that the action of $G$ on $X$ is **good** when each point in $X$ is a good point. If $\Omega$ is a $G$-invariant open set in $X$, we shall say that $\Omega$ is a **good open set** for the action $f$ when all points in $\Omega$ are good points for the $G$-action given by $f$ restricted to $\Omega$.

**Remarks.**

1. If $x \in X$ is a good point, then for any $g_0 \in G$, $g_0.x$ is also a good point: for $K$ given, choose $g_0.V$ as neighbourhood of $g_0.x$ and the compact set $L.g_0^{-1} \subset G$ to satisfy the needed conditions.

2. If $\Omega$ is a good open set, then points in $\Omega$ are not in general good points for the action on $X$.

3. If $\Omega$ is a good open set and $W \subset \Omega$ a $G$-invariant open set, then $W$ is a good open set.

4. If $M$ is a compact set of good points in $X$ for any compact set $K$ in $X$ we can find a neighbourhood $V$ of $M$ in $X$ and a compact set $L$ in $G$ such that for any point $y \in V$ and any $g \in G$ such that $g.y \in K$ there exists $\gamma \in L$ with $\gamma.y = g.y$. This is easily obtained by a standard compactness argument. We shall say that a compact set of good points is **uniformly good**.

**Lemma 2.4.2** Let $x$ be a point in $X$. Then $x$ is a good point for the $G$-action on $X$ if and only if the map $F_X : G \times X \to X \times X$ given by $(g, x) \mapsto (x, g.x)$ is semi-proper at each point of $\{x\} \times X$. As a consequence a $G$-invariant open set $\Omega$ in $X$ is a good open set for the $G$-action if and only if the map $F_\Omega : G \times \Omega \to \Omega \times \Omega$ given by $(g, x) \mapsto (x, g.x)$ is semi-proper.
Proof. Let \( x \in X \) be a good point and fix any \( z \in X \). To prove that the map \( F_X \) is semi-proper at \( (x, z) \) choose compact neighbourhoods \( V_0 \) and \( K \) of \( x \) and \( z \) in \( X \) and apply the definition of a good point to the compact set \( K \). So we can find a neighbourhood \( V \) of \( x \), that we may assume to be contained in \( V_0 \), and a compact set \( L \) in \( G \) such that for any \( y \in V \) such that \( g.y \in K \) we have a \( \gamma \in L \) with \( g.y = \gamma.y \). Then we have \( F_X(G \times X) \cap (V \times K) = F_X(L \times V_0) \cap (V \times K) \) and \( L \times V_0 \) is a compact set in \( G \times X \).

Conversely, assume that the map \( F_X \) is semi-proper at each point of \( \{x\} \times X \). Take a compact set \( K \) in \( X \) and apply the semi-properness to each point \( (x, z) \) where \( z \) is in \( K \). For each \( z \in K \) we obtain open neighbourhoods \( V_z \) and \( W_z \) of \( x \) and \( z \) in \( X \) and a compact set \( L_z \times M_z \) in \( G \times X \) such that \( F_X(G \times X) \cap (V_z \times W_z) = F_X(L_z \times M_z) \cap (V_z \times W_z) \).

Extract a finite sub-cover \( W_1, \ldots, W_N \) of \( K \) by the open sets \( W_z \) and define the compact set \( L := \bigcup_{i \in [1, N]} L_z \) and the neighbourhood \( V := \bigcap_{i \in [1, N]} V_z \) of \( x \). Then if \( y \) is in \( V \) and \( g.y \) is in \( K \) there exists \( i \in [1, N] \) such that \( g.y \in W_i \). As \( y \) is in \( V_i \) we can find a \( \gamma \in L_z \subset L \) with \( F_X(g, y) = F_X(\gamma, y) \) and this implies that \( x \) is a good point.

The second assertion is an easy consequence of the first one.

\[ \square \]

Proposition 2.4.3 Let \( G \) be a connected complex Lie group. Let \( f : G \times X \to X \) be an holomorphic action of \( G \) on a reduced complex space \( X \). Then we have the following properties:

i) If \( x \) is a good point for \( f \) the orbit \( G.x \) is a closed analytic subset of \( X \).

ii) If \( x \) is a good point for \( f \) and if \( (G.x) \cap K = \emptyset \) where \( K \subset X \) is a compact set, there exists a neighbourhood \( V \) of \( x \) in \( X \) such that \( (G.x') \cap K = \emptyset \) for any \( x' \) in \( V \) (\( x' \) is not assume here to be a good point).

iii) If \( x \) is a good point for \( f \) there exists a neighbourhood \( V \) of \( x \) such that any good point \( x' \in V \) has an orbit which is a closed analytic subset of the same dimension than \( G.x \).

iv) Let \( \Omega \) be a good connected open set in \( X \) which is normal and let \( n \) be the dimension of \( G.x \) for \( x \in \Omega \). Then there exists a holomorphic map\(^7\) \( \varphi : \Omega \to C^*_n(\Omega) \) given generically by \( \varphi(x) := G.x \) as a reduced \( n \)-cycle in \( \Omega \).

v) When we have a good open set \( \Omega \) in \( X \) which is normal, there exists a quasi-proper equidimensional holomorphic quotient of \( \Omega \) for the action restricted to \( \Omega \).

\(^7\)This means that we have an \( f \)-analytic family of \( n \)-cycles in \( \Omega \) parametrized by \( \Omega \).
PROOF. We already proved that $x$ is a good point if the map $G \to X$ given by $g \mapsto g.x$ is semi-proper in lemma 1.3.2. Now Kuhlmann’s theorem [K.64], [K.66] gives that $f_x(G) = G.x$ is a closed analytic subset of $X$. This proved i)

Assume ii) is not true; then we have a compact set $K$ such that $(G.x) \cap K = \emptyset$ and a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ converging to $x$ and such that $(\overline{G.x_\nu}) \cap K$ is not empty for each $\nu$. Fix a compact neighbourhood $\hat{K}$ of $K$ such that $(G.x) \cap \hat{K} = \emptyset$. This is possible thanks to i). Pick a point $y_\nu = \lim_{\alpha \to \infty} g_{\nu,\alpha}.x_\nu$ in $(\overline{G.x_\nu}) \cap K$ for each $\nu$. Up to pass to a subsequence we may assume that sequence $(y_\nu)$ converges to $y \in \hat{K}$ when $\nu \to +\infty$. So, for $\alpha \geq \alpha(\nu)$, we can assume that $g_{\nu,\alpha}.x_\nu$ is in $\hat{K}$. But, as $x$ is a good point, for the given $\hat{K}$ there exists a neighbourhood $V$ of $x$ and a compact set $L \subset G$ as in the definition. We may assume that $x_\nu$ is in $V$ for $\nu \geq \nu_0$ and so we may find, for $\nu \geq \nu_0, \alpha \geq \alpha(\nu)$, elements $\gamma_{\nu,\alpha} \in L$ such that $\lim_{\alpha \to \infty} \gamma_{\nu,\alpha}.x_\nu = y_\nu$. This proves ii).

Let $E := (U, B, j)$ be a $n$–scale on $\Omega$ adapted to the $n$–cycle $G.x$. Then the compact set $K := j^{-1}(\overline{U} \times \partial B)$ does not meet $G.x$, by definition of an adapted scale. Using ii), there exists a neighbourhood $V$ of $x$ such that for any $x' \in V$ we have $( \overline{G.x'} ) \cap K = \emptyset$. As for a good point $x' \in V$ we know that $G.x'$ is a closed analytic subset, the $n$–scale is then adapted to $G.x'$. This implies that the dimension of $G.x'$ is at most equal to $n$. But the semi-continuity of the dimension of the stabilizers implies that the dimension of $G.x' \cong G/\text{St}(x')$ is at least equal to $n = \dim(G/\text{St}(x))$. This proves iii).

Remark that for any $x' \in V$ such that $\overline{G.x'}$ is a closed analytic subset in $\Omega$, the previous proof shows also that $\overline{G.x'}$ is of pure dimension $n$.

To prove iv) fix a good connected open set $\Omega$ and define

$Z := \{(g, x, y) \in G \times \Omega \times \Omega / y = g.x \}$

This is a closed analytic subset in $G \times \Omega \times \Omega$. Let us show that the projection $p : Z \to \Omega \times \Omega$ is semi-proper. Pick a point $(x, y) \in \Omega \times \Omega$ and choose compact neighbourhoods $V$ and $K$ respectively of $x$ and of $y$ in $\Omega$. Using the fact that any open set $\Omega' \subset \subset \Omega$ is uniformly good, for the compact $K$ in $\Omega$ we find a compact set $L$ in $G$ such that for $x' \in \Omega'$ and $g \in G$ with $g.x' \in K$ there exists $\gamma \in L$ with $\gamma.x' = g.x'$. Choosing $\Omega'$ containing $V$ this implies that we have

$p(Z) \cap (V \times K) = p((L \times V \times K) \cap Z)$.

So the projection $p$ is semi-proper. Its image $p(Z)$ is then a closed analytic subset
of $\Omega \times \Omega$ by Kuhlmann’s theorem [K.64], [K.66]. But now the projection
\[ \pi : p(Z) \to \Omega \]
is $n$–equidimensional, thanks to iii), and has irreducible generic fibers on a normal basis $\Omega$. So its fibers (with generic multiplicity equal to 1) define an analytic family of $n$–cycles of $X$ parametrized by $\Omega$. It is clearly $f$–analytic because each fiber is irreducible\(^8\) and we have an holomorphic section because each $x$ lies in $G.x$.

To prove v) let us prove that the holomorphic map $\varphi : \Omega \to C^f(\Omega)$ classifying the fibers of $p(Z)$ is semi-proper. Fix a point $C \in C^f(\Omega)$ and a point $x_1, \ldots, x_k$ in each irreducible component of $|C|$. Let $W$ a relatively compact open neighbourhood of \{\(x_1, \ldots, x_k\}\} in $\Omega$ and let $W$ be the open set in $C^f(\Omega)$ of cycles such that each irreducible component meets $W$. Let $C'$ be in $W \cap \varphi(\Omega)$ we know that if $C' = \varphi(z)$ that $|C'| = G.z$. So $G.z$ has to meet $W$ and we can choose $y$ in the compact set $\bar{W}$ such that $C' = \varphi(y)$ and this gives the semi-properness of $\varphi$. Now the semi-proper direct image theorem 2.3.2 of [B.15] implies that the image $Q$ of $\varphi$ is a finite dimensional complex space.

Then the holomorphic map $q : \Omega \to Q$ which is a quasi-proper equidimensional holomorphic quotient for the action $f$ on $\Omega$ as each fiber of $q$ is set-theoretically a $G$–orbit.

\section{2.5 The conditions [H.1], [H.2] and [H.3].}

Now we shall consider the following conditions on the action $f$.

\begin{itemize}
  \item There exists a $G$–invariant dense open set $\Omega_1$ in $X$ which admits a GF–holomorphic quotient. \[[H.1]\]
\end{itemize}

Recall that this means that there exists a $G$–invariant geometrically $f$-flat holomorphic map $q : \Omega_1 \to Q_1$ onto a reduced complex space $Q_1$ such that each fiber of $q$ over a point in $Q_1$ is set-theoretically an orbit in $\Omega_1$.

The following stronger condition will be useful in the sequel:

\begin{itemize}
  \item There exists a $G$–invariant dense open set $\Omega$ in $X$ which is good for the action $f$ (on $\Omega$). \[[H.1str]\]
\end{itemize}

An immediate consequence of the proposition 2.4.3 is that the condition [H.1str] below is sufficient to satisfy [H.1] by taking $\Omega_1$ as the set of normal points in $\Omega$ (which is open dense and $G$–invariant in $\Omega$).

Now assume [H.1] and define $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 / y \in G.x\}$. It is a closed analytic set in $\Omega_1 \times \Omega_1$. We shall denote $\overline{\mathcal{R}}$ the closure of $\mathcal{R}$ in $X \times X$. Our second assumption will be:

\(^8\)But some multiplicities may occur.
• The subset $\mathcal{R}$ is analytic in $X \times X$ and there exists an open dense subset $\Omega_0 \subset \Omega_1$ such that for each $x \in \Omega_0$ we have

$$G.x = \mathcal{R} \cap (\{x\} \times X).$$

[H.2]

Remark that the first projection $p_1 : \mathcal{R} \cap (\Omega_0 \times X) \to \Omega_0$ is quasi-proper because we have a holomorphic section of this map (with irreducible fibers, thanks to [H.2]) which is given by $x \mapsto (x, x)$.

Assuming that $\Omega_0$ contains only normal points\footnote{This is not restrictive, as we may always assume that $X \setminus \Omega_0$ contains the non normal points in $X$. We shall always assume that $\Omega_0$ is normal in the sequel, without any more comment.} in $X$, the equidimensionality and quasi-properness on $\Omega_0$ of the projection of $\mathcal{R}$ implies that there exists a holomorphic map

$$\bar{\varphi}_0 : \Omega_0 \to C^f_n(X)$$

where the supports are given by $x \mapsto \bar{G.x}$ and where the multiplicity is generically equal to 1.

We shall denote $\Gamma \subset \Omega_0 \times C^f_n(X)$ the graph of the map $\bar{\varphi}_0$ and $\bar{\Gamma}$ its closure in $X \times C^f_n(X)$. We shall denote $\theta : \bar{\Gamma} \to X$ the map induced by the first projection.

Our last hypothesis is:

• The map $\theta : \bar{\Gamma} \to X$ is proper.

[H.3]

This hypothesis [H.3], assuming [H.1] and [H.2] is in fact equivalent to ask that the projection $\mathcal{R} \to X$ is strongly quasi-proper: this is an immediate consequence of the proposition 3.2.2 of [B.15].

The following proposition shows that these conditions [H.1], [H.2] and [H.3] are necessary for the existence of a SQP-meromorphic quotient for a completely holomorphic action of $G$ on $X$.

**Proposition 2.5.1** Assuming that the completely holomorphic action $f : G \times X \to X$ of the connected complex Lie group $G$ on the irreducible complex space $X$ has a SQP-meromorphic quotient, then the conditions [H.1], [H.2] and [H.3] are satisfied.

**Proof.** The conditions to be a SQP-meromorphic quotient gives an open set $\Omega_1$ which is dense, $G$--stable and which admits a GF holomorphic quotient for the action on $\Omega_1$. So [H.1] is clear.

Let $S$ be the graph of the equivalence relation given by $q$ on $\bar{X}$. Then the proper direct image $(\tau \times \tau)(S)$ is $\mathcal{R}$ and so the condition [H.2] is satisfied as soon as we can find an open and dense subset $\Omega_0$ in $\Omega_1$ such that for each $x \in \Omega_0$ we have the equality

$$\bar{G.x} = \mathcal{R} \cap (\{x\} \times X).$$

But this property is given by the condition ii) in the definition of a SQP-meromorphic quotient.
The composition of \( q \) with the classifying map \( Q \to C^f_n(X) \) for the fibers of \( q \) gives a holomorphic map \( \psi: \tilde{X} \to C^f_n(X) \). Composed with the direct image map, which is holomorphic (see [B.M] ch.IV; the “quasi-proper” part of this result is easy, as \( \tau \) is proper) \( \tau_*: C^f_n(\tilde{X}) \to C^f_n(X) \), we obtain the fact that \( \overline{R} \) is strongly quasi-proper on \( X \) which is the condition [H.3].

2.6 Existence of SQP-meromorphic quotient

Now we shall prove that conditions [H.1], [H.2] and [H.3] on a completely holomorphic action of a connected complex Lie group \( G \) on an irreducible complex space \( X \) are sufficient for the existence of a SQP meromorphic quotient.

**Theorem 2.6.1** Under the hypothesis [H.1], [H.2] and [H.3] there exists a proper \( G \)-equivariant modification \( \tau: \tilde{X} \to X \) with center contained in \( X \setminus \Omega_0 \) and a geometrically f-flat holomorphic map

\[
q: \tilde{X} \to Q
\]
on a reduced complex space, which give a strongly quasi-proper meromorphic quotient for the given \( G \)-action.

Of course the complex space \( \tilde{X} \) is the topological space \( \overline{\Gamma} \) with a structure of a reduced complex space such that the projection on \( X \) is a proper modification. Then the space \( Q \) is the image of \( \tilde{X} \) in \( C^f_n(X) \). So we need some semi-proper direct image theorem for such a map to prove this result. Such a result is the content of the theorem 2.3.2 of [B.15]

**Proof.** The first remark is that the hypothesis [H.3] says that the projection \( p: \tilde{\Gamma} \to X \) is a proper topological modification of \( X \). But to apply directly the part ii) of the theorem 2.3.6 of [B.13] to the projection \( p_1: \overline{R} \to X \) we need quasi-properness of this map. This is given by the proposition 3.2.2 of [B.15] as we have the condition [H.3].

Then we obtain a proper (holomorphic) modification with center in \( \Sigma \subset X \setminus \Omega_0 \), \( \tau: \tilde{X} \to X \), and a \( f \)-analytic family of cycles in \( X \) parametrized by \( \tilde{X} \) extending the family \( (G.x)_{x \in \Omega_0} \), corresponding to a “holomorphic” map extending \( \tilde{\varphi}_0: \)

\[
\tilde{\varphi}: \tilde{X} \to C^f_n(X).
\]

Now let us prove that this map \( \tilde{\varphi} \) is quasi-proper\(^{11}\). This will allow us to apply the theorem 2.3.2 of loc. cit. and to define the reduced complex space \( Q \) as the image \( \tilde{\varphi}(\tilde{X}) \). Then it will be easy to check that the map \( \tilde{\varphi}: \tilde{X} \to Q \) is a strongly quasi-proper meromorphic quotient for the \( G \)-action we consider. If \( C_0 \) is in \( C^f_n(X) \) and is not the empty cycle, choose a relatively compact open

---

\(^{10}\)The dense open subset \( \Omega_0 \subset \Omega_1 \) is defined in the condition [H.2]

\(^{11}\)This makes sense as the fibers are closed analytic subsets of \( \tilde{X} \).
set \( W \) in \( X \) such that any irreducible component of \(|C_0|\) meets \( W \). Then let \( \mathcal{W} \) be the open set in \( \mathcal{C}^\prime_n(X) \) defined by the condition that any irreducible component of \( C \) meets \( W \) for \( C \in \mathcal{W} \). Then we shall prove that there exists a compact set \( K \) in \( \tilde{X} \) such that any irreducible component of the fiber of \( \tilde{\phi} \) at a point in \( \mathcal{W} \cap \tilde{\phi}(\tilde{X}) \) meets \( K \). Let \( K := \tau^{-1}(\tilde{W}) \). If \( (y, C) \) is in \( \tilde{X} \) with \( C \in \mathcal{W} \), each irreducible component of \( C \) meets \( W \). But the fiber of \( \tilde{\phi} \) at \( C \) is equal to \(|C|\), and the quasi-properness is proved.

\[\] 3 Application.

3.1 The sub-analytic lemma.

We shall use the following lemma (see [G-M-O]) in our application.

**Lemma 3.1.1** Let \( M \) be a reduced complex space and \( Y \subset M \) a closed analytic subset with no interior point in \( M \). Let \( R \) be a closed (complex) analytic subset in \( M \setminus Y \) such that \( \bar{R} \) is a sub-analytic set in \( M \). Then \( R \) is a (complex) analytic subset in \( M \).

This important lemma is a consequence of Bishop's theorem (see [Bi.64]) and of a classical result on sub-analytic subsets (see [G-M-O]).

3.2 Proof of the theorem 1.0.1.

Now we shall assume that \( G \) is a connected complex Lie group such that \( G = K.B \) where \( B \) is a closed complex connected subgroup of \( G \) and \( K \) a compact real subgroup of \( G \).

The first condition [H.1str] for the \( G \)-action is given by the following lemma :

**Lemma 3.2.1** In the situation of the theorem 1.0.1, assume that we have a \( G \)-invariant open set \( \Omega \) which is a good open set for the \( B \)-action, then \( \Omega \) is a good open set for the \( G \)-action.

**Proof.** Consider a point \( x \in \Omega \) and a compact set \( M \in \Omega \). Then there exists a neighbourhood \( V \) of \( x \) in \( \Omega \) and a compact set \( L \) in \( B \) such that \( b.y \in M \) for some \( y \in V \) and some \( b \in B \) implies that we can find \( \beta \in L \) with \( b.y = \beta.y \). Now assume that \( M \) is \( K \)-invariant (here we use the \( G \)-invariance of \( \Omega \)) and that \( g.y \) is in \( M \) for some \( g \in G \) and some \( y \in V \). Write \( g = k.b \) for some \( k \in K \) and \( b \in B \). Then \( b.y \) is again in \( M \) so we can find \( \beta \in L \) with \( \beta.y = b.y \) and then \( g.y = k.\beta.y \) with \( k.\beta \in K.L \) which is a compact set in \( G \). So \( x \) is a good point for the \( G \)-action on \( \Omega \). \[\] 12Recall that, as a topological space, \( \tilde{X} = \tilde{\Gamma} \).
The corollary of the next lemma will give the first part of [H.2] for the \(G\)–action assuming that we have a \(G\)–invariant dense good Zariski open set \(\Omega\) for the \(B\)–action with the condition [H.2] for the \(B\)–action. In order to use the sub-analytic lemma 3.1.1 in this situation we shall need the following lemma.

**Lemma 3.2.2** Let \(\Omega\) be an open \(G\)–invariant good set for the \(B\)–action, and then also good for the \(G\)–action thanks to the previous lemma.

Let \(\chi : K \times X \times X \to X \times X\) the map given by \((k, x, y) \mapsto (k.x, y)\) and let \(p : K \times X \times X \to X \times X\) be the natural projection. Then we have

\[
p(\chi^{-1}(R_B)) = \overline{R_G}
\]

where we define

\[
R_B := \{(x, y) \in \Omega \times \Omega \mid B.x = B.y\} \quad \text{and} \quad R_G := \{(x, y) \in \Omega \times \Omega \mid G.x = G.y\},
\]

and where the closures are taken in \(X \times X\).

**Proof.** Remark first that \(p(\chi^{-1}(R_B)) = \{(x, y) \in \Omega \times \Omega \mid \exists k \in K, B.k.x = B.y\}\).

So \((x, y) \in p(\chi^{-1}(R_B))\) implies \(y \in B.k.x \subset G.x\) and also \(k.x \in B.y\); we conclude that \(x\) is in \(K.B.y = G.y\). This gives the inclusion \(p(\chi^{-1}(R_B)) \subset R_G\). The opposite inclusion is easy because \(G.x = G.y\) implies that \(x \in K.B.y\) so there exists \(k \in K\) such that \(k.x \in B.y\).

Now the map \(\chi\) and \(p\) are proper, so we obtain the inclusion \(p(\chi^{-1}(R_B)) \subset \overline{R_G}\).

Let \((x, y) := \lim_{\nu \to \infty} (x_\nu, y_\nu)\) where \((x_\nu, y_\nu)\) is in \(R_G\) for each \(\nu \in \mathbb{N}\). Then for each \(\nu\) the exists \(k_\nu \in K, b_\nu \in B\) such that \(x_\nu = k_\nu.b_\nu.y_\nu\); then \((k_\nu^{-1}, x_\nu, y_\nu) \in R_B\). Up to pass to a subsequence we may assume that the sequence \((k_\nu)\) converges to some \(k \in K\). As \((k_\nu^{-1}, x_\nu, y_\nu)\) is in \(\chi^{-1}(R_B)\) for each \(\nu\), the point \((k^{-1}, x, y)\) is in \(\overline{\chi^{-1}(R_B)} = \chi^{-1}(\overline{R_B})\)

and so \((x, y)\) is in \(p(\chi^{-1}(R_B))\) proving the opposite inclusion. 

**Corollaire 3.2.3** In the situation of the previous lemma, assume that \(X \setminus \Omega\) is a (complex) analytic subset; then if the subset \(\overline{R_B}\) is (complex) analytic in \(X \times X\), the subset \(\overline{R_G}\) is also a (complex) analytic subset of \(X \times X\).

**Proof.** Note first that the maps \(\chi\) and \(p\) are real analytic, so assuming that \(\overline{R_B}\) is analytic implies that \(p(\chi^{-1}(\overline{R_B}))\) is sub-analytic. Then, as we know that \(R_G\) is an irreducible locally closed complex analytic subset, the conclusion follows from the lemma 3.1.1, as our assumption that \(\Omega\) is a Zariski (dense) open set in \(X\) implies that \(\Omega \times \Omega\) is Zariski open (and dense) in \(X \times X\).

A first step to prove the quasi-properness of \(\overline{R_G}\) is our next result.
Lemma 3.2.4 Let assume that the $B$-action $f$ on $X$ satisfies [H.1] and [H.2]. Let $\Omega_0 \subset \Omega_1$ be an open set on which the fiber at any $x \in \Omega_0$ of $\overline{B.x}$ is equal to $\overline{B.x}$ (with some multiplicity). Then the fiber at any $x \in \Omega_0$ of $\overline{G.x}$ is equal to $\overline{G.x}$ (with some multiplicity).

PROOF. As we know that the map $x \mapsto \overline{B.x}$, with generic multiplicity 1, extends to is a $f$-analytic family of cycles of $X$ parametrized by $\Omega_0$, for each sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points in $\Omega_0$ converging to a point $x \in \Omega_0$ we have (with suitable multiplicity) $\overline{B.x} = \lim_{\nu \to \infty} \overline{B.x_\nu}$ in the topology of $\mathcal{C}_d(X)$. We shall show that this implies, also with suitable multiplicity, the equality $\overline{G.x} = \lim_{\nu \to \infty} \overline{G.x_\nu}$ in the topology of $\mathcal{C}_d(X)$. As we have $G = K.B$ with $K$ compact, for any $y \in X$ we have $\overline{G.y} = K.B.y$. So the inclusion of $\lim_{\nu \to \infty} \overline{G.x_\nu}$ in the fiber at $x$ of $\overline{G.x}$ is clear. The point is to prove the opposite inclusion. Let $y$ be a point in the fiber at $x \in \Omega_0$ of $\overline{G.x}$. It is a limit of a sequence $y_\nu \in G.x_\nu$ where $x_\nu \in \Omega_0$ converges to $x$. Write $y_\nu = k_\nu.b_\nu.x_\nu$ with $k_\nu \in K$ and $b_\nu \in B$. Up to pass to a subsequence, we may assume that the sequence $(k_\nu)$ converges to a point $k \in K$. So we have $k^{-1}.y$ which is the limit of the sequence $b_\nu.x_\nu$. We obtain that $k^{-1}.y$ is in the limit of $\overline{B.x_\nu}$ which has support equal to $\overline{B.x}$. Then $y$ is in $K.\overline{B.x} = \overline{G.x}$, concluding the proof.

Proof of the Theorem 1.0.1. The fact that the $G$-action satisfies [H.1str] is consequence of lemma 3.2.1. The analyticity of $\overline{G.x}$ in $X \times X$ is proved at corollary 3.2.3. The lemma 3.2.4 gives a dense open set $\Omega_0$ where the fiber of the projection $p_1$ of $\overline{G.x}$ at each point $x \in \Omega_0$ is equal to $\overline{G.x}$ as a set. This implies the quasi-properness of $p_1$ over $\Omega_0$, because $x$ is in $G.x$ and $G$ is connected; assuming (which is not restrictive) that $\Omega_0$ is normal, we obtain a holomorphic map

$$\Phi : \Omega_0 \longrightarrow \mathcal{C}_d(X)$$

where the support of $\Phi(x)$ is equal to $\overline{G.x}$ for each $x \in \Omega_0$ and with generic multiplicity equal to 1. This complete the proof of [H.2] for the $G$-action.

Thanks to proposition 3.2.2 of [B.15], to prove [H.3] it is enough to show that the closure of the graph $\Gamma_G$ of $\Phi$ in $X \times \mathcal{C}_d(X)$ is proper on $X$.

The projection $p_B : \overline{B} \to X$ is strongly quasi-proper so, for $V$ compact in $X$, the limits of cycles (with convenient multiplicity) $\overline{B.x}$ for $x \in V \cap \Omega_0$ stay in a compact subset $S_0$ in $\mathcal{C}_d(X)$. Then let $S := K.S_0$; this is again a compact subset in $\mathcal{C}_d(X)$. But the subset $T$ of limits of the generic fibers of the projection $p_G : \overline{G} \to X$ for $x \in V$ is a compact set of $\mathcal{C}_d^{loc}(X)$ thanks to [B.13] theorem 2.3.6 i). For $x \in V \cap \Omega_0$ we have $p_G^{-1}(x) = \overline{G.x}$, and let $T'$ be the open dense set in $T$ described by the cycles $\varphi_0(x), x \in V \cap \Omega_0$. As we know that for $x \in \Omega_0$ we have $|\varphi_0(x)| = \overline{G.x}$ we know that each of them is an union of $d$-cycles in $S$, thanks to the lemma 3.2.5 below. The proposition 2.2.3 implies that the set of limits of $\overline{G.x_\nu}$ for $x_\nu \in V \cap \Omega_0$ is a compact set in $\mathcal{C}_d(X)$, proving [H.3].

$\blacksquare$
Lemma 3.2.5 Let $K$ be a compact Lie group with a holomorphic action on a reduced complex space $M$. Let $X$ be an irreducible complex space and let $\varphi : X \to \mathcal{O}_d^f(M)$ and $\psi : X \to \mathcal{O}_d^f(M)$ two holomorphic maps. Assume that for each $x$ in a dense open subset $\Omega_0$ in $X$ we have

$$|\psi(x)| = \bigcup_{k \in K} k.|\varphi(x)| = K.|\varphi(x)| \quad (\@\@\@)$$

where $K$ acts holomorphically on $\mathcal{O}_d^f(M)$ via the action of $K$ on $M$. Then the relation $(@@@)$ holds for each $x \in X$.

**Proof.** Let $x \in X$ and choose a sequence $(x_\nu)_{\nu \geq 0}$ in $\Omega_0$ converging to $x$ and let $y$ a point in $|\psi(x)|$. Then we can choose a sequence $(y_\nu)_{\nu \geq 0}$ of points in $|\psi(x_\nu)|$ converging to $y$. As $x_\nu$ is in $\Omega_0$ we can write $y_\nu = k_\nu.z_\nu$ with $z_\nu \in |\varphi(x_\nu)|$ and $k_\nu \in K$. Up to pass to a subsequence we can assume that the sequence $(k_\nu)$ converges to some $k \in K$. So the sequence $(z_\nu)$ converges to $k^{-1}.y$ which is in $|\varphi(x)|$. This gives that $y$ is in $k.|\varphi(x)|$ and we have proved the inclusion $|\psi(x)| \subset K.|\varphi(x)|$.

Conversely, if $z$ is in $|\varphi(x)|$ and $k$ is in $K$, write again $z = \lim z_\nu$ with $z_\nu \in |\varphi(x_\nu)|$ where the sequence $(x_\nu)$ of points in $\Omega_0$ converges to $x$. We have $k.z_\nu \in |\psi(x_\nu)|$, and then $k.z$ is in $|\psi(x)|$ concluding the proof. ■
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