A Geometrical Model for the Evolution of Spherical Planetary Nebulae Based on Thin-Shell Formalism

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A spherical planetary nebula is described as a geometric model. The nebula itself is considered as a thin-shell which visualized as a boundary of two spacetimes. The inner and outer curvature tensors of the thin-shell are found in order to get an expression of the energy-momentum tensor on the thin-shell. The energy density and pressure expressions are derived using the energy-momentum tensor. The time evolution of the radius of the thin-shell is obtained in terms of the energy density. The model is tested by using a simple power function for decreasing energy density and the evolution pattern of the planetary nebula is attained.

I. INTRODUCTION

Planetary nebulae (PNe; singular, PN) are one of the last stages in the low and intermediate-mass stellar evolution which provide a transition between two drastically different phases of a dying star: a red giant and a white dwarf. A typical planetary nebula, consists of an ever expanding low-density ionized gas and a central hot star, known as the PN nucleus (PNN; a.k.a. PN central start, PNCS). The typical density of a PN is about 100 to 10,000 particles per cubic centimeter [1]. Widely different in shapes and features, morphologies of PNe are heavily influenced by the characteristics of the post-PN phases, the PNN, and the environmental factors. Although the physics behind the dynamical processes leading to this vast morphological variety is not well-understood yet, some factors such as rotation [2, 3], mass density distribution [4], rotation [5], metalicity and magneticity of the progenitor asymptotic giant branch (AGB) star [5–10], the stellar winds by the PNN itself and the neighboring stars [8, 11–13], and a possible binary-system occurrence and their corresponding tidal interactions [15–20] are enumerated in the literature [21]. With a life-span of a few tens of thousands of years, the PN-phase of a star is considerably short compared to its billion-year overall evolution period. Yet, its significance in our understanding of the evolution of the stars is singular for many reasons. Firstly, over 90% of stellar evolutions somehow experience a PN phase [22]. Secondly, PN mechanism is one through which the chemical abundance of the interstellar medium is evolved [7, 23]. Thirdly, their frequent occurrence and versatile shapes allow us to analyze them in large groups in order to develop our theoretical and experimental understanding of the way of the hydrodynamics of the stellar evolution. Furthermore, extragalactic PNe offer ways to derive stellar formation rates and metallicity gradients in galaxies [24–27]. Finally, the PN luminosity function [28–30] and the $S_{H\alpha} - r$ relation [31], can serve as an accurate distance indicators up to 20 Mpc.

With a visual classification [32], around 20% of all PNe are round [33]. However, one should beware of projection effects [8] which may cause taking a PN of a different morphology (elliptical or bipolar) for a spherical or round, when observed from a wrong angle. Considering the projection effects, it is estimated that 10 – 20% of PNe are nearly or absolutely spherical [6]. These type of PNe occur more frequently among low mass stars ($\leq 1.1M_\odot$) [34, 35]. Although spherical PNe expand almost uniformly and homogeneously, they possess microstructures, in general [4]. However, there exists a rare type of spherical PN which does not exhibit a microstructure, whatsoever. The best example of this type is Abell 39, a simple spherical shell of ionizing gas with a low brightness in the constellation of Hercules [36]. In the present study, we do not intend to study the evolution of a spherical PN by the hydrodynamical processes it goes through. Instead, our intention is to present a simple analytical model in which the time evolution of the radius, energy density and pressure of such PNe can be derived by a relativistic geometrical method. In our model, the thin outer rim of the bubble of a PN of type Abell 39 is illustrated by a thin-shell, in its general relativistic sense. This thin-shell, which is essentially a three-dimensional hypersurface, divides the spacetime into two inherently different...

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manifolds; the interior and the exterior. For this particular study, we consider an uncharged Vaidya’s metric [14], characterized by a radial flow of electrically neutral unpolarized matter radiation as the interior, and a Schwarzschild metric corresponding to a gravitational field caused by an uncharged, non-rotating spherically symmetric mass as the exterior. The two spacetimes are connected via proper junction conditions [37] which certify the energy density and angular pressure of the matter distributed over the shell. As another important consideration, note that the radius of the shell $a$ must exceed the event horizons of both interior and exterior spacetimes. As will be seen in the following lines, we demand the metric tensor to be continuous across the shell. Only it is under these conditions that one can claim the whole spacetime (including the inner and outer spacetimes plus the thin-shell itself) is a solution of Einstein field equations. These conditions, which are known as Darmois-Israel junction conditions in general relativity, firstly, demand the metric tensor to be continuous across the boundary. In this study, we model the interior spacetime by the Vaidya metric [14], where

$$\Sigma := r_{\pm} - \rho (\tau) = 0,$$

and intrinsic metric

$$ds^2_\Sigma = g_{ij} d\xi^i d\xi^j = - dr^2 + a^2 (\tau) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where $\xi^i = (\tau, \theta, \phi)$ are the coordinates on the thin-shell and $a(\tau)$ stands for the radius of the hypersurface $\Sigma$ dividing the spacetime into two distinct $3+1$-dimensional manifolds of class $C^4$. Also, $r_{\pm}$ are the radial coordinates of the outer (+) and the inner (−) spacetimes. The first junction condition requires the satisfaction of

$$(ds^2_\Sigma)^{\pm} = (ds^2_\Sigma)^{\pm} = ds^2_\Sigma$$

on the boundary. In this study, we model the interior spacetime by the Vaidya metric

$$ds^2_- = - f_- (r_-, t_-) \, dt_-^2 + 2 \epsilon \, dt_- \, dr_- + r_-^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where

$$f_- (r_-, t_-) = 1 - \frac{2M_- (t_-)}{r_-},$$

is the interior metric function, in which $M_- (t_-)$ is the time-dependent mass of the central gravitational object. Furthermore, $\epsilon$ takes on $-1$ and $+1$ for outgoing and incoming waves, respectively. The outer spacetime, on the other hand, will be of Schwarzschild-type geometry, given with the line element

$$ds^2_+ = - f_+ (r_+, t_+) \, dt_+^2 + f_+^{-1} (r_+, t_+) \, dr_+^2 + r_+^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where the metric function is

$$f_+ (r_+, t_+) = 1 - \frac{2M_+}{r_+},$$

with constant mass $M_+$. Note that as a direct consequence of the first junction condition, we have $\theta_- = \theta_+ = \theta$ and $\phi_- = \phi_+ = \phi$ at the shell. As another important consideration, note that the radius of the shell $a(\tau)$ must be chosen such that it exceeds the event horizons of both inner and outer spacetimes. As will be seen in the following lines, we always have $M_+ > M_-$, hence, we must have $a > 2M_+$ at all times.

The layout of the paper is as follows: In Section-II, the spacetimes separated by the thin-shell are given and by use of Darmois-Israel junction conditions the metric tensor on the surface of thin-shell is defined. Moreover, the inner and outer curvature tensors are calculated in order to get the energy-momentum tensor on the shell. By means of the energy-momentum tensor the energy density and the pressure of the thin-shell are calculated using the perfect fluid assumption. Furthermore, the time evolution of the radius of the thin-shell is investigated assuming a simple power function in Section-III. The time evolution of the radius of the spherical thin-shell is given at the end of this section. The paper is brought to completion with a conclusion in Section-IV.

II. THE SPACETIMES

A thin-shell can be visualized as a boundary separating two spacetimes, namely an interior spacetime from an exterior. For the thin-shell to be physical, certain junction conditions must be satisfied at the location of the thin-shell. Only it is under these conditions that one can claim the whole spacetime (including the inner and outer spacetimes plus the thin-shell itself) is a solution of Einstein field equations. These conditions, which are known as Darmois-Israel junction conditions in general relativity, firstly, demand the metric tensor to be continuous across the thin-shell. In our consideration, this thin-shell will be corresponding to a spherically symmetric PN with the defining equation

$$(1) \Sigma := r_{\pm} - a(\tau) = 0,$$

and intrinsic metric

$$ds^2_\Sigma = g_{ij} d\xi^i d\xi^j = - dr^2 + a^2(\tau) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where $\xi^i = (\tau, \theta, \phi)$ are the coordinates on the thin-shell and $a(\tau)$ stands for the radius of the hypersurface $\Sigma$ dividing the spacetime into two distinct $3+1$-dimensional manifolds of class $C^4$. Also, $r_{\pm}$ are the radial coordinates of the outer (+) and the inner (−) spacetimes. The first junction condition requires the satisfaction of

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is the interior metric function, in which $M_- (t_-)$ is the time-dependent mass of the central gravitational object. Furthermore, $\epsilon$ takes on $-1$ and $+1$ for outgoing and incoming waves, respectively. The outer spacetime, on the other hand, will be of Schwarzschild-type geometry, given with the line element

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with constant mass $M_+$. Note that as a direct consequence of the first junction condition, we have $\theta_- = \theta_+ = \theta$ and $\phi_- = \phi_+ = \phi$ at the shell. As another important consideration, note that the radius of the shell $a(\tau)$ must be chosen such that it exceeds the event horizons of both inner and outer spacetimes. As will be seen in the following lines, we always have $M_+ > M_-$, hence, we must have $a > 2M_+$ at all times.
To have a boundary which indeed distinguishes the inside from the outside, the thin-shell itself must possess an energy-momentum tensor. This tensor is given by the second junction condition, which relates the energy-momentum tensor of the matter at the thin-shell to the discontinuity in the second fundamental form across the shell. The curvature tensors of the interior and exterior spacetimes are given by [38]

\[ K_{ij}^\pm = -n_\alpha^\pm \left( \frac{\partial^2 \chi_\alpha^\pm}{\partial \xi^i \partial \xi^j} + \Gamma^\alpha_{\mu \nu} \frac{\partial \chi_\mu^\pm}{\partial \xi^i} \frac{\partial \chi_\nu^\pm}{\partial \xi^j} \right) , \]

where \( \chi_\alpha^\pm \) are the coordinates of the inner and outer spacetimes, \( n_\alpha^\pm \) are the components of the 4-normal to \( \Sigma \) given by

\[ n_\alpha^\pm = \frac{\partial_\alpha \Sigma^\pm}{[g^{\gamma \gamma'} \partial_\gamma \Sigma \partial_{\gamma'} \Sigma]} , \]

and \( \Gamma^\alpha_{\mu \nu} \) are the Christoffel symbols compatible with the bulk metrics \( g_{\mu \nu}^{\pm} \). Therefore, the second junction conditions are expressed as

\[ -8\pi S_i^j = \left[ K_i^j \right]^+ - \delta_i^j \left[ K_i^j \right]^+ , \]

where, \( K = g^{\mu \nu} K_{\mu \nu} \) is the total curvature, \( \delta_i^j \) is the Kronecker delta, and \([\ ]^+ \) indicates a jump across the shell in the quantity it embraces, e.g. \([K_{ij}]^+ = K_{ij}^+ - K_{ij}^- \). Accordingly, \( S_i^j \) is the mixed energy-momentum tensor of the matter at the shell, which for a perfect fluid picks up

\[ S_i^j = diag (-\sigma, p, p) . \]

Here, \( \sigma \) is the energy density, whereas \( p \) is the angular pressure. For the purpose of this study, the perfect fluid assumption seems reasonable since according to the Generalized Interacting Stellar Wind model (GISW) [11–13, 39] PNe evolve spherically if their pole-to-equator density contrast \( \epsilon \equiv \frac{\sigma_2}{\sigma_E} \), associated with their AGB slow wind, is unity.

Inserting (11) and (8) into (10) yields the energy density and pressure of the matter at the shell, as

\[ \sigma = \frac{1}{4\pi a} \left( \sqrt{f_- + a^2} - \sqrt{f_+ + a^2} \right) , \]

and

\[ p = \frac{1}{8\pi} \left( \frac{f_+' + 2\dot{a}}{2\sqrt{f_+ + a^2}} - \frac{1}{2} \dot{f}_- f_- - \epsilon \frac{t_+}{t_-} \right) - \frac{\sigma}{2} . \]

Herein, the overdot (\( \dot{} \)) and the prime (\( ' \)) stand for total derivatives with respect to the proper time \( \tau \) and the radial coordinates \( r_\pm \), respectively. Moreover, as an auxiliary equation, one could calculate the conservation of energy by starting from \( \nabla_j S_i^j = 0 \) and setting \( i \equiv \tau \). After some calculations, the result will come out to be

\[ \sigma' + \frac{2}{a} (\sigma + p) = \frac{1}{4\pi a} \left[ \frac{\epsilon a^2 + \frac{1}{2} \epsilon f_- + \dot{a} \sqrt{f_- + a^2} \frac{\partial f_-}{\partial t_-}}{f_- \sqrt{f_- + a^2}} \right] . \]

Note that, unlike the cases in which we have static metric functions on the two sides of the shell, here a non-zero time-dependent term appears on the right-hand side due to the dynamic nature of the Vaidya metric. However, if the interior central mass \( M_- \) does not depend on time, the right-hand side will identically amount to zero, as expected.

Besides the energy conservation relation in Eq. (14), another mechanical energy-like equation can be generated by rewriting Eq. (12) in the form

\[ \frac{1}{2} a^2 + V_{\text{eff}} = 0 , \]

where \( \frac{1}{2} a^2 \) resembles the kinetic term and the effective potential

\[ V_{\text{eff}} = \frac{1}{2} \left[ \frac{f_+ + f_-}{2} - \left( \frac{f_+ - f_-}{\kappa \sigma a} \right)^2 - \left( \frac{\kappa \sigma a}{4} \right)^2 \right] . \]
is a function of the radius of the shell and the time coordinate $t_\perp$ through $f_\perp$. The gravitational mass of the exterior metric, $M_+$, is the sum of the gravitational mass of the interior metric, $M_-$, and the mass of the shell that is the multiplication of its energy density by its surface area, i.e. $4\pi a^2\sigma$. Therefore, we have the mass relation

$$M_- = M_+ - 4\pi a^2\sigma,$$

(17)

which can be used to eliminate the time-dependent mass $M_-$ out of Eq. (16). Upon direct substitution from Eqs. (4) and (7) into Eq. (16) and then Eq. (15), one arrives at

$$\dot{a} = \sqrt{\frac{2M_+}{a} + 4\pi a\sigma (\pi a\sigma - 1)},$$

(18)

for the time-evolution of the radius of the shell.

### III. THE EVOLUTION OF THE PLANETARY NEBULA

Let us define for the energy density $\sigma$, a spherically symmetric environment given by a simple power function of the form

$$\sigma \equiv c_0 a^\mu,$$

(19)

with $c_0$ being a positive constant and $\mu$ an integer. The positivity of the constant $c_0$ is a must since we would like the fluid on the shell to have a positive energy density and satisfies the weak energy condition (WEC). In the case of a thin-shell, the WEC states $S_{ij} V^i V^j \geq 0$, in which $S_{ij}$ is the energy-momentum tensor and $V^i$ is an arbitrary timelike vector. In the context of perfect fluids, the WEC translates to two simultaneous conditions $\sigma \geq 0$ and $\sigma + p \geq 0$. The satisfaction of the WEC guarantees the ordinariness of the matter at the shell (Otherwise, the matter will be the unwanted “exotic matter”). Besides this, there is subtle condition imposing an upper bound over the value of $c_0$. It can be shown, by inserting (19) and (20) into (12), that the equation holds true only if $|c_0| \leq (2\pi a^{\mu+1})^{-1}$. Therefore, we have $0 \leq (2\pi a^{\mu+1})^{-1}$ as the permissible domain of $c_0$. However, note that the radius of the nebular shell evolves by time, i.e. $a \equiv a(\tau)$. As we now, a PN occurs in final stages of a highly evolved Sun-like star and disappears in a few thousand years, leaving a white dwarf behind. Let us assume that our model works to the moment $\tau_f$, when the radius of the nebula is $a_f$ and it is not observable in visible light anymore. This happens when the PNN cools down after the fusion has almost stopped, so that it does not emit enough ultraviolet radiation to ionize the distant nebular gas anymore. Accordingly, $M_-$ approaches a final value, say $(M_-)_f$, and the Vaidya metric becomes Schwarzschild. By this assumption, we make sure that the conservation of mass and energy are satisfied. In this limit, of course, the PN does not belong to the central mass (the white dwarf) anymore and is part of the interstellar medium. Hence, to make sure that $c_0$ remains bounded within its permissible domain at all time during the expansion, we require that

$$c_0 \in \left(0, (2\pi a^{\mu+1})^{-1}\right).$$

Moreover, as a result of our considerations of the inner and outer spacetimes, the mass of the central white dwarf $M_-$ decreases by time, whereas the reduction is added to the mass of the nebula through the outgoing waves, such that the total mass remains constant ($M_+$). Based on this, we should have $\mu \geq -2$ since for $\mu < -2$ the mass of the shell, i.e. $4\pi a^2\sigma$, decreases by time. On the other hand, unlike $M_+$, $M_-$ is not a constant. The central mass of the Vaidya metric $M_-$ (the mass of the PNN) constantly gives off outgoing waves and shrinks, while its emitted energy is absorbed by the shell and pushes it into the outer space. Hence, $\mu$ cannot be $-2$, as well, since it leads to a constant $M_-$. Therefore, we have $\mu > -2$ imposed on the values of $\mu$. Note that, $\sigma$ is the energy density of the shell not its mass density. So, although the mass density of the shell decreases as the shell expands, the same is not necessarily true for the energy density. In this seance, no upper bound is imposed over the values of $\mu$.

As the first example, let us consider an ever decreasing energy density by evoking $\mu = -1$. By looking at Eq. (18), one sees that this choice has great mathematical advantages since it simplifies Eq. (18) to

$$\dot{a} = \sqrt{\frac{2M_+}{a} - A},$$

(20)

where $A \equiv 4\pi c_0(1 - \pi c_0)$ is just a constant. Hereon, we will refer to $A$ as the evolution constant. According to the limitation over $c_0$, for the evolution parameter we always have $0 < A \leq 1$ (considering the positivity of $c_0$). This
FIG. 1: The plot of $\tilde{a}$ in terms of $\tilde{\tau}$ with $a_0 = 1$. While the rescaled radius of PN increases its speed decreases and when its speed becomes zero it instantly vanishes.

differential equation is analytically solvable, although it cannot be written explicitly for $a(\tau)$. The answer is the solution of the algebraic equation

$$\tau + \frac{\sqrt{Aa(\tau)} (2M_+ - Aa(\tau)) - M_+ \sin^{-1} \left( \frac{Aa(\tau)}{M_+} - 1 \right)}{A^{3/2}} + C = 0 \quad (21)$$

where $C$ is an integral constant that can be determined by the initial condition $a(0) = a_0$. Here, $a_0$ is the radius of the asymptotic giant branch (AGB) star when its outermost layers eject and form the expanding envelope of the PN. According to the model, although the mass $M_+$ and the evolution constant $A$ are different for different PNe, almost the same evolution pattern is expected. This, of course, can be seen by imposing the rescaling

$$\tilde{a} \equiv \frac{Aa}{M_+} \quad \text{and} \quad \tilde{\tau} + \tilde{C} \equiv \frac{A^2}{M_+} (\tau + C), \quad (22)$$

which cast the implicit equation in (21) into

$$\tilde{\tau} + \tilde{C} = \sin^{-1} (\tilde{a} - 1) - \tilde{a} \sqrt{\frac{2}{\tilde{a}} - 1}. \quad (23)$$

We add that, $\tilde{C}$ is obtained using the initial condition i.e., $\tilde{a}(0) = \tilde{a}_0$ upon which

$$\tilde{C} = \sin^{-1} (\tilde{a}_0 - 1) - \tilde{a}_0 \sqrt{\frac{2}{\tilde{a}_0} - 1}.$$  

This new equation is free of parameters $M_+$ and $A$, and reflects the general behavior of the evolution of the radius of spherical PNe. In Fig. 1, we plot the rescaled radius $\tilde{a}$ against $\tilde{\tau}$. For a particular nebula, with a specific evolution parameter $A$, we must have $\tilde{a} > 2A$ at all times, since $\tilde{a} = 2A$ corresponds to the Schwarzschild radius of the exterior spacetime.

IV. CONCLUSION

We established a geometrical model for the evolution of PN. After deriving the energy density and pressure expressions for the thin-shell we obtained the time evolution of the thin-shell’s radius. We plotted the rescaled radius as a function of rescaled time for an energy density obeys a simple power function. While the radius of thin-shell increases
its speed decreases. In its final stage the speed of PN becomes zero and it disappears instantly. Although, the model presented seems analytically reasonable it needs observational verification as well as to see a simulation of the model might give some clues about the evolution of PN.

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