Post-Newtonian expansion of a rigidly rotating disc of dust with a constant specific charge

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Abstract
We present an algorithm for obtaining the post-Newtonian expansion of the asymptotically flat solution to the Einstein–Maxwell equations describing a rigidly rotating disc of dust with a constant specific charge. Explicit analytic expressions are calculated up to the eighth order. The results are used for a physical discussion of the extreme relativistic limiting cases. We identify strong evidence for a transition to an extreme Kerr–Newman black hole.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Since the formulation of classical general relativity by Einstein in 1915, analytic solutions have made a large contribution to the understanding of its meanings and proving its properties. A crucial role in this process was played by black hole solutions, as their description requires the full generality of Einstein’s theory. Therefore, black hole solutions could be expected to exhibit inherent effects of general relativity. Indeed, the Schwarzschild solution containing a single mass parameter \( M \) was able to explain the perihelion precession of Mercury and the bending of light rays around the Sun. The Kerr solution extended the class of black hole solutions by the parameter of angular momentum \( J \), led to the discovery of ergospheres and permitted the rigorous investigation of the Lense–Thirring effect. Furthermore, adding a charge parameter \( Q \) completes the class of stationary black hole solutions by delivering the most general Kerr–Newman solution, see [1, 2].

Interestingly, static spherically symmetric ideal fluid configurations have a minimal radius, given by \( \frac{9}{8} \) times the corresponding Schwarzschild radius and thus fail to feature a continuous connection to black holes [3]. The analytic solution of the problem of a rigidly rotating uncharged disc of dust, in contrast, provides an example for a parametric transition of equilibrium configurations of ordinary matter to an extreme Kerr black hole [4], see also [5]. Rotating fluid rings admit such a black hole limit too [6, 7].
On the other hand, the so-called electrically counterpoised dust (ECD) configurations, a class of static solutions of dust with a particular constant specific charge, possess a parametric transition to the extreme Reissner–Nordström black hole, see [8]. Inspired by these opposite limiting cases, we conjecture that the dust disc with a constant specific charge varying from zero to the ECD value has a parametric transition to the extreme Kerr–Newman black hole. The parameter spaces of black holes and charged dust discs and their connections are illustrated in figure 1. In order to investigate this assumption and to support the search for the analytic solution of the charged disc, a post-Newtonian expansion is performed. We note that in the uncharged case, the analytic disc solution [4] was indeed preceded by the partly numerical but highly accurate Bardeen–Wagoner expansion [9], which could later be recovered from the analytic solution [10]. The charged disc is expected to show novel effects due to the interplay between the electromagnetic and gravitational fields.

2. Formulation of the disc problem

2.1. Model of matter

To facilitate analytic calculations, we employed the following simplifying assumptions.
Firstly, we use the model of dust, i.e. pressureless matter described by the dust part of the energy–momentum tensor in (4). Moreover, we assume a rigid rotation with a constant angular velocity $\Omega_1$. The dust we consider here contains the constant specific charge $\epsilon \in [-1, 1]$ so that the charge density $\rho_{el}$ and the baryonic mass density $\mu$ are connected via

$$\rho_{el} = \epsilon \mu$$

(we use Gauss units and $c = 1, G = 1$).

Furthermore, we assume axial symmetry and stationarity described by the Killing vectors $\eta$ and $\xi$, as well as reflectional symmetry w.r.t. an ‘equatorial plane’.

2.2. Basic field equations

We start with Einstein’s field equations

$$R_{ab} - \frac{1}{2}R g_{ab} = 8 \pi T_{ab}$$

(2)

and the covariant Maxwell equations

$$F_{[abc]} = 0, \quad F^{ab};b = 4\pi j^a,$$

(3)

wherein the energy–momentum tensor $T_{ab}$ is simply the sum of an electromagnetic part and a dust part:

$$T_{ab} = T^{(em)}_{ab} + T^{(dust)}_{ab} = \frac{1}{4\pi} (F_{ab} F^c_c - \frac{1}{4} g_{ab} F_{cd} F^{cd}) + \mu u_{a} u_{b}. $$

(4)

For the purely convective 4-current density holds $j^a = \rho_{el} u^a = \epsilon \mu u^a$. The metric can globally be written in terms of Weyl–Lewis–Papapetrou coordinates as

$$ds^2 = f^{-1} [h(d\varrho^2 + d\zeta^2) + \varrho^2 d\phi^2] - f (dt + a d\varphi)^2. $$

(5)

Moreover, a 4-potential of the form $A_\phi = (0, 0, A_\phi, A_t)$ can be introduced by $F_{ab} = A_{\alpha,b} - A_{\alpha,a} = A_{b,a} - A_{a,b}$, which automatically fulfils the homogeneous Maxwell equations. The coordinates correspond to the Killing vectors via $\partial_t = \xi$ and $\partial_\phi = \eta$, so that all the five free functions $f, h, a, A, \varrho$ depend only on the coordinates $\varrho$ and $\zeta$. In addition, $\mu$ is made up from a surface mass density by

$$\mu = \sigma(\varrho) \delta(\zeta).$$

(6)

Note that two other definitions of surface mass density will later be used: $\sigma = \sigma(\varrho) h/f$, directly linked with the boundary conditions and the coordinate-independent proper surface mass density $\sigma = \sqrt{f h}$.

The coupled nonlinear system of differential equations that emerge can be reformulated as a boundary value problem for the equations valid in the (electro-)vacuum outside the disc.

2.3. Ernst equations

In the absence of matter, the Einstein–Maxwell equations can be reduced to the Ernst equations. They are expressed in two additional potentials $\beta$ and $b$. Using the abbreviations $A_i = -a$ and $A_\phi = A$, these potentials are defined by

$$\beta_{,\varrho} = \frac{f}{\varrho} (a \alpha_{,\zeta} + A_{,\zeta}), \quad \beta_{,\zeta} = -\frac{f}{\varrho} (a \alpha_{,\varrho} + A_{,\varrho}),$$

(7)

and

$$b_{,\varrho} = -\frac{f^2}{\varrho} a_{,\zeta} + 2 (\beta \alpha_{,\varrho} - \alpha \beta_{,\varrho}), \quad b_{,\zeta} = \frac{f^2}{\varrho} a_{,\varrho} + 2 (\beta \alpha_{,\zeta} - \alpha \beta_{,\zeta}).$$

(8)
Thereby, the complex Ernst potentials $\mathcal{E}$ and $\Phi$ can be introduced:

$$\Phi = \alpha + i\beta, \quad \mathcal{E} = (f - \Phi) + ib.$$  \hspace{1cm} (9)

In these terms, the Ernst equations read [11]

$$(\Re \mathcal{E} + \Phi \Phi) \Delta \mathcal{E} = (\nabla \mathcal{E} + 2\Phi \nabla \Phi) \cdot \nabla \mathcal{E},$$  \hspace{1cm} (10)

$$(\Re \mathcal{E} + \Phi \Phi) \Delta \Phi = (\nabla \mathcal{E} + 2\Phi \nabla \Phi) \cdot \nabla \Phi,$$  \hspace{1cm} (11)

where the behaviour of $\nabla$ and $\Delta$ is analogous to that in Euclidean 3-space, with cylindrical coordinates $(\varrho, \zeta, \phi)$. The metric function $h$ is eliminated during this process of transforming the field equations to the Ernst equations and can be determined by integration afterwards. For the purpose of the post-Newtonian expansion, it is helpful to express the Ernst equations in terms of the real functions $f$, $\alpha$, $\beta$ and $b$, which leads to

$$f \Delta f = f \Delta (\alpha^2) + f \Delta (\beta^2) + \nabla f (\nabla f - 2\alpha \nabla \alpha - 2\beta \nabla \beta)$$

$$- \nabla \beta (\nabla \beta + 2\alpha \nabla \beta - 2\beta \nabla \alpha),$$  \hspace{1cm} (12)

$$f \Delta b = 2\nabla f \nabla b + 4\alpha b((\alpha \nabla \alpha)^2 - (\nabla \beta)^2) - 4(\alpha^2 - \beta^2)\nabla \alpha \nabla \beta$$

$$+ 2\nabla f (\alpha \nabla \beta - \beta \nabla \alpha) - 2\nabla b (\alpha \nabla \alpha + \beta \nabla \beta),$$  \hspace{1cm} (13)

$$f \Delta \alpha = \nabla f \nabla \alpha - \nabla \beta (\nabla b + 2\alpha \nabla \beta - 2\beta \nabla \alpha),$$  \hspace{1cm} (14)

$$f \Delta \beta = \nabla f \nabla \beta + \nabla \alpha (\nabla b + 2\alpha \nabla \beta - 2\beta \nabla \alpha).$$  \hspace{1cm} (15)

### 2.4. Boundary conditions on the disc

To describe the influence of the matter, we have to return to the full Einstein–Maxwell equations. We evaluate them in the corotating frame with $\varphi' = \varphi - \Omega t$, where the metric retains its form

$$ds^2 = f'^{-1} [h'(d\varrho^2 + d\zeta^2) + \varphi^2 d\varphi'^2] - f'(dt + a'd\varphi')^2,$$  \hspace{1cm} (16)

with the dashed functions related to the original ones by

$$f' = f \left[ (1 + \Omega \varrho)^2 - \frac{\Omega^2 \varrho^2}{f^2} \right], \quad (1 + \Omega \varrho) f = (1 - \Omega \varrho) f' \quad \text{and} \quad \frac{h}{f} = \frac{h'}{f'}.$$  \hspace{1cm} (17)

The Einstein–Maxwell equations in the corotating frame lead to the system

$$0 = -\nabla \cdot \left[ \frac{f'^2}{\varrho^2} (a' \nabla \alpha' + \nabla A') \right],$$  \hspace{1cm} (18)

$$4\pi \sigma e f'^{-1} \delta(\zeta) = \nabla \cdot \left[ \frac{a'^2}{\varrho^2} (a' \nabla \alpha' + \nabla A') - \frac{1}{f'} \nabla \alpha' \right],$$  \hspace{1cm} (19)

$$0 = \nabla \cdot \left[ \frac{f'^2}{\varrho^2} \nabla a' + 4 \frac{f'}{\varrho^2} a' (a' \nabla \alpha' + \nabla A') \right],$$  \hspace{1cm} (20)

$$8\pi f'^2 \delta(\zeta) = f' \Delta f' - (\nabla f')^2 + \frac{f'^4}{\varrho^2} (\nabla a')^2 - 2f' \left[ (\nabla \alpha')^2 + \frac{f'^2}{\varrho^2} (a' \nabla \alpha' + \nabla A')^2 \right],$$  \hspace{1cm} (21)

$$\ln h_\alpha = \frac{1}{2} \frac{f'}{\varrho} \left[ \ln f' \right]^2 - \frac{f'^2}{\varrho^2} \left( a_{.,\varrho}^2 - a_{.,\zeta}^2 \right) + 2 \frac{f'}{\varrho} \left( A_{.,\varrho}^2 - A_{.,\zeta}^2 \right)$$

$$- \frac{\varrho^2 - a^2 f'^2}{f'^2} \left( a_{.,\varrho}^2 - a_{.,\zeta}^2 \right) + 2 \frac{f'}{\varrho} \left( A_{.,\varrho} a_{.,\varrho}' - A_{.,\zeta} a_{.,\zeta}' \right).$$  \hspace{1cm} (22)
\( (\ln h')_\zeta = q \left[ 4(\ln f')_\phi (\ln f')_\zeta - \frac{f'^2}{q^2} a' a'_\zeta \right] + 4 \left[ \frac{f'}{q} (A' + d' a'_\phi) (A'_\zeta + a' a'_\zeta) - \frac{q}{f} a' a'_\zeta \right] \). \quad (23)

Integration over a small flat cylinder around a mass element of the disc (very analogous to the well known treatment of surface charge densities in electrostatics) delivers matching conditions between the areas above and beneath the disc. These matching conditions are transformed to the following four boundary conditions by the use of reflectional symmetry:

\[ \beta' = 0, \quad b' = 0, \quad a'_\zeta = -\epsilon (f^2)_{,\zeta}, \quad (f^2)_{,\phi} = -\epsilon a'_\phi. \quad (24) \]

Back in the nonrotating frame they take the form

\[ \left( \frac{\Omega q^2}{(1 + \Omega a) f^2} - a \right) a_{,\xi} = A_{,\xi}, \quad (25) \]
\[ \Omega q^2 \left( \frac{\Omega q^2}{(1 + \Omega a)^2} \right) = a_{,\xi}, \quad (26) \]
\[ \epsilon \left( \sqrt{\frac{(1 + \Omega a)^2 f - \Omega^2 q^2 f^{-1}}{\xi}} \right) = (\Omega a - a)_{,\xi}, \quad (27) \]
\[ \epsilon \left( \sqrt{\frac{(1 + \Omega a)^2 f - \Omega^2 q^2 f^{-1}}{\phi}} \right) = (\Omega a - a)_{,\phi}. \quad (28) \]

The treatment in the nonrotating frame is justified by the simple boundary conditions at spatial infinity, which are given by \( g_{ab} \to \eta_{ab} \) and \( A_a \to 0 \) in that system.

Overall we have the four differential equations (12)–(15) and the four boundary conditions (25)–(28) for the six expansion functions \( f, a, b, A, \alpha, \beta \). However, due to definitions (7) and (8), only four of these functions are independent and therefore the disc problem is well posed.

3. Ansatz for the expansion

3.1. Parameter space of the charged disc

We introduce the relativity parameter \( \gamma \), which was already successfully used for the post-Newtonian expansion of the uncharged disc [9]:

\[ \gamma = 1 - \sqrt{f_c} \quad \text{with} \quad f_c = f(\phi = 0, \zeta = 0). \quad (29) \]

The parameter \( \gamma \) is closely related to the redshift \( Z_c \) of a photon emitted at the centre of the disc and measured at infinity:

\[ \gamma = Z_c (1 + Z_c)^{-1}. \quad (30) \]

Therefore, it is not surprising that the transition to a black hole happens at \( \gamma \to 1 \) for the uncharged and the ECD case. The parameter space of the charged disc as shown in figure 1 (w.l.o.g. restricted to positive charges) is now strongly conjectured to be the area \( \gamma \in [0, 1], \epsilon \in [0, 1] \) shown in figure 2. As a third parameter, the disc radius \( \psi_0 \) is added. It naturally comes into play by introducing elliptic coordinates through

\[ \zeta = \psi_0 \xi \eta \quad \text{and} \quad \psi = \psi_0 \sqrt{(1 - \eta^2)(1 + \xi^2)} \]

\[ (\ln h')_\zeta = q \left[ 4(\ln f')_\phi (\ln f')_\zeta - \frac{f'^2}{q^2} a' a'_\zeta \right] + 4 \left[ \frac{f'}{q} (A' + d' a'_\phi) (A'_\zeta + a' a'_\zeta) - \frac{q}{f} a' a'_\zeta \right] \]. \quad (23)

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\[ \zeta = \psi_0 \xi \eta \quad \text{and} \quad \psi = \psi_0 \sqrt{(1 - \eta^2)(1 + \xi^2)} \]
and carries the whole information about the scale of the disc. By the usage of elliptic coordinates and by multiplying powers of $\varrho_0$ to unit carrying quantities, the whole disc problem can be made dimensionless. Those normalized quantities, labelled with a star, are for example

$$a^* = a/\varrho_0, \quad A^* = A/\varrho_0, \quad \Omega^* = \varrho_0 \Omega \quad \text{and} \quad \sigma^* = \varrho_0 \sigma.$$  \hfill (31)

### 3.2. Formulation of power series

Inspired by the Newtonian disc of charged dust (in the Newtonian limit we have $\gamma = -U_N (\varrho = 0, \zeta = 0) = \Omega^2/\(1 - \epsilon^2\)$ with the Newtonian potential $U_N$), the squared angular velocity is expanded as a power series in $\gamma^2$:

$$\Omega^2 = (1 - \epsilon^2)\gamma + O(\gamma^3).$$  \hfill (32)

The angular velocity $\Omega^*$ is therefore described by an odd series in $\gamma^2$:

$$\Omega^* = \sqrt{\gamma} \sum_{n=1}^{\infty} \Omega^* n^{n-1}. \hfill (33)$$

All the previously discussed functions are now either symmetric or antisymmetric with respect to a change of the sense of rotation $\Omega^* \rightarrow -\Omega^*$. As such, they can be expressed either in even or odd power series of $\gamma^2$. The lowest occurring power is determined by the Newtonian limit. On the whole, we obtain

$$f = 1 + \gamma \sum_{n=1}^{\infty} f_n \gamma^{n-1}, \quad \alpha = \gamma \sum_{n=1}^{\infty} \alpha_n \gamma^{n-1}, \quad \sigma = \gamma^2 \sum_{n=1}^{\infty} \sigma_n \gamma^{n-1},$$

$$A = \gamma^2 \sum_{n=1}^{\infty} A_n \gamma^{n-1}, \quad \beta = \gamma^2 \sum_{n=1}^{\infty} \beta_n \gamma^{n-1}, \quad \delta = \gamma^2 \sum_{n=1}^{\infty} \delta_n \gamma^{n-1}. \hfill (34)$$

The remaining coefficients are still functions of $\xi$ and $\eta$. These expansions are now inserted in the 12 equations (12)–(15), (25)–(28) plus (7) and (8). These 12 equations, which we call expansion equations, are now evaluated by equating coefficients of powers of $\gamma$ in order to obtain analytic expressions.
4. Algorithmic solution

4.1. Structural examination

The expansion equations in $k$th order read

\begin{align}
(1 + \xi^2)\beta_{k,\xi} &= -A_{k,\eta}/\varrho_0 + F_{D1k} \\
(1 - \eta^2)\beta_{k,\eta} &= +A_{k,\xi}/\varrho_0 + F_{D2k} \\
(1 + \xi^2)\eta_{k,\xi} &= +\alpha_{k,\eta}/\varrho_0 + F_{D3k} \\
(1 - \eta^2)\eta_{k,\eta} &= -a_{k,\xi}/\varrho_0 + F_{D4k} \\
\sqrt{1 - \epsilon^2(1 - \eta^2)}\alpha_{k,\xi} &= +A_{k,\xi}/\varrho_0 + F_{B1k} \\
2\sqrt{1 - \epsilon^2(1 - \eta^2)}\alpha_{k,\eta} &= -a_{k,\xi}/\varrho_0 + F_{B2k} \\
\epsilon f_{k,\xi} &= -2\alpha_{k,\xi} + F_{B3k} \\
4\sqrt{1 - \epsilon^2\eta\varrho_0}\Omega_e + f_{k,\eta} &= -2\epsilon\alpha_{k,\eta} + F_{B4k} \\
\Delta f_k &= F_{E1k} \\
\Delta b_k &= F_{E2k} \\
\Delta a_k &= F_{E3k} \\
\Delta \beta_k &= F_{E4k}. 
\end{align}

For clarity, only coefficient functions of $k$th order have been written down, and terms of lower orders in each equation are subsumed in the symbol $F$. These functions $F$ all vanish for $k = 1$ so that the first-order equations reduce to coupled Laplace equations with boundary conditions. The gravitational and the electric potential of the Newtonian disc are reproduced through $f_1 = 2U_N$ and $\alpha_1 = -U_d$ (see (A.2)). For $k > 1$, none of the functions $F$ vanish and we obtain coupled Poisson equations. The requirement for asymptotical flatness implies that all the six coefficient functions $f_k, \alpha_k, \beta_k, b_k, A_k$ and $a_k$ have to vanish at spatial infinity.

4.2. Solving the boundary value problem

A very helpful observation comes from the fact that all expansion functions are polynomials in $\eta$. Therefore, the functions $F$ (besides a factor of $(\xi^2 + \eta^2)^{-1}$ in the $F_{Eik}$) are also polynomials in $\eta$ and the Poisson inhomogeneity can thus be fragmented to Legendre polynomials. Then, the Poisson equation with an inhomogeneity consisting of a single Legendre polynomial,

\[ \Delta \psi = \frac{1}{\xi^2 + \eta^2} \left[ [(1 + \xi^2)\psi_{,\xi}]_\xi + [(1 - \eta^2)\psi_{,\eta}]_\eta \right] = \frac{I(\xi)P_n(\eta)}{\xi^2 + \eta^2}, \]

is transformed by the ansatz $\psi = A(\xi)P_n(\eta)$ into the ordinary differential equation

\[ [(1 + \xi^2)A,\xi]_\xi - n(n + 1)A = I(\xi). \]

Two independent (real) solutions to the corresponding homogeneous equation are

\[ A_1 = \imath i^2P_n(i\xi), \quad A_2 = \imath^2\imath^{-n}Q_n(i\xi), \]

where $Q_n$ are the Legendre polynomials of the second kind. Now the inhomogeneous solution can be composed as

\[ A(\xi) = -A_1(\xi) \int_{\xi_1}^{\xi} \frac{A_2(x)I(x)}{(1 + \xi^2)W} \, dx + A_2(\xi) \int_{\xi_2}^{\xi} \frac{A_1(x)I(x)}{(1 + \xi^2)W} \, dx, \quad W = A_2^2A_1 - A_1^2A_2. \]

With adequately chosen constants $c_1$ and $c_2$, the boundary conditions at $\xi \to \infty$ and $\xi \to 0$ can be fulfilled, respectively.
4.3. Completion of the expansion equations and algorithmic solution

In addition to the 12 expansion equations (35a)–(37d), two parameter relations are necessary to determine an integration constant and the scalar component \( \Omega_k \) for each order. First, from (29), we immediately obtain at \( \varrho = 0, \zeta = 0 \):

\[
\begin{align*}
    f_{1c} &= -2, \\
    f_{2c} &= 1 \quad \text{and} \quad f_{k} &= 0 \quad \text{for } k > 2. 
\end{align*}
\]  

(42)

Secondly, the following relation to the surface mass density can be derived analogously to the derivation of the boundary conditions (\( F_{\ldots} \) still collects lower order terms):

\[
\frac{1}{\eta} [f_{k,\xi} + F_{\text{REG1}k}] = 8\pi \varrho_0 \sigma_k + F_{\text{REG2}k}. 
\]  

(43)

In order to have a finite value of \( \sigma_k \) at the rim of the disc, \( \eta = 0 \), the expression \( [f_{k,\xi} + F_{\text{REG1}k}] \) has to contain a global factor \( \eta \). The integration constants of the other functions vanish due to reflectional symmetry.

The system of expansion equations can be dealt with by setting up the linear combination \( O_k = \frac{1}{2} \epsilon f_k + \alpha_k \). This provides a quantity with a given Poisson inhomogeneity and a given boundary condition:

\[
\Delta O_k = \frac{\epsilon}{2} F_{E1k} + F_{E3k}, \quad O_{k,\xi} = \frac{1}{2} F_{B3k}.
\]  

(44)

Once \( O_k \) is determined according to 4.2, \( f_k \) can be determined by (36d) and \( \alpha_k \) is immediately obtained. Hereupon \( \beta_k \) and \( b_k \) can be computed successively out of Poisson boundary value problems and \( A_k \) and \( a_k \) eventually by integration.

5. Results

With the methods described above and making use of Mathematica by Wolfram Research, coefficient functions of the six expansion functions could be calculated up to the eighth order. The first important observation is that these coefficient functions are not only polynomial in \( \eta \) but also in \( \xi \) and \( \arccot \xi \). The first two orders are listed in the appendix, and higher orders are available as data files. Noteworthy are the global prefactors in the expansion functions, which could be found in all of their coefficients:

\[
\begin{align*}
    \Omega^*_k &\propto \sqrt{1 - \epsilon^2}, \\
    \alpha_k &\propto \epsilon, \\
    \beta_k &\propto \epsilon \sqrt{1 - \epsilon^2 \eta}, \\
    b_k &\propto \sqrt{1 - \epsilon^2 \eta}, \\
    A_k^* &\propto \epsilon \sqrt{1 - \epsilon^2 (1 - \eta^2)} \quad \text{and} \\
    a_k^* &\propto \sqrt{1 - \epsilon^2 (1 - \eta^2)}. 
\end{align*}
\]  

(45)

Within the expansion, the angular velocity \( \Omega^* \) is also calculated. It is depicted in figure 3 up to the eighth order. Here, the parameter space (figure 2) is shown in the \( \gamma - \epsilon \) plane. \( \Omega^* \) vanishes at the transition to the static configurations for \( \epsilon \rightarrow 1 \) and in the extreme relativistic limit \( (\gamma \rightarrow 1) \). The latter is caused by the vanishing disc radius \( \varrho_0 \). This shows that \( \varrho_0 \) is no longer a well-chosen parameter for \( \gamma \rightarrow 1 \).

The further evaluation of the results is done by discussing the global quantities of the disc. The gravitational mass \( M \), the charge \( Q \) and the angular momentum \( J \) can be conveniently deduced from the far field behaviour of the expansion functions:

\[
\begin{align*}
    f &= 1 - \frac{2M^*}{\xi} + O(\xi^{-2}), \\
    \alpha &= \frac{Q^*}{\xi} + O(\xi^{-2}), \\
    a &= \frac{2J^*(1 - \eta^2)}{\xi} + O(\xi^{-2}).
\end{align*}
\]  

(46-48)
One compelling feature of the disc with constant specific charge is that even the baryonic mass $M_0 = \epsilon^{-1} Q$ can be derived from the asymptotical behaviour. Based on this, the relative binding energy $E_{B}^{(\text{rel})} = (M_0 - M)M_0^{-1}$ can be directly calculated (even in the limit $\epsilon \to 0$). It is depicted in figure 4 up to the seventh significant order. The first three orders read

$$E_{B}^{(\text{rel})} = \frac{1}{5}(1 - \epsilon^2)\gamma + \frac{2}{175}(2\epsilon^4 - 9\epsilon^2 + 7)\gamma^2 + \frac{\epsilon^2 - 1}{3024 \ 000 \pi^2}[179 \ 200(3\epsilon^4 - 52\epsilon^2 + 64) - \pi^2(58 \ 618\epsilon^4 - 1042 \ 821\epsilon^2 + 1275 \ 648)]\gamma^3 + \mathcal{O}(\gamma^4), \quad (49)$$

and all seven orders are given in the appendix. At this point, the accuracy of the expansion can be tested by comparison with the known analytic solution of the uncharged disc. The potentially most critical point is $\epsilon \to 0$, $\gamma \to 1$, where the transition to the extreme Kerr black hole occurs. Here, the numerically evaluated expansion gives $E_{B}^{(\text{rel})}(\epsilon = 0, \gamma = 1) = 0.3614$, which differs from the analytic value $0.373 \ 283 \ 588 \ldots$ [5] by about 3%. In the uncharged limit ($\epsilon \to 0$), the expressions for the remaining nontrivial functions $f$ and $a$ agree with an expansion of the analytic solution of the uncharged disc [10]. In addition, the results for the
global quantities for $\epsilon \to 0$ comply with those of the uncharged disc given in [5]. A non-trivial test for all $\epsilon$ consists in checking the exact relation [12]

$$M = 2\Omega J + (1 - \gamma + \epsilon\alpha_c)M_0, \quad \alpha_c = \alpha(\varrho = 0, \zeta = 0).$$

(50)

This relation is satisfied by our results for all eight available orders in $\gamma$.

Figure 5 shows the coordinate-independent surface mass density $\sigma_p$ normalized by the angular velocity $\Omega$. In contrast to the normalization by $\varrho_0$ in figure 3, this gives a plot regular at $\gamma \to 1$. The already known shift of the maximum of $\sigma_p/\Omega$ out of the disc centre appears to weaken with growing charge. For $\epsilon \to 1$, the angular velocity $\Omega$ vanishes and is hence no longer an appropriate scaling parameter.

Another interesting quantity is the disc radius $\varrho_0$, normalized by the baryonic mass $M_0$ which is depicted in figure 6. The small values at $\gamma \to 1$ and the weak dependence on $\epsilon$ indicate a transition to a black hole as in the limiting cases $\epsilon = 0$ and $\epsilon = 1$. Note that one has to distinguish the external perspective ($\varrho/M, \zeta/M$) from the internal perspective ($\varrho/\varrho_0, \zeta/\varrho_0$) in the limit $\gamma \to 1$, see the discussion in [12].

6. Conclusions

The vanishing of the quantity $\varrho_0/M_0$ for $\gamma \to 1$ can be interpreted as the contraction of a disc with given baryonic mass to the origin of the $(\varrho/M, \zeta/M)$-coordinate system in the extreme relativistic limit. This strongly indicates the occurrence of a singularity in that limiting case. To check whether this singularity is indeed related to an extreme Kerr–Newman black hole,
three other, more specific tests are carried out. For the extreme Kerr–Newman metric, the
following relation holds on the horizon \( \mathcal{H} \) (located at \( \rho / M = 0 = \zeta / M \)):

\[
n'(\mathcal{H}) = \left[ \frac{2M}{Q} - \frac{Q}{M} \right]^{-1}.
\]  

(51)

The corresponding \( \epsilon \)-dependence of \( n'(\rho / M = 0, \zeta / M = 0) \) computed out of \( E_B^{(\text{rel})} \) via 
\( M/Q = \epsilon^{-1}(1 - E_B^{(\text{rel})}) \) is depicted in figure 7 as the red line. If we transform to normalized 
coordinates by \( \rho \to \rho^*, \zeta \to \zeta^* \) (keep in mind that we expect \( \rho_0 \to 0 \) for \( \gamma \to 1 \)),
then the extreme relativistic limit yields the limiting spacetime from the internal perspective.
The relation \( n'(\rho^*) = \text{const} \) on the whole disc is strongly conjectured to be necessary and 
sufficient for obtaining the extreme Kerr–Newman metric as the external solution (i.e. for 
\( \rho^2 + \zeta^2 / M^2 > 0 \)), provided \( \rho_0 \to 0 \) as \( \gamma \to 1 \). The curves of \( n'(\rho^* = 0) \) and \( n'(\rho^* = 1) \)
are depicted in figure 7 in blue and green.

The similarity of the three curves shows that \( n' \) is indeed in good approximation 
independent of \( \rho^* \) and that its value meets the one given by (51).

Furthermore, the extreme Kerr–Newman black hole obeys the two parameter relations

\[
Q_1 := \frac{Q^2}{M^2} + \frac{J^2}{M^4} = 1 \quad \text{and} \quad Q_2 := \frac{J/M^2}{\Omega M(1 + J^2/M^4)} = 1
\]  

(52)
Figure 8. Plots of $Q_1$ for the charged disc with $n = 8$ (a) and in the limit $\gamma \to 1$ for different expansion orders $n$ (b).

Figure 9. Plots of $Q_2$ in the limit $\gamma \to 1$ for different expansion orders $n$.

with $\Omega$ meaning the angular velocity of the horizon. The first of these quotients, calculated out of the disc quantities, is shown in figure 8. For the generic disc, $Q_1$ is greater than 1 as expected, whereas in the extreme relativistic limit, $Q_1$ converges to unity for growing expansion order $n$. The last statement also holds for the second quotient $Q_2$ shown in figure 9. Therefore,
we conclude that we have found strong evidence for the transition of the charged disc to an extreme Kerr–Newman black hole in the limit $\gamma \to 1$.

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Appendix A. Coefficient functions in first order

In the appendix, we use the abbreviation $X := \arccot \xi$. Global factors are exposed at the cost of showing the fragmentation to Legendre polynomials:

$$\Omega_1^x = \sqrt{1 - \epsilon^2}$$  \hspace{1cm} (A.1)

$$f_1 = -\frac{2}{\epsilon} \omega_1 = -\frac{8}{3\pi} \frac{X}{x^3} \left[ 3x^2 + 1 \right] \frac{1}{2} \frac{X}{-\frac{3}{2} \xi} (3\eta^2 - 1)$$  \hspace{1cm} (A.2)

$$\beta_1 = -\frac{\epsilon}{4} b_1 = \frac{\epsilon \sqrt{1 - \epsilon^2}}{3\pi} \eta \left[ 9\eta^2 - (15\xi^2 + 4)\eta^2 + 3\xi (5\xi^2 + 3)\eta^2 - 3\xi^2 - 1 \right] X$$  \hspace{1cm} (A.3)

$$a_1^x = -\frac{4}{\pi} A_1^x = -\frac{\epsilon \sqrt{1 - \epsilon^2}}{\pi} (\eta^2 - 1) \left[ \xi - 13\eta^2 \xi + (3 - 15\eta^2)\xi^3 
+ (\xi^2 + 1) (3\eta^2 (5\xi^2 + 1) - 3\xi^2 + 1) X \right].$$  \hspace{1cm} (A.4)

Appendix B. Coefficient functions in second order

$$\Omega_2^x = \sqrt{1 - \epsilon^2} \frac{2\epsilon^2 - 9}{12}$$  \hspace{1cm} (B.1)

$$f_2 = \frac{1}{36\pi^2} \left[ (-72\pi + 15\pi \epsilon^2)\xi + (72 + 36\epsilon^2)\xi^2 + (-108\pi + 153\pi \epsilon^2)\xi^3 
+ \left[ -72\pi + 15\pi \epsilon^2 + (144 + 72\epsilon^2)\xi + (108\pi - 66\pi \epsilon^2)\xi^2 
+ (-144 - 72\epsilon^2)\xi^3 + (108\pi - 153\pi \epsilon^2)\xi^4 \right] X + [72 + 36\epsilon^2 
+ (-144 - 72\epsilon^2)\xi^2 + (72 + 36\epsilon^2)\xi^4] X^2 + \eta^2 \left[ (612\pi - 606\pi \epsilon^2)\xi 
+ (-432 - 216\epsilon^2)\xi^2 + (108\pi - 153\pi \epsilon^2)\xi^3 + \left[ -108\pi + 66\pi \epsilon^2 
+ (-288 - 144\epsilon^2)\xi + (-972\pi + 1116\pi \epsilon^2)\xi^2 + (864 + 432\epsilon^2)\xi^3 
+ (-108\pi + 153\pi \epsilon^2)\xi^4 \right] X + [144 + 72\epsilon^2 + (288 + 144\epsilon^2)\xi^2 
+ (-432 - 216\epsilon^2)\xi^4] X^2 \right] + \eta^4 \left[ (-660\pi + 935\pi \epsilon^2)\xi 
+ (648 + 324\epsilon^2)\xi^2 + (-1260\pi + 1785\pi \epsilon^2)\xi^3 + \left[ 108\pi - 153\pi \epsilon^2 
+ (-432 - 216\epsilon^2)\xi + (108\pi - 153\pi \epsilon^2)\xi^2 + (-1296 - 648\epsilon^2)\xi^3 
+ (1260\pi - 1785\pi \epsilon^2)\xi^4 \right] X + [72 + 36\epsilon^2 + (432 + 216\epsilon^2)\xi^2 
+ (648 + 324\epsilon^2)\xi^4] X^2 \right] \right].$$  \hspace{1cm} (B.2)
$\alpha_2 = \frac{\epsilon}{72\pi^2} \sum \left[ (81\pi - 24\pi e^2)\xi - 72\xi^2 + (27\pi - 72\pi e^2)\xi^3 + [81\pi - 24\pi e^2 - 144\xi
+ (-90\pi + 48\pi e^2)\xi^2 + 144\xi^3 + (-27\pi + 72\pi e^2)\xi^4]X + [-72 + 144\xi^2
- 72\xi^4]X^2 + \eta^4((165\pi - 440\pi e^2)\xi - 648\xi^2 + (315\pi - 840\pi e^2)\xi^3
+ [-27\pi + 72\pi e^2 + 432\xi + (-270\pi + 720\pi e^2)\xi^2 + 1296\xi^3 + (-315\pi
+ 840\pi e^2)\xi^4]X + [-72 - 432\xi^2 - 648\xi^4]X^2] + \eta^2([-34\pi
+ 336\pi e^2)\xi + 432\xi^2 + (-270\pi + 720\pi e^2)\xi^3 + [90\pi - 48\pi e^2 + 288\xi
+ (432\pi - 576\pi e^2)\xi^2 - 864\xi^3 + (270\pi - 720\pi e^2)\xi^4]X
+ (-144 - 288\xi^2 + 432\xi^4]X^2] \right],
(3.5)

$\beta_2 = \frac{\epsilon\sqrt{1 - \xi^2}}{1440\pi^2} \sum \left[ (9225\pi - 5520\pi e^2)\xi - 12960\xi^3 + (-7425\pi - 7200\pi e^2)\xi^4
+ [(3735\pi + 1200\pi e^2)\xi - 8640\xi^2 + (-6750\pi + 7920\pi e^2)\xi^3 + 25920\xi^4
+ (7425\pi + 7200\pi e^2)\xi^2 + [4320\xi + 8640\xi^3 - 12960\xi^5]X + \eta^4
\right]
\left[ -2112\pi - 2048\pi e^2 - 17280\xi + (-24255\pi - 23520\pi e^2)\xi^2 - 64800\xi^3
+ (-31185\pi - 30240\pi e^2)\xi^4 + [5760 + (7425\pi + 7200\pi e^2)\xi + 77760\xi^2
+ (34650\pi + 33600\pi e^2)\xi^3 + 129600\xi^4 + (31185\pi + 30240\pi e^2)\xi^5]X
+ [-12960\xi - 60480\xi^3 - 64000\xi^5]X^2] + \eta^2[-4320\pi + 2240\pi e^2 + 5760\xi
+ (1950\pi + 26000\pi e^2)\xi^2] - 60480\xi^3 + (34650\pi + 33600\pi e^2)\xi^4
+ [5760 + (6750\pi + 7920\pi e^2)\xi - 23040\xi^2 + (-13500\pi - 37200\pi e^2)\xi^3
- 12960\xi^4 + (34650\pi - 33600\pi e^2)\xi^5]X
+ [-8460\xi + 17280\xi^3 + 60480\xi^5]X^2] \right],
(3.6)

$\beta_2 = \frac{\epsilon\sqrt{1 - \xi^2}}{1440\pi^2} \sum \left[ (-3024\pi + 2703\pi e^2)\xi - 3825\pi e^2\xi^4 + [(1008\pi - 561\pi e^2)\xi
+ 2304\xi^2 + (3024\pi - 3978\pi e^2)\xi^3 + 6912\xi^4 - 3825\pi e^2\xi^5]X + [-1152\xi
- 2304\xi^3 + 3456\xi^5]X^2 + \eta^2[1344\pi - 1088\pi e^2 - 1536\xi + (5040\pi
- 13430\pi e^2)\xi^2 - 16128\xi^3 - 17960\pi e^2\xi^4 + (-1536 + (-3024\pi
+ 3978\pi e^2)\xi^2 + 6144\xi^4 + (-5040\pi + 19380\pi e^2)\xi^3 + 32256\xi^4
+ 17850\pi e^2\xi^5]X + [2304\pi - 4608\xi^3 - 16128\xi^5]X^2] + \eta^2[1088\pi e^2
+ 4608\pi + 12495\pi e^2\xi^2 + 17280\xi^3 + 16650\pi e^2\xi^4 + (-1536 - 3825\pi e^2\xi
- 20736\xi^2 - 17850\pi e^2\xi^3 - 34560\xi^4 - 16650\pi e^2\xi^5]X
+ [3456\xi + 16128\xi^3 + 17280\xi^5]X^2] \right],
(3.7)

$A_2^\xi = \frac{\epsilon\sqrt{1 - \xi^2}}{576\pi^2} \sum \left[ (1 - \eta^2)(-1536 + (423\pi - 72\pi e^2)\xi + 288\xi^2 + (840\pi - 632\pi e^2)\xi^3
- 864\xi^4 + (-495\pi - 480\pi e^2)\xi^5 + [423\pi - 72\pi e^2 + 576\xi + (-747\pi
+ 240\pi e^2)\xi^2 + (-675\pi + 792\pi e^2)\xi^3 + 1728\xi^4 + (495\pi + 480\pi e^2)\xi^5]X
+ [288 + 864\xi^2 - 288\xi^4 - 864\xi^6]X^2 + \eta^4(-1536 + (-3729\pi
- 3616\pi e^2)\xi - 19296\xi^2 + (-13860\pi - 13440\pi e^2)\xi^3 - 21600\xi^4
+ (-10395\pi + 10080\pi e^2)\xi^5 + (495\pi + 480\pi e^2 + 12096\pi + (7425\pi
+ 7200\pi e^2)\xi^2 + 52992\xi^3 + (17325\pi + 168000\pi e^2)\xi^4 + 43200\xi^5
+ (10395\pi + 10080\pi e^2)\xi^6]X + [-864 - 12960\pi e^2 - 33696\xi^4
- 21600\xi^6]X^2] + \eta^2(-1536 + (4014\pi + 2376\pi e^2)\xi + 4032\xi^2
\right],
(3.8)
Appendix C. Relative binding energy up to the seventh order

\[ E_B^{(\text{rel})} = (1 - \epsilon^2) \left\{ \frac{1}{5} \pi^2 + \frac{2}{175} (7 - 2\epsilon^2) \gamma^2 + \frac{1}{3024000\pi^2} [1 - 179200 (64 - 52\epsilon^2 + 3\epsilon^4) \right. \\
+ \pi^2 (1275648 - 1042821\epsilon^2 + 58618\epsilon^4)] \gamma^3 \\
- \frac{1}{119218176000\pi^2} [1 - 1433600 (4100096 + 4180767\epsilon^2 - 513631\epsilon^4) \\
+ 63360\epsilon^6] + \pi^2 (-619450073088 + 643913271759\epsilon^2 - 89452005694\epsilon^4 \\
+ 8224514048\epsilon^6)] \gamma^4 \\
+ \frac{1}{38088776613888000\pi^4} [60189671686144000 (3\epsilon^8 - 79\epsilon^6 + 556\epsilon^4) \\
- 992\epsilon^2 + 512] - 1433600\pi^2 (296565800960\epsilon^8 - 567224097024\epsilon^6 \\
+ 2399786978566584 - 40280188824129\epsilon^2 + 22326414409728) \\
+ \pi^4 (22758910736859136\epsilon^8 - 352585265954279424\epsilon^6 \\
+ 126049642449134226\epsilon^4 + 181545217870667751\epsilon^2 \\
+ 124003773931585536)] \gamma^5 + \frac{55030077220425691299848000000\pi^4}{1572597220425691299848000000\pi^4} \\
\times [\pi^4 (-497795548816521583132672\epsilon^{10})] \right\} 
\]
\[
\frac{1}{\gamma^6} \left[ 42409 186 426 629 786 850 736 209 920 000 000 \pi^2 F \\
- 23 625 990 069 063 869 264 830 660 608 000 000 \\
\times (3e^4 - 52e^2 + 64)(e^4 - 9e^2 + 8)^2 \\
+ 192 414 534 860 800 \pi^2 (138 824 602 612 452 556 800 \pi^4 + 366 983 313 237 803 008 000 \pi^2 + 31 662 317 283 432 188 592 128 \pi^4 + 128 804 662 549 475 170 301 551 \pi^6 + 254 210 776 514 571 609 476 318 \pi^4 \\
- 232 717 498 273 294 565 021 295 \pi^2 + 79 178 625 735 391 287 705 600 \pi^4 + 204 800 \pi^6 + 3 \gamma^6 + O(\gamma^8) \right] . 
\]
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