Vulnerability Parameters of Tensor Product of Cycles

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Abstract. In this paper, the vulnerability parameters, namely, connectivity, toughness, scattering number, integrity and tenacity of the graph \( C_n \times C_s \), where \( C_n \) denote the cycle of length \( n(\geq 3) \) is determined.

1. Introduction

In this paper, all graphs considered are finite, undirected and simple. Let \( G \) and \( H \) be two simple graphs. The tensor product of \( G \) and \( H \), denoted by \( G \times H \), has vertex set \( V(G \times H) = V(G) \times V(H) \) and edge set \( E(G \times H) = \{(u, v)(x, y) : ux \in E(G) \text{ and } vy \in E(H)\} \). The Cartesian product of \( G \) and \( H \), denoted by \( G \square H \), has the vertex set \( V(G \square H) = V(G) \times V(H) \) and edge set \( E(G \square H) = \{(u, v)(x, y) : u = x \text{ and } vy \in E(H) \text{ or } v = y \text{ and } ux \in E(G)\} \). The wreath product of \( G \) and \( H \), denoted by \( G \circ H \), has vertex set \( V(G \circ H) = V(G) \times V(H) \) and edge set \( E(G \circ H) = \{(u, v)(x, y) : ux \in E(G) \text{ or } u = x \text{ and } vy \in E(H)\} \). For \( S \subseteq V(G) \), \( \langle S \rangle \) denotes the subgraph of \( G \) induced by \( S \). Let \( P_m \) and \( C_m \) denote the path and cycle on \( m \) vertices, respectively and \( K_n \) and \( \overline{K}_n \) denote the complete graph and its complement on \( n \) vertices, respectively.

Let \( \omega(G) \) and \( \tau(G) \) denote the number of components and the order of a largest component of \( G \), respectively. Let \( \kappa(G) \) denote the connectivity of \( G \). A separating set or vertex cut of a connected graph \( G \) is a set \( S \subset V(G) \) such that \( G - S \) is a disconnected graph. The toughness of a graph \( G \), denoted by \( t(G) \), is defined by \( t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subset V(G) \text{ is a vertex cut of } G \right\} \). A set \( S \) of \( G \) is called a \( t(G) \)-set, if \( \frac{|S|}{\omega(G - S)} = t(G) \). The scattering number of \( G \), denoted by \( s(G) \), is defined by \( s(G) = \max \{ \omega(G - S) - |S| : S \subset V(G) \text{ is a vertex cut of } G \} \). The integrity of a graph \( G \), denoted by \( I(G) \), is defined by \( I(G) = \min \{ |S| + \tau(G - S) : S \subset V(G) \text{ is a vertex cut of } G \} \). The tenacity of a graph \( G \), denoted by \( T(G) \), is defined by \( T(G) = \min \{ \frac{|S| + \tau(G - S)}{\omega(G - S)} : S \subset V(G) \text{ is a vertex cut of } G \} \). The rupture degree of an incomplete graph \( G \) is defined by \( r(G) = \max \{ \omega(G - S) - |S| - \tau(G - S) : S \subset V(G) \text{ is a vertex cut of } G \} \). Notation and definitions which are not given here can be found in [4] or [6].

In this paper, we determine the exact values of some vulnerability parameters such as connectivity,
toughness, scattering number, integrity and tenacity of the graph \( C_r \times C_s \), where \( r, s \geq 3 \) and both \( r \) and \( s \) are odd or they are of different parity.

We quote the following theorems for our future reference.

**Theorem 1.1.** [13] The graph \( C_m \circ K_n \) has a Hamilton cycle decomposition.

In a personal communication to Paulraja, Goddard informed that the following theorem is known.

**Theorem 1.2.** The toughness of the graph \( C_m \square C_n \), \( m, n \geq 3 \), is

\[
t(C_m \square C_n) = \begin{cases} 
1, & \text{if both } m \text{ and } n \text{ are even,} \\
\frac{mn}{(m-1)(n-1)}, & \text{if both } m \text{ and } n \text{ are odd,} \\
\frac{n}{n-1}, & \text{if } m \text{ even and } n \text{ odd.}
\end{cases}
\]

**Theorem 1.3.** [19] Let \( G \) be a non-complete connected graph of order \( n \) with the scattering number \( s(G) = s \) and the toughness \( t(G) = t \). Then \( t \leq (n-s)/(n+s) \).

**Theorem 1.4.** [1] For any graph \( G \), the tenacity of \( G \), \( T(G) \geq t(G) + \frac{1}{\alpha(G)} \), where \( t(G) \) and \( \alpha(G) \) denote the toughness and the independence number of \( G \), respectively.

**Theorem 1.5.** [5] The integrity of the cycle \( C_n \) is \( I(C_n) = [2\sqrt{n}] \).

**Theorem 1.6.** [8] The tenacity of the cycle \( C_n \) is

\[
T(C_n) = \begin{cases} 
\frac{n+3}{n-1}, & \text{if } n \text{ is odd,} \\
\frac{n+2}{n}, & \text{if } n \text{ is even.}
\end{cases}
\]

2. Vulnerability Parameters of Tensor Product of Cycles

Let \( G = C_r \times C_s \), where \( r \geq 3 \) and \( s \geq 3 \). Let \( V(C_r) = \{u_0, u_1, \ldots, u_{r-1}\} \) and \( V(C_s) = \{v_0, v_1, \ldots, v_{s-1}\} \). Then \( V(G) = V(C_r \times C_s) = \bigcup_{i=0}^{r-1} S_i \bigcup_{i=0}^{s-1} S'_i \), where \( S_i = u_i \times V(C_s) \) stands for \( \{u_i, v_j\}, 0 \leq j \leq s - 1 \) and \( S'_i = V(C_s) \times v_j \) stands for \( \{u_i, v_j\}, 0 \leq i \leq r - 1 \). Clearly, \( S_i \) and \( S'_i \) are independent sets in \( G \). We call \( S_i \) a layer of \( G \) and \( S'_i \) a column of \( G \). We denote \( (u_i, v_j) \) by \( w_{ij} \); then \( S_i = \{w_{i0}, w_{i1}, \ldots, w_{i(s-1)}\} \), \( 0 \leq i \leq r - 1 \) and \( S'_i = \{w_{0j}, w_{1j}, \ldots, w_{j(s-1)}\} \), \( 0 \leq j \leq s - 1 \).

From the structure of \( G = C_r \times C_s \), we can find a set of four internally disjoint paths between any pair of distinct vertices and hence the theorem follows.

**Theorem 2.1.** For \( r, s \geq 3 \), where \( r \) and \( s \) are odd or they are of different parity, the connectivity of \( C_r \times C_s \) is 4.

If \( G \) is a bipartite graph with bipartition \( (X, Y) \), where \( X = \{x_0, x_1, \ldots, x_{n-1}\} \) and \( Y = \{y_0, y_1, \ldots, y_{n-1}\} \) and if \( G \) contains the set of edges \( F_i(X, Y) = \{x_jy_{j+i} | 0 \leq j \leq n - 1, \text{where addition in the subscript is taken modulo } n \} \), \( 0 \leq i \leq n - 1 \), then we say that \( G \) has the 1-factor of jump \( i \) from \( X \) to \( Y \). The same 1-factor is said to be of jump \( n - i \) from \( Y \) to \( X \), that is, \( F_{n-i}(Y, X) \), where we assume \( F_n(Y, X) = F_0(Y, X) \).

The following remark is a well known fact in the graph decomposition theory.
Remark 2.2. Let $G \times H$ be a connected graph with $|V(G)| = m$ and $|V(H)| = n$. Assume that the 1-factor of jump $r_i$, namely, $F_r(S_i \cup S_{i+1})$, where addition is taken modulo $m$, is contained in $(S_i \cup S_{i+1})$, where $S_i$ is the $i$th layer of $G \times H$. If $n$ and $\sum_{i=0}^{m-1} r_i$ are relatively prime, then $\bigcup_{i=0}^{m-1} F_r(S_i \cup S_{i+1})$ is a Hamilton cycle of $G \times H$.

Lemma 2.3. For odd $s \geq 3$, the toughness of $P_{2m} \times C_s$ is 1, that is $t(P_{2m} \times C_s) = 1$.

Proof. As $G = P_{2m} \times C_s$ is a bipartite graph, $t(G) \leq 1$. (1)

We shall prove that $t(G) \geq 1$. We prove this by induction on $m$.

If $m = 1$, the graph $P_2 \times C_s \simeq C_{2s}$, is the cycle of length $2s$ and hence $t(P_2 \times C_s) = 1$.

Assume the result is true for $m - 1$.

Let $G = P_{2m} \times C_s$, $G_1 = P_{2(m-1)} \times C_s$ and $G_2 = P_2 \times C_s$. Then $G$ contains disjoint copies of $G_1$ and $G_2$, where we assume the last two layers induce the graph $G_2$ and the first $2(m-1)$ layers induce the graph $G_1$. Let $S$ be an arbitrary vertex cut of $G$. Let $V(G_1) \cap S = S_1$ and $V(G_2) \cap S = S_2$. Then

$$\omega(G - S) = \omega(G - (S_1 \cup S_2)) \leq \omega(G_1 - S_1) + \omega(G_2 - S_2) \leq |S_1| + |S_2|,$$

by induction hypothesis.

Hence $\frac{|S|}{\omega(G - S)} \geq 1$. As $S$ is an arbitrary vertex cut of $G$,

$t(G) \geq 1$. (2)

By (1) and (2), $t(G) = 1$. 

Theorem 2.4. For $r, s \geq 3$, where $r$ and $s$ are odd or they are of different parity, the toughness of $C_r \times C_s$ is $t(C_r \times C_s) = \begin{cases} 1, & r \text{ and } s \text{ are of different parity}, \\ \min \left\{ 1 + \frac{1}{r}, 1 + \frac{1}{s} \right\}, & \text{if } r = 2m + 1 \text{ and } s = 2n + 1. \end{cases}$

Proof. Let $G = C_r \times C_s$, where $r \geq 3$ and $s \geq 3$. Recall that $t(G)$ is defined as $t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subset V \text{ is a vertex cut of } G \right\}$.

Case 1. $r$ is even and $s$ is odd.

As $G$ is a 4-regular bipartite graph $t(G) \leq 1$. (3)

By Remark 2.2, $G$ is hamiltonian and for a hamiltonian graph toughness is at least 1 and hence by [7], $t(G) \geq 1$. (4)
From (3) and (4), \( t(G) = 1 \).

**Case 2.** \( r \geq s \), \( r = 2m + 1 \) and \( s = 2n + 1 \).

**Subcase 2.1.** \( r = s = 2m + 1 \)

As \( C_{2m+1} \times C_{2m+1} \) is isomorphic to \( C_{2m+1} \square C_{2m+1} \), see p106 of [11] and by Theorem 1.2, 
\( t(C_{2m+1} \square C_{2m+1}) = 1 + \frac{1}{m} \) and hence the result follows.

**SubCase 2.2.** \( r > s \)

Let \( S = \bigcup_{i=0}^{m} S_{2i} \). Therefore \( |S| = (m+1)s \) and \( \omega(G - S) = rs - (m+1)s = (2m+1)s - (m+1)s = ms \). Hence

\[
t(G) \leq \frac{|S|}{\omega(G - S)} = \frac{(m+1)s}{ms} = 1 + \frac{1}{m}.
\]

(5)

On the other hand, let \( S \) be a t-set of \( G \) with \( |S| \) maximum.

**Claim 1.** A t-set \( S \) contains a layer of \( G \).

Without loss of generality, we assume that \( S_{2m+1} \subseteq S \). Let \( G' = G - S_{2m+1} \) and \( S' = S - S_{2m+1} \). Clearly, \( \omega(G' - S') = \omega(G - S) \), since \( G' = G - S_{2m+1} \) is connected; \( |S'| = |S| - s \). Therefore,

\[
\frac{|S|}{\omega(G - S)} = \frac{|S'| + s}{\omega(G' - S')} = \frac{|S'|}{\omega(G' - S')} + \frac{s}{\omega(G' - S')} \\
= \frac{|S'|}{\omega(G' - S')} + \frac{s}{\omega(G' - S')} \\
\geq 1 + \frac{1}{m}, \text{ as } \omega(G' - S') \leq ms \text{ and by Lemma 2.3.}
\]

Consequently, \( S \) gives the minimum value for \( \frac{|S|}{\omega(G - S)} \) and hence \( S \) contains a layer of \( G \).

**Claim 2.** A t-set \( S \) does not contain a column of \( G \).

Without loss of generality, we assume that \( S_{2n+1} \subseteq S \). Let \( G'' = G - S_{2n+1} \) and \( S'' = S - S_{2n+1} \). Clearly, \( \omega(G'' - S'') = \omega(G - S) \), since \( G'' = G - S_{2n+1} \) is connected; \( |S''| = |S| - r \). Therefore

\[
\frac{|S|}{\omega(G - S)} = \frac{|S''| + r}{\omega(G'' - S'')} = \frac{|S''|}{\omega(G'' - S'')} + \frac{r}{\omega(G'' - S'')} \\
= \frac{|S''|}{\omega(G'' - S'')} + \frac{r}{\omega(G'' - S'')} \\
\geq 1 + \frac{1}{n}, \text{ as } \omega(G'' - S'') \leq rn. \\
\geq 1 + \frac{1}{m}, \text{ as } r > s \text{ and hence } m > n.
\]

Consequently, \( S \) does not give the minimum value for \( \frac{|S|}{\omega(G - S)} \) and this contradicts the assumption of \( S \). Thus \( S \) cannot contain a column of \( G \).
Choose a smallest component $H$ of $G - S$ in such a way that there is a least index $p$ of $S'_j$ such that $H$ has a vertex in $S'_p$ and there is a largest index $q$ of $S'_j$ such that $H$ has a vertex in $S'_q$. Then $H$ is contained in some $P_k \times C_r$.

**Claim 3.** The toughness of $H$ is $\geq 1$.

Suppose not, then there is a set $A \subset V(G)$ such that $\frac{|A|}{\omega(G - A)} < 1$. Then $|A| < \omega(G - A)$ and hence $|A| \leq \omega(G - A) - 1$. Now $S \cup A$ is a separating set and

$$\frac{|S \cup A|}{\omega(G - (S \cup A))} = \frac{|S| + |A|}{\omega(G - S) - 1 + \omega(G - A)} \leq \frac{|S| + \omega(G - A) - 1}{\omega(G - S) - 1 + \omega(G - A)} < \frac{|S|}{\omega(G - S)}.$$

This implies that there is a separating set $S \cup A$ with $|S \cup A| > |S|$, that gives the minimum value of $t(G)$, which is a contradiction to the assumption of $S$. Hence the toughness of every such component $H \subset P_k \times C_r$.

Since $P_k \times C_r$ is a bipartite graph, $t(H) = 1$. This is true for every such component $H$ of $G - S$. Then $H$ should be a cycle or an edge. Thus there is a separating set $T$ of $G$ consisting $S$ and the vertices between $S'_p$ and $S'_q$ such that $|T| \geq |S|$ and $\frac{|T|}{\omega(G - T)} < \frac{|S|}{\omega(G - S)}$, which contradicts the assumption of the t-set $S$ of $G$.

Thus the t-set is the set $S$ that contains a layer of $G$ and $t(G) \geq 1 + \frac{1}{m}$. Hence $t(G) = \frac{m + 1}{m}$. Similarly, if $s \geq r$, then we can show that $t(G) = \frac{m + 1}{m}$. Hence the result follows.

**Theorem 2.5.** For $r, s \geq 3$, the scattering number of $C_r \times C_s$ is

$$s(C_r \times C_s) = \begin{cases} 0, & \text{if } r \text{ and } s \text{ are of different parity}, \\ -2, & \text{if both } r \text{ and } s \text{ are odd}. \end{cases}$$

**Proof.** Let $G = C_r \times C_s$, where $r, s \geq 3$. Recall that the scattering number of the graph $G$ is $s(G) = \max \{ \omega(G - S) - |S| : S \subset V(G) \text{ is a vertex cut of } G \}$.

**Case 1.** $r$ and $s$ are of different parity.

As $G$ is a regular bipartite graph with bipartition, say, $(X, Y)$, then $G - X$ contains $|Y|$ components and hence $\omega(G - X) - |X| = |Y| - |X| = 0$ and so

$$s(G) \geq 0. \quad (6)$$

Since $G = C_r \times C_s$ is a bipartite hamiltonian graph, $t(G) = 1$ so that $\min_{S \subset V} \left\{ \frac{|S|}{\omega(G - S)} \right\} = 1$ and hence $|S| \geq \omega(G - S)$, for every vertex cut $S$ of $G$. Hence $0 \geq \omega(G - S) - |S|$ for every vertex cut $S$ of $G$. Consequently,

$$0 \geq \max_{S \subset V} \{ \omega(G - S) - |S| \} = s(G). \quad (7)$$

From (6) and (7), $s(G) = 0$.

**Case 2.** Both $r$ and $s$ are odd.
Let \( r \geq s \). Let \( A = \{ w_{i-1}(j-1), w_{i-1}(j+1), w_{i+1}(j-1), w_{i+1}(j+1) \} \) for some \( i \) and \( j \). Then \( A \) is a vertex cut of \( G \) with \( |A| = 4 \) and \( \omega(G - A) = 2 \). Hence \( \omega(G - A) - |A| = 2 - 4 = -2 \), and hence,

\[
s(G) \geq -2. \tag{8}\]

Let \( r = 2m + 1 \). Let \( S \) be a vertex cut of \( G \); then \( |S| \geq 4 \), by Theorem 2.1. There is a vertex cut \( S \) with \( \omega(G - S) = 2 \),

\[
\omega(G - S) - |S| \leq 2 - 4 = -2. \tag{9}
\]

Now by Theorem 1.3, for the non-complete connected graph \( G \) of order \( n \), \( s(G) \leq n \left[ \frac{1 - \tau(G)}{1 + \tau(G)} \right] \).

Therefore

\[
s(G) \leq (2m + 1)s \left[ \frac{-1/m}{2 + 1/m} \right] = -s \leq -3, \quad \text{since } s \geq 3. \tag{10}\]

From (9) and (10), we see that \( \max \{-2, -3\} = -2 \). Therefore

\[
s(G) \leq -2. \tag{11}\]

From (8) and (11), \( s(G) = -2 \).

**Theorem 2.6.** If \( r \geq s \geq 3 \) and \( s \) is odd, then the integrity of \( C_r \times C_s \) is

\[
I(C_r \times C_s) = \begin{cases} 
    s \left\lceil \frac{r}{2} \right\rceil + 1, & \text{if } r \leq 8 \text{ or } r = 10, \\
    s \left\lceil \frac{r}{\sqrt{r}} \right\rceil + \left\lceil \frac{rs - s\sqrt{r}}{\sqrt{r}} \right\rceil, & \text{if } r = 9 \text{ or } r \geq 11.
\end{cases}
\]

**Proof.** Let \( G = C_r \times C_s \), where \( r \geq s \geq 3 \) and \( s \) odd. Recall that the integrity of the graph \( G \) is

\[
I(G) = \min \{|S| + \tau(G - S) : S \subset V(G) \text{ is a vertex cut of } G\}.
\]

Let \( S \) be an arbitrary vertex cut of \( G \). If \( \tau(G - S) = 1 \), then every component of \( G - S \) is an isolated vertex and hence \( \omega(G - S) \leq s \left\lceil \frac{r}{2} \right\rceil \), the cardinality of a maximum independent set in \( G \). Hence

\[
|S| = rs - \omega(G - S) \geq rs - s \left\lceil \frac{r}{2} \right\rceil = s\left\lceil \frac{r}{2} \right\rceil. \tag{12}\]

Consequently,

\[
|S| + \tau(G - S) \geq s \left\lceil \frac{r}{2} \right\rceil + 1.
\]

If \( \tau(G - S) > 1 \) then, \( S \) cannot contain all the vertices of each of the alternate layers, that is \( S_t, S_{t+2}, S_{t+4}, \ldots, S_{t-2} \); otherwise \( \tau(G - S) = 1 \). \( I(G) \) is minimum only if \(|S|\) and \( \tau(G - S) \) are minimum. For a given separating set \( S \), \( \tau(G - S) \) is minimum only when the components of \( G - S \) have almost equal number of vertices. This is possible only if \( G - S \) contains all the vertices of some consecutive layers of \( G \). Hence \( S \) also contains all the vertices of few layers, say \( x \) layers (not necessarily consecutive). Then \( |S| = sx \) and \( \tau(G - S) \geq \left\lceil \frac{rs - |S|}{x} \right\rceil \). Therefore,

\[
|S| + \tau(G - S) \geq |S| + \left\lceil \frac{rs - |S|}{x} \right\rceil \geq |S| + \left( \frac{rs - |S|}{x} \right) = sx + \frac{rs - sx}{x}.
\]
From (12) and (13), we see that
\[ \min_{x \geq 2} \left( s x + \frac{r s - s x}{x} \right). \]
Let \( f(x) = s x + \left( \frac{r s - s x}{x} \right) \). Using calculus, we conclude that \( f(x) \) attains its minimum at \( x = \lceil \sqrt{r} \rceil \).
Therefore
\[ |S| + \tau(G - S) \geq s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \text{.} \quad (13) \]

**Case 1.** \( r \leq 8 \) or \( r = 10 \).
From (12) and (13), we see that \( \min \left\{ s \left\lfloor \frac{r}{2} \right\rfloor + 1, s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \right\} = s \left\lfloor \frac{r}{2} \right\rfloor + 1. \) Therefore,
\[ I(G) \geq s \left\lfloor \frac{r}{2} \right\rfloor + 1 \text{.} \quad (14) \]

On the other hand, let \( A = \bigcup_{i=0}^{\left\lfloor \frac{r}{2} \right\rfloor - 1} S_{2i} \). Then \( A \) is a vertex cut of \( G \) with \( |A| = s \left\lfloor \frac{r}{2} \right\rfloor \) and \( \tau(G - A) = 1 \) and so
\[ I(G) \leq |A| + \tau(G - A) = s \left\lfloor \frac{r}{2} \right\rfloor + 1 \text{.} \quad (15) \]
Hence from (14) and (15), \( I(G) = s \left\lfloor \frac{r}{2} \right\rfloor + 1 \).

**Case 2.** \( r = 9 \) or \( r \geq 11 \).
From (12) and (13), we see that \( \min \left\{ s \left\lfloor \frac{r}{2} \right\rfloor + 1, s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \right\} = s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \). Therefore,
\[ I(G) \geq s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \text{.} \quad (16) \]

On the other hand, let \( x = \lfloor \sqrt{r} \rfloor \) and \( j = \left\lfloor \frac{x}{2} \right\rfloor \). Let \( S = S_0 \cup S_j \cup S_{2j} \cup \ldots \cup S_{(x-1)j} \) be a vertex cut of \( G \). Then \(|S| = sx\) and \( \tau(G - S) = \left( \left\lfloor \frac{x}{2} \right\rfloor - 1 \right) s \). Therefore
\[ |S| + \tau(G - S) = s x + \left( \left\lfloor \frac{r}{x} \right\rfloor - 1 \right) s \leq s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \text{.} \quad (17) \]
Hence from (16) and (17), \( I(G) = s \left\lfloor \sqrt{r} \right\rfloor + \left\lceil \frac{r s - s \left\lfloor \sqrt{r} \right\rfloor}{\left\lceil \sqrt{r} \right\rceil } \right\rceil \). \( \blacksquare \)

**Theorem 2.7.** For \( r, s \geq 3 \), where \( r \) and \( s \) are odd or they are of different parity, the tenacity of \( C_r \times C_s \) is \( T(C_r \times C_s) = \min \left\{ \frac{s \left\lfloor \frac{r}{2} \right\rfloor + 1}{s \left\lfloor \frac{r}{2} \right\rfloor + 1}, \frac{r \left\lfloor \frac{s}{2} \right\rfloor + 1}{r \left\lfloor \frac{s}{2} \right\rfloor + 1} \right\} \).

**Proof.** Let \( G = C_r \times C_s \), where \( r, s \geq 3 \). Recall that the tenacity of the graph \( G \) is \( T(G) = \min \left\{ \frac{|S| + \tau(G - S)}{\omega(G - S)} : S \subseteq V \text{ is a vertex cut of } G \right\} \).
Let $r \geq s$. Let $A = \bigcup_{i=0}^{\lceil r/2 \rceil - 1} S_{2i}$. Then $A$ is a vertex cut of $G$ with $|A| = s\lceil r/2 \rceil$ and $\tau(G - A) = 1$. Therefore,

$$T(G) \leq \frac{|A| + \tau(G - A)}{\omega(G - A)} = \frac{s\lceil r/2 \rceil + 1}{s \lceil r/2 \rceil}.$$  

(18)

**Case 1.** $r$ is even and $s$ is odd.

By Theorem 1.4, $T(G) \geq t(G) + \frac{1}{\alpha(G)}$, where $\alpha(G)$ is the maximum cardinality of an independent set of $G$. We have $t(G) = 1$, when $r$ and $s$ are of different parity. Hence

$$T(G) \geq 1 + \frac{1}{s \lceil r/2 \rceil}, \quad \text{as } \alpha(G) = s \lceil r/2 \rceil,$$

(19)

**Case 2.** Both $r$ and $s$ are odd $r \geq s$, $r = 2m + 1$ and $s$ is odd.

We have $t(G) = 1 + \frac{1}{m} = 1 + \frac{1}{\lceil r/2 \rceil}$, when $r = 2m + 1$ and $s$ is odd. Hence

$$T(G) \geq 1 + \frac{1}{\lceil r/2 \rceil} + \frac{1}{s \lceil r/2 \rceil}, \quad \text{by Theorem 1.4}$$

$$= \frac{s\lceil r/2 \rceil + 1}{s \lceil r/2 \rceil}.$$

(20)

Hence from (19) and (20),

$$T(G) \geq \frac{s\lceil r/2 \rceil + 1}{s \lceil r/2 \rceil}.$$  

(21)

From (18) and (21), we conclude that $T(G) = \frac{s\lceil r/2 \rceil + 1}{s \lceil r/2 \rceil}$, when $r \geq s$.

If $s \geq r$, then in the similar manner we conclude that $T(G) = \frac{r\lceil s/2 \rceil + 1}{r \lceil s/2 \rceil}$.

Hence $T(C_r \times C_s) = \min\left\{ \frac{s\lceil r/2 \rceil + 1}{s \lceil r/2 \rceil}, \frac{r\lceil s/2 \rceil + 1}{r \lceil s/2 \rceil} \right\}$.

3. References

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