Incentive compatible designs for tournament qualifiers with round-robin groups and repechage

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Abstract

Tournament organisers supposedly design rules such that a team cannot be better off by exerting a lower effort. The European qualifiers to recent FIFA World Cups are known to violate this requirement: a team might be eliminated if it wins its last match in the group stage, while it advances to the next round by playing a draw, provided that all other results remain the same. Inspired by this real-world case, we study the incentive compatibility of similar qualifiers with round-robin groups and a repechage. Theorems listing the sufficient and necessary conditions of strategy-proofness are proved, and applied to classify these qualifications. We design two general mechanisms for solving the problem of incentive incompatibility. The first is based on abolishing the anonymity of the matches discarded in the repechage, while the second involves a rethinking of the seeding procedure.

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1 "A single error in the original assembly of the armies can hardly ever be rectified during the entire course of the campaign." (Source: Holger H. Herwig: The Forgotten Campaign: Alsace-Lorraine August 1914, in Michael S. Neiberg (ed.): Arms and the Man: Military History Essays in Honor of Dennis Showalter, Brill Academic Publishers, Leiden, Boston, 2011.)
1 Introduction

Strategy-proofness is an extensively discussed concept in social choice theory since the famous Gibbard-Satterthwaite impossibility theorem (Gibbard, 1973; Satterthwaite, 1975), which states that any fair voting rule is susceptible to strategic voting: there always exists a voter who can achieve a more favourable outcome by being insincere. The issue of incentive compatibility seems to be probably even more important in professional sports where contestants have strong incentives to exert costly efforts, and the tournaments involve high-stake decisions that are familiar to all agents (Kahn, 2000).

All sporting contests should exclude the possibility that a contestant may be strictly better off by a weaker performance. This can immediately lead to tanking (the act of deliberately dropping points or losing in order to gain some other advantage), which is against the spirit of the game but does meet the objective of the contestant. Therefore, it always needs to be considered by administrators, for whom ensuring integrity is a significant part of their job because an ill-thought-out tournament design can easily lead to an outrage among consumers (fans) as illustrated by several historical examples (Kendall and Lenten, 2017).

The enterprise of exploring sports ranking rules from the perspective of strategy-proofness is still in its infancy, and much remains to be done. Some theoretical works find incentive incompatibility as being ranked lower in a given (group) stage might lead to facing a more preferred competitor in the subsequent (knock-out) stage, which means an advantage only in expected terms. For example, Pauly (2014) develops a mathematical model of manipulation in round-robin subtournaments and derives an impossibility theorem, while Vong (2017) considers the strategic manipulation problem in multistage tournaments and shows that only the top-ranked player can be allowed to qualify from each group in order to guarantee that all players exert full effort.

We focus on a probably more serious problem, on the possibility that a player is strictly better off by losing. In other words, a player is allowed to manipulate a match only if it is without any risk. Probably the first academic paper addressing this issue is Dagaev and Sonin (2017), where tournament systems, consisting of multiple round-robin and knock-out tournaments with noncumulative prizes, are proved to be characteristically incentive incompatible in the sense that a team may be guaranteed to gain (not only in expected terms) by losing instead of winning.

Specifically, we discuss the strategy-proofness of qualifiers to two prominent association football competitions, the FIFA World Cups (in the European Zone) and UEFA European Championships. These qualification tournaments are organised recently such that the top $p - 1$ teams from each group of size $\ell$ or $\ell + 1$ qualify for the final, while the $p$th-placed teams from each group advance to the play-offs, but some of the best $p$th-placed teams qualify or some of the worst $p$th-placed teams are eliminated. Consequently, the $p$th-placed teams – which have not played any matches against each other – should be compared in a so-called repechage group.

Our main contributions are as follows. We prove that the application of the same monotonic ranking for each group including the repechage is not sufficient to guarantee the strategy-proofness of the whole qualification system unless the set of matches considered in the repechage group is chosen appropriately. Theorems listing the sufficient and necessary conditions of incentive compatibility are provided and applied to identify nine recent qualifications that can be manipulated. Finally, we design two mechanisms for solving the problem. It turns out that strategy-proofness can be achieved without sacrificing other
important theoretical properties – contrary to other models, for example, matching markets, where incentive compatibility may lead to inefficiency and instability (Abdulkadiroğlu et al., 2009).

The main message can be interpreted intuitively: to rank a set of teams in different pools on a secondary basis along a common denominator is challenging if the size – or strength, as in the case of Swiss-system tournaments (Csató, 2017b), but that, being a softer criterion, is not an issue in the current paper – of those pools is not uniform. In finding a way around it, the criteria (each criterion and their order) for ranking these teams in the secondary sense must be identical to that used in the primary sense, in the pools themselves. Otherwise, situations susceptible to manipulation may occur.

The rest of the article is organised in the following way. Section 2 describes our starting real-world observation, the European section of the 2018 FIFA World Cup qualification, and presents an incentive incompatible scenario in this framework. Section 3 builds and analyses the formal mathematical model, which is applied to examine the strategy-proofness of some recent association football qualification tournaments in Section 4. Section 5 summarizes policy implications for organisers, while Section 6 concludes.

2 A real-world example: 2018 FIFA World Cup qualification (UEFA)

2018 FIFA World Cup qualification (UEFA) is the European section of the 2018 FIFA World Cup qualifiers of national association football teams, which are members of UEFA. Russia automatically qualified as a host, therefore – after Gibraltar and Kosovo became FIFA members in May 2016 – 54 teams entered the qualification process for the 13 slots available in the final tournament.

The qualifying format was confirmed by the UEFA Executive Committee meeting on 22-23 March 2015 in Vienna, and is as follows:

- Group stage (first round): Nine groups of six teams each, playing home-and-away round-robin matches. The winners of each group qualify for the 2018 FIFA World Cup, and the eight best runners-up advance to the play-offs (second round).

- Play-offs (second round): The eight best second-placed teams from the group stage play home-and-away matches over two legs. The four winners qualify for the 2018 FIFA World Cup.

We focus on the first round, where the tie-breaking rules are (FIFA, 2016, Article 20.6): (1) greatest number of points obtained in all group matches (with three points for a win, one point for a draw and no points for a defeat); (2) goal difference in all group matches; and (3) greatest number of goals scored in all group matches. Strangely, it is not described explicitly that greater goal difference is preferred. Further tie-breaking rules will play no role in our discussion.

Choice of the eight best second-placed teams is not addressed in FIFA (2016), and we were not able to find the relevant regulation. However, according to a FIFA (FIFA, 2017b) Media Release, which is reinforced by an earlier UEFA press release (UEFA, 2016): "[... ] the eight best runners-up will be decided by ranking criteria as stated in the 2018 FIFA World Cup Regulations, namely points, goal difference, goals scored, goals scored away from home and disciplinary ranking, with the results against teams ranked 6th not
being taken into account.” AFC (2015) provides an illustration of how to calculate the ranking of second-placed teams when some group matches are discarded.

The ranking of second-placed teams strictly follows tie-breaking in groups, with the crucial difference of discarding two matches played against the last team of the group. It turns out that this, seemingly minor, modification in the comparison of runners-up has some unintended consequences: we are able to present a possible manipulation of the European qualifiers to the 2018 FIFA World Cup.

Our starting observation has some history in the scientific research. According to our knowledge, the misaligned incentives has been revealed first by a column in the case of the European qualification for the 2014 FIFA World Cup in Brazil (Dagaev and Sonin, 2013). Dagaev and Sonin (2017, p. 22) states that: “Still, the problem of misaligned incentives is not restricted to national tournaments. For example, competition rules of the European qualification tournament for the 2014 FIFA World Cup in Brazil suffered from the same problem. And the "perverse incentives” situation was not merely a theoretical possibility. Two months before the end of the tournament, with 80% of games completed, there still was a scenario under which a team might need to achieve a draw instead of winning to go to Brazil.” According to Csató (2018), this is not only an irrelevant scenario with a marginal probability since France had an incentive to kick two own goals on its last match against Israel in the UEFA Euro 1996 qualifying tournament.

Example 2.1. There was a scenario with a (low, but) positive probability in the 2018 FIFA World Cup qualification (UEFA), after four-fifths of all matches were over, under which Bulgaria might need to play a draw instead of winning against Luxembourg on the last matchday (10 October 2017).

We generate hypothetical results for the last two matchdays, which were played between 5 October 2017 and 10 October 2017, after the example was published (Csató, 2017a). It is worth noting that all teams play one match home and one away on the last two matchdays, which is not true for two subsequent matchdays chosen arbitrarily.

Eight groups are detailed in Csató (2017a, Appendix). Table 1 shows a possible scenario in Group A. While some results of Table 1.b may be unreasonable, like Belarus defeating the Netherlands by 7-0, they are necessary to create the appropriate conditions for manipulation. Nevertheless, this set of match results had a positive probability after eight matchdays were over.

On the basis of standings in Group A-I, runners-up are ranked in Table 2. Only the eight best second-placed teams advance to the play-offs, hence Bulgaria is eliminated.

However, consider what happens if Bulgaria plays a draw of 1-1 against Luxembourg on the last matchday (10 October 2017). It is clear that this change worsens Bulgaria’s standing in the group, but it remains the runner-up with 16 points as both Bulgaria and Sweden would have the same goal difference (+4) with Bulgaria scoring more goals in all group matches (22 vs 18). On the other hand, Luxembourg overtakes Belarus thanks to its newly obtained draw because it has more points (9 vs 8). In the ranking of the second-placed teams, matches against the last team are discarded. Consequently, Bulgaria would have 13 points, placing it seventh among the runners-up according to the last row of Table 2 (it has the same goal difference as Greece with more goals scored). Thus Bulgaria would advance to the play-offs instead of Montenegro if it would concede a goal against Luxembourg.

Example 2.1 is robust with respect to Groups B-I. Considering the actual match results in these groups instead of the hypothetical ones, Slovakia is the worst second-placed
Table 1: 2018 FIFA World Cup qualification – UEFA Group A

(a) Match results of the first eight matchdays
The positions are given according to the matches already played.
The home team is in the row, the away team (represented by its position) is in the column.
The dates are given for the matches to be played on the last two matchdays in 2017.

| Position | Team     | 1   | 2    | 3    | 4    | 5    | 6     |
|----------|----------|-----|------|------|------|------|-------|
| 1        | France   | —   | 2-1  | 4-0  | 4-1  | 0-0  | 10 Oct|
| 2        | Sweden   | 2-1 | —    | 1-1  | 3-0  | 7 Oct| 4-0   |
| 3        | Netherlands | 0-1 | 10 Oct| —    | 3-1  | 5-0  | 4-1   |
| 4        | Bulgaria | 7 Oct | 3-2 | 2-0  | —    | 4-3  | 1-0   |
| 5        | Luxembourg | 1-3 | 0-1  | 1-3  | 10 Oct| —    | 1-0   |
| 6        | Belarus  | 0-0 | 0-4  | 7 Oct| 2-1  | 1-1  | —     |

(b) Hypothetical match results of the last two matchdays
The last row shows an alternative result, obtained if Bulgaria manipulates.

| Date               | Home team | Away team | Result |
|--------------------|-----------|-----------|--------|
| 7 October 2017     | Sweden    | Luxembourg| 0-4    |
| 7 October 2017     | Belarus   | Netherlands| 7-0   |
| 7 October 2017     | Bulgaria  | France    | 8-0    |
| 10 October 2017    | France    | Belarus   | 1-0    |
| 10 October 2017    | Luxembourg| Bulgaria  | 0-1    |
| 10 October 2017    | Netherlands| Sweden   | 3-0    |
| 10 October 2017*   | Luxembourg*| Bulgaria*| 1-1*   |

(c) Final standing with the runner-up results
Pos = Position; W = Won; D = Drawn; L = Lost; GF = Goals for; GA = Goals against; GD = Goal difference; Pts = Points. All teams have played 10 matches.
The penultimate row contains the second-placed team’s benchmark results, adjusted for the ranking of the runners-up (matches played against the 6th team are discarded) according to FIFA (2017b).
The last row contains the second-placed team’s alternative results, adjusted for the ranking of the runners-up (matches played against the 6th team are discarded) according to FIFA (2017b), obtained if Bulgaria manipulates.

| Pos | Team     | W | D | L | GF | GA | GD | Pts |
|-----|----------|---|---|---|----|----|----|-----|
| 1   | France   | 6 | 2 | 2 | 16 | 13 | 3  | 20  |
| 2   | Bulgaria | 6 | 0 | 4 | 22 | 17 | 5  | 18  |
| 3   | Sweden   | 5 | 1 | 4 | 18 | 14 | 4  | 16  |
| 4   | Netherlands | 5 | 1 | 1 | 19 | 18 | 1  | 16  |
| 5   | Belarus  | 2 | 2 | 6 | 11 | 17 | -6 | 8   |
| 6   | Luxembourg | 2 | 2 | 6 | 11 | 18 | -7 | 8   |
| 2   | Bulgaria | 4 | 0 | 4 | 17 | 14 | 3  | 12  |
| 2*  | Bulgaria*| 4*| 1*| 3*| 20*| 16*| 4* | 13* |

team with 12 points and a goal difference of +5 among the runners-up. Bulgaria is still eliminated by winning against Luxembourg according to Table 1.c but it advances to the play-offs if it plays a draw of 1-1.
Table 2: 2018 FIFA World Cup qualification (UEFA) – Ranking of second-placed teams

| Pos | Team          | Group | W | D | L | GF | GA | GD | Pts |
|-----|---------------|-------|---|---|---|----|----|----|-----|
| 1   | Portugal      | B     | 6 | 1 | 1 | 23 | 5  | 18 | 19  |
| 2   | Italy         | G     | 6 | 1 | 1 | 14 | 8  | 6  | 19  |
| 3   | Northern Ireland | C  | 4 | 2 | 2 | 9  | 3  | 6  | 14  |
| 4   | Wales         | D     | 3 | 5 | 0 | 8  | 5  | 3  | 14  |
| 5   | Turkey        | I     | 4 | 2 | 2 | 8  | 8  | 0  | 14  |
| 6   | Slovakia      | F     | 4 | 1 | 3 | 11 | 5  | 6  | 13  |
| 7   | Greece        | H     | 3 | 4 | 1 | 8  | 4  | 4  | 13  |
| 8   | Montenegro    | E     | 3 | 3 | 2 | 12 | 6  | 6  | 12  |
| 9   | Bulgaria      | A     | 4 | 0 | 4 | 17 | 14 | 3  | 12  |
| 7*  | Bulgaria*     | A*    | 4*| 1*| 3*| 20*| 16*| 4* | 13* |

Thus manipulation mainly depends on the events in Group A because, as it will be revealed by our mathematical model, a successful manipulation has three requirements: (1) the ranking criteria (number of points, goal difference, etc.) of a team should be better among the runners-up by exerting a lower effort in a match; (2) it should preserve the position of the manipulating team in its group; (3) it needs to result in some gain for this team with respect to qualification, for example, by advancing it to the play-offs instead of being eliminated – otherwise, it makes no sense to exert a lower effort. A situation that occurred in the UEFA European Championship 1996 qualification has satisfied the first two conditions (Csató, 2018).

3 Theoretical background

In the following, an abstract model is built for the home-and-away (double) round-robin group stage of a tournament organised in a format similar to the 2018 FIFA World Cup qualification (UEFA). It begins with several definitions for the sake of accurateness. We will see in the next section that the seemingly long preparation for the main theorems does not affect the generality of the results. For example, it is not required that a win is awarded by three points, a draw is awarded by one point, and a loss is awarded by zero points.

**Definition 3.1.** *Home-and-away round-robin tournament*: Let $X$ be a nonempty finite set of at least two teams, $x, y \in X$ be two teams and $v : X \times X \to \{(v_1; v_2) : v_1, v_2 \in \mathbb{N}\} \cup \{-\}$ be a function such that $v(x, y) = -$ if and only if $x = y$. The pair $(X, v)$ is called a *home-and-away round-robin tournament*.

In a home-and-away round-robin tournament, any team plays each other team in $X$ once at home and once at away. Function $v$ describes game results with the number of goals scored by the home and the away team, respectively.
**Definition 3.2.** Ranking in home-and-away round-robin tournaments: Let $\mathcal{X}$ be the set of home-and-away round-robin tournaments with a set of teams $X$. A ranking method $S$ is a function that maps any function $v$ of $\mathcal{X}$ into a strict order $S(v)$ on the set $X$.

Let $(X, v)$ be a home-and-away round-robin tournament, $S(v)$ be its ranking and $x, y \in X$, $x \neq y$ be two different teams. $x$ is ranked higher (lower) than $y$ if and only if $x > S(v) y$ ($x < S(v) y$).

Let $x, y \in X$, $x \neq y$ be two different teams and $v(x, y) = (v_1(x, y); v_2(x, y))$. It is said that team $x$ wins over team $y$ if $v_1(x, y) > v_2(x, y)$ (home) or $v_1(x, y) < v_2(y, x)$ (away), team $x$ loses to team $y$ if $v_1(x, y) < v_2(x, y)$ (home) or $v_1(y, x) > v_2(y, x)$ (away) and teams $x$ draws against team $y$ if $v_1(x, y) = v_2(x, y)$ or $v_1(y, x) = v_2(y, x)$.

**Definition 3.3.** Number of points: Let $(X, v)$ be a home-and-away round-robin tournament and $x \in X$ be a team. Denote by $N^w_v(x)$ the number of wins, by $N^d_v(x)$ the number of draws, and by $N^l_v(x)$ the number of losses of team $x$ in $(X, v)$, respectively. The number of points of team $x$ is $s_v(x) = \alpha N^w_v(x) + \beta N^d_v(x) + \gamma N^l_v(x)$ such that $\alpha > \beta > \gamma$.

In other words, a win means $\alpha$ points, a draw means $\beta$ point and a loss means $\gamma$ points.

**Definition 3.4.** Monotonicity of a group ranking: Let $\mathcal{X}$ be the set of home-and-away round-robin tournaments with a set of teams $X$, and $S$ be a ranking method. $S$ is called \textit{monotonic} if $s_v(x) > s_v(y)$ implies $x > S(v) y$ for any function $v$ and for any teams $x, y \in X$.

Monotonicity does not necessarily lead to a unique ranking, tie-breaking rules can be arbitrary in our model.

**Definition 3.5.** Group-based qualifier: A group-based qualifier $\mathcal{T}$ consists of $k$ groups of home-and-away round-robin tournaments with the set of teams $X^1, X^2, \ldots, X^k$ such that $X^i \cap X^j = \emptyset$ for any $i \neq j$, $1 \leq i, j \leq k$.

In order to cover the 2018 FIFA World Cup qualification (UEFA), it is allowed to compare teams from different groups in a \textit{repechage group}.

**Definition 3.6.** Repechage function: Let $\mathcal{T}$ be a group-based qualifier. A repechage function $\mathcal{G}$ associates with any set of group results $V = \{v^1, v^2, \ldots, v^k\}$ a set of teams $\mathcal{G}_1(V) \subseteq \bigcup_{i=1}^k X^i$ composing the repechage group and a set of opponents $\emptyset \neq \mathcal{G}_2(V, x) \subseteq X^i \setminus \{x\}$ for each team of the repechage group $x \in \mathcal{G}_1(V)$.

**Definition 3.7.** Impartiality of a repechage function: Let $\mathcal{T}$ be a group-based qualifier. Repechage function $\mathcal{G}$ is \textit{impartial} if:

- $|X^i \cap \mathcal{G}_1(V)| = c_i$ is fixed under any set of group results $V = \{v^1, v^2, \ldots, v^k\}$;
- $x, y \in X^i \cap \mathcal{G}_1(V)$ implies $y \in \mathcal{G}_2(V, x)$ and $x \in \mathcal{G}_2(V, y)$;
- if $x, y \in X^i \cap \mathcal{G}_1(V)$, then $z \in \mathcal{G}_2(V, x)$, $z \neq y$ implies $z \in \mathcal{G}_2(V, y)$ and $z \in \mathcal{G}_2(V, y)$, $z \neq x$ implies $z \in \mathcal{G}_2(V, x)$; and
- $x, y \in \mathcal{G}_1(V)$ implies $|\mathcal{G}_2(V, x)| = |\mathcal{G}_2(V, y)|$. 

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According to the impartiality of a repechage function: (a) the number of teams relegated to the repechage group from a given group is fixed; (b) if two teams of the repechage group have played some matches against each other, then these are considered in the repechage group; (c) if two teams are relegated to the repechage group from the same group, then their matches played against any third team should be uniformly considered or discarded in the repechage group; and (d) the number of matches taken in the repechage group into account is the same for all teams of the repechage group. The last condition ensures that the number of points is a plausible measure of performance.

**Definition 3.8.** Repechage group ranking: Let $\mathcal{T}$ be a group-based qualifier and $\mathcal{G}$ be a repechage function. A repechage ranking method $Q$ is a function that maps any set of group results $V = \{v^1, v^2, \ldots, v^k\}$ into a strict order $Q(V)$ on the set $\mathcal{G}_1(V)$.

**Definition 3.9.** Number of points in the repechage group: Let $\mathcal{T}$ be a group-based qualifier and $\mathcal{G}$ be a repechage function. The number of points in the repechage group of team $x \in \mathcal{G}_1(V)$ is:

\[
s_{\mathcal{G}, V}^{k+1}(x) = \alpha \left( \left\{ \left\{ y \in \mathcal{G}_2(V, x) : v^1_1(x, y) > v^1_2(x, y) \text{ or } v^1_1(y, x) < v^1_2(y, x) \right\} \right\} \right) + \\
+ \beta \left( \left\{ \left\{ y \in \mathcal{G}_2(V, x) : v^1_1(x, y) = v^1_2(x, y) \text{ or } v^1_1(y, x) = v^1_2(y, x) \right\} \right\} \right) + \\
+ \gamma \left( \left\{ \left\{ y \in \mathcal{G}_2(V, x) : v^1_1(x, y) < v^1_2(x, y) \text{ or } v^1_1(y, x) > v^1_2(y, x) \right\} \right\} \right).
\]

The number of points is calculated on the basis of group matches considered in the repechage group.

**Definition 3.10.** Monotonicity of a repechage ranking: Let $\mathcal{T}$ be a group-based qualifier and $\mathcal{G}$ be a repechage function. Repechage ranking $Q$ is said to be monotonic if $s_{\mathcal{G}, V}^{k+1}(x) > s_{\mathcal{G}, V}^{k+1}(y)$ implies $x \succ_{Q(V)} y$ for any set of group results $V = \{v^1, v^2, \ldots, v^k\}$ and for any teams $x, y \in \mathcal{G}_1(V)$.

**Definition 3.11.** Allocation rule: Let $\mathcal{T}$ be a group-based qualifier and $\mathcal{G}$ be a repechage function. An allocation rule $\mathcal{R}$ associates with any set of group results $V = \{v^1, v^2, \ldots, v^k\}$ a value from the set $\{0; 1; 2\}$ for each team $x \in \cup_{i=1}^k X^i$.

Consider a group-based qualifier $\mathcal{T}$, a repechage function $\mathcal{G}$, an allocation rule $\mathcal{R}$, a set of group results $V = \{v^1, v^2, \ldots, v^k\}$, and a team $x \in \cup_{i=1}^k X^i$. Team $x$ is said to be: (a) directly qualified if $\mathcal{R}(V, x) = 2$; (b) advanced to the next round (with a chance to qualify) if $\mathcal{R}(V, x) = 1$; and (c) eliminated if $\mathcal{R}(V, x) = 0$.

**Definition 3.12.** Qualification system: The triple $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ of a group-based qualifier $\mathcal{T}$, a repechage function $\mathcal{G}$, and an allocation rule $\mathcal{R}$ is a qualification system.

The outcome of any qualification system $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ is reflected by the allocation rule $\mathcal{R}$. It implies the following.

**Remark 3.1.** Let $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ be any qualification system. If there is no difference in the allocation of teams in the repechage group, that is, $\mathcal{R}(V, x) = \mathcal{R}(V, y)$ for all teams $x, y \in \mathcal{G}_1(V)$, then the qualification system can be described without a repechage group, thus $\mathcal{G}_1(V) = \emptyset$ can be assumed without loss of generality.

**Definition 3.13.** Monotonicity of a qualification system: Let $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ be a qualification system. It is called monotonic if:
• the repechage function $G$ impartial;
• there exists a common monotonic ranking $S$ in each group such that $x, y \in X^i$, $1 \leq i \leq k$ and $x \succ_{S(v^i)} y$ implies $R(V, x) \geq R(V, y)$; and
• there exists a monotonic repechage ranking $Q$ such that $x, y \in G_1(V)$ and $x \succ_{Q(V)} y$ implies $R(V, x) \geq R(V, y)$.

The idea behind a monotonic qualification system is straightforward. Impartiality of the repechage function provides the comparability of teams taken from different groups. Because of the application of a monotonic ranking in groups, teams have no incentive to exert a lower effort in any match, they cannot achieve a higher position in the group by deliberately playing worse. It is also required in the repechage group, so the repechage ranking $Q$ should be monotonic.

**Proposition 3.1.** 2018 FIFA World Cup qualification (UEFA), discussed in Section 2, fits into the model presented above: it is a monotonic qualification system.

**Proof.** There is $k = 9$ groups and given any set of group results $V = \{v^1, v^2, \ldots, v^9\}$, all criteria of Definition 3.13 hold:

- the repechage function $G$ is impartial (Definition 3.7):
  - the repechage group consists of the runners-up: $x \in X^i \cap G_1(V)$, $1 \leq i \leq 9$ if and only if $|\{y \in X^i : y \succ_{S(v^i)} x\}| = 1$, so $|X^i \cap G_1(V)| = 1$;
  - the matches played against the first five teams of the group are considered in the repechage group: for each $x \in X^i \cap G_1(V)$, $1 \leq i \leq 9$, $z \in G_2(V, x)$ if and only if $z \in X^i$ and $|\{y \in X^i : y \succ_{S(v^i)} z\}| \leq 4$;

- the group ranking $S$ is monotonic because the number of points is the first tie-breaker in groups (Definition 3.6), and:
  - the first-placed team in each group qualifies: $R(V, x) = 2$ for each $x \in X^i$ if and only if $\#y \in X^i : y \succ_{S(v^i)} x$;
  - the third-, fourth-, fifth- and sixth-placed teams in each group are eliminated: $R(V, x) = 0$ for each $x \in X^i$ if $|\{y \in X^i : y \succ_{S(v^i)} x\}| \geq 2$;

- the repechage ranking $Q$ is monotonic because the number of points is the first tie-breaker in the repechage group (Definition 3.10), and:
  - the eight best second-placed teams advance to the next round: $R(V, x) = 1$ if $x \in G_1(V)$ and $|\{y \in G_1(V) : y \succ_{Q(V)} x\}| \leq 7$; and
  - the lowest-ranked second-placed team is eliminated: $R(V, x) = 0$ if $x \in G_1(V)$ and $y \succ_{Q(V)} x$ for all $y \in G_1(V) \setminus \{x\}$.

Now we turn to the issue of incentive compatibility.
Definition 3.14. Manipulation in a group: Let \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) be a qualification system and \(V = \{v^1, v^2, \ldots, v^i, \ldots, v^k\}\) be a set of group results. A team \(x \in X^i, 1 \leq i \leq k\) can manipulate its group if there exists a set of group results \(\bar{V} = \{\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^i, \ldots, \bar{v}^k\}\) such that \(v_2^i(x, y) \leq \bar{v}_2^i(x, y)\) and \(v_1^i(y, x) \leq \bar{v}_1^i(y, x)\) for all \(y \in X^i\), furthermore, \(\mathcal{R}(\bar{V}, x) < \mathcal{R}(\bar{V}, x)\).

Manipulation in a group means that team \(x\) improves its position with respect to qualification by letting its opponents score more goals.

Definition 3.15. Strategy-proofness of a group: Let \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) be a qualification system. Consider a group with the set of teams \(X^i, 1 \leq i \leq k\). This group is said to be strategy-proof if there does not exist any set of group results \(V = \{v^1, v^2, \ldots, v^k\}\) under which any team \(x \in X^i\) can manipulate the group.

Definition 3.16. Strategy-proofness of a qualification system: A qualification system \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) is called strategy-proof if there does not exist any set of group results \(V = \{v^1, v^2, \ldots, v^k\}\) under which any team \(x \in X^1 \cup X^2 \cup \cdots \cup X^k\) can manipulate its group.

We need another notion in order to formulate the theoretical results.

Definition 3.17. Permutation of results in a group: Let \(\mathcal{T}\) be a group-based qualifier and \(V = \{v^1, v^2, \ldots, v^i, \ldots, v^k\}\) be a set of group results. Consider a group with the set of teams \(X^i, 1 \leq i \leq k\), and a permutation \(\sigma: X^i \rightarrow X^i\). Permutation of results in a group leads to the set of group results \(\{v^1, v^2, \sigma(v^i)^1, \ldots, v^k\}\) such that \(\sigma(v^i)^1(x, y) = v^i(\sigma^{-1}(x), \sigma^{-1}(y))\) for all teams \(x, y \in X^i\).

Permutation of group results is illustrated by the following example.

Example 3.1. Consider the 2018 FIFA World Cup qualification (UEFA) – which is a (monotonic) qualification system according to Proposition 3.1 –, and a set of group results \(V\). Let \(\sigma\) be the permutation swapping France and Sweden in UEFA Group A. Then France has the same number of points under \(\sigma(V)\) as Sweden under \(V\), while Sweden has the same number of points under \(\sigma(V)\) as France under \(V\). There is no other change in this or additional groups.

Our main contribution concerns the strategy-proofness of groups in a monotonic qualification system. It is summarized in Figure 1.

Theorem 3.1. Let \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) be any monotonic qualification system and \(V = \{v^1, v^2, \ldots, v^i, \ldots, v^k\}\) be any set of group results. Consider a group with the set of teams \(X^i, 1 \leq i \leq k\) such that at least one of the following conditions hold:

a) there is no team relegated to the repechage group, that is, \(|X^i \cap \mathcal{G}_1(V)| = 0\);

b) for each team \(x \in X^i \cap \mathcal{G}_1(V)\), its set of matches considered in the repechage group is independent of the group results \(V\), that is, \(\mathcal{G}_2(\mathcal{V}, x) = \mathcal{G}_2(\sigma(V), x)\) for any permutation of results in this group \(\sigma\) satisfying \(x \in (X^i \cap \mathcal{G}_1(V) \cap \mathcal{G}_1(\sigma(V)))\);

c) for each team \(x \in X^i \cap \mathcal{G}_1(V)\), its set of matches considered in the repechage group depends only on the group ranking, that is, \(\mathcal{G}_2(\bar{V}, x) = \mathcal{G}_2(\bar{V}, x)\) if \(y \succ_{S(v)} z\) implies \(y \succ_{S(\bar{v})} z\) for any sets of group results \(\bar{V}, \bar{V}\) and for all teams \(y, z \in X^i\), furthermore, \(x \prec_{S(v)} y\) implies \(y \in \mathcal{G}_2(\mathcal{V}, x)\) or \(y \notin \mathcal{G}_2(\mathcal{V}, x)\) and \(x \succ_{\bar{S}(v)} y\) implies \(y \in \mathcal{G}_2(\mathcal{V}, x)\) or \(y \notin \mathcal{G}_2(\mathcal{V}, x)\) for all teams \(y \in X^i \setminus \mathcal{G}_1(V)\).
Figure 1: A summary of the main result

**Strategy-proofness of a group in a monotonic qualification system**

Does there exist a team relegated to the repechage group?

Yes

No

*Strategy-proof* (Theorem 3.1, Condition a))

Does the set of matches carried over to the repechage group depend on the results?

Yes

No

*Strategy-proof* (Theorem 3.1, Condition b))

Does the set of matches carried over to the repechage group depend *only* on the group ranking?

Yes

No

*Both situations possible* (Proposition 3.2) (uninteresting case in practice)

Does there exist a team relegated to the repechage group such that a proper nonempty subset of its matches played against higher or lower ranked teams is considered in the repechage group?

Yes

No

*Manipulable* (Theorem 3.2)

*Strategy-proof* (Theorem 3.1, Condition c))

Then this group with the set of teams $X^i$ is strategy-proof.

For the first condition, see Remark 3.1. According to the second requirement, if a permutation of group results does not affect the relegation of a given team to the repechage group, then its set of matches carried over to the repechage group should remain the same. The last condition of Theorem 3.1 means that if a team is considered in the repechage group, then either all or none of its matches played against higher or lower teams are taken into account in the repechage group.

*Proof. a*) Monotonicity of group ranking $S$ ensures incentive compatibility.
b) Since $\mathcal{G}_2(V, x) = \mathcal{G}_2(\bar{V}, x)$ for team $x \in X^i \cap \mathcal{G}_1(V)$ under any sets of group results $V, \bar{V}$, monotonicity of group ranking $S$ and repechage ranking $Q$ ensures incentive compatibility.

c) Any team $x \in X^i \cap \mathcal{G}_1(V)$ is not able to change the set of its matches to be discarded in the repechage group, therefore monotonicity of group ranking $S$ and repechage ranking $Q$ ensures incentive compatibility.

The message of Theorem 3.1 in practice will be discussed later.

**Theorem 3.2.** Let $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ be any monotonic qualification system and $V = \{v^1, v^2, \ldots, v^i, \ldots, v^k\}$ be any set of group results. Consider a group with the set of teams $X^i$, $1 \leq i \leq k$ such that the following conditions hold:

a) there is a team relegated to the repechage group, that is, $|X^i \cap \mathcal{G}_1(V)| > 0$;

b) for each team $x \in X^i \cap \mathcal{G}_1(V)$, its set of matches considered in the repechage group depends only on the group ranking, that is, $\mathcal{G}_2(\bar{V}, x) = \mathcal{G}_2(\hat{V}, x)$ if $y \succ_S v$ implies $y \succ_S z$ for any sets of group results $\bar{V}, \hat{V}$ and for all teams $y, z \in X^i$; and

c) there exists a team $x \in X^i \cap \mathcal{G}_1(V)$ such that a proper nonempty subset of its matches played against higher or lower ranked teams are considered in the repechage group, that is:

\[
\emptyset \neq \left( \mathcal{G}_2(V, x) \cap \left\{ y \in X^i : y \succ_S v \cap (x) \right\} \right) \subset \left\{ y \in X^i : y \succ_S x \right\}, \text{ or}
\]

\[
\emptyset \neq \left( \mathcal{G}_2(V, x) \cap \left\{ y \in X^i : x \succ_S v \cap (x) \right\} \right) \subset \left\{ y \in X^i : x \succ_S y \right\}.
\]

Then this group with the set of teams $X^i$ is manipulable, consequently, the monotonic qualification system $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ violates strategy-proofness.

The requirements of Theorem 3.2 are explained in Figure 1, too, as this result goes in parallel with Theorem 3.1.

**Proof.** For the sake of simplicity, we provide an example for a group with four teams $|X^i| = 4$, where the second-placed team can manipulate its group.

**Example 3.2.** Let $X^i = \{A, B, C, D\}$. Consider the monotonic qualification system $(\mathcal{T}, \mathcal{G}, \mathcal{R})$ where $x \in X^i \cap \mathcal{G}_1(V)$ if $\left| \left\{ y \in X^i : y \succ_S v \cap (x) \right\} \right| = 1$, the runner-up is relegated to the repechage group. Since there is a difference in the allocation of teams in the repechage group (the first requirement of Theorem 3.2 holds), the results in other groups can be chosen such that the second-placed team can manipulate this group if its results considered in the repechage group improve.

Let $x \in X^i \cap \mathcal{G}_1(V)$. The following cases are possible as the second condition of Theorem 3.2 holds:

\footnote{According to Remark 3.1, it implies that there is a difference in the allocation of teams in the repechage group: if $x \in \mathcal{G}_1(V)$, then there exists at least one team $y \in \mathcal{G}_1(V)$ such that $\mathcal{R}(V, x) \neq \mathcal{R}(V, y)$.}
The runner-up is relegated to the repechage group.

The home team is in the row, the result of the match is given from its point of view.

The runner-up is relegated to the repechage group.

The runner-up is relegated to the repechage group.

(a) A situation susceptible to manipulation if the matches played against the third-placed team count (Case II), or the matches played against the fourth-placed team are discarded (Case VI) in the repechage group.

(b) A situation susceptible to manipulation if the matches played against the fourth-placed team count (Case III), or the matches played against the third-placed team are discarded (Case V) in the repechage group.

Table 3: The group of Example 3.2

The results given in Table 3.b. The number of points for each team is given in Table 3.a. The number of points for each team in the repechage group.

I. \( G_2(V, x) = \{ y \in X^i : \left| \left\{ z \in X^i : z \succ_{S(v^i)} y \right\} \right| = 0 \} \), that is, only the matches played against the top team count in the repechage group.

Condition (c) of Theorem 3.1 provides strategy-proofness: note that

\[
\left( G_2(V, x) \cap \left\{ y \in X^i : y \succ_{S(v)} x \right\} \right) = \left\{ y \in X^i : y \succ_{S(v^i)} x \right\},
\]

the third requirement of Theorem 3.2 does not hold.

II. \( G_2(V, x) = \{ y \in X^i : \left| \left\{ z \in X^i : z \succ_{S(v)} y \right\} \right| = 2 \} \), that is, only the matches played against the third-placed team count in the repechage group.

Consider the results \( v^i \) given in Table 3.a. The number of points for each team is \( s_{v^i}(A) = 4\alpha + \beta + \gamma \), \( s_{v^i}(B) = 3\alpha + 2\beta + \gamma \), \( s_{v^i}(C) = \alpha + \beta + 4\gamma \), and \( s_{v^i}(D) = \alpha + 2\beta + 3\gamma \), so \( A \succ_{S(v^i)} B \succ_{S(v^i)} D \succ_{S(v^i)} C \), \( A \succ_{S(v^i)} B \succ_{S(v^i)} C \succ_{S(v^i)} D \) due to the monotonicity of the group ranking \( S \). Furthermore, team \( B \) has \( \beta + \gamma \) points in the repechage group.

However, if \( v^i = v^i \) except for \( v^i(B, C) = \{ \text{loss} \} \) instead of \( v^i(B, C) = \{ \text{win} \} \), then \( s_{v^i}(A) = s_{v^i}(A) = 4\alpha + \beta + \gamma \), \( s_{v^i}(B) = 2\alpha + 2\beta + 2\gamma \), \( s_{v^i}(C) = 2\alpha + \beta + 3\gamma \), and \( s_{v^i}(D) = s_{v^i}(D) = \alpha + 2\beta + 3\gamma \), so \( A \succ_{S(v^i)} B \succ_{S(v^i)} C \succ_{S(v^i)} D \) due to the monotonicity of the group ranking \( S \). Team \( B \) has now \( \alpha + \gamma > \beta + \gamma \) points in the repechage group, therefore it can manipulate its group.

Note that

\[
\emptyset \neq \left( G_2(V, x) \cap \left\{ y \in X^i : x \succ_{S(v^i)} y \right\} \right) \subset \left\{ y \in X^i : x \succ_{S(v)} y \right\},
\]

the third requirement of Theorem 3.2 holds.

III. \( G_2(V, x) = \{ y \in X^i : \left| \left\{ z \in X^i : z \succ_{S(v)} y \right\} \right| = 3 \} \), that is, only the matches played against the fourth-placed team count in the repechage group.

Consider the results \( v^i \) given in Table 3.b. The number of points for each team is \( s_{v^i}(A) = 4\alpha + \beta + \gamma \), \( s_{v^i}(B) = 4\alpha + 2\gamma \), \( s_{v^i}(C) = \alpha + 2\beta + 3\gamma \), and \( s_{v^i}(D) = \alpha + \beta + 4\gamma \), so \( A \succ_{S(v^i)} B \succ_{S(v^i)} C \succ_{S(v^i)} D \) due to the monotonicity of the group ranking \( S \). Furthermore, team \( B \) has \( \alpha + \gamma \) points in the repechage group.
However, if $\bar{v}^i = v^i$ except for $\bar{v}^i(B, D) = \{\text{loss}\}$ instead of $v^i(B, D) = \{\text{win}\}$, then $s_{\bar{v}^i}(A) = s_{v^i}(A) = 4\alpha + \beta + \gamma$, $s_{\bar{v}^i}(B) = 3\alpha + 3\gamma$, $s_{\bar{v}^i}(C) = s_{v^i}(C) = 1\alpha + 2\beta + 3\gamma$, and $s_{\bar{v}^i}(D) = \alpha + \beta + 3\gamma$, so $A \succ_{v^i(S)} B \succ_{v^i(S)} D \succ_{v^i(S)} C$ due to the monotonicity of the group ranking $S$. Team $B$ has now $2\alpha > \alpha + \gamma$ points in the repechage group, therefore it can manipulate its group.

Note that

$$\emptyset \neq \left( G_2(V, x) \cap \left\{ y \in X^i : x \succ_{v^i(S)} y \right\} \right) \subset \left\{ y \in X^i : x \succ_{v^i(S)} y \right\},$$

the third requirement of Theorem 3.2 holds.

IV. $G_2(V, x) = X^i \setminus \{x\} \setminus \left\{ y \in X^i : \left| \left\{ z \in X^i : z \succ_{v^i(S)} y \right\} \right| = 0 \right\}$, that is, the matches played against the top team are discarded in the repechage group.

Condition c) of Theorem 3.1 provides strategy-proofness: note that

$$\left( G_2(V, x) \cap \left\{ y \in X^i : y \succ_{v^i(S)} x \right\} \right) = \emptyset,$$

and

$$\left( G_2(V, x) \cap \left\{ y \in X^i : x \succ_{v^i(S)} y \right\} \right) = \left\{ y \in X^i : x \succ_{v^i(S)} y \right\},$$

the third requirement of Theorem 3.2 does not hold.

V. $G_2(V, x) = X^i \setminus \{x\} \setminus \left\{ y \in X^i : \left| \left\{ z \in X^i : z \succ_{v^i(S)} y \right\} \right| = 2 \right\}$, that is, the matches played against the third-placed team are discarded in the repechage group.

Consider the results $v^i$ given in Table 3.b and the alternative results $\bar{v}^i$ from Case III. The analysis of Case III remains valid, but team $B$ has in the repechage group $2\alpha + 2\gamma$ points under $v^i$ and $3\alpha + \gamma$ points under $\bar{v}^i$, respectively, therefore it can manipulate its group.

Note that

$$\emptyset \neq \left( G_2(V, x) \cap \left\{ y \in X^i : x \succ_{v^i(S)} y \right\} \right) \subset \left\{ y \in X^i : x \succ_{v^i(S)} y \right\},$$

the third requirement of Theorem 3.2 holds.

VI. $G_2(V, x) = X^i \setminus \{x\} \setminus \left\{ y \in X^i : \left| \left\{ z \in X^i : z \succ_{v^i(S)} y \right\} \right| = 3 \right\}$, that is, the matches played against the fourth-placed team are discarded in the repechage group.

Consider the results $v^i$ given in Table 3.a and the alternative results $\bar{v}^i$ from Case II. The analysis of Case II remains valid, but team $B$ has in the repechage group $\alpha + 2\beta + \gamma$ points under $v^i$ and $2\alpha + \beta + \gamma$ points under $\bar{v}^i$, respectively, therefore it can manipulate its group.

Note that

$$\emptyset \neq \left( G_2(V, x) \cap \left\{ y \in X^i : x \succ_{v^i(S)} y \right\} \right) \subset \left\{ y \in X^i : x \succ_{v^i(S)} y \right\},$$

the third requirement of Theorem 3.2 holds.

VII. $G_2(V, x) = X^i \setminus \{x\}$, that is, no matches are discarded in the repechage group.

Condition c) of Theorem 3.1 provides strategy-proofness: note that

$$\left( G_2(V, x) \cap \left\{ y \in X^i \setminus G_1(V) : y \succ_{v^i(S)} x \right\} \right) = \left\{ y \in X^i \setminus G_1(V) : y \succ_{v^i(S)} x \right\},$$

and

$$\left( G_2(V, x) \cap \left\{ y \in X^i \setminus G_1(V) : x \succ_{v^i(S)} y \right\} \right) = \left\{ y \in X^i \setminus G_1(V) : x \succ_{v^i(S)} y \right\},$$

the third requirement of Theorem 3.2 does not hold.
Strategy-proofness is guaranteed by monotonicity in the case of two teams. Manipulation in a group with three teams can be shown by the introduction of further tie-breaking rules, it is left to the reader. Furthermore, if tie-breaking rules (goal difference, head-to-head results, fair play conduct etc.) are also considered in Definitions 3.4 and 3.10, then \( \alpha = \beta \) (a win and a draw have the same worth) or \( \beta = \gamma \) (a draw and a loss have the same worth) can be allowed.

Nevertheless, these are uninteresting cases in practice since discarding the results against an opponent if there are only two of them is difficult to justify. Even though there are tournaments with groups of three teams and a repechage group, like the 2018-19 UEFA Nations League C, matches are ignored only for groups with four teams in the ranking of third-placed teams.

It is clear that the number of teams in the group, and the position of the team relegated to the repechage group can be modified in Example 3.2 without changing the essence of the proof.

It can be realized from Figure 1 that there is a gap between Theorems 3.1 and 3.2 when the set of matches carried over to the repechage group depend on the results, but not worth) can be allowed.

An example is provided for both cases.

**Proof.** An example is provided for both cases.

**Example 3.3.** Consider the monotonic qualification system \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) of Example 3.2 with \( \mathcal{G}_2(V, x) = \left\{ y \in X^i : \left| \{ z \in X^i : z \succ_S(v^i) y \} \right| = 0 \right\} \) (Case I), that is, only the matches played against the top team count in the repechage group, but if there exists a team \( y \in X^i \) such that \( v^i_1(z, y) > v^i_1(z, y) \) and \( v^i_2(z, y) < v^i_1(z, y) \) for all \( z \in X^i, z \neq y \) (team \( y \) loses all of its matches), then \( \mathcal{G}_2(V, x) = \left\{ y \in X^i : \left| \{ z \in X^i : z \succ_S(v^i) y \} \right| = 3 \right\} \), namely, only the matches played against the fourth-placed team count in the repechage group.

The monotonic qualification system \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) of Example 3.3 is incentive compatible. First, if only the matches played against the top team count in the repechage group (there exist no team losing all of its matches), then \( \mathcal{G}_2(V, x) \) cannot be modified by the second-placed team \( x \in X^i \) through exerting a lower effort. Second, if the matches played against the fourth-placed team, which loses all of its matches, count in the repechage group, then the second-placed team \( x \in X^i \) has no incentive to cheat because it carries over the maximal number of points \((2\alpha)\) to the repechage group.
Example 3.4. Consider the monotonic qualification system \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) of Example 3.2 with \(\mathcal{G}_2(V, x) = X^i \setminus \{x\} \setminus \{y \in X^i : \{z \in X^i : z \succ_S(v') y\}\} = 3\) (Case VI), that is, the matches played against the fourth-placed team are discarded in the repechage group, but if there exists a team \(y \in X^i\) such that \(v_1^i(z, y) < v_2^i(z, y)\) and \(v_2^i(z, y) > v_1^i(z, y)\) for all \(z \in X^i\), \(z \neq y\) (team \(y\) wins all of its matches), then \(\mathcal{G}_2(V, x) = \{y \in X^i : \{z \in X^i : z \succ_S(v') y\}\} \leq 2\), namely, the matches played against the top team are discarded in the repechage group.

The monotonic qualification system \((\mathcal{T}, \mathcal{G}, \mathcal{R})\) of Example 3.4 can be manipulated similarly to Case VI in the proof of Theorem 3.2 because no team has won all of its matches there.

Note that a similar gray zone emerges in the analysis of Dagaev and Sonin (2017).

Corollary 3.1. 2018 FIFA World Cup qualification (UEFA) is incentive incompatible.

Proof. The scenario presented in the Example 2.1 shows that team Bulgaria = \(x \in X^i\) can manipulate its group since there exist sets of group results \(V = \{v^1, v^2, \ldots, v^9\}\) and \(\bar{V} = \{v^1, v^2, \ldots, v^9\}\) such that \(v^1 = v^1\) except for \(v^1_1(y, x) = 1 > 0 = v^1_2(y, x)\), where team Luxembourg = \(y \in X^i\) and \(\mathcal{R}(V, x) = 0 < 1 = \mathcal{R}(\bar{V}, x)\).

Theorem 3.2 can also be applied due to Proposition 3.1: the qualification system is monotonic, \(\mathcal{R}(V, x)\) can be 0 or 1 if team \(x \in X^i \cap \mathcal{G}_1(V)\) is in the repechage group, \(|X^i \cap \mathcal{G}_1(V)| = 1\), and for each team \(x \in X^i \cap \mathcal{G}_1(V)\), \(\mathcal{G}_2(V, x)\) depends only on the ranking in the group with the set of teams \(X^i\), furthermore, \(\emptyset \neq \{\mathcal{G}_2(V, x) \cap \{y \in X^i \setminus \mathcal{G}_1(V) : x \succ_S(v') y\}\} \subset \{y \in X^i \setminus \mathcal{G}_1(V) : x \succ_S(v') y\}\) under any set of group results \(V = \{v^1, v^2, \ldots, v^k\}\).

Theorem 3.2 proves the incentive incompatibility of 2014 FIFA World Cup qualification (UEFA), too, which has already been shown by Dagaev and Sonin (2013).

The allocation rule \(\mathcal{R}\) of the model above does not distinguish teams that advance to the subsequent round. This is not a problem if the next round is seeded randomly (like in the UEFA Euro 2000 qualifying) or on the basis of an exogenous ranking of the teams that cannot be manipulated by exerting a lower effort in some matches (like FIFA World Rankings used in the case of 2018 FIFA World Cup qualification (UEFA)). However, if, for example, the highest ranked teams of the repechage group are placed in the first pot before the seeding, then position in the repechage group counts and the qualification system may become incentive incompatible.

4 Discussion

It is known from Dagaev and Sonin (2013) and Corollary 3.1, respectively, that the 2014 and 2018 FIFA World Cup qualifications (UEFA) do not satisfy strategy-proofness. Further qualifications to the recent FIFA World Cups in the European Zone (World Cup (UEFA)) and UEFA European Championships (UEFA Euro) are analysed with respect to this property in Table 4.

The UEFA European Championship has been held every four years since 1960. The qualifications for the tournaments between 1960 and 1992 can be described without a repechage group, so they were strategy-proof due to condition a) of Theorem 3.1. The same result provides the incentive compatibility of the 2004 and 2008 qualifying.

The UEFA Euro 2020 qualifying is linked with the 2018-19 edition of the UEFA Nations League, which gives teams a secondary route to qualify for the final tournament. This
| Qualification | Groups | Teams | Gr / T | Slots | DQ | PO | Discarded matches | SP |
|---------------|--------|-------|--------|-------|----|----|-------------------|----|
| 1990 World Cup (UEFA) | 7      | 32    | 4/5, 3/4 | 13   | 1st; six 2nd | — | — | — |
| 1994 World Cup (UEFA) | 6      | 37    | 1/7, 4/6, 1/5 | 12   | 1st; 2nd | eight worst 2nd | against 5th and 6th | — |
| 1998 World Cup (UEFA) | 9      | 49    | 4/6, 5/3 | 14   | 1st; best 2nd | six worst 2nd | against 7th | — |
| 2002 World Cup (UEFA) | 8      | 51    | 3/7, 5/6 | 13   | 1st; two best 2nd | eight best 2nd | against 6th | — |
| 2006 World Cup (UEFA) | 9      | 53    | 8/6, 1/5 | 13   | 1st; best 2nd | eight best 2nd | against 6th | — |
| 2006 UEFA Euro     | 9      | 53    | 8/6, 1/5 | 13   | 1st; best 2nd | eight best 2nd | against 6th | — |
| 2010 World Cup (UEFA) | 9      | 53    | 8/6, 1/5 | 13   | 1st; best 2nd | eight best 2nd | against 6th | — |
| 2014 World Cup (UEFA) | 9      | 53    | 8/6, 1/5 | 13   | 1st; best 2nd | eight best 2nd | against 6th | — |
| 2018 World Cup (UEFA) | 9      | 54    | 9/6    | 13   | 1st; best 2nd | eight best 2nd | against 6th | — |
| 1992 UEFA Euro     | 7      | 33    | 5/5, 2/4 | 1st   | 6th | 2nd | against 5th and 6th | — |
| 1996 UEFA Euro     | 8      | 47    | 7/6, 1/5 | 15   | 1st; six best 2nd | two worst 2nd | against 5th and 6th | — |
| 2000 UEFA Euro     | 9      | 50    | 4/6, 5/5 | 14   | 1st; best 2nd | eight worst 2nd | against 6th | — |
| 2004 UEFA Euro     | 10     | 50    | 4/6, 5/5 | 14   | 1st; best 2nd | eight worst 2nd | against 6th | — |
| 2008 UEFA Euro     | 7      | 50    | 6/6, 3/5 | 14   | 1st; best 2nd | eight worst 2nd | against 6th | — |
| 2012 UEFA Euro     | 9      | 53    | 8/6, 1/5 | 13   | 1st; best 2nd | eight worst 3rd | against 6th | — |
| 2016 UEFA Euro     | 9      | 53    | 8/6, 1/5 | 13   | 1st; best 2nd | eight worst 3rd | against 6th | — |

1. The second-placed teams in the four groups containing five teams directly qualified together with the two best second-placed teams in the three groups.
2. Group 5, originally containing six teams, ended up with five after Yugoslavia was suspended.
3. One team was advanced to an intercontinental play-off.
4. The runner-up of Group 2 was drawn randomly for an intercontinental play-off.
format is not covered by our theoretical model, but it also violates strategy-proofness according to Dagaev and Sonin (2017, Proposition 2) as the qualification system essentially consists of two parallel round-robin tournaments.\(^3\)

The first incentive incompatible FIFA World Cup qualifications in the European zone was the 1998 FIFA World Cup qualification (UEFA). The 2002 FIFA World Cup qualification (UEFA) again satisfied strategy-proofness due to the lack of a repechage group (see condition a) of Theorem 3.1).

Despite the 1990 and 1994 FIFA World Cup qualifications (UEFA) were incentive compatible, their fairness seems to be questionable: the second-placed teams qualified automatically only from certain groups in the first case, and all second-placed teams qualified but group sizes varied in the second case (some qualifications before the 1990 event suffered from the same problem). It is clear that strategy-proofness is a narrower concept than fairness, see Guyon (2017) for an examination of the latter through the example of the 2016 UEFA European Championship. Hence, one can even say that administrators sacrificed fairness for the sake of strategy-proofness.

**Proposition 4.1.** Qualifications to the 1996, 2000, 2012 and 2016 UEFA European Championships as well as to the 1998, 2006, 2010, 2014 and 2018 FIFA World Cups in the European Zone were incentive incompatible.

**Proof.** It immediately follows from Theorems 3.1 and 3.2 (consider Table 4).

The list is far from exhaustive. We mention here only the elite round of the 2016 UEFA European Under-17 Championship qualification and the 2017 UEFA European Under-21 Championship because they may give inspiration for scientific papers in the future.

Proposition 4.1 carries a disconcerting message for the administrators of FIFA and UEFA: they could be responsible for a potential scandal occurring in a recent qualification. For example, in October 2017, as shown in Section 2. It would be especially disturbing because Luxembourg would have practically no incentive to interfere with the manipulation of Bulgaria in order to prevent the elimination of Montenegro, furthermore, Luxembourg might have even interested in scoring a goal to be the fifth in the group.

Fortunately, such an outcome has not materialized, and we do not know about any attempt to strategically manipulate these qualifications in the way presented above. Probably the closest case was France against Israel in the 1996 UEFA Euro qualifying, where France would have better measures among runners-up if it had scored two own goals (Csató, 2018).

### 5 Strategy-proof mechanisms

We think the lack of dishonest behaviour in the history of qualifications does not reduce the importance of strategy-proofness in practice, especially if it can be achieved without significant rule changes. For instance, in the 2018 FIFA World Cup qualification (UEFA),

\(^3\) There is some debate on tanking in the UEFA Nations League, see at https://www.reddit.com/r/soccer/comments/75vrcm/tanking_in_the_uefa_nations_league_wont_benefit/ and http://www.football-rankings.info/2017/07/uefa-nations-league-losing-could.html. However, it focuses on the observation that dropping to a lower ranked league could improve the chances of qualifying for the 2020 UEFA European Championship through the UEFA Nations League, a gain only in expected terms. According to Dagaev and Sonin (2017, Proposition 2), a team might be strictly better off by creating a vacancy in the play-offs of the UEFA Euro 2020 qualifying tournament.
the root of the problem resides in discarding the matches played against the sixth-placed teams in the ranking of runners-up. The greatest pity about this situation is that it could have been straightforward to avoid by UEFA ditching the strange policy of ignoring some group matches since all groups had six teams following the admission of Gibraltar and Kosovo.

Yet the administrators chose not to modify the rules. According to a UEFA News (UEFA, 2017), released on 10 October 2017, after the end of group stage: "[…] the exclusion of results against sixth-placed teams was retained to alleviate any possible imbalance between the qualifying groups caused by the late introductions of Gibraltar and Kosovo". While it is respectable to prevent some mathematically unprovable imbalances between the groups, this decision sacrificed the much more clear and important theoretical issue of incentive compatibility.

It seems to be necessary to suggest incentive compatible designs in order to argue against the rules of recent qualifications. Denote the number of teams by \( n \), and the number of groups by \( k \). Let \( m \equiv n \mod k \) such that \( 0 \leq m < k \) and \( \ell = (n - m)/k \in \mathbb{Z} \). We are looking for strategy-proof qualification systems on the basis of Theorem 3.1.

**Definition 5.1.** Mechanism A: eliminating the repechage group

The optimal case is to create groups of equal size, like in the UEFA Euro 2004 qualifying (see Table 4). However, it may conflict with divisibility, this solution cannot be followed if \( m > 0 \).

Another possibility is that all second- or third-placed teams either qualify or advance to the next round regardless of group sizes. For example, the UEFA Euro 2008 qualifying was strategy-proof as the top two teams in each group qualified, despite the fact that a group was larger than the others, which may be unfair (see Table 4).

Mechanism A provides incentive compatibility because of condition a) of Theorem 3.1.

**Definition 5.2.** Mechanism B: matches to be discarded in the repechage group are independent of group results

While social choice theory usually wants to avoid the violation of anonymity at all costs, it makes sense to consider this solution because of the seeding procedure. If teams are ranked on an external basis (such as FIFA World Rankings from a given month), there would be \( k \) teams in Pot 1, \( k \) teams in Pot 2, and so on, until Pot \( \ell \) with \( k < m < 2k \) (as in the UEFA Euro 2008 qualifying) or Pot \( \ell + 1 \) with \( m \leq k \) (as in the 2018 FIFA World Cup qualification (UEFA)) teams is formed. Since the last pot is responsible for the difference of group sizes, it seems to be fair to discard the matches played against the team(s) from the last pot in the repechage group.

For the 2018 FIFA World Cup qualification (UEFA), it means fixing in advance that matches played against the teams in Pot 6 (Luxembourg, Andorra, San Marino, Georgia, Kazakhstan, Malta, Liechtenstein, as well as the lately introduced Gibraltar and Kosovo) are ignored in the comparison of the runners-up. Since only Luxembourg and Georgia obtained a better (the fifth) position in the qualification, this policy does not make much difference in practice.

Nevertheless, a problem may arise when a team from the last pot is relegated to the repechage group due to its unexpectedly good performance in the qualifiers.\(^4\) The unlikely

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\(^4\) We are grateful to Dénés Puñás for spotting this issue. One of the greatest surprises of this type occurred in the 2016 UEFA Euro qualifying when Greece finished as the bottom-ranked team in Group F despite drawn from Pot 1.
scenario can be immediately solved by discarding the matches played against the team from the penultimate pot for this particular team, which does not affect strategy-proofness.

Mechanism B provides incentive compatibility because of condition b) of Theorem 3.1.

**Definition 5.3.** Mechanism C: all or none matches played against higher and lower ranked teams are considered in the repechage group

If group sizes vary because of $m > 0$, and the $p$th-placed teams of the groups are considered in the repechage group, then only their matches played against higher ranked teams can count in the repechage group as the cardinality of the set of teams that are ranked lower than them in their group is different. It means no problem when:

- a) The qualification is centred around relegation: in the 2018-19 UEFA Nations League C, 15 teams are divided into one group of three teams and three groups of four teams each such that the three fourth-placed and the worse third-placed teams are relegated to League D. In the comparison of third-placed teams, only the matches played against the teams ranked first and second in the group count. Since there is at most one team ranked lower than the team relegated to the repechage group in each group, they cannot manipulate by changing their set of matches to be ignored.

- b) A high proportion of participating teams qualify: in the 2018 European Men’s Handball Championship Qualification Phase 2, the contestants were split into seven groups of four teams each such that the top two ranked teams from each group and the best third-placed team qualified for the final tournament. In the comparison of third-placed teams, only the matches played against the teams ranked first and second in the group counted.

Mechanism C provides incentive compatibility because of condition c) of Theorem 3.1.

Mechanism C guaranteed the strategy-proofness of the 2016 UEFA European Championship with a uniform group size (Guyon, 2017), too, where third-placed teams were ranked on the basis of all group matches.

**Definition 5.4.** Mechanism D: new seeding policy

The idea behind mechanism C can be applied with a slight modification of the seeding procedure, by a bottom-up design of the pots as follows instead of the usual top-down approach. First, the worst $k$ teams form Pot $\ell + 1$, the next $k$ teams Pot $\ell$, and so on until Pot $1$ with $m$ teams is seeded. Then $m$ groups are created by drawing a team from each pot, and $k - m$ groups are created by drawing a teams from each pot except for Pot $1$. The top $p$ teams qualify from the $m$ groups of size $\ell + 1$, and the top $p - 1$ teams qualify from the $k - m$ groups of size $\ell$. The repechage group consists of the $(p + 1)$th-placed teams from the $m$ groups of size $\ell + 1$, and the $p$th-placed teams from the $k - m$ groups of size $\ell$, where they are compared on the basis of their matches played against the $\ell - p$ teams that are ranked lower in their groups. Consequently, the matches played against the already qualified teams are discarded in the repechage group, which seems to be reasonable.

Mechanism D provides incentive compatibility because of condition c) of Theorem 3.1.

It is also possible to organise a preliminary round for lower-ranked teams, either a round-robin such as in the CEV qualification for the 2018 FIVB Volleyball Men’s World Championship, or a two-leg playoff, similarly to the 2018 FIFA World Cup qualification.
– AFC First Round. However, it may be difficult to implement this solution given the constraints of the crowded match calendar.

The design of the 1990 FIFA World Cup qualification (UEFA) guarantees strategy-proofness, too: the runner(s)-up or third-placed team(s) to be directly qualified (eliminated) are chosen from the groups containing more (fewer) teams without discarding any matches. However, this format may be judged as dishonest, see Section 4.

To summarize, it is clear that mechanism A is perfect in the case of \( m = 0 \). However, its fairness seems to be questionable if \( n \) is not divisible by \( k \).

Mechanism C leads to discarding a high proportion of matches in the repechage group for the qualifications presented in Table 4, which may result in increased randomness. It means a problem in a qualification for a final tournament with huge financial issues at stake – for example, all teams were guaranteed at least USD 9.5 million each for their participation in the 2018 FIFA World Cup (FIFA, 2017a).

Consequently, we suggest considering mechanisms B or D if uniform group size cannot be guaranteed in a qualification. Both are general, that is, they can be directly applied for arbitrary values of \( k \) and \( n \), and they coincide with mechanism A if \( m = 0 \).

6 Conclusions

The optimal design of sports tournaments is an important theoretical problem of economics (Szymanski, 2003) and operations research (Scarf et al., 2009). Tournament organisers may face unpleasant situations when they miss analysing strategy-proofness as the potential costs of tanking can be enormous even if it is often a low-probability event. We have demonstrated that decision makers have chosen a risky strategy in the case of qualification tournaments to some recent FIFA World Cups and UEFA European Championships, and suggested two alternative mechanisms in order to guarantee incentive compatibility.

There are at least three possible directions for future research. First, a number of other tournament designs can be investigated from the perspective of strategy-proofness. Second, the current theory-oriented investigation can be supplemented by estimating the probability of manipulation with the use of historical and Monte-Carlo simulated data. Finally, our two general incentive compatible mechanisms can be compared.

Hopefully, this paper reinforces our view that the scientific community and the sports industry should work more closely together in studying the effects of potential rules and, especially, rule changes, even before they are implemented. For example, the governing bodies of major sports may invite academics to identify possible loopholes in proposed regulations in order to prevent future scandals.

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References

Abdulkadiroğlu, A., Pathak, P. A., and Roth, A. E. (2009). Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match. *American Economic Review*, 99(5):1954–78.

AFC (2015). *Media Release: Criteria to determine the rankings of best-placed teams among the groups*. 3 June 2015. [http://www.the-afc.com/download/criteria-to-determine-the-rankings-of-best-placed-teams-among-the-groups](http://www.the-afc.com/download/criteria-to-determine-the-rankings-of-best-placed-teams-among-the-groups).

Csató, L. (2017a). 2018 FIFA World Cup qualification can be manipulated. Manuscript. [http://unipub.lib.uni-corvinus.hu/3053/](http://unipub.lib.uni-corvinus.hu/3053/).

Csató, L. (2017b). On the ranking of a Swiss system chess team tournament. *Annals of Operations Research*, 254(1-2):17–36.

Csató, L. (2018). Was Zidane honest or well-informed? How UEFA barely avoided a serious scandal. *Economics Bulletin*, 38(1):152–158.

Dagaev, D. and Sonin, K. (2013). Game theory works for football tournaments. Manuscript. [http://voxeu.org/article/world-cup-football-and-game-theory](http://voxeu.org/article/world-cup-football-and-game-theory).

Dagaev, D. and Sonin, K. (2017). Winning by losing: Incentive incompatibility in multiple qualifiers. *Journal of Sports Economics*, forthcoming. DOI: 10.1177/1527002517704022.

FIFA (2016). *Regulations: 2018 FIFA World Cup Russia™*. 14 June – 15 July 2018. [http://resources.fifa.com/mm/document/tournament/competition/02/84/35/19/regulationsfwc2018en_neutral.pdf](http://resources.fifa.com/mm/document/tournament/competition/02/84/35/19/regulationsfwc2018en_neutral.pdf).

FIFA (2017a). FIFA Council: FIFA Council confirms contributions for FIFA World Cup participants. 27 October 2017. [http://www.fifa.com/about-fifa/news/y=2017/m=10/news=fifa-council-confirms-contributions-for-fifa-world-cup-participants-2917806.html](http://www.fifa.com/about-fifa/news/y=2017/m=10/news=fifa-council-confirms-contributions-for-fifa-world-cup-participants-2917806.html).

FIFA (2017b). Media Release: FIFA World Cup European play-off draw to take place on 17 October 2018. 6 September 2017. [http://www.fifa.com/worldcup/news/y=2017/m=9/news=fifa-world-cup-european-play-off-draw-to-take-place-on-17-october-2906954.html](http://www.fifa.com/worldcup/news/y=2017/m=9/news=fifa-world-cup-european-play-off-draw-to-take-place-on-17-october-2906954.html).

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601.

Guyon, J. (2017). What a fairer 24 team UEFA Euro could look like. Manuscript. DOI: 10.2139/ssrn.2714199.

Kahn, L. M. (2000). The sports business as a labor market laboratory. *Journal of Economic Perspectives*, 14(3):75–94.

Kendall, G. and Lenten, L. J. (2017). When sports rules go awry. *European Journal of Operational Research*, 257(2):377–394.

Pauly, M. (2014). Can strategizing in round-robin subtournaments be avoided? *Social Choice and Welfare*, 43(1):29–46.


Satterthwaite, M. A. (1975). Strategy-proofness and Arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217.

Scarf, P., Yusof, M. M., and Bilbao, M. (2009). A numerical study of designs for sporting contests. *European Journal of Operational Research*, 198(1):190–198.

Szymanski, S. (2003). The economic design of sporting contests. *Journal of Economic Literature*, 41(4):1137–1187.

UEFA (2016). News: Focus switches to World Cup qualifying. 22 August 2016. [http://www.uefa.com/european-qualifiers/news/newsid=2389887.html](http://www.uefa.com/european-qualifiers/news/newsid=2389887.html).

UEFA (2017). News: European Qualifiers: World Cup play-off places confirmed. 10 October 2017. [http://www.uefa.com/european-qualifiers/news/newsid=2506867.html](http://www.uefa.com/european-qualifiers/news/newsid=2506867.html).

Vong, A. I. K. (2017). Strategic manipulation in tournament games. *Games and Economic Behavior*, 102:562–567.