ON THE COMPLEXITY OF GRAPH COLORING
WITH ADDITIONAL LOCAL CONDITIONS

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Abstract. Let \( G = (V, E) \) be a finite simple graph. Recall that a proper coloring of \( G \) is a mapping \( \varphi : V \to \{1, \ldots, k\} \) such that every color class induces an independent set. Such a \( \varphi \) is called a semi-matching coloring if the union of any two consecutive color classes induces a matching. We show that the semi-matching coloring problem is NP-complete for any fixed \( k \geq 3 \), and we get the same result for another version of this problem in which any triangle of \( G \) is required to have vertices whose colors differ at least by three.

In this paper, we establish the algorithmic complexity of the two graph coloring problems mentioned in the abstract. They have attracted some attention in recent publications, including the paper \([2]\) by Hajiabolhassan on a semi-matching generalization of Kneser’s conjecture. The other notion, known in the literature as a local \( k \)-coloring, has been studied in several papers both from the point of view of Kneser’s conjecture and as a concept deserving an independent interest (see \([1, 4, 5]\) and references therein). As a reformulation of what is said in the abstract, one can define a local \( k \)-coloring as a mapping \( \varphi : V \to \{1, \ldots, k\} \) such that for each set \( S \) of either 2 or 3 vertices, there exist \( u, v \in S \) such that the number \( |\varphi(u) - \varphi(v)| \) is greater than or equal to the number of edges passing between the vertices in \( S \). Li, Shao, Zhu, Xu proved in \([4]\) that the local \( k \)-coloring problem is NP-complete for \( k = 4 \) and for any fixed odd \( k \geq 5 \); they posed a problem to determine the complexity in the remaining cases of \( k = 3 \) and even \( k \geq 6 \). We solve this problem, and actually we prove the NP-completeness of both the local and semi-matching coloring problems for arbitrary fixed \( k \) in a unified manner.

1. Preliminaries

Throughout the rest of our paper, we assume that \( k \geq 3 \) is a fixed integer. For ease of reference, we formulate the problems we are going to study below (these problems would become trivial if \( k \) was less than 3).

Problem 1. Given: A finite simple graph \( G \).
Question 1: Is there a local \( k \)-coloring of \( G \)?
Question 2: Is there a semi-matching \( k \)-coloring of \( G \)?

The main result of this paper is as follows.

Theorem 2. Questions 1 and 2 in Problem \([1]\) are NP-complete.

It is easy to see that both questions in Problem \([1]\) belong to NP, and we are going to prove their NP-hardness by constructing polynomial reductions from the

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problem known as NAE 3-SAT. Recall that a literal (corresponding to a set \( X \) of Boolean variables) is either an element of \( X \) or a negation of it.

**Problem 3.** (NAE 3-SAT.)

Given: A set \( X \) of variables and a set \( L \) of triples of literals.

Question: Does there exist an assignment \( X \to \{ \text{True}, \text{False} \} \) for which every triple in \( L \) has at least one false literal and at least one true literal?

2. **Earlier results that we use**

First of all, we recall that NAE 3-SAT is an NP-complete problem (see [6]), so the existence of polynomial reductions from Problem 3 to the questions in Problem 1 would indeed imply Theorem 2. Further, let us recall that the chromatic number of a graph \( G \) is the smallest integer \( \chi \) such that \( G \) admits a proper \( \chi \)-coloring; the local and semi-matching chromatic numbers are defined analogously for the corresponding types of colorings. The following appears as Theorem 2.1 in [3].

**Theorem 4.** For any integer \( \tau \geq 1 \), there exists a graph whose chromatic number, semi-matching chromatic number, and local chromatic number are all equal to \( \tau \).

**Corollary 5.** For any integer \( \tau \geq 1 \), there exists a graph \( \Gamma_\tau \) such that the chromatic number, semi-matching chromatic number, and local chromatic number of \( \Gamma_\tau \) and any graph obtained from \( \Gamma_\tau \) by removing at most two vertices are all equal to \( \tau \).

**Proof.** Take a disjoint union of three copies of the graph as in Theorem 4. \( \square \)

For any positive integers \( n, r \) satisfying \( n \geq 2r \), we denote by \( C(n, r) \) the graph obtained from a complete graph on \( n \) vertices by removing \( r \) non-adjacent edges.

**Theorem 6.** (See [1].) The local chromatic number of \( C(n, r) \) is \( \left\lfloor \frac{1.5n - 0.5}{r} \right\rfloor \).

3. **The equality gadget**

In this section, we construct polynomial reductions to local and semi-matching \( k \)-colorings from more general problems that we are ready to present. (We recall that \( k \geq 3 \) is an integer parameter fixed in advance.)

**Problem 7.**

Given: A simple graph \( G = (V, E) \); a sequence \( U \) of subsets \( U_1, \ldots, U_t \subset V \).

Question: Is there a local \( k \)-coloring of \( G \) in which, for every \( i \in \{1, \ldots, t\} \), all the vertices of \( U_i \) get the same color, and this color is either 1 or \( k \)?

Let us define a ‘semi-matching version’ of Problem 7 by replacing the word ‘local’ with the word ‘semi-matching’ in its formulation. We define the graph \( G(G, U) \) by adjoining to \( G \) the \( t \) copies of the graph \( \Gamma_{k-2} \) as in Corollary 5 and drawing the edges between every vertex in \( U_i \) and every vertex in the corresponding (that is, \( i \)th) copy of \( \Gamma_{k-2} \). Clearly, \( G \) can be computed in polynomial time as a function of \( G \) and \( U \).

**Lemma 8.** \( G \) is a reduction (i) from Problem 7 to Question 1 in Problem 1, (ii) from the semi-matching version of Problem 4 to Question 2 in Problem 1.

**Proof.** If we have a coloring of \( G \) as in Problem 7, then we can extend it to \( G \) as follows. Namely, the vertices in \( U_i \) got the same color 1 (or \( k \)), and we produce the local \((k-2)\)-coloring of the \( i \)th copy of \( \Gamma_{k-2} \) using indexes \( 3, 4, \ldots, k \) (or \( 1, 2, \ldots, k-2 \)).
2, respectively). It is straightforward to check that the resulting assignment is a local \(k\)-coloring of \(G\) (or a semi-matching coloring if we consider the semi-matching version of Problem 7).

Conversely, assume that we have a local (or semi-matching) \(k\)-coloring of \(G\). If one of the vertices in one of the \(U_i\)'s has color \(c\), then no vertex in the \(i\)th copy of \(\Gamma_{k-2}\) can have color \(c\), at most one vertex in that copy can have index \(c-1\), and at most one vertex in the copy can have index \(c+1\). Therefore, if there are two vertices in \(U_i\) of different colors, or there is a vertex of color different from 1 and \(k\), we get a proper \((k-3)\) coloring of a graph obtained from \(\Gamma_{k-2}\) by removing two vertices. This contradicts Corollary 5 and shows that the vertices in every \(U_i\) get the same color, and this color is either 1 or \(k\).

\(\square\)

4. The NAE gadget

This section is devoted to auxiliary results that will allow us to encode the NAE 3-SAT problem as an instance of Problem 7. We need to define a sequence of graphs which we denote by \(L(k)\); we set \(L(2\tau+2) = C(2\tau, \tau)\) and \(L(2\tau+3) = C(2\tau, \tau-1)\) for any integer \(\tau \geq 1\), and we define \(L(3)\) to be the graph with one vertex.

**Definition 9.** We construct the graph \(L_k(u_1, u_2, v)\) as follows. We take two disjoint copies of \(L(k)\) and three new vertices \(u_1, u_2, v\), and draw new edges from \(u_1\) to every vertex of the first copy, from \(u_2\) to every vertex of the second copy, and from \(v\) to every vertex in both copies.

**Lemma 10.** \(L_k(u_1, u_2, v)\) has no local \(k\)-coloring in which \(u_1, u_2, v\) have colors 1, 1, \(k\) or \(k, k, 1\), respectively.

**Proof.** Assume that such a coloring exists, and the vertices \(u_1, u_2, v\) are assigned colors 1, 1, \(k\) (the \(k, k, 1\) case is analogous). If there were two more vertices \(a, b\) colored either \(k-1\) or \(k\) each, then \((a, v, b)\) would be a path colored with two consecutive indexes. This contradicts the definition of a local coloring, so one of the copies of \(L(k)\) has colors 1, \ldots, \(k-2\) only. However, the union of this copy and the corresponding vertex \(u_i\) induces the subgraph which does not admit a local \((k-2)\)-coloring as we can check by Theorem 6. \(\square\)

**Lemma 11.** Any mapping \(\varphi : \{u_1, u_2, v\} \rightarrow \{1, k\}\) can be extended to a local \(k\)-coloring of \(L_k(u_1, u_2, v)\) unless \(\varphi(u_1) = \varphi(u_2) = k\) and \(k+1 - \varphi(v)\) are all equal.

**Proof.** There are only two possible cases up to symmetry.

Case 1. If \(\varphi(u_1) = \varphi(u_2) = \varphi(v) = k\), then we can use 1, \ldots, \(k-2\) to get a local \(k\)-coloring of both copies of \(L(k)\), which is possible by Theorem 6.

Case 2. Now let \(\varphi(u_1) = \varphi(v) = k\) and \(\varphi(u_2) = 1\). Again, we construct a local coloring of the first copy of \(L(k)\) using 1, \ldots, \(k-2\) only. Our strategy for the second copy depends on the parity of \(k\); if \(k = 2\tau + 2\), then we color the vertices of \(L(k) = C(2\tau, \tau)\) so that the endpoints of removed edges get indexes (2, 3), (4, 5), \ldots, \(k-2, k-1\), thus completing a local \(k\)-coloring of \(L_k(u_1, u_2, v)\).

If \(k = 2\tau + 3\), then we color the vertices of \(L(k) = C(2\tau, \tau - 1)\) so that the vertices not adjacent to removed edges get indexes 2 and \(k-1\), and the removed edges are (4, 4), (6, 6), \ldots, (2\tau, 2\tau). Actually, this assignment is a local \(k\)-coloring of \(L_k(u_1, u_2, v)\) not only when \(\tau \geq 1\) but also when \(\tau = 0\), but anyway the case of \(k = 3\) can be easily treated separately. \(\square\)
Now let us define the graph $SM_k(u_1, u_2, v)$ similarly to $L_k(u_1, u_2, v)$ but with $L(k)$ replaced by the complete graph on $k - 2$ vertices. In this case, Lemmas 10 and 11 will stay true if we replace $L$ by $SM$ and ‘local’ by ‘semi-matching’ in their formulations. We do not provide the proofs because they can be easily reconstructed from the arguments above.

5. Completing the proof

**Theorem 12.** Problem 7 is NP-hard.

*Proof.* We construct the instance $(G, U)$ of Problem 7 depending on an instance $(X, L)$ of Problem 3 as follows.

**Step 1.** For every variable $x \in X$, we add vertices corresponding to $x$ and $\overline{x}$ and connect them by an edge.

**Step 2.** For every tuple $(a, b, c) \in L$, we add a copy of the graph $L_k(u_a, u_b, v_c)$.

**Step 3.** We define $2|X|$ subsets $U_\chi$ corresponding to the variables in $X$ and their negations. Namely, we define $U_\chi$ as a union of the vertex corresponding to $\chi$ as in Step 1 and all the vertices $u_\chi$ and $v_\chi$ as in Step 2.

Clearly, the mapping $(X, L) \rightarrow (G, U)$ can be computed in polynomial time. Let us check that it is a reduction: assume that $\varphi$ is a coloring of $G$ as in Problem 7. The edges added in Step 1 guarantee that the classes $U_\chi$ and $U_{\overline{\chi}}$ should get opposite colors in $\{1, k\}$, so we can think of the color choice of every $U_\chi$ as a truth assignment of a variable $x \in X$. Since the copies of $L_k(u_a, u_b, v_c)$ are disjoint, $\varphi$ is a local coloring if and only if its restriction to every such copy is local, which is possible if and only if $a, b, c$ are not all equal, according to Lemmas 10 and 11.

Replacing every appearance of $L$ by $SM$ and ‘local’ by ‘semi-matching’ in the proof above, we get the NP-hardness of the semi-matching version of Problem 7. Applying Lemma 8, we complete the proof of Theorem 2.

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