Fast Feasible and Unfeasible Matrix Multiplication

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Abstract

Matrix-by-matrix multiplication (hereafter referred to as MM) is a fundamental operation omnipresent in modern computations in Numerical and Symbolic Linear Algebra. Its acceleration makes major impact on various fields of Modern Computation and has been a highly recognized research subject for about five decades. The researchers introduced amazing novel techniques, found new insights into MM and numerous related computational problems, and devised advanced algorithms that performed $n \times n$ MM by using less than $O(n^{2.38})$ scalar arithmetic operations versus $2n^3 - n^2$ of the straightforward MM, that is, more than half-way to the information lower bound $n^2$. The record upper bound 3 of 1968 on the exponent of the complexity MM decreased below 2.38 by 1987 and has been extended to various celebrated problems in many areas of computing and became most extensively cited constant of the Theory of Computing. The progress in decreasing the record exponent, however, has virtually stalled since 1987, while many scientists are still anxious to know its sharp bound, so far restricted to the range from 2 to about 2.3728639. Narrowing this range remains a celebrated challenge.

Acceleration of MM in the Practice of Computing is a distinct challenge, much less popular, but also highly important. Since 1980 the progress towards meeting the two challenges has followed two distinct paths because of the curse of recursion — all the known algorithms supporting the exponents below 2.38 or even below 2.7733 involve long sequences of nested recursive steps, which blow up the size of an input matrix. As a result all these algorithms improve straightforward MM only for unfeasible MM of immense size, greatly exceeding the sizes of interest nowadays and in any foreseeable future.

It is plausible and surely highly desirable that someone could eventually decrease the record MM exponent towards its sharp bound 2 even without ignoring the curse of recursion, but currently there are two distinct challenges of the acceleration of feasible and unfeasible MM.

In particular various known algorithms supporting the exponents in the range between 2.77 and 2.81 are quite efficient for feasible MM and have been implemented. Some of them make up a valuable part of modern software for numerical and symbolic matrix computations, extensively worked on in the last decade. Still, that work has mostly relied on the MM algorithms proposed more than four decades ago, while more efficient algorithms are well known, some of them appeared in 2017.

In our review we first survey the mainstream study of the acceleration of MM of unbounded sizes, cover the progress in decreasing the exponents of MM, comment on its impact on the theory and practice of computing, and recall various fundamental concepts and techniques supporting fast MM and naturally introduced in that study by 1980. Then we demonstrate how the curse of recursion naturally entered the game of decreasing the record exponents. Finally we cover the State of the Art of efficient feasible MM, including some most efficient known techniques and algorithms as well as various issues of numerical and symbolic implementation.

We hope that our review will help motivate and properly focus further effort in this highly important area.

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1 Introduction

Matrix-by-matrix multiplication (MM) is omnipresent in modern computations for Applied Mathematics, Statistics, Physics, Engineering, Combinatorics, Computer Science, Signal and Image Processing, and Bioinformatics (cf., e.g., [106], [129]) and is performed billions times per day around the globe. For a couple of specific examples, computing radiation exchange in enclosures can be written in matrix form, mainly as an $N \times N$ MM with $N$ up to 10,000 [76], and gene expression analysis in bioinformatics involves $N \times N$ MM with $N$ up to 4,000 [148].

The straightforward MM involves cubic time, namely $2n^3 - n^2$ arithmetic operations for $n \times n$, and until 1969 it was commonly believed that the order of $n^3$ operations are necessary. This belief died in 1969, when Volker Strassen decreased the exponent 3 of the complexity of MM to 2.8074 in [151]. Further decrease of the exponent towards its information lower bound 2 has instantly become the subject of worldwide interest and the goal of intensive study by literally all experts in this field.

Various novel amazing but fundamental techniques have been proposed, new insights into fast MM and some related areas of modern computations have been introduced, and the MM exponent has been decreased below 2.38, that is, more than half-way from the classical 3 to the information lower bound 2. These results have been widely recognized in the Theory of Computation, and 2.38 became the absolute constant of that field, most frequently cited there, because it bounds the exponents of the complexity of various celebrated problems in many areas of computing linked to MM.

The progress was not going smoothly. It stalled for nearly a decade after 1969, resumed in 1978 with [118], followed with new significant advances in 1979–1982 and again in 1986, and since then again virtually stalled. Even more serious problem with the record-breaking MM algorithms is the curse of recursion – they rely on long sequences of nested recursive processes, which blow up the input size. The resulting algorithms supersede the straightforward MM only when this size becomes immense – greatly exceeding the level of MM in use nowadays or even in any foreseeable future.

It is plausible and of course highly desirable that new breakthrough would eventually result in an algorithm for fast feasible MM supporting a nearly sharp exponent close to 2, but so far the acceleration of feasible MM of realistic sizes is studied as a distinct hard task for which one cannot ignore the curse of recursion. So far the known algorithms for feasible MM only support the exponents below 2.7734 (see [123]), unbeaten for 36 years since 1982 but not much recognized. Nevertheless in spite of their association with relatively large exponents, some fast algorithms for feasible MM make up a valuable part of modern software for numerical and symbolic matrix computations. Implementation work for fast feasible MM has been intensified within the last decade but still mostly relies on algorithms that are more than four decades old and have already been significantly improved (see [91], [149], and [98]).

In our present survey we first trace the history of the decrease of the exponent of MM, list some celebrated problems of the Theory of Computing linked to MM and sharing with MM the exponent of the complexity, and naturally recall the fundamental concepts and techniques for fast MM, introduced by 1981, such as recursive bilinear and trilinear algorithms, tensor decomposition, trilinear aggregation, disjoint MM, any precision approximation, and the EXPAND, PRUNE, and CONQUER techniques. We also show how the curse of recursion naturally entered the game. Finally we bring our survey closer to the earth by covering the acceleration of feasible MM of realistic sizes and some relevant methods and techniques such as the $3M$ method for complex MM, its extension to polynomial MM, computer-aided ALS search, and randomization technique for fast MM. We also discuss the implementation issues of fast MM, including numerical stability, data movement, and symbolic implementation.

We hope that our work will help motivate and properly focus further study and further progress in this highly important area of Modern Computations.
2 Straightforward MM and First Fast MM Algorithms

2.1 MM and its subproblems MV and VV. Optimality of straightforward MV and VV

$k \times m$ by $m \times n$ MM (hereafter referred to as MM($k,m,n$)) is the computation of the product $C$ of two matrices $A$ of size $k \times m$ and $B$ of size $m \times n$ whose entries can be numbers, variables, or matrices:

$$C = AB,$$

$$C_{i,h} = a_{i,1}b_{1,h} + a_{i,2}b_{2,h} + \cdots + a_{i,m}b_{m,h}, \quad i = 1, \ldots, k, \quad h = 1, \ldots, n.$$  

(2.1)

MM($k,m,1$) is matrix-by-vector multiplication (hereafter referred to as MV),

$$c_i = a_i b_1 + a_i b_2 + \cdots + a_i b_m, \quad i = 1, 2, \ldots, k;$$

and we write MM($n$) for MM($n,n,n$).

MV and MM($n$) are made up of their $n$ and $kn$ subproblems VV, respectively. Like MM both MV and VV are omnipresent in modern computations in Linear and Multilinear Algebra.

The straightforward classical algorithm for VV first computes the $m$ products $a_1 b_1, a_2 b_2, \ldots, a_m b_m$ and then sum them together. For MV and MM we can apply the same algorithm $k$ and $kn$ times, respectively. This is optimal for VV and MV – any algorithm that performs VV or MV by using only scalar arithmetic operations, that is, add, subtract, multiply, and divide, and that uses no branching must perform at least as many scalar additions and subtractions and at least as many scalar multiplications and divisions as the straightforward algorithm. The proofs rely on the techniques of active operation – basic substitution from [116], also covered and extended in [31], Section 2.3, [94], and Sections “Pan’s method” in [152] and [154].

Sparse and structured MV, also omnipresent in modern computations, can be performed much faster. In particular arithmetic time linear in $n$ up to a logarithmic or polylogarithmic factor is sufficient in order to multiply by a vector an $n \times n$ matrix with a structure of Toeplitz, Hankel, Cauchy, or Vandermonde type and even to solve a nonsingular structured linear system of $n$ equations with such a coefficient matrix versus quadratic or cubic time required for the same computations with a general matrix. See [130], [133], and the bibliography therein for computations with structured matrices (aka data sparse matrices); see [31], [11], [7], and the bibliography therein for computations with sparse matrices.

2.2 The first fast MM algorithms

The straightforward MM uses $knm$ scalar multiplications and $knm - kn$ scalar additions, that is, $n^3$ and $n^3 - n^2$ for $k = m = n$. Until 1969 the scientific world believed that this algorithm is optimal, although already in 1968 the experts knew from the following example that this was not true (see [165] and notice technical similarity to the algorithms for polynomial evaluation with preprocessing in [116] and [94]).

Example 2.1. For any even positive integer $n$, the inner product of two vectors $a = (a_i)^n_{i=1}$ and $b = (b_j)^n_{j=1}$ satisfies the following identity,

$$a^Tb = \sum_{i=1}^{n/2}(a_{2i-1} + b_{2i})(b_{2i-1} + a_{2i}) - \sum_{i=1}^{n/2}a_{2i-1}a_{2i} - \sum_{i=1}^{n/2}b_{2i-1}b_{2i}.$$  

(2.2)

By combining such identities for $n^2$ inner products defining $n \times n$ MM, perform MM($n$) by using $0.5n^3 + n^2$ scalar multiplications and $1.5n^3 + 2n^2 - 2n$ scalar additions and subtractions, thus replacing about 50% of multiplications of the straightforward MM by additions.
In the 1960s a floating point multiplication was usually two or three times slower than a floating point addition, and so the algorithm had some practical value. Not so anymore nowadays because multiplication is about as fast as addition, but the algorithm is still a historical landmark as the first fast MM. It was not fast enough in order to attract the attention of non-experts, however, and so the news that the straightforward MM is not optimal has awakened scientific world only a little later, when Volker Strassen presented the following celebrated algorithm.

**Example 2.2.** Strassen’s $2 \times 2$ MM. Compute the product $C = AB$ of a pair of $2 \times 2$ matrices, for

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = AB = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad (2.3)$$

by using the following expressions,

$$p_1 = (a_{11} + a_{22})(b_{11} + b_{22}), \quad p_2 = (a_{21} + a_{22})b_{11}, \quad p_3 = a_{11}(b_{12} - b_{22}), \quad p_4 = a_{22}(b_{21} - b_{11}),$$

$$p_5 = (a_{11} + a_{12})b_{22}, \quad p_6 = (a_{21} - a_{11})(b_{11} + b_{12}), \quad p_7 = (a_{12} - a_{22})(b_{21} + b_{22}),$$

$$c_{11} = p_1 + p_4 + p_7 - p_5, \quad c_{12} = p_3 + p_5, \quad c_{21} = p_2 + p_4, \quad c_{22} = p_1 + p_3 + p_6 - p_2.$$  

The algorithm uses 7 scalar multiplications and 18 scalar additions and subtractions versus 8 scalar multiplications and 4 scalar additions of the straightforward MM. The trade-off seems to favor the straightforward algorithm, but let all matrix entries be $2 \times 2$ matrices, reapply the algorithm for all 7 auxiliary $2 \times 2$ MMs, and arrive at $4 \times 4$ MM using 49 scalar multiplications versus 64 in the straightforward MM. Then replace the input entries of the new algorithm by $2 \times 2$ matrices and reapply the algorithm of Example 2.2 to all 49 auxiliary $2 \times 2$ MMs. Perform $d$ such recursive steps and arrive at $n \times n$ MM that uses $7^d = n^{\log_2(7)}$ scalar multiplications for $n = 2^d$ versus $8^d = n^3$ in the straightforward algorithm. Addition or subtraction of $s \times s$ matrix involves just $s^2$ scalar additions or subtractions, and the recursive algorithm based on Example 2.2 performs MM($n$) for $n = 2^d$ by applying $6 \cdot 7^d - 6 \cdot 4^d < 6n^{\log_2(7)}$ scalar additions and subtractions. This is an example of the large class of divide and conquer algorithms, which recursively reduce an original computational problem to those of smaller size. Such algorithms have been extensively used in various areas of computing, fast Fourier transform being a notable example (see Appendix B).

One can decrease the arithmetic cost to $5 \cdot 7^d - 5 \cdot 4^d < 5n^{\log_2(7)}$ by applying similar recursive divide and conquer process based on the algorithm of the following example, which performs $2 \times 2$ MM by using 7 scalar multiplications and 15 scalar additions and subtractions.

**Example 2.3.** Winograd’s $2 \times 2$ MM (cf. [74], pages 45-46).

Compute a $2 \times 2$ matrix product $C = AB$ of (2.3) by using the following expressions,

$$s_1 = a_{21} + a_{22}, \quad s_2 = s_1 - a_{11}, \quad s_3 = a_{11} - a_{21}, \quad s_4 = a_{12} - s_2,$$

$$s_5 = b_{12} - b_{11}, \quad s_6 = b_{22} - s_5, \quad s_7 = b_{22} - b_{12}, \quad s_8 = s_6 - b_{21},$$

$$p_1 = s_2s_6, \quad p_2 = a_{11}b_{11}, \quad p_3 = a_{12}b_{21},$$

$$p_4 = s_3s_7, \quad p_5 = s_1s_5, \quad p_6 = s_4b_{22}, \quad p_7 = a_{22}s_8,$$

$$t_1 = p_1 + p_2, \quad t_2 = t_1 + p_4, \quad t_3 = t_1 + p_5,$$

$$c_{11} = p_2 + p_3, \quad c_{12} = t_3 + p_6, \quad c_{21} = t_2 - p_7, \quad c_{22} = t_2 + p_5.$$  

With some additional care one can perform $n \times n$ MM for any $n$ by using $c \cdot n^{\log_2(7)}$ arithmetic operations for $c \approx 4.54$ and $c \approx 3.92$ based on Examples 2.2 and 2.3 respectively (see [74]).

### 2.3 Impact of the acceleration of MM

The news that the cubic arithmetic time of the straightforward MM is not a barrier anymore flew many times around the globe as a scientific sensation of the year of 1969, and the scientists expected that very soon the classical exponent 3 will be decreased to its information lower bound 2, that is, that very soon $n \times n$ MM will be performed in quadratic or nearly quadratic time, required already in order to access the $2n^2$ input entries as well as in order to output $n^2$ entries of the matrix. Of course this decrease was a most exciting perspective – according to [158] page 248 “such a development would trigger the greatest upheaval in the history of numerical computations.”
The scientists in many fields were excited because it was known that the acceleration of MM can be readily extended to a variety of popular and long-studied problems of Linear Algebra and Computer Science, linked to MM and sharing with MM the exponent of their complexity estimates. Their list includes Boolean MM, computation of paths and distances in graphs, parsing context-free grammars, the solution of a nonsingular linear system of equations, computations of the inverse, determinant, characteristic and minimal polynomials, and various factorizations of a matrix (see Sections 6.3–6.6, [59, 60, 102, 172, 173, 174, 89, 32, 34, 4, 66, 171, 145, 103, and 87]. In particular Strassen’s acceleration of MM in [151] implied the decrease of the known complexity exponent 3 of the cubic solution time to $\log_2(7) \approx 2.8074$ for all these computational problems.

The challenge brought MM and the complexity of algebraic computations to the limelight and motivated tremendous effort of numerous researchers around the globe, who competed for breaking the record of [151]. As Donald E. Knuth recalls, this “was not only a famous unsolved problem for many years, it also was worked on by all of the leading researchers in the field, worldwide.”

## 3 Bilinear Computational Problems and Bilinear Algorithms

A natural framework for their effort was the class of noncommutative bilinear algorithms, also called just *bilinear algorithms*. Such an algorithm solves a *bilinear computational problem* of the evaluation of a set of bilinear forms $c_h$, $h = 1, \ldots, \gamma$, in two sets of variables. MM($k, m, n$) is a special case of this problem where one evaluates a set of $kn$ bilinear forms $c_{i,h}$ such that $C = (c_{i,h})_{i=1}^{k}h=1^{n}$ = $AB$ for a pair of input matrices $A = (a_{i,j})_{i,j=1}^{k,m}$ and $B = (b_{j,h})_{j,h=1}^{m,n}$, but it is more convenient to study the general case first.

Let these sets fill two vectors $a = (a_i)_{i=1}^{\alpha}$ and $b = (b_j)_{j=1}^{\beta}$, let $T = (t_{i,j,h})_{i,j,h=1}^{\alpha,\beta,\gamma}$ denote the $\beta$-dimensional tensor filled with constants $t_{i,j,h}$ from a fixed ring (e.g., integers, rational, real, or complex numbers, or matrices filled with such numbers), and represent that problem as follows,

$$c_h(a, b) = \sum_{i,j=1}^{\alpha,\beta} t_{i,j,h} a_i b_j, \quad h = 1, \ldots, \gamma. \tag{3.1}$$

A bilinear algorithm $\mathbb{B}A$ solves this problem of size $(\alpha, \beta, \gamma)$ by successively computing

(i) two sets of $2r$ linear forms, $l_q = l_q(a)$ and $l_q' = l_q'(b)$, $q = 1, \ldots, r$, in the variables $a_1, \ldots, a_\alpha$ and $b_1, \ldots, b_\beta$, which are the coordinates of the vectors $a = (a_i)_{i=1}^{\alpha}$ and $b = (b_j)_{j=1}^{\beta}$, respectively,

(ii) $r$ pairwise products $l_1 l_1', \ldots, l_r l_r'$, and

(iii) the $\gamma$ bilinear forms $c_1(a, b), \ldots, c_\gamma(a, b)$ as $\gamma$ linear combinations of these products.

The straightforward VV and MV and the algorithms of Examples 2.2 and 2.3 for $2 \times 2$ MM are examples of bilinear algorithms for MM. At the end of this section we recall two other bilinear algorithms for two important and popular bilinear problems (see Examples 3.1 and 3.4).

The number $r$ of bilinear multiplications at stage (ii) is called the *rank* of the algorithm.

The constant coefficients in parts (i), (ii), and (iii) form three matrices $U = (u_i^{(q)})_{i,q=1}^{\alpha, r}$, $V = (v_j^{(q)})_{j,q=1}^{\beta, r}$, and $W = (w_h^{(q)})_{h,q=1}^{\gamma, r}$, such that

$$c_h = \sum_{q=1}^{r} w_h^{(q)} l_q l_q', \quad l_q = \sum_{i=1}^{\alpha} u_i^{(q)} a_i, \quad l_q' = \sum_{j=1}^{\beta} v_j^{(q)} b_j, \quad h = 1, \ldots, \gamma, \quad \text{and} \quad q = 1, \ldots, r. \tag{3.2}$$

The *rank of a bilinear computational problem* is the minimal rank of bilinear algorithms for that problem. It depends on the field of constants, e.g., can be different for real and complex constants.

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1. The algorithm designers try hard to reduce various problems of modern numerical and symbolic computations to MM, with no considerable overhead, because MM, and even the straightforward cubic time MM, has been very efficiently implemented on modern computers.
A bilinear algorithm $\mathbb{BA}$ above performs

(i) $r$ bilinear multiplications of $l_q$ by $l'_q$ for $q = 1, \ldots, r$;

(ii) $(\alpha + \beta + \gamma)r$ multiplications by scalars $a_i^{(q)}$, $v_j^{(q)}$, and $w_h^{(q)}$, for $i = 1, \ldots, \alpha$; $j = 1, \ldots, \beta$; $h = 1, \ldots, \gamma$, and $q = 1, \ldots, r$;

(iii) $(\alpha - 1)r$ additions of scaled variables $a_i$, $i = 1, \ldots, \alpha$; $(\beta - 1)r$ additions of scaled variables $b_j$, $j = 1, \ldots, \beta$, and $(r - 1)\gamma$ additions of scaled bilinear products $l_q l_q'$, $q = 1, \ldots, r$.

These upper estimates decrease in the case of sparse matrices $U$, $V$, and $W$. Let $\text{nnz}(M)$ and $n_s(M)$ denote the numbers of entries of a matrix $M$ that are nonzero and are neither of 0, 1, and $-1$, respectively. Then the above bilinear algorithm performs at most $\text{nnz}(M) + n_s(V) + n_s(W)$ scalar multiplications and at most $(\text{nnz}(M) - r) + (\text{nnz}(V) - r) + (\text{nnz}(W) - \gamma)$ scalar additions and subtractions.

Fast bilinear algorithms for the following two bilinear problems enable fast practical complex and polynomial MM, respectively (see Section 16.1).

**Example 3.1.** Multiplication of two complex numbers. Evaluate the two bilinear forms $a_1 b_1 - a_2 b_2$ and $a_1 b_2 + a_2 b_1$, which represent the real and imaginary parts, respectively, of the product of two complex numbers $a_1 + ib_1$ and $a_2 + ib_2$. The straightforward bilinear algorithm for this problem has rank 4, but here is a rank-3 algorithm:

$$l_1 l'_1 = a_1 b_1,\quad l_2 l'_2 = a_2 b_2,\quad l_3 l'_3 = (a_1 + a_2)(b_1 + b_2),\quad a_1 b_2 - a_2 b_1 = l'_1 l_2 - l_2 l'_1.$$ 

**Example 3.2.** Convolution. Compute the coefficients of the product $c(x) = \sum_{h=0}^{m+n} c_h x^h$ of two polynomials $a(x) = \sum_{i=0}^{m} a_i x^i$ and $b(x) = \sum_{j=0}^{n} b_j x^j$ or, equivalently, the convolution of the coefficients vectors of these two polynomials, $c_h = \sum_{q=0}^{k} a_q b_{h-q}$, for $h = 0, \ldots, m + n$, where $a_i = b_j = 0$ for $i > m$ and $j > n$. The straightforward algorithm solves this problem by applying $(m + 1)(n + 1)$ scalar multiplications and $(m + 1)(n + 1) - m - n - 1$ scalar additions, but FFT-based bilinear algorithm uses just $O((m + n) \log(m + n))$ arithmetic operations (see Appendix A).

We refer the reader to [78], [72], [117], [95], [85], [153], [43], [37], and [88] for the early study of bilinear algorithms and to [31] Section 2.5] for its concise exposition, and to [117] part 3 of Theorem 1], [141], and [111] part 3 of Theorem 0.1] for some results on the reduction from non-bilinear MM to bilinear MM.

## 4 Tensor Representation of Bilinear Algorithms and Tensor Product

Observe that a bilinear algorithm $\mathbb{BA}$ of rank $r$ of Section 3 can be equivalently represented as a rank-$r$ decomposition of the tensor $T = (t_{i,j,h})_{i,j,h}$:

$$t_{i,j,h} = \sum_{q=1}^{r} u_i^{(q)} v_j^{(q)} w_h^{(q)} \text{ for } i = 1, \ldots, m; \ j = 1, \ldots, n; \ h = 1, \ldots, n.$$  \hspace{1cm} (4.1)

This implies that the rank of a bilinear computational problem is precisely the rank of its tensor.

Now suppose that two tensors $T = (t_{i,j,h})_{i,j,h=1}^{\alpha,\beta,\gamma}$ and $T' = (t'_{i',j',h'})_{i',j',h'=1}^{\alpha',\beta',\gamma'}$ define two sets of bilinear forms of the sizes $(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$, respectively, and define another set of bilinear forms of the size $(\alpha\alpha', \beta\beta', \gamma\gamma')$ by the tensor product

$$T \otimes T' = (t_{i,j',h,h'})_{i,j,h,h'=1}^{k,m,n,n,k}.$$ \hspace{1cm} (4.2)

**Theorem 4.1.** Given two tensors $T = (t_{i,j,h})_{i,j,h=1}^{\alpha,\beta,\gamma}$ of rank $r$ and $T' = (t'_{i',j',h'})_{i',j',h'=1}^{k,m,n,n,k}$ of rank $r'$, such that $t_{i,j,h} = \sum_{q=1}^{r} u_i^{(q)} v_j^{(q)} w_h^{(q)}$ for all $i$, $j$, and $k$ and $t'_{i',j',h'} = \sum_{q'=1}^{r'} u_i^{(q')} v_j^{(q')} w_h^{(q')}$ for all $i'$, $j'$, and $k'$, the tensor product $T \otimes T' = (t_{i',j',h,h'})_{i',j',h,h'=1}^{k,m,n,n,k}$ has rank at most $rr'$. 

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Proof. Decompose the tensor $T \otimes T'$ by using the equations

$$t_{i,i',j,j',h,h'} = \left( \sum_{q=1}^{r} u_{i,j}^{(q)} v_{j,j'}^{(q)} w_{h,h'}^{(q)} \right) \left( \sum_{q'=1}^{r'} u_{i',j'}^{(q')} v_{j',j}^{(q')} u_{h,h'}^{(q')} \right)$$

where $u_{i,i'} = u_{i}^{(q)} u_{i'}^{(q')}$, $v_{j,j'} = v_{j}^{(q)} v_{j'}^{(q')}$, $w_{h,h'} = w_{h}^{(q)} w_{h'}^{(q')}$ for all 6-tuples $(i, i', j, j', h, h')$.  

5. Bilinear MM and the Associated Tensors

The tensor $T$ associated with the problem MM($k, m, n$) has entries with subscripts represented by three pairs of integers $(i, i')$, $(j, j')$, and $(h, h')$, rather than by three integers $i$, $j$, and $h$:  

$$T = (t_{i,i'},(j,j'),(h,h'))^k,m,n,n,k_{i,i'},j,j',h,h'=1, t_{i,i'},(j,j'),(h,h') = \delta_{i',j} \delta_{j',h} \delta_{h',i}$$

for all $i, i', j, j', h, h'$.  

Here and hereafter

$$\delta_{q,q} = 1, \delta_{q,s} = 0 \text{ if } q \neq s. \quad (5.2)$$

We can represent a bilinear algorithm of rank $r$ for the computation of the matrix product $C = AB$ by the following equations:

$$c_{i,h} = \sum_{j} a_{i,j} b_{j,h} = \sum_{q=1}^{r} u_{i,j}^{(q)} l_{q} l'_{q} \text{ for } i = 1, \ldots, k; h = 1, \ldots, n. \quad (5.3)$$

Here $l_{q}$ and $l'_{q}$ are linear forms in the entries of the matrices $A$ and $B$ (cf. (3.24)),

$$l_{q} = l_{q}(A) = \sum_{i,j=1}^{k,m} u_{i,j}^{(q)} a_{i,j} \text{ and } l'_{q} = l'_{q}(B) = \sum_{j,h=1}^{m,n} v_{j,h}^{(q)} b_{j,h} \text{, } q = 1, \ldots, r, \quad (5.4)$$

and the algorithm is defined by a triple of 3-dimensional tensors,

$$U = \left( u_{i,j}^{(q)} \right)_{i,j=1}^{k,m,r}, V = \left( v_{j,j'}^{(q)} \right)_{j,j'=1}^{m,n,r}, W = \left( w_{h,h'}^{(q)} \right)_{h,h'=1}^{n,k,r}. \quad (5.5)$$

We can rewrite the above expressions removing the links among the subscripts:

$$U = \left( u_{i,i'}^{(q)} \right)_{i,i'=1}^{k,m,r}, V = \left( v_{j,j'}^{(q)} \right)_{j,j'=1}^{m,n,r}, W = \left( w_{h,h'}^{(q)} \right)_{h,h'=1}^{n,k,r}. \quad (5.6)$$

Then simultaneous equations (5.3) and (5.4) can be equivalently rewritten as follows (cf. (35)),

$$\sum_{q=1}^{r} u_{i,i'}^{(q)} v_{j,j'}^{(q)} w_{h,h'}^{(q)} = \delta_{i,j} \delta_{j',h} \delta_{h',i} \text{ for } i, i' = 1, \ldots, k; j, j' = 1, \ldots, m; h, h' = 1, \ldots, n, \quad (5.7)$$

We can rewrite the tensor $T$ as 3-dimensional tensor by replacing every pair of subscripts by a single index, namely, $(i, i')$ by $\bar{i} = i + mi$ for $i = 1, \ldots, k$, $(j, j')$ by $\bar{j} = j + mj$ for $j = 1, \ldots, m$, and $(h, h')$ by $\bar{h} = h + kh$ for $h = 1, \ldots, n$, so that

$$t_{i,i',(j,j'),(h,h')} = t_{\bar{i},\bar{i}',\bar{j},\bar{j}',\bar{h}} \text{ for } \bar{i} = 1, \ldots, km; \bar{j} = 1, \ldots, mn; \bar{h} = 1, \ldots, nk. \quad (5.8)$$

We can similarly write

$$\sum_{q=1}^{r} u_{i,i'}^{(q)} v_{j,j'}^{(q)} w_{h,h'}^{(q)} = t_{\bar{i},\bar{i}',\bar{h}} \text{ for } \bar{i} = 1, \ldots, km; \bar{j} = 1, \ldots, mn; \bar{h} = 1, \ldots, nk. \quad (5.9)$$
6 Recursive Bilinear Algorithms for MM. Exponents of MM

The tensor product construction of equation \( \text{(4.12)} \) provides useful insight into recursive algorithm for MM. Given the problem MM\((k, m, n)\) of the computation of matrix product \( C = AB \), we can fix a triple of positive integers \( (k', m', n') \) substitute matrices of sizes \( k' \times m', m' \times n' \), and \( k' \times n' \) for the entries of the matrices \( A \), \( B \), and \( C \), respectively, and arrive at the problem MM\((kk', mm', nn')\).

Equivalently we can define this problem by its tensor, which is the product \( T \otimes T' \) associated with the two problems MM\((k, m, n)\) and MM\((k', m', n')\), respectively. Apply Theorem 4.1 and obtain

\[
\text{rank}(\text{MM}(kk', mm', nn')) \leq \text{rank}(\text{MM}(k, m, n)) \cdot \text{rank}(\text{MM}(k', m', n')).
\]

(6.1)

By recursively applying inequality \( \text{(6.1)} \) for \( k = k' = m = m' = n = n' = 2^i \), for \( i = 1, 2, \ldots \) we can bound the ranks in recursive extensions of the algorithms of Examples 2.2 and 2.3 for \( 2 \times 2 \) MM.

Next we generalize the recursive processes based on Examples 2.2 and 2.3 – we define recursive bilinear algorithms based on any bilinear algorithm for MM\((n)\) of a fixed \( n \).

**Theorem 6.1.** Given a bilinear algorithm of rank \( r \) for \( n \times n \) MM, one can perform \( K \times K \) MM for all \( K \) by using at most \( c \cdot K^{\omega_{n,r}} \) scalar arithmetic operations for a fixed \( c \) independent of \( K \) and for the exponent \( \omega_{n,r} = \log_n(r) \).

**Proof.** Substitute \( n \times n \) matrices for variables, re-apply the algorithm recursively, and in \( d \) steps, for any \( d \), arrive at a bilinear algorithm of rank \( r^d \) for \( n^d \times n^d \) MM. Then recall that a linear operation of multiplication of a \( q \times q \) matrix by a scalar as well as an addition or subtraction of a pair of \( q \times q \) matrices can be performed in \( q^2 \) scalar arithmetic operations and deduce that the arithmetic cost of performing all linear operations involved in the algorithm stays within the claimed bound. \( \square \)

By minimizing \( \omega_{n,r} = \log_n(r) \) over the ranks \( r \) of all bilinear algorithms for \( n \times n \) MM define

\[
\omega_n = \min_r \omega_{n,r}.
\]

(6.2)

Then, by minimizing \( \omega_{n,r} \) over all integers \( n \) not exceeding a fixed integer \( K \), define the exponent

\[
\omega_{\leq K} = \min_{n \leq K} \omega_n.
\]

(6.3)

For \( K = \infty \) obtain the universal or theoretical exponent of MM

\[
\omega = \inf_{n \leq \infty} \omega_n.
\]

(6.4)

Here and hereafter (except for Section 18) we consider MM over the fields of real and complex numbers, but the presented algorithms can be defined over other fields and rings as well and in some cases (to a more limited extent) over semirings (see [60], [171], [103], and the bibliography therein). Over the fields the theoretical exponent \( \omega \) only depends on the field characteristic [146] Theorem 2.8], while the hidden overhead constants can vary greatly even over the fields having the same characteristic.

7 To the exponent 2.78 by means of trilinear aggregation

Breaking Strassen’s barrier of \( \log_2(7) \approx 2.8074 \) for \( \omega \) was considered to be almost in hands in 1969, but this goal of “literally all the leading researchers in the field, worldwide” has remained a dream for almost a decade.

If one could build a recursive process on the algorithm of Example 2.1, then the dream would have come true even well before Strassen’s discovery of [151]. Indeed the algorithm of this example has rank \( r = n^3/2 + n^2 \) for any even \( n \), e.g., has rank \( r = 144 \) for \( n = 6 \). Substitute these data into the equation \( \omega = \log_6(r) \) and obtain

\[
\omega \leq \log_6(144) \approx 2.7737 < \log_2(7) \approx 2.8074.
\]
Theorem 6.1 however, cannot be applied here because MM is not commutative, and so the substitution of matrices for the variables \( a_i \) and \( b_j \) would have violated the basic identities of Example 2.1. For example, we cannot apply the equation \( v_{22}a_{22} = u_{22}v_{22} \) if \( u_{22} \) and \( v_{22} \) are matrices, e.g., if \( u_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( v_{22} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{matrix} \) because

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The algorithm of Example 2.1 based on (2.2), belongs to the class of *commutative bilinear or quadratic* algorithms. They only differ from non-commutative bilinear algorithms at the stage of nonlinear multiplications, but [83, 84, 38] proved that 7 is the sharp lower bound on the rank of 2 MM by using six bilinear multiplications, but [83, 84, 38] proved that 7 is the sharp lower bound on the rank of 2 MM.

Actually 15 is the sharp lower bound on the number of additions and subtractions in all bilinear algorithms of rank 7 for \( 2 \times 2 \) MM (cf. [140, 40]), and moreover the following theorem defines explicit expressions for all bilinear algorithms of rank 7 for \( 2 \times 2 \) MM appeared in [117, Theorem 3] (see also [58] and [133, Theorem 0.3]).

Two bilinear algorithms, both of rank \( r \), for the same problem of \( \text{MM}(k, m, n) \) (see (5.5)), defined by two triples \( \{U, V, W\} \) and \( \{U, \hat{V}, \hat{W}\} \), respectively, are said to be *equivalent* to one another if

\[
\pi_{i,j}^{(q)} = \sum_{\nu, \kappa} \sigma_{i,\nu} \nabla_{\kappa, \nu} u_{i,j}^{(t(q))}, \quad \pi_{g,h}^{(q)} = \sum_{\nu, \kappa} \lambda_{\nu, g} \mu_{h, \kappa} v_{i,j}^{(t(q))}, \quad \text{and} \quad \pi_{l,q}^{(q)} = \sum_{\nu, \kappa} \gamma_{\nu, l} \beta_{\kappa, q} w_{i,j}^{(t(q))},
\]

where the matrices in the three pairs \((\sigma_{i,\nu}, \nu, \kappa)\) and \((\gamma_{\nu, l}, \kappa, q)\) and \((\lambda_{\nu, g}, \mu_{h, \kappa})\) are inverses of one another; \( 1 \leq t(s) \leq r; t_{q_1} \neq t_{q_2} \) if \( q_1 \neq q_2 \), and all \( t(q) \) are integers.

**Theorem 7.1.** *Every bilinear algorithm of rank 7 for \( 2 \times 2 \) MM is equivalent to the algorithms of Examples 2.2 and 2.3.*

We refer the reader to the paper [108] for the current record lower bounds on the rank of \( \text{MM}(n) \) for all \( n \), to the papers [83, 84, 117, Theorem 1], [36, 38, 39, 26, 27, 142, 147], and [101] for some earlier work in this direction, and to the papers [64] and [139] for various lower and upper bounds on the arithmetic complexity and the ranks of rectangular MM of smaller sizes.

Since the study of MM(2) could not help decrease the exponent \( \log_2(7) \), the researchers tried to devise a bilinear algorithm of rank 21 for MM(3) because \( \log_3(21) < \log_2(7) \approx 2.8074 \). This turned out to be hard, and we still cannot perform \( 3 \times 3 \) MM by using less than 23 bilinear multiplications.

The exponent \( \log_2(7) \approx 2.8074 \) was decreased only in 1978, when the paper [118] presented a bilinear algorithm of rank 143,640 for \( 70 \times 70 \) MM. This implied the exponent \( \omega = \log_2(143,640) < 2.7962 \) for \( \text{MM}(n) \) and consequently for \( n \times n \) matrix inversion, Boolean \( \text{MM}(n) \), and various other well-known computational problems linked to MM and partly listed in Section 2.3.

The paper [118] has extended some novel techniques of the paper [117] of 1972, published in Russian and translated into English only in 2014 in [131]. Namely the paper [117] has accelerated the straightforward MM by combining *trilinear interpretation* of bilinear algorithms and the *aggregation* method. By following [118] we call this combination *trilinear aggregation* and briefly cover it in the next two sections. By refining trilinear aggregation of [117] the papers [118, 119, 121, 122, and 123] proposed various algorithms that further accelerated MM. In particular the paper [123] yielded the exponent

\[
\omega_{44} \leq 2.7734,
\]

and this still remains the record exponent \( \omega_{\leq K} \) for feasible MM.

The technique of trilinear aggregation has been recognized for its impact on the decrease of the MM exponent, but the paper [117] was also a landmark in the study of multilinear and tensor

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2 Until 1976 the author of [116] and [117] lived in the Soviet Union. From 1964 to 1976 he has been working in Economics in order to make his living and has written the papers [116] and [117] in his spare time.
decompositions. Such decompositions introduced by Hitchcock in 1927 received little attention except for a minor response in 1963–70 with half of a dozen papers in the psychometrics literature. The paper [117] of 1972 provided the earliest known application of nontrivial multilinear and tensor decomposition to fundamental matrix computations, now a popular flourishing area in linear and multilinear algebra with a wide range of important applications to modern computing (see [159], [96], [113], [78], and the bibliography therein).

8 Trilinear Representation and Dual Bilinear Algorithms

Trilinear representation of a bilinear algorithm enables transparent demonstration of the technique of trilinear aggregation. Otherwise it is equivalent and quite similar to its tensor representation.

Let a bilinear algorithm of rank $r$ be represented by equations (3.2). Multiply them by new variables $d_h$, sum the products in $h$, and arrive at the following representation of the algorithm as a decomposition of a trilinear form,

$$
\sum_{i,j,h=1}^{k,m,n} l_{i,j,h} a_i b_j d_h = \sum_{s=1}^{r} l_q(a) l'_q(b) l''_q(d)
$$

(8.1)

for $l_q = l_q(a)$ and $l'_q = l'_q(b)$ of (3.2), $l''_q = l''_q(d) = \sum_{h=1}^{n} w_h^{(q)} d_h$, and $q = 1, \ldots, r$. By equating the coefficients of the variables $d_h$ on both sides of this trilinear decomposition we come back to the original bilinear representation (3.2) of the same algorithm, and we can obtain its two alternative dual bilinear representations by equating the coefficients of the variables $a_i$ and $b_j$ instead.

Here is a simple example of the trilinear representation of the bilinear algorithm of Example 3.1.

**Example 8.1.** A trilinear decomposition of rank 3 for multiplication of two complex numbers.

$$
a_1 b_1 d_1 - a_2 b_2 d_1 + a_1 b_2 d_2 + a_2 b_1 d_2 = a_1 b_1 (d_1 - d_2) - a_2 b_2 (d_1 + d_2) + (a_1 + a_2)(b_1 + b_2) d_2.
$$

By equating the coefficients of the variables $d_h$ on both sides we come back to the bilinear algorithm of Example 3.1. By equating the coefficients of $a_1$ and $a_2$ on both sides of this equations or alternatively the coefficients of $b_1$ and $b_2$ on their both sides, we arrive at two alternative bilinear algorithms of rank 3 for computing the product of two complex numbers. They are close to one another but not to the algorithm of Example 3.1. We display just one of the two.

**Example 8.2.** A distinct trilinear decomposition of rank 3 for multiplying two complex numbers.

$$
l_1 = b_1, \quad l'_1 = d_1 - d_2,
$$

$$
l_2 = b_2, \quad l'_2 = d_1 + d_2,
$$

$$
l_3 = b_1 + b_2, \quad l'_3 = d_2,
$$

$$
b_1 d_1 + b_2 d_2 = l_1 l'_1 + l_3 l'_3,
$$

$$
b_1 d_2 - b_2 d_1 = l_3 l'_3 - l_2 l'_2.
$$

The book [167] demonstrates the power of duality technique in devising some efficient bilinear algorithms for FIR-filters and multiplication of complex numbers and polynomials.

In the rest of this section we apply and extend the above discussion to the special case of the algorithms for the problem MM($k,m,n$) of multiplying two matrices $A = (a_{i,j})_{i,j=1}^{k,m}$ and $B = (b_{j,h})_{j,h=1}^{m,n}$. We can represent such an algorithm by means of the following trilinear decomposition,

$$
\text{Trace}(ABD) = \sum_{i,j,h} a_{i,j} b_{j,h} d_{h,i} = \sum_{s=1}^{r} l_q(A) l'_q(B) l''_q(D) \quad \text{for} \quad l'_q(A) = \sum_{q=1}^{m,n} w_h^{(q)} d_{h,i},
$$

(8.2)

$l_q$ and $l'_q$ of (3.2), and $q = 1, \ldots, r$. Here $D = (d_{h,i})_{h,i=1}^{n,k}$ is an auxiliary $n \times k$ matrix and $\text{Trace}(M) = \sum_{i} m_{i,i}$ denotes the trace of a matrix $M = (m_{i,j})_{i,j}$. 

10
Example 8.3. A trilinear version of Strassen’s bilinear algorithm of Example 2.2 for $MM(2)$.

$$\sum_{i,j,h=1}^{2} a_{i,j}b_{j,h}d_{h,i} = \sum_{s=1}^{7} l_{i,j}^{s}l_{j,h}^{s}l_{h,i}^{s} = (a_{11} + a_{22})(b_{11} + b_{22})(d_{11} + d_{22}),$$

$$l_{3}^{11}l_{5}^{11} = (a_{21} + a_{22})b_{11}(d_{21} - d_{22}), l_{3}^{12}l_{5}^{12} = a_{11}(b_{12} - b_{22})(d_{12} + d_{22}), l_{4}^{11}l_{6}^{11} = (a_{21} - a_{11})(b_{11} + b_{22}),$$

$$l_{5}^{11}l_{6}^{12} = (a_{11} + a_{12})b_{22}(d_{12} - d_{11}), l_{5}^{12}l_{6}^{12} = (a_{11} - a_{22})(b_{12} - b_{22})(d_{12} - d_{11}).$$

We can also come back to the original bilinear algorithm of Example 2.2 for the 2 × 2 matrix product $AB$ by equating the coefficients of the variables $d_{h,i}$ on both sides of a trilinear decomposition. By equating the coefficients of the variables $a_{i,j}$ also on both sides or alternatively of $b_{j,h}$ on both sides, we can obtain two dual bilinear algorithms. In this case the three dual algorithms differ little from each other, but let us be given a bilinear algorithm of a rank $r$ we can obtain two dual bilinear algorithms. In this case the three dual algorithms differ little from each other, but let us be given a bilinear algorithm of a rank $r$ for rectangular $MM(k, m, n)$. Then we arrive at the dual algorithms of rank $r$ for the problems $MM(n, m, k)$, and $MM(k, n, m)$ as well (cf. [117] part 5 of Theorem 1), [36], [85], [139]). A bilinear algorithm for $MM(k, m, n)$ can be readily extended to three other dual algorithms of the same rank for the three other problems $MM(m, k, n)$, $MM(n, k, m)$, and $MM(k, n, m)$ because we can interchange the subscripts of the variables.

The number of linear operations (unlike the rank) can differ in the transition among the three dual algorithms, and this can possibly be exploited for minimizing this number.

Here is a useful combination of duality with tensor product construction. Given a trilinear decomposition of the trilinear form Trace($ABC$), one can perform $MM(K)$ by using $cK^\omega$ arithmetic operations for any $K$, for $\omega = \omega_{k,m,n,r} = 3\log_{k,m,n}(r)$, and for a constant $c$ independent of $K$.

9 Trilinear Aggregation and Disjoint MM

Aggregation technique is well-known in business, economics, computer science, telecommunication, natural sciences, medicine, and statistics. The idea is to mass together or cluster numerous independent but similar units into much fewer aggregates. Then the study is simplified but is supposed to characterize all the units either directly or via disaggregation techniques. Such aggregation/disaggregation processes in [109] served as a springboard for the emergence of the field of Algebraic Multigrid, now quite popular.

Aggregation/disaggregation techniques are behind the acceleration of MM in Example 2.1 preceded by similar applications of this technique to polynomial evaluation with preprocessing of coefficients [116], [24]. The papers [117] and [118] apply aggregation in order to compress the decomposition of the trilinear form Trace($ABC$) by playing with the shared subscripts of distinct variables. Other implementations of this technique appeared in [119], [120], [122], [123], and [100].

For demonstration of these techniques, consider disjoint $MM$ of computing two independent matrix products $AB$ and $UV$ represented by the trilinear form

$$\text{Trace}(ABD + UVW) = \sum_{i,j,h=1}^{k,m,n} (a_{i,j}b_{j,h}d_{h,i} + u_{j,h}w_{h,i}) .$$

For $k = m = n$, we would seek a pair of disjoint $n \times n$ matrix products, which is quite a realistic task in computational practice.

For each triple $i,j,h$ define the aggregate $(a_{i,j} + u_{j,h})(b_{j,h} + v_{h,i})(d_{h,i} + w_{i,j})$ of two monomials $a_{i,j}b_{j,h}d_{h,i}$ and $u_{j,h}v_{h,i}w_{i,j}$ and let

$$T = \sum_{i,j,h=1}^{k,m,n} (a_{i,j} + u_{j,h})(b_{j,h} + v_{h,i})(d_{h,i} + w_{i,j})$$

Theorem 8.1. Given a bilinear or trilinear algorithm of rank $r$ for $MM(k, m, n)$ and any 4-tuple of integers $k$, $m$, $n$, and $r$ such that $k,m,n > 1$, one can perform $MM(K)$ by using $cK^\omega$ arithmetic operations for any $K$, for $\omega = \omega_{k,m,n,r} = 3\log_{k,m,n}(r)$, and for a constant $c$ independent of $K$.
denote the sum of the $kmn$ aggregates. Let

$$T_1 = \sum_{i,j=1}^{k,m} a_{i,j}s_{i,j}w_{i,j}, \ T_2 = \sum_{j,h=1}^{m,n} u_{j,h}b_{j,h}r_{j,h}, \text{ and } T_3 = \sum_{h,i=1}^{n,k} q_{i,h}v_{h,i}d_{h,i}$$

denote three groups of correction terms where

$$q_{i,h} = \sum_{j=1}^{m} (a_{i,j} + u_{j,h}), \ s_{i,j} = \sum_{h=1}^{n} (b_{j,h} + v_{h,i}), \text{ and } r_{j,h} = \sum_{i=1}^{k} (d_{h,i} + w_{i,j}).$$

Then the equation

$$\text{Trace}(ABD + UVW) = T - T_1 - T_2 - T_3$$

(9.1)

defines a trilinear decomposition of rank $mnp + mn + np + pm$ (versus the rank $2mnp$ of the straightforward algorithm). Table 9.1 displays this decomposition in compressed form.

Table 9.1: Aggregation/disaggregation of a pair of MM terms.

```
| a_{i,j} | y_{j,h} | d_{h,i} |
|---------|---------|---------|
| u_{j,h} | v_{h,i} | w_{i,j} |
```

Sum the two entries in each column of the table, multiply the three products together, and obtain an aggregate. Multiply together the three entries in each row of the table and obtain the two output terms $a_{i,j}b_{j,h}d_{h,i}$ and $u_{j,h}v_{h,i}w_{i,j}$. The cross-products of other triples of the table define six correction terms. Their sum over all $n^3$ triples of indices $i, j, h$ has rank $2(km + mn + nk)$. By subtracting this sum from the sum of all $kmn$ aggregates, we decompose $2kmn$ terms of $\text{Trace}(ABD + UVW)$ into the sum of $kmn + 2(km + mn + nk)$ terms. For $m = n = p = 34$ this implies a decomposition of rank $n^3 + 6n^2$ for a pair of disjoint $MM(n)$, versus the rank $2n^3$ of the straightforward decomposition.

Demonstration of the power of trilinear aggregation can be made more transparent for disjoint $MM$, whose natural link to trilinear aggregation has been shown in [123, 126, Section 5], [124, Section 12], and [100]. The known constructions for pairs of disjoint $n \times n$ MM, however, have been extended to a single $MM(n)$ for even $n$. In particular the paper [417] presented a trilinear decomposition of rank $0.5n^3 + 3n^2$ for $MM(n)$ and any even $n$ similar to the above decomposition of $\text{Trace}(ABD + UVW)$. This implied the $MM$ exponent $\log_3(0.5n^3 + 3n^2)$, which is less than 2.85 for $n = 34$.

The paper [118] presented a trilinear decomposition of rank $(n^3 - 4n)/3 + 6n^2$ for $MM(n)$, $n = 2s$, and any positive integer $s$. For $n = 70$ this defines the $MM$ exponent 2.7962. Then again it is convenient to demonstrate this design for disjoint $MM$ associated with a decomposition of the trilinear form $\text{Trace}(XYZ + UVW + ABD)$. The basic step is the aggregation/disaggregation defined by Table 9.2.

Table 9.2: Aggregation/disaggregation of a triple of MM terms.

```
| x_{i,j} | b_{j,h} | z_{h,i} |
|---------|---------|---------|
| u_{j,h} | v_{h,i} | w_{i,j} |
| a_{h,i} | b_{i,j} | d_{j,h} |
```

Sum the $kmn$ aggregates

$$(x_{i,j} + u_{j,h} + a_{h,i})(y_{j,h} + v_{h,i} + b_{i,j})(z_{h,i} + w_{i,j} + d_{j,h}),$$

subtract order of $n^2$ correction terms, and obtain a decomposition of rank $n^3 + O(n^2)$ for

$$\text{Trace}(XYZ + UVW + ABD),$$
versus the rank $3n^3$ of the straightforward decomposition. The trace represents three disjoint problems of $MM(n)$, that is, the computation of the three independent $n \times n$ matrix products $XY$, $UV$, and $AB$, and we obtain a trilinear decomposition of rank $n^3 + O(n^2)$ for this task.

With a little more work one obtains a similar trilinear decomposition of $[[3]]$ of rank $(n^3 - 4n)/3 + 6n^2$, for any even $n$, and this implied the bound $\omega_{70} < 2.7962$. Refinements of this construction implied smaller upper bounds (see Tables 15.1 and 15.2). In particular the algorithm of [123] yielded the bound $\omega_{44} \leq 2.7734$ of (7.1).

### 10 Any Precision Approximation (APA) Algorithms

Based on the following table we arrive at the technique of Any Precision Approximation\(^3\) which is quite efficient for fast symbolic MM and for other symbolic algebraic computations.

Table 10.1: Aggregation/disaggregation of a pair of MM terms for Any Precision Approximation MM.

|   |   |   |
|---|---|---|
| $a_{i,j}$ | $b_{j,h}$ | $\lambda^2 d_{h,i}$ |
| $\lambda u_{j,h}$ | $\lambda v_{h,i}$ | $w_{i,j}$ |

For $\lambda = 1$ this table turns into Table 9.1 but for variable $\lambda$ helps us demonstrate the APA technique of [20], [24], and [21].

Let $\lambda \to 0$ and obtain trilinear decomposition

$$\text{Trace}(ABD + UVW) = \lambda^{-2}(S - T_1 - T_2 + O(\lambda)), \quad (10.1)$$

where

$$S = \sum_{i,j,h=1}^{k,m,n} (a_{i,j} + \lambda u_{j,h})(b_{j,h} + \lambda v_{h,i})(\lambda^2 d_{h,i} + w_{i,j}),$$

is the sum of $kmn$ aggregates,

$$T_1 = \sum_{i,j=1}^{k,m} a_{i,j}q_{i,j}w_{i,j}, \quad \text{and} \quad T_2 = \sum_{j,h=1}^{m,n} u_{j,h}b_{j,h}r_{j,h}$$

for

$$q_{i,j} = \sum_{h=1}^{n} (b_{j,h} + \lambda v_{h,i}) \quad \text{and} \quad r_{j,h} = \sum_{i=1}^{k} (\lambda^2 d_{h,i} + w_{i,j}).$$

The terms of order $\lambda$ vanish as $\lambda \to 0$. Counting only the remaining monomials on the right-hand side of (10.1), we define the border rank of the decomposition. It is equal to $kmn + km + kn$, versus the larger rank $kmn + km + kn + mn$ of decomposition 9.1.

Generally, given a trilinear form $T$ (e.g., given by the 3-dimensional tensor of its coefficients), multiply it by $\lambda^d$ for a fixed nonnegative integer $d$ and define a trilinear decomposition of the trilinear form $\lambda^d \cdot T$ with coefficients being polynomials in $\lambda$. Delete the terms of order $\lambda^{d+1}$ and higher and call the number of the remaining terms the border rank of the decomposition and of the associated APA algorithm. The minimal border rank over all such APA algorithms for a fixed trilinear form $\lambda^d \cdot T$ defines its border rank. All this can be readily restated for a set of bilinear forms replacing a single trilinear form $T$.

We can equate the coefficients of the variables $d_{h,i}$ and $w_{i,j}$ in the trilinear APA decomposition above and arrive at the bilinear problem of the evaluation of two disjoint matrix products $AB$ and

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\(^3\)Hereafter we use the acronym APA.
The $kmn$ trilinear aggregates turn into the $kmn$ bilinear products $(a_{i,j} + \lambda u_{j,h})(b_{j,h} + \lambda v_{h,i})$ for all $i$, $j$, and $h$. Clearly,
\[ a_{i,j} + \lambda u_{j,h} \to a_{i,j} \text{ and } b_{j,h} + \lambda v_{h,i} \to b_{j,h} \text{ as } \lambda \to 0, \]
but we must keep the terms $\lambda u_{j,h}$ and $\lambda v_{h,i}$ in the aggregates in order to compute the matrix $UV$.

This forces us to double the precision of the representation of the multiplicands $a_{i,j} + \lambda u_{j,h}$ and $b_{j,h} + \lambda v_{h,i}$ compared to the precision of the representation of the entries $a_{i,j}$, $u_{j,h}$, $b_{j,h}$, and $v_{h,i}$.

For example, suppose that $\lambda = 2^{-s}$ for a sufficiently large integer $s$ and that $a_{i,j}$, $u_{j,h}$, $b_{j,h}$, and $v_{h,i}$ are $s$-bit integers in the range $[0, 2^s)$. Then $2s$ bits are required in order to represent each of the multiplicands $a_{i,j} + \lambda u_{j,h}$ and $b_{j,h} + \lambda v_{h,i}$. If $s$ exceeds the half-length of the computer word, then using $2s$ bits in APA algorithms would move us beyond the length of the computer word, e.g., beyond the IEEE standard double precision.

This can be costly in numerical computations, but APA is valuable in symbolic computation where efficient techniques such as Chinese remainder algorithm and $p$-adic lifting facilitate computations with long numbers (cf. our Section 18).

The papers [20], [24], and [21] study border rank of MM and various other fundamental bilinear computational problems and show that border rank is quite frequently smaller than their rank.

\section{Bini’s Construction}

In the community of the Theory of Computing APA algorithms have been mostly and highly appreciated as a tool for decreasing the record upper bounds on the theoretical exponent $\omega$ of (6.4). Their power in this context is due to Bini’s theorem in [19], according to which APA decomposition of border rank $r$ and bilinear algorithms of rank $r$, both for the same bilinear problem, define the same upper estimate for the exponent $\omega$. Bini’s argument demonstrates generally fruitful idea of operating with matrix polynomials and finally recovering some scalar matrices of their coefficients. Historically this became the springboard for the derivation of the MM technique of EXPAND, PRUNE, and CONQUER.

Next we outline Bini’s argument.

Multiply trilinear decomposition (10.1) of $\text{Trace}(ABD + UVW)$ by the variable $\lambda$ and arrive at a decomposition of $\lambda \cdot \text{Trace}(ABD + UVW)$ whose coefficients are polynomials in $\lambda$ of degree at most $d = 2$. Interpolate to $\text{Trace}(ABD + UVW)$ from $d + 1$ values of the polynomial. The interpolation increases the rank by a factor of at most $(d + 1)^2$, that is, by at most a factor of $9$ for $d = 2$\footnote{By using FFT we can interpolate to a polynomial from its values at $K$th roots of unity by using $1.5K \log_2 K + K$ arithmetic operations provided that $K = 2^k > d$ for a positive integer $k$ [139, Theorem 2.2.2]; for $d = k = 2$, this implies interpolation factor 14.} and the resulting rank $9\theta \cdot (mkn + mk + kn)$ for a constant $\theta > 1$ greatly exceeds the rank $2mn$ of the straightforward algorithm for $\text{Trace}(ABD + UVW)$.

We can overcome this deficiency, however, if we recursively extend an APA algorithm to $\text{MM}(n^{2q})$ for $q = 0, 1, \ldots$. At the $q$th recursive step substitute $n^{2q} \times n^{2q}$ matrices for the input entries and observe that this squares both dimension $N$ of the $\text{MM}(N)$ and border rank $br(N)$ but only doubles the degree of the decomposition in $\lambda$. Hence in $q$ recursive steps
\[ r(N) \leq \text{INT}_q \cdot \text{br}(N), \ \text{INT}_q = 2^q(d + 1)^2, \ \text{br}(N) = (\text{br}(n))^{2^q} \]
where $r(N)$ denotes rank, $\text{br}(N)$ and $\text{br}(n)$ denote border ranks, $N = n^{2q}$, and $\text{INT}_q$ is the growth factor of the border rank in its transition to the rank in $q$ recursive steps. Therefore
\[ \omega_{N, br(N)} \leq \log_N r(N) \leq \log_N (\text{INT}_q \cdot \text{br}(N)) = \log_n (\text{INT}_q \cdot \text{br}(n)) = \log_n ((\text{INT}_q)^{1/2^q} \text{br}(n)). \]
Observe that $(\text{INT}_q)^{1/2^q} \to 1$ as $q \to \infty$, recall bound (6.4), and deduce Bini’s estimate
\[ \omega \leq \lim_{q \to \infty} \omega_{N, br(N)} \leq \lim_{q \to \infty} \log_n (\text{br}(n)). \]
12 Schönhage’s Construction. EXPAND, PRUNE, and CONQUER Algorithms

The above APA decomposition for Trace($ABD + UVW$) is associated with disjoint MM rather than MM, but Schönhage in [140] has extended Bini’s theorem by proving that the theoretical exponent \( \omega \) of MM can be bounded based on the APA decomposition for disjoint MM as follows.

**Theorem 12.1.** [140]. The theoretical exponent \( \omega \) of MM in (6.4) does not exceed \( 3\tau \) if there exists a bilinear or trilinear algorithm of rank \( r \) or an APA algorithm of border rank \( r \) for \( s \) disjoint MM problems of sizes \( (k_i, m_i, n_i) \), for \( i = 1, \ldots, s \), such that \( \sum_{i=1}^{s} (k_i m_i n_i)^{\tau} = r \).

The theorem has interesting interpretation in terms of the following Direct Sum Conjecture (first stated and then retracted by V. Strassen): the rank of \( s \) disjoint MM problems of sizes \( (k_i, m_i, n_i) \), for \( i = 1, \ldots, s \), is not less than \( \sum_{i=1}^{s} r_i \) where \( r_i \) denotes the rank of the problem \( MM(k_i, m_i, n_i) \).

This conjecture is still open but becomes wrong if in its statement border ranks replace ranks. Indeed decomposition (10.1) for disjoint pair of MM problems MM\((k, m, n)\) and MM\((m, n, k)\) has border rank \( kmn + km + kn \), that is, \( mn + m + n \) for \( k = 1 \). One can readily prove that the border rank of each of MM\((1, m, n)\) and MM\((m, n, 1)\) is \( mn \) and then observe that \( 2mn > mn + m + n \) for \( m = n > 2^{s} \) and MM\((m, n, k)\).

Nevertheless Theorem 12.1 can be equivalently stated as follows: The bound of Theorem 12.1 on the exponent \( \omega \) of (6.4) can be immediately verified if the border rank version of the Direct Sum Conjecture holds true. In [140] Schönhage proved the theorem without assuming that the Conjecture is true. His proof is the simplest in the case where \( k_i = m_i = n_i \), for \( i = 1, \ldots, s \), and \( s \) divides \( r \), that is, where we are given an APA bilinear decomposition of border rank \( r = gs \) for \( s \) disjoint problems MM\((n)\) and an integer \( g \). Then substitute \( n \times n \) matrices for the variables, reapply the algorithm for every \( s \)-tuple of bilinear multiplications, obtain an APA bilinear algorithm of border rank \( r \cdot r/s = (r/s)2^{s} \) for \( s \) disjoint problems of MM\((n^2)\), and extend this process recursively. \( q \) recursive steps define an APA bilinear algorithm of border rank \( (r/s)^{q} \) for \( s \) disjoint problems of MM\((n^q)\). Prune the input keeping just a single problem MM\((n^q)\), apply Bini’s theorem, and deduce from (6.4) that

\[
\omega \leq \log_{(n^q)} \left( s \cdot \left( \frac{q}{s} \right)^q \right) = \log_{n} \left( \frac{q}{s} \right) + \frac{1}{q} \log_{n}(s).
\]

For \( q \to \infty \) obtain Schönhage’s bound \( \omega \leq \log_{n}(r/s) \) in this special case.

We can quite readily relax the assumption that \( s \) divides \( r \); furthermore by proceeding similarly to the proof of Theorem 5.1 we yield extension to the case of \( s \cdot MM(k, m, n) \), where we are given an APA bilinear algorithm of a border rank \( r \) for \( s \) disjoint problems of rectangular MM\((k, m, n)\).

Let us extend these results to the pair of disjoint problems of MM\((k, m, n)\) and MM\((m, n, k)\) represented in Tables 9.1 and 10.1. Let a basic APA algorithm have a border rank \( r \). Then \( q \) recursive steps define an APA algorithm of border rank \( r^q \) for \( 2^q \) disjoint MM problems. Grouping together the MM problems of the same size we obtain \( q + 1 \) disjoint groups of MM problems \( T_i \cdot (k^i m^{q-i}, m^i n^{q-i}, n^i k^{q-i}) \) with binomial coefficients \( T_i = \binom{q}{i} \) for \( i = 0, 1, \ldots, q \). Choose even \( q \), write \( q = 2s \) and \( (K, M, N) = (k^s m^s, m^s n^s, n^s k^s) \), and prune the disjoint MM keeping only the term \( T_s \cdot (K, M, N) = \binom{2s}{s} \cdot (k^s m^s, m^s n^s, n^s k^s) \); restrict the given decomposition of border rank \( r^{2s} \) to this term. In this special case we have already proved Theorem 12.1 and so we obtain

\[
\omega \leq 3 \log_{R}((kmn)^{2^s}) = 3 \log_{R^{1/2^s}}(kmn), \text{ for } R = r^{2s}/\left( \binom{2s}{s} \right).
\]

Since

\[
\left( \frac{2s}{s} \right)^{(1/2^s)} \to 2 \text{ as } s \to \infty
\]

\[\text{[This counter-example to the conjecture appeared in [140]; a little more involved one, based on an APA-variant of the decomposition of Table 9.2] appeared in [119].}\]
it follows that
\[ R^{1/2s} \rightarrow r/2 \text{ as } s \rightarrow \infty, \text{ and so } \omega \leq 3 \log_{r/2}(kmn), \]
which proves the theorem in this special case.

In the general case of \( s \) disjoint MM problems of various sizes \((k_i, m_i, n_i)\), for \( i = 1, \ldots, s \), the same construction and the same proof techniques work. We again perform \( q \) recursive steps and arrive at a decomposition of border rank \( r^q \) for disjoint MM made up of the terms \( \text{MM}(K, M, N) \) where \( K = \prod_{i=1}^{s} k_i^{d_i}, M = \prod_{i=1}^{s} m_i^{d_i}, N = \prod_{i=1}^{s} n_i^{d_i} \), and \( d_1, \ldots, d_s \) range over all \( s \)-tuples of nonnegative integers summed to \( q \).

Group all MM problems of the same size together into the terms of the form \( T \cdot \text{MM}(K, M, N) \) with \( T \) denoting the coefficients of multinomial expansion. Prune the decomposition to each of these terms (that is, delete all other terms), define its APA decomposition of border rank \( r^q \). Then for every term \( T \cdot \text{MM}(K, M, N) \) we can obtain an upper bound \( \omega \leq 3 \log \left( \frac{R(KMN)}{T} \right) \) for \( R = r^q/T \). Maximize these bounds over all such terms, let \( q \rightarrow \infty \), and arrive at the claimed bound of Theorem 12.1 in the general case.

The name of EXPAND, PRUNE, and CONQUER is more adequate for these techniques than the traditional DIVIDE and CONQUER, and similarly for the derivation of Bini’s bound if we view the interpolation to a single term as pruning.

13 Faster Decrease of the Exponent of MM by Using the EXPAND, PRUNE, and CONQUER Techniques

EXPAND, PRUNE, AND CONQUER techniques enabled significant decrease of the exponent \( \omega \). Already the above APA decomposition of \( \text{Trace}(ABD + UVW) \) implies the upper bound
\[ \omega \leq 3 \log_{mkn}(0.5(kmn + km + kn)) \]
for any triple of \( k, m, \) and \( n \). Indeed for \( k = n = 7 \) and \( m = 1 \), we arrive at APA bilinear algorithm of border rank 63 for the pair of disjoint problems of \( \text{MM}(7,1,7) \) and \( \text{MM}(7,7,1) \) – of computing the outer product of two vectors of dimension 7 and of the product of a \( 7 \times 7 \) matrix by a vector, respectively. Apply Theorem 12.1 and obtain
\[ \omega \leq 3 \log_{49} 31.5 < 2.66. \] (13.1)

Refinement of this construction in [146] yielded the record estimate \( \omega < 2.548 \), and [122] promptly decreased this record bound to \( \omega < 2.522 \) by means of combining APA technique and trilinear aggregation for disjoint MM represented by \( \text{Trace}(ABC + UVW + XYZ) \). Further record of 2.496 was soon established in [54]. By continuing this line of research, the papers [155] and [55] decreased the upper estimates for the theoretical exponent of MM below 2.479 and 2.376, respectively.

The MM algorithms of these papers began with APA decomposition for disjoint MM of small sizes. It is instructive to compare the initial trilinear identities in [155] and [55] with the decomposition for disjoint MMs defined by Tables 9.1, 9.2, and 10.1. According to [55, page 255], “Strassen used the following trilinear identity, related to … trilinear aggregation of [118]:”

\[ \sum_{i=1}^{q} \left( x_0^{[2]} + \lambda x_i^{[1]} \right) \left( y_0^{[1]} + \lambda y_i^{[2]} \right) (z_i \lambda^{-1}) - x_0^{[2]} y_0^{[1]} \sum_{i=1}^{q} z_i = \sum_{i=1}^{q} \left( x_i^{[1]} y_i^{[1]} z_i + x_0^{[2]} y_0^{[2]} z_i \right) + O(\lambda). \]

This defined a basic APA algorithm of border rank \( q + 1 \) for a pair of block inner products.

[55] strengthened this construction by proposing the following basic APA algorithm of border
rank $q + 2$ for a triple of block inner products:

$$
\sum_{i=1}^{q} \lambda^{-2} (x_0^{[0]} + \lambda x_i^{[1]}) (y_0^{[0]} + \lambda y_i^{[1]}) (z_0^{[0]} + \lambda z_i^{[1]}) - \lambda^{-3} (x_0^{[0]} + \lambda^2 x_i^{[1]}) (y_0^{[0]} + \lambda^2 y_i^{[1]}) (z_0^{[0]} + \lambda^2 z_i^{[1]}) + (\lambda^3 - q \lambda^2) x_0^{[0]} y_0^{[0]} z_0^{[0]} =
$$

$$\sum_{i=1}^{q} (x_0^{[0]} y_i^{[1]} z_i^{[1]} + x_i^{[1]} y_0^{[0]} z_i^{[1]} + x_i^{[1]} y_i^{[1]} z_0^{[0]}) + O(\lambda).$$

In both papers [155] and [55] the derivation of new record upper bounds on the exponent $\omega$ from the simple basic designs above required long sophisticated recursive processes and intricate pruning based on amazing and advanced mathematical arguments. Actually paper [55] deduced “only” the record bound $\log_8(4000/27) < 2.40364$ from the above design, but then proposed some extended and more involved initial designs and decreased the bound to the famous barrier of 2.376. This record was only beaten by 0.002 in 2010 [150] and then by additional 0.001 in 2012–2014 [162]. The challenge of reaching the exponent 2 is still open – in 2018 the record bound is about 2.3728639 [104]. Moreover the study in [4] showed that the power of the extension of the approach of [55] in the directions of [150], [162], and [104] is limited, and so the decrease of the exponent below 2.37 should require some new dramatically different ideas and techniques. Likewise it was proven in [3] that the group-theoretical approach of [49], [48] to the acceleration of MM, initially considered highly promising for achieving MM in nearly quadratic time, must include some new dramatically different ideas and techniques in order to produce any competitive MM algorithm.

### 14 Some Impacts of the Study of Fast MM

The progress in decreasing the exponent $\omega$ towards its lower bound 2 has been essentially in stalemate for the last three decades, both for feasible MM at the level of about 2.7724 and for unfeasible MM at the level of about 2.38. that direction, which has virtually stalled after 1986. In spite of that disappointment we believe that overall the study of fast MM was already a success story.

- **Within less than two decades (by 1987) the straightforward upper bound 3 on the MM exponent of (6.4) decreased more than half-way to its lower bound 2 (see details in Appendix A).**

- **In order to achieve this progress researchers have found and revealed new surprising resources and have developed amazing novel techniques, all built on the top of each other, involving sophisticated combinations of trilinear aggregation, APA algorithms, disjoint MM, and EXPAND, PRUNE, and CONQUER techniques.**

- **The study of fast MM was highly important for the Theory of Computing – the exponent $\omega$ is one of the most cited quantities in that large field because progress in its estimation can be immediately extended to a great variety of well-known and intensively studied computational problems, partly listed in Section 2.3.**

- **Besides its impact on the Theory, the progress in decreasing the exponent $\omega$ strengthened the effort of researchers for the reduction of various computational problems to MM; such a reduction can be efficient even where the straightforward MM is applied.**

- **Some fast algorithms for feasible MM have been devised, developed, and implemented. Now they make a valuable part of modern software for both numerical and symbolic computations.**

Next we recall sample by-products of the study of fast MM and of its methodological impact.
• Although the origin of the field of fast Algebraic Computations can be traced back to [114], [111], [156], and [116], the studies of fast MM in [165], [117], [85], [153], [139], [140], [118], and [125] as well as APA algorithms in [19] and [20] have greatly motivated the effort and the progress in that field.

• The duality technique of [117] for generating new efficient bilinear algorithms, with applications shown in [167] is a valuable by-product of the MM research.

• The MM paper [117] was pioneering in demonstrating the power of the application of tensor decomposition to matrix computations, now a thriving and highly popular area.

Finally, in contrast to reasonable pessimism of many experts about current perspectives for further substantial decrease of the exponent $\omega$, the acceleration of feasible MM is a highly promising and dynamic area and, together with the implementation issues, is the main subject of the rest of our survey.

15 The Curse of Recursion and Fast Feasible MM

Already in 1981 it has become clear that the progress in decreasing the theoretical exponent of MM is going to be separated from the acceleration of feasible MM. Arnold Schönhage has concluded the introduction to his seminal paper [146] of 1981 as follows: “Finally it must be stressed, however, that so far all these new results are mainly of theoretical interest. The point of intersection with Strassen’s method lies beyond any practical matrix size, and . . . Pan’s estimates of 1978 for moderate values of $n$ are still unbeaten”. Schönhage’s account can be extended to the subsequent algorithms supporting record estimates for the exponent $\omega$, except that the estimate 2.7962 of 1978 for the exponents of feasible MM has successively been decreased (although by small margins) in [119], [120], [122], and [123], based purely on trilinear aggregation. By 2018 the estimate $\omega_{44} < 2.7734$ of [123] still remains the record upper bound on the exponents of feasible MM, unbeaten since 1982. 31 years later A.V. Smirnov in [149] came very close to this record by applying advanced computer-aided search: one of his algorithm supports an exponent below 2.7743 for $MM(54)$ (see Section 16.3).

Tables 15.1 and 15.2 trace the progress in estimating the record exponents of feasible MM. The overhead constants associated with the exponents are reasonably small because the supporting algorithms avoid recursive application of nested block MM and rely just on trilinear aggregation. This progress in the acceleration of feasible MM was much slower than the progress in breaking records for the theoretical exponent $\omega$, which is no surprise – the powerful resource of using unlimited recursive processes had to be excluded for devising algorithms for feasible MM.

In contrast the techniques of EXPAND, PRUNE, and CONQUER that supports Bini’s and Schönhage’s theorems as well as the derivation of all known exponents below 2.7733 involve long recursive processes, and so the associated algorithms remain inferior to the straightforward MM until the problem size is blown up and becomes immense. Due to such a curse of recursion all these record breaking works had no relevance to feasible MM of today, tomorrow, or foreseeable future.

Table 15.1: Complexity Exponents of Feasible MM

| Exponent     | 2.8074 | 2.7962 | 2.7801 | 2.7762 | 2.7734 |
|--------------|--------|--------|--------|--------|--------|
| Reference    | 151    | 118    | 120    | 122    | 123    |
| Year         | 1969   | 1978   | 1979   | 1981   | 1982   |

All in all, the concept of theoretical exponent of MM has been historically motivated but has not been related to feasible MM. The complexity exponents of feasible MM have much more relevance to the real world computations, but in the next section we significantly increase the efficiency of feasible MM without decreasing its record complexity exponent.

6 Actually also with the straightforward MM.
16 Some Ways to Acceleration of Feasible MM with No Decrease of the Complexity Exponent

16.1 Acceleration of Complex and Polynomial MM

Substitute matrices for variables of bilinear algorithms of Examples 3.1 and 3.1 and obtain efficient algorithms for Complex and Polynomial MM.

The 3M Method for Complex MM.

The rank-3 bilinear algorithm of Example 3.1 for multiplying two complex numbers saves one multiplication but uses three extra additions and subtractions in comparison to the straightforward algorithm of rank 4. Now substitute $N \times N$ matrices for the variables $a_1, a_2, b_1,$ and $b_2$ and arrive at the problem of multiplying a pair of $N \times N$ complex matrices $A_1 + iA_2$ by $B_1 + iB_2$. Then the latter algorithm, called the 3M method, involves $3N^3$ scalar multiplications and $3N^3 + 2N^2$ additions and subtractions versus straightforward $4N^3$ and $4N^3 - 2N^2$. This means saving of about 25% of all operations already for $N = 30$ (cf. [82]).

Fast Polynomial MM.

Consider $n \times n$ MM where the input matrices are filled with polynomials of degree at most $d - 1$. By applying the straightforward MM to these input matrices reduce our task to performing $n^3$ multiplications and $n^3 - n^2$ additions of polynomials of degrees at most $d$. All these polynomial additions together involve just $(n^3 - n^2)d$ scalar additions. All polynomial multiplications together involve $(2d^2 - 2d + 1)n^3$ scalar multiplications and additions if we apply straightforward polynomial multiplication, but this bound turns into $(4.5K \log_2(K) + 2K)n^3$, for $K = 2^d, 2d - 1 < K < 4d - 2$, if we apply the FFT-based convolution algorithm of Example 3.2.

This is dramatic saving if the degree bound $d$ is large, but we can save much more if we consider the input as two polynomials with matrix coefficients and apply to them fast convolution algorithm of Example 3.2. In this way we reduce our Polynomial MM to performing at most $K$ MM($n$) and at most $4.5K \log_2(K) + K$ additions, subtractions and multiplications by scalars of $n \times n$ scalar matrices for $2d - 1 < K < 4d - 2$. With the straightforward MM we solve these tasks by using at most $2n^3K + 4.5K \log_2(K)n^2$ scalar arithmetic operations overall, which additionally saves for us $(4.5K \log_2(K) + d)(n - 1)n^2$ scalar operations versus our first accelerated Polynomial MM.

We can further accelerate both Complex and Polynomial MM (decreasing all our estimates accordingly) if we incorporate fast algorithms for MM instead of applying the straightforward MM.

16.2 Randomized acceleration of feasible MM

The paper [65] presented surprising acceleration of approximate rectangular MM by means of randomization (see a short exposition in [107] Section 3.1] and see arXiv 1710.07946 and the bibliography therein for the extension to low rank approximation of a matrix, which a highly popular task linked to fundamental matrix computations and Big Data mining and analysis).

We sketch the main result of [65] by using the spectral and Frobenius matrix norms $\| \cdot \|_2$ and $\| \cdot \|_F$, respectively, in the estimates for the approximation errors.
Given two matrices $A = (a_j)^n_{j=1}$, of size $m \times n$, with columns $a_j$, and $B = (b_j^T)^n_{j=1}$, of size $n \times q$, with rows $b_j^T$, for $j = 1, \ldots, n$,

(i) first compute the so called leverage scores (aka importance sampling probabilities),

$$p_j = \frac{||a_j||_2 \ ||b_j^T||_2}{\sum_{j'=1}^n ||a_{j'}||_2 \ ||b_{j'}^T||_2}$$

for $j = 1, \ldots, n$,

(ii) then randomly select (and re-scale by $1/\sqrt{p_j}$) $c$ pairs of corresponding columns of $A$ and rows of $B$, thereby forming an $m \times c$ matrix $C$ and a $c \times n$ matrix $R$, and

(iii) finally compute an approximation $CR$ to the matrix product $AB$.

It is proven in [65] that

$$\sqrt{c} \ ||CR - AB||_F = O(||A||_F ||B||_F)$$

both in expectation and with a high probability.

16.3 Tensors at Work Again: History and Perspectives of Computer-Aided Search for Fast MM

Properly directed computer-aided search is a natural tool in the search for efficient basic designs for fast feasible MM, not necessarily directed to the decrease of the complexity exponent.

According to [143], computer-aided search has helped already in 1979, in the design of the APA algorithm of [20]. Even earlier, in [35], Richard P. Brent reduced such a search to a system of nonlinear equations [37] and proposed to apply least-squares minimization techniques for its solution.

It is more convenient to use the equivalent system of equations [53], whose solution is in turn equivalent to finding the standard canonical decomposition CANDECOMP/PARAFAC for the MM tensor (see, e.g., [96]). Unfortunately, none of the numerous techniques for 3D tensor decomposition has been successful in this particular case so far. Loosely speaking, the things are quite different from the cases favorable to the known techniques because solving Brent’s equations requires tensor data expansion rather than compression (the latter being the essence of CANDECOMP/PARAFAC technique). The main reason seems to be rather large lower bounds on the rank of MM in comparison with the rank of the general trilinear form $\sum_{i,j,k} t_{i,j,k} a_i b_j d_k$.

Now recall the uniqueness theorem of Kruskal [97], which in the case of the $MM(n)$ tensor (and with account for the full-rank of the three $n^2 \times r$ matrices involved) requires that $r < 3n^2/2$ for the canonical decomposition to be essentially unique. This is a clear contradiction with the known lower bounds on the rank $r$ of MM($n$) (e.g., $r \geq 2n^2 - 1$ of [117] or $r \geq 3n^2 - o(n^2)$ of [147] and [108]). As a result non-unique solutions of various kind can be observed (see, e.g., [13]).

Even more destructive for numerical optimization methods is the presence of infinitely growing approximate solutions, which correspond to the existence of APA algorithms. This makes the customary tool of applying unconstrained ALS optimization inefficient, and its modification or alternatives are required. (ALS is the acronym for Alternating Least Squares.)

Some success with the ALS method used for the minimization of the Euclidean norm of the residuals of Brent’s equations was reported in [88] and [112]. Namely those two studies produced two alternatives to Laderman’s bilinear algorithm of [99] for the $MM(3)$ problem with the record rank $r = 23$.

By cleverly extending Brent’s approach A.V. Smirnov in [149] achieved important progress7 He proposed a very special modification of the ALS procedure based on an adaptive ”quantization” of iterated components, which luckily resulted in finding an exact solution to $MM(3,3,6)$ with $r = 40$. This supports the MM complexity exponent $\omega_{54} < 2.7743$, a remarkably low value for such small matrix sizes, almost matching the record 2.7734 of [123] for the exponents of feasible MM.

---

7He has included [88] in his reference list but has not explained that the application of the ALS method to devising fast MM was implicit already in Brent’s paper [35] of 1970.
Unfortunately, the paper [149] contains neither estimates for the number of scalar additions and subtractions involved in its algorithms nor a constructive recipe for the implementation of their additive stages. In [18] Smirnov’s recursive algorithm based on $MM(3, 3, 6)$ has been implemented and tested for $MM(n)$ for dimensions $n$ less than 13,000. The test results are inferior to those for recursive bilinear processes based on Winograd’s and Strassen’s Examples 2.3 and 2.2, respectively. By no means this comparison is final, however. The models of communication complexity for serial and parallel computers are dynamic in time, Smirnov and other researchers have all chances to strengthen the ALS approach to fast MM, producing perhaps significant acceleration of practical MM.

Furthermore the matrices $U$, $V$, and $W$ of the algorithms of [149] are rather densely populated by nonzeros (near 50%) and may perhaps be sparsified if revised algorithms of [149] properly incorporate the techniques of TA and disjoint MM. Indeed success of TA in designing fast algorithms for feasible MM in [90] and [91] indicates potential value of that technique for simplifying computer-aided search. In particular trilinear aggregation can exploit the 6-way symmetry in order to reduce the search area for efficient MM algorithms.

17 Numerical Implementation of Fast MM

Even the straightforward MM, if it is efficiently implemented, can compete in practice with fast MM. In this section we comment on numerical implementation of that and other efficient algorithms for feasible MM. Implementation of feasible MM must be efficient in arithmetic cost, decreasing vectorization, and data locality (cf. [78, Chapter 1]). We cover all these issues, and notice that quite frequently all three goals are well compatible with each other.

So far in practical numerical computations MM is performed by means of either the straightforward algorithm or recursive bilinear $(2 \times 2)$-based MM, typically the recursive application of Winograd’s Example 2.3 or, more rarely, Strassen’s Example 2.2. The implementation of these old algorithms has been extensively worked on by many authors and makes up a valuable part of the present day MM software (cf. [14], [81], [65], [68], [69], [34], [56], [8], [18], [33], the references therein, and in [78, Chapter 1]). This work, intensified lately, is still mostly devoted to the implementation of very old algorithms, ignoring, for example, the advanced implementations of fast MM in [90] and [91] (see Section 17.1) and the significant improvement in [45] and [98] of the recursive bilinear MM based on Examples 2.2 and 2.3 by Winograd and Strassen.

We hope that our survey will motivate advancing the State of the Art both in the design of fast algorithms for feasible MM and in their efficient implementation.

17.1 Implementation of Trilinear Aggregation Algorithms

Already the first implementations in [90] and [91] of the fast algorithms for MM of moderate sizes based on trilinear aggregation showed their superiority to the alternative approaches regarding numerical stability, memory consumption, and efficiency for parallel MM. One can immediately see why so: in the 2- and 3-disjoint product algorithms in the implementation of [91], the coefficient tensors $U$, $V$, and $W$ of (5.5) and (5.6) are “supersparse”.

This explains good numerical stability of the latter algorithms according to the customary measurement by the exponent 2.322 of distinct nature, estimated in [91]. Moreover the paper [91] shows significant reduction of the workspace consumption of its algorithms in comparison with $(2 \times 2)$-based MM (that is, recursive bilinear algorithms based on Winograd’s and Strassen’s Examples 2.3 and 2.2, respectively). Namely, storage consumption decreases from $(2/3)N^2$ to $(1/4)N^2$ memory cells (corresponding to the MM exponent 2.776), that is, in $8/3$ times.

The recent important works [15] and [8], both covering the implementation and numerical stability of fast MM algorithms, have omitted proper discussion of these significant benefits for the implementation of [91], apparently leaving the challenge to the study in the future. The papers [90] and [91] had bad luck also with their exposition in the influential paper [56], where the citation “The practical implementation of Pan’s algorithm $O(n^2)$ is presented by Kaporin [Kaporin 1999; 2004]”
must be corrected into “The practical implementation of Pan’s disjoint matrix product $O(n^{2.811})$- and $O(n^{2.776})$-algorithms is presented by Kaporin [Kaporin 1999; 2004]”.

Incidentally, the test results in the interesting experimental study in [56] should be accepted with some caution because they rely on pseudo-random matrices, and such matrices tend to have too good numerical stability.

No further implementation of trilinear aggregation algorithms followed so far. This can be understood because their design is technically more involved and has been much less advertised than the recursive MM algorithms based on Example 2.3.

Clearly, further work is in order on the assessment, implementation, amelioration, and extension of such algorithms for fast feasible MM.

17.2 Some Imaginary and Real Issues

1. A considerable group of numerical analysts still believes in the folk “theorem” that fast MM is always numerically unstable, but in actual tests loss of accuracy in fast MM algorithms was limited, and formal proofs of quite reasonable numerical stability of all known fast MM algorithms is available (see [23], [90], [91], [92], and [61]).

2. The paper [9] emphasizes the importance of non-arithmetic optimization of matrix algorithms: “The traditional metric for the efficiency of a numerical algorithm has been the number of arithmetic operations it performs. Technological trends have long been reducing the time to perform an arithmetic operation, so it is no longer the bottleneck in many algorithms; rather, communication, or moving data, is the bottleneck”. This statement should be taken not too lightly, but still with a grain of salt: communication cost is limited to operating with the data in primary memory. For competent implementations of fast feasible MM their arithmetic cost is usually in a rather good accordance with vectorization and communication cost. Together (rather than in conflict) with vectorization, numerical stability, and restricting data movement, arithmetic cost is still a critical ingredient of the evaluation of practical efficiency of MM algorithms (cf. [78], Chapter 1, [18] and [8]). For a litmus test, smaller arithmetic cost of Winograd’s algorithm of Example 2.3 has made it substantially more popular among the users than Strassen’s algorithm of Example 2.2 in spite of its a little weaker numerical stability.

3. Most of nowadays computational methods are largely driven by technology. In particular presently computations in single precision are intensively promoted by the manufacturers (mainly NVIDIA) of the GP-GPU (general purpose graphic processing units). In such circumstances, floating-point computations are less preferable compared to the integer residual-based arithmetic, where rational numbers can be used, but using the integers 1, 0, and −1 is preferable.

18 Non-numerical Implementation of Fast MM

Implementation of MM in Computer Algebra has become highly efficient when it was reduced to performing MM over word size finite fields with the outputs combined by means of the Chinese Remainder Algorithm. The paper [67] spells out the following principles for computations in finite fields, which were basic for this success:

1. Reduce computations in a finite field to integer arithmetic with delayed or simultaneous modular reductions;

2. perform integer arithmetic by invoking floating point units (taking advantage of SIMD instructions and of numerical BLAS);

3. structure the computations in blocks in order to optimize the use of the memory hierarchy of the current architectures;
4. apply fast MM algorithms (so far in practice they are mostly recursive bilinear algorithm for MM based on Winograd’s $2 \times 2$ MM of Example 2.3 or less frequently based on Strassen’s $2 \times 2$ MM of 2.2, but also Kaporin’s algorithms of [90] and [91] and the Any Precision Approximation (APA) algorithm of [20] are used).

We have two further comments:

- The implementations of fast MM in [90] and [91] are particularly attractive within this framework because it uses matrices $U$, $V$ and $W$ filled with shorter numbers. Therefore one can perform the Chinese Remainder Algorithm for fewer primes and invoke it fewer times.

- We already pointed out that APA algorithms are prone to the problems of numerical stability, but have good promise for symbolic MM in Computer Algebra and for computations with integers (see such important computations in [134, Section 8]). The current level of the known implementations of APA MM, however, is rudimentary, staying at the level of the paper [20] of 1979, and must be moved forward dramatically; our survey should help to accelerate this process.

We refer the reader to [70] and the bibliography therein on further details of fast symbolic MM.

**Appendix**

**A Estimation of the Theoretical Exponent of MM**

Tables A.1 and A.2 show the dynamics of the record estimates for the theoretical exponent of $MM(n)$ since 1969. The tables link each estimate to its recorded publication in a journal, a conference proceedings, or as a research report. It displays the chronological process and reflects the competition for the decrease of estimates for the theoretical exponent, particularly intensive in 1979–1981 and 1986.

The record upper estimates for the theoretical exponent have been updated four times during the single year of 1979. At first the estimate 2.7801 appeared in February in a Research Report (see [120]). The estimate 2.7799 appeared at first as one for an APA-based estimate for the exponent of MM in [20] in June and then for the theoretical exponent of MM in [19]. The next upper estimate 2.548 of [146] was followed by 2.528 of [122], both published in the book of abstracts of the conference on the Computational Complexity in Oberwolfach, West Germany, organized by Schnorr, Schönhage and Strassen (cf. [125, page 199] and [146]).

The upper bound 2.496 of [54] was reported in October 1981 at the IEEE FOCS’81, but in Table A.1 we place it after the estimate 2.517 of the paper [144] of 1982, which was submitted in March 1980. The Research Report version of the paper [55] appeared in August of 1986, but in Table A.1 we place [55] after the paper [155], published in October of 1986 in the Proceedings of the IEEE FOCS, because the paper [155] has been submitted to FOCS’86 in the Spring of 1986 and has been widely circulated afterwards.

One could complete the historical account of Tables A.1 and A.2 by including our estimate 2.7804 (announced in the Fall of 1978, but was superseded in February 1979 in [120]) and the bound 2.5218007, which decreased our estimate 2.5218128 of 1979 and appeared at the end of the final version of [146] in 1981—that is, before the publication, but after the submission of the estimate 2.517 of [144]. Table A.2 shows the decrease of the record estimates in 1986 – 2014.

We refer the reader to [52], [103], [53], [86], [92], [103], and the bibliography therein for similar progress in asymptotic acceleration of rectangular MM and its theoretical implications.

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8Observe cross-fertilization: Schönhage’s disjoint MM and Partial MM have been motivated by trilinear aggregation of [117] and [118] and by the design of [20], respectively; in turn the papers [119] and [122] decreased the exponent 2.548 of [130] to 2.522 by combining his disjoint MM with the technique of trilinear aggregation.
Table A.1: Record upper estimates for the theoretical exponent of MM.

| Exponent | Paper | Year |
|----------|-------|------|
| 2.8074   | 154   | 1969 |
| 2.7962   | 118   | 1978 |
| 2.7801   | 120   | 1979 |
| 2.7799   | 20, 19| 1979 |
| 2.548    | 146   | 1979 |
| 2.522    | 122   |      |

| Exponent | Paper | Year |
|----------|-------|------|
| 2.517    | 144   | 1980 |
| 2.496    | 54    | 1981 |
| 2.479    | 155   | 1986 |
| 2.376    | 55, 150 | 1986 |
| 2.374    | 104   | 2010 |
| 2.373    |       | 2012 |

Table A.2: The records in 1986–2014.

| Exponent | Paper | Year |
|----------|-------|------|
| 9        | 55    | 1986 |
| 2.375477 | 150   | 2010 |
| 2.373689 | 57    | 2012 |
| 2.372927 | 162   | 2014 |
| 2.372864 | 104   |      |

B  FFT, Inverse FFT, and Convolution

Fast Fourier transform (hereafter referred to as FFT) is a celebrated example of recursive divide and conquer algorithms. It computes discrete Fourier transform at K points, that is, evaluates a polynomial \( p(x) = \sum_{i=0}^{K-1} p_i x^i \) at the Kth roots of 1. For \( K = 2^k \) it reduces the task to two such problems of half-size:

\[
p(x) = p^{(0)}(y) + xp^{(1)}(y),
\]

\[
p^{(0)}(y) = \sum_{i=0}^{K/2 - 1} p_{2i}y^i, \quad p^{(1)}(y) = \sum_{i=0}^{K/2 - 1} p_{2i+1}y^i,
\]

where \( y = x^2 \) is a \((K/2)\)nd root of 1 if \( x \) is a Kth root of 1. Recursively, the problem is reduced to four problems of quarter size and ultimately to K problems of size 1, whose solution is instant, requiring no arithmetic operations. Each of \( k = \log_2(K) \) recursive stages involves K additions/subtractions and \( K/2 \) multiplications (\( K/2 \) other multiplications are by \(-1\) and are performed as subtractions). Thus the overall arithmetic cost of FFT is \( 1.5K \log_2(K) \), versus \( K^2 \) multiplications and \( K^2 \) additions required in the straightforward algorithm. This is dramatic saving for large K, e.g., more than 50,000-fold for \( K = 1,000,000 \).

Now notice that discrete Fourier transform computes the vector of values \( \mathbf{v} \) of a polynomial \( p(x) \) at the Kth roots of unity, \( \mathbf{v} = \Omega \mathbf{p} \), for the vector \( \mathbf{p} = (p_i)_{i=0}^{K-1} \) of the coefficients of \( p(x) \) and the matrix \( \Omega = (\omega_K^{ij})_{i,j=0}^{K-1} \) filled with Kth roots of unity, where \( \omega_K \) is a primitive Kth root of 1, that is, \( \omega_K^K = 1 \), \( \omega_K^h \neq 1 \) for \( 0 < h < K \),

\[
\omega_K = \exp(2\pi i/K) \quad \text{for} \quad i = \sqrt{-1}.
\]

Inverse discrete Fourier transform computes the vector \( \mathbf{p} = \Omega^{-1} \mathbf{v} \) of the coefficients from a given vector \( \mathbf{v} \) of the values. It turned out that \( \Omega^{-1} = \frac{1}{K} \Omega^* = \frac{1}{K}(\omega_K^{-1})_{i,j=0}^{K-1} \), and one can just readily extend FFT and then perform K divisions by K.

Due to the wide range of important applications of FFT in Modern Computations, this recursive divide and conquer algorithm has become immensely popular since its publication in 1965 in [51] and was justly included into the list of the Ten Top Algorithms of the 20th century [47], even though its origin has been traced back to posthumous notes of C. F. Gauss, 1777–1855. See [24] for early history of FFT and see [1], [25], [130, Sections 2.2–2.4], [138, Chapter 12, Fast Fourier Transform], [161], and the bibliography therein for derivation of FFT and inverse FFT, their structured matrix version, generalization to the case of any integer K, numerical stability of the output vector norm, parallel implementation, and further improvements.
The straightforward algorithm for convolution involves \((m+1)(n+1)\) multiplications and \(mn\) additions, but the combination of the Toom’s seminal method of evaluation–interpolation with FFT and Inverse FFT enables dramatic acceleration.

Let \(m = n\) in order to simplify the estimates.

**Algorithm 1: Convolution via Evaluation, Interpolation, and FFT.**

**INITIALIZATION.**  
Fix the integer \(K = 2^k\) being a power of 2 in the range \(2n < K \leq 4n\), that is, \(k = 2 + \lfloor \log_2(n) \rfloor \).

**COMPUTATIONS.**

1. Compute the values \(a(\omega_K^i)\) and \(b(\omega_K^i)\) for \(\omega_K^i\) of (15.1) and \(i = 0, \ldots, K - 1\). They are the values of the polynomials \(a(x)\) and \(b(x)\) at the \(K\)th roots of 1.

2. Compute the products \(c(\omega_K^i) = a(\omega_K^i)b(\omega_K^i)\) for \(i = 0, \ldots, K - 1\). They are the values of the polynomial \(w(x)\) at the same \(K\) points.

3. Interpolate to the polynomial \(c(x)\) from its values at these points.

Stages 1 and 3 of the algorithm amount to multipoint evaluation and interpolation of polynomials at the \(K\)th roots of 1, that is, *Forward and Inverse Discrete Fourier Transforms*, respectively. FFT and Inverse FFT perform these stages by using \(K + 4.5K\log_2(K)\) arithmetic operations. Add \(K\) bilinear multiplications, performed at Stage 2, and arrive at the overall arithmetic computational cost bound \(2K + 4.5K\log_2(K)\).

Toom’s evaluation–interpolation method has a number of further applications to polynomial and rational computations. For example, it can be extended immediately to fast computation of the quotient \(q(x) = u(x)/v(x)\) of two polynomials \(u(x)\) and \(v(x)\) provided that the remainder of the division is 0; this restriction has been removed in [136].

**References**

[1] A.V. Aho, J.E. Hopcroft, J.D. Ullman, *The Design and Analysis of Algorithms*. Addison-Wesley, Reading, MA, 1974.

[2] N. Alon, Z. Galil, O. Margalit, On the Exponent of the All Pairs Shortest Path Problem, J. of Computer and System Sciences, 54, 2, 255-262, 1997.

[3] N. Alon, A. Shpilka, C. Umans, On Sunflowers and Matrix Multiplication. *Computational Complexity*, 22, 2, 219–243, 2013.

[4] A. Ambainis, Y. Filmus, F. Le Gall, Fast Matrix Multiplication: Limitations of the Laser Method. *Electronic Colloquium on Computational Complexity (ECCC)*, year 2014, paper 154. [http://eccc.hpi-web.de/report/2014/154](http://eccc.hpi-web.de/report/2014/154) Available at arXiv:1411.5414 November 21, 2014.

[5] R.R. Amossen, R. Pagh, Faster Join-projects and Sparse Matrix Multiplications. In *Proceedings of the 12th International Conference on Database Theory*, 121–126, 2009.

[6] V. Z. Arlazarov, E. A. Dinic, M. A. Kronrod, I. A. Faradzhev, On economical construction of the transitive closure of a directed graph, *Soviet Mathematics Doklady (DAN)*, 11, 5, 1209–1210, 1970.

[7] A. Azad, G. Ballard, A. Bulu, J. Demmel, L. Grigori, O. Schwartz, S. Toledo, S. Williams, Exploiting multiple levels of parallelism in sparse matrix-matrix multiplication, *SIAM Journal on Scientific Computing (SISC)*, 38, 6, C624–C651, 2016.
[8] G. Ballard, A.R., Benson, A. Druinsky, B. Lipshitz, O. Schwartz, Improving the numerical stability of fast matrix multiplication algorithms, *SIAM Journal on Matrix Analysis and Applications (SIMAX)*, 37, 4, 1382–1418, 2016.

[9] G. Ballard, E. Carson, J. Demmel, M. Hoemmen, N. Knight, O. Schwartz, Communication Lower Bounds and Optimal Algorithms for Numerical Linear Algebra. *Acta Numerica*, 23, 1–155, 2014.

[10] G. Ballard, J. Demmel, O. Holtz, B. Lipshitz, O. Schwartz, Graph Expansion Analysis for Communication Costs of Fast Rectangular Matrix Multiplication, in *Design and Analysis of Algorithms*, G. Even and D. Rawitz (eds.), *Lecture Notes in Computer Science*, Springer, Berlin–Heidelberg, 7659, 13–36, December 2012.

[11] G. Ballard, A. Druinsky, N. Knight, O. Schwartz, Hypergraph Partitioning for Sparse Matrix-Matrix Multiplication, *arXiv:1603.05627*.

[12] N. Bansal, R. Williams, Regularity lemmas and combinatorial algorithms, *Proc. 50th Ann. IEEE Symp. on Foundations of Computer Science (FOCS 2009)*, 745–754, 2009.

[13] G. Bard, Algorithms for Solving Linear and Polynomial Systems of Equations over Finite Fields with Applications to Cryptanalysis, Submitted in Partial Fulfillment for the degree of Doctor of Philosophy of Applied Mathematics and Scientific Computation. PhD Thesis, University of Maryland at College Park, April 30, 2007.

[14] D.H. Bayley, Extra High Speed Matrix Multiplication on the Cray-2. *SIAM J. on Scientific and Statistical Computing*, 9, 3, 603–607, 1988.

[15] B. Beckermann, A. Townsend, On the singular values of matrices with displacement structure, *SIAM J. on Matrix Analysis*, 38, 4, 1227–1248, 2017. Also *arXiv:1609.09494 [math.NA]*, 22 pages, 4 figures (submitted on 29 September 2016).

[16] E.G. Belaga, Some Problems in Computation of Polynomials, *Doklady of Academy of Science of USSR*, 123, 775–778, 1958.

[17] T. Bella, Y. Eidelman, I. Gohberg, V. Olshevsky, Computations with Quasiseparable Polynomials and Matrices, *Theor. Computer Science*, 409, 2, 158–179, 2008.

[18] A.R. Benson, G. Ballard, A framework for practical parallel fast matrix multiplication. In *Proceedings of the 20th ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming*, 42–53, ACM Press, New York, January 2015.

[19] D. Bini, Relations Between Exact and Approximate Bilinear Algorithms: Applications. *Calcolo*, 17, 1, 87–97, 1980.

[20] D. Bini, M. Capovani, G. Lotti, F. Romani, $O(n^{2.7799})$ Complexity for $n \times n$ Approximate Matrix Multiplication. *Information Processing Letters*, 8, 5, 234–235, June 1979.

[21] D. Bini, M. Capovani, G. Lotti, F. Romani, *Complessità Numerica*, Boringhieri publisher, 1981.

[22] D. A. Bini, G. Fiorentino, Design, Analysis, and Implementation of a Multiprecision Polynomial Rootfinder, *Numer. Algs.*, 23, 127–173, 2000.

[23] D. Bini, G. Lotti, Stability of Fast Algorithms for Matrix Multiplication. *Numerische Math.*, 36, 1, 63–72, 1980.

[24] D. Bini, G. Lotti, F. Romani, Approximate solution for the bilinear form computational problem, *SIAM J. on Computing*, 9, 4, 692–697, 1980.
[25] D. Bini, V.Y. Pan, *Polynomial and Matrix Computations, Volume 1: Fundamental Algorithms*. Birkhäuser, Boston, 1994.

[26] M. Bläser, A $5/2n^2$-Lower Bound for the Multiplicative Complexity of $n \times n$ Matrix Multiplication over Arbitrary Fields. *Proc. 40th Ann. IEEE Symp. on Foundations of Computer Science (FOCS 1999)*, 45–50, IEEE Computer Society Press, Los Alamitos, CA 1999.

[27] M. Bläser, Lower Bounds for the Multiplicative Complexity of Matrix Multiplication. *J. of Computational Complexity*, 9, 2, 73–112, 2000.

[28] J. Blasiak, T. Church, H. Cohn, J. A. Grochow, E. Naslund, W. F. Sawin, C. Umans. On cap sets and the group-theoretic approach to matrix multiplication, *arXiv:1605.06702*, 2016.

[29] M. Bodrato, A Strassen-like matrix multiplication suited for squaring and higher power computation, In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation (ISSAC '2010)*, 273–280, ACM Press, New York, 2010.

[30] A. Borodin, private communication, 1979.

[31] A. Borodin, I. Munro, *The Computational Complexity of Algebraic and Numeric Problems*. American Elsevier, New York, 1975.

[32] A. Bostan, C.-P. Jeannerod, É. Schost, Solving structured linear systems with large displacement rank. *Theoretical Computer Science*, 407, 1–3, 155–181, 2008.

[33] B. Boyer, J.-G. Dumas, Matrix Multiplication over Word-size Modular Rings Using Approximate formulae, *ACM Transactions on Mathematical Software*, 42, 3, 20:1–20:12, 2016.

[34] B. Boyer, J.-G. Dumas, C. Pernet, W. Zhou, Memory Efficient Scheduling of Strassen-Winograd’s Matrix Multiplication Algorithm. *Proc. Intern. Symposium on Symbolic and Algebraic Computation (ISSAC 2009)*, 55–62, ACM Press, New York, 2009.

[35] R. P. Brent, Algorithms for matrix multiplication. Technical report 70-157, Stanford University, Computer Science Department, 1970. Available at [http://maths-people.anu.edu.au/~brent/pub/pub002.html](http://maths-people.anu.edu.au/~brent/pub/pub002.html)

[36] R.W. Brockett, D. Dobkin, On Optimal Evaluation of a Set of Bilinear Forms. *Proc. of the 5th Annual Symposium on the Theory of Computing (STOC 1973)*, 88–95, ACM Press, New York, 1973.

[37] R.W. Brockett, D. Dobkin, On the Number of Multiplications Required for a Matrix Multiplication. *SIAM Journal on Computing*, 5, 4, 624–628, 1976.

[38] R.W. Brockett, D. Dobkin, On Optimal Evaluation of a Set of Bilinear Forms. *Linear Algebra and Its Applications*, 19, 3, 207–235, 1978.

[39] N.H. Bshouty, A Lower Bound for Matrix Multiplication. *SIAM J. on Computing*, 18, 4, 759–765, 1989.

[40] N.H. Bshouty, On the Additive Complexity of $2 \times 2$ Matrix Multiplication, *Information Processing Letters*, 56, 6, 329–335, 1995.

[41] A. Bulu, J. R. Gilbert, Parallel Sparse Matrix-Matrix Multiplication and Indexing: Implementation and Experiments, *SIAM Journal of Scientific Computing*, 34, 4, C170C191, 2012.
[42] J.R. Bunch, J.E. Hopcroft, Triangular Factorization and Inversion by Fast Matrix Multiplication. *Mathematics of Computation*, **28**, 125, 231–236, 1974.

[43] P. Bürgisser, M. Clausen, M.A. Shokrollahi, *Algebraic Complexity Theory*. Springer Verlag, 1997.

[44] J. Carrier, L. Greengard, V. Rokhlin, A Fast Adaptive Algorithm for Particle Simulation, *SIAM J. Scientific Computing*, **9**, 669–686, 1998.

[45] M. Cenk, M.A. Hasan, On the Arithmetic Complexity of Strassen-Like Matrix Multiplications, *J. of Symbolic Computation*, **80**, 2, 484–501, May–June 2017.

[46] T. M. Chan, Speeding up the Four Russian’s algorithm by about one more logarithmic factor, *Proc. ACM-SIAM Symp. on Discrete Algorithms (SODA 2015)*, 212–217, 2015.

[47] B. A. Cipra, The Best of the 20th Century: Editors Name Top 10 Algorithms, *SIAM News*, **33 4**, 2, published by the Society for Industrial and Applied Mathematics, May 16, 2000.

[48] H. Cohn, R. Kleinberg, B. Szegedy, C. Umans, Group-theoretic Algorithms for Matrix Multiplication. *Proceedings of the 46th Annual Symposium on Foundations of Computer Science (FOCS 2005)*, (Pittsburgh, PA), 379–388, IEEE Computer Society Press, 2005.

[49] H. Cohn, C. Umans, A Group-theoretic Approach to Fast Matrix Multiplication. *Proceedings of the 44th Annual Symposium on Foundations of Computer Science (FOCS 2003)*, (Cambridge, MA), 438–449, IEEE Computer Society Press, 2003.

[50] H. Cohn, C. Umans, Fast Matrix Multiplication Using Coherent Configurations. *Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2013)*, 1074–1087, 2013.

[51] J.W. Cooley, J. W. Tukey, An algorithm for the machine calculation of complex Fourier series, *Mathematics of Computation*, **19 (90)**, 297–301, 1965.

[52] D. Coppersmith, Rapid Multiplication of Rectangular Matrices. *SIAM Journal on Computing*, **11**, 3, 467–471, 1982.

[53] D. Coppersmith, Rectangular Matrix Multiplication Revisited. *Journal of Complexity*, **13**, 1, 42–49, 1997.

[54] D. Coppersmith, S. Winograd, On the Asymptotic Complexity of Matrix Multiplication, *SIAM J. on Computing*, **11**, 3, 472–492, 1982. Proceedings version in *23rd FOCS* (Nashville, TN), 82–90, IEEE Computer Society Press, 1981. doi:10.1109/SFCS.1981.27

[55] D. Coppersmith, S. Winograd, Matrix Multiplication via Arithmetic Progressions. *J. of Symbolic Computations*, **9**, 3, 251–280, 1990. Proc. version in *19th ACM Symposium on Theory of Computing (STOC 1987)*, (New York, NY), 1–6, ACM Press, New York, NY, 1987. Also Research Report RC 12104, *IBM T.J. Watson Research Center*, August 1986.

[56] P. D’Alberto, A. Nicolau, Adaptive Winograd’s Matrix Multiplication. *ACM Transactions on Mathematical Software*, **36**, 1, paper 3, 2009.

[57] A.M. Davie, A.J. Stothers, Improved Bound for Complexity of Matrix Multiplication. *Proceedings of the Royal Society of Edinburgh*, **143A**, 351–370, 2013.

[58] H.F. de Groot, On Varieties of Optimal Algorithms for the Computation of Bilinear Mappings. *Theoretical Computer Science*, **7**, 2, 127–148, 1978.
[59] C. Demetrescu, G.F. Italiano, Fully Dynamic Transitive Closure: Breaking Through the $O(n^2)$ Barrier. In Proceedings of the 41st Annual Symposium on Foundations of Computer Science (FOCS 2000), 381–389, 2000.

[60] J.W. Demmel, Numerical Linear Algebra, SIAM, Philadelphia, 1997.

[61] J. Demmel, I. Dumitriu, O. Holtz, Fast Linear Algebra Is Stable. Numerische Mathematik, 108, 1, 59–91, 2007.

[62] J. Demmel, I. Dumitriu, O. Holtz, R. Kleinberg, Fast Matrix Multiplication Is Stable. Numerische Mathematik, 106, 2, 199–224, 2007.

[63] C.C. Douglas, M. Heroux, G. Slishman, R.M. Smith, GEMMW: A Portable Level 3 BLAS Winograd Variant Of Strassen's Matrix-Matrix Multiply Algorithm. J. of Computational Physics, 110, 1, 1–10, 1994.

[64] C.-E. Drevet, Md. N. Islam, É. Schost, Optimization Techniques for Small Matrix Multiplication. Theoretical Computer Science, 412, 22, 2219–2236, 2011.

[65] P. Drineas, R. Kannan, M.W. Mahoney, Fast Monte Carlo algorithms for matrices I: Approximating matrix multiplication, SIAM Journal on Computing, 36, 1, 132–157, 2006.

[66] R. Duan, S. Pettie, Fast Algorithms for (max – min) Matrix Multiplication and Bottleneck Shortest Paths, Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2009), 384–391, 2009.

[67] J.-G. Dumas, T. Gautier, C. Pernet, Finite field linear algebra subroutines. In Teo Mora, editor, ISSAC’2002, Proceedings of the 2002 ACM International Symposium on Symbolic and Algebraic Computation, Lille, France, pages 63–74. ACM Press, New York, July 2002.

[68] J.-G. Dumas, P. Giorgi, C. Pernet, FFPACK: Finite Field Linear Algebra Package. Proc. Intern. Symposium on Symbolic and Algebraic Computation (ISSAC 2004), Jaime Gutierrez, editor, 119–126, ACM Press, New York, 2004.

[69] J.-G. Dumas, P. Giorgi, C. Pernet, Dense Linear Algebra over Word-size Prime Fields: the FFLAS and FFPACK Packages. ACM Trans. Math. Software, 35, 3, Article 19, 2009.

[70] J.-G. Dumas, V. Y. Pan, Fast Matrix Multiplication and Symbolic Computation, Available in arXiv: 1612.05766, December 2016.

[71] J.-G. Dumas, C. Pernet, Z. Sultan, Computing the Rank Profile Matrix, Proc. ACM International Symposium on Symbolic and Algebraic Computations (ISSAC 2015), 146–153, ACM Press, New York, 2015.

[72] C.M. Fiduccia, Polynomial Evaluation via the Division Algorithm: The Fast Fourier Transform Revisited. Proc. 4th Annual ACM Symposium on Theory of Computing (STOC 1972), 88–93, ACM Press, New York, 1972.

[73] C.M. Fiduccia, On Obtaining Upper Bound on the Complexity of Matrix Multiplication. In Analytical Complexity of Computations (edited by R.E. Miller, J. W. Thatcher, J. D. Bonlinger), in the IBM Research Symposia Series, pp. 31–40, Plenum Press, NY, 1972.

[74] P.C. Fischer, Further Schemes for Combining Matrix Algorithms. Proceedings of the 2nd Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science, 14, 428–436, Springer-Verlag, London, UK, 1974.
[75] M.J. Fischer, M.S. Paterson, String-Matching and Other Products. *SIAM–AMS Proc.*, 7, 113–125, 1974.

[76] B. Ghannam, K. El Khoury, M. Nemer, The Nonrecursive Plating Algorithm (NRPA) for Computing the Total Radiative Exchange Factors in Enclosures. *Numerical Heat Transfer, Part B: Fundamentals: An International Journal of Computation and Methodology*, 66, 2, 109–132, 2014. DOI: 10.1080/10407790.2014.901003

[77] J. von zur Gathen, J. Gerhard (2013). *Modern Computer Algebra*. Cambridge University Press, Cambridge, UK, third edition, 2013.

[78] G.H. Golub, C.F. Van Loan, *Matrix Computations*. Johns Hopkins University Press, Baltimore, Maryland, 2013 (4th addition).

[79] L. Greengard, V. Rokhlin, A Fast Algorithm for Particle Simulation, *Journal of Computational Physics*, 73, 325–348, 1987.

[80] Y. Han, A Θ(n^2) Time Matrix Multiplication Algorithm, preprint, 2016.

[81] N.J. Higham, Exploiting Fast Matrix Multiplication within Level 3 BLAS. *ACM Trans. on Math. Software*, 16, 4, 352–368, 1990.

[82] N.J. Higham, *Accuracy and Stability in Numerical Analysis*, SIAM, Philadelphia, 2002 (second edition).

[83] J.E. Hopcroft, L.R. Kerr, Some Techniques for Proving Certain Simple Programs Optimal. *Proceedings of the Tenth Annual Symposium on Switching and Automata Theory*, 36–45, IEEE Computer Society Press, 1969.

[84] J.E. Hopcroft, L.R. Kerr, On Minimizing the Number of Multiplications Necessary for Matrix Multiplication. *SIAM J. on Applied Math.*, 20, 1, 30–36, 1971.

[85] J.E. Hopcroft, J. Musinski, Duality Applied to Matrix Multiplication and Other Bilinear Forms. *SIAM Journal on Computing*, 2, 3, 159–173, 1973.

[86] X. Huang, V.Y. Pan, Fast Rectangular Matrix Multiplication and Applications. *Journal of Complexity*, 14, 2, 257–299, 1998. Proc. version in *Proc. Annual ACM International Symposium on Parallel Algebraic and Symbolic Computation (PASCO’97)*, 11–23, ACM Press, New York, 1997.

[87] Seung Gyu Hyun, R. Lebreton, É. Schost, Algorithms for structured linear systems solving and their implementation, *Proc. Annual ACM International Symposium on Symbolic and Algebraic Computation (ISSAC 2017)*, 205–212, ACM Press, New York, 2017.

[88] R. W. Johnson, A. M. McLoughlin, Noncommutative Bilinear Algorithms for 3 × 3 Matrix Multiplication, *SIAM J. on Computing*, 15, 2, 595–603, 1986.

[89] H. Kaplan, M. Sharir, E. Verbin, Colored Intersection Searching via Sparse Rectangular Matrix Multiplication. In *Proceedings of the 22nd ACM Symposium on Computational Geometry*, 52–60, 2006.

[90] I. Kaporin, A Practical Algorithm for Faster Matrix Multiplication. *Numerical Linear Algebra with Applications*, 6, 8, 687–700, 1999.

[91] I. Kaporin, The Aggregation and Cancellation Techniques as a Practical Tool for Faster Matrix Multiplication. *Theoretical Computer Science*, 315, 2–3, 469–510, 2004.

[92] S. Ke, B. Zeng, W. Han, V. Y. Pan, Fast Rectangular Matrix Multiplication and Some Applications. *Science in China, Series A: Mathematics*, 51, 3, 389–406, 2008.
[93] P. Kirrinis, Polynomial Factorization and Partial Fraction Decomposition by Simultaneous Newton’s Iteration, *J. of Complexity*, **14**, 378–444, 1998.

[94] D.E. Knuth, *The Art of Computer Programming: Volume 2, Seminumerical Algorithms*. Addison-Wesley, Reading, Massachusetts, 1969 (first edition), 1981 (second edition), 1997 (third edition).

[95] D.E. Knuth, *The Art of Computer Programming: Volume 3, Sorting and Searching*. Addison-Wesley, Reading, Massachusetts, 19673 (first edition), 1998 (second edition).

[96] T.G. Kolda, B.W. Bader, Tensor Decompositions and Applications. *SIAM Review*, **51**, 3, 455–500, 2009.

[97] J. B. Kruskal, Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics, *Linear Algebra and Its Applications*, **18**, 2, 95–138, 1977.

[98] E. Karstadt, O. Shwatrz, Matrix Multiplication, a Little Faster, Proc. SPAA’17, 101–110, 2017.

[99] J.D. Laderman, A noncommutative algorithm for multiplying 33 matrices using 23 multiplications, *Bull. Amer. Math. Soc.*, **82**, 1, 126–128, 1976.

[100] J. Laderman, V.Y. Pan, H.-X. Sha, On Practical Algorithms for Accelerated Matrix Multiplication. *Linear Algebra and Its Applications*, **162–164**, 557–588, 1992.

[101] J.M. Landsberg, New Lower Bound for the Rank of Matrix Multiplication. *SIAM J. on Computing*, **43**, 1, 144–149, 2014.

[102] L. Lee, Fast Context-free Grammar Parsing Requires Fast Boolean Matrix Multiplication. *Journal of the ACM (JACM)*, **49**, 1, 1–15, 2002.

[103] F. Le Gall, Faster Algorithms for Rectangular Matrix Multiplication. *Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS 2012)*, 514–523, IEEE Computer Society Press, 2012.

[104] F. Le Gall, Powers of Tensors and Fast Matrix Multiplication. *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation (ISSAC 2014)*, 296–303, ACM Press, New York, 2014.

[105] G. Lotti, F. Romani, On the Asymptotic Complexity of Rectangular Matrix Multiplication. *Theoretical Computer Science*, **23**, 171–185, 1983.

[106] R. G. Lerner, G. L. Trigg. *Encyclopaedia of Physics* (2nd ed.), VHC publishers, 1991. ISBN 3-527-26954-1.

[107] M. W. Mahoney, Randomized Algorithms for Matrices and Data, *Foundations and Trends in Machine Learning*. NOW Publishers, **3**, 2, 2011. (Abridged version in: Advances in Machine Learning and Data Mining for Astronomy, edited by M. J. Way, et al., pp. 647-672, 2012.)

[108] A. Massarenti, E. Raviolo, Corrigendum to "The rank of $n \times n$ matrix multiplication is at least $3n^2 - 2\sqrt{2n}^{3/2} - 3n$" [Linear Algebra and its Applications, **438**, 11 (2013) 4500–4509]. *Linear Algebra and its Applications*, **445**, 369–371, 2014.

[109] W. L. Miranker, V. Y. Pan, Methods of Aggregations, *Linear Algebra and Its Applications*, **29**, 231–257, 1980.

[110] R. Moenck, A. Borodin, Fast Modular Transform via Division, *Proc. 13th Ann. Symp. Switching Automata Theory*, 90–96, IEEE Comp. Soc. Press, Washington, DC, 1972.
[111] T.S. Motzkin, Evaluation of Polynomials and Evaluation of Rational Functions. *Bull. of Amer. Math. Society*, 61, 2, 163, 1955.

[112] Jinsoo Oh, Jin Kim, Byung-Ro Moon, On the inequivalence of bilinear algorithms for 3 3 matrix multiplication. *Information Processing Letters*, 113, 17, 640–645, 2013.

[113] I.V. Oseledets, E.E. Tyrtyshnikov, TT-cross Approximation for Multidimensional Arrays. *Linear Algebra Appl.*, 432, 1, 70–88, 2010.

[114] A.M. Ostrowski, On Two Problems in Abstract Algebra Connected with Horner’s Rule. In the *Studies Presented to R. von Mises*, 40–48, Academic Press, New York, 1954.

[115] V.Y. Pan, Some Schemes for the Evaluation of Polynomials with Real Coefficients, *Doklady Akademi Nauk SSSR* (in Russian), 127, 2, 266–269, 1959.

[116] V.Y. Pan, On Methods of Computing the Values of Polynomials. *Uspekhi Matematicheskikh Nauk*, 21, 1(127), 103–134, 1966. [Transl. *Russian Mathematical Surveys*, 21, 1(127), 105–137, 1966.]

[117] V.Y. Pan, On Schemes for the Evaluation of Products and Inverses of Matrices (in Russian). *Uspekhi Matematicheskikh Nauk*, 27, 5 (167), 249–250, 1972.

[118] V.Y. Pan, Strassen’s Algorithm Is Not Optimal. Trilinear Technique of Aggregating for Fast Matrix Multiplication. *Proc. the 19th Annual IEEE Symposium on Foundations of Computer Science (FOCS’78)*, 166–176, IEEE Computer Society Press, Long Beach, California, 1978.

[119] V.Y. Pan, Fields Extension and Trilinear Aggregating, Uniting and Canceling for the Acceleration of Matrix Multiplication. *Proceedings of the 20th Annual IEEE Symposium on Foundations of Computer Science (FOCS’79)*, 28–38, IEEE Computer Society Press, Long Beach, California, 1979.

[120] V.Y. Pan, New Fast Algorithms for Matrix Operations. *SIAM J. on Computing*, 9, 2, 321–342, 1980, and Research Report RC 7555, *IBM T.J. Watson Research Center*, February 1979.

[121] V.Y. Pan, The Bit-Operation Complexity of the Convolution of Vectors and of the DFT. Technical report 80-6, Computer Science Dept., SUNY, Albany, NY, 1980. (Abstract in Bulletin of EATCS, 14, page 95, 1981.)

[122] V.Y. Pan, New Combinations of Methods for the Acceleration of Matrix Multiplications. *Computers and Mathematics (with Applications)*, 7, 1, 73–125, 1981.

[123] V.Y. Pan, Trilinear Aggregating with Implicit Canceling for a New Acceleration of Matrix Multiplication. *Computers and Mathematics (with Applications)*, 8, 1, 23–34, 1982.

[124] V.Y. Pan, Fast Matrix Multiplication without APA-Algorithms, *Computers and Mathematics (with Applications)*, 8, 5, 343–366, 1982.

[125] V.Y. Pan, How Can We Speed up Matrix Multiplication? *SIAM Review*, 26, 3, 393–415, 1984.

[126] V.Y. Pan, Trilinear Aggregating and the Recent Progress in the Asymptotic Acceleration of Matrix Operations. *Theoretical Computer Science*, 33, 1, 117–138, 1984.

[127] V.Y. Pan, *How to Multiply Matrices Faster. Lecture Notes in Computer Science*, 179, Springer, Berlin, 1984.
[128] V. Y. Pan, On Computations with Dense Structured Matrices, *Math. of Computation*, 55, 191, 179–190, 1990. Proceedings version in *Proc. Intern. Symposium on Symbolic and Algebraic Computation (ISSAC’89)*, 34–42, ACM Press, New York, 1989.

[129] C. B. Parker. *McGraw Hill Encyclopaedia of Physics (2nd ed.)*, 1994. ISBN 0-07-051400-3.

[130] V. Y. Pan, *Structured Matrices and Polynomials: Unified Superfast Algorithms*, Birkhäuser/Springer, Boston/New York, 2001.

[131] V.Y. Pan, Better Late Than Never: Filling a Void in the History of Fast Matrix Multiplication and Tensor Decompositions, arXiv:1411.1972 November 2014.

[132] V.Y. Pan, Fast Approximate Computations with Cauchy Matrices and Polynomials, *Math. of Computation*, in print. Proceedings version in *Proc. of the Ninth International Computer Science Symposium in Russia (CSR’2014)*, (E. A. Hirsch et al., editors), Moscow, Russia, June 2014, *Lecture Notes in Computer Science (LNCS)*, 8476, pp. 287–300 Springer International Publishing, Switzerland (2014).

[133] V.Y. Pan, Transformations of Matrix Structures Work Again, *Linear Algebra and Its Applications*, 465, 1–32, 2015.

[134] V.Y. Pan, Matrix Multiplication, Trilinear Decompositions, APA Algorithms, and Summation, arxiv 1412.1145 CS (16 pages, 2 figures), December 3, 2014, revised on February 5, 2015.

[135] V.Y. Pan, How Bad Are Vandermonde Matrices?, *SIAM Journal of Matrix Analysis*, 37, 2, 676–694, 2016.

[136] V. Y. Pan, E. Landowne, A. Sadikou, Univariate Polynomial Division with a Reminder by Means of Evaluation and Interpolation, *Information Processing Letters*, 44, 149–153, 1992.

[137] V. Y. Pan, E. P. Tsigaridas, Nearly Optimal Computations with Structured Matrices, *Theoretical Computer Science*, Special Issue on Symbolic–Numerical Algorithms (Stephen Watt, Jan Verschelde, and Lihong Zhi, editors), 681, 117–137, 2017. [http://dx.doi.org/10.1016/j.tcs.2017.03.031](http://dx.doi.org/10.1016/j.tcs.2017.03.031)

Proceedings version in *Proc. of the International Conference on Symbolic Numeric Computation (SNC 2014)*, ACM Press, New York, 2014. Also April 18, 2014, arXiv:1404.4768 [math.NA] and [http://hal.inria.fr/hal-00980591](http://hal.inria.fr/hal-00980591)

[138] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, (3 ed.). Cambridge University Press, New York, 2007. ISBN 978-0-521-88068-8.

[139] R.L. Probert, On the Complexity of Symmetric Computations. *Canadian J. of Information Processing and Operational Research*, 12, 1, 71–86, 1974.

[140] R. L. Probert, On the Additive Complexity of Matrix Multiplication. *SIAM J. on Computing*, 5, 2, 187–203, 1976.

[141] R. Raz, On the complexity of matrix product. In *Proceedings of the Thirty-fourth Annual ACM Symposium on Theory of Computing*, 144–151, ACM Press, New York, 2002.

[142] R. Raz, A. Shpilka, Lower Bounds for Matrix Product, in Bounded Depth Circuits with Arbitrary Gates. *SIAM J. on Computing*, 32, 2, 488–513, 2003.

[143] F. Romani, private communication, 1979.
[144] F. Romani, Some Properties of Disjoint Sum of Tensors Related to MM, *SIAM J. on Computing*, 11, 2, 263–267, 1982.

[145] P. Sankowski, M. Mucha, Fast Dynamic Transitive Closure with Lookahead. *Algorithmica*, 56, 2, 180–197, 2010.

[146] A. Schönhage, Partial and Total Matrix Multiplication. *SIAM J. on Computing*, 10, 3, 434–455, 1981.

[147] A. Shpilka, Lower Bounds for Matrix Product. *SIAM J. on Computing*, 32, 5, 1185–1200, 2003. Proceedings version in *FOCS 2001*.

[148] A.A. Shabalin, Matrix eQTL: Ultra Fast eQTL Analysis via Large Matrix Operations. *Bioinformatics*, 28, 10, 1353–1358, 2012.

[149] A.V. Smirnov, The Bilinear Complexity and Practical Algorithms for Matrix Multiplication. *Computational Mathematics and Mathematical Physics*, 53, 12, 1781–1795 (Pleiades Publishing, Ltd), 2013. Original Russian Text in *Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki*, 53, 12, 1970–1984, 2013.

[150] A.J. Stothers, On the Complexity of Matrix Multiplication. Ph.D. Thesis, University of Edinburgh, 2010.

[151] V. Strassen, Gaussian Elimination Is Not Optimal. *Numerische Math.*, 13, 354–356, 1969.

[152] V. Strassen, Evaluation of Rational Functions, in *Analytical Complexity of Computations* (edited by R.E. Miller, J. W. Thatcher, and J. D. Bonlinger), pages 1-10, Plenum Press, New York, 1972.

[153] V. Strassen, Vermeidung von Divisionen. *J. Reine Angew. Math.*, 1973, 264, 184–202, 1973.

[154] V. Strassen, Some Results in Algebraic Complexity Theory, *Proceedings of the International Congress of Mathematicians*, Vancouver, 1974 (Ralph D. James, editor), Volume 1, pages 497–501, Canadian Mathematical Society, 1974.

[155] V. Strassen, The Asymptotic Spectrum of Tensors and the Exponent of Matrix Multiplication. *Proc. 27th Ann. Symposium on Foundation of Computer Science*, 49–54, 1986.

[156] J. Todd, Motivation for Working in Numerical Analysis. *Communication on Pure and Applied Math.*, 8, 1, 97–116, 1955.

[157] A. L. Toom, The Complexity of a Scheme of Functional Elements Realizing the Multiplication of Integers, *Soviet Mathematics Doklady*, 3, 714–716, 1963.

[158] N. L. Trefethen, D. Bau, III, *Numerical Linear Algebra*, SIAM, Philadelphia, 1997.

[159] E. E. Tyrtyshnikov, Tensor Approximations of Matrices Generated by Asymptotically Smooth Functions. *Mat. Sbornik*, 194, 6, 147–160, 2003.

[160] L. Valiant, General Context-Free Recognition in Less Than Cubic Time, *J. of Computer and System Sciences*, 10, 308–315, 1975.

[161] C.F. Van Loan, *Computational Frameworks for the Fast Fourier Transform*. SIAM, Philadelphia, 1992. ISBN: 0-89871-285-8.
[162] V. Vassilevska Williams, Multiplying Matrices Faster than Coppersmith–Winograd. Version available at [http://theory.stanford.edu/virgi/matrixmult-f.pdf](http://theory.stanford.edu/virgi/matrixmult-f.pdf), retrieved on January 30, 2014. Also see Proc. 44th Annual ACM Symposium on Theory of Computing (STOC 2012), 887–898, ACM Press, New York, 2012.

[163] A. Waksman, On Winograd’s Algorithm for Inner Products. *IEEE Transactions on Computers*, C-19, 4, 360–361, 1970.

[164] S. Winograd, On the Number of Multiplications Required to Compute Certain Functions. *Proc. of the National Academy of Sciences*, 58, 5, 1840–1842, 1967.

[165] S. Winograd, A New Algorithm for Inner Product. *IEEE Transaction on Computers*, C–17, 7, 693–694, 1968.

[166] S. Winograd, On the Number of Multiplications Necessary to Compute Certain Functions. *Communications on Pure and Applied Mathematics*, 23, 2, 165–179, 1970.

[167] S. Winograd, *Arithmetic Complexity of Computations*. CBMS-NSF Regional Conference Series in Applied Math., 33, SIAM, Philadelphia, 1980.

[168] Y. Xi, J. Xia, S. Cauley, V. Balakrishnan, Superfast and Stable Structured Solvers for Toeplitz Least Squares via Randomized Sampling, *SIMAX*, 35, 44–72, 2014.

[169] J. Xia, Y. Xi, M. Gu, A Superfast Structured Solver for Toeplitz Linear Systems via Randomized Sampling, *SIMAX*, 33, 837–858, 2012.

[170] H. Yu, An improved combinatorial algorithm for Boolean matrix multiplication, *Proc. Int. Conf. Automata, Language, and Programming (ICALP 2015)*, 1094–1105, 2015.

[171] R. Yuster, Efficient Algorithms on Sets of Permutations, Dominance, and Real-weighted APSP *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2009)*, 384–391, 2009.

[172] R. Yuster, U. Zwick, U. Detecting Short Directed Cycles Using Rectangular Matrix Multiplication and Dynamic Programming, In *Proceedings of the 15th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2004)*, 254–260, 2004.

[173] R. Yuster, U. Zwick, Fast Sparse Matrix Multiplication. *ACM Transactions on Algorithms*, 1, 1, 2–13, 2005.

[174] R. Yuster, U. Zwick, Answering Distance Queries in Directed Graphs Using Fast Matrix Multiplication. In *Proceedings of 46th Annual IEEE Symposium on the Foundations of Computer Science (FOCS 2005)*, 389–396, IEEE Computer Society Press, 2005.

[175] U. Zwick. All-pairs Shortest Paths Using Bridging Sets and Rectangular Matrix Multiplication. *Journal of the ACM*, 49, 3, 289–317, 2002.