IDEAL GAS OF STRINGS AND QCD AT HADRONIC SCALES*

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ABSTRACT

By using lessons learned from modern string studies, we show how interesting non-perturbative features of QCD can be learned from future heavy ion collisions even if the deconfinement density is not reached.

1. Introduction

It is the hope of many people that experimental studies of ultra-relativistic nuclear collisions in the near future could reach sufficiently high energy densities so as to reveal the physics of deconfined quarks and gluons. In this talk, I would like to suggest, based on lessons learned from modern string studies,\(^1\)\(^-\)\(^3\) that interesting non-perturbative features of QCD can already be learned even if the deconfinement energy density, \(E_d\), is not reached.

It is well understood that the character of QCD changes depending on the nature of available probes. At short distances, the basic degrees of freedom are quarks and gluons. As one moves to larger distance scales, the QCD coupling increases and one enters the non-perturbative regime. Short of resulting to lattice Monte Carlo studies, the most promising tool for a non-perturbative treatment of QCD which builds in naturally quark-gluon confinement remains the large-\(N\) expansion. In this scheme, although the vacuum of QCD at hadronic scales is complicated, model studies suggest that the effective degrees of freedom of QCD there can most profitably be expressed in terms of “extended objects”. Indeed, low-lying hadron spectrum suggests that they can be understood as “string excitations”. In high-energy soft hadronic collisions\(^4\) where the interactions are mostly peripheral, it is possible to “see” the dominant string excitations in terms of the exchanges of high-lying Regge trajectories.

We would like to suggest that, in heavy ion collisions, it is possible to probe the structure of string excitations inaccessible to other types of high energy hadronic collisions. This can be accomplished even if heavy ion collision experiments in the near future fail to reach past the deconfinement scale \(E_d\).

Assuming thermo-equilibrium can be achieved in hadronic multi-particle production, one expects that the energy density \(E\) can be parametrized monotonically in terms of a temperature, \(T\). At low temperatures, \(E(T)\) can be given effectively in terms of pion gas. At high temperature and after deconfinement, one has Stefan-

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Boltzmann law, (appropriate for gluons and light quarks), which, for SU(2) flavor, leads to $\mathcal{E} \simeq 12T^4$. Indeed, this expectation has been substantiated by lattice calculations,$^5$ as depicted below.

Fig. 1. The energy density in finite temperature QCD with Wilson fermions, taken from Ref. 5.

These “numerical experiments” suggest that the deconfinement density is of the order $\mathcal{E}_d \simeq 12T_H^4$, where $T_H \sim 200\text{MeV}$. We would like to stress that these numerical experiments also indicate the existence of another density scale, $\mathcal{E}_H \simeq cT_H^4$, where $c = 0(1)$. We believe that, for $\mathcal{E}_H < \mathcal{E} < \mathcal{E}_d$ where the notion of temperature loses much of its usefulness, the effective degrees of freedom for QCD are “string-like”. Probing this kinematical regime should reveal interesting non-perturbative features of confined QCD at hadronic scales.$^6$

2. Counting Effective Degrees of Freedom at Hadronic Scales

The basic degrees of freedom of a string theory are string excitations. Each excitation can be given a particle attribute, i.e., a mass $m$, with its center of mass energy $p_0$ and spatial momentum vector $\vec{p}$, $p_0^2 = m^2 + \vec{p}^2$. The mass is due to internal string oscillations; as such, it can take on increasing values, with a corresponding increase in degeneracies. For instance, the original dual model leads to an operator expression for the mass squared, $\alpha' m^2 = -1 + \sum_{i=1}^{3} \sum_{n=1}^{\infty} n\hat{N}_i^2$, where $\alpha'$ is the “Regge slope” and $\hat{N}_i$ is an oscillator number operator taking on eigenvalues $0, 1, 2, \ldots$. Well-known features of this model include: (i) equal-spacing rule for the mass spectrum, and (ii) exponentially increasing mass degeneracies, [i.e., for $m^2 = \alpha'N$, the degeneracy $d(m) = \sum_{n=0}^{\infty} \delta(N + 1 - \sum_{i=1}^{3} \sum_{n=1}^{\infty} n\hat{N}_i^2)$ increases exponentially with $\sqrt{N}$].

Formulating a consistent effective string theory for QCD has been one of the major challenges for string theorists.$^7$ Since we are still far from accomplishing this
task, any insight into the problem, either theoretical or experimental, could prove to be useful. A common feature of all string-like theories is the rapid increase of mass degeneracies. We shall assume that the desired effective QCD string theory has an asymptotic exponential mass degeneracy, which characterizes the growth of its effective degrees of freedom. Alternatively, one finds that the single-particle density behaves similarly at high energies

\[ f(E, V) = V \sum_i d(m_i) \int \frac{d^3k}{h^3} \delta(E - \sqrt{k^2 + m_i}) \propto E^{-(\gamma + 1)} e^{\beta_H E}. \] (1)

That is, we shall characterize a string theory minimally by two critical exponents, \( \gamma \) and \( \beta_H \), (the latter is often referred to as the inverse Hagedorn temperature.)

It can be demonstrated that, for systems which can be characterized by Eq. (1), there always exists an energy density scale, \( \varepsilon_H \), above which the conventional canonical description becomes inapplicable.\(^{[1-3]}\) A microcanonical analysis can nevertheless be carried out which is valid both above and below \( \varepsilon_H \). On the other hand, under an ideal gas approximation, the treatment is incapable of yielding the “deconfinement transition scale”, \( \varepsilon_d \), which could come about only when interactions are included. Since Monte Carlo experiments seem to indicate the existence of a sizable density interval, \([\varepsilon_H, \varepsilon_d]\), in which a large number of string states are excited, our subsequent analysis should be meaningful there.

3. Ideal Gas of Strings at High Energy Densities

For statistical systems with a finite number of fundamental degrees of freedom, it is well-known that a microcanonical and a canonical descriptions are equivalent. For strings, this equivalence breaks down at high energy densities. However, it turns out that the canonical partition function remains useful when considered as an analytic function of the inverse temperature, \( \beta = 1/T \).

The fundamental quantity in a canonical approach is the partition function:

\[ Z(\beta, V) \equiv \text{Tr} e^{-\beta H} = \sum_\alpha e^{-\beta E_\alpha}, \]

where the sum is over all possible multiparticle states of the system. For a microcanonical approach, one works with a density function, which counts the number of microstates, \( \Omega(E, V) dE \equiv \sum_\alpha \delta(E - E_\alpha) \) \( dE \). Statistical mechanics based on a microcanonical ensemble is more general, even though it is often more convenient to work with \( Z(\beta, V) \), e.g., when interactions must be included.

Representing the \( \delta \)-function by an integral along an imaginary axis, we find that \( \Omega(E, V) = \sum_\alpha \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} e^{\beta (E - E_\alpha)}. \) Note that the integral is in the form of an inverse Laplace transform over the complex-\( \beta \) plane. If one can deform the contour into a region where interchanging the order of sum and integral is allowed, one obtains

\[ \Omega(E, V) = \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} Z(\beta, V) e^{\beta E}. \] \(^{(2)}\)

The allowed region is labelled by the interception of the contour with the real axis, \( \beta_0 \). One can then recover \( Z(\beta, V) \) from \( \Omega(E, V) \) via a Laplace transform, \( Z(\beta, V) = \int_0^\infty dE \Omega(E, V) e^{-\beta E} \), which provides an alternative analytic definition for \( Z(\beta, V) \).
For conventional systems, Eq. (2) can often be approximated by a saddle point contribution at \( \hat{\beta} \). For a closed system where \( E \) is fixed, the usual notion of a temperature is given by \( \hat{\beta}^{-1}(E) \), \( \mathcal{E} = E/V \), and is related to \( E \) by the stationary condition: \( E = -\frac{\partial \log Z}{\partial \hat{\beta}} \). We shall refer to this as the “Boltzmann temperature”.

We have considered elsewhere, in a quantum statistical treatment, analytic property of \( Z(\beta, V) \) in \( \beta \) for various string theories.\(^{[1]}\) We find that, generically, \( Z(\beta, V) \) is analytic for \( \text{Re} \ \beta > 0 \) except at isolated points. For each string theory, because of the exponential growth in mass degeneracy, there is always an isolated rightmost singularity at \( \beta = \beta_H \), i.e., the inverse Hagedorn temperature for that theory. There is a finite gap in their real parts between \( \beta_H \) and the next singularity to the left, and this gap is theory-dependent but calculable.

To be more specific, for an ideal gas of strings under the Maxwell-Boltzmann (MB) statistics, one has \( Z(\beta, V) \simeq e^{\tilde{f}(\beta, V)} \), where \( \tilde{f}(\beta, V) \) is the inverse Laplace transform of the single-particle density, \( f(E, V) \). It can be shown that \( \tilde{f}(\beta, V) \) is analytic for \( \text{Re} \ \beta \) sufficiently large and its rightmost singularity is at \( \beta = \beta_H \). Given \( Z(\beta, V) \) as an analytic function of \( \beta \), \( \Omega(E, V) \) can be recovered through Eq. (2), with \( \beta_0 > \beta_H \). That is, the totality of physics of microcanonical approach for free strings has been entirely encoded in the analyticity of \( Z(\beta, V) \).\(^{[1]}\)

For strings, at low energies, Eq. (2) can be saturated by a saddle point at \( \hat{\beta}(E) \), lying to the right of \( \beta_H \). However, as the energy density \( \mathcal{E} \) is raised, one reaches a point where either the saddle point moves to the left of \( \beta_H \), or it gets close to \( \beta_H \) that the fluctuations about the saddle point become large. When this occurs, the saddle point approximation to Eq. (2) break down, and it defines the lower density scale \( E_H \) for the region of interest to heavy ion collisions spelled out earlier.

For \( \mathcal{E} > E_H \), whereas it is no longer meaningful to speak of a Boltzmann temperature, the statistical mechanics of free strings is still given unambiguously by Eq. (2). One can in fact push the contour in (2) to the left of the singular point, \( \beta_H \), by a finite distance \( \eta \), \( \eta > 0 \). As one moves past this point, one picks up an additional contribution involving the discontinuity across the cut. Denoting the discontinuity by \( \Delta Z(\beta, V), \beta < \beta_H \), the large-\( E \) behavior of \( \Omega(E, V) \) is dominated by the singularity at \( \beta_H \)

\[
\Omega(E, V) = - \int_{\beta_H - \eta}^{\beta_H} \frac{d\beta}{2\pi i} \Delta Z(\beta, V)e^{\beta E} + 0(e^{(\beta_H - \eta)E}), \quad \eta > 0.
\]

Once \( \Delta Z(\beta, V) \) is known, the dominant behavior of \( \Omega(E, V) \) can be found. Therefore, the large-\( E \) limit of a free-string theory can best be approached by working first with the canonical quantity, \( Z(\beta, V) \).\(^{[1,2]}\) (Note that earlier related works of Hagedorn and co-workers\(^{[5]}\) seem to have concentrated on the approach to the region \( \mathcal{E} \sim E_H \) from below. Our analysis, on the other hand, emphasizes on the region above \( E_H \).)

### 4. Critical Exponent \( \gamma \) and Classifications:

We shall classify different types of effective string theories for confined QCD according by the critical exponent \( \gamma \) given in Eq. (1). In standard string theories,
$2\gamma$ is equal to the number of “uncompactified” spatial dimensions. Therefore, the canonical value is $\gamma = 3/2$. However, since we are dealing with an effective string theory for QCD, we could not rule out the possibility that this critical index can take on an anomalous value. Indeed, as we suggest below, heavy ion collisions offer the unique possibility of measuring this critical exponent. This could in turn provide useful hint in our search for the realistic effective string theory for QCD.

It can be shown that, for $\gamma > 0$, $\tilde{f}(\beta)$ has a branch point at $\beta_H$ but it is bounded. For $\gamma < 0$, $\tilde{f}(\beta)$ has a divergent algebraic branch point. Finally, when $\gamma = 0$, $\tilde{f}(\beta)$ has a logarithmic branch point at $\beta_H$. Model string theories have been constructed for both cases where $\gamma > 0$ and for $\gamma = 0$. Instead of providing an exhaustive analysis, we shall assume below that $\gamma \geq 0$.

For $\gamma > 0$, $\tilde{f}(\beta)$ can be parametrized as $\tilde{f}(\beta) \sim g(\beta)(\beta - \beta_H)\gamma + \lambda(\beta)$, where $\lambda(\beta)$ is regular at $\beta_H$. It follows that $Z(\beta)$ has the same type of singularity at $\beta_H$, i.e., $\Delta Z(\beta) \sim (\beta - \beta_H)\gamma$. For $\gamma = 0$, one has $\tilde{f}(\beta) \sim -c\log(\beta - \beta_H) + \lambda(\beta)$. We shall assume below that $c = 1$, as suggested in the study of fundamental strings.[1–3]

The detailed statistical properties of a string gas turns out to be sensitive to the value of $\gamma$. This can best be brought out by studying the “inclusive distributions” for our ideal gas of strings. For instance, the single-string distribution with a definite energy $\epsilon$ under the MB statistics can be expressed as

$$D(\epsilon; E) = \frac{1}{\Omega(E)} f(\epsilon) \left[ \Omega(E - \epsilon) + \delta(E - \epsilon) \right].$$

The distribution $D(\epsilon; E)$ is normalized so that, upon integration over $\epsilon$, it yields the average number of strings, $\langle N \rangle$, in an ensemble with a total energy $E$, which can be more easily measured experimentally.

Fig. 2. Schematic plots of $\epsilon D(\epsilon; E)$ for a string gas with (a): $\gamma \geq 3/2$, and (b): $\gamma = 0$.

Exhaustive studies of this type can be found in Ref. 2, where we make use of the analytic properties of $\tilde{f}(\beta)$ and $Z(\beta)$ Here we shall mention several interesting features. For an ordinary gas of particles, $\epsilon D(\epsilon; E)$ as a function of $\epsilon$ is typically $\epsilon^\alpha e^{-\beta \epsilon}$, i.e., the usual Boltzmann distribution. In the case of string gas, for $\gamma \geq 3/2$, we find that the distribution is peaked at the two ends [Fig. (2a)] with a single energetic string soaking up most of the energy. This feature has been referred to in the past as the Frautschi-Carlitz picture. In particular, one finds that $\langle N \rangle$ becomes
energy-independent at large $E$. On the other hand, for $\gamma = 0$, we have a scale-invariant energy distribution, with strings of all energies contributing equally to the total energy. One finds that $\langle N \rangle \sim \log E$, due to the $1/\epsilon$ tail of the distribution at large energies. Experimental detection of this unique signature would be most interesting. (The situation is more complicated for $0 < \gamma < 3/2$. See Ref. 2.)

5. Comments:

While the free string gas already exhibits novel properties, it is of great interest to study the interacting gas. This could lead further insights for describing the onset of deconfinement, a feature which is absent in an ideal gas setting. Finally, I have emphasized here on how lessons learned from modern string studies could be used to study the physics of quarks and gluons at confinement scales. It is of course also true that insights from non-perturbative QCD studies, both theoretical and experimental, can also provide new hints for works on fundamental strings. Increasing future collaborative efforts in this direction could prove beneficial for both communities.

6. References

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5. H. Satz, Fortschr. Phys. 33 (1985) 4, 259-268, (in honor of R. Hagedorn).
6. Atick and Witten, (Ref. 3), suggested that the deconfinement transition in QCD should be of the first order and would take place at a temperature below $T_H$, i.e., in terms of the energy density, $\mathcal{E}_d < \mathcal{E}_H$. We would like to suggest that $\mathcal{E}_H < \mathcal{E}_d$, as indicated by the MC data.
7. For recent reviews: J. Polchinski, “Strings and QCD?”, UTTG-16-92, and D. J. Gross, “Some New/Old Approaches to QCD”, LBL 33232, PUPT 1355.