Spacing of the entropy spectrum for KS Black hole in Hořava-Lifshitz gravity

M. R. Setare *
D. Momeni †
Department of Science, Payame Noor University, Bijar, Iran

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Abstract

In this paper we present the spectrum of entropy/area for Kehagias-Sfetsos (KS) black hole in Hořava-Lifshitz (HL)gravity via quasi-normal modes (QNM) approach. We show that in the massive case the mass parameter $\mu$ disappears in the entropy spectrum and only the quasinormal modes modified by a term which is proportional to the mass square term. Our calculations show that the charge like parameter $\frac{1}{2\omega} = Q^2$ appears in the surface gravity and our calculations can be applied to any spherically symmetric spacetime which has only one physically acceptable horizon. Our main difference between our calculations and what was done in [1] is that the portion of charge and mass is included explicitly in the surface gravity and consequently in the QNM expression. Since the imaginary part of the QNM is related to the adiabatic invariance of the system and also to the entropy, surprisingly the mass parameter do not appear in the entropy spectrum. Our conclusion supported by some acclaims about that the scalar field parameters (charges ) can not change the fundamental parameters in the 4-dimensional black holes.

1 I: Introduction

The quantization of the area of a black hole in general relativity (GR) has an old history. This topic return to the Bekenstein works on the BH physics [2]. Without no doubt the first acclaim about the existence of an upper bound for the BH entropy and the corresponding analogous as a holographic model belongs to the Bekenstein. The Hawking radiation [3] was discussed in many papers and by different authors both in the context of the GR and the alternative gravity models.

Recently Hořava presents a new non relativistic model for Quantum gravity which has renormalizable and has a good UV limit for propagators [4, 5, 6]. There is no unique vacuum solution for HL theory. Also latterly, some cylindrical and plane solutions was obtained [10, 11]. More recently Majhi [1] discussed Hawking radiation and spectrum of entropy/area for the Cai-Cao and Ohta [7] spherically symmetric static black hole solution

*E-mail: rezakord@ipm.ir
†E-mail:dmomeni@phymail.ut.ac.ir
in Hořava-Lifshitz theory by using Tunneling formalism [8] and QNM [9]. According to the Blas et al arguments [12], it seems that this model must be modified by some terms to avoiding from strong coupling, instabilities, dynamical in consistencies and unphysical extra mode. One of the first exact solutions for this modified version is the work of Kiritsis [13]. Indeed the Kiritsis work contains some previous families of exact solutions as a special sub class and has a good asymptotic behaviors. The explicit form of exact solution for this modified version deal with some algebraic quadratures and lead finally to an implicit static spherically symmetric metric. But no doubt this solution generic, avoids from the trouble problems which occur in the original version of HL. The thermodynamics of the HL black holes was discussed by A. Wang et. al [20] and specifically for KS solution by M. Wang, et. al [21]. Later the Area spectrum for different BHs calculated [18].

The quantization of the black hole area in the framework of Einstein gravity, has been considered [14, 15], as a result of the absorption of a quasi-normal mode excitation. The quasi-normal modes of black holes are the characteristic, ringing frequencies which result from their perturbations [16] and provide a unique signature of these objects [17], possible to be observed in gravitational waves. In this short paper, we focused on the KS solution [19] and following the QNM method we derived an expression for entropy/area spectrum.

2 II: KS black hole solution in HL theory

Following from the ADM decomposition of the metric [22], and the Einstein equations, the fundamental objects of interest are the fields \(N(t, x), N_i(t, x), g_{ij}(t, x)\) corresponding to the lapse, shift and spatial metric of the ADM decomposition. In the (3+1)-dimensional ADM formalism, where the metric can be written as

\[
ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)
\]

and for a spacelike hypersurface with a fixed time, its extrinsic curvature \(K_{ij}\) is

\[
K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i)
\]

where a dot denotes a derivative with respect to \(t\) and covariant derivatives defined with respect to the spatial metric \(g_{ij}\), the action of Hořava-Lifshitz theory for \(z = 3\) is

\[
S = \int_M dt^3 x \sqrt{g} N (\mathcal{L}_K - \mathcal{L}_V)
\]

we define the space-covariant derivative on a covector \(v_i\) as \(\nabla_i v_j \equiv \partial_i v_j - \Gamma^l_{ij} v_l\) where \(\Gamma^l_{ij}\) is the spatial Christoffel symbol. \(g\) is the determinant of the 3-metric and \(N = N(t)\) is a dimensionless homogeneous gauge field. The kinetic term is

\[
\mathcal{L}_K = \frac{2}{\kappa^2} \mathcal{O}_K = \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2)
\]

Here \(N_i\) is a gauge field with scaling dimension \([N_i] = z - 1\).

The potential term \(\mathcal{L}_V\) of the \((3 + 1)\)-dimensional theory is determined by the principle of detailed balance [4], requiring \(\mathcal{L}_V\) to follow, in a precise way, from the gradient flow generated by a 3-dimensional action \(W_g\). This principle was applied to gravity with the
result that the number of possible terms in $\mathcal{L}_V$ are drastically reduced with respect to the broad choice available in an potential is

$$\mathcal{L}_V = \alpha_6 C_{ij} C^{ij} - \alpha_5 \epsilon_i^j R_{im} \nabla_j R^{ml} + \alpha_4 [R_{ij} R^{ij} - \frac{4\lambda - 1}{4(3\lambda - 1)} R^2] + \alpha_2 (R - 3\Lambda_W)$$

Where in it $C_{ij}$ is the Cotton tensor [4] which is defined as,

$$C^{ij} = \epsilon^{kl} \nabla_k R_{ij}$$

The kinetic term could be rewritten in terms of the de Witt metric as:

$$\mathcal{L}_K = \frac{2}{\kappa^2} K_{ij} G^{ijkl} K_{kl}$$

Where we have introduced the de Witt metric

$$G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}$$

The inverse of this metric is given by

$$G_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \tilde{\lambda} g_{ij} g_{kl}$$

Inspired by methods used in quantum critical systems and non equilibrium critical phenomena, Hořava restricts the large class of possible potentials using the principle of detailed balance outlined above. This requires that the potential term takes the form

$$\mathcal{L}_V = \frac{\kappa^2}{8} E^{ij} G_{ijkl} E^{kl}$$

Note that by constructing $E^{ij}$ as a functional derivative it automatically transverse within the foliation slice, $\nabla_i E^{ij} = 0$. The equations of motion were obtained in [23]. KS BH is a static spherically symmetric solution for HL theory which contains 2 parameter, one mass like parameter $m$ and a parameter which controls the escape from a naked singularity $\omega$ and satisfies [19]

$$\omega m^2 \geq \frac{1}{2}$$

In the usual spherical coordinates $(t, r, \theta, \phi)$ and in the Schwarzschild’s gauge the metric reads:

$$ds^2 = \text{diag}(-f, \frac{1}{f}, r^2 \Sigma_2)$$

where in it the metric gauge function is

$$f = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4m\omega r}$$

and is $\Sigma_2$ the surface element on a unit 2- sphere. As motivated by Sekiwa ”it is obvious that $1/2\omega$ is equivalent to $Q^2$ and this means that we could view $1/2\omega$ as a charge in some degree” [21] [24]. Thus The outer and inner event horizon can be compared with the outer and inner event horizon of Reissner-Nordstrom black hole. Essentially as claimed by the founders of the KS, this solution ”represents the analog of the Schwarzschild solution of GR”. 

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3 III:QNM formalism for scalar field in spherically symmetric KS backgrounds

QNM concept is based on Bohr’s correspondence principle (1923): transition frequencies at large quantum numbers should equal classical oscillation frequencies. Hence, we are interested in the asymptotic behavior (i.e., the \( n \to \infty \) limit) of the ringing frequencies. These are the highly damped black-hole oscillations frequencies, which are compatible with the statement quantum transitions do not take (See, for example, [25]). We consider a massive scalar field with a typical mass \( \mu \) satisfying the wave equation \((\nabla_\alpha \nabla^\alpha + \mu^2)\phi = 0\). The scalar perturbation fields for a massive particle outside the spherically symmetric black hole (1) are governed by a one-dimensional Schrødinger-like wave equation\[9\]:

\[
-\frac{d^2 \Psi}{dr_\ast^2} + V(r)\Psi = \omega^2 \Psi
\]

where the tortoise radial coordinate \( r_\ast \) is related to the spatial radius \( r \) (using the metric function (2)) by

\[
r_\ast = \int \frac{dr}{f}
\]

and the Potential term is given by

\[
V(r) = f(\frac{l(l+1)}{r^2} + \frac{f'}{r} - \mu^2)
\]

\( l \) is the multipole moment index. For electromagnetic, and gravitational perturbations see \[9\]. For solving the ODE (3) we must adopt a suitable boundary conditions. As one can observe that, this equation cannot be solved exactly and in general only we can determine a series solution or a Poincare’s asymptotic solution which is very good for our QNM purposes. One method is due to Medved, et. al \[9\]. Since the potential function (4) vanishes at the horizon located at the positive and real root of the algebraic equation \( f = 0 \) or the limiting point of the tortoise coordinate \( r_\ast \to -\infty \), and at spatial infinity \( r \to \infty \) or equivalently at \( r_\ast \to \infty \), it is so adequate to define the QNMs to be those modes for which we have purely ingoing plane wave at the horizon and no wave at spatial infinity. The mathematical expression which addressed correctly to this situation is,

\[
\Psi_{QNM} = e^{i\omega r_\ast}\quad \text{at}\quad r_\ast \to -\infty
\]

\[
\Psi_{QNM} = 0\quad \text{at}\quad r_\ast \to -\infty
\]

The remaining part of the calculations is straightforward. We solve the ODE (3) near horizon \( r = h \) and comparing the solution at the asymptotic region with (5),(6). By definition of the new radial coordinate \( z = r - h \) and representing the exact solution for the (3) in terms of the value of the surface gravity as \( \kappa = \frac{1}{2}f'(h) \). We consider first massless limit of the potential barrier (4).

\(^1\)Assuming a time dependence of the form \( e^{i\omega t} \) and we decompose the scalar field as \( \phi = \Psi Y_l(\theta, \varphi)e^{i\omega t} \)
3.1 Massless scalar fields $\mu^2 = 0$

If we set the mass parameter $\mu = 0$ in potential function (4), we can write the following expression for (3).\footnote{The degenerate hypergeometric functions $\Phi(a, b; x)$ and $\psi(a, b; x)$ are solutions of the degenerate hypergeometric equation. In the case $b$ is not negative integer, the function $\Phi(a, b; x)$ can be represented as Kummers series.}

$$\psi = z^\frac{\mu^2}{2} U[\alpha, \beta; \gamma z]$$

$$\kappa = 2\omega \frac{h - 2m}{1 + \omega h^2}$$

$$h = m + \sqrt{m^2 - Q^2}$$

$$\alpha = \frac{1}{4}(2 - i/\kappa(\delta(2\kappa h + l(l + 1)) - 2\omega))$$

$$\beta = 1 + i\frac{\omega}{\kappa}$$

$$\gamma = 2i\sqrt{h + l(l + 1)/\kappa/h^{3/2}}$$

$$\delta = \sqrt{\frac{\kappa}{h(\kappa l + (l + 1))}}$$

We note here that for the asymptotic boundary conditions the second function $LaguerreL[n, a, x]$ gives the generalized Laguerre polynomial and only if $a = 0$, $LaguerreL[n, x]$ is an entire function of $x$ with no branch cut discontinuities. Here is the series expansion around $z = 0$ to order 1 for generalized Laguerre polynomials $LaguerreL[n, a, bz]$.

$$LaguerreL[n, a, 0] - bLaguerreL[-1 + n, 1 + a, 0]z + Q[\varepsilon^2]$$

Since in our solution $a$ does not vanish at all, thus the only way to avoiding from a undesirable branch cut discontinuities is that setting the $c_2 = 0$. It seems that the author of the [1] completely ignore from this function. But we show that this term must be ignored by consideration of the un-physical discontinuity of the wave solution near the horizon. We know that the value of the field function must be remained finite on the horizon. The only quantity which may be diverge on this surface is the value of the stress-energy tensor. In the limit $z \to 0$ using from the Asymptotic expansion series expression for the Hypergeometric functions

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)}e^{x}x^{a-b}\left[\sum_{n=0}^{N} \frac{(b-a)n(1-a)n}{n!}x^{-n} + \varepsilon\right], x > 0$$

$$\Phi(a, b; x) = \frac{\Gamma(b)}{\Gamma(b-a)}(-x)^{-a}\left[\sum_{n=0}^{N} \frac{(a)n(a-b+1)n}{n!}(-x)^{-n} + \varepsilon\right], x < 0$$

$$\varepsilon = O(x^{-N-1})$$

\footnote{U is the confluent hypergeometric function has the integral representation}

$$U[a, b, z] = \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-zt}t^{a-1}(1+t)^{b-a-1}dt$$
and comparing the solution with our desired boundary conditions (5), (6)

$$\Psi = c_1' e^{-1/2(\beta-1)} \frac{\Gamma(\beta-1)}{\alpha(\omega)} + c_2' e^{1/2(\beta-1)} \frac{\Gamma(1-\beta)}{\alpha(-\omega)}$$  \hspace{1cm} (8)

we lead to the following relation for the QNMs:

$$\omega_n = \delta(\kappa h + l(l + 1)/2) + i\kappa(2n + 1)$$  \hspace{1cm} (9)

Immediately the imaginary part of the frequency of the QNMs is

$$\text{Im}(\nu_n) = (2n + 1)\kappa = 2\pi(2n + 1)\frac{T_H}{h},$$  \hspace{1cm} (10)

where $T_H$ is the Hawking temperature of the black hole. Finally by calculating the adiabatic invariant quantity and using of the first law of thermodynamics, the Bohr-Sommerfield quantization rule we can write the next formula for the spacing of the entropy spectrum

$$S_n = 4\pi n$$  \hspace{1cm} (11)

### 3.2 Massive scalar fields $\mu^2 \neq 0$

In case of the massive particle, the only change that must be done is a redefinition of the the set of the parameters $\alpha, \beta, ...$ in (7). By a simple computations similar to which done for a massless one, we observe that the form of the scalar perturbations (7) must be written as

$$\Psi = c_1 z^{i\mu_2} \Phi[\alpha', \beta, \gamma z] + + c_2 z^{i\mu_2} \text{Laguerre}\mathcal{L}[\alpha', \beta - 1, \gamma z]$$ \hspace{1cm} (12)

where in it the new shifted parameter $\alpha$ is defined by

$$\alpha' = \frac{i\sqrt{\kappa(h^2\mu^2 - l(l + 1) - 2h\kappa) + 2\sqrt{h(h\kappa + l(l + 1))(\kappa + i\omega)}}}{4\kappa\sqrt{h(h\kappa + l(l + 1))}}$$  \hspace{1cm} (13)

$$\alpha' = \alpha + i\frac{\delta h^2\mu^2}{4\kappa}$$

With a same discussion as (7) we can set $c_2 = 0$, and another parameters coincide with (7). Instead of the QNMs (9) we have another slightly different equation as:

$$\omega_n = \delta(\kappa h + l(l + 1)/2) - \frac{\delta h^2\mu^2}{2} + i\kappa(2n + 1)$$  \hspace{1cm} (14)

which obviously recovers the massless equation (9). Following the next steps as was written for massless case show that there is no portion of the mass parameter in the spectrum of the entropy. In the other hand, from a pure quantum mechanical point of view, the mass of the particle only changed the zero point energy and not the QNMs.

We mention here that the equispaced property remains unchanged even if we use from the tunneling mechanism as was shown for topological black holes in HL theory [1]. Although the exact value of the spacing in these two different approaches does not coincide, but their order of magnitudes are same.
4 IV: Conclusion

In the present short letter, we have studied the entropy spectrum associated with the black hole event horizon for KS black hole solutions in HL theory. We derived the spectrum of entropy/area of the new kind of the black hole in context of the HL theory, KS black hole via the QNM approach for massless and massive scalar fields perturbations. Explicitly we showed that the entropy spectrum was equispaced in the large quantum number limit as usually happens for Einstein gravity and Einstein-Gauss-Bonnet gravity. On the other hand, since the entropy was not proportional to the area, the area spectrum was not equispaced. Consequently, it has a similarity with the Einstein-Gauss-Bonnet theory, rather than the usual Einstein gravity. Also we derived the general formula for the QNM. This equation contains two Quantum numbers: first the principal quantum number \( n \) and second the multipole quantum number \( l \). The spectrum of the eigen frequencies correspond to the QNM are very like to the well known spectrum of a 3-dimensional spherically symmetric simple harmonic oscillator. As we know that if a simple charged harmonic oscillator exerted to a uniformly electric field in a specified direction the energy spectrum of the oscillator changes only up to order of a term which is proportional to the square of the field strength. This simple exact result coincides to the perturbative result for any order of the correction. The simplest contribution of the angular quantum numbers arises from the monopole term \( l = 0 \). In [1] the results were presented only for this special case and nothing was stated about the contribution of another higher multipole terms. As we can see from (9) and (14) if the multipole \( l \) increases, the \( \delta \) factor decreases and the variation of the QNM totally monotonically increases and we can argue that any multipole term bigger than monopole ones, increases the QNM. Thus we focused only on the monopole \( l = 0 \) term, we leak more information of the classical spectrum of the oscillations.

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