Lower Bounds to Quality Factor of Small Radiators through Quasistatic Scattering Modes

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Abstract—The problem of finding the optimal current distribution supported by a small radiator yielding the minimum quality ($Q$) factor of electrically small radiators is a fundamental problem in electromagnetism. $Q$ factor bounds constrain the maximum operational bandwidth of devices including antennas, metamaterials, and open optical resonators. In this manuscript, a representation of the optimal current distribution in terms of quasistatic scattering modes is introduced. Quasi-electrostatic and quasi-magnetostatic modes describe the resonances of small plasmonic and high-permittivity particles, respectively. The introduced representation leads to analytical and closed form expressions of the electric and magnetic polarizability tensors of arbitrarily shaped objects, whose eigenvalues are known to be linked to the minimum $Q$. Hence, the minimum $Q$ and the corresponding optimal current are determined from the sole knowledge of the eigenvalues and the dipole moments associated with the quasistatic modes. It is found that, when the radiator exhibits two orthogonal reflection symmetries, its minimum $Q$ factor can be simply obtained from the $Q$ factors of its quasistatic modes, through a parallel formula. When an electric type radiator supports a spatially uniform quasistatic resonance mode, or when a magnetic type resonator supports a mode of curl type, then these modes are guaranteed to have the minimum $Q$ factor.

I. INTRODUCTION

Chu [1] limit determines the minimum radiation quality ($Q$) factor of electrically small radiators. This limit applies to both self-resonant objects and non self-resonant objects, provided that a convenient tuning network is used for the latter. The minimum $Q$ factor is associated with an optimal current distribution supported by the radiator.

The search for such lower bounds originated with the works of Chu [1], Wheeler [2], and Harrington [3], and several techniques were proposed over the years by many contributors including Collin and Rothchild [4], and McLean [5]. Thal [6] noticed that the previous approaches did not account for either the energy stored within the antenna volume, or for any energy stored in the space between the actual antenna’s physical boundary and the enclosing sphere (in cases where they are not the same), thus leading to loose bounds. By including these contributions, he arrived to stricter bounds compared to his predecessors. Then, in a series of contributions, starting in Ref. [7], Gustafsson and coworkers provided shape-dependent bounds on the radiator’s minimum $Q$, linking it to the available volume in which the search of the optimal current is constrained. They also reduced the variational problem of finding the minimum $Q$ of radiators to determining the largest eigenvalues of either electric or magnetic polarizability tensors. In subsequent years, Gustafsson and co-workers refined these ideas [8], [9] exploiting the expressions for the reactive stored energy derived by Vandenbosch [10], [11], and they also included magnetic type radiators. Efficient numerical determination of these optimal currents in terms of a characteristic mode expansion was also recently demonstrated [12], [13], [14]. In this manuscript, by “minimum $Q$” we mean the shape-dependent minimum $Q$ achievable for the specific geometry of interest.

The $Q$ factor limits are also applicable to optical nanoantennas, which are often self-resonant, as in the case of plasmonic and dielectric open resonators [15], [16]. Plasmonic resonances emerge in small objects made of materials with a negative real part of the permittivity (metals), and they may be described within the quasi-electrostatic approximation of Maxwell’s equations [17], [18], [19]. Dielectric resonances emerge in small objects made of materials with a high and positive real part of permittivity, and they may be described by the quasi-magnetostatic approximation [20]. Each quasistatic resonance is associated with a current density distribution (quasistatic mode) exhibiting a radiation $Q$ factor [21], which, in the limit of high $Q$ and non-interacting modes, is equal to the inverse of the fractional bandwidth of the corresponding resonance peak. Interestingly, the recent literature on $Q$ factor bounds divides radiators into two categories (e.g., [10], [11], [22]), depending on the features of the currents they support. Radiators of the electric type support currents with zero curl, i.e., longitudinal currents, while radiators of the magnetic type support currents with zero divergence, i.e., transverse currents. This distinction naturally applies also to nanoantennas, since the plasmon resonances are driven by longitudinal currents, while the resonances in high permittivity dielectric objects are driven by transverse currents.

Given a small radiator of prescribed shape, supporting either plasmonic or dielectric resonances, it is then natural to ask i) how the quality factor of its plasmonic/dielectric modes compare to the minimum $Q$ factor admitted by that shape; ii) whether there is a link between the optimal current density distribution and the plasmonic/dielectric modes; and iii) for which geometries plasmonic/dielectric modes are associated with the minimum $Q$ factor.

In this manuscript, we introduce a representation of the optimal current in terms of quasistatic modes. This representation not only unveils the intimate connection between the optimal current yielding the minimum $Q$ factor and the modes...
supported by plasmonic and high-permittivity resonators, but it has several appealing advantages. A quasi-static mode expansion of the current density leads to analytical and closed form expressions of the electric and magnetic polarizability tensors of arbitrarily shaped objects, whose eigenvalues have been linked by Gustafsson and coworkers to the minimum $Q$ \cite{7}. This property implies that the minimum $Q$ and the corresponding optimal current distribution can be directly determined from the knowledge of the eigenvalues and the dipole moments of quasi-static scattering modes. We also show that, when a radiator exhibits two orthogonal reflection symmetries, its minimum quality factor can be simply obtained from the $Q$ factors of the quasi-static modes associated with the radiator’s shape, through a simple parallel formula. Moreover, when a resonator of the electric type supports a spatially uniform quasi-static resonance mode, this mode is guaranteed to have the minimum quality factor. Because of duality, when a resonator of the magnetic type supports quasi-magnetostatic mode of curl type, in the form $\mathbf{r} \times \mathbf{e}$ where $\mathbf{e}$ is a constant vector and $\mathbf{r}$ is the radial direction, this mode is guaranteed to have the minimum $Q$ factor. We finally expand the discussion presenting lower $Q$ bounds for translationally invariant objects, a topic that has been touched upon only superficially by previous investigators \cite{23, 24, 4}.

The manuscript is organized as follows. In Sec. \ref{sec:electrostatic_modes} we summarize the main features of the quasi-electrostatic and quasi-magnetostatic scattering modes. Then in Sec. \ref{sec:electromagnetic_scattering} we address the problem of finding the minimum $Q$ in terms of quasi-static scattering modes. In Sec. \ref{sec:examples} many examples are worked out, exemplifying the application of the introduced method to electric and magnetic radiators of arbitrary shape, three-dimensional and translationally invariant.

II. Electromagnetic Scattering Resonances of Small Objects

Consider a linear, homogeneous, isotropic, and nonmagnetic material occupying a domain, either a volume $\Omega$ or a surface $\Sigma$, of characteristic linear dimension $\ell_c$, see Fig. 1. In the former case, the material is assumed to have a relative dielectric permittivity $\varepsilon_r$, in the latter it is characterized by a surface conductivity $\sigma$. In both cases, it is surrounded by vacuum. If the size of the object is much smaller than the operating wavelength, resonant electromagnetic scattering occurs due to two different mechanisms (e.g., \cite{20, 21}), which we will discuss separately.

A. Quasi-electrostatic (plasmonic) resonances

When the real part of the permittivity is negative (e.g., metals) resonances arise from the interplay between the energy of the quasi-static electric field and the polarization energy.

1) Resonant modes of three dimensional objects: The quasi-electrostatic (QES) resonances of a three-dimensional object are associated with the eigenvalues $\chi_h$ of the integral operator that determines the electrostatic field as a function of the surface charge density on the surface $\partial \Omega$ \cite{17, 18}:

$$j_h^\| (r) = -\chi_h \nabla \int_{\partial \Omega} \frac{\hat{n} (r') \cdot j_h^\| (r')}{4\pi |r-r'|} dS',$$

where the spatial coordinates have been normalized by $\ell_c$, i.e., $r \rightarrow r'/\ell_c$. $\Omega$ is the corresponding scaled domain, $\partial \Omega$ is the boundary of $\Omega$, $\hat{n}$ is the outward-pointing normal, and $\nabla$ is the scaled gradient operator. The spectrum of the operator in Eq. \ref{eq1} is discrete \cite{18}. Each QES mode $j_h^\|$ is characterized by a real and negative eigenvalue $\chi_h$, which is size-independent \cite{18}. The modes $\{j_h^\|\}_{h \in \mathbb{N}}$ are longitudinal vector fields: they are square integrable in $\Omega$, i.e., $\in L^2 (\Omega)$, they are both curl-free and divergence free within the object, but have a nonvanishing normal component to the object surface \cite{18}, i.e.,

$$L_h^\| (\Omega) = \left\{ L^2 (\Omega) \mid \nabla \cdot w^\| = 0, \nabla \times w^\| = 0 \text{ in } \Omega \setminus \partial \Omega \right\}.$$  \label{eq2}

This normal component is related to the induced surface charge density on $\partial \Omega$, and satisfies the charge-neutrality condition

$$\int_{\partial \Omega} j_h^\| \cdot \hat{n} dS = 0.$$  \label{eq3}

Moreover, the QES modes are orthonormal:

$$\langle j_h^\|, j_{h'}^\| \rangle / \Omega = \delta_{hh'}$$  \label{eq4}

according to the scalar product

$$\langle f, g \rangle _\Omega = \int_{\Omega} f^* \cdot g dV,$$  \label{eq5}

where $\delta_{hh'}$ is the Kronecker delta. Under the normalization \cite{4} the electrostatic energy of the $h$-th QES current mode in the scaled object is given by

$$\mathcal{E}_e \left\{ j_h^\| \right\} = \frac{1}{2\varepsilon_0} \frac{1}{\chi_h}.$$  \label{eq6}

The electric dipole moment $P_h$ of the QES mode $j_h^\|$ of the scaled object is given by

$$P_h = \int_{\Omega} j_h^\| dV = \int_{\partial \Omega} (j_h^\| \cdot \hat{n}) r dS.$$  \label{eq7}

If the mode $j_h^\|$ exhibits a vanishing electric dipole moment ($P_h = 0$) it is called dark, it is bright otherwise \cite{25}.

Fig. 1. Q factor bounds on arbitrary shaped scatterers/radiators: In this work we explore and extract the optimal current distribution on an arbitrary shaped scatterer enclosed by the circumscribing "radiansphere" of radius $\ell_c$ for achieving the minimum possible $Q$. For this task we employ quasi-electrostatic and magnetostatic modes of plasmonic (metallic) or high-permittivity and explore the connection of these bounds with the shape symmetries.
Furthermore, if the shape of the resonator has two orthogonal reflection symmetries, the dipole moment of each QES mode is aligned along either one of these directions.

When the object exhibits a current mode that is spatially uniform in $\Omega$ along a given direction $c$, i.e., $j_{h}^\parallel = c$, as it happens, for instance, for a sphere or a spheroid, the orthogonality condition implies that all the remaining current modes $j_{h}^\perp$ exhibit a vanishing electric dipole moment along that specific direction, i.e.,

$$\int_{\Omega} c \cdot j_{k}^\parallel = c \cdot P_k = 0. \quad (8)$$

The resonance frequency $\omega_{h}$ of the $h$-th QES mode is the frequency at which the real part of the metal susceptibility $\chi = \varepsilon_{R} - 1$ matches the eigenvalue $\chi_{h}$:

$$\operatorname{Re} \{\chi(\omega_{h})\} = \chi_{h}. \quad (9)$$

The corresponding resonance size parameter is given by

$$x_{h} = \frac{\omega_{h}}{c_{0} \ell_{c}}. \quad (10)$$

If the $h$-th plasmonic mode is dark ($P_{h} \neq 0$), its quality factor is given by:

$$Q_{h}^\parallel = \frac{8}{(\varepsilon_{r} - \varepsilon_{h})|P_{h}|^2 x_{h}^2}. \quad (11)$$

On the contrary — if the mode is bright — the quality factor presents a more complicated expression which takes into account higher-order modes, diverging faster than $1/x_{h}^2$.

2) Resonant modes of two dimensional objects: The QES resonant modes of a two dimensional object occupying the surface $\Sigma$ are solution of the eigenvalue problem

$$j_{h}^\parallel(\mathbf{r}) = -\chi_{h} \nabla S \oint_{\Omega} \nabla S \cdot j_{h}^\parallel(\mathbf{r}') dS' \quad \forall \mathbf{r} \in \Sigma \quad (12)$$

where the spatial coordinates have been normalized by $\ell_{c}$. All considerations made for the three dimensional objects can be transplanted in this case. The resonance frequency $\omega_{h}$ of the $h$-th QES mode is the frequency at which

$$\operatorname{Im} \{\sigma(\omega_{h})\} = \frac{1}{(\varepsilon_{r} - \varepsilon_{h})} \left(\frac{\omega_{h}}{c_{0} \ell_{c}}\right). \quad (13)$$

The corresponding size parameter of the resonance is given by Eq. $\text{(10)}$. The electric dipole moment $P_h$ of the $h$-th mode is obtained by Eq. $\text{(2)}$ where the integration is now performed on the surface $\Sigma$, while the $Q$ factor is still given by Eq. $\text{(11)}$.

3) Resonant modes of translationally invariant objects: Similarly, the QES resonant modes of a translationally invariant (TI) object with cross-section $\Sigma$ are solution of the eigenvalue problem:

$$j_{h}^\parallel(\mathbf{r}) = \chi_{h} \nabla S \oint_{\partial \Sigma} j_{h}^\parallel(\mathbf{r}') \cdot \hat{n}(\mathbf{r}') \frac{\log|\mathbf{r} - \mathbf{r}'|}{2\pi} d\mathbf{l}', \quad (14)$$

where $\partial \Sigma$ is the boundary of $\Sigma$ and $\log|\mathbf{r} - \mathbf{r}'|/2\pi$ is the 2D static Green function.

The operator of Eq. $\text{(14)}$ supports an infinite countable set of real and negative eigenvalues $\{\chi_{h}\}_{h \in \mathbb{N}}$, which are mirror-symmetric $\text{[18]}$. The corresponding modes $\{j_{h}^\parallel\}$ are longitudinal, curl-free within the object, but with normal component (in the object’s cross-section) to the object’s boundary different from zero. They are orthonormal (see Eq. $\text{(4)}$) according to the scalar product $\text{[5]}$ where the integral is now extended on the cross-section $\Sigma$. The reader interested in the properties of electrostatic resonances and modes in translational invariant objects may consult Ref. $\text{[18]}$.

The electric dipole moment $P_h$ of the QES current mode $j_{h}^\parallel$ is defined as $3D$ objects in Eq. $\text{(2)}$, where the integration domain has to be replaced by the object cross-section $\Sigma$.

If the $h$-th plasmonic mode is bright ($P_{h} \neq 0$), its quality factor is given by:

$$Q_{h}^\parallel = \frac{8}{(\varepsilon_{r} - \varepsilon_{h})|P_{h}|^2 x_{h}^2}. \quad (15)$$

On the contrary, a dark mode is characterized by a $Q$ factor depending on higher-order multipoles, diverging faster than $1/x_{h}^2$.

B. Quasi-magnetostatic (dielectric) resonances

If the real part of the permittivity of the object is positive, as in the case of a dielectric, and sufficiently large, scattering resonances arise from the interplay between the magnetic field energy of the quasi-magnetostatic current density modes of the object and the polarization energy.

1) Resonant modes of three-dimensional objects: The quasi-magnetostatic (QMS) resonances of three-dimensional objects are associated with the eigenvalues $\kappa_{h}$ of the magnetostatic integral operator that gives the vector potential as a function of the current density $\text{[20], [21]}$:

$$j_{h}^\parallel(\mathbf{r}) = \kappa_{h} \int_{\Omega} \frac{j_{h}^\parallel(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} dV' \quad \forall \mathbf{r} \in \Omega, \quad (16)$$

with

$$j_{h}^\parallel \cdot \hat{n}|_{\partial \Omega} = 0. \quad (17)$$

The above equation holds in its weak form in the functional space equipped with the inner product $\text{[5]}$ and constituted by the transverse vector fields which are divergence free within $\Omega$ and have zero normal component to $\partial \Omega$, i.e.

$$\mathbf{L}^\perp_{\perp}(\Omega) = \{\mathbf{L}^2(\Omega) | \nabla \cdot \mathbf{w}^\perp = 0, \mathbf{w}^\perp \cdot \hat{n} = 0 \text{ on } \partial \Omega\}, \quad (18)$$

The spectrum of the QMS operator $\text{[16]}$ is discrete $\text{[20]}$. Each QMS current mode $j_{h}^\parallel$ is characterized by a real and positive eigenvalue $\kappa_{h}$. The current modes $\{j_{h}^\parallel\}_{h \in \mathbb{N}}$ belongs to $\mathbf{L}^\perp_{\perp}$.

Each current mode $j_{h}^\parallel$ has zero electric dipole moment. Furthermore, the QMS current density modes are orthonormal:

$$\langle j_{h}^\parallel | j_{k}^\parallel \rangle = \delta_{h,k}. \quad (19)$$

Under this normalization, the QMS energy of the $h$-th mode in the scaled object is

$$\mathcal{H}_{m} \{j_{h}^\parallel\} = \frac{\mu_{0}}{2} \frac{1}{\kappa_{h}}. \quad (20)$$

The magnetic dipole moment $M_{h}$ of the $h$-th mode $j_{h}^\parallel$ in the scaled object is given by

$$M_{h} = \frac{1}{2} \int_{\Omega} \mathbf{r} \times j_{h}^\parallel dV. \quad (21)$$
If the shape of the resonator has two orthogonal reflection symmetries, the dipole moment of each QMS mode is aligned along either one of these directions. If the resonator supports a mode of the form \( \mathbf{j}_h^\perp = \hat{r} \times \mathbf{c} \), where \( \mathbf{c} \) is a constant vector, then the orthogonality condition [19] implies that the generic remaining current mode \( \mathbf{j}_h^\perp \) has a vanishing magnetic dipole moment along \( \mathbf{c} \), i.e.,

\[
\int_\Omega \mathbf{j}_h^\perp \cdot (\hat{r} \times \mathbf{c}) \, dV = \mathbf{c} \cdot \mathbf{M}_h = 0 \tag{22}
\]

The resonance frequency of the \( h \)-th QMS mode \( \mathbf{j}_h^\perp \) is defined as the value of \( \omega \) at which

\[
\frac{\omega_h}{c_0} \ell_c = \sqrt{\frac{\kappa_h}{\text{Re} \left\{ \chi \right\}}}.
\tag{23}
\]

The corresponding value of size parameter at the resonance is given by Eq. [9]. If the \( h \)-th mode has nonvanishing magnetic dipole moment, its quality factor is given by [21]:

\[
Q_h^\perp = \frac{6\pi}{\kappa_h |\mathbf{M}_h|^2} \frac{1}{x_h^2} \tag{24}
\]

If this is not the case, the quality factor presents a more complicated expression which takes into account higher-order multipolar orders of the mode.

2) Resonant modes of surfaces: Resonant electromagnetic scattering from a non-magnetic small surface \( \Sigma \) may also occur when the imaginary part of its surface conductivity is positive and sufficiently high [20]. The corresponding QMS resonances are associated with the eigenvalues of the integral operator that relates the vector potential to the surface current density:

\[
\mathbf{j}_h^\perp (\mathbf{r}) = \kappa_h \int_\Sigma \frac{\mathbf{j}_h^\perp (\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dS',
\tag{25}
\]

All considerations made for three-dimensional objects can be also transplanted to this scenario. The QMS resonance frequency \( \omega_h \) of the \( h \)-th mode is the frequency at which [20]

\[
\text{Im} \left\{ \sigma (\omega_h) \right\} = \frac{1}{\kappa_h \omega_h \ell_c}.
\tag{26}
\]

The corresponding size parameter at the resonance is given by [10].

The magnetic dipole moment \( \mathbf{M}_h \) of the \( h \)-th QMS current mode \( \mathbf{j}_h^\perp \) can be obtained by Eq. [21], where the integration domain has to be replaced by the object cross-section \( \Sigma \). If the \( h \)-th QMS mode has nonvanishing magnetic dipole moment, its quality factor is given by:

\[
Q_h^\perp = \frac{4}{\kappa_h |\mathbf{M}_h|^2} \frac{1}{x_h^2}.
\tag{29}
\]

As for the 3D case, if the mode has vanishing magnetic dipole moment, the quality factor depends on higher-order modes, and diverges faster than \( \frac{1}{x_h^2} \).

III. LINKING OPTIMAL CURRENTS WITH QUASISTATIC CURRENT MODES

A. Minimum Q factor of radiators of the electric type

We now consider three-dimensional radiators of the electric type, supporting longitudinal current distributions, which belong to \( L_{1,2}^\perp \). Their \( Q \) factor can be expressed as \( 2\pi \) times the ratio of the electrostatic energy stored in the whole space to the energy radiated toward infinity in a period [10], [11], [9]:

\[
Q = 3 \frac{\int_{\partial \Omega} \sigma (\mathbf{r}) \int_{\partial \Omega} \sigma (\mathbf{r}') \, dSdS' \int_{\partial \Omega} \sigma (\mathbf{r}) \, dS}{\int_{\partial \Omega} \sigma (\mathbf{r}) \int_{\partial \Omega} \sigma (\mathbf{r}') |\mathbf{r} - \mathbf{r}'|^2 \, dSdS' \int_{\partial \Omega} \sigma (\mathbf{r}) \, dS} \frac{1}{x^3},
\tag{30}
\]

where \( \sigma = \mathbf{j} \cdot \mathbf{n} \). The problem of finding the minimum \( Q \) factor consists of finding the optimal current density \( \mathbf{j} \) in the functional space \( L^2_\parallel (\Sigma) \), yielding the minimum value of the functional \( x^3Q \).

The above expression for the radiation \( Q \) factor and the considerations we make in this section also hold for surface current densities supported on \( \Sigma \) provided the quantity \( \mathbf{j} \cdot \mathbf{n} \) is replaced by \( \nabla_S \cdot \mathbf{j} \).

Following [9], the minimization can be recast as the problem of finding the optimal current distribution in \( L^2_\parallel \), having a specified (i.e., constrained) squared magnitude of the dipole moment \( P^2 \), yielding the minimum electrostatic energy stored in the whole space. This constrained optimization problem can be treated through the method of Lagrange multipliers [9], searching for the stationary points of the auxiliary Lagrangian

\[
\mathcal{L}_E (\sigma, \lambda) = \int_{\partial \Omega} \sigma (\mathbf{r}) \int_{\partial \Omega} \sigma (\mathbf{r}') \, dSdS' - \lambda \left( \int_{\partial \Omega} \sigma (\mathbf{r}) \, dS \right)^2 - P^2 \tag{31}
\]

where \( \lambda \) is a Lagrange multiplier. The variational problem Eq. [31] yields the critical equations [22]:

\[
\int_{\Omega} \frac{\sigma (\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} \, dS' - \lambda \mathbf{r} \cdot \int_{\partial \Omega} \sigma (\mathbf{r}') \, dS' = 0; \tag{32}
\]
any pair \((λ, σ)\) satisfying Eq. 32 is a local minimum of the functional \(x^3Q\), and among them the absolute minimum can be found. In particular, the minimum of the \(Q\) factor is [22]:

\[
x^3Q_{\text{min}} = \frac{6π}{\lambda_{\text{e,max}}}.
\]

where \(λ_{\text{e,max}}\) is the maximum among the 3 eigenvalues of the electric polarizability tensor \(γ_e\) of the scaled object.

The polarizability tensor relates linearly \(ε_0E_0\hat{e}\) and the electric dipole moment \(\mathbf{P}\), defined as

\[
\mathbf{P} = \oint_{\partial Ω} σ(r) dS,
\]

where \(σ\) is the solution of the surface integral equation

\[
\oint_{\partial Ω} \frac{σ(r')} {4π|\mathbf{r} - \mathbf{r'}|} dS = (ε_0E_0\hat{e}) \cdot \mathbf{r} \quad \forall \mathbf{r} \in \partial Ω,
\]

and it is subjected to the charge neutrality condition.

Note that the surface integral operator in Eq. 35 is tightly related to the operator describing the electrostatic (plasmonic) modes (Eq. 1).

One of the main contributions of the present manuscript is the calculation of the polarizability tensor by using a QES modal expansion. The procedure consists in expanding the charge density \(σ\), solution of Eq. 35 in terms of the QES modes \(\mathbf{j}_h\), which constitute a complete basis for \(L_2^\| (Ω)\), namely

\[
σ(r) = \sum_{h=1}^∞ α_h \mathbf{j}_h(r) \cdot \hat{n}(r) \quad \text{on } \partial Ω. \tag{36}
\]

The charge density \(σ(r)\) constructed in this way naturally satisfies the charge neutrality condition. By using the above expansion in Eq. 35 multiplying both members by \(\mathbf{j}_h^\dagger \cdot \hat{n}\), integrating over the volume \(Ω\), and exploiting the orthogonality condition 4 we obtain the expansion coefficients \(α_h \) of the charge density distribution

\[
α_h = -\frac{γ_e}{P_{\text{opt}}} \mathbf{P}_h \cdot (ε_0E_0\hat{e}), \tag{37}
\]

and the corresponding dipole moment

\[
\mathbf{P} = -\sum_{h=1}^∞ γ_e \mathbf{P}_h \cdot (ε_0E_0\hat{e}) \hat{e}, \tag{38}
\]

where \(\otimes\) denotes the tensor product, and \(\mathbf{P}_h\) is defined in Eq. 7. The \(3 \times 3\) electric polarizability tensor \(γ_e\) is

\[
\tilde{γ}_e = -\sum_{h=1}^∞ γ_e \mathbf{P}_h \otimes \mathbf{P}_h. \tag{39}
\]

The above closed-form identity is one of the main results of this paper, as it bridges the polarizability tensor with the QES (plasmonic) modes.

In the general case, the maximum eigenvalue \(γ_{e,max}\) of \(\tilde{γ}_e\) is associated with the minimum \(Q\) factor [21]. The corresponding (normalized) eigenvector returns the direction of the dipole moment of the optimal current, which we call \(\mathbf{P}_{\text{opt}}\). The optimal current is readily obtained in terms of the QES (plasmonic) modes:

\[
\mathbf{j}_{\text{opt}}(r) = -\sum_{h=1}^∞ γ_e \mathbf{P}_h \cdot (\mathbf{P}_{\text{opt}} \cdot \mathbf{P}_h) \mathbf{j}_h(r). \tag{40}
\]

As we will see in Sec. [IV] for small radiators only a few QES modes have to be considered to have a good estimate of the minimum \(Q\) factor.

If the shape of the resonator has two orthogonal reflection symmetries, \(\tilde{γ}_e\) can be immediately put in the form of a diagonal matrix by choosing a coordinate system \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) aligned along its principal axes. The dipole moments of the QES modes are also aligned along these directions. In this case, the three occurrences of \(\tilde{γ}_e\) are given by:

\[
γ_i = \sum_{\hat{e}_i-\text{aligned}} (-λ_h) |\mathbf{P}_h|^2 \quad i = 1, 2, 3 \tag{41}
\]

where the summation only runs over the QES modes exhibiting dipole moment directed along \(\hat{e}_i\). In this case, by combining...
Eqs. [11] [33] and [11] we obtain that the minimum $Q$ along the axis $\hat{e}_i$ is obtained by the parallel formula

$$\frac{1}{(x^3 Q)_{\text{min}}} = \sum_{h \perp \hat{e}_i} \frac{1}{x^3 Q_h^i}.$$  \hfill (42)

In conclusion, if the shape of the resonator has two orthogonal reflection symmetries, the minimum $Q$ factor along a particular direction $\hat{e}_i$ can be obtained as the parallel of the $Q$ factor of the bright QES (plasmonic) modes aligned along that direction.

In addition, due to the orthogonality condition [1] when there exists a current mode which is spatially uniform along a given direction there exists one and only one QES mode with nonvanishing dipole moment along each symmetry axis, which is guaranteed to exhibit the minimum $Q$ factor.

The algorithm that we propose to determine the minimum $Q$ factor of an electric type radiator is summarized in Fig. 2. First, we preliminarily calculate the QES (plasmonic) modes associated with the assigned shape on which the search for the optimal current is performed. Then, if the object has two reflection symmetries, the minimum $Q$ along the principal axes is immediately obtained from the $Q$ factor of the QES mode oriented along that axis. If no such symmetries are present, then we analytically assemble the polarizability tensor by using the dipole moment and the eigenvalues of the modes, and eventually find its eigenvalues and eigenvectors. The minimum $Q$ and optimal currents are then immediately obtained.

**Translationally invariant objects:** In this section, we find the minimum quality factor of longitudinal current density distributions supported by translationally invariant radiators having small cross-section. The expression of the $Q$ factor is

$$Q = -\frac{8}{\pi} \int_{\Omega} \frac{\sigma (r) \sigma (r') \log |r - r'| dV dV'}{\int_{\Omega} \sigma (r) \sigma (r') |r - r'|^2 dV dV'} \frac{1}{x^2},$$  \hfill (43)

where the surface charge density per unit length (p.u.l.) $\sigma = j \cdot n$, and $n$ lies in the cross-sectional plane. Once the shape of the cross-section $\Sigma$ is assigned, the functional $x^2 Q$ can be minimized, yielding the minimum $Q$ factor. For electric type radiators, the search space is constituted by the currents belonging to $L^2 (\Sigma)$.

The same steps taken in the previous section yield the expression of the minimum $Q$ factor:

$$x^2 Q_{\text{min}} = \frac{8}{\gamma_{\text{e,max}}},$$  \hfill (44)

where $\gamma_{\text{e,max}}$ is the maximum among the eigenvalues of the $2 \times 2$ electric polarizability tensor $\gamma^e_{ij}$. By expanding the charge density p.u.l. in terms of the current modes of Eq. [4] we arrive to analytical expressions for $\gamma^e_{ij}$ identical to Eq. [39] for the optimal current identical to Eq. [40].

If the shape of the object cross-section has a reflection symmetry, $\gamma^e_{ij}$ can be immediately put in the form of diagonal matrix by choosing a co-aligned coordinate system $(\hat{e}_1, \hat{e}_2)$. In this case, the dipole moments of the QES mode are also aligned along these directions, and the minimum $Q$ along the axis $\hat{e}_i$ can be obtained through a parallel formula

$$\frac{1}{(x^2 Q)_{\text{min}}} = \sum_{h \perp \hat{e}_i} \frac{1}{x^2 Q_h^i}.$$  \hfill (45)

In addition, due to the orthogonality of QES modes, if there exists a current mode spatially uniform along a given direction, this same mode is guaranteed to be the only one with nonvanishing dipole moment along that direction. Thus, it necessarily exhibits the minimum quality factor.

**B. Minimum $Q$ factor of radiators of the magnetic type**

We now consider three-dimensional radiators of the magnetic type, supporting transverse volume currents $\mathbf{j}$, which belong to $L^2 (\Sigma)$. Their $Q$ factor can be expressed as $2\pi$ times the ratio of the magnetostatic energy stored in the whole space to the energy radiated to infinity in a period [10]. [11]. [9]

$$Q = \frac{6}{\gamma_{\text{m,max}}} \int_{\Omega} \frac{\mathbf{j} (r) \cdot \mathbf{j} (r') dV dV'}{\int_{\Omega} \frac{\mathbf{j} (r) \cdot \mathbf{j} (r') |r - r'|^2 dV dV'} x^2}. \hfill (46)$$

This expression holds also in the case of surface current density defined on $\Sigma$ provided that the volume integrals are replaced by surface integrals.

It was found [9] that the minimum $Q$ factor is obtained from the maximum eigenvalue of the magnetic polarizability tensor $\gamma^m_{ij}$ of the scaled object

$$x^3 Q_{\text{min}} = \frac{6\pi}{\gamma_{\text{m,max}}},$$  \hfill (47)

where $\gamma_{\text{m,max}}$ is the maximum among the 3 eigenvalues of the magnetic polarizability tensor $\gamma^m_{ij}$.

The polarizability tensor $\gamma^m_{ij}$, $H_0 \hat{e} \rightarrow \mathbf{M}$ (48) is a linear correspondence between $H_0 \hat{e}$ and the magnetic dipole moment of the volume current density distribution having zero-average over $\Omega$ and solving the integral equation problem [22]:

$$\int_{\Omega} \mathbf{j} (r') |r - r'| dV' = \frac{1}{2} (H_0 \hat{e}) \times r, \quad \forall r \in \Omega. \hfill (49)$$

The volume integral operator in Eq. [39] is exactly the same operator occurring in Eq. [16] which describes the magnetostatic (dielectric) modes. One of the main contributions of the present manuscript is the analytical closed-form calculation of the polarizability tensor by using the QMS modes set.

Hence, we expand the transverse current density, solution of problem [39] in terms of the QMS modes which constitutes a complete basis for $L^2 (\Sigma)$:

$$\mathbf{j}_{\text{opt}} (r) = \sum_{h=1}^{\infty} \mathbf{\beta}_h \mathbf{j}_h (r), \quad \forall r \in \Omega. \hfill (50)$$

The resulting optimal current $\mathbf{j}_{\text{opt}}$ exhibits zero average over $\Omega$ by construction. By following similar steps to the derivation
for electric type resonators, we finally obtain the $3 \times 3$ magnetic polarizability tensor $\mathcal{\tilde{\gamma}}_m$

$$\mathcal{\tilde{\gamma}}_m = \sum_{h=1}^{\infty} \kappa_h \mathbf{M}_h \otimes \mathbf{M}_h.$$  

Eq. (51) bridges the polarizability tensor with the QMS (dielectric) modes.

The maximum eigenvalue $\gamma_{m,\text{max}}$ of $\mathcal{\tilde{\gamma}}_m$ is related to the minimum $Q$ factor of a radiator of the magnetic type through Eq. (47). The corresponding (normalized) eigenvector returns the direction of the dipole moment of the optimal current, which we call $\mathbf{m}_\text{opt}$. The optimal current is readily obtained in terms of the QMS (dielectric) modes:

$$j_{\text{opt}}(r) = \sum_{h=1}^{\infty} \kappa_h (\mathbf{m}_\text{opt} \cdot \mathbf{M}_h) j^i_h(r).$$  

Furthermore, it can be proved [22] that $j_{\text{opt}}$ has support only on the radiator’s boundary. Consequently, $j_{\text{opt}}$ can be expanded in terms of either the QMS modes defined in the volume $\Omega$, solution of the problem [10] or the QMS modes of the surface $\partial \Omega$, solution of the problem [27]. As we will see in the section [IV] in many scenarios, only a few QMS modes have to be considered to have a good estimation of the minimum $Q$ factor.

If the shape of the resonator has two orthogonal reflection symmetries, $\mathcal{\tilde{\gamma}}_m$ can be put in the form of diagonal matrix by choosing a coordinate system $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ aligned along its principal axes. In this case, the dipole moments $\mathbf{M}_h$ are also aligned along either one of these directions. Thus, the three occurrences of $\mathcal{\tilde{\gamma}}_m$ are given by:

$$\gamma_i = \sum_{\hat{e}_i, \text{aligned}} \kappa_h |\mathbf{M}_h|^2, \quad i = 1, 2, 3$$  

where the summation only runs over the QMS modes exhibiting dipole moment directed along $\hat{e}_i$. In this case, the minimum $Q$ along the axis $\hat{e}_i$ is obtained by the parallel formula

$$\frac{1}{(\chi^3 Q)_{\text{min}}} = \sum_{\hat{e}_i, \text{aligned}} \frac{1}{\kappa_h^3 |\mathbf{M}_h|^2}. \quad (54)$$

In addition, due to the orthogonality of QMS modes, if there exists a current mode of the curl type, in the form $r \times c$, this same mode is guaranteed to be the only one with non vanishing magnetic dipole moment along direction $c$. Thus, it necessarily exhibits the minimum $Q$ factor.

**Translational invariant objects:** In this section, we find the minimum $Q$ factor of transverse current density distributions supported by translationally invariant radiators having a small cross-section. The expression of the quality factor is

$$Q = \frac{8}{\pi} \int_{\Sigma} j(r) \cdot \int_{\Sigma} j(r') \log |r - r'| dSdS' - \int_{\Sigma} j(r) \cdot \int_{\Sigma} j(r') |r - r'|^2 dSdS' \frac{1}{x^2}. \quad (55)$$

The same steps taken in the previous section yields the expression of the minimum $Q$ factor:

$$(\chi^3 Q)_{\text{min}} = \frac{4}{\gamma_m}, \quad (56)$$

where $\gamma_m$ is the magnetic polarizability tensor, which, for translational invariant structures, is a scalar.

By expanding the current $j$ in terms of the current modes of Eq. (27), we arrive to the closed form expressions of the scalar magnetic polarizability tensor $\gamma_m$, identical to Eq. (51) and of the optimal current $j_{\text{opt}}$ identical to Eq. (52). In this case, since all supported QMS current modes are constrained to flow in the structure cross-sectional domain, their dipole moments are directed along the translational invariance axis. Thus, the scalar magnetic polarizability tensor $\gamma_m$ is

$$\gamma_m = \sum_{h=1}^{\infty} \kappa_h |\mathbf{M}_h|^2,$$  

and the minimum $Q$ factor is always obtained by the parallel formula

$$\frac{1}{(\chi^3 Q)_{\text{min}}} = \sum_{h=1}^{\infty} \frac{1}{\kappa_h^3 |\mathbf{M}_h|^2}. \quad (58)$$

As for the 3D case, the optimal current $j_{\text{opt}}$ has support only on the object in-plane linear boundary $\partial \Sigma$, and hence it is a simple current loop. Moreover, since $j_{\text{opt}}$ is solenoidal, there exists one and only one QMS current mode defined on the boundary $\partial \Sigma$. This mode is uniform along the curvilinear abscissa and exhibits the minimum $Q$ factor supported by the structure, whose expression is given in Eq. (29).

**IV. RESULTS AND DISCUSSION**

In this section, we exemplify the outlined method, by deriving the minimum $Q$ factor of radiators of the electric (Sec. [IV-A]) and magnetic (Sec. [IV-B]) type of increasing complexity. We first consider shapes that support electric uniform modes and magnetic modes of the curl type, which are guaranteed to have the minimum $Q$ factor. Then, we consider shapes with two orthogonal reflection symmetries, when the minimum $Q$ factor can be obtained from the $Q$ factors of its quasistatic modes, through a simple parallel formula. Eventually, we consider shapes with no symmetry.

The same scheme is also applied to the case of translationally invariant radiators of the electric and magnetic type in Secs. [IV-C] and [IV-B], respectively.

From a computational standpoint, the electrostatic eigenvalue problems Eqs. (1) and (14) are solved by the surface and linear integral method outlined in Refs. [17], [27]. The QMS eigenvalue problems Eqs. (10) and (25) are solved by the volume and surface integral equation methods as documented in Refs. [20] and [26].

**A. 3D Resonators of the Electric Type**

**Shapes supporting a uniform electrostatic mode.** A homogeneous sphere supports three degenerate QES bright modes, with eigenvalue $\chi_h = -3$ where $h = 1, 2, 3$ (Fröhlich condition). The surface charge density of one of these modes is shown in Fig. 3(a) and corresponds to the spatially uniform current mode $j_{\text{opt}} = \sqrt{3}/(4\pi r^2) \hat{z}$. For the considerations made in Sec. [III-A] they are the only three bright modes (one for each coordinate axis) supported by this shape. Thus, the modes’ $Q$...
### Magnetic Type

\[ (x^3 Q)_{\min} = x_h^3 Q_h \]

| Uniform Mode | \( (x^3 Q)_{\min} \) | \( (x^3 Q)_{\min} = 3 \) |
|--------------|----------------------|----------------------|
| (a)          | -3                   | -3                   |
| (b)          | 4.07                 | 4.93                 |
| (c)          | 6.39                 | 8.78                 |
| (d)          | 10.18                | 14.7                 |

### Electric Type

\[ \gamma_e = -\sum_{h=1}^{\infty} \chi_h P_h \otimes P_h \]

\[ \gamma_m = \sum_{h=1}^{\infty} \kappa_h M_h \otimes M_h \]

\[ (x^3 Q)_{\min} = \frac{4\pi}{\max} \]

\[ (x^3 Q)_{\min} = 3\]
factor coincides with the minimum allowed $Q$ factor supported by the strong currents constrained within this shape:

\[
(x^3 Q)_{\text{min}} = x^3_h Q^h = 1.5.  \tag{59}
\]

In conclusion, the plasmonic (QES) dipole modes supported by a plasmonic sphere exhibit the minimum quality factor of an electric type radiator.

Similarly, an ellipsoid supports three uniform modes directed along the three axes, and so the considerations made in Sec. III-A they are the only bright modes. We consider a prolate and an oblate spheroid with aspect ratio $2:1$. In Fig. 3(a) we show the surface charge density of the QES current mode oriented along the major axes, with eigenvalues $\chi = -5.76$ (prolate) and $\chi = -4.23$ (oblate) [28]. Thus, the modes’ $Q$ factor coincides with the minimum allowed $Q$ factor supported by longitudinal currents constrained within this shape. Specifically, $(x^3 Q)_{\text{min}} = 3.16$ and $(x^3 Q)_{\text{min}} = 2.13$ for the prolate and oblate spheroids, respectively.

**Shapes with two reflection symmetries.** We consider a rod with radius $R$, and height $H = 4R$, aligned along $\hat{z}$. We modeled the rod as a superellipsoid, whose boundary has equation:

\[
\left(\frac{x}{R}\right)^2 + \left(\frac{y}{R}\right)^2 + \left(\frac{z}{R}\right)^{10} = 1  \tag{60}
\]

The surface mesh used for the mode calculation has 1885 nodes, and 3766 triangles. Following the algorithm outlined in Fig. 2, we compute the QES modes. The rod admits an infinite discrete set of bright modes [29]. Their quality factor can be immediately calculated using their eigenvalue and electric dipole moment by Eq. 11. Among them, in Fig. 3(b) on the right of the “=” sign, we show the four bright modes with lowest $Q$. The minimum $Q$ supported by a longitudinal current confined in this volume is obtained by Eq. 42 from the parallel of the quality factors $x^3_h Q^h$ of the modes exhibiting nonvanishing dipole moment along $\hat{z}$: it has value $(x^3 Q)_{\text{min}} = 3.98$. We note that the $Q$ factor of the first QES mode is very close to the $Q$ bound. This is because this particular mode is almost spatially constant, thus the dipole moment of the remaining modes is nearly vanishing. The relative error in the calculation of $(x^3 Q)_{\text{min}}$ by considering only the first three modes is below 0.2%.

We then consider a sphere dimer of radius $R$, aligned along the $\hat{z}$-axis with edge-edge gap $\delta = R/10$. On the left of the “=” sign in Fig. 3(c) we show the surface charge density associated with the optimal current distribution, on the right the plasmonic modes with lowest $Q$ factors. The minimum $Q$ factor is obtained by combining the $Q$ factor of the bright modes, by using a parallel formula. Differently from the rod, where the fundamental plasmonic mode exhibits a $Q$ factor very close to the minimum $Q$, for a sphere dimer the first mode exhibits a $Q$ which is quite larger than the minimum. This is because the dimer of two nearly touching spheroids supports modes which strongly deviate from the uniform distribution. The relative error in the calculation of $(x^3 Q)_{\text{min}}$ by considering only the first three modes is below 0.3%.

**Shapes with no symmetries.** We consider a block with three arms of different lengths and no reflection symmetries. We first compute the QES resonances of this object. The three bright modes with lowest $Q$ factor are shown in Fig. 3 on the right of the “=” sign. The direction of the dipole moment $\mathbf{P}_h$ of each mode is also shown in the insets. Aiming at determining the minimum $Q$ factor achievable by currents confined within this object, and considering that there are no symmetries, we have to preliminary assemble the polarizability tensor using Eq. 33 using the dipole moments of the modes $\mathbf{P}_h$. The maximum eigenvalue of $\gamma_\epsilon$ is associated with the minimum quality factor by Eq. 33. The optimal current is then obtained by using Eq. 40 and shown on the left of Fig. 3(d). The relative error in the estimation of $(x^3 Q)_{\text{min}}$ by taking into account only the four modes shown in Fig. 3(d) is 26%. We have to consider at least 25 modes to have an error below 10%.

**B. 3D Resonators of the Magnetic Type**

**Shapes supporting a QMS mode of curl type.** We now consider a radiator of the magnetic type having a form of a spherical shell (i.e., a spherical inductor). First, we compute the QMS modes associated with this shape by Eq. 25. This shape supports three degenerate current modes of curl type with nonzero magnetic dipole moment:

\[
\mathbf{j}_h = \frac{3}{2\pi} \hat{r} \times \hat{c},  \tag{61}
\]

where $\hat{c} \in \{\hat{x}, \hat{y}, \hat{z}\}$, with eigenvalue $\kappa_h = 3$, and with magnetic dipole moment oriented along three orthogonal axes. We show one of these current modes in Fig. 3(e). According to the discussion of Sec. II-B2, there is one and only one mode with co-aligned magnetic dipole moment along each coordinate axis. Thus, applying Eq. 54 this QMS mode also exhibits the minimum $Q$ factor, namely

\[
(x^3 Q)_{\text{min}} = (x^3_h Q^h) = 3  \tag{62}
\]

This is in agreement with the seminal work of Thal [6].

**Shapes with two reflection symmetries** We now consider a small radiator of the magnetic type having the shape of a solid sphere. Unlike the shell sphere, this shape does not support a mode of the form $\hat{r} \times \hat{c}$, where $\hat{c}$ are the spherical Bessel function of the first kind $j_1$. We compute the supported QMS resonances and modes, solving the eigenvalue problem 16. Among the transverse current modes, we limit our analysis to the ones having nonvanishing magnetic dipole moment along $\hat{z}$. They are an infinite countable set, and they have (normalized) analytic expression [20]:

\[
\mathbf{j}_h(r, \theta, \phi) = \frac{\sqrt{3\pi}}{2} j_1(h \pi r) \hat{r} \times \hat{z},  \tag{63}
\]

where $j_1$ are the spherical Bessel function of the first kind and order 1. The above current modes are associated with the eigenvalues:

\[
\kappa_h = (h \pi)^2, \quad h \in \mathbb{N},  \tag{64}
\]

and they have magnetic dipole moment

\[
\mathbf{M}_h = (-1)^h 2 \frac{\sqrt{3\pi}}{\kappa_h} \hat{z},  \tag{65}
\]

and quality factors

\[
(x^3_h Q^h) = \frac{(h \pi)^2}{2}.  \tag{66}
\]
The projection of the first four current modes on the equatorial section of the sphere is shown on the right of the “⇒” sign in Fig. 3(f), with their $Q$ factor. The minimum $Q$ factor $(x^2Q)_{\min}$ achievable by transverse currents confined to this volume is obtained by applying Eq. 24. This formula consists in the parallel of the $Q$ factors of magnetostatic (dielectric) modes exhibiting magnetic dipole moment along $\hat{z}$:

$$(x^2Q)^{-1}_{\min} = \sum_{h=1}^{\infty} (x_Q^\perp)^{-1}_h = \frac{2}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} = \frac{2}{\pi^2} \frac{\pi^2}{6} = \frac{1}{3}$$

(67)

In this parallel, by only considering the first 4 modes, we obtain an estimation of $(x^2Q)_{\min}$ of 13.2 with an error of 12%.

The optimal volume current is obtained applying Eq. 52

$$j_{\text{opt}}(r, \theta, \phi) = \sqrt{\frac{3}{2\pi}} \delta (r-1) \hat{r} \times \hat{z},$$

(68)

where $\delta$ is a Dirac delta function. Thus, it corresponds to a surface current localized on the sphere’s surface, which is the same optimal current found for a spherical shell. In conclusion, the minimum $Q$ factor and the corresponding optimal current are in agreement with the results achieved by Thal [6] for a spherical inductor (see previous section). Therefore, in contrast with a plasmonic homogeneous sphere which supports a mode with minimum $Q$ for electric type radiator, the first (fundamental) mode of homogeneous high-permittivity sphere does not exhibit the minimum $Q$ for magnetc-type radiators.

We then consider a spheroidal shell with aspect ratio $4:1$, with major axis aligned along $\hat{z}$. Also this shape does not support a mode of the curl type. We compute the QMS resonances by solving the eigenvalue problem 25 and we only consider the set of QMS modes exhibiting nonvanishing magnetic dipole moment along $\hat{z}$. In Fig. 3(g), we show the surface current, eigenvalue, and $Q$ factor of the modes with the lowest $Q$ factor. The minimum quality factor $(x^3Q)_{\min}$ of magnetic-type radiators supported by surface current density confined on this shape is obtained using Eq. 54 by applying a parallel formula. If the parallel is only limited to the three modes shown in Fig. 3(g), an error $<1\%$ is obtained.

**Shapes with no symmetries.** We consider a shell with no reflection symmetries, defined as the boundary of a block with three arms of different lengths. We preliminarily compute its QMS resonances and resonance modes by solving Eq. 25 their magnetic dipole moment $M_h$, by Eq. 21 and $Q$ factor by Eq. 24. The surface mesh used for the calculation has 1885 nodes, and 3766 triangles. The four modes with the lowest $Q$ factor are shown in Fig. 3(h) on the right of the equality sign, with their quality factor (above), and eigenvalue (below). Aiming at determining the minimum $Q$ factor achievable by surface currents confined on this surface, and considering that there are no symmetries, we have to preliminary assemble the magnetic polarizability tensor $\gamma_m$ using Eq. 51 from the dipole moments of the modes $M_h$. The maximum eigenvalue $\gamma_{\text{max}}$ of $\gamma_m$ is associated with the minimum quality factor by Eq. 33. The optimal current is then obtained by using Eq. 52 and shown on the left of the equality sign in Fig. 3(h). By only considering the first 3 modes, we obtain an estimation of $(x^3Q)_{\min}$ of 13.2 with an error of 12%.

**C. Translationally Invariant Resonators of the Electric Type**

**Shapes supporting a uniform QES mode.** Let us consider a cylinder of circular cross-section. Its cross-section has reflection symmetry, thus it supports one and only one uniform QES (plasmonic) mode with nonvanishing dipole moment, associated with the eigenvalue $\chi = -2$. Therefore, this current mode coincides with the optimal current supported by translationally invariant radiators of the electric type, and its $Q$ factor is

$$Q^\parallel = Q^\parallel_{\min} = \frac{4}{\pi} \frac{1}{\pi x^2}.$$  

(69)

To the authors’ best knowledge, this lower bound for the quality factor for translational invariant radiators has not been reported before.

**Shape with reflection symmetry.** We now investigate a dimer of cylinders of radius $R$, aligned along the $\hat{z}$-axis with edge-edge gap $\delta = R/10$. This system has reflection symmetries, but does not support any uniform current mode. We first compute the plasmonic modes, separate them into bright and dark modes, and compute the quality factor of the bright ones. The use of the bipolar coordinate system enables us to perform these calculations analytically. The plasmonic modes are shown on the right of Fig. 4(b) with their individual quality factors.

The minimum $Q$ factor $(x^2Q)_{\min}$ achievable by longitudinal currents supported by a dimer of cylinders of circular cross-section is obtained by Eq. 42. It consists in the parallel of the quality factors $(x^2Q^\parallel_h)$ of the infinite (countable) set of plasmonic modes exhibiting dipole moment along $\hat{z}$. In this parallel, by only considering the first 5 modes (shown in Fig. 4(b)), the relative error in the estimation of $(x^2Q)_{\min}$ is below 12%.

**D. Translationally Invariant Resonators of the Magnetic Type**

**Shapes supporting a single QMS mode.** We now assume that the search for optimal currents is constrained on the shell of translationally invariant cylinders of arbitrary cross-section. Since any supported current density has to be translationally invariant as well, it can be exhaustively specified on the closed line, i.e., the cylinder’s boundary. For the considerations made in Sec. III-B there exists one and only one supported QMS current mode, which is uniform and exhibits the minimum $Q$ factor of the structure, given in Eq. 29. As an example, we consider the shell of an infinite cylinder of circular cross-section, shown on the left of Fig. 4(c). The only QMS mode supported by this shape, has eigenvalue $\kappa = 2$, and magnetic dipole moment oriented along the cylinder axis. The resonant current density is

$$j^\perp_h = \frac{1}{\sqrt{2\pi}} \hat{z} \times \hat{r},$$

(70)

being $\hat{z}$ the cylinder axis. This QMS mode exhibits the minimum $Q$ factor, namely

$$(x^2Q)_{\min} = (x^2Q^\perp_h) = \frac{4}{\pi}. $$

(71)
### ELECTRIC TYPE

$$ (x^2 Q)^{\text{min}} = x_h^2 Q_h $$

| Uniform Mode | Magnetic Type |
|---------------|---------------|
| $ (x^2 Q)^{\text{min}} = x_h^2 Q_h $ | $ (x^2 Q)^{\text{min}} = 4 \pi $ |
| $ \chi = -2 $ | $ \kappa = 2 $ |

#### REFLECTION SYMMETRY

| Reflection Symmetry |
|----------------------|
| $ (x^2 Q)^{\text{min}} = 2.67 $ | $ (x^2 Q)^{\text{min}} = 5.78 \pi $ |
| $ (x^2 Q)^{\text{min}} = 4.30 $ | $ (x^2 Q)^{\text{min}} = 30.47 $ |
| $ (x^2 Q)^{\text{min}} = 10.74 $ | $ (x^2 Q)^{\text{min}} = 74.89 \pi $ |
| $ (x^2 Q)^{\text{min}} = 30.08 $ | $ (x^2 Q)^{\text{min}} = 74.89 \pi $ |

Fig. 4. Minimum $Q$ and corresponding optimal charge/current distribution supported by translational invariant radiators of the electric (a – b) and magnetic (c – d) type of assigned shape. Electric type radiators. (a) Optimal charge density supported by a cylinder of circular cross-section: this shape has only one uniform bright mode which coincides with the optimal charge density. (b) Optimal charge density supported by a geometry exhibiting two reflection symmetries, namely a cylinder dimer, and plasmonic modes with lowest $Q$ factor. The colormap represents the electric charge density. Magnetic type radiators. (c) Optimal current density supported by a circular and L-shaped cylinder shell, coinciding with their only uniform quasi-magnetostatic (dielectric) mode. (d) Optimal current density supported by geometries exhibiting two reflection symmetries, namely a solid cylinder. The colormap represents the magnitude of the current density, the arrow's direction.

On the right of Fig. 4(c) we consider the shell of an infinite cylinder with an L-shaped cross-section and no reflection symmetries. This shape supports a uniform QMS current mode with eigenvalue $\kappa = 4.7$, and $x^2 Q^\perp = (x^2 Q)^{\text{min}} = 3$.

**Shape with reflection symmetry.** Next, we examine a solid infinite cylinder of axis $\hat{z}$ and circular cross-section. Despite the high symmetry, this shape does not support modes of the form $\hat{r} \times \mathbf{c}$. The cylinder supports transverse current modes with nonvanishing magnetic dipole moment. They have the analytical expression

$$ J^\perp_h = \frac{J_1(Z_0 r)}{\sqrt{\pi}} \frac{1}{J_1(Z_0 h)} \hat{z} \times \hat{r}, \quad r \in [0, 1], h \in \mathbb{N}, $$

(72)

shown in Fig. 4(d) on the right of the “−” sign. They are associated with the real and positive eigenvalues $\kappa_h$

$$ \kappa_h = Z^2_{0, h}, $$

(73)

where $(\hat{r}, \hat{z})$ are the in-plane radial unit vector and the cylinder axis unit vector, respectively, $J_1$ is the Bessel function of the first kind with order 1, and $Z_{0, h}$ is the $h$-th zero of the Bessel function of the first kind with order 0.

Their magnetic dipole moment $M_h$ has the expression

$$ M_h = (-1)^{h+1} \frac{2\sqrt{\pi}}{\kappa_h} \hat{z}, $$

(74)

and, according to Eq. 29, their $Q$ factor is given by:

$$ x_h^2 Q^\perp_h = \frac{Z^2_{0, h}}{\pi}. $$

(75)

The optimal current is obtained applying Eq. 52 and is a current loop localized on the cylinder cross-section’s boundary, i.e.

$$ J_{\text{opt}} = \sqrt{\pi} \delta(r-1) \hat{z} \times \hat{r}, \quad \forall r, $$

(76)

where $\delta$ is the Dirac delta function. The optimal current $J_{\text{opt}}$ is shown in Fig. 4(d). The minimum $Q$ factor $(x^2 Q)^{\text{min}}$ achievable by transverse currents confined to the cylinder cross-section is obtained through Eq. 58 which consists in the parallel of the $Q$ factors of the infinite set of QMS modes exhibiting magnetic dipole moment along the cylinder axis $\hat{z}$:

$$ (x^2 Q)^{\text{min}} = \left( \sum_{h=1}^{\infty} \frac{1}{x^2 Q_h^\perp} \right)^{-1} = \frac{4}{\pi}, $$

(77)

where we used the identity $\sum_{h=1}^{\infty} Z^2_{0, h} = 1/4$ [30].

To the authors’ best knowledge, the lower bound in Eq. 77 (or in Eq. 71) for the quality factor of translationally invariant radiators of magnetic type has not been reported before.

### V. DESIGN OF RESONATORS SUPPORTING A MODE WITH MINIMUM $Q$

We have so far introduced a possible algorithm to determine the minimum $Q$ supported by a current density distribution constrained within the structure domain. We may also ask: how can we design a structure supporting a mode with $Q$ factor as close as possible to the minimum $Q$?
One possible strategy is to look for a structure supporting a mode as close as possible to a uniform mode for electric-type radiator, or to a mode of the form $\hat{r} \times c$ for a magnetic type radiator. We have already shown that these modes are guaranteed to exhibit the minimum $Q$.

In the following we apply this strategy to the case of a magnetic radiator confined within a spheroidal domain with aspect ratio $4 : 1$, with major axis aligned along $\hat{z}$. The minimum $Q$ supported by this domain is $(x^3 Q)_{\text{min}} = 39$, as shown in Sec. 3. In particular, in Fig. 3(g) we showed that the modes supported by a spheroidal shell with aspect ratio $4 : 1$ have individual $Q$ factor greater than 43.52. In order to reach the limit of minimum $Q$, we have to design a structure with a mode as close as possible to $\hat{r} \times c$. Thus, inspired by Refs. [6], [31] we implement a spheroidal inductor, shown in Fig. 3. Such spheroidal inductor “forces” transverse electric current to follow the strip, thus the mode is very close to be $\hat{r} \times c$. The $Q$ of the present mode is 41.7, which is closer to the minimum $Q$ than the one of the spheroidal shell’s first mode.

VI. Conclusions

Optimization problems defined on small radiators often consist in the search of the optimal current distribution which leads to the minimization of a chosen quantity. In these problems, the choice of a convenient basis for the representation of the tentative optimal current density is a fundamental and unavoidable step. Upon this choice depends not only the efficacy of the numerical computation of the optimal solution, but also the attribution of meaningful physical insights to the results. Since small electromagnetic radiators are considered, a convenient basis choice may be constituted by the quasistatic resonance modes supported by that shape, corresponding to source-free solution of the quasi-electrostatic and quasi-magnetostatic limits of the Maxwell’s equations. Specifically, quasi-electrostatic modes describe the resonances of small objects with negative permittivity (e.g., as in metals). Quasi-magnetostatic modes describe the resonances of small objects with very high and positive real part of the permittivity.

Specifically, here we address the problem of finding the optimal current yielding the minimum $Q$ factor. We demonstrated that an expansion of the current density in terms of quasistatic modes leads to analytical closed form expressions of the electric and magnetic polarizability tensors, whose eigenvalues are known to be linked to the minimum $Q$. Hence, the minimum $Q$ and the corresponding optimal current are determined from the knowledge of the eigenvalues and dipole moments associated with the quasistatic scattering modes.

We found that when the radiator exhibits two orthogonal reflection symmetries, its minimum $Q$ factor can then be simply obtained from the $Q$ factors of the quasistatic modes associated with the radiator’s shape, through a parallel formula. Moreover, when an electric radiator supports a spatially uniform quasi-electrostatic resonance mode, this mode is guaranteed to have the minimum $Q$ factor. Because of duality, when a magnetic radiator supports a quasi-magnetostatic mode of curl type, in form $\hat{r} \times c$ where $c$ is a constant vector and $\hat{r}$ is the radial direction, this mode exhibits the minimum $Q$ factor.

The introduced framework bridges a classic antenna problem to the field of open optical resonators, and may be especially appealing to researchers and engineers working in nanophotonics, where a description in terms of modes is widely used.

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