On one-loop partition functions of three-dimensional critical gravities

Thomas Zojer
Centre for Theoretical Physics, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands
E-mail: t.zojer@rug.nl

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Abstract
We calculate the one-loop partition function of three-dimensional parity even tricritical gravity. Agreement with logarithmic conformal field theory single-particle partition functions on the field theory side is found and we furthermore discover a partially massless limit of linearized six-derivative parity even gravity. Then we define a ‘truncation’ of the critical theory, at the level of the partition function, by calculating black hole determinants via summation over quasi-normal mode spectra and discriminating against those modes which are not present in the physical spectrum. This ‘truncation’ is applied to critical new massive gravity and three-dimensional parity even tricritical gravity.

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1. Introduction
Three-dimensional gravity has long been an interesting testing ground for theories of quantum gravity. But pure Einstein gravity, plus a cosmological constant, does not lead to propagating degrees of freedom in the bulk. Therefore modifications are sought. One natural way of extending the theory is to add massive gravitons. This can be achieved by adding higher-derivative terms to the action. Two concrete proposals in the literature are the parity violating topologically massive gravity (TMG) [1] with one massive (helicity) degree of freedom, and parity even new massive gravity (NMG) [2] with two massive (helicity) degrees of freedom propagating in a unitary fashion around an anti-de Sitter (AdS) background.

Lately, one center of attention in the study of three-dimensional higher-derivative gravity were so-called logarithmic or critical tunings of the parameters. The seminal example is chiral gravity [3], a fine-tuned version of TMG, which was conjectured to be dual to a logarithmic conformal field theory (LCFT) [4]. Besides the latter example the phenomenon of critical points was also studied in NMG, general massive gravity (GMG) [2, 5] and more recently in parity even tricritical gravity (PET) [6]. For a summary on the works contributing to the LCFT conjectures in three-dimensional critical gravities see e.g. [6, 7] and references therein.
One piece of evidence for the aforementioned critical gravity/LCFT conjecture was the calculation of one-loop partition functions on the gravity side and the results were shown to agree with those expected from a LCFT. This computation was performed for TMG and NMG in [8] and for GMG in [9]. In this work we calculate the one-loop partition function of PET gravity along the lines of [8]. We will scan the whole parameter space of PET gravity, covering the critical points found in [6], but also uncovering an additional special point in parameter space: the partially massless limit of linearized six-derivative parity even gravity. Focusing on the critical points we will give further evidence for the conjectured duality of PET gravity to LCFTs of rank 2 and 3.

The AdS/LCFT duality only holds for those theories where all solutions to the equations of motion are allowed. From the very beginning probably one of the most prominent questions concerning critical gravities was how (and if) one can consistently truncate (some of) the logarithmic solutions by imposing boundary conditions. For example restricting to Brown–Henneaux boundary conditions [10] for excitations on the gravity side would kill all log modes. In the prime example of TMG/chiral gravity these are the boundary conditions one has to choose to obtain chiral gravity. If one imposes log boundary conditions [11] one obtains log-TMG which is dual to a LCFT. Recently, using a scalar field toy model, another truncation of critical theories was put forward that allows some, but not all log solutions [12]. In [6] a tricritical version of this scalar field toy model was generalized to interacting spin-two fields. Calculations on the linearized level seemed to lend support to the possibility of truncating the theory. However, it was shown that the theory is flawed with a linearization instability [13] and only makes sense either as dual to a LCFT by allowing all solutions, or, imposing Brown–Henneaux boundary conditions, as a ‘trivial’ theory propagating null modes.

In the second part of this work we introduce an idea on how such a truncation via boundary conditions could be understood at the level of the partition function. To do so we will interpret the partition function as a sum over quasi-normal mode frequencies following the idea of [14]. The authors of [14] emphasize that translating from the heat-kernel to the quasi-normal mode spectrum would allow one to pick out the contribution of only one mode or only one frequency to the partition function. We try to turn this argument around by using it to dismiss the contributions of certain modes. Our ‘truncation’ will effectively be a prescription on which quasi-normal modes to keep in the spectrum and which not to keep. We obtain results concurrent with the comment made in the previous paragraph: either we keep all modes and the theory is dual to a non-unitary LCFT, or it is not logarithmic but an ‘ordinary’ CFT that propagates only null modes.

This work is organized as follows. In section 2 we calculate the partition function of PET gravity and discuss its similarity to LCFT single-particle partition functions. In section 3 we define the truncation by translating from black hole determinants to quasi-normal mode spectra and discriminating against certain modes. Finally we apply this truncation to critical NMG and PET gravity.

2. Partition function of PET gravity

In this first part of the work we calculate the gravity one-loop partition function of PET gravity. Such a calculation is conveniently carried out by splitting the metric into a background metric $g_{\mu\nu}$ and a perturbation $h_{\mu\nu}$. Our background is an AdS space-time, but, as we will argue later, this is equivalent to taking the BTZ black hole [15] as background. The partition function then consist at least of a classical part, $Z_c$, corresponding to the background $g_{\mu\nu}$, and a one-loop contribution coming from the perturbation $h_{\mu\nu}$, $Z^{1\text{-loop}}$. This one-loop contribution is precisely what we are interested in.
We start by covering the whole parameter range of $\beta$ and $b_2$ (see the action (2)), or equivalently the masses $M_+$ and $M_-$ of the two propagating massive gravitons. Subsequently, we specialize to the logarithmic points and subsection 2.2.1 is devoted to the critical loci: the tricritical point, the logarithmic line (single log) where one of the two masses $M_+$ and $M_-$ goes to zero, and the massive logarithmic line (massive log) where $M_+$ and $M_-$ degenerate with each other. Section 2.2.2 covers a special parameter limit that leads to a partially massless theory. At the critical loci the calculation confirms the conjecture that PET gravity is dual to parity even logarithmic CFTs of rank 2 and 3. This is shown in section 2.3 using the combinatorial counting argument of [8].

The calculation will follow the lead of [8] and all formulas and technicalities that are not explained in detail here can be found in that reference.

2.1. Linearized action and ghost determinants

The semi-classical one-loop contribution to the gravity partition function is given by

$$Z_{\text{PET}}^{1\text{-loop}} = \int Dh_{\mu\nu} Dh_{\mu\nu} Dh_{\mu\nu} e^{-\delta^{(2)}S_{\text{PET}}}.$$  \hspace{1cm} (1)

The action in the exponent is the linearized Euclidean action of PET gravity, given by [6]

$$\delta^{(2)}S_{\text{PET}} = \frac{1}{\kappa^2} \int d^3x \sqrt{g} \left\{ -\frac{1}{2} \bar{\sigma} f^{\mu\nu} G_{\mu\nu}(h) + k_{\mu\nu} G_{\mu\nu}(h) + 2b_2 f^{\mu\nu} G_{\mu\nu}(f) \
+ (2b_2/\ell^2 + \beta)(f^{\mu\nu} f_{\mu\nu} - f^2) - f^{\mu\nu} k_{\mu\nu} - f k \right\},$$  \hspace{1cm} (2)

with $\bar{\sigma} = \sigma + 3b_2/(2\ell^2) + \beta/(2\ell^2)$. Newton’s constant $G$ is in $\kappa^2 = 16\pi G$ and the parameters $\beta$ and $b_2$ have mass dimension $-2$ and $-4$ respectively, but are otherwise arbitrary parameters of the model. The linearized Einstein tensor $G_{\mu\nu}$—without imposing any gauge condition—takes the form

$$2G_{\mu\nu}(h) = -\nabla^2 h_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} h + 2\nabla_{(\mu} \nabla^\beta h_{\nu)\beta} - \frac{2}{\ell^2} h_{\mu\nu} - g_{\mu\nu}(\nabla_{\mu} \nabla_{\nu} h - \nabla^2 h).$$  \hspace{1cm} (3)

All fluctuations $h_{\mu\nu}$, $k_{\mu\nu}$ and $f_{\mu\nu}$ can be split into a transverse-traceless (TT), a trace, and a vector (or gauge) part:

$$h_{\mu\nu}(h^{TT}, h, \xi) = h_{\mu\nu}^{TT} + \frac{1}{2} g_{\mu\nu} h + \nabla_{(\mu} \xi_{\nu)},$$
$$k_{\mu\nu}(k^{TT}, K, v) = k_{\mu\nu}^{TT} + \frac{1}{2} g_{\mu\nu} K + \nabla_{(\mu} v_{\nu)},$$
$$f_{\mu\nu}(f^{TT}, F, u) = f_{\mu\nu}^{TT} + \frac{1}{2} g_{\mu\nu} F + \nabla_{(\mu} u_{\nu}).$$  \hspace{1cm} (4)

Note that by definition $g^{\mu\nu} h_{\mu\nu}^{TT} = \nabla^\mu h_{\mu\nu}^{TT} = 0$ and that the vector parts contribute to the traces of $g^{\mu\nu} k_{\mu\nu} = K + 2\nabla_{\mu} u^\mu$ and $g^{\mu\nu} f_{\mu\nu} = F + 2\nabla_{\mu} u^\mu$.

Each such decomposition of the fluctuations produces a ‘ghost’ factor [8] in the measure of the path-integral (1)

$$Dh_{\mu\nu} = Z_{gh} Dh_{\mu\nu} D\xi_\mu Dh.$$  \hspace{1cm} (5)

Another useful split is

$$u^\mu = u^\mu_T + \nabla^\mu \delta,$$  \hspace{1cm} (6)

where $\nabla_{\mu} u^\mu_T = 0$. This yields the ghost factor $J_1$:

$$Du_\mu = J_1 Du_T^T Dh.$$  \hspace{1cm} (7)

The values of $Z_{gh}$ and $J_1$ are given by [8]

$$Z_{gh} = \left[ \det(-\nabla^2 + 2) \right]^{1/2} \det(-\nabla^2 + 3\mu) \right]^{1/2} \quad \text{and} \quad J_1 = \left[ \det(-\nabla^2 + \mu) \right]^{1/2}.$$  \hspace{1cm} (8)

The symmetrization over indices is normalized as follows: $2\nabla_{(\mu} \nabla^{\beta) h_{\nu}^{TT}} = \nabla_{\mu} \nabla^{\beta} h_{\nu}^{TT}$.
2.2. Path-integral for PET gravity

We now consider the action (2) under the decompositions (4) and (6). All scalar, ‘vector’ (T) and ‘tensor’ (TT) modes decouple and we can write the action (2) as a sum of terms which are quadratic in the respective fluctuations. For the TT part we find

\[
\delta^{(2)}_{\text{PET}}^{\text{TT}} = \int d^3x \sqrt{\gamma} \left\{ -\frac{\tilde{\sigma}}{4} h_{\mu \nu}^{TT} \left( -\nabla^2 - \frac{2}{\ell^2} \right) h_{\mu \nu}^{TT} + \frac{1}{2} \kappa_{TT}^{\mu \nu} \left( -\nabla^2 - \frac{2}{\ell^2} \right) h_{\mu \nu}^{TT} + b_2 f_{TT}^{\mu \nu} \left( -\nabla^2 + \frac{\beta}{b_2} \right) \epsilon^T \right\}.
\]

(9)

The transverse vector part is given by

\[
\delta^{(2)}_{\text{PET}}^{T} = 2 \int d^3x \sqrt{\gamma} \left\{ \left( \frac{2b_2}{\ell^2} + \beta \right) u^\mu_{T} \left( -\nabla^2 + \frac{2}{\ell^2} \right) u_{T}^\mu - u_{T}^\mu \left( -\nabla^2 + \frac{2}{\ell^2} \right) \epsilon^T \right\},
\]

(10)

and the scalar contribution is

\[
\delta^{(2)}_{\text{PET}}^{\text{scalar}} = \int d^3x \sqrt{\gamma} \left\{ \frac{\tilde{\sigma}}{18} \left( -\nabla^2 + \frac{3}{\ell^2} \right) h - \frac{1}{3} \tilde{K} \left( -\nabla^2 + \frac{3}{\ell^2} \right) h - \frac{2b_2}{9} \tilde{F} \left( -\nabla^2 + \frac{3}{\ell^2} \right) \tilde{F} + \left( \frac{2b_2}{\ell^2} + \beta \right) \left( \frac{2}{3} \tilde{F}^2 - \frac{8}{3} \tilde{F} \nabla^2 \delta - \frac{8}{3} \delta^2 \nabla^2 \right) \right\},
\]

(11)

where \( \epsilon \) is the scalar part coming from the decomposition \( v^\mu = u^\mu_T + \nabla^\mu \epsilon \).

2.2.1. The critical/logarithmic loci. We now evaluate the path-integrals (9)–(11) for the critical values of the parameters \( \beta \) and \( b_2 \) that lead to dual LCFTs of rank 2 and 3.

Tricritical point. Let us now consider in particular the tricritical point. At this critical locus the combinations \( \tilde{\sigma} \) and \( (2b_2 / \ell^2 + \beta) \) vanish, thus the formulas (9)–(11) simplify considerably. The path-integral over the TT tensor modes is\(^2\)

\[
Z^{TT}_{\text{crit}} = \int D\gamma^{TT} Dk^{TT} Df^{TT} \varepsilon^{-\delta^{TT}_{\text{PET}}} = \left[ \left( -\nabla^2 + \frac{2}{\ell^2} \right)^{TT} \right]^{-3/2}.
\]

(12)

To perform the integral over the transverse vector and scalar fluctuations we note that the kinetic terms have the wrong sign. This can be remedied by a Gibbons–Hawking–Perry [16] rotation of the fluctuations to imaginary values, which we shall also employ repeatedly in the remainder of this work. The results are

\[
Z^{T}_{\text{crit}} = \int D\nu^T D\mu^T \varepsilon^{-\delta^{T}_{\text{PET}}} = \left[ \left( -\nabla^2 + \frac{2}{\ell^2} \right)^T \right]^{-1},
\]

(13)

and after integration over \( h, \tilde{K}, \delta, \epsilon \) and finally \( \tilde{F} \) in that order:

\[
Z^{\text{scalar}}_{\text{crit}} = \int D\ell D\tilde{\gamma} D\delta D\tilde{\delta} \varepsilon^{-\delta^{\text{scalar}}_{\text{PET}}} = \left[ \left( -\nabla^2 + \frac{3}{\ell^2} \right) \right]_{0}^{-3/2} \left[ \text{det}(\nabla^2)_{0} \right]^{-1}.
\]

(14)

\(^2\) We integrate first over \( h^{TT} \) which yields a delta function for \( k^{TT} \) and a determinant factor. The integral over \( k^{TT} \) is then done trivially and we are left with an easy integral over \( f^{TT} \). In the following we will not always try to diagonalize the action. Moreover, we will extensively make use of the delta functions emerging from ‘mixed’ path-integrals such as \( h^{TT} \nabla^2 \tilde{K}^{TT} \). For the benefit of the reader we will often denote the order of integration.
The gravity one-loop partition function for PET gravity at the tricritical point is obtained by carefully collecting all ghost determinants, see (8), and the determinants (12)–(14):

$$Z_{\text{crit PET}}^{1-\text{loop}} = Z_{\text{gh}}^{1} \cdot f_{1} \cdot Z_{\text{crit}}^{\text{TT}} \cdot Z_{\text{crit}}^{\text{scalar}} = \frac{\left[ \det \left( -\nabla^{2} + \frac{2}{\ell^{2}} \right) \right]_{1}^{1/2}}{\left[ \det \left( -\nabla^{2} - \frac{2}{\ell^{2}} \right) \right]_{2}^{1/2}}$$

$$= Z_{\text{Ein}} \cdot \left( \left[ \det \left( -\nabla^{2} - \frac{2}{\ell^{2}} \right) \right]_{2}^{1/2} \right)^{2}.$$  

(15)

with $Z_{\text{Ein}}$ being the one-loop contribution to the partition function of Einstein gravity, see equation (37). Using heat kernel techniques [17, 18] it is straightforward to obtain the result (for positive temperature $[t_2 > 0]$):

$$Z_{\text{crit PET}}^{1-\text{loop}} = \prod_{n=2}^{\infty} \frac{1}{1-q^n} \left[ \prod_{n=2}^{\infty} \prod_{m=0}^{\infty} \frac{1}{1-q^n q^m} \prod_{l=0}^{\infty} \prod_{j=2}^{\infty} \frac{1}{1-q^l q^j} \right]^{2}.$$  

(16)

Here we use the notation of [17], where $q = \exp(i\tau)$ with $\tau = \tau_1 + i\tau_2$ and $\tau_1$ and $\tau_2$ being related to the angular momentum $\theta$ and the inverse temperature $\beta$.

This can be compared to the partition function of single-particle excitations in a parity even rank-3 LCFT.

**Single log.** Now we consider the critical line in parameter space where one of the massive modes degenerates with the massless graviton, $\beta = -3b_2/\ell^2 - 2\sigma \ell^2$, which still implies $\bar{\sigma} = 0$. An interesting scaling limit is $b_2 \to \infty$ which we will treat separately in section 2.2.2.

The calculation of the TT part is simply done and yields

$$Z_{\text{log}}^{\text{TT}} = \left[ \det \left( -\nabla^{2} - \frac{2}{\ell^{2}} \right) \right]_{2}^{T} - \left[ \det \left( -\nabla^{2} - \frac{3}{\ell^{2}} - \frac{2\alpha \ell^2}{b_2} \right) \right]_{2}^{T},$$

(17)

For the vector part we integrate over $v^{T}$ to obtain a delta function for $u^{T}$ and find the same result we obtained earlier, see equation (13). For the scalar part we redefine $\varepsilon = \alpha \varepsilon'$ and $\delta' = \delta + \varepsilon'/2 + \bar{F} \ell^2/6$ with $\alpha = b_2/\ell^2 + 2\sigma \ell^2$. These redefinitions do not produce further ghost determinants. The point where $\alpha$ becomes zero is the tricritical point, which we already covered in the previous subsection. After integration over $h, \bar{K}, \varepsilon', \delta'$ and $\bar{F}$ we find the result (14). Collecting all contributions the result is

$$Z_{\text{log PET}}^{1-\text{loop}} = Z_{\text{Ein}} \cdot \left[ \det \left( -\nabla^{2} - \frac{2}{\ell^{2}} \right) \right]_{2}^{1/2} \left[ \det \left( -\nabla^{2} - \frac{3}{\ell^{2}} - \frac{2\alpha \ell^2}{b_2} \right) \right]_{2}^{T} - 1/2.$$  

(18)

Setting $b_2 = -2\sigma \ell^4$ in the final expression (18), we find that it reduces to (15). Another cross-check is the limit going to critical NMG, $b_2 \to 0$. This is not apparent from formula (18). We need to set $b_2 = 0$ in the action (9) to see that there is no contribution from the integral over $Df_{\mu}^{\text{TT}}$, i.e. we do not get the last factor in (18). Therefore we obtain exactly the same result that was obtained for critical NMG [8].

For arbitrary $b_2$ ($\neq -2\sigma \ell^4$) we find that the partition function (18) consist of the contribution of one massive mode (third term) and the contribution of critical NMG. We thus find a parity even rank-2 LCFT plus one massive mode.
The last term of equation (18) in terms of $q$s is
\[
Z_{\mathcal{M}} = \prod_{l=|\mathcal{M}|}^{\infty} \prod_{l=|\mathcal{M}|}^{\infty} \frac{1}{(1 - q^{l+1}q^{-l-1})(1 - q^{l-1}q^{l+1})},
\]
where we defined $\mathcal{M}^2 = -2\sigma \ell^2/b_2$.\(^3\)

**Massive log.** In the massive log case the parameters $\beta$ and $b_2$ are restricted by the equation
\[
\beta^2 + 4b_2\sigma + \frac{6b_2\beta}{\ell^2} + \frac{10b_2^2}{\ell^4} = 0.
\]
Here a little more effort is needed to evaluate the ‘TT’ and the scalar path-integrals. The vector components do not change; we find the result (13). To evaluate the ‘TT’ part we first rescale $2k_{\mu\nu}^\mathcal{TT} = \bar{\sigma}k_{\mu\nu}^\mathcal{TT}$ and shift $\bar{h}_{\mu\nu}^\mathcal{TT} = h_{\mu\nu}^\mathcal{TT} - 1/2k_{\mu\nu}^\mathcal{TT}$. Then $\bar{h}_{\mu\nu}^\mathcal{TT}$ decouples and we can integrate over it. It is not possible to diagonalize in the remaining variables, but we can ‘block-diagonalize’ $\bar{k}_{\mu\nu}^\mathcal{TT}$ and $f_{\mu\nu}$ in the following way. Defining
\[
A = a \bar{k}_{\mu\nu}^\mathcal{TT} + b f_{\mu\nu}^\mathcal{TT} \quad \text{and} \quad B = c \bar{k}_{\mu\nu}^\mathcal{TT} + f_{\mu\nu}^\mathcal{TT}
\]
we replace the remainder of the TT path-integral expression by
\[
A(-\nabla^2 + m_A)B + B(-\nabla^2 + m_B)B.
\]
This fixes all variables. Provided that $a$ and $b$ are not simultaneously zero the path-integral over $A$ yields a determinant depending on $m_a$ and a delta function for $B$. We find $m_A = -2/\ell^2 + M^2$, where $M^2 = M_+^2 = M_-^2$ is the mass of the two propagating massive gravitons, thus
\[
Z_{\text{inlog}}^{\mathcal{M}} = \left[ \det \left( -\nabla^2 - \frac{2}{\ell^2} \right) \right]^{-1/2} \left[ \det \left( -\nabla^2 - \frac{2}{\ell^2} + M^2 \right) \right]^{-1/2}.
\]
To obtain the result for the scalar sector we rescale $2\bar{K} = \bar{\sigma}K', \varepsilon = \alpha\varepsilon'$ with $\alpha$ as defined earlier, and shift $\bar{h} = h - K'/2$, $\delta' = \delta + \ell^2/6F$. Then we integrate over $\bar{h}$, $\varepsilon'$ and $\delta'$ to obtain a quadratic expression in $\bar{k}$ and $\bar{F}$. This we ‘block-diagonalize’ again to find exactly (14), provided $\beta \neq -2b_2/\ell^2$. Setting $\beta = -2b_2/\ell^2$, together with equation (20), implies
\[
(b_2, \beta) = (0, 0) \quad \text{or} \quad (b_2, \beta) = (-2\sigma \ell^4, 4\sigma \ell^2).
\]
The first solution is Einstein–Hilbert gravity while the second one is the tricritical point, so we already covered those.

The full one-loop gravity partition function for massive-log PET gravity reads
\[
Z_{\text{inlog PET}}^{\mathcal{M}} = Z_{\text{Einstein}} \cdot (Z_{\mathcal{M}})^2,
\]
where we defined $\mathcal{M}^2 = M^2 + 1/\ell^2$ and $Z_{\mathcal{M}}$ is given in (19).

### 2.2.2. A special point

The theory shows very interesting behavior if, in addition to the single log limit $\beta = -3b_2/\ell^2 - 2\sigma \ell^2$, we take the scaling limit $b_2 \to \infty$. In order to do so it is advised to rescale the auxiliary field $f_{\mu\nu}$. After rescaling
\[
\tilde{f}_{\mu\nu} = \sqrt{2b_2} f_{\mu\nu},
\]

\(^3\) Note that $\mathcal{M}^2$ defined in this way differs from the mass squared of the propagating massive mode—which we denote by a non-script $M^2$—by a factor $1/\ell^2$. Therefore, log-modes for which $M^2 = 0$ have $\mathcal{M}^2 = 1/\ell^2$.\(^3\)
all fields in the linearized action (2) are normalized ‘canonically’, i.e. there are no dimensionful parameters multiplying terms such as \((h/k/f)\mu\nu \mathcal{G}_{\mu\nu}(h/k/f)\). If we now take the limit \(b_2 \rightarrow \infty\), keeping \(b_2/\kappa^2 = 1/\kappa'^2\) finite, the action (2) becomes

\[
\delta^{(2)}S_{\text{pmPET}} = \frac{1}{\kappa'^2} \int d^3 x \sqrt{g} \left\{ k^{\mu\nu} \mathcal{G}_{\mu\nu}(h) + \tilde{f}^{\mu\nu} \mathcal{G}_{\mu\nu}(\tilde{f}) - \frac{1}{2\ell^2} \left( \tilde{f}^{\mu\nu} \tilde{f}_{\mu\nu} - \tilde{f}^2 \right) \right\}. \tag{27}
\]

The theory with the linearized action (27) has much more gauge symmetry then the original theory (2). First, using the self-adjointness of the tensor operator \(\mathcal{G}_{\mu\nu}\) we can write the first term as \(h^{\mu\nu} \mathcal{G}_{\mu\nu}(k)\). Thus, the vector modes \(v_\mu\) coming from the split of the auxiliary field \(k_{\mu\nu}\), see equation (4), are gauge modes similar to the vector part \(\xi_\mu\) of \(h_{\mu\nu}\). Therefore we have an additional infinitesimal vector gauge symmetry.

Secondly, we note that the \(\tilde{f}_{\mu\nu}\) terms match exactly the Lagrangian of partially massless gravity, a certain parameter limit of NMG [5]. One of the massive modes becomes partially massless in the sense of Deser and Waldron [19] and the theory is invariant under the gauge transformation

\[
\delta \tilde{f}_{\mu\nu} = \nabla_\mu \nabla_\nu \zeta - \frac{1}{\ell^2} g_{\mu\nu} \zeta, \tag{28}
\]

with an infinitesimal scalar gauge parameter \(\zeta\). Further features of ‘partially massless NMG’ were discussed in [5] and [7].

However, for NMG it was shown that this gauge invariance only appears at the linearized level [20] and does not persist in the nonlinear theory. We expect the same to be true here and it would be interesting to verify this. Furthermore, we would like to point out the relation to the ‘canonical bifurcation’ [21] effect taking place in the three-dimensional pure quadratic curvature model of [22]. In fact, for PET gravity in the above mentioned limit all conditions (i)–(iv) of [21] for the ‘canonical bifurcation’ are met.

Another remark concerns the values of the masses of the propagating modes. This is directly related to the background around which we linearize the theory. So far we have considered our background to be AdS. In [6] the limit \(b_2 \rightarrow \infty\) is not contained in the physical range of the parameters \(\beta\) and \(b_2\) because (one of) the mass squared is negative. This is a consequence of choosing an AdS background. In analogy to ‘partially massless NMG’ [5, 7] the mass squared of the partially massless mode in an AdS background is negative and saturates the Breitenlohner–Freedman bound [23].

Formally, it is easy to obtain (27) and (28) in a de Sitter background by replacing \(1/\ell^2 = -\Lambda\), with positive \(\Lambda\). Then (27) does indeed propagate partially massless modes with positive mass squared.

Because the nature of the additional symmetries at the partially massless point is not clear to us, e.g. whether they exist also at the nonlinear level, we will not discuss further the partition function of this particular theory but go on to discuss the better understood critical points of PET gravity and their LCFT duals.

### 2.3. CFT interpretation

Following the logic laid out in [8] and [9] we present the partition functions of the conjectured LCFT duals. Lacking a better knowledge/understanding of LCFT partition functions we give only the partition functions corresponding to single-particle log-excitations on the CFT side. We find perfect agreement of the results at that level.
Tricritical point. In the double-log case PET gravity is conjectured to be dual to a parity even rank-3 LCFT. For such a LCFT the single-particle contribution is

\[ Z^{1\text{-particle, Double log}} = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left( 1 + \frac{2q^2 + 2\bar{q}^2}{|1-q|^2} \right). \]  

(29)

We can thus interpret the result (16) as

\[ Z^{\log_{\text{PET}}} = Z^{1\text{-particle, Double log}} + \text{multiparticle}, \]  

(30)

where the multi-particle contribution is given by

\[ Z^{\text{multiparticle}} = \sum_{h, \bar{h}} \frac{N_{h, \bar{h}} q^h \bar{q}^{\bar{h}}}{|1-q^n|^2}. \]  

(31)

By explicit calculation of \( N_{h, \bar{h}} \) for low values of \( h \) and \( \bar{h} \), and using the combinatorial counting argument from \([8, 9]\), one can show that all \( N_{h, \bar{h}} \)s are non-negative integers. Thus, we can interpret (31) as the contribution of physical states to the partition function.

Single log. For \( \beta = -3b_2/\ell^2 - 2\sigma\ell^2 \) and \( b_2 \neq \pm 2\sigma\ell^4 \) PET gravity is conjectured to be dual to a parity even rank-2 LCFT, plus an additional massive mode. The single-particle partition function is

\[ Z^{1\text{-particle, Single log}} = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left( 1 + q^2 + \bar{q}^2 + \frac{q^{|M|+1}\bar{q}^{M-1} + q^{M-1}\bar{q}^{|M|+1}}{|1-q|^2} \right). \]  

(32)

Again, we can write

\[ Z^{\log_{\text{PET}}} = Z^{1\text{-particle, Single log}} + Z^{\text{multiparticle}}, \]  

(33)

with \( Z^{\text{multiparticle}} \) given in (31), and for given \( M \) show that all \( N_{h, \bar{h}} \)s are non-negative integers.

Massive log. In the parameter range where \( M_+ = M_- = M \), PET gravity is conjectured to be dual to a parity even rank-2 LCFT with non-vanishing central charge given by

\[ c_{R/L} = \frac{3\ell\sigma}{G} \frac{\ell^4M^4}{1 + 2\ell^2M^2 + 2\ell^4M^4} = \frac{3\ell\sigma}{G} \frac{(\ell^2M^2 - 1)^2}{1 - 2\ell^2M^2 + 2\ell^4M^4}. \]  

(34)

The single-particle CFT partition function for such a theory takes the form

\[ Z^{1\text{-particle, Massive log}} = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left( 1 + \frac{2q^{|M|+1}\bar{q}^{M-1} + 2q^{M-1}\bar{q}^{|M|+1}}{|1-q|^2} \right). \]  

(35)

Again, we can write

\[ Z^{\log_{\text{PET}}} = Z^{1\text{-particle, Massive log}} + Z^{\text{multiparticle}}, \]  

(36)

with \( Z^{\text{multiparticle}} \) given in (31), and for given \( M \) show that all \( N_{h, \bar{h}} \)s are non-negative integers.

3. Quasi-normal modes and partition functions of (truncated) critical gravities

In this section we address the question if there are other methods to calculate the gravity one-loop partition function circumventing the (direct) use of heat kernel techniques. We will focus on one specific idea which relies on the work of Denef et al [14], where they claim that the calculation of black hole determinants via heat kernel techniques is equal to summing over the quasi-normal mode spectrum of the theory. Calculations in higher spin gravity lend support to this conjecture, see e.g. [24]. However, the issue of boundary conditions does not
play such a prominent role there and the method of summing over quasi-normal mode spectra has not been applied to critical gravity theories yet.

One motivation to look for other-than heat kernel methods is the apparent difference of the results obtained for the gravity one-loop partition function of log-TMG [8] (TMG at the chiral point with log boundary conditions) and chiral gravity [25] (TMG at the chiral point with Brown–Henneaux boundary conditions). It was argued in [26] that this is an artifact of the ignorance of the AdS heat kernel to the existence of null modes and negative energy states, which should be excluded from the spectrum of physical modes. It was suggested to calculate a different heat kernel, one that would not include log-modes or null states. While this seems to be a formidable, if challenging, task we will take a different approach.

It is hard to impose boundary conditions on the heat kernel, but it is very simple to do so for quasi-normal modes: choosing the ansatz for the solution determines the boundary conditions the mode will fulfil. Here we do not refer to the boundary conditions at the black hole horizon, which need be specified for quasi-normal modes, but to the asymptotic behavior of the mode.

An analysis like the one we carried out in section 2 is usually around thermal AdS, see e.g. [17]. This is not a black hole background. However, the approach of [14] also holds for non black hole backgrounds, quasi-normal modes being replaced by normal modes [14]. On the other hand, the results obtained in section 2 would not change if we would take the background to be the BTZ black hole, instead of AdS; mainly because the BTZ is obtained by identifications of AdS [27]. Moreover, the thermal AdS background used above is also the background of the Euclidean BTZ black hole, with the identification $\tau \rightarrow -1/\tau$ [17]. Therefore, we shall blithely permit ourselves to go from one background to the other, as its suits our analysis. In this section our background will be the BTZ black hole.

Quasi-normal modes for excitations around the BTZ black hole were calculated in [28]. In the context of critical gravity this was first done for log-TMG in [29, 30] and a tricritical theory was discussed recently in [31].

In the following we will calculate the one-loop partition function of two critical gravity theories in three dimensions, NMG and PET gravity, using the conjecture of [14]. We confirm that at the (tri-)critical point the results agree with earlier calculations using heat kernel techniques. Based on that confidence we calculate different truncated one-loop partition functions by subsequently summing over different quasi-normal mode spectra, allowing log-excitations as well as not allowing them.

We will start by recalling the partition function of Einstein gravity with a negative cosmological constant. We discuss the contributions to the partition functions of Einstein gravity, NMG and PET gravity. Then we will define a truncation of the theory and apply it to critical NMG and tricritical PET gravity. Finally we summarize and comment on the results we have obtained.

3.1. Partition function of Einstein gravity

The one-loop contribution to the partition function of Einstein gravity is (see e.g. [18])

$$Z_{\text{Ein}} = \sqrt{\frac{\det(-\nabla^2 + 2)}{\det(-\nabla^2 - 2)}}. \quad (37)$$

Let us briefly comment on the two terms contributing to the partition function in (37). The determinant in the numerator is due to the gauge choice and corrects the path-integral measure (see e.g. [8]), while the determinant in the denominator stems from the gauge-fixed equations.
of motion. Thus, for a critical theory where we observe degeneration of the equations of motion, we expect to find a multiple of the contribution \( \det(-\nabla^2 - m^2) \) in the denominator.

According to [14] and [24] the determinants are evaluated using

\[
\frac{1}{2} \ln Z_{\Delta_i} = \ln \prod_{\kappa \geq 0} \left( 1 - q^{\kappa + \Delta_i} \right)^{-2(k+1)},
\]

(38)

where \( \Delta_i = 1 + |m_i| \) can be read off from the definitions

\[
\det(-\nabla^2 + m^2_1 - 2) \quad \text{and} \quad \det(-\nabla^2 + m^2_2 - 3).
\]

(39)

Comparison with (37) yields \( \Delta_1 = 3 \) and \( \Delta_2 = 2 \) to give

\[
Z_{\text{Ein}} = \prod_{n \geq 2} \frac{1}{1 - q^n}.
\]

(40)

This has a nice interpretation as the vacuum character of a CFT [32]. In the following we will refer to any theory of gravity whose one-loop contribution is of the form (40) as dual to Einstein gravity or an ‘ordinary’ CFT as opposed to a logarithmic CFT, but we make no further restrictions. As we will see a CFT with zero central charge gives rise to the same character—simply because the same modes are present in the theory—even though they are null modes.

We note here that massless spin-two quasi-normal modes are often dismissed because they are pure gauge, see e.g. [29]. But as is often the case in gravity the asymptotic behavior of the modes tells us whether it is relevant or not. The Einstein modes are large gauge transformations and as shown by Brown and Henneaux [10] they are of crucial importance as generating elements of the asymptotic symmetry group. Thus, being interested in the boundary behavior, it is logical to include those modes and that we obtain precisely the CFT vacuum character by summing over those modes that are responsible for non-trivial diffeomorphisms at the boundary.

However, we try to argue in this work that we can choose to include them in the quasi-normal mode spectrum or not. Here we do include them because they are related to non-trivial excitations in the dual conformal field theory with non-zero central charge \( (c = 3\ell^2/2G) \). In the same vein, for a critical theory we can choose not to—or, as spelled out in [26], shall not—include them in the spectrum if they correspond to null states that lead to zero central charge in the dual CFT.

It was pointed out too in [26] that even formula (40) is over-counting states if the central charge is too small. Then we find null states that are multi-particle states, combinations of states which have positive norm when considered as single-particle excitations. This lead to the restriction from (40) to a (Virasoro) minimal model character. Since in the following we are dealing with critical theories the central charge is always zero in our case. Therefore all non-log-modes are null modes.

We will now go on to our main objective, the partition functions of critical gravities.

3.2. Partition function of critical NMG and PET gravity

Let us consider the partition function of critical NMG, which is given by [8]

\[
Z_{\text{NMG}} = Z_{\text{Ein}} \cdot \frac{1}{\sqrt{\det(-\nabla^2 - 2)^T}}.
\]

(41)

Not surprisingly it contains \( Z_{\text{Ein}} \) because all solutions to Einstein gravity are also solutions to NMG. The second factor in (41) comes from the massive graviton. Here we already took the
limit to critical NMG, thus, as a consequence of the degeneration of the equations of motion, this term coincides with the spin-two contribution of Einstein gravity.

Straightforward application of formula (38) yields

\[
Z_{\text{cNMG}} = Z_{\text{Ein}} \cdot \prod_{m \geq 0}^{\infty} \frac{1 - q^{m+2}}{1 - q^{m+1} \vert_2^2} = Z_{\text{Ein}} \cdot \prod_{m \geq 2}^{\infty} \prod_{n \geq 0}^{\infty} \frac{1}{1 - q^{m+n} \vert_2^2},
\]

which, for \( q = \bar{q} \), perfectly agrees with the result in [8]. On the one hand this is a confirmation of the conjecture of [14]. On the other hand, since (42) is equal to the result that was used to support the LCFT conjecture, it tells us, that, to obtain (42), we already summed over the log quasi-normal frequencies. They are equal to the frequencies of the Einstein quasi-normal modes [29, 30] and correspond to the poles of the retarded correlators in the dual CFT [28]. In the log case these are double poles [30], so we must sum over them twice. Taking a look at (41) we see that we actually did count the spin-two frequencies twice simply because the (square root of the) determinant \( \det \left( -\nabla^2 - 2 \right)_2^{TT} \) appears twice.

The partition function of tricritical PET gravity was calculated in section 2

\[
Z_{\text{cPET}} = Z_{\text{Ein}} \cdot \frac{1}{\det(\nabla^2 - 2)_2^{TT}}.
\]

Using formula (38) we find

\[
Z_{\text{cPET}} = Z_{\text{Ein}} \cdot \left( \prod_{m \geq 0}^{\infty} \frac{1 - q^{m+2}}{1 - q^{m+1} \vert_2^2} \right)^2 = Z_{\text{Ein}} \cdot \left( \prod_{m \geq 2}^{\infty} \prod_{n \geq 0}^{\infty} \frac{1}{1 - q^{m+n} \vert_2^2} \right)^2.
\]

For \( q = \bar{q} \) this agrees with (16). Just as in (42) we summed over one factor square root of \( \det(\nabla^2 - 2)_2^{TT} \) for each pole of the retarded correlators. To obtain (44) we need three such factors and indeed one finds triple poles at a tricritical point [31].

Now that we identified where the separate terms in (41) and (43) come from we can address the issue of truncating the theory.

The calculations in (42) and (44) suggest that the conjecture of [14] holds for NMG and PET gravity at the (tri-)critical point. Based on this finding, we go on to make use of the quasi-normal mode method and excise certain modes from the spectrum by ignoring their contribution to the partition function. For example, in critical NMG an intriguing idea would be not to sum over the spin-two modes at all because they correspond either to null states or negative energy states. Of course this would alter formula (38) in a drastic way:

\[
Z_{\Delta_2} = 1.
\]

In the following we will comment on the implications of such a restriction.

3.3. Hand-picked partition functions of critical gravities

If we do not take into account the contribution from the spin-two modes the partition function of critical NMG would become

\[
Z_{\text{nomodes}}' = \prod_{m \geq 3}^{\infty} \prod_{n \geq 0}^{\infty} \frac{1 - q^{m+n} \vert_2^2}{1 - q^{m+2} \vert_2^2}.
\]

\footnote{For the non-rotating BTZ black hole \( q = \bar{q} \). This was also used to show the equality of the two approaches in [24]. We think that equality of the two calculations should also hold for the rotating BTZ but it is technically more challenging to show.}
Hence, we would be lead to conclude that the theory deprived of its spin-two modes is not trivial. However, it seems likely that the correct interpretation is that here we truncated too much and the resulting theory does not make sense. If we were to cancel only the log-modes we find, in fact, for any parity even critical gravity theory without log-modes,

\[
\tilde{Z}_\text{cNMG} = \sqrt{\frac{\det(-\nabla^2 + 2)_{T_1}}{\det(-\nabla^2 - 2)_{T_2}}} = Z_{\text{Ein}}.
\]  

(47)

Thus, we could conclude that critical NMG with Brown–Henneaux boundary conditions is dual to Einstein gravity. However, we know that the only propagating modes are not only pure gauge but also null states and the dual CFT has vanishing central charge. Therefore, according to [26] we would have to drop also the Einstein modes which, naively, brings us back to (46).

Let us now consider a six-derivative theory that offers ‘higher-rank’ criticality and more possibilities to truncate the theory, PET gravity. Truncating to the zero charge sub-sector, i.e. imposing log boundary conditions, kills the log\(^2\)-modes. If we further dispense the Einstein modes because they are null the partition function effectively reduces to

\[
\tilde{Z}_\text{PET} = \sqrt{\frac{\det(-\nabla^2 + 2)_{T_1}}{\det(-\nabla^2 - 2)_{T_2}}} = Z_{\text{Ein}}.
\]  

(48)

This suggest that tricritical gravity with the above mentioned restrictions is again dual to Einstein gravity. The propagating degrees of freedom have zero energy but non-trivial two-point functions [6]. The coexistence of the two quadratic forms, energy and correlators, yielding different results is due to a linearization instability [13] and the resulting theory does not seem sensible. If we truncate further the log-modes by imposing Brown–Henneaux boundary conditions we are again left with the infamous result (46).

3.4. Summary

We have shown the equivalence of the quasi-normal mode and heat kernel approaches to calculate the one-loop partition function for critical NMG and PET gravity. We then identified the contributions to the partition function of the different modes that on the gravity side contribute to this spectrum. This identification lead us to conclude whether or not we should include the mode when summing over the quasi-normal mode spectrum.

We applied the truncation to two critical theories, NMG and PET gravity, which are dual to a rank-2 and a rank-3 LCFT respectively. We found that a truncation of NMG by imposing Brown–Henneaux boundary conditions yields a theory dual to an ordinary CFT. PET gravity with log boundary conditions and truncating the gauge modes yields another theory dual to an ordinary CFT. In the case of PET gravity, however, the theory has a linearization instability [13]. So, while the higher-rank criticality of tricritical gravity seemed to offer the possibility of a truncation to a ‘sensible’ sub-sector, nonlinear calculations suggest that critical theories are either dual to LCFTs, or ordinary CFTs propagating null modes.

It might look as if we were deliberately canceling the annoying terms in the partition function. The main goal of this work was to motivate this cancellation by identifying each determinant with its corresponding quasi-normal mode. By deciding which mode we want to

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5 We do not make any claims about consistency of the suggested truncations. Rather, we calculate partition functions of possible truncations of critical theories that have been put forward in the literature elsewhere.
keep we are at the same time deciding which determinants actually contribute and which do not.

We did not include parity odd theories in our discussion because one of the main ingredients, equation (38), only works for parity even theories (and a non-rotating BTZ black hole background). It would be very rewarding to find an expression similar to (38) which, unlike equation (42), yields the critical NMG partition function from [8] on the nose. Furthermore, this would allow one to consider also the case of chiral gravity, or GMG at the tricritical point.

Finally, we stress again that the relation between heat kernel and quasi-normal mode approach is a conjecture. The fact that the results obtained using both formalisms agree made us confident to think about the truncation procedure. A rigorous proof of the conjecture of [14] would not only confirm our results but also be a strong motivation to consider the parity odd case mentioned in the previous paragraph.

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Appendix. Multiplicity coefficients

To show the positivity of the multiplicity coefficients $N_{h, \tilde{h}}$ we use the combinatorial counting arguments of [8, 9]. The arguments are given explicitly for the (single) log case in [8] and for the tricritical case in [9]. To proof that all multiplicity coefficients are positive the first few coefficients have to be determined explicitly. The tables in this appendix enlist those first few coefficients for the critical cases discussed in the main text. Table A1 enlists the coefficients for the double log case, table A2 the coefficients for the single log and table A3 gives the coefficients of the massive log case. In the latter two tables we fixed the mass parameter to be $\mathcal{M} = 2$, since $\mathcal{M} = 1$ would recover the double log case while $\mathcal{M} = 0$ is a partially massless mode.

| $\bar{h}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $h = 0$: | 0 | 0 | 0 | 0 | 3 | 1 | 7 | 3 |
| $h = 1$: | 0 | 0 | 0 | 0 | 1 | 3 | 3 | 9 |
| $h = 2$: | 0 | 0 | 4 | 4 | 13 | 13 | 35 | 41 |
| $h = 3$: | 0 | 0 | 4 | 4 | 13 | 23 | 47 | 77 |
| $h = 4$: | 3 | 1 | 13 | 13 | 47 | 61 | 148 | 216 |
| $h = 5$: | 1 | 3 | 13 | 23 | 61 | 115 | 238 | 422 |
| $h = 6$: | 7 | 3 | 35 | 47 | 148 | 238 | 550 | 908 |
| $h = 7$: | 3 | 9 | 41 | 77 | 216 | 422 | 908 | 1690 |

Table A1. Double log multiplicity coefficients $N_{h, \bar{h}}$ in equation (30) for $h, \bar{h} < 8$. 
Table A2. Single log multiplicity coefficients $N_{\hat{h}, \hat{h}}$ in equation (33) for $\hat{h}, \hat{\bar{h}} < 8$ and $\mathcal{M} = 2$.

| $\hat{h}$  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| $h = 0$:   | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 0 |
| $h = 1$:   | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 4 |
| $h = 2$:   | 0 | 0 | 1 | 1 | 3 | 3 | 8 | 7 |
| $h = 3$:   | 0 | 1 | 3 | 4 | 9 | 12 | 22 |
| $h = 4$:   | 1 | 0 | 3 | 4 | 11 | 14 | 31 | 41 |
| $h = 5$:   | 0 | 2 | 3 | 9 | 14 | 31 | 49 | 91 |
| $h = 6$:   | 2 | 1 | 8 | 12 | 31 | 49 | 104 | 159 |
| $h = 7$:   | 0 | 4 | 7 | 22 | 41 | 91 | 159 | 302 |

Table A3. Massive log multiplicity coefficients $N_{\hat{h}, \hat{h}}$ in equation (36) for $\hat{h}, \hat{\bar{h}} < 8$ and $\mathcal{M} = 2$.

| $\hat{h}$  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| $h = 0$:   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h = 1$:   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $h = 2$:   | 0 | 0 | 0 | 0 | 0 | 3 | 1 |
| $h = 3$:   | 0 | 0 | 0 | 0 | 0 | 1 | 3 |
| $h = 4$:   | 0 | 0 | 0 | 0 | 4 | 4 | 7 | 5 |
| $h = 5$:   | 0 | 0 | 0 | 0 | 4 | 4 | 5 | 13 |
| $h = 6$:   | 0 | 0 | 3 | 1 | 7 | 5 | 10 | 14 |
| $h = 7$:   | 0 | 0 | 1 | 3 | 5 | 13 | 14 | 38 |

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