Chromatic number of signed graphs with bounded maximum degree

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Abstract

A signed graph \((G, \Sigma)\) is a graph positive and negative (\(\Sigma\) denotes the set of negative edges). To re-sign a vertex \(v\) of a signed graph \((G, \Sigma)\) is to switch the signs of the edges incident to \(v\). If one can obtain \((G, \Sigma')\) by re-signing some vertices of \((G, \Sigma))\), then \((G, \Sigma) \equiv (G, \Sigma')\). A signed graphs \((G, \Sigma)\) admits an homomorphism to \((H, \Lambda)\) if there is a sign preserving vertex mapping from \((G, \Sigma')\) to \((H, \Lambda)\) for some \((G, \Sigma) \equiv (G, \Sigma')\).

The signed chromatic number \(\chi_s((G, \Sigma))\) of the signed graph \((G, \Sigma)\) is the minimum order (number of vertices) of a signed graph \((H, \Lambda)\) such that \((G, \Sigma)\) admits a homomorphism to \((H, \Lambda)\). For a family \(F\) of signed graphs \(\chi_s(F) = \max_{(G, \Sigma) \in F} \chi_s((G, \Sigma))\). We prove \(2\Delta/2 - 1 \leq \chi_s(G_\Delta) \leq (\Delta - 1)^2 + 2\) for all \(\Delta \geq 3\) where \(G_\Delta\) is the family of connected signed graphs with maximum degree \(\Delta\).

Keywords: signed graphs, homomorphism, signed chromatic number, maximum degree.

1 Introduction and the main result

A signed graph \((G, \Sigma)\) is a graph \(G\) with positive and negative (\(\Sigma\) denotes the set of negative edges) and \(G\) denotes the underlying graph. The set of positive edges is denoted by \(\Sigma^c\). When the set of negative edges \(\Sigma\) is known from the context, we can denote the signed graph \((G, \Sigma)\) by \((G)\). In general, the set of vertices and the set of edges of the signed graph \((G, \Sigma)\) are denoted by \(V(G)\) and \(E(G)\). To re-sign a vertex \(v\) of a signed graph \((G, \Sigma)\) is to switch the signs of the edges incident to \(v\). Two signed graphs \((G, \Sigma)\) and \((G, \Sigma')\) are equivalent if we can obtain \((G, \Sigma')\) by re-signing some vertices of \((G, \Sigma)\). Given a signed graph an adjacent vertex of \(v\) is called its neighbor. The set of all neighbors of \(v\) is denoted by \(N(v)\) while \(d(v) = |N(v)|\) is the degree of \(v\).

A 2-edge-colored homomorphism \(\psi\) of \((G, \Sigma)\) to \((H, \Lambda)\) is a vertex mapping \(\psi : V(G) \rightarrow V(H)\) such that for each edge \(uv\) of \((G, \Sigma)\) the images induces an edge \(\phi(u)\phi(v)\) in \((H, \Lambda)\) of the same sign as \(uv\). We write \((G, \Sigma) \overset{2ec}{\rightarrow} (H, \Lambda)\) whenever there exists a 2-edge-colored homomorphism of \((G, \Sigma)\) to \((H, \Lambda)\).

Given two signed graphs \((G, \Sigma)\) and \((H, \Lambda), \phi : V(G) \rightarrow V(H)\) is a homomorphism of \((G, \Sigma)\) to \((H, \Lambda)\) if there exists a \((G, \Sigma')\) equivalent to \((G, \Sigma)\) such that \(\phi\) is a 2-edge-colored homomorphism of \((G, \Sigma')\) to \((H, \Lambda)\). We write \((G, \Sigma) \rightarrow (H, \Lambda)\) whenever there exists a homomorphism of \((G, \Sigma)\) to \((H, \Lambda)\).
The signed chromatic number $\chi_s((G, \Sigma))$ or the 2-edge-colored chromatic number $\chi_2((G, \Sigma))$ of the signed graph $(G, \Sigma)$ is the minimum order (number of vertices) of a signed graph $(H, \Lambda)$ such that $(G, \Sigma) \rightarrow (H, \Lambda)$ or $(G, \Sigma) \xrightarrow{2ec} (H, \Lambda)$, respectively. The signed chromatic number $\chi_s(\mathcal{F})$ or the 2-edge-colored chromatic number $\chi_2(\mathcal{F})$ of a family $\mathcal{F}$ of signed graphs is the maximum of the signed chromatic numbers or the 2-edge-colored chromatic numbers, respectively, of the signed graphs from the family $\mathcal{F}$.

The signed graphs and their switch classes have been studied since the beginning of the last century \cite{1} \cite{5} while the homomorphism of signed graphs have been introduced and studied recently by Naserasr, Rollova and Sopena \cite{4}. Till now, the relation between the maximum degree of a signed graph and its chromatic number is not studied. We initiate it by proving the following result adapting a probabilistic proof technique used by Kostochka, Sopena and Zhu \cite{3}.

**Theorem 1.1.** If $\mathcal{G}_\Delta$ is the family of signed graphs with maximum degree at most $\Delta$, then 
\[
2^{\Delta/2-1} \leq \chi_s(\mathcal{G}_\Delta) \leq (\Delta - 1)^2 \cdot 2^{(\Delta - 1)} + 2 \text{ for all } \Delta \geq 3.
\]

Note that both lower and upper bounds are exponential in $\Delta$.

**2 Proof of Theorem 1.1**

First we will prove the lower bound.

Let $(G, \Sigma)$ be a signed graph. Let $uv$ be an positive edge and $xy$ be a negative edge. Then $u$ is a $+$-neighbor of $v$ and $x$ is a $-$-neighbor of $y$. The set of all $+$-neighbors and $-$-neighbors of a vertex $v$ is denoted by $N^+(v)$ and $N^-(v)$, respectively. Let $\vec{a} = (a_1, a_2, ..., a_j)$ be a $j$-vector such that $a_i \in \{+,-\}$ where $i \in \{1, 2, ..., j\}$. Let $J = (v_1, v_2, ..., v_j)$ be a $j$-tuple (without repetition) of vertices from $G$. Then we define the set $N^{\vec{a}}(J) = \{v \in V | v \in N^{a_i}(v_i) \text{ for all } 1 \leq i \leq j\}$. Finally, we say that $G$ has property $P_{t-1}$ if for each $j$-vector $\vec{a}$ and each $j$-tuple $J$ we have $|N^{\vec{a}}(J)| \geq \frac{1+(t-j)(t-2)}{2}$ when $j \in \{0, 1, ..., t - 1\}$.

**Lemma 2.1.** There exists a signed complete graph with property $P_{t-1}$ on $c = t(t-1)2^t$ vertices.

**Proof.** Let $(C, \Pi)$ be a random signed graph with underlying complete graph. Let $u, v$ be two vertices of $(C, \Pi)$ and the events $u \in N^a(v)$ for $a \in \{+,-\}$ are equiprobable and independent with probability $\frac{1}{2}$. We will show that the probability of $C$ not having property $P_{t-1}$ is strictly less than 1 when $|C| = c = t(t-1)2^t$. Let $P(J, \vec{a})$ denote the probability of the event $|N^{\vec{a}}(J)| < \frac{(t-j)(t-2)+1}{2}$ where $J$ is a $j$-tuple of $C$ and $\vec{a}$ is a $j$-vector for some $j \in \{0, 1, ..., t - 1\}$. Call such an event a bad event. Thus,

\[
P(J, \vec{a}) \leq \sum_{i=0}^{(t-j)(t-2)-1} \binom{c-j}{i} 2^{-ij} (1 - 2^{-j})^{c-i-j} < (1 - 2^{-j})^c \sum_{i=0}^{(t-j)(t-2)-1} \frac{c^j}{i!} (1 - 2^{-j})^{-i-j} 2^{-ij} < 2e^{-c2^{-j}} \sum_{i=0}^{(t-j)(t-2)+1} c^i < e^{-c2^{-j}} e^{\frac{(t-j)(t-2)+1}{2}}
\]

Let $P(B)$ denote the probability of the occurrence of at least one bad event. To prove this lemma it is enough to show that $P(B) < 1$. Let $T^j$ denote the set of all $j$-tuples and $W^j$ denote the set of all $j$-vectors. Then
Consider the function $f(j) = 2e^{-c2^{-j}} \frac{e^{\frac{(t-1)(t-2)}{2}+j}}{c^{\frac{(t-1)(t-2)}{2}+j}}$. Observe that $f(j)$ is the $j^{th}$ summand of the last sum from equation (2). Now

$$
\frac{f(j+1)}{f(j)} = \frac{e^{c2^{-j-1}}}{c^{\frac{t-1}{2}}} > \frac{e^{c2^{-t+1}}}{c^{\frac{t-1}{2}}} = \frac{e^{2t-1}}{(t-1)_{\frac{t-1}{2}}^2} > \left( \frac{e^{2t-1}}{t(t-1)} \cdot \frac{e^{2t-1}}{e^{2t-1}} \right)^\frac{t}{2} > 2^{10} (3)
$$

The above relation implies the following

$$
P(B) \leq \sum_{j=0}^{t-1} f(j) < \left( 1 + \frac{1}{2^{10}-1} \right) f(t-1) < 2(1.001) \left( \frac{t^3(t-1)^3}{e^{2t-1}} \cdot \frac{2^{3t}}{e^{2t-1}} \right)^\frac{t}{2} < 1 (4)
$$

This completes the proof. \(\square\)

**Lemma 2.2.** Let $(C, \Pi)$ be a signed graph with property $P_{\Delta-1}$ and $(G, \Sigma)$ be a connected signed graph with maximum degree $\Delta$ and degeneracy $(\Delta - 1)$. Then $(G, \Sigma)$ admits a homomorphism to $(C, \Pi)$.

**Proof.** Let $v_1, v_2, ..., v_k$ be the vertices of $(G, \Sigma)$ in such a way that each vertex $v_j$ has at most $(\Delta - 1)$ neighbors with lower indices. Let $(G_j, \Sigma_j)$ be the signed graph induced by the vertices $v_1, v_2, ..., v_l$ from $(G, \Sigma)$ for each $j \in \{1, 2, ..., k\}$. Now we will inductively construct a homomorphism $f : (G, \Sigma) \rightarrow (C, \Pi)$ with the following properties:

(i) The partial mapping $f(v_1), f(v_2), ..., f(v_j)$ is a homomorphism of $(G_j, \Sigma_j)$ to $(C, \Pi)$ for all $j \in \{1, 2, ..., k\}$.

(ii) For each $i > j$, all the neighbors of $v_i$ with indices less than or equal to $j$ has different images with respect to the mapping $f$.

For $j = 1$ take any partial mapping $f(v_1)$. Suppose that the function $f$ satisfies the above properties for all $i \leq j$ for some fixed $j \in \{1, 2, ..., k-1\}$. Let $A$ be the set of neighbors of $v_{j+1}$ with indices greater than $j + 1$ and $B$ be the set of vertices with indices at most $j$ and with at least one neighbor in $A$. Note that $|B| = (\Delta - 2)|A|$. Let $D$ be the set of possible options for $f(v_{j+1})$ such that the partial mapping is a homomorphism of $(G_{j+1}, \Sigma_{j+1})$ to $(C, \Pi)$. As the partial mapping can be extended also by re-signing the vertex $v_{j+1}$. Thus $|D| > |B|$. Choose any vertex from $D \setminus B$ as the image $f(v_{j+1})$. Note that this partial mapping satisfies the required conditions. \(\square\)

**Proof of Theorem** The lower bound proof of oriented chromatic number for $\mathcal{G}_\Delta$ by Kostochka, Sopena and Zhu can be easily modified to obtain the lower bound $2^{\Delta/2}$ by $\chi_2(\mathcal{G}_\Delta)$. Our lower bound follows from the relation $\chi_2((G, \Sigma)) \leq 2 \cdot \chi_s((G, \Sigma))$ for any signed graph $(G, \Sigma)$. \[4\]
For proving the upper bound, if \((G, \Sigma)\) is not \(\Delta\)-regular, but is a connected signed graph with maximum degree \(\Delta\), then \((G, \Sigma)\) is \(\Delta - 1\) degenerated then we are done by Lemma 2.1 and 2.2. Otherwise, \((G, \Sigma)\) is a \(\Delta\)-regular connected signed graph. Delete one edge \(uv\) from \((G, \Sigma)\) to obtain a connected signed graph which has maximum degree \(\Delta\) and degeneracy \(\Delta - 1\). This new signed graph admits a homomorphism \(g\) to a signed graph \((C, \Pi)\) with property \(P_{\Delta-1}\). Now modify this homomorphism by changing the images \(g(u)\) and \(g(v)\) by adding two new vertices \(g(u), g(v)\) to \(C\) and choosing signs of the edges \(uv\) and the edges between \(\{u, v\}\) and the vertices of \(C\) in such a way that our newly obtained map is also a homomorphism. \(\square\)

3 Conclusive remarks

Klostermeyer and MacGillivray [2] studied pushable chromatic number of oriented graphs. The exact same upper and lower bounds proved in Theorem 1.1 can be proved for pushable chromatic number of connected graphs with bounded maximum degree in a similar way.

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