The Diagonal Dimension of Curves

Noah Olander

Abstract

We prove Conjecture 4.16 of the paper [EL21] of Elagin and Lunts; namely, that a smooth projective curve of genus at least 1 over a field has diagonal dimension 2.

1 Introduction

Let $k$ be a field. All unadorned products will be over $k$. Recall that on $\mathbb{P}^n_k \times \mathbb{P}^n_k$ there is the famous Beilinson resolution of the diagonal

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-n) \boxtimes \Omega_{\mathbb{P}^n_k}^n(n) \to \cdots \to \mathcal{O}_{\mathbb{P}^n_k}(-1) \boxtimes \Omega_{\mathbb{P}^n_k}^1(1) \to \mathcal{O}_{\mathbb{P}^n_k} \boxtimes \mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{O}_\Delta \to 0.$$  

The existence of this resolution implies that every object of the derived category of $\mathbb{P}^n_k$ can be built from the object $G = \mathcal{O}_{\mathbb{P}^n_k} \boxtimes \mathcal{O}_{\mathbb{P}^n_k}(-1) \boxplus \cdots \boxplus \mathcal{O}_{\mathbb{P}^n_k}(-n)$ using direct sums, shifts, direct summands, and at most $n$ cones. More precisely, $D_{perf}(\mathbb{P}^n_k) = \langle G \rangle_{n+1}$. The reader can see [Rou08] for the definition of $\langle G \rangle_{n+1}$.

Now let $X$ be a smooth, separated scheme of finite type over $k$. The example of $\mathbb{P}^n_k$ suggests two notions of dimension for $X$: the Rouquier dimension of $X$, denoted $Rdim(X)$, is the least integer $n$ such that there exists an object $G \in D_{perf}(X)$ with $D_{perf}(X) = \langle G \rangle_{n+1}$. The diagonal dimension of $X$ over $k$, denoted $Ddim(X/k)$ or more often just $Ddim(X)$, is the least integer $n$ such that there exist $G, H \in D_{perf}(X)$ such that $\mathcal{O}_\Delta \in \langle G \boxtimes H \rangle_{n+1} \subset D_{perf}(X \times_k X)$. These definitions were originally given in [Rou08] and [BF12], respectively, and make sense more generally. See [EL21]. Some by now classical arguments show that

$$\dim(X) \leq Rdim(X) \leq Ddim(X) \leq 2\dim(X),$$

see [EL21].

The existence of the Beilinson resolution for $\mathbb{P}^n_k$ shows that $Rdim(\mathbb{P}^n_k) = Ddim(\mathbb{P}^n_k) = n$. In [Orl09], Orlov shows that if $X$ is a smooth curve then $Rdim(X) = 1$, and he conjectures that for any smooth, quasi-projective scheme $X$ over $k$, $Rdim(X) = \dim(X)$. If $X$ is a smooth projective curve over $k$ with $H^1(X, \mathcal{O}_X) = 0$, then $Ddim(X) = 1$. We have seen this if $X = \mathbb{P}^1_k$, and more generally it is Proposition 2. At this point, there seem to be no examples in the literature of smooth varieties $X$ with $Ddim(X) > \dim(X)$, but in [EL21], Elagin and Lunts write that they expect this inequality to hold for most smooth projective varieties. They make the following conjecture:

**Conjecture** (Elagin and Lunts). Let $X$ be a smooth projective curve over $k$ such that $H^1(X, \mathcal{O}_X) \neq 0$. Then $Ddim(X) = 2$.

In this paper, we prove the Conjecture of Elagin and Lunts, see Theorem 1. By the inequalities (1), to prove the Conjecture, we need only show $Ddim(X) > 1$. The following lemma (which the reader who does not want to learn the precise definition of $\langle \rangle_n$ should feel free to take as a black box) explains what this means:

**Lemma 1.** Let $X$ be a smooth curve over $k$. Then $Ddim(X) = 1$ if and only if there exist perfect complexes $E, F, G, H$ on $X$ and a morphism $E \boxtimes F \to G \boxtimes H$ in $D_{perf}(X \times X)$ such that $\mathcal{O}_\Delta$ is a direct summand of its cone.

**Proof.** If there exist such $E, F, G, H$ and $E \boxtimes F \to G \boxtimes H$ then $\mathcal{O}_\Delta \in \langle (E \oplus G) \boxtimes (F \oplus H) \rangle_2$, hence $1 \leq Ddim(X) \leq 1$.

Conversely, if $Ddim(X) = 1$, choose perfect complexes $K, L$ on $X$ such that $\mathcal{O}_\Delta \in \langle K \boxtimes L \rangle_2$. This means that $\mathcal{O}_\Delta$ is a direct summand of the cone of a morphism

$$\varphi: \bigoplus_{i \in I_{finite}} K \boxtimes L[i_i] \to \bigoplus_{j \in J_{finite}} K \boxtimes L[j_j].$$
Then there is the following trick: $\bigoplus_i E_i \boxtimes F_i$ is a direct summand of $(\bigoplus_i E_i) \boxtimes (\bigoplus_j F_j)$, hence we may find perfect complexes $E, F, G, H$ on $X$ and $A, B$ on $X \times X$ such that

$$E \boxtimes F = \left( \bigoplus_{i \in I} K \boxtimes L[m_i] \right) \oplus A$$
$$G \boxtimes H = \left( \bigoplus_{j \in J} K \boxtimes L[n_j] \right) \oplus B.$$

Then there is the morphism

$$\left( \varphi \begin{array}{c} 0 \\ 0 \end{array} \right) : E \boxtimes F \to G \boxtimes H.$$

The cone of this morphism contains $\text{Cone}(\varphi)$ as a direct summand, which in turn contains $O_\Delta$ as a direct summand, so we are done.

Thus we only have to prove that there do not exist such $E, F, G, H$ and $E \boxtimes F \to G \boxtimes H$. A nice exercise for the reader is to prove this under the additional assumption that $E, F, G, H$ are semistable vector bundles on $X$. The strategy of our proof is to reduce to this case. In Section 2 we reduce to the case that $E, F, G, H$ are vector bundles, and then we complete the proof in Section 3 by considering their Harder–Narasimhan filtrations. The way to make these reductions is to recognize that a perfect complex on $X$ has a lot of structure: It is a direct sum of shifts of coherent sheaves, each coherent sheaf is a sum of a vector bundle and a torsion module, and each vector bundle possesses a Harder–Narasimhan filtration. This leads to lots of structure on the box product of two perfect complexes on $X$. We are able to leverage this using Reduction Strategies 1 and 2 (see Section 2), which are easy strategies for taking a direct summand $S$ of the cone of a morphism $A \to B$ in a triangulated category and then showing that $S$ is actually a direct summand of the cone of a simpler morphism.

For people more comfortable with Abelian categories than triangulated categories, we give the following interesting corollary of Theorem 1:

**Corollary 1.** Let $X$ be a smooth projective curve over $k$ such that $H^1(X, \mathcal{O}_X) \neq 0$. Then for any resolution of $\mathcal{O}_\Delta \in \text{Coh}(X \times X)$ of the form

$$\cdots \to E_n \boxtimes F_n \to \cdots \to E_1 \boxtimes F_1 \xrightarrow{d_1} E_0 \boxtimes F_0 \to \mathcal{O}_\Delta \to 0$$

with $E_i, F_i$ vector bundles on $X$, the associated extension class $\xi \in \text{Ext}^2_{\mathcal{O}_{X \times X}}(\mathcal{O}_\Delta, \text{Ker}(d_1))$ is nonzero.

We find Corollary 1 interesting because it depends so strongly on the form of the resolution. Notice how absolutely wrong it becomes if we used instead the resolution $0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0$.

**Proof of Corollary.** This is because $\xi$ is the obstruction to the existence of a section of the map $\text{Cone}(d_1) \to \tau_{\geq 0}(\text{Cone}(d_1)) = \mathcal{O}_\Delta$, so if $\xi$ vanishes then $\mathcal{O}_\Delta$ is a direct summand of the cone of $d_1$.

The author would like to thank Johan de Jong for helpful conversations and Alex Perry for pointing him to the paper [EL21] and its interesting conjectures.

## 2 Reduction to the Vector Bundle Case

In this section, $X$ will be a smooth projective curve over $k$. Our goal is to prove Proposition 1. We begin by studying the structure of the box product $E \boxtimes F$ with $E, F$ perfect complexes on $X$ and setting up some notation regarding this.

**Lemma 2.** Let $\mathcal{E}, \mathcal{F}$ be coherent sheaves on $X$. Then

$$\text{Tor}^i_{\mathcal{O}_{X \times X}}(pr_1^*(\mathcal{E}), pr_2^*(\mathcal{F})) = 0$$

for $i > 0$.

**Proof.** By the formulæ for the stalk of pullbacks and Tor modules at a point, this immediately reduces to the following algebra problem: $A, B$ are DVRs with uniformizers $\pi_A, \pi_B$; $R$ is a regular local ring of dimension 2.
with local ring homomorphisms $A \rightarrow R$ and $B \rightarrow R$ such that $(\pi_A, \pi_B)$ is a regular system of parameters for $R$; and $M, N$ are finite modules over $A, B$. Then

$$\text{Tor}_i^R(M \otimes_A R, N \otimes_B R) = 0$$

for $i > 0$.

To prove this, use the structure theorem for finite modules over a DVR. The free parts cannot contribute to the Tor modules, hence we may assume $M = A/(\pi^n_A)$ and $N = B/(\pi^n_B)$ for $m, n > 0$. Then the Tor groups in question are

$$\text{Tor}_i^R(R/(\pi^n_A), R/(\pi^n_B)),$$

which vanish for $i > 0$ since $(\pi^n_A, \pi^n_B)$ is a regular sequence on $R$. □

Thus if $E, F$ are perfect complexes on $X$, then $E \boxtimes F$ is a decomposable object of $D(X \times X)$ (i.e., isomorphic to the direct sum of its cohomology sheaves shifted appropriately; recall that every perfect complex on $X$ is decomposable) and we have the formula

$$H^k(E \boxtimes F) = \bigoplus_{i+j=k} H^i(E) \boxtimes H^j(F)$$

with the box products on the right hand side underived. Furthermore, since a coherent sheaf on $X$ is a direct sum of a vector bundle and a torsion sheaf, we see that $H^k(E \boxtimes F)$ is a direct sum of four parts which respectively have the form:

$$\bigoplus \text{bundle} \boxtimes \text{bundle}, \bigoplus \text{bundle} \boxtimes \text{torsion}, \bigoplus \text{torsion} \boxtimes \text{bundle}, \bigoplus \text{torsion} \boxtimes \text{torsion}.$$

We will sometimes refer to the first part as $H^k(E \boxtimes F)_{\text{free}}$, the last part as $H^k(E \boxtimes F)_0$ (for 0-dimensional support), and the sum of the last three parts as $H^k(E \boxtimes F)_{\text{tors}}$.

The following two lemmas are simple calculations we will need in the proof of Proposition 1.

**Lemma 3.** Let $E, F$ be perfect complexes on $X$. Then

$$R\text{Hom}_{O_{X \times X}}(O_\Delta, E \boxtimes F) = R\Gamma(X, E \otimes_{O_X}^L F \otimes_{O_X}^LT_X)[-1],$$

and

$$R\text{Hom}_{O_{X \times X}}(E \boxtimes F, O_\Delta) = R\text{Hom}_{O_X}(E \otimes_{O_X}^L F, O_X).$$

**Proof.** The second formula is just adjunction between $R\Delta_*$ and $L\Delta^*$. We prove the first. Using the standard resolution of $O_\Delta$ by $O$ and $O(-\Delta)$, we obtain a distinguished triangle

$$R\text{Hom}_{O_{X \times X}}(O_\Delta, E \boxtimes F) \rightarrow E \boxtimes F \rightarrow E \boxtimes F(\Delta) \rightarrow .$$

Thus,

$$R\text{Hom}_{O_{X \times X}}(O_\Delta, E \boxtimes F) \cong (E \boxtimes F) \otimes_{O_{X \times X}}^L O_\Delta(\Delta)[-1] \cong R\Delta_*(E \otimes_{O_X}^L F \otimes_{O_X}^LT_X)[-1],$$

and so

$$R\text{Hom}_{O_{X \times X}}(O_\Delta, E \boxtimes F) \cong R\Gamma(X \times X, R\Delta_*(E \otimes_{O_X}^L F \otimes_{O_X}^LT_X))[-1] \cong R\Gamma(X, E \otimes_{O_X}^L F \otimes_{O_X}^LT_X)[-1].$$

□

**Lemma 4.** Let $E, F$ be torsion coherent sheaves on $X$ and $A$ a vector bundle on $X \times X$. Then

$$\text{Ext}_X^i(E \boxtimes F, A) = 0$$

for $i = 0, 1$.

**Proof.** It suffices to show $\text{Ext}_X^i(E \boxtimes F, A) = 0$ for $i = 0, 1$ by the relation between local and global Ext’s. This reduces as in Lemma 2 to a local algebra problem: $A, B$ are DVRs with uniformizers $\pi_A, \pi_B$; $R$ is a regular local ring of dimension 2 with local ring homomorphisms $A \rightarrow R$ and $B \rightarrow R$ such that $(\pi_A, \pi_B)$ is a regular system of parameters for $R$; $M, N$ are finitely generated torsion modules over $A, B$; and $F$ is a finite free $R$-module. Then

$$\text{Ext}_R^i(M \otimes_A R) \otimes_R (N \otimes_B R), F) = 0$$

for $i = 0, 1$. □
for \( i = 0, 1 \).

By the structure theorem for finite modules over a DVR, we are allowed to assume \( M = A/(\pi_A^m), N = B/(\pi_B^n) \) for \( m, n > 0 \). Then the Ext groups in question are

\[
\text{Ext}^i_R(R/(\pi_A^m, \pi_B^n), F)
\]

which vanish for \( i = 0, 1 \) since \((\pi_A^m, \pi_B^n)\) is a regular sequence on the free module \( F \). \qed

We have two strategies for taking a summand of a cone of a morphism and showing that it is actually a summand of a cone of a simpler morphism (which is exactly what we are trying to do – see Proposition 1).

**Reduction Strategy 1.** Here \( \mathcal{T} \) is a triangulated category. Suppose we have a morphism \( A \to B \) in \( \mathcal{T} \) with cone \( C \) and that \( S \) is a direct summand of \( C \), so that there are morphisms \( i: S \to C \) and \( p: C \to S \) with \( p \circ i = \text{id}_S \). Suppose that there are distinguished triangles

\[
\begin{align*}
A' &\to A \to A'' \to A'[1] \\
B' &\to B \to B'' \to B'[1]
\end{align*}
\]

in \( \mathcal{T} \).

**Lemma 5.** Notation as above. If \( \text{Hom}(B', S) = 0 \) and \( \text{Hom}(S, A''[1]) = 0 \), then actually \( S \) is a direct summand of the cone of \( A' \to B'' \).

**Proof.** Write \( D \) for the cone of \( A \to B'' \). We will first show that \( S \) is a direct summand of \( D \). By the octahedral axiom, there is a distinguished triangle

\[
B' \to C \to D \to B'[1]
\]

where the first arrow is the composition \( B' \to B \to C \). What we have to show is that the projection \( p: C \to S \) factors through \( D \). This follows via a diagram chase from the assumption that \( \text{Hom}(B', S) = 0 \).

Thus \( S \) is a direct summand of the cone of \( A \to B'' \) and a dual argument to the one in the first paragraph allows us to replace \( A \) with \( A' \). \qed

**Reduction Strategy 2.** This is a slightly more specialized situation. Again \( \mathcal{T} \) is a triangulated category and this time we have a morphism \( A \oplus B \to C \oplus D \) with matrix

\[
\begin{pmatrix}
F & 0 \\
0 & 0
\end{pmatrix},
\]

and we have a direct summand \( S \) of its cone. The cone is isomorphic to \( \text{Cone}(F) \oplus D \oplus B[1] \).

**Lemma 6.** Notation as above. Assume \( \text{Hom}(S, D) = 0 \) and \( \text{Hom}(B[1], S) = 0 \). Then \( S \) is a direct summand of \( \text{Cone}(F) \).

**Proof.** Say the inclusion \( S \to \text{Cone}(F) \oplus D \oplus B[1] \) is given by the maps \((a, b, c)\) and the retraction is given by the maps \((d, e, f)\). Then we have \( da + eb + cf = 1 \) and \( b = 0 = f \) by assumption, hence \( da = 1 \). \qed

**Proposition 1.** Suppose there are perfect complexes \( E, F, G, H \) on \( X \) and a morphism \( E \boxtimes F \to G \boxtimes H \) such that \( \mathcal{O}_\Delta \) is a direct summand of its cone. Then there are vector bundles \( E, F, G, H \) on \( X \) and a morphism \( E \boxtimes F \to G \boxtimes H \) such that \( \mathcal{O}_\Delta \) is a direct summand of its cone.

**Proof.** Step 1. \( \mathcal{O}_\Delta \) is a direct summand of the cone of \( \tau_{\leq 1}(E \boxtimes F) \to \tau_{\geq 0}(G \boxtimes H) \).

This follows from the first reduction strategy since there are no maps \( \tau_{< 0}(G \boxtimes H) \to \mathcal{O}_\Delta \) and no maps \( \mathcal{O}_\Delta \to \tau_{> 1}(E \boxtimes F)[1] \) for degree reasons.

Step 2. \( \mathcal{O}_\Delta \) is a direct summand of the cone of a map

\[
H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)[-1] \to H^0(G \boxtimes H) \oplus H^1(G \boxtimes H)[-1].
\]
Here we apply the second reduction strategy using the decompositions

\[\tau_{\leq 1}(E \boxtimes F) = (H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)[-1]) \oplus \tau_{< 0}(E \boxtimes F)\]
\[\tau_{\geq 0}(G \boxtimes H) = (H^0(G \boxtimes H) \oplus H^1(G \boxtimes H)[-1]) \oplus \tau_{> 1}(G \boxtimes H).\]

For degree reasons, the morphism between them has matrix of the form

\[
\begin{pmatrix}
  * & 0 \\
  0 & 0
\end{pmatrix},
\]

and also

\[\text{Hom}(\tau_{< 0}(E \boxtimes F)[1], \mathcal{O}_\Delta) = 0 = \text{Hom}(\mathcal{O}_\Delta, \tau_{> 1}(G \boxtimes H)),\]

so that reduction strategy gives the result.

Step 3. \(\mathcal{O}_\Delta\) is a direct summand of the cone of a map

\[H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)_0[-1] \to H^0(G \boxtimes H)_{\text{free}} \oplus H^1(G \boxtimes H)[-1].\]

First reduction strategy. This applies because

\[\text{Hom}(H^0(G \boxtimes H)_{\text{tors}}, \mathcal{O}_\Delta) = 0 = \text{Hom}(\mathcal{O}_\Delta, H^1(E \boxtimes F)/H^1(E \boxtimes F)_0).\]

To prove the first equality we have to show that if \(\mathcal{G}, \mathcal{H}\) are coherent sheaves on \(X\) with at least one of them a torsion module, then \(\text{Hom}(\mathcal{G} \boxtimes \mathcal{H}, \mathcal{O}_\Delta) = 0\). But this Hom is identified with \(\text{Hom}_{\mathcal{O}_X}(\mathcal{G} \otimes \mathcal{H}, \mathcal{O}_X)\) which vanishes since there are no maps from a torsion sheaf to a locally free sheaf. For the second equality we have to show \(\text{Hom}(\mathcal{O}_\Delta, \mathcal{E} \boxtimes \mathcal{F}) = 0\) if \(\mathcal{E}, \mathcal{F}\) are coherent sheaves on \(X\) with at least one of them locally free. By Lemma 3 this Hom is identified with \(H^{-1}(X, \mathcal{E} \otimes^L \mathcal{F} \otimes T_X) = 0\) since the tensor product is underived.

Step 4. \(\mathcal{O}_\Delta\) is a direct summand of the cone of a map

\[H^0(E \boxtimes F)_{\text{free}} \oplus H^1(E \boxtimes F)_0[-1] \to H^0(G \boxtimes H)_{\text{free}} \oplus H^1(G \boxtimes H)_0[-1].\]

Second reduction strategy using the decompositions

\[H^0(E \boxtimes F) \oplus H^1(E \boxtimes F)_0[-1] = (H^0(E \boxtimes F)_{\text{free}} \oplus H^1(E \boxtimes F)_0[-1]) \oplus H^0(E \boxtimes F)_{\text{tors}}\]
\[H^0(G \boxtimes H)_{\text{free}} \oplus H^1(G \boxtimes H)[-1] = (H^0(G \boxtimes H)_{\text{free}} \oplus H^1(G \boxtimes H)_0[-1]) \oplus H^1(G \boxtimes H)/H^1(G \boxtimes H)_0[-1].\]

The morphism between them has matrix

\[
\begin{pmatrix}
  * & 0 \\
  0 & 0
\end{pmatrix}.
\]

Most of the vanishing required to prove this is easy and left to the reader, but we will explain why

\[\text{Hom}(H^1(E \boxtimes F)_0, H^1(G \boxtimes H)/H^1(G \boxtimes H)_0) = 0.\]

It suffices to show that if \(\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}\) are coherent sheaves on \(X\) with \(\mathcal{E}, \mathcal{F}\) torsion and at least one of \(\mathcal{G}, \mathcal{H}\) a vector bundle, then \(\text{Hom}(\mathcal{E} \boxtimes \mathcal{F}, \mathcal{G} \boxtimes \mathcal{H}) = 0\). By the Künneth formula, we have

\[\text{Hom}(\mathcal{E} \boxtimes \mathcal{F}, \mathcal{G} \boxtimes \mathcal{H}) = \text{Hom}(\mathcal{E}, \mathcal{G}) \otimes_k \text{Hom}(\mathcal{F}, \mathcal{H}) = 0\]

since one of \(\mathcal{G}, \mathcal{H}\) is a vector bundle and there are no maps from a torsion sheaf to a vector bundle.

Finally, we have

\[\text{Hom}(H^0(E \boxtimes F)_{\text{tors}}[1], \mathcal{O}_\Delta) = 0 = \text{Hom}(\mathcal{O}_\Delta, H^1(G \boxtimes H)/H^1(G \boxtimes H)_0[-1])\]

for degree reasons, so that the second reduction strategy applies.

Step 5. \(\mathcal{O}_\Delta\) is a direct summand of the cone of a map

\[\varphi : H^0(E \boxtimes F)_{\text{free}} \to H^0(G \boxtimes H)_{\text{free}}.\]
This is very similar to the second reduction strategy. We note that the map
\[ H^0(E \boxtimes F)_{free} \oplus H^1(E \boxtimes F)_{0[-1]} \to H^0(G \boxtimes H)_{free} \oplus H^1(G \boxtimes H)_{0[-1]} \]
has diagonal matrix
\[
\begin{pmatrix}
\varphi & 0 \\
0 & \psi
\end{pmatrix}
\]
with respect to the direct sum decompositions: \( \text{Hom}(H^0(E \boxtimes F)_{free}, H^1(G \boxtimes H)_{0[-1]}) = 0 \) for degree reasons and \( \text{Hom}(H^1(E \boxtimes F)_{0[-1]}, H^0(G \boxtimes H)_{free}) = 0 \) by Lemma 4. Therefore, \( \mathcal{O}_\Delta \) is a direct summand of \( \text{Cone}(\varphi) \oplus \text{Cone}(\psi) \). Let the inclusion be given by maps \((a,b)\) and the retraction by maps \((c,d)\) so that \( ca + db = \text{id}_{\mathcal{O}_\Delta} \).

We are going to show that \( db \) is zero on cohomology sheaves, and then we will be done since \( ca \) will be a quasi-isomorphism. But \( \mathcal{O}_\Delta \) has only the one nonzero cohomology sheaf so actually we will be done if we can show
\[ H^0(d) : H^0(\text{Cone}(\psi)) \to \mathcal{O}_\Delta \]
is zero. This follows since \( H^0(\text{Cone}(\psi)) \) has zero-dimensional support, being a submodule of \( H^1(E \boxtimes F)_{0} \).

Step 6. Conclusion.

The Proposition now follows from the same trick as Lemma 1. Since \( H^0(E \boxtimes F)_{free} \) and \( H^0(G \boxtimes H)_{free} \) are direct sums of box products of vector bundles, there are vector bundles \( E, F, G, H \) on \( X \) and \( A, B \) on \( X \times X \) such that
\[
E \boxtimes F = H^0(E \boxtimes F)_{free} \oplus A \\
G \boxtimes H = H^0(G \boxtimes H)_{free} \oplus B.
\]
Take the morphism
\[
\begin{pmatrix}
\varphi & 0 \\
0 & 0
\end{pmatrix}
\]
between them. Then \( \mathcal{O}_\Delta \) is a direct summand of the cone of \( \varphi \) which is a direct summand of the cone of this morphism, and we are done. \( \square \)

3 The Vector Bundle Case

In this section we prove:

**Theorem 1.** Assume \( X \) is a smooth projective curve over \( k \) with \( H^1(X, \mathcal{O}_X) \neq 0 \). Then the diagonal dimension of \( X \) is 2.

We begin by reducing to the case in which \( X \) is geometrically connected over \( k \), i.e., \( H^0(X, \mathcal{O}_X) = k \).

**Lemma 7.** Let \( X \) be a smooth proper variety over \( k \). Set \( k' = H^0(X, \mathcal{O}_X) \), a finite separable field extension of \( k \). Then \( \text{Ddim}(X/k) = \text{Ddim}(X/k') \).

**Proof.** The pullback diagram
\[
\begin{array}{ccc}
X \times_{k'} X & \xrightarrow{f} & X \times_k X \\
\downarrow & & \downarrow \\
\text{Spec}(k') & \xrightarrow{\Delta_{X/k}} & \text{Spec}(k') \times_k \text{Spec}(k')
\end{array}
\]
shows that \( f \) is both an open and closed immersion, since the same is true for \( \Delta_{X/k} \). Since \( X \times_{k'} X \) is connected, \( f \) is the inclusion of a connected component of \( X \times_k X \). Since \( X \) is connected and \( f \circ \Delta_{X/k'} = \Delta_{X/k} \), we can characterize \( f \) as the inclusion of the unique connected component of \( X \times_k X \) containing the diagonal.

With this in mind, we have
\[ \mathcal{O}_{\Delta_{X/k}} \in \langle G \boxtimes_k H \rangle_{d+1} \iff \mathcal{O}_{\Delta_{X/k'}} = Lf^*(\mathcal{O}_{\Delta_{X/k}}) \in \langle Lf^*(G \boxtimes_k H) \rangle_{d+1} = \langle G \boxtimes_{k'} H \rangle_{d+1}, \]
completing the proof. \( \square \)
Let $X$ be a smooth, projective, geometrically connected curve $k$. A vector bundle $\mathcal{E}$ on $X$ has a Harder–Narasimhan filtration:

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_N = \mathcal{E}$$

and we will write $\mu_i = \mu_i(\mathcal{E}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$. The $\mu_i$ are weakly decreasing. It will sometimes be convenient to index instead by slope. For this we will write $\mathcal{E}_i = \mathcal{E}^{\mu_i}$. We will also write $\mathcal{E}_i/\mathcal{E}_{i-1} = \text{gr}^{\mu_i}(\mathcal{E})$. Given a second vector bundle $\mathcal{F}$, we get a filtration of the box-product $\mathcal{E} \boxtimes \mathcal{F}$ with

$$\text{Fil}^\gamma = \sum_{\alpha + \beta \geq \gamma} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta$$

The graded pieces of the filtration are

$$\text{gr}^\gamma = \bigoplus_{\alpha + \beta = \gamma} \text{gr}^\alpha(\mathcal{E}) \boxtimes \text{gr}^\beta(\mathcal{F}).$$

In particular, this is a filtration of $\mathcal{E} \boxtimes \mathcal{F}$ by sub-bundles.

**Proof of Theorem 1.** By Lemma 7, we may assume $X$ is geometrically connected over $k$ of genus $g \geq 1$. By Lemma 1 and Proposition 1, it suffices to show $\mathcal{O}_\Delta$ is not a direct summand of the cone of a map $\mathcal{E} \boxtimes \mathcal{F} \to \mathcal{G} \boxtimes \mathcal{H}$ for any vector bundles $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ on $X$. Suppose it is. Apply Reduction Strategy 1 using the short exact sequences

$$0 \to \sum_{\alpha + \beta \geq 2g-2} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta \to \mathcal{E} \boxtimes \mathcal{F} \to \mathcal{Q} \to 0$$

$$0 \to \sum_{\alpha + \beta > 0} \mathcal{G}^\alpha \boxtimes \mathcal{H}^\beta \to \mathcal{G} \boxtimes \mathcal{H} \to \mathcal{Q}' \to 0.$$

For this to work we need

$$\text{Hom}( \sum_{\alpha + \beta > 0} \mathcal{G}^\alpha \boxtimes \mathcal{H}^\beta, \mathcal{O}_\Delta) = 0 = \text{Hom}(\mathcal{O}_\Delta, \mathcal{Q}[1]).$$

The first equality is true because $\sum_{\alpha + \beta > 0} \mathcal{G}^\alpha \boxtimes \mathcal{H}^\beta$ has a filtration whose graded pieces are direct sums of box products $\mathcal{G}' \boxtimes \mathcal{H}'$ with $\mathcal{G}'$ and $\mathcal{H}'$ semistable bundles with positive sum of slopes, and no such box product has a morphism to $\mathcal{O}_\Delta$:

$$\text{Hom}(\mathcal{G}' \boxtimes \mathcal{H}', \mathcal{O}_\Delta) = \text{Hom}(\mathcal{G}' \otimes \mathcal{H}', \mathcal{O}_X) = \text{Hom}(\mathcal{G}', \mathcal{H}') = 0$$

since the source has higher slope than the target. The second equality is true because $\mathcal{Q}$ has a filtration whose graded pieces are direct sums of box products $\mathcal{E}' \boxtimes \mathcal{F}'$ with $\mathcal{E}'$, $\mathcal{F}'$ semistable bundles with sum of slopes $> 2g-2$, and no such box product receives a map from $\mathcal{O}_\Delta[-1]$:

$$\text{Hom}(\mathcal{O}_\Delta, \mathcal{E}' \boxtimes \mathcal{F}'[1]) = H^0(X, \mathcal{E}' \otimes \mathcal{F}' \otimes T_X) = 0$$

since $\mu(\mathcal{E}') + \mu(\mathcal{F}') + \deg(T_X) < 0$.

Therefore, $\mathcal{O}_\Delta$ is a direct summand of the cone of a morphism

$$\varphi : \sum_{\alpha + \beta \geq 2g-2} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta \to \mathcal{Q}'.$$

We now split into two cases:

**Case 1.** $g \geq 2$.

In this case $\varphi = 0$. This is because the source has a filtration whose graded pieces are direct sums of objects of the form $\mathcal{E}' \boxtimes \mathcal{F}'$ with $\mathcal{E}'$, $\mathcal{F}'$ semistable with sum of slopes $\geq 2g-2 > 0$ and the target has a filtration whose graded pieces are direct sums of objects of the form $\mathcal{G}' \boxtimes \mathcal{H}'$ with $\mathcal{G}'$, $\mathcal{H}'$ semistable with sum of slopes $\leq 0$. Furthermore we have by the K"unneth formula

$$\text{Hom}(\mathcal{E}' \boxtimes \mathcal{F}', \mathcal{G}' \boxtimes \mathcal{H}') = \text{Hom}(\mathcal{E}', \mathcal{G}') \otimes_k \text{Hom}(\mathcal{F}', \mathcal{H}')$$

which can only be non-zero if $\mu(\mathcal{E}') \leq \mu(\mathcal{G}')$ and $\mu(\mathcal{F}') \leq \mu(\mathcal{H}')$. But this contradicts the fact that $\mu(\mathcal{E}') + \mu(\mathcal{F}') > \mu(\mathcal{G}') + \mu(\mathcal{H}')$.

Hence $\mathcal{O}_\Delta$ is a direct summand of $H^0(\text{Cone}(\varphi)) = \mathcal{Q}'$ which is a vector bundle, a contradiction.
Case 2. $g = 1$.

In this case we claim that $H^0(\text{Cone}(\varphi))$ is still a vector bundle, so that this time the argument of Case 1 does not show that $\varphi$ is zero but it does show that we obtain a factorization

$$\varphi : \sum_{\alpha + \beta \geq 2, \alpha - \beta = 0} \mathcal{E}^\alpha \boxtimes \mathcal{F}^\beta \to \text{gr}^0(\mathcal{E} \boxtimes \mathcal{F}) \to \text{gr}^0(\mathcal{G} \boxtimes \mathcal{H}) \subset \mathcal{Q},$$

and it suffices to show that the cokernel of

$$\psi : \text{gr}^0(\mathcal{E} \boxtimes \mathcal{F}) \to \text{gr}^0(\mathcal{G} \boxtimes \mathcal{H})$$

is a vector bundle. The source is a direct sum of box products $\mathcal{E}' \boxtimes \mathcal{F}'$ with $\mathcal{E}', \mathcal{F}'$ semistable bundles with sum of slopes $= 0$, and similarly for the target. By the argument of Case 1 with the Künneth formula, the only nonzero components of $\psi$ are the maps $\mathcal{E}' \boxtimes \mathcal{F}' \to \mathcal{G}' \boxtimes \mathcal{H}'$ with $\mu(\mathcal{E}') = \mu(\mathcal{G}')$ and $\mu(\mathcal{F}') = \mu(\mathcal{H}')$, i.e., $\psi$ is diagonal with respect to the given direct sum decomposition. Therefore, it suffices to show that the cokernel of each nonzero component $\mathcal{E}' \boxtimes \mathcal{F}' \to \mathcal{G}' \boxtimes \mathcal{H}'$ is a vector bundle. This follows from Lemma 8 below.

**Lemma 8.** Let $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ be semistable bundles on a smooth projective geometrically connected curve $X$ over $k$ with $\mu(\mathcal{E}) = \mu(\mathcal{G})$ and $\mu(\mathcal{F}) = \mu(\mathcal{H})$. Then the kernel and cokernel of any map $\varphi : \mathcal{E} \boxtimes \mathcal{F} \to \mathcal{G} \boxtimes \mathcal{H}$ are vector bundles.

**Proof.** It suffices to prove this after pullback along the flat covering $X_\kbar \to X$. Since the pullback of a semistable bundle on $X$ to $X_\kbar$ remains semistable (see [HL10, Corollary 1.3.8]), we may assume $k = \kbar$.

We must show that the function taking a closed point $p \in X \times X$ to $\ker(\varphi \otimes k(p))$ is constant. To prove this, it is enough to show that the rank stays constant on each horizontal and vertical closed fiber $X \times \{y\}, \{x\} \times X, x, y$ closed points of $X$. The restriction of $\varphi$ to $X \times \{y\} \cong X$ is a map

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus m} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus n},$$

where $m = \text{rk}(\mathcal{F})$ and $n = \text{rk}(\mathcal{H})$. This is a map between semistable bundles of the same slope, hence it has constant rank. The same argument works for the vertical fibers and we are done.

We finish the paper by proving a converse to Theorem 1:

**Proposition 2.** Let $X$ be a smooth projective curve over $k$ such that $H^1(X, \mathcal{O}_X) = 0$. Then $\text{Ddim}(X) = 1$.

**Proof.** By Lemma 7, we may assume $X$ is geometrically connected over $k$. Then $X_\kbar \cong \mathbb{P}^1_\kbar$, i.e., $X$ is a Severi–Brauer variety of dimension 1 over $k$. There exists a vector bundle $\mathcal{E}$ on $X$ such that $\mathcal{E}_k \cong \mathbb{P}^1_k(-1) \oplus \mathbb{P}^1_k(-1)$ (take the dual of the bundle constructed in [Kol16, Baby Example 4]). Now consider the ideal sheaf $\mathcal{O}(-\Delta)$ of $\Delta \subset X \times X$. We claim that $\mathcal{O}(-\Delta)$ is a direct summand of $\mathcal{E} \boxtimes \mathcal{E}$. This will complete the proof: The cone of the composition $\mathcal{E} \boxtimes \mathcal{E} \to \mathcal{O}(-\Delta) \to \mathcal{O}_X \boxtimes \mathcal{O}_X \cong \mathcal{O}_X \boxtimes \mathcal{O}_X$ will contain $\mathcal{O}_\Delta$ as a direct summand.

Note that the base change of $\mathcal{O}(-\Delta)$ to $\kbar$ is $\mathbb{P}^1_k(-1) \boxtimes \mathbb{P}^1_k(-1)$ and so $\mathcal{E}_k \boxtimes \mathcal{E}_k \cong \mathcal{O}(\Delta)^{\oplus 4}$ but now [Kol16, Lemma 8] implies that if $\mathcal{F}$ is a vector bundle on $X \times X$ with $\mathcal{F}_k \cong \mathcal{O}(\Delta)^{\oplus 4}$, then actually $\mathcal{F} \cong \mathcal{O}(\Delta)^{\oplus d}$, and we are done.

**References**

[BF12] Matthew Ballard and David Favero. Hochschild Dimensions of Tilting Objects. *International Mathematics Research Notices*, 2012(11):2607–2645, 01 2012.

[EL21] Alexey Elagin and Valery A. Lunts. Three notions of dimension for triangulated categories. *Journal of Algebra*, 569:334–376, 2021.

[HL10] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*, volume 2nd ed of *Cambridge Mathematical Library*. Cambridge University Press, 2010.

[Kol16] János Kollár. Severi-brauer varieties; a geometric treatment, 2016.

[Orl09] Dmitri Orlov. Remarks on generators and dimensions of triangulated categories. *Moscow Mathematical Journal*, 9:143–149, 2009.

[Rou08] Raphaël Rouquier. Dimensions of triangulated categories. *Journal of K-Theory*, 1(2):193–256, 2008.

COLUMBIA UNIVERSITY DEPARTMENT OF MATHEMATICS, 2990 BROADWAY, NEW YORK, NY 10027

nolander@math.columbia.edu