THE OPTIMAL PRODUCTION AND SALES POLICY FOR A NEW PRODUCT WITH NEGATIVE WORD-OF-MOUTH

XIAOMING YAN
School of Software, Dongguan University of Technology
Dongguan 523808, Guangdong, China

PING CAO, MINGHUI ZHANG AND KE LIU
Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences
Chinese Academy of Sciences, Beijing 100190, China

(Communicated by T. C. Edwin Cheng)

Abstract. In this paper, we consider a firm which maximizes its profit by determining the production and sales policies for a new product during the lifetime of the product. Because of capacity constraint, we extend Bass demand process to a more general case including the negative effectiveness of word-of-mouth. We analyze the production and sales policies in two cases: strong negative word-of-mouth and weak negative word-of-mouth. In the case of strong negative word-of-mouth, we show that myopic policy is optimal under some mild conditions. In the case of weak negative word-of-mouth, we show that build-up policy is optimal in a special case of negligible holding cost and discount rate. However, for positive holding cost and discount rate, we compare myopic policy with build-up policy by numerical examples, and show that the build-up policy is no longer a robust approximation to the optimal policy.

1. Introduction. In January, 2008, a potential customer wanted to buy a personal computer of “HP Compaq” and consulted with his friends, who were near to him, about this kind of personal computer. However, one of his friends told him that this kind of personal computer had been sold out in these days and advised him to buy other brands of computers. His friend received the information of shortage from the web site http://www.21tx.com/notebook. Therefore, he purchased another brand of personal computer subsequently. From this customer’s experience, we can at least obtain the following conclusion: an individual, who has known the shortage of some product, maybe inform other potential customers the information of shortage, and the potential customers, who receive the information of shortage from other individuals, maybe give up purchasing the product. Because there always exists maximum production capacity for each firm and demand rate often changes from one time to another time, shortage is commonplace in real life. Hence, in reality, the phenomenon that individuals spread information of shortage is quite ordinary. In this paper, we will name this phenomenon as “negative word-of-mouth”. “Word-of-mouth” generally denotes information transfer between individuals. However,
“negative word-of-mouth”, which denotes the individuals spread the information of shortage in this study, has negative influence on the firm’s sales. Therefore, it is necessary to consider the effects of negative word-of-mouth on operational decisions when a firm has capacity constraint.

In this paper, we study joint production and sales policies for a firm with capacity constraint. Based on Bass demand [1], we consider a more general case, which includes the impact of negative word-of-mouth. When there is restriction of production capacity and the product is highly substitutable, the part of demand over the restriction will be lost. Some potential customers have worried that their demand cannot be satisfied from the lost part, therefore, they will lose also, which is the negative word-of-mouth from the lost demand. In this paper, the lost demand denotes the individuals who have known the shortage of the product and given up the adoption decisions. In the previous literature, it is always assumed that the lost demand will not cause any effect on the potential customers of the new product. In reality, the lost demand can also cause word-of-mouth just like the customers who have received the product. However, the lost demand differs from the cumulative sales (customers who have received the new product) in that the former generates negative word-of-mouth which spreads the information of shortage to potential customers, while the latter generates positive word-of-mouth that is typically assumed in the Bass model. In this paper, we study the effect of both negative and positive word-of-mouth caused by the lost sales and cumulative sales, respectively. Therefore, we consider the situation in which the rapid growth of demand for the new product is due to positive word-of-mouth and the rapid growth of lost sales is due to the impact of capacity constraint and negative word-of-mouth, respectively.

The potential customers who have received the information from lost demand will choose other product from the social system, therefore, we assume that potential customers affected by negative word-of-mouth will be lost and join in the lost demand list in this paper. Bass [1] gave a basic assumption: “The probability that an initial purchase will be made at \( T \) given that no purchase has yet been made is a linear function of the number of previous buyers”. We not only accept Bass assumption, but also assume that the probability that an initial lost demand will be made at \( T \) given that no lost demand has yet been made is a linear function of the number of previous unsatisfied demand.

When there is capacity constraint, we should distinguish the awareness process, the demand process and the cumulative sales process in the paper: the awareness process denotes the total number of customers who have been aware of the new product, which includes the customers who receive the information of the new product from mass media, positive and negative word-of-mouth; the demand process denotes the customers who go to the stores and try to buy the new product; the cumulative sales process denotes the customers who have received the new product. Therefore, the cumulative sales is bounded by the minimum of the cumulative demand and the cumulative production. The firm’s objective is to maximize its total discount profit by choosing appropriate production and sales policies during the lifetime of the product. We show that the structures of the optimal production and sales policies are dependent on the influence of negative word-of-mouth.

The rest of this paper is organized as follows. In §2, we discuss relevant literature. In §3, we present a mathematical model for determining the optimal production and sales policies. We analyze the optimal production and sales policies in the cases of
strong negative word-of-mouth and weak negative word-of-mouth in §4 and §5, respectively. In §6, we give some conclusions and future research directions.

2. Related literature. In the economics literature, Griliches [14] and Mansfield [30] have proposed diffusion models for the spread of technological innovation. These models have been adopted by Bass [1] in the marketing literature to model the adoption of innovation products. In Bass model, the instantaneous demand rate up to time $t$ is given by

$$d(t) = [N - D(t)][\alpha + \beta D(t)].$$

In expression (1), $d(t)$ is the demand rate at time $t$, $D(t) = \int_0^t d(s) ds$ is the cumulative demand up to time $t$, $N$ is the fixed market potential, $N - D(t)$ is the potential customers who have not purchased the product up to time $t$, $\alpha$ is a constant that represents the relative effects of mass media on the population, $\beta$ is a constant that represents the relative effects of word-of-mouth on the population, and the conditional likelihood of adoption is increasing linearly in the number of existing adopters, i.e. $\alpha + \beta D(t)$.

Diffusion models have been used to study advertising, pricing and sales policies by many researchers. Robison and Lakhani [36] incorporated price in Bass model and then derived the optimal pricing policy by using a static analysis, that is, the marginal revenue is equal to the marginal cost. Doland and Jeuland [10], Kalish [21] extended Robison and Lakhani’s model to a general framework that included several previous results as special cases and obtained the optimal policy by using maximum principle in control theory. Krishnan et al. [23] incorporated price decision into generalized Bass model (or GBM [2]) and analyzed the optimal pricing policy. Thompson et al. [40] analyzed an inventory model with simultaneous price and production decisions. They obtained the strong planning and strong forecast horizon which could be used to decompose the original problem into several smaller problems to solve. Cheryl [7] was concerned with a profit maximizing firm that derived the optimal price for his level of output, level of inventory and composition of productive capacity of over time. Kumar and Sethi [26] studied a problem of dynamic pricing of web content on a site where revenue is generated from subscription fee as well as advertisements. They used the optimal control theory to solve the problem and obtained the subscription fee and the advertisement level over time. He and Sethi [16] considered a supply chain in which a manufacturer sells an innovative durable product to an independent retailer over its life cycle, where the product demand follows a Bass-type diffusion process and that it is determined by the market influences, retail price of the product, and shelf space allocated to it. Several researchers in marketing have worked on modifying the Bass model to incorporate advertising. For example, Dodson and Muller [11] introduced a model of new product diffusion which included advertising and word-of-mouth, and incorporated the effects of repeat purchasing. Johansen et al. [20] studied advertising strategies in a marketing channel, formed of one manufacturer and one retailer, in a dynamic setting. Cellini and Lambertini [5] studied advertising activities in a differential oligopoly game. Dockner and Fruchtner [9], Hartl et al. [15] studied diffusion of new products in durable goods markets. Krishnan and Jain [24] used an empirically proven diffusion demand function that explicitly incorporated the advertising component. Kalish [22] introduced a framework for modeling innovation diffusion that included price, advertising and uncertainty. Mesak and Clark [34] considered a general diffusion model of innovation that included pricing and
advertising decisions. Krishnan and Jain [24] used diffusion model to study optimal
dynamic advertising policy for new products. Many other modifications of Bass
model including advertising and pricing can be found in Mahajan et al. [31] and
Mahajan et al. [33]. Bass [1] and subsequent related diffusion models formulated
the adoption of products without considering capacity constraint. However, capacity
constraint is common for availability of a new product in reality because there
often exists a maximal production rate defined by the capacity of a firm.

Fortunately, there has been several literature about the diffusion of new product
under capacity constraint. Simon and Sebastian [38] noted that supply constraints
may distort the parameter estimates obtained by Bass model. Jain et al. [19]
considered supply constraint and presented a modified Bass model where customers
who have tried to buy but have been unsuccessful are backlogging. They formulated
that the level of capacity grew with the number of backorders because it was in a
service environment and the lead time to expand capacity was short. However, in
most manufacturing environment, the capacity is always constant because the lead
time of changing capacity is long. Therefore, the customer losses are common. Ho et
al. [18] generalized Bass model by allowing for a supply constraint. In their model,
the customers who have tried to buy but have been unsuccessful join the waiting
queue and potentially abandon their adoption decision, resulting in lost sales. They
assumed that only the customers who have received the product generate positive
word-of-mouth, and lost demand does not generate any word-of-mouth. Kumar
and Swaminathan [25] considered a firm that sold an innovation product with given
market potential. In their model, the instantaneous demand at time \( t \) is given by

\[
\frac{dD(t)}{dt} = \left[ N - D(t) \right] \left[ \alpha + \beta S(t) \right],
\]

(2)

In expression (2), the cumulative sales process \( S(t) \), which denotes the actual sales
quantity up to time \( t \), satisfies capacity constraint, that is, \( S(t) \) is never more than
the minimum of cumulative demand and cumulative production. The key feature
of this model is that future demand depends not only on past demand but also
on past realized sales, that is, it is only the realized sales that has word-of-mouth.
In Kumar and Swaminathan’s model, we see that the lost demand has no effect on
other potential customers. In reality, the lost demand often causes negative word-of-
mouth. Hence, we have the necessity to study the effect of negative word-of-mouth
on the operational decisions. Kumar and Swaminathan have shown that a heuristic
“build-up” policy is a robust approximation to the optimal policy. However, when
there is negative word-of-mouth from the lost demand in our model, the build-up
policy is not always a robust approximation to the optimal policy. We present some
situations in which myopic policy is better than build-up policy.

In the rest of this section, we review some literature on negative word-of-mouth.
In the marketing literature, negative word-of-mouth, which means one possible
response telling others about the dissatisfaction, has been studied by many re-
searchers. Richins [35] examined correlates of one possible response telling others
about the dissatisfaction and identified variables that distinguished this response
from others. Mahajan et al. [32] examined a diffusion model for products in which
negative information played a dominant role, discussed its implications for opti-
mal advertising timing policy and presented an application to forecast attendance
for the movie Gandhi in the Dallas area. Singh [39] proposed a model which pre-
dicted and explained variation in voice, exit, and negative word-of-mouth behaviors.
Blodgett et al. [3] extended previous research by modeling consumer complaining
behavior as a complex, dynamic process. Charlett et al. [6] studied whether word-of-mouth has a significant effect on the attitudes and probability of purchase of potential consumers. The results of this exploratory study indicated that word-of-mouth, both positive and negative, is indeed a force that can influence the attitudes and predicted purchase behavior of consumers. Bone [4] investigated the effect of word-of-mouth communications on product judgments. Laczniak et al. [27] used attribution theory to explain consumers’ responses to negative word-of-mouth communications. Dellarocas [8] provided an overview of relevant work in game theory and economics on the topic of reputation and identified opportunities that this new area presents for operations research/management science research. Wangenheim [41] investigated postswitching negative word-of-mouth. Wetzer [42] explored the question (What do consumers want to achieve when they engage in negative word-of-mouth communication?) and revealed that consumers pursue specific goals when engaging in negative word-of-mouth and that these goals systematically differ between the specific negative emotions that are experienced. Easta et al. [13] investigated 15 studies and showed that positive word-of-mouth is more common than negative word-of-mouth in every case, moreover, the incidence ratio averages 3 to 1. Easta et al. [12] presented findings on the respondent-assessed impact of positive and negative word-of-mouth on brand purchase probability. For familiar brands, they found that: the impact of positive word-of-mouth is generally greater than negative word-of-mouth. Lau and Ng [28] examined the influence of some individual and situational factors affecting negative word-of-mouth behavior. Luo [29] sought to quantify the long-term financial impact of negative word-of-mouth. Yan and Liu [43] considered a special case in which the positive word-of-mouth and negative word-of-mouth have the same intensity. However, in this study, we incorporate operational decisions (production and sales decisions) into Bass model and mainly analyze the effect of negative word-of-mouth on the optimal production and sales policy. Moreover, since Yan and Liu [43]’s argument is no longer effective, we present a new argument to analyze the structure of the optimal policies for the generalized model in this study.

3. Model formulation. Consider a firm which plans an introduction of a new product and maximizes its total profit by adopting appropriate production and sales policies. Because we mainly analyze the effects of negative word-of-mouth on the structures of optimal policies, we assume that the production capacity q is given and keeps invariant throughout the lifetime of the product. Once the production capacity q has been installed and the diffusion demand process has started, the firm will decide its instantaneous production x(t) and its instantaneous sales s(t) at each time t. Obviously, x(t) and s(t) are bounded by the capacity and instantaneous demand at time t, respectively, that is, x(t) ≤ q and s(t) ≤ d(t), where \( d(t) = \frac{dD(t)}{dt} \), d(t) is just the number of customers who go to the store and try to buy the product at time t, and D(t) denotes the cumulative customers who have gone to the store and tried to purchase the product up to time t. At each moment, a customer who was previously not ready to adopt and has been informed from mass media or positive word-of-mouth may place an order. If the new product is available, the customer receives the product immediately. If not, he/she will abandon the adoption decision by canceling the order. Consequently, the customer population can be divided into three groups. The first group includes the potential adopters who have not been affected by mass media or word-of-mouth, we denote N − W(t),
where \( N \) is the fixed market potential and \( W(t) \), which is named awareness process, denotes the total customers who have been aware of the new product. The second group includes individuals who have placed an order and already received the new product, we denote \( S(t) \). Then \( S(t) \) also denotes the firm’s actual sales quantity up to time \( t \). The third group includes the customers who are the so-called lost demand, \( W(t) - S(t) \), which includes adopters (denoted by \( D(t) - S(t) \)) who refuse to wait and hence cancel their orders and potential customers (denoted by \( W(t) - D(t) \)) who give up their order decisions because of the impact of negative word-of-mouth from the previous lost demand. If the product capacity is unlimited, then there will be no lost demand, and the awareness process \( W(t) \), the demand process \( D(t) \) and the sales process \( S(t) \) coincide with \( D(t) \) in Bass model. In the presence of capacity constraint, potential adopters who are not able to obtain the product immediately abandon the adoption and join in the lost demand list \( W(t) - S(t) \), and potential adopters who are influenced and get the information from the lost demand also join in the lost demand list \( W(t) - S(t) \). Therefore, under capacity constraint, the awareness process and the demand process can be described as:

\[
\frac{dW(t)}{dt} = [N - W(t)] \left[ \alpha + \beta S(t) + \gamma (W(t) - S(t)) \right], \tag{3}
\]

\[
\frac{dD(t)}{dt} = [N - W(t)] \left[ \alpha + \beta S(t) \right], \tag{4}
\]

where \( \gamma \) denotes the influence factor of negative word-of-mouth on the population. The awareness process of the new product is influenced by three factors: the independent innovation dynamics \( \alpha \), the interaction dynamics between adopters \( S(t) \) and potential customers \( N - W(t) \) who haven’t been aware of the new product, and the interaction dynamics between lost demand \( W(t) - S(t) \) and potential customers \( N - W(t) \). In equation (3), the interaction dynamics between remaining potential adopters and lost demand will immediately join in the lost demand list.

The demand process itself, which defines the arrival of customer’s order, follows Kumar and Swaminathan’s model \cite{25}, that is, the demand process is only dependent on the positive word-of-mouth and mass media. In consistent with all related literature, we also assume that \( \beta N > \alpha \) holds in this paper. In order to analyze the effect of negative word-of-mouth on the optimal policies, we first present the following result which has been presented in Lemma 4.1 in Yan and Liu \cite{43}. The proofs of all results in this paper are given in the online appendix.

**Lemma 3.1.** If \( q \geq q_0 \), then \( \frac{\alpha [e^{\gamma t} - 1]}{\beta + \frac{\alpha}{\beta} e^{\gamma t}} \leq qt \) holds for all \( t \geq 0 \), where \( q_0 \) is the unique solution to

\[
\ln (\alpha + \beta N)^2 - 2\beta q_0 + (\alpha + \beta N)\sqrt{(\alpha + \beta N)^2 - 4\beta q_0} = \frac{2q_0 \frac{\alpha}{N}}{\alpha + \beta N + \frac{\alpha}{q_0}} - \frac{2\beta N + \frac{\alpha}{q_0}}{\alpha + \beta N + \sqrt{(\alpha + \beta N)^2 - 4\beta q_0}}. \tag{5}
\]

Lemma 3.1 states that if the fixed production capacity is larger than \( q_0 \), then cumulative demand process \( D(t) \) and cumulative sales process \( S(t) \) may be identical if the firm chooses appropriate production and sales policies, which just becomes Bass model. From Lemma 4.1 in Yan and Liu \cite{43}, we have known that if the fixed production capacity is larger than \( (\alpha + \beta N)^2/(4\beta) \), then cumulative demand process \( D(t) \) and cumulative sales process \( S(t) \) are not only identical if the firm chooses
appropriate production and sales policies, but the production and sales policies, which make the demand process, sales process and production process identical, are also optimal. Thus, in what follows we only consider the case of \( q \leq \frac{(\alpha + \beta N)^2}{4\beta} \).

In the following Figure 1, we present Bass demand process \( d(t) \) and the value of \( q_0 \) for a numerical example. In this numerical example, we fix \( N = 3000 \), \( \alpha = 0.03 \) and \( \beta = 0.4/3000 \). From the definition of \( q_0 \), defined in (5), and Figure 1, we obtain that \( q_0 \) in fact satisfies \( \int_0^{t_0(q_0)}[q_0 - d(t)] \, dt = \int_{t_0(q_0)}^{t(q_0)} d(t) - q_0 \, dt \), where \( t(q) \) and \( t_0(q) \) are defined in the proof of Lemma 3.1, \( \int_0^{t_0(q_0)}[q_0 - d(t)] \, dt \) denotes the area made by \( q \) \( t_0(q_0) \), \( q = q_0 \) and \( d(t) \), and \( \int_{t_0(q_0)}^{t(q_0)}[d(t) - q_0] \, dt \) denotes the area made by \( t = t_0(q_0) \), \( t = t(q_0) \), \( q = q_0 \) and \( d(t) \). Hence, from Figure 1, we can derive the following result: \( \int_0^{t_0(q)}[q - d(t)] \, dt > \int_{t_0(q)}^{t(q)}[d(t) - q] \, dt \) for \( q > q_0 \) and \( \int_0^{t_0(q)}[q - d(t)] \, dt < \int_{t_0(q)}^{t(q)}[d(t) - q] \, dt \) for \( q < q_0 \), where \( d(t) \) is the Bass demand rate at time \( t \). That is, for \( q > q_0 \), the demand can be completely satisfied by adopting appropriate production policy, but for \( q < q_0 \), there must exist lost demand.

![Figure 1. Bass demand process](image)

For any fixed value of production capacity \( q \in \left[ 0, \frac{(\alpha + \beta N)^2}{4\beta} \right] \), the firm’s purpose is to choose sales policy \( s(t) \) and production policy \( x(t) \) to maximize life-cycle discounted profit:

\[
J(q) = \max_{s(t) \geq 0, x(t) \geq 0} \int_0^\infty e^{-\theta t} \left[ ps(t) - cx(t) - h(X(t) - S(t)) \right] \, dt
\]

s.t.

\[
\frac{dW(t)}{dt} = \left[ N - W(t) \right] \left[ \alpha + \beta S(t) + \gamma(W(t) - S(t)) \right],
\]

\[
\frac{dD(t)}{dt} = \left[ N - W(t) \right] \left[ \alpha + \beta S(t) \right],
\]

\[
\frac{dX(t)}{dt} = x(t), \quad x(t) \leq q,
\]
\[ \frac{dS(t)}{dt} = s(t), \quad s(t) \leq \left[N - W(t)\right]\left[ \alpha + \beta S(t) \right], \quad (10) \]

\[ S(t) \leq X(t), \quad (11) \]

\[ W(0) = D(0) = X(0) = S(0) = 0, \quad (12) \]

where \( c \) is the fixed production cost per unit, \( p > c \) is the fixed unit selling price of the new product, \( h \) is the inventory holding cost per unit per time and \( \theta \geq 0 \) is the discount factor. The first two equations follow equations (3) and (4). Constraint (9) states that production rate is never more than production capacity. Constraint (10) states that the sales rate cannot be more than demand rate. Constraint (11) implies that \( s(t) \leq x(t) \) whenever \( S(t) = X(t) \), that is, cumulative sales quantity is never more than cumulative production quantity. Constraint (12) is assumed to be satisfied in this paper.

4. Optimal policies for the case of \( \beta \leq \gamma \). In this section, we consider the case of \( \beta \leq \gamma \), which will be called strong negative word-of-mouth in the following analysis. In this situation, we show that the optimal sales policy is myopic under some mild conditions. This result can be interpreted by the fact that because negative word-of-mouth from lost demand is stronger than positive word-of-mouth from cumulative sales, the firm should avoid shortage as far as possible. In order to analyze the structure of the optimal policies, we define the following set:

\[ E[0,t] = \left\{ f : [0, t) \rightarrow [0, \infty) \mid f(\cdot) \text{ is continuous, nonnegative, nondecreasing on } [0,t) \right\} \text{ for } 0 < t \leq \infty. \]

That is, each element in set \( E[0,t] \) is a continuous map from \([0,t)\) to \([0,\infty)\). For any feasible sales path \( s(u) \) on \([0,t)\), the corresponding cumulative sales path, \( S(u) \), can be described as \( S(u) = \int_0^u s(x) \, dx \). Then \( S(u) \) is continuous, nonnegative and nondecreasing on \([0,t)\), i.e., \( S(u) \in E[0,t) \). For notational convenience, we denote \( S_{[0,t)}(u) \) as the corresponding cumulative sales path on interval \([0,t)\) and \( S_{[0,t)}(u) = S(u) = \int_0^u s(x) \, dx \) denotes the cumulative sales up to time \( u \). In order to obtain the structure of the optimal policies, we need recur to the following definitions of orders \( \succeq \) and \( \succ \).

**Definition 4.1.** For any \( S^1_{[0,t)}, S^2_{[0,t)} \in E[0,t) \), orders \( \succeq \) and \( \succ \) are defined as follows:

1. \( S^1_{[0,t)} \succeq S^2_{[0,t)} \) if \( S_1(u) \geq S_2(u) \) for any \( u \in [0,t) \), that is, \( S^1_{[0,t)}(u) \succeq S^2_{[0,t)}(u) \) for any \( u \in [0,t) \).
2. \( S^1_{[0,t)} \succ S^2_{[0,t)} \) if \( S^1_{[0,t)}(u) \geq S^2_{[0,t)}(u) \) and there exists at least one \( u \in [0,t) \) such that \( S_1(u) > S_2(u) \).

From Definition 4.1, we can derive that not all the elements in set \( E[0,t) \) can be compared by order \( \succeq \) or \( \succ \), that is, \( \succeq \) and \( \succ \) are not complete orders on \( E[0,t) \). However, in the following analysis, we will see that orders \( \succeq \) and \( \succ \) are enough to obtain the structures of the optimal policies. For convenience in exposition, we denote \( V(t) = N - W(t) \) in the rest of this paper. Then \( V(t) \) denotes the potential customers who haven’t been aware of the new product. Before formulating the main results, we provide the following result which states that \( V(t) \) increases in the cumulative sales path \( S_{[0,t)} \) under order \( \succeq \).
Lemma 4.2. If $\beta \leq \gamma$, then $V(t)$ increases in $S_{[0,t]}$ under order $\succeq$, where $S_{[0,t]} \in E[0,t]$ denotes a cumulative sales path.

Lemma 4.2 states that the number of potential customers, who haven’t been aware of the product, increases in the cumulative sales path. That is, the more the firm sells in the previous horizon, the more potential customers who haven’t been aware of the product remain. This result is due to the condition $\gamma \geq \beta$. Based on Lemma 4.2, we obtain intuitively that the firm should satisfy the demand as soon as possible, that is, it may be optimal for the firm to adopt myopic policy to satisfy demand. In the rest of this section, we would like to prove the optimality of myopic policy in three cases: $q \leq \alpha N$, $\alpha N < q \leq q_0$ and $q_0 < q \leq (\alpha + \beta N)^2/(4\beta)$.

Proposition 1. If $\gamma \geq \beta$ and $q \leq \alpha N$, the optimal production and sales policies can be described as:

$$x_1^*(t) = s_1^*(t) = \begin{cases} q, & t \leq \delta_1 \\ d_1^*(t), & t \geq \delta_1, \end{cases}$$

where $d_1^*(t)$ is the corresponding demand process generated by policies $x_1^*(t)$ and $s_1^*(t)$, and $\delta_1$ is the unique solution to the following equation:

$$\left[ \frac{1}{N} e^{(\alpha N)\delta_1 + \frac{1}{2} (\beta - \gamma) \delta_1^2} - \gamma \int_0^{\delta_1} e^{(\alpha N)(\delta_1 - s) + \frac{1}{2} (\beta - \gamma) q (\delta_1^2 - s^2)} ds \right]^{-1} (\alpha + \beta q \delta_1) = q.$$  \hspace{1cm} (14)

Moreover, the corresponding awareness and demand processes can be described as

$$W_1^*(t) = \begin{cases} N - \left[ \frac{1}{N} e^{\int_0^t [\alpha + \gamma N + (\beta - \gamma) u] du} - \gamma \int_0^t e^{\int_s^t [\alpha + \gamma N + (\beta - \gamma) u] du} ds \right]^{-1}, & \text{if } t \leq \delta_1, \\ N - \left[ \frac{1}{N} e^{\int_0^t [\alpha + \gamma N + (\beta - \gamma) S_1^*(u)] du} - \gamma \int_0^t e^{\int_s^t [\alpha + \gamma N + (\beta - \gamma) S_1^*(u)] du} ds \right]^{-1}, & \text{if } t > \delta_1, \end{cases}$$

$$d_1^*(t) = \begin{cases} \frac{1}{N} e^{\int_0^t [\alpha + \gamma N + (\beta - \gamma) u] du} - \gamma \int_0^t e^{\int_s^t [\alpha + \gamma N + (\beta - \gamma) u] du} ds \right]^{-1} (\alpha + \beta qt), & \text{if } t \leq \delta_1, \\ \frac{1}{N} e^{\int_0^t [\alpha + \gamma N + (\beta - \gamma) S_1^*(u)] du} - \gamma \int_0^t e^{\int_s^t [\alpha + \gamma N + (\beta - \gamma) S_1^*(u)] du} ds \right]^{-1} (\alpha + \beta S_1^*(t)), & \text{if } t > \delta_1, \end{cases}$$

where $S_1^*(t) = \int_0^t s_1^*(u) du$ is the cumulative sales process generated by $s_1^*(t)$.

Proposition 1 states that under conditions $\gamma \geq \beta$ and $\alpha N \geq q$, the optimal policy is myopic. That is, the firm first produces and sells the product along the capacity and then along the demand rate whenever demand rate is less than capacity. The critical point at which the corresponding demand rate is equal to the capacity is not only unique but can also be obtained by solving equation (14). Moreover, Proposition 1 presents the structures of the corresponding awareness and demand processes under policies (13). From (15) and (16), the corresponding awareness and demand processes can be completely determined by the sales policy (13). For $t \leq \delta_1$, the corresponding awareness and demand processes have explicit forms. However,
for \( t > \delta_1 \), the value of optimal sales rate can be obtained by solving the following complex equation:

\[
\left[ \frac{1}{N} e^{\int_0^t [\alpha + \gamma N + (\beta - \gamma)S^*_1(u)] \, du} - \gamma \int_0^t e^{\int_0^s [\alpha + \gamma N + (\beta - \gamma)S^*_1(u)] \, du} \, ds \right]^{-1} (\alpha + \beta S^*_1(t)) = \frac{dS^*_1(t)}{dt}.
\]

Since it is too complicated to solve the optimal sales policy for \( t > \delta_1 \), we show the Bass demand process, the production and sales policies defined in (13), and the corresponding demand and awareness processes by a numerical example in Figure 2. In this example, we fix \( N = 3000, \alpha = 0.03, \beta = 0.4/3000, \gamma = 0.5/3000, q = 80, p = 1.2, c = 1, h = 0.001 \) and \( \theta = 0.01 \). By calculating, we obtain \( \delta_1 = 8.3 \), that is, for \( t > 8.3 \), the demand can be completely satisfied. From Figure 2, we see that the Bass demand rate is first less than the corresponding awareness rate and then larger than it. Moreover, the peak of the Bass demand rate is lower than the peak of the corresponding awareness rate. This phenomenon is due to \( \beta < \gamma \). Since the negative word-of-mouth from the lost demand is stronger than the positive word-of-mouth from the cumulative sales, the instantaneous awareness increases faster than the instantaneous Bass demand in the initial period and then the total lost demand over the life cycle of the product will be more when the capacity is smaller. Therefore, maybe it is better for the firm to increase the production capacity, which will be discussed in future research. The Bass demand rate is always larger than the corresponding demand rate over the life cycle of the product, and the peak of the Bass demand rate is higher than the peak of the corresponding demand rate. However, when time \( t \) is large enough (\( t > 20 \) in Figure 2), the differences between the corresponding awareness rate, the Bass demand rate and the corresponding demand rate approximate zero. This phenomenon maybe imply that the product will go out of production or be updated once the demand rate is very small.

**Figure 2.** Processes of optimal sales/production, Bass demand, corresponding demand and awareness under \( q = 80 \) and \( \gamma = 0.5/3000 \).
Proposition 2. If $\gamma \geq \beta$, $\alpha N < q \leq q_0$ and $(p - 2c + \frac{h}{\theta})e^{-\theta\delta_3} + c - \frac{h}{\theta} \geq 0$, then the optimal production and sales policies can be described as:

$$x^*_2(t) = \begin{cases} q, & t \leq \delta_3 \\ d^*_2(t), & t \geq \delta_3 \end{cases}$$  \hspace{1cm} (17)$$

and

$$s^*_2(t) = \begin{cases} \frac{\alpha(\alpha + \beta N)(\frac{N}{\alpha} + \beta)}{[\beta + \frac{\alpha}{N} e^{(\alpha + \beta N)t}]} e^{(\alpha + \beta N)t}, & t \leq \delta_2 \\ q, & \delta_2 \leq t \leq \delta_3 \\ d^*_2(t), & t \geq \delta_3, \end{cases}$$  \hspace{1cm} (18)$$

where $d^*_2(t)$ is the corresponding demand process generated by policies $x^*_2(t)$ and $s^*_2(t)$. $\delta_2$ is the smaller positive solution to the equation:

$$\frac{\alpha [e^{(\alpha + \beta N)}\delta_2 - 1]}{\beta + \frac{\alpha}{N} e^{(\alpha + \beta N)\delta_2}} = q\delta_2,$$  \hspace{1cm} (19)$$

and $\delta_3$ is the unique solution to the equation:

$$\left[ e^{\int_{\delta_2}^{\delta_3} [\alpha + \gamma N + (\beta - \gamma)qu] du} \cdot \left\{ \frac{1}{N - q\delta_2} - \gamma \int_{\delta_2}^{\delta_3} e^{\int_{\delta_2}^{s} [\alpha + \gamma N + (\beta - \gamma)qu] du} ds \right\} \right]^{-1} \cdot (\alpha + \beta q\delta_3) = q.$$  \hspace{1cm} (20)$$

Moreover, the corresponding awareness and demand processes can be described as

$$W^*_2(t) = \begin{cases} \frac{\alpha [e^{(\alpha + \beta N)\delta_2 - 1}]}{\beta + \frac{\alpha}{N} e^{(\alpha + \beta N)\delta_2}}, & \text{if } t \leq \delta_2, \\ N - \frac{1}{N} e^{\int_{0}^{t} [\alpha + \gamma N + (\beta - \gamma)S^*_2(u)] du} - \gamma \int_{0}^{t} e^{\int_{0}^{s} [\alpha + \gamma N + (\beta - \gamma)S^*_2(u)] du} ds \right\}^{-1}, & \text{if } t > \delta_2, \end{cases}$$

$$d^*_2(t) = \begin{cases} \frac{\alpha(\alpha + \beta N)(\frac{N}{\alpha} + \beta)}{[\beta + \frac{\alpha}{N} e^{(\alpha + \beta N)t}]} e^{(\alpha + \beta N)t}, & \text{if } t \leq \delta_2, \\ \frac{1}{N} e^{\int_{t}^{\delta_2} [\alpha + \gamma N + (\beta - \gamma)S^*_2(u)] du} - \gamma \int_{t}^{\delta_2} e^{\int_{t}^{s} [\alpha + \gamma N + (\beta - \gamma)S^*_2(u)] du} ds \right\}^{-1}, & \text{if } t > \delta_2, \end{cases}$$

where

$$S^*_2(t) = \int_{0}^{t} s^*_2(u) du = \begin{cases} \frac{\alpha [e^{(\alpha + \beta N)\delta_2 - 1}]}{\beta + \frac{\alpha}{N} e^{(\alpha + \beta N)\delta_2}}, & \text{if } t \leq \delta_2, \\ q\delta_3 + \delta_3 s^*_2(u) du, & \text{if } t > \delta_3 \end{cases}$$

is the cumulative sales process generated by $s^*_2(t)$.

Proposition 2 formulates the forms of the optimal production and sales policies under conditions $\gamma \geq \beta$, $\alpha N < q \leq q_0$ and $(p - 2c + \frac{h}{\theta})e^{-\theta\delta_3} + c - \frac{h}{\theta} \geq 0$. The optimal production policy can be described as follows: for $t \in [0, \delta_3]$, the optimal production rate equals the capacity, and for $t \in (\delta_3, \infty)$, the optimal production rate, which is less than the capacity, equals the demand rate. The optimal sales policy can be described as follows: for $t \in [0, \delta_2]$, the optimal sales rate equals the demand rate; for $t \in (\delta_2, \delta_3]$, the optimal sales rate, which equals the capacity, is less than the demand rate; and for $t \in (\delta_3, \infty)$, the optimal sales rate equals the demand rate once more. The optimality of policies (17) and (18) is due to the
condition \((p - 2c + \frac{h}{\theta})e^{-\theta \delta_3} + c - \frac{h}{\theta} \geq 0\). Next, we present two sufficient conditions under which the condition, \((p - 2c + \frac{h}{\theta})e^{-\theta \delta_3} + c - \frac{h}{\theta} \geq 0\), always holds. Note that

\[
(p - 2c + \frac{h}{\theta})e^{-\theta \delta_3} + c - \frac{h}{\theta} = (p - c)e^{-\theta \delta_3} + (c - \frac{h}{\theta})(1 - e^{-\theta \delta_3}),
\]

then the first sufficient condition is \(h \leq \theta c\). Since the dominant cost of inventory is usually the opportunity cost of the capital that has been tied up in the procurement or production of the commodity, i.e. \(h = \theta c\) (Heyman and Sobel [17], page 11), the condition, \((p - 2c + \frac{h}{\theta})e^{-\theta \delta_3} + c - \frac{h}{\theta} \geq 0\), is reasonable in a sense. The second sufficient condition is that it is profitable for the firm to produce a unit product at time zero and sell the unit product at time \(\delta_3\). That is, \(pe^{-\theta \delta_3} \geq c + \frac{h}{\theta}(1 - e^{-\theta \delta_3})\), where \(pe^{-\theta \delta_3}\) denotes the revenue of selling unit product at time \(\delta_3\), and \(\frac{h}{\theta}(1 - e^{-\theta \delta_3})\) is the holding cost of keeping unit product from time zero to \(\delta_3\). Since \(pe^{-\theta \delta_3} \geq c + \frac{h}{\theta}(1 - e^{-\theta \delta_3})\), we derive the following expression:

\[
(p - 2c + \frac{h}{\theta})e^{-\theta \delta_3} + c - \frac{h}{\theta} = pe^{-\theta \delta_3} - \left(2c - \frac{h}{\theta}\right)e^{-\theta \delta_3} + c - \frac{h}{\theta}
\geq c + \frac{h}{\theta}(1 - e^{-\theta \delta_3}) - \left(2c - \frac{h}{\theta}\right)e^{-\theta \delta_3} + c - \frac{h}{\theta}
= 2c(1 - e^{-\theta \delta_3}) > 0,
\]

that is, the condition, \((p - 2c + \frac{h}{\theta})e^{-\theta \delta_3} + c - \frac{h}{\theta} \geq 0\), holds.

Since the optimal production and sales policies, the corresponding awareness and demand processes don’t have explicit forms for \(t > \delta_3\), we present these forms by a numerical example in Figure 3. In this example, we fix \(N = 3000, \alpha = 0.03, \beta = 0.4/3000, \gamma = 0.5/3000, q = 160, p = 1.2, c = 1, h = 0.001\) and \(\theta = 0.01\). By calculating, we obtain \(\delta_2 = 3.19\) and \(\delta_3 = 8.44\). Note that the capacity in Figure 3 is larger than that in Figure 2, the difference between the corresponding demand and the corresponding awareness in Figure 3 is smaller than that in Figure 2. The corresponding demand rate in Figure 3 is much larger than that in Figure 2, but the time after which the demand is fully satisfied in Figure 3, \(\gamma_3\), is a little larger than \(\delta_3\) in Figure 2. Comparing the awareness process in Figure 3 with that in Figure 2, we obtain that increasing production capacity can make the shortage information spread slowly and the demand rate arrive quickly. In the next result, we present the optimal production and sales policies for the case of \(q_0 < q \leq (\alpha + \beta N)^2/(4\beta)\). Under some mild conditions, the optimal sales policy, the corresponding awareness and demand processes are all identical with Bass demand.

**Proposition 3.** If \(\gamma \geq \beta, q_0 < q \leq (\alpha + \beta N)^2/(4\beta)\) and \((p - c + \frac{h}{\theta})e^{-\theta t(q)} - \frac{h}{\theta} \geq 0\), then the optimal production and sales policies can be described as:

\[
x_3^*(t) = \begin{cases} 
\frac{\alpha(\alpha + \beta N)(\frac{\alpha}{N} + \beta) e^{(\alpha + \beta N)t}}{[\beta + \frac{\gamma}{N} e^{(\alpha + \beta N)t}]^2}, & 0 \leq t \leq \delta_4, \\
\frac{\alpha(\alpha + \beta N)(\frac{\alpha}{N} + \beta) e^{(\alpha + \beta N)t}}{[\beta + \frac{\gamma}{N} e^{(\alpha + \beta N)t}]^2}, & \delta_4 < t \leq t(q), \\
q, & t > t(q),
\end{cases}
\]

and

\[
s_3^*(t) = \frac{\alpha(\alpha + \beta N) \left(\frac{\alpha}{N} + \beta\right) e^{(\alpha + \beta N)t}}{[\beta + \frac{\gamma}{N} e^{(\alpha + \beta N)t}]^2},
\]
where
\[
t(q) = \frac{1}{\alpha + \beta N} \ln \frac{(\alpha + \beta N)^2 - 2\beta q + (\alpha + \beta N)\sqrt{(\alpha + \beta N)^2 - 4\beta q}}{2q N^2}
\]
and \(\delta_4\) is the unique solution to the equation:
\[
\frac{e^{(\alpha+\beta N)t(q)} - 1}{\beta + \frac{\alpha}{N} e^{(\alpha+\beta N)t(q)}} - \frac{e^{(\alpha+\beta N)\delta_4} - 1}{\beta + \frac{\alpha}{N} e^{(\alpha+\beta N)\delta_4}} = q \alpha \left[ t(q) - \delta_4 \right].
\]

Proposition 3 implies that if \(\gamma \geq \beta\), \(g_0 < q \leq (\alpha + \beta N)^2/(4\beta)\) and \((p - c + \frac{h}{\theta})e^{-\theta t(q)} - \frac{h}{\theta} \geq 0\), then the optimal sales policy, the corresponding awareness and demand processes are identical with Bass demand. Here, the critical assumption is \((p - c + \frac{h}{\theta})e^{-\theta t(q)} - \frac{h}{\theta} \geq 0\), which can be obtained by the sufficient condition: it is profitable for the firm to produce unit product at time zero and sell the unit product at time \(t(q)\). The optimal production policy can be described as follows: for \(t \in [0, \delta_4]\), the optimal production rate equals the Bass demand rate; for \(t \in (\delta_4, t(q)]\), the optimal production rate equals the capacity; and for \(t \in (t(q), \infty)\), the optimal production rate equals the Bass demand rate once more. Therefore, the optimal production and sales policies have explicit forms once \(\delta_4\) and \(t(q)\) are solved.

In Figure 4, we present these forms by a numerical example. In this example, we fix \(N = 3000\), \(\alpha = 0.03\), \(\beta = 0.4/3000\), \(\gamma = 0.5/3000\), \(q = 280\), \(p = 1.2\), \(c = 1\), \(h = 0.001\) and \(\theta = 0.01\). By calculating, we obtain \(\delta_4 = 1.5\) and \(t(q) = 8.21\). That is, the time interval on which the capacity is totally used is \((1.5, 8.21)\).

In this section, we have shown that myopic policy is not only optimal under some mild conditions but also very easy to adopt. Whether the myopic policy is still optimal under \(\beta > \gamma\) is another interesting question, which will be studied in the following section. Kumar and Swaminathan [25] have obtained that the build-up policy is a robust approximation to the optimal policy under condition \(\gamma = 0\). However, we can show that the build-up policy may be worse than the myopic policy under condition \(\gamma > 0\), in other words, the build-up policy is no longer a robust approximation to the optimal policy under condition \(\gamma > 0\).
5. Optimal policies for the case of $\beta > \gamma$. In this section, we will concentrate on analyzing the optimal production and sales policies for the case of $\beta > \gamma$. In what follows, we obtain that the optimal production and sales policies under $\beta > \gamma$ may be different from these under $\beta \leq \gamma$, that is, the myopic policy may not be optimal. First of all, we show that build-up policy is still optimal for the special case of negligible holding cost and discount rate, which can be regarded as the generalization of Theorem 2 in Kumar and Swaminathan [25]. However, compared with Kumar and Swaminathan’s argument, we present a much easier way to prove the result. Because it is very difficult to obtain the structures of optimal policies under $\beta > \gamma$, we next compare myopic policy and build-up policy by several numerical examples and obtain that the optimal policies are dependent on the values of parameters, such as production capacity $q$, influence of negative word-of-mouth $\gamma$, holding cost $h$ and selling price $p$. In order to analyze the optimal policies for the special case, we need the following result which is similar to Lemma 4.2.

Lemma 5.1. If $\beta > \gamma$, then $V(t)$ decreases in $S_{[0,t]} \in E[0,t]$ under order $\succeq$.

Lemma 5.1 implies that the more products the firm sells in the previous horizon, the less potential customers who haven’t been aware of the product remain, which is just the inverse result of Lemma 4.2. Therefore, whether the myopic policy is optimal or not is another problem which will be discussed in the following analysis. When $\gamma = 0$, Kumar and Swaminathan [25] have shown that the structure of the optimal policy is dependent on the values of parameters, and the build-up policy is a robust approximation to the optimal policy. However, when $\gamma > 0$, the build-up policy may not be a robust approximation to the optimal policy. Similar to Kumar and Swaminathan [25], we first present a result which formulates that the build-up policy is optimal under conditions $\theta = h = 0$ and $\beta > \gamma$. This result can be regarded as the generalization of Kumar and Swaminathan’s result (Theorem 2 in [25]). Since the argument used in Kumar and Swaminathan [25] is too complicated to analyze the case of $\gamma > 0$, we present a much easier way to obtain the result. Moreover, the result in Kumar and Swaminathan [25] can be very easily obtained by our argument, which is presented in the following Corollary 1.
Now, we restrict attention to the optimality of build-up policy under condition 
\( h = \theta = 0 \). Since there are no inventory holding costs and impact of discount rate,
one optimal production plan is to produce the entire market potential as soon as
possible, that is,
\[
x^*(t) = \begin{cases} 
q, & t \leq \delta^* \\
0, & t \geq \delta^* 
\end{cases}
\]
where \( \delta^* \) satisfies \( q\delta^* = S^*(\infty) \). Here, \( S^*(\infty) \) denotes the total cumulative sales
under optimal sales policy. Therefore, when \( h = \theta = 0 \) holds, the problem of
maximizing profits over the product life cycle is equivalent to minimizing lost sales
over the life cycle of the product. That is, the firm’s objective is to maximize
total cumulative sales by choosing an appropriate sales policy over the life cycle of
the product. The problem of determining the optimal sales plan now becomes the
following optimization problem:
\[
\max_{s(t) \geq 0, 0 \leq t < \infty} S(\infty)
\]
s.t.
\[
\frac{dW(t)}{dt} = [N - W(t)] [\alpha + (\beta - \gamma)S(t) + \gamma W(t)], \\
\frac{dS(t)}{dt} = s(t), \quad 0 \leq s(t) \leq [N - W(t)] [\alpha + \beta S(t)], \\
W(0) = S(0) = 0, \quad S(t) \leq qt.
\]
The above constraints can be interpreted just like constraints (7)-(12). This op-
timization problem is a generalization of Kumar and Swaminathan’s problem in
which \( \gamma = 0 \). The following result states that the build-up policy is optimal when
\( \beta > \gamma \) and \( h = \theta = 0 \).

**Theorem 5.2.** If \( \beta > \gamma \) and \( \theta = h = 0 \), then the build-up policy is optimal.

Theorem 5.2 states that the build-up policy is still optimal under conditions
\( \beta > \gamma \) and \( h = \theta = 0 \), which can be regarded as the generation of Kumar and
Swaminathan’s result. By the argument of proving Theorem 5.2, it is very easy to
obtain the following result which is just the Kumar and Swaminathan’s result.

**Corollary 1.** If \( h = \theta = \gamma = 0 \) and \( \beta > \gamma \), then the build-up policy is optimal.

The optimality of build-up policy in Theorem 5.2 and Corollary 1 is due to the
fact that the discount rate and holding cost have no effect on the profit. In reality,
however, discount factor and holding cost can not be neglected in many situations.
Thus, we have the necessity to analyze more generalized case with positive discount
rate and positive holding cost. Because it is too difficult to analyze the structure
of the optimal policy, we mainly compare myopic policy and build-up policy by
numerical examples in the rest of this section. Whether myopic policy is better than
build-up policy is dependent on the values of parameters, such as the production
capacity \( q \), the selling price \( p \), the influence factor of negative word-of-mouth \( \gamma \), the
holding cost \( h \) and so forth. In what follows, we concentrate on analyzing the effects
of parameters \( \gamma, h, q \) and \( p \) on the choice of myopic policy and build-up policy.

In these numerical examples, we fix \( N = 3000, \alpha = 0.03, \beta = 0.4/3000, c = 1 \) and
\( \theta = 0.01 \). We consider five options for the selling price \( p = 1.1, 1.2, 1.3, 1.4 \) and 1.5.
We consider high and low values for holding costs \( h = 0.01 \) and 0.001, respectively,
representing values ranging from 1% to 0.1% of unit production cost. Similarly, we
choose five values for the influence factor of negative word-of-mouth and eight values for the production capacity, i.e., $\gamma = 0, 0.05/3000, 0.1/3000, 0.2/3000$ and $0.3/3000$, $q = 60, 80, 100, 120, 140, 160, 180$ and 200. This results in 800 cases. Tables 1-9 show the dominant heuristic for a representative set of parameters. Of course, neither heuristic need be optimal.

Table 1 and Table 6 imply that when $p = 1.1$, the myopic policy is always better than the build-up policy no matter what the values of $\gamma$, $h$ and $q$ are. Therefore, the build-up policy is no longer a robust approximation to the optimal policy, which differs from Kumar and Swaminathan [25]. This result can be interpreted by the fact that when the profit of selling unit product is small (less than 10% of unit production cost), the holding cost and the discount rate may play a significantly greater role. Thus, the retailer should satisfy the demand as soon as possible although the number of customers satisfied by build-up policy is more than that of myopic policy. From Table 2, we see that when $p = 1.2$, $h = 0.001$, $q \geq 140$ and $\gamma \leq 0.1/3000$, the build-up policy is dominant, otherwise, the myopic policy is dominant. Therefore, when $p = 1.2$, $h = 0.001$ and the negative word-of-mouth is strong ($\gamma \geq 0.2/3000$), the myopic policy is always dominant no matter what the capacity is. However, when the negative word-of-mouth is weaker ($\gamma \leq 0.1/3000$), whether the myopic policy is also dominant depends on the capacity. When the capacity is larger ($q \geq 140$), the build-up policy is dominant. This result can be interpreted by the fact that when the negative word-of-mouth is strong, a lot of demand will be lost if the retailer does not sell the product in the initial period, and when the capacity is small, there will be a long time to store enough product if the build-up policy is adopted. Thus, when the capacity is small or the negative word-of-mouth is strong, the build-up policy may be dominant. From Table 6, we see that when $p = 1.2$ and $h = 0.01$, the myopic policy is always better than the build-up policy no matter what the values of $\gamma$, $h$ and $q$ are. Comparing Table 2 with Table 6, we conclude that when the holding cost $h$ increases, the possibility that the myopic policy is dominant also increases. In summary, from tables 1-9, we can obtain the following results: (1) When the production capacity, holding cost and negative word-of-mouth are fixed, the possibility that the build-up policy is dominant increases in the selling price. This result is due to the fact that the higher the selling price is, the less the relative impact of holding cost and discount rate is, therefore, the build-up policy will be dominant. (2) When the selling price, holding cost and negative word-of-mouth are fixed, the possibility that the build-up policy is dominant increases in the capacity. This result is due to the fact that when the capacity is large, it will take a short period of time to store enough product and more demand will be satisfied if the build-up policy is adopted. (3) When the selling price, production capacity and negative word-of-mouth are fixed, the possibility that the build-up policy is dominant decreases in the holding cost. This result is due to the fact that when the holding cost is large, it will take a large amount of money to store enough product if the build-up policy is adopted. (4) When the selling price, holding cost and production capacity are fixed, the possibility that the build-up policy is dominant decreases in the negative word-of-mouth. This result is due to the fact that when the negative word-of-mouth is strong, much more potential customers will be lost if the build-up policy is adopted.

In the rest of this section, we concentrate on discussing the effect of negative word-of-mouth and then compare our results with those in Kumar and Swaminathan [25] and Yan and Liu [43]. Kumar and Swaminathan [25] have shown that
the build-up policy is a robust approximation to the optimal policy when $\gamma = 0$ and $q = 100$. However, we show that the build-up policy is no longer a robust approximation to the optimal policy when the negative word-of-mouth is strong (such as $\gamma \geq 0.2/3000$) and the production capacity is small (such as $q \leq 80$). Thus, whether the build-up policy is dominant depends on the values of parameters. When the production capacity is large, holding cost is small, selling price is high and negative word-of-mouth is weak, the build-up policy may be dominant. Otherwise, the myopic policy may be dominant. That is, in real life, it is necessary for a firm to consider the negative word-of-mouth when the product is highly substitutable. Yan and Liu [43] just considered a special case in which the influence factors of positive and negative word-of-mouth have the same value. In this study, we not only present a new argument (incorporating two orders $\succeq$ and $\succ$) to obtain the structure of the optimal policy for the case of strong negative word-of-mouth but also analyze the effects of negative word-of-mouth in detail. By numerical examples, we show that when $\gamma \geq 0.2/3000$ and $q \leq 80$, the myopic policy is always dominant, that is, the myopic policy may be a robust approximation to the optimal policy.

Table 1 : Dominant heuristic when $p = 1.1$ and $h = 0.001$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|----------------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 80  | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 100 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 120 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 140 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 160 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 180 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 200 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |

Table 2 : Dominant heuristic when $p = 1.2$ and $h = 0.001$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|----------------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 80  | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 100 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 120 | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 140 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 160 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 180 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 200 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |

Table 3 : Dominant heuristic when $p = 1.3$ and $h = 0.001$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|----------------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 80  | Myopic        | Myopic               | Myopic               | Myopic               | Myopic               |
| 100 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 120 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 140 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 160 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 180 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
| 200 | Build-up      | Build-up             | Build-up             | Myopic               | Myopic               |
Table 4: Dominant heuristic when $p = 1.4$ and $h = 0.001$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|-----------------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 80  | Build-up      | Build-up              | Build-up             | Build-up             | Myopic               |
| 100 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 120 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 140 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 160 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 180 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 200 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |

Table 5: Dominant heuristic when $p = 1.5$ and $h = 0.001$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|-----------------------|----------------------|----------------------|----------------------|
| 60  | Build-up      | Build-up              | Build-up             | Build-up             | Myopic               |
| 80  | Build-up      | Build-up              | Build-up             | Build-up             | Myopic               |
| 100 | Build-up      | Build-up              | Build-up             | Build-up             | Myopic               |
| 120 | Build-up      | Build-up              | Build-up             | Build-up             | Myopic               |
| 140 | Build-up      | Build-up              | Build-up             | Build-up             | Myopic               |
| 160 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 180 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |
| 200 | Build-up      | Build-up              | Build-up             | Build-up             | Build-up             |

Table 6: Dominant heuristic when $p = 1.1$ (or $p = 1.2$) and $h = 0.01$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|-----------------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 80  | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 100 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 120 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 140 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 160 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 180 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 200 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |

Table 7: Dominant heuristic when $p = 1.3$ and $h = 0.01$.

| $q$ | $\gamma = 0$ | $\gamma = 0.05/3000$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|-----------------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 80  | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 100 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 120 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 140 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 160 | Myopic        | Myopic                | Myopic               | Myopic               | Myopic               |
| 180 | build-up      | build-up              | build-up             | build-up             | Myopic               |
| 200 | build-up      | build-up              | build-up             | build-up             | Myopic               |
6. Conclusion. In this paper, we present a new diffusion model about new product, which captures the effects of positive word-of-mouth from the cumulative sales and negative word-of-mouth from the lost demand. As we have known, it is the first time to consider the negative effectiveness of word-of-mouth from the lost demand when there is capacity constraint. Our research is motivated by the fact that in reality, negative word-of-mouth and capacity constraint are commonplace. We have shown that under some mild conditions, myopic policy is optimal for the case of $\gamma \geq \beta$. That is, when the negative word-of-mouth is stronger than the positive word-of-mouth, it is optimal for the firm to satisfy demand as far as possible. However, when $\gamma < \beta$, we show that the build-up policy is optimal for the special case of $h = \theta = 0$. For generalized case, it is very difficult to analyze the structure of optimal policy. Note that the myopic policy and the build-up policy are not only special but also convenient to adopt, therefore, we compare the myopic policy with the build-up policy by numerical example and obtain the following managerial insights: when the price is low, the myopic policy may be dominant; when the negative word-of-mouth is weak or the capacity is large, the build-up policy may be dominant.

Finally, we present five possible extensions of our model for the future research. The first possible extension is to include advertising decision which can conversely influence the parameter $\alpha$. Because the diffusions of demand and awareness will be influenced by the advertising policy, we intuitively think that some results may no longer hold with the advertising decision, for example, the structure of demand process and the structure of optimal sales policy, et. al. However, we can study the structure of optimal advertising policy and analyze the effect of advertising decision on other operations decisions in the future research. The second possible extension is to treat the production cost $c$ as the function of production level $x(t)$. In general, we assume that production cost $c(x)$ is nonnegative, convex, and increasing

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Table 8: Dominant heuristic when $p = 1.4$ and $h = 0.01$.

| $q$ | $\gamma = 0$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic               | Myopic               | Myopic               |
| 80  | Myopic        | Myopic               | Myopic               | Myopic               |
| 100 | Myopic        | Myopic               | Myopic               | Myopic               |
| 120 | Build-up      | Build-up             | Build-up             | Myopic               |
| 140 | Build-up      | Build-up             | Build-up             | Build-up             |
| 160 | Build-up      | Build-up             | Build-up             | Build-up             |
| 180 | Build-up      | Build-up             | Build-up             | Build-up             |
| 200 | Build-up      | Build-up             | Build-up             | Build-up             |

Table 9: Dominant heuristic when $p = 1.5$ and $h = 0.01$.

| $q$ | $\gamma = 0$ | $\gamma = 0.1/3000$ | $\gamma = 0.2/3000$ | $\gamma = 0.3/3000$ |
|-----|---------------|----------------------|----------------------|----------------------|
| 60  | Myopic        | Myopic               | Myopic               | Myopic               |
| 80  | Build-up      | Build-up             | Build-up             | Myopic               |
| 100 | Build-up      | Build-up             | Build-up             | Build-up             |
| 120 | Build-up      | Build-up             | Build-up             | Build-up             |
| 140 | Build-up      | Build-up             | Build-up             | Build-up             |
| 160 | Build-up      | Build-up             | Build-up             | Build-up             |
| 180 | Build-up      | Build-up             | Build-up             | Build-up             |
| 200 | Build-up      | Build-up             | Build-up             | Build-up             |
function. The third possible extension is to incorporate the capacity decision, that is, the firm first determines the production capacity before the sales horizon while weighing the benefits of producing along the larger capacity against the costs of setting up more capacity. The fourth possible extension is to treat the influence factor of the negative word-of-mouth $\gamma$ as the function of time $t$. Because shortage may be temporary and scarcely has enough time to diffusion, we maybe assume that $\gamma(t)$ decreases in $t$. That is, the negative word-of-mouth will become weaker as the time goes on. In this situation, we intuitively think that the advantage of the build-up policy will get a promotion. The fifth possible extension is to consider a case in which some customers will be backlogged. Although the structure of the optimal policy may be too complicated to analyze in this model, we can analyze the effect of the product’s substitutability in the future research.

Acknowledgments. The authors would like to thank the Editor Professor Teo, the Associate Editor and two anonymous referees for their helpful comments.

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Received December 2009; 1st revision July 2010; 2nd revision October 2010.

E-mail address: yanxiaoming325@126.com
E-mail address: cpa841004@163.com
E-mail address: zhang.minghui@163.com
E-mail address: kliu@amss.ac.cn