Application of an Improved Molière Theory to the Description of the Landau–Pomeranchuk–Migdal Effect

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Abstract. We present exact analytical and numerical results for high-energy Coulomb corrections (CCs) to the parameters of an improved Molière multiple scattering theory over a wide range of the nuclear charge number $Z$ of the target atom and show that the values of these corrections increase up to $40\% - 45\%$ for $Z = 92$. We also report our results of applying this improved Molière multiple scattering theory for calculating CCs to the quantities of the classical Migdal theory of the Landau–Pomeranchuk–Migdal (LPM) effect, and we demonstrate that the improved Migdal LPM effect theory allow one to completely eliminate the discrepancy between the predictions of the LPM effect theory and their measurement at least for high-$Z$ targets. We also obtained analytical and numerical results concerning high-energy CCs to the quantities of the quantum Migdal theory of the LPM effect in the ranges of $Z$ from 6 to 92 and the Molière expansion parameter $B$ from 4.5 to 8.5, and we managed to show that the magnitude of relative CC to the spectral radiation rate reaches $-19\%$ in these ranges considered and must be borne in mind, e.g., in the Monte-Carlo analysis of electromagnetic cascade LPM showers in extremely high-energy region.

1. Introduction

The theory of the multiple scattering of charged particles has been treated by several authors. However, the most widely used at present is the multiple scattering theory of Molière [1]. The multiple scattering theory is of interest for numerous applications related to particle transport in matter [2] and is widely used in the cascade shower theory [3]. It can also be useful in applications motivated by exploring the extremely high-energy Landau–Pomeranchuk–Migdal (LPM) showers [4] and IceCubes neutrino-induced showers with energies above 1 PeV [5, 6]. The neutrino-oscillation experiments [7], the astronomical cosmic-ray observations [8], as well as the cosmic-ray experiments in ultrahigh-energy region [9], etc., face a problem of taking into account the multiple scattering effects.

As the Molière theory can be currently used in ultrahigh-energy region, the role of the high-energy Coulomb corrections to the parameters of this theory becomes significant. Of special importance is the Coulomb correction to the screening angular parameter, as this parameter enters into other important quantities of the Molière theory and the Migdal theory of the LPM effect [10]. It is shown that the Coulomb corrections to the spectral bremsstrahlung rate of the LPM effect theory for the regime of small LPM suppression allow completely to eliminate the discrepancy between the predictions of the LPM effect theory and its measurement at least for...
high-\(Z\) targets and also to further improve the agreement between the predictions of the LPM effect theory analogue for a thin layer of matter and experimental data [10]. The aim of the present work is to estimate analytically and numerically the CCs to some important quantities in the quantum LPM effect theory, especially to the Migdal function \(G(s)\) and the spectral bremsstrahlung rate which are of special interest in describing the shower production over the energy range \(10^{16} \leq E \leq 10^{20} \text{ eV} \ [5, 6]\).

The outline of this papers is as follows. In Section 2 we present briefly our results for the high-energy Coulomb corrections to the parameters of an improved Molière multiple scattering theory [11]. Then, in Section 3 we report our results of applying this improved Molière multiple scattering theory for calculating CCs to the quantities of the classical Migdal theory of the LPM effect in the regime of small LPM suppression. In Section 4 we obtain analytical and numerical results for high-energy CCs to some quantities of the quantum Migdal theory of the LPM effect in the regime of strong LPM suppression. Finally, in Sec. 4 we state our conclusions.

2. Coulomb corrections to some parameters of the Molière theory

In this section we present an exact analytical result for the screening angular parameter \(\theta_a\) (\(\theta_a' = \sqrt{1.167} \theta_a\)) of the Molière theory valid to all orders in the Born parameter \(\xi = Z\alpha/\beta\),

\[
\Delta_{cc} [\ln (\theta_a')] = \ln (\theta_a') - \ln (\theta_a')^B = f(\xi),
\]

with the Bethe–Maximon function \(f(\xi) = \xi^2 \sum_{n=1}^{\infty} [n(n^2 + \xi^2)]^{1-1}\), instead of an approximate one for the exact \((\theta_a'^M)\) and first-order \((\theta_a^B)\) values of the screening angle \((\theta_a)\) valid to second order in the parameter \(\xi\),

\[
\theta_a^M = \theta_a^B \sqrt{1 + 3.34 \cdot \xi^2}, \quad \theta_a^B = 1.20 \cdot \alpha \cdot Z^{1/3},
\]

that was obtained in the original paper of Molière [1]. Here, \(\alpha\) is the fine structure constant and \(\beta = v/c\) is the velocity of a projectile in units of the velocity of light.

We also present the analytical and numerical results for the Coulomb corrections to the parameters \(b, B, \) and \(\bar{y}^2\) of the Molière expansion method for the angular distribution function

\[
w_m(\vartheta, L) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{B^n} w_n(\vartheta, L), \quad w_n(\vartheta, L) = \frac{1}{\vartheta^2} \int_0^{\infty} y dy J_0 \left( \frac{\vartheta}{y} \right) e^{-y^2/4} \left[ \frac{y^2}{4} \ln \left( \frac{y^2}{4} \right) \right]^n,
\]

where \(\vartheta\) is a two-dimensional particle scattering angle in the plane orthogonal to the incident particle direction, \(\bar{y}^2 = \theta_c^B B, \theta_c\) is the characteristic angle, \(L\) denotes the thickness of an absorber, \(J_0(\vartheta \eta)\) presents the Bessel function, \(y = \theta_c \eta\), and the expansion parameter \(B\) is defined by the transcendental equation \(B - \ln B = b\) with \(b = \ln(\theta_c/\theta_a')^2\).

For the Coulomb corrections to the Born parameters \(b^B, B^B, \) and \(\bar{y}^{2B}\), we get

\[
\Delta_{cc} [b] = f(\xi), \quad \Delta_{cc} [B] = \frac{f(\xi)}{1/B^B - 1}, \quad \Delta_{cc} [\bar{y}^2] = \theta_c^2 \cdot \Delta_{cc} [B].
\]

The relative Coulomb corrections become

\[
\delta_{cc} [\bar{y}^2] = \delta_{cc} [B] = f(\xi)/(1 - B^{B^B}), \quad \delta_{cc} [\theta_a] = \exp [f(\xi)] - 1.
\]

The relative difference between the approximate \(\theta_a^M\) and exact \(\theta_a\) results reads

\[
\delta_{ccM}[\theta_a] = \frac{\theta_a - \theta_a^M}{\theta_a^M} = \frac{\theta_a}{\theta_a^M} - 1 = R_{ccM}[\theta_a] - 1.
\]

\(^{1}\) The Coulomb correction \((\Delta_{cc})\) is the difference between the exact Born parameter \(\xi\) result and the Born approximation.
The LPM effect for sufficiently thick targets, the basic formulae of which in the units \( \bar{\nu} \) were presented, are: 

\[
\Delta_{cc}^{B} = \frac{\theta_{b}^{2}}{L} \cdot \Delta_{cc}[\theta_{b}], \quad \Delta_{cc}[B] = f(\xi)/(1/B^{a} - 1),
\]

as was found in the previous section. The LPM correction will be used in the next section to obtain Coulomb corrections to the quantities of the Landau–Pomeranchuk–Migdal effect theory.

3. Applying the improved Molière theory to the description of the LPM effect

Based on the Coulomb corrections found in the previous section, we first get the analytical and numerical results for the Coulomb corrections to the quantities of the classical Migdal theory of the LPM effect for sufficiently thick targets, the basic formulae of which in the units \( \bar{h} = e = 1, \quad e^{2} = 1/137 \) read [12, 13]

\[
\langle \frac{dI}{d\omega} \rangle = \Phi(s) \left( \frac{dI}{d\omega} \right)_{0}, \quad \left( \frac{dI}{d\omega} \right)_{0} = \frac{2e^{2}}{3\pi} \gamma^{2} q L, \quad q = \bar{\nu}^{2}/L,
\]

\[
\Phi(s) = 24s^{2} \left[ \int_{0}^{\infty} dx \ e^{-2sx} \cosh(x) \sin(2sx) - \frac{\pi}{4} \right], \quad s^{2} = \frac{\lambda^{2}}{\bar{\nu}^{2}}, \quad \lambda^{2} = \gamma^{-2}.
\]

Here, \( \langle dI/d\omega \rangle \) is the electron spectral bremsstrahlung intensity averaged over various trajectories of electron motion, \( (dI/d\omega)_{0} \) presents the spectral bremsstrahlung rate without accounting the multiple scattering effects in the radiation, and \( \gamma \) denotes the Lorentz factor of the scattered particle.

The analytical result for the Coulomb correction \( \Delta_{cc} \) to the Born spectral bremsstrahlung rate \( (dI/d\omega)_{0} \) is as follows:

\[
\Delta_{cc} \left[ \left( \frac{dI}{d\omega} \right)_{0} \right] = \left( \frac{dI}{d\omega} \right)_{0} - \left( \frac{dI}{d\omega} \right)_{0}^{B} = \frac{2e^{2}}{3\pi} \gamma^{2} L \cdot \Delta_{cc}[q], \quad \Delta_{cc}[q] = \frac{1}{L} \cdot \Delta_{cc} \left[ \bar{\nu}^{2} \right]
\]

with

\[
\Delta_{cc} \left[ \bar{\nu}^{2} \right] = \theta_{b}^{2} \cdot \Delta_{cc}[\theta_{b}], \quad \Delta_{cc}[B] = f(\xi)/(1/B^{a} - 1),
\]

according to (4) and (5). In doing so, \( \Delta_{cc} [(dI/d\omega)_{0}] \) and \( \delta_{cc} [(dI/d\omega)_{0}] \) become

\[
\Delta_{cc} \left[ \left( \frac{dI}{d\omega} \right)_{0} \right] = \frac{2(e \gamma \theta_{b})^{2}}{3\pi (1/B^{a} - 1)} \cdot f(\xi), \quad \delta_{cc} \left[ \left( \frac{dI}{d\omega} \right)_{0} \right] = \frac{f(\xi)}{1 - B^{a}}.
\]

Table 1. The \( Z \) dependence of the Coulomb corrections and differences defined by Eqs. (1), (4), (5), and (6) for \( \beta = 1 \) and \( B^{a} = 8.46 \).

| Target | \( Z \) | \( f(\xi) \) | \( \delta_{cc}[\theta_{a}] \) | \( \delta_{ccM}[\theta_{a}] \) | \( \delta_{cc} \left[ \bar{\nu}^{2} \right] \) | \( \Delta_{cc}[b] \) | \( \Delta_{cc}[B] \) |
|--------|--------|----------------|----------------|----------------|----------------|----------------|----------------|
| W      | 74     | 0.2813         | -0.0565        | -0.0377        | -0.2813        | -0.3190        |                |
| Pt     | 78     | 0.3067         | -0.0582        | -0.0411        | -0.3067        | -0.3478        |                |
| Au     | 79     | 0.3125         | -0.0585        | -0.0419        | -0.3125        | -0.3545        |                |
| Pb     | 82     | 0.3316         | -0.0600        | -0.0445        | -0.3316        | -0.3760        |                |
| U      | 92     | 0.3951         | -0.0622        | -0.0530        | -0.3951        | -0.4481        |                |
4. Coulomb corrections to some quantities of the quantum LPM effect theory

We have also obtained analytical and numerical results for the Coulomb corrections to the quantities of the quantum Migdal LPM theory

\[
\left\langle \frac{dI}{d\omega} \right\rangle = \frac{1}{4} \left( \frac{dI}{d\omega} \right) \left\{ \varepsilon^2 G(s) + 2[1 + (1 - \varepsilon)^2] \Phi(s) \right\},
\]

with the energy of radiation \(\varepsilon = \omega/E\) in units of the incident particle energy \(E\), Migdal’s parameter \(s = \sqrt{(\omega/q)/8}\gamma^2\), and the Migdal functions \(G(s)\) and \(\Phi(s)\)

\[
G(s) = 12\pi s^2 - 48s^2 \sum_{k=0}^{\infty} \frac{1}{(k+s+1/2)^2 + s^2}, \quad \Phi(s) = 6s - 6\pi s^2 + 24s^2 \sum_{k=1}^{\infty} \frac{2}{(k+s)^2 + s^2}
\]

for the regime of strong LPM suppression, \(G(s)_{s \to 0} \to 12\pi s^2\) and \(\Phi(s)_{s \to 0} \to 6s\) [13, 15].
Table 3. Coulomb corrections to the quantities of the quantum Migdal LPM theory, $\delta_{cc}[G(s)]$, $\delta_{cc}[\Phi(s)]$, $\delta_{cc}[(dI/d\omega)_{0}]$, and $\delta_{cc}[(dI/d\omega)]$ in the regime of strong LPM suppression

1. $B^{B} = 8.46$, $s = 0.05$, and $\beta = 1$

| Target | Z  | $f(\xi)$ | $\delta_{cc}[G(s)]$ | $\delta_{cc}[\Phi(s)]$ | $\delta_{cc}[(dI/d\omega)_{0}]$ | $\delta_{cc}[(dI/d\omega)]$ |
|--------|----|----------|---------------------|---------------------|----------------------|----------------------|
| C      | 6  | 0.0041   | -0.0005             | -0.0003             | -0.0005              | -0.0008              |
| Al     | 13 | 0.0107   | -0.0014             | -0.0007             | -0.0014              | -0.0023              |
| Fe     | 26 | 0.0430   | -0.0058             | -0.0029             | -0.0058              | -0.0094              |
| W      | 74 | 0.2810   | -0.0392             | -0.0194             | -0.0377              | -0.0621              |
| Au     | 79 | 0.3130   | -0.0438             | -0.0216             | -0.0419              | -0.0698              |
| Pb     | 82 | 0.3320   | -0.0466             | -0.0230             | -0.0445              | -0.0738              |
| U      | 92 | 0.3950   | -0.0560             | -0.0276             | -0.0530              | -0.0887              |

2. $B^{B} = 4.50$, $s = 0.05$, and $\beta = 1$

| Target | Z  | $f(\xi)$ | $\delta_{cc}[G(s)]$ | $\delta_{cc}[\Phi(s)]$ | $\delta_{cc}[(dI/d\omega)_{0}]$ | $\delta_{cc}[(dI/d\omega)]$ |
|--------|----|----------|---------------------|---------------------|----------------------|----------------------|
| C      | 6  | 0.0041   | -0.0012             | -0.0006             | -0.0012              | -0.0019              |
| Al     | 13 | 0.0107   | -0.0031             | -0.0015             | -0.0031              | -0.0050              |
| Fe     | 26 | 0.0430   | -0.0125             | -0.0062             | -0.0123              | -0.0201              |
| W      | 74 | 0.2810   | -0.0873             | -0.0427             | -0.0803              | -0.1349              |
| Au     | 79 | 0.3130   | -0.0982             | -0.0479             | -0.0894              | -0.1507              |
| Pb     | 82 | 0.3320   | -0.1049             | -0.0511             | -0.0949              | -0.1603              |
| U      | 92 | 0.3950   | -0.1273             | -0.0617             | -0.1129              | -0.1921              |

For the relative CC to the Born spectral bremsstrahlung rate (13), one can get

$$\delta_{cc}\left[(dI/d\omega)\right] = \delta_{cc}\left[(dI/d\omega)_{0}\right] + \frac{\Delta_{cc}[G(s)] + \Delta_{cc}[\Phi(s)]}{|G^{B}(s) + \Phi^{B}(s)|}$$

$$= \delta_{cc}\left[(dI/d\omega)_{0}\right] + \frac{G^{B}}{|G^{B} + \Phi^{B}|} \delta_{cc}[G] + \frac{\Phi^{B}}{|G^{B} + \Phi^{B}|} \delta_{cc}[\Phi],$$

(15)

$$\delta_{cc}\left[(dI/d\omega)_{0}\right] = \frac{\Delta_{cc}[\ln (\theta_{a}^{0})]}{1 - B^{B}}.$$  

(16)

For the regime of strong LPM suppression, we have obtained

$$\delta_{cc}[G(s)] = 1 - \frac{1}{\delta_{cc}[\theta^{2}]} + 1,$$

$$\delta_{cc}[\Phi(s)] = 1 - \frac{1}{\sqrt{\delta_{cc}[\theta^{2}]}} + 1.$$  

(17)

Table 3 and Fig. 1 demonstrate the values of $\delta_{cc}[G(s)]$, $\delta_{cc}[\Phi(s)]$ (17), $\delta_{cc}[(dI/d\omega)_{0}]$ (16), and $\delta_{cc}[(dI/d\omega)]$ (15) in the regime of strong LPM suppression over the ranges $0 \leq Z \leq 100$ and $4.5 \leq B^{B} \leq 8.5$ at $s = 0.05$. It can be seen from Table 3 that $\delta_{cc}[G(s)]$ is comparable with $\delta_{cc}[(dI/d\omega)_{0}]$ in the case of strong suppression. The value of $|\delta_{cc}[G(s)]|$ is twice as large than $|\delta_{cc}[\Phi(s)]|$ and reaches about 4.7% at $B^{B} = 8.46$ and approximately 10.5% at $B^{B} = 4.50$ for $Z = 82$ (Pb). The modulus of $\delta_{cc}[(dI/d\omega)_{0}]$ acquires a value of 7.4% at $B^{B} = 8.46$ and 16.0% at $B^{B} = 8.46$ for $Z = 82$ ($s = 0.05$). Its upper limit is 19% over the ranges considered [16].

Thus, we can conclude that such corrections as $\delta_{cc}[(dI/d\omega)]$, $\delta_{cc}[G(s)]$, and $\delta_{cc}[(dI/d\omega)_{0}]$ become significant in the regime of strong LPM suppression and must be borne in mind, e.g., in the investigations of the LPM showers in extremely high-energy region.

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Figure 1. The $Z$ dependence of relative CCs to the quantities of the quantum LPM effect theory for the regime of large LPM suppression: $1 - \delta_{CC}[\Phi(s)]$, $2 - \delta_{CC}[G(s)]$, and $3 - \delta_{CC}[\langle dI/d\omega \rangle]$ at $s = 0.05$ and $B^2 = 8.46$.

5. Conclusion
The developed approach can be useful for the analysis of cosmic-ray experiments in ultrahigh-energy region, where the LPM effect becomes significant (for instance, in the applications related to investigations of extremely high-energy LPM showers, in exploring their structure and behavior, computing their characteristics, etc. [17, 18]).

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