STATIONARY SOLUTIONS OF A FRACTIONAL LAPLACIAN
WITH SINGULAR PERTURBATION

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Abstract. In this report we extend some ideas already developed by [8, 11, 12] to the case where the singular perturbation is given by a derivative of the Dirac’s δ.

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1. Introduction

Since the seminal papers by Laskin (see [7] and the reference therein) fractional quantum mechanics has received and increasing interest from a theoretical point of view, and only recently some applications have been proposed in optics [9, 13] and in the framework of nonlinear Schrödinger equations [2, 4, 5]. In fact, most of the current studies on fractional quantum mechanics are mainly focused on the mathematical aspects; the difficulties come from the fact that the fractional Laplacian $(-\Delta)^{\alpha/2}$ is a nonlocal operator. Therefore, analysis of some simple toy models would be very useful; recently, one-dimensional fractional Laplacian perturbed by a one (or more) Dirac’s delta has been discussed [8, 11, 12].

In this short report we show how the main ideas can be extended to the case of a more singular perturbation of the one-dimensional fractional Laplacian, e.g. the $n$-th derivative of the Dirac’s delta with $n \geq 1$; the price we have of pay is to request that the power $\alpha$ of the fractional Laplacian must be greater than $2n + 1$. Hence this method does not apply to the standard Laplacian (corresponding to $\alpha = 2$) when $n$ is bigger or equal than 1. This problem does not occur for the Dirac’s delta, corresponding to $n = 0$, because it is infinitesimally bounded with respect to the Laplacian (see Theorem KLMN and Example 3 by §X.2 [13]). In fact, one could extend the result to $\alpha \leq 2n + 1$, even for $n \geq 1$, by means of a suitable renormalization procedure as done, for example, by [3] in order to define the three-dimensional Laplacian perturbed by a Dirac’s delta.

In Section 2 we give the main result, and in Section 3 we apply it to the case of $n = 0$ and $n = 1$. In fact, the result when $n = 0$ coincides with the one already given by [11, 12] up to a normalization pre-factor.

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2. Main result

Let us consider the one-dimensional fractional eigenvalue problem $H\psi = E\psi$, $\psi \in L^2(\mathbb{R}, dx)$,

$$H\psi := (-\Delta)^{\alpha/2} \psi + V_0 \delta^{(n)} \psi;$$  \hspace{2cm} (1)

where $\delta^{(n)}$ is the $n$-th derivative of the Dirac’s delta with strength $V_0 \in \mathbb{R}$. The fractional Laplacian operator is defined for any $\alpha > 0$ by means of the Fourier transform $\mathcal{F}$ of $\psi$, that is

$$\mathcal{F} \left[ (-\Delta)^{\alpha/2} \psi \right] (p) = |p|^\alpha \phi(p), \quad \phi = \mathcal{F} \psi,$$  \hspace{2cm} (2)

and the $n$-th derivative of the Dirac’s delta is defined as usual

$$\int_{\mathbb{R}} \delta^{(n)} f(x) dx = (-1)^n \frac{d^n f(0)}{dx^n}$$

for any test function $f$ and its Fourier transform is given by $\mathcal{F} \left( \delta^{(n)} \right) = (ip)^n$. Here

$$\phi(p) = [\mathcal{F} \psi](p) = \int_{\mathbb{R}} e^{-ipx} \psi(x) dx$$

and

$$\psi(x) = \left[ \mathcal{F}^{-1} \phi \right](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ipx} \phi(p) dp.$$  \hspace{2cm} (3)

Laplacian fractional operators may have equivalent definitions provided that $\alpha \in (0, 2)$; furthermore, when $\alpha$ is restricted to such an interval then properties of regularity to the equation $(-\Delta)^{\alpha/2} u = f$ occurs. In these paper we don’t restrict $\alpha$ to the interval $(0, 2)$, indeed definition (2) makes sense even for $\alpha$ large enough provided that $\phi$ rapidly decreases when $p$ goes to infinity; in fact, we will assume $\alpha$ larger that $2n + 1$.

If we consider the Fourier transform of both sides of the eigenvalue equation $H\psi = E\psi$ we have that it takes the form

$$|p|^\alpha \phi + V_0 \mathcal{F} \left( \delta^{(n)} \right) \ast \mathcal{F} (\psi) = E\phi$$

that is

$$|p|^\alpha \phi + i^n \frac{V_0}{2\pi} \int_{\mathbb{R}} (p - q)^n \phi(q) dq = E\phi.$$  \hspace{2cm} (3)

In order to solve such an equation we set

$$K_h = -i^n (-1)^{n-h} \frac{V_0}{2\pi} \left( \begin{array}{c} n \\ h \end{array} \right) \int_{\mathbb{R}} q^{n-h} \phi(q) dq, \quad h = 0, 1, 2, \ldots, n;$$  \hspace{2cm} (4)

hence it follows that (3) has solution

$$\phi(p) = \sum_{h=0}^{n} K_h \frac{p^h}{|p|^\alpha - E}.$$  \hspace{2cm} (5)

We should remark that $\phi \in L^2$ when $E < 0$ and $\alpha > n + \frac{1}{2}$; furthermore the integrals in $K_h$ converge for any $h = 0, \ldots, n$ when $\alpha > 2n + 1$. Therefore, by substituting (5) inside (4) we have that

$$K_h = \sum_{k=0}^{n} a_{h,k} K_k$$  \hspace{2cm} (6)
where (see §A.1)

\[ a_{h,k}(E) := -i^n(-1)^{n-h} V_0 \left( \frac{n}{h} \right) \int \frac{q^{n+k-h}}{|q|^\alpha + |E|} dq \]

(7)

\[ a_{h,k} := a_{h,k}(E) = -i^n(-1)^{n-h} \left( \frac{n}{h} \right) |E|^{\frac{n+k-h+1}{\alpha}} \frac{V_0[1 + (-1)^{n+k-h}]}{2\alpha \sin \left( \pi \frac{n+k-h+1}{\alpha} \right)}. \]

Then (6) can be seen as a linear system \( \sum_{n,k=0}^n \left[ a_{h,k} - \delta_h^k \right] K_k = 0 \) which has no identically zero solution provided the associated matrix has determinant equal to zero. In fact, the eigenvalue equation is finally given by

\[ \det \left[ a_{h,k}(E) - \delta_h^k \right] = 0. \]

(8)

Once one has obtained the solutions \( \hat{E} \) to equation (8), then the eigenvectors \( \psi(x) \) are given by the inverse Fourier transform of (5), that is

\[ \psi(x) = \sum_{h=0}^n \hat{K}_h \frac{1}{2\pi} \int \frac{p^h}{|p|^\alpha + |\hat{E}|} e^{ipx} dp = \sum_{h=0}^n \hat{K}_h \frac{1}{2\pi} \frac{d^h}{dx^h} F_\alpha(x) \]

(9)

where

\[ F_\alpha(x) := \int \frac{1}{|p|^\alpha + |\hat{E}|} e^{ipx} dp \]

and where \( \{ \hat{K}_h \}_{h=0}^n \) is a solution to the linear system \( \sum_{k=0}^n \left[ a_{h,k}(\hat{E}) - \delta_h^k \right] K_k = 0 \) under the normalization condition \( \int_{\mathbb{R}} |\phi(p)|^2 dp = 2\pi \) where \( \phi(p) \) is given by (5), that is

\[ 2\pi = \sum_{h,k=0}^n \overline{K}_h \hat{K}_k M_{h+k,\alpha}(\hat{E}) \]

where we define

\[ M_{h+k,\alpha}(\hat{E}) := \int \frac{p^{h+k}}{|p|^\alpha + |\hat{E}|^2} dp. \]

(10)

For an explicit expression of \( F_\alpha(x) \) and \( M_{n,\alpha}(\hat{E}) \) see, respectively, §A.3 and §A.2.

We can collect all these results within the following statement.

**Theorem 1.** Let us consider the eigenvalue equation \( H\psi = E\psi \), where \( H \) is formally defined in \( L^2(\mathbb{R}, dx) \) by (3), with \( \alpha > 2n+1 \). Then, the real and negative eigenvalues are solutions to equation (8), where \( a_{h,k} \) are defined by (7), with associated normalized eigenvectors (9).

3. **Examples**

3.1. **Dirac’s \( \delta \): \( n = 0 \).** In such a case we assume that \( \alpha > 1 \) and the eigenvalues \( \hat{E} \) are the real and negative solutions to the equation

\[ a_{0,0} - 1 = 0 \quad \text{where} \quad a_{0,0} = -\frac{V_0}{\alpha \sin \left( \frac{\alpha}{\alpha} \right)} |\hat{E}|^{\frac{1}{\alpha}}. \]
Figure 1. Here we plot the eigenvalue $\hat{E} = \hat{E}(\alpha)$. Point line corresponds to the eigenvalue in the Dirac’s delta case \cite{11}, for $\alpha > 1$. Full line corresponds to the eigenvalue in the case \cite{13} of the derivative of the Dirac’s delta, for $\alpha > 3$.

This equation has solution when $V_0 < 0$ and it is given by (see point line in Figure 1)

$$\hat{E} = - \left[ -\frac{V_0}{\alpha \sin \left( \frac{\pi}{\alpha} \right)} \right]^{\frac{\pi}{\alpha}} . \quad (11)$$

Concerning the normalized eigenvector we have that $\phi(p) = \hat{K}_0 \frac{1}{|E|^{\frac{1}{\alpha}}} |p|^{\frac{\alpha}{2}}$ where $\hat{K}_0$ is such that $|\hat{K}_0|^2 M_{0,\alpha}(\hat{E}) = 2\pi$; that is

$$\hat{K}_0 = \left[ \frac{2\pi}{M_{0,\alpha}(\hat{E})} \right]^{\frac{1}{2}} = \left[ -V_0 \frac{\alpha}{\alpha - 1} |\hat{E}| \right]^{\frac{1}{2}} ,$$

by \S A.2 Hence, by \S A.3 the normalized eigenvector is given by (see Figure 2)

$$\psi(x) = \frac{1}{2\pi} F_\alpha(x)$$

$$= \left[ -\frac{V_0 \alpha}{(\alpha - 1)|\hat{E}|} \right]^{\frac{1}{2}} \frac{1}{|x|} H_{2,3}^{1,1} \left[ |\hat{E}| |x|^{\alpha} \right] \left[ (1,1), (1,\alpha), \left( 1, \frac{\alpha}{2} \right), (1,1), \left( 1, \frac{\alpha}{2} \right) \right] . \quad (12)$$

Remark 1. In fact, this result has been already given by \cite{11}, eqns. (35) and (36), with a slightly different normalization coefficient, which coincides with the one given by \cite{12} when $\alpha = 2$. 
Here we plot the normalized eigenvectors for different values of $\alpha$ (point line corresponds to $\alpha = 1.5$, full line corresponds to $\alpha = 2$ and broken line corresponds to $\alpha = 2.5$).

3.2. Derivative of the Dirac’s $\delta$: $n = 1$. In such a case we assume that $\alpha > 3$.

One has that $a_{0,0} = a_{1,1} = 0$ and that

$$a_{0,1} = i \frac{V_0}{\alpha \sin \left(\frac{3\pi}{\alpha}\right)} |E|^{(3-\alpha)/\alpha} \quad \text{and} \quad a_{1,0} = -i \frac{V_0}{\alpha \sin \left(\frac{\pi}{\alpha}\right)} |E|^{(1-\alpha)/\alpha}$$

Hence (6) takes the form

$$\begin{cases} K_0 = a_{0,1} K_1 \\ K_1 = a_{1,0} K_0 \end{cases}$$

and the real and negative eigenvalues $\hat{E}$ are the solutions to the equation $a_{0,1} a_{1,0} = 1$, that is (see full line in Figure 1)

$$\hat{E} = -\left[ \frac{|V_0|}{\alpha \left[ \sin \left(\frac{3\pi}{\alpha}\right) \sin \left(\frac{\pi}{\alpha}\right) \right]^{1/2}} \right]^{\alpha/(\alpha-2)}$$

We may remark that the eigenvalue equation has solutions $\hat{E} := \hat{E}(V_0)$ for any $V_0 \in \mathbb{R}$, and in particular that $\hat{E}(-V_0) = \hat{E}(V_0)$.

Concerning the normalized eigenvector we have that $\phi(p) = \frac{\hat{K}_0 + p \hat{K}_1}{|p|^{\alpha} + |\hat{E}|}$ and where $\hat{K}_0 = c$ and $\hat{K}_1 = a_{1,0}(\hat{E}) c$ where $c$ is a normalization constant given by

$$c = \sqrt{2\pi} \left[ M_{0,\alpha}(\hat{E}) + |a_{1,0}(\hat{E})|^2 M_{2,\alpha}(\hat{E}) \right]^{-1/2}$$

Hence the eigenvector $\psi(x)$ is given by (see Figure 3)

$$\psi(x) = \frac{c}{2\pi} F_0^\alpha(x) + \frac{c}{2\pi i} a_{1,0}(\hat{E}) F_1^\alpha(x)$$
where \( F_0^\alpha(x) \) is an even function defined by \( F_0^\alpha(x) \) for \( x \geq 0 \), and where \( F_1^\alpha(x) \) is an odd function defined as \( F_1^\alpha(x) = \frac{dF_\alpha}{dx} \) for \( x > 0 \), i.e. (see §A.3):

\[
F_1^\alpha(x) := \frac{2\pi}{|E|}\frac{H_2^2}{H_4^4} \left[ |E|^x \left| \begin{array}{c} (1, \alpha), \quad (1, 1), \quad (1, \alpha/2), \\ (1, \alpha), \quad (1, 1), \quad (1, \alpha/2), \quad (2, \alpha) \end{array} \right| \right], \quad x > 0.
\]

**Appendix A. Some integrals**

**A.1. Integral (7).** In previous expressions we consider the integral

\[
J_{m,\alpha}(E) := \int_R \frac{q^m}{|q|^\alpha + |E|^m} dq = [1 + (-1)^m]|E|^m \hat{J}_{m,\alpha}
\]

where

\[
\hat{J}_{m,\alpha} := \int_0^{+\infty} \frac{w^m}{w^{\alpha - 1}} dw.
\]

Recalling that

\[
I_{(z, y)}(p) := \int \frac{p^m}{p^2 + 1} dp = \frac{p^{1+m}}{y + 1} F \left( \left[ 1, \frac{y}{z} + \frac{1}{z} \right], \left[ 1 + \frac{y}{z} + \frac{1}{z} \right], -p^2 \right)
\]

where \( F([a, b], [c]: z) \) is the hypergeometric function. Then

\[
\hat{J}_{m,\alpha} = \lim_{p \to +\infty} \frac{p^{1+m}}{m+1} F \left( \left[ 1, \frac{m+1}{\alpha} \right], \left[ 1 + \frac{m+1}{\alpha} \right], -p^\alpha \right)
\]
Now, recalling formula (15.3.7) by [1] which holds true when $|\arg(-z)| < \pi$, it follows that

$$F([a, b], [c]; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F([a, 1-c+a], [1-b+a]; z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F([b, 1-c+b], [1-a+b]; z^{-1}),$$

hence

$$\frac{p^{1+m}}{m+1} F\left(\left[1, \frac{m+1}{\alpha}\right], \left[1 + \frac{m+1}{\alpha}\right], -p^{\alpha}\right) = \frac{p^{1+m}}{m+1} \left(\frac{m+\alpha}{\alpha}\right) \Gamma\left(\frac{m+\alpha}{\alpha} - 1\right) \Gamma\left(\frac{m+\alpha}{\alpha} + 1\right) F\left(\left[1, 1 - \frac{m+1}{\alpha}\right], \left[2 - \frac{m+1}{\alpha}\right]; (-p)^{-\alpha}\right) + \frac{p^{1-m}}{m+1} \left(\frac{m+\alpha}{\alpha}\right) \Gamma\left(\frac{m+\alpha}{\alpha} + 1\right) F\left(\left[\frac{m+1}{\alpha}, 0\right], \left[\frac{m+1}{\alpha}\right]; (-p)^{-\alpha}\right).$$

Therefore

$$\hat{J}_{m,\alpha} = \frac{1}{\Gamma(2)} \frac{\Gamma\left(\frac{m+\alpha}{\alpha} + 1\right) \Gamma\left(\frac{m+\alpha}{\alpha} - 1\right)}{\Gamma(2)} = \frac{\pi}{\alpha \sin\left(\pi \frac{m+\alpha}{\alpha}\right)}$$

from which follows that

$$J_{m,\alpha}(E) = |E|^{\frac{m+\alpha}{\alpha}} \frac{1 + (-1)^m}{\alpha \sin\left(\pi \frac{m+1}{\alpha}\right)}$$

provided that $\alpha > m + 1$.

A.2. Integral (10). Now we consider the integral

$$M_{m,\alpha}(E) := \int_{\mathbb{R}} \frac{q^n}{||q|^\alpha + |E|^2} dq = [1 + (-1)^m]|E|^\frac{m+1-2\alpha}{\alpha} \hat{M}_{m,\alpha}$$

where

$$\hat{M}_{m,\alpha} := \int_0^{+\infty} \frac{w^n}{[w^\alpha + 1]^2} dw.$$ 

Observing that

$$\int \frac{p^{\nu}}{(p^2 + 1)^2} dp = \frac{1}{2} \frac{p^{\nu+1}}{p^2 + 1} - \frac{y + 1 - z}{z} \int \frac{p^{\nu}}{p^2 + 1} dp$$

then

$$\hat{M}_{m,\alpha} = -\frac{m+1-\alpha}{\alpha} \hat{J}_{m,\alpha} = -\frac{m+1-\alpha}{\alpha} \frac{\pi}{\alpha \sin\left(\pi \frac{m+1}{\alpha}\right)}$$

under the conditions $\alpha > m + 1$.

A.3. Integral (9). Now we consider the integral

$$F_{\alpha}(x) := \int_{\mathbb{R}} \frac{1}{|p|^\alpha + |E|} e^{ipx} dp$$

This function is an even parity function, i.e. $F_{\alpha}(-x) = F_{\alpha}(x)$. By (B12) by [11] it follows that

$$F_{\alpha}(x) = \frac{2\pi}{|E||x|} H_{2,1}^2 \left[ |E||x|^{\alpha} \begin{pmatrix} (1,1), & (1,\alpha/2), & (1,1) \end{pmatrix} \right]$$
where $H$ denotes the Fox’s special function. Here we recall that the Fox’s special functions are defined as follows [10]

$$H_{m,n}^{p,q} \left[ z \left| \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right| \right] := [\mathcal{M}^{-1}\Theta](z)$$

where $\mathcal{M}^{-1}$ denotes the inverse Mellin’s transform and where

$$\Theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{\ell=1}^{n} \Gamma(1 - a_\ell - A_\ell s)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j - B_j s) \prod_{\ell=n+1}^{p} \Gamma(a_\ell + A_\ell s)}$$

Finally, we recall the following formula concerning the derivative of the $H$-function (see formula (1.83) by [10]):

$$\frac{d}{dx} \left\{ x^s H_{p,q}^{m,n} \left[ z^h \left| \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \right| \right] \right\} = x^{s-1} H_{p+1,q+1}^{m,n+1} \left[ z^{-h} \left| \begin{array}{c} (-s, h), (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q), (-s + 1, h) \end{array} \right| \right]$$

provided that $h > 0$.

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