Frequency-comb response of a parametrically-driven Duffing oscillator to a small added ac excitation

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Abstract

Here we present a one-degree-of-freedom model of a nonlinear parametrically-driven resonator in the presence of a small added ac signal that has spectral responses similar to a frequency comb. The proposed nonlinear resonator has a spread spectrum response with a series of narrow peaks that are equally spaced in frequency. The system displays this sort of behavior after a breakup of time-inversion symmetry and when the onset of parametric instability is near in parameter space. We further show that in the frequency comb response the added ac signal can suppress the transition to parametric instability in the nonlinear oscillator. We also show that the averaging method is able to capture the essential dynamics involved.
I. INTRODUCTION

The study of the effect of small signals on non-autonomous nonlinear dynamical systems near bifurcation points was pioneered by K. Wiesenfeld and B. McNamara in the mid 80’s [1, 2]. They showed that several different dynamical systems are very sensitive to coherent perturbations near the onset of codimension-one bifurcations, such as period doubling, saddle node, transcritical, Hopf, and pitchfork (symmetry-breaking) bifurcations. They developed a general linear response theory, based on perturbation and Floquet theories, explaining the effects of small coherent signals perturbing limit cycles of nonlinear systems near the onset of bifurcation points. One of the systems to which they applied their theory was the ac-driven Duffing oscillator. It was found by them that nonlinear dynamical systems could be used as narrow-band phase sensitive amplifiers.

With the advent and development of micro-electromechanical systems (MEMS) technology in the 90’s new mechanical resonators were developed, such as the doubly clamped beam resonators that could reach very high quality factors. The dynamics of the fundamental mode of these resonators is well approximated by the Duffing equation. Furthermore, these micromechanical devices might exhibit, if properly tuned, a bistable response that can be quantitatively modeled by the bistability obtained in Duffing oscillators [3]. More recently, amplifiers that operate near the threshold of bifurcations were shown to present very high-gain amplification [4–6].

One might be interested in a very sensitive high gain amplifier with a spread spectrum, so as to sample selectively a broad band of frequencies. This is particularly interesting for applications requiring spectrum sensing, and one way to achieve this is through the use of frequency combs [7, 8]. The response of frequency combs to a narrowband small signal is given by a series of narrow peaks equally spaced in frequency. In 2017, Ganesan et al [9, 10] created the first mechanical resonators that had spectral lines similar to optical frequency-combs. Their experimental apparatus is based on two symmetrical cantilevers mechanically coupled to one another at their bases. They used two coupled nonlinear normal modes to model the dynamics in which the first mode is resonantly excited by an added external drive, while the second mode is parametrically excited by the first mode. These experiments were followed by Czaplewski et al [11] in 2018, who used another two-mode coupling nonlinear model to describe experimental data from a mechanical resonator with both flexural and torsional vibrations. In Ref. [12], the authors propose a theoretical model based on an approximation to a nonlocal Euler-Bernoulli model with many normal modes to describe frequency combs.
We present one parametrically-driven Duffing oscillator model and make a nonlinear analysis of it based on the averaging method. When there is no external added ac excitation, this nonlinear system can present a bistable region, whose onset corresponds to a saddle-node bifurcation. There is also a threshold for parametric instability, which corresponds to a Hopf bifurcation, either supercritical or subcritical. With the application of an external ac drive, the nonlinear system may present a frequency-comb-like behavior as the parametric pump amplitude is increased in the bistable region. It occurs just after a sharp increase in the spectral component corresponding to parametric instability. This corresponds to a symmetry-breaking transition, in which the number of spectral peaks is doubled with peaks equally spaced in the frequency spectrum. All these responses are captured by the first-order averaging method.

In this article we show that a weaker form of mechanical frequency combs (MFCs) can be achieved after a break up of time-inversion symmetry. In addition to that, a stronger form of MFCs occurs after a period-doubling bifurcation in the averaged dynamical system equations. In this form, there are twice as many peaks in the Duffing oscillator response to the added ac excitation as in the weaker form of the frequency comb. Furthermore, unlike the models proposed in Refs. [9–11], we show that only one parametrically-driven nonlinear mode is needed to present the frequency-comb-like behavior in the spectral response of the resonator. Different from the analysis developed by Bryant and Wiesenfeld [13], there is no period-doubling bifurcation here.

II. THE PARAMETRICALLY-DRIVEN DUFFING OSCILLATOR MODEL

The one-degree of freedom model we use to describe the dynamics of a parametrically-driven nonlinear resonator is given by

\[ \ddot{x}(t) = -\gamma \dot{x}(t) - x(t) - \alpha x^3(t) + F_p \cos(2\omega t)x(t) \]

in dimensionless units. Here \( \gamma \) is the dissipation rate, \( \alpha \) is the nonlinear coefficient, \( F_p \) is the parametric pump amplitude, and \( 2\omega \) is the parametric pump angular frequency. Assuming \( \gamma, \alpha, \) and \( F_p \) are \( O(\epsilon) \), with \( 0 < \epsilon << 1 \), we can apply the averaging method to obtain a slow autonomous dynamics. This is accomplished via the transform

\[ x(t) = u(t) \cos(\omega t) - v(t) \sin(\omega t), \]
\[ \dot{x}(t) = -\omega [u(t) \sin(\omega t) + v(t) \cos(\omega t)] . \]
After applying this change of variables and neglecting fast oscillating terms, via Poincaré weakly non-linear transformation, we obtain

\[
\begin{align*}
\dot{u} &= -\frac{1}{2\omega} \left\{ \gamma \omega u + \left[ \Omega + \frac{F_p}{2} + \frac{3\alpha}{4} (u^2 + v^2) \right] v \right\}, \\
\dot{v} &= -\frac{1}{2\omega} \left\{ \left[ -\Omega + \frac{F_p}{2} - \frac{3\alpha}{4} r^2 \right] u + \gamma \omega v \right\},
\end{align*}
\]

(3)

where \( \Omega = 1 - \omega^2 = O(\epsilon) \). The fixed points are obtained from the solution of

\[
\begin{align*}
\gamma \omega u + \left[ \Omega + \frac{F_p}{2} + \frac{3\alpha}{4} r^2 \right] v &= 0, \\
\left[ -\Omega + \frac{F_p}{2} - \frac{3\alpha}{4} r^2 \right] u + \gamma \omega v &= 0,
\end{align*}
\]

(4)

where \( r^2 = u^2 + v^2 \). The characteristic equation based on Eq. (4) can be written as

\[
\frac{F_p^2}{4} = (\gamma \omega)^2 + \left( \Omega + \frac{3\alpha}{4} r^2 \right)^2.
\]

(5)

The steady-state squared amplitude is given by

\[
r^2 = -\frac{4}{3\alpha} \left[ \Omega \pm \sqrt{\frac{F_p^2}{4} - \gamma^2 \omega^2} \right].
\]

(6)

When \( \omega > 1 \) (\( \Omega < 0 \)), we have a bistability region below the parametric instability threshold, in which the \( \pm \) solutions of Eq. (6) and the quiescent solution \( (u = v = 0 \text{ in Eq. (4)}) \) are possible. Of these three solutions, the \( + \)-branch solution is unstable, while the other two solutions are stable. From Eq. (6) one can see that the onset of bistability occurs when \( F_p = 2\gamma \omega \). When \( \omega < 1 \) (\( \Omega > 0 \)), only the quiescent solution is possible below the parametric instability threshold. Above it, the approximate amplitude of the oscillations is obtained from the minus branch of Eq. (6).

III. THE DUFFING PARAMETRIC AMPLIFIER

If we add to Eq. (1) an ac excitation, we obtain a parametric amplifier. Here we call it a Duffing amplifier (DA). It is described by the equation

\[
\ddot{x}(t) = -\gamma \dot{x}(t) - x(t) - \alpha x^3(t) + F_p \cos(2\omega t)x(t) + F_s \cos(\omega_s t + \varphi_0),
\]

(7)

where \( F_s \) is the amplitude, \( \omega_s \) is the angular frequency, and \( \varphi_0 \) is an arbitrary phase of the external ac excitation. Assuming that \( F_s = O(\epsilon), \delta = \omega_s - \omega = O(\epsilon) \) and all the other coefficients are also
small as in the previous section, we can apply the averaging method. After doing so, we find the slowly-varying dynamics

\begin{align*}
\dot{u} &= -\frac{1}{2\omega} \left\{ \gamma \omega u + \left[ \Omega + \frac{F_p}{2} + \frac{3\alpha}{4} (u^2 + v^2) \right] v \right\} + \frac{F_s}{2\omega} \sin \varphi(t), \\
\dot{v} &= -\frac{1}{2\omega} \left\{ \left[ -\Omega + \frac{F_p}{2} - \frac{3\alpha}{4} (u^2 + v^2) \right] u + \gamma \omega v \right\} - \frac{F_s}{2\omega} \cos \varphi(t),
\end{align*}

where \( \varphi(t) = \delta t + \varphi_0 \).

\section{RESULTS AND DISCUSSION}

In Fig. 1, we present the bifurcation diagram in the \( \omega \times F_p \) parameter space of the parametrically-driven Duffing oscillator, whose equation of motion is written in Eq. (1). Here, the parametric instability transition occurs when one crosses the thick continuous line in parameter space.

In frame (a) of Fig. 2, we show a time series of numerical integration of Eq. (7), which corresponds to parametric amplification in the DA with the parametric pump set at \( F_p = 0.21 \) and the ac excitation amplitude set at \( F_s = 0.01 \). The envelopes are obtained from the averaged slowly-varying dynamics given in Eqs. (8). The envelope of the pulses obeys approximately the time-inversion symmetry. In frame (b) of Fig. 2, we plot the Fourier transform (FT) corresponding to the time series shown in frame (a). The semi-analytic approximation to the numerical FT spectrum was obtained from the numerical integration of the corresponding averaged system, given in Eqs. (8), with the transformation defined in Eq. (2). The first-order harmonic balance approximation is roughly accurate, since it only predicts the signal \( (\omega_s = \omega + \delta) \) and idler \( (\omega - \delta) \) peaks in a parametric amplifier. Here, besides the signal and idler peaks, we barely see two other sidebands due to the nonlinearity of our system.

In frame (a) of Fig. 3, we show a time series of numerical integration of Eq. (7), which corresponds to parametric amplification in the DA with the parametric pump set at \( F_p = 0.21 \) and the ac excitation amplitude set at \( F_s = 0.02 \). The envelopes are obtained from the averaged slowly-varying dynamics given in Eqs. (8). Note the broken time-inversion symmetry in the time series parametric beats. In frame (b) of Fig. 3, we plot the Fourier transform (FT) corresponding to the time series shown in frame (a). The semi-analytic approximation to the numerical FT spectrum was obtained from the numerical integration of the corresponding averaged system, given in Eqs. (8), with the transformation defined in Eq. (2). The first-order harmonic balance approximation becomes inaccurate, since it only predicts the signal and idler peaks in a parametric amplifier.
Here, besides the signal and idler peaks, we also have sidebands spaced from one another by $2\delta$ and symmetrically positioned around these two central peaks. One can clearly see a frequency-comb-like spectrum with ten easily seen peaks.

In frame (a) of Fig. 4, with a higher value of the parametric pump amplitude ($F_p = 0.24$), the envelope of the DA response becomes even more asymmetric. One can see that for the same time span the number of pulses is halved as compared to the time series of Fig. 3. This indicates that there was a period-doubling bifurcation in the dynamical system of Eq. (8), when the pump amplitude was increased from $F_p = 0.21$, in Fig. 3(a), to $F_p = 0.24$. In frame (b) of Fig. 4, in addition to the signal and idler peaks, we also have a strong peak at $\omega$, which is at half the parametric drive frequency. Further new peaks can be seen at $\omega \pm 2\delta$, $\omega \pm 4\delta$, $\omega \pm 6\delta$, ... If one increases further the parametric drive amplitude, then one gets a full transition to parametric instability in which the frequency-comb like behavior disappears and one gets a very strong peak at $\nu = \omega$ with two small sidebands, the signal and the idler.

In Fig. 5, we show that a parametric instability occurs in a narrow window of amplitudes of the added ac excitation. Inside this window the averaged system of Eq. (8) presents a period-doubled solution. In this region occurs the strongest forms of frequency-comb response such as occurs in Fig. 4. For $F_s < 0.01386$ the solutions are more symmetrical such as in Fig. 2, inside the window they are least symmetrical, whereas for $F_s > 0.0181$ there is less asymmetry and the frequency-comb response is weaker, with peaks spaced by $2\delta$ (see Fig. 3).

In Figs. 6-9, we show how the seven main spectral peaks vary as a function of the parametric pump amplitude. In Figs. 6-7, we fix $\omega$ at 0.95 and we increase $F_p$ along line (a) of the bifurcation diagram of Fig. 1. We can see parametric instability suppression as evidenced by the delayed increase in the amplitude of the spectral peak at $\omega$. The sharp increase in this peak only occurs well past the parametric instability transition of Fig. 1, when there is no external drive ($F_s = 0$). One sees that the higher $F_s$ is the more suppression in parametric instability there is.

In Figs. 8-9, $\omega = 1.0625$ and we vary $F_p$ along line (b) of the bifurcation diagram of Fig. 1. One can see that the sharp increase in the spectral peak at $\omega$ of the DA response generates an even stronger frequency-comb-like behavior, with twice as many peaks, now spaced out in frequency from one another by $\delta$. By comparing the results of Figs. 8 and 9, one sees that with increasing value of $F_s$ there is also a suppression of parametric instability. In this case, the parametric instability transition still occurs in the bistable region, below the threshold to instability in the parametric oscillator (dashed line). Apparently, counterintuitively, the larger $F_s$ (the larger the
external excitation) is, the more stable the smaller amplitude of the parametrically-driven Duffing oscillator response is.

![Bifurcation Diagram](image)

**FIG. 1.** Bifurcation diagram of the parametrically-driven Duffing oscillator of Eq. (1). When the pump amplitude $F_p$ is increased past the black continuous line, a Hopf bifurcation takes place. When $\omega < 1$, a supercritical Hopf bifurcation occurs, whereas when $\omega > 1$, a subcritical Hopf bifurcation occurs. On the dashed line a saddle-node bifurcation occurs.
FIG. 2. (a) Time series of the Duffing amplifier (DA) obtained from the numerical integration of Eq. (7). (b) The corresponding Fourier Transform. The strongest peak is the signal, at $\nu = 1.05$, whereas the second strongest peak is the idler. These two peaks (signal and idler) are hallmarks of a parametric amplifier.
FIG. 3. (a) Time series of the Duffing amplifier (DA) obtained from the numerical integration of Eq. (7). (b) The corresponding Fourier Transform. One can see a frequency-comb-like behavior. The strongest peak is the signal, at $\nu = 1.05$, whereas the second strongest peak is the idler. The other peaks are due to the nonlinear nature of the amplifier and the time-inversion symmetry-breaking bifurcation.
FIG. 4. (a) Time series of the DA obtained from the numerical integration of Eq. (7). (b) The corresponding Fourier Transform. One can see a stronger frequency-comb behavior with roughly twice as many peaks as in Fig. 3(b).
FIG. 5. Time-inversion symmetry breaking bifurcation. A parametric instability occurs in the interval $0.01386 < F_s < 0.01811$. This corresponds to a period-doubled solution of the averaged system of Eqs. (8).
FIG. 6. Suppression of parametric instability. Here we plot the seven main spectral peaks of the FT as a function of pump amplitude $F_p$. In parameter space, the variation of $F_p$ occurs along line (a) of Fig. 1, although the system investigated here obeys Eq. (7), in which there is an added ac excitation. The vertical dashed line indicates the transition to parametric instability. The suppression is measured by the amplitude of the spectral peak of the Fourier transform $|\tilde{x}(\nu)|$ at $\nu = \omega$. Here the frequency-comb-like behavior reaches a maximum when the peak at $\omega$ is minimal approximately when the parametric pump amplitude $F_p = 0.36$. 
FIG. 7. Suppression of parametric instability. The vertical dashed line indicates the transition to parametric instability. Note increased suppression of parametric instability for larger value of external signal amplitude as compared with the results of previous figure.
FIG. 8. Broader frequency comb generation. The strongest frequency-comb behavior roughly occurs in the range $0.23 < F_p < 0.25$. The vertical dashed-dotted line indicates the transition to bistability and the vertical dashed line indicates the parametric instability transition in the parametrically-driven Duffing oscillator.
FIG. 9. Very narrow frequency comb generation. Note the suppression of the parametric instability transition as the amplitude of the ac excitation increases. The transition still occurs in the bistable region, but very near the parametric instability threshold (dashed line).
V. CONCLUSION

Here we investigated a one-degree-of-freedom parametrically-driven Duffing oscillator with a small added ac drive that could present suppression of parametric instability or present spectral peaks similar to recent experimental results of mechanical frequency combs [9–11]. We have seen two types of frequency comb behavior: one weaker and one stronger. The stronger frequency comb has twice as many peaks as the weaker form with peaks at half the distance from one another. We claim that the fundamental cause for the frequency-comb dynamical behavior is a time-inversion symmetry-breaking that occurs near the parametric instability transition (a Hopf bifurcation). We have also shown that the averaging method can capture this spectral response. The stronger form of the frequency comb arises after a subcritical period-doubling bifurcation of the averaged system of equations of the DA. In addition, the simple model we propose here could be used as a theoretical framework, or a normal form model, for the study of the frequency comb phenomenon in mechanical oscillators.

Furthermore, we point out that our theory is similar to the one developed by K. Wiesenfeld and McNamara in the 80’s for amplification of small signals near bifurcation points. Their theory presented in Refs. [1, 2] is a linear response theory based on Floquet theory, whereas here we present a nonlinear response theory based on the averaging method. Also, we construct an approximate analytical solution, whereas their model is generic. One would still have the difficult task of obtaining the Floquet eigenfunctions. In addition, the perturbing terms in their prototype example is a parametric drive, whereas in our models the perturbing terms are the added ac signals.

We would like to point out that Bryant and Wiesenfeld [13] (see Fig. 12) obtained an effect similar to the frequency-comb spectral peaks seen here. There are several important differences between our physical systems. Their Duffing oscillator is not driven parametrically and has a very low quality factor, which effectively reduces its dimensionality. They present a suppression of a period-doubling bifurcation, whereas here we have a suppression of parametric instability (supercritical Hopf bifurcation) due to the small added external ac signal. We saw that the suppression increases with the ac signal amplitude. In our system, the frequency-comb behavior occurs more strongly in the bistable region of the bifurcation diagram. Before the jump in amplitude of the central peak, the peaks are spaced by $2\delta$, after the jump the peaks are spaced by $\delta$. The comb peaks are stronger in a narrow region after the jump of the central peak. We have also seen that the parametric suppression is largest just before the central peak jump. That corresponds to the
largest peaks in the weaker form of the frequency comb, in which the peaks are spaced out by $2\delta$.

In the stronger form of the frequency comb, there occurs also a period-doubling bifurcation in the averaged equations, that can be seen in the period-doubling of the envelopes of the pulses. We investigate one simple nonlinear dynamical system to support our claim. The proposed system do not present mode coupling and have lower dimensions than the phenomenological models used to explain the spectral signatures of the experimental mechanical frequency combs. We believe this could lead to further research to find simpler apparatuses, with less parameters, that could deliver frequency-comb-like behavior. We also believe this work could spur more theoretical work to help understand better the connection between the frequency comb behavior and the various bifurcations that occur in conjunction.

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