STABILITY OF THE PARABOLIC POINCARÉ BUNDLE

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Abstract. Given a compact Riemann surface $X$ and a moduli space $M_\alpha(\Lambda)$ of parabolic stable bundles on it of fixed determinant of complete parabolic flags, we prove that the Poincaré parabolic bundle on $X \times M_\alpha(\Lambda)$ is parabolic stable with respect to a natural polarization on $X \times M_\alpha(\Lambda)$.

1. Introduction

Let $X$ be a smooth irreducible complex projective curve of genus $g$, with $g \geq 2$. Fix an integer $r \geq 2$ and a line bundle $L$ over $X$ of degree $d$ such that $r$ and $d$ are coprime. Let $M_{r,L}$ denote the moduli space of isomorphism classes of stable bundles of rank $r$ with $\wedge^r E \cong L$. A vector bundle $V$ over $X \times M_{r,L}$ is called a Poincaré bundle if its restriction $V|_{X \times \{E\}}$ is isomorphic to $E$ for all closed points $[E] \in M_{r,L}$. It is known that a Poincaré bundle exists; moreover, any two of them differ by tensoring with a line bundle pulled back from $M_{r,L}$. Balaji, Brambila-Paz and Newstead proved in [1] that any such Poincaré bundle is stable with respect to any ample divisor in $X \times M_{r,L}$. Recently, Biswas, Gomez and Hoffman studied in [3] the similar question for the moduli space of principal $G$-bundles.

In this short note we consider certain moduli spaces of stable parabolic bundles on $X$. Let us fix a finite set $D$ of $n$ closed points in $X$. We denote by $M_\alpha(\Lambda)$ the moduli space of stable parabolic bundles of rank $r$ on $X$ with fixed determinant $\Lambda$ having full flags at each points of $D$ and rational parabolic weights $\alpha = \{\alpha^j_i\}, 1 \leq j \leq r$ and $1 \leq i \leq n$. In this case it is known that there exists a vector bundle $\mathcal{U}_\alpha$ over $X \times M_\alpha(\Lambda)$ which has a natural parabolic structure over the divisor $D \times M_\alpha(\Lambda)$, and moreover, its restriction to each closed points $[E_\alpha]$ is isomorphic to $E_\alpha$ as parabolic bundle [2]. Any two such bundles differ by tensoring with a line bundle pulled back from $M_\alpha(\Lambda)$. We call such bundles Poincaré parabolic bundles. So, it is natural to ask whether this bundles are parabolic (slope) stable.

We prove the following:

Theorem 1.1. Let $\mathcal{U}$ be a Poincaré parabolic bundle over $X \times M_\alpha(\Lambda)$. Then $\mathcal{U}$ is a parabolic (slope) stable bundle with respect to a natural ample divisor.

We adopt the strategy of proof in [1] in the given context.

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2. Preliminaries

Let $X$ be an irreducible smooth complex projective curve of genus $g \geq 2$. Fix $n$ distinct points $x_1, \cdots , x_n$ on $X$, and denote the divisor $x_1 + \cdots + x_n$ on $X$ by $D$. Let $E$ be a holomorphic vector bundle on $X$ of rank $r$.

A quasi-parabolic structure over $E$ is a strictly decreasing filtration of linear subspaces

$$E_{x_i} = F_i^1 \supset F_i^2 \supset \cdots \supset F_i^{k_i} \supset F_i^{k_i+1} = 0$$

for every $x_i \in D$. We set

$$r^i_j := \dim F^j_i - \dim F^{j+1}_i.$$

The integer $k_i$ is called the length of the flag and the sequence $(r^i_1, r^i_2, \cdots , r^i_{k_i})$ is called the type of the flag at $x_i$. A parabolic structure on $E$ over the divisor $D$ is a quasi-parabolic structure as above together with a sequence of real numbers

$$0 \leq \alpha^i_1 < \alpha^i_2 < \cdots < \alpha^i_{k_i} < 1.$$

The parabolic degree of $E$ is defined to be

$$\text{par-deg}(E) := \deg(E) + \sum_{x_i \in D} \sum_{j=1}^{k_i} \alpha^i_j r^i_j$$

and the parabolic slope of $E$ is

$$\text{par-}\mu(E) := \frac{\text{par-deg}(E)}{r}.$$

(See [10].)

For any subbundle $F \subseteq E$, there exists an induced parabolic structure on $F$ whose quasi-parabolic filtration over $x_i$ is given by the distinct subspaces in

$$F_{x_i} = F^1_i \cap F_{x_i} \supset F^2_i \cap F_{x_i} \supset \cdots \supset F^{k_i}_i \cap F_{x_i} \supset 0,$$

where $k_0 := \max\{j \in \{1, \cdots , k_i\} \mid F^j_i \cap F_{x_i} \neq 0\}$; the parabolic weight of $F^j_i \cap F_{x_i}$ is the maximum of all $\alpha^i_j$ such that $F^j_i \cap F_{x_i} = F^j_i \cap F_{x_i}$.

A parabolic vector bundle $E$ with parabolic structure over $D$ is said to be stable (respectively, semistable) if for every subbundle $0 \neq F \subsetneq E$ equipped with the induced parabolic structure, we have

$$\text{par-}\mu(F) < \text{par-}\mu(E) \quad (\text{respectively}, \text{par-}\mu(F) \leq \text{par-}\mu(E)).$$

2.1. Poincaré parabolic bundle. Fix integers $r > 1$ and $d$ and for each $i = 1, 2, \cdots , n$ a sequence of positive integers $\{r^i_j\}_{j=1}^{k_i}$ such that $\sum_{j=1}^{k_i} r^i_j = r$ for each $i$. Then the coarse moduli space $M_X(d, r, \{\alpha^i_j\}, \{r^i_j\})$ of semistable parabolic vector bundles of rank $r$, degree $d$, flag types $\{r^i_j\}$ and parabolic weights $\{\alpha^i_j\}_{j=1}^{k_i}$ at $x_i \in D$, $1 \leq i \leq n$, is a normal projective variety [10]. The open subvariety $M_X(d, r, \{\alpha^i_j\}, \{r^i_j\})^s$ of it consisting of stable parabolic bundles is smooth.

For a scheme $S$, let $\pi_X : X \times S \to X$ and $\pi_S : X \times S \to S$ be the natural projections. For a vector bundle $U$ over $X \times S$, and $s \in S$, set $U_s := U|_{X \times \{s\}}$. Given a flag type $m_i = (r_1, \cdots , r_{k_i})$, $1 \leq i \leq n$, with $\sum_{j=1}^{k_i} r_j = r$, define $F_{m_i}$ to be the variety of
flags of type $m_i$. Furthermore, for a vector bundle $U \to S$ of rank $r$, let $\mathcal{F}_{m_i}(U) \to S$ be the bundle of flags of type $m_i$.

For each $x_i \in D$ we fix the flag type $m_i = (r_1^i, r_2^i, \ldots, r_k^i)$. A family of quasi-parabolic vector bundles parametrized by a scheme $S$ is defined to be a vector bundle $U$ over $X \times S$ together with sections $\phi_{x_i} : S \to \mathcal{F}_{m_i}(U|_{x_i \times S})$, $1 \leq i \leq n$. Note that the section $\phi_{x_i}$ corresponds to a flag of subbundles of $U|_{x_i \times S}$ with flag type $m_i$ for each $i$. A family of parabolic bundles is given by associating weights $\{\alpha_j^i\}$ to each flag of subbundles over $x_i \times S$, $x_i \in D$. We denote the family of parabolic bundles by $U_* = (U, \phi, \alpha)$ and by $U_{s,*}$ the parabolic bundle $(U_s, \phi_s, \alpha)$ above $s \in S$.

It is known that if the elements of the set $\{d, r_j^i | 1 \leq i \leq n, 1 \leq j \leq k_i\}$ have greatest common divisor equal to one then $M^*_\alpha := M_X(d, r, \{\alpha_j^i\}, \{r_j^i\})^s$ is a fine moduli space, meaning there exists a family $\mathcal{U}_\alpha := (U, \phi, \alpha)$ parametrized by $M^*_\alpha$ with the property that $\mathcal{U}_{\alpha,s}$ is a stable parabolic bundle isomorphic to $E_s$ for all $[E_s] = e \in M^*_\alpha$ [5, Proposition 3.2], [2]. Moreover, if the parabolic weights $\{\alpha_j^i\}$ are chosen to be generic, i.e., the notions of stability and semi-stability coincide, then the moduli space $M_X(d, r, \{\alpha_j^i\}, \{r_j^i\})$ is a smooth, irreducible, projective variety. We denote this variety by $M_\alpha$.

Now assume that the weights are generic, $\alpha_j^i$ are rational numbers and $r_j^i = 1$, so we are choosing full flags at each points of $D$. Note that this is the generic case. There is a well defined determinant morphism $\det : M_\alpha \to J^d(X)$, where $J^d(X)$ denotes the component of the Picard group of $X$ consisting of line bundles of degree $d$. For $\Lambda \in J^d(X)$, denote the fiber $\det^{-1}(\Lambda)$ by $M_\alpha(\Lambda)$, and the restriction of the vector bundle $\mathcal{U}$ to $X \times M_\alpha(\Lambda)$ by (with a mild abuse of notation) $\mathcal{U}$. From the earlier discussions it is clear that the vector bundle $\mathcal{U}$ over $X \times M_\alpha(\Lambda)$ gets a natural parabolic structure over the smooth divisor $D \times M_\alpha(\Lambda)$.

2.2. Strongly Parabolic Higgs Fields. In this subsection we will briefly recall some properties of strongly parabolic Higgs fields and the Hitchin map; for details see [7, Sections 2,3]. As before, we assume that the weights are generic and full flags at each points of $D$.

Let $K_X$ denotes the holomorphic cotangent bundle of $X$.

A parabolic Higgs field on a parabolic vector bundle $E_*$ is a homomorphism
\[
\Phi : E \to E \otimes K \otimes \mathcal{O}_X(D) = E \otimes K(D)
\]
such that

1. $\text{trace}(\Phi) = 0$, and
2. $\Phi$ is a strongly parabolic homomorphism, meaning for each $x_i \in D$ we have $\Phi(F_i^j) \subset F_i^{j+1} \otimes (K(D)|_{x_i})$.

The pair $(E_*, \Phi)$ is called a parabolic Higgs bundle.

A parabolic Higgs bundle $(E_*, \Phi)$ is called stable (respectively, semistable) if for all proper non-zero $\Phi$-invariant sub-bundles $F$ of $E$, we have $\text{par-}\mu(F) < \text{par-}\mu(E)$ (respectively, $\text{par-}\mu(F) \leq \text{par-}\mu(E)$). The cotangent space at $[E] \in M_\alpha(\Lambda)$ can be identified with $H^0(X, \mathcal{SParEnd}_0(E) \otimes K_X(D))$ where $\mathcal{SParEnd}_0(E)$ is the sheaf of strongly parabolic traceless endomorphisms. Then the coefficients of the characteristic polynomial of $\phi \in H^0(X, \mathcal{SParEnd}_0(E) \otimes K_X(D))$ lie in $W := \bigoplus_{j=2}^{r} H^0(X, K_X^j((j-1)D))$. 
Let \( N_\alpha(\Lambda) \) be the moduli space of isomorphism classes of strongly parabolic stable Higgs bundles with parabolic structures over \( D \) and weights \( \{\alpha_j^i\}_{j=1}^r \) at \( x_i \in D, 1 \leq i \leq n \), with fixed determinant \( \Lambda \). The total space \( T^*M_\alpha(\Lambda) \) of the cotangent bundle is an open subvariety of the moduli space \( N_\alpha(\Lambda) \). The map
\[
\begin{array}{c}
h : N_\alpha(\Lambda) \rightarrow W,
\end{array}
\]
\[
(\epsilon, \phi) \mapsto (\text{trace}(\wedge^2 \phi), \cdots, \text{trace}(\wedge^n \phi))
\]
is proper and surjective; it is called the Hitchin map. If \( s \in W \) such that the corresponding spectral curve \( X_s \) is smooth, then the fiber \( h^{-1}(s) \) is identified with the Prym variety
\[
\text{Prym}_\delta(X_s) = \{L \in J_\delta(X_s) | \text{det}(\pi_*L) \simeq \Lambda\}
\]
associated to \( X_s \), where \( \delta := d - \text{deg}(\pi_*(\mathcal{O}_{X_s})) \).

A parabolic bundle on \( X \) is called **very stable** if there is no non-trivial strongly parabolic Higgs field on it. It is known that, if the genus \( g(X) \geq 2 \), a very stable parabolic bundle is stable. There exist very stable parabolic bundles in any moduli space. In fact the subset of very stable parabolic bundles is a dense open set in \( M_\alpha(\Lambda) \). This follows from the fact the dimension of the nilpotent cone \( h^{-1}(0) \) is same as the dimension of the moduli space \( M_\alpha(\Lambda) \) [6, Corollary 3.10]. Let
\[
S' \subset S := T^*M_\alpha(\Lambda) \cap h^{-1}(0)
\]
be the open subset consisting of all \( (\epsilon, \phi) \in S \) such that \( \phi \) is nonzero. The image of \( S' \) in \( M_\alpha(\Lambda) \) under the forgetful map \( (\epsilon, \phi) \mapsto \epsilon \) will be denoted by \( B \). Note that \( B \) is the non-very stable locus in \( M_\alpha(\Lambda) \). On the other hand, there is a free action of \( \mathbb{C}^* \) on \( S' \); namely the action of any \( c \in \mathbb{C}^* \) sends any \( (\epsilon, \phi) \) to \( (\epsilon, c \cdot \phi) \). Hence we have
\[
\dim M_\alpha(\Lambda) = \dim S = \dim S' > \dim B.
\]
This implies that the complement \( M_\alpha(\Lambda) \setminus B \) is nonempty.

### 2.3. Determinant bundle.
Let \( T \) be a variety. For any coherent sheaf \( \mathcal{E} \) on \( X \times T \), flat over \( T \), let \( \det R\pi_T \mathcal{E} \) denote determinant line bundle defined as:
\[
\{\det R\pi_T \mathcal{E}\}_t := \{\det H^0(X, \mathcal{E}_t)\}^{-1} \otimes \{\det H^1(X, \mathcal{E}_t)\}
\]
for \( t \in T \) ([4], [11], [12]).

Let \( x \in X \) be a fixed closed point of \( X \). We fix rational numbers \( 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1 \) and a positive integer \( k \) such that \( \beta_j := k \cdot \alpha_j \) is an integer for each \( j = 1, \ldots, r \). Set
\[
d_j := \beta_j + 1 - \beta_j, 1 \leq j \leq r,
\]
with the assumption that \( \beta_r + 1 = 1 \). Let \( \mathcal{V}_x^\alpha = (\mathcal{V}_x^\alpha, \alpha = \{\alpha_j\}_{j=1}^r, \phi) \) be a family of rank \( r \) stable parabolic bundles over \( X \) with parabolic divisor \( \{x\} \) parametrized by a variety \( T \) and
\[
\mathcal{V}_x^\alpha|_{x \times T} = \mathcal{F}_{1,x} \supset \mathcal{F}_{2,x} \supset \cdots \supset \mathcal{F}_{r,x} \supset \mathcal{F}_{r+1,x} = (0)
\]
be the full flag of subbundles over \( x \times T \) determined by the section \( \phi \). Set \( L_j := \frac{F_{j+1,x}}{F_{j+1,x}} \). Let \( \Psi : T \rightarrow M_\alpha(\Lambda) \) be the morphism induced by this family. Define a line bundle
\[
\theta_T := (\det R\pi_T \mathcal{V}_x^\alpha)^k \otimes (\mathcal{V}_x^\alpha)^l \otimes \otimes_{j=1}^{d_j} L_j^{d_j}
\]
where \( l \) is a positive integer determined by [12, Equation (*), page 6]. Then there exists a unique (up to algebraic equivalence) ample line bundle \( \Theta_{M_\alpha} \) over \( M_\alpha(\Lambda) \) such that \( \Psi^* \Theta_{M_\alpha} = \theta_T \) [4, [12, Theorem 1.2].
3. PARABOLIC STABILITY OF THE PARABOLIC POINCARÉ BUNDLE

In this section we continue with the notation of the previous section.

Let $X$ be a smooth projective irreducible complex curve of genus $g \geq 2$, $x_1, \ldots, x_n \in X$ distinct points and $D = x_1 + \cdots + x_n$. Let $Y$ be a smooth projective irreducible complex variety.

For each point $x_i \in \text{Supp}(D)$ fix real numbers $0 \leq \alpha_i^1 < \alpha_i^2 < \cdots < \alpha_i^k_i < 1$ and $m_i = (r_i^1, \ldots, r_i^k_i)$, where each $r_i^j$ is a positive integer. Let $U$ be a rank $r$ vector bundle over $X \times Y$ with parabolic structure over the smooth divisor $D \times Y$, of flag types $m_i$ and weights $\{\alpha_i^j\}, 1 \leq i \leq n, 1 \leq j \leq k_i$. Fix ample divisors $\theta_X$ on $X$ and $\theta_Y$ on $Y$. Then for any integers $a, b > 0$, the class $a\theta_X + b\theta_Y$ is ample on $X \times Y$. Let $\Theta := a\theta_X + b\theta_Y$ for some fix integers $a, b > 0$.

**Lemma 3.1.** Suppose that for a general point $x \in X$, the vector bundle $U_x = U|_{\{x\} \times Y}$ is semi-stable with respect to $\theta_Y$ over $Y$, and for a general point $y \in Y$ the parabolic vector bundle $U_y = U|_{X \times \{y\}}$ over $X$ with parabolic divisor $D$ is semi-stable with respect to $\theta_X$. Then the parabolic vector bundle $U$ with parabolic divisor $D \times Y$ is parabolic semi-stable with respect to $\Theta$. Moreover, if $U_x$ is stable or $U_y$ is parabolic stable, then $U$ is also parabolic stable.

**Proof.** The proof essentially follows from the proof of [1, Lemma 2.2]. Let us indicate the modification needed in this case. Let $F \subset U$ be a torsionfree subsheaf. Then it has an induced parabolic structure. To compute the parabolic degrees of $U$ and $F$, one needs to compute the degree of certain vector bundles supported on the smooth divisor $D \times Y$. But this is same as computing the degree of certain subsheaves of $U$ and $F$, which can be done as in [1, Lemma 2.2].

For the rest of this section we assume that the parabolic weights are rational, generic and full flags at each points of $\text{Supp}(D)$.

Set $s = (s_2, \ldots, s_r) \in W$, and let $\pi : X_s \to X$ be the associated spectral cover. Then for $z \in X$, the fiber $\pi^{-1}(z)$ is given by the points $y \in K_X(D)|_z$ which satisfy the polynomial

$$y^r + s_2(z) \cdot y^{r-2} + \cdots + s_r(z) = 0.$$  

Let us denote this polynomial by $f$. The morphism $\pi$ is unramified over $z$ if and only if the resultant $R(f, f')$ of $f$ and its derivative $f'$ are nonzero. Since all $s_j$ vanish over $D$, the ramification locus of $\pi$ contains $D$.

**Lemma 3.2.** Let $X$ be a smooth, irreducible, projective curve of genus $g \geq 2$ and $z \notin \text{Supp}(D)$. There exists a smooth, projective spectral curve $Y$ and finite morphism $\pi : Y \to X$ of degree $r$ which is unramified over $z$.

**Proof.** Since the linear system $|K_X((j-1)D)|$ is base point free outside $D$ and $z \notin \text{Supp}(D)$, there exists $(s_2, \cdots, s_r) \in W$ such that

$$R(f, f')(s_2(z), \cdots, s_r(z)) \neq 0.$$  

Clearly this is an open condition in $W$. Thus there exists a non-empty open subset $V$ of $W$ such that for each $s \in V$, the corresponding spectral cover $X_s \to X$ is unramified.
over $z$. Now, since the genus $g \geq 2$, by [7, Lemma 3.1] the set of points in $W$ where the corresponding spectral curve is smooth is an dense open subset of $W$. Thus we can always choose a spectral curve which is smooth and unramified over $z$. \hfill \Box

**Lemma 3.3.** Let $X_s \longrightarrow X$ be a spectral curve. Let $P^\delta$ be the associated Prym variety, where $\delta := d - \deg(\pi_*(\mathcal{O}_{X_s}))$. Then there is a dominant rational map $f : P^\delta \longrightarrow M_\alpha(\Lambda).

**Proof.** Let $h'$ be the restriction of $h$ to the the total space of the cotangent bundle $T^*M_\alpha(\Lambda)$. Then for any very stable parabolic bundle $E \in M_\alpha(\Lambda)$, the restriction

$$h'_E : T^*_E M_\alpha(\Lambda) \longrightarrow W$$

of $h'$ is surjective (for a proof see [8, Lemma 1.4]). Thus, for any $s \in W$, we have $h'^{-1}(s) \cap T^*_E M_\alpha(\Lambda)$ is nonempty for every very stable parabolic bundle $E \in M_\alpha(\Lambda)$. Consequently, for all $s \in W$, the image of the map $h'^{-1}(s) \longrightarrow M_\alpha(\Lambda)$ contains the dense open set $U$ of all very stable parabolic bundles. Thus the morphism $h'^{-1}(s) \longrightarrow M_\alpha(\Lambda)$ is dominant. Since $h'^{-1}(s) \subseteq h^{-1}(s) \simeq P^\delta$ is an open set, we have a dominant rational map $f : P^\delta \longrightarrow M_\alpha(\Lambda)$. \hfill \Box

Now we discuss the ‘parabolic stability’ of $U$ with respect to a ‘naturally’ defined ample divisor on $X \times M_\alpha(\Lambda)$. For the simplicity of the exposition we assume that $D = x$ (for an arbitrary reduced divisor the same arguments will hold).

**Theorem 3.4.** Let $z \notin \text{Supp}(D)$. Then $U_z$ is semi-stable with respect the ample divisor $\Theta_{M_\alpha}$.

**Proof.** By Lemma 3.2 we get a spectral cover $\pi : Y \longrightarrow X$ which is unramified over $z$. Let $\pi^{-1}(z) = \{y_1, \ldots, y_r\}$, with $y_i$ being distinct points in $Y$.

Let $\pi \times 1 : Y \times P^\delta \longrightarrow X \times P^\delta$ denotes the product morphism. Let $\mathcal{L}$ denote the restriction of a Poincaré line bundle on $Y \times J^\delta(Y)$ to $Y \times P^\delta$. Then the direct image $(\pi \times 1)_*\mathcal{L}$ is a rank $r$ vector bundle and the $O_{X \times P^\delta}$-algebra structure on $(\pi \times 1)_*\mathcal{L}$ defines a section

$$\Phi \in H^0(X \times P^\delta, \text{End}((\pi \times 1)_*\mathcal{L}) \otimes p_X^*K_X(D)).$$

This $\Phi$ induces a parabolic structure on $(\pi \times 1)_*\mathcal{L}$ over $x \times P^\delta$. Thus we have a family of parabolic bundles parametrized by $P^\delta$. Clearly, the rational map $f : P^\delta \longrightarrow M_\alpha(\Lambda)$ is induced by the above family. Let $T^\delta$ be the open set where $f$ is defined. Then $\text{Codim}(P^\delta \setminus T^\delta) \geq 2$.

Let $\mathcal{E} := ((\pi \times 1)_*\mathcal{L})|_{X \times T^\delta}$. Since $M_\alpha(\Lambda)$ is a fine moduli space we have

$$(1 \times f)^*U \simeq \mathcal{E} \otimes p_{T^\delta}^*(L_0)$$

for some line bundle $L_0$ on $T^\delta$. Thus

$$f^*U_z \simeq \bigoplus_{i=1}^r L_{y_i} \otimes L_0$$

on $T^\delta$. Since $\text{Codim}(P^\delta \setminus T^\delta) \geq 2$ and $P^\delta$ is smooth, the line bundles $L_{y_i}$ and $L_0$ uniquely extend over $P^\delta$. The line bundles $L_{y_i}$ are already defined over $P^\delta$. Let $L_0'$ be the unique extension of $L_0$ over $P^\delta$. Since $L_{y_i}$ are algebraically equivalent, it follows that $\bigoplus_{i=1}^r L_{y_i} \otimes L_0'$ is semistable with respect to any ample line bundle on $P^\delta$. Thus if we can find an ample line bundle $H$ over $P^\delta$ such that $H|_{T^\delta} \simeq f^*(\Theta_{M_\alpha}^n)$ for some positive integer $n$, then by [1,
Lemma 2.1], $\mathcal{U}$ is semistable with respect to $\Theta^n_{M_\alpha}$. Hence it is semistable with respect $\Theta_{M_\alpha}$.

We have,

$$f^* \Theta_{M_\alpha} = \Theta^\delta = (\det R\pi T^\delta \mathcal{E})^k \otimes \det(\mathcal{E}_x)^l \otimes \otimes_{j=1}^r L_j^{d_j}.$$  

By [9, Theorem 4.3] we get that

$$(\det R\pi T^\delta \mathcal{E})^k = m \Theta^{p^\delta}_{|T^\delta}$$

for some positive integer $m$, where $\Theta^{p^\delta}$ is the restriction of the canonical theta divisor on $J^\delta(Y)$ to $P^\delta$. Let $M$ be the unique extension of $\det(\mathcal{E}_x)^l \otimes \otimes_{j=1}^r L_j^{d_j}$. Set $H := m \Theta^{p^\delta} \otimes M$. Then for some positive integer $q$, $H^q$ is ample on $P^\delta$. Thus $f^* \Theta_{M_\alpha}^\delta$ is a restriction of an ample line bundle $H$ on $P^\delta$.

As a corollary of Theorem 3.4 and Lemma 3.1 we obtain the main result:

**Theorem 3.5.** The parabolic bundle $\mathcal{U}$ over $X \times M_\alpha(\Lambda)$ is parabolic stable with respect to any integral ample divisor of the form $aD_X + b\Theta_{M_\alpha}$, where $D_X$ is an ample divisor on $X$ and $a, b > 0$.

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