Non-commutative Bloch Theory

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Abstract: For differential operators which are invariant under the action of an abelian group Bloch theory is the preferred tool to analyze spectral properties. By shedding some new non-commutative light on this we motivate the introduction of a non-commutative Bloch theory for elliptic operators on Hilbert C*-modules. It relates properties of C*-algebras to spectral properties of module operators such as band structure, weak genericity of cantor spectra, and absence of discrete spectrum. It applies e.g. to differential operators invariant under a projective group action, such as Schrödinger, Dirac and Pauli operators with periodic magnetic field, as well as to discrete models, such as the almost Matthieu equation and the quantum pendulum.

Key words. Schrödinger operator – periodic magnetic field – spectral theory – Cantor spectrum – non-commutative geometry

1. Introduction

Bloch (or Floquet) theory in its usual form has a long history already. Basically it starts from the fact that partial differential equations with constant coefficients are mapped into algebraic equations by means of the Fourier or Laplace transform. Now, if the coefficients are not constant but just periodic under an abelian (locally compact topological) group one still has the Fourier transform on such groups, mapping functions on the group \( \Gamma \) into functions on the dual group \( \hat{\Gamma} \); the original spectral problem on a non-compact manifold is mapped into a (continuous) sum of spectral problems on a compact manifold (see Section 6). This is what makes Bloch theory an indispensable tool especially for solid state physics, where one describes the motion of non-interacting electrons in a periodic solid crystal by a Schrödinger operator \(-\Delta + V\) on \( L^2(\mathbb{R}^d)\). The potential function \( V \) is the gross electric potential generated by all the crystal ions and thus is periodic under
the lattice given by the crystal symmetry. Bloch theory shows that the spectrum of the periodic Schrödinger operator has band structure in the following sense:

**Definition 1 (band structure).** A subset of the real line has band structure if it is a locally finite union of closed intervals.

Band structure is an essential ingredient of electronic transport in metals and semiconductors. By exploiting Bloch theory and the structure of the Schrödinger operator further one can see that the spectrum is purely absolutely continuous, which is sometimes included in the definition of band structure.

Measurements of crystals often require magnetic fields $b$ (2-form). In quantum mechanics, they are described by a vector potential (1-form) $a$ such that $b = da$ ($B = \text{curl} A$ for the corresponding vector fields). The magnetic Schrödinger operator then reads

$$H = - (\nabla - ia)^2 + V.$$

But, although $b$ is periodic or even constant, $a$ need not be so, and $H$ won’t be periodic. It is therefore necessary to use magnetic translations (first introduced by Zak (1968)) under which $H$ still is invariant. But now, these translations do not commute with each other in general. Therefore ordinary (commutative) Bloch theory does not apply.

Basically, the reason for this failure is that a non-abelian group has no “good” group dual: the set of (equivalence classes of) irreducible representations has no natural group structure whereas the set of one-dimensional representations is too small to describe the group — otherwise it would be abelian.

But although $\hat{\Gamma}$ does not exist any more, the algebra $C(\hat{\Gamma})$ of continuous functions continues to exist in some sense: It is given by the reduced group $C^*$-algebra of $\Gamma$ which is just the $C^*$-algebra generated by $\Gamma$ in its regular representation on itself (on $l^2(\Gamma)$).

Section 3 shows how one can re-formulate ordinary Bloch theory in a way which refrains from using the points of $\hat{\Gamma}$ and relies just on the rôle of $C(\hat{\Gamma})$. From a technical point of view this requires switching from measurable fields of Hilbert spaces to continuous fields which then can be described as Hilbert $C^*$-modules over the commutative $C^*$-algebra $C(\hat{\Gamma})$.

Having done this one can retain the setup but omit the condition of commutativity for the $C^*$-algebra $C(\hat{\Gamma})$. Thus one is lead to non-commutative Bloch theory (Section 4) dealing with elliptic operators on Hilbert $C^*$-modules over non-commutative $C^*$-algebras. The basic task is now to relate properties of the $C^*$-algebra to spectral properties of “periodic” operators. Thus one generalizes spectral results for elliptic operators on compact manifolds as well as results of ordinary Bloch theory:

**Theorem 1.** Isolated eigenvalues of $A$-elliptic operators have $A$-finite eigenprojections, their eigenspaces have finite $\tau$-dimension.

Under certain assumptions they have essential spectrum only (isolated eigenvalues have infinite multiplicity).

See Theorem 6 for exact assumptions (they are fulfilled by Schrödinger operators with periodic magnetic field).

Non-commutative Bloch theory allows to treat continuous and discrete models, i.e. differential and difference operators, on equal footing. It opens the way to apply a result of Choi and Elliott (1990) on weak genericity of Cantor spectra in discrete models to the continuous models also, i.e. to the phenomenon opposite to band structure:
**Definition 2 (Cantor set).** A Cantor set is a subset of a topological space which is nowhere dense (the closure has empty interior) and has no isolated points.

Now the $C^*$-algebras of symmetries determines which of the two opposite spectral types is present:

**Definition 3 (Kadison property).** The Kadison constant $K$ of a $C^*$-algebra $A$ together with a trace $\tau$ is defined by

$$K = \inf \{ \tau(P) \mid 0 \neq P \in A \text{ projection} \}. \quad (1)$$

We say the pair $(A, \tau)$ has the Kadison property if $K > 0$.

**Theorem 2 (band structure).** If $(A, \tau)$ has the Kadison property, then the spectrum of every symmetric $A$-elliptic operator has band structure.

(See Theorem 1.) This applies e.g. to magnetic Schrödinger operators in the case of rational magnetic flux.

Opposite to the Kadison property is the property $RRI_0$ (see Definition 3), and it is a criterion for the opposite spectral type:

**Theorem 3 (Cantor spectrum).** If $(A, \tau)$ has property $RRI_0$ then every $A$-elliptic operator can be approximated arbitrarily well (in norm resolvent sense) by one which has Cantor spectrum.

(See Theorem 5.) The important issue here is that the approximation takes place within a natural $C^*$-algebra generated by symmetries connected to the operator. Approximation within a von Neumann algebra would be pointless, of course. This theorem applies e.g. to magnetic Schrödinger operators on $\mathbb{R}^2$ in the case of irrational flux.

In Section 5 we list examples where non-commutative Bloch theory applies: gauge-periodic elliptic differential operators (Schrödinger, Pauli, Dirac with periodic magnetic field) and difference operators (almost Matthieu, quantum pendulum).

For the convenience of the reader we add an appendix on continuous fields of Hilbert spaces and on Hilbert $C^*$-modules and their GNS representation.

A short overview of this paper appeared in [Gruber (1999)](#).

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2. Commutative Bloch theory

In this section we recall the basic elements of Bloch theory for periodic operators in the geometric context of vector bundles, since even in the scalar case of a magnetic Schrödinger operator one is lead to consider possibly non-trivial complex line bundles. The standard reference for the theory of direct integrals is [Dixmier, 1957, chapter II], for Bloch theory in Euclidean space see [Reed and Simon, 1978, chapter XIII.16].

Our general assumptions are: $X$ is an oriented smooth Riemannian manifold without boundary, $\Gamma$ a discrete abelian group acting on $X$ freely, isometrically, and properly discontinuously. Furthermore, we assume the action to be cocompact in the sense that the quotient $M := X/\Gamma$ is compact.

Next, let $E$ be a smooth Hermitian vector bundle over $X$. 

Example 1 (solid crystals). The main motivating example for our setting comes from solid state physics. Here, \( X = \mathbb{R}^n \) is the configuration space of a single electron \((n = 2, 3)\). It is supposed to move in a crystal whose translational symmetries are described by a lattice \( \mathbb{Z}^n \simeq \Gamma \subset \mathbb{R}^n \), which acts on \( X \) by translations, of course. Note that this does not take into account the point symmetries. \( \Gamma \) could be extended by them but the action would not be free any more. Considering just the translations is enough to achieve the compactness of the quotient \( M \simeq T^n \).

Wave functions of electrons are just complex-valued functions on \( \mathbb{R}^n \), so we can set \( E = \mathbb{R}^n \times \mathbb{C} \). One may also include the spin of the electrons into the picture by choosing the appropriate trivial spinor bundle \( E = \mathbb{R}^n \times \mathbb{C}^k \).

Definition 4 (periodic operator). Assume there is an isometric lift \( \gamma_* \) of the action of \( \gamma \) from \( X \) to \( E \) in the following sense:

\[
\gamma_* : E_x \to E_{\gamma x} \text{ for } x \in X, \gamma \in \Gamma.
\] (2)

This defines an action \( T_\gamma \) on the sections: For \( s \in C^\infty_c(E) \) we define

\[
(T_\gamma s)(x) := \gamma_* s(\gamma^{-1} x) \text{ for } x \in X, \gamma \in \Gamma.
\] (3)

\( (T_\gamma)_{\gamma \in \Gamma} \) induces a unitary representation of \( \Gamma \) in \( L^2(E) \) since \( \gamma_* \) acts isometrically and \( T_\gamma^* = (T_\gamma)^{-1} \).

A differential operator \( D \) on \( \mathcal{D}(D) := C^\infty_c(E) \) is called periodic if, on \( \mathcal{D}(D) \), we have:

\[
\forall \gamma \in \Gamma : [T_\gamma, D] = 0
\] (4)

Example 2 (periodic Schrödinger operator). Given a manifold as described above, we may lift the action to any trivial vector bundle \( E := X \times \mathbb{C}^k \) canonically. If \( D \) is a periodic operator on \( X \) (for example any geometric operator, i.e. defined by the metric on \( X \)) and \( V \in C^\infty(X, M(k, \mathbb{C})) \) a periodic field of endomorphisms, then \( D + V \) is a periodic operator on \( E \).

In the case of a crystal, we choose the Laplacian (which describes the kinetic energy quantum mechanically) and a periodic potential \( V \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) (which describes the electric field of the ions at the lattice sites) to get the periodic Schrödinger operator \( \Delta + V \).

Example 3 (Schrödinger operator with exact periodic magnetic field). Let \( b \in \Omega^2(X) \) be a magnetic field 2-form. In dimension 3 this corresponds (by the Hodge star) to a vector field \( B \), in dimension 2 to a scalar function which may be thought of as the length and orientation of a normal vector \( B \). From physical reasons one has \( \text{div} B = 0 \), i.e. \( db = 0 \). For simplicity we assume that \( b \) is not only closed but exact, so there is \( a \in \Omega^1(X) \) with \( b = da \) \((B = \text{rot} A \text{ for the corresponding vector fields})\). This defines a magnetic Hamiltonian operator

\[
\Delta^a := (d - ia)^*(d - ia)
\] (5)

(the minimally coupled Hamiltonian), where \( d \) is the ordinary differential (corresponding to the gradient) and \( * \) the adjoint of an operator between the Hilbert spaces of \( L^2 \)-functions \( L^2(X) \) and of \( L^2 \)-1-forms \( L^2(X, \Lambda^T^*X) \).
For later convenience we set, for $\gamma \in \Gamma$ and $\omega \in \Omega(X)$, $\gamma^* \omega := (\gamma^{-1})^* \omega$, considering $\gamma^{-1}$ as a map $X \to X$ and using the usual pull-back of forms. This puts the action on forms in a notation compatible with the action on sections \((\mathfrak{F})\) from the preceding definition.

Now, if $b$ is periodic, $a$ does not need to be so: If $b \in \Omega^2(\mathbb{R}^n)$ is constant then $a$ is affine linear. So the translations are no symmetries for the magnetic Hamiltonian. Zak (1968) was the first to define the so-called magnetic translations: Since $d(a - \gamma x a) = da - \gamma x da = b - \gamma x b = 0$, one can (at least if $H^1(X) = 0$) find a function $\chi_\gamma$ with $d\chi_\gamma = a - \gamma x a$.

One may define such a function explicitly by

$$\chi_\gamma(x) := \int_{x_0}^x (a - \gamma x a)$$

which is well-defined if $H^1(X) = 0$. If we now define a gauge function $s_\gamma := e^{i \chi_\gamma}$ then

$$\gamma_{\gamma}(T_{\gamma}) = s_\gamma(T_{\gamma})(x)$$

coming from the lifted action

$$\gamma_{\gamma} : X \times \mathbb{C} \to X \times \mathbb{C},$$

$$\gamma_{\gamma}(x, c) = (\gamma x, s_\gamma(x)c).$$

The commutation relation for the magnetic translations is

$$(T_{\chi_1}T_{\chi_2} s)(x) = s_{\chi_1}(x)s_{\chi_2}(\chi_1^{-1} x)s(\chi_2^{-1} \chi_1^{-1} x)$$

$$= \exp \left( i \left( \int_{x_0}^x a - \gamma x a + \int_{\chi_1 x_0}^{\chi_1^{-1} x} a - \gamma x a \right) \right) s(\chi_2^{-1} \chi_1^{-1} x)$$

$$= \exp \left( i \left( \int_{x_0}^x a - \gamma x a + \int_{\chi_1 x_0}^{\chi_1^{-1} x} \gamma x a - (\gamma_1 \gamma_2) x a \right) \right) s(\chi_2^{-1} \chi_1^{-1} x)$$

$$= \exp \left( i \left( \int_{x_0}^{\chi_1 x_0} (\gamma_1 \gamma_2) x a - \gamma x a + \int_{x_0}^x a - (\gamma_1 \gamma_2) x a \right) \right) s(\chi_2^{-1} \chi_1^{-1} x)$$

$$= \exp \left( i \left( \int_{\gamma_1 x_0}^{\gamma x a} (\gamma_1 \gamma_2) x a - \gamma x a \right) \right) s_{\gamma_1 \gamma_2}(x)s(\chi_2^{-1} \chi_1^{-1} x)$$

$$= \Theta(\gamma_1, \gamma_2)s_{\gamma_1 \gamma_2}(x)s(\chi_2^{-1} \chi_1^{-1} x)$$

$$= \Theta(\gamma_1, \gamma_2)(T_{\gamma_1 \gamma_2} s)(x)$$
with \( \Theta(\gamma_1, \gamma_2) \in S^1 \). In general this is just a projective representation of \( \Gamma \). If \( a \) itself is periodic, then \( \chi_\gamma = 0 \) for \( \gamma \in \Gamma \), i.e. there is no gauge, and we have just ordinary translations forming a proper representation.

But even if \( a \) is not periodic it can happen that the magnetic translations commute with each other. This is called the case of integral flux since the term occurring in the exponential in line (6) is just the magnetic flux through one lattice face. A periodic \( a \) obviously gives rise to zero magnetic flux.

Furthermore, if \( V \in C^\infty(X, \mathbb{R}) \) is \( \Gamma \)-periodic it commutes with the magnetic translations as well, so \( \Delta^a + V \) is a (symmetric elliptic) periodic operator.

Finally, the very same magnetic translations can be used for the Pauli Hamiltonian and the magnetic Dirac operator.

**Remark 1 (integral flux).** In the case of integral flux mentioned above quite opposite spectral phenomena can occur: Periodic Schrödinger operators have absolutely continuous band spectrum, whereas the Landau Hamiltonian on \( \mathbb{R}^2 \) (constant magnetic field, no electric potential) exhibits pure point spectrum of infinite degeneracy. In Gruber (1999b) we show that these are indeed the only phenomena that can occur (although possibly combined) in the case of integral flux.

**Remark 2 (non-integral flux).** If the magnetic flux is rational one can find a super-lattice of \( \Gamma \), i.e. a subgroup of finite index, such that the flux is integral. The quotient will still be compact, of course, so that the rational case can be completely reduced to the integral.

If the magnetic flux is irrational there is no such super-lattice. Still, one may try to make use of the projective representation defined above. There are several approaches, similar in the objects which are used, different in the objectives that are aimed at and accordingly in the results. Our approach will mimic Bloch theory non-commutatively, see Section 3.

**Remark 3 (non-exact magnetic field).** If \( b \) is closed but not exact one first has to agree upon the quantization procedure used. \( \mathfrak{5} \) may be identified as a Bochner Laplacian for a connection with curvature \( b \), and such a connection exists if and only if \( b \) defines an integral cohomology class, i.e. \( [b] \in H^2(X, \mathbb{Z}) \). There may exist different quantizations for the same magnetic field. This is connected to the Bloch decomposition again. For this and the construction of the magnetic translations in this case see Gruber (2000).

**Lemma 1 (associated bundle).** \( E \) is the lift \( \pi^* E' \) of a Hermitian vector bundle \( E' \) over \( M \) by the projection \( \pi : X \to M \). \( E \) and \( X \) are \( \Gamma \)-principal fiber bundles over \( E' \) resp. \( M \).

To every \( \Gamma \)-principal fiber bundle and every character \( \chi \in \hat{\Gamma} \) we associate a line bundle. This gives the relations depicted in diagram \( \mathfrak{1} \) ("\( \rightsquigarrow \)" denotes association of line bundles).

In this situation we have \( E_\chi \cong E' \otimes F_\chi \).

**Proof.** \( E \) is a \( \Gamma \)-principal fiber bundle, so we can use the lifted \( \Gamma \)-action to define \( E' := E/\Gamma \). Since this action is a lift of the \( \Gamma \)-action on \( X \), \( E' \) has a natural structure
of a vector bundle over $M$. If $\pi_{E'} : E' \to M$ is the bundle projection of $E'$, then the pull back by $\pi$ is defined as

$$\pi^* E' = X \times_{\pi} E' = \{(x, e) \in X \times E' \mid \pi(x) = \pi_{E'}(e)\}.$$ 

If $\pi^E : E \to X$ is the bundle projection of $E$ and $\pi_* : E \to E'$ is the quotient map, then we get a bundle isomorphism $E \to \pi^* E'$ by

$$E \ni e \mapsto (\pi^E(e), \pi_*(e)) \in \pi^* E'.$$

Therefore, in this representation the lift $\gamma_*$ of $\gamma$ acts on $(x, e) \in \pi^* E'$ as $\gamma_*(x, e) = (\gamma x, e)$.

Sections into an associated bundle $P \times_{\rho} V$ are just those sections of the bundle $P \times V$ which have the appropriate transformation property. By construction, $E_\chi$ is a complex line bundle over $E'$, but from $E$ it inherits the vector bundle structure, so its sections fulfill:

$$C^\infty(E_\chi) \simeq C^\infty(E)^{\Gamma, \chi} = \{ s \in C^\infty(E) \mid \forall \gamma \in \Gamma : \gamma^* s = \chi(\gamma) s \} \quad \text{(7)}$$

An analogous equation holds for the line bundle $F_\chi$ over $M$. Finally, (7) shows

$$E_\chi = E \times_{\chi} \mathbb{C}$$

$$= (\pi^* E') \times_{\chi} \mathbb{C}$$

$$= (X \times_{\pi} E') \times_{\chi} \mathbb{C}$$

$$\simeq E' \otimes (X \times_{\chi} \mathbb{C})$$

$$= E' \otimes F_\chi.$$ 

Here, all equalities are immediate from the definitions, besides the last but one, which may be seen as follows:

$$(X \times_{\pi} E') \times_{\chi} \mathbb{C} = (X \times_{\pi} E' \times \mathbb{C})/\Gamma$$

with the $\Gamma$-action

$$\gamma(x, e, z) = (\gamma x, e, \chi(\gamma) z),$$

whereas

$$E' \otimes (X \times_{\chi} \mathbb{C}) = E' \otimes ((X \times \mathbb{C})/\Gamma).$$
with the $\Gamma$-action
\[
\gamma(x, z) = (\gamma x, \chi(\gamma)z).
\]
So, both bundles are quotients of isomorphic bundles with respect to the same $\Gamma$-action.

Example 4 (magnetic bundles). Consider again the case of the magnetic translations for a periodic magnetic 2-form $b \in \Omega^2(X)$, $E$ being a complex line bundle with curvature $b$ ($b \in H^2(X, \mathbb{Z})$). Hence we have $c_1(E) = [b]$ for the Chern class (up to factors of $2\pi$, depending on the convention). Since $b$ is periodic we may restrict it to a form $b_M \in \Omega^2(M)$ on the quotient. The existence of the lifted action, i.e. the fact that $E$ can be written as a pull-back $E = \pi^*E'$, corresponds to the integrality of $b_M$ from $c_1(E') = [b_M] \in H^2(M, \mathbb{Z})$. Tensoring $E'$ with the flat line bundle $F_\chi$ does not change the Chern class (up to torsion). In particular, in dimension 2 the integrality of $b_M$ is equivalent to the integrality of the flux, and $E'$ is trivial only for zero flux.

Next we want to decompose the Hilbert space $L^2(E)$ of square-integrable sections of $E$ into a direct integral over the character space $\hat{\Gamma}$. On $\hat{\Gamma}$ we use the Haar measure. From the theory of representations of locally compact groups we need the following character relations for abelian discrete $\Gamma$, i.e. for abelian, compact $\hat{\Gamma}$ (see e.g. Rudin, 1962, §1.5):

Lemma 2 (character relations). For $\gamma \in \Gamma$
\[
\int_{\hat{\Gamma}} \chi(\gamma) \, d\chi = \begin{cases} 1, & \gamma = e, \\ 0, & \gamma \neq e. \end{cases}
\] (8)

For $\chi, \chi' \in \hat{\Gamma}$
\[
\sum_{\gamma \in \Gamma} \bar{\chi}(\gamma)\chi'(\gamma) = \delta(\chi - \chi')
\] (9)
in distributional sense, i.e. for $f \in C(\hat{\Gamma})$
\[
\sum_{\gamma \in \Gamma} \int_{\hat{\Gamma}} \bar{\chi}(\gamma)\chi'(\gamma) f(\chi) \, d\chi = f(\chi').
\]

We define for every character $\chi \in \hat{\Gamma}$ a mapping $\Phi_\chi : C^\infty_c(E) \ni s \mapsto \tilde{s}_\chi \in C^\infty(E)$ by
\[
\tilde{s}_\chi(x) := \sum_{\gamma \in \Gamma} \chi(\gamma)\gamma^*s(\gamma^{-1}x).
\] (10)

Since
\[
\tilde{s}_\chi(\gamma'x) = \sum_{\gamma \in \Gamma} \chi(\gamma)\gamma^*s(\gamma^{-1}\gamma'x)
\]
\[
= \sum_{\gamma \in \Gamma} \chi(\gamma')\gamma'^{-1}\gamma(\gamma'\gamma'^{-1}\gamma)_\ast s((\gamma')^{-1}x)
\]
\[
= \chi(\gamma')\gamma'^\ast\tilde{s}_\chi(x)
\]
we have
\[ \tilde{s}_\chi \in C^\infty(E)^\Gamma : x = \{ r \in C^\infty(E) \mid \forall_{\gamma \in \Gamma} T_\gamma r = \chi(\gamma) r \} \]
which defines a section \( s_\chi \in C^\infty(E_\chi) \).

Let \( D \) be a fundamental domain for the \( \Gamma \)-action, i.e. an open subset of \( X \) such that \( \bigcup_{\gamma \in \Gamma} \gamma D = X \) up to a set of measure 0 and \( \gamma D \cap D = \emptyset \) for \( \gamma \neq e \). Then
\[
\int \| \tilde{s}_\chi \|_{L^2(E_\chi)}^2 \, d\chi = \int \int_D \sum_{\gamma_1, \gamma_2 \in \Gamma} \chi(\gamma_1^{-1} \gamma_2) \langle \gamma_1 \cdot s(\gamma_1^{-1} x) | \gamma_2 \cdot s(\gamma_2^{-1} x) \rangle_E \, d\chi \, dx
\]
\[
= \int \sum_{\gamma \in \Gamma} |s(\gamma^{-1} x)|^2 \, dx
\]
\[
= \| s \|_{L^2(E)}^2.
\]
On the one hand, this shows that we can define a measurable structure on \( \prod_{\chi \in \hat{\Gamma}} L^2(E_\chi) \) by choosing a sequence in \( C^\infty_c(E) \) which is total in \( L^2(E) \). On the other hand, we can see that the direct integral \( \int \oplus_{\hat{\Gamma}} L^2(E_\chi) \, d\chi \) is isomorphic to \( L^2(E) \) via the isometry \( \Phi \), whose inverse is given by
\[
\Phi^*: (s_\chi)_{\chi \in \hat{\Gamma}} \mapsto \int \tilde{s}_\chi(x) \, d\chi,
\]
as is easily seen from the character relations (8) and (9).

This shows

**Lemma 3 (direct integral).** The mapping defined by (10) can be extended continuously to a unitary
\[
\Phi : L^2(E) \to \int \oplus_{\hat{\Gamma}} L^2(E_\chi) \, d\chi.
\]  

For the direct integral of Hilbert spaces \( H = \int \oplus \, H_\chi \, d\chi \) the set of decomposable bounded operators \( L^\infty(\hat{\Gamma}, \mathcal{L}(H)) \) is given by the commutant \( (L^\infty(\hat{\Gamma}, \mathbb{C}))' \) in \( \mathcal{L}(H) \). Since commutants are weakly closed and \( C(\hat{\Gamma}, \mathbb{C}) \) is weakly dense in \( L^\infty(\hat{\Gamma}, \mathbb{C}) \) one has \( (L^\infty(\hat{\Gamma}, \mathbb{C}))' = (C(\hat{\Gamma}, \mathbb{C}))' \). Therefore, in order to determine the decomposable operators one has to determine the action of \( C(\hat{\Gamma}) \) on \( L^2(E) \). This is easily done using the explicit form of \( \Phi \):

**Proposition 1 (\( C(\hat{\Gamma}) \)-action).** \( f \in C(\hat{\Gamma}) \) acts on \( s \in C_c^\infty(E) \) by
\[
M_f s := \Phi^* f \Phi s,
\]
and one has
\[
(M_f s)(x) = \sum_{\gamma \in \Gamma} \hat{f}(\gamma^{-1}) T_\gamma s(x), \text{ where}
\]
\[
\hat{f}(\gamma) := \int_{\hat{\Gamma}} f(\chi) \bar{\chi}(\gamma) \, d\chi
\]
is the Fourier transform of \( f \). \( M_f \) is a bounded operator with norm \( \| f \|_\infty \).
Proof. For \( x \in X \) one has:

\[
(Mf s)(x) = (\Phi^* f \Phi s)(x) \\
= \int_{\hat{\Gamma}} (f \Phi s)_{\chi}(x) \, d\chi \\
= \int_{\hat{\Gamma}} f(\chi) \sum_{\gamma \in \Gamma} \chi(\gamma) \gamma s(\gamma^{-1} x) \, d\chi \\
= \sum_{\gamma \in \Gamma} \hat{f}(\gamma^{-1}) \gamma s(\gamma^{-1} x)
\]

Since \( f \) is a multiplication operator in each fiber it has fiber-wise norm \( \| f \|_{\infty} \), and so have \( f \) and \( Mf = \Phi^* f \Phi \).

Corollary 1 (decomposable operators). Conjugation by \( \Phi \) defines an isomorphism between decomposable bounded operators on \( \int_{\hat{\Gamma}} \mathbb{C}(\hat{\Gamma}) \mathbb{L}^2(E_{\chi}) \, d\chi \) and \( \Gamma \)-periodic bounded operators on \( \mathbb{L}^2(E) \).

Proof.

\[\Rightarrow\] A decomposable operator commutes with the \( C(\hat{\Gamma}) \)-action, especially with \( f_{\gamma} \in C(\hat{\Gamma}) \) which is defined by

\[
\hat{f}_{\gamma}(\gamma') := \begin{cases} 
1, & \text{if } \gamma = \gamma', \\
0, & \text{else.}
\end{cases}
\]

By \( \{f_{\gamma}\} \) commuting with \( f_{\gamma} \) is equivalent to commuting with \( \gamma \).

\[\Leftarrow\] To commute with the \( \Gamma \)-action means to commute with all \( f_{\gamma} \) for \( \gamma \in \Gamma \). Because of

\[
f_{\gamma}(\chi) = \chi(\gamma)
\]

the \( f_{\gamma} \) are just the characters \( \hat{\Gamma} \) of the compact group \( \hat{\Gamma} \), and by the Peter-Weyl theorem (or simpler: by the Stone-Weierstraß theorem) they are dense in \( C(\hat{\Gamma}) \). Since the operator norm of \( Mf \) and the supremum norm of \( f \) coincide the commutation relation follows for all \( f \in C(\hat{\Gamma}) \) by continuity.

An unbounded operator is decomposable if and only if its (bounded) resolvent is decomposable. For a periodic symmetric elliptic operator \( D \) we have a domain of definition \( D(D) = \mathbb{C}_{C_c}(X) \) on which \( D \) is essentially self-adjoint. This domain is invariant for \( D \) as well as for the \( \Gamma \)-action, and one has \( [D, \gamma] = 0 \) for all \( \gamma \in \Gamma \). Thus all bounded functions of \( D \) commute with the \( \Gamma \)-action, and one has:

Theorem 4 (decomposition of periodic operators). The closure \( \overline{D} \) of every periodic symmetric elliptic operator \( D \) is decomposable with respect to the direct integral of Hilbert spaces \( \int_{\hat{\Gamma}} L^2(E_{\chi}) \, d\chi \). A core for the domain of \( D_{\chi} \) is given by \( \mathbb{C}_{C_c}(E_{\chi}) \), and the action of \( D_{\chi} \) on \( \mathbb{C}_{C_c}(E_{\chi}) \) is \( C_{C_c}(E_{\chi}) \mathbb{L}^2(\hat{\Gamma}) \) is just the action of \( D \) as differential operator on \( \mathbb{C}_{C_c}(E_{\chi}) \mathbb{L}^2(\hat{\Gamma}) \). We have \( D_{\chi} = \overline{D_{\chi}} \), where

\[
D_{\chi} := D|_{\mathbb{C}_{C_c}(E_{\chi}) \mathbb{L}^2(\hat{\Gamma})}
\]

and the closures are to be taken as operators in \( L^2(E_{\chi}) \).
Proof. Given the remark above we have shown the decomposability already. 
$C^\infty(X)$ is a core for $\bar{D}$, its image under $\Phi_\chi$ is contained in $C^\infty(E)^{\Gamma,\chi}$ and is a core for $\bar{D}_\chi$, since $\Phi$ is a isometry. On this domain $\Phi$ gives the action of $\bar{D}_\chi$ as asserted in the theorem. Since $\bar{D}_\chi$ is a symmetric elliptic operator on the compact manifold $M$ it is essentially self-adjoint. $\bar{D}_\chi$ is a fiber of $\bar{D}$ (which is self-adjoint by, e.g., [Atiyah, 1976]) and therefore self-adjoint, thus both define the same unique self-adjoint extension $D_\chi$ of $\bar{D}_\chi$. 

In passing we harvest a corollary which we will not use in the sequel, but which is well known in the Euclidean setting:

**Corollary 2 (reverse Bloch property).** Every symmetric elliptic abelian periodic operator has the reverse Bloch property, i.e. to every $\lambda \in \text{spec } \bar{D}$ there is a bounded generalized eigensection $s \in C^\infty(E)$ with $Ds = \lambda s$.

**Proof.** If $\lambda \in \text{spec } \bar{D}$ then, by the general theory for direct integrals,

$$\{ \chi \in \hat{\Gamma} \mid (\lambda - \epsilon, \lambda + \epsilon) \cap \text{spec } \bar{D}_\chi \neq \emptyset \}$$

has positive measure for every $\epsilon > 0$. The fibers $\bar{D}_\chi$ are elliptic operators on a compact manifold and thus have discrete spectrum; the eigenvalues depend continuously on $\chi$ (even piece-wise real-analytically; see below). We choose a sequence $(\chi_n)_{n \in \mathbb{N}}$ with $(\lambda - 1/n, \lambda + 1/n) \cap \text{spec } \bar{D}_{\chi_n} \neq \emptyset$, so that there is an accumulation point $\chi_\infty$ ($\hat{\Gamma}$ is compact), and $\lambda \in \text{spec } \bar{D}_{\chi_\infty}$ due to continuity.

Since $\text{spec } \bar{D}_{\chi_\infty}$ is discrete $\lambda$ is an eigenvalue of $\bar{D}_{\chi_\infty}$. The lift of an eigensection (which is smooth due to ellipticity) lies in $C^\infty(E)^{\Gamma,\chi}$ and therefore is bounded. Furthermore the lift satisfies the same eigenvalue equation because of (15). 

3. Commutative Bloch theory from a non-commutative point of view

By Gelfand’s representation theorem every commutative $C^*$-algebra $A$ is isomorphic to $C^\infty(X)$, the continuous functions vanishing at infinity of a topological Hausdorff space $X$, where $X$ is the spectrum $\hat{A}$ of $A$, i.e. the set of equivalence classes of irreducible unitary representations (see e.g. [Murphy, 1990]): the $C^*$-norm is given by the supremum norm, the involution by point-wise complex conjugation. Hilbert $A$-modules are given by the sections $C_\infty(H)$ of a continuous field of Hilbert spaces over $X$, finitely generated projective $A$-modules are given by the sections $C^\infty(E)$ of a vector bundle $E$ over $X$ [Swan, 1962]. In this section we describe the corresponding structures in the case of periodic elliptic differential operators, so that we can find a formulation of Bloch theory that avoids using the points of the space $\hat{\Gamma}$ and relies solely on the algebraic structures with respect to $C(\hat{\Gamma})$.

In Proposition 1 we already determined the action of $C(\hat{\Gamma})$ on $L^2(E)$. Now we use the scalar product that is given in each fiber by the direct integral to define a $C(\hat{\Gamma})$-values scalar product:

**Definition and proposition 5 (pre-Hilbert $C(\hat{\Gamma})$-module)** For $s_1, s_2 \in C^\infty_c(E)$ we define by

$$\langle s_1 \mid s_2 \rangle(\chi) := \langle \langle \Phi s_1 \rangle_\chi \mid \langle \Phi s_2 \rangle_\chi \rangle_{L^2(E_\chi)}$$

(16)
a $C(\hat{\Gamma})$-valued scalar product that makes $C_c(E)$ into a pre-Hilbert $C^*$-module over $C(\hat{\Gamma})$; it is a submodule of the $C(\hat{\Gamma})$-module $L^2(E)$.

Proof. $C_c(E)$ is obviously a $C(\hat{\Gamma})$-submodule of $L^2(E)$. Furthermore, by definition the scalar product is

$$\langle s_1|s_2 \rangle (\chi) = \langle (\Phi s_1)\chi|(\Phi s_2)\chi \rangle_{L^2(E_\chi)}$$

$$= \sum_{\gamma,\gamma' \in \Gamma} \hat{\chi}(\gamma)\hat{\chi}(\gamma') \int_D \langle \gamma s_1(\gamma^{-1}x)|\gamma' s_2(\gamma'^{-1}x) \rangle_{E_\chi} \, dx$$

$$= \sum_{\gamma,\gamma' \in \Gamma} \chi(\gamma^{-1}\gamma') \int_{\gamma^{-1}D} \langle s_1(y)|\gamma s_2(\gamma'^{-1}y) \rangle_{E_\chi} \, dy$$

$$= \sum_{\gamma,\gamma' \in \Gamma} \chi(\gamma') \int_{\gamma^{-1}D} \langle s_1(y)|\gamma' s_2(\gamma'^{-1}y) \rangle_{E_\chi} \, dy$$

$$= \sum_{\gamma' \in \Gamma} \chi(\gamma') \langle s_1|T_{\gamma'}s_2 \rangle_{L^2(E)}$$

(17)

and therefore continuous in $\chi$, since the last sum in (17) is finite. The *-property is immediately clear, the $C(\hat{\Gamma})$-linearity of the scalar product follows from

$$\langle s_1|Mfs_2 \rangle (\chi) = \langle (\Phi s_1)\chi|(\Phi \Phi^* f s_2)\chi \rangle_{L^2(E_\chi)}$$

$$= \langle (\Phi s_1)\chi|(f s_2)\chi \rangle_{L^2(E_\chi)}$$

$$= \langle (\Phi s_1)\chi|f(\chi)s_2\chi \rangle_{L^2(E_\chi)}$$

$$= \langle (\Phi s_1)\chi|(\Phi s_2)\chi \rangle_{L^2(E_\chi)} f(\chi)$$

$$= \langle s_1|Mfs_2 \rangle (\chi)f(\chi).$$

$\square$

(17) is the Fourier transform of the map $\gamma \mapsto \langle s_1|T_\gamma s_2 \rangle$ and will lead us on the right track for the construction of a suitable Hilbert $C^*$-module in the non-commutative example of gauge-periodic elliptic operators (see Lemma 8).

In appendix B we describe how – for arbitrary (i.e. non-commutative) $C^*$-algebras – a $C^*$-valued scalar product on a $\mathcal{A}$-module together with the $C^*$-norm on $\mathcal{A}$ defines a Banach norm on the $\mathcal{A}$-module. The $C^*$-norm on $C(\hat{\Gamma})$ is the supremum norm, so that in this case the Banach norm $\|\cdot\|_{\mathcal{E}'}$ on $\mathcal{E}' := C_c(E) \ni s$ is given by

$$\|s\|_{\mathcal{E}'} := \sup_{\chi \in \hat{\Gamma}} \langle s|s \rangle (\chi).$$

We can take the closure $\mathcal{E}'$ with respect to this norm, and hence make $\mathcal{E}'$ into a $C^*$-module over $\hat{\Gamma}$:
Definition and proposition 6 (Hilbert $C(\hat{\Gamma})$-module and GNS representation) We denote the closure of $C_c(E)$ as Hilbert $C(\hat{\Gamma})$-module by $\mathcal{E}$. $\mathcal{E}$ is a submodule of the $C(\hat{\Gamma})$-module $L^2(E)$. The Haar measure on $\hat{\Gamma}$ defines a faithful trace $\tau$ on $C(\hat{\Gamma})$, and the corresponding GNS representation $\pi_{\tau}$ (see appendix B) of $\mathcal{E}$ is just the original $C(\hat{\Gamma})$-action on $L^2(E)$.

Proof. Since
\[
\| \langle s_1 | s_2 \rangle_{\mathcal{E}} \|_{L^\infty(\hat{\Gamma})} \geq \| \langle s_1 | s_2 \rangle_{\mathcal{E}} \|_{L^1(\hat{\Gamma})} \geq | \langle s_1 | s_2 \rangle_{L^2(E)} |,
\]
the closure of $C_c(E)$ in the $\mathcal{E}$-norm is a subspace of $L^2(E)$, and by definition a $C(\hat{\Gamma})$-module.

The integral with respect to a measure defines a trace. Since $\hat{\Gamma}$ is compact ($\Gamma$ is discrete) it has finite volume with respect to Haar measure, so that the trace is finite, and all $f \in C(\hat{\Gamma}) \subset L^1(\hat{\Gamma})$ are trace class. Since $\hat{\Gamma}$ has no open subsets of Haar measure zero the trace is faithful. We can compute the scalar product that is defined by $\tau$ for $s_1, s_2 \in \mathcal{E}$ as follows:
\[
\langle s_1 | s_2 \rangle_{\mathcal{E}} \overset{\text{def}}{=} \tau \langle s_1 | s_2 \rangle_{\mathcal{E}} = \int_{\hat{\Gamma}} \sum_{\gamma \in \Gamma} \chi(\gamma) \langle s_1 | T_{\gamma} s_2 \rangle_{L^2(E)} d\chi.
\]
Since $\mathcal{E} \supset C_c(E)$ is dense in $L^2(E)$ with respect to the $L^2$-norm and therefore with respect to the norm generated by $\tau$, the GNS representation space for $\tau$ is $L^2(E)$. Hence, the module structures coincide.

Proposition 2 (continuous field of Hilbert spaces over $\hat{\Gamma}$). The continuous field of Hilbert spaces over $\hat{\Gamma}$ that corresponds to $\mathcal{E}$ (see appendix A) has the fiber $L^2(E_\chi)$ over $\chi$, the continuity structure is defined by $\mathcal{E}$.

Proof. We get the fiber at $\chi$ as GNS representation space of the state $\pi_\chi : C(\hat{\Gamma}) \ni f \mapsto f(\chi)$. For the continuity structure see A.

To sum up: We have replaced the decomposition of $L^2(E)$ into a direct integral of Hilbert spaces over the space $\hat{\Gamma}$ by a Hilbert $C^\ast$-module over the $C^\ast$-algebra $C(\hat{\Gamma})$, endowed with a faithful trace whose GNS representation gives us back the original Hilbert space $L^2(E)$. In Proposition 6 we determined the $C(\hat{\Gamma})$-action and noticed that decomposable bounded operators with respect to the direct integral are just the ones commuting with this action (the periodic operators). This, decomposable operators are just the module maps on the $C(\hat{\Gamma})$-module $L^2(E)$. This includes especially the images (under the GNS representation) of module maps on $\mathcal{E}$. To conclude this section we cite a special case of Theorem 11 from Section 5 that shows that periodic elliptic differential operators define indeed regular unbounded module maps (see e.g. Lance, 1995, chapter 9 for these notions) on $\mathcal{E}$, so that the resolvent of such operators belongs to the image of the GNS representation.
Theorem 5 (decomposition of periodic operators). Let \( D \) be a periodic symmetric elliptic differential operator. Then \( D \) defines a regular operator \( D_E \) with domain of definition \( D(D_E) = C^\infty_c(E) \) in \( E \). For \( \lambda \in \mathbb{R} \) we have
\[
\pi_\tau \left( (\lambda 1_E + \overline{D_E})^{-1} \right) = (\lambda 1_{L^2(E)} + \overline{D})^{-1}.
\] (18)

4. Non-commutative Bloch theory

Motivated by the non-commutative insight gained in the previous section, we will now define a general class of abstract elliptic operators that allows for a non-commutative version of Bloch theory. This will let us read off spectral properties from properties of the \( C^* \)-algebras that are involved.

Definition 7 (\( A \)-elliptic operator). Let \( A \) be a unital \( C^* \)-algebra, \( E \) a Hilbert \( C^* \)-module over \( A \). An unbounded operator \( D \) on \( E \) is called \( A \)-elliptic if
1. \( D \) is densely defined,
2. \( D \) is regular in the sense that \( D \) has a densely defined adjoint \( D^* \) with range \( \text{ran}(1+D^*D) \) dense \( \subset E \), and
3. \( D \) has \( A \)-compact resolvent\(^1\), i.e. \((1+D^*D)^{-1} \in K_A(E)\).

Hilbert modules are understood to be Hilbert right modules, as described in appendix \( B \). Hilbert spaces are Hilbert \( \mathbb{C} \)-modules, therefore our scalar products are complex linear in the second entry and complex anti-linear in the first entry, corresponding to the convention in Mathematical Physics.

Remark 4 (module and Hilbert space operators). Given a normalized faithful trace \( \tau \) on \( A \) we can define, as described in appendix \( B \), a Hilbert space scalar product on \( E \) by
\[
\langle e_1|e_2 \rangle_\tau := \tau(\langle e_1|e_2 \rangle_E)
\]
for \( e_1, e_2 \in E \). Let \( H_\tau \) be the completion of \( E \) with respect to \( \langle \cdot | \cdot \rangle_\tau \), i.e. the corresponding GNS representation space. We write \( \langle \cdot | \cdot \rangle_{H_\tau} \) for \( \langle \cdot | \cdot \rangle_\tau \), \( L_A(E) \) is represented faithfully on \( H_\tau \). Thus the spectrum of an element \( a \) of the \( C^* \)-algebra \( L_A(E) \) coincides (as a set) with the spectrum of the operator \( \pi_\tau(a) \) on the Hilbert space \( H_\tau \):

Lemma 4 (spectrum of module and Hilbert space operators). If \( a \in L_A(E) \) then
\[
\text{spec } a = \text{spec } \pi_\tau(a).
\]
In the sequel we will identify \( E \) resp. \( L_A(E) \) with the images in \( H_\tau \) resp. \( L(H_\tau) \).

Definition and proposition 8 (\( \text{tr}_\tau \)-trace) On the \( A \)-finite operators\(^2\) \( \mathcal{F}_A(E) \) we define a faithful trace by
\[
\text{tr}_\tau(\pi^E_{x,y}) = \tau(\langle y|x \rangle_E),
\]
(19)

\(^1\) We will explain this name in the proof of Lemma 4.

\(^2\) We define \( \mathcal{F}_A(E) = \text{span} \{ \pi^E_{x,y} | x, y \in E \} \) with \( \pi^E_{x,y}(z) = x \langle y|z \rangle_E \) for \( z \in E \), so that \( K_A(E) = \overline{\mathcal{F}_A(E)} \).
the trace associated to $\tau$ in the GNS representation. We denote the corresponding trace class ideal in $L_A(E)$ by $L^1_A(E, \text{tr}_\tau)$.

**Proof.** For the generators of $F_A(E)$ one can easily show the relations

\[
\begin{align*}
(\pi^E_{x,y})^* &= \pi^E_{y,x}, & \pi^E_{x,y} &= \pi^E_{x,y}^*, \\
T \pi^E_{x,y} &= \pi^E_{Tx,y}, & \pi^E_{x,y} T &= \pi^E_{x,Ty}
\end{align*}
\]

for $x, y \in E, a \in A, T \in L_A(E)$. Thus, from the trace property of $\tau$ we have

\[
\begin{align*}
\text{tr}_\tau ((\pi^E_{x,y})^*) &= \tau (\langle x | y \rangle_E) \\
&= (\text{tr}_\tau \pi^E_{x,y})^*, \\
\text{tr}_\tau (T \pi^E_{x,y}) &= \tau (\langle y | Tx \rangle_E) \\
&= \tau (\langle T^* y | x \rangle_E) \\
&= \text{tr}_\tau (\pi^E_{x,y} T).
\end{align*}
\]

For all $z, t \in E$ we have

\[
\begin{align*}
\text{tr}_\tau (\pi^E_{x,y} \pi^E_{z,t}) &= \text{tr}_\tau \left(\pi^E_{x,y} \langle z | t \rangle_{E, t} \right) \\
&= \tau (\langle t | x \langle y | z \rangle_E \rangle_{E}) \\
&= \tau (\langle t | \pi^E_{x,y} (z) \rangle_{E})
\end{align*}
\]

so that $\text{tr}_E$ is faithful: Set $t = \pi^E_{x,y} (z)$, and note that $\tau$ is a faithful trace on $A$. \hfill \square

**Remark 5 (tr $\pi^H_{x,y}$ versus tr $\pi^E_{x,y}$).** By Definition 8 we have

\[
\begin{align*}
\text{tr}_\tau \pi^E_{x,y} &= \tau (\langle y | x \rangle_E) = \langle y | x \rangle_{H, \tau} = \text{tr} \pi^H_{x,y}
\end{align*}
\]

with the usual canonical Hilbert space trace $\text{tr}$ and the usual rank 1 operators

\[
\pi^H_{x,y} : H, \tau \ni z \mapsto x \langle y | z \rangle_{H, \tau} \in H
\]

on the Hilbert space $H, \tau$. However, $\pi^E_{x,y}$ and $\pi^H_{x,y}$ are different operators:

\[
\pi^E_{x,y} (z) = x \langle y | z \rangle_E
\]

whereas

\[
\pi^H_{x,y} (z) = x \langle y | z \rangle_{H, \tau} = x \tau (\langle y | z \rangle_E).
\]

Thus, in general $\text{tr}_\tau$ and $\text{tr}$ are indeed different traces.
Remark 6 (\(\text{tr} \pi_{x,y}^E\) versus \(\text{tr}_\tau \pi_{x,y}^E\)). Let \((e_n)_{n \in \mathbb{N}}\) be an orthonormal base\(^3\) of \(H_\tau\). Then

\[
\text{tr}_\tau \pi_{x,y}^E = \text{tr}_\tau \pi_{x,y}^{H_\tau} = \sum_{n \in \mathbb{N}} \langle e_n | \pi_{x,y}^{H_\tau} (e_n) \rangle_{H_\tau} = \sum_{n \in \mathbb{N}} \langle e_n | x(y|e_n\rangle_H,_{H_\tau} = \sum_{n \in \mathbb{N}} \langle e_n | x \rangle_{H_\tau} \langle y|e_n\rangle_{H_\tau} = \sum_{n \in \mathbb{N}} \tau(\langle e_n | x \rangle_{E}) \tau(\langle y|e_n\rangle_{E}).
\]

So, \(\text{tr} \pi_{x,y}^{H_\tau}\) and \(\text{tr} \pi_{x,y}^E\) coincide if \(\tau\) is multiplicative. But in this case \(\tau\), being a multiplicative faithful trace, is a *-isomorphism \(\mathcal{A} \to \mathbb{C}\) already, so that we just reproduce the Hilbert space trace.

In general \(\text{tr}\) will be larger than \(\text{tr}_\tau\) because

\[
\text{tr} \pi_{e_m}^E = \sum_{n \in \mathbb{N}} \tau(\langle e_n | e_m \rangle_{E} \langle e_m | e_n \rangle_{E}) \\
= \sum_{n \in \mathbb{N}} \tau(\langle e_m | e_n \rangle_{E}^* \langle e_m | e_n \rangle_{E}) \\
\geq \tau(\langle e_m | e_m \rangle_{E}^* \langle e_m | e_m \rangle_{E}) \\
\geq (\tau(\langle e_m | e_m \rangle_{E}))^2 \\
= \|e_m\|^2_{H_\tau} \\
= 1 \\
= \text{tr} \pi_{e_m}^{H_\tau} \\
= \text{tr}_\tau \pi_{e_m}^E.
\]

Here we used the Cauchy-Schwarz inequality \(\tau(a^*b) \leq \sqrt{\tau(a^*a)\tau(b^*b)}\) and the fact that the trace is normalized.

To sum up: The \(\text{tr}_\tau\)-trace is defined only on the image of the adjointable module operators in the GNS representation, and on these it is in general smaller than the Hilbert space trace.
space trace, so that the corresponding trace class ideal is larger:

\[ \pi_{\tau} \left( L^1_{A}(E, \text{tr}_{\tau}) \right) \supset \pi_{\tau} \left( L_{A}(E) \right) \cap L^1(H_{\tau}, \text{tr}) \]

**Remark 7 (tr\_\tau for standard Hilbert modules).** If \( E \) is a standard \( A \)-module \( H \otimes A \) (tensor product of Hilbert modules) with a Hilbert space \( H \) then the GNS representation space \( H_{\tau} \) of \( E \) is given by \( H_{\tau} = H \otimes h_{\tau} \) (tensor product of Hilbert spaces), where \( h_{\tau} \) is the GNS representation space of \( A \). Therefore we have for the elementary tensors \( x \otimes a, y \otimes b \in E \)

\[
\langle y \otimes b | x \otimes a \rangle_{E} = \langle y | x \rangle_{H} b^*a, \\
\pi_{x \otimes a, y \otimes b}^{H_{\tau}} = \pi_{x, y}^{H} \otimes \pi_{a, b}^{h_{\tau}}, \\
\pi_{x \otimes a, y \otimes b}^{E} = \pi_{x, y}^{H} \otimes \pi_{a, b}^{A} = \pi_{x, y}^{H} \otimes ab^*.
\]

With the standard traces \( \text{tr}_{H}, \text{tr}_{h_{\tau}} \) on the Hilbert spaces \( H, h_{\tau} \) we get

\[
\text{tr} \pi_{x \otimes a, y \otimes b}^{H_{\tau}} = \text{tr} \pi_{x \otimes a, y \otimes b}^{E} = \langle y | x \rangle_{H} \tau(b^*a) = \text{tr}_{H} \left( \pi_{x, y}^{H} \right) \text{tr}_{h_{\tau}} \left( \pi_{a, b}^{h_{\tau}} \right).
\]

Thus we arrive at

\[
\text{tr} = \text{tr}_{H} \otimes \text{tr}_{h_{\tau}}, \\
\text{tr}_{\tau} = \text{tr}_{H} \otimes \tau.
\]

**Lemma 5 (tr for tr\_\tau-trace class).** Let \( E = H \otimes A \) be as above. If \( A \) is infinite dimensional with a unitary orthonormal basis for \( h_{\tau} \), then 0 is the only \( \text{tr}_{\tau} \)-trace class operator with finite standard trace. In particular: All Hilbert \( A \)-submodules are infinite dimensional vector spaces.

**Proof.** Let \( (x_n)_{n \in \mathbb{N}} \) be an orthonormal basis of \( h_{\tau} \), consisting of unitary elements of \( A \). Then

\[
\text{tr}_{h_{\tau}} \pi_{a, b}^{A} = \sum_{n \in \mathbb{N}} \langle x_{n} | ab^* x_{n} \rangle_{h_{\tau}}, \\
= \sum_{n \in \mathbb{N}} \tau(x_{n}^* ab^* x_{n}) \\
= \sum_{n \in \mathbb{N}} \tau(ab^*).
\]

\[\square\]

**Lemma 6 (non-existence of finite dimensional modules).** If \( A \) is infinite dimensional with a unitary orthonormal basis for \( h_{\tau} \), then every projective \( A \)-module is an infinite dimensional vector space.
Proof. If $\mathcal{E}$ is a projective Hilbert $\mathcal{A}$-module then $\mathcal{E}$ is a direct summand of a free module $H \otimes \mathcal{A}$ for a suitable Hilbert space $H$, and we can apply Lemma 5. \hfill $\square$

**Lemma 7 (spectral projections).** Let $D$ be a self-adjoint $\mathcal{A}$-elliptic operator and let $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \text{spec } D, \lambda_1 \leq \lambda_2$. Then the corresponding spectral projection $[P_{[\lambda_1, \lambda_2]}]_{\mathcal{A}}$ on the interval $[\lambda_1, \lambda_2]$ is $\mathcal{A}$-compact. If $e^{-tD^2} \in L^1_{\mathcal{A}}(\mathcal{E}, tr_\tau)$ for $t > 0$ then the spectral projections are $tr_\tau$-trace class.

**Proof.**
Reduction to $D \geq 0$ If $\text{spec } D = \mathbb{R}$ there is nothing to prove. So, let $\lambda_0 \in \mathbb{R} \setminus \text{spec } D$.

We show that we can assume $D \geq 0$ for the proof of $\mathcal{A}$-compactness: Let $D' := f(D)$ with $f(x) := x - \lambda_0$ for $x \in \mathbb{R}$.

Then $0 \notin \text{spec } D'$. We set $g(x) := (1 + x^2)^{-1}$ so that

$$
(1 + D'^2)^{-1} = g \circ f(D)
= g(D)b(D) \text{ with }
$$

$$
b(x) = \frac{g \circ f(x)}{g(x)}
= \frac{1 + (x - \lambda_0)^2}{1 + x^2}.
$$

Since $b$ is continuous and bounded $b(D) \in L^1_{\mathcal{A}}(\mathcal{E})$. If $D$ is $\mathcal{A}$-elliptic, i.e. $g(D) \in K_{\mathcal{A}}(\mathcal{E})$, then we get $g(D)b(D) \in K_{\mathcal{A}}(\mathcal{E})$, i.e. $D'$ is $\mathcal{A}$-elliptic. Denote the spectral projections of $D'$ with $P'$. Then obviously $P'(\lambda) = P(\lambda + \lambda_0)$, so that it suffices to test $P'$ for $\mathcal{A}$-compactness.

Finally we set $D'' := |D'|$. Then $D''$ is $\mathcal{A}$-elliptic by definition, positive by construction, and strictly positive because $0 \notin \text{spec } D'$. If we denote the spectral projections of $D''$ by $P''$ then

$$
P''(\lambda) = 1_{(-\infty, \lambda)}(D'')
= (1_{(-\infty, \lambda)} \circ | \cdot |)(D')
= 1_{[-\lambda, \lambda]}(D')
= P'_{[-\lambda, \lambda]}.
$$

Therefore we get for $0 \leq \lambda_1 \leq \lambda_2$

$$
P''_{[\lambda_1, \lambda_2]} = P''(\lambda_2) - P''(\lambda_1)
= P'_{[-\lambda_2, \lambda_2]} - P'_{[-\lambda_1, \lambda_1]}
= P'_{[-\lambda_2, -\lambda_1] \cup (\lambda_1, \lambda_2]}.
$$

\footnote{If $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \text{spec } D$ then $P_{[\lambda_1, \lambda_2]} = P_{[\lambda_1, \lambda_2]} = P_{[\lambda_1, \lambda_2]}$.}
By assumption $0 \notin \text{spec } D'$ and therefore $P'_{[0,\infty)}, P'_{(-\infty,0]} \in \mathcal{L}_A(\mathcal{E})$, so that

\[ P'_{(\lambda_1, \lambda_2)} = P''_{(\lambda_1, \lambda_2)} P'_{[0,\infty)} \text{ in } \mathcal{K}_A(\mathcal{E}) \quad \text{and} \quad (20) \]

\[ P'_{[-\lambda_2,-\lambda_1]} = P''_{(\lambda_1, \lambda_2)} P'_{(-\infty,0]} \in \mathcal{K}_A(\mathcal{E}), \quad (21) \]

if $P''_{(\lambda_1, \lambda_2)} \in \mathcal{K}_A(\mathcal{E})$. If $\lambda_1 \leq 0 \leq \lambda_2$ we write

\[ P'_{[\lambda_1, \lambda_2]} = P'_{[\lambda_1, 0]} + P'_{(0, \lambda_2]} \]

and apply equation (20) and (21). Hence it suffices to test $P''$ for $A$-compactness.

$A$-compactness We show that every spectral projection $P_{[\lambda_1, \lambda_2]}$ for $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \text{spec } D$ can be produced by continuous functional calculus from $S := (1 + D^2)^{-1}$, so that it belongs to $\mathcal{K}_A(\mathcal{E})$. For this we note that $S^{-1} = D^2 + 1$ is densely defined ($D$ is regular), self-adjoint and bounded below by 1. Thus $\sqrt{S^{-1} - 1}$ exists, is positive and self-adjoint. By the spectral mapping theorem we have

\[ z \in \text{spec } \sqrt{S^{-1} - 1} \iff (z^2 + 1)^{-1} \in \text{spec } S \]

\[ \iff z \in \text{spec } D. \]

Therefore, the operator

\[ R_z := (z - \sqrt{S^{-1} - 1})^{-1}. \]

exists for all $z$ in the resolvent set of $D$. Since the function

\[ \lambda \mapsto (z - \sqrt{\lambda^{-1} - 1})^{-1} \]

is continuous and bounded on every closed set not containing $(z^2 + 1)^{-1}$, $R_z$ belongs to the $C^*$-algebra generated by $S$ for every $z \in \mathbb{C} \setminus \text{spec } D$ and therefore belongs to $\mathcal{K}_A(\mathcal{E})$, i.e. it is $A$-compact. Since

\[ P_{[\lambda_1, \lambda_2]} = \frac{1}{2\pi i} \oint_c R_z \, dz \]

for a suitable closed path $c$ in $\mathbb{C} \setminus \text{spec } D$ with winding number 1 fulfilling $c \cap \mathbb{R} = \{ \lambda_1, \lambda_2 \}$, $P_{[\lambda_1, \lambda_2]}$ belongs to the $C^*$-algebra generated by all $R_z$.

trace class property Let $e^{-tD^2}$ be tr-$\tau$-trace class. Since

\[ P_{[\lambda_1, \lambda_2]} = \int_{\lambda_1}^{\lambda_2} dP(\lambda) \]

\[ \leq e^{t(\lambda_2 - \lambda_1)^2} \int_{\lambda_1}^{\lambda_2} e^{-t(\lambda - \lambda_1)^2} dP(\lambda) \]

\[ \leq e^{t(\lambda_2 - \lambda_1)^2} \int_{\mathbb{R}} e^{-t(\lambda - \lambda_1)^2} dP(\lambda) \]

\[ = e^{t(\lambda_2 - \lambda_1)^2} e^{-tD^2} \]

the spectral projections inherit the trace class property from $e^{-tD^2}$.

\[ \square \]

---

5 We don’t assume positivity of $D$ any more.
If \( \lambda \) is an isolated eigenvalue then for sufficiently small \( \varepsilon > 0 \) \( P_{\lambda} := P_{[\lambda-\varepsilon, \lambda+\varepsilon]} \) is the projection on the eigenspace of \( \lambda \), independent of \( \varepsilon \). So \( P_{\lambda} \) fulfills the hypotheses of Lemma 7, and we can determine the dimension of the eigenspace:

**Theorem 6 (isolated eigenvalue).** If \( \lambda \) is an isolated eigenvalue of a self-adjoint \( A \)-elliptic operator \( D \) then the corresponding eigenspace \( E_{\lambda} \) is an (algebraically) finitely generated projective Hilbert \( A \)-module, and the projection \( P_{\lambda} \) is \( A \)-finite. If \( e^{-tD^2} \) is \( \tau \)-trace class then so is \( P_{\lambda} \), i.e. \( E_{\lambda} \) has finite \( \tau \)-dimension \( \tau P_{\lambda} \).

If \( E, A \) fulfill the hypotheses of Lemma 7 then \( E_{\lambda} \) has infinite Hilbert dimension \( \tau P_{\lambda} \) for every isolated eigenvalue \( \lambda \) of \( D \). In particular: \( D \) has essential spectrum only.

**Proof.** \( P_{\lambda} \) is the spectral projection of a self-adjoint operator and therefore self-adjoint, and \( A \)-compact by Lemma 7. Thus the eigenspace \( E_{\lambda} \) is the image of a closed adjointable projection \( P_{\lambda} \) and therefore a closed complementable \( A \)-module. Since the projection \( P_{\lambda}|_{E_{\lambda}} = 1 \) is \( A \)-compact \( E_{\lambda} \) is algebraically finitely generated and projective, because algebraically finitely generated \( A \)-modules \( E \) are just the ones with unital \( K_{\mathbb{A}}(E) \) and automatically projective (see e.g. Wegge-Olsen, 1993, Theorem 15.4.2 and Corollary 15.4.8).

If \( e^{-tD^2} \in \mathcal{L}_{\lambda}(E, \tau) \) then so is \( P_{\lambda} \) by Lemma 7, and under the same hypotheses we can apply Lemma 8.

The main idea of the following proof goes back to Sunada (1992):

**Theorem 7 (band structure).** Assume that \( K_{\mathbb{A}}(E) \) has the Kadison property with respect to \( \tau \) (see Definition 3). Then the spectrum of every self-adjoint \( A \)-elliptic operator \( D \) with \( e^{-tD^2} \in \mathcal{L}_{\lambda}(E, \tau) \) has band structure.

**Proof.** Let \( a = \lambda_0 < \ldots < \lambda_n = b \in \mathbb{R} \setminus \text{spec} D \), so that \( P_{[\lambda_i, \lambda_{i+1}]} \neq 0 \) for \( 0 \leq i \leq n-1 \), i.e. \( \text{spec} D \) has at least \( n \) components in \( [a, b] \). Then

\[
P_{[a,b]} = \sum_{i=0}^{n-1} P_{[\lambda_i, \lambda_{i+1}]}
\]

and therefore

\[
\tau P_{[a,b]} = \sum_{i=0}^{n-1} \tau P_{[\lambda_i, \lambda_{i+1}]}
\geq n c_{\mathbb{K}}.
\]

\[
\Leftrightarrow n \leq \frac{1}{c_{\mathbb{K}}} \tau P_{[a,b]},
\]

since all projections occurring in this sum are \( \tau \)-trace class by Lemma 7.

If \( c_{\mathbb{K}} = 0 \) then we cannot apply Theorem 7. Instead, spectra with the structure of a Cantor set seem possible. Examples show that the opening of gaps which are allowed depends heavily on the specific structure of the operator and cannot easily be controlled globally. To get generic results we therefore have to make sure that not only \( c_{\mathbb{K}} = 0 \), but also that the trace can be arbitrarily small on “many” projections. This is accomplished by the following theorem by Choi and Elliott (1990):
Theorem 8 (Cantor spectrum). Let \( \mathcal{A} \) be a C*-algebra with a faithful state \( \Phi \). Assume that every self-adjoint element can be approximated arbitrarily well by an element with finite spectrum on whose minimal spectral projections \( \Phi \) is arbitrarily small. Then the self-adjoint elements with Cantor spectrum are dense in all self-adjoint elements.

In particular, the algebras in Theorem 8 have real rank zero, i.e. the invertible self-adjoint elements are dense in all self-adjoint ones:

Definition and proposition 9 (real rank) Let \( \mathcal{A} \) be a unital C*-algebra. The real rank of \( \mathcal{A} \) is defined by

\[
RR(\mathcal{A}) = \min \{ m \in \mathbb{N}_0 \mid \forall n \geq m + 1 : RR_n(\mathcal{A}) \},
\]

where

\[
RR_n(\mathcal{A}) = \left\{ x \in \mathcal{A}_{sa}^n : \forall \epsilon > 0 : \exists y \in \mathcal{A}_{sa}^n : \sum_{k=1}^{n} y_k^2 \in \mathcal{A}^\times \land \left\| \sum_{k=1}^{n} (y_k - x_k)^2 \right\| < \epsilon \right\}
\]

For all \( n \in \mathbb{N}_0 \) we have \( RR_n(\mathcal{A}) \Rightarrow RR_{n+1}(\mathcal{A}) \). The following conditions are equivalent:

1. \( RR(\mathcal{A}) = 0 \)
2. \( \mathcal{A}_{sa}^\times \subset \mathcal{A}_{sa} \) dense
3. The self-adjoint elements with finite spectrum are dense in \( \mathcal{A}_{sa} \).

We say \( \mathcal{A} \) has real rank 0 with infinitesimal state (\( RRI_0 \)) if \( \mathcal{A} \) fulfills the assumptions of Theorem 8.

For the convenience of the reader we include a proof of these equivalences which are well known in the C*-community.

Proof. \( RR_n(\mathcal{A}) \Rightarrow RR_{n+1}(\mathcal{A}) \): Let \( x \in \mathcal{A}_{sa}^{n+1} \) and \( \bar{x} := (x_1, \ldots, x_n) \in \mathcal{A}_{sa}^n \). By assumption there is \( \bar{y} \in \mathcal{A}_{sa}^n \) such that

\[
\sum_{k=1}^{n} \bar{y}_k^2 \in \mathcal{A}^\times \land \left\| \sum_{k=1}^{n} (\bar{y}_k - x_k)^2 \right\| < \epsilon .
\]

For all \( k = 1, \ldots, n \) we have \( x_k^2 \geq 0 \), and \( \sum_{k=1}^{n} \bar{y}_k^2 > 0 \). We set \( y := (\bar{y}_1, \ldots, \bar{y}_n, x_{n+1}) \). Then

\[
\sum_{k=1}^{n+1} y_k^2 \geq \sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} \bar{y}_k^2 > 0
\]

\[
\Rightarrow \sum_{k=1}^{n+1} y_k^2 \in \mathcal{A}^\times
\]

and finally

\[
\left\| \sum_{k=1}^{n+1} (y_k - x_k)^2 \right\| = \left\| \sum_{k=1}^{n} (\bar{y}_k - x_k)^2 \right\| < \epsilon
\]
2 ⇒ 3: by definition.

2 ⇒ 1: because \( \text{RR}_n(A) \Rightarrow \text{RR}_{n+1}(A) \) for all \( n \in \mathbb{N}_0 \).

⇒ 2: Let \( x \in \mathcal{A}_{sa} \), \( \varepsilon > 0 \). Then there is \( y \in \mathcal{A}_{sa} \) with finite spectrum so that \( \| x-y \| < \varepsilon / 2 \). If \( y \) is invertible then there is nothing to prove, otherwise we choose \( \delta > 0 \) so that \( \text{spec} \, y \cap B_{\delta}(0) = \{0\} \). Then \( \tilde{y} := y + \frac{1}{2} \min \{\delta, \varepsilon\} \mathbf{1} \) is invertible, and

\[
\| \tilde{y} - x \| \leq \| y - x \| + \varepsilon / 2 < \varepsilon.
\]

2 ⇒ 3: Let \( x \in \mathcal{A}_{sa} \). We show first that we can approximate \( x \) by a self-adjoint element with finitely many gaps. For this we choose a sequence \( (s_i)_{i \in \mathbb{N}} \) of pair-wise distinct real numbers that are dense in the interval \( [-\|x\|, \|x\|] \). We set \( x_1 := x \) and choose inductively \( y_n \in \mathcal{A}_x \) such that

\[
\|(x_n - s_n \mathbf{1}) - y_n\| < 2^{-n} \min \{\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-1}\}, \text{ where } \\
\varepsilon_i = \text{dist} \{0, \text{spec} \, y_i\} > 0,
\]

since \( y_i \) is invertible, and we define

\[
x_{n+1} := y_n + s_n \mathbf{1}.
\]

Then, by construction

\[
\|x_{n+1} - x_n\| < 2^{-n} \min \{\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-1}\}
\]

\[
\Rightarrow \|x_{n+1} - x\| \leq \sum_{i=1}^{n} \|x_{i+1} - x_i\|
\]

\[
< \sum_{i=1}^{n} 2^{-i} \min \{\varepsilon, \varepsilon_1, \ldots, \varepsilon_{i-1}\}
\]

\[
\leq \min \{\varepsilon, \varepsilon_1, \ldots, \varepsilon_{n-1}\}.
\]

Fig. 2. A possible choice for the function \( f_n \).

Furthermore \( \text{spec} \, x_{n+1} = s_n + \text{spec} \, y_n \) so that \( B_{\varepsilon_n}(s_n) \) is in the resolvent set of \( x_n \). The rate of approximation is chosen just so that these gaps remain open (although possibly become smaller) in every step. Now we construct an approximation with
finite spectrum for each $x_{n+1}$. For this we arrange, for fixed $n$, the $s_i$, $1 \leq i \leq n$ into increasing order, say $\tilde{s}_1 < \ldots < \tilde{s}_n$, and set $\delta_n := \max\{\tilde{s}_{i+1} - \tilde{s}_i \mid 1 \leq i < n\}$. We define a continuous monotonically increasing function $f_n$ by

$$ f_n(\lambda) := \begin{cases} \frac{\tilde{s}_i + \tilde{s}_{i+1}}{2}, & \text{if } \lambda \in (\tilde{s}_i, \tilde{s}_{i+1}) \cap \text{spec } x_{n+1}, \\ \text{cont. m. i.,} & \text{else,} \end{cases} $$

such that $z_{n+1} := f_n(x_{n+1}) \in A_{sa}$ (see figure 2). $f_n$ compresses the spectrum between two gaps into one point. The spectral projections

$$ p_i = P_{[\tilde{s}_i, \tilde{s}_{i+1}]}(x_{n+1}), 0 \leq i \leq n, $$

$$ \tilde{s}_0 = -\|x_{n+1}\| - 1, $$

$$ \tilde{s}_{n+1} = \|x_{n+1}\| + 1, $$

belong to intervals with endpoints in the resolvent set so that they are in $A$. Therefore

$$ z_{n+1} = \sum_{i=0}^n \frac{\tilde{s}_i + \tilde{s}_{i+1}}{2} p_i. $$

Thus $\text{spec } z_{n+1} \subset \left\{ \frac{\tilde{s}_i + \tilde{s}_{i+1}}{2} \mid 0 \leq i \leq n \right\}$, and with the spectral family $P(\lambda)$ of $x_{n+1}$ we get

$$ \|z_{n+1} - x_{n+1}\| \leq \int |\lambda - f_n(\lambda)| \, dP(\lambda) \leq 2\delta_n \|x_{n+1}\| \Rightarrow \|z_{n+1} - x\| \leq 2\delta_n \|x_{n+1}\| + \varepsilon. $$

Since $\delta_n \to 0$ for $n \to \infty$ and $\|x_{n+1}\|$ is bounded, we can make the approximation arbitrarily good.

\[ \square \]

**Remark 8 (Kadison property and $RRI_0$).**

1. Kadison property and property $RRI_0$ are mutually exclusive since the first forbids existence of projections with arbitrarily small trace whereas the latter requires this.
2. $C^*$-algebras $A$ with $RRI_0$ can contain operators with band structure: If $A$ is the irrational rotation algebra (see below) then $A$ has $RRI_0$ by Theorem 9. But $A$ contains a subalgebra isomorphic to $C(S^1)$, consisting of operators with band spectrum only.
3. On the other hand, a $C^*$-algebra $A$ with the Kadison property cannot contain self-adjoint elements with Cantor spectrum: If $x \in A_{sa}$ has Cantor spectrum then every point in $\text{spec } x$ is an accumulation point of $\text{spec } x$ and $\mathbb{R} \setminus \text{spec } x$, so that $x$ has no band spectrum in contradiction to Theorem 7.
4. If $A_1$ has the Kadison property and $A_2$ has property $RRI_0$ then $A := A_1 \oplus A_2$ has neither of these properties.

**Remark 9 (real rank and dimension).**
1. If \( \mathcal{A} \) is commutative so that \( \mathcal{A} = C(X) \) for a topological space \( X \) then \( \text{RR}(\mathcal{A}) = \text{dim } X \) with the usual definition of dimension.

2. Therefore, \( C^* \)-algebras with real rank 0 are (non-commutative) zero-dimensional spaces. This includes finite discrete spaces. However, the additional trace condition in Theorem 8 excludes finite spaces: By the Riesz-Kakutani theorem every state on \( C(X) \) is given by an integral with respect to a normalized measure \( \mu \), i.e. \( \Phi(f) = \int f \, d\mu \) and \( \mu(X) = 1 \). Such states are faithful if and only if every open set has strictly positive measure. The trace condition requires that \( X \) has connected components with arbitrary small measure.

3. Every \( W^* \)-algebra has real rank 0, since the measurable functional calculus (as opposed to the continuous) allows to ‘cut out’ points from the spectrum arbitrarily close.

4. Property \( \text{RR}_0 \) is preserved under inductive limits, in particular \( \mathcal{A} \otimes K \) has real rank 0 if \( \text{RR}(\mathcal{A}) = 0 \).

**Example 5 (rotation algebra).** The rotation algebra \( \mathcal{A}_\theta \) is the \( C^* \)-algebra generated by two unitaries \( U, V \) and the relation \( VU = e^{2\pi i \theta} UV \) for a given \( \theta \in \mathbb{R} \). It also arises as a reduced twisted group \( C^* \)-algebra \( C^*_r(\mathbb{Z}_2, \Theta) \) for the cocycle \( \Theta \) given by \( e^{2\pi i \theta} \) since \( H^2(\mathbb{Z}_2, S^1) \simeq S^1 \). It carries a canonical trace defined by

\[
\tau(1) = 1, \tau(U) = \tau(V) = 0.
\]

The properties of this algebra depend strongly on the nature of \( \theta \):

**Theorem 9 (properties of the rotation algebra).**

1. If \( \theta = p/q \) with \( p \in \mathbb{Z}, q \in \mathbb{N} \) co-prime then the Kadison constant of \( \mathcal{A}_\theta \) and of \( \mathcal{A}_\theta \otimes K \) is \( 1/q \).

2. If \( \theta \) is irrational then \( \mathcal{A}_\theta \) and \( \mathcal{A}_\theta \otimes K \) (together with the canonical trace) have real rank 0 with infinitesimal state.

**Proof.** 1. As is well known, the spectrum of \( \mathcal{A}_\theta \) is \( T^2 \), all irreducible representations \( \pi_z \) have dimension \( q \). The canonical trace of \( a \in \mathcal{A}_\theta \) is

\[
\tau(a) = \frac{1}{q} \int_{T^2} \text{tr} \pi_z(a) \, dz
\]

with the canonical trace \( \text{tr} \) on \( M(q, \mathbb{C}) \). Minimal projections have rank 1 in the fiber, and so the Kadison constant is \( 1/q \).

2. \( \mathcal{A}_\theta \) has real rank zero by Elliott and Evans (1993). Since \( \mathcal{A}_\theta \) is simple and non-elementary also we get \( \text{RR}_0 \) from (Choi and Elliott, 1990, Corollary 8).

\[ \square \]

**Theorem 10 (Cantor spectrum).** Assume the \( C^* \)-algebra \( \mathcal{K}_\mathcal{A}(E) \) has real rank 0 with infinitesimal state. Then every self-adjoint \( \mathcal{A} \)-elliptic operator can be approximated arbitrarily close in norm resolvent sense by a self-adjoint operator with Cantor spectrum.

**Proof.** Lemma 7 and Theorem 8

\[ \square \]
5. Applications

Discrete models.

Example 6 (generalized Harper operators). Sunada (1994) defines magnetic Schrödinger Operators on graphs: Let \( X \) be a connected locally finite graph, \( \chi \) a \( \mathbb{C}^\times \)-valued (i.e. non-vanishing complex-valued) map (a weight) on the oriented edges \( E(X), o, t : E(X) \to X \) the origin and termination point mappings. We define a symmetric operator on \( l^2(X) \) by

\[
(H_\chi f)(x) = \sum_{e \in E(X)} \chi(e) f(t(e))
\]

for \( f \in l^2(X) \). Two weights \( \chi_1, \chi_2 \) are called cohomologous if there is a function \( s : X \to S^1 \) with

\[
\chi_1(e) = \chi_2(e) \frac{s(o(e))}{s(t(e))}
\]

for \( e \in E(X) \).

Furthermore, let \( \Gamma \) be a group with a properly discontinuous free action on \( X \) and such that the quotient graph is finite (say \( n \) points). A weight \( \chi \) is called gauge-invariant if \( \gamma^* \chi \) is cohomologous to \( \chi \) for all \( \gamma \in \Gamma \). Then \( \chi \) defines a cocycle \( \Theta \in Z^2(\Gamma, S^1) \) such that \( H_\chi \) commutes with the corresponding twisted right translations \( R^{\Theta_\gamma}_\gamma f(\gamma') = \Theta(\gamma', \gamma) f(\gamma' \gamma) \). Sunada constructs an injective *-homomorphism

\[
C^*_r(\Gamma, \Theta) \otimes M(n, \mathbb{C}) \to \text{End}(l^2(X)),
\]

whose image contains \( H_\chi \). On the other hand,

\[
A \otimes M(n, \mathbb{C}) = K_A(A \otimes \mathbb{C}^n)
\]

for the Hilbert \( A \)-module \( A \otimes \mathbb{C}^n \) which is the tensor product of the canonical module \( A = C^*_r(\Gamma, \Theta) \) and the Hilbert \( \mathbb{C} \)-module \( \mathbb{C}^n \). As in Theorem 7 Sunada proves band structure.

All spectral characterizations of this section apply as soon as the corresponding \( C^* \)-algebra \( C^*_r(\Gamma, \Theta) \otimes M(n, \mathbb{C}) \) fulfills the corresponding assumptions.

We get the ordinary Harper operator for \( E(X) = \Gamma = \mathbb{Z}^2 \) and a suitable graph \( X \) with coordination number 4 (square lattice), the hexagonal Harper operator and the quantum pendulum for graphs with coordination numbers 6 resp. 8. The corresponding \( C^* \)-algebras are rotation algebras, so that we have band structure for rational flux, and weak genericity of Cantor spectrum for irrational flux.

Continuous models.

Example 7 (gauge-periodic elliptic operators). In this case \( A \) will be a twisted group \( C^* \)-algebra (left translations), and the Hilbert module will be a tensor product \( \mathcal{E} = A \otimes \mathcal{H} \) with a Hilbert space \( \mathcal{H} \) such that \( \mathcal{K}_A(\mathcal{E}) \simeq A \otimes \mathcal{K}(\mathcal{H}) \). The operator \( D \) will be a differential operator which is invariant under a projective representation of a group, such as Schrödinger, Dirac and Pauli operators with periodic magnetic and electric fields.
The geometric situation we consider is similar to the case of abelian periodic operators (see Definition 4) from the introductory section. Now we allow the group to be non-commutative, and we allow the action to be represented projectively only on the bundle:

**Definition and proposition 10 (gauge-periodic operator)** Let $X$ be a smooth oriented Riemannian manifold without boundary, $\Gamma$ a discrete group acting on $X$ from the left freely, isometrically, and properly discontinuously. Furthermore, we assume the action to be cocompact in the sense that the quotient $M := X/\Gamma$ is compact. This defines, as in the abelian case, a left action of $\gamma \in \Gamma$ on smooth functions $f \in C^\infty(X)$ by

$$\gamma \ast f(x) := f(\gamma^{-1}x)$$

for $x \in X$. As before, this extends to a unitary action on $L^2(X)$.

Next, let $E$ be a smooth Hermitian vector bundle over $X$. Let $U$ be a projective representation of $\Gamma$ in the unitary operators $U(L^2(E))$ in the following sense:

$$\forall \gamma_1, \gamma_2 \in \Gamma : \exists \Theta(\gamma_1, \gamma_2) \in C(X, S^1) : U_{\gamma_1}U_{\gamma_2} = \Theta(\gamma_1, \gamma_2)U_{\gamma_1\gamma_2}. \quad (23)$$

Assume that $U$ is a (projective) lift of the $\Gamma$-action on $C^\infty(X)$, i.e.

$$\forall \varphi \in C^\infty_c(X) : \forall s \in L^2(E) : \forall \gamma \in \Gamma : U_{\gamma}(\varphi s) = (\gamma \ast \varphi)U_{\gamma}(s). \quad (24)$$

Assume that $U$ is smooth, i.e. $\forall \gamma \in \Gamma : U_{\gamma}(C^\infty(E) \cap L^2(E)) \subset C^\infty(E)$. Then $U_{\gamma}$ is $\gamma$-local, i.e.

$$\forall s \in C^\infty(E) : \text{supp}(U_{\gamma}s) \subset \gamma \text{supp } s, \quad (25)$$

and it leaves the domain $D(D) = C^\infty_c(E)$ of any differential operator $D$ on $E$ invariant. We call $D$ gauge-periodic if, on $D(D)$, one has:

$$\forall \gamma \in \Gamma : [U_{\gamma}, D] = 0 \quad (26)$$

**Proof.** Let $x \in X \setminus \text{supp } s$. Since $\text{supp } s$ is closed there is a neighborhood $O \subset X$ of $x$ and $\varphi \in C^\infty_c(X)$ with $\varphi|O = 1$, $\varphi|\text{supp } s = 0$. Then $(1 - \varphi)s = s$ and therefore

$$U_{\gamma}s = U_{\gamma}((1 - \varphi)s)$$

$$= (1 - \gamma \ast \varphi)U_{\gamma}s$$

$$= 0 \text{ on } \gamma O.$$

Since $U$ is smooth also, it leaves $C^\infty_c(E)$ invariant. \qed

**Proposition 3 (cocycle property).** $\Theta$ fulfills the cocycle property:

$$\forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma : \Theta(\gamma_1, \gamma_2)\Theta(\gamma_1\gamma_2, \gamma_3) = \Theta(\gamma_1, \gamma_2\gamma_3)\gamma_3^{-1}\theta(\gamma_2, \gamma_3) \quad (27)$$
Proposition 4 (bundle morphisms). This follows from associativity \( U_{\gamma_1} (U_{\gamma_2} U_{\gamma_3}) = (U_{\gamma_1} U_{\gamma_2}) U_{\gamma_3} \) and the projectivity condition (23):

\[
U_{\gamma_1} (U_{\gamma_2} U_{\gamma_3}) = U_{\gamma_1} \Theta(\gamma_2, \gamma_3) U_{\gamma_2 \gamma_3} \\
= \gamma_1 [\Theta(\gamma_2, \gamma_3)] U_{\gamma_1} U_{\gamma_2 \gamma_3} \\
= \Theta(\gamma_1, \gamma_2 \gamma_3) \gamma_1 [\Theta(\gamma_2, \gamma_3)] U_{\gamma_1 \gamma_2 \gamma_3} \\
(U_{\gamma_1} U_{\gamma_2}) U_{\gamma_3} = \Theta(\gamma_1, \gamma_2) U_{\gamma_1 \gamma_2} U_{\gamma_3} \\
= \Theta(\gamma_1, \gamma_2) \Theta(\gamma_1 \gamma_2, \gamma_3) U_{\gamma_1 \gamma_2 \gamma_3}
\]

\( \square \)

Remark 10 (exact cocycle and representation). \( \Theta \) therefore defines a class in the group cohomology \( H^2(\Gamma, C(X, S^1)) \) (see, e.g., [Brown, 1994]). Exact 2-cocycles have the form

\[
\Theta(\gamma, \gamma') = \sigma(\gamma) \gamma_3^2 [\sigma(\gamma')] \sigma(\gamma')^{-1}
\]

with a 1-cocycle \( \sigma \), so they define a proper representation of \( \Gamma \) by

\[
\hat{U}_\gamma := \sigma(\gamma)^{-1} U_\gamma,
\]

which also commutes with \( D \) if the cocycle is constant in \( x \in X \). Without loss of generality we assume that \( \Theta \) is normalized, i.e. \( \Theta(e, e) = 1 \).

Proposition 4 (bundle morphisms). \( U \) defines a family \( u \) of vector bundle morphisms on \( E \), \( u_\gamma : E_x \rightarrow E_{\gamma x} \). \( u \) is a projective lift of the \( \Gamma \)-action from \( X \) to \( E \), i.e.

\[
\forall \gamma_1 \gamma_2 \in \Gamma : u_{\gamma_1} u_{\gamma_2} = \Theta(\gamma_1, \gamma_2) u_{\gamma_1 \gamma_2}
\]

with the same cocycle \( \Theta \) as for \( U \). \( u \) induces \( U \) via

\[
(U_\gamma s)(x) := u_\gamma s(\gamma^{-1} x).
\]

If \( t \) is a (proper) lift of the \( \Gamma \)-action from \( X \) to \( E \) and \( T \) the induced action

\[
(T_\gamma s)(x) := t_\gamma s(\gamma^{-1} x)
\]

on \( C^\infty(X) \) then \( u \) and \( U \) can be expressed as \( u = mt \) and \( U = MT \), where \( m \) is a family of (strict) vector bundle isomorphisms.

Proof. Let \( v \in E_x \). We choose \( s \in C^\infty(E) \) with \( s(x) = v \) and set – a priori depending on \( s - u^s_\gamma(v) := (U_\gamma(s))(\gamma x) \in E_{\gamma x} \). If \( \varphi \in C^\infty(x), \varphi(x) = 1 \), we get

\[
u^s_\gamma(v) = (\gamma \star \varphi)(\gamma x) (U_\gamma) (\gamma x) = u^s_\gamma(v),
\]

i.e. \( u^s_\gamma(v) \) depends on the value of \( s \) at the point \( x \) only; hence we omit \( s \) in the notation. The morphism property follows from the corresponding property of \( U_\gamma \), and from \( (u_\gamma)^{-1} = u_{\gamma^{-1}} \).

\( u \) induces \( U \) by construction.
If there is a proper lift $t$ then $m := ut^{-1}$ defines the strict morphism we look for:

$$
\begin{array}{cccc}
E & \xrightarrow{t} & E & \xrightarrow{u} & E \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\gamma} & X & \xrightarrow{\gamma} & X
\end{array}
$$

$\square$

Remark 11 (lift of the action). If $\Theta$ is exact and $\tilde{u}$ the family of vector bundle isomorphisms belonging to $\tilde{U}$ by remark 10 then $\tilde{u}$ is a proper lift of the $\Gamma$-action from $X$ to $E$.

Proposition 5 (properties of the cocycle).

1. $\forall \gamma \in \Gamma : \Theta(\gamma, e) = \Theta(e, \gamma) = 1$
2. $\forall \gamma \in \Gamma : \Theta(\gamma, \gamma^{-1}) = \Theta(\gamma^{-1}, \gamma)$

Proof. Easy consequences of the cocycle property. $\square$

For the case of a bicharacter $\Theta$ Brüning and Sunada (1992b, 1996) describe how to construct a parametrix for elliptic gauge-periodic differential operator by lifting and translating a parametrix for a fundamental domain. The same construction works for the slightly more general case of a 2-cocycle.

From this one concludes as in the cited work:

Theorem 11 (self-adjointness). Every symmetric elliptic gauge-periodic differential operator is essentially self-adjoint on $C_c^\infty(E)$.

Similarly, a trivial extension of Brüning and Sunada (1992b, 1996) shows how to construct the heat kernel:

Theorem 12 (heat kernel). Let $D$ be a symmetric elliptic gauge-periodic differential operator, bounded below, of order $p > d = \dim X$. Then $e^{-tD}$ has, for $t > 0$, a smooth integral kernel $K_t(x, y) \in E_x \otimes E_y^*$ such that

$$
|K_t(x, y)| \leq C_1 t^{-d/p} \exp \left( -C_2 \text{dist}(x, y)^{p/(p-1)} t^{-1/(p-1)} \right)
$$

with $C_1, C_2 > 0$, uniformly on $(0, T] \times X \times X$.

Again following Brüning and Sunada (1992b) we construct a suitable decomposition of $L^2(E)$. For that we choose a fundamental domain $D$ for the $\Gamma$-action, set $\mathcal{H} = L^2(E|_D)$ and define a unitary map by

$$
\Phi : L^2(E) \rightarrow l^2(\Gamma, \mathcal{H}) \simeq l^2(\Gamma) \otimes \mathcal{H},
\Phi(s)(\gamma) = (U_{\gamma}(s))|_D.
$$
Then we have for $f \in L^2(\Gamma) \otimes \mathcal{H}$
\[
(\Phi U_\gamma \Phi^* f)(\gamma') = (U_\gamma U_\gamma \Phi^* f)|_D
= \Theta(\gamma', \gamma)(U_\gamma U_\gamma \Phi^* f)|_D
= \Theta(\gamma', \gamma)(\Phi \Phi^* f)(\gamma' \gamma)
= \Theta(\gamma', \gamma)f(\gamma' \gamma)
=: \Theta(\gamma', \gamma)R_\gamma f(\gamma') = R^\Theta_\gamma f(\gamma')
\]
with the right translation $R_\gamma$ and twisted right translation $R^\Theta_\gamma f(\gamma')$.

So it’s natural to try and define a $C^*_r(\Gamma, \Theta)$-action on $L^2(E)$ by
\[
R^\Theta_\gamma(s) = U_\gamma(s)
\]
for $s \in L^2(E)$. Here, the cocycle $\Theta$ can in general depend on $x \in X$ so that we have to find the gauge-translations in $C(X, S^1) \times_{\alpha, \theta} \Gamma$. This $C^*$-algebra has interesting structural properties but is not suitable for the applications on spectral theory developed in the previous section.

If $\Theta$ is periodic in $x \in X$ then we get a field of twisted reduced group $C^*$-algebras $C^*_r(\Gamma, \Theta_x)$, $x \in M$ over $M$. In general this field is still too ‘large’.

Therefore we require the cocycle to be constant in $x \in X$, so that we have to deal with the reduced twisted group $C^*$-algebra $C^*_r(\Gamma, \Theta)$ only. This is still general enough for the applications we are interested in: magnetic Schrödinger operators (and there Pauli and Dirac analogs).

Now note that $l^2(\Gamma)$ is the GNS representation space of $\mathcal{A} := C^*_r(\Gamma, \Theta)$ with respect to the canonical trace given by
\[
\tau(R^\Theta_\gamma) = \begin{cases} 1, & \gamma = e, \\ 0, & \text{else}, \end{cases}
\]
and that $l^1(\Gamma) \subset C^*_r(\Gamma, \Theta) \subset l^2(\Gamma)$. Therefore it’s natural to view the left Hilbert-$\mathcal{A}$ module as $\mathcal{E} := \mathcal{H} \otimes \mathcal{A}$ so that $L^2(E)$ is the Hilbert-GNS representation space of $\mathcal{E}$.

To define the scalar product we use the observations made in the commutative case (see Definition and Proposition 5).

**Lemma 8** (left pre-Hilbert $\mathcal{A}$-module).

\[
\langle s_1 | s_2 \rangle_{\mathcal{E}} = \sum_{\gamma \in \Gamma} \langle U_\gamma s_2 | s_1 \rangle_{L^2(E)} R^\Theta_\gamma
\]  

(35)

for $s_1, s_2 \in C_c(E)$ defines the structure of a left pre-Hilbert $\mathcal{A}$-module on $C_c(E)$; under the isomorphism $\Phi$ it coincides with the left tensor Hilbert $\mathcal{A}$-module structure of $\mathcal{H} \otimes \mathcal{A}$.

**Proof.** For $f_1, f_2 \in \mathcal{H}, a_1, a_2 \in \mathcal{A}$ we have by definition
\[
\langle a_1 \otimes f_1 | a_2 \otimes f_2 \rangle_{\mathcal{A} \otimes \mathcal{H}} = \langle f_2 | f_1 \rangle_{\mathcal{H}} a_1 a_2^*.
\]

\[\text{The action is naturally a left action since it is given by endomorphisms on a vector space.}\]
since a left Hilbert-$\mathcal{A}$-module is a Hilbert space with conjugated scalar product (complex linear in the first argument, anti-linear in the second). For $s_1, s_2 \in C_c(E)$ we get after identifying $\delta_{\gamma^{-1}}$ with $\Theta(\gamma, \gamma^{-1}) R^\Theta_{\gamma}$

$$\langle \Phi(s_1) | \Phi(s_2) \rangle_{A \otimes \mathcal{H}} = \sum_{\gamma, \gamma' \in \Gamma} \langle \delta_{\gamma} \otimes \Phi(s_1)(\gamma) | \delta_{\gamma'} \otimes \Phi(s_2)(\gamma') \rangle_{A \otimes \mathcal{H}}$$

$$= \sum_{\gamma, \gamma' \in \Gamma} \langle \Phi(s_2)(\gamma') | \Phi(s_1)(\gamma) \rangle_{\mathcal{H}} \bar{\Theta}(\gamma^{-1}, \gamma) \Theta(\gamma^{-1}, \gamma') R^\Theta_{\gamma^{-1}} \left(R^\Theta_{\gamma^{-1}}\right)^*$$

$$= \sum_{\gamma, \gamma' \in \Gamma} \langle \Phi(s_2)(\gamma') | \Phi(s_1)(\gamma) \rangle_{\mathcal{H}} \bar{\Theta}(\gamma^{-1}, \gamma) R^\Theta_{\gamma^{-1}} R^\Theta_{\gamma}$$

$$= \sum_{\gamma, \gamma' \in \Gamma} \langle (U_{\gamma'} s_2) | (U_{\gamma} s_1) \rangle_{\mathcal{H}} \bar{\Theta}(\gamma^{-1}, \gamma) \Theta(\gamma^{-1}, \gamma') R^\Theta_{\gamma^{-1}} R^\Theta_{\gamma}$$

$$= \sum_{\gamma, \gamma' \in \Gamma} \langle (U_{\gamma'} U_{\gamma} s_2) | (U_{\gamma} s_1) \rangle_{\mathcal{H}} \bar{\Theta}(\gamma^{-1}, \gamma') R^\Theta_{\gamma}$$

$$= \sum_{\gamma, \gamma' \in \Gamma} \langle (U_{\gamma'} s_2) | (U_{\gamma} s_1) \rangle_{\mathcal{H}} R^\Theta_{\gamma}$$

$$= \sum_{\gamma' \in \Gamma} \langle U_{\gamma'} s_2 | s_1 \rangle_{L^2(E)} R^\Theta_{\gamma'}.$$

This shows that the structures coincide. □

**Lemma 9 (left Hilbert $\mathcal{A}$-module).** The completion of the left pre-Hilbert $\mathcal{A}$-module $C_c(E)$ is isomorphic to $\mathcal{E} = \mathcal{H} \otimes \mathcal{A}$. The GNS representation of $\mathcal{E}$ with respect to the canonical trace $\tau$ on $\mathcal{A}$ is isomorphic to $L^2(E)$.

**Proof.** By equation (33) we have for $s \in C_c(E)$

$$\|s\|_{\mathcal{E}}^2 = \| \langle s | s \rangle_{\mathcal{E}} \|_{\mathcal{A}} \geq \langle s | s \rangle_{L^2(E)}.$$

Therefore, the completion of $C_c(E)$ with respect to $\| \cdot \|_{\mathcal{E}}$ is contained in the one with respect to $\| \cdot \|_{L^2(E)}$, i.e. in $L^2(E)$. But $C_c(E)$ is dense in $\mathcal{E}$.

We get the scalar product of the GNS representation with respect to $\tau$ for $s_1, s_2 \in C_c(E)$ from

$$\langle s_1 | s_2 \rangle_{\tau} = \tau(\langle s_2 | s_1 \rangle_{\mathcal{E}})$$

$$= \tau \left( \sum_{\gamma \in \Gamma} \langle U_\gamma s_1 | s_2 \rangle_{L^2(E)} R^\Theta_{\gamma} \right)$$

$$= \sum_{\gamma \in \Gamma} \langle U_\gamma s_1 | s_2 \rangle_{L^2(E)} \tau(\bar{\rho}(\delta_\gamma))$$

$$= \langle s_1 | s_2 \rangle_{L^2(E)}.$$
Since $C_c(E) \subset \mathcal{E} \subset L^2(E)$ is dense the GNS representation space is exactly $L^2(E)$. 

\[ \square \]

**Lemma 10 (A-compact operators).** The $A$-compact operators on $\mathcal{E}$ are given by

\[ K_A(\mathcal{E}) \simeq A^{\text{op}} \otimes K. \]  

(36)

Here $K$ denotes the compact operators on $\mathcal{H} = L^2(E|_T)$, and $A^{\text{op}}$ is the $C^*$-Algebra $C^*_r(\Gamma, \theta)^L$ generated by the left translations twisted with $\theta$.

**Proof.** For tensor products of left Hilbert modules we have in general

\[ K_A(A \otimes \mathcal{H}) \simeq K_A(A) \otimes K_{\mathcal{C}}(\mathcal{H}) \simeq A^{\text{op}} \otimes K. \]

The statement about $A^{\text{op}}$ is well known in the untwisted case since the opposite of left multiplication is right multiplication. It is easy to check that this holds in the twisted case also. 

\[ \square \]

Following our rationale from Section 4 we define a trace $\text{tr}_\tau$ and identify bounded module operators in $\mathcal{L}_A(\mathcal{E})$ with their images in $\mathcal{L}(L^2(E))$ under the faithful representation with respect to $\tau$.

As in Br"uning and Sunada (1992a, 1996) one shows, using Theorem 12:

**Theorem 13 (gauge-periodic operators).** Let $D$ be a symmetric gauge-periodic differential operator. The resolvent of $\bar{D}$ is $A$-compact, and $e^{-t\bar{D}^2}$ is $\text{tr}_\tau$-trace class.

**Theorem 14 (gauge-periodic module operators).** Let $D$ be a symmetric gauge-periodic differential operator. Then $D$ defines an $A$-elliptic operator $T$ such that the resolvents of $\bar{D}$ and $\bar{T}$ coincide (under the GNS representation).

**Proof.** Set $\mathcal{D}(T) := \mathcal{D}(D) = C_c^\infty(E)$. Then $\mathcal{D}(T) \subset \mathcal{E}$ dense, we set $T := D$ as operators on vector spaces.

$T$ is adjointable since $D$ is symmetric and gauge-periodic: For $s_1, s_2 \in C_c^\infty(E)$ we have

\[ \langle s_1 | Ds_2 \rangle_{\mathcal{E}} = \sum_{\gamma \in \Gamma} \langle U_\gamma s_1 | Ds_2 \rangle_{L^2(E)} R_\gamma^0 \]

\[ = \sum_{\gamma \in \Gamma} \langle DU_\gamma s_1 | s_2 \rangle_{L^2(E)} R_\gamma^0 \]

\[ = \sum_{\gamma \in \Gamma} \langle U_\gamma Ds_1 | s_2 \rangle_{L^2(E)} R_\gamma^0 \]

\[ = \langle Ds_1 | s_2 \rangle_{\mathcal{E}}. \]

Finally, $\text{ran}(1 + D^*D)$ is dense in $L^2(E)$ because $D$ is essentially self-adjoint; therefore, $T$ is regular. 

\[ \square \]

This allows to apply all of the spectral characterizations from the previous section.
Example 8 (periodic elliptic operator). A gauge-periodic operator is called periodic if the corresponding cocycle fulfills $\Theta \equiv 1$. If the group $\Gamma$ is abelian then we are back in the commutative case (see Definition 4) where ordinary Bloch theory applies. If $\Gamma$ is not abelian then it doesn’t apply, although the cocycle is trivial. But it is still covered by non-commutative Bloch theory, of course.

Example 9 (magnetic Schrödinger operator). In example 3 and remark 3 we saw that the magnetic Schrödinger operator with a magnetic field $b \in \Omega^2(X), db = 0, \left[\frac{i}{\pi} b\right] \in H^2(X, \mathbb{Z})$ is given by a (symmetric elliptic) Bochner-Laplace operator on a Hermitian line bundle $L$ over $X$ with curvature $b$. It is gauge-periodic with possibly non-constant cocycle if $H^1(X, S^1) = 0$ (see remark 3 and the work cited there). If $b$ is exact then the cocycle can be chosen to be constant. If the magnetic flux is integral ($b \in H^2(M, \mathbb{Z})$), see example 3 then the operator is periodic. If there is a periodic magnetic potential $a$ for $b = da$ (i.e. if the magnetic flux is 0) then the operator is strictly periodic in the usual sense of ordinary Bloch theory, i.e. it is a periodic operator on $L^2(X)$ (no magnetic translations, no bundles).

Example 10 (magnetic Schrödinger operator on $\mathbb{R}^2$). In the Euclidean case, if $\Gamma = \mathbb{Z}^2$ we end up with a rotation algebra $A_\theta$ where $\theta$ is given by the magnetic flux. So, from Theorem 9 we get band structure in the case of rational flux and weak genericity of Cantor spectra in the case of irrational flux. Since it is a criterion inside the algebra of symmetries it applies to the corresponding Pauli and Dirac operators as well.

Example 11 (magnetic Schrödinger operator on $\mathbb{H}^2$). To investigate the importance of the geometry it is interesting to study the hyperbolic analog, since the corresponding cocompact groups (Fuchsian groups) are non-amenable and therefore ‘opposite’ to the amenable groups in the Euclidean case. The analog of a constant magnetic field is a constant multiple of the volume form. Carey et al. (1998, 1999) compute $K$-groups and Kadison constants for twisted Fuchsian groups: Again, one has Kadison property if and only if the magnetic flux is rational. Marcolli and Mathai (1999a, b) study similar questions for good orbifolds.

Example 12 (gauge-periodic point perturbations). In Euclidean space, point perturbations provide explicitly solvable models for periodic Schrödinger operators. Brüning and Geiler (1999a, b) show how to define these types of operators more generally in our given geometric context (manifold with cocompact group action). If the point perturbation is gauge-periodic, then the perturbed operator is gauge-periodic in our sense, so that non-commutative Bloch Theory applies. In particular, periodic point perturbations of the magnetic Schrödinger operator with rational flux have band structure.

Elliptic operators on Hilbert module bundles.

Example 13: Mischenko and Fomenko (1980) extended the usual notion of an index of an operator by replacing Hilbert spaces by Hilbert modules: Let $\mathcal{A}$ be a $C^*$-algebra, $M$ a compact Riemannian manifold and $E$ a bundle over $M$ of Hilbert $\mathcal{A}$-modules (a Hilbert module bundle). On can define Sobolev norms as usual, now coming from an $\mathcal{A}$-scalar product. Thus one gets a scale of Sobolev-Hilbert $\mathcal{A}$-modules for which the
Sobolev lemma holds. Instead of the usual pseudo-differential operators whose coefficients are vector space endomorphisms one has $\mathcal{A}$-pseudo-differential operators with coefficients in the bundle $\mathcal{L}_\mathcal{A}(E) := \bigcup_{x \in M} \mathcal{L}_\mathcal{A}(E_x)$. They act in the usual way on the Sobolev-Hilbert modules. Symbols of $\mathcal{A}$-pseudo-differential operators are represented by section of $\mathcal{L}_\mathcal{A}(E)$. As in the scalar case, an elliptic operator has an $\mathcal{A}$-compact resolvent, hence it is $\mathcal{A}$-elliptic in the sense of Definition\[1\]. Furthermore, elliptic operators are $\mathcal{A}$-Fredholm and therefore have an index in $K_0(\mathcal{A})$.

A special case are the periodic elliptic operators: Let $X$ be a Riemannian manifold with properly discontinuous, isometric, cocompact action of a group $\Gamma$, and $D$ a $\Gamma$-periodic operator as in example\[8\]. Let $\rho$ be the right regular representation of $\Gamma$ on $\mathcal{A} := C_*^r(\Gamma)$. Then $X \times_\rho \mathcal{A}$ is an $\mathcal{A}$-bundle over $M$ on which $D$ acts. Besides, $\mathcal{A}$ carries the structure of a standard Hilbert-$\mathcal{A}$ module. If $D$ is elliptic then $D$ determines an elliptic operator on $X \times_\rho \mathcal{A}$.

A. Continuous fields of Hilbert spaces

We follow the classic reference Dixmier and Douady (1963).

**Definition 11 (continuous fields of Banach and Hilbert spaces).** Let $B$ be a topological space, $(E(z))_{z \in B}$ a family of Banach spaces. The linear space $\Pi := \prod_{z \in B} E(z)$ is called space of all vector fields. A continuity structure on $\Pi$ is defined by a subspace $\Lambda \subset \Pi$ such that:

1. $\Lambda$ is a $C_\infty(B)$-submodule of $\Pi$.
2. $\forall z \in B: \forall \xi \in E(z) : \exists x \in \Lambda: x(z) = \xi$
3. $\forall x \in \Lambda: (z \mapsto \|x(z)\|) \in C_\infty(B)$
4. $\forall x \in \Pi: (\forall \varepsilon > 0 : \forall z \in B : \exists x' \in \Lambda, neighborhood U \ni z : \forall z' \in U: \|x(z') - x'(z')\| < \varepsilon) \Rightarrow x \in \Lambda$

$\mathcal{E} := ((E(z))_{z \in B}, \Lambda)$ is called continuous field of Banach spaces. If the fibers $E(z)$ are Hilbert spaces we have a continuous field of Hilbert spaces. The scalar product is automatically continuous.

Condition\[4\] is a completeness condition: If a vector field $x \in \Pi$ can be locally approximated arbitrarily well by continuous vector fields then it is continuous.

**Proposition 6 (defining submodule).** Let $B, \Pi$ be as above and $\Lambda \subset \Pi$ a subspace with

1. $\forall z \in B: \{x(z) : x \in \Lambda\} =: \Lambda(z)$ dense in $E(z)$ and
2. $\forall x \in \Lambda: (z \mapsto \|x\|) \in C_\infty(B)$.

Then there is a unique continuity structure $\tilde{\Lambda}$ on $\Pi$ with $\tilde{\Lambda} \supset \Lambda$. $\tilde{\Lambda}$ is given by

$$\tilde{\Lambda} = \{x \in \Pi : \forall z \in B : \varepsilon > 0 \exists neighborhood U \ni z, x' \in \tilde{\Lambda} : \forall z' \in U: \|x(z') - x'(z')\| < \varepsilon\}.$$
Lemma 11 (continuous fields and Banach space bundles). A continuous field of Banach spaces $E$ defines a Banach space bundle over $B$ so that the continuous sections $C(E)$ are the continuous vector fields of $E$.

Proof (Sketch of the proof). As a set $E := \prod_{z \in B} E(z)$. We choose the topology so that the natural projection $\pi : E \to B$ is continuous and open. The topology is generated by the tubular neighborhoods $T(U, x, \varepsilon) := \{\xi \in E \mid \pi(\xi) \in U \land \|\xi - x(\pi(\xi))\| < \varepsilon\}$ for open sets $U \subset B$, continuous fields $x \in E$ and $\varepsilon > 0$. It is easy to check that the tubular neighborhoods generate a topology on $E$ with the desired properties. On the fibers $E(z)$ it induces the strong topology since the intersections $E(z) \cap T(U, x, \varepsilon)$ of the fibers with the tubular neighborhoods are just norm balls in the fiber.

Lemma 12 (continuous field as Hilbert $C^*$-module). A continuous field of Hilbert spaces $E = ((E(z))_{z \in B}, \Lambda)$ over $B$ defines a Hilbert $C_\infty(B)$-module structure on $\Lambda$. Vice versa: A Hilbert $C_\infty(B)$-module defines a continuous field of Hilbert spaces, and this correspondence is one-to-one.

B. Hilbert $C^*$-modules

Usually Hilbert $C^*$-modules are defined to be right modules. We define these and the left modules and list basic properties and objects connected to them.

Definition 12 ((right) Hilbert module). Let $A$ be a $C^*$-algebra. A right $A$-module $E$ is called (right) pre-Hilbert $A$-module if it is endowed with a map $\langle \cdot | \cdot \rangle : E \times E \to A$ with the following properties:

1. $\langle e | f + g \rangle = \langle e | f \rangle + \langle e | g \rangle$ for $e, f, g \in E$.
2. $\langle e | f \lambda \rangle = \langle e | f \rangle \lambda$ for $e, f \in E, \lambda \in \mathbb{C}$.
3. $\langle e | fa \rangle = \langle e | f \rangle a$ for $e, f \in E, a \in A$.
4. $\langle f | e \rangle = \langle e | f \rangle^* \lambda$ for $e, f \in E$.
5. $\langle e | e \rangle \geq 0$ in $A$ for $e \in E$, and $\langle e | e \rangle = 0 \iff e = 0$.

Then the map $E \ni e \mapsto \sqrt{\|\langle e | e \rangle\|_A}$ defines a norm on $E$. The closure of $E$ is defined as the completion of $E$ as Banach space with this norm.

$E$ is called (right) Hilbert $A$-module if $E$ is complete in this norm.

An operator $T \in \mathcal{L}(E)$ is called adjointable if there is $T^* \in \mathcal{L}(E)$ such that for all $e, f \in E: \langle e | T f \rangle = \langle T^* e | f \rangle$. The set of adjointable operators is denoted by $\mathcal{L}_A(E)$.

For $e, f \in E$ we define an operator $\pi_{e,f}$ by

$$\pi_{e,f} : E \ni x \mapsto \langle f | x \rangle \in E.$$ 

We set $\mathcal{F}_A(E) := \text{span}\{\pi_{e,f} \mid e, f \in E\}$ and call this the set of $A$-finite operators. The set $\mathcal{K}_A(E)$ of $A$-compact operators is the closure of $\mathcal{F}_A(E)$ in $\mathcal{L}_A(E)$.

The brackets indicate that by Hilbert module we mean a right Hilbert module.

---

8 A bundle has a continuous open surjection onto the base, but is not necessarily locally trivial. However, for a locally compact base and finite dimensional fibers this follows from the existence of the projection.
Definition 13 (left Hilbert module). Let $A$ be a $C^*$-algebra. A left $A$-module $E$ is called left pre-Hilbert $A$-module if it is endowed with a map $\langle \cdot | \cdot \rangle : E \times E \to A$ with the following properties:

1. $\langle e + f | g \rangle = \langle e | g \rangle + \langle f | g \rangle$ for $e, f, g \in E$.
2. $\langle \lambda e | f \rangle = \lambda \langle e | f \rangle$ for $e, f \in E, \lambda \in \mathbb{C}$.
3. $\langle ae | f \rangle = a \langle e | f \rangle$ for $e, f \in E, a \in A$.
4. $\langle f | e \rangle = \langle e | f \rangle^*$ for $e, f \in E$.
5. $\langle e | e \rangle \geq 0$ in $A$ for $e \in E$, and $\langle e | e \rangle = 0 \Leftrightarrow e = 0$.

Then the map $E \ni e \mapsto \sqrt{\langle e | e \rangle}$ defines a norm on $E$. The closure of $E$ is defined as the completion of $E$ as Banach space with this norm.

$E$ is called left Hilbert $A$-module if $E$ is complete in this norm.

An operator $T \in \mathcal{L}(E)$ is called adjointable if there is $T^* \in \mathcal{L}(E)$ such that for all $e, f \in E$: $\langle e | T f \rangle = \langle T^* e | f \rangle$. The set of adjointable operators is denoted by $\mathcal{L}_A(E)$.

For $e, f \in E$ we define an operator $\pi_{e,f}^L : E \ni x \mapsto \langle x | e \rangle f \in E$.

We set $\mathcal{F}_A(E) := \text{span} \{ \pi_{e,f}^L \mid e, f \in E \}$ and call this the set of $A$-finite operators. The set $K_A(E)$ of $A$-compact operators is the closure of $\mathcal{F}_A(E)$ in $\mathcal{L}_A(E)$.

Remark 12 (basic properties).

1. If $E$ is a pre-Hilbert $A$-module, $e \in E$, then Definition 12 implies $\langle e | e \rangle \geq 0$.

2. If $E$ is a pre-Hilbert $A$-module, $e, f \in E, a \in A$ then we have:

$$\langle ea | f \rangle = \langle f | ea \rangle^* = \langle (f | e) a \rangle^* = a^* \langle f | e \rangle^* = a^* \langle e | f \rangle$$

I.e. we have $\mathbb{C}$- and $A$-sesqui-linearity.

3. The $\mathbb{C}$-sesqui-linearity follows for unital $A$ from the $A$-sesqui-linearity.

4. For $e, f \in E$ we have $\pi_{e,f}^* = \pi_{f,e}^*$ so that indeed $\mathcal{F}_A(E) \subset \mathcal{L}_A(E)$.

5. $\mathcal{L}_A(E)$ and $K_A(E)$ are $C^*$-algebras, the former is the multiplier algebra of the latter (see e.g. Wegge-Olsen, 1993, chapter 15).

6. Everything analogous for left Hilbert modules.

7. $A \times A \ni (a, b) \mapsto a^* b \in A$ together with multiplication of $A$ on $A$ on the right gives the standard Hilbert $A$-module structure on $A$.

8. $A \times A \ni (a, b) \mapsto a b^* \in A$ together with multiplication of $A$ on $A$ on the left gives the standard left Hilbert $A$-module structure on $A$.

Definition 14 (free and projective Hilbert modules). A Hilbert $A$-module is called free if it is a free module over $A$. It is called projective if it is a direct summand of a free module.

Lemma 13 (left and right Hilbert modules). Let $A$ be a $C^*$-algebra and $(E, \langle \cdot | \cdot \rangle)$ a left pre-Hilbert $A$-module over $A$. Then

$$\langle \cdot | \cdot \rangle_{E^\text{op}} : E^\text{op} \times E^\text{op} \to A^\text{op}$$

$$(e^\text{op}, f^\text{op}) \mapsto ((f | e))^\text{op} \quad (37)$$
defines on \( E = E^{\text{op}} \) (equality as vector spaces) the structure of a pre-Hilbert \( A^{\text{op}} \)-module.

Furthermore, for a left Hilbert \( A \)-module \( (E, \langle \cdot | \cdot \rangle) \) we have \( \mathcal{F}_A(E) \simeq \mathcal{F}_{A^{\text{op}}}(E^{\text{op}}) \) and therefore \( \mathcal{K}_A(E) \simeq \mathcal{K}_{A^{\text{op}}}(E^{\text{op}}) \) and \( \mathcal{L}_A(E) \simeq \mathcal{L}_{A^{\text{op}}}(E^{\text{op}}) \).

**Proof.** It is well known that right \( A \)-modules \( E \) and left \( A^{\text{op}} \)-modules \( E^{\text{op}} \) are in one-to-one correspondence. So we just have to verify the corresponding Hilbert module structures: Let \( e^{\text{op}}, f^{\text{op}} \in E^{\text{op}} \), \( a^{\text{op}} \in A^{\text{op}} \). We denote by \( a \) and \( a^{\text{op}} \) corresponding elements in \( A^{\text{op}} \) resp. \( A \). then

\[
\langle e^{\text{op}} | f^{\text{op}} a^{\text{op}} \rangle^{\text{op}} = \langle\langle e^{\text{op}} | (af)^{\text{op}} \rangle^{\text{op}} \rangle^{\text{op}} = \langle af | e \rangle = \langle (f | e)^{\text{op}} a^{\text{op}} \rangle = \langle e^{\text{op}} | f^{\text{op}} \rangle^{\text{op}} a^{\text{op}}.
\]

Since \( E = E^{\text{op}} \) as Banach space we have \( \mathcal{L}(E) \simeq \mathcal{L}(E^{\text{op}}) \). Furthermore, for \( e, f, x \in E \)

\[
\pi_{e,f}(x) = e \langle f | x \rangle = \langle (f \langle x \rangle)^{\text{op}} e^{\text{op}} \rangle = \langle x^{\text{op}} | f^{\text{op}} \rangle^{\text{op}} e^{\text{op}} = \pi_{f^{\text{op}},e^{\text{op}}},
\]

so that \( \mathcal{F}_A(E) \) and \( \mathcal{F}_{A^{\text{op}}}(E^{\text{op}}) \) are isomorphic, and so are the corresponding closures and multiplier algebras. \( \square \)

**Remark 13 (standard module).** For the standard Hilbert \( A \)-module structure on \( A \) it is well known that \( \mathcal{F}_A(A) = A, \mathcal{K}_A(A) = A \) and \( \mathcal{L}_A(A) = \mathcal{M}(A) \). If we denote by \( A^L \) the standard left Hilbert \( A \)-module then Lemma 13 shows: \( \mathcal{F}_A(A^L) \simeq A^{\text{op}}(A^L) = A^{\text{op}} \).

**C. GNS representation for Hilbert \( C^\ast \)-modules**

Let \( A \) be a \( C^\ast \)-algebra, \( \tau \) a state on \( A \) and \( E \) a Hilbert \( A \)-module. Analogously to the well know GNS representation of Banach *-algebras we define a scalar product \( \langle \cdot | \cdot \rangle_\tau \) on \( E \) by

\[
\langle x | y \rangle_\tau := \tau(\langle x | y \rangle_E) \text{ for } x, y \in E.
\]

\( N_\tau := \{ x \in E \mid \langle x | x \rangle_\tau = 0 \} \) is the corresponding null space. Then the GNS representation space \( E_\tau \) is given by the completion of \( E/N_\tau \) with respect to \( \langle \cdot | \cdot \rangle_\tau \). \( L \in \mathcal{L}_A(E)_\tau \)

\[9\] \( A^{\text{op}} \) and \( A \) are identical as Banach spaces, and in this sense \( a^{\text{op}} = a \).

\[10\] For left Hilbert modules the scalar product must be reversed so that one gets complex linearity on the correct entry.
acts continuously on $x \in \mathcal{E}_\tau$ because
\[
\|Lx\|_2^2 = \langle Lx | Lx \rangle_\tau = \tau(\langle Lx | Lx \rangle_\mathcal{E}) = \tau(\langle x | L^*Lx \rangle_\mathcal{E}) \\
\leq \tau(\langle x | x \rangle_\mathcal{E})\|L^*L\|
\]
\[
= \|x\|_2^2 \|L\|_2.
\]
Thus we have a $*$-representation of $\mathcal{L}_\mathcal{A}(\mathcal{E})$ in $\mathcal{L}(\mathcal{E}_\tau)$.

If $\tau$ is faithful then $N_\tau = 0$ so that the representation is faithful.

If $\mathcal{E} = \mathcal{A}$ with $\langle a | b \rangle_\mathcal{E} = a^*b$ is the standard Hilbert $\mathcal{A}$-module then we get back the usual GNS representation of the multiplier algebra $\mathcal{L}_\mathcal{A}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$ and, by restriction, the GNS representation of $\mathcal{K}_\mathcal{A}(\mathcal{A}) = \mathcal{A}$.

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