Adiabatic feedback control of Hamiltonian systems.

A.E. Allahverdyan, K.G. Petrosyan and D.B. Saakian
Yerevan Physics Institute, Alikhanian Brothers St. 2, Yerevan 375036, Armenia.
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We study feedback control of classical Hamiltonian systems with the controlling parameter varying slowly in time. The control aims to change system’s energy. We show that the control problems can be solved with help of an adiabatic invariant that generalizes the conservation of the phase-space volume to control situations. New mechanisms of control for achieving heating, cooling, entropy reduction and particle trapping are found. The feedback control of a many-body system via one of its coordinates is discussed. The results are illustrated by two basic models of non-linear physics.

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Physics met control theory yet in XIX’th century: the notorious Maxwell’s demon poses a problem of building a control system that will reduce the entropy (increase the order) of a statistical system [1, 2]. Founders of cybernetics recognized the entropy reduction as one of the basic goals of control [3, 4]. This is especially relevant, since the statistical description—in particular, thermodynamical relations and concepts such as entropy—are needed not only for macroscopic systems, but also for few-body chaotic ones [5, 6]. Deep implications of the entropy-information-control relations were studied in [2, 4]. Theoretical and experimental methods of controlling physical systems have recently undergone an explosive development [7, 8, 9, 10]. The focus is on non-linear and chaotic systems in view of their numerous applications.

The potential of physical ideas applicable to the control science is, however, far from being exhausted. Here we explore a feedback control of ergodic Hamiltonian systems, where a slow motion of the controlling parameter leads to the existence of an adiabatic invariant. This fact helps to solve the control problem on a general level. Feedback means that the control parameter depends on the current state of the system, while in the non-feedback control the motion of the control parameter is prescribed. In this respect we built up on the non-feedback adiabatic control of ergodic systems, where the adiabatic invariance of the phase-space volume is well-known [11]. Despite of its general importance, from the control viewpoint this method has several drawbacks: i) The microcanonical entropy is conserved (or increases for non-adiabatic processes [12]), so that the goal of the control is not achieved. ii) The method does not function for the important case of cyclic influences of the controlling parameters, i.e., it insists on permanent modifications of the system. We show that these drawbacks are absent for adiabatic feedback control, while the main advantage—independence on the details of the system under assuming ergodicity of certain observables—is kept.

Consider a system with $N$ degrees of freedom and Hamiltonian $H(p, q, R)$. The equations of motion read

$$
p = -\partial_q H(p, q, R), \quad \dot{q} = \partial_p H(p, q, R),$$  \hspace{1cm} (1)

where $q = (q_1, ..., q_N)$ and $p = (p_1, ..., p_N)$ are, respectively, the coordinates and momenta, and where $R$ is the control parameter. The purpose of varying $R$ in time is to change system’s energy $E$ in a desired way. This setup of control is especially relevant for chaotic systems, since energy is the basic constant of motion [7, 8].

The change of $R$ can be sizable, but the speed of $R$ is small and depends on the dynamical variables $p$ and $q$ via a phase-space observable $F$ (feedback). Behind the scene of this description is a measurement of $F$ (sensor) and subsequent engineering (actuator) that leads to

$$\tau_R \dot{R} = F(z, R), \quad z \equiv (q, p),$$ \hspace{1cm} (2)

where $\tau_R$ is much larger than the characteristic time $\tau_S$ of the system (defined with respect to the $R =$const dynamics). The derivative of the control parameter depending on the state of the system is standard in control practice [7, 8]. For a general $F$ the problem is hardly tractable analytically. However, an important paper [13] shows that an one-dimensional ($N = 1$) Hamiltonian system admits an adiabatic invariant provided

$$F(z, R) = f(R) \varphi(z).$$ \hspace{1cm} (3)

Here $\varphi$ and $f$ depend on the fast and slow variables, respectively; these dependencies are thus factorized in [3]. Assuming [3], we extend this invariant to systems with ergodic observables and apply it to control problems.

The below derivation is based on the fact that $R$ and energy $E$ are slow variables, since they change on times $\sim \tau_R$. For times $\tau_R \gg \tau \gg \tau_S$ we have from [11 2 3]:

$$\frac{dE}{dt} \equiv \frac{1}{\tau} \left[ H(z_{t+\tau}, R_{t+\tau}) - H(z_t, R_t) \right]$$ \hspace{1cm} (4)

$$= \int_t^{t+\tau} \frac{ds}{\tau} \frac{dH}{dz}(z_s, R_s) = \int_t^{t+\tau} \frac{ds}{\tau} \dot{R}_s \frac{\partial H}{\partial R}(z_s, R_s),$$

$$= \frac{1}{\tau_R} f(R_t) \int_t^{t+\tau} \frac{ds}{\tau} \varphi(z) \frac{\partial H}{\partial R}(z_s, R_t) + o(\frac{\tau}{\tau_R}).$$ \hspace{1cm} (5)

The last integral refers to the dynamics with $R_t =$const. Now note the Liouville theorem $dz = dz_t$ and energy conservation $H(z_{t+\tau}) = H(z_t) = E_t$, and denote $w(z) \equiv$
\( \varphi(z) \partial_R H(z, R_t) \). Noting the microcanonical distribution
\[
\mathcal{M}(z) = \frac{\delta[E - H(z, R)]}{\partial_E \Omega(E, R)}, \quad \Omega(E) = \int dz \, \theta[E - H(z, R)],
\]
where \( \delta(x) \) and \( \theta(x) \) are, respectively, the delta and step function, and where \( \Omega \) is the phase-space volume, we get
\[
\int dz \, w(z) \mathcal{M}(z, E_t) = \frac{1}{\tau} \int_t^{t+\tau} ds \int dz \, w(z) \mathcal{M}(z, E_t),
\]
where \( w(z, s) \) is the trajectory at time \( s \) with the initial condition \( z_t \). If \( w(z, R) \) is an ergodic observable of the \( R_t = \text{const} \) dynamics, then for \( \tau \gg \tau_S \) the time-average in (7) does not depend on the initial condition \( z_t \), the integration over \( z_t \) in (7) drops out, and we get from (6) that the time-average in (6) is equal to the microcanonical average at the energy \( E_t \):
\[
\frac{dE}{d\tau} = \frac{f(R_t)}{\tau_R} \int dz \, \mathcal{M}(z, E_t, R_t) \varphi(z) \frac{\partial H}{\partial R}(z_t, R_t).
\]
Assuming ergodicity of \( \varphi(z) \) we get from (2, 3)
\[
\frac{dR}{d\tau} = \frac{1}{\tau_R} f(R_t) \int dz \, \mathcal{M}(z, E_t, R_t) \varphi(z).
\]
It is seen that (5) amounts to the conservation condition
\[
\frac{dI}{d\tau} = 0, \quad I(E, R) \equiv \int dz \, \varphi(z) \theta[E - H(z, R)],
\]
where \( I \) does not depend on the initial condition \( z_t \) of the trajectory. Given the initial values \( E_t \) and \( R_t \), the final \( E_f \) and \( R_f \) are found self-consistently from (9) and from
\[
I(E_f, R_f) = I(E_t, R_t).
\]
For \( \varphi = 1 \) (no feedback) we get from (10) the conservation of the phase-space volume \( \Omega \). For an isolated ergodic system the entropy is defined as \( S = \ln \Omega \). This definition satisfies to all reasonable features of entropy, e.g., for the temperature defined via \( 1/T = \partial_E \ln \Omega(E) \), the integration by parts leads to equipartition (8):
\[
\langle x \partial_x H \rangle = T,
\]
where \( x \) is any canonical coordinate of momentum, and \( \langle ... \rangle \) is the average over microcanonical distribution (8).

For non-adiabatic processes \( S \) normally increases, thus confirming the second law (8). In the standard thermodynamical limit \( S \) goes to the more usual expression \( \dot{S} = \ln[\partial_E \Omega] \). Neither of the above important features holds if we apply \( \dot{S} \) for a finite system.

For the feedback case \( \varphi \neq 1 \), (11) shows that \( \Omega \) is not conserved, and the entropy \( S \) can decrease, as we see below. Thus the basic goal of control will be attained.

As an application consider \( n \) particles moving between an infinite wall located at \( q = 0 \) and a piston with the controlled coordinate \( q = R > 0 \). The Hamiltonian is
\[
H = \sum_{i=1}^{n} \frac{p_i^2}{2m_i}, \quad (m_i \text{ are masses}) \quad \text{from the infinite wall interactions that lead to elastic scattering and apart from weak inter-particle couplings that ensure the ergodicity of the system (4).}
\]

The simplest type of feedback is realized via the coordinate \( q \) of one of the particles:
\[
\varphi = \varphi(q). \quad \text{The conservation of } I \text{ leads from (10)}
\]
\[
E_i^{n/2} \int_0^{R_i} dq q^{n-1} = E_i^{n/2} \int_0^{R_i} dq q^{n-1}.
\]
We start with \( n = 1 \) (Fermi accelerator), where the \( R = \text{const} \) motion is obviously ergodic. This model was intensively studied in plasma and accelerator physics, nonlinear physics and astrophysics (2, 17). A question that created the interest to the model is how to increase system’s energy \( E \) (heating)?

The non-adiabatic feedback control is able to heat up the system only upon decreasing \( R \), since the phase-space volume \( \propto \sqrt{ER} \) is conserved. This is not realistic, since involves a permanent and substantial modification of system’s parameter.

A non-adiabatic heating studied in (2, 17) amounts to an oscillating motion of the piston, e.g., \( R(t) = R_0 + a \sin \frac{\pi}{T} t \) with period \( T \). The maximal achievable energy is \( E_{\text{max}} \sim R_0 a / T \). Once \( R_i \) is fixed, \( E_{\text{max}} \) grows by increasing \( a \) (large amplitude) or by decreasing \( T \) (large piston velocity). Thus it is impossible to increase the energy as much as desired (14). The adiabatic feedback control solves this problem as follows. Take \( \varphi(q) = q - a \), where \( a > 0 \) is a constant. We get from (12)
\[
\sqrt{E} R_i(R_i - 2a) = \sqrt{E} R_i(R_i - 2a).
\]
Taking \( R_i \to 2a \) (provided \( R_i \) is not close to \( 2a \)), the final energy is increased as much as desired. Eq. (9) for the piston is \( \tau_R \frac{dR}{d\tau} = \frac{2R}{R_0} (R - 2a) \), or (taking \( f = \text{const} \))
\[
R(\tau) = 2a + (R_i - 2a)e^{f\tau / (2\tau_i)}.
\]
For \( f < 0 \), \( |f\tau| \gg \tau_R \) suffices to make \( R_i \) very close to \( 2a \). Eq. (12) also shows that for a more general feedbacks function \( \varphi \), the strong heating exists for \( \int_0^{R_i} dq \varphi(q) \to 0 \).

There are three regimes of (13) considered separately:

i) for \( R_i + R_t > 2a > R_f > R_0 \) the heating is achieved by expanding the system. Taking the feedback out and squeezing \( R \) adiabatically back to \( R_i \), we heat up the particle even more and complete the cycle. The reason of the feedback heating is seen from (2, 17): \( \dot{R} = f(a - q) \). When \( R > a \), the piston tends to move towards the approaching particle, and during the resulting collision the particle gains energy. However, \( \dot{R} > 0 \) when the particle is far from the piston, and thus in average \( \dot{R} \) increases.

ii) For \( R_i > R_f > 2a \) the particle is heated by squeezing.

iii) The case \( R_i < R_f < 2a \) and \( R_i + R_f < 2a \) is interesting, since in contrast to the above two cases the heating is not strong, but leads to the entropy decrease. Now in (13) we should take \( f > 0 \).
The ability of energy decreasing (cooling) is not less important in applications. A scenario of cooling is illustrated by in the regime $R_t + R > 2a > R > R_t$, which is realized by taking $f > 0$ in . Now $\dot{R} = f(q - a)$, and moving out of the approaching particle, the piston gains energy. In the second step we take the feedback out and expand the piston back to its original position $R_t$, thereby cooling the system even more, reducing its entropy and completing the cycle. To lower the energy as much as desired we need several cycles. This contrasts the heating situation, where one cycle sufficed. This is because $\int_0^R dq \varphi(q)$ can turn to zero, but cannot be infinite, since $\varphi$ gives the speed of $R$. Note that without feedback there is no obvious cooling cyclic process.

For a finite number of particles $n$ one can design cooling and heating processes analogously to the $n = 1$ case. Extensions to more general models of chaotic systems, e.g., billiards are straightforward.

What about controlling a many-body ergodic system, where $n \gg 1$ and the feedback still goes over one coordinate: $\varphi = \varphi(q)$? This coordinate may belong to the particle with a large mass that moves slowly (due to equipartition) and is amenable for the measurement. Now note that $\varphi(q) \sim e^{bq}$ ($|b| \sim 1$) is unphysical: if $b > 0$ this will lead to a huge speed of $L$ on the short times; see . If $b < 0$ this speed will be too small and the control takes a huge time. Then for $n \gg 1$ and $R_t > 3$ the integral is dominated by the right end: $I \sim \varphi(R) E^{n/2} R^n$. This brings for the change of the microcanonical entropy $S = n \ln[R \sqrt{E}]$: $S_t - S_1 = \ln\frac{\varphi(R_t)}{\varphi(R_1)}$. Normally $S_t - S_1 \sim 1$, though with a precise tuning we can achieve $S_t - S_1 \sim \ln n$; take $\varphi(R) = R - a$ and $R_t - a \sim n^{-1}$. Thus when controlling a macroscopic system via a single degree of freedom, the effect of feedback is small. This conclusion is more general: if the feedback uses the coordinate $q$, implies $I = \int dq \varphi(q) e^{\varphi(q)}$, where $S(q) = \ln f(q)$ is the entropy of the system with a fixed value of $q$. Once normally $S(q) \sim n$ (for $n \gg 1$), the conclusion follows along the above line.

Our last example is interesting in two respects. First this is a paradigmatic model of non-linear science and control theory: the oscillator $H = \frac{p^2}{2} - R \cos q$, where the amplitude $R > 0$ is taken as the control variable. The periodic boundary conditions are ensured by $-\pi \leq q \leq \pi$ and $\varphi(q, p) = \varphi(q + 2\pi, p)$. The model is basic for particle-wave interactions, where the potential $-R \cos q$ models a harmonic wave (in the system of reference where the wave is standing), and where $E > R$ and $E < R$ refer to the free (rotating) and captured (oscillating) particles, respectively. Second, the system is non-ergodic: for $E > R$ the phase-space of the $R恒$ dynamics consists of two ergodic components supporting the rotational motion with, respectively, $p > 0$ and $p < 0$ (the rotating particle changes the sign of its momentum, while the energy is insensitive to this sign). In contrast, the oscillatory motion for $R > E > -R$ is globally ergodic. When $R$ is time-dependent some trajectories cross the separatrix $E = R$ and move from one ergodic component to another. Although the model is non-ergodic, $I$ defined in (with the integration over the whole phase-space irrespectively of ergodic components) is conserved if the observable $w(z) = \varphi(q, p) \delta_R H(q, p) = -\varphi(q, p) \cos q$ is $p \rightarrow -p$ symmetric. Indeed, if two ergodic components are possible for a given energy, $w(z)$ does not depend on which component the particle moves. The adiabatic condition $\tau_R \gg \tau_S$ can be satisfied as well: though the $R = \text{const}$ motion on the separatrix has an infinite period (due to unstable fixed points $q = \pm \pi$), the fraction of particles trapped by the separatrix is negligible (measure zero). Thus the derivation applies. For non-feedback processes the conservation of the phase-space volume $\Omega$ was first shown in .

We now focus on the following question: how to trap particles by means of a cyclic change of the amplitude $R$? This is a version of the cooling problem. First note that the phase-space volume $\Omega$ is expressed from as (changing variables for as $\epsilon \sin \xi = \sin \frac{x}{2}$)

$$
\Omega = \sqrt{2R} \left[ \mathcal{E}(\epsilon^2) - (1 - \epsilon^2) \mathcal{K}(\epsilon^2) \right], \quad 0 < \epsilon < 1, \quad (15)
$$

$$
\Omega = \sqrt{2R} \epsilon \mathcal{E}(\epsilon^2), \quad 1 < \epsilon, \quad (16)
$$

where $\epsilon = \sqrt{E + \frac{R}{2\pi}}$, and where $\mathcal{E}(x) = \int_0^{\pi/2} d\xi (1 - x \sin^2 \xi)^{1/2}$ and $\mathcal{K}(x) = \int_0^{\pi/2} d\xi (1 - x \sin^2 \xi)^{-1/2}$ are the elliptic integrals. The two regimes and correspond, respectively, to the trapped and free motions.

As the feedback function we take $\varphi = |\cos(\frac{x}{2})| = \sqrt{1 - \sin^2(\frac{x}{2})}$, which is invariant with respect to $p \rightarrow -p$ and $q \rightarrow q + 2\pi$. The adiabatic invariant reads from

$$
I = \frac{\pi}{2} \epsilon^2 \sqrt{\frac{R}{2}}, \quad 0 < \epsilon < 1, \quad (17)
$$

$$
I = \epsilon \sqrt{\frac{R}{2}} \left[ \sqrt{1 - \frac{1}{\epsilon^2}} + \epsilon \arcsin(\frac{1}{\epsilon}) \right], \quad 1 < \epsilon. \quad (18)
$$

The two regimes have the same meaning as above. The behavior of $I$ and $\Omega$ is presented in Fig. 1. Particles with
the energy above the initial separatrix $E = E_i$ but below the final separatrix $E = E_f$ get trapped. In the non-feedback case the energy $E$ responds to the amplitude increase in a non-linear way: the low energy motion decreases its energy, as intuitively expected for almost harmonic oscillations, while the energies around the separatrix increase. In contrast, the feedback linearizes the energy response: now all energies decrease. This is because due to $R \propto |\cos(\theta)|$, the amplitude does not move when the particle is around the unstable fixed points $q = \pm \pi$, which are responsible for the strong nonlinearity.

We shall now combine the two scenarios (with and without feedback) so as to get trapping via a cyclic change of $R$. Apply first the feedback control changing $R$ from $R_i = 1$ to $R_f = 2$; see Fig. 1. Then apply the non-feedback control bringing $R$ back to its initial value $1$. It is seen from Fig. 1 and (15–18) that all energies ($E < E_m = 1.249$) get trapped after the cyclic process. Here $E_m$ is defined via solving two coupled equations: $I(E_m, R_i) = I(E^*, R_f)$ and $\Omega(E = R_i, R_i) = \Omega(E^*, R_f)$, i.e., in the feedback part of the process $E_m \to E^* = 1.129$, while in the non-feedback part $E^*$ goes precisely to the separatrix $E = E_i$. These arguments are qualitative, since in the feedback case $R_f$ is not given a priori, but comes out from the common solution of $E_i$ and can depend on $E_f$. However, the numerical results presented and explained in Table I confirm the arguments.

In conclusion, we presented an adiabatic feedback method for controlling classical Hamiltonian systems. The method is based on an adiabatic invariant. This requires ergodicity (time average is equal to the microcanonical one) of certain observables and makes explicit the change of the microcanonical entropy due to control. We illustrated the method via two basic models of the non-linear science: Fermi’s accelerator and the non-linear oscillator. For the Fermi accelerator—which was intensively studied as model of plasma heating— the adiabatic feedback method offers efficient schemes of heating that, in particular, may lead to entropy decrease. For the non-linear oscillator (wave-particle interaction) transitions are possible between ergodic and non-ergodic regimes of motion. They correspond to the particle (de)trapping by the wave. We designed schemes for particle trapping via cyclic change of the wave amplitude, and noted how the feedback changes qualitatively the response of this non-linear model. It was seen that when controlling an ergodic $n$-particle system ($n \gg 1$) via monitoring few of its coordinates, the entropy decrease $\Delta S$ due to feedback is $\Delta S \lesssim \ln n$, while the entropy itself is $S \sim n$. This is not unexpected, since, e.g., the entropy difference between an organism and a chemical substance of the same weight is also much smaller than each of those entropies. If, however, there is an efficiently controlled ($\Delta S \sim S$) macroscopic system, we foresee it to be non-ergodic, so that the adiabatic invariance is absent.

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TABLE I: Numerical illustration of the cyclic trapping method. The energies are obtained from solving numerically the oscillator equation of motion $\ddot{q} + R_i \sin q = 0$ and the feedback equation $\tau_R \dot{R} = |\cos(\theta)|$ in the time-interval from $t = 0$ till $t_f = 2 \times 10^4$. Here $R_i = R(0) = 1$ and $\tau_R = 10^{-3}$. Then a non-feedback process was applied via equation of motion $\ddot{q} + [R(t_f) - t(t_f)] \sin q = 0$ till the final time $t_f = t_i + \tau_R(R(t_i) - R_i)$, so that $R(t_f) = R_i = 1$. The adiabatic invariant $I_{S, \tau}$ is conserved: $\frac{|\dot{q}|}{\dot{R}} \sim 2 \times 10^{-4}$. Expectedly, its dependence on the initial conditions $(q(0), p(0))$ for a given $E(0)$ is very weak. The initial energies $E_i = E(0)$ are above the separatrix $E = 1$ (free motion), while the final ones are below it (trapping).

| $E_i$ | 1.26 | 1.22 | 1.2 | 1.15 | 1.1 | 1.0 |
|------|------|------|------|------|------|------|
| $E_f$ | 0.9976 | 0.9791 | 0.9709 | 0.9423 | 0.9111 | 0.8429 |

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