Radiation From a Sine-Gordon Soliton Propagating in an External Potential

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The interaction of a fast moving sine-Gordon soliton with an external periodic potential is examined. The resulting equation of motion for the collective coordinate representing the position of the soliton is given in relativistic form. We examine the radiation emitted due to the interaction of the soliton with the potential and we calculate the potential dependent part of the time evolution equation for the creation and annihilation operators for fluctuations.

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I. INTRODUCTION.

In this note we consider a sine-Gordon soliton in the presence of a periodic potential. The soliton is fast moving, thus relativistic effects shall be taken into account. In particular we expect the total energy of the system to be divided into two contributions, one equal to $M/\sqrt{1 - B^2}$ corresponding to a soliton of mass $M$ moving at velocity $B$ and another pertaining to the field of fluctuations about the soliton solution. A similar separation should hold for the total momentum of the system where the term $MB/\sqrt{1 - B^2}$ is expected to appear. Clearly expanding around the usual static solution (3) is certainly not good enough: A fast moving soliton suffers Lorentz contraction and the Lorentz factor should appear in the classical solution. The soliton interacts both with the fluctuating field and with the external potential; it will prove both natural and fruitful to treat the velocity $B$ as a dynamical variable. This is essential if recoil effects are to be taken into account. A formalism along these lines is presented in [1]; however the present note is meant to be self-contained. Note that (a) the model considered in [1] was a pure $\phi^4$ theory with no external potential (the work focused on the interaction of the soliton with the fluctuations), (b) the emphasis was on the quantization of the theory. A model of a soliton in an external potential is presented in [2] but there the aim was to examine the motion of the soliton, radiation effects being disregarded. The problem examined in the present note has been looked at in [3]; here we put an emphasis on a relativistic treatment of the problem.

II. BASICS.

The simple sine-Gordon Lagrangean is given by (for a review see [4], [5])

$$L = \int dx' \left\{ \frac{1}{2} \left( \frac{\partial \phi}{\partial t'} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x'} \right)^2 - U_{SG}(\phi) \right\}$$

where

$$U_{SG}(\phi) = \frac{\alpha}{\beta^2} (1 - \cos(\beta \phi))$$

(1)
(Lab frame coordinates are denoted by \((t', x')\)). We make the choice

\[
\alpha = \beta = 1 \tag{3}
\]

and then

\[
U_{sG}(\phi) = 1 - \cos \phi \tag{4}
\]

Eventually we will consider the soliton in an external perturbing potential of the form

\[
U(\phi) = U_{sG}(\phi) (1 + V(x')) \tag{5}
\]

\[
V(x') = \varepsilon \cos(k_0 x') \tag{6}
\]

where \(\varepsilon\) is a small quantity. The equation of motion resulting from (5) is

\[
\frac{\partial^2 \phi}{\partial t'^2} - \frac{\partial^2 \phi}{\partial x'^2} + U'_{sG}(\phi) = 0 \tag{7}
\]

The static version

\[
-\frac{d^2 \phi}{dx'^2} + U'_{sG}(\phi) = 0 \tag{8}
\]

admits (amongst others) the solution

\[
\Phi_c(x') = 4 \arctan(e^{x'}) \tag{9}
\]

where \(\Phi_c \to 0\) for \(x' \to -\infty\) and \(\Phi_c \to 2\pi\) for \(x' \to \infty\). The mass of the soliton is quite generally given by

\[
M = \int dx' \left\{ \frac{1}{2} \left( \frac{\partial \Phi_c}{\partial x'} \right)^2 + U_{sG}(\Phi_c) \right\} = \int dx' \left( \frac{\partial \Phi_c}{\partial x'} \right)^2 \tag{10}
\]

and for the particular choice (3) \(M = 8\). Fluctuations around the classical solution satisfy the equation (expand (7) around \(\Phi_c\))

\[
-\frac{d^2 f}{dx'^2} + U''_{sG}(\Phi_c) f = \omega^2 f \tag{11}
\]
with \( \omega \) the corresponding frequency. Note that the space derivative of the kink

\[
\Phi'_c(x') = \frac{2}{\cosh x'}
\]

(12)
is a solution corresponding to zero frequency (the zero mode) (in the last equation we used (9)). This simply reflects the fact that a translated kink \( \Phi_c(x' + a) \) is also a static solution.

The normal modes corresponding to a wavevector \( k \) (not to be confused with \( k_0 \) of (6) that is a characteristic of the potential) are given in [3] :

\[
\frac{1}{\omega(k)\sqrt{2\pi}} e^{ikx'} (k + i \tanh x')
\]

(13)

where \( \omega = \sqrt{k^2 + \mu^2} \), and \( \mu = 1 \) with the choice (3). We shall write the normal modes in the form

\[
f(k, x) = \tilde{A}(k, x') e^{ikx'}
\]

(14)

where the prefactor \( \tilde{A}(k, x') \) can be read off (13) modulo a norming factor; we will work with energy normalized wavefunctions

\[
\int dx f^*(k_1, x) f^*(k_2, x) = \delta(\omega_1 - \omega_2)
\]

The Hamiltonian is

\[
H = \int dx \left\{ \frac{1}{2} \left( \frac{\partial \phi_c}{\partial x'} \right)^2 + \frac{\pi_\phi^2}{2} + U_{sG}(\phi) \right\}
\]

(15)
The fields are expanded in terms of the zero mode and the travelling modes

\[
\phi(x', t') = \Phi_c(x') + q_z \Phi'_c(x') + \frac{1}{\sqrt{4\pi}} \int d\omega \sqrt{\omega} \left\{ \tilde{a}^\dagger(\omega) f(k, x') e^{i\omega t'} + \tilde{a}(\omega) f^*(k, x') e^{-i\omega t'} \right\}
\]

(16)

\[
\pi_\phi(x', t') = \frac{p_z}{M} \Phi'_c(x') + \frac{1}{\sqrt{4\pi}} \int d\omega \sqrt{\omega} \left\{ \tilde{a}^\dagger(\omega) f(k, x') e^{i\omega t'} - \tilde{a}(\omega) f^*(k, x') e^{-i\omega t'} \right\}
\]

(17)

with the commutation relations

\[
[q_z, p_z] = 1
\]

(18)
\[
\left[ \tilde{a}(\omega), \tilde{a}^\dagger(\omega) \right] = \delta(\omega - \omega')
\]

\(\phi, \Pi_\phi\) obey the standard canonical commutation relations. (NB In what follows and for the sake of brevity integration over \(\omega\) will also stand for summation over the two directions of incidence.)

Notice that \(p_z\) appears in the Hamiltonian in the form \(\frac{p_z^2}{2M}\) through the \(\frac{\pi_\phi^2}{2}\) term. Thus \(p_z\) can be identified as the momentum of the soliton. The fact that the kinetic energy of the soliton appears in the nonrelativistic form is connected to the fact that we consider fluctuations about the zero-velocity solution (9). Clearly the inclusion of relativistic effects requires expansion about a boosted version of (9). Neglecting the interaction with fluctuations the motion of the kink at a nonrelativistic speed \(u\) corresponds to

\[
q_z = ut'
\]

\[
p_z = Mu
\]

compatible with \(\pi_\phi = \frac{\partial \phi}{\partial t'}\).

We introduce the fluctuating field

\[
\chi(x', t') \equiv \phi(x', t') - \Phi_c(x')
\]

and split the Hamiltonian (15) into a part quadratic in \(\chi\) and another part involving higher order terms. Then the quadratic part (diagonalized by the normal modes) does not involve \(q_z\). Thus the momentum of the kink (identified with \(p_z\)) is conserved; it can change only through interactions either with the fluctuating field or with an external potential.

III. THE MOVING SOLITON.

A. The Hamiltonian formalism

In order to treat a fast moving soliton we employ the strategy used in (1) and in a somewhat different context in (3), namely we go over to a non-inertial frame comoving with
the soliton. The definition of the position of the soliton is of course not a straightforward affair and it will occupy us later on. The new coordinates are defined by

\[ x' = x + X(t) \ , \ t' = t \ , \ B \equiv \dot{X}(t) \]  

(22)

We redefine the field so that

\[ \Phi(x) = \phi(x + X) \]  

(23)

(Transformations (22), (23) coincide with the translation transformation implemented quantum-mechanically in [7], section 3.1). Thus

\[
\begin{align*}
\frac{\partial \phi}{\partial x'} & = \frac{\partial \Phi}{\partial x} \ , \ \frac{\partial \phi}{\partial t'} = \frac{\partial \Phi}{\partial t} - B \frac{\partial \Phi}{\partial x} \\
\end{align*}
\]

(24)

Lagrangean (1) with the potential (5) can be readily written in terms of the new coordinates

\[
L = \int dx \left\{ \frac{1}{2} \left( \frac{\partial \Phi}{\partial t} - B \frac{\partial \Phi}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 - U_{sg}(\Phi(x))(1 + V(x + X)) \right\}
\]

(25)

The momentum \( \Pi \) conjugate to \( \Phi \) is

\[ \Pi = \frac{\partial \Phi}{\partial t} - B \frac{\partial \Phi}{\partial x} \]  

(26)

whereas the momentum \( P \) conjugate to \( X \) satisfies the constraint

\[ P + \int dx \Pi \frac{\partial \Phi}{\partial x} = 0 \]  

(27)

The constraint reflects the arbitrariness in what we mean by \( X \) (i.e. the freedom to perform coordinate transformations). The canonical Hamiltonian \( H_c = pq - L \) is

\[
H_c = \int dx \left\{ \frac{\Pi^2}{2} + \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + U_{sg}(\Phi(x))(1 + V(x + X)) \right\}
\]

(28)

The numerical value of \( H_c \) coincides with the total energy of the system.

The total Hamiltonian (i.e. the quantity that yields the equations of motion) consists of the canonical Hamiltonian \( H_c \) plus the constraint multiplied by an arbitrary Lagrange multiplier \( \lambda \).
\[ H_c + B \left( P + \int dx \Pi \frac{\partial \Phi}{\partial x} \right) \]  \hspace{1cm} (29)

The identification of the Lagrange multiplier \( B \) as the velocity follows immediately by commuting \( X \) with (29) and taking into account the fact that \( X \) and \( P \) are canonically conjugate.

Since the velocity changes as a result of interactions with the fluctuations and/or the external potential it is natural to elevate it to the status of a dynamical variable and introduce its conjugate momentum \( p_B \)

\[ [B, p_B] = 1 \]  \hspace{1cm} (30)

We take \( B, p_B \) to commute with all other variables \( \Phi, \Pi, X, P \). We require \( p_B \) to vanish as a constraint and introduce the (final) total Hamiltonian

\[ H = H_c + B \left( P + \int dx \Pi \frac{\partial \Phi}{\partial x} \right) + ap_B \]  \hspace{1cm} (31)

where we added to the Hamiltonian (29) constraint \( p_B \) multiplied by a Lagrange multiplier \( a \). Commuting \( B \) with \( H \) we get

\[ \dot{B} = a \]  \hspace{1cm} (32)

Thus \( a \) is identified as the acceleration of the kink. Requiring \( p_B \) to be conserved we end up with the original constraint (27).

The situation is reminiscent of what happens in electrodynamics. The momentum conjugate to the scalar potential (the latter being the analog of \( B \) in the present case) vanishes and the requirement that the constraint be conserved leads to Gauss’s law (the analog of (27)) as a secondary constraint. The gauge invariance reflects the freedom of performing coordinate transformations of the type (22). It will be lifted when we impose a subsidiary condition (thus implicitly defining the position of the soliton and hence the meaning of a comoving frame).

If we neglect for the moment the effect of the perturbation \( V \), the field equation resulting from (25) (or (28) and (31)) for constant velocity \( B \) is
\[
\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + U'_{sG}(\Phi) - 2B \frac{\partial^2 \Phi}{\partial x \partial t} + B^2 \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad (33)
\]

A \(t\) independent solution is trivially found to be

\[
\Phi_c \left( \frac{x}{\sqrt{1 - B^2}} \right) \quad (34)
\]

where \(\Phi_c\) is the same as \(3\) (i.e. the space variable is simply rescaled by the Lorentz factor).

It is natural to expand the fields \(\Phi, \Pi\) about the classical solution \(34\) introducing the fluctuating fields \(\chi, \pi\) (the latter not to be confused with \(\pi_\phi\) of section 2). (NB in what follows \(\Phi'_c\) will refer to the derivative of \(\Phi_c\) with respect to its argument and not just with respect to \(x\).)

\[
\Phi(x, t) = \Phi_c \left( \frac{x}{\sqrt{1 - B^2}} \right) + \chi(x, t) \quad (35)
\]

\[
\Pi(x, t) = -\frac{B}{\sqrt{1 - B^2}} \Phi'_c \left( \frac{x}{\sqrt{1 - B^2}} \right) + \pi(x, t) \quad (36)
\]

The existence of the first term in \(36\) is a consequence of the \(\partial/\partial x\) term in \(26\). It follows from \(35\), \(36\) that \(\chi, \pi\) satisfy canonical commutation relations

\[
[\chi(x, t), \pi(x', t)] = 1 \quad (37)
\]

It is important however to realize that \(p_B\) does not commute with \(\chi\) and \(\pi\). Given that \(p_B\) commutes with \(\Phi, \Pi\) it follows immediately from \(35\), \(36\) that

\[
[\chi(x, t), p_B] = -\Phi'_c \left( \frac{x}{\sqrt{1 - B^2}} \right) \frac{d}{dB} \left( \frac{1}{\sqrt{1 - B^2}} \right) \quad (38)
\]

\[
[\pi(x, t), p_B] = \Phi'_c \left( \frac{x}{\sqrt{1 - B^2}} \right) \frac{d}{dB} \left( \frac{B}{\sqrt{1 - B^2}} \right) + \frac{B^2}{(1 - B^2)^2} \Phi''_c \left( \frac{x}{\sqrt{1 - B^2}} \right) \quad (39)
\]

We turn to the field momentum and express it in terms of the static solution and the fluctuations (and use \(10\)):

\[
\int dx \Pi \frac{\partial \Phi}{\partial x} = -\frac{MB}{\sqrt{1 - B^2}} + \int dx \pi \frac{\partial \chi}{\partial x} + \frac{1}{\sqrt{1 - B^2}} \int dx \left\{ \Phi'_c \pi + \frac{B}{\sqrt{1 - B^2}} \Phi''_c \chi \right\} \quad (40)
\]

It is gratifying that in the absence of fluctuations we get the expected relativistic expression for the kink’s momentum.
B. The subsidiary condition.

We have to define what we mean by the position of the soliton or in other words give a definition of the comoving frame. This is done through the imposition of a constraint (a subsidiary condition) that lifts the freedom under coordinate transformations and assigns physical meaning to the variable \( X \). The choice of the suitable constraint is dictated by three criteria each one having its own physical motivation. It is not obvious at the outset that all three can be simultaneously satisfied. The fact that they can adds physical appeal to the separation of the total field \( \Phi \) to a soliton part and a fluctuating part.

The first criterion stipulates that in the presence of fluctuations the total momentum splits neatly to the soliton and fluctuation momenta given respectively by minus the first and second terms in (40) and that the term in braces vanishes. In other words the constraint that lifts the gauge invariance is

\[
C \equiv \int dx \left\{ \Phi' \pi + \frac{B}{\sqrt{1 - B^2}} \Phi'' \chi \right\} = 0 \tag{41}
\]

The fact that \( C \) does not contribute numerically in (40) does not mean that it should be discarded; it may well contribute to the equations of motion since the LHS of (40) appears in the total Hamiltonian (31). It turns out that it does not contribute; see the remark following (43).

As a second criterion we require that the energy \( H_c \) of the system does not contain any terms linear in the fluctuations. In other words that the same state of affairs as in the case of the total momentum prevails. It turns out that this requirement is satisfied; see (43) and the remark following it.

The third criterion stipulates that constraint \( C \) commute with the quadratic part \( H_0 \) (45) of the total Hamiltonian. This motivation is linked to the acceleration of the kink. The acceleration will be determined in the next subsection (eqn (48)) by requiring that constraint \( C \) commute with the total Hamiltonian (31), and it has two physical origins: (i) The interaction of the kink with the fluctuations, i.e. terms in the Hamiltonian beyond the
quadratic in expansion (42); (ii) The interaction with the external potential $V$. Thus in the absence of (i) and (ii) the acceleration ought to vanish, i.e. $C$ should commute with $H_0$ (43). This is shown in (47) below.

We substitute expansions (35), (36) in $H_c$ (28) while expanding $U_{sG}(\Phi)$ up to terms quadratic in $\chi$. Use of (10) and straight algebra yields

$$H_c = \frac{M}{\sqrt{1 - B^2}} + \int dx \left\{ \frac{1}{2} \left( \frac{\partial \chi}{\partial x} \right)^2 + \pi^2 + \frac{1}{2} U''_{sG}(\Phi_c) \chi^2 + V\text{-independent higher orders in } \chi \right\} + \{ V(x + X)U_{sG}(\Phi_c) + V\text{-dependent higher orders in } \chi \} + \int dx \left\{ - \frac{B}{\sqrt{1 - B^2}} \Phi'_c \pi - \frac{1}{1 - B^2} \Phi'_{c} \chi + U'_{sG}(\Phi_c) \chi \right\}$$

(42)

Thus $H_c$ naturally splits to a number of contributions: The first term coincides with the covariant expression for the energy of a particle. The next three terms are quadratic in the fluctuations and enter in the calculation of the normal modes of the system. There also are terms describing the interaction of the kink with the fluctuations and/or the external potential. The last term in braces linear in fluctuations can be written using (8) in the form

$$- \frac{B}{\sqrt{1 - B^2}} \int dx \left\{ \Phi'_c \pi + \frac{B}{\sqrt{1 - B^2}} \Phi''_c \chi \right\}$$

(43)

Hence the second criterion spelt out in the beginning of this subsection is satisfied: the energy does not depend linearly on the fluctuations. Notice that (43) cancels exactly with the last term in (40) when they are both substituted in (31). Thus the total Hamiltonian $H$ is written in terms of $\chi, \pi$ in the form

$$H = M \sqrt{1 - B^2} + \int dx \left\{ \frac{1}{2} \left( \frac{\partial \chi}{\partial x} \right)^2 + \pi^2 + \frac{1}{2} U''_{sG}(\Phi_c) \chi^2 + V\text{-independent higher orders in } \chi \right\} + \{ V(x + X)U_{sG}(\Phi_c) + V\text{-dependent higher orders in } \chi \} + \int dx \pi \frac{\partial \chi}{\partial x} + ap_B + BP$$

(44)

The first terms in (40) and (42) combine to yield the first term in (44) above. This latter term yields the frequency exponential $\exp(-M \sqrt{1 - B^2} t)$ in the soliton wavefunction and is
the relativistic generalization of the phase \( \exp(+imB^2t/2) \) (with an apparently wrong sign) derived in [9] for a point particle.

We can now write down the part \( H_0 \) of the total Hamiltonian (44) quadratic in the fluctuations:

\[
H_0 = M\sqrt{1-B^2} + \int dx \left\{ \frac{1}{2} \left( \frac{\partial \chi}{\partial x} \right)^2 + \frac{\pi^2}{2} + \frac{1}{2} U''_{sG}(\Phi_c) \chi^2 \right\} + B \int dx \pi \frac{\partial \chi}{\partial x}
\]

(45)
The equation of motion describing free fluctuations derivable from the above is

\[
\frac{\partial^2 \chi}{\partial t^2} - \frac{\partial^2 \chi}{\partial x^2} + U''_{sG}(\Phi_c) \chi - 2B \frac{\partial^2 \chi}{\partial x \partial t} + B^2 \frac{\partial^2 \chi}{\partial x^2} = 0
\]

(46)

We can now check, as promised, that constraint \( C \) commutes with the quadratic Hamiltonian \( H_0 \):

\[
[C, H_0] = \int dx \Phi_c' \left\{ \frac{\partial^2 \chi}{\partial x^2} - U''_{sG}(\Phi_c) \chi + B \frac{\partial \pi}{\partial x} \right\} + \frac{B}{\sqrt{1-B^2}} \int dx \Phi_c' \left\{ \pi + B \frac{\partial \chi}{\partial x} \right\}
\]

(47)

Integrate by parts so that the above expression becomes linear in \( \chi \) and \( \pi \). The terms linear in \( \pi \) cancel immediately. The bracket multiplying \( \chi \) vanishes due to equation (8) obeyed by \( \Phi_c \).

IV. DETERMINATION OF THE ACCELERATION.

Constraint (41) must commute with the total Hamiltonian \( H \) (31) and this determines \( a \):

\[
[C, H_c] + B \left[ C, \int dx \Pi \frac{\partial \Phi}{\partial x} \right] + a [C, p_B] = 0
\]

(48)

An exact evaluation involves the fluctuating field in a somewhat cumbersome way. However in the first instance we are interested in the effect of the perturbing potential only (and disregard fluctuations). We use (35), (36), take commutators (38), (39) into account and keep terms proportional to \( V \) only:

\[
Ma \frac{d}{dB} \left( \frac{B}{\sqrt{1-B^2}} \right) = \int dx \Phi'_c \left( \frac{x}{\sqrt{1-B^2}} \right) V(x + X)U'_sG(\Phi_c)
\]

(49)
The left hand side is the relativistic expression for the force. The right hand side depends on \( X \), hence (50) cannot be readily integrated. The RHS of (50) can be written

\[- \sin (k_0 X) \int dx \Phi_c' \left( \frac{x}{\sqrt{1 - B^2}} \right) \Phi_c'' \left( \frac{x}{\sqrt{1 - B^2}} \right) \sin (k_0 x)\]

where we used in turn (8) and (3), expanded \( V \) as a sum of cosines and sines and kept the sine part (the cosine part vanishing by parity). Rescaling

\[ M \frac{d}{dt} \left( \frac{B}{\sqrt{1 - B^2}} \right) = - \sqrt{1 - B^2} \sin (k_0 X) \int_{-\infty}^{\infty} dz \Phi_c'(z) \Phi_c''(z) \sin (k_0 z \sqrt{1 - B^2}) \]  

(51)

Integrate the LHS by parts to get it in the form

\[ \frac{1}{2} k_0 \left( 1 - B^2 \right) \sin (k_0 X) \int_{-\infty}^{\infty} dz \left( \Phi_c'(z) \right)^2 \cos \left( k_0 z \sqrt{1 - B^2} \right) \]

and using (12)

\[ 2k_0 \left( 1 - B^2 \right) \sin (k_0 X) \int_{-\infty}^{\infty} dz \frac{\cos (k_0 z \sqrt{1 - B^2})}{\cosh^2 z} \]

The integral is standard and we get

\[ M \frac{d}{dt} \left( \frac{B}{\sqrt{1 - B^2}} \right) = \frac{2 \pi k_0^2 (1 - B^2)^{3/2}}{\sinh \left( \frac{\pi k_0 \sqrt{1 - B^2}}{2} \right)} \sin (k_0 X) \]  

(52)

The above expression connects relativistic acceleration and position.

V. THE NORMAL MODES.

A. The travelling modes.

1. The \((t,x)\) frame.

Solutions to (10) are of the form
\begin{equation}
A(B, K, x) \exp(-i\Omega t + iKx)
\end{equation}

and we set

\begin{equation}
g(B, K, x) = A(B, K, x) \exp(iKx)
\end{equation}

To determine the relation between \( \Omega \) and \( K \), we observe that away from the kink the modulating factor \( A \) reduces to a constant; then (54) yields

\begin{equation}
\Omega = -BK + \sqrt{K^2 + \mu^2}
\end{equation}

Note that \( K \) carries a sign depending on the direction of incidence. Also notice that \( \Omega \) is positive (at least as long as the kink does not move at superluminal speeds). Relation (54) will also be derived in the next paragraph by transforming from the inertial frame instantaneously moving with the kink. The explicit form of \( A(B, K, x) \) can be deduced by struggling with (54). We find it easier to deduce it in the next paragraph via a Lorentz transformation.

It is clear from (54) that due to the cross term in the derivatives the \( gs \) are not orthogonal. However we can still proceed and introduce creation and annihilation operators corresponding to the wavefunctions (53), the procedure now being somewhat more complicated than in standard free field theory. We define (the index in \( a^\dagger \) denoting the direction of incidence being suppressed)

\begin{equation}
a^\dagger(B, \Omega, t) = \frac{1}{\sqrt{4\pi}} \int dx g^*(B, K, x) \left\{ \sqrt{\Omega} \chi - \frac{B}{i\sqrt{\Omega}} \frac{\partial \chi}{\partial x} + \frac{\pi}{i\sqrt{\Omega}} \right\}
\end{equation}

with a corresponding expression for the complex conjugate. For \( B = 0 \) this reduces to the usual expression. To check that \( a^\dagger(B, \Omega, t) \) satisfies the usual sinusoidal time evolution equation under the influence of the quadratic Hamiltonian \( H_0 \) rewrite (45) in the form

\begin{equation}
(1 - B^2) \frac{d^2g}{dx^2} + U''_{sg}(\Phi_c) g + 2iB\Omega \frac{dg}{dx} = -\Omega^2 g
\end{equation}

Commute the RHS of (55) with \( H_0 \) (13), integrate by parts and use (56) to get in a straightforward manner
\[
\frac{d}{dt}a^\dagger(B, \Omega, t) = i\Omega a^\dagger(B, \Omega, t) \tag{57}
\]

We shall later need the bracket \([a^\dagger(B, \Omega, t), p_B]\). To this end use the definition \((55)\) and \((38), (39)\) to get
\[
[a^\dagger(B, \Omega, t), p_B] = -\frac{i}{\sqrt{4\pi\Omega}} \frac{d}{dB} \left(\frac{B}{\sqrt{1-B^2}}\right) \int dx g^*(B, K, x) \Phi'_c \left(\frac{x}{\sqrt{1-B^2}}\right) \tag{58}
\]
with the complex conjugate for \(a(B, \Omega, t)\). Notice that in the nonrelativistic version the integral in the RHS of the above equation vanishes due to the orthogonality of the eigenfunctions of \((11)\).

We expand the fluctuation field in the form
\[
\chi(x, t) = \chi_z(x, t) + \frac{1}{\sqrt{2\pi}} \int \frac{d\Omega}{\sqrt{\Omega}} \left\{ a^\dagger(B, \Omega, t)g(B, K, x) + a(B, \Omega, t)g^*(B, K, x) \right\} \tag{59}
\]
The above relation defines the field \(\chi_z(x, t)\) and is the relativistic analog of \((16)\) (with the attendant complications of non-orthogonality). To write the corresponding expansion for \(\pi(x, t)\) we are inspired (i) by the relation
\[
\frac{\partial \chi}{\partial t} = \pi + B \frac{\partial \chi}{\partial x} \tag{60}
\]
derived from \(H\), (ii) by the fact that the expansion should be valid in the special case of sinusoidal time evolution which holds when we neglect effects due to the potential or to terms of degree higher than quadratic in \(\chi\). Thus
\[
\pi(x, t) = \pi_z(x, t) + \frac{1}{\sqrt{2\pi}} \int d\Omega \left\{ \sqrt{\Omega}a^\dagger(B, \Omega, t)g(B, K, x) - \sqrt{\Omega}a(B, \Omega, t)g^*(B, K, x) \right\} + \\
+ \frac{1}{\sqrt{2\pi}} \int \frac{d\Omega}{\sqrt{\Omega}} \left\{ a^\dagger(B, \Omega, t)\frac{\partial g(B, K, x)}{\partial x} + a(B, \Omega, t)\frac{\partial g^*(B, K, x)}{\partial x} \right\} \tag{61}
\]
Again this is the definition of \(\pi_z(x, t)\). We take the fields \(\chi_z(x, t), \pi_z(x, t)\) to commute with the set of the \(a\)s and \(a^\dagger\)s. The bracket \([\chi_z(x, t), \pi_z(x', t)]\) is non-trivial in the nonrelativistic case as well. It equals
\[
\frac{1}{M} \Phi'_c(x)\Phi'_c(x')
\]
as can be seen from the zero mode part of \((16), (17)\).
2. The comoving frame.

In the absence of interactions the kink moves at uniform velocity. Denote quantities pertaining to the comoving frame by \((\sim)\) and the lab frame coordinates by prime. Consider a travelling wave of the form (13), (14)

\[
\exp \left( -i\tilde{\omega}t + i\tilde{p}\tilde{x} \right) \tilde{A} (\tilde{p}, \tilde{x})
\]  

The phase appearing in (62) is written in terms of primed coordinates

\[
-\tilde{\omega} \frac{t' - Bx'}{\sqrt{1 - B^2}} + \tilde{p} \frac{x' - Bt'}{\sqrt{1 - B^2}}
\]  

Use (22) for uniform \(B\) and rearrange

\[
-\tilde{\omega} \sqrt{1 - B^2}t + \tilde{p} + B\tilde{\omega}x
\]  

Compare with (53) to identify

\[
\Omega = \tilde{\omega} \sqrt{1 - B^2}, K = \frac{\tilde{p} + B\tilde{\omega}}{\sqrt{1 - B^2}}
\]  

Thus \((\Omega, K)\) is a rather peculiar pair: \(K\) coincides with the wavevector in the lab frame whereas \(\Omega\) is connected to the frequency in the comoving frame. Manipulation of \(\Omega, K\) as given above yields again (54). The Lorentz transformation together with (22) yield the useful relation

\[
\tilde{x} = \frac{x'}{\sqrt{1 - B^2}} \Rightarrow \tilde{x} = \frac{x}{\sqrt{1 - B^2}}
\]  

The same Lorentz transformation yields

\[
\tilde{p} = \frac{K - B\sqrt{K^2 + \mu^2}}{\sqrt{1 - B^2}}
\]  

(Recall that we our choice of units \(\mu = 1\).) Thus the mode (62) is expressed in the \((t, x)\) frame in the form

\[
\exp(-i\Omega t)g(B, K, x) = \exp(-i\Omega t + iKx)\tilde{A} \left( \tilde{p}(K), \frac{x}{\sqrt{1 - B^2}} \right)
\]
Hence (68) provides the explicit form of the solutions (53) with $\tilde{A}$ taken from (13), (14) and $\tilde{p}$ from (67). In that sense it is hardly a mystery why operators $a^\dagger(B, \Omega, t)$, $a(B, \Omega, t)$ diagonalize the total Hamiltonian: from the point of view of the comoving observer they correspond to the usual (orthogonal) modes (13), (14) of frequency $\Omega/\sqrt{1 - B^2}$.

**B. The zero mode.**

Consider an observer moving at velocity $B$ and suppose that the zero mode (19) is excited, i.e.

$$\chi(\tilde{x}, \tilde{t}) = u\tilde{t}\Phi'_c(\tilde{x})$$

where $u$ is a small parameter (the velocity of the kink with respect to the said observer). Transforming to the $(t, x)$ frame as above we get the same mode in the form

$$\chi(x, t) = ut\sqrt{1 - B^2}\Phi'_c\left(\frac{x}{\sqrt{1 - B^2}}\right) - u\frac{Bx}{\sqrt{1 - B^2}}\Phi'_c\left(\frac{x}{\sqrt{1 - B^2}}\right)$$

(69)

It can be readily shown that the above expression satisfies (46). This is precisely the form that $\chi_z(x, t)$ of (59) takes in the free case. From (60), (61) and (69) we can work out the corresponding expression for $\pi_z(x, t)$.

One can see constraint $C$ in a different light. It is easily shown that $C$ commutes with the $a$s and $a^\dagger$s and that in the special case of free fluctuations it turns out to be proportional to $u$ of (69). In other words as seen from an observer moving at velocity $B$, the small velocity $u$ associated with the excitation of the zero mode vanishes if $C$ is to hold; this is precisely what one would expect from a kink moving at velocity $B$.

**VI. RADIATION FROM AN ACCELERATED SOLITON.**

To calculate the emission of radiation at a wavevector $K$ (with respect to the lab frame) we look at the commutator $[a^\dagger(B, \Omega, t), H]$ ($H$ being the total Hamiltonian). We write

$$[a^\dagger(B, \Omega, t), H] = (I) + (II) + (III)$$

(70)
(I) is given by (57) (corresponding to free evolution). The commutator in (70) gets contributions from terms in $H$ that are of order higher than quadratic in $\chi$. However we are only interested in contributions of first order in $V$ (i.e. of order $\varepsilon$; cf (4)). Contributions (II) and (III) come from commuting (55) with the term in the second pair of braces in (44) linear in $\chi$ and with the $p_B$ term respectively.

\[
(II) = -\frac{\varepsilon}{i\sqrt{4\pi\Omega}} \int_{-\infty}^{\infty} dx g^*(B, k, x) V(x + X) U'_{sG} (\Phi_c) = \\
= -\frac{\varepsilon}{i\sqrt{4\pi\Omega}} \int_{-\infty}^{\infty} dx g^*(B, k, x) \Phi''_c \left( \frac{x}{\sqrt{1 - B^2}} \right) \cos (k_0 x + k_0 X)
\]

Recall that the $X$ has a non-trivial time dependence.

For (III) we get from (58)

\[
(III) = -a \frac{i}{\sqrt{4\pi\Omega}} \frac{d}{dB} \left( \frac{B}{\sqrt{1 - B^2}} \right) \int dx g^*(B, K, x) \Phi'_c \left( \frac{x}{\sqrt{1 - B^2}} \right)
\]

Given that $a = dB/dt$ the combination

\[
a \frac{d}{dB} \left( \frac{B}{\sqrt{1 - B^2}} \right)
\]

amounts to $1/M$ times the right hand side of (52). Notice that the RHS of (72) does not vanish since the solutions of (53) are not orthogonal. Thus

\[
(III) = -\frac{i}{M\sqrt{4\pi\Omega}} \frac{2\pi k_0^3 (1 - B^2)^{3/2}}{\sinh \left( \frac{\pi k_0 \sqrt{1 - B^2}}{2} \right)} \sin (k_0 X) \int dx g^*(B, K, x) \Phi'_c \left( \frac{x}{\sqrt{1 - B^2}} \right)
\]

VII. CONCLUSION.

This paper is meant to set the general framework for a treatment of the collective coordinate of a relativistic soliton interacting with fluctuations and an external potential. In a relativistic setting it proves natural to treat the velocity $B$ as a dynamical variable. This artificial increase in the number of degrees of freedom and the attendant gauge invariance generated by the constraint (27) are lifted by the subsidiary condition that defines the position of the soliton. Let us stress again that the formulae for the total momentum and
energy of the system (40) and (42) include the correct relativistic expressions for the soliton momentum and energy. It is equally reassuring that the left hand sides of (50) and (52) feature the time derivative of the relativistic momentum.

Concerning emission of radiation we observe that it has two physical origins. One is the interaction of the soliton with the fluctuations. This comes about when we commute the $\pi$ dependent term in (55) with the $V$ independent cubic and quartic terms in $\chi$ in the total Hamiltonian. This contribution was disregarded in the present work only because we were interested in the $V$ dependence. The acceleration also receives contributions from terms in the Hamiltonian depending on both $V$ and $\chi$, reflecting the fact that an accelerated soliton radiates; to lowest order we obtain $(II)$ calculated above. Finally observe that contribution $(III)$ stems from the $p_B$ term in the Hamiltonian, whose existence is required by the need of a relativistic treatment.

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