A REMARK ON PROJECTIVE LIMITS OF FUNCTION SPACES

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Abstract. Let $\Omega \subset \mathbb{C}^d$ be an open set and $K_m, m = 1, 2, \ldots$ an exhaustion of $\Omega$ by compact subsets of $\Omega$. We set $\Omega_m = K_m^o$ and let $X_m(\Omega_m)$ be a topological space of holomorphic functions on $\Omega_m$ between $A^\infty(\Omega_m)$ and $H(\Omega_m)$. Then we show that the projective limit of the family $X_m(\Omega_m), m = 1, 2, \ldots$, under the restriction maps is homeomorphic and linearly isomorphic to the Fréchet space $H(\Omega)$, independently of the choice of the spaces $X_m(\Omega_m)$.

1. Introduction

Let $Y$ be a space of holomorphic functions on a domain $\Omega$ in $\mathbb{C}^d, d = 1, 2, \ldots$. For most of the spaces $Y$ the following holds. If for every bump $V$ of $\Omega$ there exists a function $f_V$ in $Y$ which is not holomorphically extendable to $V$, then there exists a function $f$ in $Y$ not extendable to any $V$. Moreover, the set of such functions $f$ in $Y$ is dense and $G_\delta$ in $Y$. (see [J-P], [N])

Examples of such spaces are the space $H(\Omega)$ of all holomorphic functions in $\Omega$ endowed with the topology of uniform convergence on compacta, the spaces $A^p(\Omega)$ and $H^\infty_p(\Omega)$, Bergman spaces, Hardy spaces etc, provided that $\Omega$ satisfies some good assumptions.

Most of these spaces endowed with their natural topology, contain $A^\infty(\Omega)$ and they are contained in $H(\Omega) : A^\infty(\Omega) \subset Y \subset H(\Omega)$. Furthermore, the two inclusion maps $i_1 : A^\infty(\Omega) \to Y$, $i_2 : Y \to H(\Omega)$ with $i_1(f) = f$ and $i_2(w) = w$ are continuous. We denote by $M(\Omega)$ the set of such topological function spaces on $\Omega$.

Let $K_m, m = 1, 2, \ldots$, be an exhaustion by compact subsets of the open set $\Omega \subset \mathbb{C}^d$ and $\Omega_m = K_m^o$. Let $X_m(\Omega_m)$ be any topological function space belonging to $M(\Omega_m)$. Let $X$ denote the projective (inverse) limit of the family $X_m(\Omega_m), m = 1, 2, \ldots$ where $T_{m,n} : X_m(\Omega_m) \to X_n(\Omega_n), n \leq m$ is the restriction map $f \to f|_{\Omega_n}$.

Then we show that $X$ is a topological vector space homeomorphic and linearly isomorphic to the Fréchet space $H(\Omega)$ of all holomorphic functions.

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on Ω endowed with the topology of uniform convergence on all compact subsets of Ω, independently of the choice of the topological spaces $X_m(\Omega_m)$ from $M(\Omega_m)$. This holds even if the spaces $X_m(\Omega_m)$ are neither complete, nor vector spaces.

2. ABOUT PROJECTIVE LIMITS

We need the following particular case of projective limits (see [E]).

**Definition 2.1.** Let $(Y_n, T_n), n = 1, 2, \ldots$ be a sequence of topological spaces. For $n < m$ let $T_{m,n}: Y_m \to Y_n$ be a continuous map and let $T_{n,n}$ be the identity map from $Y_n$ to itself. We assume that for $m \leq n \leq s$ it holds $T_{s,m} = T_{n,m} \circ T_{s,n}$.

Then the projective (or inverse) limit of this family is defined to be the following subset $Y$ of the cartesian product $\prod_{n=1}^{\infty} Y_n$ where

$$Y = \{ g = (g_1, g_2, \ldots) : \text{for every } n \leq m \text{ it holds } T_{m,n}(g_m) = g_n \}$$

The topology $T_Y$ on $Y$ we consider is the relative topology from $\prod_{n=1}^{\infty} Y_n$ endowed with the product topology.

**Proposition 2.2.** [E] Under the above notation and assumption a net $g^i = (g^i_1, g^i_2, \ldots) \in Y, i \in I$ converges to $g = (g_1, g_2, \ldots) \in Y$ in the topology $T_Y$ of $Y$ if and only if $g^i_k \to g_k$ in the topology $T_k$ of $Y_k$ for all $k = 1, 2, \ldots$.

3. SOME FUNCTION SPACES

Let $\Omega$ be an open subset of $\mathbb{C}^d$, $d \geq 1$. We denote by $H(\Omega)$ the space of holomorphic functions on $\Omega$ endowed with the topology of the uniform convergence on compacta. This is a Fréchet space with seminorms $\|f\|_m = \sup_{z \in K_m} |f(z)|, m = 1, 2, \ldots$ where $K_m$ is an exhaustion of $\Omega$ by compact subsets of $\Omega$; for instance we can take (see [R], [G])

$$K_m = \left\{ z \in \Omega : \|z\| \leq m \text{ and } \text{dist}(z, \Omega^c) \geq \frac{1}{m} \right\}$$

In what follows we consider that $G$ is a bounded open subset of $\mathbb{C}^d$, although the spaces we will consider, can also be defined for an unbounded open set $G$, with some modifications in the definition of their natural topology.

The space $A^\infty(G)$ is defined as the set of holomorphic functions on $G$ such that each mixed partial derivative of $f$ extends continuously on the closure $\overline{G}$ of $G$. The natural topology of $A^\infty(G)$ is defined by the seminorms

$$\sup_{z \in G} |\frac{\partial^{a_1+\ldots+a_d} f}{\partial z_1^{a_1} \ldots \partial z_d^{a_d}}(z)|, a_1, \ldots, a_d \in \{0, 1, 2, \ldots\} , \text{ where } z = (z_1, \ldots, z_d).$$

This is also a Fréchet space.
Next we are interested in spaces $X(G)$ containing $A^\infty(G)$ and contained in $H(G)$, i.e. $A^\infty(G) \subset Y \subset H(G)$. We also require that the injections are continuous; that is, if a net $f^i \in A^\infty(G)$, $i \in I$, converges in the topology of $A^\infty(G)$ to a function $f \in A^\infty(G)$, then $f^i \to f$ in the topology of $X(G)$, as well; and if a net $w^i \in X(G)$, $i \in I$, converges to $w \in X(G)$ in the topology of $X(G)$ then $w^i \to w$ in the topology of $H(G)$. We denote by $M(G)$ the set of all topological spaces as above.

Such spaces are the spaces $A^p(G)$ consisted of $f \in H(G)$ of which every mixed partial derivative of order less than or equal to $p$ extends continuously on $G$ and the spaces $H^p(G)$ consisted of $f \in H(G)$ of which every mixed partial derivative of order less than or equal to $p$ is bounded on $G$. The above two spaces are Banach spaces, provided $p < +\infty$ and Fréchet spaces if $p = +\infty$. Their topologies are defined by the seminorms

$$\sup_{z \in G} |\frac{\partial^{a_1 + \cdots + a_d} f}{\partial z_1^{a_1} \cdots \partial z_d^{a_d}}(z)|, \quad a_1 + \cdots + a_d \leq p,$$

where $z = (z_1, \ldots, z_d)$.

Another example of such a space is the $p$-Bergman space, $0 < p < +\infty$, containing all holomorphic functions $f$ on $G$, such that $\int_G |f|^p d\nu < \infty$, where $d\nu$ denotes the Lebesgue measure on $G \subset \mathbb{C}^d$. (see [D-S])

Other examples are Hardy spaces on the disc or the ball or on more general domains (see [D], [R2], [S]), as well as subclasses of the Nevanlinna class, BMOA, VMOA, Dirichlet spaces on the disc etc.

All the previous spaces are complete spaces. However, we can also consider non complete spaces containing $A^\infty(G)$ and contained in $H(G)$ such that the two injections are continuous. For instance, consider the set $Y = A^\infty(G)$ with the relative topology induced by $H(G)$; that is, for $f_n, f \in A^\infty(G)$ the convergence in the new space is the uniform convergence on compacta $f_n \to f$. This space contains $A^\infty(G)$, is contained in $H(G)$ and the injections $i_1 : A^\infty(G) \to Y$, $i_2 : Y \to H(G)$ with $i_1(f) = f$ and $i_2(g) = g$ are continuous where $A^\infty(G)$ and $H(G)$ are endowed with their natural topologies and $Y$ with the relative topology from $H(G)$. But this space is not complete, in general. For instance, if $(G)^0 = G$ or if $G$ is a bounded, open subset of $\mathbb{C}$, then $A^\infty(G)$ with the relative topology induced by $H(G)$ is not complete.

It is also possible that a space belongs to $M(G)$ although it is not even a vector space. To give such an example we consider the set

$$Y = A^\infty(G) \cup \{f : G \to \mathbb{C} \text{ holomorphic such that } |f(z)| < 1, \text{ for all } z \in G\},$$

endowed with the topology of uniform convergence on $G$. Then $A^\infty(G) \subset Y \subset H(G)$ and if $f_n, f \in A^\infty(G)$ are such that $f_n \to f$ in the topology
of $A^\infty(G)$, it follows trivially that $f_n \to f$ in the topology of $Y$. Also if $\phi_n, \phi \in Y$ are such that $\phi_n \to \phi$ in the topology of $Y$, then obviously the convergence is uniform on each compact subset of $G$, as well.

4. The result

We start with the following definition

**Definition 4.1.** Let $G$ be a bounded open subset of $\Omega^d$. We denote by $M(G)$ the set of all topological spaces $X(G)$ satisfying $A^\infty(G) \subset X(G) \subset H(G)$ and such that the two injections $i_1: A^\infty(G) \to X(G)$, $i_1(f) = \bar{f}$ and $i_2: X(G) \to H(G)$, $i_2(g) = g$ are continuous; that is, if a net $f^i \in A^\infty(G)$, $i \in I$, converges to $f \in A^\infty(G)$, in the topology of $A^\infty(G)$, then $f^i \to f$ in the topology of $X(G)$, as well. Moreover, if a net $w^i \in X(G)$ converges to $w \in X(G)$, in the topology of $X(G)$, then $w_i \to w$ uniformly on compacta of $G$.

Let $\Omega$ be an open subset of $\mathbb{C}^d$, bounded or unbounded. Let $K_m, m = 1, 2, \ldots$ be an exhausting family of compact subsets of $\Omega$; that is, $K_m \subset K_m^{o-1}$ and $\cup_{m=1}^\infty K_m = \Omega$ (see [R], [G]). We set $\Omega_m = K_m^{o-1}$, then $\Omega_m \subset \overline{\Omega_m} \subset \Omega_{m+1}$. For each $m = 1, 2, \ldots$ let $X_m(\Omega_m)$ be a space in $M(\Omega_m)$.

For $n \leq m$ we consider the restriction map $T_{m,n}: X_m(\Omega_m) \to X_n(\Omega_n)$ given by $T_{m,n}(f) = f|_{\Omega_n}$, for every $f \in X_m(\Omega_m)$. These maps are well defined and continuous. Indeed, $X_m(\Omega_m) \subset H(\Omega_m)$ and $\overline{\Omega_m} \subset \Omega_m$; thus, for every $f \in X_m(\Omega_m)$, it follows that $T_{m,n}(f) = f|_{\Omega_n} \in A^\infty(\Omega_n) \subset X_n(\Omega_n)$, provided that $n < m$. Also if a net $f^i \in X_m(\Omega_m)$ converges to $f \in X_m(\Omega_m)$, in the topology of $X_m(\Omega_m)$, then by assumption, it converges uniformly on compacta of $\Omega_m$. Weierstrass theorem yields uniform convergence on compacta of every mixed partial derivative. Since $\overline{\Omega_m}$ is a compact subset of $\Omega_m$, it follows that $f^i|_{\Omega_n} \to f|_{\Omega_n}$ in the topology of $A^\infty(\Omega_n)$. By assumption, this implies $f^i \to f$ in the topology of $X_n(\Omega_n)$. Therefore, $T_{m,n} : X_m(\Omega_m) \to X_n(\Omega_n)$ is continuous. One can also very easily verify that $T_{m,n} = T_{s,n} \circ T_{m,s}$, for all $n \leq s \leq m$. Therefore, we can define the projective limit $X$, which is a topological space (see [E]).

**Theorem 4.2.** Under the above assumptions and notation the topological space $X$ is homeomorphic to $H(\Omega)$.

*Proof.* Let $f \in H(\Omega)$; then $f|_{\Omega_m} \in A^\infty(\Omega_m) \subset X_m(\Omega_m)$. Thus, if we set $g_m = f|_{\Omega_m}$ and $g = (g_1, g_2, \ldots) = g(f)$, then $g \in \prod_{m=1}^\infty X_m(\Omega_m)$. For $n \leq m$ one can easily verify that $T_{m,n}(g_m) = g_n$. Therefore, $g \in X$. In this way we have defined a map $\Phi : H(\Omega) \to X$, where $\Phi(f) = g(f)$.
This map is one to one. Indeed, if \( g(f) = g(w) \) it follows that \( g|_{\Omega_m} = w|_{\Omega_m} \) for all \( m = 1, 2, \ldots \). Since \( \cup_m \Omega_m = \Omega \), it follows that \( f(z) = w(z) \) for all \( z \in \Omega \); thus \( f = w \). This proves that the map \( \Phi \) is one to one.

The map is also onto. Indeed, if \( g = (g_1, g_2, \ldots) \in X \), then for every \( z \in \Omega \) there exists a least natural number \( n(z) \) so that \( z \in \Omega_m \), for all \( m \geq n(z) \), because \( \Omega_m \subset \Omega_{m+1} \) and \( \cup_m \Omega_m = \Omega \). Further, since \( T_{m,n(z)}g_m = g_{n(z)} \), it follows that \( g_m(z) = \phi_{n(z)}(z) \), for all \( m \geq n(z) \). We set \( f(z) = g_{n(z)}(z) \). We will show that this function is holomorphic on \( \Omega \). Indeed, let \( z_0 \in \Omega \); then \( z_0 \in \Omega_{n(z_0)} \). Since \( \Omega_{n(z_0)} \) is open, there exists a ball \( B(z_0, r) \) contained in \( \Omega_{n(z_0)} \). For each \( z \in B(z_0, r) \), we have \( n(z) \leq n(z_0) \) and \( f(z) = g_{n(z)}(z) = g_{n(z_0)}(z) \). But \( g_{n(z_0)} \in X_{n(z_0)}(\Omega_{n(z_0)}) \subset H(\Omega_{n(z_0)}) \). It follows that \( f \) is holomorphic in \( B(z_0, r) \). Since \( z_0 \) is arbitrary in \( \Omega \), it follows that \( f \in H(\Omega) \). It is immediate that \( \Phi(f) = g \). Since \( g \) is arbitrary in \( X \), it follows that the map \( \Phi \) is onto.

Further, \( \Phi \) is continuous. Indeed, if a net \( f^i \in H(\Omega) \), \( i \in I \) converges to \( f \in H(\Omega) \), uniformly on compacta, by Weierstrass theorem the same holds for every mixed partial derivative. Since \( \Omega_m \) is a compact subset of \( \Omega \) it follows that \( f^i|_{\Omega_m} \to f|_{\Omega_m} \) in the topology of \( A^\infty(\Omega_m) \). By assumption this implies the convergence \( f^i|_{\Omega_m} \to f|_{\Omega_m} \) in the topology of \( X_m(\Omega_m) \).

According to Proposition 2.2, it follows \( \Phi(f^i) \to \Phi(f) \) in the topology of \( X \). Thus, the map \( \Phi \) is continuous.

Finally, we will show that \( \Phi^{-1} \) is also continuous. Suppose that \( \Phi(f^i) = (f^i|_{\Omega_1}, f^i|_{\Omega_2}, \ldots) \to \Phi(f) = (f|_{\Omega_1}, f|_{\Omega_2}, \ldots) \) in the topology of \( X \). According to Proposition 2.2 this implies \( f^i|_{\Omega_m} \to f|_{\Omega_m} \) for all \( m = 1, 2, \ldots \) in the topology of \( X_m(\Omega_m) \), which, by assumption, implies the uniform convergence on compacta of \( \Omega_m \). We will show that \( f^i \to f \) uniformly on each compact set \( K \subset \Omega \). Let \( K \) be such a compact set. Since \( K \subset \Omega = \cup_{m=1}^\infty \Omega_m \) and the open sets \( \Omega_m \) satisfy \( \Omega_m \subset \Omega_{m+1} \), by compactness of \( K \) it follows that there exists \( m_0 \) such that \( K \subset \Omega_{m_0} \). But \( f^i|_{\Omega_m} \to f|_{\Omega_m} \) uniformly on each compact of \( \Omega_{m_0} \), in particular on \( K \). Thus, \( f^i \to f \) uniformly on each compact set \( K \subset \Omega \) and therefore, \( f^i \to f \) in the topology of \( H(\Omega) \). Thus, \( \Phi^{-1} \) is continuous, as well and \( \Phi \) is a homeomorphism. This completes the proof.

**Remark 4.3.** It is well known and easy to verify that the denumerable projective limit of Fréchet spaces is also a Fréchet space. In our case \( X \) is homeomorphic to the Fréchet space \( H(\Omega) \), although the spaces \( X_m(\Omega_m) \) may not be Fréchet spaces. Furthermore, independently of the fact that the spaces \( X_m(\Omega_m) \) are topological vector spaces or they do not have linear
structure, one can easily verify that $X$ is always a topological vector space and that the map $\Phi$ is a linear isomorphism.

**Remark 4.4.** Let us consider the compact set

$$K_t = \left\{ z \in \Omega : |z| \leq t, \text{dist}(z, \Omega^c) \geq \frac{1}{t} \right\}, \quad t \geq t_0,$$

for a convenient $t_0 > 0$ such that $K_{t_0}^c \neq \emptyset$. Then for $t < s$ it follows $K_t \subseteq K_s^c$ and $\cup_{t \geq t_0} K_t = \Omega$. We set $\Omega_t = K_t^c$ and let $X_t(\Omega_t)$ be any element of $M(\Omega_t)$.

One can prove that the projective limit of the family $X_t(\Omega_t)$ as $t \to +\infty$, where $T_{s,t}$ is the restriction map $f \to f|_{\Omega_t}$, $t < s$, is homeomorphic and linearly isomorphic to $H(\Omega)$. The proof is similar to that of Theorem 4.2.

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