Formation of an ordered phase in neutron star matter

M.A. Pérez García 1*, J. Díaz Alonso 1,2, N. Corte Rodríguez 1, L. Mornas 1, J.P. Suárez Curieses 1
(1) Dpto. de Física, Universidad de Oviedo. Avda. Calvo Sotelo 18, E-33007 Oviedo, Asturias, SPAIN
(2) Observatoire de Paris, D.A.R.C. (U.M.R. 8629 C.N.R.S.) F-92190 Meudon, FRANCE

August 23, 2019

Abstract

In this work, we explore the possible formation of ordered phases in hadronic matter, related to the presence of hyperons at high densities. We analyze a microscopic mechanism which can lead to the crystallization of the hyperonic sector by the confinement of the hyperons on the nodes of a lattice. For this purpose, we introduce a simplified model of the hadronic plasma, in which the nuclear interaction between protons, neutrons and hyperons is mediated by meson fields. We find that, for some reasonable sets of values of the model parameters, such ordered phases are energetically favoured as density increases beyond a threshold value.

PACS:
keywords: hyperons, crystallization, neutron stars, relativistic nuclear matter

1 Introduction

The formation of solid cores in neutron stars or the presence of solid phases in hadronic matter has been considered by many authors, invoking several mechanisms. Whereas the idea of solidification of pure neutron matter considered in the seventies [1] was abandoned, other mechanisms have received attention lately: pion condensation [2], confinement of protons [3] by the repulsive character of the in-medium n-p interaction, ordering of a mixed quark-hadron phase in a way reminiscent of the neutron drip phase in the surface layers [4], etc. (See also [5, 6, 7].)

A solid phase may have interesting observable consequences on the behavior of neutron stars [8, 9]. As a rotating star slows down, or in the course of the cooling process, a small change in the star internal pressure may trigger a sudden change of the central density and be observable as a discontinuity in the braking index, i.e. a glitch [19]. Similarly, a minicollapse following a phase transition to an ordered phase was suggested as a mechanism for anomalous gamma-ray bursts [22]. Moreover, if a sizeable fraction of the matter of the

*E-mail addresses: aperez@pinon.ccu.uniovi.es, diaz@obspm.fr, corte@pinon.ccu.uniovi.es, lysiane@fisi24.ciencias.uniovi.es, gravity@pinon.ccu.uniovi.es
neutron star interior exists in the solid state, the star supports elastic strain and develops triaxiality, leading to gravitational wave emission and precession [9]. Finally, consequences are to be expected on the mechanism of neutrino emission and neutron star cooling [10, 11].

The equation of state of hadronic matter accounting for the presence of nucleons and hyperons which interact through the exchange of several mesons has been extensively studied using phenomenological lagrangian models [12]. Such models suggest that the presence of hyperons in hadronic matter starts at a threshold density around 1.5 times nuclear saturation and increases beyond.

In the mean field approximation to the solution of these models, the hyperonic sector in the ground state is supposed to form (as the other baryonic components) an uniformly distributed Fermi fluid undergoing the action of the mesonic mean fields created by the whole baryonic distribution in the plasma. As the abundance of hyperons strongly increases with density, the contribution of this sector to the total energy of the ground state becomes very important in a Fermi fluid configuration. In these conditions, one may ask whether such a fluid is stable at high densities, or would other ground state configurations be energetically more favorable.

In this work, we shall investigate the relative stability of a new class of ordered configurations for the ground state of hadronic matter, where an hyperon sector is confined on the nodes of a periodic lattice. For this purpose, we introduce a model of the hadronic plasma where a system of baryons (protons, neutrons and hyperons) interact through the exchange of meson fields in the framework of a relativistic phenomenological lagrangian theory. The mesonic and hyperonic sectors included in the lagrangian determine the complexity of the corresponding version of the model, as well as the accuracy of its predictions. A satisfactory model should include the whole family of hyperons and the set of mesons allowing a realistic description of the strong interaction in the relevant range of densities and energies. Nevertheless, in order to elaborate the techniques for the description of eventual ordered configurations and the analysis of their stability, avoiding the complexities related to the microscopic dynamics, we shall treat here a simplified version, where the interaction is described through the exchange of scalar (σ) and vector (ω) mesons. The assumed baryonic components are protons, neutrons, and we consider only one species of electrically neutral (Λ) hyperon coupled to the mesons, as well as an electronic sector compensating the proton electric charge.

Although this simple version cannot account for all the features of the nuclear interaction, when the model is solved in the mean field approximation with the β equilibrium conditions, it allows the fitting of nuclear saturation properties at nuclear density $n_0$, and leads to a behavior of the thermodynamic functions and baryonic abundances beyond saturation density similar to what would be obtained from a more elaborate description. Consequently, it will be a useful laboratory in order to investigate the possible existence of the ordered ground state configurations, their structure and stability.

In this context, we analyze a configuration where the hyperons in the ground state of the plasma are confined on the nodes of a lattice, and inquire whether it is energetically favored, for some density ranges, with respect to the usual Fermi liquid configuration. In this ordered phase, the hyperons are treated as harmonic oscillators (gaussian clouds). They are assumed to be confined around the nodes of a cubic lattice by the potential self-consistently generated by the other hyperons in the lattice and the fluid of nucleons.

In order to avoid the complexities related to the spin-spin component of the nuclear
interaction, we analyze a particular configuration where two hyperons with antiparallel spins coexist in the ground state of the harmonic potential at every lattice site, where they behave as a bound state of hyperons. In a first approximation we neglect the interaction energy between the two hyperons at the same site. The presence of bound states of hyperons (H-dibaryons [13]) in hadronic matter at high densities has been considered in reference [14] [15]. Their stability could be enhanced if the existence of an additional attractive ΛΛ interaction should be confirmed, as the excess binding of double hypernuclei seems to indicate [16]. If this is not so, the interaction between the two independent hyperons at a node can be obtained as the potential energy between two gaussian hyperonic clouds (of the form self-consistently obtained from the lattice potential) centered on the same node. We shall see that, at least in the relevant range of densities where the solid phase is found, the balance between the contributions coming from the attractive spin-spin component of the nuclear potential and the short-range repulsive central component leads to an interaction energy between the two hyperons at a node which is attractive (and, consequently, improves the confinement) up to densities \( \sim 3n_0 \). Beyond this density it becomes repulsive, but small as compared with the depth of the confining lattice potential, and the associated corrections can be treated in a perturbative way.

The hyperons confined around the nodes of a regular lattice must induce a redistribution of the surrounding nucleons. Such a redistribution can be calculated in the random phase approximation (RPA) [24], by treating the lattice as a system of impurities perturbing the ambient mean field and the nucleon Fermi distribution or, more exactly, by solving the Dirac equation in the periodic potential of the lattice through the Bloch method [25]. Nevertheless, we shall adopt here a simpler approximation to the dynamics of the nucleons in the medium, by neglecting the spatial variations of the lattice potential and treating the nucleon as a relativistic fermion moving in the constant mean field generated by the uniform Fermi fluid of nucleons plus the spatial average of the hyperonic distribution. This approximation is the analog of the Sommerfeld’s free electron approximation in a Coulomb lattice [25]. The corrections introduced by the non-uniform character of the lattice field will be considered elsewhere.

2 The model and the solid phase

We describe the baryon interaction through the exchange of scalar \( \sigma \) mesons and vector \( \omega \) mesons as in Walecka model [26]. A leptonic sector reduced to the electrons is included in order to preserve the electric charge neutrality. The lagrangian density defining the nucleon and hyperon coupling to the mesons is

\[
L = \sum_{B=N,h} \left[ \frac{i}{2} \left( \bar{\Psi}_B \gamma^{\mu} \partial_\mu \Psi_B - \partial_\mu \bar{\Psi}_B \gamma^\mu \Psi_B \right) - m_B \bar{\Psi}_B \Psi_B + g_\sigma \bar{\Psi}_B \sigma \Psi_B + g_\omega \bar{\Psi}_B \gamma^\mu \omega_\mu \Psi_B \right] + \frac{1}{2} \left( \partial_\mu \sigma \right) \left( \partial^\mu \sigma - m_\sigma^2 \sigma \right) - \frac{1}{2} \left( F_{\omega \mu \nu} F^{\omega \mu \nu} - m_\omega^2 \omega_\mu \omega^\mu \right)
\]

where the strengths of the couplings between the nucleons and hyperons to the meson fields are different. This lagrangian is the simplest one which allows an acceptable fit of nuclear matter saturation properties.
We assume now a configuration in which the hyperons are confined on the nodes of a cubic lattice by the interaction with the nucleon background and the fields generated by the lattice itself. If the resulting potential in a lattice site, $\vec{r}_i$, is approximated by an harmonic potential, the state of each hyperon can be described as the ground state of a classical harmonic oscillator characterized by a gaussian wave function of the form ($\hbar = c = 1$)

$$\Psi_h(\vec{r}) = \left( \frac{b}{\pi} \right)^{(3/4)} \exp\left( -\frac{b}{2}(\vec{r} - \vec{r}_i)^2 \right) ; \quad b = M_h \nu_0$$

where $M_h$ is the effective hyperon mass, that is the free mass modified by the mean scalar field generated by the uniform nucleon distribution (see (23) below). The oscillator frequency $\nu_0$ is related to the central curvature of the harmonic potential, taken as the parabolic potential osculating the true lattice potential on the node. This lattice potential at every site is obtained as the superposition of the fields created by the gaussian clouds of the hyperons in the other sites.

As mentioned in the introduction, we shall consider the configuration where two hyperons with antiparallel spins fill the ground state of the harmonic potential at every site. In this way the spin-spin interaction contribution to the lattice potential which comes from every node has the same form, avoiding the difficulties related to a configuration of individual hyperons with a random or complex distribution of spins. Moreover, the dynamics of hyperons in these states can be treated as non-relativistic. Relativistic corrections will be added in a later stage in the calculation of the thermodynamic state. This is consistent with the treatment of the dynamics of the other baryons in the ordered phase and with the calculation of the Fermi liquid configuration, to which one must compare in analyzing the relative stability of the ground states.

As a first step, we calculate the field created by a gaussian cloud at a given distance of the center. In the present case is given by the potential

$$V(\vec{r}) = \int d^3 q \exp(-i \vec{q} \cdot \vec{r}) \left[ g_{\sigma h} \frac{F_\sigma(q) S_h(q)}{q^2 + m_\sigma^2} + g_{\omega h} \frac{F_\omega(q) n_h(q)}{q^2 + m_\omega^2} \right]$$

where $S_h(q)$ and $n_h(q)$ are the hyperonic scalar and particle density associated to the gaussian cloud in the $q$-space. We introduced form factors as usual in order to take into account the fact that the hyperons are composite extended particles

$$F_i(q) = \frac{\Lambda_i^2 - m_i^2}{\Lambda_i^2 + q^2} ; \quad i = \sigma, \omega$$

From this expression, we can obtain the potential energy of a point-like hyperon of given spin, at a point $\vec{r}$ in the total field created by the gaussian cloud associated to the two antiparallel-spin hyperons centered at $\vec{r} = 0$:

$$U(\vec{r}) = \frac{2\pi}{r} \int_0^\infty R n(R) dR \int_{|r+R|}^{|r-R|} x \Phi(x) dx$$

where $n(r)$ is the gaussian charge distribution of the two hyperons in the node, given by

---

1 in the non relativistic treatment, the hyperonic density $n_h$ and scalar density $S_h$, which are the sources of the vector and scalar fields respectively, coincide.
\[ n(r) = 2\Psi_h^*(r)\Psi_h(r) = 2 \left( \frac{M_h\nu_0}{\pi} \right)^{3/2} e^{-M_h\nu_0 r^2} \]  

(6)

and \( \Phi(x) \) is the expression of the elementary potential energy between two hyperon states in the one-boson exchange approximation. At zeroth order in momentum expansion and excluding the effects of the form factors, \( \Phi(x) \) reduces to the central component only, which has the usual Yukawa expression for both meson exchanges:

\[ \Phi(x) = -\frac{g^{2}_{\sigma h}}{4\pi} e^{-m_{\sigma} r} + \frac{g^{2}_{\omega h}}{4\pi} e^{-m_{\omega} r} \]  

(7)

At the first order in the momentum transfer, in the present configuration, \( \Phi(x) \) must include only the central and spin-spin components of the nuclear potential. Indeed, it can be easily seen that the spin-orbit and tensor components vanish in this ground state configuration \([18]\). We use the expressions of the Bonn potential \([23]\) coming from the exchange of \(\sigma\) and \(\omega\) mesons and including the monopolar form factors in order to account phenomenologically for the extended hadronic structure. Then

\[
\Phi(x) = -\frac{g^{2}_{\sigma h}}{4\pi} \left( 1 - \frac{1}{4} \left( \frac{m_{\sigma}}{M_h} \right)^2 \right) \left[ e^{-m_{\sigma} r} - \frac{e^{-\Lambda_{\sigma} r}}{r} - \frac{(\Lambda_{\sigma}^2 - m_{\sigma}^2) e^{-\Lambda_{\sigma} r}}{2\Lambda_{\sigma}} \right] \\
+ \frac{g^{2}_{\omega h}}{4\pi} \left( 1 + \frac{1}{3} \left( \frac{m_{\omega}}{M_h} \right)^2 \right) \left[ e^{-m_{\omega} r} - \frac{e^{-\Lambda_{\omega} r}}{r} - \frac{(\Lambda_{\omega}^2 - m_{\omega}^2) e^{-\Lambda_{\omega} r}}{2\Lambda_{\omega}} \right]
\]  

(8)

is the half of the interaction energy between two antiparallel-spin hyperons at a point and a point-like hyperon at a distance \(x\), including the central and spin-spin components as well as the form factors.

Now, the total potential energy of an hyperon around a lattice site is obtained as a superposition of the interaction potential energies with the hyperons centered on the other sites,

\[ U_{\text{sup}}(\vec{r}) = \sum_i U(|\vec{r} - \vec{r}_i|) \]  

(9)

to which the interaction energy with the fluid of nucleons must be added. For a cubic lattice, the positions of the sites are defined as

\[ \vec{r}_i = a(l\vec{i} + m\vec{j} + n\vec{k}) \]

\(l, m\) and \(n\) being null, positive or negative integers and \(a\) is the lattice constant which is related to the mean hyperonic density by \(<n_h>_{\text{s}} = 2/a^3\) (the index "s" stands for "spatial average"). The case where the three indexes vanish simultaneously is explicitly excluded and, in this way, the interaction energy between the two hyperons at each site (as well as the self interaction) is excluded. As mentioned in the introduction, the contribution of this energy to the total energy of the system, as estimated from the present approximation, is small, and will be treated as a perturbation, which will be incorporated in the final calculation of the thermodynamic state.
As we shall see later, for many reasonable choices of the model parameters, this potential has the form of a confining nearly parabolic well at every lattice site. We can describe the dynamics of the hyperon in this well by solving the Schrödinger equation, and in this way determine self-consistently the oscillator frequency which enters the expression of the potential energy in (3). Indeed, the frequency of the harmonic oscillator is related to the second derivative of the harmonic potential by

$$\nu_0 = \sqrt{\frac{\nabla^2 U(\vec{r})}{M_h}} \bigg|_{\vec{r}=0}$$ (10)

where $U$ is the parabolic fit of $U_{sup}$ around a site.

The nucleonic sector is described here as a uniform Fermi fluid in the mean field approximation [26], but undergoing the action of the spatial average of the lattice fields, besides the one of the mean fields generated by the nucleon distribution itself. In this way, we calculate the contribution of the nucleons to the total energy density of the crystallized system in the same approximation as the one used in the Fermi liquid case [17], to which we will compare in order to evaluate the relative stability of the liquid and solid configurations.

The equations for the scalar and vector fields in the medium are

$$(-\nabla^2 + m^2_\sigma) (<\sigma > + \sigma_h) = g_{\sigma N} S_N + g_{\sigma h} S_h$$ (11)

$$(\nabla^2 - m^2_\omega) (<\omega^0 > + \omega_h) = g_{\omega N} N + g_{\omega h} n_h$$ (12)

whereas the equations for the mean fields generated by the uniform nucleon (proton and neutron) distributions are

$$m^2_\sigma \left( \frac{m_N - M_N}{g_{\sigma N}} \right) + g_{\sigma h} < n_h >_s$$

$$m^2_\omega <\omega^0>_s = -g_{\omega N} \left( \frac{p_{fp}^3}{3\pi^2} + \frac{p_{fn}^3}{3\pi^2} \right)$$ (14)

where $\varepsilon_{fp,n} = \sqrt{p_{fp,n}^2 + M_N^2}$ and the effective mass of the quasi-nucleons is now given by

$$M_N = m - g_\sigma (<\sigma > + <\sigma_h>_s)$$ (15)

In the above equations $<\sigma_h>_s$ and $<n_h>_s$ are the spatial averages of the hyperonic contributions to the scalar field and the hyperon density respectively. Such spatial averages are defined as the limit of

$$<f(x)>_s = \frac{1}{V} \int d^3x \ f(x)$$ (16)

as the volume $V$ goes to infinity. The spatial averages of the lattice scalar and vector fields are related to the average hyperon density through

$$m^2_\sigma <\sigma_h>_s = g_{\sigma h} <S_h>_s$$ (17)

$$m^2_\omega <\omega_h>_s = -g_{\omega h} <n_h>_s$$ (18)

In the non-relativistic approximation for the hyperons in the lattice, the scalar density and the density of hyperons coincide.
3 The $\beta$ equilibrium

The above description allows the complete determination of the macroscopic configuration of the system at every fixed baryonic density. Indeed, we can now easily calculate the chemical potentials for each component and establish the set of equations for the beta equilibrium in the medium. The proton, neutron and electron chemical potentials in terms of their Fermi momentums are given by

\[
\mu_e = \sqrt{p_{fe}^2 + m_e^2} = \varepsilon_{fe} \tag{19}
\]
\[
\mu_n = \sqrt{p_{fn}^2 + M_N^2} - g_{\omega N}(<\omega^0> + <\omega_h>) \tag{20}
\]
\[
\mu_p = \sqrt{p_{fp}^2 + M_N^2} - g_{\omega N}(<\omega^0> + <\omega_h>) \tag{21}
\]

The hyperon chemical potential in the ordered phase is the energy gained by the system when adding a new hyperon in the lowest energy empty state, which in the postulated configuration is the energy of the first unoccupied level of the harmonic oscillator (including now the mass and well energies) plus the interaction energy of the hyperon with the mean fields created by the nucleon component:

\[
\mu_h = M_h - |Uo| + \frac{5}{2}\nu_0 - g_{\omega h} <\omega^0> \tag{22}
\]

where the effective hyperon mass is determined by the mean scalar field generated by the nucleon distribution through

\[
M_h = m_h - g_{\sigma h} <\sigma> \tag{23}
\]

The above considerations lead to the following system of equations for the beta equilibrium

\[
n_e = n_p \tag{24}
\]

from charge neutrality,

\[
\mu_n = \mu_p + \mu_e \tag{25}
\]

from neutron decay,

\[
\mu_n = \mu_h \tag{26}
\]

from hyperon decay,

\[
n_b = n_n + n_p + n_h \tag{27}
\]

from baryonic number conservation.

Moreover, the crystal cell size, $a$, is related to the macroscopic density of hyperons, $n_h$, paired in S-states on the nodes, through

\[
n_h = \frac{2}{a^3} \tag{28}
\]
These equations, coupled to the mean field equations (13, 14) and the self-consistent equations for the confining fields in the lattice nodes (10), form a system whose solution (when found, for a given set of the parameters of the model) completely determine the postulated configuration for every baryonic density. The relative stability of such ordered configuration with respect to the Fermi liquid configuration at the same baryonic density must be established by comparing the energy densities in both cases.

For the Fermi liquid configuration we can establish on similar grounds a system of equations for the β equilibrium [17]. In this case the hyperon chemical potential is given in function of the Fermi energy by

$$\mu_h = \sqrt{p_{fh}^2 + M_h^2} - g_{\omega h} < \omega^0 >$$  \hspace{1cm} (29)

where the effective hyperon mass is now given by the same equation (23) in terms of the (now unique) mean scalar field which results from the Eq. (30) below. The system of equations defining the equilibrium is now:

\begin{equation}
m_\sigma^2 < \sigma > = \frac{g_{\sigma N}}{2\pi^2} M_N [p_{fp} \varepsilon_{fp} - M_N^2 \ln \left| \frac{p_{fp} + \varepsilon_{fp}}{M_N} \right| + p_{fn} \varepsilon_{fn} - M_N^2 \ln \left| \frac{p_{fn} + \varepsilon_{fn}}{M_N} \right|] + \frac{g_{\omega h}}{2\pi^2} M_h [p_{fh} \varepsilon_{fh} - M_h^2 \ln \left| \frac{p_{fh} + \varepsilon_{fh}}{M_h} \right|]
\end{equation}

for the mean scalar field,

\begin{equation}
m_\omega^2 < \omega^0 > = -g_{\omega N} \left( \frac{p_{fp}^2}{3\pi^2} + \frac{p_{fn}^2}{3\pi^2} \right) - g_{\omega h} \frac{p_{fh}^2}{3\pi^2}
\end{equation}

for the mean vector field,

\begin{equation}
p_{fp} = p_f e
\end{equation}

from charge conservation,

\begin{equation}
\varepsilon_{fn} - g_{\omega N} < \omega^0 > = \varepsilon_{fp} - g_{\omega N} < \omega^0 > + \sqrt{p_{fe}^2 + m_e^2}
\end{equation}

from neutron decay,

\begin{equation}
\varepsilon_{fn} - g_{\omega N} < \omega^0 > = \sqrt{p_{fh}^2 + M_h^2} - g_{\omega h} < \omega^0 >
\end{equation}

from hyperon decay

\begin{equation}
\eta = \frac{p_{fp}^3}{3\pi^2} + \frac{p_{fn}^3}{3\pi^2} + \frac{p_{fh}^3}{3\pi^2}
\end{equation}

from baryonic number conservation.

The energy density in both cases is obtained as the sum of the contributions of the nucleonic and hyperonic sectors and that of the fields. The spatially averaged contribution of the fields to the total energy density in the ordered phase in the present approximation is

$$< \rho_{fields} >_s = \frac{1}{2} (m_\sigma^2 < (\sigma > + \sigma_h)^2 >_s - \frac{1}{2} m_\omega^2 < (\omega^0 > + \omega_h)^2 >_s$$
+ \frac{1}{2} < \nabla(\sigma_h) \cdot \nabla(\sigma_h) >_s \frac{1}{2} < \nabla(\omega_h) \cdot \nabla(\omega_h) >_s \quad (36)

In averaging the terms containing the spatial derivatives of the lattice fields we obtain, by integration by parts

\frac{1}{V} \int \nabla(\sigma_h) \cdot \nabla(\sigma_h) d^3 \vec{x} = \frac{1}{V} \int \nabla(\sigma_h) \cdot \nabla(\sigma_h) d^3 \vec{x} = \frac{1}{V} \int [\sigma_h \nabla^2(\sigma_h)] d^3 \vec{x} \quad (37)

and

\frac{1}{V} \int \nabla(\omega_h) \cdot \nabla(\omega_h) d^3 \vec{x} = \frac{1}{V} \int \nabla(\omega_h) \cdot \nabla(\omega_h) d^3 \vec{x} = \frac{1}{V} \int [\omega_h \nabla^2(\omega_h)] d^3 \vec{x}

In the limit of large V, the terms associated to the integrals of divergences in these formulae vanish. Using the field equations (11, 12) for the Laplace operator we obtain

\frac{1}{V} \int \nabla(\sigma_h) \cdot \nabla(\sigma_h) d^3 \vec{x} = -\frac{1}{V} \int [m_s^2 \sigma_h^2 + g_{\sigma h} \sigma_h S_h] d^3 \vec{x} \quad (38)

and finally, the expression for the spatially averaged field energy is

< \rho_{\text{fields}} >_s = \frac{g_{\sigma N}}{2} (\sigma > + \sigma_h >) S_N + \frac{g_{\omega N}}{2} (\omega > + \omega_h >) n_N

+ \frac{g_{\omega h}}{2} \sigma > \sigma_h > S_h + \frac{g_{\omega h}}{2} \omega > \omega_h > n_h + \frac{g_{\omega h}}{2} \sigma_h > \omega_h > n_h + \frac{g_{\omega h}}{2} \omega_h > \omega_h n_h \quad (40)

For the Fermi liquid configuration the field contribution to the energy density is

\rho_{\text{fields}} = \frac{1}{2} m_s^2 < \sigma >^2 - \frac{1}{2} m_\omega^2 < \omega^0 >^2 \quad (41)

The contribution to the energy density of the nucleonic Fermi fluid sector and the gas of electrons for the ordered configuration is

\rho_p + \rho_n + \rho_e = \frac{1}{8\pi^2} [\rho_f \varepsilon_f (\rho_f^2 + \varepsilon_f^2) - M_N^4 \ln \left| \frac{\rho_f + \varepsilon_f}{M_N} \right|]

+ \rho_f \varepsilon_f (\rho_f^2 + \varepsilon_f^2) - M_N^4 \ln \left| \frac{\rho_f + \varepsilon_f}{M_N} \right| + \frac{\rho_e^4}{8\pi^2} \quad (42)

whereas the spatially averaged contribution of the hyperonic sector in this case is given by the sum of the energies of the oscillators in their ground states

\rho_h = (M_h + \frac{3}{2} \nu_0 - g_{\omega h} < \omega^0 > - |U_0|) n_h \quad (43)

where the proper energy of the quasi-hyperons has been included in order to recover the (approximate) relativistic expression, consistent with the treatment of the other components.

For the Fermi liquid configuration the contribution to the energy density of the nucleons, hyperons and electrons is given by the same expression (42) where the sum must now be extended to the three sectors.

Note that in both configurations the field energy density compensates the double counting of the interaction energy density when calculated as the sum of the energies of the individual particles. The divergent contributions of the self energy of the hyperons in the ordered configuration contained in the two last terms of the field energy (40) must be discarded.
4 Energetic balance and stable configurations

The existence of a stable solution for the ordered phase is sensitive to the choice of the parameters of the model, some of which (in particular the strengths of the hyperonic couplings) are poorly known. Moreover, the nuclear interaction is treated at an elementary level and in this simplified version of the model. Consequently, no definitive conclusions about the presence of such ordered states in actual hadronic matter can be inferred from our analysis. Nevertheless, we shall explore their compatibility with our knowledge of the hadronic interactions at the level of description used in the present framework.

The values of the masses and meson-nucleon coupling constants used in the following calculations are

\[
\begin{align*}
m_N &= 939 \text{ MeV}, \quad m_\sigma = 550 \text{ MeV}, \quad m_\omega = 783 \text{ MeV} \\
m_h &= m_\Lambda = 1115 \text{ MeV}, \quad g_{\sigma N}^2 = \frac{7.29}{4\pi}, \quad g_{\omega N}^2 = \frac{10.83}{4\pi}
\end{align*}
\]  

With these values, saturation for nuclear matter is attained, in the mean field approximation, at \( p_f = 1.42 \text{ fm}^{-1} \) corresponding to a particle density \( n_0 = 0.197 \text{ fm}^{-3} \), with a binding energy \( E_b = -15.77 \text{ MeV} \). The meson-hyperon coupling constants in-medium, giving the interaction with the mean fields, are determined by fixing the ratios

\[
x_\sigma = \frac{g_{\sigma h}}{g_{\sigma N}}, \quad x_\omega = \frac{g_{\omega h}}{g_{\omega N}}
\]  

in such a way that the correct value of the \( \Lambda \) binding energy in hypernuclei will be reproduced. Figure 1 is a diagram \( x_\sigma - x_\omega \) showing the region of values of these parameters which allows the fit of the binding energy of the hyperons, extracted from spectroscopic data on single \( \Lambda^- \) hypernuclei [3], once the values of the meson-nucleon coupling constants at saturation density are fixed by the parametrization (44).

On the other hand, for the ordered configuration, the coupling of the hyperons to the scalar and vector fields governing the interaction in the lattice, which has been introduced as an interaction in vacuum (see Eq. (3)) will be fixed to different values. In reference [20] the coupling of the \( \Lambda \) hyperons to the \( \sigma \) field and the vector coupling to the \( \omega \) field are taken to be \( \frac{g_{\sigma h}}{\sqrt{4\pi}} = 2.138, \frac{g_{\omega h}}{\sqrt{4\pi}} = 2.981 \) in order to fit the scattering data in vacuum. Although our level of description of the interaction is more elementary than in the above mentioned reference, we shall keep these values in our calculations as a first approach.

Let us first consider a solution of the system for the ordered phase in the simplest case where the form factors are discarded and the first order corrections to the hyperon potential in the momentum transfer are neglected (using Eq. (7)). The parameters \( x_\sigma \) and \( x_\omega \) are now fixed to the values 0.45 and 0.483 respectively (set A), which fit the \( \Lambda^- \) hypernuclei binding energy. Simultaneously, the system for the liquid configuration is solved with the same set of parameters. In figure 2 (curve A) we show the difference between the energy densities in the liquid and the ordered configurations as a function of the baryonic density. Before the threshold density for the creation of hyperons \( (n_f = 1.75 n_0) \) the liquid phase of protons, neutrons and electrons is the only stable configuration. Beyond this threshold the
solid configuration is present and more stable than the liquid one. This relative stability increases with density.

When the first order corrections in the momentum transfer are included in the hyperonic potential \( \mathcal{V} \), but excluding the terms coming from the presence of the form factors, stable ordered configurations are also obtained for some sets of parameters. The curve B in figure 2 results from the set \( x_\sigma = 0.5; x_\omega = 0.546 \) (set B), which also fits the \( \Lambda \) binding energy in hypernuclei. Now the solid phase is less stable than the liquid one from the threshold density of appearance of the hyperonic sector \( (n_t = 1.79 n_0) \) up to a density of \( 4.3 n_0 \), around which a transition between the two phases takes place. When the form factors are included by using the full Eq. \( \mathcal{V} \) in the calculation, stable ordered configurations of the ground state are obtained when we fix the form factors \( \Lambda_\sigma = 1.5 \text{ GeV} \) and \( \Lambda_\omega = 2.0 \text{ GeV} \), and the ratios between the coupling constants to \( x_\sigma = 0.4 \), \( x_\omega = 0.42 \) (set C), which fit also the \( \Lambda \) binding energy in hypernuclei. Now, the liquid phase is more stable in a narrow range of densities between the threshold \( (n_t = 1.72 n_0) \) and \( n \sim 2.2 n_0 \) (see Fig. 2).

Both in the B and C cases, a Gibbs construction should be performed in order to determine the transition region \cite{21}. As can be seen from Fig. 2, the density around which the liquid-solid phase transition takes place is very sensitive to the level of description of the nuclear interaction (the optimum level in the present calculations corresponds to set C), but the ordered phase is always more stable than the liquid one at high enough densities.

For the solid configurations we obtain the size of the lattice cell (the lattice constant, \( a \)) as a function of the baryonic density (see figure 3). This parameter decreases monotonically with density (from infinity at the threshold) in all the meaningful range. This is related to the increasing abundance of the hyperons with baryonic density. The solution of the \( \beta \)-equilibrium system gives the abundances of all the species in each configuration, which are plotted in figures 4(A,B,C) as functions of density, for the three sets of parameters, and compared to the abundances in the corresponding liquid configurations. Before the threshold densities the hyperonic sector is absent and the neutron abundance decreases with density with a simultaneous increase of the proton and electron abundances. Beyond the threshold the regular increase of the abundance of hyperons in the solid and liquid phases is accompanied by a simultaneous reduction of the abundances of all the other fermions. Qualitatively, the behavior of the abundances for the three sets of parameters in the solid and liquid configurations is similar.

Figure 5 shows the typical shape of the confining potential around a lattice site resulting from the interaction with the hyperons located in the other sites, which is obtained self-consistently from Eqs. \( \mathcal{V} \), when solving the system of equilibrium equations. The curve (corresponding to the set A at a density \( n = 3.5 n_0 \)) gives the variation of the potential energy of a point-like hyperon with the distance to the node in the direction of a principal axis of the lattice. The potential well has a nearly spherical symmetry around the site. We have approximated this potential by a parabolic one in the calculation of the solid configuration. The reliability of such an approximation is related to the depth of the potential well, which is plotted in figure 6 as a function of density, for the three sets of parameters. We see that, beyond the threshold density, the confining energy increases regularly and the harmonic approximation becomes more accurate at high densities.

At lower densities, near the threshold \( n_t \), the abundance of hyperons is small, both in the solid and liquid configurations. Thus the hyperons in the liquid phase are not very degenerate and their dynamics is governed by small-momentum wave functions. In the solid
phase the hyperons are not strongly confined, as can be seen in figure 7, where we have plotted the ratio between the width of their gaussian wave functions to the lattice parameter (which is very large near the threshold $n_t$), versus hadronic density.

Consequently, the widths of the gaussian wave functions diverge at the threshold and there is no important difference between the physical states of the hyperons in both configurations in this region, where a very similar macroscopic behavior of the liquid configuration and the ordered one (as calculated from the harmonic approximation) should be expected. This is confirmed by the small difference between the energy densities of both configurations near the threshold for all sets of parameters, as well as by the very similar behavior of the other macroscopic functions (see below).

Let us estimate the importance of the error introduced when we neglect the interaction energy between the hyperons on the same site ($E_{cell}$), which should modify the form of the confining potential. We have calculated the ratio between $E_{cell}$ and the depth of the potential well ($|U_0|$). This ratio is shown in figure 8 as a function of the energy density for the three sets of parameters.

We see that near the threshold $n_t$, this ratio is important, but $E_{cell}$ is negative up to a density $\sim 3 n_0$. Then, by including the contributions associated to $E_{cell}$ we should improve the confinement of hyperons and the stability of the solid phase in this region. Nevertheless, the modification of the macroscopic functions associated to $E_{cell}$, is expected to be small near the threshold, owing to the small proportion of hyperons there. For higher densities the ratio remains less than 0.2 for sets B and C and reaches 0.4 at $n \sim 5 n_0$ for set A, but the associated corrections on the macroscopic quantities should be more important here, owing to the increasing abundance of hyperons with density. In order to explore the importance of these corrections on the macroscopic state we have added the density of interaction energy between the hyperons on the nodes to the total energy density in Fig. 2 (dotted lines). We see that the correction remains small up to densities $\sim 3.5 n_0$ and becomes more important beyond for all sets.

Let us now compare the behavior of the pressure in both configurations. Although in the ordered phases the stress is described by a tensor, the dominant terms come from the diagonal part of this stress tensor, which in the cubic lattice configuration is given by the pressure [27], related to the energy and particle densities through

$$p = n_b^2 \frac{\partial(\rho/n_b)}{\partial n_b}$$

Figures 9(A, B, C) are plots of the pressure in the liquid and ordered configurations as functions of the baryonic density, for the three sets of parameters. As is easily seen, the pressure in both phases has a very similar behavior up to densities $\sim 3.5 n_0$ for the three sets.

This similarity continues to higher densities for the set A, whose EOS in the solid configuration becomes slightly stiffer than the liquid one. On the other hand for sets B and C the EOS of the solid phases at high densities become dramatically softer than those of the liquid ones.

At lower densities the contribution from the hyperonic sector to the thermodynamical behavior of the system is small, as compared to the contribution from the nucleons, which is dominant and very similar in both phases. This is the reason for the similar behavior of pressure in both phases.
The behavior at higher densities can be explained in terms of the characteristics of the hyperonic interaction. There, the hyperon sector becomes dominant and behaves as a Fermi liquid in the liquid phase, where the main contribution to the pressure comes from the kinetic energy of the constituent particles. In the solid phase the pressure is mainly dominated by the total energy of the harmonic oscillators on the nodes of the lattice and their evolution with density, which is strongly dependent on the form of the OBE potential between the hyperons used in the calculation. In figure 10 we have plotted the frequency of the harmonic oscillators (\(\nu_0\), in natural units) as a function of the baryonic density, for the three sets of parameters. This frequency is proportional to the oscillator energy, which enters the total energy density expression (43), and is related to the lattice potential through Eq. (10), which is obtained from the OBE potentials associated to each set of parameters.

In the case of set A, the lattice potential obtained from the OBE potential (8) evolves with the density in such a way that the slope of the \(\nu_0 - n\) curve increases strongly at \(n \sim 3.2 n_0\) and becomes greater than those of sets B and C at high densities (see Fig. 10). As a consequence, the slope of the energy density of the solid grows nearly parallel to the liquid phase one in this density region (see Fig. 2). This explains that the pressure in both phases, which is directly related to these slopes, grows similarly in this case.

For sets B and C the lattice potential is obtained from the OBE potential (8), which includes the contributions of the spin-spin component and the first-order momentum transfer corrections to the central component. As a consequence of these modifications, the depth of the well of the OBE potential is reduced for these sets and the width of the resulting lattice potential decreases with density slower than in the case of set A, at high densities. Then, the slopes of curves B and C in Fig. 10 (as well as the slopes of the corresponding \(\rho - n\) curves) are smaller and the pressure is reduced with respect to the cases of set A and liquid phase.

## 5 Conclusions and perspectives

We have analyzed the conditions of formation of ordered phases in hadronic matter, due to the confinement of the hyperonic component, which must be present beyond a threshold density, on the nodes of a lattice. The analysis has been performed in the framework of a minimal model of the hadronic plasma which incorporates some basic features of the nuclear interaction, allowing for an acceptable fit of the nuclear saturation properties, hyperon binding energy in hypernuclei and scattering data in vacuum. In the resolution of the model for the ordered phase we have introduced some approximations, consistent with the mean-field approximation which leads to the macroscopic description of the hadronic liquid phase. In the framework of this model, we conclude that the solid phase is always energetically favored at high densities. At lower densities the relative stability of the liquid and solid configurations is very sensitive to the level of description of the hyperonic interaction. At the simplest level we obtain a solid phase which is more stable than the liquid one for all densities beyond the threshold of production of hyperons (set A). When the first-order corrections in the momentum transfer are included in the static hyperonic potential (set B), we obtain a solution where the liquid configuration is more stable than the ordered one from the threshold to a density \(\sim 4.2 n_0\). Beyond, the solid configuration becomes more stable. When we include in the description of the interaction the effects of the form factors, account-
ing for the hyperon extended structure (set C), the solid configuration becomes more stable from densities slightly beyond the threshold. Both in the cases B and C there should be a first-order phase transition region between the two configurations, to be determined through by a Gibbs construction [21]. The important softening of the hadronic EOS in presence of the solid phase, in the cases B and C at high densities, as compared to the usual liquid EOS, should introduce important modifications in the analysis of the hydrostatic equilibrium and cooling of neutron stars.

Nevertheless, owing to the simplicity of the model and the approximations introduced, these results should be considered as preliminary. In order to extract reliable conclusions as to the presence of such phases in actual hadronic matter, the present model must be improved in many aspects:

1) A more accurate description of the nuclear interaction must be introduced, by including other meson exchanges. In particular the coupling to ρ (and perhaps, π) mesons can be crucial in this context, owing to their isospin asymmetry.

2) The hyperonic sector must be extended in order to include all the hyperon states, in particular the Σ⁻. Similar extension should be introduced in the leptonic sector.

The dynamic analysis can be improved in many ways.

3) The Hartree-Fock higher order approximation can be used, instead of the mean field one, in describing the proton and neutron liquids. Exchange effects should be also included in the calculations concerning the hyperonic sector.

4) The interaction between the nucleonic sector and the hyperonic lattice, which is approximated as an uniform background in the present (Sommerfeld) approximation, can be improved by solving the Dirac equation in the periodic field created by the lattice, by generalizing the well know Bloch method for the analysis of the dynamics of electrons in metals [25].

5) Related to the last point, the screening effects due to the redistribution of the nucleons in presence of the lattice potential have been calculated in the RPA [28]. These effects introduce corrections in this potential (as well as in the energetic balance of the solid phase) and give a first simple approximation to the (more precise) calculation of the nucleon dynamics in terms of the Bloch functions. We have not yet completely explored the consequences of these corrections, but preliminary results show that the effects of the screening introduced by the redistribution of the nucleons around the lattice sites improves the confinement of the hyperons in the nodes and the stability of the solid phase.

6) The interaction energy between the two hyperons in the potential well at every lattice site has been considered here as a small perturbation. Nevertheless, we can improve the calculation if this interaction is included from the beginning in the lattice potential.

7) Finally, other configurations with single hyperons at each node (with given spatial distributions of their spins), and/or non-cubic lattice structures could be more stable than the one considered here. This program is now in progress.

The authors would like to thank S. Bonazzola, B. Carter and P. Haensel for useful discussions. M.A.P.G. acknowledges the support of Spanish M.E.C. grant AP97-76955547. This work is partially supported by project MCT-00-BFM-0357.
Figure captions.

Figure 1. Values of the rations $x_\sigma = \frac{g_{\sigma h}}{g_{\sigma N}}$ and $x_\omega = \frac{g_{\omega h}}{g_{\omega N}}$ leading to the experimental $\Lambda$-binding energy in hypernuclei.

Figure 2. Difference between the energy densities of the liquid and solid phases as functions of the baryonic density (in units of the saturation density), for the three sets of parameters.

Figure 3. Lattice parameter as a function of the baryonic density for the three sets.

Figure 4. Abundances of the fermion components in the liquid and solid phases as functions of the baryonic density, for the three sets of parameters.

Figure 5. Typical shape of the lattice potential energy (in units of the nucleon proper energy) confining the hyperons in a node as a function of the distance to the center of the node. It is obtained as the superposition of the potential energies in the field created by the hyperons confined on the neighbors nodes. The figure shows a section of this potential in the direction of a principal axis and corresponds to set A, at 3.5 times saturation density including the neighbors to the order 5.

Figure 6. Absolute value of the depth of the wells of the lattice potential confining the hyperons on the nodes as a function of the baryonic density, for the three sets of parameters.

Figure 7. Ratio between the width of the gaussian wave function of an hyperon in a lattice site and the lattice parameter as a function of the baryonic density, for the three sets of parameters.

Figure 8. Ratio between the interaction energy of two antiparallel-spin hyperons confined on a lattice site and the depth of the confining well as a function of the baryonic density, for the three sets of parameters.

Figure 9. Pressure (in MeV.fm$^{-3}$) as a function of the baryonic density (in units of the saturation density) for the liquid (dashed curves) and solid phases (continuous curves), for the three sets of parameters.

Figure 10. Frequency of the hyperonic oscillators (in units of the nucleon proper energy) in the lattice nodes as a function of the baryonic density (in units of the saturation density), for the three sets of parameters.

References

[1] V. Canuto, S.M. Chitre, Phys. Rev. D9 (1974) 1587.
G. Baym, C.J. Pethick, Ann. Rev. Nucl. Sci. 25 (1975) 27.
R. Mittet, E. Ostgaard, Nucl. Phys. A411 (1983) 417.

[2] V.R. Pandharipande, R.A. Smith, Nucl. Phys. A237 (1975) 507.
T. Matsui, K. Sakai, M. Yasuno, Prog. Theor. Phys. 61 (1979) 1093.
For a recent review see e.g Prog. Theor. Phys. Suppl. 112 (1993)

[3] M. Kutschera, W. Wojcik, Nucl. Phys. A581 (1995) 706.

[4] N.K. Glendenning, Phys. Rev. D46 (1992) 1274; Phys. Rep. 264 (1996) 143.
H. Heiselberg, C.J. Pethick, E.F. Staubo, Phys. Rev. Lett. 70 (1993) 1355.

[5] A.B. Migdal, P.J.S. Watson, Phys. Lett. B252 (1990) 32.
[6] I. Lovas, L. Molnár, K. Sailer, W. Greiner, Phys. Rev. C45 (1992) 1693.

[7] C.E. Price, J.R. Shepard, J.A. McNeil, Phys. Rev. C42 (1990) 247, C.E. Price, J.R. Shepard, J.A. McNeil, Phys. Rev. C42 (1990) 1234.

[8] J. Diaz Alonso, Astron. Astrophys. 125 (1983), 287-301.

[9] For a review on this question see P. Haensel in Relativistic Gravitation and Gravitational Radiation; Proc. of Les Houches School of Physics (1995), J.A. Marck and J.P. Lasota Edts., Cambridge University Press (1997).

[10] D.A. Baiko and P. Haensel, LANL eprint astro-ph/0004185.

[11] S. Reddy, G. Bertsch and M. Prakash, Phys. Lett. B475 (2000) 1.

[12] M. Prakash, I. Bombaci, M. Prakash, J.M. Lattimer, P. Ellis, R. Knorren, Phys. Rep. 280 (1997) 1. H. Heiselberg, M. Hjorth-Jensen, Phys. Rep. 328 (2000) 237.

[13] R. Jaffe, Phys. Lett. 38, (1977), 195.

[14] N.K. Glendenning, J. Schaffner-Bielich, Phys. Rev. C58 (1998) 1298.

[15] A. Faessler, A. Buchmann, A. Krivouchenko, Phys. Rev. C56 (1997) 1576

[16] J. Caro, C. García-Recio, J. Nieves, Nucl. Phys. A646 (1999) 299.

[17] N. Glendenning, Compact Stars, Ed. Springer-Verlag NY, (1997)

[18] C. Cohen-Tannoudji, B. Diu, F. Laloë, Quantum Mechanics, Wiley Interscience, Paris (1977).

[19] N.K. Glendenning, S. Pei, M. Hjorth-Jensen, Phys. Rev. Lett. 79 (1997) 1603.
H. Heiselberg, M. Hjorth-Jensen, Phys. Rev. Lett. 80 (1998) 5485

[20] A. Reuber, K. Holinde, J. Speth, Nucl. Phys. A570 (1994) 543-549.

[21] N.K. Glendenning, Phys. Rev. D46 (1992) 1274.

[22] T. Muto, T. Tatsumi, Prog. Theor. Phys. 83 (1990) 499.

[23] R. Machleidt, The Meson Theory of Nuclear Forces and Nuclear Structure, Adv. in Nucl. Phys. 19, J.W. Negele and E. Vogt Ed. Plenum Press, NY (1989).

[24] J. Díaz Alonso, A. Pérez Canyellas, Nucl. Phys. A526 (1991),623.

[25] N.W. Ashcroft, N.D. Mermin “Solid State Physics”. Saunders College Publishing, Fort Worth, (1976).

[26] J.D. Walecka. Ann. Phys. 83 (1974) 491.

[27] L. Landau E. Lifshitz ”Elasticity Theory”. Ed. MIR (1970).

[28] J. Díaz Alonso, L. Mornas, M.A. Pérez-García, J.P. Suárez, N. Corte. In preparation.
$E_{\text{bind}} = -28 \text{ MeV}$
Set A

$P \text{ (MeV fm}^3\text{)}$

$n/n_0$

$nt$
