Goldstone Bosons
in the Gaussian Approximation

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Abstract
The $O(N)$ symmetric scalar quantum field theory with $\lambda \Phi^4$ interaction
is discussed in the Gaussian approximation. It is shown that the
Goldstone theorem is fulfilled for arbitrary $N$.

1 Introduction
The theory of a real scalar field in $n$-dimensional Euclidean space-time with
a classical action given by

$$S[\Phi] = \int \left[ \frac{1}{2} \Phi(x)(-\partial^2 + m^2)\Phi(x) + \lambda(\Phi^2(x))^2 \right] d^n x,$$

is the most mysterious part of the standard model. Although experimentaly
not observed, the scalar Higgs field with $m^2 < 0$ and internal $O(4)$ symme-
try is necessary to give masses to interaction bosons in the Weinberg-Salam
model of weak interactions without spoiling renormalizability. Moreover,
the renormalized $\lambda \Phi^4$ theory has been almost rigorously proved [1] to be
noninteracting, in contradiction to the perturbative renormalization, which
can be performed order by order without any signal of triviality. Triviality
shows up in the leading order of the $\frac{1}{N}$ expansion [2] in the $O(N)$ symmetric
theory of $N$-component scalar field, $\Phi(x) = (\Phi_1(x), ..., \Phi_N(x))$. Other non-perturbative methods, like the Gaussian \cite{3} and post-Gaussian \cite{4,5} approximations, have therefore been applied to study renormalization of the scalar theory in the case when the number of field components is not large. However, a serious drawback of the Gaussian approximation for $N$-component field was an observation \cite{6,7} that the Goldstone theorem seemed not to be respected exactly, only in the limit of $N \to \infty$ did the would-be Goldstone bosons become massless. Here we show that this statement is not true, and is due to a faulty interpretation. In fact, the Gaussian approximation of the $O(N)$-symmetric theory yields one massive particle and $(N-1)$ massless Goldstone bosons, in agreement with the exact result of Goldstone theorem \cite{8}. There is another claim of existence of Goldstone bosons in the Gaussian approximation by Dmitrasinovic et al. \cite{9}, who found a pole in the four-point Green function of the $O(2)$-symmetric theory, which was interpreted as a bound state of two massive elementary excitations. In our work we show that massless bosons appear in the Gaussian approximation as elementary fields which are eigenvectors of the one-particle propagator matrix (two-point Green function).

It is convenient to formulate the approximation method for the effective action, $\Gamma[\phi]$, since all the Green’s functions can be obtained in a consistent way, through differentiation of the approximate expression. A full (inverse) propagator, required for one-particle states analysis, is given by the second derivative of the effective action at $\phi(x) = \phi_0$. The vacuum expectation value of the scalar field, $\phi_0$, can be obtained as a stationary point of the effective potential, $V(\phi) = -\frac{1}{\int d^n x} \Gamma[\phi]|_{\phi(x)=\phi=const}$.

We shall calculate the effective action, using the optimized expansion (OE) \cite{4}. The method consists in modifying the classical action of a scalar field \cite{11} to the form

$$S_\epsilon[\Phi,G] = \int \frac{1}{2} \Phi(x) G^{-1}(x,y) \Phi(y) \, d^n x d^n y + \epsilon \left[ \int \frac{1}{2} \Phi(x) [-\partial^2 + m^2]\delta(x-y) - G^{-1}(x,y) \Phi(y) \, d^n x d^n y + \int \lambda(\Phi^2(x))^2 \, d^n x \right],$$  \hspace{1cm} (2)

with an arbitrary free propagator $G(x,y)$. The effective action, as a series in an artificial parameter $\epsilon$, can be obtained as a sum of vacuum one-particle-irreducible diagrams with Feynman rules of the modified theory. The given order expression for the effective action is optimized, choosing $G(x,y)$ which
fulfills the gap equation
\[ \frac{\delta \Gamma_n}{\delta G^{-1}(x, y)} = 0, \] (3)
to make the dependence on the unphysical field as weak as possible.

It has been shown that in the first order of the OE for one-component field, the inverse of a free propagator can be taken in the form
\[ \Gamma(x, y) = G^{-1}(x, y) = (-\partial^2 + \Omega^2(x))\delta(x - y), \] (4)
and the Gaussian effective action (GEA) is obtained \[4\]. The effective potential, derived from the GEA for a constant background \( \varphi(x) = \phi \), coincides with the Gaussian effective potential (GEP) \[3\], obtained before by applying the variational method with Gaussian trial functionals to the functional Schrödinger equation.

Here we shall calculate the effective action for \( N \)-component field to the first order of the OE. In this case, the inverse of a trial propagator can be chosen in the form of a symmetric matrix
\[ \Gamma_{i,i}(x, y) = (-\partial^2 + M_i^2(x))\delta(x - y) \]
\[ \Gamma_{i,j}(x, y) = \Gamma_{j,i}(x, y) = M_{ij}^2(x)\delta(x - y) \] (5)
where the functions \( M_i^2(x) \) and \( M_{ij}^2(x) \) are variational parameters. The calculation of the effective action can be simplified, using the observation of Stevenson at al. \[7\] that for an \( O(N) \) symmetric theory only the shift \( \varphi(x) = (\varphi_1(x), ..., \varphi_N(x)) \) of the field sets a direction in the \( O(N) \) space. Thus, the eigendirection of a free propagator matrix will be radial and transverse, and the variational parameters for the transverse fields should be equal, because of the remaining \( O(N - 1) \) symmetry. In the coordinate system, in which the shift \( \varphi \) points in the \( i = 1 \) direction, the (inverse) trial propagator can be chosen in the form of a diagonal matrix with
\[ \Gamma_{11}(x, y) = G^{-1}(x, y) = (-\partial^2 + \Omega^2(x))\delta(x - y) \]
\[ \Gamma_{ii}(x, y) = g^{-1}(x, y) = (-\partial^2 + \omega^2(x))\delta(x - y) \quad \text{for } i \neq 1, \] (6)
and the effective action in the first order of the OE is obtained in the form
\[ \Gamma[\varphi] = -\int [\frac{1}{2} \varphi(x)(-\partial^2 + m^2)\varphi(x) + \lambda(\varphi^2(x))^2] d^n x - \frac{1}{2} Tr LnG^{-1} \]
\begin{align*}
&- \frac{N - 1}{2} Tr \ln g^{-1} + \frac{1}{2} \int (\Omega^2(x) - m^2 - 12\lambda\phi^2(x)) G(x, x) d^n x \\
&+ \frac{(N - 1)}{2} \int (\omega^2(x) - m^2 - 4\lambda\phi^2(x)) g(x, x) d^n x - 3\lambda \int G^2(x, x) d^n x \\
&- (N^2 - 1)\lambda \int g^2(x, x) d^n x - 2(N - 1) \lambda \int G(x, x) g(x, x) d^n x.
\end{align*}

Requiring the effective action to be stationarity with respect to small changes of variational parameters

\[ \frac{\delta \Gamma}{\delta \Omega^2} = \frac{\delta \Gamma}{\delta \omega^2} = 0, \]

results in gap equations

\begin{align*}
\Omega^2(x) - m^2 - 12\lambda \phi^2(x) - 12\lambda G(x, x) - 4(N - 1)\lambda g(x, x) &= 0 \\
\omega^2(x) - m^2 - 4\lambda \phi^2(x) - 4\lambda G(x, x) - 4(N + 1)\lambda g(x, x) &= 0
\end{align*}

which determine the functionals \( \Omega[\phi] \) and \( \omega[\phi] \). When limited to a constant background \( \phi = (\phi_1, ..., \phi_N) \), the GEA for \( N \)-component field gives the effective potential

\begin{align*}
V(\phi) &= \frac{m^2}{2} \phi^2 + \lambda(\phi^2)^2 + I_1(\Omega) + (N - 1)I_1(\omega) + \frac{1}{2}(m^2 - \Omega^2 + 12\lambda\phi^2)I_0(\Omega) \\
&+ \frac{N - 1}{2}(m^2 - \omega^2 + 4\lambda\phi^2)I_0(\omega) + 3\lambda I_0(\Omega)^2 + (N^2 - 1)\lambda I_0(\omega)^2 \\
&+ 2(N - 1)\lambda I_0(\Omega)I_0(\omega)
\end{align*}

with the functions \( \Omega(\phi) \) and \( \omega(\phi) \) determined by algebraic equations

\begin{align*}
\Omega^2 - m^2 - 12\lambda\phi^2 - 12\lambda I_0(\Omega) - 4\lambda(N - 1)I_0(\omega) &= 0, \\
\omega^2 - m^2 - 4\lambda\phi^2 - 4\lambda I_0(\Omega) - 4(N + 1)\lambda I_0(\omega) &= 0,
\end{align*}

where

\begin{align*}
I_1(\Omega) &= \frac{1}{2} \int \frac{d^n p}{(2\pi)^n} \ln(p^2 + \Omega^2) \\
I_0(\Omega) &= \int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2 + \Omega^2}.
\end{align*}
The same result for the $O(N)$ symmetric GEP was obtained before in the Schrödinger approach [7]. In the OE, a generalisation of the GEP to space-time dependent fields, the GEA (7), has been obtained. It enables us to derive not only the effective potential, but also one-particle-irreducible Green’s functions at arbitrary external momenta in the Gaussian approximation.

The minimum of the GEP is at $\phi_0$ fulfilling
\[
\frac{\partial V}{\partial \phi_i} = (m^2 + 4\lambda \phi^2 + 12\lambda I_0(\Omega) + 4(N - 1)\lambda I_0(\omega))\phi_i = 0; \quad (13)
\]
therefore, in the unsymmetric minimum we have
\[
B = m^2 + 4\lambda \phi^2 + 12\lambda I_0(\Omega) + 4(N - 1)\lambda I_0(\omega) = 0. \quad (14)
\]
In the GEP analysis for $N = 2$, it was pointed out by Brihaye and Consoli [6] that $\omega[\phi_0]$ is not equal to zero, which was interpreted as a violation of Goldstone theorem in the Gaussian approximation. For the same reason, Stevenson, Allès and Tarrach [7] admitted that also for a general $N$ the Gaussian approximation does not respect the Goldstone theorem. We would like to point out that this conclusion is unjustified, for $\Omega$ and $\omega$ are only variational parameters in the free propagator, and do not correspond to physical masses of scalar particles. The physical masses have to be determined as poles of a full propagator in the discussed approximation. The inverse of that propagator can be obtained as a second derivative of the GEA (7) with an implicit dependence, $\Omega^2 [\phi]$ and $\omega^2 [\phi]$, taken into account by differentiation of the gap equations (9). Upon performing the Fourier transform, the two-point vertex is calculated to be
\[
\Gamma_{11}(p) = \left. \frac{\delta^2 \Gamma}{\delta \phi_i^2} \right|_{\phi(x) = \phi_0} = p^2 + m^2 + 4\lambda \phi^2 + 12\lambda I_0(\Omega) + 4(N - 1)\lambda I_0(\omega) + 8\lambda \phi_i^2 A(p)
\]
\[
\Gamma_{ii}(p) = \left. \frac{\delta^2 \Gamma}{\delta \phi_i^2} \right|_{\phi(x) = \phi_0} = p^2 + m^2 + 4\lambda \phi^2 + 12\lambda I_0(\Omega) + 4(N - 1)\lambda I_0(\omega) + 8\lambda \phi_i^2 A(p)
\]
\[
\Gamma_{ij}(p) = \Gamma_{ji}(p) = \left. \frac{1}{2} \frac{\delta^2 \Gamma}{\delta \phi_i \delta \phi_j} \right|_{\phi(x) = \phi_0} = 8\lambda \phi_i \phi_j A(p), \quad (15)
\]
where
\[
A(p) = 1 - \frac{18\lambda I_{-1}(\Omega, p) + 2\lambda(N - 1)I_{-1}(\omega, p) + 24\lambda^2(N + 2)I_{-1}(\Omega, p)I_{-1}(\omega, p)}{1 + 6\lambda I_{-1}(\Omega, p) + 2\lambda(N + 1)I_{-1}(\omega, p) + 32\lambda^2(N + 2)I_{-1}(\Omega, p)I_{-1}(\omega, p)} \quad (16)
\]
and
\[ I_{-1}(\Omega, p) = 2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 + \Omega^2)((p + q)^2 + \Omega^2)}. \] (17)

Upon diagonalization of the matrix (15) we obtain an inverse propagator 
\[ \gamma_1(p) = p^2 + B + 8\lambda A(p)\phi^2_0, \] which corresponds to massive particle, and \((N - 1)\) inverse propagators \[ \gamma_i(p) = p^2 + B \] of Goldstone bosons, since \(B=0\) in the unsymmetric minimum (14). Therefore, for any \(N\) the Gaussian approximation of the \(O(N)\) symmetric theory does fully respect Goldstone theorem at the unrenormalized level.

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