Closed String Cohomology in Open String Field Theory

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Abstract

We show that closed string states in bosonic string field theory are encoded in the cyclic cohomology of cubic open string field theory (OSFT) which, in turn, classifies the deformations of OSFT. This cohomology is then shown to be independent of the open string background. Exact elements correspond to closed string gauge transformations, generic boundary deformations of Witten’s 3-vertex and infinitesimal shifts of the open string background. Finally it is argued that the closed string cohomology and the cyclic cohomology of OSFT are isomorphic to each other.

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1 Introduction

It has been known for a long time that Witten’s open bosonic string field theory without the inclusion of closed strings cannot be unitary. Indeed the perturbative one-loop open string diagrams contain closed string poles. One strategy is then to extend OSFT by adding closed and open-closed string vertices explicitly. This results in Zwiebach’s open-closed string field theory [1, 2]. Another strategy (although not necessarily orthogonal to the first one) is to identify the cohomology of closed string states in cubic open string field theory. To be more precise we recall that open string field theory is based on a differential graded algebra (DGA) \( \mathcal{A} = (Q, *, A) \), where the odd differential \( Q \) is the open string BRST operator. Its cohomology is that of the open string states. The idea is then to identify a second cohomology in OSFT whose elements are the closed string states. Now, it was shown [3] that self-consistent gauge invariant generalizations of cubic SFT are based on \( A_\infty \)-algebras admitting an invariant inner product \( \langle a*b, c \rangle = \langle a, b*c \rangle \). On the other hand it is well known (see e.g [4]) that infinitesimal \( A_\infty \)-deformations of a DGA preserving the invariant inner product, are classified by cyclic cohomology. This suggests that the closed string cohomology should be related to the cyclic cohomology of the differential graded algebra of cubic string field theory (see also [5]). To make this statement precise we then consider the vertices of Zwiebach’s open-closed string
field theory consisting of disks with a single closed string insertion in the bulk and an arbitrary number of open string insertions on the boundary. We then show that the condition for these vertices to be elements of the cyclic cohomology is precisely that the closed string insertion is on-shell.

Let us summarize our results. We consider arbitrary infinitesimal deformations of Witten’s cubic string field theory

$$S = \frac{1}{2} \langle \Psi, Q \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi, \Psi \rangle + \sum_{M=1}^{\infty} C_M(\Psi, \ldots, \Psi).$$

We know that they must satisfy $A_\infty$-algebraic conditions or, equivalently, the multilinear maps $C_M$ must be classified by the Hochschild cohomology. Let us briefly review what this means. First the BRST charge $Q$ acts on $C_M$ in the following way

$$(QC_M)(\Psi_1, \ldots, \Psi_M) \equiv \sum_{i=1}^{M} (-1)^{i+1+\ldots+i-1} C_M(\Psi_1, \ldots, \Psi_{i-1}, Q \Psi_i, \Psi_{i+1}, \ldots, \Psi_M).$$

And the co-boundary operator $\delta : \text{Hom}(A^{M-1}, \mathbb{C}) \to \text{Hom}(A^{M}, \mathbb{C})$ is defined by

$$(\delta C_{M-1})(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M) = \sum_{i=1}^{M-1} (-1)^i C_{M-1}(\Psi_1, \ldots, \Psi_i \star \Psi_{i+1}, \ldots, \Psi_M) + (-1)^i \Psi_i C_{M-1}(\Psi_2, \ldots, \Psi_M \star \Psi_1).$$

It turns out that $Q^2 = 0$, $\delta^2 = 0$ and $[Q, \delta] = 0$. And therefore the operator $(\delta - (-1)^{M}Q)$ squares to zero and thus defines a complex. The Hochschild cohomology is the cohomology of $(\delta - (-1)^{M}Q)$. This is almost what we need, but not quite exactly; we will need to focus on the “cyclic” elements of the cohomology. Namely we must ask that the open string maps $C_M$ satisfy

$$C_M(\Psi_2, \ldots, \Psi_M, \Psi_1) = (-1)^{M-1+\Psi_1(\Psi_2+\ldots+\Psi_M)} C_M(\Psi_1, \ldots, \Psi_M),$$

and the Hochschild cohomology restricted to the cyclic maps is called the cyclic cohomology $(HC^M)$.

The main result of this paper is that the closed string states can be identified with this cohomology. The proof goes as follows. First, remember that there is an isomorphism between cyclic cohomology and infinitesimal deformations of cubic SFT. Next, we list the possible deformations; we find three possibilities:

1. Deformations of the closed string background
2. Deformations of the open string background

3. Generic deformations of the three-point vertex

We will find that to each physical closed string state, there corresponds a closed element of the cyclic complex $CC^M$. Moreover, two closed string states are gauge-equivalent if and only if their elements in the cyclic complex are equivalent. In other words, there is a one-to-one linear map from the space of physical closed string states to the cyclic cohomology $HC^M$. In contrast, we will show that deformations of the open string background correspond to trivial elements in $HC^M$. The generic deformations of the 3-vertex are a little more subtle, but we will show that the geometric deformations are exact. And we will argue that all other deformations of the 3-vertex are either trivial or correspond to a closed string state of ghost number zero. Since the semi-relative closed string cohomology at ghost number zero and vanishing mass is trivial, we can then conclude that all elements in the cyclic cohomology must correspond to physical closed string states, and the main result of the paper follows.

We then consider finite deformations of the open string background. Namely we expand the theory around a solution of the classical equations of motion and look at the cyclic cohomology there. We are able to prove that the cyclic cohomology around any finite background is isomorphic to the original one.

What is the relevance of our results? On a conceptual level they show that OSFT already encodes the closed string states. Indeed the cohomology introduced here is defined using only the structure of OSFT, i.e. the open string BRST operator $Q$ and the product of open string fields. No extra information about closed strings enter here. On the other hand, we will argue below that Zwiebach's open-closed vertices are the only possible non-trivial generalizations of Witten's 3-vertex (assuming local insertions on the boundary). In that sense the open-closed string vertices are the only non-trivial deformations of OSFT. We will return to this point again in the conclusions. Finally, because of the fact that cyclic cohomology is invariant under deformations of the open string background, it classifies the open-closed string vertices in any open string background, in particular in the tachyon vacuum where the open string cohomology is trivial.

Plan of this paper: In section 2 we recall the relevant aspects of the construction of open-closed string field theory. In section 3 we show that the closed string cohomology is contained in the cyclic cohomology of the differential graded algebra $A$, while closed string gauge transformation are exact elements in the cyclic complex. In section 4 we then argue that the closed string cohomology and the cyclic cohomology of $A$ are in fact isomorphic to each other. In section 5 we show that the cyclic cohomology of $A$ is independent of the open string background. In section 6 we present the conclusions and some open issues. In appendix A we review some facts about $A_\infty$-algebras drawing on an analogy with algebraic topology.
2 Linearized Open-Closed SFT

Zwiebach’s quantum open-closed string field theory with manifest closed string factorization \[1, 2\] is given by the action

\[
S = \sum_{p=0, \frac{1}{2}, 1, \ldots} \hbar^p S_p,
\]

with

\[
S_p = \sum_{G,N,B} \sum_{m_1, \ldots, m_B \geq 0} g^{2p+N-2+M} S^{G,N}_{B,M,M}
\]

The \( S^{G,N}_{B,M,M} \) are contact terms described by genus \( G \) surfaces with \( N \) closed string insertions in the bulk, \( B \) boundaries and \( m_k \) open string insertions on the \( k \)-th boundary, and with \( M = m_1 + \ldots + m_B \). All the vertices in this action include strips of length \( \pi \) for the external open strings, and stubs of length \( \pi \) for the external closed strings. This is necessary at the quantum level to avoid over-counting of moduli space as, for example, an open string loop diagram with a short internal propagator is equivalent to a diagram with a long internal closed string propagator. The strips and stubs avoid the appearance of propagators of length smaller than \( 2\pi \). But this implies that Witten’s vertex is modified, and contact vertices with \( M \) open strings on the boundary of the disk must then be included for all \( M \geq 4 \).

We will be interested in the classical limit of the action \( \hbar \to 0 \). The dominant term \( S_0 \) contains the free part of the closed SFT action (which will not be relevant to us), and all purely open string vertices with strips. We will also consider the next-to-leading term \( S_2 \). It consists of the purely closed cubic vertex (which again will not be relevant), and all the contact terms described by a disk with one closed string insertion and \( M \) open string insertions on the boundary, where \( M = 0, 1, \ldots, \infty \). And we will neglect all other vertices because they are of order one or higher in \( \hbar \). To this order, it is consistent to consider the closed strings as non-dynamical; in particular the closed string tadpole \( (M = 0) \) is just a constant that we can ignore.

In this limit, it is consistent to replace the purely open string sector by the Witten vertex without external strips. To see why, we note that the over-counting mentioned above concerns diagrams with at least one internal closed string propagator. These diagrams can arise in two different ways. First as Feynman diagrams built from at least two vertices with one closed strings and open strings on the boundary. These are of order \( \hbar^\frac{3}{2} \) or higher and therefore the diagram will be of order at least one in \( \hbar \). The other possibility is that the diagram is a contact term with two or more boundaries and therefore also of order at least one in \( \hbar \). In the classical limit, we neglect these diagrams and we can thus consistently describe the purely open sector by Witten’s cubic action without strips.

We will then focus on the contact terms \( S_{1,M} \) coupling one closed string and \( M \) open strings on the disk. It is represented by the integrated correlator \( C^\phi_M(\Psi_1, \Psi_2, \ldots, \Psi_M) \) defined as in
Figure 1: The correlator $C^\phi_M(\Psi_1, \ldots, \Psi_M)$ on the disk $D$ with global coordinate $w$. The open string punctures are at the points $w_i = e^{i\alpha_i}$.

Fig.1 We have a disk with one puncture in the middle and $M$ punctures on the boundary, and local coordinates around each punctures defined by the quadratic differential solving the minimal area problem with the constraints that all non-trivial open curves have length $\geq \pi$ and all non-trivial closed curves have length $\geq 2\pi$. The vertex operators $\phi$ and $\Psi_i$ are inserted in the local coordinates. Since the moduli space of our punctured disk has dimension $M - 1$, we need to insert $M - 1$ antighosts that will determine the measure on the moduli space. We do that by inserting line integrals of $b$ around each but one of the open string vertex operators. Using the doubling trick, we can write them as

$$b(v_i) = \frac{1}{2\pi i} \oint_{P_i} b(z) v_i(z) \, dz.$$  \hspace{1cm} (3)

Let’s comment on that: The vector $v(z)$ must correspond to a tangent vector in the moduli space. This is the Schiffer variation argument that we briefly sketch. In our case, the disk with one puncture in the center and $M$ punctures on the boundary, has $M - 1$ moduli which can be taken as the positions of $M - 1$ punctures on the boundary. We know then that there exist $M - 1$ meromorphic vector fields $v_i$ defined in a neighborhood of the puncture $P_i$ which are constant near $P_i$, and which cannot be extended to the whole disk. It turns out that these vector fields generate translations in the moduli space; in other words, in our case they move the punctures. From the operator formalism point of view, these translations are generated by the line integrations of the energy-momentum tensor weighted by the vector $v_i$

$$T(v_i) = \frac{1}{2\pi i} \oint_{P_i} T(z) v_i(z) \, dz.$$  \hspace{1cm} (4)
We can now write the definition of the integrated correlator:

\[ C^\phi_M(\Psi_1, \Psi_2, \ldots, \Psi_M) \equiv \int_{T_M} ds_1 \ldots ds_{M-1} \langle \phi(0) b(v_1) \ldots b(v_{M-1}) \Psi^h_1(w_1) \ldots \Psi^h_M(w_M) \rangle_D. \]  

(5)

Here \( T_M \) is the part of moduli space of the disk with one puncture in the bulk and \( M \) punctures on the boundary, not covered by Feynman diagrams obtained by lower order vertices. The \( s_i \) are \( M - 1 \) real numbers parameterizing the moduli space. Concretely, we will take the coordinate on the disk to be

\[ w = r e^{i s}, \]  

(6)

so the \( s_i \) are the angles specifying the positions of the open string punctures, namely \( w_i = e^{i s_i} \). Finally, \( \Psi^h_i(w_i) \equiv h_i \circ \Psi(0) \) denotes the vertex operator inserted in the local coordinates with the conformal map \( h_i \). The only thing that we will need to know about \( h_i \) is that it depends on all \( s_k \ (k = 1, \ldots, M) \) and that \( h_i(0) = w_i = e^{i s_i} \).

3 Closed Strings and Cyclic Cohomology

Let us now consider Witten's open string field theory (in a flat closed string background) and let us switch on an infinitesimal closed string background \( \phi \). The theory is now described by the OCSFT action

\[ S = \frac{1}{2} \langle \Psi, Q \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi, \Psi \rangle + \sum_{M=1}^{\infty} C^\phi_M(\Psi, \ldots, \Psi) \]  

(7)

which we now view as an infinitesimal deformation of the cubic OSFT. The closed string background \( \phi \) being infinitesimal, its equation of motion is simply

\[ Q|\phi\rangle = 0. \]  

(8)

On the other hand, every on-shell physical closed string state \( \phi \) obeys Eq. (5), and can therefore be used to produce an infinitesimal background deformation. However, Eq. (8) is not the only condition that \( \phi \) must fulfill in order to be a physical state. It is well known that the condition

\[ b^-_0|\phi\rangle = 0 \]  

(9)

must be imposed as well, where \( b^-_0 \equiv b_0 - \bar{b}_0 \). Then the last condition

\[ L^-_0|\phi\rangle = 0, \]  

(10)

where \( L^-_0 \equiv L_0 - \bar{L}_0 \), follows immediately after acting with \( Q \) on Eq. (5).
We will now show that the constraint (9) implies that the multilinear functions \( C_M^\phi \) have the cyclicity symmetry

\[
C_M^\phi(\Psi_2, \ldots, \Psi_M, \Psi_1) = (-1)^{M-1+\Psi_1(\Psi_2+\cdots+\Psi_M)}C_M^\phi(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M).
\]

(11)

To show this, we note that the vectors \( v_i \) in the definition of the correlator, are given by

\[
v_i = -\partial_{s_i},
\]

(12)
in the local coordinate patch at \( s_i \), and \( v_i = 0 \) around the other punctures.\(^3\) The sum of these \( M \) vector fields can be extended to the whole disk, it is simply given by

\[
v = \sum_{i=1}^{M} v_i = -\partial_s.
\]

(13)

And remembering that the coordinate on the disk is \( w = re^{is} \), we have that

\[
\partial_s = i(w\partial_w - \bar{w}\partial_{\bar{w}}).
\]

(14)

It follows that

\[
\sum_{i=1}^{M} b(v_i) = -\oint_0 b(w)v(w)\frac{dw}{2\pi i} - \oint_0 \bar{b}(\bar{w})\bar{v}(\bar{w})\frac{d\bar{w}}{2\pi i} = i b_0^-,
\]

(15)

where it is understood that \( \bar{b}_0^- \) acts at \( w = 0 \), i.e. on the closed string field \( \phi \). Let us now consider the integrated correlator with \( \phi \) replaced by \( b_0^- \phi \), and with one less line integral of \( b \) (for concreteness we remove \( b(v_{M-1}) \)), and also we cyclically permute \( \Psi_1 \) to the right of \( \Psi_M \):

\[
\int_{T_M} ds_1 \ldots ds_{M-2} \langle (\bar{b}_0^- \phi)(0) b(v_1) \ldots b(v_{M-2}) \Psi_2^{h_1}(w_1) \ldots \Psi_M^{h_{M-1}}(w_{M-1}) \Psi_1^{h_M}(w_M) \rangle_D.
\]

(16)

Using (15), we see that this is proportional to

\[
\int_{T_M} ds_1 \ldots ds_{M-1} \langle \phi(0) b(v_1) \cdots b(v_{M-2}) \sum_{i=1}^{M} b(v_i) \Psi_2^{h_1}(w_1) \cdots \Psi_M^{h_{M-1}}(w_{M-1}) \Psi_1^{h_M}(w_M) \rangle_D
\]

= \int_{T_M} ds_1 \ldots ds_{M-1} \langle \phi(0) b(v_1) \ldots b(v_{M-2}) b(v_{M-1}) \Psi_2^{h_1}(w_1) \cdots \Psi_M^{h_{M-1}}(w_{M-1}) \Psi_1^{h_M}(w_M) \rangle_D
\]

+ \int_{T_M} ds_1 \ldots ds_{M-1} \langle \phi(0) b(v_1) \ldots b(v_{M-2}) b(v_{M-1}) \Psi_2^{h_1}(w_1) \cdots \Psi_M^{h_{M-1}}(w_{M-1}) \Psi_1^{h_M}(w_M) \rangle_D
\]

= \int_{T_M} ds_1 \ldots ds_{M-1} \langle \phi(0) b(v_M) b(v_1) \ldots b(v_{M-2}) \Psi_1^{h_M}(w_M) \Psi_2^{h_1}(w_1) \cdots \Psi_M^{h_{M-1}}(w_{M-1}) \rangle_D.
\]

(17)

\(^3\)The minus sign comes form the Schiffer variation argument.
In the first step we have replaced $ib_0^-$ acting on $\phi$, by $\sum_{l=1}^{M} b(v_l)$ acting on the open strings. And in the last line, we have moved both $b(v_M)$ and $\Psi_{1}^{1\ldots M}(w_M)$ to the left and accounted for the corresponding sign. In order to identify the last term, we note that the domain of integration $T_M$ is invariant upon cyclic permutation. Indeed, until now we have chosen to keep $w_M$ fixed to a given (unspecified) value, but the correlator is invariant under a global rotation of the disk. This means that we could equivalently have chosen to keep, say, $w_{M-1}$ fixed; we would then have to integrate over $ds_1 \ldots ds_{M-2}ds_M$. With this remark, we see that (17) is

\[
C^\phi_M(\Psi_2, \ldots, \Psi_M, \Psi_1) + (-1)^{M-2+\Psi_1(\Psi_2+\ldots+\Psi_M)} C^\phi_M(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M).
\]

Since we have kept the $\Psi_i$'s completely arbitrary, (16) is identically zero if and only if $b_0^- |\phi\rangle = 0$, and we can conclude that

\[
b_0^- |\phi\rangle = 0 \quad \text{if and only if} \quad C^\phi_M(\Psi_2, \ldots, \Psi_M, \Psi_1) + (-1)^{M-1+\Psi_1(\Psi_2+\ldots+\Psi_M)} C^\phi_M(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M).
\]  

(18)

We are now ready to look at the equation of motion and gauge symmetry of the closed string sector, and to try to relate them to algebraic conditions on the open string vertices.

Let us act with the BRST charge $Q$ on the closed string field $\phi$. Note that we don’t put any restriction on the ghost numbers of the string fields, so that inserting $Q$ in the vertex doesn’t give zero by trivial ghost number counting. This can be represented on the disk by inserting an integration of the BRST current $j$ along a closed contour around the origin (the closed string puncture). We can further deform the contour by pushing it towards the boundary of the disk and then splitting it to $M$ contours around the open string punctures and $b(v_i)$ insertions. We can cross the $b(v_i)$ insertions, thereby producing a $T(v_i)$ by virtue of

\[
\{Q, b(v_i)\} = T(v_i),
\]  

(19)

As already mentioned, $T(v_i)$ generates translations in the moduli space; more precisely:

\[
\langle T(v_i) \ldots \rangle_D = -\frac{\partial}{\partial s_i} \langle \ldots \rangle_D.
\]  

(20)
All in all, using the definition \([3]\), we have

\[
C^{Q\phi}_M(\Psi_1, \ldots, \Psi_{M-1}, \Psi_M) = \int_{T_M} ds_1 \ldots ds_{M-1} (Q(0)b(v_1) \ldots b(v_{M-1}) \Psi_1^{h_1}(w_1) \ldots \Psi_{M-1}^{h_{M-1}}(w_{M-1}) \Psi_M^{h_M}(w_M))D
\]

\[
= (-1)^{1+\phi} \int_{T_M} ds_1 \ldots ds_{M-1} \langle \phi(0) Q b(v_1) \ldots b(v_{M-1}) \Psi_1^{h_1}(w_1) \ldots \Psi_{M-1}^{h_{M-1}}(w_{M-1}) \Psi_M^{h_M}(w_M) \rangle D
\]

\[
= (-1)^{1+\phi} \int_{T_M} ds_1 \ldots ds_{M-1} \sum_{i=1}^{M-1} (-1)^i \times
\]

\[
f \frac{\partial}{\partial s_i} (\phi(0)b(v_1) \ldots b(v_{M-1}) \Psi_1^{h_1}(w_1) \ldots \Psi_{M-1}^{h_{M-1}}(w_{M-1}) \Psi_M^{h_M}(w_M)) D
\]

\[
+ (-1)^{1+\phi} \sum_{i=1}^{M} (-1)^{M-1+\Psi_1+\ldots+\Psi_{i-1}} C^{\phi}_M(\Psi_1, \ldots, Q\Psi_i, \ldots, \Psi_M). \tag{21}
\]

In the second and third lines, we have replaced the closed BRST operator \(Q\), acting on \(\phi\), with the open BRST operator acting on the boundary. In doing so, we must account for a minus sign coming from the change of orientation of the contour of integration defining \(Q\), and also for a \((-1)^{\phi}\) coming from moving \(Q\) to the right of \(\phi\). In the fourth line, we are integrating total derivatives, and we will thus have contributions only from the boundary of moduli space \(\partial T_M\). Let us be a little more explicit about these last contributions. The domain of integration of \(s_k\), when all other \(s_i\) are fixed, is some interval \((s_{k-1}, s_{k+1})\) strictly contained in the interval \((s_k, s_k)\). When \(s_k\) is at a boundary of the interval, e.g. \(s_k = s_{k}^{\leq}\), then the punctured disk is identical to the Riemann surface given by the Feynman diagram obtained by gluing the Witten cubic vertex to the disk with \(M - 1\) punctures via a propagator of length zero. Let us look at this from the point of view of overlap patterns. In Figure 2, we show the decomposition of the disk into one semi-infinite cylinder and \(M\) semi-infinite strips of width \(\pi\), arising from a Jenkins-Strebel quadratic differential \([2]\). The condition for this Riemann surface to belong to the moduli space, is that all non-trivial closed curves are not shorter than \(2\pi\) and that all non-trivial open curves are not shorter than \(\pi\). In particular the segment \(AA'B'C'C\) must have length of at least \(\pi\); this implies that the length of the segment \(BB'\), which we call \(a_{k-1,k}\) must satisfy \(a_{k-1,k} \leq \frac{\pi}{2}\). When \(s_k = s_k^{<}\), we are at a boundary of the moduli space characterized by \(a_{k-1,k} = \frac{\pi}{2}\); this means that the strings \(\Psi_{k-1}\) and \(\Psi_k\) overlap exactly on their right and left halves respectively. This is precisely the star product. In other words, the correlator is unchanged if we remove the strips corresponding to \(\Psi_{k-1}\) and \(\Psi_k\), and replace them by a strip corresponding to \(\Psi_{k-1} \ast \Psi_k\).

\(^4\)Equivalently, note that \(bpz(Q(\phi)) = (-1)^{1+\phi}\langle \phi | Q \rangle\).

\(^5\)As explained in \([2]\), \(a_{k-1,k} = 0\) is not a boundary of the moduli space.
We thus end up with
\[
\int_{\mathcal{T}_M} d^{M-1} s \sum_{l=1}^{M-1} (-1)^l \frac{\partial}{\partial s_l} \langle \phi(0) b(v_1) \ldots b(v_l) \ldots b(v_{M-1}) \Psi_1 h_1(w_1) \ldots \Psi_l h_l(w_l) \ldots \Psi_M h_M(w_M) \rangle_D
\]
\[
= C^\phi_{M-1}(\Psi_1, \Psi_2, \ldots, \Psi_M) + \sum_{l=1}^{M-1} (-1)^l C^\phi_{M-1}(\Psi_1, \ldots, \Psi_l \ast \Psi_{l+1}, \ldots, \Psi_M) \tag{22}
\]
\[
= (-1)^{\Psi_1(\Psi_2 + \ldots + \Psi_M)} C^\phi_{M-1}(\Psi_2, \ldots, \Psi_M \ast \Psi_1) + \sum_{l=1}^{M-1} (-1)^l C^\phi_{M-1}(\Psi_1, \ldots, \Psi_l \ast \Psi_{l+1}, \ldots, \Psi_M).
\]

At first sight, it may seem that each product \(\Psi_l \ast \Psi_{l+1}\) arises twice with the same sign, once when \(s_l = s_{l+1}^>\) and once when \(s_{l+1} = s_l^<\). However, the first contribution comes with a \(b\) integration around \(\Psi_{l+1}\) and no \(b\) integration around \(\Psi_l\), while the situation is opposite for the second contribution. Therefore, they add up to one term with a \(b\) integration path around both \(\Psi_{l+1}\) and \(\Psi_l\), or in other words around \(\Psi_l \ast \Psi_{l+1}\). The right hand side of Eq. (22) is recognized as the Hochschild differential (see appendix A.4)
\[
\delta : \text{Hom}(A^{M-1}, \mathbb{R}) \to \text{Hom}(A^M, \mathbb{R}) \tag{23}
\]
for \(\mathbb{Z}_2\) graded algebras. Adding the last term in (21) we then end up with
\[
C^Q\phi_M(\Psi_1, \ldots, \Psi_M) = (-1)^{1+\phi} \left( (\delta C^\phi_{M-1})(\Psi_1, \ldots, \Psi_M) - (-1)^M (QC^\phi_M)(\Psi_1, \ldots, \Psi_M) \right) \tag{24}
\]
where $QC^\phi_M$ is defined by

$$(QC^\phi_M)(\Psi_1, \ldots, \Psi_M) \equiv \sum_{i=1}^{M} (-1)^{\Psi_1+\ldots+\Psi_{i-1}} C^\phi_M(\Psi_1, \ldots, \Psi_{i-1}, Q\Psi_i, \Psi_{i+1}, \ldots, \Psi_M).$$

(25)

If we impose the condition that the closed string field be on-shell, $Q\phi = 0$, we then conclude that the maps $(C^\phi_1, C^\phi_2, \ldots, C^\phi_M, \ldots)$ are closed with respect to $(\delta - (-1)^M Q)$. If the closed string field is off-shell, $Q\phi \neq 0$ then it is not hard to show that $C^Q_M(\Psi_1, \ldots, \Psi_M)$ is cyclic, so that the operator $(\delta - (-1)^M Q)$ takes cyclic elements into cyclic elements. Moreover, the above calculation also shows that if $\phi$ is pure gauge, i.e. $\phi = Q\Lambda$ for some $\Lambda$ satisfying $b_0^- \Lambda = 0$ and $L_0^- \Lambda = 0$, then

$$C^\phi_M(\Psi_1, \ldots, \Psi_M) = (-1)^\phi \left( (\delta C^A_{M-1})(\Psi_1, \ldots, \Psi_M) - (-1)^M (QC^A_M)(\Psi_1, \ldots, \Psi_M) \right).$$

(26)

In other words, it is exact with respect to $(\delta - (-1)^M Q)$. Furthermore, $(\delta - (-1)^M Q)^2 = 0$. We conclude that the on-shell closed string states are contained in the cyclic cohomology of $(\delta - (-1)^M Q)$.

4 Generic Deformations and the Cyclic Complex

We have seen above that on-shell closed string insertions correspond to elements in the cyclic cohomology, whereas infinitesimal closed string gauge transformations correspond to exact elements in the cyclic complex. In this section we want to argue that this correspondence is in fact an isomorphism. To see this we consider below generic deformations of OSFT (or, equivalently, its corresponding DGA) that do not correspond to a closed string insertion in the bulk.

4.1 Deformation of the Open String Background

In this subsection we consider the first class of such deformations corresponding to an insertion of an open string background on the boundary of the disc with the open string being on-shell.

Let $\Psi_0$ be an infinitesimal marginal open string deformation. We express it as $\Psi_0 = O(\epsilon)$. And $\Psi_0$ is a solution of the equations of motion, therefore $Q\Psi_0 = 0 + O(\epsilon^2)$. Writing $\Psi = \Psi_0 + \tilde{\Psi}$, the action becomes

$$S' = \frac{1}{2} (\tilde{\Psi}, Q' \tilde{\Psi}) + \frac{1}{3} (\tilde{\Psi}, \tilde{\Psi}, \tilde{\Psi}) + O(\epsilon^2),$$

(27)

\(^\text{9}\text{See appendix [X] for a definition.}\)

11
where the new BRST operator $Q'$ is given by

$$Q'\tilde{\Psi} = Q\tilde{\Psi} + \Psi_0 \ast \tilde{\Psi} + (-1)^{1+\tilde{\Psi}} \tilde{\Psi} \ast \Psi_0. \tag{28}$$

And from now on we will write $\Psi$ instead of $\tilde{\Psi}$. Next, we rewrite the action $S'$ as $S$ plus an infinitesimal deformation

$$S' = S + C_2(\Psi, \Psi) + O(\epsilon^2), \tag{29}$$

where $C_2$ is defined by

$$C_2(\Psi_1, \Psi_2) = \frac{1}{2} \left( \langle \Psi_0, \Psi_1, \Psi_2 \rangle + (-1)^{1+\Psi_1}\Psi_2 \langle \Psi_0, \Psi_2, \Psi_1 \rangle \right). \tag{30}$$

This definition makes sense because when $\Psi_1 = \Psi_2 = \Psi$, with $|\Psi| = 1$, we indeed have $\Psi_0 = \Psi$. To verify this fact, one only needs the following elementary identities

$$\langle A, B, C \rangle = (−1)^{A(B+C)} \langle B, C, A \rangle \tag{31}$$

$$\langle A, B, C \rangle = \langle A \ast B, C \rangle = \langle A, B \ast C \rangle. \tag{32}$$

And we mention here one more identity that will be needed shortly

$$\langle A, B \rangle = (−1)^{AB} \langle B, A \rangle. \tag{33}$$

Let us now look at the algebraic properties of $C_2$. From the definition $\tag{30}$, we immediately see that $C_2$ satisfies the cyclicity condition

$$C_2(\Psi_2, \Psi_1) = (−1)^{1+\Psi_1}\Psi_2 C_2(\Psi_1, \Psi_2). \tag{34}$$

Furthermore, it is easy to see that the collection of maps formed by $C_2$ alone, namely $(0, C_2, 0, 0, \ldots)$, is exact. Indeed, let us define

$$D_1(\Psi) = -\frac{1}{2} \langle \Psi_0, \Psi \rangle. \tag{35}$$

We then have

$$(QD_1)(\Psi) = -\frac{1}{2} \langle \Psi_0, Q\Psi \rangle = -\frac{1}{2} \langle Q\Psi_0, \Psi \rangle = 0 + O(\epsilon^2). \tag{36}$$

And, straight from the definition of $\delta$, we find that

$$C_2 = \delta D_1. \tag{37}$$

Thus we conclude that, to order $\epsilon^1$, $(0, C_2, 0, 0, \ldots)$ is exact, namely

$$(0, C_2, 0, \ldots) = (\delta - (−1)^N Q) (D_1, 0, 0, \ldots). \tag{38}$$
We are thus left with the class of deformations of OSFT obtained by acting with some operator $O$ on the open string fields. We will treat this case in the next subsection.

4.2 Generic Deformations of the 3-Vertex

Let us assume that $C_n$ is non-vanishing for some $n > 3$. Then $C_n \neq 0$ for some $n \leq 3$ as well for, if $C_n = 0$ for $n \leq 3$ then the moduli space of perturbative OSFT is covered exactly once if and only if $C_n = 0, n > 3$. This is just the statement of consistency of Witten’s OSFT.

Let us now focus on possible deformations of the 3-vertex obtained by acting with some operator $O$ on the open string fields. Cyclicity then requires that we sum over the three graphs on the left hand side in Fig. 3. A generic deformation $C_3$ in the cyclic cohomology is then given by

$$C_3(\Psi_1, \Psi_2, \Psi_3) = \langle O\Psi_1, \Psi_2*\Psi_3 \rangle + (-1)^{O}\langle \Psi_1, O\Psi_2*\Psi_3 \rangle + (-1)^{O}(\Psi_1*\Psi_2)\langle \Psi_1, \Psi_2*O\Psi_3 \rangle$$  \hspace{1cm} (39)

subject to the equation

$$\delta C_2 + QC_3 = 0.$$  \hspace{1cm} (40)

Without restricting the generality we can assume that $bpz(O) = \pm O$. It is then not hard to see that for $B = [O, Q] \neq 0$ Eq. (40) has a solution only if $O$ is BPZ-odd and then the solution is given by

$$C_2(\Psi_1, \Psi_2) = -\langle B\Psi_1, \Psi_2 \rangle$$  \hspace{1cm} (41)

Note that the BPZ-parity of $O$ is the same as that of $B$. However, for $O$ BPZ-odd, $C_2$ and $C_3$ are exact, $C_2 = -(QD_2)$ and $C_3 = \delta D_2$ with $D_2 = \langle O\Psi_1, \Psi_2 \rangle$. The remaining possibility is then $B = 0$ and $bpz(O) = O$. Then $QC_3 = 0$ and we are left with the condition

$$\delta C_3 + QC_4 = 0$$  \hspace{1cm} (42)

with

$$\delta C_3 = -2\langle O(\Psi_1*\Psi_2), \Psi_3*\Psi_4 \rangle + 2(-1)^{O}\langle \Psi_4*\Psi_1, O(\Psi_2*\Psi_3) \rangle$$  \hspace{1cm} (43)
If \( \mathcal{O} \) is \( Q \)-exact then \( C_3 \) is again trivial in the cyclic cohomology. On the other hand if \( \mathcal{O} \) is in the cohomology of \( Q \), we claim that (42) can have a solution only if \( \mathcal{O} \) is a conformal invariant so that \( \mathcal{O} \) can be pulled in the bulk (see Fig. 4). Indeed since \( \mathcal{O} \) is not \( Q \)-exact, the only way the differential \( Q \) acting on \( C_4 \) can reproduce (43) is as a derivative on its moduli space. However, while the boundary of this moduli space may contain terms of the form \( \Psi^* \mathcal{O} \Psi \), it does not contain terms of the form appearing in (43) unless \( \mathcal{O} \) is a derivative of \( \Psi^* \). Since \( \mathcal{O} \) is BPZ-even it cannot be a derivative. On the other hand if \( \mathcal{O} \) can be pulled in the bulk as in Fig. 4 then \( [\mathcal{O}, Q] = 0 \) is equivalent to the closed string cohomology condition.

To summarize, for bpz(\( \mathcal{O} \)) = \( \mathcal{O} \) non-trivial elements in the cyclic cohomology \( HC^* \) can exist only if the closed string cohomology is non-trivial. More precisely, if we consider \( C_n \in CC^n \) then the ghost number of \( C_n \) is given by

\[
|C_n| = \begin{cases} 
|\phi| - 2 - n, & \text{disk has a closed string puncture} \\
|\mathcal{O}| - n, & \text{disk has no puncture (Fig. 3)}
\end{cases}
\]  

(44)

The first case corresponds to an insertion of an asymptotic on-shell closed string state, whereas the second corresponds to an insertion of the form \( \mathcal{O}(0)_c \). In particular, \( C_n \in HC^n \) with ghost number \(-n \) correspond either to a closed string asymptotic state with ghost number 2 or a closed string state of the form \( \mathcal{O}(0)_c \) with ghost number 0 and vanishing momentum. Since the semi-relative closed string cohomology at ghost number zero and vanishing mass contains only the vacuum (or equivalently the identity operator \( \mathcal{O} = I \)) we then conclude that all elements in the cyclic cohomology \( HC^n \) at ghost number \(-n \) must correspond to a physical closed string state. This is the main result of this paper.

For completeness we should mention that there are, in addition, deformations of OSFT by open string operators with \( C_3 = 0 \) but \( C_2 \neq 0 \). However, the corresponding connected diagrams are equivalent to those obtained by deformations of the 3-vertex so that we need not consider them separately.

### 4.3 Strips

As a concrete example of the generic deformations discussed in the last subsection we now describe a geometric deformation of the 3-vertex. In Zwiebach’s open-closed string field...
theory the vertices in the action include strips of length $\pi$ for the external open strings even in the absence of closed strings. This deformation should not be physical since it is merely a reorganization of the moduli space of the same theory. This has been shown in [7] using the BV formalism and also in [8] as a particular case of $A_\infty$-quasi-isomorphisms. To see this in our context, let us consider an infinitesimal version of Zwiebach’s open-closed theory in the absence of closed string insertions but with strips of length $\epsilon > 0$. We define $C_3^\epsilon$ pictorially in Figure 5 as the deviation from Witten’s vertex. We can easily translate this picture to the algebraic expression

$$C_3^\epsilon(\Psi_1, \Psi_2, \Psi_3) = \langle e^{-\epsilon L_0} \Psi_1, e^{-\epsilon L_0} \Psi_2, e^{-\epsilon L_0} \Psi_3 \rangle - \langle \Psi_1, \Psi_2, \Psi_3 \rangle = -\epsilon \left( \langle L_0 \Psi_1, \Psi_2, \Psi_3 \rangle + \langle \Psi_1, L_0 \Psi_2, \Psi_3 \rangle + \langle \Psi_1, \Psi_2, L_0 \Psi_3 \rangle \right) + O(\epsilon^2). \quad (45)$$

A four-vertex $C_4^\epsilon$ is now needed to produce the part of moduli space of disks with four punctures on the boundary that is missed by the Feynman diagrams constructed with $C_3^\epsilon$. The missing surfaces are readily identified as the ones whose internal propagator has length smaller than $2\epsilon$. We can thus define $C_4^\epsilon$ pictorially as in Figure 6. There are two ways to place the propagator, hence the two contributions. Note that the integration limits are reversed between the two contributions. This must be so in order to parameterize smoothly the moduli space; we start with a ‘vertical’ propagator of length $2\epsilon$, shrink it until is has length zero, then we grow a ‘horizontal’ propagator until it has length $2\epsilon$. We can again easily write the algebraic expressions of these diagrams after we notice that the short propagator entering the
definition of $C_4^\epsilon$, has a very simple expression:

$$b_0 \int_0^{2\epsilon} e^{-tL_0} dt = 2\epsilon b_0 + O(\epsilon^2).$$

(46)

We thus have

$$C_4^\epsilon = 2\epsilon (-1)^{\Psi_3+\Psi_4} \langle \Psi_1 * \Psi_2, b_0 (\Psi_3 * \Psi_4) \rangle$$

$$+ 2\epsilon (-1)^{\Psi_4+\Psi_1+\Psi_2+\Psi_3+\Psi_4+1} \langle \Psi_2 * \Psi_3, b_0 (\Psi_4 * \Psi_1) \rangle.$$  

(47)

At $O(\epsilon)$ only $C_3^\epsilon$ and $C_4^\epsilon$ give non-vanishing contribution since the volume of the moduli space of $C_3^M$ is $O(\epsilon^{M-3})$. This can be seen by considering tree-level diagrams involving $M \geq 3$ vertices. Let us now look at the cohomology class of $(0,0,C_3^\epsilon,C_4^\epsilon,0,\ldots)$. First, it is straightforward to check that $C_3^\epsilon$ and $C_4^\epsilon$ are cyclic. Next, $(0,0,C_3^\epsilon,C_4^\epsilon,0,\ldots)$ should be closed; let us check this explicitly\footnote{It is not strictly necessary to check closedness since we will show later that this set of maps is exact, hence closed. Nevertheless, it is a good consistency check to go through this illustrative calculation.} Since $Q$ commutes with $L_0$, we find almost immediately that

$$0 = (QC_3^\epsilon)(\Psi_1, \Psi_2, \Psi_3).$$

(48)

The calculations of $(\delta C_3^\epsilon)$, $(QC_4^\epsilon)$ and $(\delta C_4^\epsilon)$ are not difficult, but it might be useful to recall one more elementary identity that we need:

$$\langle OA, B \rangle = (-1)^{O A} \langle A, bpz(O)B \rangle.$$  

(49)

It will be useful when the operator $O$ is either $Q$, $L_0$ or $b_0$, and we recall that $bpz(Q) = -Q$, $bpz(L_0) = L_0$ and $bpz(b_0) = b_0$. It is then straightforward to show that

$$(\delta C_3^\epsilon)(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = 2\epsilon \left( \langle \Psi_1 * \Psi_2, L_0(\Psi_3 * \Psi_4) \rangle + \langle \Psi_2 * \Psi_3, L_0(\Psi_4 * \Psi_1) \rangle \right).$$

(50)

And the calculation of $QC_4^\epsilon$ is not much more difficult once we realize that we must anticommutate $Q$ past $b_0$ in order to cancel out pairs of terms. The only remaining terms are then the
ones containing \( \{Q, b_0\} = L_0 \). Explicitly, we have
\[
(QC_4^e)(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = 2\epsilon \left( (-1)^{\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \epsilon} \langle \Psi_1 * \Psi_2, L_0(\Psi_3 * \Psi_4) \rangle \\
+ (-1)^{\Psi_1 + \Psi_2 + \Psi_4 + \psi_1(\Psi_2 + \Psi_3 + \Psi_4)} \langle \Psi_2 * \Psi_3, L_0(\Psi_4 * \Psi_1) \rangle \right). 
\]
(51)

Now, in order to compare (50) and (51), we must have recourse to the fact that, by ghost-number counting, they are nonzero only when \(|\Psi_1| + |\Psi_2| + |\Psi_3| + |\Psi_4| = 3\). We can then conclude that
\[
\delta C_3^e - QC_4^e = 0.
\]
(52)

At last, it is also straightforward to check that
\[
\delta C_4^e = 0.
\]
(53)

And we can combine (48), (52) and (53) into the closedness relation
\[
(\delta - (-1)^NQ)(0, 0, C_3^e, C_4^e, 0, \ldots) = 0.
\]
(54)

Let us now show that \((0, 0, C_3^e, C_4^e, 0, \ldots)\) is an exact element of the cyclic cohomology. To this end we will try to find a \(D_3^e\) such that \(QD_3^e = C_3^e\) and \(\delta D_3^e = C_4^e\). It turns out that we will be lucky, and we will need neither a \(D_1^e\) nor a \(D_2^e\). It remains to write down an expression for \(D_3^e\); but if we focus on the requirement that \(QD_3^e = C_3^e\) and on the expression (45) for \(C_3^e\), we see that a natural candidate is simply the expression of \(C_3^e\) with all \(L_0\)’s replaced by \(b_0\)’s. This is almost right, we just need to tweak some signs. We claim that the expression for \(D_3^e\) is
\[
D_3^e(\Psi_1, \Psi_2, \Psi_3) \equiv -\epsilon \left( \langle b_0\Psi_1, \Psi_2, \Psi_3 \rangle + (-1)^{\Psi_1} \langle \Psi_1, b_0\Psi_2, \Psi_3 \rangle + (-1)^{\Psi_1 + \Psi_2} \langle \Psi_1, \Psi_2, b_0\Psi_3 \rangle \right).
\]
(55)

We can show that
\[
QD_3^e = C_3^e.
\]
(56)

We sketch the straightforward proof by acting with \(Q\) on the first term in the parentheses. We have
\[
\langle b_0Q\Psi_1, \Psi_2, \Psi_3 \rangle + (-1)^{\Psi_1} \langle b_0\Psi_1, Q\Psi_2, \Psi_3 \rangle + (-1)^{\Psi_1 + \Psi_2} \langle b_0\Psi_1, \Psi_2, Q\Psi_3 \rangle = \\
= \langle L_0\Psi_1, \Psi_2, \Psi_3 \rangle - (-1)^{\Psi_1} \langle b_0\Psi_1, Q(\Psi_2 \star \Psi_3) \rangle \\
+ (-1)^{\Psi_1} \langle b_0\Psi_1, (Q\Psi_2) \star \Psi_3 \rangle + (-1)^{\Psi_1 + \Psi_2} \langle b_0\Psi_1, \Psi_2 \star (Q\Psi_3) \rangle \\
= \langle L_0\Psi_1, \Psi_2, \Psi_3 \rangle,
\]
and the cancellation of terms containing \(Q\) works in the same way for the other two terms in Eq. (55). This establishes (56). Furthermore, we claim that
\[
(\delta D_3^e)(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = C_3^e(\Psi_1, \Psi_2, \Psi_3, \Psi_4).
\]
(57)
The proof is mechanical. In short, there are two kinds of terms in $\delta D^3$: Terms containing $b_0\Psi_i$ and terms containing $b_0(\Psi_i \ast \Psi_j)$. All the terms of the first kind cancel out by pairs. And the four terms of the second kind can be grouped into two terms (with a factor 2 in front) by virtue of Eq. (49), for instance

$$
\langle b_0(\Psi_1 \ast \Psi_2), \Psi_3, \Psi_4 \rangle = \langle b_0(\Psi_1 \ast \Psi_2), \Psi_3 \ast \Psi_4 \rangle = (-1)^{\Psi_1 + \Psi_2} \langle \Psi_1 \ast \Psi_2, b_0(\Psi_3 \ast \Psi_4) \rangle.
$$

At last we use the fact that everything trivially vanishes unless $|\Psi_1| + |\Psi_2| + |\Psi_3| + |\Psi_4| = 4$, and we get precisely the same expression as in Eq. (47). Therefore Eq. (57) is verified. Grouping (56) and (57), we can write

$$
(0, 0, C_3^e, C_4^e, 0, \ldots) = (\delta - (-1)^N Q) (0, 0, D_3^e, 0, 0, \ldots),
$$

which means that the geometric deformations do not contribute to the cyclic cohomology.

## 5 Background Independence

Let us expand the theory around a classical solution of the equations of motion, $\Psi \rightarrow \Psi_0 + \Psi$ with $Q\Psi_0 + \Psi_0 * \Psi_0 = 0$. The Witten action is then the same except for the appearance of a cosmological constant (irrelevant for us) and for the fact that $Q$ is replaced by a new BRST operator $Q'$ given by

$$
Q'\Psi = Q\Psi + \Psi_0 * \Psi + (-1)^{1+\Psi} \Psi * \Psi_0.
$$

We claim that the cyclic cohomology of $\langle \delta - (-1)^N Q' \rangle$ is isomorphic to the cyclic cohomology of $\langle \delta - (-1)^N Q \rangle$. Our strategy will be to find a linear bijection $h$

$$
h : \prod_{n=1}^{\infty} \text{Hom}(A^n, \mathbb{R}) \rightarrow \prod_{n=1}^{\infty} \text{Hom}(A^n, \mathbb{R})
$$

such that

$$
(\delta - (-1)^N Q') h(C_1, C_2, \ldots) = 0 \text{ iff } (\delta - (-1)^N Q) (C_1, C_2, \ldots) = 0,
$$

and such that $h(C_1, C_2, \ldots)$ is $\langle (\delta - (-1)^N Q') \rangle$-exact if and only if $(C_1, C_2, \ldots)$ is $\langle (\delta - (-1)^N Q) \rangle$-exact, where $C_n \in \text{Hom}(A^n, \mathbb{C})$. If we denote $h(C_1, C_2, \ldots)$ by $(D_1, D_2, \ldots)$, our candidate for $h$ is

$$
D_n(\Psi_1, \ldots, \Psi_n) = \sum_{k=0}^{\infty} (-1)^{(n+1)k + \frac{1}{2}k(k+1)} \sum_{f \in F_k^n} (-1)^{\sum_{i=1}^{n} (\Psi_i - 1)(k + f(i) - i)} C_{n+k}^f(\Psi_1, \ldots, \Psi_n),
$$

(62)
where $\mathcal{F}_k^n$ is the set of strictly increasing functions from $\{1, \ldots, n\}$ into $\{1, \ldots, n + k\}$ such that $f(n) = n + k$; and $C^f_{n+k}(\Psi_1, \ldots, \Psi_n)$ is an element of $\text{Hom}(A^n, \mathbb{C})$ defined by

$$C^f_{n+k}(\Psi_1, \ldots, \Psi_n) = C_{n+k}(\Psi_0, \ldots, \Psi_0, \Psi_1, \Psi_0, \ldots, \Psi_0, \Psi_2, \ldots, \Psi_n),$$

(63)

where the $\Psi_i$’s are inserted at the positions $f(i)$, and the remaining $k$ slots are filled with $\Psi_0$’s. Note that we may have one or more (or none) $\Psi_0$ on the left of $\Psi_1$, but there cannot be any on the right of $\Psi_n$. In this way we avoid over-counting since a $\Psi_0$ on the right of $\Psi_n$ can be moved to the left of $\Psi_1$ by cyclicity.

Using the fact that $D_n(\Psi_1, \ldots, \Psi_n)$ vanishes unless $(-1)^{\Psi_1+\ldots+\Psi_n} = (-1)^n$ we readily check that $D_n$ is cyclic, namely

$$D_n(\Psi_2, \Psi_3, \ldots, \Psi_n, \Psi_1) = (-1)^{\Psi_1(\Psi_2+\ldots+\Psi_n)+n+1}D_n(\Psi_1, \Psi_2, \ldots, \Psi_n).$$

(64)

It is also straightforward (although a bit lengthy) to check that

$$\delta D_{n-1} - (-1)^n Q'D_n = \sum_{k=0}^{\infty}(-1)^{(n+1)k+\frac{1}{2}k(k+1)} \sum_{f \in \mathcal{F}_k^n} (-1)^{\sum_{i=1}^{n} (\Psi_i-1)(k+f(i)-i)} \left(\delta C_{n+k-1} + (-1)^{n+k+1}QC_{n+k}\right)^f.$$

(65)

But we recognize that the last line is precisely our function $h$ defined in Eq. (62), applied on $(\delta - (-1)^N Q)(C_1, C_2, \ldots)$. We have thus found that

$$(\delta - (-1)^N Q') h = h (\delta - (-1)^N Q).$$

(66)

This equation tells us immediately that $h$ maps $(\delta - (-1)^N Q)$-closed elements to $(\delta - (-1)^N Q')$-closed elements. It also tells us that $h$ maps $(\delta - (-1)^N Q)$-exact elements to $(\delta - (-1)^N Q')$-exact elements. In order to conclude that the two cyclic cohomologies are isomorphic, we now just need to show that $h$ is invertible, so that we could write

$$(\delta - (-1)^N Q) h^{-1} = h^{-1} (\delta - (-1)^N Q').$$

(67)

Indeed, (66) and (67) show that $h$ is a linear bijection from $\text{Ker}(\delta - (-1)^N Q)$ onto $\text{Ker}(\delta - (-1)^N Q')$ and is also a linear bijection from $\text{Im}(\delta - (-1)^N Q)$ onto $\text{Im}(\delta - (-1)^N Q')$.

Let us now show that $h$ is invertible. First, we observe that

$$Q'\Psi_0 = Q\Psi_0 + \Psi_0 * \Psi_0 + \Psi_0 * \Psi_0 = \Psi_0 * \Psi_0,$$

(68)

in other words $(-\Psi_0)$ is a solution of the equations of motion with respect to $Q'$. We note also that

$$Q\Psi = Q'\Psi + (-\Psi_0) * \Psi + (-1)^{1+\Psi \Psi} (-\Psi_0).$$

(69)
Since the only facts we used in order to construct $h$ were that $\Psi_0$ obeys the equations of motion with respect to $Q$ and that $Q'\Psi = Q\Psi + \Psi_0 * \Psi + (1)_{1+\Psi}^1 \Psi_0 * \Psi_0$, we see that the inverse map $h^{-1}$ must be given by Eq. \((62)\) with $\Psi_0$ replaced with $-\Psi_0$. Since $\Psi_0$ appears $k$ times in the sum on the right-hand side of \((62)\), this amounts to simply introducing a factor of $(-1)^k$. Explicitly, we have therefore

$$C_n(\Psi_1, \ldots, \Psi_n) = \sum_{k=0}^{\infty} (-1)^{nk+\frac{1}{2}k(k+1)} \sum_{f \in F_k^n} (-1)^{\sum_{i=1}^{n}(\Psi_i^{-1})(k+f(i)-i)} D^f_{n+k}(\Psi_1, \ldots, \Psi_n). \quad (70)$$

This concludes our proof that the cyclic cohomology is background independent.

6 Conclusions

In this paper we showed that the closed string cohomology is isomorphic to the cyclic cohomology of cubic open string field theory. The latter is defined solely in terms of the structure of open string theory, i.e. the open string BRST operator and the product on the algebra of open string fields. Part of this result could have been anticipated since a generic field theory in flat space, formulated in a covariant form admits an essentially unique linearized coupling to gravity in terms of the Noether procedure. What is special about string theory, however, is that on top of that, open string theory knows about the on-shell condition of the closed string states in terms of cyclic cohomology.

The natural question that arises then is whether there is more information to be uncovered by considering second order deformations in the closed string deformation. We think the answer is positive since a general property of the cohomology ring $HC(A)$ is that it possesses a natural Lie super-algebra structure [4], more precisely an $L_\infty$-structure just like closed string field theory. So one can hope to learn more on closed strings by considering consistent deformations of OSFT. On the other hand there are certain obstructions in extending infinitesimal deformations to second order. This is, however, expected since not every marginal closed string deformation is exactly marginal.

Are there any applications of our results to open string field theory? For one thing the reformulation of closed string cohomology and in particular the open closed vertices in terms of cyclic cohomology of OSFT together with the background independence of the latter provides a formal definition of linearized open-closed string theory in any open string background. For instance, in Schnabl’s vacuum solution where the open string cohomology is empty but the closed string cohomology is not we conclude that the closed string spectrum is still encoded in OSFT.

Finally we would like to mention another possible avenue for further investigation. Within boundary string field theory it has been argued [12] that certain deformations of the closed string background are equivalent to ”collective excitations” of the open string (i.e. insertions.
of non-local boundary interactions on the boundary of the world sheet). At first sight this
seems to be in contradiction with our present result that there is an isomorphism between non-
trivial deformations of OSFT and closed string insertions in the bulk of the disk. However,
we should note that the argument given in section 4 is based on insertion of generic local
operators on the boundary. It would be interesting to relax the latter condition.

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A Geometric Interpretation

In order to gain a geometric intuition of the $A_\infty$-algebra arising in our context and Hochschild
cohomology in particular it may be helpful to draw an analogy with algebraic topology. The
relation of the closed string cohomology to the de Rham cohomology in the loop space will
be discussed in subsection A.3.

A.1 $A_\infty$-algebras

Let $M$ be a smooth manifold and $(A, d) = (\Omega^*(M), d)$ its de Rham complex. Then $(A, d, \wedge)$
defines a differential graded algebra with the operations

\begin{align*}
    d & : A \to A \\
    x & \mapsto dx \\
    \wedge & : A \otimes A \to A \\
    (x, y) & \mapsto x \wedge y
\end{align*}

(71)

The exterior derivative $d$ with $d^2 = 0$ has degree $|d| = 1$ while $\wedge$ has degree $|\wedge| = 0$. An
$A_\infty$-algebra is obtained as a certain deformation of $(A, d, \wedge)$ including higher maps $A \otimes^n \to A$.
In order to do this it is convenient to have a uniform grading for these maps. This can be
achieved as follows. We say the $x \in \Omega^p$ has degree $p$. Then $dx \in \Omega^{p+1}$ has degree $p + 1$ and
$x \wedge y \in \Omega^{p+q}$ has degree $p + q$. We then define the grading of $x$ as $\text{grad}(x) = |x| - 1$, so that
$\text{grad}(d) = \text{grad}(\wedge) = 1$. This is called a shift (or a suspension) and we denote the shifted
vector space by $A[1]$. In particular,

$$b_1(x) = dx : A[1] \to A[1]$$

$$b_2(x, y) = (-1)^{|x|+1} x \wedge y : A[1] \otimes A[1] \to A[1]$$

(72)

have both grading 1. $A_\infty$-deformations of $(A[1], b_1, b_2)$ are obtained as follows (see e.g. [10]):

We define the Bar complex

$$BA[1] = \bigoplus_{k \geq 1} A[1] \otimes k = \bigoplus_k B_k A[1]$$

(73)

and continuations $\hat{b}_i : BA[1] \to BA[1]$, $i = 1, 2$ as coderivatives with components

$$ (\hat{b}_i)_{k, k-i} = \sum_{t=0}^{k-i} 1^{\otimes k-i-t} \otimes b_i \otimes 1^{\otimes t} $$

(74)

It is not hard to see that $\hat{b} = \hat{b}_1 + \hat{b}_2$ squares to zero and hence $(BA[1], \hat{b})$ is a cochain complex. The $A_\infty$-generalization of $(BA[1], \hat{b})$ is obtained by including higher maps $b_s : A[1]^{\otimes s} \to A[1]$, $s \geq 1$ so that the coderivative with components

$$ (\hat{b})_{n,u} = \sum_{r+s+t=n \atop r+i+t=u} 1^{\otimes r} \otimes b_s \otimes 1^{\otimes t} $$

(75)

squares to zero. Now, if $\hat{b}$ is a coderivative of grade 1 on $BA[1]$ with components $b_n : A[1]^{\otimes n} \to A[1]$, then its square is a coderivative of grade 2 with components

$$ (\hat{b}^2)_n = \sum_{i+j=n+1 \atop k=0}^{n-j} b_i \cdot (1^{\otimes k} \otimes b_j \otimes 1^{\otimes n-k-j}) $$

(76)

Imposing that $\hat{b}^2 = 0$, we obtain a characterization of all differentials compatible with the $A_\infty$-structure on $BA[1]$. For example,

$$ b_1 \cdot b_2 + b_2 \cdot (b_1 \otimes 1 + 1 \otimes b_1) = 0, \quad n = 2 $$

$$ b_1 \cdot b_3 + b_3 \cdot (b_1 \otimes 1 \otimes 1) + b_3 \cdot (1 \otimes b_1 \otimes 1) + b_3 \cdot (1 \otimes 1 \otimes b_1) + b_2 \cdot (b_2 \otimes 1 + 1 \otimes b_2) = 0, \quad n = 3 $$

(77)

We can translate these relations to the maps $m_k : A^{\otimes k} \to A$ without the shift. For this we recall that if $A$ is a graded vector space over $\mathbb{C}$ then its suspension, $A[1]$ is the graded $\mathbb{C}$-vector space with a grading shifted by 1. We define a shift operator $s$ as $A[1] = sA$, or

$$(sA)_i = A_{i-1}. $$

(78)
Now, if $b_k: (sA)^\otimes k \to sA$ is a multilinear map of grading 1 one defines $m_k: A^\otimes k \to A$ of degree $|m_k| = 2 - k$ by

$$b_k = s \cdot m_k \cdot (s^{-1})^\otimes k$$

or equivalently

$$b_k(sa_1, \cdots, sa_k) = (-1)^{(k-1)|a_1|+(k-2)|a_2|+\cdots+2|a_{k-2}|+|a_{k-1}|+k(k-1)/2}sm_k(a_1, \cdots, a_k)$$

We display the first few of the $m_k$’s explicitly

$$b_1(sa) = sm_1(a)$$

$$b_2(sa_1, sa_2) = (-1)^{|a_1|+1}sm_2(a_1, a_2)$$

$$b_3(sa_1, sa_2, sa_3) = (-1)^{2|a_1|+|a_2|+3}sm_3(a_1, a_2, a_3)$$

which in turns leads to

$$m_1(m_2(a_1, a_2)) - m_2(m_1(a_1), a_2) - (−1)^{|a_1|}m_2(a_1, m_1(a_2)) = 0,$$ (82)

and

$$m_1(m_3(a_1, a_2, a_3)) + m_3(m_1(a_1), a_2, a_3) + (−1)^{|a_1|}m_3(a_1, m_1(a_2), a_3)$$

$$+(-1)^{|a_1|+|a_2|}m_3(a_1, a_2, m_1(a_3)) - m_2(m_2(a_1, a_2), a_3) + m_2(a_1, m_2(a_2, a_3)) = 0,$$ (83)

and so on. In particular, $m_2$ is associative only up to the higher homotopy map $m_3$.

The infinitesimal $A_\infty$-deformations of the DGA of OSFT relevant to our work are described in the next subsection.

A.2 Hochschild Cohomology

Let us now consider an infinitesimal $A_\infty$-deformation of the differential graded algebra $(BA[1], \hat{b})$. For that we write

$$b_1 = \hat{b}_1 + \epsilon F_1$$

$$b_2 = \hat{b}_2 + \epsilon F_2$$

$$b_k = \epsilon F_k \quad k > 2$$

The $A_\infty$-condition $\hat{b}^2 = 0$ then implies that

$$d_H(\hat{F}) = \{\hat{b}, \hat{F}\} = 0$$

Since the signs depend only on the grading modulo $\mathbb{Z}_2$, one could simplify the notation by writing $s^{-1} = s$.\footnote{Since the signs depend only on the grading modulo $\mathbb{Z}_2$, one could simplify the notation by writing $s^{-1} = s$.}
where $d_H(\hat{F})$ is the Hochschild co-boundary operator and $d_H^2(\hat{F}) \equiv 0$ follows from the Jacobi identity for $\{,\}$. We denote the corresponding complex by $CH^*(A[1], d_H)$.

We can again translate these formulas to deformations of $m_k : A^\otimes k \to A$ without the shift, that is

$$
m_1 = d + \epsilon f_1
$$

$$
m_2 = \wedge - \epsilon f_2
$$

$$
m_k = (-1)^{k+1} \epsilon f_k, \quad k > 2
$$

we find that at order $\epsilon$

$$
((-1)^{|f|} d - \delta) \hat{f} = 0
$$

where

$$
\delta f_k(x_1, \ldots, x_{k+1}) = (-1)^{|x_1|} f_k |x_1 \wedge f_k(x_2, \ldots, x_{k+1})
$$

$$
+ \sum_{i=0}^{k} (-1)^i f_k(x_1, \ldots, x_i \wedge x_{i+1} \ldots, x_{k+1})
$$

$$
+ (-1)^{n+1} f_k(x_1, \ldots, x_k) \wedge x_{k+1}.
$$

### A.3 Closed Loops

So far we have discussed the $A_\infty$-deformations of the cochain complex $(BA[1], \hat{b})$ which is the geometric analog of the differential graded algebra of open string field theory. Next we would like to draw an analogy with closed string insertions. To illustrate this relations we will consider the chain complex $(C^*(M), \partial)$ together with the cap product $\cap$, although this does not strictly speaking provide an algebra. Ignoring this fact for the time being we then consider the product

$$
C_*(\Lambda_0 M) \times C_*(M)^n
$$

where $\Lambda_0 M$ is the space of loops in $M$ with fixed base point. We can then define a map

$$
\varphi : C_*(\Lambda_0 M) \times C_*(M)^n \to C_*(M)
$$

as follows. Let $g_i$ and $h$ be homeomorphisms from a standard simplex $P$ to $M$ and $\Lambda_0 M$ respectively,

$$
g_i : (\sigma \in P) \quad \Rightarrow \quad g_i(\sigma) \in M
$$

$$
h : (\tau \in P, t \in [0,1]) \quad \Rightarrow \quad h(\tau, t) \in M
$$
For fixed $t_1 < t_2 < \cdots < t_n$ the intersection

$$
\begin{align*}
g_1(\sigma_1) &= h(\tau, t_1) \\
g_2(\sigma_2) &= h(\tau, t_2) \\
\vdots \\
g_n(\sigma_n) &= h(\tau, t_n)
\end{align*}
$$

(93)

defines an element in $C_*(M)$ that can be represented in terms of a disc diagram as in Fig. 7 and which generalizes the (transverse) intersection in $C_*(M)$. To complete this construction we should integrate over $0 \leq t_1 < t_2 < \cdots < t_n < 1$. Ignoring the issues about transversality the corresponding map then induces a chain map

$$
\varphi : C_*(\Lambda_0 M) \to CH_*(C_*(M, \partial, \cap), d_H) ; \quad d_H \varphi(c) = \varphi(\partial c) , \quad c \in C_*(\Lambda_0 M)
$$

(94)

where $d_H$ is the coboundary operator on the space $CH_*(C_*(M), \partial, \cap)$ of deformations of $(C_*(M), \partial, \cap)$.

Returning again to cochains on $M$ one can make a precise statement: The method of Chen’s iterated integrals over $0 \leq t_1 < t_2 < \cdots < t_n < 1$ (see e.g. [11]) provides an isomorphism between the homology group of $\Lambda_0 M$ and the Hochschild cohomology of $(\Omega^*(M), d, \wedge)$

$$
I_* : H_*(\Lambda_0 M) \xrightarrow{\cong} HH^*(\Omega^*(M), d, \wedge)
$$

(95)

This is the analog, in algebraic topology, of the isomorphism derived in section 4.2.

### A.4 Cyclic Cohomology

In addition to specifying the data $(A, Q, *)$, the construction of string field theory requires an invariant inner product $\langle \cdot, \cdot \rangle$ on $A$, i.e. $\langle a \ast b, c \rangle = \langle a, b \ast c \rangle$ and $\langle a, b \rangle = (-1)^{|a||b|} \langle b, a \rangle$. Note also
that $\langle a, b \rangle$ is non-vanishing only if $|a| + |b| = 3$. Deformations of $A$ preserving this inner product are governed by cyclic cohomology. To see this connection we note that in the presence of an invariant inner product, there is a natural isomorphism $\text{Hom}(A^M, A) \rightarrow \text{Hom}(A^{M+1}, \mathbb{C})$. The image of $f_M \in \text{Hom}(A^M, A)$ in $\text{Hom}(A^{M+1}, \mathbb{C})$ is then

$$C_{M+1}(f)(\Psi_1, \cdots, \Psi_{M+1}) = \langle f_M(\Psi_1, \cdots, \Psi_M), \Psi_{M+1} \rangle$$  \hspace{1cm} (96)$$

The cyclic symmetry (11) implies, in particular, that $f_2(a, b)$ preserves the inner product since

$$\langle f_2(a, b), c \rangle = C_3(a, b, c) = (-1)^{|b|+|c|}C_3(b, c, a) = (-1)^{|a|+|b|+|c|}\langle f_2(b, c), a \rangle = \langle a, f_2(b, c) \rangle$$  \hspace{1cm} (97)$$

The Hochschild co-boundary operator $\delta$ on $\text{Hom}(A^M, A)$ induces a co-boundary operator $\delta$ on $\text{Hom}(A^M, \mathbb{C})$ through $\delta C_M(f) = C_M(\delta f)$. This gives

$$(\delta C_M)(\Psi_1, \cdots, \Psi_{M+1}) = \sum_{i=1}^M (-1)^i C_M(\Psi_1, \cdots, \Psi_i \wedge \Psi_{i+1}, \cdots, \Psi_{M+1})$$

\hspace{1cm} + (-1)^{\Psi_1(\Psi_2, \cdots, \Psi_{M+1})}C_M(\Psi_2, \cdots, \Psi_M, \Psi_{M+1} \wedge \Psi_1) \hspace{1cm} (98)$$

This co-boundary operator takes cyclic elements to cyclic elements. The action of the differential $Q$ on $CC^*(A)$ is defined by

$$(QC_M)(\Psi_1, \cdots, \Psi_M) = \sum_{i=1}^M (-1)^{\Psi_1+\cdots+\Psi_{i-1}}C_M(\Psi_1, \cdots, Q\Psi_i, \cdots, \Psi_M).$$  \hspace{1cm} (99)$$

It also takes cyclic elements to cyclic elements. Since $Q^2 = \delta^2 = [Q, \delta] = 0$, the operator $(\delta - (-1)^M Q)$ squares to zero. Now, if we denote the submodule of $\text{Hom}(A^M, \mathbb{C})$ consisting of cyclic elements by $CC^M(A)$, then the cyclic cohomology of $A$ is defined by

$$HC^*(A) = \ker[(\delta - (-1)^M Q) : CC^*(A) \rightarrow CC^*(A)]/\text{Im}[\delta - (-1)^M Q] : CC^*(A) \rightarrow CC^*(A)]$$  \hspace{1cm} (100)$$

This is the cyclic cohomology that we argue to be isomorphic to the cohomology of closed strings.

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