ON RIGID STABILIZERS AND INvariant RANDOM SUBGROUPS OF GROUPS OF HOMEOMORPHISMS

TIANYI ZHENG

Abstract. A generalization of the double commutator lemma for normal subgroups is shown for invariant random subgroups of a countable group acting faithfully on a Hausdorff space. As an application, we classify ergodic invariant random subgroups of topological full groups of Cantor minimal Zd-systems. Another corollary is that for an ergodic invariant random subgroup of a branch group, a.e. subgroup H must contain derived subgroups of certain rigid stabilizers. Such results can be applied towards classification of invariant random subgroups of Grigorchuk groups.

1. Introduction

Let G be a locally compact group and denote by Sub(G) the space of closed subgroups of G equipped with the Chabauty topology. An invariant random subgroup (IRS) of G is a Borel probability measure on Sub(G) which is invariant under conjugation by G. The term IRS was coined in Abért, Glasner and Virág [AGV14]. Independently it was considered by Vershik in [Ver10, Ver12] (in terms of totally non-free actions) and Bowen in [Bow14] (in terms of random coset spaces).

Invariant random subgroups are closely related to probability measure preserving (p.m.p.) actions. Given a p.m.p. action \( G \curvearrowright (X, m) \), the pushforward of the probability measure m under the stabilizer map \( x \mapsto St_G(x) \) gives rise to an IRS, which we refer to as the stabilizer IRS of the action \( G \curvearrowright (X, m) \). It is known that all IRSs arise in this way ([AGV14 Proposition 14]), and moreover, an ergodic IRS arises as the stabilizer IRS of an ergodic p.m.p. action ([CP17 Proposition 3.5]).

The study of stabilizers of measure-preserving actions has been around for decades; it dates back to the work of Moore, see [AM66 Chapter 2].

In [BG04] it is observed that invariant random subgroups behave more similarly to normal subgroups than arbitrary subgroups. This theme has been developed tremendously in recent years, see for instances results in [ABB+17, AGV14, Ver10, Ver12], also the survey [Gel18] and references therein. Considerations in the present work are also motivated by this observation.

The following elementary lemma for normal subgroups has appeared in various forms in many contexts. We refer to it as the double commutator lemma because in its proof one takes commutators of the form \([\alpha, [\beta, \gamma]]\). It is used in [Gri00 Section 7] to show a criterion for a branch group to be just infinite. This lemma is also useful in the proofs of simplicity for certain classes of groups, see for instances [Mat06, Nek18a, Nek17].

Date: January 14, 2019.
Given $G \curvearrowright X$ and $U \subseteq X$, denote by $R_G(U)$ the rigid stabilizer of $U$, that is $R_G(U) = \{ g \in G : x \cdot g = x \text{ for all } x \in X \setminus U \}$. Throughout the paper, group actions are right actions. The commutator of two group elements is $[g, h] = ghg^{-1}h^{-1}$.

**Lemma 1.1** (The double commutator lemma, see \cite{Nek18a} Lemma 4.1). Let $G$ be a group acting faithfully by homeomorphisms on a Hausdorff space $X$, and $N$ a non-trivial normal subgroup of $G$. Then there exists a non-empty open subset $U \subseteq X$ such that $[R_G(U), R_G(U)] \leq N$.

We show that for countable groups, to some extent the double commutator lemma for normal subgroups applies to IRSs as well.

**Theorem 1.2.** Suppose $G$ is a countable group acting faithfully on a second countable Hausdorff space $X$ by homeomorphisms. Let $\mu$ be an ergodic IRS of $G$.

(i): If $\mu \neq \delta_{\text{id}}$, then for $\mu$-a.e. subgroup $H$, there exists a non-empty open set $U$ of $X$ such that $[R_G(U), R_G(U)] \leq H$.

(ii): Suppose in addition that for any open set $V$ of $X$, $R_G(V)$ has no fixed point in $V$. Then $\mu$-a.e. subgroup $H$ satisfies the following property: if $x \in X$ is not a fixed point of $H$, then there exists an open neighborhood $U$ of $x$ such that $[R_G(U), R_G(U)] \leq H$.

Roughly speaking, Theorem 1.2 implies that IRSs share the same property of containing derived subgroups of rigid stabilizers as normal subgroups do, except that IRSs are allowed to choose fixed point sets on $X$.

Theorem 1.2 can be applied towards classification of IRSs of a given group $G$ acting by homeomorphisms on $X$. We discuss two specific classes in this work: topological full groups and weakly branch groups.

Topological full groups of $\mathbb{Z}$-actions are introduced in \cite{GPS95}, in connection to the theory of topological orbit equivalence of minimal homeomorphisms of the Cantor set. Topological full groups for étale groupoids are investigated in the series of works \cite{Mat06, Mat12, Mat15}. In \cite{Mat06} it is shown that the derived subgroup of the topological full group of a minimal $\mathbb{Z}$-Cantor system is finitely generated and simple. Amenability of such groups is conjectured in \cite{GM14} and proved in \cite{JM13}, providing first examples of finitely generated infinite simple amenable groups.

When $\mathcal{G}$ is a groupoid of germs with unit space $X$ (relevant definitions are recalled in Subsection 1.1), its topological full group $F(\mathcal{G})$ is defined as the group of all homeomorphisms of $X$ whose germs belong to $\mathcal{G}$. In \cite{Nek17}, for every groupoid of germs $\mathcal{G}$, two normal subgroups $S(\mathcal{G})$ and $A(\mathcal{G})$ of the topological full group of $\mathcal{G}$ are defined, which are analogues of the symmetric and alternating groups. Following \cite{MB18}, we call $S(\mathcal{G})$ and $A(\mathcal{G})$ the symmetric and alternating full group of $\mathcal{G}$ respectively. \cite{Nek17} Theorem 1.1] states that if $\mathcal{G}$ is a minimal groupoid of germs with the space of units homeomorphic to the Cantor set (where minimal means every orbit is dense), then the alternating full group $A(\mathcal{G})$ is simple. In \cite{Nek18b}, first examples of simple groups of intermediate volume growth are constructed, these groups are alternating full groups of certain fragmentation of dihedral group actions on Cantor sets. To the best of our knowledge, so far in all known examples, $A(\mathcal{G})$ coincides with the derived subgroup of $F(\mathcal{G})$.

The IRSs of the infinite finitary permutation group $\text{Sym}_f(\mathbb{N})$ are classified in \cite{Ver12}. In a similar way one can classify IRSs of the infinite finitary alternating group $\text{Alt}_f(\mathbb{N})$, see \cite{TTD18}. The alternating group $\text{Alt}_f(\mathbb{N})$ is an example of so called finitary locally finite simple groups. Another important class of locally
finite simple groups are the inductive limits of direct product of alternating groups with respect to block diagonal embeddings (called LDA-groups), see [LP05, LDN07]. Indecomposable characters and IRSs of LDA-groups are classified in [DM13, DM17]. Independently, IRSs of inductive limits of finite alternating groups are investigated in [TTD14, TTD18].

An LDA-group can be identified as the alternating full group of its associated AF-groupoid. We are interested in IRSs of the topological full group $F(G)$ and the alternating full group $A(G)$ for more general minimal groupoids of germs. Combined with algebraic properties of the alternating full group $A(G)$, we have the following consequence of Theorem 1.2

**Theorem 1.3.** Let $G$ be a minimal groupoid of germs with unit space $X = G^{(0)}$ homeomorphic to the Cantor set. Suppose $\Gamma$ is a group such that $A(G) \leq \Gamma \leq F(G)$. Let $\mu$ be an ergodic IRS of $\Gamma$. Then $\mu$-a.e. subgroup $H$ contains the rigid stabilizer in $A(G)$ of the complement of its fixed point set, that is

$$ H \geq R_{A(G)}(X \setminus \text{Fix}_X(H)),$$

where $\text{Fix}_X(H) = \{x \in X : x \cdot h = x \text{ for all } h \in H\}$.

In the setting of Theorem 1.3, denote by $F(X)$ the space of closed subsets of $X$ equipped with the Vietoris topology. It is routine to check that the map $H \mapsto \text{Fix}_X(H)$ is Borel measurable. To classify ergodic IRSs of $A(G)$ in this situation, the task is reduced to classify possible distributions of $\text{Fix}_X(H)$, which are ergodic $G$-invariant Borel probability measures on the space $F(X)$. In particular, if the only $A(G)$-invariant ergodic measures on $F(X)$ are the $\delta$-mass at the empty set $\emptyset$ or the full set $X$, then the only IRSs of $A(G)$ are the trivial ones: $\delta_{\{id\}}$ and $\delta_{A(G)}$.

In what follows we consider examples where $A(G)$ admits invariant probability measures on $F(X)$ other than $\delta_\emptyset$ and $\delta_X$. In the special case where $A(G)$ contains an LDA-subgroup which acts minimally on $X$, one can gain information on such invariant measures with the help of the pointwise ergodic theorem for inductive limit of finite groups from [Ver74, OV96] see Corollary 5.10. The idea of applying pointwise ergodic theorem to study IRSs of the inductive limit of finite groups appeared in [TTD14, TTD18].

In the case of the topological full group $\Gamma$ of a minimal $\mathbb{Z}^d$-Cantor system, we can apply results on construction of Bratteli diagrams for minimal $\mathbb{Z}^d$ actions from [For00] to transfer information on ergodic invariant measures on $F(X)$ under action of certain LDA subgroup back to $\Gamma$. We obtain the following classification:

**Corollary 1.4.** Let $\Gamma$ be the topological full group of a minimal action of $\mathbb{Z}^d$ on the Cantor set $X$. Then the list of IRSs of the derived subgroup $\Gamma' = [\Gamma, \Gamma]$ is

(i): (atomic ones) $\delta_{\{id\}}, \delta_{\Gamma'}$.

(ii): (non-atomic ones) pushforward under the map

$$X^k \to \text{Sub}(\Gamma')$$

$$(x_1, \ldots, x_k) \mapsto \cap_{i=1}^k \text{St}_{\Gamma'}(x_i)$$

of measure $\mu_1 \times \ldots \times \mu_k$ on $X^k$, $k \in \mathbb{N}$, where each $\mu_i$ is an ergodic $\mathbb{Z}^d$-invariant measure on $X$.

Corollary 1.4 extends classification results on IRSs of LDA-groups established in the works [DM13, DM17, TTD14, TTD18] to minimal $\mathbb{Z}^d$-Cantor systems. Such
a classification shows that the alternating full group of a minimal $\mathbb{Z}^d$-action on the Cantor set $X$ does not admit non-trivial IRSs other than these stabilizer IRSs of diagonal actions on $X^k$, $k \in \mathbb{N}$. In particular, although $\Gamma'$ admits an infinite collection of non-atomic IRSs, it does not have a "zoo" of IRSs like non-abelian free groups \cite{Bow15} or the lamplighter group \cite{BGK15}. As indicated above, ingredients that go into the proof of Corollary 1.4 are: results from \cite{For00}, pointwise ergodic theorem for locally finite groups and Theorem 1.2 which in turn relies on Theorem 1.2 and general properties of $A(\mathcal{G})$ as shown in \cite{Nek17}.

Examples of topological full groups of minimal $\mathbb{Z}^d$-Cantor systems include groups of interval exchange transformations, see \cite{LMB18}, Subsection 5.3; and groups associated with the Penrose tilings introduced in \cite{CJN10}. As cited earlier, by \cite{JBM13}, the topological full group of a minimal $\mathbb{Z}$-Cantor system is amenable. For minimal $\mathbb{Z}^2$-actions, the topological full group can be non-amenable: examples which contain non-abelian free groups are constructed in \cite{EM13}.

The notion of a uniformly recurrent subgroup (URS) is introduced in \cite{GW15} as a topological analogue of invariant random subgroups. A URS of a countable group $G$ is a minimal, conjugation invariant, closed subset of $\text{Sub}(G)$. Theorem 1.2 can be compared to results on URS in \cite[Theorem 3.10]{LBMB18} and \cite[Theorem 6.1]{MB18}. Note that we don’t impose additional assumptions on the action $G \curvearrowright X$ in part (i) of Theorem 1.2. For URS, it is shown in \cite{MB18} that for any minimal groupoid of germs $\mathcal{G}$, $A(\mathcal{G})$ admit a unique URS, namely the stabilizer URS of $A(\mathcal{G}) \curvearrowright X = G^{(0)}$. In contrast, if $\mathcal{G}$ admits ergodic invariant measures on $X$, then we have an infinite collection of IRSs, including the stabilizer IRSs of diagonal actions on $X^k$. It seems to be an interesting question whether conclusion of Corollary 1.4 is true for alternating full groups of general minimal groupoids of germs.

We now turn to groups acting on spherically symmetric rooted trees. While the alternating full groups discussed in the previous paragraphs are infinite simple groups, groups acting faithfully on a rooted tree $T$ by automorphisms are residually finite.

To state the consequences of Theorem 1.2 for groups acting on rooted trees, we introduce necessary terminology and notations following references \cite{Gri00} and \cite{BGS03}. Let $d = (d_j)_{j \in \mathbb{N}}$ be a sequence of integers, $d_j \geq 2$ for all $j \in \mathbb{N}$. The spherically symmetric rooted tree $T = T_d$ with valency sequence $d$ is the tree with vertices $v = v_1 \ldots v_n$ where each $v_j \in \{0, 1, \ldots, d_j - 1\}$. The root of the tree is denoted by the empty sequence $\emptyset$. Edge set of the tree is $\{(v_1 \ldots v_n, v_1 \ldots v_{n-1})\}$. The index $n$ is called the depth or level of $v$, denoted $|v| = n$. Denote by $T^n$ the finite subtree of vertices up to depth $n$ and $L_n$ the vertices of level $n$. The boundary $\partial T_d$ of the tree $T_d$ is the set of infinite rays $x = v_1 v_2 \ldots$ with $v_j \in \{0, 1, \ldots, d_j - 1\}$ for each $j \in \mathbb{N}$. The action of $G$ on the tree $T$ extends to the boundary $\partial T$.

For each vertex $x \in T$, denote by $T_x$ the subtree rooted at $x$ and $C_x$ the cylinder set in $\partial T$ which consists of infinite rays with prefix $x$. We follow the terminology in the theory of branch groups and write

$$\text{Rist}_G(u) := R_G(C_u).$$

That is, for $u \in T$, the rigid stabilizer $R_G(C_u)$ of the cylinder set $C_u$ is called the rigid vertex stabilizer of $u$ in $G$. 

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Given a subtree $T_x$ rooted at $x$ and number $m$, denote by $\text{Rist}_m(T_x)$ the level $m$ rigid stabilizer of this subtree, that is

$$\text{Rist}_m(T_x) := \prod_{1 \leq j \leq m, \ u_j \in \{0, \ldots, d_{x+1} - 1\}} \text{Rist}_G(xu_1 \ldots u_m).$$

A group $G$ acting on the rooted tree $T$ is said to be \textit{weakly branching} if it acts level transitively and the rigid stabilizer $\text{Rist}_G(u)$ is non-trivial for any vertex $u \in T$. It is said to be a \textit{branch group} if in addition all the level rigid stabilizers $\text{Rist}_m(T)$ have finite index in $G$, $m \in \mathbb{N}$. These notions are introduced by Grigorchuk in [Gr00].

Given a closed subset $K$ of $\partial T$, we associate to it the following index set $I_K \subseteq T$. The complement $\partial T \setminus K$ can be written uniquely as a disjoint union of cylinder sets $\bigcup_{x \in I_K} C_x$, where the cylinder sets are maximal in the sense that if $x = x_1 \ldots x_t$ is in the collection $I_K$, then there exists a sibling $x' = x_1 \ldots x_{t-1}x'_t$, $x'_t \neq x_t$, such that $x' \notin I_K$. For example, if $T$ is the rooted binary tree and $K = \{1^\infty\}$, where $1^\infty$ is the right most ray, then $I_{\{1^\infty\}} = \{0, 10, \ldots, 1^{n-1}0, \ldots\}$. The following corollary is a direct consequence of Theorem 1.2.

**Corollary 1.5.** Let $G$ be a countable weakly branch group acting faithfully on a rooted spherically symmetric tree $T$. Let $\mu$ be an ergodic IRS of $G$. Then there exist numbers $(m_i)$, each $m_i \in \mathbb{N}$, such that $\mu$-a.e. subgroup $H$ satisfies

$$H \geq \bigoplus_{x \in I_{\text{Fix}(H)}} \left[ \text{Rist}_{m_i(x)}(T_x), \text{Rist}_{m_i(x)}(T_x) \right],$$

where $I_{\text{Fix}(H)}$ is the set of vertices in $T$ associated with $\text{Fix}(H) = \text{Fix}_{\partial T}(H)$ as described above.

The special case of Corollary 1.5 where $G$ is a finitary regular branch group was obtained in [BT18], where finitary means that $G$ is a subgroup of $\text{Aut}_f(T)$. Note that $\text{Aut}_f(T)$ is locally finite. Our approach is independent of [BT18] and applies to more general settings. For branch groups one can draw the following immediate consequence (Corollary 6.1): if $\mu$ is an ergodic IRS of a branch group $G$ such that $\mu$-a.e. $\text{Fix}(H)$ is clopen (possibly empty), then $\mu$ is atomic. The same conclusion holds for the Basilica group, which is weakly branching but not branching, see Corollary 6.2.

In particular, an ergodic fixed point free IRS of a just-infinite branch group $G$ is supported on finite index subgroups of $G$. The most celebrated example of a just infinite branch group is the first Grigorchuk group $\mathcal{G}$ which is defined in [Gri80], see [Gri00] for more information on just-infiniteness. The group $\mathcal{G}$ has many remarkable properties, in particular, it is torsion ([Gri80]) and shown in [Gri84] to be the first example of groups of intermediate volume growth.

The induced distribution of $\text{Fix}(H)$ is described in [BT18]. Namely, by [BT18] Lemma 2.2, 2.3, any $G$-invariant ergodic probability measure on $F(\partial T)$ arises as translation of a fixed closed set $K$ by a random element in the profinite completion $\hat{G}$, with respect to the Haar measure on $\hat{G}$. We have seen that for branch groups, $\text{Fix}(H)$ being clopen corresponds to atomic IRSs. The situation where $\text{Fix}(H)$ is a closed but not clopen subset of $\partial T$ is more complicated. Atomless IRSs of a weakly branch group with $\text{Fix}(H)$ being a finite subset of $\partial T$ are considered in [DG18]. Corollary 1.4 provides information that conditioned on the fixed point set $\text{Fix}(H)$ on $\partial T$, the subgroup $H$ must contain derived subgroups of certain level rigid
stabilizers. In other words, the conditional distribution of $H$ given $\text{Fix}(H) = K$ is pulled back from an IRS of the quotient

$$\bar{\Gamma}_{K,m} := \text{Fix}_G(K)/\bigoplus_{x \in I_K} \left[ \text{Rist}_{m|x|}(T_x), \text{Rist}_{m|x|}(T_x) \right],$$

where $\text{Fix}_G(K) = \{ g \in G : x \cdot g = x \text{ for any } x \in K \}$ is the pointwise stabilizer of the set $K$ in $G$. It is natural to ask in some specific examples, for instance the first Grigorchuk group $\mathfrak{G}$, whether one can completely classify the IRSs. For the group $\mathfrak{G}$, when $K$ is closed but not clopen, such a quotient $\bar{\Gamma}_{K,m}$ is a locally finite infinite group which admits a continuum of IRSs, see more discussion in Section 6.

We summarize the discussion above for infinite group which admits a continuum of IRSs, see more discussion in Section 6. We summarize the discussion above for $\mathfrak{G}$ as follows, answering positively [BT18, Problem 8]:

**Example 1.6.** Let $\mu$ be an ergodic invariant random subgroups of the first Grigorchuk group $\mathfrak{G}$. Then it falls into one of the following types:

(i): Fixed point free IRSs. In this case $\mu$ is supported on finite index subgroups of $\mathfrak{G}$; equivalently there exists a constant $m \in \mathbb{N}$ such that $\mu$-a.e. $H$ contains the level stabilizers $\text{St}\mathfrak{G}(m)$.

(ii): IRSs with non-empty clopen fixed point sets. In this case $\mu$ is atomic and it arises in the following way. There exists a non-empty clopen subset $C \subseteq \partial T$ and a finite index subgroup $\Gamma$ of $\text{Fix}\mathfrak{G}(C)$ such that $\mu$ is the uniform measure on $\mathfrak{G}$-conjugates of $\Gamma$.

(iii): Non-atomic IRSs. In this case there exists a closed but not clopen subset $C \subseteq \partial T$ and a sequence of integers $m$ such that the fixed point set $\text{Fix}(H)$ is a random Haar translate of $C$ and $\mu$-a.e. $H$ contains the infinite direct sum $\bigoplus_{x \in \text{Fix}(K)} \left[ \text{Rist}_{m|x|}(T_x), \text{Rist}_{m|x|}(T_x) \right]$.

The rest of the paper is organized as follows. In Section 2 we consider distributions of certain random collections of partial homeomorphisms induced by the invariant random subgroup. In Section 3 we prove the double commutator lemma for IRSs as stated in Theorem 1.2. Section 4 collects necessary definitions and basic properties of topological full groups. In Section 5 we show Theorem 1.3 on IRSs of topological full groups and consider minimal $\mathbb{Z}^{d}$-Cantor systems as main examples. Section 6 on IRSs of weakly branch groups is independent of Section 4 and 5 and can be understood right after Section 3.

**Assumption 1.7** (Standing assumption). Throughout the rest of the paper, $G$ is a countable group acting faithfully on a second countable Hausdorff space $X$ by homeomorphisms. Denote by $\mathcal{U}$ a countable base of topology of $X$.

2. Restrictions and conditional distributions

2.1. Regular conditional distributions. We follow notations of regular conditional distributions in the book [Par07, Chapter V.8]. Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$ be two Borel spaces, $\mathbb{P}$ a probability measure on $\mathcal{B}$ and $\pi : X \to Y$ a measurable map. Let $Q = \mathbb{P} \circ \pi^{-1}$ be probability measure on $\mathcal{C}$ which is the pushforward of $\mathbb{P}$. A regular conditional distribution given $\pi$ is a mapping $y \mapsto \mathbb{P}(y, \cdot)$ such that

(i) for each $y \in Y$, $\mathbb{P}(y, \cdot)$ is a probability measure on $\mathcal{B}$;

(ii) there exists a set $N \subseteq \mathcal{C}$ such that $Q(N) = 0$ and for each $y \in Y \setminus N$, $\mathbb{P}(y, X \setminus \pi^{-1}(\{y\})) = 0$;
(iii) for any $A \in \mathcal{B}$, the map $y \mapsto \mathbb{P}(y, A)$ is $\mathcal{C}$-measurable and

$$\mathbb{P}(A) = \int_Y \mathbb{P}(y, A)d\mathbb{Q}(y).$$

We will refer to these three items as properties (i),(ii),(iii) of a regular conditional distribution.

Recall that a measure space $(X, \mathcal{B})$ is called a standard Borel space if it is isomorphic to some Polish space equipped with the Borel $\sigma$-field. It is classical that if $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ are standard Borel spaces and $\pi : X \rightarrow Y$ is measurable, then there exists such a regular conditional distribution $y \mapsto \mathbb{P}(y, \cdot)$ with properties (i),(ii),(iii); and moreover it is unique: if $\mathbb{P}'(y, \cdot)$ is another such mapping, then \{y : $\mathbb{P}'(y, \cdot) \neq \mathbb{P}(y, \cdot)$\} is a set of $\mathbb{Q}$-measure 0, see [Par67] Theorem 8.1.

In our setting $G$ is a countable group, the Chabauty topology on Sub($G$) is restriction of the product topology on $(0,1)G$ to the closed subset Sub($G$). The space (Sub($G$), $\mathcal{B}$), where $\mathcal{B}$ is the Borel $\sigma$-field on Sub($G$), is a standard Borel space.

2.2. Restrictions to open subsets. Let $G \cap X$ as in the standing assumption. Let $U, V$ be two open subsets of $X$ such that there exists some $g \in G$ with $V = U \cdot g$. Given such open sets and a subgroup $H \in \text{Sub}(G)$, define the following:

$$(2.1) \quad H_{U \rightarrow V} := \{h \in H : V = U \cdot h\}.$$ 

And define the restrictions

$$(2.2) \quad \bar{H}_{U \rightarrow V} := \{h|_U : h \in H_{U \rightarrow V}\}.$$ 

Elements of $H_{U \rightarrow V}$ are viewed as partial homeomorphisms with domain $U$ and range $V$, denoted by $h|_U : U \rightarrow V$. By definition $H_{U \rightarrow V}$ is the subgroup of $H$ which consists of elements that leaves $U$ invariant, in other words the setwise stabilizer of $U$ in $H$. The group $\bar{H}_{U \rightarrow V}$ is a quotient of $H_{U \rightarrow V}$ which acts on $U$ by homeomorphisms. The following fact will be used repeatedly:

**Fact 2.1.** The set $H_{U \rightarrow V}$ is either empty or a right coset of $H_{U \rightarrow V}$. That is, if $H_{U \rightarrow V}$ is non-empty, then for any element $h \in H_{U \rightarrow V},$

$$H_{U \rightarrow V} = H_{U \rightarrow V}h.$$

Similarly, if $\bar{H}_{U \rightarrow V}$ is non-empty, then $\bar{H}_{U \rightarrow V} = \bar{H}_{U \rightarrow V}h|_U$ for any $h|_U \in \bar{H}_{U \rightarrow V}$.

**Proof.** From definitions it is clear that $H_{U \rightarrow V}h \subseteq H_{U \rightarrow V}$ for any $h \in H_{U \rightarrow V}$. In the other direction, given any element $h' \in H_{U \rightarrow V}$, we have $U \cdot h'h^{-1} = V \cdot h^{-1} = (U \cdot h)h^{-1} = U$. Therefore $h'h^{-1} \in H_{U \rightarrow V}$, which implies $H_{U \rightarrow V} = H_{U \rightarrow V}h$. The same argument shows that $\bar{H}_{U \rightarrow V} = \bar{H}_{U \rightarrow V}h|_U$.

Let $U, V \in \mathcal{U}$ be such that $U \cap V = \emptyset$ and $G_{U \rightarrow V} \neq \emptyset$. Due to Fact 2.1, given $\bar{H}_{U \rightarrow V}$, the set $H_{U \rightarrow V}$ can only take value in a countable collection, namely $\emptyset$ and right cosets of the form $\bar{H}_{U \rightarrow V}|_U$. Let $\Omega_{U, V} = \{H \in \text{Sub}(G) : H \cap G_{U \rightarrow V} \neq \emptyset\}$ be the set of subgroups of $G$ which contains some element that sends $U$ to $V$. Note that $\Omega_{U, V}$ is an open set in Sub($G$).
Let $\mu$ be an IRS of $G$. Consider a pair $U, V$ such that $\mu(\Omega_{U,V}) > 0$. Define $\mu_{U,V}$ as the probability measure on Borel subsets of $\Omega_{U,V}$

$$\mu_{U,V} : \mathcal{B}(\Omega_{U,V}) \to [0,1]$$

$$\mu_{U,V}(A) = \frac{\mu(A)}{\mu(\Omega_{U,V})}, \ A \subseteq \Omega_{U,V}.$$ 

Let $C_{U,V} := \cup_{\gamma \in G_{U \to V}} \text{Sub}(\tilde{G}_{U \to V})|_{U}$ be the union of cosets of subgroups of $\tilde{G}_{U \to V}$, represented by restrictions to $U$ of elements $\gamma \in G_{U \to V}$. Let $\mathcal{U}_U$ be a base for the Chabauty topology on $\text{Sub}(\tilde{G}_{U \to V})$. Equip $C_{U,V}$ with the topology generated by the base $\mathcal{U}_{U,V} = \cup_{\gamma \in G_{U \to V}} \{O\gamma|_{U} : O \in \mathcal{U}_U\}$. Define $\tilde{C}_{U,V}$ as the subspace of $C_{U,V} \times \text{Sub}(\tilde{G}_{U \to V})$ which consists of pairs

$$\tilde{C}_{U,V} := \{(C, A) \in C_{U,V} \times \text{Sub}(\tilde{G}_{U \to V}) : C = A\gamma|_{U} \text{ for some } \gamma \in G_{U \to V}\}.$$ 

It’s clear by definition that $\tilde{C}_{U,V}$ equipped with the Borel $\sigma$-field is a standard Borel space. By Fact 2.1 the map $H \mapsto (\tilde{H}_{U \to V}, \tilde{H}_{U \to U})$ is a map from $\Omega_{U,V}$ to $C_{U,V}$. It is routine to check that this map is measurable.

Let $\pi : \tilde{C}_{U,V} \to \text{Sub}(\tilde{G}_{U \to U})$ be the projection to the second coordinate, that is $\pi(C, A) = A$. Denote by $\mathbb{P}_{U,V}^{\mu} : \text{Sub}(\tilde{G}_{U \to U}) \times \mathcal{B}(\tilde{C}_{U,V}) \to [0,1]$ the regular conditional distribution of $(\tilde{H}_{U \to V}, \tilde{H}_{U \to U})$ given $\pi$, where the distribution of $(\tilde{H}_{U \to V}, \tilde{H}_{U \to U})$ is the pushforward of $\mu_{U,V}$ under this map $H \mapsto (\tilde{H}_{U \to V}, \tilde{H}_{U \to U})$. Recall that such regular conditional distribution exists since $\pi$ is a measurable map between standard Borel spaces.

We introduce one more piece of notation that will be also be used in the next section. For $U, V$ open sets of $X$, let $W_{V}^{U}$ be the subgroup of $G$ which consists of elements that fix $U$ pointwise and leaves $V$ invariant, that is

$$W_{V}^{U} := \{ g \in G : x \cdot g = x \text{ for any } x \in U \text{ and } V = V \cdot g \}.$$ 

Note that the set $\Omega_{U,V}$ is invariant under conjugation by $W_{V}^{U}$. Since $\mu$ is invariant under conjugation by $G$, it follows that $\mu_{U,V}$ is invariant under conjugation by $W_{V}^{U}$.

Let $\gamma \in W_{V}^{U}$. Then for such $\gamma$, it is clear from definitions that under conjugation by such an element $\gamma$, $H_{U \to V} \gamma^{-1} H_{U \to U}$ remains the same as $H_{U \to U}$ since $\gamma$ acts trivially on $U$; while $(\gamma^{-1}H_{U \to V})_{U \to V}$ is the right translate of $H_{U \to V}$ by the restriction of $\gamma$ to $V$. That is, for $\gamma \in W_{V}^{U}$,

$$\gamma^{-1}H_{U \to U} = H_{U \to U},$$

$$\gamma^{-1}H_{U \to V} = H_{U \to V}\gamma|_{V}$$

The assumption that $\mu$ is invariant under conjugation implies translation invariance properties of $\mathbb{P}_{U,V}^{\mu}(\tilde{H}_{U \to U}, \cdot)$ as stated in the following lemma.

**Lemma 2.2.** Let $\mu$ be an IRS of $G$. Let $U, V$ be open subsets of $X$ such that $\mu(\Omega_{U,V}) > 0$. Then for $\mu$-a.e. $H \in \Omega_{U,V}$, $\mathbb{P}_{U,V}^{\mu}(\tilde{H}_{U \to U}, \cdot)$ is a probability measure supported on $\{ (\tilde{H}_{U \to U} g|_{U}, \tilde{H}_{U \to U} : g \in G_{U \to V} \}$ and $\mathbb{P}_{U,V}^{\mu}(\tilde{H}_{U \to U}, \cdot)$ is invariant under right multiplication of restrictions $\gamma|_{V}$ for $\gamma \in W_{V}^{U}$; for any $g \in G_{U \to V}$ and $\gamma \in W_{V}^{U}$,

$$\mathbb{P}_{U,V}^{\mu}(\tilde{H}_{U \to U}, \{ (\tilde{H}_{U \to U} g|_{U}, \tilde{H}_{U \to U} \}) = \mathbb{P}_{U,V}^{\mu}(\tilde{H}_{U \to U}, \{ (\tilde{H}_{U \to U} g|_{U} \gamma|_{V}, \tilde{H}_{U \to U} \}).$$
Proof. The first half of the claim follows directly from property (i) and (ii) of the regular conditional distribution \( \mathbb{P}_{U,V}^{\mu_U} \) as reviewed in Subsection 2.1.

To show the second half, note that as a consequence of \( \mu_{U,V} \) being invariant under conjugation by \( \gamma, \gamma \in W_U^U \), we have that \( (\hat{H}_{U \rightarrow V}, \hat{H}_{U \rightarrow U}) \) and \( (\hat{H}_{U \rightarrow V} \gamma |_V, \hat{H}_{U \rightarrow U}) \) have the same distribution. Indeed, for any measurable bounded function \( f : \hat{G}_{U,V} \rightarrow \mathbb{R} \), we have

\[
\begin{align*}
\mathbb{E}_{\mu_{U,V}} \left[ f \left( \hat{H}_{U \rightarrow V}, \hat{H}_{U \rightarrow U} \right) \right] &= \mathbb{E}_{\mu_{U,V}} \left[ f \left( (\gamma^{-1}H\gamma)_{U \rightarrow U}, (\gamma^{-1}H\gamma)_{U \rightarrow U} \right) \right] \\
&= \mathbb{E}_{\mu_{U,V}} \left[ f \left( \hat{H}_{U \rightarrow V} \gamma |_V, \hat{H}_{U \rightarrow U} \right) \right].
\end{align*}
\]

In the last line the identities \( (\gamma^{-1}H\gamma)_{U \rightarrow U} = \hat{H}_{U \rightarrow U} \) and \( (\gamma^{-1}H\gamma)_{U \rightarrow V} = \hat{H}_{U \rightarrow V} \gamma |_V \) explained above are used. The claim follows from uniqueness of regular conditional distribution.

\( \square \)

3. Properties of IRS in connection to rigid stabilizers

In this section we prove Theorem 1.2. As explained in Lemma 2.2, conjugation invariance of the IRS distribution \( \mu \) results in translation invariance properties of certain conditional distributions. We will repeatedly use the fact that a countable orbit supporting an invariant probability measure must be finite.

3.1. From IRS to rigid stabilizers. Recall the notation \( H_{U \rightarrow V} \) and \( \hat{H}_{U \rightarrow V} \) as defined in 2.1 and 2.2. Recall that \( W_U^U \), as defined in 2.3, is the subgroup of \( G \) which consists of elements that fix \( U \) pointwise and leave \( V \) invariant. For a subgroup \( H < G \), denote by \( N_G(H) \) its normalizer in \( G \), that is \( N_G(H) = \{ g \in G : g^{-1}Hg = H \} \). A group is called an FC-group if all of its conjugacy classes are finite.

Proposition 3.1. Let \( G \curvearrowright X \) be as in the standing assumption. Let \( \mu \) be an IRS of \( G \). Then \( \mu \)-a.e. \( H \) satisfies the following property: if \( U, V \in \mathcal{U} \) are such that \( U \cap V = \emptyset \) and \( H \cap G_{U \rightarrow V} \neq \emptyset \), then:

(i): There exists an element \( \sigma \in G_{U \rightarrow V} \) and a finite index subgroup \( \Gamma \) of \( W_U^U \sigma^{-1} \) such that \( \pi_U(\Gamma) \leq \hat{H}_{U \rightarrow U} \), where \( \pi_U \) is the projection \( G_{U \rightarrow U} \rightarrow \hat{G}_{U \rightarrow U} \).

(ii): The subgroup \( K = \{ g \in N_{R_G(U)}(R_H(U)) : \pi_U(g) \in \hat{H}_{U \rightarrow U} \} \) is of finite index in \( R_G(U) \). Moreover, the quotient group \( K/K \cap H \) is an FC-group.

To prove part (i) we consider regular conditional distribution of \( (\hat{H}_{U \rightarrow V}, \hat{H}_{U \rightarrow U}) \) given \( \hat{H}_{U \rightarrow U} \). The invariance property stated in Lemma 2.2 forces the number of cosets under consideration to be finite, in order to support an invariant probability measure.

Proof of Proposition 3.1 (i). If \( \mu \) is \( \delta \)-mass at \( \{id\} \), then the claim is trivially true. We may assume \( \mu \) is not \( \delta \)-mass at \( \{id\} \). In what follows we use notations introduced in Section 2.

Take a pair of \( U, V \) such that \( \mu(\Omega_{U,V}) > 0 \) and consider the random variables \( H_{U \rightarrow V}, \hat{H}_{U \rightarrow V} \) and \( \hat{H}_{U \rightarrow U} \) as defined in 2.1, 2.2. Denote by \( \mathbb{P}_{U,V}^{\mu_U}(H_{U \rightarrow U}) \) the regular conditional distribution of \( (\hat{H}_{U \rightarrow V}, \hat{H}_{U \rightarrow U}) \) given \( \hat{H}_{U \rightarrow U} \), where \( H \) has distribution \( \mu_{U,V} \) on \( \Omega_{U,V} \).
Recall that by Lemma 2.2, $\mathbb{P}^\mu_{U,V}(\bar{H}_{U\to V}, \cdot)$ is a probability measure on a countable set. For $\mu$-a.e. $H \in \Omega_{U,V}$, we can find one coset $\bar{H}_{U\to V}\sigma\{U\}$, $\sigma \in G_{U\to V}$ depending on $\bar{H}_{U\to V}$, such that $\mathbb{P}^\mu_{U,V}(\bar{H}_{U\to V}, \{ (\bar{H}_{U\to V}\sigma\{U\}, \bar{H}_{U\to V}) \}) > 0$. If the number of right cosets $\bar{H}_{U\to V}\sigma\{U\}$, where $\gamma$ is taken over elements of $W^U_V$, is infinite, then the probability measure $\mathbb{P}^\mu_{U,V}(\bar{H}_{U\to V}, \cdot)$ cannot be invariant under right multiplication as stated in Lemma 2.2. Therefore there are only finitely many cosets of $\bar{H}_{U\to V}\sigma\{U\}$ in this collection. In other words, there are finitely many representatives $\gamma_1, \ldots, \gamma_\ell$ in $W^U_V$ such that for any $\gamma \in W^U_V$, we have $\bar{H}_{U\to V}\sigma\{U\} = \bar{H}_{U\to V}\sigma\{U\} \gamma_k$ for some $k \in \{1, \ldots, \ell\}$. It follows that for any $\gamma \in W^U_V$, there is a representative $\gamma_k$, $k \in \{1, \ldots, \ell\}$, such that $\bar{H}_{U\to V}$ contains $\sigma\{U\} (\gamma_k^{-1}) \gamma \sigma\{U\}^{-1}$. Consider the subgroup $W_1$ of $W^U_V$ generated by the collection $\gamma_k^{-1}$, where $\gamma \in W^U_V$ and $\gamma_k$ is its corresponding representative. It’s clear by definition of $W_1$ that $\bigcup_{j=1}^\ell W_1 \gamma_j = W^U_V$, therefore $W_1$ is a finite index subgroup of $W^U_V$. Recall that $\sigma$ maps $U$ to $V$, therefore $\sigma W_1 \sigma^{-1} = W^U_V \sigma^{-1}$. Let $\Gamma = \sigma W_1 \sigma^{-1}$, it is a finite index subgroup of $W^U_V \sigma^{-1}$. Elements of $\Gamma$ satisfy the property that $\bar{H}_{U\to V} = \bar{H}_{U\to V}\gamma\{U\}$, in other words, $\pi_U(\Gamma) \leq \bar{H}_{U\to V}$. Let $\Omega_{U,V}$ be the subset of $\Omega_{U,V}$ which consists of subgroups $H$ of $G$ such that statement (i) is satisfied for some $\sigma \in G_{U\to V}$ and some finite index subgroup $\Gamma \leq \pi_U W^U_V \sigma^{-1}$. We have proved that $\mu(\Omega_{U,V}) = \mu(\Omega_{U,V})$.

Finally, take the union of the measure 0 sets we want to discard. Let $\Lambda_0 = \{(U,V) \in U^2 : \mu(\Omega_{U,V}) = 0\}$ and $E = (\bigcup_{(U,V) \in \Lambda_0} \Omega_{U,V}) \cup (\bigcup_{(U,V) \notin \Lambda_0} \Omega_{U,V} \setminus \Omega_{U,V})$. This is a countable union of $\mu$-measure 0 sets, and $\text{Sub}(G) - E$ gives a full measure set in the statement of part (i).

We now turn to the proof of Proposition 3.1 (ii). Let $U$ be an open subset of $X$ and consider the subgroup $H_{U\to U}$ of $H$ as in 2.1. Elements of the subgroup $H_{U\to U}$ preserves the partition $U \sqcup U^c$ of $X$, thus $h \in H_{U\to U}$ can be recorded as a pair $(f_1, f_2)$, where $f_1 = \pi_U(h)$ and $f_2 = \pi_{U^c}(h)$. View $H_{U\to U} = \pi_U(H_{U\to U})$ as a group of homeomorphisms on $U$, then the rigid stabilizer $R_H(U)$ is a normal subgroup of $H_{U\to U}$. Note the following elementary fact about the corresponding quotient group:

**Fact 3.2.** There is an isomorphism $\phi : \pi_U^*(H_{U\to U})/R_H(U^c) \to \pi_U^*(H_{U\to U})/R_H(U)$.

**Proof.** Write $L = H_{U\to U}$. Record elements of $L$ as $(f_1, f_2)$, where $f_1 \in \pi_U(L)$ and $f_2 \in \pi_{U^c}(L)$. Let $B_L(f_2) = \{ f_1 \in \pi_U(L) : (f_1, f_2) \in L \}$. Note that $B_L(f_2)$ is a right coset of $L \cap R_G(U)$. Moreover, if $f_2$ and $f_2'$ are in the same coset of $L \cap R_G(U^c)$ in $\pi_{U^c}(L)$, then $B_L(f_2) = B_L(f_2')$. Therefore we have a map $\phi : \pi_U(L)/L \cap G_{U^c} \to \pi_U(L)/L \cap G_U$ induced by $f_2 \mapsto B_L(f_2)$.

The map $\phi$ is a homomorphism because $L \cap R_G(U)$ is normal in $\pi_U(L)$. If $B_L(f_2) = B_L(f_2')$, then there exists $f_1 \in \pi_U(L)$ such that $(f_1, f_2) \in L$ and $(f_1, f_2') \in L$. It follows that $(id, f_2^{-1} f_2') = (f_1, f_2)^{-1} (f_1, f_2') \in L$, that is $f_2$ and $f_2'$ are in the same coset of $L \cap R_G(U^c)$. In other words $\phi$ is injective. For any $f_1 \in \pi_U(L)$, the preimage of $f_1$ under $\phi$ is the coset $\{ f_2 \in \pi_{U^c}(L) : (f_1, f_2) \in L \}$. We conclude $\phi$ is an isomorphism.

By part (i) of the proposition we have that $\pi_U(\Gamma)$ is a subgroup of $\bar{H}_{U\to U}$ for some $\Gamma \leq \pi_U W^U_V \sigma^{-1}$ where $\sigma$ is some element in $G_{U\to V}$. Note that $U \cap U \cdot \sigma^{-1} = \emptyset$. 

□
Our next step is to gain some information on the intersection \( H \cap R_G(U) = R_H(U) \). Note that \( R_H(U) \) is a normal subgroup of \( H_{U \to U} \), it can also be viewed as a normal subgroup of \( H_{U \to U} \), where \( R_H(U) \) and \( H_{U \to U} \) are regarded as groups acting by homomorphisms on \( U \). We look for properties of the quotient group \( H_{U \to U}/R_H(U) \) that can be derived from conjugation invariance of the distribution \( \mu \).

**Proof of Proposition 3.1 (ii).** Let \( U, V \in U \) be such that \( U \cap V = \emptyset \) and \( \mu(\Omega_{U,V}) > 0 \).

We first show that for \( \mu \)-a.e. \( H \) in \( \Omega_{U,V} \), the subgroup \( K \) defined in the statement is of finite index in \( R_G(U) \). For \( \mu \)-a.e. \( H \) in \( \Omega_{U,V} \), let \( \Gamma = N_G(R_H(U)) \). Indeed, for \( \gamma \in \Gamma \), let \( h \in H \) be an element with \( h|_U = \gamma|_U \). Since \( \Gamma < W_{U}^{\sigma^{-1}} \), we have that such \( h \) leaves \( U \) invariant, that is \( h \in H_{U \to U} \).

Next we show the claim on finite conjugacy classes. Take an element \( \sigma \in \pi_U(G_{U \to U}) \), as in the proof of Fact 3.2 let \( B_{H_{U \to U}}(\sigma) = \{ g \in \pi_U(G_{U \to U}) : (g, \sigma) \in H_{U \to U} \} \). Recall that \( B_{H_{U \to U}}(\sigma) \) is either empty or a coset of \( R_H(U) \) in \( \pi_U(H_{U \to U}) \).

Write

\[ A_\sigma = \{ H \in \text{Sub}(G) : \sigma \in \pi_U(\{H_{U \to U}\}) \}. \]

Note that the set \( A_\sigma \) is open in \( \text{Sub}(G) \) and invariant under conjugation by the rigid stabilizer \( R_G(U) \).

Take an element \( \sigma \in \pi_U(G_{U \to U}) \) such that \( \mu(A_\sigma) > 0 \). Define a map \( \alpha_\sigma \) by

\[
\alpha_\sigma : A_\sigma \to \{0, 1\}^{G_{U \to U}},
\]

\[
\alpha_\sigma(H) = B_{H_{U \to U}}(\sigma).
\]

Similar as in Subsection 2 define the probability \( \mu_{A_\sigma} : B(A_\sigma) \to [0, 1] \) by setting \( \mu_{A_\sigma}(C) = \mu(C)/\mu(A_\sigma) \) for measurable set \( C \subseteq A_\sigma \). Since \( \mu \) is invariant under conjugation by \( G \) and \( A_\sigma \) is set-wise invariant under conjugation by \( R_G(U) \), it follows that \( \mu_{A_\sigma} \) is invariant under conjugation by \( R_G(U) \).

Consider the map

\[
A_\sigma \to \{0, 1\}^{G_{U \to U}} \times \text{Sub}(R_G(U)) \quad H \mapsto (\alpha_\sigma(H), R_H(U)).
\]

The pair \( (\alpha_\sigma(H), R_H(U)) \) takes value in the subspace \( D_U \) of \( \{0, 1\}^{G_{U \to U}} \times \text{Sub}(R_G(U)) \),

\[
D_U = \{ (C, A) \in \{0, 1\}^{G_{U \to U}} \times \text{Sub}(G_U) : C = A|_U, g \in G_{U \to U} \}.
\]

It is clear that \( D_U \) is a standard Borel space. Denote by \( \mathbb{P}_{\sigma,U}^{\mu} : \text{Sub}(R_G(U)) \times B(D_U) \to [0, 1] \) the regular conditional distribution of \( (\alpha_\sigma(H), R_H(U)) \) given the random variable \( R_H(U) \), where the distribution of the pair \( (\alpha_\sigma(H), R_H(U)) \) is the pushforward of \( \mu_{A_\sigma} \) under the map \( H \mapsto (\alpha_\sigma(H), R_H(U)) \).
Denote by $N = N_{R_G(U)}(R_H(U))$ the normalizer of $R_H(U)$ in $R_G(U)$. Then for $\gamma \in N$, we have
\begin{equation}
R_{\gamma^{-1}H\gamma}(U) = R_H(U).
\end{equation}

Since $N$ acts trivially on $U^c$, for an element $h$ of $H_{U \to U}$ recorded as $(f_1, f_2)$, where $f_1 = \pi_U(h)$, $f_2 = \pi_{U^c}(h)$, we have that $\gamma^{-1}(f_1, f_2)\gamma = (\gamma^{-1}f_1\gamma, f_2)$. Therefore for $\gamma \in N$,
\begin{equation}
\alpha(\gamma^{-1}H\gamma) = \gamma^{-1}\alpha(H)\gamma.
\end{equation}

Similar to Lemma 222, we have the following invariance property of $\mathbb{P}_{\sigma,U}^\mu$ under conjugation by $N$.

**Lemma 3.3.** For $\mu$-a.e. $H \in A_\sigma$, the regular conditional distribution $\mathbb{P}_{\sigma,U}^\mu (R_H(U), \cdot)$ of $(\alpha(H), R_H(U))$ given $R_H(U)$ is supported on the countable set $\{(R_H(U)g|_U, R_H(U)) : g \in G_{U \to U}\}$; moreover, for any $\gamma \in N = N_{R_G(U)}(R_H(U))$ and $g \in G_{U \to U}$, we have

\[
\mathbb{P}_{\sigma,U}^\mu (R_H(U), \{(R_H(U)g|_U, R_H(U))\}) = \mathbb{P}_{\sigma,U}^\mu (R_H(U), \{(R_H(U)(\gamma^{-1}g)|_U, R_H(U))\}).
\]

**Proof of Lemma 3.3.** The first claim follows from property (i),(ii) of regular conditional distribution. For any $\gamma \in N$ and bounded measurable function $f$ on $D_U$, we have

\[
E_{\mu,A_\sigma} [f(\alpha(H), R_H(U))] = E_{\mu,A_\sigma} [f(\alpha(\gamma^{-1}H\gamma), R_{\gamma^{-1}H\gamma}(U))] \text{ (by invariance of $\mu_{A_\sigma}$ under conjugation by $N$)}
\]

\[
= E_{\mu,A_\sigma} [f(\gamma^{-1}\alpha(H)\gamma, R_H(U))],
\]

where in the last line we plugged in (3.1) and (3.2). To rewrite into the form as stated, note that $\alpha(H)$ is a coset of the form $\alpha(H) = R_H(U)g|_U$, and $\gamma^{-1}\alpha(H)\gamma = \gamma^{-1}R_H(U)\gamma(\gamma^{-1}g)|_U = R_H(U)(\gamma^{-1}g)|_U$. The claim follows from uniqueness of regular conditional distribution.

With the lemma we return to the proof of the second claim of Proposition 3.1 part (ii). Given $\sigma \in \pi_U(G_{U \to U})$ with $\mu(A_\sigma) > 0$, by Lemma 3.3 for $\mu$-a.e. $H \in A_\sigma$, the conditional distribution $\mathbb{P}_{\sigma,U}^\mu (R_H(U), \cdot)$ is invariant under conjugation by $N = N_{R_G(U)}(R_H(U))$. Since the support of $\mathbb{P}_{\sigma,U}^\mu (R_H(U), \cdot)$ is countable, we have that the collection of cosets $\{R_H(U)(\gamma^{-1}g)|_U, \gamma \in N\}$ under conjugation by $N$ must be finite if $\mathbb{P}_{\sigma,U}^\mu (R_H(U), \{(R_H(U)g|_U, R_H(U))\}) > 0$. In other words, write

\[
\text{IC}(R_H(U)) := \{(R_H(U)g, R_H(U)) : g \in G_{U \to U}, |\text{Cl}_N (R_H(U)g)| = \infty\},
\]

where Cl$_N (R_H(U)g) = \{R_H(U)\gamma^{-1}g\gamma : \gamma \in N\}$.

Then the invariance property in Lemma 3.3 implies that $\mathbb{P}_{\sigma,U}^\mu (R_H(U), \text{IC}(R_H(U))) = 0$ for $\mu_{A_\sigma}$-a.e. $H$.

Let

\[
E_\sigma = \{H \in A_\sigma : \text{the } N_{R_G(U)}(R_H(U))-\text{conjugacy class of } \alpha(H) \text{ is infinite}\}.
\]

That is, $E_\sigma$ is the event that in the quotient group $\tilde{H}_{U \to U}/R_H(U)$, the element $\alpha(H)$ has infinite $N$-conjugacy class $\{\gamma^{-1}\alpha(H)\gamma, \gamma \in N\}$. The reasoning above
implies that $E_\sigma$ is a $\mu$-null set. Indeed, for $\sigma$ such that $\mu(A_\sigma) > 0$, by property (iii) of regular conditional probability, we have

$$\mu_{A_\sigma}(E_\sigma) = E_{\mu_{A_\sigma}} \left[ \mathbb{1}_{\sigma,\mu}(R_H(U), \mathbb{1}_{\mu}(R_H(U))) \right] = 0.$$ 

For $\sigma$ such that $\mu(A_\sigma) = 0$, we have $\mu(E_\sigma) = \mu(A_\sigma) = 0$.

Thus if $H \in \text{Sub}(G) - \bigcup_{\sigma \in \pi_U \cdot (G_U \setminus U)} E_\sigma$, we have that for any $\sigma \in \pi_U \cdot (G_U \setminus U)$, either $\sigma \not\in \pi_U \cdot (H_{U \rightarrow U})$ or the $N$-conjugacy orbit $\{ \gamma^{-1} \alpha_\sigma(H) \gamma \}_{\gamma \in N}$ is finite, where $N = N_{RG(U)}(R_H(U))$. Recall that by Fact 3.2 there is an isomorphism $\phi : \pi_U \cdot (H_{U \rightarrow U})/H_{U \rightarrow U} \cap R_G(G_U) \to \pi_U \cdot (H_{U \rightarrow U})/R_H(U)$ where $\phi(\bar{\sigma}) = \alpha_\sigma(H)$ for $\sigma \in \pi_U \cdot (H_{U \rightarrow U})$. In particular, since $\phi$ is onto, as sets we have

$$\bigcup_{\sigma \in \pi_U \cdot (G_U \setminus U)} \alpha_\sigma(H) = H_{U \rightarrow U}/R_H(U).$$ 

It follows that for $H \in \text{Sub}(G) - \bigcup_{\sigma \in \pi_U \cdot (G_U \setminus U)} E_\sigma$, in the quotient group $H_{U \rightarrow U}/R_H(U)$, every element has finite $N$-conjugacy orbit. Recall that by definition of $K$ we have that $K = N \cap H_{U \rightarrow U}$ and $R_H(U) = H \cap K$. Since the $N$-conjugacy orbits in $H_{U \rightarrow U}/R_H(U)$ are finite, it follows that the quotient group $K/H \cap K$ is FC. We have proved that the statement of part (ii) holds for any $H$ not in the $\mu$-null set $\bigcup_{\sigma \in \pi_U \cdot (G_U \setminus U)} E_\sigma$. 

\[ \square \]

3.2. Containment of derived subgroups of rigid stabilizers. Proposition 3.1 allows us to draw conclusions on subgroups that $H$ contains, provided some additional information on the rigid stabilizers $R_G(U)$.

We first deduce from Proposition 3.1 the following immediate corollary when every finite index subgroup of $G_U$ contains an infinite simple group. This statement will be used in applications to classify IRS of certain topological full groups.

Corollary 3.4. Suppose $G \curvearrowright X$ as in the Standing Assumption. Suppose an open set $U \subset X$ has the property that there is an infinite simple group $A_U$ such $A_U \leq R_G(U)$ and every finite index subgroup of $R_G(U)$ contains $A_U$. Then $\mu$-a.e. $H$ has the following property: if $H$ contains an element $g$ such that $U \cap U \cdot g = \emptyset$, then $H$ contains $A_U$ as a subgroup.

Proof. By part (ii) of Proposition 3.1 for $\mu$-a.e. $H$ in $\Omega_g = \{ H : H \ni g \}$, where $g$ is an element such that $U \cap U \cdot g = \emptyset$, there is a finite index subgroup $K \leq_{f.i.} R_G(U)$ such that $K/K \cap H$ is an FC group. By the assumption on $G_U$ we have that $A_U \leq K$. Then $A_U/A_U \cap H$ is isomorphic to $A_U/(K \cap H)/K \cap H$, in particular it is an FC group. Since $A_U$ is assumed to be an infinite simple group, we conclude that $A_U \cap H = A_U$, in other words, $A_U$ is contained in $H$. 

\[ \square \]

More generally we have the following lemma, which is a variation of the standard double commutator lemma mentioned in the Introduction.

Lemma 3.5. Let $G \curvearrowright X$ be as in the Standing Assumption. Let $U \in \mathcal{U}$, $K \leq_{f.i.} R_G(U)$ and $N \lhd K$ be such that the quotient $K/N$ is an FC group. Then if $x \in U$ is a point such that the orbit $x \cdot R_G(V)$ is infinite for any $V \in \mathcal{U}$ with $x \in V \subsetneq U$, then there exists a neighborhood $W$ of $x$ such that $[R_G(W), R_G(W)] \leq N$.

Proof. We will use a few times the double commutator lemma ([Nek1Sa Lemma 4.1]). We recall the argument here for the convenience of the reader: suppose
\(\Gamma \curvearrowright X\) and let \(N\) be a non-trivial normal subgroup of \(\Gamma\). If there is an element \(\gamma \in N\) and a set \(U\) such that \(U \cap U \cdot \gamma = \emptyset\), then \([R_1(U), R_1(U)] \leq N\). Indeed, given such \(\gamma, U\), take any elements \(\alpha, \beta \in \Gamma_U\), then \([[\gamma, \alpha], \beta] = [\alpha, \beta]\) and since \(N\) is normal, we have \([\gamma, \alpha], \beta \in N\). It follows that \([R_1(U), R_1(U)] \leq N\).

Replacing \(K\) by its normal core in \(R_G(U)\) if necessary, we may assume \(K\) is a normal subgroup of finite index in \(R_G(U)\). We first verify that \(K/N\) being an FC-group implies that \(x\) is not a fixed point of \(N\). Suppose on the contrary that \(x\) is fixed by \(N\). Since the orbit \(x \cdot R_G(U)\) is assumed to be infinite, \(x\) cannot be fixed by \(K\). Choose a \(\gamma \in K\) such that \(x \cdot \gamma \neq x\). Let \(V\) be a neighborhood of \(x\) such that \(V \cap V \cdot \gamma = \emptyset\). Then for any \(g \in R_G(V)\), the conjugation \(g \gamma g^{-1}\) coincides with \(g \gamma\) on \(V\). In particular the set \(\{x \cdot g \gamma g^{-1} : g \in R_K(V)\}\) has the same cardinality as \(\{x \cdot g : g \in R_K(V)\}\), where the latter is infinite by assumption. It follows that in \(K/N\) the element \(N\gamma\) has infinite conjugacy class: since \(x\) is fixed by \(N\), \(g_1\gamma g_1^{-1}\) and \(g_2\gamma g_2^{-1}\) are in different right cosets of \(N\) if \(x \cdot g_1 \neq x \cdot g_2\). Thus it contradicts with \(K/N\) being FC.

Since \(x\) is not fixed by \(N\), by the double commutator lemma for normal subgroups, we conclude that there exists a neighborhood \(W\) of \(x\) such that \(N \geq [R_K(W), R_K(W)]\). Finally, note that since \(K\) is normal in \(R_G(U)\), we have that \([R_K(W), R_K(W)]\) is normal in \(R_G(W)\). Since \(R_G(W)/[R_K(W), R_K(W)]\) is virtually abelian, thus FC, the argument in the paragraph above implies that there exists a neighborhood \(W'\) of \(x\) such that \([R_G(W'), R_G(W')] \subseteq [R_K(W), R_K(W)] \leq N\).

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \(\Omega' \subseteq \text{Sub}(G)\) be a subset of full \(\mu\)-measure such that for any \(H \in \Omega'\), the two statements of Proposition 3.1 are satisfied. Let \(H \in \Omega'\) and \(x \in X\) be a point such that \(x\) is not a fixed point of \(H\). Find a neighborhood \(x \in U \subseteq \mathcal{U}\) and \(h \in H\) such that \(U \cap U \cdot h = \emptyset\). Then as in (ii) of Proposition 3.1, we have that \(K = N_{R_G(U)}(R_H(U))\) is of finite index in \(R_G(U)\) and \(K/K \cap H\) is FC.

(i) If there exists an open set \(U\) such that \(R_G(U) = \{id\}\), then the statement is trivially true. We may assume that for any \(U \in \mathcal{U}\), \(R_G(U)\) is non-trivial. Then in this case the rigid stabilizer \(R_G(U)\) is ICC for any \(U\). To see this, we follow the argument in [Gri11, Theorem 9.17] which shows that a weakly branch group is ICC. Take \(\gamma \in R_G(U)\), \(\gamma \neq id\), and choose an open subset \(V \subseteq U\) such that \(V \cap V \cdot \gamma = \emptyset\).

Next choose an infinite sequence of mutually disjoint open subsets \(V_1, V_2, \ldots\) of \(V\) (this is possible because under our assumptions \(X\) is Hausdorff and has no isolated points). Take \(g_i \in R_G(V_i) \setminus \{id\}\) for each \(i\) and consider the collection of conjugates \(\{g_i \gamma g_i^{-1}\}_{i \in \mathbb{N}}\). The restriction of \(g_i \gamma g_i^{-1}\) to \(V\) acts as \(g_i\) in \(V_i\) and as \(id\) on \(V \setminus V_i\). Thus the set of elements \(\{g_i \gamma g_i^{-1}\}_{i \in \mathbb{N}}\) are all distinct. We conclude that \(R_G(U)\) is ICC.

Since \(K\) is a finite index subgroup of \(R_G(U)\), we have that it is ICC as well, thus \(K \cap H\) must be a non-trivial normal subgroup of \(K\). Thus the double commutator lemma implies that there exists a non-empty open subset \(V \subseteq U\) such that \([R_K(V), R_K(V)] \leq K \cap H\). Since \(K\) is of finite index in \(R_G(U)\), we have \([R_K(V), R_K(V)]\) contains a non-trivial normal subgroup of \(R_G(V)\). Therefore by the double commutator lemma again, there exists a non-empty open set \(W \subseteq V\) such that 

\[ [R_G(W), R_G(W)] \leq [R_K(V), R_K(V)] \leq H \cap K \leq H. \]
(ii) Under the additional assumption that $R_G(V)$ has no fixed point in $V$ for any $V \in \mathcal{U}$, we have that the orbit $x \cdot R_G(V)$ is infinite for any $V \in \mathcal{U}$ and $x \in V$. Indeed, first this assumption implies that $X$ has no isolated point. For any $x \in V$, first find $g_1 \in R_G(V)$ such that $x \neq x \cdot g_1$. Since $X$ is Hausdorff, we can find a neighborhood $V_1$ of $x$ such that $V_1 \cap V_1 \cdot g = \emptyset$. Continue this with choosing a $g_2 \in R_G(V_1)$ such that $x \neq x \cdot g_2$ and $V_2$ a neighborhood of $x$ such that $V_2 \cap V_2 \cdot g_2 = \emptyset$. In this way we find a sequence of group elements $g_n \in R_G(V)$ such that $x \cdot g_n$ are all distinct.

Thus we can apply Lemma 3.5 to $N = K \cap H$, which implies that these exists an open cover $V_i$ of $U$ such that $[R_G(V_i), R_G(V_i)] \leq H$ for all $V_i$. Take $W$ to be a $V_i$ such that $x \in V_i$, then we obtain the statement.

Remark 3.6. The condition that $R_G(U)$ has no fixed point in $U$ for any $U \in \mathcal{U}$ is verified in many situations. Note that if $x \in U$ is a fixed point of $R_G(U)$, then for any $g \in R_G(U)$, $\gamma \in N_G(R_G(U))$, we have $x \cdot \gamma g = x \cdot \gamma$. That is, the orbit of $x$ under $N_G(R_G(U))$ is contained in the fixed point set of $R_G(U)$. Therefore if the orbit $x \cdot G_{U \rightarrow U}$ is dense in $U$ and $R_G(U) \neq \{id\}$, then $x$ cannot be a fixed point of $R_G(U)$.

4. Preliminaries on topological full groups

This section serves as preparation for the next section where we derive consequences of Theorem 4.2 for topological full groups. Most of the materials are drawn from [Nek17]. The reader may also consult the survey [Mat16] and references therein to have a more complete view of recent development in the study of topological full groups.

4.1. Definitions. A groupoid consists of a unit space $G^{(0)}$, a set of morphisms $G$ and maps $s, r : G \rightarrow G^{(0)}$ called the source and range that specify the initial (source) and terminal (range) of a morphism. Multiplication in $G$ is partially defined: product $\gamma \delta$ of $\gamma, \delta \in G$ is defined if and only if $r(\gamma) = s(\delta)$. A topological groupoid is a groupoid equipped with a topology on it such that the operations of multiplication and taking inverse are continuous.

An important class of examples of groupoid is the transformation groupoid of an action $G \curvearrowright X$:

Example 4.1. Let $G$ be a countable group acting by homeomorphisms on a topological space $X$. Define an equivalence relation on $G \times X$ by $(g_1, x_1) \sim (g_2, x_2)$ if $x_1 = x_2$ and there exists a neighborhood $U$ of $x_1$ such that $g_1|_U = g_2|_U$. On the quotient space $G \times X/\sim$ (the space of germs), multiplication of equivalence classes $(g_1, x_1), (g_2, x_2)$ is defined if and only if $x_2 = x_1 \cdot g_1$ and the rule of multiplication is $(g_1, x_1)(g_2, x_2) = (g_1g_2, x_1)$. The groupoid $G = G \times X/\sim$ is called the groupoid of germs of the action $G \curvearrowright X$, it is also called the transformation groupoid of the action. For an open set $U \subseteq X$, the set of germs $\{(g, y), y \in U\}$ is an open set of $G$. The collection of all such open sets forms a basis of topology of $G$.

More generally one can consider groupoid of germs (also called effective étale groupoids), for which we now recall the definition.

Definition 4.2. Let $G$ be a topological groupoid.

- A $G$-bisection is a compact open subset $F \subseteq G$ such that $s : F \rightarrow s(F)$ and $r : F \rightarrow t(F)$ are homeomorphisms.
• The groupoid \( \mathcal{G} \) is said to be \( \text{étale} \) if it has a basis of topology consisting of \( \mathcal{G} \)-bisections.
• An \( \text{étale} \) groupoid \( \mathcal{G} \) is called a \textit{groupoid of germs} (or \textit{effective}) if for any non-unit \( \gamma \in \mathcal{G} \) and any bisection \( F \) that contains \( \gamma \), there exists \( \delta \in F \) such that \( s(\delta) \neq r(\delta) \).

A bisection \( F \) defines a homeomorphism from \( s(F) \) to \( r(F) \), denote by \( \tau_F \) this homeomorphism. Multiplication of bisections corresponds to composition of the associated homeomorphisms. For an \( \text{étale} \) groupoid \( \mathcal{G} \), its topological full group is defined as:

\textbf{Definition 4.3.} Let \( \mathcal{G} \) be an \( \text{étale} \) groupoid. Its \textit{topological full group} \( \mathcal{F}(\mathcal{G}) \) is the set of bisections \( F \subset \mathcal{G} \) such that \( s(F) = r(F) = \mathcal{G}^{(0)} \) with respect to multiplication of bisections.

\textbf{Example 4.4.} Let \( \mathcal{G} \) be the transformation groupoid of the action \( G \curvearrowright X \), where \( G \) is countable and \( X \) is homeomorphic to the Cantor set. Then the topological full group \( \mathcal{F}(\mathcal{G}) \) can be described as consisting of all homeomorphisms \( \gamma : X \to X \) such that for every point \( x \in X \), there is a neighborhood \( U \) and \( g \in G \) such that \( \gamma|_U = g|_U \). In other words, \( \mathcal{F}(\mathcal{G}) \) consists of homeomorphisms \( \gamma : X \to X \) such that there exists a continuous map \( c : X \to G \) such that \( x \cdot \gamma = x \cdot c(\gamma) \) for all \( x \).

More generally, when \( \mathcal{G} \) is a groupoid of germs, \( \mathcal{F}(\mathcal{G}) \) agrees with the definition of topological full group in \textbf{[Mat12, Mat15]}: it consists of all homeomorphisms \( \gamma : \mathcal{G}^{(0)} \to \mathcal{G}^{(0)} \) such that there exists a bisection \( F \subset \mathcal{G} \) such that \( \gamma = \tau_F \).

The notion of multisectsions is introduced in \textbf{Nek17}. A \textit{multisection of degree} \( d \) is a collection of \( d^2 \) bisections \( F = \{F_{i,j}\}_{i,j=1}^d \) such that

(i): \( F_{i_1,i_2}F_{i_2,i_3} = F_{i_1,i_3} \) for all \( 1 \leq i_1, i_2, i_3 \leq d \),

(ii): the bisections \( F_{i,i} \) are disjoint subsets of the unit space \( X = \mathcal{G}^{(0)} \).

The union \( \cup_{i=1}^d F_{i,i} \) is called the \textit{domain} of the multisection and the sets \( F_{i,i} \) the \textit{components} of the domain. It follows from properties (i), (ii) that \( s(F_{i,j}) = F_{i,i} \) and \( r(F_{i,j}) = F_{j,j} \) for \( 1 \leq i, j \leq d \).

Given a multisection \( F = \{F_{i,j}\}_{i,j=1}^d \) and a permutation \( \pi \in \text{Sym}(d) \), let \( F_\pi := \cup_{i=1}^d F_{i,\pi(i)} \cup (X \setminus U) \),

where \( U \) is the domain of \( F \). Then \( \pi \mapsto F_\pi \) gives an embedding of \( \text{Sym}(d) \) into \( \mathcal{F}(\mathcal{G}) \). Denote by \( S(F) \) the image of \( \text{Sym}(d) \) under this embedding, and \( A(F) \) the image of the subgroup \( \text{Alt}(d) \).

Now we have introduced enough notations to state the definitions of the groups \( S(\mathcal{G}) \) and \( A(\mathcal{G}) \).

\textbf{Definition 4.5.} Let \( \mathcal{G} \) be an \( \text{étale} \) groupoid. The group \( S_d(\mathcal{G}) \) (\( A_d(\mathcal{G}) \) resp.) is defined as the subgroup of \( \mathcal{F}(\mathcal{G}) \) which is generated by the union of all subgroups \( S(F) \), all multisections of degree \( d \). Denote by \( S(\mathcal{G}) = S_2(\mathcal{G}) \) and \( A(\mathcal{G}) = A_3(\mathcal{G}) \).

From the definitions it is clear that if \( \mathcal{H} \) is an open sub-groupoid of an \( \text{étale} \) groupoid \( \mathcal{G} \) then \( S(\mathcal{H}) \leq S(\mathcal{G}) \) and \( A(\mathcal{H}) \leq A(\mathcal{G}) \).

A Borel probability measure \( \mu \) on the unit space \( \mathcal{G}^{(0)} \) is said to be \( \mathcal{G} \)-invariant if \( \mu(\tau(F)) = \mu(s(F)) \) for any bisection \( F \).

We will only consider the special case where \( \mathcal{G} \) is a minimal groupoid of germs. Recall that a groupoid \( \mathcal{G} \) is \textit{minimal} if all orbits are dense in \( \mathcal{G}^{(0)} \). By \textbf{Nek17}
Theorem 1.1, if $G$ is a minimal groupoid of germs with $G^{(0)}$ homeomorphic to the Canter set, then $A(G)$ is simple and contained in every non-trivial normal subgroup of $F(G)$.

When $G$ is an AF-groupoid (AF is abbreviation for approximately finite), the topological full group $F(G)$ and the alternating full group $A(G)$ are well studied, see [Mat12]. AF-groupoids are associated with Bratteli diagrams. In the following example we review necessary terminologies and introduce notations for later use.

**Example 4.6.** A Bratteli diagram is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i=0}^{\infty} V_i$ and the edge set $E = \bigcup_{i=1}^{\infty} E_i$ are partitions such that $V_0 = \{v_0\}$ is a single point, $V_i$ and $E_i$ are finite sets, and moreover there are range maps $r$ and source maps $s$ from $E$ to $V$ such that $r(E_i) = V_i$ and $s(E_i) = V_{i-1}$. That is an edge $e \in E$ connects a vertex in $V_{i-1}$ to a vertex in $V_i$ for some $i$. The set $V_i$ is referred to as the $i$-th level vertices of the diagram $B$.

The space of infinite paths of $B$, denoted by $X_B$, consists of sequences $(e_1, e_2, \ldots)$ of edges such that $r(e_i) = s(e_{i+1})$ for all $i$. A finite path is a finite sequence $(e_1, \ldots, e_n)$, such that $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n$. The vertex $s(e_1)$ is called the beginning of the path and $r(e_n)$ the end of the path. Endow $X_B$ with the product topology generated by cylinder sets $U(e_1, \ldots, e_n) = \{x \in X_B : x_i = e_i, 1 \leq i \leq n\}$.

If two finite paths $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ have the same end, $r(e_n) = r(f_n)$, then this pair defines a homeomorphism between clopen subsets of $X_B$ mapping a path of the form $(e_1, \ldots, e_n, x_{n+1}, \ldots)$ to $(f_1, \ldots, f_n, x_{n+1}, \ldots)$. Let $G$ be the groupoid of germs of the semigroup generated by such local homeomorphisms. Such a groupoid associated with a Bratteli diagram is called an AF groupoid. By slight abuse of notation, we write $F(B)$ for the topological full group of the groupoid associated with the Bratteli diagram $B$, similarly for $S(B)$ and $A(B)$.

For a vertex $v \in V$, let $E(v_0, v)$ be the space of finite paths that start at $v_0$ and end at $v$. Let $d(v_0, v)$ be the cardinality of $E(v_0, v)$. From definitions, the topological full group of the Bratteli diagram $B$ is the direct limit of $G_n = \prod_{v \in V_n} \text{Sym}(E(v_0, v)) \cong \prod_{v \in V_n} \text{Sym}_{d(v_0, v)}$. The alternating full group $A(B)$ is the direct limit of $\prod_{v \in V_n} \text{Alt}(E(v_0, v)) \cong \prod_{v \in V_n} \text{Alt}_{d(v_0, v)}$.

For an infinite path $x \in X_B$, the $G$-orbit of $x$ consists of all infinite paths cofinal to $x$. The groupoid $G$ associated with the Bratteli diagram $B$ is minimal on $X_B$ if and only if the diagram $B$ is simple, which means that for any level $V_n$, there exists a level $m > n$ such that every pair of vertices $(v, w) \in V_n \times V_m$ is connected by a finite path, see for example [Put18] Theorem 3.11.

### 4.2. Rigid stabilizers.

For an open set $U \subseteq G^{(0)}$, denote by $G|_U$ the restriction of $G$ to $U$, which is the sub-groupoid

$$G|_U := \{ \gamma \in G : s(\gamma), r(\gamma) \in U \}.$$  

Similarly one can consider the alternating full group $A(G|_U)$, which by definition is generated by the union of $A(F)$, where $F$ ranges over multisections of degree $d \geq 3$ in the sub-groupoid $G|_U$. The group $A(G|_U)$ is naturally identified as a subgroup of $A(G)$ by extending elements of $A(G|_U)$ identically on the complement of $U$. We have the following identification:

**Fact 4.7.** Let $G$ be a groupoid of germs and $U$ be an open subset of $X = G^{(0)}$, then

$$A(G|_U) = R_{A(G)}(U).$$
Proof. The inclusion $A(G|_U) \leq R_{A(G)}(U)$ is clear by definitions. We need to verify the other direction. Let $g = F_\pi$ be an element in $R_{A(G)}(U)$, where $F = \{F_{i,j}\}_{i,j=1}^d$ is a multisection and $\pi \in \text{Alt}_d$.

Given the multisection $F = \{F_{i,j}\}_{i,j=1}^d$, one can define a sub-multisection $\{F_{k_i,k_j}\}_{i,j=1}^\ell$ by choosing a subset $1 \leq k_1 < \ldots < k_\ell \leq d$ and restrict to these indices. It is clear by definition that $\{F_{k_i,k_j}\}_{i,j=1}^\ell$ is a multisection of degree $\ell$. Let $\pi$ be a permutation of $\{1, \ldots, d\}$ and denote by $\text{Fix}(\pi) = \{i : 1 \leq i \leq d, \ i = \pi(i)\}$ the set of fixed points of $\pi$. List elements of $Q_\pi = \{1, \ldots, d\} \setminus \text{Fix}(\pi)$ in increasing order as $k_1, \ldots, k_\ell$, $\ell = d - |\text{Fix}(\pi)|$. Denote by $F'$ the sub-multisection $\{F_{k_i,k_j}\}_{i,j=1}^\ell$. Let $\pi'$ be the restriction of $\pi$ on $Q_\pi$. Then by definition in Lemma 4.8 we have

$$F_\pi = F'_\pi.$$ 

That is, for the group element $F_\pi \in F(G)$, one can remove the components $F_{i,i}$ in $F$ with $\pi(i) = \pi$ and consider it as $F'_\pi$, where every component in the domain is mapped to some other component.

Restrict to the sub-multisection $F' = \{F_{k_i,k_j}\}_{i,j=1}^{\ell}$, where $\ell = d - |\text{Fix}(\pi)|$, then $g = F'_\pi, \ \pi'(k_i) = \pi(k_i) \neq k_i$. Since the components of the domain are by definition disjoint, $\pi'(k_i) \neq k_i$ implies that for any point $x \in F_{k_i,k_j}, \ x \cdot F'_\pi \neq x$. Since $g \in R_{A(G)}(U)$ fixes $U^c$ pointwise, it follows that $(\bigcup_{i=1}^d F_{k_i,k_j}) \cap U^c = \emptyset$. In other words, the domain of the multisection $F'$ is contained in $U$. By definition of the restriction $G|_U$ we have that $A(F') \leq A(G|_U)$, in particular, $g \in A(G|_U)$.

\[\square\]

Similar to the alternating group on a finite set, we have the following property. When $G$ is a minimal AF-groupoid, this is shown in [DM17, Lemma 3.4].

**Lemma 4.8.** Suppose $G$ is a minimal groupoid of germs with unit space $X$ homeomorphic to the Cantor set. Then if $U$ and $V$ are clopen sets such that $U \cap V \neq \emptyset$, we have

$$R_{A(G)}(U \cup V) = \langle R_{A(G)}(U), R_{A(G)}(V) \rangle.$$

The proof of Lemma 4.8 relies on the following fact, which is a direct consequence of [Nek17, Lemma 3.2, Proposition 3.3]. Let $F = \{F_{i,j}\}_{i,j=1}^d$ be a multisection and $U$ be a clopen subset of $F_{k,k}$ for some $k \in \{1, \ldots, d\}$. Write $U_i = \tau(UF_{k,i})$ and $F_{i,j} = U_i F_{i,j}$. Then $F' = \{F'_{i,j}\}_{i,j=1}^{\ell}$ is a multisection, which is referred to as the restriction of $F$ to $U$.

**Fact 4.9.** Let $G$ be a minimal groupoid of germs with unit space $X = G^{(0)}$ homeomorphic to the Cantor set. Let $F = \{F_{i,j}\}_{i,j=1}^d$ be a multisection and $\mathcal{P} = \{U_j\}_{j=1}^q$ an open cover of the domain of $F$. Then there is a collection of multisections $F^{(k)} = \{F^{(k)}_{i,j}\}_{i,j=1}^{d_k}$, $1 \leq k \leq \ell$ such that:

(i): each component $F^{(k)}_{i,i}$ is contained in $U_j$, for some $j \in \{1, \ldots, q\}$;
(ii): for each $k$ and each set $U_j$, $\left|\{i : F^{(k)}_{i,i} \subseteq U_j\}\right| \geq 3$;
(iii): $A(F)$ is contained in the group generated by $\cup_{k=1}^\ell A(F^{(k)})$.

**Proof of Lemma 4.8.** Let $F = \{F_{i,j}\}_{i,j=1}^d$ be a multisection of degree $d$ with domain contained $U \cup V$. If the domain of $F$ is contained in $U$ (resp. $V$), then by definitions we have $A(F) \leq R_{A(G)}(U)$ (resp. $R_{A(G)}(V)$). We need to consider the case where domain of $F$ intersects both $U, V$. Partition $U \cup V$ into $U \cap V^c, U \cap V$ and $V \cap U^c$.  

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Let $F^{(k)}$, $1 \leq k \leq \ell$, be a collection of multisections satisfying the statement of Fact 4.9. We show that $A\left(F^{(k)}\right)$ is contained in $\langle R_{A(G)}(U), R_{A(G)}(V) \rangle$. To see this, let $I_i = \{i : F_{i,i} \subseteq U\}$ and let $F_U^{(k)}$ be the sub-multisection $\left\{ F_{i,j}^{(k)} \right\}_{i,j \in I_i}$. Since by definition the domain of $F_U^{(k)}$ is contained in $U$, we have that $A\left(F_U^{(k)}\right) \subseteq R_{A(G)}(U)$.

Similarly we take the sub-multisection $F_V^{(k)}$ whose domain is contained in $V$. By property (ii) in Lemma 4.9, there are at least 3 components of $F^{(k)}$ contained in the intersection $U \cap V$. Recall the elementary fact that if $X$ and $Y$ are two finite sets such that $|X|, |Y| \geq 3$ and $X \cap Y \neq \emptyset$, then the alternating group $Alt(X \cup Y)$ is generated by $Alt(X)$ and $Alt(Y)$. It follows that

$$A\left(F^{(k)}\right) = A\left(F_U^{(k)}\right), A\left(F_V^{(k)}\right) \subseteq \langle R_{A(G)}(U), R_{A(G)}(V) \rangle.$$  

We conclude that $A\left(F\right) \subseteq \langle R_{A(G)}(U), R_{A(G)}(V) \rangle$.

\[ \square \]

5. APPLICATIONS TO CLASSIFICATION OF IRSs OF TOPOLOGICAL FULL GROUPS

In this section we consider invariant random subgroups of topological and alternating full groups of a minimal groupoid of germs $G$.

5.1. Consequences of Corollary 3.4. With the alternating full group $A(G)$ in mind, we derive the following statement from Corollary 3.4. Recall that $F(X)$ denotes the space of closed subsets of $X$, equipped with the Vietoris topology.

**Proposition 5.1.** Let $G$ be a countable group acting faithfully on the Cantor set $X$ by homeomorphisms. Let $U$ be the collection of non-empty clopen subsets of $X$. Suppose there is a collection of infinite simple groups $\{A_U\}_{U \in U}$ such that each $A_U \leq R_G(U)$ and $A_U$ has no fixed point in $U$.

Suppose in addition:

(i): all finite index subgroups of $R_G(U)$ contain $A_U$;

(ii): the collection of groups $A_U$ satisfies that for any $U, V \in U$ such that $U \cap V \neq \emptyset$,

$$A_{U \cup V} = \langle A_U, A_V \rangle;$$

(iii): any ergodic $G$-invariant probability measure on the space $F(X)$ other than $\delta_X$ is supported on the subspace of closed subsets with empty interior.

Let $\mu$ be an ergodic IRS of $G$, then $\mu$-a.e. $H$ satisfies that

$$H \geq \langle \cup \{A_U : U \in U \text{ and } U \subseteq X \setminus \text{Fix}(H)\} \rangle.$$

**Proof.** Define $V_H$ to be the collection

$$V_H := \{V \subseteq X : V \text{ is open and any clopen subset } W \text{ of } V \text{ satisfies } A_W \leq H\}.$$  

In other words, an open set $V$ is in $V_H$ if and only if it can be written as an increasing union of clopen sets $\bigcup_n W_n$ where $A_{W_n} \leq H$ for each $n$. Equip $V_H$ with the partial order of set inclusions. By definition it is clear that if $\{V_i\}_{i \in I}$ is a chain in $V_H$, then the union $\cup_{i \in I} V_i$ is in $V_H$. Assumption (ii) implies that if $V$ and $V'$ are in $V_H$ and $V \cap V' \neq \emptyset$, then $V \cup V' \in V_H$ as well.

Let $M_H = \{M_j\}_{j \in J}$ be the collection of maximal elements in $V_H$. Note that any two distinct maximal subsets $M_j$ and $M_j'$ must be disjoint, otherwise $M_j \cup M_j'$ would be in $V_H$ which contradicts the maximality of $M_j, M_j'$. Since $U$ is countable, as a consequence of disjointness, we have that the index set $J$ is countable. For
Lemma 5.2. Let \( o \) ordering of \( M \) \( \nu \) equipped with the topology of weak convergence. The map \( \text{Sub}(\cdot) \) be the map defined by \( (x_j) \mapsto X \setminus \bigcup \{M_j : x_j = 1, j \in \mathbb{N}\} \). Recall that \( F(X) \) is equipped with the Vietoris topology. It is easy to verify that the map \( \tau_M \) is Borel measurable. Let \( p = 1/2, \nu \) be the Bernoulli distribution on \( \{0, 1\} \), \( \nu(\{0\}) = \nu(\{1\}) = 1/2 \), and \( \nu^\otimes \) the product measure on \( \{0, 1\}^\mathbb{N} \). Denote by \( \eta_M \) the pushforward of \( \nu^\otimes \) under \( \tau_M \). Note that the measure \( \eta_M \) does not depend on ordering of \( M \). It is routine to check measurability:

**Lemma 5.2.** Let \( \mathcal{P}(F(X)) \) be the space of Borel probability measures on \( F(X) \), equipped with the topology of weak convergence. The map \( \text{Sub}(G) \rightarrow \mathcal{P}(F(X)) \) given by \( H \mapsto \eta_{M_H} \) is Borel measurable.

**Proof of Lemma 5.2** Note that the Borel \( \sigma \)-field on \( F(X) \) is generated by sets \( \{C \in F(X) : C \subseteq U\} \), where \( U \) goes over clopen subsets of \( X \). Thus it suffices to check for any clopen set \( U \) and \( x \in [0, 1] \), the set \( \{H : \eta_{M_H}(\{C \in F(X) : C \subseteq U\} \geq x\} \) is a measurable subset of \( \mathcal{B}(G) \).

Consider all finite cover of \( U^c \) by disjoint clopen sets. This is a countable list, denote it by \( O_1, O_2, \ldots \). By definition of \( \eta_{M_H} \), we have

\[
\{H : \eta_{M_H}(\{C \in F(X) : C \subseteq U\} \geq x\} = \{H : \exists \text{ sub-collection } \{M_{j_k}\}_{k=1} \text{ of } M_H \text{ that covers } U^c \text{ and } 2^{-\ell} \geq x \}
\]

\[
= \bigcup_{i : |O_i| \leq -\log_2 x} \{H : A_V \leq H \text{ for every } V \in O_i\},
\]

where \( |O_i| \) denotes the number of sets in the cover \( O_i \). Since the set \( \{H : A_V \leq H \} \) is measurable, we conclude that \( \{H : \eta_{M_H}(\{C \in F(X) : C \subseteq U\} \geq x\} \) is measurable.

\[ \square \]

We continue the proof of the proposition. Take the probability measure on \( F(X) \) given by

\[
\nu = \int_{\text{Sub}(G)} \eta_{M_H} d\mu(H).
\]

Note that for \( g \in G \), the decomposition associated with the conjugation \( g^{-1} H g \) is \( (\text{Fix}(H) \cdot g, \{M_j \cdot g\}_{j \in \mathbb{N}}) \). Since \( \mu \) is invariant under conjugation, it follows that the measure \( \nu \) is a \( G \)-invariant probability measure on \( F(X) \):

\[
\nu \cdot g = \int_{\text{Sub}(G)} (\eta_{M_H} \cdot g) d\mu(H) = \int_{\text{Sub}(G)} (\eta_{M_H} \cdot g) d\mu(g^{-1} H g) = \int_{\text{Sub}(G)} (\eta_{M_{g^{-1} H g}}) d\mu(g^{-1} H g) = \nu.
\]

By assumption (i) on the infinite simple groups \( A_U \) and Corollary 3.4, we have that for \( \mu \) a.e. \( H \), there exists a cover \( \{U_i\}_{i \in I} \) of \( X - \text{Fix}_H \) by clopen sets such that \( A_{U_i} < H \) for every \( i \in I \). In particular each \( U_i \in \mathcal{V}_H \). The assumption that \( A_U \)
has no fixed point in \(U\) implies that a set in \(V_H\) must be disjoint from \(\text{Fix}(H)\). It follows that for \(\mu\text{-a.e.} \ H, \mathcal{M}_H\) forms a partition of \(X \setminus \text{Fix}(H)\) into disjoint open sets. Next we show that under assumption (iii), for \(\mu\text{-a.e.} \ H,\) the partition \(\mathcal{M}_H\) consists of one nonempty set, namely \(X \setminus \text{Fix}(H)\).

Consider the event \(D = \{H: H \geq A_U\text{ for all clopen } U \subseteq X \setminus \text{Fix}(H)\}\). Since \(D\) can be expressed as

\[
D = \cap_{U \in \mathcal{U}} (\{H: \text{Fix}(H) \cap U \neq \emptyset\} \cup \{H: A_U \leq H\}),
\]

it is indeed a measurable subset of \(\text{Sub}(G)\). By definition, it is clear that the set \(D\) is invariant under conjugation by \(G\). Ergodicity of \(\mu\) implies \(\mu(D) \in \{0, 1\}\). Note that \(\mu(D) = 1\) is exactly the statement of the proposition. We need to rule out the possibility that \(\mu(D) = 0\). Suppose \(\mu(D) = 0\), then it means \(\mu\text{-a.e. the partition } \mathcal{M}_H\) has at least two non-empty open sets. It follows that

\[
\nu(\{C \in F(X): C \neq X, C \text{ contains a nonempty open set}\}) \geq \frac{1}{4}.
\]

By the ergodic decomposition, this implies that there exist \(G\)-invariant ergodic probability measures supported on proper closed subsets of \(X\) with non-empty interior, which contradicts assumption (iii).

\(\square\)

### 5.2. Applications to IRSs of topological full groups and proof of Theorem 1.3

In this subsection we apply Proposition 5.1 to prove Theorem 1.3. Let \(G\) be a minimal groupoid of germs with unit space \(X = G^{(0)}\) homeomorphic to the Cantor set. Let \(\mathcal{U}\) be the collection of clopen subsets of \(X\). For \(U \in \mathcal{U}\), take \(A_U = R_{A(G)}(U) = A(G|U)\). Then by [Nek17, Theorem 1.1], \(A_U\) is a simple group and is contained in every non-trivial normal subgroup of \(F(G|U)\). Thus assumption (i) in Proposition 5.1 is verified. Assumption (ii) is satisfied because of Lemma 1.8. It remains to verify (iii):

**Lemma 5.3.** Suppose \(G\) is a minimal groupoid of germs with unit space \(X\) homeomorphic to the Cantor set. An ergodic \(A(G)\)-invariant probability measure on \(F(X)\) that is not \(\delta_X\) is supported on the subspace of closed sets with empty interior.

**Proof.** Let \(\nu\) be an ergodic \(A(G)\)-invariant probability measure on \(F(X)\) such that \(\nu \neq \delta_X\).

Let \(U, V\) be two disjoint clopen subsets of \(X\). We first show that there exist an infinite sequence of group elements \(\{g_n\}\) in \(A(G)\) and pairwise disjoint clopen subsets \(V_1, V_2, \ldots \) of \(V\) such that

\[
U \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supseteq \cdots
\]

and \(\text{supp } g_n \subseteq (V_{n-1} \cdot g_{n-1}) \cup V_n\).

Such elements can be chosen recursively by applying [Nek17, Lemma 3.2]. Indeed, suppose we have chosen \(g_1, \ldots, g_{n-1}\), take a point \(x_1 \in V \setminus \bigcup_{j=1}^{n-1} V_j\) and two points \(x_2, x_3 \in V_{n-1} \cdot g_{n-1}\) which are in the orbit of \(x_1\). Then there is a \(G\)-bisection \(H_2\) (resp. \(H_3\)) such that \(H_2\) contains an element \(\gamma_2\) (resp. \(\gamma_3\)) such that \(x_2 = x_1 \cdot \gamma_2\) (resp. \(x_3 = x_1 \cdot \gamma_3\)). Take a small clopen neighborhood \(V'_n\) of \(x_1\) in \(V \setminus \bigcup_{j=1}^{n-1} V_j\) and \(W_n\) of \(\{x_2, x_3\}\) in \(V_{n-1} \cdot g_{n-1}\) such that \(V \neq V_1 \cup \ldots \cup V_n\) and \(W_n \neq V_{n-1} \cdot g_{n-1}\). Then one can find a smaller neighborhood \(V_n\) of \(x_1\) such that \(V_n \subseteq V'_n \cap s(H_2) \cap s(H_3)\),...
moreover \( r(V_nH_2) \cup r(V_nH_3) \subseteq W_n \). Take the multisection
\[
F_n = \begin{pmatrix} V_n & V_nH_2 & V_nH_3 \\ (V_nH_2)^{-1} & (V_nH_2)^{-1}r(V_nH_2) & (V_nH_2)^{-1}V_nH_3 \\ (V_nH_3)^{-1} & (V_nH_3)^{-1}(V_nH_2) & r(V_nH_3) \end{pmatrix}.
\]

Then the \( g_n \in A(F_n) \) which acts as a 3-cycle permuting \( x_1, x_2, x_3 \) cyclically satisfies the requirements.

Write
\[
B^V_U := \{ C \in C(X) : U \subseteq C \subseteq X \setminus V \}.
\]
It follows that the translates \( \{B^V_U \cdot g_n\}_{n \in \mathbb{N}} \) are pairwise disjoint. Indeed, if \( C \in B^V_U \cdot g_i \) then \( C \cap (V \cdot g_i) = \emptyset \). For \( j > i \), \( U \cdot g_j \supseteq U \cap (\text{suppg}_{j})^{c} \), where by choice of \( g_j \), \( U \cap (\text{suppg}_{j})^{c} \) contains a non-empty subset of \( V_{j-1} \cdot g_{j-1} \). It follows that
\[
(U \cdot g_j) \cap (V \cdot g_i) \supseteq (U \cdot g_j) \cap (V \cdot g_i) \supseteq (U \cdot g_j) \cap (V_{j-1} \cdot g_{j-1}) \neq \emptyset. \]
Therefore a set \( C \in B^V_U \cdot g_j \) has non-empty intersection with \( V \cdot g_i \), thus it is not in \( B^V_U \cdot g_i \).

By invariance of \( \nu \), we have
\[
1 \geq \sum_{n \in \mathbb{N}} \nu(B^V_U \cdot g_n) = \sum_{n \in \mathbb{N}} \nu(B^V_U). \quad \text{Therefore}
\]
\[
\nu(B^V_U) = 0.
\]

Let \( F'(X) = \{ C \in F(X) : C \neq X, C \text{ contains a nonempty open subset} \} \). For each \( C \in F'(X) \), one can find clopen sets \( U \subseteq C \) and \( V \subseteq X \setminus C \), thus \( C \in B^V_U \). Therefore \( \{B^V_U\} \), where \( U, V \) are taken over disjoint clopen subsets of \( X \), form a countable cover of \( F'(X) \). Thus \( \nu(B^V_U) = 0 \) for any pair of disjoint clopen sets \( U, V \) implies \( \nu(F'(X)) = 0 \). Since \( \nu \neq \delta_X \), we conclude that \( \nu \) is supported on subsets with empty interior.

With these two lemmas we can apply Proposition 5.1 to prove Theorem 5.3.

Proof of Theorem 5.3. Take \( A_U = R_{A(G)}(U) \). As explained at the beginning of this subsection, \( \text{Nek}77 \) Theorem 1.1 implies that \( A_U \) is simple; moreover \( A_U \) has no fixed points in \( U \) since \( G \) is minimal. Lemma 4.8 and 5.3 verify condition (ii) and (iii) respectively. It follows from Proposition 5.1 that \( \mu \)-a.e. \( H \) contains the subgroup \( T \) that is generated by all \( R_{A(G)}(U) \), where \( U \) ranges over all clopen subsets that are contained in \( X \setminus \text{Fix}(H) \). It’s clear that \( T \leq R_{A(G)}(X \setminus \text{Fix}(H)) \), while the other direction \( R_{A(G)}(X \setminus \text{Fix}(H)) \leq T \) follows from definition of \( A(G) \) (c.f. Fact 4.7).

The following corollary of Theorem 5.3 is previously known as a consequence of \( \text{DMI}74 \) Theorem 2.9 on indecomposable characters.

Corollary 5.4. Let \( G \) be a minimal groupoid of germs whose unit space \( X \) is homeomorphic to the Cantor set. Assume that the action of \( A(G) \) on \( X \) is compressible in the sense that for any clopen sets \( U \) such that \( U \neq X \), there exists \( g_1, g_2 \in A(G) \) such that \( (U \cdot g_1) \cup (U \cdot g_2) \subseteq U \) and \( (U \cdot g_1) \cap (U \cdot g_2) = \emptyset \). Then \( A(G) \) does not admit any IRS other than \( \delta_{\text{id}} \) and \( \delta_{A(G)} \).

Proof. By Theorem 5.3 it suffices to show that the only \( A(G) \)-invariant probability measures on \( F(X) \) are \( \delta \)-masses at \( \emptyset \) or \( X \). Let \( \nu \) be an \( A(G) \)-invariant probability measures on \( F(X) \). For a clopen set \( U \) such that \( U \neq X \), by the compressibility assumption, we can find \( g_1, g_2 \in A(G) \) such that \( U \cdot g_1 \subseteq U \), \( U \cdot g_2 \subseteq U \) and
for any clopen set \( U \cap (U \cdot g_1) \cap (U \cdot g_2) = \emptyset \). Denote by \( C_U = \{ C \in F(X) : C \neq \emptyset, C \subseteq U \} \). Since \( \nu \) is an invariant measure, we have that \( \nu(C_U) = \nu(C_{U \cdot g_1}) = \nu(C_{U \cdot g_2}) \). On the other hand since \( C_{U \cdot g_1} \) and \( C_{U \cdot g_2} \) are disjoint, we have \( \nu(C_U) \geq \nu(C_{U \cdot g_1}) + \nu(C_{U \cdot g_2}) \). Therefore \( \nu(C_U) = 0 \) for any clopen set \( U \neq X \). The conclusion follows.

\[ \square \]

Example 5.5. Following [Mat15, Definition 4.9], we say that a clopen set \( A \subseteq G^{(0)} \) is properly infinite if there exists bisections \( F_1, F_2 \) such that \( s(F_1) = s(F_2) = A \), \( r(F_1) \cup r(F_2) \subseteq A \) and \( r(F_1) \cap r(F_2) = \emptyset \). If every clopen subset of \( X \) is properly infinite, then the groupoid \( G \) is said to be purely infinite. Examples of purely infinite minimal groupoids include the groupoid of germs of \( \Gamma \rhd X \) where the action of the countable group \( \Gamma \) on the Cantor set \( X \) is \( n \)-filling in the sense of [JR00], groupoids arising from a one-sided irreducible shift of finite type, see [Mat15] for more details. By [Mat15, Lemma 4.13] a minimal purely infinite groupoid \( G \) satisfies the assumption of Corollary 5.4. Note that by [Mat15, Theorem 4.16], in this case \( A(G) = [F(G), F(G)] \). It follows that if \( \mu \) is an ergodic IRS of \( F(G) \), where \( G \) is a minimal purely infinite groupoid of germs, then either \( \mu = \delta_{(id)} \) or \( \mu \)-a.e. \( H \) satisfies \( H \geq [F(G), F(G)] \).

The rest of the section is devoted to cases opposite to Corollary 5.4 more precisely, examples where invariant measures on \( F(X) \) can be studied through subgroups that are locally finite.

5.3. Invariant measures on the space of closed sets under the action of an LDA-group. Let \( H \) be an AF-groupoid associated with a simple Bratteli diagram \( B \). The unit space of \( H \) can be identified with the space \( X_B \) of infinite paths of the diagram \( B \). In this subsection we apply the pointwise ergodic theorem to study \( A(H) \)-invariant measures on \( F(X_B) \). The alternating full group \( A(H) \) is isomorphic to the direct limit of direct product \( \Gamma_n = \prod_{v \in V_n} \text{Alt}(E(v_0, v)) \). Apply the pointwise ergodic theorem from [Ver74, OV96] (or the general pointwise ergodic theorem for amenable groups in [Lin04]) to the locally finite group \( A(H) = \cup_{n=1}^{\infty} \Gamma_n \), we have that if \( \nu \) is an ergodic \( A(H) \)-invariant probability measure on \( F(X_B) \), then for any measurable function \( f : F(X_B) \to \mathbb{R} \),

\[
\int f d\nu = \lim_{n \to \infty} \frac{1}{|\Gamma_n|} \sum_{g \in \Gamma_n} f(C \cdot g),
\]

for \( \nu \)-a.e. \( C \).

Lemma 5.6. Let \( H \) be a minimal AF groupoid with unit space \( X = H^{(0)} \) homeomorphic to the Cantor set. Let \( \nu \) be an \( A(H) \)-invariant ergodic probability measure on \( F(X) \), \( \nu \neq \delta_0, \delta_X \). Then there exists a constant \( k \in \mathbb{N} \) such that \( \nu \) is supported on the subspace \( X^{(k)} \) of finite sets of cardinality \( k \).

Proof. Since \( H \) is assumed to be a minimal AF-groupoid, it is associated with a simple Bratteli diagram \( B = (V, E) \) with path space \( X_B \) homeomorphic to \( H^{(0)} \). We use notations and terminologies as reviewed in Example 4.9. Telescoping if necessary, we may assume \( |E(v_0, v)| \geq 5 \) for all \( v \in V_n, n \geq 1 \). Let \( \nu \) be a \( A(H) \)-invariant ergodic probability measure on \( F(X_B), \nu \neq \delta_0, \delta_X \).

We first show that if \( \nu \) is supported on infinite subsets of \( X_B \), then \( \nu(C \subseteq U) = 0 \) for any clopen set \( U \neq X \). To see this, let \( f = f_U : F(X_B) \to \mathbb{R} \) be the indicator function
function $f(C) = 1_{(C \subseteq U)}$. Then by the pointwise ergodic theorem, for $\nu$-a.e. $C$,

$$\int f \, d\nu = \lim_{n \to \infty} \frac{1}{|\Gamma_n|} \sum_{g \in \Gamma_n} f(C \cdot g)$$

$$= \lim_{n \to \infty} \frac{|\{g \in \Gamma_n : C \cdot g \subseteq U\}|}{|\Gamma_n|}.$$

We introduce the following notations. For a given clopen set $U$, let $n_0 = n_0(U)$ be the minimal number such that $U$ can be expressed as a disjoint union of cylinder sets of the form $U(e_1, \ldots, e_{n_0})$. For $v \in V_n$ and a set $A \subseteq X_B$, let $E(v_0, v : A)$ be the collection of $n$-paths

$$E(v_0, v : A) = \{(e_1, \ldots, e_n) : r(e_n) = v \text{ and } U(e_1, \ldots, e_n) \cap A \neq \emptyset\}.$$

Given a closed set $C$ and level $n \geq n_0$, write

$$\tilde{C}_n = \cup \{U(e_1, \ldots, e_n) : (e_1, \ldots, e_n) \in E(v_0, v : A)\}.$$

Recall that $\Gamma_n = \prod_{v \in V_n} \text{Alt}(E(v_0, v))$. In particular, the action of $\Gamma_n$ on $X_B$ is by permuting the $n$-prefix of the infinite paths. Since for $n \geq n_0(U)$, $U$ can be expressed as a disjoint union of cylinder sets indexed by $n$-paths, it is clear that for $g \in \Gamma_n, C \cdot g \subseteq U$ if and only if $\tilde{C}_n \cdot g \subseteq U$. Let $\tilde{P}_n = \{g \in \Gamma_n : \tilde{C}_n \cdot g = \tilde{C}_n\}$ be the setwise stabilizer of $\tilde{C}_n$ in $\Gamma_n$. Then

$$\frac{|\{g \in \Gamma_n : C \cdot g \subseteq U\}|}{|\Gamma_n|} = \frac{|\{g \in \Gamma_n : \tilde{C}_n \cdot g \subseteq U\}|}{|\Gamma_n|}$$

$$= \frac{|\{K \in \tilde{P}_n \setminus \Gamma_n : K \subseteq U\}|}{|\Gamma_n : \tilde{P}_n|},$$

(5.2)

where we identity the coset space $\tilde{P}_n \setminus \Gamma_n$ as translates of the set $\tilde{C}_n$ of the form $\tilde{C}_n \cdot g$.

Recall that the action of the alternating group $\text{Alt}(m)$ is transitive on the collection of subsets of fixed size $k$, where $1 \leq k \leq m$, $m \geq 3$. Since $\Gamma_n = \prod_{v \in V_n} \text{Alt}(E(v_0, v))$, the ratio in (5.2) can be computed as

$$\frac{|\{K \in \tilde{P}_n \setminus \Gamma_n : K \subseteq U\}|}{|\Gamma_n : \tilde{P}_n|} = \prod_{v \in V_n} \left( \frac{|E(v_0, v : U)|}{|E(v_0, v : C)|} \right).$$

(5.3)

To estimate such ratios we use the following elementary fact about path counting in a simple Bratteli diagram $B$:

**Fact 5.7.** Let $U(e_1, \ldots, e_k)$ be the cylinder set indexed by finite path $(e_1 \ldots e_k)$ and $u = r(e_k)$ be the end point of the path. Suppose that $m > k$ is a level such that all vertices in $V_k$ and $V_m$ are connected. Then for any $n \geq m$ and $v \in V_n$,

$$\frac{N(v; e_1, \ldots, e_k)}{|E(v_0, v)|} \geq \frac{1}{\max_{w \in V_m} |E(v_0, w)|}. $$

where \( N(v; e_1, \ldots, e_k) := |\{(f_1, \ldots, f_n) : f_i = e_i \text{ for } 1 \leq i \leq k \text{ and } r(f_n) = v\}|. \)

**Proof of Fact 5.7.** Recall that \( E(w, v) \) denotes the set of finite paths with start point \( w \) and end point \( v \). Path counting in the Bratteli diagram has the Markov property:

\[
|E(v_0, v)| = \sum_{w \in V_m} |E(v_0, w)| \cdot |E(w, v)|,
\]

\[
N(v; e_1, \ldots, e_k) = |\{(f_1, \ldots, f_n) : f_i = e_i \text{ for } 1 \leq i \leq k \text{ and } r(f_n) = v\}|
\]

\[
= \sum_{w \in V_m} N(w; e_1, \ldots, e_k) |E(w, v)|.
\]

By assumption of the lemma we have \( N(w; e_1, \ldots, e_k) \geq 1 \) for any \( w \in V_m \) and finite path \((e_1, \ldots, e_k)\). Comparing the two equations we have

\[
\frac{N(v; e_1, \ldots, e_k)}{|E(v_0, v)|} \geq \frac{1}{\max_{w \in V_m} |E(v_0, w)|}.
\]

\( \square \)

Now suppose the clopen set \( U \) has non-empty complement \( U^c = X_B \setminus U \). Take a cylinder set \( W = U(f_1, \ldots, f_l) \) contained in \( U^c \). Let \( m > l \) be a level such that any vertex in \( V_l \) is connected to any vertex in \( V_m \), such a level \( m \) exists because of the minimality assumption on \( \mathcal{H} \). It follows from the Fact that for all \( n \geq m \), for \( v \in V_n \),

\[
\frac{|E(v_0, v : U)|}{|E(v_0, v)|} \leq 1 - \frac{N(v; f_1, \ldots, f_l)}{|E(v_0, v)|}.
\]

\[
\leq 1 - \frac{1}{\max_{w \in V_m} |E(v_0, w)|}.
\]

Let \( c \) be the constant that \( e^{-c} = 1 - \frac{1}{\max_{w \in V_m} |E(v_0, w)|} \). Plug into \( 1 \), we conclude that for \( n \geq m \),

\[
\frac{\{|g \in \Gamma_n : C \cdot g \subseteq U\}}{\|\Gamma_n\|} \leq e^{-c \sum_{v \in V_n} |E(v_0, v : C)|}.
\]

If \( C \) is an infinite closed set, then \( \sum_{v \in V_n} |E(v_0, v : C)| \to \infty \) as \( n \to \infty \). Therefore by the pointwise ergodic theorem, if \( \nu(\{C : |C| = \infty\}) = 1 \), then for any clopen set \( U \) such that \( U \neq X, \nu \text{ a.e. } C \), we have

\[
\nu(\{C : C \subseteq U\}) = \lim_{n \to \infty} \frac{\{|g \in \Gamma_n : C \cdot g \subseteq U\}|}{\|\Gamma_n\|} \leq \lim_{n \to \infty} e^{-c \sum_{v \in V_n} |E(v_0, v : C)|} = 0.
\]

Since the sets \( \{C \in F(X_B) : C \subseteq U\} \) where \( U \) goes over all clopen proper subset of \( X \) form a countable cover of \( \{C \in F(X_B) : C \neq X, |C| = \infty\} \), we conclude that \( \nu(\{C : |C| = \infty, C \neq X\}) = 0 \). This means that \( \nu \) is \( \delta \)-mass at \( X \), which is a contradiction. By ergodicity of \( \nu \) we conclude that it must be supported on subsets of fixed size \( k \), for some \( k \in \mathbb{N} \).

\( \square \)

As in the statement of the previous lemma, let \( X^{(k)} \) be the space of subsets of size \( k \) of \( X \). We apply the pointwise ergodic theorem again to obtain further
information on $A(\mathcal{H})$-invariant probability measures on $X^{(k)}$. Recall that $M(X, \mathcal{H})$ denotes $\mathcal{H}$-invariant probability measures on $X$.

**Lemma 5.8.** Let $\mathcal{H}$ be a minimal AF groupoid with unit space $X$ homeomorphic to the Cantor set. Let $\nu$ be an ergodic $A(\mathcal{H})$-invariant probability measure on $X^{(k)}$. Then there exists an $k$-tuple of ergodic measures $(\mu_1, \ldots, \mu_k) \in M(X, \mathcal{H})^k$ such that for any Borel set $U \subseteq X$,

$$\nu(\{C \in X^{(k)} : C \subseteq U\}) = \mu_1(U) \ldots \mu_k(U).$$

**Proof.** Let $\nu$ be an ergodic $A(\mathcal{H})$-invariant probability measure on $X^{(k)}$ as given and $\eta$ be the measure on $X$ defined by

$$\eta = \int_{X^{(k)}} u_C d\nu(C),$$

where $u_C$ is the uniform measure on the finite set $C$. It’s clear by definition that $\eta$ is an $A(\mathcal{H})$-invariant probability measure on $X$. Write $\eta = \int_Y \mu_g d\theta(y)$ for its ergodic decomposition. Note that $\eta$ has at most $k$ ergodic components. Indeed, if this claim is not true, then we can find $k + 1$ disjoint measurable sets $B_1, \ldots, B_{k+1}$ in $Y$ such that $\theta(B_i) > 0$ for all $1 \leq i \leq k + 1$. Let $\eta_i = \frac{1}{\theta(B_i)} \int_{B_i} \mu_g d\theta(y)$. Since the sets $B_i$’s are disjoint, we have that the collection of measures $\eta_i, 1 \leq i \leq k + 1$ are mutually singular. Let $A_1, A_2, \ldots, A_{k+1}$ be a choice of disjoint sets in the $A(\mathcal{H})$-invariant $\sigma$-field $\mathcal{I}$ on $X$ such that $\eta_i(A_i) = 1$ for each $i$. Now consider the function $\psi : X^{(k)} \to \{0, 1\}^{k+1}$ defined as $(\psi(C))_i = 1_{\{C \cap A_i \neq \emptyset\}}$. Since each $A_i \in \mathcal{I}$, we have that $\psi$ is a measurable function invariant under $A(\mathcal{H})$. By ergodicity of $\nu$, it follows that $\nu$-a.e. $\psi$ is a constant. Because $C$ only contains $k$ points, there must exists a coordinate $j$ such that $(\psi(C))_j = 0$.

It follows that there is $j \in \{1, \ldots, k + 1\}$ such that $\nu(\{C : C \cap A_j \neq \emptyset\}) = 0$. However since $\eta = \sum_{i=1}^{k+1} \theta(B_i) \eta_i$, by relation of $\eta$ to $\nu$, we have $\nu(\{C : C \cap A_j \neq \emptyset\}) \geq \frac{1}{k} \theta(B_j) > 0$, which is a contradiction.

We have seen that there exists $\ell \in \{1, \ldots, k\}$ and ergodic $A(\mathcal{H})$-invariant probability measures $\mu_1, \ldots, \mu_\ell$ on $X$ such that the ergodic decomposition of $\eta$ is given by $\eta = \sum_{j=1}^\ell c_j \mu_j$, $c_j > 0$. Let $\Omega_j$ be a subset of $X$ provided by the pointwise ergodic theorem such that $\mu_j(\Omega_j) = 1$ and for any $x \in \Omega_j$ and any clopen set $U \subseteq X$, we have

$$\mu_j(U) = \lim_{n \to \infty} \frac{|\{g \in \Gamma_n : x \cdot g \in U\}|}{|\Gamma_n|}.$$

Let $\Omega$ be the subset of $X^{(k)}$ defined as $\Omega = \{C \in X^{(k)} : C \subseteq \bigcup_{j=1}^\ell \Omega_j\}$. Note that $\nu(\Omega') \leq k \eta(\left(\bigcup_{j=1}^\ell \Omega_j\right)^c) = 0$.

Let $\Omega'$ be the subset of $X^{(k)}$ of full $\nu$-measure provided by the pointwise ergodic theorem that is for any $K \in \Omega'$ and any clopen set $U \subseteq X$,

$$\nu(\{C : C \subseteq U\}) = \lim_{n \to \infty} \frac{|\{g \in \Gamma_n : K \cdot g \subseteq U\}|}{|\Gamma_n|}.$$

Now take a set $K = \{z_1, \ldots, z_k\} \in \Omega \cap \Omega'$. Let $U$ be a non-empty clopen set. Let $n \geq n_0(U)$ be sufficiently large such that all elements of $K$ have distinct $n$-prefix.

We use the notation $E(v_i, v : A)$ defined in (5.1) in the proof of the previous lemma. As calculated in (5.2) and (5.3), we have
We conclude that there is an index $j$ and

$$\prod_{v \in V_n} \left( \frac{|E(v_0, v : U)|}{|E(v_0, v : K)|} \right) = \prod_{v \in V_n} \left( \frac{|E(v_0, v : U)|}{|E(v_0, v : K)|} \right),$$

and

$$\prod_{i=1}^k \left( \frac{|\{g \in \Gamma_n : z_i \cdot g \subseteq U\}|}{|\Gamma_n|} \right) = \prod_{i=1}^k \prod_{v \in V_n} \left( \frac{|E(v_0, v : U)|}{|E(v_0, v : \{z_i\})|} \right).$$

Note that $|E(v_0, v : \{z_i\})|$ takes value in $\{0, 1\}$ and $|E(v_0, v : K)| = \sum_{i=1}^k |E(v_0, v : \{z_i\})|$. Thus the ratio between the two is

$$\frac{|\{g \in \Gamma_n : K \cdot g \subseteq U\}| / |\Gamma_n|}{\prod_{i=1}^k \left( \frac{|\{g \in \Gamma_n : z_i \cdot g \subseteq U\}|}{|\Gamma_n|} \right)} = \prod_{v \in V_n} \left( \frac{|E(v_0, v : U)|}{|E(v_0, v : K)|} \right) \frac{|E(v_0, v : U)| - \sum_{i=1}^k |E(v_0, v : \{z_i\})|}{|E(v_0, v : U)|} \frac{|E(v_0, v : K)|}{|E(v_0, v : U)|} \frac{1 - \sum_{i=1}^k |E(v_0, v : \{z_i\})|}{1 - \sum_{i=1}^k |E(v_0, v : \{z_i\})|} \frac{2|E(v_0, v : U)|}{|E(v_0, v : U)| - \sum_{i=1}^k |E(v_0, v : \{z_i\})|} \geq 1 - \frac{k^3}{\min_{e \in V_n} |E(v_0, v : U)|}.$$ 

It is also clear from the formula that this ratio is bounded from above by 1.

Note that $\min_{e \in V_n} |E(v_0, v : U)| \to \infty$ as $n \to \infty$ since $B$ is a simple Bratteli diagram. Indeed, the sequences $\min_{e \in V_n} |E(v_0, v)|$, $\max_{e \in V_n} |E(v_0, v)|$ are non-decreasing in $n$. If every vertex in level $n$ is connected to every vertex in level $m$ for some $m > n$, then it follows $\max_{e \in V_n} |E(v_0, v)| \leq \min_{e \in V_m} |E(v_0, v)|$. Therefore $B$ is simple, we have that $\lim_{n \to \infty} \min_{e \in V_n} |E(v_0, v)| < \infty$ is equivalent to $\lim_{n \to \infty} \max_{e \in V_n} |E(v_0, v)| < \infty$. In the case that $\max_{e \in V_n} |E(v_0, v)|$ is bounded, the path space $X_B$ is either finite (if $|V_n|$ is bounded) or countably discrete (if $|V_n|$ is unbounded). By Fact 57 we conclude that $\min_{e \in V_n} |E(v_0, v : U)| \to \infty$ as $n \to \infty$ if $X_B$ is homeomorphic to the Cantor set. Therefore

$$\lim_{n \to \infty} \frac{|\{g \in \Gamma_n : K \cdot g \subseteq U\}| / |\Gamma_n|}{\prod_{i=1}^k \left( \frac{|\{g \in \Gamma_n : z_i \cdot g \subseteq U\}|}{|\Gamma_n|} \right)} = 1.$$ 

Since $K \in \Omega \cap \Omega'$, from the way $\Omega$ and $\Omega'$ are chosen, we have for each $1 \leq i \leq k$, there is an index $j_i \in \{1, 2, \ldots, \ell\}$ such that for any clopen set $U$,

$$\lim_{n \to \infty} \frac{|\{g \in \Gamma_n : K \cdot g \subseteq U\}| / |\Gamma_n|}{\nu(C : C \subseteq U)} = \nu(C : C \subseteq U),$$

$$\lim_{n \to \infty} \frac{|\{g \in \Gamma_n : z_i \cdot g \subseteq U\}| / |\Gamma_n|}{\mu_{j_i}(U)} = 1.$$ 

We conclude that

$$\nu(C : C \subseteq U) = \mu_{j_1}(U) \cdot \mu_{j_2}(U) \ldots \mu_{j_k}(U).$$
Remark 5.9. Lemma 5.6 and 5.8 imply that an ergodic $A(H)$-invariant measure on $F(X)$ other than $\delta_0$ or $\delta_X$ must be pushforward of a product measure $\mu_1 \times \ldots \times \mu_k$ on $X^k$ under $(x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\}$ for some $k$-tuple of ergodic invariant measures.

(i): Since the group $A(H)$ is an infinite simple locally finite group, by Theorem 2.4 the ergodic measures in $M(X, H)$ are weakly mixing. Therefore the converse direction of Lemma 5.8 is true: for any tuple $(\mu_1, \ldots, \mu_k) \in M(X, H)^k$ of ergodic invariant measures, the measure $\nu$ defined by $\nu(\{C \in X^{(k)} : C \subseteq U\}) = \mu_1(U) \ldots \mu_k(U)$ is an ergodic $A(H)$-invariant measure on $X^{(k)}$.

(ii): When there are only finitely many ergodic invariant measures in $M(X, H)$, one can also deduce Lemma 5.6 and 5.8 as a consequence of the classification of IRSs of $A(H)$ in DM17.

(iii): Invariant measures on Bratteli diagrams have been investigated extensively in BKMS10 BKMS13. For instance, sufficient conditions for unique ergodicity or admitting finitely many ergodic invariant measures are provided there.

5.4. Examples of classifications of IRSs of alternating full groups. Combine Theorem 1.3 and Lemma 5.8 we have the following:

Corollary 5.10. Let $G$ be a minimal groupoid of germs with unit space $X = G^{(0)}$ homeomorphic to the Cantor set. Assume that there is an open sub-groupoid $H \subseteq G$ such that $H$ is a minimal AF-groupoid with unit space $H^{(0)} = G^{(0)}$. Let $\mu$ be an ergodic IRS of $A(G)$, $\mu \neq \delta_{(id)}$, $\delta_{A(G)}$. Then there exists an integer $k \in \mathbb{N}$ and an $A(G)$-invariant ergodic probability measure $\nu$ on $X^{(k)}$, such that $\mu$ is the pushforward of $\nu$ under the map

$$X^{(k)} \to \text{Sub}(A(G))$$

$$\{x_1, \ldots, x_k\} \mapsto \cap_{i=1}^k \text{St}_{A(G)}(x_i).$$

Proof. Since $H$ is an open sub-groupoid of $G$ with $H^{(0)} = G^{(0)}$, we have $A(H) \leq A(G)$. Let $\nu$ be an $A(H)$-invariant ergodic probability measure on $F(X)$, $\nu \neq \delta_0$, $\delta_X$. Then it follows from Lemma 5.6 that there exists a constant $k \in \mathbb{N}$ such that $\nu$ is supported on the subspace $X^{(k)}$ of finite sets of cardinality $k$. Indeed, since $\nu$ is also $A(H)$-invariant, by the ergodic decomposition theorem, we have $\nu = \int \eta d\lambda(\theta)$, where $\eta$’s are ergodic $A(H)$-invariant probability measures. Since the cardinality function $|C|$ is $A(G)$-invariant, it follows there exists $k \in \mathbb{N} \cup \{0, \infty\}$ such that $\nu$ and $\eta$’s are concentrated on the subspace $X^{(k)}$. Then Lemma 5.6 implies that $k \in \mathbb{N}$.

Let $\nu$ be the distribution of $\text{Fix}(H)$, where $H$ is an ergodic IRS of $A(G)$ with distribution $\mu$. Since the action of $A(G)$ on $X$ is faithful, $\nu = \delta_X$ implies $\mu = \delta_{(id)}$. By Theorem 1.3 $\mu$-a.e. $H \geq R_{A(G)}(X \setminus \text{Fix}(H))$, thus $\nu = \delta_0$ implies $\mu = \delta_{A(G)}$. When $\mu \neq \delta_{(id)}$, $\delta_{A(G)}$, as explained above, $\nu$ is supported on finite sets of cardinality $k$, for some $k \in \mathbb{N}$. In this case $R_{A(G)}(X \setminus \{x_1, \ldots, x_k\}) = \{g \in A(G) : x_i \cdot g = x_i\}$ for all $1 \leq i \leq k\}$, the statement follows.

Under the assumption of Corollary 5.10 IRSs of $A(G)$ other than $\delta_{(id)}$, $\delta_{A(G)}$ are in one-to-one correspondence with $A(G)$-invariant ergodic probability measures on the space of non-empty finite subsets of $X$. It seems an interesting question whether
the conclusion of Corollary 5.10 is true for general minimal groupoid of germs, that is, without the assumption of containing an open minimal AF sub-groupoid.

In the rest of this subsection we consider the topological full group of a minimal \( \mathbb{Z}^d \) action on the Cantor set, where Corollary 5.10 applies.

For minimal Cantor \( \mathbb{Z} \)-actions, a systematic study of topological orbit equivalence was completed in the works [HPS92, GPS95]. In [For00] it is shown for \( d \geq 2 \) that the orbit equivalence relation of a free minimal \( \mathbb{Z}^d \)-action on the Cantor set contains a "large" AF sub-equivalence relation. In [GMPS08, GMPS10] it is shown (for \( d = 2 \) and \( d \geq 3 \) resp.) that every minimal action of \( \mathbb{Z}^d \) action on the Cantor set is topologically orbit equivalent to an AF relation. For our purpose, it suffices to apply results from [For00] so that Corollary 5.10, Lemma 5.6 and Lemma 5.8 can be used.

We are now ready to prove the classification result stated in Corollary 1.4.

**Proof of Corollary 1.4.** Let \( \mathcal{G} \) be the transformation groupoid of the action \( \varphi : \mathbb{Z}^d \to \text{Homeo}(X) \). First assume that the action \( \varphi \) is minimal and free. For \( d = 1 \), by the model theorem (see [HPS92]), up to topological conjugacy, every minimal action of \( \mathbb{Z} \) on a Cantor set arises as the Bratteli-Vershik map of some properly ordered, simple Bratteli diagram. For \( d \geq 2 \), by [For00, Theorem 1.2] there exists an open AF-subgroupoid \( \mathcal{H} \) of \( \mathcal{G} \) with the same unit space such that \( \mathcal{H} \) is associated with a simple Bratteli diagram \( B \), and the set \( Y \), which consists of points whose \( \mathcal{H} \)-orbit is not equal to its \( \mathcal{G} \)-orbit, is a set of zero measure with respect to any \( \mathcal{H} \)-invariant measure on \( X \). The set \( Y \) can be chosen to be \( \mathbb{Z}^d \)-invariant and \( \mathcal{H} \)-invariant.

Therefore the conditions in Corollary 5.10 are satisfied by \( \mathcal{G} \), we have that for \( A(\mathcal{G}) \), ergodic IRSs other than \( \theta_{\{d\}} \), \( \delta_{A(\mathcal{G})} \) are in one-to-one correspondence to ergodic invariant probability measures on \( X^{(k)} \), \( k \in \mathbb{N} \). It remains to show that an ergodic invariant measure on \( X^{(k)} \) must be of the form in the statement. Observe that \( M \left( X^{(k)}, A(\mathcal{G}) \right) = M \left( X^{(k)}, A(\mathcal{H}) \right) \) for any \( k \in \mathbb{N} \). Indeed, let \( \nu \) be an \( A(\mathcal{H}) \)-invariant measure on \( X^{(k)} \). Since the set \( Y \) has measure zero with respect to any invariant measure, we have that \( \nu \left( \{ C : C \cap Y = \emptyset \} \right) = 1 \). Let \( W \) be an open set in \( X^{(k)} \) and \( W' = \{ C \in W : C \cap Y = \emptyset \} \), let \( g \in A(\mathcal{G}) \) be a group element. Then there exists a countable partition of \( W' \) into measurable subsets \( W' = \bigcup_{h \in A(\mathcal{H})} B_h \) such that for \( C \in B_h \), \( C \cdot g = C \cdot h \). Then \( \nu \left( W \cdot g \right) = \nu \left( W' \cdot g \right) = \sum_h \nu \left( B_h \cdot h \right) = \sum \nu \left( B_h \right) = \nu \left( W' \right) \). Thus \( \nu \) is \( A(\mathcal{G}) \)-invariant. Apply Lemma 5.8 to \( A(\mathcal{H}) \)-invariant measures, we obtain the statement.

As pointed out in [GMPS08, GMPS10] the freeness assumption on the \( \mathbb{Z}^d \) action can be dropped for the following reason. Given a minimal \( \mathbb{Z}^d \)-action on \( X \), it can be seen as a free action of a quotient group \( \mathbb{Z}^d / H \). As shown in [GMPS10, Theorem A.1], if the free abelian rank of \( \mathbb{Z}^d / H \) is \( d' \), \( d' \geq 1 \), then there exists a free action of \( \mathbb{Z}^d \) on \( X \) with the same topological full group. Thus a minimal action of \( \mathbb{Z}^d \) can be reduced to the case of a free minimal action.

\( \square \)

**Example 5.11.** An interval exchange map \( f \) on the unit interval \( I = [0,1) \) is obtained by cutting \( I \) into subintervals and reordering them. The map \( f \) is a translation on each sub-interval. Following notations of [JMBMdlS18], we identify the end points \( a, b \) and regard \( f \) as a right-continuous permutation of \( \mathbb{R}/\mathbb{Z} \). The set of translation length (or angle), \( \{ x \cdot f - x : x \in \mathbb{R}/\mathbb{Z} \} \) is finite. Dynamic properties
of interval exchange maps have attracted a lot of research in the past decades, see for instance the survey [Via06] and references therein.

The connection between interval exchange groups and topological full groups is explained in [JMBMdS18]. Let $\Lambda$ be a finitely generated infinite subgroup of $\mathbb{R}/\mathbb{Z}$ and $\Sigma = \{x_1, \ldots, x_r \}$ be a finite subset of $\mathbb{R}/\mathbb{Z}$. The group $\text{IET}(\Lambda, \Sigma)$ is defined as the collection of interval exchange transforms $f$ such that the extrema of the defining intervals of $f$ lie in the cosets $x_i + \Lambda$, $x_i \in \Sigma$ and $x \cdot f - x \in \Lambda$ for all $x \in \mathbb{R}/\mathbb{Z}$. In [JMBMdS18 Proposition 5.11] it is shown that there is a minimal Cantor $\Lambda$-system $\varphi : \Lambda \to \text{Homeo}(X)$ such that its topological full group is isomorphic to $\text{IET}(\Lambda, \Sigma)$. Thus we can apply Corollary [6.3] to $\text{IET}(\Lambda, \Sigma)$, regarded as the topological full group of a minimal Cantor $\Lambda$-system.

Assume that $\Sigma + \Lambda$ contains some irrational numbers and $0 \in \Sigma + \Lambda$. Then $\text{IET}(\Lambda, \Sigma)$ contains irrational rotations and the only invariant measure on $\mathbb{R}/\mathbb{Z}$ is the Lebesgue measure. Corollary [6.3] implies that when $\Sigma + \Lambda$ contains an irrational number and $0 \in \Sigma + \Lambda$, the list of IRSs of the derived subgroup of $\text{IET}(\Lambda, \Sigma)$ is

- (atomic) $\delta_{\{\mu\}}$, $\delta_{\Gamma}$;
- (non-atomic) stabilizer IRSs of diagonal actions on $((\mathbb{R}/\mathbb{Z})^k, m^k)$, where $m$ is the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$, $k \in \mathbb{N}$.

In [Nek18b] Nekrashevych constructs first examples of simple groups of intermediate growth. These groups are obtained as alternating full groups of fragmentations of certain non-free expansive actions of dihedral group on the Cantor set. Corollary [6.10] can be applied to classify IRSs of some examples of such groups. For instance, consider the explicit example of a group $F$ which is obtained by fragmentation of the golden mean dihedral action in [Nek18b Section 8]. It contains an LDA subgroup which acts minimally on the Cantor set (denoted by $A_\omega$ in [Nek18b Subsection 8.2]). Similar to Corollary [6.3] one can classify IRSs of $F'$, we don’t spell out the details here.

6. Applications to weakly branch groups

Let $T$ be a rooted spherically symmetric tree and $G$ a subgroup of $\text{Aut}(T)$. In this case, $G$ is residually finite and let $\overline{G}$ be its profinite completion. Equip $X = \partial T$ with distance $d(x, y) = 2^{-|x \wedge y|}$, where $x \wedge y$ denotes the longest common prefix of $x$ and $y$. Then $G$ acts faithfully on $(\partial T, d)$ by isometry.

Corollary [6.3] is an immediate consequence of Theorem [6.2].

Proof of Corollary [6.3]. Let $G$ be a weakly branch group acting on the rooted tree $T$. Recall that by definition of weakly branching, it means $G$ acts level transitively on $T$ and the rigid stabilizer $\text{Rist}_G(u) = R_G(C_u)$ is nontrivial for every vertex $u \in T$.

First note that $\text{Rist}_G(u)$ does not have any fixed point in the cylinder set $C_u$. Indeed the stabilizer of the vertex $u$, $\text{St}_G(u)$, is in the normalizer of $\text{Rist}_G(u)$ in $G$. By level transitivity of action of $G$, we have that $\text{St}_G(u)$ acts level transitively on the subtree $T_u$ rooted at $u$. Thus the orbit $x \cdot \text{St}_G(u)$ is dense in $C_u$ for any $x \in C_u$. By Remark [3.6] $\text{Rist}_G(u)$ cannot have any fixed point in $C_u$.

Given a subgroup $H$, write $K = \text{Fix}(H)$ and $\partial T - K = \cup_{x \in I_K} C_x$ as the decomposition into disjoint union of maximal cylinders. Given a length $\ell \in \{|x| : x \in I_K\}$, consider all vertices $x$ in the index set $I_K$ such that $|x| = \ell$. Define $m_\ell(H)$ as the
smallest integer $m$ such that
\[
H \geq \prod_{x \in I_K, |x| = \ell} [\text{Rist}_m(T_x), \text{Rist}_m(T_x)].
\]

Note that $m_\ell(H)$ is invariant under conjugation by $G$: $m_\ell(H) = m_\ell(g^{-1} H g)$. Indeed, $\text{Fix}(g^{-1} H g) = \text{Fix}(H) \cdot g$ and the set $I_{K,g} = I_K \cdot g$. Thus we have that $g^{-1} H g \geq \prod_{x \in I_{K,g}, |x| = \ell} [\text{Rist}_m(T_x), \text{Rist}_m(T_x)]$, with $m = m_\ell(H)$.

Since the set $\{|x|, x \in I_K\}$, $K = \text{Fix}(H)$ is invariant under conjugation by $G$, it follows from ergodicity of $\mu$ that it is a.e. a fixed set $P$. By Theorem 1.2 we have that for $\mu$-a.e. $H$, there is a countable open cover $\{V_i\}_{i \in I}$ of $\partial T - \text{Fix}(H)$ such that $[G_{V_i}, G_{V_i}] \leq H$ for each $i \in I$. For each cylinder $C_x$ in the decomposition of $\partial T - \text{Fix}(H)$, since $C_x$ is compact we can take a finite sub-cover of $C_x$ from $\{V_i\}$ and for each $V_i$ in the sub-cover, shrink it to $V_i \cap C_x$. It follows that for each $C_x$, there exists a finite level $m_x$ such that $H \geq [\text{Rist}_{m_x}(T_x), \text{Rist}_{m_x}(T_x)]$. It is clear by definition of $m_\ell(H)$ that for each $\ell \in P$, $m_\ell(H) \leq \max_{x \in I_{K,g}, |x| = \ell} m_x$, thus $m_\ell(H)$ is a finite integer. Finally since $m_\ell(H)$ is a conjugation invariant function, ergodicity of $\mu$ implies that $m_\ell(H)$ is $\mu$-a.e. a finite constant. We have proved the statement.

Given a set $C \subseteq \partial T$, write $\text{Fix}_G(C) = \{g \in G : x \cdot g = x \text{ for all } x \in C\}$ for its pointwise stabilizer in $G$.

The distribution of $\text{Fix}(H)$ in the space $F(\partial T)$ of closed subsets of $\partial T$ is known. In our setting $G$ is a group acting faithfully and level transitively on a rooted tree $T$. Then $G$ is residually finite, denote by $\hat{G}$ its profinite completion and $\eta$ the Haar measure on the profinite group $\hat{G}$. Then by [BT18, Lemma 2.4], for an ergodic IRS $\mu$ of $G$, there exists a closed set $K \subseteq \partial T$ such that
\[
\mu(\text{Fix}(H) \in B) = \eta(\{g \in \hat{G} : K \cdot g \in B\}) \text{ for any } B \in B(F(\partial T)).
\]
That is, the distribution of $\text{Fix}(H)$ is the pushforward of the Haar measure on $\hat{G}$ under $\hat{g} \mapsto K \cdot \hat{g}$, where $K$ is a deterministic closed set in $\partial T$. When $K$ is not clopen, the orbit of $K$ under $\hat{g}$ is infinite. In this case as considered in [BT18], one can take the IRS of $G$ defined by $\{\text{Fix}_G(K \cdot \hat{g})\}$ where the distribution of $\hat{g}$ is the Haar measure $\eta$. When $G$ is weakly branch, this construction gives a continuum of atomless IRSs when $K$ varies over closed but not clopen subsets of $\partial T$.

Denote by $P_\mu : F(\partial T) \times B(\text{Sub}(G)) \to [0, 1]$ the regular conditional distribution of $H$ given its fixed point set $\text{Fix}(H)$. Then Theorem 1.3 states that there exists a sequence of positive integers $m = (m_i)$ such that for $\mu$-a.e. $H$, $P_\mu(\text{Fix}(H), \cdot)$ is pulled back from an IRS of the quotient group $\hat{G}_{\text{Fix}(H),m}$, where
\[
\hat{G}_{C,m} := \text{Fix}_G(C) / \bigoplus_{x \in I_C} [\text{Rist}_{m_{\ell|x}}(T_x), \text{Rist}_{m_{\ell|x}}(T_x)].
\]

Recall that a group $G$ acting on the rooted tree $T$ is said to be a branch group if it acts level transitively and the level rigid stabilizers $\text{Rist}_m(T)$ have finite index in $G$ for all $m \in \mathbb{N}$. A group is just infinite if it is infinite and all of its proper quotient groups are finite.

**Corollary 6.1.** Let $\mu$ be an ergodic IRS of a branch group $G$ such that $\mu$-a.e. $\text{Fix}(H)$ is clopen (possibly empty). Then $\mu$ is atomic. Suppose in addition $G$ is just infinite, then $\mu$-a.e. $H$ is a finite index subgroup of $\text{Fix}_G(C)$, where $C = \text{Fix}(H)$ (which is assumed to be clopen a.e.).

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Proof. Since $G$ is assumed to be a branch group, we have that $\text{Rist}_G(x)$ has no fixed point in the cylinder $C_x$, indeed the orbit $z \cdot \text{Rist}_G(x)$ for infinite for any $z \in C_x$. Then Corollary 1.5 applies. Since $\mu$-a.e. $\text{Fix}(H)$ is clopen, it follows that the distribution of $\text{Fix}(H)$ is uniform over translates of a clopen set $K$ and the index set $I_K$ is finite. Let $m$ be the sequence of integers provided by Corollary 1.5. Since $I_K$ is finite and $G$ is branch, we have that the quotient group $\text{Fix}_G(K)/\bigoplus_{x \in I_K} [\text{Rist}_{m|x}(T_x), \text{Rist}_{m|x}(T_x)]$ is virtually abelian. Since a virtually abelian group has only countably many subgroups, by Corollary 1.5 we conclude that $\mu$ is atomic.

Suppose in addition $G$ is just infinite, then by the characterization in [Gri00] we have that $\text{Rist}_m(T_x)$ has finite abelianization for any $x \in T$ and $m \in \mathbb{N}$. It follows that in this case $\text{Fix}_G(C)/\bigoplus_{x \in I_K} [\text{Rist}_{m|x}(T_x), \text{Rist}_{m|x}(T_x)]$ is finite. Since $\mu$-a.e. $H$ contains $\bigoplus_{x \in I_{\text{Fix}(H)}} [\text{Rist}_{m|x}(T_x), \text{Rist}_{m|x}(T_x)]$, the statement follows. \hfill \Box

Recall that according to [Gri00], a group acting on a rooted tree $T$ is said to have the congruence property if any finite index subgroup contains a level stabilizer $\text{St}_G(n)$ for some $n \in \mathbb{N}$. This definition is in analogy to the classical congruence property for arithmetic groups, where level stabilizers replace congruence modulo ideals. By Corollary 6.1 we have that if $G$ is a just infinite branch group which satisfies the congruence property, then an ergodic fixed point free IRS of $G$ is atomic and contains a level stabilizer (which is a congruence subgroup in this context) almost surely.

The most celebrated example of just infinite branch groups is the first Grigorchuk group $\mathfrak{G}$, which is constructed in [Gri80]. The group $\mathfrak{G}$ is generated by (for notation wreath recursion see [BGS03 Chapter 1])

$$a = \varepsilon, \quad b = (a, c), \quad c = (a, d), \quad d = (1, b),$$

where $\varepsilon$ is the root permutation which permutes the two subtrees of the root. The group $\mathfrak{G}$ is branch, just infinite and has the congruence property (see [Gri00]). Therefore by Corollary 6.1 for an ergodic fixed point free IRS $\mu$ of $\mathfrak{G}$, there is a level $n$ such that $\mu$-a.e. $H$ contains the level stabilizer $\text{St}_{\mathfrak{G}}(n)$.

The phenomenon that ergodic IRSs with clopen fixed point sets must be atomic occurs in some examples of weakly branch groups as well. We illustrate it on the Basilica group, which is an example of weakly branch groups that are not branch. The Basilica group $\mathfrak{B}$ is introduced in [GZ02]. It is generated by two automorphisms

$$a = (1, b), \quad b = (1, a)\varepsilon.$$ 

It is shown in [GZ02] that $\mathfrak{B}$ weakly branches over its commutator subgroup $[\mathfrak{B}, \mathfrak{B}]$, and $\mathfrak{B}$ is of exponential growth, non-elementary amenable.

**Corollary 6.2.** Suppose $\mu$ is an ergodic IRS of the Basilica group $\mathfrak{B}$ such that $\mu$-a.e. $\text{Fix}(H)$ is clopen (possibly empty). Then $\mu$ is atomic.

**Proof.** We will use the following algebraic fact about $\mathfrak{B}$: $\mathfrak{B}'' = \gamma_3(\mathfrak{B}) \times \gamma_3(\mathfrak{B})$, see [GZ02]. It means that the one step recursion $g = (g_0, g_1)$ of an element $g \in \mathfrak{B}''$ satisfies that $g_0, g_1 \in \gamma_3(\mathfrak{B})$. It follows that $\mathfrak{B}/\mathfrak{B}''$ is virtually step 2-nilpotent.

We first verify that for $K$ clopen, the group $\bar{\Gamma}_K = \text{Fix}_\mathfrak{B}(K)/\bigoplus_{x \in I_K} \text{Rist}_{m|x}(T_x)'$ is a sub-quotient of a finitely generated virtually nilpotent group. Recall the notation of $I_K$, $I_K$ is finite in this case. Let $n = \max\{|x| + m_{|x|} : x \in I_K\}$
$Q_n$ be the subgroup $\prod_{a \in L_n} (B, \mathcal{B})_a$. Then the quotient group $\mathcal{B}/Q'_n$ is virtually step-2 nilpotent. It is clear by the choice of $n$ that $\Gamma_{K, m}$ is a quotient group of $\text{Fix}(B)(K)/(\text{Fix}(B)(K) \cap Q'_n)$, thus a sub-quotient of a virtually step-2 nilpotent group $\mathcal{B}/Q'_n$.

It follows then $\Gamma_{K, m}$ is a finitely generated virtually nilpotent group. Thus it only has countably many subgroups and all its IRSs are atomic. Corollary then implies the statement.

We now discuss the case where $\text{Fix}(H)$ is a general closed subset of $\partial T$. Let $G$ be a branch group acting faithfully on $T$ and $\mu$ an ergodic IRS of $G$. The quotient group $\Gamma_{K, m}$ as in (6.1) may admit a continuum of non-atomic IRSs. For example, consider a branch group acting on the rooted binary tree and an IRS where $\text{Fix}(H)$ is a single point. In this case the distribution of $\text{Fix}(H)$ is a uniform point on $\partial T$ with respect to Hausdorff measure. According to Theorem there is a sequence of integers $m = (m_i)$ such that the conditional distribution of $H$ given $\text{Fix}(H) = \{x\}$ is pulled back from an IRS of the quotient $\Gamma_{x, m} = \text{St}_G(x) / \bigoplus_{i=1}^\infty [\text{Rist}_{m_i}(T_{x_{1} \ldots x_{i-1} \hat{x}_i}), \text{Rist}_{m_i}(T_{x_{1} \ldots x_{i-1} \hat{x}_i})]$. Note that by definitions, the group $\Gamma_{x, m}$ contains the infinite direct sum of finite groups

$$L_{x, m} = \bigoplus_{i=1}^\infty \text{Rist}_G(x_1 \ldots x_{i-1} \hat{x}_i)/ [\text{Rist}_{m_i}(T_{x_{1} \ldots x_{i-1} \hat{x}_i}), \text{Rist}_{m_i}(T_{x_{1} \ldots x_{i-1} \hat{x}_i})]$$

as a normal subgroup. For the first Grigorchuk group $G$, the parabolic subgroup $\text{St}_G(1^\infty)$ is described explicitly in [BG02] Theorem 4.4, it has the structure of an iterated semi-direct product. More generally from such explicit descriptions one can see that sub-quotients of the form $\Gamma_{K, m}$ in (6.1) are locally finite. To further understand IRSs of such quotients one would need more algebraic information on the groups under consideration, which is beyond the scope of this work.

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Tianyi Zheng, Department of Mathematics, UC San Diego, 9500 Gilman Dr, La Jolla, CA 92037, USA. Email address: tzheng2@math.ucsd.edu.

Tianyi Zheng, Department of Mathematics, UC San Diego, 9500 Gilman Dr, La Jolla, CA 92037, USA. Email address: tzheng2@math.ucsd.edu.