Statistical approach to study neutrino interactions.

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Abstract

Neutrino oscillation parameters can be understood in a better way by building a more complete picture of neutrino interactions. This poses a series of important theoretical and experimental challenges because of the elusive nature of neutrino and an inherent difficulty in its detection. This work is an attempt to study neutrino interactions through a purely theoretical approach using fluctuation theory. Though experiments are trying hard to shed more light on neutrino interactions, our knowledge of neutrino oscillation parameters is not precise at different energy levels. In this context, one cannot help but give a bold try to marvel at how far our theoretical frameworks can extend.

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1 Introduction

For decades there had been a problem that the number of measured neutrinos coming from the sun due to nucleosynthesis is inconsistent with the prediction from the standard solar model \([1]\). Similarly, the observed deficit of muon-neutrinos produced by cosmic ray high energy proton-proton collisions in the atmosphere remained a puzzle for long \([2]\). In 1998, it was revealed at Super-Kamiokande, Japan, that neutrinos can oscillate between their flavours \([3]\). First proposed by Pontecorvo in 1958, neutrino oscillations is a quantum mechanical phenomenon \([4]\) which arises since the mass eigen-states of neutrinos \((\nu_1, \nu_2, \nu_3)\) and their flavour eigen-states \((\nu_\mu, \nu_\tau, \nu_e)\) are not identical, due to which there is a flip in their flavour or mass-mixing as these travel large distances. These are related as:

\[ \nu_1 = \Sigma U_{1m} \nu_m \quad (1.1) \]

where \(m = 1, 2, 3\) and \(l = e, \mu, \tau\). \(U_{1m}\) is a unitary matrix, called Pontecorvo-Maki-Nakagawa-Sakata or PMNS matrix \([5]\). It is a fundamental result that neutrinos cannot oscillate unless these have mass. The discovery of neutrino oscillation resolved the anomalies of both solar and atmospheric neutrino deficits. The discovery of neutrino oscillations is one of the most exciting recent developments in particle physics. Current and future neutrino experiments are aiming to make precise measurements of the oscillation parameters. Improving our understanding of neutrino interactions is crucial to these precision studies of neutrino oscillations \([6]\).

2 Gaussian Distribution and mean square fluctuations of neutrinos

Suppose a \(|\nu_\alpha\rangle\) flavor neutrino beam is generated at the source at space-time point \((0,0)\) and is directed along X-axis to a detector at some distance \(L\) from the generation point, then the probability for finding a different flavor state \(|\nu_\beta\rangle\) at some space-time point \((x,t)\) in the beam at the detector will be given by square of the amplitude term \(|<\nu_\beta(x,t) |\nu_\alpha(0,0)>|^2\)

Hence the oscillation probability is

\[ P(\nu_\alpha \rightarrow \nu_\beta) = |<\nu_\beta(x,t) |\nu_\alpha(0,0)>|^2 \quad (2.2) \]

In terms of the mixing matrix relation, between the flavour states \(|\nu_\alpha\rangle\ and \(|\nu_\beta\rangle\ and mass eigen-states \(|\nu_1\rangle\ and \(|\nu_2\rangle\), the above equation may be written as

\[ P(\nu_\alpha \rightarrow \nu_\beta) = 2 \cos^2 \theta \sin^2 \theta (1 - \cos(\phi_2 - \phi_1)) \quad (2.3) \]

where \(\phi_1\) and \(\phi_2\) are the phases of the wave-packets of each of the mass eigen-states. The eigen-states of Hamiltonian are \(|\nu_1\rangle\ and \(|\nu_2\rangle\ with eigen-values \(m_1\) and \(m_2\) for neutrinos at rest. Neutrinos are produced in weak interactions in weak eigen-states of definite lepton number \(||\nu_e\rangle, |\nu_\mu\rangle\ or |\nu_\tau\rangle\) that are not energy eigen-states. These two sets of states are related to each other by a unitary matrix \(U\) given by \([19]\)

\[ U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

Where \(\theta\) is an unspecified parameter known as the mixing angle and is to be measured experimentally \([20]\). Now using the trigonometric identities \(\cos \theta \sin \theta = \frac{1}{2} \sin(2\theta)\) and \(2 \sin^2 \theta = 1 - \cos(2\theta)\), we find that

\[ P(\nu_\alpha \rightarrow \nu_\beta) = \sin^2(2\theta) \sin^2 \left(\frac{\phi_2 - \phi_1}{2}\right) \quad (2.4) \]
Using the expression for phase, where $i$ denotes the flavour state ($e, \mu$ or $\tau$),

$$\phi_i = E_i t - p_i x$$  \hspace{1cm} (2.5)

the phase difference is given by

$$\phi_2 - \phi_1 = (E_2 - E_1)t - (p_2 - p_1)x$$  \hspace{1cm} (2.6)

Now we take into account relativistic neutrinos of momentum $p$, which can be written as

$$p_i = E_i \sqrt{1 - \frac{m^2_i}{E^2_i}}$$  \hspace{1cm} (2.7)

Furthermore, for relativistic neutrinos, it is a very reasonable assumption that $t = x = L$ \cite{21}, so

$$\phi_2 - \phi_1 = \left(\frac{m^2_1}{2E_1} - \frac{m^2_2}{2E_2}\right) L = \frac{\Delta m^2 L}{2E}$$  \hspace{1cm} (2.8)

Now, from equation (2.5), we get

$$E_i = \frac{\phi_i}{t} + \frac{p_i x}{t}$$  \hspace{1cm} (2.9)

Again assuming $t = x = L$, we write

$$E_i = \frac{\phi_i}{L} + p_i$$  \hspace{1cm} (2.10)

Equation (2.7) can be alternately written as

$$p^2_i = E^2_i - m^2_i$$  \hspace{1cm} (2.11)

Using the same in equation (2.10), we get the energy eigen values of neutrinos as

$$E_i = \frac{\phi_i}{2L} + \frac{m^2 L}{2\phi_i}$$  \hspace{1cm} (2.12)

Where we obtain $E_i$ as a function of two variables $\phi_i$ and $m_i$, thus

$$E_i = E_i(\phi_i, m_i)$$  \hspace{1cm} (2.13)

Expanding $\Delta E_i$ as a series upto the second-order terms, we get

$$\Delta E_i = \frac{\partial E_i}{\partial \phi_i} \Delta \phi_i + \frac{\partial E_i}{\partial m_i} \Delta m_i + \frac{1}{2} \left[ \frac{\partial^2 E_i}{\partial \phi_i^2} (\Delta \phi_i)^2 + \frac{\partial^2 E_i}{\partial \phi_i \partial m_i} \Delta \phi_i \Delta m_i + \frac{\partial^2 E_i}{\partial m_i^2} (\Delta m_i)^2 \right]$$  \hspace{1cm} (2.14)

Under extremum conditions, first derivatives of energy vanishes and we are left with

$$\Delta E_i = \frac{1}{2} \left[ \frac{\partial^2 E_i}{\partial \phi_i^2} (\Delta \phi_i)^2 + \frac{\partial^2 E_i}{\partial \phi_i \partial m_i} \Delta \phi_i \Delta m_i + \frac{\partial^2 E_i}{\partial m_i^2} (\Delta m_i)^2 \right]$$  \hspace{1cm} (2.15)

Now given the trend exhibited by neutrino interaction cross-sections obtained from various experimental and simulation results \cite{22} shown in figure 1, it is very reasonable to express neutrino oscillation probability distribution as a Guassian function where we know that the probability $W$ of a fluctuation is given by \cite{23}

$$W \propto \exp -\frac{R_{\text{min}}}{T}$$  \hspace{1cm} (2.16)

Where $R_{\text{min}}$ is the work needed to carry out the change or fluctuation at a temperature $T$.

In our case

$$\frac{R_{\text{min}}}{T} = \frac{\Delta E_i}{T}$$  \hspace{1cm} (2.17)

Thus the probability of fluctuation is

$$w \propto \exp -\frac{\Delta E_i}{T}$$  \hspace{1cm} (2.18)
Using equation (2.15) in equation (2.18) we obtain

\[ W \propto \exp -\frac{1}{2T} \left[ \frac{\delta^2 E_i}{\delta \phi_i^2} (\Delta \phi_i)^2 + \frac{\delta^2 E_i}{\delta \phi_i \delta m_i} \Delta \phi_i \Delta m_i + \frac{\delta^2 E_i}{\delta m_i^2} (\Delta m_i)^2 \right] \]  

(2.19)

Now the general Gaussian distribution is given as

\[ W = \sqrt{\frac{\beta}{(2\pi)^{n/2}}} \exp -\frac{1}{2 \beta_{ij} x_i x_j} \]  

(2.20)

where \( \beta \) is the determinant of \( \beta_{ij} \).

For two variables, the gaussian distribution can be written as

\[ W = \sqrt{\frac{\beta}{2\pi}} \exp -\frac{1}{2} \left[ \frac{(\Delta x_1)^2}{\langle (\Delta x_1)^2 \rangle} + \frac{(\Delta x_2)^2}{\langle (\Delta x_2)^2 \rangle} \right] \]  

(2.21)

where \( \langle (\Delta x_1)^2 \rangle = \frac{1}{\beta_{11}} \) and \( \langle (\Delta x_2)^2 \rangle = \frac{1}{\beta_{22}} \) which for the present case can be written as

\[ W = \sqrt{\frac{\beta}{2\pi}} \exp -\frac{1}{2} \left[ \frac{(\Delta \phi_1)^2}{\langle (\Delta \phi_1)^2 \rangle} + \frac{(\Delta m_1)^2}{\langle (\Delta m_1)^2 \rangle} \right] \]  

(2.22)

where

\[ \langle (\Delta \phi_1)^2 \rangle = \frac{\Gamma}{\beta_{\phi \phi}}, \]

(2.23)

\[ \langle (\Delta m_1)^2 \rangle = \frac{\Gamma}{\beta_{m m}}, \]

(2.24)

and

\[ \beta = \beta_{\phi \phi} \beta_{m m} \]  

(2.25)

Further we assume that \( \Delta \phi \) and \( \Delta m \) are statistically independent so that \( \langle \Delta \phi \Delta m \rangle = 0 \). After calculations we find the mean square fluctuations as

\[ \langle (\Delta \phi_1)^2 \rangle = \frac{\phi_1^2}{m_1^2 L} \]  

(2.26)

\[ \langle (\Delta m_1)^2 \rangle = \frac{\phi_1^2}{L} \]  

(2.27)
and the determinant $\beta$ as

$$\beta = \frac{m_i^2 L^2}{\phi_i^2 T^2} \quad (2.28)$$

Thus the Gaussian distribution becomes

$$W = \frac{m_i L}{2\pi \phi_i T} \exp \left( -\frac{1}{2T} \left[ \frac{m_i^2 L}{\phi_i^2} (\Delta \phi_i)^2 + \frac{L}{\phi_i} (\Delta m_i)^2 \right] \right) \quad (2.29)$$

Comparing the mean square fluctuations we find an interesting result i.e

$$\frac{\langle (\Delta m_i)^2 \rangle}{m_i^2} = \frac{\langle (\Delta \phi_i)^2 \rangle}{\phi_i^2} = \frac{\phi_i T}{m_i L} \quad (2.30)$$

The phase of the wave-packet of each of the mass eigen-states is something that differentiates neutrino mixing from real rotations. It is only present if all the three flavours mix. Alternatively, one can state that if there is no phase difference, there is no flavour oscillation or mass mixing. Hence the mean square fluctuations in phase and mass should be equivalent as implied by equation (2.30).

Now using this in equation (2.29) we can also write the probability as

$$W = \frac{m_i L}{2\pi \phi_i T} \exp \left( -\frac{1}{T} \frac{m_i^2 L}{\phi_i^2} (\Delta \phi_i)^2 \right) \quad (2.31)$$

This is a simple and compact expression for neutrino oscillation probability in vacuum. For a given length of flight $L$ of neutrinos at a temperature $T$, one can find the oscillation probability from the following plot

![Figure 2: Theoretical plot of neutrino oscillation probability](image)

Now according to the most general expression for neutrino oscillation probability given by the following expression, we find that neutrino oscillations are sensitive only to the difference in the squares of their masses [24].

$$P(\nu_\alpha \to \nu_\beta) = \sin^2 (2\theta) \sin^2 \left( \frac{1.27 \Delta m^2 L}{E} \right) \quad (2.32)$$

As of 2020, [25] the best-fit value of the difference of the squares of the masses of mass eigenstates 1 and 2 is $\Delta m^2_{21} = 0.000074$ eV$^2$, while for eigenstates 2 and 3, it is $\Delta m^2_{32} = 0.00251$ eV$^2$. Since $\Delta m^2_{32}$ is the difference of two squared masses, at least one of them must have a value which is the square root of this value. From this we infer that there exists at least one neutrino mass eigenstate with a mass of at least 0.05 eV [26].

Now when this result is compared to the theoretical plot shown in figure 2, it appears quite consistent. This can further throw a hint at neutrino masses.

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[24]: Reference to the most general expression for neutrino oscillation probability

[25]: Reference to the best-fit value of the difference of the squares of the masses of mass eigenstates

[26]: Reference to the lower limit of neutrino mass eigenstates
3 Riemannian curvature of the \((m_i, \Phi_i)\) space and the neutrino interactions

We see that the probability distribution given by equation (2.31) depends on two parameters viz; \(m_i\) and \(\Phi_i\), therefore we choose to work with \((m_i, \Phi_i)\) space. The Gaussian distribution obtained in the previous section can alternatively be written as

\[
W \propto \exp\left(-\frac{1}{2} \left[ g_{\Phi\Phi} (\Delta \Phi_i)^2 + g_{mm} (\Delta m_i)^2 \right] \right) \tag{3.33}
\]

where we assume a space with coordinates \(\Phi_i\) and \(m_i\) and define the line element in this space as

\[
\Delta l^2 = g_{\Phi\Phi} (\Delta \Phi_i)^2 + g_{mm} (\Delta m_i)^2 \tag{3.34}
\]

and the metric elements of this space are expressed as

\[
g_{\Phi\Phi} = \frac{1}{T} \frac{\partial^2 E_i}{\partial \Phi_i^2} = \frac{m_i^2 L}{\Phi_i^2 T} \tag{3.35}
\]

\[
g_{mm} = \frac{1}{T} \frac{\partial^2 E_i}{\partial m_i^2} = \frac{L}{\Phi_i T} \tag{3.36}
\]

and

\[
g = g_{\Phi\Phi} g_{mm} = \frac{m_i^2 L^2}{\Phi_i^4 T} \tag{3.37}
\]

Knowing the metric elements of a space, we calculate the curvature using the formula

\[
R = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial m_i} \left( \frac{1}{\sqrt{g}} \frac{\partial g_{\Phi\Phi}}{\partial m_i} \right) + \frac{\partial}{\partial \Phi_i} \left( \frac{1}{\sqrt{g}} \frac{\partial g_{mm}}{\partial \Phi_i} \right) \right] \tag{3.38}
\]

After some calculations we find

\[
R = 0 \tag{3.39}
\]

This is quite an interesting result. The curvature is zero when there are no mutual interactions and this is in agreement with the fact that neutrinos are indeed very feebly interacting particles. This can be gauged from the fact that whereas the photons emitted from the solar core may require 40,000 years to diffuse to the outer layers of the Sun, neutrinos generated in the fusion reactions at the stellar core cross this distance practically unimpeded at nearly the speed of light [27]. Neutrinos interact so weakly with the matter that these can go through the whole of earth without deviation.

Furthermore, the order of a system is measured in terms of correlation function. As for instance, the interactions can be determined by the density correlation function which is related to number fluctuations as [23]

\[
\int \zeta dV = \frac{\langle (\Delta N_i)^2 \rangle}{N_i} - 1 \tag{3.40}
\]

For no mutual interactions i.e ideal approximation

\[
\frac{\langle (\Delta N_i)^2 \rangle}{N_i} - 1 = 0 \tag{3.41}
\]

This gives

\[
\langle (\Delta N_i)^2 \rangle = N_i \tag{3.42}
\]

Now using mass in place of density, we may consider the assumption

\[
\frac{\langle (\Delta m_i)^2 \rangle}{m_i^2} = \frac{\langle (\Delta N_i)^2 \rangle}{N_i^2} = \frac{\Phi_i T}{m_i^2 L} \tag{3.43}
\]

from where we get

\[
N_i = \frac{m_i^2 L}{\Phi_i T} \tag{3.44}
\]
Equation (3.43) further relates the mass mixing with the number fluctuation of neutrinos so that we can write the extended result as

\[ \frac{\langle (\Delta m_i)^2 \rangle}{m_i^2} = \frac{\langle (\Delta N_i)^2 \rangle}{N_i^2} = \frac{\langle (\Delta \phi_i)^2 \rangle}{\phi_i^2} \]  

(3.45)

This is obvious since a change in the phase or mass produces a neutrino of different flavor and consequently alters the number of neutrinos corresponding to a given flavor. All these are well established results and raise the scope of fluctuation theory as an important tool in studying neutrino interactions.

4 Conclusions

We have tried to ground our theory in a domain where little or no data exists, and have sought other means to assess its validity. The oscillation probability given by equation (2.4) is derived on the basis of two important assumptions;that the neutrino flavour and mass states are mixed and that a coherent superposition of mass states is created at the weak vertex [28]. This coherent superposition reflects the fact that we can’t experimentally resolve which mass state was created at the vertex. If one could find that, then it would also let us know the mass of the neutrino state that propagates to the detector and then there would be no idea of superposition. The mass and phase fluctuations are therefore identical which is also implied by our result in equation (2.30).

An interesting class of theories called grand unified theories suggests that all interactions (strong, electromagnetic, and weak) are unified at a large energy scale and that the neutrino masses are inversely proportional to this scale. A way to probe this very high energy scale is to search for very small neutrino masses. One such attempt could be through the gaussian distribution approach which has been attempted in the present study.

Similarly a zero Rieman curvature for neutrino interactions as shown in equation (3.39) leads to ideal gas like approximation for neutrinos to study their associated thermodynamic properties. Further, the neutrino number fluctuations expressed in equation (3.43) as a consequence of change in mass or phase is also an obvious result from the existing knowledge of neutrinos confirmed to us once again through the fluctuation theory.

5 Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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