Classical c=1 Tachyon Scattering and 1/2 BPS Correlators

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We study the correlator of chiral primary operators in \( \mathcal{N}=4 \) super Yang-Mills theory in large \( N \) limit. Through the free fermion picture, we map the gauge group rank and R-charges in SYM to the Fermi level and tachyon momenta, respectively, in the c=1 matrix model. By doing so, it is seen that half-BPS correlators are reproduced by tree-level tachyon scattering amplitudes.

§1. Introduction and summary

As is pointed out by Okuyama in Ref. 1), it is possible to use the Das-Jevicki-Sakita term in computing the two-point function of chiral primary operators in \( \mathcal{N}=4 \) SYM. This elucidates the relation with the two-dimensional Yang-Mills theory. Motivated by Ref. 4), in which 2d YM is related to the c=1 matrix model through the collective field theory established by Das and Jevicki, it is thus tempting to formulate a certain correspondence between the c=1 matrix model and \( \mathcal{N}=4 \) SYM.

We find that the “S-matrix” structure in extremal two-point functions is essentially the same as that of the c=1 matrix model up to a phase and non-perturbative terms; that is, they are diagonalized by the relativistic fermion basis. In large \( N \) limit, these half-BPS correlators can then be reproduced by the tree-level c=1 tachyon scattering, where non-perturbative effect is dropped out. This is carried out by mapping the gauge group rank \( N \) and R-charges in SYM to the Fermi level \( \mu \) and tachyon momenta, respectively.

The reason can be intuitively understood as follows. Since half-BPS chiral primary operators correspond to certain \( N \)-fermion quantum mechanical states in a harmonic oscillator potential, we are thus effectively comparing two kinds of ground Fermi liquids in the phase space, i.e. on the SYM side, it is a disk of radius \( \sqrt{2N} \), while in the c=1 case, the profile is determined by the hyperbola \( p^2-x^2 \leq -2\mu \). This makes clear \( \mu \leftrightarrow N \). Also, by means of the bosonization, the map between R-charges and tachyon momenta can be accounted for due to their relation to momenta of the aforementioned 2d relativistic fermions.

The outline of this note is as follows. In §2, we recall some basics of the complex matrix model. In §3, we briefly review the c=1 matrix model and identify classical

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tachyon scattering amplitudes with half-BPS correlators.

§2. Complex matrix model and 1/2-BPS correlators

We first recall some ingredients in the computation of extremal correlators in $\mathcal{N}=4$ SYM, following Ref. 1) – 3). This enables us to see how the “S-matrix” extracted from the two-point function can be diagonalized by Schur polynomials.

We focus on $Z = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, where $\phi_1$ and $\phi_2$ are two of the six adjoint scalar fields in $\mathcal{N}=4$ SYM. As known from the non-renormalization theorem, the extremal correlator

$$\langle \prod_{i=1}^{S} \text{Tr} Z^{J_i}(y) \prod_{j=1}^{P} \text{Tr} Z^{J_j}(x_1) \cdots \prod_{k=1}^{Q} \text{Tr} Z^{J_k}(x_r) \rangle$$

(2.1)

has no dependence on the gauge coupling. When the theory is compactified on $R \times S^3$, the lowest KK modes depend only on time, and it is possible to use the complex matrix model, whose action is given by

$$\int dt \text{Tr} [\dot{Z}^\dagger(t) \dot{Z}(t) - Z^\dagger(t) Z(t)],$$

(2.2)

to evaluate Eq. (2.1) in the free field limit. Let us also explain why the Hamiltonian $H$ corresponding to Eq. (2.2) can be diagonalized using Schur polynomials. Due to the VanderMonde determinant arising from the measure $dZdZ^\dagger$, we can absorb it into the wave function $\Psi$ to redefine

$$\Psi \rightarrow \Delta \Psi, \quad H \rightarrow \Delta H \frac{1}{\Delta}, \quad \Delta = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).$$

(2.3)

The eigenstate of $H$ is of the form

$$\Psi = \chi_0 \det_{i,j} \lambda_j^{n_i}, \quad \chi_0 = e^{-\sum_i \lambda_i^* \lambda_i}, \quad i, j = 1, \cdots, N,$$

(2.4)

which is just the Slater determinant of $N$ fermions in a harmonic oscillator potential. Note that the ground state is $\Psi_0 = \chi_0 \Delta$, and the inner product now becomes $\langle \Psi | \Psi \rangle = \int d\lambda^* d\lambda \; \Psi^* \Psi$ without the factor $|\Delta|^2$. A tool of particular use is the Weyl character formula

$$S(\vec{r}) = \frac{\det_{1 \leq i,j \leq N} \lambda_j^{N-i+r_i}}{\Delta},$$

(2.5)
where \( S(\vec{r}) \) is the Schur polynomial. Here, \( \vec{r} = (r_1, \ldots, r_N) \) represents the row lengths of a Young diagram \( R \), which assigns a representation of \( U(N) \) or the symmetric group \( S_n \) (\( n = \sum r_i \)). Setting \( n_i = N - i + r_i \) and using Eq. (2.5), we can rewrite an excited state \( \Psi \) as a product of the ground state and a Schur polynomial, i.e. \( \Psi = \Psi_0 S(\vec{r}) \). The energy eigenvalue of this excited state is

\[
\sum_{i=1}^{N} n_i = n + \frac{N(N + 1)}{2},
\]

where \( n \) stands for the \( U(1) \) R-charge.

Let us return to the two-point function

\[
\langle \prod_{k=1}^{P} \text{Tr} \ Z_k^{\dagger} J_k(t) \prod_{l=1}^{Q} \text{Tr} \ Z_l^{\dagger} J_l(t') \rangle = Ge^{iJ(t' - t)}, \quad J = \sum_l J_l = \sum_k J_k,
\]

where \( G \) (the “S-matrix” of scattering \( \{J_k\} \rightarrow \{J_l\} \)) is expressed as

\[
G(\{J_k\}; \{J_l\}) = \int dZdZ^\dagger e^{-2\text{Tr}(Z^\dagger Z)} 
\prod_{k=1}^{P} \text{Tr} \ Z_k^{\dagger} \prod_{l=1}^{Q} \text{Tr} \ Z_l^{\dagger},
\]

From Eq. (2.5), it is found that \( G \) is diagonalized as

\[
G^{\text{diag}} = \int d\lambda d\lambda' \Psi_{\{n_i\}}^* \Psi_{\{n'_i\}} = \int dZdZ^\dagger e^{-2\text{Tr}(Z^\dagger Z)} S_S(Z^\dagger)S_R(Z) = t(R)\delta_{SR},
\]

where \( S_R(Z) = \langle R|Z \rangle \) denotes the Schur polynomial. Moreover, \( t(R) \) in Eq. (2.9) has been determined by Jevicki et al. in Ref. 3) to be

\[
t(R) = \frac{\text{dim} R(N)}{d(R)} = \prod_{\Box(i,j)} (N - i + j), \quad d(R) = \prod_{\Box(i,j)} \frac{1}{h_{i,j}},
\]

where \( \Box(i, j) \) and \( h_{i,j} \) label the location and the hook length of the box \( \Box \), respectively, in the Young diagram.

For later convenience, we show that the above \( |R\rangle \) can be written in terms of relativistic fermions. As in Ref. 3), we can rewrite the Schur polynomial as

\[
S_R(Z) = \sum_{\vec{k}} \langle R|\vec{k}\rangle \langle \vec{k}|Z \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \text{Tr}(\sigma Z),
\]

\(^*\) We have omitted an overall factor resulting from off-diagonal elements of \( Z \).
where the total box number \( n \) is related to \( \vec{k} = \{ k_\ell \} \) by \( n = \sum \ell k_\ell \), while \( \chi_R(\sigma) \) denotes the character of the permutation \( \sigma \). Then, introducing the coherent state representation

\[
|Z\rangle = e^{\sum_{n>0} \frac{1}{n} (\text{Tr} Z^n)\alpha_{-n}} |0\rangle,
\]

we have

\[
\langle \vec{k} | Z \rangle = \prod_\ell (\text{Tr} Z^\ell)^{k_\ell}, \quad |\vec{k}\rangle = \prod_\ell (\alpha_{-\ell})^{k_\ell} |0\rangle, \quad [\alpha_m, \alpha_n] = m\delta_{m+n,0}.
\]

By further applying the fermionization, i.e.

\[
\alpha_n = \sum_{r \in \mathbb{Z}+1/2} b_r c_{n-r}, \quad \{ c_r, b_s \} = \delta_{r+s,0},
\]

the linear combination \( \sum_{\vec{k}} \langle R|\vec{k}\rangle \langle \vec{k}| \) gives

\[
|R\rangle = \prod_{i=1}^{\text{diag}(R)} c_{-r_i+i-\frac{1}{2}} b_{-h_i+i-\frac{1}{2}} |0\rangle, \quad b_s |0\rangle = c_s |0\rangle = 0, \quad s > 0.
\]

Here, \( r_i \) (\( h_i \)) is the \( i \)-th row (column) length of the Young diagram \( R \), while \( \text{diag}(R) \) is the diagonal box number. These fermions are identically those appearing in 2d YM on a cylinder if we identify \( c_{-r_i+i-\frac{1}{2}} \) with the \( i \)-th particle at level \( (r_i - i + 1) \) above the Fermi level \( n_F \), and \( b_{-h_i+i-\frac{1}{2}} \) with the \( i \)-th hole at level \( (h_i - i) \) below \( n_F \). Due to (2.14), discrete \( R \)-charges (quantum numbers of \( \alpha \)'s) can thus be mapped to momenta of 2d relativistic free fermions.

The authors of Ref. 5) summarized the leading planar result for \( G \) given in (2.8) with a graphical method. For example, the \( 1 \rightarrow 4 \) case is shown to be

\[
G(J; J_1, J_2, J_3, J_4) = J_1 J_2 J_3 J_4 (J - 1)(J - 2) N^{-3}.
\]

We will see below that Eq. (2.16) can be reproduced by the tree-level tachyon scattering in the \( c=1 \) matrix model.

§3. \( c=1 \) matrix model and scattering of tachyons

Let us briefly review some basics of the \( c=1 \) matrix model. The Lagrangian is defined as

\[
\int dt \text{Tr} \left[ \frac{i}{2} (D_t \Phi)^2 + \frac{1}{2} \Phi^2 \right], \quad D_t \Phi = \partial_t \Phi + [A_t, \Phi],
\]

where \( \Phi \) is an \( N \times N \) Hermitian matrix and the non-dynamical gauge field \( A_t \) is introduced in order to project the wave function \( \Psi(\Phi) \) onto the singlet sector. It
is well known that Eq. (3.1) is equivalent to a fermion liquid in an upside-down harmonic oscillator potential. The Hamiltonian is given by

\[ H = \frac{1}{2\pi} \int dx \int_{p_0}^{p_1} dp \frac{1}{2}(p^2 - x^2), \]  

(3.2)

where we have \( p_\pm(x, t) = \pm \sqrt{x^2 - 2\mu} \), and \(-\mu\) denotes the Fermi level below the tip of the potential. Defining \( x = -e^{-q} \) (i.e. \( x \in [-\infty, 0] \) and \( q \in [-\infty, \infty] \)), we can further set

\[ p_\pm(q, t) = \pm e^{-q} \mp \epsilon_\pm(q, t) e^q, \quad \epsilon_\pm = \sqrt{\pi(\pm \partial_q S - \partial_q S)}, \]  

(3.3)

so that

\[ H = \frac{1}{2} \int dq \left[ \Pi^2_S + (\partial_q S)^2 + e^{2q} \mathcal{O}(S^3) \right]. \]  

(3.4)

That is, \( S \) describes fluctuations (ripples) on the Fermi surface.

For \( q \to -\infty \), \( S(q, t) \) behaves asymptotically like a massless field and can be expanded as

\[ S(q, t) = \int_{-\infty}^{\infty} \frac{d\xi}{2\sqrt{\pi|\xi|}} \left( a_\xi e^{-i|\xi|t+i\xi q} + a_\xi^\dagger e^{i|\xi|t-i\xi q} \right), \quad [a_\xi, a_\xi^\dagger] = |\xi|\delta(\xi - \xi'). \]  

(3.5)

Then, by defining that

\[ |\xi; in\rangle = a_\xi^\dagger|0\rangle, \quad \xi > 0, \]  

(3.6)

the tachyon scattering amplitude can be computed as follows.\(^*\) According to Polchinski,\(^7\) the tree-level tachyon scattering can be calculated by means of the hidden \( \mathcal{W}_\infty \) symmetry. Due to the Liouville wall, incoming and outgoing modes are related as

\[ a_\xi^\dagger = \left( \frac{1}{2\mu} \right)^{-i\xi} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{i}{\mu} \right)^{n-1} \frac{\Gamma(1 - i\xi)}{\Gamma(2 - n - i\xi)} \int_{-\infty}^{0} d^n \xi \prod_{\ell=1}^{n} (a_{\xi_\ell} - a_{\xi_\ell}) \delta(\sum_{\ell=1}^{n} |\xi_\ell| - \xi), \]  

(3.7)

where the sign of \(|\xi_\ell|\)'s is plus (minus) if the creation (annihilation) operator is chosen in front of the delta function. The \( 1 \to n \) amplitude is (up to leg factors)

\[ \langle \xi_1 \cdots \xi_n; out|\xi; in\rangle = \left( \frac{i}{\mu} \right)^{n-1} \frac{\Gamma(1 + i\xi)}{\Gamma(2 - n + i\xi)} 2^n \delta(\xi_1 + \cdots + \xi_n - \xi) \prod_{\ell=1}^{n} \xi_\ell, \]  

\[ \frac{\Gamma(1 + i\xi)}{\Gamma(2 - n + i\xi)} = (i\xi)(i\xi - 1) \cdots (i\xi - n + 2). \]  

\(^*\) Here, the name “tachyon” arises from the fact that usually \( e^{2q}S(q, t) \) is regarded as a tachyon field \( T(q, t) \) (up to a phase) in the dual Liouville theory, see Ref. 6).
It is also well known that the S-matrix of the c=1 matrix model can be diagonalized using fermionic fields \( b(z) \) and \( c(z) \) (where \( z \) is the complex coordinate)

\[
b(z) = \sum_{r \in \mathbb{Z}+1/2} b_r z^{-r-1/2}, \quad c(z) = \sum_{r \in \mathbb{Z}+1/2} c_r z^{-r-1/2}, \quad \{c_r, b_s\} = \delta_{r+s,0}, \tag{3.9}
\]

which are related to the above non-relativistic fermions by the second quantization.\(^8\)

The reason is that the incoming mode \( b_{-r} \) differs from the outing mode \( (Rb)_{-r} \) by a reflection factor \( R \), i.e.

\[
R_r = i \sqrt{1 + ie^{-\pi(\mu+ir)}} \sqrt{\frac{\Gamma(\frac{1}{2} - i\mu + r)}{\Gamma(\frac{1}{2} + i\mu - r)}}. \tag{3.10}
\]

In other words, the c=1 matrix model is free in terms of the \( b, c \) system, which is the fermionization version of the asymptotical \( S(q,t) \) via

\[
c \sim e^{i \int dq' (\Pi S - \partial_q S)}, \quad b \sim e^{-i \int dq' (\Pi S - \partial_q S)}. \tag{3.11}
\]

Just as done in (2.14), \( \xi \)'s of tachyons can be thus mapped to momenta of these 2d relativistic fermions.

Following Refs. 10\(^{)\text{ and 11), we see that } |R \rangle \text{'s in Eq. (2.15) form a diagonal basis of the c=1 S-matrix such that}

\[
\langle R | S_{c=1} | R \rangle = \prod_{i=1}^{\text{diag}(R)} \frac{\Gamma(i\mu + r_i - i + 1) \cos[\pi/2 (r_i - i + i\mu)] \cos[\pi/2 (h_i - i - i\mu)]}{\Gamma(i\mu - h_i + i) \sin[\pi(i\mu - h_i + i)]} + \mathcal{O}(e^{-\mu})
\]

\[
\approx e^{-\frac{i\mu}{2} \sum_{i=1}^{\text{diag}(R)} (r_i + h_i - 2i)} \frac{\text{dim}(\mu)}{d(R)}. \tag{3.12}
\]

For large \( \mu \), up to a pure phase, the diagonal element in the last line is just the aforementioned \( t(R) \), if we replace \( \mu \) with \( N \).

Based on the two diagonalized S-matrix elements in Eqs. (2.9) and (3.12), we observe that through the prescription, i.e. \( i\xi \to J \) and \( \mu \to N \), Eqs. (2.16) and (3.8) are identical, up to a delta function and an irrelevant phase. Therefore, we arrive at the same conclusion as the authors of Ref. 5\(^{)\text{, in which the Euclideanized AdS droplet approach is used to show this equivalence. The above observation can be understood as follows. For chiral primary operators (i.e. their conformal dimensions are equal to the R-charges), which can be treated as } N \text{-fermion states in a harmonic oscillator potential, in the phase space we are thus equivalently comparing the following two kinds of fermion liquids:}

\[
x^2 - p^2 \geq 2\mu,
\]

\[
x^2 + p^2 \leq 2N. \tag{3.13}
\]
The Fermi surface in the first case is determined by a hyperbola, while that in the second one is a circle of radius $\sqrt{2N}$. In addition, we recall that both $\xi$ (in the $c=1$ matrix model) and $J$ (R-charge in SYM) are related to momenta of 2d relativistic fermions, so $i\xi \rightarrow J$ can be interpreted as a result from the sign change of the "non-relativistic" $p^2$ term in Eq. (3.13).

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Note added: Near the completion of this work, we received a new preprint hep-th/0612262 from Jevicki and Yoneya, who reach the same conclusion through a detailed analysis of the Euclideanized AdS droplet scattering.