Neutrino mixing matrix and masses at a particular point of the generalized Fridberg-Lee model

N. Razzaghi$	extsuperscript{1}$

$\textsuperscript{1}$Department of Physics, Qazvin Branch, Islamic Azad University, Qazvin, Iran

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We propose the generalized Friedberg-Lee model neutrino mass model at a particular point (point D at which $\alpha = \beta = -\frac{\pi}{4}$) where at this point, the generalized Friedberg-Lee model is converted to the Democratic mass matrix with the $S_3$ symmetry. The Democratic texture has an experimentally unfavored degenerate mass spectrum on the base of the tribimaximal mixing matrix ($U_{TBM}$). We modify the Democratic mass matrix, at point D, to obtain a nondegenerate mass spectrum by adding the breaking mass term as preserving the twisted Fridberg-Lee symmetry. Although the mixing matrix is still ($U_{TBM}$), where leads to $\theta_{13} = 0$ which is not consistent with the results from Daya Bay and RENO experiments that have established a nonzero value for $\theta_{13}$.

Preserving the leading behavior of $U$ as tribimaximal, and we apply the Broken Democratic neutrino mass texture as a mass matrix at point D. Subsequently, we characterize a minimal perturbation mass matrix which is responsible for a nonzero $\theta_{13}$ along with CP violation parameters, besides the solar neutrino mass splitting has been resulted from it. Let us mention that, unlike other investigations, the perturbation matrix is not adopted on an ad hoc basis, but is generated only in one step by the rules of perturbation method that we will describe. Subsequently, we develop the following results to the literature: (a) we obtain the corresponding neutrino mixing matrix of the generalized Fridberg-Lee model at point D with $\theta_{23} = \frac{\pi}{4}$ and non-zero $\delta$; (b) the ordering of the neutrino masses is inverted; (c) we also obtain the allowed range of the mass parameters, the Dirac phase and the Jarlskog parameter which are consistence with the available experimental data.

I. INTRODUCTION

The results of the neutrino oscillation experiments [1, 2] have found that neutrinos have masses. The present remarkable achievements $3\sigma$ global fits that led to the existing and known neutrino oscillation parameters can be summarized as follows [1]:

\[
\begin{align*}
\delta m^2[10^{-5}eV^2] &= (6.94 - 8.14), \\
|\Delta m^2[10^{-3}eV^2]| &= (2.47 - 2.63) - (2.37 - 2.53), \\
\sin^2\theta_{12} &= (0.271 - 0.369), \\
\sin^2\theta_{23} &= (0.434 - 0.610) - (0.433 - 0.608), \\
\sin^2\theta_{13} &= (0.02000 - 0.02405) - (0.02018 - 0.02424), \\
\delta &= (128^\circ - 359^\circ) - (200^\circ - 353^\circ),
\end{align*}
\]

(1.1)

Multiple sets of allowed ranges are stated, and the left columns corresponds to normal hierarchy and the right columns to inverted hierarchy. $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - m_2^2$.

The measurement indicate that $\theta_{13}$ is non-zero by more than $5\sigma$ [2] and is small compared to the other neutrino mixing angles.

The lepton mixing matrix in the standard parametrization is [3]:

\[
U_{PMNS} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & 0 \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}
\end{pmatrix}
\begin{pmatrix}
  e^{i\theta} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & e^{i\sigma}
\end{pmatrix},
\]

(1.2)

*Electronic address: n.razzaghi@qiau.ac.ir*
where \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \) for \( i, j = 1, 2, 3 \) \( (i < j) \); \( \delta \) is known as the Dirac phase, analogous to the CKM phase, and \( \rho \) and \( \sigma \) are known as the Majorana phases and are applicable to Majorana neutrinos.

A Dirac mass term for the neutrinos and charged leptons is written as

\[
\mathcal{L}_m = -\bar{\ell}_L M_e \ell_R - \bar{\nu}_L M_D \nu_R + h.c.,
\]  

(1.3)

Friedberg and Lee (FL) proposed a successful phenomenological model of neutrino mass\(^\text{(4)}\) with a suitable flavor symmetry for Dirac neutrinos. In this model the charged-lepton mass matrix is diagonal. Therefore, neutrino mixing matrix can simply be described by a \( 3 \times 3 \) unitary matrix \( U \) that transforms the neutrino mass eigenstates into flavor eigenstates, \((\nu_e, \nu_\mu, \nu_\tau)\). In the pure FL model, one of the neutrino masses is exactly zero, which is partly responsible for the smallness of the neutrino masses. Furthermore, assuming \( \mu - \tau \) symmetry, the matrix \( U \) reduces to the tribimaximal mixing matrix\(^\text{(2)}\).

The Dirac neutrino mass operator of the FL model can be written as

\[
\mathcal{M}_{FL} = a (\bar{\nu}_\tau - \bar{\nu}_\mu) (\nu_\tau - \nu_\mu) + b (\bar{\nu}_\mu - \bar{\nu}_e) (\nu_\mu - \nu_e) + c (\bar{\nu}_e - \bar{\nu}_\tau) (\nu_e - \nu_\tau) + m_0 (\bar{\nu}_e \nu_e + \bar{\nu}_\mu \nu_\mu + \bar{\nu}_\tau \nu_\tau).
\]  

(1.4)

All the parameters in this model \((a, b, c \text{ and } m_0)\) are assumed to be real. For \( m_0 = 0 \), this Lagrangian has the following symmetry \( \nu_e \rightarrow \nu_e + z, \nu_\mu \rightarrow \nu_\mu + z, \) and \( \nu_\tau \rightarrow \nu_\tau + z \), where \( z \) is an element of the Grassman algebra. In a special case where \( z = \text{constant} \), we get a FL symmetry\(^\text{(3)}\), whose kinetic term is also invariant. However the other terms of the electroweak Lagrangian do not have such symmetry. The \( m_0 \) term explicitly breaks this symmetry.

However, we should mention that the FL symmetry leads into a magic matrix and this property is not broken by the \( m_0 \) term. The magic symmetry has many demonstrations which we will talk about in details. It has also been argued that the FL symmetry is the residual symmetry of the neutrino mass matrix after the \( SO(3) \times U(1) \) flavor symmetry breaking\(^\text{(3)}\). The mass matrix can be displayed by,

\[
M_{FL} = \begin{pmatrix}
-b + c + m_0 & -b & -c \\
-b & a + b + m_0 & -a \\
-c & -a & a + c + m_0
\end{pmatrix},
\]  

(1.5)

where \( a \propto (Y_{\mu\tau} + Y_{\tau\mu}), b \propto (Y_{e\mu} + Y_{\mu e}) \) and \( c \propto (Y_{e\tau} + Y_{\tau e}) \) and \( Y_{\alpha\beta} \) denotes the Yukawa coupling constant\(^\text{(1)}\). It is obvious that \( M_{FL} \) has an exact \( \mu - \tau \) symmetry only if \( b = c \). Setting \( b = c \) and using the hermiticity of \( M_{FL} \), a straightforward diagonalization procedure yields \( U_{TBM}^T M_{FL} U_{TBM} = \text{Diag} \{m_1, m_2, m_3\} \), where

\[
m_1 = 3b + m_0 \quad m_2 = m_0 \quad m_3 = 2a + b + m_0.
\]  

(1.6)

Therefore, the well-known tribimaximal (TBM) neutrino mixing matrix can be reproduced\(^\text{(2)}\), which can be written as

\[
U_{TBM} = \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{\sqrt{6}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\
\frac{\sqrt{3}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]  

(1.7)

The exact tribimaximal mixing matrix \( U_{TBM} \) sets \( (U_{e3})_{TBM} = 0 \) independently of the model. The values of the mixing angles in the mixing matrix \( U_{TBM} \) agree with the current existing neutrino oscillation parameters except \( U_{e3} \).

The role of a non-zero \( \theta_{13} \) or equivalently \( U_{e3} \), is numerus. It is necessary for CP violation in neutrino oscillations and can explain leptogenesis. Of course, for CP violation, both \( \theta_{13} \) and the complex phase \( \delta \) should be non-zero. Moreover, \( \theta_{13} \neq 0 \) corresponds to the quark sector since the mixing between the three generations is a confirmed result, although the mixing angles in the lepton sectors are very small in comparison to the quark sector. It is obvious that Eq. (1.7) implies the following forms of the neutrino mass matrix in the flavor basis,

\[
\mathcal{M} = \frac{m_1}{6} \begin{pmatrix}
4 & -2 & -2 \\
-2 & 1 & 1 \\
-2 & 1 & 1
\end{pmatrix} + \frac{m_2}{3} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} + \frac{m_3}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix},
\]  

(1.8)

\(^1\) The proportionality constant is the expectation value of the Higgs field.
where $m_i (i = 1, 2, 3)$ are the neutrino mass eigenvalues. The second term in Eq. (1.8) is noteworthy. This form of neutrino mass matrix is called the democratic matrix [7]. Democratic matrix is a phenomenological model of Dirac neutrino mass with $S(3)_L \times S(3)_R$ flavor symmetry. A democratic basis is adopted to produce a flavor–democratic mass matrix in which all elements of the matrix are equal. In this basis, the $S_3$ operations generate the permutations of three objects, e.g., the exchanging the first and second indices. Invariance under such transformations requires the universal size of the three-generation couplings when belonging to a three-dimensional representation of $S_3$ and the Higgs field is in the singlet.

The smallness of $\theta_{13}$ compared to the other mixing angles persuade us to modify the neutrino mixing matrix by a small perturbation in the basic tribimaximal structure and could lead to a realistic neutrino mixing matrix. We focus our interest on the neutrino mass model proposed by Friedberg and Lee (FL) in a very special case, which leads us to obtain the Democratic mass matrix in the flavor basis. The democratic matrix is suitable for generating $\theta_{13} \neq 0$ from an initial tribimaximal form. However, the result of its mass spectrum with the $U_{TBM}$ is experimentally unfavorable at first glance. Therefore, the main objective of this paper is to find the experimentally favored mass spectrum of the Democratic matrix together with the realistic neutrino mixing matrix through a small perturbation in the basic tribimaximal framework. There are many neutrino mass models from which the tribimaximal form of the neutrino mixing matrix [8] can be obtained. Moreover, many theoretical and phenomenological works have discussed massive neutrino models to generate $\theta_{13} \neq 0$ in different ways starting from an initial tribimaximal form [9]. To have CP-violation in the standard parametrization given in Eq. (1.2), the necessary conditions are $\delta \neq 0$ and $\theta_{13} \neq 0$.

The Jarlskog rephasing invariant parameter $J$ [10],

$$J = \text{Im}(U_{11}U_{12}^\ast U_{13}^\ast U_{22}),$$  

(1.9)

is relevant for CP violation in lepton number conservation processes like neutrino oscillations. Oscillation experiments cannot distinguish between the Dirac and Majorana neutrinos. Numerous theoretical and phenomenological works have discussed massive neutrino models breaking $\mu - \tau$ symmetry as a prelude to CP violation.

In this paper, we generalize the FL model by introducing complex parameters. We work on the massive FL model and set some obvious constraints that mass eigenvalues become real. In generalized FL model, there is a confined region of the parameter space where CP violation arises. However, we focus on a notable point, point D, on the border of this region that the neutrino mass matrix is changed to the Democratic neutrino mass matrix in the parameter space. The democratic neutrino mass matrix is not consistent with the experimental data $m_1 = m_3 = 0$. Therefore, we consider a case with $m_1 \neq 0$ by adding the symmetry breaking term based on FL symmetry. Consequently, the universality of the experimental data imposes some constraints on the elements of the mass matrix in the basic structure of $U_{TBM}$. Subsequently, we obtain our unperturbed neutrino mass matrix with $\mu - \tau$ symmetry and magic property in the flavor basis. However, we find that the solar mass splitting is absent, hence we generate this splitting by a small perturbation, which is responsible for $U_{e3} \neq 0$. To do this, we produce the perturbation mass matrix by applying the perturbation theory in the mass basis with complex elements, which breaks mildly the $\mu - \tau$ symmetry and this may generate $U_{13} \neq 0$ along with CP violation parameter. So we obtain nonzero values for both $\theta_{13}$ and $\delta$ which, together with a realistic neutrino mixing matrix, leads to a CP violation.

The outline of the paper is as follows. In section 2, we present our model and display the condition of a considerable point in the generalized FL model, and then present the results of complex perturbation analysis described above. Moreover, we obtain the perturbation mass matrix generating CP violation, solar mass splitting and we get a realistic neutrino mixing matrix. In section 3, we display that our model at point D is, in general, consistent with the available experimental data. Afterwards, we find the allowable ranges for all the parameters of the model. Ultimately, not only do we check the consistency of all of the results with the available experimental data, but also present our predictions for the actual masses, the Dirac phase, and the Jarlskog parameter. In section 4, we summarize and present the results. In Appendix A, we briefly introduce Twisted Friedberg-Lee symmetry.

II. THE MODEL

In this section, we generalize the FL model by adding complex Yukawa coupling constants to procure CP violation, which is achieved by obtaining $U_{e3} \neq 0$. However, we suppose that the eigenvalues of the generalized mass matrix are

\footnotesize
\begin{enumerate}
\item Detection of neutrinoless double beta decay would provide direct evidence for lepton number non-conservation and the Majorana nature of neutrinos.
\end{enumerate}
real. We figure out that only one specific choice allows for the softly breaking of \( \mu - \tau \) symmetry, i.e. \((a \in \mathbb{R}; \ b, c \in \mathbb{C}\) and \(b = c^*\)) which leads to a non-hermitian mass matrix. To simplify the notation, we define the parameters as follows: \( \Re (b) = \Re (c) = b_r \) and \( \Im (b) = -\Im (c) = B^3 \).

The neutrino mass matrix \( M'_\nu \) is given by

\[
M'_\nu = \begin{pmatrix}
2b_r + m_0 & -b_r & -b_r \\
-b_r & a + b_r + m_0 & -a \\
-b_r & -a & a + b_r + m_0
\end{pmatrix} + iB \begin{pmatrix}
0 & -1 & 1 \\
-1 & 1 & 0 \\
1 & 0 & -1
\end{pmatrix}.
\]

(2.1)

Note that \( M'_\nu \) is a symmetric matrix, therefore it could also be used as a Majorana mass matrix. In addition \( M'_\nu \) and \( M_{\mu \nu} \) are magic matrices since one of the eigenstates is \((\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{3}} \ \frac{1}{\sqrt{3}})\). We choose it to be \( |\nu_2\rangle \) to be consistent with Eq. (1.7). A naive diagonalization of \( M'_\nu \) yields,

\[
\begin{align*}
\tilde{m}_1 &= (a + 2b_r + m_0) + \sqrt{(a - b_r)^2 - 3B^2} \\
\tilde{m}_2 &= m_0, \\
\tilde{m}_3 &= (a + 2b_r + m_0) - \sqrt{(a - b_r)^2 - 3B^2}.
\end{align*}
\]

(2.2)

The results given in Eq. (2.2) are correct only in the limit \( B \to 0 \), when we compare the results in this limit with those in Eq. (1.3), we find that \( a < b \).

Since \( M'_\nu \) is a non-hermitian matrix, we require two distinguished unitary matrices \( U \) and \( V \) to diagonalize it.\(^4\) that \( V = U^* \). The resulting valid diagonal matrix is acquired by \( M'_{\text{diag}} = U^\dagger M'_\nu V \) and its elements are:

\[
\begin{align*}
m'_1 &= iB(a - b_r) + 3B^2 + (a + 2b_r + m_0)^2 - (a - b_r + iB) \sqrt{3B^2 + (a + 2b_r + m_0)^2} \\
m'_2 &= m_0, \\
m'_3 &= iB(a - b_r) + 3B^2 + (a + 2b_r + m_0)^2 + (a - b_r + iB) \sqrt{3B^2 + (a + 2b_r + m_0)^2},
\end{align*}
\]

(2.3)

where \( m'_1 \) and \( m'_3 \) are complex.\(^5\) The most general form of the diagonal neutrino mass matrix can be written as,

\[
M'_{\text{diag}} = e^{i\alpha}c^{i\beta}c^{i\lambda}s^{i\mu}M'_{\text{diag}}^\text{real}.
\]

(2.4)

\(\alpha\) is an overall phase, in our model it automatically turns out to be zero. We would not use the overall phase even if it were not zero. Due to the fact that \( m'_2 \) is real, we acquire \( \beta = \gamma \). This leads that the \( \arg(m'_1) = -\arg(m'_3) = 2\beta^2 \).

Using this condition in Eq. (11) we get

\[
B = \pm \sqrt{\frac{-(2a + b_r + m_0)(3b_r + m_0)}{3}}
\]

(2.5)

To require that the quantity \( B \) take on real values, one must constrain \( b_r \) as \(-\frac{m_0}{3} \leq b_r \leq -(2a + m_0)\), where the lower bound of \( b_r \) is a check on the condition \( m_1 > 0 \) in Eq. (1.3). Since \( U \) and \( V \) should approach \( U_{TBM} \) (given by Eq. (1.4)) in the limit \( B \to 0 \), we get \( 2b_r + a + m_0 \geq 0 \). Using (2.5) and \( a < b \), we get \( 3a + m_0 \leq 0 \). Finally, from the allowed regions for the parameters of the model and considering the overall symmetry of the F.L model we get \( b_r < 0^7 \). We have to take into account that a CP violation can only occur in a restricted range in the plane \( a-b_r \) with \( B \neq 0 \). In the figure (1) we have shown the allowed region of the parameter space in which the CP violation occurs.

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\(^3\) The parameters that distinguish the measure of CP violation and \( \mu - \tau \) symmetry breaking are proportional to \( B \) and so we expect them to be small.

\(^4\) These matrices can be obtained by diagonalizing \( M'_\nu M'_\nu^\dagger \) and \( M'_\nu^\dagger M'_\nu \), separately. \( U \) and \( V \) are the conventional transformation matrices for left-handed and right-handed neutrinos, respectively.

\(^5\) In the Dirac case \(^1\), we can extract the phases and transform them into the mass eigenstates.

\(^6\) Note that ignoring the overall phase is equivalent to the following: \( \text{Det}(M'_{\text{diag}}) \) is real and \( \text{Det}(U) = 1 \).

\(^7\) This conclusion is consistent with the results of the experiments on the oscillation of solar neutrinos, which indicate that \( m_2 > m_1 \).
Figure 1: CP violation is only possible in the right-angled triangle in the $\alpha - \beta$ parameter space, where $\alpha \equiv \frac{am_0}{m_0}$ and $\beta \equiv \frac{br}{m_0}$. The specified triangle shows the allowed region within our model. The red point shows our remarkable point D at $(-\frac{1}{3}, -\frac{1}{3})$. (The line above the base of the triangle is given by $2br + a + m_0 = 0$).

A detailed study of this region, its corresponding CP violation, achievable consequences, and its agreement with experimental data has been presented in Ref. [13].

In Figure (1) there is a remarkable point (red point) which displays $a = b_r = -\frac{m_0}{3}$ where at it $B = 0$. Therefore at first glance, it seems we cannot have CP violation at it. We have called this notable Point, point D.

The study of point D is significant because at this point we have $\Delta m^2 = 0$.

In the following, we focus our study only at point D (see Figure (1)) where the generalized FL neutrino mass matrix, $M'_\nu$ in Eq. (2.1), is reduced to the following specific form:

$$M'_{\nu|\text{point } D} = (3b_r + m_0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - b_r \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= (3b_r + m_0)1 - b_r D. \quad (2.6)$$

The first term in Eq. (2.6) certainly vanishes and $M'_\nu$ reduces to $-b_r D$, $D$ denotes democratic mass matrix. Therefore $M'_\nu$ obtains the democratic form that can be realized by the $S_3$ family of symmetric permutations [7]. Here we separately impose the $S_3$ permutation symmetry for the left- and right- handed neutrinos as flavor symmetry, which are indicated by $S_{3L}$ and $S_{3R}$, respectively. Therefore it leads to $M'_\nu$ in Eq. (2.6) invariant under $S_{3L} \times S_{3R}$ symmetry.

$M'_\nu$ in Eq. (2.6) can be diagonalized by $U_{TB}M'$ that yields $m_2 = m_0$ and $m_1 = m_3 = 0$. Since such a neutrino mass matrix is experimentally unfavorable, we should rescue $M'_\nu$ by replacing it with an alternative one (in which $m_1$ or $m_3$ does not vanish) by using a possible procedure. With that in mind, let us consider a non-zero $m_1$ case below by adding the symmetry-breaking term based on FL symmetry.

The combination of FL symmetry with $\mu - \tau$ symmetry is a kind of familiar translational symmetry called twisted FL symmetry [14]. The twisted FL symmetry for Dirac neutrinos is separately imposed on left-handed and right-handed neutrinos as follows:

$$\nu_Li \rightarrow \nu'_L i = S^L_i \nu_L j + \Lambda_L j z$$

$$\nu_Ri \rightarrow \nu'_R i = S^R_j \nu_R j + \Lambda_R j z, \quad (2.7)$$

where $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)^T$ are c-numbers, $z$ is a space-time independent Grassmann parameter such that $z^2 = 0$, and $S$ is the permutation matrix between the second and third families, as follows

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \quad (2.8)

Therefore, As mentioned in Appendix A under these transformations, the breaking mass matrix for the $S_{3L} \times S_{3R}$ flavor symmetry of $M'_\nu$ in Eq. (2.6) obtain as follows

$$M^B_{\nu} = g \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}. \quad (2.9)$$

We added the breaking mass matrix, $M^B_{\nu}$ in Eq. (2.9), to $M'_\nu$ in Eq. (2.6) as a breaking mass matrix for the $S_{3L} \times S_{3R}$
flavor symmetry. Consequently, the total neutrino mass matrix at point D is written as follows

$$M_{\nu}^{BD}_{\text{point D}} = \frac{m_0}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + g \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}. \quad (2.10)$$

We call it Broken Democratic (BD) neutrino mass matrix and its allowable neutrino mass spectrum is given by

$$\tilde{m}_1 = 6g, \quad \tilde{m}_2 = m_0, \quad \tilde{m}_3 = 0. \quad (2.11)$$

Note that, up to now, the achievements are: (i) the twisted FL symmetry is employed as a breaking mass matrix for the $S_3L \times S_3R$ flavor symmetry, (ii) the total mass matrix in Eq. (2.10) violates the $S_3L \times S_3R$ symmetry but preserves the $\mu - \tau$ symmetry and magic property. Therefore, still $U_{TBM}$ can be realized as the mixing matrix and (iii) the neutrino masses ordering is inverted.

Thus, the main objective will be to find a small perturbation mass matrix together along with a realistic neutrino mixing matrix that is corresponding $M_{\nu}^{BD}$ in Eq. (2.10) on the basic tribimaximal mixing matrix. Therefore, according to Eq. (1.8), a general mass matrix $M$ satisfies

$$U_{TBM} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \quad (2.12)$$

where

$$A = m - \frac{\Delta_{31}}{3},$$
$$B = \frac{\Delta_{31} - \Delta_{32}}{3},$$
$$C = \frac{\Delta_{31}}{2}. \quad (2.13)$$

and we have set

$$m = \frac{\sum m_i}{3},$$
$$\Delta_{3j} \equiv (m_3 - m_j), \quad \text{for} \quad j = 1, 2. \quad (2.14)$$

Substituting the $\tilde{m}_i$s in Eq. (2.11) into (2.14), we obtain $m = \frac{m_0}{3} + 2g$, $\Delta_{32} \equiv -m_0$ and $\Delta_{31} \equiv -6g$. We work in a flavor basis in which mixing in the lepton sector is determined entirely by the neutrino mass matrix.

Experimental data has definitely confirmed that $\delta m^2 = m_2^2 - m_1^2 > 0$ and it is too small. Substituting $\tilde{m}_i$s in Eq. (2.11) into $\delta m^2$, we obtain $|g| < \frac{m_0}{3}$. Then assuming $\Delta_{32} \simeq \Delta_{31} \equiv \Delta$, we obtain $\Delta \simeq -m_0 \simeq -6g$, which indicates $g \simeq \frac{m_0}{6}$, and $\Delta$ is negative$^8$. In the following by using these approximations the neutrino mass matrix $M_{\nu}^{BD}$ in Eq. (2.10) in the flavor basis rewritten as follows

$$M_{\nu}^{(0)}_{\text{point D}} \simeq \begin{pmatrix} m - \frac{\Delta}{3} & 0 & 0 \\ 0 & m + \frac{\Delta_3}{3} & -\frac{\Delta}{2} \\ 0 & -\frac{\Delta}{2} & m + \frac{\Delta}{3} \end{pmatrix}, \quad (2.15)$$

here in $M_{\nu}^{(0)}$ is the unperturbed neutrino mass matrix with magic $\mu - \tau$ symmetry (at point D). Therefore, The mixing matrix corresponding to $M_{\nu}^{(0)}$ is still $U_{TBM}$.

The eigenvalues of $M_{\nu}^{(0)}$ in Eq. (2.15) are obtain as follows,

$$m_1^{(0)} \simeq m_2^{(0)} = m - \frac{\Delta}{3} = m_0, \quad \text{and} \quad m_3^{(0)} = m + \frac{2\Delta}{3} = 0. \quad (2.16)$$

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$^8$ We should note that $\Delta$ is an important quantity due to the scale for atmospheric neutrino oscillations is set by it.
\( m_1^{(0)}, \) and \( m_2^{(0)} \) are real and positive, although at this level, the solar mass splitting is absent. Also \( m_3 = 0 \), similar to the result obtained in [13], therefore the ordering of neutrino masses is still inverted.

We should note that, up to now, the shortcomings are: (i) the absence of the solar neutrino mass splitting, and (ii) the mixing matrix is still \( U_{TBM} \). Thus, we will aim to obtain the solar mass splitting by the same mass perturbation that is the cause of \( \theta_{13} \neq 0 \) and CP violation. Finally, in this procedure CP violation conditions necessarily mandate that \( \mu - \tau \) symmetry should be broken. An interesting question is whether \( \theta_{23} = 45^\circ \) holds after the \( \mu - \tau \) symmetry breaking.

In summary, up to now, we have proposed that the generalized FL neutrino mass matrix in Eq. (2.1) at point D (see Figure (11)) has a combination in which \( m_1, m_3 \), and CP violation parameters are vanishing, while \( \theta_{23} = 45^\circ \). Moreover, \( \delta m^2 \) does not vanish but is not consistent with experimental data. However, the solar mixing angle \( \theta_{12} \) can be chosen by the mixing angles of the tribimaximal mixing matrix. This is a promising estimate of the observed data although some characteristics are missing at point D. Therefore, at this notable point the neutrino mass matrix in Eq. (2.6) has two mass eigenvalues, \( m_1 \) and \( m_3 \), which are degenerate, hence it is highly distinctive from the obtained results of the generalized FL model in [13] and the experimental data [1], in which \( m_1 \neq m_3 \). In the following, we obtain the allowable neutrino mass spectrum of the generalized FL model at point D by adding the breaking mass matrix for the \( S_{3L} \times S_{3R} \) flavor symmetry in to the neutrino mass matrix in Eq. (2.6). Then, according to the available experimental data for the mass of neutrinos, we apply some approximations to the elements of the mass matrix \( M^B_{\nu} \) in Eq. (2.10). Therefore, we obtain the mass matrix \( M^0_\nu \) in Eq. (2.15) that its mass spectrum shows the absence of the solar neutrino mass splitting. In the next stage, we will consider obtaining a perturbation matrix, which could generate not only CP violation parameters in the neutrino mixing matrix, namely, \( U_{13} \), \( (\theta_{13} \) and \( \delta) \) but also \( \delta m^2 \) which provides only minor corrections to the mixing angle \( \theta_{12} \) (but not to \( \theta_{23} \)). We believe, because of the small \( \theta_{13} \) and \( \delta m^2 \), the perturbation method is a more meticulous method than others for finding the correction of the tribimaximal mixing matrix. Moreover, to the best of our knowledge, this kind of generalized FL model at a specific point has not been performed in the literature yet.

As mentioned before, our method for establishing the structure of the neutrino mass matrix, at point D, is to apply perturbation theory. Thus, we have \( M_{\nu, \text{point D}} = M_{\nu}^0 + M_{\nu}^P \), where \( M_{\nu}^0 \) is a perturbation matrix and \( M_{\nu}^P \ll M_{\nu}^0 \). Both \( M_{\nu}^0 \) and \( M_{\nu}^P \) will be symmetric and could, in general, be complex. However, in our case \( M_{\nu}^0 \) in Eq. (2.10) is a Hermitian matrix (real and symmetric). The following will consider the case of complex \( M_{\nu}^P \). Therefore, in this situation, we have \( \theta_{13} \neq 0 \) and \( \delta \neq 0 \) which means CP is violated and the solar neutrino masses are split.

The eigenstates of \( M_{\nu}^0 \) in the mass basis, the unperturbed mass eigenstates, are as follows,

\[
|\nu_1^{(0)}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\nu_2^{(0)}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\nu_3^{(0)}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

(2.17)

in which \( |\nu_1^{(0)}\rangle \), and \( |\nu_2^{(0)}\rangle \) are degenerate. We investigate \( M_{\nu}^P \) such that the first two mass eigenstates in Eq. (2.17) are its non-degenerate eigenstates, namely, \( (\nu_i^{(0)}|M_{\nu}^P|\nu_j^{(0)}) = m_i^{(1)} \delta_{ij} \) where \( (i, j = 1, 2) \), with \( m_1^{(1)} \neq m_2^{(1)} \). Therefore, in the mass basis, we take \( (M_{\nu}^P)_{33} = 0 \) and consequently consider only \( (M_{\nu}^P)_{13} \) and \( (M_{\nu}^P)_{23} \). So the basis vectors \( |\nu_1^{(0)}\rangle \), and \( |\nu_2^{(0)}\rangle \) are chosen with the aim that they regenerate the correct solar mixing and the physical basis is fixed by perturbation. Needless to say, when eigenstates in Eq. (2.17) expressed in the flavor basis are the columns of \( U_{TBM} \) in Eq. (1.7). Therefore in the flavor basis, eigenstates are as follows,

\[
|\nu_1^{(0)}\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{1}{6}} \sqrt{\frac{1}{\sqrt{3}}} \end{pmatrix}, \quad |\nu_2^{(0)}\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \sqrt{\frac{1}{\sqrt{3}}} \\ \frac{1}{\sqrt{3}} \sqrt{\frac{1}{\sqrt{3}}} \end{pmatrix}, \quad |\nu_3^{(0)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(2.18)

our goal is to obtain the third perturbed mass eigenstate in the CP violated case that when expressed in the flavor basis is the third column of the mixing matrix in Eq. (1.2). Thus we can obtain the elements of the perturbation mass matrix \(^9\).

we suppose \( M_{\nu}^P \) which is symmetric, to be a complex matrix, and therefore not Hermitian, and this is true as well as for the total mass matrix \( M_{\nu} = M_{\nu}^0 + M_{\nu}^P \). Therefore, this is accomplished by obtaining nonzero values for \( \theta_{13} \) and \( \delta \).

\(^9\) we do not set a perturbation mass matrix by hand; instead, we obtain it by using the third perturbed mass eigenstate
The columns of the mixing matrix $U$ in the Eq. (1.2) must be eigenvectors of $M^\nu_0 M^\nu_0 + M^\nu_1 M^\nu_1$, where we have dropped the term $O(M^\nu)^2$. In the following, we should mention that unperturbed $M^\nu_0$ is Hermitian and its eigenstates are the same as the columns that produce $U_{TBM}$, in the Eq. (1.3), and its eigenvalues are $|m_1^{(0)}|^2$, $|m_2^{(0)}|^2$, and $|m_3^{(0)}|^2$. For the perturbation expansion we keep terms up to linear order in $s_{13}$. To first order we have

$$|\nu_3^{>0}| = |\nu_3^{(0)}| + \sum_{j \neq 3} C_{3j}|\nu_j^{(0)}|.$$  \hspace{1cm} (2.19)

Where,

$$C_{3j} = -C_{3j}^* = \left( |m_3^{(0)}|^2 - |m_j^{(0)}|^2 \right)^{-1} M_{j3}, \quad (j \neq 3)$$  \hspace{1cm} (2.20)

where $M_{j3} = <\nu_j^{(0)}| (M^\nu_0 M^\nu_1 + M^\nu_1 M^\nu_0) |\nu_3^{(0)}>$ and the coefficients $C_{3j}$ are complex and proportional to $M_{j3}$ in the mass basis [13].

Evidently, $|\nu_3|$ in Eq. (2.19) should correspond to the third column of the neutrino mixing matrix in Eq. (1.2). we can quickly determine $C_{31}$ and $C_{32}$ by using Eq. (2.19) in the flavor basis. Therefore explicitly we obtain the matrix equation, as follows

$$
\begin{pmatrix}
s_{13} e^{-i\delta} \\
s_{23} c_{13} \\
c_{23} s_{13}
\end{pmatrix} = \begin{pmatrix}
0 \\
-\sqrt{2} \\
\sqrt{2}/2
\end{pmatrix} + \begin{pmatrix}
0 & \sqrt{2} e^{-i\delta} & -\sqrt{2} e^{-i\delta} \\
\sqrt{2} e^{-i\delta} & 0 & \sqrt{2} e^{-i\delta} \\
-\sqrt{2} e^{-i\delta} & -\sqrt{2} e^{-i\delta} & 0
\end{pmatrix}.
$$  \hspace{1cm} (2.21)

Readily one obtains, to order linear in $s_{13}$, $C_{31} = -\sqrt{2}s_{13} e^{-i\delta}$ and $C_{32} = \sqrt{2}s_{13} e^{-i\delta}$, where maximality of the 2-3 mixing angle has been used ($\theta_{23} = 45^\circ$). Therefore, in the mass basis by using Eq. (2.11) and Eq. (2.20) we obtain, $(M^\nu_{13}) = m_0 \sqrt{2} s_{13} e^{-i\delta}$ and $(M^\nu_{23}) = -m_0 \sqrt{2} s_{13} e^{-i\delta}$.

Up to now, we have focused on deducing $\theta_{13} \neq 0$ via a perturbation method starting from the basic tribimaximal neutrino mixing matrix. Now, we attend the solar mass splitting. In our framework of minimal perturbation matrix, we choose $(M^\nu_{12}) = (M^\nu_{21}) = 0$. The first-order corrections to the neutrino masses are obtained from $m_1^{(1)} \delta_{1j} = <\nu_j^{(0)}| M^\nu \nu_j^{(0)} >$. We seek that the first-order of neutrino mass corrections as

$$m_1^{(1)} = m_3^{(1)} = 0 \quad \text{and} \quad m_2^{(1)} \neq 0.$$  \hspace{1cm} (2.22)

Therefore, in the mass basis, this implies that $(M^\nu_{22}) \neq 0$ and other diagonal elements of the perturbation matrix to be zero. Such a correction certifies to split solar neutrino masses, in which $m_2^{(1)} = m_2 - m_1$, and $\delta m^2$ to be positive.

Consequently, in the mass basis, we have the final form of the perturbation mass matrix, at point D, as follows

$$M^\nu_{13}\big|_{\text{point D}} = m_0 \begin{pmatrix}
s_{13} e^{-i\delta} \\
s_{23} c_{13} \\
c_{23} s_{13}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \sqrt{2} e^{-i\delta} \\
0 & \sqrt{2} e^{-i\delta} & -\sqrt{2} e^{-i\delta} \\
\sqrt{2} e^{-i\delta} & -\sqrt{2} e^{-i\delta} & 0
\end{pmatrix},$$  \hspace{1cm} (2.23)

where $\varepsilon \equiv \frac{m_1^{(1)}}{m_0 s_{13}}$ is a dimensionless parameter which relates the solar mass splitting, $m_2^{(1)}$, to $s_{13}$. In the next section, we will utilize $\varepsilon$ to compute the order of $s_{13}$. In general $m_2^{(1)}$, the solar mass splitting, can be complex as; $m_2^{(1)} \equiv |m_2^{(1)}| \exp(i\phi)$. Therefore, we can write $m_2 = m_2^{(0)} + m_2^{(1)} \equiv |m_2| \exp(i\phi)$ and obtain neutrino mass spectrum at point D as follows

$$m_1^{(0)} = |m_1^{(0)}| = m_0, \quad |m_3^{(0)}| = 0,$$

$$m_2^{(1)} = \left[(m_1^{(0)})^2 + (m_2^{(0)})^2 + m_1^{(0)} m_2^{(0)} \cos \varphi \right]^{1/2}, \quad \phi = \tan^{-1} \left[ \frac{|m_1^{(0)}| \sin \varphi}{m_1^{(0)} + |m_2^{(0)}| \cos \varphi} \right].$$  \hspace{1cm} (2.24)

For the Dirac neutrinos, the phase $\phi$ can be removed but for the Majorana neutrinos this phase remains as Majorana phase and contribute to CP violation.
Now, let us rewrite $M_{\nu}^P$ [which is given by Eq. (2.23) and it was calculated in the mass basis] in the flavor basis by employing the relation between mass and flavor basis as

$$M_{\nu}^{P(f)}_{\text{point D}} = m_0 \, s_{13} \left[ e^{-i\delta} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \frac{\varepsilon}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right],$$

(2.25)

where the first and the second terms of $M_{\nu}^{P(f)}$ are responsible for 1-3 mixing angle and for solar neutrino mass splitting, respectively. We observe that $M_{\nu}^{P(f)}$ in Eq. (2.25) violates magic and $\mu - \tau$ symmetry of the total neutrino mass matrix. Therefore, by using degenerate perturbation theory [16], to order linear in $s_{13}$, and $M_{\nu}^P$ in Eq. (2.24) we obtain the neutrino mixing matrix with CP violation parameters in the lepton sector at point D, as follows

$$U|_{\text{point D}} = U_{\text{TBM}} + s_{13} e^{i\delta} \begin{pmatrix} 0 & 0 & e^{-2i\delta} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} & 0 \end{pmatrix},$$

(2.26)

$U$ is unitary up to order $s_{13}$.

In [17], with a different point of view, this same structure for the neutrino mixing matrix has been discussed and the consistency with the observed mixing angles noted.

A rephasing-invariant measure of CP violation in neutrino oscillation is the universal parameter $J$ [10] given in Eq. (1.9), where its form is independent of the choice of the Dirac or Majorana neutrinos.

Using Eq. (1.9) and Eq. (2.26) the expression for $J$ simplifies to,

$$J|_{\text{point D}} = -\frac{1}{3\sqrt{2}} s_{13} \sin \delta,$$

(2.27)

as mentioned before, to have CP violation in the lepton sector, both $s_{13}$ and $\delta$ must take nonzero values.

### III. COMPARISON WITH EXPERIMENTAL DATA

In this section, we compare the results obtained from our generalized FL model at remarkable point D (while $B = 0$) with the available experimental data in Eq. (1.1). It is important to note that in the generalized FL model [13] almost there are some numerical predictions for neutrino parameters while $B \neq 0$. Therefore, in this section, we compare our herein results with the corresponding ones obtained in [13], too. Let us accomplish this in two steps.

In the first step, we compute the allowed ranges for the parameters of the neutrino mass matrix along with the perturbation term. We do this, by comparing and mapping our herein results with neutrino mass constraints obtained from the available experimental data for $\delta m^2$ and $\Delta m^2$ in Eq. (1.1) respectively.

Therefore, the allowed ranges for $m_0$ and $g$ are obtained as,

$$m_0 \approx \pm (4.87 - 5.03) \times 10^{-2} eV,$$

$$g \approx \pm (0.812 - 0.838) \times 10^{-2} eV,$$

(3.1)

which agree well with the allowed ranges for $m_1$ and $m_0$ that obtained in [13]. Also, we see that we can have $m_0 > 0$ and $m_0 < 0$.

In Figure 2 we have shown the limits imposed by $\delta m^2$ and $m_0 > 0$ on $m_2^{(1)}$ and in Figure 3 have shown the limitations for $m_0 < 0$. In Figure 2 and 3 we have plotted the overlap of $\delta m^2$ in Eq. (2.24) by using the experimental data in Eq. (1.1) with the allowed ranges of $m_0$ in Eq. (3.1) onto the $m_2^{(1)}$ and $\varphi$ perturbation parameter space. Therefore, when $m_0 > 0$, our results for the perturbation parameters are given by

$$m_2^{(1)} \approx (0.073 - 0.9) \times 10^{-2} eV,$$

(3.2)

and when $m_0 < 0$, our results are as follows

$$m_2^{(1)} \approx (0.073 - 0.9) \times 10^{-2} eV.$$  

(3.3)
Figure 2: (color online). In this figure, the whole region of the $m(1)^2 - \varphi$ plane which is allowed by our model along with the allowed range of $m_0 > 0$ is shown. Each color curve implies a value of $m_0$ in the range $(4.87 - 5.03)10^{-2}eV$ in the $m(1)^2 + 2m_0m(1)^2 \cos \varphi$. When $m_0 > 0$, the overlap region of the experimental values for $\Delta m_{21}^2$ with our model are two tiny regions. These regions are the semi-symmetry of each other.

A point that should not be ignored about the allowed obtained ranges of $m(1)^2$ in Eq. (3.2) and Eq. (3.4) is that the obtained values of $m(1)^2$ must predict the values of $\sin \theta_{13}$, that to be consistent with the experimental data in Eq. (1.1). To check this issue, we examine the parameter $\varepsilon = \frac{m(1)^2}{m_0 \sin \theta_{13}}$. We assume that all nonzero components of the perturbation matrix in Eq. (2.23) should be in a similar order. We may then expect $\varepsilon \sim O(1)$, therefore by using the order of $m_0$ and $m(1)^2$ from the previous equations we must obtain $\sin \theta_{13} \sim O(10^{-1})$. We do this and find that the whole range of $m(1)^2$ in Eq. (3.2) and Eq. (3.4) are not consistent with the experimental order of $\sin \theta_{13}$. Therefore, the consistent range of $m(1)^2$ with the all of the experimental data for both cases, $m_0 > 0$ and $m_0 < 0$, is as follows,

\[
|m(1)^2| \approx (0.487 - 0.9)10^{-2}eV,
\]
\[
\varphi \approx \{(83.12^\circ - 91.72^\circ) - (269.43^\circ - 275.16^\circ)\}, \quad \text{when } m_0 > 0
\]
\[
\approx \{(88.85^\circ - 97.45^\circ) - (263.69^\circ - 269.43^\circ)\}, \quad \text{when } m_0 < 0.
\]

Therefore, we obtain $\sin \theta_{13} \sim O(10^{-1})$ based on the values of $m_0$, $m(1)^2$ (respectively in Eq. (3.2) and Eq. (3.4)) and considering $\varepsilon \sim O(1)$. Further, employing Eq. (2.26), and considering only the order of $\sin \theta_{13}$, we obtained the allowed range of $\sin^2 \theta_{23}$ as follows,

\[
\sin^2 \theta_{23} = \frac{|U_{23}|^2}{1 - |U_{13}|^2} \approx (0.505 - 0.549),
\]

which agrees well with the experimental data.

Therefore, not only we have obtained all the parameters of the model at the remarkable point D, but also make predictions for neutrino masses, see Eq. (3.6). According to our predictions in Eq. (3.6), we find that both obtained results related to $m_0 > 0$ and $m_0 < 0$ are in agreement with the available experimental data, see Eq. (1.1).
Figure 3: (color online). In this figure, the whole region of the $m_2^{(1)} - \varphi$ plane which is allowed by our model along with the allowed range of $m_0 < 0$ is shown. Each color curve implies a value of $m_0$ in the range $-(4.87 - 5.63) \times 10^{-2} eV$ in the $m_2^{(1)} + 2m_0m_2^{(1)} \cos \varphi$. When $m_0 < 0$, the overlap region of the experimental values for $\Delta m^2_{31}$ with our model is a tiny region.

\begin{align*}
m_1 &= m_0 \approx \pm(4.87 - 5.03) \times 10^{-2} eV, \\
|m_2| &\approx \{(4.95 - 5.08), (4.94 - 5.10)\} \times 10^{-2} eV, \\
&\approx \{(4.96 - 5.09), (4.95 - 5.12)\} \times 10^{-2} eV, \\
\phi &\approx \{((5.60^\circ) - (10.19^\circ)), ((-5.63^\circ) - (-10.16^\circ))\}, \\
&\approx \{((-5.59^\circ) - (-10.18^\circ)), ((5.61^\circ) - (10.12^\circ))\}, \\
m_3 &= 0, \\
\delta m^2 &\approx \{(5.46 - 7.86), (6.87 - 7.10)\} \times 10^{-5} eV^2, \\
&\approx \{(6.07 - 8.85), (7.86 - 9.13)\} \times 10^{-5} eV^2, \\
|\Delta m^2| &\approx (2.37 - 2.53) \times 10^{-3} eV^{2.10}. \quad (3.6)
\end{align*}

Note that as mentioned before, $\phi$ is the origin of the Majorana phases which is generated from the perturbation. We could dispense with the overall phase, $e^{i\phi}$. Therefore, for $m_0 < 0$ we would have two phases that emerge in the mass eigenvalues in Eq. (3.6) which are $\rho = (\frac{\pi}{2} - \frac{\varphi}{2})$ and $\sigma = -\frac{\varphi}{2}$. Whereas, for $m_0 > 0 \rho = \sigma = -\frac{\varphi}{2}$. For the Dirac neutrinos, these phases can be removed, and for the Majorana neutrinos, these phases remain as Majorana phases and contribute to CP violation in the lepton sector.\textsuperscript{11} \cite{18}.

In the following, in the second step, by employing and referring to the obtained values for the Jarlskog parameter which is restricted by $0.027 \leq J \leq 0.044$ in \cite{13}, we find the allowed ranges for $J$ and $\delta$. We do this by inserting the experimental data of $\sin \theta_{13}$ in to Eq. (2.27) and restricted $J$ in Eq. (2.27) to the allowed ranges of it in \cite{13}. Therefore, we obtain the allowed region of $J$ and $\delta$ in our work as follows

\begin{align*}
J &\approx (0.027 - 0.036), \\
\delta &\approx (229.30^\circ - 312.42^\circ). \quad (3.7)
\end{align*}

\textsuperscript{11} All of our model predictions for Dirac and Majorana neutrinos are the same, except Majorana phases in general.
An important experimental result for the sum of the three light neutrino masses has just been reported by the Planck measurements of the cosmic microwave background (CMB) at 95% CL \[19\], which is

\[ \sum m_\nu < 0.12eV(\text{Planck}+\text{WMAP}+\text{CMB}+\text{BAO}). \] (3.8)

This sum in our model is

\[ \sum m_\nu \approx \{ (0.0982 - 0.1011)eV - (0.0981 - 0.1013)eV \}, \quad \text{when } m_0 > 0 \]
\[ \approx \{ (0.0983 - 0.1012)eV - (0.0982 - 0.1015)eV \}, \quad \text{when } m_0 < 0. \] (3.9)

, which is exactly consistent with the constraint of the above.

### IV. CONCLUSIONS

In this paper we have recommenced a generalization of Friedberg-Lee neutrino mass model, in which CP violation is viable. Therefore, the elements of neutrino mass matrix could be complex. Of course with the constraint that the mass eigenvalues be real. We obtain and indicate the region in our parameter space where CP violation is possible. It is viable. Therefore, the elements of neutrino mass matrix could be complex. Of course with the constraint that the mass comes from the value of at point D whether or not is compatible with experimental data. In our work, the most restricting experimental data is Planck measurements of the cosmic microwave background (CMB) at 95% CL \[19\], which is consistent with the experimental data reported by Planck(+WMAP+CMB+BAO)experiment.

Therefore, we improve our model to obtain the experimentally favored neutrino mass spectrum by adding the breaking mass term which preserve the twisted FL symmetry to the mass matrix at point D. Hence, we obtain a neutrino mass spectrum without degeneracy and the corresponding mixing matrix is \( U_{TBM} \), also our method predicts the massless third generation neutrino and inverted mass hierarchy. In the following, our method is based on the \( U_{TBM} \) in which the mixing angles (except \( \theta_{13} \)) is consistent with the experimental data. The tribimaximal mixing matrix led us to generate a neutrino mass matrix, \( M_0^{\nu}\big|_{\text{pointD}} \), constrained by the elements of \( U_{TBM} \) and the available experimental data.

Therefore, \( M_0^{\nu} \) is unperturbed mass matrix and loses the solar neutrino mass splitting whereas it preserves magic and \( \mu - \tau \) symmetry features. In the following, by employing perturbation method, we produce a perturbation matrix, at point D, which breaks both the magic and \( \mu - \tau \) symmetry, therefore CP violation is happen. Our investigation proceeded as follows, we obtained the complex elements of the perturbation mass matrix in both the mass and flavor bases. In addition, We regenerate a realistic neutrino mixing matrix, at point D, with non-zero the Dirac phase. However, in this case, the \( \mu - \tau \) symmetry is softly broken, but still we have \( \theta_{23} = 45^\circ \).

For the purpose of getting our predictions regarding neutrino masses at point D, we compared the results of our phenomenological model with the available experimental data. We have display that how our phenomenological model at point D whether or not is compatible with experimental data. In our work, the most restricting experimental data comes from the value of \( m_1 = \sqrt{\Delta m^2} \) and the order of \( \sin \theta_{13} \). Mapping the allowed range of the experimental data of \( \delta m^2 \) onto the allowed region of our parameter space can designate valid values for our parameters. Also, we have shown that only by applying the order of \( \sin \theta_{13} \) can restrict the allowed region of the predicted solar mass splitting. Afterward, we can put the values of three masses, \( m_1 \approx \pm(4.87 - 5.03)\times 10^{-2}eV, \quad |m_2| \approx \{ (4.95 - 5.08), (4.94 - 5.10) \} \times 10^{-2} eV \} \cup \{ (4.96 - 5.09), (4.95 - 5.12) \} \times 10^{-2} eV \}, \quad \text{and } m_3 = 0 \). The compatibility of the allowed ranges of our parameters with the current experimental data shows that our model has inverted hierarchy as, \( m_3 = 0 \). We also obtain predictions for the CP violation parameters \( \delta \) and \( J \) (at point D). These are \( \delta \approx (229.30^\circ - 312.42^\circ) \), and \( J \approx (0.027 - 0.036) \). Our predictions are agree with the observational data reported by Planck(+WMAP+CMB+BAO)experiment.

In our method, at an unusual point D, the mass matrix was saved from having degenerate unfavored excremental eigenvalues and the least possible perturbation matrix was obtained by employing a fundamental process, and not only added by hand. Our predictions for the neutrino masses and CP violation parameters could be assayed in future neutrino experiments.

### V. APPENDIX A : TWISTED FRIEDBERG-LEE SYMMETRY

In this appendix, we give a detailed discussion how to obtain the neutrino mass matrix used in the main part from the twisted FL symmetry for Dirac neutrinos \[14\].
Let us consider the Dirac neutrino case

\[ -\mathcal{L}_D = \bar{\nu}_{Li} M^D_{ij} \nu_{Rj} + \text{h.c.} . \]  

(5.1)

For Dirac neutrinos, in general, the twisted FL symmetry can be imposed on the left- and right-handed neutrinos separately as

\[
\nu_{Li} \rightarrow \nu'_{Li} = S_{ij}^L \nu_{Lj} + \Lambda_{Lj} z ,
\]

(5.2)

\[
\nu_{Ri} \rightarrow \nu'_{Ri} = S_{ij}^R \nu_{Rj} + \Lambda_{Rj} z .
\]

(5.3)

Two independent \( \mu - \tau \) permutation symmetries make the Dirac mass matrix as

\[
M^D = \begin{pmatrix}
D & -2C & -2C \\
-2B & -A & -A \\
-2B & -A & -A
\end{pmatrix}
\]

(5.4)

while the translational symmetries lead to the conditions

\[
M^D_{ij} \Lambda_{Rj} = \begin{pmatrix}
D & -2C & -2C \\
-2B & -A & -A \\
-2B & -A & -A
\end{pmatrix}
\begin{pmatrix}
\Lambda_{R1} \\
\Lambda_{R2} \\
\Lambda_{R3}
\end{pmatrix} = 0 ,
\]

(5.5)

and

\[
\Lambda_{Li} \ M^D_{ij} = \begin{pmatrix}
\Lambda_{L1} & \Lambda_{L2} & \Lambda_{L3}
\end{pmatrix}
\begin{pmatrix}
D & -2C & -2C \\
-2B & -A & -A \\
-2B & -A & -A
\end{pmatrix}
= 0 .
\]

(5.6)

The resulting form of the mass matrix depends on the correlations among \( \Lambda_{Li} \) and \( \Lambda_{Ri} \) again. In this letter, we have assumed the uniform translation, that is \( \Lambda_{Li} \propto (1,1,1) \) and \( \Lambda_{Ri} \propto (1,1,1) \). Then, the mass matrix of the Dirac neutrino takes the form

\[
M^D = C \begin{pmatrix}
4 & -2 & -2 \\
-2 & 1 & 1 \\
-2 & 1 & 1
\end{pmatrix} .
\]

(5.7)

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