Bundled string solutions of the Bethe ansatz equations in the non-Hermitian spin chain

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Abstract. The asymmetric simple exclusion process (ASEP) is a paradigmatic stochastic integrable system that describes nonequilibrium transport phenomena. In this paper, we investigate the solutions of the Bethe equations for the ASEP in the thermodynamic limit. The Bethe equations for many Hermitian integrable systems, such as the spin-1/2 Heisenberg chain, have regular solutions called string solutions. However, in the case of the ASEP, the pattern of the string solution changes owing to its non-hermiticity. We call this new type of string solutions “bundled string solutions”. We introduce and formulate the bundled string solutions, and then derive the Bethe-Takahashi equation for the ASEP.

1. Introduction

Understanding many-body systems out of equilibrium is one of the most important challenges in modern physics. In most cases, it is difficult to analyze many-body systems without approximations. Therefore, integrable systems play a fundamental role in the investigation of many-body systems out of equilibrium. Recently, research on nonequilibrium physics for quantum integrable systems has been rapidly developing. For example, many aspects of relaxation dynamics and nonequilibrium steady states for isolated quantum systems have been revealed. Integrable systems do not relax to the usual thermodynamic ensembles, but to the generalized Gibbs ensemble (GGE) because of their extensive number of conserved charges \cite{1, 2}. Thermodynamic quantities in the GGE are obtained by the thermodynamic Bethe ansatz (TBA) \cite{3, 4, 5, 6}. The Quench Action provides a method for calculating time dependent expectation values of observables after quantum quench \cite{7, 8}. The generalised hydrodynamics (GHD), which is a powerful hydrodynamic framework based on GGE, enables us to investigate
non-equilibrium steady states \[10\] \[11\] \[12\]. In addition to these developments for isolated quantum systems, non-equilibrium steady states of open quantum systems have been investigated based on the Lindblad master equation \[13\].

Another approach for pursuing nonequilibrium physics is investigating stochastic integrable systems. Various nonequilibrium phenomena are described by the Markov process. The Markov process shares many aspects of its mathematical structure with the quantum mechanics. The time development of the Markov process is described by a master equation

\[
\frac{d}{dt} |\psi(t)\rangle = \mathcal{M} |\psi(t)\rangle,
\]

where \(|\psi(t)\rangle\) is a state vector and \(\mathcal{M}\) is a Markov matrix (Hamiltonian). This can be interpreted as the imaginary time Schrödinger equation. Therefore, many techniques used for quantum integrable systems are also applicable to stochastic integrable systems.

The asymmetric simple exclusion process (ASEP), which is a continuous Markov process describing nonequilibrium transport phenomena, is a typical example of a stochastic integrable system \[9\]. As in the case of quantum systems, the Markov matrix of the ASEP is described by the spin operators and diagonalized by the Bethe ansatz \[14\] \[15\]. Unlike the usual Hamiltonian of the quantum mechanics, the Markov matrix can be non-Hermitian. The non-hermicity of the Markov matrix reflects the fact that the system is out of equilibrium. The Markov matrix of the ASEP contains an asymmetric parameter \(\epsilon\), which reflects the degree of hermicity breaking, as we shall show in the later section. When \(\epsilon = 0\), the Markov matrix of the ASEP is Hermitian. In this case, the Markov matrix of the ASEP is identical to the Hamiltonian of the spin-1/2 Heisenberg XXX chain. When \(\epsilon \neq 0\), the Markov matrix of the ASEP is non-Hermitian. In this case, the Markov matrix of the ASEP corresponds to the Hamiltonian of the spin-1/2 Heisenberg XXZ chain with twisted boundary terms in the framework of the quantum system \[14\] \[16\].

As we mentioned above, many techniques, such as the TBA, the Quench Action and the GHD, have been developed for quantum integrable systems. It is expected that extending these techniques to non-Hermitian systems will contribute to understanding the nonequilibrium physics of the Markov process and quantum systems with boundary terms. As a first step for this aim, we investigate the Bethe equations of the ASEP, looking ahead to extend the TBA for this model.

The Bethe equations of the spin-1/2 Heisenberg chain have a series of regular complex solutions called the string solutions in the thermodynamic limit \[4\] \[5\]. The string solutions play a key role in the TBA formalism of the spin-1/2 Heisenberg chain. In the paper, we present new types of the string solutions for the non-Hermitian system. We found that the Bethe equations of the ASEP also have a series of regular solutions in the thermodynamic limit. These solutions are different from the traditional string solutions. The strings are deformed by the non-hermicity of the Hamiltonian. We called these solutions “the bundled string solutions”.

\[\text{Bundled string solutions in the non-Hermitian spin chain}\]
The paper is organized as follows. In section 2, we introduce and formulate the ASEP. The relationship between the ASEP and the spin-1/2 Heisenberg chain is also explained. In section 3, we analyze the Bethe equations. First, we review the traditional string solutions of the spin-1/2 Heisenberg XXX chain, and then introduce the bundled string solutions of the ASEP. After the formulation of the bundled string solutions is presented, dual-conjugacy, which is a new type of symmetry of non-Hermitian systems, is explained. In section 4, we show the numerical solutions of the Bethe equations of the ASEP in small systems. In section 5, we derive the Bethe-Takahashi equation for the ASEP, which plays a crucial role in the TBA formalism. Finally, in section 6, we present our conclusions.

2. Asymmetric simple exclusion process (ASEP)

In this section, we explain the asymmetric simple exclusion process (ASEP). The settings and formulation of the ASEP are presented. The ASEP can be interpreted as an extension of the spin-1/2 Heisenberg chain for the non-Hermitian system. The relation between the ASEP and the spin-1/2 Heisenberg chain is also explained here.

2.1. Formulation of the ASEP

The asymmetric simple exclusion process (ASEP) is a continuous-time Markov process that describes nonequilibrium transport phenomena in a one-dimensional lattice. The update rule is simple. Each particle hops to the nearest right (left) site with a hopping rate \( p \) (\( q \)), \( (p + q = 1) \). Particles have an exclusion volume effect, i.e. each cell contains only a single particle at most. Therefore, if the destination is not vacant, hopping does not occur. A schematic drawing is shown in Fig. 1. Despite of its simplicity, the ASEP describes interesting nonequilibrium phenomena, such as the boundary-induced phase transition \[17, 18\]. Therefore, this model has been applied to various transport phenomena, such as biophysical transport \[19, 20\], traffic flow \[21, 22\] and surface growth phenomena \[23\]. In addition, the ASEP also has fascinating mathematical features. As we mentioned above, the ASEP is a stochastic integrable system. The Markov matrix is diagonalized exactly by the Bethe ansatz \[14, 15\] and stationary states are constructed by the matrix product ansatz \[24\].

The ASEP is formulated by the second quantization method. We consider a periodic boundary condition. The number of lattice sites is \( L \). We introduce the variable \( n_j \), which denotes the number of particles at site \( j \). A local state of site \( j \) is denoted by the two-dimensional vector \(|n_j\rangle\). The basis of a local states is \((|1\rangle = (0, 1)^T, |0\rangle = (1, 0)^T)\). If a particle exists at site \( j \), \(|n_j\rangle = |1\rangle\). Otherwise, \(|n_j\rangle = |0\rangle\). The basis of the configuration space is \(|n\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_L\rangle\). The state of the system at time \( t \) is given by

\[
|\psi(t)\rangle = \sum_n \psi(n, t)|n\rangle, \tag{2}
\]
Figure 1. Schematic drawing of the ASEP on a ring. Each particle hops to the nearest right (left) site with a hopping rate $p$ ($q$) if the site is empty.

where $\psi(n,t)$ is the probability that the system is in a state $|n\rangle$ at time $t$. The Markov matrix (Hamiltonian) of the ASEP $\hat{H}_{\text{ASEP}}$ is given by

$$\hat{H}_{\text{ASEP}} = \sum_{j=1}^{L} \left\{ p\hat{S}_j^+\hat{S}_{j+1}^- + q\hat{S}_j^-\hat{S}_{j+1}^+ - p\hat{n}_j(1 - \hat{n}_{j+1}) - q(1 - \hat{n}_j)\hat{n}_{j+1} \right\}, \quad (3)$$

where $\hat{S}_j^\pm$ and $\hat{n}_j$ are the ladder operators and the particle number operator, respectively, which are defined by the Pauli matrices $\hat{\sigma}_x,y,z$ as $\hat{S}_j^\pm = \frac{1}{2}(\hat{\sigma}_j^x \pm i\hat{\sigma}_j^y)$ and $\hat{n}_j = \frac{1}{2}(1 - \hat{\sigma}_j^z)$. The time development of this state is described by the following master equation,

$$\frac{d}{dt}|\psi(t)\rangle = \hat{H}_{\text{ASEP}}|\psi(t)\rangle. \quad \text{(4)}$$

This master equation is the imaginary-time Schrödinger equation, so we call the Markov matrix $\hat{H}_{\text{ASEP}}$ Hamiltonian.

2.2. Relation to the spin-1/2 Heisenberg chain

The ASEP can be regarded as an extension of the spin-1/2 Heisenberg chain for the non-Hermitian system [14, 16]. The relation between the ASEP and the spin-1/2 Heisenberg chain is important when we look for the string solutions of the ASEP, as we shall show in section 3. Here, we review the correspondence between the ASEP and the spin-1/2 Heisenberg chain.

When the hopping rates of the ASEP are symmetric $p = q = 1/2$, the Hamiltonian $\hat{H}_{\text{ASEP}}$ [3] is

$$\hat{H}_{\text{ASEP}} = \sum_{j=1}^{L} \left\{ \hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + \hat{S}_j^z\hat{S}_{j+1}^z - \frac{L}{4} \right\} = \hat{H}_{\text{XXX}}, \quad (5)$$

where the spin operators are given by $\hat{S}_j^{x,y,z} = \frac{1}{2}\hat{\sigma}_j^{x,y,z}$. This Hamiltonian is the same as the one of the spin-1/2 Heisenberg XXX chain. Therefore, when hopping rates are
symmetric, the ASEP is equivalent to the spin-1/2 Heisenberg XXX chain. In this case, the Hamiltonian $\hat{H}_\text{ASEP}$ is Hermitian and the system does not relax to non-equilibrium steady states.

When the hopping rates of the ASEP are asymmetric ($p \neq q$), the Hamiltonian $\hat{H}_\text{ASEP}$ is non-Hermitian. In this case, we can map the Hamiltonian $\hat{H}_\text{ASEP}$ to the spin-1/2 Heisenberg XXZ chain with twisted boundary terms. We define the operator

$$U_i = I \otimes \cdots \otimes I \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha^{i-1} \end{array} \right) \otimes I \otimes \cdots \otimes I,$$

(6)

where

$$\alpha = \sqrt{\frac{q}{p}}$$

(7)

and $I$ is the $2 \times 2$ identity matrix. Consider a similarity transformation

$$\mathcal{H} = (\alpha + \alpha^{-1}) \left( \prod_{i=1}^{L} U_i \right) \left( \hat{H}_\text{ASEP} + \frac{L}{4} \mathcal{I} \right) \left( \prod_{i=1}^{L} U_i \right)^{-1},$$

(8)

where $\mathcal{I}$ is the $2^L \times 2^L$ identity matrix. Then, we obtain the following Hamiltonian

$$\mathcal{H} = 2 \sum_{j=1}^{L-1} \left\{ \hat{S}_j^x \hat{S}_{j+1}^x + \hat{S}_j^y \hat{S}_{j+1}^y + \frac{\alpha + \alpha^{-1}}{2} \hat{S}_j^z \hat{S}_{j+1}^z \right\} + \alpha L \hat{S}_1^+ \hat{S}_1^- + \alpha^{-L} \hat{S}_L^+ \hat{S}_L^- + (\alpha + \alpha^{-1}) \hat{S}_1^z \hat{S}_1^z.$$

(9)

This is the Hamiltonian of the spin-1/2 Heisenberg XXZ chain with the twisted boundary conditions

$$\hat{S}_{L+1}^+ = \alpha L \hat{S}_1^+, \quad \hat{S}_{L+1}^- = \alpha^{-L} \hat{S}_1^-, \quad \hat{S}_{L+1}^z = \hat{S}_1^z.$$

(10)

Consider the asymmetricity $\epsilon$ for the hopping rates of the ASEP

$$\epsilon = p - q.$$

(11)

When $\epsilon = 0$, the twisted boundary terms disappear, and the Hamiltonian $\hat{H}_\text{ASEP}$ is Hermitian and identical to that of the spin-1/2 Heisenberg XXX chain with a periodic boundary condition. When $\epsilon \neq 0$, the twisted boundary terms appear, and the Hamiltonian $\hat{H}_\text{ASEP}$ becomes non-Hermitian. In this sense, the ASEP is interpreted as an extension of the spin-1/2 Heisenberg chain for the non-Hermitian system.

### 3. String solutions

This section presents the bundled string solution, which is a new type of string solutions deformed by the non-hermicity. We analyze the Bethe equations here. As shown in section 2, the ASEP is regarded as an extension of the spin-1/2 Heisenberg chain for the non-Hermitian system. The bundled string solutions are also interpreted as the
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deformed string solutions of the spin-1/2 Heisenberg chain owing to non-hermicity. Therefore, we first review the traditional string solutions of the spin-1/2 Heisenberg XXX model. We then introduce and formulate the bundled string solutions of the ASEP. In addition, dual-conjugacy, which is a symmetry of the Bethe equations caused by the non-hermicity, is also introduced.

3.1. String solutions of the spin-1/2 Heisenberg XXX chain

The spin-1/2 Heisenberg XXX chain is one of the most famous quantum integrable systems that was exactly solved by Bethe [25]. The Bethe equations of the spin-1/2 Heisenberg XXX chain have a series of regular solutions called string solutions in the thermodynamic limit [4, 5]. Although the validity of the string hypothesis is still controversial [26, 27, 28], the string solutions lead us to further analyze integrable systems, such as the TBA. As shown in section 2, when the hopping rate of the ASEP is symmetric \( p = q \), the ASEP is equivalent to the spin-1/2 Heisenberg XXX chain. The bundled string solutions are related to the string solutions of the spin-1/2 Heisenberg XXX chain, as will be shown later. Here, we review the string solutions of the spin-1/2 Heisenberg XXX chain.

The Bethe equations of the spin-1/2 Heisenberg XXX chain with a set of \( N \) rapidities \( \{\lambda_j\} = \{\lambda_1, \lambda_2, \cdots, \lambda_N\} \) are given by

\[
\left( \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \right)^L = \prod_{l \neq j} \frac{\lambda_j - \lambda_l + i}{\lambda_j - \lambda_l - i}, \quad j = 1, \cdots, N. \tag{12}
\]

The energy eigenvalue is described as

\[
E = - \sum_{j=1}^{N} \frac{2}{4\lambda_j^2 + 1}. \tag{13}
\]

To take the logarithm of the Bethe equations \([12]\) and to introduce the Bethe quantum numbers that correspond to the set of rapidities, we define functions \( \varphi_n(x) \) and \( \theta_n(x) (n \in \mathbb{N}) \) as

\[
\varphi_n(x) = -e^{-i\theta_n(x)} = \frac{x + \frac{i}{2}n}{x - \frac{i}{2}n},
\]

\[
\theta_n(x) = 2\tan^{-1}\left( \frac{2x}{n} \right). \tag{14}
\]

Then the logarithm of the Bethe equations \([12]\) are

\[
\theta_1(\lambda_j) = \frac{2\pi}{L} I_j + \frac{1}{L} \sum_{l \neq j}^N \theta_2(\lambda_j - \lambda_l), \quad j = 1, \cdots, N, \tag{15}
\]

where \( \{I_j\} = \{I_1, I_2, \cdots, I_N\} \) are called the Bethe quantum numbers. They are integers when \( L - N \) is odd, and half-integers when \( L - N \) is even.
In the thermodynamic limit, the Bethe equations of the spin-1/2 Heisenberg XXX spin chain have a series of regular solutions called the string solutions. Here we derive these. Assume a complex solution \( \lambda_j \):

\[
\lambda_j = a_j + ib_j, \quad a_j, b_j \in \mathbb{R}. \tag{16}
\]

If \( \text{Im} \lambda_j = b_j > 0 \), the left-hand side (LHS) of eq. (12) diverges to infinity in the thermodynamic limit \( L \to \infty \). Therefore, the denominator of the right-hand side (RHS) of eq. (12) should be close to zero:

\[
\lambda_j - \lambda_l - i \sim 0, \quad \text{if } b_j > 0. \tag{17}
\]

Similarly, if \( \text{Im} \lambda_j = b_j < 0 \), the LHS of eq. (12) becomes zero in the thermodynamic limit \( L \to \infty \). Therefore, the numerator of the RHS of eq. (12) should be close to zero:

\[
\lambda_j - \lambda_l + i \sim 0, \quad \text{if } b_j < 0. \tag{18}
\]

Eq. (17) implies that, if \( \lambda_j(b_j > 0) \) is a solution of the Bethe equations (12), then \( \lambda_j + i \) is also a Bethe root. Similarly, eq. (18) implies that, if \( \lambda_j(b_j < 0) \) is a solution of the Bethe equation (12), \( \lambda_j - i \) is also a Bethe root. These algorithms inductively produce a series of solutions of the Bethe equations (12) along with the imaginary axis. These solutions are called the string solutions, and the straight lines parallel to the imaginary axis, on which the string solutions are located, are called the strings. A schematic drawing of the string solutions for the spin-1/2 Heisenberg XXX chain is shown on the right side of Fig. 2. An \( n \)-string is given by

\[
\lambda^{n,j}_\alpha = \lambda_\alpha^n + i(b^{n,1}_\alpha - j), \quad j = 1, \cdots, n. \tag{19}
\]

Here, \( n \) is the string length that corresponds to the number of the Bethe roots on the string, \( \alpha \) is the index of the strings, \( \lambda_\alpha^n \) is the string center, and \( b^{n,1}_\alpha \) is the maximum value of the imaginary part of the Bethe roots on the string. The Bethe equations of the spin-1/2 Heisenberg XXX model have the self-conjugacy, which is a symmetry of the Bethe roots based on the hermicity of the Hamiltonian \([31]\). The set of the Bethe roots is identical to its complex conjugate \( \{ \lambda_j \} = \{ \overline{\lambda_j} \} \) (where the overline means complex conjugate: \( x + iy = x - iy \)). This imposes constraints on the imaginary parts of the string solutions. \( b^{n,1}_\alpha \) is determined by the self-conjugacy as

\[
b^{n,1}_\alpha = \frac{n - 1}{2}. \tag{20}
\]

Therefore, \( n \)-string solutions of the Heisenberg XXX model are described as

\[
\lambda^{n,j}_\alpha = \lambda_\alpha^n + \frac{i}{2}(n + 1 - 2j), \quad j = 1, \cdots, n. \tag{21}
\]
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Figure 2. Schematic drawings of the string solutions. The right figure shows the 3-string solutions of the Heisenberg XXX model. The left figure shows the 3-bundled string solutions of the ASEP. In the symmetric limit $\epsilon \to 0$, the bundled strings correspond to the traditional strings.

3.2. Bundled string solutions of the ASEP

Here, we present the bundled string solutions, which are new types of the string solutions deformed by the non-hermicity of the Hamiltonian. First, we consider the transformation of the Bethe equations of the ASEP. After transformation, we introduce the bundled string solutions. Many parts of the discussion for formulating the string solutions are parallel to that of the spin-1/2 Heisenberg XXX model. We also introduce the dual-conjugacy, which is an extension of the self-conjugacy for non-Hermitian systems.

The Bethe equations of the ASEP with a set of $N$ rapidities $\{z_j\} = \{z_1, z_2, \cdots, z_N\}$ are given by

$$z_j^L = (-1)^{N-1} \prod_{i \neq j}^N \frac{p - z_j + q z_j z_i}{p - z_i + q z_j z_i}, \quad j = 1, \cdots, N. \quad (22)$$

where $p, q$ are hopping rates ($p + q = 1$). The eigenvalue of the Hamiltonian is given by

$$E = p \sum_{j=1}^N \frac{1}{z_j} + q \sum_{j=1}^N z_j - N. \quad (23)$$

Introducing an asymmetric parameter $\epsilon \in [0, 1)$, we rewrite the hopping rate as

$$p = \frac{1 + \epsilon}{2}, \quad q = \frac{1 - \epsilon}{2}. \quad (24)$$

The asymmetric parameter $\epsilon$ reflects the non-hermicity of the Hamiltonian. When $\epsilon = 0$ ($p = q$), the system is Hermitian and relaxes to an equilibrium state. In this case, the Hamiltonian of the ASEP is the same as that of the spin-1/2 Heisenberg XXX chain. When $\epsilon > 0$ ($p > q$), the system is non-Hermitian and relaxes to a nonequilibrium steady
state. Note that we ignore the case of $\epsilon = 1$ ($p = 1, q = 0$). In this case, the system is completely asymmetric and is called the TASEP (totally asymmetric simple exclusion process). As we shall show in section 4, the Bethe equations of the TASEP are singular and do not have the bundled string solutions. Therefore, we do not consider $\epsilon = 1$ in this section. The Bethe equations of the TASEP are investigated in refs. [29, 30]. Then the Bethe equations (22) are rewritten as

$$z_j^L = (-1)^{N-1} \prod_{l \neq j}^N \frac{(1 + \epsilon) - 2z_j + (1 - \epsilon)z_j z_l}{(1 + \epsilon) - 2z_l + (1 - \epsilon)z_j z_l}, \quad j = 1, \cdots, N. \quad (25)$$

We change the rapidities $\{z_j\}$ as

$$z_j = \frac{\lambda_j + i}{\lambda_j - i}, \quad (26)$$

then the Bethe equations become

$$\left(\frac{\lambda_j + i}{\lambda_j - i}\right)^L = \prod_{l \neq j}^N \frac{(1 - \epsilon)\lambda_j - (1 + \epsilon)\lambda_l + i}{(1 + \epsilon)\lambda_j - (1 - \epsilon)\lambda_l - i}, \quad j = 1, \cdots, N. \quad (27)$$

By comparing the Bethe equations of this form (27) with eq. (12), we can see how the asymmetric parameter $\epsilon$ deforms the Bethe equation of the spin-1/2 Heisenberg XXX chain (12). The eigenvalue of the Hamiltonian is

$$E = \sum_{j=1}^N \left(-\frac{2}{4\lambda_j^2 + 1} - i \frac{4\lambda_j \epsilon}{4\lambda_j^2 + 1}\right). \quad (28)$$

We set a parameter $\gamma$ and a constant $C$ as

$$\gamma = \frac{1 + \epsilon}{1 - \epsilon}, \quad (29)$$

$$C = \sqrt{\frac{(1 - \epsilon)^2}{2\epsilon}}. \quad (30)$$

To take a logarithm of the Bethe equations (27) and to introduce the Bethe quantum numbers that correspond to the rapidities, we define functions $\varphi_{n,\epsilon}(x, y)$ and $\theta_{n,\epsilon}(x, y)$ ($n \in \mathbb{N}$):

$$\varphi_{n,\epsilon}(x, y) = -e^{-i\theta_{n,\epsilon}(x, y)} = \frac{x - \gamma^\frac{n}{2} y - i\frac{\gamma^\frac{n}{2} - \gamma^{-\frac{n}{2}}}{2\epsilon}}{x - \gamma^{-\frac{n}{2}} y - i\frac{\gamma^{-\frac{n}{2}} - \gamma^{\frac{n}{2}}}{2\epsilon}},$$

$$\theta_{n,\epsilon}(x, y) = 2\tan^{-1} \left[ \frac{2x - (\gamma^\frac{n}{2} + \gamma^{-\frac{n}{2}})y + i\frac{2 - (\gamma^\frac{n}{2} + \gamma^{-\frac{n}{2}})}{2\epsilon}}{(\gamma^\frac{n}{2} - \gamma^{-\frac{n}{2}})(\frac{1}{2\epsilon} + iy)} \right]. \quad (31)$$
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\( \theta_{n,\epsilon}(x, y) \) is an extension of the function \( \theta_n(x) \) of the spin-1/2 Heisenberg spin chain (14) for the ASEP. Then the logarithms of the Bethe equations of (27) are written as

\[
\begin{align*}
\theta_{1,\epsilon}(\lambda_j, i(1/2 \epsilon - C)) & = \frac{2\pi}{L} I_j + \frac{1}{L} \sum_{l \neq j} \left[ \theta_{2,\epsilon}(\lambda_j, \lambda_l) - i \log \gamma \right] \\
& = \frac{2\pi}{L} I_j + \frac{1}{L} \sum_{l \neq j} \theta_{2,\epsilon}(\lambda_j, \lambda_l) \quad j = 1, \ldots, N,
\end{align*}
\]

(32)

where \( \{I_j\} \) are the Bethe quantum numbers, which are integers when \( L - N \) is odd, and half-integers when \( L - N \) is even. Here we introduce

\[
\theta_{2,\epsilon}(x, y) = \phi_{2,\epsilon}(x, y) - i \log \gamma = 2 \tan^{-1} \left( \frac{x - y}{1 + i \epsilon(x + y)} \right).
\]

(33)

These equations (27)-(32) can be interpreted as the deformed equations of the spin-1/2 Heisenberg XXX chain (12)-(15) because of the asymmetricity \( \epsilon \). When \( \epsilon = 0 \), they are identical to those of the spin-1/2 Heisenberg XXX chain.

Let us investigate the Bethe equations (27) in the thermodynamic limit and introduce the bundled string solutions. Assume a complex solution \( \lambda_j \):

\[
\lambda_j = a_j + ib_j, \quad a_j, b_j \in \mathbb{R}.
\]

(34)

If \( \text{Im} \lambda_j = b_j > 0 \), the LHS of eq. (27) diverges to infinity in the thermodynamic limit \( L \to \infty \). Therefore, the denominator of the RHS of eq. (27) should be close to zero:

\[
(1 + \epsilon)\lambda_j - (1 - \epsilon)\lambda_l - i \sim 0, \quad \text{if} \ b_j > 0.
\]

(35)

Similarly, when \( \text{Im} \lambda_j = b_j < 0 \), the LHS of eq. (27) becomes zero in the thermodynamic limit \( L \to \infty \). Therefore, the numerator of the RHS of eq. (27) should be close to zero:

\[
(1 - \epsilon)\lambda_j - (1 + \epsilon)\lambda_l + i \sim 0, \quad \text{if} \ b_j < 0.
\]

(36)

Eq. (35) implies that, if \( \lambda_j(b_j > 0) \) is a solution of the Bethe equation (27), then the following is also a Bethe root:

\[
\lambda_l \sim \frac{1 + \epsilon}{1 - \epsilon} \lambda_j - i \frac{1}{1 - \epsilon}
\]

\[
\sim \frac{1 + \epsilon}{1 - \epsilon} a_j - i \frac{1}{1 - \epsilon} \{1 - (1 + \epsilon)b_j\}, \quad \text{if} \ b_j > 0.
\]

(37)

Similarly, eq. (36) that, if \( \lambda_j(b_j < 0) \) is a solution of the Bethe equation (27), then the following is also a Bethe root:

\[
\lambda_l \sim \frac{1 - \epsilon}{1 + \epsilon} \lambda_j + i \frac{1}{1 + \epsilon}
\]

\[
\sim \frac{1 - \epsilon}{1 + \epsilon} a_j + i \frac{1}{1 + \epsilon} \{1 + (1 - \epsilon)b_j\}, \quad \text{if} \ b_j < 0.
\]

(38)
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These algorithms inductively produce a series of solutions of the Bethe equation (27). We call this series of solutions “bundled string solutions”. The meaning of “bundled” is explained later. To have a finite number of solutions, the conditions

\begin{align}
\text{Im}\lambda_l &< b_j \quad \text{if } b_j > 0 \quad (39) \\
\text{Im}\lambda_l &> b_j \quad \text{if } b_j < 0 \quad (40)
\end{align}

should be satisfied. The second condition (39) is always satisfied. The first condition requires that

\[ b_j < \frac{1}{2\epsilon}. \quad (41) \]

Therefore, the bundled string solutions are restricted to this region on the complex plane.

We rewrite the Bethe roots \( \{\lambda_j\} \) on an \( n \)-bundled string to

\[ \lambda_{n,j}^\alpha = a_{n,j}^\alpha + ib_{n,j}^\alpha, \quad j = 1, \cdots n \quad \text{(42)} \]

where \( \alpha \) is an index of a bundled string, \( n \) is the string length that corresponds to the number of the Bethe roots on a bundled string, and \( j \) is an index of a Bethe root on a bundled string. The indices \( \{j\} \) are ordered in descending order of the imaginary parts of the Bethe roots, as shown on the left side of Fig. 2.

From eqs. (37), (38), we obtain recursion relations of the bundled string solutions \( \lambda_{n,j}^\alpha \):

\begin{align}
a_{n,j}^{\alpha+1} &= \frac{1 + \epsilon}{1 - \epsilon} a_{n,j}^{\alpha}, \quad (43) \\
b_{n,j}^{\alpha+1} &= \frac{1 + \epsilon}{1 - \epsilon} b_{n,j}^{\alpha} - \frac{1}{1 - \epsilon}. \quad (44)
\end{align}

Solving eqs. (43) and (44), then we obtain

\begin{align}
a_{n,j}^{\alpha} &= a_{n,1}^{\alpha} \gamma^j \quad \text{(45)} \\
b_{n,j}^{\alpha} &= \left( b_{n,1}^{\alpha} - \frac{1}{2\epsilon} \right) \gamma^{j-1} + \frac{1}{2\epsilon}. \quad (46)
\end{align}

Therefore, bundled string solutions are described as

\[ \lambda_{n,j}^\alpha = a_{n,j}^\alpha + ib_{n,j}^\alpha = a_{n,1}^\alpha \gamma^j + i\left( b_{n,1}^{\alpha} - \frac{1}{2\epsilon} \right) \gamma^{j-1} + \frac{1}{2\epsilon}, \quad j = 1, \cdots, n. \quad \text{(47)} \]

From eqs. (45) and (46), we obtain

\[ b_{n,j}^{\alpha} = \left( b_{n,1}^{\alpha} - \frac{1}{2\epsilon} \right) \frac{a_{n,j}^\alpha}{a_{n,1}^\alpha} + \frac{1}{2\epsilon}. \quad \text{(48)} \]

This is an equation of a line with a slope of \( (b_{n,1}^{\alpha} - 1/2\epsilon) / a_{n,1}^{\alpha} \) and an intercept of \( 1/2\epsilon \). Eq. (48) represents an equation of the bundled strings and the bundled string solutions.
are located on this line. Eq. (48) implies that all the bundled strings are crossed at the point \((0, 1/2\epsilon)\). This point is a knot of the bundled strings. That is why we name these strings “bundled strings”. In the symmetric limit \(\epsilon \to 0\), the knot of the bundled strings goes to infinity \((0, 1/2\epsilon) \to (0, \infty)\), and the string becomes parallel to the imaginary axis, as shown in Fig. 2. This is consistent with the traditional strings of the spin-1/2 Heisenberg XXX chain.

In the case of the Heisenberg XXX model, the Bethe equations (12) have the self-conjugacy. A set of the Bethe roots satisfies \(\{\lambda_j\} = \{\bar{\lambda}_j\}\), and the imaginary parts of the string solutions are constrained. Similarly, the Bethe equations of the ASEP have a symmetry, which is an extension of the self-conjugacy for non-Hermitian systems. We call this symmetry “Dual-conjugacy”. Let us explain what the dual-conjugacy means in the following.

Consider a system of the ASEP with asymmetric rate \(\epsilon\). We define a dual system of this system as a system of the ASEP with asymmetric rate \(-\epsilon\), as shown in Fig. 3. The dual system is almost the same as the original one, except for the direction of the asymmetry. We denote the set of the Bethe roots of the original system as \(\{\lambda_j(\epsilon)\}\) and that of the dual system as \(\{\lambda_j(-\epsilon)\}\). The dual-conjugacy means

\[
\{\lambda_j(\epsilon)\} = \{\bar{\lambda}_j(-\epsilon)\}.
\] (49)

In other words, a set of the Bethe roots \(\{\lambda_j(\epsilon)\}\) is identical to the complex conjugate of the set of the Bethe roots for the dual system \(\{\lambda_j(-\epsilon)\}\). The relation between the bundled string solutions of the original system and that of its dual one is shown in Fig. 4. The dual-conjugacy is numerically confirmed, as we shall show in the next section. We can also immediately confirm that, when \(\lambda_j(\epsilon)\) is a Bethe root, \(\bar{\lambda}_j(-\epsilon)\) is also a Bethe root by transforming \(\epsilon \to -\epsilon\) and considering complex conjugate for the Bethe equation (27). From eq. (28), if the dual-conjugacy is satisfied, the energy eigenvalue of the dual system is a complex conjugate of that of the original one

\[
E(\epsilon) = \overline{E(-\epsilon)}.
\] (50)

As in the case of the spin-1/2 Heisenberg XXX chain, the imaginary parts of the bundled string solutions are determined by the dual-conjugacy. From the dual-conjugacy (see Fig. 4)

\[
b^{n,1}_\alpha(\epsilon) = -b^{n,n}_\alpha(-\epsilon),
\] (51)

we obtain

\[
b^{n,1}_\alpha(\epsilon) = \frac{1 - \gamma^{1-n}}{2\epsilon(1 + \gamma^{1-n})}.
\] (52)

In the symmetric limit \(\epsilon \to 0\), eq. (52) corresponds to eq. (20). From eq. (47) and eq. (52), the n-bundled solutions are described as

\[
\lambda^{n,j}_\alpha = a^{n,1}_\alpha \gamma^{j-1} + i \frac{1}{2\epsilon} \left(1 - \frac{2\gamma^{j-n}}{1 + \gamma^{1-n}}\right).
\] (53)
4. Examples of the bundled string solutions

In section 3, the bundled string solution was introduced. In this section, we present numerical solutions of the Bethe equations of the ASEP (27) in small systems to confirm the bundled string solutions. The system sizes considered are \( L = 9 \) and 12, and the number of particles is \( N = 2 \) or 3. The asymmetric parameters \( \epsilon \) were set from 0.0 to 0.7 in increments of 0.1. The results are listed in Tabs. 1–4 and shown in Figs. 5–8. In Figs. 5–8, the blue dots show the Bethe roots, and the red cross mark shows the knot of the bundled strings \((0, 1/2\epsilon)\). In addition, we also show numerical solutions of the Bethe equations of the ASEP (27) with an asymmetric parameter \( \epsilon \) being negative to confirm the dual-conjugacy. In this case, we set the system size to \( L = 9 \) and the number of particles to \( N = 3 \). The results are presented in Tab. 5 and Fig. 9.

As we discussed in section 3, the bundled string solutions are obtained in the thermodynamic limit \( L \to \infty \). Although it is difficult to numerically solve the Bethe equations of such a large system, we can observe a series of solutions, which is similar...
to the bundled string solutions, even in the numerical solutions of small systems. Figs. 5 and 6 show the 2-bundled string solutions. These solutions exhibit the characteristics of the bundled solutions well. When $\epsilon = 0.0$, the string is parallel to the imaginary axis. As $\epsilon$ become large, the bundled strings lean and the imaginary parts of the knot become small. Figs. 7 and 8 show the 3-bundled string solutions. When $\epsilon$ is small, these solutions exhibit the characteristics of the bundled solutions. However, when $\epsilon$ increases, the strings tend to bend. In other words, these solutions are no longer being located on the straight line. This deviation of the bundled string solutions is caused by the finite size effect. We see a crossover with the singularity of $\epsilon = 1$. As previously mentioned, the Bethe equations (27) have a singularity in the case of the TASEP. When $\epsilon = 1$, the Bethe equations (27) become

$$
\left(\frac{\lambda_j + i \frac{\epsilon}{2}}{\lambda_j - i \frac{\epsilon}{2}}\right)^L = \prod_{t \neq j}^{N} \frac{-2\lambda_t + i}{2\lambda_j - i}, \quad j = 1, \ldots, N.
$$

(54)

This equation is quite different from eq. (27), from which we obtain the bundled string solutions. It is not possible to apply eq. (54) to the discussion for formulating the bundled string solution in section 3. Moreover, $\gamma = \frac{1 + \epsilon}{1 - \epsilon}$ diverges when $\epsilon \rightarrow 1$. Therefore, in the totally asymmetric case ($\epsilon = 1$), the bundled string solutions do not exist. In the numerical calculation, we consider the Bethe equations of the small systems. The bundled string solutions are justified in the thermodynamic limit. In Figs. 7 and 8 we observe a crossover between $\epsilon \neq 1$ and $\epsilon = 1$. In finite systems, the bundled strings are rounded as $\epsilon$ increases. In the thermodynamic limit, the bundled strings become straight lines except for $\epsilon = 1$. In fact, the curvature of the string with a system size of $L = 9$ (Fig. 7) for large $\epsilon$ is greater than that of the system of size $L = 12$ (Fig. 8).

Tab. 5 and Fig. 9 give the bundled string solutions of a dual system. We can confirm the dual-conjugacy (49) and eq. (50) by comparing Tab. 3 with Tab. 5.
Table 1. Numerical solutions of 2-bundled strings: \((L, N) = (9, 2)\)

| \(\epsilon\) | Bethe roots \(\lambda\) | Eigenvalue \(E\) | \(\{I_j\}\) |
|---|---|---|---|
| 0.0 | \(1.13277 - 0.562861i\) | \(-0.391662 + 0.0i\) | \(\{3, 4\}\) |
| 0.1 | \(1.22560 - 0.658595i\) | \(-0.394303 - 0.100870i\) | \(\{3, 4\}\) |
| 0.2 | \(1.33500 - 0.764237i\) | \(-0.402098 - 0.203236i\) | \(\{3, 4\}\) |
| 0.3 | \(1.46560 - 0.872171i\) | \(-0.414546 - 0.308724i\) | \(\{3, 4\}\) |
| 0.4 | \(1.62239 - 0.968565i\) | \(-0.430420 - 0.419040i\) | \(\{3, 4\}\) |
| 0.5 | \(1.80250 - 1.03354i\) | \(-0.447377 - 0.535335i\) | \(\{3, 4\}\) |
| 0.6 | \(1.98738 - 1.05174i\) | \(-0.462518 - 0.657057i\) | \(\{3, 4\}\) |
| 0.7 | \(2.15261 - 1.02705i\) | \(-0.474155 - 0.781985i\) | \(\{3, 4\}\) |

Table 2. Numerical solutions of 2-bundled strings: \((L, N) = (12, 2)\)

| \(\epsilon\) | Bethe roots \(\lambda\) | Eigenvalue \(E\) | \(\{I_j\}\) |
|---|---|---|---|
| 0.0 | \(1.65109 + 0.619417i\) | \(-0.237741 + 0.0i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.1 | \(1.53397 + 0.530122i\) | \(-0.239068 - 0.086031i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.2 | \(1.43826 + 0.458396i\) | \(-0.242971 - 0.172805i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.3 | \(1.36440 + 0.397982i\) | \(-0.249173 - 0.261085i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.4 | \(1.30989 + 0.344692i\) | \(-0.257098 - 0.351611i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.5 | \(1.27179 + 0.297169i\) | \(-0.265776 - 0.444888i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.6 | \(1.24697 + 0.255826i\) | \(-0.274005 - 0.540840i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.7 | \(1.23187 + 0.221317i\) | \(-0.280877 - 0.638723i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
| 0.8 | \(1.21598 + 0.190005i\) | \(-0.287893 - 0.741977i\) | \(\{\frac{9}{2}, \frac{11}{2}\}\) |
Table 3. Numerical solutions of 3-bundled strings: \((L,N) = (9,3)\)

| \(\epsilon\) | \(\text{Bethe roots } \lambda\) | \(\text{Eigenvalue } E\) | \(\{I_j\}\) |
|--------------|----------------------------------|--------------------------|-----------|
| 0.0          | 0.830574 + 0.999371i             | -0.478659 + 0.0i         | \(\{5/2, 7/2, 9/2\}\) |
|              | 0.866025 + 0.0i                  |                           |           |
|              | 0.830574 - 0.999371i             |                           |           |
| 0.1          | 0.724088 + 0.78023i              | -0.490926 - 0.109513i    | \(\{5/2, 7/2, 9/2\}\) |
|              | 0.891437 - 0.177978i             |                           |           |
|              | 1.01526 - 1.24521i               |                           |           |
| 0.2          | 0.670062 + 0.608186i             | -0.521265 - 0.235151i    | \(\{5/2, 7/2, 9/2\}\) |
|              | 0.979505 - 0.349248i             |                           |           |
|              | 1.29341 - 1.41706i               |                           |           |
| 0.3          | 0.645634 + 0.489088i             | -0.550836 - 0.379026i    | \(\{5/2, 7/2, 9/2\}\) |
|              | 1.13961 - 0.477639i              |                           |           |
|              | 1.48832 - 1.42591i               |                           |           |
| 0.4          | 0.628612 + 0.407087i             | -0.575499 - 0.528840i    | \(\{5/2, 7/2, 9/2\}\) |
|              | 1.34781 - 0.511426i              |                           |           |
|              | 1.48985 - 1.46382i               |                           |           |
| 0.5          | 0.615165 + 0.345515i             | -0.599235 - 0.682058i    | \(\{5/2, 7/2, 9/2\}\) |
|              | 1.51202 - 0.437303i              |                           |           |
|              | 1.44732 - 1.63424i               |                           |           |
| 0.6          | 0.605678 + 0.296873i             | -0.622539 - 0.839645i    | \(\{5/2, 7/2, 9/2\}\) |
|              | 1.59853 - 0.337067i              |                           |           |
|              | 1.44205 - 1.86116i               |                           |           |
| 0.7          | 0.599897 + 0.257816i             | -0.644156 - 1.00177i     | \(\{5/2, 7/2, 9/2\}\) |
|              | 1.63789 - 0.246215i              |                           |           |
|              | 1.46457 - 2.09972i               |                           |           |
Table 4. Numerical solutions of 3-bundled strings: \((L, N) = (12, 3)\)

| \(\epsilon\) | Bethe roots \(\lambda\) | Eigenvalue \(E\) | \(\{I_j\}\) |
|---|---|---|---|
| 0.0 | \(1.38995 + 1.06992i\) | \(-0.312580 + 0.0i\) | \(\{4, 5, 6\}\) |
|   | \(1.46743 + 0.0i\) | \(1.38995 - 1.06992i\) | |
| 0.1 | \(1.22211 + 0.851035i\) | \(-0.317756 - 0.108379i\) | \(\{4, 5, 6\}\) |
|   | \(1.49369 - 0.157964i\) | \(1.60977 - 1.32214i\) | |
| 0.2 | \(1.10800 + 0.681986i\) | \(-0.331030 - 0.222435i\) | \(\{4, 5, 6\}\) |
|   | \(1.57610 - 0.300740i\) | \(1.85709 - 1.55046i\) | |
| 0.3 | \(1.03603 + 0.559742i\) | \(-0.34675 - 0.343999i\) | \(\{4, 5, 6\}\) |
|   | \(1.71176 - 0.399141i\) | \(2.04978 - 1.71410i\) | |
| 0.4 | \(0.988098 + 0.472543i\) | \(-0.361633 - 0.470224i\) | \(\{4, 5, 6\}\) |
|   | \(1.86910 - 0.425516i\) | \(2.13333 - 1.87895i\) | |
| 0.5 | \(0.953582 + 0.406979i\) | \(-0.375959 - 0.599114i\) | \(\{4, 5, 6\}\) |
|   | \(2.00206 - 0.387010i\) | \(2.15763 - 2.10250i\) | |
| 0.6 | \(0.928375 + 0.354803i\) | \(-0.390104 - 0.730449i\) | \(\{4, 5, 6\}\) |
|   | \(2.09126 - 0.317790i\) | \(2.17415 - 2.37352i\) | |
| 0.7 | \(0.910401 + 0.311946i\) | \(-0.403756 - 0.864317i\) | \(\{4, 5, 6\}\) |
|   | \(2.14309 - 0.242643i\) | \(2.19926 - 2.66822i\) | |
Table 5. Numerical solutions of 3-bundled strings of a dual system: \((L, N) = (9, 3)\)

| \(\epsilon\) | \(\text{Bethe roots } \lambda\) | \(\text{Eigenvalue } E\) | \(\{I_j\}\) |
|------------|-----------------|---------------------|---------|
| 0.0        | 0.830574 + 0.999371\(i\) | -0.478659 + 0.0\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 0.866025 + 0.0\(i\)       |                     |         |
| -0.1       | 1.01526 + 1.24521\(i\)   | -0.490926 + 0.109513\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 0.891437 + 0.177978\(i\) |                     |         |
|            | 0.724088 - 0.780230\(i\) |                     |         |
| -0.2       | 1.29341 + 1.41706\(i\)   | -0.521265 + 0.235151\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 0.979505 + 0.349248\(i\) |                     |         |
|            | 0.670062 - 0.608186\(i\) |                     |         |
| -0.3       | 1.48832 + 1.42591\(i\)   | -0.550836 + 0.379026\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 1.13961 + 0.477639\(i\)  |                     |         |
|            | 0.645634 - 0.489088\(i\) |                     |         |
| -0.4       | 1.48985 + 1.46382\(i\)   | -0.575499 + 0.528840\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 1.34781 + 0.511426\(i\)  |                     |         |
|            | 0.628612 - 0.407087\(i\) |                     |         |
| -0.5       | 1.44732 + 1.63424\(i\)   | -0.599235 + 0.682058\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 1.51202 + 0.437303\(i\)  |                     |         |
|            | 0.615165 - 0.345515\(i\) |                     |         |
| -0.6       | 1.44205 + 1.86116\(i\)   | -0.622539 + 0.839645\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 1.59853 + 0.337067\(i\)  |                     |         |
|            | 0.605678 - 0.296873\(i\) |                     |         |
| -0.7       | 1.46457 + 2.09972\(i\)   | -0.644156 + 1.00177\(i\) | \(\{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\) |
|            | 1.63789 + 0.246215\(i\)  |                     |         |
|            | 0.599897 - 0.257816\(i\) |                     |         |
Bundled string solutions in the non-Hermitian spin chain

Figure 5. 2-bundled-string solutions: \((L, N) = (9, 2), \{I_j\} = \{3, 4\}\). The blue dots show the Bethe roots, and the red cross mark shows the knot of the bundled strings \((0, 1/2\varepsilon)\).
Bundled string solutions in the non-Hermitian spin chain

Figure 6. 2-bundled-string solutions: \((L, N) = (12, 3)\), \(\{I_j\} = \{\frac{9}{7}, \frac{11}{7}\}\), The blue dots show the Bethe roots, and the red cross mark shows the knot of the bundled strings \((0, 1/2\epsilon)\).
Bundled string solutions in the non-Hermitian spin chain

Figure 7. 3-bundled-string solutions: \((L, N) = (9, 3), \{I_j\} = \{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\). The blue dots show the Bethe roots, and the red cross mark shows the knot of the bundled strings \((0, 1/2\epsilon)\).
**Figure 8.** 3-bundled-string solutions: \((L, N) = (12, 3), \{I_j\} = \{4, 5, 6\},\) The blue dots show the Bethe roots, and the red cross mark shows the knot of the bundled strings \((0, 1/2 \epsilon).\)
Bundled string solutions in the non-Hermitian spin chain

Figure 9. 3-bundled-string solutions of the dual system: \((L, N) = (9, 3)\), \(\{I_j\} = \{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}\). The blue dots show the Bethe roots, and the red cross mark shows the knot of the bundled strings \((0, -1/2\epsilon)\).
5. Bethe-Takahashi equation for the bundled strings

The thermodynamic Bethe ansatz (TBA) was originally introduced for the Lieb-Liniger model by Yang and Yang [3]. The Bethe roots of the Lieb-Liniger model with repulsive interactions are real number. Thermodynamic quantities of the Lieb-Liniger model are obtained by considering statistics of the quasi particles that are corresponding to the Bethe roots. Although the Bethe roots of the spin-1/2 Heisenberg chain are complex numbers, it is also possible to apply the framework of TBA by considering the strings as quasi particles. The reduced Bethe ansatz equations for the strings are called the Bethe-Takahashi equation. The Bethe-Takahachi equation plays a central role in the TBA formalism for the spin-1/2 Heisenberg chain. We succeeded in extending the Bethe-Takahashi equation for the bundled string solutions. In this section, we first review the derivation of the Bethe-Takahashi equation for the spin-1/2 Heisenberg XXX chain, and then explain the derivation of the Bethe-Takahashi equation for the ASEP.

5.1. Bethe-Takahashi equation for the spin-1/2 Heisenberg XXX chain

We briefly review the derivation of the Bethe-Takahashi equation for the spin-1/2 Heisenberg XXX chain [4, 5]. Consider that \(N\) rapidities are assigned into \(N\) strings of length \(j\), i.e. \(\sum_j jN_j = N\). Rewrite the Bethe equations for the spin-1/2 Heisenberg \(XXX\) chain (12) on an \(n\)-string by string solutions \(\{\lambda_{n,j}^\alpha\}\)

\[
\left(\frac{\lambda_{n,j}^\alpha + \frac{i}{2}}{\lambda_{n,j}^\alpha - \frac{i}{2}}\right)^L = \left(\prod_{\beta \neq \alpha}^{m} \frac{\lambda_{n,j}^\alpha - \lambda_{\beta,k}^m + i}{\lambda_{n,j}^\alpha - \lambda_{\beta,k}^m - i}\right) \left(\prod_{j' \neq j} \frac{\lambda_{n,j}^\alpha - \lambda_{n,j'}^\alpha + i}{\lambda_{n,j}^\alpha - \lambda_{n,j'}^\alpha - i}\right), \quad j = 1, \ldots, n.
\]  

Multiplying the Bethe equations on an \(n\)-string, then we obtain a following equation

\[
\prod_{j=1}^{n} \left(\frac{\lambda_{n,j}^\alpha + \frac{i}{2}}{\lambda_{n,j}^\alpha - \frac{i}{2}}\right)^L = \prod_{j=1}^{n} \prod_{\beta \neq \alpha}^{m} \frac{\lambda_{n,j}^\alpha - \lambda_{\beta,k}^m + i}{\lambda_{n,j}^\alpha - \lambda_{\beta,k}^m - i},
\]  

where we use

\[
\prod_{j=1}^{n} \prod_{j' \neq j} \frac{\lambda_{n,j}^\alpha - \lambda_{n,j'}^\alpha + i}{\lambda_{n,j}^\alpha - \lambda_{n,j'}^\alpha - i} = 1.
\]  

From eq. (21), we can calculate the products of eq. (56) and obtain the Bathe-Takahashi equation

\[
(\varphi_n(\lambda_{n}^\alpha))^L = \prod_{\beta \neq \alpha}^{m} \Phi_{nm}(\lambda_{n}^\alpha - \lambda_{\beta}^m),
\]  

where

\[
\Phi_{nm}(x) = \begin{cases} 
\varphi_{n+m}\varphi_{n-m} \left[\left(\varphi_{n-m+2}\right)^2(\varphi_{n-m-2})^2\cdots(\varphi_{n+m-2})^2\right] & n \neq m \\
\varphi_{2n} \left[(\varphi_4)^2(\varphi_{4})^2\cdots(\varphi_{2n-2})^2\right] & n = m.
\end{cases}
\]
From eq. (14) and eq. (58), the logarithmic form of the Bethe-Takahashi equation is described as

$$\theta_n(\lambda_\alpha) = \frac{2\pi}{L} I_n^\alpha + \frac{1}{N} \sum_{\beta \neq \alpha} \Theta_{nm}(\lambda_\alpha - \lambda_\beta^m),$$

(60)

where

$$\Theta_{nm}(x) = \begin{cases} 
\theta_{|n-m|} + 2\theta_{|n-m|+2} + 2\theta_{|n-m|+4} + \cdots + 2\theta_n + 2\theta_m & n \neq m \\
2\theta_2 + 2\theta_4 + \cdots + 2\theta_{2(n-2)} + 2\theta_n & n = m,
\end{cases}$$

(61)

and $I_n^\alpha$ is a Bethe quantum number, which is an integer when $L - N_n$ is odd and a half-integer when $L - N_n$ is even.

### 5.2. Bethe-Takahashi equation for the ASEP

We extend the Bethe-Takahashi equation for the ASEP based on the bundled string solutions. The outline of the derivation of the Bethe-Takahashi equation for the ASEP is similar to that for the spin-1/2 Heisenberg XXX chain.

We consider that $N$ rapidities are assigned into $N_j$ strings of length $j$, i.e. $\sum_j jN_j = N$. Rewrite the Bethe equations for the ASEP (27) on an $n$-bundled string by bundled string solutions $\{\lambda_{n,j}^\alpha\}$ as

$$\left(\frac{\lambda_{n,j}^\alpha + \frac{i}{2}}{\lambda_{n,j}^\alpha - \frac{i}{2}}\right)^L = \prod_{k=1}^m \prod_{\beta \neq \alpha} \frac{(1 - \epsilon)\lambda_{n,j}^\alpha - (1 + \epsilon)\lambda_{m,k}^\alpha + i}{(1 + \epsilon)\lambda_{n,j}^\alpha - (1 - \epsilon)\lambda_{m,k}^\alpha - i} \prod_{j' \neq j} \left(\frac{(1 - \epsilon)\lambda_{n,j}^{n,j'} - (1 + \epsilon)\lambda_{n,j'}^\alpha + i}{(1 + \epsilon)\lambda_{n,j}^{n,j'} - (1 - \epsilon)\lambda_{n,j'}^{n,j'} - i}\right), \quad j = 1, \cdots, n.$$ 

(62)

Multiplying the Bethe equations on an $n$-bundled string, then we obtain a following equation.

$$\prod_{j=1}^n \left(\frac{\lambda_{n,j}^\alpha + \frac{i}{2}}{\lambda_{n,j}^\alpha - \frac{i}{2}}\right)^L = \prod_{j=1}^n \prod_{\beta \neq \alpha} \frac{(1 - \epsilon)\lambda_{n,j}^\alpha - (1 + \epsilon)\lambda_{m,k}^\alpha + i}{(1 + \epsilon)\lambda_{n,j}^\alpha - (1 - \epsilon)\lambda_{m,k}^\alpha - i},$$

(63)

where we use

$$\prod_{j=1}^n \prod_{j' \neq j} \frac{(1 - \epsilon)\lambda_{n,j}^{n,j'} - (1 + \epsilon)\lambda_{n,j'}^\alpha + i}{(1 + \epsilon)\lambda_{n,j}^{n,j'} - (1 - \epsilon)\lambda_{n,j'}^{n,j'} - i} = 1.$$ 

(64)

Here, we introduce variables $\nu$ as

$$\nu = \lambda - \frac{i}{2\epsilon}.$$ 

(65)

From eq. (47), the bundled string solutions $\{\nu_{n,j}^\alpha\}$ are given by

$$\nu_{n,j}^\alpha = \nu_{n,1}^{n,j} 1^{-j},$$

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where

$$\nu_{\alpha}^{n,1} = a_{\alpha}^{n,1} + i \left( b_{\alpha}^{n,1} - \frac{1}{2\epsilon} \right). \quad (67)$$

We define a string center of \( n \)-bundled strings \( \nu_{\alpha}^{n} (\lambda_{\alpha}^{n}) \) as

$$\nu_{\alpha}^{n} = \sqrt{\nu_{\alpha}^{n,1} \nu_{\alpha}^{n,n}} = \lambda_{\alpha}^{n} - \frac{i}{2\epsilon}. \quad (68)$$

Then, the bundled string solutions \( \{\nu_{\alpha}^{n,j}\} \) are denoted by

$$\nu_{\alpha}^{n,j} = \nu_{\alpha}^{n} \gamma_{j}^{-\frac{n+1}{2}}. \quad (69)$$

By changing variables \( \lambda \) to \( \nu \), we calculate the product of eq. (63). From eq. (30), (31), (65) and eq. (69), the LHS of eq. (63) is

$$\prod_{j=1}^{n} \left( \frac{\lambda_{\alpha}^{n,j} + i \frac{1}{2}}{\lambda_{\alpha}^{n,j} - i \frac{1}{2}} \right)^L = \prod_{j=1}^{n} \left( \frac{\nu_{\alpha}^{n,j} + i C \gamma_{\frac{1}{2}}}{\nu_{\alpha}^{n,j} + i C \gamma_{\frac{1}{2}}} \right)^L$$

$$= (\varphi_{n,\epsilon}(\nu_{\alpha}^{n}, -i C))^L$$

$$= (\varphi_{n,\epsilon}(\lambda_{\alpha}^{n}, i(1/2\epsilon - C)))^L,$$

where we define a function \( \tilde{\varphi}_{n,\epsilon}(x, y) \) as

$$\varphi_{n,\epsilon}(x, y) = \varphi_{n,\epsilon}(x + i/2\epsilon, y + i/2\epsilon)$$

$$= \frac{x - y \gamma_{\frac{1}{2}}}{x - y \gamma_{-\frac{1}{2}}}. \quad (71)$$

In order to calculate the RHS of eq. (63), we consider the product over \( k \) at first.

$$\prod_{k=1}^{m} \frac{(1 - \epsilon)\lambda_{\alpha}^{n,j} - (1 + \epsilon)\lambda_{\beta}^{m,k} + i}{(1 + \epsilon)\lambda_{\alpha}^{n,j} - (1 - \epsilon)\lambda_{\beta}^{m,k} - i} = \prod_{k=1}^{m} \frac{\nu_{\alpha}^{n,j} - \gamma \nu_{\beta}^{m,k}}{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,k}}$$

$$= \gamma^{-m} \frac{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,m}}{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,0}} \frac{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,m+1}}{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,1}} \quad (72)$$

Then we calculate the product over \( j \) of eq. (63).

$$\prod_{j=1}^{n} \left[ \gamma^{-m} \frac{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,m}}{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,0}} \frac{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,m+1}}{\nu_{\alpha}^{n,j} - \nu_{\beta}^{m,1}} \right] = \prod_{j=1}^{n} \left[ \gamma^{-m} \frac{\nu_{\alpha}^{n} - \nu_{\beta}^{m,n}}{\nu_{\alpha}^{n} - \nu_{\beta}^{m,0}} \frac{\nu_{\alpha}^{n} - \nu_{\beta}^{m,n+1}}{\nu_{\alpha}^{n} - \nu_{\beta}^{m,1}} \right]$$

$$= \Phi_{nm,\epsilon}(\nu_{\alpha}^{n}, \nu_{\beta}^{n})$$

$$= \Phi_{nm,\epsilon}(\lambda_{\alpha}^{n}, \lambda_{\beta}^{n}). \quad (73)$$
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where we define

\[
\Phi_{nm,\epsilon}(x,y) = \begin{cases}
\gamma^{-nm} \varphi_{n+m,\epsilon} \varphi_{|n-m|,\epsilon} \left[ (\varphi_{|n-m|+2,\epsilon})^2 (\varphi_{|n-m|+4,\epsilon})^2 \cdots (\varphi_{n+m-2,\epsilon})^2 \right] & n \neq m \\
\gamma^{-n^2} \varphi_{2n,\epsilon} \left[ (\varphi_{2,\epsilon})^2 (\varphi_{4,\epsilon})^2 \cdots (\varphi_{2n-2,\epsilon})^2 \right] & n = m \end{cases},
\]

(74)

\[
\tilde{\Phi}_{nm,\epsilon}(x,y) = \Phi_{nm,\epsilon}(x + i/2\epsilon, y + i/2\epsilon).
\]

(75)

From eq. (74), (70) and (73), we obtain the Bethe-Takahashi equation for the ASEP

\[
(\varphi_{n,\epsilon}(\lambda_{\alpha}^n, i(1/2\epsilon - C)))^L = \prod_{\beta \neq \alpha} \Phi_{nm,\epsilon}(\lambda_{\alpha}^n, \lambda_{\beta}^m).
\]

(76)

From eq. (63), (70) and (73), the logalithmic form of the Bethe-Takahashi equation for the ASEP is described as

\[
\theta_{n,\epsilon}(\lambda_{\alpha}^n, i(1/2\epsilon - C)) = \frac{2\pi}{L} I^n_\alpha + \frac{1}{L} \sum_{\beta \neq \alpha} \Theta_{nm,\epsilon}(\lambda_{\alpha}^n, \lambda_{\beta}^m),
\]

(77)

where we define

\[
\Theta_{nm,\epsilon}(x,y) = \begin{cases}
\theta_{|n-m|,\epsilon} + 2\theta_{|n-m|+2,\epsilon} + 2\theta_{|n-m|+4,\epsilon} \\
+ \cdots + 2\theta_{n+m-2,\epsilon} + \theta_{n+m,\epsilon} - i n m \log \gamma & n \neq m \\
2\theta_{2,\epsilon} + 2\theta_{4,\epsilon} + \cdots + 2\theta_{2n-2,\epsilon} + \theta_{2n,\epsilon} - i n^2 \log \gamma & n = m,
\end{cases}
\]

(78)

and \(I^n_\alpha\) is a Bethe quantum number, which is an integer when \(L - N_n\) is odd and a half-integer when \(L - N_n\) is even.

6. Conclusion

In this paper, we investigated the Bethe equations for the ASEP and presented a new type of the string solutions called the bundled string solutions. The Hamiltonian of the ASEP is a non-Hermitian matrix. The non-hermicity of the ASEP deforms the traditional strings of the spin-1/2 Heisenberg XXX chain. Although the strings of the spin-1/2 Heisenberg XXX chain are parallel to the imaginary axis in the complex plane, the bundled strings are not parallel to each other. All bundled string solutions are bundled at the knot \((0, 1/2\epsilon)\). We also presented the dual-conjugacy, which is an extension of the self-conjugacy for non-Hermitian systems. This symmetry indicates that a system of the ASEP is connected to its dual system by complex conjugate. The dual-conjugacy determined the imaginary part of the bundled string solutions. The bundled string solutions and the dual-conjugacy were confirmed by numerical solutions of small systems. In the TBA formalism for the spin-1/2 Heisenberg chain, the Bethe-Takahashi equation, which is the reduced Bethe equations for strings, play a central role. We extended the Bethe-Takahashi equation for the ASEP based on the bundled string solutions.
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These results are the first step in investigating the non-Hermitian integrable systems, such as stochastic integrable systems and quantum integrable systems with boundary terms. In future work, we plan to construct a framework to calculate the physical quantities and dynamics of non-Hermitian integrable systems based on the bundled string solutions.

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References

[1] Rigol M, Dunjko V, Yurovsky V and Olshanii M 2007 Phys. Rev. Lett. 98 050405
[2] Ilievski E, Nardis J D, Wouters B, Caux J S, Essler F H L and Prosen T 2015 Phys. Rev. Lett. 115 157201
[3] Yang C N and Yang C P 1969 J. Math. Phys. 10, 1115
[4] Takahashi M 1971 Prog. Theor. Phys. 46 401
[5] Takahashi M, Thermodynamics of One-dimensional Solvable Models (Cambridge University Press, Cambridge, England, 2005).
[6] Mossel J and Caux J S 2012 J. Phys. A: Math. Theor. 45 255001
[7] Caux J S and Essler F H L 2013 Phys. Rex. Lett. 110 257203
[8] Caux J S 2016 J. Stat. Mech. 064006
[9] Derrida B 1998 Phys. Rep 301 65
[10] Castro-Alvaredo O A, Doyon B and Yoshimura T 2016 Phys. Rev. X 6 041065
[11] Bertini B, Collura M, Nardis J D and Fagotti M 2016 Phys. Rev. Lett. 117 207201
[12] Nardis J D, Bernard D and Doyon B 2019 SciPost Phys. 6 049
[13] Prosen T 2011 Phys. Rev. Lett. 106 217206
[14] Golinelli O and Mallick K 2006 J. Phys. A 39 12679
[15] Gier J d and Essler F H L 2005 Phys. Rev. Lett 95 240601
[16] Essler F H L and Rittenberg V 1996 J. Phys. A: Math. Gen. 29 3375
[17] Blythe R A, Evans M R, and Essler F H L 2000 J. Phys. A 33 2313
[18] Krug J, 1991 Phys. Rev. Lett 67 1882
[19] MacDonald C T, Gibbs J H and Pipkin A C 1968 Biopolymers 6 1
[20] Kluempp S and Lipowsky R 2003 J. Stat. Phys 113 233
[21] Schadschneider A 2000 Physica. A 285 101
[22] Schadschneider A, Chowdhury D and Nishinari K, Stochastic Transport in Complex Systems: From Molecules to Vehicles (Elsevier Science, Amsterdam, 2010).
[23] Takeuchi K A 2018 Physica. A 504 77
[24] Blythe R A and Evans M R 2007 J. Phys. A 40 R333
[25] Bethe H A 1931 Z. Phys. 71 205
[26] Essler F H L, Korepin V E and Schoutens K 1992 J. Phys. A: Math. Gen. 25 4115
[27] Hagemans R and Caux J S 2007 J. Phys. A: Math. Theor. 40 14605
[28] Deguchi T and Giri P R 1992 J. Phys. A: Math. Theor. 25 4115
[29] Golinelli O and Mallick K 2005 J. Phys. A: Math. Gen. 38 1419
[30] Golinelli O and Mallick K 2005 J. Stat. Phys. 120 779
[31] Vladimirov A A 1986 Theor. Math. Phys. 66 102