1. Introduction

Let $X$ be a smooth projective curve over $\mathbb{C}$, $\mathcal{O}_X(1)$ a very ample line bundle on $X$, and $M_1, M_2$ two line bundles on $X$ so that $M_1 \otimes M_2 \simeq K_X^{-1}$. An ADHM sheaf $\mathcal{E}$ on $X$ with twisting data $(M_1, M_2)$ is a coherent $\mathcal{O}_X$-module $E$ decorated by morphisms

$$
\Phi_i : E \otimes_X M_i \to E, \quad \phi : E \otimes_X M_1 \otimes_X M_2 \to \mathcal{O}_X, \quad \psi : \mathcal{O}_X \to E
$$

with $i = 1, 2$, satisfying the ADHM relation

$$(1.1) \quad \Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) + \psi \circ \phi = 0.\quad$$

An ADHM sheaf $\mathcal{E}$ will be said to be of type $(r, e) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ if $E$ has rank $r \in \mathbb{Z}_{\geq 0}$ and degree $e \in \mathbb{Z}$. 

2. Stack Function Algebras for ADHM Quiver Sheaves

2.1. Brief Review of Joyce Theory

2.2. Application to ADHM Quiver Sheaves

2.3. Stack function identities

3. Wallcrossing Formulas

3.1. ADHM invariants via weighted Euler characteristic

3.2. Counting invariants and wallcrossing

4. Comparison with Kontsevich-Soibelman Formula

Appendix A. Bell Polynomials

References
A triple \((E, \Phi_1, \Phi_2)\) with \(\Phi_1, \Phi_2\) morphisms of \(O_X\)-modules as above satisfying relation (1.1) for \(\phi = 0, \psi = 0\), will be called a Higgs sheaf on \(X\) with coefficient sheaf \(M_1 \oplus M_2\).

The following construction results concerning moduli spaces of ADHM sheaves were proved in the first part of this work [3].

- There exists a stability condition for ADHM sheaves depending on a real parameter \(\delta \in \mathbb{R}\) [3 Def. 2.1], [3 Def. 2.2] so that for fixed \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) the are finitely many critical stability parameters dividing the real axis into chambers. The set of \(\delta\)-semistable ADHM sheaves is constant within each chamber, and strictly semistable objects may exist only if \(\delta\) takes a critical value. The origin \(\delta = 0\) is a critical value for all \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\).

- For fixed \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) and \(\delta \in \mathbb{R}\) there is an algebraic moduli stack of finite type over \(\mathbb{C}\) \(\mathcal{M}_\delta^r(X, r, e)\) of \(\delta\)-semistable locally free ADHM sheaves. If \(\delta \in \mathbb{R}\) is noncritical, \(\mathcal{M}_\delta^r(X, r, e)\) is a separated algebraic space of finite type over \(\mathbb{C}\) equipped with a perfect obstruction theory [3 Thm 1.2], [3 Thm 1.4].

- For fixed \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) and \(\delta \in \mathbb{R}\) there is a natural algebraic torus \(S = \mathbb{C}^*\) action on the moduli stack \(\mathcal{M}_\delta^r(X, r, e)\) which acts on \(\mathbb{C}\)-valued points by scaling the morphisms \((\Phi_1, \Phi_2) \rightarrow (t^{-1}\Phi_1, t\Phi_2), t \in S\). If \(\delta\) is noncritical [3 Thm 1.5] proves that the stack theoretic fixed locus \(\mathcal{M}_\delta^r(X, r, e)^S\) is a proper algebraic space over \(\mathbb{C}\). Therefore residual ADHM invariants \(A_{\delta}^S(r, e)\) are defined by equivariant virtual integration in each stability chamber [3 Def. 1.8].

- For \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}\) there exists a critical value \(\delta_M \in \mathbb{R}_{>0}\) so that for any \(\delta > \delta_M, \mathcal{M}_\delta^r(X, r, e)\) is isomorphic to the moduli space of stable pairs of Pandharipande and Thomas [12] on the total space of the rank two bundle \(M_1^{-1} \oplus M_2^{-1}\) on \(X\). This identification includes the equivariant perfect obstruction theories establishing an equivalence between local stable pair theory and asymptotic ADHM theory (see [4 Thm. 1.11] and [4 Cor. 1.12] for precise statements.)

The present paper represents the second part of this work. Its main goal is to derive wallcrossing formulas for the ADHM invariants \(A_{\delta}^S(r, e)\) using Joyce’s stack function algebra theory and the theory of generalized Donaldson-Thomas invariants of Joyce and Song. Moreover, it will be also shown that these formulas imply the BPS rationality conjecture formulated by Pandharipande and Thomas in [12] for local stable pair invariants of curves.

Similar results have been obtained by Toda [15] for stable pair invariants of smooth projective Calabi-Yau threefolds defined via the the stack theoretic topological Euler character introduced by Joyce in [8, 6]. Moreover the wallcrossing formula relating stable pair and Donaldson-Thomas theory has been derived for the same type of invariants in [14, 13]. The moduli spaces involved in the local construction considered here are under better technical control, making the theory of Joyce and Song applicable to virtual residual stable pair invariants.

1.1. **Main results.** Let \(\delta_c \in \mathbb{R}_{\geq 0}\) be a critical stability parameter of type \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z},\) possibly zero, and \(\delta_+, \delta_- \in \mathbb{R}\) be stability parameters so that there are no critical stability parameters of type \((r, e)\) in the interval \([\delta_-, \delta_+].\) In order to simplify the formulas, we will denote the numerical invariants by \(\alpha = (r, e),\) and
use the notation
\[ \mu_{\delta}(\alpha) = \frac{e + \delta}{r}, \quad \mu(\alpha) = \frac{e}{r} \]
for any \( \alpha = (r, e) \) with \( r \geq 1 \), and any \( \delta \in \mathbb{R} \).

For fixed \( \alpha = (r, e), \delta \geq 0 \) and \( l \in \mathbb{Z}_{\geq 2} \) let \( S_{\delta}(\alpha) \) be the set of all ordered decompositions
\[ (1.5) \quad \alpha = \alpha_1 + \cdots + \alpha_l, \quad \alpha_i = (r_i, e_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}, \quad i = 1, \ldots, l \]
satisfying
\[ (1.3) \quad \mu(\alpha_1) = \cdots = \mu(\alpha_{l-1}) = \mu_{\delta}(\alpha_l) = \mu_{\delta}(\alpha). \]

Note that the union \( S_{\delta}(\alpha) = \bigcup_{l \geq 2} S_{\delta}(\alpha) \) is a finite set for fixed \( \delta \geq 0 \). Then the following theorem is proven in section 3.2.

**Theorem 1.1.** (i) The following wallcrossing formula holds for \( \delta > 0 \)
\[ (1.4) \quad A^S_{\delta}(\alpha) - A^S_{\delta}(\alpha) = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{\delta}^{(l)}(\alpha)} \prod_{j=1}^{l-1} (-1)^{c_j - r_j(g-1)} (e_j - r_j(g-1)) H^S(\alpha_j)]. \]

(ii) The following wallcrossing formula holds for \( \delta = 0 \).
\[ (1.5) \quad A^S_{\delta}(\alpha) - A^S_{\delta}(\alpha) = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{\delta}^{(l)}(\alpha)} \prod_{j=1}^{l-1} (-1)^{c_j - r_j(g-1)} (e_j - r_j(g-1)) H^S(\alpha_j)] + \sum_{l \geq 1} \frac{1}{l!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{\delta}^{(l)}(\alpha)} \prod_{j=1}^{l} (-1)^{c_j - r_j(g-1)} (e_j - r_j(g-1)) H^S(\alpha_j)] \]

Moreover, if \( g \geq 1 \), the right hand sides of equations (1.4), (1.5) vanish.

Here \( H^S(\alpha) \) are generalized Donaldson-Thomas type invariants for Higgs sheaves with numerical invariants \( \alpha = (r, e) \) on \( X \) defined in section 3.2.

For fixed \( r \in \mathbb{Z}_{\geq 1} \), and fixed \( \delta \in \mathbb{R}_{>0} \setminus \mathbb{Q} \) let
\[ (1.6) \quad Z_\delta(q)_r = \sum_{e \in \mathbb{Z}} q^{e \cdot r(g-1)} A^S_{\delta}(r, e), \quad Z_\infty(q)_r = \sum_{e \in \mathbb{Z}} q^{e \cdot r(g-1)} A^S_\infty(r, e). \]

Note that \( \delta \) is noncritical of any type \( (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) since it is irrational, therefore all \( A^S_{\delta}(r, e) \) are virtual residual invariants. According to [3] Cor. 1.12 \( Z_\infty(q)_r \) is the generating function of degree \( r \) local stable pair invariants of the data \( \mathcal{X} = (X, M_1, M_2) \). Using theorem 1.1 the following rationality result is proven in section 3.3.

**Theorem 1.2.** For any \( r \in \mathbb{Z}_{\geq 1} \), and any \( \delta \in \mathbb{R}_{>0} \setminus \mathbb{Q} \), \( Z_\delta(q)_r, Z_\infty(q)_r \) are Laurent expansions of rational functions of \( q \). Moreover, the rational function corresponding to \( Z_\infty(q)_r \) is invariant under \( q \leftrightarrow q^{-1} \). If \( g \geq 1 \), \( Z_\delta(q)_r = Z_\infty(q)_r \) is a polynomial in \( q, q^{-1} \) invariant under \( q \leftrightarrow q^{-1} \).
Theorem (1.1) and theorem (1.2) are proven in sections (2), (3) by explicit computations in the stack theoretic Ringel-Hall algebras defined by Joyce [5]-[9], which yield wallcrossing formulas for Donaldson-Thomas invariants using the theory of Joyce and Song [10].

In section (4) it is shown that the wallcrossing formula (1.4) is in agreement with the wallcrossing formula of Kontsevich and Soibelman [11]. An analogous computation shows that (1.4) is also in agreement with the semi-primitive wallcrossing formula derived by Denef and Moore [2, Eqn. 5.6] from supergravity considerations.

Acknowledgements. D.-E. D. would like to thank Arend Bayer, Ugo Bruzzo, Daniel Jafferis, Dominic Joyce, Jan Manschot, Greg Moore, Sven Meinhardt, Kento Nagao, Alexander Schmitt, Andrei Teleman, Yokinobu Toda, and especially Ron Donagi, Liviu Nicolaescu and Tony Pantev for very helpful discussions, and correspondence and to Ionut Ciocan-Fontanine, Bumsig Kim and Davesh Maulik for collaboration on a related project. We are especially grateful to Dominic Joyce for pointing out Lemma (3.1), which resulted in significant simplifications of the original proofs. The work of D.-E. D. is partially supported by NSF grants PHY-0555374-2006 and PHY-0854757-2009. The work of W.-y. C. was supported by DOE grant DE-FG02-96ER40959.

2. Stack Function Algebras for ADHM Quiver Sheaves

This section explains how the formalism of stack functions and Ringel-Hall algebras constructed by Joyce in [5]-[9], [8] can be applied to ADHM quiver sheaves on a smooth projective curve \( X \) over \( \mathbb{C} \). Note that a detailed exposition of Joyce’s results can be found for example in [15, Sect. 2], so we will restrict ourselves to a brief recollection of the main steps of the construction.

2.1. Brief Review of Joyce Theory. Let \( \mathfrak{F} \) be an algebraic stack locally of finite type over \( \mathbb{C} \) with affine geometric stabilizers (that is, the automorphisms groups of \( \mathbb{C} \)-valued points of \( \mathfrak{F} \) are affine algebraic groups over \( \mathbb{C} \).) The space of stack functions of \( \mathfrak{F} \) is a \( \mathbb{Q} \)-vector space constructed as follows [6, Sect. 2.3].

- Consider pairs \( (\mathfrak{X}, \varrho) \) where \( \mathfrak{X} \) is an algebraic \( \mathbb{C} \)-stack of finite type with affine geometric stabilizers and \( \varrho : \mathfrak{X} \to \mathfrak{F} \) is a finite type morphism of algebraic stacks.
- Two such pairs are said to be equivalent, \( (\mathfrak{X}, \varrho) \sim (\mathfrak{X}', \varrho') \), if there is an isomorphism of stacks \( \mathfrak{X} \cong \mathfrak{X}' \) so that the obvious triangle diagram is commutative. Denote equivalence classes by \( [(\mathfrak{X}, \varrho)] \).
- Suppose \( (\mathfrak{X}, \varrho) \) is a pair as above, and \( \mathfrak{Y} \hookrightarrow \mathfrak{X} \) is a closed substack. Then the pair \( (\mathfrak{X}, \varrho) \) yields two pairs \( (\mathfrak{Y}, \varrho|_{\mathfrak{Y}}) \) and \( (\mathfrak{X} \setminus \mathfrak{Y}, \varrho|_{\mathfrak{X} \setminus \mathfrak{Y}}) \). The stack function space \( \text{SF}(\mathfrak{F}) \) is the \( \mathbb{Q} \)-vector space generated by equivalence classes \( [(\mathfrak{X}, \varrho)] \) subject to the relation

\[
[(\mathfrak{X}, \varrho)] = [(\mathfrak{Y}, \varrho|_{\mathfrak{Y}})] + [(\mathfrak{X} \setminus \mathfrak{Y}, \varrho|_{\mathfrak{X} \setminus \mathfrak{Y}})].
\]

\( \text{SF}(\mathfrak{F}) \subseteq \text{SF}(\mathfrak{F}) \) is the linear subspace generated by equivalence classes of pairs \( [(\mathfrak{X}, \varrho)] \) with \( \varrho \) representable.

A central element in Joyce’s theory is the existence of an associative algebra structure on the \( \mathbb{Q} \)-vector space \( \text{SF}(\mathfrak{F}) \) when \( \mathfrak{F} \) is the moduli space of all objects in a \( \mathbb{C} \)-linear abelian category \( \mathcal{C} \) satisfying certain assumptions [5 Assumption 7.1], [5 Assumption 8.1]. The basic assumptions require \( \mathcal{C} \) to be noetherian and artinian.
and all morphisms spaces in \( \mathcal{C} \) to be finite dimensional complex vector spaces. Natural \( \mathbb{C} \)-bilinear composition maps of the form
\[
\text{Ext}^i(B, C) \times \text{Ext}^j(A, B) \to \text{Ext}^{i+j}(A, C)
\]
are required to exist for \( 0 \leq i, j \leq 1 \), \( i + j = 0,1 \) and all \( A, B, C \) objects of \( \mathcal{C} \). Moreover, a quotient \( K(\mathcal{C}) \) of the Grothendieck group \( K_0(\mathcal{C}) \) by some fixed subgroup is also required, with the property that \( [A] = 0 \) in \( K(\mathcal{C}) \) implies \( A = 0 \) in \( \mathcal{C} \). The cone spanned by classes of objects of \( \mathcal{C} \) in \( K(\mathcal{C}) \) will be denoted by \( \overline{C}(\mathcal{C}) \). The complement of the class \( [0] \in \overline{C}(\mathcal{C}) \) will be denoted by \( C(\mathcal{C}) \).

The remaining assumptions in \cite{5} Assumption 7.1, \cite{5} Assumption 8.1 will not be listed in detail here. Essentially, one requires the existence of Artin moduli stacks \( \mathcal{D}(\mathcal{C}), \mathcal{E}_\Gamma(\mathcal{C}) \), locally of finite type over \( \mathbb{C} \), parameterizing all objects of \( \mathcal{C} \), respectively three term exact sequences
\[
(2.1) \quad 0 \to A' \to A \to A'' \to 0
\]
in \( \mathcal{C} \). Moreover there also exist natural projections
\[
(2.2) \quad p, p', p'' : \mathcal{E}_\Gamma(\mathcal{A}) \to \mathcal{D}(\mathcal{C})
\]
which are 1-morphisms of Artin stacks of finite type. There should also exist natural disjoint union decompositions
\[
\mathcal{D}(\mathcal{C}) = \bigsqcup_{\alpha \in \mathcal{C}(\mathcal{C})} \mathcal{D}(\mathcal{C}, \alpha)
\]
\[
(2.3) \quad \mathcal{E}_\Gamma(\mathcal{C}) = \bigsqcup_{\alpha, \alpha', \alpha'' \in \mathcal{C}(\mathcal{C})} \mathcal{E}_\Gamma(\mathcal{C}, \alpha, \alpha', \alpha'')
\]
compatible with the forgetful morphisms \( (2.2) \). All this data should satisfy additional natural conditions which will not be explicitly stated here.

Granting assumptions \cite{5} Assumption 7.1, \cite{5} Assumption 8.1, one can define a \( \mathbb{Q} \)-bilinear operation \( * : \mathcal{SF}(\mathcal{D}(\mathcal{C})) \times \mathcal{SF}(\mathcal{D}(\mathcal{C})) \to \mathcal{SF}(\mathcal{D}(\mathcal{C})) \) \cite{6} Def. 1 as follows. Given two stack functions \( [(\mathcal{X}_i, f_i)] \in \mathcal{SF}(\mathcal{D}(\mathcal{C})) \) set
\[
(2.4) \quad [(\mathcal{X}_2, f_2)] * [(\mathcal{X}_1, f_1)] = [(p', p'')^*(\mathcal{X}_1 \times \mathcal{X}_2), p \circ f_j],
\]
where the stack function in the right hand side of equation \( (2.4) \) is determined by the following diagram
\[
\begin{array}{ccc}
\mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{(p', p'')} & \mathcal{E}_\Gamma(\mathcal{C}) & \xrightarrow{p} & \mathcal{D}(\mathcal{C}) \\
\downarrow f_1 \times f_2 & & & & \downarrow (p', p'') \\
\mathcal{X}_1 & \xrightarrow{f_1} & \mathcal{X}_2 & \xrightarrow{f_2} & \mathcal{D}(\mathcal{C}) \times \mathcal{D}(\mathcal{C})
\end{array}
\]
According to \cite{6} Thm. 5.2, \( \mathcal{SF}(\mathcal{D}(\mathcal{C})) \), \( *, \delta_{[0]} \) is an associative algebra with unity, where \( \delta_{[0]} = [(\text{Spec}(\mathcal{C}), 0)] \) is the stack function determined by the zero object in \( \mathcal{C} \).

For further reference, note that the construction of the associative stack function algebra can be also applied with no modification to an exact subcategory \( \mathcal{A} \) of \( \mathcal{C} \) (assuming that \( \mathcal{C} \) satisfies the above assumptions.)

Note also that an important element in the proof of wallcrossing formulas will be a refinement of the stack function algebra, the Ringel-Hall Lie algebra \( \mathcal{SF}^{ind}(\mathcal{D}(\mathcal{A})) \). This is a Lie algebra over \( \mathbb{Q} \) whose underlying vector space is the linear subspace of the stack function algebra spanned by stack functions with algebra stabilizers.
supported on virtually indecomposable objects. We will not review all the relevant definitions here since they will not be needed in the rest of the paper. We refer to [5 Sect 5.1], [6 Sect. 5.2] for details. The important result for us [5 Thm. 5.17] is that this linear subspace is closed under the Lie bracket determined by the associative product , therefore it has a Lie algebra structure.

2.2. Application to ADHM Quiver Sheaves. Let \( X \) be a smooth projective curve over \( \mathbb{C} \). Let \( M_1, M_2 \) be fixed line bundles on \( X \) equipped with a fixed isomorphism \( M_1 \otimes_X M_2 \cong K_X^{-1} \). Recall that an abelian subcategory \( \mathcal{C}_X \) of ADHM quiver sheaves with twisting data \((M_1, M_2)\) has been defined in \[3\ Sect. 3.1\]. For completeness, recall that the objects of \( \mathcal{C}_X \) are ADHM quiver sheaves on \( X \) with \( E_\infty = V \otimes \mathcal{O}_X \), where \( V \) is a finite dimensional complex vector space. Morphisms are natural morphisms of ADHM quiver sheaves with component at \( \infty \) of the form \( f \otimes 1_{\mathcal{O}_X} \), where \( f \) is a \( \mathbb{C} \)-linear map.

Since the objects of \( \mathcal{C}_X \) are decorated pairs of coherent \( \mathcal{O}_X \)-modules, the basic assumptions recalled in the previous section hold for \( \mathcal{C}_X \). The quotient \( K(\mathcal{C}_X) \) of the Grothendieck group of \( \mathcal{C}_X \) is isomorphic to the lattice \( \mathbb{Z}^3 \). The class of an object \( \mathcal{E} \) of \( \mathcal{C}_X \) is given by the triple \((r, e, v) = (r(\mathcal{E}), d(\mathcal{E}), v(\mathcal{E})) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} \), where \( r(\mathcal{E}) \), \( d(\mathcal{E}) \) are the rank, respectively degree of the underlying \( \mathcal{O}_X \)-module \( E \), and \( v(\mathcal{E}) \) is the dimension of \( V \).

Let \( \mathcal{A}_X \) be the exact full subcategory of \( \mathcal{C}_X \) consisting of locally free ADHM quiver sheaves on \( X \). According to \[3\ Lemma 5.2\], there is a locally finite type algebraic moduli stack \( \mathfrak{Ob}(X) \) with affine geometric stabilizers parameterizing all objects of \( \mathcal{A}_X \). Moreover, \[3\ Lemma 5.2\], there also exists an algebraic moduli stack \( \mathfrak{Ex}(X) \) of three term exact sequences of objects of \( \mathcal{A}_X \), which is locally of finite type over \( \mathbb{C} \).

Let \( \mathfrak{H} = \mathfrak{Ob}(X) \) in the construction described in the previous section. The remaining conditions in \[3\ Assumption 7.1\] follow by analogy with \[3\ Thm. 10.10\], \[6\ Thm 10.12\] since the objects of \( \mathcal{A}_X \) are decorated sheaves on \( X \). In conclusion, the construction of the associative product in \[6\ Def. 5.1\] carries over to the present situation. Therefore we obtain again an associative algebra with unity \((\mathfrak{SE}(\mathfrak{Ob}(X)), *, [0])\) over \( \mathbb{Q} \). The construction of the Ringel-Hall Lie algebra of virtually indecomposable representable stack functions with algebra stabilizers also carries over to the present case, resulting in a Lie algebra \( \mathfrak{SE}_{\text{ind}}(\mathfrak{Ob}(X)) \).

2.3. Stack function identities. According to \[3\ Cor. 5.6\], for any stability parameter \( \delta \in \mathbb{R} \) and any splitting type \( t \in T \) there are open immersions

\[
\begin{align*}
\mathfrak{Ob}^*_{\delta}(X, r, e, 1) & \hookrightarrow \mathfrak{Ob}^*(X) \hookrightarrow \mathfrak{Ob}(X) \\
\mathfrak{Ob}^*_{\delta}(X, r, e, 0) & \hookrightarrow \mathfrak{Ob}^*(X) \hookrightarrow \mathfrak{Ob}(X).
\end{align*}
\]

(2.6)

The corresponding elements of the stack function algebra will be denoted by

\[
\delta_\alpha(\alpha), b(\alpha) \in \mathfrak{SE}(\mathfrak{Ob}(X)),
\]

where \( \alpha = (r, e) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \).

Let \( \delta_\epsilon \in \mathbb{R}_{>0} \) be a critical stability parameter for ADHM sheaves on \( X \) of type \( \alpha = (r, e) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \). According to \[3\ Lemm. 4.13\], any \( \delta_\epsilon \)-semistable object of \( \mathcal{A}_X \) with \( v = 1 \) has a one-step Harder-Narasimhan filtration with respect to \( \delta_\pm \) stability, where \( \delta_- < \delta_\epsilon, \delta_+ > \delta_\epsilon \) are noncritical stability parameters sufficiently close to \( \delta_\epsilon \). More precisely, let \( \epsilon_\pm \in \mathbb{R}_{>0} \) be positive real numbers as in \[3\ Lemm. 4.13\], for

\[
\delta_\epsilon = \delta_n. \quad \delta_+ \in (\delta_\epsilon, \delta_n + \epsilon_+) \quad \delta_- \in (\delta_\epsilon - \epsilon_-, \delta_n)
\]

be noncritical stability parameters
Obviously, \( \Omega \in \mathbb{R} \) will be denoted by \( \partial_+ \) and \( \partial_- \). Given any numerical type \( \alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \), set and

\[
\mu_1(\alpha) = \frac{e}{r}, \quad \mu_2(\alpha) = \mu_1(\alpha) + \frac{\delta}{r}, \quad \mu_3(\alpha) = \mu_1(\alpha) + \frac{\delta e}{r}
\]

provided that \( r(t) \neq 0 \).

Then the following lemma holds.

**Lemma 2.1.** The following relations hold in \( \mathbf{SF}(\mathcal{D}b(\mathcal{X})) \)

\[
\partial_+(\alpha) = \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \atop \mu_1(\alpha_1) = \mu_2(\alpha_2) = \mu_3(\alpha)} \mathbf{h}(\alpha_2) \ast \partial_+(\alpha_1) \quad \partial_-(\alpha) = \sum_{\alpha_1, \alpha_2 \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \atop \mu_1(\alpha_1) = \mu_2(\alpha_2) = \mu_3(\alpha)} \partial_-(\alpha_1) \ast \mathbf{h}(\alpha_2)
\]

where the sums in the right hand sides of equations (2.7) are finite.

**Proof.** Given [3 Lemm. 4.13], theorem [9 Thm 5.11] applies to the present case, yielding formulas (2.7). Finiteness is obvious from the summation conditions.

The important point in the following is that relations (2.7) can be inverted according to [9 Thm 5.12]. In order to write down the inverse relations, for any \( l \in \mathbb{Z}_{\geq 1} \) and any \( 1 \leq j \leq l \) define

\[
\mathcal{S}_{\delta}(\alpha) = \left\{ (\alpha_1, \ldots, \alpha_l) \in \mathbf{Z}_{\geq 1} \times \mathbf{Z} \left| \sum_{i=1}^{l} \alpha_i = \alpha, \mu(\alpha_i) = \mu_1(\alpha), \ 1 \leq i \leq l, \ i \neq j, \ \mu(\alpha_j) = \mu_1(\alpha) \right. \right\}
\]

Obviously, \( \mathcal{S}_{\delta}(\alpha) \) is a finite set for fixed \( \alpha, l, j \).

Then [9 Thm 5.12] implies the following

**Lemma 2.2.** The following relations hold in \( \mathbf{SF}(\mathcal{D}b(\mathcal{X})) \)

\[
\partial_+(\alpha) = \partial_+(\alpha) + \sum_{l \geq 2} (-1)^{l-1} \sum_{(\alpha_1, \ldots, \alpha_l) \in \mathcal{S}_{\delta}(\alpha)} \mathbf{h}(\alpha_1) \ast \cdots \ast \mathbf{h}(\alpha_{l-1}) \ast \partial_+(\alpha_l)
\]

\[
\partial_-(\alpha) = \partial_-(\alpha) + \sum_{l \geq 2} (-1)^{l-1} \sum_{(\alpha_1, \ldots, \alpha_l) \in \mathcal{S}_{\delta}(\alpha)} \partial_-(\alpha_1) \ast \mathbf{h}(\alpha_2) \ast \cdots \ast \mathbf{h}(\alpha_l)
\]

where the sums in the right hand sides of equations (2.8) are finite.

**Proof.** We will check only the first equation in (2.8), since the second is entirely analogous. According to [9 Thm 5.12], inverting the first relation in (2.7) yields

\[
\partial_+(\alpha) = \sum_{l \geq 1} (-1)^{l-1} \sum_{j=1}^{l} \sum_{(\alpha_1, \ldots, \alpha_l) \in \mathcal{S}_{\delta}(\alpha)} \mathbf{h}(\alpha_1) \ast \cdots \ast \partial_+(\alpha_j) \ast \cdots \ast \mathbf{h}(\alpha_l)
\]

where

\[
\mu_1^{\mu}(\alpha_1 + \cdots + \alpha_k) = \left\{ \begin{array}{ll}
\mu(\alpha_1 + \cdots + \alpha_k) & \text{for } k < j \\
\mu(\alpha_1 + \cdots + \alpha_k) & \text{for } k \geq j
\end{array} \right.
\]
the parameter $\delta$ (2.12) relation holds [7, Thm 8.2]

\[
\begin{aligned}
\mu_{+}^{l}(\alpha_{k+1} + \cdots + \alpha_{l}) &= \\
&= \begin{cases} \\
\mu(\alpha_{k+1} \cdots + \alpha_{l}) & \text{for } k \geq j \\
\mu_{+}(\alpha_{k+1} + \cdots + \alpha_{l}) & \text{for } k < j
\end{cases}
\end{aligned}
\]

for any $l \geq 2$ and any $1 \leq k \leq l - 1$. However, using the relations

\[
\mu(\alpha_{i}) = \mu_{c}(\alpha_{j}) = \mu_{c}(\alpha)
\]

in (2.8) and $\delta_{+} > \delta_{c}$, it is straightforward to prove that the inequality

\[
\mu_{+}^{l}(\alpha_{1} + \cdots + \alpha_{k}) < \mu_{+}^{l}(\alpha_{k+1} + \cdots + \alpha_{l})
\]

is satisfied if and only if $j = l$.

\[\square\]

Lemmas (2.1), (2.2) imply the following corollary, which follows by direct substitution.

**Corollary 2.3.** Under the conditions of lemmas (2.1), (2.2) the following relations hold in the stack function algebra $\underline{SE}(\mathfrak{Ob}(\mathcal{X}))$.

\[
\delta_{+}(\alpha) - \delta_{-}(\alpha) = \sum_{l \geq 2} (-1)^{l} \sum_{(\alpha_{1}, \ldots, \alpha_{l}) \in S_{\mathfrak{u}}^{(l, l)}(\alpha)} h(\alpha_{1}) \star \cdots \star [\delta_{-}(\alpha_{1}), h(\alpha_{1} - 1)]
\]

(2.10)

Next note that since the sum in the right hand side of equation (2.10) is finite, the parameter $\delta_{-} \in \mathbb{R}_{>0}$ can be chosen to be noncritical with respect to all types $\alpha_{i}$ so that $\delta_{-}(\alpha_{i}) \neq 0$. This implies that any $\mathbf{S}$-fixed $\delta_{-}$-semistable object of splitting type $\alpha_{i}$ is $\delta_{-}$-stable. Since $\delta_{\pm}$ have been chosen noncritical of type $(r, c)$ the same holds for $\delta_{\pm}$-semistable objects of splitting type $\alpha$. In particular the automorphism group of all such objects is isomorphic to $\mathbb{C}^{\times}$, according to [3, Lemm. 3.7]. Given the definition of virtually indecomposable objects with algebra stabilizers [6, Sect. 5.1-5.2], this implies that the stack functions $\delta_{\pm}(\alpha)$, $\delta_{-}(\alpha_{i})$ belong to the Lie algebra $\underline{SF}_{\mathfrak{ind}}^{\mathfrak{ind}}(\mathfrak{Ob}(\mathcal{X}))$ for all possible splitting types $\alpha_{i}$ in the right hand side of equation (2.10).

However, the stack functions $h(\alpha_{i})$ in the same equation do not satisfy this property for arbitrary splitting type $\alpha_{i}$, since strictly semistable Higgs sheaves will be present. Then one has to use [7, Thm. 8.7] in order to construct virtually indecomposable log stack functions $g(\alpha)$ as follows

\[
g(\alpha) = \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{\alpha_{1}, \ldots, \alpha_{l} \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \\
\alpha_{1} + \cdots + \alpha_{l} = \alpha \\
\mu(\alpha_{i}) = \mu(\alpha), 1 \leq i \leq l}} h(\alpha_{1}) \star \cdots \star h(\alpha_{l})
\]

(2.11)

where the sum in the right hand side is finite. Then [7, Thm. 8.7] implies that $g(\alpha)$ is an element of the Lie algebra $\underline{SF}_{\mathfrak{ind}}^{\mathfrak{ind}}(\mathfrak{Ob}(\mathcal{X}))$. Moreover, the following inverse relation holds [7, Thm 8.2]

\[
h(\alpha) = \sum_{l \geq 1} \frac{1}{l!} \sum_{\substack{\alpha_{1}, \ldots, \alpha_{l} \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \\
\alpha_{1} + \cdots + \alpha_{l} = \alpha \\
\mu(\alpha_{i}) = \mu(\alpha), 1 \leq i \leq l}} g(\alpha_{1}) \star \cdots \star g(\alpha_{l})
\]

(2.12)

where the sum in the right hand side is again finite.
Lemma 2.4. The following relation holds in $\text{SF}_{\text{cl}}(\text{Ob}(\mathcal{X}))$

\begin{equation}
\mathcal{D}_+ (\alpha) - \mathcal{D}_- (\alpha) = \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(a_1,\ldots,a_l) \in S^{(l)}(\alpha)} [g(a_1),\ldots,g(a_{l-1}),\mathcal{D}_-(\alpha_l)] \cdot \cdot \cdot
\end{equation}

\textbf{Proof.} Expanding the commutators in each term in the right hand side of equation (2.13)

\begin{equation}
\mathcal{D}_+ (\alpha) - \mathcal{D}_- (\alpha) = \sum_{k=0}^{l-1} \sum_{i_1< \ldots < i_k} \sum_{j_1,\ldots,j_{l-1-k} \in \{2,\ldots,l\} \setminus \{i_1,\ldots,i_k\}} (-1)^k g(\alpha_{i_1}) \cdot \cdot \cdot \cdot g(\alpha_{i_k}) \cdot \mathcal{D}_-(\alpha_l) \cdot g(\alpha_{j_{l-1-k}}) \cdot \cdot \cdot g(\alpha_{j_1})
\end{equation}

where, by convention, $\{i_1,\ldots,i_k\} = \emptyset$, $\{j_1,\ldots,j_{l-1-k}\} = \{1,\ldots,l-1\}$ if $k = 0$, respectively $\{i_1,\ldots,i_k\} = \{1,\ldots,l-1\}$, $\{j_1,\ldots,j_{l-1-k}\} = \emptyset$ if $k = l-1$. Summing over all values of $(a_1,\ldots,a_l) \in S^{(l)}(\alpha)$ for fixed $l \geq 2$ yields

\begin{equation}
\frac{(-1)^{l-1}}{(l-1)!} \sum_{(a_1,\ldots,a_l) \in S^{(l)}(\alpha)} \sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} g(a_1) \cdot \cdot \cdot g(a_k) \cdot \mathcal{D}_-(\alpha_l) \cdot g(\alpha_{k+1}) \cdot \cdot \cdot g(\alpha_{l-1}).
\end{equation}

employing similar conventions. Substituting (2.12) in (2.10), we obtain

\begin{equation}
\mathcal{D}_+ (\alpha) - \mathcal{D}_- (\alpha) = \sum_{p \geq 2} \binom{-1}{p} \sum_{(a_1,\ldots,a_p) \in S^{(p)}(\alpha)} \sum_{m_1 \geq 1} \sum_{(\beta_1,\ldots,\beta_{m_1}) \in S^{(m_1)}(\alpha)} \cdots \sum_{m_p \geq 1} \sum_{(\beta_{p-1},\ldots,\beta_{m_p}) \in S^{(m_p)}(\alpha)} \frac{1}{m_1! \cdot \cdot \cdot m_p!} g(\beta_{1,1}) \cdot \cdot \cdot g(\beta_{1,m_1}) \cdot g(\beta_{2,1}) \cdot \cdot \cdot g(\beta_{2,m_2}) \cdot \cdot \cdot
\end{equation}

\begin{equation}
\cdot \mathcal{D}_-(\beta_p, g(\beta_{p-1,1}) \cdot \cdot \cdot g(\beta_{p-1,m_p-1}))
\end{equation}

where

$S^{(m)}(\alpha) = \{(\beta_1,\ldots,\beta_m) \in (\mathbb{Z}_{\geq 1} \times \mathbb{Z})^m \mid \beta_1 + \cdot \cdot \cdot + \beta_m = \alpha, \ \mu(\beta_1) = \cdot \cdot \cdot = \mu(\beta_m)\}$

for any $m \geq 1$ and any $\alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$.

The right hand side of (2.15) can be rewritten as

\begin{equation}
\mathcal{D}_+ (\alpha) - \mathcal{D}_- (\alpha) = \sum_{p \geq 2} \binom{-1}{p} \sum_{m_1,\ldots,m_{p-1} \geq 1} \sum_{(\beta_{1,1},\ldots,\beta_{1,m_1},\ldots,\beta_{p-1,1},\ldots,\beta_{p-1,m_p-1},\beta_1) \in S^{(l)}(\alpha)} \frac{1}{m_1! \cdot \cdot \cdot m_{p-1}!} g(\beta_{1,1}) \cdot \cdot \cdot g(\beta_{1,m_1}) \cdot g(\beta_{2,1}) \cdot \cdot \cdot g(\beta_{2,m_2}) \cdot \cdot \cdot
\end{equation}

\begin{equation}
\cdot \mathcal{D}_-(\beta_1, g(\beta_{p-1,1}) \cdot \cdot \cdot g(\beta_{p-1,m_p-1}))
\end{equation}

where $l = m_1 + \cdot \cdot \cdot + m_{p-1} + 1$.

Note that for fixed $(p,l)$ in the right hand side of (2.16) we sum over ordered sequences $(m_1,\ldots,m_{p-1}) \in \mathbb{Z}_{\geq 0}^{p-1}$ satisfying $m_1 + \cdot \cdot \cdot + m_{p-1} = l - 1$. For $p \geq 3$
there are exactly two monomials associated to each such ordered sequence, namely
\[ g(\beta_1) \cdots g(\beta_{l-1}) \cdot d_-(\beta_l) \quad \text{and} \quad g(\beta_1) \cdots g(\beta_k) \cdot d_-(\beta)l_1 \cdots g(\beta_{l-1}) \]
with \(1 \leq k = m_{p-2} \leq l - 1\). The same statement holds for \(p = 2\), except that the second monomial in the above equation reads \(d_-(\beta_l) \cdot g(\beta_{k+1}) \cdots g(\beta_{l-1})\).

Given an arbitrary monomial of the form
\[(2.17) \quad g(\beta_1) \cdots g(\beta_{l-1}) \cdot d_-(\beta_l) \]
with fixed \(l \geq 2\) and fixed \((t_1, \ldots, t_l) \in S_{k,l}(\alpha)\) there is an obvious one-to-one correspondence between ordered sequences \((m_1, \ldots, m_{p-1})\) and partitions of the ordered sequence \((\beta_1, \ldots, \beta_{l-1})\) of the form
\[(2.18) \quad (\beta_1, \ldots, \beta_{m_1} \mid \ldots \mid \beta_{l-m_p} \ldots \beta_l) \].

Moreover, the sequence \((m_1, \ldots, m_{p-1})\) also determines a length \((p - 1)\) unordered partition \(\lambda_{(m_1, \ldots, m_{p-1})} = (1^{j_1}, 2^{j_2}, \ldots, s^{j_s})\) of \((l - 1)\), which will be called the underlying partition of the sequence \((m_1, \ldots, m_{p-1})\). The factor
\[
\frac{1}{m_1! \cdots m_{p-1}!} = \frac{1}{(1!)^{j_1} \cdots (s!)^{j_s}}
\]
depends only on the underlying partition \(\lambda_{(m_1, \ldots, m_{p-1})}\). Conversely, for a fixed length \((p - 1)\) partition \(\lambda = (1^{j_1}, 2^{j_2}, \ldots, s^{j_s})\) of \((l - 1)\), there are
\[
\frac{(p - 1)!}{j_1! j_2! \cdots j_s!}
\]
distinct ordered sequences \((m_1, \ldots, m_{p-1})\) as above with underlying partition \(\lambda\). Each such sequence corresponds to a partition of the set \((\beta_1, \ldots, \beta_{l-1})\) of the form \((2.18)\).

Similar arguments apply to any monomial of the form
\[(2.19) \quad g(\beta_1) \cdots g(\beta_k) \cdot d_-(\beta_l) \cdot g(\beta_{k+1}) \cdots g(\beta_{l-1}) \]
with \(1 \leq k \leq l - 1\). For \(p \geq 3\), there is a one-to-one correspondence between ordered sequences \((m_1, \ldots, m_{p-2})\) with \(m_{p-2} = k\) and partitions of the ordered sequence \((\beta_1, \ldots, \beta_k)\) of the form
\[(2.20) \quad (\beta_1, \ldots, \beta_{m_1} \mid \ldots \mid \beta_{m_{p-2}+1} \cdots \beta_k) \]
Moreover, an ordered sequence \((m_1, \ldots, m_{p-2})\) as above also determines a length \((p - 2)\) partition of \(k\), \(\lambda_{(m_1, \ldots, m_{p-2})} = (1^{j_1}, \ldots, s^{j_s})\). The following relation holds
\[
\frac{1}{m_1! \cdots m_{p-2}!} = \frac{1}{(1!)^{j_1} \cdots (s!)^{j_s}}.
\]

Conversely, for a length \((p - 2)\) partition of \(k\), \(\lambda = (1^{j_1}, \ldots, s^{j_s})\) there are
\[
\frac{(p - 2)!}{j_1! \cdots j_s!}
\]
distinct ordered sequences \((m_1, \ldots, m_{p-2})\) with underlying partition \(\lambda\).
In conclusion, the right hand side of (2.16) can be further rewritten as follows
\[ (2.21) \quad d_+ (\alpha) - d_- (\alpha) = \]
\[ \sum_{l \geq 2} \sum_{(\beta_1, \beta_2, \ldots, \beta_l) \in \beta_{k,l}^0} \sum_{k=0}^{l-1} c_k (\beta_1, \ldots, \beta_l) g(\beta_1) \ast \cdots \ast g(\beta_k) \ast d_- (\beta_l) \ast g(\beta_{k+1}) \ast \cdots \ast g(\beta_{l-1}) \]
where the coefficients \( c_k (\beta_1, \ldots, \beta_l) \) are given by
\[ c_l-1 (\beta_1, \ldots, \beta_l) = - \sum_{p \geq 2} (-1)^p \sum_{\lambda \in \mathcal{P}_{p-1}(l-1)} \frac{(p-1)!}{j_1! j_2! \cdots j_s! (1!)^{j_1} \cdots (s!)^{j_s}} \]
\[ c_k (\beta_1, \ldots, \beta_l) = \frac{1}{(l-k-1)!} \sum_{p \geq 3} (-1)^p \sum_{\lambda \in \mathcal{P}_{p-2}(k)} \frac{(p-2)!}{j_1! j_2! \cdots j_s! (1!)^{j_1} \cdots (s!)^{j_s}} \]
for \( 1 \leq k \leq l-2, \ l \geq 3 \), and
\[ c_0 (\beta_1, \ldots, \beta_l) = \frac{1}{(l-1)!} \]
if \( k = 0 \). Here, where \( \mathcal{P}_{p-1}(l-1) \) denotes the set of length \((p-1)\) partitions of \((l-1)\), \( \mathcal{P}_{p-2}(k) \) denotes the set of length \((p-2)\) partitions of \(k\).

Next note that the coefficients \( c_k (\beta_1, \ldots, \beta_l) \) may be expressed in terms of Bell polynomials
\[ c_{l-1} (\beta_1, \ldots, \beta_l) = \frac{1}{(l-1)!} \sum_{p \geq 2} (-1)^{p-1} (p-1)! B_{l-1,p-1}(1,1,\ldots,1) \]
respectively
\[ c_k (\beta_1, \ldots, \beta_l) = \frac{1}{k! (l-k-1)!} \sum_{p \geq 2} (-1)^{p-2} (p-2)! B_{l-k-1,p-2}(1,1,\ldots,1) \]
for \( 1 \leq k \leq l-2, \ l \geq 3 \). Some basic facts on Bell polynomials are recalled for convenience in appendix A. Then a special case of the Faà di Bruno formula (see equation (A.3)) yields
\[ c_{l-1} (\beta_1, \ldots, \beta_l) = \frac{(-1)^{l-1}}{(l-1)!} \quad c_k (\beta_1, \ldots, \beta_l) = \frac{(-1)^{l-k-1}}{k! (l-k-1)!} = \frac{(-1)^{l-k}}{(l-1)!} \frac{1}{k} \]
Therefore, taking into account equation (2.14), the final formula for the difference \( d_+ (\alpha) - d_- (\alpha) \) is indeed (2.13).

\[ \square \]

Analogous arguments yield an identity relating the stack functions \( d_{\pm} (\alpha) \) where \( \delta_+ \in \mathbb{R}_{>0}, \delta_- \in \mathbb{R}_{\leq 0} \) are stability parameters sufficiently close to the origin. More precisely, take \( \delta_+ < \epsilon_+ , \ \delta_- > \epsilon_- \), where \( \epsilon_\pm \) are as in [4, Lemn. 4.15]. Let \( \mathfrak{d} \) be the stack function determined by the moduli stack of objects of \( \mathcal{A}_X \) of type \((r,e,v) = (0,0,1)\). Note that any such object is isomorphic to \( O = (0, \mathbb{C}, 0, 0, 0, 0) \) and the moduli stack in question is isomorphic to the quotient stack \([s/\mathbb{C}^\times]\). Let \( S_0^{(l)} (\alpha) \) be the set obtained by setting \( \delta_c = 0 \) in equation in (2.8), which then
becomes independent of $1 \leq j \leq l$. Then, in complete analogy with lemmas (2.1), (2.2), (2.3).

**Lemma 2.5.** The following identity holds in the Ringel-Hall Lie algebra $SF^{\text{ind}}_{\delta}(\mathcal{D}b(X))$

\[
\delta_+ (\alpha) - \delta_- (\alpha) = \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_1,...,\alpha_l) \in S_0^{(l)}(\alpha)} [g(\alpha_1),...,[g(\alpha_{l-1}), \delta_- (\alpha),] \cdots] \\
+ \sum_{l \geq 1} \frac{(-1)^{l}}{l!} \sum_{(\alpha_1,...,\alpha_l) \in S_0^{(l)}(\alpha)} [g(\alpha_1),...,[g(\alpha_l), \mathcal{O},] \cdots]
\]

where the sum in the right hand side of equation (2.22) is finite.

### 3. Wallcrossing Formulas

In this section we prove theorems (1.1) and (1.2).

#### 3.1. ADHM invariants via weighted Euler characteristic.

In order to derive wallcrossing formulas using the formalism of Joyce and Song, the ADHM invariants must be first expressed in terms of Behrend’s weighted Euler characteristic. This is the content of the following lemma, which is due to Dominic Joyce.

**Lemma 3.1.** Let $\delta \in \mathbb{R}_{>0}$ be a noncritical stability parameter of type $(r, e)$. Then

\[
A^S_\delta (r, e) = \chi^B(\mathcal{M}_\delta^{ss}(X, r, e))
\]

where the right hand side of equation (3.1) is Behrend’s weighted Euler characteristic of the algebraic space $\mathcal{M}_\delta^{ss}(X, r, e)$.

**Proof.** Recall that the ADHM invariant $A^S_\delta (r, e)$ is defined by virtual integration on the fixed locus $\mathcal{M}_\delta(X, r, e)^S$

\[
A^S_\delta (r, e) = \int_{[\mathcal{M}_\delta^{ss}(X, r, e)]^S} \epsilon_\delta(\mathcal{M}_\delta^{ss}(X, r, e)^S / \mathcal{M}_\delta^{ss}(X, r, e))^{-1}
\]

The virtual cycle of the fixed locus is determined by the fixed part of the perfect tangent-obstruction theory of the moduli space restricted to the fixed locus. Since the perfect tangent-obstruction theory of $\mathcal{M}_\delta(X, r, e)$ is $S$-equivariant symmetric it follows that the induced tangent-obstruction theory of the fixed locus is symmetric. Therefore the resulting virtual cycle is a 0-cycle.

The virtual normal bundle $N_{\delta}^{vir}(\mathcal{M}_\delta(X, r, e)^S / \mathcal{M}_\delta(X, r, e))$ is determined by the $S$-moving part of the perfect obstruction theory of $\mathcal{M}_\delta(X, r, e)$ restricted to $\mathcal{M}_\delta(X, r, e)^S$, which is also $S$-equivariant symmetric. By construction, the virtual normal bundle $N_{\delta}^{vir}(\mathcal{M}_\delta(X, r, e)^S / \mathcal{M}_\delta(X, r, e))$ is an equivariant K-theory class of the form

\[
\mathbb{E}_1^m - \mathbb{E}_2^m
\]

where $\mathbb{E}_1^m$ is an equivariant locally free sheaf on $\mathcal{M}_\delta(X, r, e)^S$ and $\mathbb{E}_2^m$ is its (equivariant) dual. Moreover, the character decomposition of $\mathbb{E}_1^m$ does not contain the trivial character.

Since the virtual cycle of the fixed locus is a 0-cycle, it suffices to compute the equivariant Euler class $\epsilon_\delta(N_{\delta}^{vir}(\mathcal{M}_\delta(X, r, e)^S / \mathcal{M}_\delta(X, r, e))))$ of the restriction of the virtual normal bundle to a closed point $\mathcal{M}$ of the fixed locus. Let $\mathcal{E}$ be the $S$-fixed $\delta$-stable
ADHM sheaf on $X$ corresponding to $\mathcal{M}$. Then, given the construction of the perfect tangent-obstruction theory in [3 Sect. 5.4] it follows that

$$N^\text{vir}_{\mathcal{M}^S(X, r, e)}|_m = \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})^m - \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})^m$$

where $\operatorname{Ext}^k(\mathcal{E}, \mathcal{E})^m$, $k = 1, 2$ denotes the moving part of the ext group $\operatorname{Ext}^k(\mathcal{E}, \mathcal{E})$ in the abelian category $A_X$. Moreover, using [3 Prop. 3.15], it is straightforward to check that there is an equivariant isomorphism $\operatorname{Ext}^2(\mathcal{E}, \mathcal{E})^m \simeq (\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})^m)'$. This implies that

$$\tag{3.3} e_S(N^\text{vir}_{\mathcal{M}^S(X, r, e)}|_{\mathcal{M}^S(X, r, e)})|_m = (-1)^{\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})^m}. $$

Since the virtual normal bundle is a K-theory class of the form (3.2), the right hand side of equation (3.3) must be independent of $E$ when $\mathcal{M}$ varies within a connected component $\Xi$ of the fixed locus. This can be in fact confirmed by a direct computation based on the locally free complex given in [3, Prop. 3.15], but the details will not be needed in the following. Let $\sigma(\Xi)$ denote the common value of $(-1)^{\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})^m}$ for all closed points $\mathcal{M} \in \Xi$. Then we obtain

$$\tag{3.4} A^S_{\delta}(r, e)_1 = \sum_{\Xi} \sigma(\Xi) \int_{[\Xi]^\text{vir}} 1 = \sum_{\Xi} \sigma(\Xi) \chi^B(\Xi)$$

where $\chi^B(\Xi)$ denotes the weighted Euler character of the connected component $\Xi$ of the fixed locus. Next we claim that for any $\Xi$

$$\tag{3.5} \sigma(\Xi) \chi^B(\Xi) = \chi(\Xi, \nu|\Xi),$$

where $\nu$ is Behrend’s constructible function of the moduli space $\mathcal{M}^S_{\delta}(X, r, e)$.

Let $\mathcal{E}$ be an $S$-fixed $\delta$-stable ADHM sheaf corresponding to a closed point $\mathcal{M} \in \Xi$ as above. Then [3 Thm. 7.1] implies that the moduli space $\mathcal{M}^S_{\delta}(X, r, e)$ is analytically locally isomorphic near $\mathcal{M}$ to the critical locus of a holomorphic function $\Phi : U \to C$, where $U \subset \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})$ is an analytic open neighborhood of the origin. Moreover, given the construction in [3 Sect. 7], $U, \Phi$ can be naturally chosen so that $U$ is preserved by the induced $S$-action on $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})$, and $\Phi$ is $S$-invariant. In particular, $\Phi$ yields a holomorphic function $\Phi^S$ on the fixed locus $U^S \subset U$ so that $\Xi$ is analytically locally isomorphic to the critical locus of $\Phi^S$. Then

$$\nu(\mathcal{M}) = (-1)^{\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})}(1 - \chi_{\text{top}}(MF(\Phi, 0)))$$

where $MF(\Phi, 0)$ is the Milnor fiber of $\Phi$ at $0 \in U$, and $\chi_{\text{top}}$ denotes the topological Euler characteristic. Furthermore

$$\nu_{\Xi}(\mathcal{M}) = (-1)^{\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})'}(1 - \chi_{\text{top}}(MF(\Phi^S, 0))).$$

where $\nu_{\Xi}$ is Behrend’s constructible function of the fixed locus $\Xi$, and

$$\chi_{\text{top}}(MF(\Phi, 0)) = \chi_{\text{top}}(MF(\Phi^S, 0)).$$

Therefore

$$\nu([\mathcal{E}]) = (-1)^{\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E})^m} \nu_{\Xi}(\mathcal{E}),$$

which implies that

$$\tag{3.6} \sigma(\Xi) \chi^B(\Xi) = \chi(\Xi, \nu|\Xi).$$

Since $\Xi$ are the connected components of the $S$-fixed locus, equation (3.6) then follows easily.
3.2. Counting invariants and wallcrossing. Let \( E_1, E_2 \) be two locally free ADHM sheaves on \( X \) of numerical types \((r_1, e_1, 1), (r_2, e_2, 0)\). Let
\[
\chi(E_1, E_2) = \dim \text{Ext}^0(E_1, E_2) - \dim \text{Ext}^1(E_1, E_2) \\
- \dim \text{Ext}^0(E_2, E_1) + \dim \text{Ext}^1(E_2, E_1).
\]
According to [3, Lemm. 7.3],
\[
\chi(E_1, E_2) = e_2 - r_2(g - 1)
\]
depends only of the numerical types of the two objects.

Now let \( L(X) \) be the \( \mathbb{Q} \)-vector space spanned by the formal symbols \( \lambda^\alpha, \lambda^{(\alpha, 1)}, \alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \). Then the following antisymmetric bilinear form
\[
\left[ \lambda^{\alpha_1}, \lambda^{\alpha_2} \right]_{\leq 1} = 0
\]
\[
= (-1)^{r_2 - r_2(g - 1)(e_2 - r_2(g - 1))} \lambda^{(\alpha_1 + \alpha_2, 1)}
\]
\[
\left[ \lambda^{(\alpha_1, 1)}, \lambda^{(\alpha_2, 2)} \right]_{\leq 1} = 0
\]
defines a \( \mathbb{Q} \)-Lie algebra structure on \( L(X) \).

Let \( \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X))_{\leq 1} \) be the truncation of the \( \mathbb{Q} \)-vector space \( \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X)) \) to stack functions \([([X], \vartheta])\) so that \( \vartheta \) factors through the open immersion \( \mathcal{O}b(X)^{S_1} \hookrightarrow \mathcal{O}b(X)^{S} \). Using the Lie algebra structure \([ , ]\) on \( \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X)) \), we define a truncated Lie algebra structure \([ , ]_{\leq 1}\) on \( \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X)_{\leq 1}^S) \) which is equal to \([ , ]\) if the arguments satisfy \( v_1 + v_2 \leq 1 \) and vanishes identically if both arguments are elements with \( v = 1 \).

Now let \( \nu \) denote the Behrend constructible function of the algebraic stack \( \mathcal{O}b(X)_{\leq 1}^S \) defined in [10 Prop. 4.4]. Then, given [3 Thm 7.2], [3 Lemm. 7.3] and [3 Thm. 7.4], the following theorem holds by analogy with [10 Thm. 5.12].

**Theorem 3.2.** There exists a Lie algebra morphism
\[
\Psi : \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X))_{\leq 1} \to L(X)_{\leq 1}
\]
which maps an element of \( \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X))_{\leq 1} \) of numerical type \( \alpha \), respectively \( (\alpha, 1) \), \( \alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) to \( Q^\alpha \), respectively \( Q^{(\alpha, 1)} \).

Moreover, suppose \([([X], \vartheta])\) is an element of \( \mathbb{SF}^{\text{ind}}(\mathcal{O}b(X))_{\leq 1} \) of type \((\alpha, 1)\), where \( X \to X \) is a \( \mathbb{C}^* \)-gerbe over an algebraic space \( X \) of finite type over \( \mathbb{C} \), and \( \vartheta : X \to \mathcal{O}b(X)_{\leq 1}^S \) is an open immersion. Then
\[
\Psi([([X], \vartheta)]) = -\chi^B(X)\lambda^{(\alpha, 1)}
\]
were \( \chi^B(X) \) is Behrend’s weighted Euler characteristic of the algebraic space \( X \).

Recall that according to [3 Cor. 5.5] for any noncritical stability parameter of type \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) and the moduli stack \( \mathcal{O}b_{\delta}^{ss}(\mathcal{X}, r, e, 1) \) is a \( \mathbb{C}^* \)-gerbe over the algebraic moduli space \( \mathbb{M}^{ss}_\delta(\mathcal{X}, r, e) \) of \( \delta \)-semistable ADHM sheaves of type \((r, e) \). Then theorem 3.2 lemma 3.1 imply

**Corollary 3.3.** Let \( \delta \in \mathbb{R}_{>0} \) be a noncritical stability parameter of type \((r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \). Then
\[
\Psi(\mathcal{O}_\delta(\alpha)) = -A^S_{\delta}(r, e)\lambda^{(\alpha, 1)}.
\]
In order to formulate a wallcrossing result for ADHM invariants, one has to also define Higgs sheaf invariants by

\[ \Psi(g(\alpha)) = H^S(r, e) \lambda^\alpha \]

for any \( \alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \).

By analogy with [10], define the invariants \( H^S(r, e) \) by the multicovery formula

\[ H^S(r, e) = \sum_{m \geq 1} \frac{1}{m^2} H^S(r/m, e/m). \]

Conjecturally, \( H^S(r, e) \) are \( \mathbb{Z} \)-valued invariants.

**Proof of Theorem (1.1).** Formulas (1.4) and (1.5) follow by a simple computation applying the Lie algebra morphism \( \Psi \) of theorem (3.2) to the stack function identities derived in lemmas (2.4), respectively (2.5).

The proof of theorem (1.2) will require more general wallcrossing formulas relating \( \delta \)-ADHM invariants, for a generic stability parameter \( \delta \in \mathbb{R}_{>0} \) to those corresponding to small values of the stability parameter. More precisely, for fixed \( \alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) let \( \delta \in \mathbb{R}_{>0} \setminus \mathbb{Q} \) be a fixed noncritical stability parameter of type \( (r, e) \). Let \( \delta_{0-} \in \mathbb{R}_{<0}, \delta_{0+} \in \mathbb{R}_{>0} \) be stability parameters so that there are not critical stability values of type \( (r, e) \) in the intervals \([\delta_{0-}, 0), (0, \delta_{0+}]\). Then the following lemmas prove wallcrossing formulas for the differences \( A^S_\delta(\alpha) - A^S_{\delta_{0+}}(\alpha), A^S_{\delta_0}(\alpha) - A^S_{\delta_{0-}}(\alpha) \).

Recall according to [3, Cor. 2.8], for fixed \( r \geq 1 \) there exists a (non-unique) integer \( c(r) \in \mathbb{Z} \) so that \( A^S_\delta(r, e) = 0 \) for any \( e < c(r) \) and any noncritical stability parameter \( \delta \in \mathbb{R}_{>0} \). Obviously, there exist integers \( c(r') \) satisfying this property for each \( 1 \leq r' \leq r \) so that

\[ \frac{c(r')}{r'} = \frac{c(r)}{r} \]

for all \( 1 \leq r' \leq r \). Let \( \mu_0(r) \) denote the common value of the ratios (3.13).

For any \( l \in \mathbb{Z}_{\geq 2} \) let \( S(l)_{+, \delta}(\alpha) \) be the set of all ordered decompositions

\[ \alpha = \alpha_1 + \cdots + \alpha_l, \quad \alpha_i = (r_i, e_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}, \quad 1 \leq i \leq l \]

satisfying

\[ \mu_0(\alpha) \leq \mu(\alpha_1) \leq \mu(\alpha_i) < \mu_3(\alpha) \]

for all \( 2 \leq i \leq l \). Similarly, let \( S(l)_{-, \delta}(\alpha) \) be the set of ordered decompositions satisfying

\[ \mu_0(\alpha) \leq \mu(\alpha_1) \leq \mu(\alpha_i) < \mu_3(\alpha) \]

For any \( l \geq 1 \), let \( S(l)_{\delta}(\alpha) \) denote the set of ordered decompositions satisfying

\[ \mu_0(\alpha) \leq \mu(\alpha_i) < \mu_3(\alpha) \]

for all \( 1 \leq i \leq l \). Let \( S(l)_{+, \delta}(\alpha), S(l)_{-, \delta}(\alpha), S(l)_{\delta}(\alpha) \) be the sets of ordered partitions of \( \alpha \) satisfying slope inequalities of the form

\[ \mu_0(\alpha) \leq \mu(\alpha_1) < \mu(\alpha_i), \quad \mu_0(\alpha) \leq \mu(\alpha_1) \leq \mu(\alpha_i), \quad \mu_0(\alpha) \leq \mu(\alpha_i) \]

respectively. Formally, this is equivalent to taking the limit \( \delta \to \infty \) in the above definitions. Then, using theorem (1.1), the following lemmas hold.
Lemma 3.4. For any \( \alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \), with \( \mu(\alpha) \geq \mu_0(r) \),

\[
A^S_\infty(\alpha) - A^S_{\infty+}(\alpha) = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S^S_{\infty+}(\alpha)} \prod_{j=2}^{l} \left[ (-1)^{e_j-r_j(g-1)}(e_j - r_j(g-1))H^S(\alpha_j) \right].
\]

\[
A^S_\infty(\alpha) - A^S_{\infty-}(\alpha) = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S^S_{\infty-}(\alpha)} \prod_{j=2}^{l} \left[ (-1)^{e_j-r_j(g-1)}(e_j - r_j(g-1))H^S(\alpha_j) \right]
\]

\[+ \sum_{l \geq 1} \frac{1}{l!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S^S_{\infty-}(\alpha)} \prod_{j=1}^{l} \left[ (-1)^{e_j-r_j(g-1)}(e_j - r_j(g-1))H^S(\alpha_j) \right]\]

where only finitely many terms in the right hand sides of equations (3.15), (3.16) are nontrivial. Moreover analogous formulas hold for \( A^S_\infty(\alpha) - A^S_{\infty+}(\alpha), A^S_\infty(\alpha) - A^S_{\infty-}(\alpha) \).

Proof. For any \( n \in \mathbb{Z}_{\geq 1} \) and any collection of \( n \) positive integers \( (l_1, \ldots, l_n) \in \mathbb{Z}_{\geq 1}^n \), define

\[
S^{(l_1, \ldots, l_n)}_{\infty+}(\alpha) = \left\{(\alpha_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in (\mathbb{Z}_{\geq 1} \times \mathbb{Z})^{(l_1+\cdots+l_n+1)} \mid \sum_{i=1}^{n} \sum_{j=1}^{l_i} \eta_{i,j} = \alpha, \mu_0(r) \leq \mu(\alpha_1) < \mu(\eta_{1,1}) = \cdots = \mu(\eta_{1,l_1}) < \mu(\eta_{2,1}) = \cdots = \mu(\eta_{2,l_2}) < \cdots < \mu(\eta_{n,1}) = \cdots = \mu(\eta_{n,l_n}) < \mu_\delta(\alpha) \right\}
\]

Then it straightforward to check that the union

\[
\bigcup_{n \geq 1} \bigcup_{l_1, \ldots, l_n \geq 1} S^{(l_1, \ldots, l_n)}_{\infty+}(\alpha)
\]

is a finite set.

Let \( (\alpha_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in S^{(l_1, \ldots, l_n)}_{\infty+}(\alpha) \) be an arbitrary element, for some \( n \geq 1 \) and \( l_1, \ldots, l_n \geq 1 \). Let \( \mu_i, 1 \leq i \leq n \) denote the common value of the slopes \( \mu(\eta_{i,j}), 1 \leq j \leq l_i \). If \( n \geq 2 \), let also

\[
\alpha_i = \alpha_1 + \eta_{1,1} + \cdots + \eta_{i-1, l_{i-1}}
\]

for \( 2 \leq i \leq n \). For all \( n \geq 1 \) set \( \alpha_{n+1} = \alpha \). Define the stability parameters \( \delta_i \), \( 1 \leq i \leq n \) by

\[
\mu_\delta_1(\alpha_1) = \mu_1
\]

\[
\mu_\delta_i(\alpha_i) = \mu_i, \quad 2 \leq i \leq n \quad (\text{if } n \geq 2).
\]
By construction, $\delta_i$ is a critical stability parameter of type $\alpha_{i+1}$ for all $1 \leq i \leq n$. Given the slope inequalities in (3.17), it is straightforward to check that

$$0 < \delta_1 < \delta_2 < \cdots < \delta_n < \delta.$$

Moreover, since the set (3.18) is finite, the set

$$\Delta_n = \bigcup_{n \geq 1} \bigcup \{\delta_1, \ldots, \delta_n\}$$

is also finite. Therefore there exists $\delta_{0+} \in \mathbb{R}_{>0}$ so that $\delta_{0+} > \min \Delta_n$ and there are no critical stability parameters of type $\alpha$ in the interval $(0, \delta_{0+})$.

Conversely, suppose $\delta_{0+} < \delta_1 < \cdots < \delta_n < \delta$, $n \geq 1$, is a sequence of stability parameters so that there exists $(\alpha_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in (\mathbb{Z}_+ \times \mathbb{Z})^{(l_1+\cdots+l_n+1)}$ for some $l_1, \ldots, l_n \geq 1$ satisfying the following conditions

(a) $\alpha_1 + \eta_{1,1} + \cdots + \eta_{n,l_n} = \alpha$
(b) Conditions (3.19) hold.

Then a direct computation shows that

$$(\alpha_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in T(l_1, \ldots, l_n)$$

and $\delta_1, \ldots, \delta_n \in \Delta_n$.

Then successive applications of the wallcrossing formula (1.4) yields

$$A^S_\delta(\alpha) - A^S_{\delta_0+}(\alpha) = \sum_{n \geq 1} \sum_{l_1, \ldots, l_n \geq 1} \prod_{i=1}^n \frac{1}{l_i!}$$

(3.20)

$$\left\{ \sum_{(\alpha_1, \eta_{1,1}, \ldots, \eta_{1,l_1}, \ldots, \eta_{n,1}, \ldots, \eta_{n,l_n}) \in S^{(l_1, \ldots, l_n)}_\delta(\alpha)} A^S_{\delta_0+}(\alpha_1) \right\}$$

$$\prod_{i=1}^n \prod_{j=1}^l \left( (-1)^{e_{i,j}-r_{i,j}}(g-1)(e_{i,j} - r_{i,j}(g-1)) H^S(\eta_{i,j}) \right)$$

Recall that $A^S_{\delta_0+}(\alpha_1) = 0$ if $\mu(\alpha_1) < \mu_0(r)$. Note that in each term in the right hand side of (3.20) the only factor depending on $\alpha_1$ is $A^S_{\delta_0+}(\alpha_1)$. Then equation (3.15) follows by simple combinatorics. Equation (3.16) also follows by an analogous argument substituting formula (1.5) in (3.20).

□

Next recall that [3, Lemm. 2.3] implies

$$A^S_\delta(r,e) = A^S_\delta(r, -e + 2r(g-1))$$

(3.21)

for any $(r, e) \in \mathbb{Z}_+ \times \mathbb{Z}$, and any noncritical stability parameter $\delta \in \mathbb{R}_{>0}$. Similarly, it is straightforward to prove that

$$H^S(r,e) = H^S(r, -e + 2r(g-1))$$

(3.22)

for any $(r, e) \in \mathbb{Z}_+ \times \mathbb{Z}$.
Moreover, using again corollary [3, Cor. 2.8], it follows that
\[ A_{S_0}(r, e) = 0 \] for any \( e > -c(r) + 2r(g-1) \). Let \( \overline{\tau}(r) = -c(r) + 2r(g-1) \) and \( \overline{\tau_0}(r) = -\mu_0(r) + 2(g-1) \). Note that equation (3.13) implies
\[ \frac{\overline{\tau}(r')}{r'} = \frac{\overline{\tau}(r)}{r} = \overline{\tau_0}(r) \]
for any \( 1 \leq r' \leq r \).

Next, using (3.21), (3.22), theorem (1.1) implies

**Lemma 3.5.** The following holds for any \( \alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \) with \( \mu(\alpha) < \mu_0(r) \)
\[ (3.24) \]
\[ A_{S_0}^\dagger(\alpha) = \sum_{l \geq 1} \frac{1}{l!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{0}^{(l)}(\alpha)} \prod_{j=1}^{l} (\frac{(-1)^{-r_j - r_j(g-1)}(-e_j + r_j(g-1))}{H_{S_j}(\alpha_j)}) \]

**Proof.** Substituting equations (3.21), (3.22) in the wallcrossing formula (1.5) and making obvious redefinitions yields
\[ (3.25) \]
\[ A_{S_0}^\dagger(\alpha) - A_{S_0}^\ddagger(\alpha) = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{0}^{(l)}(\alpha)} A_{S_0}^\dagger(\alpha_1) \prod_{i=2}^{l} (\frac{(-1)^{-e_i - r_i(g-1)}(-e_i + r_i(g-1))}{H_{S_i}(\alpha_i)}) \]
\[ + \sum_{l \geq 1} \frac{1}{l!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{0}^{(l)}(\alpha)} \prod_{i=1}^{l} (\frac{(-1)^{-e_i - r_i(g-1)}(-e_i + r_i(g-1))}{H_{S_i}(\alpha_i)}) \]

If \( \mu(\alpha) < \mu_0(r) \) it follows that \( A_{S_0}^\dagger(\alpha) = 0, A_{S_0}^\ddagger(\alpha_1) = 0 \) for all values of \( \alpha_1 \) in the right hand side of equation (3.25). Therefore formula (3.24) follows. \hfill \Box

Moreover, for any \( l \geq 2 \) let \( S_{l, -\delta}(\alpha) \) denote the set of all ordered decompositions of the form (3.14) satisfying
\[ \mu_{-\delta}(\alpha) < \mu(\alpha_i) < \mu(\alpha_1) \leq \overline{\tau_0}(r) \]
for all \( 2 \leq i \leq l \). For any \( l \geq 1 \) let \( S_{<\mu_0(r), -\delta}(\alpha) \) denote the set of ordered decompositions satisfying
\[ \mu_{-\delta}(\alpha) < \mu(\alpha_i) < \mu_0(r) \]
Finally, let \( S_{l, -\infty}(\alpha) \), \( S_{<\mu_0(r), -\infty} \) be respectively defined by the slope inequalities
\[ \mu(\alpha_i) < \mu(\alpha_1) \leq \overline{\tau_0}(r), \quad \mu(\alpha_i) < \mu_0(r) \]
Lemma 3.6. The following formula holds for any $\alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with $\mu(\alpha) \leq \overline{\mu}_0(r)$

\begin{equation}
A_{-\delta}^S(\alpha) - A_{0-}^S(\alpha) = \sum_{l \geq 2} \frac{1}{(l - 1)!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{l, -\delta}^S(\alpha)} A_{0-}^S(\alpha_1) \prod_{j=2}^l \{(-1)^{e_j - r_j(g - 1)}(-e_j + r_j(g - 1))H^S(\alpha_j)\}
+ \sum_{l \geq 2} \frac{1}{l!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{l, -\delta}^S(\alpha)} \prod_{j=1}^l \{(-1)^{e_j - r_j(g - 1)}(-e_j + r_j(g - 1))H^S(\alpha_j)\}
- \sum_{l \geq 2} \frac{1}{l!} \sum_{(\alpha_1, \ldots, \alpha_l) \in S_{l, -\delta}^S(\alpha)} \prod_{j=1}^l \{(-1)^{e_j - r_j(g - 1)}(-e_j + r_j(g - 1))H^S(\alpha_j)\}
\end{equation}

where the number of nontrivial terms in the right hand side of equation (3.26) is finite. An analogous formula holds for the difference $A_{-\infty}^S(\alpha) - A_{0-}^S(\alpha)$.

Proof. Using equations (3.24), (3.22), equation (3.20) yields

\begin{equation}
A_{-\delta}^S(\alpha) - A_{0-}^S(\alpha) = \sum_{n \geq 1} \sum_{l_1, \ldots, l_n \geq 1} \frac{1}{l_1!} \sum_{(\alpha_1, \eta_{l_1}, \ldots, \eta_{l_n}, l_1, \ldots, n, \eta_{l_1}, \ldots, \eta_{l_n}) \in S_{l_1, \ldots, l_n, \delta}^S(\alpha)} A_{0-}^S(\alpha_1) \prod_{i=1}^n \prod_{j=1}^{l_i} \{(-1)^{e_i - r_i(g - 1)}(e_i - r_i(g - 1))H^S(\eta_{i,j})\}
\end{equation}

where $S_{l_1, \ldots, l_n, \delta}^S(\alpha)$, $l_i \geq 1$, $1 \leq i \leq n$, $n \geq 1$, is the set of all ordered decompositions

$\alpha = \alpha_1 + \eta_{l_1} + \cdots + \eta_{l_i} + \cdots + \eta_{n, l_n}$

satisfying

$\mu_{-\delta}(\alpha) < \mu(\eta, 1) = \cdots = \mu(\eta_{l_1}) < \cdots < \mu(\eta_{l_n}) = \cdots = \mu(\eta_{l_n, l_n}) < \mu(\alpha_1) \leq \overline{\mu}_0(r)$

Now we substitute equation (3.21) in all terms in the right hand side of equation (3.27) with $\mu(\alpha_1) < \mu_0(r)$. Then equation (3.20) follows by simple combinatorics. □

Proof of Theorem (1.2). Recall that $A_{\delta}^S(r, e) = 0$ for all $e < c(r)$, for any fixed $r \geq 1$. Then, using formula (3.16) the generating function $Z_{\delta}(q)_r$ can be rewritten as follows

\begin{equation}
Z_{\delta}(q)_r = Z_{0-}(q)_r + \sum_{l \geq 2} \frac{1}{(l - 1)!} \sum_{r_1 + \cdots + r_l = r, r_1, \ldots, r_l \geq 1} Z_{-\delta}(q)(r_1, \ldots, r_l) + \sum_{l \geq 1} \frac{1}{l!} \sum_{r_1, \ldots, r_l \geq r} Z_{H, \delta}(q)(r_1, \ldots, r_l)
\end{equation}
where

\[
Z_0^-(q) = \sum_{e(r) \leq e \leq \overline{r}(r)} q^{e-r(g-1)} A_{0}^{S} (r, e)
\]

\[
Z_{-\delta}(q_{(r_1, \ldots, r_l)}) = \sum_{e \in \mathbb{Z}} q^{e-r(g-1)} \sum_{e_1, \ldots, e_l \in \mathbb{Z}, e_1 + \ldots + e_l = e} A_{0}^{S} (r_1, e_1)
\]

\[
\prod_{i=1}^{l} \left( (1)^{e_i-r_i(g-1)} (e_i-r_i) H^S (r_i, e_i) \right)
\]

\[
Z_{H, \delta}(q_{(r_1, \ldots, r_l)}) = \sum_{e \in \mathbb{Z}} q^{e-r(g-1)} \sum_{e_1, \ldots, e_l \in \mathbb{Z}, e_1 + \ldots + e_l = e} A_{0}^{S} (r_1, e_1)
\]

\[
\prod_{i=1}^{l} \left( (1)^{e_i-r_i(g-1)} (e_i-r_i) H^S (r_i, e_i) \right)
\]

Note that the range of summation over degrees in \((3.29)\), \((3.30)\) follows from equation \((3.16)\) taking into account relations \((3.13)\), \((3.23)\) and the fact that \(A_{0}^{S}(r, e) = 0\) for all \(e > \overline{r}(r)\).

For any fixed \(r \geq 1, a \in \mathbb{R}, r_1, \ldots, r_l \geq 1, l \geq 1, r_1 + \ldots + r_l = r\) and \(\delta > 0\) let

\[
F_{\delta}(r, a, l, r_1, \ldots, r_l)(q) = \sum_{e \in \mathbb{Z}} q^{e-r(g-1)} \sum_{e_1, \ldots, e_l \in \mathbb{Z}, e_1 + \ldots + e_l = e} A_{0}^{S} (r_1, e_1)
\]

\[
\prod_{i=1}^{l} \left( (1)^{e_i-r_i(g-1)} (e_i-r_i) H^S (r_i, e_i) \right)
\]

Then equations \((3.29)\), \((3.30)\) are equivalent to

\[
Z_{-\delta}(q_{(r_1, \ldots, r_l)}) = \sum_{e_1 \in \mathbb{Z}} q^{e-r_1(g-1)} A_{0}^{S} (r_1, e_1) F_{\delta} \left( r-r_1, \frac{e_1}{r_1}, l-1, \ldots, r_l \right)(q)
\]

\[
Z_{H, \delta}(q_{(r_1, \ldots, r_l)}) = F_{\delta} \left( r, \mu_0(r), l, r_1, \ldots, r_l \right)(q).
\]

In equation \((3.34)\), \(\mu_0(r)\) is the common value of the ratios \(c(r')/r'\), \(1 \leq r' \leq r\), according to equation \((3.13)\).

The next step of the proof establishes rationality of the series \((3.32)\). Given equations \((3.28)\), \((3.29)\), \((3.30)\), \((3.32)\), this implies that \(Z_{\delta}(q)\) is a rational function of \(q\) for any \(r \geq 1\).

For fixed \(r \geq 1, a \in \mathbb{R}, r_1, \ldots, r_l \geq 1, l \geq 1, r_1 + \ldots + r_l = r, e \in \mathbb{Z}\) and \(\delta > 0\) let \(E(r, a, l, r_1, \ldots, r_l, e, \delta)\) denote the set of ordered partitions \(e = e_1 + \ldots + e_l, e_i \in \mathbb{Z}\), \(1 \leq i \leq l\), so that

\[
a \leq \frac{e_i}{r_i} < \frac{e+\delta}{r}
\]
for all $1 \leq i \leq l$. Note that there is a natural injective map

$$
\iota_e : E(r, a, l, r_1, \ldots, r_l, e, \delta) \to E(r, a, l, r_1, \ldots, r_l, e + r, \delta)
$$

$$
\iota_e(e_1, \ldots, e_l) = (e_1 + r_1, \ldots, e_l + r_l).
$$

Next it will be proven that $\iota_e$ is an isomorphism if $e \geq ar + (r - 1)\delta$. Obviously, $\iota_e$ is an isomorphism if any element $(e_1, \ldots, e_l) \in E(r, a, l, r_1, \ldots, r_l, e + r, \delta)$ satisfies $e_i - r_i \geq ar_i$ for all $1 \leq i \leq l$. If this is satisfied, one can construct an obvious inverse for $\iota_e$. Suppose there exists such an element which violates this condition for some $1 \leq i \leq l$. Therefore $e_i - r_i < ar_i$. Then inequalities (3.35) imply

$$
e + r - (a + 1)r_i < e + r - e_i < \frac{r - r_i}{r}(e + r + \delta),
$$

which yields

$$
e < ra + \frac{r - r_i}{r_i} \delta \leq ra + (r - 1)\delta.
$$

Therefore if $e \geq ra + (r - 1)\delta$, such elements cannot exist, and the above claim is proven.

Now note that the invariants $H^S(r, e), (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ satisfy the relations

$$
(3.36)
$$

$$
H^S(r, e) = H^S(r, e + r)
$$

for any $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. This follows from the observation that taking tensor product by any fixed degree 1 line bundle $L$ on $X$ yields an of moduli stacks $\mathcal{N}igg_{ss}(X, r, e) \simeq \mathcal{N}igg_{ss}(X, r, e + r)$. Then the series (3.32) can be rewritten as follows

$$
F_\delta(r, a, l, r_1, \ldots, r_l)(q) = F_\delta(r, a, l, r_1, \ldots, r_l)_{<ar+(r-1)\delta}(q)
$$

$$
+ \sum_{e \in \mathbb{Z}_{\geq 1}} \sum_{ar+(r-1)\delta \leq e \leq (a+1)r+(r-1)\delta-1 \leq e \leq (e+\delta)/r, \ 1 \leq i \leq l} \prod_{i=1}^{l} \left( -q \right)^{e_i - r_i} H^S(r_i, e_i) \sum_{n \geq 0} (e_i + nr_i - r_i(g - 1))(-q)^{nr_i}
$$

$$
(3.37)
$$

where $F_\delta(r, a, l, r_1, \ldots, r_l)_{<ar+(r-1)\delta}(q)$ is the truncation of the right hand side of equation (3.32) to terms with $e < ar + (r - 1)\delta$. Note that the summation conditions in (3.37) imply that all nontrivial terms satisfy $e > ra - \delta$. Therefore $F_\delta(r, a, l, r_1, \ldots, r_l)_{<ar+(r-1)\delta}(q)$ is a finite sum i.e. a polynomial in $q^{-1}, q$. Moreover, the second term in the right hand side of (3.32) is also a finite sum. Since, obviously,

$$
\sum_{n \geq 0} (e_i + nr_i - r_i(g - 1))(-q)^{nr_i} = \left( q \frac{d}{dq} + e_i - r_i(g - 1) \right) \frac{1}{1 - (-q)^r},
$$

it follows that $F_\delta(r, a, l, r_1, \ldots, r_l)(q)$ is indeed a rational function of $q$.

Next note that analogous computations yield the following formula for the asymptotic series $Z_\infty(q)_r$.

$$
Z_\infty(q)_r = Z_0(q)_r + \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{r_1 + \cdots + r_l = r} Z_-(q)(r_1, \ldots, r_l) + \sum_{l \geq 1} \frac{1}{l!} \sum_{r_1 + \cdots + r_l = r} Z_H(q)(r_1, \ldots, r_l)
$$

$$
(3.38)
$$
where $Z_{-0}(r)$ has been defined in (3.29), and

\begin{equation}
Z_{-}^{(q)}(r_1, \ldots, r_l) = \sum_{e_1 \in \mathbb{Z}} q^{e_1 - r_1(g - 1)} A_{-}^{S}(r_1, e_1) \prod_{i=2}^{l} F_{1}^{(r_i, \frac{r_1 e_1}{r_1})}(q)
\end{equation}

\begin{equation}
Z_{H}^{(q)}(r_1, \ldots, r_l) = \prod_{i=1}^{l} F^{(r_i, c(r_i))}(q).
\end{equation}

For any fixed $r \in \mathbb{Z}_{\geq 1}$, $a \in \mathbb{R}$, the series $F(r, a)(q)$ is defined by

\begin{equation}
F(r, a)(q) = \sum_{c \in \mathbb{Z}} q^{-r(g - 1)}(-1)^{e-r(g - 1)(e - r(g - 1))} H^{S}(r, e).
\end{equation}

This follows by formally taking the limit $\delta \to +\infty$ in the above derivation, which simplifies the summation conditions as shown in lemma (3.4). Since this is a straightforward exercise, details will be omitted. Moreover, a simple summation as above yields

\begin{equation}
F(r, a)(q) = \sum_{e} (-q)^{-r(g - 1)} H^{S}(r, e) \left( q \frac{d}{dq} + v - r(g - 1) \right) \frac{(-q)^{r(m(a, v) + 1)}}{1 - (-q)^{r}}
\end{equation}

where $m(a, v) = \max \{ m \in \mathbb{Z} | m < (a - v)/r \}$.

In order to prove invariance under $q \leftrightarrow q^{-1}$ note that

\[ Z_{\infty}(q^{-1}) = Z_{-\infty}(q) \]

for any $r \geq 1$, where

\[ Z_{-\infty}(q) = \sum_{e \in \mathbb{Z}} q^{-r(g - 1)} A_{-\infty}^{S}(r, e). \]

Recall that $A_{-\infty}^{S}(r, e) = 0$ for all $e > e(r)$, for any fixed $r \geq 1$. Then, using lemma (3.3), $Z_{-\infty}(q)$ takes the following form

\begin{equation}
Z_{-\infty}(q) = Z_{0}(q) + \sum_{l \geq 2} \frac{1}{(l - 1)!} \sum_{r_1 + \ldots + r_l = r} Z_{l}'(q)(r_1, \ldots, r_l) + \sum_{l \geq 1} \frac{1}{l!} \sum_{r_1, \ldots, r_l \geq 1} Z_{l}'(q)(r_1, \ldots, r_l)
\end{equation}

where

\begin{equation}
Z_{l}'(q)(r_1, \ldots, r_l) = \sum_{e_1 \in \mathbb{Z}} q^{e_1 - r_1(g - 1)} A_{0}^{S}(r_1, e_1) \prod_{i=2}^{l} F_{l}^{(r_i, \frac{r_1 e_1}{r_1})}(q)
\end{equation}

\begin{equation}
Z_{l}'(q)(r_1, \ldots, r_l) = \prod_{i=1}^{l} F^{(r_i, c(r_i))}(q)
\end{equation}

and

\begin{equation}
F^{(r, a)}(q) = \sum_{e \in \mathbb{Z}, e \leq a} q^{-r(g - 1)}(-1)^{e-r(g - 1)(e - r(g - 1))} H^{S}(r, e). \]
Again, a simple computation shows that $F'(r, a)(q)$ is a rational function, and moreover it is equal to $F(r, a)(q)$ for any $r, a$. Then equations (3.38), (3.39), (3.40), respectively (3.43), (3.44), (3.45) imply $Z_{\infty}(q)_r = Z_{\infty}(q)_r$ for any $r \geq 1$.

\section{Comparison with Kontsevich-Soibelman Formula}

In this section we specialize the wallcrossing formula of Kontsevich and Soibelman \cite{KS} to ADHM invariants, and prove that it implies equation (1.4). Recall that locally free ADHM quiver sheaves on $X$ have a numerical invariants of the form $(r, e, v) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. The pair $(r, e)$ is denoted by $a$ in theorem (1.1). Let $e_\alpha = \lambda^\alpha$, $f_\alpha = \lambda^{(\alpha, 1)}$, $\alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ be alternative notation for the generators of the Lie algebra $L(X)_{\geq 1}$. Therefore

\begin{align}
[e_{\alpha_1}, e_{\alpha_2}]_{\leq 1} &= 0 \\
[f_{\alpha_1}, f_{\alpha_2}]_{\leq 1} &= 0 \\
[e_{\alpha_1}, f_{\alpha_2}]_{\leq 1} &= (-1)^{e_2 - r_2(g - 1)}(e_2 - r_2(g - 1))f_{\alpha_1 + \alpha_2}
\end{align}

Let $\delta_c \in \mathbb{R}_{> 0}$ be a critical stability parameter of type $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ as in theorem (1.1). Then there exist $\alpha, \beta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, with

\begin{align}
\mu_c(\alpha) &= \mu(\beta) = \mu_c(\alpha)
\end{align}

so that any $\eta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with

\begin{align}
\mu_c(\eta) &= \mu_c(\alpha)
\end{align}

is uniquely written as

\begin{align}
\eta &= \alpha + q\beta, \quad q \in \mathbb{Z}_{\geq 0}
\end{align}

and any $\rho \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ with

\begin{align}
\mu(\eta) &= \mu_c(\alpha)
\end{align}

is uniquely written as

\begin{align}
\rho &= q\beta, \quad q \in \mathbb{Z}_{\geq 0}.
\end{align}

Therefore $\alpha$ and $\beta$ generate a subcone of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}$ consisting of elements of $\delta_c$-slope equal to $\mu_c(\alpha)$.

For any $q \in \mathbb{Z}_{\geq 0}$ define to be the following formal expressions

\begin{align}
U_{\alpha + q\beta} &= \exp(f_{\alpha + q\beta}) \\
U_{q\beta} &= \exp(\sum_{m \geq 1} \frac{e_{mq\beta}}{m^2})
\end{align}

In this context, the wallcrossing formula of Kontsevich and Soibelman \cite{KS} reads

\begin{align}
\prod_{q \geq 0, \eta \downarrow} U_{\alpha + q\beta}^A \prod_{q \geq 0, \eta \uparrow} U_{q\beta}^T &= \prod_{q \geq 0, \eta \downarrow} U_{q\beta}^H \prod_{q \geq 0, \eta \uparrow} U_{\alpha + q\beta}^A \prod_{q \geq 0, \eta \downarrow} U_{\alpha + q\beta}^A \prod_{q \geq 0, \eta \uparrow} U_{\alpha + q\beta}^A
\end{align}

where an up, respectively down arrow means that the factors in the corresponding product are taken in increasing, respectively decreasing order of $q$.

In the following we will prove that equation (4.4) implies the wallcrossing formula (1.4). First note that given equation (4.4), the formal operators $U$ commute within each product over $q$ in equation (4.3). Therefore (4.4) can be rewritten as

\begin{align}
\prod_{q \geq 0} U_{\alpha + q\beta}^A \exp(\sum_{m \geq 1} \sum_{q \geq 0} \mathcal{H}^S(mq\beta) \frac{e_{mq\beta}}{m^2}) = \exp(\sum_{m \geq 1} \sum_{q \geq 0} \mathcal{H}^S(mq\beta) \frac{e_{mq\beta}}{m^2}) \prod_{q \geq 0} U_{\alpha + q\beta}^A
\end{align}
This formula can be rewritten in terms of the rational invariants \( H^S(\alpha) \) using (3.12). We obtain
\[
(4.5) \prod_{q \geq 0} U_{\alpha+q\beta}^A \exp\left( \sum_{q \geq 0} H^S(q\beta) e_{q\beta} \right) = \exp\left( \sum_{q \geq 0} H^S(q\beta) e_{q\beta} \right) \prod_{q \geq 0} U_{\alpha+q\beta}^A.
\]
Let us denote by
\[
\mathbb{H} = \sum_{q \geq 0} H^S(q\beta) e_{q\beta}.
\]
Therefore we obtain
\[
(4.6) \prod_{q \geq 0} U_{\alpha+q\beta}^A = \exp(\mathbb{H}) \prod_{q \geq 0} U_{\alpha+q\beta}^A \exp(-\mathbb{H}).
\]
Using again the Lie algebra structure (4.1), note that
\[
\prod_{q \geq 0} U_{\alpha+q\beta}^A = \exp\left( \sum_{q \geq 0} A^S(\alpha + q\beta) \right) e_{\alpha+q\beta}.
\]
Therefore equation (4.6) simplifies to
\[
(4.7) \exp\left( \sum_{q \geq 0} A^S(\alpha + q\beta) e_{\alpha+q\beta} \right) = \exp(\mathbb{H}) \exp\left( \sum_{q \geq 0} A^S(\alpha + q\beta) e_{\alpha+q\beta} \right) \exp(-\mathbb{H}).
\]
Now let us recall the following form of the BCH formula:
\[
\exp(A) \exp(B) \exp(-A) = \exp\left( \sum_{n=0}^{\infty} \frac{1}{n!} (Ad(A))^n B \right)\]
\[
= \exp(B + [A, B] + \frac{1}{2} [A, [A, B]] + \cdots)
\]
Using this formula in (4.7), we obtain
\[
(4.9) \exp\left( \sum_{q \geq 0} A^S(\alpha + q\beta) e_{\alpha+q\beta} \right) = \exp\left( \sum_{l \geq 1} \sum_{q_1, \ldots, q_l \geq 0} \frac{1}{l!} \prod_{i=1}^{l} (-1)^{\chi(q_i, \alpha)} \chi(q_i, \alpha) H^S(q_i\beta) f_{\alpha+(q_1+\cdots+q_l)\beta} \right)
\]
Finally, identifying the coefficients of a given Lie algebra generator \( f_{\alpha+p\beta} \) we obtain the wallcrossing formula (1.4).

**APPENDIX A. BELL POLYNOMIALS**

In this section we summarize some basic facts concerning Bell polynomials used in the proof of lemma (2.4) following [1, Sect. 3.3-3.4].

Let \( P_k(n) \) be the set of unordered length \( k \geq 1 \) partitions of a positive integer \( n \geq 1 \). A partition \( \lambda \in P_k(n) \) is determined by a sequence \( (j_1, \ldots, j_{n-k+1}) \) of non-negative integers satisfying
\[
j_1 + 2j_2 + \cdots = n, \quad j_1 + j_2 + \cdots = k.
\]
Then we write \( \lambda = (1^{j_1}, 2^{j_2}, \ldots) \). For us, the Bell polynomial \( B_{n,k}(x_1, \ldots, x_{n-k+1}) \) will be defined by the following formula

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{\lambda \in P_k(n)} \frac{n!}{j_1! j_2! \cdots j_{n-k+1}! (1!)^{j_1} (2!)^{j_2} \cdots ((n-k+1)!)^{j_{n-k+1}}} x_1^{j_1} x_2^{j_2} \cdots x_{n-k+1}^{j_{n-k+1}}
\]

The power series version of Faà di Bruno’s formula is the following identity (see [1, Thm.A, Sect. 3.4]. Let

\[
f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n \quad g(x) = \sum_{n=1}^{\infty} \frac{b_n}{n!} x^n
\]

be formal power series with complex coefficients. Then

\[
g(f(x)) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} B_{n,k}(a_1, \ldots, a_{n-k+1}) x^n.
\]

Now let

\[
f(x) = e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad g(x) = -\frac{x}{1+x} = \sum_{n=1}^{\infty} (-1)^n x^n
\]

Then

\[
g(f(x)) = -1 - e^{-x} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^n,
\]

and equation (A.2) yields

\[
\sum_{k=1}^{n} (-1)^k k! B_{n,k}(1, \ldots, 1) = (-1)^n.
\]

REFERENCES

[1] L. Comtet. Advanced combinatorics. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.
[2] F. Denef and G. W. Moore. Split states, entropy enigmas, holes and halos. arXi.org:hep-th/0702146.
[3] D.-E. Diaconescu. Chamber structure and wallcrossing in the ADHM theory of curves I. arXiv:0904.4451.
[4] D. E. Diaconescu. Moduli of ADHM sheaves and local Donaldson-Thomas theory. arXiv.org:0801.0820.
[5] D. Joyce. Configurations in abelian categories. I. Basic properties and moduli stacks. Adv. Math., 203(1):194–255, 2006.
[6] D. Joyce. Configurations in abelian categories. II. Ringel-Hall algebras. Adv. Math., 210(2):635–706, 2007.
[7] D. Joyce. Configurations in abelian categories. III. Stability conditions and identities. Adv. Math., 215(1):153–219, 2007.
[8] D. Joyce. Motivic invariants of Artin stacks and ‘stack functions’. Q. J. Math., 58(3):345–392, 2007.
[9] D. Joyce. Configurations in abelian categories. IV. Invariants and changing stability conditions. Adv. Math., 217(1):125–204, 2008.
[10] D. Joyce and Y. Song. A theory of generalized Donaldson-Thomas invariants. arxiv.org:0810.5645.
[11] M. Kontsevich and Y. Soibelman. Stability structures, Donaldson-Thomas invariants and cluster transformations. arXiv.org:0811.2435.
[12] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *Invent. Math.*, 178(2):407–447, 2009.

[13] J. Stoppa and R. Thomas. Hilbert schemes and stable pairs: GIT and derived category wall crossings. arXiv.org:0903.1444.

[14] Y. Toda. Curve counting theories via stable objects I. DT/PT correspondence. arXiv.org:0902.4371.

[15] Y. Toda. Generating functions of stable pair invariants via wall-crossings in derived categories. arXiv.org:0806.0062.