The Annular Report on Non-Critical String Theory

Emil J. Martinec\textsuperscript{1}

\textit{Enrico Fermi Inst. and Dept. of Physics}
\textit{University of Chicago}
\textit{5640 S. Ellis Ave., Chicago, IL 60637, USA}

Abstract

Recent results on the annulus partition function in Liouville field theory are applied to non-critical string theory, both below and above the critical dimension. Liouville gravity coupled to $c \leq 1$ matter has a dual formulation as a matrix model. Two well-known matrix model results are reproduced precisely using the worldsheet formulation: (1) the correlation function of two macroscopic loops, and (2) the leading non-perturbative effects. The latter identifies the eigenvalue instanton amplitudes of the matrix approach with disk instantons of the worldsheet approach, thus demonstrating that the matrix model is the effective dynamics of a D-brane realization of $d \leq 1$ non-critical string theory. In the context of string theory above the critical dimension, \textit{i.e.} $d \geq 25$, Liouville field theory realizes two-dimensional de Sitter gravity on the worldsheet. In this case, appropriate D-brane boundary conditions on the annulus realize the S-matrix for two-dimensional de Sitter gravity.

\textsuperscript{1}e-martinec@uchicago.edu
1 Introduction

Non-critical string theory has been a thread running through the remarkable progress achieved by string theory over the years, beginning with Polyakov’s seminal work \[1, 2\]. The study of the Liouville theory

\[
S_L = \frac{1}{2\pi} \int dt d\sigma \sqrt{-\hat{g}} \left( \frac{1}{2} (-\hat{\nabla} \varphi)^2 - \frac{\partial}{\partial t} \hat{R} \varphi - 2\pi \mu e^{2\varphi} \right)
\]

introduced there was a motivation for the ground-breaking work of \[3\] on two-dimensional conformal field theory, as well as the modern treatment of covariant string perturbation theory \[4, 5, 6\]. The first progress toward a non-perturbative formulation of string theory came with the development of matrix models \[7, 8, 9\] of strings in \(d \leq 1\) target space dimensions (see \[10\] for a comprehensive review). These matrix models foreshadowed the development of D-branes \[11\] and the AdS/CFT correspondence \[12\], in that closed strings are represented as excitations at large \(N\) of the matrix collective field. Indeed, we will see below that the matrix eigenvalues are directly related to D-branes in Liouville theory.\[2\]

Noncritical string theory in \(d > 25\) is less studied \[26\]-\[32\]. It was shown recently in \[33\] that supercritical string backgrounds provide models of quantum cosmology that rather accurately model the physics of de Sitter space and inflation, in a context where quantum gravity is under rather good control. One of the main motivations for the present work was to undertake the quantization of these models.

In spite of all the progress the Liouville theory has inspired over the years, an understanding of its detailed properties – for instance, a prescription to compute correlation functions – is of relatively recent vintage \[34, 35\] (for a review and further references, see \[40\]). A treatment of conformal boundary conditions and boundary correlation functions is even more recent \[22, 23, 24\].

Our eventual goal is to apply this recent progress in Liouville theory to the quantization of two-dimensional de Sitter gravity, and in particular to give a prescription for what could be called the \(2d\) de Sitter S-matrix \[11, 42\]. There has been much speculation recently on what are the proper ingredients for a quantum theory of gravity with positive cosmological constant, see for example \[43, 44, 41, 42\], even to the extent of questioning whether such a theory exists \[45, 46, 47\]. Below we will show that, at least in two dimensions, there is a perfectly acceptable, conventional theory of quantum de Sitter gravity.

By way of preparation, we take what seems like a detour into the world of \(d \leq 1\) string theory, and reproduce several results obtained quite some time ago using the matrix model representation, here using Liouville theory on the annulus

\[2\]This connection appeared already in \[13\] (see also \[14\]-\[21\]), and seems to be a part of folklore; however, there have been relatively few quantitative comparisons of Liouville D-brane amplitudes and the matrix model. The disk one-point functions are computed in \[22, 23, 24\], and the open string disk two-point function is analyzed in \[25\].

\[3\]Although the roots of the method go back to the early literature, see \textit{e.g.} \[36, 37, 38, 39\].
with boundary cosmological constant(s), coupled to appropriate boundary states of the matter CFT. The first reproduced result is the correlation function of two macroscopic loops, that is, worldsheet boundaries of fixed length

$$\ell = \oint ds e^{b \varphi}$$

with appropriate Dirichlet boundary conditions on the matter CFT. We will find in section 2 precise agreement with the matrix model results of Moore and Seiberg [48] for this quantity.

The second reproduced result is the strength of the leading non-perturbative effects in $d < 1$ string theory [49, 50, 51, 52]. In the matrix model representation, these come from ‘eigenvalue instantons’, subsidiary stationary points of the matrix integral with one matrix eigenvalue lying a finite distance from the endpoint of the large N eigenvalue distribution. We will again find complete agreement with the matrix model results in section 3 using a different set of Liouville boundary states introduced in [24] (following early work of [53, 54, 55]). The wavefunctions of these latter boundary states are concentrated in the strong coupling region $\varphi \to \infty$ of $c \leq 1$ non-critical string theory, and hence are the appropriate states to use to compute such non-perturbative effects. This result identifies the eigenvalue instantons of the matrix model with D-instantons of the worldsheet approach, demonstrating (in agreement with folklore) that the matrix model was in hindsight the first instance of an AdS/CFT type of duality.

With the confidence inspired by these calculations, we turn in section 4 to the super-critical Liouville theory appropriate to strings in $d \geq 25$ dimensions (always excluding the Liouville direction in the definition of $d$). Here the classical solutions of Liouville theory describe two-dimensional de Sitter space – Liouville gravity coupled to $c \geq 25$ matter is de Sitter gravity plus matter. The previous calculations turn out to be less of a detour than one might have thought. The conformal (Carter-Penrose) diagram of classical global 2d de Sitter space is a finite cylinder or annulus, and the Liouville field takes the value $\phi = \infty$ on the conformal boundary. Thus the boundary states of [24] used in the D-instanton calculation are also the appropriate boundary states for the description of asymptotically de Sitter worldsheets above the critical dimension, and the annulus worldsheet is the appropriate topology for spacetimes that are asymptotically de Sitter in both the past and the future. One-point functions on the disk describe spacetimes with only one asymptotic de Sitter region, and a big bang or big crunch in the past or future. We give a prescription for the computation of the transition amplitudes for 2d spacetimes that are asymptotically de Sitter, and calculate them for a large class of initial conditions.

It should perhaps also be mentioned that Liouville theory enters critical string theory in a central way, in the description of dynamics in the throat of NS5-branes [53, 57, 58, 59] and in $AdS_3$ [60]: the results below have interesting implications for these applications as well, which we hope to explore elsewhere.

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4Passage to the super-critical dimension involves a Wick rotation $\phi = i \varphi$. 
Note added: While we were writing up our results, the work [61] appeared, having overlap with the material of section 3.

2 Minimal models on the annulus

We wish to calculate the partition function on the annulus of Liouville field theory coupled to $c \leq 1$ matter. Our immediate goal is to reproduce the correlation function of two macroscopic loops of length $\ell$, $\ell'$, and matter momentum $p$, obtained by Moore and Seiberg [48] using the matrix integral representation of these models:

$$\langle W(\ell, p) W(\ell', -p) \rangle = \int_0^\infty \frac{dE}{2\pi} G(E, p) \psi_E(\ell) \psi_E(\ell')$$

$$\psi_E(\ell) = \frac{1}{\pi} \sqrt{E \sinh(\pi E)} K_{iE}(M\ell)$$

$$G(E, p) = \begin{cases} \frac{\pi}{\cosh(\pi E)} & \text{for } c = 0 \\ \frac{\pi}{\cosh(\pi E) + \frac{1}{2}} & \text{for } c = \frac{1}{2} \\ \frac{1}{E^2 + p^2 \sinh(\pi E)} & \text{for } c = 1 \end{cases}$$

The quantity $M$ is proportional to $\sqrt{\mu}$; one is free to absorb the constant into the definition of $\ell$, but we will not do so. For $c = 0$, there are no matter degrees of freedom; the only allowed matter momentum is $p = 0$. Similarly, for the Ising model $c = \frac{1}{2}$, there are two possible Dirichlet boundary states of fixed boundary spin, and hence two values of the loop momentum. For $c = 1$, the momentum $p \in \mathbb{R}$ if the target is non-compact.\(^5\) It was pointed out in [62] that if one makes the $c = 1$ result periodic in momentum space

$$\sum_{n=-\infty}^{\infty} \frac{1}{E^2 + (p + 2n)^2 \sinh(\pi E)} = \frac{\pi}{[\cosh(\pi E) - \cos(\pi p)]},$$

all the amplitudes have the same basic structure of momentum space propagator. This form of the $c = 1$ propagator is appropriate for a target space which is a one-dimensional lattice $\mathbb{Z}$ rather than the continuous line $\mathbb{R}$.

In the worldsheet representation of string perturbation theory, the amplitudes\(^6\) arise from the annulus topology. In the standard conformal gauge, the annulus partition function for 2d gravity coupled to matter is of the form

$$Z = \int_0^\infty d\tau \ Z_{\text{ghost}} \ Z_{\text{Liouville}} \ Z_{\text{matter}}.$$ 

Here $\tau$ is the annulus modulus in the closed string channel (see figure 1).

\(^5\)For $c = 0$ matter, this result was derived implicitly in [89] and explicitly in [90]; the latter also gives general results for $n$ loops.

\(^6\)We have changed the normalization of the $c = 1$ propagator relative to the answer in [48] by a factor of two, in order to make it consistent with the $c < 1$ formulae.
The various constituent partition functions are known. First, the partition function for the Faddeev-Popov ghosts of conformal gauge is \[ Z_{\text{ghost}} = \eta(q)^2, \] where \( \eta(q) \) is the usual Dedekind eta function, \( \eta(q) = q^{1/24} \prod_n (1 - q^n) \), and as usual \( q = \exp[-2\pi \tau] \).

In conformal gauge, 2d gravity dynamics is that of the scale factor of the metric, and is governed by the Euclidean Liouville action

\[ L = \frac{1}{4\pi} \int d^2 z \left( (\partial \varphi)^2 + Q R \varphi + 4\pi \mu e^{2b\varphi} \right) + \oint ds \left( \frac{1}{2\pi} Q K \varphi + \mu_B e^{b\varphi} \right), \] where we have included the boundary interactions with the extrinsic curvature \( K \) as well as a boundary cosmological constant \( \mu_B \) (the latter can be different on each disconnected boundary component). The Liouville partition function was determined up to normalization in recent work of Fateev, the Zamolodchikovs, and Teschner \[ Z_{\text{Liouville}} = \int d\nu \Psi_{\nu}(\sigma)\Psi_{-\nu}(\sigma')\chi_{\nu}(q) \] \[ \Psi_{\nu}(\sigma) = \frac{\Gamma(1 + 2i\nu b)\Gamma(1 + 2i\nu/b)\cos(2\pi\sigma\nu)}{2^{1/4}(-2i\pi\nu)} \mu^{-i\nu/b}. \]

The notation is as follows: the parameters defining the theory are characterized by

\[ c_L = 1 + 6Q^2, \quad Q = b^{-1} + b; \]
\[ \mu = \pi \mu \gamma(b^2); \]
\[ \mu_B = \frac{\sqrt{\mu} \cosh(\pi b \sigma)}{\sqrt{\sin(\pi b^2)}} = \frac{1}{\pi} \Gamma(1 - b^2)\sqrt{\mu} \cosh(\pi b \sigma); \]

with \( \gamma(x) = \Gamma(x)/\Gamma(1 - x) \). The quantity \( \mu \) is the (KPZ) scaling parameter of the correlation functions. Finally, the character of a non-degenerate Liouville primary is

\[ \chi_{\nu}(q) = \frac{q^{\nu^2}}{\eta(q)}. \]
We consider first $c = 1$ matter, then the slightly more complicated case of $c < 1$.

### 2.1 $c = 1$

For $c = 1$ matter on a continuous, non-compact target, the matter partition function is

$$Z_{\text{matter}} = \int_{-\infty}^{\infty} dp \frac{1}{\sqrt{2}} \frac{q^{p^2/4}}{\eta(q)} e^{ip(x-x')} \quad \text{(12)}$$

The factor of $1/\sqrt{2}$ is the square of the Dirichlet boundary entropy [64].

Combining the component partition functions, the factors of $\eta(q)$ cancel. The integral over $\tau$ yields

$$Z(\sigma, x|\sigma', x') = \int dp \int d\nu \frac{e^{ipx \cos(2\pi\nu)} \cdot e^{-ipx' \cos(2\pi\nu')} \sinh^2(2\pi\nu)[(2\nu)^2 + p^2]}{\pi \cos(2\pi\nu)} \quad \text{(13)}$$

where we have set $b = 1$ for $c_{\text{matt}} = 1$.

To compare to the result (3), we note that $K_{2i\nu/b}(M\ell) = \int_0^\infty d(\pi b\sigma) e^{-M\ell \cosh(\pi b\sigma)} \cos(2\pi\nu\sigma)$

$$\frac{\pi \cos(2\pi\nu\sigma)}{(2\nu/b) \sinh(2\pi\nu/b)} = \int_0^\infty dl e^{-M\ell \cosh(\pi b\sigma)} K_{2i\nu/b}(M\ell) \quad \text{(14)}$$

Note that, since the boundary interaction is $\oint \mu B e^{b\phi} = \mu B \ell$, we can now identify the parameter $M$ in (3) as $M = \pi^{1-b^2} \Gamma(1-b^2) \sqrt{\mu}$, which reduces to $\sqrt{\mu}/\pi$ in the semi-classical limit $b \to 0$.

The correlator (3) can be Laplace transformed to a function of the boundary cosmological constants $\mu_B = M \cosh(\pi b\sigma)$ (see equation (10)) using the second relation of (14); the partition function in the worldsheet approach is most naturally given as a function of $\mu_B$. The two-loop correlator (3) becomes

$$\langle W(\sigma, p)W(\sigma', -p) \rangle_{c=1} = \int_0^\infty dE \frac{\cos(\pi\sigma E) \cdot \cos(\pi\sigma' E)}{\sinh^2(\pi E)} \left[ E^2 + p^2 \right] \quad \text{(15)}$$

Identifying $E = 2\nu$, this is exactly the result (13).

### 2.2 $c < 1$

Let us now turn to $c < 1$ minimal models coupled to gravity. The annulus partition function of $c < 1$ matter is known from work of Saleur and Bauer [65]. The $c \leq 1$ models are parametrized by

$$c_m = 1 - 6\bar{Q}^2 \quad \text{,} \quad Q = \bar{b}^{-1} - \bar{b} \quad \text{,} \quad \bar{b} = \sqrt{q/p} \quad \text{(16)}$$

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7 The analysis of [64] employs the convention $\alpha' = \frac{1}{2}$; in the present work we are setting $\alpha' = 1$.

8 The correlator as a function of loop length is in fact better defined, and regulates the divergence at small $E$ in equation (13); a better comparison would therefore take the inverse transform of (13) and compare it directly to (3).
for \( p, q \in \mathbb{Z} \). Coupling to gravity sets \( b = \tilde{b} \). The unitary matter models have \( p = q \pm 1 \) and can be thought of as the conformal continuum limit of a lattice height model, where the heights are associated to the nodes of an ADE Dynkin diagram \[66\]. In the context of coupling to quantum gravity, these models were studied using matrix model techniques, in a series of works by Kostov and collaborators \[67\]-\[76\], \[62\], \[25\]. The basic idea is to put a gas of loops on a fluctuating random lattice; the loops are the domain boundaries between different heights on the Dynkin diagram. In \[25\] (see also \[71\]), some of the boundary correlators are worked out, and the phase of the loop gas model that connects to the continuum path integral is identified – the Liouville theory with action \[7\] is described by a dilute gas of loops, and the boundary interaction in \[7\] is appropriate to a Dirichlet boundary condition on the heights. The dilute loop gas corresponds to the condition \( q < p \). The matter annulus partition function in this phase, with fixed heights \( a, c \) on the two boundaries, is given by \[65\]

\[
Z_m(a|c) = \sum_{r=1}^{q-1} \sum_{j \in \exp} S_c^{(j)} S_d^{(j)} \frac{\sin(\frac{\pi j}{q})}{\sin(\frac{\pi j}{q})} \chi_{r,1}(q')
\]

where \( j \) runs over the exponents of the Dynkin diagram, which we take to be the \( A_{q-1} \) diagram. The matter wavefunctions

\[
S_a^{(j)} = \sqrt{2/q} \sin(\pi ja/q)
\]

are eigenfunctions of the \( A_{q-1} \) Cartan matrix, which is the discrete Laplace operator on the Dynkin diagram.

Now, the above partition function is written in the open string channel \( q' = \exp[-2\pi/\tau] \); we need to rewrite it in the closed string channel in which the Liouville and ghost partition functions \[8\] are written. For this purpose, we may use a formula from \[77\]

\[
\chi_{r,s}(q') = \frac{1}{\eta(q')} \sum_{k=-\infty}^{\infty} \exp[-2\pi \tau l^2/(4pq)] \sin(\frac{\pi rl}{q}) \sin(\frac{\pi sl}{p})^2
\]

writing \( l = 2kq + k' \), the sum over \( r \) gives

\[
\sum_{r=1}^{q-1} \sin(\frac{\pi rl}{q}) \sin(\frac{\pi rl}{q}) = \frac{q}{2} (\delta_{k',j} - \delta_{k',-j})
\]
and so

\[ Z_m(a|c) = \sum_{j=1}^{q-1} \frac{S_c^{(j)} S_a^{(j)}}{\sin \left( \frac{\pi j}{q} \right)} \left( \frac{q}{2p} \right)^{\frac{j}{2}} \sum_{k=-\infty}^{\infty} \left[ q^{(2kq+j)^2/4pq} \sin \left( \frac{(2kq+j)\pi}{p} \right) - q^{(2kq-j)^2/4pq} \sin \left( \frac{(2kq-j)\pi}{p} \right) \right] . \] (21)

We are now ready to combine the component partition functions and integrate over \( \tau \). The factors of \( \eta(q) \) cancel, and one is left with simply the exponential factors – the Gaussian sum of powers of \( q \) in (21), and a factor \( q^{\nu^2} \) from (14). The result of the \( \tau \) integral is

\[ Z(a|c) = \sum_{j=1}^{q-1} \int_0^\infty \frac{d\nu}{\sqrt{2 \sinh(2\pi \nu/b) \sinh(2\pi \nu b)}} \cos(2\pi \sigma \nu) \cos(2\pi \sigma' \nu) \] \[ \times \frac{S_c^{(j)} S_a^{(j)}}{\sin \left( \frac{\pi j}{q} \right)} \left( \frac{q}{2p} \right)^{\frac{j}{2}} \frac{2}{2\pi} \left\{ \sum_{k=1}^{\infty} \left[ \sin \left( \frac{(2kq+j)\pi}{p} \right) \frac{(2kq+j)^2}{4pq} + \nu^2 \right] - (j \to -j) \right\} + \sin \left[ \frac{2\pi j}{p} \right] \} . \] (22)

One is now instructed to expand out the argument of the sine in the numerator of the sum over \( k \), and also to expand the denominators into a sum over simple poles and recombine terms. The expression in curly brackets in (22) becomes

\[ \sum_{k=1}^{\infty} \frac{2k \sin \left( \frac{2kq \pi}{p} \right)}{2i \nu \cdot 2\beta q} \left[ \frac{k^2 + \left( \frac{\nu}{2\beta q} + i \frac{\pi}{2q} \right)^2}{k^2 + \left( \frac{\nu}{2\beta q} - i \frac{\pi}{2q} \right)^2} - \frac{2k \sin \left( \frac{2kq \pi}{p} \right)}{2k^2 + \left( \frac{\nu}{2\beta q} - i \frac{\pi}{2q} \right)^2} \right] \] \[ + \sum_{k=1}^{\infty} \sin \left[ \frac{2\pi k}{p} \right] \left[ \frac{\sin \left( \frac{2\pi k}{p} \right)}{k^2 + \left( \frac{\nu}{2\beta q} + i \frac{\pi}{2q} \right)^2} + \frac{\sin \left( \frac{2\pi \nu}{b} \right)}{k^2 + \left( \frac{\nu}{2\beta q} - i \frac{\pi}{2q} \right)^2} \right] + \frac{\sin \left[ \frac{2\pi j}{p} \right]}{\beta^2 j^2 + \nu^2} \] (23)

where we have defined \( \beta^2 = (4pq)^{-1} \). Now we make use of the identities

\[ \sum_{k=1}^{\infty} \frac{k \sin(kx)}{k^2 + \alpha^2} = \frac{\pi}{2} \frac{\sin[\alpha(\pi - x)]}{\sinh[\alpha\pi]} ; \] \[ \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh[\alpha(\pi - x)]}{\sinh[\alpha\pi]} - \frac{1}{2\alpha^2} ; \] (24)

after some algebra involving repeated use of trigonometric identities, one finds that (23) reduces to

\[ \frac{\pi}{b \nu} \frac{\sin(2\pi \nu/b) \sin \left( \frac{j\pi}{q} \right)}{\cosh(2\pi \nu/b) \cos \left( \frac{j\pi}{q} \right)} \] (25)

(recall \( b = \sqrt{q/p} \)). Replacing the curly brackets in (22) by this expression, we have

\[ Z(a, \sigma|c, \sigma') = \sum_{j=1}^{q-1} \int \frac{d\nu}{b} \frac{S_c^{(j)} \cos[2\pi \sigma \nu] \cdot S_a^{(j)} \cos[2\pi \sigma' \nu]}{(2\nu/b) \sinh(2\pi \nu/b) \cdot [\cosh(2\pi \nu/b) - \cos(\pi j/q)]} . \] (26)
Now, this is indeed the right answer! The correlator (3) should again be Laplace transformed to a function of the boundary cosmological constants $\mu_B = \frac{\sqrt{\mu \cosh(\pi b \sigma)}}{\sqrt{\sin(\pi b^2)}}$ (see equation (10)); the correlator becomes

$$
\langle W(\sigma, p) W(\sigma', -p) \rangle = \int_0^\infty dE \frac{\pi \cos(2\pi b \sigma E) \cdot \cos(2\pi b \sigma' E)}{2\pi E \sinh(\pi E) [\cosh(\pi E) - \cos(\pi p)]}. \tag{27}
$$

Identifying $E = 2\nu/b, p = j/q$, this is exactly the result (26). In particular, pure 2d gravity is the minimal model $p = 3, q = 2$ coupled to gravity. Thus $\cos[\pi j/q] = 0$ and we recover the result (3). Similarly, for the Ising model $p = 4, q = 3$ one has $\cos[\pi j/q] = \pm \frac{1}{2}$ for $j = 1, 2$; again one recovers the result (3).

### 3 Branes at infinity

Thus, matrix model results are fully compatible with the Liouville bootstrap of \cite{22, 23, 24}. The generic, non-degenerate Virasoro representation of the Liouville theory yields the character (11). One then reconstructs the annulus partition function (8) from the wavefunction (9); the latter corresponds to the boundary state

$$
|B_\sigma\rangle = \int_{-\infty}^{\infty} d\nu e^{2\pi i \nu \sigma} \Psi_\nu(\sigma) |\nu\rangle, \tag{28}
$$

where $|\nu\rangle$ are Ishibashi states \cite{22, 23, 24, 78, 79}. In the semiclassical limit $b \to 0$, these states implement the natural boundary condition

$$
\partial_n \varphi = QK + 2\pi \mu_B b e^{b \varphi} \tag{29}
$$

derived from the variational principal on (7). The work of \cite{24} also introduced another set of boundary states (for early work, see \cite{53, 54, 55})

$$
|m, n\rangle = \int d\nu \Psi_\nu(m, n) |\nu\rangle \tag{30}
$$

$$
\Psi_\nu(m, n) = 2 \sinh(2\pi m \nu/b) \sinh(2\pi n \nu/b) \left[ \frac{\Gamma(1 + 2i\nu b) \Gamma(1 + 2i\nu/b)}{2^{1/4} (-2i\pi \nu)} \mu^{-i\nu/b} \right]
$$

associated to degenerate representations of conformal dimension

$$
h_{m,n} = \frac{1}{4} Q^2 - \frac{1}{4} (m/b + nb)^2 \tag{31}
$$

having a null vector on level $mn$. The (normalized) disk one point function of Liouville exponentials with this boundary state is

$$
\langle e^{2\alpha \varphi} \rangle = \frac{U_{m,n}(\alpha)}{(1 - zz)^{2h_\alpha}} \tag{32}
$$

$$
U_{m,n}(\alpha) = \frac{\Psi_{i(Q/2-\alpha)}(m, n)}{\Psi_{iQ/2}(m, n)}
$$
where \( h_\alpha = \frac{1}{4}Q^2 + \nu^2 \). In particular, one has

\[
\langle e^{2b\varphi} \rangle = \frac{Q}{\pi \mu b} \frac{1}{(1 - z\bar{z})^2} \tag{33}
\]

which is the constant negative curvature metric on the Poincaré disk. The boundary state (30) is thus concentrated in the region of \( \varphi \to \infty \). The amplitudes (32) have been compared [24] with the perturbative expansion of Liouville theory in the semiclassical limit \( b \to 0 \); agreement was found for \( m = n = 1 \) but not otherwise. Thus, in Euclidean Liouville theory, the \( m = n = 1 \) boundary state satisfies all the necessary criteria to be identified with the boundary state at conformal infinity of Euclidean \( AdS_2 \). In particular, its null vector at level \( L_{-1} \) is associated to \( SL(2, \mathbb{R}) \) invariant boundary condition for Euclidean \( AdS_2 \). The one-point functions (32) for \( m > 1 \) are singular in this limit, indicating that there is no associated semi-classical geometry for these states. The geometrical interpretation of the states \( m = 1, n > 1 \) has until now also remained unclear; in analogy to D-branes in current algebra models, one expects them to be some sort of bound states of the basic \( m=n=1 \) branes.

An important feature of the boundary states (30) is the spectrum of open string vertex operators that couple to them. It was demonstrated in [24] that the transform to the open string channel of the annulus partition function

\[
Z_L(m, n| m', n') = \int d\nu \chi_\nu(q) \Psi_\nu(m, n) \Psi_{-\nu}(m', n') \tag{34}
\]

contains only open string characters of the degenerate Virasoro representations \( (m'', n'') \) that appear in the fusion algebra of the degenerate representations \( (m, n) \) and \( (m', n') \)

\[
m'' \in \{|m' - m| + 1, \ldots, m + m' - 1\} \quad , \quad n'' \in \{|n' - n| + 1, \ldots, n + n' - 1\} \tag{35}
\]

There are thus only a finite number of Virasoro primaries flowing in the Liouville open string channel with these boundary conditions.

The boundary states (30) are formally related to the ‘standard’ boundary state (28) as

\[
|m, n\rangle = |B_{\sigma(m, n)}\rangle - |B_{\sigma(m, -n)}\rangle \tag{36}
\]

\[
\sigma(m, n) = i \left( \frac{m}{b} + nb \right) .
\]

Using (10), we have

\[
\mu_B(m, n) = (-1)^n \sqrt{\mu} \frac{\cos(\pi nb^2)}{\sqrt{\sin(\pi b^2)}} ; \tag{37}
\]

note that the two values \( \sigma(m, \pm n) \) contributing to (36) yield the same value of \( \mu_B \), even though the corresponding boundary states are distinct.
In the matrix model formulation of \cite{67-76,25} of strings propagating along a Dynkin diagram, the boundary cosmological constant parameter $\mu_B$ of Liouville theory can be identified with the coordinate $z$ of the complex eigenvalue plane of the matrices \cite{80,48,10}. In the matrix model, strings are excitations of the large $N$ collective field of the matrices. The string background corresponds to a matrix eigenvalue distribution supported along the real axis, on the interval

$$z \in (-\infty, -\mu_B(\sigma = 0)) = (-\infty, -M).$$

(38)

The disk partition function and all disk correlation functions are meromorphic functions of $z$ with a cut along the real axis extending along the interval \cite{38}. For instance, consider the amplitude for a macroscopic loop (disk partition function at fixed boundary length $\ell$) located at a fixed node of the Dynkin diagram of the unitary height model; it has the factorized form \cite{69}

$$\langle W(a, \ell) \rangle = w(\ell) \sin\left(\frac{\pi a}{q}\right),$$

(39)

where the Laplace transform of the loop amplitude $w(\ell)$ yields

$$\hat{w}(z) = \int_0^\infty d\ell e^{-z\ell} w(\ell) = \text{const.} \times [(z + \sqrt{z^2 - M^2})^g + (z - \sqrt{z^2 - M^2})^g]$$

(40)

with $g = b^{-2}$. The quantity $\hat{w}(z)$ indeed has a cut on the physical sheet along the interval \cite{38}, and the discontinuity along the cut gives the density of eigenvalues in the matrix model \cite{68}. The cut is nicely parametrized by \cite{10}, i.e. $\mu_B \rightarrow \sigma$ is a uniformizing map that smooths the branch point. In terms of $\sigma$,

$$\hat{w}(\sigma) = \text{const.} \times \cosh\left(\frac{\pi\sigma}{b}\right)$$

(41)

is analytic in $\sigma$.

The pair of boundary states in \cite{36} live at points near the endpoint of the cut in the eigenvalue distribution, on the $m$th sheet.\footnote{The authors of \cite{24} were unable to associate a semiclassical, geometrical interpretation to the states with $m > 1$; perhaps this is related to the fact that they are not on the first, physical sheet of the $z$ plane.} The singularity at $z = -\mu_B(\sigma = 0)$ is associated to surfaces of diverging boundary length, and thus $\varphi \rightarrow \infty$ near the boundary.

We can now give another interpretation to the $m, n$ boundary states – they are associated to D-instantons in $c \leq 1$ non-critical string theory, and the eigenvalues of the matrix model are essentially the D-branes of Liouville theory in a non-critical gauge/gravity correspondence. One might regard the $m, n$ boundary states \cite{36} as D-instantons of Liouville theory located in the region of large positive $\varphi$ (semiclassically, $\varphi \rightarrow \infty$), corresponding to stationary matrix eigenvalues located outside the cut \cite{38}.
The leading non-perturbative effects in \( c \leq 1 \) non-critical string theory come precisely from eigenvalue instanton amplitudes in the matrix model representation \([49, 50, 51]\), see also \([52, 16, 18]\). In the continuum worldsheet approach, the D-instanton amplitude is a disk partition function with appropriate boundary conditions. We can obtain this amplitude from the annulus partition function by factorization on the identity in the closed string channel; this gives the absolute square of the disk instanton amplitude. The matter partition function of interest is \([17]\). We wish to factorize it on the (gravitationally dressed) identity operator contribution in the closed string channel; to do this, let us couple the matter partition function \([21]\) to the ghost partition function \([6]\) and the Liouville partition function with boundary conditions corresponding to the \((m,n)\)-type states \([30]\) on both boundaries of the annulus \([24]\).

\[
Z_L(mn|m'n') = \int_{-\infty}^{\infty} d\nu \chi_\nu(q) \frac{2 \sinh(2\pi m\nu/b) \sinh(2\pi n\nu/b) \cdot 2 \sinh(2\pi m'\nu/b) \sinh(2\pi n'\nu/b)}{\sqrt{2} \sinh(2\pi \nu/b) \sinh(2\pi \nu b)}. \tag{42}
\]

The integral over \( q \) and resulting sum over descendants proceeds as before, yielding

\[
Z(a,mn|m',n') = \sum_{j=1}^{p-1} \int_{-\infty}^{\infty} d\nu \frac{2S_c^{(j)}(\nu) \sinh(2\pi m\nu/b) \sinh(2\pi n\nu/b) \cdot 2S_c^{(j)}(\nu) \sinh(2\pi m'\nu/b) \sinh(2\pi n'\nu/b)}{(2\nu/b) \sinh(2\pi \nu/b) \cdot [\cosh(2\pi \nu/b) - \cos(\pi j/q)]}. \tag{43}
\]

To pick out the contribution of the identity operator, deform the contour of integration to pick up the poles on the positive imaginary axis. The leading pole at \( \nu = ij/\sqrt{4pq} \) is the contribution of the \( j \)th order parameter in the closed string channel, with \( j = 1 \) corresponding to the (gravitationally dressed) lowest dimension operator, and \( j = p - q \) for the (gravitationally dressed) identity operator;\(^{10}\) the higher poles at fixed \( j \) give the contributions of ‘gravitational descendants’ of the \( j \)th order parameter in the terminology of matrix models. Setting \( j = p - q \) for the contribution of the identity operator, and furthermore putting \( m = m' \), \( n = n' \), and \( a = c \), the residue of the leading pole gives the square of the disk one-point function of the cosmological constant operator (the ‘puncture operator’ in matrix model terminology) with the given boundary conditions. One finds (again recalling \( b = \sqrt{q/p} \))

\[
Z_{\text{disk}}(a; m, n) = \frac{\partial \Gamma_{\text{inst}}}{\partial \mu} = \left[ 2\sqrt{2} \frac{\sin(\frac{\pi a}{q})}{\sin(\frac{\pi j}{q})} \sin(\frac{\pi j m}{q}) \sin(\frac{\pi j n}{p}) \right] \mu^{\frac{j}{2j}} \tag{44}
\]

up to an overall phase. We have also restored the power of \( \mu \) that drops out of the \( I_\nu K_\nu \) expansion of the two-loop correlator \([31, 48]\), as one sees from the wavefunction \([30]\).

\(^{10}\)More generally, \( j = |pr - qs| \) for the contribution of the \((r,s)\) matter primary.
This result should be compared with the instanton contributions to the matrix model evaluated in [52], [16], [18]:

$$\Gamma_{\text{inst}} = \left[ 2\sqrt{2} \sin\left(\frac{\pi k}{q}\right) \sin\left(\frac{\pi l}{p}\right) \right] \frac{2q}{2q + 1} \mu^{\frac{2q+1}{2q}}$$

(we have divided the result quoted in section 5.3 of [52] by a factor $\sqrt{2}$ to restore the conventional normalization of the matrix model susceptibilities $u_j = \frac{\partial}{\partial \mu} \langle O_j \rangle$). The result (45) holds for unitary matter models $p = q + 1$ coupled to gravity. The comparison for non-unitary minimal models is complicated by the fact that the identity operator is not the leading contribution to the factorization of the annulus amplitude – there are operators of negative dimension that contribute stronger singularities in the matter partition sum, and so one must be careful in the identification of the contribution of the puncture operator. For simplicity, let us therefore in the comparison restrict consideration to the unitary models $p = q + 1$ (so that $j = p - q = 1$) in (44). One finds agreement with (45) after integrating (44) with respect to the cosmological constant, and setting $m = 1$.

To summarize, we have succeeded in reproducing a number of basic properties of non-critical string theory, originally derived in the context of matrix models, using the continuum worldsheet approach. This reinforces our understanding of how these two approaches are dual representations of the same theory. In particular, we have obtained a precise identification of single eigenvalues of the matrix model with D-branes of the worldsheet approach, demonstrating that the matrix models of non-critical string theory are in fact instances of the AdS/CFT correspondence. Further investigation might allow a rather detailed duality map to be developed, which would certainly elucidate the general matrix/string duality. Instead, we now proceed to analytically continue the result (42) for the Liouville annulus amplitude, in order to obtain some new results – namely, a prescription for the dynamics of two-dimensional de Sitter quantum gravity.

4 The de Sitter regime

We now turn to string theory above the critical dimension, which was shown in [33] to rather accurately model the cosmology of de Sitter space and inflation. The annulus amplitude in this case realizes the conformal (Carter-Penrose) diagram of certain regimes of the cosmology, see for example figure 2.

4.1 de Sitter Liouville

Let us begin with a review of the classical theory, following [33]. The notational conventions of the present paper conform to the conventions of [22], [23], [24], which are different from those of [33]; the translation between these two sets of conventions...
may be found in appendix A. Working above the critical dimension involves a continuation of $b$ to imaginary values; therefore let us define

$$\beta = -ib, \quad \phi = i\varphi .$$

The general classical solution of the Liouville theory (1) can be expressed locally as

$$e^{2\beta \phi} = -\frac{1}{\pi \mu \beta^2} \frac{\partial A(x^+) \partial B(x^-)}{[A(x^+) - B(x^-)]^2},$$

where $x^\pm = t \pm \sigma$. The gauge constraints

$$T_{\pm \pm}^{\text{Liouville}} + T_{\pm \pm}^{\text{matter}} = 0 \quad (48)$$

are a set of non-linear differential equations that determine $A(x^+)$ and $B(x^-)$ in terms of the matter stress-energy. One may broadly characterize the solutions according to the monodromy of the functions $A$ and $B$

$$A(x^+ + 2\pi) = \frac{aA(x^+) + b}{cA(x^+) + d}, \quad B(x^- - 2\pi) = \frac{aB(x^-) + b}{cB(x^-) + d}; \quad (49)$$

the solution is said to be of the elliptic, parabolic, or hyperbolic class, depending on whether the trace of the monodromy matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has $TrM < 2$, $TrM = 2$, or $TrM > 2$, respectively. In string theory terms, these solutions correspond to
tachyonic, massless, and massive strings in the target space, respectively; they also correspond to positive, zero, or negative Liouville energy, see figure 3.

The elliptic category of classical solutions to de Sitter Liouville theory are asymptotically de Sitter in both past and future; they reach infinite scale factor at finite conformal time parameter in both past and future. An example of such a space is the metric

$$ ds^2 = e^{2\beta \phi}(-dt^2 + d\sigma^2) = \frac{1}{\pi \mu \beta^2} \frac{\epsilon^2}{\sin^2(\epsilon t)}(-dt^2 + d\sigma^2) \quad , $$ (50)

where $t \in (0, \pi/\epsilon)$ and $\sigma \in (0, 2\pi)$; the conformal diagram of this geometry is shown in figure 2. In the parametrization (47), one has $A(x^+) = \tan(\frac{1}{2} \epsilon x^+)$ and $B(x^-) = -\tan(\frac{1}{2} \epsilon x^-)$. The Hamiltonian (Virasoro, BRST) constraints fix the parameter $\epsilon$ in terms of the matter energy

$$ \frac{\epsilon^2}{2\beta^2} + E_{\text{matter}} - \frac{d - 1}{12} = 0 \quad . $$ (51)

Global de Sitter space corresponds to $\epsilon = 1$; increasing $E_{\text{matter}}$ decreases $\epsilon$ until one reaches the threshold of cosmological singularity formation at $\epsilon = 0$, where the Liouville solution is then of the parabolic class. A slight further increase in $E_{\text{matter}}$ changes the global geometry; $\epsilon = i\epsilon$ becomes purely imaginary. The Liouville solution is then of the hyperbolic class, and exhibits a Milne-type cosmological singularity. For more details on the classical solutions, see [33].

The fact that $\phi \to \infty$ on the asymptotically de Sitter conformal boundary of the classical spacetime suggests that the appropriate quantum analogue uses the
boundary states (30). Roughly, the boundary states (30) in this regime of central charge $c$ describe S-branes located in the vicinity of $\phi = \infty$. The fact that the conformal diagram of the classical spacetime is a finite cylinder suggests that we consider the annulus amplitude with these boundary states.

4.2 Liouville theory as a 2d dS CFT

Consider the Liouville annulus partition function (42) coupled to $c = d$ matter and ghosts; for example, one might take the matter to be $d$ free scalar fields. The combined partition function is

$$Z(p, mn | -p, m'n') = \int_0^\infty d\tau \int_{-\infty}^\infty d\nu \int_0^\infty d\omega \rho(\omega) \frac{q^{1/4}p^2 + \nu^2}{\sqrt{2} \eta(q)} \frac{2 \sin(2\pi m\nu/\beta) \sin(2\pi n\nu/\beta) \cdot 2 \sin(2\pi m'\nu/\beta) \sin(2\pi n'\nu/\beta)}{2 \sin(2\pi \nu/\beta) \sin(2\pi \nu/\beta)},$$

with $\nu = \epsilon/2\beta$. More generally, the matter plus ghost contribution will have the structure

$$\frac{q^{1/4}p^2}{\sqrt{2} \eta(q)} \cdot 2 \frac{\sin(2\pi \nu/\beta) \sin(2\pi \nu/\beta)}{2 \sin(2\pi \nu/\beta) \sin(2\pi \nu/\beta)}.$$

For simplicity, consider $m' = n' = 1$. The amplitude becomes

$$Z(p; m, n) = \int_0^\infty d\tau \int_{-\infty}^\infty d\nu \int_0^\infty d\omega \rho(\omega) q^{1/4}p^2 + \nu^2 + \omega - \frac{\epsilon^2}{4\tau} \cdot 2\sqrt{2} \sin(2\pi m\nu/\beta) \sin(2\pi n\nu/\beta)$$

The integral over $\nu$ is Gaussian; note that it has saddle points located at

$$\nu = \pm \frac{i(m/\beta \pm n\beta)}{2\tau},$$

and giving the result

$$Z(p; m, n) = \int_0^\infty d\tau \int_0^\infty d\omega \rho(\omega) q^{1/4}p^2 + \omega - \frac{\epsilon^2}{4\tau} \cdot 2\sqrt{2} \times \left( \exp\left[ -\frac{\pi}{2\tau} (m/\beta - n\beta)^2 \right] - \exp\left[ -\frac{\pi}{2\tau} (m/\beta + n\beta)^2 \right] \right)$$

The integral over $\tau$ is manifestly divergent for $\frac{1}{4}p^2 + \omega < \frac{\epsilon^2}{4\tau^2}$, for a trivial reason – we have been attempting to use Euclidean worldsheets to describe an intrinsically Lorentzian process, namely what is from the target space (string) perspective the propagation of closed string tachyons. One has improperly attempted to rotate the integral over the Schwinger parameter from an oscillating exponential to a real exponential. Therefore let us return the $\tau$ integral back to the imaginary axis from which it originated, defining $\tau = i\hat{\tau}$. The resulting integral over $\hat{\tau}$ is elementary, and
yields
\[
Z(p, mn) = \int_0^\infty d\omega \rho(\omega) \left( \frac{i/\sqrt{2}}{\frac{\omega}{24} - \frac{1}{4}p^2 - \omega} \right) \times \exp\left( 2\pi i (m/\beta - n\beta) \left[ \frac{\omega}{24} - \frac{1}{4}p^2 - \omega \right]^2 \right) - (n \to -n) .
\]

Note that the dominant contribution to the \( \hat{\tau} \) integral comes from the saddle point
\[
\hat{\tau} = \frac{(m/\beta \pm n\beta)}{2 \left( \frac{\omega}{24} - \frac{1}{4}p^2 - \omega \right)^{\frac{1}{2}}},
\]
or in other words
\[
\nu^2 = \frac{\omega}{24} - \frac{1}{4}p^2 - \omega
\]
which is the mass shell condition (51) for the propagating string mode. Plugging into (55), we see that the classical geometry (50) is recovered in the limit \( \beta \to 0 \), provided once again that \( m = 1 \). It may be that further consideration of the loop perturbation expansion in Liouville theory around the classical solution (50) might only match the exact answer (57) for \( m = n = 1 \), as in [22].

In the semi-classical limit \( \beta \to 0 \), one expects that fluctuations around the classical geometry are suppressed, at least for smooth spacetimes. Consider a double saddle point approximation to the \( \nu \) and \( \hat{\tau} \) integrals; as mentioned above, the saddle points are located at (55) and (58). The width of these saddles is (dropping subleading terms in \( \beta \))
\[
\delta \frac{\tau}{\tau} \sim \delta \frac{\nu}{\nu} \sim \frac{\beta}{\pi^2 m^2 (1 - (p^2 + 4\nu \beta)^2)^{1/4}}.
\]

Thus, as long as we are far from the threshold of the cosmological singularity at \( \nu = 0 \) (i.e. \( p^2 + 4\omega = \beta^{-2} \)), fluctuations in the geometry are indeed suppressed; we recover the classical de Sitter geometry (50), (51) via the strongly peaked saddle point (55), (58). Near the threshold at \( \nu = 0 \), there are large fluctuations in the path integral and the classical geometry cannot be trusted.\(^{11}\)

### 4.3 Wavefunctions

We now turn to a discussion of de Sitter wavefunctions. The transform (14) of the \( c \leq 1 \) Liouville wavefunctions [2] in \( \sigma \) space yields the wavefunctions for the boundary length \( \ell = \oint e^{b\varphi} \), namely the Bessel functions \( K_i E_i(M\ell) \). The analogue for de Sitter Liouville should reproduce the corresponding de Sitter wavefunctions,

\(^{11}\)Note, however, that well above the big bang/crunch threshold, fluctuations are again suppressed, but the saddle point is off in the complex plane. This fact will be important below when we discuss factorization of the annulus.
which in the minisuperspace approximation can be taken to be Hankel functions (see e.g. \[33\]). The analogue of the transform \((14)\) for Hankel functions is

\[
\frac{i\pi}{2} e^{-\pi E/2} H_{iE}^{(1)}(M\ell) = \int_0^\infty du e^{iM\ell \cosh u} \cos(Eu)
\]

\[
\frac{-2i e^{\pi E/2}}{E \sinh(\pi E)} \cos(Eu) = \int_0^\infty \frac{d\ell}{\ell} e^{iM\ell \cosh \ell} H_{iE}^{(1)}(M\ell) .
\]

Indeed, in the continuation \((61)\), the boundary interaction which is the kernel of the integral transform \((14)\) continues as

\[
\exp[-\mu_B \ell] \rightarrow \exp\left[\pm i \left(\sqrt{\mu} \cosh(\pi \beta s) / \sin(\pi \beta^2) \right) \ell\right] ,
\]

where we have also chosen to continue \(s = i\sigma\); the wavefunction \((33)\) rotates as

\[
\Psi_{i\nu}(\sigma) \rightarrow \Gamma(1-2i\nu/\beta)\Gamma(1+2i\nu/\beta) \cos(2\pi s\nu) \mu^{-i\nu/\beta} ,
\]

and similarly for the \((m,n)\) boundary state wavefunctions \((30)\). From \((61)\), the transform of the wavefunction \((63)\) to \(\ell\) space can be written

\[
\Psi_{i\nu}(\ell) = \left[\frac{\Gamma(1-2i\nu/\beta)}{2^{1/4} \pi \beta} \Gamma(1+2i\nu/\beta) \right]^{1/2} \left(\frac{\mu}{\Gamma(1+\beta^2)}\right)^{1/2} (-\mu)^{-i\nu/\beta} \sqrt{\pi(2\nu/\beta)} \sinh(2\pi \nu/\beta) H_{2i\nu/\beta}^{(1)}(M\ell)
\]

with \(M = \pi \sqrt{\mu / \sin(\pi \beta^2)} = \Gamma(1+\beta^2) \sqrt{-\mu}\). These are indeed, up to normalization, the appropriate minisuperspace de Sitter wavefunctions in the semi-classical limit \(\beta \to 0\). These wavefunctions arise in the disk one-point functions of vertex operators with hyperbolic monodromy, which are the continuation of equation \((32)\). Thus the appropriately normalized disk one-point functions provide the transition amplitudes for the hyperbolic class of geometries that are only asymptotically de Sitter in the future (or past), and have a big bang singularity in the past (or future). The vertex operator at the puncture in the disk specifies the state of the universe at the big bang (or crunch).

### 4.4 Comments

At this point a number of remarks about the result \((57)\) are in order.

- The integral over \(\omega\) is divergent. Consider \(m = n = 1\); then the first exponential in square brackets is

\[
\exp\left[-2\pi \sqrt{\left(\frac{c-25}{6}\right)\left(L_0 - \frac{c-1}{24}\right)}\right] .
\]

In the semi-classical limit, this is exactly the suppression factor for the production of closed strings found in \([33, 32]\); it was also found there that this
suppression factor was overwhelmed by the density of states $\rho(\omega)$, leading to a divergence in the production of string modes. Now, standard D-brane boundary states such as the free field Dirichlet/Neumann states

$$|B\rangle = \exp\left[\pm \sum_{n>0} \frac{1}{n} \alpha_{-n} \bar{\alpha}_{-n}\right]|0\rangle \quad (66)$$

do not couple to all closed string states, but rather only to those with a matching spectrum of left- and right-moving modes. These have only an open string degeneracy

$$\rho(\omega) \sim \exp\left[+2\pi \sqrt{\left(\frac{c-1}{6}\right)\left(L_0 - \frac{c-1}{24}\right)}\right], \quad (67)$$

but this is still slightly overcomes the suppression factor (65). However, we are not interested in actually carrying out the sum over states; rather, we regard the propagator (57) as a generating function for de Sitter spacetimes. We can select the initial and final states of interest by appropriate change in the choice of matter wavefunctions on the two boundaries. We have used D-brane wavefunctions, which overlap with a substantial number of (closed string) matter states. To pick out a particular (closed string) state of the matter from this set, one should use the corresponding wavefunction of that state rather than the D-brane boundary state.

• The majority of matter states do not belong to the partition sum (52); absent are the matter states whose left- and right-moving excitations are different. For a general matter state, it seems appropriate to use the Ishibashi state associated to the Virasoro module of a matter highest weight state. The matter contribution to the annulus amplitude is then a Virasoro character such as (11), and as in section 2, longitudinal modes cancel among Liouville, matter, and ghost sectors. The annulus amplitude will then take the form of the integrand in (57).

• The equivalence of the exponential factor in (57) with the pair production probability amplitude (65) is not an accident. The production probability amplitude for massive string modes is essentially the above-barrier reflection amplitude in the Liouville potential of figure 3, see for instance [83]. We see that the annulus amplitude (52) computes the probability that the universe starts off at infinite scale factor and returns to infinite scale factor, even if the energy is such that the geometry classically wants to climb onto the flat part of the Liouville potential and shrink to zero scale factor (the time reverse of the negative energy solution of figure 3).

• An alternative, standard way to obtain general amplitudes uses perturbations of the state at the conformal boundary [84, 85] (for a review, see [86]; for a
discussion in the de Sitter context, see [41, 42]). In fact, the AdS version of our discussion involves exactly this prescription. Matter perturbations build up an arbitrary state of the matter CFT, which is then dressed by the Liouville gravity/ghost sector. The $m = n = 1$ Liouville state has only the Liouville identity operator (and its Virasoro descendants) at the boundary,\(^{12}\) i.e. the state is always asymptotically locally (anti)de Sitter.

For the general $(m, n)$ vacua with $m, n \neq 1$, there is the further possibility of adding boundary vertex operators $B_i$ on either end of the annulus

$$\left\langle \prod_{i=1}^{N_{in}} \oint_{\partial_{in}} B_i \prod_{j=1}^{N_{out}} \oint_{\partial_{out}} B_j \right\rangle. \quad (68)$$

For example, $B = e^{\alpha \phi + ip \cdot x} \partial^k X$ adds or subtracts an excitation of the $k^{th}$ mode of the free matter field $X$. Similarly, we can perturb the disk amplitudes that give the transition amplitudes for spaces that are asymptotically de Sitter in only the past or future. Now, the boundary condition associated to the $(m, n)$ boundary state admits nontrivial correlations with only a finite number of Liouville boundary primaries, see equation (35). The Liouville exponent $\alpha$ of matter boundary vertex operators is generically not compatible with the null vector condition that leads to $\sigma = \sigma(m, n)$ in (36), i.e. $\alpha \not\in \{\alpha_{m,n} = (1-n)\beta -(1-m)/\beta\}$. Of course, one could always adjust the matter momentum $p$ in the free field case in order to satisfy the constraints with $\alpha = \alpha_{m,n}$ for some $m, n$, but this seems rather restrictive. In any event, there are only a rather small number of perturbations by integrated boundary vertex operators of the general $(m, n)$ vacua, and boundary perturbations of the standard global vacuum state with $m = n = 1$ cannot involve the Liouville field.

- Recently, the mere existence of de Sitter quantum gravity has come into question [45, 46, 47]. The complaints come in several versions:

  - (1) Correlation functions cannot decay completely to zero, contradicting locality;
  - (2) Poincaré recurrences lead to bizarre and unacceptable cosmology;
  - (3) Finiteness of the density of states is incompatible with the classical $SO(d, 1)$ isometry of the $dS_d$ vacuum.

The first two of these do not preclude the existence of de Sitter quantum gravity; they are more aesthetic in nature. The last assertion seems more serious.

The isometry in question for $dS_2$ is $SO(2, 1) \simeq SL(2, \mathbb{R})$. This $SL(2, \mathbb{R})$ is

\(^{12}\)For instance, taking a Liouville closed string vertex operator to the boundary yields only the identity operator and its descendants as the vertex interacts with its image across the boundary, c.f. [24].
the symmetry of the global $\epsilon = 1$ vacuum solution \[50\].\(^{13}\) In the quantum theory, this becomes the $SL(2, \mathbb{R})$ symmetry of the $m = n = 1$ Liouville open string vacuum state (which formally has a null vector at level one, i.e. is annihilated by $L_{-1}$ in addition to $L_1$ and $L_0$); thus this symmetry is preserved in the quantum theory. It would be interesting to understand the relation between the arguments of \[15\] \[16\] \[17\] and the construction of quantum de Sitter space in two dimensions that we have presented here. One difference of the present analysis with that of \[15\] \[16\] \[17\] is the assumption in those works of a well-defined Hilbert space and quantum mechanics for the ‘static patch’ of de Sitter space; the constructions of the present article uniformly employ a global description.

- We have implicitly assumed that the quantum matter states of interest are built on the conformal vacuum of the matter CFT. Recently, there has been some interest in considering other classes of de Sitter invariant matter states \[87\] \[88\], in particular in connection with possible ‘trans-Planckian’ effects in inflationary cosmology. These states $|\alpha\rangle$ are defined by the conditions

\[
(\alpha_n \cosh \alpha - \alpha_{-n} \sinh \alpha)|\alpha\rangle = 0
\]

\[
(\tilde{\alpha}_n \cosh \alpha - \tilde{\alpha}_{-n} \sinh \alpha)|\alpha\rangle = 0 , \quad (69)
\]

or in other words

\[
|\alpha\rangle = \exp\left[\tanh \alpha \sum_{n>0} \frac{1}{n} (\alpha^2_n + \tilde{\alpha}^2_n)\right]|0\rangle \quad (70)
\]

for the oscillator modes of a free field (compare \[66\]). It seems problematic to consistently couple such states, which (like D-brane boundary states) lie outside of the CFT Hilbert space, to Liouville gravity. In particular, they do not seem to admit a sensible action of the conformal algebra, so it seems difficult to satisfy the BRST constraints. In any event, two dimensional de Sitter gravity seems a fruitful laboratory for the search for novel properties of quantum matter and gravity in inflating spacetimes.

- The homogeneous classical solution to de Sitter Liouville theory in the negative energy, hyperbolic class of monodromies \[19\] is

\[
e^{2\beta \phi} = \frac{1}{\pi \mu^{\prime 2} \varepsilon^2} \frac{\varepsilon^2}{\sinh^2(\varepsilon t)} . \quad (71)
\]

If one takes this solution seriously for all times $t \in (-\infty, \infty)$, the $t < 0$ region describes a big bang at $t = -\infty$ that expands to an asymptotically

\(^{13}\) Another $SL(2, \mathbb{R})$ is the monodromy \[19\] of the classical solutions. In the quantum theory, this becomes a hidden, internal quantum group symmetry $U_q(\mathfrak{sl}(2, \mathbb{R}))$ \[11\] \[12\] \[13\] \[14\] (for reviews, see \[19\] \[20\] \[21\] \[22\]). Liouville chiral vertex operators lie in well-defined representations of this symmetry, which is thus maintained (although $q$-deformed) in the quantum theory.
de Sitter geometry at $t \to 0^-$, and $t > 0$ describes a big crunch that starts at asymptotically de Sitter space at $t \to 0^+$ and crunches at $t = +\infty$. If we formally compactify $t$ by adding the point at infinity, one might ask whether the resulting crunch/bang spacetime appears in the annulus amplitude (57). In other words, does the passage from positive to negative Liouville energy look like figure 4?

![Figure 4: Naively, passage from the elliptic to the hyperbolic class of Liouville solution leads to a big crunch/big bang geometry.](image)

The exponential suppression of the amplitude at negative Liouville energy (i.e. matter energy larger than $\frac{c-1}{24}$), together with its consistent interpretation in terms of above barrier reflection in the de Sitter Liouville potential, seems to indicate that the answer is no. Is there any way to obtain the crunch/bang geometry of figure 4? Naively, such a geometry should arise if we pinch the annulus, factorizing on the disk one-point amplitudes (the unnormalized version of (32)) for localized vertex operators $e^{2\alpha \phi}$, $\alpha \in \frac{Q}{2} + i\mathbb{R}$, that create the negative energy Liouville states of interest. The fact that the spatial circle shrinks to a point at the location of the vertex operator is associated to an infinitely long propagation in conformal time; the Liouville vertex operator specifies the nature of the wavefunction of the geometry in the region of small scale factor. Ordinarily, the pinch of the annulus does correspond to the limit $\hat{\tau} \to \infty$, where closed string propagation goes on shell; and the disk one-point amplitudes are the residue of the closed string propagator pole. However, this conventional intuition actually breaks down here. To see the source of the problem, let us perform the $\nu$ and $\hat{\tau}$ integrals in the opposite order. Doing the $\hat{\tau}$ integral in (52) first reveals the pole structure in the amplitude

$$Z(p, mn) = i \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \int_{0}^{\infty} d\omega \rho(\omega) \frac{1}{\tau p^2 + \nu^2 + \omega - \frac{c-1}{24}} \times \frac{2 \sin(2\pi m\nu/\beta) \sin(2\pi n\nu/\beta) \cdot 2 \sin(2\pi m'\nu/\beta) \sin(2\pi n'\nu/\beta)}{\sqrt{2} \sin(2\pi \nu/\beta) \sin(2\pi \nu)}.$$
Now, for $\frac{1}{2}p^2 + \omega > \frac{c-1}{24}$, we might try to close the contour at infinity to pick out the residue of the pole at $\nu = i\hat{\nu}$, where $\hat{\nu} = (\frac{1}{4}p^2 + \omega - \frac{c-1}{24})^{1/2}$, to get

$$\frac{2 \sinh(2\pi m\hat{\nu}/\beta) \sinh(2\pi n\hat{\nu}\beta) \cdot 2 \sinh(2\pi m'\hat{\nu}/\beta) \sinh(2\pi n'\hat{\nu}\beta)}{\sqrt{2} \cdot 2\hat{\nu} \sinh(2\pi \hat{\nu}/\beta) \sinh(2\pi \hat{\nu}\beta)}.$$

(73)

The product of the disk one-point functions with $(m, n)$ and $(m', n')$ boundary conditions is indeed of this form. The problem, of course, is that closing the contour in this way is improper due to the presence of the sine functions; the correct closing of the contour always selects the exponentially damped contribution, as in (57), rather than the sinh that would be required to reproduce the factorization on the crunch/bang geometry. Thus the annulus amplitude can only yield the above-barrier reflection amplitude, and does not tell us how to propagate through a big crunch geometry and emerge on the other side into a big bang.

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One might ask why and how big bang or big crunch singularities are regularized in two-dimensional gravity. It is essentially a result of conformal invariance; we are specifying the particular UV completion of the theory that defines the physics in the neighborhood of the ‘singularity’, which from the point of view of two-dimensional CFT is a boundary condition at a puncture describing the ‘big bang’ state there. The partition functions of such geometries are disk amplitudes such as

$$\langle e^{2\alpha\phi} O_{\text{matter}} \rangle_{(m, n)}$$

(74)

with asymptotically de Sitter boundary conditions [30]; the Liouville part of these amplitudes is the continuation of (32), associated to the wavefunction [64]. This is not a trivial statement; when one couples 2d gravity non-trivially to matter, as in the inflationary cosmologies studied in [33], the dynamics changes dramatically from small scale (large negative Liouville field $\phi$) to large scale (large positive $\phi$). At large positive $\phi$, there can be all kinds of complicated matter interactions, dressed by gravity. At large negative $\phi$, however, the dynamics must converge to some particular matter CFT coupled to Liouville, and in addition all the Liouville dressings to relevant matter interactions $e^{2\alpha\phi} O_{\text{matter}}$ shut off (irrelevant matter interactions typically do not lead to a well-defined continuum theory). Thus the theory at small scales is specified by the choice of a conformal field theory. Presumably in higher dimensions it is the UV completion of quantum gravity plus matter, i.e. string theory, that plays the same role in taming cosmological singularities.

Finally, in the light of the connection between $c_{\text{mat}} \leq 1$ noncritical string theory and matrix models, it is rather intriguing that the de Sitter Liouville gravity dynamics obtained in the regime $c_{\text{mat}} \geq 25$ is so closely parallel. It hints at a dual formulation of two-dimensional quantum de Sitter gravity in terms of a matrix model. The asymptotic de Sitter boundary condition is implemented by what could
reasonably be described as ‘S-branes at infinity’ in $\phi$ space.\textsuperscript{14} One might hope that these are the worldsheet reflection of matrix degrees of freedom, just as in the $c < 1$ disk instanton amplitudes of section 3 and that careful consideration of de Sitter Liouville amplitudes of the sort that we have constructed here will lead us to the appropriate matrix theory.

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A Conventions

The present paper follows the conventions of \cite{22, 23, 24}, which differ from those of \cite{10, 33}. Let us denote quantities appearing in the latter works by the subscript $GM$. The present paper, as well as \cite{22, 23, 24}, sets $\alpha' = 1$, whereas $\alpha'_{GM} = 2$. The parameters of the Liouville field theory are related by

\begin{equation}
\begin{align*}
b &= \frac{\gamma_{GM}}{\sqrt{2}} \\
\mu &= \frac{\mu_{GM}}{8\pi\gamma_{GM}^2}
\end{align*}
\end{equation}

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\textsuperscript{14}Which in some sense ‘live’ on the boundary of spacetime, much as in the AdS/CFT correspondence.
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