Entanglement in Fermionic Chains and Bispectrality

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Abstract

Entanglement in finite and semi-infinite free Fermionic chains is studied. A parallel is drawn with the analysis of time and band limiting in signal processing. It is shown that a tridiagonal matrix commuting with the entanglement Hamiltonian can be found using the algebraic Heun operator construct in instances when there is an underlying bispectral problem. Cases corresponding to the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ as well as to the q-deformed algebra $\mathfrak{so}_q(3)$ at $q$ a root of unity are presented.

This paper is dedicated to Roman Jackiw with admiration and gratitude on the occasion of his 80th birthday.

1 Introduction

Throughout his career Roman Jackiw has achieved a number of important scientific advances and in the process he has brought many modern geometrical, topological and representation theoretic results to bear on the elaboration and understanding of physical theories. He has hence much contributed to increasing the level of interactions between physicists and mathematicians. We here wish to thankfully pay tribute to him by discussing how symmetry and algebraic considerations can contribute to entanglement studies in light of a parallel with long-studied issues in signal processing. We hope that this report will hence capture some of the bridge building spirit of Roman’s insightful and inspiring papers.

A fundamental feature of quantum theories, entanglement enables correlations and is a key resource in applications to information. It is therefore relevant to obtain quantitative evaluation of this property and this is being much explored using the notion of entropy. This paper belongs to that class of studies.
and focuses on systems consisting of free Fermionic chains that have been much looked at \cite{1, 2}, because of their simplicity.

Basically, for these systems, the entanglement entropy is determined by the eigenvalues of a truncated correlation matrix. However a significant difficulty in carrying their computation arises for large chains because the spectra of these correlation matrices usually accumulates near certain points thereby rendering the numerical analysis problematic. As a matter of fact this entanglement problem proves closely analogous \cite{3, 4} to the classical question of the time and band limiting of signals where the corresponding calculation difficulty is circumvented thanks to the discovery made by Slepian et al. \cite{5, 6} of an operator easy to handle numerically that commutes with the limiting operator.

The main goal of the present paper is to explain how this efficient processing of signals can be adapted in certain cases to the entropy analysis of Fermionic chains.

Although this is still not fully understood, the circumstances for the existence of the commuting operator are perceived to stem from bispectrality situations \cite{8, 9} where the functions involved depend on two variables and satisfy a pair of eigenvalue problems such that in the first equation the operator acts on one variable and the eigenvalue is a function of the other variable and vice-versa in the second equation. The hypergeometric polynomials of the Askey scheme \cite{10, 11}, offer examples of such bispectral problems: they are eigenfunctions of a differential or difference operator in the variable with the eigenvalues depending on the degree and, their orthogonality requires that they satisfy a recurrence relation which is viewed as an eigenvalue equation for an operator acting on the degree taken as a discrete variable with the eigenvalue in this case solely depending on the standard variable. One notes that there are two pictures for the pair of operators: the variable picture and the degree picture much like the coordinate and momentum representations in Quantum Mechanics.

That bispectrality has something to do with the existence of a commuting operator in problems of the time and band limiting class was revisited recently in Ref. \cite{12}. Assume that the limiting takes place by restricting the range of the two variables associated to the problem. A first observation is that the bilinear combinations of the two bispectral operators provide generalizations of the Heun operator which itself actually arises in the particular case of the Jacobi polynomials. The reader will recall that the usual Heun operator defines the Fuchsian second order differential equation with four regular singularities. To each bispectral problem is thus associated what has been called an algebraic Heun operator. Once this is recognized, it is easy to determine how these generalized Heun operators should be specialized so as to commute with the projectors on the restrained domains and as a consequence with the limiting operator.

Basically, determining the entanglement of Fermionic chains amounts: i. to taking the chain in some state which we will assume to be the ground state, ii. to dividing the system into two parts, and iii. to examining how these two parts are coupled in the chosen state. The analogy with time and band limiting arises as energy is gapped by the Fermi sea filling and space is chopped through
the partitioning of the chain. In cases where the Fermionic chain Hamiltonian exhibits bispectral features, we shall show how algebraic Heun operators provide matrices that commute with the entanglement Hamiltonian and have nice properties from the point of view of numerical analysis.

This paper enlarges and complements our recent article [13] on this topic where the emphasis in the characterization of the chains and their properties was put on the associated orthogonal polynomials. Here the focus is on bispectrality and algebras. The parallel with time and band limiting will be explained with the help of a review of the classic results in this field and the connection with algebraic Heun operators will be illustrated in this context first. Supplementing the set of chains considered in [13], we shall discuss a semi-infinite chain as well as a finite one based on a representation of a $q$-deformed algebra at $q$ a root of unity that has as special case the uniform chain treated in [2] and [13].

The presentation will proceed along the following lines. The free Fermionic Hamiltonians and their diagonalization are described in Section 2 that will also establish notation. Section 3 introduces the restricted correlation matrix as the central quantity for the study of entanglement. Section 4 momentarily leaves the topic of Fermionic chains to offer a short overview of the classical problem of limiting in time a signal which is banded in frequency. It shall explain how the Heun operator associated to the Fourier bispectral problem leads to the second order differential operator that commutes with the integral operator that effects the limiting in this case. Section 5 returns to Fermionic chains in light of this understanding and discusses generally when the Hamiltonians are characterized by a bispectral problem. For finite chains this will involve Leonard pairs which are known to be in correspondence with the families of orthogonal polynomials of the terminating branch of the Askey scheme. Section 6 derives the tridiagonal matrices that commute with the chopped correlation matrix from the algebraic Heun operators attached to Hamiltonians with bispectral underpinnings. Special bispectral situations that will be considered as examples shall be arising from the representation theory of Lie and $q$-algebras. This will be the contents of Sections 7, 8, and 9. Section 7 will reproduce results from [13] by discussing the chain based on $su(2)$. Section 8 will treat the case of the semi-infinite chain associated to $su(1, 1)$. Section 9 will focus on the chain whose couplings are given by the representation matrices of the non-standard deformation $so_q(3)$ of $so(3)$ at $q$ root of unity. This will have as a special case the uniform chain treated in Refs. [2] and [13]. Section 10 shall bring the paper to a close with concluding remarks.
2 Free-Fermion Hamiltonian and its diagonalization

We consider the following open quadratic free-Fermion inhomogeneous Hamiltonian with nearest neighbour interactions and with magnetic fields

\[ \hat{H} = \sum_{n=0}^{N-1} (J_n c_n^\dagger c_{n+1} + J_n^* c_{n+1}^\dagger c_n) - \sum_{n=0}^{N} B_n c_n^\dagger c_n, \]  

(1)

where \( B_n \) (resp. \( J_n \)) are real (resp. complex) parameters, \( J_n^* \) is the complex conjugate of \( J_n \) and \( \{c_m^\dagger, c_n\} = \delta_{m,n} \). For the sake of simplicity of the following computations, we enumerate the sites of the lattice from 0 to \( N \). We can also consider the case \( N \to +\infty \) which corresponds to a semi-infinite chain (see Section 8 for an example).

In order to diagonalize \( \hat{H} \), it is convenient to rewrite it as follows

\[ \hat{H} = (c_0^\dagger, \ldots, c_N^\dagger) \hat{H} \begin{pmatrix} c_0 \\ \vdots \\ c_N \end{pmatrix}. \]  

(2)

The \((N+1) \times (N+1)\) matrix \( \hat{H} \) is an Hermitian tridiagonal matrix given by

\[ \hat{H} = \sum_{n=0}^{N} \left( J_{n-1}|n-1\rangle\langle n| - B_n|n\rangle\langle n| + J_n^*|n+1\rangle\langle n| \right), \]  

(3)

with the convention \( J_N = J_{-1} = 0 \). The set \( \{|0\rangle, |1\rangle, \ldots, |N\rangle\} \) of elements in \( \mathbb{C}^{N+1} \) denotes the canonical orthonormal basis and will be called the position basis. The spectral problem for \( \hat{H} \) reads

\[ \hat{H} |\omega_k\rangle = \omega_k |\omega_k\rangle, \]  

(4)

where

\[ |\omega_k\rangle = \sum_{n=0}^{N} \phi_n(\omega_k) |n\rangle. \]  

(5)

We order the \( N+1 \) eigenvalues \( \omega_0, \omega_1, \ldots, \omega_N \) so that \( \omega_k < \omega_{k+1} \). We also normalize the eigenvectors \( |\omega_0\rangle, |\omega_1\rangle, \ldots, |\omega_N\rangle \) so that they form an orthonormal basis of \( \mathbb{C}^{N+1} \), to be called the momentum basis. Having diagonalized \( \hat{H} \), we see that the Hamiltonian \( \hat{H} \) can be rewritten as

\[ \hat{H} = \sum_{k=0}^{N} \omega_k \tilde{c}_k^\dagger \tilde{c}_k, \]  

(6)

where the annihilation operators \( \tilde{c}_k \) are defined by

\[ \tilde{c}_k = \sum_{n=0}^{N} \phi_n^*(\omega_k) c_n, \]  

(7)
and the corresponding formulas for the creation operators \( \hat{c}_k^\dagger \) are given by the Hermitian conjugation of (7). These operators obey the anticommutation relations

\[
\{ \hat{c}_k^\dagger , \hat{c}_p \} = \{ \hat{c}_p , \hat{c}_k \} = 0 , \quad \{ \hat{c}_k^\dagger , \hat{c}_k \} = \delta_{k,p} .
\]

(8)

One can invert relation (7) to get

\[
c_n = \sum_{k=0}^{N} \phi_n(\omega_k) \hat{c}_k .
\]

(9)

The eigenvectors of \( \hat{H} \) are therefore given by

\[
| \Psi \rangle = \hat{c}_k^\dagger \cdots \hat{c}_{k_1}^\dagger |0\rangle ,
\]

(10)

where \( k_1 < \ldots < k_r \in \{ 0 , \ldots , N \} \), and the vacuum state \( |0\rangle \) is annihilated by all the annihilation operators

\[
\hat{c}_k |0\rangle = 0 , \quad k = 0 , \ldots , N .
\]

(11)

The corresponding energy eigenvalues of \( \hat{H} \) are simply given by

\[
E = \sum_{i=1}^{r} \omega_{k_i} .
\]

(12)

3 Correlations and the entanglement Hamiltonian

For the sake of concreteness, we shall consider entanglement in the ground state described below. We shall further review how the reduced density matrix for the first \( \ell + 1 \) sites of the chain is determined by the 1-particle correlation matrix and equivalently by the entanglement Hamiltonian.

3.1 Defining the ground state

The fact that the ground state is constructed by filling the Fermi sea leads to a restriction in energy. Indeed, the ground state \( |\Psi_0\rangle \) of the Hamiltonian (1) is given by

\[
|\Psi_0\rangle = \hat{c}_0^\dagger \cdots \hat{c}_K^\dagger |0\rangle ,
\]

(13)

where \( K \in \{ 0, 1, \ldots , N \} \) is the greatest integer below the Fermi momentum, such that

\[
\omega_K < 0 , \quad \omega_{K+1} > 0 .
\]

(14)

Let us remark that \( K \) can be modified by adding a constant term to the external magnetic fields \( B_n \). We shall in fact choose this constant magnetic field so as to ensure that \( \omega_k \neq 0 \) for any \( k \) in order to avoid dealing with a degenerate ground state.
The correlation matrix $\hat{C}$ in the ground state is an $(N+1) \times (N+1)$ matrix with the following entries

$$\hat{C}_{mn} = \langle \Psi_0 | c^\dagger_m c_n | \Psi_0 \rangle .$$

Expressing everything in terms of annihilation and creation operators using (9) and (10), and then using the anticommutation relations (8) and the property (11) of the vacuum state, we obtain

$$\hat{C}_{mn} = \sum_{k=0}^{K} \phi^*_m(\omega_k)\phi_n(\omega_k), \quad 0 \leq n, m \leq N .$$

It is then manifest that

$$\hat{C} = \sum_{k=0}^{K} |\omega_k\rangle\langle \omega_k| ,$$

namely, that $\hat{C}$ is the projector onto the subspace of $\mathbb{C}^{N+1}$ spanned by the vectors $|\omega_k\rangle$ with $k = 0, ..., K$ running over the labels of the excitations in the ground state.

### 3.2 Entanglement entropy

In order to examine entanglement, we must first define a bipartition of our free-Fermionic chain. As subsystem (part 1) we shall take the first $\ell + 1$ consecutive sites, and shall find how it is intertwined with the rest of the chain in the ground state $|\Psi_0\rangle$. To that end, we need the reduced density matrix

$$\rho_1 = \text{tr}_2 |\Psi_0\rangle \langle \Psi_0 | ,$$

where part 2, the complement of part 1, is comprised of the sites $\{\ell + 1, \ell + 2, ..., N\}$; from this quantity one can compute for instance the von Neumann entropy

$$S_1 = - \text{tr}(\rho_1 \log \rho_1) .$$

The explicit computations of this entanglement entropy amounts to finding the eigenvalues of $\rho_1$. It has been observed that this reduced density matrix $\rho_1$ is determined by the spatially “chopped” correlation matrix $C$, which is the following $(\ell + 1) \times (\ell + 1)$ submatrix of $\hat{C}$:

$$C = |\hat{C}_{mn}|_{0 \leq m, n \leq \ell} .$$

The argument which we take from Ref. [14] (see also Ref. [15]) goes as follows. Because the ground state of the Hamiltonian $\hat{H}$ is a Slater determinant, all correlations can be expressed in terms of the one-particle functions, i.e. in terms of the matrix elements of $\hat{C}$. Restricting to observables $A$ associated to part 1, since the expectation value of $A$ is given by $\langle A \rangle = \text{tr}(\rho_1 A)$, the factorization property will hold according to Wick’s theorem if $\rho_1$ is of the form

$$\rho_1 = \kappa \exp(-\mathcal{H}) ,$$

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with the entanglement Hamiltonian $H$ given by

$$H = \sum_{m,n=0}^{\ell} h_{mn} c_m^\dagger c_n.$$  

(22)

The hopping matrix $h = |h_{mn}|_{0 \leq m,n \leq \ell}$ is defined so that

$$C_{mn} = \text{tr}(\rho_1 c_m^\dagger c_n), \quad m,n \in \{0, 1, \ldots, \ell\},$$

(23)

holds, and one finds through diagonalization that

$$h = \log[(1 - C)/C].$$

(24)

We thus see that the $2^{(\ell+1)} \times 2^{(\ell+1)}$ matrix $\rho_1$ is obtained from the $(\ell+1) \times (\ell+1)$ matrix $C$ or equivalently, from the entanglement Hamiltonian $H$.

Introducing the projectors

$$\pi_1 = \sum_{n=0}^{\ell} |n\rangle\langle n| \quad \text{and} \quad \pi_2 = \sum_{k=0}^{K} |\omega_k\rangle\langle \omega_k| = \hat{C},$$

(25)

the chopped correlation matrix can be written as (see for instance Refs. [16, 17])

$$C = \pi_1 \pi_2 \pi_1.$$  

(26)

To calculate the entanglement entropies one therefore has to compute the eigenvalues of $C$. As explained in Ref. [1], this is not easy to do numerically because the eigenvalues of that matrix are exponentially close to 0 and 1. We shall show in the following how to go about this problem by drawing on methods developed in signal processing.

4 A review of time and band limiting

We here digress to underscore that the treatment of time and band limiting problems is of relevance for the characterization of entanglement in Fermionic chains. To make that clear, we shall review aspects of the classic problem of optimizing the concentration in time of a band limited signal. In the first part of this section we shall show that the limiting integral operator can also be expressed in terms of projectors exactly as in (26). The diagonalization of this operator that would give the optimization solution is also plagued by computational difficulties. In the second part of the section we shall indicate how the underlying bispectrality provides a way to overcome this numerical analysis challenge by allowing to identify a differential operator that commutes with the limiting one.

Let $f(t)$ be a signal limited to the band of frequencies $[-W,W]$:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-W}^{W} dp \; e^{ipt} F(p) \in B_W,$$

(27)
and call $B_W$ the space of such functions taken to be real. It is natural to want a signal of finite duration, that is to ask that $f(t)$ vanishes outside the interval $-T < t < T$: \[ f \neq 0 \text{ only for } -T < t < T. \] (28)

It is however readily realized that this is impossible: since $f(t) \in B_W$, it is entire in complex $t$-plane; therefore if $f(t) = 0$ for any interval, it follows that $f(t)$ is identically zero ($f(t) \equiv 0$). In the 1960s and 1970s Slepian, Landau, Pollak from Bell labs (see the reviews [5, 6]) considered how to approximate the situation wished for and asked the question: Which band-limited signal $\in B_W$ is best concentrated in the time interval $-T < t < T$, second best concentrated etc.? In other words which functions $f(t) \in B_W$ are maximizing

\[ \alpha^2(T) = \frac{\int_{-T}^{T} f^2(t) dt}{\int_{-\infty}^{\infty} f^2(t) dt}. \] (29)

\[ = 2 \int_{-W}^{W} dp' \int_{-W}^{W} dp'' \left[ \frac{\sin((p' - p'')T)}{(p' - p'')^2} \right] F(p'')F^*(p') \int_{-W}^{W} dp' F(p')F^*(p'). \] (30)

As is well known from the calculus of variations, the answer to that question is provided by the solutions of

\[ GF(p) = \lambda F(p), \] (31)

where the integral operator $G$ is defined by

\[ GF(p) = \int_{-W}^{W} dp' K(p - p')F(p'), \] (32)

with $K(p - p')$ the sinc kernel

\[ K(p - p') = \frac{\sin((p - p')T)}{\pi(p - p')}. \] (33)

Let us remark that $GF(p)$ is zero if $p$ is not between $-W$ and $W$ as the functions $F(p)$ we start with.

In principle this should settle the concentration problem. However the spectrum of $G$ accumulates sharply at the origin and this makes the numerical computations intractable. Slepian, Landau, Pollak [5, 7] quite remarkably found a way out by showing that there exists a second order differential operator $D$ that commutes with the integral operator $G$. This is important because $D$ has common eigenfunctions with $G$ and second order differential operator are typically well behaved numerically. It is interesting to mention that $D$ actually arises in separating the Laplacian in prolate spheroidal coordinates. Let us indicate how
this commuting operator is obtained using the bispectral framework of Fourier transform.

We shall first note that $G$ can be written in a form similar to that given in (26) for the chopped correlation matrix. Consider projectors on an interval:

$$\pi_L^f(x) = \begin{cases} f(x) & -L < x < L \\ 0 & \text{otherwise} \end{cases}$$

(34)

$$= \left[ \Theta(x + L) - \Theta(x - L) \right] f(x) ,$$

(35)

with $\Theta(x)$ the step function. Let $\mathcal{F} : f(t) \mapsto F(p)$ denote the Fourier transform and $\mathcal{F}^{-1}$ its inverse. Take the following projectors in Fourier (or band) space:

$$\pi_W^p$$ and the Fourier transformed \(\hat{\pi}_T^p = \mathcal{F}\pi_T^p\mathcal{F}^{-1} \). (36)

It is straightforward to see that

$$G = \pi_W^p \hat{\pi}_T^p \pi_W^p .$$

(37)

Operators $X$ and $Y$ form a bispectral pair if they have common eigenfunctions $\psi(x,n)$ such that

$$X \psi(x,n) = \omega(x) \psi(x,n) ,$$

(38)

$$Y \psi(x,n) = \lambda(n) \psi(x,n) ,$$

(39)

with $X$ acting on the variable $n$ and $Y$, on the variable $x$. When forming products of these operators $X$ and $Y$, we shall understand that they are both taken in same representation “$n$” or “$x$”. The functions $\psi(t,p) = e^{ipt}$ in Fourier transforms satisfy

$$- \frac{d^2}{dt^2} \psi(t,p) = p^2 \psi(t,p), \quad - \frac{d^2}{dp^2} \psi(t,p) = t^2 \psi(t,p) ,$$

(40)

and are thus associated to a most simple bispectral problem: the functions $\psi(t,p)$ are eigenfunctions of an operator acting on $t$ with eigenvalues depending on $p$ and vice-versa. All the orthogonal polynomials of the Askey scheme are solutions of bispectral problems defined by the differential/difference equation and the recurrence relation.

How does this help find the differential operator that commutes with the limiting operator $G$?

To each bispectral problem, one can attach an Algebraic Heun Operator \[18\] defined as the most general operator $W_H$ which is bilinear in the bispectral operators $X$ and $Y$:

$$W_H = \tau_1 \{X,Y\} + \tau_2 [X,Y] + \tau_3 X + \tau_4 Y + \tau_0 ,$$

(41)

with $\tau_i, i = 0, 1 \ldots, 4$, constants and $\{X,Y\} = XY + YX$. The name comes from the fact that the standard Heun operator results when this construct is
applied to the bispectral operators of the Jacobi polynomials, namely the hypergeometric operator and multiplication by the variable $x$. We claim that the commuting operators belong to that class of operators. Let us return to the Fourier case where in the “frequency” representation

$$X = -\frac{d^2}{dp^2}, \quad Y = p^2.$$  \hspace{1cm} (42)

In this representation, taking $\tau_0 = 0$ and $\tau_1 = -1/2$, the algebraic Heun operator which we will now denote by $D$ takes the form:

$$D = \frac{1}{2} \left\{ \frac{d^2}{dp^2}, p^2 \right\} + \tau \left[ -\frac{d^2}{dp^2}, p^2 \right] - \mu \frac{d^2}{dp^2} + \nu p^2$$  \hspace{1cm} (43)

$$= (p^2 - \mu) \frac{d^2}{dp^2} + (2 - 4\tau)p \frac{d}{dp} + \nu p^2 - 2\tau + 1.$$  \hspace{1cm} (44)

Given (43), such an operator will commute with $G$ if it commutes with both $\pi_p W$ and $\tilde{\pi}_p t$. Consider a general second order differential operator written as

$$D = A(p) \frac{d^2}{dp^2} + B(p) \frac{d}{dp} + C(p).$$  \hspace{1cm} (45)

Let us look first at the projector onto the semi-infinite interval $[W, \infty)$

$$\tilde{\pi}_p W = \Theta(p - W).$$  \hspace{1cm} (46)

It is easy to see that $[D, \tilde{\pi}_p W] = 2A(p)\delta(p-W)\frac{d}{dp} + (-A'(p) + B(p))\delta(p-W) = 0$ if $A(W) = 0$ and $A'(W) = B(W)$. Now recall that

$$\pi_p W = \Theta(p + W) - \Theta(p - W).$$

In this case $[D, \pi_p W] = 0$ is satisfied if

$$A(\pm W) = 0 \quad \text{and} \quad A'(\pm W) = B(\pm W).$$  \hspace{1cm} (47)

Applying these conditions to $D$ as given by (44) is readily seen to imply that

$$\mu = W^2 \quad \text{and} \quad \tau = 0.$$  \hspace{1cm} (48)

Now if in addition $[D, \tilde{\pi}_p t] = 0$, we would have $[D, G] = 0$. Clearly $[D, \tilde{\pi}_p t] = [D, \mathcal{F} \pi_t \mathcal{F}^{-1}] = 0$ is tantamount to $[\mathcal{F}^{-1} D \mathcal{F}, \pi_t] = 0$, namely to the condition that the Fourier transform $\tilde{D} = \mathcal{F}^{-1} D \mathcal{F}$ of $D$ commutes with a projector in $t$ with parameter $T$ that is similar to $\pi_p W$. Under the Fourier transform: $p^2 \rightarrow -\frac{d^2}{dt^2}, \quad \frac{d}{dp} \rightarrow t^2$ and $\tilde{D}$ is obtained from $D$ by exchanging $p$ and $t$ as well as $\mu$ and $\nu$ and by taking $\tau$ into $-\tau$. It is then obvious that the condition $[\tilde{D}, \pi_t] = 0$ is satisfied by taking

$$\nu = T^2 \quad \text{and again} \quad \tau = 0.$$  \hspace{1cm} (49)
It thus follows that the second order differential operator that commutes with the limiting integral operator is simply obtained from the algebraic Heun operator \( H \) by imposing the conditions \( \tau = 0, \mu = W^2 \) and \( \nu = T^2 \) on the parameters.

The parallel with the study of entanglement in Fermionic chains is quite clear. Taking the chain in its ground state (or in any other reference state) involves restricting the energies and corresponds to band limiting. Associated to that is the projector \( \pi_2 \) in (25). Establishing the bipartition truncates space and this is akin to time limiting. Attached to this is the projector \( \pi_1 \) in (25).

The task is to solve the eigenvalue problem for the chopped correlation matrix \( C = \pi_1 \pi_2 \pi_1 \) which looks very much like the limiting operator \( G \) as given in (37) (the picture is actually the dual one here). On the basis of this similarity, we may therefore hope that there could be a tridiagonal matrix - the discrete analog of a second order differential operator - that would commute with both \( \pi_1 \) and \( \pi_2 \) and hence with \( C \) so as to ease the numerical analysis. Recalling that the existence of the commuting operator was predicated on the fact that there was an underlying bispectral problem, we shall discuss next what this requirement entails for the specifications of the Fermionic chains that shall henceforth be considered.

5 A bispectral framework for Fermionic chains

In order to identify Fermionic chains that are based on bispectral problems, let us recall that two natural bases, the position basis \( \{|n\rangle\} \) and the momentum basis \( \{\omega_k\} \), are associated to the chains. The \((N + 1) \times (N + 1)\) matrix \( \hat{H} \) that defines the Hamiltonian is irreducible tridiagonal in the first of these bases and diagonal in the second. (By irreducible it is understood that there are no zeros on the sub- and super-diagonals.) From (4) we have

\[
\langle n|\hat{H}|\omega_k\rangle = \omega_k \langle n|\omega_k\rangle
\]

and thus in view of (3) the wavefunctions \( \phi_n(\omega_k) = \langle n|\omega_k\rangle \) satisfy the eigenvalue equation

\[
\omega_k \phi_n(\omega_k) = J_n \phi_{n+1}(\omega_k) - B_n \phi_n(\omega_k) + J_{n-1} \phi_{n-1}(\omega_k), \quad 0 \leq n \leq N.
\]

We wish the functions \( \phi_n(\omega_k) \) to be solutions of a bispectral problem. To that end we need to adjoin to \( \hat{H} \) a companion operator \( \hat{X} \) with the property of being diagonal in the basis \( \{|n\rangle\} \) and irreducible tridiagonal in the basis \( \{\omega_k\} \). In other words we need an \( \hat{X} \) such that

\[
\hat{X} = \sum_{n=0}^{N} \lambda_n |n\rangle \langle n|,
\]

and

\[
\hat{X} = \sum_{k=0}^{N} \left( J_{k-1} |\omega_{k-1}\rangle \langle \omega_k| - \bar{T}_k |\omega_k\rangle \langle \omega_k| + J_k |\omega_{k+1}\rangle \langle \omega_k| \right),
\]
with the convention $J_{-1} = J_{N+1} = 0$. It then follows that
\[ \langle n|\hat{X}|\omega_k \rangle = \lambda_n \langle n|\omega_k \rangle \] becomes the difference equation
\[ \lambda_n \phi_n(\omega_k) = J_k \phi_{n+1}(\omega_k) - B_k \phi_n(\omega_k) + J_{k-1} \phi_{n-1}(\omega_k), \quad 0 \leq k \leq N. \] Equations (51) and (55) provide a bispectral problem for $\phi_n(\omega_k)$ which is a discrete version of the bispectral problem (40) at the root of the previous section.

When $N$ is finite, the couple of operators $\hat{H}$ and $\hat{X}$ form by definition a Leonard pair \[19\]. One can deduce that the eigenvalues $\{\omega_k\}$ of $\hat{H}$ are pairwise distinct and similarly for the eigenvalues $\{\lambda_n\}$ of $\hat{X}$ (see Lemma 1.3. in \[20\]). Leonard pairs have been classified \[19\] and shown to be in correspondence with the orthogonal polynomial families of the truncating part of the Askey tableau. As a matter of fact, all discrete hypergeometric polynomials of that scheme, not only the finite classes, provide admissible $\hat{H}$ and $\hat{X}$ through their recurrence relation and difference equation.

Summing up, the Fermionic chains susceptible of admitting a commuting tridiagonal matrix are those whose specifications are dictated by a duo of operators $\hat{H}$ and $\hat{X}$ with the special properties described above. Operators that would qualify are for instance two generators of the Askey-Wilson algebra or, for $q = 1$, of the Racah algebra; these are quadratic algebras which respectively describe the bispectral properties of the polynomials sitting at the top of the Askey scheme. As particular and simpler cases, a moment’s thought will make one realize that two generators of rank-one Lie or $q$-deformed Lie algebras will meet the requirement that one of these elements will be represented by an irreducible tridiagonal matrix in the eigenbasis of the other and vice-versa. These are the situations on which we will focus in Sections 7, 8 and 9.

Given such bispectral contexts, the time and band limiting experience has taught us that nice commuting operators can be simply obtained from the associated algebraic Heun operator. This is what we will explain in the next section before we come to examples.

6 Algebraic Heun operators and commuting matrices

Looking for a tridiagonal matrix $T$ that commutes with $C$, in the spirit of Section 4 we introduce the “discrete - discrete” version of the algebraic Heun operator (44). As per (41), we take this operator to be \[18\] the following bilinear combination of the two operators that define the bispectral problem:
\[ \hat{T} = \{\hat{X}, \hat{H}\} + \tau [\hat{X}, \hat{H}] + \mu \hat{X} + \nu \hat{H}. \] At this point the parameters $\tau, \mu, \nu$ are free. (Note that allowing for redefinition by an irrelevant overall factor, the coefficient of $\{\hat{X}, \hat{H}\}$ has been set equal to
1.) It is immediate to see that $\hat{T}$ is tridiagonal in both the position basis
\[ \hat{T}|n⟩ = J_n (\lambda_n (1 + \tau) + \lambda_n (1 - \tau) + \nu) |n - 1⟩ \]
+ \[(\mu \lambda_n - 2B_n \lambda_n - \nu B_n) |n⟩ \]
+ \[J_n (\lambda_n (1 - \tau) + \lambda_n (1 + \tau) + \nu) |n + 1⟩ , \] \hspace{1cm} (57)
and the momentum basis
\[ \hat{T}|ω_k⟩ = J_k(ω_k(1 + \tau) + ω_k(1 - τ) + μ) |ω_k - 1⟩ \]
+ \[(\nu ω_k - 2Bk ω_k - μ Bk) |ω_k⟩ \]
+ \[J_k(ω_k(1 + \tau) + ω_{k+1}(1 - τ) + μ) |ω_{k+1}⟩ . \] \hspace{1cm} (58)
As a matter of fact, it has been shown in Ref. [21] that $\hat{T}$ is the most general operator which is tridiagonal in both bases in finite-dimensional situations.

Let $\hat{T}_{mn} = ⟨m|\hat{T}|n⟩$ and define the “chopped” matrix $T$ by
\[ T = |\hat{T}_{mn}|_{0 ≤ m,n ≤ ℓ} . \] \hspace{1cm} (59)
Following the results of Refs. [18, 22], we know that $T$ and $C$ will commute, \[ [T,C] = 0 , \] \hspace{1cm} (60)
if the parameters in $\hat{T}$ [56] are given by
\[ \tau = 0 , \quad μ = - (ω_K + ω_{K+1}) \quad \text{and} \quad ν = - (λ_ℓ + λ_{ℓ+1}) . \] \hspace{1cm} (61)
Indeed, with the particular value of $ν$ given by [61], we see that the matrix $\hat{T}$ leaves the subspace $\{|n⟩, n = 0,1,\ldots,ℓ\}$ invariant. Therefore, $T$ commutes with $π_1$. Similarly, with $μ$ specified by [61], $\hat{T}$ leaves the subspace $\{ω_k⟩, k = 0,1,\ldots,K\}$ invariant and $T$ commutes with $π_2$. Finally, in view of (26), it is easy to see that (60) holds.

The main result of this section is that the tridiagonal matrix $T$ [59] i.e.
\[ T = \begin{pmatrix}
  d_0 & t_0 & t_1 & t_2 & \cdots & \cdots & \cdots & \cdots \\
  t_0 & d_1 & t_1 & t_2 & \cdots & \cdots & \cdots & \cdots \\
  t_1 & t_1 & d_2 & t_2 & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  t_{ℓ-2} & t_{ℓ-1} & d_{ℓ-1} & t_1 & t_2 & \cdots & \cdots & \cdots \\
  t_{ℓ-1} & t_{ℓ-1} & d_ℓ & t_1 & t_2 & \cdots & \cdots & \cdots \\
  d_ℓ & d_ℓ & d_ℓ & d_ℓ & d_ℓ & \cdots & \cdots & \cdots \\
\end{pmatrix} , \] \hspace{1cm} (62)
whose nonzero matrix elements are given by (see [57])
\[ t_n = J_n (\lambda_n + λ_{n+1} - λ_ℓ - λ_{ℓ+1}) , \] \hspace{1cm} (63)
\[ d_n = -B_n (2λ_n - λ_ℓ - λ_{ℓ+1} - λ_n (ω_K + ω_{K+1}) \] \hspace{1cm} (64)
commutes with the correlation matrix [60]. A key ingredient obviously is the operator $\hat{X}$ defined in [52]. In the following sections, we apply this construction to examples of both finite and semi-infinite free Fermionic chains.

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If \( t_n \neq 0 \) (which is the case in the examples below), \( T \) is non-degenerate (see e.g. Lemma 3.1 in Ref. [19]) and the commuting matrices \( T \) and \( C \) have a unique set of common eigenvectors. Since \( T \) is tridiagonal, its eigenvectors can be readily computed numerically. By acting with \( C \) on these eigenvectors, the eigenvalues of \( C \) can be easily obtained. The eigenvalues of the entanglement Hamiltonian \( \mathcal{H} \), and therefore the entanglement entropy of the model, can then also be straightforwardly determined.

7 The chain based on \( \mathfrak{su}(2) \)

In this section, and the subsequent ones, we use unitary representations of Lie and \( q \)-deformed algebras to identify appropriate pairs of bispectral Hermitian operators \( \hat{H} \) and \( \hat{X} \). We then construct the Heun operator to obtain explicit examples of matrices \( T \) that commute with the respective entanglement Hamiltonians.

We begin with the simplest case, that is, \( \mathfrak{su}(2) \). The spin \( s \) \((s \in \mathbb{Z}/2)\) representation of \( \mathfrak{su}(2) \) is given by

\[
s^x = \frac{1}{2} \sum_{n=0}^{2s} \sqrt{(n+1)(2s-n)} \left( |n\rangle \langle n+1| + |n+1\rangle \langle n| \right),
\]

\[
s^y = -\frac{i}{2} \sum_{n=0}^{2s} \sqrt{(n+1)(2s-n)} \left( |n\rangle \langle n+1| - |n+1\rangle \langle n| \right),
\]

\[
s^z = -\sum_{n=0}^{2s} (n-s) |n\rangle \langle n|.
\]

We choose

\[
\hat{H} = \cos(\theta)s^z - \sin(\theta)s^x - b,
\]

where \( b \) and \( \theta \) are real constants. In view of (3), we study a chain with \( N = 2s \) and with parameters (see (1)) given by

\[
B_n = \cos(\theta)(n-s) + b, \quad J_n = -\frac{1}{2} \sin(\theta)\sqrt{(n+1)(2s-n)}.
\]

To diagonalize \( \hat{H} \), we observe that \( \hat{H} = U(s^z - b)U^\dagger \) with \( U = e^{i\theta s^y} \), and hence

\[
\hat{H}|\omega_k\rangle = \omega_k|\omega_k\rangle \quad \text{with}
\]

\[
|\omega_k\rangle = U|2s-k\rangle \quad \text{and} \quad \omega_k = k - s - b,
\]

where \( k = 0, 1, \ldots, 2s \). The integer \( K \) in (13) is the unique integer satisfying

\[
s + b - 1 \leq K < s + b. \quad \text{Let us mention that} \quad \phi_n(\omega_k) = \langle n |\omega_k\rangle = \langle n |U|k\rangle \quad \text{are given in terms of the Krawtchouk polynomials in this case.}
\]

\[1\] We choose \( b \) such that \( K \in \{0, 1, \ldots, N\} \). The other case \( K < 0 \) (resp. \( K > N \)) corresponds to an empty (resp. full) ground state which is not interesting from the point of view of the entanglement entropy.
The operator \( \hat{X} \) [52] can be chosen as \( \hat{X} = s^2 \), which is diagonal in the position basis with \( \lambda_n = s - n \). We observe that
\[
s^2 = U (\cos(\theta)s^2 + \sin(\theta)s^x)U^\dagger. \tag{71}
\]
Hence, in the momentum basis, in light of the first equation in (70), \( \hat{X} \) is given by
\[
\hat{X} = \cos(\theta) \sum_{k=0}^{2s} (k-s)|\omega_k\rangle\langle\omega_k| + \frac{1}{2} \sin(\theta) \sum_{k=0}^{2s} \sqrt{(k+1)(2s-k)} \left( |\omega_k\rangle\langle\omega_{k+1}| + |\omega_{k+1}\rangle\langle\omega_k| \right). \tag{72}
\]
Comparing with the general form [53] for \( \hat{X} \) in the momentum basis, we have
\[
B_k = -\cos(\theta)(k-s), \quad J_k = \frac{1}{2} \sin(\theta) \sqrt{(k+1)(2s-k)}. \tag{73}
\]
The Heun operator associated to the Lie algebra \( su(2) \) has been studied previously in [23]. We conclude that the matrix \( T \) is given by (62) with
\[
t_n = \sin(\theta)(n-\ell)\sqrt{(n+1)(2s-n)}, \quad d_n = [\cos(\theta)(n-s) + b] (2n-2\ell-1) + (s-n)(2s-2K+2b-1). \tag{74}
\]

8 The chain based on \( su(1,1) \)

In this section, we focus on the irreducible discrete series unitary representation of the Lie algebra \( su(1,1) \) given by (see e.g. [24])
\[
\sigma^x = \frac{1}{2} \sum_{n=0}^{\infty} \sqrt{(n+1)(\kappa+n)} \left( |n\rangle\langle n+1| + |n+1\rangle\langle n| \right), \tag{76}
\]
\[
\sigma^y = \frac{i}{2} \sum_{n=0}^{\infty} \sqrt{(n+1)(\kappa+n)} \left( |n\rangle\langle n+1| - |n+1\rangle\langle n| \right), \tag{77}
\]
\[
\sigma^z = \sum_{n=0}^{\infty} \left( n + \frac{\kappa}{2} \right) |n\rangle\langle n|, \tag{78}
\]
where \( \kappa \) is a real positive parameter. Indeed, one can show that
\[
[\sigma^x, \sigma^y] = -i\sigma^z, \quad [\sigma^z, \sigma^x] = i\sigma^y, \quad [\sigma^z, \sigma^y] = -i\sigma^x. \tag{79}
\]
We choose for \( \hat{H} \)
\[
\hat{H}^{\text{ell}} = \cosh(\theta)\sigma^z - \sinh(\theta)\sigma^x + b, \tag{80}
\]
where \( b \) and \( \theta \) are real. The superscript “ell” stands for elliptic. To justify this name, we recall that a rotation by an element of the group \( SU(1,1) \) of a generic
element \( l_x \sigma^x + l_y \sigma^y + l_z \sigma^z \) preserves the non-definite form \( l_x^2 + l_y^2 - l_z^2 \). For the Lie element \( \cosh(\theta) \sigma^x - \sinh(\theta) \sigma^z \) in \( SU(2) \), this non-definite form is negative with the element thus belonging to the elliptic orbit.

We are therefore studying in this section a chain with an infinite number of sites. In view of (3), the parameters of the Hamiltonian \( H \) defined by (1) are given by

\[
B_{n}^{\text{ell}} = - \cosh(\theta) \left( n + \frac{\kappa}{2} \right) - b, \quad J_{n}^{\text{ell}} = - \frac{1}{2} \sinh(\theta) \sqrt{(n+1)(\kappa+n)}. \tag{81}
\]

To obtain the eigenvalues and eigenvectors of \( H^{\text{ell}} \) \( SU(2) \), we note here that \( H^{\text{ell}} = U(\sigma^z + b)U^\dagger \) with \( U = e^{i\theta \sigma^y} \), and find that \( H^{\text{ell}} | \omega_k \rangle = \omega_k | \omega_k \rangle \) with

\[
| \omega_k \rangle = U | k \rangle \quad \text{and} \quad \omega_k = k + \frac{\kappa}{2} + b. \tag{82}
\]

for \( k = 0, 1, \ldots \). Let us mention that the wavefunctions \( \phi_n(\omega_k) \) are expressed in terms of the Meixner polynomials in this case.

The operator \( \hat{X} \) is taken to be \( \hat{X} = \sigma^z \), and is diagonal in the position basis with \( \lambda_n = n + \frac{\kappa}{2} \). We observe that

\[
\hat{X}^{\text{ell}} = U (\cosh(\theta) \sigma^x + \sinh(\theta) \sigma^z) U^\dagger. \tag{83}
\]

Proceeding as for the \( su(2) \) model and referring to [63], we observe that the expression of \( \hat{X} \) in the momentum basis involves the following coefficients:

\[
\overline{B}_k^{\text{ell}} = - \cosh(\theta) \left( k + \frac{\kappa}{2} \right), \quad \overline{T}_k^{\text{ell}} = \frac{1}{2} \sinh(\theta) \sqrt{(k+1)(\kappa+k)}. \tag{84}
\]

The Heun operator associated to the Lie algebra \( su(1, 1) \) has been studied previously in [23]. We conclude that the matrix \( T \) in this case is given by (62) with

\[
t_n = - \sinh(\theta)(n - \ell) \sqrt{(n+1)(\kappa+n)}, \tag{85}
\]

\[
d_n = \cosh(\theta)(n + \frac{\kappa}{2} + b) \left( 2n - 2\ell - 1 \right) - (n + \frac{\kappa}{2})(\kappa + 2K + 2b + 1). \tag{86}
\]

9 The chain based on \( so_q(3) \) at \( q \) root of unity

In this section, we offer a final explicit example based on an irreducible unitary representation of the \( q \)-deformed Lie algebra \( so_q(3) \) at \( q \) root of unity. Let \( N \) be a positive integer and \( d = 1, 2, \ldots N - 1 \). There is a \( (d+1) \times (d+1) \) irreducible representation of \( so_q(3) \) with \( q = \exp(2i\pi/N) \) given by [25]

\[
K_1 = \frac{1}{2} \sum_{n=0}^{d-1} \frac{\sin \left( \frac{\pi(n+\ell)}{N} \right) \sin \left( \frac{\pi(n-n)}{N} \right)}{\cos \left( \frac{\pi(d-2n-2)}{2N} \right) \cos \left( \frac{\pi(d-2n)}{2N} \right)} \left( | n \rangle \langle n + 1 | + | n + 1 \rangle \langle n | \right), \tag{87}
\]

\[
K_0 = \sum_{n=0}^{d} \sin \left( \frac{\pi(2n-d)}{2N} \right) | n \rangle \langle n |. \tag{88}
\]
We define $K_2 = e^{i\pi/(2N)} K_0 K_1 - e^{-i\pi/(2N)} K_1 K_0$. Then, one gets
\begin{align}
e^{i\pi/(2N)} K_1 K_2 - e^{-i\pi/(2N)} K_2 K_1 &= -\sin^2 \left( \frac{\pi}{N} \right) K_0, \quad (89) \\
e^{i\pi/(2N)} K_2 K_0 - e^{-i\pi/(2N)} K_0 K_2 &= -\sin^2 \left( \frac{\pi}{N} \right) K_1, \quad (90)
\end{align}

thus realizing the defining relations of $\mathfrak{so}_q(3)$ (we have changed the normalisation of the generators $K_i$ for later convenience).

We take for $\hat{H}$
\[ \hat{H}^{so} = K_1 + b, \quad (91) \]
where $b$ is a real constant. This defines a chain with $d + 1$ sites. In view of (3), the couplings of the Hamiltonian $\hat{H}$ defined by (1) are in this case given by
\begin{align}
B_n^{so} &= -b, \quad J_n^{so} = -\frac{1}{2} \sqrt{\frac{\sin \left( \frac{\pi(n+1)}{N} \right) \sin \left( \frac{\pi(d-n)}{N} \right)}{\cos \left( \frac{\pi(d-2n-2)}{2N} \right) \cos \left( \frac{\pi(d-2n)}{2N} \right)}}. \quad (92)
\end{align}

Let us remark that when the number of sites is related to the order of the unity root, i.e. when $d = N - 2$, these reduce to
\begin{align}
B_n^{so} &= -b, \quad J_n^{so} = -\frac{1}{2}. \quad (93)
\end{align}

Hence, the model treated here generalizes the homogeneous chain studied in [13].

Note that the $q$-commutation relations (90) of $\mathfrak{so}_q(3)$ are symmetric under the exchange $K_0 \leftrightarrow K_1$; hence, in the present representation where this permutation is unitarily realized, $K_1$ has the same spectrum as $K_0$. Therefore, $\hat{H}$ given by (91) is diagonalized as follows: for $k = 0, 1, \ldots, d$,
\[ \hat{H}^{so} |\omega_k \rangle = \omega_k |\omega_k \rangle, \quad \omega_k = \sin \left( \frac{\pi}{2} \frac{(2k - d)}{2N} \right) + b. \quad (94) \]

Let us mention that the wavefunctions $\phi_n(\omega_k)$ involve the $q$-ultraspherical polynomials at $q$ a root of unity. It is interesting to realize that the finite Chebychev polynomials that occur in the uniform chain are a special case of these $q$-polynomials.

The operator $\hat{X}$ can be chosen as $\hat{X}^{so} = K_0$, which is diagonal in the position basis with $\lambda_n = \sin \left( \frac{\pi}{2} \frac{(2n - d)}{2N} \right)$. In the momentum basis, this operator $\hat{X}$ is also tridiagonal and reads
\begin{align}
B_k^{so} &= 0, \quad J_k^{so} = \frac{1}{2} \sqrt{\frac{\sin \left( \frac{\pi(k+1)}{N} \right) \sin \left( \frac{\pi(d-k)}{N} \right)}{\cos \left( \frac{\pi(d-2k-2)}{2N} \right) \cos \left( \frac{\pi(d-2k)}{2N} \right)}}. \quad (95)
\end{align}
We conclude that the matrix $T$ is given by (62) with

$$t_n = 2 \cos \left( \frac{\pi}{2N} \right) \frac{\sin \left( \frac{\pi(\ell - n)}{2N} \right) \cos \left( \frac{\pi(\ell + n - d + 1)}{2N} \right)}{\sin \left( \frac{2(n+1)}{N} \right) \sin \left( \frac{2d-n}{N} \right) \cos \left( \frac{2(d-2n-2)}{2N} \right) \cos \left( \frac{2(d-2n+1)}{2N} \right)} ,$$

$$d_n = -2 \cos \left( \frac{\pi}{2N} \right) \left[ b \sin \left( \frac{\pi(2\ell - d + 1)}{2N} \right) + \sin \left( \frac{\pi(2n - d)}{2N} \right) \sin \left( \frac{\pi(2K - d + 1)}{2N} \right) \right] .$$

This coincides with the matrix found in [2] and [13] when $d = N - 2$.

## 10 Concluding remarks

This paper has discussed entanglement in free Fermionic chains and focused in particular on the challenges associated to the diagonalization of the entanglement Hamiltonian. It has underscored in this respect the connection that these studies bear with the classic treatment of time and band limiting in signal processing. This article has illustrated how the methods developed in the latter context can be usefully imported in the entanglement analyses of Fermionic chains. The key feature that has thus been adapted is the existence of a second order differential (or difference) operator that commutes with the non-local limiting operator. In time this remarkable fact has been understood to arise from an underlying bispectral situation, and recently [18] the related algebraic Heun operator construct was seen to lead to these commuting operators. This was reviewed here and was seen to be transposable to the entanglement of Fermionic chains.

The specifications of chains which have a bispectral underpinning have been characterized. Involved are two operators ($\hat{H}$ and $\hat{X}$) which are diagonal in the momentum and position bases respectively and tridiagonal in the other. They define the bispectral problem that the wavefunctions satisfy. Attached to chains of that type are algebraic Heun operators that readily yield a tridiagonal matrix that commutes with the restricted correlation matrix which is the fundamental operator that needs to be diagonalized. It was pointed out that the bispectral operators generate algebraic structures of interest and are connected to orthogonal polynomials. With that perspective, three pairs of bispectral operators were identified from representations of the Lie and q-deformed algebras $su(2)$, $su(1,1)$ and $so_q(3)$. The corresponding free Fermionic chains were introduced and the commuting matrices presented. The first model gave an example of a finite chain, the second of a semi-infinite one and the third based on representations of $so_q(3)$ at $q$ a root of unity offered a one-parameter generalization of the chain with uniform couplings.
A number of interesting questions are pending and deserve further investigations. In all our considerations, the bipartition of the chains has been defined by considering one part as the subset of sites consisting of consecutive nodes starting with the first one. It would obviously be of relevance to extend the approach to other space limiting. Studies of entanglement of Fermions (and Bosons) on different graphs have been undertaken \cite{26, 27}. We plan on examining how the considerations developed in this paper could extend in that context. It would also be nice to carry this out in field theory especially in the Schrodinger representation (see in particular \cite{28}) that Roman Jackiw has at times advocated \cite{29, 30}.

**Acknowledgements** The authors are grateful to the editors for the invitation to contribute to this volume in honour of Roman Jackiw. N. Crampé warmly thanks the Centre de Recherches Mathématiques (CRM) for hospitality and support during his visit to Montreal in the course of this investigation. The research of L. Vinet is supported in part by a Discovery Grant from the Natural Science and Engineering Research Council (NSERC) of Canada.

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