Weak order in averaging principle for stochastic differential equations with jumps

Bengong Zhang · Hongbo Fu · Li Wan · Jicheng Liu

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Abstract The present article deals with the averaging principle for a two-time-scale system of jump-diffusion stochastic differential equation. Under suitable conditions, the weak error is expanded in powers of timescale parameter. It is proved that the rate of weak convergence to the averaged dynamics is of order 1. This reveals the rate of weak convergence is essentially twice that of strong convergence.

Keywords Jump-diffusion · averaging principle · invariant measure · weak convergence · asymptotic expansion

Mathematics Subject Classification (2000) 60H10 · 70K70

1 Introduction

We consider a two-time-scale system of jump-diffusion stochastic differential equation in form of

\[ dX^\varepsilon_t = a(X^\varepsilon_t, Y^\varepsilon_t)dt + b(X^\varepsilon_t)dB_t + c(X^\varepsilon_t-)dP_t, \quad X^\varepsilon_0 = x, \]  

(1.1)

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\[ dy^\varepsilon_t = \frac{1}{\varepsilon} f(x^\varepsilon_t, y^\varepsilon_t) dt + \frac{1}{\sqrt{\varepsilon}} g(x^\varepsilon_t, y^\varepsilon_t) dW_t + h(x^\varepsilon_t, y^\varepsilon_t) dN^\varepsilon_t, 
\]
\[ y^\varepsilon_0 = y, \quad (1.2) \]

where \( X^\varepsilon_t \in \mathbb{R}^n, Y^\varepsilon_t \in \mathbb{R}^m \), the drift functions \( a(x, y) \in \mathbb{R}^n, f(x, y) \in \mathbb{R}^m \), the diffusion functions \( b(x) \in \mathbb{R}^{n \times d_1}, c(x) \in \mathbb{R}^n, g(x, y) \in \mathbb{R}^{m \times d_2} \) and \( h(x, y) \in \mathbb{R}^m \). \( B_t \) and \( W_t \) are the vectors of \( d_1, d_2 \)-dimensional independent Brownian motions on a complete stochastic base \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \), respectively. \( P_t \) is a scalar simple Poisson process with intensity \( \lambda_1 \), and \( N^\varepsilon_t \) is a scalar simple Poisson process with intensity \( \lambda^\varepsilon_2 \). The positive parameter \( \varepsilon \) is small and describes the ratio of time scales between \( X^\varepsilon_t \) and \( Y^\varepsilon_t \). Systems (1.1)-(1.2) with two time scales occur frequently in applications including chemical kinetics, signal processing, complex fluids and financial engineering.

With the separation of time scale, we can view the state variable of the system as being divided into two parts: the “slow” variable \( X^\varepsilon_t \) and the “fast” variable \( Y^\varepsilon_t \). It is often the case that we are interested only in the dynamics of slow variable and possesses the essential features of the system, is highly desirable. Such a simplified equation is often constructed by averaging procedure as in [2, 20] for deterministic ordinary differential equations, as well as the further development [13, 10, 14, 15, 25] for stochastic differential equations with continuous Gaussian processes. As far as averaging for stochastic dynamical systems in infinite dimensional space is concerned, it is worthwhile to quote the important works of [3, 5, 26] and also the works of [9, 10, 21]. For related works on averaging for multivalued stochastic differential equations we refer the reader to [12, 22].

In order to derive the averaged dynamics of the system (1.1)-(1.2), we introduce the fast motion equation with a frozen slow component \( x \in \mathbb{R}^n \) in form of

\[ dy^x_t = f(x, y^x_t) dt + g(x, y^x_t) dW_t + h(x, y^x_t) dN_t, 
\]
\[ y^x_0 = y, \quad (1.3) \]

whose solution is denoted by \( Y^x_t(y) \). Under suitable conditions on \( f, g \) and \( h \), \( Y^x_t(y) \) induces a unique invariant measure \( \mu^x(dy) \) on \( \mathbb{R}^m \), which is ergodic and ensures the averaged equation:

\[ d\bar{X}_t = \bar{a}(\bar{X}_t) dt + b(\bar{X}_t) dB_t + c(\bar{X}_t) dP, \quad \bar{X}_0 = x, \]

where the averaging nonlinearity is defined by setting

\[ \bar{a}(x) = \int_{\mathbb{R}^m} a(x, y) \mu^x(dy) = \lim_{t \to +\infty} \mathbb{E}a(x, Y^x_t(y)). \]

In [11], it was shown that under the above conditions the slow motion \( X^\varepsilon_t \) converges strongly to the solution \( \bar{X}_t \) of the above averaged equation with jumps. The order of convergence \( \frac{1}{2} \) in strong sense was provided in [17]. To our best knowledge, there is no existing literature to address the weak order in averaging principle for jump diffusion stochastic differential systems. In fact,
it is fair to say that the weak convergence in stochastic averaging theory of systems driven by jump noise is not fully developed yet, although some strong approximation results on the rate of strong convergence were obtained [1,23,24].

Therefore, we aim to study this problem in this paper. Here we are interested in the rate of weak convergence of the averaging dynamics to the true solution of slow motion $X^\epsilon_t$. In other word, we will determine the order, with respect to timescale parameter $\epsilon$, of weak deviation between original solution to slow equation and the solution of the corresponding averaged equation. The main technique we adapted is to find an expansion with respect to $\epsilon$ of the solutions of the Kolmogorov equations associated with the jump diffusion system. The solvability of the Poisson equation associated with the generator of frozen equation provides an expression for the coefficients of the expansion. As a result, the boundedness for the coefficients of expansion can be proved by smoothing effect of the corresponding transition semigroup in the space of bounded and uniformly continuous functions, where some regular conditions is needed on drift and diffusion term.

Our result shows that the weak convergence rate to be 1 even when there are jump components in the system. It is the main contribution of this work. We would like to stress that asymptotic method was first applied by Bréhier [3] to an averaging result for stochastic reaction-diffusion equations in the case of Gaussian noise of additive type was included only in the fast motion. However, the extension of this argument is not straightforward. The method used in the proof of weak order in [3] is strictly related to the differentiability in time of averaged process. Therefore, once the noise is introduced in the slow equation, difficulties will arise and the procedure becomes more complicated. Our result in this paper bridges such a gap, in which the slow and the fast motions are both perturbed by noise with jumps.

The rest of the paper is structured as follows. Section 2 is devoted to notations, assumptions and summarize preliminary results. The ergodicity of fast process and the averaged dynamics of system with jumps is introduced in Section 3. Then the main result of this article, which is derived via the asymptotic expansions and uniform error estimates, is presented in Section 4. Finally, we give the appendix in section 5.

It should be pointed out that the letter $C$ below with or without subscripts will denote generic positive constants independent of $\epsilon$ in the whole paper.

### 2 Assumptions and preliminary results

For any integer $d$, the scalar product and norm on $d$–dimensional Euclidean space $\mathbb{R}^d$ are denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ and $\| \cdot \|_{\mathbb{R}^d}$, respectively. For any integer $k$, we denote by $C^k_b(\mathbb{R}^d, \mathbb{R})$ the space of all $k$–times differentiable functions on $\mathbb{R}^d$, which have bounded uniformly continuous derivatives up to the $k$-th order.

In what follows, we shall assume that the drift and diffusion coefficients arising in the system fulfill the following conditions.
where \( L \) and \( y \) verge to its equilibrium state.

phasor the dependence on the initial data, is denoted by 
\( (X \in y) \)

condition and it is assumed in order to have the regularizing effect of the 
\( b, c, f, g \)

have

Lemma 2.1

directional derivatives exist and are controlled:

\[
\|D_x a(x, y) \cdot k_1\|_{\mathbb{R}^n} \leq L\|k_1\|_{\mathbb{R}^n},
\]

\[
\|D_y a(x, y) \cdot l_1\|_{\mathbb{R}^n} \leq L\|l_1\|_{\mathbb{R}^m},
\]

\[
\|D_{xx} a(x, y) \cdot (k_1, k_2)\|_{\mathbb{R}^n} \leq L\|k_1\|_{\mathbb{R}^n}\|k_2\|_{\mathbb{R}^n},
\]

\[
\|D_{yy} a(x, y) \cdot (l_1, l_2)\|_{\mathbb{R}^n} \leq L\|l_1\|_{\mathbb{R}^m}\|l_2\|_{\mathbb{R}^m},
\]

where \( L \) is a constant independent of \( x, y, k_1, k_2, l_1 \) and \( l_2 \). For differentiability of mappings \( b, c, f, g \) and \( h \) we possess the analogous results. For examples, we have

\[
\|D_{xx} b(x) \cdot (k_1, k_2)\|_{\mathbb{R}^n} \leq L\|k_1\|_{\mathbb{R}^n}\|k_2\|_{\mathbb{R}^n}, \quad k_1, k_2 \in \mathbb{R}^n,
\]

\[
\|D_{xy} f(x, y) \cdot (l_1, l_2)\|_{\mathbb{R}^n} \leq L\|l_1\|_{\mathbb{R}^m}\|l_2\|_{\mathbb{R}^m}, \quad l_1, l_2 \in \mathbb{R}^m.
\]

Remark 2.1 Notice that from (A1) it immediately follows that the following directional derivatives exist and are controlled:

As far as the assumption (A2) is concerned, it is a sort of non-degeneracy condition and it is assumed in order to have the regularizing effect of the Markov transition semigroup associated with the fast dynamics. Assumption (A3) is the dissipative condition which determines how the fast equation converges to its equilibrium state.

As assumption (A1) holds, for any \( \epsilon > 0 \) and any initial conditions \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \), system (1.1)-(1.2) admits a unique solution, which, in order to emphasize the dependence on the initial data, is denoted by \((X^\epsilon_t(x, y), Y^\epsilon_t(x, y))\). Moreover the following lemma holds (for a proof see e.g. [14]).

**Lemma 2.1** Under the assumptions (A1), (A2) and (A3), for any \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and \( \epsilon > 0 \) we have

\[
\mathbb{E}\|X^\epsilon_t(x, y)\|_{\mathbb{R}^n} \leq C_T(1 + \|x\|_{\mathbb{R}^n}^2 + \|y\|_{\mathbb{R}^n}^2), \quad t \in [0, T]
\]

and

\[
\mathbb{E}\|Y^\epsilon_t(x, y)\|_{\mathbb{R}^n} \leq C_T(1 + \|x\|_{\mathbb{R}^n}^2 + \|y\|_{\mathbb{R}^m}^2), \quad t \in [0, T].
\]
3 Frozen equation and averaged equation

Fixing \( \epsilon = 1 \), we consider the fast equation with frozen slow component \( x \in \mathbb{R}^n \),

\[
\begin{aligned}
    dY_t^x(y) &= f(x, Y_t^x(y))dt + g(x, Y_t^x(y))dW_t + h(x, Y_t^x(y))dN_t, \\
    Y_0^x &= y.
\end{aligned}
\] (3.1)

Under assumptions (A1)-(A3), such a problem has a unique solution, which satisfies [17]:

\[
E \|Y_t^{x}(y)\|_{\mathbb{R}^m}^2 \leq C(1 + \|x\|_{\mathbb{R}^n}^2 + e^{-\beta t} \|y\|_{\mathbb{R}^m}^2), \ t \geq 0. \tag{3.2}
\]

Let \( Y_t^{x}(y') \) be the solution of problem (3.1) with initial value \( Y_0^{x} = y' \), the Itô formula implies that for any \( t \geq 0 \),

\[
E \|Y_t^{x}(y) - Y_t^{x}(y')\|_{\mathbb{R}^m}^2 \leq \|y - y'\|_{\mathbb{R}^m}^2 e^{-\beta t}. \tag{3.3}
\]

Moreover, as discussion in [17] and [11], equation (3.1) admits a unique ergodic invariant measure \( \mu^{x} \) satisfying

\[
\int_{\mathbb{R}^m} \|y\|_{\mathbb{R}^m}^2 \mu^{x}(dy) \leq C(1 + \|x\|_{\mathbb{R}^n}^2). \tag{3.4}
\]

Then, by averaging the coefficient \( a \) with respect to the invariant measure \( \mu^{x} \), we can define an \( \mathbb{R}^n \)-valued mapping

\[
\bar{a}(x) := \int_{\mathbb{R}^m} a(x, y) \mu^{x}(dy), x \in \mathbb{R}^n.
\]

Due to assumption (A1), it is easily to check that \( \bar{a}(x) \) is 2-times differentiable with bounded derivatives, and hence it is Lipschitz-continuous such that

\[
\|\bar{a}(x_1) - \bar{a}(x_2)\|_{\mathbb{R}^n} \leq C\|x_1 - x_2\|_{\mathbb{R}^n}, \ x_1, x_2 \in \mathbb{R}^n.
\]

According to invariant property of \( \mu^{x} \), (3.4) and assumption (A1), we have

\[
\begin{aligned}
\|\mathbb{E}a(x, Y_t^{x}(y)) - \bar{a}(x)\|_{\mathbb{R}^n}^2 &= \|\int_{\mathbb{R}^m} \mathbb{E}(a(x, Y_t^{x}(y)) - a(x, Y_t^{x}(z)))\mu^{x}(dz)\|_{\mathbb{R}^n}^2 \\
&\leq \int_{\mathbb{R}^m} \mathbb{E}\|Y_t^{x}(y) - Y_t^{x}(z)\|_{\mathbb{R}^m}^2 \mu^{x}(dz) \\
&\leq e^{-\beta t} \int_{\mathbb{R}^m} \|y - z\|_{\mathbb{R}^m}^2 \mu^{x}(dz) \\
&\leq Ce^{-\beta t}(1 + \|x\|_{\mathbb{R}^n}^2 + \|y\|_{\mathbb{R}^m}^2). \tag{3.5}
\end{aligned}
\]

Now we can introduce the effective dynamical system

\[
\begin{aligned}
    d\bar{X}_t(x) &= \bar{a}(\bar{X}_t(x))dt + b(\bar{X}_t(x))dB_t + c(\bar{X}_t(x))dP_t, \\
    \bar{X}_0 &= x.
\end{aligned}
\] (3.6)
As the coefficients $\bar{a}, b$ and $c$ are Lipschitz-continuous, this equation admits a unique solution such that

$$\mathbb{E}\|\bar{X}_t(x)\|_{\mathbb{R}^n}^2 \leq C_T (1 + \|x\|_{\mathbb{R}^n}^2), \quad t \in [0, T].$$

(3.7)

With the above assumptions and notations we have the following result, which is a direct consequence of Lemma 4.1, Lemma 4.2 and Lemma 4.5.

**Theorem 3.1** Assume that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then, under assumptions $(A1)$, $(A2)$ and $(A3)$, for any $T > 0$ and $\phi \in C^3_b(\mathbb{R}^n, \mathbb{R})$, there exists a constant $C_{T, \phi, x, y}$ such that

$$|\mathbb{E}\phi(X_t^\epsilon(x, y)) - \mathbb{E}\phi(X_T(x))| \leq C_{T, \phi, x, y} \epsilon.$$

As a consequence, it can be claimed that the weak order in averaging principle for jump-diffusion stochastic systems is 1.

4 Asymptotic expansion

Let $\phi \in C^3_b(\mathbb{R}^n, \mathbb{R})$ and define a function $u'(t, x, y) : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$u'(t, x, y) = \mathbb{E}\phi(X_t^\epsilon(x, y)).$$

We are now ready to seek an expansion formula for $u'(t, x, y)$ with respect to $\epsilon$ with the form

$$u'(t, x, y) = u_0(t, x, y) + \epsilon u_1(t, x, y) + r'(t, x, y),$$

(4.1)

where $u_0$ and $u_1$ are smooth functions which will be constructed below, and $r'$ is the remainder term. To this end, let us recall the Kolmogorov operator corresponding to the slow motion equation, with a frozen fast component $y \in \mathbb{R}^m$, which is a second order operator taking form

$$L_1 \Phi(x) = \left( a(x, y), D_x \Phi(x) \right)_{\mathbb{R}^n} + \frac{1}{2} Tr \left[ D^2_{xx} \Phi(x) \cdot b(x) b^T(x) \right] + \lambda_1 (\Phi(x + c(x)) - \Phi(x)), \quad \Phi \in C^2_b(\mathbb{R}^n, \mathbb{R}).$$

For any frozen slow component $x \in \mathbb{R}^m$, the Kolmogorov operator for equation (3.1) is given by

$$L_2 \Psi(y) = \left( f(x, y), D_y \Psi(y) \right)_{\mathbb{R}^m} + \frac{1}{2} Tr \left[ D^2_{yy} \Psi(y) \cdot g(x, y) g^T(x, y) \right] + \lambda_2 (\Psi(y + h(x, y)) - \Psi(y)), \quad \Psi \in C^2_b(\mathbb{R}^m, \mathbb{R}).$$

We set

$$L' := L_1 + \frac{1}{\epsilon} L_2.$$
It is known $u^\epsilon(t, x, y)$ solves the equation
\[
\begin{aligned}
\frac{\partial}{\partial t} u^\epsilon(t, x, y) &= \mathcal{L}^\epsilon u^\epsilon(t, x, y), \\
u^\epsilon(0, x, y) &= \phi(x),
\end{aligned}
\tag{4.2}
\]
Also recall the Kolmogorov operator associated with the averaged equation (3.6) is defined as
\[
\mathcal{L}\phi(x) = \left(\bar{a}(x), D_x \phi(x)\right)_{\mathbb{R}^n} + \frac{1}{2} Tr \left[D_x^2 \phi(x) \cdot b(x) b^T(x)\right]
+ \lambda_1 (\phi(x + c(x)) - \Phi(x)), \quad \Phi \in C_b^2(\mathbb{R}^n, \mathbb{R}).
\]
If we set
\[
\bar{u}(t, x) = \mathbb{E}\phi(\bar{X}_t(x)),
\]
we have
\[
\begin{aligned}
\frac{\partial}{\partial t} \bar{u}(t, x) &= \mathcal{L}\bar{u}(t, x), \\
\bar{u}(0, x) &= \phi(x).
\end{aligned}
\tag{4.3}
\]

4.1 The leading term

Let us begin with constructing the leading term. By substituting expansion (4.1) into (4.2), we see that
\[
\frac{\partial u_0}{\partial t} + \epsilon \frac{\partial u_1}{\partial t} + \frac{\partial r^\epsilon}{\partial t} = \mathcal{L}_1 u_0 + \epsilon \mathcal{L}_1 u_1 + \mathcal{L}_1 r^\epsilon
+ \frac{1}{\epsilon} \mathcal{L}_2 u_0 + \mathcal{L}_2 u_1 + \frac{1}{\epsilon} \mathcal{L}_2 r^\epsilon.
\]
By equating powers of $\epsilon$, we obtain the following system of equations:
\[
\mathcal{L}_2 u_0 = 0, \tag{4.4}
\]
\[
\frac{\partial u_0}{\partial t} = \mathcal{L}_1 u_0 + \mathcal{L}_2 u_1. \tag{4.5}
\]
According to (4.4), we can conclude $u_0$ does not depend on $y$, that is
\[
u_0(t, x, y) = u_0(t, x).
\]
We also impose the initial condition $u_0(0, x) = \phi(x)$. Note that $\mathcal{L}_2$ is the generator of a Markov process defined by equation (3.1), which admits a unique invariant measure $\mu^\epsilon$, we have
\[
\int_{\mathbb{R}^m} \mathcal{L}_2 u_1(t, x, y) \mu^\epsilon(dy) = 0. \tag{4.6}
\]
Thanks to (4.5), this yields
\[
\frac{\partial u_0}{\partial t}(t, x) = \int_{\mathbb{R}^m} \frac{\partial u_0}{\partial t}(t, x) \mu^\epsilon(dy)
\]
\[
\begin{align*}
\int_{\mathbb{R}^m} \mathcal{L}_1 u_0(t, x) \mu^x(dy) \\
= \int_{\mathbb{R}^m} \left( a(x, y), D_x u_0(t, x) \right) \mu^x(dy) \\
+ \frac{1}{2} \text{Tr} \left[ D_{xx}^2 u_0(t, x) \cdot b(x)b^T(x) \right] \\
+ \lambda_1 (u_0(x + c(x)) - u_0(x)) \\
= \mathcal{L} u_0(t, x),
\end{align*}
\]

so that \( u_0 \) and \( \bar{u} \) are described by the same evolutionary equation. By uniqueness argument, we easily have the following lemma:

**Lemma 4.1** Under assumptions \((A1), (A2)\) and \((A3)\), for any \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and \( T > 0 \), we have \( u_0(T, x, y) = \bar{u}(T, x) \).

### 4.2 Construction of \( u_1 \)

According to Lemma 4.1, (4.3) and (4.5), we get

\[
\begin{align*}
\mathcal{L} \bar{u} &= \mathcal{L}_1 \bar{u} + \mathcal{L}_2 u_1, \\
\mathcal{L}_2 u_1(t, x, y) &= \left( \bar{a}(x) - a(x, y), D_x \bar{u}(t, x) \right)_{\mathbb{R}^n} \\
&= -\rho(t, x, y), \quad (4.7)
\end{align*}
\]

where \( \rho \) is of class \( C^2 \) with respect to \( y \), with uniformly bounded derivatives. Moreover, for any \( t \geq 0 \) and \( x \in \mathbb{R}^n \), the equality \( \mathcal{L} \bar{u} \) guarantees that

\[
\int_{\mathbb{R}^m} \rho(t, x, y) \mu^x(dy) = 0.
\]

For any \( y \in \mathbb{R}^m \) and \( s > 0 \) we have

\[
\frac{\partial}{\partial s} \mathcal{P}_s \rho(t, x, y) = \left( f(x, y), D_y[\mathcal{P}_s \rho(t, x, y)] \right)_{\mathbb{R}^m} \\
+ \frac{1}{2} \text{Tr} \left[ D_{yy}^2 \mathcal{P}_s \rho(t, x, y) \cdot g(x, y)g^T(x, y) \right] \\
+ \lambda_2 \left( \mathcal{P}_s [\rho(t, x, y + h(x, y))] - \mathcal{P}_s [\rho(t, x, y)] \right), \quad (4.8)
\]

here

\[
\mathcal{P}_s [\rho(t, x, y)] := E \rho(t, x, Y_{s}^x(y)).
\]

Recalling that \( \mu^x \) is the unique invariant measure corresponding to Markov process \( Y_{s}^x(y) \) defined by equation (3.1), from Lemma 4.1 we infer that

\[
\left| E \rho(t, x, Y_{s}^x(y)) - \int_{\mathbb{R}^m} \rho(t, x, z) \mu^x(dz) \right|
\]
\[
\int_{\mathbb{R}^m} E[\rho(t, x, Y^x_s(y)) - \rho(t, x, Y^x_s(z))] \mu^x(dz) \\
\leq \int_{\mathbb{R}^m} E \left( a(x, Y^x_s(z)) - a(x, Y^x_s(y)), D_x \bar{u}(t, x) \right) \mu^x(dz) \\
\leq C \int_{\mathbb{R}^m} E[|Y^x_s(z) - Y^x_s(y)|] \mu^x(dz).
\]

Now it follows from (3.3) and (3.4) that
\[
\int_{\mathbb{R}^m} E[\rho(t, x, Y^x_s(y)) - \int_{\mathbb{R}^m} \rho(t, x, z) \mu^x(dz)] \\
\leq C(1 + ||x||_{\mathbb{R}^n} + ||y||_{\mathbb{R}^m}) e^{-\frac{\beta}{2} s},
\]
which implies
\[
\lim_{s \to +\infty} E\rho(t, x, Y^x_s(y)) = \int_{\mathbb{R}^m} \rho(t, x, z) \mu^x(dz) = 0.
\]

With the aid of the above limit, we can deduce from (4.8) that
\[
\left( f(x, y), D_y \int_0^{+\infty} [\mathcal{P}_s \rho(t, x, y)] ds \right)_{\mathbb{R}^m} \\
+ \frac{1}{2} Tr \left[ D^2_{yy} \int_0^{+\infty} [\mathcal{P}_s \rho(t, x, y)] \cdot g(x, y) g^T(x, y) ds \right] \\
+ \lambda_2 \left( \int_0^{+\infty} \mathcal{P}_s [\rho(t, x + h(x, y))] ds - \int_0^{+\infty} \mathcal{P}_s [\rho(t, x, y)] ds \right) \\
= \int_0^{+\infty} \frac{\partial}{\partial s} \mathcal{P}_s [\rho(t, x, y)] ds \\
= \lim_{s \to +\infty} \mathcal{P}_s [\rho(t, x, Y^x_s(y))] - \rho(t, x, y) \\
= \int_{\mathbb{R}^m} \rho(t, x, z) \mu^x(dz) - \rho(t, x, y) \\
= -\rho(t, x, y),
\]
which implies
\[
\mathcal{L}_2 \left( \int_0^{+\infty} \mathcal{P}_s \rho(t, x, y) ds \right) = -\rho(t, x, y).
\]

Therefore,
\[
u_1(t, x, y) := \int_0^{+\infty} \mathcal{P}_s \rho(t, x, Y^x_s(y)) ds \quad (4.9)
\]
is the solution to equation (4.7).

**Lemma 4.2** Under assumptions (A1), (A2) and (A3), for any \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and \( T > 0 \), we have
\[
|\nu_1(t, x, y)| \leq C_T (1 + ||x||_{\mathbb{R}^n} + ||y||_{\mathbb{R}^m}), \quad t \in [0, T]. \quad (4.10)
\]
Proof By (4.9), we have
\[ u_1(t, x, y) = \int_0^{+\infty} \mathbb{E}\left( \bar{a}(x) - a(x, Y^x_s(y)), D_x \bar{u}(t, x) \right) ds, \]
so that
\[ |u_1(t, x, y)| \leq \int_0^{+\infty} \| \bar{a}(x) - \mathbb{E}[a(x, Y^x_s(y))] \|_{\mathbb{R}^n} \cdot \| D_x \bar{u}(t, x) \|_{\mathbb{R}^n} ds. \]
Therefore, from Lemma 5.5 and (3.5), we get
\[ |u_1(t, x, y)| \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}) \int_0^{+\infty} e^{-\frac{s}{2}} ds \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}). \]

4.3 Determination of remainder \( r^\epsilon \)

We now turn to the construction for remainder term \( r^\epsilon \). It is known that
\[ (\partial_t - L^\epsilon) u^\epsilon = 0, \]
which, together with (4.4) and (4.5), implies
\[ (\partial_t - L^\epsilon) r^\epsilon = -(\partial_t - L^\epsilon) u_0 - \epsilon(\partial_t - L^\epsilon) u_1 \]
\[ = -(\partial_t - \frac{1}{\epsilon} L_2 - L_1) u_0 - \epsilon(\partial_t - \frac{1}{\epsilon} L_2 - L_1) u_1 \]
\[ = \epsilon(L_1 u_1 - \partial_t u_1). \]

(4.11)

In order to estimate the remainder term \( r^\epsilon \) we need the following two lemmas.

Lemma 4.3 Under assumptions (A1), (A2) and (A3), for any \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \) and \( T > 0 \), we have
\[ \left| \frac{\partial u_1}{\partial t} (t, x, y) \right| \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}). \]

Proof In view of (11), we get
\[ \frac{\partial u_1}{\partial t} (t, x, y) = \int_0^{+\infty} \mathbb{E}\left( \bar{a}(x) - a(x, Y^x_s(y)), \frac{\partial}{\partial t} D_x \bar{u}(t, x) \right) ds. \]

By Lemma 5.6 introduced in Section 5, we have
\[ \left| \frac{\partial u_1}{\partial t} (t, x, y) \right| \leq \int_0^{+\infty} \mathbb{E}\left( \| \bar{a}(x) - a(x, Y^x_s(y)) \|_{\mathbb{R}^n} \cdot \| D_x \bar{u}(t, x) \|_{\mathbb{R}^n} \right) ds \]
\[ \leq C_T \int_0^{+\infty} \mathbb{E}\| \bar{a}(x) - a(x, Y^x_s(y)) \|_{\mathbb{R}^n} ds, \]
so that from (4.6), we have
\[ \left| \frac{\partial u_1}{\partial t} (t, x, y) \right| \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}). \]
Lemma 4.4 Under assumptions (A1), (A2) and (A3), for any \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \) and \( T > 0 \), we have

\[
|L_1 u_1(t, x, y)| \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}), \quad t \in [0, T].
\]

Proof Recalling that \( u_1(t, x, y) \) is the solution of equation (4.7) and equality (4.9) holds, we have

\[
L_1 u_1(t, x, y) = \left( a(x, y), D_x u_1(t, x, y) \right)_{\mathbb{R}^n} + \frac{1}{2} \text{Tr} D^2_{xx} u_1(t, x, y) \cdot b(x) b^T(x) + \lambda [u_1(t, x + c(x), y) - u_1(t, x, y)],
\]

and then, in order to prove the boundedness of \( L_1 u_1 \), we have to estimate the three terms arising in the right hand side of above equality.

**Step 1:** Estimate of \( (a(x, y), D_x u_1(t, x, y))_{\mathbb{R}^n} \).

For any \( k \in \mathbb{R}^n \), we have

\[
D_x u_1(t, x, y) \cdot k = \int_0^{+\infty} \left( \frac{D_x (\bar{a}(x) - \mathbb{E} a(x, Y^x_s(y))) \cdot k, D_x \bar{u}(t, x)}{\mathbb{R}^n} \right) ds + \int_0^{+\infty} \left( \frac{\bar{a}(x) - \mathbb{E} a(x, Y^x_s(y)), D^2_{xx} \bar{u}(t, x) \cdot k}{\mathbb{R}^n} \right) ds =: I_1(t, x, y, k) + I_2(t, x, y, k).
\]

By Lemma 5.1 and 5.4 we infer that

\[
|I_1(t, x, y, k)| \leq \|D_x \bar{u}(t, x)\|_{\mathbb{R}^n} \int_0^{+\infty} \|D_x (\bar{a}(x) - \mathbb{E} a(x, Y^x_s(y))) \cdot k\|_{\mathbb{R}^n} ds
\]

\[
\leq C_T \|k\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}) \int_0^{+\infty} e^{-\frac{t}{2}} ds
\]

\[
\leq C_T \|k\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}).
\]

By Lemma 5.2 and inequality (3.9), we obtain

\[
|I_2(t, x, y, k)| \leq C_T \|k\|_{\mathbb{R}^n} \int_0^{+\infty} \|\bar{a}(x) - \mathbb{E} a(x, Y^x_s(y))\|_{\mathbb{R}^n} ds
\]

\[
\leq C_T \|k\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}) \int_0^{+\infty} e^{-\frac{t}{2}} ds
\]

\[
\leq C_T \|k\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}).
\]

This, together with (4.10), implies

\[
\|D_x u_1(t, x, y) \cdot k\| \leq C_T \|k\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}),
\]

and then, as \( a(x, y) \) is bounded, it follows

\[
\left| \left( a(x, y), D_x u_1(t, x, y) \right)_{\mathbb{R}^n} \right| \leq C_T \|a(x, y)\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m})
\]
With a similar argument we can also show that

\[ \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}). \]

**Step 2:** Estimate of \( \text{Tr} \left[ D_{xx}^2 u_1(t, x, y) \cdot b(x)b^T(x) \right] \).

Since \( u_1(t, x, y) \) is given by the representation formula, for any \( k_1, k_2 \in \mathbb{R}^n \) we have

\[
D_{xx}^2 u_1(t, x, y) \cdot (k_1, k_2) \\
= \int_0^{+\infty} \mathbb{E} \left( D_{xx}^2 (\tilde{a}(x) - a(x, y_s^x(y))) \cdot (k_1, k_2), D_x \tilde{a}(t, x) \right)_{\mathbb{R}^n} ds \\
+ \int_0^{+\infty} \mathbb{E} \left( D_x (\tilde{a}(x) - a(x, y_s^x(y))) \cdot k_1, D_{xx}^2 \tilde{a}(t, x) \cdot k_2 \right)_{\mathbb{R}^n} ds \\
+ \int_0^{+\infty} \mathbb{E} \left( D_x (\tilde{a}(x) - a(x, y_s^x(y))) \cdot k_2, D_{xx}^2 \tilde{a}(t, x) \cdot k_1 \right)_{\mathbb{R}^n} ds \\
+ \int_0^{+\infty} \mathbb{E} \left( \tilde{a}(x) - a(x, y_s^x(y)), D_{xx}^3 \tilde{a}(t, x) \cdot (k_1, k_2) \right)_{\mathbb{R}^n} ds \\
:= \sum_{i=1}^4 J_i(t, x, y, k_1, k_2).
\]

Thanks to Lemma 5.7 and Lemma 5.5 we get

\[
|J_1(t, x, y, k_1, k_2)| \\
\leq \int_0^{+\infty} \left| \mathbb{E} \left( D_{xx}^2 (\tilde{a}(x) - a(x, y_s^x(y))) \cdot (k_1, k_2), D_x \tilde{a}(t, x) \right)_{\mathbb{R}^n} \right| ds \\
\leq C_T \left( 1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m} \right) \|k_1\|_{\mathbb{R}^n} \|k_2\|_{\mathbb{R}^n} \int_0^{+\infty} e^{-\frac{s}{T^*}} ds \\
\leq C_T \left( 1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m} \right) \|k_1\|_{\mathbb{R}^n} \|k_2\|_{\mathbb{R}^n}. \tag{4.14}
\]

By Lemma 5.4 and 3.3 we infer that

\[
|J_2(t, x, y, k_1, k_2)| \\
\leq \int_0^{+\infty} \left| \mathbb{E} \left( D_x (\tilde{a}(x) - a(x, y_s^x(y))) \cdot k_1, D_{xx}^2 \tilde{a}(t, x) \cdot k_2 \right)_{\mathbb{R}^n} \right| ds \\
\leq C_T \left( 1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m} \right) \|k_1\|_{\mathbb{R}^n} \|k_2\|_{\mathbb{R}^n} \int_0^{+\infty} e^{-\frac{s}{T^*}} ds \\
\leq C_T \left( 1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m} \right) \|k_1\|_{\mathbb{R}^n} \|k_2\|_{\mathbb{R}^n}. \tag{4.15}
\]

With a similar argument we can also show that

\[
|J_3(t, x, y, k_1, k_2)| \\
\leq C_T \left( 1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m} \right) \|k_1\|_{\mathbb{R}^n} \|k_2\|_{\mathbb{R}^n}. \tag{4.16}
\]

By making use of Lemma 7.3 and 3.3, we get

\[
|J_4(t, x, y, k_1, k_2)|
\]
\[ \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n} \cdot (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^n}) \int_0^{+\infty} e^{-\frac{t}{2}} ds \]

\[ \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^n}). \quad (4.17) \]

In view of the above estimates (4.13), (4.15), (4.16) and (4.17), we can conclude that there exists a constant \( C_T \) such that

\[ |D^2_{xx} u_1(t, x, y) \cdot (k_1, k_2)| \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^n}), \quad t \in [0, T], \]

which means that for fixed \( y \in \mathbb{R}^m \) and \( t \in [0, T] \),

\[ \|D^2_{xx} u_1(t, x, y)\|_{L(\mathbb{R}^n, \mathbb{R})} \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}), \]

where \( \| \cdot \|_{L(\mathbb{R}^n, \mathbb{R})} \) denotes the usual operator norm on Banach space consisting of bounded and linear operators from \( \mathbb{R}^n \) to \( \mathbb{R} \). As the diffusion function \( g \) is bounded, we get

\[ \text{Tr}\left(D^2_{xx} u_1(t, x, y)gg^T\right) \leq C_T \|D^2_{xx} u_1(t, x, y)\|_{L(\mathbb{R}^n, \mathbb{R})} \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}). \]

**Step 3:** Estimate of \( \lambda_1[u_1(t, x + c(x), y) - u_1(t, x, y)] \).

By Lemma 4.2 and boundedness condition of \( c(x) \), we directly have

\[ |\lambda_1[u_1(t, x + c(x), y) - u_1(t, x, y)]| \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}), \quad t \in [0, T]. \]

Finally, it is now easy to gather all previous estimates for terms in (4.12) and conclude

\[ |L_1 u_1(t, x, y)| \leq C_T (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}), \quad t \in [0, T]. \]

**Lemma 4.5** Under the conditions of Lemma 4.3 for any \( T > 0, x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \), we have

\[ |r^T(T, x, y)| \leq C_T \epsilon (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}). \]

**Proof** By a variation of constant formula, we write the equation (4.11) in its integral form

\[ r^T(T, x, y) = E[r^T(0, X^T_T(x, y), Y^T_T(x, y)) | r^T(0, x, y)] \]

\[ + \epsilon \int_0^T E(\mathcal{L}_1 u_1 - \frac{\partial u_1}{\partial s})(s, X^T_{T-s}(x, y), Y^T_{T-s}(x, y)) ds. \]

Since \( u^c \) and \( \bar{u} \) satisfy the same initial condition, we have

\[ |r^T(0, x, y)| = |u^c(0, x, y) - \bar{u}(0, x) - cu_1(0, x, y)| \]

\[ = \epsilon |u_1(0, x, y)|, \]
so that, thanks to (4.10), (2.1) and (2.2) we have
\[ E[r^\epsilon(0, X_T^\epsilon(x, y), Y_T^\epsilon(x, y)) \leq C(1 + \|x\|_{R^n} + \|y\|_{R^m}). \] (4.18)

Using Lemma 4.3 and Lemma 4.4 yields
\[ E[(L_1 u_1 - \frac{\partial u_1}{\partial s})(s, X_{T-s}^\epsilon(x, y), Y_{T-s}^\epsilon(x, y))] \leq CE(1 + \|X_{T-s}^\epsilon(x, y)\| + \|Y_{T-s}^\epsilon(x, y)\|), \]
and, according to (2.1) and (2.2), this implies that
\[ E[\int_0^T (L_1 u_1 - \frac{\partial u_1}{\partial s})(s, X_{T-s}^\epsilon(x, y), Y_{T-s}^\epsilon(x, y)) ds] \leq CT(1 + \|x\|_{R^n} + \|y\|_{R^m}). \]
The last inequality together with (4.18) yields
\[ |r^\epsilon(T, x, y)| \leq \epsilon CT(1 + \|x\|_{R^n} + \|y\|_{R^m}). \]

5 Appendix

In this appendix we collect some technical results to which we appeal in the proofs of the main results in Section 4.

**Lemma 5.1** For any \( T > 0 \), there exists a constant \( C_T > 0 \) such that for any \( x, k \in \mathbb{R}^n \) and \( t \in [0, T] \), we have
\[ |D_x u(t, x) \cdot k| \leq C_T \|k\|_{\mathbb{R}^n}. \]

**Proof** Observe that for any \( k \in \mathbb{R}^n \),
\[ D_x u(t, x) \cdot k = E\left[D\phi(\bar{X}_t(x)) \cdot \eta^{k,x}_t\right] = E\left(\phi'(\bar{X}_t(x)), \eta^{k,x}_t\right)_{\mathbb{R}^n}, \]
where \( \eta^{k,x}_t \) denotes the first mean-square derivative of \( \bar{X}_t(x) \) with respect to \( x \in \mathbb{R}^n \) along the direction \( k \in \mathbb{R}^n \), then we have
\[ \begin{cases} 
  d\eta^{k,x}_t = D_x a(\bar{X}_t(x)) \cdot \eta^{k,x}_t dt + D_x b(\bar{X}_t(x)) \cdot \eta^{k,x}_t dB_t \\
  + D_x c(\bar{X}_{T-t}(x)) \cdot \eta^{k,x}_{T-t} dP_t, \\
  \eta^{k,x}_0 = k.
\end{cases} \]

This means that \( \eta^{k,x}_t \) is the solution of the integral equation
\[ \eta^{k,x}_t = k + \int_0^t D_x a(\bar{X}_s(x)) \cdot \eta^{k,x}_s ds + \int_0^t D_x b(\bar{X}_s(x)) \cdot \eta^{k,x}_s dB_s \]

\[ E[r^\epsilon(0, X_T^\epsilon(x, y), Y_T^\epsilon(x, y)) \leq C(1 + \|x\|_{R^n} + \|y\|_{R^m}). \] (4.18)
+ \int_0^t \bar{D}_s c(\bar{X}_s(x)) \cdot \bar{\eta}^{k,x}_s \, dP_s$

and then thanks to assumption (A1), we get

$$\mathbb{E}[\|\bar{\eta}^{k,x}_t\|_{\mathbb{R}^n}^2] \leq C_T \|k\|_{\mathbb{R}^n}^2 + C_T \int_0^t \mathbb{E}[\|\eta^{k,x}_s\|_{\mathbb{R}^n}^2] \, ds.$$  

Then by Gronwall lemma it follows that

$$\mathbb{E}[\|\eta^{k,x}_t\|_{\mathbb{R}^n}^2] \leq C_T \|k\|_{\mathbb{R}^n}^2, \quad t \in [0, T],  

(5.1)$$

so that

$$|\bar{D}_s \bar{u}(t, x) \cdot k| \leq C_T \|k\|_{\mathbb{R}^n}.$$

Next, we introduce an analogous result for the second derivative of $\bar{u}(t, x)$.

**Lemma 5.2** For any $T > 0$, there exists a constant $C_T > 0$ such that for any $x, k_1, k_2 \in \mathbb{R}^n$ and $t \in [0, T]$, we have

$$|D^2_{xx} \bar{u}(t, x) \cdot (k_1, k_2)| \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n}.$$

**Proof** For any $k_1, k_2 \in \mathbb{R}^n$, we have

$$D^2_{xx} \bar{u}(t, x) \cdot (k_1, k_2) = \mathbb{E}[\phi''(\bar{X}_t(x)) \cdot (\eta^{k_1,x}_t, \eta^{k_2,x}_t) + \phi'(\bar{X}_t(x)) \cdot \xi^{k_1,k_2}_t]'',  

(5.2)$$

where $\xi^{k_1,k_2}_t$ is the solution of the second variation equation corresponding to the averaged equation, which may be rewritten in the following form:

$$\xi^{k_1,k_2}_t = \int_0^t \left[ D_s \bar{a}(\bar{X}_s(x)) \cdot \xi^{k_1,k_2}_s + D^2_{xx} \bar{a}(\bar{X}_s(x)) \cdot (\eta^{k_1,x}_s, \eta^{k_2,x}_s) \right] \, ds$$

$$+ \int_0^t \left[ D^2_{xx} b(\bar{X}_s(x)) \cdot (\eta^{k_1,x}_s, \eta^{k_2,x}_s) + D_s b(\bar{X}_s(x)) \cdot \xi^{k_1,k_2}_s \right] \, dB_s$$

$$+ \int_0^t \left[ D^2_{xx} c(\bar{X}_s(x)) \cdot (\eta^{k_1,x}_s, \eta^{k_2,x}_s) + D_s c(\bar{X}_s(x)) \cdot \xi^{k_1,k_2}_s \right] \, dP_s.$$

Thus, by assumption (A1) and (5.1) we have

$$\mathbb{E}[\|\xi^{k_1,k_2}_t\|_{\mathbb{R}^n}^2] \leq C_T \int_0^t \left( \mathbb{E}[\|\eta^{k_1,x}_s\|_{\mathbb{R}^n}^2]^{1/2} \mathbb{E}[\|\eta^{k_2,x}_s\|_{\mathbb{R}^n}^2]^{1/2} + \mathbb{E}[\|\eta^{k_1,x}_s\|_{\mathbb{R}^n}^2]^{1/2} \mathbb{E}[\|\xi^{k_1,k_2}_s\|_{\mathbb{R}^n}^2]^{1/2} \right) \, ds$$

$$\leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n} + C_T \int_0^t \mathbb{E}[\|\xi^{k_1,k_2}_s\|_{\mathbb{R}^n}^2] \, ds.$$  

By applying the Gronwall lemma we have

$$\mathbb{E}[\|\xi^{k_1,k_2}_t\|_{\mathbb{R}^n}^2] \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n}.$$

Returning to (5.2), we can get

$$|D^2_{xx} \bar{u}(t, x) \cdot (k_1, k_2)| \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n}.$$
By using the analogous arguments used before, we can prove the following estimate for the third order derivative of \( \tilde{a}(t, x) \) with respect to \( x \).

**Lemma 5.3** For any \( T > 0 \), there exists a constant \( C_T > 0 \) such that for any \( x, k_1, k_2, k_3 \in \mathbb{R}^n \) and \( t \in [0, T] \), we have

\[
|D^3_{xxx}(\tilde{a}(t, x) \cdot (k_1, k_2, k_3))| \leq C_T \|k_1\|_{\mathbb{R}^n} \cdot \|k_2\|_{\mathbb{R}^n} \cdot \|k_3\|_{\mathbb{R}^n}.
\]

The following lemma states boundedness for the first derivative of \( \tilde{a}(x) - \mathbb{E}a(x, Y^x_t(y)) \) with respect to \( x \).

**Lemma 5.4** There exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^m \), \( k \in \mathbb{R}^n \) and \( t > 0 \) it holds

\[
\|D_x(\tilde{a}(x) - \mathbb{E}a(x, Y^x_t(y))) \cdot k\|_{\mathbb{R}^n} \leq Ce^{\frac{-Ct}{2}}\|k\|_{\mathbb{R}^n} (1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}).
\]

**Proof** The proof is a modification of the proof of [3, Proposition C.2]. For any \( t_0 > 0 \), we set

\[
\tilde{a}_{t_0}(x, y, t) = \tilde{a}(x, y, t) - \tilde{a}(x, y, t + t_0),
\]

where

\[
\tilde{a}(x, y, t) := \mathbb{E}a(x, Y^x_t(y)).
\]

Then we have

\[
\lim_{t_0 \to +\infty} \tilde{a}_{t_0}(x, y, t) = \mathbb{E}a(x, Y^x_t(y)) - \tilde{a}(x).
\]

By Markov property, we have

\[
\tilde{a}_{t_0}(x, y, t) = \tilde{a}(x, y, t) - \mathbb{E}a(x, Y^x_{t+t_0}(y))
= \tilde{a}(x, y, t) - \mathbb{E}\tilde{a}(x, Y^x_{t_0}(y), t)
\]

Due to assumption (**A1**), for any \( k \in \mathbb{R}^n \) we have

\[
D_x\tilde{a}_{t_0}(x, y, t) \cdot k = D_x\tilde{a}(x, y, t) \cdot k - \mathbb{E}D_x(\tilde{a}(x, Y^x_{t_0}(y), t)) \cdot k
= \tilde{a}'(x, y, t) \cdot k - \mathbb{E}\tilde{a}'(x, Y^x_{t_0}(y), t) \cdot k
- \mathbb{E}\tilde{a}'(x, Y^x_{t_0}(y), t) \cdot (D_yY^x_{t_0}(y) \cdot k),
\]

where the symbols \( \tilde{a}' \) and \( \tilde{a}'_t \) denote the directional derivatives with respect to \( x \) and \( y \), respectively. Note that the first derivative \( \xi^{x,y,k}_{t_0} = D_xY^x_t(y) \cdot k \), at the point \( x \) and along the direction \( k \in \mathbb{R}^n \), is the solution of equation

\[
d\xi^{x,y,k}_{t_0} = \left( f'_x(x, Y^x_t(y)) \cdot k + f'_y(x, Y^x_t(y)) \cdot \xi^{x,y,k}_{t_0} \right) dt
+ \left( g'_x(x, Y^x_t(y)) \cdot k + g'_y(x, Y^x_t(y)) \cdot \xi^{x,y,k}_{t_0} \right) dW_t
+ \left( h'_x(x, Y^x_t(y)) \cdot k + h'_y(x, Y^x_t(y)) \cdot \xi^{x,y,k}_{t_0} \right) dN_t.
\]
with initial data $\zeta_0^{x,y,k} = 0$. Hence, by assumption (A1), it is straightforward to check

$$\mathbb{E}\|\zeta_t^{x,y,k}\|_{\mathbb{R}^m} \leq C\|k\|_{\mathbb{R}^m}$$  \hspace{1cm} (5.4)

for any $t \geq 0$. Note that for any $y_1, y_2 \in \mathbb{R}^m$, we have

$$\|\tilde{a}(x, y_1, t) - \tilde{a}(x, y_2, t)\|_{\mathbb{R}^n} = \|\mathbb{E}a(x, Y_t^{x}(y_1)) - \mathbb{E}a(x, Y_t^{x}(y_2))\|_{\mathbb{R}^n} \leq C\mathbb{E}\|Y_t^{x}(y_1) - Y_t^{x}(y_2)\|_{\mathbb{R}^m} \leq Ce^{-\frac{\beta t}{2}}\|y_1 - y_2\|_{\mathbb{R}^m},$$

where (5.3) was used to obtain the last inequality. This means that

$$\|\tilde{a}'_y(x, y, t) \cdot l\|_{\mathbb{R}^m} \leq Ce^{-\frac{\beta t}{2}}\|l\|_{\mathbb{R}^m}, \quad l \in \mathbb{R}^m.$$  \hspace{1cm} (5.5)

From (5.4) and (5.5), we obtain

$$\mathbb{E}\|\tilde{a}'_y(x, Y_t^{x}(y), t) \cdot (D_x Y_t^{x}(y) \cdot k)\|_{\mathbb{R}^m} = \mathbb{E}\|\tilde{a}'_y(x, Y_t^{x}(y), t) \cdot (\zeta_t^{x,y,k})\|_{\mathbb{R}^m} \leq Ce^{-\frac{\beta t}{2}}\|k\|_{\mathbb{R}^m}.$$  \hspace{1cm} (5.6)

Then, by easy calculations, we have

$$\tilde{a}'_y(x, y_1, t) \cdot k - \tilde{a}'_y(x, y_2, t) \cdot k = \mathbb{E}(a'_y(x, Y_t^{x}(y_1))) \cdot k - \mathbb{E}(a'_y(x, Y_t^{x}(y_2))) \cdot k + \mathbb{E}(a'_y(x, Y_t^{x}(y_1))) \cdot (\zeta_t^{x,y_1,k} - a'_y(x, Y_t^{x}(y_2))) \cdot \zeta_t^{x,y_2,k}$$

$$= \mathbb{E}(a'_y(x, Y_t^{x}(y_1))) \cdot k - \mathbb{E}(a'_y(x, Y_t^{x}(y_2))) \cdot k + \mathbb{E}(a'_y(x, Y_t^{x}(y_1))) \cdot (\zeta_t^{x,y_1,k} - \zeta_t^{x,y_2,k})$$

$$= \sum_{i=1}^{m} \mathcal{N}_i(t, x, y_1, y_2, k).$$  \hspace{1cm} (5.7)

Now, we estimate the three terms in the right hand side of above equality. Concerning $\mathcal{N}_1(t, x, y_1, y_2, k)$ we have

$$\|\mathcal{N}_1(t, x, y_1, y_2, k)\|_{\mathbb{R}^m} \leq \mathbb{E}\|a'_y(x, Y_t^{x}(y_1)) \cdot k - a'_y(x, Y_t^{x}(y_2)) \cdot k\|_{\mathbb{R}^m} \leq C\mathbb{E}\|Y_t^{x}(y_1) - Y_t^{x}(y_2)\|_{\mathbb{R}^m} \cdot \|k\|_{\mathbb{R}^m} \leq Ce^{-\frac{\beta t}{2}}\|y_1 - y_2\|_{\mathbb{R}^m} \cdot \|k\|_{\mathbb{R}^m}.$$  \hspace{1cm} (5.8)

Next, by assumption (A1) we get

$$\|\mathcal{N}_2(t, x, y_1, y_2, k)\|_{\mathbb{R}^m}$$
For the third term, by making use of assumption (A1) again, we can infer that

\[
\|N_3(t, x, y_1, y_2, k)\|_{\mathbb{R}^n} \\
\leq \mathbb{E}[|a'_y(x, Y^x_t(y_1)) - a'_y(x, Y^x_t(y_2))| \cdot \zeta_{3.1.4}^{x, y_1, k} \cdot \zeta_{3.1.4}^{x, y_2, k} \cdot \|\nabla_x^{x, y_1, k} - \nabla_x^{x, y_2, k}\|_{\mathbb{R}^m}] \\
\leq C\mathbb{E}\left[\|\zeta_{3.1.4}^{x, y_1, k} - \zeta_{3.1.4}^{x, y_2, k}\|_{\mathbb{R}^m}\right] \\
\leq Ce^{-\frac{\beta}{2}t}\|y_1 - y_2\|_{\mathbb{R}^m} \cdot \|k\|_{\mathbb{R}^n}. \quad (5.9)
\]

Now, returning to (5.7) and taking into account of (5.8), (5.9) and (5.10), we get

\[
\|\bar{a}'(x, y, t) \cdot k - \bar{a}'(x, y, t) \cdot k\| \\
\leq Ce^{-\frac{\beta}{2}t}\|y_1 - y_2\|_{\mathbb{R}^m} \cdot \|k\|_{\mathbb{R}^n},
\]

which leads to

\[
\|\bar{a}'(x, y, t) \cdot h - \mathbb{E}\bar{a}'(x, Y^x_t(y), t) \cdot k\|_{\mathbb{R}^n} \\
\leq Ce^{-\frac{\beta}{2}t}(1 + \|y\|_{\mathbb{R}^m} + \|Y^x_t(y)\|_{\mathbb{R}^m}) \cdot \|k\|_{\mathbb{R}^n} \\
\leq e^{-\frac{\beta}{2}t}(1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}) \cdot \|k\|_{\mathbb{R}^n}, \quad (5.11)
\]

where we used the inequality (5.12). Returning to (5.3), by (5.6) and (5.11) we conclude that

\[
\|D_x\bar{a}_{t_0}(x, y, t) \cdot k\|_{\mathbb{R}^n} \leq Ce^{-\frac{\beta}{2}t}\left(1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}\right)\|k\|_{\mathbb{R}^n}.
\]

Taking the limit as \(t_0 \to +\infty\) we obtain

\[
\|D_x(\bar{a}(x) - \mathbb{E}a(x, Y^x_t(y)))\|_{\mathbb{R}^n} \leq Ce^{-\frac{\beta}{2}t}\|k\|_{\mathbb{R}^n} \left(1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}\right).
\]

Proceeding with similar arguments above we obtain the following higher order differentiability.

**Lemma 5.5** There exists a constant \(C > 0\) such that for any \(x, k_1, k_2 \in \mathbb{R}^n, y \in \mathbb{R}^m\) and \(t > 0\) it holds

\[
\|D_{xx}^2(\bar{a}(x) - \mathbb{E}a(x, Y^x_t(y)))(k_1, k_2)\|_{\mathbb{R}^n} \\
\leq Ce^{-\frac{\beta}{2}t}\|k_1\|_{\mathbb{R}^n}\|k_2\|_{\mathbb{R}^n} \left(1 + \|x\|_{\mathbb{R}^n} + \|y\|_{\mathbb{R}^m}\right).
\]

Finally, we introduce the following auxiliary result.

**Lemma 5.6** There exists a constant \(C > 0\) such that for any \(x, k \in \mathbb{R}^n, y \in \mathbb{R}^m\) and \(t > 0\) it holds

\[
\|\frac{\partial}{\partial t}D_x\bar{a}(t, x) \cdot k\|_{\mathbb{R}^n} \leq C\|k\|_{\mathbb{R}^n}.
\]
Proof For simplicity of presentation, we will prove it for the 1-dimensional case. The multi-dimensional situation can be treated similarly, only notations are somewhat involved. In this case we only need to show

$$\left| \frac{\partial}{\partial t} \frac{\partial}{\partial x} \bar{u}(t, x) \right| \leq C. \quad (5.12)$$

Actually, for any $\phi \in C^3_b(\mathbb{R}, \mathbb{R})$ we have

$$\frac{\partial}{\partial x} \bar{u}(t, x) = \frac{\partial}{\partial x} \mathbb{E}[\phi(\bar{X}_t(x))] = \mathbb{E} \left( \phi'(\bar{X}_t(x)) \cdot \frac{\partial}{\partial x} \bar{X}_t(x) \right).$$

If we define

$$\zeta_t^x := \frac{\partial}{\partial x} \bar{X}_t(x),$$

we have

$$\zeta_t^x = 1 + \int_0^t \bar{a}'(\bar{X}_s(x)) \cdot \zeta_s^x ds + \int_0^t \bar{b}'(\bar{X}_s(x)) \cdot \zeta_s^x dB_s + \int_0^t \bar{c}'(\bar{X}_s-(x)) \cdot \zeta_s^x dP_s.$$ 

The boundedness of $\bar{a}'$, $\bar{b}'$ and $\bar{c}'$ guarantees

$$\mathbb{E}|\zeta_t^x|^2 \leq C_T, \quad t \in [0, T]. \quad (5.13)$$

By using Itô formula we have

$$\mathbb{E}[\phi'(\bar{X}_t(x)) \cdot \zeta_t^x]$$

$$= \phi'(x) + \mathbb{E} \int_0^t [\phi'(\bar{X}_s(x)) \bar{a}'(\bar{X}_s(x)) \zeta_s^x + \zeta_s^x \phi''(\bar{X}_s(x)) \bar{a}(\bar{X}_s(x))] ds$$

$$+ \mathbb{E} \int_0^t [\phi'(\bar{X}_s(x)) \zeta_s^x \phi''(\bar{X}_s(x))] b(\bar{X}_s(x)) ds$$

$$+ \frac{1}{2} \mathbb{E} \int_0^t \zeta_s^x \phi'''(\bar{X}_s(x)) b^2(\bar{X}_s(x)) ds$$

$$+ \lambda_1 \mathbb{E} \int_0^t \phi'(\bar{X}_s(x)) c'(\bar{X}_s-(x)) \zeta_s^x ds$$

$$+ \lambda_1 \mathbb{E} \int_0^t \zeta_s^x [\phi'(\bar{X}_s-(x) + c(\bar{X}_s-(x))) - \phi'(\bar{X}_s-(x))] ds$$

$$+ \lambda_1 \mathbb{E} \int_0^t c'(\bar{X}_s-(x)) \zeta_s^x [\phi'(\bar{X}_s-(x) + c(\bar{X}_s-(x))) - \phi'(\bar{X}_s-(x))] ds.$$ 

Since $\phi$ belongs to $C^3_b(\mathbb{R}, \mathbb{R})$, from the assumption (A1) it follows that for any $t \in [0, T]$,

$$\left| \frac{\partial}{\partial t} \frac{\partial}{\partial x} \bar{u}(t, x) \right| = \left| \frac{\partial}{\partial t} \mathbb{E}[\phi'(X_t(x)) \cdot \zeta_t^x] \right| 
\leq C \mathbb{E}|\zeta_t^x|,$$

then, by taking (5.13) into account, one would easily arrive at (5.12).
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Competing interests

The authors declare that they have no competing interests.

Authors contributions

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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