THE MULTICOLOUR EAST MODEL

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ABSTRACT. We consider the multicolour East model, a model of glass forming liquids closely related to the East model on \( \mathbb{Z}^d \). The state space \((G \cup \{\star\})^{\mathbb{Z}^d}\) consists of \(|G| \leq 2^d\) different vacancy types and the neutral state \(\star\). To each \(h \in G\) we associate unique facilitation mechanisms \(\{c_h^x\}_{x \in \mathbb{Z}^d}\) that correspond to rotated versions of the East model constraints. If \(c_h^x\) is satisfied, the state on \(x\) can transition from \(h\) to \(\star\) with rate \(p \in (0,1)\) or vice versa with rate \(q_h \in (0,1)\), where generally \(q_h \neq q_{h'}\) if \(h' \neq h\). Notably, vertices in the state \(h\) cannot transition directly to \(h' \neq h\) and neighbouring \(h'\)-vacancies do not contribute in satisfying \(c_h^x\). Thus, there is a novel blocking mechanism between vacancies of differing type. We find sufficient conditions on the model geometry to have a positive spectral gap and prove that with \(|G| = 2^d\) the model is not ergodic. For \(d = 2\) we prove that the model with \(|G| \leq 3\) has positive spectral gap and we find sufficient conditions on the transition rates for the spectral gap to be given in the leading order by the spectral gap of the East model on \(\mathbb{Z}^2\) with parameter \(q_{\text{min}} = \min_{h \in G} q_h\) in the limit \(q_{\text{min}} \to 0\). In particular, we prove this when there are \(h \in G\) with \(q_h \gg q_{\text{min}}\) by explicitly constructing mechanisms on which the frequent vacancy types cooperate to facilitate the East movement of the least frequent vacancies.

1. INTRODUCTION

In [7], Chandler and Garrahan introduce a coarse-grained model inspired by the complex dynamics of glass-forming liquids. It is best described as a generalization of the East process, so let us first recall the dynamics of the East process on \(\mathbb{Z}^d\) (see e.g. [1, 6, 14]). The East process is an interacting particle system on \(\mathbb{Z}^d\) with single parameter \(q \in (0,1)\). Each vertex \(x \in \mathbb{Z}^d\), with rate one and independently across \(\mathbb{Z}^d\), is resampled from \(\{0,1\}\) according to the \(\text{Ber}(p)\)-measure, \(p = 1 - q\), iff in the current configuration there is at least one vacancies (i.e. a state “0”) among the neighbours \(y\) of \(x\) of the form \(y = x - e\) where \(e\) are canonical base vectors of \(\mathbb{Z}^d\) the set of which we denote by \(B\). We say that the vacancies have north and east as their propagation directions.

The model by Chandler and Garrahan is an interacting particle system on \(\mathbb{Z}^d\) with state space \((G \cup \{\star\})^{\mathbb{Z}^d}, |G| = 2^d\), and two parameters \(q, \xi \in (0,1)\), that is informally described as follows. To each element of \(G\) we associate a unique set of propagation directions corresponding to one of the \(2^d\) possible rotations of the East propagation directions. On each vertex there are two Poisson clocks: one that gives diffusive rings with rate \(\xi\) and one that gives directed rings with rate \(1 - \xi\). Both of these rings come with their own facilitation mechanisms. A diffusive ring on \(x \in \mathbb{Z}^d\) is legal if there is a neighbour \(y\) of \(x\) in the state \(h\) such that \(x - y\) corresponds to one of the associated propagation directions of \(h\), we say that \(x\) is \(h\)-facilitated. On a legal diffusive ring, if
the state of $x$ is in $G$, there is a transition to $\star$ with rate $p := 1 - q$ and if the state of $x$ is $\star$ there is a transition to any of the states in $G$ with rate $q/2^d$ respectively. Legality for directed rings is a bit more tricky. At a directed ring on $x$, if $x$ is in the neutral state $\star$, it can transition to $h \in G'$ with rate $q/2^h$, where $G'$ is the set of all vacancy types $h$ such that $x$ is $h$-facilitated. If $x$ is in the state $h$ it can transition to the neutral state $\star$ with rate $p$ iff it is $h$-facilitated. Thus, while in the diffusive rings it suffices to be facilitated at all to be able to transition from a state in $G$ to $\star$ and vice versa, directed rings require the transitions and the facilitation to be by the same vacancy type, so that the various vacancy types block each other.

In this paper we consider the limit case $\xi = 0$ where there are no diffusive rings. This model behaves like multiple rotated versions of the East model evolving at the same time with a shared “1” state, represented by the neutral state $\star$. The original model is not ergodic (see Theorem [11A]) so we consider the model with only a subset of all possible $2^d$ rotations and allow for varying transition rates $q_h$ for the various states $h \in G$. We dub this model the multicolour East model (MCEM). The MCEM is reversible with respect to the product measure $\mu$ that locally assigns a state with its corresponding probability $\{q_h\}_{h \in G}$ or $p := 1 - \sum_{h \in G} q_h$ for the neutral state, so that the transition rates correspond to the equilibrium densities of the respective states.

In Theorem 1 we give sufficient conditions on the geometry of the vacancy types so that the MCEM on $\mathbb{Z}^d$ has positive spectral gap, which in particular implies that the two-dimensional MCEM with $|G| \leq 3$ has positive spectral gap and with $|G| = 4$ is not ergodic, thus fully classifying the ergodicity landscape in two dimensions. In Theorem 2 we then give sufficient conditions on the equilibrium densities and the geometry for the spectral gap of the two-dimensional MCEM to be given by the two-dimensional East model spectral gap in the leading order in the limit $q_{\text{min}} := \min_{h \in G} q_h \to 0$. In particular, we prove this for cases where there are one or two vacancy types with much larger equilibrium densities than $q_{\text{min}}$. This result might be surprising at first, as one could expect the East model dynamics of the least frequent vacancies to be blocked by the more frequent ones, leading to a spectral gap that is given by these blocking dynamics. In fact, we prove that the frequent vacancies cooperate in a way to facilitate the two-dimensional East movement of the least frequent vacancies, so that the blocking is negligible.

As far as the author is aware this is the first time this model is treated in mathematical literature, but the physical motivation and dynamics bear close resemblance to those of kinetically constrained models (KCM) [118]. In fact, this paper lines up well with current research on KCM on $\mathbb{Z}^d$ which looks at ergodicity and in particular ergodicity breaking transitions [1117] and at the spectral gap behaviour [1015].

1.1. Construction. We start by constructing the state space and together with the associated propagation directions.

**Definition 1.1** (Vacancy types and their constraints). The set of vacancy types is a finite set $\mathcal{V}$ of cardinality $2^d$. We identify $\mathcal{V}$ with the hypercube $H_d := \{0, 1\}^d \subset \mathbb{Z}^d$ and refer to the vacancy type corresponding to the vertex $h \in H_d$ as the vacancy of type $h$ or the $h$-vacancy. We say that $v$ is a propagation direction for the $h$-vacancy, and write $v \in P(h)$,
if $\|v\| = 1$ and $h + v \in H_d$. For $h \in H_d$ we say that $x \prec^{(h)} y$ if $x \cdot v \leq y \cdot v$ for every $v \in \mathcal{P}(h)$.

Given $G \subset H_d$, which we identify with a collection of vacancy types in $\mathcal{V}$, we define the vertex state space $S(G)$ as the union of $G$ together with the neutral state denoted by $\star$. For $\omega \in S(G)^{\mathbb{Z}^d}$, $h \in G$ and $x \in \mathbb{Z}^d$ the constraint $c^h_x(\omega)$ is given by

$$
c^h_x(\omega) = \begin{cases} 1 & \text{if } \exists v \in \mathcal{P}(h) : \omega_{x-v} \text{ is a } h\text{-vacancy}, \\ 0 & \text{otherwise.} \end{cases}
$$

See Fig. 1 for an illustration of $H_2$ and $H_3$ with the associated propagation directions for each vacancy type.

**Remark 1.2.** If $G = \{(0,0,\ldots,0)\}$, we can identify $\star$ with $1$ and $\{0,0,\ldots,0\}$ with $0$ to recover the state space of the $d$-dimensional East model with the corresponding constraints on $\mathbb{Z}^d$.

**Notation warning:** In the sequel, for a given $\omega \in S(G)^{\mathbb{Z}^d}$ and $h \in G$, we will often write $\omega_x = h$ meaning that $\omega_x$ is a vacancy of type $h$.

For $G \subset H_d$ we call vectors $q = \{q_h : h \in G\}$ with $q_h > 0$ for $h \in G$, and $\sum_{h \in G} q_h < 1$, valid parameter sets and write $p = 1 - \sum_{h \in G} q_h$. Given a valid parameter set $q$ let $\nu$ denote the probability measure on $S(G)$ that assigns probability $p$ to the state $\star$ and $q_h$ to $h$ for all $h \in G$. For any $\Lambda \subset \mathbb{Z}^d$ define the state space $\Omega_\Lambda = S(G)^\Lambda$ and the measure $\mu_\Lambda : = \otimes_{x \in \Lambda} \nu$, where we recall the notational convention that we leave away $\Lambda$ if $\Lambda = \mathbb{Z}^d$. We also omit the dependence on $q$ and $G$ in the notation of $p$, $\nu$ and $\Omega_\Lambda$ since they will be clear from context.

For subsets $V \subset \Lambda \subset \mathbb{Z}^d$ and configurations $\omega \in \Omega_V$, $\omega' \in \Omega_{\Lambda \setminus V}$ in $\Omega_\Lambda$ we write $\omega \cdot \omega' \in \Omega_{\Lambda}$ for the state given by $\omega$ on $V$ and $\omega'$ on $\Lambda \setminus V$. We say a function $f : \Omega \to \mathbb{R}$ is local if the value $f(\omega)$ only depends on the state of finitely many vertices.

**Definition 1.3** (The $G$-MCEM process). Given a subset $G \subset H_d$ and a valid parameter set $q$ we define the continuous time $G$-MCEM process on $\mathbb{Z}^d$ via the infinitesimal generator,
which we define through its action on local functions $f : \Omega \to \mathbb{R}$, as
\[
L f(\omega) = \sum_{h \in G} \sum_{x \in \mathbb{Z}^d} c^h_x(\omega) [\mathbbm{1}_{\omega_x = h} \mathbbm{1}_{\omega_x = h} q_h + \mathbbm{1}_{\omega_x = h} p] \nabla^{(h)}_x f(\omega),
\]
where
\[
\nabla^{(h)}_x f(\omega) := \begin{cases} 
  f(h \cdot \omega^{Z \setminus \{x\}}) - f(\omega) & : \text{if } \omega_x = \ast, \\
  f(\ast \cdot \omega^{Z \setminus \{x\}}) - f(\omega) & : \text{if } \omega_x = h, \\
  0 & : \text{else.}
\end{cases}
\]

We write $\omega(t)$ for the state at time $t$ and $E_\eta$ and $P_\eta$ for the corresponding expectation and law for the process started at $\eta \in \Omega$.

**Remark 1.4.** It might be surprising that the sum of the rates $q_h + p$ is strictly smaller than 1. In fact, here the missing rate $1 - q_h - p$ is hidden in $\nabla^{(h)}_x f(\omega) = 0$ if $\omega_x \notin \{h, \ast\}$, thus we could have added a term $\mathbbm{1}_{\omega_x \notin \{h, \ast\}} (1 - q_h - p)$ for the transition in which nothing happens.

**Remark 1.5.** Notice that in the G-MCEM process a state can transition from $\ast$ to $h$ iff there is a vector $v \in P(h)$ such that $x - v$ has an $h$-vacancy justifying the name propagation direction for $v$. In particular, an $h$-vacancy at $x$ can only influence those vertices $y$ such that $x \prec^{(h)} y$. Further, there is no transition from one vacancy type to another. The process, in order to change the state of a vertex from one vacancy type to another, first has to go through the neutral state $\ast$ (justifying its name). In particular, when $|G| \geq 2$ an $h$-vacancy can be blocked by a cluster of nearby vacancies of type in $G \setminus \{h\}$. This blocking interaction is the main hurdle in bounding the spectral gap.

The generated process is reversible with respect to $\mu$, indeed for $x \in \mathbb{Z}^d$ and $\omega' \in \Omega_{\mathbb{Z}^d \setminus \{x\}}$
\[
\sum_{\omega \in \Omega_x} \mu_x(\omega) f(\omega \cdot \omega') [\mathbbm{1}_{\omega_x = \ast} q_h + \mathbbm{1}_{\omega_x = h} p] \nabla^{(h)}_x g(\omega \cdot \omega')
\]
\[
= pq_h (f(\ast \cdot \omega') - f(h \cdot \omega')) (g(h \cdot \omega') - g(\ast \cdot \omega')),
\]
and thus, since $c^h_x$ does not depend on the state of $x$ we have
\[
\mu(fLg) = \mu(gLf),
\]
from which reversibility follows since $f, g$ were arbitrary. The associated Dirichlet form is then
\[
D(f) := \mu(-fLf) = \sum_{h \in G} \sum_{x \in \mathbb{Z}^d} pq_h \mu \left[ c^h_x \mathbbm{1}_{\omega_x \in \{\ast, h\}} (f(\ast \cdot \omega) - f(h \cdot \omega))^2 \right] \tag{1.1}
\]
and we define the spectral gap as
\[
\gamma(G; q) = \gamma(G) := \inf_{f \in \text{Dom}(L)} \frac{D(f)}{\text{Var}(f)} . \tag{1.2}
\]

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1See [11, 13] on how to construct a continuous time Markov process starting from the action of the generator on local functions.
Using [12], Section IV, Theorem 4.13 the MCEM process is ergodic with stationary measure $\mu$, if 0 is a simple eigenvalue of $L$ and thus in particular if the spectral gap is positive.

**Remark 1.6.** We will sometimes write $L_\Lambda$ and $D_\Lambda$ in which the sum $\sum_{x \in \mathbb{Z}^d}$ is replaced with a sum over $\sum_{x \in \Lambda}$ and the measure $\mu$ with the measure $\mu_\Lambda$. These are functions on $\Omega_{\Lambda^c}$ and thus either appear with a configuration $\omega \in \Omega_{\Lambda^c}$ or inside an average w.r.t. $\mu_{\Lambda^c}$.

### 1.1.1. Graphical construction

An alternative to the construction via the infinitesimal generator is via a graphical construction. Put a marked Poisson process on each vertex in $\mathbb{Z}^d$. The $k$-th ring at the vertex $x \in \mathbb{Z}^d$ occurs at time $t_{x,k}$ and for each ring we have the mark $U_{x,k} \sim \mu$ so that $U_{x,k} \in \mathcal{S}(G)$ and $\{U_{x,k}\}_{x,k}$ is an i.i.d. family. Consider a starting state $\omega(0) \in \Omega$ and denote by $\omega(t)$ the state at time $t \in \mathbb{R}_+$. With $t_{x,k}$ an infinitesimally smaller time than $t_{x,k}$, the graphically constructed process evolves as follows:

(i) At $t_{x,k}$ we say that we have a $U_{x,k}$-legal ring if any of the following conditions is satisfied

(a) $U_{x,k} = \star$ and there is an $h \in G$ such that $\omega_x(t_{x,k}^-) = h$ and $c_{x,k}^h(\omega(t_{x,k}^-)) = 1$, or

(b) $U_{x,k} \neq \star$, $\omega_x(t_{x,k}^-) = \star$ and $c_{x,k}^h(\omega(t_{x,k}^-)) = 1$, or

(c) $U_{x,k} = \omega_x(t_{x,k}^-)$ and $c_{x,k}^h(\omega(t_{x,k}^-)) = 1$ (i.e. nothing changes).

(ii) If $t_{x,k}$ is an $U_{x,k}$-legal ring, we set $\omega_x(t_{x,k})$ equal to $U_{x,k}$.

Showing that this construction is well defined on $\mathbb{Z}^d$ and leads to the same process as the one constructed above through the infinitesimal generator is analogous to the proof presented in [11] for the North-East model.

### 2. Results

The first result gives sufficient conditions for ergodicity of the $G$-MCEM. Recall for this that any $G \subset H_2$ inherits the graph structure of $\mathbb{Z}^d$.

**Theorem 1.** Consider all the following $G$-MCEM with an arbitrary valid parameter set $q$.

(A) If $G = H_2$ then the $G$-MCEM process is not ergodic.

(B) Suppose $G \subset H_2$ is such that either condition holds:

(B.i) there is a canonical base vector $e \in B$ of $\mathbb{Z}^d$ such that for any two $h, h' \in G$ we have $h \cdot e = h' \cdot e$.

(B.ii) there is a superset $G' \subset H_2$ of $G$ such that $G'$ is isomorphic to a star-graph. Then the $G$-MCEM process has a positive spectral gap.

**Example 1.** Any $G \subset H_3$ that is a subset of a single face satisfies (B.i) and any $G \subset H_2$ with $|G| < 4$ satisfies (B.ii). In particular note that this gives complete information about ergodicity in $d = 2$ but leaves gaps for $d \geq 3$.

For $d = 2$ we even find sufficient conditions on the geometry of $G$ and the parameter set so that the limiting behaviour of the spectral gap is given by the East model. Given a valid parameter set $q$ we define $q_{\min} = \min_{h \in G} q_h$, $q_{\max} = \max_{h \in G} q_h$ and if $|G| = 3$ we write $q_{\text{med}}$ for the $q_h \in G$ that is not in \{ $q_{\max}$, $q_{\min}$\}. We further define $\theta_h = \theta_{q_h} :=$
\( \log_2(q_h) \) and write \( \gamma_2 = \gamma_2(q) \) for the spectral gap of the two-dimensional East model with vacancy density \( q \) and \( \gamma(G; q) \) the spectral gap of the G-MCEM with parameter set \( q \).

**Theorem 2.** Fix \( \Delta > 0 \) and consider a G-MCEM on \( \mathbb{Z}^2 \) with \( |G| \in \{2, 3\} \) and a valid parameter set \( q \) such that \( p > \Delta \). Then,

\[
\lim_{q_{\min} \to 0} \frac{\gamma(G; q)}{\gamma_2(q_{\min})} = 1
\]

in the following cases.

- Any 2-subset \( G \subset H_2 \) and either one of the following conditions holds:
  1. \( \lim_{q_{\min} \to 0} q_{\max} \theta_{q_{\min}}^3 = 0 \),
  2. \( \lim_{q_{\min} \to 0} q_{\max} \theta_{q_{\min}}^3 / \log_2(\theta_{q_{\min}}) = \infty \).

- Any 3-subset \( G \subset H_3 \) and either one of the following conditions holds:
  1. \( \lim_{q_{\min} \to 0} q_{\max} \theta_{q_{\min}}^3 = 0 \),
  2. \( \lim_{q_{\min} \to 0} q_{\max} \theta_{q_{\min}}^3 / \log_2(\theta_{q_{\min}}) = \infty \) and \( \lim_{q_{\min} \to 0} q_{\med} \theta_{q_{\min}}^6 = 0 \),
  3. \( G \) is such that the vacancies associated to \( q_{\med} \) and \( q_{\max} \) share a propagation direction and \( \lim_{q_{\min} \to 0} q_{\med} \theta_{q_{\min}}^6 > 0 \).

**Remark 2.1.** The cases are ordered from the easiest to the hardest regime. The cases (2.i) and (3.i) are the easiest since in these cases even the highest density \( q_{\max} \) is relatively low so that most vacancies in equilibrium are surrounded by large neutral state patches. Thus for these cases it is natural to conjecture that the spectral gap of the G-MCEM should be given by the two-dimensional East model spectral gap. This also includes the case by [7] in which all vacancy type had the same density.

The next harder case is if there is one vacancy type that is frequent in equilibrium, i.e. case (2.ii) and (3.ii). The conclusion of Theorem 2 still presents itself as a natural conjecture if we consider that any vacancy of the frequent type will see large patches of either neutral vertices or its own vacancy type. Thus, any vacancy of the frequent type that blocks the infrequent vacancies is likely to be removable by close vacancies of the same type allowing the infrequent vacancies to evolve according to their respective two-dimensional East model dynamics.

The hardest case is (3.iii), when two vacancy types are frequent. In this case the frequent vacancy types might block each other and we only manage to find configurations that remove the blocking frequent vacancies if they share a propagation direction.

**Remark 2.2.** It is possible to relax the requirement that \( p > \Delta \) at the cost of an additional factor of \( 1/p \) or \( 1/p^2 \) in \( \gamma(G; q) \) which represents the average waiting time for a vertex to get back into the neutral state. If \( p \to 0 \) then vertices rarely reach the neutral state and there can be no transition from one vacancy type to another explaining the extra cost in the spectral gap. As we have no tight bounds on the \( p \) dependency, i.e. whether it should be \( 1/p \) or \( 1/p^2 \) or even worse, we limit the discussion to the case \( p > \Delta \).
3. Key tools

We recall past results together with smaller Lemmas that enter the proofs of Theorem 1 and Theorem 2. If $G$ and $q$ are not explicitly stated then they, and correspondingly the state space $\Omega$, local equilibrium $\nu$ and particle density $p$, are arbitrarily fixed.

3.1. A constrained Poincaré inequality for product measures. Define the support $\text{Supp}(A)$ of an event $A \subset \Omega$ as the set of vertices the event depends on.

**Definition 3.1 (Exterior condition).** Given an increasing and exhausting collection of subsets $\{V_n\}_{n \in Z}$ of $Z^d$ (i.e. $V_n \subset V_{n+1}$ for all $n$ and $\cup_n V_n = Z^d$), let the exterior of $x \in V_n$ be the set $\text{Ext}_x := \cup_{j=n}^{\infty} V_{j+1} \setminus V_j$. We then say that the family of events $\{A_x\}_{x \in Z^d}$ satisfies the exterior condition w.r.t. $\{V_n\}_{n \in Z}$ if $\text{Supp}(A_x) \subset \text{Ext}_x$ for all $x \in Z^d$.

Let $\{A_x^{(i)}\}_{i \in \{1, \ldots, k\}}$ and write $\text{Supp}(A_x^{(i)}) = \bigcup_{i \in I} \text{Supp}(A_x^{(i)})$ for nonempty subsets $I \subset \{1, \ldots, k\}$.

**Theorem 3.2 (Exterior condition theorem, [15, Theorem 2]).** Assume that

$$
(2^k - 1) \sup_{x \in Z^d} \sum_{J \subset \{1, \ldots, k\}} \sum_{x \in Z^2} \mu \left( \prod_{j \in J} \left( 1 - \mathbb{1}_{A_x^{(j)}} \right) \right) < 1/4.
$$

(3.1)

Suppose in addition that there exists an exhausting and increasing family $\{V_n\}_{n \in Z}$ of subsets of $Z^d$ such that, for any $i \in \{1, \ldots, k\}$, the family $\{A_x^{(i)}\}_{x \in Z^d}$ satisfies the exterior condition w.r.t. $\{V_n\}_{n \in Z}$. Then, for any local function $f : \Omega \to \mathbb{R}$ we have

$$
\text{Var}(f) \leq 4 \sum_{x} \mu \left( \sum_{j=1}^{k} \mathbb{1}_{A_x^{(j)}} \right) \text{Var}_x(f).
$$

(3.2)

In particular, the same conclusion holds if, instead of Eq. (3.1) we have that

$$
\lim_{q_{\min} \to 0} \max_{j \in \{1, \ldots, k\}} \sup_{x \in Z^d} |\text{Supp}(A_x^{(j)})| \sup_{x \in Z^d} \mu \left( 1 - \mathbb{1}_{A_x^{(j)}} \right) = 0.
$$

(3.3)

**Proof.** Eq. (3.3) implies Eq. (3.1) and the proof how Eq. (3.1) implies Eq. (3.2) is in [15]. Note that [15] made the statement with KCM in mind, but the proof only uses that $\mu$ is a product measure so applies equally to MCEM.

3.2. Monotonicity in $G$ of the spectral gap. Naturally one conjectures that the more vacancy types are added to the $G$-MCEM the lower the spectral gap should be as the model gets progressively more jammed through the interaction of the various vacancy types. Indeed, the next result shows this is the case.

**Lemma 3.3.** For any $G' \subset G \subset H_d$ and valid parameter set $q$ for the $G$-MCEM we have

$$
\gamma(G, q) \leq \gamma(G', q')
$$

with $q' = \{q_h : h \in G'\}$ and in particular

$$
\gamma(G, q) \leq \gamma_d(q_{\min}).
$$
Proof. Let $G' \subset G \subset H_d$ and fix a parameter set $q$ for the $G$-MCEM. Recall that $S(G) = G \cup \{\star\}$. Define the projection $\varphi$ on $S(G)$ to $S(G')$ that maps $G'$ onto itself and $S(G) \setminus G'$ to $\star$. We then have, through the variational characterisation of the spectral gap Eq. (1.2), that

$$\gamma(G, q) = \inf_{f \in \text{Dom}(L(G, q))} \frac{D(f)}{\text{Var}(f)} \leq \inf_{g \in \text{Dom}(L(G', q))} \frac{D(g \circ \varphi)}{\text{Var}(g \circ \varphi)},$$

where (exceptionally) we write $L(G, q)$ for the generator of the $G$-MCEM to make the $G$-dependence explicit in this proof. Write $\nu'$ for the measure on $S(G')$ that assigns probability $q_h$ to $h \in G'$ and $p' := 1 - \sum_{h \in G'} q_h$ to $\star$ and let $\mu'$ be the product measure of $\nu'$. Since $\mu(g \circ \varphi) = \mu'(g)$ and $(g \circ \varphi)(\cdot)^2 = g^2 \circ \varphi(\cdot)$ we get $\text{Var}(g \circ \varphi) = \text{Var}_{\mu'}(g)$. For the Dirichlet form we get (recall Eq. (1.1))

$$D(g \circ \varphi) = \sum_{h \in G} \sum_{x \in \mathbb{Z}^d} \mu'[c^h_x q_h p'(\nabla^{(h)}(g \circ \varphi))^2] \leq \sum_{h \in G'} \sum_{x \in \mathbb{Z}^d} \mu'[c^h_x q_h p'(\nabla^{(h)}(g))^2],$$

where we used that the constraints $c^h_x$ only check whether a qualified neighbour is $h$ or not, and thus is the same for the $G$-MCEM and the $G'$-MCEM if $h \in G'$. Further $\nabla^{(h)}(g \circ \varphi)(\omega) = 0$ if $\omega_x \not\in G'$ and so we could replace $\mu$ with $\mu'$. The r.h.s. is equal to the Dirichlet form of the $G'$-MCEM with parameter set $q$ so we get the first part of the claim.

The second part follows analogously by mapping the $h$ with the lowest equilibrium density to 0 and all the other states to 1 thus recovering the spectral gap $\gamma_d(q_{\min})$ of the East model with vacancy density $q_{\min}$. \hfill \Box

3.3. Variance as transition terms and the path method. Given the valid parameter set $q$, recall the measure $\nu$ on $S(G) = \{\star\} \cup G$ that assigns probability $q_h$ to $h \in G$ and $p = 1 - \sum_{h \in G} q_h$ to $\star$.

**Lemma 3.4** (Variance as transition terms). For any function $f : S(G) \to \mathbb{R}$ we find

$$\text{Var}_{\nu}(f)(\omega) \leq 2 \sum_{h \in G} q_h (\nabla^{(h)}(f))^2(\omega).$$

**Proof.** Writing $p = q_\star$ in this proof we have

$$\frac{1}{2} \sum_{\omega, \omega' \in S(G)} q_\omega q_{\omega'} (f(\omega) - f(\omega'))^2 = \sum_{\omega \in S(G)} q_\omega f(\omega)^2 - \left(\sum_{\omega \in S(G)} q_\omega f(\omega)\right)^2 = \text{Var}_{\nu}(f).$$

We recover the left hand side of the claim by applying Cauchy-Schwarz, giving

$$(f(\omega) - f(\omega'))^2 \leq 2 \left((f(\omega) - f(\star))^2 + (f(\star) - f(\omega'))^2\right)$$

and thus the claim. \hfill \Box
We say that a family of configurations \( \{(\omega^{(i)})\}_{i \in \mathbb{N}} \) is a legal path if each transition from \( \omega^{(i)} \) to \( \omega^{(i+1)} \) is legal for the G-MCEM. Recall also that \( x \prec_{(h)} y \) for \( h \in H_d \) if \( x \cdot v \leq y \cdot v \) for any \( v \in \mathcal{P}(h) \).

Our second tool, the path method, is a well known trick in estimating the spectral gap see for example [4 Proposition 6.6] or [9] for uses in other contexts. Recall for this the notation of \( \mathcal{D}_\Lambda \) introduced in Remark 1.6 where the sum over all vertices in \( \mathbb{Z}^d \) is replaced by the sum over \( \Lambda \subset \mathbb{Z}^d \) and the equilibrium measure \( \mu \) by \( \mu_\Lambda \).

**Lemma 3.5 (The path method).** Let \( \omega, \eta \in \Omega \) and let \( \Gamma = (\omega^{(1)}, \ldots, \omega^{(n)}) \) be a legal path such that \( \omega^{(1)} = \omega \) and \( \omega^{(n)} = \eta \) and let \( \Lambda \subset \mathbb{Z}^d \) consist of those vertices \( x \) such that \( \omega^{(i)} \neq \omega^{(i+1)} \) for some \( i \in [n] \). Then, for any \( f : \Omega \to \mathbb{R} \)

\[
\mu_\Lambda(\omega)(f(\omega) - f(\eta))^2 \leq \frac{n}{\min(q, p)} \max_{i \in [n]} \frac{\mu_\Lambda(\omega)}{\mu_\Lambda(\omega^{(i)})} \mathcal{D}_\Lambda(f)(\omega).
\]

**Proof.** Write \( f(\omega) - f(\eta) = \sum_{i \in [n-1]} f(\omega^{(i)}) - f(\omega^{(i+1)}) \) as a telescopic sum and use Cauchy-Schwarz to get

\[
\mu_\Lambda(\omega)(f(\omega) - f(\eta))^2 \leq \frac{n}{\min(q, p)} \max_{i \in [n]} \frac{\mu_\Lambda(\omega)}{\mu_\Lambda(\omega^{(i)})} \sum_{i \in [n-1]} \mu_\Lambda(\omega^{(i)})(f(\omega^{(i)}) - f(\omega^{(i+1)}))^2
\]

\[
\leq \frac{n}{\min(q, p)} \max_{i \in [n]} \frac{\mu_\Lambda(\omega)}{\mu_\Lambda(\omega^{(i)})} \mathcal{D}_\Lambda(f)(\omega),
\]

where in the last inequality we used that for \( \omega^{(i)} \to \omega^{(i+1)} \) to be a legal transition there is exactly one \( x \) such that \( \omega^{(i)}_x \neq \omega^{(i+1)}_x \). \( \square \)

In the proofs of part (B) and (C) of Theorem 1 we do not explicitly mention the length of the involved paths as the important thing is that they are finite not how they scale. In Theorem 2 instead it is of crucial importance to know the order of magnitude.

### 4. Proof of Theorem 1

Part (A) of Theorem 1 can be proven without any of the tools just introduced, while the two subparts of part (B) require some more involved construction.

#### 4.1. Proof of Theorem 1(A).

If \( G = H_d \) say that \( \omega \in \Omega_{H_d} \) is a blocked state if \( \omega_{1-h} = h \) for each \( h \in H_d \). By construction, there is no legal transition from a blocked state to a non-blocked state since to transition the \( h \)-vacancy at \( 1-h \) to \( \ast \) you need another \( h \)-vacancy inside \( H_d \) but every vertex in \( H_d \) is already occupied by a different vacancy type. Let \( A \) be the event that \( \omega \upharpoonright_{H_d} \) is a blocked state. Then \( 1_A \) is not a constant function but \( \mathcal{D}(1_A) = 0 \) while \( \mu(1_A) > 0 \) so that \( \text{Var}_\mu(1_A) > 0 \). Thus, 0 is not a simple eigenvalue of the generator and we get the claim by [12, Section IV, Theorem 4.13]. \( \square \)

#### 4.2. Vacancies with a common direction: Proof of Theorem 1(B.i).

By Lemma 3.3 we assume without loss of generality that \( G = \{h_j : j \in \{0, 1\}^{d-1} \simeq H_{d-1}\} \) where \( h_j = (j_1, \ldots, j_{d-1}, 0) \in H_{d-1} \otimes \{0\} \subset H_d \). For any \( j \in H_{d-1} \) we have \( e_d \in \mathcal{P}(h_j) \) and for \( i \in [d-1] \) we have \( (-1)^j e_i \in \mathcal{P}(h_j) \).
We start by identifying a configuration on $H_d$ that allows us to remove any vacancies in $H_d + ke_d$ for $k \geq 2$ and for which we can apply the exterior condition theorem, Theorem 3.2. Then we use the path method to conclude.

Recall from the construction of the MCEM that each vacancy type $h \in G$ is associated to a corner $x \in H_d$ of the hypercube. We say that a configuration $\omega \in \Omega$ is $H_d$-good if $\omega_x$ for $x \in H_d$ is either in the state of its associated vacancy type or in the neutral state if there is no associated vacancy type, i.e. if $\omega_h = h$ for every $h \in G$ and $\omega_x = \ast$ for $x \in H_d \setminus \{ G \}$ (see the top left of Fig. 2). We start by identifying a configuration on $\Omega$ such that $\omega$ is $H_d$-good, then any vertex $v \in H_d$ with $v \cdot e_d = 1$ is in the neutral state, i.e. $\omega_{H_d-1 \otimes \{ 1 \}} \equiv \ast$. Given an $H_d$-good $\omega$ and any vacancy type $h \in G$ there is a legal path starting from $\omega$ and ending in a state $\eta$ with $\eta_x = h$ for $x \in H_d-1 \otimes \{ 1 \}$ and $\eta_x = \omega_x$ otherwise. Indeed, assume $h = (0,0,\ldots,0)$, then we can put $h$ on $e_d = h + e_d$ since $e_d \in \mathcal{P}(h)$. Subsequently, we can put $h$ on any $e_d + e_i$ for $i \neq d$ since $\mathcal{P}(h)$ consists of all positive propagation directions. Iterate this procedure adding another $e_j$ with $j \neq i, d$ and so on until all of $H_d-1 \otimes \{ 1 \}$ is in state $h$. By construction this is possible for any $h \in G$.

Then, there is a legal path starting from $\omega$ ending in a state $\eta$ such that $\eta_{H_d-1 \otimes \{ 2 \}} \equiv \ast$. Indeed, this is a consequence of $e_d$ being a propagation direction of any vacancy type $h$ and the fact that we can bring $h$ to any vertex in $H_d-1 \otimes \{ 1 \}$ as discussed in the previous paragraph. By reversibility, this implies that we can construct a legal path that puts $H_d-1 \otimes \{ 2 \}$ into any state.

For any $k \in \mathbb{N}$, $k \geq 2$ we can iterate this argument to find a legal path from $\omega$ to $\sigma$ where $\sigma_x = \ast$ if $x \in \cup_{j \in [2,k]}(H_d-1 \otimes \{ j \})$ and $\sigma_x = \omega_x$ otherwise. By reversibility we can thus find a legal path to any $\sigma$ that agrees with $\omega$ outside of $\cup_{j \in [2,k]}(H_d-1 \otimes \{ j \})$.

For $x \in \mathbb{Z}^d$ we say that $\omega$ is $(H_d + x)$-good if $\eta$ given by $\eta_y = \omega_{y+x}$ is $H_d$ good. Let $V_n = \{ x \in \mathbb{Z}^d : x \cdot e_d \geq -n \}$ for $n \in \mathbb{Z}$ so that $\{ V_n \}_{n \in \mathbb{Z}}$ is an increasing and exhausting family of subsets of $\mathbb{Z}^d$. With $A_{x,j} := \{ \omega : \omega \text{ is } (H_d + x - (j + 1)e_d)\text{-good} \}$ we find that the family $A_{x,j}(N) = \cup_{j \in [N]}A_{x,j}$ for $N \geq 1$ satisfies the exterior condition with respect to $\{ V_n \}_{n \in \mathbb{Z}}$. The support of $A_{x,j}(N)$ increases linearly in $N$ but the equilibrium failure probability decreases exponentially in $N$. Thus, we can choose $N$ large enough for Eq. (3.1) to hold and with Lemma 3.4 we find a constant $C(q)$ such that

$$\text{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^d} \mu\left(1_{A_{x,j}(N)} \text{Var}_x(f)\right) \leq C(q) \sum_{x \in \mathbb{Z}^d} \sum_{j \in [N]} \sum_{h \in G} \sum_{\omega \in A_{x,j}} \mu(\omega)(\nabla_x^{(h)} f(\omega))^2.$$  

Fix some $x \in \mathbb{Z}^d$, $j \in [N]$, $h \in G$ and $\omega \in A_{x,j}$ and assume w.l.o.g. that $\omega_x = h$. By the above observations and translation invariance of the dynamics we find a legal path $(\sigma^{(1)}, \ldots, \sigma^{(m)})$ with $m = O(N^2 2^{2d})$, $\sigma^{(1)} = \omega$ and $\sigma^{(m)} = \sigma$ where $\sigma$ is the state given by $\sigma_x = \ast$ and $\sigma_{\mathbb{Z}^d \setminus \{ x \}} = \omega_{\mathbb{Z}^d \setminus \{ x \}}$. Using the path method gives

$$\mu(\omega)(\nabla_x^{(h)} f(\omega))^2 \leq C(q,m) \mu_{\mathbb{Z}^d \setminus A_{x,j}}(\omega) D_{A_{x,j}}(f)(\omega)$$
Figure 2. Path from proof of part (B.i) for \( d = 3 \). The first image (top left) shows the part on \( H_d \) of a \( H_d \)-good configuration with the propagation directions of the involved vacancies. To remove the \((0, 0, 1)\)-vacancy on \((0, 1, 2)\) we use the red path from the second image. Iterating this procedure to put \( * \) on all the black vertices (of initially arbitrary state) at \((\cdot, \cdot, 2)\) (third image). This procedure iterates to any \((\cdot, \cdot, k)\) for \( k \geq 2 \) (fourth picture).

where \( \Lambda_j(x) \) is the smallest box containing both the support of \( A_{x,j} \) and the origin. Using that \( \Lambda_j(x) \) is finite for any \( x \) and \( j \) we get

\[
\text{Var}(f) \leq C(q, m) \sum_{x \in \mathbb{Z}^d} \sum_{j \in [N]} \sum_{h \in G} \sum_{\omega \in A_{x,j}} \mu_{\mathbb{Z}^d \setminus \Lambda_j(x)}(\omega) \mathcal{D}_{\Lambda_j(x)}(f)(\omega)
\]

\[
\leq C(q, m) \sum_{x \in \mathbb{Z}^d} \sum_{j \in [N]} \mu(-f \mathcal{L}_{\Lambda_j(x)} f)
\]

\[
\leq C(q, m) \mathcal{D}(f),
\]

where the \( C(q, m) \) may be different from line to line (as is the convention for constants throughout this paper). By the variational characterisation of the spectral gap we thus have

\[
\gamma(G, q) > 1/C(q, m).
\]

which is the claim. \(\Box\)

4.3. \( G \) as a star graph: Proof of Theorem 1(B.ii). By Lemma 3.3, assume w.l.o.g. that \( G = \{h_c, h_1, \ldots, h_d\} \) where \( h_c = 0 \) is the central vertex of \( G \) and \( h_i = e_i, \; i \in [d] \). We have \( \mathcal{P}(h_c) = \{e_1, \ldots, e_d\} \) and \( \mathcal{P}(h_i) = \{e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_d\} \) so that the direction \(-e_i\) is unique to \( h_i \).

For \( x \in \mathbb{Z}^d \) and \( N_1, \ldots, N_d \in \mathbb{N} \) we say that the set \( x + \bigotimes_{i \in [d]} \{0, \ldots, N_i\} \) is a box with side lengths \( (N_1, \ldots, N_d) \) and origin \( x \). We call \( x + (N_1, \ldots, N_d) \) the top right corner of
B. Let $\Lambda$ be the equilateral box of side length 2 and origin at 0, i.e. $\Lambda = \{0, 1, 2\}$. We call the vertex $x \in \Lambda$ a corner of $\Lambda$ if $x_i \in \{0, 2\}$ for all $i \in [d]$ and write $F_i = \{x \in \Lambda: x_i = 0\}$. For a configuration $\omega \in \Omega$ we say that $\Lambda$ is good if $\omega_{2h} = h$ for every $h \in G$ and $\omega_x = \ast$ for $x \in \cup_i F_i \setminus \{2h: h \in G\}$. Analogously define good boxes $\Lambda + x$ for any $x \in \mathbb{Z}^d$.

**Lemma 4.1.** If $\omega \in \Omega$ is such that $\Lambda$ is good and for each $i \in [d]$ there is a smallest $k_i \geq 2$ with $\omega_{v+k_i e_i} = h_i$, where we write $v = \sum_{i \in [d]} e_i$, then there is a legal path starting at $\omega$ and ending at a configuration $\sigma$ such that

(i) $\Lambda + v$ is good in $\sigma$, and

(ii) $\sigma_x = \ast$ for any $x$ between $\Lambda$ and $v + k_i e_i$, i.e. any $x \in \cup_i \{v + je_i: 3 \leq j \leq k_i - 1\}$ if $k_i \geq 4$, and

(iii) $\omega$ and $\sigma$ agree otherwise.

**Proof.** We start by showing that there is a legal path that puts any state on $\Lambda \setminus \cup_i F_i$ and then we show how to use this to get that $\Lambda + v$ is good. The steps are visualised for $d = 2$ in Fig. 3.

Relax $\Lambda \setminus \cup_i F_i$: Fix an $\omega$ as in the claim. Consider the vertex $h_i + v = v \in \Lambda$. For any $i \in [d]$ we have $v - e_i \in F_i \setminus \{2h: h \in G\}$ and thus $\omega_{v-e_i} = \ast$. Now let $j \in [d], j \neq i$. Since $-e_j$ is the unique negative unit vector in $\mathcal{P}(h_j)$, there is a path from $e_j$ to $v - e_i$ contained in $F_i \setminus \{2h: h \in G\}$ consisting only of steps in $\mathcal{P}(h_i)$. Similar considerations apply to paths from the origin containing only steps in $\mathcal{P}(h_i)$. Thus, recalling that $e_i \in \mathcal{P}(h_i)$, if $\Lambda$ is good there is a legal path that removes any non-$h_i$-vacancy from $v$. Since $i$ was arbitrary any vacancy type on $v$ can be removed and by reversibility also any vacancy type can be put. Hence, we can also remove any non $h_i$-vacancy from $v + e_i$.

Now use that $v + e_i - e_j$ for $j \neq i$ is in $F_j$, if $d > 2$ it is even in $F_j \setminus \{2h: h \in G\}$. If $v + e_i - e_j$ is in $F_j$, there is a path contained in $F_j \setminus \{2h: h \in G\}$ from $e_i$ to $v + e_i - e_j$ consisting only of steps in $\mathcal{P}(h_i)$ so that there is a legal path that removes any vacancy from $v + e_i$. Since $i$ was arbitrary again we find a legal path that can put or remove any vacancy from $v + e_i$ for a canonical base vector $e \in B$. Analogously, it follows by induction in $n$ that we can remove or put any vacancy type on $x \in \Lambda \setminus \cup_i F_i$ with $\|x - v\|_1 = n$, where we just proved the base case $n = 1$.

Make $\Lambda + v$ good: For $i \in [d]$ we want to find a legal path which puts $h_i$ on $2e_i + v$ and, if $k_i \geq 4$, also puts the neutral state $\ast$ on $\{v + je_i: 3 \leq j \leq k_i - 1\}$. Then remove any other vacancies from $\cup_i \{F_i + v\} \setminus \{2h + v: h \in G\}$.

Let $i \in [d]$ and assume w.l.o.g. that $k_i > 2$ since otherwise the $h_i$-vacancy is already at the correct position for $\Lambda + v$ to be good. We already know that we can put any vacancy on $v$ and since $e_i \in \mathcal{P}(h)$ for every $h \in G \setminus \{h_i\}$ there is a legal path that removes any vacancy from $\{v + je_i: j \in [1, k_i - 1]\}$ (that are by assumption not $h_i$-vacancies since $k_i$ is the smallest integer such that $\omega_{v+k_ie_i} = h_i$). Then, use that $-e_i \in \mathcal{P}(h_i)$ to bring the $h_i$ from $v + k_i e_i$ to $v + 2e_i$ and put $\ast$ in between $v + 2e_i$ and $v + k_i e_i$.  

\footnote{We use the term path to mean paths on the lattice $\mathbb{Z}^d$ and the term legal path to mean paths of configurations in $\Omega$ that are legal in the $G$-MCEM.}
The Lemma tells us that we can move a good $\Lambda$ in the direction $v$, given enough non-central vacancies outside of $\Lambda$. To satisfy the exterior condition this is too lose a condition as we cannot always assume that we find these vacancies for each step. The next Lemma gives another construction that does not require new vacancies after every step.
Lemma 4.2. Fix an $N \in 2\mathbb{N}$, $N \geq 4$. Let $\omega \in \Omega$ be such that $\Lambda$ is good and for each $i \in [d]$ there is a $k_i \in [N, 3N/2]$ with $\omega_{v + k_i e_i} = h_i$ and such that $\omega_y = \star$ for each $y \in \{v + ne_i : n \in [k_i - 1]\}$. Then, there is a legal path starting at $\omega$ and ending at $\sigma$ such that $\Lambda + (N - 2)v$ is good and that agrees with $\omega$ otherwise.

Proof. Fig. 4 illustrates a state $\omega$ as in the claim and the steps of the following proof. We start by clearing the line $\{2v + je_i : j \in [N - 1]\}$ of any vacancies and then move the good box by $v$ so that we recover the initial situation and can iterate the argument.

Fix an $i, j \in [d]$ s.t. $i \neq j$. We can bring the $h_i$-vacancy from $v + k_i e_i$ to $v + ne_i$ for any $n \in [N]$. Thus, we can remove any $h_i$-vacancy from $\{v + ne_i : n \in [N]\}$. Since $v + e_j \in \Lambda \setminus \cup_i F_i$, we can put any $h$-vacancy on it. Thus, there is a legal path to remove any vacancy from $\{v + ne_i + e_j : n \in [N]\}$ using that $e_i \in \mathcal{P}(h)$ for $h \neq h_i$. The chosen $j \neq i$ was arbitrary so that this works for any pair $i, j$. 

**Figure 4.** The first image (top left) shows an example $\omega$ as in Lemma 4.2 for two dimensions using the colour and shape code from Fig. 2. The second image shows the paths used to get rid of any vacancies on $\{2v + je_i : j \in [N - 1]\}$. The third image then shows the paths to move the good box and put the $(0, 1)$- resp. $(1, 0)$-vacancy on the respective $\{2v + (N - 1)e_i\}$. The fourth image then shows how the resulting state is the same as in Lemma 4.2 translated by $v$ so we can iterate the proof by setting $N \mapsto N - 1$. 
Analogously to the proof of Lemma 4.1 this is the starting case for the proof that we can remove any vacancy from \( \{v + ne_i + \sum_{j \in I} e_j : n \in [N - 1], I \subset [d] \setminus [i], |I| = m \} \) for any \( m \leq d - 1 \) by induction in \( m \).

In particular, we can remove any vacancy from \( y \in 2v + (n - 1)e_i \) for \( n \in [N] \) since \( y = v + ne_i + \sum_{j \in [d]\setminus\{e_i\}} e_j \) is covered by the case \( m = d - 1 \). The choice of \( i \in [d] \) was arbitrary so that we can construct a legal path that ends in a state \( \sigma(1) \) on which \( \cup_i \{2v + ne_i : n \in [N - 2]\} \) is in the neutral state and \( 2v + (N - 1)e_i \) has an \( h_i \)-vacancy.

Further, by Lemma 4.1 there is a path starting at \( \sigma(1) \) and ending in a state \( \sigma(2) \) in which \( \Lambda + v \) is good and which does not change \( \cup_i \{2v + ne_i : n \in [N - 1]\} \). The state \( \sigma(2) \) is now in the configuration of the claim for \( N - 1 \) so that we can iterate the proof until we find a legal path that ends in a state with \( \Lambda + (N - 2)v \) good.

There is a useful property on boxes that allows us to use both of the previous Lemmas.

**Definition 4.3.** We say that a subset \( R \subset \mathbb{Z}^d \) is colourful in a configuration \( \omega \in \Omega \) if for each vacancy type \( h \in G \) on each straight line connecting two opposite boundaries of \( R \) there is an \( x \) such that \( \omega_x = h \).

For \( N \in 2\mathbb{N} \) define the event \( \mathcal{E}(N) \) (see Fig. 5) as the set of configurations \( \omega \) such that

(E.i) Let \( B \) be the box with side lengths \( (N, \ldots, N) \) and origin \( -Nv \). The subset of \( B \) given by the box with side lengths \( (N - 1, \ldots, N - 1) \) and origin \( -(N - 1)v \) excluding the origin \( 0 \) is colourful.

(E.ii) For each vertex \( x \in B \) such that there is an \( i \in [d] \) with \( x \cdot e_i = -N \) there is a \( k \in [N] \) and a good box \( \Lambda \) such that \( x - kv \) is the top right corner of \( \Lambda \).

(E.iii) For each \( i \in [d] \) the box with origin \( -2N e_i \) and side lengths given by \( N \) in the \( i \)-direction and \( N/2 \) otherwise is colourful.

Let \( \mathcal{E}_x(N) \) be the correspondingly translated event for \( x \in \mathbb{Z}^d \). On \( \mathbb{Z}^d \) consider the \( d - 1 \)-dimensional hyperplane \( U_0 \) perpendicular to \( v \) that goes through the origin. By construction we have \( \text{Supp}(\mathcal{E}(N)) \subset V_0' \) where \( V_0 := \bigcup_{i=1}^\infty (U_0 + \ell v) \). And the family \( \{\mathcal{E}_x(N)\}_{x \in \mathbb{Z}^d} \) satisfies the exterior condition w.r.t. \( \{V_n\}_{n \in \mathbb{Z}} \) where \( V_n = V_0 - nv \). The failure probability of \( \mathcal{E}_x(N) \) can be upper bounded by a series of simple union bounds to get a bound that is exponentially decreasing with \( N \) while the support grows polynomially in \( N \).

We can thus choose \( N \) large enough (depending on \( q \)) such that Theorem 3.2 then gives

\[
\text{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^d} \mu(1_{\mathcal{E}_x(N)} \text{Var}_x(f)).
\]

The proof then concludes analogously to the proof of (B.i) once we have defined the paths that allow us to remove or put any vacancy type on \( x \). W.l.o.g. consider only the case \( x = 0 \) and fix \( \omega \in \mathcal{E}(N) \). Consider a \( y \) in the box \( B \) from (E.i) such that \( y \cdot e_i = -N \) for some \( i \in [d] \). By (E.ii) there is a \( k \in [N] \) such that \( y - kv \) is in a good box \( \Lambda \). For any \( j \in [d] \) we are guaranteed to hit one of the boxes from (E.iii) or the subset of \( B \) given in (E.i) when going in the direction \( e_j \) from \( y - kv \). Since these sets are colourful, we can apply Lemma 4.1 to propagate the good box. Since this works for arbitrary \( k \),
we can propagate it until it intersects $B$ and put the neutral state on $y$ in a legal path of finite length. These transitions are independent of the state of $y$, so we find a legal path starting at $\omega$ and ending in a state $\sigma$ where $\sigma_y = \star$ and $\sigma_z = \omega_z$ for $z \neq y$.

The vector $y$ was arbitrary so that we can iterate this argument until the conditions to apply Lemma 4.2 are satisfied. Notice that for this we need to propagate and leave the good box around $-Nv$ which follows by the same argument. The statement then follows analogously to (B.i).

\[ \Box \]

5. \textbf{Spectral gap bounds for the two-dimensional MCEM: Proof of Theorem 2}

The upper bound in Theorem 2 follows by Lemma 3.3. The steps to prove the corresponding lower bound are analogous to the proof of Theorem 1(B). The main difference is that we have to be careful about the cost of our intermediate steps. Before we were fine estimating the spectral gap by any positive constant. Now we want to show that in the highest order the spectral gap is given by the spectral gap of the East model on $\mathbb{Z}^d$ which we explicitly recall here as it is central to the proof.

\textbf{Theorem 5.1 (\cite{[3]} Theorem 1)}. As $q \rightarrow 0$ the spectral gap $\gamma_d(q)$ of the East model on $\mathbb{Z}^d$ with parameter $q$ is given as

\[ \gamma_d(q) = 2^{-\frac{q^2}{2\theta_q^2}} (1 + \omega(1)), \]

where $\theta_q := |\log_2(q)|$.

5.1. \textbf{Preliminary constructions}. Note that by Lemma 3.3 the cases (3.i) and (3.iii) imply the cases (2.i) and (2.ii). Using this and symmetry considerations, w.l.o.g. we can assume in the following that $G = \{(1, 1), (0, 0), (0, 1)\}$. We call the associated
We do this by defining events \( A \) of the more intuitive direction from only using steps in \( P \). The spectral gap is then to identify events each vacancy type separately. The main difficulty in finding good lower bounds on the overcounting term can be absorbed into the \( \Lambda \) for some \( h \in \mathbb{Z}^2 \). Let us start by outlining the construction used in the proof of part (3.i) and (3.ii).

Fix a \( G \subset H_2 \) and consider a family \( \{A_x\}_{x \in \mathbb{Z}^2} \) of events that satisfies the requirements of the exterior condition theorem so that with Lemma 3.4 we have

\[
\text{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^2} \mu(\mathbb{1}_{A_x} \text{Var}_x(f)) \leq 4 \sum_{h \in G} \sum_{x \in \mathbb{Z}^2} \mu[\mathbb{1}_{A_x} \mathbb{1}_{\text{pq}}(\nabla_x f)]^2.
\]

Since \( \mathbb{1}_{\text{pq}}(\nabla_x f)^2 = \text{Var}_x(f \mathbb{1}_{\{x,h\}}) =: \text{Var}_x(f \mathbb{1}_{\{x,h\}}) \) we can treat the transition for each vacancy type separately. The main difficulty in finding good lower bounds on the spectral gap is then to identify events \( A_x \) that satisfy the exterior condition, have a low failing probability and such that for each \( h \in G \) we have

\[
\mu[\mathbb{1}_{A_x} \text{Var}_x(f | \{x,h\})] \leq 2 \gamma_{\min}^{(1+\varepsilon)/4} \mu[D_{\Lambda_h}(f)],
\]

for some \( \Lambda_h \) such that the overlap of the various \( \Lambda_h \) for the different \( x \) (and thus the overcounting term) can be absorbed into the \( \varepsilon \) in \( 2 \gamma_{\min}^{(1+\varepsilon)/4} \) for \( \gamma_{\min} \) small enough.

We do this by defining events \( A_x^{(h)} \) for each \( h \in G \) and setting \( A_x = \bigcap_{h \in G} A_x^{(h)} \). Each \( A_x^{(h)} \) is defined such that it allows the rewriting of the local variance with indicator \( \mathbb{1}_{A_x^{(h)}} \) to a Dirichlet form by using a mixture of auxiliary models that behave like the standard one- or two-dimensional East model and the path method. Let us start by outlining the construction used in the proof of part (3.i) and (3.ii).

5.1.1. Geometric construction. Let us start with some deterministic constructions for \( h \in G \).

**Definition 5.2** \((h\text{-paths})\). For \( h \in H_2 \) we say that \( \Gamma = (x_1, \ldots, x_n) \subset \mathbb{Z}^2 \) is an \( h \)-path if \( x_i - x_{i+1} \in \mathcal{P}(h) \) for \( i \in [n-1] \), i.e. starting from \( x_n \) we can reach \( x_1 \) staying on \( \Gamma \) and only using steps in \( \mathcal{P}(h) \).

**Remark 5.3.** Note that we want \( x_i - x_{i+1} \) to be a propagation direction of \( \mathcal{P}(h) \) instead of the more intuitive direction from \( x_i \) to \( x_{i+1} \) (i.e. \( x_{i+1} - x_i \)). Defining it this way we can find an \( h \)-path starting from some vertex \( x \in \mathbb{Z}^d \) and ending in a vertex containing an \( h \)-vacancy which can then travel on the \( h \)-path back to \( x \).

We build the \( h \)-grid first for \( B \)-vacancies and then explain how to generalise to \( h \in \{A,C\} \). We do the construction incrementally by starting with a base cell for \( B \)-vacancies.

**Definition 5.4** \((B\text{-Base cell} Q)\). Let \( \ell \in 8\mathbb{N} \). Define \( D^{(1)} \subset \mathbb{Z}^2 \) as the \( B \)-path starting at \( e_1 + 3e_2 \) that first does an \( e_1 \)-step, then zigzags north and east for 2 steps respectively until \( \ell \) steps east have been made with the last step being a single one. Then define \( D^{(2)} \subset \mathbb{Z}^2 \) as the path starting again at \( e_1 + 3e_2 \) which starts with 4 steps north, goes one step east and...
then zigzags 8 steps north and one step east until \( \ell \) steps north have been made with the last step 4 long instead of 8. Then, define \( D^{(3)} = D^{(1)} + \ell/8e_1 + \ell e_2 \) and \( D^{(4)} = D^{(2)} + \ell e_1 + \ell e_2 \), i.e. the paths \( D^{(1)} \) resp. \( D^{(2)} \) shifted to start at the end point of \( D^{(2)} \) resp. \( D^{(1)} \). We then define the \( B \)-base cell \( Q \) with side length \( \ell \) as the set of vertices enclosed by and including the boundaries \( D^{(i)} \) for \( i \in [4] \). We refer to \( D^{(i)} \) as the bottom, left, top and right boundary of \( Q \) for \( i = 1, 2, 3, 4 \) respectively (see left side of Fig. 6).

\[
\begin{align*}
D^{(1)} & = D^{(3)} + \ell/8e_1 + \ell e_2, \\
D^{(2)} & = D^{(4)} + \ell e_1 + \ell e_2.
\end{align*}
\]

**Figure 6.** Left and right: \( B \)-Base cell \( Q \) with side length \( \ell = 16 \) (see Definition 5.4). Left: Notation as introduced in Definition 5.4. Right: Base cell \( Q \) with cross as in Definition 5.5 with horizontal interior crossing in blue and vertical interior crossing in red. The colours only serve to better distinguish the horizontal from the vertical path.

For the rest of this section fix a side length \( \ell \). In this base cell we define the notion of interior crossing paths in the horizontal and vertical direction.

**Definition 5.5** (Interior \( B \)-crossings and cross). Let \( Q \) be the \( B \)-base cell. We say that a \( B \)-path \( (x^{(1)}, \ldots, x^{(n)}) \subset Q \) is a vertical interior \( B \)-crossing for \( Q \) if \( x^{(1)} \in D^{(1)} \), \( x^{(n)} \in D^{(3)} \) and \( x^{(i)} \notin \bigcup_{i \in [4]} D^{(i)} \) for \( i \in [2, n-1] \). Similarly, we say that it is a horizontal interior \( B \)-crossing if \( x^{(1)} \in D^{(2)} \), \( x^{(n)} \in D^{(4)} \) and \( x^{(i)} \notin \bigcup_{i \in [4]} D^{(i)} \) for \( i \in [2, n-1] \) (see right side of Fig. 6). We call a pair \( C_0 = (C_0^{(v)}, C_0^{(h)}) \) of a vertical interior crossing and horizontal interior crossing of \( Q \) a cross in \( Q \).
We translate the cell $Q$ to construct larger square grids of cells.

**Definition 5.6 ($Q_{i,j}$).** Let $b_1 = \ell (e_1 + e_2)$ and $b_2 = (\ell/8)e_1 + \ell e_2$. For $i, j \in \mathbb{Z}$ we then let $Q_{i,j} = Q_0 + ib_1 + jb_2$. Given a square side length $N \in \mathbb{N}$ we define the rectangle of grids $Q^{(B)}$ as

$$Q^{(B)} = \bigcup_{(i,j) \in [0,N]^2} Q_{i,j}.$$

**Remark 5.7.** Notice that $Q_{0,0} = Q$ and that neighbouring cells share a boundary.

In what follows consider the square side length $N \in \mathbb{N}$ fixed. On sets of neighbouring cells we introduce a notion of hard interior $B$-crossing, as opposed to the local one which only dealt with paths in one cell.

**Definition 5.8 ($B$-strips and hard interior $B$-crossing).** For $i \in [0,N]$ we call the set of cells

$$Q^{(v)}_i = \bigcup_{j \in [0,N]} Q_{i,j}$$

**Figure 7.** Left: $Q_{i,j}$ for $i \in [0,2]$ and $j \in [0,2]$, side length $\ell = 8$. The vertical strip $Q^{(v)}_1$ and the horizontal strip $Q^{(h)}_2$ are shaded in gray. Right: A grid $C$ with $N = 2$. The hard vertical interior crossings are red and hard horizontal interior crossings are blue and the intersections points $X(C)$ are black.
the $i$-th vertical $B$-strip and for $j \in [0, N]$ we define the $j$-th horizontal $B$-strip as

$$Q_j^{(h)} = \bigcup_{i \in [0, N]} Q_{i,j}.$$ 

A $B$-path $\Gamma \subset Q_i$ is a hard vertical interior $B$-crossing of $Q_i$ if $\Gamma \cap Q_{i,j}$ is a vertical interior $B$-crossing of $Q_{i,j}$ for any $j \in [0, N]$. Analogously for hard horizontal interior $B$-crossings (see Fig. 7).

The set of hard interior crossings induce a grid $C$.

**Definition 5.9 (B-grids).** For $i \in [0, N]$ let $C_i^{(v)}$ be a hard vertical interior crossing for the $i$-th vertical $B$-strip and for $j \in [0, N]$ let $C_j^{(h)}$ be a hard horizontal interior crossing of the $j$-th horizontal strip. We call $C = (C^{(v)}, C^{(h)})$ a $B$-grid of $Q(B)$ where $C^{(v/h)} = \{C_i^{(v/h)}\}_{i \in [0, N]}$. Given a grid $C$ of $Q(B)$ we call $C_{i,j} = (C_i^{(v)}, C_j^{(h)})$ the cross induced in $Q_{i,j}$.

The intersection points of the induced crosses in each $Q_{i,j}$ form a set that is isomorphic to an equilateral box in $\mathbb{Z}^2$.

**Definition 5.10 (Intersection points associated to grid).** Given a $B$-grid $C$ of $Q(B)$ we denote by $x_{i,j}$ the highest point in $C_i^{(v)} \cap C_j^{(h)}$ in the $\prec$-partial order\(^3\) and call it an intersection point of $C$. We write $X(C)$ for the set of intersection points. We call $x_{i,j}$ and $x_{i',j'}$ neighbours in $X(C)$ if $(i, j)$ and $(i', j')$ are neighbours in $[0, N]^2$. Analogously we call $x_{i',j'}$ an oriented neighbour of $x_{i,j}$ in $X(C)$ if $x_{i',j'}$ and $x_{i,j}$ are neighbours in $X(C)$ such that $(i', j') \prec (i, j)$. We call $x_{i+1,j}$ (if it exists) the east neighbour of $x_{i,j}$ in $X(C)$ and $x_{i,j+1}$ (if it exists) the north neighbour of $x_{i,j}$ in $X(C)$ and analogously for the south and west neighbours.

**Remark 5.11.** The $\prec$-ordering is only partial but since we look at intersection points of $B$-paths there is always a unique highest point on $C_i^{(v)} \cap C_j^{(h)}$, and since $C$ induces a cross in each $Q_{i,j}$, $x_{i,j}$ is well defined for any $i, j \in [0, N]$.

The $A$-base cell is defined analogously by exchanging the role of $e_1$ with $-e_2$ and $e_2$ with $-e_1$ and for the $C$-base cell exchange $e_1$ with $-e_1$. Do the analogous exchanges in the following definitions for $h$-crossings and $h$-grids. When changing the base vectors like this the horizontal $A$-crossing would cross the base cell vertically so change the names appropriately.

Further, this construction can be translated to be based at any $x \in \mathbb{Z}^2$ by replacing the origin in the definitions with $x$. We will denote this as an explicit argument so $Q_{i,j}(x) := Q_{i,j} + x$. Since by translation invariance we can apply the results for the origin to any $x \in \mathbb{Z}^2$ this notation is rarely used.

Consider the set $V_0$ given by the points $x \in \mathbb{Z}^2$ such that $-x_1 + x_2 \leq 0$ (i.e. the set that is ‘below’ the main diagonal going through the origin) and for $n \in \mathbb{Z}$ let $V_n = V_0 + (-n, n)$, then $\{V_n\}_{n \in \mathbb{Z}}$ is an increasing and exhausting set of $\mathbb{Z}^2$. The following Lemma is the principal reason to construct the $h$-grid as we did.

---

\(^3\)The $\prec$-partial order corresponds to the usual order where $x \prec y$ if $x_i \leq y_i$ for all $i \in [d]$, we write $\prec$ to make it easier to generalise to $A$- and $C$-grids.
Lemma 5.12. Let $A_x$ be an event with support in $\cup_{h \in [A, B, C]} Q_x^{(h)}$, then the family $\{A\}_{x \in \mathbb{Z}^2}$ satisfies the exterior condition w.r.t. $\{V_n\}_{n \in \mathbb{Z}}$.

Proof. Follows from the construction of the grids. \qed

5.1.2. Crossing probabilities and grid relaxation. Let $q$ be a parameter set for the $ABC$-model and set $\ell = \lfloor \theta_B^2/2 \rfloor$, $N = 2^{\lfloor \theta_B/2 + \log_2(\theta_B) \rfloor}$ as the parameters for any base cells and grids. The goal for this section is to define an event so that we can use the exterior condition theorem, Theorem 3.2.

For a fixed $\omega \in \Omega$ let $\Gamma = \Gamma(\omega)$ be a closed non-backtracking dual path connecting the left to the right boundary and $n_n, n_e, n_s, n_w$ be the amount of north, east, south and west steps in it respectively. $\Gamma$ being closed then implies the existence of at least $(n_e + n_s)/2$ $A$- or $C$-vacancies, only half since if an east step follows a south step they
have the same associated vertex. Further note that by construction of $Q_i^{(v)}$ every eighth step north an additional step east or south has to be made to reach the right boundary while any step west immediately implies another step east. So, $\Gamma$ being closed implies the existence $\Theta(|\Gamma|)$ $A$- or $C$-vacancies. Let $\Pi_{x^*}$ be the set of dual paths starting at $x^* \in D^{(2,*)}$ and ending at $D^{(4,*)}$. We then have for some constants $\kappa, C$,

$$
\mu(\mathcal{A}^c) \leq \sum_{x^* \in D^{(2,*)}} \sum_{\Gamma \in \Pi_{x^*}} \mu(\Gamma \text{ is closed}) \\
\leq \sum_{x^* \in D^{(2,*)}} \sum_{\Gamma \in \Pi_{x^*}} (q_A + q_C)^{\Theta(|\Gamma|)} \\
\leq \sum_{x^* \in D^{(2,*)}} \sum_{k=\kappa \ell}^{\infty} 3^k (q_A + q_C)^{\Theta(k)} \\
\leq CN \ell 2^{-\ell}
$$

where we chose $q_B$ small enough and use that $q_A + q_C \to 0$ as $q_B \to 0$. The proof for horizontal strips is analogous and the claim follows.

With this we can calculate the failing probability of finding a $B$-traversable grid is a simple union bound.

**Corollary 5.14.** Let $E^{(B,1)}$ be the event that

4Note, we are not saying that there are only $\Theta(|\Gamma|)$ vacancies, but that the directly implied amount is of this order
Then, for parameter sets such that

\[ (q_A + q_c) \to 0 \]

as \( \nu \to 0 \) we have

\[
\lim_{\nu \to 0} Q(B) \mu(1 - 1_{E(B, 1)}) = 0.
\]

This gives us a \( B \) vacancy on an intersection point and the necessary \( B \)-traversable paths to bring it into \( Q_{0,0} \). The intersection point \( x_{0,0} \) is still random though so we require another set of \( B \)-traversable paths to bring the \( B \)-vacancy to a deterministic point. The following result gives this with another set of simple estimates.

**Lemma 5.15.** Let \( E^{(B, 2)} \) be the event that the boundary \( D^{(1)}_{0,0} \) is \( B \)-traversable. Then we have for parameter sets such that \( \ell^2(q_A + q_c) \to 0 \) as \( \nu \to 0 \) that

\[
\lim_{\nu \to 0} \text{Supp}(E^{(B, 2)}) \mu(1 - 1_{E(B, 2)}) = 0.
\]

To show that \( E^{(B)} = E^{(B, 1)} \cap E^{(B, 2)} \) allows us to find an inequality like Eq. (5.1) we need to introduce another tool.

**Lemma 5.16 (Extending the variance).** Let \( A = A_1 \cap A_2 \cap A_3 \) be an event on \( \Omega \), let \( V_i := \text{Supp}(A_i) \) for \( i \in [3] \) and assume that \( V_i \cap V_j = \emptyset \) for any pair \( i \neq j \). Then, for any \( f \in L^2(\mu) \) and for the conditional variance \( \text{Var}_x(f | A) = \mu_x(f^2 | A) - (\mu_x(f | A))^2 \) we find

\[
\mu(1_A \text{Var}_V(f | A)) \leq \mu(1_A \text{Var}_V(f | A))
\]

for \( V = V_1 \cup V_2 \).

**Remark 5.17.** The usual use case is that we have an event \( A \) with a large support that we split into two smaller events \( A_1, A_2 \) and the ‘rest’ \( A_3 \) which is why \( V \) only contains \( V_1 \) and \( V_2 \).

**Proof.** Write \( A' = A_1 \cap A_3 \) and calculate directly

\[
\mu(1_A \text{Var}_{V_1}(f | A)) = \mu_{V_2}(A_2) \mu_{V_2}[1_{A'} \mu_{V_2}[(\mu_{V_1}(f^2 | A) - (\mu_{V_1}(f | A))^2) | A_2] \]
\]

\[
\leq \mu_{V_2}(A_2) \mu_{V_2}[1_{A'}(\mu_V(f^2 | A) - (\mu_V(f | A))^2)]
\]

\[
= \mu(1_A \text{Var}_V(f | A)),
\]

where in the first inequality we used Jensen’s inequality and in the last equality we used that \( \text{Var}_V(f | A) \) does not depend on spins in \( V_2 \) anymore.

Any configuration in \( E^{(B)} \) potentially contains many conforming \( B \)-grids so let us introduce a partial order on them. Let \( \Gamma = (x^{(1)}, \ldots, x^{(n)}) \) and \( \Gamma' = (y^{(1)}, \ldots, y^{(m)}) \) be two hard interior crossings of the same strip. If there is no crossing point say that \( \Gamma \) is smaller than \( \Gamma' \) if \( x^{(1)} \prec y^{(1)} \). If they cross in a single point \( x^{(i)} = y^{(j)} \) and \( x^{(i+1)} \prec y^{(j+1)} \) then we say that \( \Gamma \) is smaller than \( \Gamma' \).

This generalises to a partial order on any family of hard interior crossings of the same strip with multiple crossing points if the above condition is fulfilled after every crossing point. Note that this is only a partial order but there is a unique smallest
crossing. For $\omega \in \mathcal{C}^B$ we write $\mathcal{G}(\omega)$ for the $B$-grid with the smallest crossings in each strip conforming to $\mathcal{C}^B$.

A final remark about notation: We will write $\mu^{(h)}(\cdot) := \mu(\cdot | \{*, h\})$ and $\text{Var}^{(h)}(\cdot) := \text{Var}(\cdot | \{*, h\})$ for the measure resp. variance conditioned to be in the state space $\{*, h\}$. Recall further that $\mathcal{Q}^B$ is the grid of base cells with parameters $N, \ell$ of which the smallest vertex in the $\prec$-partial order (i.e. the closest vertex to the origin) is $z_B := e_1 + 3e_2$.

**Lemma 5.18.** Let $f \in L^2(\mu)$. For any $\varepsilon > 0$ we find a $q(\varepsilon) > 0$ such that

$$\mu_{\mathcal{Q}^B}(\mathbb{1}_{\mathcal{E}(B)} \text{Var}^{B}_{z_B}(f)) \leq 2^{\mathbb{R}^B(1+\varepsilon)/4} \sum_{y \in \mathcal{Q}^B} \mu_{\mathcal{Q}^B}\left[\mathbb{q}_{B} p(\nabla^{B}_{y} f)^2\right],$$

for $q_B < q(\varepsilon)$.

**Proof.** For simplicity we write $\mu_{\mathcal{Q}^B} = \mu$ in this proof. There might be many intersection points $x_{i,j} \in X(\mathcal{G}(\omega))$ such that $\omega_{x_{i,j}+e} = B$ for some $e \in \mathcal{B}$. Introduce the constraint $c^{(G)}_{x_{i,j}}$ that there is an $e \in \mathcal{B}$ such that $\omega_{x_{i,j}+e} = B$ and denote by $\xi(\omega)$ the vertex in $X(\mathcal{G}(\omega))$ with the highest coordinate in the lexicographic order such that $c^{(G)}_{\xi(\omega)} = 1$. Since this uniquely identifies a grid and an intersection point we have

$$\mu\left[\mathbb{1}_{\mathcal{E}(B)} \text{Var}^{B}_{z_B}(f)\right] = \sum_{C \text{ grid}} \sum_{x \in X(C)} \mu\left[\mathbb{1}_{\mathcal{G} = C, \xi = x, \mathcal{E}(B, 2)} \text{Var}^{B}_{z_B}(f)\right].$$

Let us upper bound a generic summand $\mu\left[\mathbb{1}_{\mathcal{G} = C, \xi = x, \mathcal{E}(B, 2)} \text{Var}^{B}_{z_B}(f)\right]$ and assume without loss of generality that $x = x_{N,N-1}$. Let $V \subset X(C)$ be a subset with $x_{0,0}, x_{N,N-1} \in V$. The event $\mathcal{G} = C$ on $V \cap z_B$ reduces to requiring $B$-traversability so that we can extend the variance (Lemma 5.16)

$$\mu\left[\mathbb{1}_{\mathcal{G} = C, \xi = x, \mathcal{E}(B, 2)} \text{Var}^{B}_{z_B}(f)\right] = \mu\left[\mathbb{1}_{\mathcal{G} = C, \xi = x, \mathcal{E}(B, 2)} \text{Var}^{B}_{V \cup z_B}(f)\right].$$

We now want to consider two separated blocks. One in which we consider the relaxation on $V$ given that $\xi = x$ and the other were we consider $z_B$ given that $x_{0,0}$ has a $B$-vacancy. We can do this using the block relaxation lemma, that we explicitly state here again for completeness sake as we frequently cite it throughout this paper so that we state it in a more general form. Consider two sets $V_1, V_2 \subset \mathbb{Z}^d$ together with some state space $\Omega^{(i)}$ and measures $\nu^{(i)}$ on $V_i$ for $i \in [2]$. Write $V = V_1 \cup V_2$ and $\nu = \nu^{(1)} \otimes \nu^{(2)}$. Consider an event $A$ on $\Omega^{(1)}$ such that $\nu^{(1)}(A) > 0$.

**Lemma 5.19** (Block relaxation Lemma, [11] Proposition 4.4]). In the above situation we have for $f : \Omega^{(1)} \otimes \Omega^{(2)} \rightarrow \mathbb{R}$

$$\text{Var}_\nu(f) \leq \frac{2}{\nu^{(1)}(A)} \nu(\text{Var}^{(1)}(f) + \mathbb{1}_A \text{Var}^{(2)}(f)).$$

Using this we have

$$\mu\left[\mathbb{1}_{\mathcal{G} = C, \xi = x, \mathcal{E}(B, 2)} \text{Var}^{B}_{V \cup z_B}(f)\right]$$

\begin{equation}
\leq \frac{2}{q_B} \mu\left[\mathbb{1}_{\mathcal{G} = C, \xi = x, \mathcal{E}(B, 2)} \left(\mathbb{1}_{\omega_{0,0} = B} \text{Var}^{B}_{z_B}(f) + \text{Var}^{B}_{V}(f)\right)\right].
\end{equation}
Consider the auxiliary model on \( \Gamma \). Contrary to the one-dimensional East model where \( B_{\omega} \) state is the particle state and note that with \( \Gamma \) being \( B \)-traversable. Thus, we can extend the variance again to get

\[
\mu \left[ 1_{G=C,\xi=x,\mathcal{E}(B,2),\omega_{z_B}=B} \text{Var}_{z_B}(f) \right] \leq \mu \left[ 1_{G=C,\xi=x,\mathcal{E}(B,2),\omega_{x_{0,0}=B}} \text{Var}^{(B)}_{\Gamma}(f) \right]. \tag{5.3}
\]

Consider the auxiliary model on \( \Gamma \) with equilibrium measure \( \mu^{(B)} \) that is given by the one-dimensional East model where \( B \)-vacancies are the vacancy state and the neutral state is the particle state and note that with \( \omega_{x_{0,0}} = B \) this has ergodic boundary conditions. Since \( |\Gamma| = \Theta(\ell) \) we find a constant \( \kappa > 0 \) with [3] Theorem 2 such that

\[
1_{\omega_{x_{0,0}}=B} \text{Var}^{(B)}_{\Gamma}(f) \leq 2^{\kappa \theta_B \log_2(\theta_B)} \sum_{y \in \Gamma} \mu^{(B)} \left( c^B_y p/q_B (\nabla y f)^2 \right),
\]

for \( q_B \) small enough, where we used that the one-dimensional constraints on \( \Gamma \) lower bound the two-dimensional constraints \( c^B_x \) for \( x \in \Gamma \) and that \( p > \Delta \) to bound the \( 1/(q_B + p) \) term coming from the conditional density of vacancies and particles in the East model. Inserting back into Eq. (5.3) gives terms like

\[
\sum_{y \in \Gamma} \mu \left[ 1_{G=C,\xi=x,\mathcal{E}(B,2),c^B_y p/q_B (\nabla y f)^2} \right] \leq \sum_{y \in Q_{0,0}} \mu^{(B)} \left( c^B_y p/q_B (\nabla y f)^2 \right).
\]

Contrary to \( \Gamma \), \( Q_{0,0} \) is not dependent on the specific \( C \) and \( \xi \) anymore so that we can resolve the sum over them to get that the first summand Eq. (5.2) gives a contribution of

\[
2^{\kappa \theta_B \log_2(\theta_B)} \sum_{y \in Q_{0,0}} \mu^{(B)} \left( c^B_y q_B (\nabla y f)^2 \right).
\]

For the second summand in Eq. (5.2) note that we have not yet specified the subset \( V \subset X(C) \). \( X(C) \) is isomorphic to an equilateral box in \( \mathbb{Z}^2 \) and the dynamics with the constraints \( c^G_y \) are equivalent to a two-dimensional East process on that box. Thus by [4] Proposition 3.5)(i) we find a subset \( \{x_{0,0}, x_{N,N-1}\} \subset V \subset X(C) \) such that

\[
\text{Var}^{(B)}_V(f) \leq 2^\theta_B^{D_b(1+\varepsilon)/4} \sum_{y \in V} \mu^{(B)} \left( c^G_y q_B (\nabla y f)^2 \right)
\]

for \( q_B \) small enough. Given the events \( \{G = C\} \cap \{\xi = x\} \) we can again extend to \( B \)-traversable paths this time between points on \( X(C) \) and using completely analogous calculations to the first summand we get.

\[
\mu \left[ 1_{G=C,\xi=x,\mathcal{E}(0,2)} \text{Var}^{(B)}_{V}(f) \right] \leq 2^{\kappa \theta_B \log_2(\theta_B)} \sum_{y \in C} \mu^{(B)} \left[ 1_{G=C,\xi=x,\mathcal{E}(0,2),c^B_y q_B (\nabla y f)^2} \right]
\]

\[
\leq 2^{\kappa \theta_B \log_2(\theta_B)} \sum_{y \in Q(\ell)} \mu^{(B)} \left[ 1_{G=C,\xi=x,\mathcal{E}(B,2),c^B_y q_B (\nabla y f)^2} \right].
\]

Resolve the sum over \( C \) and \( \xi \) again and note that \( Q_{0,0} \) is counted twice leading to an additional term of the order \( O(\ell^2) \) that we absorb into \( \kappa \) to get the claim. \( \square \)
Remark 5.20. For simplicity we limited the discussion in this section to $B$-grids. The results generalise to $h$-grids with $q_h$ going to 0 and the conditions for $q_A + q_C$ are substituted with conditions for $1 - q_h - p$.

5.2. Low vacancy density: Proof of Theorem 2(3.i). Fix an $\varepsilon > 0$, let $q$ be a parameter set such that $\min_{h \in G} q_h = q_B$ and $(q_A + q_C)\theta_B^3 \to 0$. By Lemma 3.3 we have $\gamma(G, q) \leq \gamma_2(q_B)$, so using Theorem 5.1 we need to show that there is a $\delta > 0$ so that for $q_B < \delta$ we have

$$\gamma(G, q) \geq 2^{-\theta_B^3(1+\varepsilon)/4}.$$ 

For $h \in G$ let $\mathcal{E}^{(h)} = \mathcal{E}^{(h,1)} \cap \mathcal{E}^{(h,2)}$ be the events from Section 5.1 and let $\mathcal{E}^{(h)}_x$ be the corresponding translated event and let $z_h$ be the analogous vertices to $z_B$. Using the results from Section 5.1 we can get $h$-vacancies to $z_h$. We thus need an event that allows us to bring the vacancies back to the origin.

Let $\mathcal{E}^{(0)}_x$ be the event that there is no vacancy on $\{x + ie_1 : i \in [3]\}$ and $\mathcal{E}^{(h)}_x := \mathcal{E}^{(0)}_x \cap \bigcap_{h \in G} \mathcal{E}^{(h)}_{\omega_h}$. By construction the family $\{\mathcal{E}^{(h)}_x \}_{x \in \mathbb{Z}^2}$ satisfies the exterior condition with respect to the exhausting and increasing family of sets $\{V_n\}_{n \in \mathbb{Z}}$ given in Lemma 5.12. By assumption on the parameter set and Corollary 5.14 and Lemma 5.15, Eq. (5.1) holds for the family $\{\mathcal{E}^{(h)}_x \}_{x \in \mathbb{Z}^2}$ for $q_B$ small enough. Thus, we can apply the exterior condition theorem, Theorem 3.2 and Lemma 3.4 to get

$$\Var(f) \leq 4 \sum_{x \in \mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{E}^x} \Var_x(f)) \leq C \sum_{x \in \mathbb{Z}^2} \sum_{h \in G} \mu(\mathbb{1}_{\mathcal{E}^{(0)}_x \cap \mathcal{E}^{(h)}_{\omega_h}} \Var^{(h)}_x(f)). \quad (5.4)$$

Let us consider w.l.o.g. only the term for $h = B$ and $x = 0$ and leave away the subscript $x$. Recall that we write $z_B = e_1 + 3e_2$ which by $\mathcal{E}^{(B,2)}$ is $B$-traversable so that we can extend the variance, Lemma 5.16 and apply the block relaxation Lemma, Lemma 5.19

$$
\mu(\mathbb{1}_{\mathcal{E}^{(0)}_0 \cap \mathcal{E}^{(B)}_1} \Var^{(B)}_0(f)) \leq \mu(\mathbb{1}_{\mathcal{E}^{(0)}_0 \cap \mathcal{E}^{(B)}_1} \Var^{(B)}_{\omega_B = 0} (f)) \\
\leq \frac{C}{q_B} \mu(\mathbb{1}_{\mathcal{E}^{(0)}_0 \cap \mathcal{E}^{(B)}_1} \mu^{(B)}_0(\mathbb{1}_{\omega_B = 0} \Var^{(B)}_0 (f) + \Var^{(B)}_1 (f))) \\
= \frac{C}{q_B} \mu(\mathbb{1}_{\mathcal{E}^{(0)}_0 \cap \mathcal{E}^{(B)}_1} (\mathbb{1}_{\omega_B = 0} \Var^{(B)}_0 (f) + \Var^{(B)}_1 (f))),
$$

where in the last equality we used that $\operatorname{Supp}(\mathcal{E}) \cap \{0\} = \emptyset$ and the tower property. By Lemma 5.18 we can upper bound the second summand by

$$\mu(\mathbb{1}_{\mathcal{E}^{(0)}_0 \cap \mathcal{E}^{(B)}_1} \Var^{(B)}_{\omega_B} (f)) \leq 2^{\theta_B^3(1+\varepsilon)/4} \mu(D_{Q_B}(f)),
$$

where we added the missing $A$- and $C$-transition terms to get a contribution to the Dirichlet form. Using an analogous estimate to the bounding of the one-dimensional terms in the proof of Lemma 5.18 we can show that the first summand is of lower order than the contribution of the second summand. By translation invariance we get analogous terms for any $x \in \mathbb{Z}^2$. When taking the sum over $x$ we need to account for the overcounting, which we recall is how many times a single vertex $y \in \mathbb{Z}^2$ appears in the various Dirichlet forms that we get in the above way for the different $x \in \mathbb{Z}^2$. In this case, for any $y \in \mathbb{Z}^2$ there are $O(|Q_B|)$ different $x \in \mathbb{Z}^2$ such that $y \in Q^{(B)}_x$ we
can absorb\(^5\) the overcounting into \(\varepsilon\) for \(q_B\) small enough. Thus for \(h = B\) the r.h.s. in Eq. (5.4) is upper bounded by
\[
\sum_{x \in \mathbb{Z}^2} \mu(1_{E^0(x)} \cap E^B(x)) \mathcal{V}_{0}^{(B)}(f) \leq 2^q \varepsilon^2 (1+\varepsilon)^{1/4} \mathcal{D}(f),
\]
for \(q_B\) small enough. The calculation works analogously for each \(h \in \mathcal{G}\) and we get the claim for the chosen \(q\) by arbitrariness of \(\varepsilon\). Further, the proof also works analogously for any \(q\) such that \(q_{\text{max}} q_{\text{min}}^3 \to 0\) as \(q_{\text{min}} \to 0\) giving part (3.i) of Theorem 2
\(\square\)

5.3. **Single frequent vacancy type: Proof of Theorem 2 (3.ii).** Throughout this section assume that \(q\) is a parameter set such that \(q_{\text{min}} = q_B, q_{\text{max}} q_{\text{med}}^3 / \log_2(\theta_B) \to \infty\) and \(q_{\text{med}} q_B^0 \to 0\) as \(q_B \to 0\) where we recall that \(q_{\text{med}}\) is the remaining element of \(q \setminus \{q_{\text{max}}, q_{\text{min}}\}\).

In this case the assumptions on \(q\) in Corollary 5.14 do not hold anymore. We resolve this problem by working on boxes and defining traversable configurations on them that do not exclude the frequent vacancy type. We then show that on this coarse grained lattice we can apply the results from Section 5.1 again and conclude the proof by using auxiliary models and the path method to go from the coarse grained lattice back to \(\mathbb{Z}^2\).

We start with the proofs for the case where \(q_{\text{min}} = q_B\). We will see later that this is sufficient as the proofs for \(q_{\text{min}} \in \{q_A, q_C\}\) are analogous.

5.3.1. **The case \(q_{\text{max}} = q_A\).** Assume for this subsection that \(q_{\text{max}} = q_A\) and \(q_{\text{med}} = q_C\). We start by defining the coarse graining and the states on the coarse-grained lattice.

**Definition 5.21.** For \(j \in \mathbb{Z}^2\) and \(L = \lfloor \theta_B^3 \rfloor\) let \(A_j = (L+1)j + \{0, \ldots, L-1\}^2\) be an equilateral box of side length \(L-1\) and origin \((L+1)j\) and let \(W_j\) be the outline of it, i.e. the shortest cycle containing \((L+1)j + \{0, (L-1)e_1, (L-1)e_2, (L-1)(e_1 + e_2)\}\). Let the enlargement \(E W_j\) of \(W_j\) be the union of \(W_j\) with the set \(\Lambda \setminus A_j\) where \(\Lambda\) is an equilateral box of side length \(L\), origin \((L+1)j\) and denote the top right corner of \(E W_j\) by \(x_j = (L+1)j + L(e_1 + e_2)\). For an \(\omega \in \Omega\) we call \(E W_j\)

- **B-traversable** (see Fig. 9) if
  - \(\omega_x \in \{*, A\}\) for any \(x \in W_j\),
  - \(\omega_x \in \{*, A, B\}\) for any \(x \in E W_j \setminus W_j\) and
  - for any \(i \in [2]\) there is at least one \(x \in (L+1)j + \{e_i, 2e_i, \ldots, (L-1)e_i\}\) such that \(\omega_x = A\).

Let \(q_{BT} := \mu_{E W_j}(B\text{-traversable})\).

- **B-super** if \(E W_j\) is B-traversable and \(\omega_{x_j} = B\). Let \(q_{BS} := \mu_{E W_j}(B\text{-super})\).

- **B-evil** if it is not B-traversable. Let \(q_{BE} := \mu_{E W_j}(B\text{-evil})\).

\(^5\)We often use the term absorb in this context, where we either mean make the constant larger/smaller or here specifically, where \(\varepsilon\) is fixed, do the whole proof for \(\varepsilon/2\) and only in the final step write \(\varepsilon\) upper bounding any lower order term by \(2^q \varepsilon^{1/4}\).

\(^6\)We use super and evil instead of the more common good and bad to avoid confusion in the notation with \(G \subset H_d\) and \(B\)-vacancies.
Figure 9. Illustration of a $B$-traversable $EW_j$. Note that the $A$-vacancies on the bottom and left boundary can be anywhere on that boundary.

**Attention:** Previously, if we said that $EW_j$ was $B$-traversable, we meant that $\omega_x \in \{\ast, B\}$ for any $x \in EW_j$ instead of the above definition. In the context in which $q_A \gg q_B$ this notion of $B$-traversability has a very small equilibrium probability so it is not useful for the proof of part (3.ii). We justify the recycling of the name since the two notions of traversability play analogous roles. In Section 5.1 we looked for grids of paths with vertices only in $\{\ast, B\}$. In this section we look for grids where each vertex is a $EW_j$ that is $B$-traversable in the above sense.

The next result shows that we can use the results from Section 5.1 on the coarse-grained lattice if the $B$-super boxes play the role of $B$-vacancies and $B$-evil boxes the role of $A$ and $C$ vacancies. The proof is a simple union bound.

**Lemma 5.22.** For $EW_j$ as in Definition 5.21 we have

$$\frac{\theta_{qBS}}{\theta_B} \to 1, \quad \theta_{qBS}^3 q_{BE} \to 0,$$

as $q_B \to 0$.

**Remark 5.23.** While $q_A \theta_B^3 \to \infty$ as $q_B \to 0$ we thus find a renormalisation such that we again have the equivalent of $(q_A + q_C) \theta_B^3 \to 0$ from Section 5.1 on the renormalised lattice.

For $j \in \mathbb{Z}^2$ let $\Omega_j = S(G)^{EW_j}$, $\mu_j^*(\cdot) = \mu_{EW_j}(\cdot | B\text{-traversable})$ and let $\text{Var}^*_j(f)$ be the associated variance. We will only use the letter $j$ in bold font to refer to indices of $EW_j$ and thus say interchangeably that $j$ or $EW_j$ is $B$-traversable, $B$-super or $B$-evil. Let us come to the analogue statement of Lemma 5.18 for which we need to define the analogue of the events $E^{(B,j)}$ for the lattice of boxes. We define $Q^{(B, \ast)} = \{EW_j : j \in Q^{(B)}\}$ for $Q^{(B)}$ with side length $\ell = [\theta_{qBS}^{3/2}]$ and square side length $N = 2^{[\theta_{qBS}^{3/2} + \log_2(\theta_{qBS})]}$. The vector $z_B = e_1 + 3e_2$ we now write as $j_B$. 
Let $\mathcal{E}^{(B,1,+)}$ be the event that we find a $B$-grid $\mathcal{C}$ in $Q^{(B)}$ such that $EW_{\mathcal{C}}$ is $B$-traversable for any $\mathcal{C}$ and such that there is an intersection point $j_{i,j} \in X(\mathcal{C})$ with $i,j > N/2$ and $e \in B$ such that $EW_{j_{i,j}+\mathcal{C}}$ is $B$-super.

Let $\mathcal{E}^{(B,2,+)}$ be event that for each $j$ on the boundary $D_{0,0}^{(1)}$, $EW_{\mathcal{C}}$ is $B$-traversable. We write $\mathcal{E}^{(B,+)} = \mathcal{E}^{(B,1,+)} \cap \mathcal{E}^{(B,2,+)}$. The support is included in $Q^{(B,+)}$, i.e. $\text{Supp}(\mathcal{E}^{(B,+)}) \subset Q^{(B,+)}$ and $\mathcal{E}^{(B,+)}$ satisfies the exterior condition with respect to the same $\{V_n\}_{n \in \mathbb{Z}}$ as in Lemma 5.12.

The auxiliary model for which we state the analogue of Lemma 5.18 is given by the constraints $c_j^{s,B}$ defined as the indicator over the event that there exists an $e \in B$ such that $j+e$ is $B$-super. Analogous to the $\mu^{(h)}$ notation we write $\mu^{(AB)}(\cdot) = \mu(\cdot \mid \{*, A, B\})$ and $\text{Var}^{(AB)}(f) := \text{Var}(f \mid \{*, A, B\})$.

**Corollary 5.24.** For any $\varepsilon > 0$ we find a $\delta > 0$ such that

$$\mu(\mathbb{1}_{\mathcal{E}^{(B,+)}} \text{Var}_{x_{j_B}}^{(AB)}(f)) \leq 2\theta_2^{1+\varepsilon}/4 \sum_{j \in Q^{(B)}} \mu[\mathbb{1}_{j \text{-traversable}} c_j^{s,B} \text{Var}_{x_j}^{(AB)}(f)]$$

for $q_B < \delta$.

**Remark 5.25.** Notice that on the right hand side we only take the variance over the top-right corner points of each box instead of the variance over $EW_{\mathcal{C}}$. This is because with our conditions we cannot relax all of $EW_{\mathcal{C}}$, since the $A$-vacancies in $B$-traversable boxes may not reach all vertices in $EW_{\mathcal{C}}$.

**Proof.** As in the proof of Lemma 5.18 let $\mathcal{G}$ be the smallest $B$-grid with $B$-traversable crossings and $\xi$ the vertex with the highest coordinate in the $\ll$-partial order such that if $j_{i,j} = \xi$ then there is an $e \in B$ with a $B$-vacancy on $x_{j_{i,j}+e}$. Then,

$$\mu(\mathbb{1}_{\mathcal{E}^{(B,+)}} \text{Var}_{x_{j_B}}^{(AB)}(f)) = \sum_{\mathcal{C} \text{-grid}} \sum_{y \in X(\mathcal{C})} \mu[\mathbb{1}_{\mathcal{E}^{(C,+)}} \mathcal{E}^{(B,+)} \text{Var}_{x_{j_B}}^{(AB)}(f)].$$

To save some space let us write $\hat{\mathcal{E}} := \{\mathcal{G} = \mathcal{C} \cap \{\xi = j'\} \cap \mathcal{E}^{(B,+)}\}$. We upper bound a generic summand so fix a $\mathcal{C}$ and a $j'$. Consider the subset $C^{(TR,+)} := \{x_{j} : j \in \mathcal{C}\}$ of top right corners of $EW_{\mathcal{C}}$. The event $\mathcal{F} = \mathcal{C}$ reduces to $x_{j_B} \in \{A, B, *\}$ on any $x_{j_B} \in C^{(TR,+)}$.

In an analogous proof to Lemma 5.18 we find

$$\mu[\mathbb{1}_{\mathcal{E}} \text{Var}_{x_{j_B}}^{(AB)}(f)] \leq 2\theta_2^{1+\varepsilon}/4 \sum_{y \in C^{(TR,+)}} \mu[\mathbb{1}_{\mathcal{E}^{(G)}} \text{Var}_{y}^{(AB)}(f)]$$

for $q_B$ small enough where $\mathcal{E}^{(G)}$ is the constraint that there is an $e \in B$ such that $j+e \in \mathcal{G}$ and $x_{j_B}+e$ has a $B$-vacancy. Given $\mathcal{E}$ any $EW_{\mathcal{C}}$ for $j \in \mathcal{G}$ is $B$-traversable so that $\mathbb{1}_{\mathcal{E}^{(G)}} \leq \mathbb{1}_{\mathcal{E}^{(B,+)}}$. We can thus upper bound the sum in the r.h.s. by

$$\sum_{y \in C^{(TR,+)}} \mu[\mathbb{1}_{\mathcal{E}^{(G)}} \text{Var}_{y}^{(AB)}(f)] \leq \sum_{j \in Q^{(B,+)}} \mu[\mathbb{1}_{j \text{-traversable}} c_j^{s,B} \text{Var}_{x_j}^{(AB)}(f)].$$

We get the claim after resolving the sum over $\mathcal{G}$ and $\xi$ and taking into account the overcounting which we can absorb into the $\varepsilon$. \qed
Given a $B$-traversable box with a neighbouring $B$-super box we want to recover from a generic term in the r.h.s. in Corollary 5.24 a Dirichlet form of the $ABC$-model. To that end, let us isolate two generic situations first. The first explains how to use the $A$-vacancies on $W_j$ to relax $E W_j$.

**Lemma 5.26.** Let $C_2, C_1 = O(\theta_B^3)$ be two constants and consider two paths

$$
\Gamma_1 = \{0, e_1, \ldots, C_1 e_1\}, \\
\Gamma_2 = \{-C_2 e_1 - e_2, -(C_2 - 1)e_1 - e_2, \ldots, C_1 e_1 - e_2\}.
$$

On these paths define the event $A$ that on $\Gamma_1$ we find no $C$-vacancies, on $\Gamma_2$ no $B$- or $C$-vacancies, there is an $A$-vacancy on $\Gamma_2 \setminus (\Gamma_1 - e_2)$ and $\omega(\Gamma_{1+1} e_1) = B$ (see Fig. 10). Then we find a constant $\kappa > 0$ such that for any $y \in \Gamma_1$

$$
\mu(\mathbb{1}_A \text{Var}^{(AB)}(f)) \leq 2\kappa \theta_B \log 2(\theta_B) \sum_{x \in \Gamma_1} \mu(\mathbb{1}_A \text{Var}^{(AB)}(f))
$$

for $q_B$ small enough. For each $x$ we can extend the variance and use block relaxation, Lemma 5.19, to get

$$
\mu(\mathbb{1}_A c^B_x \text{Var}^{(AB)}(f)) \leq C q_A \mu\left[\mathbb{1}_A (\text{Var}^{(AB)}(f) + \text{Var}^{(A)}(f))\right],
$$

for $q_B$ small enough. For the first summand we can write the variance as transition terms (Lemma 3.4) and use that $\mathbb{1}_A \omega_{x-e_2} = A \leq c^A_x$ to recover a term of the Dirichlet form. For the second summand we can use the enlargement trick ([13, Lemma 3.6]) so that

$$
\mu(\mathbb{1}_A \text{Var}^{(AB)}(f)) \leq 2\kappa \theta_B \log 2(\theta_B) \sum_{x \in \Gamma_1} \mu(D_{\{x\} \cup \Gamma_2}).
$$

The overcounting is of order $O(\theta_B^3)$ and can thus be absorbed into the $\kappa$ and we recover the claim.

Being able to relax $E W_j$ means that we can move the $B$-vacancy freely on it using the block relaxation Lemma. The second of our isolated results moves the $B$-vacancy from a neighbouring $B$-super box to a $B$-traversable box. This requires the path method.
since \( x_j \) does not neighbour a vertex in \( W_j \) and so we can not use Lemma 5.26 together with the block relaxation Lemma to move a \( B \)-vacancy here.

**Lemma 5.27.** Consider the set \( V = \{0, -e_2, -e_1 - e_2\} \) and the event \( A \) given by the \( \omega \in \Omega \) such that \( \omega_{-e_1-e_2} = A, \omega_{e_1-e_2} = \star \) and \( \omega_{e_1} = B \). Further define the event \( A' \) given by the configurations \( \omega \) such that \( \omega_x \in \{\star, A, B\} \) for \( x \in \{-e_2, 0, e_1\} \) and \( \omega_x \in \{\star, A\} \) for \( x \in \{-e_1-e_2, e_1-e_2\} \). Then, for \( q_B \) small enough we find a constant \( \kappa \) such that

\[
\mathbb{P}_V \left[ \mathbb{1}_A \mathbb{V}_0^{(AB)}(f) \mid A' \right] \leq 2^{\kappa \theta_B} \sum_{x \in V} \mathcal{D}_V(f).
\]

**Proof.** The path to use the path method with, and thus the proof, is apparent from Fig. 11. \( \square \)

Armed with these results we can upper bound the right hand side in Corollary 5.24. For this we introduce the notation \( EW(V) = \cup_{j \in V} EW_j \) for any subset \( V \subset \mathbb{Z}^2 \).

**Lemma 5.28.** Let \( j \in \mathbb{Z}^2 \) and \( V = \{j, j + e_1, j + e_2\} \). We find a constant \( \kappa > 0 \) such that

\[
\mathbb{P} \left[ \mathbb{1}_{j \text{ B-traversable}} \mathcal{C}_j^{*B} \mathbb{V}_{x_j}^{(AB)}(f) \mid A' \right] \leq 2^{\kappa \theta_B} \log_2(\theta_B) \mathcal{D}_{EW(V)}(f)
\]

for \( q_B \) small enough.

**Proof.** W.l.o.g. consider only the case \( j = 0 \) and where the constraint on the l.h.s. of the claim is replaced by \( \tilde{c} = \mathbb{1}_{EW_{e_1} \text{ is } B \text{-super}} \). Let \( U = x_0 + \{0, e_1, -e_2 - e_1, -e_2, -e_2 + e_1\} \) and let \( A \) be the event from Lemma 5.27 translated by \( x_0 \), so that \( \text{Supp}(A) \subset U \). Analogously define \( A' \) as the translated version of \( A' \) from Lemma 5.27. The event that \( EW_0 \) and \( EW_{e_1} \) are \( B \)-traversable on \( U \) reduces to \( A' \). Thus, we can extend the
variance, Lemma 5.16 and use the block relaxation Lemma 5.19
\[ \mu \left[ \mathbb{E}_{B \text{-traversable \ Var}_{x_0}^{(AB)}(f)} \right] \leq \mu \left[ \mathbb{E}_{B \text{-traversable \ Var}_U(f \mid A')} \right] \leq 2^{\theta_B} \mu \left[ \mathbb{E}_{B \text{-traversable \ \left( \mathbb{I}_A \ Var}_{x_0}^{(AB)}(f) + \ Var}_{U \setminus \{x_0\}}(f \mid A') \right] \]
for $q_B$ small enough. The first summand can be upper bounded using Lemma 5.27. For the second summand write
\[ \mu \left[ \mathbb{E}_{B \text{-traversable \ Var}_{U \setminus \{x_0\}}(f \mid A')} \right] \leq \mu \left[ \mathbb{E}_{B \text{-traversable \ } \sum_{y \in U \setminus \{x_0\}}} \ Var_y(f \mid A') \right]. \]
For $y \in (E W_0 \setminus W_0) \cup \{E W_{e_1} \setminus W_{e_1}\}$ we can use Lemma 5.26 and for the others we can use the enlargement trick ([3, Lemma 3.6]) to get the claim. \qed

Combining the previous results we thus have
\[ \mu(\mathbb{I}_{\mathcal{E}^{(h,s)}} \ Var_{x_{B}}^{(AB)}(f)) \leq 2^{\theta_B(1+\varepsilon)/4} \mu(\mathcal{D}_{Q^{(h,s)}}(f)), \]
for $q_B$ small enough. For $C$-vacancies we can use the same construction of $E W_j$ and $W_j$ with length parameter $[\theta_C^3]$ and define $C$-traversable, -super, and -evil by replacing the $B$-vacancies with $C$-vacancies. Recall that we assume $q_A \theta_C^3 / \log_2(\theta_B) \to \infty$ as $q_B \to 0$ so that the results follow analogously for $C$-vacancies with minor adjustments. We omit details here that lead to the result that
\[ \mu(\mathbb{I}_{\mathcal{E}^{(c,s)}} \ Var_{x_{C}}^{(AC)}(f)) \leq 2^{\theta_B(1+\varepsilon)/4} \mu(\mathcal{D}_{Q^{(c,s)}}(f)), \]
for $j_C = -3e_1 + e_2$ and $q_B$ small enough.
As in the low vacancy density case we need a final event that brings the $B$- resp. $C$-vacancy from $Q^{(h,s)}(x)$ to $x$. To that end, let us define some paths (see Fig. 12).

- Let $\Gamma^{(B)}$ be a shortest path starting at $e_2$ and ending at the first vertex neighbour- ing $E W_{j_B} \setminus W_{j_B}$ that first goes straight up and then right.
- Let $\Gamma^{(B,\text{left})}$ be the path that starts at $[\theta_C^3] e_1 + e_2$ and is straight until $-e_1 + e_2$ and then equal to $(\Gamma^{(B)} - e_1) \setminus \Gamma^{(B)}$. Let $\Gamma^{(B,\text{right})}$ be the path starting at $e_1 + [\theta_C^3] e_2$ that goes straight up until it hits $\Gamma^{(B)} - e_2$, which it then follows to the right.
- Let $\Gamma^{(C)}$ be the shortest path that starts at $-e_1$ and goes straight left and then up that ends up at a vertex neighbour- ing $E W_{j_C} \setminus W_{j_C}$. Let $x^{(C)}$ be the point where the path switches from going left to going up.
- Let $\Gamma^{(C,\text{left})}$ be the union of $\Gamma^{(C)} - e_1 - e_2$ and $\{x^{(C)} - [\theta_C^3] e_1, \ldots, x^{(C)} - e_1\}$. Notice that since $\Gamma^{(B,\text{left})}$ starts at $[\theta_C^3] e_1 + e_2$ the various paths do not intersect. We define $\mathcal{E}^{(0)}$ as the $\omega \in \Omega$ such that

- $\omega_x \in \{\ast, A, B\}$ for any $x \in \Gamma^{(B)}$.
- $\omega_x \in \{\ast, A\}$ for any $x \in \Gamma^{(B,\text{left})} \cup \Gamma^{(B,\text{right})}$ and there is at least one $A$-vacancy on $\Gamma^{(B,\text{left})} \setminus \{\Gamma^{(B)} - e_1\}$ and on $\Gamma^{(B,\text{right})} \setminus \{\Gamma^{(B)} - e_2\}$.
- $\omega_x \in \{\ast, A, C\}$ for any $x \in \Gamma^{(C)}$. \
\[ \mathcal{Q}(B,*) \quad \mathcal{Q}(C,*) \]

\[ \Gamma(B) \quad \Gamma(C) \]

\[ \Gamma^{(B, \text{left})} \quad \Gamma^{(C, \text{left})} \quad \Gamma^{(B, \text{right})} \]

**Figure 12.** Image of the various \( \Gamma \) paths and the exemplary \( A \)-vacancies where \( \mathcal{E}^{(0)} \) requires them. Note the different sizes of \( \mathcal{Q}(B,*) \) and \( \mathcal{Q}(C,*) \).

\[ \omega_x \in \{*, A \} \text{ for any } x \in \Gamma^{(C, \text{left})} \text{ at least one } A \text{-vacancy on } \Gamma^{(C, \text{left})} \setminus (\Gamma(C) - e_1 - e_2). \]

The support of \( \mathcal{E}^{(0)} \) by construction has no intersection with \( \mathcal{Q}(B,*) \) and \( \mathcal{Q}(C,*) \). Let \( \mathcal{E}^{(*)} := \mathcal{E}^{(0)} \cap \mathcal{E}(B,*) \cap \mathcal{E}(C,*) \) and let \( \mathcal{E}_{x}^{(*)} \) be the translated version by \( x \in \mathbb{Z}^2 \). Then the family \( \{ \mathcal{E}_{x}^{(*)} \}_{x \in \mathbb{Z}^2} \) satisfies the exterior condition w.r.t. to the same family of sets as given in Lemma 5.12. Using the assumptions on \( q \) it is straightforward to check that

\[ \lim_{q_B \to 0} \text{Supp}(\mathcal{E}^{(0)}) \mu(1 - \mathcal{E}^{(0)}) = 0. \]

Combining this with Lemma 5.22 and the results from Section 5.1.2 we can apply the exterior condition theorem, Theorem 3.2. Further, \( \mathcal{E}^{(0)} \) fulfills that analogous role to the eponymous event in Section 5.2 as we see in the next Lemma.

**Lemma 5.29.** Let \( A \) be the event defined by the intersection

\[ A := \mathcal{E}^{(0)} \cap \{ EW_{j_B} B\text{-super} \} \cap \{ EW_{j_C} C\text{-super} \}. \]

Then,

\[ \mu(\mathbb{1}_A \text{Var}_D(f)) \leq 2^{\kappa \theta_B \log_2(\theta_B)} D_{\text{Supp}(A)}(f). \]

for \( q_B \) small enough.

**Sketch of the proof.** We only give a sketch since the employed techniques are always the same. Extending the variance, Lemma 5.16 and using block relaxation, Lemma 5.19.
gives
\[ \mu(\mathbb{1}_A \text{Var}_0(f)) \leq \frac{C}{q_B} \mu\left(\mathbb{1}_A \left(\text{Var}_{e_2}^{(AB)}(f) + \mathbb{1}_{\omega e_2 = B} \text{Var}_0(f)\right)\right). \]

For the first summand, given \( \mathcal{A} \), we can use a combination of the block relaxation Lemma \((\text{Lemma 5.19})\) and Lemma \((\text{Lemma 5.26})\) to get an appropriate upper bound. For the second summand we can repeat the calculation for the \( C \)-vacancy side to get a term
\[ \frac{C}{q_A q_B q_C} \mu(\mathbb{1}_{A \omega e_2 = B, \omega - 2e_1 = C, \omega - 2e_1 - e_2 = A} \text{Var}_0(f)). \]

Write \( \text{Var}_0(f) \) as a sum of transition terms using Lemma \((\text{Lemma 3.4})\) for the \( B \)-transition use that \( \mathbb{1}_{\omega e_2 = B} \leq e_0 \) and for the \( A \) and \( C \) transition terms we can use the path method recalling that for \( \omega \in \mathcal{A} \) we have \( \omega - e_1 \in \{\ast, A, C\} \) (analogously to, for example, the situation in Fig. \((\text{I})\)). The claim follows. \( \square \)

Since the intersection of the various grids is negligible this gives us the proof for the case of \( q_A = q_{\text{max}} \). We omit the proof as it is a straightforward implication of the above Lemma with the block relaxation Lemma.

**Proposition 5.30.** For parameter sets as fixed in the beginning of the section with \( q_{\text{max}} = q_A \) we have
\[ \lim_{q_B \to 0} \frac{\gamma(G, q)}{\gamma_2(q_B)} \geq 1. \]

We never explicitly used that \( q_C > q_B \) so the same result also holds in the case \( q_B = q_{\text{med}} \) and \( q_C = q_{\text{min}} \). Further, by symmetry this also covers the case \( q_B = q_{\text{max}} \). The case \( q_C = q_{\text{max}} \) is analogous. Indeed, above \( EW_j \) extended \( W_j \) to the north and east (i.e. the boxes shared their origin). In the case \( q_C = q_{\text{max}} \) we do the completely analogous construction only that \( W_j \) and \( EW_j \) share the north-west corner. As everything else works analogously we omit details here. \( \square \)

### 5.4. Single low density vacancy type: Proof of Theorem 2(3.iii).

In this section consider again the \( G \)-MCEM with \( G = \{A, B, C\} \) this time with a parameter set \( q \) such that \( q_{\text{min}} = q_B \) and \( \lim \inf_{q_B \to 0} q_{\text{med}} > 0 \), i.e. there is a constant \( \lambda > 0 \) with \( q_A, q_C > \lambda \) for \( q_B \) small enough. This covers case \((3.iii)\) since both \( A \)- and \( C \)-vacancies share the direction \( e_1 \), the other case in which \( q_{\text{min}} = q_A \), is equivalent to the present case by symmetry.

Using that both \( A \)- and \( C \)-vacancies have a high equilibrium density, we find configurations that clear any non-\( B \)-vacancy in the \( e_1 \)-direction. As in previous proofs we work with block lattices. In this section we let \( \{W_j\}_{j \in \mathbb{Z}^2} \) be the block lattice given by boxes of side lengths \((0, 2)\) so that
\[ W_j = (j_1, 3j_2) + \{0, e_2, 2e_2\}. \]

We call \((j_1, 3j_2)\) the **lower vertex** of \( W_j \), \((j_1, 3j_2 + 2)\) the **upper vertex**, the set of lower and upper vertices we then call the **outer vertices** and \((j_1, 3j_2 + 1)\) the **central vertex**. The associated local state space is \( \Omega_j^* := \{0, 1\}^{W_j} \), the equilibrium measure is \( \mu_j^* = \mu_{W_j}^* \) and the variance is \( \text{Var}_j^*(f) = \text{Var}_{W_j}(f) \). For \( \omega \in \Omega_j^* \) we say that \( W_j \)
- is \( B \)-traversable, if there is no \( B \) on the outer vertices.
• is $B$-super, if it is $B$-traversable and the central vertex is $B$.
• is $AC$-traversable, if there is no $B$ on $W_j$.
• is $AC$-super, if it is $AC$-traversable, the lower vertex is $A$ and the upper vertex is $C$.

**Remark 5.31.** To justify the above definitions and the recycling of the traversable and super names let us give a high level overview of what we do with these states to prepare the reader for the detailed calculations. Recall that $A$-vacancies propagate north and east, while $C$-vacancies propagate south and east. In an $AC$-super box the central vertex is always facilitated for any transition from $A$ or $C$ to the neutral state and vice versa. By $A$- and $C$-vacancies sharing the east propagation direction this extends to any vertex in an $AC$-traversable box to the east of an $AC$-super box (see Lemma 5.33).

Further, if there are any $B$-traversable or $B$-super boxes to the East of an $AC$-super box, following at least one $AC$-traversable box, we can also remove any non $B$-vacancy from the central vertex. This is what allows us in Lemmas 5.34 and 5.36 to propagate the $B$-vertices from $B$-super boxes on paths of $B$-traversable boxes given an appropriate configuration of $AC$-super and traversable boxes.

As in the previous proofs our goal is to define a set of events $\{E_j\}_{j \in \mathbb{Z}^2}$ on which we can use the exterior condition theorem and where $E_x$ allows us to recover a Dirichlet form of the $ABC$-model starting from a term like $\mu(1_{E_x} \operatorname{Var}_{W_j}(f))$ at a cost $2^{\theta_B (1+\varepsilon)/4}$ for $q_B$ small enough.

For this we cannot use $Q^{(B)}$ anymore since there is no obvious relaxation scheme that allows us to transport $B$-vacancies on coarse-grained $B$-paths (as in the proof of part (3.ii)). Since the $A$-vacancies have a high frequency we also do not have to make a construction that stays above the diagonal as in Lemma 5.12 to satisfy the exterior condition. We can work with the set $V_0$ given by the vertices ‘below’ the line that goes through the origin and $2^{\theta_B} \mathbf{e}_1 + \mathbf{e}_2$ and define $V_n = V_0 + n\mathbf{e}_2$ for any $n \in \mathbb{Z}^2$ so that $\{V_n\}_{n \in \mathbb{N}}$ is an increasing and exhausting set. This allows us to construct a lattice of straight lines of side length at most $2^{\theta_B}$ in the positive quadrant and still put a condition on the line going in the $-e_1$ direction from the origin while satisfying the exterior condition.

Let $\ell = \lceil \theta_B^{3/2} \rceil$ and $N = 2^{\lceil \theta_B^{3/2} \rceil}$. For $i \in [0, N]$ we call the box of side lengths $(\ell - 1, N \ell - 1)$ with origin at $i\mathbf{e}_1 + \mathbf{e}_2$ the $i$-th vertical strip $Q_i^{(v)}$. For $j \in [0, N]$ we call the box with side lengths $(N \ell - 1, \ell - 1)$ and origin at $(j \ell + 1)\mathbf{e}_2$ the $j$-th horizontal strip $Q_j^{(h)}$. We denote by $Q_{i,j}$ the equilateral box of side length $\ell - 1$ given by $Q_i^{(v)} \cap Q_j^{(h)}$. The union $Q^{(B)}$ of all strips is an equilateral box of side length $N \ell - 1$ and origin $\mathbf{e}_2$.

The dynamics to propagate $B$-vacancies on horizontal and vertical paths is different. $A$- and $C$-vacancies only share the $\mathbf{e}_1$ direction so that $AC$-super boxes can only propagate in an $\mathbf{e}_1$ directions, which means that for each row we want to move a $B$-super box vertically, we need an $AC$-super box somewhere that removes any $A$- or $C$-vacancies. To propagate $B$-super boxes horizontally a single $AC$-super box suffices. Thus vertically we need boxes that guarantee us the $AC$-super vertices.
Definition 5.32 (Vertical crossing). Consider a box $\Lambda \subset Q_i^{(v)}$ of side lengths $\left(\lceil \theta_B^{5/4} \rceil - 1, N\ell - 1 \right)$ with origin $j_0$. Let $\partial^{(r)} \Lambda$ be the right boundary of $\Lambda$, i.e. the $j \in \Lambda$ such that $j \cdot e_1 = j_0 \cdot e_1 + \lceil \theta_B^{5/4} \rceil - 1$. For $\omega \in \Omega_{Q_i}^*$, $\Lambda$ is a vertical crossing of $Q_i^{(v)}$ if

- $W_j$ is $B$-traversable for any $j \in \partial^{(r)} \Lambda$.
- $W_j$ is $AC$-traversable for any $j \in \Lambda \setminus \partial^{(r)} \Lambda$.
- There is at least one $j$ per row in $\Lambda \setminus (\partial^{(r)} \Lambda \cup (\partial^{(r)} \Lambda - e_2))$ such that $W_j$ is $AC$-super.

The main idea behind this definition is the following: To propagate a $B$-super box on the right boundary, we use that on each row there is an $AC$-super box on a line of $AC$-traversable boxes. This $AC$-super box can remove any $A$- or $C$-vacancy from the $AC$-traversable part and then in particular also from the $B$-traversable part on the right boundary of $\Lambda$, which then allows the $B$-vacancy in the $B$-super box to move down. Let us isolate this horizontal motion of $AC$-super boxes. Recall for this, that we write $\mu^{(AC)}$ and $\text{Var}^{(AC)}$ to denote the measure resp. variance conditioned on there only being $A$ and $C$ vacancies and that by definition

$$\text{Var}^*_j(f \mid AC\text{-traversable}) = \text{Var}^{(AC)}_{W_j}(f).$$

Using the path method the next lemma is straight forward.

Lemma 5.33. We find a constant such that

$$\mu(1_{W_{-e_1}} \text{ AC-super}\text{-Var}_0^*(f \mid AC\text{-traversable}) \leq C \mu(D_{W_{-e_1} \cup W_0}(f)).$$

With this we can show how $B$-super boxes propagate vertically on vertical crossings.

Lemma 5.34 (Vertical propagation). Let $\Lambda \subset Q_i^{(v)}$ as in Definition 5.32 and let $j^{(1)} \in \partial^{(r)} \Lambda \cap Q_{i,0}$. Let $A^{(v)}$ be the event given by the $\omega$ such that $\Lambda$ is a vertical crossing of $Q_i^{(v)}$. 

**Figure 13.** One of the $N$ horizontal sections of a vertical crossing (see Definition 5.32). Three vertically aligned vertices (e.g. the rectangles drawn) are one box $W_j$. The right box is on the right boundary and thus by assumption $B$-traversable, so that there is no condition on the central vertex (black dot). All other vertices have no $B$-vacancy. The left box is the $AC$-super box implied by the definition of vertical crossings.
and there is a \( j^{(2)} \in \partial^{(v)} \Lambda \cap Q_{i,1} \) such that \( W_{j^{(2)}} \) is \( B \)-super. Then,

\[
\mu \left( \mathbb{1}_{A^{(v)}} \operatorname{Var}_{j^{(1)}}^{*} (f \mid A^{(v)}) \right) \leq 2^{\kappa_{B} \log_{2}(\theta_{B})} \mu \left( D_{Q_{1,0} \cup Q_{1,1}} (f) \right).
\]

**Proof.** W.l.o.g. assume that the right boundary \( \partial^{(v)} \Lambda \) of \( \Lambda \) is on the vertical axis such that \( j^{(1)} = 0 \) and assume also w.l.o.g. that the \( B \)-super \( j^{(2)} \) implied by \( A^{(v)} \) is on the furthest vertex in \( \partial^{(v)} \Lambda \cap Q_{i,1} \) from the origin, i.e., \( j^{(2)} = (2l - 1)e_{2} \). Let \( \Gamma = \{0, e_{2}, \ldots, j^{(2)} - e_{2}\} \) be the part of the right boundary starting at \( j^{(1)} \) and stopping right before \( j^{(2)} \). Let \( c_{j}^{(v)} \) be the constraint given by the indicator over the event that \( W_{j+e_{2}} \) is \( B \)-super if \( j \neq j^{(2)} - e_{2} \) and 1 if \( j = j^{(2)} - e_{2} \).

Consider the auxiliary process on \( \Gamma \) with the constraints \( c_{j}^{(v)} \) that, if \( W_{j} \) is unconstrained, samples from all \( B \)-traversable states on \( W_{j} \). The equilibrium measure of this process is given by \( \mu_{B}^{(v, BT)} := \otimes_{j \in \Gamma} \mu_{j}^{(v)} | B \)-traversable). Since \( \mu_{j}^{(v, BT)} (B \text{-super} | B \text{-traversable}) = q_{B} \) the spectral gap of this process is equal to the spectral gap of the one-dimensional East model with vacancy density \( q_{B} \) on \( \Gamma \) with good boundary conditions.

Hence, we can extend the variance (Lemma 5.16) and use [3] Theorem 2] to get

\[
\mu \left( \mathbb{1}_{A^{(v)}} \operatorname{Var}_{j^{(1)}}^{*} (f \mid A^{(v)}) \right) \leq \mu \left( \mathbb{1}_{A^{(v)}} \operatorname{Var}_{\mu_{B}^{(v, BT)}} (f) \right)
\leq 2^{\kappa_{B} \log_{2}(\theta_{B})} \sum_{j \in \Gamma} \mu \left( \mathbb{1}_{A^{(v)}} c_{j}^{(v)} \operatorname{Var}_{\mu_{B}^{(v, BT)}} (f) \right).
\]

Consider the summand for \( j = 0 \) and let \( \omega \in A^{(v)} \). Let \( V^{(0)} = \bigcup_{j=0}^{3} j_{e}^{2} \) be the union of vertices in \( W_{0} \) together with the lower vertex of \( W_{e_{1}} \) and recall that by \( c_{0}^{(v)} \) the vertex \( 4e_{2} \) has a \( B \)-vacancy. Further, let \( V^{(i)} = \bigcup_{j=0}^{3} W_{j_{e_{i}} + j_{e_{2}}} \) for \( i \in [2] \) and let \( V = V^{(0)} \cup V^{(2)} \). Recall that by \( A^{(v)} \) the boxes in \( V^{(i)} \) are \( AC \)-traversable and define further \( \tilde{A} \) as the event that \( W_{j} \) is \( AC \)-super for \( j \in V^{(2)} \). We can extend the variance to \( V \) and use the block relaxation Lemma (Lemma 5.19) to get

\[
\mu \left( \mathbb{1}_{A^{(v)}} c_{0}^{(v)} \operatorname{Var}_{\mu_{0}^{(v, BT)}} (f) \right) \leq \mu \left( \mathbb{1}_{A^{(v)}} c_{0}^{(v)} \operatorname{Var}_{\mu_{0}^{(v)}} (f | A^{(v)}) \right)
\leq C \mu \left[ \mathbb{1}_{A^{(v)}} c_{0}^{(v)} \left( \mathbb{1}_{\tilde{A}} \operatorname{Var}_{V^{(0)}} (f | A^{(v)}) \right) + \operatorname{Var}_{V^{(2)}} (f | A^{(v)}) \right].
\]

We upper bound the two summands separately. For the first term we get

\[
\mu \left( \mathbb{1}_{A^{(v)}} c_{0}^{(v)} \operatorname{Var}_{V^{(0)}} (f | A^{(v)}) \right) \leq 2^{\kappa_{B}} \mu \left( D_{\cup_{i \in \{0, 2\}} V^{(i)}} (f) \right).
\]

This is done through the path method analogous to Lemma 5.33 with the additional step of defining the paths for the \( B \)-vacancy on \( 4e_{2} \) to move downwards after clearing any \( A \) or \( C \) vacancy on \( V^{(0)} \) using the \( AC \)-super states in \( V^{(2)} \).

For the second summand first split up the variance

\[
\operatorname{Var}_{V^{(2)}} (f | A^{(v)}) \leq \mu_{V^{(2)}} (\operatorname{Var}_{V_{2e_{2}}}^{*} (f | A^{(v)}) + \operatorname{Var}_{e_{2} + e_{1}}^{*} (f | A^{(v)}) | A^{(v)})
\]

and consider the variance over \( W_{-2e_{2}} \). The upper bound for the second term follows analogously.
Consider an auxiliary process with the constraints \( c_j(h) \) given by the indicator over the event that \( W_j - e_1 \) is \( AC \)-super. If \( W_j \) is unconstrained in this process, sample it from all \( AC \)-traversable states at a legal ring. This process has the same spectral gap as the East model with vacancy density \( \frac{q_A q_C}{(q_A+q_C)^2} \). Using that \( A(h) \) implies that there is an \( AC \)-super box to the left of \( W_{e_2} \), we can use the enlargement trick ([3, Lemma 3.6], which immediately generalises to this case), to get

\[
\mu(\mathbb{1}_{A(h)} \text{Var}^{\neq}_{2e_2}(f \mid A^{(v)})) \leq C \sum_{j=2}^{\lfloor \theta B/4 \rfloor - 2} \mu(c_{-je_1} \text{Var}^*_{je_1}(f \mid AC\text{-traversable}))
\]

where in the second inequality we used Lemma 5.33. Combining the two estimates gives the claim after taking into account that the vertices in \( V^{(2)} \) are counted twice which we absorb into \( \kappa \).

\[\square\]

Remark 5.35. Notice that here we lose the indicator over \( A^{(v)} \) since it requires there to be no \( B \)-vacancy between the central vertices but the path method adds these transitions. This will be important later, as keeping the indicators was important for taking the sum over the possible grids \( C \).

The horizontal paths will consist of \( B \)-traversable \( W_j \) that connect the vertical crossings. We isolate here the result that allows us to propagate a central \( B \) on these horizontal paths.

The basic situation is as follows. Let \( \Gamma = \Gamma^{(l)} \cup \Gamma^{(r)} \) with \( \Gamma^{(l)} = [-\lfloor \theta B/4 \rfloor e_1, \ldots, -e_1] \) and \( \Gamma^{(r)} = [0, \ldots, \ell e_1 - 1] \). Let \( A(h) \) be the event that \( w_j \) for \( j \in \Gamma^{(l)} \) is \( AC \)-traversable, that there is an \( AC \)-super \( W_{-ie_1} \) for \( i \leq -3 \), that \( W_j \) for \( j \in \Gamma^{(r)} \) is \( B \)-traversable and that \( W_{e_1} \) is \( B \)-super (see Fig. 14).

Lemma 5.36 (Horizontal propagation). For \( \Gamma \) and \( A(h) \) as above we find a constant \( \kappa \) such that

\[
\mu(\mathbb{1}_{A(h)} \text{Var}^*_0(f \mid A^{(h)})) \leq 2^{\kappa \theta B/2} \mu(D_{W^{(1)}}(f)),
\]
where \( W(\Gamma) = \bigcup_{j \in \Gamma} W_j \)

Proof. Split \( W(\Gamma^{(r)}) \) into \( W^{(ro)} \cup W^{(rc)} \) of respectively the set of outer and central vertices. Define the event \( \mathcal{A} \) that there are only \( C \)-vacancies on the upper vertices of \( W^{(ro)} \) and only \( A \)-vacancies the lower vertices. Then, we can extend the variance (Lemma \ref{lemma:variance}) and use the block relaxation Lemma (Lemma \ref{lemma:block}) to find a constant \( \kappa \) such that

\[
\mu(\mathbbm{1}_{A^{(h)}} \text{Var}_0^*(f \mid A^{(h)})) \leq \mu(\mathbbm{1}_{A^{(h)}} \text{Var}^*_{\Gamma^{(r)}}(f \mid A^{(h)})) \\
\leq 2^{\kappa \theta_B^2} \mu\left(\mathbbm{1}_{A^{(h)}} \left(\mathbbm{1}_{\bar{A}} \text{Var}_{W^{(ro)}}(f) + \text{Var}_{W^{(rc)}}^{(AC)}(f)\right)\right).
\]

Consider the first summand. On \( A^{(h)} \) there is a \( B \)-vacancy to the right of \( W^{(rc)} \), so consider the auxiliary model with the standard \( B \)-vacancy constraints \( e_x^B \) that samples from \( \mu_x \) at a legal ring on \( x \in W^{(rc)} \). Given \( \mathbbm{1}_{A^{(h)}} \) this auxiliary model on \( W^{(rc)} \) has good boundary conditions and the same spectral gap as the East model with vacancy density \( q_B \) so that by \cite[Theorem 2]{article}

\[
\mu(\mathbbm{1}_{A^{(h)}}, \mathbbm{1}_{\bar{A}} \text{Var}_{W^{(rc)}}(f)) \leq 2^{\kappa \theta_B \log_2(\theta_B)} \sum_{x \in W^{(rc)}} \mu(\mathbbm{1}_{\bar{A}} c_x^B \text{Var}_x(f)).
\]

Now write the variances as transition terms using Lemma \ref{lemma:transition} and use that with \( \bar{A} \) and \( c_x^B \) every \( x \in W^{(rc)} \) is unconstrained for every transition so that

\[
\sum_{x \in W^{(rc)}} \mu(\mathbbm{1}_{\bar{A}} c_x^B \text{Var}_x(f)) \leq C \mu(D_{W^{(rc)}}(f)).
\]

For the second summand in Eq. \eqref{eq:variance} write \( \text{Var}_{W^{(rc)}}^{(AC)}(f) \) as a sum of transition terms for \( A \)- and \( C \)-vacancy transitions. We saw in Lemma \ref{lemma:transition} how an \( AC \)-super state can put any state on an \( AC \)-traversable state to its right. Given an \( AC \)-super and then an \( AC \)-traversable state we can thus put any state in \( \{ \ast, A, C \} \) onto the upper or lower vertices of boxes right to them, if they don’t contain \( B \)-vacancies. The legal path dynamic is completely analogous to the one in Lemma \ref{lemma:transition} so we omit the details. The lengths of the paths are \( O(|W^{(ru)}|) = O(\theta_B^3/2) \), so the path method gives an upper bound of the order \( 2^{\kappa \theta_B^2} \) and the claim follows.

We now come to the grids we use in this section (see Fig. \ref{fig:grid}.)

**Definition 5.37 (Grid).** Call a union of \( \mathcal{C} = \bigcup_{i \in [N]} \mathcal{C}_i^{(h)} \cup \mathcal{C}_i^{(v)} \) a grid if \( \mathcal{C}_i^{(h)} \subset Q_i^{(h)} \) is a box of side length \( (N,\ell - 1, 0) \) and \( \mathcal{C}_i^{(v)} \subset Q_j^{(v)} \) is a box with side lengths \( (\theta_B^{5/4} - 1, N,\ell - 1) \). We call the grid good if \( W_j \) is \( B \)-traversable for any \( j \in [N] \) and \( \mathcal{C}_i^{(v)} \) is a vertical crossing for each \( j \in [N] \).

We have that \( |\mathcal{C}_i^{(h)} \cap \mathcal{C}_j^{(v)}| = O(\theta_B^{5/4}) \) and that on a grid we require this part to be \( B \)-traversable, \( AC \)-traversable and to contain an \( AC \)-super box at the same time. This is well-defined since \( AC \)-super states are a subset of \( AC \)-traversable states which in turn are subsets of \( B \)-traversable states.
Figure 15. Grid $\mathcal{C}$ as defined in Definition 5.37. The horizontal configurations from Lemma 5.36 are in blue, the vertical crossings from Definition 5.32 in red, a bit thicker to represent the horizontal extension of $\frac{5}{4}$. The black circles form the set $X(\mathcal{C})$. The striped area indicates the area on which we cannot condition by the exterior condition theorem together with the exhausting family of sets $\{V_n\}_{n \in \mathbb{Z}}$ defined above Definition 5.32.

For a grid $\mathcal{C}$ let $X(\mathcal{C})$ be the vertices given by $j_{i,j} = \partial^{(r)}C_i^{(v)} \cap C_j^{(h)}$ for $i,j \in [N-1]$, where we recall that $\partial^{(r)}$ is the right boundary. We define the event $\mathcal{E}^{(1)}$ as the $\omega \in \Omega^*$ such that there is a good grid $\mathcal{C}$ and there are $i,j \in [N-1]$ with $i,j > N/2$ such that $W_{j_{i,j}}$ is $B$-super. The next lemma is again a straightforward union bound.

Lemma 5.38. For any $\varepsilon > 0$ we find a $q(\varepsilon)$ such that
\[(N\ell)^2 \mu(1 - \mathbb{1}_{\mathcal{E}^{(1)}}) \leq \varepsilon\]
if $q_B < q(\varepsilon)$.

Combining these events we can bring a $B$-super vertex to $e_2$ for the respective good grid given by $\mathcal{E}^{(1)}$. As before, we need to bring the $B$-super box to a deterministic vertex. Since the grid this time starts at $e_2$ we can immediately bring it back to the origin. Let $\mathcal{E}^{(2)}$ be the event that $W_j$ is

- $AC$-traversable for $j$ either in $\Gamma^{(1)} := \{-[\frac{5}{4}]e_1, \ldots, -e_1\}$ or $\Gamma^{(2)} := \Gamma^{(1)} + e_2$ and there is at least one $j$ in both $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with $W_j$ $AC$-super.
- $B$-traversable for $j$ in $\Gamma^{(3)} := \{e_2, \ldots, (\ell - 1)e_2\}$ (i.e. the left boundary of $Q_{0,0}$).

In an analogous calculation to Lemma 5.38 we get.

Lemma 5.39. For any $\varepsilon > 0$ we find a $q(\varepsilon)$ such that
\[\ell \mu(1 - \mathbb{1}_{\mathcal{E}^{(2)}}) \leq \varepsilon\]
If \( q_B < q(\varepsilon) \).

Let \( \mathcal{E} := \mathcal{E}^{(1)} \cap \mathcal{E}^{(2)} \) and let \( \mathcal{E}_x \) be the event translated by \( x \in \mathbb{Z}^2 \). \( \{\mathcal{E}_x\}_x \) satisfies the exterior condition w.r.t. the \( \{V_n\}_{n \in \mathbb{Z}} \) defined above and thus using Lemmas 5.38 and 5.39 we get that we can apply the exterior condition theorem, Theorem 3.2, with this family of events. We come to the proof of part (3.iii).

**Proof of Theorem 2(3.iii).** By the exterior condition theorem we have

\[
\text{Var}(f) \leq 4 \sum_{x \in \mathbb{Z}^2} \mu(\mathbb{1}_{\mathcal{E}_x} \text{Var}_x(f)).
\]

Let us upper bound the summand for \( x = 0 \). First use that \( \text{Supp}(\mathcal{E}_0) \cap W_0 = \emptyset \) to extend the variance (Lemma 5.16)

\[
\mu(\mathbb{1}_{\mathcal{E}_0} \text{Var}_0(f)) \leq \mu(\mathbb{1}_{\mathcal{E}_0} \text{Var}_0^*(f)).
\]

For \( \omega \in \mathcal{E} \) let \( \mathcal{G}(\omega) \) denote the unique good grid in \( \omega \) consisting of the lowest horizontal paths and vertical crossings in the \( < \)-order that make a good grid. Further let \( \xi \in X(\mathcal{G}) \) be the largest intersection point that is \( B \)-super in the lexicographic order. Let \( \mathcal{E}_{C,j,i} \) be the event \( \mathcal{E} \) with \( \mathcal{G} = C \) and \( \xi = j_i \). We have

\[
\mu(\mathbb{1}_{\mathcal{E}_0} \text{Var}_0^*(f)) = \sum_{C \text{ grid } n,m \in [N]} \sum_{\mathcal{E}_{C,j,n,m}} \mu(\mathbb{1}_{\mathcal{E}_{C,j,n,m}} \text{Var}_0^*(f)).
\]

Further let \( \mathcal{E}_{C,j,n,m}^{(i,j)} \) for \( (i, j) \in [0, n] \times [0, m-1] \) be the part of the event \( \mathcal{E}_{C,j,n,m} \) that depends on the vertices outside the \( i \)-th vertical strip and \( j \)-th horizontal strip, if \( i > n \) or \( j > m - 1 \) let \( \mathcal{E}_{C,j,n,m}^{(i,j)} = \mathcal{E}_{C,j,n,m} \). We have

\[
\sum_{C \text{ grid } n,m \in [N]} \sum_{\mathcal{E}_{C,j,n,m}} \mathbb{1}_{\mathcal{E}_{C,j,n,m}} \leq 2\ell
\]

since only the grid outside of the \( Q_i^{(h)} \) and \( Q_j^{(v)} \) is fixed and inside these strips there are at most \( \ell \) choices of straight horizontal paths or boxes that could be vertical crossings respectively (in the latter case \( \ell \) is a rough estimate of \( \ell / [g_{B,1/4}^2] \)).

Fix a grid \( C \) and \( n, m \in [N] \), extend the variance (Lemma 5.16) and use the block relaxation Lemma (Lemma 5.19) to get

\[
\mu(\mathbb{1}_{\mathcal{E}_{C,j,n,m}} \text{Var}_0^*(f)) \leq 2^{\rho B} \mu[\mathbb{1}_{\mathcal{E}_{C,j,n,m}} (\mathbb{1}_{W_{j_0,0}} B\text{-super \text{Var}_0^*(f) + Var}_{j_0,0}^{(s,BT)}(f))] .
\] (5.6)

We extend the variance in the first summand to \( \{0, e_2\} \) and then use the block relaxation Lemma again:

\[
\mu(\mathbb{1}_{\mathcal{E}_{C,j,n,m}, W_{j_0,0}} B\text{-super \text{Var}_0^*(f)})
\]

\[
\leq 2^{\rho B} \mu[\mathbb{1}_{\mathcal{E}_{C,j,n,m}, W_{j_0,0}} B\text{-super} (\mathbb{1}_{W_{e_2}} B\text{-super \text{Var}_0^*(f) + Var}_{e_2}^{(s,BT)}(f))].
\] (5.7)

For the second summand in Eq. (5.7) there is a unique shortest path \( \Gamma \) from \( e_2 \) to \( j_{0,0} \) first on the bottom boundary of \( D_{0,0} \) and then following the grid \( C \). Through a combination of extending the variance, the block relaxation Lemma, Lemmas 5.34 and 5.36
we get

\[ \mu \left[ \mathbb{1}_{\mathcal{C}_{j,n,m}, W_{j,0}} B \text{-super } \mathcal{V}_{\mathcal{E}_2}^{(\kappa, BT)}(f) \right] \leq 2^{e^B \beta^{3/2}} \mu \left[ \mathbb{1}_{\mathcal{C}_{j,n,m}, W_{0}} D_{\Gamma \cup \text{Supp}(\mathcal{E}^{(2)})}(f) \right]. \]

Analogously for the first term in Eq. (5.7) using Lemma 5.34. We can then take the sum over \( C, n \) and \( m \) and absorb the overcounting of the vertices in \( \text{Supp}(\mathcal{E}^{(2)}) \) into \( \kappa \) above for \( q_B \) small enough.

For the second summand in Eq. (5.6) we use completely analogous techniques to the proofs of the two-dimensional relaxation on the grids in part (i) and (ii), where here the \( B \)-super state corresponds to the vacancy state and the \( B \)-traversable state to the particle state of the auxiliary two-dimensional East model on the intersection points. Recovering the spectral gap of the \( ABC \)-model follows the same one-dimensional techniques from the first summand of Eq. (5.6). □

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REFERENCES

[1] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, Kinetically constrained spin models, Probab. Theory Rel. 140 (2008), no. 3-4, 459–504.
[2] P. Chleboun, A. Faggionato, and F. Martinelli, Mixing time and local exponential ergodicity of the East-like process in \( \mathbb{Z}^d \), Annales de la Faculté des sciences de Toulouse : Mathématiques Ser. 6, 24 (2015), no. 4, 717–743, DOI 10.5802/afst.1461.
[3] P. Chleboun, A. Faggionato, and F. Martinelli, Relaxation to equilibrium of generalized East processes on \( \mathbb{Z}^d \) : Renormalization group analysis and energy-entropy competition, The Annals of Probability 44 (2016), no. 3, 1817–1863.
[4] Y. Couzinié and F. Martinelli, On a front evolution problem for the multidimensional East model, arXiv, 2021.
[5] Y. Couzinié, The multidimensional East model: a multicolour model and a front evolution problem, Ph.D. thesis, Roma Tre University, 2022. Available on my website https://yannick-couzinie.github.io/theses/phd-thesis.
[6] A. Faggionato, F. Martinelli, C. Roberto, and C. Toninelli, The East model: recent results and new progresses, arXiv preprint arXiv:1205.1607 (2012).
[7] J. P. Garrahan and D. Chandler, Coarse-grained microscopic model of glass formers, Proceedings of the National Academy of Sciences 100 (2003), no. 17, 9710–9714.
[8] J. P Garrahan, P. Sollich, and C. Toninelli, Kinetically constrained models, Dynamical heterogeneities in glasses, colloids, and granular media 150 (2011), 111–137.
[9] E. Giné, G. R Grimmett, and L. Saloff-Coste, Lectures on Probability Theory and Statistics: Ecole D’Eté de Probabilités de Saint-Flour XXVI-1996, Springer, 2006.
[10] I. a. M. Hartarsky Fabio and Toninelli, Universality for critical KCM: finite number of stable directions, The Annals of Probability 49 (2021), no. 5, 2141–2174.
[11] G. Kordzakhia and S. P. Lalley, Ergodicity and mixing properties of the Northeast model, Journal of applied probability 43 (2006), no. 3, 782–792.
[12] T. M. Liggett, Interacting particle systems, Vol. 2, Springer, 1985.
[13] T. M. Liggett, Continuous time Markov processes: an introduction, Vol. 113, American Mathematical Soc., 2010.
[14] L. Marêché, *Exponential convergence to equilibrium for the $d$-dimensional East model*, Electronic Communications in Probability **24** (2019), 1–10.

[15] F. Martinelli and C. Toninelli, *Towards a universality picture for the relaxation to equilibrium of kinetically constrained models*, The Annals of Probability **47** (2019), no. 1, 324–361.

[16] F. Martinelli, A. Shapira, and C. Toninelli, *Diffusive scaling of the Kob-Andersen model in $\mathbb{Z}^d$*, Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, 2020, pp. 2189–2210.

[17] A. Shapira, *Kinetically constrained models with random constraints*, The Annals of Applied Probability **30** (2020), no. 2, 987–1006.

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