Efficient 2D $C_0$ FE based HOZT for analysis of singly curved laminated composite shell structures under point load

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Abstract. An efficient Higher order zigzag theory (HOZT) is proposed in this paper for analysis of laminated composite singly curved shell structures acted by point load on surface. The formulation includes effects of transverse displacement during analysis. Constant transverse displacement is assumed in face sheets while parabolic variation of transverse displacement field is assumed for central layer. Nine noded $C_0$ finite element (FE) with eleven degrees of freedom per node is used for carrying out solution. Stress continuity at interfaces and zero stress condition at top and bottom face of shell are also included during formulation. A number of problems are solved for cylindrical shell panels subjected to point load is analysed. Present results are found in accordance with results those already available in literature.

1. Introduction

Because of excellent properties possessed by composite materials such as high stiffness to weight ratio, high strength to weight ratio, resistance to chemical attacks, thermal and acoustic transmission, these materials are gaining popularity in many industries such as aeronautical, automobile, civil, marine, defence industries etc. Though these materials have many advantages, but also present many technical difficulties during their analysis. These problems include continuity of transverse stresses along interfaces, zero stress condition at top and bottom and lower value of shear modulus. Laminated shell structures are one of the most widely used structural configuration in industry. Since laminated shells are used in weight sensitive industries and can undergo large deformations and rotations. Therefore, this highlights the inclusion of transverse deformations during analysis in order to predict their more accurate behaviour under service conditions. Finite element (FE) is one of the most widely used method for carrying out analysis of laminated shell structures. Brank et al. [1] solved number of problems on curved surface subjected to point load. Saigal et al. [2] used multi-layered four noded finite shell element with six degrees of freedom per node for analysing laminated composite structures subjected to point loads. Cubic B-spline interpolation functions along with central displacement control method is employed for analysing singly curved laminated composite shell with point load at its centre. Due to Kirchhoff-Love approximation included in formulation of analysis it is not able to predict good results even for thin structures. Lee and Kanok-Nukulchai [3] employed element based Lagrangian formulation for large deformation analysis of laminated shells. Arc-length control method is employed for tracing load displacement paths and assumed natural strain method for avoiding shear or membrane locking effects. Laschet and Jeusette [4] carried out analysis of thin laminated cylindrical shells using 3D isoparametric multi-layered FE along with incremental/iterative method for carrying out analysis in non-linear range. Based on higher order shear deformation theories, Balah and Al-Ghamedy [5] carried out analysis of laminated composite structures using four noded FE in non-linear range. Han et al. [6] proposed first order based nine-noded shear deformable
Lagrangian FE for analysing thin laminated composite plates and shells under point loads. Assumed natural strain method is also applied to avoid shear locking or membrane locking effects. Carrera and Giunta [7] and Giunta et al. [8] used Carrera unified formulation for analysing doubly curved simply supported laminated composite shells under point loads using Navier’s solution. Present paper represents the analysis of singly curved laminated composite shells subjected to point loads. Linear static analysis is carried out by using higher order zigzag theory (HOZT) including effects of transverse deformations. Main advantage of inclusion of transverse deformations is able to predict more accurate results especially in case of thick shells. Displacement field chosen is combination of cubic variation of over entire thickness and linear zigzag function having different slope at each layer. Constant variation of transverse displacement is taken for face layer and parabolic variation of transverse displacement is taken for core. Present theory incorporates stress continuity at interface along the thickness of structure. Stress free condition at top and bottom of shell is also satisfied. Nine noded C0 FE having eleven degrees of freedom per node is used for carrying out solution. The main advantage of proposed theory is that it does not require any penalty function to be multiplied with the stiffness matrix and also eliminates C1 continuity requirements. Accuracy of present FE based HOZT model is depicted by solving the available problems in literature and thereby comparing the obtained results.

2. Modelling

Considering the middle plane of the shell as reference plane (Fig. 1), the in-displacement fields can be written as

$$U_x = u^0 + z\theta^x + \sum_{j=1}^{n_u-1} (z - z_j^{u})H(z - z_j^{u})P_j^{xu} + \sum_{k=1}^{n_r-1} (z - z_k^{r})H(-z + z_k^{r})Q_k^{x1} + z^2\alpha^x + z^3\beta^x \quad (1)$$

$$V_y = v^0 + z\theta^y + \sum_{j=1}^{n_u-1} (z - z_j^{u})H(z - z_j^{u})P_j^{yu} + \sum_{k=1}^{n_r-1} (z - z_k^{r})H(-z + z_k^{r})Q_k^{y1} + z^2\alpha^y + z^3\beta^y \quad (2)$$

**Figure 1.** Variation on in-plane displacement field through the thickness of laminated shell.

Transverse displacement field is assumed to vary linear in top and bottom layers and quadratic in core layer written as
\[ W_k = l_1 w^u + l_2 w^0 + l_3 w^l \] (for core) = \( w^u \) (for upper face layer) = \( w^l \) (for lower face layer) \hspace{1cm} (3)

Constitutive relationship for an orthotropic lamina (for \( i^{th} \) layer) with an arbitrary orientation in accordance to global axis (\( x-, y-, z- \)-axis system) can be written as

\[ \{\sigma\} = [\bar{Q}]_i \{\epsilon\} \hspace{1cm} (4) \]

Linear strain-displacement relation as per Sander’s approximation are

\[ \epsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{w_x}{R}, \epsilon_{yy} = \frac{\partial v_y}{\partial y}, \epsilon_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial w_z}{\partial y}, \gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}, \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial w_z}{\partial y} \] \hspace{1cm} (5)

Using the condition for transverse shear stress continuity at interfaces and transverse shear stress free condition at top and bottom surface of shell, in addition to condition of \( u = u^l, v = v^l \) for top surface of shell and \( u = u^l, v = v^l \) for bottom surface of shell, unknown quantities \( (\alpha^x, \alpha^y, \beta^x, \beta^y, p_{1u}^u, p_{2u}^u, p_{n_u-1}^u, Q_0^l, Q_1^l, Q_{n_l-1}^l, P_1^u, P_2^u, P_{n_u-1}^u, Q_1^l, Q_2^l, \ldots, Q_{n_l-1}^l, \frac{\partial w_u}{\partial x}, \frac{\partial w_u}{\partial y}, \frac{\partial w_l}{\partial x}, \frac{\partial w_l}{\partial y}) \) can be written as

\[ \{B\} = [A]\{\omega\} \hspace{1cm} (6) \]

where,

\[ \{B\} = \{a^x \beta^y \alpha^x \beta^y p_{1u}^u, p_{2u}^u, p_{n_u-1}^u, Q_0^l, Q_1^l, Q_{n_l-1}^l, P_1^u, P_2^u, p_{n_u-1}^u, Q_1^l, Q_2^l, \ldots, Q_{n_l-1}^l, \frac{\partial w_u}{\partial x}, \frac{\partial w_u}{\partial y}, \frac{\partial w_l}{\partial x}, \frac{\partial w_l}{\partial y}\}^T \]

\& \{\omega\} = \{u^0, v^0, \theta^x \theta^y \theta^x u^l v^0 u^l v^l\}^T.

The elements of \( \{B\} \) are function of material properties. The problem associated with \( C_l \) continuity requirements can be efficiently handled in this way as no derivative occurs in \( \{\omega\} \) or require no penalty function.

Using Eq. (6), Eqs. (1) and (2) can be re-written as

\[ U_x = c_1 u^0 + c_2 v^0 + c_3 \theta^x + c_4 \theta^y + c_5 u^u + c_6 v^u + c_7 u^l + c_8 v^l \] \hspace{1cm} (7)

\[ V_y = d_1 u^0 + d_2 v^0 + d_3 \theta^x + d_4 \theta^y + d_5 u^u + d_6 v^u + d_7 u^l + d_8 v^l \] \hspace{1cm} (8)

\( c_i \) and \( d_i \) are coefficients which are function of material properties, unit step function and thickness coordinates.

Using Eqs. (3), (7) and (8), displacement vector \( \{\delta\} \) can be written as

\[ \{\delta\} = \{u^0, v^0, w^0, \theta^x \theta^y \theta^x u^l v^0 u^l v^l\} \] \hspace{1cm} (9)

The strain-displacement relationship as in Eq. (5) can be stated as

\[ \{\epsilon\}_{6 \times 1} = [H]_{6 \times 33}\{\epsilon\}_{33 \times 1} \] \hspace{1cm} (10)

Total potential energy of the shell subjected to external load of intensity \( q \) can be written as

\[ \Pi_s = U_s - W_{\text{external}} \] \hspace{1cm} (11)

\[ U_s = \frac{1}{2} \iint \{\epsilon\}^T [D]\{\epsilon\} \, dx \, dy \] \hspace{1cm} \text{is the strain energy and} \hspace{1cm} \[ W_{\text{external}} = \iint wq \, dx \, dy \] \hspace{1cm} \text{is the amount of work done by external load.}

\[ [D] = \sum_{k=1}^n [H]^T [\bar{Q}]_i [H] \, dz \] \hspace{1cm} (12)

Using nine-noded isoparametric finite element (FE) having eleven degrees of freedom per node, strain vector in Eq. (10) can be written as

\[ \{\epsilon\} = [B]\{\delta\} \] \hspace{1cm} (13)

Where \( \{\delta\} = \{u^0, v^0, w^0, \theta^x \theta^y \theta^x u^l v^0 u^l v^l\}^T \) and using FE, generalized displacement vector corresponding to node \( "i" \) and number of nodes per element (\( n=9 \) in present analysis) can be stated as
\[ \{\delta\} = \sum_{i=1}^{N} [N]_i \{\delta\}_i \]

Potential energy of element as per Eq. (11) can be re-written as

\[
\Pi_e = \frac{1}{2} \iint \{\delta\}^T [B]^T [D] [B] \{\delta\} \, dx \, dy - \frac{1}{2} \iint \{\delta\}^T [B]^T [N^c]^T q \, dx \, dy
\]

\[ = \frac{1}{2} \{\delta\}^T [K]_e \{\delta\} - \frac{1}{2} \{\delta\}^T [P]_e \]

(14)

where

\[ [K]_e = \iint [B]^T [D] [B] \, dx \, dy \]  

(15)

\[ [P]_e = \iint [B]^T [N^c]^T q \, dx \, dy \]  

(16)

At equilibrium, \( \Pi_e \) in Eq. (14) has to be minimized with respect to \( \{\delta\} \).

\[ [K]_e \{\delta\} = [P]_e \]  

(17)

In the similar way, global load matrix and global stiffness matrices are formed by including all elements. Then boundary conditions are included. Numerical code for the above-mentioned methodology is written in MATLAB.

3. Results

Under this section results are reported for laminated composite shells analysed using proposed models acted upon by point load. The obtained results are compared with those available in literature. Material properties used during present studies are:

\[ E_1 = 3E_2, \quad E_2 = \text{open}, \quad E_3 = E_2, \quad G_{12} = G_{13} = G_{23} = 0.2E_2, \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \]

3.1. Thick hinged laminated composite cylindrical shell panel

Multi-layered laminated composite cylindrical shell hinged along straight edges and free along curved edges subjected to point load is investigated. Total thickness of shell is taken as 12.6 mm in order to study the applicability of proposed model in case of thick shells. Thickness of all layers are equal. Shell is hinged along straight edges and free along circular edges. Point load is subjected to centre of the shell. Radius of curvature of shell equals 2540 mm, length of shell equals 508 mm and angle subtended by shell at centre equals 0.2 radians. Firstly, convergence study is carried out on shell having lamination scheme [90°/0°/90°]. Results are reported in figure 2. It can be seen that results converge at mesh size of 14x14. For further studies, mesh size of 14x14 is taken.

Two cases of lamination schemes [90°/0°/90°] (case 1) and [0°/90°/0°] (case 2) are taken in order to study the influence of lamination schemes on the behaviour of shells. The obtained results are compared with those given by Saigal et al. [2] and Laschet and Jeusette [4]. Results are reported in figure 3. It can be seen that lamination scheme has great influence on the behaviour of shell. Shell having case 2 configuration is able to withstand higher loads that is, possess more flexural stiffness as compared to shell having case 1 configuration. Also, the obtained results are compared with those calculated using ABAQUS/CAE with element named as S8R5 in ABAQUS library.

Results predicted by Saigal et al. [2] presents much stiffer response as compared to present results. Behaviour predicted by Laschet and Jeusette [4] and present results are almost similar. However, present results are more accurate because it includes effects of transverse shear deformations which are more pronounced in case of thick shells (length to thickness ratio=40). Because of this reason maxim error of 10% is observed when compared with Saigal et al. [2]. However, snap through response is observed for both types of shells.

Figure 4 shows results for same shell with two layers having configuration [45°/-45°] and are compared with response obtained by Laschet and Jeusette [4].
3.2. Thin hinged laminated composite cylindrical shell panel

Same shell configuration as taken in previous example with thickness 6.35 mm is used for study in order to study behaviour of thin cylindrical shells with lamination scheme [90°/0°/90°]. Obtained results are compared with those obtained by Brank et al. [1] using four noded shell FE based on 3D continuum theory. It can be seen from figure 5 that the proposed model is able to predict results which are close to 3D solutions.

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**Figure 2.** Convergence study for central displacement of three layered laminated composite cylindrical shell.

**Figure 3.** Behavior of three layered laminated composite cylindrical shell having lamination scheme [90°/0°/90°] and [0°/90°/0°].
Figure 4. Behavior of three layered laminated composite cylindrical shell having lamination scheme 
$[45^\circ/-45^\circ]$.

Maximum error in case of present results is found to be 5%. Result for same problem with lamination 
scheme $[0^\circ/90^\circ/0^\circ]$ is reported in figure 6. From figures 5 & 6, it can be seen that thin shell exhibit 
both snap through and snap back limit points with complex equilibrium path along with number 
of limit path as compare to thick shell. Hence, thin shells achieve equilibrium path after exhibiting 
several limit points. However, lamination scheme $[0^\circ/90^\circ/0^\circ]$ exhibit simpler path for attaining 
equilibrium when compared to lamination scheme $[90^\circ/0^\circ/90^\circ]$. This highlights the thorough study of 
thin laminated cylindrical shells when subjected to point loads.

Figure 5. Behavior of three layered thin laminated composite cylindrical shell having lamination 
scheme $[90^\circ/0^\circ/90^\circ]$. 
Figure 6. Behavior of three layered thin laminated composite cylindrical shell having lamination scheme $[0^\circ/90^\circ/0^\circ]$.

4. Discussion

Behaviour of laminated composite cylindrical shells under point load is entirely different than usual uniformly distributed or sinusoidal load acting on the structure. Point or localised loads mainly affects the structure underneath the point of action of load. From above studies, it can be seen that the present model predicts results in good agreement when compared with available results. Since the proposed model includes transverse deformation effects, behaviour of thick shells can also be predicted in good agreement. It is shown above that in case of thin shells, the behaviour is more complicated before achieving the stable configuration. Therefore, thin shells must be analysed carefully under service conditions during analysis stage. Lamination scheme is the one of the major factors affecting the behaviour of laminated composite shell as seen while solving above problems. In $[90^\circ/0^\circ/90^\circ]$ thin lamination scheme, both snap through and snap back phenomenon are observed while in case of same lamination scheme for thick shells, snap through phenomenon is observed. Thus, thickness of shell also affects the behaviour of shell.

5. Conclusions

The proposed C₀ FE model based on HOZT is applied for predicting the behaviour of laminated composite cylindrical shells subjected to localised loading at surface. It can be seen that with the help of proposed model, the obtained results are in accordance with already available results. Also, the proposed model is capable to predict more accurate results especially for thick shells. Both snap through and snap back phenomenon are observed for thin shells while only snap through phenomenon is observed in case of thick shells. The behaviour of shell is widely affected by the lamination scheme for shell having same number of layers. Hence, in such cases the lamination scheme must be analysed carefully and proper scheme must be chosen.
## 6. Appendix A

### Notation

| Symbol       | Description                                                                 |
|--------------|-----------------------------------------------------------------------------|
| \( u_0, v_0 \) | In-plane displacement of any point on the mid surface                      |
| \( \theta^x, \theta^y \) | Rotation of normal to the mid plane about y- and x- axis respectively         |
| \( n_u, n_l \) | Number of upper and lower layers respectively                                |
| \( \alpha^x, \alpha^y, \beta^x, \beta^y \) | Higher-order unknowns                                                        |
| \( p_j^{ xu}, p_j^{ yu}, Q_j^{ xl}, Q_j^{ yl} \) | Slopes of \( j^{th} \) layer corresponding to upper and lower layer respectively |
| \( H(z - z^u_j), H(-z + z^l) \) | Unit step functions                                                          |
| \( w^u, w^0, w^l \) | Transverse displacement at top, middle and lower layer of the core         |
| \( l_1, l_2, l_3 \) | Lagrangian interpolation functions in the thickness coordinate as in Appendix B |
| \( \bar{\sigma}, [\bar{Q}], \{ \bar{\epsilon} \} \) | Stress vector, transformed rigidity matrix and strain vector respectively for \( i^{th} \) lamina |
| \( A_1 \) | Tracer by which analysis can be reduced to shear deformable Love’s first approximation |
| \( [N^c] \) | Shape function matrix in which terms are associated with corresponding nodal displacements |
| \( [K^c], [P^c] \) | Elemental stiffness matrix and nodal load vector respectively               |
| \( \{ \epsilon \} \) | Modified strain vector                                                       |
| \( [B] \) | Strain displacement matrix written in Cartesian coordinate                  |

## 7. References

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