Vortices in rotating Bose-Einstein condensates confined in homogeneous traps

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Abstract

We investigate analytically the thermodynamical stability of vortices in the ground state of rotating 2-dimensional Bose-Einstein condensates confined in asymptotically homogeneous trapping potentials in the Thomas-Fermi regime. Our starting point is the Gross-Pitaevskii energy functional in the rotating frame. By estimating lower and upper bounds for this energy, we show that the leading order in energy and density can be described by the corresponding Thomas-Fermi quantities and we derive the next order contributions due to vortices. As an application, we consider a general potential of the form\[ V(x, y) = (x^2 + \lambda^2 y^2)^{s/2} \] with slope \( s \in [2, \infty) \) and anisotropy \( \lambda \in (0, 1] \) which includes the harmonic (\( s = 2 \)) and 'flat' (\( s \to \infty \)) trap, respectively. For this potential, we derive the critical angular velocities for the existence of vortices and show that all vortices are single-quantized. Moreover, we derive relations which determine the distribution of the vortices in the condensate i.e. the vortex pattern.

Key words:
Static properties of condensates; thermodynamical, statistical and structural properties, Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices and topological excitations
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1 Introduction

Many efforts have been made in understanding ultra-cold quantum gases, especially since the experimental achievement of Bose-Einstein condensates (BECs) in 1995. A particular interesting subject is the study of rotating BECs. When the trap is subjected to an external rotation the condensate does not

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rotate like a solid body. Instead, beyond a critical angular velocity quantized vortices appear manifesting the genuine quantum character of the system. Indeed, vortices in BECs were observed in 1999 for the first time (see Refs. [29] and [27,28]). Theoretical studies were already presented before (see e.g. [32] for one of the earliest papers on the subject) and have since then grown to a substantial branch of its own (see e.g. [38,11,12,15-17,25-36]). A general treatment of BECs can be found in the textbooks of [30] and [31].

Most of the theoretical studies have been undertaken in the framework of the Gross-Pitaevskii (GP) theory whose validity as an approximation of the quantum mechanical many-body ground state was established in [22] for the non-rotating case and in [24] for rotating systems. Particular attention has been put on the so-called Thomas-Fermi (TF) regime of strong coupling. This is especially true for the study of vortex structures (see the monograph [1]). In [18,19] a rigorous analysis of vortices for BECs in harmonic anisotropic trap potentials was achieved for a GP-type functional in the TF limit. A previous analysis was developed in [35] in the context of superfluids. The methodology of those papers originates from [7] where a rigorous analysis of vortices in Ginzburg-Landau models of vanishing magnetic field in the regime which corresponds to the TF limit was developed. In [33] and [34], general results on symmetry breaking which are not limited to the TF regime were proven in traps of arbitrary shape.

Within the GP theory, the properties of vortices are determined by two physical parameters apart from the external trap, namely angular velocity and interaction strength between the particles. In this paper, we consider the ground state of rotating 2D Bose-Einstein condensates which are trapped in asymptotically homogeneous anisotropic potentials rotating with angular velocity $\Omega$. The aim of this investigation consists of deducing analytically the Gross-Pitaevskii energy and density in presence of vortices and deriving their properties in the TF regime. We consider thermodynamical conditions for vortex existence, i.e. we are looking for angular velocities which reduce the total energy in such a way that vortices are energetically favoured to appear. This work was originally inspired by the papers of [4] and [8] which consider anisotropic harmonic potentials. There and for instance in Refs. [15,25,32], it was established by numerical methods that vortices are single-quantized. We show here by analytical estimates, in particular, that this is true for a very large class of trapping potentials. In fact, the majority of studies uses numerical and variational methods for a limited number of trap potentials (e.g. harmonic or harmonic-plus-quartic) whereas we derive analytical formulae for a very large class of potentials. Thereby, we try to present the analysis in such a way that both the physical ideas and mathematical estimates are brought out in a clear way.

This paper is organized as follows: In Section 2, we state the setting and present the main result. We decompose the condensate wave function in a vortex-free part and a vortex-carrying part. This allows a splitting of the un-
derlying energy functional in separate contributions which can be estimated subsequently. In Section 3, we study the leading asymptotics of the energy and density. In Section 4, we justify a model for the structure and number of vortex cores which is compatible with the considered order of magnitude of the angular velocities. Sections 5-7 contain lower and upper bound estimates of the vortex-carrying energy contributions in terms of the winding number of the vortices and the coupling parameter. In Section 8, we specify an external potential which is of a general anisotropic homogeneous form. For this potential, we deduce the critical angular velocity for the appearance of one or a finite number of vortices. The leading orders of the energy in presence of vortices are calculated and it is shown that all vortices have winding number one, i.e. they are all single-quantized. Furthermore, we deduce relations which determine the distribution of the vortices in the condensate, i.e. the vortex pattern. Finally, in Section 9 we present the conclusions.

2 Setting and main result

Our starting point is the 2D Gross-Pitaevskii energy functional in the reference frame rotating (uniformly) with \( \tilde{\Omega} \) (see e.g. [8,30,31]):

\[
\mathcal{E}^{GP}[u] = \int_{\mathbb{R}^2} \left[ \frac{\hbar^2}{2m}|\nabla u|^2 + V(r)|u|^2 + \frac{Ng}{2}|u|^4 - iu^* \hbar \tilde{\Omega} \cdot (\nabla u \times r) \right].
\] (1)

Indeed, it is only meaningful to consider this reference frame as far as the (temporal) stability of structures is concerned which appear due to the rotation: The external trap is time-independent and the states are stationary with respect to that frame (see e.g. Ref. [8]). The function \( u(r) \) is a complex field (the complex conjugate is denoted as \( u^* \)). We write the associated polar decomposition as \( u = |u|e^{iS_u} \) where \( |u|^2 \) is proportional to the density of condensed particles with normalization

\[
\int_{\mathbb{R}^2} |u|^2 = 1
\]

and \( S_u \) is the phase function. The minimizer of (1) is called the order parameter or 'wave function of the condensate' in the rotating frame. The external trap potential is denoted by \( V \) and \( N \) is the number of particles with mass \( m \). The third term in (1) describes the effective interaction between the particles where the coupling constant in 2D is given by \( g = \sqrt{8\pi} \hbar^2 a/(mh) \) with the 3D scattering length \( a \) and the thickness \( h \) of the system in the strongly confined direction which we choose to be the \( z \)-axis, so that the system is effectively 2D (in the \( x \)-\( y \)-plane). We denote \( \times \) as the vector product in \( \mathbb{R}^3 \), \( r = (x,y,0) \) and \( \tilde{\Omega} = (0,0,\tilde{\Omega}) \) is the angular velocity vector assuming that the gas rotates around the \( z \)-axis. An important parameter, consisting of the
scattering length and a density, is given by the ‘healing length’ \( \xi \). It is defined originally by setting \( \hbar^2/(2m\xi^2) = 2\pi \rho a h^2/m \) where the r.h.s is the energy per particle for gases in a box in the limit of dilute systems \( \rho a^3 \to 0 \) with density \( \rho \), so \( \xi = 1/\sqrt{4\pi a \rho} \). For inhomogeneous and rotating systems, the healing length may be defined accordingly by using an appropriate (mean) value for the density. In particular, the healing length determines the effective radius of a vortex core in rotating systems.

In 2 dimensions, the ratio between the healing length and the characteristic length of the system \( L \), which is set by the external trap or box respectively, is

\[
\varepsilon^2 \sim \frac{\xi^2}{L^2} = \frac{\hbar^2}{\sqrt{2\pi} N g m}
\]

where we introduce the dimensionless parameter \( \varepsilon \). In this paper, we will be concerned with the TF limit where this ratio tends to zero (meaning physically that \( 0 < \xi \ll L \) or \( 0 < \varepsilon \ll 1 \) respectively). However, when performing the TF limit in a naive way for external potentials, where the gas can spread out indefinitely, one obtains a trivial result, namely the minimizer goes to zero and the energy to infinity. In order to obtain a non-trivial limit, it is then necessary to rescale all lengths by an \( \varepsilon \)-dependent factor (see also [9]):

Suppose \( V \) is homogeneous of order \( s \), i.e. \( V(\gamma r) = \gamma^s V(r) \) for \( \gamma > 0 \). We rescale the energy functional (1) by setting \( r = k r' \) and \( u(r) = u'(r')/k \) with \( Ng/2 = h^2/(4\varepsilon^2 m) \) and \( k = (h^2/(4\varepsilon^2 m))^{1/(s+2)} \). Then we have

\[
\mathcal{E}_{GP}[u] = \frac{1}{k^2} \int_{\mathbb{R}^2} \left[ \frac{\hbar^2}{2m} |\nabla u'|^2 + k^{s+2} V(r') |u'|^2 + \frac{Ng}{2} |u'|^4 - iu' \ast k^2 \Omega \cdot (\nabla u' \times r') \right] d^2r'
\]

with \( \int |u'|^2 = 1 \). Choosing \( h = 1 = m \) and inserting \( k \), (2) becomes \( \mathcal{E}_{GP}[u] = (16\varepsilon^4)^{1/(s+2)} \mathcal{E}_{GP}[u'] \) with the energy on the r.h.s. (omitting the primes)

\[
\mathcal{E}_{GP}[u] = \int_{\mathbb{R}^2} \left[ \frac{1}{2} |\nabla u|^2 + \frac{|u|^2}{4\varepsilon^2} (V + |u|^2) - iu^\ast \Omega(\varepsilon) \cdot (\nabla u \times r) \right]
\]

and the scaled angular velocity \( \Omega(\varepsilon) \) is related to the original unscaled one by

\[
\Omega(\varepsilon) = \tilde{\Omega}/(16\varepsilon^4)^{1/(s+2)}.
\]

For brevity, we will also write \( \Omega \) but it should be kept in mind that \( \Omega \) depends on \( \varepsilon \) after scaling. In the forthcoming, we study the functional in (3) which can be also written in the following form

\[
\mathcal{E}_{GP}[u] = \int_{\mathbb{R}^2} \left[ \frac{1}{2} |(\nabla - i(\Omega \times r))u|^2 + \frac{|u|^2}{4\varepsilon^2} (V + |u|^2) - \frac{1}{2} \Omega^2 r^2 |u|^2 \right]
\]

and \( r := |r| \). Critical points of \( \mathcal{E}_{GP}[u] \) are solutions of the following associated Euler-Lagrange equation, called Gross-Pitaevskii equation

\[
\Delta u = \frac{u}{2\varepsilon^2} (V + 2|u|^2 - 4\varepsilon^2 \mu_{GP}) + 2i(\Omega \times r) \cdot \nabla u
\]
where the GP chemical potential $\mu^{\text{GP}}$ is fixed by the normalization. Denoting a minimizer of (3) as $u_\varepsilon$, it is given by

$$\mu^{\text{GP}} = \mathcal{E}^{\text{GP}}[u_\varepsilon] + \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^2} |u_\varepsilon|^4. \tag{7}$$

The corresponding amplitude squared $|u_\varepsilon|^2$ will be referred to as Gross-Pitaevskii density. Inserting $u = |u|e^{iS_u}$ into (6) results in hydrodynamic-like relations for the density and the velocity:

$$\Delta |u| - |u|(\nabla S_u)^2 + 2|u|(|\Omega \times \mathbf{r}) \cdot \nabla S_u - \frac{|u|}{2\varepsilon^2} (V + 2|u|^2 - 4\varepsilon^2\mu^{\text{GP}}) = 0,$$

$$\nabla \cdot [|u|^2(\nabla S_u - \Omega \times \mathbf{r})] = 0.$$

The GP functional (3) for $\Omega = 0$ describes the gas without rotation

$$\mathcal{E}^{\text{GP}}[f] = \int_{\mathbb{R}^2} \left[ \frac{1}{2}(\nabla f)^2 + \frac{f^2}{4\varepsilon^2}(V + f^2) \right], \tag{8}$$

with $f$ a real, positive function. The minimizer of (8) will be denoted as $f_\varepsilon$. The normalization condition $\int_{\mathbb{R}^2} f^2 = 1$ fixes the associated chemical potential $\nu^{\text{GP}}$ which is given by

$$\nu^{\text{GP}} = \mathcal{E}^{\text{GP}}[f_\varepsilon] + \frac{1}{4\varepsilon^2} \int f_\varepsilon^4 \tag{9}$$

and which is of the order $1/\varepsilon^2$. The functional (8) tends for $\varepsilon \to 0$ to a Thomas-Fermi type functional

$$\mathcal{E}^{\text{TF}}[\rho] = \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^2} \rho(V + \rho), \tag{10}$$

which is a functional for the density $\rho = f^2$ alone. It can be shown (see e.g. Ref. [22]) that it has a unique positive minimizer, the Thomas-Fermi density,

$$\rho^{\text{TF}} = \frac{1}{2}[4\varepsilon^2 \mu^{\text{TF}} - V]_+ =: \frac{1}{2}[\mu - V]_+ \tag{11}$$

where $[.]_+$ denotes the positive part and $\mu := 4\varepsilon^2\mu^{\text{TF}}$. The TF chemical potential $\mu^{\text{TF}}$ (or $\mu$ respectively) is determined by

$$\int_{\mathcal{D}} \rho^{\text{TF}} = 1 \tag{12}$$

where

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : \rho^{\text{TF}} > 0\}$$

is the Thomas-Fermi domain whose shape depends on the external potential $V$. Moreover,

$$\mu^{\text{TF}} = \mathcal{E}^{\text{TF}}[\rho^{\text{TF}}] + \frac{1}{4\varepsilon^2} \int (\rho^{\text{TF}})^2,$$
which is of the order $1/\varepsilon^2$ whereas $\mu := 4\varepsilon^2\mu_{\text{TF}}$ is of the order of a constant independent of $\varepsilon$.

### 2.1 Splitting of the GP energy functional

In the TF regime where $\varepsilon$ is small, vortex cores are small compared to the characteristic length scale of the system, producing narrow 'holes' which effectively shrink as $\varepsilon \to 0$. It is argued in Section 4 that vortices appear at a critical angular velocity of the order $\Omega \simeq C|\ln \varepsilon|$ with $C$ a positive constant (independent of $\varepsilon$) depending on the external trap. Explicit expressions for $C$ will be determined in the forthcoming analysis (see also Refs. [4] and [8] for the harmonic trap case).

In the minimization of (8), i.e. (3) with $\Omega = 0$, one considers all functions in the subspace of angular momentum zero and the density profile is given by $f_{\varepsilon}^2$. Considering (3) with $\Omega > 0$ we will see that, as long as $\Omega \leq C|\ln \varepsilon|$ asymptotically, the overall density can still be described by the vortex-free density $f_{\varepsilon}^2$ in good approximation. However, in a non-isotropic potential $V$ there appears a phase $S$ (depending on $V$), i.e. the vortex-free function is then more generally $f e^{iS}$. Since this function has no vortex, the phase $S$ is non-singular and (9) gives

$$\Delta f = -f \nabla S \cdot [2(\Omega \times \mathbf{r}) - \nabla S] + \frac{f}{2\varepsilon^2}(V + 2f^2 - 4\varepsilon^2\tilde{\nu}_{\text{GP}})$$

and

$$\nabla \cdot [f^2(\nabla S - \Omega \times \mathbf{r})] = 0$$

where $\tilde{\nu}_{\text{GP}}$ is the associated chemical potential. A solution without vortex is a minimizer of the problem $\min \{\mathcal{E}_{\text{GP}}[f e^{iS}] : f e^{iS} \in H^1 \text{ with } f > 0, \int f^2 = 1\}$ (see also [18]) with

$$\mathcal{E}_{\text{GP}}[f e^{iS}] = \int_{\mathbb{R}^2} \left[ \frac{1}{2}(\nabla f)^2 + \frac{f^2}{4\varepsilon^2}(V + f^2) + \frac{1}{2}f^2[(\nabla S)^2 - 2\nabla S \cdot (\Omega \times \mathbf{r})] \right].$$

Later on in this paper we are going to consider external traps of the form

$$V(x, y) = (x^2 + \lambda^2 y^2)^{s/2}$$

with slope $s \in [2, \infty)$ and $\lambda \in (0, 1]$ describing the anisotropy. It is a fairly general potential which includes also the important special cases of the harmonic ($s = 2$) and flat ($s \to \infty$) trap which are extensively used in experiments. The corresponding phase to this potential is

$$S = \frac{\lambda^2 - 1}{\lambda^2 + 1} \Omega xy$$

which vanishes for the isotropic case $\lambda = 1$. This expression for $S$ was also deduced for the harmonic trap in Refs. [4] and [8]. Note, however, that it is
not dependent on the slope parameter $s$. We also see that the terms in (15) involving $\nabla S$ are at most of the order $|\ln \varepsilon|^2$ for $\Omega < |\ln \varepsilon|$ and hence of much lower order than the remaining part described by (8) which is $\sim 1/\varepsilon^2$.

We now decompose the order parameter $u$ of (3) into the vortex-free part $f e^{iS}$ and a part which carries the vorticity. A similar splitting can be found in Refs. [4,5,8,20] and more recently in Refs. [6,14,37]. Writing $u = |u|e^{iS_u} = f e^{i\delta} v = f|v|e^{i(S + S_v)}$ with $|u| = f|v|$ and $S_u = S + S_v$, the contribution $v = |v|e^{iS_v}$ accounts for the presence of vortices. In a vortex point, the amplitude vanishes, i.e. $|u| = |v| = 0$ since $f \neq 0$ and the phase fulfills the usual circulation condition which is a quantization condition because $u$ (resp. $v$) is a complex field:

$$\oint_C \nabla S_u \cdot \tau = \oint_C (\nabla S_v + \nabla S) \cdot \tau = 2\pi d + 0$$

since $S$ has no singularity and $\tau$ is a unit tangent vector to the curve $C$ encircling the vortex with winding number $d$. Without the presence of vortices, there would be $u = f e^{iS}$ with density $|u|^2 = f^2$ and the phase $S_u$ would be non-singular. Inserting the decomposition $u = f e^{iS_v}$ in the energy functional (8) results in the following splitting (see also [4,5]) where the integrals are over $\mathbb{R}^2$: The first term becomes

$$\int \frac{1}{2} |\nabla (f e^{iS_v})|^2 = \int \left[ \frac{1}{2} f^2 |\nabla v|^2 + \frac{1}{2} |v|^2 [ (\nabla f)^2 + f^2 (\nabla S)^2 ] + \frac{1}{4} \nabla (f^2) \cdot \nabla |v|^2 \\
+ \frac{1}{2} f^2 \nabla \cdot (iv \nabla v^* - iv^* \nabla v) \right],$$

the second one is simply

$$\int \frac{|f e^{iS_v}|^2}{4\varepsilon^2} (V + |f e^{iS_v}|^2) = \int \frac{f^2 |v|^2}{4\varepsilon^2} (V + f^2 |v|^2)$$

and for the rotation term we get

$$- \int if e^{-iS_v} v^* \Omega \cdot (\nabla (f e^{iS_v}) \times r) = \int if^2 v^* \nabla v \cdot (\Omega \times r) - \int f^2 |v|^2 \nabla S \cdot (\Omega \times r).$$

Putting the terms together and separating the vortex-free part of the energy (15), we have

$$\mathcal{E}^{GP}[u] = \mathcal{E}^{GP}[f e^{iS}] + \int |v|^2 - 1 \times$$

$$\left[ \frac{1}{2} (\nabla f)^2 + \frac{1}{2} f^2 (\nabla S)^2 - f^2 \nabla \cdot (\Omega \times r) + \frac{V f^2}{4\varepsilon^2} \right] +$$

$$+ \int \frac{1}{4} \nabla (f^2) \cdot \nabla |v|^2 + \int \frac{1}{2} f^2 |\nabla v|^2 + \int \frac{f^4}{4\varepsilon^2} (|v|^4 - 1) +$$

$$+ \int \left[ \frac{1}{2} f^2 \nabla S (iv \nabla v^* - iv^* \nabla v) + if^2 v^* \nabla v \cdot (\Omega \times r) \right].$$
The third term of this expression becomes

\[
\int \frac{1}{4} \nabla (f^2) \cdot \nabla |v|^2 = \int \frac{1}{4} \nabla (f^2) \cdot \nabla (|v|^2 - 1) = - \int \frac{1}{4} (|v|^2 - 1) \Delta (f^2) \\
= - \int \frac{1}{2} (|v|^2 - 1) f \Delta f - \int \frac{1}{2} (|v|^2 - 1) (\nabla f)^2 \\
= \int (|v|^2 - 1) \left[ f^2 \nabla S \cdot (\Omega \times r) - \frac{1}{2} f^2 (\nabla S)^2 - \frac{f^2}{4\epsilon^2} (V + 2 f^2 - 4\epsilon^2 \tilde{v}_{GP}) - \frac{1}{2} (\nabla f)^2 \right],
\]

where we used (13) for \( \Delta f \).

Moreover, for the fifth term in (18) we use the identity

\[
\int \frac{f^4}{4\epsilon^2} (|v|^4 - 1) = \int \frac{f^4}{2\epsilon^2} (|v|^2 - 1) + \int \frac{f^4}{2\epsilon^2} (1 - |v|^2)^2.
\]

Inserting the last two equations into (18) we get the following splitting of the functional in (3)

\[
E_{GP}[u] = E_{GP}[fe^iS] + \int_{\mathbb{R}^2} \left[ \frac{f^2}{2} |\nabla v|^2 + \frac{f^4}{4\epsilon^2} (1 - |v|^2)^2 \right] - \int_{\mathbb{R}^2} i f^2 v^* \nabla v \cdot (\nabla S - \Omega \times r) \\
=: E_{GP}[fe^iS] + \mathcal{G}_f[v] - R_f[v] \\
\tag{19}
\]

where we used \( \tilde{v}_{GP} \int f^2 (|v|^2 - 1) = 0 \) because of the normalization conditions and the last term in (18) was written in a more convenient form using

\[
\int \left[ \frac{1}{2} f^2 \nabla S \cdot (i v \nabla v^* - i v^* \nabla v) + f^2 i v^* \nabla v \cdot (\Omega \times r) \right] \\
= \int f^2 \nabla S \cdot (i v, \nabla v) - \int \left[ f^2 (\Omega \times r) \cdot (i v, \nabla v) + \frac{f^2}{2} i (\Omega \times r) \cdot \nabla (|v|^2) \right] \\
= \int f^2 (i v, \nabla v) \cdot (\nabla S - \Omega \times r) = - \int f^2 i v^* \nabla v \cdot (\nabla S - \Omega \times r)
\]

where \((u, v) := (uv^* + u^* v)/2\). The terms apart from the vortex-free energy in (19) describe the contribution of the vorticity to the energy: The second term \( \mathcal{G}_f[v] \) looks formally like a 'weighted' Ginzburg-Landau (GL) energy functional without magnetic field and accordingly will be called GL-type energy in the forthcoming and \( R_f[v] \) is the rotation energy.

Using the splitting (19), vortices of \( u \) (if present) are vortices of \( v \) and they are described via the functionals \( \mathcal{G}_f[v] - R_f[v] \).

### 2.2 Main result

We have the following main result:

**Main result:** Let \( u_\epsilon \) be a minimizer of (3) and \( f_\epsilon \) a minimizer of (15) for
\[ V \text{ in (16) and } S \text{ in (17) and under the normalization constraints. Let } C \text{ and } \delta \text{ be positive constants independent of } \varepsilon \text{ with } 0 < \delta \ll 1 \text{ and let } o(1) \text{ denote a quantity which goes to zero as } \varepsilon \to 0. \]

For some integer \( n \geq 1 \) and

\[ \Omega_n = C_1 \ln |\varepsilon| + (n - 1) \ln |\ln |\varepsilon|| =: \Omega_1 + C_1(n - 1) \ln |\ln |\varepsilon|| \tag{20} \]

with

\[ C_1 := \frac{s + 2 \mu^2/\lambda}{s \mu^2/s + 1 + \lambda^2}; \]

we have the following results:

i) If \( \Omega \leq \Omega_1 - C_1 \delta \ln |\ln |\varepsilon|| \) and \( \varepsilon \) sufficiently small, then \( u_\varepsilon \) has no vortices in \( \mathcal{D} \setminus \partial \mathcal{D} \) and the Gross-Pitaevskii energy is

\[ \mathcal{E}^{GP}[u_\varepsilon] = \mathcal{E}^{GP}[f_\varepsilon e^{iS}] + C. \tag{21} \]

ii) If \( \Omega_n + C_1 \delta \ln |\ln |\varepsilon|| \leq \Omega \leq \Omega_{n+1} - C_1 \delta \ln |\ln |\varepsilon|| \) for \( n \geq 1 \), then, for \( \varepsilon \) sufficiently small, \( u_\varepsilon \) has \( n \) vortices with winding number one located in \( r_1, ..., r_n \in \mathcal{D} \setminus \partial \mathcal{D} \), \( r_i = (x_i, y_i), i = 1, ..., n \). Setting \( \tilde{r}_i = (\tilde{x}_i, \tilde{y}_i) \) with \( \tilde{x}_i = x_i \sqrt{\Omega}, \tilde{y}_i = y_i \lambda \sqrt{\Omega} \), the configuration \((\tilde{r}_1, ..., \tilde{r}_n)\) minimizes the function

\[ w(a_1, ..., a_n) = -\frac{\pi \mu}{4} \sum_{i \neq j} \ln[(X_i - X_j)^2 + \lambda^{-2}(Y_i - Y_j)^2] + \frac{\pi \mu}{1 + \lambda^2} \sum_{i=1}^{n} (X_i^2 + Y_i^2) - \frac{\pi \ln \Omega}{4 \Omega^{s/2}} \sum_{i=1}^{n} (X_i^2 + Y_i^2)^{s/2} \]

with \( a_i = (X_i, Y_i), i = 1, ..., n \) and the Gross-Pitaevskii energy is

\[ \mathcal{E}^{GP}[u_\varepsilon] = \mathcal{E}^{GP}[f_\varepsilon e^{iS}] + \frac{\pi}{2} \mu n \left( \ln |\varepsilon| - \frac{2s}{(1 + \lambda^2)(s + 2) \mu^2/\lambda \Omega} \right) + \frac{\pi}{4} \mu n(n - 1) \ln \Omega + w(\tilde{r}_1, ..., \tilde{r}_n) + C + o(1). \tag{22} \]

The proof is split into several estimates which are shown in the following sections. There, positive constants are denoted by \( C \) (sometimes carrying primes) and they may change from line to line.

3 The leading order in energy and density

In this section, we show the leading asymptotics for the GP energy and density. We will see, in particular, that it is not affected by vortices whose influence can only be seen in the next lower order. The leading term in the energy comes from the TF contribution in [10] which is \( \sim 1/\varepsilon^2 \) whereas vortices contribute to the order \( \Omega \sim |\ln |\varepsilon|| \) (see also Section 4). However, the determination of the
precise expressions in (22) requires a more detailed analysis which is carried out in Sections 5-8.

For the following estimates, we introduce the function

\[ b(r) := \frac{1}{2}(\mu - V(r)) \]  

(23)

whose positive part is the TF density, i.e. \([b(r)]_+ := \rho_{TF}\).

**Estimate 1:** For \(\Omega(\varepsilon)\) satisfying \(C_V|\ln \varepsilon| \leq \Omega(\varepsilon) < C|\ln \varepsilon|\) where \(C_V\) depends on the parameters of the external potential \(V\), \(C > C_V\), and for \(\varepsilon\) sufficiently small,

\[ \mathcal{E}^{GP}[u_\varepsilon] = \mathcal{E}^{TF}[\rho_{TF}] + C|\ln \varepsilon| \]  

(24)

and

\[ \int_{\mathbb{R}^2} (|u_\varepsilon|^2 - \rho_{TF})^2 = o(1). \]

**Proof:**

This can be shown similar as Prop. 2.3 in [9]. The lower bound can be trivially obtained by neglecting the first positive term in (5)

\[ \mathcal{E}^{GP}[u_\varepsilon] \geq \mathcal{E}^{TF}[\rho_{TF}] - C\Omega(\varepsilon)^2. \]

The upper bound can be obtained by using \(\mathcal{E}^{GP}[u_\varepsilon] \leq \mathcal{E}^{GP}[u_\varepsilon]|_{\Omega=0}\) and \(\sqrt{\rho_{TF}}\) as a trial function,

\[ \mathcal{E}^{GP}[u_\varepsilon]|_{\Omega=0} \leq \mathcal{E}^{TF}[\rho_{TF}] + C|\ln \varepsilon|. \]

Concerning the density asymptotics, we estimate the following: using the negativity of \(b(r)\) outside the TF domain, we have

\[ \int_{\mathbb{R}^2} (|u_\varepsilon|^2 - \rho_{TF})^2 \leq \int_{\mathbb{R}^2} \left(|u_\varepsilon|^4 - 2b(r)|u_\varepsilon|^2 + (\rho_{TF})^2\right). \]

On the other hand, we deduce

\[ 4\varepsilon^2 \mathcal{E}^{TF}[|u_\varepsilon|^2] = \int_{\mathbb{R}^2} \left[|u_\varepsilon|^4 + |u_\varepsilon|^2V\right] = \int_{\mathbb{R}^2} |u_\varepsilon|^4 - 2\int_{\mathbb{R}^2} b(r)|u_\varepsilon|^2 + \mu, \]

that is

\[ \int_{\mathbb{R}^2} (|u_\varepsilon|^2 - \rho_{TF})^2 \leq 4\varepsilon^2 \mathcal{E}^{TF}[|u_\varepsilon|^2] + \int_{\mathbb{R}^2} (\rho_{TF})^2 - \mu \]

\[ = 4\varepsilon^2 (\mathcal{E}^{TF}[|u_\varepsilon|^2] - \mathcal{E}^{TF}[\rho_{TF}]) \leq C\varepsilon^2 |\ln \varepsilon| \]

and the last inequality follows from (24). Thus, the GP density approaches the TF density for \(\varepsilon \to 0\) showing Estimate 1. (A similar result is also true for higher angular velocities as is shown in [9]). In the same way, we see the following result which is used to show Estimate 3 below

\[ \int_{\mathbb{R}^2} (|u_\varepsilon|^2 - \rho_{TF})^2 + \int_{\mathbb{R}^2 \setminus \mathbb{D}} |u_\varepsilon|^4 = 4\varepsilon^2 \mathcal{E}^{TF}[|u_\varepsilon|^2] + \int_{\mathbb{D}} (\rho_{TF})^2 - \mu \]
\[
\leq 4\varepsilon^2 (\mathcal{E}^{\text{TF}}[|u_\varepsilon|^2] - \mathcal{E}^{\text{TF}}[\rho^{\text{TF}}]) \leq C\varepsilon^2 |\ln\varepsilon|,
\]
so that
\[
\int_{\mathbb{R}^2 \setminus D} |u_\varepsilon|^4 \leq C\varepsilon^2 |\ln\varepsilon|.
\tag{25}
\]

Now we return to the non-rotating ground state described by (8). We have the following point-wise estimate for \(f_\varepsilon\) within the TF domain:

**Estimate 2:** Let \(f_\varepsilon\) be a minimizer of (8) under the normalization constraint. It is the unique positive solution of
\[
\Delta f = \frac{f}{2\varepsilon^2} (V + 2f^2 - 4\varepsilon^2 \nu^{\text{GP}}) \quad \text{in} \quad \mathbb{R}^2
\tag{26}
\]
with the chemical potential \(\nu^{\text{GP}}\) in (9). If \(\varepsilon\) is sufficiently small, then
\[
|\sqrt{\rho^{\text{TF}}(r)} - f_\varepsilon(r)| \leq C\varepsilon^{1/3} \sqrt{\rho^{\text{TF}}(r)}
\tag{27}
\]
for \(r \in D^{\text{in}} := \{r \in \mathbb{R}^2 : V(r) \leq \mu - \varepsilon^{1/3}\}\). That is, we may replace the vortex-free density \(f_\varepsilon^2\) by the Thomas-Fermi density \(\rho^{\text{TF}}\) within a region almost as large as the Thomas-Fermi domain making only an error of order \(o(1)\).

**Proof:**
As is shown in Ref. [22], there exists a unique minimizer for the functional (8). Since each minimizer fulfills (26) (which is the corresponding Euler-Lagrange equation) and \(\mathcal{E}^{\text{GP}}[f_\varepsilon] = \mathcal{E}^{\text{GP}}[|f_\varepsilon|]\) the positivity of the minimizer \(f_\varepsilon\) follows. Now we look at (27). It can be shown similar as in Refs. [2, 5] by using suitable sub- and supersolutions: We consider a disc \(B_\delta(r_0)\) around \(r_0 \in D' := \{r \in \mathbb{R}^2 : V(r) \leq \mu - t, t > 0\}\) with radius \(\delta < t\) and construct a subsolution \(w(r) = \sqrt{\rho}\tanh q\) with \(r \in B_\delta(r_0), q := \frac{\delta^2 - |r-r_0|^2}{\delta \varepsilon}\) and \(\rho := \min_{B_\delta(r_0)} \rho^{\text{TF}}\). Using \(w(r)\), we see that \(\Delta w \geq \frac{w}{\delta^2} (V + 2w^2 - 4\varepsilon^2 \nu^{\text{GP}})\) is fulfilled since \(4\varepsilon^2 \nu^{\text{GP}} > \mu + 2[\rho \tanh^2 q - \rho^{\text{TF}}]\) for \(\varepsilon\) sufficiently small. On \(\partial B_\delta(r_0)\) there is \(|r - r_0| = \delta\) and \(w|_{\partial B_\delta} = 0 < f_\varepsilon\). So \(w\) is a subsolution for (26) in \(B_\delta(r_0)\) and
\[
\sqrt{\rho} - f_\varepsilon(r_0) \leq \sqrt{\rho} - w(r_0) = \frac{2\sqrt{\rho} e^{-2\delta/\varepsilon}}{1 + e^{-2\delta/\varepsilon}} \leq 2\sqrt{\rho} e^{-2\delta/\varepsilon} \leq 2\sqrt{\rho^{\text{TF}}} e^{-2\delta/\varepsilon}.
\tag{28}
\]

Since \(\rho^{\text{TF}}\) is smooth in \(D \setminus \partial D\), we can approximate \(\rho^{\text{TF}}(r_0)\) by \(\rho\), making a small error of the order \(o(1)\). So,
\[
\frac{\sqrt{\rho^{\text{TF}}(r_0)} - f_\varepsilon(r_0)}{\sqrt{\rho^{\text{TF}}(r_0)}} \leq \frac{\sqrt{\rho^{\text{TF}}(r_0)} - \sqrt{\rho}}{\sqrt{\rho^{\text{TF}}(r_0)}} + \frac{2\sqrt{\rho}}{\sqrt{\rho^{\text{TF}}(r_0)}} e^{-2\delta/\varepsilon} \leq C\left(\frac{\delta}{\sqrt{t}} + e^{-2\delta/\varepsilon}\right).
\]
δ must be chosen such that $e^{-2\delta/\varepsilon}$ is exponentially small. We choose $\delta = \varepsilon^{2/3}$ and $t = \varepsilon^{1/3}$ as in [2]. Likewise we construct a supersolution $p(\mathbf{r}) = \sqrt{m} \coth[\text{arcoth}(\sqrt{M/m}) + \sqrt{mq}]$ with $\mathbf{r} \in B_\delta(\mathbf{r}_0)$, $m := \max_{B_\delta(\mathbf{r}_0)} \rho^{\text{TF}}$, $M = \max_{\mathcal{D}} \rho^{\text{TF}}$ and $q$ as above. Using $p(\mathbf{r})$, we see that $\Delta p \leq \frac{p}{\varepsilon^2} \left( V + 2p^2 - 4\varepsilon^2 \nu^{\text{GP}} \right)$ is fulfilled since $4\varepsilon^2 \nu^{\text{GP}} \leq \mu + 2(m \coth^2(\text{arcoth}(\sqrt{M/m}) + \sqrt{mq}) - \rho^{\text{TF}})$ for $\varepsilon$ sufficiently small. On $\partial B_\delta(\mathbf{r}_0)$ there is $|\mathbf{r} - \mathbf{r}_0| = \delta$ and $p|_{\partial B_\delta} = \sqrt{M} \geq f_\varepsilon$. So $p(\mathbf{r})$ is a supersolution for (26) in $B_\delta(\mathbf{r}_0)$. Proceeding as before, we get

$$\frac{f_\varepsilon(\mathbf{r}_0) - \sqrt{\rho^{\text{TF}}(\mathbf{r}_0)}}{\sqrt{\rho^{\text{TF}}(\mathbf{r}_0)}} \leq \frac{p(\mathbf{r}_0) - \sqrt{\rho^{\text{TF}}(\mathbf{r}_0)}}{\sqrt{\rho^{\text{TF}}(\mathbf{r}_0)}} \leq C \left( \frac{\delta}{\sqrt{t}} + e^{-2\delta/\varepsilon} \right).$$

Choosing $\delta$ and $t$ appropriately again, we get (27) for any $\mathbf{r} \in \mathcal{D}^{\text{in}}$.

It is intuitively clear that only the vortex-free density $f_\varepsilon^2$ and not the 'full' GP density $|u_\varepsilon|^2$ can satisfy a pointwise estimate as above: $u_\varepsilon$ may have vortices whereas $\rho^{\text{TF}}$ carries no vorticity at all. However, what can be shown is the fact that, in the TF regime, where $\varepsilon \to 0$, $|u_\varepsilon|^2$ is exponentially small outside the TF domain (see also Refs. [18] and [9]):

**Estimate 3:** For $\mathbf{r} \in \Theta_\varepsilon := \{ \mathbf{r} \in \mathbb{R}^2 : V(\mathbf{r}) > \mu + \varepsilon^{1/3} \}$ and $\varepsilon$ sufficiently small, there is

$$|u_\varepsilon(\mathbf{r})|^2 \leq C \varepsilon^{1/6} \ln \varepsilon^{1/2} \exp \left( \frac{b(\mathbf{r})}{C \varepsilon^{2/3}} \right),$$

where $b(\mathbf{r})$ is defined in (23).

**Proof:**

By using (3) we have

$$-\frac{1}{2} \Delta |u_\varepsilon|^2 = -|\nabla u_\varepsilon|^2 - \frac{V}{2\varepsilon^2} |u_\varepsilon|^2 - \frac{|u_\varepsilon|^4}{\varepsilon^2} + 2\mu^{\text{GP}} |u_\varepsilon|^2 - i(u_\varepsilon(\Omega \times \mathbf{r}) \cdot \nabla u_\varepsilon^* + u_\varepsilon^*(\Omega \times \mathbf{r}) \cdot \nabla u_\varepsilon).$$

The estimate

$$2\Omega(\varepsilon) |i u_\varepsilon^* \nabla u_\varepsilon \times \mathbf{r}| \leq |\nabla u_\varepsilon|^2 + \Omega(\varepsilon)^2 |\mathbf{r}|^2 |u_\varepsilon|^2$$

leads to

$$-\frac{1}{2} \Delta |u_\varepsilon|^2 \leq \left[ \Omega(\varepsilon)^2 |\mathbf{r}|^2 \varepsilon^2 - \frac{V}{2} - |u_\varepsilon|^2 + 2\mu^{\text{GP}} \varepsilon^2 \right] \frac{|u_\varepsilon|^2}{\varepsilon^2} \text{ in } \mathbb{R}^2.$$

From (7) and Estimate 1 follows

$$\varepsilon^2 \mu^{\text{GP}} = \varepsilon^2 \mathcal{E}^{\text{GP}}[u_\varepsilon] + \frac{1}{4} \int_{\mathbb{R}^2} |u_\varepsilon|^4 \leq \varepsilon^2 \mathcal{E}^{\text{TF}}[\rho^{\text{TF}}] + o(1) + \frac{1}{4} \int_{\mathbb{R}^2} |u_\varepsilon|^4$$

$$= \varepsilon^2 \mu^{\text{TF}} + o(1) + \frac{1}{4} \int_{\mathbb{R}^2} (|u_\varepsilon|^4 - (\rho^{\text{TF}})^2) \leq \varepsilon^2 \mu^{\text{TF}} + o(1).$$
Thus
\[-\frac{1}{2}\Delta |u_\varepsilon|^2 \leq \left[ \Omega(\varepsilon)^2 |r|^2 \varepsilon^2 - \frac{V}{2} + 2\varepsilon^2 \mu_{\mathrm{TF}} + o(1) - |u_\varepsilon|^2 \right] \frac{|u_\varepsilon|^2}{\varepsilon^2} \leq C \frac{b(\varepsilon)}{\varepsilon^2} |u_\varepsilon|^2 < 0\]
in $\Theta_\varepsilon$ where $b(\varepsilon) < -\varepsilon^{1/3}$. That is $|u_\varepsilon|^2$ fulfills
\[-\varepsilon^2 \Delta |u_\varepsilon|^2 - C' b(\varepsilon) |u_\varepsilon|^2 \leq 0 \text{ in } \Theta_\varepsilon.\] (29)

So $|u_\varepsilon|^2$ is subharmonic in $\Theta_\varepsilon$ for $\varepsilon$ sufficiently small. That means, there is for all $r = |r|$ with $B_\varrho(r) \subset \Theta_\varepsilon$ that
\[|u_\varepsilon(r)|^2 \leq \frac{1}{\pi \varrho^2} \int_{B_\varrho(r)} |u_\varepsilon|^2 \leq \frac{1}{\sqrt{\pi \varrho}} \left( \int_{r \in \Theta_\varepsilon} |u_\varepsilon|^4 \right)^{1/2} \leq \frac{C}{\varrho} \varepsilon^{1/2} |\ln \varepsilon|^{1/2}\]
using (25). If we now take $r \in \Sigma_\varepsilon := \{ r \in \mathbb{R}^2 : V(r) \geq \mu + \frac{\varepsilon^{1/3}}{2} \}$ and choose $\varrho = \varepsilon^{1/3}$ we get
\[|u_\varepsilon(r)|^2 \leq C \varepsilon^{1/6} |\ln \varepsilon|^{1/2}\]
so that $|u_\varepsilon(r)|^2 \to 0$ in $\Sigma_\varepsilon$ for $\varepsilon \to 0$. Moreover, from (29) it follows that $|u_\varepsilon|^2$ is a subsolution of
\[
\left\{
\begin{aligned}
-\Delta w + C'' \varepsilon^{-5/3} w &= 0 \text{ in } \Sigma_\varepsilon \\
w &= C \varepsilon^{1/6} |\ln \varepsilon|^{1/2} \text{ on } \partial \Sigma_\varepsilon.
\end{aligned}
\right.\] (30)

On the other hand, one can verify that
\[
\tilde{u} = C \varepsilon^{1/6} |\ln \varepsilon|^{1/2} \exp \left( \frac{b(\varepsilon)}{C \varepsilon^{2/3}} \right)
\]
is a supersolution of (30). Therefore $0 \leq |u_\varepsilon(r)|^2 \leq \tilde{u}$ for $r \in \Theta_\varepsilon$.

So, since $|u_\varepsilon|^2$ is exponentially small in $\varepsilon$ outside of the TF domain $D$, the above energy splitting (19) can be put now into the form
\[
E_{\mathrm{GP}}[u_\varepsilon] = E_{\mathrm{GP}}[f_\varepsilon e^{iS}] + \int_D \left[ \frac{f_\varepsilon^2}{2} |\nabla v_\varepsilon|^2 + \frac{f_\varepsilon^4}{4 \varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right] - \int_D i f_\varepsilon^2 v_\varepsilon^* \nabla v_\varepsilon : (\nabla S - \Omega \times r) + o(1)\]
\[=: E_{\mathrm{GP}}[f_\varepsilon e^{iS}] + G_f[v_\varepsilon] - R_f[v_\varepsilon] + o(1)\] (31)
where $v_\varepsilon = u_\varepsilon / f_\varepsilon e^{iS}$ and $o(1) \to 0$ as $\varepsilon \to 0$. Thus in the following, it suffices to restrict our considerations to the Thomas-Fermi domain $D$. 

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4 Vorticity in the Thomas-Fermi regime

We know from experiments that angular momentum is quantized in the form of vortices when the gas is subjected to an external rotation. Hence we may approximate the vorticity field by $N_\nu$ isolated point vortices. However, it is a difficult task in general to prove the validity of this approximation rigorously from more basic properties. It has been shown in the work of [19] for BECs in harmonic anisotropic traps that the vorticity is indeed concentrated in a finite (independent of $\varepsilon$) number of vortex cores if one assumes that the angular velocity is bounded by $\Omega \leq C|\ln \varepsilon|$ asymptotically. This has been achieved by using a number of technical vortex core constructions. We will not generalize these methods to the more general traps considered here but instead we like to argue by physical reasoning how the number of vortices scales with $\Omega(\varepsilon)$.

In experiments $\Omega$ and $\varepsilon$ are independent parameters. Usually, the interaction between the particles is tuned and afterwards $\Omega$ is increased (independently of $\varepsilon$) beyond the critical value. So in principle one could study the whole parameter domain spanned by $\Omega$ and (here) positive $\varepsilon$. However, we restricted to the TF regime where the scaled angular velocity $\Omega = \Omega(\varepsilon)$ depends on $\varepsilon$ in such a way that for $\varepsilon \to 0$, $\Omega \to \infty$ and hence we cover only a fraction of the possible parameter domain. Which dependence of $\Omega(\varepsilon)$ may occur in the TF regime?

There are essentially three regimes in $\Omega$ for non-harmonic traps where interesting effects appear (see [9]), namely $\Omega \sim |\ln \varepsilon|$, $\Omega \sim 1/\varepsilon$, $\Omega \gg 1/\varepsilon$ (the first regime also applies to harmonic traps). One may ask for a connection between different vortex core sizes, the magnitude of $\Omega(\varepsilon)$ and the kind of defects appearing in the condensate. For $\Omega \sim |\ln \varepsilon|$, one may deduce similar estimates for the vortex energy using core sizes of the order $\sigma = \kappa \varepsilon$ or $\sigma = \varepsilon^\alpha$ with constants $\kappa, \alpha > 0$ and the choice is fixed by technical reasons. However, in the fast rotating regimes, the size of the defects seems to be much more restrictive. As is shown in [9], for $\Omega \sim 1/\varepsilon$ there appears a 'hole' around the origin and the core size of the vortices itself is of the order $\sqrt{\varepsilon}$. For even larger velocities $\Omega \gg 1/\varepsilon$, the condensate is expelled to a small layer at the boundary and there remains a 'giant vortex' state filling out almost all of the condensate.

\[1\]

---

\[1\] One may argue that vortices with larger core radii, say e.g. $\sigma \sim 1/|\ln \varepsilon|$ could in principle exist at lower angular velocities of the order $\Omega \sim \ln |\ln \varepsilon|$. However, the characteristic length, where perturbations of the condensate wave function are smoothed out, is given by the healing length $\xi$ or $\varepsilon$ respectively. Hence we expect the cores to be of the order $\varepsilon$ and larger cores are not stable in the setting described here. Moreover, for angular velocities of the order $|\ln \varepsilon|$ we may also not expect the appearance of pathological cases like non-isolated vortices forming dense 1-dimensional structures because they would have a much higher energy than would be favourable at this order of $\Omega$. It seems that the underlying equations are too regular to support such kinds of defects even at much higher angular velocities.
In the regime of large vorticity one may also consider a kind of correspondence principle for a large number of vortices which is argued by Feynman [13] in the context of rotating superfluid $^4$He: a dense lattice of uniform distributed vortices should 'mimic' solid-body rotation on average, although the flow is strictly irrotational away from the vortex cores. The circulation around a closed contour $C$ which encloses a large number of vortices $N_v$ is $\Gamma = f_C \nabla S_u \cdot \tau = 2\pi d N_v$ for vortices with winding number $d$. On the other hand, if the vortex lattice mimics solid-body rotation there is $\Gamma = 2\Omega A$ where $A$ is the area enclosed by the contour $C$. In this approximation, the vortex density per area is $n_v = N_v / A = \Omega / (\pi d)$ and the area per vortex is $1/n_v = \pi d / \Omega$ and so decreases with increasing $\Omega$. The crucial ingredient in this argument is the assumption of a uniform distribution of vortices. But this is justified only if the number of vortices is very large, i.e. if $\Omega$ is very large which means for the TF regime that, indeed, $N_v, \Gamma$, and $\Omega$ have to increase as $\varepsilon \to 0$: Uniform distribution means that $A$ is finitely large, i.e. bounded from below by a positive constant (independent of $\varepsilon$). Actually, $A$ is the whole condensate domain and the contour $C$ is the boundary of that domain. So, from $\Gamma = 2\Omega A$ we see that $\Gamma \simeq \Omega$, i.e. the circulation is of the same order than the angular velocity if the vortices are distributed uniformly.

On the other hand, considering the case that $N_v$ and $\Gamma$ respectively can be bounded from above by a finite constant (independent of $\varepsilon$), then vortices cannot be distributed uniformly but instead they form a polygonal lattice (see e.g. the pictures in [28]). So the above argument concerning solid-body rotation gives only an upper bound for $\Gamma$. However, this bound is still quite good in experimental realizations as is demonstrated in [10]. In order to estimate roughly the order of magnitude of $\Omega$ for a finite number of vortices to appear, one may calculate the GP energy of a single vortex. This has been done most often in the approximation of a homogeneous system (see e.g. Refs. [30,31]) or for a condensate in harmonic traps (see e.g. Refs. [26,4]). In any case, the leading contribution comes from the angular kinetic energy. This can be already seen heuristically by considering a vortex of circulation $\Gamma = 2\pi d$ and core radius $\sigma \sim \varepsilon$ which is located at the origin of a flat trap with radius $R$. Writing the vortex in the form $v(r, \theta) = g(r)e^{id\theta}$ with $g(r) \sim r^d$ if $0 \leq r \leq \varepsilon R$ and $g(r) \sim R^{-1}$ if $\varepsilon R \leq r \leq R$, the kinetic energy is then $\int |\nabla v|^2 \sim R^{-2} (d^2 |\ln \varepsilon| + C)$, whereas the rotation term gives $-iv^* \Omega(\varepsilon) \cdot (\nabla v \times r) = -d\Omega(\varepsilon)$. Thus one may expect a vortex of winding number $d$ to appear when

$$\int ((|\nabla v|^2 - iv^* \Omega(\varepsilon) \cdot (\nabla v \times r)) \sim \frac{d^2}{R^2} |\ln \varepsilon| - d\Omega(\varepsilon) < 0 \quad \text{i.e. } \Omega(\varepsilon) > C \frac{d}{R^2} |\ln \varepsilon|$$

and the constant $C$ is fixed by the external potential accordingly. Hence one vortex or a finite number of them are favourable to exist if the angular velocity is of the order $\Omega(\varepsilon) \simeq C |\ln \varepsilon|$. $^2$ Furthermore, from $\Gamma < 2\Omega A \leq C$ we get

$^2$ For $|\ln \varepsilon| \ll \Omega(\varepsilon) \ll 1/\varepsilon$ the number of vortices is no longer bounded as $\varepsilon \to 0$ but the density is still not affected in leading order, see [9].
$A \leq C/\Omega$, that is the vortices are enclosed within a disc centered at the origin having a radius of the order

$$r_v \leq \frac{C}{\sqrt{\Omega}} \approx \frac{C'}{\sqrt{|\ln \varepsilon|}}.$$  \hspace{1cm} (32)

So with regard to the above discussion, we model the vorticity in terms of a finite number of vortices within the TF domain denoting their positions as $r_i = (x_i, y_i) \in \mathcal{D} \setminus \partial \mathcal{D}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$. The vortex cores are modelled as non-overlapping discs $B_i = B(r_i, \sigma)$ with core radius $\sigma \sim \varepsilon$, all contained within $\mathcal{D}$:

$$B(r_i, \sigma) \subset \mathcal{D} \text{ for all } i, \quad B(r_i, \sigma) \cap B(r_j, \sigma) = \emptyset \text{ for all } i \neq j$$  \hspace{1cm} (33)

assuming that $|r_i - r_j| > 2\sigma$. Otherwise, their energy would surpass the order of $|\ln \varepsilon|$ and would hence not be favourable for the angular velocities considered here. In a vortex point $r_i$, the condensate wave function vanishes $|u|(r_i) = |v|(r_i) = 0 \forall i$ and the circulation condition can be written as

$$\int_{\partial B_i} \nabla S_u \cdot \tau = \int_{\partial B_i} \nabla S_v \cdot \tau = \int_{\partial B_i} \frac{\partial S_v}{\partial \tau} = 2\pi d_i \forall i,$$  \hspace{1cm} (34)

where $\tau$ is the unit tangent vector to $B_i$ and $d_i$ is the degree of the vortex in $r_i$. The domain outside the cores is denoted as

$$\tilde{\mathcal{D}} := \mathcal{D} \setminus \bigcup_i B_i.$$  

In that region, there holds $|u| \to f$, i.e. $|v| \to 1$ and we may thus approximate

$$0 \leq |v| \leq 1 - o(1) \text{ in } B(r_i, \sigma)$$  \hspace{1cm} (35)

and

$$|v| = 1 - o(1) \text{ in } \tilde{\mathcal{D}}$$  \hspace{1cm} (36)

where $o(1)$ goes to zero for $\sigma \to 0$ (i.e. $\varepsilon \to 0$). The detailed form of the error in $o(1)$ depends on the steepness of the radial falloff of the vortex core profile. For the core radii we are going to use, namely $\sigma = \varepsilon^\alpha, \alpha > 0$, the error due to the core profile is negligible within the orders considered.

5 Lower bound for the Ginzburg-Landau-type energy $G_f[v]$ 

In this section, we consider the functional

$$G_f[v_\varepsilon] = \int_{\mathcal{D}} \left[ \frac{f_\varepsilon^2}{2} |\nabla v_\varepsilon|^2 + \frac{f_\varepsilon^4}{4\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right]$$  \hspace{1cm} (37)
which is part of the energy splitting (31). Because of (27), we can replace $f_\varepsilon$ by $\sqrt{\rho_{\text{TF}}}$ in (37) and the error is of the order $o(1)$. Using the polar decomposition $v_\varepsilon = |v_\varepsilon|e^{iS_\varepsilon}$, (37) is equivalent to

$$
G_f[v_\varepsilon] = \int_D \left[ \frac{\rho_{\text{TF}}}{2} ((\nabla|v_\varepsilon|)^2 + |v_\varepsilon|^2(\nabla S_\varepsilon)^2) + \frac{(\rho_{\text{TF}})^2}{4\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right] - o(1) .
$$

Minimizing this at fixed $\rho_{\text{TF}}$ results in

$$
-\Delta |v_\varepsilon| + |v_\varepsilon|(\nabla S_\varepsilon)^2 - \frac{(\rho_{\text{TF}})^2}{\varepsilon^2}|v_\varepsilon|(1 - |v_\varepsilon|^2) = 0
$$

and

$$
\nabla \cdot \left[ |v_\varepsilon|^2 \nabla S_\varepsilon \right] = 0.
$$

We have the following estimate:

**Estimate 4:** Let $f_\varepsilon$ be a minimizer of (15), $u_\varepsilon$ a minimizer of (3) and $v_\varepsilon = u_\varepsilon/f_\varepsilon e^{iS_\varepsilon}$. Let $\sigma = C_\varepsilon^\alpha$ with constants $C, \alpha > 0$ and let $v_\varepsilon$ satisfy (33) - (36) in presence of vortices in $r_i$ having winding numbers $d_i, i = 1, ..., n$. Then, for $\varepsilon$ sufficiently small and $\Omega \leq C|\ln \varepsilon|$ asymptotically, the GL-type energy can be bounded from below by

$$
G_f[v_\varepsilon] \geq \pi |\ln \sigma| \sum_{i=1}^{n} d_i^2 \rho_{\text{TF}}(r_i) + \pi \ln \frac{\sigma}{\varepsilon} \sum_{i=1}^{n} |d_i|\rho_{\text{TF}}(r_i) - 

- \pi \sum_{i \neq j} d_id_j \ln |r_i - r_j|\rho_{\text{TF}}(r_i) + o(1).
$$

**Proof:**

First we are going to estimate $G_f[v_\varepsilon]$ in the vortex-free domain where $|v_\varepsilon| = 1 - o(1)$. Then (38) reduces to

$$
G_f[v_\varepsilon]|_D = \frac{1}{2} \int_D \rho_{\text{TF}}(\nabla S_{v_\varepsilon})^2 = \frac{1}{2} \int_D \rho_{\text{TF}} V^2
$$

up to an error of order $o(1)$ and we use the relation $\nabla S_{v_\varepsilon} = V$ where $V$ is the (linear) superfluid velocity of the condensate (see (39)). Indeed, it is shown in Ref. [23] that BECs are 100 % superfluid in their ground state. Minimizing the functional with respect to $V$ gives

$$
\rho_{\text{TF}} \nabla \cdot V + \nabla \rho_{\text{TF}} \cdot V = 0 \text{ in } \tilde{D}.
$$

Because of the circulation condition (34)

$$
2\pi d_i = \int_{\partial B_i} \nabla S_{v_\varepsilon} \cdot \tau = \int_{\partial B_i} V \cdot \tau = \int_{B_i} \nabla \times V \cdot d\mathbf{o},
$$

there is

$$
\nabla \times V(r) = 2\pi d_i \delta(r - r_i) \text{ in } B_i.
$$
denoting the oriented surface element as \( d\mathbf{o} \). Since the cores \( B_i \) are small and the TF density \( \rho^{TF} \) is smooth in \( \mathcal{D} \setminus \partial\mathcal{D} \), \( \rho^{TF} \) is nearly constant within them, and we may approximate

\[
\nabla \cdot \mathbf{V} = 0 \text{ in } B_i. \tag{44}
\]

This allows to define a stream function \( \psi_\varepsilon \), which is the dual to the phase \( S_{\psi_\varepsilon} \), so \( \nabla S_{\psi_\varepsilon} = \nabla \times \psi_\varepsilon \). Using this form for \( \mathbf{V} \) together with \( \text{(43)} \) and \( \text{(14)} \), the stream function becomes

\[
\psi_\varepsilon(\mathbf{r}) = -d_i \ln |\mathbf{r} - \mathbf{r}_i| \text{ in } B_i. \tag{45}
\]

Now we calculate the integral

\[
\frac{1}{2} \int_{\mathcal{D}} \rho^{TF}\mathbf{V}^2 = \frac{1}{2} \int_{\mathcal{D}} \rho^{TF}(\mathbf{V} \times \nabla \psi_\varepsilon) \cdot d\mathbf{o} = \frac{1}{2} \int_{\mathcal{D}} \nabla \times [\rho^{TF}\psi_\varepsilon \mathbf{V}] \cdot d\mathbf{o} - \frac{1}{2} \int_{\partial\mathcal{D}} \psi_\varepsilon \nabla \rho^{TF} \cdot \mathbf{V} - \frac{1}{2} \int_{\partial\mathcal{D}} \psi_\varepsilon \rho^{TF}(\nabla \cdot \mathbf{V})
\]

\[
= \frac{1}{2} \int_{\partial\mathcal{D}} \rho^{TF} \psi_\varepsilon \mathbf{V} \cdot \tau - \frac{1}{2} \int_{\mathcal{D}} \psi_\varepsilon [\nabla \rho^{TF} \cdot \mathbf{V} + \rho^{TF}(\nabla \cdot \mathbf{V})]
\]

\[
= \frac{1}{2} \int_{\partial\mathcal{D}} \rho^{TF} \psi_\varepsilon \mathbf{V} \cdot \tau + \frac{1}{2} \sum_{i=1}^{n} \int_{\partial B_i} \rho^{TF} \psi_\varepsilon \mathbf{V} \cdot \tau
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \rho^{TF}(\mathbf{r}_i) \int_{\partial B_i} \psi_\varepsilon \mathbf{V} \cdot \tau + o(1), \tag{46}
\]

where we used Stokes theorem, \( \text{(42)} \) and \( \rho^{TF} = 0 \) on \( \partial\mathcal{D} \). Using again the fact that the cores \( B_i \) are small and \( \rho^{TF} \) is smooth in \( \mathcal{D} \setminus \partial\mathcal{D} \), we replace it by its value in the core center \( \rho^{TF}(\mathbf{r}_i) \) and the error is of the order \( o(1) \).

If we insert \( \psi_\varepsilon \) from \( \text{(15)} \) we get

\[
\frac{1}{2} \sum_{i} \rho^{TF}(\mathbf{r}_i) \int_{\partial B_i} \psi_\varepsilon \mathbf{V} \cdot \tau = -\frac{1}{2} \sum_{i} \rho^{TF}(\mathbf{r}_i) d_i \int_{\partial B_i} \ln |\mathbf{r} - \mathbf{r}_i| \mathbf{V} \cdot \tau
\]

\[
- \frac{1}{2} \sum_{i} \rho^{TF}(\mathbf{r}_i) \sum_{j \neq i} d_j \int_{\partial B_i} \ln |\mathbf{r} - \mathbf{r}_j| \mathbf{V} \cdot \tau + o(1). \tag{47}
\]

The first term on the r.h.s. describes the 'diagonal part' \( (i = j) \) and the second one the 'non-diagonal part' \( (i \neq j) \). Using \( \ln |\mathbf{r} - \mathbf{r}_i| = \ln \sigma \) and \( |\mathbf{r} - \mathbf{r}_j| = |\mathbf{r}_i - \mathbf{r}_j| + o(1) \) for \( \mathbf{r} \in \partial B_i \), there simply remains in each case the circulation condition which gives \( 2\pi d_i \). So we have

\[
\frac{1}{2} \sum_{i} \rho^{TF}(\mathbf{r}_i) \int_{\partial B_i} \psi_\varepsilon \mathbf{V} \cdot \tau = -\pi \ln \sigma \sum_{i} d_i^2 \rho^{TF}(\mathbf{r}_i) - \pi \sum_{i \neq j} d_id_j \ln |\mathbf{r}_i - \mathbf{r}_j| \rho^{TF}(\mathbf{r}_i). \tag{48}
\]

Since \( \sigma \ll 1 \), \( -\ln \sigma \) can be replaced by \( |\ln \sigma| \) and we recover the first and third term in \( \text{(40)} \). In this result, we see the familiar energy dependence on the winding number squared \( d^2 \) and the logarithmic divergence due to the vortex cores \( |\ln \sigma| \) (see also the approach in Ref. \[26\] for the harmonic trap.
case). If there are more vortices present than one, their interaction energy is modelled by
\[
W(r_1, \ldots, r_n) = -\pi \sum_{i \neq j} d_id_j \ln |r_i - r_j| \rho_{TF}^2 (r_i)
\] (49)
in (48). It has the form of a Coulombian interaction in 2-dimensional systems where vortices with the same sign of the winding number repel each other and vortices with opposite sign attract each other. In Ref. [7], the analogue to this function is called renormalized energy because it remains after the core energy, which is the leading order, is separated: \( W < |\ln \sigma| \) as long as \( |r_i - r_j| > 2\sigma \).

We also see that \( W \) is bounded from below by a constant if all winding numbers have the same sign.

It remains to find a lower bound of \( G_f[v_\epsilon] \) in the vortex cores \( B_i \):
\[
G_f[v_\epsilon] \big|_{\bigcup_i B_i} = \sum_i \int_{B_i} \left[ \frac{\rho_{TF}^2}{2} |\nabla v_\epsilon|^2 + \frac{(\rho_{TF})^2}{4\epsilon^2} (1 - |v_\epsilon|^2)^2 \right] - o(1)
\geq \sum_i \int_{B_i} \frac{\rho_{TF}^2}{2} |\nabla v_\epsilon|^2 - o(1) = \sum_i \frac{\rho_{TF}^2 (r_i)}{2} \int_{B_i} |\nabla v_\epsilon|^2 - o(1)
\] (50)
where we used again the fact that the TF density \( \rho_{TF} \) varies only of the order \( o(1) \) in the small discs \( B_i \). The integral over \( B_i \) can be estimated as follows:
\[
\int_{B_i} |\nabla v_\epsilon|^2 \geq \int_{B_i \setminus B_\epsilon} |\nabla v_\epsilon|^2 \geq \int_\epsilon^\sigma \int_0^{2\pi} \frac{1}{r^2} \left| \frac{\partial v}{\partial \phi} \right|^2 rdrd\phi
\]
with polar coordinates \((r, \phi)\) on the annulus, \( B_\epsilon \) a disc with radius \( \epsilon \) centered at \( r_i \), and we use the polar decomposition \( v(r, \phi) = |v|(r)e^{id_i\phi} \) for a vortex with winding number \( d_i \) in the disc \( B_i \) with radius \( \sigma \). Using \( \int_0^{2\pi} |\partial v/\partial \phi| \geq 2\pi|d_i| \) and Cauchy-Schwartz inequality, we get
\[
\int_\epsilon^\sigma \int_0^{2\pi} \frac{1}{r} \left| \frac{\partial v}{\partial \phi} \right|^2 rdrd\phi \geq 2\pi |d_i| \ln \frac{\sigma}{\epsilon}.
\] (51)
Combining (47), (48), (50) and (51), we complete the proof of (40).

6 The rotation energy \( R_f[v] \)

The estimate for the rotation term
\[
R_f[v_\epsilon] = \int_D if^2 v_\epsilon^* \nabla v_\epsilon \cdot (\nabla S - \Omega \times r)
\] (52)
in (31) proceeds similar as in [35]. Because of (27), we replace \( f \) by \( \sqrt{\rho_{TF}} \) in (52) and the error is of the order \( o(1) \). Then we get from (14)

\[
\nabla \cdot [\rho_{TF} (\nabla S - \Omega \times r)] = 0 \text{ in } \tilde{D}
\]

(53)

from where we see that there is a real function \( \chi(x,y) \) satisfying

\[
\rho_{TF} (\nabla S - \Omega \times r) = \Omega \nabla \perp \chi
\]

(54)

with \( \nabla \perp \chi = (-\partial_y \chi, \partial_x \chi) \), \( \partial_x = \frac{\partial}{\partial x} \), etc. We impose the accompanying boundary condition \( \chi = 0 \) on \( \partial \tilde{D} \). To determine the auxiliary function \( \chi \) we use (54) and rewrite it as

\[
(\nabla S - \Omega \times r) \perp = \frac{\Omega}{\rho_{TF}} \nabla \chi
\]

where \( r \perp = (-y,x) \) if \( r = (x,y) \). Applying the operator \( \nabla \), we get

\[
\partial_x (\partial_y S - \Omega x) + \partial_y (\partial_x S - \Omega y) = \Omega \nabla \cdot \left( \frac{\nabla \chi}{\rho_{TF}} \right)
\]

or

\[
\nabla \cdot \left( \frac{\nabla \chi}{\rho_{TF}} \right) = -2.
\]

(55)

We have the following estimate for the rotation term:

**Estimate 5:** Let \( f_\varepsilon \) be a minimizer of (13), \( u_\varepsilon \) a minimizer of (3), \( v_\varepsilon = u_\varepsilon / f_\varepsilon e^{iS} \) and \( \chi \) the solution of (55). Let \( \sigma = C\varepsilon^\alpha \) with constants \( C, \alpha > 0 \) and let \( v_\varepsilon \) satisfy (33) - (36) in presence of vortices in \( r_i \) having winding numbers \( d_i \), \( i = 1, \ldots, n \). Then for \( \varepsilon \) sufficiently small and \( \Omega \leq C|\ln \varepsilon| \) asymptotically, the rotation energy is

\[
R_f[v_\varepsilon] = 2\pi \Omega \sum_{i=1}^{n} d_i \chi(r_i) + o(1).
\]

(56)

**Proof:**

We can see that the contribution in the vortex cores becomes small:

\[
\left| \sum_{i=1}^{n} \int_{B_i} i \rho_{TF} v_\varepsilon \nabla v_\varepsilon \cdot (\nabla S - \Omega \times r) \right| \leq \sum_{i=1}^{n} \int_{B_i} |\rho_{TF}| |v_\varepsilon||\nabla v_\varepsilon||\nabla S - \Omega \times r|
\]

\[
\leq n\mu \Omega \left( \int_{B_i} |v_\varepsilon|^2 \right)^{1/2} \left( \int_{B_i} |\nabla v_\varepsilon|^2 \right)^{1/2} \leq C\sigma |\ln \varepsilon|^{3/2} \leq o(1)
\]

where we used that \( \int_{B_i} |\nabla v_\varepsilon|^2 | \leq C|\ln \varepsilon| \) which will be shown in (59).

For estimating the rotation term outside of the vortex discs, we use again \( |v_\varepsilon| = 1 - o(1) \) to get
\[
\int_{\hat{D}} i \rho^{TF} v_{\varepsilon} \nabla v_{\varepsilon} \cdot (\nabla S - \Omega \times r) = \Omega \int_{\hat{D}} i v_{\varepsilon} \nabla v_{\varepsilon} \cdot \nabla \chi = -\Omega \int_{\hat{D}} \nabla S_v \cdot \nabla \chi \\
= -\Omega \int_{\partial \hat{D}} \nabla \chi \cdot \frac{\partial S_v}{\partial r} + \Omega \sum_{i=1}^{n} \int_{\partial B_i} \chi \frac{\partial S_v}{\partial r} \\
= \Omega \sum_{i=1}^{n} \chi(r_i) \int_{\partial B_i} \frac{\partial S_v}{\partial r} + o(1) = 2\pi \Omega \sum_{i=1}^{n} d_i \chi(r_i) + o(1)
\]  
(57)

where we used (54), \(\nabla \cdot \nabla S_v = 0\), \(\chi = 0\) on \(\partial D\) and \(34\). Furthermore, since the cores \(B_i\) are small and \(\chi\) is smooth in \(D \setminus \partial D\), we replace it by its value in the core center \(\chi(r_i)\) and the error is of the order \(o(1)\). We thus arrive at (56).

The expressions (40) and (56) for the vortex contributions suggest that the lowest vortex energy is attained for vortices with winding number \(d_i = 1\) for all \(i\), which will be explicitly shown in Section 8.2. In estimating (40), we used \(\nabla \times V = 2\pi d_i \delta(r - r_i)\) in the vortex core \(B_i\), which is valid for any vortex core radius \(\sigma\). On the other hand, due to the characteristic scale \(\varepsilon\) the core is not much larger than \(\sigma \sim \varepsilon^\alpha, \alpha > 0\). Then, the gradient term of \(G_f[v]\) outside the core dominates over the contribution of the core itself. Concerning the interaction energy, one could first of all ask which terms of \(G_f[v] - R_f[v]\) in the splitting of the functional \(E_{GP}[u]\) in (19) will contribute to the interaction between vortices. We have just seen that the rotation energy outside of vortex cores is of the 'diagonal' form given in (57), whereas it is of order \(o(1)\) in the cores. So the interaction must be modelled by the GL-type energy \(G_f[v]\).

In addition, the interaction is only relevant in the domain outside the cores. There we have \(|v| \approx 1\) and only the gradient term of \(G_f[v]\) plays the significant role. In particular, the form of the core and interaction energy in (40) was deduced by minimizing (41) with respect to \(\nabla S_v = V\). The core energy dominates as long as \(\sigma \ll |r_i - r_j|\) for all \(i \neq j\), i.e. as long as the vortex core size is much smaller than the distance between vortices. This is vastly fulfilled in the regime \(\varepsilon \to 0\) and \(\Omega \approx C|\ln \varepsilon|\) since then there is \(\sigma \approx C\varepsilon^\alpha, \alpha > 0\) whereas \(|r_i - r_j| \geq C/\sqrt{|\ln \varepsilon|}\) which will be shown in Section 8.3.

Remark: Our analysis may be compared with the works of [4] and [18]. We start from the original energy functional (11) and rescale it to arrive at (2) and (3) respectively. In [18,19], a functional is used, motivated in [4] and justified by the normalization condition, which already in the beginning looks like a GL-type functional. The so encountered additional term is ‘thrown away’ because it does not contribute to the vortex energy. But one has to keep in mind that the true leading order is \(1/\varepsilon^2\) which can be explicitly seen in (3). In the estimate of the lower bound of (37), we consider equations (41) and (42). The behaviour of \(V\) in the discs is derived from the quantization condition (34) and is given in (45) in terms of the stream function. These ingredients are used
in the estimate of (46). Instead, methods of Ref. [7] are adapted in [19] to the functional studied by considering a ‘linear problem’ as in [7]. By introducing a suitable function, it gives eventually the interaction energy between vortices. Concerning the forthcoming estimates, we will benefit from inequality (62). The corresponding equality for \( s = 2 \) is used in [4,18]. In [18,19], it is assumed that \( \Omega \leq C|\ln \varepsilon| \) asymptotically. Otherwise, there is no a priori assumption on the fine structure of vorticity. We have argued in Section 4 that the assumption of a non-zero circulation which is bounded from above by a natural number independent of \( \varepsilon \) leads to an angular velocity of the order \( \Omega \sim |\ln \varepsilon| \).

So, our assumptions on the vortex fine structure, concerning number and size of vortex cores, are actually compatible with this order of \( \Omega \).

7 Upper bound for \( \mathcal{G}_f[v] - \mathcal{R}_f[v] \)

The energy without vortex is always larger or equal to \( \mathcal{E}^{GP}[u_\varepsilon] \) and it can be used as a trial function for the whole energy (see also the proof of Estimate 1):

\[
\mathcal{E}^{GP}[u_\varepsilon] = \mathcal{E}^{GP}[f_\varepsilon e^{iS}] + \mathcal{G}_f[v_\varepsilon] - \mathcal{R}_f[v_\varepsilon] + o(1) \leq \mathcal{E}^{GP}[f_\varepsilon e^{iS}] + C + o(1)
\]

so \( \mathcal{G}_f[v_\varepsilon] - \mathcal{R}_f[v_\varepsilon] \leq C + o(1) \). A more precise upper bound is obtained as follows:

**Estimate 6:** Let \( f_\varepsilon \) be a minimizer of (15), \( u_\varepsilon \) a minimizer of (3) and \( v_\varepsilon = u_\varepsilon / f_\varepsilon e^{iS} \). Then, for \( \varepsilon \) sufficiently small and \( \Omega \leq C|\ln \varepsilon| \) asymptotically, the vortex energy \( \mathcal{G}_f[v_\varepsilon] - \mathcal{R}_f[v_\varepsilon] \) can be bounded from above by

\[
\mathcal{G}_f[v_\varepsilon] - \mathcal{R}_f[v_\varepsilon] \leq \pi |\ln \varepsilon| \sum_{i=1}^k d_i \rho^{TF}(r_i) - 2\pi \Omega \sum_{i=1}^k d_i \chi(r_i) + W(r_1, \ldots, r_k) + C + o(1)
\]

(58)

where \( d_i \geq 1 \) for all \( i \), \( W(r_1, \ldots, r_k) \) from (49) and \( i, j = 1, \ldots, k \).

**Proof:**

We fix \( k \geq 1, k \in \mathbb{N} \) vortex positions \( r_1, \ldots, r_k \) in \( D \), each is center of a disc with fixed radius \( R > 0 \) but small such that the discs are completely contained in \( D \) and do not overlap i.e. \( \bar{B}(r_i, R) \subset D \) and \( \bar{B}(r_i, R) \cap \bar{B}(r_j, R) = \emptyset \) for all \( i \neq j \). We use the trial function \( \hat{v} = |\hat{v}| e^{iS_\varepsilon} \) with \( |\hat{v}| = 1 \) in \( D \) and

\[
\hat{v}(r) = \begin{cases} \frac{r - r_i}{|r - r_i|} & \text{for } |r - r_i| \geq \varepsilon \\ \frac{r - r_i}{\varepsilon} & \text{otherwise} \end{cases}
\]
in the discs $B_i(r_i, R)$. Since $|\nabla \hat{v}|^2 = (\nabla \hat{S}_v)^2$ in $\tilde{D}$ and the phase is smooth and bounded outside the discs with finite size, there is
\[ G_f[\hat{v}]|_{\tilde{D}} = \int_{\tilde{D}} \frac{\rho_{\text{TF}}}{2}(\nabla \hat{S}_v)^2 \leq C. \]

However, in the discs there is
\[ G_f[\hat{v}]|_{\bigcup_i B_i} = \sum_{i=1}^{k} \frac{\rho_{\text{TF}}(r_i)}{2} \int_{B_i} |\nabla \hat{v}|^2 + \sum_{i=1}^{k} \frac{(\rho_{\text{TF}})^2(r_i)}{4\varepsilon^2} \int_{B_i} (1 - |\hat{v}|^2)^2 + o(1) \]
where
\[ \int_{B_i} |\nabla \hat{v}|^2 - 4\pi = \int_{B_i \setminus B_\varepsilon} |\nabla \hat{v}|^2 = \int_{B_i \setminus B_\varepsilon} \frac{1}{|r - r_i|^2} = \int_0^{2\pi} \int_\varepsilon^{R} \frac{r dr d\phi}{r^2 - 2r \varepsilon \cos \phi + \varepsilon^2} = \pi \ln(r^2 - \varepsilon^2)|^R_\varepsilon \leq \pi \ln r^2|^R_\varepsilon = 2\pi |\ln \varepsilon| + 2\pi \ln R, \] (59)

and the other contributions are at most of the order of a constant. With the above trial function, the rotation energy in $\tilde{D}$ is the same as in (56) apart from the sum running from $i = 1$ to $k$. The contribution inside the vortex discs is simply
\[ \left| \sum_{i=1}^{k} \int_{B(r_i, R)} i\rho_{\text{TF}} \hat{v}^* \nabla \hat{v} \cdot (\nabla S - \Omega \times r) \right| \leq \sum_{i=1}^{k} \int_{B(r_i, R)} |\rho_{\text{TF}}||\hat{v}||\nabla \hat{v}||\nabla S - \Omega \times r| \]
\[ \leq k\mu \Omega \left( \int_{B_i} |\hat{v}|^2 \right)^{1/2} \left( \int_{B_i} |\nabla \hat{v}|^2 \right)^{1/2} \]
\[ \leq k\mu \Omega (\pi(R^2 - \varepsilon^2) + 3\pi\varepsilon^2/2)^{1/2} (2\pi |\ln \varepsilon| + 2\pi \ln R + 4\pi)^{1/2} \leq C\varepsilon |\ln \varepsilon|^{3/2}. \]

Then, to recover (58) we finally use the fact that $W(r_1, ..., r_k)$ can be bounded from below by a constant.

8 The anisotropic homogeneous trap

In the above estimates (24), (40), (56) and (58), the external trap potential $V$ enters via the Thomas-Fermi density $\rho_{\text{TF}}$ which was not specified until now. These estimates are valid as long as $V$ satisfies (asymptotical) homogeneity (see also the remark at the end of this section). As an application, we consider now the potential in (16). The associated TF density is
\[ \rho_{\text{TF}}(x, y) = \frac{1}{2} \left( \mu - (x^2 + \lambda^2 y^2)^{s/2} \right). \] (60)
From (12) we have $\mu = \left(\frac{s+2}{s}\right)^{s/(s+2)}$. For $s \to \infty$, $\mu \to 2\lambda/\pi$, hence $\mu$ is always smaller than one. The auxiliary function $\chi$ is determined from (55) to

$$\chi(x, y) = \frac{1}{1 + \lambda^2} \left[ \frac{1}{s+2} (x^2 + \lambda^2 y^2)^{(s+2)/2} - \mu \frac{2}{s+2} (x^2 + \lambda^2 y^2) + \frac{s}{2(s+2)} \mu^{(s+2)/s} \right].$$

It can be estimated from above in terms of the TF density $\rho_{\text{TF}}$ by

$$\chi(x, y) \leq \frac{1}{1 + \lambda^2} \left[ \frac{s^{2/s}}{s+2} \left(\rho_{\text{TF}}(x, y)^{(2+s)/s}, \right.\right.$$}

where strict equality only holds for the harmonic trap $s = 2$. This upper bound will be very useful in Section 8.2 where the winding numbers of vortices are derived. The phase $S$ can be determined by inserting (61) and (60) into (54) and is already given in (17).

### 8.1 The energy with one vortex

The upper bound of the energy using (31) and (58) is

$$\mathcal{E}_{\text{GP}}[u_\varepsilon] \leq \mathcal{E}_{\text{GP}}[f_\varepsilon e^{iS}] + \pi |\ln \varepsilon| \sum_{i=1}^k d_i \rho_{\text{TF}}(r_i) - 2\pi \Omega \sum_{i=1}^k d_i \chi(r_i) + C + o(1).$$

For a trial function having one vortex with winding number $d = 1$ at the origin we get

$$\mathcal{E}_{\text{GP}}[u_\varepsilon] \leq \mathcal{E}_{\text{GP}}[f_\varepsilon e^{iS}] + \frac{\pi}{2} \mu |\ln \varepsilon| - \frac{\pi s \mu^{(s+2)/s}}{(1 + \lambda^2)(s+2)} \Omega + C + o(1).$$

The energy $\mathcal{E}_{\text{GP}}[u_\varepsilon]$ will be smaller than the vortex-free energy $\mathcal{E}_{\text{GP}}[f_\varepsilon e^{iS}]$ if the r.h.s. of (63) is smaller or equal to $\mathcal{E}_{\text{GP}}[f_\varepsilon e^{iS}] - o(1)$. Equivalently, the angular velocity must fulfill

$$\Omega \geq \Omega_1 + C + o(1)$$

where

$$\Omega_1 = \frac{s + 21 + \lambda^2}{s \mu^{2/s} - \frac{2}{2} |\ln \varepsilon| =: C_1 |\ln \varepsilon|.$$

(Equation (64) has to be multiplied by $(16\varepsilon^4)^{1/(s+2)}$ in order to obtain the unscaled angular velocity $\Omega_1$, see (11)). So for $\Omega \geq \Omega_1 + C + o(1)$, minimizers of $\mathcal{E}_{\text{GP}}[u]$ will have vortices, or in other terms: $\Omega_1$ is the leading order in the angular velocity where the solution with one vortex having $d = 1$ starts to be globally thermodynamically stable. We may also see the following: Consider $(x, y) \in D$, let $s > 2$ and denote $\delta E = \pi |\ln \varepsilon| \rho_{\text{TF}}(x, y) - 2\pi \Omega \chi(x, y) + C$. Then one can see that at the origin $\nabla(\delta E)(0, 0) = 0$ and $\Delta(\delta E)(0, 0) =$
$4\pi \mu \Omega > 0$, i.e. $(0, 0)$ is a local minimum for all $\Omega$. However, $(0, 0)$ is a global minimum for $\delta E(0, 0) < 0$ or $\Omega_1 + C + o(1) < \Omega$ with $\Omega_1$ in (64). However, for the harmonic trap $s = 2$ one has $\nabla(\delta E)(0, 0) = 0$ and $\Delta(\delta E)(0, 0) = -2\pi |\ln \varepsilon|(1 + \lambda^2) + 4\pi \mu \Omega$. For $\Delta(\delta E)(0, 0) < 0$, the origin is a local maximum; for $\Delta(\delta E)(0, 0) > 0$ the origin is a local minimum. So there is an angular velocity for local thermodynamical stability which is $\Omega > \frac{1 + \lambda^2}{2\mu} |\ln \varepsilon| = \Omega_1/2$ and $\Omega_1$ in (64) with $s = 2$ which was also shown in Ref. [4].

8.2 All vortices are single-quantized

**Estimate 7:** If $\sigma = \varepsilon^\alpha$, $0 < \alpha < 1$, $\varepsilon$ sufficiently small and $\Omega \leq \Omega_1 + C F(\varepsilon)$ with $F(\varepsilon)$ of lower order than $|\ln \varepsilon|$, then $d_i = 1$ for all $i$.

**Proof:**
We use $\sigma = \varepsilon^\alpha$ with $0 < \alpha < 1$ in (40) and (56), $W \geq C$ and

$$\Omega \leq C_1 |\ln \varepsilon| + C_2 F(\varepsilon)$$

with $C_1$ from (64) and $C_2$ is another positive constant. This upper bound for $\Omega$ is suggested by (64) and $F(\varepsilon)$ is assumed to be of lower order than $|\ln \varepsilon|$. In the next section, we will see that $F(\varepsilon) = |\ln \varepsilon|$. So

$$\pi |\ln \sigma| \sum_{i=1}^{n} d_i^2 \rho^{TF}(r_i) + \pi \ln \sigma \sum_{i=1}^{n} d_i \rho^{TF}(r_i) - 2\pi \Omega \sum_{i=1}^{n} d_i \chi(r_i) \leq C$$

or

$$\pi |\ln \sigma| \sum_{i} (d_i^2 - d_i) \rho^{TF}(r_i) + \pi |\ln \varepsilon| \sum_{i} d_i \rho^{TF}(r_i) \leq C + 2\pi \Omega \sum_{i} d_i \chi(r_i)$$

$$\leq C + \frac{2\pi \Omega}{1 + \lambda^2} \frac{s}{2^s/\mu} \sum_{i} d_i \rho^{TF}(r_i)$$

$$\leq C + (C_1 |\ln \varepsilon| + C_2 F(\varepsilon)) \frac{2\pi}{1 + \lambda^2} \frac{s}{2^s/\mu} \sum_{i} d_i \rho^{TF}(r_i)$$

$$\leq C + \pi |\ln \varepsilon| \sum_{i} d_i \rho^{TF}(r_i) + C F(\varepsilon) \sum_{i} d_i \rho^{TF}(r_i). \quad (65)$$

Here we used (62) and $(\rho^{TF})^{(s+2)/s} \leq \left(\frac{\mu}{2}\right)^{2/s} \rho^{TF}$. We also note that the last inequality in (65) is valid only for $C_1$ from (64).

So we see that (65) reduces to

$$\sum_{i=1}^{n} (d_i^2 - d_i) \rho^{TF}(r_i) \leq o(1)$$

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for $\varepsilon$ sufficiently small. Therefore, if the vortices are not located at the boundary of the Thomas-Fermi domain where $\rho_{\text{TF}}$ vanishes, there must be $d_i = 1$ for all $i$ for sufficiently small $\varepsilon$.

### 8.3 The energy with $n$ vortices

Since all vortices have winding number one, we see from the lower and upper bounds of $E_{\text{GP}}[u_\varepsilon]$ in (40), (56) and (58) that they coincide in their orders (up to a constant). By applying the transformation $\tilde{r}_i = (\tilde{x}_i, \tilde{y}_i)$ with $\tilde{x}_i = x_i \sqrt{\Omega}, \tilde{y}_i = y_i \lambda \sqrt{\Omega}$, the vortex interaction energy $W(r_1, \ldots, r_n)$ in (49) can be decomposed as follows:

$$W(r_1, \ldots, r_n) = -\pi \sum_{i \neq j} \ln |r_i - r_j| \rho_{\text{TF}}(r_i)$$

$$= -\frac{\pi}{4} \sum_{i \neq j} \ln (|x_i - x_j|^2 + |y_i - y_j|^2)(\mu - (x_i^2 + \lambda^2 y_i^2)^{s/2})$$

$$= \frac{\pi}{4} \mu n(n - 1) \ln \Omega - \frac{\pi}{4 \Omega^{s/2}} \sum_i (\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2} - \frac{\pi \mu}{4} \sum_{i \neq j} \ln \left( (\tilde{x}_i - \tilde{x}_j)^2 + \frac{1}{\lambda^2} (\tilde{y}_i - \tilde{y}_j)^2 \right)$$

$$+ \frac{\pi}{4 \Omega^{s/2}} \sum_{i \neq j} \ln \left( (\tilde{x}_i - \tilde{x}_j)^2 + \frac{1}{\lambda^2} (\tilde{y}_i - \tilde{y}_j)^2 \right) (\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2}$$

$$= \frac{\pi}{4} \mu n(n - 1) \ln \Omega - \frac{\pi}{4} \mu \sum_{i \neq j} \ln \left( |\tilde{x}_i - \tilde{x}_j|^2 + \frac{1}{\lambda^2} |\tilde{y}_i - \tilde{y}_j|^2 \right) + o(1). \quad (66)$$

The first term contains the relevant order $\ln \Omega$ whereas the remainder is of the order of a constant. The rotation term (56) becomes

$$-2\pi \Omega \sum_{i=1}^n \chi(r_i) =$$

$$- \frac{2\pi \Omega}{1 + \lambda^2} \sum_i \left[ \frac{1}{s + 2} (x_i^2 + \lambda^2 y_i^2)^{(s+2)/2} - \frac{\mu}{2} (x_i^2 + \lambda^2 y_i^2) + \frac{s}{2(s + 2)} \mu^{(s+2)/s} \right]$$

$$- \frac{\pi \mu n}{(1 + \lambda^2)(s + 2)} \Omega^{(s+2)/s} + \frac{\pi \Omega \mu}{1 + \lambda^2} \sum_i (x_i^2 + \lambda^2 y_i^2)$$

$$- \frac{2\pi \Omega}{1 + \lambda^2} \sum_i (x_i^2 + \lambda^2 y_i^2)^{(s+2)/2}$$

$$= - \frac{\pi \mu n}{(1 + \lambda^2)(s + 2)} \Omega^{(s+2)/s} + \frac{\pi \mu}{1 + \lambda^2} \sum_i (\tilde{x}_i^2 + \tilde{y}_i^2)$$

$$- \frac{2\pi}{1 + \lambda^2} \frac{1}{\Omega^{s/2}} \sum_i (\tilde{x}_i^2 + \tilde{y}_i^2)^{(s+2)/2} \quad (67)$$
where in the last step the variable transformation was applied. The remaining part of the lower bound of $G_f[u_\varepsilon]$ in (40) is (with $d_i = 1 \ \forall i$)

$$\pi \ln \varepsilon^\alpha \left| \sum_i \rho_{TF}(r_i) \right| + \pi \ln \left( \frac{\varepsilon^\alpha}{\varepsilon} \right) \sum_i \rho_{TF}(r_i) = \pi \ln \left| \sum_i \rho_{TF}(r_i) \right|$$

for small $\varepsilon$ and using $\sigma = \varepsilon^\alpha$. With $u_\varepsilon$ and $f_\varepsilon$ as above, we thus recover (22) for the Gross-Pitaevskii energy in presence of $n$ vortices

$$E_{GP}[u_\varepsilon] = E_{GP}[f_\varepsilon e^{iS}] + \frac{\pi}{2} \mu n \left( \left| \ln \varepsilon \right| - \frac{2s}{(1 + \lambda^2)(s + 2)} \mu^{2/s} \Omega \right) +$$

$$+ \frac{\pi}{4} \mu n (n - 1) \ln \Omega + w(\tilde{r}_1, ..., \tilde{r}_n) + C + o(1) \quad (68)$$

with

$$w(\tilde{r}_1, ..., \tilde{r}_n) = -\frac{\pi \mu}{4} \sum_{i \neq j} \ln \left( (\tilde{x}_i - \tilde{x}_j)^2 + \frac{1}{\lambda^2} (\tilde{y}_i - \tilde{y}_j)^2 \right) +$$

$$+ \frac{\pi \mu}{1 + \lambda^2} \sum_i (\tilde{x}_i^2 + \tilde{y}_i^2) - \frac{\pi \ln \Omega}{4 \Omega^{s/2}} \sum_i (\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2}$$

where we put all terms proportional to $\Omega^{-m}$, $m > 0$ into $o(1)$ since $\Omega \leq C|\ln \varepsilon|$ asymptotically.

A necessary condition for the minimizing configuration to have more than one vortex is

$$\min_{U_n} E_{GP}[u] \leq \min_{U_1} E_{GP}[u] \quad n \geq 2 \quad (69)$$

where $U_n$ is the set of functions with $n$ vortices having winding number one each and $U_1$ is the set of functions with one vortex at the origin with winding number one. Using (62), we first want to deduce a rough estimate for the critical angular velocity $\Omega_n$ for $n$ vortices to appear. To this aim, we neglect in (68) the term coming from the interaction and we take the energy with all vortices close to the origin, i.e. we approximate $\rho_{TF}(r_i) \approx \mu/2$ for all $i$, which is a more stringent condition on the l.h.s. of (69). Indeed, from (32) we expect the vortices to be near to the origin. Hence

$$\frac{\pi}{2} \left| \ln \varepsilon \right| \mu (n-1) + \frac{\pi}{4} \mu n (n-1) \ln \Omega_n + \frac{\pi s \mu^{(s+2)/s}}{(1 + \lambda^2)(s + 2)} \Omega_1 \leq \frac{\pi s \mu^{(s+2)/s}}{(1 + \lambda^2)(s + 2)} \Omega_n + C,$$

and

$$\frac{1 + \lambda^2 s + 2}{2 s \mu^{2/s}} \left| \ln \varepsilon \right| \frac{n - 1}{n} + \frac{1 + \lambda^2 s + 2}{2 s \mu^{2/s}} \frac{n - 1}{2} \ln \Omega_n + \frac{1}{n} \Omega_1 \leq \Omega_n + C.$$
which can be put into the form

$$\Omega_{1} + C_{1} \frac{n - 1}{2} \ln |\ln \varepsilon| + C_{1} \frac{n - 1}{2} \ln C_{1} - C \leq \Omega_{n},$$

with $C_{1}$ and $\Omega_{1}$ from (64). We thus see that the critical angular velocity has to be at least of the order $C_{1} |\ln \varepsilon| + C' \ln |\ln \varepsilon|$ (we neglect the constant term). This order of magnitude for $\Omega$ is assumed in Ref. [19] from the outset before the number of vortices is rigorously derived. Using now the ansatz

$$\Omega = \Omega_{1} + C_1 \nu(\varepsilon) \ln |\ln \varepsilon|$$

(70)

with

$$(k - 1) + \delta \leq \nu(\varepsilon) \leq k - \delta$$

for an integer $k \geq 0$ counting the number of vortices and $0 < \delta \ll 1$ a fixed constant independent of $\varepsilon$, we see the following: Inserting this form of $\Omega$ in our energy estimate (68) we get for the upper bound

$$\mathcal{E}^{\text{GP}}[u_{\varepsilon}] \leq \mathcal{E}^{\text{GP}}[f_{\varepsilon}e^{iS}] + \frac{\pi}{2} \mu k |\ln \varepsilon| - \frac{\pi s k \mu^{1+2/s}}{(1 + \lambda^2)(s + 2)} \Omega + \frac{\pi}{4} \mu k (k - 1) \ln C_{1} + C$$

or

$$\mathcal{G}_{f}[v_{\varepsilon}] - \mathcal{R}_{f}[v_{\varepsilon}] \leq - \frac{\pi}{2} \mu \nu(\varepsilon) \ln |\ln \varepsilon| + \frac{\pi}{4} \mu k (k - 1) \ln |\ln \varepsilon| +$$

$$+ \frac{\pi}{4} \mu k (k - 1) \ln C_{1} + C$$

(71)

respectively. Considering the case $k = 0$ (no vortices), i.e. $\nu(\varepsilon)$ in (70) satisfies $-1 + \delta \leq \nu(\varepsilon) \leq -\delta$, we see from (71) and (64) that $\Omega \leq \Omega_{1} - C_{1} \delta \ln |\ln \varepsilon|$ and $\mathcal{G}_{f}[v_{\varepsilon}] - \mathcal{R}_{f}[v_{\varepsilon}] = C$, showing (21).

Considering $k = 1$ (one vortex), i.e. $\delta \leq \nu(\varepsilon) \leq 1 - \delta$ and comparing its energy with the lower bound of $\mathcal{G}_{f}[v_{\varepsilon}]$ and $\mathcal{R}_{f}[v_{\varepsilon}]$ we have

$$- \frac{\pi}{2} \mu \nu(\varepsilon) \ln |\ln \varepsilon| + C \geq \pi |\ln \varepsilon| \sum_{i=1}^{n} \rho^{\text{TF}}(r_{i}) - 2\pi \Omega \sum_{i=1}^{n} \chi(r_{i}) + C$$

$$\geq \pi \sum_{i=1}^{n} \rho^{\text{TF}}(r_{i})(-\nu(\varepsilon) \ln |\ln \varepsilon| + C) \geq - \frac{\pi}{2} \mu \nu(\varepsilon) \ln |\ln \varepsilon|$$

so

$$1 - o(1) \leq n,$$

that is, there is at least one vortex for

$$\Omega_{1} + C_{1} \delta \ln |\ln \varepsilon| \leq \Omega \leq \Omega_{1} + C_{1} (1 - \delta) \ln |\ln \varepsilon| = \Omega_{2} - C_{1} \delta \ln |\ln \varepsilon|$$

(using (20)) and $\varepsilon$ sufficiently small. Now we compare the lower and upper bound of the energy if $k > 1$ is arbitrary large: the upper bound is in (71),
whereas the lower bound is
\[ G_f[v_\varepsilon] - R_f[v_\varepsilon] \geq -\frac{\pi}{2} \mu n \nu(\varepsilon) \ln |\ln \varepsilon| + \frac{\pi}{4} \mu n(n - 1) \ln |\ln \varepsilon| + \frac{\pi}{4} \mu(n - 1) \ln C_1 + C. \] (72)

Comparison of (71) and (72) gives
\[ -n \nu(\varepsilon) + \frac{1}{2} n(n - 1) + o(1) \leq -k \nu(\varepsilon) + \frac{1}{2} k(k - 1) + o(1). \]

Assuming now that \( n \leq k - 1 \), we have
\[ \nu(\varepsilon)(k - n) \leq \frac{1}{2} (k - n)(k + n - 1) + o(1), \]
so
\[ (k - 1) + \delta \leq \nu(\varepsilon) \leq \frac{1}{2} (k + n - 1) + o(1) \leq k - 1 + o(1) \]
which is a contradiction for \( \varepsilon \) sufficiently small, since \( \delta \) is a fixed constant.

On the other hand, assuming \( n \geq k + 1 \) we have
\[ \nu(\varepsilon)(n - k) \geq \frac{1}{2} (n - k)(k + n - 1) + o(1), \]
so
\[ k - \delta \geq \nu(\varepsilon) \geq \frac{1}{2} (k + n - 1) + o(1) \geq k + o(1) \]
which is again a contradiction for \( \varepsilon \) sufficiently small. So we see that there are exactly \( n \equiv k \) vortices for \( \varepsilon \) sufficiently small and from this follows
\[ G_f[v_\varepsilon] - R_f[v_\varepsilon] = -\frac{\pi}{2} \mu n \nu(\varepsilon) \ln |\ln \varepsilon| + \frac{\pi}{4} \mu n(n - 1) \ln |\ln \varepsilon| + \frac{\pi}{4} \mu(n - 1) \ln C_1 + C \]
and
\[ \Omega_n + C_1 \delta \ln |\ln \varepsilon| \leq \Omega \leq \Omega_{n+1} - C_1 \delta \ln |\ln \varepsilon| \]
by using (20). This completes the proof of the main result stated in the end of Section 2.

Special cases:

For the harmonic trap \( s = 2 \) there is
\[ \Omega_n = \frac{1 + \lambda^2}{\mu} \left[ |\ln \varepsilon| + (n - 1) \ln |\ln \varepsilon| \right]. \]

From (4), the unscaled angular velocity is then
\[ \tilde{\Omega}_n = \frac{2}{\mu} (1 + \lambda^2) \varepsilon \left[ |\ln \varepsilon| + (n - 1) \ln |\ln \varepsilon| \right] \]
which may be compared to the results in Refs. [4] and [8]. For the flat trap $s \to \infty$, there is

$$\Omega_n = \frac{1 + \lambda^2}{2} \left[ |\ln \varepsilon| + (n - 1) |\ln \varepsilon| \right]$$

and the same is true for $\tilde{\Omega}_n$ since $\tilde{\Omega}_n \to \Omega_n$ for $s \to \infty$. (We put in mind that for the flat trap, the scaled energy functional converges to the original one, i.e. $E^{GP}[u'] \to E^{GP}[u]$, see Section 2). By comparing the first critical angular velocities, we see the following: For the flat trap $\tilde{\Omega}_1 \sim |\ln \varepsilon|$, resembling the corresponding velocity for the rotating bucket and this is no surprise since the flat trap approximates the bucket. On the other hand, there is $\tilde{\Omega}_1 \sim \varepsilon |\ln \varepsilon|$ for the harmonic trap which is much smaller. The ratio of the unscaled first critical angular velocities is thus

$$\frac{\tilde{\Omega}_1(s \to \infty)}{\tilde{\Omega}_1(s = 2)} = \frac{\mu}{4\varepsilon}.$$  

8.4 The vortex pattern

The minimization of the energy in (68) with respect to the coordinates $\tilde{\mathbf{r}}_i = (\tilde{x}_i, \tilde{y}_i)$ determines the distribution of the vortices in the condensate and therefore the resulting pattern which appears for a given number $n$ of vortices. The energy in (68) is minimal with respect to the coordinates if $w(\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_n)$ is minimal. Setting $\nabla w = 0$, we obtain

$$\frac{\pi \mu}{2} \sum_{i \neq j} \frac{\tilde{x}_i - \tilde{x}_j}{(\tilde{x}_i - \tilde{x}_j)^2 + \lambda^{-2}(\tilde{y}_i - \tilde{y}_j)^2} + \frac{\pi s \ln \Omega}{2\Omega^{s/2}} \sum_i \tilde{x}_i(\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2-1} = \frac{2\pi \mu}{1 + \lambda^2} \sum_i \tilde{x}_i$$ (73)

and

$$\frac{\pi \mu}{2\lambda^2} \sum_{i \neq j} \frac{\tilde{y}_i - \tilde{y}_j}{(\tilde{x}_i - \tilde{x}_j)^2 + \lambda^{-2}(\tilde{y}_i - \tilde{y}_j)^2} + \frac{\pi s \ln \Omega}{2\Omega^{s/2}} \sum_i \tilde{y}_i(\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2-1} = \frac{2\pi \mu}{1 + \lambda^2} \sum_i \tilde{y}_i.$$ (74)

Multiplying (73) and (74) with $\tilde{x}_i$ and $\tilde{y}_i$ respectively and adding them together gives

$$\sum_i (\tilde{x}_i^2 + \tilde{y}_i^2) = \frac{1 + \lambda^2}{4} \frac{n(n - 1)}{2} + \frac{1 + \lambda^2}{4\mu} \frac{s \ln \Omega}{\Omega^{s/2}} \sum_i (\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2}. \quad (75)$$

On the other hand, multiplying them with $\tilde{y}_i$ and $-\lambda^2 \tilde{x}_i$ respectively and adding them gives

$$(1 - \lambda^2) \sum_i \tilde{x}_i \tilde{y}_i = \frac{1 + \lambda^2}{4\mu} \frac{s \ln \Omega}{\Omega^{s/2}} (1 - \lambda^2) \sum_i \tilde{x}_i \tilde{y}_i (\tilde{x}_i^2 + \tilde{y}_i^2)^{s/2-1}. \quad (76)$$
The relations (75) and (76) are constraints for the (non-dimensionalized) vortex positions. They simplify considerably for the harmonic trap $s = 2$ (and only for this trap!). They were already deduced in Ref. [4] and we only state them for completeness: $\sum_i \tilde{x}_i = \sum_i \tilde{y}_i = 0$ and

$$\sum_i (\tilde{x}_i^2 + \tilde{y}_i^2) = \frac{n(n-1)}{4} \left( \frac{2}{1+\lambda^2} - \frac{\ln \Omega}{\Omega \mu} \right) (1 - \lambda^2) \sum_i \tilde{x}_i \tilde{y}_i = 0.$$ 

For the anisotropic case $\lambda \neq 1$, the last relation leads to $\sum_i \tilde{x}_i \tilde{y}_i = 0$. For $n = 2$ vortices, one already sees that $\tilde{x}_1 = -\tilde{x}_2$ and the same for the $\tilde{y}$-coordinates. Similarly one can proceed for $n > 2$ vortices (see Ref. [4] for a more detailed discussion).

However, for anharmonic traps with $s > 2$ the above relations are more complicated but one may proceed in a similar way than for harmonic traps. What can be seen immediately is the fact that, as $\varepsilon \to 0$, (75) and (76) reduce to

$$\sum_{i=1}^n (\tilde{x}_i^2 + \tilde{y}_i^2) = \frac{1 + \lambda^2 n(n-1)}{4} + o(1)$$

and

$$(1 - \lambda^2) \sum_{i=1}^n \tilde{x}_i \tilde{y}_i = o(1)$$

where $o(1) \sim \frac{\ln \Omega}{\Omega s^2}$. Remarkably, the distribution of vortices in anharmonic traps with $s > 2$ differs only in this lower order from each other.

**Remark:** The analysis in the foregoing sections holds generally for asymptotically homogeneous traps according to Def.1.1 in Ref. [22] which is as follows: $V$ is asymptotically homogeneous of order $s > 0$ if there is a function $U$ with $U(\mathbf{r}) \neq 0$ for $\mathbf{r} \neq 0$ such that

$$\frac{\gamma^{-s} V(\gamma \mathbf{r}) - U(\mathbf{r})}{1 + |U(\mathbf{r})|} \to 0 \quad \text{as } \gamma \to \infty$$

and the convergence is uniform in $\mathbf{r}$. $U$ is clearly uniquely determined and homogeneous of order $s$, i.e. $U(\gamma \mathbf{r}) = \gamma^s U(\mathbf{r})$ for all $\gamma \geq 0$.

In the case that $V$ itself is homogeneous, there is $V \equiv U$. But if $V$ for instance is a harmonic-plus-quartic potential, $U$ contains only the quartic contribution. Consider for example the following trap

$$V(x, y) = (x^2 + \lambda^2 y^2)[1 + \zeta (x^2 + \lambda^2 y^2)]$$

(77)

with $\zeta \in (0, 1)$ independent of $\varepsilon$ describing the degree of anharmonicity. This trap is *asymptotically* homogeneous of order $s = 4$. But since $V$ is not homogeneous, equation (3) is not exactly right. However, using the above definition for asymptotically homogeneous potentials, (3) can be used if $V(\mathbf{r})$ is replaced.
by $U(r) + o(1)$. So only the leading contribution, i.e. the asymptotically homogeneous one in the potential is 'visible'. In order to see the anharmonic contribution of (77) in our regime, one would have to introduce an additional scaling parameter, i.e. $\zeta$ would have to depend on $\varepsilon$ (see for instance Ref. [2]).

9 Conclusions

In this paper, we studied the Gross-Pitaevskii (GP) energy and density for Bose-Einstein condensates confined in asymptotically homogeneous traps which are subjected to an external rotation in the Thomas-Fermi (TF) limit when the coupling parameter goes to infinity. We derived by analytical estimates the leading order of the GP energy and density, which are given by the corresponding TF quantities, and the next orders due to vortices. In deriving the contributions of the vortices, we estimated the relation between the vortex core sizes and the considered magnitude of angular velocity. As an example, we considered a very general anisotropic homogeneous potential for which we calculated the critical angular velocities for a finite number of vortices together with the associated GP energy. We have shown that all vortices inside the Thomas-Fermi domain are single-quantized and arranged in a polygonal lattice whose shape can be deduced explicitly by a few simple constraints satisfied by the vortex positions. In fact, the results may be used to compare with experiments when the latter involve asymptotically homogeneous traps in the TF regime. In this paper, we considered the above trap for the reason of explicitness and because it incorporates the harmonic and flat trap for which most experimental results are available, but any trap potential satisfying asymptotical homogeneity could be used.

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