Random walks on multiplex networks

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We consider a discrete-time random walk model on multiplex networks, in which at each time step the walker first undergoes a random jump between neighboring nodes in the same layer, and then tries to hop from one node to one of its replicas in another layer. We derive the so-called supra-Markov matrix that governs the evolution of the occupation probability of the walker. The occupation probability at stationarity is different from the weighted average over the counterparts on each layer, unless the transition probabilities between layers vanish. However, they are approximately equal when the transition probabilities between layers are very small, which is given by the first-order degenerate perturbation theory. Moreover, we compute the mean first passage time (MFPT) and the graph MFPT (GrMFPT) that is the average of the MFPT over all pairs of distinct nodes. Interestingly, we find that the GrMFPT can be smaller than that of any layer taken in isolation. The result embodies the advantage of global search on multiplex networks.

I. INTRODUCTION

In the past two decades, we have witnessed the power of network science on modeling dynamical processes in complex systems made of large numbers of interacting elements. However, the recent availability of massive data sets of social, technological and biological systems has suggested that many real-world complex systems are usually composed of interwined multilayer networks. As an example, transportation networks between cities are formed by different types of networks, such as highway network, railway network, and airline network, etc. Another example is represented by online social networks where each layer corresponds to a different social structure (e.g., Facebook and Twitter) and users play the role of nodes. It has been recognized that the multilayer networks can not only induce some novel features different from the single-layer networks, such as complexity, diversity and fragility, but also offer a better understanding for dynamical behaviors, including epidemic spreading, vaccination, synchronization, evolution of cooperation, opinion formation, transportation. Multiplex is a particular type of multilayer network in which each agent participates in different layers simultaneously, just as our previous example in the case of online social networks. A mathematical framework has been developed to generalize several important network descriptors and dynamical processes on multiplex networks.

In the realm of dynamical processes on networks, random walk is a very simple but important model. It not only lies in the heart of many transport processes on networked systems such as the spreading of information or epidemics, human mobility, but also finds a broad range of applications in community detection, ranking and searching on the web.

In this context, two important physical quantities are relevant. One is the occupation probability at stationarity, which quantifies the frequency to visit each node in the long time limit. The other one is the mean first-passage time (MFPT), which is the average time from one node to another one for the first time. First passage underlies a wide variety of important problems like epidemic extinction, neuronal firing, consensus formation. For a single-layer network, the computations of two quantities are well-established. The stationary occupation probability of the walker at a node is proportional to its degree or strength. The MFPT can be calculated by some different approaches, such as the approach through the adjoint equation, the renewal approach combined with spectral decomposition of the transition matrix.

However, much less is known about random walk on multiplex networks. Gómez et al. studied a linear diffusion on multiplex networks and showed that the multiplex structure is able to exhibit a super-diffusive behavior. This means that the time scale to relax to the steady state on a multiplex network is smaller than that of any layer taken in isolation. The diffusive properties are related to a structural transition of the multiplex from a decoupled regime to a systemic regime. Cencetti and Battiston studied the effect of edge overlap between different layers on the super-diffusive phenomenon. They found that super-diffusion emerges the earliest in systems with poorly diffusive layers, and that its onset is independent of the presence of overlap, which only influences the maximum relative intensity of the phenomenon.

Tejedor et al. studied the diffusion processes on directed multiplex networks where at least one layer consists of a directed graph. Interestingly, they found that the rate of convergence to steady state is a nonmonotonic function of coupling strength between layers. There is an intermediate degree of coupling for which the diffusion is the fastest. Domenico et al. considered different types of random walk on multilayer networks and inves-
igated the time-dependent average fraction of distinct nodes that are visited by walker at least once (in any layer) \[47\]. They then examined the coverage as a function of time when some nodes are deleted to model the resilience of multilayer networks to random node failures. Battiston et al. explored how to extend biased random walks to the case of multiplex networks, and they computed the stationary occupation probability and entropy rate for several different classes of walks \[48\]. Guo et al. studied Lévy random walks on multiplex networks and found that the efficiency of such a navigation strategy varies nonmonotonically with an index parameter in Lévy flights when the interlayer coupling is weak enough \[49\]. This result is in contrast to the case in a single-layer network \[50\].

In the present work, we propose a discrete-time random walk model on multiplex networks. At each time step, the walker first performs a random walk between neighboring nodes in the same layer and then tries to switch between replica nodes among different layers. The motivation of the model is to imitate, for example, the information dissemination on online social networks. A user can send a piece of message to one of his friends by Skype, and then his friend further forwards this message by other online social media softwares, such as WeChat, WhatsApp, etc. For the model, we derive the so-called supra-Markov matrix governing the evolution of the occupation probability of the walker at each node. We show that the stationary occupation probability on a multiplex network cannot simply reduced to a conglomerate of additive processes on each single-layer network. Unless the interlayer coupling is weak enough, they are approximately equal to each other from the degenerate perturbation theory. Moreover, we compute the graph mean first passage time (GrMFPT), i.e., the average of MFPT over all pairs of distinct nodes. Interestingly, we find that the GrMFPT can vary monotonically or nonmonotonically with the transition probability between different layers. The GrMFPT can be smaller than that of any layer in isolation, and even that of an aggregate network by all layers.

\[\text{II. MODEL}\]

We consider a walker that performs discrete-time random walks on a multiplex network. The network is consisted of \(L\) layers. Each layer contains the same number of nodes, \(N\), and there exists a one-to-one correspondence between nodes in different layers. The topology in layer \(\alpha\) is described by an adjacency matrix \(A^{(\alpha)}\) \((\alpha = 1, \ldots, L)\), whose entries \(A_{ij}^{(\alpha)}\) are defined as \(A_{ij}^{(\alpha)} = 1\) if there is an edge from node \(i\) to node \(j\) in the \(\alpha\)th layer, and \(A_{ij}^{(\alpha)} = 0\) otherwise. For the sake of simplicity, we consider the case where all connections are undirected and each intralayer network is strongly connected. At each time step, the walker first jumps from node \(i\) in the \(\alpha\)th layer (denoted by \(i_{\alpha}\)) to one of its neighborhood in the same layer, saying \(j_{\alpha}\), and then tries to make an interlayer switch (see Fig. 1 for an illustration). The transition probability in the same layer can be written as \(W_{ij}^{(\alpha)} = A_{ij}^{(\alpha)}/d_i^{(\alpha)}\), where \(d_i^{(\alpha)}\) is the degree of node \(i\) in the \(\alpha\)th layer. The transitions across layers can be described by a Markov matrix \(\pi\) whose entry \(\pi_{\alpha\beta}\) gives the transition probability from the \(\alpha\)th layer to the \(\beta\)th layer, where we have assumed that the transition probability between layers is independent of the node’s label.

It is obvious that our model is equivalent to standard random walks on a single-layer network when the topologies of all the layers are the same. Our goal is whether our model produces some unexpected results when the topology of each layer is different. It is also worth to mention that in our model the way to random walks is similar to some previous works \[17\, 18\, 51\, 52\], but the definition of the transition matrix is essentially different from the others. In the forthcoming section, we will see that our model in the steady state does not satisfy the so-called detailed balance condition.

\[\text{III. STATIONARY OCCUPATION PROBABILITY}\]

\[\text{A. General theory}\]

Let us denote by \(P_{j_{\beta}}(t|i_{\alpha})\) the probability to find the walker at node \(j_{\beta}\) at time \(t\) providing that the walker has started from node \(i_{\alpha}\) at \(t = 0\). The master equation governing the time evolution of \(P_{j_{\beta}}(t|i_{\alpha})\) reads,

\[P_{j_{\beta}}(t|i_{\alpha}) = \sum_{\gamma=1}^{L} \pi_{\gamma\beta} \sum_{k=1}^{N} P_{k_{\gamma}}(t-1|i_{\alpha}) W_{k_{\gamma}j_{\beta}}^{(\gamma)}. \quad (1)\]
Let us indicate with a row vector $P^{(\beta)} = (P_{1 \beta}, \ldots, P_{N \beta})$ with respect to layer $\beta$, and thus Eq. (1) can be written as the matrix form,

$$P^{(\beta)} (t|i_\alpha) = \sum_{\gamma=1}^L \pi_{\gamma \beta} P^{(\gamma)} (t-1|i_\alpha) W^{(\gamma)}. \quad (2)$$

Furthermore, we introduce a supra-vector $P = (P^{(1)}, \ldots, P^{(L)}) = (P_1, \ldots, P_N, \ldots, P_{1L}, \ldots, P_{NL})$, such that Eq. (2) can be rewritten as

$$P (t|i_\alpha) = P (t-1|i_\alpha) W, \quad (3)$$

where

$$W = \begin{pmatrix} \pi_{11} W^{(1)} & \cdots & \pi_{1L} W^{(1)} \\ \vdots & \ddots & \vdots \\ \pi_{L1} W^{(L)} & \cdots & \pi_{LL} W^{(L)} \end{pmatrix}. \quad (4)$$

Note that $W$ is also a Markov matrix satisfying the sum of entries of each row is always equal to one. We call $W$ the supra-Laplacian matrix, which is borrowed from the supra-Lapacian matrix proposed in [42] to model the diffusion process on a multiplex network, and the normalized supra-Laplacian matrix proposed subsequently in [47] to model continuous time random walks on a multiplex network. In the limit of $t \to \infty$, Eq. (3) gives the stationary equation,

$$P (\infty) = P (\infty) W, \quad (5)$$

where we have dropped the conditional probability since in the long time limit $P(\infty)$ does not depend on the initial condition. Eq. (5) implies that $P(\infty)$ is the left eigenvector of $W$ corresponding to the unit eigenvalue. We have checked that the detailed balance condition and thus the stationary equation, $\sum \pi_i W(i|j) = \sum \pi_j W(j|i)$, satisfied by $W$, holds. The detailed balance condition, i.e., $P_{i\alpha} (\infty) W_{\alpha j \beta} \neq P_{j\beta} (\infty) W_{\beta i \alpha}$. (6)

Based on $P(\infty)$, we can compute the stationary occupation probability of the walker at node $i$ regardless of the layer,

$$P_i (\infty) = \sum_{\alpha=1}^L P_{i\alpha} (\infty). \quad (7)$$

It was known that on a monoplex network the stationary occupation probability at each node equals to the normalized left eigenvector $u^{(o)}$ corresponding to the unit eigenvalue [21]. For the standard random walk on networks, one has $u^{(o)} = d^{(o)} / E^{(o)}$, where $E^{(o)}$ is the number of edges in the oth layer. That is to say, the stationary occupation probability at each node on a monoplex network is proportional to the degree of the node. On the other hand, the stationary occupation probability at each layer is given by the left eigenvector of the transition matrix $\pi$ between layers corresponding to the unit eigenvalue, i.e., $\mu \pi = \mu$ with $\mu = (\mu_1, \cdots, \mu_L)$ satisfying $\sum_{\alpha=1}^L \mu_\alpha = 1$.

An intuition is that the stationary occupation probability at each node on a multiplex network may be given by averaging the stationary occupation probabilities on monoplex networks, with the weight equals to the stationary probability $\mu_\alpha$ on each layer, i.e.,

$$G_i (\infty) = \sum_{\alpha=1}^L \mu_\alpha d^{(o)}/E^{(o)}. \quad (8)$$

where we have used the letter “$G$” to avoid confusion with the results on multiplex networks. In fact, the intuition is untenable unless the transition probabilities between layers vanish. In Fig. 2 we show the stationary occupation probabilities at five nodes in Fig. 1. For the sake of simplicity, we have considered that the transition probabilities between two layers are set to the same, $\pi_{12} = \pi_{21} = p$. Solid lines correspond to the theoretical values obtained from Eq. (5) and Eq. (7), dotted lines to the results from the perturbed theory in Eq. (22), and solid circles to the results from Monte Carlo simulations.

![FIG. 2. The stationary occupation probability at each node on a multiplex network shown in Fig. 1. Here the transition probabilities between two layers are set to the same, $\pi_{12} = \pi_{21} = p$. Solid lines correspond to the theoretical values obtained from Eq. (5) and Eq. (7), dotted lines to the results from the perturbed theory in Eq. (22), and solid circles to the results from Monte Carlo simulations.](image)

Without loss of generality, we only consider the case when the underlying network has two layers. The transition matrix between layers is given by

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} = \begin{pmatrix} 1 - \pi_{12} & \pi_{12} \\ \pi_{21} & 1 - \pi_{21} \end{pmatrix}. \quad (9)$$
The left eigenvector of $\pi$ corresponding to the unit eigenvalue is simply given by $\mu = \frac{\pi_{21}}{\pi_{21} + \pi_{21}}$. 

B. The limit $\pi_{12} \to 0$ and $\pi_{21} \to 0$

For $\pi_{12} = \pi_{21} = 0$, the system can be decoupled into two subsystems, where the supra-Markov matrix can be written as $W = W_0$, where

$$W_0 = \begin{pmatrix} W^{(1)} & W^{(2)} \end{pmatrix}$$

is a block diagonal matrix. Let us denote by $\lambda_1^{(1)}$, $\lambda_1^{(2)}$, ..., $\lambda_N^{(1)}$, $\lambda_N^{(2)}$ the $i$th eigenvalue of $W^{(1)}$ and $W^{(2)}$, and by $u_1^{(1)}$, $u_1^{(2)}$, ..., $u_N^{(1)}$, $u_N^{(2)}$ the corresponding left and right eigenvectors, respectively. Since $W^{(1)}$ and $W^{(2)}$ are both Markov matrices, we have $1 = \lambda_1^{(1)} < \lambda_2^{(1)} \leq \cdots \leq \lambda_N^{(1)}$ and $1 = \lambda_1^{(2)} < \lambda_2^{(2)} \leq \cdots \leq \lambda_N^{(2)}$. The left and right eigenvectors corresponding to $\lambda_1^{(1,2)} = 1$ are $u_1^{(1,2)} = \begin{pmatrix} d_1^{(1,2)} & \cdots & d_N^{(1,2)} \end{pmatrix}$ and $v_1^{(1,2)} = (1, \ldots, 1)$, where $E^{(1,2)}$ is the number of edges in the first (second) layer. The eigenvalues of $W_0$ are the set formed by the union of the eigenvalues of $W^{(1)}$ and $W^{(2)}$. They are $\{\lambda_1^{(1)}$, $\lambda_1^{(2)}$, $\lambda_2^{(1)}$, $\lambda_2^{(2)}$, $\ldots$, $\lambda_N^{(1)}$, $\lambda_N^{(2)}\}$, and the corresponding left eigenvectors are $\begin{pmatrix} u_1^{(1)} & 0 & \cdots & 0 \end{pmatrix}$, $\begin{pmatrix} u_1^{(2)} & 0 & \cdots & 0 \end{pmatrix}$, ..., $\begin{pmatrix} 0 & u_1^{(2)} & \cdots & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & \cdots & u_1^{(2)} \end{pmatrix}$, respectively.

In the limits, $\pi_{12} \to 0$ and $\pi_{21} \to 0$, we will use the perturbation theory to approximate calculate the left eigenvector of $W$ corresponding to the unit eigenvalue, that is the stationary occupation probability of the walker on multiplex network. To the end, we first rewrite Eq. (10) as

$$W = W_0 + \Delta W,$$

where

$$\Delta W = \begin{pmatrix} -\pi_{12} W^{(1)} & \pi_{12} W^{(1)} \\ \pi_{21} W^{(1)} & -\pi_{21} W^{(2)} \end{pmatrix}$$

is considered as a perturbed matrix. Since $\lambda = 1$ is an eigenvalue of $W_0$ with algebraic multiplicity two, we will use the degenerate perturbation theory to estimate the eigenvector corresponding to $\lambda = 1$, as done in quantum mechanics. To the end, we first write the left eigenvector of $W$ as a linear combination of the unperturbed system $W_0$, i.e.,

$$u_1 = c_1 \begin{pmatrix} u_1^{(1)} & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & u_1^{(2)} \end{pmatrix}.$$ 

In terms of characteristic equation,

$$u_1 (W_0 + \Delta W) = (\lambda + \Delta \lambda) u_1$$

Since $u_1 W_0 = \lambda u_1$, Eq. (13) simplifies to

$$u_1 \Delta W = u_1 \Delta \lambda$$

Right multiplying respectively by $\begin{pmatrix} v_1^{(1)} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ v_1^{(1)} \end{pmatrix}$ both sides of Eq. (15), we obtain homogeneous equations,

$$\begin{pmatrix} H_{11} - \Delta \lambda & H_{12} \\ H_{21} & H_{22} - \Delta \lambda \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$H_{11} = \begin{pmatrix} u_1^{(1)} & 0 \end{pmatrix} \Delta W \begin{pmatrix} v_1^{(1)} \\ 0 \end{pmatrix} = -\pi_{12},$$

$$H_{12} = \begin{pmatrix} 0 & u_1^{(2)} \end{pmatrix} \Delta W \begin{pmatrix} v_1^{(1)} \\ 0 \end{pmatrix} = \pi_{21},$$

$$H_{21} = \begin{pmatrix} u_1^{(1)} & 0 \end{pmatrix} \Delta W \begin{pmatrix} 0 \\ v_1^{(2)} \end{pmatrix} = \pi_{12},$$

$$H_{22} = \begin{pmatrix} 0 & u_1^{(2)} \end{pmatrix} \Delta W \begin{pmatrix} 0 \\ v_1^{(2)} \end{pmatrix} = -\pi_{21}.$$

Nontrivial solutions for Eq. (16) requires that

$$\det \begin{pmatrix} H_{11} - \Delta \lambda & H_{12} \\ H_{21} & H_{22} - \Delta \lambda \end{pmatrix} = 0.$$ 

Substituting Eq. (17) into Eq. (18), one obtain

$$\Delta \lambda_1 = 0 \quad \text{or} \quad \Delta \lambda_2 = -\pi_{12} - \pi_{21},$$

which leads to $\lambda_1 = \lambda + \Delta \lambda_1 = 1$ and $\lambda_2 = \lambda + \Delta \lambda_2 = 1 - \pi_{12} - \pi_{21}$. We can seek that the perturbed left eigenvector corresponding to $\lambda_1 = 1$, which is the approximate stationary occupation probability. Substituting $\Delta \lambda_1 = 0$ into Eq. (16), we obtain

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \propto \begin{pmatrix} \pi_{21} \\ \pi_{12} \end{pmatrix}.$$ 

Substituting Eq. (19) into Eq. (20) and then using the normalization condition, we obtain

$$u_1 = \frac{1}{\pi_{12} + \pi_{21}} \begin{pmatrix} \pi_{21} u_1^{(1)} \\ \pi_{12} u_1^{(2)} \end{pmatrix},$$

which gives the stationary occupation probability in the first-order perturbation theory,

$$P_1(\infty) = \frac{1}{\pi_{12} + \pi_{21}} \left( \frac{\pi_{21} u_1^{(1)}}{2E^{(1)}} + \frac{\pi_{12} u_1^{(2)}}{2E^{(2)}} \right).$$

We see that the stationary occupation probability in Eq. (22) by the perturbation theory is the same as the result in Eq. (8).
C. The limit \( \pi_{12} = \pi_{21} = 1 \)

In the opposite limit \( \pi_{12} = \pi_{21} = 1 \), the supra-Markov matrix defined in Eq. (11) is given by

\[
W = \begin{pmatrix}
0 & W^{(1)} \\
W^{(2)} & 0
\end{pmatrix}.
\]  

(23)

Letting \( P = (P^{(1)}, P^{(2)}) \) be the stationary occupation probability in the limit, and using \( PW = P \), we have

\[
P^{(1)} = P^{(1)}W^{(1)}W^{(2)},
\]

\[
P^{(2)} = P^{(2)}W^{(2)}W^{(1)}.
\]

(24)

This implies that \( P^{(1)} \) and \( P^{(2)} \) are the left eigenvectors of the Markov matrices \( W^{(1)}W^{(2)} \) and \( W^{(2)}W^{(1)} \) corresponding to the unit eigenvalue, respectively. Generally speaking, \( P^{(1)} \) and \( P^{(2)} \) are linearly independent, unless \( W^{(1)} \) and \( W^{(2)} \) are exchangeable in matrix multiplications.

IV. MEAN FIRST PASSAGE TIME

Let us consider the first passage problem in a multiplex network. Assuming the walker starts from a node \( i \) in the \( o \)th layer, i.e., \( i_o \), how long does it arrive another node \( j \) for the first time, regardless of the layer of the target node? Let us denote by \( T_{i_o,j} \) the MFPT from node \( i_o \) to node \( j \) (any layer), which satisfies the following equation,

\[
T_{i_o,j} = 1 + \sum_{k \neq j} W_{i_o,k_j}T_{k_j,j}.
\]

(25)

Furthermore, the MFPT can be taken the average over the layer of starting node, with the weight proportional to the stationary occupation probability in each layer, given by

\[
T_{ij} = \frac{\sum_{\alpha=1}^{L} P_{i_o}(\infty)T_{i_o,j}}{\sum_{\alpha=1}^{L} P_{i_o}(\infty)}.
\]

(26)

To measure the global search efficiency, we define the GrMFPT, which is the average of the MFPT over all pairs of distinct nodes \( L \),

\[
\text{GrMFPT} = \frac{2}{N(N-1)} \sum_{i \neq j} T_{ij}.
\]

(27)

To illustrate our results, we consider different two-layer multiplex topologies, where each layer can be an Erdős–Rényi (ER) random network [51] with the average degree \( \langle k \rangle = 3.48 \) or a Barabási–Albert (BA) scale-free network [55] with the average degree \( \langle k \rangle = 2 \). For each layer, the size of network is fixed at \( N = 50 \). For simplicity, we set \( \pi_{12} = \pi_{21} = p \). In Fig. 3, we show the GrMFPT as a function of \( p \) for three different multiplex networks. Interestingly, the GrMFPT shows a monotonic or nonmonotonic change with \( p \). For the case of ER+BA or BA+BA topology, there exists an optimal value of \( p \) for which the GrMFPT is a minimum. To validate the theoretical results, we have performed Monte Carlo simulations. The simulation results (see symbols in Fig. 3) obtained by averaging over \( 10^3 \) realizations for each pair of distinct nodes, which agree well with theoretical ones.

For comparison, we also compute the GrMFPT on each monoplex network, given by two top horizontal lines in Fig. 3. The MFPT on each layer can be computed by numerically iterating the following equations,

\[
T_{ij}^{(\alpha)} = 1 + \sum_{k \neq j} W_{ik}^{(\alpha)}T_{kj}^{(\alpha)},
\]

(28)

Form Fig. 3, one finds that a wide range of \( p \) exists for which the GrMFPT on the multiplex network is always less than those in two monoplex networks. This implies that the search on a multiplex network is more advantageous than any one of monoplex networks when the coupling between layers is properly set up.

We further consider the superposition of two-layer networks. The way to superposition is as follows. Two nodes \( i \) and \( j \) are considered to be connected only if they are connected in any layer, in the sense that the aggregate network is still unweighted and undirected. The entries of the adjacency matrix of such an aggregate network are given by

\[
A_{ij}^S = \min \left\{ 1, \sum_{\alpha=1}^{L} A_{ij}^{(\alpha)} \right\}.
\]

(29)

The GrMFPT on the aggregate network is also shown in Fig. 3 by the lowest horizontal line. We can see that for the case of ER+BA or BA+BA topology the GrMFPT on the multiplex network is larger than that in the aggregate network, but the minimal GrMFPT on the multiplex network is close to the GrMFPT in the aggregate network. While for the case of ER+ER topology, the GrMFPT on the multiplex is even smaller than that in the corresponding aggregate network when the transition probability between two layers is large enough.

V. CONCLUSIONS

To conclude, we have proposed a random walk model on multiplex networks. At each time step, the walker hops from one node to one of its neighbors in the same layer, as the standard random walk in a monoplex network, and then tries to switch between replicas of one node belong to different layers. We have focused on the stationary occupation probability of the walker at each node and the GrMFPT. The former is given by the left eigenvector of the supra-Markov matrix corresponding to the unit eigenvalue. We show that the model at steady state does not satisfy the detailed balance and the stationary occupation probability does not equal to the superposition of counterparts in all layers. On the other hand, the GrMFPT shows a nontrivial dependence on
the transition probability between different layers. In a wide range of parameter, the GrMFPT is smaller than that of any layer in isolation, and can even smaller than that of an aggregate network by all layers. The result shows that the efficiency of search on real-world complex systems may benefit from the multiplexity.

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