Corrections of order $\alpha_s^3$ to $R_{\text{had}}$ in pQCD with light gluinos

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Abstract

By a completely independent analytic calculation made in general covariant gauge we confirm the known result for the massless $\mathcal{O}(\alpha_s^3)$ correction to $R_{\text{had}}$ and check explicitly its gauge independence. We extend the calculations to include contributions due to a hypothetical existence of a colour octet of (light) neutral fermions known also under the name of gluinos. In numerical form and for five active flavours we found (in $\overline{\text{MS}}$ scheme) that

$$R(s) = \frac{11}{3} [1 + \alpha_s/\pi + (1.409 - 0.346n_{\tilde{g}})(\alpha_s/\pi)^2 + (-12.805 - 3.006n_{\tilde{g}} - 0.0466n_{\tilde{g}}^2)(\alpha_s/\pi)^3]$$

where $n_{\tilde{g}}$ is the number of such colour octets.

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1 Introduction

The simplest inclusive reaction involving quarks—their production through a decay of a heavy virtual photon or a Z boson—is a process of fundamental importance for QCD as the theory of strong interactions. Firstly, it provides us with a beautiful confirmation of the existence of colour [1]. Secondly, the precision measurements of \( \Gamma(Z \to \text{hadrons}) \) have developed into an important experimental tool for a reliable determination of \( \alpha_s(M_Z) \) [2]. Thirdly, by comparing \( \alpha_s(M_Z) \) with the value of \( \alpha_s \) as obtained from the measurements of \( R(s) \) at lower energies one could in principle directly test the evolution of the strong coupling constant [3].

Impressive wealth of theoretical results about \( R(s) \) is now available in the QCD framework (see e.g. a recent review [4] for a detailed discussion). In the massless approximation, valid in the high energy limit, corrections to \( R \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-) \) have been calculated up to order \( \alpha^3 \) [5, 6]. For precision measurements the dominant mass corrections can be included through an expansion in \( m^2 q / s \). Terms up to order \( \alpha^3 m^2 q / s \) [7] and \( \alpha^2 m^4 q / s^2 \) [8] are available at present, providing an acceptable approximation from the high energy region down to intermediate energy values. At order \( \alpha^2 \) even full account of the quark mass effects has been recently made on the basis of a semi-analytic approach [9]. Among massless calculations the one in the order \( \alpha^3 \) is probably most involved. Let us discuss it separately.

The results of the reevaluation of massless next-next-to-leading \( \alpha^3 \) correction to \( R(s) \) (first computed erroneously in Ref. [10] because of bugs in a computer program) were published in Refs. [5, 6] five years ago. Both calculations have produced identical results and are much alike in many respects:

- the same theoretical tools as well as computer programs have been used;
- the simplest gauge condition – the Feynman one – has been employed;
- An important subclass of all diagrams (which includes, in fact, the most complicated ones) – the QED type diagrams (that is those not containing three and four-gluon vertices) – had been computed first and published in Ref. [11] by a collaboration of authors of Refs. [5, 6].

A golden rule well-known among multi-loop people says that a result of a multi-loop calculation can be trusted and considered as the result only if it is confirmed by an independent calculation preferably made by a different group and with the use of the general covariant gauge. Therefore, in view of the importance of the \( O(\alpha^3) \) correction for both theory and phenomenology it is necessary to check those by a really independent calculation.

In the present paper we report about such an attempt. Our work confirms the results of Refs. [5, 6] and explicitly demonstrate their gauge independence. We have also computed extra diagrams which include virtual loops of Majorana fermions transforming as an octet with respect to the colour \( SU_c(3) \) group. Note that the case of Majorana fermions considered as \textit{gluinos}, that is as superpartners for gluons [12], has some phenomenological relevance at present. On one side, light gluino are not completely excluded by the current experimental data [13, 14, 15]. On the other side, overall consistency of
various determinations of $\alpha_s(M_Z)$ seems to improve if the light gluino with mass of a few GeV does exist [16, 17, 18].

2 Preliminaries

From purely theoretical point of view $R(s)$ is an extremely suitable object to deal with within pQCD. This is because all relevant information is contained in the current correlation function

$$\Pi_{\mu\nu}(q) = (4\pi)^2 i \int dx e^{iqx} \langle 0 | T [ j^em_\mu(x) j^em_\nu(0) ] | 0 \rangle = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(-q^2),$$

with the hadronic EM current $j^em_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f$, and $Q_f$ being the EM charge of the quark $f$. The optical theorem relates the inclusive cross-section and thus the function $R(s)$ to the discontinuity of $\Pi$ in the complex plane

$$R(s) = \frac{3}{4\pi} \text{Im} \Pi(-s - i\delta).$$ (2)

Experimental $e^+e^-$ data are taken in the physical regime of timelike momentum transfer $q^2 > 0$. This region is influenced by threshold and bound state effects which make the use of perturbative QCD questionable. However, perturbative QCD is strictly applicable for large spacelike momenta ($q^2 = -Q^2 < 0$), since this region is far away from non-perturbative effects due to hadron thresholds and resonance effects [19]. Therefore, reliable theoretical predictions can be made for $\Pi(Q^2)$ with $Q^2 > 0$. To compare theoretical predictions and experimental results for time-like momenta, one has to perform suitable averaging procedures [20, 21]. For large positive $s$ one may appeal to the experimentally observed smoothness of $R$ as a function of $s$ and to the absence of any conceivable non-perturbative contribution.

The renormalization mode of the polarization operator $\Pi(Q^2)$ reads (see, e.g. Ref. [4])

$$\Pi(Q^2/\mu^2, \alpha_s) = Z^{\text{em}} + \Pi_0(Q^2, \alpha_s^0),$$ (3)

where $\alpha_s = g^2/(4\pi)$ is the strong coupling constant. Within the $\overline{MS}$ scheme [22] (here and below we are using a convenient combination $a_s = \alpha_s/\pi$)

$$Z^{\text{em}} = \sum_{1 \leq j \leq i} (Z^{\text{em}})_{ij} \frac{a_s^{i-j}}{\epsilon^j},$$ (4)

with the coefficients $(Z^{\text{em}})_{ij}$ being pure numbers and $D = 4 - 2\epsilon$ standing for the space-time dimension. As a result we arrive at the following renormalization group (RG) equation for the polarization operator

$$\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s)a_s \frac{\partial}{\partial a_s} \right) \Pi = \gamma^{\text{em}}(a_s)$$ (5)

or, equivalently, ($L_Q = \ln \frac{\mu^2}{Q^2}$)

$$\frac{\partial}{\partial L_Q} \Pi = \gamma^{\text{em}}(a_s) - \left( \beta(a_s)a_s \frac{\partial}{\partial a_s} \right) \Pi.$$ (6)
Here the photon anomalous dimension and the $\beta(a_s)$-function are defined in the usual way

\[
\gamma_{em} = \mu^2 \frac{d}{d\mu^2} (Z_{em}) - \epsilon Z_{em} = - \sum_{i \geq 0} (i + 1)(Z_{em})_i a_s^i, \tag{7}
\]

\[
\mu^2 \frac{d}{d\mu^2} a_s = \alpha_s \beta(a_s) \equiv - \sum_{i \geq 0} \beta_i a_s^{i+2}. \tag{8}
\]

The relation (7) explicitly demonstrates the main computational advantage of finding first the polarization function $\Pi(Q^2)$ against a direct calculation of $R(s)$ in the case of massless QCD\textsuperscript{1}. Indeed, in order $a_s^n$ the derivative $\frac{d}{dL_{Q^2}} \Pi$ and, consequently, $R(s)$ depends on the very function $\Pi$ which is multiplied by at least one factor of $a_s$. This means that one needs to know $\Pi$ up to order $a_s^{n-1}$ only to unambiguously reconstruct all $Q$-dependent terms in $\Pi$ to order $a_s^n$, provided, of course, the beta function and anomalous dimension $\gamma_{em}$ is known to order $a_s^n$. On the other hand, the calculation of an anomalous dimension or a beta-function is known to be much easier than computing a correlator of the same order in the coupling constant (see below).

On specifying the order of $n = 3$ we conclude that for computing the $\mathcal{O}(a_s^3)$ contribution to $R(s)$ one needs to know the following ingredients:

- the $\beta$ function to $a_s^2$: it is available through independent calculations of Refs. [23];
- the polarization function to $a_s^2$ including the constant part not depending on $\log Q^2$: the logarithmic contributions were originally obtained in Refs. [24, 25] while the constants were first published in Refs. [5, 6], albeit not in an explicit form;
- the photon anomalous dimension $\gamma_{em}$ to $a_s^3$: it is known from works [3, 8] where it was computed with the use of Feynman gauge.

Thus, in order to check the results of Refs. [3, 8] by an independent calculation one should compute $\Pi$ in order $a_s^2$ (three-loop diagrams) and $\gamma_{em}$ in order $a_s^3$ (four-loop diagrams). The corresponding massless Feynman integrals depend only on one external momentum, $q$ in our notation, and will be conventionally referred to as $p$-integrals in what follows.

### 3 Calculation of $p$-integrals

It should be stressed that the first, three-loop calculation is definitely much less involved than the second, four-loop one. There are three reasons for it. First, the total number of diagrams is by order of magnitude less. Second and most important, there exists an elaborated algorithm — the method of integration by parts of Ref. [26, 27] — which allows one to analytically evaluate divergent as well as finite parts of any three-loop $p$-integral. Third and also very important fact is that the algorithm has been neatly and reliably\textsuperscript{2}.

\textsuperscript{1}To our knowledge essentially identical observation was first made in Ref. [24]

\textsuperscript{2} The package has been extensively tested in various ways and a probability of the existence of a bug in it seems to be extremely small. In fact, the calculation we are describing here may also be considered as another highly non-trivial check of the program.
implemented in the language of FORM \[28\] as the package named MINCER in Ref. \[28\]. An important feature of the algorithm is its ability to evaluate bare dimensionally regulated diagrams. This allows a convenient two-step evaluation scheme: first calculation of bare diagrams followed by a renormalization procedure. The latter eventually reduces to a straightforward substitution of the bare coupling constants expressed through the renormalized ones multiplied by proper renormalization constants. Needless to say that such a substitution is now routinely done with the help of algebraic manipulation programs like FORM.

The situation with four-loop diagrams is quite different. At present there is simply no way to directly compute the divergent part of a bare four-loop p-integral, not speaking about its constant part (see in this connection a discussion in Ref. \[30\]). The best what can be done is the use of the method of Infrared Rearrangement (IRR). The method is based on an important observation of Ref. \[31\] that in the \(\overline{\text{MS}}\)-scheme any UV counterterm has to be polynomial in momenta and masses. The observation was effectively employed in Ref. \[32\] to simplify considerably the calculation of UV counterterms. The trick essentially amounts to an appropriate transformation of the IR structure of Feynman integrals by setting zero some external momenta and masses (in some cases after some differentiation is performed with respect to the latter). As a result the calculation of UV counterterms is much simplified by reducing the problem to evaluating p-integrals. The method of IRR was ultimately refined and freed from unessential qualifications with inventing a so-called \(R^\ast\)-operation — a generalization of the BPHZ \(R\)-operation to subtract UV and IR divergences — in Refs. \[33, 34\].

The following theorem has been proven in Ref. \[34\] by the explicit construction of the corresponding algorithm:

**Any UV counterterm for any \((h+1)\)-loop Feynman integral can be expressed in terms of pole and finite parts of some appropriately constructed \(h\)-loop p-integrals.**

The above theorem coupled with the integration by parts method solves *at least in principle* the task of analytical evaluation of \(\gamma_{\text{em}}\) to \(a_3^3\). It should be noted that the \(R^\ast\) -operation is absolutely essential to prove the Theorem, though in most (but not in all) practical cases one could proceed without it. However, such a practice, in fact, forces diagram-wise renormalization mode, what, in turns, brings down a heavy penalty of manual treatment of hundreds of diagrams.

Indeed, in genuine four-loop calculations the reduction to three-loop p-integrals is far from being trivial and includes a lot of manipulations. Typical steps here are

- **a** to reduce the initial Feynman integral to logarithmically divergent ones via a proper differentiation with respect masses and external momenta;
- **b** to identify UV and IR divergent subgraphs of the resulting integral;
- **c** to remove in a recursive way the corresponding UV and IR divergences;
- **d** to compute resulting p-integrals.

Among these steps only the calculation of p-integrals can be at present completely performed by a computer. All others, especially **b** and **c** are difficult to computerize. As a
result in works [11, 5, 6] an inherently time-consuming and error-prone way of manually handling every separate diagram (including its full UV renormalization via \( R \)-operation) has been employed.

In present work we will use the full power of the \( R^* \)-operation to simplify the steps b and c so far that both UV and IR renormalizations can be done in a global form and, consequently, can be simply performed by computer.

4 IRR in a global form

We start from the Dayson–Schwinger equation for the correlator [1] written in the bare form

\[
\Pi^0_{\alpha\alpha}(q, a_s^0) = - \int dp \frac{(4\pi)^2}{(2\pi)^D} Tr[\gamma_\alpha G^0(p + q, a_s^0)\Gamma^0_\alpha(p, q, a_s^0)G^0(p, a_s^0)]. \tag{9}
\]

Here \( G^0 \) and \( \Gamma^0_\alpha \) are the full quark propagator and the EM current vertex function respectively; below the integration with respect to the loop momentum \( p \) with the weight function \((4\pi)^2/(2\pi)^D\) will not be explicitly displayed. The renormalized version of (9) can be written in two different but eventually equivalent forms: in terms of renormalization constants or through \( R \)-operation. The first form reads

\[
\Pi_\alpha(q, a_s) = Z_{em} q^2 (1 - D) - \frac{Z^2_V}{Z^2_2} Tr[\gamma_\alpha G^0(p + q, a_s^0)\Gamma^0_\alpha(p, q, a_s^0)G^0(p, a_s^0)]. \tag{10}
\]

Here \( Z_2 \) is the quark wave function renormalization constant; \( Z_V \) is the renormalization constant of the vector current defined as

\[
\bar{\psi}\gamma_\alpha\psi = Z_V \bar{\psi}\gamma_\alpha\psi^0
\]

where the current inside squared brackets is the renormalized one. The QED Ward identity implies the equality \( Z_V = Z_2 \), whence the equivalence of (3) and (10) follows.

The second representation is

\[
\Pi_\alpha(q, a_s) = Z_{em} q^2 (1 - D) - R'Tr[\gamma_\alpha G^0(p + q, a_s^0)\Gamma^0_\alpha(p, q, a_s^0)G^0(p, a_s^0)], \tag{11}
\]

where \( R' \) stands for the “incomplete” \( R \)-operation which, when applied to a Feynman integral, subtracts only all its UV subdivergences not touching the overall one.

From the finiteness of the renormalized correlator we have two ways of finding \( Z_{em} \), viz.

\[
Z_{em} = -K_\epsilon \left\{ \frac{1}{2D(D-1)} \left( \frac{1}{Z_2} \boxq Tr[\gamma_\alpha G^0(p + q, a_s^0)\Gamma^0_\alpha(p, q, a_s^0)G^0(p, a_s^0)] \right) \right\} \tag{12}
\]

\[
+ \frac{\delta Z_V}{Z_2} \boxq Tr[\gamma_\alpha G^0(p + q, a_s^0)\Gamma^0_\alpha(p, q, a_s^0)G^0(p, a_s^0)] \tag{13}
\]

\[\text{The calculation of Ref. [5] did not use the } R^* \text{-operation at all while that of Ref. [6] employed it only for a few diagrams.}\]

\[\text{For simplicity we set the } \text{’t} \text{ Hooft-Veltman unit of mass } \mu \text{ equal to 1 below.}\]
where \( K_\epsilon f(\epsilon) \) stands for the singular part of the Laurent expansion of \( f(\epsilon) \) in \( \epsilon \) near \( \epsilon = 0 \) and \( \delta Z_V = Z_V - 1 \). In Eqs. (13) and (14) we have let a Dalambertian with respect to the external momenta \( q \) act on quadratically divergent diagrams to transform them to the logarithmically divergent ones. We also have introduced an auxiliary mass dependence to a quark propagator — the one entering into the “left” vertex \( \gamma_\alpha \) — by making the following change

\[
\gamma_\alpha \rightarrow \tilde{\gamma}_\alpha = \frac{\gamma_\alpha p^2}{(p^2 - m_0^2)}.
\]

(15)

Note, please, that the auxiliary mass dependence has caused somewhat more complicated structure of UV renormalizations in the right hand side of Eq. (13). In the case of the second equation corresponding modifications are taken into account automatically by \( R' \)-operation.

Now, to IRR. The idea of the method is quite simple: since the renormalization constant \( Z_{\text{em}} \) does not depend on the momentum \( q \) dimensionfull one could significantly simplify the calculation of \( Z_{\text{em}} \) by nullifying the momentum Eqs. (13) and (14). The only requirement which must be respected is the absence of any IR singularities in the resulting bubble integrals. Unfortunately, a mass introduced to a propagator is not always sufficient to suppress all IR divergences. For instance, if \( q = 0 \) then there appear completely massless tadpoles in the second term on the rhs of Eq. (13). On the other hand, in the diagram-wise calculations of Refs. [24, 11, 5, 6], based on Eq. (14), it proves possible to tune the position of the massive propagator for any given diagram in such a way to suppress all IR singularities.

Our idea, instead, is to use Eq. (13) supplemented by the corresponding IR subtractions as prescribed by the \( R' \)-operation formalism [34]. The problem is facilitated by the fact that, as shown in Ref. [35] the IR counterterm constants for a given diagram may be determined in terms of some properly chosen combination of UV ones. The only remaining task is to write the IR subtractions in a global way. We have done it with the following result:

\[
Z_{\text{em}} = -K_\epsilon \left\{ \frac{1}{2D(D-1)} \Box_q Tr[\gamma_\alpha G^0(p + q, a_s)\Gamma_\alpha^0(p, q, a_s) G^0(p, a_s)] \right\}_q = 0
- \frac{1}{Z_2} \frac{1}{4D} Tr[\delta \Gamma_\alpha^0(0, 0, a_s)\gamma_\alpha] Z_{\text{em}} - \frac{\delta Z_V}{Z_2} Z_{\text{em}} \right\}.
\]

(16)

Several comments are in order regarding this formula.

First, Eq. (16) is, rigorously speaking, applicable as it stands only to the so-called non-singlet diagrams, that is to those where both EM currents belong to one and the same quark loop. The four singlet diagrams, violating this requirement, appear first in order \( a_s^3 \). A full derivation of (16) with modifications necessary to include singlet case will be presented elsewhere.

Second, by \( \delta \Gamma_\alpha^0(p, q, a_s) \) we denote the vertex function of the electromagnetic current with the tree contribution removed. The “tilde” atop the index \( \alpha \) again means that in every diagram the quark propagator entering to the vertex \( j_\alpha \) is softened at small momenta.
by means of the auxiliary mass $m_0$ according to Eq. (15). The bare coupling constant $a_s^0$ is to be understood as $a_s = Z_a a_s$, with $Z_a$ being the coupling constant renormalization constant.

Finally, an inspection of (16) immediately shows that, in order to find the $(n + 1)$-loop correction to $Z_{em}^{\text{ren}}$, one needs only to know the renormalization constants $Z_2$ and $Z_{em}^{\text{ren}}$ to order $a_s^n$ as well as the bare Green functions

$$G^0(p, a_s^0), \quad \frac{\partial}{\partial q^\beta} [\Gamma^0_\alpha(p, q, a_s^0)] \big|_{q = 0} = 0, \quad \Box_q [\Gamma^0_\alpha(p, q, a_s^0)] \big|_{q = 0} = 0, \quad \delta \Gamma^0_\alpha(0, 0, a_s^0)$$

up to (and including) $n$-loops, that is to order $(a_s^n)$. Thus, we have achieved our aim and obtained a general formula for $Z_{em}^{\text{ren}}$ in terms of bare $p$-integrals with explicitly resolved UV and IR subtractions.

\section{Results and discussion}

We have computed with the program MINCER the unrenormalized three-loop Green functions (17) as well as the quark wave function renormalization constant $Z_2$ to order $a_s^3$. The calculations have been performed in the general covariant gauge with the gluon propagator $(g_{\mu\nu} - \xi q^\mu q^\nu)/q^2$. We have also taken into account the singlet diagrams as well as extra diagrams with some of virtual quarks replaced by colour octet neutral fermions. In the minimal supersymmetric standard model such a fermion known as gluino appears as the superpartner of the gluon. The total calculational time with an DEC workstation exceeds 200 hours for the general gauge; for the Feynman one it is reduced to about 20 hours.

Then we have used Eqs. (16) and (8) to find $\gamma_{em}$ to order $a_s^3$. We have used the following values for the coefficients of the beta-function in QCD with gluinos \cite{36}

$$\beta_0 = \frac{1}{4} \left[ \frac{11}{3} C_A - \frac{4}{3} T_n f - \frac{2}{3} C_A n_\tilde{g} \right],$$

$$\beta_1 = \frac{1}{16} \left[ \frac{34}{3} C_A^2 - 4 C_F T_n f - \frac{20}{3} C_A T_n f - \frac{16}{3} C_F^2 n_\tilde{g} \right].$$

Here $C_A$ and $C_F$ are the Casimir operators of the adjoint and quark (defining) representations of the colour group; $T$ is the normalization of the trace of generators of quark representation $T_R(t^a t^b) = T \delta^{ab}$; $n_f$ is the number of quark flavours; $d[R]$ is the dimension of the quark representation of the colour group and $n_\tilde{g}$ is the number of neutral colour octets which we take either zero or one.

Our results for $\gamma_{em}$ and $R(s)$ read

$$\gamma_{em} = d[R] \sum_f Q_f^2 \left\{ \frac{4}{3} + a_s C_F ight. \right.$$  

$$+ a_s^2 \left[ C_F^2 \left( -\frac{1}{8} \right) + C_F C_A \left( \frac{133}{144} \right) + C_F T_n f \left( -\frac{11}{36} \right) + C_F C_A n_\tilde{g} \left( -\frac{11}{72} \right) \right]$$
\begin{align*}
+ a_s^3 & \left[ C_F^3 \left(-\frac{23}{32}\right) + C_F^2 C_A \left(\frac{215}{216} - \frac{11}{18} \zeta(3)\right) + C_F C_A^2 \left(\frac{5815}{15552} + \frac{11}{18} \zeta(3)\right) \\
& + C_F^2 T_n f \left(-\frac{169}{216} + \frac{11}{9} \zeta(3)\right) + C_F C_A T n f \left(-\frac{769}{3888} - \frac{11}{9} \zeta(3)\right) \\
& + C_F T^2 n_f^2 \left(-\frac{77}{972}\right) + C_F C_A n_f T n_g \left(-\frac{77}{972}\right) + C_F^2 C_A n_g \left(\frac{19}{216} + \frac{1}{9} \zeta(3)\right) \\
& + C_F C_A^2 n_g \left(-\frac{77}{3888}\right) \right] \\
+ a_s^3 & \left(\sum_f Q_f \right)^2 \frac{d_{abc}d_{abc}}{256} \left(\frac{176}{9} - \frac{128}{3} \zeta(3)\right),
\end{align*}

(19)

\begin{align*}
R(s) &= d[R] \sum_f Q_f^2 \left\{ 1 + a_s(\mu) r_1 + a_s^2(\mu) \left[ s_2 + \ln \frac{\mu^2}{s} (s_1 \beta_0) \right] \\
& + a_s^3(\mu) \left[ s_3 + \ln \frac{\mu^2}{s} (2s_2 \beta_0 + s_1 \beta_1) + \ln \frac{\mu^2}{s} (s_1 \beta_0^2) \right] \right\} \\
& + a_s^3 \left(\sum_f Q_f \right)^2 \frac{d_{abc}d_{abc}}{1024} \left[ \frac{176}{3} - 128\zeta(3)\right],
\end{align*}

(20)

\begin{align*}
r_1 &= C_F \left[\frac{3}{4}\right], \quad r_2 = C_F^2 \left[\frac{-3}{32}\right] + C_F C_A \left[\frac{123}{32} - \frac{11}{4} \zeta(3)\right] + C_F T n_f \left[-\frac{11}{8} + \zeta(3)\right] \\
& + C_F C_A n_g \left[-\frac{11}{16} + \frac{1}{2} \zeta(3)\right],
\end{align*}

\begin{align*}
r_3 &= C_F^3 \left[\frac{-69}{128}\right] + C_F^2 C_A \left[-\frac{127}{64} - \frac{143}{16} \zeta(3) + \frac{55}{4} \zeta(5)\right] \\
& + C_F C_A T n_f \left[-\frac{485}{27} \pi^2 + \frac{112}{9} \zeta(3) + \frac{5}{6} \zeta(5)\right] + C_F T^2 n_f^2 \left[\frac{151}{54} - \frac{1}{36} \pi^2 - \frac{19}{9} \zeta(3)\right] \\
& + C_F C_A n_f T n_g \left[\frac{151}{54} - \frac{1}{36} \pi^2 - \frac{19}{9} \zeta(3)\right] + C_F^2 C_A n_g \left[\frac{9}{16} + \frac{13}{8} \zeta(3) - \frac{5}{2} \zeta(5)\right] \\
& + C_F C_A^2 n_g \left[-\frac{33767}{3456} + \frac{11}{2} \pi^2 + \frac{251}{36} \zeta(3) + \frac{5}{12} \zeta(5)\right] + C_F C_A^2 n_g \left[\frac{151}{216} - \frac{1}{36} \pi^2 - \frac{19}{36} \zeta(3)\right],
\end{align*}

(21)

where $d_{abc} = 2Tr(\{t^a t^b t^c\})$.

We observe that neither $\gamma_{\text{ren}}$ no $R(s)$ depend on the gauge fixing parameter $\xi$ as it must be. If $n_g$ is set to zero then $R(s)$ is in complete agreement with the results of Refs. \[. \[. \]

For the standard QCD $SU_c(3)$ colour group values $C_F = 4/3, C_A = 3, T = 1/2$ and $d_{abc}d_{abc} = 40/3$ we get for $R(s)$ with $\mu^2 = s$

\begin{align*}
R(s) &= 3 \sum_f Q_f^2 \left\{ 1 + a_s + a_s^2 \left[\frac{365}{24} - 11 \zeta(3) - \frac{11}{12} n_f + \frac{2}{3} \zeta(3) n_f - \frac{11}{4} n_g + 2 \zeta(3) n_g\right] \\
& + a_s^3 \left[\frac{87029}{288} - \frac{121}{48} \pi^2 - \frac{1103}{4} \zeta(3) + \frac{275}{6} \zeta(5) - \frac{7847}{216} n_f + \frac{11}{36} \pi^2 n_f\right]
\end{align*}

8
\[
\begin{align*}
&\frac{262}{9} \zeta(3) n_f - \frac{25}{9} \zeta(5) n_f + \frac{151}{162} n_f^2 - \frac{1}{108} \pi^2 n_f^2 - \frac{19}{27} \zeta(3) n_f^2 \\
&- \frac{32903}{288} n_{\tilde{g}} + \frac{11}{12} \pi^2 n_{\tilde{g}} + \frac{277}{3} \zeta(3) n_{\tilde{g}} - \frac{25}{3} \zeta(5) n_{\tilde{g}} + \frac{151}{27} n_f n_{\tilde{g}} \\
&- \frac{1}{18} \pi^2 n_f n_{\tilde{g}} - \frac{38}{9} \zeta(3) n_f n_{\tilde{g}} + \frac{151}{18} n_{\tilde{g}}^2 - \frac{1}{12} \pi^2 n_{\tilde{g}}^2 - \frac{19}{3} \zeta(3)n_{\tilde{g}}^2 \bigg) \\
&+ a_s^3 \left( \sum_f Q_f \right)^2 \left( \frac{55}{72} - \frac{5}{3} \zeta(3) \right)
\end{align*}
\] (22)

or, in the numerical form,

\[
R(s) = 3 \sum_f Q_f^2 \left\{ 1 + a_s + a_s^2 \left( 1.98571 - 0.115295 n_f - 0.345886 n_{\tilde{g}} \right) \\
+ a_s^3 \left( -6.63694 - 1.20013 n_f - 0.00518 n_f^2 - 2.85053 n_{\tilde{g}} \\
- 0.03107 n_f n_{\tilde{g}} - 0.04661 n_{\tilde{g}}^2 \right) \right\} \\
- a_s^3 \left( \sum_f Q_f \right)^2 1.2395.
\] (23)

At last, for the phenomenologically relevant case of \( n_f = 5 \) we obtain

\[
R(s) = \frac{11}{3} \left[ 1 + a_s a_s^2 (1.409 - 0.346 n_{\tilde{g}}) + a_s^3 (-12.805 - 3.006 n_{\tilde{g}} - 0.0466 n_{\tilde{g}}^2) \right].
\] (24)

Thus, the \( \mathcal{O}(a_s^3) \) gluino contribution to \( R(s) \) has the same (negative) sign as in the \( a_s^2 \) order. We should also add that for a meaningful phenomenological discussion of the gluino contribution to \( R(s) \) one should also take into account the running of the coupling constant in the next-next-to leading order. This requires the knowledge of the gluino contribution to the three-loop coefficient \( \beta_2 \) which, to our knowledge, is not yet available in the literature (the purely QCD contribution to \( \beta_2 \) is known from \cite{37, 38}).

To summarize: we have suggested a new convenient way to compute the UV renormalization constant of the correlator of vector quark currents. Our final formula (16) directly expresses the constant in terms of unrenormalized p-integrals, with all UV and IR subtractions being implemented in a global form. The formula is useful in carrying out completely automatic calculations. In our previous work \cite{39} similar formula has been obtained for the case of the correlator of scalar currents.

Using the formula and the FORM version of MINCER \cite{29} we have computed the \( \mathcal{O}(a_s^3) \) correction to \( R(s) \) in pQCD including light gluino. In a particular case of the standard QCD we have reproduced the result of Refs. \cite{4, 6}. This gives also an extra support for the non-accidental nature of the findings of Ref. \cite{40}. An important feature of our calculation was the use of the general covariant gauge, which has allowed us to demonstrate for the first time the gauge independence of \( R(s) \) at \( \mathcal{O}(a_s^3) \).

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