Limits on diffusive shock acceleration in dense and incompletely ionised media

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Received August 21, 1995; Accepted October 12, 1995

Abstract. The limits imposed on diffusive shock acceleration by upstream ion-neutral Alfvén wave damping, and by ionisation and Coulomb losses of low energy particles, are calculated. Analytic solutions are given for the steady upstream wave excitation problem with ion-neutral damping and the resulting escaping upstream flux calculated. The time dependent problem is discussed and numerical solutions presented. Finally the significance of these results for possible observational tests of shock acceleration in supernova remnants is discussed.

Key words: Acceleration of particles – plasmas – shock waves – cosmic rays – ISM: supernova remnants – gamma rays: theory

1. Introduction

Diffusive shock acceleration is generally believed to be an important astrophysical mechanism for producing high energy charged particles (Blandford & Eichler \cite{Blandford1987}, Berezko & Krymsky \cite{Berezko1988}, Jones & Ellison \cite{Jones1991}). In particular, operating at the strong shocks of Galactic supernova remnants, it is thought to be responsible for producing the Galactic cosmic rays i.e., those of energy less than about $10^{14}$ eV. However this hypothesis has not been convincingly confirmed by any direct observational test. Possibilities which have been suggested include the observation of high energy gamma-ray emission from supernova remnants, or detection of shock precursor structures via Balmer line observations (Aharonian et al. \cite{Aharonian1994} and Raymond \cite{Raymond1991}).

A key element in the effective operation of the acceleration process is the resonant excitation of scattering waves by the accelerated particle pressure gradient ahead of the accelerating shock resulting in much smaller values of the particle diffusion coefficient near the shock than in the general medium. This was first emphasised by Bell \cite{Bell1978} who gave an analysis of the process in the steady state and also pointed out the importance of ion-neutral friction in damping the waves and quenching the acceleration of high energy particles.

Another important aspect of diffusive shock acceleration is that it can only operate in conjunction with an injection mechanism, which must accelerate particles directly out of the ‘thermal’ plasma up to a velocity of several times the thermal speed. However in dense media low energy injected particles are subject to Coulomb and ionisation energy losses. The acceleration process has to be fast enough at energies somewhat above ‘thermal’ to compete with these collisional processes, otherwise the shock will only be able to accelerate the few ambient pre-existing high energy particles.

These effects obviously have important implications for observational diagnostics such as the gamma ray luminosity of supernova remnants and, in the case of non-radiative shocks, the detailed line profiles of the faint Balmer line emission. The gamma-ray emission will only be easily detected if the shock is propagating into a relatively dense medium with a number density in excess of $0.1 \text{ cm}^{-3}$ (Aharonian et al. \cite{Aharonian1994} and the Balmer diagnostics require that the medium into which the shock is propagating be only about 90\% ionised (Raymond \cite{Raymond1991}). Rather generally one can anticipate that most observational tests for cosmic ray acceleration in supernova remnants will require some significant matter density to yield a detectable signal. However too high a density may quench the acceleration and destroy the very effect one is trying to detect.

In this paper we first develop the theory of wave damping and excitation in cosmic ray shocks in more detail than has been done previously and derive corrected estimates for the energy at which this process quenches the acceleration. Then we consider the conditions under which collisional processes can suppress the natural shock injection process. Finally we discuss the implications of these results for the observability of cosmic ray acceleration effects in supernova remnants.

2. Wave excitation and damping

The model we study is identical to that considered by Bell although we use a slightly different notation. We ignore the reaction of the accelerated particles on the flow, and consider only their reaction on the scattering waves, which we take to be Alfvén waves, moving at speed $V$ along the magnetic field relative to a steady upstream flow of speed $U$ and density $\rho$ in the shock rest frame. Transmission of the waves through the shock and their damping downstream has been considered by Achterberg and Blandford \cite{Achterberg1986} and is not addressed here. For simplicity, we will consider the case of parallel shocks, in which the magnetic field lies along the normal to the shock surface.
Particles of momentum \( p \), charge \( e \) and pitch \( \mu = \cos \theta \) (where \( \theta \) is the angle between the particle’s momentum vector and the mean magnetic field direction) resonantly interact with waves of spatial wavenumber along the field \( k = 1/(\mu r_k) \) where \( r_k = p/(eB) \) is the gyroradius of the particle in the field of strength \( B \) (there is an implicit assumption here, that the waves are ‘slow’ and the particles ‘fast’). Although there is a contribution to the scattering of particles of momentum \( p \) by waves on all spatial scales smaller than the particle gyroradius, it is clear that the bulk of the scattering is from waves close to \( r_k \) in scale. In common with most discussions of this topic we assume that particles of momentum \( p \), which from now on we take to mean particles in a (natural) logarithmic interval of momentum, interact only with waves in a logarithmic interval of \( k \) space where \( k \) and \( p \) are implicitly related by \( kp = \Omega(eB) \). This is often called “sharpening the resonance.”

3. Equations of the model

The accelerated particles diffuse against the flow so that the spatial and temporal evolution of the isotropic part of the particle phase space density \( f(t, x, p) \) is given by the advection-diffusion equation

\[
\frac{\partial f}{\partial t} + (U - V) \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( \kappa \frac{\partial f}{\partial x} \right)
\]

with a diffusion coefficient \( \kappa \) which we take, from quasi-linear theory, to have the form

\[
\kappa = \frac{\kappa_B I}{T}, \quad \kappa_B = \frac{r_k U}{3}
\]

where \( I \) is the dimensionless resonant wave intensity per logarithmic wavenumber and \( \kappa_B \) is the so-called Bohm diffusion coefficient corresponding to a free mean path of order the particle gyroradius. The total wave energy density is given by

\[
\frac{\langle (B^2) \rangle}{2\mu_0} = \frac{\langle B^2 \rangle}{2\mu_0} \int I(k) \, d\ln(k).
\]

Physically all this says is that the larger the fluctuations \( \delta B \) imposed on the mean field \( B \), the stronger the scattering, until when \( \delta B \approx B \) the field is totally disordered and particle trajectories lose coherence on scales as short as the gyroradius. Note that the advection is with the waves which we assume to be moving backwards at velocity \( V \) relative to the background plasma, and thus to be moving at velocity \( U - V \) relative to the shock (which we locate at \( x = 0 \)).

In scattering off the waves the particles do work at the rate \( V \nabla P \) where \( \nabla P \) is the accelerated particle pressure gradient. The resonance sharpening simplification allows us to apply this to a particular set of resonant particles and waves to give the wave energy equation,

\[
\frac{\partial I}{\partial t} + (U - V) \frac{\partial I}{\partial x} = V \frac{\partial P}{\partial x} - \gamma I
\]

where \( I \) is the (dimensional) wave energy density, \( P \) the resonant particle pressure and \( \gamma \) a damping coefficient. It is convenient to write this in nondimensional form, and scale \( I \) to the background field energy density \( B^2/2\mu_0 \) and \( P \) to the ram pressure of the background flow of ionised plasma, \( \rho_i U^2 \), where \( \rho_i \) is the ion mass density. We let \( \mathcal{P} \) denote the dimensionless pressure per logarithmic momentum interval,

\[
\mathcal{P} = \frac{4\pi^3}{3} \frac{\rho_i U^2}{\langle (p^2) \rangle} f(p)
\]

so that the total accelerated particle pressure is

\[
\int \frac{\rho_i U^2}{3} 4\pi^3 f(p) \, dp = \rho_i U^2 \int \mathcal{P}(p) \, d\ln(p)
\]

Expressed in terms of the dimensionless quantities \( \mathcal{P} \) and \( I \) we then obtain the fundamental pair of coupled equations

\[
\frac{\partial \mathcal{P}}{\partial t} + (U - V) \frac{\partial \mathcal{P}}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\kappa_B \partial \mathcal{P}}{I} \right)
\]

and

\[
\frac{\partial I}{\partial t} + (U - V) \frac{\partial I}{\partial x} = \frac{2U^2}{V} \frac{\partial \mathcal{P}}{\partial x} - \gamma I.
\]

We assume that the main process contributing to the wave damping is ion neutral friction. If the neutrals do not move coherently with the ions, the resulting wave damping rate \( \gamma \) has the value

\[
\gamma = n_n \langle \sigma v \rangle
\]

where \( n_n \) is the neutral number density and \( \langle \sigma v \rangle \) is the mean of the collision velocity times the charge exchange cross section; Kulsrud & Cesarsky (1971) give the approximation

\[
\langle \sigma v \rangle \approx 8.4 \times 10^{-9} \left( \frac{T}{10^4 \text{ K}} \right)^{0.4} \text{ cm}^3\text{s}^{-1}
\]

for temperatures \( T \) in the range \( 10^2 \) K to \( 10^7 \) K. The assumption that the neutrals do not participate in the coherent oscillations of the ions in the Alfvén wave implies that the period of the wave must be short relative to the momentum transfer time scale from the ions to the neutrals. By assumption the wavelength is of order the gyroradius of a resonant particle, thus this condition implies

\[
\frac{r_k}{V} < \frac{1}{n_i \langle \sigma v \rangle}
\]

where \( n_i \) is the ion number density. Numerically, this condition translates, for accelerated protons of energy \( E \), to

\[
\frac{E}{1 \text{ GeV}} < 8 \left( \frac{T}{10^4 \text{ K}} \right)^{-0.4} \left( \frac{n_i}{1 \text{ cm}^{-3}} \right)^{-3/2} \left( \frac{B}{1 \mu\text{G}} \right)^2.
\]

At higher energies the resonant waves are of sufficiently long period for the neutrals to participate coherently in the wave motion. This will lead to decreased wave damping and a reduction in the effective Alfvén speed. The full dispersion relation for Alfvén waves in a partially ionised medium is discussed by Kulsrud & Pearce (1969) and also by Völk et al. (1981). However for our purposes what is required is not really the dispersion relation for freely propagating waves but the damping of driven modes. An elementary calculation of the energy dissipation in a frictionally coupled system of ions and neutrals when the ions are forced at frequency \( \omega \) gives

\[
\gamma = \frac{\omega^2}{\omega^2 + \omega_i^2} \omega_i n_i = n_{i,n} \langle \sigma v \rangle
\]
Thus, as one would expect, at high frequencies the damping is constant and equal to $\omega_i$. However as $\omega \to \omega_i$ the damping rate drops to $\omega_i^2/2$ and at lower frequencies decreases as $\omega^2$.

Because the resonant wave frequency is inversely proportional to particle momentum $p$ at sufficiently high particle energies the damping rate of the resonant waves will decrease as $p^{-2}$.

4. Parameters

Looking at this system we see that it explicitly contains only two dimensionless combinations of parameters. One is the ratio $V/U$ which is the inverse Alfvén Mach number of the shock. The other is $\gamma \kappa s/U^2$ which is essentially the ratio of the acceleration time scale to the damping time scale. However the boundary conditions on $I$ and $P$ introduce additional dimensionless parameters into the problem.

The ratio $V/U$, for astrophysical shocks, is expected to lie in the range $10^{-3}$ to 1. In fact the assumption made above, that the amplified waves are those propagating backwards relative to the flow breaks down unless $V < U$. For most of the calculations reported here we have used $V/U = 10^{-2}$.

The dimensionless wave damping parameter due to ion-neutral friction has the approximate numerical value at energies below the limit given by (12)

$$ \frac{\gamma \kappa B}{U^2} = 25.2 \left( \frac{n_n}{1 \text{ cm}^{-3}} \right) \left( \frac{T}{10^4 \text{ K}} \right)^{0.4} \left( \frac{E}{1 \text{ TeV}} \right) \left( \frac{B}{1 \mu G} \right)^{-1} \left( \frac{U}{10^2 \text{ km s}^{-1}} \right)^{-2} \tag{14} $$

where $n_n$ is the neutral number density, $T$ the temperature and $E$ the accelerated particle energy (for relativistic protons). Because the acceleration time scale increases with energy the effects of wave damping become more significant as the energy increases to the value given by (12). At higher energies the damping rate decreases faster than the acceleration rate. Thus the energy given by (12) represents a critical threshold at which the wave damping effects are strongest. If the acceleration can reach this energy, then the ion-neutral wave damping alone cannot restrict further acceleration and the upper cut-off will be determined by other factors such as shock geometry or age.

The dimensionless resonant particle pressure at the shock can be estimated by assuming that the shock acceleration is efficient, so that the total particle pressure is comparable to the ram pressure of the flow, and that the spectrum is roughly that predicted by test-particle theory for a strong shock, namely $f(p) \propto p^4$ up to a maximum momentum $p_{\text{max}}$ (which may be of order $10^{14}$ eV for supernova remnant shocks). This gives

$$ P \approx \frac{1}{\ln(p_{\text{max}}/mc)} \approx 0.1 \cdot 0.05 \tag{15} $$

as the (steady state) value at $x = 0$, which is the right hand boundary condition on $P$. In general one could have a population of particles advected into the shock from far upstream ($x \to -\infty$), corresponding to a finite far upstream value of $P$, however the physically interesting case is where all the particles are produced at the shock so that the left hand boundary condition is $P \to 0$ as $x \to -\infty$ (note that if $I$ also tends to zero this is compatible with a nonvanishing escaping flux of particles upstream).

For the resonant wave intensity, we have to specify a left hand boundary condition. One tempting possibility is to suppose that all the waves are created by the resonant particles, so that $I \to 0$ as $x \to -\infty$. However this is a rather singular limit as we will see in the next section. In general one has to suppose that some “seed” field of waves exists in the far upstream plasma and is advected into the shock where the backward propagating ones are then amplified by the resonant particles.

5. Steady analytic solutions

If we set the time derives to zero in Eqs. (1) and (8), these can be linearised by introducing a new independent variable $\tau$ similar to the “optical depth” variable used in radiative transfer:

$$ \tau \ = \ \int_{x_-}^x dx' \frac{U - V}{\kappa(x')} \tag{16} $$

$$ = \int_{x_-}^x dx' I(x') \frac{U - V}{\kappa_B} , $$

where $x_-$ is an arbitrary point in the flow. We then find

$$ (U - V) \frac{dP}{d\tau} = \frac{d}{d\tau} \left( (U - V) \frac{dP}{d\tau} \right) \tag{17} $$

and

$$ (U - V) \frac{dI}{d\tau} = \frac{2U^2}{V} \frac{dP}{d\tau} - \frac{\gamma \kappa B}{U - V} . \tag{18} $$

Equation (17) immediately yields a conserved quantity – the flux $\phi$ of particles:

$$ \phi = \frac{d}{d\tau} \left( (U - V) \frac{dP}{d\tau} \right) \tag{19} $$

and imposing the boundary condition $P = P_-$ at $x = x_-$ leads to the solution

$$ P = \frac{\phi}{U - V} + \left( P_- - \frac{\phi}{U - V} \right) e^\tau \tag{20} $$

This may be substituted into Eq. (13), leading to the solution

$$ I = I_- - \frac{\kappa B \gamma}{(U - V)^2} \tau + \frac{2U^2(P_-(U - V) - \phi)}{V(U - V)^2} (e^\tau - 1) . \tag{21} $$

Although it is in general both inconvenient and unnecessary to evaluate $P$ and $I$ as functions of $x$, in the case of no damping $\gamma = 0$, the procedure is straightforward. Defining $\alpha = 2U^2 |P_-(U - V) - \phi|/[(U - V)^2V]$ and $\beta = (\alpha - I_-)(U - V)(x - x_-)/\kappa_B$, one finds

$$ e^\tau = \frac{\alpha - I_-}{\alpha - I_- e^\beta} \tag{22} $$

so that

$$ P = \frac{\phi}{U - V} + \frac{\alpha V(U - V)(\alpha - I_-)}{2U^2(\alpha - I_- e^\beta)} \tag{23} $$

and

$$ I = I_- e^\beta \left( \frac{\alpha - I_-}{\alpha - I_- e^\beta} \right) . \tag{24} $$

These solutions diverge at a point $x_1$ given by

$$ x_1 = x_- + \kappa_B \ln(\alpha/I_-)/[(\alpha - I_-)(U - V)] , \tag{25} $$
which must therefore be chosen to lie in the downstream region. In the special case \( \alpha = 0 \), we recover the solution found by Lagage & Cesarsky (1983) by allowing \( x_+ \to -\infty \), whilst keeping \( x_- \) finite:
\[
I = I_- \left\{ 1 - \exp \left[ \frac{(U-V)I_-(x-x_+)}{\kappa B} \right] \right\}^{-1},
\]
(26)
which, in the limit \( I_- \to 0 \), reduces to the somewhat simpler solution found by Bell (1978):
\[
I = \frac{\kappa B}{(U-V)(x_1-x)}.
\]
(27)
This is also a good approximation to the more general solution Eq. (24) for \( I > I_- \). We remark at this point that the transformation Eq. (23) has eliminated the uninteresting solution \( P = \)constant, \( I = 0 \). Nevertheless, \( I_- \to 0 \) is a singular limit of the full set (4) and (6).

When damping is included, we note from Eq. (23) that there exists a point where the wave growth is just balanced by damping and \( \partial I/\partial \tau = 0 \). Following Bell (1978), we assume that at this point the particles escape freely into the upstream plasma. As usual in the diffusion approximation, we can implement this boundary condition by demanding that the density \( P \) vanish there. Let us choose \( x = x_- \) and \( \tau = 0 \) at the free escape boundary. Since the derivative of \( I \) there is by definition zero and we also demand \( P_- = 0 \), the escaping particle flux is determined independently of \( I_- \) to be:
\[
\dot{\phi} = -\frac{\kappa B V}{2U^2}.
\]
(28)
As pointed out by Bell, the escaping flux will steepen the spectrum if it is comparable to the flux advected away downstream. Thus, wave damping leads to a cut-off in the spectrum where
\[
\frac{\gamma \kappa B V}{U^2} \approx \frac{U-V}{4} P_0
\]
(29)
where \( P_0 \) is the value of \( P \) at the shock. Note however that Bell (1978), and following him Draine and McKee (1993), overestimate the escaping flux by a factor \( c/(U-V) \) where \( c \) is the speed of light in vacuo. This error resulted from the use of the solution given in Eq. (23), in which the particle flux is identically zero everywhere, as a basis for the estimate of the escaping flux.

6. Stability and time dependence

If we examine the system of equations and think about the physics involved, it is clear that the wave excitation is strongest in those regions which already have an enhanced level of wave activity, and thus smaller values of the diffusion coefficient and steeper pressure gradients. It follows that the stability of the system is rather questionable; there is a clear possibility of some form of modulational instability. It is also possible that the steady solutions are not in fact the physically realistic ones. Before examining these questions numerically, it is interesting to look at the case of small high-frequency perturbations, which can be examined analytically.

We begin by linearising about a smooth background solution, \( \bar{P} \) and \( \bar{I} \), on which is superimposed a small fluctuating component, \( \tilde{P}, \tilde{I} \). The linearised equations are
\[
\frac{D\tilde{P}}{Dt} = \frac{\partial}{\partial x} \left( \kappa_B \frac{\partial \tilde{P}}{\partial x} - \kappa_B \frac{P}{I} \frac{\partial \tilde{I}}{\partial x} \right)
\]
(30)
and
\[
\frac{D\tilde{I}}{Dt} = \frac{2U^2}{1} \frac{\partial \tilde{P}}{\partial x} - \gamma \tilde{I}
\]
(31)
where
\[
\frac{D}{Dt} = \frac{\partial}{\partial \tau} + (U-V) \frac{\partial}{\partial x}
\]
(32)
denotes the advective derivative. We now assume that the fluctuations are of high spatial and temporal frequency and make a formal expansion in inverse powers of the wavenumber \( k \),
\[
\tilde{P} = e^{i\theta(x,t)} \sum_{n=0}^{\infty} k^{-n} \tilde{P}_n
\]
(33)
\[
\tilde{I} = e^{i\theta(x,t)} \sum_{n=0}^{\infty} k^{-n} \tilde{I}_n
\]
(34)
where \( \theta(x,t) \) is the rapidly varying phase, \( \partial \theta/\partial x = -k \) and \( \partial \theta/\partial t = \omega = O(k) \). Substituting and collecting similar powers of \( k \) we obtain, to order \( k^2 \) and \( k \),
\[
\tilde{P}_0 = 0
\]
(35)
\[
\tilde{P}_1 = i \tilde{I}_0 \frac{\partial \tilde{P}}{\partial x}
\]
(36)
\[
\frac{D\tilde{I}_0}{Dt} = -i \frac{2U^2}{1} \tilde{P}_1 - \gamma \tilde{I}_0
\]
(37)

\[
\tilde{I}_2 = 0
\]
(38)
\[
\frac{D\tilde{I}_0}{Dt} = \frac{2U^2}{1} \tilde{I}_0 \frac{\partial \tilde{P}}{\partial x} - \gamma \tilde{I}_0
\]
(39)
\[
\frac{1}{\tilde{I}_0} \frac{D\tilde{I}_0}{Dt} \frac{1}{\tilde{I}_0} \frac{D\tilde{I}_0}{Dt} = \frac{1}{\tilde{I}_0} \frac{D\tilde{I}_0}{Dt} \tilde{I}_0
\]

It follows that, to lowest order, small-scale perturbations in the resonant wave intensity are advected in at velocity \( U-V \) and grow (or decay) at exactly the same rate as the general background wave intensity. While this is not an instability, it does mean that the system “remembers” all the fluctuations in the initial seed wave field and is neutrally stable; in this lowest order linear high frequency analysis the relative amplitude of perturbations neither grows nor decays. Clearly a numerical study of non-linear finite size perturbations is desirable.

There is no particular problem in developing a time-dependent numerical scheme for solving this system of equations. However some thought has first to be given to a time-dependent right hand boundary condition for \( P \). The problem arises because we are trying to look at one small part of the general process of shock acceleration in isolation. Essentially we have to model the rest of the shock acceleration process in a simple, but physically plausible way, in the boundary conditions we use. We assume that there is a steady flux, of particles into the momentum interval we are considering from accelerations we use. We assume that there is a steady flux, of particles downstream scattering, so we simply assume that there is efficient scattering downstream and that the accelerated particles fill a downstream phase space “reservoir” of size \( 4\pi p^2 L \) with
\( L \approx \kappa_b / W_2 \) where \( W_2 \) is the downstream advection velocity (we avoid discussion of whether this is the plasma velocity minus the Alfvén velocity). From this they can diffuse into the upstream region and they can be removed either by advection downstream, or by being accelerated to even higher energies.

We need an equation for the resonant particle pressure, \( P \), at the shock and in the downstream “reservoir”. The required equation is simply energy conservation as applied to the reservoir. If we denote by \( W_1 = U - V \) the advection velocity into the shock and by \( E \) the energy density in resonant particles, then the energy flux in resonant particles into the reservoir from upstream is \( W_1 (E + P) - \kappa_b \partial E / \partial x \), the flux out downstream is \( W_2 (E + P) \), the flux in from lower momenta is \( \epsilon_0 (W_1 - W_2) / 3 \) where \( \epsilon_0 \) is the energy density per logarithmic interval at lower energies and the flux out to higher energies is \( \epsilon (W_1 - W_2) / 3 \).

Thus

\[
L \frac{dE}{dt} = W_1 (E + P) - \kappa_b \frac{\partial E}{\partial x} - W_2 (E + P) - \frac{W_1 - W_2}{3} (E - \epsilon_0) \tag{40}
\]

Note that, because we are only considering particles in a specific momentum interval, this is not a two-fluid approximation.

To simplify matters we assume that \( W_2 = W_1 / 4 \) and that we are dealing with relativistic particles where \( E = 3P \). The right hand boundary condition for \( P \) then becomes simply

\[
L \frac{dP}{dt} = W_2 (P_0 + 3P) - \kappa_b \frac{\partial P}{\partial x} \tag{41}
\]

where \( P_0 \) is the value to which \( P \) tends if the solution approaches the standard steady result (note that in a steady solution \( \kappa_b \frac{\partial P}{\partial x} = 4W_2 P \)).

7. Numerical solutions

It is not hard to solve the fundamental system of equations (41) numerically. We have used a uniform grid and fixed time step with \( \Delta x / \Delta t = (U - V) \) so that the advection is exact. The diffusion is treated implicitly giving a simple tridiagonal system and the wave intensity is defined on a staggered mesh between each pair of \( P \) values. The wave damping is also treated implicitly so that the scheme is absolutely stable for all diffusion coefficients and damping rates. For more elaborate calculations there would be definite advantages in going to variable grid spacings and time steps, and perhaps a second order scheme (our scheme is only of first order). However for investigating stability there are distinct advantages in using a simple first order scheme with no advection errors.

We first demonstrate that this program can reproduce the analytic results for steady solutions. In Figure 1 we show the solution attained at time 100 (in units of \( \kappa_b / U^2 \)) starting from \( P = 0 \) and \( I = 0.1 \). The parameters have been chosen so that in Bell’s solution the right-hand value of \( I \) is unity. The left-hand boundary conditions are zero net flux for \( P \) and fixed \( I = 0.1 \) for the “seed” waves. In Figure 2 we show \( 1 / I \) over an extended range and for various upstream seed values. Note that with an upstream value of 0.01 the plot of \( 1 / I \) is an almost perfect straight line, as required by Bell’s solution, and that the solutions for other seed values tend asymptotically to this solution in the manner described by Lagage and Cesarsky’s solution.

In Figure 3 we show the effect of including modest damping on the solutions of Figure 2. As is clear physically, and obvious from the analysis of section 4, far upstream the damping dominates. However the various solutions do converge to a single asymptotic solution near the shock. This is rather easier to see in Figure 4 where the final sections of the solutions of Figure 2 are plotted in the style of Figure 1. The left-hand boundary condition for these solutions was taken to be the constant escaping flux given by equation (28). It is then easy to see that the right-hand boundary condition (41), in the steady state, implies

\[
P = P_0 - \frac{\gamma \kappa_b}{U^2} \frac{V}{2W_2}
\tag{43}
\]
Fig. 3. The reciprocal of the resonant wave intensity, $1/I$, with modest damping after time $200k_B/U^2$ in the region from $-100k_B/U$ to 0. The parameters are as in Fig. 2 except that the damping rate $\gamma = 0.1$.

which for the parameters used here gives $P = 0.003$ or $P/P_0 = 0.6$, as shown by the numerical solutions. We note that this argument gives an approximate form for the spectrum at energies below the cut-off given by equation (29).

Fig. 4. The resonant wave intensity $I$ (dashed lines) and $P/P_0$ (solid lines) near the shock for the six solutions of Fig. 3.

However it is most unlikely that the upstream “seed” field will be absolutely steady. In Fig. 5 we show the solution from Fig. 3 corresponding to a steady left-hand boundary condition of $I = 0.01$ and also that of the strongly modulated boundary condition, $I = 0.01(1 + 0.9 \sin t)$. More revealingly, in Fig. 6 we plot the ratio of these two solutions. It is remarkable that, even at this very high level of modulation, the solution behaves exactly as predicted by the analytic theory. Only very close to the shock is there some evidence of a slight increase in the perturbation amplitude. It is also noteworthy that the steady solution corresponding to the “average” value of $I$ is a good approximation to the “average” time-dependent solution. Calculations with different periods and amplitudes yield similar results; although locally very different, the large-scale structure appears to be well represented by the steady solution corresponding to the “average” value of $I$.

Fig. 5. The reciprocal of the wave intensity with $V/U = 0.01$, $\gamma = 0.1$, $P_0 = 0.005$ after time $200k_B/U^2$ with a sinusoidally modulated and a steady left-hand boundary condition for $I$. The solid curve represents the solution with $I = 0.01(1 + 0.9 \sin t)$ as boundary condition, the dashed curve that with $I = 0.01$.

Fig. 6. The ratio of the two solutions shown in Fig. 5.

In fact one has to be rather careful about exactly what sort of average is meant in the above statements. If one is spatially averaging along the particle gradient, the appropriate mean diffusion coefficient is the harmonic mean, corresponding to the simple arithmetic mean of the wave intensity. If one is averaging in the perpendicular plane, the appropriate mean is the arithmetic mean of the diffusion coefficient. More generally one has to consider the geometry of the fluctuations as in percolation theory. It may be helpful to think of the electrical analogy of resistance networks connected in series and in parallel. To some extent, the fact that the arithmetic mean of the wave intensity generates a steady solution which is a very
good approximation to the “average” non-steady solution, is an artifact of the strictly one-dimensional model we are using.

Although many extension or generalisations of diffusive shock acceleration have been considered, as far as we are aware there has been no work on shock acceleration with time-dependent scattering. If one considers the standard microscopic arguments leading to the steady-state spectrum for test-particle acceleration, it is clear that the time-averaged spectrum at the shock will be the same power-law spectrum, independent of the details of the time-dependent scattering, as long as three conditions are met. All particles heading upstream must return to the shock, which will be the case for negligible wave damping or if there exists sufficient upstream wave power. The angular distribution of the accelerated particles at the shock front must be close to isotropic, which is generally required for the diffusive description to be applicable. And, crucially, the mean flux advected downstream must equal the downstream advection velocity times the mean density at the shock. This last assumption appears open to question although it is in fact used in our approximate right-hand boundary condition. This may be an interesting way to produce non-standard spectral indices.

8. Summary of wave damping effects

In the preceding sections we have generalised the analytic theory of steady resonant wave excitation to include wave damping and obtained an analytic formula for the resulting escaping particle flux. In reality the solutions will be time-dependent, but analytic arguments and numerical experiments show that “on average” the steady solutions can still give a good representation of the system (although this would probably change if more complicated non-linear processes were considered).

The limit we derive for the maximum energy to which particles can be accelerated before the escaping upstream flux kills the acceleration differs substantially from that quoted in Draine and McKee (1993). Their formula (4.13) gives

\[ E_{\text{acc}} < 3 \times 10^{-3} \frac{v_\text{sh}^4 x_1^{1/2} P_0}{(1 - x_1) n_i^{1/2} T_4^{0.4}} \]  

(44)

where in their notation \( E_{\text{acc}} \) is the upper cut-off energy for a proton in units of GeV, \( v_\text{sh} \) is the shock speed in units of \( 10^3 \text{ cm s}^{-1} \), \( n_i \) is the total number density and \( x_1 \) the ionisation fraction. Note that their dimensionless particle pressure parameter \( \phi \) is related to ours by

\[ \phi = x_1 \int_{\rho_0} \rho_0 d \ln \rho \]  

(45)

Our result (44) says that if the upper cut-off energy \( E \) is determined by the ion-neutral wave damping in the upstream region, then numerically

\[ \frac{E}{\text{TeV}} < \left( \frac{U}{10^9 \text{ km s}^{-1}} \right)^3 \left( \frac{T}{10^4 \text{ K}} \right)^{-0.4} \times \left( \frac{n_i}{1 \text{ cm}^{-3}} \right)^{-1} \left( \frac{n_0}{1 \text{ cm}^{-3}} \right)^{0.5} \left( \frac{P_0}{0.1} \right) \]  

(46)

where \( n_i \) is the neutral number density and \( n_0 \) the ion density in the medium into which the shock is propagating. Our limit is a factor \( c/v_\text{sh} = 3 \times 10^8 / v_\text{sh} \) times higher. In fact, as discussed above, the damping will become unimportant above energies given by (42). The condition that the upper cut-off energy exceed that at which the neutrals start to coherently move with the ions is

\[ \frac{P_0}{0.1} \left( \frac{U}{10^9 \text{ km s}^{-1}} \right)^3 > 8 \times 10^{-3} \left( \frac{n_0}{1 \text{ cm}^{-3}} \right) \times \left( \frac{n_i}{1 \text{ cm}^{-3}} \right)^{-2} \left( \frac{B}{1 \mu G} \right)^2. \]  

(47)

If this condition is satisfied upstream ion-neutral wave damping places no restriction on shock acceleration.

9. Collisional losses near injection

At low energy, the most important loss processes for cosmic rays are ionisation and Coulomb losses. If the theory of diffusive acceleration at shocks is to be viable, it must overcome these for all energies above that of injection. The constraints that this imposes on the density, temperature and magnetic field of the ambient medium as well as on the energy of injection can be estimated simply by comparing the loss-rates with the acceleration rate.

Standard treatments of diffusive acceleration give the acceleration rate in terms of the rate of change of momentum,

\[ \dot{p} = \frac{(u_1 - u_2)}{3 \kappa_1 u_1 + \kappa_2 u_2}, \]  

(48)

where the subscript refers to the upstream (1) or downstream (2) medium, \( \kappa \) is the diffusion coefficient and \( u \) the fluid speed. However loss rates are normally discussed in terms of the particle kinetic energy; defining the energy acceleration rate \( t_\text{acc}^{-1} \), as the rate of change of kinetic energy divided by the kinetic energy we get

\[ t_\text{acc}^{-1} = \frac{u_1^2 \gamma \beta^2}{\kappa_1 (\gamma - 1) r[1 + (r \kappa_2 / \kappa_1)]}, \]  

(49)

where \( r = u_1 / u_2 \) is the compression ratio of the shock, \( \beta e \) is the speed of the particle and the Lorentz factor is \( \gamma = (1 - \beta^2)^{-1/2} \). The diffusion coefficient can be conveniently parameterised in terms of the Bohm value in a magnetic field \( B \):

\[ \kappa_1 = k_1 \kappa_B \]  

(50)

\[ \kappa_2 = k_1 \beta^2 \gamma m_\mu c^2 \]  

(51)

leading to

\[ t_\text{acc}^{-1} = 1.1 \times 10^{-7} k_1^{-1} \left( \frac{u_1}{1000 \text{ km s}^{-1}} \right)^2 \left( \frac{B}{1 \mu G} \right) \times \left[ \frac{(r - 1)}{r[1 + (r \kappa_2 / \kappa_1)]} \right] \left( \frac{\gamma - 1}{\gamma} \right)^{1 - 1}. \]  

(52)

At non-relativistic energies the acceleration rate is proportional to \( \beta^{-2} \).

Expressions for the loss processes have been given by Mannheim & Schlickeiser 1991. For Coulomb collisions with free electrons of temperature \( T = T_4 \times 10^4 \text{ K} \) and density \( x_i n_H \text{ cm}^{-3} \), where \( x_i \) is the ionisation fraction and \( n_H \) the total (ionised, neutral and molecular) density of hydrogen atoms, one finds

\[ t_C^{-1} = 3.3 \times 10^{-16} x_i n_H \frac{\beta^2}{(\gamma - 1)(\beta_{th}^2 + \beta^2)} \text{ s}^{-1}, \]  

(53)
where $\beta_{i,\text{e}}$ is the thermal speed of the electrons: $\beta_{i,\text{e}} = 2 \times 10^{-3} \gamma^{1/2}$. A similar expression is valid for losses due to the ionisation of neutral material. Using a composition appropriate to the interstellar medium, Mannheim & Schlickeiser (1994) find

$$t_{\text{acc}}^{-1} \approx 3.9 \times 10^{-16} (1 - x_i) n_{\text{H}} \left( \frac{\beta^2}{(\gamma - 1)(\beta_0^3 + 2\beta^3)} \right) s^{-1},$$

where $\beta_0$ is the speed of an electron in the ground state of the hydrogen atom: $\beta_0 \approx 0.01$.

The Coulomb and ionisation losses are roughly constant for particles of speed below $\beta_{i,\text{e}}$ and $\beta_{0\text{e}}$ respectively and then fall as $\beta^{-3}$ at higher velocities, see Fig. 7. Comparing this behaviour with that of $t_{\text{acc}}^{-1}$ it is clear that the ionisation and Coulomb losses will not suppress the shock acceleration at any energy if the acceleration exceeds the loss rate for particles of speed $\beta_{i,\text{e}}$ and $\beta_{0\text{e}}$ respectively. This is the case if

$$k_{\text{i}}^{-1} \left( \frac{u_1}{10^3 \text{km s}^{-1}} \right)^2 \left( \frac{B}{1 \mu\text{G}} \right) \frac{1}{n_{\text{H}}} \gg 10^{-6} \text{Max} \left[ x T_4^{-1/2}, (1 - x_i) \right].$$

Provided particles are injected at a speed several times the ion thermal velocity, so that the distribution function can be approximately isotropic in both the upstream and downstream frames (Kirk & Schneider 1988, Malkov & Völk 1992) and the self-excited waves are not damped by thermal ions, we can expect that $k_1 \gtrsim 1$. In this case, the constraint is not restrictive for the interstellar medium.

10. Conclusions and implications for observations

With respect to observational tests of shock acceleration in supernova remnants, the limit imposed by upstream wave damping on the maximum particle energies is significant, but not serious, for the observability of supernova remnants in gamma-rays using the atmospheric Cherenkov technique. The limit can of course be circumvented by locating the particle acceleration and the gamma-ray production target in different regions or phases. In a clumpy medium, one could imagine accelerating in the low-density interclump phase with the dense clumps behind the shock providing the target material. Or one can consider the possibility of a SNR exploding near a molecular cloud (Aharonian et al. 1994). In addition one should consider the pre-ionisation of the upstream medium by the soft X-ray and UV radiation from behind the shock, an effect which will also reduce the effect of ion-neutral damping.

One other possibility for obtaining observational evidence for shock acceleration in SNRs is the use of Balmer diagnostics in non-radiative shocks (Raymond 1991, Smith et al. 1994) to probe the structure of the cosmic ray precursor. This technique essentially determines the upstream plasma temperature and velocity averaged over the ion-neutral charge exchange length

$$L_{\text{exch}} = \frac{U_{\text{i}}}{n_i (\sigma v)}$$

which is numerically

$$4 \times 10^{-3} \left( \frac{U}{10^3 \text{km s}^{-1}} \right) \left( \frac{n_i}{1 \text{cm}^{-3}} \right)^{-1} \left( \frac{T}{10^4 \text{K}} \right)^{-0.4} \text{ pc}.$$ (57)

This has to be compared with the characteristic scale associated with particles of energy $E$, $\kappa_{\text{H}}(E)/U$. The two are equal at

$$\left( \frac{E}{1 \text{TeV}} \right) = 0.04 \left( \frac{U}{10^3 \text{km s}^{-1}} \right)^2 \left( \frac{n_i}{1 \text{cm}^{-3}} \right)^{-1} \times \left( \frac{T}{10^4 \text{K}} \right)^{-0.4} \left( \frac{B}{1 \mu\text{G}} \right).$$ (58)

The balmer diagnostic technique can in principle detect that part of the precursor structure produced by particles above this energy.

In conclusion, the main result of this paper is that a more detailed analysis of the constraints on particle acceleration in dense media does not support the pessimistic view expressed by Draine and McKee (1993): “shocks that are optically observable, either as nonradiative shocks in a partially neutral medium or as radiative shocks, are generally unable to accelerate particles to extremely relativistic energies”. As far as the constraints imposed by upstream ion-neutral wave damping, and ionisation and Coulomb losses are concerned optically visible SNR shocks should be capable of accelerating cosmic rays either to the revised cutoff (49), or, if the condition (47) is satisfied to the maximum energy allowed by the shock geometry and age.
Acknowledgement

This work was supported by the Commission of the European Communities under HCM contract ERBCHRXCT940604.

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