A New Approach For Weighted Hardy’s Operator In VELS

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Abstract

A considerable number of research has been carried out on the generalized Lebesgue spaces $L^{p(x)}$ and boundedness of different integral operators therein. In this study, a new approach for weighted increasing near the origin and decreasing near infinity exponent function that provides a boundedness of the Hardy’s operator in variable exponent space is given.

Keywords: Hardy inequality, Variable exponent, Boundedness.
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1 Introduction

Variable exponent studies have been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with a non-standard growth condition, we refer to monographs [7, 11, 12, 13, 15, 18, 21, 23, 29, 30, 31]. The boundedness problems for weighted Hardy’s operator in variable exponent Lebesgue spaces $L^{p(x)}$ are well studied, we refer to monographs [4, 9, 14, 16, 17, 19, 20, 22, 24, 26, 27, 28]. In this connection, a necessary and sufficient condition that assumes a log-regularity of exponent function near origin and infinity has been proved in [1, 5, 6, 8, 10, 17, 23]. For a compactness problem of main integral operators in variable exponent Lebesgue space, we refer to [2, 3, 25] and especially for Hardy’s operator [4].

In this paper, we establish an integral-type necessary and sufficiency condition on a monotone weighted exponent function $p : (0, \infty) \to (1, \infty)$ governing the boundedness

$$\left\| \frac{1}{x} v(x) \int_0^x f(t) w(t) dt \right\|_{L^{p(x)}(0, \infty)} \leq C \left\| f \right\|_{L^{p(x)}(0, \infty)}$$

(1)

of Hardy’s operator $\frac{1}{x} v(x) \int_0^x f(t) w(t) dt$ in $L^{p(x)}(0, \infty)$.

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The following main result has been obtained in this study.

**Theorem 1.1.** Let \( p : (0, \infty) \to (1, \infty) \) be a measurable function monotony increasing on some neighborhood of origin \((0, \varepsilon)\) and decreasing on some neighborhood of infinity \((N, \infty)\) such that \( p^+ < \infty \) and \( p(\infty) > 1 \). Then it holds the inequality

\[
\left\| \frac{1}{x} v(x) \int_0^x f(t)w(t) \, dt \right\|_{L_p^p(0, \infty)} \leq C_1 \| f \|_{L_p^p(0, \infty)}
\]

for any positive measurable function \( f(x) \) on \((0, \infty)\) with a positive constant \( C_1 \) depending on constant \( C_2 \) below and \( p^+, p(\infty) \) if and only if the condition

\[
\int_b^\infty x^{-\frac{1}{p(x)}} \, dx \leq C_2 b^{-\frac{1}{p(\infty)}}, \quad b > 0.
\]  

(2)

is satisfied.

We use the following notation. By \( C, C_i \) we denote a positive constant depending on \( p(\infty), p^+ \) and \( C_2 \) from the conditions (2). We use also notation \( p^+ = \sup \{ p(x) : x \in (0, \infty) \} \) and \( p^- = \inf \{ p(x) : x \in (0, \infty) \} \). Recall the norm in variable exponent Lebesgue space \( L_p(0, \infty) \) given as \( \| f \| = \inf \left\{ \lambda > 0 : \| f \|_{L_p^\lambda(0, \infty)} \leq 1 \right\} \) makes it as a Banach space, where the modular \( L_p(0, \infty) (f) = \int_0^\infty |f(t)|^p(t) \, dt \). Denote \( \| f \|_{L_p^\lambda(0, \infty)} \) or \( \| f \|_{L_p^\lambda(0, \infty)} \) for the \( L_p^\lambda(0, \infty) \) variable exponent Lebesgue norm of function \( f \). For a function \( p : [0, \infty) \to [1, \infty) \) denote \( p'(x) \) the function satisfying \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \) and \( p' = \infty \) if \( p = 1 \). Denote by \( \chi_E \) the characteristic function of set \( E \subset R \). The weight functions \( v \) and \( w \) are assumed to be measurable and having non-negative finite values almost everywhere in \((0, \infty)\).

## 2 Auxiliary Statements

In this section, we prove some auxiliary assertions in order to prove the main result of this paper.

**Lemma 2.1.** Let the condition (2) be satisfied for a monotone increasing near the origin on \((0, \varepsilon)\) and decreasing near infinity on \((N, \infty)\) function \( p : (0, \infty) \to (1, \infty) \) such that \( p^+ < \infty \). Then there exists a positive constant \( C_3 \) depending on \( C_2, p^+ \) such that

\[
[p(2b) - p(b)] \ln \frac{1}{b} \leq C_3 \text{ for } b \in (0, \delta)
\]

(3)

and

\[
[p(b) - p(2b)] \ln b \leq C_4 \text{ for } b \in (N, \infty).
\]

(4)

**Proof.** The proof for \( b \in (0, 1) \) similar to those in (see [10] the Lemma 4.1 therein). For the completeness of presentation, we consider here the both cases \( b \in (0, \varepsilon) \) and \( b \in (N, \infty) \).

Let the case \( b \in (0, 1) \) be considered. Write the condition (2) over interval \((0, 1)\) :

\[
\int_b^1 x^{-\frac{1}{p(x)}} \, dx \leq C_2 b^{-\frac{1}{p(\infty)}}, \quad 0 < b < \frac{1}{4}.
\]

By monotony increasing of \( p, \quad x^{-\frac{1}{p(x)}} \geq (4b)^{-\frac{1}{p(2b)}} \) for \( x \in (2b, 4b) \), where \( 0 < b < \frac{1}{4} \). From this, it follows that

\[
C_2 b^{-\frac{1}{p(\infty)}} \geq \int_b^{2b} x^{-\frac{1}{p(x)}} \, dx \geq (4b)^{-\frac{1}{p'(2b)}} \ln 2.
\]
Therefore, for $0 < b < \frac{1}{4}$, we have

$$
\left( \frac{1}{b} \right)^{\frac{1}{p(b)} - \frac{1}{2}} \leq \frac{C_2}{\ln 2} 4^{\frac{1}{p(2b)}} \leq 4C_2 = C_3.
$$

This proves (3).

Let $b > N$ and $p(x)$ decreases near infinity, say for $x > M$. Then $x^{-\frac{1}{p(x)}} \geq x^{-\frac{1}{p(2b)}}$ for $x \in (b, 2b)$ and $b > N$. Therefore, from condition (2) over $(N, \infty)$, it follows that,

$$
\int_b^{2b} x^{-\frac{1}{p(2b)}} \, dx \leq C_2b^{-\frac{1}{p(b)}}, \quad b > N.
$$

Then

$$
C_2b^{-\frac{1}{p(b)}} \geq \int_b^{2b} x^{-\frac{1}{p(2b)}} \, dx \geq \int_b^{2b} x^{-\frac{1}{p(2b)}} \, dx \geq a^{-\frac{1}{p(2b)}} \ln 2.
$$

and a decreasing of $p$ over $(b, 2b)$ provides

$$
x^{-\frac{1}{p(2b)}} \geq 2^{-\frac{1}{p(2b)}} b^{-\frac{1}{p(2b)}} \geq b^{-\frac{1}{p(2b)}} \ln 2.
$$

Therefore, for $b > N$, we have

$$
b^{\frac{p(N)}{p(2b)}} \leq \frac{C_2}{\ln 2} = C_5 \quad \text{or} \quad b^{p(b) - p(2b)} \leq C_6.
$$

Whence,

$$
b^{p(b) - p(2b)} \leq C_6, \quad b > N
$$

where $C_6 = C_5^{p(N)^2}$. Therefore (4) satisfied by a constant $C_3 = (p^+)^2 \ln C_5$. This proves Lemma 2.1

**Lemma 2.2.** Let the condition (2) be satisfied for a monotone increasing near origin on $(0, \epsilon)$ and decreasing near infinity on $(N, \infty)$ function $p : (0, \infty) \rightarrow (1, \infty)$ such that $p^+ < \infty$. Then there exists an $\delta > 0$ depending on $C_2, p^+$ such that the function $x^{\delta - \frac{1}{p(\epsilon)}}$ is almost decreasing near origin $(0, \epsilon)$ and infinity $(N, \infty)$.

**Proof.** Set $g(x) = \int_0^x \frac{1}{p(t)} \, dt$, $x > 0$. From (2), it follows that

$$
g(x) \leq -C_2 x g'(x), \quad 0 < x < \epsilon.
$$

First, integrating this inequality over $(t_1, t_2)$ with $0 < t_1 < t_2 < \epsilon$, we get

$$
t_2^{\frac{1}{p(\epsilon)}} g(t_2) \leq C_2 t_1^{\frac{1}{p(\epsilon)}} + \frac{1}{t_2^{\frac{1}{p(\epsilon)}}}.
$$

(5)

On the other hand, using the conditions (2) and (3) for $0 < t_2 < \epsilon$, we get

$$
g(t_2) \geq \frac{2\epsilon}{t_2} x^{-\frac{1}{p(\epsilon)}} \, dx \geq \left( \frac{1}{2} \right)^{\frac{1}{p(\epsilon)}} \ln 2 \left( \frac{1}{t_2} \right)^{\frac{1}{p(\epsilon)}} \geq \frac{\ln 2}{2} t_2^{-\frac{1}{p(\epsilon)}}.
$$
Inserting this in (5) for all $0 < t_1 < t_2 < \varepsilon$ we get
\[
t_2^{-\frac{1}{p(t_2)} + \frac{1}{t_2}} \leq \frac{2C_2}{\ln^2 t_1} - \frac{1}{p(t_1)} + \frac{1}{t_1}.
\]

Now, let $N < t_1 < t_2 < \infty$. We will show that again (5) holds for all $N < t_1 < t_2 < \infty$. From (3), it follows that
\[
g(x) \leq -C_2 x g'(x), \quad N < x < \infty.
\]
Integrating this inequality over $(t_1, t_2)$ for $N < t_1 < t_2 < \infty$ from (2), it follows that
\[
g(x) \leq -C_2 x g'(x), \quad N < x < \infty.
\]
Integrating this inequality over $(t_1, t_2)$, for $N < t_1 < t_2 < \infty$, we get
\[
t_2^{-\frac{1}{p(t_2)} + \frac{1}{t_2}} g(t_2) \leq C_2 t_2^{-\frac{1}{p(t_2)} + \frac{1}{t_2}}.
\] (6)

On the other hand, using conditions (2) and (4), we get
\[
g(t_2) \geq \int_{t_1}^{t_2} x^{-\frac{1}{p(x)}} dx \geq \ln 2 \left(\frac{1}{2t_2}\right)^{\frac{1}{p(t_2)}} \geq 2^{-\frac{1}{p(t_2)}} \ln 2 \ t_2^{-\frac{1}{p(t_2)}}, \quad t_2 \geq N.
\]
Inserting this in (6), we obtain,
\[
t_2^{-\frac{1}{p(t_2)} + \frac{1}{t_2}} \leq \frac{2C_2}{\ln^2 t_1} - \frac{1}{p(t_1)} + \frac{1}{t_1}
\]
for all $N < t_1 < t_2 < \infty$. Whence, the function $x^{\frac{\delta}{p(x)}}$ is almost decreasing on $(N, \infty)$. Therefore, we have proved that
\[
t_2^{-\frac{1}{p(t_2)}} + \delta \leq C_7 t_1^{-\frac{1}{p(t_1)} + \delta}
\]
for all $0 < t_1 < t_2 < \varepsilon$ and $N < t_1 < t_2 < \infty$. The constants $C_7 = \frac{2C_2}{\ln^2}, \quad \delta = \frac{1}{t_2}$.

This proves Lemma 2.2

**Lemma 2.3.** Let the condition (2) be satisfied for a monotone increasing near origin on $(0, \varepsilon)$ and decreasing near infinity on $(N, \infty)$ function $p : (0, \infty) \rightarrow (1, \infty)$ such that $p^2 < \infty$. Then for $0 < x < \varepsilon$ or for $x > N$ and $t \in (2^{-n-1}x, 2^{-n}x)$, the following inequality holds:
\[
x^{-\frac{1}{p(x)}} \leq C_7 2^{-n \delta} t^{-\frac{1}{p(t)}}.
\]

**Proof.** By applying Lemma 2.2 for $2^{-n-1}x < t < 2^{-n}x$, $0 < x < \varepsilon$ or $x > N$, we have
\[
x^{\frac{\delta}{p(x)}} \leq C_7 \ t^{-\frac{1}{p(t)}},
\]
since in both cases $t < x$. Indeed, using that $2^{-n-1}x < t < 2^{-n}x$ and Lemma 2.2, we get
\[
x^{-\frac{1}{p(x)}} \leq C_7 \left\{\frac{t}{x}\right\}^{\frac{\delta}{p(x)}} t^{-\frac{1}{p(t)}} \leq C_7 2^{-n \delta} t^{-\frac{1}{p(t)}}
\]
or
\[
x^{-\frac{1}{p(x)}} \leq C_7 2^{-n \delta} t^{-\frac{1}{p(t)}} \quad \text{for} \quad 0 < x < \varepsilon \quad \text{or} \quad x > N.
\]
This completes the proof of Lemma 2.3

By applying preceding lemmas, we easily get the following assertion.
Lemma 2.4. Let the condition (2) be satisfied for a monotone increasing near origin on \((0, \varepsilon)\) and decreasing near infinity on \((N, \infty)\) function \(p : (0, \infty) \to (1, \infty)\) such that \(p^+ < \infty\). Then for \(0 < x < \varepsilon\) or \(x > N\) and \(t \in (2^{-n-1}x, 2^{-n}x)\), the following estimate holds

\[
(2^{-n}x)^{\frac{1}{p(x)}} \leq C_8 t^{\frac{1}{p(0)}},
\]

where \(p_{x,n} = \inf\{p(s) : s \in (2^{-n-1}x, 2^{-n}x)\}\).

Proof. Using the assertions of Lemma 2.1 and condition (2), we get

\[
(2^{-n}x)^{\frac{1}{p(x)}} \leq (2rt)^{\frac{1}{p(t)}} \leq 2r^{\frac{1}{p(t)}}
\]

\[
\leq 2t^{\frac{1}{p(0)}} r^{\frac{1}{p(0)}} \leq 2t^{\frac{1}{p(0)}} r^{\frac{p_{x,n} - p(t)}{p_{x,n}(0)}} \leq 2t^{\frac{1}{p(0)}} \left(\frac{p(t) - p_{x,n}}{p_{x,n}(0)}\right) \leq C_8 t^{\frac{1}{p(0)}}.
\]

3 Proof of Theorem 1.1

3.1 Sufficiency

Let the condition (2) be satisfied. We will show that inequality (1) holds. Take a measurable positive function \(f\) with

\[
\|f\|_{p(.)} \leq 1.
\]

In order to prove sufficiency, we have to show that \(\|\frac{1}{x} v(x) \int_0^x f(t)w(t)dt\|_{p(.)} \leq 1\) or the same is to show that \(I_{p(.)} (\frac{1}{x} v(x) \int_0^x f(t)w(t)dt) \leq 1\) (see, e.g. in [10]).

Using Minkowski’s inequality for \(p(.)\)-norms we have

\[
\left\|\frac{1}{x} v(x) \int_0^x f(t)w(t)dt\right\|_{p(.)[0,\infty)} \leq \left\|\frac{1}{x} v(x) \int_0^x f(t)w(t)dt\right\|_{p(.)[0,N)} + \left\|\frac{1}{x} v(x) \int_0^x f(t)w(t)dt\right\|_{p(.)[N,\infty)}
\]

For the first summand it follows from the results of the paper [10] (for \(\alpha = 0\) therein) that it may be estimated as

\[
\left\|\frac{1}{x} v(x) \int_0^x f(t)w(t)dt\right\|_{p(.)[0,N)} \leq C_1 \|f\|_{p;(0,N)} \leq 1
\]

since an integration interval \((0,N)\) is finite in this part and condition (2) is satisfied.

Now, we pass to an estimation for a second summand in (9), i.e. we get an estimate for the term \(i = \left\|x^{-1} \int_0^x f(t)dt\right\|_{L^{p(.)}[N,\infty)}\). By using Minkowski’s inequality for \(p(.)\)-norms, it follows that,

\[
\left\|x^{-1} v(x) \int_0^x f(t)w(t)dt\right\|_{L^{p(.)}[N,\infty)} = \left\|x^{-1} v(x) \int_0^x f(t) \chi_{\{t>N\}} (t)w(t)dt\right\|_{L^{p(.)}[N,\infty)}
\]

\[
+ \left\|x^{-1} v(x) \int_0^N f(t)w(t)dt\right\|_{L^{p(.)}[N,\infty)}.
\]
Since \( p^- > 1 \) and \( \|f\|_{p^-} \leq 1 \) it follows that

\[
\int_0^N f(t) dt \leq \|f\|_{p^-} \|\chi_{\{0 < t < N\}}\|_{p^-} \leq \|\chi_{\{0 < t < N\}}\|_{p^-} \leq N^{\frac{1}{p^-}}.
\]

Whence

\[
\left\| x^{-1}v(x) \int_0^N f(t)w(t) dt \right\|_{L^{p^-}(N,\infty)} \leq \|\chi_{\{0 < t < N\}}\|_{p^-} \|x^{-1}\|_{L^{p^-}(N,\infty)} \leq C_0(N, p^+, p(\infty))
\]

(11)

since \( p(\infty) > 1 \),

\[
I_{p^-}\left(r^{-1}f_{N,(\infty)}(t)\right) = \int_N^\infty \frac{dt}{t^{p^-}} \leq \int_N^\infty \frac{dt}{t^{p(\infty)}} = \frac{N^{1-p(\infty)}}{p(\infty) - 1}
\]

and therefore,

\[
\left\| x^{-1}\right\|_{L^{p^-}(\infty)} \leq N^{1-p(\infty)}(p(\infty) - 1)^{-\frac{1}{p(\infty)}}.
\]

Thus, in order to get an estimation for \( i \) it suffices to estimate

\[
\left\| x^{-1}v(t) \int_0^x f(u)\chi_{\{t > N\}}(t)w(t) dt \right\|_{L^{p^-}(N,\infty)}.
\]

We have

\[
\left\| x^{-1}v(x) \int_0^x f(t)w(t) dt \right\|_{L^{p^-}(N,\infty)} \leq \left\| x^{-1} \sum_{n=0}^\infty v(x) \int_2^{-n-1} f(t)\chi_{\{t > N\}}(t)w(t) dt \right\|_{L^{p^-}(N,\infty)}
\]

(12)

\[
\leq \sum_{n=0}^\infty \left\| x^{-1}v(x) \int_2^{-n-1} f(t)\chi_{\{t > N\}}(t)w(t) dt \right\|_{L^{p^-}(N,\infty)}.
\]

We shall estimate every summand on the right hand side. By using Lemma 2.3 for \( N < t < x < \infty \), it follows that

\[
x^{-\frac{1}{p(t)}} \leq C_7 2^{-n\delta} x^{-\frac{1}{p(t)}}.\]

Therefore

\[
x^{-\frac{1}{p(t)}} x^{-\frac{1}{p(t)}} \int_2^{-n-1} f(t)\chi_{\{t > N\}}(t) dt
\]

\[
\leq C_7 2^{-n\delta} x^{-\frac{1}{p(t)}} \int_2^{-n-1} f(t)\chi_{\{t > N\}}(t) dt.
\]

By using Hölder’s inequality for \( p(x) \)-norms and assumption (8), we get

\[
\int_2^{-n-1} f(t)\chi_{\{t > N\}}(t) dt
\]

\[
\leq C_0 \|f(t)\chi_{\{t > N\}}(t)\|_{L^{p^-}(2^{-n-1}, 2^{-n})} \times \|\chi_{\{t > N\}}(t)\|_{L^{p^-}(2^{-n-1}, 2^{-n})}
\]
\[ \leq \left\| \chi_{\{t>N\}}(\cdot) \right\|_{L^{p'(1)}(2^{-n-1}x,2^{-n}x)}. \]

**Lemma 3.1.** There exists a positive constant \( C_9 > 1 \) depending on \( C_2, p^+, p(\infty) \), such that
\[ \left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x,2^{-n}x)}(\cdot) \right\|_{L^{p'(1)}} \leq C_9 y^{\frac{1}{p(\infty)}}, \quad \forall y \in (2^{-n-1}x,2^{-n}x) \quad (13) \]
for \( x > N \).

**Proof.** We prove estimation (13) from the opposite. Let inequality (13) be violated. We show that a contradiction occurs. Let
\[ \left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x,2^{-n}x)}(\cdot) \right\|_{L^{p'(1)}} > C_9 y^{\frac{1}{p(\infty)}}. \quad (14) \]
From the definition of \( p'(\cdot) \)-norms, it follows that for any sufficiently small \( \delta > 0 \) the inequality holds
\[ \int_{2^{-n-1}x}^{2^{-n}x} \left( \frac{1}{\left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x,2^{-n}x)}(\cdot) \right\|_{L^{p'(1)}} - \delta} \right)^{p'(t)} dt > 1. \quad (15) \]
Indeed, if (15) does not hold, we get a contradiction, that is,
\[ \left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x,2^{-n}x)}(\cdot) \right\|_{L^{p'(1)}} \]
is a list number satisfying
\[ \int_{2^{-n-1}x}^{2^{-n}x} \left( \frac{1}{x} \right)^{p'(t)} dt \leq 1. \]
We set \( \delta = \frac{1}{2} \left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x,2^{-n}x)}(\cdot) \right\|_{L^{p'(1)}} \) in (15), to get
\[ \int_{2^{-n-1}x}^{2^{-n}x} \left( \frac{2}{\left\| \chi_{\{t>N\}}(\cdot) \chi_{(2^{-n-1}x,2^{-n}x)}(\cdot) \right\|_{L^{p'(1)}}} \right)^{p'(t)} dt > 1. \]
By applying here (14), it follows that
\[ \int_{2^{-n-1}x}^{2^{-n}x} \left( \frac{2}{C_9 y^{\frac{1}{p(\infty)}}} \right)^{p'(t)} dt > 1, \quad \forall y \in (2^{-n-1}x,2^{-n}x), \quad x > N. \]
For any points \( t,y \) lying in \( (2^{-n-1}x,2^{-n}x) \) it follows from Lemma 2.1 that \( y^{\frac{1}{p'(t)}} \) is comparable with \( t^{\frac{1}{p(\infty)}} \). This means, there exist two positive constants \( C_{10}, C_{11} \) depending on \( C_2, p^+ \) such that
\[ C_{10} t^{\frac{1}{p'(t)}} \leq y^{\frac{1}{p'(t)}} \leq C_{11} t^{\frac{1}{p'(t)}}. \]
Therefore,
\[ \int_{2^{-n-1}x}^{2^{-n}x} \left( \frac{2}{C_{10} C_{11} t^{\frac{1}{p(\infty)}}} \right)^{p'(t)} dt > 1, \quad x > N. \]
In order to carry out it, we get an estimation for the proper modular. By choosing constant $C_9$ sufficiently large, we get a contradiction with (16). This proves estimate (13). Thus we have proved that

$$\int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) dt \leq C_9 y^{\frac{1}{p(x)}} \quad \forall y \in (2^{-n-1}x, 2^{-n}x), \quad x > N.$$  \hspace{1cm} (17)

This proves Lemma 3.1.

Now, we shall derive an estimate for

$$\left\| \frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right\|_{L^p(N, \infty)}.$$  

In order to carry out it, we get an estimation for the proper modular

$$I_{p(x)}(N, \infty) \left( \frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right) = I_{p(x)}(N, \infty) \left( \chi_{\{t>M\}}(t) w(t) dt \right).$$  \hspace{1cm} (18)

It follows from Lemma 2.3 that

$$x^{-\frac{1}{p(x)}} \leq C_7 2^{-n\delta} t^{-\frac{1}{p(y)}}$$

for $\forall y, t \in (2^{-n-1}x, 2^{-n}x)$ and $x > N$. By using the last estimate the expression (18) is exceeded

$$\int_{N}^{\infty} \left( C_7 2^{-n\delta} y^{-\frac{1}{p(y)}} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right)^{p(x)} \frac{dx}{x}$$

$$= \int_{N}^{\infty} C_7^{p(x)} 2^{-n\delta} y^{-\frac{p(x)}{p(y)}} \left( \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{\{t>N\}}(t) w(t) dt \right)^{p(x)} \frac{dx}{x}$$

By applying here Lemma 2.1, we get that, the term $y^{-\frac{p(x)}{p(y)}}$ is comparable with $t^{-\frac{p(x)}{p(y)}}$ in the integral terms, i.e.

$$\left( y^{-\frac{1}{p(y)}} \cdot t^{-\frac{1}{p(y)}} \right)^{p^+} \leq C_{12} \quad \text{for all } t, y \text{ lying in } (2^{-n-1}x, 2^{-n}x).$$
By a use of (13) a parentheses term in the preceding integral is less than 1. By decreasing the power $p(x)$ on the power of parentheses to $p_{x,n}$, we will increase the fraction. Then the last expression is less then

$$
C_{12}2^{-n\delta p} \int_{N}^{\infty} \left( \frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{[t>N]}(t)w(t)dt}{C_M^{p_M}} \right)^{\frac{p_{x,n}}{p_{x,n}}} \frac{dx}{x} \leq C_{12}2^{-n\delta p} \int_{N}^{\infty} \left( \frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{[t>N]}(t)w(t)dt}{C_M^{p_M}} \right)^{\frac{p_{x,n}}{p_{x,n}}} \frac{dx}{x}. \tag{19}
$$

Using Holder’s inequality, it follows that

$$
v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t) \chi_{[t>N]}(t)w(t)dt \leq \left( v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} \chi_{[t>N]}(t)w(t)dt \right)^{\frac{1}{p_{x,n}}} (2^{-n-1}x)^{\frac{1}{(p_{x,n})}}
$$

with $x \geq N2^{n+1}$. By inserting this in the interior integral (19), that is exceeded by

$$
C_{12}2^{-n\delta p} \int_{N}^{\infty} \left( \frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} \chi_{[t>N]}(t)w(t)dt}{C_M^{p_M}} \right)^{\frac{p_{x,n}}{p_{x,n}}} \frac{dx}{x} \leq \frac{1}{x} \int_{N}^{\infty} \left( \frac{v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} \chi_{[t>N]}(t)w(t)dt}{C_M^{p_M}} \right)^{\frac{p_{x,n}}{p_{x,n}}} \frac{dx}{x}.
$$

Since $2^{-n-1}x \leq t$ and by using Lemma 2.4 and Lemma 2.1, it follows that, the last expression is exceeded by

$$
\int_{N}^{\infty} \left( \frac{1}{x} \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} \chi_{[t>N]}(t)w(t)dt \right)^{p(x)} \frac{dx}{x} \leq C_{13}2^{-n\delta p} \int_{N}^{\infty} \left( v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)^{p_{x,n}} \chi_{[t>N]}(t)w(t)dt \right) \frac{dx}{x}. \tag{20}
$$

Now, we estimate the interior integral through $I_{p(\cdot)}(f)$. The interior integral here is exceeded

$$
\int_{B_{x,\alpha} \cap \{ t: f(t) > \frac{1}{1+t^2} \}} \left( \frac{f(t) \chi_{[t>N]}(t)}{1+t^2} \right)^{p_{x,n}} \left( \frac{1}{1+t^2} \right)^{p_{x,n}} dt + \int_{B_{x,\alpha} \cap \{ t: f(t) \leq \frac{1}{1+t^2} \}} (f(t))^{p_{x,n}} dt
$$
\[
\leq \int_{B_{x,n}} \left( \frac{f(t) \chi_{\{t>\gamma\}}(t)}{1+t^2} \right)^{p_{\gamma,n}} \left( \frac{1}{1+t^2} \right)^{\tilde{p}_{\gamma,n}} dt + \int_{B_{x,n}} \left( \frac{\chi_{\{t>\gamma\}}(t)}{1+t^2} \right)^{\tilde{p}_{\gamma,n}} \left( \frac{1}{1+t^2} \right)^{p_{\gamma,n}} dt
\]
\[
\leq \int_{B_{x,n}} (f(t))^{p(t)} (1+t^2)^{p(t)-p_{\gamma,n}} \chi_{\{t>\gamma\}}(t) dt + \int_{B_{x,n}} \left( \frac{\chi_{\{t>\gamma\}}(t)}{1+t^2} \right)^{\tilde{p}_{\gamma,n}} \left( \frac{1}{1+t^2} \right)^{p_{\gamma,n}} dt
\]
\[
\leq C_{14} \int_{B_{x,n}} f(t)^{p(t)} \chi_{\{t>\gamma\}}(t) dt + \int_{B_{x,n}} \left( \frac{\chi_{\{t>\gamma\}}(t)}{1+t^2} \right) dt; \quad B_{x,n} = (2^{-n-1}x, 2^{-n}x)
\]

since the assumption (2) is made using (4) from Lemma 2.1 and \( p_{\gamma,n} \geq 1 \), it follows that
\[
(1+t^2)^{p(t)-p_{\gamma,n}} \leq C_{15}.
\]

Evidently, \( \left( \frac{1}{1+t^2} \right)^{\tilde{p}_{\gamma,n}} \leq \frac{1}{1+t^2} \). Therefore, we get an estimate
\[
\int_{B_{x,n}} (f(t))^{p(t)} \chi_{\{t>\gamma\}}(t) dt \leq C_{15} \int_{B_{x,n}} f(t)^{p(t)} \chi_{\{t>\gamma\}}(t) dt + \int_{B_{x,n}} \left( \frac{\chi_{\{t>\gamma\}}(t)}{1+t^2} \right) dt. \tag{21}
\]

Inserting this inequality in (20) and applying Fubini’s therom, it follows that
\[
I_{p(.), (N, \infty)} \left( \frac{1}{x} v(x) \int_{B_{x,n}} f(t) \chi_{\{t>\gamma\}}(t) w(t) dt \right)
\]
\[
\leq C_{15} 2^{-n\delta p^-} \int_{N} \left( \frac{v(x)}{2^{-n-1}x} \int_{2^{-n}x} f(t)^{p(t)} \chi_{\{t>\gamma\}}(t) w(t) dt + v(x) \int_{B_{x,n}} \left( \frac{\chi_{\{t>\gamma\}}(t)}{1+t^2} \right) w(t) dt \right) dx \frac{1}{x}
\]
\[
= C_{15} 2^{-n\delta p^-} v(x) \int_{2^{-n}N} \left( \int_{2^{-n-1}x} f(t)^{p(t)} \chi_{\{t>\gamma\}}(t) w(t) dt + v(x) \int_{2^{-n}x} \left( \frac{f(t)^{p(t)} w(t) dt}{1+t^2} \right) \right) dx \frac{1}{x}
\]
\[
\leq C_{15} 2^{-n\delta p^-} \ln 2 \left( \int_{2^{-n}N} f(t)^{p(t)} \chi_{\{t>\gamma\}}(t) dt + \ln 2 \int_{2^{-n}N} \chi_{\{t>\gamma\}}(t) dt \right)
\]
\[
= C_{15} 2^{-n\delta p^-} \ln 2 \left( \int_{N} f(t)^{p(t)} dt + \int_{N} dt \right) \leq C_{15} 2^{-n\delta p^-} \ln 2 \left( 1 + \frac{n}{2} \right) = C_{16} 2^{-n\delta p^-}.
\]

Wanneer,
\[
I_{p(.), (N, \infty)} \left( \frac{1}{x} v(x) \int_{B_{x,n}} f(t) \chi_{\{t>\gamma\}}(t) w(t) dt \right) \leq C_{16} 2^{-n\delta p^-}. \tag{22}
\]
From this it follows that
\[
\left\| \frac{1}{x} v(x) \int_{2^{-n-1}x}^{2^{-n}x} f(t)p(t) \chi_{[t>N]}(t)w(t) dt \right\|_{L^p(N,\infty)} \leq C_{17} 2^{-n \frac{np^-}{p^+}}. \tag{23}
\]
Inserting (23) in (12), we get
\[
\left\| x^{-1} v(x) \int_{N}^{x} f(x) dx \right\|_{L^p(1,\infty)} \leq \sum_{n=1}^{\infty} C_{17} 2^{-n \frac{np^-}{p^+}} = C_{18}.
\]
Further, substitute this estimate and (11) into (10), we complete the proof of the sufficiency part of Theorem 1.1.

3.2 Necessity.

Let us note, for an increasing near origin exponent functions \( p(.) \) it was proved in [19] that a necessity condition for inequality (1) to hold in the class of measurable positive functions \( f \) with support in finite interval \((0,N)\) is the same condition (2) over the points \( b \in (0,N) \) (observe not over all axes \((0,\infty)\)). To finish the necessity in Theorem 1.1 it remains to get this condition over the points \( b > N \), where we shall essentially use the decreasing of exponent near infinity.

Below, we shall prove the necessity of condition (2) over points \( x > N \) for decreasing near infinity exponent functions. We insert a function
\[
f_0(x) = x^{-\frac{1}{p^+}} \chi_{(b,2b)}(x), \quad x > N
\]
into inequality (1.2) with a parameter \( b > N \) be fixed. It is clear that, \( I_{p(.)}(f_0) = \ln 2 \). It follows from the inequality (1) that
\[
I_{p(.)} \left( \frac{1}{x} \int_{b}^{x} f_0(x) dx \right) \leq C_{19}.
\]
From this it follows
\[
\int_{2b}^{4b} \left( x^{-\frac{1}{p^+}} \int_{b}^{2b} t^{-\frac{1}{p^+}} dt \right)^{p(x)} \frac{dx}{x} \leq C_{19}. \tag{24}
\]
By monotony of \( p \) in \((N,\infty)\) it follows the functions \( x^{-\frac{1}{p^+}} \) and \( t^{-\frac{1}{p^+}} \) are decreasing therein. This yields
\[
x^{-\frac{1}{p^+}} \geq (4b)^{-\frac{1}{p^+}}, \quad 2b < x < 4b
\]
and
\[
t^{-\frac{1}{p^+}} \geq (2b)^{-\frac{1}{p^+}}, \quad b < t < 2b.
\]
By taking into account this inequalities, it follows from (24) that
\[
\int_{2b}^{4b} \left( (4b)^{-\frac{1}{p^+}} (2b)^{-\frac{1}{p^+}} \right)^{p(x)} \frac{dx}{x} \leq C_{19}.
\]
This inequality yields
\[
\left( \frac{1}{2} (2b)^{-\frac{1}{p^+}} - \frac{1}{p^+} \right)^p \ln 2 \leq C_{19}
\]
if the parenthesis term is greater 1. If not, we have
\[
\frac{1}{2} (2b)^{-\frac{1}{p^+}} - \frac{1}{p^+} \leq 1,
\]
i.e. it follows that

\[(2b) ^{\frac{1}{p'(2b)}} - \frac{1}{p(2b)} \leq C_{20},\]

or

\[(2b) ^{\frac{p(2b) - p(2b)}{p'(2b)}} \leq C_{20}.

Replacing \(2b\) by \(b\), we get the inequality

\[b ^{\frac{p(b) - p(2b)}{p'(2b)}} \leq C_{20}, \quad b > 2N.\]

Whence,

\[p^{p(b) - p(2b)} \leq C_{21}, \quad C_{21} = C_{20}^{(p^+)^2}, \quad b > 2N,

or

\[\left[ p(b) - p(2b) \right] \ln b \leq C_{22}, \quad C_{22} = (p^+)^2 \ln C_{21}. \quad (25)\]

Now, having property (25) and using inequality (1), we shall derive condition (2) for \(b > N\), too. In connection, we need some assertions below, e.g. following lemma is similar to those in [10].

**Lemma 3.2.** For any \(\frac{1}{2}x < y < 2x\) with \(x > N\) the estimates

\[C_{23} x ^{\frac{1}{p'(x)}} \leq y ^{\frac{1}{p'(y)}} \leq C_{24} x ^{\frac{1}{p'(y)}}\]

are satisfied with positive constants \(C_{23}, C_{24}\) depending on \(C_{22}\).

**Proof.** By using estimate (25) and decreasing of \(\frac{1}{p'(x)}\), it follows that

\[y ^{\frac{1}{p'(y)}} \leq \left(\frac{x}{2}\right) ^{\frac{1}{p'(y)}} \leq x ^{\frac{1}{p'(y)}} - \frac{1}{p'(2x)} \cdot x ^{\frac{1}{p'(y)}} 2 \frac{1}{p'(y)}

\leq 2x ^{p(x) - p(2x)} \cdot x ^{-\frac{1}{p'(y)}} \leq 2C_{21} x ^{-\frac{1}{p'(y)}}.

By the same way,

\[x ^{-\frac{1}{p'(y)}} \leq \left(\frac{y}{2}\right) ^{-\frac{1}{p'(y)}} \leq y ^{-\frac{1}{p'(y)}} - \frac{1}{p'(2y)} \cdot y ^{-\frac{1}{p'(y)}} 2 \frac{1}{p'(y)}

\leq 2y ^{p(y) - p(2y)} \cdot y ^{-\frac{1}{p'(y)}} \leq 2C_{21} y ^{-\frac{1}{p'(y)}}.

This proves Lemma 3.2.

**Lemma 3.3.** The function \(x ^{-\frac{1}{p'(y)}}\) almost decreases on \((N, \infty)\).

**Proof.** Take any \(N < t_1 \leq t_2 < \infty\), we show that there exists a constant \(C_{25} > 1\) depending on the constant \(C_2\) of inequality (1) and \(p^+\) such that

\[t_2 ^{-\frac{1}{p'(t_2)}} \leq C_{25} t_1 ^{-\frac{1}{p'(t_1)}}.

We fix any \(t_1 = b > 2N\) and choose \(n \in N\) such that \(2^{n-1}a < t_2 \leq 2^n b\). Then

\[
C_2 = \int_{2b}^{\infty} \left( x ^{-\frac{1}{p'(x)}} \int_b^{2b} t ^{-\frac{1}{p'(t)}} dt \right) ^{p(x)} \frac{dx}{x},
\]
by using Lemma 3.2, 

\[ \geq \sum_{k=1}^{\infty} \int_{2^{k-1}b}^{2^{k}b} \left( x^{\frac{1}{p(t)}} \int_{b}^{2b} \left( x^{\frac{1}{p(t)}} \right) dx \right) \frac{p(x)}{x} \, dx, \]

by using Lemma 3.2, we obtain

\[ \geq \sum_{k=1}^{\infty} \int_{2^{k-1}b}^{2^{k}b} \left[ C_{24}^{-1} b \frac{1}{p(t)} (2^k b)^{-\frac{1}{p(t)}} x^{-\frac{1}{p(t)}} \right] \frac{p(x)}{x} C_{24}^{p'} \, dx, \]

where \( \sum_{k \in N} \) is a summation over \( k \in N \) such that the inequality

\[ b \frac{1}{p(t)} (2^k b)^{-\frac{1}{p(t)}} \leq 1 \]

holds, and \( \sum_{k \in N'} \) is a summation of the opposite case. Therefore, for any \( k \in N \) one gets

\[ b \frac{1}{p(t)} (2^k b)^{-\frac{1}{p(t)}} \leq C_{25}. \]

This yields

\[ t_2^{-\frac{1}{p(t)}} \cdot t_1^{-\frac{1}{p(t)}} \leq C_{26} \]

i.e. function \( x^{-\frac{1}{p(t)}} \) is almost decreasing. Since \( x^{-\frac{1}{p(t)}} \) is almost decreasing on \([N, \infty)\), we get

\[ C_{2} \geq \int_{b}^{\infty} C_{26}^{-1} b \frac{1}{p(t)} x^{-\frac{1}{p(t)}} \, dx \]

or

\[ \int_{b}^{\infty} x^{-\frac{p(x)}{p(t)}} \, dx \leq C_{27} b^{\frac{p(x)}{p(t)}}, \quad b > N. \]  

(26)

Now, let \( N < t_1 < t_2 < \infty \). We show that condition (26) entails (2). We take any \( N < t_1 < t_2 < \infty \) and set

\[ K(x) = \int_{x}^{\infty} t^{-\frac{p(x)}{p(t)}} \, dt, \quad x > N. \]

From (26) it follows that

\[ K(x) \leq -C_{27} x K'(x), \quad N < x < \infty. \]

Integrating this inequality over \((t_1, t_2)\), for \( N < t_1 < t_2 < \infty \) and using (26), we get

\[ t_2^{\frac{p(x)}{p(t)}} K(t_2) \leq C_{27} t_1^{-\frac{p(x)}{p(t)}} + \frac{C_{27}}{t_2}, \quad t_2 \geq N. \]

(27)

On the other hand, by using monotone decreasing of \( x^{-\frac{p(x)}{p(t)}} \), the conditions (25) and (26), it follows that

\[ k(t_2) \geq \int_{t_2}^{2t_2} x^{-\frac{p(x)}{p(t)}} \, dx \geq \ln 2 \left( \frac{1}{2t_2} \right)^{\frac{p(x)}{p(t)}} \geq 2^{-\frac{p(x)}{p(t)}} \ln 2 \, t_2^{-\frac{p(x)}{p(t)}}, \quad t_2 \geq N. \]
By inserting this in (27) one gets,
\[ t_2^{-\frac{1}{p'(x)}} \leq C_{28} t_1^{-\frac{1}{p'(x)}} \]
for all \( N < t_1 < t_2 < \infty \). Whence, the function \( x^{-\frac{1}{p'(x)}} \) is almost decreasing on \((N, \infty)\). Therefore, we have proved that
\[ t_2^{-\frac{1}{p'(x)} + \delta_1} \leq C_{28} t_1^{-\frac{1}{p'(x)} + \delta_1} \]
for all \( N < t_1 < t_2 < \infty \); a constant \( \delta_1 = \frac{\delta}{p'} \).

Therefore \( x^{-\frac{1}{p'(x)} + \delta_1} \) is almost decreasing with \( \delta_1 = \frac{\delta}{p'} \), \( \delta = \frac{1}{C_{27}} \). From this it follows
\[
\int_b^\infty x^{-\frac{1}{p'(x)}} \frac{dx}{x} \leq C_{28} \int_a^\infty b^{\frac{1}{p'(x)}} \frac{dx}{x^1 + \delta} \\
\leq C_{28}\frac{b^\delta}{\delta} \int_b^\infty \frac{dx}{x^1 + \delta} = \frac{C_{28}}{\delta} b^{\frac{1}{p'(0)}}, \quad b > N
\]
This proves the necessity of condition (2) near infinity.

Conclusion

A new method of weighted increasing near origin and decreasing near infinity exponent function that provides a boundedness of the Hardy’s operator in variable exponent Lebesgue space was obtained. The method we use here leads us to the most general sense. We don’t have to work in a particular interval. This method can be applied to different operators. And this method brings important facilities in the study of operator theory.

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