Stability Theory in $\ell_1$ for Nonlinear Markov Chains and Stochastic Models for Opinion Dynamics over Influence Networks

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Abstract—We study Markov chain models where the transition mechanism depends nonlinearly on the current state. One specific choice for such a model, where the state represents “belief,” was proposed in [1] to model opinion dynamics and is referred to as the DeGroot-Friedkin model. Herein, we consider a general class of such nonlinear Markov chain models and develop a theory for assessing stability. Our approach relies on establishing that the differential of the nonlinear dynamics (under suitable analyticity conditions) is contractive in the $\ell_1$ metric. We apply the theory to two type of nonlinear random walks, i.e., nonlinearly adapting the transition probabilities, where the adaptation is exponential and linear, respectively. The latter includes the DeGroot-Friedkin model and generalizations. We also discuss continuous-time generalization as well as interacting (particle) models and discuss their relevance with regard to modeling social dynamics over influence networks. Finally, we view the nonlinear adaptation of the transition mechanism as feedback and quantify the effect of external bias on the stationary distribution.

Keywords: $\ell_1$-stability of stochastic models, Markov chains, nonlinear feedback Markov models, opinion dynamics, influence networks, reflected appraisal, interacting potential.

I. INTRODUCTION

Models of social interactions and the formation of opinions in large groups have been receiving increasing attention in recent years (see [1], [2], [3], [4], [5], [6] and the references therein). As the basis for social exchanges, an averaging mechanism has been postulated in the literature, whereby the outcome represents a weighted sum of individual preferences or beliefs. In turn, the averaging mechanism itself is modified by the outcome of past interactions, reflecting relative increase or decrease in the confidence and influence of particular individuals. Such feedback models can be traced to [7], [8], [9], [10].

Averaging schemes leading to consensus are broadly relevant in coordination of dynamical systems such as co-operating drones or ground robots, sensor networks, formation flight, and distributed frequency regulation in power grid. The distinguishing feature of social interaction models has been the postulate of a suitable nonlinear effect that enhances or, perhaps, diminishes the influence of particular individual in the group. The purpose of this paper is first to step back, abstract the general framework as one dealing with nonlinear random walks and nonlinear Markov chains, and present a framework for stability analysis of such models. To this end, we develop a stability theory for certain stochastic feedback models by resorting to the $\ell_1$ metric. The key element of our approach is to consider the differential of stochastic maps and assess whether these are contractive in $\ell_1$.

More specifically, we consider a discrete-time (or rather, discrete-indexed, since the index may represent issue being con-

sidered) Markov chain $\{X_t \mid t \in \mathbb{Z}_+\}$ taking values on a finite state-space $\mathcal{X} = \{1, 2, \ldots, n\}$. We denote by $p(t)$ the marginal probability vector, i.e., its entry $p_i(t) = \Pr(X_t = i)$ is the occupation probability of state $i$ at iteration index $t$, and postulate a transition mechanism that depends nonlinearly on the occupation probability (a.k.a. belief state) of the Markov chain according to the rule:

\[ \Pi_{ij} := \Pr(X_{t+1} = j \mid X_t = i) = \rho_i(t)\delta_{ij} + (1 - \rho_i(t))C_{ij}, \quad (1a) \]

\[ \rho_i(t) := r(p_i(t)), \quad (1b) \]

where $[C_{ij}]_{i,j=1}^n$ is a row-stochastic matrix $\delta_{ij}$ equals one for $i = j$ and zero otherwise, and $r()$ is a differentiable function

\[ r : [0,1] \mapsto [0,1]. \]

In general, the mapping $r()$ needs to be neither onto nor invertible (nor independent of $i$, as taken at the early part of the paper, for simplicity); typical examples include

\[ r(x) = x, 1 - x, 1 - e^{-\gamma x}, \text{ or } e^{-\gamma x}, \text{ for some } \gamma > 0, \text{ etc.} \]  

Equation (1a) represents a model for a “lazy” random walk where the transition probabilities $C_{ij}$ are modified to increase/decrease the “prior” return-probability from $C_{ii}$ to $p_i + C_{ii}(1 - p_i)$, in a way that depends on the probability of the corresponding state, since $p_i(t) = r(p_i(t))$. For this reason, we refer to $r()$ as the reinforcement function. Thus, the essence of the above model is that the random walk adapts the return probability of each state so as to promote or discourage residence in states with high marginal probability. A schematic is given in Figure 1.

An alternative interpretation of the time $t$-marginal probabilities is as confidence or influence which, accordingly, is modified (constructively or destructively) by the likelihood of the particular state of the process. It has been argued, for instance, that high confidence and success in an argument, begets higher confidence.

The matrix $C = [C_{ij}]_{i,j=1}^n$, in the context of opinion dynamics, encodes the influence of neighboring nodes. Of particular interest will be the exponentially-scaled transition kernel (introduced here)

\[ \Pi_{ij}(x) = (1 - e^{-\gamma x})\delta_{ij} + e^{-\gamma x}C_{ij}, \quad \text{and its “opposite”} \]  

\[ \Pi_{ij}(x) = e^{-\gamma x}\delta_{ij} + (1 - e^{-\gamma x})C_{ij}, \quad (3a) \]

as well as linearly-scaled kernels

\[ \Pi_{ij}(x) = \gamma x_i\delta_{ij} + (1 - \gamma x_i)C_{ij}, \quad \text{and} \]  

\[ \Pi_{ij}(x) = (1 - \gamma x_i)\delta_{ij} + \gamma x_i C_{ij}, \quad (4a) \]

that have considered before, see e.g., [1]. Notice that in all these cases, the matrices $\Pi = [\Pi_{ij}]_{i,j=1}^n$ and $\Pi$ are row-stochastic (i.e., rows sum to one). Those two models will be analyzed in some detail as they provide rather insightful examples of the dynamics that one can expect of such models. We highlight ranges of parameters where globally stable behavior is observed and where the Markov chain tends towards a stationary distribution and, others, where

\[ \Pi_{ij} \geq 0 \text{ for all } (i,j) \text{ and } \sum_{j} C_{ij} = 1. \]
multiple equilibria, periodic orbits, or chaotic behavior is observed. Local stability of equilibria (i.e., local stationarity of distributions on the nonlinear Markov chain), if that is the case, can be assessed numerically using our theory in Section II.

Thus, the evolution of the marginal probability (column) vector \( p(t) \) corresponding to (1) as well as (3-4) is as follows:

\[
p(t + 1) = \Pi(p(t))^\dagger p(t),
\]

with

\[
\Pi(p(t)) = \left( \text{diag}(r(p(t))) + C^\dagger(I - \text{diag}(r(p(t)))) \right)\, \delta.
\]

starting from a given \( p_0 \in S_{n-1} \), where \( S_{n-1} \) denotes the probability simplex \( \{ x \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1 \} \) and prime “†” denotes transposition or adjoint.

A closely related alternative model for the evolution of influence and opinion dynamics that has appeared in the literature, is to postulate the transition mechanism:

\[
p(t + 1) = \left[ \Pi(p(t))^\dagger \right]_{FP}
\]

where the notation \( \left[ \Pi(p(t))^\dagger \right]_{FP} \) represents the mapping \( \Pi^\dagger \mapsto q \in S_{n-1} \) of an irreducible (row) stochastic matrix \( \Pi \) to its corresponding Frobenius-Perron eigenvector, i.e., to the unique probability (column) vector \( q \) that satisfies \( \Pi^\dagger q = q \). The relation between the two update-mechanisms, (5a) and (6), can be understood by virtue of the fact that \( (\Pi(p(t))^\dagger)^\dagger p(t) \) is approximately equal to the right Frobenius-Perron eigenvector of \( (\Pi(p(t)))^\dagger \) for sufficiently large \( k \), and hence a suitable modification of the dynamics in (5a) (i.e., by introducing a suitably high exponent) approximates the dynamics in (6). We will not be concerned with the update mechanism in (6), as our primary interest is in the general transition mechanism (5a). It is reasonable to expect that stochastic maps in either form (5a), or (6), for specific choices of kernel \( \Pi_{ij}(\cdot) \) and generalizations (see Sections V, IV), have appealing properties as models for opinion dynamics.

The exposition in our manuscript proceeds as follows. In Section II we show that if the Jacobian of a nonlinear stochastic map is stochastic, then the map is contractive in \( \ell_1 \) on the probability simplex. This result is apparently new and especially applicable to the type of nonlinear stochastic maps considered herein. In Sections III, IV we analyze the two models given in equations (1, 4) in detail. Finally, in Sections V, VII we develop the continuous-time counterpart of the models as well as models that introduce local coupling in the reinforcement mechanism and, accordingly, comment on extensions of the theory.

II. \( \ell_1 \)-contractiveness of Stochastic Maps

Consider a stochastic evolution on the probability simplex \( S_{n-1} \) that is modeled by the following nonlinear map

\[
f : p \mapsto f(p) := \Pi(p)^\dagger p = q,
\]

where \( \Pi(p) = (\Pi_{ij}(p_1, \ldots, p_n))_{i,j=1}^n \) is a (row) stochastic matrix that depends nonlinearly on the probability (column) vector \( p \), and is differentiable in \( p \). We will show that under suitable conditions, which often hold for the dynamical processes we consider, the map is strictly contractive in \( \ell_1 \), i.e., we show that under these conditions, if \( p^b \) and \( p^a \) are two probability vectors then

\[
\|p^b - p^a\|_1 > \|\Pi(p^a)^\dagger p^b - \Pi(p^b)^\dagger p^a\|_1.
\]

Denote by \( T \) the tangent space of the probability simplex, i.e., \( T := \{ \delta \in \mathbb{R}^n \mid 1^\dagger \delta = 0 \} \) with \( 1 \) denoting the column vector of ones. The Jacobian of \( f(\cdot) \) is

\[
df : T \rightarrow T : (\delta_j)_{j=1}^n \mapsto \left( \sum_{i=1}^n \Pi_{ij} \delta_i \right)_{j=1}^n
\]

starting from a given \( p_0 \in S_{n-1} \), where \( S_{n-1} \) denotes the probability simplex \( \{ x \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1 \} \) and prime “†” denotes transposition or adjoint.

Thus, \( df \) can be written in a matrix form as

\[
df : \delta \mapsto \delta + \left[ \frac{\partial \Pi^\dagger}{\partial p_1}, \ldots, \frac{\partial \Pi^\dagger}{\partial p_n} \right] \delta = Q \delta.
\]

Trivially,

\[
1^\dagger Q^\dagger = 1^\dagger \Pi^\dagger + \left[ \frac{\partial \Pi^\dagger}{\partial p_1}, \ldots, \frac{\partial \Pi^\dagger}{\partial p_n} \right] = 1^\dagger
\]

since \( 1^\dagger \Pi^\dagger = 1^\dagger \) and the partials are zero. However, in general, \( Q \) may not be stochastic due to the fact that elements of \( Q \) are not guaranteed to be non-negative.

Theorem 1. Let a matrix \( \left[ \Pi_{ij}(p) \right]_{i,j=1}^n \) be row-stochastic and a differentiable function of the probability vector \( p \), and suppose that the Jacobian of the map \( f(\cdot) \) in (7) has strictly positive entries. Then, \( f \) is strictly contractive in the \( \ell_1 \)-metric.

Proof. Consider two probability vectors \( p^a \) and \( p^b \) and let \( \alpha := (p^b - p^a)_+ \) be the vector with the positive entries of the difference \( p^b - p^a \), and let \( \beta := -(p^b - p^a)_- \) contain the entries that appear with negative sign. Thus,

\[
p^b - p^a = \alpha - \beta,
\]

2The theory applies to more general differentiable \( f(p) \) as well.
but in this representation $\alpha$ and $\beta$ have non-negative entries and have no common support, i.e., $\alpha, \beta = 0$ as they are not simultaneously $\neq 0$. On the other hand

\[
\|p^b - p^a\|_1 = \sum_i |p_i^b - p_i^a| = \|\beta - \alpha\|_1 = \sum_i \beta_i + \sum_i \alpha_i = \|\beta\|_1 + \|\alpha\|_1
\]

as both $\alpha, \beta$ have positive entries.

Now consider a path $p(\lambda) = (1 - \lambda)p^a + \lambda p^b$ for $\lambda \in [0,1]$ and consider comparing the distance between $p^b$ and $p^a$ to the length of the path

\[
q(\lambda) = \Pi(p(\lambda))^\dagger p(\lambda), \; \lambda \in [0,1].
\]

Clearly,

\[
dq(\lambda) = (\beta - \alpha)d\lambda,
\]

and thus

\[
\int_{\lambda=0}^{1} \|dq(\lambda)\|_1 = \int_{\lambda=0}^{1} \|\beta - \alpha\|_1 d\lambda = \|\beta - \alpha\|_1.
\]

On the other hand

\[
\int_{\lambda=0}^{1} \|dq(\lambda)\|_1 = \int_{\lambda=0}^{1} \|Q(p(\lambda)\|_1 d\lambda < \int_{\lambda=0}^{1} \left( \|Q(p(\lambda)\|_1 + \|Q(p(\lambda)\|_1 \right) d\lambda
\]

\[
= \int_{0}^{1} (\|\beta\|_1 + \|\alpha\|_1) d\lambda = \|p^b - p^a\|_1.
\]

To see why the inequality \(8\) holds, note that for each $\lambda$,

\[
v^\beta := Q(p(\lambda))\beta \quad \text{and} \quad v^\alpha := Q(p(\lambda))\alpha
\]

are vectors with nonnegative entries. Then, $\|v^\beta\|_1$ and $\|v^\alpha\|_1$ (within the integral in \(9\)) are sums of the entries of $v^\beta$, $v^\alpha$, respectively, while $\|v^\beta - v^\alpha\|_1$ is strictly smaller than the sum $\|v^\beta\|_1 + \|v^\alpha\|_1$ due to cancellations between respective entries of $v^\beta$ and $v^\alpha$. The fact that both $v^\beta_i$ and $v^\alpha_i$ have non-zero values for some $i$, implies strict inequality

\[
\|v^\beta - v^\alpha\|_1 < \|v^\beta\|_1 + \|v^\alpha\|_1,
\]

which follows from the fact that $Q$ has strictly positive entries. Then also \(8\) implies that $\|Q(p(\lambda))\|_1 \|\beta\|_1$ and, similarly, $\|Q(p(\lambda))\|_1 \|\alpha\|_1$, establishing the equality in \(10\).

Finally, the metric property of $\|\cdot\|_1$ implies that

\[
\|q(1) - q(0)\|_1 \leq \int_{\lambda=0}^{1} \|dp(\lambda)\|_1,
\]

where $q(1) = \Pi(p)^{b}p^b$ and $q(0) = \Pi(p)^{a}p^a$. Hence, \[\Pi(p)^{b}p^b - \Pi(p)^{a}p^a\|_1 < \|p^b - p^a\|_1.
\]

This concludes the proof.

Stronger statements follow as easy extensions of the above theorem and are listed below. Notice that all statements hold without restriction as to the functional form of $\Pi(p)$ other than being row-stochastic, differentiable, and satisfying the stated conditions on the Jacobian of corresponding maps, i.e., all results hold for functional forms that are more general than the exponential and linear models considered herein.

**Corollary 2.** Let matrix $[\Pi_{ij}(p)]_{i,j=1}^{n}$ be row-stochastic and differentiable in $p$, and suppose that the Jacobian of the map $f(\cdot)$ in \(7\) has non-negative entries. Then, $f$ is contractive in the $\ell_1$-metric.

**Corollary 3.** Let matrix $[\Pi_{ij}(p)]_{i,j=1}^{n}$ be row-stochastic and differentiable in $p$, and suppose for a suitable integer $m$, the differential (Jacobian) of the $m$th iterate

\[
f^m(p) := f(f(\ldots f(p)))
\]

has strictly positive entries for all $p$. Then, $f$ has a unique fixed point.

**Proof.** It is clear from the theorem that $f^m$, which is stochastic and satisfies the conditions, has a unique attractive fixed point. Suppose that this fails to be a fixed point $p^*$ of $f$ (since we no longer assume that $df$ has non-negative entries). In that case, there must be an $m$-periodic orbit of $f$ that includes $p^*$. Let $p^*, f(p^*), \ldots, f^{m-1}(p^*)$ be the points of the cycle ($m$-orbit). Clearly, $f^m(p) = p$ for $p = f(p^*)$ and since $f^m$ is contractive, $p = p^*$.

**Corollary 4.** Let matrix $[\Pi_{ij}(p)]_{i,j=1}^{n}$ be row-stochastic and differentiable in $p$, and that $p^*$ is a fixed point of the map $f$ in \(7\), i.e., $p^* = \Pi(p^*)p^*$. Suppose that for a suitable integer $m$ the Jacobian of the map $f$ evaluated at $p^*$ is such that, for a suitable integer $m$,

\[
(df|_{p^*})^m
\]

has strictly positive entries. Then $p^*$ is a locally attracting equilibrium.

**Proof.** The expression $(df|_{p^*})^m$ is precisely the Jacobian of the $m$th iterate, i.e.,

\[
(df|_{p^*})^m = df(df(\ldots df(p^*))|_{p^*})
\]

By continuity, the entries of $df(df(\ldots df(p^*))$ will remain positive in a neighborhood of $p^*$. The claim follows as in Corollary 3.

**Corollary 5.** Let matrix $[\Pi_{ij}(p)]_{i,j=1}^{n}$ be row-stochastic and differentiable in $p$, and that $p^i$, for $i = 0, 1, 2, \ldots, m - 1$, is a periodic orbit for $f$ in \(7\), i.e.,

\[
p^{i+1 \text{mod}(m)} = \Pi(p^{i \text{mod}(m)}).
\]

Suppose that the product of the Jacobians

\[
(df|_{p^{i+1 \text{mod}(m)}}) \ldots (df|_{p^{i \text{mod}(m)}})
\]

has strictly positive entries for some $i$. Then, the periodic orbit is locally attractive.

**Proof.** Under the stated condition, $p^i$ is an attractive fixed point for the $m$th iterate, $f(f(\ldots f(p^i)))|_{p^i}$. The fact that the orbit is locally attractive now follows from the fact that

\[
(df|_{p^{i+k \text{mod}(m)}}) \ldots (df|_{p^{i \text{mod}(m)}})
\]

has $\ell_1$-gain strictly less than 1 for $k = 1, 2, \ldots, m - 1$. 


In light of the potential applications of Theorem 1 in the analysis of the stability of equilibria of a nonlinear Markov chain, we provide a bound on the induced $\ell_1$-gain of the Jacobian

$$\|df\|_{\ell_1}(1) := \max\{\|Q^T\delta\|_1, \|\delta\|_1 = 1\},$$

which is naturally restricted to operate on tangent vectors (hence, $T\delta = 0$ as a constraint). As before, at any point $p$, the Jacobian is represented by a matrix $Q(p)^T$. Theorem 6. Let $f$ be a differentiable stochastic map as in (1). For any $p^k, p^\delta \in S_{n-1}$,

$$\|f(p^k) - f(p^\delta)\|_1 \leq \sup_{p \in S_{n-1}} \|df(p)|_{\ell_1}(1)\|p^k - p^\delta\|_1,$$

and, in general,

$$\|df\|_{\ell_1}(1) = \frac{1}{2} \max_{j,k} \sum_{i=1}^n \left| (Q(p))_{ji} - (Q(p))_{ki} \right|.$$

Proof. The first claim is straightforward. The induced norm can also be readily computed by noting that a $\delta \in T$ with $\|\delta\|_1 = 1$ can always be written as before $\delta = \frac{1}{2}(\beta - \alpha)$ with $\alpha, \beta$ having positive entries and $\|\alpha\|_1 = \|\beta\|_1 = 1$. Then,

$$\|df\|_{\ell_1}(1) = \frac{1}{2} \max\{\|Q^T\beta - Q^T\alpha\|_1 | \alpha, \beta \in S_{n-1}\}.$$ (12)

The claim follows by convexity; the maximal value is attained when $\alpha, \beta$ each have a single nonzero element (which selects a corresponding row of $Q$).

Remark 7. The above theorem allows us to ensure that specific equilibria $p^\star$ are attractive in $\ell_1$ by explicitly establishing that $\|df(p^\star)\|_{\ell_1} < 1$ or that $\|df_m(p)\|_{\ell_1} < 1$ for a suitable $m$. Thus, in particular, it allows strengthening the statements of Corollaries 2 through 5 accordingly. The $\ell_1$-distance to the equilibrium, when $f$ is locally $\ell_1$-contractive, serves as a naturally Lyapunov function. Evidently, the spectral radius of $df$ can also be used since, being less than one, is necessary and sufficient for local stability. So one distinguishing feature of the $\ell_1$-theory is that it provides easy conditions for global stability.

Remark 8. To recap, the essence of the idea underlying this section is that contractiveness of the map, and the consequent stability and existence of a fixed point or periodic orbits for iterations $p_{\text{next}} = f(p)$ or, of higher order iterants, may be deduced from the infinitesimal properties of $f$ in the $\ell_1$-metric. More specifically, $df$ which will be stochastic in general, unless it has non-negative entries. Negative entries appear when the derivatives of $\Pi(p)^T = (f_{ij}(p))_{ji}$ are large in absolute value and have negative sign. The idea of looking at $\delta f$ is quite general and will be applied in representative cases below in the process of exploring specific nonlinear Markov chain models.

III. EXPONENTIAL-INFLUENCE MODELS

In this section we present analysis of models where the reinforcement function $r(x)$ that dictates adaptation of the return probability to the various states is either $1 - e^{-\gamma x}$ or $e^{-\gamma x}$, for some $\gamma > 0$. The first choice satisfies $r(0) = 0$ and $r'(0) = 1$.

Thus, it suggests strengthening return probabilities for states with relatively large marginal probability at corresponding times $t$. The second choice has $r(0) = 1$ and $r'(0) = -\gamma$, and suggests an opposite tendency.

Throughout we assume that $C$ is an irreducible acyclic row-stochastic matrix, and we denote by $e$ the unique (positive) Frobenius-Perron left eigenvector, i.e., $e$ satisfies

$$C^T e = e,$$

and is normalized so that $1^T e = 1$. In fact, because of the irreducibility assumption, $e$ has positive entries.

A. Case $r(x) = 1 - e^{-\gamma x}$ for $\gamma \leq 1$.

We begin by analyzing the case $\gamma = 1$. We sum up the main conclusions in the following proposition.

Proposition 9. Consider the map $p(t) \mapsto f(p(t)) = p(t+1)$ where

$$f(p(t)) = \left(\text{diag}(1 - e^{-p(t)}) + C^T(e^{-p(t)})\right) p(t).$$ (13)

The following hold:

i) $f(\cdot)$ is contractive in $\ell_1$,

ii) starting from an arbitrary $p(0) \in S_{n-1}$, the limit

$$p^\star = \lim_{t \to \infty} p(t)$$

exists and is unique,

iii) $p^\star$ is the unique fixed point of the map $f(\cdot)$,

iv) the entries of $p^\star$ satisfy $e^{-p^\star}; p^\star_i = \kappa e_i$, for some $\kappa > 0$.

Proof. The Jacobian $df$ is of the form

$$\delta \mapsto \left(\text{diag}(r(p) + p \circ r'(p)) + C^T(I - \text{diag}(r(p) + p \circ r'(p)))\right) \delta = \left(\text{diag}(1 - e^{-p} + p \circ e^{-p}) + C^T\text{diag}(e^{-p} - p \circ e^{-p})\right) \delta,$$

where $r$ denotes the entry-wise multiplication of vectors, and for a vector $v = (v_i)_i$, $e_i$ denotes the vector with entries $e_i$. Since both functions $1 - e^{-x} + xe^{-x}$ and $e^{-x} - xe^{-x}$ take positive values on $[0, 1]$, $Q(p)^T := \left(\text{diag}(1 - e^{-p} + p \circ e^{-p}) + C^T\text{diag}(e^{-p} - p \circ e^{-p})\right)$ is a (column) stochastic matrix. Thus, $f$ is contractive. Further, $Q$ inherits irreducibility from $C$ since it has the same pattern of positive entries; in addition it is acyclic, irrespective of $C$, because its diagonal is not zero. Hence, independently of $p$, there exists integer $m$ such that

$$Q(f(...f(p))) \vdots Q(f(p)) \vdots Q(p)$$ (14)

has all entries positive. The expression in (14) is precisely the differential of the $m$th iterant (cf. (1)). Hence, the $m$th iterant is strictly contractive. By Corollary 4 $f$ has a unique attractive fixed point. This fixed point can be obtained by solving

$$p = \left(\text{diag}(1 - e^{-p}) + C^T\text{diag}(e^{-p})\right) p.$$ (15)

 Naturally, the rates also depend on the choice of $C$. 

Rearranging terms we see that $p e^{-p}$ is proportional to $e$ (the Frobenius-Perron vector of $C$), and therefore,

$$p_i e^{-p_i} = \kappa c_i, \ i = 1, \ldots, n. \tag{16}$$

The function $xe^{-x}$ is monotonic on $[0, 1]$ and hence, for any $\kappa \leq \frac{e^{-1}}{\max\{c_i\}} =: \kappa_{\text{max}},$

there is a unique solution $\{p_i \mid i = 1, \ldots, n\}$ of (16). Let now $s(\kappa) := \sum_s p_i$. The function $s(\kappa)$ is monotonically increasing as a function of $\kappa$ and has $s(0) = 0$. For $\kappa = \kappa_{\text{max}}$ one of the $p_i$'s is equal to 1 and hence $s(\kappa_{\text{max}}) \geq 1$. Thus, the equation $s(\kappa) = 1$ has a unique solution that corresponds to the probability vector $p$ that satisfies (15). This is the limit $p^*$ and the expression for it is as claimed. □

The theorem holds verbatim for $\gamma \leq 1$. The essence is the same. Again $\frac{df^m(\cdot)}{dt}$ is strictly contractive and the fixed-point equations admit a unique solution. We summarize.

**Proposition 10.** For any $\gamma \in [0, 1]$ consider

$$p(t) \mapsto f(p(t)) = p(t + 1), \text{ where} \tag{17a}$$

$$f(p(t)) = \left(\text{diag}(1 - e^{-\gamma p(t)}) + C^\top \text{diag}(e^{-\gamma p(t)})\right) p(t). \tag{17b}$$

The map $f$ is contractive in $\ell_1$ and, starting from an arbitrary $p(0) \in S_{n-1}$, the limit $p^* = \lim_{t \to \infty} p(t)$ exists, is unique, and its entries satisfy $e^{-\gamma p^*_i} = \kappa c_i$ for some $\kappa > 0$. 

**Remark 11.** It is natural to consider a DeGroot-Friedkin-like modification of the above, given by the evolution

$$p(t + 1) = \left(\text{diag}(1 - e^{-\gamma p(t)}) + C^\top \text{diag}(e^{-\gamma p(t)})\right)_{\text{FP}}. \tag{18}$$

It is easy to show that there is a unique fixed point and that this coincides with that in Theorem 9. The stationarity conditions are exactly the same. Contraction follows similarly by noting that the pertinent stochastic matrix is irreducible and the iteration in (18) is equivalent to taking the limit

$$p(t + 1) = \lim_{m \to \infty} \left(\text{diag}(1 - e^{-\gamma p(t)}) + C^\top \text{diag}(e^{-\gamma p(t)})\right)^{\text{FP}} p(t). \tag{19}$$

**B. Case $r(x) = 1 - e^{-\gamma x}$ for $\gamma > 1$.**

The case $\gamma > 1$ is substantially different than before. Here, there can be several attractive points of equilibrium for the nonlinear dynamics in (17) and even more complicated nonlinear behavior. In fact, we believe that such a behavior may be more appropriate for models of opinion dynamics as it is reasonable to expect a different outcome depending on the starting point (that encapsulates confidence/beliefs of individuals). We illustrate the behavior with two numerical examples for 3-state Markov chains to highlight differences with the case when $\gamma \leq 1$.

1) *Example:* We consider the dynamics in (17) for a 3-state Markov chain (i.e., $n = 3$) with $\gamma = 4$ and

$$C = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}. \tag{19}$$

The left Frobenius-Perron eigenvector of $C$ is $(2/3, 1/6, 1/6)\top$. The fixed-point conditions for possible stationary distributions become

$$e^{-4p_1^*} p_1^* = \frac{2}{3}, \quad e^{-4p_2^*} p_2^* = \frac{1}{6}, \quad 2p_1^* + p_2^* = 1. \tag{20}$$

Upon eliminating $\kappa$ between the first two, and substituting $p_1$ in terms of $p_2$, we obtain

$$\frac{1 - 2p_2^*}{p_2^*} e^{-4(1 - 3p_2^*)} = 4. \tag{20}$$

This equation has the unique solution

$$p^* := (0.9904, 0.0048, 0.0048)\top.$$ 

It turns out that this is a locally attractive fixed point. This can be verified by evaluating the Jacobian of $f$ at $p^*$ as

$$\frac{df}{\|f\|} p^* = \begin{bmatrix} 1.0113 & 0.3849 & 0.3849 \\ -0.0056 & 0.2303 & 0.3849 \\ -0.0056 & 0.3849 & 0.2303 \end{bmatrix}.$$ 

Even though the Jacobian has negative entries it is still strictly contractive. Indeed, we explicitly evaluate the induced gain using Theorem 6 and this is

$$\|df|\| = \frac{1}{2} \max\{1.2528, 1.2528, 0.3092\} = 0.6264 < 1.$$ 

Thus $p^*$ is a stable fixed point. This analysis is consistent with simulations shown in Fig. 2. In the figure we depict trajectories (in different color) starting from random initial conditions that clearly tend to $p^*$.

2) *Example:* Once again we consider a 3-state Markov chain with $\gamma = 4$, but this time we take

$$C = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}. \tag{21}$$

The fixed-point equations have 7 solutions (taking into account symmetries). Out of those, three are attractive fixed points with coordinates cyclically selected from $\{1 - a, a/2, a/2\}$ for $a = 0.046$. 

![Fig. 2: Convergence of trajectories to a unique fixed point for the 3-state exponential model (17) with $\gamma = 4$ and influence matrix $C$ given by (19).](image-url)
The remaining four are unstable fixed points. One is at the center $(1/3, 1/3, 1/3)^T$ (due to symmetry), and the rest have coordinates cyclically selected from \( \{1 - a, a/2, a/2\} \) for \( a = 0.874 \). Just like the previous example, we can verify stability by computing the Jacobian \( df \) at fixed points. For instance, for the fixed point \( p_6^* = (0.954, 0.023, 0.023)^T \), we have

\[
\begin{bmatrix}
1.0620 & 0.4141 & 0.4141 \\
-0.0310 & 0.1718 & 0.4141 \\
-0.0310 & 0.4141 & 0.1718
\end{bmatrix},
\]

and

\[
\|df\|_{\infty} = \frac{1}{2} \max\{1.2958, 1.2958, 0.4846\} = 0.6479 < 1.
\]

Applying Theorem 6 we conclude that \( p_6^* \) is a stable fixed point. For another fixed point \( p_6^* = (0.1260, 0.4370, 0.4370)^T \), we have

\[
\begin{bmatrix}
0.7004 & -0.0651 & -0.0651 \\
0.1498 & 1.1302 & -0.0651 \\
0.1498 & -0.0651 & 1.1302
\end{bmatrix},
\]

and

\[
\|df\|_{\infty} = \frac{1}{2} \max\{1.9608, 1.9608, 2.3907\} = 1.1954 > 1.
\]

Numerical evidence shown in Fig. 3 confirms that \( p_6^* \) is stable and \( p_6^* \) is unstable. Convergence of trajectories depends on the initial conditions with respect to the basins of attraction for the three stable fixed points. The qualitative behavior of the trajectories around the four unstable and three stable fixed points is illustrated in Fig. 4.

C. Case \( r(x) = e^{-\gamma x} \) for \( \gamma \leq 1 \).

In this case there is a unique fixed point and it is always globally attractive. We summarize our conclusions as follows:

**Proposition 12.** For any \( \gamma \in [0, 1] \) consider

\[
p(t) \to f(p(t)) = p(t + 1), \quad \text{where} \quad f(p(t)) = \begin{pmatrix}
\text{diag}(e^{-\gamma p(t)}) + C^T \text{diag}(I - e^{-\gamma p(t)})
\end{pmatrix} p(t). \tag{22a}
\]

The map \( f \) is contractive in \( \ell_1 \) and, starting from an arbitrary \( p(0) \in S_{n-1} \), the limit \( p^* = \lim_{t \to \infty} f(t) \) exists, is unique, and its entries satisfy \( \left(1 - e^{-\gamma p_i}\right) p_i^* = \kappa c_i \), for some \( \kappa > 0 \).

**Proof.** First, the Jacobian matrix \( Q(p)^T \) is of the form

\[
\text{diag}(e^{-\gamma p} - \gamma p \circ e^{-\gamma p}) + C^T \text{diag}(I - e^{-\gamma p} + \gamma p \circ e^{-\gamma p}).
\]

Notice that \( Q(p)^T \) is differentiable in \( p \), and for \( \gamma \leq 1 \), it is a (column) stochastic matrix with non-negative entries. Therefore, by Corollary 2 the map \( Q(p)^T \) is contractive in \( \ell_1 \) and inherits irreducibility from \( C^T \). Following a similar line of argument as in Proposition 2 uniqueness of the fixed point for map \( Q(p)^T \) is guaranteed. Next, we write stationarity conditions

\[
p^* = \begin{pmatrix}
\text{diag}(e^{-\gamma p^*}) + C^T \text{diag}(I - e^{-\gamma p^*})
\end{pmatrix} p^*,
\]

equivalently,

\[
(1 - e^{-\gamma p^*}) \circ p^* = C^T (1 - e^{-\gamma p^*}) \circ p^*.
\]

to obtain that

\[
(1 - e^{-\gamma p_i}) p_i^* = \kappa c_i, \quad i = 1, \ldots, n, \tag{23}
\]

where \( c_i \) denotes the \( i \)-th entry of the Frobenius-Perron vector of \( C \) and \( \kappa = \sum_{i=1}^n (1 - e^{-\gamma p_i}) p_i^* \).

D. Case \( r(x) = e^{-\gamma x} \) for \( \gamma > 1 \).

In this case too there exists a unique fixed point in any dimension (any \( n \)). This follows easily as the fixed-point conditions are the same,

\[
(1 - e^{-\gamma p_i}) p_i^* = \kappa c_i.
\]

Then, for all \( \gamma > 0 \), \( (1 - e^{-\gamma x}) x \) is a monotonically increasing starting at 0 for \( x = 0 \). Solving for a given \( \kappa \), the sum \( \sum_{i=1}^n p_i^* (\kappa) \) is also monotonically increasing function of \( \kappa \) and its value exceeds 1 for a suitable \( \kappa \). Thus, there is a unique solution \( p_i^* (\kappa) \) which is a probability vector (and the \( p_i^* \)'s sum up to 1).

However, interestingly, the nonlinear dynamics now display diverse behaviors. Below we give three examples. In the first two the unique fixed point is attractive, but they differ, in that assurances for stability are drawn (for the second example) by computing the
norm of the differential of higher iterants (2nd in this case). In the third example we observe a 2-periodic attractive orbit.

1) Example: We consider a 3-state Markov chain with $\gamma = 4$, and

$$C = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}. \quad (24)$$

Since $C$ is doubly stochastic, the unique fixed point for (23) is $p^* = (1/3, 1/3, 1/3)^T$, and we have

$$df|_{p^*} = \begin{bmatrix} -0.0880 & 0.5440 & 0.5440 \\ 0.5440 & -0.0880 & 0.5440 \\ 0.5440 & 0.5440 & -0.0880 \end{bmatrix}.$$  

and

$$\|df|_{p^*}\|_1 = \frac{1}{2} \max\{1.2640, 1.2640, 1.2640\} = 0.6320 < 1.$$

Using Theorem 5, we conclude that $p^*$ is a stable fixed point.

2) Example: For $\gamma = 4$, now take

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$  

The unique fixed point is again $p^* = (1/3, 1/3, 1/3)^T$. Here, $df|_{p^*} = \begin{bmatrix} -0.0880 & 0.5440 & 0.5440 \\ 0 & 0.4560 & 0.5440 \\ 1.0880 & 0 & -0.0880 \end{bmatrix},$

and

$$\|df|_{p^*}\|_1 = 1.1760.$$  

However,

$$\|df^2|_{p^*}\|_1 = 0.7911.$$  

This ensures local attractiveness.

Fig. 5: For the 3-state exponential model (22) with $\gamma = 4$ and influence matrix $C$ given by (23), trajectories converge to the unique stable fixed point $p^* = (1/3, 1/3, 1/3)^T$.

3) Example: Once again we consider a 3-state Markov chain with $\gamma = 4$, but we now take

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0.8 & 0.2 & 0 \\ 0.8 & 0.2 & 0 \end{bmatrix}. \quad (25)$$

Uniqueness of a fixed point is guaranteed. This turns out to be

$$p^* = (0.4173, 0.1537, 0.4298)^T.$$  

It turns out that

$$df|_{p^*} = \begin{bmatrix} -0.1261 & 0.6333 & 0.9031 \\ 0 & 0.2084 & 0.2258 \\ 1.1261 & 0.1583 & -0.1289 \end{bmatrix}.$$  

has $\ell_1$-norm equal to 1.255, and so do the differentials of higher order iterants. However, a stable 2-periodic orbit now appears alternating between

$p^a = (0.1943, 0.1042, 0.7015)^T$ and $p^b = (0.6450, 0.2005, 0.1545)^T$.

The periodic orbit is locally attractive. The Jacobians at these two points are

$$df|_{p^a} = \begin{bmatrix} 0.1024 & 0.4923 & 0.8873 \\ 0.3846 & 0.2218 & 0.8976 \\ 0.1231 & -0.1092 \end{bmatrix}$$  

and

$$df|_{p^b} = \begin{bmatrix} -0.1197 & 0.7290 & 0.6352 \\ 0 & 0.8888 & 0.1588 \\ 1.1197 & 0.1822 & 0.2060 \end{bmatrix},$$  

respectively, and it can be verified that the norm of their product is $\|df|_{p^a}df|_{p^b}\|_1 = 0.8750$. Interestingly, $\|df|_{p^b}df|_{p^a}\|_1 = 0.7120$, which is different, but $< 1$ too (as expected). Stability can be ascertained by Corollary 5.
IV. DEGROOT-FRIEDKIN MODEL AND ITS VARIANTS

We now consider the two classes of nonlinear Markov chains with \( r(x) = \gamma x \) and \( 1 - \gamma x \), for \( 0 < \gamma \leq 1 \). The bounds \( 0 < \gamma \leq 1 \) ensure that \( \Pi(p) \) (in (5a)) remains stochastic for all values of the probability vector \( p \) and any \( C \). For small values of \( \gamma \), \( \gamma x \approx 1 - e^{-\gamma t} \) and, evidently, these models approximate the corresponding exponential models of Section [M].

A. Case \( r(x) = \gamma x \)

The case where \( r(x) = x \) and \( C \) is restricted to be doubly stochastic has been studied in [11] and referred to as a modified/one-step DeGroot-Friedkin model. Existence and stability of the fixed point was analyzed and, in particular, it was conjectured that the equilibrium is stable for any irreducible row stochastic matrix \( C \) (see [11]). Herein, we consider the general case where \( r(x) = \gamma x \). For this class of models, very much as in the case of the exponential models, we can ascertain \( \ell_1 \) strict-contractiveness for a range of values for \( \gamma \), while for other values, we can ascertain stability on a case by case basis.

We begin with the following proposition for general irreducible stochastic \( C \) and \( \gamma \leq \frac{1}{2} \).

**Proposition 13.** For \( \gamma \leq \frac{1}{2} \) consider

\[
\begin{align*}
\mathbf{p}(t) &\mapsto \mathbf{f}(\mathbf{p}(t)) = \mathbf{p}(t + 1), \quad \text{where} \\
\mathbf{p}(t + 1) &= \left( \text{diag}(\gamma \mathbf{p}(t)) + C^\dagger \text{diag}(1 - \gamma \mathbf{p}(t)) \right) \mathbf{p}(t). 
\end{align*}
\]

The map \( \mathbf{f} \) is strictly contractive in \( \ell_1 \), the iteration for \( t = 0, 1, \ldots \) converges a unique fixed point \( \mathbf{p}^\star = \lim_{t \to \infty} \mathbf{p}(t) \), and \n
\[
(1 - \gamma \mathbf{p}_i^\star) \mathbf{p}_i^\star = \kappa c_i, \quad \text{for a suitable } \kappa > 0.
\]

**Proof.** As before, the Jacobian \( df \) is now

\[
\delta \mapsto \left( \text{diag}(2\gamma \mathbf{p}) + C^\dagger \text{diag}(1 - 2\gamma \mathbf{p}) \right) \delta.
\]

For \( 0 < \gamma \leq \frac{1}{2} \), \( Q(p) \) is element-wise non-negative and inherits irreducibility from \( C^\dagger \). Corollary 2 ensures that \( \mathbf{f} \) is contractive in \( \ell_1 \) and a unique fixed point \( \mathbf{p}^\star \) exists. The fixed-point condition is now

\[
C^\dagger (1 - \gamma \mathbf{p}^\star) \circ \mathbf{p}^\star = (1 - \gamma \mathbf{p}^\star) \circ \mathbf{p}^\star,
\]

and therefore, the fixed point \( \mathbf{p}_i^\star \) satisfies \((1 - \gamma \mathbf{p}_i^\star) \mathbf{p}_i^\star = \kappa c_i \) with \( \kappa = 1 - \gamma \| \mathbf{p}^\star \|_2 \).

For the range \( \gamma \in [\frac{1}{2}, 1] \) all-encompassing conclusions cannot be drawn and examples have to be worked out on a case by case basis. In passing, we note that for \( \gamma = 1 \), trivially, the vertices of \( S_{n-1} \) are fixed points; in general, when \( \gamma \neq 1 \) this is not the case. Also, when \( C \) is doubly stochastic and \( \gamma \neq 1 \), \( \frac{1}{n} \mathbf{1} \) is the unique fixed point of \( [26] \).

**B. Case \( r(x) = 1 - \gamma x \)**

We first establish that the corresponding map admits a unique fixed point for any \( \gamma > 0 \), and show that it is \( \ell_1 \)-contractive for \( \gamma \leq \frac{1}{2} \).

**Proposition 14.** Consider

\[
\begin{align*}
\mathbf{p}(t) &\mapsto \mathbf{f}(\mathbf{p}(t)) = \mathbf{p}(t + 1) \quad \text{where} \\
\mathbf{p}(t + 1) &= \left( \text{diag}(1 - \gamma \mathbf{p}(t)) + C^\dagger \text{diag}(\gamma \mathbf{p}(t)) \right) \mathbf{p}(t). 
\end{align*}
\]

For any \( \gamma > 0 \), there is a unique fixed point \( \mathbf{p}^\star \), where

\[
\mathbf{p}_i^\star = \frac{\sqrt{c_i}}{n}, \quad i = 1, \ldots, n.
\]

For \( 0 < \gamma \leq \frac{1}{2} \), \( f \) is \( \ell_1 \)-contractive and in this case \( \mathbf{p}^\star \) is an attractive fixed point.

**Proof.** The fixed-point condition

\[
\gamma \mathbf{p}^\star \circ \mathbf{p}^\star = C^\dagger \gamma \mathbf{p}^\star \circ \mathbf{p}^\star
\]

implies that \( \mathbf{p}_i^\star \) must equal \( \kappa \sqrt{c_i} \), for each \( i \) and some \( \kappa > 0 \). Thus, the fixed point is always unique and is as claimed. For \( 0 < \gamma \leq \frac{1}{2} \), the Jacobian \( df \)

\[
\delta \mapsto \left( \text{diag}(1 - 2\gamma \mathbf{p}) + C^\dagger \text{diag}(2\gamma \mathbf{p}) \right) \delta
\]

is element-wise non-negative, inherits irreducibility from \( C^\dagger \), and as before, \( f \) is \( \ell_1 \)-contractive.

Once again, for \( \gamma \in [\frac{1}{2}, 1] \), analysis can be done on a case by case basis and no general conclusion can be drawn.

V. CONTINUOUS-TIME MARKOV PROCESSES

It is quite natural to extend the framework presented to the setting of continuous-time Markov chains. Indeed,

\[
\dot{\mathbf{p}}(t) = L^\dagger (I - \text{diag}(r(\mathbf{p}(t)))) \mathbf{p}(t),
\]

where \( L = C - I \) is a Laplacian matrix satisfying \( L \mathbf{1} = 0 \), is the continuous-time analog of (5). It is clear that \((I - \text{diag}(r(\mathbf{p}(t))))L \)

is a Laplacian matrix as \((I - \text{diag}(r(\mathbf{p}(t))))L \mathbf{1} = 0 \). The scaling by the diagonal matrix \( \text{diag}(r(\mathbf{p}(t))) \) plays a similar role. It promotes or discourages staying at a state \( i \) in accordance with the current value of the corresponding occupation probability \( p_i \). A relation can be drawn by noting that for a small \( h \),

\[
\exp[hL^\dagger (I - \text{diag}(r(\mathbf{p}(t))))] \approx I + h L^\dagger (I - \text{diag}(r(\mathbf{p})))
\]

\[
= (1 - h) I + h \left( \text{diag}(r(\mathbf{p})) + C^\dagger (I - \text{diag}(r(\mathbf{p}))) \right).
\]

The special case when \( r(\mathbf{p}) = \mathbf{p} \) was recently considered in [2].

For general \( r(\cdot) \) as in (2), the existence of fixed points can be ascertained along similar lines as in the discrete-time setting. For instance, if \( \mathbf{p}^\star \) is a fixed point, it must satisfy

\[
(1 - r(\mathbf{p}^\star)) \circ \mathbf{p}^\star = \kappa c,
\]

where \( c \) is the eigenvector of \( C \) associated with eigenvalue 1 (equivalently, \( c \) is the eigenvector of \( L \) associated with eigenvalue 0).
Analysis of the various cases proceeds similarly (e.g., see Theorem 2). In particular, regarding stability of fixed points, if $\delta \in \mathcal{T}$ is a small perturbation of $p$,

$$\delta \approx [L^t \{I - \text{diag}(r(p))\} - L^t \text{diag}(r(p)p)\delta].$$

It follows

$$\delta(dt) \approx [I + (L^t \{I - \text{diag}(r(p))\} - L^t \text{diag}(r(p)p))dt]\delta(0).$$

Reasoning as in Section II, we can establish $\ell_1$-contractiveness of the above map and ascertain the global or local stability of fixed points accordingly. As an example, we note that for $r(x) = 1 - e^{-\gamma x}$ or $r(x) = \gamma x$ and $\gamma < 1$ the fixed point is globally attractive. The details are omitted.

VI. GROUPING AND CAHOOTS

It is quite interesting to speculate about the effect of colluding sub-group in opinion forming. Indeed, everyday experience suggests that opinion is often reinforced within groups of like-minded individuals that opinion is often reinforced within groups of like-minded individuals that draw confidence upon the collective wisdom, or lack of. To account for such interactions, we use a stochastic matrix $W$ to model the joint influence between group members by weighing their collective states via $r(Wp)$, which should be contrasted with individual-reinforcement of opinion/confidence modeled by $r(p)$. This is independent and in addition to $C$, which is used to model information flow over the total influence network. A reasonable choice for $W$ is to be block diagonal where the blocks correspond to different subgroups of interacting individuals. The special case where $W$ is identity matrix reduces to the earlier setting.

In fact, what we propose herein is an “interacting particle” analogue for nonlinear Markov chains, modeled as follows:

$$p(t + 1) = \Pi(p(t))^\dagger p(t) = (\text{diag}(r(Wp(t))) + C^\dagger(I - \text{diag}(r(Wp(t))))p(t). \quad (31)$$

In particular, using a fixed-point argument as in II, we establish existence results for the cases $r(x) = x$ and $r(x) = 1 - e^{-x}$, and a general stochastic matrix $W$.

**Proposition 15.** Let $r(x) = x$ or $r(x) = 1 - e^{-x}$, and $W$ a stochastic matrix. Assume that $c_k < \frac{1}{2}$ for all $k$. The Markov nonlinear model \((31)\) has at least one fixed point in the interior of probability simplex $S_{n-1}$.

**Proof.** Any fixed point of \((31)\) must satisfy

$$p_j = F_j(p) := \frac{1}{1 + \sum_{k \neq j} \frac{c_k}{c_j(1 - r_k)}}. $$

Since

$$\sum_{k \neq j} \frac{c_k}{c_j(1 - r_k)} > \sum_{k \neq j} \frac{c_k}{c_j} > 1,$$

there exists $\epsilon > 0$ small enough such that

$$\left(\sum_{k \neq j} \frac{c_k}{c_j(1 - r_k)} - 1\right)\epsilon - \sum_{k \neq j} \frac{c_k}{c_j(1 - r_k)}\epsilon^2 > 0.$$

It follows

$$\frac{1}{1 + \sum_{k \neq j} \frac{c_k}{c_j(1 - r_k)}} < 1 - \epsilon.$$

Combining the above we obtain

$$F_j(p) \leq \frac{1}{1 + \sum_{k \neq j} \frac{c_k}{c_j(1 - r_k)}} < 1 - \epsilon.$$

The “nonlocal interaction” matrix $W$ may in general introduce negative off-diagonal elements in $d$. The theory in Section II applies on a case by case basis, but no general conclusion can be drawn at this point regarding global stability of particular class of models as we did earlier. Indeed, for $r(x) = 1 - e^{-x}$, a matrix representation of the differential \((8)\) becomes

$$Q(p) = \text{diag}(I - e^{-Wp}) + C^\dagger \text{diag}(e^{-Wp}) + (I - C^\dagger) \text{diag}(p \circ e^{-Wp})W.$$

This, in general, has negative entries, which however doesn’t imply that the fixed point is unstable. The theory in Section II applies and attractiveness of equilibria can be ascertained by e.g., explicitly computing the $\ell_1$-gain of $df_\mathcal{T}$.

Below is an example in $S_2$. We take $r(x) = 1 - e^{-Wx}$,

$$C = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.4 & 0.2 \end{bmatrix} \quad (32)$$

and

$$W = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (33)$$

Numerically (Fig. 7), we see that the system has a unique fixed point, $p^* = (0.6975, 0.1744, 0.1282)^T$, which is stable. This results
The perturbation $\xi$ of the output is bounded by
\[ \| (I - Q(p^*)^\dagger)^{-1} \|_{(1)} \| (Q(p^*)^\dagger - \Pi(p^*)^\dagger) \|_{(1)} \| \delta \|_1. \]
The term $\| (Q(p^*)^\dagger - \Pi(p^*)^\dagger) \|_{(1)}$ can be computed using (12).
Similarly, we can determine $\| (I - Q(p^*)^\dagger)^{-1} \|_{(1)}$ since, its inverse,
\[ \| (I - Q(p^*)^\dagger)^{-1} \|_{(1)} \] is precisely
\[ \min \{ \| (I - Q(p^*)^\dagger) \delta \|_1 \mid \delta \in T \text{ and } \| \delta \|_1 = 1 \}. \]

VIII. CONCLUDING REMARKS

The theory allows drawing general results on global attractiveness of equilibria of nonlinear Markovian models in certain cases. Besides the intrinsic importance in studying nonlinear Markovian dynamics, the subject is of great interest in modeling dynamical interactions over social networks. We expect that the development herein, i.e., both the theory as well as the new class of exponential models that we present, will provide the impetus for further advances and, in particular, for addressing uncertainty and the effect of disturbances in such models.

IX. ACKNOWLEDGMENTS

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