On the chromatic number of random regular graphs*

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Determining the chromatic number of random graphs is one of the longest-standing challenges in probabilistic combinatorics. For the Erdős-Rényi model, the single most intensely studied model in the random graphs literature, the question dates back to the seminal 1960 paper that started the theory of random graphs [4].

Apart from $G_{ER}(n, m)$, the model that has received the most attention certainly is the random regular graph $G(n, d)$. We provide an almost complete solution to the chromatic number problem on $G(n, d)$, at least in the case that $d$ remains fixed as $n \to \infty$. The strongest previous result on the chromatic number of $G(n, d)$ is due to Kemkes, Pérez-Giménez and Wormald [5]. They proved that w.h.p. for $k \geq 3$ if $d \in ((2k - 3) \ln(k - 1), (2k - 2) \ln(k - 1))$ then $\chi(G(n, d)) = k$ and if $d \in [(2k-2) \ln(k-1), (2k-1) \ln k]$ then $\chi(G(n, d)) \in \{k, k+1\}$. These bounds imply that $G(n, d)$ is k-colorable w.h.p. if $d < (2k - 2) \ln(k - 1)$, while $G(n, d)$ fails to be k-colorable w.h.p. if $d > (2k - 1) \ln k$. Our main result is

**Theorem 1** There is a sequence $(\varepsilon_k)_{k \geq 3}$ with $\lim_{k \to \infty} \varepsilon_k = 0$ such that the following is true.

1. If $d \leq (2k - 1) \ln k - 2 \ln 2 - \varepsilon_k$, then $G(n, d)$ is $k$-colorable w.h.p.
2. If $d \geq (2k - 1) \ln k + \varepsilon_k$, then $G(n, d)$ fails to be $k$-colorable w.h.p.

This implies that for every integer $k$ exceeding a certain constant $k_0$ we identify a number $d_{k-col}$ such that $G(n, d)$ is $k$-colorable w.h.p. if $d < d_{k-col}$ and non-$k$-colorable w.h.p. if $d > d_{k-col}$.

The best current results on coloring $G_{ER}(n, m)$ as well as the best prior result on $\chi(G(n, d))$ are obtained via the second moment method [1, 3, 5]. So are the present results. Recently, Coja-Oghlan and Vilenchik [3] improved the result from [1] on the chromatic number of $G_{ER}(n, m)$. This improvement is obtained by considering a different random variable, namely the number $Z_{k,good}$ of “good” $k$-colorings instead of $Z_{k-col}$ the number of all $k$-colorings. The definition of this random variable draws on intuition from non-rigorous statistical mechanics work on random graph coloring [6, 8]. Crucially, the concept of good colorings facilitates the computation of the second moment. Theorem 1 provides a result matching [3] for $G(n, d)$. Following [5], we combine the second moment bound from [3] with small subgraph conditioning. The previous lower bound on the chromatic number of $G(n, d)$ is based on a simple first moment argument over the number of $k$-colorings. The bound that can be obtained in this way, attributed to Molloy and Reed [7], is that $G(n, d)$ is non-$k$-colorable w.h.p. if $d > (2k - 1) \ln k$. By contrast, the second assertion in Theorem 1 marks a strict improvement. The proof is via an adaptation of techniques developed in [2] for the random $k$-NAESAT problem. Extending this argument to the chromatic number problem on $G(n, d)$ requires substantial technical work.

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