On the last digits of tetrations of base $2^k$ and $5^k$

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Abstract
In this paper will be proved the existence of a formula to reduce a
tetration of base $2^k$ and $5^k$ mod $10^n$. Indeed, last digits of a tetra-
tion are the same starting from a certain hyper-exponent; In order to
compute the last digits of those expressions we reduce them mod $10^n$.
Lots of different formulas will be derived, for different cases of $k$ (where
$k$ is the exponent of the base of the tetration). This kind of operation is
fascinating, because the tetration grows very fast. But using these for-
mulas we can actually have informations about the last digits of those
expressions.

1 Introduction
We said that tetrations with hyper-exponent $m$ and base $l^k$ end with the same
digits for all $m$ greater or equal than a certain number, more precisely we will
show how by varying $m$ , also the last digits that can repeat themselves are
changing. For example, consider the infinite tetration of base $2^{625}$; starting
from the seventh tetration of $2^{625}$, all the successive tetrations have the same
last 30 digits. The seventh tetration of $2^{625}$ is about $10^{10^{10^{10^{10^{10^{10}}}}}}$
In the proof the Carmichael function will be fundamental [3]; we call the Carmichael
function $\lambda(n)$ the smallest exponent $u$ such that:

$$a^u \equiv 1 \mod (n)$$

You can actually compute $\lambda(n)$ for all different values of $n$ [1]:

$$\lambda(n) = [\lambda(2^{k_0}), \phi(p_1^{k_1}), \phi(p_2^{k_2}), \phi(p_3^{k_3}) \ldots]$$

Where:

$$n = 2^{k_0} p_1^{k_1} p_2^{k_2} p_3^{k_3} \ldots$$

And $[x, y, z \ldots]$ represents the lowest common multiple between $x, y, z \ldots$
The proof is divided in two cases: when the base of the tetration is of the form
$2^k$ and when is of the form $5^k$.
Here below there is a list of the formulas that will be derived.

$m^2 \equiv n+2 \quad \mod (10^n) [1]$
∀ \( m \in \mathbb{N}, m \geq n + 2, \forall n \in \mathbb{N} \)

\[ m(2^{2k}) \equiv n + 1 (2^{2k}) \mod (10^n) \] [2]

∀ \( m \in \mathbb{N}, m \geq n + 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \neq 2h \land k \neq 5h \)

\[ m(2^{3k}) \equiv n + 1 (2^{3k}) \mod (10^n) \] [3]

∀ \( m \in \mathbb{N}, m \geq n + 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \neq 2h \land k \neq 5h \)

\[ m(4^{4k}) \equiv n - 1 (4^{4k}) \mod (10^n) \] [4]

∀ \( m \in \mathbb{N}, m \geq n - 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \neq 5h \)

\[ m(2^{5^tk}) \equiv (\lceil \frac{n}{5^t} \rceil + 1)(2^{5^tk}) \mod (10^n) \] [5]

∀ \( m \in \mathbb{N}, m \geq \lceil \frac{n}{5^t} \rceil + 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \forall t \in \mathbb{N}, k \neq 5h \)

\[ m(5^{2^tk}) \equiv (\lceil \frac{n}{5^t} \rceil - 1)(5^{2^tk}) \mod (10^n) \] [6]

Where \( \lceil n \rceil \) is the ceiling function of \( n \) and represents the nearest integer to \( n \), greater or equal to \( n \); and \( ^a n \) represent the \( a \)-th tetration of \( n \), or \( n \ldots n \) \( a \) times.

2 \( 2^k \) case

2.1 Last digits of \( m 2 \)

In this section will be proved the first of the 6 formulas seen before.

Consider the infinite tetration of base 2, how much last digits \( 2^{2^\ldots} \) has in common with the last digits of \( b \cdot 2 \), \( b \in \mathbb{N} \). Consider the following modular congruence:

\[ m 2 \equiv b 2 \mod (10^n) \]

With \( m \geq b \).

But we have that \( 10^n = 2^n \cdot 5^n \), so: \( m 2 \equiv b 2 \equiv 0 \mod (2^n) \) because \( n \leq m - 12 \).

So we’ll have that:

\[ m 2 \equiv b 2 \mod (5^n) \]

Since \( (2, 5^n) = 1 \), where \( \langle x, y \rangle \) represents the greatest common divisor between \( x \) and \( y \), certainly \( \exists u_0 \) such that \( 2^{u_0} \equiv 1 \mod (5^n) \)

The minimum value that \( u_0 \) can assume is exactly equal to \( \lambda(5^n) \).

But \( \lambda(5^n) = \phi(5^n) = 5^n - 5^{n-1} = 4 \cdot 5^{n-1} \)

Where \( \phi(n) \) [2] is the Euler totient function, which counts the positive integers up to a given integer \( n \) that are relatively prime to \( n \).
We have that \(2^{4 \cdot 5^{n-1}} \equiv 1 \mod (5^n)\), so we can reduce the second floor of the tetration \(\mod 4 \cdot 5^{n-1}\). So we’ll have that:

\[
m^{-1}2 \equiv b^{-1}2 \mod (4 \cdot 5^{n-1})
\]

But \(4 \mid m^{-1}2\), so we’ll have that:

\[
m^{-1}2 \equiv b^{-1}2 \mod (5^{n-1})
\]

Repeating this process we’ll obtain:

\[
m^{-2}2 \equiv b^{-2}2 \mod (5^{n-2})
\]
\[
m^{-3}2 \equiv b^{-3}2 \mod (5^{n-3})
\]

\[\vdots\]
\[
m^{-n}2 \equiv b^{-n}2 \mod (5^{n-n})
\]

Repeat this process \(n\)-times. We’ll obtain the \(p\)-uple composition of \(\lambda(5^n)\):

\[
\lambda_p(5^n) = \lambda(\lambda(\lambda(\ldots(5^n))))
\]

Where \(p\) represents the number of compositions of the Carmichael function. We know that \(\lambda_n(5^n) = 4\), in fact \(\lambda_n(5^n)\) is the first value of \(\lambda_p(n)\) such that \(v_5[\lambda_p(n)] = 0\) where \(v_5(n)\) is the 5-adic valuation of \(n\), or rather the maximum exponent that given to 5 divides \(n\).

This is true because in general \(\lambda_p(5^n) = 4 \cdot 5^{n-p}\)

We can prove this by induction:

Consider the \(p+1\) case:

\[
\lambda_{p+1}(5^n) = \lambda(4 \cdot 5^{n-p}) = \left[\lambda(2^2), \phi(5^{n-p})\right] = \left[2, 5^{n-p-1}(5-1)\right] = 4 \cdot 5^{n-p-1}
\]

Which is the thesis of our induction argument.

So if \(n = p\):

\[
\lambda_n(5^n) = 4 \cdot 5^{n-n} = 4
\]

This value plays a fundamental role, because it means that we can reduce the \(n\)-th floor \(\mod 4\). So we’ll have:

\[
m^{-n}2 \equiv b^{-n}2 \mod 4
\]

And:

\[
m^{-n+1}2 \equiv b^{-n+1}2 \equiv 1 \mod 5
\]

We can conclude that exists a number \(b\) such that this modular congruence is true. We can conclude that all tetration after the floor:

\[
\left\lfloor \min(b) \right\rfloor - th
\]
they are equal to \(\left\lfloor \min(b) \right\rfloor\)-th tetration \(\mod 5^n\), and so \(\mod 10^n\) since the \(\left\lfloor \min(b) \right\rfloor\)-th floor is congruent to 1 \(\mod 5\). It follows immediately that they have the same last n digits in common.

We have to find \(\left\lfloor \min(b) \right\rfloor\) such that this is true. Notice that:

\[
2^{2^2} = 16 \equiv 1 \mod 5
\]

This happens if \(b = n + 2\), so:

\[
m-n+1 \equiv n^2-n+2 \equiv 3^2 \equiv 1 \mod 5
\]

And to conclude:

\[
m \equiv n^2 \mod (10^n)
\]

\(\forall m \in \mathbb{N}, m \geq n + 2, \forall n \in \mathbb{N}\)

### 2.2 Last digits of \(m(2^{2k})\)

In this section will be proved the second of the 6 formulas seen before.

Consider the infinite tetration of base \(2^{2k}\). Exists a number \(\left\lfloor \min(b) \right\rfloor\) such that the following modular congruence is true:

\[
m(2^{2k}) \equiv b(2^{2k}) \mod (10^n)
\]

with \(m \geq b\)

As the case analyzed before:

\[
m(2^{2k}) \equiv b(2^{2k}) \mod (5^n)
\]

Since \((2, 5^n) = 1\), where \((x, y)\) represents the greatest common divisor between \(x\) and \(y\), certainly \(\exists u_0\) such that \(2^{u_0} \equiv 1 \mod (5^n)\)

The minimum value that \(u_0\) can assume is exactly equal to \(\lambda(5^n)\).

But \(\lambda(5^n) = \phi(5^n) = 5^n - 5^{n-1} = 4 \cdot 5^{n-1}\)

\[
m-1(2^{2k}) \equiv b-1(2^{2k}) \mod (4 \cdot 5^{n-1})
\]

But \(4 \mid b-1\) 2 , so we’ll have that:

\[
m-1(2^{2k}) \equiv b-1(2^{2k}) \mod (5^{n-1})
\]

And as before, analyzing every floor \(\mod \lambda(5^n)\) we arrive to the \(n - 1\)-th floor, which we can reduce \(\mod 5\).

\[
(2^{2k})^{(2^{2k})} \equiv (4^k)^{(4^k)} \equiv [(-1)^k]4^k \equiv 1 \mod 5
\]

So:

\[
m-n+1(2^{2k}) \equiv b-n+1(2^{2k}) \equiv 1 \mod 5
\]

It is true for \(b = n + 1\), indeed the second tetration of \(2^{2k}\) is congruent to 1 \(\mod 5\). So we proved that:

\[
m(2^{2k}) \equiv n+1(2^{2k}) \mod (10^n)[2]
\]
2.3 Last digits of $m(2^{3k})$

In this section will be proved the fourth of the 6 formulas seen before. Consider the infinite tetration of base $2^{3k}$. Exists a number $\min(b)$ such that the following modular congruence is true:

$$m(2^{3k}) \equiv b(2^{3k}) \mod (10^n)$$

with $m \geq b$

As the case analyzed before:

$$m(2^{3k}) \equiv b(2^{3k}) \mod (5^n)$$

Since $(2, 5^n) = 1$, where $(x, y)$ represents the greatest common divisor between $x$ and $y$, certainly $\exists u_0$ such that $2^{u_0} \equiv 1 \mod (5^n)$

The minimum value that $u_0$ can assume is exactly equal to $\lambda(5^n)$.

But $\lambda(5^n) = \phi(5^n) = 5^n - 5^{n-1} = 4 \cdot 5^{n-1}$

$$m^{-1}(2^{3k}) \equiv b^{-1}(2^{3k}) \mod (4 \cdot 5^{n-1})$$

But $4 \mid b^{-1}$, so we’ll have that:

$$m^{-1}(2^{3k}) \equiv b^{-1}(2^{3k}) \mod (5^{n-1})$$

And as before, analyzing every floor $\mod \lambda_p(5^n)$ we arrive to the $n-1$-th floor, which we can reduce $\mod 5$.

$$(2^{3k})(2^{3k}) \equiv (8^k)(8^k) \equiv [(-2)^k]^8 \equiv 1 \mod 5$$

So:

$$m-n+1(2^{3k}) \equiv b-n+1(2^{3k}) \equiv 1 \mod 5$$

It is true for $b = n + 1$, indeed the second tetration of $2^{3k}$ is congruent to 1 $\mod 5$. So we proved that:

$$m(2^{3k}) \equiv n+1(2^{3k}) \mod (10^n)[3]$$

$\forall m \in \mathbb{N}, m \geq n + 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \neq 2h \land k \neq 5h$

2.4 Last digits of $m(2^{4k})$

In this section will be proved the third of the 6 formulas seen before. Consider the infinite tetration of base $2^{4k}$. Exists a number $\min(b)$ such that the following modular congruence is true:

$$m(2^{4k}) \equiv b(2^{4k}) \mod (10^n)$$
with $m \geq b$
As the case analyzed before:

$$m^{(2^{3k})} \equiv b^{(2^{3k})} \mod (5^n)$$

Since $(2, 5^n) = 1$, where $(x, y)$ represents the greatest common divisor between $x$ and $y$, certainly $\exists u_0$ such that $2^{u_0} \equiv 1 \mod (5^n)$

The minimum value that $u_0$ can assume is exactly equal to $\lambda(5^n)$.

But $\lambda(5^n) = \phi(5^n) = 5^n - 5^{n-1} = 4 \cdot 5^{n-1}$

$$m^{-1}(2^{3k}) \equiv b^{-1}(2^{3k}) \mod (4 \cdot 5^{n-1})$$

But $4 \mid b^{-1}$ 2 , so we’ll have that:

$$m^{-1}(2^{3k}) \equiv b^{-1}(2^{3k}) \mod (5^{n-1})$$

And this time, analyzing every floor mod $\lambda_p(5^n)$ we arrive to the $n-2$-th floor, which we can reduce mod 25.

$$(2^{4k})^{(2^{4k})} \equiv (16^{k})^{(16^{k})} \equiv 16^k \mod 25$$

So:

$$m-n+2(2^{4k}) \equiv b-n+2(2^{4k}) \equiv 16^k \mod 25$$

It is true for $b = n - 1$, in fact the $(m - n + 2)$-th tetration is congruent to the second tetration of $2^{4k}$ which is congruent to $16^k$ mod 25, which is exactly equal to the base of the tetration. So:

$$m(2^{4k}) \equiv n-1(2^{4k}) \mod (10^n)[4]$$

$\forall m \in \mathbb{N}, m \geq n - 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \neq 5h$

### 2.5 Last digits of $m(2^{5^i k})$

In this section will be proved the last formula regarding tetration of base $2^{5^i k}$ with $k \neq 5h$, the most complicated case. With this particular base the tetration will repeat the last n digits much faster, so with relatively small hyper-exponents it is possible to repeat a large number of last digits. Consider the infinite tetration of base $2^{5^i k}$. Exists a number $\left\lfloor \min(b) \right\rfloor$ such that the following modular congruence is true:

$$m(2^{5^i k}) \equiv b(2^{5^i k}) \mod (10^n)$$

with $m \geq b$
As the case analyzed before:

$$m(2^{5^i k}) \equiv b(2^{5^i k}) \mod (5^n)$$
Since \((2, 5^n) = 1\), where \((x, y)\) represents the greatest common divisor between \(x\) and \(y\), certainly \(\exists u_0\) such that \(2^{u_0} \equiv 1 \mod (5^n)\)

The minimum value that \(u_0\) can assume is exactly equal to \(\lambda(5^n)\).

But \(\lambda(5^n) = \phi(5^n) = 5^n - 5^{-1} = 4 \cdot 5^{n-1}\)

This time we cannot proceed to the \(n\)-th floor because we have to make an important consideration. Indeed:

\[
\lambda(5^n) = 4 \cdot 5^{n-1}
\]

And so:

\[
2^{4 \cdot 5^{n-1}k} \equiv 1 \mod (5^n)
\]

And since:

\[
2^{(5^{n-1})} = \left[2^{5^t}\right]^{5^{(n-1)-t}}
\]

We’ll have that:

\[
\left\{\left[2^{5^t}\right]^{5^{(n-1)-t}}\right\}^{4k} \equiv 1 \mod (5^n)
\]

But \(2^{5^t}\) is the base of the tetration, so for example, the following modular congruence is not true:

\[
\left\{\left[32\right]^{5^{(n-1)}}\right\}^{4k} \not\equiv 1 \mod (5^n)
\]

But for \(t = 1\) the following one is true:

\[
\left\{\left[32\right]^{5^{(n-2)}}\right\}^{4k} \equiv 1 \mod (5^n)
\]

It is possible to reduce different floors in the following way:

- First floor: \(\mod 5^n\)
- Second floor: \(\mod 5^{n-(t+1)}\)
- Third floor: \(\mod 5^{n-2(t+1)}\)
- Fourth floor: \(\mod 5^{n-3(t+1)}\)

\[
\ldots
\]

- \(q\)-th floor: \(\mod 5^{n-(q-1)(t+1)}\)
- Second last floor: \(\mod 5\)
- Last floor: \(\mod 4\).

Consider the tetration of base \(2^{5^2}\mod (5^7)\):

\[
\lambda(5^7) = 4 \cdot 5^6
\]

So:

\[
2^{4 \cdot 5^8} \equiv 1 \mod (5^7)
\]

\[
\left\{\left[2^{5^2}\right]^{5^{(7-1)-2}}\right\}^{4} \equiv 1 \mod (5^7)
\]

\[
\left[2^{5^2}\right]^{4 \cdot 5^4} \equiv 1 \mod (5^7)
\]
So it’s possible to reduce the second floor \( \mod 5^4 \), so we have that:
\[
\lambda(5^4) = 4 \cdot 5^3
\]
But \( 4 \mid 2^{4 \cdot 5^3} \), so:
\[
2^{4 \cdot 5^3} \equiv 1 \mod (5^4)
\]
And then:
\[
\left\{ \left[ 2^{5^t} \right]^{5^{(4-1-2)}} \right\}^4 \equiv 1 \mod (5^4)
\]
So it’s possible to reduce the third floor \( \mod 5 \). If you repeat this process another time it follows that:
\[
\lambda(5) = 4
\]
So:
\[
2^4 \equiv 1 \mod 5
\]
So you can reduce the last floor \( \mod 4 \). But \( 2^{5^t} \) is congruent to \( 0 \mod 4 \), so we’ll have that:
\[
(2^{5^t})^{2^{5^t}} \equiv (2^{5^t})^0 \equiv 1 \mod 5
\]
The last floor is exactly equal to \( \left\lceil \frac{n}{t+1} \right\rceil + 1 \) where \( \lceil n \rceil \) represents the nearest integer to \( n \), greater or equal to \( n \).
In this case the last floor was the fourth (for \( n = 7 \) and \( t = 2 \)), indeed \( \left\lceil \frac{7}{2+1} \right\rceil + 1 = 3 + 1 = 4 \)
So:
\[
(m - \left\lceil \frac{n}{t+1} \right\rceil + 1)(2^{5^t}) \equiv (b - \left\lceil \frac{n}{t+1} \right\rceil + 1)(2^{5^t}) \equiv 1 \mod 5
\]
is true for \( b = \left\lceil \frac{n}{t+1} \right\rceil + 1 \), in fact the second tetration of \( 2^{5^t} \) is congruent to \( 1 \mod 5 \) for every integer number \( t \). So:
\[
m(2^{5^t}) \equiv (\left\lceil \frac{n}{t+1} \right\rceil + 1)(2^{5^t}) \mod (10^n)[5]
\]
\( \forall m \in \mathbb{N}, m \geq \left\lceil \frac{n}{t+1} \right\rceil + 1, \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \forall t \in \mathbb{N}, k \neq 5h \)

### 3 \( 5^k \) case

In this section will be proved the sixth formula about tetrations of base \( 5^{2^t} \).

Consider the infinite tetration of base \( 5^{2^t} \). Exists a number \( \left\lfloor \min(b) \right\rfloor \) such that the following modular congruence is true:
\[
m(5^{2^t}) \equiv b(5^{2^t}) \mod (10^n)
\]
with \( m \geq b \)
And in this case:
\[
m(5^{2^t}) \equiv b(5^{2^t}) \mod (2^n)
\]
Because $5^n \mid m(5^{2^k})$ Since $(5^{2^k}, 2^n) = 1$, where $(x, y)$ represents the greatest common divisor between $x$ and $y$, certainly $\exists u_0$ such that $5^{u_0} \equiv 1 \mod (2^n)$

The minimum value that $u_0$ can assume is exactly equal to $\lambda(2^n)$.

But $\lambda(2^n) = 2^{n-2}$ for $n \geq 3$.

So:

$$5^{2^{n-2}} \equiv 1 \mod (2^n)$$

And furthermore:

$$5^{(2^{n-2})} = \left[5^{2^t}\right]^{2^{(n-2-t)}}$$

But $5^{2^t}$ is the base of the tetration we’re analyzing.

So:

$$\left\{\left[5^{2^t}\right]^{2^{(n-2-t)}}\right\}^k \equiv 1 \mod (2^n)$$

It is possible to reduce different floors in the following way:

First floor: $\mod 2^n$
Second floor: $\mod 2^{n-(t+2)}$
Third floor: $\mod 2^{n-2(t+2)}$
Fourth floor: $\mod 2^{n-3(t+2)}$

... $q$-th floor: $\mod 2^{n-(q-1)(t+2)}$
Third last floor: $\mod 8$ if $n$ is odd, $\mod 4$ if $n$ is even.
Last second floor: $\mod 2$
Last floor: $\mod 1$.

Notice that exists a floor which, depending on the parity of $n$, is reducible $\mod 8$ or $\mod 4$; in fact if $n$ is even, that floor will be reducible $\mod 4$; while if $n$ is odd that floor will be reducible $\mod 8$.

Consider the tetration of base $5^{2^t} \mod 2^7$:

$$\lambda(2^7) = 2^5$$

So:

$$5^{2^5} \equiv 1 \mod (5^7)$$

$$\left[5^{2^5}\right]^{2^7-2^2-2} \equiv 1 \mod (5^7)$$

So the second floor can be reduced $\mod 2^3$

$$\lambda(2^3) = 2$$

So the third floor can be reduced $\mod 2$

And the last, fourth floor can be reduced $\mod 1$.

The number of floors until the last one, which is reducible $\mod 1$, is equal to $\left\lceil \frac{n}{t+2} \right\rceil + 2$. But:

$$(5^{2^t})^{(5^{2^t})} \equiv (5^{2^n})^0 \equiv 1 \mod 2$$
We are searching the minimum value of $b$ such that the following modular congruence is true mod 2:

$$(m-\left\lfloor \frac{m}{t+2} \right\rfloor +2)(5^{2^t}) \equiv (b-\left\lfloor \frac{b}{t+2} \right\rfloor +2)(5^{2^t}) \equiv 1 \mod 2$$

So $b = \left\lfloor \frac{b}{t+2} \right\rfloor - 1$

Notice that:

$5^{2^t} \equiv 1 \mod 2$

So $\min(b) = \left\lfloor \frac{b}{t+2} \right\rfloor - 1$ because:

$$(m-\left\lfloor \frac{m}{t+2} \right\rfloor +2)(5^{2^t}) \equiv (\left\lfloor \frac{b}{t+2} \right\rfloor -1 -\left\lfloor \frac{b}{t+2} \right\rfloor +2)(5^{2^t}) \equiv 1(5^{2^t}) \equiv 1 \mod 2$$

So:

$m(5^{2^t}) \equiv (\left\lfloor \frac{b}{t+2} \right\rfloor-1)(5^{2^t}) \mod (10^n)[6]$

$\forall m \in N, m \geq \left\lfloor \frac{b}{t+2} \right\rfloor - 1, \forall n \in N, \forall k \in N, \forall t \in N, k \neq 2h$

4 Analysis of the results and some examples

$m(2^{5^t}) \equiv (\left\lfloor \frac{b}{t+2} \right\rfloor+1)(2^{5^t}) \mod (10^n)[5]$

$\forall m \in N, m \geq \left\lfloor \frac{b}{t+2} \right\rfloor + 1, \forall n \in N, \forall k \in N, \forall t \in N, k \neq 5h$

$m(5^{2^t}) \equiv (\left\lfloor \frac{b}{t+2} \right\rfloor -1)(5^{2^t}) \mod (10^n)[6]$

$\forall m \in N, m \geq \left\lfloor \frac{b}{t+2} \right\rfloor - 1, \forall n \in N, \forall k \in N, \forall t \in N, k \neq 2h$

It's easy to see the simmetry between those expressions, and notice that for same values of $n$ and $t$:

$$\left\lfloor \frac{n}{t+1} \right\rfloor + 1 \geq \left\lfloor \frac{n}{t+2} \right\rfloor - 1$$

This implies that tetrations of base $5^{2^t}$ will begin to repeat their last digits with a smaller hyper-exponent respect to tetrations of base $2^{5^t}$, using same values of $n$ and $t$. Furthermore the "fastest" tetration of base $2^k$ in repeating last digits is the one which base is of the form $2^{5^t}$.

4.1 Example 1

Consider the $127$-th tetration of base $2^{5^3}$ We know that, using the fifth formula, the last 20 digits are the same starting from $m = 5$, where $m$ is the hyper-exponent of this tetration.

$$12^{7}(2^{5^3}) \equiv (\left\lfloor \frac{m}{t+1} \right\rfloor+1)(2^{5^3}) \mod (10^{20}) \equiv (5)(2^{9375}) \mod (10^{20})$$

$$^{(5)}(2^{9375}) \equiv 48108335298171109376 \mod (10^{20})$$
(6) \((2^{9375}) \equiv 4810835298171109376 \mod (10^{20})\)

(7) \((2^{9375}) \equiv 4810835298171109376 \mod (10^{20})\)

\[\ldots\]

But, as a consequence that \(b\) was minimum:

(4) \((2^{9375}) \not\equiv 4810835298171109376 \mod (10^{20})\)

4.2 Example 2

Consider the 145-th tetration of base 5\(^{2^{13}}\cdot 7\). We know that, using the fifth formula, the last 37 digits are the same starting from \(m = 2\), where \(m\) is the hyper-exponent of this tetration.

\[145 (5^{2^{13}}) \equiv (\lceil \frac{37}{13+2}\rceil + 1) (5^{2^{13}}) \mod (10^{37}) \equiv (2) (5^{57344}) \mod (10^{37})\]

(2) \((5^{57344}) \equiv 8756454076865338720381259918212890625 \mod (10^{37})\)

(3) \((5^{57344}) \equiv 8756454076865338720381259918212890625 \mod (10^{37})\)

(4) \((5^{57344}) \equiv 8756454076865338720381259918212890625 \mod (10^{37})\)

But:

(1) \((5^{57344}) \not\equiv 8756454076865338720381259918212890625 \mod (10^{37})\)

5 Conclusions

Tetrations are fascinating, and with this theorem we can have more informations about these giant numbers without computing them. It’s possible to use these results on a software in order to reduce tetrations \(\mod 10^n\) faster.

References

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