CRITICAL SETS OF RANDOM SMOOTH FUNCTIONS ON COMPACT MANIFOLDS

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ABSTRACT. We prove a Chern-Lashof type formula computing the expected number of critical points of smooth function on a smooth manifold $M$ randomly chosen from a finite dimensional subspace $V \subseteq C^\infty(M)$ equipped with a Gaussian probability measure. We then use this formula this formula to find the asymptotics of the expected number of critical points of a random linear combination of a large number eigenfunctions of the Laplacian on the round sphere, tori, or a products of two round spheres. In the case $M = S^1$ we show that the number of critical points of a trigonometric polynomial of degree $\leq \nu$ is a random variable $Z_\nu$ with expectation $E(Z_\nu) \sim 2\sqrt{0.6}\nu$ and variance $\text{var}(Z_\nu) \sim c\nu$ as $\nu \to \infty$, $c \approx 0.35$.

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INTRODUCTION

Suppose that $M$ is a compact, connected smooth manifold of dimension $m$. Given a finite dimensional vector space $V \subseteq C^\infty(M)$ of dimension $N$ we would like to know the average (expected) size of the critical set of a function $v \in V$. For the applications we have in mind $N \gg m$. We will refer to $V$ as the sample space and we will denote by $\mu(v)$ the number of critical points of the function $v \in V$.

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More explicitly, we fix a Euclidean inner product $h$ on $V$ and we denote by $S(V)$ the unit sphere in $V$. We define expected number of critical points $\mu(M, V, h)$ of a random function in $V$ to be the quantity

$$\mu(M, V, h) : = \frac{1}{\text{area}(S(V))} \int_{S(V)} \mu(v)|dS_h(v)|$$

$$= \frac{1}{(2\pi \sigma^2)^{\frac{N}{2}}} \int_V e^{-\frac{|v|^2}{2\sigma^2}} \mu(v) |dv_h(v)|, \ \forall \sigma > 0. \quad (\mu_0)$$

In other words, $\mu(M, V, h)$ is the expectation of the random variable $Z_{V, h}$

$$S(V) \ni v \mapsto Z_{V, h} := \mu(v), \quad (Z)$$

where $S(V)$ is equipped with the probability measure determined by the suitably rescaled area density determined by the metric $h$.

Let us observe that we can cast the above setup in the framework of Gaussian random fields, [1, 9]. Fix an orthonormal basis $(\Psi_\alpha)_{1 \leq \alpha \leq N}$ of $(V, h)$, and a Gaussian probability measure on $V$,  

$$\gamma = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|v|^2}{2}} |dv|.$$

Then the functions $\xi_\alpha : V \to \mathbb{R}, \ v \mapsto \xi_\alpha(v) = (v, \Psi_\alpha)$, are independent, normally distributed random variables with mean 0 and variance 1, and the equality

$$v(x) = \sum_\alpha \xi_\alpha(v) \Psi_\alpha(x)$$

defines an $\mathbb{R}$-valued centered Gaussian random field on $M$ with covariance kernel

$$K_V(x, y) = \sum_\alpha \Psi_\alpha(x) \Psi_\alpha(y), \ \forall x, y \in M.$$

Thus, we are seeking the expectation of the number of critical points of a sample function of this field. However, in this paper, most of the time, this point of view, will only stay in the background. A notable exception is Theorem 6.1 whose proof relies in an essential way on results from the theory of stationary gaussian processes.

It is possible that all the functions in $V$ have infinite critical sets, in which case the integrals in $(\mu_0)$ are infinite. To avoid this problem we impose an ampleness condition on $V$. More precisely, we require that for any point $x \in M$, and any covector $\xi \in T_x^* M$ there exists a function $v \in V$ whose differential at $x$ is $\xi$. As explained in [28, 9]), this condition implies that almost all functions $v \in V$ are Morse functions and thus have finite critical sets.

The above ampleness condition can be given a different interpretation by introducing the evaluation map

$$ev = ev^V : M \to V^\vee := \text{Hom}(V, \mathbb{R}), \ x \mapsto ev_x,$$

where for any $x \in M$ the linear map $ev_x : V \to \mathbb{R}$ is given by

$$ev_x(v) = v(x), \ \forall v \in V.$$ 

The ampleness condition is equivalent with the requirement that the evaluation map be an immersion.

This places us in the setup considered by J. Milnor [24] and Chern-Lashof [9]. These authors investigated immersions of a compact manifold $M$ in an Euclidean space $E$, and they computed the average number of critical points of the pullback to $M$ of a random linear function on $E$. That is precisely our problem with $E = V^\vee$ since a function $v \in V$ can be viewed canonically as a linear function on $V^\vee$.

The papers [9, 24] contain a philosophically satisfactory answer to our initial question. The expected number of critical points is, up to a universal factor, the integral over $M$ of a certain scalar
called the total curvature of the immersion and canonically determined by the second fundamental form of the immersion.

Our interests are a bit more pedestrian since we are literally interested in estimating the expected number of critical points when \( \dim V \to \infty \). In our applications, unlike the situation analyzed in [9, 24], the metric on \( M \) is not induced by a metric on \( V \), but the other way around. We typically have a natural Riemann metric on \( M \) and then we use it to induce a metric on \( V \), namely, the restriction of the \( L^2 \)-metric on \( C^\infty(M) \) defined by our Riemann metric on \( M \). The first theoretical goal of this paper is to rewrite the results in [9, 24] in a computationally friendlier form.

More precisely, we would like to describe a density \( |d\mu| \) on \( M \) such that

\[
\mu(M, V, h) = \frac{1}{\text{area}(S(V))} \int_M |d\mu| = \frac{1}{\sigma_{N-1}} \int_M |d\mu|,
\]

where \( |d\mu| \) captures the infinitesimal behavior of the family of functions \( V \).

In Corollary 1.3 we describe such a density on \( M \) by relying on a standard trick in integral geometry. Our approach is different from the probabilistic method used in the proof of the closely related result, [8, Thm. 4.2], which is a higher dimensional version of a technique pioneered by M. Kac and S. Rice, [1, 21, 32].

It is easier to explain Corollary 1.3 if we fix a metric \( g \) on \( M \). The density \( |d\mu| \) can be written as \( |d\mu| = \rho_g \, dV_g \), for some smooth function \( \rho_g : M \to [0, \infty) \). For \( x \in M \), the number \( \rho_g(x) \) captures the average infinitesimal behavior of the family \( V \) at \( x \). Here is the explicit description of \( \rho_g(x) \).

Denote by \( K_x \) the subspace of \( V \) consisting of the functions that admit \( x \) as a critical point. Let \( S(K_x) \) denote the unit sphere in \( K_x \) defined by the metric \( h \) on \( V \). Any function \( v \in K_x \) has a well-defined Hessian at \( x \), \( \text{Hess}_x(v) \), that can be identified via the metric \( g \) with a symmetric linear operator

\[
\text{Hess}_x(v, g) = \text{Hess}_x(v, g) : T_xM \to T_xM.
\]

We set

\[
\Delta_x(V) := \int_{S(K_x)} |\det \text{Hess}_x(v)| \, |dS(v)| = \frac{2}{\Gamma(N)} \int_{K_x} |\det \text{Hess}_x(v)| e^{-|v|^2} \, |dv|.
\]

The differential of the evaluation map at \( x \) is a linear map \( A^1_x : T_xM \to V^\vee \), and we denote by \( J_g(A^1_x) \) its Jacobian, i.e., the norm of the induced linear map \( \Lambda^m A^1_x : \Lambda^m T_xM \to \Lambda^m V^\vee \). Then

\[
\rho_g(x) = \frac{\Delta_x(V, g)}{J(A^1_x)},
\]

and thus

\[
\mu(M, V, h) = \frac{1}{\sigma_{N-1}} \int_M \Delta_x(V) |dV_g(x)|
\]

\[
= (2\pi)^{-m/2} \int_M \frac{1}{J_g(A^1_x)} \left( \int_{K_x} |\det \text{Hess}_x(v)| \frac{e^{-|v|^2}}{(2\pi)^{\dim K_x/2}} \, |dV_K(v)| \right) |dV_g(x)|.
\]

We want to emphasize that the density \( \rho_g |dV_g| \) is independent of the metric \( g \), but it does depend on the metric \( h \) on \( V \). In particular, the expectation \( \mu(M, V, h) \) does depend on the choice of metric \( h \). For all the applications we have in mind, the metric \( h \) on \( V \) is obtained from a metric \( g \) on \( M \) in the fashion explained above. In this case we will use the notation \( \mu(M, V, g) \). Remark 2.2 contains a rather dramatic illustration of what happens when \( h \) is induced by a Sobolev metric other than \( L^2 \).
The equality \((\mu_1)\) can be used in some instances to compute the variance of the number of critical points of a random function in \(V\). More precisely, to a sample space \(V \subset C^\infty(M \times M)\) we associate a sample space \(V_\Delta \subset C^\infty(M \times M)\), the image of \(V\) via the diagonal injection
\[
\Delta : C^\infty(M) \to C^\infty(M \times M), \quad \Delta v(x, y) = v(x) + v(y), \quad \forall v \in V, \quad x, y \in M.
\]
We equip \(V_\Delta\) with the metric \(h_\Delta\) so that the map \(\Delta : (V, h) \to (V_\Delta, h_\Delta)\) is an isometry. Since \(\mu(\Delta(v)) = \mu(v)^2\) we deduce that
\[
E(Z_{V, h}) = E(Z_{V_\Delta, h_\Delta}),
\]
where \(E\) denotes the expectation of a random variable, and \(Z_{\cdot, \cdot}\) are defined as in \((Z)\).

We also want to point out that if we remove the absolute value from the integrand \(\text{Hess}(v)\) in \((\mu_1)\), then we obtain a Gauss-Bonnet type theorem
\[
\chi(M) = \frac{1}{\pi} \int_M \frac{1}{J_g(A^\perp_{\gamma})) \left( \int_{K_\gamma} \det \text{Hess}_\gamma(v) e^{-|v|^2} |dv| \right) |dV_g(x)|,
\]
where \(\chi(M)\) denotes the Euler characteristic of \(M\).

Most of our applications involve sequences of subspaces \(V_n \subset C^\infty(M)\) such that \(\dim V_n \to \infty\), and we investigate the asymptotic behavior of \(\mu(M, V_n, g)\) as \(n \to \infty\), where \(g\) is a metric on \(M\). One difficulty in applying \((\mu_1)\) comes from the definition \((\Delta)\) which involves integrals over spheres of arbitrarily large dimensions. There is a simple way of dealing with this issue when \(V\) is 2-jet ample, that is, for any \(x \in M\), and any 2-jet \(j_x\) at \(x\), there exists \(v \in V\) whose 2-jet at \(x\) is \(j_x\).

Denote \(\text{Sym}^M_x\), the space of selfadjoint linear operators \((T_x M, g) \to (T_x M, g)\). In this case, the linear map \(\text{Hess} : K_x \to \text{Sym}^M_x\) is onto. The pushforward by \(\text{Hess}_x\) of the Gaussian probability measure \(\gamma_x\) on \(K_x\)
\[
\gamma_x = \frac{e^{-|v|^2}}{(2\pi)^{\dim K_x} dV_{K_x}(v)},
\]
is a (centered) Gaussian probability measure \(\hat{\gamma}_x^V\) on \(\text{Sym}^M_x\); see [20, §16]. In particular, \(\hat{\gamma}_x^V\) is uniquely determined by its covariance matrix. This is a symmetric, positive definite linear operator
\[
\hat{c}_x^V : \text{Sym}^M_x \to \text{Sym}^M_x.
\]
We can then rewrite \((\mu_1)\) as
\[
\mu(M, V, h) = (2\pi)^{-m/2} \int_M J_g(A^\perp_{\gamma}) \left( \int_{\text{Sym}^M_x} \det H ||d\hat{\gamma}_x^V(H)|| |dV_g(x)| \right).
\]
This is very similar to the integral formula employed by Douglas-Shiffman-Zelditch, \([13, 14]\), in their investigation of critical sets of compact holomorphic sections of (ample) holomorphic line bundles.

In concrete situations a more ad-hoc method may be more suitable. Suppose that for every \(x\) we can find a subspace \(L_x \subset K_x\) of dimension \(\ell(x)\), such that for any \(v \in K_x\), \(v \perp L_x\) we have \(\text{Hess}_x(v) = 0\). Noting that \(\dim K_x = \dim V - \dim M = N - m\) and \(\dim L_x = N - m - \ell(x)\) we obtain
\[
\int_{K_x} e^{-|v|^2} \det \text{Hess}_x(v) |dV| = \left( \int_{L_x} e^{-|u|^2} |dV(u)| \right) \times
\]
\[
\times \left( \int_{L_x} e^{-|w|^2} \det \text{Hess}_x(w) |dV| \right).
\]
Using (σ) we can now rewrite (µ₁) as

\[
\mu(M, V; h) = \pi^{-\frac{d}{2}} \int_M \frac{e^{-|w|^2}}{J_d(A_\mu^1)} |dV_\sigma(x)|.
\]

Our first application of formula (µ₂) is in the case \( M = S^{d-1} \) and \( g \) is the round metric \( g_d \) of radius 1 on the \((d - 1)\) sphere. The eigenvalues of the Laplacian \( \Delta_d \) on \( S^{d-1} \) are

\[
\lambda_n(d) = n(n + d - 2), \quad n = 0, 1, 2, \ldots.
\]

For any nonnegative integer \( n \), and any positive real number \( \nu \), we set

\[
y_{n,\nu} := \ker(\Delta_d - \lambda_n(d)) \quad V_\nu(d) := \bigoplus_{n \leq \nu} eY_{n,d}.
\]

In Theorem 2.1 and Corollary 5.2 we show that for any \( d \geq 2 \) there exists a universal constant \( K_d > 0 \) such that

\[
\mu(S^{d-1}, V_\nu(d), g_d) \sim K_d \dim V_\nu(d) \sim \frac{2K_d}{(d-1)!} \nu^{d-1} \text{ as } \nu \to \infty.
\]

We denote by \( \zeta_n \) the expected number \( \zeta_n \) of nodal domains of a random spherical harmonic of degree \( n \) then, according to the recent work of Nazarov and Sodin, [26], there exists a positive constant \( a \) such that

\[
\zeta_n \sim an^2 \text{ as } n \to \infty.
\]

We next consider various spaces of trigonometric polynomials on an \( L \)-dimensional torus \( \mathbb{T}^L \). To a finite subset \( \mathcal{M} \subset \mathbb{Z}^L \) we associate the space \( V(M) \) of trigonometric polynomials on \( \mathbb{T}^L \) spanned by the “monomials”

\[
\cos(m_1 \theta_1 + \cdots + m_L \theta_L), \quad \sin(m_1 \theta_1 + \cdots + m_L \theta_L), \quad (m_1, \ldots, m_L) \in \mathcal{M},
\]

and in Theorem 3.1 we give a formula for the expected number \( \mu(M) \) of critical points of a trigonometric polynomial in \( V(M) \). We consider the special case when

\[
\mathcal{M} = \mathcal{M}_\nu^L := \{(m_1, \ldots, m_L) \in \mathbb{Z}^L; \ |m_i| \leq \nu, \ \forall i = 1, \ldots, L \}
\]

and in Theorem 3.2 we show that as \( \nu \to \infty \) we have

\[
\mu(\mathcal{M}_\nu^L) \sim \left( \frac{\pi}{6} \right)^{\frac{L}{2}} (|\det X|)_\infty \times \dim V(\mathcal{M}_\nu^L).
\]
Above, $\langle |\det X| \rangle_\infty$ denotes the expected value of the absolute value of random symmetric $L \times L$ matrix, where the space $\text{Sym}_L$ of such matrices is equipped with a certain gaussian probability measure that we describe explicitly. In particular, when $L = 1$, we have

$$\mu(M^1_\nu) \sim \sqrt{\frac{3}{5}} \dim V(M^1_\nu) = 2\nu \sqrt{\frac{3}{5}}, \quad (E)$$

while for $L = 2$ we have

$$\mu(M^2_\nu) \sim z_2 \dim V(M^2_\nu).$$

The proportionality constant $z_2$ can be given an explicit, albeit complicated description in terms of elliptic functions. In particular,

$$z_2 \approx 0.4717\ldots$$

In the case $L = 1$ we were able to prove a bit more. We denote by $Z_\nu$ the number of critical points of a random trigonometric polynomial in $M^1_\nu$. Then $Z_\nu$ is a random variable with expectation $E(Z_\nu)$ satisfying the asymptotic behavior $(E)$. In Theorem 6.1 we prove that its variance satisfies the asymptotic behavior

$$\text{var}(Z_\nu) \sim \delta_\infty \nu, \quad (V)$$

where $\delta_\infty$ is a positive constant ($\delta_\infty \approx 0.35$) described explicitly by an integral formula, (6.1).

We also compute the average number of critical points of a real trigonometric polynomial in two variables of the form

$$\{ a \cos x + b \sin x + c \cos y + d \sin y + p \cos(x + y) + q \sin(x + y) \}.$$

This family of trigonometric polynomials was investigated by V.I. Arnold in [6] where he proves that a typical polynomial of this form has at most 8 critical points. In Theorem 3.6 we prove that the average number of critical points of a trigonometric polynomial in this family is $\frac{4\pi}{3} \approx 4.188$. Note that the minimum number of critical points of Morse function on the 2-torus is 4, and the above average is very close to this minimal number.

We then consider products of spheres $S^{d_1-1} \times S^{d_2-1}$ equipped with the product of the round metrics $g_{d_1} \times g_{d_2}$. In Theorems 4.4 and 5.6 we show that, for any $d_1, d_2 \geq 2$, there exists a constant $K_{d_1,d_2} > 0$ such that, for any $r \geq 1$, as $\nu \to \infty$, we have

$$\mu(S^{d_1-1} \times S^{d_2-1}, V_{\nu^r}(d_1) \otimes V_{\nu^r}(d_2)) \sim K_{d_1,d_2}(\dim V_{\nu^r}(d_1) \otimes V_{\nu^r}(d_2))^{\varpi(d_1,d_2,r)}, \quad (C)$$

where

$$\varpi(d_1,d_2,r) = \begin{cases} 1, & (d_1 - 2)(d_2 - 2) = 0, \\ \frac{(d_1 - 3)r + d_2 + 1}{(d_1 - 1)r + d_2 - 1}, & (d_1 - 2)(d_2 - 2) \neq 0. \end{cases} \quad (\varpi)$$

Let us point out that for $d_1, d_2 > 2$, the function $r \mapsto \varpi(d_1, d_2, r), r \geq 1$, is decreasing, nonnegative,

$$\lim_{r \to \infty} \varpi(d_1, d_2, r) = \frac{d_1 - 3}{d_2 - 1} = \kappa(d_1, d_2) \quad \text{and} \quad \varpi(d_1, d_2, r = 1) = 1.$$

In particular,

$$\varpi(d_1, d_2, r) < 1, \quad \forall r > 1.$$

More surprisingly,

$$\varpi(d_1, d_2, r) = \varpi(d_2, d_1, r) = 1 \quad \text{if} \quad (d_1 - 2)(d_2 - 2) = 0,$$

but this symmetry is lost if $(d_1 - 2)(d_2 - 2) \neq 0$.

We find the asymmetry displayed in $(C) + (\varpi)$ very surprising and we would like to comment a bit on this aspect.
Observe that the union of the increasing family of subspaces $W_{\nu,r} = V_{\nu}^r(d_1) \otimes V_{\nu}^r(d_2)$ is dense in the Fréchet topology of $C^\infty(M)$, $M = S^{d_1 - 1} \times S^{d_2 - 1}$. The space $C^\infty(M)$ carries a natural stratification, where the various strata encode various types of degeneracies of the critical sets of functions on $M$. The top strata are filled by (stable) Morse function. This stratification traces stratifications on each of the subspaces $W_{\nu,r}$ and, as $\nu \to \infty$, the combinatorics of the induced stratification on $W_{\nu,r}$ captures more and more of the combinatorics of the stratification of $C^\infty(M)$.

The equality (C) shows that if $r' > r \geq 1$, the functions in $W_{\nu,r'}$ have, on average, relatively more critical points than the functions in $W_{\nu,r}$. This suggest that the subspace $(W_{\nu,r})$ captures more of the stratification of $C^\infty(M)$ than $W_{\nu,r'}$, and in this sense it is a more efficient approximation. The best approximation would be when $r = 1$, i.e., when the two factors $S^{d_2 - 1}$ participate in the process as equal spectral partners. Note that this asymmetric behavior is not present when one of the factors is $S^1$.

This heuristic discussion suggests the following concepts. Suppose that $M$ is a compact, connected Riemann manifold of dimension $m$. Define an approximation regime on $M$ to be a sequence of finite dimensional subspaces $W_* = (W_\nu)_{\nu \geq 1}$ of $C^\infty(M)$ such that

$$W_1 \subset W_2 \subset \cdots$$

and their union is dense in the Fréchet topology of $C^\infty(M)$. For any Riemann metric $g$ on $M$, we define the upper/lower complexities of such a regime to be the quantities

$$\kappa^*(W_*, g) := \limsup_{\nu \to \infty} \frac{\log \mu(M, W_\nu, g)}{\log \dim W_\nu}, \quad \kappa_*(W_*, g) := \liminf_{\nu \to \infty} \frac{\log \mu(M, W_\nu, g)}{\log \dim W_\nu}.$$

Intuitively, the approximation regimes with high upper complexity offer better approximations of $C^\infty(M)$. Finally, set

$$\kappa^*(M) := \sup_{W_*, g} \kappa^*(W_*, g), \quad \kappa_*(M) := \inf_{W_*, g} \kappa_*(W_*, g).$$

The above results imply that

$$\kappa^*(S^{d_2 - 1}), \quad \kappa^*(S^{d_1 - 1} \times S^{d_2 - 1}) \geq 1, \quad \forall d_1, d_2 \geq 2,$$

$$\kappa_*(S^{d_1 - 1} \times S^{d_2 - 1}) \leq \frac{d_1 - 3}{d_2 - 1}, \quad \forall d_1, d_2 \geq 3.$$

In particular, this shows that for any $d \geq 3$, we have

$$\kappa_*(S^2 \times S^{d_2 - 1}) = 0.$$

In Example 5.4 we\footnote{The construction of the approximation regime in Example 5.4 was worked out during a very lively conversation with my colleague Richard Hind who was confident of its existence.} construct an approximation regime $(W_n)_{n \geq 1}$ on $S^1$ such that

$$\lim_{n \to \infty} \frac{\log \mu(S^1, W_n)}{\log \dim W_n} = \infty,$$

so that $\kappa^*(S^1) = \infty$.

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Notations

(i) $i := \sqrt{-1}$.

(ii) We will denote by $\sigma_n$ the “area” of the round $n$-dimensional sphere $S^n$ of radius 1, and by $\omega_n$ the “volume” of the unit ball in $\mathbb{R}^n$. These quantities are uniquely determined by the equalities (see [27, Ex. 9.1.11])

$$\sigma_{n-1} = n\omega_n = 2\frac{n^{n/2}}{\Gamma(n/2)}, \quad \Gamma(1/2) = \sqrt{\pi}, \quad (\sigma)$$

where $\Gamma$ is Euler’s Gamma function.

(iii) For any Euclidean space $V$, we denote by $S(V)$ the unit sphere in $V$ centered at the origin and by $B(V)$ the unit ball in $V$ centered at the origin.

(iv) If $V_0$ and $V_1$ are two Euclidean spaces of dimensions $n_0, n_1 < \infty$ and $A : V_0 \to V_1$ is a linear map, then the Jacobian of $A$ is the nonnegative scalar $J(A)$ defined as the norm of the linear map

$$\Lambda^k A : \Lambda^k V_0 \to \Lambda^k V_1, \quad k := \min(n_0, n_1).$$

More concretely, if $n_0 \leq n_1$, and $\{e_1, \ldots, e_{n_0}\}$ is an orthonormal basis of $V_0$, then

$$J(A) = \left( \det G(A) \right)^{1/2}, \quad (J_-)$$

where $G(A)$ is the $n_0 \times n_0$ Gramm matrix with entries

$$G_{ij} = \langle Ae_i, Ae_j \rangle_{V_1}.$$

If $n_1 \geq n_0$ then

$$J(A) = J(A^\dagger) = \left( \det G(A^\dagger) \right)^{1/2}, \quad (J_+)$$

where $A^\dagger$ denotes the adjoint (transpose) of $A$. Equivalently, if $d\text{Vol}_1 \in \Lambda^{n_1} V_1^\ast$ denotes the metric volume form on $V_1$, and $d\text{Vol}_A$ denotes the metric volume form on $\text{ker} A$, then $J(A)$ is the positive number such that

$$d\text{Vol}_0 = \pm d\text{Vol}_A \wedge A^* d\text{Vol}_1. \quad (J'_+)$$

(v) For any nonnegative integer $d$, we denote by $[x]_d$ the degree $d$ polynomial

$$[x]_d := x(x-1) \cdots (x-d+1),$$

and by $B_d(x)$ the degree $d$ Bernoulli polynomial defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{d \geq 0} B_d(x) \frac{t^d}{d!}.$$

The $d$-th Bernoulli number is $B_d := B_d(0)$, while the leading coefficient of $B_d(x)$ is equal to 1, and,

$$\frac{B_{d+1}(\nu + 1) - B_{d+1}}{d + 1} = \sum_{n=1}^{\nu} n^d, \quad \forall \nu \in \mathbb{Z}_{>0}. \quad (S)$$

More generally, for any smooth function $f : (0, \infty) \to \mathbb{R}$ and any positive integers $\nu, m$, we have the Euler-Maclaurin summation formula, (see [3, Thm D.2.1] or [33, §7.21]),

$$\sum_{n=1}^{\nu-1} f(n) = \int_1^{\nu} f(x)dx + \sum_{k=1}^{m} \frac{b_k}{k!} \left( f^{(k-1)}(\nu) - f^{(k-1)}(1) \right) + \frac{(-1)^{m-1}}{m!} \int_1^{\nu} \bar{B}_m(x) f^{(m)}(x)dx, \quad (EM)$$
where $b_k$ denotes the $k$-th Bernoulli number, $b_k := B_k(0)$, and $\bar{B}_m$ denotes the associated periodic function

$$
\bar{B}_m(x) := B_m(x - \lfloor x \rfloor), \quad \forall x \in \mathbb{R}.
$$

We will use one simple consequence of the Euler-Maclaurin summation formula. Suppose that $f(x)$ is a rational function of the form

$$
f(x) = \frac{P_0(x)}{P_1(x)},
$$

where $P_0(x)$ and $P_1(x)$ are polynomials with leading coefficients 1 and of degrees $d_0 > d_1$. We further assume that $f$ has no poles at nonnegative integers. Then

$$
\sum_{n=1}^{\nu} f(n) \sim \frac{1}{d_0 - d_1 + 1} \nu^{d_0 - d_1 + 1} \quad \text{as} \quad \nu \to \infty.
$$

\(S_\infty\)

1. AN ABSTRACT RESULT

Suppose that $(M, g)$ is compact, connected Riemann manifold of dimension $m$. We denote by $|dV_g|$ the induced volume density.

Let $V \subset C^\infty(M)$ be a vector subspace of finite dimension $N$. We set $V^\vee := \text{Hom}(V, \mathbb{R})$, and we fix a Euclidean metric $\bar{h} = (\cdot, \cdot)$ on $V$. We denote by $S(V)$ the unit sphere in $V$ with respect to this metric and by $|dS|$ the area density on $S(V)$. The goal of this section is to give an integral geometric description of the quantity

$$
\mu(M, V) = \mu(M, g, V, \bar{h}) := \frac{1}{\text{area}(S(V))} \int_{S(V)} \mu_M(v) |dS(v)|.
$$

The significance of $\mu(M, V)$ is clear: it is the expected number of critical points of a random function $v \in S(V)$.

To formulate our main result we need to introduce some notation. We form the trivial vector bundle $V_M := V \times M$. Observe that the dual bundle $V_M^\vee = V^\vee \times M$ is equipped with a canonical section

$$
\ev : M \to V^\vee, \quad M \ni x \mapsto \ev_x \in V^\vee, \quad \ev_x(v) = v(x), \quad \forall v \in V.
$$

Using the metric identification $V^\vee \to V$ we can regard $\ev$ as a map $M \to V$. More explicitly, if $(\Psi_\alpha)_{1 \leq \alpha \leq N}$ is an orthonormal basis of $V$, then

$$
\ev_x = \sum_{\alpha} \Psi_\alpha(x) \cdot \Psi_\alpha \in V.
$$

We have an adjunction morphism

$$
A : V \times M \to T^*M, \quad V \times M \ni (v, x) \mapsto A_x v := d_x v \in T^*_x M,
$$

where $d_x$ denotes the differential of the function $v$ at the point $x \in M$. We will assume that the vector space $V$ satisfies the ampleness condition

$$
\forall x \in M \quad \text{the linear map} \quad V \ni v \mapsto A_x v \in T^*_x M \quad \text{is surjective}. \quad (1.1)
$$

The assumption (1.1) is equivalent to the condition:

$$
\text{the evaluation map} \quad \ev : M \to V^\vee \quad \text{is an immersion}. \quad (1.2)
$$

As explained in [28, §1.2], the condition (1.1) implies that for generic $v \in V$, the restriction of the function $v$ to $K$ is a Morse function. We denote by $\mu_M(v)$ its number of critical points.

For every $x \in M$, we denote by $K_x$ the kernel of the map $A_x$. The ampleness condition (1.1) implies that $K_x$ is a subspace of $V$ of codimension $m$. Observe that the collection of spaces $(K_x)_x$
is naturally organized as a codimension \(m\)-subbundle \(K \to M\) of \(V_M\), namely the kernel bundle of \(A\).

Consider the dual bundle morphism \(A^\dagger : TM \to V^\vee \times M\). Using the metric identification \(V^\vee \to V\) we can regard \(A^\dagger\) as a bundle morphism \(A^\dagger : TM \to V_M\). Its range is \(K^\perp\), the orthogonal complement to the kernel of \(A\). Note that if \(\{\Psi_{\alpha}\}_{1 \leq \alpha \leq N}\) is an orthonormal frame of \(V\), \(x_0 \in M\), and \(X \in T_{x_0} M\), then
\[
A_{x_0}^\dagger X = \sum_{\alpha=1}^N \left( (X \cdot \Psi_{\alpha})(x_0) \right) \cdot \Psi_{\alpha} \in V.
\]
The trivial bundle \(V_M\) is equipped with a trivial connection \(D\). More precisely, we regard a section of \(u\) of \(\mathbf{V}_M\) as a smooth map \(u : M \to V\). Then, for any vector field \(X\) on \(M\), we define \(D_X u\) as the smooth function \(M \to V\) obtained by derivating \(u\) along \(X\). Note that \(A^\dagger = D e v\).

We have an orthogonal direct sum decomposition \(\mathbf{V}_M = K^\perp \times K\). For any section \(u\) of \(\mathbf{V}_M\), we denote by \(u^\perp\) the component of \(u\) along \(K^\perp\), and by \(u^0\) its component along \(K\). The shape operator of the subbundle \(K^\perp\) is the bundle morphism \(\Xi : TM \otimes K^\perp \to K\) defined by the equality
\[
\Xi(X, u) := (D_X u)^0, \quad \forall X \in C^\infty(TM), \quad u \in C^\infty(K^\perp).
\]
For every \(x \in M\), we denote by \(\Xi_x\) the induced linear map \(\Xi_x : T_x M \otimes K^\perp_x \to K_x\). If we denote by \(\text{Gr}_m(V)\) the Grassmannian of \(m\)-dimensional subspaces of \(V\), then we have a Gauss map
\[
M \ni x \mapsto \mathcal{G}(x) := K^\perp_x \subset \text{Gr}_m(V).
\]
For \(x \in M\), the shape operator \(\Xi_x\) can be viewed as a linear map
\[
\Xi_x : T_x M \to \text{Hom}(K^\perp_x, K_x) = T_{K^\perp_x} \text{Gr}_m(V),
\]
and, as such, it can be identified with the differential of \(\mathcal{G}\) at \(x\), [27, §9.1.2]. Any \(v \in K_x\) determines a bilinear map
\[
\Xi_x \cdot v : T_x M \otimes K^\perp_x \to \mathbb{R}, \quad \Xi_x \cdot v(e, u) = \Xi_x(e, u) \cdot v,
\]
where, for simplicity, we have denoted by \(\cdot\) the inner product in \(V\). By choosing orthonormal bases \((e_i)\) in \(T_x M\) and \((u_j)\) of \(K_x\) we can identify this bilinear form with an \(m \times m\)-matrix. This matrix depends on the choices of bases, but the absolute value of its determinant is independent of these bases. It is thus an invariant of the pair \((\Xi_x, v)\) that we will denote by \(|\det \Xi_x \cdot v|\).

**Theorem 1.1.**
\[
\mu(M, V) = \frac{1}{\sigma_{N-1}} \int_M \left( \int_{S(K_{x})} |\det \Xi_x \cdot v| |dS(v)| \right) |dV_g(x)|. \tag{1.3}
\]

**Proof.** We denote by \(E_x\) the intersection of \(K_x\) with the sphere \(S(V)\) so that \(E_x\) is a geodesic sphere in \(S(V)\) of dimension \((N - m - 1)\). Now consider the incidence set
\[
E_M := \{(x, v) \in M \times S(V); \; A_x v = 0\} = \{(x, v) \in M \times S(V); \; v \in E_x\}.
\]
We have natural (left/right) smooth projections
\[
M \xleftarrow{\lambda} E_M \xrightarrow{\rho} S(V).
\]
The left projection \(\lambda : E_M \to M\) describes \(E_M\) as the unit sphere bundle associated to the metric vector bundle \(K_M\). In particular, this shows that \(E_M\) is a compact, smooth manifold of dimension...
For generic \( v \in S(V) \) the fiber \( \rho^{-1}(v) \) is finite and can be identified with the set of critical points of \( v : M \to \mathbb{R} \). We deduce
\[
\mu(M, V) = \frac{1}{\text{area}(S(V))} \int_{S(V)} \#\rho^{-1}(v) |dS(v)|. \tag{1.4}
\]
Denote by \( g_E \) the metric on \( E_M \) induced by the metric on \( M \times S(V) \) and by \( |dV_E| \) the induced volume density. The area formula (see [16, §3.2] or [22, §5.1]) implies that
\[
\int_{S(V)} \#\rho^{-1}(w)|dS(v)| = \int_E J_{\rho}(x, v) |dV_E(x, v)|, \tag{1.5}
\]
where the nonnegative function \( J_{\rho} \) is the Jacobian of \( \rho \) defined by the equality
\[
\rho^*|dS| = J_{\rho} \cdot |dV_E|.
\]
To compute the integral in the right-hand side of (1.5) we need a more explicit description of the geometry of \( E_M \).

Fix a local orthonormal frame \((e_1, \ldots, e_m)\) of \( TM \) defined in a neighborhood \( N \) in \( M \) of a given point \( x_0 \in M \). We denote by \((e^1, \ldots, e^m)\) the dual co-frame of \( T^*M \). Set
\[
f_i(x) := A_{x}^i e_i(x) \in V, \quad i = 1, \ldots, m, \quad x \in N.
\]
More explicitly, \( f_i(x) \) is defined by the equality
\[
(f_i(x), v)_V = \partial_{e_i} v(x), \quad \forall v \in V. \tag{1.6}
\]
Fix a neighborhood \( \mathcal{U} \subset \lambda^{-1}(N) \) in \( M \times S(V) \) of the point \((x_0, v_0)\), and a local orthonormal frame \( u_1(x, v), \ldots, u_{N-1}(x, v) \) over \( \mathcal{U} \) of the bundle \( \rho^*TS(V) \to M \times S(V) \) such that the following hold.

- The vectors \( u_1(x, v), \ldots, u_m(x, v) \) are independent of the variable \( v \) and form an orthonormal basis of \( K_{x,v} \). (E.g., we can obtain such vectors from the vectors \( f_1(x), \ldots, f_m(x) \) via the Gramm-Schmidt process.)
- For \((x, v) \in \mathcal{U}\), the space \( T_{x}E_{x} \) is spanned by the vectors \( u_{m+1}(x, v), \ldots, u_{N-1}(x, v) \).

The collection \( u_1(x), \ldots, u_m(x) \) is a collection of smooth sections of \( V_M \) over \( N \). For any \( x \in N \) and any \( e \in T_xM \), we obtain the vectors (functions).
\[
D_e u_1(x), \ldots, D_e u_m(x) \in V.
\]
Observe that
\[
E_M \cap \mathcal{U} = \{(x, v) \in \mathcal{U}; \quad U_i(x, v) = 0, \quad \forall i = 1, \ldots, m \}, \tag{1.7}
\]
where \( U_i \) is the function \( U_i : N \times V \to \mathbb{R} \) given by
\[
U_i(x, v) := (u_i(x), v)_V.
\]
Thus, the tangent space of \( E_M \) at \((x, v)\) consists of tangent vectors \( \dot{x} \oplus \dot{v} \in T_xM \oplus T_vS(V) \) such that
\[
dU_i(\dot{x}, \dot{v}) = 0, \quad \forall i = 1, \ldots, m.
\]
We let \( \omega_U \) denote the \( m \)-form
\[
\omega_U := dU_1 \wedge \cdots \wedge dU_m \in \Omega^m(\mathcal{U}),
\]
and we denote by \( ||\omega_U|| \) its norm with respect to the product metric on \( M \times S(V) \). Denote by \( |\hat{dV}| \) the volume density on \( M \times S(V) \) induced by the product metric. The equality (1.7) implies that
\[
|\hat{dV}| = \frac{1}{||\omega_U||} |\omega_U \wedge dV_E|.
\]
Hence
\[ J_\rho |\widehat{dV}| = \frac{1}{\|\omega_U\|} |\omega_U \wedge \rho^* dS|. \]

We deduce
\[ J_\rho (x_0, v_0) = J_\rho (x_0, v_0) |\widehat{dV}| (e_1, \ldots, e_m, u_1, \ldots, u_{N-1}) \]
\[ = \frac{1}{\|\omega_U\|} |\omega_U \wedge \rho^* dS|(e_1, \ldots, e_m, u_1, \ldots, u_{N-1}) = \frac{1}{\|\omega_U\|} \left[ \omega_U (e_1, \ldots, e_m) \right] (x_0, v_0). \]

Hence,
\[ \int_{S(V)} \#\rho^{-1}(w)|dS(v)| = \int_E \frac{\Delta_U}{\|\omega_U\|} |dV_E(x, v)|, \quad (1.8) \]

Lemma 1.2. We have the equality \( J_\lambda = \frac{1}{\|\omega_U\|} \), where \( J_\lambda \) denotes the Jacobian of the projection \( \lambda : E_M \to M \).

Proof. Along \( \mathcal{U} \) we have
\[ |\widehat{dV}| = \frac{1}{\|\omega_U\|} |\omega_U \wedge dV_E| \]
while \( (J'_\lambda) \) implies that
\[ |dV_E| = \frac{1}{J_\lambda} |dV_g \wedge dS_{E^x}|. \]

Therefore, suffices to show that along \( \mathcal{U} \) we have
\[ |\widehat{dV}| = |\omega_U \wedge dV_g \wedge dS_{E^x}|, \]
i.e.,
\[ |\omega_U \wedge dV_g \wedge dS_{E^x}(e_1, \ldots, e_m, u_1, \ldots, u_{N-1})| = 1. \]

Since \( dU_i(u_k) = 0 \), \( \forall k \geq m + 1 \) we deduce that
\[ |\omega_U \wedge dV_g \wedge dS_{E^x}(e_1, \ldots, e_m, u_1, \ldots, u_{N-1})| = |\omega_U(u_1, \ldots, u_m)|. \]

Thus, it suffices to show that
\[ |\omega_U(u_1, \ldots, u_m)| = 1. \]

This follows from the elementary identities
\[ dU_i(u_j) = (u_i, u_j) v = \delta_{ij}, \quad \forall 1 \leq i, j \leq m, \]
where \( \delta_{ij} \) is the Kronecker symbol.

Using Lemma 1.2 in (1.8) and the co-area formula we deduce
\[ \int_{S(V)} \#\rho^{-1}(w)|dS(v)| = \int_M \left( \int_{E^x} \Delta_U(x, v) |dS_{E^x}(v)| \right) |dV_g(x)|. \quad (1.9) \]

Observe that at a point \((x, v) \in \lambda^{-1}(N) \subset E_M\) we have
\[ dU_i(e_j) = (D_{e_j} u_i(x), v) v. \]

We can rewrite this in terms of the shape operator \( \Xi_x : T_x M \otimes K_{x}^\perp \to K_x \). More precisely,
\[ dU_i(e_j) = (\Xi_x(e_j, u_i), v) v. \]

Hence,
\[ \Delta_U(x, v) = |\det \Xi_x \cdot v|, \]
We conclude that
\[
\int_{S(\mathbf{V})} \# \rho^{-1}(v) |dS(v)| = \int_M \left( \int_{E_x} \left| \det \Xi_x \cdot v \right| |dS_{E_x}(v)| \right) |dV_M(x)|.
\]
This proves (1.3) \(\square\)

The story is not yet over. We want to rewrite the right-hand side of (1.3) in a more computationally friendly form, preferably in terms of differential-integral invariants of the evaluation map. The starting point is the observation that the left-hand side of (1.3) is plainly independent of the metric \(g\) on \(M\). This raises the hope that if we judiciously choose the metric on \(M\) we can obtain a more manageable expression for \(\mu(M, \mathbf{V})\). One choice presents itself. Namely, we choose the metric \(\bar{g}\) on \(M\) uniquely determined by requiring that the bundle morphism
\[
\mathcal{A}^\dag : (TM, \bar{g}) \to \mathbf{V} \times M
\]
is an isometric embedding. Equivalently, \(\bar{g}\) is the pullback to \(M\) of the metric on \(\mathbf{V}\) via the immersion \(ev : M \to \mathbf{V}^\vee \cong \mathbf{V}\). More concretely, for any \(x \in M\) and any \(X, Y \in T_xM\), we have
\[
\bar{g}_x(X, Y) = (\mathcal{A}^\dag_x X, \mathcal{A}^\dag_x Y)_{\mathbf{V}}.
\]

With this choice of metric, Theorem 1.1 is precisely the main theorem of Chern and Lashof, [9].

Fix \(x \in M\) and a \(\bar{g}\)-orthonormal frame \((e_i)_{1 \leq i \leq m}\) of \(TM\) defined in a neighborhood \(\mathcal{N}\) of \(x\). Then the collection \(u_j = \mathcal{A}^\dag e_j, 1 \leq j,\) is a local orthonormal frame of \(K^\perp\) on \(\mathcal{N}\). The shape operator has the simple description
\[
\Xi_x(e_i, u_j) = (D e_i \mathcal{A}^\dag e_j)^0.
\]

Fix an orthonormal basis \((\Psi_\alpha)_{1 \leq \alpha \leq N}\) of \(\mathbf{V}\) so that every \(v \in \mathbf{V}\) has a decomposition
\[
v = \sum_\alpha v_\alpha \Psi_\alpha, \quad v_\alpha \in \mathbb{R}.
\]

Then, for any \(y \in \mathcal{N}\), we have
\[
\mathcal{A}^\dag e_j(y) = \sum_\alpha (\partial e_j \Psi_\alpha)_y \Psi_\alpha, \quad D e_i \mathcal{A}^\dag e_j(y) = \sum_\alpha (\partial^2 e_i e_j \Psi_\alpha)_y \Psi_\alpha,
\]
and
\[
((D e_i \mathcal{A}^\dag e_j)_y, v)_\mathbf{V} = \sum_\alpha v_\alpha (\partial^2 e_i e_j \Psi_\alpha)_y = \partial^2 e_i e_j v(y).
\]

If \(v \in K_x\), then the Hessian of \(v\) at \(x\) is a well-defined, symmetric bilinear form \(\text{Hess}_x(v) : T_xM \times T_xM \to \mathbb{R}\), i.e., an element of \(T^*_xM \otimes T^*_xM\). Using the metric \(\bar{g}\) we can identify it with a linear operator
\[
\text{Hess}_x(v, \bar{g}) : T_xM \to T_xM.
\]

If we fix a \(\bar{g}\)-orthonormal frame \((e_i)\) of \(T_xM\), then the operator \(\text{Hess}_x(v, \bar{g})\) is described by the symmetric \(m \times m\) matrix with entries \(\partial^2 e_i e_j v(x)\). We deduce that
\[
|\det \Xi_x \cdot v| = |\det \text{Hess}_x(v, \bar{g})|, \quad \forall v \in E_x.
\]

In particular, we deduce that
\[
\mu(M, \mathbf{V}) = \frac{1}{\sigma_{N-1}} \int_M \left( \int_{E_x} \left| \det \text{Hess}_x(v, \bar{g}) \right| |dS_x(v)| \right) |dV_{\bar{g}}(x)|. \quad (1.10)
\]

Finally, we want to express (1.10) entirely in terms of the adjunction map \(A\). For any \(x \in M\) and any \(v \in K_x\), we define the density
\[
\rho_{x, v} : \Lambda^m T_xM \to \mathbb{R},
\]
that this density depends on the metric on
we deduce
We have thus proved the following result.
Observe that for any \( \bar{g} \)-orthonormal frame of \( T_x M \) we have
If we integrate \( \rho_{x,v} \) over \( v \in S(K_x) \), we obtain a density
that varies smoothly with \( x \), and thus it defines a density \( |d\mu(x, V)| \) on \( M \). We want to emphasize
that this density depends on the metric on \( V \) but it is independent on any metric on \( M \). We will refer to it as the density of \( V \).
If we fix a different metric \( g \) on \( M \), then we can express \( |d\mu(-, V)| \) as a product
where \( \rho_g = \rho_g(v) : M \to \mathbb{R} \) is a smooth nonnegative function.
To find a more useful description of \( \rho_g \), we choose local coordinates \( (x^1, \ldots, x^m) \) near \( x \) such that
\((\partial_{x^i}) \) is a \( g \)-orthonormal basis of \( T_x M \). Then
\[
\rho_{x,v}(\partial_{x^1} \wedge \cdots \wedge \partial_{x_m}) = \left| \det \left( \partial_{x^i}^2 v(x) \right) \right|_{1 \leq i,j \leq m} \cdot \left( \det \left( (A^1 \partial_{x^i}, A^j \partial_{x^j}) v \right) \right)_{1 \leq i,j \leq m}^{-1/2}.
\]
Observe that the matrix \( (\partial_{x^i}^2 v(x))_{1 \leq i,j \leq m} \) describes the Hessian operator
\[
\text{Hess}_x(v, g) : T_x M \to T_x M
\]
induced by the Hessian of \( v \) at \( x \) and the metric \( g \).
The scalar \( \left( \det \left( (A^1 \partial_{x^i}, A^j \partial_{x^j}) v \right) \right)_{1 \leq i,j \leq m}^{1/2} \) is precisely the Jacobian of the dual adjunction map \( A_x^1 : T_x M \to V \) defined in terms of the metric \( g \) on \( T_x M \) and the metric on \( V \). We denote it by \( J(A_x^1, g) \). We set
\[
\Delta_x(V, g) := \int_{S(K_x)} |\det \text{Hess}_x(v, g)| |dS_x(v)|.
\]
Since
\[
|dV_g(x)|(\partial_{x^1} \wedge \cdots \wedge \partial_{x_m}) = 1,
\]
we deduce
\[
\rho_{g,v}(x) = \Delta_x(V, g) \cdot J(A_x^1, g)^{-1}.
\] (1.11)
We have thus proved the following result.

**Corollary 1.3.** Suppose \((M, g)\) is a compact, connected Riemann manifold and \( V \subset C^\infty(M) \) is a vector subspace of dimension \( N \). Fix an Euclidean inner product \( h \) on \( V \) with norm \( |\cdot|_h \). Then
\[
\mu(M, g, V, h) = \frac{1}{\text{area}(S(V))} \int_{|v|_h = 1} |\mu(v)| |dS_h(v)| = \frac{1}{\sigma_{N-1}} \int_M \frac{\Delta_x(V, g)}{J(A_x^1, g, h)} |dV_g(x)|, \tag{1.12}
\]
where \( |dS_h| \) denotes the area density on the unit sphere \( \{|v|_h = 1\} \), and \( J(A_x^1, g, h) \) denotes the Jacobian of the dual adjunction map \( A_x^1 : T_x M \to V \) computed in terms of the metrics \( g \) on \( T_x M \) and \( h \) on \( V \).

We will refer to the quantity \( \mu(M, g, V, h) \) as the expectation of the quadruple \((M, g, V, h)\).
Remark 1.4. Let us observe that we have proved a little bit more. To every Morse function \( v \in V \) we associate the measure
\[
\mu_v = \sum_{dv(x) = 0} \delta_x,
\]
where \( \delta_x \) Denotes the Dirac measure concentrated at \( x \). For every continuous function \( f : M \to \mathbb{R} \) we set
\[
\mu(v, f) := \int_M f d\mu_v = \sum_{dv(x) = 0} f(x),
\]
and we denote by \( \mu(v, f) \) the expectation of the random variable \( S(V) \ni v \mapsto \mu(v, f) \),
\[
\mu(v, f) = \frac{1}{\text{area}(S(V))} \int_{S(V)} \mu(v, f) |dS(v)|.
\]
Arguing exactly as in the proof of Corollary 1.3 we deduce that, for any Riemann metric \( g \) on \( M \) we have
\[
\mu(v, f) = \frac{1}{\text{area}(S(V))} \int_{|v|_h = 1} \mu(v) |dS_h(v)| = \frac{1}{\sigma_{N-1}} \int_M \frac{\Delta_x(V, g)}{J(A^1_x, g, h)} f(x) |dV_g(x)|.
\]
The resulting density on \( M \)
\[
\mu_v := \frac{\Delta_x(V, g)}{\sigma_{N-1} J(A^1_x, g, h)} |dV_g(x)|
\]
\[
= \frac{1}{\pi^{N-2} J(A^1_x, g, h)} \left( \int_{K_x} e^{-|u|^2} \det \text{Hess}_x(u) |dV_{K_x}(u)| |dV_g(x)| \right)
\]
is called the expected density of critical points of a function in \( V \). As explained in the introduction, if \( V \) is 2-jet ample, then the above Gaussian integral over \( K_x \) can be reduced to a Gaussian integral over \( \text{Sym}(T_x M) \). In this case, the resulting formula is a special case of [8, Thm.4.2] that was obtained by a different approach, more probabilistic in nature.

Remark 1.5 [(A Gauss-Bonnet type formula)]. With a little care, the above arguments lead to a Gauss-Bonnet type theorem. More precisely, if we assume that \( M \) is oriented, then, under appropriate orientation conventions, the Morse inequalities imply that the degree of the map \( \rho : E_M \to S(V) \) is equal to the Euler characteristic of \( M \). If instead of working with densities, we work with forms, then we conclude that
\[
\chi(M) = \frac{1}{\sigma_{N-1}} \int_M \frac{\chi_x(V, g)}{J(A^1_x, g, h)} dV_g(x),
\]
where
\[
\chi_x(V, g) := \int_{S(K_x)} \det \text{Hess}_x(v, g) dS_x(v).
\]
When \( M \) is a submanifold of the Euclidean space \( V \), and we identify \( V \) with \( V^\vee \subset C^\infty(V) \), then the above argument yields the Gauss-Bonnet theorem for submanifolds of a Euclidean space.

We say that a quadruple \((M, g, V, h)\) as in Corollary 1.3 is homogeneous with respect to a compact Lie group \( G \) if the following hold.

- The group \( G \) acts transitively and isometrically on \( M \).
- For any function \( v \in V \), and any \( g \in G \), the pullback \( g^*v \) is also a function in \( V \).
- The metric \( h \) is invariant with respect to the induced right action of \( G \) on \( V \) by pullback.
For homogeneous quadruples formula (1.12) simplifies considerably because in this case the function $\rho_{g, V}$ is constant. We deduce that in this case we have
\[
\mu(M, g, V, h) = \frac{\Delta_{x_0}(V, g)}{\sigma_{N-1}(A^1_{x_0}, g, h)} \cdot \text{vol}_g(M),
\]
where $x_0$ is an arbitrary point in $M$.

Let us observe that to any triple $(M, g, V)$, $V \subset C^\infty(M)$, we can associate in a canonical fashion a quadruple $(M, g, V, h_g)$, where $h_g$ is the inner product on $V$ induced by the $L^2(M, |dV_g|)$ inner product on $C^\infty(M)$. The expectation of such a triple is, by definition, the expectation of the associated quadruple. We will denote it by $\mu(M, g, V)$. We say that a triple $(M, g, V)$ is homogeneous if the associated quadruple is so.

2. RANDOM POLYNOMIALS ON SPHERES

As is well known, the spectrum of the Laplacian on the unit sphere $(S^{d-1}, g_d) \subset \mathbb{R}^d$ is
\[
\{\lambda_n(d) = n(n + d - 2); \ n \geq 0\}.
\]
We denote by $Y_{n,d}$ the eigenspace corresponding to the eigenvalue $\lambda(d)$. As indicated in Appendix B, the space $Y_{n,d}$ has dimension
\[
M(n, d) = \frac{2n + d - 2}{n + d - 2} \binom{n + d - 2}{d - 2} \sim 2 \frac{n^{d-2}}{(d-2)!} \quad \text{as} \ n \to \infty,
\]
and can be explicitly described as the space of restrictions to $S^{d-1}$ of harmonic homogeneous polynomials of degree $n$ on $\mathbb{R}^d$. For any positive integer $\nu$, we set
\[
V_\nu = V_\nu(d) := \bigoplus_{n=0}^{\nu} Y_{n,d}.
\]
The space $V_\nu(d)$ can be identified with the space of restrictions to $S^{d-1}$ of polynomials of degree $\leq \nu$ in $d$ variables. Note that
\[
N_\nu := \dim V_\nu(d) \sim \frac{2\nu^{d-1}}{(d-1)!} \quad \text{as} \ \nu \to \infty.
\]
The resulting triple $(S^{d-1}, g_d, V_\nu)$ is homogeneous, and we denote by $\mu(S^{d-1}, V_\nu)$ its expectation. The goal of this section is to describe the asymptotics of $\mu(S^{d-1}, V_\nu(d))$ as $\nu \to \infty$ in the case $d \geq 3$. The simpler case $d = 2$ will be analyzed separately in Corollary 5.2.

**Theorem 2.1.** For any $d \geq 3$ there exists a positive constant $K = K_d$ that depends only on $d$ such that
\[
\mu(S^{d-1}, V_\nu(d)) \sim K_d \dim V_\nu(d) \quad \text{as} \ \nu \to \infty.
\]
In particular,
\[
\log \mu(S^{d-1}, V_\nu(d)) \sim \log \dim V_\nu(d) \quad \text{as} \ \nu \to \infty.
\]

**Proof.** For simplicity, we will write $V_\nu$ instead of $V_\nu(d)$. We will rely on some classical facts about spherical harmonics surveyed in Appendix B. For any integer $d \geq 2$, we denote by $B_{n,d}$ the canonical orthonormal basis of $Y_{n,d}$ constructed by the inductive process outlined in Appendix B and described in more detail below.
According to Corollary 1.3, it suffices to describe the density of $V_\nu$ at the North Pole $p_0 = (0, 0, \ldots, 0, 1) \in S^{d-1}$. Denote by $K_\nu(p_0)$ the subspace of $V_\nu$ consisting of functions for which $p_0$ is a critical point. Note that

$$\dim K_\nu(p_0) = \dim V_\nu - \dim S^{d-1} = N_\nu - (d - 1).$$

Near $p_0$ we use $x' = (x_1, \ldots, x_{d-1})$ as local coordinates so that

$$x_d = \sqrt{(1 - |x'|^2) = 1 - \frac{1}{2}|x'|^2 + \text{higher order terms}.} \tag{2.3}$$

Note that, at $p_0$, the tangent vectors $\partial_{x_1}, \ldots, \partial_{x_{d-1}}$ form an orthonormal frame of $T_{p_0}S^{d-1}$.

For any function $f \in C^\infty(S^{d-1})$, we denote by $\text{Hess}(f)$ the Hessian of $f$ at $p_0$, i.e., the $(d - 1) \times (d - 1)$ symmetric matrix with entries

$$\text{Hess}(f)_{ij} = \partial^2_{x_i x_j} f(p_0), \quad 1 \leq i, j \leq d - 1.$$ 

We set

$$B_{j,d} := \{1, \ldots, M(j, d - 1)\},$$

and we parametrize the basis $B_{j,d-1}$ as

$$B_{j,d-1} = \{Y_{j,\beta}; \ \beta \in B_{j,d}\},$$

where $Y_{j,\beta}$ is a homogeneous harmonic polynomial of degree $j$ in the variables $x' = (x_1, \ldots, x_{d-1})$.

For any integers $j, n, 0 \leq j \leq n$, and any $\beta \in B_{j,d}$, we define $Z_{n,j,\beta} \in C^\infty(S^{d-1})$ by

$$Z_{n,j,\beta}(x) := C_{n,j,d} P_{n,d}^{(j)}(x_d) Y_{j,\beta}(x'), \ \forall x \in S^{d-1}, \tag{2.4}$$

where $P_{n,d}^{(j)}$ denotes the $j$-th order derivative of the Legendre polynomial $P_{n,d}$ defined by (B.2), while the universal constant $C_{n,j,d}$ is described in (B.4). Then, for fixed $n$, the collection of functions

$$\{Z_{n,j,\beta} \in C^\infty(S^{d-1}); \ 0 \leq j \leq n, \ \beta \in B_{j,d}\}$$

is the orthonormal basis $B_{n,d}$. Any $v \in V_\nu$ admits a decomposition

$$v = \sum_{n=0}^{\nu} \sum_{j=0}^{n} \sum_{\beta \in B_{j,d}} v_{n,j,\beta} Z_{n,j,\beta}, \quad v_{n,j,\beta} \in \mathbb{R},$$

so that

$$\text{Hess}(v) = \sum_{n=0}^{\nu} \sum_{j=0}^{n} \sum_{\beta \in B_{j,d}} v_{n,j,\beta} \text{Hess}(Z_{n,j,\beta}).$$

From the description (2.4) we deduce that

$$\text{Hess}(Z_{n,j,\beta}) = 0, \ \forall j \geq 3. \tag{2.5}$$

Next, we observe that when $j = 0$ we have $M(0, d - 1) = 1$ and $B_{0,d-1}$ consists of the constant function $\sigma_{d-2}^{-1/2}$. We deduce

$$B_{0,d-1} = \{\sigma_{d-2}^{-1/2}\}, \quad B_{j,d} = \{1\},$$

$$\text{Hess}(Z_{n,0,1}) = C_{n,0,d} \sigma_{d-2}^{-1/2} \text{Hess}(P_{n,d}(x_d)).$$

Using the equalities

$$P_{n,d}(t) = P_{n,d}(1) + P_{n,d}'(1)(t - 1) + \text{higher order terms},$$

we deduce

$$P_{n,d}(x_d) = P_{n,d}(1) + P_{n,d}'(1)(x_d - 1) + \text{higher order terms}.$$
\[ (2.3) \quad P_n(1) = \frac{P_n'(1)}{2} |x'|^2 + \text{higher order terms} \]
\[ (B.3) \quad 1 - \frac{1}{4(n + 1)} \left( n + \frac{d - 3}{2} \right) |x'|^2 + \text{higher order terms}. \]

This shows that
\[ \text{Hess}(P_n,d(x)) = -\frac{1}{2}(n + 1) \left( n + \frac{d - 3}{2} \right) \mathbb{1}_{d-1}, \tag{2.6} \]
where \( \mathbb{1}_{d-1} \) denotes the identity \((d - 1) \times (d - 1)\)-matrix. Hence
\[ Z_{n,0,d} \in K_{\nu}(p_0), \tag{2.7} \]
\[ \text{Hess}(Z_{n,0,1}) = -\frac{1}{2} \sigma_{d-2}^{-1/2} C_{n,0,d}(n + 1) \left( n + \frac{d - 3}{2} \right) \mathbb{1}_{d-1}. \tag{2.8} \]

Similarly
\[ P_n'(x) = P_n'(1) - \frac{1}{2} P_n''(1) |x'|^2 + \text{higher order terms}, \]
which implies that
\[ \text{Hess}(Z_{n,1,\beta}) = 0, \quad \forall n, \quad \beta \in B_1 = \{1, \ldots, d - 1\}. \tag{2.9} \]

For any \( \beta \in B_{2,d} \), we denote by \( H_\beta \) the Hessian of \( Y_\beta(x') \) at \( x' = 0 \in \mathbb{R}^{d-1} \). We deduce that
\[ Z_{n,2,\beta} \in K_{\nu}(p_0), \tag{2.10} \]
\[ \text{Hess}(Z_{n,2,\beta}) = C_{n,2,d} P_{n,d}(1) H_\beta. \tag{2.11} \]

Using (2.5), (2.9) and (2.11) we conclude that
\[ \text{Hess}(v) = \sum_{n=2}^{\nu} \sum_{\beta \in B_{2,d}} v_{n,2,\beta} \text{Hess}(Z_{n,2,\beta}) + \sum_{n=0}^{\nu} v_{n,0,1} \text{Hess}(Z_{n,0,1}) \]
\[ = \sum_{\beta \in B_{2,d}} \left( \sum_{n=2}^{\nu} v_{n,2,\beta} C_{n,2,d} P_{n,d}(1) \right) H_\beta \]
\[ - \frac{1}{2} \sigma_{d-2}^{-1/2} \left( \sum_{n=0}^{\nu} v_{n,0,1} C_{n,0,d}(n + 1) \left( n + \frac{d - 3}{2} \right) \right) \mathbb{1}_{d-1}. \]

The last equality can be rewritten in a more convenient form as follows. Define
\[ a_0 = a_0(\nu) := -\frac{1}{2} \sigma_{d-2}^{-1/2} \sum_{n=0}^{\nu} C_{n,0,d}(n + 1) \left( n + \frac{d - 3}{2} \right) Z_{n,0,1} \in V_{\nu}, \tag{2.12} \]
and for \( \beta \in B_{2,d} \), set
\[ a_\beta = a_\beta(\nu) := \sum_{n=2}^{\nu} C_{n,2,d} P_{n,d}(1) Z_{n,2,\beta} \in V_{\nu}. \tag{2.13} \]

We deduce that
\[ \text{Hess}(v) = (v, a_0) \mathbb{1}_{d-1} + \sum_{\beta} (v, a_\beta) H_\beta. \]

Note that the vectors \( a_0, a_\beta \) are mutually orthogonal, and they span a vector space \( L_{\nu} \) of dimension
\[ \ell = \ell(d) = M(2, d - 1) + 1 \tag{B.1} \]
\[ = \binom{d}{2}. \]
Moreover, the conditions (2.7) and (2.10) imply that \( L_\nu \subset K_\nu(p_0) \). Define
\[
e_0 := \frac{1}{|a_0|} a_0, \quad e_\beta := \frac{1}{|a_\beta|} a_\beta, \quad \beta \in B_{2,d},
\]
and
\[
r_0 = r_0(\nu) = |a_0|^2 = \frac{1}{4} \sigma_{d-2}^{-1} \sum_{n=0}^{\nu} C_{n,0,d}^2 (n + 1)^2 \left( n + \frac{d - 3}{2} \right)^2,
\]
and
\[
r_\beta = r_\beta(\nu) = |a_\beta|^2 = \sum_{n=2}^{\nu} C_{n,2,d}^2 P_{n,d}^{(2)}(1)^2, \quad \beta \in B_{2,d}.
\]

Note that the collection \( \{ e_0, e_\beta, \beta \in B_{2,d} \} \) is an orthonormal basis of \( L_\nu \). For any \( v \in K_\nu(p_0) \), we denote by \( \bar{v} \) its orthogonal projection onto \( L_\nu \), and we set
\[
\bar{v}_0 := \langle v, e_0 \rangle, \quad \bar{v}_\beta := \langle v, e_\beta \rangle, \quad \beta \in B_{2,d}.
\]
We deduce that for any \( v \in K_\nu(p_0) \), we have
\[
\text{Hess}(v) = \text{Hess}(\bar{v}) = r_0^{1/2} \bar{v}_0 \mathbb{1}_{d-1} + \sum_{\beta} r_\beta^{1/2} \bar{v}_\beta H_\beta.
\]

For \( v \in K_\nu(p_0) \) we set
\[
Q_\nu(v) := |\det \text{Hess}(v)|.
\]
Note that \( Q_\nu(v) \) is positively homogeneous of degree \( d - 1 \). Using Lemma A.1 in the special case
\[
n = \dim K_\nu(p_0) = N_\nu - (d - 1), \quad n_1 = \dim L_\nu = \ell, \quad n_0 = N_\nu - (d - 1) - \ell,
\]
we deduce
\[
\Delta_x(V_\nu) = \int_{S(K_\nu(p_0))} Q_\nu(v) |dS(v)| = \sigma_{N_\nu - \ell - d} \int_{B(L_\nu)} (1 - |\bar{v}|^2)^{N_\nu - \ell - d - 1} Q_\nu(\bar{v}) |dV(\bar{v})|.
\]
Using Lemma A.2 we deduce
\[
\int_{B(L_\nu)} (1 - |\bar{v}|^2)^{N_\nu - \ell - d - 1} Q_\nu(\bar{v}) |dV(\bar{v})| = \frac{\Gamma(\ell + d + 1)}{2 \Gamma^{2} \left( \frac{N_\nu - d + 1}{2} \right)} \int_{S(L_\nu)} Q_\nu(\bar{v}) |dS(\bar{v})|.
\]
Using the equality \( (\sigma) \), we conclude
\[
\int_{S(K_\nu(p_0))} Q_\nu(v) |dS(v)| = \frac{\pi^{N_\nu - \ell - d + 1}}{\Gamma \left( \frac{N_\nu - d}{2} \right)} I_\nu.
\]
Next, we compute the Jacobian of the adjunction map \( A^+_\nu \) at \( p_0 \). We use the coordinates \( x' \) near \( p_0 \). For \( i = 1, \ldots, d - 1 \) we have
\[
A^+_\nu \partial_{x_i} = \sum_{n=0}^{\nu} \sum_{j=0}^{n} \sum_{\beta \in B_{1,d}} \partial_{x_i} (Z_{n,j,\beta}(p_0)) Z_{n,j,\beta} = \sum_{n=1}^{\nu} \sum_{\beta \in B_{1,d}} \partial_{x_i} (Z_{n,1,\beta}(p_0)) Z_{n,1,\beta}.
\]
Using (B.6), we deduce that \( B_{1,d} = \{ 1, \ldots, d - 1 \} \) and for any \( \beta \in B_{1,d} \) we have
\[
Y_\beta = \sigma_{d-3}^{-1/2} C_{1,0,d-1} x_\beta, \quad Z_{n,1,\beta} = \sigma_{d-3}^{-1/2} C_{1,0,d-1} C_{n,1,d} P_{n,d}(x_d) x_\beta.
\]
We deduce that
\[
A^+_\nu \partial_{x_i} = \sigma_{d-3}^{-1/2} C_{1,0,d-1} \sum_{n=1}^{\nu} C_{n,1,d} P_{n,d}(1) Z_{n,1,i}.
\]
This shows that the vectors $A_{p_0}^i \partial_x i$, $i = 1, \ldots, d - 1$, are mutually orthogonal and they have identical lengths

$$|A_{p_0}^i \partial_x| = r(\nu)^{1/2}, \quad r(\nu) = \sigma_{d-3}^{-1} C_{1,0,d-1}^{2} \sum_{n=1}^{\nu} (C_{n,1,d} P_{n,d}(1))^{2}. \quad (2.18)$$

We deduce that the Jacobian of $A_{p_0}^i$ is

$$J_\nu = r(\nu)^{d-1}. \quad (2.19)$$

The equalities (1.14), (2.16) and (2.19) now imply that

$$\mu(S^{d-1}, V_\nu) = \frac{\sigma_{d-1}}{\sigma_{\nu-1}} \cdot \frac{\pi^{\nu-\frac{d-1}{2}} \Gamma(\frac{\nu+d}{2})}{r(\nu)^{d-1} \Gamma(\frac{\nu}{2})} I_\nu.$$

Using (\sigma) we can simplify this to

$$\mu(S^{d-1}, V_\nu) = \frac{\Gamma(\frac{\nu+d}{2})}{(\pi r(\nu))^{d-1} \Gamma(\frac{\nu}{2})} I_\nu. \quad (2.20)$$

To obtain the asymptotics of $\mu(S^{d-1}, V_\nu)$ as $\nu \to \infty$ we need to understand the asymptotics of the quantities

$$r(\nu), \quad r_0(\nu), \quad r_\beta(\nu), \quad \beta \in B_{2,d-1}.$$

To achieve this, note first that (B.3) and (B.4) imply that

$$C_{n,1,d} P_{n,1,d}(1)^2 = \frac{1}{2^{d-2} \Gamma(\frac{d-1}{2})^2} \frac{(2n + d - 2)(n + d - 3)_{d-3}}{n+d-2_{d-1}} \frac{(n+1)^2}{4} \left( \frac{n + d - 3}{2} \right)^2$$

$$= \frac{1}{2^{d-1} \Gamma(\frac{d-1}{2})^2 B_{d-1}(n)}, \quad (r)$$

where $A_{2d-1}(x)$ (respectively $B_{d-1}(x)$) is a monic polynomial of degree $(2d-1)$ (respectively $d-1$).

Using (S\infty), we deduce that

$$\sum_{n=1}^{\nu} C_{n,1,d} P_{n,1,d}(1)^2 \sim \frac{1}{2^{d-1}(d+1) \Gamma(\frac{d-1}{2})^2 B_{d-1}(n)} \nu^{d+1}, \quad \text{as } \nu \to \infty,$$

so that

$$r(\nu) \sim \frac{C_{1,0,d-1}^2}{2^{d-1} \Gamma(\frac{d-1}{2})^2 \sigma_{d-3}(d+1) \nu^{d+1}}, \quad \text{as } \nu \to \infty, \quad (2.21)$$

where

$$C_{1,0,d-1}^2 = \frac{(d-1)(d-3)!}{2^{d-3} \Gamma(\frac{d-2}{2})^2}.$$

Invoking (B.3) and (B.4) again we deduce that

$$C_{n,2,d} P_{n,2,d}(1)^2 = \frac{1}{2^{d-2} \Gamma(\frac{d-1}{2})^2 \Gamma(\frac{d-1}{2})^2} \frac{(2n + d - 2)(n + d - 3)_{d-3}}{n+d-1_{d-1}} \left( \frac{1}{4} \left( \frac{n+2}{2} \right) \left( \frac{n + d - 3}{2} \right) \right)^2$$

$$= \frac{1}{2^{d+3} \Gamma(\frac{d-1}{2})^2 B_{d+1}(n)}, \quad (r)\beta$$
where $A_{2d+3}(x)$ (respectively $B_{d-1}(x)$) is a monic polynomial of degree $(2d+3)$ (respectively $d+1$). Using $(S_{\infty})$ we deduce that

$$\forall \beta \in B_{2d-1}; \quad r_\beta(\nu) = \sum_{n=2}^{\nu} C_{n,2,d}^2 P_{n,2,d}(1)^2 \sim \frac{1}{2^{d+3}(d+3)!} \nu^{d+3}, \quad \text{as } \nu \to \infty. \quad (2.22)$$

Using (B.4), we deduce

$$C_{n,0,d}^2(n+1)^2 \left( n + \frac{d-3}{2} \right)^2 = \frac{(2n+d-2)[n+d-3]}{2^{d-2}\Gamma(\frac{d-1}{2})^2} \left( n + \frac{d-3}{2} \right)^2 = \frac{1}{2^{d-3}\Gamma(\frac{d-1}{2})^2} A_{d+2}(n), \quad (r_0)$$

where $A_{d+2}(x)$ denotes a monic polynomial of degree $d+2$. Invoking $(S_{\infty})$ again we deduce that as $\nu \to \infty$ we have

$$r_0(\nu) = \frac{1}{4\sigma_{d-2}} \sum_{n=0}^{\nu} C_{n,0,d}^2(n+1)^2 \left( n + \frac{d-3}{2} \right)^2 \sim \frac{1}{2^{d-1}\Gamma(\frac{d-1}{2})^2 \sigma_{d-2}(d+3)} \nu^{d+3}. \quad (2.23)$$

Define

$$\bar{r}_0 = \lim_{\nu \to \infty} \nu^{-(d+3)} r_0(\nu), \quad \bar{r}_\beta = \lim_{\nu \to \infty} \nu^{-(d+3)} r_\beta(\nu), \quad \bar{r} = \lim_{\nu \to \infty} \nu^{-(d+1)} r(\nu).$$

The precise values of these constants can be read off (2.21)-(2.23). Denote by $L_{\infty}$ the Euclidean space of dimension $\ell = \binom{d}{2}$ with Euclidean coordinates $u_0, u_\beta, \beta \in B_{2d},$ and we set

$$A_{\nu}(u) = r_0(\nu)^{1/2} u_0 \mathbb{1}_{d-1} + \sum_{\beta} r_\beta(\nu)^{1/2} u_\beta H_\beta, \quad A_{\infty}(u) = \bar{r}_0^{1/2} u_0 \mathbb{1}_{d-1} + \sum_{\beta} \bar{r}_\beta^{1/2} u_\beta H_\beta. \quad (2.24)$$

We can now rewrite (2.20) as follows

$$\mu(S^{d-1}, V_\nu) = \frac{\Gamma(\frac{\ell+d}{2})}{(r(\nu)^{\frac{d}{2}} \Gamma(\frac{d}{2})^\ell) S(L_{\infty})} | \det A_{\nu}(u) | |dS(u)|. \quad (2.25)$$

The estimates (2.22) and (2.23) show that as $\nu \to \infty$, we have

$$| \det A_{\nu}(u) | \sim \nu^{(d+3)(d-1)/2} | \det A_{\infty}(u) |,$$

uniformly with respect to $u \in S(L_{\infty})$. We deduce that as $\nu \to \infty$, we have

$$\mu(S^{d-1}, V_\nu) \sim \frac{\Gamma(\frac{\ell+d}{2})}{(\pi \bar{r}^{\frac{d}{2}} \Gamma(\frac{d}{2})^\ell)} \nu^{d-1} \int_{|u|=1} | \det A_{\infty}(u) | |dS(u)|. \quad (2.25)$$

This proves (2.2) where

$$K_d = \frac{2 \Gamma(\frac{\ell+d}{2})}{(\pi \bar{r}^{\frac{d}{2}} \Gamma(\frac{d}{2})(d-1)!)} \int_{|u|=1} | \det A_{\infty}(u) | |dS(u)|, \quad \ell = \binom{d}{2}. \quad (2.25)$$

Remark 2.2. We want to analyze what happens to the above expectation if we change the $L^2$-metric product on $V_\nu(d)$ to a new Euclidean metric so that the resulting quadruple $(S^{d-1}, g, V_\nu(d), h)$ continues to be homogeneous with respect to the action of $SO(d)$. 

\[\square\]
To perform such changes we use the fact that each of the spaces $\mathcal{Y}_{n,d}$ is an irreducible representation of $SO(d)$. Any sequence $w = (w_n)_{n \geq 0}$ of positive real numbers determines a Euclidean metric $\| - \|_w$ on $V_\nu(d)$ as follows. If
\[ v = \sum_{n=0}^{\nu} v_n \in V_\nu(d), \ v_n \in \mathcal{Y}_{n,d}, \]
then we set
\[ \| v \|^2_w := \sum_{n=0}^{\nu} \frac{1}{w_n^2} \| v_n \|^2_{L^2(S^{d-1})}. \]
In the sequel, we will choose the weights $w$ of the form
\[ w_{p,v} = (\nu + 1)^p, \quad p \in \mathbb{R}. \] (2.26)
The corresponding metric $\| - \|_{w_p}$ is (equivalent to) the metric of the Sobolev Hilbert space $H^{-p}$ consisting of distributions with “derivatives up to order $-p$ in $L^2$.

The quadruple $(S^{d-1}, g, V_\nu(d), \| - \|_w)$ is homogeneous and we denote by $\mu_p(S^{d-1}, V_\nu)$ its expectation. The collection
\[ \{ w_n Z_{n,j,\beta} \in C^\infty(S^{d-1}); \quad 0 \leq j \leq n \leq \nu, \ \beta \in B_{j,d} \} \]
is an orthonormal basis of $V_\nu$ with respect to the inner product $h_w$ associated to $\| - \|_w$. Any $v \in V_\nu$ admits a decomposition
\[ v = \sum_{n=0}^{\nu} \sum_{j=0}^{n} \sum_{\beta \in B_{j,d}} v_{n,j,\beta} w_n Z_{n,j,\beta}, \ v_{n,j,\beta} \in \mathbb{R}. \]
The arguments in the proof of Theorem 2.1 show that for $v \in V_\nu$, we have
\[ \text{Hess}(v) = \sum_{\beta \in B_{2,d}} (\sum_{n=2}^{\nu} w_n v_{n,2,\beta} C_{n,2,d} P_{n,d}^{(2)}(1)) H_\beta \]
\[ - \frac{1}{2} \sigma_{d-2}^{-1/2} \left( \sum_{n=0}^{\nu} w_n v_{n,0,1} C_{n,0,d}(n+1) \left( n + \frac{d-3}{2} \right) \right) \cdot 1_{d-1}. \]
In particular, if $\text{Hess}(v) \neq 0$, then the North Pole is a critical point of $v$, i.e., $v \in K_\nu(p_0)$. Define
\[ a_0 = a_0(v, w) := -\frac{1}{2} \sigma_{d-2}^{-1/2} \sum_{n=0}^{\nu} w_n C_{n,0,d}(n+1) \left( n + \frac{d-3}{2} \right) Z_{n,0,1} \in V_\nu, \] (2.27)
and for $\beta \in B_{2,d}$ set
\[ a_\beta = a_\beta(v, w) := \sum_{n=2}^{\nu} w_n C_{n,2,d} P_{n,d}^{(2)}(1) Z_{n,2,\beta} \in V_\nu. \] (2.28)
We deduce that
\[ \text{Hess}(v) = (v, a_0) 1_{d-1} + \sum_{\beta} (v, a_\beta) H_\beta. \]
Define
\[ e_0 := \frac{1}{|a_0|} a_0, \ e_\beta := \frac{1}{|a_\beta|} a_\beta, \ \beta \in B_{2,d}, \] (2.29)
\[ r_0 = r_0(v, w) := |a_0|^2, \ r_\beta = r_\beta(v, w) := |a_\beta|^2, \ \beta \in B_{2,d}. \]
For any $v \in K_\nu(p_0)$, we set
\[ Q_\nu(v) := |\det \text{Hess}(v)|, \ \bar{v}_0 = (v, e_0), \ \bar{v}_\beta = (v, e_\beta), \ \beta \in B_{2,d}, \]
and we deduce
\[ \text{Hess}(\nu) = \text{Hess}(\bar{\nu}) = r^{1/2} \bar{v}_0 \mathbb{I}_{d-1} + \sum_{\beta} r^{1/2}_\beta \bar{v}_\beta H_\beta, \tag{2.30} \]
\[ \int_{S(\nu, r\nu)} Q_\nu(\nu) |dS(\nu)| = \frac{\pi N_{\nu \cdot \nu - d + 1}}{\Gamma(\frac{d + 1}{2})} \int_{S(\nu, r\nu)} Q_\nu(\bar{v}) |dS(\bar{v})|, \tag{2.31} \]
where \( \bar{v} \) denotes the orthogonal projection of \( v \) onto the space \( L_\nu \) spanned by \( e_0, e_\beta \). Similarly, we have
\[ A_{p_0}^{1} \partial_{\xi_i} = \sigma_{d-3}^{-1/2} C_{1,0,d-1} \sum_{n=1}^{\nu} w_n C_{n,1,d} P'_{n,d}(1) Z_{n,i}. \tag{2.32} \]
This shows again that the vectors \( A_{p_0}^{1} \partial_{\xi_i}, i = 1, \ldots, d - 1 \), are mutually orthogonal and they have identical length
\[ |A_{p_0}^{1} \partial_{\xi_i}| = r(\nu)^{1/2}, \quad r(\nu) = \sigma_{d-3}^{-1} C_{1,0,d-1} \sum_{n=1}^{\nu} (w_n C_{n,1,d} P'_{n,d}(1))^2. \tag{2.33} \]
We deduce that the Jacobian of \( A_{p_0}^{1} \) is
\[ J_\nu = r(\nu)^{d-1}. \tag{2.34} \]
If the exponent \( p \) in (2.26) is nonnegative, then using \( (r), (r_{\beta}), (r_0) \) and the Euler-Maclaurin summation formula \( (S_{\infty}) \), we deduce as before that as \( \nu \to \infty \) we have
\[ r(\nu, w) \sim K_1 \nu^{d+1+2p}, \quad r_\beta(\nu, w) \sim K_2 \nu^{d+1+2p}, \quad r_0(\nu, w) \sim K_3 \nu^{d+3+2p}, \tag{2.35} \]
where above and in the sequel we will use the symbols \( K_1, K_2, \ldots \), to denote positive constants that depend only on \( d \) and \( p \). This shows that
\[ \mu_p(S^{d-1}, V_\nu) \sim K_4 \text{dim } V_\nu. \]
If the exponent \( p \) in (2.26) is \( \ll 0 \), then a similar argument shows that
\[ \mu_p(S^{d-1}, V_\nu) \sim K_5, \quad \text{as } \nu \to \infty. \]
\[ \square \]

We conclude this section with a computation suggested by the recent results of Nazarov-Sodin, [26].

**Theorem 2.3.** We denote by \( Y_n \) the eigenspace corresponding to the eigenvalue \( \lambda_n = n(n+1) \) of the Laplacian on \( S^2 \), and we set \( \mu(Y_n) := \mu(S^2, Y_n) \). Then\(^2\)
\[ \mu(Y_n) \sim \frac{2}{\sqrt{3}} n^2 \quad \text{as } \quad n \to \infty. \tag{2.36} \]

**Proof.** The computation is very similar to the computations in Theorem 2.1, but much simpler. We continue to use the notations in the proof of that theorem. In particular, \( K_n \) denotes the space of harmonic polynomials in \( Y_n \) that admit the North Pole \( p_0 = (0,0,1) \in \mathbb{R}^3 \) as a critical point.

An orthonormal basis of \( Y_n \) is given by the
\[ Z_{n,j,\beta}, \quad 0 \leq j \leq n, \quad \beta \in B_j := B_{j,3}. \]

\(^2\)Let us point out that \( \frac{2}{\sqrt{3}} \approx 1.154 \).
Any \( y \in Y_n \) admits a decomposition
\[
y = \sum_{j=0}^{n} \sum_{\beta \in \mathcal{B}_j} y_{j,\beta} Z_{n,j,\beta}.
\]
We conclude as in the proof of Theorem 2.1 that
\[
\text{Hess}(y) = \sum_{\beta \in B_2} y_{2,\beta} \text{Hess}(Z_{n,2,\beta}) + y_{0,1} \text{Hess}(Z_{n,0,1}).
\tag{2.37}
\]
We have
\[
\text{Hess}(Z_{n,0,1}) = -\frac{1}{2\sigma_1^{1/2}} C_{n,0,3} n(n+1) \mathbb{1}_2 = -\frac{1}{2(2\pi)^{1/2}} n(n+1) \sqrt{n+1} \mathbb{1}_2.
\]
In this case, the basis \( B_2 \) consists of two elements, \( B_2 = \{1, 2\} \), and we have
\[
Z_{n,2,1} = C_{n,2,3} P_{n,3}^{(2)}(x_3) Y_1(x_1, x_2), \quad Z_{n,2,2} = C_{n,2,3} P_{n,3}^{(2)}(x_3) Y_2(x_1, x_2),
\]
where
\[
Y_1(x_1, x_2) = C_1 (x_1^2 - x_2^2), \quad Y_2(x_1, x_2) = C_2 x_1 x_2,
\]
and the constants \( C_1, C_2 \) are found from the identities
\[
1 = C_1^2 \int_{S^1} (x_1^2 - x_2^2)^2 |ds| = C_2^2 \int_{S^1} x_1^2 x_2^2 |ds|.
\]
Note that
\[
\int_{S^1} x_1^2 x_2^2 |ds| = \frac{1}{4} \int_{S^1} \sin^2(2\theta) |d\theta| = \frac{\pi}{4}, \quad \text{and} \quad \int_{S^1} (x_1^2 - x_2^2)^2 |ds| = \int_{S^1} \cos^2(2\theta) |d\theta| = \pi,
\]
so that
\[
C_1 = \pi^{-1/2}, \quad \text{and} \quad C_2 = \frac{2}{\pi^{1/2}}.
\]
We set
\[
H_1 := \text{Hess}(Y_1) = \frac{2}{\pi^{1/2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H_2 := \text{Hess}(Y_2) = \frac{2}{\pi^{1/2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
We deduce that
\[
\text{Hess}(y) = y_{2,1} C_{n,2,3} P_{n,3}^{(2)}(1) H_1 + y_{2,2} C_{n,2,3} P_{n,3}^{(2)}(1) H_2 - y_{0,1} \frac{1}{2\sigma_1^{1/2}} C_{n,0,3} n(n+1) \mathbb{1}_2
\]
\[
= \frac{2}{\pi^{1/2}} C_{n,2,3} P_{n,3}^{(2)}(1) \left( y_{2,1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y_{2,2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) - y_{0,1} \frac{1}{2(2\pi)^{1/2}} n(n+1) \left( n + \frac{1}{2} \right)^{1/2} \mathbb{1}_2.
\]
We set
\[
a(n) := C_{n,2,3} P_{n,3}^{(2)}(1) = \frac{1}{8} \left( \left( n + \frac{1}{2} \right) [n + 2] \right)^{1/2}, \quad b(n) := \frac{1}{4\sqrt{2}} n(n+1) \left( n + \frac{1}{2} \right)^{1/2}.
\tag{2.38}
\]
Note that
\[
a(n)^2 \sim \frac{n^5}{64}, \quad b(n)^2 \sim \frac{n^5}{32} \quad \text{as} \quad n \to \infty.
\]
To ease the presentation, we set
\[
u_1 := y_{2,1}, \quad u_2 := y_{2,2}, \quad u = y_{0,1},
\]
and we deduce that
\[
\text{Hess}(y) = \frac{2}{\pi^{1/2}} \begin{bmatrix} au_1 - bu & au_2 \\ au_2 & -au_1 - bu \end{bmatrix}.
\]
Denote by $L$ the space spanned by $Z_{n,2,1}, Z_{n,2,2}, Z_{n,0,1}$, $\ell := \dim L = 3$. It is contained in $K_n$, and if $y \perp L$, then $\text{Hess}(y) = 0$. For

$$y = u_1Z_{n,2,1} + u_2Z_{n,2,2} + uZ_{n,0,1} \in L,$$

we have

$$|\det \text{Hess}(y)| = \frac{4}{\pi} |b^2u^2 - a^2u_1^2 - a^2u_2^2| \sim \frac{n^5}{16\pi} |2u^2 - u_1^2 - u_2^2|, \ n \to \infty.$$

Arguing as in (2.17), we deduce that

$$A_{p_0}^1 \partial_{x_i} = (\partial_{x_i} Z_{n,1,i}(0)) Z_{n,1,i}, \ i = 1, 2.$$

Recall that

$$Z_{n,1,i} = C_0 C_{n,1,3} P'_{n,3}(x_3) x_i, \ 1 = C_0^3 \int_{S^1} x_i^2 |ds| = \pi C_0^2,$$

so that

$$|A_{p_0}^1 \partial_{x_i}| = C_0 C_{n,1,3} P'_{n,3}(1) Z_{n,1,i} \quad (B.4) = \pi^{-1/2} \left( \frac{2n + 1}{2n(n + 1)} \right)^{1/2} \times \frac{1}{2} n(n + 1)$$

$$= \frac{1}{(4\pi)^{1/2}} \left( \left( n + \frac{1}{2} \right) n(n + 1) \right)^{1/2}.$$

Hence

$$J(A_{p_0}^1) = \frac{n(n + 1)(n + \frac{1}{2})}{4\pi} \sim \frac{n^3}{4\pi} \text{ as } n \to \infty.$$

Putting together all of the above and invoking (\mu_2), we deduce that

$$\frac{\mu(y_n)}{n^2} \sim \text{area}(S^2) \times \pi^{-\frac{11}{2}} \times \frac{1}{4} \int_{\mathbb{R}^3} e^{-\left( (u^2 + u_1^2 + u_2^2) \right)} |2u^2 - (u_1^2 + u_2^2)| |dudu_1du_2|$$

$$= \frac{1}{\pi^{3/2}} \int_{\mathbb{R}^3} e^{-\left( (u^2 + u_1^2 + u_2^2) \right)} |2u^2 - (u_1^2 + u_2^2)| |dudu_1du_2|.$$

Using cylindrical coordinates $(u, r, \theta)$, $u_1 = r \cos \theta, u_2 = r \sin \theta$, we deduce

$$\int_{\mathbb{R}^3} e^{-\left( (u^2 + u_1^2 + u_2^2) \right)} |2u^2 - (u_1^2 + u_2^2)| |dudu_1du_2| = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\left( (u^2 + r^2) \right)} |2u^2 - r^2| r dr d\theta d\phi$$

$$= 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\left( (u^2 + r^2) \right)} |2u^2 - r^2| r dr d\phi = 4\pi \int_0^{\infty} \left( \int_0^{\infty} e^{-\left( (u^2 + r^2) \right)} |2u^2 - r^2| r dr d\phi \right)$$

$$= I.$$

Hence

$$\mu(y_n) \sim 4\pi^{-1/2} n^2.$$

To proceed further, we use polar coordinates $u = t \cos \varphi, r = t \sin \varphi, 0 < \varphi < \frac{\pi}{2}, t \geq 0$ and we deduce

$$I = \int_0^{\infty} \left( \int_0^{\pi/2} |2 \cos^2 \varphi - \sin^2 \varphi| \sin \varphi d\varphi \right) e^{-t^2} t^4 dt$$

$$= \left( \int_0^{\infty} e^{-t^2} t^4 dt \right) \cdot \left( \int_0^{\pi/2} |3 \cos^2 \varphi - 1| \sin \varphi d\varphi \right)$$

(Use the substitutions $s = t^2, x = \cos \varphi$)

$$= \frac{1}{2} \left( \int_0^{\infty} e^{-s} s^{3/2} ds \right) \cdot \left( \int_0^{1} |3x^2 - 1| dx \right) = \frac{1}{2} \Gamma(5/2) \cdot \frac{4}{3\sqrt{3}} = \frac{\pi^{1/2}}{2\sqrt{3}}.$$
Hence
\[ \mu(y_n) \sim \frac{2}{\sqrt{3}} n^2 \text{ as } n \to \infty. \]

**Remark 2.4.** Let us observe that for \( n \) very large, a typical spherical harmonic \( y \in Y_n \) is a Morse function on \( S^2 \) and 0 is a regular value. The nodal set \( \{ y = 0 \} \) is disjoint union of smoothly embedded circles. According to the classical theorem of Courant [10, §VI.6], the complement of the nodal set has at most \( n^2 \) connected components called nodal domains. We denote by \( D_y \) the collection of nodal domain, and we set
\[ \delta(y) := \#D_y \leq n^2, \quad \delta_n := \pi^{\frac{\dim Y_n}{2}} \int_{y_n} e^{-|y|^2} \delta(y) |dV(y)| = \frac{1}{\operatorname{area}(S(Y_n))} \delta(y) |dS(y)|. \]
In [26], it is shown that there exists a positive constant \( a > 0 \) such that
\[ \delta_n \sim an^2 \text{ as } n \to \infty. \]
Additionally, for large \( n \), with high probability, \( \delta(y) \) is close to \( an^2 \) (see [26] for a precise statement).

Denote by \( p(y) \) the number of local minima and maxima of \( y \), and by \( s(y) \) the number of saddle points. Then
\[ \mu(y) = p(y) + s(y), \quad p(y) - s(y) = \chi(S^2) = 2. \]
This proves that
\[ p(y) = \frac{1}{2} (\mu(y) + 2). \]
For every nodal region \( D \), we denote by \( p(y, D) \) the number of local minima and maxima of \( y \) on \( D \). Note that \( p(y, D) > 0 \) for any \( D \) and thus the number \( p(y) = \sum_{D \in D_y} p(y, D) \) can be viewed as a weighted count of nodal domains. We set
\[ p(y_n) := \pi^{\frac{\dim Y_n}{2}} \int_{y_n} e^{-|y|^2} p(y) |dV(y)|. \]
Theorem 2.3 implies that
\[ p(y_n) \sim \frac{1}{\sqrt{3}} n^2 \text{ as } n \to \infty. \]
Since \( \delta(y) \leq p(y) \), this shows that \( a \leq \frac{1}{\sqrt{3}}. \)

**Remark 2.5.** We can use Remark 1.5 as a simple test for the accuracy of the computations in Theorem 2.3. As explained in Remark 1.5, the Euler characteristic of \( S^2 \) is described by an integral very similar to the one describing \( \mu(y_n) \). More precisely, we should have
\[ \pm 2 = \pm \chi(S^2) = \text{area}(S^2) \times \pi^{-\frac{3+2}{2}} \times \frac{1}{J(A_{\nu_0}^1)} \times \int_L e^{-|y|^2} \det \operatorname{Hess}(y) |dV(y)|. \]
The term \( J \) can be computed as follows.
\[ J = \frac{16}{\pi^{1/2} n(n+1)(n+2)} \int_L e^{-|y|^2} \det \operatorname{Hess}(y) |dV(y)| \]

\(^3\)A simple application of the maximum principle shows that on each nodal domain, all the local extrema of \( y \) are of the same type: either all local minima or all local maxima. Thus \( p(y, D) \) can be visualized as the number of peaks of \( |y| \) on \( D \).
\[
\begin{align*}
&= \frac{64}{\pi^{3/2}n(n+1)(n+\frac{1}{2})} \int_{\mathbb{R}^3} e^{-(u^2+u_1^2+u_2^2)} (b(n)^2u^2 - a(n)^2(u_1^2 + u_2^2)) \, |d\mu_1 d\mu_2| \\
&= \frac{1}{\pi^{3/2}} \int_{\mathbb{R}^3} e^{-(u^2+u_1^2+u_2^2)} (2\beta(n)u^2 - \alpha(n)(u_1^2 + u_2^2)) \, |d\mu_1 d\mu_2|,
\end{align*}
\]
where
\[
\beta(n) = n(n+1), \quad \alpha(n) = (n+2)(n-1).
\]
Arguing exactly as in the proof of Theorem 2.3, we deduce
\[
\begin{align*}
\int_{\mathbb{R}^3} e^{-(u^2+u_1^2+u_2^2)} & (2\beta(n)u^2 - \alpha(n)(u_1^2 + u_2^2)) \, |d\mu_1 d\mu_2| \\
&= 4\pi \int_0^{\infty} \int_0^{\infty} e^{-(u^2+r^2)} (2\beta(n)u^2 - \alpha(n)r^2) r \, dr du \\
&= 4\pi \int_0^{\infty} e^{-t^2} \int_0^{\pi/2} e^{-t^2} t^2 (2\beta(n)\cos^2 \varphi - \alpha(n)\sin^2 \varphi) \sin \varphi \, dt d\varphi \\
&= 4\pi \times \frac{1}{2} \int_0^{\infty} e^{-s^2/2} ds \times \int_0^{1} (2\beta(n) + \alpha(n)) x^2 - \alpha(n) \, dx \\
&= \frac{3\pi^{3/2}}{2} \times \frac{2}{3} (\beta(n) - \alpha(n)) = 2\pi^{3/2}.
\end{align*}
\]
This confirms the prediction in Remark 1.5, namely, \(J = \pm 2\).

Remark 2.6. Most of the arguments in the proof of Theorem 2.3 work with minor changes for spherical harmonics of an arbitrary number of variables and lead to the conclusion
\[
\mu(Y_{n,d}) \sim \mathcal{Z}_d n^{d-1} \quad \text{as} \quad n \to \infty,
\]
but the constant \(\mathcal{Z}_d\) is a bit more mysterious. Here are the details.

If
\[
y = \sum_{j=0}^{n} \sum_{\beta \in \mathcal{B}_{j,d}} y_{j,\beta} Z_{n,j,\beta} \in Y_{n,d},
\]
then
\[
\text{Hess}(y) = \sum_{\beta \in \mathcal{B}_{n,d}} y_{2,\beta} \text{Hess}(Z_{n,2,\beta}) + y_{0,1} \text{Hess}(Z_{n,0,1}).
\]
From (2.8), we deduce
\[
\text{Hess}(Z_{n,0,1}) = \frac{1}{2} \sigma_{d-2}^{-1/2} C_{n,0,d}(n+1) \left( n + \frac{d-3}{2} \right) \mathbb{1}_{d-1}
\]
\[
= -\frac{1}{2} \sigma_{d-2}^{-1/2} \times \left( \frac{(2n+d-2)[n+d-3]_{d-3}}{2d-2} \right)^{1/2} \times \frac{(n+1)(n+d-3)}{\Gamma(\frac{d-1}{2})} \mathbb{1}_{d-1}
\]
\[
\sim -\frac{1}{2 \sigma_{d-2}^{1/2}} \frac{n+d-3}{\Gamma(\frac{d-1}{2})} \mathbb{1}_{d-1} \quad \text{as} \quad n \to \infty.
\]

\(= \alpha(d)\)
As in (2.4), we have
\[ Z_{n,2,\beta}(x) := C_{n,j,d} P_{n,d}^{(2)}(x_d) Y_{\beta}(x'), \quad x = (x', x_d). \]
If we denote by \( H_{\beta} \) the Hessian of \( Y_{\beta} \) at \( x' = 0 \), we deduce
\[ \text{Hess}(Z_{n,2,\beta}) = C_{n,2,d} P_{n,d}^{(2)}(1) H_{\beta}. \]
Using (r_{\beta}), we deduce
\[ \text{Hess}(Z_{n,2,\beta}) \sim \frac{1}{2} \frac{n^{d+2}}{\Gamma(\frac{d-1}{2})} H_{\beta} \text{ as } n \to \infty. \]
Arguing as in the proof of (2.17) we deduce
\[ A_{p_0}^i \partial_{x_i} = \partial_{x_i} Z_{n,1,i}(P_0) Z_{n,1,i}, \]
where
\[ Z_{n,1,i} = C_{n,1,d} P_{n,d}'(x_d) \cdot C_d x_i, \quad \int_{\mathbb{S}^{d-2}} C_d x_i^2 |dS(x')| = 1. \]
Hence
\[ A_{p_0}^i \partial_{x_i} = C_d C_{n,1,d} P_{n,d}'(1) Z_{n,1,i}. \]
We have
\[ C_{n,1,d} \sim \frac{1}{2} \frac{n^{d-4}}{\Gamma(\frac{d-1}{2})} \text{ as } n \to \infty. \]
Using (B.3) we obtain
\[ P_{n,d}'(1) \sim \frac{1}{2} n^2 \text{ as } n \to \infty. \]
Using (B.8) we deduce
\[ C_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d-1}{2}}}. \]
Hence,
\[ |A_{p_0}^i \partial_{x_i}| \sim \frac{C_d}{2^{\frac{1}{2}} \Gamma(\frac{d-1}{2})} n^{\frac{d}{2}} \text{ as } n \to \infty, \]
so that
\[ J(A_{p_0}^i) \sim \left( \frac{C_d}{2^{\frac{1}{2}}} \right)^{d-1} \frac{n^{\frac{d(d-1)}{2}}}{\Gamma(\frac{d-1}{2})(d-1)}. \]
Denote by \( L \) the subspace of \( y_{n,d} \) spanned by the orthonormal collection of spherical harmonics
\[ \{ Z_{n,0,1}, Z_{n,2,\beta}, \beta \in B_{2,d} \}. \]
It has dimension
\[ \dim L = N_{d-1} := \dim \text{Sym}(T_{p_0} \mathbb{S}^{d-1}) = \binom{d}{2}. \]
Using (\( \mu_2 \)), Corollary 1.3 and the above computations we deduce
\[ \frac{\mu(y_{n,d})}{n^{d-1}} \sim \sigma_{d-1} \times \pi^{-\frac{N_{d-1}+d-1}{2}} \left( \frac{C_d}{2^{\frac{1}{2}}} \right)^{-(d-1)} \times \int_L e^{-|y|^2} \left| \det A_{\infty}(y) \right| |dV(y)|, \]
where for
\[ y = y_1 Z_{N,0,1} + \sum_{\beta \in B_{2,d}} y_{\beta} Z_{n,2,\beta} \in L, \]
we have

\[ A_\infty(y) = y_1 a(d) 1_{d-1} + \sum_{\beta} y_\beta b(d) H_\beta. \]

We interpret \( A_\infty \) as an isometry from \( L \) to the space \( \text{Sym}_{d-1} = \text{Sym}(T_{p_0}S^{d-1}, g) \) such that the collection

\[ a(d) 1_{d-1}, \ b(d) H_\beta \]

is an orthonormal basis of \( \text{Sym}_{d-1} \). We denote by \( g_{a,b} \) this \( O(d-1) \)-invariant metric on \( \text{Sym}_{d-1} \) and by \( |dV_{a,b}| \) the associated volume density. We deduce

\[
\int_L e^{-|y|^2} |\det A_\infty(y)| |dV(y)| = \int_{\text{Sym}_{d-1}} e^{-|A_{a,b}^2|} |\det(A)| |dV_{a,b}(A)|.
\]

On \( \text{Sym}_{k-1} \) we have a canonical \( O(d-1) \) metric \( |\cdot|_* \) defined by

\[ |A|^2_*, = \text{tr} A^2. \]

We denote by \( |dV_*| \) the associated volume density. From (C.3), we deduce

\[ |dV_{a,b}| = \gamma_d |dV_*|, \quad \gamma = \frac{1}{a((d-1)\lambda/2)(bR)^{d-1}}, \quad R^2 = \frac{4G(d+\beta)}{\pi^{d-2}}. \]

From (C.2), we deduce

\[ |A|^2_{a,b} = \alpha \text{tr} A^2 + \beta (\text{tr} A)^2, \]

where

\[
\alpha = \frac{1}{b^2 R^2} = \frac{2^{d+1}\pi \frac{d-1}{2}}{\Gamma\left(\frac{d+3}{2}\right)},
\]

\[
\beta = \frac{1}{d-1} \left( \frac{1}{(d-1)\lambda^2} - \frac{1}{b^2 R^2} \right) = \frac{1}{(d-1)} \left( \frac{2^{d-1} \sigma_{d-2}}{(d-1)^2} - \frac{2^{d+1} \pi \frac{d-1}{2}}{\Gamma\left(\frac{d+3}{2}\right)} \right)
\]

\[
= \frac{2^d \pi \lambda^2}{(d-1)} \left( \frac{1}{\Gamma\left(\frac{d-1}{2}\right)} - \frac{2}{\Gamma\left(\frac{d+3}{2}\right)} \right) = \frac{2^d \pi \lambda^2}{(d-1)} \left( \frac{1}{2} - \frac{4}{(d+1)(d-1)} \right)
\]

\[
= \frac{d-7}{8(d-1)} \alpha.
\]

We deduce

\[
\int_{\text{Sym}_{d-1}} e^{-|A|^2_{a,b}} |\det(A)| |dV_{a,b}(A)| = \gamma_d \int_{\text{Sym}_{d-1}} e^{-\alpha \text{tr} A^2 - \beta (\text{tr} A)^2} |\det A| |dV_*(A)|.
\]

As explained in Appendix C, the last integral can be further simplified to

\[
\int_{\text{Sym}_{d-1}} e^{-\alpha \text{tr} A^2 - \beta (\text{tr} A)^2} |\det A| |dV_*(A)|
\]

\[
= \mathcal{Z}_d \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - \frac{\beta}{2\alpha} (\text{tr} x^2)} \prod_{i=1}^{d-1} |x_i| \cdot \prod_{1 \leq i < j \leq d-1} |x_i - x_j| |dV(x)|,
\]

where \( \text{tr}(x) := x_1 + \cdots + x_{d-1} \), and \( \mathcal{Z}_d \) is a positive constant that can be determined explicitly. The integral \( I_d \) seems difficult to evaluate. The trick used in [18] does not work when \( d > 7 \), since in that case \( \beta > 0 \). The asymptotics of \( I_d \) as \( d \to \infty \) are very intriguing.
3. RANDOM TRIGONOMETRIC POLYNOMIALS WITH GIVEN NEWTON POLYHEDRON

Fix a positive integer \(L\) and denote by \(\mathbb{T}^L\) the \(L\)-dimensional torus \(\mathbb{T}^L := \mathbb{R}^L/(2\pi \mathbb{Z})^L\) equipped with the induced flat metric. Let \(\vec{\theta} = (\theta_1, \ldots, \theta_L)\) denote the angular coordinates induced from the canonical Euclidean coordinates on \(\mathbb{R}^L\). For any \(\vec{m} \in \mathbb{Z}^L\) we set

\[
p(\vec{m}) = \begin{cases} 
\frac{2^{1/2}}{(2\pi)^{L/2}}, & |\vec{m}| \neq 0 \\
\frac{1}{(2\pi)^{L/2}}, & |\vec{m}| = 0,
\end{cases}
\]

\[
\widehat{A}_{\vec{m}} := p(\vec{m}) \cos \left( \sum_{j=1}^L m_j \theta_j \right), \quad \widehat{B}_{\vec{m}} := p(\vec{m}) \sin \left( \sum_{j=1}^L m_j \theta_j \right).
\]

The lattice \(\mathbb{Z}^L\) is equipped with the lexicographic order \(<\), where we define \(\vec{m} < \vec{n}\) if the first non zero element in the sequence \(n_1 - m_1, \ldots, n_L - m_L\) is positive. We define \(\mathcal{C}_L\) to be the positive cone \(\mathcal{C}_L := \{\vec{m} \in \mathbb{Z}^L; \vec{0} \prec \vec{m}\}\).

The collection \(\{\widehat{A}_{\vec{m}}\} \cup \{\widehat{A}_{\vec{n}}; \vec{m} \in \mathcal{C}_L\} \cup \{\widehat{B}_{\vec{m}}; \vec{m} \in \mathcal{C}_L\}\) is an orthonormal basis of \(L^2(\mathbb{T}^L)\). A finite set \(\mathcal{M} \subset \mathcal{C}_L\) is called symmetric if for any permutation \(\varphi\) of \(\{1, \ldots, L\}\) we have

\[
(m_1, \ldots, m_L) \in \mathcal{M} \iff (m_{\varphi(1)}, \ldots, m_{\varphi(L)}) \in \mathcal{M} \cup -\mathcal{M}.
\]

For example, the set \(\{(2, -1), (1, -1), (1, -2)\} \subset \mathcal{C}_2\) is symmetric.

For any finite set \(\mathcal{M} \subset \mathcal{C}_L\) we define

\[
\mathbf{V}(\mathcal{M}) := \text{span}\{\widehat{A}_{\vec{m}}, \widehat{B}_{\vec{n}}; \vec{m}, \vec{n} \in \mathcal{M}\},
\]

the scalars

\[
a_{jk} = a_{jk}(\mathcal{M}) = \frac{2}{(2\pi)^L} \sum_{\vec{m} \in \mathcal{M}} m_j m_k,
\]

and the vectors

\[
\vec{a}_{jk} = \vec{a}_{jk}(\mathcal{M}) = -\frac{2^{1/2}}{(2\pi)^{L/2}} \sum_{\vec{m} \in \mathcal{M}} m_j m_k \widehat{A}_{\vec{m}} \in \mathbf{V}(\mathcal{M}).
\]

If \(\mathcal{M}\) is symmetric then the scalars \(a_{jj}\) are independent of \(j\) and we denote their common value by \(\alpha(\mathcal{M})\). Similarly, the scalars \(a_{jk}, j \neq k\), are independent of \(j \neq k\), and we denote their common value by \(b(\mathcal{M})\).

**Theorem 3.1.** Suppose \(\mathcal{M} \subset \mathcal{C}_L\) is a symmetric finite subset of cardinality \(N > L\). We set

\[
a := a(\mathcal{M}), \quad b := b(\mathcal{M}), \quad \vec{a}_{ij} = \vec{a}_{ij}(\mathcal{M}).
\]

Then the following hold.

(a) The sample space \(\mathbf{V}(\mathcal{M})\) is ample if and only if \(a \neq b\).

(b) Suppose that

the vectors \(\vec{a}_{ij}\) are linearly independent. (♯)
Denote by $\text{Sym}_L$ the Euclidean space of symmetric $L \times L$ matrices with orthonormal basis $(H_{ij})_{i,j \leq L}$, where $H_{jk}$ is the symmetric $L \times L$ matrix with nonzero entries only in at locations $(j,k)$ and $(k,j)$, and those entries are 1. We denote by $\langle -, - \rangle$ the resulting inner product on $\text{Sym}_L$. Then

$$\mu(M) = \frac{1}{(2\pi)^{L^2}/2} \int_{\text{Sym}_L} e^{-\frac{1}{2}(C^{-1}X,X)} \det X \, |dX|,$$

where for $X = \sum_{i,j \leq L} x_{ij} H_{ij}$

$$|dX| = \prod_{i \leq j} dx_{ij},$$

and $C : \text{Sym}_L \to \text{Sym}_L$ is the symmetric linear operator described in the orthonormal basis $(H_{ij})$ by the matrix

$$C_{ij,k\ell} = \langle \tilde{a}_{ij}, \tilde{a}_{k\ell} \rangle = \frac{2}{(2\pi)^L} \sum_{m \in M} m_i m_j m_k m_\ell.$$

**Proof.** We will compute $\mu(M)$ via the identity (1.14). Observe first that $V(M)$ is invariant under the action of $\mathbb{T}^L$ on itself, and the induced action on $V(M)$ is by isometries. Let $p = (0, \ldots, 0) \in \mathbb{T}^d$, and denote by $K_p$ the subspace of $V(M)$ consisting of trigonometric polynomials that admit $p$ as a critical point. Set $\partial_j := \partial_{\theta_j}$, $f_j := \partial_j |_{p}$, $j = 1, \ldots, d$. We have

$$A_p^\dagger f_j = \sum_{m \in M} \left( \partial_j \hat{A}_m(\tilde{\theta}) \hat{A}_m + \partial_j \hat{B}_m(\tilde{\theta}) \hat{B}_m \right) |_{\tilde{\theta} = 0} = \sum_{m \in M} m_j p(m) \hat{B}_m.$$

We have

$$G_{jk} := \langle A_p^\dagger f_j, A_p^\dagger f_k \rangle = \frac{2}{(2\pi)^L} \sum_{m \in M} m_j m_k = a_{jk}(M).$$

Since $M$ is symmetric we deduce that $A_p^\dagger$ is described by the symmetric $L \times L$ matrix $G_L(a,b)$ whose diagonal entries are all equal to $a$, and all the off-diagonal entries are equal to $b$. We denote by $\Delta_L(a,b)$ its determinant. We deduce

$$\Delta_L(a,b) = (a - b)^{L-1}(a + (L - 1)b),$$

so that the Jacobian of $A_p^\dagger$ is

$$J(A_p^\dagger) = \Delta(a,b)^{1/2}. \tag{3.5}$$

Observe that $V(M)$ is ample if and only if the Jacobian of the adjunction map $A_p^\dagger$ is nonzero, i.e., if and only if $\Delta_L(a,b) \neq 0$. This proves part (a).

If

$$v = \sum_{m \in M} (a_m \hat{A}_m + b_m \hat{B}_m),$$

then

$$\partial_j \partial_k v(p) = \frac{-2^\dagger}{(2\pi)^{L/2}} \sum_{m \in M} a_m m_j m_k.$$

We deduce that

$$\text{Hess}_p(v) = \frac{-2^\dagger}{(2\pi)^{L/2}} \sum_{j \leq k} \left( \sum_{m \in M} a_m m_j m_k \right) H_{jk} \tag{3.6}$$

---

*4 If $C_L$ denotes the $L \times L$ matrix with all entries 1, then $G_L(a,b) = (a - b)1_L + bC_L$. The matrix $C_L$ has rank 1 and a single nonzero eigenvalue equal to $L$. This implies (3.4).*
Using the notations (3.1) and (3.2) we can rewrite the equality (3.6) as

$$\text{Hess}_p(v) = \sum_{i \leq j} (v, \bar{a}_{ij}) H_{ij}.$$ 

Using Lemma A.3 and the equality \( \dim V(M) = 2N \) we deduce that

$$\int_{S(K_p)} |\det \text{Hess}(v)| |dS(v)| = \frac{2}{\Gamma(N)} \int_{K_p} e^{-|v|^2} |\det \text{Hess}(v)| |dV(v)|$$

$$= \frac{2 - \frac{L+1}{2} \frac{\dim K_p}{2}}{\Gamma(N)} \int_{K_p} e^{-\frac{|v|^2}{2}} |\det \text{Hess}(v)| \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{\dim K_p}{2}}} |dV(v)|.$$ 

We performed all this yoga to observe that \( I(M) \) is an integral with respect to a Gaussian density over \( K_p \). Denote by \( \mathcal{T} = \mathcal{T}_M \) the linear map

$$\mathcal{T} : K_p \rightarrow \text{Sym}_L, \quad v \mapsto \text{Hess}_p(v) = \sum_{i \leq j} (v, \bar{a}_{ij}) H_{ij}.$$ 

Since the vectors \( \bar{a}_{ij} \) are assumed to be linearly independent, the map \( \mathcal{T} \) is surjective. Clearly, \( \det \text{Hess}(v) \) is constant along the fibers of \( \mathcal{T} \). As is well known (see e.g. [20, §16]) the pushforward of a Gaussian measure via a surjective linear map is also a Gaussian measure. Thus the density

$$|d\gamma_M| := \mathcal{T}^* \left( \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{\dim K_p}{2}}} |dV(v)| \right)$$

is a Gaussian density on \( \text{Sym}_L \). Since the density \( \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{\dim K_p}{2}}} |dV(v)| \) is centered, i.e., its expectation is trivial, we deduce that its pushforward by \( \mathcal{T} \) is also centered. The Gaussian density \( |d\gamma_M| \) is thus determined by its covariance operator

$$C = C_M : \text{Sym}_L \rightarrow \text{Sym}_L$$

described in the orthonormal basis \((H_{ij})\) by the matrix

$$C_{ij;k\ell} = (\bar{a}_{ij}, \bar{a}_{k\ell}) = \frac{2}{(2\pi)^L} \sum_{\bar{m} \in M} m_i m_j m_k m_{\ell}.$$ 

The symmetry condition on \( M \) imposes many relations between these numbers. We deduce

$$I(M) = \frac{1}{(2\pi)^{\frac{L(L+1)}{2}} (\det C)^{1/2}} \int_{\text{Sym}_L} e^{-\frac{1}{2} (C^{-1}X,X)} |\det X| |dX|, \quad s(L) := \dim \text{Sym}_L.$$ 

Using (1.14) we deduce

$$\mu(M) = \frac{\text{vol} (\mathbb{T}^L)}{\sigma_{2N-1} \Delta_L(a,b)^{1/2}} \times \frac{2 - \frac{L+1}{2} \frac{\dim K_p}{2}}{\Gamma(N)} I(M)$$

$$= \frac{\text{vol} (\mathbb{T}^L)}{\Delta_L(a,b)^{1/2}} \times \frac{2 - \frac{L}{2} \frac{\dim K_p}{2}}{\pi^N} I(M) = \frac{(2\pi)^{\frac{L}{2}}}{\Delta_L(a,b)^{1/2}} I(M).$$

This proves (3.3). \( \square \)
We will put the above theorem to work in several special cases. Let us observe that the assumption (#) is automatically satisfied if \( M \) contains the points 
\[(1, 0, \ldots, 0), \quad (1, 1, 0, \ldots, 0).\]
Indeed, the symmetry of \( M \) implies that all the functions \( \cos(\theta_i) \) and \( \cos(\theta_i + \theta_j), \quad 1 \leq i \neq j \leq L, \)
belong to \( V(M) \) and the hessians of these functions span the whole space of \( L \times L \) matrices.

Suppose now that \( M = M^L_\nu := \Lambda^L_\nu \cap \mathbb{E} \), where \( \nu \) is a (large) positive integer, and \( \Lambda_\nu \) is the cube
\[
\Lambda^L_\nu := \{ \vec{m} \in \mathbb{Z}^L; \quad |m_j| \leq \nu, \quad \forall j = 1, \ldots, L \}.
\]
Let us observe that
\[
\Lambda^L_\nu = M^L_\nu \cup (-M^L_\nu) \cup \{ \vec{0} \}. \tag{3.7}
\]
Among other things, this proves that \( M_\nu \) is symmetric. We want to investigate the behavior of \( \mu(M^L_\nu) \) as \( \nu \to \infty \). To formulate our next result we need to introduce additional notation.

Let us observe that we have an orthogonal decomposition
\[
\text{Sym}_L = D_L \oplus D_L^\perp, \tag{3.8}
\]
where \( D_L \) consists of diagonal matrices
\[
D_L = \text{span}\{ H_{ii}; \quad 1 \leq i \leq L \}
\]
and
\[
D_L^\perp = \text{span}\{ H_{ij}; \quad 1 \leq i < j \leq L \}.
\]
For any real numbers \( a, b \) we denote by \( G_L(a, b) \) the \( L \times L \)-matrix with entries
\[
g_{ij} = \begin{cases} a, & i = j \\ b, & i \neq j. \end{cases}
\]

**Theorem 3.2.** Let
\[
M^L_\nu := \{ \vec{m} \in \mathbb{E}_L; \quad |m_j| \leq \nu, \quad \forall 1 \leq j \leq L \}.
\]
Then, as \( \nu \to \infty \) we have
\[
\mu(M^L_\nu) \sim \left( \frac{\pi}{6} \right)^L \langle |\det X| \rangle_{\mathcal{C}_\infty} \dim V(M^L_\nu), \tag{3.9}
\]
where \( \langle |\det X| \rangle_{\mathcal{C}_\infty} \) the expectation of \( |\det X| \) with respect to the centered gaussian probability measure on \( \text{Sym}_L \) with covariance matrix that has the block description
\[
\mathcal{C}_\infty = G_L \left( \frac{9}{5}, 1 \right) \oplus \mathbb{I}_{(\ell)}
\]
with respect to the decomposition \( (3.8) \).

**Proof.** Let us first compute
\[
a(\nu) = a(M^L_\nu) = \frac{2}{(2\pi)^L} \sum_{\vec{m} \in M^L_\nu} m_1^2 \tag{3.7} = \frac{1}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1^2 = \frac{|\Lambda^L_\nu|}{(2\pi)^L} \sum_{|m_1| \leq \nu} m_1^2 |\Lambda^L_\nu|^{-1}
\]
\[
= \frac{2(2\nu + 1)L - 1}{(2\pi)^L} \sum_{k=1}^L k^2 = \frac{2(2\nu + 1)L - 1}{3(2\pi)^L} B_3(\nu + 1) \sim \frac{1}{3\pi L} \nu^{L+2} \quad \text{as} \quad \nu \to \infty.
\]
Similarly, we have
\[
b(\nu) = b(M^L_\nu) = \frac{1}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1 m_2.
\]
The last sum is 0 due to the invariance of $\Lambda^L_\nu$ with respect to the reflection 
$$(m_1, m_2, \ldots, m_L) \longleftrightarrow (-m_1, m_2, \ldots, m_L).$$
Thus, in this case
$$\Delta_L(a, b) = a(\nu)L^2 \sim \frac{1}{3L^{L+2}} \nu^{L(L+2)} \text{ as } \nu \to \infty. \quad (3.10)$$
To compute the covariance operator $C$ we observe first that, in view of the symmetry of $M_\nu$ it suffices to compute only the entries $C_{11;ij}, \ i \leq j \text{ and } C_{12;ij}, \ i < j$.

We have
$$C_{11;11} = \frac{1}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1^4 = \frac{2|\Lambda^L_\nu|}{(2\pi)^L} \sum_{k=1}^{\nu} k^4 = \frac{2(2\nu + 1)L^{-1}}{5(2\pi)^L} B_5(\nu + 1) \sim \frac{1}{5\pi L^3} \nu^{L+4}.$$

For $i > 1$ we have
$$C_{11;ii} = \frac{1}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1^2 m_i^2 = \frac{1}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1^2 m_i^2 = \frac{|\Lambda^L_\nu|}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1^2 m_i^2 = \frac{2|\Lambda^L_\nu|}{(2\pi)^L} \left( \sum_{k=1}^{\nu} k^2 \right)^2 = \frac{4(2\nu + 1)L^{-2}}{9(2\pi)^L} B_3(\nu + 1) \sim \frac{1}{9\pi L^3} \nu^{L+4}.$$

Using the invariance of $\Lambda_\nu$ with respect to the reflections
$$(m_1, \ldots, m_i, \ldots, m_L) \leftrightarrow (m_1, \ldots, -m_i, \ldots, m_L) \quad (3.11)$$
we deduce that for any $i < j$ we have
$$C_{11;ij} = 0.$$

To summarize, we have shown that
\begin{align*}
x_\nu &= C_{ii;ii} = C_{11;11} \sim \frac{1}{5\pi L^3} \nu^{L+4} \quad (3.12a) \\
y_\nu &= C_{ii;jj} = C_{11;jj} \sim \frac{1}{9\pi L^3} \nu^{L+4} \forall 1 \leq i < j. \quad (3.12b) \\
C_{ii;jk} &= 0, \forall i, \ j < k. \quad (3.12c)
\end{align*}

Next, we observe that
$$C_{12;12} = \frac{1}{(2\pi)^L} \sum_{\vec{m} \in \Lambda^L_\nu} m_1^2 m_i^2 = \frac{4(2\nu + 1)L^{-2}}{9(2\pi)^L} B_3(\nu + 1) = y_\nu \sim \frac{1}{9\pi L^3} \nu^{L+4}.$$

Using the reflections (3.11) we deduce that
$$C_{12;ij} = 0, \forall i < j, \ (i, j) \neq (1, 2).$$

With respect to the decomposition (3.8) the covariance operator has a block decomposition
$$C = \begin{bmatrix} \mathcal{G} & \mathcal{F} \\ \mathcal{F}^* & \mathcal{H} \end{bmatrix},$$
where $\mathcal{F} : \mathcal{D}_L^\perp \to \mathcal{D}_L$. The above computations show that
$$\mathcal{F} = 0, \ \mathcal{H} = y_\nu \mathcal{1}_{\mathcal{D}_L^\perp} = y_\nu \mathcal{1}_{(L^2)}.$$
The operator $\mathcal{S}$ is described in the basis $(H_{il})$ of $\mathbb{D}_L$ by the matrix $G_L(x_\nu, y_\nu)$. We deduce that

$$ C = G_L(x_\nu, y_\nu) \oplus y_\nu \mathbb{1}_{(L)} = y_\nu \times \left( G_L(z_\nu, 1) \oplus \mathbb{1}_{(L)} \right), \quad z_\nu = \frac{x_\nu}{y_\nu}. $$

Using (3.12a) and (3.12b) we deduce that

$$ \lim_{\nu \to \infty} z_\nu = \frac{9}{5}. $$

We conclude that

$$ \det C \sim y_\nu^{(L) + L} \det G_L \left( \frac{9}{5}, 1 \right) \sim \left( \frac{4}{5} \right)^{L-1} \left( \frac{4}{5} + L \right) y_\nu^{(L) + L}, \quad \text{as } \nu \to \infty. \quad (3.13) $$

Using (3.3), (3.10) we deduce

$$ \mu(M^L_\nu) = \frac{3 \pi L^2}{(2\pi)^{3/2}} \frac{1}{y_\nu^{L/2}} \frac{1}{y_\nu} \left( \frac{L}{\nu} \right)^{L(L+1)/2} \frac{1}{(\nu^L)} \frac{1}{(\nu^{L+1})} \left( \det \mathcal{C}_\nu \right)^{1/2} \int_{\text{Sym}_L} e^{-\frac{1}{2\nu^2} (\mathcal{C}_\nu^{-1})^T x X | dX | \det X | dX |}, $$

making the change in variables $X = y_\nu^{1/2} Y$ we deduce

$$ \mu(M^L_\nu) = \frac{3 \pi L^2}{(2\pi)^{3/2}} \frac{1}{y_\nu^{L/2}} \frac{1}{y_\nu} \left( \frac{L}{\nu} \right)^{L(L+1)/2} \frac{1}{(\nu^L)} \frac{1}{(\nu^{L+1})} \left( \det \mathcal{C}_\nu \right)^{1/2} \int_{\text{Sym}_L} e^{-\frac{1}{2\nu^2} (\mathcal{C}_\nu^{-1})^T y Y | dY | | dY |}, $$

As $\nu \to \infty$ we have

$$ \mathcal{C}_\nu \to \mathcal{C}_\infty := G_L \left( \frac{9}{5}, 1 \right) \oplus \mathbb{1}_{(L)}. $$

Using (3.12b) we deduce that as $\nu \to \infty$ we have

$$ \mu(M_\nu) \sim Z_L v^L, \quad Z_L = \frac{1}{\frac{3}{2} \pi \left( \frac{L}{\nu} \right)^{L(L+1)/2} \left( \det \mathcal{C}_\infty \right)^{1/2} \int_{\text{Sym}_L} e^{-\frac{1}{2} (\mathcal{C}_\infty^{-1})^T y Y \det Y | dY | | dY |}. $$

Since

$$ \dim V(M^L_\nu) \sim (2\nu)^L \quad \text{as } \nu \to \infty, $$

we deduce

$$ \mu(M^L_\nu) \sim \left( \frac{\pi}{6} \right)^{L/2} \times \frac{1}{\left( \frac{2\pi}{2} \right)^{L(L+1)/2} \left( \det \mathcal{C}_\infty \right)^{1/2} \int_{\text{Sym}_L} e^{-\frac{1}{2} (\mathcal{C}_\infty^{-1})^T y Y \det Y | dY | | dY |}. $$

This proves (3.9). \hfill \Box

Let us apply the above result in the case $L = 1$. In this case $\mathcal{M}_\nu$ consists of trigonometric polynomials of degree $\leq \nu$ on $S^1$, and $\text{Sym}_L = \mathbb{R}$. In this case we have

$$ \langle | \det X | \rangle_{\mathcal{C}_\infty} = \frac{\sqrt{5}}{3} \times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{5x^2}{18}} |x| dx = \frac{2\sqrt{5}}{3} \times \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{5x^2}{18}} x dx = \left( \frac{6}{\pi} \right)^{1/2} \sqrt{3/5}. $$

We deduce the following result.
Corollary 3.3.

\[
\mu(M^1_{\nu}) \sim 2 \sqrt{\frac{3}{5}} \nu, \quad \text{as } \nu \to \infty.
\] (3.14)

When \( L = 2 \), the computations are a bit more complicated, but we can still be quite explicit.

Corollary 3.4.

\[
\mu(M^2_{\nu}) \sim Z_2 \dim V(M^2_{\nu}), \quad Z_2 \approx 0.4717, \quad \text{as } \nu \to \infty.
\] (3.15)

Proof. We decompose the operators \( X \in \text{Sym}_2 \) as

\[
X = xH_{11} + yH_{22} + zH_{12}
\]

so that \( \det X = (xy - z^2) \). We write \( a := \frac{2}{5}, \ b := 1 \). Then

\[
\langle |\det X| \rangle_{C_{\infty}} = \frac{1}{(2\pi)^{3/2}(a^2 - b^2)^{1/2}} \int_{\mathbb{R}^3} e^{-\frac{1}{2(a^2 - b^2)(ax^2 + ay^2 - 2bxy) - \frac{1}{2}z^2} |xy - z^2|} \, dx \, dy \, dz =: I(a, b).
\]

As shown in Proposition A.4, the integral \( I(a, b) \) can be reduced to a 1-dimensional integral

\[
I(a, b) = \sqrt{2\pi(a^2 - b^2)} \left( \int_0^{2\pi} \frac{2c^{3/2}}{(c + 2)^{1/2}} \, d\theta - 2\pi a + 2\pi \right),
\]

where

\[
c(\theta) := (a - b \cos 2\theta).
\]

We deduce

\[
\langle |\det X| \rangle_{C_{\infty}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{2c^{3/2}}{(c + 2)^{1/2}} \, d\theta - a + 1 \approx 1.7207...
\]

and

\[
\frac{\pi}{6} \times \langle |\det X| \rangle_{C_{\infty}} \approx 0.4717...
\]

\[\square\]

Remark 3.5. The antiderivative of \( \frac{c^{3/2}}{(c + 2)^{1/2}} \) can be expressed in a rather complicated fashion in terms of elliptic integrals. \[\square\]

Still in the case \( L = 2 \), suppose that

\[
\mathcal{M} = \{(1, 0), (0, 1), (1, 1)\}.
\] (3.16)

The space \( V(\mathcal{M}) \) was investigated in great detail by V.I. Arnold, [4, 5, 6].

Theorem 3.6. If \( \mathcal{M} \) is given by (3.16), then

\[
\mu(\mathcal{M}) = \frac{4\pi}{3} \approx 4.188.
\]
Proof. We rely on Theorem 3.1, or rather its proof. In this case $L = 2$, $\dim V(M) = 6$. The collection $\{A_{1,0}, A_{1,1}, A_{0,1}\}$ is an orthonormal system, and we denote by $L$ the vector space they span. Note that $L \subset K_p$, and $\text{Hess}(v) = 0$ if $v \in L^\perp \cap K_p$. We have

$$a = \frac{1}{2\pi^2} \sum_{\tilde{m} \in \mathcal{M}} m_1^2 = \frac{1}{\pi^2}, \quad b = \frac{1}{2\pi^2} \sum_{\tilde{m} \in \mathcal{M}} m_1 m_2 = \frac{1}{2\pi^2}.$$ 

Then

$$a - b = \frac{1}{2\pi^2}, \quad a + (L - 1)b = \frac{3}{2\pi^2}, \quad J(A_p^1) = (a - b) L^\perp (a + (L - 1)b)^{1/2} = \sqrt{3} \frac{1}{2\pi^2}.$$ 

We decompose $v \in L$ as

$$v = x \tilde{A}_{1,0} + y \tilde{A}_{0,1} + z \tilde{A}_{1,1},$$

and we have

$$\text{Hess}_p(v) = -\frac{2^{1/2}}{2\pi} \begin{bmatrix} x + z & z & y + z \end{bmatrix},$$

$$|\det \text{Hess}_p(v)| = \frac{1}{2\pi^2} |x y + y z + z x|, \quad |u|^2 = x^2 + y^2 + z^2.$$ 

Using $(\mu_2)$ we deduce

$$\mu(M) = \frac{\text{vol}(S^1 \times S^1)}{\pi^{5/2}} J(A_p^1) \times \frac{1}{2\pi^2} \int_{\mathbb{R}^3} e^{-(x^2 + y^2 + z^2)} \left|xy + yz + zx\right| |dS(x, y, z)|$$

$$= \frac{4}{\pi^{1/2}} \sqrt{3} \int_{\mathbb{R}^3} e^{-(x^2 + y^2 + z^2)} \left|xy + yz + zx\right| |dxdydz|.$$ 

The quadratic form $Q(x, y, z) = xy + yz + zx$ can be diagonalized via an orthogonal change of coordinates. The matrix describing $Q$ in the orthonormal coordinates $x, y, z$ is the symmetric matrix

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

and its eigenvalues are $1, -\frac{1}{2}, -\frac{1}{2}$. Thus, for some Euclidean coordinates $u, v, w$, we have

$$Q = \frac{1}{2} \left(2u^2 - v^2 - w^2\right),$$

and therefore,

$$\mu(M) = \frac{2}{\pi^{1/2}} \sqrt{3} \int_{\mathbb{R}^3} e^{-(u^2 + v^2 + w^2)} |2u^2 - v^2 - w^2| |dudvdw|.$$ 

The above integral can be computed using cylindrical coordinates $(u, r, \theta)$,

$$r = (v^2 + w^2)^{1/2}, \quad v = r \cos \theta, \quad w = r \sin \theta.$$

We deduce

$$I = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2 + r^2)} |2u^2 - r^2| dr dud\theta = 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} e^{-(u^2 + r^2)} |2u^2 - r^2| dr dud$$

$$(u = t \cos \varphi, r = t \sin \varphi \leq \varphi \leq \pi, \ t \geq 0)$$

$$= 2\pi \int_0^{\infty} \left( \int_0^{\pi} |2 \cos^2 \varphi - \sin^2 \varphi| \sin \varphi d\varphi \right) e^{-t^2} t^4 dt$$

$$= 2\pi \int_0^{\infty} \left( \int_0^{\pi} |2 \cos^2 \varphi - \sin^2 \varphi| \sin \varphi d\varphi \right) e^{-t^2} t^4 dt.$$
\[ x = \cos \varphi = 2\pi \left( \int_0^\infty e^{-t^2} dt \right) \cdot \left( \int_{-1}^1 |3x^2 - 1| dx \right) = \pi \left( \int_0^\infty e^{-s} s^{3/2} ds \right) \cdot \left( \int_{-1}^1 |3x^2 - 1| dx \right) = \pi \cdot \Gamma(5/2) \cdot \left( \frac{8\sqrt{3}}{9} \right) = \frac{2\pi^{3/2} \sqrt{3}}{3}. \]

Hence,

\[ \mu(M) = \frac{1}{\sqrt{3}} I = \frac{4\pi}{3} \approx 4.188. \]

\[ \square \]

**Remark 3.7.** The typical trigonometric polynomial \( t \in T(M) \) is a Morse function on \( S^1 \times S^1 \), and thus it has an even number of critical points. Morse inequalities imply that it must have at least 4 critical points. We see that the expected number of critical points of a polynomial in \( V(M) \) is very close to this minimum, and that \( V(M) \) must contain Morse functions with at least 6 critical points. Arnold proved in [6] that the typical function in \( T(M) \) has at most 8 critical points.

A later result of Arnold, [5, Thm. 1] states that a generic trigonometric polynomial in \( V(M) \) has at most 6 critical points. However, there is an elementary, but consequential error in the proof of this theorem. More precisely, a key concept in the proof is a (real) linear operator that associates to each holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) a new function holomorphic function \( \tilde{f} \) defined by \( \tilde{f}(z) := \overline{f(z)} \).

Arnold states that if \( z_0 \in \mathbb{C} \) is a critical point of \( f \), i.e., \( \frac{df}{dz}(z_0) = 0 \), then it is also a critical point of \( \tilde{f} \). Clearly this is true only if \( z_0 \) is real. For example, \( z_0 = i \) is a critical point of \( f(z) = (z - i)^2 \), but it is not a critical point of \( \tilde{f}(z) = (z + i)^2 \).

\[ \square \]

4. A PRODUCT FORMULA

Suppose that \((M, g, V)\) is a homogeneous triple, \( m = \dim M \). We say that it is special if it admits a core, i.e., a quadruple \((p, S, L, w)\), where \( p \) is a point in \( M \), \( f = \{f_1, \ldots, f_m\} \) is an orthonormal frame of \( T_p M \), \( L \) is a subspace of \( V \) and \( w \in V \) such that the following hold.

- **P 1.** The vectors \( A_p f_r \), \( r = 1, \ldots, m \) are mutually orthogonal. For any \( v \in V \) we denote by \( \text{Hess}(v) \) the Hessian of \( v \) at \( p \) computed using the frame \( f \).
- **P 2.** The subspace \( L \) is contained in \( K_p = \ker A_p \), and for any \( v \in L^\perp \) we have \( \text{Hess}(v) = 0 \).
- **P 3.** \( w \in K_p \cap L^\perp \), \( w(p) \neq 0 \) and \( |w| = 1 \). We set \( \tilde{L} := L \oplus \text{span} \{w\} \).
- **P 4.** \( \tilde{L}^\perp \subset \ker \text{ev}_p \).

**Remark 4.1.** (a) The importance of a core stems from the fact that in applications we often have

\[ \dim L \ll \dim V. \]

We regard \( \text{Hess} \) as a linear map

\[ K_p \to \text{Sym}(T_p M) := \text{symmetric linear maps } T_p M \to T_p M. \]

We observe that \( L \supset (\ker \text{Hess})^\perp \), so we would expect the dimension of \( L \) to be at least as big as \((m+1) = \dim \text{Sym}(T_p M) \). In many applications, \( \dim L \) is only slightly bigger than \((m+1) \).

(b) The conditions \( P_3, P_4 \) can be somewhat relaxed. We can define a core to be a subspace \( \tilde{L} \subset K_p \) that contains \( \text{ev}_p \in V \) and satisfies \( P_2 \). For example, if \( \text{ev}_p \in K_p \), we can choose \( \tilde{L} \) to be the sum between the line spanned by \( \text{ev}_p \) and the orthogonal complement of \( \ker \text{Hess} \), but this space may be difficult to get a handle on in practice. For reasons having to do with the applications we have in mind, we prefer to work with the above more flexible definition. 

\[ \square \]
Suppose that \((p, g, L, w)\) is a core of the special triple \((M, g, V)\). A basis of \(V\) is said to be adapted to the core if it can be represented as collection of functions \(Y_j \in V\), \(j \in J\), where \(J\) is a set of cardinality \(\dim V\) equipped with a partition

\[
J = \{c\} \sqcup I \sqcup I^* \sqcup R_m
\]

such that the following hold.

- The collection \((Y_j)_{j \in J}\) is an orthonormal basis of \(V\).
- The collection \((Y_j)_{j \in I}\) is an orthonormal basis of \(L\).
- \(Y_c = w\).
- The collection \((Y_j)_{j \in I^*}\) is an orthonormal basis of \(K_p\).
- \(R_m = \{1, \ldots, m\}\) and

\[
Y_r = \frac{1}{|A_{p}^\dagger f_r|} A_{p}^\dagger f_r, \quad \forall r \in R_m.
\]

For such a basis, we write \(\hat{I} := \{c\} \cup I\), so that the collection \((Y_j)_{j \in \hat{I}}\) is an orthonormal basis of \(\hat{L}\).

**Proposition 4.2.** (a) Suppose that \((M, g, V)\) is a special triple and \((p, f, L, w)\) is a core of this triple. Set \(m := \dim M\), \(\ell = \dim L\) and \(N = \dim V\). Then

\[
\mu(M, g, V) = \frac{\operatorname{vol}_g(M) \Gamma\left(\frac{\ell+m}{2}\right)}{2\pi^{\ell+m} \prod_{r=1}^{m} |A_{p}^\dagger f_r|} \int_{S(L)} |\det H(v)| |dS(v)| \tag{4.1a}
\]

\[
= \frac{\operatorname{vol}_g(M)}{\pi^{\ell+m} \prod_{r=1}^{m} |A_{p}^\dagger f_r|} \int_{L} e^{-|u|^2} |\det H(u)| |dV(u)|. \tag{4.1b}
\]

(b) Suppose \((M_1, g_1, V_1), \alpha = 1, 2\), are special triples with cores \((p_\alpha, f_\alpha, L_\alpha, w_\alpha)\). Then the triple \((M_1 \times M_2, g_1 \oplus g_2, V_1 \oplus V_2)\) is special. The core is defined by the datum \((p, f, L, w)\), where \(p := (p_1, p_2), \ f := f_1 \cup f_2, \ w(x_1, x_2) := w_1(x_1)w_2(x_2)\), and

\[
L := (L_1 \ast L_2) \oplus K_{p_1}^\perp \otimes K_{p_2}^\perp,
\]

where \(L_1 \ast L_2\) denotes is the orthogonal complement of \(w\) in \(\hat{L}_1 \otimes \hat{L}_2\). Moreover

\[
|\mathbf{e}v_{p_1, p_2}| = |\mathbf{e}v_{p_1}| \cdot |\mathbf{e}v_{p_2}|, \tag{4.2}
\]

\[
J(A_{p_1, p_2}) = J(A_{p_1}^\dagger) \cdot J(A_{p_2}^\dagger) \cdot \frac{(|\mathbf{e}v_{p_1}| \cdot |\mathbf{e}v_{p_2}|)^{m_1+m_2}}{|\mathbf{e}v_{p_1}|^{m_1} \cdot |\mathbf{e}v_{p_2}|^{m_2}}, \tag{4.3}
\]

where \(J(S)\) denotes the Jacobian of a linear map between two Euclidean vector spaces.

**Proof.** (a) Note that for any \(v \in K_p\), the Hessian \(H(v)\) of \(v\) at \(p\) depends only on the projection \(\bar{v}\) of \(v\) on \(L\). Using \((A.2)\) we deduce

\[
\int_{S(K_p)} |\det H(v)| |dS(v)| = \sigma_{N-m-\ell-1} \int_{B(L)} (1 - |x|^2)^{N-m-\ell-2} |\det H(x)| |dV(x)|
\]

\[
= \frac{\sigma_{N-m-\ell-1} \Gamma\left(\frac{\ell+m}{2}\right) \Gamma\left(\frac{N-m-\ell}{2}\right)}{2\pi^{\ell+m} \Gamma\left(\frac{N}{2}\right)} \int_{S(L)} |\det H(x)| |dS(x)|
\]

\[
= \frac{\pi^{\frac{N-m-\ell}{2}} \Gamma\left(\frac{\ell+m}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} \int_{S(L)} |\det H(x)| |dS(x)|.
\]
We deduce that

\[
\mu(M, g, V) = \frac{\text{vol}_g(M)}{\sigma_{N-1}} \left| \det H(v) \right| \int_{S(K_p)} |dS(v)|
\]

\[
= \frac{\text{vol}_g(M) \Gamma \left( \frac{r-m}{2} \right)}{2\pi \sum_{i=1}^{m-m} |A_{i,p}f_r|} \int_{S(L)} |\det H(v)| |dS(v)|.
\]

(b) Choose an orthonormal basis \( (Y^\alpha_j)_{j \in J_\alpha} \) of \( V_\alpha \) adapted to the core \( (p_\alpha, f^\alpha, L_\alpha, w_\alpha) \), where

\[
J_\alpha = \{ c_\alpha \} \sqcup I_\alpha \sqcup I^*_\alpha \sqcup R_{m_\alpha}.
\]

The collection

\[
\{ Y^1_{j_1} Y^2_{j_2} \}_{(j_1, j_2) \in J_1 \times J_2}
\]

is an orthonormal basis of \( V_1 \otimes V_2 \).

Observe that \( \text{Hess}(Y^1_{j_1} Y^2_{j_2}) \), the Hessian of \( Y^1_{j_1} Y^2_{j_2} \) at \( p = (p_1, p_2) \), admits a block decomposition

\[
\begin{bmatrix}
Y^2_{j_2}(p_2) \text{Hess}(Y^1_{j_1}) & A \\
A^T & Y^1_{j_1}(p_1) \text{Hess}(Y^2_{j_2})
\end{bmatrix},
\]

where for any \((r_1, r_2) \in R\), the \((r_1, r_2)\) entry of the matrix \( A \) is

\[
A_{r_1 r_2} = \partial f^*_1 Y^1_{j_1}(p_1) \partial f^*_2 Y^2_{j_2}(p_2), \ (r_1, r_2) \in R_{m_1} \times R_{m_2}.
\]

The properties \( P_2 \) and \( P_4 \) show that if \( \text{Hess}(Y^1_{j_1} Y^2_{j_2}) \neq 0 \), then

either \((j_1, j_2) \in \bar{I}_1 \times \bar{I}_2 \setminus \{(c_1, c_2)\} \) or \((j_1, j_2) \in R_{m_1} \times R_{m_2} \).

This shows that if \( v \perp L \), then \( \text{Hess}(v) = 0 \) at \((p_1, p_2)\). The condition \( L \subset K_{p_1, p_2} \) follows from the properties \( P_2 \) and \( P_4 \) of special triples.

Next observe that for any \( r_1 \in R_{m_1} \), we have

\[
A^\dagger_{(p_1, p_2)} f^1_{r_1} = \sum_{j_1 \in J_1, j_2 \in J_2} \partial f^*_1 Y^1_{j_1}(p_1) Y^2_{j_2}(p_2) Y^1_{j_1} Y^2_{j_2}
\]

\[
= \sum_{j_2 \in J_2} (\partial f^*_1 Y^1_{r_1}(p_1) ) Y^2_{j_2}(p_2) Y^1_{j_2} Y^2_{j_2}.
\]

This proves that \( A^\dagger_{(p_1, p_2)} f^1_{r_1}(p_1, p_2) = 0 \) (since \( Y^1_{r_1}(p_1) = 0 \) by \( P_4 \)) and

\[
|A^\dagger_{(p_1, p_2)} f^1_{r_1}| = |\partial f^*_1 Y^1_{r_1}(p_1)|^2 \sum_{j_2=0}^{k_2} |Y^2_{j_2}(p_2)|^2
\]

\[
= |A^\dagger_{p_1} f^1_{r_1}|^2 \sum_{j_2=0}^{k_2} |Y^2_{j_2}(p_2)|^2 = |A^\dagger_{p_1} f^1_{r_1}|^2 \cdot |e v_{p_2}|^2.
\]

We have an analogous formula for \( A^\dagger_{(p_1, p_2)} f^2_{r_2}, r_2 \in R_{m_2} \). The conclusions of part (b) of Proposition 4.2 are now obvious.

\[\square\]

**Example 4.3.** An important example of special triple is \((S^{d-1}, g_d, V_\nu)\), \( d \geq 3 \), where \((S^{d-1}, g_d)\) is the round sphere of radius 1, and \( V_\nu \) is the space spanned by the eigenfunctions of the Laplacian corresponding to eigenvalues \( \lambda_n = n(n + d - 2), n \leq \nu \). A core can be constructed as follows.
As distinguished point, we choose the North Pole \( p_0 = (0, \ldots, 0, 1) \). Near \( p_0 \) we use \( x' = (x_1, \ldots, x_{d-1}) \) as local coordinates, and we set
\[
f_r = \partial_{x_r} \in T_{p_0}S^{d-1}.
\]
We choose \( w \) to be the constant function \( \sigma_{d-1}^{-1/2} \). Finally, the subspace \( L \) is the space spanned by the functions \( e_0, e_\beta \) defined by (2.12), (2.13) and (2.14), so that
\[
I = \{0\} \cup B_{2,d}.
\]
The properties \( P_1, P_2 \) and \( P_3 \) of a core are obvious. To prove property \( P_4 \), we have to show that if a function \( v \in V_\nu \) is orthogonal to \( w, e_0, \ldots, e_\beta, \beta \in B_{2,d}, \) then \( v(p_0) = 0 \). The function \( v \) admits a decomposition
\[
v = \sum_{n=0}^{\nu} \sum_{j=0}^{n} \sum_{\beta \in B_j,d} v_{n,j,\beta} Z_{n,j,\beta}.
\]
Since \( Z_{n,j,\beta}(p_0) = 0 \) if and only if \( j > 0 \), we deduce that
\[
v(p_0) = \sum_{n=0}^{\nu} v_{n,0,1} Z_{n,0,1}(p_0).
\]
Note that \( w = Z_{0,0,1} \). Since \( v \perp w \), we deduce \( v_{0,0,1} = 0 \), and therefore,
\[
v(p_0) = \sum_{n=1}^{\nu} v_{n,0,1} Z_{n,0,1}(p_0).
\]
We now remark that (2.12) can be rewritten as
\[
a_0 = -\frac{1}{2} \sum_{n=1}^{\nu} Z_{n,0,1}(p_0) Z_{n,0,1}.
\]
We deduce that
\[
0 = -(v, 2a_0) = \sum_{n=1}^{\nu} v_{n,0,1} Z_{n,0,1}(p_0) = v(p_0).
\]
Note that (2.15) implies
\[
\text{Hess}(e_0) = r_0(\nu)^{1/2} \mathbb{1}_{d-1}, \quad \text{Hess}(e_\beta) = r_\beta(\nu)^{1/2} H_\beta.
\]
From (2.17) we deduce
\[
(A_{p_0}^\dagger f_r) = \sigma_{d-3}^{-1/2} C_{1,0,d-1} \sum_{n=1}^{\nu} C_{n,1,d} P_{n,d}(1) Z_{n,1,r}.
\]
Using (2.18) and (2.21), we deduce that this triple has the additional property that
\[
|A_{p_0}^\dagger f_1|^2 = \cdots = |A_{p_0}^\dagger f_{d-1}|^2 = r(d, \nu) \sim \frac{1}{2^{d-1} \Gamma(d-1)^2 \sigma_{d-3}(d+1)} \nu^{d+1} \quad \text{as} \quad \nu \to \infty.
\]
Let us compute the length of the evaluation functional \( ev_{p_0} = ev_{p_0,\nu} : V_\nu \to \mathbb{R} \). We will use the notations in the proof of Theorem 2.1. We have
\[
|ev_{p_0,\nu}|^2 = \sum_{n=0}^{\nu} \sum_{j=0}^{n} \sum_{\beta \in B_j,d} |C_{n,j,d} P_{n,d}(1) Y_{j,\beta}(0)|^2 = \sum_{n=0}^{\nu} \sum_{\beta \in B_{0,d}} |C_{n,0,d} P_{n,d}(1) Y_{0,\beta}(0)|^2.
\]
In this case, the basis $\mathcal{B}_{0,d-1}$ consists of a single constant function $Y_0 = Y_{0,\beta} = \sigma_{d-2}^{-1/2}$ and

$$P_{n,d}(1) = 1, \quad C_{n,0,d}^2 = \frac{(2n + d - 2)[n + d - 3]}{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)}.$$ 

Hence,

$$Y_{0,\beta}(0)^2 C_{n,0,d}^2 = \frac{(2n + d - 2)[n + d - 3]}{(2\pi)^{d-1}},$$

and we conclude that

$$|\text{ev}_{p_0,\nu}|^2 = \frac{1}{(2\pi)^{d-1}} \sum_{n=0}^{\nu} (2n + d - 2)[n + d - 3] \sim \frac{2}{(2\pi)^{d-1}(d-1)} \nu^{d-1} \text{ as } \nu \to \infty. \quad (4.7)$$

For $r = 1, \ldots, d - 1$, we set

$$U_r := \frac{1}{|A^\dagger_{p_0} f_r|} A^\dagger_{p_0} f_r. \quad (4.8)$$

The computations in the proof of Theorem 2.1 imply that

$$\partial_{x_1} U_1(p_0) = \partial_{x_2} U_2(p_0) = \ldots = \partial_{x_{d-1}} U_{d-1}(p_0) = c(d,\nu) = r(d,\nu)^{1/2}. \quad (4.9a)$$

$$\partial_{x_i} U_j(p_0) = 0, \quad \forall i, j = 1, \ldots, d - 1, \quad i \neq j. \quad (4.9b)$$

\[\square\]

**Theorem 4.4.** Assume $d_1, d_2 \geq 3$ and fix $r \geq 1$. Then there exists a positive constant $K$ that depends only on $d_1$ and $d_2$ such that,

$$\mu(S^{d_1-1} \times S^{d_2-1}, V_{\nu_1}(d_1) \otimes V_{\nu_2}(d_2)) \sim K (\dim V_{\nu_1}(d_1) \otimes V_{\nu_2}(d_2))^{\varpi(d_1,d_2,r)}$$

if $\nu_1, \nu_2 \to \infty$ and $\nu_1 \nu_2^{-r}$ converges to a positive constant. The exponent $\varpi(d_1,d_2,r)$ is described in (\textcircled{2}).

$$\varpi(d_1,d_2,r) = \frac{2(d_1 - 3)r + 2d_2 + 2}{(2d_1 - 1)r + 2d_2 - 2}.$$ 

**Proof.** Choose cores $(p_\alpha, f_\alpha, L_\alpha, w_\alpha)$ of $(S^{d_{\alpha}-1}, g_0, V_{\nu_\alpha})$ as indicated in Example 4.3. Next, choose bases adapted to these cores

$$(Y^\alpha_j)_{j \in I_\alpha}, \quad J_\alpha = \{e_\alpha\} \cup I_\alpha \cup I^\alpha_\star \cup R_{d_{\alpha}-1}, \quad I_\alpha = \{0\} \cup B_{2,d_{\alpha}},$$

and set

$$(\hat{I}_1 \times \hat{I}_2) \setminus \{(c_1, c_2)\}, \quad R = R_{d_1-1} \times R_{d_2-1}, \quad I = S \cup R.$$ 

Recall that for $i_\alpha \in I_\alpha = \{0\} \cup B_{2,d_\alpha}$, we have

$$Y^\alpha_{i_\alpha} = \begin{cases} e_0(\nu_\alpha), & i_\alpha = 0 \\ e_\beta(\nu_\alpha), & i_\alpha = \beta \in B_{2,d_\alpha}, \end{cases}$$

where the functions $e_0, e_\beta$ are defined by (2.14). Moreover, for $r_\alpha \in R_{d_{\alpha}-1} = \{1, \ldots, d_{\alpha} - 1\}$, we have $Y^\alpha_{r_\alpha} = U_{r_\alpha}$, where $U_{r_\alpha}$ is defined by (4.8).

We construct a core $(p, f, L, w)$ of $(S^{d_1-1} \times S^{d_2-1}, V_{\nu_1} \otimes V_{\nu_2})$ as in Proposition 4.2. Note that the collection

$$\{ Y_{i_1,j_2} = Y^1_{i_1} Y^2_{i_2}; \quad (i_1, i_2) \in I \}$$
is an orthonormal basis of \( L \). For \( v \in V_{i_1} \otimes V_{i_2} \) we denote by \( \text{Hess}(v) \) the Hessian matrix of \( v \) at \( p \) computed using the frame \( f \). Note that if \( (i_1, i_2) \in S \), then

\[
\text{Hess}(Y_{i_1, i_2}) = \begin{bmatrix}
Y_{i_2}^2(p_2) \text{Hess}(Y_{i_1}^1) & 0 \\
0 & Y_{i_1}^1(p_1) \text{Hess}(Y_{i_2}^2)
\end{bmatrix}.
\]

Using (4.9a) and (4.9b) we deduce that for \( (r_1, r_2) \in R \) we have

\[
\text{Hess}(Y_{r_1, r_2}) = \begin{bmatrix}
0 & c(d_1, \nu_1)c(d_2, \nu_2)\Delta_{r_1,r_2} \\
c(d_1, \nu_1)c(d_2, \nu_2)\Delta_{r_1,r_2} & 0
\end{bmatrix}
\]

or

\[
= c(d_1, \nu_1)c(d_2, \nu_2)\begin{bmatrix}
\Delta_{r_1,r_2} & 0 \\
0 & \Delta_{r_1,r_2}
\end{bmatrix},
\]

where \( \Delta_{r_1,r_2} \) denotes the \( (d_1 - 1) \times (d_2 - 1) \) matrix whose entry on the \((r_1, r_2)\) position is 1, while the other entries are 0. Thus, if

\[
v = \sum_{(i_1, i_2) \in I} v_{i_1, i_2} Y_{i_1, i_2} \in L,
\]

then

\[
\text{Hess}(v) = \sum_{(i_1, i_2) \in S} v_{i_1, i_2} \text{Hess}(Y_{i_1, i_2}) + c(d_1, \nu_1)c(d_2\nu_2) \sum_{(r_1, r_2) \in R} v_{r_1, r_2} \hat{\Delta}_{r_1,r_2}.
\]

(4.10)

To make further progress, we need to choose the basis \( \{Y_{\alpha}^\beta\}_{\alpha, \beta} \) of \( L_\alpha \) as indicated in Example 4.3. Using the notations in the proof of Theorem 2.1 we let

\[
I_\alpha = \{0\} \cup B_{2,d_\alpha}
\]

and the functions \( Y_0^\alpha \) respectively \( Y_{\beta}^\alpha, \beta \in B_{2,d_\alpha} \) are equal to the functions \( e_0 \) and respectively \( e_{\beta} \) defined by (2.14), (2.12), (2.13). More precisely, for \( \alpha = 1, 2 \), we have

\[
Y_0^\alpha = \frac{1}{|a_0^\alpha|} a_0^\alpha,
\]

where

\[
a_0^\alpha = -\frac{1}{2} \sigma_{d_\alpha}^{-1/2} \sum_{n=0}^{\nu_\alpha} C_{n,0,d_\alpha} n \left( n + \frac{d_\alpha - 3}{2} \right) Z_{n,0,1},
\]

and for \( \beta \in B_{2,d_\alpha} \), we have

\[
Y_{\beta}^\alpha = \frac{1}{|a_{\beta}^\alpha|} a_{\beta}^\alpha,
\]

where

\[
a_{\beta}^\alpha = \sum_{n=2}^{\nu_\alpha} C_{n,2,d} P_{n,d}^{(2)}(1) Z_{n,2,\beta}.
\]

We set

\[
r_0(\nu_\alpha) := |a_0^\alpha|^2, \quad r_{\beta}(\nu_\alpha) := |a_{\beta}^\alpha|^2.
\]
Using (2.22) and (2.23) we deduce that for any \( d \geq 2 \) there exist explicit positive constants \( \tilde{r}_0(d) \), \( \tilde{r}_1(d) \) such that

\[
\begin{align*}
    r_0(\nu_\alpha) &\sim r_0(d_\alpha)\nu_\alpha^{-d_\alpha+3}, \quad \text{as} \quad \nu_\alpha \to \infty, \\
    r_\beta(\nu_\alpha) &\sim r_1(d_\alpha)\nu_\alpha^{-d_\alpha+3}, \quad \text{as} \quad \nu_\alpha \to \infty.
\end{align*}
\]  
(4.11)

Let us observe that for any \( \beta \in B_{2,d_\alpha} \) we have

\[
    Y_\beta^\alpha(p_\alpha) = 0, \quad \text{Hess}(Y_\beta^\alpha) = r_\beta(\nu_\alpha)^{1/2}H_\beta \sim r_1(d_\alpha)\nu_\alpha^{-d_\alpha+3}H_\beta, \quad \text{as} \quad \nu_\alpha \to \infty.
\]  
(4.12)

Moreover,

\[
    \text{Hess}(Y_0^\alpha) = r_0(\nu_\alpha)^{1/2}1_{d_\alpha-1} \sim r_0(d_\alpha)^{1/2}\nu_\alpha^{-d_\alpha+3}1_{d_\alpha-1}, \quad \text{as} \quad \nu_\alpha \to \infty.
\]  
(4.13)

Next,

\[
    Y_0^\alpha(p_\alpha) = -\frac{1}{2r_0(\nu_\alpha)^{1/2}\sigma_{d_\alpha-2}}\sum_{n=0}^{\nu_\alpha}C_{n,0,d_\alpha}n\left(n + \frac{d_\alpha - 3}{2}\right)Z_{n,0,1}(p_0),
\]

where

\[
    Z_{n,0,1}(p_\alpha) = C_{n,0,d_\alpha}\sigma_{d_\alpha-2}^{-1/2}.
\]

We deduce

\[
    Y_0^\alpha(p_\alpha) = -\frac{1}{2r_0(\nu_\alpha)^{1/2}\sigma_{d_\alpha-2}}\sum_{n=0}^{\nu_\alpha}C_{n,0,d_\alpha}n\left(n + \frac{d_\alpha - 3}{2}\right)
\]

\[
= -\frac{1}{2^{d-1}r_0(\nu_\alpha)^{1/2}\sigma_{d_\alpha-2}\Gamma\left(d_\alpha-2\right)}\sum_{n=0}^{\nu_\alpha}n\left(n + \frac{d_\alpha - 3}{2}\right)(2n + d_\alpha - 2)[n + d_\alpha - 3]_{d_\alpha-3}
\]

\[
= -\frac{1}{2(4\pi)^{d-1}r_0(\nu_\alpha)^{1/2}}\sum_{n=0}^{\nu_\alpha}n\left(n + \frac{d_\alpha - 3}{2}\right)(2n + d_\alpha - 2)[n + d_\alpha - 3]_{d_\alpha-3}.
\]

Note that

\[
    \sum_{n=0}^{\nu_\alpha}n\left(n + \frac{d_\alpha - 3}{2}\right)(2n + d_\alpha - 2)[n + d_\alpha - 3]_{d_\alpha-3} \sim \frac{2}{d_\alpha + 1}\nu_\alpha^{d_\alpha+1} \quad \text{as} \quad \nu_\alpha \to \infty.
\]

Using (4.11a), we deduce that

\[
    Y_0^\alpha(p_\alpha) \sim -\frac{1}{(4\pi)^{d_\alpha-1}(d_\alpha + 1)r_0(d_\alpha)^{1/2}}\nu_\alpha^{-d_\alpha} \quad \text{as} \quad \nu_\alpha \to \infty.
\]  
(4.14)

For \( i \in I_\alpha \), we define the symmetric \((d_\alpha - 1) \times (d_\alpha - 1)\)-matrix

\[
    H_i^\alpha := \begin{cases} 
        I_{d_\alpha-1}, & i = 0 \\
        H_\beta, & i = \beta \in B_{2,d_\alpha}.
    \end{cases}
\]

(4.15)

Putting together all of the above, we deduce that for \((i_1, i_2) \in S\) and \(\nu_1, \nu_2 \to \infty\) we have

\[
    Y_{i_2}^2(p_2) \text{Hess}(Y_{i_1}^1) \sim \begin{cases} 
        0, & i_2 \in B_{2,d_\alpha} \text{ or } i_1 = c_1 \\
        A\nu_1^{d_\alpha+3}H_{i_1}^\alpha, & i_2 = c_2 \text{ and } i_1 \in I_1 \\
        B_{i_1}\nu_2^{d_\alpha+3}H_{i_2}^\alpha, & i_2 = 0 \text{ and } i_1 \in I_1.
    \end{cases}
\]  
(4.16a)
Using (4.3), (4.7) and (4.6) we deduce that the Jacobian $J(v_1, v_2)$ of $A_{(p_1, p_2)}$ satisfies the asymptotic estimate

$$J(v_1, v_2) = J(A_{(p_1)}) \cdot |e v_{p_1}|^{d_2 - 1} \cdot |e v_{p_1}| \sim CF v_1 \frac{(d_1 - 1)}{2} \frac{(d_2 - 1)}{2} (v_1 v_2) \frac{(d_1 - 1)(d_2 - 1)}{2},$$

where $F$ is a positive constant. Assume now that

$$\nu_1 \sim t^{2\kappa_1}, \ \nu_2 \sim t^{2\kappa_2}, \ \ t \to \infty,$$

i.e., $v_1, v_2$ go to infinity in such a fashion that

$$\frac{\nu_1}{\nu_2} \to 1, \ \ r := \frac{\kappa_1}{\kappa_2}.$$

The assumption $r \geq 1$ implies that

$$\kappa_1 \geq \kappa_2 > 0.$$  \hspace{1cm} (4.18)

We have

$$\nu^{(d_\alpha)/2} \sim t^{p_\alpha}, \ p_\alpha := 3 \kappa_\alpha d_\alpha + 3 \kappa_\alpha, \ \alpha = 1, 2,$$

$$\nu_1^{\frac{d_2 - 1}{2}} \nu_2^{\frac{d_1 + 3}{2}} \sim Ct^{\omega_{11}}, \ \ \omega_{11} = \kappa_1 d_1 + \kappa_2 d_2 + 3 \kappa_1 - \kappa_2,$$

$$\nu_1^{\frac{d_2 - 1}{2}} \nu_2^{\frac{d_1 + 3}{2}} \sim Ct^{\omega_{22}}, \ \ \omega_{22} = \kappa_1 d_1 + \kappa_2 d_2 + 3 \kappa_2 - \kappa_1,$$

$$\nu_1^{\frac{d_1 + 1}{2}} \nu_2^{\frac{d_2 + 1}{2}} \sim Ct^{\omega_{12}}, \ \ \omega_{12} = \kappa_1 d_1 + \kappa_2 d_2 + \kappa_1 + \kappa_2,$$

$$\nu_1^{\frac{(d_1 - 1)}{2}} \nu_2^{\frac{(d_2 - 1)}{2}} (v_1 v_2) \frac{(d_1 - 1)(d_2 - 1)}{2} \sim Ct^q,$$

where

$$q = \kappa_1 (d_1^2 - 1) + \kappa_2 (d_2^2 - 1) + (\kappa_1 + \kappa_2)(d_1 - 1)(d_2 - 1) = \kappa_1 d_1^2 + \kappa_2 d_2^2 + (\kappa_1 + \kappa_2)(d_1 d_2 - d_1 - d_2).$$

From (4.18) we deduce

$$\omega_{22} - \omega_{11} = -4(\kappa_1 - \kappa_2) \leq 0, \ \ \omega_{12} - \omega_{11} = -2(\kappa_1 - \kappa_2) \leq 0,$$

$$p_2 - \omega_{11} = -(d_1 + 3) \kappa_1 + 4 \kappa_2 < -4(\kappa_1 - \kappa_2),$$

$$p_1 - \omega_{11} = -\kappa_2 (d_2 - 1) < 0.$$

Using (4.10), (4.15), (4.16a), (4.16b) and (4.17) we deduce that

$$H(v) \sim t^{\omega_{11}} \begin{bmatrix} (B(v) + o(1)) & Et^{\omega_{12} - \omega_{11}} E \left( \sum_{r_1, r_2} v_{r_1} r_2 \Delta_{r_1} v_{r_1} + o(1) \right) \\ Et^{\omega_{12} - \omega_{11}} \left( \sum_{r_1, r_2} v_{r_1} r_2 \Delta_{r_1} v_{r_1} + o(1) \right) & D_0 v_0 t^{\omega_{12} - \omega_{11}} (1 - d_1 - o(1)) \end{bmatrix} =: A(t).$$
where \( o(1) \) denotes a quantity that converges to 0 as \( t \to \infty \), uniformly with respect to \( v \in S(L) \), and

\[
B(v) = \sum_{i_1} B_{i_1} v_{i_1,0} H_{i_1}^1,
\]

where \( B_{i_1} \) are defined as in (4.16a). We set

\[
S(t) := \begin{bmatrix} 1_{d_1-1} & 0 \\ 0 & t^{2(\kappa_1 - \kappa_2)} \end{bmatrix}.
\]

Observe that

\[
A(t) = S(t) \cdot \begin{bmatrix} (B(v) + o(1)) & E\left( \sum_{r_1,r_2} v_{r_1,r_2} \Delta_{r_1,r_2} + o(1) \right) \\ E\left( \sum_{r_1,r_2} v_{r_1,r_2} \Delta_{r_1,r_2} + o(1) \right) & D_0 v_{0,0} \left( \mathbb{1}_{d_{2-1}} + o(1) \right) \end{bmatrix} \cdot S(t).
\]

We deduce that

\[
\det H(v) \sim t^{(d_1 + d_2 - 2)\omega_{11}} \det S(t)^2 \cdot \det \left[ \begin{array}{cc} B(v) & E \sum_{r_1,r_2} v_{r_1,r_2} \Delta_{r_1,r_2} \\ E \sum_{r_1,r_2} v_{r_1,r_2} \Delta_{r_1,r_2} & D_0 v_{0,0} \mathbb{1}_{d_{2-1}} \end{array} \right] =: H_\infty(v).
\]

Let us point out that \( \det H_\infty(v) \) is not identically zero. To see this, it suffices to choose \( v \) such that \( v_{0,0} = 1 \) and all the other coordinates \( v_{i_1,i_2} \) are trivial. In this case (4.15) and (4.16a) imply that

\[
H_\infty(v) = \begin{bmatrix} B_0 & 0 \\ 0 & D_0 \mathbb{1}_{d_{2-1}} \end{bmatrix}.
\]

It follows that

\[
\mu(S^{d_1-1} \times S^{d_2-1}, V_{t^{2\kappa_1}} \otimes V_{t^{2\kappa_2}}) \sim C t^{(d_1 + d_2)\omega_{11} - 4d_2(k_1 - k_2) - q}.
\]

An elementary computation shows that

\[
(d_1 + d_2 - 2)\omega_{11} - 4d_2(k_1 - k_2) - q = 2d_1 \kappa_1 + 2d_2 \kappa_2 - 6\kappa_1 + 2\kappa_2.
\]

On the other hand,

\[
\dim V_{t^{2\kappa_1}} \otimes V_{t^{2\kappa_2}} = \dim V_{t^{2\kappa_1}} \times \dim V_{t^{2\kappa_2}} \sim K_{d_1,d_2} t^{2\kappa_1 d_1 + 2\kappa_2 d_2 - 2\kappa_1 - 2\kappa_2}.
\]

The desired conclusion follows by observing that

\[
\frac{2d_1 \kappa_1 + 2d_2 \kappa_2 - 6\kappa_1 + 2\kappa_2}{2\kappa_1 d_1 + 2\kappa_2 d_2 - 2\kappa_1 - 2\kappa_2} = \frac{2(d_1 - 3)r + 2d_2 + 2}{2(d_1 - 1)r + 2d_2 - 2} = \omega(d_1,d_2,r), \quad r = \frac{\kappa_1}{\kappa_2}.
\]

\( \square \)
5. Random Polynomials on $S^1 \times S^{d-1}$, $d \geq 3$

For any $m \in \mathbb{Z}$ define $X_m : S^1 \to \mathbb{R}$ by

$$
\Phi_m(\theta) = \begin{cases} 
(2\pi)^{-1/2}, & m = 0 \\
\pi^{-1/2} \cos(m\theta), & m < 0 \\
\pi^{-1/2} \sin(m\theta), & m > 0. 
\end{cases}
$$

(5.1)

The collection $(\Phi_m)_{m \in \mathbb{Z}}$ is an orthonormal Hilbert basis of $L^2(S^1, d\theta)$. For any positive integer $\nu > 0$, we set

$$
T_\nu := \text{span} \{ \Phi_m; |m| \leq \nu \}.
$$

In other words, $T_\nu$ is the space of trigonometric polynomials of degree $\leq \nu$.

**Lemma 5.1.** Let $g_0$ denote the natural metric on $S^1$ of length $2\pi$. Then the triple $(S^1, g_0, T_\nu)$ is special. Moreover

$$
|e\nu_0|^2 = \frac{1}{\pi} \left( \nu + \frac{1}{2} \right).
$$

(5.2)

**Proof.** As base point we choose $p_0 = 0$ and the frame is $f = \{ \partial_\theta \}$. We denote by $K_0$ the space of trigonometric polynomials that have 0 as a critical point. Also, for any trigonometric polynomial $t \in T_\nu$ we denote by $\text{Hess}(t)$ the Hessian of $t$ at 0, i.e., the $1 \times 1$ matrix $\text{Hess}(t) := \partial^2_\theta t(0) \mathbb{1}$. Note that

$$
A_0^\dagger \partial_\theta = -\pi^{-1/2} \sum_{m>0} m \Phi_m.
$$

In particular,

$$
J_\nu := |A_0^\dagger \partial_\theta| = \pi^{-1/2} \left( \sum_{m=1}^\nu m^2 \right)^{1/2} \sim (3\pi)^{-1/2} \nu^{3/2} \text{ as } \nu \to \infty.
$$

(5.3)

We set

$$
p_\nu := \frac{1}{|A_0^\dagger \partial_\theta|} A_0^\dagger \partial_\theta,
$$

(5.4)

A simple computation shows that

$$
\partial_\theta p_\nu(0) = J_\nu.
$$

(5.5)

Next observe that

$$
\text{Hess}(\Phi_m) = \begin{cases} 0, & m \geq 0 \\
-m^2 \Phi_m(0) \mathbb{1}, & m < 0. 
\end{cases}
$$

Thus, if

$$
t = \sum_{|m| \leq \nu} t_m \Phi_m \in T_\nu,
$$

then

$$
\text{Hess}(t) = -\pi^{-1/2} \left( \sum_{m<0} m^2 t_m \right) \mathbb{1}.
$$

We now introduce

$$
a = a_\nu = -\pi^{-1/2} \sum_{m<0} m^2 \Phi_m, \quad e = e_\nu = \frac{1}{|a_\nu|} a_\nu,
$$

so that

$$
\text{Hess}(t) = (t, a) \mathbb{1} = |a_\nu|(t, e_\nu) \mathbb{1}.
$$

(5.6)
We set
\[
\nu := |a_\nu|^2 = \frac{1}{\pi} \sum_{m=1}^{\nu} m^4 = \frac{1}{5\pi} B_5(\nu + 1) \sim \frac{1}{5\pi} \nu^5 \quad \text{as } \nu \to \infty,
\]
and we observe that
\[
\det \text{Hess}(e_\nu) = \nu^{1/2} = \frac{(5\pi)^{-1/2} B_5(\nu + 1)^{1/2}}{\nu^{1/2}} \sim (5\pi)^{-1/2} \nu^{5/2} \quad \text{as } \nu \to \infty. \tag{5.7}
\]
We set \( c(\nu) = e_\nu(0)\), and we observe that
\[
c(\nu) = -\frac{1}{\nu^{1/2}} \sum_{m<0} m^2 \Phi_m(0) \sim -\frac{5^{1/2}}{3\pi^{1/2}} \nu^{1/2} \quad \text{as } \nu \to \infty. \tag{5.8}
\]
We see that we can choose as core the quadruple \( (p_0, \partial \theta, L_\nu, Y_0) \), where \( L_\nu \) denotes the 1-dimensional space spanned by \( e_\nu \). The equality (5.2) follows from the identity \( \sum_{m \leq 0} m^2 \Phi_m(0) \Phi_m \).

\[\text{Corollary 5.2.} \quad \text{There exists a universal positive constant } K \text{ such that}
\]
\[
\mu(S^1, T_\nu) \sim 2\nu \sqrt{\frac{3}{5}} \sim \sqrt{\frac{3}{5}} \dim T_\nu \quad \text{as } \nu \to \infty.
\]
This agrees with our previous estimate (3.14).

\[\text{Proof.} \quad \text{We use Proposition 4.2(a), and we have}
\]
\[
\mu(S^1, T_\nu) = \frac{\text{vol}(S^1) \Gamma(1)}{2\pi |A|^{1/2}} \cdot 2|\text{Hess}(e(\nu))|
\]
\[
\sim 2\sqrt{\frac{3}{5}} \nu \quad \text{as } \nu \to \infty.
\]

\[\text{Remark 5.3.} \quad \text{Let us mention that, according to J. Dunnage, } [15], \text{ the expected number of real zeros of a random trigonometric polynomial in}
\]
\[
T_0^0 = \text{span}\{ \Phi_m; 0 < |m| \leq \nu \}.
\]

\[
equipped with the } L^2\text{-metric is } \sim \frac{2}{\sqrt{\nu}} \nu \quad \text{as } \nu \to \infty. \quad \text{The number of zeros of such a polynomial is a random variable } \zeta_\nu, \text{ and its asymptotic behavior as } \nu \to \infty \text{ has been recently investigated in great detail by A. Granville and I. Wigman, } [19]. \text{ Let us observe the operator } \partial \theta \text{ induces a linear isomorphism}
\]
\[
\partial \theta : T_\nu \to T_\nu^0.
\]
From this point of view we see that the expected number of critical points of a random trig polynomial in \( T_\nu^0 \), equipped with the \( L^2\)-metric is equal to the expected number of critical zeros of a random trig polynomial in \( T_\nu^0 \), \emph{equipped with the Sobolev norm } \( \| u \|_{H^{-1}} \). \text{ In Section 6 we will describe the asymptotic behavior as } \nu \to \infty \text{ of the variance of the number of critical points of a random trigonometric polynomial in } T_\nu^0. \]

\[\text{Example 5.4 (Approximation regimes with large upper complexity).} \quad \text{Suppose}
\]
\[
\phi : \{ 0, 1, 2, \ldots \} \to \{ 0, 1, 2, \ldots \}
\]
is a bijection such that \( \phi(0) = 0 \). Define
\[
T_\nu^\phi := \text{span}\{ \Phi_{\pm \phi(m)}; 0 \leq m \leq \nu \}.\]
We denote by \(\mu(T^\varphi_\nu)\) the expected number of critical points of a random trigonometric polynomial in \(T^\varphi_\nu\). A simple modification of the arguments used in the proofs of Lemma 5.1 shows that

\[
\mu(T^\varphi_\nu) = 2 \left( \frac{\sum_{k=1}^\nu \varphi(k)\varphi(k)}{\sum_{k=1}^\nu \varphi(k) - \sum_{k=1}^\nu \varphi(k)} \right)^{1/2}. 
\] (5.9)

We want to construct a permutation \(\varphi\) such that

\[
\limsup_{\nu \to \infty} \frac{\log \mu(T^\varphi_\nu)}{\log \dim T^\varphi_\nu} = \infty.
\]

To do this we fix a very fast increasing sequence of positive integers \((\ell_n)_{n \geq 0}\) such that

\[
\ell_0 = 0, \quad \frac{\ell_{n+1}}{\ell_n} = 2^n, \quad \forall n \geq 1.
\]

For \(n \geq 0\) we set

\[
S_n := \{\ell_n + 1, \ell_n + 2, \ldots, \ell_{n+1}\}.
\]

We consider the bijection \(\phi: \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}\) uniquely determined by the following requirements.

- \(\phi(0) = 0\)
- \(\phi(S_n) = S_n, \forall n \geq 0\)
- The restriction of \(\varphi\) to \(S_n\) is strictly decreasing so that
  \[
  \phi(\ell_n + 1) = \ell_{n+1}, \quad \phi(\ell_n + 2) = \ell_{n+1} - 1 \quad \text{etc.}
  \]

We set

\[
\nu_n := \ell_n + 1, \quad W_n := T^\phi_{\nu_n}.
\]

Note that the collection \((W_n)_{n \geq 0}\) is an approximation regime in the sense defined in the introduction, and

\[
\dim W_n = 2\ell_n + 3.
\]

We claim that

\[
\lim_{n \to \infty} \frac{\log (\mu(W_n))}{\log \dim W_n} = \infty. \tag{5.10}
\]

Indeed, for any positive integer \(k\), we have

\[
\sum_{m=1}^{\nu_n} \phi(m)^k = \sum_{m=1}^{\ell_n} m^k + \ell_{n+1}^k = P_{k+1}(\ell_n) + \ell_{n+1}^{k-2} ,
\]

where, according to (S),

\[
P_k(x) = \frac{1}{k+1} (B_{k+1}(x) - B_k(x))
\]

is a universal polynomial of degree \(k + 1\). Using (5.9) we deduce

\[
\frac{1}{4} \mu(W_n)^2 = \frac{P_5(\ell_n) + \ell_n^{2n+2}}{P_3(\ell_n) + \ell_n^{2n+1}} \sim \ell_n^{2n+1} \quad \text{as} \quad n \to \infty.
\]

Hence,

\[
\log \mu(W_n) \sim 2^n \log \ell_n,
\]

which proves the claim (5.10).

---

\(^5\)This construction was worked out during a lively conversation with my colleague Richard Hind.
Remark 5.5. Let us observe that for any positive \( \nu \), the space

\[
T^L_\nu := \bigotimes^L T_\nu \subset C^\infty(S^1 \times \cdots \times S^1)
\]

contains the space \( V(M^d_\nu) \) of Section 3 as a codimension one subspace. The orthogonal complement of \( V(M^d_\nu) \) in \( T^L_\nu \) is the 1-dimensional space spanned by the constant functions. \( \square \)

Theorem 5.6. For any \( d \geq 3 \) there exists a universal positive constant \( K = K_d \) such that

\[
\mu(S^1 \times S^{d-1}, T_\rho \otimes V_\nu(d)) \sim K_d (\dim T_\rho \otimes V_\nu(d)) \quad \text{as} \quad \rho, \nu \to \infty.
\]

Proof. We will again rely on Proposition 4.2. We consider the core \((0, \partial_0, L_\rho, w_\rho)\) of \((S^1, g_{S^1}, T_\rho)\) described in Lemma 5.1 and the core \((p_0, f_\Lambda, L_\nu, w_\nu)\) of \((S^{d-1}, g_{S^{d-1}}, V_\nu)\) described in Example 4.3. We form the core \((p, f', L_{\rho,\nu}, w)\) of \((S^1 \times S^{d-1}, g_{S^1} + g_{S^{d-1}}, T_\rho \otimes V_\nu)\) following the prescriptions in the proof of Proposition 4.2. We have

\[
L_\rho = \text{span} \{ e_\rho \}, \quad e_\rho = \frac{1}{|a_\rho|}, \quad a_\rho = -\pi^{-1/2} \sum_{m=-1}^{-\rho} m^2 \Phi_m(\theta), \quad w_\rho(\theta) = (2\pi)^{-1/2}.
\]

As in the proof of Theorem 4.4, we choose a basis

\[
(Y_j)_{j \in J}, \quad J_\alpha = \{ \ast \} \sqcup I \sqcup I^* \sqcup R_{d-1}, \quad I = \{ 0 \} \sqcup B_{2,d},
\]

adapted to the core \((p_0, f_\Lambda, L_\nu, w_\nu)\). We have

\[
Y_\ast_d = \sigma_{d-1}^{-\frac{1}{2}}.
\]

For \( i \in I = \{ 0 \} \sqcup B_{2,d} \), we have

\[
Y_i = \begin{cases} 
  e_0(\nu), & i = 0 \\
  e_\beta(\nu), & i = \beta \in B_{2,d},
\end{cases}
\]

where the functions \( \{ e_0(\nu), e_\beta(\nu) \} \) are defined by (2.14). For \( r \in R_{d-1} = \{ 1, \ldots, d-1 \} \) we have

\[
Y_r = U_r := \frac{1}{|A_{p_0} \partial_{x_r}|} A_{p_0}^{\dagger} \partial_{x_r}, \quad A_{p_0}^{\dagger} \partial_{x_r} = \sigma_{d-1}^{-1/2} C_{1,0,d-1} \sum_{n=1}^\nu C_{n,1,d} P'_{n,d}(1) R_{n,1,r}.
\]

We can now write down an orthonormal basis of \( L_{\rho,\nu} \),

\[
\{ A_{i,1} := w_\rho Y_i; \quad i \in I \} \cup \{ Z := e_\rho Y_\ast \} \cup \{ A_{i,2} := e_\rho Y_i; \quad i \in I \} \cup \{ B_r := p_\rho Y_r; \quad r \in R_{d-1} \},
\]

where \( p_\rho \) is given by (5.4). For any function \( v \in V := T_\rho \otimes V_\nu \) we denote by \( \text{Hess}(v) \) its Hessian at \((0, p_0)\). We have

\[
\text{Hess}(A_{i,1}) = \begin{bmatrix} 0 & 0 \\ 0 & (2\pi)^{-1/2} \text{Hess}(Y_i) \end{bmatrix}, \quad \text{Hess}(Z) = \begin{bmatrix} \sigma_{d-1}^{-1/2} \text{Hess}(e_\rho) & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
\text{Hess}(A_{i,2}) = \begin{bmatrix} \text{Hess}(e_\rho) Y_i(p_0) & 0 \\ 0 & e_\rho(0) \text{Hess}(Y_i) \end{bmatrix},
\]

\[
\text{Hess}(p_\rho Y_r) = \begin{bmatrix} p_\rho'(0) \partial_{x_r} Y_r(p_0) \Delta_r & 0 \\ 0 & p_\rho'(0) \partial_{x_r} Y_r(p_0) \Delta_r \end{bmatrix},
\]

where \( \Delta_r \) is the \( (d-1) \)-dimensional space of linear functions on \( S^{d-1} \).
where $\Delta_r$ denotes the $1 \times (d-1)$ matrix with a single nonzero entry equal to 1 in the $r$-th position, and $\Delta_r^\top$ denotes its transpose. In the sequel, the symbols $C_0, C_1, \ldots$, will indicate positive constants that depend only on $d$.

Using (5.5), (5.7) and (5.8) we deduce that as $\rho \to \infty$, we have

$$p'_\rho(0) \sim C_0 \rho^{3/2}, \quad \text{Hess}(e_\rho) \sim C_1 \rho^5, \quad e_\rho(0) \sim -C_2 \rho^{1/2}.$$ 

Using (2.22), (2.23) and (4.4) we deduce that as $\nu \to \infty$, we have

$$\text{Hess}(Y_0) \sim C_3 \nu^{d+3} \mathbb{1}_{d-1}, \quad \text{Hess}(Y_\beta) \sim C_4 \nu^{d+3} H_\beta \beta \in B_{2,d}.$$ 

For $i \in I$ we set

$$H_i := \begin{cases} \mathbb{1}_{d-1}, & i = 0, \\ H_\beta, & i = \beta \in B_{2,d}. \end{cases}$$

We have

$$Y_\beta(p_0) = 0, \quad \forall \beta \in B_{2,d},$$

while (4.14) implies that as $\nu \to \infty$, we have

$$Y_0(p_0) \sim -C_5 \nu^{d+1}.$$ 

Finally, using (2.18) and (4.9a), we deduce that as $\nu \to \infty$ we have

$$\partial_x Y_r(p_0) \sim C_6 \nu^{d+1}.$$ 

Putting together all of the above, we deduce that if

$$v = \sum_{i \in I} v_i, A_{i,1} + \sum_{i \in I} v_i, A_{i,2} + z Z + \sum_{i \in R} v_i B_r \in L_{\rho, \nu},$$

then, as $\rho, \nu \to \infty$, we have

$$\text{Hess}(v) \sim \begin{bmatrix} -C_7 v_0, 2 \rho^5 \nu^{d+1} / 2 + C_8 z \rho^5 \quad C_9 \sum_{r \in R} \rho^{3/2} \nu^{d+1} / 2 v_r \Delta_r \\ C_9 \sum_{r \in R} \rho^{3/2} \nu^{d+1} / 2 v_r \Delta_r \quad \nu^{d+3} (B_0(v) + \rho^{1/2} B_1(v)) \end{bmatrix},$$

where

$$B_0(v) := C_1 v_{0,1} \mathbb{1}_{d-1} + \sum_{i \in B_{2,d}} C_{11} v_{i,1} H_i, \quad B_1(v) := C_{10} v_{0,2} \mathbb{1}_{d-1} - \sum_{i \in B_{2,d}} C_{11} v_{i,2} H_i.$$ 

Factoring out $\nu^{(d-1)/2}$ and then $\rho^{5/2}$ from the first row and the first column, we deduce that

$$\det \text{Hess}(v) \sim \nu^{(d-1)/2} \rho^5 \det \begin{bmatrix} -C_7 v_{0,2} + C_8 z & C_9 \sum_{r \in R} \rho^{-1} v_r \Delta_r \\ C_9 \sum_{r \in R} \rho^{-1} v_r \Delta_r \quad \nu^2 (B_0(v) + \rho^{1/2} B_1(v)) \end{bmatrix}$$

(factor out $\nu$ from the last $(d-1)$ rows and the last $(d-1)$ columns)

$$= \nu^{(d+1)(d-1)/2} \rho^5 \det \begin{bmatrix} -C_7 v_{0,2} + C_8 z & C_9 \sum_{r \in R} \rho^{-1} v_r \Delta_r \\ C_9 \sum_{r \in R} \rho^{-1} v_r \Delta_r \quad B_0(v) + \rho^{1/2} B_1(v) \end{bmatrix}$$

(factor out $\rho^{1/4}$ from the last $(d-1)$ rows and the last $(d-1)$ columns)

$$= \nu^{(d+4)(d-1)/2} \rho^{5/4} \det \begin{bmatrix} -C_7 v_{0,2} + C_8 z & C_9 \sum_{r \in R} \rho^{-5/4} v_r \Delta_r \\ C_9 \sum_{r \in R} \rho^{-5/4} v_r \Delta_r \quad B_1(v) + \rho^{-1} B_0(v) \end{bmatrix}.$$
\[
\sim \nu^{\frac{(d+4)(d-1)}{2}} \rho^{5+\frac{d-1}{2}} \det \begin{bmatrix}
-C_7v_0 + C_8z & 0 \\
0 & B_1(\nu)
\end{bmatrix}.
\]

To compute the Jacobian \( J_{\rho,\nu} \) of the adjunction map \( A^\dagger_{p_0} : T_{(0,p_0)}(S^1 \times S^{d-1}) \to T_\rho \otimes V_\nu \) we use Proposition 4.2(a). We will denote by \( K_0, K_1, \ldots \) positive constants that depend only on \( d \).

According to (5.2) and (5.3), the Jacobian \( J_\rho \) of \( A^\dagger_{0} : T_{S^1} S^{d-1} \to T_\rho \) and the evaluation functional \( ev^\rho_0 : T_\rho \to \mathbb{R} \) satisfy the \( \rho \to \infty \) asymptotics

\[
|ev^\rho| \sim K_0 \rho^{1/2}, \quad J_\rho \sim K_1 \rho^{3/2}.
\]

Using (4.6) and (4.7) we deduce that the Jacobian \( J_\nu \) of \( A^\dagger_0 : T_{p_0} S^{d-1} \to T_\rho \) and the evaluation functional \( ev^\nu = ev^\nu_{p_0} : V_\nu \to \mathbb{R} \) satisfy the \( \nu \to \infty \) asymptotics

\[
J_\nu \sim K_2 \nu^{\frac{(d-1)(d+1)}{2}}, \quad |ev^\nu| \sim K_3 \nu^{\frac{d-1}{2}}.
\]

Using (4.3), we conclude that

\[
J_{\rho,\nu} = J_\rho \cdot J_\nu \cdot |ev^\rho|^{d-1} \cdot |ev^\nu| \sim K_4 \rho^{3} \nu^{\frac{(d-1)(d+2)}{2}}.
\]

We conclude that as \( \rho, \nu \to \infty \), we have

\[
\mu(S^1 \times S^{d-1}, T_\rho \otimes V_\nu) \sim K_5 \rho \nu^{d-1} \sim K_6 \dim(T_\rho \otimes V_\nu).
\]

6. THE VARIANCE OF THE NUMBER OF CRITICAL POINTS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

The statistics of the zero set of a random trigonometric polynomial is equivalent with the statistics of the zero set of the gaussian field

\[
\eta_\nu(t) = \frac{1}{\sqrt{\pi \nu}} \sum_{m=1}^{\nu} (a_m \cos mt + b_m \sin mt),
\]

where \( a_m, b_m \) are independent normally distributed random variables with mean 0 and variance 1. This is a stationary gaussian process with covariance function

\[
\sigma_\nu(t) = \frac{1}{\pi \nu} \sum_{m=1}^{\nu} \cos mt.
\]

The statistics of the critical set of a random sample function of the above process is identical to the statistics of the zero set of a random sample function of the stationary gaussian process

\[
\xi_\nu(t) = \frac{d\eta_\nu(t)}{dt} = \frac{1}{\sqrt{\pi \nu}} \sum_{m=1}^{\nu} (-ma_m \sin mt + mb_m \cos mt).
\]

Equivalently, consider the gaussian process

\[
\Phi_\nu(t) = \frac{1}{\sqrt{\pi \nu^2}} \sum_{m=1}^{\nu} \left( mc_m \cos \left( \frac{t}{\nu} \right) + md_m \sin \left( \frac{t}{\nu} \right) \right),
\]

where \( c_m, d_m \) are independent random variables with identical standard normal distribution, and the random variable

\[
Z_\nu := \text{the number of zeros of } \Phi_\nu(t) \text{ in the interval } [-\pi \nu, \pi \nu].
\]
Note that the expectation of $Z_\nu$ is precisely the expected number of critical points of a random trigonometric polynomial in the space
\[ \text{span}\{\cos(m\theta), \sin(m\theta); \ m = 1, \ldots \nu \}, \]
equipped with the inner product
\[ (u, v) = \int_0^{2\pi} u(\theta)v(\theta) d\theta. \]
The covariance function of $\Phi_\nu$ is
\[ R_\nu(t) = \frac{1}{\pi \nu^3} \sum_{m=1}^{\nu} m^2 \cos \left( \frac{mt}{\nu} \right). \]
The Rice formula, [11, Eq. (10.3.1)], implies that the expectation of $Z_\nu$ is
\[ E(Z_\nu) = 2\nu \left( \frac{\lambda_2(\nu)}{\lambda_0(\nu)} \right)^{\frac{1}{2}}, \]
where
\[ \lambda_0(\nu) = R_\nu(0) = \frac{1}{\pi \nu^3} \sum_{m=1}^{\nu} m^2 \]
and
\[ \lambda_2(\nu) = -R''_\nu(0) = \frac{1}{\pi \nu^5} \sum_{m=1}^{\nu} m^4. \]
This is in perfect agreement with our earlier computations. We let $E(\zeta)$, and respectively $\text{var}(\zeta)$, denote the expectation, and respectively the variance, of a random variable $\zeta$. The following is the main result of this section.

**Theorem 6.1.** Set
\[ \bar{\lambda}_0 := \lim_{\nu \to \infty} \lambda_0(\nu) = \frac{1}{3}, \quad \bar{\lambda}_2 := \lim_{\nu \to \infty} \lambda_2(\nu) := \frac{1}{5}, \]
Then for any $t \in \mathbb{R}$ the limit $\lim_{\nu \to \infty} R_\nu(t)$ exists, it is equal to
\[ R_\infty(t) = \frac{1}{t^3} \int_0^t \tau^2 \cos \tau d\tau = \int_0^1 \lambda^2 \cos(\lambda t) d\lambda, \quad \forall t \in \mathbb{R}, \]
and
\[ \lim_{\nu \to \infty} \frac{1}{\nu} \text{var}(Z_\nu) = \delta_\infty := \frac{2}{\pi} \int_{-\infty}^{\infty} \left( f_\infty(t) - \frac{\bar{\lambda}_2}{\bar{\lambda}_0} \right) dt + 2 \sqrt{\frac{\bar{\lambda}_2}{\bar{\lambda}_0}}, \]
where
\[ f_\infty(t) = \frac{(\bar{\lambda}_0^2 - R_{\infty}^2)\bar{\lambda}_2 - \bar{\lambda}_0(R'_{\infty})^2}{(\bar{\lambda}_0^2 - R_{\infty}^2)^{\frac{3}{2}}} \left( \sqrt{1 - \rho_{\infty}^2} + \rho_{\infty} \arcsin \rho_{\infty} \right), \]
and
\[ \rho_{\infty} = \frac{R''_{\infty}(\bar{\lambda}_0^2 - R_{\infty}^2) + (R'_{\infty})^2 R_{\infty}}{(\bar{\lambda}_0^2 - R_{\infty}^2)\lambda_2 - \bar{\lambda}_0(R'_{\infty})^2}. \]
Moreover, the constant $\delta_\infty$ is positive.\[^6\]

**Proof.** We follow a strategy inspired from [19]. The variance of $Z_\nu$ can be computed using the results in [11, §10.6]. We introduce the gaussian field
\[ \Psi(t_1, t_2) := \begin{bmatrix} \Phi_\nu(t_1) \\ \Phi'_\nu(t_2) \\ \Phi_\nu(t_2) \\ \Phi'_\nu(t_2) \end{bmatrix}. \]
\[^6\]Numerical experiments indicate that $\delta_\infty \approx 0.35$. \]
Its covariance matrix depends only on \( t = t_2 - t_1 \). We have (compare with \([19, \text{Eq. (17)})\]

\[
\Xi(t) = \begin{pmatrix}
\lambda_0 & R_{\nu}(t) & 0 & R'_{\nu}(t) \\
R_{\nu}(t) & \lambda_0 & -R'_{\nu}(t) & 0 \\
0 & -R'_{\nu}(t) & \lambda_2 & -r''_{\nu}(t) \\
R'_{\nu}(t) & 0 & -R''_{\nu}(t) & \lambda_2
\end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}.
\]

As explained in \([30], \text{to apply \([11, \S 10.6)\) we only need that } \Xi(t) \text{ is nondegenerate. This is established in the next result whose proof can be found in Appendix D.}

**Lemma 6.2.** The matrix \( \Xi(t) \) is nonsingular if and only if \( t \not\in 2\pi\nu\mathbb{Z} \).

For any vector

\[
x := \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} \in \mathbb{R}^4
\]

we set

\[
p_{t_1, t_2}(x) := \frac{1}{4\pi^2 (\det \Xi)^{1/2}} e^{-\frac{1}{2} \langle \Xi^{-1} x, x \rangle}.
\]

Then, the results in \([9, \S 10.6)\) show that

\[
E(Z_\nu^2) - E(Z_\nu) = \int_{I_\nu \times I_\nu} \left( \int_{\mathbb{R}^2} |y_1, y_2| : p_{t_1, t_2}(0, 0, y_1, y_2) |dy_1 dy_2| \right) |dt_1 dt_2|.
\]

As in \([19]\) we have

\[
\Xi(t)^{-1} = \begin{pmatrix} * & * \\ * & \Omega^{-1} \end{pmatrix}, \Omega = C - B^T A^{-1} B.
\]

More explicitly,

\[
\Omega = C - B^T A^{-1} B
\]

\[
= \begin{pmatrix} \lambda_2 & -R''_{\nu}(t) \\ -R'_{\nu}(t) & \lambda_2 \end{pmatrix} - \frac{1}{\lambda_0^2 - R_{\nu}(t)^2} \begin{pmatrix} 0 & -R'_{\nu}(t) \\ R'_{\nu}(t) & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_0 & -R_{\nu}(t) \\ -R_{\nu}(t) & \lambda_0 \end{pmatrix} \cdot \begin{pmatrix} 0 & R'_{\nu}(t) \\ -R''_{\nu}(t) & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} \lambda_2 & -R''_{\nu} \\ -R''_{\nu} & \lambda_2 \end{pmatrix} - \frac{(R''_{\nu})^2}{\lambda_0^2 - R_{\nu}(t)^2} \begin{pmatrix} \lambda_0 & R_{\nu} \\ R_{\nu} & \lambda_0 \end{pmatrix} = \mu \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix},
\]

where

\[
\mu = \mu_{\nu} = \frac{(\lambda_0^2 - R_{\nu}^2)\lambda_2 - \lambda_0(R''_{\nu})^2}{\lambda_0^2 - R_{\nu}^2}, \quad \rho = \rho_{\nu} = \frac{R''_{\nu}(\lambda_0^2 - R_{\nu}^2) + (R''_{\nu})^2 R_{\nu}}{(\lambda_0^2 - R_{\nu}^2) \lambda_2 - \lambda_0(R''_{\nu})^2}.
\]

We want to emphasize, that in the above equalities the constants \( \lambda_0 \) and \( \lambda_2 \) do depend on \( \nu \), although we have not indicated this in our notation.

**Remark 6.3.** The nondegeneracy of \( \Xi \) implies that \( \mu_{\nu}(t) \neq 0 \) and \( |\rho_{\nu}(t)| < 1 \), for all \( t \not\in 2\pi\nu\mathbb{Z} \).

We obtain as in \([19, \text{Eq. (24)})\]

\[
\det \Xi = \det A \cdot \det \Omega = \mu^2 (\lambda_0^2 - R_{\nu}^2)(1 - \rho^2), \quad \Omega^{-1} = \frac{1}{\mu(1 - \rho^2)} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.
\]
We can now rewrite the equality (6.2) as \( (t = t_2 - t_1) \)

\[
E(Z^2_\nu) - E(Z_\nu) = \int_{I_\nu \times I_\nu} \left( \int_{\mathbb{R}^2} |y_1 y_2| e^{-\frac{y_1^2 + 2y_1 y_2 + y_2^2}{2(1 - \rho^2)}} \frac{dy_1 dy_2}{4\pi^2} \right) \mu \sqrt{\frac{\mu_1 d\mu_2}{(\lambda_0^2 - R^2_\nu)(1 - \rho^2)}}
\]

From [7, Eq. (A.1)] we deduce that

\[
\int_{\mathbb{R}^2} |y_1 y_2| e^{-\frac{y_1^2 + 2y_1 y_2 + y_2^2}{2(1 - \rho^2)}} \frac{dy_1 dy_2}{4\pi^2} = 1 - \rho^2 \left( 1 + \frac{\rho}{\sqrt{1 - \rho^2}} \arcsin \rho \right).
\]

Hence

\[
E(Z^2_\nu) - E(Z_\nu) = \frac{1}{\pi^2} \int_{I_\nu \times I_\nu} \left\{ \frac{\mu}{\lambda_0^2 - R^2_\nu} \left( \sqrt{1 - \rho^2 + \rho \arcsin \rho} \right) \right\} dt_1 dt_2
\]

\[
= \frac{1}{\pi^2} \int_{I_\nu \times I_\nu} \left\{ \frac{(\lambda_0^2 - R^2_\nu) \lambda_2 - \lambda_0 (R_\nu')^2}{(\lambda_0^2 - R^2_\nu)^2} \left( \sqrt{1 - \rho^2 + \rho \arcsin \rho} \right) \right\} dt_1 dt_2. \tag{6.3}
\]

The function \( f_\nu(t) = f_\nu(t_2 - t_1) \) is doubly periodic with periods \( 2\pi \nu, 2\pi \nu \) and we conclude that

\[
E([Z_\nu]_2) := E(Z^2_\nu) - E(Z_\nu) = \frac{2\nu}{\pi} \int_{-\pi \nu}^{\pi \nu} f_\nu(t) dt. \tag{6.4}
\]

We conclude that

\[
\var(Z_\nu) = E([Z_\nu]_2) + E(Z_\nu) - (E(Z_\nu))^2 = E([Z_\nu]_2) - [E(Z_\nu)]_2
\]

\[
= \frac{2\nu}{\pi} \int_{-\pi \nu}^{\pi \nu} \left( f_\nu(t) - \frac{\lambda_2}{\lambda_0} \right) dt + 2\nu \sqrt{\frac{\lambda_2}{\lambda_0}}. \tag{6.5}
\]

To complete the proof of Theorem 6.1 we need to investigate the integrand in (6.4). This requires a detailed understanding of the behavior of \( R_\nu \) as \( \nu \to \infty \). It is useful to consider more general sums of the form

\[
A_{\nu, r}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r \cos \frac{mt}{\nu}, \quad B_{\nu, r}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r \sin \frac{mt}{\nu}, \quad r \geq 1.
\]

Note that if we set \( z := \cos \frac{mt}{\nu} + i \sin \frac{mt}{\nu} \). We have

\[
A_{\nu, r}(t) + iB_{\nu, r}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r z^m.
\]

Observe that

\[
R^{(k)}_\nu(t) = -\frac{1}{\pi \nu^{k+3}} \text{Re} \left( \frac{1}{t^{k+2}} C_{\nu, k+2}(t) \right). \tag{6.6}
\]

We set

\[
A_{\infty, r}(t) = \frac{1}{\nu^{r+1}} \int_0^t \tau^r \cos \tau d\tau, \quad B_{\infty, r}(t) = \frac{1}{\nu^{r+1}} \int_0^t \tau^r \sin \tau d\tau.
\]

Observe that \( R_\nu = A_{\nu, 2} \) and \( R_\infty = A_{\infty, 2} \). We have the following result.
Lemma 6.4.
\[ |A_{\nu,r}(t) - A_{\infty,r}(t)| + |B_{\nu,r}(t) - B_{\infty,r}(t)| = O\left(\frac{\max(1,t)}{\nu}\right), \quad \forall t \geq 0, \quad (6.7) \]
where, above and in the sequel, the constant implied by the O-symbol is independent of \( t \) and \( \nu \). In particular
\[ \lim_{\nu \to \infty} \frac{1}{\nu^{r+1}} C_{\nu,r}(t) = C_r(t) := \frac{1}{t^{r+1}} \int_0^t \tau^r e^{i\tau} d\tau, \quad \forall t \geq 0. \quad (6.8) \]

Proof. We have
\[ A_{\nu}(t) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} m^r \cos \left( \frac{mt}{\nu} \right) = \frac{1}{\nu^{r+1}} \sum_{m=1}^{\nu} \left\{ \left( \frac{mt}{\nu} \right)^r \cos \left( \frac{mt}{\nu} \right) \right\} \cdot \left( \frac{t}{\nu} \right). \]

The term \( S_{\nu}(t) \) is a Riemann sum corresponding to the integral
\[ \int_0^t f(\tau) d\tau, \quad f(\tau) := \tau^r \cos \tau, \]
and the subdivision
\[ 0 < \frac{t}{\nu} < \cdots < \frac{(\nu - 1)t}{\nu} < t. \]
of the interval \([0, t]\). A simple application of the mean value theorem implies that there exist points
\[ \theta_m \in \left[ \frac{(m - 1)t}{\nu}, \frac{mt}{\nu} \right] \]
such that
\[ \int_0^t f(\tau) d\tau = \sum_{m=1}^{\nu} f(\theta_m) \frac{t}{m}. \]
We deduce that
\[ \int_0^t f(\tau) d\tau - S_{\nu}(t) = \frac{t}{\nu} \sum_{m=1}^{\nu} \left( f(\theta_m) - f \left( \frac{mt}{\nu} \right) \right) \]
Now set
\[ M(t) := \sup_{0 \leq \tau \leq t} |f'(\tau)|. \]
Observe that
\[ M(t) = \begin{cases} O(t^{r-1}), & 0 \leq t \leq 1 \\ O(t^r), & r > 1. \end{cases} \]
We deduce
\[ \left| S_{\nu}(t) - \int_0^t f(\tau) d\tau \right| \leq M(t) \cdot \frac{t^2}{\nu}. \]
This, proves the \( A \)-part of \((6.7)\). The \( B \)-part is completely similar. \( \Box \)

We need to refine the estimates \((6.7)\). Recall that \([m]_r := m(m - 1) \cdots (m - r + 1), r \geq 1. \) We will express \( C_{\nu,r}(t) \) in terms of the sums
\[ D_{\nu,r}(t) := \sum_{m=1}^{\nu} [m]_r z^m = z^r \frac{d^r}{dz^r} \left( \sum_{k=1}^{\nu} z^k \right) = z^r \frac{d^r}{dz^r} \left( \frac{z - z^{r+1}}{1 - z} \right) = z^r \frac{d^r}{dz^r} \left( \frac{1 - z^{r+1}}{1 - z} \right). \]
Using the classical formula
\[ m^r = \sum_{k=1}^{r} S(r, k)[m]_k, \]
where \( S(r, k) \) are the Stirling numbers of the second kind, we deduce,
\[ C_{\nu,r}(\zeta) = \sum_{k=1}^{r} S(r, k) D_{\nu,k}(\zeta) = \sum_{k=1}^{r} S(r, k) z^r \frac{d^k}{dz^k} \left( \frac{1 - z^{\nu+1}}{1 - z} \right). \]  

(6.9)

Lemma 6.5. Set \( \theta := \frac{t}{2\nu} \), and \( f(\theta) = \frac{\sin \theta}{\theta} \). Then
\[ \frac{t^{r+1}}{\nu^{r+1}} D_{\nu,r}(t) = i^r r! \left( \frac{2 \sin \left( \frac{(\nu+1)t}{2\nu} \right)}{\nu^{r+1}} \cdot e^{\frac{i(\nu+1)t}{2\nu}} - e^{it} \sum_{j=1}^{r} i^{r+1-j} \binom{\nu+1}{j} t^j \cdot \left( \frac{e^{i\theta}}{f(\theta)} \right)^{r+1-j} \right) \]

(6.10)

Proof. We have
\[ D_{\nu,r}(t) = z^r \sum_{j=0}^{r} \binom{r}{j} \frac{d^j}{dz^j} (1 - z^{\nu+1}) \frac{d^{r-j}}{dz^{r-j}} (1 - z)^{-1} \]
\[ = r! \frac{z^r (1 - z^{\nu+1})}{(1 - z)^{r+1}} - \sum_{j=1}^{r} \binom{r}{j} [\nu+1]_j (r-j)! \frac{z^{\nu+1+r-j}}{(1 - z)^{1+r-j}} \]
\[ = r! \frac{z^r (1 - z^{\nu+1})}{(1 - z)^{r+1}} - z^r r! \sum_{j=1}^{r} \binom{\nu+1}{j} \left( \frac{z}{1 - z} \right)^{r+1-j}. \]

Using the identity
\[ 1 - e^{i\alpha} = \frac{2}{i} \sin \left( \frac{\alpha}{2} \right) e^{\frac{i\alpha}{2}} \]
we deduce
\[ \frac{z}{1 - z} = \frac{ie^{\frac{i\theta}{2\nu}}}{2 \sin \left( \frac{\theta}{2\nu} \right)}, \]
and
\[ D_{\nu,r}(t) = i^r r! \frac{\sin \left( \frac{(\nu+1)t}{2\nu} \right)}{\nu^{r+1}} e^{\frac{i(\nu+1+2\nu)t}{2\nu}} - e^{it} r! \sum_{j=1}^{r} i^{r+1-j} \binom{\nu+1}{j} \left( 2 \sin \theta \right)^{r+1-j} \]
\[ = i^r r! \left( \frac{2 \sin \left( \frac{(\nu+1)t}{2\nu} \right)}{(2 \sin \theta)^{r+1}} \cdot e^{\frac{i(\nu+1)t}{2\nu}} - e^{it} \sum_{j=1}^{r} i^{r+1-j} \binom{\nu+1}{j} \left( \frac{e^{i\theta}}{2 \sin \theta} \right)^{1+r-j} \right). \]

Multiplying both sides of the above equality by \( \left( \frac{t}{\nu} \right)^{r+1} \) we get (6.10).

Lemma 6.5 coupled with the fact that the function \( f(\theta) \) is bounded on \([0, \frac{\pi}{2}]\) yield the following estimate.

\[ \frac{t^{r+1}}{\nu^{r+1}} D_{\nu,r}(t) = O(1), \quad \forall \nu, \quad 0 \leq t \leq \pi \nu. \]

(6.11)
Using (6.11) and the identity $S(r, 1) = 1$ in (6.9) we deduce that there exists $K = K_r > 0$ such that for any $\nu > 0$ and any $t \in [0, \pi \nu]$ we have
\[
\left| \frac{t^{r+1}}{\nu^{r+1}} (C_{\nu,r}(t) - D_{\nu,r}(t)) \right| \leq K_r \sum_{j=0}^{r-1} \left| \frac{t^{r+1}}{\nu^{r+1}} D_{\nu,j}(t) \right| \leq K_r \sum_{j=0}^{r-1} \left( \frac{1}{\nu} \right)^r \leq K_r \frac{t}{\nu},
\]
so that
\[
\left| \frac{1}{\nu^{r+1}} (C_{\nu,r}(t) - D_{\nu,r}(t)) \right| \leq K_r \frac{1}{\nu^r}.
\]
Using Lemma 6.5 we deduce
\[
\lim_{\nu \to \infty} \frac{t^{r+1}}{\nu^{r+1}} \Re D_{\nu,r}(t) = I_r(t) := \tilde{i}^r r! \left( 2 \sin \left( \frac{t}{2} \right) - e^{\frac{it}{2}} - e^{it} \sum_{j=1}^{r} i^{1-j} \frac{t^j}{j!} \right).
\]
uniformly for $t$ on compacts. The estimate (6.12) implies that
\[
\lim_{\nu \to \infty} \frac{t^{r+1}}{\nu^{r+1}} \Re D_{\nu,r}(t) = I_r(t).
\]
We have the following crucial estimate whose proof can be found in Appendix D.

**Lemma 6.6.** For every $r \geq 0$ there exists $C_r > 0$ such that for any $\nu > 0$ we have
\[
\left| \frac{1}{\nu^{r+1}} D_{\nu,r}(t) - \frac{1}{t^{r+1}} I_r(t) \right| \leq \frac{C_r}{\nu} t^{r+1} - 1, \quad \forall 0 < t \leq \pi \nu.
\]

Using Lemma 6.6 in (6.12) we deduce
\[
\left| \frac{1}{\nu^{r+1}} C_{\nu,r}(t) - \frac{1}{t^{r+1}} I_r(t) \right| \leq \frac{C_r}{\nu} t^{r+1} - 1, \quad \forall 0 < t \leq \pi \nu.
\]
\[
I_r(t) = t^{r+1} C_r(t) = \int_0^t \tau^r e^{i \tau} d\tau.
\]
Using (6.7) and (6.14a) we deduce that for any nonnegative integer $r$ there exists a positive constant $K = K_r > 0$ such that
\[
| C_{\nu,r}(t) - C_r(t) | \leq \frac{K_r}{\nu} t^{r+1} - 1.
\]
Coupling the above estimates with (6.7) we deduce
\[
C_{\nu,r}(t) = C_r(t) + O \left( \frac{1}{\nu} \right), \quad \forall 0 \leq t \leq \nu,
\]
where the constant implied by the symbol $O$ depends on $k$, but it is independent of $\nu$. The last equality coupled with (6.6) implies that
\[
R_{\nu}^{(k)}(t) = R_{\infty}^{(k)}(t) + O \left( \frac{1}{\nu} \right), \quad \forall 0 \leq t \leq \nu.
\]
We deduce that, for any $t > 0$ we have
\[
\lim_{\nu \to \infty} f_{\nu}(t) = f_{\infty}(t)
\]
where $f_{\nu}$ is the function defined in (6.3), while
\[
f_{\infty}(t) = \frac{(\lambda_0^2 - R_{\infty}^2) \lambda_2 - \lambda_0 R_{\infty}^2}{(\lambda_0^2 - R_{\infty}^2)^2} \left( \sqrt{1 - \rho_{\infty}} + \rho_{\infty} \arcsin \rho_{\infty} \right),
\]
where
\[ \bar{\lambda}_0 = \lambda_0(\infty) = \lim_{\nu \to \infty} \lambda_0(\nu) = \frac{1}{3}, \quad \bar{\lambda}_2 = \lim_{\nu \to \infty} \lambda_2(\nu) = \frac{1}{5}, \]
\[ \rho_\infty(t) = \lim_{\nu \to \infty} \rho_\nu(t) = \frac{R''_\infty(\bar{\lambda}_0^2 - R^2_\infty) + (R'_\infty)^2 R_\infty}{(\bar{\lambda}_0^2 - R^2_\infty)\bar{\lambda}_2 - \lambda_0(R'_\infty)^2}. \]

We have the following result whose proof can be found in Appendix D.

**Lemma 6.7.**

\[
|R_\infty(t)| < R_\infty(0), \quad |R''_\infty(t)| < |R''_\infty(0)|, \quad \forall t > 0, \tag{6.19a}
\]
\[ R_\infty(t), \quad R'_\infty(t), \quad R''_\infty(t) = O\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty. \tag{6.19b} \]
\[ (\bar{\lambda}_0^2 - R^2_\infty)\bar{\lambda}_2 - \lambda_0(R'_\infty)^2 > 0, \quad \forall t > 0. \tag{6.19c} \]
\[ R_\infty(t) = \frac{1}{3} - \frac{1}{10}t^2 + \frac{1}{168}t^4 + O(t^6), \quad R'_\infty(t) = -\frac{1}{5}t + \frac{1}{42}t^3 + O(t^5), \quad \text{as} \quad t \to 0. \tag{6.19d} \]

We set
\[ \delta(t) := \max(t, 1), \quad t \geq 0. \]

We find it convenient to introduce new functions
\[ G_\nu(t) := \frac{1}{R_\nu(0)}R_\nu(t) = \frac{1}{\bar{\lambda}_0(\nu)}R_\nu(t), \quad H_\nu(t) = \frac{1}{R''_\nu(0)}R''_\nu(t) = -\frac{1}{\bar{\lambda}_2(\nu)}R''_\nu(t). \]

Using these notations we can rewrite (6.19c) as
\[
(1 - G^2_\infty) - \frac{\bar{\lambda}_0}{\bar{\lambda}_2}(G'_\infty)^2 > 0, \quad \forall t > 0. \tag{6.20} \]

The equalities (6.19d) imply that
\[ \eta(t) = \frac{3}{4375}t^4 + O(t^6), \quad \forall |t| \ll 1. \tag{6.21} \]

Then
\[ f_\nu(t) = \frac{\bar{\lambda}_2(\nu)}{\bar{\lambda}_0(\nu)} \times E_\nu(t) \times \left(\frac{1}{\sqrt{1 - \rho_\nu^2}} + \rho_\nu \arcsin \rho_\nu\right), \]

where
\[ E_\nu(t) := \frac{(1 - G_\nu(t)^2) - \frac{\lambda_0(\nu)}{\bar{\lambda}_2(\nu)}(G'_\nu(t))^2}{(1 - G_\nu(t)^2)^{3/2}}, \]

and
\[ \rho_\nu = \frac{R''_\nu(\lambda_0(\nu)^2 - R^2_\nu) + (R'_\nu)^2 R_\nu}{(\bar{\lambda}_0^2(\nu) - R^2_\nu)\bar{\lambda}_2 - \lambda_0(R'_\nu)^2} = -\frac{H_\nu(1 - G^2_\nu) + \frac{\lambda_0(\nu)}{\bar{\lambda}_2(\nu)}(G'_\nu)^2 G_\nu}{(1 - G^2_\nu) - \frac{\lambda_0(\nu)}{\bar{\lambda}_2(\nu)}(G'_\nu)^2}. \]

**Lemma 6.8.** Let \( \kappa \in (0, 1) \). Then
\[ E_\nu(t) = O(t), \quad \forall 0 \leq t \leq \nu^{-\kappa}, \]

where the constant implied by \( O \)-symbol is independent of \( \nu \) and \( t \in [0, \nu^{-\kappa}) \), but it could depend on \( \kappa \).
Proof. Observe that for $t \in [0, \nu^{-\kappa}]$ we have

$$G_{\nu}(t) = 1 - \frac{\lambda_2(\nu)}{2\lambda_0(\nu)}t^2 + O(t^4), \quad G'_{\nu}(t) = \frac{\lambda_2(\nu)}{\lambda_0(\nu)}t + O(t^3)$$

so that

$$(1 - G_{\nu}(t))^2 = \left(\frac{\lambda_2(\nu)}{\lambda_0(\nu)}t\right)^3 \times (1 + O(t)),$$

and

$$(1 - G_{\nu}(t))^2 - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G'_{\nu}) = O(t^4)$$

so that $\mathcal{E}_\nu(t) = O(t)$.

\[\square\]

**Lemma 6.9.** Let $\kappa \in (0, 1)$. Then

$$\mathcal{E}_\nu = \mathcal{E}_\infty \times \left(1 + O\left(\frac{(G'_{\infty})^2}{\nu \eta(t)} + \frac{1}{\nu \gamma(t) \delta(t)} + \frac{1}{\nu \delta(t) \eta(t)} + \frac{1}{\nu^2 \eta(t)}\right)\right), \quad (6.22)$$

and

$$\mathcal{E}_\nu(t) = \mathcal{E}_\infty(t) + O\left(\frac{(G'_{\infty})^2}{\nu \gamma(t)^{3/2}} + \frac{\eta(t)}{\nu \delta(t) \gamma(t)^{3/2}} + \frac{1}{\nu \delta(t) \gamma(t)^{3/2}} + \frac{1}{\nu^2 \gamma(t)^{3/2}}\right), \quad (6.23)$$

where

$$\eta(t) := (1 - G_{\nu}^2) - \frac{\lambda_0}{\lambda_2}(G'_{\nu})^2 \quad \text{and} \quad \gamma(t) = 1 - G_{\infty}^2.$$

**Proof.** Observe that

$$G_{\nu}^2 = \left(G_{\infty} + O\left(\frac{1}{\nu}\right)\right)^2 \quad (6.19b) = G_{\infty}^2 + O\left(\frac{1}{\nu \delta(t)}\right), \quad (6.24)$$

so that

$$1 - G_{\infty}^2(t)^2 = (1 - G_{\infty}^2(t)) \left(1 + O\left(\frac{1}{\nu \delta(t) \gamma(t)}\right)\right).$$

For $t > \nu^{-\kappa}$ we have

$$\frac{1}{\nu \delta(t) \gamma(t)} = O\left(\frac{1}{\nu^{1-\kappa}}\right) = o(1) \quad \text{uniformly in } t > \nu^{-\kappa} \text{ as } \nu \to \infty.$$

Hence

$$(1 - G_{\nu}^2)^{-3/2} = (1 - G_{\infty}^2)^{-3/2} \left(1 + O\left(\frac{1}{\nu \delta(t) \gamma(t)}\right)\right), \quad \gamma(t) := 1 - G_{\infty}^2.$$

Next observe that

$$\lambda_0(\nu) = \frac{B_3(\nu + 1)}{3\nu^3} = \frac{1}{3} + \nu^{-1} + O(\nu^{-2}), \quad \lambda_2(\nu) = \frac{B_5(\nu + 1)}{5\nu^5} = \frac{1}{5} + \nu^{-1} + O(\nu^{-2}) \quad (6.26)$$

and

$$\frac{\lambda_0(\nu)}{\lambda_2(\nu)} = \frac{\frac{1}{3} + \nu^{-1} + O(\nu^{-2})}{\frac{1}{5} + \nu^{-1} + O(\nu^{-2})} = \frac{5}{3} - \frac{10}{3} \nu^{-1} + O(\nu^{-2}).$$
Using (6.24) and the above estimate we deduce
\[
(1 - G_\nu(t)^2)^2 - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\nu')^2 = (1 - G_\infty^2)^2 - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\infty')^2 + 10 \frac{\lambda_0(\nu)}{3\nu}(G_\infty')^2 + O \left( \frac{1}{\nu \delta(t)} \right) + O \left( \frac{1}{\nu^2} \right)
\]
\[
= \left( (1 - G_\infty^2) - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\infty')^2 \right) \cdot \left( 1 + O \left( \frac{(G_\infty')^2}{\nu \eta(t)} + \frac{1}{\nu \delta(t) \eta(t)} + \frac{1}{\nu^2 \eta(t) \delta(t)} \right) \right).
\]
(6.27)

Using (6.25) we deduce (6.22). The estimate (6.23) follows (6.22) by invoking the definitions of \(\gamma(t)\) and \(\eta(t)\).

\[\text{Lemma 6.10. Let } \kappa \in (0, \frac{1}{2}). \text{ Then}\]
\[
\rho_\nu = \rho_\infty + O \left( \frac{1}{\nu \eta(t)} + \frac{(G_\infty')^2}{\nu \eta(t) \delta(t)} + \frac{1}{\nu \delta(t)^2} + \frac{1}{\nu \delta(t) \eta(t)} + \frac{1}{\nu^2 \eta(t) \delta(t)} \right), \quad \forall t > \nu^{-\kappa}, \quad (6.28)
\]

where the constant implied by \(O\)-symbol is independent of \(\nu\) and \(t > \nu^{-\kappa}\), but it could depend on \(\kappa\).

\[\text{Proof. The estimates (6.19b), (6.19d) and (6.21) imply that for } t > \nu^{-\kappa} \text{ we have}\]
\[
\frac{(G_\infty')^2}{\nu \eta(t)} + \frac{1}{\nu \delta(t) \eta(t)} + \frac{1}{\nu^2 \eta(t) \delta(t)} = O \left( \frac{1}{\nu^{1-4\kappa}} \right) = O(1),
\]
uniformly in \(t > \nu^{-\kappa}\). We conclude from (6.27) that
\[
\left( (1 - G_\nu(t)^2)^2 - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\nu')^2 \right)^{-1} = \left( (1 - G_\infty^2)^2 - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\infty')^2 \right)^{-1}
\]
\[
\times \left( 1 + O \left( \frac{(G_\infty')^2}{\nu \eta(t)} + \frac{1}{\nu \delta(t) \eta(t)} + \frac{1}{\nu^2 \eta(t) \delta(t)} \right) \right).
\]
(6.29)

Since \(H_\nu = H_\infty + O(\nu^{-1})\) we deduce
\[
-H_\nu(1 - G_\nu)^2 + \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\nu')^2 G_\nu = -H_\infty(1 - G_\infty)^2 + \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\infty')^2 G_\infty + O \left( \frac{1}{\nu} \right)
\]

Recalling that
\[
\rho_\nu = \frac{-H_\nu(1 - G_\nu)^2 + \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\nu')^2 G_\nu}{(1 - G_\nu^2)^2 - \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\nu')^2}
\]
and \(-H_\infty(1 - G_\infty)^2 + \frac{\lambda_0(\nu)}{\lambda_2(\nu)}(G_\infty')^2 G_\infty = O(\delta^{-1})\) we see that (6.28) follows from (eq: est22).

Consider the function
\[
A(u) = \sqrt{1 - u^2} + u \arcsin u, \quad |u| \leq 1.
\]

Observe that
\[
\frac{dA}{du} = O(1), \quad \forall |u| \leq 1, \quad (6.30a)
\]
\[
\frac{dA}{du} = \arcsin u = O(u) \quad \text{as } u \to 0. \quad (6.30b)
\]

Now fix an exponent \(\kappa \in (0, \frac{1}{2})\). We discuss separately two cases.
Lemma 6.11. If \( \nu^k < t < \pi \nu \). Then in this range we have
\[
\frac{1}{\delta}, \rho_\infty = O(t^{-1}), \quad \rho_\nu = \rho_\infty + o(1), \quad \frac{1}{\eta(t)} = O(1),
\]
and using (6.30b) and (6.28) we deduce
\[
A(\rho_\nu) = A(\rho_\infty) + A'(\rho_\infty)(\rho_\nu - \rho_\infty) + O\left((\rho_\mu - \rho_\infty)^2\right) = A(\rho_\infty) + O\left(\frac{1}{t\nu} + \frac{1}{\nu^2}\right). \quad (6.31)
\]

2. \( \nu^{-\kappa} < t < \nu^\kappa \). The equality (6.21) shows that in this range we have
\[
\frac{1}{\eta(t)} = O(\nu^{-4\kappa}), \quad \frac{1}{\delta(t)} = O(1)
\]
so that and (6.28) implies that
\[
\rho_\nu - \rho_\infty = O\left(\frac{1}{\nu^{1-4\kappa}}\right).
\]
Using (6.30a) we deduce
\[
A(\rho_\nu) - A(\rho_\infty) = O\left(\frac{1}{\nu^{1-4\kappa}} + \frac{1}{\nu^{1-4\kappa} \delta(t)^2} + \frac{1}{\nu^{2-8\kappa} \delta(t)}\right). \quad (6.32)
\]
Set
\[
\Delta_\nu := \mathcal{C}_\nu A(\rho_\nu) - \mathcal{C}_\infty A(\rho_\infty), \quad q_\nu := \left(\frac{\lambda_2(\nu)}{\lambda_0(\nu)}\right), \quad q_\infty = \left(\frac{\bar{\lambda}_2}{\bar{\lambda}_0}\right).
\]
Then using (6.26) we deduce that
\[
q_\nu - q_\infty = \frac{6}{5} \nu^{-1} + O(\nu^{-2}), \quad q_\nu = q_\infty \left(1 + 2\nu^{-1} + O(\nu^{-2})\right).
\]
Then
\[
(f_\nu(t) - f_\nu) - (f_\infty(t) - f_\infty) = q_\nu \left(\mathcal{C}_\infty A(\rho_\infty) - 1 + \Delta_\nu\right) - q_\infty \left(\mathcal{C}_\infty A(\rho_\infty) - 1\right)
\]
\[
= (q_\nu - q_\infty) \left(\mathcal{C}_\infty A(\rho_\infty) - 1\right) + q_\infty \Delta_\nu.
\]
To prove (6.1) we need to prove the following equality.
\[
(q_\nu - q_\infty) \int_0^{\pi \nu} (\mathcal{C}_\nu(t) A(\rho_\infty(t)) - 1) dt, \quad q_\infty \int_0^{\pi \nu} \Delta_\nu(t) dt = o(1) \text{ as } \nu \to \infty. \quad (6.33)
\]
We can dispense easily of the first integral above since \(\mathcal{C}_\infty(t) A(\rho_\infty(t)) - 1\) is absolutely integrable on \([0, \infty)\) and \(q_\nu - q_\infty = O(\nu^{-1})\).

The second integral requires a bit of work. More precisely, we will show the following result.

Lemma 6.11. If \(0 < \kappa < \frac{1}{2}\), then
\[
\int_0^{\nu^{1-\kappa}} \Delta_\nu(t) dt, \quad \int_{\nu^{1-\kappa}}^{\nu^{1-\kappa}} \Delta_\nu(t) dt, \quad \int_{\nu^{1-\kappa}}^{\nu^{1-\kappa}} \Delta_\nu(t) dt = o(1) \text{ as } \nu \to \infty. \quad (6.34)
\]

Proof. We will discuss each of the three cases separately.

1. \(0 < t < \nu^{1-\kappa}\). The easiest way to prove that \(\int_0^{\nu^{1-\kappa}} \Delta_\nu(t) dt \to 0\) is to show that
\[
\mathcal{C}_\nu(t) A(\rho_\nu(t)) = O(1), \quad 0 < t < \nu^{1-\kappa}.
\]
This follows using Lemma 6.8 and observing that the function \(A(u)\) is bounded.
Lemma A.1. Let \( r = \frac{1}{\eta} = O(\nu^{-2\kappa}), \frac{1}{\gamma(t)} = O(\nu^{-2\kappa}), \frac{1}{\delta(t)} = O(1), \) \( C_{\infty}^\nu(t) = O(1). \) Using (6.23), (6.32) we deduce

\[
\mathcal{C}_\nu(t) = \mathcal{C}_\infty(t) + O \left( \frac{1}{\nu^{1-3\kappa}} \right), \quad A(\rho_\nu) = A(\rho_\infty) + O \left( \frac{1}{\nu^{1-4\kappa}} \right).
\]

Hence

\[
\Delta_\nu = O \left( \frac{1}{\nu^{1-4\kappa}} \right) \quad \text{and} \quad \int_{\nu^{-\kappa}}^{\nu^\kappa} \Delta_\nu(t) dt = O \left( \frac{1}{\nu^{1-5\kappa}} \right) = o(1).
\]

2. \( \nu^{-\kappa} < t < \nu^\kappa. \) In this range we have

\[
\frac{1}{\eta} = O(\nu^{-4\kappa}), \quad \frac{1}{\gamma(t)} = O(\nu^{-2\kappa}), \quad \frac{1}{\delta(t)} = O(1), \quad C_{\infty}^\nu(t) = O(1).
\]

Using (6.23), (6.32) we deduce

\[
\mathcal{C}_\nu(t) = \mathcal{C}_\infty(t) + O \left( \frac{1}{\nu^{1-3\kappa}} + \frac{1}{\nu^2} \right), \quad A(\rho_\nu) = A(\rho_\infty) + O \left( \frac{1}{\nu^t} + \frac{1}{\nu^2} \right)
\]

so that

\[
\Delta_\nu(t) = O \left( \frac{1}{\nu^t} + \frac{1}{\nu^2} \right), \quad \int_{\nu^{-\kappa}}^{\nu^\kappa} \Delta_\nu(t) dt = O \left( \frac{1}{\nu} + \frac{\log \nu}{\nu} \right) = o(1).
\]

3. \( \nu^\kappa < t < \pi \nu. \) For these values of \( t \) we have

\[
\eta(t), \frac{1}{\eta(t)} = O(1), \quad \frac{1}{\delta(t)} = O(t^{-1}).
\]

Using (6.23) and (6.31) we deduce

\[
\mathcal{C}_\nu(t) = \mathcal{C}_\infty(t) + O \left( \frac{1}{\nu^t} + \frac{1}{\nu^2} \right), \quad A(\rho_\nu) = A(\rho_\infty) + O \left( \frac{1}{\nu^t} + \frac{1}{\nu^2} \right)
\]

so that

\[
\Delta_\nu(t) = O \left( \frac{1}{\nu^t} + \frac{1}{\nu^2} \right), \quad \int_{\nu^{-\kappa}}^{\pi \nu} \Delta_\nu(t) dt = O \left( \frac{1}{\nu} + \frac{\log \nu}{\nu} \right) = o(1).
\]

\( \square \)

The fact that \( \delta_\infty \) defined as in (6.1) is positive follows by arguing exactly as in [19, §3.2]. This completes the proof of Theorem 6.1. \( \square \)

Remark 6.12. The proof of Lemma 6.11 shows that for any \( \varepsilon > 0 \) we have

\[
\text{var}(Z_\nu) = \nu \delta_\infty + O(\nu^\varepsilon) \quad \text{as} \quad \nu \to \infty.
\]

(6.35)

Numerical experiments suggest that \( \delta_\infty \approx 0.35. \)

\( \square \)

**APPENDIX A. SOME ELEMENTARY INTEGRALS**

Suppose \( W \) is an oriented Euclidean vector space equipped with an orthogonal decomposition

\[
W = W_0 \oplus W_1, \quad \dim W_i = n_i, \quad i = 0, 1, \quad \dim W = n = n_0 + n_1.
\]

For any \( w \in W \) we denote by \( w_i \) its orthogonal projection on \( W_i, \) \( i = 0, 1, \) so that \( w = w_0 + w_1. \)

We set \( r_i(w) := |w_i|, \) \( i = 0, 1. \)

Lemma A.1. Let \( \varphi_i : W_i \to \mathbb{R}, \) \( i = 0, 1, \) be locally integrable functions, such that \( \varphi_0 \) is positively homogeneous of degree \( k_0 \geq 0, \) and set \( \varphi(w_0, w_1) := \varphi_0(w_0)\varphi_1(w_1). \) Then

\[
\int_{S(W)} \varphi(w) |dS(w)| = \int_{S(W_0)} \varphi_0(w_0) |dS(w_0)| \times \int_{B(W_1)} \varphi_1(w_1)(1 - r_1^2)^{k_0+n_0-2} |dV(w_1)|.
\]

(A.1)

In particular, if \( \varphi_0 = 1, \) then

\[
\int_{S(W)} \varphi(w) |dS(w)| = \sigma_{n_0-1} \int_{B_1(W_1)} \varphi(w_1)(1 - r_1^2)^{n_0-2} |dV(w_1)|.
\]

(A.2)
Proof. The key trick behind the equality (A.2) is the co-area formula. Denote by $\pi$ the orthogonal projection onto $W_1$, i.e.,

$$\pi : W \to W_1, \ w_0 + w_1 \mapsto w_1.$$ 

This induces a smooth map $\pi : S(W) \to B(W_1)$. We denote by $\Sigma_{w_1}$ the fiber of this map over $w_1$. We observe that $\Sigma_{w_1}$ is the sphere in $W_0$ of radius $(1 - r_1^2)^{1/2}$ and centered at the origin. Denote by $J_\pi : S(W) \to \mathbb{R}$ the relative Jacobian of the map $\pi$ defined as in [22, §5.1.1].

Fix a point $w = (w_0, w_1) \in \Sigma_{w_1}$. Next, choose an orthonormal basis $e_1, \ldots, e_{n_1}$ of $W_1$ such that $e_1 = \frac{1}{r_1} w_1$.

The orthogonal complement $N_w \Sigma_{w_1}$ of $T_w \Sigma_{w_1}$ in $T_w S(W)$ consists of vectors that are orthogonal on $W_0$ and on the unit vector $w = w_0 + w_1$. We deduce that the collection

$$f_1 = -r_1^2 w_0 + r_0^2 w_1, \ f_2 = e_2, \ldots, f_{n_1} = e_{n_1},$$

is an orthogonal basis of $N_w \Sigma_{w_1}$. Note that

$$|f_1| = r_1^4 r_0^2 + r_0^4 r_1^2 = r_1^2 r_0^2.$$ 

We obtain an orthonormal basis by replacing $f_1$ with the vector

$$\tilde{f}_1 = \frac{1}{|f_1|} f_1 = -\frac{r_1}{r_0} w_0 + \frac{r_0}{r_1} w_1.$$ 

The orthogonal projection onto $W_1$ of the orthonormal basis $\tilde{f}_1, f_2, \ldots, f_{n_1}$ is the orthogonal basis

$$r_0 e_1, e_2, \ldots, e_{n_1},$$

whose determinant is $r_0$. This shows that

$$J_\pi (w) = r_0 (w) = (1 - r_1 (w)^2)^{1/2}, \ \forall w \in S(W).$$ 

The coarea formula [22, Thm. 5.3.9] implies that

$$\int_{S(W)} \varphi (w) |dS(w)| = \int_{B(W_1)} \left( \int_{\Sigma_{w_1}} \frac{1}{J_\pi(w_0, w_1)} \varphi (w_0, w_1) |dS(w_1)| \right).$$ 

$$= \left( \int_{S(W_0)} \varphi (w_0) |dS(w_0)| \right) \int_{B(W_1)} \varphi_1 (w_1) r_0^{k_0 + n_0 - 2} |dV(w_1)|$$ 

$$= \left( \int_{S(W_0)} \varphi (w_0) |dS(w_0)| \right) \int_{B(W_1)} \varphi_1 (w_1) (1 - r_1^2)^{k_0 + n_0 - 2} |dV(w_1)|.$$

$\square$

Suppose that $L$ is a Euclidean vector space of dimension $\ell$, and $Q : L \to \mathbb{R}$ is a continuous, positively homogeneous function of degree $k > 0$. For any positive integer $n$ we set

$$I_n(Q) := \int_{B(L)} |Q(x)| (1 - |x|^2)^{n/2} |dV(x)|, \ J_n(Q) := \int_{S(L)} |Q(x)| |dS(x)|,$$

where $S(L)$ denotes the unit sphere in $L$ centered at the origin, and $B(L)$ denotes the unit ball in $L$ centered at the origin.

Lemma A.2.

$$I_n(Q) = \frac{\Gamma \left( \frac{\ell + k}{2} \right) \Gamma \left( \frac{n}{2} + 1 \right)}{2 \Gamma \left( \frac{n + \ell + k}{2} + 1 \right)} J_n(Q). \tag{A.3}$$
Proof. We have
\[ I_n(Q) = J_n(Q) \int_0^1 s^{\ell+k-1}(1-s)^{n/2}ds = \frac{1}{2} J_n(Q) \int_0^1 s^{\ell+k-1}(1-s)^{n/2}ds \]
\[ = \frac{1}{2} J_n(Q) B \left( \frac{\ell+k}{2}, \frac{n}{2} + 1 \right), \]
where \( B(p, q) \) denotes the Eulerian integral
\[ B(p, q) = \int_0^1 (1-s)^{p-1}s^{q-1}ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \]
\[ \square \]

The integrals of homogeneous functions over the unit sphere can be reduced to gaussian integrals of these polynomials. More precisely we have the following result.

**Lemma A.3.** Suppose that \( W \) is an Euclidean space of dimension \( N \) and \( f : W \to \mathbb{R} \) is a locally integrable positively homogeneous function of degree \( \ell \geq 0 \). Then
\[ \int_{S(W)} f(x)|dS(x)| = \frac{2}{\Gamma(\frac{N+\ell}{2})} \int W e^{-|x|^2} f(x) |dV(x)|, \] (A.4)
or equivalently,
\[ \frac{1}{\text{area } S(W)} \int_{S(W)} f(x)|dS(x)| = \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N+\ell}{2})} \int W f(x) e^{-|x|^2} \frac{1}{\pi^{\frac{N}{2}}} |dV(x)|. \] (A.5)

**Proof.** We have
\[ \int W e^{-|x|^2} f(x) |dV(x)| = \int_0^\infty \left( \int_{S(W)} f(x)|dS(x)| \right) e^{-r^2} r^{N+\ell-1}dr \]
\[ = \left( \int_{S(W)} f(x)|dS(x)| \right) \int_0^\infty e^{-s} s^{\frac{N+\ell}{2}-1}ds \frac{\Gamma(\frac{N+\ell}{2})}{2} \int_{S(W)} f(x)|dS(x)|. \]
\[ \square \]

**Proposition A.4.** Suppose \( a \) and \( b \) are nonnegative real numbers such that \( a > b \). Then
\[ I(a, b) := \int_{\mathbb{R}^3} e^{-\frac{1}{2(a^2-b^2)}(ax^2+ay^2-2axy)-\frac{1}{2}z^2|xy-z^2|}dxdydz \]
\[ = \sqrt{2\pi(a^2-b^2)} \left( \int_0^{2\pi} \frac{2c^{3/2}}{(c+2)^{1/2}}d\theta - 2\pi a + 2\pi \right), \]
where \( c(\theta) := (a - b \cos 2\theta). \)

**Proof.** Let \( \{i, j, k\} \) be the canonical orthonormal basis of \( \mathbb{R}^3 \). Define a new orthonormal basis \( e_1, e_2, e_3 \) of \( \mathbb{R}^3 \) by setting
\[ e_1 = \frac{1}{\sqrt{2}}(i+j); \quad e_2 = \frac{1}{\sqrt{2}}(i-j); \quad e_3 = k. \]
If we let \( (u, v, w) \) denote the coordinates with respect to this new orthonormal frame, then from the equality
\[ u e_1 + v e_2 + w e_3 = xi + yj + zk \]
we deduce
\[ z = w, \ x = \frac{1}{\sqrt{2}}(u + v), \ y = \frac{1}{\sqrt{2}}(u - v) \]
\[ xy = \frac{1}{2}(u^2 - v^2), \ ax^2 + ay^2 - 2bxy = (a - b)u^2 + (a + b)v^2. \]

We deduce that
\[ I = \frac{1}{2} \int_{\mathbb{R}^3} e^{-\frac{1}{2}((a-b)u^2+(a+b)v^2) - \frac{1}{2}w^2} |u^2 - v^2 - 2w^2| |dudvdw| \]

We now make the change in variables
\[ u = \sqrt{2(a + b)}u, \ v = \sqrt{2(a - b)}v, \ w = \sqrt{2}w, \]
to deduce
\[ I = 2 \sqrt{2(a^2 - b^2)} \int_{\mathbb{R}^3} e^{-(u^2 + v^2 + w^2)} |(a + b)u^2 + (a - b)v^2 - 2w^2| |dudvdw| =: I_1 \]

We now change to cylindrical coordinates,
\[ w = w, \ u = r \cos \theta, \ v = r \sin \theta, \]
so that
\[ u^2 + v^2 + w^2 = r^2 + w^2, \]
\[ (a + b)u^2 + (a - b)v^2 - 2w^2 = r^2((a + b) \cos^2 \theta + (a - b) \sin^2 \theta) - 2w^2 = r^2(a - b \cos 2\theta) - 2w^2. \]

We have
\[ I_1 = \int_0^{2\pi} d\theta \int_{-\infty}^\infty dw \int_{-\infty}^\infty e^{-r^2}|r^2(a - b \cos 2\theta) - 2w^2| r dr \]
\[ = \frac{1}{2} \int_0^{2\pi} d\theta \int_{-\infty}^\infty e^{-w^2} dw \int_0^\infty e^{-s} s(a - b \cos 2\theta) - 2w^2| ds \]

At this point we observe that for any \( c, d > 0 \) we have
\[ \int_0^\infty e^{-x}|cx - d| dx = 2ce^{-\frac{d}{c}} + d - c. \]

Hence, if we set \( c = c(\theta) = (a - b \cos 2\theta) \) we deduce
\[ I_1 = \frac{1}{2} \int_0^{2\pi} d\theta \int_{-\infty}^\infty e^{-w^2} \left( 2ce^{-\frac{2w^2}{c}} + 2w^2 - c \right) dw =: J(c) \]

We have
\[ J(c) = 2c \int_{\mathbb{R}} e^{-\frac{c+2}{c}w^2} dw + 2 \int_{\mathbb{R}} e^{-w^2} dw - c \int_{\mathbb{R}} e^{-w^2} dw \]
\[ = \frac{2c^{3/2}}{(c + 2)^{1/2}} \pi^{1/2} + 2 \Gamma(3/2) - c \pi^{1/2} = \pi^{1/2} \left( \frac{2c^{3/2}}{(c + 2)^{1/2}} - c + 1 \right). \]

We deduce that
\[ I = \sqrt{2(a^2 - b^2)} \int_0^{2\pi} J(c) d\theta = \sqrt{2\pi(a^2 - b)^2} \int_0^{2\pi} \left( \frac{2c^{3/2}}{(c + 2)^{1/2}} - c + 1 \right) d\theta. \]

The conclusion of the proposition follows by observing that \( \int_0^{2\pi} c(\theta) d\theta = a. \)
APPENDIX B. BASIC FACTS ABOUT SPHERICAL HARMONICS

We survey here a few classical facts about spherical harmonics that we needed in the main body of the paper. For proofs and more details we refer to our main source, [25].

We denote by $\mathcal{H}_{n,d}$ the space of homogeneous, harmonic polynomials of degree $n$ in $d$ variables. We regard such polynomials as functions on $\mathbb{R}^d$, and we denote by $\mathcal{Y}_{n,d}$ the subspace of $C^\infty(S^{d-1})$ spanned by the restrictions of these polynomials to the unit sphere. We have

$$\dim \mathcal{H}_{n,d} = \dim \mathcal{Y}_{n,d} = M(n,d) = \binom{d+n-1}{n} - \binom{d+n-3}{n-2} = \frac{2n+d-2 \binom{n+d-2}{d-2}}{n+d-2} \sim 2n^{d-2} \frac{n^2}{(d-2)!} \quad \text{as} \quad n \to \infty.$$ 

Observe that

$$M(0,d) = 1, \quad M(1,d) = d, \quad M(2,d) = \binom{d+1}{2} - 1. \quad (B.1)$$

The space $\mathcal{Y}_{n,d}$ is the eigenspace of the Laplace operator on $S^{d-1}$ corresponding to the eigenvalue $\lambda_n(d) = n(n+d-2)$.

We want to describe an inductive construction of an orthonormal basis of $\mathcal{Y}_{n,d}$. We start with the case $d = 2$. For any $m \in \mathbb{Z}$, we set

$$\varphi_m(\theta) = \begin{cases} \cos(m\theta), & m \leq 0 \\ \sin(m\theta), & m > 0 \end{cases}, \quad t_m = \|\varphi_m\|_{L^2} = \begin{cases} (2\pi)^{1/2}, & m = 0 \\ \pi^{1/2}, & m > 0 \end{cases}, \quad \Phi_m = \frac{1}{t_m} \varphi_m.$$ 

Then $\mathcal{B}_{0,2} = \{\Phi_0\}$ is an orthonormal basis of $\mathcal{Y}_{0,2}$, while $\mathcal{B}_{n,2} = \{\Phi_{-n}, \Phi_n\}$ is an orthonormal basis of $\mathcal{Y}_{n,2}, n > 0$.

Assuming now that we have produced orthonormal bases $\mathcal{B}_{n,d-1}$ of all the spaces $\mathcal{Y}_{n,d-1}$, we indicate how to produce orthonormal bases in the harmonic spaces $\mathcal{Y}_{n,d}$. This requires the introduction of the Legendre polynomials and their associated functions.

The Legendre polynomial $P_{n,d}(t)$ of degree $n$ and order $d$ is given by the Rodriguez formula

$$P_{n,d}(t) = (-1)^n R_n(d) (1 - t^2)^{-\frac{d-3}{2}} \left( \frac{d}{dt} \right)^n (1 - t^2)^{n + \frac{d-3}{2}}, \quad (B.2)$$

where $R_n(d)$ is the Rodriguez constant

$$R_n(d) = 2^{-n} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(n + \frac{d-1}{2})} = 2^{-n} \frac{1}{\left[n + \frac{d-1}{2}\right]_n},$$

where we recall that $[x]_k := x(x-1) \cdots (x-k+1)$. Equivalently, they can be defined recursively via the relations

$$P_{0,d}(t) = 1, \quad P_{1,d}(t) = t,$$

$$(n + d - 2) P_{n+1,d}(t) - (2n + d - 2) t P_{n,d}(t) + n P_{n-1,d}(t) = 0, \quad n > 0.$$ 

In particular, this shows that

$$P_{2,d}(t) = \frac{1}{d-1} (dt^2 - 1).$$

The Legendre polynomials are normalized by the equality

$$P_{n,d}(1) = 1, \quad \forall d \geq 2, \quad n \geq 0.$$
More generally, for any \( n > 0, d \geq 2 \), and any \( 0 < j \leq n \), we have
\[
P_{n,d}^{(j)}(1) = (-1)^n R_n(d) \left( \begin{array}{c} n+j \\ j \end{array} \right) \left\{ \frac{D_1(t)(1-t)^{n+d-3/2}}{(1-t)^{d-3/2}} - \frac{D_1^j(t)(1+t)^{n+d-3/2}}{(1+t)^{d-3/2}} \right\} \bigg|_{t=1}, \quad D_t := \frac{d}{dt},
\]
\[
= 2^{n-j} R_n(d) \left( \begin{array}{c} n+j \\ j \end{array} \right) \left[ n + \frac{d-3}{2} \right]_n \left[ n + \frac{d-3}{2} \right]_j,
\]
which implies
\[
P_{n,d}^{(j)}(1) = 2^{-j} \left( \begin{array}{c} n+j \\ j \end{array} \right) \left[ n + \frac{d-3}{2} \right]_j.
\] (B.3)

For any \( d \geq 3, n \geq 0 \) and \( 0 \leq j \leq n \), we define the \textit{normalized associated Legendre functions}
\[
\hat{P}_{n,d}^{j}(t) := C_{n,j,d} (1 - t^2)^j / 2^{j} P_{n,d}^{(j)}(t),
\]
where
\[
C_{n,j,d} := \frac{[n + d - 3]!_d - 3}{\Gamma(d - \frac{1}{2})} \frac{(2n + d - 2)}{2^{d-2} [n + d + j - 3]_{2j + d - 3}} \frac{1}{(n + j)!}.
\] (B.4)

When \( d = 3 \), the above formulae take the form
\[
\hat{P}_{n,3}^{j}(t) = \sqrt{\frac{(n + \frac{1}{2})!(n - j)!}{(n + j)!}} (1 - t^2)^j P_{n,3}^{(j)}(t).
\] (B.5)

For any \( 0 \leq j \leq n \), and any \( d > 2 \) we define a linear map
\[
\mathcal{T}_{n,j,d} : y_{j,d-1} \rightarrow y_{n,d}, \quad Y \mapsto \mathcal{T}_{n,j,d}[Y],
\]
\[
\mathcal{T}_{n,j,d}[Y](x) = \hat{P}_{n,d}^{j}(x_d) \cdot Y \left( \frac{1}{\|x'\|} x' \right), \quad \forall x \in S^{d-1}, x' = (x_1, \ldots, x_{d-1}) \neq 0.
\]

Note that for \( x = (x', x_d) \in S^{d-1} \) we have
\[
\|x'\| = (1 - x_d)^{1/2} \quad \text{and} \quad \hat{P}_{n,d}^{j}(x_d) = C_{n,j,d} (1 - x_d^2)^j / 2^{j} P_{n,d}^{(j)}(x_d) = C_{n,j,d}\|x'\| / \|P_{n,d}^{(j)}(x_d),
\]
so that
\[
\mathcal{T}_{n,j,d}[Y](x) = C_{n,j,d} \hat{P}_{n,d}^{j}(x_d) \tilde{Y}(x'), \quad \forall x = (x', x_d) \in S^{d-1},
\]
where \( \tilde{Y} \) denotes the extension of \( Y \) as a homogeneous polynomial of degree \( j \) in \( (d - 1) \)-variables. The sets \( \mathcal{T}_{n,j,d}[B_{j,d-1}] \), \( 0 \leq j \leq n \) are disjoint, and their union is an orthonormal basis of \( y_{n,d} \) that we denote by \( B_{n,d} \).

The space \( y_{0,d} \) consists only of constant functions and \( B_{0,d} = \{ \sigma_{d-1}^{-\frac{1}{2}} \} \). The orthonormal basis \( B_{1,d} \) of \( y_{1,d} \) obtained via the above inductive process is
\[
B_{1,d} = \{ C_0 x_i, \quad 1 \leq i \leq d \} = \{ \sigma_{d-2}^{-\frac{1}{2}} C_{1,0,d} x_i; \quad 1 \leq i \leq d \}. \quad \text{(B.6)}
\]

The orthonormal basis \( B_{2,d} \) of \( y_{2,d} \) is
\[
C_1 (dx_i^2 - r^2), \quad 1 \leq i < d, \quad C_2 x_i x_j, \quad 1 \leq i < j \leq d, \quad \text{(B.7)}
\]
where \( r^2 = x_1^2 + \cdots + x_d^2 \), and the positive constants \( C_0, C_1, C_2 \) are found from the equalities
\[
C_0^2 \int_{S^{d-1}} x_1^2 |dS(x)| = C_1^2 \int_{S^{d-1}} (d^2 x_1^2 - 2dx_1^2 + 1) |dS(x)| = C_2^2 \int_{S^{d-1}} x_1^2 x_2^2 |dS(x)| = 1,
\]
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aided by the classical identities, \([27, \text{Lemma 9.3.10}],\)

\[
\int_{S^{d-1}} x_1^{2h_1} \cdots x_d^{2h_d} |dS(x)| = \frac{2\Gamma(\frac{2h_1+1}{2}) \cdots \Gamma(\frac{2h_d+1}{2})}{\Gamma(\frac{2h+d}{2})}, \quad h = h_1 + \cdots + h_d.
\] (B.8)

APPENDIX C. INVARIANT INTEGRALS OVER THE SPACE OF SYMMETRIC MATRICES

In the main body of the paper we encountered many integrals of the form

\[
\int_{\text{Sym}_N} |\det A||d\gamma(A)|,
\]

where \(\text{Sym}_N\) is the space of symmetric \(N \times N\) matrices, and \(\gamma\) is a Gaussian probability measure on \(\text{Sym}_N\). In this appendix, we want show that in certain cases we can reduce this integral to an integral over a space of much lower dimension using a basic trick in random matrix theory. We set

\[
D_N := \dim \text{Sym}_N = \left(\begin{array}{c} N + 1 \\ 2 \end{array}\right), \quad \text{Sym}_N^0 := \{ S \in \text{Sym}_N; \ tr S = 0 \}.
\]

Note first that we have a canonical \(O(N)\)-invariant metric \(g_*\) on \(\text{Sym}_N\) with norm \(|-|_*\) given by

\[
|A|_* := \left(\tr A^2\right)^{1/2}.
\]

Using the canonical basis of \(\mathbb{R}^k\) we can describe each \(A \in \text{Sym}_N\) as a linear combination

\[
A = \sum_{i \leq j} a_{ij} H_{ij},
\]

where \(H_{ij}\) is the symmetric \(k \times k\) matrix whose \((i, j)\) and \((j, i)\) entries are 1, while the remaining entries are 0. With respect to the coordinates \((a_{ij})\) we have

\[
g_* = \sum da_i^2 + 2 \sum da_{ij}^2.
\]

The collection \((H_{ij})_{1 \leq i \leq j \leq N}\) is an orthonormal basis with respect to the metric \(g_*\). The volume density \(|dV|_*\) determined by the metric \(g_*\) has the description

\[
|dV|_* = 2^{\frac{D_N-N}{2}} \prod_{i \leq j} \left|da_{ij}\right|.
\]

Via the metric on \(\mathbb{R}^N\) we can identify \(\text{Sym}_N\) with the vector space of homogeneous polynomials of degree 2 in \(N\)-variables. More precisely, to such a polynomial \(P\) we associate the matrix \(\text{Hess}(P)\), the Hessian of \(P\) at the origin. The subspace \(\text{Sym}_N^0\) corresponds to the space \(\mathcal{H}_{2,N}\) of homogeneous, harmonic polynomials of degree 2 on \(\mathbb{R}^N\).

The orthogonal group \(O(N)\) acts by conjugation on \(\text{Sym}_N\), and \(\text{Sym}_N\) decomposes into irreducible components

\[
\text{Sym}_N = \mathbb{R} \langle 1_N \rangle \oplus \text{Sym}_N^0,
\]

where \(\mathbb{R} \langle 1_N \rangle\) denotes the one-dimensional space spanned by the identity matrix \(1_N\).

We fix an \(O(N)\)-invariant metric on \(\text{Sym}_N\). The irreducibility of \(\text{Sym}_N^0\) implies that such a metric is uniquely determined by two constants \(a, b > 0\) so that the collection

\[
a1_N, \quad b\text{Hess}(Y), \quad Y \in B_{2,N}
\]

is an orthonormal basis. We denote by \(|-|_{a,b}\) the norm of this metric. We want to express \(|A|_{a,b}\) in terms of \(\tr A^2\) and \((\tr A)^2\).
Note first that
\[ |\mathbb{1}_N|_{a,b}^2 = \frac{1}{a^2} = \frac{1}{Na^2} |\mathbb{1}_N|_{\star}^2. \]
The irreducibility of \( \text{Sym}_k^0 \) implies that there exists a universal constant \( R = R_N > 0 \) such that for any homogeneous harmonic polynomial \( P \) of degree 2 in \( N \) variables we have
\[ |\text{Hess}(P)|_{\star}^2 = R^2 \int_{S^{N-1}} P(x)^2 |dS(x)|. \]
If we take \( P = x_1 x_2 \), we deduce
\[ 2 = R^2 \int_{S^{N-1}} x_1^2 x_2^2 |dS(x)|, \]
and using (B.8), we deduce
\[ R^2 = \frac{\Gamma(N+1)}{\Gamma(3/2)^2 \Gamma(1/2)^{N-2}} = \frac{4 \Gamma(N+4)}{\pi^{N/2}}. \tag{C.1} \]
We see that for any \( P \in \mathcal{H}_{2,N} \)
\[ |Y|_{a,b}^2 = \frac{1}{b^2} \|Y\|_{L^2(S^{N-1})}^2 = \frac{1}{b^2 R^2} |\text{Hess}(Y)|_{\star}^2. \]
In particular, we deduce that
\[ | - |_{\star} = | - |_{a_*, b_*}, \ a_* = \frac{1}{N}, \ b_* = \frac{1}{R}. \]
In general, if \( A \in \text{Sym}_N \), then we have a decomposition
\[ A = \frac{1}{N} (\text{tr} A) \mathbb{1}_N + \left( A - \frac{1}{N} (\text{tr} A) \mathbb{1}_N \right) \]
that is orthogonal with respect to both \( | - |_{\star} \) and \( | - |_{a,b} \). We deduce
\[ |A|_{a,b}^2 = \frac{1}{N} (\text{tr} A) \mathbb{1}_N \bigg|_{a,b}^2 + \left( A - \frac{1}{N} (\text{tr} A) \mathbb{1}_N \right) \bigg|_{a,b}^2 \]
\[ = \frac{1}{N^2 a^2} (\text{tr} A)^2 + \frac{1}{b^2 R^2} \text{tr} \left( A^2 - \frac{2}{N} (\text{tr} A) A + \frac{1}{N^2} (\text{tr} A)^2 \mathbb{1}_N \right) \]
\[ = \frac{1}{N} \left( \frac{1}{Na^2} - \frac{1}{b^2 R^2} \right) (\text{tr} A)^2 + \frac{1}{b^2 R^2} \text{tr} A^2. \tag{C.2} \]
Note that the quantities \( \alpha, \beta \) depend on \( a, b \) and the dimension \( N \).
If \( |dV_{\star}| \) denotes the volume density determined by the metric \( | - |_{\star} \) and \( |dV_{a,b}| \) denotes the volume density associated to the metric \( | - |_{a,b} \), then we have
\[ |dV_{a,b}| = C_N(a,b) |dV_{\star}|, \ C_N(a,b) := \frac{1}{a^{N/2} (b R)^{D_N-1}}. \tag{C.3} \]
Suppose now that \( f : \text{Sym}_N \to \mathbb{R} \) is a continuous \( O(N) \)-invariant function that is homogeneous of degree \( \ell > 0 \). We want to find a simpler expression for the integral
\[ J_{a,b}(f) := \int_{\text{Sym}_N} e^{-|A|_{a,b}^2} f(A) |dV_{a,b}(A)|. \]
This can be reduced to a situation frequently encountered in random matrix theory. We have
\[ J_{a,b}(f) = C_N(a,b) \int_{\text{Sym}_N} e^{-\alpha |A|_{a,b}^2} f(A) |d\nu_{a,b}(A)| = C_N(a,b) \int_{\text{Sym}_N} e^{-\alpha \text{tr} A^2 - \beta (\text{tr} A)^2} f(A) |d\nu_{\star}(A)| \]
where \( \alpha, \beta \) are universal constants.
\[ A = \frac{1}{\sqrt{2\alpha}} B \]

\[ = \frac{\pi^N D_N}{\alpha^{D_N}} C_N(a, b) \int_{\text{Sym}_N} \frac{e^{-\frac{\alpha}{2\sqrt{2\alpha}} (\text{tr } B)^2 f \left( \frac{1}{\sqrt{2\alpha}} B \right)}}{(2\pi)^{D_N}} dV_\star \]

Observe that the function \( B \mapsto \Phi_{\alpha, \beta}(B) \) is also \( O(N) \) invariant. We denote by \( \mathcal{D}_N \subset \text{Sym}_N \) the subspace consisting of diagonal matrices. We identify in particular, we have a special case when the multidimensional integral \([23, \text{Chap. 6}], [12, \text{Prop. 4.1.1}] \) or \([12, \text{Thm. 2.50}] \) we deduce that

\[ \int_{\text{Sym}_N} \Phi_{\alpha, \beta}(B) \frac{e^{-\frac{1}{2} \text{tr } B^2}}{(2\pi)^{D_N}} dV_\star \]

where

- \( \Delta(x) \) is the discriminant \( \Delta(x_1, \ldots, x_N) = \prod_{1 \leq i < j \leq N} (x_i - x_j) \)
- The constant \( Z_N \) is given by the integral

\[ Z_N = \int_{\mathcal{D}_N} |\Delta(B)| \cdot \frac{e^{-\frac{1}{2} \text{tr } B^2}}{(2\pi)^{D_N}} dV(B) = (2\pi)^{-\frac{D_N - N}{2}} \int_{\mathbb{R}^N} |\Delta(x)| \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{N}{2}}} dV(x). \]

Putting together all of the above, we deduce

\[ J_{a,b}(f) = \frac{\pi^{\frac{D_N}{2}} C_N(a, b)}{Z_N \alpha^{D_N}} \int_{\mathbb{R}^N} e^{-\frac{1}{2} |x|^2 - \frac{\beta}{2\alpha}(x_1 + \cdots + x_N)^2} f \left( \frac{x}{\sqrt{2\alpha}} \right) |\Delta(x)| |dV(x)|. \]

In particular, we have

\[ \int_{\text{Sym}_N} e^{-|A|^{2,b}_\alpha} |\det(A)| |dV_{a,b}(A)| \]

\[ = \frac{\pi^{\frac{D_N}{2}} C_N(a, b)}{Z_N (2\alpha)^{\frac{D_N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2} - \frac{\beta}{2\alpha} (\sum_{i=1}^N x_i)^2} \prod_{i=1}^N |x_i| |\Delta(x)| |dV(x)|, \]

where \( \alpha, \beta \) are defined by \((\text{C.2})\) and \( C_N(a, b) \) by \((\text{C.3})\).

Let us point out that, up to a universal multiplicative constant, the measure \( e^{-\frac{1}{2} \text{tr } A^2} |dV_\star(A) \) is the probability distribution of the real gaussian ensemble, \([2, 12]\). As explained in \([12, \text{Chap. 3}]\), the multidimensional integral \((\text{C.5})\) can be reduced to computations of 1-dimensional integrals in the special case when \( \beta = 0 \), i.e., \( k\alpha^2 = b^2R^2 \). As explained in \([17, \S 1.5], [18]\), the case \( \beta < 0 \) can be reduced to computations of 1-point correlations of the Gaussian ensemble of \((k + 1) \times (k + 1)\)-matrices. In turn, these can be reduced to computations of 1-dimensional integrals \([12, \S 4.4], [17, \text{Chap. 6}], [23, \text{Chap. 7}]\).

**Appendix D. Some elementary estimates**

**Proof of Lemma 6.2.** Consider the complex valued random process

\[ \mathcal{F}_\nu(t) := \frac{1}{\sqrt{\pi \nu^3}} \sum_{m=1}^\nu m z_m e^{i\nu t}, \quad z_m = c_m - id_m. \]
The covariance function of this process is
\[
\mathcal{R}_\nu(t) = E(\mathcal{F}_\nu(t)\mathcal{F}_\nu(0)) = \frac{1}{\pi \nu^3} \sum_{m=1}^\nu m^2 e^{\text{int} \frac{m}{\nu}}.
\]
Observe that \(\text{Re} \mathcal{F}_\nu = \Phi_\nu\). Note that the spectral measure of the process \(\mathcal{F}_\nu\) is
\[
d\sigma_\nu = \frac{1}{\pi \nu^3} \sum_{m=1}^\nu m^2 \delta_{\frac{m}{\nu}},
\]
where \(\delta_{t_0}\) denotes the Dirac measure on \(\mathbb{R}\) concentrated at \(t_0\). We form the covariance matrix of the gaussian vector valued random variable
\[
\begin{bmatrix}
\mathcal{F}_\nu(0) \\
\mathcal{F}_\nu(t) \\
\mathcal{F}'_\nu(0) \\
\mathcal{F}'_\nu(t)
\end{bmatrix} 
\rightarrow 
\begin{bmatrix}
\mathcal{R}_\nu(0) & \overline{\mathcal{R}_\nu(t)} & \mathcal{R}_\nu(0) & \overline{\mathcal{R}_\nu(t)} \\
\mathcal{R}_\nu(t) & \mathcal{R}_\nu(0) & -i\overline{\mathcal{R}'_\nu(t)} & -i\mathcal{R}_\nu(0) \\
\mathcal{R}'_\nu(0) & i\overline{\mathcal{R}'_\nu(t)} & \mathcal{R}_\nu''(0) & \overline{\mathcal{R}_\nu''(0)} \\
\mathcal{R}'_\nu(t) & i\mathcal{R}_\nu'(0) & -\mathcal{R}_\nu''(t) & -\overline{\mathcal{R}_\nu''(t)}
\end{bmatrix}.
\]
Observe that \(\text{Re} \mathcal{X}(t) = \text{Re} \Xi(t)\). If we let
\[
\mathcal{z} = \begin{bmatrix}
u_0 \\
v_0 \\
u_1 \\
u_1
\end{bmatrix} \in \mathbb{C}^4
\]
Then, as in [9, Eq. (10.6.1)] we have
\[
\langle \mathcal{X}_\nu \mathcal{z}, \mathcal{z} \rangle = \frac{1}{\pi \nu^3} \sum_{m=1}^\nu m^2 \left| u_0 + v_0 e^{\frac{\text{int}}{\nu}} + i \frac{m}{\nu} \left( u_1 + v_1 e^{\frac{\text{int}}{\nu}} \right) \right|^2
\]
We see that
\[
\langle \mathcal{X}_\nu \mathcal{z}, \mathcal{z} \rangle = 0 \iff \left( u_0 + v_0 e^{\frac{\text{int}}{\nu}} + i \frac{m}{\nu} \left( u_1 + v_1 e^{\frac{\text{int}}{\nu}} \right) \right) = 0, \quad \forall m = 1, \ldots, \nu.
\]
We see that if the linear system (D.1) has a nontrivial solution \(\mathcal{z}\) then the complex \(4 \times 4\) matrix
\[
A_\nu(t) := \begin{bmatrix}
1 & \zeta & 1 & \zeta \\
1 & \zeta^2 & 2 & 2\zeta^2 \\
1 & \zeta^3 & 3 & 3\zeta^3 \\
1 & \zeta^4 & 4 & 4\zeta^4 \\
\end{bmatrix}, \quad \zeta = e^{\frac{\text{int}}{\nu}},
\]
must be singular, i.e., \(\det A_\nu(t) = 0\). We have
\[
\det A_\nu(t) = \det \begin{bmatrix}
1 & \zeta & 1 & \zeta \\
0 & \zeta^2 - \zeta & 1 & 2\zeta^2 - z \\
0 & \zeta^3 - \zeta & 2 & 3\zeta^3 - z \\
0 & \zeta^4 - z & 3 & 4\zeta^4 - z \\
\end{bmatrix} = \zeta^2 \det \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & \zeta - 1 & 1 & 2\zeta - 1 \\
0 & \zeta^2 - 1 & 2 & 3\zeta^2 - 1 \\
0 & \zeta^3 - 1 & 3 & 4\zeta^3 - 1 \\
\end{bmatrix}
\]
\[
= \zeta^2 \det \begin{bmatrix}
\zeta - 1 & 1 & 2\zeta - 1 \\
\zeta^2 - 1 & 2 & 3\zeta^2 - 1 \\
\zeta^3 - 1 & 3 & 4\zeta^3 - 1 \\
\zeta^4 - z & 3 & 4\zeta^4 - z \\
\end{bmatrix} = \zeta^2 \det \begin{bmatrix}
\zeta - 1 & 1 & \zeta \\
\zeta^2 - 1 & 2 & 2\zeta^2 \\
\zeta^3 - 1 & 3 & 3\zeta^3 \\
\zeta^4 - z & 3 & 4\zeta^4 - z \\
\end{bmatrix}
\]
\[
= \zeta^3 \det \begin{bmatrix}
\zeta - 1 & 1 & 1 \\
\zeta^2 - 1 & 2 & 2\zeta \\
\zeta^3 - 1 & 3 & 3\zeta^2 \\
\zeta^4 - z & 3 & 4\zeta^3 - z \\
\end{bmatrix} = \zeta \det \begin{bmatrix}
\zeta - 1 & 1 & 0 \\
\zeta^2 - 1 & 2 & 2\zeta^2 - 2 \\
\zeta^3 - 1 & 3 & 3\zeta^3 - 3 \\
\zeta^4 - z & 3 & 4\zeta^3 - z \\
\end{bmatrix}
\]
Since \(|\zeta| = 1\), we have that \(\det A_r(t) = 0\) if and only if \(t \in 2\pi \nu \mathbb{Z}\). \(\square\)

**Proof of Lemma 6.6.** Recall that \(\theta := \frac{t}{2\nu}\), \(f(\theta) := \frac{\sin \theta}{\theta}\). By (6.10) we have

\[
\frac{t^{r+1}}{i^{r+1} \nu^{r+1}} D_{\nu,r}(t) = r! \left( \frac{2 \sin \left( \frac{(\nu+1)t}{2\nu} \right)}{f(\theta)^{r+1}} \cdot e^{i\frac{(\nu+1)t}{2\nu}} - e^{it} \sum_{j=1}^{r} j^{r-j} \frac{1}{\nu^j} \cdot \left( e^{\frac{i\theta}{f(\theta)}} \right)^{r-j} \right).
\]

Using (6.13) we deduce that

\[
\left| \frac{t^{r+1}}{i^{r+1} \nu^{r+1}} D_{\nu,r}(t) - \frac{1}{i} I_r(t) \right| \leq 2r! t^{r+1} \left| \frac{\sin \left( \frac{(\nu+1)t}{2\nu} \right)}{f(\theta)^{r+1}} e^{i\frac{t}{2}} - \sin \left( \frac{t}{2} \right) \right|
\]

\[
+ r! \sum_{j=1}^{r} j^{r-j} \left( \frac{e^{\frac{i\theta}{f(\theta)}}}{f(\theta)} \right)^{r-j} - \frac{1}{j!}.
\]

In the sequel we will use Landau’s symbol \(O\). These implied constants will be independent of \(\nu\). Also we will denote by the same symbol \(C_r\) constants independent of \(\nu\) put possibly dependent on \(r\). Throughout we assume \(0 < t \leq \pi \nu\). Then \(0 < \theta < \frac{\pi}{2}\) and for \(0 \leq j \leq r\) we have

\[
e^{i\theta} = 1 + O(\theta), \quad \frac{\left( \frac{\nu+1}{\nu} \right)}{\nu^j} = 1 + O\left( \frac{1}{\nu} \right), \quad \sin \left( \frac{(\nu+1)t}{2\nu} \right) = \sin \left( \frac{t}{2} \right) + O(\theta),
\]

\[
e^{\frac{i\theta}{f(\theta)}} = \frac{\theta \cos \theta + i \sin \theta}{\sin \theta} = 1 + O(\theta).
\]

Hence

\[
\left| \sin \left( \frac{(\nu+1)t}{2\nu} \right) e^{i\frac{t}{2}} - \sin \left( \frac{t}{2} \right) \right| = O(\theta), \quad (D.3)
\]

while for any \(1 \leq j \leq r\) we have

\[
\left| \frac{\left( \frac{\nu+1}{\nu} \right)}{\nu^j} \left( e^{\frac{i\theta}{f(\theta)}} \right)^{r-j} - \frac{1}{j!} \right| = O \left( \frac{1}{\nu} + \theta \right). \quad (D.4)
\]

Using (D.3) and (D.4) in (D.2) we deduce that

\[
\left| \frac{t^{r+1}}{i^{r+1} \nu^{r+1}} D_{\nu,r}(t) - \frac{1}{i} I_r(t) \right| = O \left( \theta + \left( \theta + \frac{1}{\nu} \sum_{j=1}^{r-1} t^j \right) \right) = O \left( \frac{1}{\nu} \sum_{j=1}^{r+1} t^j \right).
\]
Hence
\[
\left| \frac{1}{i^r} D_{t^r} I_r(t) - \frac{1}{i^r} I_{t^r}(t) \right| = O \left( \frac{1}{t^{r+1}} \right).
\]

\[\square\]

**Proof of Lemma 6.7.** We have
\[
R^{(k)}_{\infty}(t) = \pm \frac{1}{t^{k+3}} \int_0^t \tau^{k+2} u(\tau) d\tau, \quad u(\tau) = \begin{cases} \sin \tau, & k \in 1 + 2\mathbb{Z} \\ \cos \tau, & k \in 2\mathbb{Z}. \end{cases}
\]

Note that
\[
|R^{(k)}_{\infty}(0)| = \frac{1}{k+3} = \frac{1}{t^{k+3}} \int_0^t \tau^{k+2} d\tau.
\]

The inequality (6.19a) now follows from the inequality $|u(\tau)| \leq 1, \forall \tau$.

For any positive integer $r$ we denote by $j_r$ the $r$-th jet at 0 of a one-variable function. We can rewrite (6.13) as follows:
\[
\frac{1}{i^r} I_r(t) = r! \left( 2 \sin \left( \frac{t}{2} \right) e^{it} - ie^{it} \sum_{j=1}^r (-i)^j j! \right) = r! \left( 2 \sin \left( \frac{t}{2} \right) e^{it} - ie^{it} \cdot j_r(e^{-it} - 1) \right)
\]
\[
= r! \left( 2 \sin \left( \frac{t}{2} \right) e^{it} + ie^{it} - ie^{it} \cdot j_r(e^{-it}) \right)
\]
\[
= r! \left( -i(e^{it} - e^{-it}) e^{it} + ie^{it} - ie^{it} \cdot j_r(e^{-it}) \right) = ir! \left( 1 - e^{it} \cdot j_r(e^{-it}) \right)
\]

Hence
\[
\text{Re} \left( \frac{1}{i^r} I_r(t) \right) = \text{Im} \left( e^{it} \cdot j_r(e^{-it}) \right) \quad \text{and} \quad \frac{1}{t^{r+1}} I_r(t) = O(t^{-1}) \quad t \to \infty.
\]

This proves (6.19b).

The spectral measure
\[
d\sigma_\nu = \frac{1}{\pi \nu^3} \sum_{m=1}^{\nu} m^2 \delta_{\frac{m}{\nu}}
\]

of the process $\mathcal{F}_\nu$ converges weakly as $\nu \to \infty$ to the measure
\[
d\sigma_\infty = \frac{1}{\pi} \chi_{[0,1]} t^2 dt,
\]

where $\chi_{[0,1]}$ denotes the characteristic function of $[0, 1]$. Indeed, an argument identical to the one used in the proof of Lemma 6.4 shows that for every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ we have
\[
\lim_{\nu \to \infty} \int_{\mathbb{R}} f(t) d\sigma_\nu(t) = \int_{\mathbb{R}} f(t) d\sigma_\infty.
\]

The complex valued stationary Gaussian process $\mathcal{F}_\infty$ on $\mathbb{R}$ with spectral measure $d\sigma_\infty$ has covariance function
\[
\mathcal{R}_\infty = \frac{1}{\pi} \int_0^1 i^2 e^{it} dt.
\]

Note that $\text{Re} \mathcal{R}_\infty = R_\infty$. The results in [9, §10.6] show that the covariance matrix
\[
\mathcal{X}_\infty = \begin{bmatrix} R_\infty(0) & \overline{R}_\infty(t) & \overline{R}_\infty(0) & \overline{R}_\infty(t) \\ R_\infty(t) & R_\infty(0) & -iR_\infty'(t) & -iR_\infty'(0) \\ R_\infty'(0) & iR_\infty'(t) & -R_\infty'(0) & -R_\infty'(t) \\ R_\infty'(t) & -iR_\infty'(0) & -R_\infty'(t) & -R_\infty'(0) \end{bmatrix} = \lim_{\nu \to \infty} \mathcal{X}_\nu,
is nondegenerate. The equality \( \det \Re \mathcal{X}_\infty(t) \neq 0, \forall t \in \mathbb{R} \) implies as in Remark 6.3 that \( \mu_\infty(t) \neq 0, |\rho_\infty(t)| < 1, \forall t \in \mathbb{R} \), where

\[
\mu_\infty = \frac{(\lambda_0^2 - R_\infty^2)\lambda_2 - \bar{\lambda}_0 R_\infty'}{\lambda_0^2 - R_\infty^2}, \quad \rho_\infty = \frac{R_\infty''(\lambda_0^2 - R_\infty^2) + (R_\infty')^2 R_\infty}{(\lambda_0^2 - R_\infty^2)\lambda_2 - \lambda_0 (R_\infty')^2}.
\]

This proves (6.19c). The equality (6.19d) follows from the Taylor expansion of \( R_\infty \).

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