Geometric generalizations in Kresin-Maz’ya Sharp Real-Part Theorems *†

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Abstract
In the present article we give geometric generalizations of the estimates from Chapters 5, 6, 7 from [4], while extending their sharpness to new cases.

1 Preliminaries

G.Kresin and V.Maz’ya, in their recently published, remarkable research monograph [4], have collected in one place generalizations and different modifications of theorems, which the authors called real-part theorems honoring the known theorem of Hadamard (1892). All of them are formulated with sharp constants.

In the present article, our starting point is the content of the three chapters of the above monograph, namely, Chap.5, Estimates for the derivatives of analytic functions, Chap.6, Bohr’s type real estimates, Chap.7, Estimates for the increment of derivatives of analytic functions. Methods, used in [1], allow a geometric generalization for some of the results in those chapters and the sharpness of the results is

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extended to new cases. We remark, that there are two methods to prove the sharpness of the corresponding estimates. The first one, which is shorter, uses a number of facts from the monograph [4], the second, somewhat longer, but independent of [4]. To illustrate the point we use the approach in §2, and the second one in §3.

2 Estimates for the derivatives of holomorphic functions

We begin by formulating a theorem, inspired by Th.5.1, from [4]

**Theorem 2.1** (G.Kresin-V.Maz’ya) Let \( f \) be holomorphic in \( D_R = \{ z \in \mathbb{C} : |z| < R \} \) and let \( z = re^{i\theta} \), \( a = r_a e^{i\theta} \), \( 0 \leq r_a \leq r < R \). Then the inequality

\[
|f^{(n)}(z)| \leq \frac{2n!R(R - r_a)}{(R - r)^{n+1}(R + r_a)}Q_a(f)
\]

holds for every \( n \geq 1 \) with the best possible constant, where \( Q_a(f) \) is each of the following expressions

i) \( \sup_{|\zeta| < R} \Re f(\zeta) - \Re f(a) \). In this case the claim of the Theorem 2.1 is a generalization of the Hadamard real-part theorem.

ii) \( \sup_{|\zeta| < R} |\Re f(\zeta)| - |\Re f(a)| \). In this case the claim of the Theorem 2.1 is a generalization of the Landau type inequality.

ii) \( \sup_{|\zeta| < R} |f(\zeta)| - |f(a)| \). In this case the claim of the Theorem 2.1 is a generalization of the Landau inequality.

iv) \( \Re f(a) \), if \( \Re f > 0 \) on \( D_R \). In this case the claim of the Theorem 2.1 is a generalization of the Caratheodory inequality.

We remark that the estimate (2.1) has a meaning in the cases (i), (ii), (iii) only if the corresponding \( \sup \) is finite. In the case \( a = z \) the inequalities are due to S.Ruscheweyh [5]. Other references on different type of inequalities preceding the inequality (2.1) are to be found in [4].

Denote now by \( \tilde{G} \) the convex hull of the domain \( G \subset \mathbb{C} \). A point \( p \in \partial G \) is called a point of convexity if \( p \in \partial \tilde{G} \). A point of convexity \( p \) is called regular if there exists a disk \( D' \subset G \) so that \( p \in D' \).

**Theorem 2.2** Let \( f \) be holomorphic function in \( D_R \), \( f(D_R) \subset G \), where \( G \) is a domain \( \mathbb{C} \), so that \( \tilde{G} \neq \mathbb{C} \). Let also \( z = re^{i\theta} \), \( a = r_a e^{i\theta} \) be complex numbers so that \( 0 \leq r_a \leq r < R \). Then the inequality

\[
|f^{(n)}(z)| \leq \frac{2n!R(R - r_a)}{(R - r)^{n+1}(R + r_a)} \text{dist}(f(a), \partial \tilde{G})
\]

holds for \( n \geq 1 \). If \( \partial \tilde{G} \) contain at least one regular point of convexity, then the constant in (2.2) is sharp.
Remark 2.1 We point out that the cases (i) and (ii) in Theorem 2.1 are the cases when
\[ G = \{ z \in \mathbb{C} : \Re z < \sup_{|\zeta|<R} \Re f(\zeta) \} \]
is half-plane or
\[ G = \{ z \in \mathbb{C} : |z| < \sup_{|\zeta|<R} |f(\zeta)| \} \]
is a strip. The case (iii) in Theorem 2.1 corresponds to the set
\[ G = \{ z \in \mathbb{C} : |z| < \sup_{|\zeta|<R} |f(\zeta)| \} \]
being a disc. The case (iv) of this theorem corresponds to the right half-plane case
\[ G = \Pi = \{ z \in \mathbb{C} : \Re f(z) > 0 \}. \]

Proof: Let us consider the case (iv) in Theorem 2.1. If \( \Re f(z) > 0 \), \( z \) on \( D_R \), then for every \( n \geq 1 \) one has
\[
|f^{(n)}(z)| \leq \frac{2n!R(R - r_a)}{(R - r)^{n+1}(R + r_a)} \text{dist}(f(a), \partial \Pi), \tag{2.3}
\]
where \( \Pi \) is the right half-plane. By translation and rotation, this half plane can be transformed into any half-plane \( \Pi_1 \). The same transformations can be applied to holomorphic in \( D_R \) functions, that is \( f(z) \to (f(z) + c)e^{i\phi} \). Under such transformations of both, the half-plane and the functions, at the same time the conclusion (2.3) does not alter.

Now, let \( G \) be a domain in \( \mathbb{C} \), such that \( \tilde{G} \neq \mathbb{C} \). Let also the distance from the right hand-side of the inequality (2.2) is realized at the point \( p \in \partial \tilde{G} \). Then there exist a line of support to \( \tilde{G} \) at \( p \), bounding the half-plane \( \Pi_1 \), such that \( G \subset \Pi_1 \). For this particular half-plane \( \Pi_1 \) we apply the inequality (2.3) and using the fact
\[
\text{dist}(f(a), \partial \tilde{G}) = \text{dist}(f(a), \partial \Pi_1) = \text{dist}(f(a), p) = \text{dist}(f(a), \Pi_1),
\]
we obtain (2.2).

Assume now that \( \partial G \) contains at least one regular point of convexity \( p \in \partial G \cap \partial \tilde{G} \cap \partial D' \), where \( D' \) is some disc \( D' \subset G \). Let \( D_\beta \) be a disk, in which the functions from (iii) of Theorem 2.1 take their values:
\[
D_\beta = \{ z \in \mathbb{C} : |z| < \beta = \sup_{|\zeta|<R} |f(\zeta)| \}. \]
For this disc $D_\beta$ the constant in (2.1) is sharp. That is, there exists a family of functions, which we take from §5.7,4, 

$$g_\xi(z) = \frac{\xi}{z - \xi} + \frac{|\xi|^2}{|\xi|^2 - R^2},$$

(2.4) 

depending on complex parameter $\xi = \rho e^{i\phi}$, $\rho > R$, for which this constant is attained. Put $z = x < 0$, $\xi = \rho > 0$, the sharpness of the constant in [4] was proved by passing to the limit when $\rho \downarrow R$. Then the first summand in (2.4) is negative and remains bounded while $\rho \downarrow R$. The second summand in (2.4) tends to $+\infty$ while $\rho - \rightarrow R$. Hence, when $\rho$ is sufficiently close to $R$, the function $g_\rho(x)$ is positive. This implies that $g_\rho(a)$ is also positive.

Of crucial importance in our reasoning is the fact that one can transform the disk $D_\beta$ into the disc $D'$ by using homothety and translation. At the same time, we apply both transformations to the family of functions $g_\xi(z) - \rightarrow \tilde{g}_\xi(z) = \alpha g_\xi(z) + c$. Then the inequality (2.1) in the case (iii) of the Theorem 2.1 does not change. Furthermore, if the for the family of functions $\{g_\xi(z)\}_\xi$ the sharpness of the constant was attained before the applications of the transformations under the assumption $f(D_R) \subset D_\beta$, then the family of functions obtained after the transformation illustrates the sharpness of the constant under the assumption $f(D_R) \subset D'$.

In the §5.7 of [4], for the proof of the sharpness of the given constant only the modula of $|g_\xi(z)|$ and $|g_\xi(a)|$ were used. Therefore, instead of the family $\{g_\rho(z)\}_\rho$ one can use the family $\{e^{i\phi}g_\rho(z)\}_\rho$ in order to show the required sharpness of the constant, provided that $\phi$ is chosen in a such a manner that the point $e^{i\phi}g_\rho(a)$ on the radius emanating from the center of the disc $D_\beta$ to the point $p' \in \partial D_\beta$, where $p'$ is the pre-image of the point $p$ under the above mentioned homothety and translation of the disc $D_\beta$ into the disc $D'$. Then, after the transformation, the point $\tilde{g}_\rho(a)$ will lie on the radius emanating from the center of the disc $D'$ to the point $p$, and therefore

$$\text{dist}(\tilde{g}_\rho(a), \partial \tilde{G}) = \text{dist}(\tilde{g}_\rho(a), p).$$

This completes the proof of the theorem. ♦

For $a = 0$ one has the following

**Corollary 2.1** Let $f$ be holomorphic in the disc $D_R$ and $f(D_R) \subset G$, where $G$ is a domain in $\mathbb{C}$ whose convex hull $\tilde{G}$ is not equal to $\mathbb{C}$. Then the inequality

$$|f^{(n)}(z)| \leq \frac{2n!R}{(R - r)^{n+1}} \text{dist}(f(0), \partial \tilde{G})$$

(2.5) 

holds for every $n \geq 1$. If $\partial \tilde{G}$ contains at least one regular point of convexity, then the constant in (2.5) is sharp.
3 Bohr’s type real part estimates

The results, contained in the Theorems 6.1-6.4 in [4], are collected in the following

Theorem 3.1 (G.Kresin-V.Maz’ya) Let the function

\[ f(z) = \sum_{n=0}^{\infty} c_n z^n \]  

be holomorphic in the disc \( D_R \) and \( q > 0, m \geq 1, |z| = r < R \). Then the inequality

\[ \left( \sum_{n=m}^{\infty} |c_n z^n|^q \right)^{\frac{1}{q}} \leq \frac{2r_m}{R^{m-1}(R^q - r^q)^\frac{1}{q}} \mathcal{R}(f) \]  

holds with the best possible constant in the cases when \( \mathcal{R}(f) \) is each one of the following expressions:

i) \( \sup_{|\zeta|<R} (\Re(f(\zeta)) - \Re(f(0))) \).

ii) \( \sup_{|\zeta|<R} (|\Re(f(\zeta)| - |\Re(f(0))|) \).

iii) \( \sup_{|\zeta|<R} (|f(\zeta)| - |f(0)|) \).

iv) \( \Re(f(0)) \), if \( \Re(f) > 0 \) on \( D_R \).

We remark here that the case (iii) of the above theorem gives for \( m = q = 1 \) the classical theorem of Bohr, [2] (for related references see [1]) for \( r = R^\frac{3}{5} \). Similarly to the previous section, we state the geometric generalization of this theorem.

Theorem 3.2 Let the function (3.1) be holomorphic in the disc \( D_R \), \( q > 0, m \geq 1, |z| = r < R \), and \( f(D_R) \subset G \), where \( G \) be a domain in \( \mathbb{C} \) and \( \tilde{G} \neq \mathbb{C} \). Then the inequality

\[ \left( \sum_{n=m}^{\infty} |c_n z^n|^q \right)^{\frac{1}{q}} \leq \frac{2r_m}{R^{m-1}(R^q - r^q)^\frac{1}{q}} \text{dist}(c_0, \partial \tilde{G}), \]  

holds. If the boundary \( \partial G \) contains at least one regular point of convexity, then the constant in (3.3) is sharp.

Proof: The estimate (3.3) is proven in exactly the same way as in the previous section. The sharpness of the constant in (3.3) can also be proven in the same manner, but we prefer to give an independent proof.

The main point of our approach is the simple observation that convex hull of the domain between to discs, when the smaller one is contained in the larger one and their boundaries have exactly one common point, is the larger disc.
To be more specific, for \( a > 0 \) we consider
\[
D_1 = \{ z \in \mathbb{C} : |z - ai| < a \}
\]
\[
D_2 = \{ z \in \mathbb{C} : |z - 2ai| < 2a \}
\]
be two discs. It is clear that \( \partial D_1 \cap \partial D_2 = \{0\} \). Define the domain
\[
G = D_1^c \cap D_2,
\]
where, as usual, \( D_1^c \) denotes the complement of the disc \( D_1 \) in \( \mathbb{C} \).

It is obvious that the convex hull \( \tilde{G} \) of \( G \) is equal to \( D_2 \). The conformal map \( f(\zeta) = \frac{1}{\zeta} \) maps \( G \) onto a strip
\[
T = \{ z \in \mathbb{C} : -\frac{i}{2a} < \Im z < -\frac{i}{4a} \}
\]
The \( \partial T \) consists of two parallel lines, on which \( z \) moves in the opposite directions. The point \( z = \infty \) is a double point. The width of the strip is equal to \( \frac{1}{4a} \). The map \( w = f_1(z) = z + \frac{i}{2a} \) shifts the strip up. The new strip \( T + \frac{i}{2a} \) has the real axis as its lower bound, while the width remains the same. Then the map \( \omega = f_2(w) = e^{4a\pi w} \) transforms con-formally the strip onto the upper half plane
\[
H = \{ \omega \in \mathbb{C} : \Im \omega > 0 \}
\]
Finally, we transfer \( H \) by translation \( \psi = f_3(\omega) = \omega + p \), where \( p \in \mathbb{C}, \Im p < 0 \). All the above maps are invertible. Thus we have a map
\[
F : H + p \rightarrow G
\]
\[
F(\psi) = \frac{1}{\frac{4a\pi}{4a\pi} \ln(\psi - p) - \frac{i}{2a}},
\]
which is the inverse of the composition of \( f_i \). Its expansion in the disc \( D(0, |p|) \) is given by
\[
c_n = \frac{a_n - c_0 b_n - c_1 b_{n-1} - \cdots - c_{n-1} b_1}{b_0}, \text{ where}
\]
\[
a_n = 0, \ \forall n \geq 1, \ a_0 = 1,
\]
\[
b_n = \left. \left( \frac{1}{4a\pi} \ln(\psi - p) - \frac{i}{2a} \right)^{(n)} \right|_{\psi=0}, \ n \geq 1,
\]
and \( c_0 = F(0) \). Or equivalently,

\[
c_n = \frac{(-1)^n b^n_1}{b_0^{n+1}}, \text{ where } b_1 = \frac{1}{4a\pi (-p)}, \quad b_0 = \frac{1}{4a\pi} \ln(-p) - \frac{i}{2a}
\]

Thus

\[
c_n = \frac{4a\pi}{p^n(\ln(-p) - \frac{i}{2a})}, \quad n \geq 1
\]

If \( p = -i \) then \( c_0 = \frac{1}{4a\pi \ln(i) - \frac{i}{2a}} = \frac{8a}{3}i \) and hence

\[
\text{dist}(c_0, \tilde{G}) = \frac{4a}{3}
\]

Assume now, that the best constant in (3.3) is denoted by \( C(r) \). For \( |z| = r \), \( 0 < r < p \), one has

\[
\sum_{n=1}^{\infty} |c_n z^n|^q = \sum_{n=1}^{\infty} \left( \frac{4\pi a}{|\ln(-p) - \frac{i}{2a}|} \right)^q \frac{r^{mq} p^q}{p^{mq}(p^q - r^q)}
\]

The last power series is geometric one with ratio \( (\frac{r}{p})^q \). Therefore, for \( m \geq 1 \), one has

\[
\left( \frac{4\pi a}{|\ln(-p) - \frac{i}{2a}|} \right)^q \sum_{n=m}^{\infty} \left( \frac{r}{p} \right)^{mq} p^q
\]

Thus, the estimate from above is

\[
\frac{8a}{3} \frac{r^m}{p^{m-1}} \frac{1}{((p^q - r^q))^\frac{q}{2}} \leq C(r) \text{dist}(c_0, \tilde{G}) = C(r) \frac{4a}{3}
\]

or

\[
\frac{2r^m}{R^{m-1}((R^q - r^q))^\frac{q}{2}} \leq C(r)
\]

Thus the exactness is proven for \( m = 1, 2, \ldots \).

This particular example shows that in the case of bounded, simply connected domain \( G \), whose boundary contains a point of regular convexity, the constant in
The Theorem 3.2 is sharp. Actually if $\zeta_0$ is a point of regular convexity, then it means that $\zeta_0 \in \partial \tilde{G}$ and there is a disc $U$ of radius $\rho$, contained in $G$ so that $\partial U \cap \partial G = \{\zeta_0\}$. We inscribe in the disc $U$, the disc $U_1$ whose center lies in the diameter of $U$, whose end-points are $\zeta_0, \zeta'_0$. The radius of the disc $U_1$ is $\frac{1}{2}\rho$ and its center $k_1$ satisfies $|k_1 - \zeta'_0| = \frac{\rho}{2}$. Then for the domain $\mathcal{U} = U \cap U_1^c$ we repeat the construction of the above example. The key fact here is that $\tilde{\mathcal{U}} = U$ and that the distance $d(c_0, \tilde{\mathcal{U}})$ is realized at the point $\zeta_0 \in \tilde{G}$. So the sharpness of the constant is proven.$\Box$

Furthermore, the Theorem 6.5, [4], states

**Theorem 3.3** (G.Kresin-V.Maz'ya) Let $f(z)$ be a function holomorphic in the disc $\mathcal{D}_R$ and assume that in the neighborhood of the point $a \in \mathcal{D}_R$ the expansion

$$f(z) = \sum_{k=0}^{\infty} c_k(a)(z-a)^k$$

(3.4)

is valid. Then for every $z \in \mathcal{D}_R$, $|z-a| = r < d_a = \text{dist}(a, \partial \mathcal{D}_R)$ the following inequality

$$\sum_{k=1}^{\infty} |c_k(a)(z-a)^k| \leq \frac{2Rr}{(2R-d_a)(d_a-r)} Q_a(f)$$

holds with the best possible constant and where $Q_a(f)$ is each of the following expressions (i), (ii), (iii), (iv) from the Theorem 2.1.

Similarly one can prove

**Theorem 3.4** Let $f(z)$ be a function holomorphic in the disc $\mathcal{D}_R$ and assume that in the neighborhood of the point $a \in \mathcal{D}_R$ the expansion (3.4) is valid. Assume also that $f(\mathcal{D}_R) \subset G$, where $G$ is a domain in $\mathbb{C}$ such that $\tilde{G} \neq \mathbb{C}$. Then for every $z \in \mathcal{D}_R$, $|z-a| = r < d_a = \text{dist}(a, \partial \mathcal{D})$ the following inequality

$$\sum_{k=1}^{\infty} |c_k(a)(z-a)^k| \leq \frac{2Rr}{(2R-d_a)(d_a-r)} \text{dist}(f(a), \partial \tilde{G})$$

(3.5)

holds. If $\partial G$ contains at least one regular point of convexity, then the constant in (3.5) is sharp.

4 Estimates for the increment of derivatives of holomorphic functions

In this section we will be using the notation $\Delta g(z) = g(z) - g(0)$ to describe the increment of a function $g$ at $z = 0$. We will formulate the results of the Corollaries 7.2-7.5 from the book [4] in the following manner.
**Theorem 4.1** (G.Kresin-V.Maz’ya) Let \( f(z) \) be a function holomorphic in the disc \( D_R \). Then, for any fixed \( z, |z| = r < R \), the inequality
\[
|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n!(R^{n+1} - (R - r)^{n+1})}{(R - r)^{n+1}R^n} \mathcal{R}(f)
\]
holds with the best constant for every \( n \geq 0 \) and where \( \mathcal{R}(f) \) is each of the expression (i)-(iv) from the Theorem 3.1.

Analogously to the previous sections one can prove the following

**Theorem 4.2** Let \( f(z) \) be a function holomorphic in the disc \( D_R \). Assume also that \( f(D_R) \subset G \), where \( G \) is a domain in \( \mathbb{C} \) such that \( \tilde{G} \neq \mathbb{C} \). Then for every fixed \( z \in D_R, |z| = r \) the following inequality
\[
|f^{(n)}(z) - f^{(n)}(0)| \leq \frac{2n!(R^{n+1} - (R - r)^{n+1})}{(R - r)^{n+1}R^n} \text{dist}(f(0), \partial \tilde{G})
\]
holds for every \( n \geq 0 \). If \( \partial G \) contains at least one regular point of convexity, then the constant in (4.1) is the best one.

We conclude the article with the following

**Remark 4.1** 1) It seems that it is possible to formulate a geometric variant of the results from [3].

2) Do the constants in the above cited geometric generalizations of results of Kresin-Maz’ya remain sharp, if one assumes that \( \partial G \) does not contain any regular point of convexity?

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