THE NEUMANN EIGENVALUE PROBLEM FOR THE
∞-LAPLACIAN

L. ESPOSITO, B. KAWOHL, C. NITSCH, AND C. TROMBETTI

Abstract. The first nontrivial eigenfunction of the Neumann eigenvalue problem
for the $p$-Laplacian, suitable normalized, converges to a viscosity solution of an
eigenvalue problem for the $\infty$-Laplacian. We show among other things that the
limiting eigenvalue, at least for convex sets, is in fact the first nonzero eigenvalue of
the limiting problem. We then derive a number consequences, which are nonlinear
analogues of well-known inequalities for the linear (2-)Laplacian.

Keywords: Neumann eigenvalues, viscosity solutions, infinity Laplacian
2010 MSC: 35P30, 35P15, 35J72, 35D40, 35J92, 35J70

1. Introduction and statements

In this paper we study the $\infty$-Laplacian eigenvalue problem under Neumann boundary
conditions

\[
\begin{align*}
\min\{|\nabla u| - \Lambda u, -\Delta_\infty u\} &= 0 \quad \text{in } \{u > 0\} \cap \Omega \\
\max\{-|\nabla u| - \Lambda u, -\Delta_\infty u\} &= 0 \quad \text{in } \{u < 0\} \cap \Omega \\
-\Delta_\infty u &= 0, \quad \text{in } \{u = 0\} \cap \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

A solution $u$ to this problem has to be understood in the viscosity sense, and the Neumann
eigenvalue $\Lambda$ is some nonnegative real constant. For $\Lambda = 0$ problem (1) has constant
solutions. We consider those as trivial. Our main result is

Theorem 1. Let $\Omega$ be a smooth bounded open convex set in $\mathbb{R}^n$ then a necessary condition
for the existence of nonconstant continuous solutions $u$ to (1) is

\[
\Lambda \geq \Lambda_\infty := \frac{2}{\text{diam}(\Omega)}.
\]

Here $\text{diam}(\Omega)$ denotes the diameter of $\Omega$. Moreover problem (1) admits a Lipschitz solution
when $\Lambda = \frac{2}{\text{diam}(\Omega)}$.

If $\Omega$ is merely bounded, connected and has Lipschitz boundary, then the notion of
diameter can be generalized as in Definition 2. In that case solutions of (2) exist, see
Section 2 or [16]. However, it is still unclear whether $\Lambda_\infty$ is always the first eigenvalue.

Theorem 1 has a number of interesting consequences, one of which we list right here.
By the isodiametric inequality we may conclude

Corollary 1. If $\Omega^*$ denotes the ball of same volume as $\Omega$, then the Szegö-Weinberger
inequality $\Lambda_\infty(\Omega) \leq \Lambda_\infty(\Omega^*)$ holds.
For the case of the ordinary Laplacian \((p = 2)\) this result was shown in [17] and [19]. For the 1-Laplacian case and convex plane \(\Omega\) we refer to [9]. While the Faber-Krahn inequality \(\lambda_p(\Omega^*) \leq \lambda_p(\Omega)\) holds for any \(p\), the Szegö-Weinberger inequality has resisted attempts to be generalized to general \(p\), and for general \(p\) we are unaware of any results in this direction.

The reason why we call problem (1) \(\infty\)-Laplacian eigenvalue problem under Neumann boundary conditions is that (1) can be derived as the limit \(p \to \infty\) of Neumann eigenvalue problems for the \(p\)-Laplacian

\[
\begin{aligned}
-\Delta_p u &= \Lambda_p |u|^{p-2} u \quad \text{in } \Omega \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

whenever \(\Omega\) is a bounded open Lipschitz set of \(\mathbb{R}^n\).

For the Dirichlet \(p\)-Laplacian eigenvalue problem on open bounded sets \(\Omega \subset \mathbb{R}^n\)

\[
\begin{aligned}
-\Delta_p v &= \lambda_p |v|^{p-2} v \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

the same limit was studied by Juutinen, Lindqvist and Manfredi in [13, 12]. They formulate and fully investigate the so-called Dirichlet \(\infty\)-Laplacian eigenvalue problem employing the notion of viscosity solutions. Recall for instance that, when \(\lambda_p\) denotes for all \(p \geq 1\) the first nontrivial eigenvalue of (4), the limit yields

\[
\lim_{p \to \infty} \lambda_p = \lambda_{\infty} := \frac{1}{R(\Omega)},
\]

where \(R(\Omega)\) denotes inradius, i.e. the radius of the largest ball contained in \(\Omega\). Moreover, they identify the limiting eigenvalue problem as

\[
\begin{aligned}
\min \{|\nabla v| - \lambda v, -\Delta_{\infty} v\} &= 0 \quad \text{in } \Omega \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

in the sense that nonnegative normalized eigenfunctions of (4) converge, up to a subsequence, to a positive Lipschitz function \(v_{\infty}\) which solves (5) in the viscosity sense with \(\lambda(\Omega) = \lambda_{\infty}(\Omega)\). Finally they also show that the infinity Laplacian eigenvalue problem (5) admits nontrivial solutions if and only if \(\lambda \geq \lambda_{\infty}\) and positive solutions if and only if \(\lambda = \lambda_{\infty}\). Therefore they call \(\lambda_{\infty}\) the principal eigenvalue of the \(\infty\)-Laplacian eigenvalue problem under Dirichlet boundary condition.

In the Neumann case (see [16]) and for any bounded connected \(\Omega\) with Lipschitz boundary the limiting problem \(p \to \infty\) for (3) is given by (1).

In analogy to the Dirichlet case, the first nontrivial eigenvalues of (3) satisfy

\[
\lim_{p \to \infty} \Lambda_p = \Lambda_{\infty}.
\]

Our result proves that on the class of convex sets the first nontrivial Neumann \(p\)-Laplacian eigenvalues converge to the first nontrivial Neumann \(\infty\)-Laplacian eigenvalue, namely \(\Lambda = \Lambda_{\infty}\) is in fact the first nontrivial eigenvalue in (1).

Therefore we can point out some consequences.
Corollary 2. For convex $\Omega$ the first positive Neumann eigenvalue $\Lambda_\infty(\Omega)$ is never larger than the first Dirichlet eigenvalue $\lambda_\infty(\Omega)$. Moreover $\lambda_\infty(\Omega) = \Lambda_\infty(\Omega)$ if and only if $\Omega$ is a ball.

The inequality $\Lambda_2(\Omega) < \lambda_2(\Omega)$ follows from a combination of the Szegő-Weinberger and the Faber-Krahn inequalities, see e.g. the books by Bandle or Kesavan [3, 14]. The strict inequality $\Lambda_p(\Omega) < \lambda_p(\Omega)$ for general $p$ and any convex $\Omega$ has been recently proved in [2].

Corollary 3. For convex $\Omega$ any Neumann eigenfunction associated with $\Lambda_\infty(\Omega)$ cannot have a closed nodal domain inside $\Omega$.

Since a Neumann eigenfunction $u$ for the $\infty$-Laplacian is in general just continuous, a closed nodal line inside $\Omega$ means that there exists an opens subset $\Omega' \subset \Omega$ such that $u > 0$ in $\Omega'$ (or $< 0$ in $\Omega'$) and $u = 0$ on $\partial\Omega'$. Assuming that such a nodal line exists, we can use standard arguments. We observe that $u$ is also a Dirichlet eigenfunction on $\Omega'$ with same eigenvalue. We get

$$\frac{2}{\text{diam}(\Omega')} = \Lambda_\infty(\Omega) = \lambda_\infty(\Omega') = \frac{1}{\text{rad}(\Omega')} \geq \frac{2}{\text{diam}(\Omega')}$$

and notice that the last inequality is strict for all sets other than balls. This proves the Corollary.

Next we recall that the Payne-Weinberger inequality states that on any convex subset $\Omega \subset \mathbb{R}^n$ the first non trivial Neumann eigenvalue for the Laplacian is bounded from below by the quantity $\frac{\pi^2}{\text{diam}(\Omega)^2}$. Recently such an estimate has been generalized to the first non trivial Neumann $p$-Laplacian eigenvalues in [7,8,18] to get

$$\Lambda_p \geq (p - 1)\frac{2\pi}{p \text{ diam}(\Omega) \sin \frac{\pi}{p}}. \quad (7)$$

As $p \to \infty$ the right hand side in this Payne-Weinberger inequality (7) converges

$$\lim_{p \to \infty} (p - 1)^{1/p} \left( \frac{2\pi}{p \text{ diam}(\Omega) \sin \frac{\pi}{p}} \right) = \frac{2}{\text{diam}(\Omega)},$$

and in view of (6) we may therefore conclude that

Corollary 4. The Payne-Weinberger inequality (7) for the first Neumann eigenvalue of the $p$-Laplacian becomes an identity for $p = \infty$.

As a byproduct of our proofs we obtain also the following result, which is related to the hot-spot conjecture. The hot spot conjecture, see [4], says that a first nontrivial Neumann eigenfunction for the linear Laplace operator on a convex domain $\Omega$ should attain its maximum or minimum on the boundary $\partial\Omega$ and the proof of Lemma 1 will show that $u_\infty$ has this property as well. But there may be more than one eigenfunction associated to $\Lambda_\infty$.

Corollary 5. If $\Omega$ is convex and smooth, then any first nontrivial Neumann eigenfunction, i.e. any viscosity solution to (1) for $\Lambda = \Lambda_\infty$ attains both its maximum and minimum only on the boundary $\partial\Omega$. Moreover the extrema of $u$ are located at points that have maximal distance in $\Omega$.

The proof of our main result, Theorem 1, will be a combination of Theorem 2 in Section 2 on the limiting problem as $p \to \infty$ and Proposition 1 in Section 3. Corollary 5 will be derived at the very end of this paper.
2. The limiting problem as \( p \to \infty \)

**Definition 1.** Let \( \Omega \) be a bounded open connected domain in \( \mathbb{R}^n \). The intrinsic diameter of \( \Omega \), denoted by \( \text{diam}(\Omega) \), is defined as

\[
\text{diam}(\Omega) := \sup_{x,y \in \Omega} d_\Omega(x,y)
\]

with \( d_\Omega \) denoting geodetic distance in \( \Omega \).

Consider the eigenvalue problem

\[
\Lambda_p = \min \left\{ \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^{p-2} v \, dx} : v \in W^{1,p}(\Omega), \int_\Omega |v|^{p-2} v \, dx = 0 \right\}.
\]

Let \( u_p \) be a minimizer of (9) such that \( \|u_p\|_p = 1 \), where \( \|f\|_p = \frac{1}{|\Omega|} \int_\Omega |f|^p \, dx \).

For every \( p > 1 \) \( u_p \) satisfies the Euler equation

\[
\begin{aligned}
-\divergence(\nabla u_p|^{p-2}\nabla u_p) &= \Lambda_p |u_p|^{p-2} u_p \quad \text{in } \Omega \\
|\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \nu} &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

and

**Lemma 1.** Let \( \Omega \) be a connected bounded open set in \( \mathbb{R}^n \) with Lipschitz boundary, then

\[
\lim_{p \to +\infty} \Lambda_p = \Lambda_\infty := \frac{2}{\text{diam}(\Omega)},
\]

here \( \text{diam}(\Omega) \) denotes the intrinsic diameter as defined in (8).

**Proof.** **Step 1** \( \limsup_{p \to +\infty} \Lambda_p \leq \frac{2}{\text{diam}(\Omega)} \).

We start proving that \( \Lambda_\infty \leq 2/\text{diam}(\Omega) \). Let \( x_0 \in \Omega \). We choose \( c_p \in \mathbb{R} \) such that \( w(x) = d_\Omega(x, x_0) - c_p \) is a good test function in (9), that is

\[
\int_\Omega |w|^{p-2} w \, dx = 0.
\]

Using this test function in (9) we get (recalling that \( |\nabla d_\Omega(x, x_0)| \leq 1 \) a.e. in \( \Omega \))

\[
\Lambda_p \leq \frac{1}{\left( \frac{1}{|\Omega|} \int_\Omega |d_\Omega(x, x_0) - c_p|^p \right)^{1/p}}.
\]

Now we observe that \( 0 \leq c_p \leq \text{diam}(\Omega) \) and thus up to a subsequence \( c_p \to c \), with \( 0 \leq c \leq \text{diam}(\Omega) \), then we obtain

\[
\liminf_{p \to +\infty} \left( \frac{1}{|\Omega|} \int_\Omega |d(x, x_0) - c_p|^p \right)^{1/p} = \sup_{x \in \Omega} |d_\Omega(x, x_0) - c| \geq \text{diam}(\Omega)/2
\]

and then from (12) the Step 1 is proved.

**Step 2** \( \liminf_{p \to +\infty} \Lambda_p \geq \frac{2}{\text{diam}(\Omega)} \).

By definition we get

\[
\left( \frac{1}{|\Omega|} \int_\Omega |\nabla u_p(x)|^p \, dx \right)^{1/p} = \Lambda_p.
\]
Let us fix $m > n$. For $p > m$ by Hölder inequality we have

$$\left( \frac{1}{|\Omega|} \int_{\Omega} |\nabla u_p(x)|^m dx \right)^{1/m} \leq \Lambda_p.$$ 

We can deduce that $\{u_p\}_{p \geq m}$ is uniformly bounded in $W^{1,m}(\Omega)$ and then assume that, up to a subsequence, $u_p$ converges weakly in $W^{1,m}(\Omega)$ and in $C^{0}(\Omega)$ to a function $u_{\infty} \in W^{1,m}(\Omega)$. For $q > m$, by semicontinuity and Hölder inequality, we get

$$\frac{\|\nabla u_{\infty}\|_q}{\|u_{\infty}\|_q} \leq \liminf_{p \to \infty} \left( \frac{1}{|\Omega|} \int_{\Omega} |\nabla u_p(x)| q \, dx \right)^{1/q} \leq \liminf_{p \to \infty} \left( \frac{1}{|\Omega|} \int_{\Omega} |\nabla u_p(x)| p \, dx \right)^{1/p}.$$

Thus

$$\|\nabla u_{\infty}\|_q \leq \|u_{\infty}\|_q \liminf_{p \to \infty} \Lambda_p$$

letting $q \to \infty$ we get

$$\frac{\|\nabla u_{\infty}\|_\infty}{\|u_{\infty}\|_\infty} \leq \liminf_{p \to \infty} \Lambda_p.$$

Now we observe that condition $\int_{\Omega} |u_p|^{p-2} u_p = 0$ leads to

$$\sup u_{\infty} = -\inf u_{\infty},$$

infact we have

$$0 \leq \|(u_{\infty})^+\|_{p-1} - \|(u_{\infty})^-\|_{p-1} = \|(u_{\infty})^+\|_{p-1} - \|(u_p)^+\|_{p-1} + \|(u_p)^-\|_{p-1} - \|(u_{\infty})^-\|_{p-1}$$

$$\leq \|(u_{\infty})^+\|_{p-1} - \|(u_p)^+\|_{p-1} + \|(u_{\infty})^-\|_{p-1} - \|(u_p)^-\|_{p-1}$$

$$\leq \|(u_{\infty})^+ - (u_p)^+\|_{p-1} + \|(u_{\infty})^- - (u_p)^-\|_{p-1}.$$ 

Letting $p \to \infty$ we obtain (15). Using the following inequality (see for instance [5], p.269)

$$|u_{\infty}(x) - u_{\infty}(y)| \leq d_{\Omega}(x, y) \|\nabla u_{\infty}\|_\infty \leq \text{diam}(\Omega) \|\nabla u_{\infty}\|_\infty,$$

we can conclude the proof by (14) observing that

$$2\|u\|_{\infty} = \sup u_{\infty} - \inf u_{\infty} \leq \text{diam}(\Omega) \|\nabla u_{\infty}\|_{\infty}.$$

\[\Box\]

**Remark 1.** Our proof shows that $u_{\infty}$ increases with constant slope $\Lambda_{\infty} \|u_{\infty}\|_{\infty}$ along the geodesic between two point spanning diam$(\Omega)$. In a rectangle this would be a diagonal.
Before proving Theorem 2 we recall the definition of viscosity super (sub) solution to

\[
\begin{align*}
F(u, \nabla u, \nabla^2 u) &= \min\{|\nabla u| - \Delta |u|, -\Delta u\} = 0 \quad \text{in } \{u > 0\} \cap \Omega \\
G(u, \nabla u, \nabla^2 u) &= \max\{|\nabla u| - |\nabla u|, -\Delta u\} = 0 \quad \text{in } \{u < 0\} \cap \Omega \\
H(\nabla^2 u) &= -\Delta u = 0 \quad \text{in } \{u = 0\} \cap \Omega \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(17)

**Definition 2.** An upper semicontinuous function \( u \) is a viscosity subsolution to (17) if whenever \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) are such that

\[ u(x_0) = \phi(x_0), \quad \text{and } u(x) < \phi(x) \text{ if } x \neq x_0, \quad \text{then} \]

\[ F(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \leq 0 \quad \text{if } u(x_0) > 0 \]

(18)

\[ G(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \leq 0 \quad \text{if } u(x_0) < 0 \]

(19)

\[ H(\nabla^2 \phi(x_0)) \leq 0 \quad \text{if } u(x_0) = 0, \]

(20)

while if \( x_0 \in \partial \Omega \) and \( \phi \in C^2(\hat{\Omega}) \) are such that

\[ u(x_0) = \phi(x_0), \quad \text{and } u(x) < \phi(x) \text{ if } x \neq x_0, \quad \text{then} \]

\[ \min\{F(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0)\} \leq 0 \quad \text{if } u(x_0) > 0 \]

(21)

\[ \min\{G(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0)\} \leq 0 \quad \text{if } u(x_0) < 0 \]

(22)

\[ \min\{H(\nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0)\} \leq 0 \quad \text{if } u(x_0) = 0. \]

(23)

**Definition 3.** A lower semicontinuous function \( u \) is a viscosity supersolution to (17) if whenever \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) are such that

\[ u(x_0) = \phi(x_0), \quad \text{and } u(x) > \phi(x) \text{ if } x \neq x_0, \quad \text{then} \]

\[ F(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \geq 0 \quad \text{if } u(x_0) > 0 \]

(24)

\[ G(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \geq 0 \quad \text{if } u(x_0) < 0 \]

(25)

\[ H(\nabla^2 \phi(x_0)) \geq 0 \quad \text{if } u(x_0) = 0, \]

(26)

while if \( x_0 \in \partial \Omega \) and \( \phi \in C^2(\hat{\Omega}) \) are such that

\[ u(x_0) = \phi(x_0), \quad \text{and } u(x) > \phi(x) \text{ if } x \neq x_0, \quad \text{then} \]

\[ \max\{F(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0)\} \geq 0 \quad \text{if } u(x_0) > 0 \]

(27)

\[ \max\{G(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0)\} \geq 0 \quad \text{if } u(x_0) < 0 \]

(28)

\[ \max\{H(\nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0)\} \geq 0 \quad \text{if } u(x_0) = 0. \]

(29)
Definition 4. A continuous function $u$ is a solution to (17) iff it is both a supersolution and a subsolution to (17).

Remark 2. It is instructive to use the definition for checking that the one-dimensional function $u(x) = x_1$ on the square $\Omega = (-1,1) \times (-1,1)$ is a viscosity solution of (17). In fact, $u \in C^2(\Omega)$, and $-\Delta_\infty u = 0$ in $\Omega$.

So the first PDE in (17) is satisfied if also $1 = |\nabla u| \geq \Lambda u$ on $\{u > 0\}$, and that implies $\Lambda \leq 1$.

The Neumann boundary condition is satisfied in classical sense on horizontal parts of $\partial \Omega$. However, for Neumann condition to hold in the viscosity sense on the right part, we must verify

$$\min \{ \min \{ |\nabla \phi| - \Lambda \phi, -\Delta_\infty \phi \} , \partial \phi / \partial \nu \}(x_0) \leq 0$$

for any $C^2$ test function $\phi$ touching $u$ in $x_0 \in \partial \Omega$ from above, and

$$\max \{ \min \{ |\nabla \psi| - \Lambda \psi, -\Delta_\infty \psi \} , \partial \psi / \partial \nu \}(x_0) \geq 0$$

for any smooth test function $\psi$ touching $u$ from below.

Recall $|\nabla u| = \partial u / \partial \nu = 1$ everywhere. Therefore only the very first constraint is active on the boundary and implies

$$\Lambda \geq 1.$$\ntext{This shows that $u(x) = x_1$ is a viscosity solution to (17) with eigenvalue $\Lambda = 1$, but}

$$\Lambda = 1 > \frac{1}{\sqrt{2}} = \frac{2}{\text{diam}(\Omega)} = \Lambda_\infty.$$\n
In what follows we will use the notation

$$F_p(u, \nabla u, \nabla^2 u) = -(p-2)|\nabla u|^{p-4}\Delta_\infty u - |\nabla u|^{p-2}\Delta u - \Lambda_p^p|u|^{p-2}u$$

with

$$\Delta_\infty u = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_j}.$$

Lemma 2. Let $u \in W^{1,p}(\Omega)$ be a weak solution to

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = \Lambda_p^p|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega \end{cases}$$

then $u$ is a viscosity solution to

$$\begin{cases} F_p(u, \nabla u, \nabla^2 u) = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega. \end{cases}$$

Proof. That $u$ is a viscosity solution to the differential equation $F_p = 0$ in $\Omega$ was shown in [13], Lemma 1.8. It remains to show that the Neumann boundary condition is satisfied in the viscosity sense as defined for instance in [10]. Let $x_0 \in \partial \Omega$, $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $\phi(x) < u(x)$ when $x \neq x_0$. Assume by contradiction that

$$\max \{ |\nabla \phi(x_0)|^{p-2}\frac{\partial \phi}{\partial \nu}(x_0), F_p(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)) \} < 0.$$
Then there exists a ball \( B_r(x_0) \), centered at \( x_0 \) with radius \( r > 0 \), such that (32) holds true \( \forall x \in \Omega \cap B(x_0, r) \). Denoted by \( 0 < m = \inf_{\Omega \cap B_r(x_0)} (u(x) - \phi(x)) \) and by \( \psi(x) = \phi(x) + \frac{m}{2} \).

Using \((\psi - u)^+\) as test function in the weak formulation we have both
\[
\int_{\psi > u} |\nabla \psi|^{p-2} \nabla \psi \nabla (\psi - u) \, dx < \Lambda_p^p \int_{\psi > u} |\phi|^{p-2} (\psi - u) \, dx
\]
and
\[
\int_{\psi > u} |\nabla u|^{p-2} \nabla u \nabla (\psi - u) \, dx = \Lambda_p^p \int_{\psi > u} |u|^{p-2} u (\psi - u) \, dx.
\]
Subtraction yields the contradiction
\[
C \int_{\psi > u} |\nabla (\psi - u)|^p \, dx \leq \int_{\psi > u} (|\nabla \psi|^{p-2} \nabla \psi - |\nabla u|^{p-2} \nabla u, \nabla (\psi - u)) \, dx
\]
\[
< \Lambda_p^p \int_{\psi > u} (|\phi|^{p-2} \phi - |u|^{p-2} u) (\psi - u) \, dx < 0.
\]

\[\Box\]

**Theorem 2.** Let \( \Omega \) be an open bounded connected set of \( \mathbb{R}^n \). If \( u_\infty \) and \( \Lambda_\infty \) are defined as above then \( u_\infty \) satisfies (17) in the viscosity sense with \( \Lambda = \Lambda_\infty \).

**Proof.** First we observe that in fact there exists a subsequence \( u_{p_i} \) uniformly converging to \( u_\infty \) in \( \Omega \). Now let us prove that \( u_\infty \) is a viscosity super solution to (17) in \( \Omega \). Let \( x_0 \in \Omega \) and let \( \phi \in C^2(\Omega) \) be such that \( \phi(x_0) = u_\infty(x_0) \) and \( \phi(x) < u_\infty(x), x \in \Omega \setminus \{x_0\} \).

Since \( u_{p_i} \to u_\infty \) uniformly in \( B_r(x_0) \) one can prove that \( u_{p_i} - \phi \) has a local minimum in \( x_i \), with \( \lim_i x_i = x_0 \). Recalling that \( u_{p_i} \) is a viscosity solution to (31), choosing \( \psi(x) = \phi(x) - \phi(x_i) + u_{p_i}(x_i) \) as test function we obtain
\[
-[(p_i - 2)|\nabla \phi(x_i)|^{p_i-4} \Delta \phi(x_i) + |\nabla \phi(x_i)|^{p_i-2} \Delta \phi(x_i)] \geq \Lambda_{p_i}^p |u_{p_i}(x_i)|^{p_i-2} u_{p_i}(x_i).
\]

Three cases can occur.

- **\( u_\infty(x_0) > 0 \).** In this case (34) implies that \( |\nabla \phi(x_i)| > 0 \), hence dividing (34) by \( |\nabla \phi(x_i)|^{p_i-4} (p_i - 2) \) we have
\[
\frac{|\nabla \phi(x_i)|^{p_i-2} \Delta \phi(x_i)}{p_i - 2} - \Delta_\infty \phi(x_i) \geq \left( \frac{\Lambda_{p_i} u_{p_i}(x_i)}{a_i} \right)^{p_i-4} \Lambda_{p_i}^{p_i} u_{p_i}^{p_i}(x_i).
\]

Letting \( p_i \) go to \( +\infty \) we have \( \frac{\Lambda_\infty \phi(x_0)}{|\nabla \phi(x_0)|} \leq 1 \) and \( -\Delta_\infty \phi(x_0) \geq 0 \) hence
\[
\min\{|\nabla \phi(x_0)|, -\Lambda_\infty \phi(x_0), -\Delta_\infty \phi(x_0)\} \geq 0.
\]

- **\( u_\infty(x_0) < 0 \).** Also in this case (31) implies that \( |\nabla \phi(x_i)| > 0 \), and dividing by \( |\nabla \phi(x_i)|^{p_i-4} (p_i - 2) \) we have again (35). If \( \frac{\Lambda_\infty \phi(x_0)}{|\nabla \phi(x_0)|} < 1 \), let \( p_i \) go to \( \infty \), we have \( -\Delta_\infty \phi(x_0) \geq 0 \), otherwise \( \frac{\Lambda_\infty \phi(x_0)}{|\nabla \phi(x_0)|} \geq 1 \). In both cases we have
\[
\min\{\Lambda_\infty \phi(x_0), -|\nabla \phi(x_0)|, -\Delta_\infty \phi(x_0)\} \geq 0.
\]

- **\( u_\infty(x_0) = 0 \).** If \( |\nabla \phi(x_0)| = 0 \) then, by definition, we have \( -\Delta_\infty \phi(x_0) = 0 \). If \( |\nabla \phi(x_0)| > 0 \) then \( \lim_i \frac{\Lambda_{p_i} u_{p_i}(x_i)}{|\nabla \phi(x_i)|} = 0 \) hence (35) implies
\[
-\Delta_\infty \phi(x_0) \geq 0.
\]
It remains to prove that $u_\infty$ satisfies the boundary conditions in the viscosity sense. Assume that $x_0 \in \partial \Omega$ and let $\phi \in C^2(\Omega)$ be such that $\phi(x_0) = u_\infty(x_0)$ and $\phi(x) < u_\infty(x)$ \quad x \in \Omega \setminus \{x_0\}$. Using again the uniform convergence of $u_{p_i}$ to $u_\infty$ we obtain that $u_{p_i} - \phi$ has a minimum point $x_i \in \Omega$, with $\lim_i x_i = x_0$.

If $x_i \in \Omega$ for infinitely many $i$ arguing as before we get
\begin{align*}
\min\{|\nabla \phi(x_0)| - \Lambda_\infty |\phi(x_0)|, -\Delta_\infty \phi(x_0)\} &\geq 0, \quad \text{if } u(x_0) > 0, \\
\max\{|\Lambda_{\infty} |\phi(x_0)| - |\nabla \phi(x_0)|, -\Delta_\infty \phi(x_0)\} &\geq 0, \quad \text{if } u(x_0) < 0, \\
-\Delta_\infty \phi(x_0) &\geq 0, \quad \text{if } u(x_0) = 0.
\end{align*}

If $x_i \in \partial \Omega$, since $u_{p_i}$ is viscosity solution to (17), for infinitely many $i$ we have
\[|\nabla \phi(x_i)|^{p_i - 2} \frac{\partial \phi}{\partial \nu}(x_i) \geq 0\]
which concludes the proof.

Arguing in the same way we can prove that $u_\infty$ is a viscosity subsolution to (17) in $\Omega$.

\[\square\]

3. $\Lambda_\infty$ is the first non trivial eigenvalue

**Proposition 1.** Let $\Omega$ be a smooth bounded open convex set in $\mathbb{R}^n$. If for some $\Lambda > 0$ problem (17) admits a nontrivial eigenfunction $u$, then $\Lambda \geq \Lambda_\infty$.

The main idea is to use a test function involving the distance from a suitable point $x_0 \in \Omega$. This function is smooth everywhere except $x_0$. For the nonconvex case one may want to use intrinsic distance instead, which however is not of class $C^2$, as pointed out in [1].

**Lemma 3.** Let $\Omega$, $\Lambda$ and $u$ be as in the statement of Proposition 1. Let $\Omega_1$ be an open connected subset of $\Omega$ such that $u \geq m$ in $\overline{\Omega}_1$ for some positive constant $m$. Then $u > m$ in $\Omega_1$.

**Proof.** Let $x_0$ be any point in $\Omega_1$. Our aim is to show that $u(x_0) > m$. Obviously, for any given $R > 0$ such that $B_R(x_0) \subset \Omega_1$ we have $u \neq m$ in $B_R(x_0)$ otherwise we have in $B_R(x_0)$ that $|\nabla u| - \Lambda |u| < 0$ (in the viscosity sense) which violates the first equation in (17). This means that for any $R > 0$ such that $B_R(x_0) \subset \Omega_1$ it is possible to find $x_1 \in B_{R/4}(x_0)$ such that $u(x_1) > m$. The continuity of $u$ implies that for some $\varepsilon > 0$ small enough, there exists $r \leq \text{dist}(x_0, x_1)$ such that $u > m + \varepsilon$ on $\partial B_r(x_1)$. Therefore the function

\[v(x) = m + \frac{\varepsilon}{\frac{R}{2} - r} \left( \frac{R}{2} - |x - x_1| \right) \quad \text{in } B_{R/2}(x_1) \setminus B_r(x_1)\]

is such that
\[-\Delta_\infty v = 0 \quad \text{in } B_{R/2}(x_1) \setminus B_r(x_1)\]

Since
\[-\Delta_\infty u \geq 0 \quad \text{in } B_{R/2}(x_1) \setminus B_r(x_1)\]
in the viscosity sense, and
\[u \geq v \quad \text{on } \partial B_{R/2}(x_1) \cup \partial B_r(x_1)\]
the comparison principle, see Theorem 2.1 in [11], implies that \( u \geq v > m \) in \( B_{R/2}(x_1) \setminus B_r(x_1) \) and therefore \( u(x_0) > m \).

**Lemma 4.** Let \( \Omega, \Lambda \) and \( u \) be as in the statement of Proposition 4. Then \( u \) certainly changes sign.

**Proof.** Since \( u \) is a nontrivial solution to (1), we can always assume, possibly changing the sign of the eigenfunction \( u \), that it is positive somewhere. We shall prove that the minimum of \( u \) in \( \bar{\Omega} \) is negative. We argue by contradiction and we assume that the minimum \( m \) is nonnegative. In view of Lemma 3, a positive minimum can not be attained in \( \Omega \). On the other hand zero as well can not be attained as minimum in \( \Omega \). If so, since \( u \neq 0 \), there would exist a point \( x_0 \in \Omega \) and a ball \( B_R(x_0) \subset \Omega \) such that \( u(x_0) = 0 \) and \( \max_{B_{R/4}(x_0)} u > 0 \). Let \( x_1 \in B_{R/4}(x_0) \) be such that \( u(x_1) > 0 \). The continuity of \( u \) implies that there exists \( r \leq \text{dist}(x_0, x_1) \) such that \( u > u(x_1)/2 \) on \( \partial B_r(x_1) \). Therefore the function

\[
v(x) = \frac{u(x_1)}{R - 2r} \left( \frac{R}{2} - |x - x_1| \right)
\]

is such that

\[-\Delta_{\infty} v = 0 \quad \text{in} \ B_{R/2}(x_1) \setminus B_r(x_1).
\]

Since

\[-\Delta_{\infty} u \geq 0 \quad \text{in} \ B_{R/2}(x_1) \setminus B_r(x_1)
\]

in the viscosity sense, and

\[u \geq v \quad \text{on} \ \partial B_{R/2}(x_1) \cup \partial B_r(x_1)
\]

the comparison principle, see Theorem 2.1 in [11], implies that \( u \geq v > 0 \) in \( B_{R/2}(x_1) \setminus B_r(x_1) \) and therefore \( u(x_0) > 0 \).

Therefore the only possibility is that there exists \( x_0 \in \partial \Omega \) nonnegative minimum point of \( u \). We shall prove that \( \frac{\partial u}{\partial n}(x_0) < 0 \) in the viscosity sense in contradiction to (24) - (26).

Indeed there certainly exist \( \bar{x} \in \Omega \) and \( r > 0 \) such that the ball \( B_r(\bar{x}) \subset \Omega \) is inner tangential to \( \partial \Omega \) at \( x_0 \) and \( \partial B_r(\bar{x}) \cap \partial \Omega = \{x_0\} \). Then the function

\[
v(x) = u(\bar{x}) - \frac{(u(\bar{x}) - u(x_0))}{r} (|x - \bar{x}|) \quad \text{in} \ B_r(\bar{x}) \setminus \{\bar{x}\}
\]

satisfies

\[-\Delta_{\infty} v = 0 \quad \text{in} \ B_r(\bar{x}) \setminus \{\bar{x}\}
\]

since

\[-\Delta_{\infty} u \geq 0 \quad \text{in} \ B_r(\bar{x}) \setminus \{\bar{x}\}
\]

in the viscosity sense, and

\[u \geq v \quad \text{on} \ \partial B_r(\bar{x}) \cup \{\bar{x}\}.
\]

Using again the comparison principle, see Theorem 2.1 in [11], we get \( u \geq v \) in \( \bar{\Omega} \). Therefore the function

\[
\phi = u(\bar{x}) - (u(\bar{x}) - u(x_0)) \left( \frac{|x - \bar{x}|}{r} \right)^{1/2}
\]

is such that \( \phi \in C^2(\Omega - \{\bar{x}\}) \),

\[\phi < v \leq u \quad \text{in} \ B_r(\bar{x}) \setminus \{\bar{x}\},
\]
Therefore (a comparison) Theorem 2.1 in [11] ensures that

\[ u(x_0) = \phi(x_0). \]

However

\[ \max \{ F(\phi(x_0), \nabla \phi(x_0), \nabla^2 \phi(x_0)), \frac{\partial \phi}{\partial \nu}(x_0) \} < 0 \]

contradicts [24]-[26]. \( \square \)

**Proof of Proposition 1.** Let \( u \) be a non trivial eigenfunction of (17) and let us denote by \( \Omega_+ = \{ x \in \Omega : u(x) > 0 \} \) and by \( \Omega_- = \{ x \in \Omega : u(x) < 0 \} \). Lemma [4] ensures that they are both nonempty sets. Let us normalize the eigenfunction \( u \) such that

\[ \max_\Omega u = \frac{1}{\Lambda} \]

Then \( \Lambda u \leq 1 \) which implies that

\[ \min \{|\nabla u| - 1, -\Delta_\infty u\} \leq 0 \quad \text{in} \ \Omega_+ \]

in the viscosity sense.

For every \( x_0 \in \Omega \setminus \Omega_+ \) and for every \( \epsilon > 0 \) and \( \gamma > 0 \) the function \( g_{\epsilon, \gamma}(x) = (1 + \epsilon)|x - x_0| - \gamma|x - x_0|^2 \) belongs to \( C^2(\Omega \setminus B_\rho(x_0)) \) for every \( \rho > 0 \). If \( \gamma \) is small enough compared to \( \epsilon \), it verifies

\[ \min \{|\nabla g_{\epsilon, \gamma}| - 1, -\Delta_\infty g_{\epsilon, \gamma}\} \geq 0 \quad \text{in} \ \Omega_+. \]

Therefore (a comparison) Theorem 2.1 in [11] ensures that

\[ m = \inf_{x \in \Omega_+} (g_{\epsilon, \gamma}(x) - u(x)) = \inf_{x \in \partial \Omega_+} (g_{\epsilon, \gamma}(x) - u(x)). \]

Now \( \partial \Omega_+ \) contains certainly points in \( \Omega \) and possibly on \( \partial \Omega \). To rule out that the infimum in the right hand side of (39) is attained on \( \partial \Omega \), assume that there exists \( \bar{x} \in \partial \Omega \cap \partial \Omega_+ \) such that \( g_{\epsilon, \gamma}(\bar{x}) - u(\bar{x}) = m \) and choose \( g_{\epsilon, \gamma} - m \) as test function in (21). By construction for every \( x \in \partial \Omega \cap \partial \Omega_+ \) and \( \gamma < \frac{\epsilon}{2 \text{diam}(\Omega)} \) it results that

\[ |\nabla g_{\epsilon, \gamma}(x)| = 1 + \epsilon - 2\gamma|x - x_0| > 1, \]

\[ \frac{\partial g_{\epsilon, \gamma}}{\partial \nu}(x) = ((1 + \epsilon) - 2\gamma|x - x_0|) \left( \frac{x - x_0}{|x - x_0|} , \nu(x) \right) > 0, \]

and

\[ -\Delta_\infty g_{\epsilon, \gamma} = 2\gamma|\nabla g_{\epsilon, \gamma}|^2 > 0 \]

which give a contradiction to (21). Together with (39) this implies that

\[ m = \inf_{x \in \Omega_+} (g_{\epsilon, \gamma}(x) - u(x)) = \inf_{x \in \partial \Omega_+ \cap \Omega} (g_{\epsilon, \gamma}(x) - u(x)) \geq 0. \]

Letting \( \epsilon \) and \( \gamma \) go to zero we have that

\[ |x - x_0| \geq u(x) \quad \forall x \in \{ y : u(y) \geq 0 \}, \quad \forall x_0 \in \{ y : u(y) \leq 0 \} \]

hence

\[ d^+ = \sup_{x \in \Omega_+} \text{dist}(x, \{ u = 0 \}) \geq \frac{1}{\Lambda}. \]

Arguing in the same way we obtain
\[ d^- = \sup_{x \in \Omega_-} \text{dist}(x, \{ u = 0 \}) \geq \frac{1}{\Lambda} \]

hence

\[ \text{diam}(\Omega) \geq d^+ + d^- \geq \frac{2}{\Lambda} \]

which concludes the proof of our proposition. \( \square \)

Corollary 5 follows now easily. Returning to \cite{10} pick \( x = \mathfrak{x} \) as the point in which \( u \) attains its maximum and correspondingly \( x = \mathfrak{y} \) as the point in which \( u \) attains its minimum. Then \( d(\mathfrak{x}, \Omega_-) \geq \frac{1}{\Lambda} \) and \( d(\mathfrak{y}, \Omega_+) \geq \frac{1}{\Lambda} \), so that \( \text{diam}(\Omega) \geq |\mathfrak{x} - \mathfrak{y}| \geq \frac{2}{\Lambda} \). Since \( \Lambda = \Lambda_\infty \), equality holds and the max and min of \( u \) are attained in boundary points which have farthest distance from each other.

References

[1] S.B. Alexander, I.D. Berg, R.L. Bishop, The Riemannian obstacle problem, Illinois J. of Math. 33 (1987), 167–184.
[2] L. Brasco, C. Nitsch, C. Trombetti An inequality à la Szegö–Weinberger for the \( p \)-Laplacian, (manuscript) arXiv:1407.7422
[3] C. Bandle, Isoperimetric inequalities and applications. Monographs and Studies in Mathematics, 7. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.
[4] K. Burdzy, Neumann eigenfunctions and Brownian couplings. Potential theory in Matsue, Adv. Stud. Pure Math. 44 Math. Soc. Japan, Tokyo, (2006), 11–23.
[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations Springer Universitext 223 Heidelberg (2010)
[6] M. G. Crandall, H. Ishii, P. L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bulletin of the AMS. 27 (1992), 1–67.
[7] L. Esposito, C. Nitsch and C. Trombetti, Best constants in Poincaré inequalities for convex domains, J. Conv. Anal. 20 (2013) 253–264.
[8] V. Ferone, C. Nitsch and C. Trombetti, A remark on optimal weighted Poincaré inequalities for convex domains, Rend. Lincei Mat. Appl. 23 (2012) 467–475.
[9] L. Esposito, V. Ferone, B. Kawohl, C. Nitsch, C. Trombetti, The longest shortest fence and sharp Poincaré-Sobolev inequalities, Arch. Ration. Mech. Anal. 206 (2012) 821–851.
[10] J. García - Azorero, J. J. Manfredi, I. Peral, J. D. Rossi, The Neumann problem for the \( \infty \)-Laplacian and the Monge-Kantorovich mass transfer problem, Nonlinear Anal 66 (2007), 349–366.
[11] R. Jensen, Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient, Arch. Rational Mech. Anal. 123 (1993), 51–74.
[12] P. Juutinen, P. Lindqvist, On the higher eigenvalues for the \( \infty \)-eigenvalue problem, Arch. Rational Mech. Anal. 148 (1999), 89–105.
[13] P. Juutinen, P. Lindqvist, J. Manfredi, The \( \infty \)-eigenvalue problem, Arch. Rational Mech. Anal. 148 (1999), 89–105.
[14] S. Kesavan, Symmetrization & applications. Series in Analysis, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.
[15] L.E. Payne and H.F. Weinberger, An optimal Poincaré inequality for convex domains, Arch. Rational Mech. Anal. 5 (1960), 286–292.
[16] J.D. Rossi, N. Saintier, On the first nontrivial eigenvalue of the \( \infty \)-Laplacian with Neumann boundary conditions, Houston J. Math., to appear.
[17] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. 3 (1954), 343–356.

[18] D. Valtorta, *Sharp estimate on the first eigenvalue of the p-Laplacian*, Nonlinear Anal. 75 (2012), 4974–4994.

[19] H.F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Rational Mech. Anal. 5 (1956), 633–636.

Dipartimento di Matematica e Informatica, via Ponte Don Melillo, 84084 Fisciano (SA)
*E-mail address:* luesposi@unisa.it

Mathematisches Institut, Universität zu Köln, 50923 Köln
*E-mail address:* kawohl@math.uni-koeln.de

Dipartimento di Matematica e Applicazioni, via Cintia, 80126 Napoli
*E-mail address:* c.nitsch@unina.it

Dipartimento di Matematica e Applicazioni, via Cintia, 80126 Napoli
*E-mail address:* cristina@unina.it