Fractional dynamical systems and applications in mechanics and economics

Mihai Boleantu
Dept. of Economics, West University of Timisoara,
email: mihaiboleantu01@yahoo.com

Dumitru Opris
Dept. of Mathematics, West University of Timisoara,
email: miticaopris@yahoo.com

Abstract

Using the fractional integration and differentiation on \( \mathbb{R} \) we build the fractional jet fibre bundle on a differentiable manifold and we emphasize some important geometrical objects. Euler-Lagrange fractional equations are described. Some significant examples from mechanics and economics are presented.

Mathematics Subject Classification: 26A33, 53C60, 58A05, 58A40
Keywords: fractional derivatives, fractional bundle, Euler-Lagrange fractional equations

1 Introduction

The operators of fractional differentiation have been introduced by Leibnitz, Liouville, Riemann, Grunwal and Letnikov [6]. The fractional derivatives and integrals are used in the description of some models in mechanics, physics [6], economics [11] and medicine [11]. The fractional variational calculus [11] is an important instrument in the analysis of such models. The Euler-Lagrange equations are non-autonomous fractional differential equations in those models.

In this paper we present the fractional jet fibre bundle of order \( k \) on a differentiable manifold as being \( J^\alpha k(\mathbb{R}, M) = \mathbb{R} \times Osc^\alpha k(M), \alpha \in (0, 1), k \in \mathbb{N}^* \).
The fibre bundle $J^{\alpha k}$ is built in a similar way as the fibre bundle $E^k$ by R. Miron [9]. Among the geometrical structures defined on $J^\alpha(\mathbb{R}, M)$ we consider the dynamical fractional connection and the fractional Euler-Lagrange equations associated with a function defined on $J^{\alpha k}(\mathbb{R}, M)$.

In section 2 we describe the fractional operators on $\mathbb{R}$ and some of their properties which are used in the paper. In section 3 we describe the fractional osculator bundle of order $k$. In section 4 the fractional jet fibre bundle $J^\alpha(\mathbb{R}, M)$ is defined, the fractional dynamical connection is built and the fractional Euler-Lagrange equations are established using the notion of fractional extremal value and classical extremal value on $J^{\alpha k}(\mathbb{R}, M)$. In section 5 we consider some examples and applications.

2 Elements of fractional integration and differentiation on $\mathbb{R}$

Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $\alpha \in (0, 1)$. The left-sided (right-sided) fractional derivative of $f$ is the function

$$(-D^\alpha_t f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)-f(a)}{(t-s)^\alpha} ds,$$

$$ (+D^\alpha_t f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(b)-f(s)}{(s-t)^\alpha} ds,$$

where $t \in [a, b)$ and $\Gamma$ is Euler’s gamma function.

**Proposition 1.** (see [6]) The operators $-D^\alpha_t$ and $+D^\alpha_t$ have the properties:

1. If $f_1$ and $f_2$ are defined on $[a, b]$ and $-D^\alpha_t$, $+D^\alpha_t$ exists, then

$$-D^\alpha_t (c_1 f_1 + c_2 f_2)(t) = c_1 (-D^\alpha_t f_1)(t) + c_2 (-D^\alpha_t f_2)(t).$$

2. If $\{\alpha_n\}_{n \geq 0}$ is a real number sequence with $\lim_{n \to \infty} \alpha_n = 1$ then

$$\lim_{n \to \infty} (-D^\alpha_t f)(t) = (-D^1_t f)(t) = \frac{d}{dt} f(t).$$

3. a) If $f(t) = c$, $t \in [a, b]$, $c \in \mathbb{R}$ then

$$(-D^\alpha_t f)(t) = 0.$$  

b) If $f(t) = t^\gamma$, $t \in (a, b]$, $\gamma \in \mathbb{R}$, then

$$(-D^\alpha_t f)(t) = \frac{t^{\gamma-\alpha} \Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}.$$  

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c) If \( f(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} \), then 
\[ (-D_t^\alpha f)(t) = 1. \]  
(6)

4. If \( f_1 \) and \( f_2 \) are analytic functions on \([a, b]\) then 
\[ (-D_t^\alpha(f_1f_2))(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-D_t^{\alpha-k}f_1)(t) \frac{d^k}{(dt)^k}f_2(t), \]  
where \( \frac{d^k}{(dt)^k} = \frac{d}{dt} \circ \frac{d}{dt} \circ ... \circ \frac{d}{dt} \).  
(7)

5. It also holds true 
\[ \int_a^b f_1(t)(-D_t^\alpha f_2)(t)dt = -\int_a^b f_2(t)(+D_t^\alpha f_1)(t)dt. \]  
(8)

6. a) If \( f : [a, b] \to \mathbb{R} \) admits fractional derivatives of order \( a\alpha \), \( a \in \mathbb{N} \), then 
\[ f(t + h) = E_\alpha((ht)^a - D_t^\alpha)f(t), \]  
where \( E_\alpha \) is the Mittag-Leffler function given by 
\[ E_\alpha(t) = \sum_{a=0}^{\infty} \frac{t^{\alpha a}}{\Gamma(1 + \alpha a)}. \]  
(9)

b) If \( f : [a, b] \to \mathbb{R} \) is analytic and \( 0 \in (a, b) \) then the fractional McLaurin series is 
\[ f(t) = \sum_{a=0}^{\infty} \frac{t^{\alpha a}}{\Gamma(1 + \alpha a)}(-D_t^{\alpha a} f)(t) \big|_{t=0}. \]  
(11)

The physical and geometrical interpretation of the fractional derivative on \( \mathbb{R} \) is suggested by the interpretation of the Stieltjes integral, because the integral used in the definition of the fractional derivative is a Riemann-Stieltjes integral [10].

By definition, the left-sided (right-sided) fractional derivative of \( f \), of order \( \alpha, m = [\alpha] + 1 \), is the function 
\[ D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left( \frac{d}{dt} \right)^m \int_{-\infty}^t f(s) - f(0) \frac{(t-s)^{\alpha}}{(s-t)^\alpha} ds, \quad 0 \in (-\infty, t) \]
\[ ^*D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left( -\frac{d}{dt} \right)^m \int_t^{\infty} f(s) - f(0) \frac{(s-t)^{\alpha}}{(s-t)^\alpha} ds, \quad 0 \in (t, \infty). \]  
(12)

If \( \text{supp} f \subset [a, b] \), then \( D_t^\alpha f = -D_t^\alpha f, \quad ^*D_t^\alpha f = +D_t^\alpha f. \)

Let us consider the seminorms 
\[ |x| J^\alpha_{L^2(\mathbb{R})} = \| D_t^\alpha x \|_{L^2(\mathbb{R})} \]
\[ |x| J^\alpha_{R^2(\mathbb{R})} = \| ^*D_t^\alpha x \|_{L^2(\mathbb{R})}, \]
and the norms
\[ \| x \|_{L^2(R)} = \left( \| x \|_{L^2(R)}^2 + |x|^2 \right)^{1/2}, \]
\[ \| x \|_{J^\alpha R_\alpha} = \left( \| x \|_{L^2(R)}^2 + |x|^2 \right)^{1/2}, \]
and \( J^\alpha_{oL}(R), J^\alpha_{oR}(R) \) the closures of \( C_0^\infty(R) \) with respect to the two norms from above, respectively. In \([6]\) it is proved that the operators \( D^\alpha_t \) and \( D^\alpha_t \) satisfy the properties:

**Proposition 2.** Let \( I \subset \mathbb{R} \) and let \( J^\alpha_{oL}(I) \) and \( J^\alpha_{oR}(I) \) be the closures of \( C_0^\infty(I) \) with respect to the norms from above. For any \( x \in J^\beta_{oL}(I), 0 < \alpha < \beta \), the following relation holds:

\[ D^\beta_t x(t) = D^\alpha_t D^\beta_\alpha x(t). \]

For any \( x \in J^\beta_{oR}(I), 0 < \alpha < \beta \), it also holds

\[ D^\beta_\alpha x(t) = D^\alpha_\alpha D^\beta_\alpha x(t). \]

In the following we shall consider the fractional derivatives defined above.

### 3 The fractional osculator bundle of order \( k \) on a differentiable manifold

Let \( \alpha \in (0, 1] \) be fixed and \( M \) a differentiable manifold of dimension \( n \).

Two curves \( \rho, \sigma : I \to \mathbb{R} \), with \( \rho(0) = \sigma(0) = x_0 \in M, 0 \in I \), have a fractional contact \( \alpha \) of order \( k \in \mathbb{N}^* \) in \( x_0 \), if for any \( f \in F(U) \), \( x_0 \in U \), \( U \) a chart on \( M \), it holds

\[ D^\alpha_a t (f \circ \rho)_{t=0} = D^\alpha_a t (f \circ \sigma)_{t=0} \quad (13) \]

where \( a = \overline{1, k} \). The relation \((13)\) is an equivalence relation. The equivalence class \([\rho]_{x_0}^{\alpha k}\) is called the fractional \( k \)-osculator space of \( M \) in \( x_0 \) and it will be denoted by \( Osc^\alpha_{x_0}(M) \). If the curve \( \rho : I \to M \) is given by \( x^i = x^i(t), t \in I, i = \overline{1, n} \), then, considering the formula \((11)\), the class \([\rho]_{x_0}^{\alpha k}\), may be written as

\[ x^i(t) = x^i(0) + \frac{t^\alpha}{\Gamma(1 + \alpha)} D^\alpha t x^i(t)_{t=0} + ... + \frac{t^{\alpha k}}{\Gamma(1 + \alpha k)} D^\alpha t x^i(t)_{t=0}, \quad (14) \]

where \( t \in (-\varepsilon, \varepsilon) \). We shall use the notation

\[ x^i(0) = x^i, \quad y^{(aa)} = \frac{1}{\Gamma(1 + \alpha a)} D^\alpha a x^i(t)_{t=0}, \quad (15) \]
for $i = \overline{1, n}$ and $a = \overline{1, k}$.

By definition, the fractional osculator bundle of order $r$ is the fibre bundle $(\text{Osc}^\alpha(M), M)$ where $\text{Osc}^\alpha(M) = \bigcup_{x_0 \in M} \text{Osc}^\alpha_{x_0}(M)$ and $\pi^\alpha_0 : \text{Osc}^\alpha(M) \to M$ is defined by $\pi^\alpha_0([\rho]^\alpha_{x_0}) = x_0$, $(\forall)[\rho]^\alpha_{x_0} \in \text{Osc}^\alpha(M)$.

For $f \in \mathcal{F}(U)$, the fractional derivative of order $\alpha$, $\alpha \in (0, 1)$, with respect to the variable $x^i$, is defined by

$$
(D^\alpha_{x^i} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x^i} \int_a^x f(t^1, \ldots, t^{i-1}, s, t^{i+1}, \ldots, t^n) - f(x^1, \ldots, x^{i-1}, a^i, x^{i+1}, \ldots, x^n) \, ds,
$$

where $x^i$ are the coordinate functions on $U$, $\frac{\partial}{\partial x^i}$, $i = \overline{1, n}$, is the canonical base of the vector fields on $U$ and $U_{ab} = \{ x \in U, a^i \leq x^i \leq b^i, i = \overline{1, n} \} \subset U$.

Let $U, U' \subset M$ be two charts on $M, U \cap U' \neq \emptyset$ and consider the change of variable

$$
\bar{x}^i = \bar{x}^i(x^1, \ldots, x^n)
$$

with $\det \left( \frac{\partial \bar{x}^i}{\partial x^i} \right) \neq 0$. Let $\{dx^i\}_{i=1, n}$ be the canonical base of 1-forms of $\mathcal{D}^1(U)$

and let us define the 1-forms $d(x^i)^\alpha = \alpha(x^i)^{\alpha-1} dx^i, i = \overline{1, n}$. The exterior differential $d^\alpha : \mathcal{F}(U \cap U') \to \mathcal{D}^1(U \cap U')$ is defined by

$$
d^\alpha = d(x^i)^\alpha D^\alpha_{x^j} = d(\bar{x}^i)^\alpha D^\alpha_{\bar{x}^j}.
$$

Using (18) and the property $D^\alpha_{x^i} \left( \frac{(x^i)^\alpha}{\Gamma(1+\alpha)} \right) = 1$, it follows that

$$
d(x^i)^\alpha = \frac{1}{\Gamma(1 + \alpha)} D^\alpha_{x^i} (x^i)^\alpha d(x^i)^\alpha.
$$

Using the notation

$$
J^\alpha_i(x, \bar{x}) = \frac{1}{\Gamma(1 + \alpha)} D^\alpha_{x^i} (x^i)^\alpha,
$$

from (19) we get

$$
d(x^i)^\alpha = J^\alpha_i(x, \bar{x}) d(x^i)^\alpha.
$$

From (20) it follows that

$$
J^\alpha_i(x, \bar{x}) = \delta^\alpha_i,
$$

Consider $x^i = x^i(t)$ and $\bar{x}^i(t) = \bar{x}^i(x(t))$, $i = \overline{1, n}$, $t \in I$. Applying the operator $D^\alpha_{\bar{x}}$ we get

$$
(D^\alpha_{\bar{x}} \bar{x}^i)(t) = D^\alpha_{\bar{x}}\bar{x}^i(x)(D^\alpha_{\bar{x}} x^j)(t) = J^\alpha_j(\bar{x}, x)(D^\alpha_{\bar{x}} x^j)(t).
$$
Considering the notation from (15) we have
\[ y^{i(\alpha)} = J_{j}^{\alpha}(\bar{x}, x)y^{j(\alpha)}. \] (24)

Also, from (15) we deduce
\[ D_{t}^{\alpha} y^{i(\alpha)} = \frac{\Gamma(\alpha a)}{\Gamma(\alpha(a - 1))} y^{i(\alpha)}, \] (25)

where \( i = 1, \ldots, n \). Applying the operator \( D_{t}^{\alpha} \) in the relation (24) we find
\[ \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} \bar{y}^{i(\alpha)} = \Gamma(1 + \alpha)J_{j}^{\alpha}(\bar{y}^{a(\alpha-1)}, x)y^{j(\alpha)} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} J_{j}^{\alpha}(y^{(a-1)}, y^{\alpha})y^{j(2\alpha)} + \ldots + \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha)}, \] (26)

where \( a = 1, k \).

**Proposition 3.** (see [2], [5])

a) The coordinate transformation on \( \text{Osc}^{(\alpha k)}(M), \) \((x^{i}, y^{i(\alpha)}, ..., y^{i(k\alpha)}) \rightarrow (\bar{x}^{i}, \bar{y}^{i(\alpha)}, ..., \bar{y}^{i(k\alpha)})\) are given by the formulas (17) and (26).

b) The operators \( D_{x}^{\alpha} \) and the 1-forms \((dx)^{i\alpha}, i = 1, n\), transform by the formulas
\[ D_{x}^{\alpha} = J_{j}^{\alpha}(\bar{x}, x)D_{x}^{\alpha} \]
\[ d(\bar{x}^{i\alpha}) = J_{j}^{\alpha}(\bar{x}, x)d(x^{j})^{\alpha}. \] (27)

### 4 The fractional jet bundle of order \( k \) on a differentiable manifold; geometrical objects

By definition, the \( k \)-order fractional jet bundle is the space \( J^{\alpha k}(\mathbb{R}, M) = \mathbb{R} \times \text{Osc}^{k\alpha}(M) \). A system of local coordinates on \( J^{\alpha k}(\mathbb{R}, M) \) will be denoted by \((t, x, y^{(\alpha)}, y^{(2\alpha)}, ..., y^{(k\alpha)})\). Consider the projections \( \pi_{0}^{\alpha k} : J^{\alpha k}(\mathbb{R}, M) \rightarrow M \) defined by
\[ \pi_{0}^{\alpha k}(t, x, y^{(\alpha)}, ..., y^{(k\alpha)}) = x. \] (28)
Let $U, U' \subset M$ be two charts on $M$ with $U \cap U' \neq \emptyset$, $(\pi_0^\alpha)^{-1}(U), (\pi_0^\alpha)^{-1}(U') \subset J^\alpha(\mathbb{R}, M)$ the corresponding charts on $J^\alpha(\mathbb{R}, M)$ and, respectively, the corresponding coordinates $(x^i), (\bar{x}^i)$ and $(t, x^i, y^{(\alpha)}), (t, \bar{x}^i, \bar{y}^{(\alpha)})$. The transformations of coordinates are given by

$$
\bar{x}^i = \bar{x}^i(x^1, ..., x^n) \\
\bar{y}^{(\alpha)} = \bar{J}(x, \bar{x})y^{(\alpha)}.
$$

(29)

Consider the functions $(t)^\alpha, (x^i)^\alpha, (y^{(\alpha)})^\alpha \in \mathcal{F}((\pi_0^\alpha)^{-1}(U))$, the 1-forms $\frac{1}{\Gamma(1+\alpha)}d(t)^\alpha, \frac{1}{\Gamma(1+\alpha)}(x^i)^\alpha$, $\frac{1}{\Gamma(1+\alpha)}d((y^{(\alpha)})^\alpha) \in D^1((\pi_0^\alpha)^{-1}(U))$ and the operators $D_t^\alpha, D_{x^i}^\alpha, D_{y^{(\alpha)}}^\alpha$ on $(\pi_0^\alpha)^{-1}(U), i = 1, n$. The following relations hold:

$$
D_t^\alpha(\frac{1}{\Gamma(1+\alpha)}t^\alpha) = 1, \quad D_{x^i}^\alpha(\frac{1}{\Gamma(1+\alpha)}(x^i)^\alpha) = \delta_i^j, \\
D_{y^{(\alpha)}}^\alpha(\frac{1}{\Gamma(1+\alpha)}(y^{(\alpha)})^\alpha) = \delta_i^j, \quad \frac{1}{\Gamma(1+\alpha)}d(t^\alpha)(D_t^\alpha) = 1, \\
\frac{1}{\Gamma(1+\alpha)}d((x^i)^\alpha)(D_{x^i}^\alpha) = \delta_i^j, \quad \frac{1}{\Gamma(1+\alpha)}d((y^{(\alpha)})^\alpha)(D_{y^{(\alpha)}}^\alpha) = \delta_i^j.
$$

(30)

On $J^\alpha(\mathbb{R}, M)$ we may define the canonical structures

$$
\theta_1 = d(t^\alpha) \otimes (D_t^\alpha + y^{(\alpha)}D_{x^i}^\alpha) \\
\theta_2 = \theta^i \otimes D_{x^i}^\alpha, \quad \theta^i = \frac{1}{\Gamma(1+\alpha)}(d(x^i)^\alpha - y^{(\alpha)}d(t)^\alpha) \\
\bar{\theta}_1 = \bar{\theta}^i \otimes \bar{D}_{y^{(\alpha)}}^\alpha \\
V_i = D_{y^{(\alpha)}}^\alpha.
$$

(31)

Using (29) it is easy to show that the structures (31) have geometrical character. The space of the operators generated by the operators $\{D_t^\alpha, D_{x^i}^\alpha, D_{y^{(\alpha)}}^\alpha\}, i = 1, n$, will be denoted by $\chi^\alpha((\pi_0^\alpha)^{-1}(U))$. For $\alpha \to 1$ the space of these operators represents the space of the vector fields on $\pi_0^{-1}(U)$.

A vector field $\bar{\Gamma} \in \chi^\alpha((\pi_0^\alpha)^{-1}(U))$ is called $FODE$ (fractional ordinary differential equation) iff

$$
d(t)^\alpha(\bar{\Gamma}) = 1 \\
\theta^i(\bar{\Gamma}) = 0,
$$

(32)

for $i = 1, n$. In local coordinates $FODE$ is given by

$$
\bar{\Gamma} = D_t^\alpha + y^{(\alpha)}D_{x^i}^\alpha + F^i D_{y^{(\alpha)}}^\alpha,
$$

(33)
where \( F^i \in C^\infty((\pi_0^\alpha)^{-1}(U)), \ i = 1, n. \) The integral curves of the field \( FODE \) are the solutions of the fractional differential equation (EDF)

\[
D_t^{2\alpha} x^i(t) = F^i(t, x(t), D_t^\alpha x(t)), \quad i = 1, n.
\]

The fractional dynamical connection on \( J^\alpha(\mathbb{R}, M) \) is defined by the fractional tensor fields \( \tilde{H} \) of type \((1,1)\) which satisfy the conditions

\[
\begin{align*}
\tilde{\theta}_1 & \circ \tilde{H} = 0 \\
\tilde{\theta}_2 & \circ \tilde{H} = \tilde{\theta}_2 \\
\tilde{H} \bigg|_{\tilde{V}} & = -i d \bigg|_{\tilde{V}},
\end{align*}
\]

where \( \tilde{V} \) is formed by operators generated by \( \{D_\alpha^\alpha \} \). In the chart \((\pi_0^\alpha)^{-1}(U)\) the fractional tensor field \( \tilde{H} \) has the expression

\[
\begin{align*}
\tilde{H} & = (\frac{1}{\Gamma(1+\alpha)} \left( -y^{i(\alpha)} D_x^\alpha + H_j D_y^{\alpha} \right) \otimes d(t)^\alpha + \\
& \quad (H_j^i dt^\alpha + H_j^i d(x^j)^\alpha + H_j^i d(y^{i(\alpha)})) \otimes D_x^\alpha + \\
& \quad (H^i d(t)^\alpha + H_j^i d(x^j)^\alpha + H_j^i d(y^{i(\alpha)}) \otimes D_y^{\alpha}).
\end{align*}
\]

The tensor field \( \tilde{H} \) has a geometrical character, fact which results by using the relations (29), and is called a \( d^\alpha \)-tensor field. Using the relations (30) and (31) we get

**Proposition 4.**

a) The fractional dynamical connection \( \tilde{H} \), in the chart \((\pi_0^\alpha)^{-1}(U)\), is given by

\[
\tilde{H} = \left( \frac{1}{\Gamma(1+\alpha)} \right) \left( -y^{i(\alpha)} D_x^\alpha + H_j D_y^\alpha \right) \otimes d(t)^\alpha + \\
(D_x^\alpha + H_j^i D_x^\alpha) \otimes d(x^j)^\alpha - D_y^\alpha \otimes d(y^{i(\alpha)})^\alpha.
\]

b) The fractional dynamical connection \( \tilde{H} \) defines a \( f(3, -1) \) fractional structure on \( J^\alpha(\mathbb{R}, M) \), i.e.,

\[
(\tilde{H})^3 = \tilde{H}.
\]

c) The fractional tensor fields \( \tilde{l} \) and \( \tilde{m} \) which are defined by

\[
\begin{align*}
\tilde{l} & = \tilde{H} \circ \tilde{H} \\
\tilde{m} & = - \tilde{H} \circ \tilde{H} + I,
\end{align*}
\]
where $I$ is the identity map, satisfy the relations
\[
\begin{align*}
\alpha I \circ I & = \alpha I, \quad \alpha m \circ \bar{m} = \bar{m} \circ \alpha I, \quad \alpha I + \bar{m} = I \\
\alpha (D^\alpha_I) & = -y^{(\alpha)}(t)D_{x_1} + (y^{(\alpha)}(t)H^j_i + H^j)D_{y^{(\alpha)}} \\
\alpha (D^\alpha_{x_1}) & = D^\alpha_I, \quad \alpha (D^\alpha_{y^{(\alpha)}}) = D^\alpha_{y^{(\alpha)}} \\
\bar{m}(D^\alpha_I) & = D^\alpha_I + y^{(\alpha)}(t)D^\alpha_{x_1} + (y^{(\alpha)}(t)H^j_i + H^j)D^\alpha_{y^{(\alpha)}} \\
\bar{m}(D^\alpha_{x_1}) & = 0, \quad \bar{m}(D^\alpha_{y^{(\alpha)}}) = 0.
\end{align*}
\] (39)

d) The fractional vector field $\bar{\Gamma} \in \chi^\alpha(J^\alpha(\mathbb{R}, M))$ given by
\[
\bar{\Gamma} = \bar{m}(D^\alpha_I) = D^\alpha_I + y^{(\alpha)}(t)D^\alpha_{x_1} + (y^{(\alpha)}(t)H^j_i + H^j)D^\alpha_{y^{(\alpha)}}
\] (40)
defines a field FODE associated to the fractional dynamical connection. The integral curves are the solutions of the EDF
\[
D^\alpha_{x^i}(t) = D^\alpha_{x^i}(t)H^j_i + \bar{\Gamma}(1 + \alpha)H^j
\] (41)
where $H^j_i$ and $\bar{\Gamma}^j$ are functions of $(t, x(t), y^{(\alpha)}(t))$.

Let $L \in C^\infty(J^\alpha(\mathbb{R}, M))$ be a fractional Lagrange function. By definition, the Cartan fractional 1-form is the 1-form $\bar{\theta}_L$ given by
\[
\bar{\theta}_L = Ld(t)^\alpha + \bar{S}(L).
\] (42)
We call the Cartan fractional 2-form, the 2-form $\bar{\omega}_L$ given by
\[
\bar{\omega}_L = d^\alpha \bar{\theta}_L
\] (43)
where $d^\alpha$ is the fractional exterior differential:
\[
d^\alpha = d(t)^\alpha D^\alpha_I + d(x^i)^\alpha D^\alpha_{x_i} + d(y^{(\alpha)})^\alpha D^\alpha_{y^{(\alpha)}}.
\] (44)

In the chart $(\pi\alpha_0)^{-1}(U)$, $\bar{\theta}_L$ and $\bar{\omega}_L$ are given by
\[
\begin{align*}
\bar{\theta}_L & = (L - \frac{1}{\Gamma(1+\alpha)}y^{(\alpha)}(t)L)d(t)^\alpha + \frac{1}{\Gamma(1+\alpha)}D^\alpha_{y^{(\alpha)}}(L)d(x^i)^\alpha \\
\bar{\omega}_L & = A_i d(t)^\alpha \wedge d(x^i)^\alpha + B_i d(t)^\alpha \wedge d(y^{(\alpha)})^\alpha + \\
& + A_{ij} d(x^i)^\alpha \wedge d(x^j)^\alpha + B_{ij} d(x^i)^\alpha \wedge d(y^{(\alpha)})^\alpha,
\end{align*}
\] (45)
where
\[
A_i = \frac{1}{\Gamma(1+\alpha)} D_t^\alpha D^\alpha_{y^i(\alpha)}(L) + \frac{1}{\Gamma(1+\alpha)} y^j(\alpha) D^\alpha_{x^2} D^\alpha_{y^i(\alpha)}(L) - D^\alpha_{x^2}(L)
\]
\[
B_i = \frac{1}{\Gamma(1+\alpha)} D^\alpha_{y^i(\alpha)} (y^j(\alpha) D_j^\alpha(\alpha)(L))
\]
\[
A_{ij} = D^\alpha_{x^i} D^\alpha_{y^j(\alpha)}(L), \quad B_{ij} = - D^\alpha_{y^j(\alpha)} D^\alpha_{y^i(\alpha)}(L).
\]

**Proposition 5.** If \( L \) is regular (i.e., \( \det \left( \frac{\partial^2 L}{\partial y^i(\alpha) \partial y^j(\alpha)} \right) \neq 0 \) then there exists a fractional field FODE \( \Gamma^\alpha_L \) such that \( i_{\Gamma^\alpha_L} \omega^\alpha_L = 0 \). In the chart \( (\pi^\alpha_0)^{-1}(U) \) we have
\[
\Gamma^\alpha_L = D^\alpha_t + y^j(\alpha) D_{x^j} + M^i D^\alpha_{y^i(\alpha)},
\]
where
\[
M^i = g^{ik}(D^\alpha_k(L) - d^\alpha_t (\frac{\partial^\alpha t}{\partial y^i(\alpha)}))
\]
\[
d^\alpha_t = D^\alpha_t + y^i(\alpha) D^\alpha_{x^i} \quad (g^{ik}) = (D^\alpha_{y^i(\alpha)} D^\alpha_{y^j(\alpha)}(L))^{-1}.
\]

An important structure on \( J^\alpha(\mathbb{R}, M) \) is described by the fractional Euler-Lagrange equations. Let \( c : t \in [0, 1] \rightarrow (x^i(t)) \subseteq M \) be a parameterized curve, such that \( Imc \subseteq U \subseteq M \). The extension of the curve \( c \) to \( J^\alpha(\mathbb{R}, M) \) is the curve \( c^\alpha : t \in [0, 1] \rightarrow (t, x^i(t), y^i(\alpha)(t)) \subseteq J^\alpha(\mathbb{R}, M) \). Consider \( L \in C^\infty(J^\alpha(\mathbb{R}, M)) \). The action of \( L \) along the curve \( c^\alpha \) is defined by
\[
\mathcal{A}(c^\alpha) = \int_0^1 L(t, x(t), y^\alpha(t)) dt.
\]

Let \( c_\varepsilon : t \in [0, 1] \rightarrow (x^i(t, \varepsilon)) \subseteq M \) be a family of curves, where \( \varepsilon \) is sufficiently small so that \( Imc_\varepsilon \subseteq U \), \( c_0(t) = c(t) \), \( D^\alpha_\varepsilon c_\varepsilon(0) = D^\alpha_\varepsilon c_\varepsilon(1) = 0 \). The action of \( L \) along the curves \( c_\varepsilon \) is
\[
\mathcal{A}(c^\alpha_\varepsilon) = \int_0^1 L(t, x(t, \varepsilon), y^\alpha(t, \varepsilon)) dt,
\]
where \( y^i(\alpha)(t, \varepsilon) = \frac{1}{\Gamma(1+\alpha)} D^\alpha_t x^i(t, \varepsilon) \). The action (50) has a fractional extremal value if
\[
D^\alpha_{\varepsilon} \mathcal{A}(c^\alpha_\varepsilon) |_{\varepsilon=0} = 0.
\]
The action (50) has an extremal value if
\[
D^1_{\varepsilon} \mathcal{A}(c^\alpha_\varepsilon) |_{\varepsilon=0} = 0.
\]
Using the properties of the fractional derivative we obtain
Proposition 6. a) A necessary condition for the action (50) to reach a fractional extremal value is that \(c(t)\) satisfies the fractional Euler-Lagrange equations

\[
D^\alpha_L - d^\alpha_t(D^\alpha_{y^\alpha} L) = 0
\]

where \(i = 1, n\).

b) A necessary condition for the action (50) to reach an extremal value is that \(c(t)\) satisfies the Euler-Lagrange equations

\[
D^1_L - d^1_t(D^1_{y^\alpha} L) = 0
\]

where \(i = 1, n\).

The equations (53) may be written in the form

\[
D^\alpha_L - d^\alpha_t(D^\alpha_{y^\alpha} L) - y^{i(2\alpha)} D^\alpha_{y^\alpha} (D^\alpha_{y^\alpha} L) = 0,
\]

for \(i = 1, n\). The equations (54) may be written as

\[
\frac{\partial L}{\partial x^i} - d^i_t \left( \frac{\partial L}{\partial y^\alpha} \right) - y^{i(2\alpha)} \frac{\partial^2 L}{\partial y^\alpha \partial y^\alpha} = 0,
\]

where \(i = 1, n\). Let us denote by

\[
g^\alpha_{ij} = D^\alpha_{y^\alpha} (D^\alpha_{y^\alpha} L),
\]

and by \( (g^\alpha_{ij})^{-1} = (g^\alpha_{ij})^{-1} \), if \( \det(g^\alpha_{ij}) \neq 0 \). From (55) and from Proposition 5, we get the fractional field \( FODE \Gamma^L_{\alpha} \) associated to \( L \).

Let \( c : t \in [0, 1] \rightarrow (x^i(t)) \subset U \) be a parameterized curve. The extension of \( c \) to \( J^\alpha_k(\mathbb{R}, M) \) is the curve \( c^\alpha_k : t \in [0, 1] \rightarrow (t, x^i(t), y^\alpha(t)) \in J^\alpha_k(\mathbb{R}, M) \), \( a = 1, k \). Let \( L : J^\alpha_k(\mathbb{R}, M) \rightarrow \mathbb{R} \) be a Lagrange function. The action of \( L \) along the curve \( c^\alpha_k \) is

\[
\mathcal{A}(c^\alpha_k) = \int_0^1 L(t, x^i(t), y^\alpha(t))dt.
\]

Let \( c_\varepsilon : t \in [0, 1] \rightarrow (x^i(t, \varepsilon)) \in M \) be a family of curves, where the absolute value of \( \varepsilon \) is sufficiently small so that \( Im c_\varepsilon \subset U \subset M \), \( c_0(t) = c(t) \), \( D^\alpha_{y^\alpha} c(\varepsilon) |_{\varepsilon=0} = D^\alpha_{y^\alpha} c(\varepsilon) |_{\varepsilon=1} = 0 \). The action of \( L \) on the curve \( c_\varepsilon \) is given by

\[
\mathcal{A}(c_\varepsilon^\alpha_k) = \int_0^1 L(t, x(t, \varepsilon), y^\alpha(t, \varepsilon))dt
\]
where $y^{(aa)}(t, \varepsilon) = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t, \varepsilon)$, $a = \frac{1}{k}$. The action (59) has a fractional extremal value if

$$D_\varepsilon^\alpha (A(c^{\alpha k}')) |_{\varepsilon=0} = 0.$$  \hspace{1cm} (60)

The action (59) has an extremal value if

$$D_\varepsilon^1 (A(c^{\alpha k}')) |_{\varepsilon=0} = 0.$$  \hspace{1cm} (61)

**Proposition 7.** a) A necessary condition for the action (58) to reach a fractional extremal value is that $c(t)$ satisfies the fractional Euler-Lagrange equations

$$D_\alpha^\alpha L + \sum_{a=1}^{k} (-1)^a d^\alpha_t (D_{y^{(aa)}}^\alpha L) = 0,$$  \hspace{1cm} (62)

where

$$d^\alpha_t = D_t^\alpha + y^{(\alpha)} D_{x^i}^\alpha + y^{(2\alpha)} D_{y^{(\alpha)}}^\alpha + ... + y^{(\alpha a)} D_{y^{(\alpha(a-1))}}^\alpha,$$  \hspace{1cm} (63)

and $i = \frac{1}{k} n$.

b) A necessary condition that the action (58) reaches an extremal value is that $c(t)$ satisfies the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} + \sum_{a=1}^{k} (-1)^a d^\alpha_t (D_{y^{(aa)}}^\alpha L) = 0,$$  \hspace{1cm} (64)

where

$$d^\alpha_t = D_t^1 + y^{(\alpha)} D_{x^i}^1 + ... + y^{(\alpha a)} D_{y^{(\alpha(a-1))}}^1.$$  \hspace{1cm} (65)

**Example.** Consider the fractional differential equation

$$\frac{c^{\Gamma(1+\gamma)}}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}(t) f(t) + a_1 \Gamma(1+2\alpha) y^{(2\alpha)} + a_2 \Gamma(1+3\alpha) y^{(3\alpha)} = 0.$$  \hspace{1cm} (66)

The equation (66) is the fractional Euler-Lagrange equation (62) for the function

$$L = \frac{c^{\Gamma(1+\gamma)}}{1+\gamma-\alpha} x^{\gamma} - a_1 \Gamma(1+2\alpha)(y^{\alpha})^\alpha + a_2 \Gamma(1+3\alpha)(y^{2\alpha})^\alpha.$$  \hspace{1cm}

The equation (66) is the fractional Euler-Lagrange equation (64) for the function

$$L = \frac{c^{\Gamma(1+\gamma)} x^{\gamma-\alpha+1}}{\Gamma(1+\gamma-\alpha)(1+\gamma-\alpha)} f - \frac{a_1}{2} \Gamma(1+2\alpha)(y^{\alpha})^2 + \frac{a_2}{2} \Gamma(1+3\alpha)(y^{2\alpha})^2.$$  \hspace{1cm}
5 Examples and applications

1. The nonhomogeneous Bagley-Torvik equation

The dynamics of a flat rigid body embedded in a Newton fluid is described by the equation

\[ aD_t^2 x(t) + bD_t^{3/2} x(t) + cx(t) - f(t) = 0, \quad (67) \]

where \(a, b, c \in \mathbb{R}\) and the initial conditions are \(x(0) = 0, D_t^1 x(0) = 0\). The equation (67) is a fractional differential equation on the bundle \(J^\alpha(\mathbb{R}, \mathbb{R})\) for \(\alpha = \frac{1}{4}\). Indeed, let’s consider the fractional differential equation

\[ aD_t^8 x(t) + bD_t^{6\alpha} x(t) + cx(t) - f(t) = 0, \quad (68) \]

with \(\alpha > 0\). For \(\alpha = \frac{1}{4}\) the equation (68) reduces to (67). With the notations (15), the equation (68) becomes

\[ a\Gamma(1 + 8\alpha)y^{(8\alpha)}(t) + b\Gamma(1 + 6\alpha)y^{(6\alpha)}(t) + cx(t) - f(t) = 0. \quad (69) \]

On the bundle \(J^{4\alpha}(\mathbb{R}, \mathbb{R})\) let us consider the Lagrange function

\[ L(t, x, y^{(3\alpha)}, y^{(4\alpha)}) = \frac{1}{2}cx^2 - fx - \frac{b}{2}\Gamma(1 + 6\alpha)(y^{(3\alpha)})^2 + \frac{a}{2}\Gamma(1 + 8\alpha)(y^{(4\alpha)})^2. \quad (70) \]

Using the relation (65), the Euler-Lagrange equation for (70) is

\[ D_x^1 L - D_t^{3\alpha}(D_y^{(3\alpha)} L) + D_t^{4\alpha}(D_y^{(4\alpha)} L) = \]

\[ cx - f + b\Gamma(1 + 6\alpha)D_t^{3\alpha}y^{(3\alpha)} + a\Gamma(1 + 8\alpha)D_t^{4\alpha}y^{(4\alpha)} = \]

\[ cx - f + b\Gamma(1 + 6\alpha)y^{(6\alpha)} + a\Gamma(1 + 8\alpha)y^{(8\alpha)} = 0. \quad (71) \]

**Proposition 8.** The equation (67) represents the Euler-Lagrange equation on the bundle \(J^{4\alpha}(\mathbb{R}, \mathbb{R})\) for \(\alpha = \frac{1}{4}\), with the Lagrange function given by

\[ L(t, x, y^{(3/2)}, y^{(2)}) = \frac{1}{2}cx^2 - fx - \frac{b}{2}\Gamma(5/2)(y^{(3/2)})^2 + \]

\[ \frac{a}{2}\Gamma(3)(y^{(2)})^2. \quad (72) \]

2. Differential equations of order one, two and three which admit fractional Lagrangians

The following differential equations don’t have classical Lagrangians such that the Euler-Lagrange equation represents the given equation:

\[ \dot{x}(t) + V_1(t, x) = 0, \quad V_1(t, x) = \frac{\partial U_1(t, x)}{\partial x}, \quad (73) \]
\[ \ddot{x}(t) + a_1 \dot{x}(t) + V_2(t, x) = 0, \quad V_2(t, x) = \frac{\partial U_2(t, x)}{\partial x}, \] (74)

\[ \ddot{x}(t) + a_2 \dot{x}(t) + a_1 \dot{x}(t) + V_3(t, x) = 0, \quad V_3(t, x) = \frac{\partial U_3(t, x)}{\partial x}. \] (75)

Let us associate the fractional equations from below to the equations (73), (74) and (75), respectively:

\[ D^{2\alpha}_t x(t) + V_1(t, x) = 0, \] (76)

\[ D^{4\alpha}_t x(t) + a_1 D^{2\alpha}_t x(t) + V_2(t, x) = 0, \] (77)

\[ D^{6\alpha}_t x(t) + a_2 D^{4\alpha}_t x(t) + a_1 D^{2\alpha}_t x(t) + V_3(t, x) = 0. \] (78)

**Proposition 9.**

a) Let \( J^\alpha(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \) be the fractional bundle and consider \( L : J^\alpha(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \) given by

\[ L(t, x, y^{(\alpha)}) = U_1(t, x) - \frac{1}{2} \Gamma(1 + 2\alpha)(y^{(\alpha)})^2. \] (79)

The Euler-Lagrange equation of (79) is

\[ \frac{\partial L}{\partial x} - D^\alpha_t \left( \frac{\partial L}{\partial y^{(\alpha)}} \right) = \frac{\partial U_1(t, x)}{\partial x} + \Gamma(1 + 2\alpha)y^{(2\alpha)} = V_1(t, x) + D^{2\alpha}_t x(t) = 0. \] (80)

b) Let \( J^{2\alpha}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \) be the fractional bundle and the Lagrangian \( L : J^{2\alpha}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \) given by

\[ L(t, x, y^{(\alpha)}, y^{(2\alpha)}) = U_2(t, x) - \frac{1}{2} a_1 \Gamma(1 + 2\alpha)(y^{(\alpha)})^2 + \frac{1}{2} \Gamma(1 + 4\alpha)(y^{(2\alpha)})^2. \] (81)

The Euler-Lagrange equation of (81) is

\[ \frac{\partial L}{\partial x} - D^\alpha_t \left( \frac{\partial L}{\partial y^{(\alpha)}} \right) + D^{2\alpha}_t \left( \frac{\partial L}{\partial y^{(2\alpha)}} \right) = V_2(t, x) + a_1 \Gamma(1 + 2\alpha)y^{(2\alpha)} + a_2 \Gamma(1 + 4\alpha)y^{(4\alpha)} = V_2(t, x) + a_1 D^{2\alpha}_t x(t) + D^{4\alpha}_t x(t) = 0. \] (82)

c) Let \( J^{3\alpha}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \) be the fractional bundle and \( L : J^{3\alpha}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \) given by

\[ L(t, x, y^{(\alpha)}, y^{(2\alpha)}, y^{(3\alpha)}) = V_3(t, x) - \frac{a_1}{2} \Gamma(1 + 2\alpha)(y^{(\alpha)})^2 + \frac{a_2}{2} \Gamma(1 + 4\alpha)(y^{(2\alpha)})^2 - \frac{1}{2} \Gamma(1 + 6\alpha)(y^{(3\alpha)})^2. \] (83)
The Euler-Lagrange equation of (83) is

\[
\frac{\partial L}{\partial x} - D_t^\alpha \left( \frac{\partial L}{\partial y'} \right) + D_t^{2\alpha} \left( \frac{\partial L}{\partial y''} \right) - D_t^{3\alpha} \left( \frac{\partial L}{\partial y'''(t)} \right) = V_3(t, x) + \\
a_1 \Gamma(1 + 2\alpha) y^{(2\alpha)} + a_2 \Gamma(1 + 4\alpha) y^{(4\alpha)} + \Gamma(1 + 6\alpha) y^{(6\alpha)} = 0.
\]

(84)

\[
V_3(t, x) + a_1 D_t^{2\alpha} x(t) + a_2 D_t^{4\alpha} x(t) + D_t^{6\alpha} x(t) = 0.
\]

For \( \alpha = \frac{1}{2} \) we obtain the fractional Lagrangians that describe the equations (73), (74), (75), respectively

\[
L(t, x, y^{(1/2)}) = U_1(t, x) - \frac{1}{2} \Gamma(2)(y^{(1/2)})^2
\]

\[
L(t, x, y^{(1/2)}, y^{(1)}) = U_2(t, x) - \frac{1}{2} a_1 \Gamma(2)(y^{(1/2)})^2 + \frac{1}{2} \Gamma(3)(y^{(1)})^2
\]

\[
L(t, x, y^{(1/2)}, y^{(1)}, y^{(3/2)}) = U_3(t, x) - \frac{a_1}{2} \Gamma(2)(y^{(1/2)})^2 + \\
\frac{a_2}{2} \Gamma(2)(y^{(1)})^2 - \frac{1}{2} \Gamma(4)(y^{(3/2)})^2.
\]

(85)

In the category of the equations (74) and (75) there are:

a) the nonhomogeneous classical friction equation

\[
m \ddot{x}(t) + \gamma \dot{x}(t) - \frac{\partial U(t, x)}{\partial x} = 0,
\]

(86)

b) the nonhomogeneous model of Phillips

\[
\ddot{x}(t) + a_1 \dot{x}(t) + b_1 x(t) + f(t) = 0,
\]

(87)

c) the nonhomogeneous business cycle with innovation

\[
\ddot{y}(t) + a_2 \dot{y}(t) + a_1 \dot{y}(t) + b_1 x(t) + f(t) = 0.
\]

(88)

Conclusions

The paper presents the main differentiable structures on \( J^\alpha(\mathbb{R}, M) \), in order to describe fractional differential equations and ordinary differential equations, using Lagrange functions defined on \( J^\alpha(\mathbb{R}, M) \).

With the help of the methods shown, there may be analyzed other models, such as those found in [4] and [11].

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