Relativistic electrons on a rotating spherical magnetic dipole: surface orbitals

James M. Gelb
Department of Physics
University of Texas at Arlington
Arlington, Texas 76019

Kaundinya S. Gopinath and Dallas C. Kennedy
Department of Physics
University of Florida at Gainesville
Gainesville, Florida 32611

(December 25, 2021)

PACS numbers: 97.60.Gb, 97.60.Jd, 71.70.Di, 73.20.At, 03.65.Sq

1. INTRODUCTION

Neutron star pulsars are known to have surface magnetic fields of up to $10^{12}$−$13$ Gauss [1−3]. The magnetic field lines dragged in by neutron star collapse are presumably squeezed by magnetic flux conservation in the contracting plasma. However, that assumption does not answer the question of how the magnetic field sustains itself at later times. The crust, and at least part of the interior of a neutron star, have electrons and protons. Their internal macroscopic currents might also affect the field, if not actually create it, but these internal currents are in turn strongly affected by the intense field. We assume that such currents are essentially electronic. If the density of electrons is low and/or the magnetic field large, a phenomenon akin to the quantum Hall effect [4] should be expected.

We select a simplified system retaining some features of a realistic neutron star. This system is chosen to isolate quantum Hall-like surface states: a charged particle constrained to move on the surface of a sphere of radius $R$ threaded by an intense magnetic dipole field. The treatment is restricted to surface states as their distinctive properties may someday be observable. The sphere rotates with angular velocity $\Omega$, not necessarily parallel to the magnetic axis. Our main discussion assumes that the two axes are parallel in order to estimate the effect of rotation. This is followed by the general case of tilted axes, which is not greatly different for a moderate rotation rate. The problem can be treated relativistically by use of the appropriate metric for the rotating frame. The surface orbits serve as a prelude to the three-dimensional tilted, rotating system, which we treat in a separate paper [5].

Neglecting rotation, the distinct physical regimes can be characterized by comparison of three dimensionless parameters:

$$\beta_0 \equiv |q|B_0R/(2mc^2)$$
$$\epsilon \equiv E/mc^2 \equiv \beta_0\eta$$
$$l_\phi \equiv 2cP_\phi/(qB_0R^2)$$

which are, respectively, the magnetic field strength $B_0$ in rescaled units at the magnetic poles; particle energy $E$ in units of the rest mass; and particle canonical azimuthal angular momentum $P_\phi$ in rescaled units about the rotation axis. For electrons on a neutron star with the strongest measured fields, $\beta_0 \approx (0.1)R/(2\lambda_C) \sim 2.5 \times 10^{15}$, where $\lambda_C$ is the electron Compton wavelength $= 2\pi\hbar/(mc)$. The relevant energy scale is then set not by $mc^2$, but by $\eta = \epsilon/\beta_0$.

The regime of $\eta \gtrsim 1$ is the ultrahigh energy case, $E \approx 10^{21}$ eV or higher, depending on $B_0$ and $R$ but not on $m$. The magnetic field $\beta_0$ sets the scale for $P_\phi$: when $|l_\phi| \sim 1$, the ultrahigh $P_\phi$ case, the effect of $P_\phi$ can overcome the inhibiting effect of the field. When $\eta$ and $l_\phi$ are small, the charged particle has no allowed region of motion on the spherical surface except very close to the rotational and magnetic poles. A large $l_\phi$ allows narrow regions away from the poles, but only $\eta \gtrsim 1$ allows charged particle motion over a substantial portion of the sphere.

The rotational angular velocity is rescaled to $\tilde{\omega} = \Omega R/c$, which is $\lesssim 0.1$ in realistic cases [4, 5]. We take $\tilde{\omega} = 0.1$ throughout as illustrative, being an order-of-magnitude upper limit on pulsar rotation [6].

The simplest treatments, in the limit of infinite field strength, of charged particles trapped on spheres by intense magnetic fields result in particles frozen in place in the crust [2−4]. The treatment here adds the feature of expanding the particle motion in inverse powers of the field strength. Although electrons are stripped from the neutron star surface by the rotation-induced electric field, the bulk of electrons remain in the crust to prevent significant charge separation, with the surface sheathed by a thin space charge. The surface space charge is stabilized by the Coulomb force (with the positive crystal)
opposing the induced electric field. Only a small fraction of electrons are accelerated into the stellar wind [3]. The application of this paper is to quantum single- and many-body states where radiation emission is neglected. This is exact for charged particles in their ground states or in excited one-body states unable to decay by Pauli exclusion blocking in the presence of other fermions. We also seek a general classification of possible orbitals, based on the kinematic parameters $\epsilon, \eta,$ and $t_0$ and rotation $\dot{\omega}$.

2. CLASSICAL KINEMATICS

We explicitly show all factors of $c$ and, in Sect. IV, of $\hbar$. Thus $x_\mu = (x_0, \mathbf{x}) = (ct, \mathbf{x})$. The metric has dimensions with signature $(+ ---)$.

2.1 Generally covariant Lagrangian

The general relativistic action of a particle of mass $m$ and charge $q$ in an external electromagnetic and gravitational field is

$$S = -mc \int \sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu} + \frac{q}{c} \int A_\mu(x)dx^\mu \tag{1}$$

with fixed endpoints in parameter-independent form [8]. The path parameter is the proper time $\tau$. Then if

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}, \tag{2}$$

the general relativistic Lagrangian of a particle in the external fields $A_\mu$ and $g_{\mu\nu}$ is

$$L = -mc\sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} + \frac{q}{c} A_\mu \dot{x}^\mu. \tag{3}$$

The canonical momenta are given by

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -mc g_{\mu\nu} \dot{x}^\nu + \frac{q}{c} A_\mu. \tag{4}$$

Using the identity

$$g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\nu, \tag{5}$$

Eq. (4) gives the constraint

$$g^{\mu\nu}(P_\mu - \frac{q}{c} A_\mu)(P_\nu - \frac{q}{c} A_\nu) = (mc)^2. \tag{6}$$

The equations of motion are given by

$$\frac{\partial L}{\partial x^\mu} - \frac{d P_\mu}{d\tau} = 0. \tag{7}$$

2.2 Two-dimensional rotating sphere

We choose axes so that the magnetic dipole is along the $\theta = 0$ direction. The threading dipole magnetic field has polar strength $B_0$:

$$A_\theta = 0 \tag{8}$$

$$A_\varphi = 0 \tag{9}$$

$$A_\phi = (B_0 R^2/2) \sin^2 \theta, \tag{10}$$

where $A_\phi$ is defined in the rotating spherical coordinates.

The rotation axis is tilted at an angle $\theta_0$ with respect to the magnetic dipole in the $\phi = 0, \pi$ plane (Fig. 1). (This apparently strange choice of coordinates is motivated by the fact that the magnetic field still dominates the rotational effects.) The metric in a spherical polar coordinate system $(r, \theta, \phi)$ rotating with the sphere is given by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2(1 - \frac{\omega^2}{c^2})dt^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) - 2cr\omega \sin \theta dt d\phi - 2cr\omega d\theta d\phi. \tag{11}$$

The vector $\omega$ is defined from the rotational angular velocity vector $\Omega$ by $\omega = \Omega \times r/c$, with components

$$\omega_\theta = \dot{\omega} \sin \theta_0 \cos \theta - \sin \theta_0 \sin \phi \cos \phi \quad ,$$

$$\omega_\phi = -\dot{\omega} \sin \theta_0 \cos \phi \quad ,$$

$$\omega^2 = \omega_\theta^2 + \omega_\phi^2. \tag{12}$$

These are the appropriate generalizations to the case $\theta_0 \neq 0$ [8].

FIG. 1. The sphere of radius $R$, threaded by a magnetic dipole field $\mathbf{M}$, and rotating with angular velocity $\Omega$ which is tilted by angle $\theta_0$ with respect to dipole.

The Lagrangian is expressed in terms of the proper time $\tau$, with $x^\mu \equiv dx^\mu/d\tau$. Eq. (7) are three equations, one for each momentum component, suppressing radial motion. Since the Lagrangian does not explicitly depend on $t$, we have

$$\frac{\partial L}{\partial t} = 0 \quad , \tag{13}$$

which implies
Thus the energy \( E = P_0 \) is conserved. The equations of motion for \( \phi \) and \( \theta \) are non-trivial:

\[
\frac{\partial L}{\partial \phi} - \frac{dP_\phi}{d\tau} = 0
\]

and

\[
\frac{\partial L}{\partial \theta} - \frac{dP_\theta}{d\tau} = 0 .
\]

The constraint (6) is

\[
g^{00} P_0^2 + 2g^{0i} P_0 \left( P_i - \frac{q}{c} A_i \right) + g^{ij} \left( P_i - \frac{q}{c} A_i \right) \left( P_j - \frac{q}{c} A_j \right) = (mc)^2 ,
\]

including both \( g^{00} \) and \( g^{ij} \) terms. The contravariant metric components are

\[
g^{00} = 1/c^2 , \quad g^{rr} = -1/r^2 , \quad g^{\theta \theta} = -(1 - \omega_\phi^2)/r^2 , \quad g^{\phi \phi} = -(1 - \omega_\phi^2)/r^2 \sin^2 \theta , \quad g^{0\phi} = -\omega_\phi/(cr \sin \theta) , \quad g^{\theta 0} = -\omega_\phi/(cr) ,
\]

where \( r = R \) for our case. Along the actual worldpath in spacetime, the constraint \( g_{\mu\nu}x^\mu x^\nu = c^2 \) obtains; this condition is valid after varying the action and simplifies the equations of motion.

### 3. CLASSICAL ANALYSIS: ZERO TILT

A simplified treatment of rotation assumes zero tilt, \( \theta_0 = 0 \). Eq. (8-10) define the dipole field, with \( A_\phi \) defined in the rotating spherical coordinates. If \( \theta_0 = 0 \), the Lagrangian does not explicitly depend on \( \phi \), and

\[
\frac{dP_\phi}{d\tau} = 0 \quad .
\]

yielding another constant of motion, the canonical azimuthal angular momentum \( P_\phi \), along with \( E \). Further, we define

\[
l_\theta = 2cP_\phi/(qBR^2) = \left(1/\beta_0\right) \left\{ -1 + e^2 - 2\omega \beta_0 [l_\phi - \sin^2 \theta] - \left[1/\sin^2 \theta - \omega^2\right] \beta_0^2 [l_\phi - \sin^2 \theta]^2 \right\}^{1/2} .
\]

The opposite sign of the square root, not shown, is also valid. We consider Eq. (22) in three different limits of physical interest, with \( q > 0 \). The \( q < 0 \) case can be obtained from the \( q > 0 \) case by reversing the sign of \( l_\phi \) and \( \omega \).

While the \( \phi(t) \) motion is trivial, the polar motion \( \theta(t) \) is not. The two angular motions, both periodic, decouple from one another because \( l_\phi \) is conserved. Without radial motion, the polar motion alone is a closed one-dimensional system. The two periods, \( \tau_\phi \) and \( \tau_\theta \), are in general not equal or even commensurate: their ratio \( \tau_\phi/\tau_\theta \) is not necessarily a rational number \( k_\phi/k_\theta \), where \( (k_\phi, k_\theta) \) are a pair of integers with no common divisors. If the periods are commensurate, then the orbits can be arbitrarily complex, but close after a time \( \tau_{\text{closure}} = k_\theta \tau_\theta = k_\phi \tau_\phi \); otherwise the orbits never close. We compute below the magnetic flux \( \Phi \) enclosed by a pole orbit (see III.C), but the flux is well-defined only if the orbit is closed. For very large fields, nonetheless, the variation of \( \theta \) is \( \mathcal{O}(\epsilon/\beta_0) \) and tiny in these two cases (ultrahigh \( P_\phi \) and localized pole orbits), and we define the enclosed flux by one complete revolution of \( \phi \) from 0 to \( 2\pi \) at the approximately constant polar angle \( \theta_{\text{mid}} \):

\[
\Phi(\theta_{\text{mid}}) = \pi BR^2 \left[ 1 - \cos 2\theta_{\text{mid}} \right] = \left(2\pi \hbar/c\right) \left(\frac{mcR}{\hbar} \right) \beta_0 \left[ 1 - \cos 2\theta_{\text{mid}} \right] .
\]

In general, this flux is macroscopically large, a product of two large dimensionless factors and the small elementary flux quantum.

In a uniform field, the particle’s cyclotron radius and magnetic flux enclosed are constant. The enclosed magnetic flux is still conserved, as an adiabatic invariant, for slowly-varying fields \( \mathbf{B} \). A charged particle on a sphere in the present configuration would, in general, see a rapidly varying field. But if the field is intense \( (\beta_0 \to \infty) \) and the energy small \( (\eta \ll 1) \), the variation of the particle’s orbit from constant \( \theta \) is higher order in \( 1/\beta_0 \), and the enclosed magnetic flux is quasi-invariant (see also Sect. V).

#### 3.1 Ultrahigh energy orbits

In the limit of very high energy, i.e., \( \epsilon \gg 1 \), Eq. (22) becomes

\[
l_\theta = \sqrt{\eta^2 + 2\omega \eta \sin^2 \theta - \left[1 - \omega^2 \sin^2 \theta\right] \sin^2 \theta} \quad ,
\]

where \( P_\phi \) has been neglected. That is, \( \eta \sim \mathcal{O}(1) \) and \( |l_\phi| \ll 1 \). In Fig. \( \mathbf{3} \) we plot the right hand side versus \( \cos \theta \) and various values of \( \eta \). We find that, for values of energy \( \eta < \) a critical value \( \eta_c(\omega) \), there are four turning points enclosing two distinct allowed regions for the particle between the poles and the equator, with two of the four turning points very near the poles. (These polar turning points are nonzero because of the centrifugal
effect of \( P_{\phi} \) and vanish as \( P_{\phi} \) vanishes.) As we increase the energy, the allowed regions expand and, for \( \eta > \eta_c \), merge into a single region occupying essentially the whole surface of the sphere. In the latter case, two of the turning points merge and disappear. The turning points in the ultrahigh energy limit are:

\[
\sin^2 \theta_{\mp} = \frac{1 - 2\bar{\omega}\eta \pm \sqrt{1 - 4\bar{\omega}\eta}}{2\bar{\omega}}. \tag{25}
\]

Only one of these roots, \( \sin^2 \theta_- \), is physical in the limit \( \Omega \to 0; \sin^2 \theta_- \to \eta^2 \). If \( \Omega = 0 \), then \( \eta_c = 1 \).

**3.2 Ultrahigh \( P_{\phi} \) orbits**

Taking \( |l_{\phi}| \) in this case to be \( \sim \mathcal{O}(1) \), Eq. (22) can be re-expressed as

\[
l_{\theta} = \frac{1}{\beta_0} \left\{ -1 + \epsilon^2 - 2\bar{\omega}\epsilon \beta_0 |l_{\phi} - \sin^2 \theta| - \left[ 1/|\sin^2 \theta - \bar{\omega}^2| \right] \beta_0^2 |l_{\phi} - \sin^2 \theta|^2 \right\}^{1/2}. \tag{26}\]

The ultrahigh magnetic field introduces a very large negative quantity into the square root. We expand results in inverse powers of \( \beta_0 \). For a given \( P_{\phi} \) or \( l_{\phi} \), the two allowed regions are very narrow in \( \theta \) and, to \( \mathcal{O}(1) \) in \( 1/\beta_0 \), given by the two distinct values of \( \bar{\theta} \):

\[
\sin \bar{\theta} = \sqrt{l_{\phi}}, \tag{27}
\]

where \( \bar{\theta} \) is defined over the range 0 to \( \pi \), and \( \eta \ll 1 \); that is, \( \epsilon \) has been neglected compared to \( l_{\phi} \). Note that \( 0 \leq l_{\phi} \leq 1 \). To \( \mathcal{O}(1/\beta_0) \), the endpoints of the allowed regions are given by

\[
\cos 2\theta = \cos 2\bar{\theta} + \delta_{1,2}/\beta_0, \tag{28}
\]

where

\[
\delta_{1,2} = -\bar{\omega}\epsilon \pm [\bar{\omega}\epsilon]^2 + \left[ 1/|l_{\phi} - \bar{\omega}^2| (\epsilon^2 - 1) \right]^{1/2} \frac{2[1/|l_{\phi} - \bar{\omega}^2|]}{2[1/|l_{\phi} - \bar{\omega}^2|]}. \tag{29}
\]

The turning points near the poles are given by

\[
\theta = \tilde{\theta} \pm \Delta \theta/2 = \sqrt{|l_{\phi}|} \left[ 1 \pm 1/2 \sqrt{(\epsilon^2 - 1)/|l_{\phi}^2|} \right], \quad (l_{\phi} \geq 0) \tag{31}
\]

and the same replacing \( \theta \) by \( \pi - \theta \). In both cases, as the field grows, the turning points approach zero angle.
with the rotation axis and the annular width vanishes. (The $O(l_{\phi})$ effects are too small to show in Fig. 1.) The magnetic flux enclosed by an orbit at the rotational poles is given by

$$\Phi(\theta \simeq 0, \pi) = 2\pi R^2 B_0 \bar{\theta}^2 = \left(\frac{4\pi \hbar c}{|q|}\right) \left(\frac{mcR}{\hbar}\right) \beta_0 l_{\phi}$$

(32)

and the same replacing $\bar{\theta}$ by $\pi - \bar{\theta}$. Note that $\Phi \propto l_{\phi} \propto P_{\phi}/q$. The expression (32) for the magnetic flux does not have to be macroscopically large, as it consists of two large and one small dimensionless factors times the elementary flux quantum. If $l_{\phi}$ is small enough, $\Phi$ can be microscopic, signaling a quantum Hall-like state.

4. SEMICLASSICAL QUANTIZATION: ZERO TILT

Although we do not carry out the full quantum analysis, a semiclassical treatment brings out many of the desired features. For periodic classical orbits, semiclassical quantization is most easily implemented with the Wilson-Sommerfeld (W-S) or Bohr-Sommerfeld conditions [9]. This procedure is valid for $\theta$ quantization as $\theta$ motion is a complete subsystem alone if $\theta_0 = 0$. The closure or non-closure of the classical orbits is then irrelevant.

The condition on the validity of the W-S procedure is that the particle's de Broglie wavelength be much smaller than the length scale over which the background field varies. In our case,

$$\frac{2\pi \hbar}{|p|} \ll R$$

a condition that holds except where $|p| \to 0$. Only infinitesimally close to the $\theta$ turning points does this condition break down, as the three-momentum $|p|$ is otherwise large and $R$ enormous in any case. The W-S semiclassical quantization is validated by the WKB approximation, with this restriction [8].

The classical analysis already implies orbitals reminiscent of a quantum lattice: quasi-free conduction bands (ultrahigh energies) and localized states (polar and azimuthal orbitals). The azimuthal (ultrahigh $P_{\phi}$) rings are localized in one, but not two, dimensions. They conduct along one direction but are trapped in the other. The pole orbitals are similarly localized, but are also confined in absolute position to be near the poles.

The classically allowed regions are those in which the quantum wavefunctions are oscillatory rather than exponential. Because of the Heisenberg uncertainty principle constraint, the quantization conditions, together with the classical spherical cyclotron relations, lead to a tightly constrained set of semiclassical orbitals. In a uniform, planar magnetic system (the Landau system), these orbitals would be equivalent to harmonic oscillator states [10].

We now impose W-S quantization. Since $P_{\phi}$ is conserved, the azimuthal quantization is trivial:

$$P_{\phi} = n_{\phi} \hbar$$

(33)

using the Bohr form of the W-S rule, valid for circular motion. For $P_{\phi} \gg \hbar$, $n_{\phi}$ is very large and the $\phi$ motion is essentially classical. We assume this to be the case, except very near the rotational poles; in that case, the orbital size vanishes as $P_{\phi} \to 0$, so that $P_{\phi}$ must be small.

4.1 Localized pole states: flux quantization

The $\theta$ motion is periodic but non-circular and requires the alternate form of the W-S rule,

$$2 \int_{\theta_1}^{\theta_2} P_{\theta} \, d\theta = (n_{\theta} + 1/2) \hbar$$

(34)

for turning points $\theta_1$ and $\theta_2$. The replacement $n \to n + 1/2$ preserves the exact quantization for simple harmonic motion. This integral is evaluated here in a simple rectangular approximation (Simpson’s rule). With eqs. (30,31,34), we obtain

$$E \simeq \left(mc^2\right) \sqrt{1 + \left((n_{\theta} + 1/2)(\hbar \beta_0/(2mcR))\right)}$$

(35)

Fig. 5 shows $E/mc^2$ as a function of $B_0$ for $n_{\theta} = 0, 1, 2$. The dimensions are set by the critical field $B_c$, the field strength with $mc^2$ of energy within a Compton cube of volume $(2\pi \hbar/mc)^3$ : $B_c = (m^2/\pi) \sqrt{c^5/\hbar^3}$. Fields with this strength or greater introduce effects of quantum field theory such as vacuum polarization. The energy step size is controlled by the tiny ratio of Compton wavelength $\hbar/mc$ to $R$, multiplied by the enormous magnetic field $\beta_0$. Electrons in excited states radiate until they reach the ground state ($n_{\theta} = 0$).

FIG. 5. Quantized pole states: energy $E/mc^2$ as a function of pole field strength $B_0$ for quantum states $n_{\theta} = 0, 1, 2$. For the electron, magnetic field units in $B_c = (m^2/\pi)(c^5/\hbar^3)^{1/2} = 1.3 \times 10^{12}$ Gauss and $q^2/(\hbar c) = 1/137$. 

5
The magnetic flux enclosed by a semiclassical orbit, being an adiabatic quasi-invariant, is quantized and is given by

\[ \Phi(\theta \approx 0, \pi) \approx 2\pi B_0 R^2 \theta^2, \]  \quad (36)

so that

\[ \Phi(\theta \approx 0, \pi) \approx \frac{4\pi c P_\theta}{q} = \left( \frac{4\pi \hbar c}{q} \right) \left( \frac{mcR}{\hbar} \right) \beta_0 \propto n_\phi, \]  \quad (37)

analogous to the quantum Hall effect. Any \( n_\theta \) dependence in \( \Phi \) is a correction to Eq. (37) of relative order \( \mathcal{O}(1/\beta_0) \) and arises from the breakdown of exact invariance of \( \Phi \).

### 4.2 Localized azimuthal rings

In the limit of high azimuthal angular momentum \( |l_\phi| \sim \mathcal{O}(1) \), using eqs. (26-29,34), we obtain

\[ E \simeq (mc^2) \sqrt{1 + (n_\theta + 1/2)(h\beta_0/(2mcR\sqrt{l_\phi}))}. \]  \quad (38)

The same combination of tiny ratio multiplied by very large \( \beta_0 \) that occurred in Eq. (35) appears again, and excited states are again unstable to radiation. Note, the magnetic flux enclosed by a semiclassical orbit is not interesting in the quantum regime, as it is macroscopically large.

### 4.3 Poleward conduction: quasi-continuum

In the limit of ultrahigh energies using eqs. (24,25,34), we obtain

\[ E \simeq C(n_\theta + 1/2) (hc/R), \]  \quad (39)

where \( C \approx 4, 1, \) or 1/2 for \( \eta \lesssim, \sim, \) or \( \gtrsim 1 \). Note that the energies scale as harmonic oscillator energies, where the frequency is set by the size of the sphere, not the magnetic field. As in Sect. III.A, we have assumed \( P_\phi \) to be negligibly small. In realistic cases, \( hc/R \ll mc^2 \) and \( E \), so that the energy levels are very closely spaced. Such levels form a quasi-continuum that allows the charged particles to conduct almost freely along the polar directions, subject only to external crystal resistance and radiation losses. For these ultrahigh energies, \( n_\theta \) is so large that the motion is essentially classical. The energy levels depend only weakly on \( \Omega \), for small \( \Omega R/c \).

The magnetic flux \( \Phi \) is not interesting in this case because the allowed region is a large part or all of the sphere. Flux quantization is not relevant because the energy is independent of the magnetic field \( B_0 \), apart from the overall constant \( C \). Only if the energy \( E \) is taken \( \ll \mathcal{O}(q|B_0 R/2|) \) do \( B_0 \) and \( P_\theta \) become important again; this is the localized pole orbit case.

### 4.4 Density of states

The density of states is important in any charged fermion system for determining the conductivity. The semiclassical density of states \( dN/dE \) is

\[ dN = \frac{d\theta}{dP_\theta} \frac{d\phi}{dP_\phi} \frac{dP_\phi}{dE}, \]  \quad (40)

and the differential surface area element is

\[ dS = R^2 \sin \theta \, d\theta \, d\phi \]  \quad (41)

Thus we have for the the number of states per unit area per unit azimuthal angular momentum per unit energy

\[ \frac{d^3N}{dS \, dP_\theta \, dE} = \frac{dP_\theta}{dE} \left( \frac{1}{4\pi^2 h^4 R^2 \sin \theta} \right). \]  \quad (42)

Eq. (22) can be rewritten as

\[ P_\theta/mcR = [(E/mc^2)^2 - (E/mc^2)\alpha_1(\theta; P_\phi, \Omega) - \alpha_0(\theta; P_\phi, \Omega)]^{1/2}, \]  \quad (43)

with

\[ \alpha_1 = 2(\Omega/mc^2)[P_\phi - (qB_0 R^2/2c) \sin^2 \theta], \]  \quad (44)

\[ \alpha_0 = 1 + (1/2 \sin^2 \theta - (\Omega R/c)^2 [P_\phi - (qB_0 R^2/2c) \sin^2 \theta]^2/(mcR)^2. \]  \quad (45)

Therefore

\[ \frac{dP_\theta}{dE} = (mcR/P_\theta)[E/mc^2 - \alpha_1/2](R/c). \]  \quad (46)

The factor \( dP_\theta/dE \) is shown as a function of \( \eta \) in Fig. 3 for \( |l_\phi| \ll 1 \), and as a function of \( E/mc^2 \) in Fig. 4 for \( l_\phi \sim 0.1 \). For low energies, \( E \to mc^2 \), this function approaches

\[ [1 - \alpha_1 - \alpha_0]^{-1/2} \left[ 1 - \alpha_1/2 - \frac{\alpha_0 + \alpha_1^2/4}{1 - \alpha_1 - \alpha_0 (K/mc^2)} \right], \]  \quad (47)

where \( K = E - mc^2 \). In the ultrarelativistic limit, \( E \gg mc^2 \) and/or \( \eta \gtrsim 1 \), \( dP_\phi/dE \) approaches a constant, \( R/c, \) a characteristic of two-dimensional Landau states. Note the large peak in \( dP_\theta/dE \) at low \( \epsilon \) or \( \eta \), the semiclassical edge of the discretized quantum regime.

![FIG. 6. Density of states factor \( dP_\theta/dE \), in units of \( R/c \), as a function of energy \( \eta \) for various values of \( \theta \) and \( |l_\phi| \ll 1 \).](image)
FIG. 7. Density of states factor \( dP_\theta/dE \), in units of \( R/c \), as a function of energy \( E/mc^2 \) for \( l_\phi = 0.1 \)

In the quantum limit,

\[
dN = d\phi \left( \frac{d \phi \ dP_\phi}{2\pi \hbar} \right), \tag{48}
\]
treating \( P_\phi \) as continuous. Equation (42) becomes:

\[
\frac{d^2N}{d\mathcal{L} \ dP_\phi \ dE} = d\phi \left( \frac{1}{2\pi \hbar R^2 \sin \theta} \right), \tag{49}
\]
where \( d\mathcal{L} = R \sin \theta \ d\phi \), a unit of azimuthal arc length. The function \( d\psi/dE \) can be obtained from eqs. (34, 35, or 38). For the localized pole states of Eq. (35), take \( E^2 = (mc^2)^2 + (n_\theta + 1/2)\varepsilon_1^2 \), where \( \varepsilon_1^2 = \varepsilon_1^2(qB_0, R, P_\phi) \). Then

\[
\frac{d\psi}{dE} = \frac{d\psi}{dK} = \frac{2E}{\varepsilon_1^2} = \frac{2(mc^2 + K)}{\varepsilon_1^2} \tag{50}
\]
for \( E^2 \geq (mc^2)^2 + \varepsilon_1^2/2 \).

Related to the density of states is the degeneracy factor for a given \( n_\theta \) level. Semiclassically, this degeneracy arises from shifting an orbital center and is identical to the degeneracy of planar Landau states \cite{4}. The number of degenerate states available per \( dS \) per \( dP_\phi \) at state \( n_\theta \) is

\[
\frac{d^2N}{dS \ dP_\phi}(n_\theta) = \left( \frac{\varepsilon_1/4\pi^{2}cR\sqrt{\varepsilon_1}}{\sqrt{n_\theta + 1/2}} \right). \tag{51}
\]

5. NON-ZERO TILT: APPROXIMATE ORBITALS

5.1 Adiabatic quasi-invariance

As in Sects. II and III, the energy \( E = P_\theta \) is exactly conserved, but \( P_\phi \) or \( l_\phi \) no longer is, unless \( \omega \sin \theta_0 \) is zero. \( P_\theta \) or \( l_\phi \) was never conserved, but in with zero tilt, its motion was exactly separable with \( l_\phi \) conserved. That separability is also lost if \( \omega \sin \theta_0 \neq 0 \).

When \( \theta_0 \neq 0 \), the non-conservation of \( l_\phi \) is apparent from the equation of motion (10):

\[
\frac{dl_\phi}{d\tau} = \frac{\text{sgn}(q)}{eR\beta_0} \left[ \frac{1}{2} \frac{\partial \omega_\phi}{\partial \phi} + \frac{\omega_\phi}{\partial \phi} (c\dot{\phi}i \sin \theta) + \frac{\partial \omega_\phi}{\partial \phi} (c\dot{\phi} \theta) \right], \tag{52}
\]
using the general metric for \( \theta_0 \neq 0 \). The non-conservation of \( l_\phi \) is \( \mathcal{O}(1/\beta_0) \), as well as requiring \( \omega \sin \theta_0 \neq 0 \). In the limit of ultra-intense magnetic field \( \beta_0 \), the motion is not qualitatively different from the zero-tilt case. The orbits are more involved and possibly chaotic, but so focused by the intense field that, to zeroth order in \( 1/\beta_0 \), the results of Sects. II and III with \( \theta_0 = 0 \) are still qualitatively valid.

The approximate conservation of \( l_\phi \) validates a perturbative treatment of the classical orbits. In Sect. II, where \( l_\phi \) is exactly conserved, the action variable

\[
J_\phi = \oint d\phi \ P_\phi \tag{53}
\]
is an adiabatic invariant \cite{8,11}. This allows for the semiclassical quantization of \( P_\phi \) as a trivial step in that case as well as the exactly separable treatment of \( P_\theta \). We thus expect any effects of the non-zero tilt to be suppressed by powers of \( \bar{\omega} \) and \( 1/\beta_0 \).

5.2 Classical motion

Since the \( \phi \) motion to zeroth order in \( 1/\beta_0 \) is periodic, global expressions involving the \( \phi \) coordinate (such as \( E \), but not the orbit \( \phi(\tau), \theta(\tau) \)) can be averaged over the full circle \( \phi \in [0, 2\pi] \) with \( l_\phi \) treated as constant, in order to determine the first-order corrections \cite{11}. The \( \mathcal{O}(1/\beta_0) \) correction to \( dl_\phi/d\tau = 0 \) is obtained from averaging (14) over \( \phi \):

\[
\left\langle \frac{dl_\phi}{d\tau} \right\rangle _\phi = \frac{\text{sgn}(q)}{mcR\beta_0} \frac{\partial L_\phi}{\partial \phi} = 0 , \tag{54}
\]
where all the terms are \( \propto \cos \phi, \sin \phi, \) or \( \cos \phi \sin \phi \) and average to zero. Consequently, the non-conservation of \( l_\phi \), on average, actually starts at \( \mathcal{O}(1/\beta_0^2) \).

We consider the same set of limiting cases as in Sect. II. In the limit of ultrahigh energy, \( \beta_0 \gg l_\phi \), the constraint (19) with Eq. (20) becomes:

\[
l_\phi (1 - \omega_\phi^2)/\beta_0 = \omega_\phi \pm \sqrt{(\omega_\phi^2 - 1) + \omega_\phi \sin \theta - (1 - \omega_\phi^2) \sin^2 \theta} , \tag{55}
\]
retaining all powers of \( \eta \). Figs. 8 and 9 show the allowed regions for several combinations of \( E \) and \( \theta_0 \), where the allowed regions are defined by the requirement that \( l_\theta \) in (54) be real. Note that the allowed regions of Figs. 8 and 9 are no longer functions of \( \theta_0 \) alone, unlike the case of eqs. (24,25). The axial symmetry about the dipole is lost, but for moderate values of \( \bar{\omega} \), the asymmetry is not extreme. For large tilts \( \theta_0 \rightarrow \pi \), the energy \( \eta \) must be somewhat higher than in the zero-tilt case to make the entire sphere allowed. If \( \phi \) averaging is applied to the constraint (19), the axial symmetry about the dipole is restored.
FIG. 8. Allowed regions of real $l_{\theta}$ in the ultrahigh energy limit, for $\eta = 0.5$. Surface coordinates are colatitude $\theta$ and longitude $\phi$ in degrees. (a) $\theta_0 = 45^\circ$. (b) $\theta_0 = 135^\circ$.

The limit where $l_\phi$ cannot be neglected, but $\eta \ll l_\phi$, is the ultrahigh $P_\phi$ case; $l_\theta$ is then zero except within the narrow allowed regions. If the tilt is zero, eqs. (26-29), the particle orbits are almost infinitely narrow circles at constant $\theta$. The $\theta$ motion is $O(1/\beta_\theta)$ relative to the $\phi$ motion. With non-zero tilt and $l_\phi$ approximately conserved, the quasi-circular orbits occur at the same angle as the no tilt case, $\sin \bar{\theta} = \sqrt{l_\phi}$, for $l_\phi \geq 0$, with $O(1/\beta_\phi)$ and $O(\bar{\omega} \sin \theta_0/\beta_0)$ corrections.

The same argument holds for the small angle case, which is related to the ultrahigh $P_\theta$ limit in the limit $l_\phi \to 0$, with $\eta$ still negligible. The approximate conservation of $l_\phi$ still holds in the limit of large $\beta_\phi$, with additional corrections of relative order $\bar{\omega} \sin \theta_0$.

The magnetic flux enclosed by an orbit is, to leading order in $1/\beta_0$, just as in Sect. II, eqs. (32,36,37). The averaged rotational corrections are of order $O(\bar{\omega}^2)$.

5.3 Semiclassical quantization

In the present case, the quantization of $l_\phi$ in the Wilson-Sommerfeld condition (33) is only approximate, but valid through $O(1/\beta_\phi)$.

The ultrahigh energy case is not qualitatively different if $\theta_0 \neq 0$. As seen from the classical orbits, some regions of the sphere are forbidden for large tilt and moderate $\eta$. The quantum energy levels are still approximately the harmonic oscillator levels given by Eq. (39), depending only weakly on the field.

The ultrahigh $P_\phi$ case results in the energy levels:

$$E/mc^2 \approx \left\{ 1 - (\bar{\omega}/2) \right\} \sin^2 \theta_0 + l_\phi \cos^2 \theta_0 + (1 - (5\bar{\omega}/4) \sin^2 \theta_0 + \bar{\omega}^2 l_\phi \cos^2 \theta_0) \times (n_\theta + 1/2) h\beta_0/(2mcR\sqrt{\bar{\omega}}) \right\}^{1/2},$$

(56)

to leading order in $\beta_0$ and $\bar{\omega}$, with non-zero tilt. Note that the rotational corrections enter at $O(\bar{\omega}^2)$. The corrections multiplying $\eta$ can either raise or lower the energy, but the first correction (the pure centrifugal effect) always lowers the energy.

The localized pole states are also corrected by rotational effects. Their levels are:

$$E/mc^2 \approx \left\{ 1 - (\bar{\omega}/4) \sin^2 \theta_0 + (1 - (3\bar{\omega}/4) \sin^2 \theta_0) (n_\theta + 1/2) h\beta_0/(2mcR) \right\}^{1/2},$$

(57)

and reproduce Eq. (35) if $\bar{\omega} \to 0$. For both this and the ultrahigh $P_\phi$ cases, the rotational corrections begin at $O(\bar{\omega}^2)$ because of the angular averaging. However, corrections of $O(\bar{\omega}^2/\beta_0)$ may enter.

For moderate rotational velocities $\bar{\omega} = \Omega R/c \lesssim 0.1$, the azimuthally-averaged corrections are $O(\bar{\omega}^2)$. Corrections not explicitly treated are: $O(1/\beta_\phi)$, $O(\bar{\omega}/\beta_0)$, and $O(\bar{\omega} \sin \theta_0/\beta_0)$ (Fig. [11]). The multiplicative rotational corrections to the magnetic field in the energy arise as an effective electric field induced, as seen by an inertial observer, by the rotation. Purely rotational (centrifugal) corrections also enter.
FIG. 10. Phase space for the ultrahigh $P_0$ case, with negligible $\eta$. Effects of $O(1/\beta_0)$ are greatly exaggerated. (a) Phase space for $\phi$ (degrees), showing small variation of $l_\phi$ for the case $\theta_0 \neq 0$. The allowed region (phase space trajectory) itself has finite $O(1/\beta_0)$ thickness (not shown). (b) Phase space for $\theta$ (degrees) showing small variation of $\theta$ and small values of $l_\theta$.

6. CONCLUSION

We have examined only the simplified scenario of charged particles on a rotating spherical surface with an intense dipole magnetic field. The resulting energy levels and magnetic flux quantization, for very high field strength, are roughly similar to the relativistic Landau and quantum Hall systems. The effect of realistic rotation velocities ($\Omega R/c \approx 0.1$ or less) is small to moderate in the two-dimensional case. For these small values of $\hat{\omega}$, the case of a tilted axis is not qualitatively different, although non-zero tilt creates a non-trivial geometry and complicates the dynamics.

There are a number of related issues in this idealized situation not treated here. A treatment of the three-dimensional orbitals requires inclusion of the radial motion and radial dependence of the vector potential $A$ and has been considered for particles confined within the crust in a separate work [3]. A fully quantum treatment of the orbitals necessitates the Dirac equation and the inclusion of spin, although the basic features of the Dirac orbitals are expected to be outlined by the W-S method.

Application of our results to a neutron star demands further realism. In particular, the effect of the lattice of nuclei on non-localized electrons and nuclei must be included [5,12]. The lattice structure of a neutron star crust must be disordered in its surface layers, although the fermion temperature is small compared to the Fermi energy [3]. These lattice effects modify the magneto-sphere charge cloud, the conduction bands, and the flux quantization. The lattice must be included to give a full picture of the crust’s charge and current distribution. The magnetic field itself is also modified if the currents contribute significantly to the field.

Finally, observational questions remain. These include verification, if possible, of flux quantization and determination of what roles it and the special conduction band structure play in the formation and evolution of a neutron star’s magnetic field and crust.

ACKNOWLEDGMENTS

The authors are indebted to Sudheer Maremanda (U.T. Arlington) for preparing the figures. We thank Pradeep Kumar (Univ. Florida) for the original suggestion of this problem and Bernard Whiting (Univ. Florida) for helpful suggestions. This work was supported at the Univ. of Florida Institute for Fundamental Theory and U.S. Department of Energy Contract DE-FG05-86-ER40272; and at the Univ. of Texas at Arlington by the Research Enhancement Program.
The graph shows the behavior of $(P_\theta/mcR)/(\beta_0\eta)$ as a function of $\cos\theta$ for different values of $\eta$: $\eta=0.5$, $\eta=1$, $\eta=2$, and $\eta=4$. The y-axis represents the normalized momentum component, and the x-axis represents the cosine of the angle $\theta$. The term $\beta_0\eta$ is a normalization factor. The graph indicates an ultrahigh energy scenario.
Pole States

\[ \frac{E}{mc^2} \text{ versus } \frac{B_0}{B_c} \]

- \( n_\theta = 0 \)
- \( n_\theta = 1 \)
- \( n_\theta = 2 \)
\( \bar{\theta} = \sin^{-1} \sqrt{l_\phi} \)

Ultrahigh \( P_\phi, l_\phi = 0.1 \)
$\theta_o = 45^0, \eta = 0.5$
$\theta_o = 135^0, \eta = 0.5$
\[ \theta_o = 45^0, \eta = 1.0 \]
$\theta_0 = 135^0, \eta = 1.0$
$l_\phi$
\[ O\left( \frac{1}{\beta_0} \right) \]