POSITIVE CONJUGACY CLASSES IN WEALEN GROUPS

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1. Let $W$ be a Weyl group. In this paper we introduce the notion of positive conjugacy class of $W$. This generalizes the notion of elliptic regular conjugacy class in the sense of Springer [Sp].

Let $w \mapsto |w|$ be the length function on $W$. Let $S = \{ s \in W; |s| = 1 \}$.

Let $v$ be an indeterminate. Recall that the Iwahori-Hecke algebra of $W$ is the associative $\mathbb{Q}(v)$-algebra $H$ which, as a $\mathbb{Q}(v)$-vector space has basis $\{ T_w; w \in W \}$ and has multiplication given by $T_w T_{w'} = T_{ww'}$ if $|ww'| = |w| + |w'|$ and $(T_s + 1)(T_s - v^2) = 0$ if $s \in S$; note that $T_1$ is the unit element of $H$. This is a split semisimple algebra. Let $q = v^2$.

For $w, w'$ in $W$ let $N_{w,w'}$ be the trace of the $\mathbb{Q}(v)$-linear map $H \to H$, $h \mapsto T_w h T_{w'} - 1$. We have $N_{w,w'} \in \mathbb{Z}[q]$ and

\[ N_{w,w'} = \sum_{E \in \text{Irr}W} \text{tr}(T_w, E_v) \text{tr}(T_{w'}, E_v) \]

where $\text{Irr}W$ is the set of irreducible $\mathbb{Q}[W]$-modules up to isomorphism and for $E \in \text{Irr}W$, $E_v$ denotes the corresponding simple $H$-module. Note that when $v$ is specialized to 1, $H$ becomes the group algebra $\mathbb{Q}[W]$ of $W$ and $N_{w,w'}$ specializes to $N_{w,w'}(1)$, the number of elements $y \in W$ such that $wy = yw'$. In particular, if $w \in W$, $N_{w,w}$ specializes to $n_w$, the order of the centralizer of $w$ in $W$; thus the polynomial $N_{w,w}$ can be viewed as a $q$-analogue of the number $n_w$.

If $C$ is a conjugacy class in $W$ we denote by $C_{\text{min}}$ the set of all $y \in C$ such that $C \to \mathbb{N}$, $w \mapsto |w|$, reaches its minimum at $y$. By a result of Geck and Pfeiffer [GP,3.2.9], for any $E \in \text{Irr}W$, $w \mapsto \text{tr}(T_w, E_v)$ is constant on $C_{\text{min}}$. Using this and (a) we see that $w \mapsto N_{w,w}$ is constant on $C_{\text{min}}$. We say that $C$ is positive if $C \neq \{ 1 \}$ and for some/any $w \in C_{\text{min}}$ we have $N_{w,w} \in \mathbb{N}[q]$. (We then also say that any element $w \in C_{\text{min}}$ is positive.)

2. For any $f \in \mathbb{Z}[q]$ we write $f = \sum_{i \geq 0} f_i q^i$ where $f_i \in \mathbb{Z}$. For $w \in W$ let $S_w$ be the set of all $s \in S$ such that $s$ appears in some/any reduced expression for $w$.

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Let $\mathcal{L}_w = \{ s \in S ; |sw| < |w| \}$, $\mathcal{R}_w = \{ s \in S ; |ws| < |w| \}$. For $a, a'$ in $W$ we write

$$T_a T_{a'} = \sum_{b \in W} \phi(a, a', b) T_b$$

where $\phi(a, a', b) \in \mathbb{Z}[q]$. We show:

(a) $\phi(a, a', b)_{|a|} \geq 0$, $\phi(a, a', b)_i = 0$ for $i > |a|$. If $\phi(a, a', b)_{|a|} \neq 0$ then either

$$a' = b, \quad S_a \subset \mathcal{L}_{a'} \quad \text{or} \quad |b| < |a'|.$$  

If $a' = b$, $S_a \subset \mathcal{L}_{a'}$ then $\phi(a, a', b)_{|a|} = 1$.

We argue by induction on $|a|$. When $|a| = 0$ the result is obvious. Assume now that $|a| \geq 1$. We write $a = a_1 s$ where $s \in S$, $|a_1| = |a| - 1$. If $|sa'| = |a'| + 1$ then $\phi(a, a', b) = \phi(a_1, sa', b)$ and the induction hypothesis shows that $\phi(a, a', b)_i = 0$ for $i > |a|$. Since $s \in S_a$, $s \notin \mathcal{L}_{a'}$, we see that the desired result holds. Next we assume that $|sa'| = |a'|$. Then $\phi(a, a', b) = q \phi(a_1, sa', b) + (q - 1) \phi(a_1, a', b)$. From the induction hypothesis we see that $\phi(a_1, a', b)_i = 0$ if $i > |a|$, that $\phi(a_1, sa', b)|_{a_1} = \phi(a_1, a', b)|_{a_1} + \phi(a_1, a', b)|_{a_1} \geq 0$ and that if $\phi(a, a', b)_{|a|} \neq 0$ then either $\phi(a, sa', b)|_{a_1} \neq 0$ or $\phi(a_1, a', b)|_{a_1} \neq 0$, so that we are in one of the cases (i)-(iv) below.

(i) $sa' = b$,

(ii) $|b| < |sa'|$;

(iii) $a' = b, \quad S_a \subset \mathcal{L}_{a'}$;

(iv) $|b| < |a'|$.

In case (i),(ii),(iii) we have $|b| < |a'|$; in case (iii) have $a' = b$ and $S_a \subset \mathcal{L}_{a'}$ (since $S_a = S_a \cup \{s\}$), as desired. Now assume that $a' = b, \quad S_a \subset \mathcal{L}_{a'}$. It remains to show that $\phi(a_1, sa', b)|_{a_1} + \phi(a_1, a', b)|_{a_1} = 1$. By the induction hypothesis we have $\phi(a_1, a', b)|_{a_1} = 1$ (since $S_a \subset \mathcal{L}_{a'}$) $\phi(a_1, sa', b)|_{a_1} = 0$ (since $sa' \neq b$ and $|b| \neq |a'|$). This completes the proof.

The following result is proved in the same way as (a).

(b) $\phi(a, a', b)|_{a'} \geq 0$, $\phi(a, a', b)_i = 0$ for $i > |a'|$. If $\phi(a, a', b)|_{a'} \neq 0$ then either

$$a = b, \quad S_a \subset \mathcal{R}_{a'} \quad \text{or} \quad |b| < |a|.$$  

If $a = b$, $S_{a'} \subset \mathcal{R}_{a'}$ then $\phi(a, a', b)|_{a'} = 1$.

For $a, a', a''$ in $W$ we have $T_a T_{a'} T_{a''} = \sum_{b \in W} f(a, a', a'', b) T_b$ where $f(a, a', a'', b) \in \mathbb{Z}[q]$. Let $n = |a| + |a''|$. We show:

(c) $f(a, a', a'', a')_n = 0, f(a, a', a'', a')_i = 0$ for $i > n$. If $f(a, a', a'', a')_n \neq 0$ then $S_a \subset \mathcal{L}_{a'}$ and $S_{a''} \subset \mathcal{R}_{a'}$. Conversely, if $S_a \subset \mathcal{L}_{a'}$ and $S_{a''} \subset \mathcal{R}_{a'}$ then $f(a, a', a'', a')_n = 1$.

We have $f(a, a', a'', a') = \sum_{c \in W} \phi(a, a', c) \phi(c, a'', a')$. Hence for $i \geq 0$ we have $f(a, a', a'', a')_i = \sum_{c \in W: j \geq 0, j' \geq 0, j+j' \geq i} \phi(a, a', c) \phi(c, a'', a')_{j+j'}$. Using (a) and (b) in the last sum we can take $j \geq |a|, j' \geq |a''|$. Hence if $i > n = |a| + |a''|$ then $f(a, a', a'', a')_i = 0$ and $f(a, a', a', a')_n = \sum_{c \in W} \phi(a, a', c) \phi(c, a'', a')_{|a|+|a''|} \geq 0$.

Assume now that $f(a, a', a'', a')_n \neq 0$. Then in the last sum we can assume that $c = a'$, $S_a \subset \mathcal{L}_{a'}$ or $|c| < |a'|$ and $a' = c, S_{a''} \subset \mathcal{R}_{c}$ or $|a''| < |c|$.

Thus we can assume that $c = a'$, $S_a \subset \mathcal{L}_{a'}$ and $S_{a''} \subset \mathcal{R}_{a'}$ and using again (a),(b) we have $f(a, a', a'', a')_n = 1$.

For $w, w'$ in $W$ we set $n = |w| + |w'|$; we show:

(d) $N_{w, w'}^w = \#(a' \in W ; S_w \subset \mathcal{L}_{a'}, S_{w'} \subset \mathcal{R}_{a'}) > 0, N_{w, w'}^w = 0$ for $i > n$.

We have $N_{w, w'}^w = \sum_{a' \in W} f(w, a', w', a')$ and the result follows from (c). (We use that if $a' = w_0$, the longest element of $W$ then $S_w \subset \mathcal{L}_{a'} = S, S_{w'} \subset \mathcal{R}_{a'} = S$.)
From (d) we deduce:

(e) Let \( w, w', n \) be as in (d). Assume that either \( S_w = S \) or \( S_{w'} = S \). then \( N_{w,w'}^n = 1 \).

Indeed, if \( a' \in W \) satisfies \( S \subset L_{a'} \) or \( S \subset R_{a'} \) then \( a' = w_0 \).

We state the following result.

(f) Let \( w, w', n \) be as in (d). For \( i = 0, 1, \ldots, n \) we have \( N_{i}^{w, w'} = (-1)^{n}N_{n-i}^{w, w'} \).

Let \( v : \mathbb{Q}(v) \to \mathbb{Q}(v) \) be the field automorphism such that \( \bar{v} = v^{-1} \). For \( E \in \text{Irr} W \) let \( E^\dagger \in \text{Irr} W \) be the tensor product of \( E \) with the sign representation of \( W \). It is known that for \( w \in W \) we have

\[
\text{tr}(T_w, E_v^\dagger) = (-v^2)^{n} \text{tr}(T_{w^{-1}}, E_v) = (-v^2)^{n} \text{tr}(T_w, E_v).
\]

It follows that

\[
N_{w,w'}^{n} = \sum_{E \in \text{Irr} W} \text{tr}(T_w, E_v^\dagger)\text{tr}(T_{w'}, E_v^\dagger)
\]

\[
= \sum_{E \in \text{Irr} W} (-v^2)^{n} \text{tr}(T_w, E_v)(-v^2)^{n} \text{tr}(T_{w'}, E_v)
\]

\[
= \sum_{E \in \text{Irr} W} (-v^2)^{n} \text{tr}(T_w, E_v)\text{tr}(T_{w'}, E_v).
\]

We see that

\[
N_{w,w'}^{n} = (-q)^{n}N_{w,w'}^{n}
\]

and (f) follows.

3. We now assume that \( W \) is irreducible. Let \( \nu = |w_0| \) where \( w_0 \) is the longest element of \( W \). An element \( w \in W \) (or its conjugacy class) is said to be elliptic if its eigenvalues in the reflection representation of \( W \) are all \( \neq 1 \). For any \( d \in \{2, 3, 4, \ldots\} \) let \( C_d \) be the set of all elliptic elements \( w \in W \) which have order \( d \) and are regular in the sense of Springer [Sp]. It is known [Sp] that \( C_{d}^{d} \) is either empty or a single conjugacy class in \( W \). Let \( D = \{d \in \{2, 3, \ldots\}; C_{d}^{d} \neq \emptyset\} \). It is known [Sp] that if \( d \in D \) and \( w \in C_{d_{\text{min}}} \) then \( |w| = 2 \nu / d \). Let \( h \) be the Coxeter number of \( W \). We have \( h \in D \).

According to [Sp], the set \( D \) is as follows:

Type \( A_n (n \geq 1) \): \( D = \{n + 1\} \).

Type \( B_n (n \geq 2) \): \( D = \{d \in \{2, 4, 6, \ldots\}; 2n / d = \text{integer}\} \).

Type \( D_n (n \text{ even}, n \geq 4) \):
\( D = \{d \in \{2, 4, 6, \ldots\}; (2n - 2) / d = \text{odd integer or} 2n / d = \text{integer}\} \).

Type \( D_n (n \text{ odd}, n \geq 5) \):
\( D = \{d \in \{2, 4, 6, \ldots\}; (2n - 2) / d = \text{odd integer}\} \).

Type \( E_6 \): \( D = \{3, 6, 9, 12\} \).

Type \( E_7 \): \( D = \{2, 6, 14, 18\} \).

Type \( E_8 \): \( D = \{2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30\} \).
Type $F_4$: $\mathcal{D} = \{2, 3, 4, 6, 8, 12\}$.

Type $G_2$: $\mathcal{D} = \{2, 3, 6\}$.

We note the following properties:

(a) If $2 \in \mathcal{D}$, $d \in \mathcal{D}$ is even and $w \in C^d_{\min}$ then $w^{d/2} = w_0$, $(d/2)|w| = |w_0|$ hence $T_{w^{d/2}} = \mathbb{T}_{w_0}$.

(b) If $d = h$, $w \in C^d_{\min}$ then $T_w = \mathbb{T}_{w_0}$.

(c) If $d \in \mathcal{D}$, $h/d \in \mathbb{N}$ and $y \in C^h_{\min}$ then $y^{h/d} \in C^d_{\min}$ and $(h/2)|y| = |y^{h/d}|$ hence $T_{y^{h/d}} = \mathbb{T}_{y^{h/d}}$.

The equation $w^{d/2} = w_0$ in (a) holds by examining the characteristic polynomial of $w$ and $w^{d/2}$ in the reflection representation of $W$; then (a) follows. The equality in (b) can be deduced from [Bo, Ch.V, §6, Ex.2]. The equation $w^{h/d} \in C^d_{\min}$ in (c) holds by examining the characteristic polynomial of $w$ and $w^{h/d}$ in the reflection representation of $W$; then (c) follows.

For any $E \in \text{Irr} W$ we define $a_E \in \mathbb{N}$ as in [L3, 4.1]. Let $\bar{a}_E = \nu - a_E + a_{E^\dagger}$.

(d) $T_{w_0}^2 = \nu^{2\bar{a}_E} 1 : E_v \rightarrow E_v$.

This can be deduced from [L3, (8.12.2)]; a closely related statement was first proved by Springer, see [GP, 9.2.2].

We show:

(e) Let $E \in \text{Irr} W$ and let $d \in \mathcal{D}, w \in C^d_{\min}$. Then all eigenvalues of $T_w : E_v \rightarrow E_v$ (in an algebraic closure of $\mathbb{Q}(\nu)$) are roots of $1$ times $\nu^{2\bar{a}_E}/d$.

If $d$ is as in (a) then the result follows from (a) and (d). If $d = h$ then the result follows from (b) and (d). If $d, y$ are as in (c) then the result follows from (c) and the previous sentence. From the description of $\mathcal{D}$ for various types we see that if $d \in \mathcal{D}$ is not as in (a) then it is as in (c). This proves (e). (A closely related result can be found in [GP, 9.2.5].)

From (e) we deduce:

(f) In the setup of (e), $\text{tr}(T_w, E_v)$ equals $\nu^{2\bar{a}_E/d} \text{tr}(w, E)$; this is $0$ if $2\bar{a}_E/d \notin \mathbb{Z}$.

(The idea of the proof leading to (f) appeared in [L3, p.320].) Using (f) and 1(a) we deduce:

(g) If $d \in \mathcal{D}, w \in C^d_{\min}$, then

$$N^{w,w} = \sum_{E \in \text{Irr} W} q^{2\bar{a}_E/d} \text{tr}(w, E)^2.$$  

In particular, we have $N^{w,w} \in \mathbb{N}[q]$ and $w$ is positive.

4. Using 1(a) and the CHEVIE package [Ch] one can find a list of positive conjugacy classes in $W$ (assumed to be irreducible of low rank). I thank Gongqin Li for help with programming in GAP. The list of positive conjugacy classes in $W$ which are not regular elliptic for $W$ of type $E_6, E_7, E_8, F_4, G_2, B_5, B_6$ is as follows. (We specify a conjugacy class by the characteristic polynomial of one of its elements in the reflection representation. We denote by $\Phi_k$ the $k$-th cyclotomic polynomial; thus $\Phi_2 = q + 1, \Phi_3 = q^2 + q + 1$, etc.)
Type $E_6$: none.
Type $E_7$: $\Phi_{12} \Phi_6 \Phi_2, \Phi_{10} \Phi_6 \Phi_2, \Phi_{10} \Phi_3^2, \Phi_8 \Phi_4 \Phi_2, \Phi_4^2 \Phi_3^2$.
Type $E_8$: $\Phi_{18} \Phi_6, \Phi_{18} \Phi_2^2, \Phi_9 \Phi_3, \Phi_{14} \Phi_2^2$.
Type $F_4$: none.
Type $G_2$: none.
Type $B_3$: $\Phi_8 \Phi_2, \Phi_6 \Phi_2^2, \Phi_7^2 \Phi_2$.
Type $B_6$: $\Phi_{10} \Phi_2^2, \Phi_8 \Phi_4, \Phi_8 \Phi_3^2, \Phi_6 \Phi_3^2$.

In each of these examples any positive element of $W$ is elliptic; we expect this to be true in general. The example of $B_6$ suggests that if $W$ is of type $B_n$ with $2n = 4 + 8 + \cdots + 4k$, then an element of $W$ with cycle type $(4)(8)\ldots(4k)$ might be positive.

5. Let $k$ be an algebraic closure of the finite field $F_q$ with $q$ elements. Let $G$ be a connected reductive group over $k$ with a fixed $F_q$-split rational structure and whose Weyl group is $W$. Let $F : G \to G$ be the corresponding Frobenius map. For $w \in W$ let $X_w$ be the variety of Borel subgroups $B$ of $G$ such that $B$ and $F(B)$ are in relative position $w$, see [DL, 1.3]. The finite group $G^F = \{g \in G; F(g) = g\}$ acts on $X_w$ by conjugation. For $w, w' \in W$ we denote by $X_{w,w'} = G^F \setminus (X_w \times X_{w'})$ the space of $G^F$-orbits for the diagonal action of $G^F$ on $X_w \times X_{w'}$. Now $(B, B') \mapsto (F(B), F(B'))$ induces a map $X_{w,w'} \to X_{w,w'}$ (denoted again by $F$) which is the Frobenius map for an $F_q$-rational structure on $X_{w,w'}$. By [L2, 3.8] for any integer $e \geq 1$ we have

$\#(\xi \in X_{w,w'}; F^e(\xi) = \xi) = N_{w,w'}^e(q^e)$.

6. In the remainder of this paper we assume that $G$ in no.5 is simply connected and $W$ is irreducible. In the case where $w$ is a Coxeter element of minimal length of $W$, the left hand side of 5(a) (with $w = w'$) has been computed in [L1, p.158]. This gives the following formulas for $N_{w,w}$.

Type $A_n (n \geq 1)$: $q^{2n} + q^{2n-2} + \cdots + q^2 + 1$.
Type $B_n (n \geq 2)$: $q^{2n} + 2q^{2n-2} + 2q^{2n-4} + \cdots + 2q^2 + 1$.
Type $D_n (n \geq 4)$: $q^{2n} + q^{2n-2} + 2q^{2n-4} + 2q^{2n-6} + \cdots + 2q^4 + q^2 + 1$.
Type $E_6$: $q^{12} + q^{10} + 2q^8 + 4q^6 + 2q^4 + q^2 + 1$.
Type $E_7$: $q^{14} + q^{12} + 2q^{10} + 4q^8 + 2q^7 + 4q^6 + 2q^4 + q^2 + 1$.
Type $E_8$: $q^{16} + q^{14} + 2q^{12} + 4q^{10} + 2q^9 + 10q^8 + 2q^7 + 4q^6 + 2q^4 + q^2 + 1$.
Type $F_4$: $q^8 + 2q^6 + 6q^4 + 2q^2 + 1$.
Type $G_2$: $q^4 + 2q^2 + 1$.

Let $N_G$ be the variety consisting of all pairs $(g, g')$ where $g$ runs through the standard Steinberg cross section of the set of regular elements of $G$ and $g'$ is an element in the centralizer of $g$ in $G$ modulo the centre of $G$. (This variety, introduced in [L1, p.158], makes sense even if $k$ is replaced by the complex numbers. It plays a role in [FT] where it is called the universal centralizer.) According to [L1, p.158], the number of $F_q$-rational points of $N_G$ is equal to $N_{w,w}^e(q)$ hence it is given by the formulas above with $q = q$. 

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