Reachability of Black-Box Nonlinear Systems after Koopman Operator Linearization

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Presenter: Kostiantyn Potomkin
Presentation outline

• Motivation
• Koopman Operator
• Challenges
• Verifying linear systems with nonlinear observables
• Evaluation
• Conclusions
Motivation
Reliability and Safety
Koopman Operator
Example

Nonlinear dynamics:

\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= x_2 - x_1^2
\end{align*}
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Substitution:

\[ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix} \]
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Koopman linear system:

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\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix}
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Koopman operator:

\[
\mathcal{K}_t = e^{\tilde{\mathcal{K}}t}
\]
Example

\[ y_1(0) \in [1, 3] \]
\[ y_2(0) \in [0, 2] \]
\[ y_3(0) = y_1(0)^2 \in [1, 9] \]
Red dots – linear system, green curves – trajectories of the original nonlinear system, blue sets – output of Flow*
Challenges
• Obtain a Koopman linearized model of the nonlinear dynamics with a good approximation of the original system (ideally no approximation).
• Add a support of nonlinear initial state sets to state-of-the-art linear reachability algorithms.
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Verifying linear systems with nonlinear observables
Direct Encoding

\[ y = g(x_0) \]

State space of the observables

\[ y_t = K_t y \]

Original state space

\[ x_t = My_t \]
Overapproximating Nonlinear Constraints with Intervals

\[ y_1(0) \in [1, 3] \]
\[ y_3(0) = y_1(0)^2 \in [1, 9] \]
Hyperplane Backpropagation

\[ q^T y \leq r \]
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Hyperplane Backpropagation

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Zonotope Domain Splitting
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Evaluation
Implementation

- SMT Solver: dReal
- Programming Language: Julia
- Koopman Linearization: DataDrivenDiffEq.jl - github.com/SciML/DataDrivenDiffEq.jl
### Benchmarks

| Model Name                        | Number of original state variables | Number of observables |
|-----------------------------------|-------------------------------------|------------------------|
| Roessler                          | 3                                   | 70                     |
| Steam                             | 3                                   | 71                     |
| Coupled Van der Pol oscillator    | 4                                   | 131                    |
| Biological                        | 7                                   | 104                    |

From HyPro benchmark repository: [https://ths.rwth-aachen.de/research/projects/hypro/benchmarks-of-continuous-and-hybrid-systems/](https://ths.rwth-aachen.de/research/projects/hypro/benchmarks-of-continuous-and-hybrid-systems/)
Evaluation

Computational time (seconds) comparing Flow*, Direct Encoding and the Zonotope Domain Splitting. The dReal tool timed out on all models.

- dReal TO’s on all original nonlinear models
- Flow* outperforms Direct Encoding on most of the instances.
- Zonotope Domain Splitting outperforms all other tools on most of the instances.

| Name           | i   | Flow*    | Direct   | Zono   |
|----------------|-----|----------|----------|--------|
| Roessler       | 0   | 55.28    | 181.06   | 9.53   |
|                | 10  | 78.33    | 177.92   | 5.01   |
|                | 20  | 55.29    | 174.63   | 3.5    |
| Steam          | 0   | 61.06    | 197.08   | 182.62 |
|                | 5   | 285.20   | 59.53    | 37.27  |
|                | 10  | 77.68    | 29.21    | 18.52  |
| Coupled VP     | 1   | 251.11   | 788.45   | 0.57   |
|                | 8   | 497.61   | 680.61   | 53.91  |
|                | 16  | 1665.16  | 557.24   | 18.52  |
| Biological     | 1   | 260.69   | 470.59   | 0.59   |
|                | 5   | 250.26   | 426.37   | 49.41  |
|                | 10  | 238.56   | 427.00   | 179.25 |
Conclusions
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• We presented novel techniques to efficiently handle non-linear initial sets which demonstrate competitive results.
• Koopman operator can be used as part of reachability analysis workflow.